Abstract. This paper introduces a new duality theory that generalizes the classical Fenchel conjugation to functions defined on Riemannian manifolds. This notion of conjugation even yields a more general Fenchel conjugate for the case where the manifold is a vector space. We investigate its properties, e.g., the Fenchel–Young inequality and the characterization of the convex subdifferential using the analogue of the Fenchel–Moreau Theorem. These properties of the Fenchel conjugate are employed to derive a Riemannian primal-dual optimization algorithm, and to prove its convergence for the case of Hadamard manifolds under appropriate assumptions. Numerical results illustrate the performance of the algorithm, which competes with the recently derived Douglas–Rachford algorithm on manifolds of nonpositive curvature. Furthermore we show that our novel algorithm numerically converges on manifolds of positive curvature.

Key words. convex analysis, Fenchel conjugate function, Riemannian manifold, Hadamard manifold, primal-dual algorithm, Chambolle–Pock algorithm, total variation

AMS subject classifications.
et al., 2008. In addition, higher order differences or differentials can be taken into account, see for example Chan, Marquina, Mulet, 2000; Papafitsoros, Schönlieb, 2014 or most prominently the total generalized variation (TGV) Bredies, Kunisch, Pock, 2010. These models use the idea of the pre-dual formulation of the energy functional and Fenchel duality to derive efficient algorithms. Within the image processing community the resulting algorithms of primal-dual hybrid gradient type are often referred to as the Chambolle–Pock algorithm Chambolle, Pock, 2011.

In the recent years, optimization on Riemannian manifolds has gained a lot of interest. Starting in the 1980s and 90s with Udriște, 1994, optimization on Riemannian manifolds and corresponding algorithms have been investigated. For a comprehensive textbook on optimization on matrix manifolds, see Absil, Mahony, Sepulchre, 2008. With the emergence of manifold-valued imaging, for example in InSAR imaging Bürgmann, Rosen, Fielding, 2000, data consisting of orientations for example in electron backscattered diffraction (EBSD) Adams, Wright, Kunze, 1993; Kunze et al., 1993 or for diffusion tensors in magnetic resonance imaging (DT-MRI), for example discussed in Pennec, Fillard, Ayache, 2006, the development of optimization techniques and/or algorithms on manifolds (especially for non-smooth functionals) has gained a lot of attention. Within these applications, the same tasks appear as for classical, Euclidean imaging, such as denoising, inpainting or segmentation. Both Lellmann et al., 2013 as well as Weinmann, Demaret, Storath, 2014 introduced the total variation as a prior in a variational model for manifold-valued images. While the first extends a lifting approach previously introduced for cyclic data in Strekalovskiy, Cremers, 2011 to Riemannian manifolds, the latter introduces a cyclic proximal point algorithm (CPPA) to compute a minimizer of the variational model. Such an algorithm was previously introduced by Báčák, 2014 on CAT(0) spaces based on the proximal point algorithm introduced by Ferreira, Oliveira, 2002 on Riemannian manifolds. Based on these models and algorithms, higher order models have been derived Bergmann, Laus, et al., 2014; Báčák et al., 2016; Bergmann, Fitschen, et al., 2018; Bredies, Holler, et al., 2018. Using a relaxation, the half-quadratic minimization Bergmann, Chan, et al., 2016, also known as iteratively reweighted least squares (IRLS) Grohs, Sprecher, 2016, has been generalised to manifold-valued image processing tasks and employs a quasi-Newton method. Finally, the parallel Douglas–Rachford algorithm (PDRA) was introduced on Hadamard manifolds Bergmann, Persch, Steidl, 2016 and its convergence proof is, to the best of our knowledge, limited to manifolds with constant nonpositive curvature. Numerically, the PDRA still performs well on arbitrary Hadamard manifolds. However, for the classical Euclidean case the Douglas–Rachford algorithm is equivalent to applying the alternating directions method of multipliers (ADMM) Gabay, Mercier, 1976 on the dual problem and hence is also equivalent to the algorithm of Chambolle, Pock, 2011.

In this paper we introduce a new notion of Fenchel duality for Riemannian manifolds, which allows us to derive a conjugate duality theory for convex optimization problems posed on such manifolds. In the absence of a global concept of convexity on general Riemannian manifolds, our approach is local in nature. On so-called Hadamard manifolds, however, it is global.

The work closest to ours is Ahmadi Kakavandi, Amini, 2010, who introduce a Fenchel conjugacy-like concept on Hadamard metric spaces, using a quasilinearization map in terms of distances as the duality product. In contrast, our work makes use of intrinsic tools from differential geometry such as geodesics, tangent and cotan-
gent vectors to establish a conjugation scheme which extends the theory from locally convex vector spaces to Riemannian manifolds. We investigate the application of the correspondence of a primal problem

\[
\text{Minimize } F(p) + G(\Lambda p)
\]

to a suitably defined dual and derive a primal-dual algorithm on Riemannian manifolds. In the absence of a concept of linear operators between manifolds we follow the approach of Valkonen, 2014 and state an exact and a linearized variant of the newly established Riemannian Chambolle–Pock algorithm (RCPA). We then study its convergence on Hadamard manifolds.

As an example, we detail the algorithm for the anisotropic and isotropic total variation with squared distance data term, i.e., the ROF model on Riemannian manifolds. After illustrating the correspondence to the Euclidean (classical) Chambolle–Pock algorithm, we compare the numerical performance of the RCPA to the CPPA and the PDRA. While the latter has only been shown to converge on Hadamard manifolds of constant curvature, it performs quite well on Hadamard manifolds in general. On the other hand, the CPPA is known to converge possibly arbitrarily slowly; even in the Euclidean case. We illustrate that our linearized algorithm competes with the PDRA, but also performs similarly even on manifolds with non-negative curvature, like the sphere.

The remainder of the paper is organized as follows: Section 2 introduces proper results from convex analysis in \( \mathbb{R}^d \). In an effort to make the paper self-contained, we also briefly recall concepts from differential geometry. Section 3 is devoted to the development of a complete conjugation scheme for functions defined on manifolds. In this section we extend some classical results from convex analysis and locally convex vector spaces to manifolds, like the Fenchel–Moreau Theorem (also known as the Biconjugation Theorem) and useful characterizations of the subdifferential in terms of the conjugate function. In Section 4 we formulate the primal-dual hybrid gradient method (RCPA) for general optimization problems on manifolds involving non-linear operators. In this section we present an exact and a linearized formulation of our novel method and prove, under suitable assumptions, convergence for the linearized formulation to a minimizer (of a linearized problem) on arbitrary Hadamard manifolds. Section 5 focuses on the analysis of several total variation models on manifolds. In Section 6 we carry out numerical experiments to illustrate the performance of our novel primal-dual algorithm. Finally, we end with some conclusions and further remarks on future research in Section 7.

2. Preliminaries on Convex Analysis and Differential Geometry. In this section we review some preliminaries from convex analysis in \( \mathbb{R}^d \) as well as concepts of differential geometry. We also revisit the intersection of both topics, convex analysis on Riemannian manifolds, including subdifferential calculus.

2.1. Convex Analysis. In this subsection let \( f : \mathbb{R}^d \to \mathbb{R} \), where \( \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\} \) denotes the extended real line. For standard definitions like closedness, properness, lower semicontinuity (lsc) and convexity of \( f \) we refer the reader, e.g., to the textbooks Rockafellar, 1970; Bauschke, Combettes, 2011.
THEOREM 2.1 (Rockafellar, 1970, Thm. 12.1). A closed convex function \( f : \mathbb{R}^d \to \mathbb{R} \) is the pointwise supremum of the collection of all affine functions \( h : \mathbb{R}^d \to \mathbb{R} \) such that \( h \leq f \).

COROLLARY 2.2 (Rockafellar, 1970, Cor. 12.1.2). Given a proper, convex function \( f : \mathbb{R}^d \to \mathbb{R} \), there exists some \( b \in \mathbb{R}^d \) and \( \beta \in \mathbb{R} \) such that \( f(x) \geq \langle x, b \rangle - \beta \) for every \( x \in \mathbb{R}^d \).

DEFINITION 2.3. The Fenchel conjugate of a function \( f : \mathbb{R}^d \to \mathbb{R} \) is defined as the function \( f^* : \mathbb{R}^d \to \mathbb{R} \) such that

\[
(2.1) \quad f^*(x) := \sup_{x \in \mathbb{R}^d} \{ \langle x^*, x \rangle - f(x) \}.
\]

We recall the properties of the classical Fenchel conjugate function in the following lemma.

LEMMA 2.4 (Rockafellar, 1970, Chap. 12). Let \( f, g : \mathbb{R}^d \to \mathbb{R} \) be proper, convex functions and \( \alpha, \lambda \in \mathbb{R} \). Then the following statements hold.

i) \( f^* \) is a closed convex function.
ii) If \( f(x) \leq g(x) \) for all \( x \in \mathbb{R}^d \), then \( f^*(x^*) \geq g^*(x^*) \) for all \( x^* \in \mathbb{R}^d \).
iii) If \( g(x) = f(x) + \alpha \) for all \( x \in \mathbb{R}^d \), then \( g^*(x^*) = f^*(x^*) - \alpha \) for all \( x^* \in \mathbb{R}^d \).
iv) If \( g(x) = \lambda f(x) \) for all \( x \in \mathbb{R}^d \), then \( g^*(x^*) = \lambda f^*(x^*)/\lambda \) for all \( x^* \in \mathbb{R}^d \).
v) If \( g(x) = f(x) + b \) for all \( x \in \mathbb{R}^d \), then \( g^*(x^*) = f^*(x^*) - \langle x^*, b \rangle \).
vii) The Fenchel-Young inequality holds, i.e., for all \( x, x^* \in \mathbb{R}^d \) we have

\[
(2.2) \quad \langle x^*, x \rangle \leq f(x) + f^*(x^*).
\]

We now recall some results related to the definition of the subdifferential of a proper, convex function.

DEFINITION 2.5 (Rockafellar, 1970, Chap. 23). Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a proper, convex function. Its subdifferential is defined as

\[
(2.3) \quad \partial f(x) := \{ x^* \in \mathbb{R}^d \mid f(z) \geq f(x) + \langle x^*, z - x \rangle \text{ for all } z \in \mathbb{R}^d \}.
\]

THEOREM 2.6 (Rockafellar, 1970, Thm. 23.5). Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a proper, convex function and \( x \in \mathbb{R}^d \). Then, \( x^* \in \partial f(x) \) holds if and only if

\[
(2.4) \quad f(x) + f^*(x^*) = \langle x^*, x \rangle.
\]

COROLLARY 2.7 (Rockafellar, 1970, Cor. 23.5.1). Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a proper, convex function and \( x^* \in \mathbb{R}^d \). Then \( x \in \partial f^*(x^*) \) holds if and only if \( x^* \in \partial f(x) \).

The Fenchel biconjugate \( f^{**} : \mathbb{R}^d \to \mathbb{R} \) of a function \( f : \mathbb{R}^d \to \mathbb{R} \) is given by

\[
(2.5) \quad f^{**}(x) = (f^*)^*(x) = \sup_{x^* \in \mathbb{R}^d} \{ \langle x^*, x \rangle - f^*(x^*) \}.
\]

Finally, we conclude this section with the following result known as the Fenchel–Moreau or Biconjugation Theorem.

THEOREM 2.8 (Rockafellar, 1970, Thm. 12.2). Given a proper function \( f : \mathbb{R}^d \to \mathbb{R} \), the equality \( f^{**}(x) = f(x) \) holds if and only if \( f \) is a closed convex function.
2.2. Differential Geometry. This section is devoted to the collection of necessary concepts from differential geometry. For details concerning the subsequent definitions, see for example do Carmo, 1992; Lee, 2003; Jost, 2017.

Suppose that $M$ be a $d$-dimensional smooth manifold. The tangent space at $p \in M$ is a vector space of dimension $d$ and it is denoted by $T_pM$. Elements of $T_pM$, i.e., tangent vectors, will be denoted by $X_p$ and $Y_p$ etc. or simply $X$ and $Y$ when the base point is clear from the context. The disjoint union of all tangent spaces, i.e.,

\[(2.6) \quad TM := \bigcup_{p \in M} T_pM,\]

is called the tangent bundle of $M$. It is a smooth manifold of dimension $2d$.

The dual space of $T_pM$ is denoted by $T^*_pM$ and it is called the cotangent space to $M$ at $p$. The disjoint union

\[(2.7) \quad T^*M := \bigcup_{p \in M} T^*_pM\]

is known as the cotangent bundle. Elements of $T^*_pM$ are called cotangent vectors to $M$ at $p$ and they will be denoted by $\xi_p$ and $\eta_p$ or simply $\xi$ and $\eta$. The natural duality product between $X \in T_pM$ and $\xi \in T^*_pM$ is denoted by $\langle \xi, X \rangle = \xi(X) \in \mathbb{R}$.

We suppose that $M$ is equipped with a Riemannian metric, i.e., a smoothly varying family of inner products on the tangent spaces $T_pM$. The metric at $p \in M$ is denoted by $(\cdot, \cdot)_p : T_pM \times T_pM \to \mathbb{R}$. The induced norm on $T_pM$ is denoted by $\|\cdot\|_p$. The Riemannian metric furnishes a linear bijective correspondence between the tangent and cotangent spaces via the Riesz map and its inverse, the so-called musical isomorphisms; see Lee, 2003, Chap. 8. They are defined as

\[(2.8) \quad \flat : T_pM \ni X \mapsto X^\flat \in T^*_pM\]

satisfying

\[(2.9) \quad \langle X^\flat, Y \rangle = (X, Y)_p \quad \text{for all} \quad Y \in T_pM,\]

and its inverse,

\[(2.10) \quad \sharp : T^*_pM \ni \xi \mapsto \xi^\sharp \in T_pM\]

satisfying

\[(2.11) \quad (\xi^\sharp, Y)_p = \langle \xi, Y \rangle \quad \text{for all} \quad Y \in T_pM.\]

The musical isomorphisms further introduce an inner product and an associated norm on the cotangent space $T^*_pM$, which we will also denote by $(\cdot, \cdot)_p$ and $\|\cdot\|_p$, since it is clear which inner product or norm we refer to based on the respective arguments.

A curve $c : [0, 1] \to M$ satisfying $\|c\|_c(t)$ is constant for $t \in [0, 1]$ is called a geodesic curve, or simply a geodesic. We say that a geodesic connects $p$ to $q$ if $c(0) = p$ and $c(1) = q$ holds. Notice that a geodesic connecting $p$ to $q$ need not always exist, and it if exists, it need not be unique. If a geodesic connecting $p$ to $q$ exists, there also
exists a shortest geodesic among them, which may in turn not be unique. If it is, we
denote the unique minimal geodesic connecting \( p \) and \( q \) by \( \gamma_{\widetilde{p,q}} \).

Minimal geodesics introduce a notion of metric \( d_M \) on \( M \) via their length, which
may assume the value \( +\infty \) if \( p \) and \( q \) are not connected by a geodesic. We denote by
\begin{equation}
\mathcal{B}_r(p) := \{ y \in M \mid d_M(p,q) < r \}
\end{equation}
the open metric ball of radius \( r > 0 \) with center \( p \in M \).

We denote by \( \gamma_{p,X} : (-\varepsilon,\varepsilon) \to M \) a geodesic starting at \( p \) with \( \dot{\gamma}_{p,X}(0) = X \)
where \( X \in T_pM \). We denote the subset of \( T_pM \) where these geodesics are well defined
until \( t = 1 \) by \( \mathcal{G}_p \). A Riemannian manifold is said to be complete if \( \mathcal{G}_p = T_pM \).

The exponential map is defined as the function \( \exp_p : \mathcal{G}_p \to M \) with \( \exp_p X := \gamma_{p,X}(1) \).
Note that \( \exp_p(tX) = \gamma_{p,X}(t) \) holds for every \( t \in [0,1] \). We further introduce
the set \( \mathcal{G}'_p \subset M \) such that \( \exp_p : \mathcal{G}'_p \to \exp_p(\mathcal{G}_p) \) is a diffeomorphism. The logarithmic map
is defined as the inverse function of the exponential map, i.e., \( \log_p : \exp_p(\mathcal{G}'_p) \to \mathcal{G}'_p \subset T_pM \).

By a Euclidean space we mean \( \mathbb{R}^d \) (where \( T_p\mathbb{R} = \mathbb{R} \) holds), equipped with the
Riemannian metric given by the Euclidean inner product. In this case, \( \exp_p X = p+X \)
and \( \log_p q = q - p \) hold.

Finally, given \( p,q \in M \) and \( X \in \mathcal{T}_pM \), we denote by \( \mathcal{P}_{p\to q}X \) the so-called parallel
transport of \( X \) along a unique minimal geodesic \( \gamma_{\tilde{p},\tilde{q}} \). Using the musical isomorphisms
presented above, we can also parallelly transport cotangent vectors along geodesics
according to
\begin{equation}
\mathcal{P}_{p\to q}\xi_p := \left( \mathcal{P}_{p\to q}(\xi_p^1) \right)^b.
\end{equation}

In the particular case where the sectional curvature of the manifold is nonpositive
everywhere, all geodesics connecting distinct points are unique. If furthermore, the
manifold is complete, the manifold is called Hadamard manifold, see Bačák, 2014,
Def. 1.2.3. Then the exponential and logarithmic maps are defined globally.

### 2.3. Convex Analysis on Riemannian Manifolds.
Throughout this section, \( M \) is assumed to be Riemannian manifold. In this subsection we recall the basic
concepts of convex analysis on manifolds. The central idea is to replace straight lines
in the definition of convex sets in Euclidean vector spaces by geodesics.

**Definition 2.9 (Sakai, 1996, Def. IV.5.1).** A subset \( C \subset M \) of a Riemannian manifold \( M \) is said to be strongly convex if for all \( p,q \in C \) minimal geodesics exist, are unique, and lie completely in \( C \).

**Definition 2.10.** Let \( C \subset M \). We introduce the tangent subset \( \exp_p^{-1}(C) \subset \mathcal{T}_pM \), \( p \in C \), as
\[
\exp_p^{-1}(C) := \{ X \in \mathcal{T}_pM \mid \exp_p X \in C \}.
\]

Note that if \( C \) is strongly convex, the exponential and logarithmic map introduce
bijections between \( C \) and \( \exp_p^{-1}(C) \) for any \( p \in C \).
A function $F: \mathcal{M} \to \mathbb{R}$ is proper if $\text{dom} F := \{x \in \mathcal{M} \mid F(x) < \infty\} \neq \emptyset$ and $F(x) > -\infty$ holds for all $x \in \mathcal{M}$.

The following definition states the important concept of convex functions on Riemannian manifolds. From now on until the end of Section 2, suppose that $\mathcal{C}$ is a strongly convex subset of $\mathcal{M}$.

**Definition 2.11 (Sakai, 1996, Def. 5.9).** A proper function $F: \mathcal{M} \to \mathbb{R}$ is called convex on a strongly convex set $\mathcal{C} \subset \mathcal{M}$ if for all $p, q \in \mathcal{C}$ the composition $F \circ \gamma_{p,q}$ is a convex function on $[0,1]$ in the classical sense. Similarly $F$ is called strictly or strongly convex if $F \circ \gamma_{p,q}$ fulfills these properties.

The epigraph of a function $f: \mathcal{M} \to \mathbb{R}$ is defined as

$$\text{epi } F := \{(x, \alpha) \in \mathcal{M} \times \mathbb{R} \mid F(x) \leq \alpha\}.$$  
(2.14)

Similarly, $\mathcal{M}$ can be replaced here by a subset of $\mathcal{M}$. A function $F: \mathcal{C} \to \mathbb{R}$ is convex if and only if $\text{epi } F \cap (\mathcal{C} \times \mathbb{R})$ is a strongly convex set in the product Riemannian manifold $\mathcal{M} \times \mathbb{R}$.

A proper function $F: \mathcal{C} \to \mathbb{R}$ is called lower semicontinuous (lsc) if $\text{epi } F$ is closed. Equivalently, the composition $F \circ \exp_{m}^{-1}: \exp_{m}^{-1}(\mathcal{C}) \to \mathbb{R}$ is lsc in the classical sense for any $m \in \mathcal{C}$.

We now recall the notion of the subdifferential of a convex function defined on a Riemannian manifold.

**Definition 2.12 (Udrişte, 1994, Def. 4.4).** For a proper, convex function $F: \mathcal{C} \to \mathbb{R}$ on the strongly convex set $\mathcal{C} \subset \mathcal{M}$, the subdifferential $\partial_{\mathcal{M}} F$ at a point $p \in \mathcal{M}$ is given by

$$\partial_{\mathcal{M}} F(p) := \{\xi \in (\exp_{p}^{-1}(\mathcal{C}))^{\flat} \mid F(q) \geq F(p) + \langle \xi, \log_{p} q \rangle \text{ for all } q \in \mathcal{C}\}.$$  
(2.15)

In the above notation the index $\mathcal{M}$ refers to the fact that it is the Riemannian subdifferential; the set $\mathcal{C}$ should always be clear from the context.

**Definition 2.13.** Suppose that $\mathcal{C} \subset \mathcal{M}$ is strongly convex and closed. The projection of $p \in \mathcal{M}$ onto $\mathcal{C}$ is defined as the set-valued function $P_{\mathcal{C}}(\cdot): \mathcal{M} \to \mathcal{C}$ such that

$$P_{\mathcal{C}}(p) := \text{Arg min}_{q \in \mathcal{C}} d_{\mathcal{M}}(p, q).$$  
(2.16)

Note that although the projection mapping is possibly set-valued, we still have a generalization of the classical variational inequality which characterizes certain solutions of (2.16); see for instance Bauschke, Combettes, 2011, Thm. 3.14 for the Hilbert space case.

**Proposition 2.14.** Suppose that $\mathcal{C} \subset \mathcal{M}$ is strongly convex and closed and $m \in \mathcal{C}$. Then there exists an open ball $\mathcal{B}_{r}(m)$ such that for any $p \in \mathcal{B}_{r}(m)$, $m \in P_{\mathcal{C}}(p)$ holds if and only if

$$\langle \log_{m} p, X \rangle_{m} = 0 \text{ for all } X \in \exp_{m}^{-1}(\mathcal{C} \cap \mathcal{B}_{r}(m)).$$  
(2.17)
Proof. The proof can be directly adapted from Li et al., 2009, Thm. 3.2, where it was shown even for weakly convex sets. We use the fact that \( \log_m \) is well defined near \( m \).

As a corollary, we apply the result above by using the epigraph of an appropriate function as the set \( C \) to project onto.

Corollary 2.15. Suppose that \( C \subset M \) is strongly convex and closed and \( m \in C \). Moreover, suppose that \( F: C \to \mathbb{R} \) is proper, convex and lsc. Then there exists an open ball \( B_r(m) \) such that the following holds. Whenever \( p \in C \cap B_r(m) \) and \( \alpha < F(p) \), then \( (m, \beta) \in P_{\text{opt}} F(p, \alpha) \) if and only if \( \alpha < F(m) = \beta \) and

\[
(2.18) \quad (\log_m p, X)_m \leq (\beta - \alpha)(F(\exp_m X) - \beta) \quad \text{for all } X \in \exp_m^{-1}(C \cap B_r(m)).
\]

Proof. The proof can be performed employing Proposition 2.14 in the same way as in Bauschke, Combettes, 2011, Cor. 9.18, where it is stated for functions defined on Hilbert spaces.

Remark 2.16. Due to Li et al., 2009, Cor. 3.1 the open ball \( B_r(m) \) can be chosen in a way that \( C \subset B_r(m) \). For the ease of notation we will assume such inclusion from now on in the rest of the paper. Hence (2.18) becomes

\[
(2.19) \quad (\log_m p, X)_m \leq (\beta - \alpha)(F(\exp_m X) - \beta) \quad \text{for all } X \in \exp_m^{-1}(C).
\]

We further recall the definition of the proximal map, that was generalized to Hadamard manifolds in Ferreira, Oliveira, 2002.

Definition 2.17. Let \( M \) be a Riemannian manifold, \( F: M \to \mathbb{R} \) be proper, and \( \lambda > 0 \). The proximal map is defined as

\[
(2.20) \quad \text{prox}_\lambda F(p) := \underset{q \in M}{\text{Arg min}} \left\{ \frac{1}{2} d^2_M(p, q) + \lambda F(q) \right\}.
\]

Notice that on Hadamard manifolds, the proximal map is single-valued; see Bácák, 2014, Chap. 2.2 for details. The following lemma is used later on to characterize the proximal map using the subdifferential on Hadamard manifolds.

Lemma 2.18 (Ferreira, Oliveira, 2002, Thm. 5.1). Let \( F: M \to \mathbb{R} \) be a proper, convex function on the Hadamard manifold \( M \). Then, the equality \( q = \text{prox}_\lambda F(p) \) is equivalent to

\[
(2.21) \quad \frac{1}{\lambda}(\log_q p)^\flat \in \partial_M F(q).
\]

Remark 2.19. When \( F: C \to \mathbb{R} \) is defined on a strongly convex set \( C \subset M \), the characterization Lemma 2.18 continues to hold. The main reason is that Proposition 2.14 replaces Ferreira, Oliveira, 2002, Cor. 3.1 in this setting.

3. Fenchel Conjugation Scheme on Manifolds. In this section we present a novel Fenchel conjugation scheme for functions defined on manifolds \( F: M \to \mathbb{R} \). We generalize ideas from Bertsekas, 1978, who defined local conjugation on manifolds embedded in \( \mathbb{R}^d \).

Throughout this section, suppose that \( M \) is a Riemannian manifold and \( C \subset M \) is strongly convex. The definition of the Fenchel conjugate of \( F \) is motivated by Theorem 2.1.
**Definition 3.1.** Suppose that \( F: \mathcal{C} \to \mathbb{R} \) and \( m \in \mathcal{C} \). The \( m \)-Fenchel conjugate of \( F \) is defined as the function \( F^*_m: \mathcal{T}_m^* \mathcal{M} \to \mathbb{R} \) by

\[
F^*_m(\xi_m) := \sup_{X \in \exp_m^{-1}(\mathcal{C})} \{ \langle \xi_m, X \rangle - F(\exp_m X) \}, \quad \xi_m \in \mathcal{T}_m^* \mathcal{M}
\]

Remark 3.2. Note that the Fenchel conjugate \( F^*_m \) depends on both the strongly convex set \( \mathcal{C} \) and on the base point \( m \). Observe as well that when \( \mathcal{M} \) is a Hadamard manifold, it is possible to have \( \mathcal{C} = \mathcal{M} \). In the particular case of the Euclidean space \( \mathcal{M} = \mathcal{C} = \mathbb{R}^d \), Definition 3.1 becomes

\[
F^*_m(\xi) = \sup_{X \in \mathbb{R}^d} \{ \langle \xi, X \rangle - F(m + X) \} = \sup_{Y \in \mathbb{R}^d} \{ \langle \xi, Y - m \rangle - F(Y) \} = F^*(\xi) - \langle \xi, m \rangle
\]

for \( \xi \in \mathbb{R}^d \). Hence, taking \( m \) to be the zero vector we recover the classical (Euclidean) conjugate \( F^* \) from Definition 2.3.

We first establish a result regarding the properness of the \( m \)-conjugate function.

**Lemma 3.3.** Suppose that \( F: \mathcal{C} \to \mathbb{R} \) and \( m \in \mathcal{C} \). If \( F^*_m \) is proper, then \( F \) is also proper.

**Proof.** Applying Definition 3.1 we get

\[
F^*_m(\xi_m) := \sup_{X \in \exp_m^{-1}(\mathcal{C})} \{ \langle \xi_m, X \rangle - F(\exp_m X) \}.
\]

Since \( F^*_m \) is proper by assumption we can suppose without loss of generality that \( \xi_m \in \text{dom} F^*_m \). Hence

\[
\sup_{X \in \exp_m^{-1}(\mathcal{C})} \{ \langle \xi_m, X \rangle - F(\exp_m X) \} < +\infty,
\]

so there must exist at least one \( \bar{X} \in \exp_m^{-1}(\mathcal{C}) \) such that \( F(\exp_m \bar{X}) \in \mathbb{R} \). This shows that \( F \neq +\infty \). On the other hand, let \( p \in \mathcal{C} \) and take \( X := \log_m p \). If \( F(p) \) were equal to \(-\infty\), then \( F^*_m(\xi_m) = +\infty \) for any \( \xi_m \in \mathcal{T}_m^* \mathcal{M} \), which would contradict the properness of \( F^*_m \).

**Definition 3.4.** Suppose that \( F: \mathcal{C} \to \mathbb{R} \) and \( m, m' \in \mathcal{C} \). Then the \( (mm') \)-Fenchel biconjugate function \( F^{***}_{mm'}: \mathcal{C} \to \mathbb{R} \) is defined as

\[
F^{***}_{mm'}(p) = \sup_{\xi_m \in \mathcal{T}_m^* \mathcal{M}} \{ \langle \xi_m, \log_{m'} p \rangle - F^*_m(\exp_{m'} \bar{m} \xi_m) \}, \quad p \in \mathcal{C}.
\]

**Lemma 3.5.** Suppose that \( F: \mathcal{C} \to \mathbb{R} \) and \( m \in \mathcal{C} \). Then \( F^{***}_{mm}(p) \leq F(p) \) holds for all \( p \in \mathcal{C} \).

**Proof.** Applying (3.2), we have

\[
F^{***}_{mm}(p) = \sup_{\xi_m \in \mathcal{T}_m^* \mathcal{M}} \{ \langle \xi_m, \log_m p \rangle - F^*_m(\xi_m) \}
= \sup_{\xi_m \in \mathcal{T}_m^* \mathcal{M}} \{ \langle \xi_m, \log_m p \rangle - \sup_{X \in \exp_m^{-1}(\mathcal{C})} \{ \langle \xi_m, X \rangle - F(\exp_m X) \} \}
= \sup_{\xi_m \in \mathcal{T}_m^* \mathcal{M}} \{ \langle \xi_m, \log_m p \rangle + \inf_{X \in \exp_m^{-1}(\mathcal{C})} \{ -\langle \xi_m, X \rangle + F(\exp_m X) \} \}
\leq \sup_{\xi_m \in \mathcal{T}_m^* \mathcal{M}} \{ \langle \xi_m, \log_m p \rangle - \langle \xi_m, \log_m p \rangle + F(\exp_m(\log_m p)) \}
= F(p).
\]
The following lemma collects some properties that Definition 3.1 inherits from the classical convex case presented in Lemma 2.4.

**Lemma 3.6.** Let \( F, G : C \to \mathbb{R} \) be two functions and suppose that \( m \in C \) and \( \alpha \in \mathbb{R}, \lambda > 0 \) holds. Then the following statements hold.

i) The \( m \)-conjugate \( F_m^* \) is proper, convex and lsc.

ii) If \( F(p) \geq G(p) \) for all \( p \in C \), then \( F_m^*(\xi_m) \leq G_m^*(\xi_m) \) for all \( \xi_m \in T_m^*M \).

iii) If \( G(p) = F(p) + \alpha \) for all \( p \in C \), then \( G_m^*(\xi_m) = F_m^*(\xi_m) - \alpha \) for all \( \xi_m \in T_m^*M \).

iv) If \( G(p) = \lambda F(p) \) for all \( p \in C \), then \( G_m^*(\xi_m) = \lambda F_m^*(\xi_m) \) for all \( \xi_m \in T_m^*M \).

v) It holds \( F_{mmm}^*(\xi_m) = F_m^*(\xi_m) \) for all \( \xi_m \in T_m^*M \).

**Proof.** Let us start with ii) since i) is a direct application of Bauschke, Combettes, 2011, Prop. 13.11 to functions defined on a manifold \( M \). If \( F(p) \geq G(p) \) for all \( p \in C \), then it also holds \( F(\exp_m X) \geq G(\exp_m X) \) for every \( X \in \exp_m^{-1}(C) \). Then we have for any \( \xi_m \in T_m^*M \) that

\[
F_m^*(\xi_m) = \sup_{X \in \exp_m^{-1}(C)} \{\langle \xi_m, X \rangle - F(\exp_m X)\} \\
\leq \sup_{X \in \exp_m^{-1}(C)} \{\langle \xi_m, X \rangle - G(\exp_m X)\} = G_m^*(\xi_m).
\]

Similarly, we prove iii). Let us suppose now that \( G(p) = F(p) + \alpha \) for all \( p \in C \). Then \( G(\exp_m X) = F(\exp_m X) + \alpha \) for every \( X \in \exp_m^{-1}(C) \). Hence, for any \( \xi_m \in T_m^*M \) we obtain

\[
G_m^*(\xi_m) = \sup_{X \in \exp_m^{-1}(C)} \{\langle \xi_m, X \rangle - G(\exp_m X)\} \\
= \sup_{X \in \exp_m^{-1}(C)} \{\langle \xi_m, X \rangle - F(\exp_m X) + \alpha\} \\
= \sup_{X \in \exp_m^{-1}(C)} \{\langle \xi_m, X \rangle - F(\exp_m X)\} - \alpha = F_m^*(\xi_m) - \alpha.
\]

Let us now prove iv). Suppose that \( \lambda > 0 \) and \( G(\exp_m X) = \lambda F(\exp_m X) \) for every \( X \in \exp_m^{-1}(C) \). Then we have for any \( \xi_m \in T_m^*M \) that

\[
G_m^*(\xi_m) = \sup_{X \in \exp_m^{-1}(C)} \{\langle \xi_m, X \rangle - G(\exp_m X)\} \\
= \sup_{X \in \exp_m^{-1}(C)} \{\langle \xi_m, X \rangle - \lambda F(\exp_m X)\} \\
= \lambda \sup_{X \in \exp_m^{-1}(C)} \{\langle \xi_m, \frac{X}{\lambda} \rangle - F(\exp_m X)\} = \lambda F_m^*(\xi_m).
\]

Finally, we prove v). Applying Lemma 3.5 to \( F \), we get \( F_{mmm}^* \leq F^* \), and due to ii) we get \( F_{mmm}^* \geq F_m^* \). The converse inequality is ensured by a direct application of Lemma 3.5 to \( F_m^* \). \( \Box \)

**Proposition 3.7.** Suppose that \( F : C \to \mathbb{R} \) and \( m \in C \). Then

\[
F(p) + F_m^*(\xi_m) \geq \langle \xi_m, \log_m p \rangle
\]

holds for all \( p \in C \) and \( \xi_m \in T_m^*M \).
Proof. Suppose that \( \xi_m \in T^*_m \mathcal{M} \) and \( p \in \mathcal{C} \) holds and set \( X := \log_m p \). From Definition 3.1 we obtain
\[
F^*_m(\xi_m) \geq \langle \xi_m, \log_m p \rangle - F(\exp_m(\log_m p)),
\]
which implies (3.3) after rearranging terms.

We continue introducing the manifold counterpart of the Fenchel–Moreau Theorem, see Theorem 2.8.

**Theorem 3.8.** Let \( F: \mathcal{C} \to \mathbb{R} \) be a proper, convex function and \( m \in \mathcal{C} \). Then there exists an open ball \( \mathcal{B}_r(m) \) containing \( \mathcal{C} \) such that the following holds.

i) \( F(p) = F^**_{mm}(p) \) for all \( p \in \mathcal{C} \) if and only if \( F \) is lsc on \( \mathcal{C} \).

ii) \( F^**_{mm} \) is proper if \( F \) is.

**Proof.** Due to Remark 2.16, there exists a set \( \mathcal{C} \subset \mathcal{B}_r(m) \). We will prove the result along the lines of Bauschke, Combettes, 2011, Thm. 13.32. First of all, let us suppose that \( F(p) = F^**_{mm}(p) \) for all \( p \in \mathcal{C} \). Due to Lemma 3.6 i), \( F \) is proper, convex, and lsc since it is an \( m \)-Fenchel conjugate.

Let us prove the converse. To this end, let \( \mathcal{C} \subset \mathcal{B}_r(m) \) be a ball as in Corollary 2.15. Let \( F \) be a proper, convex, lsc function on \( \mathcal{C} \). Fix any \( p \in \mathcal{C} \) and \( \alpha < F(m) \), and set \( \eta \in \text{Pep}_F(p, \alpha) \).

Taking the supremum over \( X \in \exp^{-1}_m(\mathcal{C}) \), we get
\[
F^*_m(\xi'_m) \leq -\beta + \langle \xi'_m, \eta \rangle \quad \text{for all } \eta \in \exp^{-1}_m(\mathcal{B}).
\]

In particular, it also holds for \( \eta_m = \xi_m \) and since \( \beta > \alpha \),

\[
F^*_m(\xi'_m) \leq -\beta + \langle \frac{1}{1-\alpha} \xi_m, \eta_m \rangle.
\]

Now, we use the property of the musical isomorphisms, (2.8) and (2.10), to obtain the equalities \( (X, Y)_m = (X^\flat, Y^\flat)_m \) and \( (\xi_m, \eta_m)_m = (\xi^\flat_m, \eta^\flat_m)_m \). Then, using the fact that the parallel transport does not change the value of the Riemannian metric, we get
\[
(\xi^\flat_m, \eta^\flat_m)_m = \langle \xi'_m, \log_m p \rangle.
\]

Then, (3.7) becomes
\[
F^*_m(\xi'_m) \leq -\beta + \langle \xi'_m, \log_m p \rangle.
\]

Rearranging terms, we get
\[
\beta \leq \langle \xi'_m, \log_m p \rangle - F^*_m(\xi'_m) \leq F^**_{mm}(p).
\]

If \( \beta > \alpha \), we obtain
\[
F^**_{mm}(p) \geq \beta > \alpha.
\]
Let us show that \( F_{mm}^*(p) = F(p) \). Since \( \text{dom } F \cap C \neq \emptyset \), we first look at \( p \in \text{dom } F \cap C \). Since \( (m, \beta) \in P_{\text{epi } F}(p, \alpha) \), \( \beta = F(m) > \alpha \), and due to Lemma 3.5 and (3.8) it follows that \( F(p) \geq F_{mm}^*(p) \geq \alpha \). Since \( \alpha < F(p) \) and the fact that \( \alpha \) can be chosen arbitrarily, we finally get \( F_{mm}^*(p) = F(p) \) on \( \text{dom } F \cap C \).

Let us suppose that \( p \notin \text{dom } F \) but \( p \in C \). Notice that \( p \) must be always in \( \mathcal{B} \) since \( \log_m p \) has to be well defined. If \( \beta > \alpha \), by (3.8) we get \( F_{mm}^* > \alpha \) and since \( \alpha \) can be chosen arbitrarily, \( F_{mm}(p) = F(p) = +\infty \). Otherwise, if \( \beta = \alpha \) and due to \( (p, \alpha) \notin \text{epi } F \) but \( (m, \beta) \in \text{epi } F \), we have \( d_M(p, m) > 0 \) so the tangent vector \( \log_m p \in \exp_{m}^{-1}(C) \) has positive length, i.e., \( \|\log_m p\|_m > 0 \). Now, fix \( \xi_m \in \text{dom } F_m^* \) with \( \xi_m \in \exp_{m}^{-1}(\mathcal{B}) \). Observe that if \( \xi_m \in \exp_{m}^{-1}(\mathcal{B}) \) with \( m \neq m' \), the biconjugate would be \( F_{mm'}^* \) and we want to recover \( F_{mm}^* \). Then, by Definition 3.1 and using Proposition 2.14 we obtain

\[
\langle \mu_m, X \rangle - F(\exp_m X) \leq F_m^*(\mu_m), \tag{3.9}
\]

\[
\langle \xi_m, X \rangle \leq 0
\]

for all \( X \in \exp_{m}^{-1}(C) \) for which we recall that \( \xi_m \in \exp_{m}^{-1}(\mathcal{B}) \) such that \( \xi_m^* = \log_m p \). Taking \( c > 0 \) and combining the two inequalities from (3.9) we get

\[
\langle \mu_m, X \rangle + \langle c \xi_m, X \rangle - F(\exp_m X) \leq F_m^*(\mu_m), \tag{3.10}
\]

also for every (fixed) \( \mu_m \in \text{dom } F_m^* \) with \( \mu_m \in \exp_{m}^{-1}(\mathcal{B}) \). We analyze (3.10) with the particular choice \( \mu_m = \xi_m \in \exp_{m}^{-1}(\mathcal{B}) \) and we get for all \( X \in \exp_{m}^{-1}(C) \) that

\[
\langle \xi_m (1 + c), X \rangle - F(\exp_m X) \leq F_m^*(\xi_m)
\]

\[
= F_m^*(\xi_m) + \langle (1 + c) \xi_m, \log_m p \rangle
\]

\[
- \langle (1 + c) \xi_m, \log_m p \rangle.
\]

Taking the supremum over \( X \in \exp_{m}^{-1}(C) \) yields

\[
F_m^*((1 + c) \xi_m) \leq F_m^*(\xi_m) + \langle (1 + c) \xi_m, \log_m p \rangle - \langle (1 + c) \xi_m, \log_m p \rangle,
\]

and rearranging the above inequality, we obtain

\[
\langle (1 + c) \xi_m, \log_m p \rangle - F_m^*((1 + c) \xi_m) \geq \langle (1 + c) \xi_m, \log_m p \rangle - F_m^*(\xi_m),
\]

which implies that

\[
F_{mm}^*(p) \geq \langle (1 + c) \xi_m, \log_m p \rangle - F_m^*(\xi_m).
\]

Analyzing the summand \( \langle (1 + c) \xi_m, \log_m p \rangle \), we get

\[
\langle (1 + c) \xi_m, \log_m p \rangle = (1 + c) \langle \xi_m, \log_m p \rangle = (1 + c) \langle \xi_m^*, \log_m p \rangle_m
\]

\[
= (1 + c) \langle \log_m p, \log_m p \rangle_m = (1 + c) \|\log_m p\|_m > 0.
\]

Thus, the positivity of \( \langle (1 + c) \xi_m, \log_m p \rangle \) implies that \( F_{mm}^*(p) = F(p) = \infty \) also in the last case where \( p \notin \text{dom } F \) but \( p \in C \). Finally, the last part of the theorem is a straightforward application of Lemma 3.5.

We now address the manifold counterpart of Theorem 2.6.
**Theorem 3.9.** Let \( F: \mathcal{C} \to \mathbb{R} \) be a proper, convex function and \( m, p \in \mathcal{C} \). Then, \( \xi_p \in \partial_M F(p) \) holds if and only if

\[
(3.11) \quad F(p) + F_m^*(\mathcal{P}_{p \to m} \xi_p) = \langle \mathcal{P}_{p \to m} \xi_p, \log_m p \rangle.
\]

**Proof.** From (3.11), we get

\[
F(p) + F_m^*(\mathcal{P}_{p \to m} \xi_p) = \langle \mathcal{P}_{p \to m} \xi_p, \log_m p \rangle = \langle \mathcal{P}_{m \to p} \mathcal{P}_{p \to m} \xi_p, \mathcal{P}_{m \to p} \log_m p \rangle = \langle \xi_p, -\log_p m \rangle = -\langle \xi_p, \log_p m \rangle
\]

and hence

\[
(3.12) \quad F(p) + \langle \xi_p, \log_p m \rangle = -F_m^*(\mathcal{P}_{p \to m} \xi_p).
\]

On the other hand, for every \( m \in \mathcal{M} \),

\[
-F(m) = \langle \mathcal{P}_{p \to m} \xi_p, 0_m \rangle - F(\exp_m 0_m) \leq \sup_{X \in \exp_m(\mathcal{C})} \{ \langle \mathcal{P}_{p \to m} \xi_p, X \rangle - F(\exp_m X) \} = F_m^*(\mathcal{P}_{p \to m} \xi_p),
\]

and hence

\[
(3.13) \quad F(m) \geq -F_m^*(\mathcal{P}_{p \to m} \xi_p).
\]

Combining (3.12) and (3.13), we get that

\[
(3.14) \quad F(p) + \langle \xi_p, \log_p m \rangle = -F_m^*(\mathcal{P}_{p \to m} \xi_p) \leq F(m),
\]

holds true for all \( m \in \mathcal{C} \), or equivalently, \( \xi_p \in \partial_M F(p) \) according to Definition 2.12. \( \square \)

Given \( F: \mathcal{C} \to \mathbb{R} \) and \( m \in \mathcal{C} \), we can state the subdifferential from Definition 2.12 for the conjugate function \( F_m^*: T_m^* \mathcal{M} \to \mathbb{R} \),

\[
(3.15) \quad \partial_M F_m^*(\xi_m) := \{ X \in T_m^* \mathcal{M} : F_m^*(\eta_m) \geq F_m^*(\xi_m) + \langle X, \eta_m - \xi_m \rangle \text{ for all } \eta_m \in T_m^* \mathcal{M} \}.
\]

Before providing the manifold counterpart of Corollary 2.7, let us show how Theorem 3.9 looks like for \( F_m^* \) instead of \( F \).

**Corollary 3.10.** Let \( F: \mathcal{C} \to \mathbb{R} \) be a proper, convex and lsc function and \( m, p \in \mathcal{C} \). Then, for all \( \zeta_m \in T_m^* \mathcal{M} \) it holds

\[
(3.16) \quad \log_m p \in \partial_M F_m^*(\zeta_m) \iff F_m^*(\zeta_m) + F(p) = \langle \zeta_m, \log_m p \rangle.
\]

**Proof.** Similarly as in the proof of Theorem 3.9, for every \( \eta_m \in T_m^* \mathcal{M} \), it holds

\[
\log_m p \in \partial_M F_m^*(\zeta_m) \iff F_m^*(\eta_m) \geq F_m^*(\zeta_m) + \langle \log_m p, \eta_m - \zeta_m \rangle \iff -F_m^*(\zeta_m) + \langle \log_m p, \zeta_m \rangle \geq \langle \log_m p, \eta_m \rangle - F_m^*(\eta_m).
\]

Taking the supremum over \( \eta_m \), we get

\[
\log_m p \in \partial_M F_m^*(\zeta_m) \iff -F_m^*(\zeta_m) + \langle \log_m p, \zeta_m \rangle \geq F_{mm}^{**}(p) \iff \langle \log_m p, \zeta_m \rangle \geq F_m^*(\zeta_m) + F_{mm}^{**}(p).
\]
Hence, applying Proposition 3.7 and using that \( F \) is proper, convex, and lsc, Theorem 3.8 implies that
\[
\langle \log m p, \zeta_m \rangle \geq F^*_m(\xi_p) + F(p) \geq \langle \log m p, \zeta_m \rangle,
\]
and we finally get the right-hand-side of (3.16). The converse follows in a straightforward way since in particular we have
\[
\langle \log m p, \zeta_m \rangle \geq F^*_m(\zeta_m) + F^{**}_{mm}(p),
\]
which completes the proof.

To conclude this section the following result shows the symmetric relation between the conjugate function and the subdifferential when the involved function is proper, convex, and lsc. This generalizes Corollary 2.7.

Corollary 3.11. Let \( F: \mathcal{C} \to \mathbb{R} \) be a proper, convex and lsc function and \( m, p \in \mathcal{C} \). Then it holds
\[
(3.17) \quad \xi_p \in \partial_M F(p) \iff \log m p \in \partial_M F^*_m(\mathcal{P}_{p \to m} \xi_p).
\]

Proof. The proof is a straightforward combination of Theorem 3.9 and taking as a particular cotangent vector \( \eta_m = \mathcal{P}_{p \to m} \xi_p \) in Corollary 3.10. \( \square \)

4. Optimization on Manifolds. In this section we derive a primal-dual optimization algorithm to solve minimization problems on manifolds of the form
\[
(4.1) \quad \text{Minimize} \quad F(p) + G(\Lambda p), \quad \ p \in \mathcal{C}.
\]

Here \( \mathcal{C} \subset \mathcal{M} \) and \( \mathcal{D} \subset \mathcal{N} \) are strongly convex sets, \( F: \mathcal{C} \to \mathbb{R} \) and \( G: \mathcal{D} \to \mathbb{R} \) are proper functions, \( \Lambda: \mathcal{M} \to \mathcal{N} \) is a general operator, and \( \Lambda(\mathcal{C}) \subset \mathcal{D} \). We emphasize that convexity of \( F \) or \( G \) is not necessarily assumed.

Our algorithm requires a choice of primal and dual base points \( m \in \mathcal{C} \) and \( n \in \mathcal{D} \). The role of \( m \) is to serve as a possible linearization point for \( \Lambda \), while \( n \) is the base point of the Fenchel conjugate for \( G \). More generally, the points can be allowed to change during the iterations. We emphasize this possibility by writing \( m^{(k)} \) and \( n^{(k)} \) when appropriate.

In the particular case that the functions involved in the primal problem (4.1) are proper, convex, and lsc the following saddle-point formulation is equivalent to (4.1),
\[
(4.2) \quad \text{Minimize} \quad \max_{\xi_n \in \exp^{-1}(\mathcal{D})} \langle \log n, \Lambda(p), \xi_n \rangle + F(p) - G^*_n(\xi_n), \quad \ p \in \mathcal{C}
\]

The proof of equivalence uses Theorem 3.8.

The solution of problem (4.2) by primal-dual optimization algorithms is challenging due to the lack of a vector space structure, which implies in particular the absence of a concept of linearity of \( \Lambda \). We follow the approach of Valkonen, 2014 to tackle the nonlinearity of the operator \( \Lambda \). More precisely, we linearize the operator \( \Lambda \) locally by its first order approximation \( DA(m): \mathcal{T}_m \mathcal{M} \to \mathcal{T}_{\Lambda(m)} \mathcal{N} \), i.e., we use
\[
(4.3) \quad \Lambda(p) \approx \exp_{\Lambda(m)} DA(m)[\log m p].
\]
Since $D\Lambda : T\mathcal{M} \to T\mathcal{N}$ is a linear operator between tangent bundles, we can employ the classical adjoint operator defined as $D\Lambda(m)^* : T^*_\mathcal{N} \to T^*_m \mathcal{M}$. We further point out that we can work algorithmically with cotangent vectors $\xi_n \in T^*_m \mathcal{N}$ with a fixed base point $n$ since we can obtain a cotangent vector $\xi_{\Lambda(m)} \in T^*_{\Lambda(m)} \mathcal{N}$ from it by parallel transport, i.e., $\xi_{\Lambda(m)} = \parallel_{\Lambda(m)} \xi_n$. The duality pairing reads as

$$ (D\Lambda(\log m)[p], P_{n \to \Lambda(m)} \xi_n) = (\log_m p, (D\Lambda(\log m))^*[P_{n \to \Lambda(m)} \xi_n]) $$

for every $p \in \mathcal{M}$ and $\xi_n \in \exp_n^{-1}(\mathcal{D})$. The first-order optimality conditions for $(\hat{p}, \hat{\xi}_n) \in \mathcal{C} \times \exp_n^{-1}(\mathcal{D})$ to solve (4.2) read as

$$ P_{m \to \hat{p}}(-(D\Lambda(\log m))^*[P_{n \to \Lambda(m)} \hat{\xi}_n]) \in \partial_\mathcal{M} F(\hat{p}), $$

$$ \log_m \Lambda(\hat{p}) \in \partial G_n^*(\hat{\xi}_n). $$

**Remark 4.1.** In the specific case that $\mathcal{X} = \mathcal{M}$ and $\mathcal{Y} = \mathcal{N}$ are Hilbert spaces, $F : \mathcal{X} \to \mathcal{R}$ is $C^1$, $\Lambda : \mathcal{X} \to \mathcal{Y}$ is a linear operator, and either $(D\Lambda(m))^*$ has empty null space or $\text{dom} \ G = \mathcal{Y}$, we observe (similar to Valkonen, 2014) that the optimality conditions (4.5) simplify to

$$ -\Lambda^* \hat{\xi} \in \partial F(\hat{p}), $$

$$ \Lambda \hat{p} \in \partial G_n^*(\hat{\xi}), $$

where $\hat{p} \in \mathcal{X}$ and $\hat{\xi} \in T^*_m \mathcal{N} = \mathcal{Y}^*$.

### 4.1. Exact Riemannian Chambolle–Pock

In this subsection we develop the exact Riemannian Chambolle–Pock algorithm summarized in Algorithm 4.1. The name “exact”, introduced by Valkonen, 2014, refers to the fact that the operator $\Lambda$ in the dual step is used in its exact form and only the primal step employs the adjoint $(D\Lambda(m))^*$ of the linearized operator.

Let us motivate the formulation of Algorithm 4.1. We start from the second inclusion of (4.5). We obtain for any $\tau > 0$ the equivalent condition

$$ \hat{\xi}_n + \tau (\log_m \Lambda(\hat{p}))^* \in \hat{\xi}_n + (\tau \partial G_n^*(\hat{\xi}_n))^* = (I + (\tau \partial G_n^*))^*(\hat{\xi}_n). $$

---

**Algorithm 4.1** Exact Riemannian Chambolle–Pock for (4.2)

**Input:** $m \in \mathcal{C}$, $n \in \mathcal{D}$, $p(0) \in \mathcal{C}$, and $\xi_n(0) \in T^*_n \mathcal{N}$, and parameters $\sigma_0, \tau_0, \theta_0, \gamma$

1: $k \leftarrow 0$, $p(0) \leftarrow p(0)$

2: while not converged do

3: $\xi_n(k+1) \leftarrow \text{prox}_{\tau_k \log_m \Lambda} \left( \xi_n(k) + \tau_k (\log_m \Lambda(p(k)))^* \right)$,

4: $p(k+1) \leftarrow \text{prox}_{\sigma_k \log \Lambda} \left( \exp_{p(k)} \left( P_{m \to p(k)} \left( -\sigma_k (D\Lambda(m))^*[P_{n \to \Lambda(m)} \xi_{n(k+1)}] \right) \right) \right)$,

5: $\theta_k = (1 + 2\gamma \sigma_k)^{-\frac{1}{2}}$, $\sigma_{k+1} \leftarrow \sigma_k \theta_k$, $\tau_{k+1} \leftarrow \tau_k / \theta_k$

6: $\hat{p}(k+1) \leftarrow \exp_{p(k+1)} \left( -\theta_k \log_{p(k+1)} p(k) \right)$

7: $k \leftarrow k + 1$

8: end while

**Output:** $p(k)$
Similarly, for every $\sigma > 0$ we get from the first inclusion in (4.5),
\begin{equation}
- \frac{1}{\sigma} \sigma \mathcal{P}_{m \rightarrow \hat{p}}(D\Lambda(m))^* \mathcal{P}_{n \rightarrow \Lambda(m)} \mathcal{P}_{n \rightarrow \Lambda(m)} \xi_n \in \partial_M F(\hat{p}).
\end{equation}

Lemma 2.18 suggests the following alternating algorithmic scheme
\begin{equation}
\xi^{(k+1)}_n = \text{prox}_{\tau G^*_n \mathcal{P}_{m \rightarrow \hat{p}} \mathcal{P}_{n \rightarrow \Lambda(m)} \mathcal{P}_{n \rightarrow \Lambda(m)} \xi^{(k)}_n},
\end{equation}
\begin{equation}
p^{(k+1)} = \text{prox}_{\sigma F} \mathcal{P}_{m \rightarrow p^{(k)}} - (\sigma(D\Lambda(m))^* \mathcal{P}_{n \rightarrow \Lambda(m)} \xi^{(k+1)}_n)\mathcal{P}_{m \rightarrow \hat{p}} \mathcal{P}_{n \rightarrow \Lambda(m)} \xi^{(k+1)}_n\bigg),
\end{equation}
where
\begin{equation}
\tau^{(k)}_n = \xi^{(k)}_n + \tau \left( \log_n \Lambda(\hat{p}^{(k)}) \right)\mathcal{P}_{n \rightarrow \Lambda(m)} \xi^{(k+1)}_n,
\end{equation}
\begin{equation}
\tilde{\xi}^{(k+1)}_n = \exp_{\mathcal{P}_{m \rightarrow \hat{p}}} \left( -\theta \log_{p^{(k)}} p^{(k+1)} \right),
\end{equation}
i.e., we perform an over-relaxation on the primal variable. From these equations we can further employ the acceleration as described in Chambolle, Pock, 2011, Sec. 5, which is already reflected in Algorithm 4.1.

4.2. Linearized Riemannian Chambolle–Pock. Analogously to Valkonen, 2014 the second possibility is not only to linearize the operator in the primal step, where we require the adjoint, but also to linearize it in the dual step.

From the linearization (4.4) we derive the linearized version of the saddle-point representation (4.2) as
\begin{equation}
\text{Minimize } \max_{\xi_n \in \exp^{-1} D} \langle D\Lambda(m)[\log_m p], \mathcal{P}_{n \rightarrow \Lambda(m)} \xi_n \rangle + F(p) - G^*_n(\xi_n), \quad p \in \mathcal{C}
\end{equation}
From here we obtain both the optimality conditions as well as the algorithmic scheme completely analogously to the previous section.

Both the exact and the linearized variants can also be stated to over-relax either the primal variable as in Algorithm 4.1 or the dual variable. In total this yields four possibilities — exact vs. linearized, and primal vs. dual over-relaxation —, from which we only present one further version of these, namely the linearized version with dual variable over-relaxation. We further include the possible change of the base points in Algorithm 4.2.

Until now there was no restriction on how to choose the base points. We could change these in every iteration. Letting $m^{(k)}$ depend on $k$ changes the linearization point of the operator, while allowing $n^{(k)}$ to change introduces different $n^{(k)}$-Fenchel conjugates $G^*_n(\xi_n)$, and also incurs a parallel transport on the dual variable.

Reasonable choices for the base points include, e.g., to set both $m^{(k)} = m$ and $n^{(k)} = \Lambda(m)$, for $k \geq 0$ and some $m \in \mathcal{M}$. This choice eliminates the parallel transport in the dual update step as well as the most inner parallel transport of the primal update step. Another choice is, to fix just $n$ and set $m^{(k)} = p^{(k)}$, which eliminates the parallel transport in the primal update step. It further eliminates both parallel transports of the dual variable in steps 6 and 7.
Algorithm 4.2 Linearized Riemannian Chambolle–Pock for (4.12)

Input: $m^{(k)} \in \mathcal{C}$, $n^{(k)} \in \mathcal{D}$, $p^{(0)} \in \mathcal{C}$, $\xi^{(0)}_n \in T_{n^{(0)}} \mathcal{N}$, and parameters $\sigma_0, \tau_0, \theta_0, \gamma$

1: $k \leftarrow 0$, $\bar{p}^{(0)} \leftarrow p^{(0)}$
2: while not converged do
3: $p^{(k+1)} \leftarrow \text{prox}_{\sigma_k F} \left( \exp_{\bar{p}^{(k)}} \left( p^{(k)} - \sigma_k (D\Lambda(m^{(k)}))^\ast \left[ p^{(k)} - \Lambda(m^{(k)}) \tau_n (\xi^{(k)}) + \Lambda(\sigma_k m^{(k)}) \tau_n (\xi^{(k)}) \right] \right) \right)$
4: $\xi^{(k+1)}_n \leftarrow \text{prox}_{\tau_n G^*} \left( \xi^{(k)}_n + \tau_k \left( p^{(k)} - \Lambda(\sigma_k m^{(k)}) \tau_n (\xi^{(k)}) \right) \right)$
5: $\theta_k = \left( 1 + 2\gamma \sigma_k \right)^{-\frac{1}{2}}$, $\sigma_{k+1} \leftarrow \sigma_k \theta_k$, $\tau_{k+1} \leftarrow \frac{\tau_k}{\theta_k}$
6: $\hat{\xi}^{(k+1)}_n \leftarrow \frac{p^{(k)} - p^{(k)}}{\theta_k}$
7: $\xi^{(k+1)}_n \leftarrow \frac{\xi^{(k)}_n}{\theta_k}$
8: $k \leftarrow k + 1$
9: end while

Output: $\bar{p}^{(k)}$

4.3. Relation to the Chambolle–Pock Algorithm on Hilbert Spaces. In this section we confirm that both Algorithm 4.1 and Algorithm 4.2 boil down to the classical Chambolle–Pock method in Hilbert spaces. To this end, suppose in this section that $\mathcal{M} = \mathcal{X}$ and $\mathcal{N} = \mathcal{Y}$ are finite-dimensional Hilbert spaces with inner products $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$, respectively, and that $\Lambda : \mathcal{X} \rightarrow \mathcal{Y}$ is a linear operator. In Hilbert spaces, geodesics are straight lines. Moreover, $\mathcal{X}$ and $\mathcal{Y}$ can be identified with their tangent spaces at arbitrary points, the exponential map equals addition, and the logarithmic map equals subtraction. In addition, all parallel transports are identity maps.

We are now showing that Algorithm 4.1 reduces to the classical Chambolle–Pock method when $n = 0 \in \mathcal{Y}$ is chosen. The same holds true for Algorithm 4.2 since $\Lambda$ is already linear. Notice that the iterates $p^{(k)}$ belong to $\mathcal{X}$ while the iterates $\xi^{(k)}_n$ belong to $\mathcal{Y}^\ast$. We can drop the fixed base point $n = 0$ from their notation. Also notice that $G^*_0$ agrees with the classical Fenchel conjugate and will be denoted by $G^* : \mathcal{Y} \rightarrow \overline{\mathbb{R}}$.

We only need to consider steps 3, 4 and 6 in Algorithm 4.1. The dual update step becomes

$$\xi^{(k+1)}_n \leftarrow \text{prox}_{\tau_n G^*} \left( \xi^{(k)}_n + \tau_k (\hat{\xi}^{(k)} + \Lambda(\sigma_k m^{(k)} \tau_n (\xi^{(k)})) \right).$$

Here $\hat{\xi}^* : \mathcal{Y} \rightarrow \mathcal{Y}^\ast$ denotes the Riesz isomorphism for the space $\mathcal{Y}$. Next we address the primal update step, which reads

$$p^{(k+1)} \leftarrow \text{prox}_{\sigma_k F} \left( p^{(k)} - \sigma_k (\Lambda^\ast \xi^{(k+1)})^\ast \right).$$

Here $\Lambda^\ast : \mathcal{X}^\ast \rightarrow \mathcal{X}$ denotes the inverse Riesz isomorphism for the space $\mathcal{X}$. Finally, the (primal) extrapolation step becomes

$$\bar{p}^{(k+1)} \leftarrow \frac{p^{(k+1)} - \theta_k (p^{(k)} - p^{(k+1)})}{\theta_k} = p^{(k+1)} + \theta_k (p^{(k+1)} - p^{(k)}).$$

The steps above agree with Chambolle, Pock, 2011, Alg. 1 (with the roles of $F$ and $G$ reversed).
4.4. Convergence of the Linearized Chambolle–Pock Algorithm. In the following we adapt the proof of Chambolle, Pock, 2011 to solve the linearized saddle-point problem (4.12). We restrict the discussion to the case where \( \mathcal{M} \) and \( \mathcal{N} \) are Hadamard manifolds and \( \mathcal{C} = \mathcal{M} \) and \( \mathcal{D} = \mathcal{N} \). Moreover, we fix \( m \) and \( n := \Lambda(m) \) during the iteration and set the acceleration parameter \( \gamma \) to zero and \( \theta_k \equiv 1 \).

Before presenting the main result of this section and motivated by the condition introduced after Valkonen, 2014, eq. (2.4), we introduce the following constant

\[
L := \|D\Lambda(m)\|,
\]

i.e., the operator norm of \( D\Lambda(m) : T_m\mathcal{M} \to T_n\mathcal{N} \).

**Theorem 4.2.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be two Hadamard manifolds and \( F : \mathcal{M} \to \mathbb{R}, \ G : \mathcal{N} \to \mathbb{R} \) be proper, convex, lsc, and \( \Lambda : \mathcal{M} \to \mathcal{N} \). Fix \( m \in \mathcal{M} \) and \( n := \Lambda(m) \in \mathcal{N} \). Suppose that the linearized saddle-point problem (4.12) has a saddle-point \( (\bar{p}, \bar{\xi}_n) \). Choose \( \sigma, \tau \) such that \( \sigma \tau L_2 < 1 \), with \( L \) defined in (4.13), and let the iterates \( (\xi^{(k)}_n, p^{(k)}, \check{\xi}^{(k)}_n) \) be given by Algorithm 4.2. Suppose that there exists \( K \in \mathbb{N} \) such that for all \( k \geq K \), the following holds:

\[
C(k) := \frac{1}{\sigma} d^2_M(p^{(k)}, \bar{p}) + \langle \check{\xi}^{(k)}_n, D\Lambda(m)[\xi_k] \rangle \geq 0,
\]

where \( \check{p}^{(k)} \) is defined in (4.10),

\[
\zeta_k := \mathcal{P}_{p^{(k)} \to m} \left( \log p^{(k)} p^{(k+1)} - \log \bar{p} \right) - \log_m p^{(k+1)} + \log_m \bar{p},
\]

with \( \check{\xi}^{(k)}_n = 2\xi^{(k)}_n - \xi^{(k-1)}_n \). Then the following statements are true.

i) The sequence \( (p^{(k)}, \xi^{(k)}_n) \) remains bounded, i.e.,

\[
(4.15) \frac{1}{2\tau} \|\xi_n - \xi^{(k)}_n\|_n^2 + \frac{1}{2\sigma} d^2_M(p^{(k)}, \bar{p}) \leq \frac{1}{2\tau} \|\xi_n - \xi^{(0)}_n\|_n^2 + \frac{1}{2\sigma} d^2_M(p^{(0)}, \bar{p}).
\]

ii) There exists a saddle-point \( (p^*, \xi^*_n) \) such that \( p^{(k)} \to p^* \) and \( \xi^{(k)}_n \to \xi^*_n \).

**Remark 4.3.** A main difference of Theorem 4.2 to the Hilbert space case is the condition on \( C(k) \). Restricting this theorem to the setting of subsection 4.3, the parallel transport and the logarithmic map simplify to the identity and subtraction, respectively. Then

\[
\zeta_k = p^{(k+1)} - p^{(k)} - \bar{p} + p^{(k)} - p^{(k+1)} + m + \bar{p} - m = \bar{p}^{(k)} - p^{(k)}
\]

and hence \( C(k) \) simplifies to

\[
C(k) = \sigma \|D\Lambda(m)^*[\check{\xi}^{(k)}_n]\|_{\mathcal{Y}}^2 - \sigma \langle \check{\xi}^{(k)}_n, D\Lambda(m)([D\Lambda(m)]^*[\check{\xi}^{(k)}_n]) \rangle = 0
\]

for any \( \check{\xi}^{(k)}_n \), so condition (4.14) vanishes.

**Proof of Theorem 4.2.** Recall that we assume \( \Lambda(m) = n \) in the following. Following along the lines of Chambolle, Pock, 2011, Thm. 1, we first write the iterations from Algorithm 4.2 in a general form

\[
\begin{align*}
p^{(k+1)} &= \text{prox}_{\sigma F}(\bar{p}) \quad \bar{p}^{(k)} := \exp_{p^{(k)}} \left( \mathcal{P}_{m \to p^{(k)}} (-\sigma D\Lambda(m)^*[\check{\xi}^{(k)}_n]) \right) \\
\xi^{(k+1)} &= \text{prox}_{\tau G}(\check{\xi}^{(k)}_n) \quad \check{\xi}^{(k)}_n := \xi^{(k)}_n + \tau (D\Lambda(m)[\log_m \bar{p}])
\end{align*}
\]
for some $\tilde{\xi}_n$ and $p$ to be specified later on. Applying Lemma 2.18, we get

$$\frac{1}{\sigma} (\log_{p^{(k+1)}} \tilde{p}^{(k)}) \in \partial_{\mathcal{M}} F(p^{(k+1)}),$$

(4.17) \(\left(\frac{\xi_n^{(k)} - \xi_n^{(k+1)}}{\tau} \right)^2 + DL(m)[\log_m \tilde{p}] \in \partial G^*_n(\xi_n^{(k+1)}).

Due to Definition 2.5 and Definition 2.12, we obtain that for every $\xi$ and, noticing that

$$\text{we rephrase the last term as}

\frac{1}{\sigma} (\log_{\tilde{p}^{(k)}} p^{(k)}, \log_{\tilde{p}^{(k)}} p) \geq \frac{1}{2\sigma} d^2_{\mathcal{M}}(\tilde{p}^{(k)}, p^{(k)}) + \frac{1}{2\sigma} d^2_{\mathcal{M}}(p, p^{(k+1)}) - \frac{1}{2\sigma} d^2_{\mathcal{M}}(\tilde{p}^{(k)}, p).

Rearranging the law of cosines for the triangle $\Delta = (p^{(k)}, \tilde{p}^{(k)}, p)$ yields

$$-\frac{1}{2\sigma} d^2_{\mathcal{M}}(\tilde{p}^{(k)}, p) \geq \frac{1}{2\sigma} d^2_{\mathcal{M}}(\tilde{p}^{(k)}, p^{(k)}) - \frac{1}{2\sigma} d^2_{\mathcal{M}}(p^{(k)}, p) - \frac{1}{\sigma} (\log_{\tilde{p}^{(k)}} p^{(k)}, \log_{\tilde{p}^{(k)}} p).

We rephrase the last term as

$$-\frac{1}{\sigma} (\log_{\tilde{p}^{(k)}} p^{(k)}, \log_{\tilde{p}^{(k)}} p) = -\frac{1}{\sigma} (P_{\tilde{p}^{(k)} \to p^{(k)}} \log_{\tilde{p}^{(k)}} p^{(k)}, P_{\tilde{p}^{(k)} \to p^{(k)}} \log_{\tilde{p}^{(k)}} p)

= -\frac{1}{\sigma} (\log_{\tilde{p}^{(k)}} \tilde{p}^{(k)}, P_{\tilde{p}^{(k)} \to p^{(k)}} \log_{\tilde{p}^{(k)}} p)

= -(DL(m)[\tilde{\xi}_n]^{k}, P_{p^{(k)} \to \tilde{p}^{(k)}} P_{\tilde{p}^{(k)} \to p^{(k)}} \log_{\tilde{p}^{(k)}} p)

= -\langle \tilde{\xi}_n, DL(m)[P_{p^{(k)} \to m} P_{\tilde{p}^{(k)} \to p^{(k)}} \log_{p^{(k)}} p] \rangle.

We insert the estimates above into the first inequality in (4.18) to obtain

$$F(p) \geq F(p^{(k+1)}) + \frac{1}{2\sigma} d^2_{\mathcal{M}}(\tilde{p}^{(k)}, p^{(k+1)}) + \frac{1}{2\sigma} d^2_{\mathcal{M}}(p, p^{(k)}) + \frac{1}{2\sigma} d^2_{\mathcal{M}}(\tilde{p}^{(k)}, p^{(k)})

- \frac{1}{\sigma} (\log_{p^{(k)}} \tilde{p}^{(k)}, \log_{p^{(k)}} p^{(k+1)})

\text{Considering now the geodesic triangle } \Delta = (\tilde{p}^{(k)}, p^{(k)}, p^{(k+1)}) \text{ we get}

$$\frac{1}{2\sigma} d^2_{\mathcal{M}}(p^{(k+1)}, \tilde{p}^{(k)}) \geq \frac{1}{2\sigma} d^2_{\mathcal{M}}(p^{(k)}, p^{(k+1)}) + \frac{1}{2\sigma} d^2_{\mathcal{M}}(p^{(k)}, \tilde{p}^{(k)})

- \frac{1}{\sigma} (\log_{p^{(k)}} \tilde{p}^{(k)}, \log_{p^{(k)}} p^{(k+1)}),

\text{and, noticing that}

$$-\frac{1}{\sigma} (\log_{p^{(k)}} \tilde{p}^{(k)}, \log_{p^{(k)}} p^{(k+1)}) = \langle \tilde{\xi}_n, DL(m)[P_{p^{(k)} \to m} P_{\tilde{p}^{(k)} \to p^{(k)}} \log_{p^{(k)}} p] \rangle,$$
we write
\[ \mathcal{F}(p) \geq \mathcal{F}(p^{(k+1)}) + \frac{1}{2\sigma} d_M^2(p^{(k+1)}, p) - \frac{1}{2\sigma} d_M^2(p^{(k)}, p) + \frac{1}{2\sigma} d_M^2(p^{(k)}, p^{(k+1)}) \]
\[ + \frac{1}{\sigma^2} \mathcal{R}_n(p^{(k)}, p^{(k)}) \]
\[ + \langle \hat{\xi}_n, D\Lambda(m)[\mathcal{P}_{p^{(k)}} \to m \log_{\mathcal{P}^{(k)}} p^{(k+1)} - \mathcal{P}_{p^{(k)}} \to m \mathcal{P}_{p^{(k)}} \to m \log_{\mathcal{P}^{(k)}} p \rangle. \]

Adding this inequality with the second inequality from (4.18), we get
\[ \frac{1}{2\pi} \|\xi_n - \xi_n^{(k)}\|^2 + \frac{1}{2\pi} d_M^2(p^{(k)}, p) \]
\[ \geq \langle \mathcal{D}(\Lambda)(\log_m p^{(k+1)}), \xi_n \rangle + \mathcal{F}(p^{(k+1)}) - G_n^*(\xi_n) \]
\[ - \langle \mathcal{D}(\Lambda)(\log_m p), \xi^{(k+1)} \rangle + F(p) - G_n^*(\xi^{(k+1)}) \]
\[ + \frac{1}{2\pi} \|\xi_n - \xi_n^{(k+1)}\|^2 + \frac{1}{2\pi} \|\xi^{(k)} - \xi^{(k+1)}\|^2 \]
\[ + \frac{1}{2\pi} d_M^2(p^{(k+1)}, p) + \frac{1}{2\pi} d_M^2(p^{(k)}, p^{(k+1)}) \]
\[ (4.19a) + \frac{1}{\sigma^2} d_M^2(p^{(k)}, p^{(k)}) \]
\[ (4.19b) + \langle \hat{\xi}_n, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \mathcal{D}(\Lambda)(\log_m p) \rangle \]
\[ (4.19c) + \langle \hat{\xi}_n^{(k+1)} - \hat{\xi}_n, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ (4.19d) - \langle \xi_n^{(k+1)} - \hat{\xi}_n, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ (4.19e) - \langle \hat{\xi}_n, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle. \]

Choosing now \( \hat{\mu} = p^{(k+1)} \) and \( \hat{\xi}_n = 2\xi_n^{(k)} - \xi_n^{(k-1)} \) (4.19c) vanishes. We continue with (4.19d) and estimate it according to
\[ -\langle \xi_n^{(k+1)} - \hat{\xi}_n, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ = -\langle \xi_n^{(k+1)} - \xi_n^{(k)} - (\xi_n^{(k)} - \xi_n^{(k-1)}), \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ = -\langle \xi_n^{(k+1)} - \xi_n^{(k)}, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ + \langle \xi_n^{(k)} - \xi_n^{(k-1)}, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ - \langle \xi_n^{(k-1)} - \xi_n^{(k)}, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ - \mathcal{L} \|\xi_n^{(k)} - \xi_n^{(k-1)}\|_n \|\log_m p^{(k+1)} - \log_m p\|_n. \]

Using that \( 2ab \leq \alpha a^2 + b^2 / \alpha \) holds for every \( a, b \geq 0 \) and \( \alpha > 0 \), and choosing in particular \( \alpha = \frac{\sqrt{2}}{\sqrt{\pi}} \), we get
\[ -\langle \xi_n^{(k+1)} - \hat{\xi}_n, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ \geq -\langle \xi_n^{(k+1)} - \xi_n^{(k)}, \mathcal{D}(\Lambda)(\log_m p^{(k+1)}) - \log_m p \rangle \]
\[ + \langle \xi_n^{(k)} - \xi_n^{(k-1)}, \mathcal{D}(\Lambda)(\log_m p^{(k)}) - \log_m p \rangle \]
\[ - \frac{\mathcal{L} \sqrt{2}}{2\sqrt{\pi}} d_M^2(p^{(k+1)}, p^{(k)}) - \frac{\mathcal{L} \sqrt{2}}{2\sqrt{2}} \|\xi_n^{(k-1)} - \xi_n^{(k)}\|^2_n, \]
\[ (4.20). \]
where $L$ is the constant defined in (4.13). We notice now that (4.19a), (4.19b) and (4.19c) correspond to $C(k + 1)$, hence we simplify the inequality of interest by writing $C(k + 1)$ instead and also noticing that the first two lines on the right hand side of (4.20) are the primal dual gap, denoted in the following by $\text{PDG}(k + 1)$.

With these modifications to (4.19a)–(4.19c), we get

$$
\frac{1}{2\tau} \| \xi_n - \xi_n^{(k)} \|_n^2 + \frac{1}{2\sigma} d_M^2(p^{(k)}, p) \\
\geq \text{PDG}(k + 1) + C(k + 1) + \left( \frac{1}{2\sigma} - \frac{L\sqrt{\tau}}{2\sqrt{\gamma}} \right) d_M^2(p^{(k)}, p^{(k+1)}) + \frac{1}{2\sigma} d_M^2(p^{(k+1)}, p) \\
+ \frac{1}{2\tau} \| \xi_n - \xi_n^{(k+1)} \|_n^2 + \frac{1}{2\tau} \| \xi_n^{(k)} - \xi_n^{(k+1)} \|_n^2 - \frac{L\sqrt{\tau}}{2\sqrt{\gamma}} \| \xi_n^{(k)} - \xi_n^{(k+1)} \|_n^2 \\
- \langle \xi_n^{(k+1)} - \xi_n^{(k)}, D\Lambda(m)[\log_m p^{(k+1)} - \log_m p] \rangle \\
+ \langle \xi_n^{(k)} - \xi_n^{(k-1)}, D\Lambda(m)[\log_m p^{(k)} - \log_m p] \rangle.
$$

(4.21)

We continue to sum (4.21) from 0 to $N - 1$, where we set $\xi_n^{(-1)} := \xi_n^{(0)}$ in coherence with the initial choice $\xi_n^{(0)} = \xi_n^{(0)}$. For every $p$ and $\xi_n$, we get

$$
\frac{1}{2\tau} \| \xi_n - \xi_n^{(0)} \|_n^2 + \frac{1}{2\sigma} d_M^2(p^{(0)}, p) \\
\geq \sum_{k=0}^{N-1} \text{PDG}(k + 1) + \sum_{k=0}^{N-1} C(k + 1) + \frac{1}{2\tau} \| \xi_n - \xi_n^{(N)} \|_n^2 + \frac{1}{2\sigma} d_M^2(p^{(N)}, p) \\
+ \left( \frac{1}{2\sigma} - \frac{L\sqrt{\tau}}{2\sqrt{\gamma}} \right) \sum_{k=1}^{N} d_M^2(p^{(k)}, p^{(k-1)}) + \left( \frac{1}{2\tau} - \frac{L\sqrt{\tau}}{2\sqrt{\gamma}} \right) \sum_{k=1}^{N-1} \| \xi_n^{(k)} - \xi_n^{(k-1)} \|_n^2 \\
+ \frac{1}{2\tau} \| \xi_n^{(N-1)} - \xi_n^{(N)} \|_n^2 - \langle \xi_n^{(N)} - \xi_n^{(N-1)}, D\Lambda(m)[\log_m p^{(N)} - \log_m p] \rangle.
$$

(4.22)

We further develop the last term in (4.22) and get

$$
-\langle \xi_n^{(N)} - \xi_n^{(N-1)}, D\Lambda(m)[\log_m p^{(N)} - \log_m p] \rangle \\
\geq -L\| \xi_n^{(N-1)} - \xi_n^{(N-1)} \|_n d_M^2(p^{(N)}, p) \\
\geq -\frac{L\alpha}{2} \| \xi_n^{(N)} - \xi_n^{(N-1)} \|_n^2 + \frac{L}{2\alpha} d_M^2(p^{(N)}, p).
$$

Choosing $\alpha = 1/(\tau L)$, it follows

$$
-\langle \xi_n^{(N)} - \xi_n^{(N-1)}, D\Lambda(m)[\log_m p^{(N)} - \log_m p] \rangle \\
\geq -\frac{1}{2\tau} \| \xi_n^{(N)} - \xi_n^{(N-1)} \|_n^2 - \frac{\tau L^2}{2} d_M^2(p^{(N)}, p).
$$
Hence (4.22) becomes
\begin{align*}
\frac{1}{2\tau} \|\xi_n - \xi_n^{(0)}\|^2 + \frac{1}{2\sigma} d_M^2(p^{(0)}, p) \\
\geq \sum_{k=0}^{N-1} \text{PDG}(k + 1) + \sum_{k=0}^{N-1} C(k + 1) \\
+ \frac{1}{2\tau} \|\xi_n - \xi_n^{(N)}\|^2 + \left(\frac{1}{2\tau} - \frac{L\sqrt{\sigma}}{2\sqrt{\tau}}\right) \sum_{k=1}^{N} \|\xi_n^{(k)} - \xi_n^{(k-1)}\|^2 \\
+ \left(\frac{1}{2\sigma} - \frac{\tau L^2}{2}\right) d_M^2(p^{(N)}, p) + \left(\frac{1}{2\sigma} - \frac{L\sqrt{\tau}}{2\sqrt{\sigma}}\right) \sum_{k=1}^{N} d_M^2(p^{(k)}, p^{(k-1)}).
\end{align*}
\hspace{1cm} (4.23)

Taking in particular \((p, \xi_n) = (\hat{p}, \hat{\xi}_n)\), the combination of the feasibility of the saddle-point \((\hat{p}, \hat{\xi}_n)\) together with (4.14) and the inequality \(1 > \sigma \tau L^2\) implies that the sequence \(\{p^{(k)}, \xi_n^{(k)}\}\) is bounded, which is the statement i).

Part ii) follows completely analogously to the steps of Chambolle, Pock, 2011, Thm. 1(c) adapted to (4.21).

5. ROF Models on Manifolds. A starting point of the work of Chambolle, Pock, 2011 is the ROF \(\ell^2\)-TV denoising model Rudin, Osher, Fatemi, 1992, which was generalized to manifolds in Lellmann et al., 2013 for the so-called isotropic and anisotropic cases. This class of \(\ell^2\)-TV models can be formulated in the discrete setting as follows: let \(F = (f_{i,j})_{i,j} \in \mathcal{M}^{d_1 \times d_2}\), \(d_1, d_2 \in \mathbb{N}\) be a manifold-valued image, i.e., its pixel \(f_{i,j}\) take values in a manifold \(\mathcal{M}\). Then the manifold-valued \(\ell^2\)-TV energy functional reads
\begin{align*}
\mathcal{E}_q(P) := \sum_{i,j=1}^{d_1, d_2} d_{\mathcal{M}}^2(f_{i,j}, p_{i,j}) + \alpha \nabla P_{g,q,1}, \quad P = (p_{i,j})_{i,j} \in \mathcal{M}^{d_1 \times d_2},
\end{align*}
\hspace{1cm} (5.1)

where \(q \in \{1, 2\}\). Moreover, \(\nabla : \mathcal{M}^{d_1 \times d_2} \to \mathcal{TM}^{d_1 \times d_2 \times 2}\) denotes the generalization of the finite difference operator, which is defined as
\begin{align*}
(\nabla P)_{i,j,k} &= \begin{cases} 
0 \in \mathcal{T}_{p_{i,j}} \mathcal{M} & \text{if } i = d_1 \text{ and } k = 1, \\
0 \in \mathcal{T}_{p_{i,j}} \mathcal{M} & \text{if } j = d_2 \text{ and } k = 2, \\
\log_{p_{i,j}} p_{i+1,j} & \text{if } i < d_1 \text{ and } k = 1, \\
\log_{p_{i,j}} p_{i,j+1} & \text{if } j < d_2 \text{ and } k = 2.
\end{cases}
\end{align*}
\hspace{1cm} (5.2)

The corresponding norm is then given by
\begin{align*}
\|\nabla P\|_{g,q,1} &= \sum_{i,j=1}^{d_1, d_2} \left(\|(\nabla P)_{i,j,1}\|_g + \|(\nabla P)_{i,j,2}\|_g\right)^{\frac{q}{2}}.
\end{align*}
\hspace{1cm} (5.3)

For simplicity of notation we do not explicitly state the base point in the Riemannian metric but denote the norm on \(\mathcal{T}\mathcal{M}\) by \(\|\cdot\|_g\). Depending on the value of \(q \in \{1, 2\}\), we call the energy functional (5.1) isotropic when \(q = 2\) and anisotropic for \(q = 1\). Note that previous algorithms like CPPA Weinmann, Demaret, Storath, 2014 or Douglas–Rachford (DR) Bergmann, Persch, Steidl, 2016 are only able to tackle the
anisotropic case \( q = 1 \) due to a missing closed form of the prox for the isotropic TV summands. A relaxed version of the isotropic case can be computed using the half-quadratic minimization \cite{BergmannChenetal}. Looking at the optimality conditions of the isotropic or anisotropic energy functional, the authors in \cite{BergmannTenbrinck} derived and solved the corresponding \( q \)-Laplace equation. This can be generalized even to the cases \( q > 0, q \not\in \{1, 2\} \).

The minimization of (5.1) fits into the setting of the model problem (4.1). Indeed, \( \mathcal{M} \) is replaced by \( \mathcal{M}^{d_1 \times d_2}, \mathcal{N} = \mathcal{T}\mathcal{M}^{d_1 \times d_2} \), \( \Lambda = \nabla \) and \( G = \alpha \| \cdot \|_{g,q,1} \). We apply Algorithm 4.2 to solve the linearized saddle-point problem (4.12). This procedure will yield an approximate minimizer of (5.1). To this end we require both the Fenchel conjugate and its proximal map of \( G \). Further shortening notation we denote \( G \nabla = 2018 \), Sect. 4.2. Defining \( N \) computed using the so-called adjoint Jacobi fields, see e.g. Bergmann, Gousenbourger, Tenbrinck, 2018 derived and solved the corresponding \( q \)-Laplace equation. This can be generalized even to the cases \( q > 0, q \not\in \{1, 2\} \).

The corresponding proximal maps read

\[
\text{prox}_{rG^*_{2}}(\Xi) = \left( \max \{ 1, \| \Xi_{i,j,k} \|_g \} \right)^{-1} \Xi_{i,j,k}
\]

and

\[
\text{prox}_{rG^*_{\infty}}(\Xi) = \left( \max \{ 1, \| \Xi_{i,j,k} \|_g \} \right)^{-1} \Xi_{i,j,k}
\]

Finally, to derive the adjoint of \( D\Lambda(m) \), let \( P \in \mathcal{M}^{d_1 \times d_2} \) and \( X \in T_P \mathcal{M}^{d_1 \times d_2} \). Applying the chain rule it is not difficult to prove that

\[
(D\nabla(P)[X])_{i,j,k} = D_1 \log_{p_{i,j}}(p_{i,j+e_k})[X_{i,j}] + D_2 \log_{p_{i,j}}(p_{i,j+e_k})[X_{i,j+e_k}].
\]

In the above formula \( e_k \) represents the vector used to reach a neighbor. The symbols \( D_1 \) and \( D_2 \) represent the differentiation of the logarithmic map w.r.t. the base point and its argument, respectively. We notice that \( D_1 \log_{p_{i,j}}(p_{i,j+e_k}) \) and \( D_2 \log_{p_{i,j}}(\cdot) \) can be computed by a straightforward application of Jacobi fields; see for example Bergmann, Fitschen, et al., 2018, Lem. 4.ii) and iii).

With \( (D\nabla)(\cdot)[\cdot] : \mathcal{T}\mathcal{M}^{d_1 \times d_2} \to \mathcal{T}\mathcal{N} \) given by Jacobi fields its adjoint can be computed using the so-called adjoint Jacobi fields, see e.g. Bergmann, Gousenbourger, 2018, Sect. 4.2. Defining \( N_{i,j} \) to be the set of neighbors of the pixel \( p_{i,j} \), for every \( X \in \mathcal{M}^{d_1 \times d_2} \), the two cases of main interest, namely \( q = 1 \), hence \( q^* = \infty \), and \( q = q^* = 2 \). The dual functions read

\[
G^*_{2}(\Xi) = \| \Xi \|_{g,2,\infty} = \max_{i \in \{1, \ldots, d_1\}, j \in \{1, \ldots, d_2\}} \| |\Xi_{i,j,.}|_g \|_2,
\]

\[
G^*_{\infty}(\Xi) = \| \Xi \|_{g,\infty,\infty} = \max_{i \in \{1, \ldots, d_1\}, j \in \{1, \ldots, d_2\}} \| |\Xi_{i,j,.}|_g \|_g, \Xi \in \mathcal{T}^*\mathcal{N}.
\]

The 2-norm within \( G^*_{2} \) extends over \( k \in \{1, 2\}, \) i.e., over the directional differences.

The corresponding proximal maps read

\[
\text{prox}_{rG^*_{2}}(\Xi) = \left( \max \{ 1, \| \Xi_{i,j,.} \|_g \} \right)^{-1} \Xi_{i,j,.}
\]

and

\[
\text{prox}_{rG^*_{\infty}}(\Xi) = \left( \max \{ 1, \| \Xi_{i,j,k} \|_g \} \right)^{-1} \Xi_{i,j,k}.
\]
\[ \mathcal{T}_P \mathcal{M}^{d_1 \times d_2} \text{ and } \eta \in \mathcal{T}_V^* \mathcal{N} \] we have

\[
\langle D \nabla (P)(X), \eta \rangle = \sum_{i,j,k} \langle D \nabla (P)(X)[i,j,k], \eta[i,j,k] \rangle = \sum_{i,j,k} \langle D_1 \log p_{i,j} + e_k[X_{i,j+k}], \eta[i,j,k] \rangle + \sum_k \langle D_2 \log p_{i,j+k} + e_k[X_{i,j+k}], \eta[i,j,k] \rangle
\]

\[
= \sum_{i,j} \langle X_{i,j}, D_1^* \log p_{i,j} + e_k[\eta[i,j]], \eta[i,j,k] \rangle + \sum_{k} \langle X_{i,j+k}, D_2^* \log p_{i,j+k} + e_k[\eta[i,j,k]], \eta[i,j,k] \rangle
\]

\[
= \sum_{i,j} \langle X_{i,j}, (D^* \nabla (P)(\eta))[i,j] \rangle,
\]

which leads to the component-wise entries in the linearized adjoint

\[
(D^* \nabla (P)(\eta))_{i,j} = \sum_k D_1^* \log p_{i,j+k} |_{\eta[i,j,k]} + \sum_{(i',j') \in N(i,j)} D_2^* \log p_{i,j+k} |_{\eta[i',j',k]}
\]

As before, we recall that \(D_1^* \log \cdot (p_{i,j+k})\) and \(D_2^* \log \cdot (p_{i,j+k})\) also follow from Bergmann, Fitschen, et al., 2018, Sect. 4.

6. Numerical Experiments. The numerical experiments are implemented in the toolbox MANOPT.jl\(^1\) (Bergmann, 2019) in Julia\(^2\). They were run on a MacBook Pro, 2.5 Ghz Intel Core i7, 16 GB RAM, with Julia 1.1. All our examples are based on the linearized saddle-point formulation (4.12) for \(\ell^2\)-TV.

6.1. A Signal with Known Minimizer. The first example uses signal data \(\mathcal{M}^{d_1}\) instead of an image, where the data space is \(\mathcal{M} = S^2\). This gives us the opportunity to consider the same problem also on the embedding manifold \((\mathbb{R}^3)^{d_1}\) in order to illustrate the difference between the manifold-valued and Euclidean settings. We construct the data \((f_i)\) such that the unique minimizer of (5.1) is known in closed form. Therefore a second purpose of this problem is to compare the numerical solution obtained by Algorithm 4.2, i.e., an approximate saddle-point of the linearized problem (4.12), to the solution of the original saddle-point problem (4.2). Third, we wish to explore how the value \(C(k)\) from (4.14) behaves numerically.

The piecewise constant signal is given by

\[
f \in \mathcal{M}^{30} \quad f_i = \begin{cases} p_1 & \text{if } i \leq 15, \\ p_2 & \text{if } i > 15. \end{cases}
\]

for two values \(p_1, p_2 \in \mathcal{M}\) specified below.

Notice that since \(d_2 = 1\), the isotropic and anisotropic models (5.1) coincide. The exact minimizer \(\hat{p}\) of (5.1) is piecewise constant with the same structure as the data \(f\).

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\(^1\)available at http://www.manoptjl.org, thanks to N. Boumal, developer of the Matlab toolbox manopt, see https://manopt.org, for allowing us to use the name.

\(^2\)https://julialang.org
(a) Signal $f$ of unit vectors.

(b) Minimizer with values in $\mathcal{M} = \mathbb{S}^2$.

(c) Minimizer with values in $\mathcal{M} = \mathbb{R}^3$.

(d) Signal of $\mathcal{P}(3)$ matrices.

(e) Minimizer on $\mathcal{M} = \mathcal{P}(3)$.

Fig. 1: Computing the minimizer of the manifold-valued $\ell^2$-TV model for a signal of unit vectors shown in (a) with respect to both manifolds $\mathbb{R}^3$ and $\mathbb{S}^2$ with $\alpha = 5$: (b) on $(\mathbb{S}^2)^3$ and (c) on $(\mathbb{R}^3)^3$. The known effect, loss of contrast is different for both cases, since on $\mathbb{S}^2$ the vector remain of unit length. The same effect can be seen for a signal of spd matrices, i.e., $\mathcal{P}(3)$; see (d) and (e).

Its values are $\hat{p}_1 = \gamma_{p_1,p_2}(\delta)$ and $\hat{p}_2 = \gamma_{p_2,p_1}(\delta)$ where $\delta = \min\{d_M(p_1,p_2)\alpha \frac{1}{\sqrt{2}}, \frac{1}{2}\}$. Notice that the notion of geodesics are different for both manifolds under consideration, and thus $\hat{p}_{\mathbb{R}^3}$ is different from $\hat{p}_{\mathbb{S}^2}$.

In the following we use $\alpha = 5$ and $p_1 = \frac{1}{\sqrt{2}}(1,1,0)^T, p_2 = \frac{1}{\sqrt{2}}(1,-1,0)^T$. The data $f$ is shown in Figure 1a.

We ran the linearized Riemannian Chambolle–Pock Algorithm 4.2 with relaxation parameter $\theta = 1$ on the dual variable as well as $\sigma = \tau = \frac{1}{2}$, and $\gamma = 0$, i.e., without acceleration, as well as $\rho^{(0)} = f$ and $\xi^{(0)}_n$ as the zero vector. The stopping criterion is set to 500 iterations and as $m$ we use the mean of the data, which is just $m = \gamma_{p_1,p_2}(\frac{1}{\sqrt{2}})$. For the Euclidean case $\mathcal{M} = \mathbb{R}^3$, we obtain a shifted version of the original Chambolle–Pock algorithm, since $m \neq 0$.

While the algorithm on $\mathcal{M} = \mathbb{S}^2$ takes about 0.85 seconds, the Euclidean algorithm takes about 0.44 seconds, which is most likely due to the exponential, logarithmic map as well as the parallel transport on $\mathbb{S}^2$, which involve sines and cosines. The results obtained by the Euclidean algorithm is $2.18 \cdot 10^{-12}$ away in Euclidean norm from the analytical minimizer $\hat{p}_{\mathbb{R}^3}$. Notice that the convergence of the Euclidean algorithm is covered by the theory in Chambolle, Pock, 2011. Moreover, notice that in this setting, $\Lambda$ is a linear map between vector spaces. During the iterations, we confirmed that the value of $C(k)$ is numerically zero ($\pm 5.5511 \cdot 10^{-17}$), as expected from Remark 4.3.

Although Algorithm 4.2 on $\mathcal{M} = \mathbb{S}^2$ is based on the linearized saddle-point problem (4.12) instead of (4.2), we observed that it converges to the exact minimizer $\hat{p}_{\mathbb{S}^2}$ of (5.1). Therefore it is meaningful to plug in $\hat{p}_{\mathbb{S}^2}$ into the formula (4.14) to evaluate $C(k)$ numerically. The numerical values observed throughout the 500 iterations are in the interval $[-4.0 \cdot 10^{-13}, 4.0 \cdot 10^{-9}]$. We interpret this as confirmation that $C(k)$ is non-negative in this case. However, even with this observation the convergence of
Algorithm 4.2 is not covered by Theorem 4.2 since \( S^2 \) is not a Hadamard manifold. Quite to the contrary, it has constant positive sectional curvature.

The results are shown in Figure 1c and Figure 1b, respectively. They illustrate the well known loss of contrast and reduction of jump heights. This leads to shorter vectors in \( \hat{p}_{R^3} \), while, of course, their unit length is preserved in \( \hat{p}_{S^2} \).

We also constructed a similar signal on \( M = \mathcal{P}(3) \), the manifold of symmetric positive definite (SPD) matrices with affine metric; see Pennec, Fillard, Ayache, 2006. This is a Hadamard manifold with non-constant curvature. Let \( I \in \mathbb{R}^{3\times3} \) denote the unit matrix and

\[
\begin{align*}
    p_1 &= \exp_I \left( \frac{2}{\|X\|_I} X \right), \quad &p_2 &= \exp_I \left( -\frac{2}{\|X\|_I} X \right), \quad \text{with} \quad X = \frac{1}{2} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 6 \end{pmatrix} \in T_I \mathcal{P}(3).
\end{align*}
\]

In this case, the run time is 5.94 seconds, which is due to matrix exponentials and logarithms as well as singular value decompositions that need to be computed. Here, \( C(k) \) turns out to be numerically zero (\( \pm 8 \cdot 10^{-15} \)) and the distance to the analytical minimizer \( \hat{p}_{\mathcal{P}(3)} \) is \( 1.08 \cdot 10^{-12} \). The original data \( f \) and the result \( \hat{p}_{\mathcal{P}(3)} \) (again with a loss of contrast as expected) are shown in Figure 1d and Figure 1e, respectively.

6.2. A Comparison of Algorithms. As a second example we compare our Algorithm 4.2 to the cyclic proximal point algorithm (CPPA) Bačák, 2014, which was first applied to \( \ell^2 \)-TV problems in Weinmann, Demaret, Storath, 2014. It is known to be a robust but generally slow method. We also compare the proposed method with the parallel Douglas–Rachford algorithm (PDRA), which was introduced in Bergmann, Persch, Steidl, 2016.

As an example, we use the anisotropic \( \ell^2 \)-TV model (5.1) on images of size 32 \( \times \) 32 with values in the manifold of 3 \( \times \) 3 SPD matrices \( \mathcal{P}(3) \) as in the previous subsection. The original data is shown in Figure 2a. No exact solution is known for this example. We use a value of \( \alpha = 6 \). To generate a reference solution we allowed the CPPA with step size \( \lambda_k = \frac{4}{k} \) to run for 4,000 iterations. This required 1484.13 seconds and it yields a value of the objective function (5.1) of approximately 232.419. The result is shown in Figure 2b.

We compare CPPA to PDRA as well as to our Algorithm 4.2 using the value of the cost function and the run time as criteria. The PDRA was run with parameters \( \eta = 0.58, \lambda = 0.93 \), which where used by Bergmann, Persch, Steidl, 2016 for a similar example. It took 477.99 seconds to perform 122 iterations in order to reach the same value of the cost function as obtained by CPPA. The main bottleneck is the approximate evaluation of the involved mean, which has to be computed in every iteration. Here we performed 20 gradient descent steps for this purpose.

For Algorithm 4.2 we set \( \sigma = \tau = 0.4 \) and \( \gamma = 0.2 \). We choose the base point \( m \in \mathcal{P}(3)^{32\times32} \) to be the constant image of unit matrices so that \( n = \Lambda(m) \) consists of zero matrices. We initialize the algorithm with \( p^{(0)} = f \) and \( \xi^{(0)} \) as the zero vector. Our algorithm stops after 113 iterations, which take 96.20 seconds, when the value of (5.1) was below the value obtained by the CPPA. While the CPPA requires about half a second per iteration, our method requires a little less than a second per iteration, but it also requires only a fraction of the iteration count of CPPA. The behavior of
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(a) Original Data.  
(b) Minimizer.  
(c) Cost function.

Fig. 2: Development of the three algorithms Cyclic Proximal Point (CPPA), parallel Douglas–Rachford (PDRA) as well as the linearized Riemannian Chambolle–Pock Algorithm 4.2 (lRCPA) starting all from the original data in (a) reaching the final value (image) in (b) is shown in (c), where the iterations on the x-axis are in log-scale.

The cost function is shown in Figure 2c, where the iterates are in log scale, since the “tail” of CPPA is quite long.

6.3. Dependence on the Point of Linearization. We mentioned previously that Algorithm 4.2 depends on the base points $m$ and $n$ and it cannot, in general, be expected to converge to a saddle point of (4.2) since it is based on the linearized saddle-point problem (4.12). In this experiment we illustrate the dependence of the limit of sequence of primal iterates on the base point $m$.

As data $f$ we use the S2Whirl image designed by Johannes Persch in Laus et al., 2017, adapted to MANOPT.jl, see Figure 3a. We set $\alpha = 1.5$ in the manifold-valued anisotropic TV (5.1). We employed Algorithm 4.2 with $\sigma = \tau = 0.35$ and $\gamma = 0.2$ and ran it for 300 iterations. The initial iterate is $p^{(0)} = f$ and $\xi^{(0)}_n$ as the zero vector.

We compare two different base points $m$. The first base point is the constant image whose value is the mean of all data pixels. The second base point is the constant image whose value is $p = (1, 0, 0)^T$ (“west”). The final iterates are shown in Figures 3b and 3c, respectively. The development of the cost function during the iterations is given in Figure 3d. Both runs yield piecewise constant solutions, but since their linearizations of $\Lambda$ are using different base points and hence yield different
7. Conclusions. This paper introduces a novel concept of Fenchel duality for manifolds employing the tangent bundle. We investigate properties of this novel duality concept and study corresponding primal-dual formulations of non-smooth optimization problems on manifolds. This leads to the first primal-dual algorithm on manifolds, both in its exact and its linearized form, which we call exact and linearized Riemannian Chambolle–Pock algorithm. The convergence proof for the linearized version is given on arbitrary Hadamard manifolds under a suitable assumption. This extends the previous convergence proof for a comparable method, namely the Douglas–Rachford algorithm, where the proof is restricted to Hadamard manifolds of constant curvature. Numerical results not only illustrate that the linearized Riemannian Chambolle–Pock algorithm performs as well as state of the art methods on Hadamard manifolds, but it also performs similarly well on manifolds with positive sectional curvature. Note that here it also has to deal with the absence of a global convexity concept of the functional.

Preliminary experiments indicate that the exact Riemannian Chambolle–Pock Algorithm 4.1, which keeps the operator step exact and only linearizes the adjoint
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step, performs even slightly better than the fully linearized Algorithm 4.2. A more thorough investigation as well as a convergence proof for the exact variant are points for future research. Another point of future research is an investigation of the choice of the base points \( m \in \mathcal{M} \) and \( n \in \mathcal{N} \) on the convergence, especially when the base points vary during the iterations.

Furthermore, our novel concept of duality also permits both a definition of infimal convolution as well as a direct possibility to introduce the total generalized variation. In what way these novel priors correspond to existing ones, is another issue of ongoing research. Furthermore, the investigation of both a convergence rate as well as properties on manifolds with non-negative curvature are also open.

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