Feedback maximum principle
for ensemble control of local continuity equations.
An application to supervised machine learning

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\textbf{Abstract}—We consider an optimal control problem for a system of local continuity equations on a space of probability measures. Such systems can be viewed as macroscopic models of ensembles of non-interacting particles or homotypic individuals, representing several different "populations". For the stated problem, we propose a necessary optimality condition, which involves feedback controls inherent to the extremal structure, designed via the standard Pontryagin’s Maximum Principle conditions. This optimality condition admits a realization as an iterative algorithm for optimal control. As a motivating case, we discuss an application of the derived optimality condition and the consequent numeric method to a problem of supervised machine learning via dynamic systems.

I. INTRODUCTION

This article is devoted to a particular (relatively simple) class of optimal control problems for transport equations in the space of probability measures. In the last few years, such systems have become a very popular area of the modern applied mathematics, mainly thanks to the recent progress in the analysis on metric spaces of probability measures, and in the optimal transportation theory\cite{1}–\cite{3}. Among prominent applications of such developments is the analysis of various models of collective behavior, pedestrian dynamics and crowd motion control (see, e.g., the survey \cite{4} and citations therein).

Our study is to some extent aligned with articles \cite{5}–\cite{7}, which present another attractive field of application of differential equations in spaces of measures, namely, in the area of artificial neural networks (NNs); it was whipped up by recent works \cite{8}–\cite{11} that promote an approach to supervised machine learning via dynamical systems. This approach delegates the role of the “machine”, supposed to solve a problem of classification or regression, to a controlled ODE, which can be interpreted as an “infinitely deep” NN.

To give the intuition, we treat the binary classification problem, where a machine is trained to “weed out the bad from the good”, based on given sample data. The idea is, roughly speaking, to regard the set of samples as a crowd of individuals representing different “populations”, and try to find a common control that steers these two populations to different targets. Imagine, for example, a hillock of wheat on a plate, contaminated by chaff. Can we separate the wheat from the chaff by shaking the plate in a horizontal plane? Theoretically, the answer is positive, if we idealize the cereals and the particles of chaff up to material points $x_i \in \mathbb{R}^2$ and $y_j \in \mathbb{R}^2$, assume that they do not influence one another, and rightly choose the way of shaking.

Thanks to the known results on the controllability of ODE’s implied by the classical Chow–Rashevskii theorem from the sub-Riemannian geometry, for any $T > 0$, and any finite sets of points $x_i \in \mathbb{R}^n$ and $y_j \in \mathbb{R}^n$, there is a finite number $m$ of vector fields $f_k$ and a smooth control $u = (u^1, …, u^m)$ such that the respective solution $t \mapsto X_T[u](\eta)$ to the Cauchy problem

$$\dot{x} = \sum_{k=1}^{m} f^k(x) u^k, \quad x(0) = \eta,$$

enjoys the property:

$$X_T[u](x_i) = \xi_i, \quad X_T[u](y_j) = \zeta_j, \quad \forall i, j.$$

Here, $\{\xi_i\} \subset \mathbb{R}^n$ and $\{\zeta_j\} \subset \mathbb{R}^n$ are finite sets of arbitrary pairwise different points having the cardinalities $N$ and $M$ of $\{x_i\}$ and $\{y_j\}$, respectively.

In the above speculative example, $\{\xi_i\}$ and $\{\zeta_j\}$ can be grouped arbitrarily close to two distant points, $\xi \in \mathbb{R}^2$ and $\zeta \in \mathbb{R}^2$, which means that the cereals and the chaff are indeed separated into two hillocks, i.e., they are \textit{classified}.

In practice, to provide the above \textit{generic universal interpolation} property, one can take seven vector fields $f_k$, which mimic residual NNs \cite{9}. As soon as one has a control at hand with such a property, it can be applied to new data, beyond the training sample, and, if the points of these new data were close enough to some used in the training, it is natural to expect an accurate classification due to the continuity of the map $\eta \mapsto X_T[u](\eta)$.

The beautiful result on the controllability, however, does not hint, how to design the actual control, which solves our problem:

$$\sum_i (X_T[u](x_i) - \xi)^2 + \sum_j (X_T[u](y_j) - \zeta)^2 \to \min.$$
If the numbers of points $x_i$ and $y_j$ were few, we could recruit standard tools of the optimal control theory (Pontryagin’s Maximum Principle and Dynamic Programming), as well as existing numeric algorithms, see, e.g., [8], [12]. However, if the cardinalities $N$ and $M$ of the sets $\{x_i\}$ and $\{y_j\}$ are relatively large, the classical methods are either not applicable anymore or numerically inefficient.

In this article, we propose a method for numeric implementation of such problems for the case $N, M \to \infty$. Our approach combines the “averaging principle” from the control of multi-agent systems with the technique of “feedback control variations” involving the constructions of Pontryagin’s Maximum Principle, adapted from [13], [14].

By the first principle, the sets $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^M$ are replaced by empirical probability measures $\frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ and $\frac{1}{M} \sum_{j=1}^M \delta_{y_j}$, while the respective system of $N + M$ control ODEs turns into a system of two PDEs, namely, local transport equations. For the resulted distributed control problem, we design a deterministic optimization technique, which uses the classical Pontryagin’s Maximum Principle, albeit in a non-standard way, to generate feedback controls. As we show, this approach serves to escape local extrema, and therefore has a potential to find a global solution.

II. PROBLEM STATEMENT

A. Notations

Given a metric space $X$, we use the standard notation $C([0, T]; X)$ for the spaces of continuous maps $[0, T] \to X$ (endowing it with the usual sup-norm), and $L_p([0, T]; \mathbb{R}^n)$, $p = 1, \infty$, for the Lebesgue quotient spaces of summable and bounded measurable functions $[0, T] \to \mathbb{R}^n$, respectively.

By $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$, we denote the set of probability measures on the vector space $\mathbb{R}^n$, and by $\mathcal{P}_1(\mathbb{R}^n)$ the subset of $\mu \in \mathcal{P}$ having finite first moment, i.e., such that $m_1(\mu) = \int_{\mathbb{R}^n} |x| \, d\mu(x) < \infty$. Recall that $\mathcal{P}_1$ is a complete separable metric space when it is endowed with the 1-Kantorovich (Wasserstein) distance

$$ W_1(\mu, \nu) = \sup \left\{ \int_{\mathbb{R}^n} \varphi \, d(\nu - \mu) \mid \varphi \in C(\mathbb{R}^n; \mathbb{R}), \text{Lip} (\varphi) \leq 1 \right\}. $$

Here, Lip$(\varphi)$ is the minimal Lipschitz constant of $\varphi$.

Given $\mu \in \mathcal{P}$ and a Borel measurable map $F : \mathbb{R}^n \to \mathbb{R}^n$, $F_\mu \mu$ denotes the push-forward $\mu \circ F^{-1}$ of $\mu$ through $F$.

$\mathcal{L}^n$ stands for the usual Lebesgue measure on $\mathbb{R}^n$.

B. Finite-Dimensional Model

To simplify the presentation, we will consider the case of two classes of individuals, “$x$” and “$y$”, whose states at time $t$ are represented by vectors $x_i(t), y_j(t) \in \mathbb{R}^n$, $i = 1, N$, $j = 1, M$ (hereinafter $i = 1, N$ abbreviates $i \in \{1, \ldots, N\}$).

An essential hypothesis, we are to accept here, is that all $x_i$ are homotypic and so are all $y_j$. In other words, $x_i$ and $x_k$ are indistinguishable to us, though we keep to distinguish $x_i$ from $y_j$, for any $i$ and $j$.

The representatives of both classes find themselves in a common environment, but may have different properties, which means that the action of the media on each class may be different. This action results in a drift of points $x_i$ and $y_j$ in accordance with different vector fields $f$ and $g$.

Given a fixed time interval $[0, T]$, the individuals $x_i$ and $y_j$ are aimed at minimizing the values $\ell_1(x_i(T))$ and $\ell_2(y_j(T))$, where the cost functions $\ell_{1, 2} : \mathbb{R}^n \to \mathbb{R}$ are the same for all representatives of a class.

Playing the part of a guide, we can influence the media (i.e., all the individuals of both classes, simultaneously) via a vector field $v : \mathbb{R}^n \times U \to \mathbb{R}^n$ depending on a control parameter $u$ from a given set $U \subset \mathbb{R}^n$. As feasible control signals, we admit (equivalence classes of) measurable functions $[0, T] \to U$. The space $U = L_\infty([0, T]; U)$ of controls is endowed with the weak-* topology $\sigma(L_\infty, L_1)$.

Thus, we come to the optimal control problem $(P_{N, M})$:

$$ \inf \left( \frac{1}{N} \sum_{i=1}^N \ell_1 (x_i(T)) + \frac{1}{M} \sum_{j=1}^M \ell_2 (y_j(T)) \right) $$

subject to

$$ x_i(0) = x_i^0, \quad x_i = f(x_i) + v_u(x_i), \quad i = 1, N, \quad y_j(0) = y_j^0, \quad y_j = g(y_j) + v_u(y_j), \quad j = 1, M, $$

$$ t \in [0, T], \quad u \in U, $$

where $x_i^0$ and $y_j^0$ are given initial positions of the individuals. Note that $(P_{M, N})$ can be viewed as a version of the mean field social control problem (see, e.g., [15], [16]).

C. Standing Assumptions

We make the standard regularity hypotheses (H1):

$U \subset \mathbb{R}^m$ is compact; vector fields $F_u(x) = f(x) + v_u(x)$ and $G_u(x) = g(x) + v_u(x)$ are continuous in $(x, u)$, and there exist $C > 0$ such that

$$ |F_u(x)| \leq C(1 + |x|) \quad \text{and} \quad |F_u(x) - F_u(y)| \leq C|x - y| $$

for any $u \in U$ and all $x, y \in \mathbb{R}^n$ (similarly for $G_u$); $\ell_{1, 2}$ are continuous.

D. Problem Reformulation in Terms of Probability Measures

Now we shall pass to the so-called mean field limit of the ODE (1) and (2). For this, consider the curves $t \mapsto \mu^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)}$ and $t \mapsto \nu^M_t = \frac{1}{M} \sum_{j=1}^M \delta_{y_j(t)}$ in $\mathcal{P}_1(\mathbb{R}^n)$. Assume that

$$ \mu^N_0 = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^0} \to \vartheta_1 \quad \text{and} \quad \nu^M_0 = \frac{1}{M} \sum_{j=1}^M \delta_{y_j^0} \to \vartheta_2 \quad (4) $$

for some $\vartheta_i \in \mathcal{P}_1$, $i = 1, 2$. Then the curves $t \mapsto \mu^N_t$ and $t \mapsto \nu^M_t$ converge $^3$ in $C([0, T]; \mathcal{P}_1)$ to distributional solutions $t \mapsto \mu_t$ and $t \mapsto \nu_t$ of the continuity equations

$$ \partial_t \mu_t + \nabla \cdot \left( [f + v_u(t)] \mu_t \right) = 0, \quad \mu_0 = \vartheta_1, \quad (5) $$

$$ \partial_t \nu_t + \nabla \cdot \left( [g + v_u(t)] \nu_t \right) = 0, \quad \nu_0 = \vartheta_2. \quad (6) $$

Here, $\partial_i$ abbreviates $\frac{\partial}{\partial x_i}$, “$\cdot$” denotes the scalar product, and $\nabla \equiv \frac{\partial}{\partial x}$. Note that equations (5) and (6) are independent

$^2$This convergence follows from (4) and the continuous dependence of the solutions to (5) and (6) on the initial measures $\vartheta_i$, $i = 1, 2$ (see, e.g., [17, Lemma 2.8]).
and paired only by the control function $u$. As it is standard, solutions to PDEs (5) and (6) are understood in the sense of distributions, and can be represented as
\[ \mu_\ell = (X_\ell)_t \partial_1, \quad \nu_\ell = (Y_\ell)_t \partial_2, \quad t \in [0, T], \]
where $t \mapsto X_\ell(x) \equiv X_\ell[u](x) \in C([\mathbb{R}^n; \mathbb{R}^n])$ and $t \mapsto Y_\ell(x)$ denote the flows of the characteristic ODEs (1) and (2).

Thus, we come to the optimal control problem (P) in the space of probability measures:
\[ \inf_{u \in U} \left( \int_0^T \ell_1 \, dt \right) \mu_T[u] + \int_0^T \ell_2 \, dnu_T[u] \] subject to (5) and (6).

Hereinafter, we abbreviate $f = \int f \, d\mu$ and drop arguments of integrands for brevity.

E. Pontryagin’s Maximum Principle

To approach problem (P), we will recruit the following assertion, which is a version of the paradigmatic Pontryagin’s Maximum Principle (PMP).

Proposition 1: Assume that hypotheses $(H_1)$ hold together with the following additional assumption
\[(H_2): F_u \text{ and } G_u \text{ are continuously differentiable in } x.\]
Let $\sigma \equiv (\mu, \nu, u)$ be an optimal control process. Then, for $L^1$-almost all (a.a.) $t \in [0, T]$, $u$ satisfies the maximum condition
\[ u(t) \in \arg \max_{u \in U} \left( \int \nabla p \cdot v_u \, d\mu_t + \int \nabla q \cdot v_u \, d\nu_t \right), \] where $p$ and $q$ are solutions of the dual transport equations
\begin{align*}
\partial_t p + \nabla p \cdot (f + v_u(t)) &= 0, \quad p_T = -\ell_1, \quad \text{(7)} \\
\partial_t q + \nabla q \cdot (g + v_u(t)) &= 0, \quad q_T = -\ell_2. \quad \text{(8)}
\end{align*}

Solutions to (8), (9) are also understood in the sense of distributions and are constructed by the method of characteristics as $p_t = -\ell_1 \circ (X_t)^{-1}$, $q_t = -\ell_2 \circ (Y_t)^{-1}$, $t \in [0, T]$.

The proof of Proposition 1 follows closely the one in [18, Theorem 2]. This assertion gives a necessary condition for local extremum within the class of so-called needle variations of control function. Here, one can point out that (P) is a nonlinear (and nonconvex) problem, even if $v_u$ is linear in $u$: although the cost is linear in the measure, the dynamics contains the product $u\mu$. This means that PMP shall not be a sufficient optimality condition. In other words, as it is inherited from control of ODEs, Proposition 1 does not select out global solutions to problem (P), and, therefore, it has a potential for “improvement”. Towards this, we propose to adopt the approach [13], [14] from the finite-dimensional control theory, which enables to extract additional information from the extremal condition (7).

The idea is to use the “local information” (a reference control process), to generate feedback signals of the “extremal structure”. Realization of such feedbacks through a standard sampling scheme leads to a class of control variations, which appears to be somewhat richer than the usual class of needle variations. The optimality qualification within this new class of variations results in a necessary condition that remains within the formalism of PMP, but demonstrates a greater potential to discard non-optimal extrema.

III. FEEDBACK MAXIMUM PRINCIPLE

To simplify the exposition of the main idea, we now focus on the case of a single population by setting $\ell_1 = \ell, \ell_2 \equiv 0$. Then, our problem reduces to
\[ I[u] = \int_0^T \ell \, dt \rightarrow \min_{u \in U} \text{subject to (5)}. \]

Similarly to [18], one can derive the following exact formula for the increment of the cost functional in terms of solutions to (5) and (8):
\[ \Delta_u I = \int_0^T \, dt \int \nabla p_\ell \cdot (v_u(t) - v_a) \, d\mu_t. \]

This formula entails the specification of the control function $u$ in the ensemble-feedback form
\[ u_\ell[\mu] \in U_\ell^p(\mu) \equiv \arg \max_{u \in U} \int \nabla p_\ell(x) \cdot v_u(x) \, d\mu(x), \]

in which case one could expect an “improvement” of the reference control $\bar{u}: \Delta_u I < 0$. However, the realization of (10) as a control law requires the same accuracy as in the case of ODEs, due to the, possibly, discontinuous character of the ensuing vector fields.

A. Ensemble-Feedback Controls and Sampling Solutions

In this section, we shall discuss the concept of solution to feedback controlled PDE (5). Consider two types of mappings, $u : [0, T] \times \Pi \rightarrow U$ and $u : [0, T] \times \Pi \rightarrow \mathcal{U}$, that could be used to construct sampling solutions of the respective transport equations (to be discussed below); $u$ could be arbitrary, while $u$ is assumed to be Borel measurable. We denote the sets of such functions by $U$ and $\mathcal{U}$, respectively. Feedbacks of the class $U$ generate piecewise constant controls in accordance with the classical Krasovskii-Subbotin scheme, while the ones of the class $\mathcal{U}$ produce piecewise open-loop controls (short-term programs) in the spirit of the so-called model predictive control.

Now, we are going to present two sampling schemes, entailed by the proposed notions of feedback control, which are designed by usual Euler polygons.

1) $u$-Sampling Scheme: Given $u \in U$, and a partition $\pi = \{t_k\}_{k=0}^K$ of the interval $[0, T]$, $t_0 = 0$, $t_{k-1} < t_k$, $k = 0, K$, $t_K = T$, the polygonal arc $t \mapsto \mu_t^u[\mu] \in \mathcal{P}$ is defined via step-by-step integration of PDE (5) over $[t_{k-1}, t_k]$:
\[ \mu_t^u[\mu] = \mu_0^u, \quad t \in [t_{k-1}, t_k), \quad k = 0, K, \]
\[ \mu_t^0 \equiv \partial, \quad \mu_t^{k+1} \equiv \mu_t^0 \mu_t^{k+1} \right |_{t_{k-1}, \mu_t^{k+1}}[\mu_t^{k+1}] \right |_{t_{k-1}, \mu_t^{k+1}}[\mu_t^{k+1}], \quad k = 1, K, \]
and $\mu[u](\tau, \vartheta)$ is a distributional solution of the continuity equation with control $u$, and the initial condition $\mu^0 = \partial$.

Along with the polygonal arc, we compute a piecewise constant control
\[ u^* = u_{t_{k-1}}[\mu_{t_{k-1}}^{k+1}] \text{ on } [t_{k-1}, t_k), \quad k = 0, K. \]

Now, $u$-sampling solution is introduced as any partial limit in $C([0, T]; \mathcal{P}_1)$ of a sequence of the above polygons as$\lim_{\varepsilon \to 0} \max_{1 \leq k \leq K} (t_k - t_{k-1}) \rightarrow 0$; $\mathcal{S}_1(\mathcal{P}_1)$ stands for the set of all such solutions, produced by $u$. 
2) u-Sampling Scheme: For \( u \in \mathfrak{U} \), we design a polygonal arc \( t \mapsto \mu^t \in \mathcal{P} \) by step-by-step integration (11), where \( \mu^k, k = 1, K \), are defined by

\[
\mu^0 = \emptyset, \quad \mu^k = \mu \bigl[ u(t) \mu^{k-1} \bigr] (t_{k-1}, t_k),
\]

and the associated control is a concatenation

\[
u^*(t) = u_k \mu^{k-1}, \quad t \in [t_{k-1}, t_k), \quad k = 1, K.
\]

Up to the mentioned difference, the notion of u-sampling solution and the set \( \mathcal{S}_2(u) \) are introduced as above.

The following assertion is implied by a generalized version of the Arzela-Ascoli theorem, if we notice that, thanks to \( (H_1) \), polygonal arcs \( \mu^T \in C([0, T]; \mathcal{P}_1) \) are uniformly Lipschitz continuous with a constant depending only on \( C, T \) and \( m_1(\mathcal{P}) \) [19, Lemma 3] (and therefore, by the Gronwall’s lemma, are equicontinuous, and uniformly bounded).

**Proposition 2:** Under assumptions \( (H_1), \mathcal{S}_1(u) \neq \emptyset \) and \( \mathcal{S}_2(u) \neq \emptyset \), for any \( u \in \mathcal{U} \) and \( u \in \mathfrak{U} \).

Simple examples [20, Appendix A] show that \( \mathcal{S}_1(u) \) and \( \mathcal{S}_2(u) \) are not proper subsets of each other, even in the case of ODEs.

Given \( u \) or \( u \), along with respective sampling solutions, one can consider the associated classical feedback solutions, defined analogously to the Carathéodory feedback solutions of ODEs, [13]. Notice that, in general, there could be no classical feedback solutions at all.

**B. Optimality Qualification via Feedbacks**

We are ready to present a necessary optimality condition, which employs feedback controls of the PMP-extremal structure, proposed by formula (10). For brevity, we will operate with feedbacks of the type \( \mathcal{U} \) though it is possible to take the ones of the sort \( \mathfrak{U} \), or both.

Consider a reference process \( \bar{\sigma} = (\bar{\mu}, \bar{u}) \), whose optimality is to be checked. Taking \( w \in \mathcal{U} \), let \( \mathcal{S}(w) \) denote the union of the set \( \mathcal{S}_1(w) \), and the, possibly empty, set of the respective classical feedback solutions. Introduce the following accessory problem associated to \( \bar{\sigma} \):

\[
(AP_{\bar{\sigma}}) \quad \text{Minimize} \quad \int \ell d\xi, \quad \xi \in \mathcal{S}(w), \quad w = w_2[\mu] \in U^p_1(\mu).
\]

Here, \( U^p_1 \) is defined as in (10). Note that \( (AP_{\bar{\sigma}}) \) is a variational problem, and does not involve a control anymore. The following assertion presents a variational necessary optimality condition for a single-population version of problem \( (P) \), which we call the feedback maximum principle (FMP).

**Theorem 1:** Assume that \( (H_1) \) and \( (H_2) \) hold. Then the optimality of \( \bar{\sigma} = (\bar{\mu}, \bar{u}) \) for \( (P) \) implies the optimality of \( \bar{\mu} \) for \( (AP_{\bar{\sigma}}) \).

The proof is a combination of two simple facts:

i) An optimal process \( \bar{\sigma} \) is PMP extremal, and therefore \( \bar{\mu} \) is admissible for \( (AP_{\bar{\sigma}}) \) as a classical feedback solution (note that \( \bar{u}(t) \in U^p_1(\bar{\mu}) \) for \( L^1 \)-a.a. \( t \in [0, T] \)), and

ii) Sampling solutions can, by definition, be uniformly approximated by usual distributional solutions corresponding to open-loop controls.

Indeed, if one is managed to find a feedback control \( \bar{w} \in U^p \), and a sampling solution \( \tilde{\xi} \in \mathcal{S}_1(\bar{w}) \) such that \( \int \ell d\bar{\mu} > \int \ell d\tilde{\xi} \), the same inequality also holds for polygonal arcs \( \xi^\pi \) approximating \( \tilde{\xi} \) and corresponding to certain piecewise constant open-loop control \( \bar{w}^\pi \in \mathfrak{U} \), for sufficiently small \( \text{diam}(\pi) \): \( \int \ell d\bar{\mu} > \int \ell d\xi^\pi = \int \ell d\mu(\bar{w}^\pi) \), which contradicts the optimality of \( \bar{\sigma} \).

One has to admit that, in the presented variational form, the result is not easy to grasp. To benefit from FMP, one can use its counter-positive version, what means to treat Theorem 1 as a sufficient condition for non-optimality via the qualification condition:

\[
\int \ell d\bar{\mu} \leq \int \ell d\xi, \quad \forall \xi \in \mathcal{S}_1(w) \quad \forall w \in U^p.
\]

In other words, given a reference process \( \bar{\sigma} = (\bar{\mu}, \bar{u}) \) (which could be a local extremum) with a dual state \( \bar{p} \), one can try to “improve” \( \bar{\sigma} \) by testing it against feedbacks, running through the set \( U^p \).

In the following simple (but eloquent) example, PMP can not distinguish the “worst” process from the “best” one, while condition (16) does by “playing” with the non-uniqueness of signals of the extremal structure.

**Example 1 (Weeding out PMP extremum by FMP):**

Consider a version of our problem in \( \mathbb{R}^2 \) with \( \ell(a, b) = b \), \( f(a, b) = 0, u_\ell(a, b) = (u, -au) \), and the following data: \( T = 1, U = [-1, 1] \), and \( \bar{\theta}(a, b) = \delta(a) \otimes \eta(b) (\otimes \text{denotes the product measure}) \) with an arbitrary \( \eta \in \mathcal{P}(\mathbb{R}) \).

As one can easily check, the constant control \( \bar{u} \equiv 0 \) is PMP-extremal with the co-state \( \bar{p}(a, b) = -b \). The extremal multifunction is specified as \( U^p_1(\bar{\mu}) = \text{sign} \int a d\mu(a, b); \text{sign} s = s/|s| \text{ for } s \neq 0, \text{and sign} 0 = [-1, 1] \text{ by definition}.

Let us apply Theorem 1. Since \( \int a d\bar{\theta} = \int a d\delta_0(a) \), we can choose any control \( u \in \text{sign} 0 = [-1, 1] \text{ as the initial guess}. \) In our option, take \( u = 1 \). This control shifts the mass from the axis \( a = 0 \) to the right \( (a > 0) \); both sampling schemes leave the strategy \( u = 1 \) intact and lead to a single solution \( t \mapsto \mu_t = \left[(a, b) \mapsto (a + t, b - at - t^2/2)\right]_{t^2} \bar{\theta} \), which gives \( \int b d\eta(b) - \delta^2 = I[u] < I[\bar{u}] = \int b d\eta(b). \)

One can check that this is, in fact, a global solution, and the choice \( u = -1 \) leads to the same result. Furthermore, any \( u \in [-1, 1] \setminus \{0\} \) also improves the reference extremal \( (\bar{u}, \bar{\mu}) \), and, therefore, qualifies it as non-optimal.

C. Iterations of Feedback Maximum Principle. Strategy for Numeric Implementation

FMP can be implemented iteratively in the following manner, taken desired accuracies \( \varepsilon_i > 0, i = 1, 2 \):

0. Given an initial guess \( \bar{u} \in \mathfrak{U} \), compute \( \bar{p} \) as a solution to (8) with \( u = \bar{u} \).
1. Choose a feedback control \( w \in U^p \).
2. Fix a parameter \( \pi \in [0, T] \), compute the polygonal arc \( \mu^\pi[w] \) together with the open-loop control \( u^\pi \).
3. Check the “improvement” property \( I[w^\pi] - I[\bar{u}] < \varepsilon_1 \). If it holds then update the control: \( \bar{u} = u^\pi \). Otherwise, check \( \text{diam}(\pi) < \varepsilon_2 \). If “yes”, return to step 2. Else, go to step 1.
In practice, control variations provided by feedbacks \( w \in U^P \) could be too strong, such that the associated feedback solution would not ensure the improvement property. To fix this problem, on Step 2, one can use a “localized” feedback

\[
\omega[t] = \alpha \bar{u}(t) + (1 - \alpha) w[t] \tag{17}
\]

involving extra parameter \( \alpha \in (0, 1) \) to be adjusted.

Let us stress two features of the proposed conceptual numeric method. Notice that the control at each step is designed based on the aggregated, macroscopic information, which is an integral characteristic of the ensemble of particles. In practice, this means that there is no need to trace actual positions of all individuals, as it is enough to follow their “averaged representative”, and this provides an essential dimensionality reduction.

Another point to be mentioned is that a realization of the proposed algorithm requires numeric solution of Cauchy problems only. The latter is essentially simpler from the computational viewpoint than a shooting method for a boundary-value problem, or dynamic programming via the Hamilton-Jacobi equation, which is known to be a more complicated task than the original optimal control problem.

IV. MACHINE LEARNING

Now, let us return to the multi-population framework and revise the correlation problem discussed in the Introduction.

For the binary case (an extension to several classes is trivial) this problem is exactly of the form \((P_{N,M})\), if we regard the collections of training samples associated to different labels as representatives of different populations, specify the dynamics as \( f = g \equiv 0, v_u(x) = \sum f^k(x) \cdot u^k \), with \( f^k : \mathbb{R}^n \rightarrow \mathbb{R}^n \) providing the generic universal interpolation, and take \( \ell_1(x) = |x - \zeta|^2, \ell_2(x) = |x - \zeta|^{2/3} \) with two (distant) target points \( \zeta \neq \zeta \in \mathbb{R}^n \). Thus, all the results of this paper are applicable to the classification problem.

Above, we regarded the set of training samples as a crowd of individuals. However, a sample itself could be viewed as a population. In the following example, we learn a machine to classify simple geometric forms having only single representative of each a class as a training set. Such problems of “machine learning over small data”, which sound deliberately pointless within the paradigm of NNs, still can be dealt with by the discussed control-theoretical approach. For this, planar figures are treated as distributions on a plane, and the type of a figure marks out the respective population. If the data are properly prepared, the “classification effect” provided by a control is naturally extrapolated from a single sample measure (one copy of a figure) to all \( W_1 \)-close measures (sufficiently similar figures) thanks to the continuity of the map \((\vartheta, u) \mapsto \mu[u](\vartheta)\).

**Example 2 (Control system as a classifier for MNIST):**

Consider a very standard NN exercise: to design a classifier of handwritten digits, using the canonical database MNIST. To our preference, monochromatic pictures representing “3” and “6” were selected. As “right answers”, two sample images (one for “3” and one for “6”) were randomly chosen. The pictures are treated as discrete measures \( \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \) on the unit square, where \( N \) and \( x_i \) correlate with the power of white color. To learn the resulted finite-dimensional control system, a problem of the form \((P_{N,M})\) was solved numerically, taken \( n = 3, \xi = -\zeta = (3, 0), f^1 \equiv (0, 1), f^2(a, b) = (a + b, a), f^3(a, b) = (\sin(a), a - b) \), and \( U = \{u^1, u^2, u^3 \setminus |u^1, u^2, u^3| \leq 1\} \). The obtained control (Fig. 1) appeared to provide an accurate classification in a series of tests with randomly chosen candidates.

![Fig. 1. Numeric control for distinguishing “3” and “6” in MNIST.](Image)

Above, the problem was still finite-dimensional. In the next example, our algorithm separates two infinite populations which constitute geometric patterns (a “cross” and a “ring”) corrupted by a Gaussian blur.

**Example 3 (Classification of “uncertain” planar figures):**

The same \( f^k \) and \( U \) are used as in Example 2. Taken \( \xi = (1, 1), \zeta = -\xi, T = 2 \), Cauchy problems \((5), (6) \) and \((8), (9) \) were solved in the spatial computational domain \( (a, b) \in [-5, 5]^2 \) with Dirichlet boundary conditions corresponding to the vanishing measures on the boundaries. A standard upwind scheme (the donor-cell upwind method \([21]\)) is used, backward for \((8), (9) \) and forward in time for \((5), (6) \). The mesh spacing in both spatial directions is \( h = 5 \cdot 10^{-2} \), the time step is \( \tau = 2 \cdot 10^{-3} \), \( \alpha \) in (17) is chosen to be 0.75.

The initial measures \( \mu_0 \) and \( \nu_0 \) are depicted in Fig. 2, representing a cross and a ring, correspondingly. The initial guess for the control is constant: \( \bar{u} \equiv (1, 0, 0) \). After 30 iteration of the algorithm, we assume that the problem is solved. The terminal distributions \( \mu_T, \nu_T \) and the respective control \( u = (u^1, u^2, u^3) \) are shown in Figs 3 and 4.

V. CONCLUSION

A natural direction of future work would be an extension of the discussed framework to the case of nonlocal vector fields involving a convolutional term \( K * \mu \), which brings certain analogy with convolutional NNs. The PMP for ensemble control of such nonlinear transport equations does exist \([22]\), but the Hamiltonian system appears to be inseparable into the direct and dual subsystems, which makes the realization of our approach a challenging issue.

In what concerns applications to machine learning, the real potential of the presented approach needs investigation via numeric experiments, and this is another part of our future study. At present, as a mission of control-theoretical considerations of machine learning tasks, we see the new
light shed on mathematical principles behind the machinery of the artificial intelligence.

REFERENCES

[1] L. Ambrosio, N. Gigli, and G. Savaré, Gradient flows in metric spaces and in the space of probability measures, 2nd ed. Basel: Birkhäuser, 2008.

[2] F. Santambrogio, Optimal transport for applied mathematicians : calculus of variations, PDEs, and modeling, 1st ed. Cham, Switzerland: Birkhäuser.

[3] C. Villani, Optimal transport. Old and new, ser. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 2009, vol. 338. [Online]. Available: http://dx.doi.org/10.1007/978-3-540-71050-9

[4] B. Piccoli and F. Rossi, “Measure-theoretic models for crowd dynamics,” in Crowd Dynamics, Volume 1. Modeling and Simulation in Science, Engineering and Technology, B. N. Gibelli L., Ed. Birkhäuser, Cham, 2018, pp. 137–165.

[5] P. Chaudhari, A. Oberman, S. Osher, S. Soatto, and G. Carlier, “Deep relaxation: partial differential equations for optimizing deep neural networks,” Research in the Mathematical Sciences, vol. 5, no. 3, p. 30, Jun 2018. [Online]. Available: https://doi.org/10.1007/s40687-018-0148-y

[6] S. Sonoda and N. Murata, “Transport analysis of infinitely deep neural network,” Journal of Machine Learning Research, vol. 20, no. 2, pp. 1–52, 2019. [Online]. Available: http://jmlr.org/papers/v20/16-243.html

[7] E. Weinan, J. Han, and Q. Li, “A mean-field optimal control formulation of deep learning,” Research in the Mathematical Sciences, vol. 6, no. 1, p. 10, Dec 2018. [Online]. Available: https://doi.org/10.1007/s40687-018-0172-y

[8] M. Benning, E. Celledoni, M. Ehrhardt, B. Owren, and C.-B. Schönlieb, “Deep learning as optimal control problems: models and numerical methods,” Journal of Computational Dynamics, vol. 6, no. 2, pp. 171–198, Dec. 2019.

[9] C. Cuchiero, M. Larsson, and J. Teichmann, “Deep neural networks, generic universal interpolation, and controlled odes,” SIAM Journal on Mathematics of Data Science, vol. 2, no. 3, pp. 901–919, 2020. [Online]. Available: https://doi.org/10.1137/19M1284117

[10] J. H. Seidman, M. Fazlyab, V. M. Preciado, and G. J. Pappas, “Robust deep learning as optimal control: Insights and convergence guarantees,” in Proceedings of the 2nd Annual Conference on Learning for Dynamics and Control, L4DC 2020, Online Event, Berkeley, CA, USA, 11-12 June 2020, ser. Proceedings of Machine Learning Research, A. M. Bayen, A. Jadbabaie, G. J. Pappas, P. A. Parrilo, B. Recht, B. Recht, C. J. Tomlin, and M. N. Zeilinger, Eds., vol. 120. PMLR, 2020, pp. 884–893. [Online]. Available: http://proceedings.mlr.press/v120/seidman20a.html

[11] E. Weinan, “A proposal on machine learning via dynamical systems,” Communications in Mathematics and Statistics, vol. 5, no. 1, pp. 1–11, Mar 2017. [Online]. Available: https://doi.org/10.1007/s40304-017-0103-z

[12] Q. Li, L. Chen, C. Tai, and E. Weinan, “Maximum principle based algorithms for deep learning,” J. Mach. Learn. Res., vol. 18, no. 1, p. 5998–6026, Jan. 2017.

[13] V. A. Dykhta, “Weakly monotone solutions of the Hamilton-Jacobi inequality and optimality conditions with positional controls,” Automation and Remote Control, vol. 75, no. 5, pp. 829–844, May 2014. [Online]. Available: https://doi.org/10.1134/S0005117914050038

[14] ——, “Nonstandard duality and nonlocal necessary optimality conditions in nonconvex optimal control problems,” Automation and Remote Control, vol. 75, no. 11, pp. 1906–1921, Nov 2014. [Online]. Available: https://doi.org/10.1134/S0005117914100222

[15] M. Huang, P. E. Caines, and R. P. Malhame, “Social optimum in mean field LQG control: Centralized and decentralized strategies,” IEEE Transactions on Automatic Control, vol. 57, no. 7, pp. 1736–1751, 2012.

[16] B.-C. Wang, H. Zhang, and J.-F. Zhang, “Mean field linear–quadratic control: Uniform stabilization and social optimality,” Automatica, vol. 121, p. 109088, 2020. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0005109820302867

[17] N. Pogodaev, “Optimal control of continuity equations,” NoDEA Nonlinear Differential Equations Appl., vol. 23, no. 2, Art. 21, 24, 2016.

[18] N. Pogodaev and M. Staritsyn, “On a class of problems of optimal impulse control for a continuity equation,” Trudy Instituta Matematiki i Mekhaniki UrO RAN, vol. 24, 2016.

[19] N. Pogodaev, “Program strategies for a dynamic game in the space of measures,” Optimization Letters, vol. 13, no. 8, pp. 1913–1925, Nov 2019. [Online]. Available: https://doi.org/10.1007/s11590-018-1318-y

[20] M. Staritsyn and S. Sorokin, “On feedback strengthening of the maximum principle for measure differential equations,” Journal of Global Optimization, vol. 76, p. 587–612, 2020. [Online]. Available: https://doi.org/10.1007/s10898-020-07908

[21] R. J. LeVeque, Finite Volume Methods for Hyperbolic Problems, ser. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2002.

[22] N. Pogodaev and M. Staritsyn, “Impulsive control of nonlocal transport equations,” Journal of Differential Equations, vol. 269, no. 4, pp. 3585–3623, 2020. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S002203962030108X