Derived decompositions of abelian categories, I.

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Abstract

Derived decompositions of abelian categories are introduced in internal terms of abelian subcategories. They are used to construct semi-orthogonal decompositions (or in other terminology, Bousfield localizations, or hereditary torsion pairs) in derived categories of abelian categories. A sufficient condition is given for abelian categories to have derived decompositions. It is also necessary if abelian categories have enough projectives and injectives. Derived decompositions completely characterize semi-orthogonal decompositions of bounded derived categories. Applications are given to localizing subcategories, homological ring epimorphisms, commutative noetherian rings and nonsingular rings. Moreover, a derived stratification of module categories over commutative noetherian rings of Krull dimension at most 1 is presented.

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1 Introduction

Semi-orthogonal decompositions (or hereditary torsion pairs in the terminology of [6]) have been applied in a number of branches of mathematics. For example, in homotopy and triangulated categories, they were also named as Bousfield localizations ([22, Section 9.1]) and applied to get $t$-structures of triangulated categories (see [5]), and in algebraic geometry they were used to study Fourier-Mukai transforms on derived categories of coherent sheaves of smooth projective varieties (see [16, Chapter 11], [8]). The precise definition of semi-orthogonal decompositions is as follows:

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2010 Mathematics Subject Classification: Primary 16G10, 18E10; Secondary 18E30, 13E05, 18E40.
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Definition 1.1. Let \( \mathcal{D} \) be a triangulated category with the shift functor \([1]\). A pair \( (\mathcal{X}, \mathcal{Y}) \) of full subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of \( \mathcal{D} \) is called a semi-orthogonal decomposition of \( \mathcal{D} \) if

1. \( \mathcal{X} \) and \( \mathcal{Y} \) are triangulated subcategories of \( \mathcal{D} \).
2. \( \text{Hom}_{\mathcal{D}}(X, Y) = 0 \) for all \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \).
3. For each object \( D \in \mathcal{D} \), there exists a triangle in \( \mathcal{D} \)

\[ X_D \rightarrow D \rightarrow Y^D \rightarrow X_D[1] \]

with \( X_D \in \mathcal{X} \) and \( Y^D \in \mathcal{Y} \).

Semi-orthogonal decompositions are also called hereditary torsion pairs in triangulated categories (see [22, Section 9.1], [6, Chapter I.2]). In the course of studying semi-orthogonal decompositions, the following fundamental question seems to remain:

**Question.** Given an abelian category \( A \), how can we construct semi-orthogonal decompositions of the \( * \)-bounded derived category \( \mathcal{D}^*(A) \) of \( A \) for \( * \in \{b, +, -, \emptyset\} \)?

By definition, semi-orthogonal decompositions are defined at the level of derived categories, but it would be useful to have a characterization of such decompositions directly at the level of given abelian categories themselves. So we modify slightly Definition 1.1 and introduce the notion of derived decompositions of abelian categories.

**Definition 1.2.** Let \( A \) be an abelian category, and let \( \mathcal{X} \) and \( \mathcal{Y} \) be full subcategories of \( A \). For \( * \in \{b, +, -, \emptyset\} \), \( \mathcal{D}^*(A) \) denotes the \( * \)-bounded derived category of \( A \). The pair \( (\mathcal{X}, \mathcal{Y}) \) is called a \( \mathcal{D}^* \)-decomposition of \( A \) if

1. \( \mathcal{X} \) and \( \mathcal{Y} \) are abelian subcategories of \( A \), and the inclusions \( \mathcal{X} \subseteq A \) and \( \mathcal{Y} \subseteq A \) induce fully faithful functors \( \mathcal{D}^*(\mathcal{X}) \rightarrow \mathcal{D}^*(A) \) and \( \mathcal{D}^*(\mathcal{Y}) \rightarrow \mathcal{D}^*(A) \), respectively.
2. \( \text{Hom}_{\mathcal{D}^*(A)}(X, Y[n]) = 0 \) for all \( X \in \mathcal{X} \), \( Y \in \mathcal{Y} \) and \( n \in \mathbb{Z} \).
3. For any object \( M^* \in \mathcal{D}^*(A) \), there is a triangle in \( \mathcal{D}^*(A) \)

\[ X_{M^*} \rightarrow M^* \rightarrow Y^{M^*} \rightarrow X_{M^*}[1] \]

such that \( X_{M^*} \in \mathcal{D}^*(\mathcal{X}) \) and \( Y^{M^*} \in \mathcal{D}^*(\mathcal{Y}) \).

In this paper we present sufficient and necessary conditions of when \( \mathcal{D}^* \)-decompositions (thus also semi-orthogonal decompositions) of abelian categories exist. This is given in entirely internal terms of the abelian categories. We then apply our characterization to construct \( \mathcal{D}^* \)-decompositions for a wide variety of situations, including homological ring epimorphisms, localizing subcategories and commutative noetherian rings. Also, we introduce the notion of derived indecomposable, abelian categories and the notion of derived stratification of abelian categories (see Section 2.3). We show that the module category over a commutative noetherian ring of Krull dimension at most 1 has a derived stratification.

Compared with stratifications by bounded derived categories of rings (see [1]), derived decompositions of abelian categories may shed a new light on stratifying bounded derived categories of rings, providing a way to approach the derived category of an abelian category by those of its smaller abelian subcategories.

Our main result reads as follows.

**Theorem 1.3.** Let \( A \) be an abelian category, \( \mathcal{X} \) and \( \mathcal{Y} \) full subcategories of \( A \) and \( * \in \{b, +, -, \emptyset\} \).

1. The pair \( (\mathcal{X}, \mathcal{Y}) \) is a \( \mathcal{D}^* \)-decomposition of \( A \) if the following conditions hold:
   a. \( \text{Ext}^n_A(X, Y) = 0 \) for any \( n \geq 0 \), \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \).
   b. For each object \( M \in A \), there is a long exact sequence

\[ 0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0 \]

in \( A \) with \( X_M, X^M \in \mathcal{X} \) and \( Y_M, Y^M \in \mathcal{Y} \).
(c) For each object $M \in \mathcal{A}$, there is a monomorphism $M \rightarrow I$ in $\mathcal{A}$ such that $X^I = 0$ in (b).
(d) For each object $M \in \mathcal{A}$, there is an epimorphism $P \rightarrow M$ in $\mathcal{A}$ such that $Y_P = 0$ in (b).

(2) Suppose that $\mathcal{A}$ has enough projectives and injectives. Then $(\mathcal{X}, \mathcal{Y})$ is a $\mathcal{D}^*$-decomposition of $\mathcal{A}$ if and only if (a) and (b) together with (c') and (d') hold, where
(c') if $M$ is injective, then $X^M = 0$ in (b).
(d') If $M$ is projective, then $Y_M = 0$ in (b).

Remark that the condition (b) was used in [20] to study the telescope conjecture for hereditary rings. Except this, (b) has not been used elsewhere in the literature before our investigation in this paper.

For bounded derived categories, we can characterize which semi-orthogonal decompositions are induced from derived decompositions (see Proposition 3.20 for details). For convenience, a $\mathcal{D}^b$-decomposition of $\mathcal{A}$ is termed derived decomposition of $\mathcal{A}$ in the sequel.

Theorem 1.3(2) implies that if $\mathcal{A}$ has enough projectives and injectives, then the existences of $\mathcal{D}^*$-decompositions of $\mathcal{A}$ for all $* \in \{b, +, -, \emptyset\}$ are equivalent. In particular, we have the consequence.

Corollary 1.4. Let $\lambda : R \rightarrow S$ be a homological ring epimorphism. Define $\mathcal{Y} := \{Y \in \text{R-Mod} \mid \text{Hom}_R(S, Y) = 0 = \text{Ext}^1_R(S, Y)\}$ and $\mathcal{Z} := \{Z \in \text{R-Mod} \mid S \otimes_R X = 0 = \text{Tor}^1_R(S, X)\}$. Then

(1) $(\text{S-Mod}, \mathcal{Y})$ is a derived decomposition of $	ext{R-Mod}$ if and only if $\text{projdim}_R(S) \leq 1$ and $\text{Hom}_R(\text{Coker}(\lambda), \text{Ker}(\lambda)) = 0$.
(2) $(\mathcal{Z}, \text{S-Mod})$ is a derived decomposition of $	ext{R-Mod}$ if and only if $\text{flatdim}_R(S) \leq 1$ and $\text{Coker}(\lambda) \otimes_R I = 0$ for any injective $\text{R}$-module $I$.
(3) If (1) and (2) are satisfied, then, for any $* \in \{b, +, -, \emptyset\}$, we have $\mathcal{D}^*(\mathcal{Y}) \xrightarrow{\cong} \mathcal{D}^*(\mathcal{Z})$ and there exists a recollement:

Applying Theorem 1.3 to commutative rings, we have the following corollary. For notation and notions, we refer the reader to Section 4.3.

Corollary 1.5. Let $R$ be a commutative noetherian ring.

(1) Suppose that $\Phi$ is a specialization noetherian subset of $\text{Spec}(R)$. If the Krull dimension of $R$ is at most 1, then $\left(\text{Supp}^{-1}(\Phi), \text{Supp}^{-1}(\Phi^c)\right)$ is a derived decomposition of $\text{R-Mod}$, where $\Phi^c := \text{Spec}(R) \setminus \Phi$, and $\text{Supp}^{-1}(\Phi)$ is the category of those $\text{R}$-modules $M$ such that their supports are contained in $\Phi$.
(2) Let $\Sigma$ be a multiplicative subset of $R$, $S$ the localization of $R$ at $\Sigma$ and $\Phi := \{p \in \text{Spec}(R) \mid p \cap \Sigma = \emptyset\}$. Then $\left(\text{Supp}^{-1}(\Phi), \text{S-Mod}\right)$ is a derived decomposition of $\text{R-Mod}$.

Further applications of Theorem 1.3 to localizing subcategories and nonsingular rings, are given by Proposition 4.9 and Corollary 4.10, respectively. For a commutative noetherian ring of Krull dimension at most 1, we show that its module category always admits a derived stratification (see Corollary 4.15).

The article is outlined as follows: In Section 2 we fix notation and recall definitions needed in proofs. In Section 3 we prove Theorem 1.3. The proof is divided into two parts. The first one is for the proof of Theorem 1.3(1), while the second one is for that of Theorem 1.3(2). In Section 4 we apply Theorem 1.3 to construct derived decompositions of module categories from various aspects: ring epimorphisms, localizing subcategories, commutative noetherian rings and nonsingular rings, and therefore prove the two corollaries.

In the second paper we shall construct complete cotorsion pairs from derived decompositions and apply them to infinitely generated tilting modules.

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2 Notation and definitions

In this section we first fix some notation and recall definitions of semi-orthogonal decompositions (or hereditary torsion pairs), cotorsion pairs and complete Ext-orthogonal pairs.

2.1 Notation for derived categories

Let $\mathcal{A}$ be an additive category.

A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is always assumed to be closed under isomorphisms. For an object $X \in \mathcal{A}$, $\text{add}(X)$ (respectively, $\text{Add}(X)$) denotes the full subcategory of $\mathcal{A}$ consisting of all direct summands of finite (respectively, arbitrary) coproducts of copies of $X$ (if arbitrary coproducts exist).

Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor from $\mathcal{A}$ to another additive category $\mathcal{A}'$. The kernel and image of $F$ are defined as $\text{Ker}(F) := \{ X \in \mathcal{A} \mid FX \cong 0 \}$ and $\text{Im}(F) := \{ Y \in \mathcal{A}' \mid \exists X \in \mathcal{A}, FX \cong Y \}$, respectively. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. The kernel, image and cokernel of $f$, whenever they exist, will be denoted by $\text{Ker}(f)$, $\text{Im}(f)$ and $\text{Coker}(f)$, respectively.

By a complex $X^\bullet$ over $\mathcal{A}$ we mean a sequence of morphisms $d^i$ between objects $X^i$ in $\mathcal{A}$: $\cdots \to X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots$, such that $d^i d^{i+1} = 0$ for all $i \in \mathbb{Z}$. We write $X^\bullet = (X^i, d^i)_{i \in \mathbb{Z}}$ and call $d^i$ the $i$-th differential of $X^\bullet$. For a fixed $n \in \mathbb{Z}$, we denote by $X^\bullet[n]$ the complex obtained from $X^\bullet$ by shifting $n$ degrees, that is, $(X^\bullet[n])^i = X^{n+i}$ with the $i$-th differential $(-1)^i d^{n+i}$, and by $H^n(X^\bullet)$ the $n$-th cohomology of $X^\bullet$.

Let $\mathcal{C}(\mathcal{A})$ be the category of all complexes over $\mathcal{A}$ with chain maps as morphisms, and $\mathcal{K}(\mathcal{A})$ the homotopy category of $\mathcal{C}(\mathcal{A})$. We denote by $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$ the bounded complex and homotopy categories of $\mathcal{A}$, respectively.

From now on, let $\mathcal{A}$ be an abelian category.

By $\mathcal{P}(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$ we denote the unbounded and bounded derived categories of $\mathcal{A}$, respectively. Throughout the paper, we always identify $\mathcal{D}^b(\mathcal{A})$ with the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of all complexes with finitely many non-zero cohomologies because they are equivalent as triangulated categories. Further, by $\mathcal{D}^+(\mathcal{A})$ and $\mathcal{D}^-(\mathcal{A})$ we denote the bounded-below and bounded-above derived categories of $\mathcal{A}$, respectively.

For any $X, Y \in \mathcal{A}$ and $i \in \mathbb{Z}$, we write $\text{Ext}^i_A(X,Y)$ for $\text{Hom}_{\mathcal{D}(\mathcal{A})}(X,Y[i])$. Note that $\text{Ext}_A^i(X,Y) = \text{Hom}_A(X,Y)$ and $\text{Ext}_A^i(X,Y) = 0$ whenever $i < 0$. For each $i \geq 1$, $\text{Ext}_A^i(X,Y)$ can be identified with the set of equivalence classes of long exact sequences $0 \to Y \to E_1 \to \cdots \to E_i \to X \to 0$ in $\mathcal{A}$ (see [17, XI] for details).

The following facts are standard in homological algebra.

(1) Suppose that $\mathcal{A}$ has enough projectives with $\mathcal{P}(\mathcal{A})$ the category of all projective objects of $\mathcal{A}$. Further, let $\mathcal{K}^{-,b}(\mathcal{P}(\mathcal{A}))$ be the full subcategory of $\mathcal{K}(\mathcal{A})$ consisting of bounded-above complexes with all terms in $\mathcal{P}(\mathcal{A})$ and finitely many nonzero cohomologies. Then there is a triangle equivalence between $\mathcal{K}^{-,b}(\mathcal{P}(\mathcal{A}))$ and $\mathcal{D}^b(\mathcal{A})$. In this case, $\text{Ext}_A^i(X,Y)$ is isomorphic to the usual $i$-th extension group of $X$ and $Y$, defined by projective resolutions of $X$.

(2) Dually, suppose that $\mathcal{A}$ has enough injectives with $\mathcal{I}(\mathcal{A})$ the category of all injective objects of $\mathcal{A}$. Then there is a triangle equivalence between $\mathcal{K}^{+,b}(\mathcal{I}(\mathcal{A}))$ and $\mathcal{D}^b(\mathcal{A})$, where $\mathcal{K}^{+,b}(\mathcal{I}(\mathcal{A}))$ is defined similarly. In this situation, $\text{Ext}_A^i(X,Y)$ can be calculated by taking injective resolutions of $Y$.

A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called an abelian subcategory of $\mathcal{A}$ if $\mathcal{B}$ is an abelian category and the inclusion $\mathcal{B} \to \mathcal{A}$ is an exact functor between abelian categories. This is equivalent to saying that $\mathcal{B}$ is closed
under taking kernels and cokernels in $A$. The full subcategories $\{0\}$ and $A$ are called the trivial abelian subcategories of $A$.

For $n \in \mathbb{N}$ and a full subcategory $B$ of $A$, we define the full subcategories of $A$:

$$
i^n B := \{ X \in A \mid \text{Ext}^n_A(X,Y) = 0, Y \in B \},$$

$$\perp^n B := \{ X \in A \mid \text{Ext}^n_A(X,Y) = 0, Y \in B, j > n \},$$

$$\perp B := \{ X \in A \mid \text{Ext}^n_A(X,Y) = 0, Y \in B, j \geq 0 \}.$$

Similarly, $B^{\perp n}$, $B^{\perp 0}$ and $B^\perp$ are defined. Recall that $i^0 B$ is said to be left perpendicular to $B$ in $A$, while $B^\perp$ is said to be right perpendicular to $B$ in $A$ (see [14]).

Lemma 2.1. If $B$ is a abelian subcategory of an abelian category $A$ such that the inclusion $i: B \to A$ induces a fully faithful functor $\mathcal{D}^b(i): \mathcal{D}^b(B) \to \mathcal{D}^b(A)$, then $\text{Im}(\mathcal{D}^b(i))$ consists of all bounded complexes $X^\bullet$ with $H^n(X^\bullet) \in B$ for all $n \in \mathbb{Z}$.

Proof. Let $\mathcal{K}$ be the full subcategory of $\mathcal{D}^b(A)$ consisting of complexes with all cohomologies in $B$. Note that $\text{Im}(\mathcal{D}^b(i))$ is a full triangulated subcategory of $\mathcal{D}^b(A)$ containing $B$ and being closed under isomorphisms. Since $B \subseteq A$ is an abelian subcategory, $\text{Im}(\mathcal{D}^b(i)) \subseteq \mathcal{K}$. Now, let $X^\bullet \in \mathcal{K}$. We can show by induction on the number of nonzero terms of $X^\bullet$ that $X^\bullet$ belongs to the smallest full triangulated subcategory containing $H^n(X^\bullet)$ for all $n \in \mathbb{Z}$. Clearly, this subcategory is contained in $\text{Im}(\mathcal{D}^b(i))$. Thus $X^\bullet \in \text{Im}(\mathcal{D}^b(i))$ and $\mathcal{K} \subseteq \text{Im}(\mathcal{D}^b(i))$. Hence $\text{Im}(\mathcal{D}^b(i)) = \mathcal{K}$. $\square$

By a ring we mean an associative ring with identity. We denote by $R$-Mod the category of all unitary left $R$-modules. For an $R$-module $M$, we denote by $\text{projdim}(R M)$, $\text{injdim}(R M)$ and $\text{flatdim}(R M)$ the projective, injective and flat dimensions of $M$, respectively. As usual, we simply write $\mathcal{C}(R)$, $\mathcal{K}(R)$ and $\mathcal{D}(R)$ for the complex, homotopy and derived categories of $R$-Mod, respectively.

Let $\lambda : R \to S$ be a homomorphism of rings. We denote by $\lambda_* : S$-Mod $\to$ $R$-Mod the restriction functor induced by $\lambda$, and by $\mathcal{D}(\lambda_*): \mathcal{D}(S) \to \mathcal{D}(R)$ the derived functor of $\lambda_*$. If $\lambda$ is fully faithful, then $\lambda$ is called a ring epimorphism. If $\mathcal{D}(\lambda_*)$ is fully faithful, then $\lambda$ is called a homological ring epimorphism. Note that $\lambda$ is a homological ring epimorphism if and only if the multiplication $S \otimes_R S \to S$ is an isomorphism and $\text{Torr}^R_n(S,S) = 0$ for all $n \geq 1$. In this case, we always identify $S$-Mod with $\text{Im}(\lambda_*)$, and $\mathcal{D}(S)$ with $\text{Im}(\mathcal{D}(\lambda_*))$.

2.2 Derived stratifications of abelian categories

Semi-orthogonal decompositions are also called hereditary torsion pairs in triangulated categories (see [22, Section 9.1], [6, Chapter I.2]). They are closely related to half recollements of triangulated categories. In fact, a pair $(\mathcal{X}, \mathcal{Y})$ of full triangulated subcategories of $\mathcal{D}$ is a semi-orthogonal decomposition of $\mathcal{D}$ if and only if there exists a lower half recollement among $\mathcal{X}$, $\mathcal{D}$ and $\mathcal{Y}$, in the sense that there are four triangle functors demonstrated in the diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{i} & \mathcal{D} \\
\xrightarrow{j} & & \xrightarrow{l} \\
\mathcal{Y} & & \mathcal{J} \\
\end{array}
\]

such that

1. $i$ and $j$ are canonical inclusions;
2. both $i(R)$ and $l(J)$ are adjoint pairs;
3. $Li = 0$ (and thus also $Rj = 0$);
4. for each object $D \in \mathcal{D}$, there exists a distinguished triangle in $\mathcal{D}$

\[
iR(D) \to D \to jL(D) \to iR(D)[1]
\]
where \(iR(D) \rightarrow D\) is the counit adjunction and \(D \rightarrow jL(D)\) is the unit adjunction. In this case, there are equivalences of triangulated categories: \(\mathcal{D}/\mathcal{X} \sim \mathcal{Y}\) and \(\mathcal{D}/\mathcal{Y} \sim \mathcal{X}\).

Observe that the conditions in Definition 1.1 is weaker than the ones given in [4, 16, 24] because \(i\) may not have a left adjoint, nor \(j\) have a right adjoint. But, if \(i\) does have a left adjoint (or equivalently, \(L\) has a fully faithful left adjoint), then the lower half recollement can be completed to a recollement among triangulated categories \(\mathcal{X}, \mathcal{D}\) and \(\mathcal{Y}\) in the sense of Beilinson, Bernstein and Deligne (see [5] for definition).

For the convenience of general considerations, we restate derived decompositions in a slightly different way.

**Definition 2.2.** Let \(\mathcal{A}, \mathcal{X}\) and \(\mathcal{Y}\) be abelian categories, and let \(i: \mathcal{X} \rightarrow \mathcal{A}\) and \(j: \mathcal{Y} \rightarrow \mathcal{A}\) be exact functors. The pair \((\mathcal{X}, \mathcal{Y})\) is called a derived decomposition (or \(D\)-decomposition) of \(\mathcal{A}\) with respect to \(i\) and \(j\) if the two conditions hold:

1. The derived functors \(D^b(i): D^b(\mathcal{X}) \rightarrow D^b(\mathcal{A})\) and \(D^b(j): D^b(\mathcal{Y}) \rightarrow D^b(\mathcal{A})\) induced from \(i\) and \(j\) are fully faithful.
2. \((\text{Im}(D^b(i)), \text{Im}(D^b(j)))\) is a semi-orthogonal decomposition of \(D^b(\mathcal{A})\).

Clearly, (D1) implies that both \(i\) and \(j\) are fully faithful, and thus the images of \(i\) and \(j\) are abelian subcategories of \(\mathcal{A}\).

If \((\mathcal{X}, \mathcal{Y})\) is a derived decomposition of \(\mathcal{A}\), then \(\mathcal{X}\) and \(\mathcal{Y}\) are called derived factors of \(\mathcal{A}\). Moreover, there are exact sequences of derived categories:

\[
D^b(\mathcal{X}) \xrightarrow{D^b(i)} D^b(\mathcal{A}) \xrightarrow{p} D^b(\mathcal{Y}), \quad D^b(\mathcal{Y}) \xrightarrow{D^b(j)} D^b(\mathcal{A}) \xrightarrow{q} D^b(\mathcal{X})
\]

where \(p\) and \(q\) are compositions of the canonical quotient functors with an equivalence functor, respectively.

As in [1] for triangulated categories, we now introduce the notion of derived stratifications for abelian categories.

**Definition 2.3.** (1) An abelian category \(\mathcal{A}\) is derived indecomposable if it does not have any non-trivial derived decompositions, that is, only \((\mathcal{A}, 0)\) and \((0, \mathcal{A})\) are the derived decompositions of \(\mathcal{A}\).

(2) A derived stratification of an abelian category \(\mathcal{A}\) is a sequence of derived decompositions:

1. A derived decomposition \((\mathcal{X}_0, \mathcal{X}_1)\) of \(\mathcal{A}\) if \(\mathcal{A}\) is not derived indecomposable,
2. A derived decomposition \((\mathcal{X}_{00}, \mathcal{X}_{01})\) of \(\mathcal{X}_0\) if \(\mathcal{X}_0\) is not derived indecomposable,
3. A derived decomposition \((\mathcal{X}_{10}, \mathcal{X}_{11})\) of \(\mathcal{X}_1\) if \(\mathcal{X}_1\) is not derived indecomposable,
4. Derived decompositions of \(\mathcal{X}_{ij}\) with \(0 \leq i, j \leq 1\) if \(\mathcal{X}_{ij}\) are not derived indecomposable. Continuing this procedure of decompositions, until one arrives at all derived factors being derived indecomposable. This procedure may continue to infinitum.

All the derived indecomposable, abelian categories appearing in this procedure are called composition factors of the stratification. The cardinality of the set of all composition factors (counting the multiplicity) is called the length of the stratification. If this procedure stops after finitely many steps, we say that this stratification is finite or of finite length.

Examples of derived indecomposable abelian categories can be found in Section 4.3

### 2.3 Ext-orthogonal pairs and cotorsion pairs in abelian categories

Derived decompositions of abelian categories are associated with both complete cotorsion pairs and complete Ext-orthogonal pairs in abelian categories. The notion of complete cotorsion pairs is classical and has been widely applied to relative homological algebra and generalized tilting theory (see [10, 6, 15]), while the notion of complete Ext-orthogonal pairs seems only to be employed in dealing with the telescope conjecture.
for hereditary rings (see [20]). We will show in the next section that the latter may be useful in derived decompositions.

Throughout this section, $A$ is an abelian category, and $(\mathcal{X}, \mathcal{Y})$ is a pair of full subcategories of $A$.

**Definition 2.4.** (1) The pair $(\mathcal{X}, \mathcal{Y})$ is called a cotorsion pair in $A$ if $\mathcal{X} = \text{coker} \mathcal{Y}$ and $\mathcal{Y} = \text{ker} \mathcal{X}$; and a complete cotorsion pair in $A$ if it is a cotorsion pair and, for each object $M \in A$, there are short exact sequences in $A$

$$0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

(2) [20, Definition 2.1] The pair $(\mathcal{X}, \mathcal{Y})$ is said to be Ext-orthogonal in $A$ if $\mathcal{X} = ^{10} \mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{1}$; and a complete Ext-orthogonal in $A$ if it is Ext-orthogonal and satisfies the gluing condition $(\text{GC})$: For each object $M \in A$, there exists a five-term exact sequence in $A$

$$\varepsilon_M: 0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

Note that the two conditions $(a)$ and $(b)$ in Theorem 1.3 are equivalent to saying that $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $A$.

**Lemma 2.5.** Let $A$ be an abelian category, and let $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $A$. If $\mathcal{X} \subseteq \mathcal{Y}$ and $(\mathcal{X}, \mathcal{Y})$ satisfies $(\text{GC})$, then

1. $\mathcal{X} = ^{10} \mathcal{Y} \cap 1 \mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{10} \cap \mathcal{X}^{1}$.
2. $(\mathcal{X}, \mathcal{Y})$ is complete Ext-orthogonal.
3. Both $\mathcal{X}$ and $\mathcal{Y}$ are abelian subcategories of $A$.

**Proof.** (1) It follows from $\mathcal{X} \subseteq \mathcal{Y}$ that $\mathcal{X} \subseteq ^{10} \mathcal{Y} \cap 1 \mathcal{Y}$. Now, we show the converse of this inclusion. Let $M \in ^{10} \mathcal{Y} \cap 1 \mathcal{Y}$. Since $(\mathcal{X}, \mathcal{Y})$ satisfies $(\text{GC})$, there is a five-term exact sequence $\varepsilon_M$ for $M$ in $A$. In particular, both $Y_M$ and $Y^M$ belong to $\mathcal{Y}$. It then follows from $\text{Hom}_A(M, Y^M) = 0$ that there is a short exact sequence $0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow 0$ in $A$. Further, this sequence splits due to $\text{Ext}^1_A(M, Y_M) = 0$. Thus $X_M = Y_M \oplus M$. Since $\text{Hom}_A(X_M, Y_M) = 0$, we have $Y_M = 0$ and $M \cong X_M \in \mathcal{X}$. So $^{10} \mathcal{Y} \cap 1 \mathcal{Y} \subseteq \mathcal{X}$. Thus the first equality in (1) holds. Similarly, the second equality in (1) holds.

(2) It suffices to show both $\mathcal{X} = ^{1} \mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{1}$. But this follows from (1) and the inclusion $\mathcal{X} \subseteq ^{1} \mathcal{Y}$.

(3) We only prove that $\mathcal{X}$ is an abelian subcategory of $A$. The conclusion on $\mathcal{Y}$ can be proved dually.

Clearly, $\mathcal{X}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms in $A$. Thus $\mathcal{X}$ is an abelian subcategory of $A$ if and only if $\mathcal{X}$ is closed under cokernels (or equivalently, kernels) in $A$. In the following, we show that $\mathcal{X}$ is closed under cokernels in $A$.

Let $f : M \rightarrow N$ be a morphism in $A$ with $M, N \in \mathcal{X}$. Then there is a canonical four-term exact sequence

$$0 \rightarrow \text{Ker}(f) \rightarrow M \overset{f}{\rightarrow} N \rightarrow \text{Coker}(f) \rightarrow 0$$

in $A$. On the one hand, from $M, N \in \mathcal{X} \subseteq ^{10} \mathcal{Y}$ and the fact that $^{10} \mathcal{Y}$ is closed under quotients in $A$, it follows that both $\text{Im}(f)$ and $\text{Coker}(f)$ lies in $^{10} \mathcal{Y}$. On the other hand, for $Y \in \mathcal{Y}$, by applying $\text{Hom}_A(\_, Y)$ to the short exact sequence $0 \rightarrow \text{Im}(f) \rightarrow N \rightarrow \text{Coker}(f) \rightarrow 0$, with $N \in \mathcal{X} = ^{10} \mathcal{Y}$, we get $\text{Hom}_A(\text{Im}(f), Y) = \text{Ext}^1_A(\text{Coker}(f), Y)$. This yields $\text{Ext}^1_A(\text{Coker}(f), Y) = 0$, and consequently, $\text{Coker}(f) \in ^{10} \mathcal{Y} \cap 1 \mathcal{Y}$. By (1), $\text{Coker}(f) \in \mathcal{X}$ and thus $\mathcal{X}$ is closed under cokernels in $A$. □

**Lemma 2.6.** [20, Lemma 2.9] Let $(\mathcal{X}, \mathcal{Y})$ be an Ext-orthogonal pair in an abelian category $A$ and $M$ an object in $A$. Suppose that there is an exact sequence

$$\varepsilon_M: 0 \rightarrow Y_M \overset{\varepsilon_M^2}{\rightarrow} X_M \overset{\varepsilon_M^1}{\rightarrow} M \overset{\varepsilon_M^0}{\rightarrow} Y^M \overset{\varepsilon_M^1}{\rightarrow} X^M \rightarrow 0$$

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in \( A \) with \( X_M, X^M \in \mathcal{X} \) and \( Y_M, Y^M \in \mathcal{Y} \).

(1) There are isomorphisms of abelian groups for all \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \):

\[
(\varepsilon_M^{-1})^* : \text{Hom}_A(X, X_M) \xrightarrow{\cong} \text{Hom}_A(X, M) \quad \text{and} \quad (\varepsilon_M^0)^* : \text{Hom}_A(Y^M, Y) \xrightarrow{\cong} \text{Hom}_A(M, Y).
\]

(2) If \( \varepsilon_N : 0 \to Y_N \xrightarrow{\varepsilon_N^2} X_N \xrightarrow{\varepsilon_N^1} N \xrightarrow{\varepsilon_N^0} Y^N \xrightarrow{\varepsilon_N^1} X^N \to 0 \) is an exact sequence in \( A \) with \( X_N, X^N \in \mathcal{X} \) and \( Y_N, Y^N \in \mathcal{Y} \), then each morphism \( f : M \to N \) extends uniquely to a morphism \( \varepsilon_f : \varepsilon_M \to \varepsilon_N \) of exact sequences:

\[
\begin{array}{c}
\varepsilon_M : \\
0 \to Y_M \xrightarrow{\varepsilon_M^2} X_M \xrightarrow{\varepsilon_M^1} M \xrightarrow{\varepsilon_M^0} Y^M \xrightarrow{\varepsilon_M^1} X^M \to 0 \\
\varepsilon_f \\
0 \to Y_f \xrightarrow{\varepsilon_f^2} X_f \xrightarrow{\varepsilon_f^1} f \xrightarrow{\varepsilon_f^0} Y^f \xrightarrow{\varepsilon_f^1} X^f \\
\varepsilon_N : \\
0 \to Y_N \xrightarrow{\varepsilon_N^2} X_N \xrightarrow{\varepsilon_N^1} N \xrightarrow{\varepsilon_N^0} Y^N \xrightarrow{\varepsilon_N^1} X^N \to 0.
\end{array}
\]

(3) Any exact sequence \( 0 \to Y \to X \to M \to Y' \to X' \to 0 \) in \( A \) with \( X, X' \in \mathcal{X} \) and \( Y, Y' \in \mathcal{Y} \) is isomorphic to \( \varepsilon_M \).

Now, let \((\mathcal{X}, \mathcal{Y})\) be a complete Ext-orthogonal pair in an abelian category \( A \). For each object \( M \in A \), we fix an exact sequence in \( A \)

\[
(*) \quad \varepsilon_M : 0 \to Y_M \xrightarrow{\varepsilon_M^2} X_M \xrightarrow{\varepsilon_M^1} M \xrightarrow{\varepsilon_M^0} Y^M \xrightarrow{\varepsilon_M^1} X^M \to 0
\]

such that \( X_M, X^M \in \mathcal{X} \) and \( Y_M, Y^M \in \mathcal{Y} \). In particular, if \( M \in \mathcal{X} \), then \( \varepsilon_M^{-1} : X_M \to M \) is an isomorphism and \( Y^M \cong 0 \); if \( M \in \mathcal{Y} \), then \( \varepsilon_M^0 : M \to Y^M \) is an isomorphism and \( X_M \cong 0 \).

By Lemma 2.5, both \( \mathcal{X} \) and \( \mathcal{Y} \) are abelian subcategories of \( A \) closed under direct summands. Let \( i : \mathcal{X} \to A \) and \( j : \mathcal{Y} \to A \) be the inclusions. Then \( i \) and \( j \) are exact functors. Moreover, \( i \) has a right adjoint \( r : A \to \mathcal{X} \) and \( j \) has a left adjoint \( \ell : A \to \mathcal{Y} \), which are defined as follows:

For each \( M \in A \) and for a morphism \( f : M \to N \) in \( A \),

\[
r(M) = X_M, \quad r(f) = X_f : X_M \to X_N \quad \text{and} \quad \ell(M) = Y^M, \quad \ell(f) = Y^f : Y^M \to Y^N.
\]

These are well defined by Lemma 2.6. For the adjoint pair \((i, r)\) of functors, the unit adjunction of \( X \in \mathcal{X} \) is given by the inverse of the isomorphism \( \varepsilon_X^{-1} : r(X) \to X \), and the counit adjunction of \( M \in A \) is given by \( \varepsilon_M^{-1} : i\ell(M) \to M \). Similarly, the unit and counit adjunctions associated with \((\ell, j)\) can be defined by \( \varepsilon_M^0 \).

Now, we can form the diagram of functors between abelian categories:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{i} & A \\
\downarrow{r} & & \downarrow{\ell} \\
\mathcal{Y} & \xleftarrow{j}
\end{array}
\]

where \( r \) is left exact and \( \ell \) is right exact. In general, neither \( r \) nor \( \ell \) is exact. So \((\downarrow)\) is neither a localization nor a colocalization sequence of abelian categories, and therefore, it may not be completed into a recollement of abelian categories. However, since \( i \) and \( j \) are exact, they induce derived functors between bounded derived categories: \( D^b(i) : D^b(\mathcal{X}) \to D^b(A) \) and \( D^b(j) : D^b(\mathcal{Y}) \to D^b(A) \). With the notation in \((\downarrow)\) , the sequence \((*)\) can be rewritten as follows:

\[
(*) \quad \varepsilon_M : 0 \to Y_M \xrightarrow{\varepsilon_M^2} r(M) \xrightarrow{\varepsilon_M^1} \ell(M) \xrightarrow{\varepsilon_M^0} X^M \to 0.
\]

Finally, we consider the following full subcategories of \( A \) defined via the sequence \((*)\):

\[
A_{r\text{-adj}} := \{ M \in A \mid X^M = 0 \} \quad \text{and} \quad A_{l\text{-adj}} := \{ M \in A \mid Y_M = 0 \}.
\]

The following properties of the two subcategories will be used in the proof of Theorem 1.3(1).
Lemma 2.7. (1) $A_{r_{adj}}$ is closed under extensions and quotients in $A$.

(2) $A_{r_{adj}}$ is closed under extensions and subobjects in $A$.

(3) The restriction of $r$ to $A_{r_{adj}}$ is exact, that is, if $0 \to M^{-2} \to M^{-1} \to M^0 \to 0$ is an exact sequence in $A$ with $M^i \in A_{r_{adj}}$ for $-2 \leq i \leq 0$, then $0 \to r(M^{-2}) \to r(M^{-1}) \to r(M^0) \to 0$ is an exact sequence in $\mathcal{X}$.

(4) The restriction of $\ell$ to $A_{r_{adj}}$ is exact, that is, if $0 \to M^{-2} \to M^{-1} \to M^0 \to 0$ is an exact sequence in $A$ with $M^i \in A_{r_{adj}}$ for $-2 \leq i \leq 0$, then $0 \to \ell(M^{-2}) \to \ell(M^{-1}) \to \ell(M^0) \to 0$ is an exact sequence in $\mathcal{Y}$.

Proof. We only prove (1) and (3) since (2) and (4) can be proved dually.

Let $0 \to M^{-2} \xrightarrow{f} M^{-1} \xrightarrow{g} M^0 \to 0$ be an exact sequence in $A$. We regard it as a complex $M^* \in \mathcal{C}^b(A)$ with $M^0$ in degree 0. It follows from Lemma 2.6(2) that the sequence $(\ast)$, associated with $M^i$ for $-2 \leq i \leq 0$, induces an exact sequence of complexes over $A$:

$$0 \to Y_{M^*} \xrightarrow{e_M^2} r(M^*) \xrightarrow{e_M^1} M^* \xrightarrow{e_M^0} \ell(M^*) \xrightarrow{e_M^1} X_{M^*} \to 0,$$

where $r(M^*)$ means $(r(M^i))_{i \in \mathbb{Z}}$. Recall that $r$ is a left exact functor and $\ell$ is a right exact functor. Thus the complexes $r(M^*)$ and $\ell(M^*)$ are exact everywhere except in the degrees 0 and $-2$, respectively. Since $M^*$ is an exact sequence, (1) holds.

(3) Suppose $M^i \in A_{r_{adj}}$ for $-2 \leq i \leq 0$. Then $X_{M^*} = 0$. To show that $r(M^*)$ is an exact sequence, it suffices to show that the homomorphism $r(g) : r(M^{-1}) \to r(M^0)$ is surjective (or equivalently, $H^0(r(M^*)) = 0$).

Let $c(M^*)$ be the cokernel of the chain map $e_M^2$. Taking cohomologies on the sequence $0 \to Y_{M^*} \to r(M^*) \to c(M^*) \to 0$, we get a long exact sequence in $A$:

$$\cdots \xrightarrow{H^{-1}(r(M^*))} \xrightarrow{H^{-1}(c(M^*))} \xrightarrow{H^0(Y_{M^*})} \xrightarrow{H^0(r(M^*))} \xrightarrow{H^0(c(M^*))} \cdots.$$

Note that $H^{-1}(r(M^*)) = 0$ and $H^0(Y_{M^*}) \simeq \text{Coker}(Y_g)$, where $Y_g : Y_{M^*} \to Y_{M^0}$ is induced from $g$. Since the sequence $M^*$ is exact and $0 \to c(M^*) \to M^* \to \ell(M^*) \to 0$ is exact in $\mathcal{C}^b(A)$, we have $H^i(c(M^*)) \simeq H^{i-1}(\ell(M^*))$ for any $i \in \mathbb{Z}$. This implies $H^0(c(M^*)) \simeq H^{-1}(\ell(M^*)) = 0$ and $H^{-1}(c(M^*)) \simeq H^{-2}(\ell(M^*)) \simeq \text{Ker}(Y_f)$, where $Y_f : \ell(M^{-2}) \to \ell(M^{-1})$ is induced from $f$. Consequently, there is a short exact sequence in $A$:

$$0 \to \text{Ker}(Y_f) \to \text{Coker}(Y_g) \to H^0(r(M^*)) \to 0.$$

Clearly, $\text{Ker}(Y_f), \text{Coker}(Y_g) \in \mathcal{Y}$ and $H^0(r(M^*)) \in \mathcal{X}$ since $\mathcal{Y}$ and $\mathcal{X}$ are abelian full subcategories of $A$ by Lemma 2.5(3). Thus $H^0(r(M^*)) \in \mathcal{X} \cap \mathcal{Y}$. It follows from $\mathcal{X} = \perp \mathcal{Y}$ that $H^0(r(M^*)) = 0$. So (3) holds. □

3 Derived decompositions of abelian categories

In this section we will prove Theorem 1.3. In particular, we show that a complete Ext-orthogonal pair is a derived decomposition of an abelian category with enough projectives and injectives if and only if the five-term exact sequences for both projective and injective objects are reduced to four terms.

3.1 Proof of Theorem 1.3(1)

Complete Ext-orthogonal pairs and derived decompositions are defined in terms of abelian categories. We will show that derived decompositions induce complete Ext-orthogonal pairs.

Proposition 3.1. Let $A$ be an abelian category and let $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $A$. Given $\ast \in \{b, +, -, \emptyset\}$, if $(\mathcal{X}, \mathcal{Y})$ is a $\mathcal{D}^\ast$-decomposition of $A$, then it is a complete Ext-orthogonal pair in $A$. 

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Proof. Suppose that \((\mathcal{X}, \mathcal{Y})\) is a \(\mathcal{D}^*\)-derived decomposition of \(\mathcal{A}\). By Definition 1.2(D2), \(\mathcal{X} \subseteq \mathcal{Y}\), and by Definition 1.2(D3), for each \(M \in \mathcal{A}\), there is a triangle \(X^* \to M \to Y^* \to X^*[1]\) in \(\mathcal{D}^*(\mathcal{A})\) such that \(X^* \in \mathcal{D}^*(\mathcal{X})\) and \(Y^* \in \mathcal{D}^*(\mathcal{Y})\), where \(\mathcal{D}^*(\mathcal{X})\) and \(\mathcal{D}^*(\mathcal{Y})\) can be regarded as triangulated subcategories of \(\mathcal{D}^*(\mathcal{A})\) by (D1) in Definition 1.2. By taking cohomologies on this triangle, one gets the long exact sequence in \(\mathcal{A}\)

\[
0 \to H^{-1}(Y^*) \to H^0(X^*) \to M \to H^0(Y^*) \to H^1(X^*) \to 0.
\]

Recall that \(\mathcal{X}\) is abelian subcategories of \(\mathcal{A}\). Since \(X^* \in \mathcal{D}^*(\mathcal{X})\), we have \(H^i(X^*) \in \mathcal{X}\) for all \(i \in \mathbb{Z}\). Particularly, both \(H^0(X^*)\) and \(H^1(X^*)\) lie in \(\mathcal{X}\). Similarly, both \(H^{-1}(Y^*)\) and \(H^0(Y^*)\) lie in \(\mathcal{Y}\). Thus the pair \((\mathcal{X}, \mathcal{Y})\) satisfies the gluing condition. Now, by Lemma 2.5(2), \((\mathcal{X}, \mathcal{Y})\) is a complete Ext-orthogonal pair in \(\mathcal{A}\). \(\square\)

Having shown that \(\mathcal{D}^*\)-decompositions are complete Ext-orthogonal pairs, we now consider its converse:

Given a complete Ext-orthogonal pair \((\mathcal{X}, \mathcal{Y})\) in an abelian category \(\mathcal{A}\), when is it a \(\mathcal{D}^*\)-decomposition of \(\mathcal{A}\) for \(\ast \in \{b, +, -, \emptyset\}\)?

To address this question, we use derived categories of exact categories. Recall that an exact category (in the sense of Quillen) is an additive category \(\mathcal{E}\) endowed with a class of conflations closed under isomorphism and satisfying certain axioms (for example, see [18, Section 4]). In case that \(\mathcal{E}\) is an abelian category, the class of conflations coincides with the class of short exact sequences.

Let \(\mathcal{E}\) be a full subcategory of the abelian category \(\mathcal{A}\). Suppose that \(\mathcal{E}\) is closed under extensions in \(\mathcal{A}\), that is, for any exact sequence \(0 \to X \to Y \to Z \to 0\) in \(\mathcal{A}\) with \(X, Z \in \mathcal{E}\), we have \(Y \in \mathcal{E}\). Then \(\mathcal{E}\) endowed with the short exact sequences of \(\mathcal{A}\) having their terms in \(\mathcal{E}\) is an exact category and the inclusion \(\mathcal{E} \subseteq \mathcal{A}\) is a fully faithful exact functor. So \(\mathcal{E}\) is called a fully exact subcategory of \(\mathcal{A}\).

A complex \(X^* \in \mathcal{C}(\mathcal{E})\) is said to be strictly exact if it is exact in \(\mathcal{C}(\mathcal{A})\) and all of its boundaries belong to \(\mathcal{E}\). Let \(\mathcal{H}_{ac}(\mathcal{E})\) be the full subcategory of \(\mathcal{H}(\mathcal{E})\) consisting of those complexes which are isomorphic to strictly exact complexes. Then \(\mathcal{H}_{ac}(\mathcal{E})\) is a full triangulated subcategory of \(\mathcal{H}(\mathcal{E})\). The unbounded derived category of \(\mathcal{E}\), denoted by \(\mathcal{D}(\mathcal{E})\), is defined to be the Verdier quotient of \(\mathcal{H}(\mathcal{E})\) by \(\mathcal{H}_{ac}(\mathcal{E})\). Similarly, the bounded-below, bounded-above and bounded derived categories \(\mathcal{D}^+(\mathcal{E})\), \(\mathcal{D}^-\mathcal{D}(\mathcal{E})\) and \(\mathcal{D}^{b}(\mathcal{E})\) can be defined through bounded-below, bounded-above and bounded complexes over \(\mathcal{E}\), respectively. Moreover, the canonical functor \(\mathcal{D}^+(\mathcal{E}) \to \mathcal{D}(\mathcal{E})\) is fully faithful for any \(\ast \in \{+, -, b\}\). If \(\mathcal{E}\) is closed under cokernels in \(\mathcal{A}\), then a complex \(X^* \in \mathcal{C}^{b}(\mathcal{E})\) is strictly exact if and only if it is exact in \(\mathcal{C}(\mathcal{A})\).

For more details on derived categories of exact categories, we refer the reader to [18]. The following result follows from [18, Theorem 12.1].

Lemma 3.2. Let \(\mathcal{E}\) be a full subcategory of an abelian category \(\mathcal{A}\), satisfying the two conditions:

(i) If \(0 \to X \to Y \to Z \to 0\) is an exact sequence in \(\mathcal{A}\) with \(X \in \mathcal{E}\), then \(Y \in \mathcal{E}\) if and only if \(Z \in \mathcal{E}\).

(ii) For each object \(M \in \mathcal{A}\), there is a long exact sequence in \(\mathcal{A}\):

\[
0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0
\]

for a nonnegative integer \(n\) such that \(E_i \in \mathcal{E}\) for all \(0 \leq i \leq n\). Then the inclusion \(\mathcal{E} \subseteq \mathcal{A}\) induces a triangle equivalence \(\mathcal{D}^+(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^+(\mathcal{A})\) which can be restricted to an equivalence \(\mathcal{D}^{b}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^{b}(\mathcal{A})\).

Remark 3.3. If (ii) in Lemma 3.2 is strengthened as

(ii') There is a nonnegative integer \(n\) such that, for each object \(M \in \mathcal{A}\), there is a long exact sequence \(0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0\) in \(\mathcal{A}\) with \(E_i \in \mathcal{E}\) for all \(0 \leq i \leq n\), then the inclusion \(\mathcal{E} \subseteq \mathcal{A}\) induces a triangle equivalence \(\mathcal{D}(\mathcal{E}) \to \mathcal{D}(\mathcal{A})\) which restricts to an equivalence \(\mathcal{D}^{+}(\mathcal{E}) \to \mathcal{D}^{+}(\mathcal{A})\) for any \(\ast \in \{+, -, b\}\).

For a proof of this result and its dual statement, we refer the reader to [25, Proposition A.5.6]
**Proof of Theorem 1.3(1).** It follows from (a), (b) and Lemma 2.5 that \((\mathcal{X}, \mathcal{Y})\) is a complete Ext-orthogonal pair in \(\mathcal{A}\) and that both \(\mathcal{X}\) and \(\mathcal{Y}\) are abelian subcategories of \(\mathcal{A}\). In particular, \((D2)\) in Definition 1.2 holds.

Now, we keep all the notation introduced in Section 2.3. Under the assumptions of (a), (b) and (c), we show that the functor \(D^*(i) : D^*(\mathcal{X}) \to D^*(\mathcal{A})\), induced from the inclusion \(i : \mathcal{X} \to \mathcal{A}\), is fully faithful.

By Lemma 2.7(1), \(A_{r-adj}\) is closed under extensions in \(\mathcal{A}\) and thus a fully exact subcategory of \(\mathcal{A}\). Moreover, \(i\) has a right adjoint \(r : \mathcal{A} \to \mathcal{X}\) which is an exact functor when restricted to \(A_{r-adj}\) by Lemma 2.7(3). Thus \(r : A_{r-adj} \to \mathcal{X}\) induces a derived functor \(D^*(r) : D^*(A_{r-adj}) \to D^*(\mathcal{X})\). Since the functor \(i\) is fully faithful, the composition of \(i\) and \(r\) is isomorphic to the identity functor of \(\mathcal{X}\). Thus \((D^*(i), D^*(r))\) is an adjoint pair and the composition of \(D^*(i)\) with \(D^*(r)\) is isomorphic to the identity functor of \(D^*(\mathcal{X})\). This implies that \(D^*(i) : D^*(\mathcal{X}) \to D^*(A_{r-adj})\) is fully faithful. Further, for each object \(M \in \mathcal{A}\), it follows from (c) that there is a monomorphism \(M \to I\) in \(\mathcal{A}\) such that \(I \in A_{r-adj}\). Since \(A_{r-adj}\) is closed under quotients in \(\mathcal{A}\) by Lemma 2.7(1), there is an exact sequence \(0 \to M \to I \to J \to 0\) in \(\mathcal{A}\) with \(I, J \in A_{r-adj}\). By Remark 3.3, the inclusion \(A_{r-adj} \subseteq \mathcal{A}\) of exact categories induces a triangle equivalence \(D^*(A_{r-adj}) \cong D^*(\mathcal{A})\) for any \(* \in \{+, -, 0\}\). Consequently, \(D^*(i) : D^*(\mathcal{X}) \to D^*(\mathcal{A})\) is fully faithful.

Dually, under (a), (b) and (d), the functor \(D^*(j) : D^*(\mathcal{Y}) \to D^*(\mathcal{A})\), induced from the inclusion \(j : \mathcal{Y} \to \mathcal{A}\), is fully faithful. This follows from Lemmas 2.7(2) and 2.7(4), and the dual statement of Lemma 3.2.

Thus Definition 1.2(D1) is satisfied. We identify \(D^*(\mathcal{X})\) and \(D^*(\mathcal{Y})\) with \(\text{Im}(D^*(i))\) and \(\text{Im}(D^*(j))\), respectively. It remains to check Definition 1.2(D3).

Let \(N^* \in D^*(\mathcal{A})\). Since \(D^*(A_{r-adj}) \overset{\cong}{\longrightarrow} D^*(\mathcal{A})\), there is a complex \(M^* := (M^i)_{i \in \mathbb{Z}} \in \mathcal{C}^*(A_{r-adj})\) such that \(N^* \simeq M^*\) in \(D^*(\mathcal{A})\). Moreover, since \((\mathcal{X}, \mathcal{Y})\) is a complete Ext-orthogonal pair in \(\mathcal{A}\), it follows from Lemma 2.6(2) that each morphism \(f : M \to N\) in \(\mathcal{A}\) extends uniquely to a morphism \(\varepsilon_f : \varepsilon_M \to \varepsilon_N\) of exact sequences (see Lemma 2.6 for notation). Applying this to the differentials of \(M^*\) yields a long exact sequence

\[
\varepsilon_{M^*} : 0 \longrightarrow Y_{M^*} \xrightarrow{\varepsilon_{M^*}^2} X_{M^*} \xrightarrow{\varepsilon_{M^*}^1} M^* \xrightarrow{\varepsilon_{M^*}^0} Y_{M^*} \xrightarrow{\varepsilon_{M^*}^1} X_{M^*} \longrightarrow 0
\]

in \(\mathcal{C}^*(\mathcal{A})\) such that \(X_{M^*}, Y_{M^*} \in \mathcal{C}^*(\mathcal{X})\) and \(Y_{M^*}, Y_{M^*} \in \mathcal{C}^*(\mathcal{Y})\). Since \(M^i \in A_{r-adj}\) for any \(i \in \mathbb{Z}\), we have \(X_{M^*} = 0\). This implies \(X_{M^*} = 0\) and thus the exact sequence \(\varepsilon_{M^*}\) is of the form:

\[
\varepsilon_{M^*} : 0 \longrightarrow Y_{M^*} \xrightarrow{\varepsilon_{M^*}^2} X_{M^*} \xrightarrow{\varepsilon_{M^*}^1} M^* \xrightarrow{\varepsilon_{M^*}^0} Y_{M^*} \longrightarrow 0.
\]

Let \(Z_{M^*}\) be the mapping cone of the chain map \(\varepsilon_{M^*}^1\). Then \(X_{M^*} \to M^* \to Z_{M^*} \to X_{M^*}[1]\) is a distinguished triangle in \(D^*(\mathcal{A})\). We show \(Z_{M^*} \in \mathcal{C}^*(\mathcal{Y})\).

Since \(\varepsilon_{M^*}^1\) is the composite of the surjection \(X_{M^*} \to \text{Coker}(\varepsilon_{M^*}^2)\) with the injection \(\text{Coker}(\varepsilon_{M^*}^2) \to M^*\), there is a triangle \(Y_{M^*}[1] \to Z_{M^*} \to Y_{M^*[2]}\) in \(D^*(\mathcal{A})\) (constructed from the octahedral axiom of triangulated categories). As \(D^*(\mathcal{Y})\) is a full triangulated subcategory of \(D^*(\mathcal{A})\), it follows from \(Y_{M^*}, Y_{M^*[2]} \in D^*(\mathcal{Y})\) that \(Z_{M^*} \in \mathcal{D}^*(\mathcal{Y})\).

Since \(N^* \simeq M^* \in D^*(\mathcal{A})\), there is a triangle \(X_{M^*} \to N^* \to Z_{M^*} \to X_{M^*[1]}\) satisfying that \(X_{M^*} \in \mathcal{D}^*(\mathcal{X})\) and \(Z_{M^*} \in \mathcal{D}^*(\mathcal{Y})\). This shows Definition 1.2(D3). Thus \((\mathcal{X}, \mathcal{Y})\) is a derived decomposition of \(D^*(\mathcal{A})\). \(\square\)

As a consequence of Theorem 1.3(1), we have

**Corollary 3.4.** Let \(\mathcal{A}\) be an abelian category, and let \(\mathcal{X}\) and \(\mathcal{Y}\) be full subcategories of \(\mathcal{A}\) with \(\mathcal{X} \subseteq \mathcal{Y}\). If for each object \(M \in \mathcal{A}\), there is a short exact sequence \(0 \to X \to M \to Y \to 0\) in \(\mathcal{A}\) with \(X \in \mathcal{X}\) and \(Y \in \mathcal{Y}\), then \((\mathcal{X}, \mathcal{Y})\) is a \(D^*\)-decomposition of \(\mathcal{A}\) for any \(* \in \{b, +, -, 0\}\).

**3.2 Proof of Theorem 1.3(2)**

Throughout this section, \(\mathcal{A}\) is an abelian category and \((\mathcal{X}, \mathcal{Y})\) is a complete Ext-orthogonal pair in \(\mathcal{A}\). We keep all the notation in Section 2.3.
Lemma 3.5. Let $M \in \mathcal{A}$ and $N \in \mathcal{Y}$.

(1) There is a long exact sequence of extension groups for $n \in \mathbb{Z}$:

$$
\cdots \rightarrow \text{Ext}^n_{\mathcal{A}}(Y_M, N) \rightarrow \text{Ext}^n_{\mathcal{A}}(\ell(M), N) \rightarrow \text{Ext}^n_{\mathcal{A}}(M, N) \rightarrow \text{Ext}^{n-1}_{\mathcal{A}}(Y_M, N) \rightarrow \cdots
$$

(2) If $M \in ^{1+}\mathcal{Y}$, then $\ell(M) \in ^{1+}\mathcal{Y}$.

(3) If $M \in ^{1-0}\mathcal{Y}$, then $\text{Ext}^n_{\mathcal{A}}(Y_M, N) \cong \text{Ext}^n_{\mathcal{A}}(\ell(M), N)$ for all $n \geq 2$.

Proof. (1) We define $C = \text{Im}(\varepsilon_{\mathcal{A}})$ and $K = \text{Im}(\varepsilon_{\mathcal{M}})$ in $\varepsilon_M$. Then we have two exact sequences $0 \rightarrow Y_M \rightarrow r(M) \rightarrow C \rightarrow 0$ and $0 \rightarrow K \rightarrow \ell(M) \rightarrow X^M \rightarrow 0$. It follows from the first exact sequence that $\text{Ext}^n_{\mathcal{A}}(Y_M, N) \cong \text{Ext}^n_{\mathcal{A}}(C, N)$ for all $n \in \mathbb{Z}$. Similarly, it follows from the second exact sequence that $\text{Ext}^n_{\mathcal{A}}(\ell(M), N) \cong \text{Ext}^n_{\mathcal{A}}(K, N)$ for all $n \in \mathbb{Z}$. We then apply $\text{Ext}^n_{\mathcal{A}}(-, N)$ to the exact sequence $0 \rightarrow C \rightarrow M \rightarrow K \rightarrow 0$ and get the desired exact sequence.

(2) and (3) follow from (1). □

From Lemma 3.5, we have the following

Corollary 3.6. Assume that $\mathcal{A}$ has enough projectives. Let $P \in \mathcal{P}(\mathcal{A})$ and $N \in \mathcal{Y}$. Then

(1) $\ell(P) \in ^{1+}\mathcal{Y}$ and $\text{Ext}^n_{\mathcal{A}}(Y_P, N) \cong \text{Ext}^n_{\mathcal{A}}(\ell(P), N)$ for all $n \geq 2$.

(2) $\ell(P) \in ^{1-0}\mathcal{Y}$ if and only if $Y_P = 0$.

Proof. If $P \in \mathcal{P}(\mathcal{A})$, then $P \in ^{1-0}\mathcal{Y}$. Now, (1) follows from Lemma 3.5(2)-(3). Further, by (1), $\ell(P) \in ^{1-0}\mathcal{Y}$ if and only if $\text{Ext}^n_{\mathcal{A}}(Y_P, Z) = 0$ for all $Z \in \mathcal{Y}$ and for all $m \geq 0$. Due to $Y_P \in \mathcal{Y}$, $\text{Hom}_{\mathcal{A}}(Y_P, Y_P) = 0$ implies $Y_P = 0$. Thus (2) holds. □

We need the following result from [14, Proposition 4.3].

Lemma 3.7. Let $\mathcal{B}$ be an abelian full subcategory of $\mathcal{A}$ and let $\lambda : \mathcal{B} \rightarrow \mathcal{A}$ be the inclusion. Then $\mathcal{D}^b(\mathcal{A}) \rightarrow \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{B})$ is fully faithful if and only if, for any $X, Y \in \mathcal{B}$ and for any $n \in \mathbb{N}$, the homomorphism $\phi^n_{X,Y} : \text{Ext}^n_{\mathcal{B}}(X, Y) \rightarrow \text{Ext}^n_{\mathcal{A}}(X, Y)$ induced from $\mathcal{D}^b(\mathcal{A})$ is an isomorphism.

The following result, which will be used in Section 4, is implied by Lemmas 3.7 and 3.5(1). Here, we omit its proof.

Corollary 3.8. (1) Suppose that $\mathcal{D}^b(j) : \mathcal{D}^b(\mathcal{Y}) \rightarrow \mathcal{D}^b(\mathcal{A})$ is fully faithful. If $I \in \mathcal{I}(\mathcal{Y})$, then $\text{Ext}^n_{\mathcal{A}}(M, I) = 0$ for any $M \in \mathcal{A}$ and $n \geq 2$.

(2) Suppose that $\mathcal{D}^b(i) : \mathcal{D}^b(\mathcal{X}) \rightarrow \mathcal{D}^b(\mathcal{A})$ is fully faithful. If $P \in \mathcal{P}(\mathcal{X})$, then $\text{Ext}^n_{\mathcal{A}}(P, M) = 0$ for any $M \in \mathcal{A}$ and $n \geq 2$.

Lemma 3.9. Let $\mathcal{B}$ be an abelian full subcategory of $\mathcal{A}$.

(1) Assume that $\mathcal{B}$ has enough injectives. Then the derived functor $\mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A})$ induced from the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is fully faithful if and only if $\mathcal{I}(\mathcal{B}) = \mathcal{B} \cap ^{1-0}\mathcal{B}$.

(2) Assume that $\mathcal{B}$ has enough injectives. Then the derived functor $\mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A})$ induced from the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is fully faithful if and only if $\mathcal{I}(\mathcal{B}) = \mathcal{B} \cap \mathcal{B}^{1-0}$.

Proof. We only prove (1) since (2) can be proved dually. Note that the inclusions $\mathcal{B} \cap ^{1-0}\mathcal{B} \subseteq \mathcal{B} \cap ^{1+}\mathcal{B} \subseteq \mathcal{D}(\mathcal{B})$ always hold.

Let $\lambda : \mathcal{B} \rightarrow \mathcal{A}$ be the inclusion. By Lemma 3.7, to prove (1), it is enough to show that, for any $X, Y \in \mathcal{B}$ and for any $n \in \mathbb{N}$, the homomorphism $\varphi^n_{X,Y} : \text{Ext}^n_{\mathcal{B}}(X, Y) \rightarrow \text{Ext}^n_{\mathcal{A}}(X, Y)$ is an isomorphism. Clearly, $\varphi^n_{X,Y}$ is the identity map since $\mathcal{B}$ is a full subcategory of $\mathcal{A}$. So it suffices to check that $\varphi^n_{X,Y}$ is an isomorphism for $n \geq 1$.

Suppose that $\mathcal{D}^b(\lambda)$ is fully faithful. If $X \in \mathcal{P}(\mathcal{B})$, then $\text{Ext}^n_{\mathcal{B}}(X, Y) = 0$ for all $n \geq 1$, and therefore $\text{Ext}^n_{\mathcal{A}}(X, Y) = 0$. Thus $X \in ^{1-0}\mathcal{B}$ and $\mathcal{P}(\mathcal{B}) \subseteq \mathcal{B} \cap ^{1-0}\mathcal{B}$. So the necessity of Lemma 3.9(1) holds.
Conversely, suppose \( P(\mathcal{B}) = \mathcal{B} \cap \text{add}^{>0} \mathcal{B} \). This implies that \( \varphi_{X,Y}^n \) is an isomorphism for \( X \in P(\mathcal{B}) \) because \( \text{Ext}^n_B(X,Y) = 0 = \text{Ext}^n_A(X,Y) \). Since \( \mathcal{B} \) has enough projectives, any object \( X \in \mathcal{B} \) has a projective resolution in \( \mathcal{B} \). This resolution is also exact in \( \mathcal{A} \) because \( \mathcal{B} \) is an abelian full subcategory of \( \mathcal{A} \). Now, for a fixed object \( Y \in \mathcal{B} \), we apply the functors \( \text{Ext}^n_B(-,Y) \) and \( \text{Ext}^n_A(-,Y) \) for \( i \in \mathbb{N} \) to this resolution and then get two long exact sequences of extension groups, linked by commutative diagrams. Note that \( \text{Ext}^i_A(P,Y) = 0 \) for all \( i \geq 1 \) and \( P \in P(\mathcal{B}) \). By induction on \( n \) and by the Five-Lemma, we can show that \( \varphi_{X,Y}^n \) are isomorphisms for \( n \geq 1 \). Thus \( D^b(\mathcal{A}) \) is fully faithful. □

**Lemma 3.10.** (1) If \( \mathcal{A} \) has enough projectives, then so does \( \mathcal{Y} \), and \( P(\mathcal{Y}) = \text{add} \{ \ell(P) \mid P \in P(\mathcal{A}) \} \).

(2) If \( \mathcal{A} \) has enough injectives, then so does \( \mathcal{X} \), and \( I(\mathcal{X}) = \text{add} \{ r(I) \mid I \in I(\mathcal{A}) \} \).

**Proof.** (1) Since \( \ell \) is an exact functor and \( (\ell, j) \) is an adjoint pair, \( \ell \) is a right exact functor and preserves projective objects. This means \( \ell(P) \in P(\mathcal{Y}) \) for \( P \in P(\mathcal{A}) \). Given any object \( Y \in \mathcal{Y} \), since \( \mathcal{A} \) has enough projectives, there exists an epimorphism \( \pi : Q \to j(Y) \) in \( \mathcal{A} \) with \( Q \in P(\mathcal{A}) \). Hence \( \ell(\pi) : \ell(Q) \to \ell(j(Y)) \) is an epimorphism in \( \mathcal{Y} \). As \( \ell(j(Y)) \cong Y \), \( \ell(\pi) \) is an epimorphism from \( \ell(Q) \) to \( Y \). This shows that \( \mathcal{Y} \) has enough projectives. Moreover, if \( Y \in P(\mathcal{Y}) \), then \( Y \) is a direct summand of \( \ell(Q) \). This shows (1).

(2) can be proved dually. □

The next result characterizes when \( D^b(i) \) and \( D^b(j) \) are fully faithful.

**Lemma 3.11.** (1) If \( \mathcal{A} \) has enough projectives, then \( D^b(j) : D^b(\mathcal{Y}) \to D^b(\mathcal{A}) \) is fully faithful if and only if \( Y_P = 0 \) for any object \( P \in P(\mathcal{A}) \).

(2) If \( \mathcal{A} \) has enough injectives, then \( D^b(i) : D^b(\mathcal{X}) \to D^b(\mathcal{A}) \) is fully faithful if and only if \( X_i = 0 \) for any object \( I \in I(\mathcal{A}) \).

**Proof.** We show (1) by Lemma 3.9(1). By Lemmas 3.10(1) and 3.9(1), the functor \( D^b(j) \) is fully faithful if and only if \( \ell(P) \in P(\mathcal{Y}) \) for all \( P \in P(\mathcal{A}) \). But the latter is equivalent to saying \( Y_P = 0 \) by Corollary 3.6(2). Thus (1) holds. Dually, (2) can be proved by Lemma 3.9(2). □

**Proof of Theorem 1.3(2).** The sufficiency is a direct consequence of Theorem 1.3(1). To show the necessity, we suppose that \( (\mathcal{X}, \mathcal{Y}) \) is a \( \mathcal{D}^* \)-decomposition of \( \mathcal{A} \). Then (a) and (b) hold by Proposition 3.1. If \( \mathcal{A} \) has enough injectives, then (c) and (c') are equivalent. Similarly, if \( \mathcal{A} \) has enough projectives, then (d) and (d') are equivalent. Note that \( \mathcal{D}^*(i) : \mathcal{D}^*(\mathcal{X}) \to \mathcal{D}^*(\mathcal{A}) \) can be restricted to bounded derived categories. By Definition 1.2(D1), \( \mathcal{D}^b(i) : \mathcal{D}^b(\mathcal{X}) \to \mathcal{D}^b(\mathcal{A}) \) is fully faithful. Similarly, \( \mathcal{D}^b(j) : \mathcal{D}^b(\mathcal{Y}) \to \mathcal{D}^b(\mathcal{A}) \) is fully faithful. Since \( \mathcal{A} \) has enough projectives and injectives, it follows from Lemma 3.11 that (c') and (d') in Theorem 1.3(2) hold. □

### 3.3 Derived decompositions versus semi-orthogonal decompositions

In this section we establish relations between derived decompositions of abelian categories and semi-orthogonal decompositions of several derived categories.

The following result is an easy observation from Definitions 1.1 and 1.2.

**Lemma 3.12.** Let \( \mathcal{A} \) be an abelian category, \( \mathcal{X} \) and \( \mathcal{Y} \) abelian subcategories of \( \mathcal{A} \) and \( \ast \in \{ b, +, -, \varnothing \} \). Suppose that the inclusions \( i : \mathcal{X} \subseteq \mathcal{A} \) and \( j : \mathcal{Y} \subseteq \mathcal{A} \) induce fully faithful functors \( \mathcal{D}^*(i) : \mathcal{D}^*(\mathcal{X}) \to \mathcal{D}^*(\mathcal{A}) \) and \( \mathcal{D}^*(j) : \mathcal{D}^*(\mathcal{Y}) \to \mathcal{D}^*(\mathcal{A}) \), respectively. If \( (\text{Im}(\mathcal{D}^*(i)), \text{Im}(\mathcal{D}^*(j))) \) is a semi-orthogonal decomposition of \( \mathcal{D}^*(\mathcal{A}) \), then \( (\mathcal{X}, \mathcal{Y}) \) is a \( \mathcal{D}^* \)-decomposition of \( \mathcal{A} \).

To obtain the converse of Lemma 3.12, we consider abelian categories with additional properties.

**Definition 3.13.** Let \( \mathcal{A} \) be an abelian category.
Suppose that $D$ is a derived decomposition of $X$; and bicomplete if it is complete and cocomplete.

(2) $A$ satisfies AB4 if it is cocomplete such that coproducts of short exact sequences (indexed over sets) are exact. Dually, $A$ satisfies AB4' if it is complete and products of short exact sequences (indexed over sets) are exact.

If $A$ satisfies AB4, then $D(A)$ has coproducts indexed over sets, and therefore the coproducts of distinguished triangles in $D(A)$ are distinguished triangles. Moreover, $D(A)$ itself is the smallest full triangulated subcategory of $D(A)$ containing $A$ and being closed under coproducts. Dually, if $A$ satisfies AB4', then $D(A)$ has products indexed over sets, and therefore the products of triangles in $D(A)$ are again triangles.

Note that if a cocomplete abelian category has enough injectives, then it satisfies AB4. Dually, if a complete abelian category has enough projectives, then it satisfies AB4'. Examples of abelian categories with both AB4 and AB4' are the module categories of rings, and the categories of additive functors from essentially small triangulated categories to the category of abelian groups (see [22, Chapter 6] for details).

Lemma 3.14. Let $A$ be an abelian category and let $\mathcal{X}$ and $\mathcal{Y}$ be abelian subcategories of $A$. Assume that

1. $\mathcal{X}$ satisfies AB4 and the inclusion $i : \mathcal{X} \subseteq A$ preserves coproducts,
2. $\mathcal{Y}$ satisfies AB4' and the inclusion $j : \mathcal{Y} \subseteq A$ preserves products,
3. $\text{Hom}_{D(A)}(X, Y[n]) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $n \in \mathbb{Z}$.

Then $\text{Hom}_{D(A)}(i(X^*), j(Y^*)) = 0$ for all $X^* \in D(\mathcal{X})$ and $Y^* \in D(\mathcal{Y})$.

Proof. For any $Y \in \mathcal{Y}$, let $\mathcal{Y}(Y)$ be the full subcategory of $D(\mathcal{X})$ consisting of objects $X^*$ such that $\text{Hom}_{D(A)}(i(X^*), j(Y)[n]) = 0$ for all $n \in \mathbb{Z}$. Then $\mathcal{Y}(Y)$ is a full triangulated subcategory of $D(\mathcal{X})$. By (1), $D(\mathcal{X})$ has coproducts and $\mathcal{Y}(Y) \subseteq D(\mathcal{X})$ is closed under coproducts. Moreover, $\mathcal{X} \subseteq \mathcal{Y}(Y)$ by (3). Note that $D(\mathcal{X})$ is the smallest full triangulated subcategory of $D(\mathcal{X})$ containing $\mathcal{X}$ and being closed under coproducts. Thus $\mathcal{Y}(Y) = D(\mathcal{X})$. It follows that $\text{Hom}_{D(A)}(i(X^*), j(Y)[n]) = 0$ for all $X^* \in D(\mathcal{X})$ and $n \in \mathbb{Z}$. Dually, when $X^* \in D(\mathcal{X})$ is fixed, one can apply (2) and (3) to prove $\text{Hom}_{D(A)}(i(X^*), j(Y^*)) = 0$ for all $Y^* \in D(\mathcal{Y})$. □

Proposition 3.15. Suppose that $A$ is an abelian category satisfying AB4 and AB4'. Let $\mathcal{X}$ and $\mathcal{Y}$ be abelian subcategories of $A$ and let $* \in \{+, -, \mathcal{D}\}$. Then the following are equivalent:

1. $(\mathcal{X}, \mathcal{Y})$ is a $D^*$-decomposition of $A$.
2. The inclusions $i : \mathcal{X} \subseteq A$ and $j : \mathcal{Y} \subseteq A$ induce faithful exact functors $D^*(i) : D^*(\mathcal{X}) \to D^*(A)$ and $D^*(j) : D^*(\mathcal{Y}) \to D^*(A)$, respectively, and $(\text{Im}(D^*(i)), \text{Im}(D^*(j)))$ is a semi-orthogonal decomposition of $D^*(A)$.

Proof. (2) implies (1) by Lemma 3.12. Suppose (1) holds. Then $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $A$ by Proposition 3.1. In particular, $\mathcal{X} = \perp \mathcal{Y}$. It follows that $\mathcal{X}$ is closed under coproducts in $A$ and $i : \mathcal{X} \subseteq A$ preserves coproducts. Since $A$ satisfies AB4, $\mathcal{X}$ also satisfies AB4. Dually, $\mathcal{Y}$ is closed under products in $A$ and satisfies AB4'. Now, (2) holds by Definitions 1.2 and 1.1 and Proposition 3.14. □

As a consequence of Theorem 1.3(2) and Proposition 3.15, we can construct half recollements of derived categories from derived decompositions.

Corollary 3.16. Let $A$ be a bicomplete abelian category with enough projectives and injectives. If $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $A$, then there exists a lower half recollement

$\begin{align*}
D^*(\mathcal{X}) & \xrightarrow{D^*(i)} D^*(A) \xrightarrow{D^*(j)} D^*(\mathcal{Y}) \\
\mathbb{R}^*(r) & \xrightarrow{\mathbb{R}^*(i)} \mathbb{R}^*(A) \xrightarrow{\mathbb{R}^*(j)} \mathbb{R}^*(\mathcal{Y})
\end{align*}$

for $* \in \{+, -, \mathcal{D}\}$, where $\mathbb{R}^*(r)$ and $\mathbb{L}^*(\ell)$ denote the right- and left-derived functors of $r$ and $\ell$, respectively.
Remark 3.17. Consider the statements:

1. \((\mathcal{X}, \mathcal{Y})\) is a \(\mathcal{D}\)-decomposition of \(\mathcal{A}\);
2. \((\mathcal{X}, \mathcal{Y})\) is a \(\mathcal{D}^{\ast}\)-decomposition of \(\mathcal{A}\) for any \(\ast \in \{+, -\}\);
3. \((\mathcal{X}, \mathcal{Y})\) is a derived decomposition of \(\mathcal{A}\).

Then (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3). Thus the existence of \(\mathcal{D}^{b}\)-decompositions is the weakest condition among those other type of derived decompositions introduced in Definition 1.2. This is why we sometimes pay more attention to the existence of such decompositions.

To show the above implications, we consider the triangle given in Definition 1.2(D3). If \(H^n(M^\ast) = H^{n+1}(M^\ast) = 0\) for some integer \(n\), then \(H^{n+1}(X_{M^\ast}) \approx H^n(Y_{M^\ast})\). Clearly, \(H^{n+1}(X_{M^\ast}) \in \mathcal{X}\) and \(H^n(Y_{M^\ast}) \in \mathcal{Y}\) since \(\mathcal{X}\) and \(\mathcal{Y}\) are abelian subcategories of \(\mathcal{A}\). However, \(\mathcal{X} \cap \mathcal{Y} = \{0\}\) by Definition 1.2(D2). Thus \(H^{n+1}(X_{M^\ast}) = H^n(Y_{M^\ast}) = 0\).

An open question is: does (3) always imply (1)? Theorem 1.3(2) tells us this is the case if \(\mathcal{A}\) has enough projectives and injectives.

3.4 One-to-one correspondence between derived and homological decompositions

In this section we establish a one-to-one correspondence between derived decompositions of abelian categories and homological decompositions, a subclass of semi-orthogonal decompositions, of bounded derived categories of abelian categories.

Let \(\mathcal{A}\) be an abelian category. Given a full subcategory \(\mathcal{X}\) of \(\mathcal{D}^{b}(\mathcal{A})\), we denote by \(\mathcal{X}_0\) the full subcategory of \(\mathcal{A}\) consisting of 0-th cohomologies of all objects in \(\mathcal{X}\). Conversely, given a full subcategory \(\mathcal{X}\) of \(\mathcal{A}\), we denote by \(\mathcal{D}^{b}(\mathcal{A})\) consisting of complexes such that all of their cohomologies in \(\mathcal{X}\).

Definition 3.18. Let \(\mathcal{A}\) be an abelian category. A pair \((\mathcal{X}, \mathcal{Y})\) of full subcategories of \(\mathcal{D}^{b}(\mathcal{A})\) is called a homological decomposition if the following three conditions hold:

(a) \((\mathcal{X}, \mathcal{Y})\) is a semi-orthogonal decomposition of \(\mathcal{D}^{b}(\mathcal{A})\).
(b) \(\mathcal{X}_0 \subseteq \mathcal{X}\) and the inclusion \(\mathcal{X}_0 \subseteq \mathcal{A}\) induces a fully faithful functor \(\mathcal{D}^{b}(\mathcal{A}) \rightarrow \mathcal{D}^{b}(\mathcal{A})\).
(c) \(\mathcal{Y}_0 \subseteq \mathcal{Y}\) and the inclusion \(\mathcal{Y}_0 \subseteq \mathcal{A}\) induces a fully faithful functor \(\mathcal{D}^{b}(\mathcal{A}) \rightarrow \mathcal{D}^{b}(\mathcal{A})\).

Lemma 3.19. Let \(\mathcal{X}\) be a full triangulated subcategory of \(\mathcal{D}^{b}(\mathcal{A})\). If \(\mathcal{X}_0 \subseteq \mathcal{X}\), then \(\mathcal{X}_0\) is an abelian subcategory of \(\mathcal{A}\) and \(\mathcal{X} = \mathcal{D}^{b}_{\mathcal{X}_0}(\mathcal{A})\).

Proof. Since \(\mathcal{X}\) is a full triangulated subcategory of \(\mathcal{D}^{b}(\mathcal{A})\), it follows that \(H^n(X^\ast) = H^0(X^\ast[n]) \in \mathcal{X}_0\) for any \(n \in \mathbb{Z}\) and \(X^\ast \in \mathcal{X}\). Suppose \(\mathcal{X}_0 \not\subseteq \mathcal{X}\). Then \(H^n(X^\ast) \in \mathcal{X}\). Let \(f : X^{-1} \rightarrow X^0\) be a morphism in \(\mathcal{A}\) with \(X^{-1}, X^0 \in \mathcal{X}_0\). Now, regarding \(f\) as a complex in degrees -1 and 0, we obtain \(f \in \mathcal{X}\). Consequently, as homologies of \(f\), \(\text{Ker}(f)\) and \(\text{Coker}(f)\) belong to \(\mathcal{X}_0\). Thus \(\mathcal{X}_0 \subseteq \mathcal{A}\) is an abelian subcategory.

Note that \(\mathcal{X} \subseteq \mathcal{D}^{b}_{\mathcal{X}_0}(\mathcal{A})\), while the converse inclusion is due to the fact that each object \(M^\ast\) in \(\mathcal{D}^{b}(\mathcal{A})\) belongs to the smallest full triangulated subcategory containing \(H^n(M^\ast)\) for all \(n\).

Proposition 3.20. For an abelian category \(\mathcal{A}\), there exists a one-to-one correspondence

\[
\{\text{derived decompositions of } \mathcal{A}\} \leftrightarrow \{\text{homological decompositions of } \mathcal{D}^{b}(\mathcal{A})\},
\]

\[
(\mathcal{X}, \mathcal{Y}) \mapsto (\mathcal{D}^{b}_{\mathcal{X}_0}(\mathcal{A}), \mathcal{D}^{b}_{\mathcal{Y}_0}(\mathcal{A})),
\]

\[
(\mathcal{X}_0, \mathcal{Y}_0) \mapsto (\mathcal{X}, \mathcal{Y}),
\]

where \((\mathcal{X}, \mathcal{Y})\) is a homological decomposition of \(\mathcal{D}^{b}(\mathcal{A})\) and \((\mathcal{X}_0, \mathcal{Y}_0)\) is a derived decomposition of \(\mathcal{A}\).
Proof. Given a homological decomposition \((X, Y)\) of \(D^b(A)\), it follows from Definition 3.18 and Lemma 3.19 that \(X = D^b_{X_0}(A)\) and \(Y = D^b_{Y_0}(A)\). Since the inclusion \(u: X_0 \subseteq A\) induces a fully faithful functor \(D^b(X_0) \to D^b(A)\), we have \(\text{Im}(D^b(u)) = D^b_{X_0}(A)\) by Lemma 2.1. Similarly, \(\text{Im}(D^b(v)) = D^b_{Y_0}(A)\) where \(v: Y_0 \to A\) is the inclusion. Thus \((\text{Im}(D^b(u)), \text{Im}(D^b(v)))\) is a semi-orthogonal decomposition of \(D^b(A)\). By definition 2.2, \((X_0, Y_0)\) is a derived decomposition of \(A\).

Given a derived decomposition \((X, Y)\) of \(A\), by Definition 2.2 and Lemma 2.1, the pair \((D^b_{X}(A), D^b_{Y}(A))\) is a semi-orthogonal decomposition of \(D^b(A)\). Clearly, \((D^b_{X}(A))_0 = X \subseteq D^b_{X}(A)\) and \((D^b_{Y}(A))_0 = Y \subseteq D^b_{Y}(A)\). By Definitions 3.18 and 2.2, \((D^b_{X}(A), D^b_{Y}(A))\) is a homological decomposition of \(D^b(A)\). Thus the correspondence is well defined. By Lemma 3.19, the correspondence is one-to-one. □

4 Constructing derived decompositions of module categories

In this section we first apply Theorem 1.3 to show that homological ring epimorphisms provide derived decompositions (see Proposition 4.1), and then prove that localizing subcategories and right perpendicular subcategories in abelian categories also give rise to derived decompositions (see Proposition 4.9). Finally, we construct derived decompositions for module categories over left nonsingular rings and commutative noetherian rings (see Corollaries 4.10 and 4.14, respectively). This construction shows also that the module category of a commutative ring with the Krull dimension at most 1 admits a derived stratification (see Corollary 4.15).

4.1 Homological ring epimorphisms

In this section we show that homological ring epimorphisms produce not only derived decompositions, but also derived equivalences and recollements.

Throughout this section, we assume that \(\lambda: R \to S\) is a homological ring epimorphism. Define

\[
A := R\text{-Mod}, \quad X := S\text{-Mod}, \\
Y := \{Y \in R\text{-Mod} \mid \text{Hom}_R(S, Y) = 0 = \text{Ext}_1^R(S, Y)\}, \\
Z := \{Z \in R\text{-Mod} \mid S \otimes_R Z = 0 = \text{Tor}_1^R(S, Z)\}.
\]

Proposition 4.1. (1) \((X, Y)\) is a complete Ext-orthogonal pair in \(A\) if and only if \(\text{projdim}_R(S) \leq 1\).

(2) \((X, Y)\) is a derived decomposition of \(A\) if and only if \(\text{projdim}_R(S) \leq 1\) and \(\text{Hom}_R(\text{Coker}(\lambda), \text{Ker}(\lambda)) = 0\).

Proof. (1) Since \(\lambda\) is a ring epimorphism, the restriction functor \(\lambda_*: X \to A\) is fully faithful. So, we identify \(X\) with the image of \(\lambda_*\). Further, since \(\lambda\) is homological, the derived functor \(D^b(\lambda_*): D^b(S) \to D^b(R)\) is fully faithful. Note that \(S\in \mathcal{P}(\mathcal{X})\), the category of projective \(S\)-modules. If \((X, Y)\) is a complete Ext-orthogonal pair in \(A\), then \(\text{projdim}_R(S) \leq 1\) by Corollary 3.8(2). This shows the necessity of (1).

To show the sufficiency of (1), we assume \(\text{projdim}_R(S) \leq 1\). Then \(Y = S^!\). It follows from [14, Proposition 1.1] that \(Y\) is an abelian full subcategory of \(A\). Since \(X^!\) contains \(y_S\) and is closed under direct sums in \(A\), it must contain all projective \(S\)-modules. Moreover, each object of \(X\) admits a projective resolution by projective \(S\)-modules. Consequently, for any \(X \in \mathcal{X}\), \(Y \in \mathcal{Y}\) and \(n \in \mathbb{N}\), \(\text{Hom}_{D^b(R)}(X, Y[n]) = \text{Hom}_{D^b(R)}(\Omega_S^*(X), Y) = 0\), where \(\Omega_S^*(X)\) denotes an \(n\)-th syzygy module of \(S\). This implies \(X \subseteq X^!\). By Lemma 2.5(2), to show (1), it suffices to prove that \((X, Y)\) satisfies (GC).

The functor \(D^b(\lambda_*): D^b(S) \to D^b(R)\) has a right adjoint functor \(\mathbb{R}\text{Hom}_R(S, -): D^b(R) \to D^b(S)\). Let \(\varepsilon: D^b(\lambda_*) \mathbb{R}\text{Hom}_R(S, -) \to \text{Id}_{D^b(R)}\) be the counit adjunction. Then, for each \(R\)-module \(M\), there exists a distinguished triangle in \(D^b(R)\):

\[
(*): D^b(\lambda_*) \mathbb{R}\text{Hom}_R(S, M) \xrightarrow{\varepsilon_M} Y_M^\bullet \to D^b(\lambda_*) \mathbb{R}\text{Hom}_R(S, M)[1].
\]
Since \( \lambda \) is homological, \( \mathbb{D}^b(\lambda) \) is fully faithful. So \( \mathbb{R}\text{Hom}_R(S, \epsilon_M) = \text{isomorphism in } \mathbb{D}^b(S) \). This means \( Y_M^\bullet \in \mathcal{Y} = \text{Ker}(\mathbb{R}\text{Hom}_R(S, -)) \subseteq \mathbb{D}^b(R) \). Thus

\[
Y_M^\bullet \in \mathcal{Y} = \{ Y^\bullet \in \mathbb{D}^b(R) \mid \text{Hom}_{\mathbb{D}^b(R)}(S, Y^\bullet[n]) = 0 \text{ for all } n \in \mathbb{Z} \} \quad \text{and} \quad \mathcal{Y} = \mathcal{Y} \cap R\text{-Mod}.
\]

Taking cohomologies on the triangle (†) yields an exact sequence of \( R \)-modules:

\[
0 \longrightarrow H^{-1}(Y_M^\bullet) \longrightarrow \text{Hom}_R(S, M) \xrightarrow{\text{Hom}_R(\lambda, M)} M \twoheadrightarrow H^0(Y_M^\bullet) \longrightarrow \text{Ext}^1_R(S, M) \longrightarrow 0
\]

where \( M \) is identified with \( \text{Hom}_R(R, M) \). Clearly, both \( \text{Hom}_R(S, M) \) and \( \text{Ext}^1_R(S, M) \) belong to \( \mathcal{X} \). On the other hand, since \( \text{projdim}(R_S) \leq 1 \), the \( R \)-module \( S \) is isomorphic in \( \mathbb{D}^b(R) \) to a two-term complex of projective \( R \)-modules and there is the following exact sequence by [9, Lemma 3.4]:

\[
0 \longrightarrow \text{Hom}_{\mathbb{D}^b(R)}(S, H^{n-1}(Y^\bullet[1])) \longrightarrow \text{Hom}_{\mathbb{D}^b(R)}(S, Y^\bullet[n]) \longrightarrow \text{Hom}_{\mathbb{D}^b(R)}(S, H^n(Y^\bullet)) \longrightarrow 0.
\]

This shows \( \mathcal{Y} = \{ Y^\bullet \in \mathbb{D}^b(R) \mid H^n(Y^\bullet) \in \mathcal{Y} \text{ for all } n \in \mathbb{Z} \} \). It then follows from \( Y_M^\bullet \in \mathcal{Y} \) that \( H^i(Y_M^\bullet) \in \mathcal{Y} \) for any \( i \in \mathbb{Z} \). Now, we define \( X_M := \text{Hom}_R(S, M) \), \( X^\bullet := \text{Ext}^1_R(S, M) \), \( Y_M := H^{-1}(Y_M^\bullet) \) and \( Y^\bullet := H^0(Y_M^\bullet) \). This shows the sufficiency of (1).

(2) Clearly, \( \mathcal{A} \) has enough projectives and injectives. If \( M \) is injective, then \( X^M = 0 \). Note that \( Y_M \cong \text{Ker}(\text{Hom}_R(\lambda, M)) = \text{Hom}_R(\text{Coker}(\lambda), M) \) as \( R \)-modules. By (1) and Theorem 1.3, \( (\mathcal{A}, \mathcal{Y}) \) is a derived decomposition of \( \mathcal{A} \) if and only if \( \text{projdim}(R_S) \leq 1 \) and \( \text{Hom}_R(\text{Coker}(\lambda), M) = 0 \) whenever \( M \) is projective. To check \( \text{Hom}_R(\text{Coker}(\lambda), M) \) is a projective modules \( M \), we only need to show \( \text{Hom}_R(\text{Coker}(\lambda), R) = 0 \). because \( \text{Hom}_R(\text{Coker}(\lambda), -) \) commutes with products and each projective \( R \)-module can be embedded into a product of copies of \( R \). However, since \( \lambda \) is a ring epimorphism, \( \text{Hom}_R(\text{Coker}(\lambda), S) = 0 \). This implies \( \text{Hom}_R(\text{Coker}(\lambda), R) \ncong \text{Hom}_R(\text{Coker}(\lambda), \text{Ker}(\lambda)) \). Thus (2) holds. □

When dealing with flat dimensions, we have

**Proposition 4.2.** (1) \( (\mathcal{Z}, \mathcal{X}) \) is a complete Ext-orthogonal pair in \( \mathcal{A} \) if and only if \( \text{flatdim}(S_R) \leq 1 \).

(2) \( (\mathcal{Z}, \mathcal{X}) \) is a derived decomposition of \( \mathcal{A} \) if and only if \( \text{flatdim}(S_R) \leq 1 \) and \( \text{Coker}(\lambda) \otimes_R I = 0 \) for any injective \( R \)-module \( I \).

**Proof.** The proof of this result is similar to the one of Proposition 4.1. For the convenience of the reader, we only sketch a few key points of the proof.

Let \( J := \text{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}) \). Then \( J \) is an injective cogenerator in \( S\text{-Mod} \). Further, \( \text{injdim}(R_J) = \text{flatdim}(S_R) \) because a right \( R \)-module \( N \) is flat if and only if \( R\text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) \) is injective. So the necessity of (1) follows from Corollary 3.8(1). If \( \text{flatdim}(S_R) \leq 1 \), then

(1) \( \mathcal{Z} \) is abelian full subcategory of \( \mathcal{A} \).

(2) For any \( M^\bullet \in \mathcal{D}(R) \), \( S \otimes_R M^\bullet = 0 \) if and only if \( H^n(M^\bullet) \in \mathcal{Z} \) for all \( n \in \mathbb{Z} \).

(3) For any \( M \in \mathcal{A} \), there is an exact sequence of \( R \)-modules:

\[
0 \longrightarrow \text{Tor}^R_1(S, M) \longrightarrow X_M \longrightarrow M \xrightarrow{\lambda \otimes_R M} S \otimes_R M \longrightarrow X^M \longrightarrow 0
\]

such that \( X_M, X^M \in \mathcal{Z} \). Clearly, \( X^M \cong \text{Coker}(\lambda) \otimes_R M \) as \( R \)-modules. Now, all other assertions in Proposition 4.2 can be concluded from Theorem 1.3. □

As a consequence of Proposition 4.2, we have the following result on localizations of commutative rings.

**Corollary 4.3.** Let \( R \) be a commutative noetherian ring, \( \Phi \) a multiplicative subset of \( R \), \( S \) the localization of \( R \) at \( \Phi \) and \( \mathcal{U} := \{ X \in R\text{-Mod} \mid S \otimes_R X = 0 \} \). Then \( (\mathcal{U}, S\text{-Mod}) \) is a derived decomposition of \( R\text{-Mod} \).
Proof. Let \( \lambda : R \to S \) be the localization of \( R \) at \( \Phi \). Then \( S \) is commutative and flat as an \( R \)-module, and therefore \( \lambda \) is a homological ring epimorphism. By Proposition 4.2(2), it suffices to show \( \text{Coker}(\lambda) \otimes_R I = 0 \) (or equivalently, \( \lambda \otimes_R I \) is surjective) for any injective \( R \)-module \( I \).

Since \( R \) is a commutative noetherian ring, each injective \( R \)-module is a direct sum of indecomposable injective \( R \)-modules (see [10, Theorem 3.3.10]). So we only need to check the surjection of \( \lambda \otimes_R I \) whenever \( I \) is indecomposable. By [10, Theorem 3.3.7], there is a prime ideal \( p \) of \( R \) such that \( I \) is isomorphic to the injective envelope \( E(R/p) \) of the \( R \)-module \( R/p \). Moreover, by [10, Theorem 3.3.8(6)], \( \lambda \otimes_R E(R/p) \) is an isomorphism if \( \Phi \cap p = \emptyset \); and \( S \otimes E(R/p) = 0 \) if \( \Phi \cap p = \emptyset \). This implies that \( \lambda \otimes_R E(R/p) \) is always surjective, and therefore \( \lambda \otimes_R I \) is surjective. Thus Corollary 4.3 follows from Proposition 4.2(2). □

Next, we show that the derived decompositions in Propositions 4.1 and 4.2 provide also lower half recollements of derived categories.

For the ring epimorphism \( \lambda : R \to S \), we get naturally a complex \( Q^* : 0 \to R \xrightarrow{\lambda} S \to 0 \) of \( R \)-\( R\)-bimodules with \( R \) and \( S \) in degrees \(-1 \) and \( 0 \), respectively. Let

\[
F := Q^*[1] \otimes_R^L - : \mathcal{D}(R) \to \mathcal{D}(R), \quad G := \mathbb{R}\text{Hom}_R(Q^*[1], -) : \mathcal{D}(R) \to \mathcal{D}(R)
\]

and let \( \text{Tria}(RQ^*) \) be the smallest full triangulated subcategory of \( \mathcal{D}(R) \) containing \( Q^* \) and being closed under direct sums. Then \( (F, G) \) is an adjoint pair of triangle functors and the restriction of \( G \) to \( \text{Tria}(RQ^*) \) is fully faithful (see [23, Section 4]).

In the case (2) of Proposition 4.1, it follows from Corollary 3.16 that there is a lower half recollement of derived categories:

\[
\begin{array}{ccc}
\mathcal{D}^*(S) & \xrightarrow{\mathbb{R}\text{Hom}_R(S, -)} & \mathcal{D}^*(R) \\
\xrightarrow{\mathcal{D}^*(\lambda_*)} & & \xrightarrow{\mathcal{D}^*(\ell_*)} \\
& \xrightarrow{\mathcal{D}^*(j)} & \mathcal{D}^*(Y)
\end{array}
\]

for \( * \in \{b, +, -, \emptyset\} \), where \( \ell : \mathcal{R}\text{-Mod} \to \mathcal{Y} \) is a left adjoint of the inclusion \( j : \mathcal{Y} \to \mathcal{R}\text{-Mod} \). We claim that \( \ell \) is the composition of the functors:

\[
\mathcal{R}\text{-Mod} \to \mathcal{D}^*(R) \xrightarrow{G} \mathcal{D}^*(R) \xrightarrow{H^0} \mathcal{R}\text{-Mod}.
\]

In fact, there is a canonical triangle \( Q^*[1] \xrightarrow{\sigma} R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^* \) in \( \mathcal{D}^b(R \otimes \mathcal{Z}\text{-Op}) \) which induces a sequence of triangle functors from \( \mathcal{D}^*(R) \) to \( \mathcal{D}^*(R) \)

\[
\begin{array}{c}
\mathcal{D}^*(\mathcal{Z}) \\
\xrightarrow{\mathcal{D}^*(\lambda_*)} \\
\xrightarrow{\mathcal{D}^*(\ell_*)} \\
\mathcal{D}^*(S)
\end{array}
\]

such that their operations on a fixed object in \( \mathcal{D}^*(R) \) yield a triangle in \( \mathcal{D}^*(R) \). Clearly, \( \mathbb{R}\text{Hom}_R(R, -) \) can be identified with the identity functor of \( \mathcal{D}^*(R) \) up to natural isomorphism. So, for an \( R \)-module \( M \), by taking cohomologies on \( (\_\_\_\_\_\_) \), we get a long exact sequence of \( R \)-modules:

\[
0 \to H^{-1}(G(M)) \to \text{Hom}_R(S, M) \xrightarrow{\text{Hom}_R(\lambda, M)} M \to H^0(G(M)) \to \text{Ext}^1_R(S, M) \to 0.
\]

Now, as in the proof of Proposition 4.1(1), both \( \text{Hom}_R(S, M) \) and \( \text{Ext}^1_R(S, M) \) belong to \( \mathcal{X} \) and both \( H^{-1}(G(M)) \) and \( H^0(G(M)) \) belong to \( \mathcal{Y} \). Thus \( \ell(M) = H^0(G(M)) \) by the definition of \( \ell \).

Similarly, in the case (2) of Proposition 4.2, we obtain a lower half recollement:

\[
\begin{array}{ccc}
\mathcal{D}^*(\mathcal{Z}) & \xrightarrow{\mathcal{D}^*(\lambda_*)} & \mathcal{D}^*(S) \\
\xrightarrow{\mathcal{D}^*(\ell_*)} & & \xrightarrow{\mathcal{D}^*(j)} \\
\mathcal{D}^*(R) & \xrightarrow{\mathcal{D}^*(\lambda_*)} & \mathcal{D}^*(Y)
\end{array}
\]
where \( r := H^0 F (-) : R - \text{Mod} \rightarrow \mathcal{Z} \) is a right adjoint of the inclusion \( i : \mathcal{Z} \rightarrow R - \text{Mod} \). The five-term exact sequence of \( R \)-modules is given by

\[
0 \rightarrow \text{Tor}_1^R(S, M) \rightarrow H^0(F(M)) \rightarrow M \overset{\lambda \otimes_R M}{\rightarrow} S \otimes_R M \rightarrow H^1(F(M)) \rightarrow 0.
\]

for \( M \in R - \text{Mod} \).

The following result is an immediate consequence of Propositions 4.1(2) and 4.2(2), which supplies recollements and triangle equivalences of bounded derived categories.

**Corollary 4.4.** Suppose

(a) \( \text{projdim}(r S) \leq 1 \) and \( \text{Hom}_R(\text{Coker}(\lambda), \text{Ker}(\lambda)) = 0 \), and

(b) \( \text{flatdim}(S_R) \leq 1 \) and \( \text{Coker}(\lambda) \otimes_R I = 0 \) for any injective \( R \)-module \( I \). Then

1. there is a recollement of \( * \)-bounded derived categories of abelian categories for \( * \in \{b, +, -, \emptyset\} \):

\[
\begin{array}{c}
\mathcal{D}^*(S) \\
\mathcal{D}^*(R) \\
\mathcal{D}^*(Y)
\end{array} 
\xleftarrow{\lambda (s)} \xrightarrow{l(i)} \xrightarrow{\mathcal{S}} \xrightarrow{\mathcal{S}} \xrightarrow{\mathcal{S}} \xrightarrow{\mathcal{S}} \xrightarrow{\mathcal{S}}
\]

2. The functor \( \mathbb{R}^* (r) \mathcal{D}^*(j) : \mathcal{D}^*(Y) \rightarrow \mathcal{D}^*(\mathcal{Z}) \) is a triangle equivalence with a quasi-inverse functor \( \mathbb{L}^* (i) \mathcal{D}^*(j) \) for \( * \in \{b, +, -, \emptyset\} \).

An example of Corollary 4.4 reads as follows: Let \( R \) be a \( 1 \)-Gorenstein ring (that is, a commutative noetherian ring such that the injective dimension of \( R \) is at most 1) and let \( \Phi \) be the set of all non-zero divisors of \( R \). Then \( \lambda \) is always injective and \( r S \) is flat, injective and of projective dimension at most 1, satisfying (a) and (b). In the case \( \mathbb{Z} \in \mathbb{Q} \), we get a recollement \( (\mathcal{D}^*(\mathbb{Q}), \mathcal{D}^*(\mathbb{Z}), \mathcal{D}^*(Y)) \) and an equivalence \( \mathcal{D}^*(Y) \cong \mathcal{D}^*(Z) \).

**Remark 4.5.** Corollary 4.4(2) deals with arbitrary homological ring epimorphisms. It generalizes the Matlis equivalences of derived categories of abelian categories in [26, Theorem 7.6] for localizations \( \lambda : R \rightarrow S \) of a commutative ring \( R \) since (a) and (b) in Corollary 4.4 are satisfied under the assumptions in [26, Theorem 7.6] by [26, Lemma 6.1]. Very recently, the derived equivalences of bounded derived categories in Corollary 4.4(2) has been extended to the ones of derived categories of other types in [3, Corollary 18.5]. Thus Corollary 4.4(2) coincide with [3, Corollary 18.5].

### 4.2 Localizing subcategories

In this section we construct derived decompositions from localizing subcategories.

Let \( \mathcal{A} \) be an abelian category and \( \mathcal{X} \) a full subcategory of \( \mathcal{A} \). We say that \( \mathcal{X} \) is a **Serre** subcategory if it is closed under subobjects, quotients and extensions. In particular, \( \mathcal{X} \) is an abelian subcategory of \( \mathcal{A} \), and the **quotient category** \( \mathcal{A}/\mathcal{X} \) (in the sense of Gabriel, Grothendieck, Serre) is defined by inverting all these morphisms in \( \mathcal{A} \) that have kernels and cokernels in \( \mathcal{X} \). The quotient category has the same objects as \( \mathcal{A} \) and is again an abelian category. Moreover, there is a canonical exact functor \( q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{X} \) (called the quotient functor) such that the kernel of \( q \) is exactly \( \mathcal{X} \).

A Serre subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is called a **localizing** subcategory of \( \mathcal{A} \) if \( q \) has a right adjoint \( s : \mathcal{A}/\mathcal{X} \rightarrow \mathcal{A} \) (called the section functor). This is equivalent to saying that \( q \) restricts to an equivalence of additive categories from \( \mathcal{X}^{\mathcal{A} / \mathcal{X}} := \mathcal{X}^{\mathcal{A}} \cap \mathcal{X}^{\mathcal{X}} \) to \( \mathcal{A}/\mathcal{X} \) (see [12, Chap. III.2] and [14, Proposition 2.2]). In this case, \( \mathcal{X} = \mathcal{X}^{\mathcal{A} / \mathcal{X}} \). Note that \( \mathcal{X}^{\mathcal{A} / \mathcal{X}} \) is closed under extensions and kernels in \( \mathcal{A} \) (see, for example, [14, Proposition 1.1]), but it may not be an abelian subcategory of \( \mathcal{A} \) in general.

If \( \mathcal{A} \) is a Grothendieck category (that is, an abelian category with a generator and coproducts such that direct limits of exact sequences are exact), then a Serre subcategory of \( \mathcal{A} \) is localizing if and only if it is closed under coproducts in \( \mathcal{A} \) (see [14, Proposition 2.5]).
Lemma 4.6. Let $A$ be an abelian category and $\mathcal{X}$ a localizing subcategory of $A$ with $\mathcal{Y} := \mathcal{X}^\perp$. Then

1. $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $A$ if and only if $\mathcal{Y} = \mathcal{X}^{\perp,0,1}$ if and only if the section functor $s: A/\mathcal{X} \to A$ is exact.

2. $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $A$ if and only if both $\mathcal{Y} = \mathcal{X}^{\perp,0,1}$ and the derived functor $D^b(i): D^b(\mathcal{X}) \to D^b(A)$, induced from the inclusion $i: \mathcal{X} \to A$, is fully faithful.

Proof. (1) Since $\mathcal{X}$ is a localizing subcategory of $A$, it follows from [14, Proposition 2.2] that, for each object $M \in A$, there is an exact sequence $0 \to X_1 \to M \to M_1 \to 0$ in $A$ with $X_1, X_2 \in \mathcal{X}$ and $M_1 \in \mathcal{X}^{\perp,1}$. Clearly, $\mathcal{X} \subseteq \mathcal{Y}$. By Lemma 2.5, $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $A$ if and only if $\mathcal{Y} = \mathcal{X}^{\perp,1}$. It remains to show that $\mathcal{Y} = \mathcal{X}^{\perp,1}$ if and only if $s$ is exact.

Let $B := A/\mathcal{X}$ and $q_1: \mathcal{X}^{\perp,0,1} \to B$ the restriction of the canonical functor $q: A \to B$ to $\mathcal{X}^{\perp,0,1}$. It is known that $s$ is always fully faithful and isomorphic to the composition of the quasi-inverse of $q_1$ with the inclusion $\mathcal{X}^{\perp,1} \subseteq A$ (see, for example, [14, Proposition 2.2]). If $\mathcal{Y} = \mathcal{X}^{\perp,1}$, then $\mathcal{Y}$ is an abelian subcategory of $A$ since $\mathcal{X}^{\perp,1}$ is closed under extensions and kernels in $A$. In this case, $q_1$ is an equivalence of abelian categories, and thus $s$ is exact.

Conversely, suppose that $s$ is an exact functor. Since both $q$ and $s$ are exact, they induce derived functors $D^b(q): D^b(A) \to D^b(B)$ and $D^b(s): D^b(B) \to D^b(A)$ such that $(D^b(q), D^b(s))$ is an adjoint pair and $D^b(s)$ is fully faithful. Picking an object $Y \in \mathcal{X}^{\perp,1}$, we then have $Y = s(Z)$ for some $Z \in B$ since $\text{Im}(s) = \mathcal{X}^{\perp,1}$. For any $X \in \mathcal{X}$ and $n \in \mathbb{N}$,

$$\text{Ext}^n_A(X, Y) = \text{Ext}^n_A(X, s(Z)) = \text{Hom}_{D^b(A)}(X, s(Z)[n]) \cong \text{Hom}_{D^b(B)}(q(X), Z[n]) = \text{Hom}_{D^b(B)}(0, Z[n]) = 0.$$

This implies both $\mathcal{X}^{\perp,1} \subseteq \mathcal{Y}$ and $\mathcal{X}^{\perp,1} = \mathcal{Y}$.

(2) The necessity of the conditions in (2) is a consequence of Definition 1.2, Proposition 3.1 and (1). Now, we show the sufficiency of the conditions in (2).

Suppose that the functor $D^b(i)$ is fully faithful and $\mathcal{Y} = \mathcal{X}^{\perp,1}$. Let $\mathcal{Y} := \text{Ker}(D^b(q))$ and let $\eta: \text{Id}_{D^b(A)} \to D^b(q)D^b(q)$ and $\varepsilon: D^b(q)D^b(s) \to \text{Id}_{D^b(B)}$ be the unit and counit of the adjoint pair $(D^b(q), D^b(s))$, respectively. The full faithfulness of $D^b(s)$ is equivalent to saying that the morphism $\varepsilon_{Z^*}: D^b(q)D^b(s)(Z^*) \to Z^*$ is an isomorphism for each $Z^* \in D^b(B)$ (see [21, Theorem 1, p.90]). Now, for each $L^* \in D^b(A)$, we take a triangle $N^* \to L^* \xrightarrow{\eta_{L^*}} D^b(q)D^b(s)(L^*) \to N^*[1]$ in $D^b(A)$. It then follows from $D^b(q)(\eta_{L^*})D^b(s)(L^*) = \text{id}_{D^b(q)D^b(s)}D^b(s)(L^*)$ (see [21, (9), p.85]) that $D^b(q)(\eta_{L^*})$ is an isomorphism and so $N^* \in \mathcal{Y}$. Further, for any $X^* \in \mathcal{X}$ and $Y^* \in \text{Im}(D^b(s))$, the adjunction isomorphism of the pair $(D^b(q), D^b(s))$ implies $\text{Hom}_{D^b(A)}(X^*, Y^*; s) = 0$. Consequently, $(\mathcal{X}, \text{Im}(D^b(s)))$ is a semi-orthogonal decomposition of $D^b(A)$. Since $q$ is exact, $\mathcal{X}$ coincides with the full triangulated subcategory of $D^b(A)$ consisting of complexes $X^* \in D(A)$ such that $H^n(X^*) \in \mathcal{X}$ for all $n$. Further, since $D^b(i)$ is fully faithful, $\mathcal{Y} = \text{Im}(D^b(i))$ by Lemma 2.1. Recall that $s$ is fully faithful and $\text{Im}(s) = \mathcal{Y}$. Thus $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $A$. □

Corollary 4.7. Let $A$ be an abelian category and $\mathcal{X}$ be a localizing subcategory of $A$ with $\mathcal{Y} := \mathcal{X}^\perp$. Assume $\text{Ext}^2_A(X, M) = 0$ for $X \in \mathcal{X}$ and $M \in A$. Then $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $A$.

Proof. We first prove $\text{Ext}^n_A(X, M) = 0$ for all $n \geq 2, X \in \mathcal{X}$ and $M \in A$. This particularly implies the equality $\mathcal{Y} = \mathcal{X}^{\perp,1}$. For $n = 2$, this is true by assumption. For $n \geq 3$, we need the following general result:

If $M^* \in D^b(A)$ satisfies both $H^n(M^*) = 0$ for all $m \neq 0$ and $H^0(M^*) \in \mathcal{X}$, then there is a two-term complex $E^*: 0 \to E^{-1} \to E^0 \to 0$ in $D^b(A)$ and a quasi-isomorphism from $E^*$ to $M^*$. Moreover, if, in addition, $M^* \in D^b(\mathcal{X})$, then $E^* \in D^b(\mathcal{X})$.

Indeed, let $M^* := (M^m, d^m)_{m \in \mathbb{Z}}$ with $d^m: M^m \to M^{m+1}$ and $N := H^0(M^*)$. Then the inclusion from $\tau_{\leq 0}(M^*)$ into $M^*$ is a quasi-isomorphism, where the truncated complex $\tau_{\leq 0}(M^*)$ at degree 0 is of the form $\cdots \to M^{-2} \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} \text{Ker}(d^0) \to 0$. Since $\text{Ext}^2_A(N, \text{Im}(d^{-2})) = 0$, we apply $\text{Ext}^1_A(N, -)$ to the exact sequence $0 \to$
Let there is another exact sequence

Suppose that conclude from the above general result that $\text{Ext}^1_A(N, d^{-1}) \to \text{Ext}^1_A(N, M^{-1}) \to \text{Ext}^1_A(N, \text{Im}(d^{-1}))$.

As the exact sequence $\delta: 0 \to \text{Im}(d^{-1}) \to \text{Ker}(d^0) \to N \to 0$ corresponds to an element in $\text{Ext}^1_A(N, \text{Im}(d^{-1}))$, there is another exact sequence $\delta: 0 \to M^{-1} \xrightarrow{\delta} E^0 \to N \to 0$ which is sent to $\delta$ by taking the push-out of $(g, d^{-1})$. This also provides an associated morphism $h: E^0 \to \text{Ker}(d^0)$ such that $gh = d^{-1}: M^{-1} \to \text{Ker}(d^0)$.

Let $E^0: 0 \to E^1 \xrightarrow{g} E^0 \to 0$ with $E^1 := M^{-1}$, and let $f^*: (f^m)_{m \in \mathbb{Z}}$ with $f^{-1} = 1d_{M^{-1}}$. $f^0 = h$ and $f^m = 0$ for $m \neq 0, 1$. Then $f^*: E^* \to \tau_{0}(M^*)$ is a quasi-isomorphism, and therefore the composition of $f^*$ with the inclusion $\tau_{0}(M^*) \to M^*$ is a quasi-isomorphism. Since $\mathcal{X}$ is an abelian full subcategory of $\mathcal{A}$ and is closed under extensions, we have $E^* \in \mathcal{E}^{b}(\mathcal{X})$ if $M^* \in \mathcal{E}^{b}(\mathcal{X})$.

Recall that $\text{Ext}^0_A(X, M)$ stands for $\text{Hom}_{\mathcal{A}}(X, M[n])$. By calculating this Hom-space in $\mathcal{D}^{b}(\mathcal{A})$, we conclude from the above general result that $\text{Ext}^0_A(X, -) = 0 = \text{Ext}^n_A(X, -)$ for all $n \geq 2$ and $X \in \mathcal{X}$.

Thus the functor $\mathcal{D}^{b}(i): \mathcal{D}^{b}(\mathcal{X}) \to \mathcal{D}^{b}(\mathcal{A})$, induced from the inclusion $i: \mathcal{X} \to \mathcal{A}$, preserves Ext for all $n \geq 2$. It also preserves Hom and Ext of $\mathcal{X}$ is an abelian full subcategory of $\mathcal{A}$ and is closed under extensions. Hence $\mathcal{D}^{b}(i)$ induces $\text{Ext}^n_A(X, X') \xrightarrow{\mathcal{D}^{b}(i)} \text{Ext}^n_A(X, X')$ for all $n \geq 0$ and $X, X' \in \mathcal{X}$. This implies that $\mathcal{D}^{b}(i)$ is fully faithful by Lemma 3.7. Now, Corollary 4.7 is a consequence of Lemma 4.6(2). □

Lemma 4.8. Suppose that $\mathcal{A}$ is an abelian category such that each object of $\mathcal{A}$ has an injective envelope. Let $\mathcal{X}$ be a Serre subcategory of $\mathcal{A}$ and $\mathcal{E} := \mathcal{X}^{0,1} \cap \mathcal{I}(\mathcal{A})$. Then $\mathcal{X}^{1}$ (respectively, $\mathcal{X}^{1,0}$) consists of all objects $M$ which has a minimal injective resolution $0 \to M \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ with $I_i \in \mathcal{E}$ for all $i \geq 0$ (respectively, $i = 0, 1$).

Proof. We first prove that $\mathcal{X}^{1,0}$ is closed under injective envelope in $\mathcal{A}$, that is, if $Z \in \mathcal{X}^{1,0}$, then the injective envelope $E(Z)$ of $Z$ belongs to $\mathcal{X}^{1,0}$.

Let $Z \in \mathcal{X}^{1,0}$ and assume contrarily that there is a nonzero morphism $f: X \to E(Z)$ in $\mathcal{A}$ for some $X \in \mathcal{X}$. Then $\text{Im}(f) \neq 0$ and there is a monomorphism $g: \text{Im}(f) \to E(Z)$. Let $h: Z \to E(Z)$ be an injective envelope of $Z$. Taking the pull-back of $(g, h)$ yields another two monomorphisms $K \to Z$ and $K \to \text{Im}(f)$ in $\mathcal{A}$. As $E(Z)$ is the injective envelope of $Z$, we have $K \neq 0$. By assumption, $\mathcal{X}$ is closed under subobjects and quotients. Hence, with $X$ also $\text{Im}(f)$ and $K$ lie in $\mathcal{X}$. It follows from $Z \in \mathcal{X}^{1,0}$ that $K = 0$, a contradiction. This shows $E(Z) \in \mathcal{X}^{1,0}$. Hence $\mathcal{X}^{1,0}$ is closed under injective envelope in $\mathcal{A}$.

If $Z \in \mathcal{X}^{1,0}$, then $E(Z) \in \mathcal{E}$. Moreover, there are inclusions of categories: $\mathcal{E} \subseteq \mathcal{X} \subseteq \mathcal{X}^{1,0} \subseteq \mathcal{A}$. Recall that $\mathcal{X}^{1}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms in $\mathcal{A}$, and that $\mathcal{X}^{1,0}$ is closed under extensions and kernels in $\mathcal{A}$. Now, it is easy to verify Lemma 4.8. □

The following result furnishes a way to get derived decompositions from localizing subcategories.

Proposition 4.9. Let $\mathcal{A}$ be an abelian category such that each of its objects has an injective envelope. If $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$ with $\mathcal{Y} := \mathcal{X}^{1}$, then the following are equivalent:

1. $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $\mathcal{A}$.
2. Each morphism $I^0 \to I^1$ between injective objects in $\mathcal{A}$ with $I^1 \in \mathcal{Y}$ can be completed to an exact sequence $I^0 \to I^1 \to I^2$ such that $I^2$ is injective and $I^2 \in \mathcal{Y}$.
3. The image of each morphism from an injective object in $\mathcal{A}$ to an object in $\mathcal{Y}$ belongs to $\mathcal{Y}$.

Proof. Since $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$, the proof of Lemma 4.6(1) shows that the five-term exact sequence associated with an object $M \in \mathcal{A}$ becomes

\[ 0 \to r(M) \xrightarrow{\varepsilon^{M}_1} M \xrightarrow{\varepsilon^{M}_2} \ell(M) \to X^M \to 0, \]

with $r(M), X^M \in \mathcal{X}$ and $\ell(M) \in \mathcal{X}^{1,0}$.

1. $\Rightarrow$ 2. Since $\mathcal{A}$ has enough injectives and $\mathcal{D}^{b}(i): \mathcal{D}^{b}(\mathcal{X}) \to \mathcal{D}^{b}(\mathcal{A})$ is fully faithful, we see from Lemma 3.11(2) that $X^M = 0$ whenever $M \in \mathcal{I}(\mathcal{A})$. Let $f: I^0 \to I^1$ be a morphism of injective objects in $\mathcal{A}$ with $I^1 \in \mathcal{Y}$. Then there is an exact sequence $0 \to r(I^0) \to I^0 \to \ell(I^0) \to 0$ in $\mathcal{A}$. We always have $\text{Hom}_{\mathcal{A}}(r(I^0), I^1) = 21$.
0, due to \( r(I^0) \in \mathcal{X} \) and \( I^1 \in \mathcal{Y} \). Consequently, \( f \) is the composition of the morphism \( \varepsilon^0 : I^0 \to \ell(I^0) \) with another morphism \( g : \ell(I^0) \to I^1 \). This implies \( \text{Coker}(f) \simeq \text{Coker}(g) \). Since \( \mathcal{Y} \) is an abelian subcategory of \( \mathcal{A} \), \( \text{Coker}(g) \in \mathcal{Y} \), and thus also \( \text{Coker}(f) \in \mathcal{Y} \). Let \( I^2 \) be the injective envelope of \( \text{Coker}(f) \). Then \( I^2 \in \mathcal{Y} \) by Lemma 4.8. Now, we extend \( f \) to an exact sequence \( I^0 \to I^1 \to I^2 \).

(2) \( \Rightarrow \) (3): Thanks to Lemma 4.8, the assumption (2) implies \( \mathcal{Y} = \mathcal{X}^{1,0,1} \). Since \( \mathcal{Y} \) is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms in \( \mathcal{A} \) and since \( \mathcal{X}^{1,0,1} \) is closed under kernels in \( \mathcal{A} \) by [14, Proposition 1.1], \( \mathcal{Y} \) is an abelian subcategory of \( \mathcal{A} \). Let \( h : I^0 \to Y \) be morphism in \( \mathcal{A} \) with \( Y \in \mathcal{Y} \) and \( I^0 \) an injective object. Further, let \( I^1 \) be the injective envelope of \( Y \) with a monomorphism \( s : Y \to I^1 \). Then \( I^1 \in \mathcal{Y} \), according to Lemma 4.8. Moreover, by (2), the composition \( h \) with \( s \) can be completed to an exact sequence \( I^0 \xrightarrow{hs} I^1 \xrightarrow{\ell} I^2 \) in \( \mathcal{A} \) such that \( I^2 \) is injective and \( I^2 \in \mathcal{Y} \). Thus \( \text{Im}(hs) = \text{Ker}(t) \in \mathcal{Y} \). Since \( \text{Im}(h) \simeq \text{Im}(hs) \), \( \text{Im}(h) \in \mathcal{Y} \), and therefore (3) follows.

(3) \( \Rightarrow \) (1): By Lemma 4.8, \( \mathcal{Y} = \mathcal{X}^{1,0,1} \). Thus, by Lemma 4.6(2), to show (1), it suffices to prove that \( \mathcal{D}(i) : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{A}) \) is fully faithful.

In fact, given an injective object \( I \) of \( \mathcal{A} \), since \( \ell(I) \in \mathcal{Y} \), (3) implies that the image of \( \varepsilon^1_i : I \to \ell(I) \) belongs to \( \mathcal{Y} \). Further, since \( \varepsilon^1_i : I \to \ell(I) \) is the unit adjunction of \( I \), \( \text{Im}(\varepsilon^1_i) = \ell(I) \). This shows \( X^1 = 0 \). Now, by Lemma 3.11, \( \mathcal{D}(i) \) is fully faithful since \( \mathcal{A} \) has enough injectives. Thus (1) holds. □

### 4.3 Nonsingular rings and commutative noetherian rings

In this section we will apply Proposition 4.9 to construct derived decomposition from left nonsingular and commutative noetherian rings, respectively. Thus we prove Corollary 1.5 in the introduction.

First, we recall some notions on left nonsingular rings (see [13, Chapter 1]).

Let \( R \) be a ring and \( M \) be an \( R \)-module with a submodule \( N \). Recall that \( M \) is an essential extension of \( N \) (or \( N \) is an essential submodule of \( M \)) if every nonzero submodule of \( M \) has nonzero intersection with \( N \). Recall that the injective envelope of \( N \) is just an essential extension \( M \) of \( N \) with \( M \) an injective module. As before, \( M \) is denoted by \( E(N) \). The set of all essential submodules of \( _R R \) is denoted by \( \mathcal{J}(R) \). A class \( \mathcal{U} \) of \( R \)-modules is said to be closed under essential extensions in \( R\text{-Mod} \) provided that \( M \in \mathcal{U} \) whenever \( M \) is an essential extension of a module \( N \in \mathcal{U} \).

For an \( R \)-module \( M \), we define \( Z(M) := \{ x \in M \mid IX = 0 \text{ for some } I \in \mathcal{J}(R) \} \). This is a submodule of \( M \) and called the singular submodule of \( M \). The module \( M \) is called a singular module if \( Z(M) = M \); and a nonsingular module if \( Z(M) = 0 \). The ring \( R \) is said to be left nonsingular if \( _R R \) is a nonsingular module. Examples of left nonsingular rings include left semi-hereditary rings, direct products of integral domains, semiprime left Goldie rings and commutative semiprime rings (see [13] for more examples).

**Corollary 4.10.** Let \( R \) be a ring, \( \mathcal{X} \) be the full subcategory of singular modules in \( R\text{-Mod} \), and \( \mathcal{Y} \) be the full subcategory of \( R\text{-Mod} \) consisting of all direct summands of arbitrary products of copies of \( E(_R R) \). If \( R \) is left nonsingular, then \( (\mathcal{X}, \mathcal{Y}) \) is a derived decomposition of \( R\text{-Mod} \).

To show Corollary 4.10, we need the following basic properties of singular and nonsingular modules (see [13, Propositions 1.20 and 1.22] for proofs).

**Lemma 4.11.** Let \( R \) be a ring, \( \mathcal{X} \) the full subcategory of singular modules in \( R\text{-Mod} \), and \( M \) an \( R \)-module.

1. \( M \in \mathcal{X} \) if and only if \( M \) is isomorphic to the quotient \( X/Y \) of an essential extension \( Y \subseteq X \).
2. \( M \in \mathcal{X}^{1,0} \) if and only if \( M \) is nonsingular.
3. \( \mathcal{X} \) is closed under submodules, quotients and direct sums in \( R\text{-Mod} \); and \( \mathcal{X}^{1,0} \) is closed under submodules, direct products, extensions and essential extensions in \( R\text{-Mod} \).

In general, the full subcategory \( \mathcal{X} \) of singular \( R \)-modules may not be closed under extensions in \( R\text{-Mod} \). Nevertheless, the next lemma, taken from [13, Propositions 1.23 and 2.12], provides a positive situation.
Lemma 4.12. Let $R$ be a left nonsingular ring and $\mathcal{X}$ be the full subcategory of singular $R$-modules. Then

1. $\mathcal{X}$ is closed under extensions and essential extensions in $R\text{-Mod}$.
2. $\mathcal{X}^{\bot 0}(\mathcal{X}^{\bot 0}) = \mathcal{I}^{0} E(R)$.
3. Let $M$ be an $R$-module. Then $M/\text{Z}(M) \in \mathcal{X}^{\bot 0}$. Moreover, $M$ is nonsingular if and only if $M$ can be embedded in a direct product of copies of $E(R)$.

Remark 4.13. If $R$ is a left nonsingular ring, then the subcategory $\mathcal{X}$ of singular $R$-modules is a localizing subcategory of $R\text{-Mod}$ by Lemmas 4.11 and 4.12. Hence $(\mathcal{X}, \mathcal{X}^{\bot 0})$ is a hereditary torsion pair in $R\text{-Mod}$ (see [6, Section 1, p.13] for definition). Moreover, $E(R)$ is nonsingular, while $E(R)/R$ is singular.

Proof of Corollary 4.10. Let $R$ be a left nonsingular ring. We show $\mathcal{Y} = \mathcal{X}^{\bot 0} \cap \mathcal{I}(R\text{-Mod}) = \mathcal{X}^{\bot} = \mathcal{X}^{\bot 0,1}$.

Since $E(R)$ is injective and nonsingular, $\mathcal{Y} \subseteq \mathcal{X}^{\bot 0} \cap \mathcal{I}(R\text{-Mod})$. The converse inclusion $\mathcal{X}^{\bot 0} \cap \mathcal{I}(R\text{-Mod}) \subseteq \mathcal{Y}$ follows from Lemmas 4.11(2) and 4.12(3). Thus $\mathcal{Y} = \mathcal{X}^{\bot 0} \cap \mathcal{I}(R\text{-Mod})$.

Clearly, $\mathcal{Y} \subseteq \mathcal{X}^{\bot} \subseteq \mathcal{X}^{\bot 0,1}$. So, to show the other equalities, it is enough to show $\mathcal{X}^{\bot 0,1} \subseteq \mathcal{Y}$. To this purpose, we first prove a general result:

$\langle * \rangle$ If $0 \to M \xrightarrow{f} I \xrightarrow{g} J$ is an exact sequence of $R$-module such that $I$ is injective and $J$ is nonsingular, then $M$ is injective.

In fact, since $f$ is injective and $gI$ is injective, there is another injective $R$-module $K$ such that $\text{Coker}(f) \simeq (E(M)/M) \oplus K$. Moreover, since $J$ is nonsingular, $\text{Im}(g)$ is also nonsingular by Lemma 4.11(3). It follows from $\text{Coker}(f) = \text{Im}(g)$ that $E(M)/M$ is nonsingular. However, $E(M)/M$ is singular by Lemma 4.11(1). This implies that $M = E(M)$ is injective.

By Lemma 4.8, the category $\mathcal{X}^{\bot 0,1}$ consists of all $R$-modules $Y$ which has a minimal injective presentation $0 \to Y \to I_0 \to I_1 \to \text{add}(I_0) \subseteq Y$. Since $I_0$ is injective and $I_1$ is nonsingular, it follows from $\langle * \rangle$ that $Y$ is injective. Thus $Y \subseteq \text{add}(I_0) \subseteq Y$. This shows $\mathcal{X}^{\bot 0,1} \subseteq \mathcal{Y}$.

As a consequence of $\langle * \rangle$, there is an isomorphism $I \cong M \oplus \text{Im}(g)$. Thus $\text{Im}(g)$ is injective. This implies $\text{Im}(g) \subseteq \text{add}(J)$. Now, if $J \subseteq Y$, then $\text{Im}(g) \subseteq Y$. Hence Corollary 4.10 follows from Proposition 4.9(3). $\square$

In the sequel, we deal with commutative noetherian rings.

Let $R$ be a commutative noetherian ring and $\text{Spec}(R)$ be the prime spectrum of $R$. For a multiplicative subset $\Sigma$ of $R$, we denote by $\Sigma^{-1} R$ the localization of $R$ at $\Sigma$. For $p \in \text{Spec}(R)$, let $R_p$ be the localization of $R$ at the set $R \setminus p$. We always identify $\text{Spec}(\Sigma^{-1} R)$ with the subset of all prime ideals $p$ of $R$ satisfying $p \cap \Sigma = \emptyset$. We also regard $(\Sigma^{-1} R)\text{-Mod}$ as a full subcategory of $R\text{-Mod}$ in the sense that an $R$-module $M$ belongs to $(\Sigma^{-1} R)\text{-Mod}$ if and only if $M \simeq (\Sigma^{-1} R) \otimes_R M$ as $R$-modules. Let $\text{Ass}(M)$ be the set of prime ideals $p$ of $R$ such that $R_p$ is isomorphic to a submodule of $M$. Thus $\text{Ass}(M) = \text{Ass}(E(M))$, where $E(M)$ is an injective envelope of $M$. The support of $M$, denoted by $\text{Supp}(M)$, is by definition the set of prime ideals $p$ of $R$ satisfying $\text{Tor}_i^R(R_p/pR_p, M) \neq 0$ for some $i \in \mathbb{N}$ (see [11]). In general, $\text{Ass}(M) \subseteq \text{Supp}(M) \subseteq \{ p \in \text{Spec}(R) \mid M_p \neq 0 \}$. The second inclusion is an equality if the module $M$ is finitely generated. Note that $\text{Supp}(M)$ is the union of the subsets $\text{Ass}(I)$ of $\text{Spec}(R)$, where $I$ runs over all those injective $R$-modules that appear in a minimal injective resolution of $M$ (see [11, Remark 2.9] or [19, Lemma 3.3]). In particular, if $M$ is injective, then $\text{Ass}(M) = \text{Supp}(M)$.

The following hold for a commutative noetherian ring $R$:

(a) Each injective $R$-module is a direct sum of indecomposable injective $R$-modules.
(b) $\{ E(R/p) \mid p \in \text{Spec}(R) \}$ is a complete set of non-isomorphic indecomposable injective $R$-modules.
(c) $\text{Hom}_R(E(R/p), E(R/q)) = 0$ if and only if $p \in q$ in $\text{Spec}(R)$ (see [10, Theorems 3.3.7 and 3.3.8]).

Let $\mathcal{S}$ be a full subcategory of $R\text{-Mod}$ and let $\Phi$ be a subset of $\text{Spec}(R)$. We define

$$\text{Supp}(\mathcal{S}) := \bigcup_{M \in \mathcal{S}} \text{Supp}(M) \quad \text{and} \quad \text{Supp}^{-1}(\Phi) := \{ M \in R\text{-Mod} \mid \text{Supp}(M) \subseteq \Phi \}.$$
Spec(R). The inverse of Supp is just given by Supp\(^{-1}\). This was extended in [19, Theorem 3.1] to a bijection (with the same maps) between the set of abelian full subcategories of R-Mod closed under extensions and arbitrary direct sums, and the set of coherent subsets of Spec(R).

A subset \(\Phi\) of Spec(R) is said to be specialization closed provided that if \(p, q \in \text{Spec}(R)\) and \(p \subseteq q\), then \(p \in \Phi\) implies \(q \in \Phi\); coherent provided that each homomorphism \(I^0 \to I^1\) between injective modules with \(\text{Ass}(I^0) \cup \text{Ass}(I^1) \subseteq \Phi\) can be completed to an exact sequence \(I^0 \to I^1 \to I^2\) such that \(I^2\) is injective and \(\text{Ass}(I^2) \subseteq \Phi\) (see [19, Section 3]). Examples of coherent subsets are specialization closed subsets and Spec(S\(^{-1}\)R). For further information on coherent subsets, we refer the reader to [19, Section 4].

An application of Proposition 4.9 is the following

**Corollary 4.14.** Let \(R\) be a commutative noetherian ring, \(\Phi\) be a specialization closed subset of Spec(R), and \(\Phi^c := \text{Spec}(R) \setminus \Phi\). Then the pair \((\text{Supp}^{-1}(\Phi), \text{Supp}^{-1}(\Phi^c))\) is a derived decomposition of R-Mod if and only if \(\Phi^c\) is coherent.

**Proof.** Let \(\mathcal{X} := \text{Supp}^{-1}(\Phi)\) and \(\mathcal{X}^{-1}(\Phi^c) := \{M \in \text{R-Mod} \mid \text{Ass}(M) \subseteq \Phi^c\}\). We first show that \(\mathcal{X}^{-1}(\Phi^c)\).

Let \(u \in \mathcal{X}\) and \(v \in \mathcal{X}^{-1}(\Phi^c)\). Then \(\text{Supp}(E(U)) \subseteq \text{Supp}(U) \subseteq \Phi\) and \(\text{Ass}(E(V)) = \text{Ass}(V) \subseteq \Phi^c\). If \(\text{Hom}_R(E(U), E(V)) \neq 0\), then there is a non-zero homomorphism from a direct summand \(E(R/p)\) of \(E(U)\) to a direct summand \(E(R/q)\) of \(E(V)\), where \(p \in \Phi\) and \(q \in \Phi^c\). In this case, we have \(p \subseteq q\). This is contradictory to the assumption that \(\Phi\) is specialization closed. Thus \(\text{Hom}_R(E(U), E(V)) = 0\). This implies \(\text{Hom}_R(U, V) = 0\) and shows \(\text{Ass}^{-1}(\Phi^c) \subseteq \mathcal{X}^{-1}(\Phi^c)\). To verify \(\mathcal{X}^{-1}(\Phi^c) \subseteq \text{Ass}^{-1}(\Phi^c)\), we take \(w \in \mathcal{X}^{-1}(\Phi^c)\) and \(a \in \text{Ass}(W)\). Then \(f/a\) is isomorphic to a nonzero submodule of \(W\). If \(a \in \Phi\) and \(f\) is specialization closed, then \(\text{Supp}(R/a) = \{b \in \text{Spec}(R) \mid a \subseteq b\} \subseteq \Phi\). This implies \(R/a \in \mathcal{X}\) and \(\text{Hom}_R(R/a, W) = 0\), a contradiction. Thus \(a \in \Phi^c\).

Let \(\mathcal{Y} := \mathcal{X}^{-1}\) and \(\mathcal{E} := \mathcal{X}^{-1}(\Phi) \cap \mathcal{F}(\text{R-Mod})\). Then \(\mathcal{E} = \text{Ass}^{-1}(\Phi) \cap \mathcal{F}(\text{R-Mod}) = \mathcal{Y} \cap \mathcal{F}(\text{R-Mod})\). By Lemma 4.8, \(\mathcal{Y}\) consists of all \(R\)-modules \(M\) which has a minimal injective resolution \(0 \to M \to I_0 \to I_1 \to I_2 \to \cdots\) such that \(I_i \in \mathcal{E}\) for all \(i \geq 0\). Since \(\text{Supp}(M) = \bigcup_{i \geq 0} \text{Ass}(I_i)\), \(\mathcal{Y} = \text{Supp}^{-1}(\Phi^c)\). Observe that, for any \(\Psi \subseteq \text{Spec}(R)\), \(\text{Supp}^{-1}(\Psi)\) is always closed under direct sums in \(\text{R-Mod}\) because \(\text{Supp}(\bigoplus_{j \in J} M_j) = \bigcup_{j \in J} \text{Supp}(M_j)\) for any family \(\{M_j\}_{j \in J}\) of \(R\)-modules with \(J\) an index set. In particular, \(\mathcal{Y}\) is closed under direct sums in \(\text{R-Mod}\).

If \((\mathcal{X}, \mathcal{Y})\) is a derived decomposition of \(\text{R-Mod}\), then \(\mathcal{Y}\) is an abelian full subcategory of \(\text{R-Mod}\) and closed under both extensions and direct sums, and therefore \(\Phi^c\) is coherent.

Conversely, suppose that \(\Phi^c\) is coherent. Let \(f : I^0 \to I^1\) be a homomorphism between injective \(R\)-modules with \(I^1 \in \mathcal{Y}\). By Proposition 4.9(2), we need to extend \(f\) to an exact sequence \(I^0 \to I^1 \to I^2\) in \(\text{R-Mod}\) with \(I^2 \in \mathcal{E}\). This can be done if \(I^2 \in \mathcal{E}\) since \(\Phi^c\) is coherent. For the general case, we decompose \(I^0\) into a direct sum of indecomposable injective modules. Recall that \(\{E(R/p) \mid p \in \text{Spec}(R)\}\) is a complete set of isomorphism classes of indecomposable injective \(R\)-modules and that \(\text{Ass}(E(R/p)) = \text{Supp}(E(R/p)) = \{p\}\). Consequently, \(E(R/p)\) belongs to either \(\mathcal{X}\) or \(\mathcal{Y}\). This yields a decomposition \(I^0 = X \oplus Y\) with \(X \in \mathcal{X}\) and \(Y \in \mathcal{Y}\). Since \(\text{Hom}_R(X, I^1) = 0\), \(f = 0\) and \(g : Y \to I^1\) is the restriction of \(f\) to \(Y\). Clearly, \(g\) is a homomorphism between modules in \(\mathcal{E}\). Now, we first extend \(g\) and then \(f\) to an exact sequence \(I^0 \to I^1 \to I^2\) in \(\text{R-Mod}\) with \(I^2 \in \mathcal{E}\). Thus \((\mathcal{X}, \mathcal{Y})\) is a derived decomposition of \(\text{R-Mod}\).

In Corollary 4.14, when \(\Phi^c\) is coherent, \(\mathcal{Y} := \text{Supp}^{-1}(\Phi^c)\) is an abelian full subcategory of \(\text{R-Mod}\) closed under direct sums, and the inclusion \(j : \mathcal{Y} \to \text{R-Mod}\) has a left adjoint \(\ell : \text{R-Mod} \to \mathcal{Y}\). Let \(S = \text{End}_R(\ell(R))\) and let \(\lambda : R \to S\) be the ring homomorphism induced from the functor \(\ell\). Thanks to [14, Proposition 3.8], \(\lambda\) is a ring epimorphism, inducing an equivalence of abelian categories: \(S-\text{Mod} \overset{\sim}{\to} \mathcal{Y}\), and \(S\) is a flat \(R\)-module since \(\ell\) is exact. Thus \(\lambda\) is a flat ring epimorphism (see also [2] for further details). Consequently, \(S\) is also a commutative noetherian ring. So we can apply Corollary 4.14 (for example, via localizations) to \(S\) and obtain a derived decomposition of \(S-\text{Mod}\). By iterating this procedure, we can stratify \(\text{R-Mod}\) as a sequence of derived decompositions. In particular, when the Krull dimension of \(R\) is at most 1, a derived stratification (see Definition 2.3) of \(\text{R-Mod}\) can be constructed explicitly.
Proof of Corollary 1.5. By [19, Theorem 1.2], if the Krull dimension of $R$ is at most 1, then every subset of $\text{Spec}(R)$ is coherent. Thus (1) follows from Corollary 4.14. For (2), let $S$ be the localization of $R$ at $\Sigma$ and recall that $\text{Spec}(S)$ is identified with the subset of all prime ideals $p$ of $R$ satisfying $p \cap \Sigma = \emptyset$. Thus $\Phi^c = \text{Spec}(S)$ and is coherent. Further, $\text{Supp}^{-1}(\Phi^c) = S-\text{Mod}$. Now (2) is a consequence of Corollary 4.14. It also follows from Corollary 4.3. □

**Corollary 4.15.** Suppose that $R$ is a commutative noetherian ring of Krull dimension at most 1. Let $\text{Max}(R)$ be the set of maximal ideals of $R$ and let $\text{Min}(R)$ be the set of prime ideals of $R$ which are not maximal. Then

1. $(\text{Supp}^{-1}(\text{Max}(R)), \text{Supp}^{-1}(\text{Min}(R)))$ is a derived decomposition of $R-\text{Mod}$.

2. There are equivalences of abelian categories:

$$\text{Supp}^{-1}(\text{Max}(R)) \xrightarrow{\cong} \prod_{p \in \text{Max}(R)} \text{Supp}^{-1}(\{m\}) \quad \text{and} \quad \text{Supp}^{-1}(\text{Min}(R)) \xrightarrow{\cong} \prod_{p \in \text{Min}(R)} R_p-\text{Mod},$$

where $\prod$ denotes the direct product of abelian categories.

3. Both $\text{Supp}^{-1}(\{m\})$ and $R_p-\text{Mod}$ are derived indecomposable for any $m \in \text{Max}(R)$ and $p \in \text{Min}(R)$.

*Proof.* (1) Clearly, $\text{Max}(R)$ is a specialization closed subset of $\text{Spec}(R)$. Since the Krull dimension of $R$ is at most 1, the statement (1) follows from Corollary 1.5.

(2) To show the equivalences, we first establish the following

**Lemma 4.16.** Let $\Phi$ be a subset of $\text{Spec}(R)$ with the property $(\diamond)$: for $p, q \in \Phi$, $p \subsetneq q$ implies $p = q$. Then there is an equivalence of abelian categories: $\prod_{p \in \Phi} \text{Supp}^{-1}(\{p\}) \xrightarrow{\cong} \text{Supp}^{-1}(\Phi)$.

*Proof.* In fact, given an $R$-module $M$ with a minimal injective resolution $0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$, we always have $\text{Supp}(M) = \bigcup_{i \geq 0} \text{Ass}(I_i) = \bigcup_{i \geq 0} \text{Supp}(I_i)$. Consequently, $M \in \text{Supp}^{-1}(\Phi)$ if and only if $I_i \in \text{Supp}^{-1}(\Phi)$ for all $i \geq 0$. Recall that, for each $a \in \text{Spec}(R)$, if $a \not\subsetneq b \in \text{Spec}(R)$, then $\text{Hom}_R(E(R/a), E(R/b)) = 0$. Moreover, $R_a-\text{Mod}$ is regarded as an abelian full subcategory of $R-\text{Mod}$ and $E(R/a) \in R_a-\text{Mod}$. So, the property $(\diamond)$ implies

1. $M \in \text{Supp}^{-1}(\Phi)$ if and only if $M \simeq \bigoplus_{p \in \Phi} M_p$ with $M_p \in \text{Supp}^{-1}(\{p\})$, where $M_p$ stands for the localization of $M$ at $p$;

2. If $p, q \in \Phi$ and $p \subsetneq q$, then $\text{Hom}_R(X, Y) = 0$ for all $X \in \text{Supp}^{-1}(\{p\})$ and $Y \in \text{Supp}^{-1}(\{q\})$.

By (1) and (2), the functor $\oplus: \prod_{p \in \Phi} \text{Supp}^{-1}(\{p\}) \to \text{Supp}^{-1}(\Phi)$, given by taking direct sums in $R$-Mod, is an equivalence of abelian categories. □

Now, since the Krull dimension of $R$ is at most 1, both $\text{Max}(R)$ and $\text{Min}(R)$ have the property $(\diamond)$. Moreover, if $p$ is a minimal prime ideal of $R$, then it follows from $\text{Supp}(M) = \bigcup_{i \geq 0} \text{Ass}(I_i)$ that $\text{Supp}^{-1}(\{p\}) = R_p-\text{Mod}$. Note that $\text{Min}(R)$ consists of all minimal prime ideals of $R$ which are not maximal. Now, the existence of equivalences in Corollary 4.15 follows from Lemma 4.16.

(3) If $(\mathcal{X}, \mathcal{Y})$ is an Ext-orthogonal pair of an abelian category $\mathcal{A}$ with arbitrary direct sums, then $\mathcal{X} = \perp \mathcal{Y}$ and $\mathcal{X}$ is closed under both extensions and arbitrary direct sums (see the dual of [21, Theorem 1, p. 116]). It follows from [19, Theorem 3.1] that, for any $a \in \text{Spec}(R)$, the abelian category $\text{Supp}^{-1}(\{a\})$ does not contain non-trivial abelian full subcategory which is closed under extensions and arbitrary direct sums. This implies that the abelian $\text{Supp}^{-1}(\{a\})$ is derived indecomposable. Thus (3) holds. □

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