Cut Diagrams for High Energy Scatterings

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Abstract

A new approach is introduced to study QCD amplitudes at high energy and comparatively small momentum transfer. Novel cut diagrams, representing resummation of Feynman diagrams, are used to simplify calculation and to avoid delicate cancellations encountered in the usual approach. Explicit calculation to the 6th order is carried out to demonstrate the advantage of cut diagrams over Feynman diagrams.

1 Introduction

The rapidity-gap events recently observed at HERA [1] revived the community’s interest in Regge-pole description of high energy scattering. The word ‘Pomeron’ seldom heard in recent years is once again found in the lexicon of experimentalists. It is perhaps then a good time to have a new look at the connection between perturbative QCD and Regge pole. Non-perturbative effect may be important for momentum transfer of the order of $\Lambda_{QCD}$ or smaller, but we shall avoid it by going to a larger momentum transfer if necessary.

It was proposed by Low and Nussinov [2] some years ago that the Pomeron may simply reflect a two-gluon exchange in QCD. Diagrammatic calculations have been carried out [4, 7, 8] in the leading log approximation to substantiate this proposal and to study other aspects of the high energy near forward scattering amplitudes. We shall follow the classical approach of implementing physical ($s$-channel) unitarity by summing up Feynman diagrams in the Feynman gauge [3, 4]. However, other approaches are available. The use of physical gauge and dispersion relation [5] can sometimes make things more transparent. One can also emphasize on the complex angular momentum aspect by concentrating on $t$-channel towers and their interactions as guided by $t$-channel unitarity [6]. We shall speak no further of these alternate approaches except to remark that our cut diagram method, to be discussed below, can be thought of an intermediate link between the classical Feynman diagram sum and the dispersion relation approaches.

Due to the gauge and non-abelian nature of the theory, perturbative calculations are lengthy and complicated. Even by ignoring self energy and vertex corrections, as well as renormalization effects on the grounds that they will not alter the qualitative nature of high energy scattering we are trying to learn, even by ignoring quark pair productions as a first step, a complete QCD calculation can be carried out only up to the 6th order. In the much...
simpler case of QED, an 8th order calculation \[9\] was reported to have taken sixteen months and two thousand pages to complete. In the case of QCD, it is much worse, a complete calculation of the 8th order is not available, though partial calculations have been carried out \[7\].

What makes the calculation so complicated is the large number of diagrams that has to be tackled, and the inevitable cancellations between them. The computation of each diagram is fairly straightforward, though somewhat lengthy at higher orders. After the individual diagrams are calculated, care must also be exercised to add them up because of the presence of many delicate cancellations. Leading-log dependences on energy get subtracted away, complicated functions of momentum transfer also disappear. What emerges at the end is a product that is surprisingly simple. To the extent that it has been verified, high energy near forward scattering is described by multiple reggeized gluon exchange, supplemented by elementary gluon production off the reggeons and by s-channel unitarity \[3, 7\].

This outcome has been verified up to the 6th order \[3, 4, 5\]. As mentioned above, many delicate cancellations take place in the sum to enable this simple picture to emerge. A complete calculation in the 8th order is not available, but if one assumes these cancellations which take place up to the 6th order also occur in the 8th and higher orders, then this reggeized picture has been verified up to the 10th order \[7\]. This gives a strong support to the conjecture that it is true to all perturbative orders.

While the conjecture is attractive, it is impossible to verify or refute without a new method to simplify the complicated calculations. We report in this article such a new method, in which high energy scatterings are computed via cut diagrams rather than the normal Feynman diagrams. These cut diagrams are not the Cutkosky cut diagrams. The sum of all cut diagrams here is equal to the sum of all Feynman diagrams, and not just their discontinuities. These cut diagrams can be regarded as a resummation of the Feynman diagrams in which many of the delicate cancellations encountered in the latter have been built in and explicitly avoided. Consequently, the simplicity of the final sum is revealed already in individual cut diagrams, and not masked by terms to be cancelled as is the case with Feynman diagrams. Besides, individual cut diagrams are easier to calculate than individual Feynman diagrams.

In this paper we shall introduce the formalism of cut diagrams, as well as an explicit calculation to the 6th order to demonstrate its effectiveness. We shall discuss quark-quark scattering throughout but allow the quarks to carry any \(SU(N)\) color charge. The result should be equally valid for gluon-gluon and other scatterings because high energy processes are insensitive to the spins of the colliding particles. We will study pure \(SU(N)\) QCD and ignore the production of quark pairs. If one is bothered by considering quark-quark scattering while ignoring quark productions, one can consider instead gluon-gluon scattering, and as mentioned above, the result would be the same.

In a subsequent paper \[11\] we shall use the cut diagrams to discuss the reggeization conjecture in higher orders.

In Sec. 2 the result of Feynman diagram calculations to \(O(g^6)\) is reviewed. How these diagrams combine to give a sum much simpler than the individuals will be discussed in some
Figure 1: Quark-quark scattering in QCD up to the 6th order. The thick lines at the top and bottom of each diagram are the fermion lines, and the thin lines are gluon lines.

detail. In Sec. 3, we begin to examine more carefully these cancellations by using sum rules. In Sec. 4, the method of cut diagrams is explained, and in Sec. 5, this method is applied to 4th and 6th order calculations to demonstrate the savings effected by this method.

2 Elastic scattering up to $O(g^6)$

Leading-log calculations in QCD, at high c.m. energy $\sqrt{s}$ and comparatively small momentum transfer $\sqrt{-t} = \sqrt{\Delta^2}$, are discussed, among others[13], in the book of Cheng and Wu [3]. Citations to the original literature can also be located there. We review in this section the calculation of quark-quark scattering to the 6th order done in this standard way.

The relevant Feynman diagrams are given in Fig. 1. The 2nd order diagram is labelled A, the 4th order diagrams labelled B1 and B2, and the 6th order diagrams C1 to C21. This last labelling is identical to the ones used in Fig. 12.7 of Ref. [3][11].

Under the interchange of the Mandelstam variables $s = (p_1 + p_2)^2$ and $u = (p_1 - p_2)^2$, the spacetime parts of these diagrams are either self conjugate, or one goes into another. For example, A$\leftrightarrow$A, B2$\leftrightarrow$B1, C2$\leftrightarrow$C1, C15$\leftrightarrow$C16, C17$\leftrightarrow$C19, and C18$\leftrightarrow$C20, under $s\leftrightarrow u$.

Normalizing the Dirac spinors to $\bar{\psi}\psi = 1$, and designating the fermion mass and gauge coupling constant as $m$ and $g$, with $\beta = g^2/2\pi$, the T-matrix element $T = (g^2 s/2m^2)\mathcal{M}$ receives the following contributions from the individual diagrams to $\mathcal{M} [3,4][12]$: 

$$A = -I_1 \cdot G_1$$
$$B_1 = -\beta \ln \left( se^{-\pi t} \right) I_2 \cdot G_2$$
$$B_2 = +\beta (\ln s) I_2 \cdot (G_2 + cG_1)$$
\[ \bar{C}_1 = \beta^2 \ln^2 \left( s e^{-\pi i} \right) \left[ \frac{1}{2} \Delta^2 I_2^2 - J_2 I_2 \right] \cdot G_3 \]
\[ \bar{C}_2 = -\beta^2 (\ln s) \left[ \frac{1}{2} \Delta^2 I_2^2 - J_2 I_2 \right] \cdot (G_3 + c^2 G_1) \]
\[ C_3 = \beta^2 \left( \ln^2 \left( s e^{-\pi i} \right) - \ln^2 s \right) \frac{1}{4} J_2 I_2 \cdot (G_3 - c G_2) \]
\[ C_4 = C_5 = C_6 \]
\[ C_7 = -\beta^2 \ln \left( s e^{-\pi i} \right) \frac{1}{4} J_2 I_2 \cdot (-c G_2) \]
\[ C_8 = C_9 = C_{10} \]
\[ C_{11} = \beta^2 (\ln s) \frac{1}{4} J_2 I_2 \cdot (-c G_2 - c^2 G_1) \]
\[ C_{12} = C_{13} = C_{14} \]
\[ C_{15} = -\beta^2 (\ln s) 2 J_3 \cdot G_4 \]
\[ C_{16} = -\beta^2 (\ln s) 2 J_3 \cdot (G_4 - G_3 + 3c G_2 + c^2 G_1) \]
\[ C_{17} = \beta^2 (\ln s) (J_3 + \pi i I_3) \cdot (G_4 + c G_2) = C_{18} \]
\[ C_{19} = \beta^2 (\ln s) (J_3 - \pi i I_3) \cdot (G_4 - G_3 + 2c G_2) = C_{20} . \tag{2.1} \]

Part of \( C_{21} \) has been combined with \( C_1 \) to form \( \bar{C}_1 \), and the remaining part of \( C_{21} \) has been combined with \( C_2 \) to form \( \bar{C}_2 \). Note that the result for different diagrams is consistent with their \( s \leftrightarrow u \) character, under whose exchange the \( T \) amplitude responds with the swap \( se^{-\pi i} \leftrightarrow s \). See App. A and B for a brief discussion of how these amplitudes are computed.

The two-dimensional vector \( \Delta \) in the transverse direction is the momentum transfer. The functions \( J_n(\Delta) \) and \( I_n(\Delta) \) are defined as follows:

\[ I_1(\Delta) = \frac{1}{\Delta^2} \]
\[ I_n(\Delta) = \int \left( \prod_{i=1}^{n} \frac{d^2 q_i}{(2\pi)^2 q_i^2} \right) \frac{1}{q_i^2} \left( \sum_{i=1}^{n} q_i - \Delta \right) \]
\[ J_2(\Delta) = \int \frac{d^2 q_i}{(2\pi)^2 q_i^2} \frac{1}{q_i^2} \]
\[ J_3(\Delta) = \int \left( \prod_{i=1}^{3} \frac{d^2 q_i}{(2\pi)^2 q_i^2} \right) \ln \frac{q_2^2}{q_3^2} \cdot (2\pi)^2 \delta^2 \left( \sum_{i=1}^{n} q_i - \Delta \right) . \tag{2.2} \]

The functions \( I_2, I_3, J_2, J_3 \) are denoted respectively by \( I, I_1, K, I_2 \) in Ref. [3]. The infrared divergences of these integrals can be regulated by a mass, either put in by hand or via the Higgs mechanism. This regulation discussed in the literature [3–9] does not affect the following discussions so we shall ignore it. There is however also an ultraviolet divergence in the integral defining \( J_2(\Delta) \), but it turns out that this function disappears in the sum of the sixth order diagrams so it causes no trouble either.

The factors \( G_i \) in (2.1) are the color factors, with \( c = N/2 \) for \( SU(N) \) colors. As defined in [3], \( G_1, G_2, G_3, G_4 \) are respectively the color factors for the diagrams A, B1, C1, and C15.
They can be computed by an elegant graphical method \cite{3} from the \( SU(N) \) commutation relation and identities

\[ [t_a, t_b] = if_{abc} t_c , \quad f_{abc}f_{abd} = 2c\delta_{cd} , \quad i^3 f_{ady}f_{bed}f_{cge} = ci f_{abc} . \] (2.3)

Hence the combination of color factors \( G_i \) given in (2.1) for the various diagrams remain valid whatever the color of the quark is, although \( G_i \) themselves would be different for different color of the quark. See App. A for a brief discussion on the computation of the color factors.

The sum of all the terms in (2.1), from \( A \) to \( C \), is

\[
\mathcal{M} = \left[ \frac{1}{2} \Delta^2 \left[ 1 - \alpha \ln s + \frac{1}{2} \alpha^2 \ln^2 s \right] \right] \cdot G_1 + \frac{1}{2} g^2 i (I_2 - 2\beta c I_3 \ln s) \cdot G_2
\]

\[ + \ g^2 i \beta \ln s \left[ I_3 - \frac{1}{2} \Delta^2 I_2^2 \right] \cdot G_3 + g^4 \left( I_2^2 - \frac{1}{2} \Delta^2 I_2^2 \right) \cdot G_4 , \] (2.4)

with

\[ \overline{\alpha}(\Delta) \equiv \beta c \Delta^2 I_2(\Delta) . \] (2.5)

It is important to note the various cancellations taking place to make the sum (2.4) vastly simpler than the individual terms appearing in (2.1). For example,

1. In the fourth order, the leading term proportional to \( \ln s \) is cancelled out between \( B_1 \) and \( B_2 \) in the color amplitude proportional to \( G_2 \), though not in \( G_1 \).

2. In the sixth order, the leading \( \ln s \) contributions to \( G_4 \) from \( C_{15} \) to \( C_{20} \) also add up to zero. The expressions given in (2.1) are not accurate enough to deal with the subleading terms. The term in (2.4) proportional to \( G_4 \) is obtained separately from the eikonal formula.

3. As a result of these cancellations, the energy dependence and the \( SU(N) \) (or \( c \)) dependence of the \( G_1 \) amplitude is \( (g^2 c \ln s)^m \), and those of \( G_2, G_3, G_4 \) are respectively \( g^2 (g^2 c \ln s)^m \), \( g^2 (g^2 c \ln s)^m \), \( g^4 (g^2 c \ln s)^m \). These dependences can be summarized all at once by introducing a different notation for the color factors. We shall use the notation \( F_{i,j} \) to denote a color factor with \( i \) parallel vertical lines connecting the two fermions, and \( j \) parallel horizontal lines joining any two of the vertical gluon lines. We shall also write \( F_{i,0} \) simply as \( F_i \). The relations with the color factors \( G_i \) are \( G_1 = F_1, G_2 = F_2, G_3 = F_{2,1}, \) and \( G_4 = F_3 \). The \( g, c \) and \( \ln s \) dependences of \( F_{i,j} \) in (2.4) are then given by \( g^{2(i-1)} (g^2 c \ln s)^{m-i} c^{-j} \) for a diagram of order \( 2(m+i) \). We shall refer to such dependences as Regge-like, for the Reggeization of the scattering amplitude to be discussed later relies critically on this feature of the scattering amplitude. Note from (2.1) that contributions from individual diagrams are not Regge-like. Only the sum is.
4. Simplification in transverse-momentum dependences also occur in the sum. The simple integrals \( I_n \) survive, but the complicated integral \( J_3 \) and the divergent integral \( J_2 \) do not appear in the sum. This cancellation is highly nontrivial because both of them contribute different amounts to different color amplitudes. More specifically,

5. The function \( J_3(\Delta) \) appears in all the color amplitudes \( G_1, G_2, G_3 \) and \( G_4 \) in diagrams \( C_{15} \) to \( C_{20} \). Those in \( G_2, G_3, G_4 \) actually get cancelled out in the sum, but its presence in the \( G_1 \) amplitude survives. However, since this term is of order \( g^6 \ln s \), it is negligible compared to terms of order \( g^6 \ln^2 s \) appearing in the \( G_1 \) amplitudes of \( C_{22} \) and \( C_{11} \) to \( C_{14} \), it can be ignored in the leading-log result displayed in (2.4).

6. \( J_2(\Delta) \) appears in the color amplitudes \( G_1, G_2 \) and \( G_3 \) in individual diagrams \( C_1 \) to \( C_{14} \) and all these appearances get cancelled out.

As a result of these cancellations, \( \mathcal{M} \) acquires a very simple interpretation in terms of reggeized gluon exchanges. These exchanges are constructed in such a way to ensure \( s \)-channel unitarity [3,7].

Let us denote the reggeon propagator by

\[
R_1(\Delta, s) = \frac{1}{\Delta^2} \exp(-\alpha(\Delta) \ln s) .
\]

(2.6)

This reduces to the (transverse part of the) ordinary propagator \( I_1(\Delta) = \Delta^{-2} \) for small \( g^2 c \ln s \). Similarly, let us denote the reggeized version of \( I_n(\Delta) \) by

\[
R_n(\Delta, s) = \int \left( \prod_{i=1}^{n} \frac{d^2 q_i}{(2\pi)^2} R_1(q_i, s) \right) \cdot (2\pi)^2 \delta^2 \left( \sum_{i=1}^{n} q_i - \Delta \right) ,
\]

(2.7)

indicating the exchange of \( n \) reggeons. Then to order \( g^6 \) in \( \mathcal{T} \), we can write

\[
\mathcal{M} = -R_1(\Delta, s) \cdot \mathbf{F}_1 + i \frac{g^2}{2!} [R_2(\Delta, s) \cdot \mathbf{F}_2 + R_{2,1}(\Delta, s) \cdot \mathbf{F}_{2,1}] + \frac{g^4}{3!} R_3(\Delta, s) \cdot \mathbf{F}_3 .
\]

(2.8)

In other words, the \( \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_{2,1}, \) and \( \mathbf{F}_3 \) components looked precisely like diagrams A, B1, C1, and C15 respectively, but with the vertical gluons replaced by their reggeized version whose propagators are given in (2.6), and with all longitudinal-momentum integrations omitted. To interpret it this way for \( R_{2,1} \mathbf{F}_{2,1} \) we need to know the Lipatov-Dickinson vertex [8] describing how elementary gluons are produced and absorbed from the reggeized gluons.

This remarkable simplicity and regularity led to the conjecture [3, 7] that the reggeized formula (2.8), suitably generalized, is the correct high energy limit to all perturbative orders. This conjecture is very difficult to verify on account of the sheer complexity in higher order calculations. For QCD in the 8th order it is simply not manageable without simplifying assumptions. If one assumes all cancellations occurred up to \( O(g^6) \) will also occur in higher
orders, the final result can be extracted from a relatively small set of diagrams, then it is
reported that this reggeization conjecture is true to 8th and 10th orders \[7\]. Even so these
heroic calculations are so lengthy and complicated that to our knowledge the full details
have never been published.

3 High energy kinematics

We shall discuss in the next two sections a method of using cut diagrams to sum up the
Feynman diagrams, a method in which most of the cancellations discussed in the last section
are automatically built in. This simplification shortens the computations and makes it
possible to study higher order diagrams. In this paper we shall discuss how it simplifies the
calculations up to $O(g^6)$. In a subsequent paper \[10\] we shall discuss how it helps to verify
part of the reggeization conjecture to all orders. To prepare for the grounds for both, we
discuss here the relevant kinematical features of high energy scattering which enables this
new method to work.

We will assume the colliding beams in their c.m. system to be directed along the $z$
direction. In lightcone coordinates, $p_\pm = p^0 \pm p^3$, the components of a four-vector are
labelled in the order $p^\mu = (p_+, p_-, p_\perp)$, with the two-dimensional vector $p_\perp$ lying in the
transverse $x$–$y$ plane. In this notation, the incoming fermion momenta are $p_1 = (\sqrt{s}, 0, 0)$
and $p_2 = (0, \sqrt{s}, 0)$, in which their mass $m$ has been neglected. The outgoing fermion
momenta are approximately given by $p_1' = (\sqrt{s}, 0, \Delta)$ and $p_2' = (0, \sqrt{s}, -\Delta)$. See Fig. 1.

Suppose $n$ gluons are hooked up to the upper fermion line as shown in Fig. 2. The in itial
and final fermions are on-shell but the gluons can be off-shell, though with an amount of en-
ergy far less than $\sqrt{s}$. At high energy, the numerator of the propagator can be approximated
by

$$\gamma p = 2m \sum_\lambda u_\lambda(p_1) \bar{u}_\lambda(p_1)$$  \hspace{1cm} (3.1)

provided the Dirac spinors are normalized to $\bar{u}_\lambda(p) u_\lambda(p) = \delta_{\lambda\lambda'}$. With that, the dominant
current $\bar{u}_\lambda(p_1) \gamma^a u_\lambda(p_1)$ at high energy is just its translational part $\delta_{\lambda\lambda'} p_1^a / m$. This shows
that the spin content at high energy is unimportant. All that it does is to enforce helicity
conservation of the fermion, and to produce a factor $2p_1$ at each vertex together with an over-
all normalization factor of $1/2m$. For most of the discussions below we shall ignore this QED
factor and concentrate on the contribution from the denominators, \textit{i.e.}, the corresponding scalar theory.

The denominator of the $i$th inverse propagator is

$$(p_1 + \sum_{j=1}^i q_j)^2 - m^2 + i\epsilon \simeq s(\sum_{j=1}^i x_j + i\epsilon)$$  \hspace{1cm} (3.2)
where \( x_i = q_i - 1/\sqrt{s} \). The scalar amplitude in Fig. 2 is given by \( s^{-n} \) times

\[
a[12\cdots n] \equiv -2\pi i \delta \left( \sum_{j=1}^{n} x_j \right) - \prod_{i=1}^{n-1} \frac{1}{\sum_{j=1}^{i} x_j + i\epsilon} \cdot \prod_{i=1}^{n} \frac{1}{\sum_{j=1}^{i} x_{v_j} + i\epsilon}.
\] (3.3)

Note that a momentum conservation \( \delta \)-function for the negative components (together with an explicit factor \(-2\pi i\)) has been incorporated. In eq. (3.3) the ordering of the gluon lines from left to right is \([123\cdots n]\). If they are ordered differently, say \([v_1v_2\cdots v_n] \equiv V\), then the corresponding amplitude is

\[
a[v_1v_2\cdots v_n] \equiv a[V] \equiv -2\pi i \delta \left( \sum_{j=1}^{n} x_{v_j} \right) \cdot \prod_{i=1}^{n-1} \frac{1}{\sum_{j=1}^{i} x_{v_j} + i\epsilon} \cdot \prod_{i=1}^{n} \frac{1}{\sum_{j=1}^{i} x_{v_j} + i\epsilon}.
\] (3.4)

4 Sum Rules

It is possible to compute sums of Feynman diagrams without much of the delicate cancellations discussed in Sec. 2, by using the cut diagrams which we shall describe in this section and the next. The derivation of cut diagrams relies on two exact combinatorial formulas for the quantity \( a[V] \) in (3.4): the factorization formula and the multiple commutator formula derived in Ref. [14]. We shall discuss the former in this section, and the latter in the next section.

Consider an ordering of \( n_i \) gluon lines: \([v_1v_2\cdots v_{n_i}] \equiv V_i\). We shall use the notation \( \{V_1; V_2; \cdots; V_m\} \) to denote the set of all orderings of the \( M \equiv \sum_{i=1}^{m} n_i \) gluon lines, provided the relative orderings of lines within each \( V_i \) are maintained. The number of orderings in this set is given by the multinomial coefficient \( M!/\prod_{j=1}^{n} n_j! \). For example, if \( V_1 = [135], V_2 = [24], \) then \( \{V_1; V_2\} \equiv \{135; 24\} \) consists of the \( 5!/3!2! = 10 \) orderings \([13524],[13542],[13254],[13245],[12354],[12345],[12435],[21354],[21345],[21435], \) and \([24135]\).

We shall use the notation

\[
a\{V_1; V_2; \cdots; V_m\} = \sum_{V \in \{V_1; V_2; \cdots; V_m\}} a[V]
\] (4.1)
to denote the sum of all amplitudes for the gluon orderings in the set. The factorization formula \[14\] then states that

\[ a\{V_1; V_2; \ldots, V_m\} = \prod_{i=1}^{m} a[V_i] \quad (4.2) \]

In particular, if each set \(V_i = [v_i]\) consists of only one gluon line labelled by \(v_i\), then \(\{V_1; V_2; \ldots; V_m\}\) is the set of all orderings of the \(m\) gluon lines. In that case the factorization formula reduces to the well-known eikonal formula \[3, 15\]. Other special cases of this formula have also been discovered before \[7, 16\].

It is useful to adopt an alternative notation for the right hand side of (4.2) to denote \(\prod_{i=1}^{m} a[V_i]\) simply as \(a[V_1|V_2| \cdots |V_m]\). This notation is suggestive because the vertical bar can be interpreted graphically as a cut in the fermion propagator between the last gluon line of \(V_i\) and the first gluon line of \(V_{i+1}\). For a cut propagator, instead of the usual factor \((\sum_{j=1}^{n} x_{v_j} + i\epsilon)^{-1}\), we have \(-2\pi i \delta(\sum_{j=1}^{n} x_{v_j})\). This notation is also convenient because it makes (4.2) deceptively simple. It now reads \(a\{V_1; V_2; \ldots; V_m\} = a[V_1|V_2| \cdots |V_m]\); we simply have to change the semicolons to vertical bars.

Cut propagators are not limited to tree diagrams like Fig. 2. The offshell gluons can be connected to other diagrams to form a composite diagram that inherits the original cuts. The cut diagrams so formed are similar to but different from the Cutkosky cut diagrams. Similar because we have the same factors for the cut propagators. Different because the cuts here occur only on fermion lines whereas in a Cutkosky diagram they can occur on any line. Moreover, via (4.2), our cut diagram represents a sum of \(M/\prod_{j=1}^{m} n_j!\) (uncut) Feynman diagrams, with their real and imaginary parts fully included, which is unlike the Cutkosky diagrams in which only the imaginary part or the discontinuity is represented.

It is clear from (4.2) that the factorization formula can be thought of as a sum rule, to represent sums of Feynman diagrams as cut diagrams. As will be discussed in Appendix B, a cut diagram is easier to compute than an uncut diagram. In this way not only it is unnecessary to compute the individual diagrams first, the cut diagram representing the sum is actually easier to compute than just one single Feynman diagram.

We shall now apply the factorization formula to compute sums of amplitudes quoted in Sec. 2. We shall see that the \(\ln s\) factor and the \(J_i\) functions that get cancelled out in the sum never once appear in the cut diagrams.

In what follows we shall use the notation \(\langle B1 \rangle\) to denote the spacetime part of \(B1\) (without the color factor \(G_2\)). Similar notation will be used for the spacetime part of other diagrams as well.

In the 4th order, the \(G_2\) coefficient in \(M\) is given by \(\langle B1 \rangle + \langle B2 \rangle\). Using the eikonal formula \(a[12] + a[21] = a[1|2]\) on the upper fermion line (see Fig. 3), \(\langle B1 \rangle + \langle B2 \rangle\) is reduced to a cut diagram that can be easily calculated (see Appendix B for all the calculations), yielding the correct result \(g^4 I_2/2\) given in (2.1) and (2.4). Similarly, the sum rule \(a\{1; 2; 3\} = a[1|2|3]\) applied to the six horizontal ladder diagrams in the 6th order, yields a cut diagram (see Fig. 4) which gives the sum of \(\langle C15 \rangle\) to \(\langle C20 \rangle\) to be \(g^4 I_3/6\). This is the correct coefficient of \(G_4\) given in (2.4). Note that neither \(\ln s\) nor \(J_3\) ever appears.
Figure 3: An illustration of summing Feynman diagrams to obtain cut diagrams.

\[
\begin{align*}
\begin{array}{c}
1 \quad 2 \\
\end{array} & + 
\begin{array}{c}
2 \quad 1 \\
\end{array} = 
\begin{array}{c}
1 \quad 2 \\
\end{array}
\end{align*}
\]

Figure 4: Another sum rule.

\[
\begin{align*}
\begin{array}{c}
1 \quad 2 \quad 3 \\
\end{array} & + 
\begin{array}{c}
1 \quad 2 \quad 3 \\
\end{array} + 
\begin{array}{c}
1 \quad 2 \quad 3 \\
\end{array} + 
\begin{array}{c}
1 \quad 2 \quad 3 \\
\end{array} + 
\begin{array}{c}
1 \quad 2 \quad 3 \\
\end{array} = 
\begin{array}{c}
1 \quad 2 \quad 3 \\
\end{array}
\end{align*}
\]

Figure 5: Yet another sum rule.

\[
\begin{align*}
\begin{array}{c}
3 \quad 2 \quad 1 \\
\end{array} & + 
\begin{array}{c}
3 \quad 1 \quad 2 \\
\end{array} + 
\begin{array}{c}
2 \quad 3 \quad 1 \\
\end{array} = 
\begin{array}{c}
3 \quad 1 \quad 2 \\
\end{array}
\end{align*}
\]

Figure 6: An illustration of how sum of Feynman diagrams can be turned into sum of cut diagrams.

\[
\begin{align*}
\begin{array}{c}
2 \quad 1 \quad 3 \\
\end{array} & + 
\begin{array}{c}
2 \quad 3 \quad 1 \\
\end{array} + 
\begin{array}{c}
1 \quad 3 \quad 2 \\
\end{array} = 
\begin{array}{c}
2 \quad 1 \quad 3 \\
\end{array} - 
\begin{array}{c}
2 \quad 1 \quad 3 \\
\end{array} + 
\begin{array}{c}
1 \quad 3 \quad 2 \\
\end{array}
\end{align*}
\]
Let us now look at the cancellation of $J_3(\Delta)$ in the $G_2$ and $G_3$ color factors, as discussed in point (5) of Sec. 2. According to (2.1), the coefficient of $-G_3$ from diagrams C15 to C20 is given by $\langle C16 \rangle + \langle C19 \rangle + \langle C20 \rangle$. Using $a\{31;2\} = a\{31\}2$ on the upper fermion line as in Fig. 5, this sum is given by a cut diagram which can be evaluated to be $-g^4 i \ln sI_3/2\pi$. Again $J_3$ does not appear in the cut diagram. Similarly, the coefficient of $cG_2$ from diagrams C15 to C20 is given by (2.1) to be $3\langle C16 \rangle + \langle C17 \rangle + \langle C18 \rangle + 2\langle C19 \rangle + 2\langle C20 \rangle$. This is equal to twice the previous sum, plus $\langle C16 \rangle + \langle C17 \rangle + \langle C18 \rangle$. To use the sum rule, this requires an expression (see Fig. 6) for $a\{321\} + a\{213\} + a\{132\}$. It is easy to see that this is equal to $a\{21;3\} - a\{23;1\} + a\{13;2\}$, and by the factorization formula, also equal to $a\{21\}3 - a\{23\}1 + a\{13\}2$. There may seem to be very little gained by replacing the sum of three terms by the sum of another three terms, but this is not so. We are replacing the sum of three uncut diagrams by the sum of three cut diagrams. The computation of cut diagrams is much easier than the computation of uncut diagrams. For one thing the complicated function $J_3$ that appears on all the three uncut diagrams but disappears from their sum never appears in any of the three cut diagrams. Evaluating the cut diagrams, the coefficient of $cG_2$ from C15 to C20 is $-g^4 ic \ln sI_3/2\pi$, agreeing with the answer given in (2.4).

5 Cut Diagrams

The method introduced in the last section has a serious shortcoming. It does not tell us which cut diagrams to compute without detailed considerations of the kind carried out there. In this section we discuss a remedy for this shortcoming with the help of the multiple commutator formula [12].

So far we have been treating Fig. 2 mostly as a QED or a scalar amplitude. For QCD the non-abelian color matrices $t_a$ have to be incorporated. Instead of (3.4) the amplitude is now

$$A[v_1v_2\cdots v_n] = a[v_1v_2\cdots v_n]t[v_1v_2\cdots v_n] \equiv a[V]t[V] \equiv A[V] ,$$

(5.1)

where $t[V] = t_{v_1}t_{v_2}\cdots t_{v_n}$. What we want is a formula for the sum of the $n!$ permuted gluon orderings, $A = \sum_{V \in S_n} A[V]$. For QED, where we can take all $t_a = 1$, this is simply the eikonal formula, so what we need is the non-abelian generalization of it. This is the multiple commutator formula

$$A \equiv \sum_{V \in S_n} a[V]t[V] = \sum_{V \in S_n} a[V_c]t[V'_c] .$$

(5.2)

It expresses the sum of $a[V]t[V]$ in terms of sums over the corresponding cut amplitude $a[V_c]t[V'_c]$. Compared to the eikonal formula this looks complicated; instead of a single term on the right hand side we have now a sum over $n!$ terms. The complication is inevitable because we are attempting to sum up amplitudes for every color. However, we shall see that many of these terms are actually zero, and moreover, the cut diagrams on the right
are considerably simpler to evaluate than the uncut diagrams on the left. Again delicate
cancellations will largely be avoided as before.

What remains to be described is what the cut diagram $V_c$ that corresponds to the
Feynman diagram $V$ is, as well as what the amplitudes $a[V_c]$ and $t[V'_c]$ are. Given a
$V = [v_1 v_2 \cdots v_n]$, start from the rightmost number $v_n$ and proceed leftward until one comes
to the first number less than $v_n$. Put a cut just to the right of this number. Then start
from this number and proceed leftward again until one comes to the first number that is less
than this number, and another cut is put just to the right of this new minimum number.
Continue this way until the end and we have constructed the cut diagram $V_c$. For example,
for $n = 2$, the 2 cut diagrams are $[12]_c = [1|2]$ and $[21]_c = [21]$. For $n = 3$, the six cut
diagrams are $[123]_c = [1|2|3], [213]_c = [21|3], [312]_c = [31|2], [132]_c = [1|32], [231]_c = [231],$
and $[321]_c = [321]$.

To each cut diagram we associate a spacetime cut amplitude $a[V_c]$ as described in the
last section. Namely, it is given by (3.4) except the propagator at a cut is replaced by
$-2\pi i \delta(\sum_j x_{v_j})$.

The complementary diagram $V'_c$ of a cut diagram $V_c$ is obtained as follows. If a cut
appears between two numbers in $V_c$, then there will be no cut between the same two numbers
in $V'_c$, and vice versa. For $n = 2$, the complementary cut diagrams are $[12]' = [12]$ and
$[21]' = [21]$. For $n = 3$, the complementary cut diagrams are $[123]' = [123], [213]' = [213],$
$[312]' = [312], [132]' = [132], [231]' = [231],$ and $[321]' = [321]$.

When no cut appears in $V'_c$ the color factor $t[V'_c]$ is simply $t[v_1 v_2 \cdots v_n] = t_{v_1} t_{v_2} \cdots t_{v_n}$. If a cut appears between $v_i$ and $v_{i+1}$, then the product
$t_{v_i} t_{v_{i+1}}$ is replaced by their commutators $[t_{v_i}, t_{v_{i+1}}]$. If two or more consecutive cuts appears, then the corresponding product of $t$’s
is replaced by multiple commutators. For example, $t[2|13] = [t_2, t_3] t_5, t[2|3|1] = [t_2, [t_3, t_1]],$
and $t[4|3|2|15] = [t_4, [t_3, [t_2, t_1]]] t_5$.

6 Cut amplitudes to $O(g^6)$

We shall compute the cut diagrams to $O(g^6)$ to demonstrate the simplifications obtained
therefrom.

Using (5.2) on the upper fermion, we obtain a set of cut diagrams as shown in Fig. 7. This set is not unique as others can be obtained from a different labelling of the gluon lines.

In the horizontal ladder diagrams B1, B2, C15–C20, we choose the planar diagrams B1
and C15 to be the ones whose upper lines are completely cut. This fixes the labelling [12]
for B1 and [123] for C15 as shown, and it in turn determines how diagrams B2 and C16 to
C20 are to be cut. Using computational methods discussed in App. A and B, these diagrams
yield

$$B1_c = + \frac{1}{2} g^2 i L_2 \cdot G_2$$

$$B2_c = + \beta (\ln s) i L_2 \cdot c G_1$$
Similarly, we choose to cut the line of the planar diagram $C_1$ to obtain

$$C_{15_c} = + g^4 \frac{1}{6} I_3 \cdot G_4$$
$$C_{16_c} = - \beta^2 (\ln s) 2 J_3 \cdot c^2 G_1$$
$$C_{17_c} = C_{18_c} = 0$$
$$C_{19_c} = 0$$
$$C_{20_c} = - g^2 \beta i (\ln s) I_3 \cdot (c G_2 - G_3) \quad (6.1)$$

The twelve diagrams $C_3$ to $C_{20}$ can be divided into four groups of three, each giving identical contributions, so it is necessary to consider only one of these groups. The group of $C_3$, $C_7$, $C_{12}$ have been chosen for that purpose. There is a symmetry between gluon lines 2 and 3 so we may double this group and consider it as a sum of six Feynman diagrams. By applying the multiple commutator formula, the six cut diagrams shown in Fig. 7 are obtained. Their values are

$$C_{3_c} = \frac{1}{2} (C_{3_c}' + C_{3_c}'') = - g^2 \beta i (\ln s) \frac{1}{4} J_2 I_2 \cdot G_3$$
$$C_{7_c} = \frac{1}{2} (C_{7_c}' + C_{7_c}'') = 0$$
$$C_{12_c} = \frac{1}{2} (C_{12_c}' + C_{12_c}'') = \beta^2 (\ln^2 s) \frac{1}{4} J_2 I_2 \cdot (- c^2 G_1) \quad (6.2)$$

The expressions in (6.1)–(6.3) should be compared with those of the uncut diagrams, eq. (2.1). It should also be compared with the sum found in eq. (2.4). Several points can be noted from these comparisons:
1. In $s$ factors that get cancelled in the sum of the Feynman amplitude (see points (1) to (3) in Sec. 2) never even appear in (6.1)–(6.3). Cancellation of this kind is automatically built into the cut diagram formalism.

2. The transverse function $J_3$ appears only in $C_{16c}$ in eq. (6.1). This expression survives the sum but can be ignored compared to the contribution from $C_{22c}$. In other words, as opposed to the Feynman amplitude (2.1) where $J_3$ appears in many places but most of them are cancelled out at the end (see points (4) and (5) in Sec. 2), in the cut amplitude $J_3$ does not appear except when it survives the sum.

3. The cut amplitude is not as successful in cancelling the transverse function $J_2$ (point (6) of Sec. 2), although there is still an improvement here over (2.1) in that $J_2$ appears in fewer places. In fact, it appears in $C_{3c}$ to $C_{20c}$ only when absolutely needed to cancel its previous appearance in $C_{1c}$ and $C_{2c}$. In order for $J_2$ to disappear completely it is necessary to combine diagrams with triple and four gluon vertices together using the Lipatov-Dickinson vertex $[8]$. The technique of cut diagrams by itself, which deals mainly with the fermion lines, is not sufficient for that purpose.

4. Other than the $J_2$ complication mentioned above, the summands of the final answer (2.4) appear directly in the cut amplitudes. In that sense the cut amplitudes are as economical and as simple as they can ever be. In particular, the Regge-like feature mentioned in point (3) of Sec. 2 is present already in individual cut diagrams.

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A Color factors

The method to compute the color factors graphically $[3]$ is briefly reviewed here. It is suitable for both cut and uncut diagrams.

The basic tool is the graphical identities depicted in Figs. 8 and 9, which are nothing but the commutation relation and identities in (2.3). The commutation relation is valid for all representations of the color matrix $t^a$, hence Fig. 8 remains true when the quark line is replaced by a gluon line.

As an example, the computation of the color factor for the uncut diagram $C_{19}$ in eq. (2.1) is carried out in Fig. 10, and the computation of the color factor for the cut diagram $C_{19c}$ in eq. (6.1) is carried out in Fig. 11. For the latter, we have to remember to use the
Figure 8: Graphical representation of the commutation relation of color matrices. Thick lines are fermions and thin lines are gluons. The color factor at a gluon-fermion vertex is $t_a$ of eq. (2.3), and the color factor for a triple gluon vertex whose color indices $a, b, c$ in the diagram are in clockwise order is $if_{abc}$. A cut represents a commutator between two color matrices.

$$
\begin{align*}
\begin{array}{c}
\text{Thick lines are fermions and thin lines are gluons.}
\end{array}
\end{align*}
$$

$G_4 + 2c G_2 - G_3$

Figure 9: Graphical representation of the last two identities in equation (2.3). $c = N/2$ for $SU(N)$ color.

$$
\begin{align*}
\begin{array}{c}
\text{Figure 10: A sample calculation of the color factor of a Feynman diagram.}
\end{array}
\end{align*}
$$
Figure 11: A sample calculation of the color factor of a cut diagram.

\[
\begin{align*}
\text{Figure 12: Flow diagrams.}
\end{align*}
\]

complementary cut diagram for the color factor, and not the cut diagram itself shown in Fig. 7.

The color factors in eqs. (2.1), (6.1)–(6.3) can all be obtained in this way.

\section*{B \ Spacetime amplitudes}

High energy limit of a Feynman diagram can be computed either in the Feynman-parameter representation\cite{13}, or directly in momentum space using lightcone coordinates\cite{3, 4, 5}. We follow the latter approach and review briefly the main points. For a more detailed discussion see Ref. \cite{3}. It serves well to remember at this point that the bottom fermion carries mostly the `'−' momentum and the top fermion line carries mostly the `'+' momentum.

The procedure to follow for the computation is simple: (i) carry out the `'+' momentum integration using residue calculus; (ii) then carry out the `'−' momentum integration to produce the \( \ln s \) dependence. The transverse momentum integrations are never explicitly carried out, which is why the answers are all expressed in terms of functions like \( I_n(\Delta) \) and \( J_n(\Delta) \) given in eq. (2.2).

Step (i) requires a knowledge of the location of poles for each propagator, which in turn depends on the choice of loop momenta \( k_a \). Suppose \( q = P + \sum_{a=1}^{d} c_a k_a \) is the momentum flowing through a propagator, with \( P \) being some combination of external momenta and \( c_a \) some integer coefficients. The propagator \( D^{-1} = (q^2 - m^2 + i\epsilon)^{-1} = (q_+ - q_-^2 - m^2 + i\epsilon)^{-1} \) has a pole in the integration variable \( k_{a+} \) if \( c_a \neq 0 \). This pole is located in the upper half plane if \( c_a q_- \) is negative, and in the lower half plane if it is positive. A way to keep track of the sign is to draw a flow diagram for the `'−' momentum, with arrows indicating the
direction of the ‘−’ flow. See Fig. 12 for some illustrative examples. The arrows around a loop pointing one way (clockwise or counter clockwise) have their poles in one half plane, and those pointing the other way have poles in the opposite half plane. The arrows in the flow diagrams must obey momentum conservation, and one may also assume they do not go around in closed loops. Otherwise the poles will all be in the same half plane, and the integral is zero if the contour is closed in the opposite half plane. With these constraints we see in Fig. 12 that a one-loop diagram allows only one flow path, but for two and more loops there is bound to be more than one flow diagrams because these constraints simply cannot fix the direction of the ‘−’ flow on a boundary line of two loops.

Each flow diagram corresponds to a range of $k_a$ variables. By definition, the ‘−’ variables along the direction of the flow are always non-negative. We shall always close integration contours in the lower half planes, and indicate the poles so enclosed by a cross (x) in the flow diagram. For a scalar diagram, the $\mathcal{T}$-matrix element is equal to the product of $I$ propagators

$$D^{-1} = \left(q^2 - m^2 + i\epsilon\right)^{-1},$$

integrated over the $\ell$ loop momenta $d^4k_a = \frac{1}{2}dk_+dk_-d^2k_\perp$, with an extra numerical factor $-\left[i/(2\pi)^4\right]^{\ell}$. Each ‘+’ integration produces a factor $-2\pi i$, the $\mathcal{T}$-matrix is equal to

$$\mathcal{T} = -\sum \int Dk_\perp \left(\prod_{a=1}^\ell \frac{dx_a}{4\pi}\right) \frac{1}{\prod_{i=1}^I D_i},$$

(B.1)

where $x_a = k_a/\sqrt{s}$ is the scaled ‘−’ momenta,

$$Dk_\perp \equiv \prod_{a=1}^\ell \frac{d^2k_{a\perp}}{(2\pi)^2}$$

(B.2)

is the measure for transverse momentum integration, and $D_i$ is either the propagator evaluated at the x poles or the residue of the x pole divided by $\sqrt{s}$. Each summand in (B1) corresponds to a flow diagram with a pole taken from the lower half plane (an x pole) of each loop. A flow diagram may have more than one set of x poles, in which case the sum is taken over all possible sets. For example, the one-loop diagram B1 in Fig. 1 has only one flow path (Fig. 12(a)) and one set of x poles. The two-loop diagram C17 of Fig. 1 has two flow paths, Figs. 12(b) and 12(c). The first flow diagram has two sets of x poles, and the latter flow diagram has one set of x poles. We shall see later that in the leading log approximation, we may discard the $q_+$ dependence on the upper fermion line and the accompanied poles, in which case 12(b) is also left with one set of x poles.

Equation (B1) is also valid for cut diagrams, provided the propagator factor $D^{-1} = (q^2 - m^2 + i\epsilon)^{-1}$ of a cut line is replaced by $-2\pi i\delta(q^2 - m^2)$.

Let $q_i = \sqrt{s}z_i$, $i = 1, 2, \cdots, I$. Every $z_i$ is a linear combination of $x_a$, and in the case of a propagator along the bottom fermion line, $x_0 \equiv 1$ is also involved. The sign of each $z_i$ is fixed by the direction of arrows in the flow diagram. If there are $\chi$ cut lines, then the last $\chi$ $x_a$ will be chosen to be equal to $|z_i|$ of these cut lines. The $I$ indices $i$ will now be divided into three sets, $a, b$ for indices from 1 to $\ell$ labelling the internal lines with an x pole, $u$ for indices
labelling propagators on the top fermion line, and $m$ for the rest. Then $q_{a^+} = q_{a\perp}^2/\sqrt{s}z_a$, and $q_{m^+}, q_{u^+}$ can be expressed as linear combinations of $q_{a^+}$, e.g.,

$$q_{m^+} = \sum_a c_{ma}q_{a^+} = z_m \sum_b c_{mb}q_{b\perp}^2/z_b$$

(B.3)

Within the leading log approximation, one has

$$D_a = z_a$$

$$D_u \simeq sz_u \ (\text{uncut}) , \quad D_u = -2\pi i\delta(sz_u) \ (\text{cut})$$

$$D_m = \sum_b c_{mb}z_m^2 q_{b\perp}^2 - q_{m\perp}^2 + i\epsilon \ .$$

(B.4)

The ‘$-$’ momentum flows mainly along the bottom fermion line, with very little seeping out to avoid a substantial mixing with the ‘$+$’ momentum coming from the top fermion line, for a finite mixture of these two at a propagator would make it proportional to $s$ and therefore negligible. This means the dominant contribution to $\mathcal{T}$ comes from regions where all $x_a$ are small. Since $x_a = 0$ for $a = \ell - \chi + 1, \ldots, \ell$, on account of the $\delta$-functions of the cut lines, the region of ‘$-$’ integration can be roughly divided into regions where the remaining $\ell' = \ell - \chi x_a$’s are strongly ordered, and regions where two or more of these $x_a$’s are of the same order of magnitude. We shall label any one of the latter regions by $S$, and the former regions by $R$, as in (B.4).

The $x$-dependence of $\Pi_i D_i$ in $R[12 \cdots \ell']$ is of the form $\prod_{a=1}^\ell x_a^{m_a}$, so the $x$-integral encountered in (B1) is

$$\int_2^1 dx_\ell' \int_{x_\ell'}^{x_{\ell' - 1}} dx_{\ell'-1} \cdots \int_{x_2}^{x_1} dx_1 \sim \frac{1}{sM} (\ln s)^B ,$$

(B.5)

where $M = \sum_{a=1}^\ell (m_a - 1)$ and $B$ is determined by how many times the sum $\sum_{b=1}^\ell (m_a - 1)$ reaches zero by varying $b$ from 1 to $\ell'$. Clearly $B \leq \ell'$, and the only way for $B = \ell'$ is to have all $m_a = 1$, in which case we will call the $\ln s$ dependence of $\mathcal{T}$ saturated. Otherwise it is said to be unsaturated. For the uncut diagrams, $\chi = 0$ and $\ell' = \ell$. We see in (2.1) that all the diagrams except for the 6th order horizontal ladder diagrams C15 to C20 are saturated. For the cut diagrams in (6.1) to (6.3), only $C_{16c}$ is unsaturated, but this diagram does not contribute to the final sum because of its subdominance.

The integral in an $S$ region is like one in an $R$ region with $\ell'$ reduced. For example, suppose $x_{\ell' - 2}$ to $x_{\ell'}$ are roughly equal but $x_1$ to $x_{\ell''}$ ($\ell'' = \ell' - 2$) are strongly ordered as in $R[12 \cdots \ell''] = R''$. Then the volume element in the last three variables in spherical coordinates is $r^2 dr d\Omega_2$, so the integration region is effectively $R''$ but with $x_{\ell''}$ replaced by $r$. Since $\ell'' < \ell$, integrals in $S$ will never lead to saturation.

Cut diagrams are easier to compute than uncut Feynman diagrams for three reasons. First and most trivial, a cut line contains a $\delta(sz)$ which makes the corresponding ‘$-$’ integration simpler to carry out. Second and more importantly, the $\delta$-function demands the absence of the ‘$-$’ momentum across this line, so the cut line is cut also in the sense of
being an open electric circuit. This generally reduces the number of possible flow diagrams and makes the corresponding integral easier to saturate. For example, each of the two-loop diagrams in Fig. 13 has only one flow path. Thirdly, the flow pattern often leads to vanishing cut diagrams. For example, both diagrams 13(c) and 13(d) are zero from the ‘+’ integration around the loop (12345) because the ‘−’ flows around that loop are all in the same direction. This accounts for the equality \( C_{17c} = C_{18c} = 0 \) in eq. (6.1).

The cost of this simplicity is the presence of tricky diagrams, which are cut diagrams that are apparently logarithmic divergent at large \( k_{a+} \). This happens whenever there is a loop with only one arrow present. Arrows on the top fermion line should not be counted for this purpose because their approximate propagators \( \frac{1}{(sz + i\epsilon)} \) no longer carry any \( k_{a+} \). Figs. 13(a) and 13(b) are examples of such diagrams. This apparent divergence is produced by the approximation of replacing the inverse propagators of an upper line by \( sz + i\epsilon \), thus losing some \( q^+ \) factors needed for convergence. To regulate it we must replace the \( \delta \)-function of the cut line by a smeared \( \delta \)-function, thereby allowing a small amount of ‘−’ momentum to flow through, and in the process restoring the lost \( q^+ \) factor. If we do so to 13(a), the \( k^+ \) integration is no longer divergent, it will have exactly the same flow path as in 12(a) and this is indicated in 13(a) by the dotted arrows, so a pole can be taken at the bottom fermion line as shown. The result is given by (B1) to be \( \left( \frac{g^4i}{2}\right)I_2 \cdot \frac{1}{2} \), the extra \( \frac{1}{2} \) is there because the \( x \) integration is bounded by the flow path in 12(a) to be between 0 and 1, so only half of the \( \delta(sx) \) is integrated. When multiplied by the QED vertex and normalization factor \( \frac{2s}{2m}^2 \), one obtains \( B_{1c} \) in (6.1).

There is another way to compute Fig. 13(a). This is to recognize the fact that the cut makes it symmetric in lines 1 and 2, so we may replace the diagram by half the sum of it and its crossed diagram. Using the sum rule (3.4), \( a[12] + a[21] = a[1|2] \), a cut can be produced at the bottom line, and the resulting factor \( -2\pi i\delta(\sqrt{s}q_1) \) makes the \( q_1^+ \) integration convergent. See Fig. 14. Moreover, the result of this ‘+’ integration is \( -2\pi i \), exactly the same as if we were to do it by residue calculus. The extra factor of \( \frac{1}{2} \) obtained in the last paragragaph now emerges because the double-cut diagram on the right of Fig. 14 is the sum of two diagrams.

Now we come to the tricky diagram 13(b), which unlike the one-loop case is much more difficult to compute by regulating the \( \delta \)-functions. The reason is that there are now two cut lines, the relative magnitude of the small ‘−’ flows matters, and that produces once again two flow diagrams like 12(b) and 12(c). The poles are no longer at the bottom fermion line and the

\[
\text{Figure 13: Sample cut diagrams to illustrate their computation.}
\]

\[
\text{Figure 14: Sample cut diagrams to illustrate their computation.}
\]
computation is no simpler than the uncut diagrams. We must then compute it by the second method. With the double cuts in 13(b), it is symmetric in all the $q_i$ variables. Insisting on this symmetry, 13(b) is equal to $1/3!$ times the sum of 6 diagrams, obtained by permuting the bottom gluon lines in all possible ways. From the factorization formula $a\{1; 2; 3\} = a[1|2|3]$ used on the bottom line, the result is equal to a diagram with all its fermion propagators cut. So 13(b) is given by (B1) to be

$$g_6 \left( \frac{-2\pi i}{2} \right)^2 I_3 \frac{1}{(4\pi s)^2} 3! = g_6 I_3 / 24 s^2.$$  

Incorporating the extra QED factor $-\left( \frac{2s}{2m} \right)^3 / (2m)^2$ for the $T$ matrix, we obtain the contribution of $C_{15c}$ to $M \equiv (2m^2 / g^2 s) T$ shown in (6.1).

We conclude this appendix by discussing the remaining expressions in eqs. (6.1)–(6.3). We shall use the notation $\langle B_2c \rangle$ to denote the spacetime contribution to $B_2c$, etc. Then $\langle B_2c \rangle = \langle B2 \rangle, \langle C12'c \rangle = \langle C_12''\rangle = \langle C12 \rangle, \langle C16c \rangle = \langle C16 \rangle$, and $\langle C19c \rangle = \langle C19 \rangle$ can be obtained from (2.1). The zero of $C19$ in (6.1) is due to the vanishing of the color factor as shown in Fig. 11. Next, we compute $\langle C3'c \rangle = \langle C3'\rangle$. The flow path of this is shown in Fig. 13(f), with the dotted arrows indicating the small regulating current which is allowed to flow only in the direction shown. The calculation is identical to the uncut diagram [3], except that the factor $-\frac{1}{2} \ln^2 s$ is replaced by $\frac{1}{2} (\ln s) (-2\pi i)$ $(-2\pi i$ from the cut, $\ln s$ because only one uncut line is left on top, and $\frac{1}{2}$ because only half of the $\delta$-function is integrated). The result can then be read off from (2.1) to be

$$\langle C3'c \rangle = \langle C3''c \rangle = -g^2 \beta i (\ln s) \frac{1}{4} J_2 I_2 ,$$

which gives rise to the expression in (6.3).

Finally, we must show that $C7'c = 0$ and $C7''c = 0$. The former can be seen from the flow path in Fig. 13(g), where the arrows around the small triangle goes around in a closed loop. The latter is so because the scalar diagram in Fig. 7 for $C7''c$ is symmetrical in lines 2 and 3 but the triple gluon vertex is antisymmetrical in these two lines.

Figure 14: One-loop tricky diagram and its computation.
References

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[1] ZEUS Collaboration (M. Derrick et al), Phys. Lett. B315 (1993) 481; Phys. Lett. B332
(1994) 228; H1 Collaboration (T. Ahmed et al), Nucl. Phys. B429 (1994) 477; Phys. Lett.
B348 (1995) 681.

[2] F.E. Low, Phys. Rev. D12 (1975) 163; S. Nussinov, Phys. Rev. Lett. 34 (1975) 1286.

[3] H. Cheng and T.T. Wu, ‘Expanding Protons: Scattering at High Energies’, (M.I.T.
Press, 1987).

[4] N.T. Nieh and Y.P. Yao, Phys. Rev. Lett. 32 (1974) 1074; Phys. Rev. D13 (1976) 1082;
B. McCoy and T.T. Wu, Phys. Rev. Lett. 35 (1975) 604; Phys. Rev. D12 (1975) 3257;
D13 (1976) 1076; L. Tyburski, ibid. 13 (1976) 1107; L.L. Frankfurt and V.E. Sherman,
Yad. Fiz. 23 (1976) 1099 [Sov. J. Nucl. Phys. 23 (1976) 581]; A.L. Mason, Nucl. Phys.
B117 (1976) 493.

[5] L.N. Lipatov, Yad. Fiz. 23 (1976) 642 [Sov. J. Nucl. Phys. 23 (1976) 338]; Ya. Ya.
Balitskii and L.N. Lipatov, Yad. Fiz. 28 (1978) 1597 [Sov. J. Nucl. Phys. 28 (1978) 822].

[6] J. Bartels, Phys. Lett. 68B (1977) 258; Nucl. Phys. B151 (1979) 293; J.B. Bronzan and
R.L. Sugar, Phys. Rev. D17 (1978) 585; V.N. Gribov, Zh. Eksp. Teor. Fiz. 53 (1967)
654; L.V. Gribov, E.M. Levin, and M.G. Ryskin, Phys. Rep. 100 (1983) 1.

[7] C.Y. Lo and H. Cheng, Phys. Rev. D13 (1976) 1131; D15 (1077) 2959; H. Cheng, J.A.
Dickinson, and K. Olaussen, Phys. Rev. D23 (1981) 534.

[8] E.A. Kuraev, L.N. Lipatov, and V.S. Fadin, Zh. Eksp. Teor. Fiz. 71 (1976) 840
[Sov. Phys. JETP 44 (1976) 443]; ibid. 72 (1977) 377 [ibid. 45 (1977) 199]; J.A. Dickin-
son, Phys. Rev. 16 (1977) 1863.

[9] H. Cheng and T.T. Wu, Phys. Rev. Lett. 22 (1969) 666.

[10] Y.J. Feng, O. Hamidi-Ravari, and C.S. Lam, to be published.

[11] Note however that for later convenience the femion arrows in Fig. 1 and all subsequent
figures are drawn from right to left, reflecting the order one writes matrices as well as
initial and final states on a piece of paper, whereas in Ref. [3] they are drawn in a more
conventional way from left to right.

21
There is a misprint in Fig. 12.7 of Ref. [3]. The $G_2$ portion of diagram 16 should have a coefficient $3n/2$ rather than $n^2/2$.

B.W. Lee and R.F. Sawyer, Phys. Rev. 127 (1962) 2266; J.C. Polkinghorne, J. Math. Phys. 4 (1963) 503; P.G. Federbush and M.T. Grisaru, Annals of Physics 22 (1963) 263, 299; J.D. Bjorken and T.T. Wu, Phys. Rev. 130 (1963) 2566; G. Tiktopoulos, Phys. Rev. 131 (1963) 480, 2372; R.J. Eden, P.V. Landshoff, D.I. Olive, and J.C. Polkinghorne, ‘The Analytic $S$-Matrix’, (Cambridge University Press, 1966); J.V. Greenman, J. Math. Phys. 7 (1967) 1782; 8 (1967) 26; C.S. Lam, Nuovo Cimento 62A (1969) 97.

C.S. Lam and K.F. Liu, ‘A multiple commutator formula for the sum of Feynman diagrams’, McGill/96-12, hep-ph 9604337, to be published.

H. Cheng and T.T. Wu, Phys. Rev. 186 (1969) 1611; M. Levy and J. Sucher, Phys. Rev. 186 (1969) 1656.

C.Y. Lo, Nucl. Phys. B157 (1979) 212; Phys. Rev. D23 (1981) 508.