More General Queries and Less Generalization Error in Adaptive Data Analysis

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March 18, 2015

Abstract

Adaptivity is an important feature of data analysis—typically the choice of questions asked about a dataset depends on previous interactions with the same dataset. However, generalization error is typically bounded in a non-adaptive model, where all questions are specified before the dataset is drawn. Recent work by Dwork et al. (STOC ’15) and Hardt and Ullman (FOCS ’14) initiated the formal study of this problem, and gave the first upper and lower bounds on the achievable generalization error for adaptive data analysis.

Specifically, suppose there is an unknown distribution $\mathcal{P}$ and a set of $n$ independent samples $x$ is drawn from $\mathcal{P}$. We seek an algorithm that, given $x$ as input, “accurately” answers a sequence of adaptively chosen “queries” about the unknown distribution $\mathcal{P}$. How many samples $n$ must we draw from the distribution, as a function of the type of queries, the number of queries, and the desired level of accuracy?

In this work we make two new contributions towards resolving this question:

1. We give upper bounds on the number of samples $n$ that are needed to answer statistical queries that improve over the bounds of Dwork et al.

2. We prove the first upper bounds on the number of samples required to answer more general families of queries. These include arbitrary low-sensitivity queries and the important class of convex risk minimization queries.

As in Dwork et al., our algorithms are based on a connection between differential privacy and generalization error, but we feel that our analysis is simpler and more modular, which may be useful for studying these questions in the future.

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1 Introduction

A common problem in empirical research is testing the significance of multiple “hypotheses” on a finite sample of data drawn from some population. An observation is deemed significant if it is unlikely to have occurred by chance alone, and a “false discovery” occurs if the analyst incorrectly declares an observation significant. Unfortunately, false discovery has been identified as a substantial problem in the scientific community (see e.g. [Ioa05]). This problem persists despite decades of research by Statisticians on methods for preventing false discovery, such as the widely used Bonferroni Correction [Bon36, Dun61] and the Benjamini-Hochberg Procedure [BH95].

False discovery is often attributed to misuse of statistics. However, an alternative explanation is that the prevalence of false discovery arises from the inherent adaptivity in the data analysis process—the choice of hypotheses to test depends on previous interactions with the data (see e.g. [Ioa05, GL13]). This problem was first formally modeled and studied in a striking recent paper by Dwork, Feldman, Hardt, Pitassi, Reingold, and Roth [DFH+15], who gave the first non-trivial algorithms for provably ensuring statistical validity in adaptive data analysis. The problem was also studied through a computational lens by [HU14, SU14], who showed that there is an inherent computational barrier to preventing false discovery in adaptive settings.

In this work, we extend the results of Dwork et al. [DFH+15] along two axes. First, we prove quantitatively stronger bounds on the statistical accuracy that can be achieved. Second, we significantly generalize the types of statistics that can be computed in adaptive settings. Like the algorithms of Dwork et al., all of our algorithms rely on the connection between differential privacy [DMNS06] and statistical validity, and our analysis generalizes, strengthens, and (in our opinion) simplifies this connection.

1.1 Overview of Results

Following these previous papers, we formalize the problem of adaptive data analysis as follows: There is a distribution \( \mathcal{P} \) over some finite universe \( \mathcal{X} \), and an oracle \( \mathcal{O} \) that does not know \( \mathcal{P} \) but is given \( n \) samples from \( \mathcal{P} \). Using its sample, the oracle must answer queries on \( \mathcal{P} \). Here, a query \( q \) maps a distribution \( \mathcal{P} \) to a real-valued answer, and we assume the queries come from some family \( \mathcal{Q} \). The oracle’s answer \( a \) to a query \( q \) is \( \alpha \)-accurate if \( |a - q(\mathcal{P})| \leq \alpha \) with high probability. Importantly, the oracle’s goal is to provide answers that “generalize” to the underlying distribution, rather than answers that are specific to the sample.

We model adaptivity by allowing the analyst to make a sequence of queries \( q_1, q_2, \ldots, q_k \in \mathcal{Q} \) to the oracle, who responds with answers \( a_1, a_2, \ldots, a_k \). In the adaptive setting, the query \( q_j \) may depend on the previous queries and answers \( q_1, a_1, \ldots, q_{j-1}, a_{j-1} \) arbitrarily. We say the oracle is \( \alpha \)-accurate given \( n \) samples for \( k \) adaptively chosen queries if, when given \( n \) samples from an arbitrary distribution \( \mathcal{P} \), the oracle accurately responds to any adaptive analyst that makes at most \( k \) queries with high probability.

Dwork et al. [DFH’15] considered the family of statistical queries [Kea93]. A statistical query \( q \) is specified by a function \( p : \mathcal{X} \rightarrow [0, 1] \) and is defined as \( q(\mathcal{P}) = \mathbb{E}_{z \sim \mathcal{P}}[p(z)] \). A simple analysis shows that, when \( k \) queries are specified non adaptively, (i.e. independent of previous answers), then simply evaluating the query on the sample is \( \alpha \)-accurate with high probability as long as the number of samples \( n \) satisfies \( n \geq \log k/\alpha^2 \). However, when the queries can be chosen adaptively, the empirical average performs exponentially worse, requiring \( n \geq k/\alpha^2 \) to guarantee accuracy.
Dwork et al. [DFH+15], showed how to achieve dramatically better accuracy guarantees for statistical queries, namely that \( n \gtrsim \min \left\{ \sqrt{\frac{k}{\alpha^2}}, \frac{\sqrt{\log |X| \log k}}{\alpha^2} \right\} \) samples suffice, which can be exponentially smaller than what is needed when outputting the sample mean. However, these bounds suffer from a large dependence on the accuracy level \( \alpha \).

Our first contribution is to give a tighter and simpler analysis of (a slight variant of) their algorithms that yields the best known accuracy guarantees for adaptive statistical queries. In particular, we show that \( \tilde{O} \left( \sqrt{\frac{k}{\alpha^2}} \right) \) samples suffice, which is the optimal dependence on \( \alpha \).

Although statistical queries are a surprisingly general primitive for data analysis, we would like to be able to ask more general queries on the distribution \( P \) that capture a wider variety of machine learning and data mining tasks. Specifically, in this work we consider two general types of queries: low-sensitivity queries and search queries.

Low-sensitivity queries are a generalization of statistical queries that are specified by an arbitrary function \( p : X^n \rightarrow \mathbb{R} \) satisfying \( |p(x) - p(x')| \leq 1/n \) for every \( x, x' \in X^n \) differing on exactly one element. The query applied to the population is defined to be \( q(P) = \mathbb{E}_{x \sim P^n} [p(x)] \).

Search queries are a broad generalization of low-sensitivity queries to arbitrary output domains. The query is specified by a loss function \( L : X^n \times \Theta \rightarrow \mathbb{R} \) that is low-sensitivity in its first parameter, and the goal is to output \( \theta \in \Theta \) that is “best” in the sense that it minimizes the average loss. Specifically, \( q(P) = \arg\min_{\theta \in \Theta} \mathbb{E}_{z \sim P^n} [L(z; \theta)] \). An important special case is when \( L \) is convex in the parameter \( \theta \), which captures fundamental tasks in machine learning such as linear and logistic regression.

Our second contribution is to give the first bounds on the sample complexity required to answer a large number of adaptively chosen low-sensitivity queries and search queries. Our sample complexity bounds are summarized in Figure 1.

\[
\begin{array}{|c|c|c|}
\hline
\text{Query Type} & \text{Sample Complexity} & \text{Time per Query} \\
\hline
\text{Statistical (} k \ll n^2 \text{)} & \tilde{O} \left( \frac{\sqrt{k}}{\alpha^2} \right) & \text{poly}(n, \log |X|) \\
\text{Statistical (} k \gg n^2 \text{)} & \tilde{O} \left( \frac{\sqrt{\log k \log |X|}}{\alpha^2} \right) & \text{poly}(n, |X|) \\
\text{Low Sensitivity (} k \ll n^2 \text{)} & - & \tilde{O} \left( \frac{\sqrt{k}}{\alpha^2} \right) & \text{poly}(n, \log |X|) \\
\text{Low Sensitivity (} k \gg n^2 \text{)} & - & \tilde{O} \left( \frac{\sqrt{\log |X| \log k}}{\alpha^2} \right) & \text{poly}(|X|^2) \\
\text{Convex Min. (} k \ll n^2 \text{)} & - & \tilde{O} \left( \frac{\sqrt{k}}{\alpha^2} \right) & \text{poly}(n, d, \log |X|) \\
\text{Convex Min. (} k \gg n^2 \text{)} & - & \tilde{O} \left( \frac{\sqrt{dk + \log k}}{\alpha^2} \sqrt{\log |X|} \right) & \text{poly}(n, d, |X|) \\
\hline
\end{array}
\]

\( k \) = number of queries, \( n \) = number of samples, \( \alpha \) = desired accuracy, \( X \) = universe of possible samples, \( d \) = dimensionality of space being minimized over.

Figure 1: Summary of Results.
On Optimality: Although our bounds improve on [DFH+15] for statistical queries, and are the first non-trivial bounds for other families of queries, we cannot prove that any of these bounds are optimal. Even for non-adaptive statistical queries, \( n \gtrsim \log(k)/\alpha^2 \) samples are necessary, and [HU14, SU14] showed that \( n \gtrsim \min\{\sqrt{k}, \sqrt{\log|\mathcal{X}|}/\alpha\} \) samples are necessary to answer adaptively chosen statistical queries. These lower bounds are information theoretic and apply to computationally unbounded oracles.

However, it is known that, for every family of queries we consider, the differentially private mechanisms we use achieve optimal sample complexity [BUV14, BST14]. It is also known that the privacy parameters we require cannot be improved. Thus, any improvement to our bounds (beyond logarithmic factors) must come from using an oracle that is not differentially private.

On Computational Complexity: Throughout, we will assume that the analyst only issues queries \( q \) such that \( q(x) \) can be evaluated in time \( \text{poly}(n, \log|\mathcal{X}|) \), and are not the bottleneck in computation. Thus the empirical answer can be given in time \( \text{poly}(n, \log|\mathcal{X}|) \). When \( k \ll n^2 \) our algorithms have similar running time. However, when answering \( k \gg n^2 \) queries, our algorithms suffer running time \( \text{poly}(n, |\mathcal{X}|) \). Since the oracle’s input is of size \( n \cdot \log|\mathcal{X}| \), these algorithms cannot be considered computationally efficient. For example, if \( \mathcal{X} = \{0, 1\}^d \) for some dimension \( d \), then in the non adaptive setting \( \text{poly}(n, 2^d) \) running time would suffice, whereas our algorithms require \( \text{poly}(n, 2^d) \) running time. Unfortunately, this running time is known to be optimal, as [HU14, SU14] (building on hardness results in privacy [Ull13]) showed that, assuming exponentially hard one-way functions exist, any \( \text{poly}(n, 2^{o(d)}) \) time oracle that answers \( k = \omega(n^2) \) statistical queries is not even 1/3-accurate.

1.2 Overview of Techniques

Following Dwork et al. [DFH+15], the main technique we use is a connection between differential privacy and generalization. Intuitively, differential privacy guarantees that the distribution of outputs given by the oracle does not depend “too much” on any one of the samples it is given. Differential privacy can be seen as a strong stability guarantee that behaves well under adaptive data analysis. Thus, this work fits within the rich line of work in machine learning connecting algorithmic stability and generalization (cf. [BE02, SSSS10]).

We show that a differentially private algorithm that provides answers to adaptive queries that are close to the empirical value on the sample gives answers that generalise to the underlying distribution. In particular, for the class of low-sensitivity queries, we obtain the following theorem.

Theorem 1.1 (Main “Transfer Theorem”). Let \( O \) be an oracle that takes as input a sample \( x \in \mathcal{X}^n \) and answers \( k \) adaptively chosen low-sensitivity queries. Suppose \( O \) satisfies the following.

1. The answers \( O \) gives are \( \alpha \)-accurate with respect to the sample \( x \). That is,

\[
\mathbb{E}\left[ \max_{j \in [k]} |q_j(x) - a_j| \right] \leq \alpha,
\]

where \( q_1, \ldots, q_k : \mathcal{X}^n \to \mathbb{R} \) are the low-sensitivity queries that are asked and \( a_1, \ldots, a_k \in \mathbb{R} \) are the answers given. The expectation is taken only over \( O \)’s random coins—the sample \( x \), and the sequence of queries \( q_1, \ldots, q_k \) can be arbitrary.

2. \( O \) satisfies \((\alpha, \alpha)\)-differential privacy. (Definition 2.6)
Then there exists an oracle $O'$ that takes as input a sample $x' \leftarrow \mathcal{P}^{n'}$ where $\mathcal{P}$ is an arbitrary distribution over $\mathcal{X}$, and answers $k$ adaptively chosen low-sensitivity queries such that, if $n' = O(n \cdot \log(1/\beta))$, then $O$ gives $O(\alpha)$-accurate answers with respect to the population $\mathcal{P}$ with high probability. That is,

$$
P \left[ \max_{j \in k} \left| \mathbb{E}_{x \sim \mathcal{P}^{n'}} [q_j(x)] - a_j \right| \leq O(\alpha) \right] \geq 1 - \beta,
$$

where the probability is taken only over the choice of $x$ and $O'$'s random coins. Moreover, the running time of $O'$ is roughly the same as that of $O$.

The results in Figure 1 follow from combining Theorem 1.1 (or an analogous theorem for minimization queries) with known differentially private algorithms.

Compared to the results of Dwork et al., Theorem 1.1 requires a weaker privacy guarantee: we require $(\alpha, \alpha)$-differential privacy, instead of (roughly) $(\alpha, (\beta/k)^{1/\alpha})$-differential privacy. It also generalizes this phenomenon beyond statistical queries.

Our analysis differs from that of Dwork et al. in two key ways. First, Dwork et al. analyze the algorithm one query at a time. That is, they show that if the oracle is differentially private for suitable parameters, then the oracle answers each query $q_j$ accurately. In order to argue that the oracle answers every query accurately (even with just a non-zero probability), they need to prove that each query is answered accurately with very high probability. To do so, they rely on an involved analysis that is heavily tailored to statistical queries and introduces a poor dependence on the privacy parameters.

We give a simple proof that, if the oracle is differentially private, then it answers every query accurately with at least constant probability. As a feature of its simplicity, the analysis generalizes to richer families of queries. Our proof introduces the concept of a monitoring algorithm. Roughly, the monitoring algorithm simulates the interaction between the oracle and the adversary and outputs a query based on the entire interaction, thereby reducing analyzing the entire interaction to analyzing just the monitoring algorithm. In our setting, we will have the monitoring algorithm output the least accurate query in the whole interaction. Thus, if there were even one query in the interaction that is inaccurate, then the monitoring algorithm would output a single query that is inaccurate. By a simple argument, there is at least a constant probability that the monitor’s query is answered accurately, so there is also a constant probability that every query in the interaction is answered accurately.

The next step is to amplify the probability of success. (Dwork et al. avoid this step.) In the non-adaptive setting, amplification can be easily achieved by running the oracle a few times on independent samples, and taking some sort of median of its answers. However, in the adaptive setting, even if each sample is independent, the answers of the oracle may not be independent. Instead, we have the oracle split its sample into a small number of subsamples. To answer any given query, it will use only one of the subsamples. We then design a “censor” algorithm that can identify when the oracle gives an incorrect answer, and when this occurs, the oracle will throw out that subsample and use a new one. For any given subsample, the oracle will answer every query correctly with constant probability, therefore with high probability we only need to throw out a small number of subsamples, and thus are unlikely to exhaust the supply of subsamples.

1.3 Related Work in Differential Privacy

Each of our results requires instantiating the oracle with a suitable differentially private algorithm. For statistical queries, the optimal mechanisms are the well known Gaussian Meccha-
nism (slightly refined by [SU15]) when \( k \) is small and the Private Multiplicative Weights Mechanism [HR10] when \( k \) is large. For arbitrary low-sensitivity queries, the Laplace Mechanism is again optimal when \( k \) is small, and for large \( k \) we can use the Median Mechanism [RR10].

When considering arbitrary search queries over an arbitrary finite range, the optimal algorithm is the Exponential Mechanism [MT07]. The first efficient differentially private algorithms for the important special case of convex minimization queries over an infinite domain were given by Dwork and Lei [DL09] and by Chaudhuri, Monteleone, and Sarwate [CMS11], leading to a long line of subsequent work. For the case of small \( k \), we use the worst-case optimal algorithm of Bassily, Smith, and Thakurta [BST14]. When \( k \) is large, we can use an algorithm of [Ull15] that accurately answers exponentially many such queries.

2 Preliminaries

2.1 Queries

Given a distribution \( \mathcal{P} \) over \( \mathcal{X} \) or a sample \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n \), we would like to answer queries about \( \mathcal{P} \) or \( x \) from some family \( Q \). We will work with several families of queries.

- **Statistical Queries:** These queries are specified by a predicate \( q : \mathcal{X} \rightarrow [0, 1] \), and (abusing notation) are defined as
  \[
  q(\mathcal{P}) = \mathbb{E}_{z \leftarrow \mathcal{P}}[q(z)], \quad q(x) = \frac{1}{n} \sum_{i \in [n]} q(x_i).
  \]
  The error of an answer \( a \) to a statistical query \( q \) with respect to \( \mathcal{P} \) or \( x \) is defined to be
  \[
  \text{err}_x(q, a) = a - q(x) \quad \text{and} \quad \text{err}_\mathcal{P}(q, a) = a - q(\mathcal{P}).
  \]

- **\( \Delta \)-Sensitive Queries:** For \( \Delta \in [0, 1], n \in \mathbb{N} \), these queries are specified by a function \( q : \mathcal{X}^n \rightarrow \mathbb{R} \) satisfying \( |q(x) - q(x')| \leq \Delta \) for every pair \( x, x' \in \mathcal{X}^n \) differing in only one entry. We define (abusing notation)
  \[
  q(\mathcal{P}) = \mathbb{E}_{z \leftarrow \mathcal{P}}[q(z)].
  \]
  The error of an answer \( a \) to a \( \Delta \)-sensitive query \( q \) with respect to \( \mathcal{P} \) or \( x \) is defined to be
  \[
  \text{err}_x(q, a) = a - q(x) \quad \text{and} \quad \text{err}_\mathcal{P}(q, a) = \mathbb{E}_{z \leftarrow \mathcal{P}}[\text{err}_z(q, a)] = a - q(\mathcal{P}).
  \]
  We denote the set of all \( \Delta \)-sensitive queries by \( Q_\Delta \). If \( \Delta = O(1/n) \) we say the query is low sensitivity. Note that \( 1/n \)-sensitive queries are a strict generalization of statistical queries.

- **Minimization Queries:** These queries are specified by a loss function \( L : \mathcal{X}^n \times \Theta \rightarrow \mathbb{R} \). We require that \( L \) has sensitivity \( \Delta \) with respect to its first parameter, that is,
  \[
  \sup_{\theta \in \Theta, x, x' \in \mathcal{X}^n \text{ s.t. } d_{Ham}(x, x') = 1} |L(x; \theta) - L(x'; \theta)| \leq \Delta.
  \]
  Here \( \Theta \) is an arbitrary set of items (sometimes called “parameter values”) among which we aim to chose the parameter with minimal loss, either with respect to a particular input data set \( x \), or with respect to expectation over a distribution \( \mathcal{P} \).
The error of an answer \( \theta \in \Theta \) to a minimization query \( L : \mathcal{X}^n \times \Theta \rightarrow \mathbb{R} \) with respect to \( x \) is defined to be
\[
\text{err}_x(L, \theta) = L(x, \theta) - \min_{\theta' \in \Theta} L(x, \theta')
\]
and, with respect to \( P \), is
\[
\text{err}^P(L, \theta) = \mathbb{E}_{z \leftarrow P^n}[\text{err}_z(L, \theta)] = \mathbb{E}_{z \leftarrow P^n}[L(z, \theta)] - \mathbb{E}_{z \leftarrow P^n}[\min_{\theta' \in \Theta} L(z, \theta')].
\]
Note that \( \min_{\theta' \in \Theta} \mathbb{E}_{z \leftarrow P^n}[L(z, \theta')] \geq \mathbb{E}_{z \leftarrow P^n}[\min_{\theta' \in \Theta} L(z, \theta')] \), whence
\[
\mathbb{E}_{z \leftarrow P^n}[L(z, \theta)] - \min_{\theta' \in \Theta} \mathbb{E}_{z \leftarrow P^n}[L(z, \theta')] \leq \text{err}^P(L, \theta).
\]
Note that minimization queries (with \( \Theta = \mathbb{R} \)) generalize low-sensitivity queries: Given a \( \Delta \)-sensitive \( q : \mathcal{X}^n \rightarrow \mathbb{R} \), we can define \( L(x, \theta) = |\theta - q(x)| \) to obtain a minimization query with the same answer.
We denote the set of minimization queries by \( Q_{\text{min}} \). We highlight two special cases:

- **Minimization for Finite Sets** We denote by \( Q_{\text{min}, D} \) the set of minimization queries where \( \Theta \) is finite with size at most \( D \).

- **Convex Minimization Queries** If \( \Theta \subset \mathbb{R}^d \) is closed and convex and \( L(x; \cdot) \) is convex on \( \Theta \) for every data set \( x \), then the query can be answered nonprivately up to any desired error \( \alpha \), in time polynomial in \( d \) and \( \alpha \). We denote the set of all convex minimization queries by \( Q_{\text{CM}} \).

### 2.2 Oracles for Adaptive Queries

Our goal is to design an oracle \( O \) that answers queries on \( P \) using only independent samples \( x_1, \ldots, x_n \leftarrow_P P \). Our focus is the case where the queries are chosen adaptively and adversarially.

Specifically, \( O \) is a stateful algorithm that holds a collection of samples \( x_1, \ldots, x_n \in \mathcal{X} \), takes a query \( q \) from some family \( Q \) as input, and returns an answer \( a \). We require that when \( x_1, \ldots, x_n \) are independent samples from \( P \), the answer \( a \) is "close" to \( q(P) \) in a sense that is appropriate for the family of queries. Moreover we require that this condition holds for every query in an adaptively chosen sequence \( q_1, \ldots, q_k \). Formally, we define the accuracy game between an \( O \) and a stateful adversary \( A \) in Figure 2.

\[\begin{align*}
    \mathcal{A} & \text{ chooses a distribution } P \text{ over } \mathcal{X}.
    
    \text{Sample } x_1, \ldots, x_n \leftarrow_P P, \text{ let } x = (x_1, \ldots, x_n). \text{ (Note that } \mathcal{A} \text{ does not know } x. \)
    
    \text{For } j = 1, \ldots, k \quad \\
    \quad \mathcal{A} \text{ outputs a query } q_j \in Q.
    
    \mathcal{O}(x, q_j) \text{ outputs } a_j.
    
    \text{(As } \mathcal{A} \text{ and } \mathcal{O} \text{ are stateful, } q_j \text{ and } a_j \text{ may depend on the history } q_1, a_1, \ldots, q_{j-1}, a_{j-1}. \)
\end{align*}\]

Figure 2: The Accuracy Game \( \text{Acc}_{n,k,Q}[O, A] \)
**Definition 2.1 (Accuracy).** An oracle $O$ is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q$ given $n$ samples in $\mathcal{X}$ if for every adversary $A$,

$$\mathbb{P}_{\text{Acc}_{n,k,Q}[O,A]} \left[ \max_{j \in [k]} \left| \text{err}^P(q_j, a_j) \right| \leq \alpha \right] \geq 1 - \beta.$$ 

We will make use of average accuracy.

**Definition 2.2 (Average Accuracy).** An oracle $O$ is $\alpha$-accurate on average with respect to the population for $k$ adaptively chosen queries from $Q$ given $n$ samples in $\mathcal{X}$ if for every adversary $A$,

$$\mathbb{E}_{\text{Acc}_{n,k,Q}[O,A]} \left[ \max_{j \in [k]} \left| \text{err}^P(q_j, a_j) \right| \right] \leq \alpha.$$ 

We will also use a definition of accuracy relative to the sample given to the oracle (Figure 3).

A chooses $x = (x_1, \ldots, x_n) \in \mathcal{X}^n$.
For $j = 1, \ldots, k$ 
$A$ outputs a query $q_j \in Q$.
$O(x, q_j)$ outputs $a_j$.
($q_j$ and $a_j$ may depend on the history $q_1, a_1, \ldots, q_{j-1}, a_{j-1}$ and on $x$.)

**Figure 3: The Sample Accuracy Game SampAcc_{n,k,Q}[O,A]**

**Definition 2.3 (Sample Accuracy).** An oracle $O$ is $(\alpha, \beta)$-accurate with respect to samples of size $n$ from $\mathcal{X}$ for $k$ adaptively chosen queries from $Q$ if for every adversary $A$,

$$\mathbb{P}_{\text{SampAcc}_{n,k,Q}[O,A]} \left[ \forall j \in [k] \left| \text{err}_x(q_j, a_j) \right| \leq \alpha \right] \geq 1 - \beta.$$ 

**Definition 2.4 (Average Sample Accuracy).** An oracle $O$ is $\alpha$-accurate on average with respect to samples of size $n$ from $\mathcal{X}$ for $k$ adaptively chosen queries from $Q$ if for every adversary $A$,

$$\mathbb{E}_{\text{SampAcc}_{n,k,Q}[O,A]} \left[ \max_{j \in [k]} \left| \text{err}_x(q_j, a_j) \right| \right] \leq \alpha.$$ 

By Markov’s inequality, $\alpha'$-accuracy on average implies $(\alpha, \beta)$ accuracy for suitable choices of $\alpha, \beta$.

**Fact 2.5.** For all $\alpha, \beta > 0$, if $O$ is $\beta$-accurate on average with respect to the population (resp. sample) for $k$ adaptively chosen queries from $Q$ given $n$ samples in $\mathcal{X}$, then $O$ is $(\alpha, \beta)$-accurate with respect to the population (resp. sample) for $k$ adaptively chosen queries from $Q$ given $n$ samples in $\mathcal{X}$.

Because of the dependence on $\beta$ in the assumption, we only apply this result when $\beta = \Omega(1)$. 

7
2.3 Differential Privacy

A key tool in our analysis is differential privacy \cite{DMNS06}. Informally, an algorithm is differentially private if it is randomized and the distribution of its outputs does not depend “much” on any one element of its input sample. We say that two samples \( x, x' \in X^n \) are neighbors if they differ on exactly one entry, and denote this relation by \( x \sim x' \).

**Definition 2.6** (Differential Privacy). Let \( M : X^n \to R \) be a randomized algorithm. We say \( M \) is \((\epsilon, \delta)\)-differentially private if for every \( x \sim x' \) and every \( R \subseteq R \),

\[
\Pr[M(x) \in R] \leq e^\epsilon \cdot \Pr[M(x') \in R] + \delta.
\]

A useful fact about differential privacy is that it is robust to post-processing.

**Lemma 2.7** ([DMNS06]). If \( M : X^n \to R \) is \((\epsilon, \delta)\)-differentially private, and \( f : R \to R' \) is any (possibly randomized) function, then \( f \circ M : X^n \to R' \) is also \((\epsilon, \delta)\)-differentially private.

3 From Privacy to Accuracy for Adaptive Queries

We begin by showing that differentially private algorithms that answer adaptive low-sensitivity queries and are accurate with respect to the sample, are accurate with respect to the population.

3.1 Differential Privacy Implies Low Expected Error

We start with the following key lemma, which analyzes the expectation of \( q(x) \) versus \( q(P) \) when \( q \) is allowed to depend on \( x \) in a differentially private manner.

**Lemma 3.1.** Let \( M : X^n \to Q_\Delta \) be \((\epsilon, \delta)\)-differentially private where \( Q_\Delta \) is the class of \( \Delta \)-sensitive queries \( q : X^n \to R \). Let \( P \) be a distribution on \( X \) and let \( x \leftarrow_r P^n \). Then

\[
\left| \mathbb{E}_{x,M} [q(P) \mid q = M(x)] - \mathbb{E}_{x,M} [q(x) \mid q = M(x)] \right| \leq 2\Delta(e^\epsilon - 1 + \delta)n.
\]

**Proof.** The proof proceeds via a hybrid argument. Let \( x' \leftarrow_r P^n \) be independent of \( x \). For \( \ell \in [n] \cup \{0\} \), define \( x^{\ell} \in X^n \) by

\[
x^{\ell}_i = x_i \text{ if } i > \ell \quad \text{ and } \quad x^{\ell}_i = x'_i \text{ if } i \leq \ell.
\]

Then \( x^0 = x \), \( x^n = x' \), and \( x^{\ell} \sim x^{\ell-1} \) are neighboring databases for every \( \ell \in [n] \). For \( \ell \in [n] \), define \( B_\ell : X^n \times X^n \to R \) by

\[
B_\ell(y,z) = q(z) - q(z_{-\ell}) + \Delta \text{ for } q = M(y),
\]

where \( z_{-\ell} \) is \( z \) with the \( \ell \)-th row replaced by some arbitrary fixed element of \( X \).
Since \( x_{\ell}^{\ell} = x_{\ell-1}^{\ell-1} \), we have
\[
\begin{align*}
&\left| \mathbb{E} [q(P) \mid q = \mathcal{M}(x)] - \mathbb{E} [q(x) \mid q = \mathcal{M}(x)] \right| \\
&= \left| \mathbb{E} [q(x') \mid q = \mathcal{M}(x)] - \mathbb{E} [q(x) \mid q = \mathcal{M}(x)] \right| \\
&\leq \sum_{\ell \in [n]} \left| \mathbb{E} [q(x') \mid q = \mathcal{M}(x)] - \mathbb{E} [q(x') \mid q = \mathcal{M}(x)] \right| \\
&= \sum_{\ell \in [n]} \left| \mathbb{E} [(q(x') - q(x_{\ell}) + \Delta) - (q(x_{\ell}) - q(x_{\ell}^{\ell-1}) + \Delta) \mid q = \mathcal{M}(x)] \right| \\
&= \sum_{\ell \in [n]} \left| \mathbb{E} [B_{\ell}(x,x^\ell) - B_{\ell}(x,x^{\ell-1})] \right|.
\end{align*}
\]

Thus, it suffices to show that \( \left| \mathbb{E} [B_{\ell}(x,x^\ell) - B_{\ell}(x,x^{\ell-1})] \right| \leq 2\Delta(e^\varepsilon - 1 + \delta) \) for all \( \ell \in [n] \).

Since \( q \) is \( \Delta \)-sensitive, for every \( \ell, y, z \), we have \( 0 \leq B_{\ell}(y, z) \leq 2\Delta \). Moreover, by construction, \( B_{\ell}(y, z) \) is a \((\varepsilon, \delta)\)-differentially private function of \( y \). We will use these two facts to prove the desired bound.

We need one more observation to complete the proof. Note that each \( x^\ell \) has the same marginal distribution, namely \( n \) independent samples from \( P \). Thus, \( x^\ell \) has the same marginal distribution as \( x \) (although they are not independent). Moreover, for every \( \ell \in [n] \cup \{0\} \), the pair \((x^\ell, x)\) has the same distribution as \((x, x^\ell)\): Each corresponding pair of rows \((x^\ell_i, x_i)\) is independent from the other rows. For \( i \leq \ell \), \( x^\ell_i \) and \( x_i \) are independent samples from \( P \). For \( i > \ell \), \( x^\ell_i = x_i \) and they are a sample from \( P \).

Consider the random variables \( B_{\ell}(x,x^\ell) \) and \( B_{\ell}(x,x^{\ell-1}) \) for some \( \ell \in [n] \). Now we use privacy and the earlier observation:
\[
B_{\ell}(x,x^\ell) \sim B_{\ell}(x^\ell,x) \sim_{(\varepsilon, \delta)} B_{\ell}(x^{\ell-1},x) \sim B_{\ell}(x,x^{\ell-1}),
\]

where \( \sim \) denotes having the same distribution and \( \sim_{(\varepsilon, \delta)} \) denotes having \((\varepsilon, \delta)\)-indistinguishable distributions.\(^1\) Thus \( B_{\ell}(x,x^{\ell-1}) \) and \( B_{\ell}(x,x^\ell) \) are \((\varepsilon, \delta)\)-indistinguishable.

We have
\[
\begin{align*}
\mathbb{E}[B_{\ell}(x,x^{\ell-1})] &\leq \int_{0}^{2\Delta} \mathbb{P}[B_{\ell}(x,x^{\ell-1}) \geq t] \, dt \\
&\leq \int_{0}^{2\Delta} \left( e^\varepsilon \mathbb{P}[B_{\ell}(x,x^\ell) \geq t] + \delta \right) \, dt \\
&= e^\varepsilon \int_{0}^{2\Delta} \mathbb{P}[B_{\ell}(x,x^\ell) \geq t] \, dt + 2\Delta \delta \\
&= e^\varepsilon \mathbb{E}[B_{\ell}(x,x^\ell)] + 2\Delta \delta \\
&\leq \mathbb{E}[B_{\ell}(x,x^\ell)] + 2\Delta(e^\varepsilon - 1 + \delta),
\end{align*}
\]

as \( \mathbb{E}[B_{\ell}(x,x^\ell)] \leq 2\Delta \). This gives one half of the result, the other side is symmetric. \(\square\)

\(^1\)In the spirit of differential privacy, distributions \( A \) and \( B \) over \( \mathcal{R} \) are \((\varepsilon, \delta)\)-indistinguishable if for every (measurable) \( R \subseteq \mathcal{R} \), \( \mathbb{P}[A \in R] \leq e^\varepsilon \cdot \mathbb{P}[B \in R] + \delta \) and vice versa.
Now that we have Lemma 3.1, we can prove the following result that differentially private oracles that are accurate with respect to their sample are also accurate with respect to the population from which that sample was drawn.

**Theorem 3.2.** Assume that $\mathcal{O}$ is

1. $(\varepsilon, \delta)$-differentially private for $\varepsilon = \delta = \alpha/10 \Delta n \leq 1/4$, and
2. $\alpha/2$-accurate on average with respect to samples of size $n$ from $\mathcal{X}$ for $k$ adaptively chosen queries from $Q \subseteq Q_\Delta$ (the family of $\Delta$-sensitive queries on $\mathcal{X}$).

Then $\mathcal{O}$ is $\alpha$-accurate on average with respect to the population for $k$ adaptively chosen queries from $Q$ given $n$ samples from $\mathcal{X}$.

**Proof.** Suppose, for the sake of contradiction, that $\mathcal{O}$ is not $\alpha$-accurate on average with respect to the population for $k$ adaptively chosen queries from $Q$ given $n$ samples from $\mathcal{X}$. That is, there exists an adversary $A$ such that

$$\mathbb{E}_{\text{Acc}_{n,k,Q}(\mathcal{O}, A)} \left[ \max_{j \in [k]} \| \text{err}^P(q_j, a_j) \| \right] > \alpha.$$

Fix such an adversary $A$ and the distribution $P$ that it outputs. Now define a meta algorithm $\mathcal{M}$, which we call the monitor. Note that $\mathcal{M}$ has access to the “true” answers with respect to $P$.

**Algorithm $\mathcal{M} = \mathcal{M}(\mathcal{O}, A, P) : \mathcal{X}^n \rightarrow Q_\Delta$.**

1. Draw a sample $x \in \mathcal{X}^n$ from $P$ and let $\mathcal{O} = \mathcal{O}(x, \cdot)$ have this sample.
2. Simulate $A$ and $\mathcal{O}$ interacting.
3. Let $q_1, \ldots, q_k \in Q$ be the queries given by $A$.
4. Let $a_1, \ldots, a_k \in \mathbb{R}$ be the corresponding answers given by $\mathcal{O}$.
5. Pick the $j \in [k]$ that maximizes $\| \text{err}^P(q_j, a_j) \|$.
6. If $a_j - q_j(P) \geq 0$, output $q_j$ and halt.
7. If $a_j - q_j(P) < 0$, output $-q_j$ and halt.

(Note that $Q_\Delta$ is closed under negation.)

**Claim 3.3.** $\mathcal{M}$ is $(\varepsilon, \delta)$-differentially private.

**Proof.** This follows because $\mathcal{O}$ is $(\varepsilon, \delta)$-differentially private and $\mathcal{M}$ is a post-processing of the output of $\mathcal{O}$ (Lemma 2.7).

**Claim 3.4.**

$$\mathbb{E}_{x,\mathcal{M}} [q(x) - q(P) | q = \mathcal{M}(x)] > \alpha/2.$$

**Proof.** Let $q$ be the query that $\mathcal{M}(x)$ returns and $a$ the corresponding answer from $\mathcal{O}$. By the assumption that $\mathcal{O}$ is not $\alpha$-accurate on average with respect to the population,

$$\mathbb{E}_{x,\mathcal{M}} [a - q(P)] > \alpha.$$
The assumption that $O$ is $\alpha/2$-accurate on average with respect to the sample implies
$$\mathbb{E}_M[|a - q(x)|] \leq \alpha/2.$$ Thus
$$\mathbb{E}_{x,M}[q(x) - q(P) | q = M(x)] = \mathbb{E}_{x,M}[a - q(P)] - \mathbb{E}_{x,M}[a - q(x)] > \alpha - \alpha/2 = \alpha/2,$$ as required. \hfill \Box

Now combine Claim 3.4 and Lemma 3.1 to get
$$\frac{\alpha}{2} < \mathbb{E}_{x,M}[q(x) - q(P) | q = M(x)] \leq 2\Delta n(e^\varepsilon - 1 + \delta).$$

Now we use the assumption that $\varepsilon, \delta = \alpha/10\Delta n \leq 1/4$. We have $e^\varepsilon - 1 \leq \frac{5}{4} \varepsilon \leq \alpha / 8\Delta n$. Thus $2\Delta n(e^\varepsilon - 1 + \delta) \leq 2\Delta n(\alpha / 8\Delta n + \alpha / 10\Delta n) < \alpha/2$, a contradiction. This completes the proof. \hfill \Box

### 3.2 Amplifying the Success Probability

Theorem 3.2 only gives an average error guarantee, where in applications we would like a high confidence bound on the error.

By Markov’s Inequality (Fact 2.5) the maximum error is bounded with at least constant probability. And we will amplify this probability by “serial repetition.” We will run the oracle $O$ along with a “censor” $C$ that ensures that the oracle is giving accurate answers. If $C$ detects that $O$ has given an inaccurate answer, then a fresh sample is drawn and a new instance of $O$ is started. Since $O$ has a constant probability of giving good answers for all $k$ queries, the number of times a fresh sample is needed is bounded by $O(\log(1/\beta))$ with probability at least $1 - \beta$, as required. The challenge is constructing a suitable censor $C$, that can detect when the oracle is inaccurate using only a small number of samples. We define such a censor as follows.

**Definition 3.5 (Bounded Censor).** An oracle $C$ is a $\beta$-sound $c$-bounded censor for $k$ sensitivity-$\Delta$ queries with threshold $\alpha$ given $n$ samples in $X$ if, for every adversary $A$, the algorithm answers every query $q_j$ with $a_j \in \{\top, \bot, \star\}$ and
$$\mathbb{P}_{\text{Acc}_{n,k,0}\beta}[C, A] \left[ \forall j \in [k] \left( \begin{array}{ll}
  a_j = \top & \implies |q_j(P) - \alpha| > \frac{\varepsilon}{2} \\
  a_j = \bot & \implies |q_j(P) - \alpha| \leq \alpha \\
  a_j = \star & \iff |\{i \in [j-1] : a_i = \top\}| \geq c
\end{array} \right) \right] \geq 1 - \beta.$$

Intuitively, a bounded censor is an algorithm which answers $k$ sensitivity-$\Delta$ queries and approximately determines whether or not they are above a threshold. It outputs $\top$ if the answer is above the threshold and $\bot$ if the answer is below the threshold. However, the bounded censor may cease to give useful answers after $c$ query answers have been above the threshold. Notice that a bounded censor is a weaker object than an accurate oracle, so there is no circularity in our approach of using a bounded censor to boost the accuracy of the oracle.

First, we will show that a bounded censor with suitable parameters, plus an oracle that is accurate on average, implies the existence of an oracle that is accurate with high probability. We do so by way of the construction in Figure 4.

Note that the semantics of $\mathcal{F}^O,C$ are a bit different from the semantics of an oracle defined in Section 2.2. For notational convenience, we have written $\mathcal{F}$ as if it has access to a sampling
Draw a sample \( x' \leftarrow \mathcal{P}'' \) and initialize \( C \) with \( x' \).
Draw sample \( x \leftarrow \mathcal{P}' \) and initialize \( O \) with \( x \).

For \( j = 1, \ldots, k \):
- Receive the next query \( q_j \in Q \) from \( A \).
- (†) Let \( a_j = O(x, q_j) \).
- Let \( \hat{a}_j(z) = \text{err}_z(q_j, a_j) \).
- Let \( \hat{a}_j(x') = C(x', \hat{a}_j) \).
- If \( \hat{a}_j = \top \), draw a fresh sample \( x \leftarrow \mathcal{P}' \), reinitialize \( O \) with \( x \), and goto (†).
- If \( \hat{a}_j \notin \{\top, \bot\} \), halt.
- Otherwise, answer with \( a_j \).

![Figure 4: The Oracle \( \mathcal{F}^{O,C} \)](image)

oracle for \( \mathcal{P} \), rather than a finite sample. Moreover, a new sample of size \( n \) is drawn every time \( \hat{a}_j = \top \), which can potentially occur \( k \) times, meaning that the number of samples drawn can be as large as \( n' + n \cdot k \). However, since our analysis shows that with probability at least \( 1 - \beta \), \( \hat{a}_j = \top \) at most \( c - 1 \) times. Thus, we could easily rewrite the algorithm to have a finite sample of size \( n' + n \cdot c \), which is split into one sample of size \( n' \) and \( c \) samples of size \( n \), and the algorithm will halt if it exhausts its set of samples, which occurs with probability at most \( \beta \).

Theorem 3.6. Assume that

1. \( O \) is \( \alpha/4 \)-accurate on average with respect to the population for \( k \) adaptively chosen queries from \( Q \subseteq Q_\Delta \) given \( n \) samples in \( X \), and
2. \( C \) is a \( \beta/2 \)-sound \( c \)-bounded censor for \( k + c \) sensitivity-\( \Delta \) queries with threshold \( \alpha \) given \( n' \) samples in \( X \), where \( c = \lceil \log_2(4/\beta) \rceil \).

Then \( \mathcal{F}^{O,C} \) is \((\alpha, \beta)\)-accurate with respect to the population for \( k \) adaptively chosen queries from \( Q \) given \( n' + n \cdot c \) samples in \( X \).

We will show (in Theorem 3.7) that we only need \( n' = O(n + \log(1/\beta) \cdot \log k) \). Thus Theorem 3.6 states that we can amplify from \( \alpha/4 \) expected accuracy to \((\alpha, \beta)\)-accuracy with a \( \log(1/\beta) \) factor increase in sample complexity.

Proof of Theorem 3.6. By Definition 3.5 and the fact that

\[
\mathbb{E}_{z \leftarrow \mathcal{P}''}[\hat{a}_j(z)] = \text{err}_z^\mathcal{P}(q_j, a_j),
\]

with probability at least \( 1 - \beta/2 \), for all \( j \in [k] \),

1. \( \hat{a}_j = \bot \) implies \( |\text{err}_z^\mathcal{P}(q_j, a_j)| \leq \alpha \),
2. \( \hat{a}_j = \top \) implies \( |\text{err}_z^\mathcal{P}(q_j, a_j)| > \frac{\alpha}{2} \), and
3. \( \hat{a}_j = \star \) implies \( |\{ j \in [k] : \hat{a}_j = \top \}| \geq c \).

We condition on the event that these statements hold. By condition (i), for each \( j \in [k] \), \( |\text{err}_z^\mathcal{P}(q_j, a_j)| \leq \alpha \), so \( \mathcal{F}^{O,C} \) only outputs accurate answers, as required.
By condition (iii), \( \mathcal{F}^{O,C} \) only halts early if the number of times \( \top \) is returned by \( C \) exceeds \( c - 1 \). Note that halting early ensures that the number of samples needed by \( \mathcal{F}^{O,C} \) is bounded by \( n' + n \cdot c \), as required.

It remains to show that the number of times \( \top \) is returned by \( C \) is at most \( c - 1 \) with probability at least \( 1 - \beta/2 \). By condition (ii), \( \top \) is only returned by \( C \) only when \( \| err_P(q_j, a_j) \| > \alpha^2 \). However, by Fact 2.5, \( O \) is \((\alpha/2, 1/2)\)-accurate with respect to the population for \( k \) adaptively chosen queries from \( Q \) given \( n \) samples in \( X \). This means that the probability that \( \| err_P(q_j, a_j) \| > \alpha^2 \) for any \( j \in [k] \) is at most \( 1/2 \), where the probability is over only the coins of \( O \) and the choice of \( x \). Every time \( \top \) is returned by \( C \), the coins of \( O \) and its sample \( x \) are redrawn independently. Thus, the probability that \( \top \) is returned \( m \) or more times is at most \( 2^{-m} \). In particular, the probability that \( \top \) is returned \( c - 1 \) or more times is at most \( \beta/2 \), as required. \( \square \)

### 3.2.1 The Bounded Censor Algorithm

Now we turn our attention to constructing an appropriate bounded censor \( C \), which is given in Figure 5. This construction is inspired by the sparse vector algorithm [DNR+09, HR10, DNPR10, RR10] and the analysis is inspired by [Rot14].

**Algorithm**

\[
\begin{align*}
C &= C(\alpha, n, c, k). \\
&\text{Draw a sample } x \in X^n \text{ from } P. \\
&t \leftarrow 0. \\
&\text{For each } j = 1, \ldots, k: \\
&\quad \text{Receive query } q_j : X^n \to \mathbb{R}. \\
&\quad \text{If } t \geq c, \text{ return } a_j = \star. \\
&\quad \text{If } t < c \text{ and } |q_j(x)| \leq \frac{3}{4} \alpha, \text{ return } a_j = \bot. \\
&\quad \text{If } t < c \text{ and } |q_j(x)| > \frac{3}{4} \alpha, \text{ return } a_j = \top \text{ and set } t \leftarrow t + 1.
\end{align*}
\]

Figure 5: The Bounded Censor Algorithm

**Theorem 3.7.** The Bounded Censor algorithm \( C(\alpha, n, c, k) \) in Figure 5 is a \( \beta \)-sound \( c \)-bounded censor for \( k \) sensitivity-\( \Delta \) queries with threshold \( \alpha \) given \( n \) samples in \( X \) assuming

\[
n \geq \frac{8(n\Delta)^2}{\alpha^2} (\log(2/\beta) + (c + 1) \cdot \log(k + 1)).
\]

In the common scenario where we have \( \Delta = O(1/n) \), the bounded censor \( C \) only requires

\[
n \geq O\left(\frac{\log(1/\beta) + c \cdot \log k}{\alpha^2}\right).
\]

When \( C \) is invoked in Theorem 3.6, we have \( c = O(\log(1/\beta)) \). So the bounded censor only needs \( n \geq O(\log(1/\beta) \cdot \log(k)/\alpha^2) \), which is small enough not to be a bottleneck in any of our bounds.

**Proof of Theorem 3.7.** Fix an adversary \( A \). Without loss of generality, we may assume that \( A \) is deterministic. By Definition 3.5, we must verify that

\[
\mathbb{P}_{\text{Acc}_{n,k,0}[C,A]} \left[ \forall j \in [k] \left( \begin{array}{c}
   a_j = \top \implies |q_j(P)| > \frac{\alpha}{2} \\
   a_j = \bot \implies |q_j(P)| \leq \alpha \\
   a_j = \star \implies |\{ i \in [j-1] : a_i = \top \}| \geq c
\end{array} \right) \right] \geq 1 - \beta.
\]
It suffices to show that, with probability at least $1 - \beta$, for every query $q$ that $A$ could potentially ask$^2$, $|q(x) - q(P)| \leq \alpha/4$. We will prove in two steps: (i) We show that for any fixed single query $q$ we have $|q(x) - q(P)| \leq \alpha/4$ with high probability over the sample $x$. And (ii) we bound the number of queries that $A$ could potentially ask and apply a union bound. Note that we must take a union bound over all the queries that $A$ could potentially ask, not just the queries that $A$ does ask in a particular instance of $\text{Acc}_{n,k,Q_\alpha}[C,A]$. 

For (i), we use the following standard concentration inequality.

**Lemma 3.8 (McDiarmid’s Inequality [McD89]).** Let $X_1, \cdots, X_n \in \mathcal{X}$ be independent random variables and $f : \mathcal{X}^n \to \mathbb{R}$. Suppose $f$ is $\Delta$-sensitive—that is,

$$
\forall x \in \mathcal{X}^n \forall i \in [n] \forall y \in \mathcal{X} \left| f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) - f(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n) \right| \leq \Delta.
$$

Then

$$
\forall \lambda > 0 \quad \mathbb{P} \left[ f(X_1, \cdots, X_n) - \mathbb{E}[f(X_1, \cdots, X_n)] > \lambda \right] \leq \exp \left( \frac{-2\lambda^2}{n\Delta^2} \right).
$$

Thus, for any $q \in Q_\Delta$, we have

$$
\mathbb{P}_{X \sim \text{Po}(n)} \left[ |q(x) - q(P)| > \alpha/4 \right] \leq 2 \cdot \exp \left( \frac{-\alpha^2}{8\Delta^2 n} \right). \quad (1)
$$

For (ii): Since $A$ is deterministic, the queries it asks during $\text{Acc}_{n,k,Q_\alpha}[C,A]$ are determined by the answers given by $C$. In particular, each query $A$ potentially asks can be specified by a string $s \in \bigcup_{\ell=1}^k \{\top, \bot\}^\ell$ of length at most $k$ with at most $c$ occurrences of $\top$. This string represents the answers given by $C$ up until the time the query is asked. (Note that we omit the final $\star$ symbol, as it is immaterial.) By counting the number of such strings, we see that the number of possible queries $A$ can ask is bounded by $(k + 1)^{c+1}$. Thus we take a union bound over the $(k + 1)^{c+1}$ potential queries of $A$.

By applying this union bound to the bound in (1), the overall failure probability of $C$ is bounded by

$$
2 \cdot \exp \left( \frac{-\alpha^2}{8\Delta^2 n} \right) \cdot (k + 1)^{c+1} \leq \beta,
$$

as required. \qed

Combining our theorem about accuracy on average (Theorem 3.2) with our amplification technique from the previous section (Theorems 3.6 and 3.7) gives the following bounds.

**Theorem 3.9.** Assume that $O$ is

1. $(\epsilon, \delta)$-differentially private for $\epsilon = \delta = \alpha/40\Delta n$, and
2. $\alpha/8$-accurate on average with respect to samples of size $n$ from $\mathcal{X}$ for $k$ adaptively chosen queries from $Q \subseteq Q_\Delta$ (the family of $\Delta$-sensitive queries on $\mathcal{X}$).

Then there exists an oracle $O'$ that is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q$ given $O((n + (n\Delta/n)^2 \cdot \log k) \cdot \log(1/\beta))$ samples from $\mathcal{X}$. Moreover, the running time of $O'$ is at most a $O(\log(1/\beta))$ factor more than that of $O$.

---

$^2$Excluding the queries that are only asked after the $t \geq c$ condition is reached.
4 From Function Approximation to Optimization Queries

We now extend our results for low-sensitivity queries to more general minimization queries.

4.1 Oracles for Adaptive Minimization Queries

Analogous to Theorem 3.2, we can show that an oracle that answers minimization queries on its input and is also differentially private gives answers that generalize to the whole distribution.

Theorem 4.1. Assume that the oracle \( O \) is

- \((\varepsilon, \delta)\)-differentially private for \( \varepsilon = \delta = \frac{a}{20\Delta n} \leq \frac{1}{2} \) and,
- \( \alpha/2 \)-accurate on average with respect to samples of size \( n \) from \( X \) for \( k \) adaptively chosen queries from \( Q \subseteq Q_{\text{min}} \) (the family of \( \Delta \)-sensitive convex minimization queries on \( X \)).

Then \( O \) is \( \alpha \)-accurate on average with respect to the population for \( k \) adaptively chosen queries from \( Q \) given \( n \) samples from \( X \).

Proof. Suppose, for the sake of contradiction, that \( O \) is not \( \alpha \)-accurate on average with respect to the population for \( k \) adaptively chosen queries from \( Q \) given \( n \) samples from \( X \). That is, there exists an adversary \( A \) such that

\[
\mathbb{E}_{x, M}[\max_{j \in [k]} |\text{err}_P(L_j, \theta_j)|] > \alpha.
\]

Fix such an adversary \( A \) and the distribution \( P \) that it outputs. Now define a monitor \( M \) as follows.

Algorithm \( M = M(O, A, P) : \mathcal{X}^n \rightarrow \mathcal{L} \times \Theta \):

1. Draw a sample \( x \in \mathcal{X}^n \) from \( P \) and let \( O = O(x, \cdot) \) have this sample.
2. Simulate \( A \) and \( O \) interacting.
3. Let \( L_1, \ldots, L_k \) be the loss functions given by \( A \).
4. Let \( \theta_1, \ldots, \theta_k \) be the corresponding answers given by \( O \).
5. Choose \( j \in [k] \) that maximizes \( \text{err}_P(L_j, \theta_j) \).
6. Output \( (L_j, \theta_j) \).

As for Theorem 3.2, we prove the theorem via a series of claims. We omit proofs where they are sufficiently similar to those in the proof of Theorem 3.2.

Claim 4.2. \( M \) is \((\varepsilon, \delta)\)-differentially private.

Claim 4.3.

\[
\mathbb{E}_{x, M}[^{err}_P(L, \theta) \mid (L, \theta) = M(x)] > \alpha.
\]

Proof. This follows from our supposition that \( O \) is not accurate with respect to the population and the fact that \( \text{err}_P(L, \theta) \geq 0 \) for all \( L \) and \( \theta \). \( \square \)

3Recall that for many natural minimization queries, the sensitivity \( \Delta \) scales as \( 1/n \), so that \( \Delta n \) is a constant.
Claim 4.4. For all $x \in \mathcal{X}^n$,
$$
\mathbb{E}_{\mathcal{M}} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] \leq \frac{\alpha}{2}.
$$

Proof. This follows from our assumption that $\mathcal{O}$ is accurate with respect to the sample. \qed

Claim 4.5. Define $q : \mathcal{X}^n \rightarrow \mathbb{R}$ by $q(x) = \text{err}_x(L, \theta)$ for fixed values of $L : \mathcal{X}^n \times \Theta \rightarrow \mathbb{R}$ and $\theta \in \Theta$. Then we have $q$ is $2\Delta$-sensitive.

Proof. Let $x, x' \in \mathcal{X}^n$ be neighboring and $\hat{\theta} = \arg\min_{\theta' \in \Theta} L(x, \theta')$. Since $L$ is $\Delta$-sensitive,

$$
q(x) = L(x, \theta) - \min_{\theta' \in \Theta} L(x, \theta') = L(x, \theta) - L(x, \hat{\theta}) \\
\leq L(x', \theta) - L(x', \hat{\theta}) + 2\Delta \\
\leq L(x', \theta) - \min_{\theta' \in \Theta} L(x', \theta') + 2\Delta \\
= q(x') + 2\Delta.
$$

Claim 4.6.

$$
\mathbb{E}_{\mathcal{M}, x} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] \leq \mathbb{E}_{\mathcal{M}, x} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] + 4\Delta n (e^\varepsilon - 1 + \delta).
$$

Proof.

$$
\mathbb{E}_{\mathcal{M}, x} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] = \mathbb{E}_{\mathcal{M}, x} \left[ \mathbb{E}_{x' \sim P_n} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] \right] \\
\leq \mathbb{E}_{\mathcal{M}, x} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] \\
+ \mathbb{E}_{\mathcal{M}, x, x'} \left[ \text{err}_x(L, \theta) - \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] \\
\leq \mathbb{E}_{\mathcal{M}, x} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] + 4\Delta n (e^\varepsilon - 1 + \delta),
$$

where the final inequality follows from Lemma 3.1 and Claims 4.2 and 4.5. \qed

Combining Claims 4.3, 4.4, and 4.6 gives

$$
\alpha < \mathbb{E}_{\mathcal{M}, x} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] \\
\leq \mathbb{E}_{\mathcal{M}, x} \left[ \text{err}_x(L, \theta) \mid (L, \theta) = \mathcal{M}(x) \right] + 4\Delta n (e^\varepsilon - 1 + \delta) \\
\leq \alpha/2 + 4\Delta n (e^\varepsilon - 1 + \delta).
$$

By assumption $e^\varepsilon - 1 \leq \frac{5}{4} \varepsilon \leq \alpha/16\Delta n$ and, hence, $4\Delta n (e^\varepsilon - 1 + \delta) \leq \alpha/4 + \alpha/5 < \alpha/2$, which completes the proof by contradiction. \qed
4.2 Amplifying the Success Probability

Similar to Theorem 3.6, we can amplify the success probability of an algorithm for minimization queries.

**Theorem 4.7.** Assume that

1. $O$ is $\alpha/4$-accurate on average with respect to the population for $k$ adaptively chosen queries from $Q \subseteq Q_{\text{min}}$ given $n$ samples in $\mathcal{X}$, and
2. $C$ is a $\beta/2$-sound $c$-bounded censor for $k + c$ sensitivity-$\Delta$ queries with threshold $\alpha$ given $n'$ samples in $\mathcal{X}$, where $c = \lceil \log_2(4/\beta) \rceil$.

Then $\mathcal{F}^{O,C}$ is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q$ given $n' + n \cdot c$ samples in $\mathcal{X}$.

The proof is almost identical to that of Theorem 3.6. The only difference is that, for a $\Delta$-sensitive loss function $L_j$, the query to the censor $\hat{q}_j(z) = \text{err}_z(L_j, \theta_j)$ is now a $2\Delta$-sensitive real-valued query (Claim 4.5). Combining Theorems 4.1, 4.7, and 3.7 gives the following.

**Theorem 4.8.** Assume that $O$ is

1. $(\varepsilon, \delta)$-differentially private for $\varepsilon = \delta = \alpha/80 \Delta n$, and
2. $\alpha/8$-accurate on average with respect to samples of size $n$ from $\mathcal{X}$ for $k$ adaptively chosen queries from $Q \subseteq Q_{\text{min}}$.

Then there exists an oracle $O'$ that is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q$ given $O(n + (n\Delta/n)^2 \cdot \log k \cdot \log(1/\beta))$ samples from $\mathcal{X}$. Moreover, the running time of $O'$ is at most a $O(\log(1/\beta))$ factor more than that of $O$.

5 Applications

5.1 Low-Sensitivity and Statistical Queries

We now plug known differentially private mechanisms in to Theorem 3.9 to obtain oracles that provide strong error guarantees with high probability for both low-sensitivity and statistical queries.

**Corollary 5.1** (Theorem 3.9 and [DMNS06, SU15]). There is an oracle $O$ that is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q_{\Delta}$ where $\Delta = O(1/n)$ given $n$ samples from $\mathcal{X}$ for

$$n \geq O\left(\frac{\sqrt{k \cdot \log k \cdot \log(1/\alpha) \cdot \log(1/\beta)}}{\alpha^2}\right)$$

The oracle runs in time $\text{poly}(n, \log |\mathcal{X}|, \log(1/\beta))$ per query.

**Corollary 5.2** (Theorem 3.9 and [RR10]). There is an oracle $O$ that is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q_{\Delta}$ where $\Delta = O(1/n)$ given $n$ samples from $\mathcal{X}$ for

$$n = O\left(\frac{\log |\mathcal{X}| \cdot \log k \cdot \sqrt{\log(1/\alpha) \cdot \log(1/\beta)}}{\alpha^3}\right)$$
The oracle runs in time $\text{poly}(|X|^n)$ per query. The case where $\Delta$ is not $O(1/n)$ can be handled by rescaling the output of the query.

**Corollary 5.3** (Theorem 3.9 and [HR10]). There is an oracle $O$ that is $\alpha$-accurate on average with respect to the population for $k$ adaptively chosen queries from $Q_{SQ}$ given $n$ samples from $X$ for

$$n = O\left(\sqrt{\log |X| \cdot \log k \cdot \log(1/\alpha) \cdot \log(1/\beta)} \right)^{\alpha^3}$$

The oracle runs in time $\text{poly}(n,|X|)$ per query.

### 5.2 Optimization Queries

The results of the Section 4 can be combined with existing differentially private algorithms for minimizing “empirical risk” (that is, loss with respect to the sample $x$) to obtain algorithms for answering adaptive sequences of minimization queries. We provide a few specific instantiations here, based on known differentially private mechanisms.

#### 5.2.1 Minimization Over Arbitrary Finite Sets

**Corollary 5.4** (Theorem 4.8 and [MT07]). Let $\Theta$ be a finite set of size at most $D$. Let $Q \subset Q_{\text{min}}$ be the set of sensitivity-$1/n$ loss functions bounded between 0 and $C$. Then there is an oracle $O$ that is $(\alpha,\beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q_{\text{min}}$ given

$$n \geq O\left(\frac{\log(DC/\alpha) \cdot \sqrt{k \cdot \log(1/\alpha) \cdot \log(1/\beta)}}{\alpha^2}\right)$$

samples from $X$. The running time of the oracle is dominated by $O((k + \log(1/\beta)) \cdot D)$ evaluations of the loss function.

#### 5.2.2 Convex Minimization

We state bounds for convex minimization queries for some of the most common parameter regimes in applications. In the first two corollaries, we consider 1-Lipschitz loss functions over a bounded domain.

**Corollary 5.5** (Theorem 4.8 and [BST14]). Let $\Theta$ be a closed, convex subset of $\mathbb{R}^d$ set such that $\max_{\theta \in \Theta} \|\theta\|_2 \leq 1$. Let $Q \subset Q_{\text{min}}$ be the set of convex 1-Lipschitz loss functions that are $1/n$-sensitive. Then there is an oracle $O$ that is $(\alpha,\beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q$ given

$$n = O\left(\sqrt{dk \cdot \log(1/\beta)} \right)^{\alpha^2}$$

samples from $Q$. The running time of the oracle is dominated by $k \cdot n^2$ evaluations of the gradient $\nabla L$.

---

4A loss function $L: X \times \mathbb{R}^d \to \mathbb{R}$ is 1-Lipschitz if for every $\theta, \theta' \in \mathbb{R}^d$, $x \in X$, $|L(\theta, x) - L(\theta', x)| \leq \|\theta - \theta'\|_2$. 

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18
Corollary 5.6 (Theorem 4.8 and [Ull15]). Let $\Theta$ be a closed, convex subset of $\mathbb{R}^d$ such that $\max_{\theta \in \Theta} \|\theta\|_2 \leq 1$. Let $Q \subset Q_{\min}$ be the set of convex 1-Lipschitz loss functions that are $1/n$-sensitive. Then there is an oracle $O$ that is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q$ given

$$n = \tilde{O}\left(\frac{\sqrt{\log |X|} \cdot \left(\sqrt{d} + \log k\right) \cdot \log(1/\beta)}{\alpha^3}\right)$$

samples from $X$. The running time of the oracle is dominated by $\text{poly}(n, |X|)$ and $k \cdot n^2$ evaluations of the gradient $\nabla L$.

In the next two corollaries, we consider 1-strongly convex\(^5\), Lipschitz loss functions over a bounded domain.

Corollary 5.7 (Theorem 4.8 and [BST14]). Let $\Theta$ be a closed, convex subset of $\mathbb{R}^d$ such that $\max_{\theta \in \Theta} \|\theta\|_2 \leq 1$. Let $Q \subset Q_{\min}$ be the set of 1-strongly convex, 1-Lipschitz loss functions that are $1/n$-sensitive. Then there is an oracle $O$ that is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q$ given

$$n = \tilde{O}\left(\frac{\sqrt{d k}}{\alpha^{3/2} \cdot \log(1/\beta)}\right)$$

samples from $X$. The running time of the oracle is dominated by $k \cdot n^2$ evaluations of the gradient $\nabla L$.

Corollary 5.8 (Theorem 4.8 and [Ull15]). Let $\Theta$ be a closed, convex subset of $\mathbb{R}^d$ such that $\max_{\theta \in \Theta} \|\theta\|_2 \leq 1$. Let $Q \subset Q_{\min}$ be the set of 1-strongly convex 1-Lipschitz loss functions that are $1/n$-sensitive. Then there is an oracle $O$ that is $(\alpha, \beta)$-accurate with respect to the population for $k$ adaptively chosen queries from $Q$ given

$$n = \tilde{O}\left(\frac{\sqrt{\log |X|} \cdot \left(\frac{\sqrt{d}}{\alpha^{5/2}} + \frac{\log k}{\alpha^3}\right) \cdot \log(1/\beta)}{\alpha^3}\right)$$

samples from $X$. The running time of the oracle is dominated by $\text{poly}(n, |X|)$ and $k \cdot n^2$ evaluations of the gradient $\nabla L$.

Acknowledgements

We thank Aaron Roth for suggesting the technique that we used to prove Theorem 3.7. We would also like to thank Mark Bun, Moritz Hardt, and Salil Vadhan for helpful discussions.

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\(^5\)A loss function $L : X \times \mathbb{R}^d \rightarrow \mathbb{R}$ is 1-strongly convex if for every $\theta, \theta' \in \mathbb{R}^d, x \in X$,

$$L(\theta', x) \geq L(\theta, x) + \langle \nabla L(\theta, x), \theta' - \theta \rangle + \frac{1}{2} \cdot \|\theta - \theta'\|_2^2,$$

where the (sub)gradient $\nabla L(\theta, x)$ is taken with respect to $\theta.$
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