Higher-order resonances and instability of high-frequency WKB solutions

Yong Lu*

Abstract

This paper focuses on the destabilizing role of resonances in high-frequency WKB solutions. Specifically, we study higher-order resonances associated with higher-order harmonics generated by nonlinearities. We give examples of systems and solutions for which such resonances generate instantaneous instabilities, even though the equations linearized around the leading WKB terms are initially stable, meaning in particular that the key destabilizing terms are not present in the data.

Contents

1 Introduction
   1.1 Klein-Gordon systems ........................................................................ 2
   1.2 Structure of the paper ........................................................................ 5
   1.3 Background ......................................................................................... 6
   1.4 Higher-order resonances and instability ............................................. 8

2 Description of the results
   2.1 Notations .......................................................................................... 13
   2.2 Statement of the results ...................................................................... 14

3 Proof of Proposition 2.1
   3.1 WKB expansion .................................................................................. 16
   3.2 The approximate solution and Proof of Proposition 2.1 ................. 18

4 Proof of Theorem 2.3
   4.1 Preparation ...................................................................................... 19
   4.2 Duhamel representation and an upper bound .................................. 28
   4.3 Existence in logarithmical time and upper bound ............................. 30
   4.4 Lower bound .................................................................................... 32

*Mathematical Institute, Charles University, Sokolovská 83, 186 75 Praha, Czech Republic, luyong@karlin.mff.cuni.cz.
1 Introduction

We study highly-oscillating solutions to hyperbolic systems based on Maxwell’s
equations. Considerable progress has recently been made in this line of research,
especially following the works of Joly, Métivier and Rauch in the nineties (see for
instance [4, 9, 10], and [5] for an overview and further references). The underlying
physical problems deal with light-matter interactions.

The specific systems under study here have the form

\[ \partial_t U + \frac{1}{\varepsilon} A_0 U + \sum_{1 \leq j \leq d} A_j \partial_{x_j} U = \frac{1}{\sqrt{\varepsilon}} B(U, U), \]  

and the data have the form

\[ U(0, x) = \Re \left( a(x) e^{ik \cdot x/\varepsilon} \right) + \sqrt{\varepsilon} \varphi(x). \]

The small parameter \( \varepsilon > 0 \) is the wavelength of light. The initial wavenumber \( k \) is
a given vector in \( \mathbb{R}^d \). The constant matrix \( A_0 \) is non-zero and skew-symmetric. Its
presence in the equations implies that the dispersion relation is non-homogeneous.
This is a typical feature of systems describing light-matter interactions. The matrices \( A_j \) are constant and symmetric. Explicit examples of such operators are given
in (1.8) below. We denote for any \( \xi \in \mathbb{C}^d \):

\[ A(\xi) := \sum_{1 \leq j \leq d} A_j \xi_j. \]

The data (1.2) are oscillating, with frequencies \( O(1/\varepsilon) \) that are naturally of the
same order of magnitude as the characteristic frequencies of the hyperbolic operator
in \((1.1)\); this means that the light may propagate in the medium. The amplitude of the initial oscillations is \(O(1)\) with respect to \(\varepsilon\).

In \((1.1)\), the map \(B\) is bilinear \(\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\). Its specific form derives from a phenomenological description of nonlinear interactions \([18]\), which may also include higher-order (order three, four, ...) interactions.

A key point here is the large prefactor \(1/\sqrt{\varepsilon}\) in front of the nonlinearity. Since we will consider solutions of amplitude \(O(1)\) with respect to \(\varepsilon\), this means that nonlinear effects play a role in the propagation of the initial oscillations in short time \(O(\sqrt{\varepsilon})\). In other words, the propagation is not weakly nonlinear in time \(O(1)\).

It has indeed been observed that the weakly nonlinear regime fails to describe nonlinear effects in time \(O(1)\) for a number of physical systems: Maxwell-Bloch (by Joly, Métivier, and Rauch \([10]\)), Maxwell-Euler (by Texier in \([21]\)), and Maxwell-Landau-Lifshitz (by the author in \([14]\)).

For systems and data of the form \((1.1)-(1.2)\), a systematic study of resonances and stability of WKB solutions was given by Texier and the author in \([15]\). In particular, the article \([15]\) contains a detailed account of how resonances may destabilize precise WKB solutions.

By WKB solutions we mean truncated power series in \(\varepsilon\) which approximately solve \((1.1)\). Each term in the series is a trigonometric polynomial in \(\theta := (k \cdot x - \omega t)/\varepsilon\), where \(\omega\) is an appropriate characteristic temporal frequency, in the sense that

\[
\det (-i\omega + A(ik) + A_0) = 0.
\]

That is, a WKB solution is \(U^a\) such that

\[
U^a = \sum_{n=0}^{2K_a} \varepsilon^{n/2} U_n, \quad U_n = \sum_{p \in \mathcal{H}_n} e^{ip\theta} U_{n,p}, \quad K_a \in \mathbb{Z}_+, \quad \mathcal{H}_n \subset \mathbb{Z},
\]

where amplitudes \(U_{n,p}(t,x)\) are not highly-oscillating, and

\[
\partial_t U^a + \frac{1}{\varepsilon} A_0 U^a + \sum_{1 \leq j \leq d} A_j \partial_{x_j} U^a = \frac{1}{\sqrt{\varepsilon}} B(U^a, U^a) + \varepsilon^{K_a} R^\varepsilon,
\]

\[
U^a(0, x) = U(0, x) + \varepsilon^K \psi^\varepsilon(x),
\]

where \(|R^\varepsilon|_{L^\infty} + |\psi^\varepsilon|_{L^\infty}\) is bounded uniformly in \(\varepsilon\). Parameters \(K_a\) and \(K\) describe the level of precision of the WKB solution \(U^a\).

If \(\omega\) is a characteristic temporal frequency, satisfying \((1.3)\), then symmetries in the equations typically imply that some \(p\omega\), with \(p \in \mathbb{Z}, p \neq 1\), are characteristic as well, meaning

\[
\det (-ip\omega + A(ipk) + A_0) = 0.
\]

We denote \(\mathcal{H}_0 := \{ p \in \mathbb{Z}, \det (-ip\omega + A(ipk) + A_0) = 0 \}\). Since \(A_0 \neq 0\), this set is typically finite. In \((1.4)\), the leading term \(U_0\) appears as a sum over \(\mathcal{H}_0\). The
coefficients $U_{0,p}$ are said to be harmonics of the leading coefficient $U_0$ in the WKB solution. We suppose ker $(-ip\omega + A(ipk) + A_0)$ is of dimension one with a generator $e_p$. Then $U_{0,p}$ satisfies the so-called polarization condition for any $p \in \mathcal{H}_0$:

\begin{equation}
U_{0,p}(t,x) = g(t,x) e_p, \quad \text{for some scalar function } g(t,x).
\end{equation}

Resonances are frequencies that satisfy some functional relations involving the eigenvalues of the hyperbolic operator in (1.1) and the initial wavenumber $k$, of the form

\begin{equation}
\lambda_j(\xi + pk) = \lambda_j'(\xi) + p\omega, \quad \text{for some } \xi \in \mathbb{R}^d,
\end{equation}

where $\lambda_j$ and $\lambda_j'$ solve the dispersion relation.

The main result of [15] gives structural relations, involving the hyperbolic operator and the nonlinearity $B$, which imply that arbitrarily small initial perturbations may be instantaneously amplified. Small perturbations means $K$ large in (1.5). That is, no matter how large $K$ and $K_\alpha$ are in (1.5), the WKB solution $U^\alpha$ may deviate instantaneously from the exact solution to (1.1), under conditions put forward in [15]. In applications, WKB solutions are commonly used to simulate the interactions, since they satisfy model equations which are considerably cheaper to simulate than the original system (1.1) based on Maxwell. The result of [15] shows precisely how in some instances the WKB computations may fail to accurately describe the interactions, hence should not be used for simulations.

In [15] the assumption is made that no higher-order harmonics in the leading term of WKB solutions, such as $2(\omega, k)$, $3(\omega, k)$, etc., which are created by the nonlinearity. We relax this assumption here, and specifically focus on the destabilizing role played by resonances associated with higher-order harmonics. These resonances are related to (1.7) with $p = 2, 3, ...$

Instead of building a complete theory, as in [15], here we focus on one example, explicitly given in (1.8) below, comprising coupled Klein-Gordon operators. Such operators were shown to derive from Euler-Maxwell in [20, 3, 15]; they were also the focus of article [6]. The nonlinearity in (1.8) is tailored for the instability phenomenon that we want to observe; there is no doubt that the phenomenon is not limited to this specific form, but could occur for a variety of operators and bilinear terms of the form (1.1).

Our point is that, for physical systems based on Maxwell’s equations, resonances associated with higher-order harmonics may destabilize precise WKB solutions. We give a precise description of the destabilization process. This is a purely nonlinear mechanism: the higher-order harmonics are generated by the semilinear source terms; in particular, they are not present in the equation at $t = 0$. In our example, the equations linearized around the initial oscillations are indeed stable.

There is an analogy with recent work of Lerner, Morimoto and Xu [12], and later Lerner, Nguyen and Texier [13]. These articles, [12] and [13], study the phenomenon of loss of hyperbolicity, for which a model equation is $(\partial_t + it\partial_x)u = 0$: hyperbolicity
holds at $t = 0$, but strong instabilities occur for $t > 0$. Similarly, the initial linearized equations are stable here. Higher-order harmonics are $O(t)$, and generate instabilities. There is a form of degeneracy analogous to the one in $\partial_t + it\partial_x$. As a result, the instability is slow to develop: it occurs in time $O(\varepsilon^{1/2} |\ln \varepsilon|)$, as opposed to time $O(\varepsilon^{1/4} |\ln \varepsilon|)$ in [15].

1.1 Klein-Gordon systems

For notational simplicity, we restrict to the following, one-dimensional system:

\[
\begin{align*}
\partial_t u + \begin{pmatrix}
0 & \partial_x \\
\partial_x & 0 \\
0 & -\varepsilon^{-1} \alpha_0 \omega_0 & 0
\end{pmatrix} u = \begin{pmatrix}
0 \\
\varepsilon^{-1/2} (u_3 + v_3) v_3
\end{pmatrix}, \\
\partial_t v + \begin{pmatrix}
0 & \theta_0 \partial_x \\
\theta_0 \partial_x & 0 \\
0 & -\varepsilon^{-1} \omega_0 & 0
\end{pmatrix} v = \begin{pmatrix}
0 \\
\varepsilon^{-1/2} (-u_2^2 + v_2^2)
\end{pmatrix},
\end{align*}
\]

where $x \in \mathbb{R}$, $U := (u, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, and $\varepsilon$ is a small parameter. The constant $\theta_0$ is assumed to satisfy $0 < \theta_0 < 1$. This means that the Klein-Gordon operators have different velocities (as in the operators derived from Euler-Maxwell, which feature one velocity equal to the speed of light, and one velocity equal to the ratio of the electronic thermal velocity to the speed of light). We assume $\omega_0 > 0$. We also assume that the masses are distinct; this means $\alpha_0 \neq 1$. We assume $5/2 < \alpha_0 < 3$ for two technical reasons. The first is to make sure that higher-order harmonics exist in the leading term of WKB solution; precisely, that is to guarantee that equation (2.7) has solution $(\omega, k)$. The second reason introducing this condition on $\alpha_0$ is to reduce the number of resonances as in Section 5.3 of [15]. This allows us to focus on the resonances associated with higher-order harmonics.

We consider highly oscillating initial datum of the form:

\[
(1.9) \quad u(0, x) = \sqrt{\varepsilon} u_r(\varepsilon, x), \quad v(0, x) = v^0(\varepsilon, x),
\]

where for any fixed $\varepsilon > 0$, we suppose sufficient Sobolev regularity for $v^0$, $u_r$, and $v_r$. The initial spatial wave number $k \in \mathbb{R}$ will be chosen such that (2.6) is satisfied. This initial datum will be chosen in (2.10) and in Section 4.4 in such a way that the slow instability develops. The notation $a^*$ denotes the complex conjugate of $a$.

The system (1.8) is symmetric hyperbolic, with bilinear source term. For any fixed $\varepsilon > 0$, the local existence, uniqueness and regularity in smooth Sobolev spaces $H^s$ with $s > d/2$ are classical; however, the large nonlinear source term of order $O(1/\sqrt{\varepsilon})$ causes the classical existence time to be $O(1/\varepsilon)$ (see for instance Chapter 7 of [17]). We study the system in high frequency limit $\varepsilon \to 0$.
1.2 Structure of the paper

This paper is organized as follows. We mention previous results in [10] and [15] in Section 1.3; we compare these with our results in Section 1.4. In Section 2, we give some notations and state our main results. Section 3 and 4 comprise the proofs. In Section 3, we construct a WKB solution, and in Section 4 we show its instability. In Appendix A we recall some concepts about pseudodifferential operators. The most technical parts of the instability proof are given in Appendix B and Appendix C.

1.3 Background

In this section, we describe briefly the main results in [10] and [15].

1.3.1 WKB solution, weak and strong transparency, stability

We rewrite the system (1.1) as

\[
\partial_t U + \frac{1}{\varepsilon} A_0 U + \sum_{1 \leq j \leq d} A_j \partial x_j U - \frac{1}{\sqrt{\varepsilon}} B(U, U) = 0.
\]

We assume that the spectral decomposition

\[
A(\xi) + A_0 / i = \sum_{j=1}^{J} \lambda_j(\xi) \Pi_j(\xi)
\]

is smooth, meaning that the eigenvalues \( \lambda_j \) and the eigenprojectors \( \Pi_j \) are smooth. The notations \( \lambda_j \) and \( \Pi_j \) are used temporarily in Introduction.

We study the stability of WKB approximate solutions of form (1.4), in particular with amplitude \( O(1) \). We plug (1.4) into (1.10), then the left-hand side has the form \( \sum_{n \geq -2} \varepsilon^{n/2} \Phi_n(t, x, \theta) \) with

\[
\Phi_n := (\partial_t + A(\partial_x)) U_n + (-\omega \partial_\theta + A(k \partial_\theta) + A_0) U_{n+2} - \sum_{n_1 + n_2 = n+1} B(U_{n_1}, U_{n_2}).
\]

Solving (1.10) amounts to solve \( \Phi_n = 0 \) for all \( n \in \mathbb{Z} \), \( n \geq -2 \). This is generally not possible because there are infinity of \( n \). At least, we can solve (1.10) approximately by solving \( \Phi_n = 0 \) up to some nonnegative order: if we solve \( \Phi_n = 0 \) up to some order \( N \geq 0 \), the WKB solution \( U^a \) solves (1.10) with a remainder of order \( O(\varepsilon^{(N+1)/2}) \) which goes to zero. This is the typical way to construct a WKB solution (see Section 3 for more details).

**Definition 1.1.** We say \( U^a \) in (1.4) to be a WKB solution to (1.10) of order \( N \) provided \( \Phi_0 = \Phi_1 = \cdots = \Phi_N = 0 \).
Remark that the leading term $U_0 = \sum_{p \in \mathbb{N}_0} e^{ip\theta} U_{0,p}$ plays a special role in the WKB expansion. Indeed, a WKB solution $U^a$ is approximated by its leading term

$$|U^a - U_0(t, x, (kx - \omega t)/\varepsilon)| = O(\sqrt{\varepsilon}) \rightarrow 0.$$  

Considering the initial datum (1.2), the initial values of $U_{0,p}$ are chosen as

$$U_{0,0}(0, x) = a(x), \quad U_{0,-1}(0, x) = (a(x))^*, \quad U_{0,n}(0, x) = 0, \quad \text{for } p \not\in \{-1, 1\}.$$

In the leading term, $U_{0,1}$ and $U_{0,-1}$ are said to be fundamental harmonics (or phases), while $U_{0,p}$ with $|p| > 1$ are said to be higher-order harmonics (or phases).

We would like to show that the high-order harmonics can destabilize the WKB solution, in spite of their initial values being null. To achieve this, we need the higher-order harmonics to be non-null when $t > 0$. The higher-order harmonics are generated by the nonlinearity $B$ and the fundamental harmonics. In solving $\Phi_n = 0$, there arises an equation of the form

$$(\partial_t + \Pi(3\tilde{\beta})A(\partial_x)\Pi(3\tilde{\beta})){U_{0,3}} = B(U_{0,1}, U_{1,2}) + \cdots,$$

where $\tilde{\beta} := (\omega, k)$, $\Pi(\tau, \xi)$ denotes the orthogonal projector onto ker $(-i\tau + A(i\xi) + A_0)$ and $U_{1,2}$ solves

$$(-2\omega + A(2k) + A_0)U_{1,2} = B(U_{0,1}, U_{0,1}).$$

In our context, $(-2\omega + A(2k) + A_0)$ is an invertible matrix, then $U_{1,2}$ can be written as a bilinear form of $U_{0,1}$. This gives

$$(1.11) \quad (\partial_t + \Pi(3\tilde{\beta})A(\partial_x)\Pi(3\tilde{\beta})){U_{0,3}} = Q(U_{0,1}) + \cdots,$$

where $Q$ is cubic in $U_{0,1}$. Then a non-null solution $U_{0,3}(t, \cdot)$ is expected for $t > 0$.

In [10], Joly, Métivier and Rauch introduced the weak transparency condition:

**Weak transparency** For any $p, p_1 \in \mathbb{Z}$ and any $U, V \in \mathbb{C}^n$, one has

$$(1.12) \quad |\Pi(p_1\tilde{\beta})B(\Pi(p_1 - p)\tilde{\beta})U, \Pi(p\tilde{\beta})V)| = 0.$$  

Under this weak transparency assumption, we can show the existence of WKB solution of any order in time $O(1)$ (see Proposition 6.19 in [15]).

Concerning the stability of the WKB solution, the following strong transparency condition was introduced in [10]:

**Strong transparency** There exists a constant $C$ such that for any $p \in \mathbb{Z}$, $1 \leq j, j' \leq J$, $\xi \in \mathbb{R}^d$ and $U, V \in \mathbb{C}^n$, one has

$$(1.13) \quad |\Pi_j(\xi + pk)B(\Pi(p\tilde{\beta})U, \Pi_{j'}(\xi)V)| \leq C|\lambda_j(\xi + pk) - \lambda_{j'}(\xi) - \rho\omega| \cdot |U| \cdot |V|.$$  

Remark that the strong transparency condition is strictly stronger than the weak transparency condition.

Typically, strong transparency implies stability, via normal form reductions (see [10] [14] [15]). For (1.3)-(1.7), we show that the weak transparency is satisfied and a WKB solution can be constructed. However, the strong transparency is not satisfied. This implies instabilities of the WKB solution.
1.3.2 Absence of strong transparency and instability

In [15], Texier and the author consider systems of the form (1.1) for which the weak transparency condition is satisfied while the strong transparency condition is not. This indicates that, approximate solutions can be constructed through WKB expansion, but the normal form reduction method cannot be applied to show the stability of such WKB solutions.

The absence of strong transparency means that there exists \((j, j', p)\) such that (1.13) is not satisfied. Denote \(J_0\) the set containing all such index \((j, j', p)\) and \(R_{jj', p}\) the \((j, j', p)\)-resonant set defined as

\[
R_{jj', p} := \{ \xi \in \mathbb{R}^d, \lambda_j(\xi + pk) = p\omega + \lambda_{j'}(\xi) \}.
\]

If \(R_{jj', p}\) is empty, by the regularity of \(\lambda_j\) and \(\Pi_{j, j = 1, \cdots, J}\), condition (1.13) is satisfied for the index \((j, j', p)\). Then for any \((j, j', p) \in J_0\), \(R_{jj', p}\) is not empty, and the following quantity is well defined:

\[
\Gamma := \sup_{(j, j', p) \in J_0} |g_p(0, x_p)|^2 \sup_{\xi \in R_{jj', p}} \text{tr} \left( \Pi_j(\xi + pk)B(\varepsilon_p)\Pi_{j'}(\xi)B(\varepsilon_{-p})\Pi_j(\xi + pk) \right),
\]

where \(g_p\) come from the polarization condition (1.6) and \(x_p\) is the point where \(|g_p(0, \cdot)|\) admits its maximum.

In [15], it is shown that the stability of the WKB solution is determined by the sign of \(\Gamma\):

- If \(\Gamma < 0\), the perturbation system is symmetrizable and the WKB solution is stable.
- If \(\Gamma > 0\), it is shown that the WKB solution is unstable. The instability analysis consists first in reducing, via normal form reductions, the problem to the study of interaction systems of the form

\[
\partial_t u + \frac{1}{\sqrt{\varepsilon}} \text{op}_\varepsilon(M_0)u = f,
\]

where \(f\) is small of order \(O(\varepsilon^\kappa)\) for some \(\kappa > 0\) and contains in particular nonlinear terms, and \(\text{op}_\varepsilon(M_0)\) is the semiclassical pseudo-differential operator associated with a matrix-valued symbol \(M_0\) which is of order zero, essentially independent of \(t\) and has the form

\[
M_0 := \begin{pmatrix}
i\lambda_j - ip\omega & -\sqrt{\varepsilon}b_{jj'} \\-\sqrt{\varepsilon}b_{j'j} & i\lambda_{j'}
\end{pmatrix}
\]

with

\[
b_{jj'} = \Pi_j(\xi + pk)B(U_{0, p}(0, x))\Pi_{j'}(\xi), \quad b_{j'j} = \Pi_{j'}(\xi)B(U_{0, -p}(0, x))\Pi_j(\xi + pk)
\]

the interaction coefficients associated with resonance \(\lambda_j(\xi + pk) = p\omega + \lambda_{j'}(\xi)\).

The analysis of the interaction systems relies on a Duhamel representation formula introduced by Texier in [22]. The analysis of [22] shows that a solution operator
for the interaction systems can be constructed as a pseudo-differential operator with leading symbol the symbolic flow of (1.15), defined by

\[(1.16) \quad \partial_t S_0(\tau; t) + \frac{1}{\sqrt{\varepsilon}} M_0 S_0(\tau; t) = 0, \quad S_0(\tau; \tau) = \text{Id}.\]

In fact, the index in (1.14) is the maximal real part of \(\text{sp} (M/\sqrt{\varepsilon})\). The ordinary differential equation (1.16) is autonomous and has solution

\[(1.17) \quad S_0(\tau; t) = \exp \left( -\frac{M_0(t - \tau)}{\sqrt{\varepsilon}} \right).\]

From this explicit expression, a good upper bound for \(S_0\) can be obtained. By *good* upper bound we mean an upper rate of growth that is arbitrarily close to a lower rate of growth. Indeed, \(\Gamma > 0\) implies that some eigenvalue of \(M_0\) has positive real part. Then \(S_0\) is exponentially growing in time. Via the Duhamel Lemma of Appendix C, an instability result for the WKB solution ensues.

1.4 Higher-order resonances and instability

In this section, we compare our results to the results of [10] and [15]. In particular, we point out the main new difficulties compared to [15].

1.4.1 Transparency and loss of hyperbolicity

In [10] and [15], there exist interaction coefficients which are non-transparent (meaning that the strong transparency condition is not satisfied) both initially and for positive time. This implies a loss of hyperbolicity around a resonance, both initially and for positive time. A simple example model is the non-degenerate Cauchy-Riemann equation

\[\partial_t u + \frac{i \partial_x}{\sqrt{\varepsilon}} u = 0, \quad u(0) = u_0,\]

of which the solution is

\[\hat{u}(t, \xi) = \exp \left( \frac{t \xi}{\sqrt{\varepsilon}} \right) \hat{u}_0(\xi).\]

Then the instability is recorded in time \(O(\sqrt{\varepsilon} |\ln \varepsilon|)\) for frequencies \(O(1)\).

The present situation is analogous to the degenerate Cauchy-Riemann equation

\[\partial_t u + \frac{it \partial_x}{\sqrt{\varepsilon}} u = 0, \quad u(0) = u_0.\]

When \(t = 0\), the equation is hyperbolic. When \(t > 0\), the hyperbolicity is lost. The solution is

\[\hat{u}(t, \xi) = \exp \left( \frac{t^2 \xi}{2\sqrt{\varepsilon}} \right) \hat{u}_0(\xi).\]

The instability develops in time \(O(\varepsilon^{1/4} |\ln \varepsilon|^{1/2})\) for frequencies \(O(1)\).
1.4.2 Stability index

For system (1.8)-(1.9), we will show that for any non-transparent index \((j, j', p) \in J_0\), there holds \(U_{0,p}(0, \cdot) = 0\). Then the stability index \(\Gamma\) defined in (1.14) is zero. This case is not covered by the analysis of [15]. Here instability relies on condition (1.18)

\[
\sup_{(j, j', p) \in J_0} \sup_{\xi \in \mathbb{R}} |\partial_t g_p(0, y_p)|^2 \text{tr} \left( \Pi_j(\xi + pk) B(\xi) \Pi_{j'}(\xi) B(\xi - p) \Pi_j(\xi + pk) \right) > 0.
\]

Recall that \(g_p\) is the function introduced in the polarization condition (1.6). The stability index \(\Gamma_1\) is defined as the square root of the left-hand side of (1.18) where \(y_p\) is the point at which \(|\partial_t g_p(0, \cdot)|\) admits its maximum. In our analysis, we have \(p \in \{-3, 3\}\) corresponding to higher-order harmonics. Parameter \(\Gamma_1\) can be explicitly calculated, in terms of the system and the datum. Indeed, in the WKB expansion, we find that \(g_3\) satisfies a transport equation with a cubic source term in \(g_1\) and \(g_{-1}\) (see (1.11)). Then we can calculate \(\partial_t g_3(0, \cdot)\) through the equation and initial data \(g_{\pm}(0, \cdot)\) which can be obtained from (1.9).

1.4.3 Bounds for the symbolic flow

The analysis here relies partly on the Duhamel representation formula introduced in [22]. Contrary to [15], here we need to consider the upper bound for a symbolic flow which is solution to a non-autonomous system of the following form (see (B.9)):

\[
(1.19) \quad \partial_t \tilde{S}_0(\tau; t) + \frac{1}{\varepsilon^{3/4}} M_0(t) \tilde{S}_0(\tau; t) = 0, \quad \tilde{S}_0(\tau; \tau) = \text{Id}.
\]

We remark that in Section 1.4.3, the time \(t\) is rescaled by \(\varepsilon^{1/4}\) so that the instability is now expected in time \(O(|\ln \varepsilon|^{1/2})\).

The matrix \(M_0(t)\) is of the form (the index \((j, j', p)\) is chosen accordingly):

\[
M_0(t) := \begin{pmatrix}
i\lambda_j - ip\omega & -\varepsilon^{3/4} t \tilde{b}_{jj'} \\
-\varepsilon^{3/4} t \tilde{b}_{jj'} & i\lambda_{j'}
\end{pmatrix},
\]

where

\[
\tilde{b}_{jj'} = \Pi_j(\xi + pk) B(\partial_t U_{0,p}(0, x)) \Pi_{j'}(\xi), \quad \tilde{b}_{jj'} = \Pi_{j'}(\xi) B(\partial_t U_{0,-p}(0, x)) \Pi_j(\xi + pk).
\]

The two blocks \(t \tilde{b}_{jj'}\) and \(t \tilde{b}_{j'j}\) come from the interaction coefficients

\[
\Pi_j(\xi + pk) B(U_{0,p}(\varepsilon^{1/4} t, x)) \Pi_{j'}(\xi), \quad \Pi_{j'}(\xi) B(U_{0,-p}(\varepsilon^{1/4} t, x)) \Pi_j(\xi + pk)
\]

and the Taylor formula with respect to \(t\):

\[
U_{0,p}(\varepsilon^{1/4} t) = U_{0,p}(0) + \varepsilon^{1/4} t \partial_t U_{0,p}(0) + O(\varepsilon^{1/2} t^2),
\]

10
where $U_{0,p}(0,\cdot) = 0$ and the remainder $O(\varepsilon^{1/2}t^2)$ contributes a term of order $O(\varepsilon^{1/4})$ and is neglected (see (B.5)-(B.9)).

The goal is to obtain a good upper bound for $\tilde{S}_0$ of the form

\begin{equation}
|\tilde{S}_0(\tau; t)| \leq C \exp\left((t^2 - \tau^2)\gamma^+/2\right).
\end{equation}

Here $\gamma^+$ is such that $\gamma^+ t$ is the maximum of the real parts of the eigenvalues of $-M_0(t)/\varepsilon^{3/4}$. By direct calculation, the eigenvalues of $M_0(t)$ are (see also (1.11)):

\begin{equation}
i\lambda_j - ip\omega, \quad i\lambda_j',
\end{equation}

with

\begin{equation}
\nu_\pm := \frac{i}{2} (\lambda_j - p\omega + \lambda_j') \pm \frac{1}{2} (4\varepsilon^{3/2}t^2 \text{tr}(\tilde{b}_{12}\tilde{b}_{21}) - (\lambda_j - p\omega - \lambda_j')^2)^{1/2}.
\end{equation}

This implies that, at resonances $\lambda_j - p\omega = \lambda_j'$, the real parts of the eigenvalues of $-M_0(t)/\varepsilon^{3/4}$ admit their maximum $t\sqrt{\text{tr}(\tilde{b}_{12}\tilde{b}_{21})}$ provided $\text{tr}(\tilde{b}_{12}\tilde{b}_{21}) > 0$. The positivity of $\text{tr}(\tilde{b}_{12}\tilde{b}_{21})$ is guaranteed by the positivity of $\Gamma_1$. Precisely, $\gamma^+$ is the maximum of $\sqrt{\text{tr}(\tilde{b}_{12}\tilde{b}_{21})}$ over a sufficient small neighbourhood of the resonant sets due to the localization (see Section 4.1.4).

We say upper bound (1.20) is good because it has almost the same growth rate as the lower bound that we can obtain, which is $\exp\left((t^2 - \tau^2)\gamma^-/2\right)$ with $\gamma^-$ sufficiently close to $\gamma^+$.

In addition to the difficulties already present in [15]: fast oscillations $O(\varepsilon^{-3/4})$ and small distance $O(\varepsilon^{3/4})$ between resonances and crossing points of the eigenvalues of $M_0(t)$, the issue here is that the equation (1.19) is non-autonomous, implying that we do not have an explicit formula like (1.17) for the solution $\tilde{S}_0$. In particular, the argument in [15] to show the uniform bound for $\tilde{S}_0$ cannot be applied.

To show upper bound (1.20), the idea is to diagonalize $M_0(t)$, wherever possible:

\[ M_0(t) = \sum_j \gamma_j(t) \Theta_j(t), \quad \gamma_j \text{ are eigenvalues, } \Theta_j \text{ are eigenprojectors.} \]

Remark that notations $\gamma_j$ and $\Theta_j$ are used temporarily in Introduction. Applying $\Theta_j(t)$ onto (1.19) gives

\[ \partial_t (\Theta_j(t)\tilde{S}_0) + \frac{1}{\varepsilon^{3/4}} \gamma_j(t)(\Theta_j(t)\tilde{S}_0) = (\partial_t \Theta_j(t))\tilde{S}_0, \quad (\Theta_j S_0)(\tau; \tau) = \Theta_j(\tau). \]

Then we can write an explicit formula:

\[ \Theta_j(t)\tilde{S}_0(\tau; t) = \exp\left(-\varepsilon^{-3/4} \int_\tau^t \gamma_j(t')dt'\right) \Theta_j(\tau) \]

\[ + \int_\tau^t \exp\left(-\varepsilon^{-3/4} \int_s^t \gamma_j(s)ds\right) (\partial_t \Theta_j(t'))\tilde{S}_0(\tau; t')dt'. \]

11
It is shown that the maximum of the real parts of $\varepsilon^{-3/4} \gamma_j$ for all $j$ is $\gamma^+ t$. Then $\varepsilon^{-3/4} (\gamma_j(t) + \gamma_j(t)^{\star})/2 \leq \gamma^+ t$ and

$$
|\Theta_j(t)\tilde{S}_0(\tau; t)| \leq \exp\left(\left(\frac{t^2}{2}\right)\frac{\gamma^+}{2}\right)|\Theta_j(\tau)|
$$

(1.22)

$$
+ \int_{\tau}^{t} \exp\left(\left(\frac{t^2}{2}\right)\frac{\gamma^+}{2}\right)(\partial_t \Theta_j(t'))\tilde{S}_0(\tau; t')dt'.
$$

In the context of (1.8)-(1.9), we find $\partial_t \Theta_j(t) \leq C(1+1/t)$. Then for $t$ large, $\partial_t \Theta_j(t)$ is bounded. The problem is that for $t$ near 0, $\partial_t \Theta_j(t)$ is unbounded and is of order $1/t$, which implies that the integral on the right-hand side of (1.22) is not well defined when taking $\tau = 0$. As in the proof of Lemma 3.1 we overcome this difficulty by introducing the following change of variable for some small $c_0$:

$$
\tilde{S}_1(\tau; t) := \tilde{S}_0(\tau; t + c_0).
$$

Then for $\tilde{S}_1$, we can obtain a similar inequality as (1.22), in which the term $\partial_t \Theta_j(t')$ is replaced by $\partial_t \Theta_j(t' + c_0)$ which becomes uniformly bounded with bound $C/c_0$.

For small time in $[0, c_0]$, it is enough to use a rough estimate (Lemma B.4) for $\tilde{S}_0$. As we remarked right after (1.19), the instability is expected in time $O(|\ln \varepsilon|^{1/2})$, so $[0, c_0]$ is indeed a small time interval.

After considering $\Theta_j \tilde{S}_0$ for all $j$, by the identity $\sum_j \Theta_j = \text{Id}$ we finally obtain

$$
|\tilde{S}_0(\tau; t)| \leq C \exp\left(\left(\frac{t^2}{2}\right)\frac{\gamma^+}{2}\right) \exp(Ct), \quad \text{for some constant } C > 0.
$$

We remark that, for the time we consider of order $O(|\ln \varepsilon|^{1/2})$, the term $\exp(Ct)$ is negligible compared to the main growth term $\exp\left(\left(\frac{t^2}{2}\right)\frac{\gamma^+}{2}\right)$.

For the case where $M_0(t)$ is not diagonalizable, we show there exists an invertible matrix $P$ which is independent of $t$, and $|P| + |P^{-1}| \leq C(c_0)$ such that $|PM_0(t)P^{-1} + (PM_0(t)P^{-1})^{\star}|/2 \leq c_0 t\varepsilon^{3/4}$ with $c_0$ small and $C(c_0)$ a constant bounded for $c_0$ away from zero (we can choose $c_0$ as small as we want; here we only need to fix $c_0$ such that $c_0 \leq \gamma^+$). Then for the new unknown $\tilde{S}_0^{(1)} := P \tilde{S}_0$ which satisfies

$$
\partial_t \tilde{S}_0^{(1)} + \frac{1}{\varepsilon^{3/4}}(PM_0(t)P^{-1})\tilde{S}_0^{(1)} = 0, \quad \tilde{S}_0^{(1)}(\tau; \tau) = P,
$$

we have

$$
|\tilde{S}_0^{(1)}| \leq |P| \exp\left(c_0(t^2 - \tau^2)/2\right).
$$

This implies

$$
|\tilde{S}_0(\tau; t)| \leq |P| |P^{-1}| \exp\left(c_0(t^2 - \tau^2)/2\right) \leq C \exp\left(\left(\frac{t^2}{2}\right)\frac{\gamma^+}{2}\right).
$$

Rigorous arguments are given in Appendix B.
2 Description of the results

In this paper, we focus on one spatial dimension \( d = 1 \). However, we still use \( d \) on some occasions, when it is useful to stress the dependence on the dimension. If there is no specific definition, \( C \) denotes a constant independent of \((x, \xi, t, \tau, \varepsilon, c_0)\) where \( c_0 \) is a small constant introduced in Section B.2.2 and fixed in Section B.2.7. However the value of \( C \) could change from line to line.

2.1 Notations

We introduce the notations

\[
L(\omega_0, \partial) := \partial_t + \begin{pmatrix} 0 & \partial_x & 0 \\ \partial_x & 0 & \alpha_0 \omega_0 \\ 0 & -\alpha_0 \omega_0 & 0 \end{pmatrix}, \quad M(\omega_0, \partial) := \partial_t + \begin{pmatrix} 0 & \theta_0 \partial_x & 0 \\ \theta_0 \partial_x & 0 & \omega_0 \\ 0 & -\omega_0 & 0 \end{pmatrix},
\]

where \( \partial = (\partial_t, \partial_x) \). Then the system \( \text{(1.8)} \) of coupled Klein-Gordon systems can be written as

\[
\begin{align*}
L(\omega_0, \varepsilon, \partial) u &= \frac{1}{\sqrt{\varepsilon}} F(u + v, v), \\
M(\omega_0, \varepsilon, \partial) v &= \frac{1}{\sqrt{\varepsilon}} (G(u, u) + H(v, v)),
\end{align*}
\]

where \( F, G, H : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) are symmetric bilinear forms defined for any \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) as

\( F(u, v) := (0, u_3 v_3, 0), \quad G(u, v) := (0, -u_2 v_2, 0), \quad H(u, v) := (0, u_2 v_2, 0). \)

The characteristic varieties are the sets of time-space frequencies that define plane-wave solutions of \( L \) and \( M \):

\[
\text{Char} L := \{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}, \det L(\omega_0, -i\tau, i\xi) = 0 \}, \\
\text{Char} M := \{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}, \det M(\omega_0, -i\tau, i\xi) = 0 \}.
\]

They both admit global smooth parameterizations, by \( \{0, \pm \lambda\} \) and \( \{0, \pm \mu\} \) respectively, where

\[
\lambda(\xi) := \sqrt{\alpha_0^2 \omega_0^2 + |\xi|^2}, \quad \mu(\xi) := \sqrt{\omega_0^2 + \theta_0^2 |\xi|^2}.
\]

For any \( (\tau, \xi) \in \mathbb{R} \times \mathbb{R} \), we denote by \( P(\tau, \xi) \) and \( Q(\tau, \xi) \) the projectors onto the kernel of \( L(\omega_0, -i\tau, i\xi) \) and \( M(\omega_0, -i\tau, i\xi) \), respectively. Then we have the following smooth spectral decompositions:

\[
\begin{align*}
L(\omega_0, 0, i\xi) &= i\lambda(\xi) P_+ (\xi) - i\lambda(\xi) P_- (\xi) + 0 \cdot P_0 (\xi), \\
M(\omega_0, 0, i\xi) &= i\mu(\xi) Q_+ (\xi) - i\mu(\xi) Q_- (\xi) + 0 \cdot Q_0 (\xi),
\end{align*}
\]
where the eigenvalues are given by (2.4) and the eigenprojectors are

\[ P_+(\xi) := P(\lambda(\xi), \xi), \quad P_-(\xi) := P(-\lambda(\xi), \xi), \quad P_0(\xi) := P(0, \xi), \]
\[ Q_+(\xi) := Q(\mu(\xi), \xi), \quad Q_-(\xi) := Q(-\mu(\xi), \xi), \quad Q_0(\xi) := Q(0, \xi). \]

Given a characteristic phase \((\tau, \xi) \in \text{Char}\ L\), given \(p \in \mathbb{Z}\), the phase \((p\tau, p\xi)\) belongs to \(\text{Char}\ L\) if and only if \(p \in \{-1, 0, 1\}\). The same is true of \(\text{Char}\ M\). Also, by choice of \(\theta_0\) and \(a_0\), the intersection \(\text{Char}\ L \cap \text{Char}\ M\) is equal to \{(0, \xi), \xi \in \mathbb{R}\}.

### 2.2 Statement of the results

For initial datum (1.9), we choose \(k \neq 0\) such that for some \(\omega \neq 0\), there holds

\[ \beta = (\omega, k) \in \text{Char}\ M, \quad 3\beta = (3\omega, 3k) \in \text{Char}\ L. \]

By (2.4), equation (2.6) amounts to

\[ k^2 = \left(1 - \frac{\alpha_0^2}{9}\right)(1 - \theta_0^2)^{-1}\omega_0^2, \quad \omega^2 = k^2 + \frac{\alpha_0^2}{9}\omega_0^2. \]

We choose initial amplitude \(v^0\) satisfying the polarization condition:

\[ v^0 \in \ker M(\omega_0, -i\omega, ik). \]

We suppose the regularity \(v^0 \in H^s\) with \(s\) sufficient large as in Remark 2.2.

In Section 3, we show that the weak transparency condition is satisfied, then we construct an approximate solution by WKB expansion:

**Proposition 2.1.** Under the choice of \((\omega, k)\) as in (2.6), the polarization condition (2.8) and the regularity assumption \(v^0 \in H^s\) with \(s\) large, for any \(K\), there exists \((v^a, v^a)\) that solves

\[\begin{cases}
L\left(\frac{\omega_0}{\varepsilon}, \partial\right)u^a = \frac{1}{\varepsilon}F(u^a + v^a, v^a) + \varepsilon K_1 r^\xi,
M\left(\frac{\omega_0}{\varepsilon}, \partial\right)v^a = \frac{1}{\varepsilon}(G(u^a, u^a) + H(v^a, v^a)) + \varepsilon K_2 r^\xi,
u^a(0, x) = \sqrt{\varepsilon}u^0(0, x),
v^a(0, x) = \Re \left(v^0(0, x)e^{ikx/\varepsilon}\right) + \sqrt{\varepsilon}v^\xi(0, x),
\end{cases}\]

in some time interval \([0, \tilde{T}]\) with \(\tilde{T} > 0\) independent of \(\varepsilon\), and for \(j = 1, 2\),

\[ r^\xi_j(t, x) = R_j(t, x, \frac{kr - \omega t}{\varepsilon}), \quad R_j(t, x, \theta) \in L^\infty([0, \tilde{T}]_t, H^1(T_\theta, H^{s-K}(\mathbb{R}))) \]

Moreover, \((u^a, v^a)\) has the following expansion:

\[ u^a = \Re \left(u_{03}(t, x)e^{3i(kx-\omega t)/\varepsilon}\right) + \sqrt{\varepsilon}u^\xi, \quad v^a = \Re \left(v_{01}(t, x)e^{i(kx-\omega t)/\varepsilon}\right) + \sqrt{\varepsilon}v^\xi, \]

\[14\]
where the leading amplitudes have initial data

\[ u_{03}(0, x) = 0, \quad v_{01}(0, x) = v^0(x). \]

The correctors are of the form

\[ (u^\xi_r(t, x), v^\xi_r(t, x)) = \left( U^a_r(t, x, \frac{kx - \omega t}{\varepsilon}), V^a_r(t, x, \frac{kx - \omega t}{\varepsilon}) \right) \]

with

\[ (U^a_r, V^a_r)(t, x, \theta) \in L^\infty([0, T], H^1(T, H^{s-K^a}(\mathbb{R}^d))). \]

**Remark 2.2.** Precisely, we choose the Sobolev regularity index \( s > d/2 + K + (d + 2) + (q_0 + 3)/4 \) according to the \( H^{s-1} \) estimate \( (3.12) \) of \( \partial_t q \), the need for the smallness of the right-hand side of \( (4.34) \), and the estimate for \( \partial_x^a S_q, |\alpha| \leq d + 1, q \leq q_0 \) where \( S_q \) defined as in \( (C.5) \) and \( q_0 \) is the order for the expansion in constructing solution operator in Appendix C (see \( (C.15) \)).

In Section 4, we show the WKB solution \((u^a, v^a)\) obtained in Proposition 2.1 is unstable. We consider initial data of the form

\[ u(0) = \sqrt{\varepsilon} u^0(0) + \varepsilon^K \phi_1^\varepsilon, \quad v(0) = \Re \left( v^0(x)e^{ikx/\varepsilon} \right) + \sqrt{\varepsilon} v^0_{1\varepsilon}(0) + \varepsilon^K \phi_2^\varepsilon \]

corresponding to small perturbations of the WKB solution of Proposition 2.1.

**Theorem 2.3.** There exists \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \), for some initial perturbations satisfying \( \sup_{0 < \varepsilon < \varepsilon_0} \| (\phi_1^\varepsilon, \phi_2^\varepsilon) \|_{L^1 \cap L^\infty} < \infty \), the solution \((u, v)\) to \((1.8)\) issued form the initial datum \((2.10)\) is unstable, in the following two senses:

- for any \( K_0 + 1/4 > K > d/2 + 1/4 \), there exists a unique solution \( u \in \mathcal{C}_0^0([0, T_0 \ln^1 \varepsilon \ln^1 \eta], H^\infty) \) for some \( s_0 > d/2 \) and all \( \varepsilon \)-independent \( T_0 < T^*_0 \) where

  \[ T^*_0 := \sqrt{2(K - d/2 - 1/4)}/\Gamma_1 \]

  with \( \Gamma_1 \) precisely given in \((1.24)\). Moreover, for any \( \kappa_0 > d/2 + 1/4 \), there holds for \( T_0 \) close to \( T^*_0 \):

  \[ \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\kappa_0} \| (u - u^a, v - v^a) \|_{L^2} = \infty; \]

- for any \( K_0 + 1/4 > K > 1/4 \) and solution \( u \in L^\infty([0, T_1 \ln^1 \varepsilon \ln^1 \eta] \times \mathbb{R}) \) for any \( \varepsilon \)-independent \( T_1 < T^*_1 \) where

  \[ T^*_1 := \sqrt{2(K - 1/4)}/\Gamma_1, \]

  for any \( \kappa_1 > 1/4 \) there holds for \( T_1 \) close to \( T^*_1 \):

  \[ \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\kappa_1} \| (u - u^a, v - v^a) \|_{L^2 \cap L^\infty} = \infty. \]
We make two remarks about Theorem 2.3:

• first remark: We can think of $K_a$ being equal to $K$ and large. Then the initial perturbation is very small, and the WKB solution almost solves the system (1.8) of coupled KG equations. The point we make in Theorem 2.3 is that even though the WKB solution is very close to solving the exact system, it is somehow not close to the exact solution. Parameter $\kappa_0$ measure how 'far' the WKB solution strays from the exact solution: the distance between $(u,v)$ and $(u_a,v_a)$ goes from $\varepsilon K$, very small ($K$ arbitrarily large), to much larger than $\varepsilon^{\kappa_0}$ (with $\kappa_0$ fixed, depending on the dimension), in short time $O(\varepsilon^{1/4} \ln \varepsilon^{1/2})$.

• second remark: The second result (2.14) is obtained by assuming the existence of solution $u$ on $[0,T_1^* \varepsilon^{1/4} \ln \varepsilon^{1/2}]$ with any $T_1^* < T^*$. The existence time we obtain here is $[0,T_0^* \varepsilon^{1/4} \ln \varepsilon^{1/2}]$ with $T_0^* < T_0$ and clearly $T_0^* < T^*$. Hence an open problem is to show the existence in longer time.

3 Proof of Proposition 2.1

The aim of this section is to construct an approximate solution through WKB expansion. At the same time, this gives a proof for Proposition 2.1.

3.1 WKB expansion

We describe approximate solutions to (2.2) in the forms of WKB expansions for profiles with $\theta = (kx - \omega t)/\varepsilon$:

\[(u,v)(t,x) = \sum_{n=0}^{2K_a} \varepsilon^{n/2} (u_n, v_n)(t,x) = \sum_{n=0}^{2K_a} \varepsilon^{n/2} \sum_{p \in \mathcal{H}_n} e^{i\varphi \theta} (u_{n,p}, v_{n,p})(t,x),\]

where $2K_a \in \mathbb{Z}_+$ determines the precision of the expansion. For $(u,v)$ in (3.1),

\[(F(u+v,v),G(u,u),H(v,v)) = \left[ \sum_{n=0}^{2K_a} \varepsilon^{n/2} (F_n, G_n, H_n) \right].\]

We let $(u)_p$ denote the $p$-harmonic of a trigonometric polynomial $u$ in $\theta$. For example, in (3.1), $(u_n,v_n)_p = (u_{n,p},v_{n,p})$. We denote by $\mathbb{P}$ and $\mathbb{Q}$ the operators acting diagonally on trigonometric polynomials, defined as

\[\mathbb{P}u = \sum_p e^{i\varphi \theta} P(p\tilde{\beta})(u)_p, \quad \mathbb{Q}u = \sum_p e^{i\varphi \theta} Q(p\tilde{\beta})(u)_p,\]

where $P(p\tilde{\beta})$ and $Q(p\tilde{\beta})$ are orthogonal projectors onto $\text{ker} L(ip\tilde{\beta})$ and $\text{ker} M(ip\tilde{\beta})$ respectively with the definitions

\[L(ip\tilde{\beta}) := L(\omega_0,(-ip\omega,ipk)), \quad M(ip\tilde{\beta}) := M(\omega_0,(-ip\omega,ipk)),\]
for which we use the notation (2.1).

We now plug (3.1) into (2.2) and write a cascade of WKB equations, the first of which comprises all terms of order $O(\varepsilon^{-1})$:

$$L(\tilde{\beta}\partial_{\theta})u_0 = 0, \quad M(\tilde{\beta}\partial_{\theta})v_0 = 0.$$  

(3.2)

By (2.6), equation (3.2) implies

$$u_0 = u_{0,-3}\varepsilon^{-3\theta} + u_{0,0} + u_{0,3}\varepsilon^{3\theta}, \quad v_0 = v_{0,-1}\varepsilon^{-i\theta} + v_{0,0} + v_{0,1}\varepsilon^{i\theta}$$

with $u_{0,p} \in \ker L(ip\tilde{\beta}), v_{0,p} \in \ker M(ip\tilde{\beta})$. Direct calculation gives

$$v_{0,1} = f\tilde{e}_1, \quad u_{0,3} := g\tilde{e}_3,$$

where $f$ and $g$ are scalar functions determined later by (3.10); $\tilde{e}_1$ and $\tilde{e}_3$ are constant vectors, which form the bases of vector spaces $\ker M(i\tilde{\beta})$ and $\ker L(i\tilde{\beta})$ respectively:

$$\tilde{e}_1 := \left(\frac{\theta_0 k}{\omega}, 1, \frac{i\omega_0}{\omega}\right), \quad \tilde{e}_3 := \left(-\frac{k}{\omega}, 1, \frac{\alpha_0 \omega_0}{3\omega}\right).$$

For negative $p$, reality requires $u_{0,p} = \bar{u}_{0,-p}$, $v_{0,p} = \bar{v}_{0,-p}$. We define $\tilde{e}_p := (\tilde{e}_{-p})^*$ for $p \in \{-3, -1\}$.

The $O(\varepsilon^{-1/2})$ terms in the expansion are

$$L(\tilde{\beta}\partial_{\theta})u_1 = F_0, \quad M(\tilde{\beta}\partial_{\theta})v_1 = G_0 + H_0.$$  

(3.5)

A consequence of (3.5) is the following necessary condition

$$P(p\tilde{\beta})(F_0)_p = 0, \quad Q(p\tilde{\beta})(G_0 + H_0)_p = 0,$$

(3.6)

for all $p, p' \in \mathbb{Z}$. By the properties of the characteristic varieties, the choice of $\beta$ (2.0) and the structure of the bilinear terms (2.3), condition (3.6) is equivalent to the following compatibility condition for all $p \in \mathbb{Z}$:

$$P(p\tilde{\beta}) \sum_{p_1 + p_2 = p} F((P + Q)(p_1\tilde{\beta})\cdot, (Q(p_2\tilde{\beta})\cdot) = 0,$$

$$Q(p\tilde{\beta}) \sum_{p_1 + p_2 = p} \left(G(P(p_1\tilde{\beta})\cdot, P(p_2\tilde{\beta})\cdot) + H(Q(p_1\tilde{\beta})\cdot, Q(p_2\tilde{\beta})\cdot)\right) = 0.$$  

(3.7)

In our context, we find out (3.7) is satisfied. This is in fact the weak transparency condition. Since there is no mean mode in initial datum (1.9), we simply take $u_{0,0} = v_{0,0} = 0$. We remark that this choice of zero mean mode is not necessary to construct an approximate solution by WKB expansion.

The equation (3.5) also implies

$$1 - P(p\tilde{\beta})u_{1,p} = L(ip\tilde{\beta})^{-1}(F_0)_p, \quad (1 - Q(p\tilde{\beta}))v_{1,p} = M(ip\tilde{\beta})^{-1}(G_0 + H_0)_p,$$

(3.8)
where $L^{(-1)}$ and $M^{(-1)}$ denote partial inverses, naturally defined on the orthogonal complements of ker $L$ and ker $M$.

The $O(\varepsilon^0)$ terms are

\[(3.9) \quad L(\tilde{\beta}\partial_\theta)u_2 + L(0, \partial)u_0 = F_1, \quad M(\tilde{\beta}\partial_\theta)v_2 + M(0, \partial)v_0 = G_1 + H_1.\]

From (3.9) we deduce

\[(3.10) \quad P(3\tilde{\beta})L(0, \partial)P(3\tilde{\beta})u_{0,3} = P(3\tilde{\beta})(F_1)_3, \quad Q(\tilde{\beta})M(0, \partial)Q(\tilde{\beta})v_{0,1} = Q(\tilde{\beta})(G_1 + H_1)_1.\]

By (3.6), (3.8) and symmetry of $F$, the source term in (3.10) is

\[P(3\tilde{\beta})(F_1)_3 = 2P(3\tilde{\beta})F(M(2i\tilde{\beta})^{-1}H(v_{0,1}, v_{0,1}), v_{0,1}) + (F_1)_3(u_{0,3}, v_{0,1}),\]

where $F_1$ is a polynomial in $(u_{0,3}, v_{0,1})$. Since there is no third order harmonic in the leading terms of the initial data (1.9), we choose always $u_{0,3}(0, \cdot) = 0$. Then the initial value $F_1(0, \cdot) = 0$. By (3.6) and (3.8), the source term in (3.10) (ii) is also a polynomial in $(u_{0,3}, v_{0,1})$ that admits zero initial value. By reality of the nonlinear terms, and symmetry of $L$ and $M$, the system in $(u_{0,-3}, v_{0,-1})$ is conjugated to (3.10).

The differential operators in the left-hand side of (3.10) are transport operators at the group velocities $\partial_\xi \lambda(3\tilde{\beta})$ and $\partial_\xi \mu(\tilde{\beta})$ respectively (for a proof of this fact, see [19]), where $\lambda$ and $\mu$ are defined in (2.4).

A consequence of the compatibility condition (3.7) is that the formal WKB equations lead to closed transport equations in $(u_{0,3}, v_{0,1})$ with polynomial source terms. This implies that, given initial data $(u_{0,3}, v_{0,1})(0) = (0, v^0) \in H^s$, $s > d/2$, system (3.10) and its conjugate system admit a unique solution $(u_{0,3}, v_{0,1})$ on $[0, T]$ for some $T > 0$ independent of $\varepsilon$. Moreover, there holds the estimate:

\[(3.11) \quad \|(u_{0,3}, v_{0,1})\|_{L^\infty([0, T], H^s)} + \|\partial_t (u_{0,3}, v_{0,1})\|_{L^\infty([0, T], H^{s-1})} \leq C(1 + T).\]

In particular, since $u_{0,3}(0, \cdot) \equiv 0$, together with (3.3), we have $g(0, \cdot) \equiv 0$ and

\[(3.12) \quad \|g\|_{L^\infty([0, T], H^s)} + \|\partial_t g\|_{L^\infty([0, T], H^{s-1})} \leq CT.\]

One key point here is that, by the structure of (2.4), the third-order harmonics $u_{0,3}$ grow in time on $[0,t]$ for some $0 < t \leq T$. Indeed, by $(u_{0,3}, v_{0,1})(0) = (0, v^0)$, (3.10), and polarization condition $v^0 = Q(\beta)v^0$, there holds for nonzero $v^0$ that

\[(3.13) \quad (\partial_t u_{0,3})(0, x) = 2P(3\tilde{\beta})F(M(2i\tilde{\beta})^{-1}H(v^0, v^0), v^0)(x) \neq 0.\]

The equation (3.3) also implies

\[
(1 - P(p\tilde{\beta}))u_{2,p} = L(ip\tilde{\beta})^{-1}(v^0, v^0)(x) = 0,
\]

\[
(1 - Q(p\tilde{\beta}))v_{2,p} = M(ip\tilde{\beta})^{-1}(v^0, v^0)(x) = 0.
\]
3.2 The approximate solution and Proof of Proposition 2.1

With the compatibility condition (3.7), a similar proof as Theorem 2.3 in [10], or an application of Section 6.6 in [15], we can continue the WKB expansion in Section 3.1 up to any order. This implies that, for any \( K_a > 0 \), there exists a WKB solution \((u^a, v^a)\) of the form (3.1), such that on \([0, T]\) for some \( T > 0 \) independent of \( \varepsilon \):

\[
\begin{align*}
L\left( \frac{\omega_0}{\varepsilon}, \partial \right) u^a &= \frac{1}{\sqrt{\varepsilon}} F(u^a + v^a, v^a) + \varepsilon^{K_a} r_1^\varepsilon, \\
M\left( \frac{\omega_0}{\varepsilon}, \partial \right) v^a &= \frac{1}{\sqrt{\varepsilon}} (G(u^a, u^a) + H(v^a, v^a)) + \varepsilon^{K_a} r_2^\varepsilon.
\end{align*}
\]

The other results in Proposition 2.1 are obtained from the standard WKB expansion.

4 Proof of Theorem 2.3

We show in this section that the WKB solution \((u^a, v^a)\) constructed in Section 3 is unstable in the sense stated in Theorem 2.3.

4.1 Preparation

By symmetry of the hyperbolic operator, for \( \varepsilon > 0 \) the solution \((u, v)\) to the initial value problem (2.2), (2.10) is defined on time interval \([0, T(\varepsilon)]\) for some \( T(\varepsilon) > 0 \).

By (2.2), (2.9) and (2.10), the unknown perturbation \((\dot{u}, \dot{v})\) defined as (4.1)

\[
(u, v) =: (u^a, v^a) + \varepsilon^\kappa (\dot{u}, \dot{v}) \quad \text{for some } 1/4 < \kappa \leq \min\{K, K_a + 1/4\}
\]

satisfies (4.2)

\[
\begin{align*}
L\left( \frac{\omega_0}{\varepsilon}, \partial \right) \dot{u} &= \frac{1}{\sqrt{\varepsilon}} F(u^a) \dot{v} + F(v^a) \dot{u} + 2F(v^a) \dot{v} + \frac{\varepsilon^K}{\sqrt{\varepsilon}} F(\dot{u} + \dot{v}, \dot{v}) - \varepsilon^{K_a - \kappa} r_1^\varepsilon, \\
M\left( \frac{\omega_0}{\varepsilon}, \partial \right) \dot{v} &= \frac{2}{\sqrt{\varepsilon}} (G(u^a) \dot{u} + H(v^a) \dot{v}) + \frac{\varepsilon^K}{\sqrt{\varepsilon}} (G(\dot{u}, \dot{u}) + H(\dot{v}, \dot{v})) - \varepsilon^{K_a - \kappa} r_2^\varepsilon, \\
\dot{u}(0, x) &= \varepsilon^{K - \kappa} \phi_1^\varepsilon(x), \quad \dot{v}(0, x) = \varepsilon^{K - \kappa} \phi_2^\varepsilon(x),
\end{align*}
\]

where we define \( B(a)B := B(a, b) \) for any \( B \in \{F, G, H\} \).

4.1.1 Spectral decompositions, resonances and transparencies

According to (2.5), we write the decompositions in semiclassical Fourier multipliers:

\[
\begin{align*}
L\left( \frac{\omega_0}{\varepsilon}, \partial_t, \partial_x \right) := \partial_t + \frac{i}{\varepsilon} \left( \text{op}_\varepsilon(\lambda_+) \text{op}_\varepsilon(P_+) + \text{op}_\varepsilon(\lambda_-) \text{op}_\varepsilon(P_-) + \text{op}_\varepsilon(\lambda_0) \text{op}_\varepsilon(P_0) \right), \\
M\left( \frac{\omega_0}{\varepsilon}, \partial_t, \partial_x \right) := \partial_t + \frac{i}{\varepsilon} \left( \text{op}_\varepsilon(\mu_+) \text{op}_\varepsilon(Q_+) + \text{op}_\varepsilon(\mu_-) \text{op}_\varepsilon(Q_-) + \text{op}_\varepsilon(\mu_0) \text{op}_\varepsilon(Q_0) \right)
\end{align*}
\]
with
\[ \lambda_+ = -\lambda_- := \lambda, \quad \mu_+ = -\mu_- := \mu, \quad \lambda_0 = \mu_0 := 0. \]
To unify the notations, we denote for \( j \in \{+, -, 0\} \):
\[ \lambda_j^L := \lambda_j, \quad \lambda_j^M := \mu_j, \quad \Pi_j^L := P_j, \quad \Pi_j^M := Q_j. \]
By Proposition 2.1, we have for \( B \in \{F, G, H\} \):
\[ (4.3) \quad B(u^a) = \sum_{p=\pm 3} e^{ip\theta} B(u_{0,p}) + \sqrt{\varepsilon} B(u_{r}), \quad B(v^a) = \sum_{p=\pm 1} e^{ip\theta} B(v_{0,p}) + \sqrt{\varepsilon} B(v_{r}). \]
This indicates that the singular linear source terms (of order \( 1/\sqrt{\varepsilon} \)) are essentially those associated with the leading terms of the WKB solutions: \( u_{0,\pm 3} \) and \( v_{0,\pm 1} \).
As we introduced in Introduction, resonances are zero points to the factors appearing in strong transparency condition. They actually correspond to crossing of eigenmodes, and are defined as obstruction to the solvability of homological equations that arise in normal form change of variables used to prove the stabilities of WKB solutions in for example [10]. They play an important role in the well-posedness analysis. We give the precise definitions in our setting:

**Definition 4.1** (Resonances and interaction coefficients). Given \( i, j \in \{+,-,0\}^2 \), \( p \in \{-3,-1,1,3\} \) and \( (\delta, \sigma) \in \{L,M\}^2 \), we define the resonance set
\[ \mathcal{R}_{ij,p}^{\delta,\sigma} := \{ \xi \in \mathbb{R}, \ p\omega = \lambda^\delta_i(\xi + pk) - \lambda^\sigma_j(\xi) \}. \]
For bilinear form \( B \in \{F, G, H\} \), the families of matrices \( \Pi^{\delta}_i(\xi + pk)B(\varepsilon_p)\Pi^{\sigma}_j(\xi) \) and \( \Pi^{\sigma}_j(\xi)B(\varepsilon_{-p})\Pi^{\delta}_i(\xi + pk) \), indexed by \( \xi \in \mathbb{R} \), are called the interaction coefficients associated with \( (i,j,p,\delta,\sigma) \). The scalar function \( \xi \rightarrow \lambda^\delta_i(\xi + pk) - \lambda^\sigma_j(\xi) - p\omega \) is called the resonant phase associated with \( (i,j,p,\delta,\sigma) \).

We recall \( e_p \) come from the polarization condition \((3.3)\) and are given in \((3.4)\).

**Definition 4.2** (Transparencies). An interaction coefficient \( \Pi^\delta_i(\cdot + pk)B(\varepsilon_p)\Pi^\sigma_j(\cdot) \) or \( \Pi^\sigma_j(\cdot)B(\varepsilon_{-p})\Pi^\delta_i(\cdot + pk) \) is said to be transparent if the associated resonant phase can be factored out, which means there holds for all \( \xi \in \mathbb{R} \) the bound
\[ |\Pi^\delta_i(\xi + pk)B(\varepsilon_p)\Pi^\sigma_j(\xi)| \leq C|\lambda^\delta_i(\xi + pk) - p\omega - \lambda^\sigma_j(\xi)|, \]
or
\[ |\Pi^\sigma_j(\xi)B(\varepsilon_{-p})\Pi^\delta_i(\xi + pk)| \leq C|\lambda^\delta_i(\xi + pk) - p\omega - \lambda^\sigma_j(\xi)|. \]

**Remark 4.3.** (i), Interaction coefficients \( \Pi^\delta_i(\xi + pk)B(\varepsilon_p)\Pi^\sigma_j(\xi) \) and \( \Pi^\sigma_j(\xi)B(\varepsilon_{-p})\Pi^\delta_i(\xi + pk) \) share the same resonant phase \( \lambda^\delta_i(\xi + pk) - p\omega - \lambda^\sigma_j(\xi) \) and resonant set \( \mathcal{R}_{ij,p}^{\delta,\sigma} \).
Equivalently, \( \mathcal{R}_{ij,p}^{\delta,\sigma} \) and \( \mathcal{R}_{ji,-p}^{\sigma,\delta} \) share essentially (a shift of \( \xi \) with \( pk \) or \(-pk \)) the
same interaction coefficients, so it is enough to consider one of these two resonant sets and study the transparency property of the associated interaction coefficients.

(ii) If $\mathcal{R}_{ij,p}^{\sigma,\delta}$ is empty, by the smoothness of the eigenmodes (2.4), there holds for some $c > 0$ and all $\xi \in \mathbb{R}^1$ the lower bound:

$$|\lambda_j^\delta(\xi + pk) - p\omega - \lambda_j^\sigma(\xi)| \geq c.$$  

Then the related interaction coefficients $\Pi_j^\delta(\cdot + pk)B(\bar{\varepsilon}_p^\delta)\Pi_j^\sigma(\cdot)$ and $\Pi_j^\sigma(\cdot)B(\bar{\varepsilon}_p)\Pi_j^\delta(\cdot + pk)$ are all transparent. This is true because the interaction coefficients are uniformly bounded in $\xi$, see later (4.5), (4.6), (4.7) and (4.8).

We now calculate all the non-empty resonance sets and the transparency property for the related interaction coefficients. Remark that, we only need to calculate the interaction coefficients appearing in (4.2).

Associated with the fundamental harmonics $v_{0,\pm 1}$ (namely $e_{\pm 1}$), with our choice $0 < \theta_0 < 1$ and $2.5 < \alpha_0 < 3$, essentially there are four non-empty resonance sets:

$$\mathcal{R}^{M,L}_{+0,1} = \mathcal{R}^{M,M}_{+0,1} = \{ \xi \in \mathbb{R}, \ \omega = \mu_+(\xi + k)\} = \{0, -2k\},$$

(4.4)

$$\mathcal{R}^{L,L}_{0,-1} = \mathcal{R}^{M,M}_{0,-1} = \{ \xi \in \mathbb{R}, \ \omega = 0 - \mu_-(-\xi - k)\} = \{0, 2k\}.$$

To check the transparency property of the interaction coefficients, we need to calculate the interaction coefficients appearing in (4.2).

(i) for $j \in \{+, -\}, \delta \in \{L, M\}$, there holds

(4.5)

$$\Pi_j^\delta(\xi)V = \frac{(V, \Omega_j^\delta(\xi))\Omega_j^\delta(\xi)}{|\Omega_j^\delta(\xi)|^2} = \frac{(V, \Omega_j^\delta(\xi))\Omega_j^\delta(\xi)}{2},$$

where

(4.6)

$$\Omega_j^L(\xi) := \left(\frac{-\xi}{\lambda_j^L(\xi)}, 1, \frac{i\alpha_0\omega_0}{\lambda_j^L(\xi)}\right), \quad \Omega_j^M := \left(\frac{-\theta_0\xi}{\mu_j^M(\xi)}, 1, \frac{i\omega_0}{\mu_j^M(\xi)}\right).$$

(ii) for $j = 0$, $\delta \in \{L, M\}$, there holds

(4.7)

$$\Pi_0^\delta(\xi)V = \frac{(V, \Omega_0^\delta(\xi))\Omega_0^\delta(\xi)}{|\Omega_0^\delta(\xi)|^2},$$

where

(4.8)

$$\Omega_0^L(\xi) := \left(1, 0, -\frac{i\xi}{\alpha_0\omega_0}\right), \quad \Omega_0^M := \left(1, 0, -\frac{i\theta_0\xi}{\omega_0}\right).$$

To calculate the interaction coefficients related to the non-empty resonance sets in (4.4), by (4.2) and (4.3), it is sufficient to calculate

(4.9)

$$P_0(\xi)F(\bar{e}_-^1)Q_+(\xi + k), \quad Q_0(\xi)H(\bar{e}_-^1)Q_+(\xi + k), \quad Q_+(\xi + k)H(\bar{e}_1)Q_0(\xi), \quad P_0(\xi)F(\bar{e}_1)Q_-(\xi - k), \quad Q_0(\xi)H(\bar{e}_1)Q_-(\xi - k), \quad Q_-(\xi - k)H(\bar{e}_-^1)Q_0(\xi).$$

With our choice of the bilinear forms as in (2.3), we find the interaction coefficients in (4.9) are all identically zero. Together with Remark 4.3 (ii), we have:
Lemma 4.4. All the interaction coefficients related to the fundamental harmonics $v_{\pm 1}$ are transparent.

Associated with the third order harmonics $u_{0, \pm 3}$ (namely $\varepsilon_{\pm 3}$), the non-empty resonance sets are

\begin{align}
\mathcal{R}^{L,L}_{+0,3} &= \mathcal{R}^{L,M}_{+0,3} = \{ \xi \in \mathbb{R}, \ 3\omega = \lambda_{+}(\xi + 3k) \} = \{ 0, -6k \}, \\
\mathcal{R}^{L,L}_{0,-3} &= \mathcal{R}^{M,L}_{0,-3} = \{ \xi \in \mathbb{R}, \ 3\omega = 0 - \lambda_{-}(\xi - 3k) \} = \{ 0, 6k \}, \\
\mathcal{R}^{M,M}_{+0,3} &= \mathcal{R}^{M,M}_{+0,3} = \{ \xi \in \mathbb{R}, \ 3\omega = \mu_{+}(\xi + 3k) \} = \{ -\xi_1 - 3k, \xi_1 - 3k \}, \\
\mathcal{R}^{M,M}_{0,-3} &= \mathcal{R}^{M,M}_{0,-3} = \{ \xi \in \mathbb{R}, \ 3\omega = 0 - \mu_{-}(\xi - 3k) \} = \{ \xi_1 + 3k, -\xi_1 + 3k \}, \\
\mathcal{R}^{M,M}_{-0,3} &= \{ \xi \in \mathbb{R}, \ 3\omega = \mu_{-}(\xi) - \lambda_{-}(\xi - 3k) \} = \{ -\xi_2, -\xi_3 \}.
\end{align}

(4.10)

where $\xi_1 = \sqrt{9\omega^2 - \omega_0^2}/\theta_0$ such that $\mu(\xi_1) = 3\omega$, $\xi_2$ and $\xi_3$ are solutions to

\[ 3\omega = \sqrt{(\xi + 3k)^2 + \alpha_0^2\omega_0^2} - \sqrt{\theta_0^2\xi_2^2 + \omega_0^2}. \]

For (4.12), the interaction coefficients related to the non-empty resonance sets in (4.10) and the corresponding resonances are

\begin{align*}
P_{+}(\xi + 3k)F(\bar{\varepsilon}_3)Q_0(\xi), & \quad Q_0(\xi)G(\bar{\varepsilon}_3)P_{+}(\xi + 3k), \quad \xi = 0, -6k; \\
P_{-}(\xi - 3k)F(\bar{\varepsilon}_3)Q_0(\xi), & \quad Q_0(\xi)G(\bar{\varepsilon}_3)P_{-}(\xi - 3k), \quad \xi = 0, 6k; \\
P_0(\xi)F(\bar{\varepsilon}_3)Q_{+}(\xi + 3k), & \quad Q_{+}(\xi + 3k)G(\bar{\varepsilon}_3)P_0(\xi), \quad \xi = -\xi_1 - 3k, \xi_1 - 3k; \\
P_0(\xi)F(\bar{\varepsilon}_3)Q_{-}(\xi - 3k), & \quad Q_{-}(\xi - 3k)G(\bar{\varepsilon}_3)P_0(\xi), \quad \xi = \xi_1 + 3k, -\xi_1 + 3k; \\
P_{+}(\xi + 3k)F(\bar{\varepsilon}_3)Q_{+}(\xi), & \quad Q_{+}(\xi)G(\bar{\varepsilon}_3)P_{+}(\xi + 3k), \quad \xi = \xi_2, \xi_3; \\
P_{-}(\xi - 3k)F(\bar{\varepsilon}_3)Q_{-}(\xi), & \quad Q_{-}(\xi)G(\bar{\varepsilon}_3)P_{-}(\xi - 3k), \quad \xi = -\xi_2, -\xi_3.
\end{align*}

By direct calculation, the above interaction coefficients are all transparent except the following ones around the following resonance points:

\begin{align}
P_{+}(\xi + 3k)F(\bar{\varepsilon}_3)Q_0(\xi), & \quad \xi = -6k; \quad P_{-}(\xi - 3k)F(\bar{\varepsilon}_3)Q_0(\xi), \quad \xi = 6k; \\
P_{+}(\xi + 3k)F(\bar{\varepsilon}_3)Q_+\xi), & \quad Q_{+}(\xi)G(\bar{\varepsilon}_3)P_{+}(\xi + 3k), \quad \xi = \xi_2, \xi_3; \\
P_{-}(\xi - 3k)F(\bar{\varepsilon}_3)Q_{-}(\xi), & \quad Q_{-}(\xi)G(\bar{\varepsilon}_3)P_{-}(\xi - 3k), \quad \xi = -\xi_2, -\xi_3.
\end{align}

(4.11)

We denote the set of all the non-transparent resonance points appeared in (4.11) by

\[ \mathcal{R} := \{-6k, 6k, \xi_2, \xi_3, -\xi_2, -\xi_3\}. \]

We choose $\theta_0$ and $\alpha_0$ such that the resonance points in $\mathcal{R}$ are pairwise distinct. This is true except for finite choices of $\theta_0$ and $\alpha_0$.

We will show that these non-transparent interaction coefficients in (4.11) will cause the solution $(\hat{u}, \hat{v})$ to be amplified instantaneously.
4.1.2 Projections and frequency shifts

We observe that the leading terms of the WKB solutions are highly oscillating, in the sense that they have prefactors $e^{i p \theta}$ with $\theta = (k x - \omega t)/\varepsilon, p \in \{-3, -1, 1, 3\}$. These highly oscillating factors will cause technical difficulties in our analysis. Indeed, we focus on the non-transparent interaction coefficients by localizing the frequencies near resonances. This localization is done by applying a semiclassical Fourier multiplier $\op \varepsilon (\chi)$ to the related interaction coefficients (see Section 4.1.4 for further details). The symbol $\chi (\xi)$ is a cut-off function compactly supported in a neighborhood of some resonance set. If there is a highly oscillating multiplier $e^{i p \theta}$ to some interaction coefficient $b_{jj'}$, then

$$\op \varepsilon (\chi) (e^{i p \theta} b_{jj'}) = e^{i p \theta} \op \varepsilon (\chi) b_{jj'} + [\op \varepsilon (\chi), e^{i p \theta}] b_{jj'},$$

where the commutator $[\op \varepsilon (\chi), e^{i p \theta}]$ is $O(1)$ due to the high oscillation. We need the commutator to be of order at least $O(1/\sqrt{\varepsilon})$ because the linear source terms in (4.2), where the interaction coefficients come from, have prefactor $1/\sqrt{\varepsilon}$.

Thus we would like to eliminate those highly oscillating prefactors $e^{i p \theta}$ of the non-transparent interaction coefficients. To achieve this, we introduce the following frequency shifts by defining $U := (u_+, u_-, u_0, v_+, v_-, v_0) \in \mathbb{R}^{18}$ as

$$\begin{align*}
  u_+ &:= e^{-i 3 \theta} \op \varepsilon (P_+) \dot{u}, & u_- &:= e^{i 3 \theta} \op \varepsilon (P_-) \dot{u}, & u_0 &:= \op \varepsilon (P_0) \dot{u}, \\
  v_+ &:= \op \varepsilon (Q_+) \dot{v}, & v_- &:= \op \varepsilon (Q_-) \dot{v}, & v_0 &:= \op \varepsilon (Q_0) \dot{v}.
\end{align*}$$

(4.12)

The perturbation unknown $(\dot{u}, \dot{v})$ can be reconstructed from $U$ via

$$\dot{u} = e^{i 3 \theta} u_+ + e^{-i 3 \theta} u_- + u_0, \quad \dot{v} = v_+ + v_- + v_0.$$

From (4.2) and (4.12), the new unknown $U$ solves

$$\partial_t U + i \frac{\varepsilon}{\varepsilon} \op \varepsilon (A) U = \frac{1}{\sqrt{\varepsilon}} \op \varepsilon (B) U + F.$$  

(4.13)

The symbol of the propagator is the diagonal matrix

$$A := \text{diag} \left( \lambda_+ (\cdot + 3k) - 3\omega, \lambda_- (\cdot - 3k) + 3\omega, 0, \mu_+, \mu_-, 0 \right).$$

The symbol of the singular source term is

$$B := \begin{pmatrix}
  B_{[P, P]} & B_{[P, Q]} \\
  B_{[Q, P]} & B_{[Q, Q]}
\end{pmatrix},$$

where the blocks are:

$$B_{[P, P]} := \sum_{p = \pm 1} e^{i p \theta} \begin{pmatrix}
  P_{+, p + 3} F(v_0 p) P_{+, 3} & e^{-i 3 \theta} P_{+, p - 3} F(v_0 p) P_{-, 3} & e^{i 3 \theta} P_{+, p} F(v_0 p) P_{0} \\
  e^{i 6 \theta} P_{-, p + 3} F(v_0 p) P_{-, 3} & P_{-, p - 3} F(v_0 p) P_{-, 3} & e^{i 3 \theta} P_{-, p} F(v_0 p) P_{0} \\
  e^{i 3 \theta} P_{0, p + 3} F(v_0 p) P_{+, 3} & e^{-i 3 \theta} P_{0, p - 3} F(v_0 p) P_{-, 3} & P_{0, p} F(v_0 p) P_{0}
\end{pmatrix};$$

23
\[ B_{[P,Q]} := B_{[Q,P]}^{1} + B_{[Q,P]}^{2} \]

\[ B_{[P,Q]}^{1} := \sum_{p = \pm 3} e^{ip\theta} \left( e^{-i3\theta} P_{+,p} F(u_{0p}) Q_{+} - e^{-i3\theta} P_{-,p} F(u_{0p}) Q_{-} \right) + \left( e^{-i3\theta} P_{+,p} F(u_{0p}) Q_{+} - e^{-i3\theta} P_{-,p} F(u_{0p}) Q_{-} \right), \]

\[ B_{[Q,P]}^{2} := 2 \sum_{p = \pm 1} e^{ip\theta} \left( e^{-i3\theta} P_{+,p} F(v_{0p}) Q_{+} - e^{-i3\theta} P_{-,p} F(v_{0p}) Q_{-} \right) + \left( e^{-i3\theta} P_{+,p} F(v_{0p}) Q_{+} - e^{-i3\theta} P_{-,p} F(v_{0p}) Q_{-} \right) ; \]

\[ B_{[Q,P]} := 2 \sum_{p = \pm 1} \left( e^{ip\theta} Q_{j1+p} H(v_{0p}) Q_{j2} \right)_{j1,j2 \in \{+, -, 0\}}, \]

where we use the notations

\[ P_{j,q}(\xi) := P_{j}(\xi + qk), \quad Q_{j,q}(\xi) := Q_{j}(\xi + qk), \quad q \in \mathbb{Z}, \quad j \in \{+, -, 0\}. \]

In \[ \textbf{(4.13)} \], the remainder \( F \) is the sum of the quadratic terms, the contribution of the higher-order WKB terms and remainder terms arising from compositions of pseudo-differential operators; for precise information, one may check Section 3.1.2 in \[ \textbf{[15]} \]. Similarly as Lemma 3.1 in \[ \textbf{[15]} \], we have here:

**Lemma 4.5.** There holds for all multiple index \( \alpha \in \mathbb{N}^{d} \) with \( |\alpha| \leq d/2 + d + 2 + (q_{0} + 3)/4 : \)

\[ |(\varepsilon \partial_{x})^{\alpha} F(t, \cdot)|_{L^{2}_{x}} \leq C \varepsilon^{\alpha-1/2} |(\dot{u}, \dot{v})(t, \cdot)|_{L^{\infty}_{x}} |(\varepsilon \partial_{x})^{\alpha} U(t, \cdot)|_{L^{2}_{x}} + C \varepsilon^{K_{s} - \kappa}. \]

### 4.1.3 Normal form reduction

By Lemma \[ \textbf{(4.14)} \], the interaction coefficients in the blocks \( B_{[P,P]}^{1}, B_{[P,Q]}^{2} \) and \( B_{[Q,P]}^{1} \) are all transparent.

We then separate the non-transparent interaction coefficients \[ \textbf{(4.13)} \] from \( B_{[P,Q]}^{1} \) and \( B_{[Q,P]}^{1} \) as we write:

\[ B_{[P,Q]}^{1} = B_{[P,Q]}^{nt} + B_{[P,Q]}^{1,t}, \quad B_{[Q,P]}^{1} = B_{[Q,P]}^{nt} + B_{[Q,P]}^{1,t}, \]

where

\[ B_{[P,Q]}^{nt} := \begin{pmatrix} P_{+,3} F(u_{03}) Q_{+} & 0 & 0 & P_{+,3} F(u_{03}) Q_{0} \\ 0 & P_{-,3} F(u_{0,-3}) Q_{-} & P_{-,3} F(u_{0,-3}) Q_{0} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ B_{[Q,P]}^{nt} := 2 \begin{pmatrix} Q_{+} G(u_{0,-3}) P_{+,3} & 0 & 0 & 0 \\ 0 & Q_{-} G(u_{0,3}) P_{-,3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]
The index $t$ means transparent and $nt$ means non-transparent. We remark that, with our choice of shift of frequencies in (4.12), the non-transparent interaction coefficients, which are all included in $B_{[P,Q]}^n$ and $B_{[Q,P]}^n$, are no longer multiplied by highly oscillating terms $e^{i\theta}$.

According to (4.11), we consider frequency cut-off functions near resonance sets for any $(i,j) \in \{(+,0),(-,0),(+,+),(-,-)\}$:

\begin{equation}
\chi_{[i,j]} \in \mathcal{C}_c^\infty (\mathcal{R}_{[i,j]}^h), \quad \chi_{[i,j]} \equiv 1 \text{ on } \mathcal{R}_{[i,j]}^{h/2}
\end{equation}

where $h > 0$ is a small constant fixed later on and $\mathcal{R}_{[i,j]}^r$ are balls of radial $r$ centered by the resonance points appeared in (4.11):

\begin{align}
\mathcal{R}_{[+,0]}^r &= B(-6k,r); \quad \mathcal{R}_{[-,0]}^r = B(6k,r); \\
\mathcal{R}_{[+,+]}^r &= (B(\xi_2,r) \cup B(\xi_3,r)); \quad \mathcal{R}_{[-,-]}^r := (B(-\xi_2,r) \cup B(-\xi_3,r)).
\end{align}

We choose $h$ small such that

\[ B(\xi,h) \cap B(\eta,h) = \emptyset, \quad \forall \xi,\eta \in \mathcal{R}, \xi \neq \eta, \]

which holds true because the resonance points in $\mathcal{R}$ are pairwise distinct. We then decompose the non-transparent interaction coefficients $B_{[P,Q]}^n$ and $B_{[Q,P]}^n$ as follows:

\[ B_{[P,Q]}^n = B_{[P,Q]}^{nt,1} + B_{[P,Q]}^{nt,2}, \quad B_{[Q,P]}^n = B_{[Q,P]}^{nt,1} + B_{[Q,P]}^{nt,2}, \]

where

\begin{align*}
B_{[P,Q]}^{nt,1} &= (\chi_{[+,+]}B_{+,+}^{(1)} + (\chi_{[+,+]}B_{+,+}^{(1)} + (\chi_{[+,+]}B_{+,+}^{(1)})B_{[-,-]}^{(1)} + (\chi_{[+,+]}B_{+,+}^{(1)})B_{[-,-]}^{(1)}, \\
B_{[Q,P]}^{nt,1} &= (\chi_{[+,+]}B_{+,+}^{(2)} + (\chi_{[+,+]}B_{+,+}^{(2)} + (\chi_{[+,+]}B_{+,+}^{(2)})B_{[-,-]}^{(2)}
\end{align*}

with

\begin{align*}
B_{+,+}^{(1)} &= \begin{pmatrix} P_{+,3}F(u_{03})Q_+ & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{+-}^{(1)} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_{-,3}F(u_{03})Q_- & 0 \end{pmatrix}, \\
B_{+,+}^{(2)} &= 2 \begin{pmatrix} Q_+G(u_{03})P_{+,3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{+-}^{(2)} := 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q_-G(u_{03})P_{-,3} & 0 \end{pmatrix}, \\
B_{+,+}^{(1)} &= \begin{pmatrix} 0 & 0 & P_{+,3}F(u_{03})Q_0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{+-}^{(1)} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P_{-,3}F(u_{03})Q_0 \end{pmatrix}.
\end{align*}

Since the cut-off functions are identically one near resonances, the parts $B_{[P,Q]}^{nt,2}$ and $B_{[Q,P]}^{nt,2}$ are supported away from resonances, then are transparent. The following
proposition states that the operator with symbol

\[ \mathcal{D} := \left( \begin{array}{ccc} \mathcal{B}_{[P,P]} & \mathcal{B}_{[P,Q]}^2 & \mathcal{B}_{[P,Q]}^{1,t} \\ \mathcal{B}_{[Q,P]}^t & \mathcal{B}_{[Q,Q]}^{[nt,2]} & \mathcal{B}_{[Q,Q]}^{[nt,2]} \end{array} \right) \]

can be eliminated, via a normal form reduction, from the evolution equation (4.13).

**Proposition 4.6.** There exists \( \Lambda \in S^0 \), with \( \partial_t \Lambda \in S^0 \) uniformly in \( t \in [0, \bar{T}] \), where \( \bar{T} > 0 \) independent of \( \varepsilon \) is from Proposition (2.1), such that

\[ (4.17) \quad \varepsilon [\partial_t, \text{op}_\varepsilon(\Lambda)] + i [\mathcal{A}, \text{op}_\varepsilon(\Lambda)] = \text{op}_\varepsilon(\mathcal{D}) + \varepsilon R(0), \]

where \( R(0)(t) \) a semiclassical pseudodifferential operator form \( L^2_x \) to \( L^2_x \) and is uniformly bounded with respect to \( t \in [0, \bar{T}] \) and in \( \varepsilon > 0 \). The symbol set \( \mathcal{S}^m \), \( m \in \mathbb{R} \) are defined in Appendix A.

**Proof.** We write \( \mathcal{D} = \sum_{|q| \leq 6} e^{i q t} \mathcal{D}_q \) and look for solution \( \Lambda \) to (4.17) of the form

\[ \Lambda = \sum_{|q| \leq 6} e^{i q t} \Lambda_q. \]

By direct calculation, we deduce that in order to solve (4.17), it is sufficient to solve

\[ i \left( -q \omega + \lambda_j^\delta_{j,\alpha} - \lambda_{j'}^\delta_{j',\alpha} \right) \Lambda_{(j,j',\delta,\sigma)} = (\mathcal{D}_q)_{(j,j',\delta,\sigma)} \]

for all index \( |q| \leq 6 \), \( (j, j') \in \{+, -, 0\}^2 \), \( (\delta, \sigma) \in \{L, M\}^2 \). By transparencies and the definitions of the cut-off functions, the solutions \( \Lambda_{(j,j',\delta,\sigma)} \) are well defined. For more details, one can check the proof of Proposition 3.3 in [15]. \( \square \)

By Proposition A.2 with \( \Lambda \) given in Proposition 4.6, the operator \( \text{op}_\varepsilon(\Lambda)(t) \) is bounded from \( L^2_x \) to \( L^2_x \) uniformly in \( t \in [0, \bar{T}] \) and in \( \varepsilon > 0 \). In particular, for \( \varepsilon \) small, \( \text{Id} + \sqrt{\varepsilon} \text{op}_\varepsilon(\Lambda) \) is invertible. We consider the change of variable

\[ \tilde{U}(t) := \left( \text{Id} + \sqrt{\varepsilon} \text{op}_\varepsilon(\Lambda(\varepsilon^{1/4} t)) \right)^{-1} U(\varepsilon^{1/4} t) \]

corresponding to a normal form reduction and a rescaling in time. Then we have:

**Corollary 4.7.** The equation in \( \tilde{U} \) is

\[ \partial_t \tilde{U} + \frac{i}{\varepsilon^{3/4}} \text{op}_\varepsilon(\mathcal{A}) \tilde{U} = \frac{1}{\varepsilon^{1/4}} \text{op}_\varepsilon(\tilde{\mathcal{B}}) \tilde{U} + \tilde{F}, \quad \tilde{\mathcal{B}} := \left( \begin{array}{cc} 0 & \mathcal{B}_{[P,Q]}^{[nt,1]} \\ \mathcal{B}_{[Q,P]}^{[nt,1]} & 0 \end{array} \right) (\varepsilon^{1/4} t), \]

where for all multiple index \( \alpha \in \mathbb{N}^d \) with \( |\alpha| \leq d/2 + d + 2 + (q_0 + 3)/4 \):

\[ |\varepsilon^{\alpha} \tilde{F}(t, \cdot)|_{L^2_x} \leq C \varepsilon^{\alpha - 1/4} |\tilde{U}(t, \cdot)|_{L^2_x} + C \varepsilon^{K_\alpha + 1/4 - \kappa}. \]

The proof is identical to the proof for Corollary 3.5 in [15], and is omit here.
4.1.4 Space-frequency localization

We define the following two quantities for the frequencies near non-transparent resonance sets:

\begin{align}
\gamma_1(\xi) &:= 2\text{tr}\left(P_+(\xi + 3k)F(\varepsilon_3)Q_+(\xi)G(\varepsilon_3)P_+(\xi + 3k)\right), \quad \text{for } \xi \in \mathcal{R}^h_{[+,+]}, \tag{4.19} \\
\gamma_2(\xi) &:= 2\text{tr}\left(P_-(\xi - 3k)F(\varepsilon_3)Q_-(\xi)G(\varepsilon_3)P_-(\xi - 3k)\right), \quad \text{for } \xi \in \mathcal{R}^h_{[-,-]}.
\end{align}

By (4.5) and (4.6), we have

\begin{equation}
\gamma_1(\xi) = \frac{\alpha_0 \omega_0^2}{6\omega \mu_+(\xi)}, \quad \gamma_2(\xi) = \frac{-\alpha_0 \omega_0^2}{6\omega \mu_-(\xi)}. \tag{4.20}
\end{equation}

Since \( \mu_+ = -\mu_- = \mu \) with \( \mu \) defined in (2.21) which is always positive, there holds

\begin{equation}
\gamma_1(\xi) = \gamma_2(-\xi) = \frac{\alpha_0 \omega_0^2}{6\omega \mu(\xi)} > 0. \tag{4.21}
\end{equation}

Equation (4.19)-(4.20) and the fact \( \mathcal{R}^h_{[+,+]} = -\mathcal{R}^h_{[-,-]} \) imply that \( \gamma_1 \) and \( \gamma_2 \) have the same maximum and minimum over their domains of definition.

Denote \( \xi_0, \xi'_0 \) the points in resonance set \( \mathcal{R}_{+,+} = \{\xi_2, \xi_3\} \) such that

\begin{equation}
\mu(\xi_0) = \min\{\mu(\xi_2), \mu(\xi_3)\}, \quad \mu(\xi'_0) = \max\{\mu(\xi_2), \mu(\xi_3)\}. \tag{4.22}
\end{equation}

This implies \( \gamma_1(\xi_0) = \gamma_2(-\xi_0) \geq \gamma_1(\xi'_0) = \gamma_2(-\xi'_0) > 0 \). For \( h \) small, by the continuity of \( \gamma_1 \) and \( \gamma_2 \), there holds the lower bound:

\begin{equation}
\inf_{\mathcal{R}^h_{[+,+]}} \gamma_1(\xi) := \inf_{\mathcal{R}^h_{[-,-]}} \gamma_2(\xi) \geq \gamma_1(\xi'_0)/2 > 0. \tag{4.23}
\end{equation}

By (3.12) and (3.13), \( \partial_t g(0, x) \) is not null, and is continuous and decaying at spatial infinity by Sobolev embedding. Then there exists \( x_0 \in \mathbb{R} \) such that

\[ |\partial_t g(0, x_0)| = \sup_{x \in \mathbb{R}} |\partial_t g(0, x)| > 0. \]

Then, \( \Gamma_1 \) introduced as the square root of the left-hand side of (1.15) is actually

\begin{equation}
\Gamma_1 = \sup_{x \in \mathbb{R}} |\partial_t g(0, x)| \sup_{\xi \in \{\xi_2, \xi_3\}} (\gamma_1(\xi))^{1/2} = |\partial_t g(0, x_0)| (\gamma_1(\xi_0))^{1/2} > 0, \tag{4.24}
\end{equation}

where the computation of \( \partial_t g(0, x) \) (and furthermore \( \Gamma_1 \)) does not require any knowledge of the solution, only a knowledge of the datum and the equations.

We consider extensions of the frequency cut-off functions in (4.14), by choosing \( \chi^{(0)}_{[i,j]} \) and \( \chi^{(1)}_{[i,j]} \) in \( C^\infty_c(\mathcal{R}^h_{[i,j]}) \) for any \( (i,j) \in \{(+,+), (-,-), (+,+), (-,+), (-,-)\} \) such that

\[ \chi^{(0)}_{[i,j]}|_{\text{supp } \chi_{[i,j]}} = \chi^{(1)}_{[i,j]}|_{\text{supp } \chi^{(0)}_{[i,j]}} = 1. \]

27
We then define the sums for \( \theta \in \{0, 1\} \):

\[
\chi := \chi_{[+,+]} + \chi_{[-,-]} + \chi_{[+0]} + \chi_{[-0]}, \quad \chi_{\theta} := \chi_{[+,+]} + \chi_{[-,-]} + \chi_{[+\theta]} + \chi_{[-\theta]}.
\]

We also consider the space cut-off functions \( \varphi, \varphi_0, \varphi_1 \in C^\infty_c(\mathbb{R}_x) \), such that \( \varphi \equiv 1 \) in a neighborhood of \( x_0 \), and \( \varphi_0|_{\text{supp } \varphi} = \varphi_1|_{\text{supp } \varphi_0} \equiv 1 \). We will choose and fix \( \varphi \) later on such that (4.33) is satisfied.

For any \( (i, j) \in \{(+, +), (-, -), (+, 0), (-, 0)\} \), we let

\[
(4.25) \quad V_{i,j} := \text{op}_\varepsilon(\chi_{i,j}^{(1)})(\varphi_0 U), \quad W_1 := \text{op}_\varepsilon(\chi_0)(1 - \varphi_0)U, \quad W_2 := (1 - \text{op}_\varepsilon(\chi_0))U
\]

so that

\[
\hat{U} = V_{+,+} + V_{-,-} + V_{+,0} + V_{-,0} + W_1 + W_2.
\]

Similarly as Lemma 3.8 in [15], we have

**Lemma 4.8.** The system in \( \mathbf{V} := (V_{+,+}, V_{-,-}, V_{+,0}, V_{-,0}, W_1, W_2) \) is

\[
\begin{aligned}
\partial_t V_{i,j} + \frac{1}{\varepsilon^{3/4}} \text{op}_\varepsilon^\theta(M_{i,j})V_{i,j} &= F_{i,j}, \\
\partial_t W_1 + \frac{i}{\varepsilon^{3/4}} \text{op}_\varepsilon(A)W_1 &= \text{op}_\varepsilon(B_1)W_1 + FW_1, \\
\partial_t W_2 + \frac{i}{\varepsilon^{3/4}} \text{op}_\varepsilon(A)W_2 &= FW_2,
\end{aligned}
\]

with symbols

\[
(4.27) \quad M_{i,j} := i\chi_{[i,j]}^{(1)}A - \varepsilon \chi_{[i,j]} \left( \begin{array}{c} 0 \\ B_{i,j}^{(1)} \\ 0 \end{array} \right) B_{i,j}^{(-1)} \left( \varepsilon^{1/4} t \right), \quad B_1 := (1 - \varphi)\chi_1 t \partial_t \mathcal{B}(0, x, \xi),
\]

where \( B_{i,j}^{(1)} \) and \( B_{i,j}^{(2)} \) are defined in (4.16), for which we introduce \( B_{+,0}^{(2)} = B_{-,0}^{(2)} = 0 \). The source term \( F_{\mathbf{V}} := (F_{i,j}, F_{W_1}, F_{W_2}) \) satisfies for \( \alpha \in \mathbb{N}^d \) with \( |\alpha| \leq 2d/2 + 2 + (q_0 + 3)/4 \):

\[
(4.28) \quad (\varepsilon \partial_x)^{\alpha}(F_{\mathbf{V}}(t)) \leq C\varepsilon^{\alpha + 1/4}(\varepsilon^{1/4} t)(\varepsilon^{1/4} t)(\varepsilon \partial_x)^{\alpha} V(t) \leq C\varepsilon^{K_a + 1/4 - \kappa}.
\]

### 4.2 Duhamel representation and an upper bound

Our goal in this Section is to write an integral representation formula for \( V_{i,j} \) by using Theorem [C,8] from Appendix [C] and then derive an upper bound for \( |V|_{L^2} \).

Assumption [C,1] of Theorem [C,8] is satisfied: support because of \( \varphi_1 \) and \( \chi_1 \); regularity and bound for \( \mathbf{M}_{0,d}(M_{i,j}) \) simply by (2.4) and (3.11).

For any \( (i, j) \in \{(+, +), (-, -), (+, 0), (-, 0)\} \), the symbolic flow \( S_{i,j}^{(i,j)} \) of \( M_{i,j} \) is defined as the solution to the initial-value problem

\[
(4.29) \quad \partial_t S_{i,j}^{(i,j)}(\tau; t) + \varepsilon^{-3/4} M_{i,j}(\tau; t) S_{i,j}^{(i,j)}(\tau; t) = 0, \quad S_{i,j}^{(i,j)}(\tau; \tau) = \text{Id}.
\]

The following proposition gives a pointwise bound for the symbolic flow. This ensures that Assumption [C,2] is satisfied. The proof is postponed to Appendix [B].
Proposition 4.9. For all $0 \leq \tau \leq T_1 |\ln \varepsilon|^{1/2}$ with any $T_1 > 0$ independent of $\varepsilon$, for all $(x, \xi)$, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d + 1 + (q_0 + 3)/4$, there holds for any $(i,j) \in \{(+, +), (-, -), (+, 0), (-, 0)\}$:

$$|\partial^\gamma_x S^i_j(\tau; t)| \leq C |\ln \varepsilon|^{\alpha/2} \exp(C(1 + |\alpha|)|\ln \varepsilon|^{1/2}) \exp((t^2 - \tau^2)\gamma^+/2),$$

where

$$\gamma^+ := \sup_{\mathcal{R}_h^{\pm}} \frac{\gamma_1(\xi)^{1/2}}{\sup_{\mathcal{R}_h^{\pm}} \gamma_2(\xi)^{1/2}}.$$

Recall that $\gamma_1$ and $\gamma_2$ are given in (4.19). Together with (4.24), we have

$$\gamma^+ \to \Gamma_1 \text{ as } h \to 0.$$

For equation (4.26), as long as

$$F_{V_{i,j}} \in L^\infty([0, t_0], H^s), \text{ for some } s > d/2,$$

Theorem C.8 implies the representation for all $0 \leq t \leq t_0$:

$$V_{i,j}(t) = |\partial_t g(0, x_0)| \sup_{\mathcal{R}_h^{\pm}} \gamma_1(\xi)^{1/2} = |\partial_t g(0, x_0)| \sup_{\mathcal{R}_h^{\pm}} \gamma_2(\xi)^{1/2}.$$

(4.30)

where $S^{i,j} := \sum_{0 \leq q \leq q_0} S^i_j$, with the leading term $S^i_j$ solution of (4.29), and the correctors $S^i_j$, $q \geq 1$ defined as in (C.3). The order $q_0$ of the expansion is a function of $\gamma^+$ and $T_1$, as seen in (C.15). The source term $F_{V_{i,j}}$ can be expressed in terms of $F_{V_{i,j}}$ and $V_{i,j}(0)$ as in (C.12). The bound (C.13) implies

$$|\tilde{F}_{V_{i,j}}(t)|_{L^2} \lesssim |F_{V_{i,j}}(t)|_{L^2} + \varepsilon \|V_{i,j}(0)\|_{L^2},$$

where $\zeta$ is defined in (C.15) and is strictly positive. The notation $\lesssim$ is introduced in (C.8) and satisfies the property stated in Remark C.7. Then for $V := (V_{+, +}, V_{-, -}, V_{+}, V_{-})$, by Lemma 4.8 Proposition 4.9 and Lemma C.5 we have

$$|V(t)|_{L^2} \lesssim e^{t^2 \gamma^+/2} |V(0)|_{L^2} + \int_0^t e^{(t^2 - t')^2 \gamma^+/2} \left( e^{t^2 - t' \gamma^+/2} |\tilde{V}(t')|_{L^2} + \varepsilon \frac{K_0 + 1/4}{t'}, \varepsilon \frac{K_0 + 1/4}{t'}} \right) dt'.$$

(4.33)

By symmetry of $\mathcal{A}$, we implement $L^2$ estimate in (4.26) and (4.27). This yields for $W := (W_1, W_2)$,

$$\partial_t (|W(t)|_{L^2}^2) \leq |\partial_t g(0)| \sup_{|\alpha| \leq d} \left( \partial^\gamma_x ((1 - \varphi)\partial_t g(0, \cdot)) \right)_{L^\infty}.$$
By our choice for the Sobolev regularity $s > d/2 + K_a + d + 2 + (q_0 + 3)/4$, we have $\partial_t g(0, \cdot) \in H^{s-1} \subset H^{3/2+d+1+(q_0+3)/4}$, then its spatial derivatives up to order $d + 1$ tend to zero at infinity by Sobolev embedding. We now choose and fix $\varphi$ such that $\varphi = 1$ on $B(x_0, r_0)$ with $r_0 > 0$ sufficient large such that

$$\sup_{|x| \leq d} |\partial_x^\alpha ((1 - \varphi) \partial_t g(0, \cdot))|_{L^\infty} \leq \frac{\Gamma_1}{C}.$$  

Then

$$|W(t)|_{L^2}^2 \lesssim |W(0)|_{L^2}^2 + \int_0^t \Gamma_1 t'' |W(t'')|_{L^2}^2 + \left(\varepsilon^{\kappa - 1/4} |(\hat{u}, \hat{v})(\varepsilon^{1/4} t''|_{L^\infty} \right) \right) |V(t'')|_{L^2} + \varepsilon^{K_a + 1/4 - \kappa}) |W(t'')|_{L^2} dt''.$$  

By (4.33) and (4.36) and the fact $\Gamma_1 \leq \gamma^+$, Gronwall's inequality implies

$$|V(t)|_{L^2} \lesssim \exp \left( t^2 \gamma^+ / 2 + C \varepsilon^{\kappa - 1/4} \exp (C| \ln \varepsilon |^{1/2}) | \ln \varepsilon |^C |(\hat{u}, \hat{v})(\varepsilon^{1/4} t)|_{L^\infty} \right) \times (|V(0)|_{L^2} + \varepsilon^{K_a + 1/4 - \kappa}).$$  

### 4.3 Existence in logarithmical time and upper bound

In this section, we will prove existence and uniqueness of solution $V$ in time of order $O(|\ln \varepsilon|^{1/2})$ to (4.26) issued from $V(0)$ such that $\|V(0)\|_{\varepsilon, s} \leq C \varepsilon^{K - \kappa}$, where the semi-classical Sobolev norm is defined by

$$\|u\|_{\varepsilon, s}^2 := \int (1 + \varepsilon |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$  

We already have the $L^2$ estimate for $V$ in (4.37). Define the semi-classical Fourier multiplier $\Lambda^s := \text{op}_\varepsilon \left((1+|\xi|^2)^s/2\right)$, $s > d/2$. Denoting $(V_{i,j,s}, W_{1,s}, W_{2,s}) = \Lambda^s (V_{i,j}, W_1, W_2)$, then

$$\begin{cases}
\partial_t V_{i,j,s} + \frac{1}{\varepsilon^{3/4}} \text{op}_\varepsilon (M_{i,j}) V_{i,j,s} = -\frac{1}{\varepsilon^{3/4}} \Lambda^s \text{op}_\varepsilon (M_{i,j}) V_{i,j} + \Lambda^s W_{i,j}, \\
\partial_t W_{1,s} + \frac{i}{\varepsilon^{3/4}} \text{op}_\varepsilon (A) W_{1,s} = \text{op}_\varepsilon (B_1) W_{1,s} + \Lambda^s \text{op}_\varepsilon (B_1) W_1 + \Lambda^s W_{1,s}, \\
\partial_t W_{2,s} + \frac{i}{\varepsilon^{3/4}} \text{op}_\varepsilon (A) W_{2,s} = \Lambda^s W_{2,s}. 
\end{cases}$$

By Proposition A.24 and Proposition A.25 we deduce the commutator estimates:

$$\|\Lambda^s \text{op}_\varepsilon (M_{i,j}) V_{i,j}(t)\|_{L^2} \leq C \varepsilon \|V_{i,j}(t)\|_{\varepsilon, s}, \quad \|\Lambda^s \text{op}_\varepsilon (B_1) W_{1}(t)\|_{L^2} \leq C \varepsilon \|W_{1}(t)\|_{\varepsilon, s}.$$  

By following the proof of $L^2$ estimate (4.37), as long as (4.31) is true, we have

$$\|V(t)\|_{\varepsilon, s} \lesssim \exp \left( t^2 \gamma^+ / 2 + C \varepsilon^{\kappa - 1/4} \exp (C| \ln \varepsilon |^{1/2}) | \ln \varepsilon |^C |(\hat{u}, \hat{v})(\varepsilon^{1/4} t)|_{L^\infty} \right) \times (\|V(0)\|_{\varepsilon, s} + \varepsilon^{K_a + 1/4 - \kappa}).$$  

Then we deduce the following existence and uniqueness result:
Proposition 4.10. Let $V(0) \in H^s$ satisfying $\|V(0)\|_{\epsilon,s} \leq C\epsilon^{K-\kappa}$ with $s > d/2$, $K_a + 1/4 > K > d/2 + 1/4$ and $1/4 < \kappa \leq K$. Then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, for any $\epsilon$-independent $T < T_0^\gamma$ there exists a unique solution $V \in C^0([0, T|\ln \epsilon|^{1/2}], H^s)$ to (4.26) issued form $V(0)$, and there holds

\begin{equation}
\|V(t)\|_{\epsilon,s} \lesssim \epsilon^{K-\kappa} \epsilon^{t\gamma^+}/2, \quad \sup_{0 \leq t \leq T} \|V(t)\|_{\epsilon,s} \leq \epsilon^{-\kappa+d/2+1/4+\epsilon},
\end{equation}

where

\begin{equation}
\epsilon \colon = \frac{2(K-d/2-1/4)/\Gamma_1 - T^2}{5} > 0.
\end{equation}

Proof. We introduce

$T := \sup \{ t > 0, \text{ for all } 0 \leq t' \leq t, \text{ there holds } \|V(t')\|_{\epsilon,s} \leq \epsilon^{-\kappa+d/2+1/4+\epsilon} \}$.

By local-in-time existence and continuity, the sup is well defined. By Sobolev embedding in semi-classical norms, there holds

\begin{equation}
\sup_{0 \leq t \leq T} \|(\dot{u}, \dot{v})(\epsilon^{1/2} t)\|_{L^\infty} \leq C\epsilon^{-d/2} \sup_{0 \leq t \leq T} \|(\dot{u}, \dot{v})(\epsilon^{1/2} t)\|_{\epsilon,s}
\end{equation}

\begin{equation}
\leq C\epsilon^{-d/2} \sup_{0 \leq t \leq T} \|V(t)\|_{\epsilon,s} \leq C\epsilon^{-\kappa+1/4+\epsilon}.
\end{equation}

For $0 \leq t \leq T$, equation (4.31) holds true by estimate (4.28). Then upper bound (4.38) holds true for such time. Moreover, with (4.41), there holds for $0 \leq t \leq T$:

$$\|V(t)\|_{\epsilon,s} \lesssim \exp \left( t^{2\gamma^+/2} + C\epsilon^{4} \exp \left( C|\ln \epsilon|^{1/2} \right) \ln \epsilon \right) \times (\epsilon^{K-\kappa} + \epsilon^{K_a+1/4-\kappa}) .$$

Since $\epsilon > 0$, we have for $\epsilon$ small:

\begin{equation}
\|V(t)\|_{\epsilon,s} \lesssim \epsilon^{K-\kappa} \epsilon^{t\gamma^+/2}.
\end{equation}

Recall $\gamma^+ \to \Gamma_1$ as $h \to 0$. Then for $h$ small, we have $T^2 \leq 2(K-d/2-1/4)/\gamma^+ - 4\epsilon$. Then by (4.42) we obtain for $0 \leq t \leq T|\ln \epsilon|^{1/2}$, provided $T|\ln \epsilon|^{1/2} \leq T$:

$$\|V(t)\|_{\epsilon,s} \lesssim \epsilon^{K-\kappa} \epsilon^{-K+d/2+1/4+2\epsilon} = \epsilon^{-\kappa+d/2+1/4+2\epsilon} .$$

By Remark C.7, for $\epsilon$ small, there holds

$$\|V(t)\|_{\epsilon,s} \leq \epsilon^{-\kappa+d/2+1/4+3\epsilon/2} = \epsilon^{\epsilon/2 - \kappa+d/2+1/4+\epsilon} .$$

Finally, the classical continuation argument implies the existence time $T \geq T|\ln \epsilon|^{1/2}$.

\[\square\]
4.4 Lower bound

For the initial datum (2.10) and the definition of perturbation 4.11, we have

\[ \dot{u}(0, x) = \varepsilon^{K-\kappa} \phi_1^\varepsilon(x), \quad \dot{\varepsilon}(0, x) = \varepsilon^{K-\kappa} \phi_2^\varepsilon(x). \]

We choose here \( \phi_1^\varepsilon \) and \( \phi_2^\varepsilon \) are of the forms

\[ \phi_1^\varepsilon(x) := e^{ix(\xi_0+3k)/\varepsilon} \Psi(x)(0, 0), \quad \phi_2^\varepsilon(x) := 0 \]

where \( \xi_0 \) is determined by (4.22). \( \Psi \) is a spatial cut-off function with small support around \( x_0 \), such that \( \Psi(x_0) = 1 \) and \( \varphi \Psi \equiv \Psi \) where \( \varphi \) is the spatial cut-off function introduce in Section 4.1.4 and

\[ \varepsilon_0 \in \text{Image} P_+(\xi_0 + 3k)F(\tilde{e}_3)Q_+(\xi_0)G(\tilde{e}_3)P_+(\xi_0 + 3k). \]

Since \( \text{rank } P_+(\xi_0 + 3k) = \text{rank } P_+(\xi_0 + 3k)F(\tilde{e}_3)Q_+(\xi_0)G(\tilde{e}_3)P_+(\xi_0 + 3k) = 1 \), (4.45) is equivalent to \( \varepsilon_0 \in \text{Image } P_+(\xi_0 + 3k) \). More precisely, we choose \( \varepsilon_0 \) such that \( P_+(\xi_0 + 3k)\varepsilon_0 = \varepsilon_0 \).

Lemma 4.11. With the choices in (4.43), (4.44) and (4.45), the initial value of \( V_{+,+} \) satisfies

\[ V_{+,+}(0, x) = \varepsilon^{K-\kappa} \left( e^{ix\xi_0/\varepsilon} \Psi(x)\varepsilon_0, 0, \ldots, 0 \right) + \varepsilon^{K-\kappa+1/2} \tilde{V}_\varepsilon(x) \]

for some \( \tilde{V}_\varepsilon(x) \) such that \( \sup_{\varepsilon>0} |\tilde{V}_\varepsilon|_{L^2} < \infty \).

Proof. By (4.12), (4.43) and (4.44), the datum for \( U = (u_+, u_-, u_0, v_+, v_-, v_0) \) is

\[
\begin{cases}
  u_+(0, x) = \varepsilon^{K-\kappa} e^{ix\xi_0/\varepsilon} P_+(\varepsilon D_x + \xi_0 + 3k)(\Psi(x)\varepsilon_0), \\
  u_-(0, x) = \varepsilon^{K-\kappa} e^{ix\xi_0+6kx/\varepsilon} P_-(\varepsilon D_x + \xi_0 + 3k)(\Psi(x)\varepsilon_0), \\
  u_0(0, x) = \varepsilon^{K-\kappa} e^{ix\xi_0+3kx/\varepsilon} P_0(\varepsilon D_x + \xi_0 + 3k)(\Psi(x)\varepsilon_0), \\
  (v_+, v_-, v_0)(0, x) = 0.
\end{cases}
\]

We compute

\[ P_+(\varepsilon D_x + \xi_0 + 3k) = P_+(\xi_0 + 3k) + \varepsilon \int_0^1 (\partial_\xi P_+) (s \varepsilon D_x + \xi_0 + 3k) ds. \]

The choice of \( \varepsilon_0 \) such that \( P_+(\xi_0 + 3k)\varepsilon_0 = \varepsilon_0 \) gives us

\[ u_+(0, x) = \varepsilon^{K-\kappa} e^{ix\xi_0/\varepsilon} (\Psi(x)\varepsilon_0) + \varepsilon^{K-\kappa+1} \tilde{u}_+(x), \]

where

\[ \tilde{u}_+(x) = e^{ix\xi_0/\varepsilon} \int_0^1 (\partial_\xi P_+) (s \varepsilon D_x + \xi_0 + 3k)(\Psi(x)\varepsilon_0) ds \in L^2 \text{ bounded uniformly in } \varepsilon. \]
By similar calculation as above and the orthogonality of the eigenprojectors, we have:

\[(u_-, u_0)(0, x) = \varepsilon^{K-\kappa+1} (\tilde{u}_-(x), \tilde{u}_0(x))\]

with \((\tilde{u}_-(x), \tilde{u}_0(x))\) uniformly bounded with respect to \(\varepsilon\) in \(L^2\).

By (4.18), we can write

\[
\tilde{U}(0, x) = U(0, x) + \varepsilon^{1/2}\tilde{U}(x)
\]

with \(\tilde{U}(x)\) uniformly bounded in \(L^2\) with respect to \(\varepsilon\). Then by (4.25), the initial value of \(V_{+,+}\) appears as

\[
V_{+,+}(0, x) = \varepsilon^{K-\kappa} \left( \text{op}_\varepsilon(\chi^{(0)}_{[+,+]}) (\varphi e^{ix\xi_0/\varepsilon}\tilde{\psi}_0, 0, \ldots, 0) \right) + \varepsilon^{K-\kappa+1/2}\tilde{V}(x)
\]

with \(\tilde{V}(x)\) uniformly bounded in \(L^2\) with respect to \(\varepsilon\). By our choice such that \(\Psi \varphi \equiv \Psi\), and the fact \(\chi^{(0)}_{[+,+]}(\xi_0) = 1\) we have

\[
\text{op}_\varepsilon(\chi^{(0)}_{[+,+]}) (e^{ix\xi_0/\varepsilon}\tilde{\psi}_0) = e^{ix\xi_0/\varepsilon}\chi^{(0)}_{[+,+]}(\xi_0) \tilde{\psi} = e^{ix\xi_0/\varepsilon}\tilde{\psi} + \varepsilon\tilde{\Psi}
\]

with \(\tilde{\Psi}\) uniformly bounded in \(L^2\) with respect to \(\varepsilon\). This completes the proof. \(\Box\)

**Lemma 4.12.** For the datum \(V_{+,+}(0, \cdot)\) described in Lemma 4.11 there holds for some small \(\rho > 0\), for some small \(c > 0\) and some large \(C > 0\):

\[
|\text{op}_\varepsilon(S^{+,+})(0; t)V_{+,+}(0, \cdot)|_{L^2(B(x_0, \rho))}
\geq \varepsilon^{K-\kappa} \left( c^2 \rho^d \exp \left( t^2 \gamma^- / 2 \right) - C \varepsilon^{1/4} |\ln \varepsilon| C \exp \left( C |\ln \varepsilon|^{1/2} \right) \exp \left( t^2 \gamma^+ / 2 \right) \right),
\]

where

\[
\gamma^- := \min_{|x-x_0| \leq \rho} |\partial_t g(0, x)| (\gamma_1(\xi_0))^{1/2}.
\]

By (4.30) and (4.24), there holds

\[
\lim_{\rho \to 0^+} \gamma^- = \lim_{h \to 0^+} \gamma^+ = \Gamma_1.
\]

**Proof.** To simplify the notation, in this proof, we omit the index \((+, +)\) by denoting \(S = S^{+,+} \) and \(V = V_{+,+}\). We let \(V_0 := \Psi(x)(\tilde{\psi}_0, 0, 0, 0, 0)\). By Remark A.1

\[
\text{op}_\varepsilon(S(0; t))(e^{ix\xi_0/\varepsilon} V_0) = e^{ix\xi_0/\varepsilon} \int e^{ix \xi} S(0; t, x, \xi_0 + \varepsilon \xi) \tilde{V}_0(\xi) d\xi,
\]

where \(S(0; t, x, \xi) := \left( \mathcal{F}^{-1} \psi \ast \tilde{S}(0; t) \right) \left( \frac{x}{\varepsilon}, \xi \right)\), with \(\tilde{S}(x, \xi) := S(\varepsilon x, \xi)\). Direct calculation gives

\[
e^{-ix\xi_0/\varepsilon} \text{op}_\varepsilon(S(0; t))(e^{ix\xi_0/\varepsilon} V_0(0)) = S(0; t, x, \xi_0) V_0(x) + \varepsilon^{1/4} \tilde{V}_0(t, x),
\]
where

\[ |\tilde{V}_0(t, \cdot)|_{L^2} \lesssim e^{2\gamma^+ / 2}. \]

We now show a lower bound for the leading term \( S(0; t, x, \xi_0)V_0(x) \). By (B.11), (B.8), (B.9), the construction of solution operator \( \text{op}_\xi^0(S) \) in Appendix C (specifically, by definition of \( S \) just above Lemma C.6) and the upper bounds for the correctors in Lemma C.5, and the fact that the frequency cut-off functions \( \chi_{[+,+]}^{(0)} \), \( \chi_{[+,+]}^{(1)} \) and \( \chi_{[+,+]}^{(2)} \) are of value one near \( \xi_0 \), we have

\[ S(0, t, x, \xi_0) = S_r(0, t, x, \xi_0) + e^{1/4}|\tilde{S}_r(0, t, x, \xi_0)|, \quad |\tilde{S}_r(0, t, x, \xi_0)| \lesssim e^{2\gamma^+ / 2}, \]

where the block diagonal matrix

\[ S_r(0; t, x, \xi_0) = \text{diag}\left( \tilde{S}_0(0; t, x, \xi_0), e^{-\varepsilon^{-3/4}it\lambda_2(x)}\text{Id}_3, \text{Id}_3 \right) \]

with \( \tilde{S}_0(0; t, x, \xi_0) \) the solution to (B.9) at point \( (0; t, x, \xi_0) \). At \( \xi = \xi_0 \), there holds \( \lambda_1 = \mu \), then the solution \( \tilde{S}_0 \) for (B.9) has the following explicit expression:

\[ \tilde{S}_0(0; t, x, \xi_0) = \exp \left( -e^{-3/4}it\lambda_1 t \right) \exp \left( t^2\varphi_1(x)|\partial_t g(0, x)|\bar{M}_0/2 \right), \]

where

\[ \bar{M}_0(x, \xi_0) := \begin{pmatrix} 0 & \frac{\partial_t g(0, x)}{|\partial_t g(0, x)|} & \frac{\partial_t g(0, x)}{|\partial_t g(0, x)|} \\ \frac{\partial_t g(0, x)}{|\partial_t g(0, x)|} & 0 & 0 \\ \frac{\partial_t g(0, x)}{|\partial_t g(0, x)|} & 0 & 0 \end{pmatrix}, \quad \begin{cases} \tilde{b}_1(\xi) := P_+(\xi + 3k)F(\bar{c}_3)Q_+(\xi), \\ \tilde{b}_2(\xi) := 2Q_+(\xi)G(\bar{c}_3)P_+(\xi + 3k), \end{cases} \]

where we forcibly define \( \frac{\partial_t g(0, x)}{|\partial_t g(0, x)|} \) if \( |\partial_t g(0, x)| = 0 \). We then rewrite

\[ \bar{M}_0 = G(x) \begin{pmatrix} 0 & \tilde{b}_1(\xi_0) & 0 \\ \tilde{b}_2(\xi_0) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} G(x)^{-1}, \quad G(x) := \begin{pmatrix} \frac{\partial_t g(0, x)}{|\partial_t g(0, x)|} & \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}, \]

where \( G(x) \) is unitary for all \( x \). By Lemma B.7 since \( \text{rank} \tilde{b}_1 \leq 1, \text{rank} \tilde{b}_2 \leq 1 \), the eigenvalues of \( \bar{M}_0 \) are \( 0, \pm (\text{tr} \tilde{b}_1 \tilde{b}_2(\xi_0))^{1/2} \). By (4.19), (4.20) and (4.21), we have \( \text{tr} \tilde{b}_1 \tilde{b}_2(\xi_0) = \gamma_1(\xi_0) > 0 \) implying that the eigenvalues of \( \bar{M}_0 \) are strictly separated. Moreover, again by \( \text{rank} \tilde{b}_1 \leq 1, \text{rank} \tilde{b}_2 \leq 1 \), we can conclude that the geometric multiplicity and the algebra multiplicity for \( \bar{M}_0 \) are always equal. Then we have the smooth spectral decomposition

\[ \bar{M}(x, \xi_0) = (\text{tr} \tilde{b}_1 \tilde{b}_2(\xi_0))^{1/2}(\Pi_+ - \Pi_-)(x, \xi_0), \]

and there holds for all \( x \):

\[ |\Pi_+(x, \xi_0)| + |\Pi_-(x, \xi_0)| \leq C. \]
The eigenspace associated with the positive eigenvalue \((\text{tr } \bar{b}_1 \bar{b}_2)^{1/2}\) is of dimension one, described by vectors \((r_{12}, (\text{tr } \bar{b}_1 \bar{b}_2(\xi_0))^{-1/2} \bar{b}_2(\xi_0)r_{12})\) with \(r_{12} \in \text{Image } \bar{b}_1 \bar{b}_2(\xi_0) = \text{Image } P_+(\xi_0 + 3k)\). Then by our choice (4.45) for \(\tilde{e}_0 \in \text{Image } P_+(\xi_0 + 3k)\), there holds \(\Pi_+(x, \xi_0)(\tilde{e}_0, 0) \neq 0\) for all \(x\). This gives for \(x\), \(|x - x_0| \leq \rho\) with \(\rho\) small such that \(\Psi(x) = 1:\)

\[
\left| \tilde{S}_0(0; t, x, \xi_0)(\Psi(x)\tilde{e}_0) \right| \geq \exp \left( t^2 \gamma^- / 2 \right) |\Pi_+ \tilde{e}_0| - \exp \left( - t^2 \gamma^- / 2 \right) |\Pi_- \tilde{e}_0| \geq c \exp \left( t^2 \gamma^- / 2 \right) - C,
\]

for some \(c > 0\). Then by (4.49) and (4.51), we obtain

\[
|S_r(0; t, x, \xi_0)V_0(x)|_{L^2(B(x_0, \rho))} \geq c \rho^d \exp \left( t^2 \gamma^- / 2 \right) - C.
\]

Then (4.47) and (4.48) imply

\[
|\text{op}_e^x(S(0; t)(e^{ix\xi_0/\epsilon}V_0)|_{L^2(B(x_0, \rho))} \geq c \rho^d \exp \left( t^2 \gamma^- / 2 \right) - C \epsilon^{1/2} |\ln \epsilon| \exp \left( C |\ln \epsilon|^{1/2} \right) \exp \left( t^2 \gamma^+ / 2 \right).
\]

By Lemma 4.12 where we show \(V(0, \cdot) = \epsilon^{K - \kappa} e^{ix\xi_0/\epsilon}V_0 + \epsilon^{K - \kappa + 1/2} \tilde{V}_\epsilon(x)\), we can complete the proof. \(\square\)

### 4.5 Proof of the deviation estimate (2.12)

Given \(d/2 + 1/4 < \kappa_0 < K < K_a + 1/4\), we choose \(\kappa = d/2 + 1/4 + \iota_0\) in (4.11) where \(\iota_0 = (\kappa_0 - d/2 - 1/4)/2 > 0\). Clearly, \(\kappa_0 = d/2 + 1/4 + 2\iota_0 > \kappa\).

With the initial datum given by (4.43), (4.44) and (4.45), the assumptions in Proposition 4.10 are all satisfied. Then for \(T_0^*\) defined in (2.11), there exists a unique solution to (4.26) in time interval \([0, T]\) \(|\ln \epsilon|^{1/2}\) for any \(T < T_0^*\), and the solution satisfies the estimates in (4.39).

From the Duhamel representation (4.32) and Lemma C.5 we deduce the lower bound for \(V_{+,+}\)

\[
|V_{+,+}(t)|_{L^2} \geq \left| \text{op}_e^x(S^{+,+}(0; t))V_{+,+}(0)|_{L^2(B(x_0, \rho))} - \int_0^t |\text{op}_e^x(S^{+,+}(t'; t))\tilde{F}_{V_{+,+}}(t')|_{L^2} dt'.
\]

By (4.28), Proposition 4.10 and Lemma C.5 we find

\[
|\text{op}_e^x(S^{+,+}(t'; t))\tilde{F}_{V_{+,+}}(t')|_{L^2} \lesssim \epsilon^{K - \kappa} (\epsilon^{K - 1/4 - d/2} + \epsilon^{K_a + 1/4 - K}) \epsilon^{t^2 \gamma^+ / 2}.
\]

By Lemma 4.12 we have for all \(0 \leq t \leq T\) \(|\ln \epsilon|^{1/2}\) with \(T < T_0^*:\)

\[
|V_{+,+}(t)|_{L^2} \geq \epsilon^{K - \kappa} \epsilon^{t^2 \gamma^- / 2} \left( C \rho^d - C \epsilon^{t^*} |\ln \epsilon|^C \exp(C |\ln \epsilon|^{1/2}) \epsilon^{t^2 (\gamma^+ - \gamma^-) / 2} \right),
\]

where \(t^* := \min\{\kappa - d/2 - 1/4, K_a + 1/4 - K\} > 0\).
By (4.46), we can choose $\rho$ and $h$ small such that
\[(T_0^*)^2(\gamma^+ - \gamma^-)/2 < \epsilon^*, \quad (T_0^*)^2\Gamma_1/2 \geq (T_0^*)^2 \Gamma_1/2 - \epsilon_0/2 = K - d/2 - 1/4 - \epsilon_0/2.\]
We then choose $T$ close to $T_0^*$ such that
\[T^2\gamma^-/2 \geq (T_0^*)^2\gamma^-/2 - \epsilon_0/2 = K - d/2 - 1/4 - \epsilon_0.\]
Together with (4.52), for $\varepsilon$ small, we obtain
\[
\|V_{+,+}(T|\varepsilon|^{1/2})\|_{L^2} \geq \frac{c\rho d}{2}e^{d/2+1/4+\epsilon_0-\kappa} = \frac{c\rho d}{2}.
\]
Back to the original variable and original time scaling, we have
\[
\|(u,v) - (u^a, v^a)))(T|\varepsilon|^{1/4} |\varepsilon|^{1/2})\|_{L^2} \geq \frac{c\rho d}{2}e^{\kappa} = \frac{c\rho d}{2}e^{\kappa_0-\epsilon_0}.
\]
This gives (2.12) by multiplying $e^{-\kappa_0}$ and taking the supreme in $\varepsilon \in (0, \varepsilon_0)$.

### 4.6 Proof of the deviation estimate (2.14)

For any $1/4 < \kappa_1 < 1/2$, let $\kappa = \kappa_1$ in (4.1). We work by contradiction we suppose that (2.14) does not hold. This provides a uniform $L^\infty$ bound for any $T_1 < T_1^*$:
\[
(4.53) \quad \sup_{0 < \varepsilon < \varepsilon_0} \sup_{0 \leq t \leq T_1^{1/4} |\varepsilon|^{1/2}} |(\hat{u}, \hat{v})(t)|_{L^2 \cap L^\infty} < \infty.
\]
We use (4.54) in (4.37), to find the upper bound
\[
|V(t)|_{L^2} \lesssim \exp(t^2\gamma^+/2 + C\varepsilon^{\kappa-1/4}e^{C|\ln \varepsilon|^{1/2}}|\ln \varepsilon|^{C})(|V(0)|_{L^2} + \varepsilon^{K_a+1/4-\kappa}).
\]
Since $1/4 < \kappa < 1/2$ and $K < K_a + 1/4$, by the argument in Remark C.7, for small $\varepsilon$, the above upper bound implies
\[
|V(t)|_{L^2} \lesssim e^{t^2\gamma^+/2}(|V(0)|_{L^2} + \varepsilon^{K_a+1/4-\kappa}) \lesssim e^{K-K_a}e^{t^2\gamma^+/2}.
\]
Then from the Duhamel representation (4.32) and Lemma C.5, we deduce
\[
(4.54) \quad |V_{+,+}(t)|_{L^2} \geq e^{K-K_a}e^{t^2\gamma^+/2}(c\rho d - C\varepsilon^{\epsilon^*} |\ln \varepsilon|^C \exp(C |\ln \varepsilon|^{1/2})e^{t^2(\gamma^+ - \gamma^-)/2}),
\]
where $\epsilon^* = \min\{\kappa - 1/4, K_a + 1/4 - K\} > 0$.

We now choose $h > 0$ and $p > 0$ small enough, $T_0^*$ close enough to $T_1^*$ so that
\[
(T_0^*)^2(\gamma^+ - \gamma^-)/2 < \epsilon^*/2, \quad T_0^2\gamma^-/2 \geq (T_0^*)^2 \Gamma_1/2 - \epsilon_1 = K - 1/4 - \epsilon_1,
\]
where $\epsilon_1 := (K - 1/4)/2 > 0$. Then for $\varepsilon$ small, (4.51) implies
\[
|V_{+,+}(T_0^{1/4} |\varepsilon|^{1/2})|_{L^2} \geq \frac{c\rho d}{2}e^{-\kappa+1/4+\epsilon_1} = \frac{c\rho d}{2}e^{-\epsilon_1}.
\]
This implies
\[
\sup_{0 < \varepsilon < \varepsilon_0} \sup_{0 \leq t \leq (T_1^{1/4} |\varepsilon|^{1/2})} |V_{+,+}(t)|_{L^2} = \infty,
\]
which contradicts to (4.53) because $|V_{+,+}(t)|_{L^2} \leq C|(\hat{u}, \hat{v})(\varepsilon^{1/4} t)|_{L^2}$.  

36
A Symbols and operators

In this appendix, we recall some definitions and properties for the symbols and pseudo-differential (including para-differential) operators that we used in this paper. This is a short version of Section 6.1 in [15], where one can find the details and proofs which we omit here.

Given \( m \in \mathbb{R} \), we denote \( S^m \) the set of matrix-valued symbols with finite spatial regularity \( a \in C^s(\mathbb{R}^d_x; C^\infty(\mathbb{R}^d_\xi)) \), such that for all \( \alpha \in \mathbb{N}^d \) with \( |\alpha| \leq \bar{s} \), for all \( \beta \in \mathbb{N}^d \), there exists some \( C_{\alpha\beta} > 0 \) such that for all \( (x, \xi) \),

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} (\xi)^{m-|\beta|}, \quad (\xi) := (1 + |\xi|^2)^{1/2}.
\]

The regularity index \( \bar{s} \) is determined by the regularity of the approximate solution \((u^a, v^a)\). Motivated by Remark 2.2, we let \( \bar{s} > d/2 + d + 1 + (q_0 + 3)/4 \) in the definition of \( S^m \).

Given \( a \in S^m \), the definitions for the associated family of pseudo-differential operators and para-differential operators in both classical and semi-classical quantization are given in Section 6.1 of [15], and we do not repeat here; one can also check Bony [1] and Hörmander [7]. We recall the following remark:

**Remark A.1.** The classical symbol of \( \text{op}_\psi^a \) is

\[
(x, \xi) \mapsto (\mathcal{F}^{-1} \psi \ast \tilde{a}) \left( \frac{x}{\varepsilon}, \varepsilon \xi \right) = \int \mathcal{F}^{-1} \psi(y, \varepsilon \xi) a(x - \varepsilon y, \varepsilon \xi) \, dy.
\]

For the following proposition, the first result is deduced from Theorem 18.8.1 in Hörmander [7] and a simple proof for the second result can be found in Hwang [8].

**Proposition A.2 (Action).** Given \( a \in S^m \), we have

- There holds for all \( u \in L^2 \) the bound
  \[
  |\text{op}_\varepsilon^a u|_{L^2} + |\text{op}_\varepsilon^\psi u|_{L^2} \leq C_d \sum_{|\alpha| \leq d+1} \sup_{\xi \in \mathbb{R}^d} |\partial_x^\alpha a(\cdot, \xi)|_{L^1(\mathbb{R}^d_x)}|u|_{L^2}.
  \]

- For all \( m, s \in \mathbb{R}, k \in \mathbb{N} \), there exists \( C = C(m, s, k) > 0 \) such that for all \( u \in H^s_{\varepsilon+m} \) there holds
  \[
  \|\text{op}_\varepsilon^a u\|_s \leq CM^m_{d,d}(a)\|u\|_{\varepsilon,s+m}, \quad \|\text{op}_\varepsilon^\psi u\|_s \leq CM^m_{0,d}(a)\|u\|_{\varepsilon,s+m},
  \]
  where
  \[
  M^m_{k,k'}(a) := \sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d} \langle \xi \rangle^{-(m-|\beta|)} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)|.
  \]

We give two para-linearization estimates which are derived from Proposition 5.2.2 and Theorem 5.2.8 in [17].

37
**Proposition A.3.** For any \( r \in \mathbb{N}^s, s \geq r \), given \( a \in H^s \), there exists \( C > 0 \) such that for all \( u \in L^\infty \),
\[
\| (a - \text{op}_\varepsilon^\psi(a))u \|_{\varepsilon,s} \leq C \| (\varepsilon \partial_x)^\tau a \|_{\varepsilon,s-r} \| u \|_{L^\infty},
\]
and for all \( u \in L^2 \),
\[
\| (a - \text{op}_\varepsilon^\psi(a))u \|_{\varepsilon,s} \leq C \| (\varepsilon \partial_x)^\tau a \|_{L^\infty} \| u \|_{L^2},
\]
We give the composition estimate which is derived from Theorem 6.1.4 of [17]:

**Proposition A.4** (Composition of para-differential operators). There holds for all \( m_1, m_2, r \in \mathbb{N}^s \),
\[
\text{op}_\varepsilon^\psi(a_1) \text{op}_\varepsilon^\psi(a_2)u = \text{op}_\varepsilon^\psi(1^{m_1} a_1 a_2)u + \varepsilon^r R^\psi_\varepsilon(a_1, a_2),
\]
with the notation
\[
a_1^{m_1} a_2 = \sum_{|\alpha| < r} \varepsilon^{|\alpha|} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\varepsilon^\alpha a_1 \partial_\varepsilon^\alpha a_2,
\]
and for some \( d^r \leq 2d + r + 1 \), for all \( s \in \mathbb{R} \), some \( C = C(m_1, m_2, d, s, r) > 0 \), for all \( u \in H^{s+m_1+m_2-r} \), there holds
\[
\| R^\psi_\varepsilon(a_1, a_2)u \|_{\varepsilon,s} \leq C \left( M_{m_1}^{m^r_{d^r}}(a_1)M_{m^2_{d^r}}^{m^r_{d^r}}(a_2) + M_{m_1^r}^{m^r_{d^r}}(a_1)M_{m^2_{d^r}}^{m^r_{d^r}}(a_2) \right) \| u \|_{\varepsilon,s+m_1+m_2-r}.
\]

For the composition of a Fourier multiplier and a scalar function, we have the following proposition. The proof is rather direct.

**Proposition A.5.** Given any \( u \in H^s \), \( s > 0 \) and any semi-classical Fourier multiplier \( \sigma(\varepsilon D_x) \) the symbol of which satisfies
\[
\sigma(\xi) \in C^1(\mathbb{R}^d), \quad \| \sigma(\nabla \sigma) \|_{L^\infty} < +\infty,
\]
we have
\[
\| \sigma(\varepsilon D_x)u \|_{\varepsilon,s} \leq C \| \sigma \|_{L^\infty} \| u \|_{\varepsilon,s}.
\]

Given any scalar function \( g(x) \in H^{s+d/2+1+\eta} \) for some \( \eta > 0 \), we have the estimate:
\[
\| [\sigma(\varepsilon D_x), g(x)]u \|_{\varepsilon,s} \leq \varepsilon C \| g \|_{H^{s+d/2+1+\eta}} \| \nabla \sigma \|_{L^\infty} \| u \|_{\varepsilon,s}.
\]
The constant \( C \) is independent of \( \sigma \) and \( g \).

## B Bounds for the symbolic flow

Our goal in Appendix B is to prove Proposition 4.9. We first consider the case \((i, j) = (+, +)\) and we reproduce below the proposition.

**Proposition B.1.** For all \( 0 \leq \tau \leq t \leq T \), \( |\ln \varepsilon|^{1/2} \) and all \( \alpha \in \mathbb{N}^d \) with \( |\alpha| \leq d+1+q_0+3/4 \), the solution to (1.29) with \((i, j) = (+, +)\) satisfies
\[
|\partial_\varepsilon^{\alpha} \varphi^+(\tau; t)| \leq C \| \ln \varepsilon \|^{1/2} \exp(C(1 + |\alpha|)|\ln \varepsilon|^{1/2}) \exp((t^2 - \tau^2)\gamma^+/2).
\]
B.1 Preparation

By (4.16) and (4.27), the matrix $M_{++,}$ is

\[
M_{++,} = \chi^{(1)}_{[+,+]} \begin{pmatrix}
\lambda_1 & 0 & 0 & -\sqrt{\varepsilon} b_{12} & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 & 0 \\
-\sqrt{\varepsilon} b_{21} & 0 & 0 & i\mu & 0 & 0 \\
0 & 0 & 0 & 0 & -i\mu & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \in \mathbb{C}^{18 \times 18}
\]

denoting

\[
\lambda_1(\cdot) = \lambda(\cdot + 3k) - 3\omega, \quad \lambda_2(\cdot) = -\lambda(\cdot - 3k) + 3\omega
\]

and

\[
b_{12}(t, x, \xi) := \chi_{[+,+]}(\xi)\varphi_1(x)P_+(\xi + 3k)F(u_{0,3}(\varepsilon^{1/4}t, x))Q_+(\xi) \in \mathbb{C}^{3 \times 3},
\]

\[
b_{21}(t, x, \xi) := 2\chi_{[+,+]}(\xi)\varphi_1(x)Q_+(\xi)G(u_{0,3}(\varepsilon^{1/4}t, x))P_+(\xi + 3k) \in \mathbb{C}^{3 \times 3}.
\]

Up to a change of order for the volumes of $S^{+,+}_0$, it is equivalent to rewrite

\[
M_{++,} = \chi^{(1)}_{[+,+]} \begin{pmatrix}
\lambda_1 & -\sqrt{\varepsilon} b_{12} & 0 & 0 & 0 & 0 \\
-\sqrt{\varepsilon} b_{21} & i\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \in \mathbb{C}^{18 \times 18}.
\]

By reality of $\lambda$ and $\mu$, and the fact that $b_{12}$ and $b_{21}$ vanish identically outside $\text{supp} \varphi_1 \times \text{supp} \chi_{[+,+]}$, and that $\chi_{[+,+]}^{(1)}(\xi) = 1$ for any $\xi \in \text{supp} \chi_{[+,+]}$; it suffices to prove the following estimate for $(x, \xi) \in \text{supp} \varphi_1 \times \text{supp} \chi_{[+,+]}$:

\[
|\partial^2_\tau \tilde{S}(\tau; t)| \leq C|\ln \varepsilon|^{\alpha/2} \exp \left( C(1 + |\alpha|) |\ln \varepsilon|^{1/2} \right) \exp \left( (t^2 - \tau^2)^{\gamma^+} / 2 \right),
\]

where $\tilde{S}$ solves

\[
\partial_t \tilde{S} + \varepsilon^{-3/4} \tilde{M} \tilde{S} = 0, \quad \tilde{S}(\tau; \tau) = \text{Id}
\]

with $\tilde{M}$ defined as

\[
\tilde{M} := \begin{pmatrix}
\lambda_1 & -\sqrt{\varepsilon} b_{12} \\
-\sqrt{\varepsilon} b_{21} & i\mu
\end{pmatrix} \in \mathbb{C}^{6 \times 6}.
\]

By Taylor expansion with integral form remainder, there holds

\[
u_{0,3}(\varepsilon^{1/4}t) = u_{0,3}(0) + \varepsilon^{1/4}t \partial_t u_{0,3}(0) + \varepsilon^{1/2}t^2 \int_0^1 (\partial^2_t u_{0,3})(s\varepsilon^{1/4}t, x)(1 - s)ds.
\]
Recall \( u_{0,3}(t, x) = g(t, x)\bar{e}_3 \) and \( u_{0,3}(0, \cdot) \equiv 0 \), we have

\[
\tilde{M} = M_0 + \varepsilon M_1, \quad M_0 := \begin{pmatrix} i\lambda_1 & -\varepsilon^{3/4} t \tilde{b}_{12} \\ -\varepsilon^{3/4} t \tilde{b}_{21} & i\mu \end{pmatrix},
\]

\[
|\partial^\beta_t \partial^\alpha_x M_1(t, \cdot)|_{L^\infty_x} \leq C t^2 |\partial^2_x \partial^\alpha_x g|_{L^\infty_{t,x}}
\]

with

\[
\tilde{b}_{12}(x, \xi) := \chi_{[+,+]}(\xi) \varphi_1(x) \partial_t g(0, x) P_+(\xi + 3k) F(\bar{e}_3) Q_+(\xi),
\]

\[
\tilde{b}_{21}(x, \xi) := 2 \chi_{[+,+]}(\xi) \varphi_1(x) \partial_t \bar{g}(0, x) Q_+(\xi) G(\bar{e}_3) P_+(\xi + 3k).
\]

Then we can rewrite (B.5) in the form

\[
\partial_t \tilde{S} + \varepsilon^{3/4} M_0 \tilde{S} = -\varepsilon^{1/4} M_1 \tilde{S}, \quad \tilde{S}(\tau; \tau) = \text{Id}.
\]

To show the estimate (B.4), we start from considering the the following simpler equation, in which the small source term in (B.8) is not included:

\[
\partial_t \tilde{S}_0 + \varepsilon^{3/4} M_0 \tilde{S}_0 = 0, \quad \tilde{S}_0(\tau; \tau) = \text{Id}.
\]

**Proposition B.2.** For all \( 0 \leq \tau \leq t \) and all \((x, \xi) \in \text{supp} \varphi_1 \times \text{supp} \chi_{[+,+]}\), the solution \( \tilde{S}_0 \) to (B.9) satisfies

\[
|\tilde{S}_0(\tau; t)| \leq C \exp ((t^2 - \tau^2)\gamma^+/2) \exp (C t).
\]

The proof is given in the next section. We immediately have a corollary:

**Corollary B.3.** For all \( 0 \leq \tau \leq t \leq T_1 |\ln \varepsilon|^{1/2} \) and all \((x, \xi) \in \text{supp} \varphi_1 \times \text{supp} \chi_{[+,+]}\), the solution \( \tilde{S}_0 \) to (B.9) satisfies

\[
|\tilde{S}_0(\tau; t)| \leq C \exp(C |\ln \varepsilon|^{1/2}) \exp ((t^2 - \tau^2)\gamma^+/2).
\]

**B.2 Proof of Proposition B.2**

We prove Proposition B.2 step by step in the following subsections. We recall that it suffices to consider \((x, \xi) \in \text{supp} \varphi_1 \times \text{supp} \chi_{[+,+]}\), and we will not repeat this restriction in the following statements in Section B.2.

**B.2.1 A rough estimate**

**Lemma B.4.** There holds for all \( 0 \leq \tau \leq t < \infty \):

\[
|\tilde{S}_0(\tau; t)| \leq \exp((t^2 - \tau^2)b^+/2), \quad b^+ := \sup_{x, \xi} \left( \frac{|\tilde{b}_{12}| + |\tilde{b}_{21}|}{2} \right).
\]
Remark B.5. In general, $b^+$ is strictly larger than $\gamma^+$. Then this estimate is worse than the estimate in Proposition B.2, so we call it a rough estimate.

To prove Lemma B.4, we introduce:

**Lemma B.6.** Suppose $M(t)$ a continuous matrix in $\mathbb{C}^{n \times n}$. The solution $y(\tau; t)$ to

$$\partial_t y + M(t)y = 0, \quad y(\tau; \tau) = 1$$

satisfies

$$|y(\tau; t)| \leq \exp \left( \int_{\tau}^{t} \frac{|M(t') + M(t')^*|}{2} dt' \right).$$

**Proof of Lemma B.6.** We denote $(\cdot, \cdot)$ the inner product in $\mathbb{C}^{n}$. Then

$$\partial_t (|y|^2) = \partial_t (y, y) = (\partial_t y, y) + (y, \partial_t y) = -(M + M^*)y, y \leq |M + M^*| \cdot |y|^2.$$

Gronwall’s inequality implies

$$|y(\tau; t)|^2 \leq \exp \left( \int_{\tau}^{t} |M(t') + M(t')^*| dt' \right).$$

This completes the proof. \(\square\)

Lemma B.4 can now be proved immediately: by (B.9) and Lemma B.6

$$|\tilde{S}_0(\tau; t)| \leq \exp \left( \int_{\tau}^{t} \frac{|M_0 + M_0^*|}{2 \varepsilon^{3/4}} dt' \right) \leq \exp \left( \int_{\tau}^{t} t'b^+ dt' \right) = \exp((t^2 - \tau^2)b^+/2).$$

**B.2.2 Spectral of $M_0$**

The eigenvalues of $M_0$ play an important role in estimating $\tilde{S}_0$. We recall the following lemma in linear algebra:

**Lemma B.7.** Suppose $A, B, C, D$ are $n \times n$ matrices, if $A$ is invertible and $AC = CA$, we have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Then the eigen-polynomial of $M_0$ is

$$\det \begin{pmatrix} x - i\lambda_1 & \varepsilon^{3/4}t\tilde{b}_{12} \\ \varepsilon^{3/4}t\tilde{b}_{21} & x - i\mu \end{pmatrix} = \det ((x - i\lambda_1)(x - i\mu) - \varepsilon^{3/2}t^2\tilde{b}_{21}\tilde{b}_{12}).$$

Then $x \in \text{sp}(M_0)$ if and only if $(x - i\lambda_1)(x - i\mu) \in \text{sp}(\varepsilon^{3/2}t^2\tilde{b}_{21}\tilde{b}_{12})$. By the fact $\text{rank}(\tilde{b}_{12}\tilde{b}_{21}) \leq 1$, the only possible nonzero eigenvalue for $\tilde{b}_{21}\tilde{b}_{12}$ is $\text{tr}(\tilde{b}_{12}\tilde{b}_{21})$. Then the eigenvalues of $M_0$ are

$$\begin{align*}
\lambda_1, & \quad i\mu, \\
\nu_\pm & := \frac{i}{2}(\lambda_1 + \mu) \pm \frac{1}{2}(4\varepsilon^{3/2}t^2\text{tr}(\tilde{b}_{12}\tilde{b}_{21}) - (\lambda_1 - \mu)^2)^{1/2}.
\end{align*}$$

By (1.19) and (1.21), there holds always $\text{tr}(\tilde{b}_{12}\tilde{b}_{21}) \geq 0$. We consider the following subcases related to a small number $0 < c_0 < 1$ to be fixed later on.
B.2.3 The case $\text{tr} (\tilde{b}_{12} \tilde{b}_{21}) < c_0$.

Lemma B.8. If $\text{tr} (\tilde{b}_{12} \tilde{b}_{21}) < c_0$, there holds
\[
|\tilde{S}_0(\tau;t)| \leq \exp \left( \left( t^2 - \tau^2 \right) C \sqrt{c_0}/2 \right).
\]

Proof. By (B.7) and (4.19), there holds
\[
\text{tr} (\tilde{b}_{12} \tilde{b}_{21}) = |\varphi_1(x) \chi_{[+]}(\xi) \partial_t g(0,x)|^2 \gamma_1(\xi).
\]
By the lower bound of $\gamma_1$ in (4.23), for the case $\text{tr} (\tilde{b}_{12} \tilde{b}_{21}) < c_0$, there holds
\[
|\varphi_1(x) \chi_0(\xi) \partial_t g(0,x)| \leq \sqrt{2c_0/\gamma_1(\xi)}.
\]
This implies $|\tilde{b}_{12}| + |\tilde{b}_{21}| \leq C \sqrt{c_0}$. By Lemma B.4, the estimate in Lemma B.8 holds.

B.2.4 The case $\text{tr} (\tilde{b}_{12} \tilde{b}_{21}) \geq c_0$ and around the coalescence locus

By (B.10), the coalescence locus $\nu_+ = \nu_-$ occurs if and only if
\[
|\lambda_1 - \mu| = 2\varepsilon^{3/4} t \sqrt{\text{tr} (\tilde{b}_{12} \tilde{b}_{21})}.
\]
We consider the following subset of $\text{supp} \varphi_1 \times \text{supp} \chi_{[+]}$ near the coalescence locus:
\[
G_1 := \{(x,\xi) : \text{tr} (\tilde{b}_{12} \tilde{b}_{21}) \geq c_0, \quad |\lambda_1 - \mu| - 2\varepsilon^{3/4} t \sqrt{\text{tr} (\tilde{b}_{12} \tilde{b}_{21})} \leq c_0 \varepsilon^{3/4} t \}.
\]

Lemma B.9. For all $(x,\xi) \in G_1$, there holds for all $0 \leq \tau \leq t < \infty$:
\[
|\tilde{S}_0(\tau;t)| \leq \frac{C}{\sqrt{c_0}} \exp \left( \frac{C c_0^{3/2} (t^2 - \tau^2)/2}{\varepsilon^{3/4}} \right).
\]

Proof. We will only consider the case when $\lambda_1 \geq \mu$. For $(x,\xi) \in G_1$. The case when $\lambda \leq \mu$ can be treated similarly. We write the decomposition
\[
M_0 = \frac{i(\lambda_1 + \mu)}{2} + \begin{pmatrix} i(\lambda_1 - \mu) / 2 & -\varepsilon^{3/4} t \tilde{b}_{12} \\ -\varepsilon^{3/4} t \tilde{b}_{21} & -i(\lambda_1 - \mu) / 2 \end{pmatrix}
\]
\[
= \frac{i(\lambda_1 + \mu)}{2} + \varepsilon^{3/4} t N_{01} + \varepsilon^{3/4} c_1(t,x,\xi) t N_{02},
\]
where $c_1(t,x,\xi) = \frac{\lambda_1 - \mu - 2\varepsilon^{3/4} t \sqrt{\text{tr} (\tilde{b}_{12} \tilde{b}_{21})}}{\varepsilon^{3/4} t} \in [-c_0, c_0]$ and
\[
N_{01} := \begin{pmatrix} i \sqrt{\text{tr} (\tilde{b}_{12} \tilde{b}_{21})} & -\tilde{b}_{12} \\ -\tilde{b}_{21} & -i \sqrt{\text{tr} (\tilde{b}_{12} \tilde{b}_{21})} \end{pmatrix}, \quad N_{02} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]
By Lemma B.7, the eigen-polynomial of $N_{01}$ is
\[ x^2 \left( x - i \sqrt{\text{tr} (\tilde{b}_{12\tilde{b}_{21}})} \right)^2 \left( x + i \sqrt{\text{tr} (\tilde{b}_{12\tilde{b}_{21}})} \right)^2. \]
This implies its eigenvalues are $0, \pm i \sqrt{\text{tr} (\tilde{b}_{12\tilde{b}_{21}})}$. We now introduce the Schur decomposition:

**Lemma B.10.** For any matrix $A$ of order $n \times n$, there exists a unitary $Q$ and an upper triangular $T$, such that $Q^* A Q = T$. Precisely, if we denote $T = (t_{jk})_{n \times n}$, then $t_{jk} = 0$ provided $j > k$ and $t_{11}, t_{22}, \ldots, t_{nn}$ are the eigenvalues of $A$.

Then there exists a unitary $Q_1$ and an upper triangular $N_{01}^{(1)}$ such that $N_{01} = Q_1^* N_{01}^{(1)} Q_1$, where $(N_{01}^{(1)})_{jk} = 0$ for $j > k$ and $(N_{01}^{(1)})_{jj} \in \left\{ 0, \pm i \sqrt{\text{tr} (\tilde{b}_{12\tilde{b}_{21}})} \right\}$ are eigenvalues of $N_{01}$. Let
\[
P_{01} := \text{diag}(c_0^{1/2}, c_0^{1/3}, \ldots, c_0^{1/6}), \quad P_{01}^{-1} = \text{diag}(c_0^{-1/2}, c_0^{-1/3}, \ldots, c_0^{-1/6}, 1).
\]
Define $N_{01}^{(2)} := P_{01} N_{01}^{(1)} P_{01}^{-1}$, then
\[
(N_{01}^{(2)})_{jk} = \begin{cases} 0, & j > k, \\ \pm i \sqrt{\text{tr} (\tilde{b}_{12\tilde{b}_{21}})} & j = k, \\ \frac{1}{c_0^{1/2}} (N_{01}^{(1)})_{jk} & j = k = 6. \\
\frac{1}{c_0^{1/2}} (N_{01}^{(1)})_{jk} & j < k, \\
\frac{1}{c_0^{1/2}} (N_{01}^{(1)})_{jk} & j < k < 6. 
\end{cases}
\]

We change the base as we define $\tilde{S}_{0}^{(1)} := P_{01} Q_1 \tilde{S}_0$. Then $\tilde{S}_{0}^{(1)}$ solves
\[
\partial_\tau \tilde{S}_{0}^{(1)} + \varepsilon^{-3/4} M_{0}^{(1)} \tilde{S}_{0}^{(1)} = 0, \quad \tilde{S}_{0}^{(1)}(\tau; \tau) = P_{01} Q_1,
\]
where
\[
M_{0}^{(1)} := \frac{i(\lambda_1 + \mu)}{2} + \varepsilon^{3/4} t N_{01}^{(2)} + \varepsilon^{3/4} c_1(t, x, \xi) t(P_{01} Q_1) N_{02}(P_{01} Q_1)^{-1}.
\]
By (B.11) and the reality of $\lambda_1, \mu$ and $\sqrt{\text{tr} (\tilde{b}_{12\tilde{b}_{21}})}$, there holds
\[
\varepsilon^{-3/4} \left| M_{0}^{(1)} + (M_{0}^{(1)})^* \right| \leq C t \left( \frac{1}{c_0^{1/2}} + c_0^{1/2} \right).
\]
By Lemma B.6, we have
\[
|\tilde{S}_{0}^{(1)}(\tau; t)| \leq |P_{01} Q_1| \exp \left( C c_0^{1/2} (t^2 - \tau^2)^{\frac{1}{2}} \right).
\]
Then
\[
|\tilde{S}_0(\tau; t)| = |(P_{01} Q_1)^{-1} \tilde{S}_{0}^{(1)}(\tau; t)| \leq \frac{C}{\sqrt{c_0}} \exp \left( C c_0^{1/2} (t^2 - \tau^2)^{\frac{1}{2}} \right).
\]

\[ \square \]
B.2.5 The case \( \text{tr} (\tilde{b}_{12} \tilde{b}_{21}) \geq c_0 \) and around the resonances

Resonance happens when \( \lambda_1 = \mu \). We now consider the following subset of \( \text{supp} \varphi_1 \times \text{supp} \chi_{[\tau, \tau']} \) around the resonances:

\[
G_2 := \{ (x, \xi) : \text{tr} (\tilde{b}_{12} \tilde{b}_{21}) \geq c_0, \ |\lambda_1 - \mu| \leq c_0 t^{3/4} \} \setminus G_1.
\]

Lemma B.11. For all \((x, \xi) \in G_2\), there holds for all \(0 \leq \tau \leq t < \infty\):

\[
|\tilde{S}_0(\tau; t)| \leq \frac{C}{c_0} \exp(b c_0 t) \exp \left( \frac{(t^2 - \tau^2) \gamma^+}{2} \right) \exp \left( \frac{C (t - \tau)c_0^2}{2} \right).
\]

Proof. For \((x, \xi) \in G_2\), we consider the decomposition:

\[
M_0 = i\lambda_1 + e^{3/4 t} M_0^{(1)}, \quad M_0^{(1)} := \begin{pmatrix} 0 & -\tilde{b}_{12} \\ -\tilde{b}_{21} & ic_2(t, x, \xi) \end{pmatrix},
\]

where

\[
c_2(t, x, \xi) := \frac{\mu - \lambda_1}{t^{3/4}} \in [-c_0, c_0].
\]

By Lemma B.7, the eigenvalues of \(M_0^{(1)}\) are

\[
0, \ ic_2(t, x, \xi), \ \kappa_\pm = \frac{ic_2(t, x, \xi)}{2} \pm \frac{1}{2} \sqrt{-c_2^2(t, x, \xi) + 4 \text{tr} (\tilde{b}_{12} \tilde{b}_{21})}.
\]

Since \(\text{tr} (\tilde{b}_{12} \tilde{b}_{21}) \geq c_0 > c_0^2\), there holds

\[
|\kappa_+ - \kappa_-| \geq \sqrt{3} c_0, \ |\kappa_+ - 0| = |\kappa_- - 0| = |\kappa_+ - ic_2(t, x, \xi)| = |\kappa_- - ic_2(t, x, \xi)| \geq \frac{\sqrt{3}}{2} c_0.
\]

For the eigen-spaces \(\ker(M_0^{(1)} - 0)\) and \(\ker(M_0^{(1)} - ic_2(t, x, \xi))\), direct calculation gives

\[
M_0^{(1)} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \iff \tilde{b}_{21}(\xi) w_1 = w_2 = 0,
\]

\[
(M_0^{(1)} - ic_2(t, x, \xi)) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \iff \tilde{b}_{12}(\xi) w_2 = w_1 = 0.
\]

Here, \(\text{tr} (\tilde{b}_{12} \tilde{b}_{21}) \geq c_0 > 0\), there holds \(\text{rank} \tilde{b}_{12}(\xi) = \text{rank} \tilde{b}_{21}(\xi) = 1\). Since \(\tilde{b}_{12}\) and \(\tilde{b}_{21}\) are both of order \(3 \times 3\), we have

\[
\dim \ker(M_0^{(1)} - 0) = \dim \ker(M_0^{(1)} - ic_2(t, x, \xi)) = 2.
\]

Together with \(|\kappa_+ - \kappa_-| \geq \sqrt{3} c_0 > 0\), we have that, for matrix \(M_0^{(1)}\), the geometry multiplicity and the algebra multiplicity are equal, both of which are 6. Together
with the fact that the eigenvalues are all separated with a minimum distance $\sqrt{3}c_0/2$, we can always diagonalize $M_0^{(1)}$, with the spectral decomposition:

\begin{equation}
M_0^{(1)} = 0 \Pi_{00} + ic_2(t, x, \xi)\Pi_{c_2} + \kappa_+ \Pi_+ + \kappa_- \Pi_-,
\end{equation}

where there holds for all $(x, \xi) \in G_2$:

$|\Pi_{00}| + |\Pi_{c_2}| + |\Pi_+| + |\Pi_-| \leq C/c_0$.

Spectral decomposition (B.12) and the following equality

\begin{equation}
\Pi_{00} = \Pi_{00} + \Pi_{c_2} + \Pi_+ + \Pi_- = \text{Id}
\end{equation}

imply

\begin{align*}
\Pi_+ &= \frac{M_0^{(1)} - \kappa_-}{\kappa_+ - \kappa_-} + \frac{\kappa_- \Pi_{00}}{\kappa_+ - \kappa_-} - \frac{(ic_2(t, x, \xi) - \kappa_-)\Pi_{c_2}}{\kappa_+ - \kappa_-}, \\
\Pi_- &= \frac{M_0^{(1)} - \kappa_+}{\kappa_- - \kappa_+} + \frac{\kappa_+ \Pi_{00}}{\kappa_- - \kappa_+} - \frac{(ic_2(t, x, \xi) - \kappa_+)\Pi_{c_2}}{\kappa_- - \kappa_+}.
\end{align*}

It is easy to find that $\Pi_{00}$ and $\Pi_{c_2}$ are independent of $t$. Then by applying $\Pi_{00}$ to (B.9), we obtain

$\partial_t(\Pi_{00}\tilde{S}_0) + \varepsilon^{-3/4}i\lambda_1(\Pi_{00}\tilde{S}_0) = 0$, $\partial_t(\Pi_{c_2}\tilde{S}_0) + (\varepsilon^{-3/4}i\lambda_1 + itc_2(t, x, \xi)(\Pi_{c_2}\tilde{S}_0) = 0$,

which implies

$\|(\Pi_{00}\tilde{S}_0)(\tau; t)\| = \|(\Pi_{00}\tilde{S}_0)(\tau; \tau)\|$, $\|(\Pi_{c_2}\tilde{S}_0)(\tau; t)\| = \|(\Pi_{c_2}\tilde{S}_0)(\tau; \tau)\|.$

Applying $\Pi_+(t)$ and $\Pi_-(t)$ to (B.9) gives

\begin{equation}
\partial_t(\Pi_+\tilde{S}_0) + (\varepsilon^{-3/4}i\lambda_1 + t\kappa_+(\Pi_+\tilde{S}_0) = (\partial_t\Pi_+)(\tilde{S}_0), \\
\partial_t(\Pi_-\tilde{S}_0) + (\varepsilon^{-3/4}i\lambda_1 + t\kappa_-(\Pi_-\tilde{S}_0) = (\partial_t\Pi_-)(\tilde{S}_0).
\end{equation}

From (B.12), direct calculation gives

$|\partial_t\Pi_+| + |\partial_t\Pi_-| \leq \frac{C}{c_0}(1 + \frac{1}{\tau}).$

This means that $(\partial_t\Pi_+)$ and $(\partial_t\Pi_-)$ are unbounded for $t$ near 0. To show the upper bound, we first consider for large $t > c_0 > 0$; for small $t$, we use the rough estimate in Lemma (B.4). To be precise, we consider $\tilde{S}_1(\tau; t) := \tilde{S}_0(\tau; t + c_0)$. Then

$\partial_t\tilde{S}_1(\tau; t) + \varepsilon^{-3/4}M_0(t + c_0)\tilde{S}_1(\tau; t) = 0$, $\tilde{S}_1(\tau; \tau) = \tilde{S}_0(\tau; \tau + c_0)$.

By applying the eigenprojectors, we have

\begin{equation}
|(\Pi_{00}\tilde{S}_1)(\tau; t)| = |(\Pi_{00}\tilde{S}_1)(\tau; \tau)|, \quad |(\Pi_{c_2}\tilde{S}_1)(\tau; t)| = |(\Pi_{c_2}\tilde{S}_1)(\tau; \tau)|,
\end{equation}
Then Gronwall’s inequality gives
\[ \partial_t (\Pi_+ (t + c_0) \tilde{S}_1) + (\varepsilon^{-3/4}i\lambda_1 + t\kappa_+) (\Pi_+ (t + c_0) \tilde{S}_1) = (\partial_t \Pi_+ (t + c_0)) \tilde{S}_1, \]
\[ \partial_t (\Pi_- (t + c_0) \tilde{S}_1) + (\varepsilon^{-3/4}i\lambda_1 + t\kappa_-) (\Pi_- (t + c_0) \tilde{S}_1) = (\partial_t \Pi_- (t + c_0)) \tilde{S}_1, \]
where there holds
\[ |\partial_t \Pi_+ (t + c_0)| + |\partial_t \Pi_- (t + c_0)| \leq \frac{C}{c_0} (1 + \frac{1}{t + c_0}) \leq \frac{C}{c_0}. \]
Since the real parts \( \Re(\varepsilon^{-3/4}i\lambda_1 + t\kappa_+) \leq t\gamma^+, \Re(\varepsilon^{-3/4}i\lambda_1 + t\kappa_-) \leq 0 \), together with equation \((B.15)\), we have
\[ |\Pi_+ (t + c_0) \tilde{S}_1(\tau; t)| \leq \exp \left( ((t^2 - \tau^2)\gamma^+/2) |\Pi_+ (\tau + c_0) \tilde{S}_1(\tau; \tau) | + \frac{C}{c_0} \int_\tau^t \exp \left( (t'^2 - \tau^2)\gamma^+/2 \right) |\tilde{S}_1(t'; \tau)| dt' \right), \]
\[ |\Pi_- (t + c_0) \tilde{S}_1(\tau; t)| \leq |\Pi_- (\tau + c_0) \tilde{S}_1(\tau; \tau)| + \frac{C}{c_0} \int_\tau^t |\tilde{S}_1(t'; \tau)| dt'. \]
By Lemma \((B.3)\)
\[ |\tilde{S}_1(\tau; \tau)| = |\tilde{S}_0(\tau; \tau + c_0)| \leq \exp \left( ((\tau + c_0)^2 - \tau^2) c_0^2 / 2 \right) \leq \exp \left( b^+ (c_0 \tau + c_0^2 / 2) \right). \]
We choose \( c_0 \) small such that
\[ \exp(b^+ c_0^2 / 2) \leq 2. \]
Then \( |\tilde{S}_1(\tau; \tau)| \leq 2 \exp(b^+ c_0 \tau). \) Together with \((B.14)\) and \((B.16)\), we deduce
\[ |\tilde{S}_1(\tau; t)| \leq \frac{C}{c_0} \exp \left( (t^2 - \tau^2)\gamma^+/2 \right) \exp(b^+ c_0 \tau) \]
\[ + \frac{C}{c_0} \int_\tau^t \exp \left( (t'^2 - \tau^2)\gamma^+/2 \right) |\tilde{S}_1(t'; \tau)| dt'. \]
Gronwall’s inequality gives
\[ |\tilde{S}_1(\tau; t)| \leq \frac{C}{c_0} \exp(b^+ c_0 \tau) \exp \left( (t^2 - \tau^2)\gamma^+/2 \right) \exp \left( C(t - \tau) / c_0^2 \right). \]
Back to \( \tilde{S}_0 \), for \( t \geq \tau + c_0 \), we have
\[ |\tilde{S}_0(\tau; t)| = |\tilde{S}_1(\tau; t - c_0)| \leq \frac{C}{c_0} \exp(b^+ c_0 \tau) \exp \left( (t^2 - \tau^2)\gamma^+/2 \right) \exp \left( C(t - \tau) / c_0^2 \right). \]
For \( 0 \leq \tau \leq t \leq \tau + c_0 \), Lemma \((B.4)\) gives \( |\tilde{S}_0(\tau; t)| \leq 2 \exp(b^+ c_0 \tau) \). Then we get the estimate in Lemma \((B.11)\).
B.2.6 The case \( \text{tr} (\tilde{b}_{12} \tilde{b}_{21}) \geq c_0 \) and away both from the coalescence locus and the resonance

We consider the following subset of \( \text{supp} \varphi \times \text{supp} \chi_{[+,+]} \):

\[
G_3 := \left\{ (x, \xi) : \text{tr} (\tilde{b}_{12} \tilde{b}_{21}) \geq c_0, \ |\lambda_1 - \mu| \geq c_0 \varepsilon^{3/4} t, \ |\lambda_1 - \mu| - 2 \varepsilon^{3/4} t \sqrt{\text{tr} (\tilde{b}_{12} \tilde{b}_{21})} \geq c_0 \varepsilon^{3/4} t \right\}.
\]

In this case we can always diagonalize \( M_0 \). A similar argument as in the previous section gives:

**Lemma B.12.** For all \( (x, \xi) \in G_3 \), there holds for all \( 0 \leq \tau \leq t < \infty \):

\[
|\tilde{S}_0(\tau; t)| \leq \frac{C}{c_0} \exp(b^+ c_0 \tau) \exp \left( (t^2 - \tau^2) (\gamma^+/c_0) / 2 \right) \exp \left( C (t - \tau) / c_0^2 \right).
\]

B.2.7 Proof of Proposition B.2—Summary

We choose and fix \( c_0 \) small such that

\[
\exp(c_0^2/2) \leq 2, \quad \text{for (B.17)}; \quad Cc_0^{1/2} \leq \gamma^+, \quad \text{for Lemma B.9 and Lemma B.12}
\]

Then by Lemma B.8 Lemma B.9 Lemma B.11 and Lemma B.12 for all \( (x, \xi) \in \text{supp} \varphi \times \text{supp} \chi_{[+,+]} \), there holds

\[
|\tilde{S}_0(\tau; t)| \leq \frac{C}{c_0} \exp(b^+ c_0 \tau) \exp \left( (t^2 - \tau^2) \gamma^+/2 \right) \exp \left( C (t - \tau) / c_0^2 \right).
\]

With the new constant \( C = C / c_0^2 \), we obtain the estimate in Proposition B.2.

B.3 Proof of Proposition B.1

To prove Proposition B.1 it is sufficient to prove (B.4) where \( \tilde{S} \) is solution of (B.8).

By (B.8) and (B.9), we have

\[
\tilde{S}(\tau; t) = \tilde{S}_0(\tau; t) - \varepsilon^{1/4} \int_{\tau}^{t} \tilde{S}_0(t', t) M_1 \tilde{S}(\tau; t') dt'.
\]

By Corollary B.3 there holds for any \( 0 \leq \tau \leq t \leq T_1 |\ln \varepsilon|^{1/2} \):

\[
|\tilde{S}(\tau; t)| \leq |\tilde{S}_0(\tau; t)| + \varepsilon^{1/4} \int_{\tau}^{t} |\tilde{S}_0(t', t)||M_1||\tilde{S}(\tau; t')| dt' \\
\leq C \exp(C |\ln \varepsilon|^{1/2}) \exp \left( (t^2 - \tau^2) \gamma^+/2 \right) \\
+ C \exp(C |\ln \varepsilon|^{1/2}) \varepsilon^{1/4} \int_{\tau}^{t} \exp \left( ((t^2 - t'^2) \gamma^+/2) |\tilde{S}(\tau; t')| dt' \right.
\]

47
Let
\[
S^b(\tau; t) := \frac{|\tilde{S}(\tau; t)|}{\exp((t^2 - \tau^2)\gamma^+/2)}.
\]
Then
\[
S^b(\tau; t) \leq C \exp(C|\ln \varepsilon|^{1/2}) + C\varepsilon^{1/4} \exp(C|\ln \varepsilon|^{1/2}) \int_{\tau}^{t} S^b(\tau; t') dt'.
\]
For \(\varepsilon\) small, there always holds
\[
\varepsilon^{1/4} \exp(C|\ln \varepsilon|^{1/2}) = \varepsilon^{1/5} \exp\left(|\ln \varepsilon|^{1/2}(C - |\ln \varepsilon|^{1/2}/20)\right) \leq \varepsilon^{1/5}.
\]
Then by Gronwall’s inequality,
\[
S^b(\tau; t) \leq C \exp(C|\ln \varepsilon|^{1/2}) \exp(4C\varepsilon^{1/5}) \leq 2C \exp(C|\ln \varepsilon|^{1/2}).
\]
Back to \(\tilde{S}\),
\[(B.18) \quad |\tilde{S}(\tau; t)| \leq 2C \exp(C|\ln \varepsilon|^{1/2}) \exp((t^2 - \tau^2)\gamma^+/2),\]
which is \((B.4)\) with \(\alpha = 0\). To show the higher-order estimates, we apply \(\partial_j\) to \((B.8)\) and obtain
\[
\partial_t \partial_j \tilde{S} + \varepsilon^{-3/4} M_0 \partial_x \tilde{S} = \varepsilon^{-3/4}(\partial_j M_0) \tilde{S} + \varepsilon^{1/4}(\partial_x M_1) \tilde{S} - \varepsilon^{1/4} M_1 \partial_x \tilde{S}.
\]
By the definitions of \(M_0\) and \(M_1\) in \((B.6)\) and Proposition \((B.2)\) observing that \(\partial_x \tilde{S}(\tau; \tau) = 0\), we have
\[
|\partial_x \tilde{S}(\tau; t)| \leq C(1 + \varepsilon^{1/4}) \exp(C|\ln \varepsilon|^{1/2}) \int_{\tau}^{t} \exp\left((t^2 - t^2)\gamma^+/2\right) |\tilde{S}(\tau; t')| dt' + C\varepsilon^{1/4} \exp(C|\ln \varepsilon|^{1/2}) \int_{\tau}^{t} \exp\left((t^2 - t^2)\gamma^+/2\right) |\partial_x \tilde{S}(\tau; t')| dt'.
\]
By \((B.18)\), the above equation implies
\[
|\partial_x \tilde{S}(\tau; t)| \leq C(t - \tau) \exp(2C|\ln \varepsilon|^{1/2}) \exp\left((t^2 - \tau^2)\gamma^+/2\right) + C\varepsilon^{1/4} \exp(C|\ln \varepsilon|^{1/2}) \int_{\tau}^{t} \exp\left((t^2 - t^2)\gamma^+/2\right) |\partial_x \tilde{S}(\tau; t')| dt'.
\]
Then Gonwall’s inequality gives
\[
|\partial_x \tilde{S}(\tau; t)| \leq C|\ln \varepsilon|^{1/2} \exp(2C|\ln \varepsilon|^{1/2}) \exp\left((t^2 - \tau^2)\gamma^+/2\right).
\]
Since \(\partial^2_x M_0\) and \(\partial^2_x M_1\) are uniformly bounded for all \(|\alpha| \leq d + 1 + (q_0 + 3)/4\), then by induction, we have for any \(\alpha \in \mathbb{N}^d\) with \(|\alpha| \leq d + 1 + (q_0 + 3)/4\):
\[
|\partial^2_x \tilde{S}(\tau; t)| \leq C|\ln \varepsilon|^{\alpha/2} \exp(C(1 + |\alpha|) \ln \varepsilon^{1/2}) \exp\left((t^2 - \tau^2)\gamma^+/2\right).
\]
We complete the proof of Proposition \([B.1]\).
B.4 Upper bound for $S^{-,-}$

By a similar argument as the proof of Proposition [3.1] and [4.30], we have

**Proposition B.13.** For all $0 \leq \tau \leq t \leq T_1|\ln \varepsilon|^{1/2}$, all $(x, \xi)$ and all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d + 1 + (q_0 + 3)/4$, the solution to (4.29) with $(i, j) = (−, −)$ satisfies the bound

$$|\partial_x^\alpha S^{-,-}_0(\tau; t)| \leq C|\ln \varepsilon|^{\alpha/2} \exp(C(1 + |\alpha|)|\ln \varepsilon|^{1/2}) \exp((t^2 − \tau^2)^{\gamma^+}/2).$$

B.5 Upper bound for $S^{+,0}$

**Proposition B.14.** For any $0 < \tilde{c}_0 < 1$, all $0 \leq \tau \leq t \leq T_1|\ln \varepsilon|^{1/2}$, all $(x, \xi)$ and all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d + 1 + (q_0 + 3)/4$, the solution to (4.29) with $(i, j) = (+, 0)$ satisfies the bound

$$|\partial_x^\alpha S^{+,0}_0(\tau; t)| \leq \frac{1}{\tilde{c}_0}|\ln \varepsilon|^{\alpha/2} \exp(\tilde{b}^+(0)(t^2 − \tau^2)/2),$$

where

$$\tilde{b}^+ := \sup_{x, \xi} |\chi_{[+, 0]}(\xi)\varphi_1(x)\partial_t g(0, x)P_+(\xi + 3k)F(\varepsilon_3)Q_0(\xi)|.$$

**Proof.** By (4.16) and (4.27), the matrix $M_{+,0}$ is:

$$M_{+,0} = \begin{pmatrix}
             i\chi_{[+, 0]}^{(1)}\lambda_1 & 0 & 0 & 0 & 0 & −\sqrt{\varepsilon}b \\
             0 & i\chi_{[+, 0]}^{(1)}\lambda_2 & 0 & 0 & 0 & 0 \\
             0 & 0 & 0 & 0 & 0 & 0 \\
             0 & 0 & 0 & i\chi_{[+, 0]}^{(1)}\mu & 0 & 0 \\
             0 & 0 & 0 & 0 & −i\chi_{[+, 0]}^{(1)}\mu & 0 \\
             0 & 0 & 0 & 0 & 0 & 0
           \end{pmatrix}$$

denoting

$$b(x, \xi) := \chi_{[+, 0]}(\xi)\varphi_1(x)\partial_t g(0, x)P_+(\xi + 3k)F(\varepsilon_3)Q_0(\xi) \in \mathbb{R}^{3 \times 3}.$$

By the argument in Section [B.1] and Section [B.3] to prove Proposition [B.14] it suffices to prove the following lemma:

**Lemma B.15.** For the solution to

$$\partial_t \bar{S}_1 + \varepsilon^{-3/4} \tilde{M}_1 \bar{S}_1 = 0, \quad \bar{S}_1(\tau; \tau) = \text{Id},$$

where

$$\tilde{M}_1 := \begin{pmatrix}
             i\lambda_1 & −\varepsilon^{3/4}\bar{b} \\
             0 & 0
           \end{pmatrix}, \quad \bar{b} := \chi_{[+, 0]}(\xi)\varphi_1(x)\partial_t g(0, x)P_+(\xi + 3k)F(\varepsilon_3)Q_0(\xi),$$

we have the following estimate for any $0 < \tilde{c}_0 < 1$ and all $0 \leq \tau < t < \infty$:

$$|\bar{S}_1(\tau; t)| \leq \frac{1}{\tilde{c}_0} \exp(\tilde{b}^+(0)(t^2 − \tau^2)/2).$$
Proof of Lemma B.15: Let \( P_0 := \left( \begin{array}{cc} \tilde{c}_0 & 0 \\
0 & 1 \end{array} \right) \), \( \tilde{S}_2 := P_0 \tilde{S}_1 \). Then \( \tilde{S}_2 \) solves

\[
\partial_t \tilde{S}_2 + \varepsilon^{-3/4} \tilde{M}_2 \tilde{S}_2 = 0, \quad \tilde{S}_1(\tau; \tau) = \left( \begin{array}{cc} \tilde{c}_0 & 0 \\
0 & 1 \end{array} \right), \quad \tilde{M}_2 := \left( \begin{array}{cc} i\lambda_1 & -\varepsilon^{3/4} \tilde{c}_0 t \bar{b} \\
0 & 0 \end{array} \right).
\]

By Lemma B.6 we have here

\[
|\tilde{S}_2(\tau; t)| \leq \exp \left( \tilde{b}^+_{[+0]} \tilde{c}_0 (t^2 - \tau^2) / 2 \right).
\]

Then

\[
|\tilde{S}_1(\tau; t)| = |P_0^{-1} \tilde{S}_2(\tau; t)| \leq \frac{1}{\tilde{c}_0} \exp \left( \tilde{b}^+_{[+0]} \tilde{c}_0 (t^2 - \tau^2) / 2 \right).
\]

\[ \square \]

B.6 Upper bound for \( S^{+,0} \)

The same argument as in Section B.5 gives

**Proposition B.16.** For any \( 0 < \tilde{c}_0 < 1 \), for all \( 0 \leq \tau \leq t \leq T_1 \left| \ln \varepsilon \right|^{1/2} \), all \((x, \xi)\) and all \( \alpha \in \mathbb{N}^d \) with \( \alpha \leq d + 1 + (q_0 + 3)/4 \), the solution to (4.29) with \((i, j) = (-, 0)\) satisfies the bound

\[
|\partial^{\alpha}_{x} S^+_{0}(\tau; t)| \leq \frac{1}{\tilde{c}_0} \left| \ln \varepsilon \right|^{n/2} \exp \left( \tilde{b}^+_{[-0]} \tilde{c}_0 (t^2 - \tau^2) / 2 \right),
\]

where

\[
\tilde{b}^+_{[-0]} := \sup_{x, \xi} \left| \chi_{[-0]}(\xi) \varphi_1(x) \partial_t \bar{g}(0, x) P_{-} (\xi - 3k) F(\varepsilon_{-3}) Q_{0}(\xi) \right|.
\]

B.7 Proof of Proposition 4.9

Proposition 4.9 is concluded by Proposition B.1, Proposition B.13, Proposition B.14 and Proposition B.16 with the choice \( \tilde{c}_0 \) in Proposition B.14 and Proposition B.16 small so that \( \tilde{c}_0 \tilde{b}^+_{[\pm, 0]} \leq \gamma^+ \).

C An integral representation formula

We adapt to the present context an integral representation formula introduced in [22]. Consider the initial value problem

(C.1) \[ \partial_t u + \frac{1}{\varepsilon^{3/4}} \text{op}_\psi^+(M) u = f, \quad u(0) = u_0, \]

where \( u_0 \in L^2(\mathbb{R}^d) \), \( f \in L^\infty([0, T_1 \left| \ln \varepsilon \right|^{1/2}], L^2(\mathbb{R}^d)) \), for some \( T_1 > 0 \) independent of \( \varepsilon \). We assume that \( M = M(\varepsilon, t, x, \xi) \) is a matrix-valued, time-dependent symbol that satisfies the following assumption:
Assumption C.1. For some fixed $R_1, R_2 > 0$, for all $t$ and $\varepsilon > 0$, $M$ satisfies

$$M = 0, \text{ for } |\xi| \geq R_1; \quad M = \text{diag} \{i\lambda_1(\xi), \cdots, i\lambda_N(\xi)\}, \text{ for } |x| \geq R_2,$$

where $\lambda_1, \cdots, \lambda_N$ are smooth real valued scalar functions dependent only on $\xi$. We also suppose that $M$ satisfies the bound

$$\sup_{0 \leq t \leq T_1} \sup_{|\varepsilon| \leq R, (x, \xi) \in \mathbb{R}^{d+1}} |\partial_{\xi}^\alpha \partial_{\xi}^\beta M(\varepsilon, t, x, \xi)| \leq C_{\alpha, \beta}, \quad \alpha, \beta \in \mathbb{N}^d, \quad |\alpha| \leq d/2 + d + 1 + (q_0 + 3)/4.$$

Then we have the following lemma:

For the flow $S_0$ of $\varepsilon^{-3/4}M$, defined for $0 \leq \tau \leq t \leq T_1 |\ln \varepsilon|^{1/2}$ by

$$(C.2) \quad \partial_t S_0(\tau; t) + \frac{1}{\varepsilon^{3/4}} MS_0(\tau; t) = 0, \quad S_0(\tau; \tau) = \text{Id},$$

we assume an exponential growth in time:

Assumption C.2. There holds for some $\gamma^+ > 0$, all $0 \leq \tau \leq t \leq T_1 |\ln \varepsilon|^{1/2}$, and all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d + 1 + (q_0 + 3)/4$:

$$|\partial_{\xi}^\alpha S_0(\tau; t)| \leq C |\ln \varepsilon|^{\alpha/2} \exp(C \left(1 + |\alpha|\right) |\ln \varepsilon|^{1/2}) \exp \left((t^2 - \tau^2)\gamma^+/2\right).$$

We introduce correctors $\{S_q\}_{1 \leq q \leq q_0}$, with $q_0$ large determined by (C.15), defined as the solutions of

$$(C.3) \quad \partial_t S_q + \frac{1}{\varepsilon^{3/4}} MS_q + \sum_{1 \leq |\alpha| \leq (q+3)/4} \frac{(-i)^{|\alpha|}}{|\alpha|!} \partial_{\xi}^\alpha M \partial_x^{\gamma} S_{q+3-4|\alpha|} = 0, \quad S_q(\tau; \tau) = 0.$$

Then we have the following lemma:

Lemma C.3. There holds, for all $q \in [0, q_0]$, all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d + 1$, all $\beta \in \mathbb{N}^d$, all $0 \leq \tau \leq t \leq T_1 |\ln \varepsilon|^{1/2}$, the bounds

$$(C.4) \quad |\partial_{\xi}^\alpha \partial_{\xi}^\beta S_q(\tau; t)| \leq C \varepsilon^{-3|\beta|/4} |\ln \varepsilon|^{\alpha + \beta + q/2}
\times \exp \left(C \left(1 + |\alpha + \beta| + q\right) |\ln \varepsilon|^{1/2}\right) \exp \left((t^2 - \tau^2)\gamma^+/2\right).$$

Proof. By (C.2) and (C.3), there holds for $q \geq 1$

$$(C.5) \quad S_q(\tau; t) = \sum_{1 \leq |\alpha| \leq (q+3)/4} \frac{(-i)^{|\alpha|}}{|\alpha|!} \int_{\tau}^{t} S_0(\tau'; t) \partial_{\xi}^\alpha M(t') \partial_x^{\gamma} S_{q+3-4|\alpha|}(\tau; t') dt'.$$

From here, we see that the bound (C.4) which holds true for $\beta = 0$ by Assumption C.2 propagates from $q$ to $q + 1$. By induction in $\beta$, we obtain the desired result.

By Assumption C.1 and C.3, for $x \geq R_2$, we have
Lemma C.4. If \( x \geq R_2 \), there holds
\[
S_q(\tau; t, x, \xi) = \delta_q(\tau; t, x) := \exp \left( i(t - \tau) \frac{\lambda_1(\xi)}{\varepsilon^{3/4}} \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & \lambda_2(\xi) & & \\
\vdots & \ddots & \ddots & \\
0 & 0 & \cdots & \lambda_N(\xi)
\end{pmatrix} \right).
\]

Then we have the operator norm

Lemma C.5. There holds, for all \( q \in [0, q_0] \), all \( 0 \leq \tau \leq T_1|\ln \varepsilon|^{1/2} \), and all \( u \in L^2 \), the bound
\[
|\text{op}_\varepsilon^\psi(S_q(\tau; t)) u|_{L^2} \leq C|\ln \varepsilon|^{(d+1+q)/2} \times \exp \left( C(2 + d + q)|\ln \varepsilon|^{1/2} \right) \exp \left( (t^2 - \tau^2)\gamma^+/2 \right) \|u\|_{L^2}.
\]

Proof. From Lemma C.3, the compactness of the support of \( M \) on \( \xi \) and Lemma C.4 we deduce the bound
\[
\sum_{|\alpha| \leq d+1} \sup_{\xi \in \mathbb{R}^d} \left| \partial^{\alpha}_\xi (S_q - \delta_q) \right|_{L^2} \leq C|\ln \varepsilon|^{(d+1+q)/2} \times \exp \left( C(d + 1 + q)|\ln \varepsilon|^{1/2} \right) \exp \left( (t^2 - \tau^2)\gamma^+/2 \right),
\]
where \( \delta_q \) is given in Lemma C.4 and \( \text{op}_\varepsilon^\psi(\delta_q) \) is a unitary Fourier multiplier. We then conclude by Proposition A.2. \( \square \)

Let \( S := \sum_{0 \leq q \leq q_0} \varepsilon^{q/4} S_q \). The following Lemma expresses the fact that \( \text{op}_\varepsilon^\psi(S) \) is an approximate solution operator:

Lemma C.6. Under Assumptions C.1 and C.2, there holds
\[
\text{op}_\varepsilon^\psi(\partial_t S) + \frac{1}{\varepsilon^{3/4}} \text{op}_\varepsilon^\psi(M) \text{op}_\varepsilon^\psi(S) = \rho_0,
\]
where for \( 0 \leq \tau \leq t \leq T_1|\ln \varepsilon|^{1/2} \),

\[
\|\rho_0\|_{L^2} \lesssim \varepsilon^{q_0/16 - 3(d+1)/2} \exp \left( (t^2 - \tau^2)\gamma^+/2 \right),
\]
where the notation \( a \lesssim b \) means
\[
a \leq C|\ln \varepsilon|^C \exp(C|\ln \varepsilon|^{1/2}), \quad \text{for some constant } C > 0.
\]

Remark C.7. If \( a \lesssim \varepsilon^{\zeta} b \) for some \( \zeta > 0 \), there holds for small \( \varepsilon \):
\[
a \leq \varepsilon^{\zeta/2} b,
\]
Indeed, \( a \leq C\varepsilon^{\zeta}|\ln \varepsilon|^C \exp(C|\ln \varepsilon|^{1/2}) \) can be rewritten as
\[
a \leq C\varepsilon^{\zeta/2} (\varepsilon^{3/4} |\ln \varepsilon|^C) \exp \left( |\ln \varepsilon|^{1/2} (-\zeta|\ln \varepsilon|^{1/2}/4 + C) \right) b.
\]
This implies \( a \leq \varepsilon^{\zeta/2} b \) for \( \varepsilon \) small.
Proof. By definition of $S$ and (C.3),

\[(C.9) \quad - \partial_t \text{op}_\varepsilon^\psi (S) = I + II,\]

with the notations

\[I := \sum_{0 \leq q \leq q_0} \varepsilon^{(q/4 - 3/4)} \text{op}_\varepsilon^\psi (MS_q),\]

\[II := \sum_{1 \leq q \leq q_0} \sum_{1 \leq |\alpha| \leq [(q+3)/4]} \varepsilon^{q/4} \frac{(-i)^{|\alpha|}}{|\alpha|!} \text{op}_\varepsilon^\psi \left( \partial_\xi^\alpha M \partial_x^\alpha S_{q+3-4|\alpha|} \right).\]

By Proposition A.4,

\[\text{op}_\varepsilon^\psi (MS_q) = \text{op}_\varepsilon^\psi (M) \text{op}_\varepsilon^\psi (S_q) - \sum_{1 \leq |\alpha| \leq [(q_0-q+3)/4]} \varepsilon^{q/4} \frac{(-i)^{|\alpha|}}{|\alpha|!} \text{op}_\varepsilon^\psi \left( \partial_\xi^\alpha M \partial_x^\alpha S_q \right) - \varepsilon^{1+[(q_0-q+3)/4]} \text{op}_\varepsilon^\psi (M, S_q),\]

so that

\[I = \varepsilon^{-3/4} \text{op}_\varepsilon^\psi (M) \text{op}_\varepsilon^\psi (S) - \sum_{0 \leq q \leq q_0-1} \sum_{1 \leq |\alpha| \leq [(q_0-q+3)/4]} \varepsilon^{(q/4 - 3/4 + |\alpha|)} \frac{(-i)^{|\alpha|}}{|\alpha|!} \text{op}_\varepsilon^\psi \left( \partial_\xi^\alpha M \partial_x^\alpha S_q \right) + \rho_0,\]

with

\[(C.10) \quad \rho_0 := \sum_{0 \leq q \leq q_0-1} \varepsilon^{(1+q)/4 + [(q_0-q+3)/4]} \text{op}_\varepsilon^\psi (M, S_q).\]

Changing variables in the double sum, we find

\[I = \varepsilon^{-3/4} \text{op}_\varepsilon^\psi (M) \text{op}_\varepsilon^\psi (S) - \sum_{1 \leq q' \leq q_0} \sum_{1 \leq |\alpha| \leq [(q'+3)/4]} \varepsilon^{q'/4} \frac{(-i)^{|\alpha|}}{|\alpha|!} \text{op}_\varepsilon^\psi \left( \partial_\xi^\alpha M \partial_x^\alpha S_{q'+3-4|\alpha|} \right) + \rho_0,\]

hence

\[(C.11) \quad I + II = \varepsilon^{-1/4} \text{op}_\varepsilon^\psi (M) \text{op}_\varepsilon^\psi (S) + \rho_0.\]

Identities (C.9) and (C.11) prove (C.6). From Proposition A.4 and Assumption C.1 we deduce

\[\| R_1^{\psi, q_0-q+3/4} (M, S_q) \|_{L^2 \to L^2} \leq C M_1^{\psi, q_0-q+3/4, 2d+2+[(q_0-q+3)/4]} (S_q),\]

and with Lemma C.3 this implies

\[\| R_1^{\psi, q_0-q+3/4} (M, S_q) \|_{L^2 \to L^2} \leq \varepsilon^{-3(2d+2+[(q_0-q+3)/4])} \exp \left( \left( t^2 - \tau^2 \right)^{\gamma^+ / 2} \right).\]
Then by (C.10), we have
\[ \|\rho_0\|_{L^2 \to L^2} \lesssim \varepsilon^{q_0/16-3(d+1)/2} \exp \left( (t^2 - \tau^2)\gamma^+ / 2 \right). \]

We finally give the representation formula:

**Theorem C.8.** Under Assumptions C.1 and C.2, the Cauchy problem (C.1) with source \( f \in L^\infty([0,T_1|\ln \varepsilon|^{1/2}], L^2) \) and datum \( u_0 \in L^2 \) has a unique solution \( u \in L^\infty([0,T_1|\ln \varepsilon|^{1/2}], L^2) \) given by
\[
(C.12) \quad u = \text{op}^\psi(\psi(0; t))u_0 + \int_0^t \text{op}^\psi(\psi(t'; t))(\text{Id} + \varepsilon \xi F_1(t'))(f(t') + \varepsilon \xi F_2(t')u_0) \, dt',
\]
for some \( \zeta > 0 \), where for some \( N(\xi) > 0 \), for all \( 0 \leq t \leq T_1|\ln \varepsilon|^{1/2} \), there holds
\[
(C.13) \quad \|F_1(t)\|_{L^2 \to L^2} + \|F_2(t)\|_{L^2 \to L^2} \leq |\ln \varepsilon|^{N(\xi)} \exp(N(\xi)|\ln \varepsilon|^{1/2}).
\]

**Proof.** By Lemma C.6 and direct calculation, the map
\[ u := \text{op}^\psi(\psi(0; t))u_0 + \int_0^t \text{op}^\psi(\psi(t'; t))g(t') \, dt' \]
 solves (C.1) in time interval \( [0,T_1|\ln \varepsilon|^{1/2}] \) if and only if there holds for such time:
\[
(C.14) \quad (\text{Id} + r(t))g = f(t) - \rho_0(0; t)u_0,
\]
where \( r \) is the linear integral operator
\[
r(t) : v \to \int_0^t \rho_0(\tau; t)v(\tau) \, d\tau,
\]
with \( \rho_0 \) the remainder in Lemma C.6 satisfying bound (C.7). We choose the expansion index \( q_0 \) large enough so that
\[
(C.15) \quad \zeta := \frac{q_0}{16} - \frac{3d + 3}{2} - \frac{T_1^2\gamma^+}{2} > 0.
\]
Then there holds for all \( 0 \leq t \leq T_1|\ln \varepsilon|^{1/2} \):
\[
(C.16) \quad \sup_{0 \leq t \leq T_1|\ln \varepsilon|^{1/2}} \|r(t)v(t)\|_{L^2} \lesssim \varepsilon^\zeta \sup_{0 \leq t \leq T_1|\ln \varepsilon|^{1/2}} \|v(t)\|_{L^2}.
\]

By (C.10), for \( \varepsilon \) small, \( \text{Id} + r(t) \) is invertible in \( L(L^\infty([0,T_1|\ln \varepsilon|^{1/2}], L^2)) \) which denotes the vector space of linear operators form \( L^\infty([0,T_1|\ln \varepsilon|^{1/2}], L^2) \) to itself. Again by (C.16), \( (\text{Id} + r(t))^{-1} \) is also bounded in \( L(L^\infty([0,T_1|\ln \varepsilon|^{1/2}], L^2)) \), uniformly in \( \varepsilon \). Hence, (C.14) can be solved in \( L^\infty([0,T_1|\ln \varepsilon|^{1/2}], L^2) \), and we obtain (C.12) with
\[
\varepsilon \xi F_1(t) := (\text{Id} + r(t))^{-1} - \text{Id}, \quad \varepsilon \xi F_2(t) := -\rho_0(0; t).
\]

Bound (C.13) follows from (C.16) and \( \|\rho_0\|_{L^2 \to L^2} \lesssim \varepsilon^\zeta \). Uniqueness is a direct consequence of the linearity of equation (C.1) and boundedness of \( \text{op}^\psi(M) \) in \( L(L^2) \).
References

[1] J. M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non-linéaires, Ann. Scient. E.N.S., 14 (1981), 209–246.

[2] T. Colin, D. Lannes, Long-wave short-wave resonance for nonlinear geometric optics, Duke Math. J., vol. 107 (2001), 351-419.

[3] T. Colin, G. Ebrard, G. Gallice, B. Texier, Justification of the Zakharov model from Klein-Gordon-Waves systems, Comm. Part. Diff. Eq. 29 (2004), no. 9-10, 1365-1401.

[4] P. Donnat, J.-L. Joly, G. Métivier, J. Rauch, Diffractive nonlinear geometric optics, Séminaire Equations aux Dérivées Partielles, Ecole Polytechnique, Palaiseau, 1995-1996, pp. XVII 1-23.

[5] E. Dumas, About nonlinear geometric optics, Bol. Soc. Esp. Mat. Apl. SeMA No. 35 (2006), 7-41.

[6] P. Germain, Global existence for coupled Klein-Gordon equations with different speeds, Annales de l'Institut Fourier 61 (2011), no. 6, 2463-2506.

[7] L. Hörmander, The analysis of linear partial differential operators III. Grundlehren der Mathematischen Wissenschaften 274, Springer Verlag, 1985.

[8] I. L. Hwang, The $L^2$-boundedness of pseudodifferential operators. Trans. Amer. Math. Soc. 302 (1987), no. 1, 55–76.

[9] J.-L. Joly, G. Métivier, J. Rauch, Diffractive nonlinear geometric optics with rectification, Indiana U. Math. J., vol. 47 (1998), 1167-1241.

[10] J.-L. Joly, G. Métivier, J. Rauch, Transparent nonlinear geometric optics and Maxwell-Bloch equations, J. Diff. Eq., vol. 166 (2000), 175-250.

[11] D. Lannes, Dispersive effects for nonlinear diffractive geometrical optics with rectification, Asymptotic Analysis 18 (1998), 111-146.

[12] N. Lerner, Y. Morimoto, C.-J. Xu, Instability of the Cauchy-Kovalevskaya solution for a class of nonlinear systems. Amer. J. Math. 132 (2010), no. 1, 99-123.

[13] N. Lerner, T. Nguyen, B. Texier, The onset of instability for quasi-linear systems, in preparation.

[14] Y. Lu, High-frequency limit of the Maxwell-Landau-Lifshitz system in the diffractive optics regime, Asymptotic Analysis 82 (2013), 109-137.

[15] Y. Lu, B. Texier, A stability criterion for high-frequency oscillations, 2013, arXiv:1307.4196.

[16] G. Métivier Remarks on the well-posedness of the nonlinear Cauchy problem, Geometric analysis of PDE and several complex variables, 337-356, Contemp. Math., 368, Amer. Math. Soc., Providence, RI, 2005.

[17] G. Métivier Para-differential Calculus and Applications to the Cauchy Problem for Nonlinear Systems, Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, 5. Edizioni della Normale, Pisa, 2008. xii+140 pp.

[18] Y. Nishiura, M. Mimura, Layer oscillations in reaction-diffusion systems, SIAM J. Appl. Math. 49 (1989), no. 2, 481-514.

[19] B. Texier, The short wave limit for nonlinear, symmetric hyperbolic systems, Adv. Diff. Eq. 9 (2004), no. 1, 1-52.

[20] B. Texier, WKB asymptotics for the Euler-Maxwell equations, Asymptotic Analysis 42 (2005), no. 3-4, 211-250.

[21] B. Texier, Derivation of the Zakharov equations, Archive for Rational Mechanics and Analysis 184 (2007), 121-183.

[22] B. Texier, Approximations of pseudo-differential flows, 2014, arXiv:1402.6868.