LETTERS AND COMMENTS

Reply to comments on ‘Energy in one-dimensional linear waves in a string’

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Received 18 May 2011
Published 9 September 2011
Online at stacks.iop.org/EJP/32/L43

Abstract

In this reply we respond to comments made by Repetto et al and by Butokov on our letter (Burko 2010 Eur. J. Phys. 31 L71–7), in which we discussed two different results for the elastic potential energy of a string element. One derived from the restoring force on a stretched string element and the other from the work done to bring the string to a certain distorted configuration. We argue that one cannot prefer from fundamental principles the former over the latter (or vice versa), and therefore one may apply either expression to situations in which their use contributes to insight. The two expressions are different by a boundary term which has a clear physical interpretation. For the case of standing waves, we argue that the latter approach has conceptual clarity that may contribute to physical understanding.

Our letter [1] has been commented on by Repetto, Roatta and Welti (henceforth RRW) [2] and by Butokov [3]. In this short reply we respond to the criticisms included therein. Both RRW and Butokov critique our use of equation (4) in [1] for the energy of a string element and argue that one may only use the conventional expression, specifically equation (3) therein. We undertake here the viewpoint adopted by Morse and Feshbach [4] that the potential energy of a string element is not unique, because the question of the energy of the endpoints of the element under consideration cannot be uniquely determined. Both RRW and Butokov appear to disagree with this viewpoint, and require that equation (3) in [1] is used exclusively.

In practice, both RRW and Butokov assume the uniqueness of equation (3) in [1] (equivalent to equation (2.5) in [2] or equation (2) in [3]), and then simply draw the elementary conclusions from their assumption, which in fact are included in sections 1 and 2 of [1]. There is therefore nothing new or enlightening in either comment. Specifically, RRW ignore the non-uniqueness of the potential energy term of a string element. Naturally, since RRW and Butokov assume equation (3) in [1], they obtain conclusions consistent with it. We note in passing that the Lagrangian formalism in [5] (reference [5] in [2]) assumes a specific form for the potential energy term, and therefore cannot be used to determine which potential energy term one is to use.

Unlike RRW, Butokov does also consider equation (4) in [1], yet argues that it cannot be used to describe the energy content of a string element because of the quasi-staticity assumed in its derivation, and that this expression may consequently be used only for the entire string.
We disagree with that viewpoint. The result for potential energy of a string element does not depend on the assumption of quasi-staticity. Indeed, if the concept of potential energy—the energy content of a physical system due to its position and configuration in space—is to have any physical meaning, it cannot depend on how the system got to that configuration in space. Therefore, the work done on a string element to bring it to a particular configuration must be the same whether the distortion of the string was done quasi-statically or otherwise. Quasi-staticity is indeed assumed in the derivation for simplicity, yet the result cannot depend on quasi-staticity. The same principle is implied in the calculation of the potential energy (of any type, not just elastic) of any physical system or field. We therefore contend that equation (4) in [1] indeed describes correctly the energy content of a string element when the endpoints of the element are included. Below, we bring evidence from energy conservation that must indeed be the case.

Notably, there is no fundamental reason to prefer equation (3) of [1] over equation (4) (or vice versa). As summarized by Morse and Feshbach [4], ‘each answer is equally good for the complete string, for when the endpoints of the string element are the ends of the string (which are rigidly or freely supported so that no energy is given to the supports), both results are the same. This is the only case which should give a unique answer. Since expression (2.1.11) (equivalent to equation (3) in [1]) is simpler, we shall use it in our subsequent discussions’. For a travelling wave, there is much conceptual clarity in using equation (3) in [1]. We argue that much insight into standing waves in a string can be gained by considering equation (4) in [1]; there is no fundamental reason not to use it, and the simplicity alluded to by Morse and Feshbach is a subjective aesthetic consideration: we argue that for standing waves it may in fact be simpler sometimes to use the latter.

In [1] we studied the consequences of assuming equation (4) in [1] for the case of a standing wave. We believe that there is much aesthetic clarity in these consequences, specifically that the total energy of a string element including its endpoints is constant. This is the context in which our statements in [1] should be understood: when the endpoints of the string element are included, energy conservation requires that the total energy is constant; variable energy would violate energy conservation. (Notably, this is no longer the case when the endpoints are excluded.) Both RRW and Butokov dogmatically require the consequences of assuming equation (3) in [1], and therefore the adoption of equation (4) therein appears to them to be erroneous, especially when recalling statements appearing in many general physics texts, that for simplicity or conventionality make a similar assumption. It is important to recall, however, the non-uniqueness of the potential energy of a string element discussed in [4]; ‘this expression for the energy of only a portion of a string is not unique, for the question of the energy of the “ends” of the chosen portion cannot be uniquely determined.’ It is therefore a mistake to discard the latter as both RRW and Butokov do.

To demonstrate some of the insight that might be gained by using equation (4) in [1] and at the same time also demonstrate its reasonability, consider the power radiated along a string element, and let the wavefunction of the standing wave be given by \( y(x, t) = y_0 \sin kx \sin \omega t \). (All quantities are defined in [1].) We would like to make use of this opportunity to correct a typo in [1]: the last displayed equation on page L75 should read

\[
E_{21}^{t+dx} - E_{11}^{t+dx} = \frac{1}{2} \tau \left[ \frac{\partial y}{\partial x} \right]_{t}^{t+dx} = \mu \frac{y_0^2 \omega^2}{2} (\sin^2 kx - \cos^2 kx) \sin^2 \omega t \, dx = -\frac{1}{2} \mu \frac{y_0^2 \omega^2}{2} \cos 2kx \sin^2 \omega t \, dx.
\]
Note that the time rate of change of this boundary term is given to linear order in $dx$ by

$$\frac{d}{dt} \left\{ -\frac{1}{2} \tau \left[ y \left( \frac{\partial y}{\partial x} \right) \right]_{x}^{x+dx} \right\} = -\frac{1}{2} \mu \omega^{3} y_{0}^{2} \cos 2kx \sin 2\omega t \, dx,$$

and that

$$\frac{d}{dt} E_{1} \mid_{x}^{x+dx} = \frac{1}{2} \mu \omega^{3} y_{0}^{2} \cos 2kx \sin 2\omega t \, dx = -\frac{d}{dt} \left\{ -\frac{1}{2} \tau \left[ y \left( \frac{\partial y}{\partial x} \right) \right]_{x}^{x+dx} \right\}.$$

The meaning of this result is that the change in the energy of the differential string element is accounted for by the change of the energy of the two endpoints of the element. Indeed, conservation of energy requires that the sum of the time rates of change of the string element energy and the endpoint energy vanishes. This is the reason why, when one includes the two endpoints of the element in the energy calculations, the total energy is constant. We believe that this viewpoint is insightful. Unlike claims by RRW and Butokov, there is no violation of local energy conservation using this approach: the rate by which the string element changes is completely accounted for by the change in the energy of the endpoints. The alternative viewpoint—which is physically equivalent yet both RRW and Butokov demand be exclusively adopted—is that the time rate of change of the string element is given by $(d/dt) E_{1} \mid_{x}^{x+dx}$. The point we are making is that the latter viewpoint is not unique, and one may also make use of equation (4) in [1] when it may lead to insight or clarification.

**Acknowledgments**

This work has been supported in part by NASA/SSC, grant no. NNX07AL52A, and by NSF grants no. PHY–0757344 and DUE–0941327.

**References**

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