Resolution of the Plane-symmetric Einstein-Maxwell fields with a generalization of the Lambert W function

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Abstract

Here, the Einstein-Maxwell field equation with plane-symmetry is resolved and a solution involving a generalized Lambert W function is obtained. This generalized Lambert W function is studied and its derivative, Taylor series, Mellin transform and approximation formula are derived and its real branches are determined. Implications of these analytic properties to the solution function are also presented.

1. Introduction

The Einstein’s field equations are ten equations, contained in the tensor (see [1])

\[ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + g_{\mu \nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu \nu}, \quad (1) \]

where \( R_{\mu \nu} \) is the Ricci curvature tensor, \( R \) is the scalar curvature, \( g_{\mu \nu} \) is the metric tensor, \( \Lambda \) is the Cosmological constant, \( G \) is Newton’s gravitational constant, \( c \) is speed of light and \( T_{\mu \nu} \) is the Stress energy tensor. These equations describe the gravity as a result of spacetime being curved by mass and energy.

The basic characteristic of Einstein’s theory is the Lorentz transformation which relates scientific measurements in one frame of reference to another. Maxwell’s equations together with the transformations of the special theory of relativity, are referred to as the Einstein-Maxwell equations [2].

A plane-symmetric Einstein-Maxwell field with \( \Lambda = 0 \) is given by (see [3])

\[ ds^2 = r^2(dx^2 + dy^2) - 2dudr - \left[ (\varepsilon^2 (u) / r^2 - 2m(u) / r) \right] du^2. \quad (2) \]

For \( \varepsilon \neq 0 \), both \( \varepsilon \) and \( m \) are constant, and the metric is either static, or spatially homogeneous. By a transformation of the \( r \)-coordinate, the line element (2) can be transformed into the form [3–53–5]

\[ ds^2 = Y^2(z)(dx^2 + dy^2) + \frac{1}{2} Y'(z)(dz^2 - dr^2), \quad (3) \]

where \( Y(z) \) is determined implicitly by the transcendental equation

\[ (Y - A)^2 + 2A^2 \log(Y + A) = -Cz, \quad (4) \]

and \( A, C \) are constants and \( Y'(z) \) denotes derivative of \( Y \) with respect to \( z \).

In the next section, (4) is transformed into the form

\[ ye^{\eta y^2 + \eta} = x, \quad (5) \]

which resembles the defining equation of the classical Lambert function. The function \( y = \gamma(x) \) that satisfies (5) will be called the quadratic Lambert function.

There are already existing generalizations of the Lambert W function in the literature. One generalization is the solution to transcendental algebraic equations of the form (see [6, 7])
where $P_N(x)$ and $Q_M(x)$ are polynomials in $x$ of degrees $N$ and $M$, respectively and $c$ is constant. Observe that (6) can be generalized further into the form

$$e^{\pm cx} = F(u),$$

where $F(u)$ is not restricted to rational functions.

Letting $u = ay^2 + y$, then (5) can be written as

$$e^{-u} = -1 \pm \frac{\sqrt{4au + 1}}{2ax},$$

which shows that (5) is a special case of (7). However, we cannot say that the general form (7) is solvable with a generalized Lambert function because we have not yet explored such general case. Although the proposed generalization (5) can be considered a special case of the more general form (7), this generalization in particular, is not yet considered in the literature.

The generalized forms

$$\frac{x - t}{x - s} e^x = a; \quad (x - t_1)(x - t_2)e^x = a$$

with parameters $t, s, t_1, t_2$ were studied in [8]. Some applications of the generalized Lambert functions defined as the solutions to the equations of the form (9) were presented in [9].

Motivated by the above mentioned resolution of the plane symmetric Einstein-Maxwell field equation we proceed to study the quadratic Lambert W function. Formal notation of the function is given in section 2 and its analytic properties such as derivative, Taylor series and Mellin transform are derived. The real branches are discussed in section 3 and an asymptotic approximation formula is proved in section 4. A discussion on implications of the results from sections 2–4 is given in section 5. Finally, conclusion is given in section 6.

2. The quadratic Lambert function and some basic properties

Denote by $W(a; x)$ the quadratic Lambert function which is the solution to (5). The proposition that follows shows that (4) can be transformed to (5) and a solution is expressed in terms of the quadratic Lambert function.

**Proposition 1.** The solution of equation (4) can be expressed by the quadratic Lambert function as follows,

$$Y(z) = -\frac{A}{2} W\left(\frac{1}{8}; -\frac{2}{A} \exp\left(-\frac{C}{2A^2} - 2\right)\right) - A.$$

**Proof.** Substituting $E = Y + A$ to (4) and performing simple algebraic manipulation yield

$$E \exp\left(\frac{1}{2A^2} E^2 - \frac{2}{A}\right) = \exp\left(-\frac{C}{2A^2} - 2\right).$$

With $F = -\frac{1}{2}E$, the last equation transforms into

$$F \exp\left(\frac{1}{8} + F\right) = \frac{2}{A} \exp\left(-\frac{C}{2A^2} - 2\right).$$

The left hand side is already of the form (5). This means that

$$F = W\left(\frac{1}{8}; -\frac{2}{A} \exp\left(-\frac{C}{2A^2} - 2\right)\right).$$

To find the solution $Y(z)$ of (4), note that $Y = E - A = -\frac{A}{2}F - A$.

With the result in proposition 1, knowing the behavior of the quadratic Lambert $W(a; x)$ will enable us to describe the behavior of the solution $Y(z)$ of (4). Thus, we proceed to establish some analytic properties of the quadratic Lambert function.

As $W(a; x)$ is an inverse function, its derivative can be found readily.

**Theorem 2.** The derivatives of the quadratic Lambert W function are

$$\frac{\partial}{\partial x} W(a; x) = \frac{1}{e^{aW^2(a; x)} + W(a; x)(1 + 2aW^2(a; x) + W(a; x))},$$

(10)
\[
\frac{\partial}{\partial a} W(a; x) = - \frac{W^3(a; x)}{1 + W(a; x) + 2aW^2(a; x)}. \tag{11}
\]

**Proof.** The simple fact \((f^{-1})' = 1/(f' \circ f^{-1})\) is used to derive (10). To prove (11), take the derivative of the defining equation (5) with respect to \(a\). Without writing out the arguments:
\[
W' e^{aW^2 + W} + We^{aW^2 + W} (W^2 + 2WW' + W') = 0.
\]

Here the prime refers to the derivative with respect to \(a\). The exponential terms cancel out, and a rearrangement gives (11). \( \square \)

We expect \(W(a; x)\) close to \(W(x)\) if \(a\) is small. When \(a \to 0\), (11) says that the second Taylor coefficient of \(W(a; x)\) around \(a = 0\) is
\[
- \frac{W''(x)}{1 + W(x)}.
\]

Hence, up to the first order term,
\[
W(a; x) = W(x) - \frac{W''(x)}{1 + W(x)}a + O(a^2).
\]

The Taylor series around \(x = 0\) can be explicitly determined as the following theorem shows.

**Theorem 3.** The Taylor series of the quadratic Lambert \(W\) function around \(x = 0\) is given as
\[
W(a; x) = \frac{1}{a} \sum_{k=1}^{\infty} \frac{(ax)^k}{k!} \sum_{m=0}^{k-1} \frac{(-k/m)^m}{m!} \left( \frac{m}{k-1-m} \right).
\]

**Proof.** By the Lagrange inversion,
\[
W(a; x) = \sum_{k=1}^{\infty} \frac{x^k}{k!} \lim_{\epsilon \to 0} \frac{d^{k-1}}{dx^{k-1}} \left( \frac{x}{xe^{a(x^2 + x)}} \right)^k.
\]

Calculating the \((k-1)\)th derivative yields
\[
\frac{d^{k-1}}{dx^{k-1}} \left( \frac{x}{xe^{a(x^2 + x)}} \right)^k = \frac{d^{k-1}}{dx^{k-1}} \sum_{m=0}^{\infty} \frac{(-k)^m}{m!} (ax^2 + x)^m = \sum_{m=0}^{\infty} \frac{(-k)^m}{m!} \frac{d^{k-1}}{dx^{k-1}} (ax^2 + x)^m.
\]

The binomial theorem gives that
\[
\frac{d^{k-1}}{dx^{k-1}} (ax^2 + x)^m \sum_{i=0}^{m} \binom{m}{i} d^i (i + m)x^{i+m-k}.
\]

(Here \(x^k = x(x - 1) \cdots (x - k + 1)\) is the falling factorial with \(x^0 = 1\).) Taking \(x \to 0\), the terms that survive are only those for which \(i + m = k - 1\). Since \(i \leq m\), \(0 \leq i + m \leq 2m\) must hold, it must be that \(m \leq \frac{k-1}{2}\). Thus,
\[
\frac{d^{k-1}}{dx^{k-1}} \left( \frac{x}{xe^{a(x^2 + x)}} \right)^k = \sum_{m=0}^{k-1} \frac{(-k)^m}{m!} \left( \frac{m}{k-1-m} \right) a^{k-1-m}(k-1)^{k-1} (k \geq 1).
\]

A simplification and rearrangement gives the desired formula. \( \square \)

Few terms of the Taylor series are given below,
\[
W(a; x) = x - x^2 + \frac{1}{2} (3 - 2a)x^3 + \frac{4}{3} (3a - 2)x^4 + \frac{5}{24} (12a^2 - 60a + 25)x^5 - \cdots. \tag{13}
\]

Integral transformations of functions are often important in applications. The Mellin transform in particular, has applications in geophysics [10] and time series models [11]. The Mellin transform of the quadratic Lambert is derived in the next theorem.

**Theorem 4.** The Mellin transform of the quadratic Lambert function is
\[
\mathcal{M} W(a; x)(s) = \frac{(-as)^{1/2}}{4as^2} \left[ s^2 \Gamma \left( \frac{s+1}{2} \right) \text{Ei} \left( \frac{s+2}{2} , \frac{3}{2} \right) ; \frac{s}{4a} \right] + 2 \sqrt{-as} \left[ s + \frac{1}{2} , \frac{3}{2} ; \frac{s}{4a} \right].
\]

for \(a > 0\) and \(-1 < s < 0\).

**Proof.** By the definition of Mellin transform,
\[
\mathcal{M} W(a; x)(s) = \int_{0}^{\infty} t^{s-1} W(a; t) dt \tag{14}
\]
Let $u = W(a; t)$, so that $t = ue^{au^2 + u}$. Then $dt = e^{au^2 + u}(1 + u + 2au^2)du$ and (14) becomes

$$\left(\mathcal{L}W(a; x)\right)(s) = \int_0^\infty t^{s-1}W(a; t)\,dt = \int_0^\infty (ue^{au^2 + u})^{-1}u e^{au^2 + u}(1 + u + 2au^2)\,du = \int_0^\infty u e^{(au^2 + u)}(1 + u + 2au^2)\,du.$$ 

The integrals

$$I_{s,a,b} = \int_0^\infty u^{s+b}e^{(au^2 + u)}\,du$$

can be expressed in terms of the confluent hypergeometric function as

$$I_{s,a,b} = \frac{1}{2}(-as)^{(-b-s-2)}\left[\sqrt{-as}\Gamma\left(b + s + 1, \frac{1}{2}\right) + s\Gamma\left(b + s + 2, \frac{3}{2}; \frac{s}{4a}\right)\right].$$

Using this expression for $I_{s,a,b}$ with $b \in \{0, 1, 2\}$ and some algebra will give the formula in the theorem. The signs of $s$ and $a$ must be opposite in order to have convergent integrals. Moreover, as $W(a; t)$ behaves around $t = 0$ as $t$, so the integrand in (14) behaves as $t$ around zero. This explains why $s > -1$ is necessary.

**Remark 5.** The Laplace transform of the Lambert $W$ function would result in iterated-exponential integrals (integrals with integrands of the form $e^{e^t}$) which do not have a closed form. One therefore takes $W \circ \exp$ instead. However, the Laplace transform of the quadratic Lambert,

$$\left(\mathcal{L}W(a; x)\right)(s) = \int_0^\infty e^{-st}W(a; t)\,dt$$

seems to be a harder nut. In order to determine it, one needs to find a closed form of the integrals

$$\int_0^\infty u^{s+b}e^{-(au^2 + u)}\,du,$$

where $b \in \{0, 1, 2\}$. This comes from the substitution made in the proof of the previous theorem. In particular, when $t = 0$, then $ue^{au^2 + u} = 1$, and the lower limit $W(a; 1)$ of the integral is the solution of this equation. No closed form of these integrals is known.

### 3. The real branches of the quadratic Lambert function

It is important to know how many real solutions (5) has, and how this number depends on the parameter $a$. This analysis is done below.

We begin by finding the singularities of the derivative $W'(a; x)$. Note that $W'(a; x)$ can be written as

$$W'(a; x) = \frac{W(a; x)}{x[2aW^2(a; x) + W(a; x) + 1]}.$$ (15)

The denominator is zero if and only if $x = 0$ or $W(a; x) = \frac{-1 + \sqrt{1 - 8a}}{4a}$, $\frac{-1 - \sqrt{1 - 8a}}{4a}$. Let $\alpha_a = \frac{-1 + \sqrt{1 - 8a}}{4a}$, $\beta_a = \frac{-1 - \sqrt{1 - 8a}}{4a}$ and $f_a(x) = xe^{ax^2 + x}$. Depending on the value of the parameter $a$, the quadratic Lambert $W(a;x)$ has one, two or three real branches. These branches are described in more detail in the following theorem.

**Theorem 6.** The branches of the quadratic Lambert function are as follows.

I. If $a = 1/8$, then $\alpha_a = \beta_a = -2$. Also $\text{sgn}(W(a; x)) = \text{sgn}(x)$. Moreover, $W(a; x) = 0$ if and only if $x = 0$. Thus, the singularity $x = 0$ is a removable singularity and we have only two branches for this case:

1. $W_0(a; x): \left[f_a(-2), +\infty\right) \rightarrow \left[-2, +\infty\right)$ is a strictly increasing function,
2. $W_1(a; x): (-\infty, f_a(-2)) \rightarrow (-\infty, -2)$ is also a strictly increasing function.

These two functions are differentiable on the interior of their domains.

II. If $0 < a < 1/8$, then $\alpha_a > \beta_a$ and both $\alpha_a$ and $\beta_a$ are negative real numbers. We have three branches for this case:

1. $W_0(a; x): \left[f_a(\alpha_a), +\infty\right) \rightarrow \left[\alpha_a, +\infty\right)$ is a strictly increasing function.
2. $W_1(a; x): [f_a(\beta_a), f_a(\alpha_a)] \rightarrow [\beta_a, \alpha_a]$ is a strictly decreasing function.
3. $W_2(a; x): (-\infty, f_a(\beta_a)] \to (-\infty, \beta_a]$ is a strictly increasing function. These three functions are differentiable on the interior of their domains. Also $\text{sgn}(W(a; x)) = \text{sgn}(x)$.

III. If $a < 0$, then $\alpha_a < 0$, $\beta_a > 0$. We also have three branches for this case.

1. $W_1(a; x): [f_+(\alpha_a), +\infty) \to [\beta_a, +\infty)$ is a strictly increasing function.

2. $W_0(a; x): [f_-(\alpha_a), f_+(\beta_a)] \to [\alpha_a, \beta_a]$ is a strictly decreasing function.

3. $W_2(a; x): (-\infty, f_-(\alpha_a)] \to (-\infty, \alpha_a]$ is a strictly increasing function.

These three functions are differentiable on the interior of their domains.

IV. If $a > 1/8$, $W'(a; x)$ is undefined only at $x = 0$ but because $W(a; 0) = 0$, this singularity is removable. In this case we only have one branch given by $W(a; x): \mathbb{R} \to \mathbb{R}$ is a strictly increasing function which is differentiable for all $x$.

Proof. In each case, the domains are determined by looking at the singularities of $W'(a; x)$. We will prove Case II only. The other cases may be done similarly. Let $u = W(a; x)$. Since $x = 0$ is a removable singularity, we have only three possibilities $u > \alpha_a$, $\beta_a < u < \alpha_a$, and $u < \beta_a$. These constitute the branches of the function. The domains can then be readily determined to be those specified in the theorem. To see whether the function is increasing or decreasing on the branch, the first derivative test is used.

We can write $W'(a; x) = \frac{W(a; x)}{x(u - \alpha_a)(u - \beta_a)}$. Note that $\frac{W(a; x)}{x} > 0$ for all $x$. When $u \geq \alpha_a$, then $u - \alpha_a > 0$, $u - \beta_a > 0$, hence $W'(a; x) > 0$ for such $x$ in the interior of the corresponding domain. Thus, the function is strictly increasing. When $\beta_a < u < \alpha_a$, then $u - \beta_a > 0$ and $u - \alpha_a < 0$. Hence $W'(a; x) < 0$ for all $x$ in the interior of the corresponding domain. Consequently, the function is strictly decreasing. Lastly, when $u < \beta_a$, we have $u - \beta_a < 0$ and $u - \alpha_a < 0$, hence $W'(a; x) > 0$ for all $x$ in the interior of the corresponding domain. Thus $W(a; x)$ is strictly increasing.

4. Asymptotics of $W(a; x)$

In this section asymptotic formulas of the quadratic Lambert function are derived valid for positive values of $a$.

Theorem 7. For $a > 0$,

\[
W(a; x) \sim -\frac{1}{2a} + \frac{1}{2a} \sqrt{2a \log \left(\frac{x^2}{a \log(x)}\right)} + 1, \quad (x \to \infty) \tag{16}
\]

and

\[
W(a; x) \sim -\frac{1}{2a} - \frac{1}{2a} \sqrt{2a \log \left(\frac{x^2}{a \log(x)}\right)} + 1, \quad (x \to -\infty). \tag{17}
\]

Proof. Applying the logarithm to both sides of (5) yields

\[
y^2 \left(\frac{\log(y)}{y^2} + a + \frac{1}{y}\right) = \log(x).
\]

Since $y \to \infty$ as $x \to \infty$, the left-hand side \~$\sim ay^2$. Thus,

\[
y = \frac{1}{a} \log(x). \tag{18}
\]

Using (18), $W(a; x)$ can be written

\[
W(a; x) = \sqrt{\frac{1}{a} \log(x)} + u(x), \tag{19}
\]

where $u(x)$ is to be determined. Then, (5) gives

\[
\sqrt{\frac{1}{a} \log(x)} \left[1 + \frac{u(x)}{\sqrt{\frac{1}{a} \log(x)}}\right] e^{au(x) + \left(\frac{1}{2a} \sqrt{\frac{1}{a} \log(x)} + 1\right) u(x) + \frac{1}{4a} \log(x)} = 1.
\]
Since \(|u(x)| \ll \frac{1}{\sqrt{a}} \log(x)\) if \(x\) is large, approximately, we obtain
\[
e^{au(x) + \left(2a \sqrt{\frac{1}{a}} \log(x) + 1\right) u(x) + \frac{1}{2} \log(x)} = \frac{1}{\sqrt{a}} \log(x).
\]
Applying the logarithm to both sides yields
\[
aux^2(x) + \left(2a \sqrt{\frac{1}{a}} \log(x) + 1\right) u(x) + \frac{1}{2} \log(x) = -\log\left(\frac{1}{\sqrt{a}} \log(x)\right).
\]
By completing the square, we obtain
\[
\left[ u(x) + \frac{2a \sqrt{\frac{1}{a}} \log(x) + 1}{2a} \right]^2 = \frac{1}{4a^2} + \frac{1}{a} \log\left(\frac{x}{\sqrt{\frac{1}{a}} \log(x)}\right).
\]  \tag{20}
Taking the positive square root, (20) becomes
\[
u(x) = -\frac{1}{\sqrt{a}} \log(x) - \frac{1}{2a} + \frac{1}{2a} \sqrt{2a \log\left(\frac{x^2}{\frac{1}{a} \log(x)}\right)} + 1.
\]  \tag{21}
Substituting (21) to (19), (16) is obtained.

For the second part of the theorem, let \(x \to -\infty\). Then, \(y \to -\infty\). Write \(x = -|x|\) and \(y = -|y|\) in (5) to yield
\[
|y|e^{ay} - e^{-|y|} = |x|.
\]
Since \(\frac{|x|}{|y|} < 1\), the left-hand side \(\sim e^{ay}\). Thus, approximately, \(e^{ay} = |x|\). Applying the logarithm to both sides gives
\[
y^2 = \frac{1}{a} \log(|x|).
\]  \tag{22}
Since \(y \to -\infty\), we only take the negative square root and (22) becomes
\[
y = -\frac{1}{\sqrt{a}} \log(|x|).
\]  \tag{23}
Using this, write \(W(a, x)\) as
\[
W(a, x) = -\frac{1}{\sqrt{a}} \log(|x|) + u(x).
\]  \tag{24}
By the same argument in deriving (16), we obtain
\[
u(x) = \frac{1}{\sqrt{a}} \log(|x|) - \frac{1}{2a} - \frac{1}{2a} \sqrt{2a \log\left(\frac{x^2}{\frac{1}{a} \log(|x|)}\right)} + 1.
\]  \tag{25}
Substituting (25) to (24), we get (17).

\begin{remark}
For the case \(a < 0\) and \(x \to -\infty\), write \(a = -|a|\) and \(x = -|x|\). Since \(y < 0\) if \(x < 0\), write \(y = -|y|\). Then the defining formula \(ye^{ay^2 + y} = x\) becomes
\[
\frac{|y|}{e^{||y||^2}} e^{-|a||y|^2} = |x|.
\]
Note that \(\frac{|y|}{e^{||y||^2}} < 1, \quad \forall \ y\). Thus, the left-hand side is less than \(e^{-|a||y|^2}\) while the right-hand side is equal to \(|x|\). If \(x \to -\infty\), then \(|x| \to +\infty\). That is, the right-hand side \(\to +\infty\). However, there is no way that the left-hand side which is less than \(e^{-|a||y|^2}\) can go to \(+\infty\). Thus, there is no asymptotic solution for this case.
\end{remark}

Remark 9. Computation using Mathematica and (16) will give
\[
W(3; 6) \approx \text{QuadraticLambert}[3, 6] = 0.69674341.
\]
It can be verified that
\[
0.69674341e^{3(0.696 743 41)^2 - 0.696 743 41} \approx 6.000 000 048 \approx 6.
\]
Similarly, using (17),
\[
W(0.5; -4) \approx \text{QuadraticLambert}[0.5, -4] = -2.4169
\]
and it can be verified that

$$(-2.4169)e^{0.5(-2.4169)^2-2.4169} = -3.9999 \sim 4.$$  

\[5\text{. Implications to the Solution } Y(z)\]

Let \( u = u(z) = -\frac{2}{3\varepsilon} \exp \left( -\frac{Cz}{2\varepsilon} \right) \). The result in proposition 1 becomes

$$Y(z) = -A \left[ 1 + \frac{1}{2} W \left( \frac{1}{8}; \frac{1}{u} \right) \right].$$

It follows from theorem 2 that

$$\frac{dY}{dz} = -A \left[ \frac{1}{2} \frac{d}{du} W \left( \frac{1}{8}; \frac{1}{u} \right) \right] = -A \left[ \frac{C e^{-\frac{Cz}{2\varepsilon}} W \left( \frac{1}{z}; u \right)}{2u A e^{-\frac{Cz}{2\varepsilon}} \left( 1 + 2\frac{1}{8} W \left( \frac{1}{8}; \frac{1}{u} \right) + W \left( \frac{1}{8}; \frac{1}{u} \right) \right)} \right].$$

The Taylor series expansion of \( W \left( \frac{1}{z}; u \right) \) in (13) will give the following Taylor series expansion of \( Y(z) \),

$$Y(z) = -A - \frac{A}{2u} + \frac{A}{2} u^2 - \frac{11A}{16} u^3 + \frac{13A}{12} u^4 - \frac{1415A}{768} u^5 - \cdots.$$

Since \( a = \frac{1}{8} \), theorem 6 implies that there are only two branches of the quadratic Lambert \( W \left( \frac{1}{z}; u \right) \). Consequently, \( Y(z) \) has only two branches, namely,

$$Y_0(z) = -A \left[ 1 + \frac{1}{2} W_0 \left( \frac{1}{8}; \frac{1}{u} \right) \right],$$

where \( W_0 \left( \frac{1}{8}; u \right) \) is strictly increasing function on \((-2, +\infty) \rightarrow \left[-2, +\infty\right)\), and

$$Y_1(z) = -A \left[ 1 + \frac{1}{2} W_1 \left( \frac{1}{8}; \frac{1}{u} \right) \right],$$

where \( W_1 \left( \frac{1}{8}; u \right) \) is strictly increasing function on \((-\infty, -2) \rightarrow \left(-\infty, -2\right)\). For positive constants \( A \) and \( C \), when \( z \rightarrow +\infty, u \rightarrow -\frac{2}{A e^2} \). Consequently,

$$Y(z) \rightarrow -A \left[ 1 + \frac{1}{2} W \left( \frac{1}{8}; \frac{-2}{A e^2} \right) \right].$$

When \( z \rightarrow -\infty, u \rightarrow -\infty \). Using the approximation formula in (17),

$$Y(z) \sim -A \left[ 1 - 4 - 4 \left\{ \frac{1}{4} \log \left( \frac{u^2}{1 + u} \right) \right\} + 1 \right] \sim A \left\{ 3 + 4 \left\{ \frac{1}{4} \log \left( \frac{u^2}{1 + u} \right) \right\} + 1 \right\}.$$

However, although the Mellin transform of the quadratic Lambert \( (\mathcal{M}W (a; x)) (s) \) exists for \( a > 0 \) and \(-1 < s < 0\) as given in theorem 4, the Mellin transform of \( Y(z) \) does not exist for any value of \( s \).

\[6\text{. Conclusion}\]

It is shown that the quadratic Lambert function has application to Einstein–Maxwell fields and such application is used as motivation of the study. Considered as a new function some analytic properties of the quadratic Lambert function are proved. These properties have implications to the solution \( Y(z) \) to equation (4) as presented in section 5. It is also interesting to study the integral of the quadratic Lambert function and its applications to nonlinear differential equations.

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