DIFFERENTIAL STRUCTURE ON $\kappa$-MINKOWSKI SPACE, AND $\kappa$-POINCARÉ ALGEBRA

STJEPAN MELJANAC AND SAŠA KREŠIĆ-JURIĆ

Abstract. We construct realizations of the generators of the $\kappa$-Minkowski space and $\kappa$-Poincaré algebra as formal power series in the $h$-adic extension of the Weyl algebra. The Hopf algebra structure of the $\kappa$-Poincaré algebra related to different realizations is given. We construct realizations of the exterior derivative and one-forms, and define a differential calculus on $\kappa$-Minkowski space which is compatible with the action of the Lorentz algebra. In contrast to the conventional bicovariant calculus, the space of one-forms has the same dimension as the $\kappa$-Minkowski space.

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1. Introduction

Recent years have witnessed a growing interest in applications of noncommutative (NC) geometry to a possible unification of quantum field theory and gravity [1]–[15]. Current progress in high-energy physics relies in great part on ideas which embody a modification in the description of spacetime as a continuous geometrical structure. This modification is a natural consequence of the appearance of a new fundamental length scale known as Planck length [2], [3]. The Planck length plays a fundamental role in loop quantum gravity where a quantization process leads to the area and volume operators having discrete spectra. The minimal values of the corresponding eigenvalues are proportional to the square and cube of the Planck length, respectively [16], [17]. As a new fundamental, observer-independent quantity, the Planck length is incorporated into kinematic theory within the framework of the doubly special relativity (DSR) theory [18], [19]. In DSR there exist two observer-independent scales, velocity (identified with the speed of light) and length or mass (expected to be the Planck length or Planck mass). The Minkowski...
spacetime is deformed into a noncommutative space for which a mathematical model is provided by the $\kappa$-Minkowski space $[18]-[22]$. The symmetry algebra for doubly special relativity is obtained by deforming the ordinary Poincaré algebra into a Hopf algebra known as $\kappa$-Poincaré algebra $[21]-[27]$. Different representations (bases) $[21]$ of the $\kappa$-Poincaré algebra correspond to different versions of the DSR theory. However, the resulting spacetime algebra is independent of the representation $[21], [27]$. Recently, the $\kappa$-Minkowski NC space in bicrossproduct basis was shown to emerge from considerations of a NC differential structure on a pseudo-Riemannian manifold $[28], [29]$.

As a part of a general effort to understand the structure of NC spaces, in this paper we shall be interested in developing differential calculus on the $\kappa$-Minkowski space. The $\kappa$-Minkowski space was studied by different groups, from both the mathematical and physical points of view. The construction of differential calculus on the $\kappa$-Minkowski space was considered by Sitarz in Ref. $[30]$. He has shown that there is no fourdimensional bicovariant differential calculus which is Lorentz covariant. If one requires that both conditions are met, then the space of one-forms becomes five dimensional. His work was subsequently generalized to $n$ dimensions by Gonera et. al. $[31]$. A drawback of this approach is that if one-forms are to be generated by an action of the exterior derivative on the NC coordinates, then there should be exactly $n$ forms obtained in this way. There have been several attempts to circumvent this problem in the Euclidean and Minkowski space $[32]-[33]$. In Ref. $[34]$ the authors have constructed a noncommutative version of one-forms on the $\kappa$-Euclidean space as deformations of ordinary one-forms. The NC forms are obtained by an action of a deformed exterior derivative on NC coordinates. In Ref. $[33]$, Bu et. al. constructed a differential algebra on the $\kappa$-Minkowski space from Jordanian twist of the Weyl algebra and showed that the algebra is closed in four dimensions. In their approach they extended the $\kappa$-Poincaré algebra with a dilatation operator and used a coproduct of the Lorentz generators which is different from the one used in Ref. $[30]$.

The present paper is a continuation of previous work on differential forms discussed in Ref. $[34]$. Here we construct a differential algebra on the $\kappa$-Minkowski space in which the number of NC one-forms $\xi_\mu$ is equal to the number of NC coordinates $\hat{x}_\mu$. This is a key difference between our approach and the one presented in Ref. $[30]$. The differential algebra is compatible with an action of the Lorentz generators $M_{\mu\nu}$ and has the property that the commutator $[\xi_\mu, \hat{x}_\nu]$ is closed in the space spanned by the one-forms $\xi_\mu$ alone. The
closedness of the commutator is important since in this case any $k$-form can be written as a linear combination of forms of the type $f_p(\hat{x})\xi_{\mu_1}\xi_{\mu_2}...\xi_{\mu_p}$, $0 \leq p \leq k$, where $f_p(\hat{x})$ is a monomial in $\hat{x}_\mu$.

The paper is organized as follows. In section 2 we introduce the algebra generated by the $\kappa$-Minkowski coordinates $\hat{x}_\mu$ and Lorentz generators $M_{\mu\nu}$. We extend this algebra by the momentum operators $p_\mu$ such that $M_{\mu\nu}$ and $p_\mu$ generate the $\kappa$-deformed Poincaré algebra. We illustrate by examples that this extension is not unique, and in a special case it leads to the undeformed Poincaré algebra. We then study realizations of the coordinates $\hat{x}_\mu$ and generators $M_{\mu\nu}$ as formal power series in the $h$-adic extension of the Weyl algebra. We find a large class of such realizations requiring that the commutator $[M_{\mu\nu}, \hat{x}_\lambda]$ is of Lie type. These realizations generalize the results from Refs. [35] and [36]. Of particular importance is the noncovariant realization used in the construction of the differential algebra in section 4. In section 3 we give a brief description of the Hopf algebra structure of the $\kappa$-Poincaré algebra based on the realizations found in section 2. Section 4 deals with differential algebra on the $\kappa$-Minkowski space. We find realizations of the exterior derivative $\hat{d}$ whose action on NC coordinates $\hat{x}_\mu$ leads to deformed one-forms $\xi_\mu$. The one-forms $\xi_\mu$ are constructed as elements of the $h$-adic extension of a super Weyl algebra. The algebra found using these realizations has the important property that the commutator $[\xi_\mu, \hat{x}_\nu]$ is closed in the vector space spanned by one-forms $\xi_\mu$ alone. Since $\hat{x}_\mu$, $M_{\mu\nu}$ and $\xi_\mu$ belong to an associative algebra, all graded Jacobi identities are automatically satisfied. The Jacobi identities allow us to define an action of $M_{\mu\nu}$ on the algebra generated by $\hat{x}_\mu$ and $\xi_\mu$ which is compatible with the structure of this algebra. This action is different from the one found by Sitarz in Ref. [30] and it does not require introduction of an additional one-form. However, when restricted to the coordinates of the $\kappa$-Minkowski space it agrees with the action found in Ref. [30]. We note that in our approach the exterior derivative is not Lorentz-invariant and one-forms do not transform vector-like under the action of the Lorentz generators.

2. $\kappa$-Minkowski space with Lorentz and $\kappa$-Poincaré algebra

In this section we consider the $\kappa$-Minkowski space with $\kappa$-Poincaré algebra, and their realizations as formal power series in the $h$-adic extension of the Weyl algebra. This construction was introduced in Refs. [35] and [37] for the $\kappa$-deformed Euclidean space.
The $\kappa$-Minkowski space is an algebra generated by NC coordinates $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1}$ satisfying the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \quad a \in \mathbb{R}^n. \quad (1)$$

The coordinates $\hat{x}_\mu$ generate a Lie algebra with structure constants $C^\lambda_{\mu\nu} = a_\mu \delta_{\nu\lambda} - a_\nu \delta_{\mu\lambda}$ describing a deformation of the ordinary Minkowski space. One may view $\hat{x}_\mu$ as deformations of ordinary commutative coordinates $x_\mu$ in the sense that $\hat{x}_\mu \to x_\mu$ as $a \to 0$. Let $\mathcal{L}$ denote the Lorentz algebra generated by $M_{\mu\nu}$, \[ [M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda}, \quad (2) \]

where $\eta = \text{diag}(-1, 1, \ldots, 1)$ is the Minkowski metric. The $\kappa$-Minkowski space $\mathcal{M}_\kappa$ and Lorentz algebra $\mathcal{L}$ can be embedded into a Lie algebra $\mathfrak{g}_\kappa$ which contains $\mathcal{M}_\kappa$ and $\mathcal{L}$ as Lie subalgebras, and $\mathfrak{g}_\kappa = \mathcal{M}_\kappa \oplus \mathcal{L}$ as vector spaces. The correct form of the mixed commutator for $M_{\mu\nu}$ and $\hat{x}_\lambda$ in the Euclidean case was found in Ref. [35]. For the Minkowski metric it is given by

$$[M_{\mu\nu}, \hat{x}_\lambda] = \eta_{\nu\lambda} \hat{x}_\mu - \eta_{\mu\lambda} \hat{x}_\nu - ia_\mu M_{\nu\lambda} + ia_\nu M_{\mu\lambda}. \quad (3)$$

Since all Jacobi identitites for $M_{\mu\nu}$ and $\hat{x}_\lambda$ hold, Eqs. (1)–(3) define a Lie algebra structure on $\mathfrak{g}_\kappa$. The algebra $\mathfrak{g}_\kappa$ can be extended further by momentum generators $p_\mu$ satisfying the commutation relations

$$[p_\mu, p_\nu] = 0, \quad (4)$$

$$[p_\mu, \hat{x}_\nu] = -iH_{\mu\nu}(p), \quad (5)$$

$$[M_{\mu\nu}, p_\lambda] = G_{\mu\nu\lambda}(p), \quad (6)$$

where $H_{\mu\nu}$ and $G_{\mu\nu\lambda}$ are real-analytic functions of $p$ which generally depend on the deformation parameter $a$. We require that $H_{\mu\nu}$ and $G_{\mu\nu\lambda}$ satisfy the classical limit conditions

$$\lim_{a \to 0} H_{\mu\nu} = \eta_{\mu\nu}, \quad \lim_{a \to 0} G_{\mu\nu\lambda} = \eta_{\nu\lambda} p_\mu - \eta_{\mu\lambda} p_\nu. \quad (7)$$

Relations (1), (4) and (5) define a deformed Heisenberg algebra $\mathcal{H}_\kappa$, while relations (2), (4) and (6) define a $\kappa$-deformed Poincaré algebra $\mathcal{P}_\kappa$.

The choice of deformations $H_{\mu\nu}$ and $G_{\mu\nu\lambda}$ must be compatible with the requirement that $\hat{x}_\mu$, $M_{\mu\nu}$ and $p_\mu$ satisfy the Jacobi relations. The Jacobi identities for $\hat{x}_\mu$ and $p_\mu$
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imply that $H_{\mu\nu}$ satisfy a system of partial differential equations (PDE’s)

$$\sum_{\alpha=0}^{n-1} \left( \frac{\partial H_{\lambda\mu}}{\partial p_\alpha} H_{\alpha\mu} - \frac{\partial H_{\lambda\nu}}{\partial p_\alpha} H_{\alpha\nu} \right) = a_\mu H_{\lambda\nu} - a_\nu H_{\lambda\mu}. \quad (8)$$

Similarly, the Jacobi identities for $M_{\mu\nu}$ and $p_\mu$ hold if and only if

$$\sum_{\alpha=0}^{n-1} \left( G_{\mu\nu\alpha} \frac{\partial G_{\lambda\rho\sigma}}{\partial p_\alpha} - G_{\lambda\rho\alpha} \frac{\partial G_{\sigma\mu\nu}}{\partial p_\alpha} \right) = \eta_{\nu\lambda} G_{\mu\rho\sigma} - \eta_{\mu\lambda} G_{\nu\rho\sigma} - \eta_{\nu\rho} G_{\mu\lambda\sigma} + \eta_{\mu\rho} G_{\nu\lambda\sigma}. \quad (9)$$

Furthermore, the remaining Jacobi identity for $\hat{x}_\mu$, $p_\nu$ and $M_{\lambda\rho}$ can be used to derive a system of PDE’s relating the functions $H_{\mu\nu}$ and $G_{\mu\nu\lambda}$.

It is important to note that the extension of $g_\kappa$ by the momentum generators $p_\mu$ is not unique since the differential equations for $H_{\mu\nu}$ and $G_{\mu\nu\lambda}$ admit an infinite family of solutions. For example, one solution is given by

$$H_{\mu\nu}(P) = \eta_{\mu\nu} \left( aP + \sqrt{1 + a^2 P^2} \right) - a_\mu P_\nu, \quad (10)$$

$$G_{\mu\nu\lambda}(P) = \eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu, \quad (11)$$

where the scalar product in (10) is taken with respect the Minkowski metric ($aP = -a_0 P_0 + \sum_{i=1}^{n-1} a_i P_i$). By straightforward computation one can check that all Jacobi relations for $\hat{x}_\mu$, $M_{\mu\nu}$ and $P_\mu$ are satisfied. In view of Eq. (11) this particular solution yields the undeformed Poincaré algebra. Another solution with $a = (a_0, 0, \ldots, 0)$ is given by

$$H_{00}(p) = -\psi(A), \quad H_{0j}(p) = 0, \quad (12)$$

$$H_{i0}(p) = -a_0 p_i \gamma(A), \quad H_{ij}(p) = \delta_{ij} \varphi(A), \quad (13)$$

and

$$G_{000}(p) = -\frac{\psi(A)}{\varphi(A)} p_0, \quad (14)$$

$$G_{00j}(p) = \delta_{ij} \varphi(A) \left( \frac{1 - e^{\varphi(A)}}{a_0} - \frac{a_0}{2} \square e^{\varphi(A)} \right) - a_0 \frac{\gamma(A)}{\varphi(A)} p_0 p_j, \quad (15)$$

$$G_{ij0}(p) = 0, \quad G_{ijk}(p) = \delta_{jk} p_i - \delta_{ik} p_j, \quad (16)$$

where $A = a_0 p_0$, $\psi$ and $\varphi$ are arbitrary real-analytic functions such that $\psi(0) = \varphi(0) = 1$, $\varphi'(0)$ is finite and $\gamma = \frac{\varphi'}{\varphi} + 1$. The function $\Psi(A)$ is defined by Eq. (35) and the deformed Laplace operator $\square$ is given by Eq. (37) (with $p_\mu = -i \partial_\mu$). As we shall see shortly, these deformations are related to realizations discussed in section 2.1.
The deformed Heisenberg algebra \( H_\kappa \) acts on the subalgebra \( M_\kappa \subset H_\kappa \) as follows. Let 1 denote the unit in \( M_\kappa \) and define the action \( \triangleright: H_\kappa \times M_\kappa \to M_\kappa \) by

1. \( \hat{x}_\mu \triangleright f(\hat{x}) = \hat{x}_\mu f(\hat{x}) \),
2. \( p_\mu \triangleright f(\hat{x}) = [p_\mu, f(\hat{x})] \triangleright 1, \quad p_\mu \triangleright 1 = 0 \),
3. \( (ab) \triangleright 1 = a \triangleright (b \triangleright 1) \) for all \( a, b \in H_\kappa \),

for any monomial \( f(\hat{x}) \in M_\kappa \). The rules (i) and (iii) imply that \( f(\hat{x}) \triangleright g(\hat{x}) = f(\hat{x})g(\hat{x}) \) for all monomials \( f(\hat{x}), g(\hat{x}) \in M_\kappa \). Consider now the action of the momentum generator \( p_\mu \). Since \( \lim_{a \to 0} H_{\mu \nu}(p) = \eta_{\mu \nu} \) we have \( H_{\mu \nu}(p) = \eta_{\mu \nu} + o(p) \). Hence, in view of Eq. (5) the action of \( p_\mu \) on monomials of order one yields

\[
p_\mu \triangleright \hat{x}_\nu = [p_\mu, \hat{x}_\nu] \triangleright 1 = -i(\eta_{\mu \nu} + o(p)) \triangleright 1 = -i\eta_{\mu \nu}.
\]

Similarly, for monomials of order two we have

\[
p_\mu \triangleright (\hat{x}_\mu \hat{x}_\lambda) = [p_\mu, \hat{x}_\mu \hat{x}_\lambda] \triangleright 1 = ([p_\mu, \hat{x}_\nu] \hat{x}_\lambda + \hat{x}_\nu [p_\mu, \hat{x}_\lambda]) \triangleright 1
= \left( -i(\eta_{\mu \nu} + o(p)) \hat{x}_\lambda - i\hat{x}_\nu (\eta_{\mu \lambda} + o(p)) \right) \triangleright 1
= -i(\eta_{\mu \nu} \hat{x}_\lambda + \eta_{\mu \lambda} \hat{x}_\nu) - i(o(p) \hat{x}_\lambda) \triangleright 1.
\]

This leads to a deformed Leibniz rule for the action of \( p_\mu \). We recognize \(-i(o(p) \hat{x}_\lambda) \triangleright 1 \) as a deformation of the standard Leibniz rule \(-i(\eta_{\mu \nu} \hat{x}_\lambda + \eta_{\mu \lambda} \hat{x}_\nu) \). The deformation obviously depends on the function \( H_{\mu \nu}(p) \). To illustate the point consider the deformed Heisenberg algebra \( \{H_\kappa\} \) with \( \varphi - \psi = 1 + i a_0 \partial_0 \). Then one finds

\[
p_\mu \triangleright (\hat{x}_\nu \hat{x}_\lambda) = -i(\eta_{\mu \nu} \hat{x}_\lambda + \eta_{\mu \lambda} \hat{x}_\nu) + a_0 \eta_{\mu \nu} \eta_{\mu \lambda}.
\]

Therefore, the coproduct \( \Delta p_\mu \) induced by the Leibniz rule is also deformed. In the classical limit as \( a \to 0 \) the Heisenberg algebra \( H_\kappa \) becomes undeformed and \( p_\mu \) obeys the standard Leibniz rule \( p_\mu \triangleright (\hat{x}_\nu \hat{x}_\lambda) = -i(\eta_{\mu \nu} \hat{x}_\lambda + \eta_{\mu \lambda} \hat{x}_\nu) \). Hence, \( \lim_{a \to 0} \Delta p_\mu = \Delta_0 p_\mu \) where \( \Delta_0 p_\mu = 1 \otimes p_\mu + p_\mu \otimes 1 \) is the primitive coproduct. Deformations of the Leibniz rule and coproduct described above are discussed in Refs. \[35, 37\] and \[38\].

2.1. Realizations. In this section we shall study deformations of the Heisenberg and Poincaré algebras using realizations of the generators as formal power series in the \( h \)-adic extension of the Weyl algebra. We want to represent coordinates \( \hat{x}_\mu \) as deformations of commutative coordinates \( x_\mu \) depending on the parameter \( a \in \mathbb{R}^n \) in Eq. (1). Let \( \mathcal{A}_a \) be the Weyl algebra over the field of complex numbers \( \mathbb{C} \) generated by \( x_\mu \) and the differential
operators $\partial_\mu \equiv \frac{\partial}{\partial x_\mu}$, $\mu = 0, 1, \ldots, n - 1$. The generators of $\mathcal{A}_n$ satisfy the commutation relations

$$[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0, \quad [\partial_\mu, x_\nu] = \eta_{\mu\nu}. \quad (20)$$

Let $\mathcal{A}_n[[a]]$ denote the $h$-adic extension of $\mathcal{A}_n$. The elements of $\mathcal{A}_n[[a]]$ are formal power series in $a_0, a_1, \ldots, a_{n-1}$ with coefficients in $\mathcal{A}_n$. Consider a representation of $\hat{x}_\mu$ as an element of $\mathcal{A}_n[[a]]$ given by

$$\hat{x}_\mu = \sum_{\alpha=0}^{n-1} x^\alpha \phi_{\alpha\mu}(\partial), \quad (21)$$

where $x^\alpha = \sum_\beta x_\beta \eta_{\beta\alpha}$, and $\phi_{\alpha\mu}$ is a formal power series in $a_\mu$ with coefficients in the ring of differential operators $\partial_\mu$. We require that $\hat{x}_\mu \to x_\mu$ as $a \to 0$ which implies that $\lim_{a \to 0} \phi_{\alpha\mu} = \eta_{\alpha\mu}$. A representation (21) is called a $\phi$-realization of the NC coordinates $\hat{x}_\mu$. This realization is compatible with commutation relations (1) if and only if $\phi_{\mu\nu}$ satisfy the system of PDE’s

$$\sum_{\beta=0}^{n-1} \left( \frac{\partial \phi_{\alpha\mu}}{\partial \partial_\beta} \phi_{\beta\nu} - \frac{\partial \phi_{\alpha\nu}}{\partial \partial_\beta} \phi_{\beta\mu} \right) = ia_\mu \phi_{\alpha\nu} - ia_\nu \phi_{\alpha\mu}. \quad (22)$$

Given the complexity of Eqs. (22) the system is often simplified by assuming that $\phi_{\mu\nu}$ are functions of the commuting variables $A = ia\partial$ and $B = a^2 \partial^2$ where the scalar product is taken with respect to the Minkowski metric ($uv = -u_0v_0 + \sum_{i=1}^{n-1} u_iv_i$). A large class of such realizations in the Euclidean case was found in Refs. [35], [37] and [38].

Consider realizations of the Lorentz generators $M_{\mu\nu}$ and momenta $p_\mu$ given by

$$M_{\mu\nu} = \sum_{\alpha=0}^{n-1} x^\alpha \Gamma_{\mu\nu\alpha}(\partial) \quad \text{and} \quad p_\mu = -i\partial_\mu \quad (23)$$

where $\Gamma_{\mu\nu\alpha}$ is a formal power series in $a_\mu$ with coefficients in the ring of differential operators $\partial_\mu$. In the classical limit we require that $\lim_{a \to 0} \Gamma_{\mu\nu\alpha} = \eta_{\alpha\mu} \partial_\nu - \eta_{\alpha\nu} \partial_\mu$. The functions $\Gamma_{\mu\nu\alpha}$ are uniquely determined by the commutation relations (2)–(3) and the realization (21). Substituting the realizations for $\hat{x}_\mu$, $M_{\mu\nu}$ and $p_\mu$ into Eqs. (5) and (6) we find

$$\phi_{\mu\nu}(\partial) = H_{\mu\nu}(-i\partial), \quad \Gamma_{\mu\nu\lambda}(\partial) = -iG_{\mu\nu\lambda}(-i\partial). \quad (24)$$

Thus, if the realization of the momentum generator is fixed by $p_\mu = -i\partial_\mu$ there is a one-to-one correspondence between the realizations of the generators $\hat{x}_\mu$ and $M_{\mu\nu}$ and deformations of the algebras $\mathcal{H}_\kappa$ and $\mathcal{M}_\kappa$. In the rest of the paper we assume that the momenta have the fixed realization $p_\mu = -i\partial_\mu$. 

A key tool in the construction of differential forms to be discussed in section 4 is the shift operator. The shift operator $Z$ is an element of $\mathcal{A}_n[[a]]$ defined by the commutation relations

$$[Z, \hat{x}_\mu] = ia_\mu Z, \quad [Z, \partial_\mu] = 0.$$  \hfill (25)

The first relation in Eq. (25) implies that conjugation by $Z^n$ shifts the coordinate $\hat{x}_\mu$ by the amount $ina_\mu$,

$$Z^n \hat{x}_\mu Z^{-n} = \hat{x}_\mu + ina_\mu, \quad n \in \mathbb{Z}. \hfill (26)$$

The shift operator also satisfies the relation $\hat{x}_\mu Z \hat{x}_\nu = \hat{x}_\nu Z \hat{x}_\mu$.

2.1.1. **Natural realization.** Different realizations are obtained by choosing different admissible functions $\phi_{\mu\nu}$. Alternatively, starting from a fixed realization $\phi_{\mu\nu}$ one can introduce a change of generators of the Weyl algebra, $x_\mu \mapsto X_\mu(x, \partial)$ and $\partial_\mu \mapsto D_\mu(\partial)$, to obtain new realizations. A class of such transformations called similarity transformations was described in Ref. [38].

In this paper we shall consider two types of realizations, noncovariant [35] and a special type of covariant realizations known as the natural realization [37]. The variables used to express these two types of realizations in Eq. (21) will be denoted by $(x_\mu, \partial_\mu)$ and $(X_\mu, D_\mu)$, respectively. The natural realization is given by

$$\hat{x}_\mu = X_\mu Z^{-1} + i(aX)D_\mu,$$ \hfill (27)

where $Z^{-1}$ is the inverse shift operator

$$Z^{-1} = -iaD + \sqrt{1 - a^2}D^2.$$ \hfill (28)

One can show that if the NC coordinates are given by Eq. (27), then the Lorentz generators have the standard representation

$$M_{\mu\nu} = X_\mu D_\nu - X_\nu D_\mu.$$ \hfill (29)

Thus, Eqs. (27)–(29) provide the natural realization of the algebra (1)–(3). Since the realization of the momentum generators is given by $P_\mu = -iD_\mu$, $M_{\mu\nu}$ and $P_\mu$ generate the undeformed Poincaré algebra. Note that the natural realization corresponds to our first example of algebra deformation (10)–(11). This example is rather special since in a generic realization the Poincaré algebra is deformed.
2.1.2. Noncovariant realizations. In the rest of the paper we shall restrict our attention to deformations of the Minkowski space when $a = (a_0, 0, \ldots, 0)$. Then the commutation relations (1) and (3) yield

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_j] = i a_0 \hat{x}_j, \quad (30)$$

$$[M_{i0}, \hat{x}_0] = -\hat{x}_i + i a_0 M_{i0}, \quad [M_{i0}, \hat{x}_k] = -\delta_{ik} \hat{x}_0 + i a_0 M_{ik}, \quad (31)$$

$$[M_{ij}, \hat{x}_0] = 0, \quad [M_{ij}, \hat{x}_k] = \delta_{jk} \hat{x}_i - \delta_{ik} \hat{x}_j. \quad (32)$$

By convention the greek indices run through the set $\{0, 1, \ldots, n-1\}$, and the latin indices run through the subset $\{1, 2, \ldots, n-1\}$.

A family of noncovariant realizations of $\hat{x}_\mu$ satisfying the algebra (30) is given by

$$\hat{x}_0 = x_0 \psi(A) + i a_0 \left( \sum_{k=1}^{n-1} x_k \partial_k \right) \gamma(A), \quad \hat{x}_i = x_i \phi(A), \quad (33)$$

$$\gamma = \frac{\phi'}{\phi} \psi + 1, \quad (34)$$

where $A = -ia_0 \partial_0$. This family is parametrized by two real-analytic functions $\phi$ and $\psi$ satisfying the initial conditions $\phi(0) = \psi(0) = 1$ and $\phi'(0)$ is finite. The shift operator in the noncovariant realization is found to be

$$Z = e^{\Psi(A)}, \quad \Psi(A) = \int_0^A \frac{dt}{\psi(t)}. \quad (35)$$

For a given realization (33) we want to find a realization of the Lorentz generators such that $M_{\mu\nu}$ generate the undeformed Lorentz algebra (2) and $[M_{\mu\nu}, \hat{x}_\lambda]$ is given by (31)–(32). The realization of $M_{\mu\nu}$ can be found from the natural realization (29) using the transformation of variables $(x_\mu, \partial_\mu) \mapsto (X_\mu, D_\mu)$ which connect the noncovariant and natural realizations of $\hat{x}_\mu$. One can show that $D_\mu$ is given in terms of $\partial_\mu$ according to

$$D_0 = \frac{e^{-\Psi(A)} - 1}{ia_0} + \frac{ia_0}{2} \Box, \quad D_i = \partial_i \frac{e^{-\Psi(A)}}{\phi(A)}, \quad (36)$$

where $\Box$ is the deformed Laplace operator

$$\Box = \Delta e^{-\Psi(A)} \frac{2}{ia_0} \sinh^2 \left( \frac{1}{2} \Psi(A) \right), \quad \Delta = \sum_{i=1}^{n-1} \partial_i^2. \quad (37)$$
The deformed Laplace operator satisfies the commutation relation $[\Box, \hat{x}_\mu] = 2D_\mu$. Moreover, the transformation of $X_\mu$ is given by

$$X_0 = \left[x_0\psi(A) + ia_0 \sum_{k=1}^{n-1} x_k \partial_k \gamma(A)\right] \frac{1}{1 + \frac{a_0^2}{2} \Box}, \quad (38)$$

$$X_i = x_i \varphi(A) e^{\Psi(A)} + ia_0 \left[ x_0 \psi(A) + ia_0 \sum_{k=1}^{n-1} x_k \partial_k \gamma(A)\right] \frac{1}{1 + \frac{a_0^2}{2} \Box} \varphi(A). \quad (39)$$

Substituting Eqs. (36) and (38)–(39) into Eq. (29) we obtain the noncovariant realization of $M_{\mu\nu}$:

$$M_{i0} = x_i \varphi(A) \left( \frac{1 - e^{\Psi(A)}}{ia_0} + \frac{ia_0}{2} e^{\Psi(A)} \right) - \left[ x_0 \psi(A) + ia_0 \sum_{k=1}^{n-1} x_k \partial_k \gamma(A)\right] \frac{\partial_i}{\varphi(A)}, \quad (40)$$

$$M_{ij} = x_i \partial_j - x_j \partial_i. \quad (41)$$

The realizations (36)–(37) and (40)–(41) generalize the results found in Refs. [35] and [39]. For example, if $\psi = 1$ or $\psi = 1 + 2A$ we obtain the realizations found in Ref. [35], and if $\psi = 1 + rA$, $r \neq 0$, and $\gamma = \text{const.}$ we reproduce the realizations found in Ref. [39] (with $\tau = 1$). The noncovariant realization corresponds to the algebra deformation in example (12)–(16).

3. Hopf algebra structure of $\kappa$-Poincaré algebra

In this section we give a brief description of the Hopf algebra structure of the $\kappa$-Poincaré algebra $\mathcal{P}_\kappa$. In the algebra sector the Lorentz generators satisfy the standard relations (2), and the commutator $[M_{\mu\nu}, p_\lambda]$ in Eq. (6) is assumed to be deformed by Eqs. (14)-(16). The reason for considering this Hopf algebra structure is its relation to the differential algebra on the $\kappa$-Minkowski space discussed in section 4. The coproduct and antipodes of $p_\mu$ and $M_{\mu\nu}$ can be conveniently expressed in terms of the shift operator $Z = e^{\Psi(A)}$ where $A = a_0 p_0$ (c.f. Eq. [35]). Since $\Delta Z = Z \otimes Z$ (see Ref. [37]) we find

$$\Delta p_0 = \frac{1}{a_0} \Psi^{-1}(\ln(Z \otimes Z)) \quad (42)$$

where $\ln(Z \otimes Z) = \ln(Z) \otimes 1 + 1 \otimes \ln(Z)$. Similarly, one can show that (see Refs. [35] and [38])

$$\Delta p_i = \varphi(a_0 \Delta p_0) \left( \frac{p_i}{\varphi(a_0 p_0)} \otimes 1 + Z \otimes \frac{p_i}{\varphi(a_0 p_0)} \right). \quad (43)$$
Furthermore, one finds that the coproducts of the Lorentz generators are given by

$$\Delta M_{i0} = M_{i0} \otimes 1 + Z \otimes M_{i0} - a_0 \sum_{j=1}^{n-1} \frac{p_j}{\varphi(a_0 p_0)} \otimes M_{ij},$$  

(44)

$$\Delta M_{ij} = M_{ij} \otimes 1 + 1 \otimes M_{ij}.$$  

(45)

The counits for all the generators are undeformed. From the definition of antipode \[40\] and using Eqs. (42)–(45) we find

$$S(p_0) = \frac{1}{a_0} \Psi^{-1}\left(\ln(Z^{-1})\right),$$  

(46)

$$S(p_i) = -p_i \frac{\varphi(S(a_0 p_0))}{\varphi(a_0 p_0)} Z^{-1},$$  

(47)

$$S(M_{i0}) = -Z^{-1} M_{i0} - a_0 Z^{-1} \sum_{j=1}^{n-1} \frac{p_j}{\varphi(a_0 p_0)} M_{ij},$$  

(48)

$$S(M_{ij}) = -M_{ij}.$$  

(49)

The antipode of the shift operator is given by $S(Z) = Z^{-1}$. The coalgebra structure as well as the antipodes are deformed in all realizations, and particularly in the natural and noncovariant realizations considered here. In the special case when $\varphi = \psi = 1$ we obtain

$$\Delta p_0 = p_0 \otimes 1 + 1 \otimes p_0, \quad \Delta M_{i0} = M_{i0} \otimes 1 + Z \otimes M_{i0} - a_0 \sum_{j=1}^{n-1} p_j \otimes M_{ij},$$  

(50)

$$\Delta p_i = p_i \otimes 1 + Z \otimes p_i, \quad \Delta M_{ij} = M_{ij} \otimes 1 + 1 \otimes M_{ij}.$$  

(51)

Similarly, the antipodes yield

$$S(p_0) = -p_0, \quad S(M_{i0}) = -Z^{-1} M_{i0} - a_0 Z^{-1} \sum_{j=1}^{n-1} p_j M_{ij},$$  

(52)

$$S(p_i) = -Z^{-1} p_i, \quad S(M_{ij}) = -M_{ij}.$$  

(53)

Relations (42)–(49) describe the Hopf algebra structure of $\mathcal{P}_\kappa$ in different bases corresponding to different choices of $\varphi$ and $\psi$. For example, the choice $\varphi = \psi = 1$ described above corresponds to the bicrossproduct basis \[25\], \[30\], while $\varphi = e^{-A}$ and $\psi = 1$ corresponds to the left ordering \[32\], \[35\]. Similarly, $\varphi = A/(e^A - 1)$ and $\psi = 1$ corresponds to the Weyl symmetric ordering \[32\], \[33\], \[37\]. Furthermore, if $\varphi = \psi = 1 - A$ (resp. $\varphi = 1$, $\psi = 1 + A$) we obtain a basis that corresponds to the left (resp. right) covariant realization in Refs. \[33\], \[37\] and \[39\]. The coproduct and antipode for the generators $P_\mu = -iD_\mu$ in the natural realization are given in Refs. \[37\] and \[41\]. The Hopf algebra structure of
\(\mathcal{P}_\kappa\) in the natural realization (27) is related to the classical basis of \(\mathcal{P}_\kappa\) [21], [41]. We note that the coproducts for \(P_\mu\), \(N_i\) and \(M_i\) used in Ref. [30] correspond to the coproducts for \(p_\mu\) and \(M_{\mu\nu}\) when \(\varphi = \psi = 1\) in Eqs. (12)–(15).

4. Differential forms on \(\kappa\)-Minkowski space

Differential calculus on the \(\kappa\)-deformed Euclidean and Minkowski spaces were considered by several authors in Refs. [30], [32]–[34]. In Ref. [30] Sitarz has shown that there is no four-dimensional bicovariant differential calculus on the \(\kappa\)-Minkowski space \(M_\kappa\) which is Lorentz covariant. If both conditions are satisfied this leads to a contradiction with the mixed Jacobi identity for NC coordinates and one-forms. In order to avoid the problem Sitarz has constructed a differential calculus in which the space of one-forms is five-dimensional. However, in an \(n\)-dimensional spacetime one should expect exactly \(n\) one-forms generated by the action of exterior derivative on the coordinates. In this work we take a different approach based on realizations introduced in section 2. We show that one can define \(n\)-dimensional differential algebra on \(M_\kappa\) which is consistent with an action of the Lorentz algebra in the sense that all graded Jacobi identities involving the NC coordinates, Lorentz generators and one-forms are satisfied.

Let \(\hat{x}_\mu\) be the coordinates on \(M_\kappa\) satisfying relations (1), and suppose \(\hat{x}_\mu\) are represented in a \(\phi\)-realization [21]. We introduce deformed exterior derivative \(\hat{d}\) and one-forms \(\xi_\mu\) by

\[
\hat{d} = \sum_{\alpha,\beta=0}^{n-1} dx^\alpha \partial_\beta k_{\alpha\beta}(\partial), \quad \xi_\mu = \sum_{\alpha=0}^{n-1} dx^\alpha h_{\alpha\mu}(\partial)
\]

(54)

where \(dx^\alpha = \sum_\beta dx_\beta \eta_{\beta\alpha}\) and \(k_{\mu\nu}, h_{\mu\nu}\) are formal power series in \(a_\mu\) with coefficients in the ring of differential operators \(\partial_\mu\). Differential forms \(dx_\mu\) satisfy the commutation relations

\[
[dx_\mu, x_\nu] = [dx_\mu, \partial_\nu] = 0 \quad \text{and} \quad \{dx_\mu, dx_\nu\} = 0.
\]

(55)

The algebra generated by \(x_\mu, \partial_\mu\) and \(dx_\mu\) is a Lie superalgebra graded by the degree of \(dx_\mu\) (\(\deg(x_\mu) = \deg(\partial_\mu) = 0\) and \(\deg(dx_\mu) = 1\)). The matrix \([h_{\mu\nu}]\) is assumed to be regular. Furthermore, we assume that \(k_{\mu\nu} \rightarrow \delta_{\mu\nu}\) and \(h_{\mu\nu} \rightarrow \eta_{\mu\nu}\) as \(a \rightarrow 0\), hence in the classical limit we have \(\hat{d} \rightarrow d = \sum_\alpha dx^\alpha \partial_\alpha\) and \(\xi_\mu \rightarrow dx_\mu\) as \(a \rightarrow 0\). Let us define an action of the exterior derivative on monomials \(f(\hat{x})\) by \(\hat{d} \cdot f = [\hat{d}, f]\). We note that in the classical limit we have \(d \cdot x_\mu = [d, x_\mu] = dx_\mu\). Hence, in the noncommutative case we require that \(\xi_\mu\) and \(\hat{d}\) are related by

\[
\xi_\mu = [\hat{d}, \hat{x}_\mu].
\]

(56)
Using realizations (54) and the fundamental relation (56) we want to construct a differential calculus on $M_\kappa$ that satisfies the following properties:

(i) $\hat{d}^2 = 0$,

(ii) one-forms anti-commute, $\{\xi_\mu, \xi_\nu\} = 0$ where $\{\xi_\mu, \xi_\nu\} = \xi_\mu \xi_\nu + \xi_\nu \xi_\mu$,

(iii) $\hat{d}$ satisfies the undeformed Leibniz rule

$$\hat{d} \cdot (\hat{f} \hat{g}) = (\hat{d} \cdot \hat{f}) \hat{g} + \hat{f} (\hat{d} \cdot \hat{g}) \quad (57)$$

where $\hat{f}$ and $\hat{g}$ are monomials in $\hat{x}_\mu$.

(iv) the commutator $[\xi_\mu, \hat{x}_\nu]$ is closed in the vector space spanned by one-forms alone,

$$[\xi_\mu, \hat{x}_\nu] = \sum_{\lambda=0}^{n-1} i K^\lambda_{\mu\nu} \xi_\lambda, \quad K^\lambda_{\mu\nu} \in \mathbb{R}. \quad (58)$$

We note that the commutator $[\xi_\mu, \hat{x}_\nu]$ depends on the realizations of $\hat{x}_\mu$ and $\xi_\mu$, and need not be closed in $\xi_\mu$.

A generalization of the above construction to higher-order forms was presented in detail in Ref. [34]. Here we only state that a $k$-form is a finite linear combination of monomials in $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1}$ and $\xi_0, \xi_1, \ldots, \xi_{n-1}$ such that there are precisely $k$ one-forms in each monomial. One can extend $\hat{d}$ to a linear map $\hat{d}: \hat{\Omega}^k \to \hat{\Omega}^{k+1}$ where $\hat{\Omega}^k$ is the space of $k$-forms. In general a $k$-form cannot be written such that all $\xi_\mu$'s are placed to the far right unless Eq. (58) holds. If this is true, than any $k$-form is a linear combination of forms of the type $\hat{f}_p(\hat{x})\xi_{\mu_1}\xi_{\mu_2} \ldots \xi_{\mu_p}$, $0 \leq p \leq k$. Furthermore, in this case one can define an extended star-product of (classical) differential forms [34].

Relation (56) is equivalent to a system of PDE’s relating the functions $k_{\mu\nu}, h_{\mu\nu}$ and $\phi_{\mu\nu}$. Solutions of such a system in different realizations were discussed in Ref. [34]. Without any further requirements on $k_{\mu\nu}$ and $h_{\mu\nu}$ the exterior derivative and one-forms satisfy the properties (i)-(iii). We note that consistency of Eqs. (1) and (56) requires that $\xi_\mu$ and $\hat{x}_\mu$ satisfy the compatibility condition

$$[\xi_\mu, \hat{x}_\nu] - [\xi_\nu, \hat{x}_\mu] = i (a_\mu \xi_\nu - a_\nu \xi_\mu). \quad (59)$$

This condition places certain restrictions on the realizations of $\xi_\mu$. For example, $h_{\mu\nu} = \delta_{\mu\nu}$ is not an admissible realization since in this case $[\xi_\mu, \hat{x}_\nu] = 0$ for all $\mu, \nu = 0, 1, \ldots, n-1$ contradicting Eq. (59).

Let us consider condition (iv). In general, $K^\lambda_{\mu\nu}$ is a formal power series in $\partial_\mu$ and it depends on the realizations of $\hat{x}_\mu$ and $\xi_\mu$. Using Eq. (59) one can decompose $K^\lambda_{\mu\nu}$ into
symmetric and antisymmetric parts

\[ K^\lambda_{\mu\nu} = A^\lambda_{\mu\nu} + S^\lambda_{\mu\nu}, \]

where \( A^\lambda_{\mu\nu} = \frac{1}{2}(a_\mu \delta_{\nu\lambda} - a_\nu \delta_{\mu\lambda}) \) and

\[ S^\lambda_{\mu\nu} = -\frac{i}{2} \sum_{\alpha, \beta = 0}^{n-1} h^{-1}_{\lambda\alpha} \left( \frac{\partial h_{\mu\alpha}}{\partial \beta} \phi_{\beta\nu} + \frac{\partial h_{\alpha\nu}}{\partial \beta} \phi_{\beta\mu} \right). \]

Here \( h^{-1}_{\mu\nu} \) denotes the \((\mu, \nu)\) element of the inverse matrix \([h_{\mu\nu}]^{-1}\). Thus, in order to satisfy condition (iv) we need to find \( h_{\mu\nu} \) such that the symmetric part \( S^\lambda_{\mu\nu} \) is constant. Solving the above problem in full generality is fairly complicated. However, by way of a concrete example we show that such solutions exist. For a given noncovariant realization of \( \hat{x}_\mu \) we will construct \( \hat{d} \) such that the one-forms given by Eq. (56) have the desired properties.

Assume the following Ansatz for \( \hat{d} \):

\[ \hat{d} = -dx_0 \partial_0 K_1(A) + \left( \sum_{k=1}^{n-1} dx_k \partial_k \right) K_2(A), \quad A = -ia_0 \partial_0. \]  

Using the realization (33)–(34) for \( \hat{x}_\mu \) we find

\[ \xi_0 = [\hat{d}, \hat{x}_0] = dx_0(AK'_1 + K_1)\psi + ia_0 \left( \sum_{k=1}^{n-1} dx_k \partial_k \right) (\psi K'_2 + \gamma K_2), \]

\[ \xi_i = [\hat{d}, \hat{x}_i] = dx_i K_2 \varphi, \]

where \( K'_i = \frac{dK_i}{dA} \). We want to find \( K_1 \) and \( K_2 \) such that \( \xi_0 = dx_0 Z^{-s} \) and \( \xi_i = dx_i Z^{-t} \) for some \( s, t \in \mathbb{R} \), where the shift operator \( Z \) is given by Eq. (35). From Eqs. (63) and (64) we obtain a system of differential equations

\[ (AK'_1 + K_1)\psi = Z^{-s}, \quad \psi K'_2 + \gamma K_2 = 0, \quad \varphi K_2 = Z^{-t}. \]

Since \( \gamma = \psi \varphi' / \varphi + 1 \), the last two equations are compatible if and only if \( t = 1 \). Hence,

\[ K_2(A) = \frac{Z^{-1}}{\varphi(A)}. \]

Solving the differential equation for \( K_1 \) and taking into account the initial condition \( \lim_{a_0 \to 0} K_1(A) = 1 \) yields

\[ K_1(A) = \frac{1 - Z^{-s}}{sA}, \quad s \neq 0. \]

In the limit \( s \to 0 \) the solution is given by

\[ K_1(A) = \frac{1}{A} \int_0^A \frac{dt}{\psi(t)}. \]
Thus, we obtain a one-parameter family of exterior derivatives

\[ \hat{d} = -dx_0 \partial_0 + \left( \sum_{k=1}^{n-1} dx_k \partial_k / (sA) \right) Z^{-1}. \]  

(69)

and corresponding one-forms

\[ \xi_0 = dx_0 Z^{-s}, \quad \xi_i = dx_i Z^{-1}. \]  

(70)

Since the shift operator satisfies \([Z^\alpha, \hat{x}_\mu] = \alpha_i a_\mu Z^\alpha, \alpha \in \mathbb{R}\), it follows that the commutators \([\xi_\mu, \hat{x}_\nu]\) are closed:

\[ [\xi_0, \hat{x}_0] = -sia_0 \xi_0, \quad [\xi_i, \hat{x}_0] = -ia_0 \xi_i, \]  

(71)

\[ [\xi_0, \hat{x}_j] = 0, \quad [\xi_i, \hat{x}_j] = 0. \]  

(72)

We point out that the algebra generated by \(\hat{x}_\mu\) and \(\xi_\mu\) is closed for all noncovariant realizations \((33)–(34)\) and that all graded Jacobi identities for this algebra hold.

Let us now consider the commutation relations for \(M_{\mu\nu}\) and \(\xi_\lambda\). Using the natural realization \((27)\) one can express the Lorentz generators as

\[ M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) Z \]  

(73)

which yields

\[ [M_{\mu\nu}, \xi_\lambda] = [\hat{x}_\mu, \xi_\lambda] D_\nu Z - [\hat{x}_\nu, \xi_\lambda] D_\mu Z. \]  

(74)

Thus, one may use the commutation relations \((71)–(72)\) to find

\[ [M_{i0}, \xi_0] = -sia_0 \xi_0 \frac{\partial_i}{\varphi(A)}, \quad [M_{ij}, \xi_0] = 0, \]  

(75)

\[ [M_{i0}, \xi_k] = -ia_0 \xi_k \frac{\partial_i}{\varphi(A)}, \quad [M_{ij}, \xi_k] = 0. \]  

(76)

The algebra generated by \(\hat{x}_\mu, \xi_\mu\) and \(M_{\mu\nu}\) is not closed because the commutator \([M_{i0}, \xi_\mu]\) is given in terms of an infinite power series in \(\partial_\mu\). However, since \(\hat{x}_\mu, M_{\mu\nu}\) and \(\xi_\mu\) belong to an associative algebra generated by \(x_\mu, \partial_\mu\) and \(dx_\mu\) all graded Jacobi identities hold.

Thus, one can define an action of \(M_{\mu\nu}\) on the differential algebra \(\mathcal{D}_\kappa\) defined by relations \((30), (71)\) and \((72)\) as follows. First define the action of \(\hat{x}_\mu\) and \(\xi_\mu\) on \(\mathcal{D}_\kappa\) simply by \(\hat{x}_\mu \triangleright f(\hat{x}, \xi) = \hat{x}_\mu f(\hat{x}, \xi)\) and \(\xi_\mu \triangleright f(\hat{x}, \xi) = \xi_\mu f(\hat{x}, \xi)\) for all \(f(\hat{x}, \xi) \in \mathcal{D}_\kappa\). Furthermore, define \(M_{\mu\nu} \triangleright 1 = 0\). Since the commutator \([M_{\mu\nu}, \xi_\lambda]\) depends on \(\partial_\mu\) we also need to set \(\partial_\mu \triangleright 1 = 0\). Now we define the action of \(M_{\mu\nu}\) on \(\mathcal{D}_\kappa\) by

\[ M_{\mu\nu} \triangleright f(\hat{x}, \xi) = (M_{\mu\nu} f(\hat{x}, \xi)) \triangleright 1. \]  

(77)
The action (77) is completely specified by the action of \( \hat{x}_\mu, \xi_\mu, M_{\mu\nu} \) and \( \partial_\mu \), and the commutation relations between \( M_{\mu\nu}, \hat{x}_\mu \) and \( \xi_\mu \). Also, due to the Jacobi identities the action (77) is compatible with the commutation relations (71)–(72). Since \( M_{\mu\nu} \triangleleft 1 = 0 \), Eq. (77) can be written in equivalent form

\[ M_{\mu\nu} \triangleright f(\hat{x},\xi) = \left[ M_{\mu\nu}, f(\hat{x},\xi) \right] \triangleleft 1. \tag{78} \]

When the action is restricted to monomials in \( \hat{x}_\mu \), due to commutation relations (3) one obtains a polynomial in \( \hat{x}_\mu \), written symbolically

\[ M_{\mu\nu} \triangleright f(\hat{x}) = g(\hat{x}), \tag{79} \]

and the result is clearly independent of realizations. For example, the action of \( M_{\mu\nu} \) on \( \hat{x}_\mu \) yields

\[ M_{i0} \triangleright \hat{x}_0 = -\hat{x}_i, \quad M_{ij} \triangleright \hat{x}_0 = 0, \tag{80} \]

\[ M_{i0} \triangleright \hat{x}_k = -\delta_{ik}\hat{x}_0, \quad M_{ij} \triangleright \hat{x}_k = \delta_{jk}\hat{x}_i - \delta_{ik}\hat{x}_j. \tag{81} \]

The above result agrees with the action of the Lorentz generators on \( \kappa \)-Minkowski space obtained by Sitarz [30]. Furthermore,

\[ M_{\mu\nu} \triangleright f(\xi) = 0 \tag{82} \]

for any monomial \( f(\xi) \) in \( \xi_\mu \). Since a basis of \( D_\kappa \) consists of the monomials

\[ \hat{x}_0^{k_0} \hat{x}_1^{k_1} \ldots \hat{x}_{n-1}^{k_{n-1}} \xi_0^{l_0} \xi_1^{l_1} \ldots \xi_{n-1}^{l_{n-1}}, \quad k_i \geq 0, \quad l_i = 0, 1, \tag{83} \]

it suffices to calculate the action of \( M_{\mu\nu} \) on the product of monomials \( f(\hat{x})g(\xi) \). Using Eqs. (79) and (82) one can show that

\[ M_{\mu\nu} \triangleright (f(\hat{x})g(\xi)) = (M_{\mu\nu} \triangleright f(\hat{x}))g(\xi) \tag{84} \]

which is again independent of realization. Thus, the action of \( M_{\mu\nu} \) on the entire differential algebra \( D_\kappa \) is realization independent.

The action (84) can be expressed in terms of the quantum adjoint action

\[ \text{ad}(M_{\mu\nu})(f(\hat{x})) = \sum M_{\mu\nu(1)} f(\hat{x}) S(M_{\mu\nu(2)}) \tag{85} \]

where we use the Sweedler notation for the coproduct \( \Delta M_{\mu\nu} = \sum M_{\mu\nu(1)} \otimes M_{\mu\nu(2)} \). Since the rotation generators are undeformed (cf. Eqs. (43) and (49)), we have

\[ \text{ad}(M_{ij})(f(\hat{x})) = [M_{ij}, f(\hat{x})]. \tag{86} \]
The coproduct and antipode for boosts given by Eqs. (44) and (48) yield
\begin{align}
\text{ad}(M_0)(f(\hat{x})) &= M_0 f(\hat{x}) - Z f(\hat{x}) Z^{-1} M_0 \\
+ ia_0 Z f(\hat{x}) Z^{-1} \sum_{j=1}^{n-1} \frac{\partial_j}{\varphi(A)} M_{ij} - ia_0 \sum_{j=1}^{n-1} \frac{\partial_j}{\varphi(A)} f(\hat{x}) M_{ij},
\end{align}
where we have used \( p_\mu = -i \partial_\mu \). If \( f(\hat{x}) \) is a monomial of degree \( m \), then Eq. (20) implies
\[ Z f(\hat{x}) Z^{-1} = f(\hat{x} + i a) = f(\hat{x}) + r_{m-1}(\hat{x}) \]
where \( r_{m-1}(\hat{x}) \) is a monomial of degree \( m - 1 \). Substituting this into Eq. (87) we obtain
\begin{align}
\text{ad}(M_0)(f(\hat{x})) &= [M_0, f(\hat{x})] - r_{m-1}(\hat{x}) M_0 \\
+ ia_0 (f(\hat{x}) + r_{m-1}(\hat{x})) \sum_{j=1}^{n-1} \frac{\partial_j}{\varphi(A)} M_{ij} - ia_0 \sum_{j=1}^{n-1} \frac{\partial_j}{\varphi(A)} f(\hat{x}) M_{ij},
\end{align}
Since \( \partial_\mu \triangleright 1 = 0 \) and \( M_{\mu\nu} \triangleright 1 = 0 \), it follows that
\[ M_{\mu\nu} \triangleright f(\hat{x}) = [M_{\mu\nu}, f(\hat{x})] \triangleright 1 = \text{ad}(M_{\mu\nu})(f(\hat{x})) \triangleright 1,
\]
and consequently
\[ M_{\mu\nu} \triangleright f(\hat{x}) g(\xi) = (\text{ad}(M_{\mu\nu})(f(\hat{x})) \triangleright 1) g(\xi). \]

A few comments about the action (77) are in order. Although the action is independent of the realizations introduced in section 2, this is generally not true if the commutator \([M_{\mu\nu}, \hat{x}_\lambda]\) given by Eq. (3) is modified. The action (77) is different from the one introduced by Sitarz [30], but both actions agree when restricted to the coordinates of the \( \kappa \)-Minkowski space \( \mathcal{M}_\kappa \) (c.f. Eqs. (80)–(81)). We remarked earlier that when the action in [30] is extended in a covariant way from \( \mathcal{M}_\kappa \) to the differential algebra \( \mathcal{D}_\kappa \) one obtains a contradiction with the mixed Jacobi identity for \( \hat{x}_\mu, \hat{x}_\nu \) and \( \xi_\lambda \). In order to resolve the contradiction Sitarz introduced an additional one-form \( \phi \) which is Lorentz invariant, \( M_{\mu\nu} \triangleright \phi = 0 \), thus making the space of one-forms \( (n + 1) \)-dimensional. In Ref. [33] the same problem was resolved by extending the \( \kappa \)-Poincaré algebra with a dilatation operator and using a different coproduct for \( M_{\mu\nu} \). In this case the commutator \([M_{\mu\nu}, \hat{x}_\lambda]\) is different from the one in Eq. (3). The coproduct for \( P_\mu \) in Ref. [33] corresponds to the coproduct for \( \partial_\mu \) in the left-covariant realization in Ref. (87) and to the special case of the noncovariant realization (33)–(34) with \( \varphi = \psi = 1 - A \). We note that the requirement \([M_{\mu\nu}, \hat{d}] = 0\) is equivalent to the bicovariance requirement in Ref. [30]. In our work, however, \([M_{\mu\nu}, \hat{d}] \neq 0\) for any choice of \( \varphi \) and \( \psi \) in the noncovariant realization. We also note that this is true even in the classical limit since \([M_{\mu\nu}, \hat{d}] \to dx_\mu \partial_\nu - dx_\nu \partial_\mu \neq 0\) as \( a \to 0 \).
We conclude the discussion with the following remarks. In this paper we have general-
ized the realizations of \( \hat{x}_\mu \) and \( M_{\mu\nu} \) introduced originally in Ref. [35]. We have given
the Hopf algebra structure of the \( \kappa \)-Poincaré algebra \( P_\kappa \) when the deformation of the al-
gebra sector is given by Eqs. (14)-(16). In addition, we found realizations of the exterior
derivative \( \hat{d} \) and one-forms \( \xi_\mu \) such that \([\xi_\mu, \hat{x}_\nu]\) is closed in the vector space spanned by
one-forms alone. The requirement that both commutators \([M_{\mu\nu}, \hat{x}_\lambda]\) and \([\xi_\mu, \hat{x}_\nu]\) be closed
is not compatible with Lorentz invariance of \( \hat{d} \) since \([M_{\mu\nu}, \hat{d}] \neq 0 \). A different construction
of the Lorentz generators satisfying the invariance condition \([M_{\mu\nu}, \hat{d}] = 0 \) will be presented
elsewhere. Full treatment of this construction as well as its applications to Snyder space
[42]-[45], scalar field theory, statistics and twist operators [33], [36], [39], [46], [47] will be
given in future work.

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(S. Meljanac) Rudjer Bošković Institute, Bijenička cesta b.b., 10000 Zagreb, Croatia
E-mail address: meljanac@irb.hr

(S. Krešić-Jurić) Faculty of Natural and Mathematical Sciences, University of Split, Teslina 12, 21000 Split, Croatia
E-mail address: skresic@pmfst.hr