THE ISOMORPHISM CONJECTURE FOR ARTIN GROUPS

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ABSTRACT. We prove the Farrell-Jones fibered isomorphism conjecture for several classes of Artin groups of finite and affine types. As a consequence, we compute explicitly the surgery obstruction groups of the finite type pure Artin groups.

1. INTRODUCTION

In this article we are concerned about proving the fibered isomorphism conjecture of Farrell and Jones ([11]), for a class of Artin groups of both finite (also called spherical) and affine types. The classical braid group is an example of a finite type Artin group (type $A_n$), and we considered this group in [13] and [28].

The isomorphism conjecture is an important conjecture in Geometry and Topology, and much work has been done in this area in recent times (e.g. [2], [21], [22], [28]). Among other conjectures, the Borel and the Novikov conjectures are consequences of the isomorphism conjecture. Although, these two conjectures deal with finitely presented torsion free groups, the isomorphism conjecture is stated for any group. Furthermore, it provides a better understanding of the $K$- and $L$-theory of the group. It is well-known that the fibered isomorphism conjecture implies vanishing of the lower $K$-theory ($Wh(-)$, $\tilde{K}_0(-)$ and $K_{-i}(-)$ for $i \geq 1$) of any torsion free subgroup of the group. In fact, it is still an open conjecture that this should be the case for all torsion free groups.

Following Farrell-Jones and Farrell-Hsiang, in all of recent works on the isomorphism conjecture, geometric input on the group plays a significant role. Here also, we follow a similar path to prove the conjecture. We prove the following theorem.

**Theorem 1.1.** Let $\Gamma$ be an Artin group of type $A_n$, $B_n (= C_n)$, $D_n$, $F_4$, $G_2$, $I_2(p)$, $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ or $\tilde{D}_n$. Then, the fibered isomorphism conjecture wreath product with finite groups is true for any subgroup of $\Gamma$. That is, the fibered isomorphism conjecture is true for $H \wr G$, for any subgroup $H$ of $\Gamma$ and for any finite group $G$.

See Remark 2.1 for an extension of the theorem to the Artin groups corresponding to the finite complex reflection groups of type $G(de, e, r)$. Also, Theorem 1.1 is known for right-angled Artin groups, since right-angled Artin groups are $CAT(0)$ ([2]).

As an application, in Theorem 2.4, we extend our earlier computation of the surgery groups of the classical pure braid groups in [28], to the finite type pure Artin groups appearing in Theorem 1.1.

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For the classical braid group case as in [13] and [28], the crucial idea was to see that the geometry of a certain class of 3-manifolds is involved in the building of the group. In the situation of the Artin groups considered in this paper, we find that the geometry of orbifolds is implicit in the group, and we exploited this information to prove the conjecture. More precisely, some of the Artin groups are understood as a subgroup of the orbifold fundamental group of the configuration space of unordered \(n\)-tuples of distinct points, on some 2-dimensional orbifold. In the classical braid group case, the complex plane played the role of this 2-dimensional orbifold.

The proofs need vanishing of the higher orbifold homotopy groups of the configuration spaces involved. For example, we prove that this configuration space, which itself is a high dimensional orbifold, has an orbifold covering space which is a contractible manifold, provided the 2-dimensional orbifold has a similar orbifold covering space (Theorem 2.2). This result is new in the literature. In the case, when the orbifold is the complex plane, the result is well known ([10]). It is also known that the finite type Artin groups are fundamental groups of aspherical manifolds ([9]). In the affine type Artin group case, this is still a conjecture.

Here we remark that our results, as well as their proofs, for the Farrell-Jones fibered isomorphism conjecture are valid for the more general version of the conjecture as stated in [2], with coefficients in an additive category.

The rest of the paper is organized as follows. In Section 2 we recall some background on Artin groups and state our main results. Here, we define a new class of groups (orbi-braid) based on some intrinsic properties of the orbifold fundamental group of the pure braid space (Definition 2.1) of a 2-dimensional orbifold (Theorem 4.2). Section 3 contains some study of homotopy theory of orbifolds and configuration spaces which help in proving Theorem 2.2. Theorem 2.2 provides the necessary condition for the induction argument to work in the proofs. We describe some work from [1] in Section 5, which is the key input for this work. Section 6 contains the proofs of the main results. The computation of the surgery groups of pure Artin groups is given in Section 7. Finally, in Section 8 we recall the statement of the isomorphism conjecture and all the basic facts needed to prove our results.

## 2. Artin groups and statements of results

Artin groups are an important class of groups, and appear in different areas of Mathematics.

We are interested in those Artin groups, which appear as extensions of Coxeter groups by the fundamental groups of hyperplane arrangement complements in \(\mathbb{C}^n\).

The Coxeter groups are generalization of reflection groups, and is yet another useful class of groups. Several Coxeter groups appear as Weyl groups of simple Lie algebras. In fact, all the Weyl groups are Coxeter groups.

Next, we give a description of the Coxeter groups in terms of generators and relations.

Let \(K = \{s_1, s_2, \ldots, s_k\}\) be a finite set, and \(m : K \times K \to \{1, 2, \ldots, \infty\}\) be a function with the property that \(m(s, s) = 1\), and \(m(s', s) = m(s, s') \geq 2\) for \(s \neq s'\). The Coxeter group associated to the system \((K, m)\) is by definition the following group.

\[
W_{(K,m)} = \{ K \mid (ss')^{m(s,s')} = 1, \ s, s' \in S \text{ and } m(s, s') < \infty \}.
\]
A complete classification of finite, irreducible Coxeter groups is known (see [8]). Here, irreducible means the corresponding Coxeter diagram is connected. In this article, without any loss, by a Coxeter group we will always mean an irreducible Coxeter group. See Remark 2.5. We reproduce the list of all finite Coxeter groups in Table 1. These are exactly the finite reflection groups. For a general reference on this subject we refer the reader to [18].

Also, there are infinite Coxeter groups which are affine reflection groups. See Table 2, which shows a list of only those we need for this paper. For a complete list see [18]. In the tables the associated Coxeter diagrams are also shown.

The symmetric groups $S_n$ on $n$ letters and the dihedral group $I_2(p)$ are examples of Coxeter groups. These are the Coxeter group of type $A_n$ and $I_2(p)$ respectively in
the table. The Artin group associated to the Coxeter group $W_{(K,m)}$ is, by definition,
\[ A_{(K,m)} = \{ K \mid ss's' \cdots = s's's' \cdots ; s, s' \in K \}, \]
here the number of times the factors in $ss's' \cdots$ appear is $m(s, s')$; e.g., if $m(s, s') = 3$, then the relation is $ss's = s's's'$. $A_{(K,m)}$ is called the Artin group of type $W_{(K,m)}$. There is an obvious surjective homomorphism $A_{(K,m)} \to W_{(K,m)}$. The kernel $\mathcal{P}A_{(K,m)}$ (say) of this homomorphism is called the associated pure Artin group. In the case of type $A_n$, the (pure) Artin group is also known as a classical (pure) braid group. When a Coxeter group is a finite or an affine reflection group, the associated Artin group is called of finite or affine type, respectively.

Now, given a finite Coxeter group in $GL(n, \mathbb{R})$, consider the hyperplane arrangement in $\mathbb{R}^n$ fixed pointwise by the group. Next, complexify $\mathbb{R}^n$ to $\mathbb{C}^n$ and consider the corresponding complexified fixed hyperplanes. We call these complex hyperplanes in this arrangement, reflecting hyperplanes associated to the finite reflection group. The fundamental group of the complement, $\mathcal{P}A_{(K,m)}$ (say), of the arrangement is identified with the pure Artin group associated to the reflection group ([3]). The reflection group acts freely on this complement. The quotient space has fundamental isomorphic to the Artin group, associated to the reflection group ([3]). That is, we get an exact sequence of the following type.

\[ 1 \longrightarrow \mathcal{P}A_{(K,m)} \longrightarrow A_{(K,m)} \longrightarrow W_{(K,m)} \longrightarrow 1. \]

We call the space $(\mathcal{P}A_{(K,m)})/PA_{(K,m)}$ the (pure) Artin space of the Artin group of type $W_{(K,m)}$.

In this article, this geometric interpretation of the Artin groups is relevant for us.

Next, our aim is to talk some generalization of this interpretation, replacing $\mathbb{C}$ by some 2-dimensional orbifold, in the classical braid group case. These generalized groups are related to some of the Artin groups. This connection is exploited here to help extend our earlier result in [13] and [28], to the Artin groups of types $A_n$, $B_n (= C_n)$, $D_n$, $A'_n$, $B'_n$, $C'_n$ and $D'_n$. Then, using a different method we treat the cases $F_4$, $G_2$ and $I_2(p)$.

In [13] and [28], we proved the Farrell-Jones fibered isomorphism conjecture for $H \wr F$, where $H$ is any subgroup of the classical braid group and $F$ is a finite group. In such a situation, we say that the fibered isomorphism conjecture wreath product with finite groups or $FIC\wr F$ is true for $H$.

The FICwF version of the conjecture was introduced in [21], and its general properties were proved in [27] (or see [23]). The importance of this version of the conjecture was first observed in [13] and [21]. See Section 8, for more on this subject.

**Remark 2.1.** Recall that, a complex reflection group is a subgroup of $GL(n, \mathbb{C})$ generated by complex reflections. A complex reflection is an element of $GL(n, \mathbb{C})$ which fixes a hyperplane in $\mathbb{C}^n$ pointwise. There is a classification of finite complex reflection groups in [32]. There are two classes.

1. $G(de, e, r)$, where $d, e, r$ are positive integers.
2. 34 exceptional groups denoted by $G_4, \ldots G_{37}$.

There are associated Artin groups of the finite complex reflection groups. In a recent preprint ([(7), Proposition 4.1]) it is shown that $A_{G(de, e, r)}$ is a subgroup of $A_{B_\cdot}$. (In [7], these are called braid groups associated to the reflection groups).
Therefore, using Lemma 8.1 and Theorem 1.1, FICwF follows for the class of groups $\mathcal{A}_{G(d,e,c,r)}$.

Now, we are in a position to state the results we prove in this paper. We begin with a corollary of the main result. Another corollary (Theorem 1.1) is already mentioned in the Introduction.

**Definition 2.1.** For a topological space $X$, we define the pure braid space of $X$ on $n$ strings by the following.

$$\text{PB}_n(X) = X^n - \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i, j \in \{1, 2, \ldots, n\}\}.$$ 

The symmetric group $S_n$ acts freely on the pure braid space. The quotient space $\text{PB}_n(X)/S_n$ is denoted by $\text{B}_n(X)$, and is called the braid space of $X$. When $X$ is a 2-dimensional orbifold, then the orbifold fundamental group $\pi_1^{orb}(\text{B}_n(X))$ is called the braid group on the orbifold $X$. Similarly, $\pi_1^{orb}(\text{PB}_n(X))$ is called the pure braid group on the orbifold $X$.

Recall that, an orbifold is called good if it has an orbifold covering space which is a manifold. See Section 3 for some background on orbifolds.

**Definition 2.2.** We define a good, connected 2-dimensional orbifold $S$ to be non-spherical, if it does not have $S^2$ as an orbifold covering space.

**Theorem 2.1.** Let $S$ be a non-spherical orbifold. Then, the FICwF is true for any subgroup of the braid group on the orbifold $S$.

In the process of proving Theorem 2.1 we need to prove the following theorem. This result was unknown in the literature, and is of independent interest.

**Theorem 2.2.** Let $S$ be a non-spherical orbifold. Then, the universal orbifold covering space of the pure braid space $\text{PB}_n(S)$ of $S$ is a contractible manifold.

**Proof.** See Theorem 4.1 and Corollary 4.1. □

To state our main theorem we need to define the following class of groups, which is a generalization of the class of groups defined in [[22], Definition 1.2.1].

**Definition 2.3.** A discrete group $\Gamma$ is called orbi-braid if there exists a finite filtration of $\Gamma$ by subgroups; $1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$ such that the following conditions are satisfied.

1. $\Gamma_i$ is normal in $\Gamma$ for each $i$.
2. For each $i$, $\Gamma_{i+1}/\Gamma_i$ is isomorphic to the orbifold fundamental group $\pi_1^{orb}(S_i)$, of a connected 2-dimensional orbifold $S_i$.
3. For each $\gamma \in \Gamma$ of infinite order, and $i \in \{1, 2, \ldots n-2\}$ the following conditions are satisfied.
   (a) There is a finite sheeted, connected orbifold covering space $q : \tilde{S}_i \to S_i$, so that $\tilde{S}_i$ is a manifold.
   (b) The conjugation action $c_{\gamma^k}$ of $\gamma^k$, for some $k \geq 1$, on $\Gamma_{i+1}/\Gamma_i$ leaves the subgroup $q_* (\pi_1(\tilde{S}_i)) < \pi_1^{orb}(S_i)$ ($q_*$ is injective by Lemma 3.2) invariant, where $\pi_1^{orb}(S_i)$ is identified with $\Gamma_{i+1}/\Gamma_i$ via a suitable isomorphism.
   (c) There is a homeomorphism $f : \tilde{S}_i \to \tilde{S}_i$ so that the induced homomorphism $f_\#$ on $\pi_1(\tilde{S}_i)$ is equal to $c_{\gamma^k}\mid_{\pi_1(\tilde{S}_i)}$ in $\text{Out}(\pi_1(\tilde{S}_i))$.

In such a situation we say that the group $\Gamma$ has rank $\leq n$. 


We will show, in Theorem 4.2, that the pure braid groups on most 2-dimensional orbifolds are orbi-braid.

Our main theorem is the following.

**Theorem 2.3.** Let $\Gamma$ be a group which contains an orbi-braid subgroup of finite index. Then, the FICwF is true for any subgroup of $\Gamma$.

The motivation behind the isomorphism conjecture are the Borel and the Novikov conjectures, which claim that any two closed aspherical homotopy equivalent manifolds are homeomorphic, and the homotopy invariance of rational Pontryagin classes of aspherical manifolds, respectively. There are two classical exact sequences in $K$- and $L$-theory which summarizes the history of the two conjectures.

\[ A_K : H_s(B\Gamma, \mathbb{K}) \to K_s(\mathbb{Z}[\Gamma]) \]

and

\[ A_L : H_s(B\Gamma, \mathbb{L}) \to L_s(\mathbb{Z}[\Gamma]) \]

In the case of torsion free groups, it is conjectured that the above two maps are isomorphisms. This is also implied by the isomorphism conjecture, which is stated for any discrete group ([11], §1.6.1)).

Therefore, we have the following corollary.

**Corollary 2.1.** For any torsion free subgroup of $\Gamma$, where $\Gamma$ is as in Theorem 1.1, 2.1 or 2.3, the above two assembly maps are isomorphisms.

Below we mention a well known consequence of the isomorphism of the $K$-theory assembly map. See ([11], §1.6.1].

**Corollary 2.2.** The Whitehead group $Wh(-)$, the reduced projective class group $\tilde{K}_0(-)$ and the negative $K$-groups $K_{-i}(-)$, for $i \geq 1$, vanish for any torsion free subgroup of the groups considered in Theorems 1.1, 2.1 and 2.3.

**Remark 2.2.** It is conjectured that the Whitehead group and $\tilde{K}_0(-)$ of a torsion free group vanish. Also, W.-c. Hsiang conjectured that $K_{-i}(-)$, for $i \geq 2$, should vanish for any group ([17]).

Finally, we state a corollary regarding an explicit computation of the surgery groups $L_*(\mathcal{P})$ of the pure Artin groups of finite type. This comes out of the isomorphisms of the two assembly maps $A_K$ and $A_L$, for the groups stated in Theorem 1.1. The calculation was done in [28] for the classical pure braid group case. Together with the isomorphisms of $A_K$ and $A_L$, the proof basically used the homotopy type of the suspension of the pure braid space and the knowledge of the surgery groups of the trivial group.

**Theorem 2.4.** The surgery groups of finite type pure Artin groups take the following form

\[
L_i(\mathcal{P}A) = \begin{cases} 
\mathbb{Z} & \text{if } i = 4k, \\
\mathbb{Z}^N & \text{if } i = 4k + 1, \\
\mathbb{Z}_2 & \text{if } i = 4k + 2, \\
\mathbb{Z}_2^N & \text{if } i = 4k + 3.
\end{cases}
\]

where, $N$ is the number of reflecting hyperplanes associated to the finite Coxeter group, as given in the following table.
The isomorphism conjecture for Artin groups 7

\[ \mathcal{P}, A = \text{pure Artin group of type } N \]

| A_{n-1} | \frac{2(n-1)}{n} |
| B_n (= C_n) | n^2 |
| D_n | n(n-1) |
| F_4 | 24 |
| I_2(p) | p |
| G_2 | 6 |

Table 3

Remark 2.3. Recall here, that there are surgery groups for different kinds of surgery problems, and they appear in the literature with the notations \( L_i^* \), where \( * = h, s, \langle -\infty \rangle \) or \( \langle j \rangle \) for \( j \leq 0 \). But, all of them are naturally isomorphic for groups \( G \), if the Whitehead group \( Wh(G) \), the reduced projective class group \( \tilde{K}_0(\mathbb{Z}G) \), and the negative \( K \)-groups \( K_{-i}(\mathbb{Z}G) \), for \( i \geq 1 \), vanish. This can be checked by the Rothenberg exact sequence \([31], 4.13\)

\[
\cdots \to L_i^{(j+1)}(R) \to L_i^{(j)}(R) \to \tilde{H}^i(\mathbb{Z}/2; \tilde{K}_j(R)) \to L_i^{(j+1)}(R) \to L_i^{(j)}(R) \to \cdots \]

Where \( R = \mathbb{Z}[G] \), \( j \leq 1 \), \( Wh(G) = \tilde{K}_1(R) \), \( L_i^{(1)} = L_i^h \), \( L_i^{(2)} = L_i^s \) and \( L_i^{(-\infty)} \) is the limit of \( L_i^{(j)} \). Therefore, because of Corollary 2.2, we use the simplified notation \( L_i(-) \) in the above corollary.

Remark 2.4. The same calculation holds for the other pure Artin groups corresponding to the finite type Coxeter groups and finite complex reflection groups (see Remark 2.1) also, provided we know that \( A_k \) and \( A_L \) are isomorphisms. We further need the fact that the Artin spaces are aspherical ([9]), which implies that the Artin groups are torsion free. The last fact can also be proved group theoretically ([14]). The Artin groups corresponding to the finite complex reflection groups \( G(d,e,r) \) is also torsion free, since \( A_{G(d,e,r)} \) is a subgroup of \( A_{B_r} \) ([7], Proposition 4.1).

We conclude this section with the following useful remark.

Remark 2.5. If we do not assume a Coxeter group \( C \) to be irreducible, then \( C \simeq C_1 \times \cdots \times C_k \), where \( C_i \) is an irreducible Coxeter group for \( i = 1, 2, \ldots, k \). Hence, \( A_C \simeq A_{C_1} \times \cdots \times A_{C_k} \). Therefore, by Lemma 8.6, FICwF is true for \( A_C \), if it is true for \( A_{C_i} \) for each \( i \).

3. SOME HOMOTOPY THEORY OF ORBIFOLDS

In this section we give a short background on orbifolds, and then prove some basic facts. Also, we recall some homotopy theory of orbifolds from [5].

The concept of orbifold was first introduced in [29], and was called ‘V-manifold’. Later, it was revived in [33], with the new name orbifold, and orbifold fundamental group (denoted by \( \pi_1^{orb}(X) \)) of a connected orbifold was defined. Orbifold homotopy groups (denoted by \( \pi_n^{orb}(X) \)) are defined as the ordinary homotopy groups of the classifying space of the topological groupoid associated to the orbifold. This was developed in [15]. More recently, in [5], all these were generalized in the more general category of orbispaces (see [15], Appendix §6), their homotopy groups were defined and the homotopy theory of orbispaces was developed. See Theorems 4.1.12 and 4.2.7 in [5]. We will be using this homotopy theory for orbispaces, and apply it in our situation of orbifolds.
Recall that an orbifold is a topological space, which at every point looks like the quotient space of an Euclidean space $\mathbb{R}^n$, for some $n$, by some finite group action. This finite group is called the local group. The image of the fixed point set of this finite group action is called the singular set. Points outside the singular set are called regular points. In case the local group is cyclic (of order $k$) acting by rotation about the origin on the Euclidean space, the image of the origin is called a cone point of order $k$. If the local group at some point is trivial then it is called a manifold point. As in the case of manifold, one can also talk of the dimension of a connected orbifold. One obvious example of an orbifold is the quotient of a manifold by a finite group. More generally, the quotient of a manifold by a properly discontinuous faithful action of some discrete group ([33], Proposition 5.2.6) is an orbifold. The notion of orbifold covering space, orbifold fiber bundle etc. were defined in [33]. We refer the reader to this source for the basic materials and examples.

In general, an orbifold need not have a manifold as an orbifold covering space, but, if this is the case then the orbifold is called good. One can show that a good compact orbifold has a finite sheeted orbifold covering space, which is a manifold. In the case of closed 2-dimensional orbifolds, only the sphere with one cone point and the sphere with two cone points of different orders are not good orbifolds. See the figure below. Also, see [[33], Theorem 5.5.3].

![Figure 1: Bad orbifolds](image)

A boundary of the underlying space of an orbifold has two types, one which we call manifold boundary and the other orbifold boundary. These are respectively defined as points on the boundary (of the underlying space) with local group trivial or non-trivial.

We begin with the following two lemmas. The proof of the first lemma follows from the classification of 2-manifolds.

**Lemma 3.1.** Let $S$ be a non-spherical orbifold. Then, the universal orbifold covering space of $S$ is homeomorphic to a submanifold of $\mathbb{R}^2$.

**Remark 3.1.** We make here the obvious remark, that if $S$ is non-spherical then $S - \{\text{finitely many points}\}$ is also non-spherical.

The following second lemma follows from standard covering space theory for orbifolds. We will be using this lemma frequently throughout the paper without referring to it.

**Lemma 3.2.** Let $q : \tilde{S} \to S$ be a finite sheeted orbifold covering map between two connected orbifolds. Then, the induced map $q_* : \pi_{1,\text{orb}}^\text{orb}(\tilde{S}) \to \pi_{1,\text{orb}}^\text{orb}(S)$ is an injection, and the image $q_*(\pi_{1,\text{orb}}^\text{orb}(\tilde{S}))$ is a finite index subgroup of $\pi_{1,\text{orb}}^\text{orb}(S)$.

**Proof.** See [[6], Corollary 2.4.5].
Recall that, the quotient map from an orbifold by a finite group action is always an orbifold covering map, and the quotient space is called a global quotient.

Hence, we have the following consequence. We need this corollary to deduce the FICwF for $\pi_{orb}^1(B_n(S))$, after proving the FICwF for $\pi_{orb}^1(PB_n(S))$.

**Corollary 3.1.** Let $S$ be a connected 2-dimensional orbifold. Then, $q_*$ is injective and $q_*(\pi_{orb}^1(PB_n(S)))$ is a finite index subgroup of $\pi_{orb}^1(B_n(S))$. Here, $q$ denotes the global quotient map

$$q : PB_n(S) \to PB_n(S)/S_n = B_n(S).$$

Now, recall that a connected space is called aspherical if all its homotopy groups, except the fundamental group, vanish. The following is a parallel concept in the category of orbifold (or orbispaces).

**Definition 3.1.** We call a connected orbifold (or orbispace) $X$ orbi-aspherical if $\pi_i^{orb}(X) = 0$ for all $i \geq 2$.

Consequently, a good orbifold is orbi-aspherical if it has a contractible manifold universal covering space.

Next, we give a necessary condition on the orbifold fundamental group of a 2-dimensional orbifold to ensure that the orbifold is good. This result will be needed to deduce the FICwF for the orbifold fundamental groups of 2-dimensional orbifolds. Also, we will need the lemma to begin an induction step to prove Theorem 2.3.

**Lemma 3.3.** Let $S$ be a 2-dimensional orbifold with finitely generated, infinite orbifold fundamental group. Then, $S$ is good.

**Proof.** Let $S$ be a 2-dimensional orbifold as in the statement. Recall that, there are only three kinds of singularities in a 2-dimensional orbifold: cone points, reflector lines (or mirror) and corner reflectors, as there are only three kinds of finite subgroup of $O(2)$. See [[33], Proposition 5.4.2]. After going to a finite sheeted covering we can make sure that it has only cone points. So, assume that $S$ has only cone points.

First, we consider the case when $S$ is noncompact. Since the orbifold fundamental group is finitely generated we can find a compact suborbifold $S' \subset S$ with circle boundary components so that the interior of $S'$ is homeomorphic to $S$ as an orbifold. Clearly, all these circle boundary components consists of manifold points of $S$.

Now, let $DS'$ be the double of $S'$. Then $DS'$ has infinite orbifold fundamental group. Hence, from the classification of 2-dimensional orbifolds using geometry (see [[33], Theorem 5.5.3]), it follows that $DS'$ is a good orbifold. Since, $DS'$ has even number of cone points, and in the case of two cone points they have the same orders. See Figure 1. Clearly, then $S'$, and hence $S$, also has an orbifold covering space, which is a manifold.

Next, if $S$ is compact then the same argument as in the previous paragraph, shows that $S$ is good.

This completes the proof of the Lemma. \qed

We now recall a couple of homotopy theoretic results for orbifolds from [5]. These results were proven in the general category of orbispaces, but here we reformulate them for orbifolds.
Lemma 3.4. ([5], Theorem 4.1.4) or ([6], Corollary 2.4.5) Let \( q : M \to N \) be an orbifold covering map between two connected orbifolds \( M \) and \( N \). Then, \( q_* : \pi^n_{orb}(M) \to \pi^n_{orb}(N) \) is an isomorphism for all \( n \geq 2 \).

Lemma 3.5. ([5], Theorem 4.2.5) Let \( f : M \to N \) be an orbifold fibration between two connected orbifolds \( M \) and \( N \), with fiber \( F \) over some regular point of \( N \). Then, there is a long exact sequence of orbifold homotopy groups.

\[
\cdots \to \pi^n_{orb}(N) \to \pi^n_{orb}(F) \to \pi^n_{orb}(M) \to \pi^n_{orb}(N) \to \cdots
\]

An immediate corollary is the following. This result is parallel to the 2-manifold situation, where we know that any 2-manifold of genus \( \geq 1 \) is aspherical.

Lemma 3.6. A non-spherical orbifold is orbi-aspherical.

Proof. The proof is immediate from Lemmas 3.1 and 3.4.

In this paper we will avoid mentioning base points in the notations of homotopy groups and fundamental groups, as we will be considering connected orbifolds (or manifolds) only. But, we always take a regular point as a base point.

4. Topology of Orbifold Braid Spaces

In this section we establish some results on the topology of the pure braid spaces of an orbifold. Then, we prove that the pure braid group on a non-spherical orbifold is orbi-braid.

Theorem 4.1. Let \( S \) be a non-spherical orbifold. Then, \( PB_n(S) \) is orbi-aspherical.

Proof. Recall that \( PB_n(S) = S^n - \bigcup_{i<j;i,j=1}^n H_{ij} \), where \( H_{ij} = \{(x_1, x_2, \ldots, x_n) \in S^n \mid x_i = x_j\} \).

Consider the projection \( p : PB_n(S) \to PB_{n-1}(S) \) to the last \((n-1)\) coordinates. \( p \) is a locally trivial orbifold fibration with generic fiber \( S' = S - \{(n-1)\text{ regular points}\} \).

We can now prove the theorem by induction on \( n \). For \( n = 1 \), \( PB_n(S) = S \). Hence, the induction starts by Lemma 3.6. So, we assume the statement for dimension \( n - 1 \). Now, applying Lemma 3.5 to the fibration \( p \), and the induction hypothesis we complete the proof of the theorem. We only need to note here that, \( S' \) is non-spherical by Remark 3.1 and hence, Lemma 3.6 is applicable on \( S' \).

Corollary 4.1. Let \( S \) be a non-spherical orbifold. Then, the universal orbifold covering space of \( PB_n(S) \) is a contractible manifold.

Proof. By hypothesis, \( S \) has a finite sheeted orbifold covering space \( \tilde{S} \) (say), where \( \tilde{S} \) is a manifold. Therefore, \( \tilde{S}^n - (q^n)^{-1}(\bigcup_{i<j;i,j=1}^n H_{ij}) \) is a manifold, where \( q^n : \tilde{S}^n \to S^n \) is the \( n \)-times product of the covering map \( q : \tilde{S} \to S \). Hence, we have an orbifold covering map

\[
\tilde{S}^n - (q^n)^{-1}(\bigcup_{i<j;i,j=1}^n H_{ij}) \to PB_n(S),
\]

whose total space is a manifold. Next, applying Theorem 4.1 and Lemma 3.4 we get that all the orbifold homotopy groups of the universal cover of \( PB_n(S) \) is trivial. Now, since (by uniqueness) the universal cover is a manifold, its ordinary homotopy groups are isomorphic to the orbifold homotopy groups. Therefore, the ordinary homotopy groups are also trivial, and hence the universal cover is contractible.
The vanishing of the higher orbifold homotopy groups of the pure braid space of a non-spherical orbifold, induces a strong geometric structure on its orbifold fundamental group. This is the crucial part of this paper. In the following theorem we establish this fact.

**Theorem 4.2.** Let S be a non-spherical orbifold. Then, the group $\pi_1^{orb}(PB_n(S))$ is orbi-braid.

**Proof.** The proof is by induction on $n$. For $n = 1$, we have $PB_n(S) = S$. Clearly, it is an orbi-braid group of rank $\leq 1$. Note that, Condition (3) in the definition of the orbi-braid group is vacuously true in this case.

So, we assume that $PB_k(S)$ is an orbi-braid group of rank $\leq k$ for $k \leq n - 1$. We need to show that $PB_n(S)$ is an orbi-braid group.

Consider the projection $p : PB_n(S) \to PB_{n-1}(S)$, to the last $(n-1)$ coordinates. We know that $p$ is a locally trivial orbifold fibration, with generic fiber $S' = S - \{(n-1) \text{ regular points}\}$. Also, by Theorem 4.1, $PB_k(S)$ is orbi-asperpherical for all $k$. Therefore, we have the following exact sequence by Lemma 3.5.

$$1 \to \pi_1^{orb}(S') \to \pi_1^{orb}(PB_n(S)) \to \pi_1^{orb}(PB_{n-1}(S)) \to 1.$$ 

By the induction hypothesis, $\pi_1^{orb}(PB_{n-1}(S))$ is an orbi-braid group of rank $\leq n - 1$. Hence, it has a filtration

$$< 1 > = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_{n-2} < \Gamma_{n-1} = \pi_1^{orb}(PB_{n-1}(S))$$

satisfying the three conditions in the definition of the orbi-braid group.

Now, consider the following filtration of $\pi_1^{orb}(PB_n(S))$ by pulling back the above filtration, by $p_* : \pi_1^{orb}(PB_n(S)) \to \pi_1^{orb}(PB_{n-1}(S))$.

$$< 1 > = \Delta_0 < \Delta_1 < \cdots < \Delta_{n-1} < \Delta_n = \pi_1^{orb}(PB_n(S)).$$

Here $\Delta_1 = \pi_1^{orb}(S')$ and $\Delta_k = p_*^{-1}(\Gamma_{k-1})$.

Note that, Conditions (1) and (2) are already satisfied for the above filtration of $\pi_1^{orb}(PB_n(S))$. Condition (3) is satisfied for all $\gamma \in \Delta_n$ and for $i \geq 2$. We only need to check Condition (3) for $\gamma \in \Delta_n$ and for $i = 1$.

Recall, the notation we used in the proof of Corollary 4.1 and consider the following commutative diagram.

$$\begin{align*}
\tilde{S}^n - (q^n)^{-1}\left(\bigcup_{i<j;i,j=1}^{n}H_{ij}\right) & \xrightarrow{p} \tilde{S}^{n-1} - (q^{n-1})^{-1}\left(\bigcup_{i<j;i,j=1}^{n-1}H_{ij}\right) \quad \text{for } q^n
\\
PB_n(S) & \xrightarrow{p} PB_{n-1}(S).
\end{align*}$$

Here, $q : \tilde{S} \to S$ is a finite sheeted orbifold covering map, where $\tilde{S}$ is a 2-manifold. We denote by the same notation $p$ the projection to the last $(n - 1)$ coordinates of $S^n$ and $\tilde{S}^n$. Also, the same notation $q$ is used for any restriction of the covering map. We now introduce the new notation;

$$PB_k^q(\tilde{S}) = (\tilde{S})^k - (q^k)^{-1}\left(\bigcup_{i<j;i,j=1}^{k}H_{ij}\right).$$

Then, the above diagram reduces to the following commutative diagram.
Recall that the bottom right horizontal map is a locally trivial orbifold fibration. Since \(q^n\) and \(q^{n-1}\) are both covering maps, the top right horizontal map is a locally trivial fibration of manifolds. Here, \(\tilde{S}'\) is the fiber of this fibration. The map \(\tilde{S}' \to S'\) is a finite sheeted orbifold covering map, where \(\tilde{S}'\) is a manifold.

Now, note that by Lemmas 3.4 and 4.1 the spaces involved in the above diagram are all orbi-aspherical. Therefore, applying Lemma 3.5 and the orbifold fundamental group functor on the above commutative diagram, we get the following.

\[
\begin{array}{c}
\pi_1(\tilde{S}') \\
\downarrow \quad \downarrow q^n \\
\pi_1(\tilde{S}^g) \\
\downarrow \quad \downarrow q^n \quad \downarrow q^{n-1} \\
\pi_1(PB_n^g(S)) \\
\downarrow \quad \downarrow q^n \quad \downarrow q^{n-1} \\
\pi_1(PB_n(S)) \\
\downarrow \quad \downarrow q^n \quad \downarrow q^{n-1} \\
\pi_1(PB_n(S)) \\
\end{array}
\]

Since, the spaces on the top row are all manifolds the diagram reduces to the following.

\[
\begin{array}{c}
\pi_1(\tilde{S}') \\
\downarrow \quad \downarrow \quad \downarrow \\
\pi_1(\tilde{S}^g) \\
\downarrow \quad \downarrow \quad \downarrow \\
\pi_1(PB_n^g(S)) \\
\downarrow \quad \downarrow \quad \downarrow \\
\pi_1(PB_n(S)) \\
\end{array}
\]

Now, we can check Condition (3) for \(i = 1\). Consider the conjugation action of \(\gamma \in \Delta_n = \pi_1^{orb}(PB_n(S))\) on \(\Delta_1 = \pi_1^{orb}(S')\). Note that, there is a positive integer \(k\), so that \(g^k\) lies in the image of \(q^n\). Therefore, the conjugation action by \(\gamma^k\) leaves the image of \(\pi_1(\tilde{S}')\) invariant. Now, since \(PB_n^g(\tilde{S}) \to PB_n^{g-1}(\tilde{S})\) is a locally trivial fibration of manifolds, the monodromy action of \(\pi_1(PB_n^{g-1}(\tilde{S}))\) on \(\tilde{S}'\) gives a realization of the conjugation action of \(\gamma^k\), by a homeomorphism of \(\tilde{S}'\).

This completes the proof. \(\square\)

We will need the following lemma, for the proof of Theorem 1.1. The lemma is easily deduced from the lifting orbi-braid structure argument as in the proof of Theorem 4.2 above.

**Lemma 4.1.** Let \(f : M \to N\) be a locally trivial fibration between two connected aspherical manifolds, with fiber a connected 2-manifold. If \(\pi_1(N)\) is orbi-braid, then so is \(\pi_1(M)\).
Proof. The proof is essentially the same (in fact easier) as the proof of Theorem 4.2. We only need to replace $PB_n(S)$ by $M$, $PB_{n-1}(S)$ by $N$ and assume $q^n = q^{n-1} = id$. □

5. Artin groups and their orbifold braid representations

This section is devoted to recall an interesting connection between braid groups on an orbifold and Artin groups. More precisely, finding braid type representation of elements of an Artin group. In Remark 2.1, we already mentioned braid type representation of the Artin groups associated to a certain class of finite complex reflection groups. In all these cases one has to replace the complex plane by some suitable space. For example, in the case of finite complex reflection groups of type $G(de, e, r)$, one has to consider the braid group on the punctured plane (annulus). See Remark 2.1 and Theorem 5.1. For some of the other Artin groups, we describe it below.

Let $S$ be a 2-dimensional orbifold with only cone points. We can consider $B_n(S)$ as an orbifold as well as its underlying topological space.

Therefore, although the fundamental group of the underlying topological space of $B_n(S)$ has the classical pictorial braid representation, there is something more to it due to its orbifold structure. We point out here a similar pictorial braid representation of the orbifold fundamental group of $B_n(S)$ from [1].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{orbifold_braid.png}
\caption{Orbifold braid}
\end{figure}

In this orbifold situation the movement of the braids is restricted, because of the presence of the cone points. Therefore, one has to define new relations among braids, respecting the singular set in the orbifold. We produce one situation to see how this is done. The above figure represents $S \times I$ and the thick line is $o \times I$, where $o$ is a cone point of order $p$.

Now, if a string in the braid wraps the thick line $p$ times then, it is equal to the second picture. This is because, if a loop circles $p$ times around the cone point $o$, then the loop gives the trivial element in the orbifold fundamental group of $S$. Therefore, both braids represents the same element in the orbifold fundamental group of $B_n(S)$. For more details see [1].

This representation of an element in $\pi_1^{orb}(B_n(S))$, as a braid helps to relate the orbifold braid group of $S$ with some of the Artin groups. For a given Artin group one has to choose a suitable orbifold, as in the following table in Theorem 5.1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$A$ & $\pi_1^{orb}(B_n(S))$ \\
\hline
\end{tabular}
\caption{Orbifold braid group}
\end{table}

Theorem 5.1. ([1]) Let $A$ be an Artin group, and $S$ be an orbifold as described in the following table. Here, all the cone points have order 2. Then, $A$ is a (normal) subgroup of the braid group $\pi_1^{orb}(B_n(S))$ on the orbifold $S$. The third column gives the quotient group $\pi_1^{orb}(B_n(S))/A$. 

Artin group of type | Orbifold $S$ | Quotient group | $n$
---|---|---|---
$A_{n-1}$ | $\mathbb{C}$ with a puncture | $<1>$ | $n > 1$
$B_n (= C_n)$ | $\mathbb{C}$ with a puncture | $<1>$ | $n > 1$
$D_n$ | $\mathbb{C}$ with a cone point | $\mathbb{Z}/2$ | $n > 1$
$\tilde{A}_{n-1}$ | $\mathbb{C}$ with a puncture | $\mathbb{Z}$ | $n > 2$
$\tilde{B}_n$ | $\mathbb{C}$ with a cone point and a puncture | $\mathbb{Z}/2$ | $n > 2$
$\tilde{C}_n$ | $\mathbb{C}$ with two punctures | $<1>$ | $n > 1$
$\tilde{D}_n$ | $\mathbb{C}$ with two cone points | $\mathbb{Z}/2 \times \mathbb{Z}/2$ | $n > 2$

Table 4

Proof. See [1] for the proof.

Remark 5.1. It is not yet known if any of the other Artin groups have similar orbifold braid representation. But, all the finite type pure Artin groups considered in this paper are orbi-braid. One may ask a question here. Does any of the other pure Artin groups have an orbi-braid structure?

6. Proofs of the theorems 1.1, 2.1 and 2.3

Proof of Theorem 2.3. By Lemma 8.1, it is enough to prove the FICwF for $\Gamma$. Next, by Lemma 8.3, it is enough to consider $\Gamma$ to be an orbi-braid group.

So, let $\Gamma$ be an orbi-braid group of rank $n$. We need to prove the FICwF for $\Gamma$. The proof is by induction on $n$.

Case $n = 1$. In this case $\Gamma \simeq \pi_1^{orb}(S)$ for some 2-dimensional orbifold $S$. The proof is completed using Lemma 8.5.

Case $n \geq 2$. Assume that the FICwF is true for any orbi-braid group of rank $\leq n - 1$. Consider the following exact sequence.

$$1 \rightarrow \Gamma_1 \rightarrow \Gamma_n \stackrel{p}{\rightarrow} \Gamma_n/\Gamma_1 \rightarrow 1.$$  

We would like to apply Lemma 8.2 to the map $p$. It is easily seen that $\Gamma_n/\Gamma_1$ is an orbi-braid group of rank $\leq n - 1$. Therefore, by the induction hypothesis, it follows that the FICwF is true for $\Gamma_n/\Gamma_1$. Furthermore, by Case $n = 1$, the FICwF is true for $\Gamma_1$. Hence, we need to prove the FICwF for $p^{-1}(V)$, for any infinite cyclic subgroup $V$ of $\Gamma_n/\Gamma_1$. Let $\alpha$ be a generator of $V$. By Condition (3a) in the definition of orbi-braid group there is a finite index subgroup $\pi_1(\tilde{S})$, the fundamental group of a 2-manifold $\tilde{S}$, of $\Gamma_1$ left invariant by the conjugation action of $\alpha^k$ for some $k \geq 1$. And by Condition (3b), this action is induced by a homeomorphism $f$ (say) of $\tilde{S}$. Therefore, we get that $p^{-1}(V)$ contains $\pi_1(M)$ as a subgroup of finite index, where $M$ is a 3-manifold homeomorphic to the mapping torus of $f$. By Lemma 8.5, the FICwF is true for $\pi_1(M)$, and therefore, for $p^{-1}(V)$ also by Lemma 8.3.

This completes the proof of the Theorem. 

□
Proof of Theorem 1.1. By Lemma 8.1, it is enough to prove the FICwF for the Artin groups of the stated types.

First, we give the proof for all the cases except for $F_4, I_2(p)$ and $G_2$.

In all the other cases, the Artin group is a subgroup of the orbifold braid group of a certain orbifold, as described in Theorem 5.1. Therefore, by Lemma 8.1 again, and by Theorems 2.3 and 4.2 we only need to check that the orbifolds in Table 4 are good, and does not have $S^2$ as an orbifold covering space. The second condition is obvious. So, we only need to prove that these orbifolds are good. Clearly, the only situation we need to consider is of the two cone points case. So, let $S$ be the orbifold in Table 4 with two cone points. Take a disc $D$ on $S$, which contains the cone points. Then, the double of $D$ is a sphere with four cone points. But, a sphere with more than three cone points is good. See [33], Theorem 5.5.3. Therefore, the inverse image of the interior of $D$ under an orbifold covering map of the double of $D$, whose total space is a manifold, produces the required orbifold covering space of $S$.

Next, we deduce the proof of Theorem 1.1 in the $F_4$ case. The idea is to see the pure braid space of type $F_4$ as a locally trivial fibration over another space, whose fundamental group is orbi-braid, and the fiber is a smooth 2-manifold. Then, Lemma 4.1 will be applicable.

We begin with the following lemma.

Lemma 6.1. Let $Z_n = \{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq 0, z_i \neq z_j \text{ for } i \neq j \}$. Then, $Z_n$ is aspherical and $\pi_1(Z_n)$ is an orbi-braid group for all $n$.

Proof. The proof that $Z_n$ is aspherical follows easily by an induction argument, on taking successive projections and using the long exact sequence of homotopy groups for a fibration.

The proof that $\pi_1(Z_n)$ is an orbi-braid group, follows from Lemma 4.1, using successive projection and an induction argument. □

The $F_4$ case. For the following discussion we refer to [[4], Proposition 2].

The pure Artin group of type $F_4$ is isomorphic to the fundamental group of the following hyperplane arrangement complement in $\mathbb{C}^4$.

$$PA_{F_4} = \{ (y_1, y_2, y_3, y_4) \in \mathbb{C}^4 \mid y_i \neq 0 \text{ for all } i; y_i \pm y_j \neq 0 \text{ for } i \neq j; y_1 \pm y_2 \pm y_3 \pm y_4 \neq 0 \}.$$  

The signs in the last formula appear in arbitrary combinations. Next, consider the map $PA_{F_4} \to \mathbb{Z}_3$ defined by the following formula.

$$z_i = y_1 y_2 y_3 y_4 (y_i^2 - y_i) .$$

This map is a locally trivial fibration, with fiber a 2-manifold. Hence, by Lemmas 4.1 and 6.1, $PA_{F_4}$ is an orbi-braid group. This completes the proof of the FICwF for the pure Artin group of type $F_4$, by Theorem 2.3. Next, apply Lemma 8.3 to complete the proof for the Artin group of type $F_4$.

The $I_2(p)$-case. First, note that this is a rank 2 case. In this case any hyperplane arrangement complement has orbi-braid fundamental group. We prove this in the next lemma.

Lemma 6.2. Any hyperplane arrangement complement in $\mathbb{C}^2$ has orbi-braid fundamental group.
Proof. If there is only one (complex) line in the arrangement, then the complement has infinite cyclic fundamental group, hence there is nothing prove in this case. So, assume that the arrangement has \( n \) lines, for \( n \geq 2 \). After a linear change of coordinates it is easy to make the equations of the lines look like the following.

\[
x = 0, y = 0, y - c_1 x = 0, y - c_2 x = 0, \ldots, y - c_{n-2} x = 0.
\]

Where, \( c_i \)'s are distinct non-zero constants. Next, one just constructs a map (in fact a coordinate projection), from the arrangement complement to \( \mathbb{C}^* \), which is a locally trivial fibration with fiber \( \mathbb{C} - \{(n - 1) \text{ points}\} \). Then, applying Lemma 4.1 we complete the proof.

Once again, we apply Lemmas 8.3 and 2.3 to prove that the FICwF is true for the Artin group of type \( I_2(p) \).

Finally, since \( G_2 = I_2(6) \), the proof of the theorem is completed.

An alternate proof of the FICwF in the \( D_n \) case. As this alternate proof goes in the same line, as the proof of the \( F_4 \) case, we just point out the differences.

First, note that the pure Artin space of type \( D_n \) has the following form.

\[
PA_{D_n} = \{(y_1, \ldots, y_n) \in \mathbb{C}^n \mid y_i \pm y_j \neq 0 \text{ for } i \neq j\}.
\]

Next, the map \( PA_{D_n} \to \mathbb{Z}_{n-1} \) defined by \( z_i = y_i^2 - y_j^2 \) is a locally trivial fibration. The proof now follows from Lemmas 4.1, 2.3 and 8.3.

Proof of Theorem 2.1. First, by Lemma 8.1, it is enough to prove the FICwF for the braid group on \( S \). Next, using Corollary 3.1, we see that the pure braid group on \( S \) is isomorphic to a finite index subgroup of the braid group on \( S \). Therefore, using Lemma 8.3, it is enough to prove the FICwF for the pure braid group on \( S \). The theorem is now a combination of Theorems 4.2 and 2.3.

7. Computation of surgery groups

This section is devoted to some application related to computation of the \( L \)-theory of some of the discrete groups considered in this paper.

Since, the finite type pure Artin groups are torsion free, Corollary 2.2 is applicable to the finite type pure Artin groups considered in Theorem 1.1. A parallel to this \( K \)-theory vanishing result is the computation of the \( L \)-theory.

For finite groups, the computation of the \( L \)-theory is well established ([16]). The infinite groups case needs different techniques and is difficult, even when we have the isomorphism of the \( L \)-theory assembly map. For a survey on known results and techniques, on computation of surgery groups for infinite groups, see [27].

In the case of classical pure braid group, we did the computation in [28]. Here, we extend it to the pure Artin groups of the finite type Artin groups considered in Theorem 1.1.

The main idea behind the computation is the following lemma. This lemma was stated and proved in [28], in the case of hyperplane arrangements. We recall the proof here with some more elaboration.
Lemma 7.1. The first suspension $\Sigma(\mathbb{C}^n - \bigcup_{j=1}^N A_j)$ of the complement of a hyperplane arrangement $A = \{A_1, A_2, \ldots, A_N\}$ in $\mathbb{C}^n$, is homotopically equivalent to the wedge of spheres $\vee_{j=1}^N S_j$, where $S_j$ is homeomorphic to the 2-sphere $S^2$ for $j = 1, 2, \ldots, N$.

Proof. In [30], (2) of Proposition 8, it is proved that $\Sigma(\mathbb{C}^n - \bigcup_{j=1}^N A_j)$ is homotopically equivalent to the following space. $\Sigma(\bigvee_{p \in P}(S^{2n-d(p)-1} - \Delta P_{<p}))$.

We recall the notations in the above display from [30], p. 464. $P$ is in bijection with $A$ under a map $f$ (say). Also, $P$ is partially ordered by the rule that $p < q$ for $p, q \in P$ if $f(q)$ is a subspace of $f(p)$. (Recall that, in [30], $A$ was, more generally, an arrangement of affine linear subspaces of $\mathbb{C}^n$.) $\Delta P$ is a simplicial complex, whose vertex set is $P$ and chains in $P$ define simplices. $\Delta P_{<p}$ is the subcomplex of $\Delta P$ consisting of all $q \in P$, so that $q < p$. $d(p)$ denotes the dimension of $f(p)$. The construction of the embedding of $\Delta P_{<p}$ in $S^{2n-d(p)-1}$ is also a part of [30], Proposition 8.

Now, in the situation of the Lemma, obviously, $d(p) = 2n - 2$ and $\Delta P_{<p} = \emptyset$, for all $p \in P$. Finally, note that the suspension of a wedge of spaces is homotopically equivalent to the wedge of the suspensions of the spaces. This completes the proof of the Lemma. □

We need one more result to prove Theorem 2.4.

Proposition 7.1. Let $A = \{A_1, A_2, \ldots, A_N\}$ be a hyperplane arrangement in $\mathbb{C}^n$. Assume that the two assembly maps $A_K$ and $A_L$ in Section 2 are isomorphisms, for the group $\Gamma = \pi_1(\mathbb{C}^n - \bigcup_{j=1}^N A_j)$. Then, the surgery groups of $\Gamma$ are given by the following.

$$L_i(\Gamma) = \begin{cases} \mathbb{Z} & \text{if } i \equiv 0 \mod 4 \\ \mathbb{Z}^N & \text{if } i \equiv 1 \mod 4 \\ \mathbb{Z}_2 & \text{if } i \equiv 2 \mod 4 \\ \mathbb{Z}^N_2 & \text{if } i \equiv 3 \mod 4. \end{cases}$$

Proof. Using Lemma 7.1, the proof is same as the proof of [[28], Theorem 2.2]. We recall the main ideas behind the proof. First, by Lemma 7.1, the first suspension of $\mathbb{C}^n - \bigcup_{j=1}^N A_j = X$ (say) is homotopically equivalent to the wedge of $N$ many 2-spheres. Hence, any generalized homology theory $h_*$, applied over $X$ satisfies the following equation.

$$h_i(X) = h_i(*) \oplus h_{i-2}(*)^N.$$  

Where $*$ denotes a single point space. The proof is now completed using the known computation of the surgery groups of the trivial group, and using the fact that surgery groups are 4-periodic. That is, $L_i(\Gamma) \simeq L_{i+4}(\Gamma)$ for any group $\Gamma$. □

Now, we are in a position to prove Theorem 2.4.

Proof of Theorem 2.4. For the proof, we need to use Corollary 2.1 and Proposition 7.1, together with the calculation of the number of reflecting hyperplanes associated to the finite Coxeter groups. For different cases, the number of reflecting hyperplanes is given in the following table. We refer the reader to [4], for the equations
of the hyperplanes given in the table. Or see [[18], p. 41-43] for a complete root structure.

| $\mathcal{P}A =$ | Reflecting hyperplanes in $\mathbb{C}^n = \{ (z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C} \}$ | Number of reflecting hyperplanes |
|------------------|-------------------------------------------------|---------------------------------|
| pure Artin group type | $A_{n-1}$: $z_i = z_j$ for $i \neq j$ | $\frac{2(n-1)}{n^2}$ |
| $B_n (= C_n)$ | $z_i = 0$ for all $i$; $z_i = z_j$, $z_i = -z_j$ for $i \neq j$ | $n^2$ |
| $D_n$ | $z_i = z_j$, $z_i = -z_j$ for $i \neq j$ | $n(n-1)$ |
| $F_4$ | $n = 4$; $z_i = 0$ for all $i$; $z_i = z_j$, $z_i = -z_j$ for $i \neq j$; $z_1 \pm z_2 \pm z_3 \pm z_4 = 0$ | 24 |
| $I_2(p)$ | $n = 2$; roots are in one-to-one correspondence with the lines of symmetries of a regular $p$-gon in $\mathbb{R}^2$. See [[18], page 4]. Hence $p$ hyperplanes. | $p$ |
| $G_2 = I_2(6)$ | | 6 |

This completes the proof of the theorem. \(\square\)

8. APPENDIX: THE ISOMORPHISM CONJECTURE AND RELATED BASIC RESULTS

In this section, we recall the statement of the Farrell-Jones fibered isomorphism conjecture, and related basic results. We state the original version of the conjecture for clarity of exposition, and also as we will be using it for some explicit computation. We remark here that the proofs of our results are still valid, without any modification, for the general version of the conjecture as stated in [2], that is, the Farrell-Jones conjecture with coefficients in additive categories.

Before getting into the technical formulation of the conjecture, we mention here that the conjecture says; if we can compute the $K$- and $L$-theory of all the virtually cyclic subgroups of a group, then the respective theory can be computed for the group also.

The following version of the conjecture is taken from [12]. The primary source is [11].

Let $\mathcal{S}$ be one of the following three functors from the category of topological spaces to the category of spectra: (a) the stable topological pseudo-isotopy functor $\mathcal{P}();$ (b) the algebraic $K$-theory functor $\mathcal{K}();$ and (c) the $L$-theory functor $\mathcal{L}^{−\infty}().$

Let $\mathcal{M}$ be the category of continuous surjective maps. The objects of $\mathcal{M}$ are continuous surjective maps $p : E \rightarrow B$ between topological spaces $E$ and $B.$ And a morphism between two maps $p : E_1 \rightarrow B_1$ and $q : E_2 \rightarrow B_2$ is a pair of continuous maps $f : E_1 \rightarrow E_2,$ $g : B_1 \rightarrow B_2,$ so that the following diagram commutes.

$\begin{array}{ccc} 
E_1 & \xrightarrow{f} & E_2 \\
\downarrow{p} & & \downarrow{q} \\
B_1 & \xrightarrow{g} & B_2. 
\end{array}$

There is a functor defined by Quinn ([19]), from $\mathcal{M}$ to the category of $\Omega$-spectra, which associates to the map $p : E \rightarrow B$ the spectrum $\mathbb{H}(B, S(p)),$ with the property that $\mathbb{H}(B, S(p)) = S(E),$ when $B$ is a single point. For an explanation of $\mathbb{H}(B, S(p))$
see [11, Section 1.4]. Also, the map \( \mathbb{H}(B, S(p)) \to S(E) \) induced by the morphism: id: \( E \to E; B \to * \) in the category \( M \), is called the Quinn assembly map.

Let \( \Gamma \) be a discrete group. Let \( E \) be a \( \Gamma \)-space, which is universal for the class of all virtually cyclic subgroups of \( \Gamma \), and denote \( E/\Gamma \) by \( B \). For definition of universal space see [11, Appendix]. Let \( X \) be a space on which \( \Gamma \) acts freely and properly discontinuously, and \( p: X \times_{\Gamma} E \to E/\Gamma = B \) be the map induced by the projection onto the second factor of \( X \times \mathcal{E} \).

The fibered isomorphism conjecture states that the map

\[
\mathbb{H}(B, S(p)) \to S(X \times_{\Gamma} \mathcal{E}) = S(X/\Gamma)
\]

is a (weak) equivalence of spectra. The equality is induced by the map \( X \times_{\Gamma} \mathcal{E} \to X/\Gamma \), and using the fact that \( S \) is homotopy invariant.

Let \( \tilde{Y} \) be a connected CW-complex, and \( \Gamma = \pi_1(Y) \). Let \( X \) be the universal cover \( \tilde{Y} \) of \( Y \), and the action of \( \Gamma \) on \( X \) is the action by a group of covering transformations. If we take an aspherical CW-complex \( Y' \) with \( \Gamma = \pi_1(Y') \), and \( X \) is the universal cover \( \tilde{Y}' \) of \( Y' \), then, by ([11, corollary 2.2.1]), if the fibered isomorphism conjecture is true for the space \( \tilde{Y}' \), it is true for \( \tilde{Y} \) also. Thus, whenever we say that the fibered isomorphism conjecture is true for a discrete group \( \Gamma \), we mean it is true for the Eilenberg-MacLane space \( K(\Gamma, 1) \) and the functor \( S() \).

The main advantage of the fibered isomorphism conjecture is that it has hereditary property. That is, if the conjecture is true for a group, then it is true for any of its subgroup also. The following version of the conjecture was defined in [21], and has several advantages. For example, it passes to finite index overgroups.

In this article we use the following definition of the FICwF.

**Definition 8.1.** ([21, Definition 2.1]) A group \( \Gamma \) is said to satisfy the fibered isomorphism conjecture wreath product with finite groups, or in short FICwF is satisfied for \( \Gamma \), if the fibered isomorphism conjecture for the \( K \)- and \( L \)-theory functors are true for the wreath product \( G \wr F \), for any finite group \( F \).

Note that, the fibered isomorphism conjecture is true for all virtually cyclic groups, and hence for all finite groups. The FICwF is also true for all virtually cyclic groups, but this needs a proof of the fibered isomorphism conjecture for the fundamental groups of closed flat manifolds, which is true by Theorem 8.1 below. Also, see [13, Fact 3.1].

Now, we state a series of lemmas to recall some of the basic results in this area. These results have already been used widely in the literature, so we give some backgrounds, the statements and refer the reader to the original sources.

We begin with an obvious consequence of the hereditary property of the fibered version of the conjecture.

**Lemma 8.1.** If the FICwF is true for a group \( G \), then the FICwF is true for any subgroup of \( G \) also.

The following (inverse) lemma is an important basic result in this subject. This lemma helps in induction steps, for proving the conjecture in several new classes of groups. In its original form (where \( V \) was assumed to be virtually cyclic) it appeared in [11, Proposition 2.2]. The following improved version was first proved in [24], Lemma 3.4 (also see [27, Proposition 3.5.2] or [23, Proposition 5.2]), and it makes induction steps easier to deal with.
Lemma 8.2. ([24], Lemma 3.4 or [25], Lemma 2.3) Consider a surjective homomorphism of groups $f : G \to H$, with kernel $K$. Assume that the FICwF is true for $K$, $H$ and for $f^{-1}(V)$, for any infinite cyclic subgroup $V$ of $H$. Then the FICwF is true for $G$.

Now, we exhibit the special property of the FICwF. This property helps in deducing the conjecture for a group, if the group is commensurable with another group for which the FICwF is already known. This fact was very crucial in the proof of the conjecture for 3-manifold groups ([21] and [22]) and for classical braid groups ([13] and [28]), among other applications.

Lemma 8.3. ([Lemma 3.4, [24]] or [Lemma 2.3, [25]]) Let $G$ be a finite index subgroup of a group $K$, and the FICwF is true for $G$. Then, the FICwF is true for $K$.

Filtering and then taking limit type argument was used in [20], in the context of proving the isomorphism of the classical $L$-theory assembly map for 3-manifold groups, which has infinite first homology. The following lemma gives the necessary tool for its application in the fibered version of the isomorphism conjecture. The above argument was another crucial ingredient in the proof of the FICwF for 3-manifold groups.

Lemma 8.4. ([Theorem 7.1, [12]]) Let $\{G\}_{i \in I}$ be a directed system of groups and assume that the FICwF is true for each $G_i$, then the FICwF is true for the direct limit of $\{G\}_{i \in I}$ also.

Finally, we do a little generalization of some existing result.

Lemma 8.5. The FICwF is true for the orbifold fundamental group of any 2-dimensional orbifold, and of any good 3-dimensional orbifold.

Proof. Let $S$ be a 2-dimensional orbifold. Since the FICwF is true for finite groups, we can assume that $\pi_{orb}^1(S)$ is infinite. Furthermore, we can also assume that $\pi_{orb}^1(S)$ is finitely generated by Lemma 8.4. Therefore, by Lemma 3.3, $S$ is good and hence has finite sheeted orbifold covering $\tilde{S}$, which is a manifold. Clearly, $\tilde{S}$ has infinite fundamental group. Now, note that a 2-manifold with infinite fundamental group either has free fundamental group, or is a closed 2-manifold of genus $\geq 1$.

In the second case of a good 3-dimensional orbifold, the orbifold fundamental group has got a finite index 3-manifold subgroup (Lemma 3.2).

So, using Lemmas 8.3 and 8.4 we have to show that the FICwF is true for finitely generated free groups, fundamental group of closed 2-manifold of genus $\geq 1$, and for 3-manifold groups. In the first case, the group can be embedded into the fundamental group of a closed 2-manifold of genus $\geq 1$. Now, since a closed 2-manifold of genus $\geq 1$ supports a non-positively curved Riemannian metric, using the following result and Lemma 8.1, we are left with the 3-manifold groups case only.

Theorem 8.1. The FICwF is true for the fundamental groups of closed non-positively curved Riemannian manifolds.

Proof. Let $G$ be the fundamental group of a closed nonpositively curved Riemannian manifold. $L$-theory and lower $K$-theory case of the fibered isomorphism conjecture for $G$ was proved in [2]. The lower $K$-theory case was extended to all of
The FICwF for $L$-theory and $K$-theory can then be deduced from [[28], Lemma 3.1 and Corollary 3.1].

Now, to complete the proof of Lemma 8.5 we need to consider the 3-manifold groups case. For this, we apply [[26], Theorem 2.1] in the $L$-theory case. The $K$-theory case is the combination of [[24], Theorem 2.2], [34] and [2].

We conclude with the following easy lemma.

**Lemma 8.6.** ([[21], Lemma 5.1]) If the FICwF is true for two groups $G_1$ and $G_2$, then the FICwF is true for the direct product $G_1 \times G_2$.

**Proof.** We can assume that both the groups $G_1$ and $G_2$ are infinite. Next, applying Lemma 8.2 twice, to the two projections of the direct product $G_1 \times G_2$, it is enough to prove that the FICwF is true for the free abelian group on two generators. But, a two generator free abelian group is isomorphic to the fundamental group of a flat torus. Hence, Theorem 8.1 is applicable here. This completes the proof of the lemma.

**Remark 8.1.** For the proof of Theorem 1.1, we do not need the full strength of Theorem 8.5 in the 3-manifold case. We only need the fact that the FICwF is true for the fundamental group of the mapping torus of a non-compact 2-manifold. In this particular case, one just need to use the proof of [[28], Lemma 3.4], which in turn uses Theorem 8.1. We conclude by remarking that, Theorem 1.1 can be proved for the pseudoisotopy version of the conjecture also.
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