Anyon Condensation and Persistent Currents

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Condensation and persistent currents in 1+1 dimensional anyon systems are discovered.

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The recent interest in quantum many body systems in one space dimension is essentially related to some applications in modern condensed matter physics. Among others, they include fractional quantum Hall effect\(^4\), high temperature super-conductivity\(^2\), quasi-one-dimensional organic metals, quantum dots, quantum wires, etc. Some of the just mentioned systems exhibit properties, which require new physical concepts. Signatures of generalized, anyonic statistics (see e.g.\(^3\) and references therein) have been observed for instance in experiments with quantum Hall bars and in one-dimensional organic metals. This letter represents a short account of our recent results\(^4\) concerning the behaviour of anyon systems in one space dimension is essentially related to some applications in modern condensed matter physics.

Let us summarize first the basic points of our approach. We denote by \(\varphi\) a free massless scalar field and its dual \(\tilde{\varphi}\) in 1+1 dimensions, which satisfy the equations of motion

\[
g^{\mu\nu}\partial_\mu \partial_\nu \varphi(x) = g^{\mu\nu}\partial_\mu \partial_\nu \tilde{\varphi}(x) = 0 ,
\]

\((g = \text{diag} (1, -1))\), the duality constraint

\[
\partial_\mu \tilde{\varphi}(x) = \varepsilon_{\mu\nu} \partial^\nu \varphi(x) ,
\]

\((\varepsilon_{01} = 1)\) and the standard equal-time canonical commutation relations. As it is well known, all these assumptions are fulfilled by

\[
\varphi(x) = \frac{1}{2} \left[ \varphi_R(x^-) + \varphi_L(x^+) \right] ,
\]

\[
\tilde{\varphi}(x) = \frac{1}{2} \left[ \varphi_R(x^-) - \varphi_L(x^+) \right] ,
\]

where \(x^\pm = x^0 \pm x^1\) and

\[
\varphi_R(\zeta) \equiv \sqrt{2} \int_0^\infty \frac{dk}{2\pi} \left[ a^*(k) e^{ik\zeta} + a(k) e^{-ik\zeta} \right] ,
\]

\[
\varphi_L(\zeta) \equiv \sqrt{2} \int_{-\infty}^0 \frac{dk}{2\pi} \left[ a^*(k) e^{-ik\zeta} + a(k) e^{ik\zeta} \right] ,
\]

with

\[
[a(k) , a(p)] = [a^*(k) , a^*(p)] = 0 ,
\]

\[
[a(k) , a^*(p)] = |k|^{-1} 2\pi \delta(k - p) .
\]

The distribution \(|k|^{-1}\), whose explicit form is not essential for our discussion, satisfies \(|k| |k|^{-1} = 1\). The parameter \(\lambda \gg 0\) has a well understood infrared origin\(^5\). From Eqs.\(^6\) one gets

\[
[\varphi_1(\zeta_1) , \varphi_2(\zeta_2)] = -i \varepsilon(\zeta_1) \delta_{z_1, z_2} ,
\]

where \(Z = L, R, \zeta_{12} \equiv \zeta_1 - \zeta_2\) and

\[
\delta_{z_1, z_2} = \begin{cases} 1 & \text{if } Z_1 = Z_2 , \\ 0 & \text{if } Z_1 \neq Z_2 . \end{cases}
\]

We denote by \(A_z\) the algebra generated by \(\varphi_z\) and observe that

\[
\alpha_{\mu z} : \varphi_z(\zeta) \mapsto \varphi_z(\zeta) - \frac{1}{\sqrt{\pi}} \mu_z \zeta , \quad \mu_z \in \mathbb{R} ,
\]

is an automorphism of \(A_z\). The parameters \(\mu_z\) will play the role of chemical potentials associated with the chiral charges

\[
q_z = \frac{1}{2} \int_{-\infty}^{\infty} d\zeta j_z(\zeta) , \quad j_z(\zeta) = \frac{1}{\sqrt{\pi}} \partial \varphi_z(\zeta) .
\]

Eq.\(^6\) implies

\[
[q_{z_1}, \varphi_{z_2}(\zeta)] = \frac{1}{i \sqrt{\pi}} \delta_{z_1, z_2} , \quad [q_{z_1}, q_{z_2}] = 0 .
\]

We now turn to generalized statistics, which can be introduced at purely algebraic level. In fact, let us consider the set of fields parametrized by \(\xi = (\sigma , \tau) \in \mathbb{R}^2\) and defined by

\[
A(x; \xi) \equiv z(\lambda ; \xi) \exp \left[ i \frac{\pi}{2} (\tau q_R - \sigma q_L) \right] \times \exp \left\{ i \sqrt{\pi} \left( \sigma \varphi_R(x^-) + \tau \varphi_L(x^+) \right) \right\} ,
\]

where \(z(\lambda ; \xi)\) is a suitable normalization factor and the normal ordering \(\{ \ldots \} \) is taken with respect to \(\{a(k), a^*(k)\}\). Using Eqs.\(^7\), one finds for space-like separated points \((x_1^2 < 0)\)

\[
A(x_1, \xi) A(x_2, \xi) = \exp[-i \pi (\sigma^2 - \tau^2) \varepsilon(x_1^1 - x_2^1)] A(x_2, \xi) A(x_1, \xi) .
\]
Therefore, $A(x; \xi)$ is an anyon field whose statistics parameter $\bar{\xi}$ is
\begin{equation}
\bar{\theta}(\xi) = \pi^2 - \tau^2.
\end{equation}

Bose or Fermi statistics are recovered when $\bar{\theta}$ is an even or odd integer respectively. The remaining values of $\bar{\theta}$ lead to Abelian braid statistics. What is crucial in implementing of generalized statistics is the relative non-locality of $\varphi$ and $\bar{\theta}$.

Let us concentrate on the family $\{A(x; \xi) : \xi \in \Xi\}$ with $\Xi = \{\xi_1, -\xi_1, ..., \xi_n, -\xi_n\}$. In we have constructed a representation $\mathcal{T}(\Xi; \mu_L, \mu_R)$ describing these fields in thermal equilibrium with inverse temperature $\beta$ and chemical potentials $\mu_L, \mu_R$. The construction is in two steps. Using the conventional thermal representation $\mathcal{T}_k$ of the commutation relations $\mathcal{F}$ and the automorphism $\alpha_{\mu_Z}$, we first derive a thermal representation $\mathcal{T}_k(\mu_Z)$ of $\mathcal{A}_x$. The representation $\mathcal{T}_k(\mu_Z)$ has indefinite metric and depends on the parameter $\lambda$ and a bosonic chemical potential $\mu_Z$, introduced for technical reasons. The second step consists in determining $\mathcal{T}(\Xi; \mu_L, \mu_R) = \mathcal{T}_k(\mu_Z) \otimes \mathcal{T}_n(\mu_X)$ by appropriate selection rules. Referring for all details to [5], we stress that $\mathcal{T}(\Xi; \mu_L, \mu_R)$ is both $\lambda$- and $\mu_Z$-independent and has positive metric. The basic correlation functions characterizing this physical representation are
\begin{equation}
\langle A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta = \left( \frac{1}{2\pi} \right)^{\frac{n}{2}} \sum_{i=1}^n (\sigma_i^2 + \tau_i^2) \times 
\exp \left[ \frac{i}{2} \sum_{i,j=1}^n (\tau_i \sigma_j - \tau_j \sigma_i) - i \mu_R \sum_{i=1}^n \sigma_i x_i^+ - i \mu_L \sum_{i=1}^n \tau_i x_i^+ \right] 
\prod_{i,j=1}^n \left[ \frac{i}{\beta} \sinh \left( \frac{\pi}{\beta} x_i^+ - i \epsilon \right) \right] \sigma_i \sigma_j \left[ \frac{i}{\beta} \sinh \left( \frac{\pi}{\beta} x_j^+ - i \epsilon \right) \right] \tau_i \tau_j 
\end{equation}
where $\xi_i \in \Xi$ and
\begin{equation}
\sum_{i=1}^n \sigma_i = \sum_{i=1}^n \tau_i = 0.
\end{equation}

All correlators violating (17) vanish, whereas the insertion of $m$ current operators $j_{x_j}$ in (16) is obtained by iteration, using
\begin{align}
\langle j_{x_1}(\zeta_1) j_{x_2}(\zeta_2) \cdots j_{x_m}(\zeta_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta 
&= - \left\{ \frac{\mu_Z}{\beta} - \frac{\mu_Z}{\beta} \sum_{j=1}^m \frac{\beta}{\beta} \coth \left( \frac{\pi}{\beta} x_j^+ - i \epsilon \right) \right\} 
\langle j_{x_2}(\zeta_2) \cdots j_{x_m}(\zeta_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta 
&= \left[ \frac{\mu_Z}{\beta} + \frac{\mu_Z}{\beta} \sum_{j=1}^m \frac{\beta}{\beta} \coth \left( \frac{\pi}{\beta} x_j^+ - i \epsilon \right) \right] 
\langle j_{x_1}(\zeta_1) j_{x_2}(\zeta_2) \cdots j_{x_j}(\zeta_j) \cdots j_{x_m}(\zeta_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta,
\end{align}

where
\begin{equation}
\xi^Z = \left\{ \begin{array}{ll}
\sigma & \text{if } Z = R, \\
\tau & \text{if } Z = L,
\end{array} \right.
\end{equation}
and the hat in the right hand side of (18) indicates that the corresponding current must be omitted. Eq. (18) implies
\begin{equation}
\langle j_{x_2}(\zeta_2) \rangle_{\mu_L, \mu_R}^\beta = - \frac{\mu_Z}{\pi},
\end{equation}
which is the origin of non-vanishing charge density and persistent current.

The explicit form of the correlators is quite remarkable and deserves a comment. Without current insertions, the equal-time $n$-point function (16) is a finite temperature and density generalization of the Jastrow-Laughlin wave function [5]. We recall that the latter describes the $n$-particle ground state in several one-dimensional models, showing a Tomonaga-Luttinger liquid structure. In that context, the current insertions in (16) are associated with the charged excitations of the liquid. The expectation values (15-18) are invariant under the transformations:
\begin{equation}
K \equiv H - \mu_L Q_L - \mu_R Q_R,
\end{equation}
and
\begin{align}
A(x; \xi) &\to A(x^0 + t, x^1; \xi), \\
j_{x_2}(\zeta_2) &\to j_{x_2}(\zeta_2 + t), \\
A(x; \xi) &\to e^{-i s_n^2} A(x; \xi), \\
j_{x_2}(\zeta_2) &\to j_{x_2}(\zeta_2),
\end{align}

one can prove [5] that the correlation functions (16-18) obey the Kubo-Martin-Schwinger condition relative to $\alpha_s$ with (inverse) temperature $\beta$. Clearly, this result is fundamental for the physical interpretation of the representation $\mathcal{T}(\Xi; \mu_L, \mu_R)$.

The equations of motion, induced by the correlation functions (18), are
\begin{equation}
\frac{\partial}{\partial x^2} \langle j_{x_2}(\zeta_2) A(x; \xi) \rangle = \frac{\partial}{\partial x^2} A(x; \xi).
\end{equation}

where the normal product $:\cdots: \equiv \lim_{y \to x} \{ j_{x_2}(y^Z) A(x; \xi) \}$ and
\begin{equation}
\langle j_{x_2}(\zeta_2) \rangle_{\mu_L, \mu_R}^\beta = - \frac{\mu_Z}{\pi},
\end{equation}

The chiral components of the energy-momentum tensor are
\begin{equation}
\Theta_{x_2}(\zeta_2) = \frac{1}{2} \lim_{\zeta \to \zeta} \left[ \frac{i}{\pi} \langle j_{x_2}(\zeta') j_{x_2}(\zeta) \rangle + \frac{1}{2} (\zeta' - \zeta - i \epsilon)^2 \right].
\end{equation}
Expressed in terms of $\Theta_x$, the Hamiltonian $H$ reads

$$H = \int d\zeta \left[ \Theta_R(\zeta) + \Theta_L(\zeta) \right].$$

In order to get a deeper insight into the physical properties of the field $A(x; \xi)$, it is instructive to derive the relative momentum distribution. For this purpose we consider the Fourier transform

$$\tilde{W}^\beta(\omega, k; \xi) = \int d^2 x e^{i\omega x - i k x} \langle A^\dagger(x; \xi) A(x_2; \xi) \rangle^\beta_{\mu_L, \mu_R}.$$ 

Inserting the explicit expression (see Eq. (16)) of the two-point function, one gets

$$\tilde{W}^\beta(\omega, k; \xi) = \frac{1}{2} \times$$

$$\theta \left( \frac{\omega}{2} + k + \mu_\sigma; \sigma^2, \beta \right) \theta \left( \frac{\omega}{2} - k + \mu_\tau; \tau^2, \beta \right),$$

(22)

Observing that the contributions of the left- and right-moving modes factorize, we consider first the cases

$$\tilde{W}^\beta(\omega, k; (\sigma, 0)) = 2\pi \delta(\omega - k) \theta \left( \frac{\omega}{2} + k + \mu_\sigma; \sigma^2, \beta \right),$$

(24)

and

$$\tilde{W}^\beta(\omega, k; (0, \tau)) = 2\pi \delta(\omega + k) \theta \left( \frac{\omega}{2} + k + \mu_\tau; \tau^2, \beta \right).$$

(25)

Eqs. (24,25) have a simple physical interpretation: the $\delta$-factors fix the dispersion relations, whereas the $\theta$-factors give the momentum distributions. From the exchange relation (14) we already know that the fields $A(x; (\pm 1, 0))$ and $A(x; (0, \pm 1))$ have Fermi statistics. In agreement with this fact, one gets from (24,25)

$$\tilde{W}^\beta(\omega, k; (\pm 1, 0)) = 2\pi \delta(\omega - k) \frac{1}{1 + e^{-\beta(k + \mu_L)}},$$

(26)

$$\tilde{W}^\beta(\omega, k; (0, \pm 1)) = 2\pi \delta(\omega + k) \frac{1}{1 + e^{-\beta(k + \mu_L)}}.$$ 

(27)

We thus recover the familiar Fermi distribution (with Fermi momenta $k_F = \mp \mu_\sigma$ and $k_F = \pm \mu_\tau$), which represents an useful check. Turning back to the general expression (24,25), we see that for $\sigma > 0$ the distribution $\tilde{W}$ is a smooth positive function, whose asymptotic behaviour is encoded in

$$\tilde{W}(\omega, k; (\sigma, 0)) \sim \frac{1}{\Gamma(\alpha)} \left( \frac{k}{2\pi} \right)^{\alpha-1}, \quad k \to \infty,$$

(28)

$$\tilde{W}(\omega, k; (0, \tau)) \sim \frac{e^{\beta k}}{\Gamma(\alpha)} \left( \frac{k}{2\pi} \right)^{\alpha-1}, \quad k \to -\infty.$$ 

(29)

FIG. 1. The distribution $g(k; \alpha, \beta = 1)$ for $\alpha = 1$ (dashed line) and $\alpha = 0.1$ (continuous line).

In the range $\alpha \geq 1$, $g$ is monotonically increasing on the whole line $k \in \mathbb{R}$. When $0 < \alpha < 1$, $g$ increases monotonically for $k \leq 0$ and, according to Eqs. (28,29), admits at least one local maximum for $k > 0$. Let us denote the position of the first one (when $k$ moves from 0 to $\infty$) by $k_c(\alpha, \beta)$. It is easily seen that $k_c(\alpha, \beta) = \beta^{-1} s_c(\alpha)$, where $s_c(\alpha)$ is a solution of certain functional equation, which is not displayed for the sake of conciseness. We have some numerical evidence that the maximum $k_c(\alpha, \beta)$ is unique. The plots of $g$ in Fig. 1 indicate an interesting condensation-like behaviour around $k_c$. For comparison we have plotted there for $\beta = 1$ the cases $\alpha = 1$ (Fermi distribution) and $\alpha = 0.1$. The phenomenon is clearly marked for small values of the temperature and/or of the parameter $\alpha$ in the domain $0 < \alpha < 1$. We find quite remarkable that for any fixed temperature, one can achieve an arbitrary sharp anyon condensation, taking a sufficiently small value of $\alpha$. Concerning the behaviour of $\tilde{W}(\omega, k; \xi)$ when both $\sigma \neq 0$ and $\tau \neq 0$, the above analysis implies condensation at

$$\omega = k_c(\sigma, \beta) + k_c(\tau^2, \beta) - \mu_\sigma - \mu_\tau,$$

(30)

$$k = k_c(\sigma^2, \beta) - k_c(\tau^2, \beta) - \mu_\sigma + \mu_\tau,$$

(31)

provided that $0 < \sigma^2 < 1$ and $0 < \tau^2 < 1$. We stress that the condensation effect we discovered, does not contradict the Hohenberg-Mermin-Wagner (HMW) theorem on the absence of condensation in one space dimension. The point is that anyonic statistics can be equivalently described by a suitable exchange interaction with two- and three-body potentials, determined by the corresponding exchange factor. When these potentials are confining, some assumptions of the HMW theorem are violated and condensation may occur even in one dimension. Summarising, we have shown that the right (left) moving modes of the anyon field $A(x; \xi)$ condensate in the range $0 < \sigma^2 < 1$ ($0 < \tau^2 < 1$).

The thermal representations $\mathcal{T}(\Xi; \mu_L, \mu_R)$ have universal character. As a concrete application, let us consider the 2-d Thirring model. Classically, the model describes a two-component field $\Psi$ satisfying the equation of mo-
\[ i\gamma^5 \partial_\nu \Psi(x) = g\pi \overline{\Psi(x)} \gamma_\nu \Psi(x) \gamma^5 \Psi(x) , \]

where \( g \in \mathbb{R} \) is the coupling constant and

\[ \Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

Both the vector and the chiral current,

\[ J_\nu(x) = \overline{\Psi(x)} \gamma_\nu \Psi(x), \quad J_5^\nu(x) = \overline{\Psi(x)} \gamma_\nu \gamma^5 \Psi(x), \]

(\( \gamma^5 = \gamma^0 \gamma^1 \)) are conserved and satisfy the duality relation

\[ J_5^\nu(x) = \epsilon_{\nu\mu} J^\mu(x). \]

The problem of quantizing this system at finite temperature and chemical potentials \( \mu \) and \( \mu^5 \), associated with the charges \( Q \) and \( Q^5 \) and generated by the currents \([33] \), has been solved in \([33] \). We have constructed there an anyonic solution (see also \([10] \)) with statistical parameter \( \vartheta > -g \). The basic building block of our construction is the representation \( T(\Xi_\nu; \mu_L, \mu_R) \) where \( \Xi_\nu = \{ \xi, -\xi, \xi', -\xi' \} \) with \( \xi = (\sigma, \tau), \xi' = (\tau, \sigma) \) and

\[ \mu_R = \frac{1}{\sigma + \tau} \mu + \frac{1}{\sigma - \tau} \mu_5 , \quad \mu_L = \frac{1}{\sigma + \tau} \mu - \frac{1}{\sigma - \tau} \mu_5 . \]

(35)

It turns out that \( \Psi_1(x) = A(x; \xi) \) and \( \Psi_2(x) = A(x; \xi') \), the quantum counterpart of Eq. \((22) \) being Eq. \((21) \). Finally, in terms of \( g \) and \( \vartheta \) one has

\[ \sigma = \pm \frac{g + 2\vartheta}{2\sqrt{g + \vartheta}}, \quad \tau = \pm \frac{g}{2\sqrt{g + \vartheta}}. \]

(36)

Referring again for details to \( [33] \), we observe that the correlation functions of the Thirring field \( \Psi \) and of both vector and axial currents follow directly from Eqs. \((14 \& 15) \). In particular, for the expectation value of the current \( J_\nu(x) \) one finds

\[ \langle J_0(x) \rangle_{\mu, \mu_5}^\beta = \frac{\mu}{\pi (g + \vartheta)}, \quad \langle J_1(x) \rangle_{\mu, \mu_5}^\beta = \frac{\mu_5}{\pi \vartheta}. \]

(37)

The expression for the charge density resolves some discrepancies present in the literature, confirming the result of \([33] \) and extending it to the case \( \vartheta \neq 1 \). The appearance of a persistent current \( \langle J_1(x) \rangle_{\mu, \mu_5}^\beta \) in the finite temperature Thirring model represents to our knowledge a novel feature. Let us recall in this respect that persistent currents of quantum origin have been experimentally observed (\([11], [12] \)) in mesoscopic rings placed in an external magnetic field. Such fields are absent in the two-dimensional world, but chiral symmetry, combined with duality still allow for a non-vanishing \( \langle J_1(x) \rangle_{\mu, \mu_5}^\beta \).

Notice that the persistent current grows like \( \vartheta^{-1} \) for small values of \( \vartheta \).

Concerning the energy-momentum tensor \( T_{\mu\nu}(x) \) of the Thirring model, we find that

\[ T_{00}(x) = T_{11}(x) = \Theta_\mu(x^+ + \Theta_\mu(x^-), \]

\[ T_{01}(x) = T_{10}(x) = \Theta_\mu(x^+) - \Theta_\mu(x^-), \]

which lead to the energy and momentum densities

\[ \langle T_{00}(x) \rangle_{\mu, \mu_5}^\beta = \frac{\pi}{6\vartheta^2} + \frac{g + \vartheta}{2\pi} \left[ \frac{\mu^2}{(g + \vartheta)^2} + \frac{\mu_5^2}{\vartheta^2} \right], \]

\[ \langle T_{01}(x) \rangle_{\mu, \mu_5}^\beta = \frac{\mu_5}{\pi \vartheta}, \]

and determine the equation of state \([3] \).

In conclusion, we have shown that finite temperature and density anyon systems in 1+1 dimensions exhibit condensation and persistent currents. These phenomena deserve further attention both from the conceptual and applicative point of view.

1. R. B. Laughlin, Phys. Rev. Lett. 50 1395 (1983).
2. P. W. Anderson, Phys. Rev. Lett. 64 1839 (1990).
3. F. Wilczek, Fractional Statistics and Anyon Superconductivity (World Scientific, Singapore, 1990).
4. A. Liguori, M. Mintchev, L. P. Alvarez-Estrada, and A. Gómez Nicola, Phys. Rev. Lett. 90 2074 (1998).
5. T. G. Stoof, in Cargèse Lectures in Theoretical Physics (Gordon and Breach, New York, 1964).
6. A. Liguori and M. Mintchev, Commun. Math. Phys. 169 635 (1995).
7. O. Bratteli and D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics, Vol.2 (Springer, Berlin, 1996).
8. V. Bagnato and D. Kleppner, Phys. Rev. A 44 7439 (1991).
9. P. Nozières, in Bose-Einstein Condensation (Cambridge University Press, Cambridge, 1995).
10. N. Iliev and W. Thirring, Anyons and the Bose-Fermi duality in finite-temperature Thirring model, hep-th/9906202.
11. L. P. Levy, G. Dolan, J. Dunsmuir and H. Bouchiat, Phys. Rev. Lett. 64 2074 (1990).
12. D. Mailly, C. Chapelier and A. Benoît, Phys. Rev. Lett. 70 2020 (1993).
13. R. F. Alvarez-Estrada and A. Gómez Nicola, Phys. Rev. D 57, 3618 (1998).