Quantum Circuit for Calculating Mean Values Via Grover-like Algorithm

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Abstract

In this paper, we give a quantum circuit for calculating the mean value of a function $A(x^n) \in \mathbb{C}$, where $x^n \in \{0, 1\}^n$. Known classical algorithms for calculating the mean value of a structureless function $A(x^n)$ take $O(2^n)$ steps. Our quantum algorithm is based on a Grover-like algorithm and it takes $O(\sqrt{2^n})$ steps. Our algorithm differs significantly from previously proposed quantum algorithms for calculating the mean value of a function via Grover’s algorithm.
1 Introduction

In this paper, we give a quantum circuit for calculating the mean value of a function \( A(x^n) \in \mathbb{C} \), where \( x^n \in \text{Bool}^n \). Classical algorithms for calculating the mean value of a structureless function \( A(x^n) \) take \( O(2^n) \) steps. Our quantum algorithm is based on the original Grover’s algorithm (see Ref.[1]) or some variant thereof (such as AFGA, described in Ref.[2]), and it takes \( O(\sqrt{2^n}) \) steps.

Previous papers (see Refs.[1, 3, 4]) have proposed algorithms for finding the mean value of a function via Grover’s algorithm. Our algorithm differs significantly from theirs. One big difference is that our algorithm encodes the final answer in the amplitude of a state whereas theirs encodes it in the quantum numbers of a state. A good analogy is to say that we use something akin to amplitude modulation (AM radio, where the signal is encoded in the amplitude) whereas they use something akin to frequency modulation (FM radio, where the signal is encoded in the frequencies).

This paper assumes that the reader has already read most of Ref.[5] by Tucci. Reading that previous paper is essential to understanding this one because this paper applies AM techniques (what we call targeting two hypotheses and blind targeting) described in that previous paper.

2 Notation and Preliminaries

Most of the notation that will be used in this paper has already been explained in previous papers by Tucci. See, in particular, Sec.2 (entitled “Notation and Preliminaries”) of Ref.[5].

3 Quantum Circuit For Calculating Mean Values

In this section, we will give a quantum circuit for calculating the mean value of a probability amplitude \( A(x^n) \) where \( x^n \in \text{Bool}^n \). Our algorithm can also be used to find the mean value of more general functions using the method given in Appendix C of Ref.[5].

For \( x^n \in \text{Bool}^n \), and a normalized \( n \)-qubit state \( |\psi\rangle \), define

\[
|\psi\rangle_{\alpha^n} = \sum_{x^n} A(x^n) |x^n\rangle_{\alpha^n} , \tag{1}
\]

and

\[
\overline{A} = \frac{1}{2^n} \sum_{x^n} A(x^n) . \tag{2}
\]

Note that this function \( A() \) is not completely general since
\[ \sum_{x^n} |A(x^n)|^2 = 1. \] (3)

We will assume that we know how to compile \(|\psi\rangle_{\alpha^n}\) (i.e., that we can construct it starting from \(|0^n\rangle_{\alpha^n}\) using a sequence of elementary operations. Elementary operations are operations that act on a few (usually 1, 2 or 3) qubits at a time, such as qubit rotations and CNOTS.) Multiplexor techniques for doing such compilations are discussed in Ref.\[6\]. If \(n\) is very large, our algorithm will be useless unless such a compilation is of polynomial efficiency, meaning that its number of elementary operations grows as \(\text{poly}(n)\).

For concreteness, we will use \(n = 3\) henceforth in this section, but it will be obvious how to draw an analogous circuit for arbitrary \(n\).

![Figure 1: Circuit for generating \(|s\rangle\) used in AFGA to calculate mean value of \(A(x^3)\).](image)

We want all horizontal lines in Fig. I to represent qubits. Let \(\alpha = \alpha^3\), \(\beta = \beta^3\). Define

\[ T(\alpha, \beta) = \prod_{j=0}^{2} \{H(\alpha_j)H(\beta_j)\} , \] (4)

\[ \pi(\alpha) = \prod_{j=0}^{2} P_0(\alpha_j) , \] (5)

and

\[ \pi(\beta) = \prod_{j=0}^{2} P_0(\beta_j) . \] (6)
Our method for calculating the mean value of $A(x^3)$ consists of applying the algorithm AFGA\(^1\) of Ref.[2] in the way that was described in Ref.[5], using the techniques of targeting two hypotheses and blind targeting. As in Ref.[5], when we apply AFGA in this section, we will use a sufficient target $|0\rangle_\omega$. All that remains for us to do to fully specify our circuit for calculating the mean value of $A(x^3)$ is to give a circuit for generating $|s\rangle$.

A circuit for generating $|s\rangle$ is given by Fig. 1. Fig.1 is equivalent to saying that

$$
|s\rangle_{\mu,\nu,\omega} = \sigma_X(\omega)\pi(\beta)\pi(\alpha) \frac{1}{\sqrt{2}} \begin{bmatrix}
T(\alpha, \beta) |\psi\rangle_\alpha |0^3\rangle_\beta \\
|1\rangle_\gamma \\
|1\rangle_{\mu_0} \\
|1\rangle_\omega
\end{bmatrix} + 
\begin{bmatrix}
|\psi\rangle_\alpha |0^3\rangle_\beta \\
|0\rangle_\gamma \\
|0\rangle_{\mu_0} \\
|1\rangle_\omega
\end{bmatrix}.
$$

(7)

Claim 1

$$
|s\rangle_{\mu,\nu,\omega} = z_1 |\psi_1\rangle_{\mu} + z_0 |\psi_0\rangle_{\mu} + |\chi\rangle_{\mu,\nu},
$$

for some unnormalized state $|\chi\rangle_{\mu,\nu}$, where

$$
|\psi_1\rangle_{\mu} = |0^3\rangle_{\alpha} |1\rangle_{\mu_0},
|\psi_0\rangle_{\mu} = |0^3\rangle_{\alpha} |0\rangle_{\mu_0},
|0\rangle_{\nu} = |0^3\rangle_{\beta} |1\rangle_\gamma,
$$

(9)

$$
z_1 = \frac{1}{\sqrt{2}} \left[ \frac{1}{2^3} \sum_{x^3} \langle x^3 | \psi \rangle \right],
$$

(10)

$$
z_0 = \frac{1}{\sqrt{2}} \langle 0^3 | \psi \rangle,
$$

(11)

$$
\frac{|z_1|}{|z_0|} = \sqrt{\frac{P(1)}{P(0)}}.
$$

(12)

proof:

Recall that for any quantum systems $\alpha$ and $\beta$, any unitary operator $U(\beta)$ and any projection operator $\pi(\alpha)$, one has

$$
U(\beta)\pi(\alpha) = (1 - \pi(\alpha)) + U(\beta)\pi(\alpha).
$$

(13)

\(^1\)As discussed in Ref.[5], we recommend the AFGA algorithm, but Grover’s original algorithm (see Ref.[1]) or any other Grover-like algorithm will also work here, as long as it drives a starting state $|s\rangle$ to a target state $|t\rangle$. 

4
Applying identity Eq. (13) with $U = \sigma_X(\omega)$ yields:

$$|s\rangle = \sigma_X(\omega)\pi(\beta)\pi(\alpha)|s'\rangle$$  \hspace{1cm} (14)

$$= \sigma_X(\omega)\pi(\beta)\pi(\alpha)|s'\rangle + |\chi\rangle_{\mu,\nu} |1\rangle_{\omega}$$  \hspace{1cm} (15)

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \pi(\beta)\pi(\alpha)T(\alpha, \beta) |\psi\rangle_{\alpha} [0^3]_{\beta} & A(0^3) |0^3\rangle_{\alpha} \\ |0\rangle_{\gamma} & |0\rangle_{\gamma} \end{bmatrix} + |\chi\rangle_{\mu,\nu} |1\rangle_{\omega}.$$  \hspace{1cm} (16)

Applying identity Eq. (13) with $U = \sigma_X(\beta_j)$ yields:

$$\pi(\beta)\pi(\alpha)T(\alpha, \beta) |\psi\rangle_{\alpha} [0^3]_{\beta} = |0^3\rangle_{\alpha} \frac{1}{2^3} \sum_{x^3} A(x^3).$$  \hspace{1cm} (17)

QED

Note that the amplitude $z_0$ for the null hypothesis is $|\psi\rangle$ dependent. This is contrary to the Mobius Transform algorithm of Ref. [7] where the $z_0$ is $|\psi^-\rangle$ independent.

Note also that for this method of finding the mean value of $A(x^n)$ to work well, $\overline{A}$ and $A(0^n)$ must be of “comparable” size, for if $|z_0|^2 << |z_1|^2$, then, by Eq. (12), $P(0) << P(1)$, and it will take an unreasonable amount of time to get a bunch of null events. Of course, the circuit for generating $|s\rangle$ can be changed easily so that $z_0$ is proportional to $A(y^n)$ instead of $A(0^n)$, where $y^n$ is any element of $\text{Bool}^n$.

References

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