Generalization of the Multiplicative and Additive Compounds of Square Matrices and Contraction Theory in the Hausdorff Dimension

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Abstract—The $k$ multiplicative and $k$ additive compounds of a matrix play an important role in geometry, multilinear algebra, the asymptotic analysis of nonlinear dynamical systems, and in bounding the Hausdorff dimension of fractal sets. These compounds are defined for the integer values of $k$. Here, we introduce generalizations called the $\alpha$ multiplicative and $\alpha$ additive compounds of a square matrix, with $\alpha$ real. We study the properties of these new compounds and demonstrate an application in the context of the Douady and Oesterlé theorem. Our results lead to a generalization of contracting systems to $\alpha$-contracting systems, with $\alpha$ real. Roughly speaking, the dynamics of such systems contracts any set with the Hausdorff dimension larger than $\alpha$. For $\alpha = 1$, they reduce to standard contracting systems. We demonstrate our theoretical results by designing a state-feedback controller for a classical chaotic system, guaranteeing the well-ordered behavior of the closed-loop system.

Index Terms—Additive compound matrix, contraction theory, fractal sets, multiplicative compound matrix, nonlinear dynamical systems, ribosome flow model, Thomas’ cyclically symmetric attractor.

I. INTRODUCTION

The $k$ multiplicative and $k$ additive compound matrices have found numerous applications in multilinear algebra, geometry, graph theory, and dynamical system theory (see, e.g., [10], [35], [36], and [45]). Recently, they have also been used to generalize several important notions from systems and control theory [3].

The purpose of this article is to introduce a generalization of the $k$ multiplicative and $k$ additive compounds of a square matrix, called the $\alpha$ multiplicative and $\alpha$ additive compounds, where $\alpha \geq 1$ is allowed to be a real number. For $k < \alpha < k + 1$, the $\alpha$ compounds may be interpreted as a weighted interpolation of the $k$ and $k + 1$ compounds. When $\alpha$ is an integer, this (almost) reduces to the standard $k$ compounds.

Our generalization is motivated by the Hausdorff dimension of a set and, in particular, the seminal Douady and Oesterlé theorem [8], which provides an upper bound for the Hausdorff dimension of a set that is negatively invariant under a $C^1$ mapping. As an application, we show that the $\alpha$ compounds can be used to provide elegant and intuitive expressions for the basic terms that appear in this theorem.

The new compounds naturally lead to the new notion of $\alpha$-contracting systems, with $\alpha$ real, which generalizes the notion of $k$-contracting systems with $k$ an integer [22], [28], [57]. We analyze the properties of $\alpha$-contracting systems and demonstrate their applications. For $\alpha = 1$, these reduce to the standard contracting systems. Our results show that if an $n$-dimensional dynamical system contracts $n$-dimensional volumes, then there exists a minimal real value $\alpha^* \in [1, n]$ such that the system is $\alpha$ contracting for any $\alpha > \alpha^*$. Roughly speaking, an $\alpha$-contracting system contracts any set with a Hausdorff dimension larger than $\alpha$. This generates (in a given metric) a continuum of contraction instead of the standard binary view, namely, that a system is either contracting or not contracting.

We demonstrate these results using several applications. First, we show that the classical linear consensus algorithm is $(1 + s)$-contracting for any $s > 0$. This is in agreement with the fact that the set of equilibrium points of the dynamics is a line, i.e., has a Hausdorff dimension one. We also show that a certain nonlinear system is $(1 + s)$-contracting for any $s > 0$. In this case, the set of equilibrium points is a one-dimensional (1-D) curve. As a control-theoretic application, we consider Thomas’ cyclically symmetric attractor [51]. We use our theoretical results to design a state-feedback controller for this popular chaotic system guaranteeing that the closed-loop system has a well-ordered (and, in particular, nonchaotic) behavior. The design is based on first showing that the chaotic system is $(2 + s)$-contracting, where $s \in (0, 1)$ and its exact value depends on the parameters of the chaotic system, and then designing a controller guaranteeing that the closed-loop system is 2-contracting.
The rest of this article is organized as follows. Section II reviews several known definitions and results that will be used later on. Section III introduces the \( \alpha \) multiplicative and \( \alpha \) additive compounds of a square nonsingular matrix and analyzes the properties of these compounds. Section IV introduces \( \alpha \)-contracting systems. Section V describes a control-theoretic application of \( \alpha \)-contraction. Finally, Section VI concludes this article.

**Notation:** Let \( I_n \) denote the \( n \times n \) identity matrix. Denote
\[
\begin{align*}
\mathbb{R}_{\leq 0} & := \{ a \in \mathbb{R} \mid a \leq 0 \} \\
\mathbb{R}_{> 0} & := \{ a \in \mathbb{R} \mid a > 0 \}.
\end{align*}
\]
For a square matrix \( X \), \( \text{spec}(X) \) is the set of eigenvalues of \( X \). We define the set of matrices
\[
\Omega_n := \{ X \in \mathbb{C}^{n \times n} \mid \text{spec}(X) \cap \mathbb{R}_{\leq 0} = \emptyset \}
\]
i.e., if \( X \in \Omega_n \), then any real eigenvalue of \( X \) is positive.

Let \( | \cdot | : \mathbb{C}^n \to \mathbb{R}_{\geq 0} \) denote a vector norm. The induced matrix norm \( || \cdot || : \mathbb{C}^{n \times n} \to \mathbb{R}_{\geq 0} \) is \( ||A|| := \max_{||x||=1} ||Ax|| \). For \( p \in [1, \infty] \), we use
\[
||x||_p := (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}
\]
to denote the \( L_p \) vector norm of \( x \in \mathbb{C}^n \), and let \( || \cdot ||_p \) denote the induced matrix norm. A norm \( | \cdot | : \mathbb{C}^n \to \mathbb{R}_{\geq 0} \) is called monotonic if for any \( x, y \in \mathbb{C}^n \) such that \( |x_i| \leq |y_i| \) for all \( i \in \{1, \ldots, n\} \), we have \( |x| \leq |y| \) [5].

Let \( \lambda_i(A) \in \mathbb{C}, \sigma_i(A) \in \mathbb{R}_{\geq 0}, \ i = 1, \ldots, n \), denote the eigenvalues and singular values of \( A \in \mathbb{C}^{n \times n} \), respectively, and ordered such that
\[
\Re(\lambda_1(A)) \geq \Re(\lambda_2(A)) \geq \cdots \geq \Re(\lambda_n(A))
\]
and
\[
\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0.
\]
For an integer \( k \in \{1, 2, \ldots, n\} \), let \( Q^{k,n} \) denote the sequence of \( k \)-tuples of distinct numbers from \( \{1, \ldots, n\} \) in lexicographic order. Note that there are \( \binom{n}{k} \) such \( k \)-tuples. Let \( Q^k_\ell \) denote the \( \ell \)th tuple in \( Q^{k,n} \). For example
\[
Q^{2,3} = \{ (1, 2), (1, 3), (2, 3) \}
\]
and \( Q^{2,3}_2 = \{1, 3\} \).

## II. PRELIMINARIES

To make this article more self-contained, we briefly review several topics that are needed to define, analyze, and apply the \( \alpha \) compounds of a matrix and \( \alpha \)-contracting systems.

### A. \( k \) Compound Matrices

Consider a matrix \( A \in \mathbb{C}^{n \times m} \) and fix an integer \( k \in \{1, \ldots, \min\{m, n\}\} \). The \( k \) multiplicative compound matrix of \( A \), denoted \( A^{(k)} \), is the \( \binom{n}{k} \times \binom{m}{k} \) matrix that includes all the minors of order \( k \) of \( A \) organized in a lexicographic order. For example, if \( n = 3, m = 2, \) and \( k = 2 \), then \( A^{(2)} \in \mathbb{R}^{3 \times 1} \) and is given by
\[
A^{(2)} = \begin{bmatrix}
A(\{1, 2\}|\{1, 2\}) \\
A(\{1, 3\}|\{1, 2\}) \\
A(\{2, 3\}|\{1, 2\})
\end{bmatrix}
\]
where \( A(\phi|\psi) \) denotes the minor of \( A \) obtained by taking the rows indexed by \( \phi \) and the columns indexed by \( \psi \). Note that this implies that \( A^{(1)} = A \), and if \( m = n \), then \( A^{(n)} = \det(A) \).

In addition, this implies that \( (I_n)^{(k)} \) is the identity matrix \( \mathcal{I} \), with \( r := \binom{n}{k} \).

The Cauchy–Binet formula [9] asserts that for any \( A \in \mathbb{C}^{n \times m}, B \in \mathbb{C}^{m \times r}, \) and any \( k \in \{1, \ldots, \min\{n, m, r\}\} \),
\[
(AB)^{(k)} = A^{(k)}B^{(k)}.
\]
This justifies the term multiplicative compound. For example, if \( A, B \) are \( n \times n \), then (5) with \( k = n \) reduces to the familiar formula \( \det(AB) = \det(A)\det(B) \).

When \( n = m, \) i.e., \( A \) is a square matrix, the \( k \) additive compound matrix of \( A \) is defined by
\[
A^{[k]} := \frac{d}{d\varepsilon} (I_n + \varepsilon A)^{(k)} |_{\varepsilon = 0}.
\]
This implies that
\[
(I_n + \varepsilon A)^{(k)} = I^{(k)} + \varepsilon A^{[k]} + o(\varepsilon)
\]
i.e., \( A^{[k]} \) is the coefficient of the first-order term in the Taylor series of \( (I_n + \varepsilon A)^{(k)} \). For example, if \( A \in \mathbb{C}^{n \times n} \) is diagonal, denoted \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \), and \( k = 2 \), then
\[
(I_n + \varepsilon A)^{(2)} = (\text{diag}(1 + \varepsilon \lambda_1, 1 + \varepsilon \lambda_2, \ldots, 1 + \varepsilon \lambda_{n-1}, 1 + \varepsilon \lambda_n))^{(2)} = I^{(2)} + \varepsilon (\text{diag}(\lambda_1 + \lambda_2, \ldots, \lambda_{n-1} + \lambda_n) + o(\varepsilon))
\]
so
\[
A^{[2]} = \text{diag}(\lambda_1 + \lambda_2, \ldots, \lambda_{n-1} + \lambda_n).
\]

Note that every eigenvalue of \( A^{[2]} \) is the sum of two eigenvalues of \( A \).

It can be shown [35] that (5) and (7) imply that
\[
(A + B)^{[k]} = A^{[k]} + B^{[k]}
\]
for any \( A, B \in \mathbb{C}^{n \times n} \). This justifies the term additive compound.

The definition of the \( k \) additive compound in (6) is implicit, but \( A^{[k]} \) can be described explicitly in terms of the entries \( a_{ij} \) of \( A \). For \( A \in \mathbb{C}^{n \times n} \) and \( k \in \{1, \ldots, n\} \), recall that the \( i \)th entry of \( A^{(k)} \) is \( A(Q_k^{k,n}|Q_k^{k,n}) \). Thus, we can label each entry of \( A^{[k]} \) by \( A(\phi^k|\psi^k) \), where \( \phi, \psi \in Q_k^{k,n} \).

**Proposition 1 (see [10] and [45]):** Let \( A \in \mathbb{R}^{n \times n} \) and fix \( k \in \{1, \ldots, n\} \), and \( \phi, \psi \in Q_k^{k,n} \). Denote the entries of \( \phi | \psi \) by \( 1 \leq i_1 < \cdots < i_k \leq n \) \( 1 \leq j_1 < \cdots < j_k \leq n \). The entry \( A(\phi|\psi) \) is:
\[
\begin{align*}
1) & \sum_{\ell=1}^{k} a_{i_\ell j_\ell} \\
2) & (-1)^{m+1} a_{i_j j_i} \\
3) & 0
\end{align*}
\]
if all the indices in \( \phi \) and \( \psi \) coincide except for a single index \( i \neq j \); otherwise.

**Remark 1:** To explain this, consider, for example, the case \( k = 2 \). Let \( B := (I_n + \varepsilon A)^{(2)} \), and let
\[
\delta_{pq} := \begin{cases} 
1, & \text{if } p = q \\
0, & \text{otherwise}
\end{cases}
\]
Fix $1 \leq i_1 < i_2 \leq n$ and $1 \leq j_1 < j_2 \leq n$. Then
\[
B_{\{i_1,i_2\}\{j_1,j_2\}} = b_{i_1,j_2} b_{i_2,j_1} - b_{i_1,j_1} b_{i_2,j_2}
= (\delta_{i_1,j_1} + \varepsilon a_{i_1,j_1})(\delta_{i_2,j_2} + \varepsilon a_{i_2,j_2})
- (\delta_{i_1,j_2} + \varepsilon a_{i_1,j_2})(\delta_{i_2,j_1} + \varepsilon a_{i_2,j_1})
= c + (\delta_{i_1,j_1} a_{i_2,j_2} + \delta_{i_2,j_2} a_{i_1,j_1} - \delta_{i_1,j_2} a_{i_2,j_1} - \delta_{i_2,j_1} a_{i_1,j_2}) \varepsilon + o(\varepsilon)
\]
where $c$ is a constant that does not depend on $\varepsilon$. Thus, (6) gives
\[
A^{[2]}_{\{i_1,i_2\}\{j_1,j_2\}} = \delta_{i_1,j_1} a_{i_2,j_2} + \delta_{i_2,j_2} a_{i_1,j_1}
- \delta_{i_1,j_2} a_{i_2,j_1} - \delta_{i_2,j_1} a_{i_1,j_2}
\]
and it is straightforward to see that this agrees with the expression given in Proposition 1.

Note that Proposition 1 implies, in particular, that $A^{[1]} = A$, and that $A^{[n]} = \text{trace}(A)$.

Compound matrices have found numerous applications. We quickly review some examples. The $k$-multiplicative compound has an important geometric interpretation. Let $a^1, \ldots, a^k$ be $k$ vectors in $\mathbb{R}^n$, and let
\[
P(a^1, \ldots, a^k) := \left\{ \sum_{i=1}^k r_i a^i \mid r_i \in [0, 1] \right\}
\]
denote the parallelotope with vertices $0, a^1, \ldots, a^k$ (see Fig. 1).

Let $A := [a^1 \ldots a^k] \in \mathbb{R}^{n \times k}$, and note that $A^{(k)}$ has dimensions $(n \times k)$; therefore, it is a column vector. Then, the volume of $P$ is
\[
\text{volume}(P(a^1, \ldots, a^k)) = |A^{(k)}|_2
\]
(see [11, Ch. IX] and [13]). In particular, when $k = n$, this reduces to the well-known formula
\[
\text{volume}(P(a^1, \ldots, a^n)) = |\det \left( \begin{bmatrix} a^1 & \ldots & a^n \end{bmatrix} \right)|.
\]

We now describe some applications of compound matrices in dynamical systems theory. In this context, the relevant case is square matrices. Suppose that $X : \mathbb{R} \to \mathbb{R}^{n \times n}$ is the solution of the linear matrix differential equation
\[
\frac{d}{dt} X(t) = A(t) X(t), \quad X(t_0) = X_0
\]
with $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ continuous. Then, as shown in [35]
\[
\frac{d}{dt} X^{(k)}(t) = A^{[k]}(t) X^{(k)}(t), \quad X^{(k)}(t_0) = (X_0)^{(k)}
\]
for any $k \in \{1, \ldots, n\}$. In other words, all the $k$ minors of $X$, stacked in the matrix $X^{(k)}$, also follow a linear dynamics, with the matrix $A^{[k]}$. Roughly speaking, (10) describes the evolution of volumes of parallelotopes under a linear dynamics.

Note that if $A$ is time invariant and $t_0 = 0$, then the solution of (9) is $X(t) = \exp(A t) X_0$, so $(X(t))^{(k)} = (\exp(A t))^{(k)}(X_0)^{(k)}$, and combining this with (10) gives
\[
(\exp(A t))^{(k)} = \exp(A^{[k]} t) \text{ for all } t \in \mathbb{R}.
\]
Equation (10) has important applications in the analysis of time-varying nonlinear dynamical systems in the form
\[
\dot{x}(t) = f(x(t)), \quad f(x) \text{ differentiable on } x
\]
We assume throughout that the vector field $f$ is differentiable on $x$, and that $f(t, x)$ and the Jacobian
\[
J_f(t,x) := \frac{\partial}{\partial x} f(t,x)
\]
are continuous in $(t,x)$. Let $x(t,a)$ denote the solution at time $t \geq 0$ of (12) with $x(0) = a$. The variational equation associated with (12) along the trajectory $x(t,a)$ is
\[
\dot{y}(t) = J_f(t,x(t,a)) y(t).
\]
This analysis of this linear time-varying equation plays an important role in the asymptotic analysis of (12). Combining this with (10) has far-reaching applications in the theory of non-linear dynamical systems [35]. Recent applications include the following:

1) totally positive differential systems [33], [45], that is, systems where the transition matrix corresponding to the variational equation (14) is TP (see also [6]);
2) $k$-cooperative dynamical systems [53], that is, systems where $J_f^{[k]}$ is a Metzler matrix;
3) $k$-contracting systems [57], that is, systems where $J_f^{[k]}$ is infinitesimally contracting;
4) the notion of a discrete-time $k$-diagonally stable dynamical system, that is, a system of the form $x(k+1) = A x(k)$, and there exists a positive-definite diagonal matrix $D$, such that $(A^{(k)})^T D A^{(k)} - D$ is negative definite [58].

See also [15] and [16] for some recent work that studies systems with an input and output using compound matrices.

### B. Real Power of a Square Nonsingular Matrix

The definition of the $\alpha$ compound, with $\alpha$ real, requires computing the real power of a matrix. Any complex number $a \in \mathbb{C} \setminus \{0\}$ can be written in the polar representation $a = |a| \exp(i \theta(a))$, where $j := \sqrt{-1}$, $|a| > 0$ is the modulus of $a$, and $\theta(a) \in (-\pi, \pi]$ is the argument of $a$. Then, for any $\alpha \in \mathbb{R}$, we have
\[
a^\alpha := |a|^\alpha \exp(j(\alpha \theta(a) + 2\pi k))
\]
where we again take the principal branch, i.e., the integer $k$ is chosen such that $\alpha \theta(a) + 2\pi k \in (-\pi, \pi]$. For example, for $a = -5 + j \epsilon$, with $\epsilon \in \mathbb{R}$, and $\alpha = 1/2$, we have $|a^\alpha| = (25 + \ldots)$.
Recall that $\theta(\cdot)$ is not continuous along the negative semiaxis, and that $a^\alpha$ may be a complex (nonreal) number even when $a \in \mathbb{R}$.

The next result describes properties of $A^\alpha$. The proof follows from the results in [17, Ch. 1].

Proposition 2: Let $A \in \mathbb{C}^{n \times n}$ be nonsingular with Jordan canonical form (16), given in the Jordan matrix $J$. Then, for all $\alpha \in \mathbb{R}$, $A^\alpha := T J^\alpha T^{-1}$

where

$$J^\alpha := \text{diag}(J_1^\alpha, \ldots, J_p^\alpha)$$

with

$$J_i^\alpha := \begin{cases} 1 \\ \ell_i^{-1} \\ \vdots \\ \ell_i^{-\alpha} \\ \ell_i^{-1} \\ \vdots \\ \ell_i^{-\alpha} \\ \ell_i^{-1} \end{cases} \in \mathbb{C}^{m_i \times m_i},$$

with $\sum_{i=1}^{p} m_i = n$, and every $\ell_i$, $i = 1, \ldots, p$, is an eigenvalue of $A$. The matrix $J$ is unique, up to the ordering of the blocks $J_i$.

Since the real power of square matrices is a particular class of a matrix function [12, 17], it is defined based on the general definition given in [17, Def. 1.2].

Definition 1 (Real power of a nonsingular square matrix): Consider a nonsingular matrix $A \in \mathbb{C}^{n \times n}$, given in the Jordan canonical form (16), where every $J_i$ has the form (17). Let $\alpha \in \mathbb{R}$. Then

$$A^\alpha := T J^\alpha T^{-1}$$

where

$$J^\alpha := \text{diag}(J_1^\alpha, \ldots, J_p^\alpha)$$

with

$$J_i^\alpha := \begin{cases} 1 \\ \ell_i^{-\alpha} \\ \vdots \\ \ell_i^{-\alpha} \\ \ell_i^{-\alpha} \end{cases} \in \mathbb{C}^{m_i \times m_i},$$

with $\Sigma_{i=1}^{p} m_i = n$, and every $\ell_i$, $i = 1, \ldots, p$, is an eigenvalue of $A$. The matrix $J$ is unique, up to the ordering of the blocks $J_i$.

The next result describes properties of $A^\alpha$. The proof follows from the results in [17, Ch. 1].

Proposition 2: Let $A \in \mathbb{C}^{n \times n}$ be nonsingular with Jordan canonical form (16). Fix $\alpha \in \mathbb{R}$. Then, we have the following.

a) The eigenvalues of $A^\alpha$ are $\ell_i^\alpha$, $i = 1, \ldots, p$.

b) $(XAX^{-1})^\alpha = X A^\alpha X^{-1}$, for any nonsingular matrix $X \in \mathbb{C}^{n \times n}$.

c) $(A^T)^\alpha = (A^\alpha)^T$.

d) If $A$ is real and $A \in \Omega_n$, then $A^\alpha$ is real.

C. Kronecker Product and Kronecker Sum of Matrices

The Kronecker product of $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{p \times q}$ is

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{bmatrix}. \quad (21)$$

Hence, $A \otimes B \in \mathbb{C}^{(np) \times (mq)}$.

The Kronecker sum of two square matrices $X \in \mathbb{C}^{n \times n}$ and $Y \in \mathbb{C}^{m \times m}$ is

$$X \oplus Y := X \otimes I_m + I_n \otimes Y. \quad (22)$$

Several properties of the Kronecker product and Kronecker sum that are used later on are summarized in Appendix A.

D. Matrix Measures

Given an induced matrix norm $\| \cdot \| : \mathbb{C}^{n \times n} \to \mathbb{R}_{\geq 0}$, the associated matrix measure $\mu : \mathbb{C}^{n \times n} \to \mathbb{R}$ is

\[ \mu(A) := \lim_{\varepsilon \to 0} \frac{\| I_n + \varepsilon A \| - 1}{\varepsilon}. \quad (24) \]

Matrix measures (also called logarithmic norms) play an important role in numerical linear algebra [50] and in systems and control theory [2, 26]. The reason for this is twofold. First, for the system (9), if there exists a matrix measure such that

$$\mu(A(t)) \leq \eta \text{ for all } t \geq 0 \quad (25)$$

then

$$\|X(t)\| \leq \exp(\eta t) \|X(0)\| \text{ for all } t \geq 0.$$

In particular, if $\eta < 0$, this implies exponential convergence to zero with rate $\eta$. Second, let $\mu_p$ denote the matrix measure induced by the $L_p$ vector norm, with $p \in \{1, 2, \infty\}$. Then, for any $A \in \mathbb{C}^{n \times n}$ and $k \in \{1, \ldots, n\}$, there exist explicit formulas for $\mu_p(A^{[k]})$ (see Appendix B). These formulas with $k = 1$ provide simple to check sufficient conditions for (25) to hold.

The next result provides a useful expression for the matrix measure of a Kronecker sum of matrices (see Appendix C for the proof). This result will be used below to determine the matrix measure for the generalized additive compound of a matrix.

Theorem 1: Let $\| \cdot \|$ be an induced matrix norm such that

\[ \|A \otimes B\| = \|A\| \|B\| \text{ for any } A, B. \quad (26) \]

Then, the associated matrix measure $\mu$ satisfies

$$\mu(X \oplus Y) = \mu(X) + \mu(Y) \quad \text{for any } X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{m \times m}.\quad$$

E. $k$-Contraction

For an integer $k \in \{1, \ldots, n\}$, the system (12) is said to be (infinitiesimally) $k$-contracting if there exists a vector norm such
that the associated matrix measure $\mu$ satisfies

$$\mu(J_f^k(t,x)) \leq -\eta < 0$$

for any $t \geq 0$ and any $x$ in the state space [57] (see also [22] and [28]). Roughly speaking, this implies that the dynamics contracts $k$-dimensional volumes. For example, consider the system $\dot{x}(t) = A(t)x(t)$, and suppose that it is $n$-contracting. Since $A(t) = \text{trace}(A(t))$, this implies that $\text{trace}(A(t)) \leq -\eta < 0$ for all $t \geq 0$. Combining this with (10) implies the following. Fix $n$ initial conditions $a_i, i = 1, \ldots, n$, and let

$$X(t) := \begin{bmatrix} x(t,a^1) & x(t,a^2) & \cdots & x(t,a^n) \end{bmatrix}.$$  

Then, $|\det(X(t))| \leq \exp(-\eta t)|\det(X(0))|$. Therefore, $n$-dimensional volumes contract at an exponential rate. Following the terminology used in physics, we say that (12) is dissipative if it is $n$-contracting.

We will use the following result, whose proof follows from the formulas in Appendix B.

**Proposition 4** (see [57]): Let $A \in \mathbb{C}^{n \times n}$ and $p \in \{1, 2, \infty\}$. Suppose that there exists an integer $\ell \in \{1, \ldots, n\}$ such that

$$\mu_p(A^{[\ell]}) \leq 0$$

and any $\ell$-contracting systems, with $A^\ell = \text{trace}(A^\ell)$, are dissipative. For example, the Cantor set $E$ is a dissipative system.

**Remark 2**: We believe that Proposition 4 holds for any matrix measure. However, we have a proof only for the $L_p$ norms, with $p \in \{1, 2, \infty\}$.

### F. Hausdorff Dimension

In this section, we review the Hausdorff dimension of a set, following [20], [38], and [48].

Let $K$ be a set in $\mathbb{R}^n$. For $\epsilon > 0$, an $\epsilon$ cover of $K$ is a cover of $K$ by a countable union of balls, where each ball has a radius smaller than or equal to $\epsilon$. Note that the covering may include balls of different radii, but all are bounded by $\epsilon$. For $d \geq 0$, the $d$-measured volume of $\epsilon$-coverings of $K$ is

$$\zeta(K,\epsilon,d) := \inf \left\{ \sum_i r_i^d : \text{there exists an } \epsilon \text{ cover of } K \right\}.$$  

Note also that if $K$ is compact, then it would suffice to use finite coverings.

By definition, $\zeta(K,\epsilon,d)$ is nonincreasing in $\epsilon$. The Hausdorff $d$-measure of $K$ is

$$m(K,d) := \lim_{\epsilon \downarrow 0} \zeta(K,\epsilon,d)$$

where the limit may be infinite.

For any $s > 0$, we have

$$\zeta(K,\epsilon,d+s) \leq \epsilon^s \zeta(K,\epsilon,d)$$

implying that if $m(K,d) < \infty$, then $m(K,d+s) = 0$ for any $s > 0$. The Hausdorff dimension of $K$ is

$$\dim_H K := \inf \{ d \geq 0 | m(K,d) = 0 \}.$$  

Intuitively, if we try to cover a square (which is a 2-D set) by 1-D balls (i.e., lines), then we need an infinite and uncountable set of lines, but a cover with 2-D balls requires a finite number of balls. Therefore, $\dim_H K$ is exactly the dimension for which the “volume” of $K$ becomes finite.

For smooth shapes, or shapes with a small number of “corners,” the Hausdorff dimension is an integer agreeing with the more standard topological dimension. For example, suppose that $K$ is an $\ell$-dimensional cube in $\mathbb{R}^n$. Intuitively speaking, for any $\epsilon > 0$, we require $\Theta((1/\epsilon)^\ell)$ balls of radius $\epsilon$ to cover $K$. Hence

$$\zeta(K,\epsilon,d) \approx (1/\epsilon)^d \epsilon^d.$$ (30)

As $\epsilon \downarrow 0$, the right-hand side of (30) goes to $\infty$ if $d < \ell$, and to zero if $d > \ell$. It is not difficult to see that using balls with varying sizes does not change the analysis, so

$$\dim_H K = \ell.$$  

However, for fractal sets (e.g., sets that contain strange attractors of chaotic dynamical systems), the Hausdorff dimension is typically not an integer. In this context, the Hausdorff dimension is useful in quantifying sets of Lebesgue measure zero, which are nevertheless “substantial.”

For example, the Cantor set $E \subset [0, 1]$ is defined inductively as follows. Let $E_0 := \{0, 1\}$. For $j \geq 1$, the set $E_j$ is obtained by removing the open middle third in any interval in $E_{j-1}$, so

$$E_1 = [0, 1/3] \cup [2/3, 1]$$

and $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Each Cantor set is $E := \cap_{j=0}^\infty E_j$. Each $E_j$ is the union of $2^j$ intervals of length $3^{-j}$. The topological dimension of $E$ is, thus, $\lim_{j \to \infty} (2^j/3^j) = 0$. It was shown in [56] that $\dim_H E = \log(2)/\log(3) \approx 0.631$. Intuitively, this implies that the Cantor set is less than a line, but more than a discrete set of points.

The next result summarizes some useful properties of $\dim_H$.

**Proposition 5** (see [20, Ch. 3]): The Hausdorff dimension satisfies the following properties.

1) $\dim_H \emptyset = 0$.

2) Monotonicity: If $A, B \subseteq \mathbb{R}^n$ with $A \subseteq B$ then $\dim_H A \leq \dim_H B$.

3) Countable subadditivity: If $A_i \subseteq \mathbb{R}^n$, $i = 1, 2, \ldots$, then

$$\dim_H(\cup_i A_i) = \sup_i \{ \dim_H(A_i) \}.$$  

The next sections describe our main results.

### III. $\alpha$ Compounds

In this section, we define the new notions of the $\alpha$ multiplicative and $\alpha$ additive compounds of a square matrix and analyze the properties of these compounds. In Section IV, we show how this leads to the new notion of $\alpha$-contracting systems, with $\alpha$ real.
A. α Multiplicative Compound

Consider a noninteger real number \( \alpha \in (1, n) \setminus \mathbb{Z} \). From here on, we decompose \( \alpha \) as

\[
\alpha = k + s, \quad k \in \{1, 2, \ldots, n - 1\}, \quad s \in (0, 1).
\]

**Definition 2**: Let \( A \in \mathbb{C}^{n \times n} \) be nonsingular. The \( \alpha \) multiplicative compound matrix of \( A \) is

\[
A^{(\alpha)} := \left( A^{(k)} \right)^{1-s} \otimes \left( A^{(k+1)} \right)^{s}.
\]  

(31)

Note that \( A^{(\alpha)} \in \mathbb{C}^{r \times r} \), where \( r := \binom{n}{k} \binom{n+1}{k+1} \), and that \( A^{(\alpha)} \) may be complex (nonreal) even if \( A \) is real. Since \( A \) is nonsingular, \( A^{(\ell)} \) is nonsingular for all \( \ell \in \{1, \ldots, n\} \), so \( (A^{(k)})^{1-s} \) and \( (A^{(k+1)})^{s} \) in (31) are well defined.

For example, for \( \alpha = 2.5 \), we have \( k = 2 \) and \( s = 1/2 \), so \( A^{(2.5)} = (A^{(2)})^{1/2} \otimes (A^{(3)})^{1/2} \), which can be interpreted as a “multiplicative interpolation,” with equal weights, between \( A^{(2)} \) and \( A^{(3)} \).

**Example 1**: Suppose that \( D = \text{diag}(d_1, \ldots, d_4) \) is nonsingular. Fix \( \alpha \in (2, 3) \), so \( k = 2 \) and \( s = \alpha - 2 \in (0, 1) \). Then

\[
D^{(\alpha)} = (D^{(2)})^{1-s} \otimes (D^{(3)})^{s} = \text{diag}((d_1d_2)^{1-s}, (d_1d_3)^{1-s}, \ldots, (d_3d_4)^{1-s}) \otimes \text{diag}((d_1d_2d_3)^{s}, (d_1d_2d_4)^{s}, (d_1d_3d_4)^{s}, (d_2d_3d_4)^{s})
\]

so any eigenvalue of \( D^{(\alpha)} \) is a “multiplicative interpolation” between eigenvalues of \( D^{(2)} \) and \( D^{(3)} \).

In contrast to (5), it is straightforward to show that the formula

\[
(AB)^{(\alpha)} = A^{(\alpha)}B^{(\alpha)}
\]

does not hold in general. However, it holds in some special cases. Suppose, for example, that both \( A \) and \( B \) are diagonalizable, and that \( A \) commutes with \( B \). Then, the same holds for \( A^{(\ell)} \) and \( B^{(\ell)} \). This implies that \( A^{(\ell)} \) and \( B^{(\ell)} \) are simultaneously diagonalizable. Then, it is easy to show that for any \( s \in (0, 1) \), we have

\[
(A^{(\ell)} B^{(\ell)})^{s} = (A^{(\ell)})^{s} (B^{(\ell)})^{s}.
\]

This implies that \( (AB)^{(\alpha)} = (A^{(\alpha)}B^{(\alpha)}) \).

**Remark 3**: If \( \alpha \) is allowed to be an integer, say, \( \alpha = k \), then \( s = 0 \) and (31) becomes

\[
A^{(\alpha)} = (A^{(k)})^{1} \otimes (A^{(k+1)})^{0} = A^{(k)} \otimes I,
\]

where \( r := \binom{n}{k+1} \). This is not equal to \( A^{(k)} \) (but, ignoring multiplicity, it has the same eigenvalues as \( A^{(k)} \)). Therefore, Definition 2 only considers the case where \( \alpha \) is not an integer. For the integer case, we will just use the standard definition for the \( k \) multiplicative compounds. A similar property holds also for the \( \alpha \) additive compounds defined below.

**Remark 4**: Suppose that \( A \in \mathbb{C}^{n \times n} \) is nonsingular. Using Property (c) in Proposition 2, Property (e) in Lemma 3, and the fact that \( (X^{(\ell)}T) = (X^{T})^{(\ell)} \) for any \( X \in \mathbb{C}^{n \times n} \) and \( \ell \in \{1, \ldots, n\} \) yields

\[
(A^{(\alpha)}T)^{T} = \left( (A^{(k)})^{1-s} \otimes (A^{(k+1)})^{s} \right)^{T} = \left( (A^{(k)})^{1-s} \right)^{T} \otimes \left( (A^{(k+1)})^{s} \right)^{T} = (A^{T})^{(k)}^{1-s} \otimes (A^{T})^{(k+1)}^{s} = (A^{T})^{(\alpha)}.
\]

In particular, if \( A = A^{T} \), then \( A^{(\alpha)} = (A^{(\alpha)})^{T} \).

An alternative possible definition of the \( \alpha \) multiplicative compound matrix is

\[
A^{(\alpha)}_{\text{alt}} := (A^{(1-s)})^{(k)} \otimes (A^{s})^{(k+1)}.
\]

(32)

The next result shows that under a certain condition, (31) and (32) are equivalent.

**Theorem 2**: Consider a nonsingular matrix \( A \in \mathbb{C}^{n \times n} \) and fix \( \alpha \in (1, n) \setminus \mathbb{Z} \). If for any \( k \in \{1, \ldots, n\} \), we have \( A^{(k)} \in \Omega_{r} \), where \( r := \binom{n}{k} \), then

\[
A^{(\alpha)} = A^{(\alpha)}_{\text{alt}}.
\]

(33)

**Proof**: Fix \( k \in \{1, \ldots, n\} \) and \( s \in (0, 1) \). It is enough to show that

\[
(A^{(k)})^{s} = (A^{s})^{k}.
\]

(34)

The proof consists of two steps.

**Step 1**: Consider the case when \( A \) is diagonalizable and \( A^{(k)} \in \Omega_{r} \) for all \( k \). Then, there exists a nonsingular \( T \in \mathbb{C}^{n \times n} \) and a diagonal matrix \( D \in \mathbb{C}^{n \times n} \) such that \( A = TDT^{-1} \), which is also the Jordan canonical form of \( A \). Then, \( (A^{(k)})^{s} = (T^{(k)}D^{(k)})^{(T^{-1})(k)s} \). Using the fact that \( (T^{-1}(k)) = (T^{-1}(k))^{-1} \) and Property (b) in Proposition 2 gives

\[
(A^{(k)})^{s} = T^{(k)}(D^{(k)})^{s}(T^{(k)})^{-1}
\]

Since \( D^{(k)} \) is also diagonal, \( (D^{(k)})^{s} = (D^{s})^{k} \). Hence

\[
(A^{(k)})^{s} = T^{(k)}(D^{s})^{k}(T^{-1}(k))^{k} = (TD^{s}T^{-1})^{k} = (A^{s})^{k}.
\]

We conclude that (34) holds when \( A \) is diagonalizable.

**Step 2**: Pick \( k \in \{1, \ldots, n\} \). The condition \( A^{(k)} \in \Omega_{r} \), where \( r := \binom{n}{k} \), implies that the mapping \( A^{(k)} \rightarrow (A^{(k)})^{s} \) is continuous \([18, \text{Th. 6.2.27}]\). Furthermore, the multiplicative compound of \( A \) is continuous w.r.t. the entries of \( A \). Therefore, the mappings \( A \rightarrow (A^{(k)})^{s} \) and \( A \rightarrow (A^{s})^{k} \) are continuous in this case. Recall that diagonalizable matrices are dense in \( \mathbb{C}^{n \times n} \) (see, e.g., \([23, \text{Corollary 7.3.3}]\)). Thus, there exists a sequence of nonsingular and diagonalizable matrices \( B_{i}, i = 1, 2, \ldots \), such that \( \lim_{i \rightarrow \infty} B_{i} = A \). By Step 1, \( (B_{i}^{(k)})^{s} = (B_{i}^{s})^{k} \) and using continuity yields (34).

The next result describes the spectral properties of \( A^{(\alpha)} \).

**Lemma 1**: Fix a nonsingular matrix \( A \in \mathbb{R}^{n \times n} \) and \( \alpha \in (1, n) \setminus \mathbb{Z} \). Write \( \alpha = k + s \), with \( k \) an integer and \( s \in (0, 1) \).

i) The eigenvalues of \( A^{(\alpha)} \) are

\[
\prod_{i \in Q_{\ell}^{k,n}} \lambda_{i}(A)^{1-s} \prod_{i \in Q_{\ell+1}^{s,n}} \lambda_{i}(A)^{s}
\]

for \( \ell \in \{1, \ldots, \binom{n}{k}\} \), \( j \in \{1, \ldots, \binom{n+1}{k+1}\} \).

ii) The eigenvalues of \( (AT)^{\alpha} \) are

\[
\prod_{i \in Q_{\ell}^{k,n}} (\sigma_{i}(A))^{2(1-s)} \prod_{i \in Q_{\ell+1}^{s,n}} (\sigma_{i}(A))^{2s}
\]

for \( \ell \in \{1, \ldots, \binom{n}{k}\} \), \( j \in \{1, \ldots, \binom{n+1}{k+1}\} \).

**Proof**: It is well known \([35]\) that \( \eta \in \mathbb{C} \) is an eigenvalue of \( A^{(k)} \) iff it is the product of \( k \) eigenvalues of \( A \), that is,
there exists \( t \in \{1, \ldots, \binom{n}{k}\} \) such that \( \eta = \prod_{i \in Q_t} \lambda_i(A) \). By Definition 1
\[
\eta^{1-s} = \left( \prod_{i \in Q_t} \lambda_i(A) \right)^{1-s}
\]
is an eigenvalue of \((A^{(k)})^{1-s}\). Similarly, every eigenvalue of \((A^{(k+1)})^{s}\) has the form
\[
\left( \prod_{i \in Q_{k+1}} \lambda_i(A) \right)^{s}
\]
for some \( j \in \{1, \ldots, \binom{n}{k+1}\} \). Using Property (g) in Lemma 3 proves Property (i).

To prove Property (ii), note that \((A^TA)^{(\alpha)} = ((A^TA)^{(k)})^{1-s} \otimes ((A^TA)^{(k+1)})^{s}\). The eigenvalues of \(A^TA\) are \( ||A||_2^2 = 0, i = 1, \ldots, n\), and using Property (i) yields Property (ii).

Remark 5: Fix a real \( \alpha \geq 1 \), and define \( \omega_n : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}_{\geq 0}\) by
\[
\omega_n(A) := \sigma_1(A) \cdots \sigma_k(A)(\sigma_{k+1}(A))^s.
\]
This function plays a crucial role in the Douady and Oesterlé theorem [8] (see Section IV). Combining Property (ii) of Lemma 1 with the ordering of the eigenvalues and singular values implies that
\[
\lambda_1((A^TA)^{(\alpha)}) = (\sigma_1(A) \cdots \sigma_k(A))^{2(1-s)}
\]
\[
\lambda_2((A^TA)^{(\alpha)}) = (\sigma_1(A) \cdots \sigma_{k+1}(A))^{2s}
\]
\[
= (\omega_n(A))^2.
\]
(35)

Since \(A^TA\) is symmetric, it follows from Remark 4 that so is \((A^TA)^{(\alpha)}\). Hence, \(\lambda_1((A^TA)^{(\alpha)}) = \sigma_1((A^TA)^{(\alpha)}) = ||(A^TA)^{(\alpha)}||_2\), so we conclude that
\[
||(A^TA)^{(\alpha)}||_2 = (\omega_n(A))^2.
\]
(36)

Thus, the \(\alpha\) multiplicative compound provides a matrix norm expression for \(\omega_n\). This was our original motivation for introducing the \(\alpha\) multiplicative compound. Note also that in general
\[
(\sigma_1(A^{(\alpha)}))^2 = \lambda_1((A^{(\alpha)})^T A^{(\alpha)})
\]
\[
\neq \lambda_1((A^TA)^{(\alpha)}).
\]

B. \(\alpha\) Additive Compound

We now turn to define a generalization of the \(k\) additive compound. The definition of the \(\alpha\) additive compound matrix follows (6).

Definition 3: Let \( A \in \mathbb{R}^{n \times n} \) and \( \alpha \in (1, n) \setminus \mathbb{Z} \). The \(\alpha\) additive compound matrix of \( A \) is
\[
A^{(\alpha)} := \frac{d}{d\varepsilon}(I_n + \varepsilon A)^{(\alpha)}|_{\varepsilon=0}.
\]
(37)

Note that for any \( \varepsilon > 0 \) sufficiently small and any \( k \in \{1, 2, \ldots, n\}, (I_n + \varepsilon A)^{(k)} \) is nonsingular and \((I_n + \varepsilon A)^{(k)} \in \mathbb{R}^{r \times r} \cap \Omega_r\), where \( r := \binom{n}{k} \). Hence, Proposition 2 and Definition 2 guarantee that \(A^{(\alpha)}\) is well defined and is a real matrix.

Note also that (37) implies that
\[
(I_n + \varepsilon A)^{(\alpha)} = I_r + \varepsilon A^{(\alpha)} + o(\varepsilon)
\]
where \( r := \binom{n}{k+1} \).

Example 2: Consider \( D = \text{diag}(d_1, \ldots, d_n) \). Fix \( \alpha \in (2, 3)\), so that \( k = 2 \) and \( s = \alpha - 2 \in (0, 1) \). Let \( h_1(\varepsilon) := 1 + \varepsilon d_1 \).

By Example 1
\[
(I_4 + \varepsilon D)^{(\alpha)} = \text{diag}(h_1(\varepsilon)h_2(\varepsilon)h_3^k(\varepsilon), h_1(\varepsilon)h_2(\varepsilon)h_4^k(\varepsilon), \ldots, h_s^k(\varepsilon)h_3(\varepsilon)h_4(\varepsilon))
\]
\[
= I_4 + \varepsilon \text{diag}(d_1 + d_2 + sd_3, d_1 + d_2 + sd_4, \ldots, sd_2 + d_3 + d_4) + o(\varepsilon)
\]
and (37) yields
\[
D^{(\alpha)} = \text{diag}
\]
\[
(d_1 + d_2 + sd_3, d_1 + d_2 + sd_4, \ldots, sd_2 + d_3 + d_4).
\]

Note that every eigenvalue of \(D^{(\alpha)}\) is an “additive interpolation” of three eigenvalues of \(D\).

The next result provides an expression for \(A^{(\alpha)}\) in terms of \(A^{(k)}\) and \(A^{(k+1)}\).

Theorem 3: Fix \( A \in \mathbb{R}^{n \times n} \) and \( \alpha \in (1, n) \setminus \mathbb{Z} \). Then
\[
A^{(\alpha)} = (1 - s)A^{(k)} \oplus (s A^{(k+1)}).
\]
(39)

Note that this also shows that \(A^{(\alpha)}\) is real, as \(A^{(k)}\) is real for any \( k \in \{1, \ldots, n\}\).

To prove this result, we require an auxiliary result.

Lemma 2: Consider a matrix-valued mapping \( A : (a, b) \rightarrow \mathbb{C}^{n \times n} \), where \( a < b \). Assume that \( A(\varepsilon) \) is nonsingular and \( A(\varepsilon) \in \Omega_r \) for all \( \varepsilon \in (a, b) \), and, furthermore, that it has constant (generalized) eigenvectors, that is, it can be written in the Jordan canonical form
\[
A(\varepsilon) = T J(\varepsilon) T^{-1}, \ J(\varepsilon) = \text{diag}(J_1(\varepsilon), J_2(\varepsilon), \ldots, J_p(\varepsilon))
\]
where \( J_i : (a, b) \rightarrow \mathbb{C} \setminus \{0\} \) is \( C^1 \). Then, for any \( \alpha \in \mathbb{R} \) and \( \varepsilon \in (a, b) \),
\[
\frac{d}{d \varepsilon} A^{\alpha}(\varepsilon) = \alpha(A(\varepsilon))^{\alpha-1} \frac{d}{d \varepsilon} A(\varepsilon).
\]
(40)

The proof of Lemma 2 is placed in Appendix C.

Proof of Theorem 3: Consider the case where \( A \) is diagonalizable, that is, \( A = T D T^{-1} \), where \( T \in \mathbb{C}^{n \times n} \) is nonsingular, and \( D \in \mathbb{C}^{n \times n} \) is a diagonal matrix. Let \( B(\varepsilon) := I_n + \varepsilon A \). Fix \( k \in \{1, \ldots, n\} \). Note that \((B^{(\varepsilon)})^{(k)} \in \Omega_r\), \( r := \binom{n}{k}\), for any \( \varepsilon > 0 \) sufficiently small. Then
\[
(B^{(\varepsilon)})^{(k)} = (T (I_n + \varepsilon D) T^{-1})^{(k)}
\]
\[
= T^{(k)} (I_n + \varepsilon D)^{(k)} (T^{-1})^{(k)}.
\]
Since \( D \) is diagonal, so is \((I_n + \varepsilon D)^{(k)}\). Therefore, \((B^{(\varepsilon)})^{(k)}\) satisfies the conditions in Lemma 2. We use Lemma 2 to determine the derivative of \((B^{(\varepsilon)})^{(\alpha)}\) with respect to \( \varepsilon \). To simplify the notation, we write \( B \) for \(B^{(\varepsilon)}\). Then
\[
\frac{d}{d \varepsilon} B^{(\alpha)} = \frac{d}{d \varepsilon} \left( (B^{(k)})^{1-s} \otimes (B^{(k+1)})^{s}\right)
\]
\[
= \left( \frac{d}{d \varepsilon} (B^{(k)})^{1-s} \right) \otimes (B^{(k+1)})^{s}
\]
\[
+ (B^{(k)})^{1-s} \otimes \left( \frac{d}{d \varepsilon} (B^{(k+1)})^{s}\right)
\]
Using (39) gives
\[
\lambda_1(A) = (1 - s) \sum_{i=1}^k \lambda_i(A) + s \sum_{i=1}^{k+1} \lambda_i(A) = \sum_{i=1}^k \lambda_i(A) + s \lambda_{k+1}(A). \tag{41}
\]
It is well known [35] that for any \( A, T \in \mathbb{C}^{n \times n} \) with \( T \) nonsingular, and any integer \( \ell \in \{1, \ldots, n\} \), we have
\[
(TAT^{-1})[\ell] = T[\ell] A[\ell] (T^{-1})[\ell] = T[\ell] A[\ell] (T^{-1})[\ell]. \tag{42}
\]
The next result describes how the \( \alpha \) additive compound behaves under a similarity transformation.
\[ \textbf{Theorem 5}: \text{Let } A, T \in \mathbb{C}^{n \times n}, \text{ with } T \text{ nonsingular, and pick } \alpha \in (1, n) \backslash \mathbb{Z}. \text{ Then}
\]
\[
(TAT^{-1})[\alpha] = (T[k] \otimes T[k+1]) A[\alpha] (T[k] \otimes T[k+1])^{-1}. \tag{43}
\]
\[ \textbf{Proof}: \text{Let } B := TAT^{-1}. \text{ Using (39) and (42) yields}
\]
\[
\lambda_1(B[A] = (1 - s) (T[k] \otimes T[k+1]) (A[k] \otimes I_{r_1}) + s (T[k] \otimes T[k+1]) (I_{r_2} \otimes A[k+1]) + s I_{r_2} \otimes A[k+1].
\]
and this completes the proof.

\[ \textbf{Remark 7}: \text{If } T \in \mathbb{R}^{n \times n} \text{ is nonsingular, then } T^{(k+1/2)} = (T[k])^{1/2} \otimes (T[k+1/2])^{1/2}, \text{ and thus}
\]
\[
(T[k+1/2])^2 = (T[k])^{1/2} \otimes (T[k+1/2])^{1/2} \otimes (T[k+1/2])^{1/2} = (T[k])^{1/2} \otimes (T[k+1/2])^{1/2}.
\]
Therefore, (43) can be rewritten in the more compact form
\[
(TAT^{-1})[\alpha] = (T[k+1/2])^2 A[\alpha] (T[k+1/2])^{-2}. \tag{44}
\]

\[ \textbf{C. Matrix Measures of the } \alpha \text{ Additive Compound}
\]
Consider a diagonal matrix \( D = \text{diag}(d_1, \ldots, d_n) \in \mathbb{C}^{n \times n} \). It is well known [5] that if \( \| \cdot \| \) is a monotonic norm, then the
induced matrix norm satisfies \( \|D\| = \max(|d_1|, \ldots, |d_n|) \). This implies that the induced matrix measure satisfies
\[
\mu(D) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(|I_n + \varepsilon D| - 1)
\]
\[
= \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( \max \{ |1 + \varepsilon d_i| \} - 1 \right)
\]
\[
= \max \{ \Re(d_i) \}. 
\] (44)

Therefore, for any \( \ell \in \{1, \ldots, n\} \), we have
\[
\mu(D[\ell]) = \max_{\{i_1, \ldots, i_{\ell-1}\} \subseteq Q^{\ell,n}} \left( \sum_{p=1}^n \Re(d_{i_p}) \right)
\] (45)

and combining this with (39) yields
\[
\mu(D^{[\alpha]}((1 - s) \mu(D^{[k]})) = (1 - s) \mu(D^{[k]}) + s \mu(D^{[k+1]}). 
\] (46)

The next result shows that this useful expression holds for general matrices.

**Theorem 6:** Let \( \mu_p \) denote a matrix measure induced by some \( L_p \) norm with \( p \in [1, \infty] \). For any \( A \in \mathbb{R}^{n \times n} \) and any \( \alpha \in (0, 1) \), we have
\[
\mu_p(A^{[\alpha]}) = (1 - s) \mu_p(A^{[k]}) + s \mu_p(A^{[k+1]}). 
\] (47)

**Proof:** From Proposition 3, (26) holds for all \( L_p \) norms. Since \( A^{[\alpha]} = (1 - s)A^{[k]} + s A^{[k+1]} \), Theorem 1 yields
\[
\mu_p(A^{[\alpha]}) = \mu_p((1 - s)A^{[k]} + s A^{[k+1]})
\]
\[
= (1 - s)\mu_p(A^{[k]}) + s \mu_p(A^{[k+1]})
\]
the where last equality follows from the homogeneity of the matrix measure and the fact that \( s \in (0, 1) \). \( \Box \)

The next example demonstrates Theorem 6 in the case \( n = 2 \).

**Example 3:** Let \( A \in \mathbb{R}^{2 \times 2} \). Fix \( \alpha \in (1, 2) \). Then, \( \alpha = k + s \), with \( k = 1 \) and \( s \in (0, 1) \), so
\[
A^{[\alpha]} = ((1 - s)A^{[1]} + s A^{[2]})
\]
\[
= ((1 - s)A^{[1]} + s A^{[2]})
\]
\[
= ((1 - s)A^{[1]} + s A^{[2]})
\]
where for any matrix measure \( \mu \) and any \( c \in \mathbb{R} \), \( \mu(A + c I_2) = \mu(A) + c \) (see, e.g., [7]). Thus
\[
\mu(A^{[\alpha]}) = \mu((1 - s)A^{[1]} + s A^{[2]})
\]
\[
= (1 - s)\mu(A^{[1]}) + s \mu(A^{[2]}).
\]
Note that for this particular example, (47) holds for all matrix measures. \( \Box \)

**IV. APPLICATION: \( \alpha \)-CONTRACTING SYSTEMS**

We now describe an application of the \( \alpha \) compounds in the context of the Douady and Oesterlé theorem [8]. For a modern treatinent of this theorem and its numerous extensions and applications, see the recent monograph [20]. Some connections between contracting systems and the Douady and Oesterlé theorem have already appeared in the note [28].

In this section, \( \alpha \in [1, n] \), and the special case where \( \alpha \) is an integer is also allowed. The standard \( k \) compound matrices will be used if \( \alpha \) is an integer. The Hausdorff dimension of a set \( K \subset \mathbb{R}^n \) is denoted by \( \dim_H K \). Let \( D \subset \mathbb{R}^n \) be an open set, and let \( g : D \to \mathbb{R}^n \) be a \( C^1 \) mapping. Let
\[
J_g(x) := \frac{\partial}{\partial x} g(x).
\]

A set \( K \subset D \) is said to be negatively invariant under \( g \) if \( K \subset g(K) \). Intuitively speaking, \( g \) “increases” \( K \). The next result is the Douady and Oesterlé theorem [8], stated as in [48].

**Theorem 7:** Suppose that \( K \subset D \) is compact and negatively invariant under \( g \). Fix \( \alpha \in [1, n] \), and write \( \alpha = k + s \), with \( k \in \{1, 2, \ldots, n - 1\} \) and \( s \in [0, 1) \). Let \( \sigma_i(J_g(x)) \) denote the singular values of \( J_g(x) \) ordered as in (4), and let
\[
\omega(K, \alpha, g) := \max_{x \in K} (\sigma_1(J_g(x)) \cdots \sigma_k(J_g(x)) \sigma_{k+1}(J_g(x)))^s.
\]
If
\[
\omega(K, \alpha, g) < 1.
\] (48)

Then, \( \dim_H K < \alpha \).

Intuitively speaking, condition (48) implies that the mapping \( g \) is “contracting in dimension \( \alpha \),” uniformly in \( K \), that is, if \( g \) “increases” \( K \), then necessarily \( \dim_H K < \alpha \). In particular, if \( \dim_H K \geq \alpha \), then \( K \not\subset g(K) \).

The next example illustrates Theorem 7.

**Example 4:** Let \( g : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear mapping given by \( g(x) = \text{diag}(1, 1/2, 1/4) x \). The singular values of the Jacobian of \( g \) are 1, 1/2, 1/4, and (48) holds for any \( \alpha > 1 \). Thus, Theorem 7 implies that for any compact set \( K \subset \mathbb{R}^3 \) such that \( K \subset g(K) \), we have \( \dim_H K \leq 1 \). For example, the set \( S := [0, 1] \times \{0\} \times \{0\} \) satisfies \( K \subset g(S) \) and \( \dim_H K = 1 \).

Using the \( \alpha \) multiplicative compound, we can express Theorem 7 in a more elegant form. Indeed, it follows from (36) that
\[
(\omega(K, \alpha, g))^2 = \max_{x \in K} \| (J_g^T(x)J_g(x))^{(\alpha)} \|_2
\]
so the condition in Theorem 7 becomes
\[
\max_{x \in K} \| (J_g^T(x)J_g(x))^{(\alpha)} \|_2 < 1.
\]
This provides a more intuitive description for “contracting in dimension \( \alpha \)” of a mapping \( g \).

Theorem 7 has been used to upper bound the Hausdorff dimension of invariant sets (and, in particular, attractors) of dynamical systems. Our results allow us to restate and generalize these results in a more intuitive fashion using the \( \alpha \) additive compound.

Consider the time-varying dynamical system (12). Let \( x(t, t_0, x_0) \) denote the solution of (12) at time \( t \) with \( x(t_0) = x_0 \). We assume from here on that \( t_0 = 0 \), and let \( x(t, x_0) := x(t, 0, x_0) \). We also assume that the dynamics admits an invariant set \( D \subset \mathbb{R}^n \), that is, for any \( x_0 \in D \), we have \( x(t, x_0) \in D \) for all \( t \geq 0 \). Let \( J_f(t, x) := \frac{\partial}{\partial x} f(t, x) \), and consider the matrix differential equation
\[
X(t) = J_f(x(t, t_0), x_0)(X(t), X(0) = X_0.
\]
We begin with an auxiliary result.

**Proposition 6:** Let \( K \subset \mathbb{R}^n \) be a compact invariant set of (12). Fix \( \alpha \in [1, n] \) and let \( \alpha = k + s \), with \( k \in \mathbb{R} \).
\{1, 2, \ldots, n-1\}$ and $s \in [0, 1)$. For a matrix measure $\mu$ associated with an $L_p$ norm $\| \cdot \|$ with $p \in [1, \infty]$ and $t \geq 0$, let
\[
\gamma_{J_f}(t) := \max_{x_0 \in \mathcal{K}} \int_0^t \mu(J_f^{[0]}(x(\tau, x_0))) \, d\tau.
\]
(49)

Then
\[
\|X^{(k)}(t)\|^{1-s} \|X^{(k+1)}(t)\|^s 
\leq \exp(\gamma_{J_f}(t)) \|X_0^{(k)}\|^{1-s} \|X_0^{(k+1)}\|^s
\]
for any $X_0$.

Proof: Pick $\ell \in \{1, \ldots, n\}$. Since $\frac{d}{dt} X^{(\ell)} = J_f^{[\ell]} X^{(\ell)}$,
\[
\|X^{(\ell)}(t)\| \leq \exp(\int_0^t \mu(J_f^{[\ell]}(\tau)) \, d\tau) \|X^{(\ell)}(0)\|.
\]
Applying this bound to $\|X^{(k)}(t)\|^{1-s} \|X^{(k+1)}(t)\|^s$, and using (47) completes the proof.

Note that if $\mu(J_f^{[0]}(x)) \leq -\eta < 0$ for all $x \in K$, then
Proposition 6 implies that
\[
\|X^{(k)}(t)\|^{1-s} \|X^{(k+1)}(t)\|^s 
\leq \exp(-\eta t) \|X_0^{(k)}\|^{1-s} \|X_0^{(k+1)}\|^s
\]
for all $t \geq 0$.

A set $K \subseteq D$ is called a strongly invariant set of (12) if for any $t \geq 0$, we have
\[
K = \{x(t, a) \mid a \in K\}. \quad (50)
\]

For example, an equilibrium or a limit cycle are strongly invariant sets.

We can now bound the Hausdorff dimension of strongly invariant sets of (12), thus extending a result in [48]. For generality, contraction theory typically uses contraction metrics [26] and associated scaled norms. Consider a $C^1$ scaling matrix $\Theta : K \rightarrow \mathbb{R}^{n \times n}$ satisfying
\[
\det(\Theta(z)) \neq 0 \text{ for all } z \in K. \quad (51)
\]
Let $\Theta_f(z)$ denote the matrix obtained by replacing every entry $\theta_{ij}(z)$ in $\Theta(z)$ by the value $(\frac{\partial \theta_{ij}(z)}{\partial x})_{\bar{z}} f(z)$ and define the so-called generalized Jacobian [26] as
\[
\bar{J} := \Theta_f \Theta^{-1} + J_f \Theta^{-1}.
\]

Note that if $\Theta(z) = I_n$, for all $z$, then $\bar{J} = J_f$. The next result bounds the Hausdorff dimension of a strongly invariant set using the generalized Jacobian $\bar{J}$.

**Theorem 8:** Let $K \subseteq \mathbb{R}^n$ be a compact and strongly invariant set of the time-varying system (12). Fix $\alpha \in [1, n]$ and let $\alpha = k + s$, with $k \in \{1, 2, \ldots, n-1\}$ and $s \in [0, 1)$. Assume that there exists a matrix measure $\mu$ associated with an $L_p$ norm $\| \cdot \|$, with $p \in [1, \infty]$, and a time $\tau > 0$ such that
\[
\gamma_{J_f}(\tau) := \max_{x_0 \in \mathcal{K}} \int_0^\tau \mu(J^{[\alpha]}(x(\tau, x_0))) \, d\tau
\]
\[
< 0. \quad (52)
\]

Then, $\dim_H K < \alpha$.

Proof: Define $g : \mathbb{R}_{\geq 0} \times K \rightarrow K$ by $g(t, x_0) := x(t, x_0)$.

Then, $J_g(t, x_0) := \frac{\partial}{\partial x_0} g(t, x_0)$. Let
\[
Y(t, x_0) := \Theta(x(t, x_0)) \frac{\partial}{\partial x_0} g(t, x_0).
\]
(53)

To simplify the notation, we sometimes write $\Theta(x)$ or $\Theta(t)$ for $\Theta(x(t, x_0))$. By (53)
\[
\dot{Y} = \Theta \frac{\partial}{\partial x_0} g + \Theta \frac{\partial}{\partial x_0} f(t, x) \frac{\partial}{\partial x_0} g
\]
\[
= (\Theta_f \Theta^{-1} + \Theta J_f \Theta^{-1}) Y.
\]

Thus, $Y(t, x_0)$ is the solution at time $t$ of the matrix differential equation $\dot{Y} = Y J_f$, initialized with $Y(0) = \Theta(x_0)$. Let $c(x_0) := \|\Theta(x_0)\|^{1-s} \|\Theta(x_0)\|^s$. Proposition 6 and (52) imply that
\[
\|Y^{(k)}(\tau)\|^{1-s} \|Y^{(k+1)}(\tau)\|^s \leq c(x_0) \exp(\gamma_{J_f}(\tau)),
\]
for any $x_0 \in K$. Hence, for any integer $\ell \geq 1$,
\[
\|\Theta^{(k)}(\tau)J_f^{(k)}(\tau)\|^{1-s} \|\Theta^{(k+1)}(\tau)J_f^{(k+1)}(\tau)\|^s 
\leq c(x_0) \exp(\gamma_{J_f}(\tau)). \quad (54)
\]

Recall that if $\| \cdot \| : \mathbb{C}^{n \times n} \rightarrow \mathbb{R}_{>0}$ is an induced matrix norm and $P \in \mathbb{C}^{n \times n}$ is nonsingular, then the $P$-weighted induced matrix norm is $\|M\|_P := \|PM P^{-1}\|$. Equation (54) yields
\[
\|J_f^{(k)}(\tau)\|^{1-s} \|J_f^{(k+1)}(\tau)\|^s 
\]
\[
\leq c(x_0) \exp(\gamma_{J_f}(\tau))^s \|\Theta^{(k)}(\tau)\|^{1-s} \|\Theta^{(k+1)}(\tau)\|^{-s}.
\]

Since $K$ is compact, we can make the right-hand side of this equation arbitrarily small by taking $\ell$ large enough. Using the equivalence of norms implies that there exists an integer $\ell$ such that $\|J_f^{(k)}(\tau)\|^{s} \|J_f^{(k+1)}(\tau)\| \leq \ell < 1$. Let $\sigma_i$, $i = 1, \ldots, n$, denote the singular values of $J_f(\tau)$. Then, we conclude that
\[
\sigma_1 \cdots \sigma_k \sigma_{k+1} < 1.
\]

Since $g(\bar{\ell}, K) = K$, Theorem 7 implies that $\dim_H K < \alpha$.

**Remark 8:** In Theorem 7, Proposition 6, and Theorem 8, we consider $\alpha \in [1, n]$, as we assume that $\alpha = k + s$, $k \in \{1, 2, \ldots, n-1\}$, and $s \in [0, 1)$. However, the case $\alpha = n$ is also allowed. For example, $\omega(K, \alpha, g)$ in Theorem 7 with $\alpha = n$ reduces to
\[
\omega(K, \alpha, g) = \max_{x \in K} (\sigma_1 (J_g(x)) \cdots \sigma_n (J_g(x)))^s. \quad (55)
\]

From now on, we consider for simplicity the nonscaled case, i.e., $\bar{J} = J_f$. Of course, a sufficient condition for (52) to hold is that $\mu(J^{[\alpha]}(x)) < 0$ for all $x \in K$, which naturally leads to the following new definition.

**Definition 4:** Let $\mu$ be a matrix measure induced by an $L_p$ norm, with $p \in [1, \infty]$. Suppose that the trajectories of (12) evolve on a state space $D$. Pick a real $\alpha \in [1, n]$. System (12) is called (infinitesimally) $\alpha$-contracting w.r.t. the norm $\| \cdot \|_P$ if
\[
\mu_p(J_g^{[\alpha]}(t, x)) \leq -\eta < 0 \text{ for all } t \geq 0, \quad x \in D. \quad (55)
\]

**Remark 9:** If $\alpha$ in Definition 3 is allowed to be an integer, i.e., $\alpha = k + s$, with $s = 0$, then
\[
A^{[\alpha]} = \left( (1-s) A^{[k]} \right) \oplus \left( s A^{[k+1]} \right)
\]
\[
= A^{[k]} \oplus 0
\]
and Proposition 3 and Theorem 1 imply that
\[
\mu_p(A^{[\alpha]}) = \mu_p(A^{[k]}).
\]
Thus, we recover the $L_p$ matrix measure of the $k$ additive compound.

**Remark 10:** An important property of contracting systems is that various compositions of contracting systems yield a contracting system [26, 28, 41, 46]. The subadditivity of the matrix measure and Property (a) in Theorem 4 suggest that this remains valid for interconnections of $\alpha$-contracting systems. As a simple example, consider the interconnected system

$$
\dot{x}(t) = c_1(t)f(t, x) + c_2(t)g(t, x)
$$

(56)

with $c_1(t) \geq 0$ for any $t \geq 0$. The Jacobian of this system is $c_1 J_f + c_2 J_g$, and

$$
\mu \left( c_1 J_f + c_2 J_g \right) = \mu \left( c_1 J_f^{[\alpha]} + c_2 J_g^{[\alpha]} \right) 
\leq c_1 \mu \left( J_f^{[\alpha]} \right) + c_2 \mu \left( J_g^{[\alpha]} \right)
$$

Thus, it is straightforward to provide sufficient conditions for $\alpha$-contraction of (56) in terms of the sub-systems $\dot{x}(t) = f(t, x)$ and $\dot{x}(t) = g(t, x)$.

The next result follows immediately from Theorems 7 and 8.

**Corollary 1:** Suppose that (12) is $\alpha$-contracting. Then, any compact and strongly invariant set has Hausdorff dimension smaller than $\alpha$.

**Example 5:** Consider the system $\dot{x}(t) = A(t)x(t)$, with $A(t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -t \end{pmatrix}$. Take $\alpha = 2 + s$ with $s \in (0, 1)$. By Proposition 1, $A^{[2]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -t & -1 \\ 0 & 1 & -t \end{pmatrix}$ and $A^{[3]} = -t$. Hence, the $\alpha$-additive compound is

$$
A^{[\alpha]} = ((1-s)A^{[2]} + (sA^{[3]}) = \begin{pmatrix} -st & 0 & 0 \\ 0 & -t & s-1 \\ 0 & 1-s & -t \end{pmatrix}
$$

Note that $\mu_2(A^{[2]}) = 0$, and $\mu_2(A^{[3]} = -st < 0$ for any $t > 0$. That is, this system is $(2 + s)$-contracting for any $s \in (0, 1)$. Theorem 8, thus, guarantees that any compact and strongly invariant set $K$ satisfies $\dim_H K < 2 + s$. Since $s \in (0, 1)$ can be arbitrarily small

$$
\dim_H K \leq 2.
$$

(57)

For example, the set $K := \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq c, x_3 = 0 \}$ with any $c \geq 0$ is compact, strongly invariant, and satisfies (57). □

The next result shows that if the system is $\alpha$-contracting w.r.t. $| \cdot |_p$, for some $p \in \{1, 2, \infty\}$, then it is also $\alpha$-contracting w.r.t. the same norm for any $\alpha > \alpha$.

**Theorem 9:** Consider the system (12). Suppose that condition (55) holds for some matrix measure $\mu$ induced by an $L_p$ norm, with $p = \{1, 2, \infty\}$, and some $\alpha \in (1, n)$. Then, (12) is $\beta$-contracting for any $\beta \in (\alpha, n]$.

**Proof:** Consider first the case where $\alpha$ is an integer, that is, $\alpha = k \in \{1, \ldots, n-1\}$. Then, (55) becomes

$$
\mu_p(J_f^{[k]}(t, x)) \leq -\eta < 0 \text{ for all } t \geq 0, x \in D.
$$

Fix arbitrary $x \in D$ and $t \geq 0$. To simplify the notation, we write $J_f$ for $J_f(t, x)$. Proposition 4 ensures that $\mu_p(J_f^{[k+1]}(x)) \leq \mu_p(J_f^{[k]}(x)) + \varepsilon \mu_p(J_f^{[k+1]}(x)) \leq (1-\varepsilon)\mu_p(J_f^{[k]}) + \varepsilon \mu_p(J_f^{[k+1]})$.

$$
\mu_p(J_f^{[k]}(x)) \leq (1-\varepsilon)\mu_p(J_f^{[k]}) + \varepsilon \mu_p(J_f^{[k+1]})
$$

(58)

Indeed, if $\mu_p(J_f^{[k]}) \geq 0$, then $\mu_p(J_f^{[k+1]}) \leq -\eta/s < 0$, so (58) holds, and if $\mu_p(J_f^{[k]}) < 0$, then (58) follows from Proposition 4. Hence, for any $\varepsilon \in (0, 1-s)$, we have

$$
\mu_p(J_f^{[\alpha]}(x)) = (1-s-\varepsilon)\mu_p(J_f^{[k]}) + (s+\varepsilon)\mu_p(J_f^{[k+1]})
$$

$$
= \mu_p(J_f^{[\alpha]}) - \varepsilon \left( \mu_p(J_f^{[k]}) - \mu_p(J_f^{[k+1]}) \right)
$$

$$
\leq \mu_p(J_f^{[\alpha]})
$$

and this completes the proof. □

Theorem 9 implies the following result.

**Corollary 2:** Consider the dynamical system (12). Suppose that condition (55) holds for a matrix measure $\mu$ induced by an $L_p$ norm, with $p = \{1, 2, \infty\}$, and some $\alpha \in (1, n)$. Then, there exists a minimal real value $\alpha^* \in [1, \infty)$ such that (12) is $\beta$-contracting for any $\beta > \alpha^*$.

In other words, contraction is not a binary property, but rather any dissipative system is located on a continuous axis of contraction level. It is important to note that the value $\alpha^*$ depends on the norm that induces the matrix measure. This is also true of standard contraction, where the analysis of contraction critically depends on using the “right” norm.

Several recent papers considered systems that are, in some sense, on “the verge of contraction” [19], [26], [29], [32], [34], [49]. Such systems are referred to as semi-contracting [26], [54], or sometimes weakly contracting [19] (note that this terminology is used instead for $k$-contraction in [28]). Since 1-contraction corresponds to contracting systems, we can expect semi-contracting systems to be $\alpha$-contracting for some $\alpha > 1$.

The next result formalizes the notion of a system “on the verge of contraction” in terms of $\alpha$-contraction and extends a result of [54]. We consider a time-invariant dynamical system

$$
\dot{x} = f(x)
$$

(59)

with $f \in C^1$. Let $J_f(x) := \frac{df}{dx}(x)$.

Recall that a set $E$ is called *pathwise connected* [55, Def. 27.1] if for any two points $a, b \in E$, there exists a continuous function $g : [0, 1] \to E$ such that $g(0) = a$ and $g(1) = b$.
**Theorem 10:** Fix $p \in \{1, 2, \infty\}$. Suppose that there exists $s^* \in (0, 1)$ such that the system (59) is $(1 + s)$-contracting w.r.t. the norm $|| \cdot ||_p$ for any $s \in (0, s^*)$, i.e.,

$$
\mu_p(J_f^{1+s}(x)) < 0 \text{ for all } x \in \mathbb{R}^n.
$$

(60)

Then, for any $a, b \in \mathbb{R}^n$, we have

$$
|x(t, a) - x(t, b)|_p \leq |a - b|_p \text{ for all } t \geq 0.
$$

(61)

If (59) has at least one bounded trajectory then the following properties hold:

a) All the trajectories of (59) are bounded.

b) Every trajectory of (59) converges to an equilibrium point.

c) If (59) has multiple equilibrium points, then the equilibrium set is pathwise connected.

**Proof:** We claim that

$$
\mu_p(J_f(x)) \leq 0 \text{ for all } x \in \mathbb{R}^n.
$$

(62)

Indeed, if this is not true, then there exists $a \in \mathbb{R}^n$ such that $\mu_p(J_f(a)) > 0$. Then

$$
\mu_p(J_f^{1+s}(a)) = (1 - s)\mu_p(J_f(a)) + s\mu_p(J_f^2(a)) > 0
$$

for any sufficiently small $s > 0$, and this contradicts (60). We conclude that (62) holds, and this implies (61).

Now, let $x_0 \in \mathbb{R}^n$ be such that $x(t, x_0)$ is bounded for all $t \geq 0$. Combining this with (61) proves (a). Since the system is $(1 + s)$-contracting, Theorem 9 implies that the system is 2-contracting. Thm. 2.5 in [24] implies that every nonempty omega limit set of the dynamics is a single equilibrium. Since all the trajectories are bounded, this proves (b).

Let $E$ denote the set of equilibrium points of (59). Assume that $e^0, e^1 \in E$, and let $\gamma(r) := (1 - r)e^0 + re^1$, with $r \in [0, 1]$. We already know that $x(t, \gamma(r))$ converges to an equilibrium. Let $e^* := \lim_{t \to \infty} x(t, \gamma(r))$. Pick $r_1, r_2 \in [0, 1]$. Using (61) yields

$$
|x(t, \gamma(r_1)) - x(t, \gamma(r_2))|_p \leq |\gamma(r_1) - \gamma(r_2)|_p = |(r_1 - r_2)(e^0 - e^1)|_p
$$

and taking $t \to \infty$ gives

$$
|e^{r_1} - e^{r_2}|_p \leq |r_1 - r_2||e^0 - e^1|_p.
$$

(63)

Define $g : [0, 1] \to E$ by $g(r) := e^r$. Then, $g(0) = e^0, g(1) = e^1$, and (63) implies that $g$ is (Lipschitz) continuous. We conclude that the set of equilibrium points $E$ is pathwise connected.

We demonstrate Theorem 10 in the important example of studying synchronization using contraction theory [37], [52] and in analyzing a model from systems biology called the ribosome flow model on a ring (RFMR) [39], [60].

**Example 6:** Consider the linear time-invariant system

$$
\dot{x} = -Lx
$$

(64)

where $L$ is the Laplacian of a (directed or undirected) weighted graph with a globally reachable vertex. We claim that (64) is not 1-contracting w.r.t. any norm. Yet, for any $c \in (0, 1)$, there exists a vector norm $|| \cdot ||$ such that (64) is $(1 + c)$-contracting w.r.t. to $|| \cdot ||$.

Indeed, for any $c \in \mathbb{R}$, we have that $c1_n$ is an equilibrium of (64), so the system cannot be 1-contracting w.r.t. any norm.

On the other hand, the eigenvalues $\lambda_i(A)$, ordered as in (3), satisfy $\lambda_1 = 0$ and $\text{Re}(\lambda_2) < 0$. Fix $\varepsilon \in (0, 1)$. By (41)

$$
\text{Re}(\lambda_1(A^{1+\varepsilon})) = \text{Re}(\lambda_1(A) + \varepsilon\lambda_2(A)) = \varepsilon \text{Re}(\lambda_2(A)) < 0
$$

so $A^{1+\varepsilon}$ is Hurwitz, and it is well known [2], [25] that there exists a matrix measure $\mu$, induced by a weighted $L_2$ norm, such that $\mu(A^{1+\varepsilon}) = \varepsilon \text{Re}(\lambda_2(A))$. By Theorems 5 and 10, we conclude that all trajectories are bounded, every trajectory converges to an equilibrium point, and all equilibrium points are pathwise connected.

**Example 7 (RFMR with dimension $n = 3$):** Consider the nonlinear system

$$
\begin{align*}
\dot{x_1} &= \beta_3x_3(1 - x_1) - \beta_1x_1(1 - x_2) \\
\dot{x_2} &= \beta_1x_1(1 - x_2) - \beta_2x_2(1 - x_3) \\
\dot{x_3} &= \beta_2x_2(1 - x_3) - \beta_3x_3(1 - x_1)
\end{align*}
$$

(65)

where $\beta_1, \beta_2, \text{ and } \beta_3$ are positive constants. This model has been used to study the flow of ribosomes along a circular mRNA molecule during the process of translation [39], [60]. In this context, $x_i(t)$ is the density of ribosomes at time $t$ in the $i$th site along the mRNA, normalized such that $x_i(t) = 0$ [$x_i(t) = 1$] corresponds to site $i$ being empty [completely full].

The trajectories of (65) evolve on the closed unit cube

$$
D := \{ x \in \mathbb{R}^3 \mid x_i \in [0, 1], i = 1, 2, 3 \}.
$$

Therefore, $\text{int}(D)$, the interior of $D$, is also positively invariant. The equilibrium set of (65) is

$$
E := \{ e \in D \mid |\beta_1e_1(1 - e_2) - \beta_2e_2(1 - e_3) - \beta_3e_3(1 - e_1)| = 0 \}.
$$

This is not a singleton, and thus, the RFMR is not 1-contracting w.r.t. any norm.

The Jacobian of (65) is

$$
J_f(x) =
\begin{bmatrix}
-\beta_1(1 - x_2) & \beta_1x_1 & \beta_3(1 - x_1) \\
\beta_1(1 - x_2) & -\beta_2(1 - x_3) & \beta_2x_2 \\
\beta_3x_3 & \beta_2(1 - x_3) & -\beta_3(1 - x_1)
\end{bmatrix}
- \text{diag}(\beta_3x_3, \beta_1x_1, \beta_2x_2).
$$

Note that $J_f$ is Metzler for any $x \in D$, and that the column sum of each column is zero. Let $\mu_1$ denote the matrix measure associated with the $L_1$ norm. It is straightforward to verify that

$$
\mu_1(J_f(x)) = 0, \text{ for all } x \in D
$$

$$
\mu_1(J_f^2(x)) < 0, \text{ for all } x \in \text{int}(D).
$$

By Theorem 6, we have

$$
\mu_1(J_f^{1+s}(x)) < 0, \text{ for all } s \in (0, 1), x \in \text{int}(D).
$$

That is, this system is $(1 + s)$-contracting on $\text{int}(D)$ w.r.t. $|| \cdot ||$. Theorem 10 implies that every trajectory emanating from $\text{int}(D)$ converges to $E$, which is indeed pathwise connected. □
V. Designing a Controller for a Chaotic System

This section demonstrates an application of our theoretical results to the control of a chaotic system. A popular chaotic system, introduced by Thomas [51] (see also the recent review [4]), is the cyclically symmetric attractor

\[
\begin{align*}
\dot{x}_1 &= \sin(x_2) - bx_1 \\
\dot{x}_2 &= \sin(x_3) - bx_2 \\
\dot{x}_3 &= \sin(x_1) - bx_3
\end{align*}
\]  

(66)

where \( b > 0 \) is the dissipation constant. Note that the convex set \( D := \{ x \in \mathbb{R}^3 : |x|_\infty \leq 1 \} \) is an invariant set of the dynamics.

This system undergoes a series of bifurcations as \( b \) decreases. For \( b > 1 \), the origin is the single stable equilibrium. When \( b = 1 \), it undergoes a pitchfork bifurcation, splitting into two attractive fixed points. As \( b \) is decreased further to \( b \approx 0.32899 \), these undergo a Hopf bifurcation, creating a stable limit cycle. The limit cycle undergoes a period doubling cascade and becomes chaotic at \( b \approx 0.208186 \).

Fig. 2 depicts the solution of the system emanating from \([-1 \ 1 \ 1]^T\) for

\[
b = 0.193186.
\]  

(67)

Note the symmetric strange attractor.

Let \( f \) denote the vector field in (66). The Jacobian is

\[
J_f(x) = \begin{bmatrix}
-b & \cos(x_2) & 0 \\
0 & -b & \cos(x_3) \\
\cos(x_1) & 0 & -b
\end{bmatrix}
\]

and thus

\[
J_f^{[2]}(x) = \begin{bmatrix}
-2b & \cos(x_2) & 0 \\
0 & -2b & \cos(x_3) \\
-\cos(x_1) & 0 & -2b
\end{bmatrix}
\]

and \( J_f^{[3]} = \text{trace}(J_f(x)) = -3b \). This implies that the system is 3-contracting (that is, dissipative, w.r.t. any norm, for any \( b > 0 \).)

Let \( \alpha = 2 + s \), with \( s \in (0, 1) \). Then

\[
J_f^{[\alpha]}(x) = (1-s)J_f^{[2]}(x) + sJ_f^{[3]}(x)
\]

\[
= \begin{bmatrix}
-(2+s)b & (1-s)\cos(x_3) & 0 \\
0 & -(2+s)b & (1-s)\cos(x_2) \\
-(1-s)\cos(x_1) & 0 & -(2+s)b
\end{bmatrix}.
\]

This implies that

\[
\mu_1(J_f^{[\alpha]}(x)) \leq 1 - 2b - s(b+1) \text{ for all } x \in D.
\]

We conclude that for any \( b \in (0,1/2) \), the system is \((2+s)\)-contracting for any \( s > \frac{1-2b}{1+b} \).

We now demonstrate how our results can be applied to design a partial-state controller for the system guaranteeing that the closed-loop system has a “well-ordered” behavior. Suppose that the closed-loop system is

\[
\dot{x} = f(x) + g(x)
\]

where \( g \) is the controller. Let \( \alpha = 2 + s \), with \( s \in (0, 1) \). The Jacobian of the closed-loop system is \( J_{cl} := J_f + J_g \), so

\[
\mu_1(J_{cl}^{[\alpha]}(x)) = \mu_1(J_f^{[\alpha]} + J_g^{[\alpha]})
\]

\[
\leq \mu_1(J_f^{[\alpha]}) + \mu_1(J_g^{[\alpha]})
\]

\[
\leq 1 - 2b - s(b+1) + \mu_1(J_g^{[\alpha]}).
\]

This implies that the closed-loop system is \( \alpha \)-contracting if

\[
\mu_1(J_g^{[\alpha]}(x)) < s(b+1) + 2b - 1 \text{ for all } x \in D.
\]  

(68)

Consider, for example, the controller

\[
g(x_1, x_2) = \text{diag}(c, c, 0)x, \text{ with } c < 0.
\]

Then

\[
J_g^{[\alpha]} = c \text{diag}(2, 1+s, 1+s)
\]

and for any \( c < 0 \), condition (68) becomes

\[
(1+s)c < s(b+1) + 2b - 1.
\]  

(69)

This provides a simple recipe for determining the gain \( c \) so that the closed-loop system is \((2+s)\)-contracting. For example, when \( s \to 0 \), (69) yields

\[
c < 2b - 1
\]

and this guarantees that the closed-loop system is 2-contracting. Recall that in a 2-contracting system, every nonempty omega limit set is a single equilibrium [24, Th. 2.5], thus ruling out chaotic attractors and even nontrivial limit cycles [24].

Fig. 3 depicts the behavior of the closed-loop system with \( b \) as in (67) and \( c = 2b - 1.1 \). The closed-loop system is, thus, 2-contracting, and as expected, every solution converges to an equilibrium.

VI. Conclusion

The \( k \) multiplicative and \( k \) additive compounds of a matrix play an important role in geometry, multilinear algebra, dynamical systems, and more. These compounds are based on \( k \times k \) minors and are, thus, defined for integer values of \( k \) only. The \( k \) compounds have recently been used to study an extension of contracting systems to \( k\)-contracting systems [57].

Here, we generalized \( k \) compounds to \( \alpha \) compounds, with \( \alpha \) real, and analyzed the properties of these compounds. As an application, we showed that these compounds provide a direct and
intuitive interpretation of important functions, e.g., \( \omega(K, \alpha, g) \) appearing in the seminal work of Douady and Oesterlé. We also introduced the new notion of \( \alpha \)-contracting systems, with \( \alpha \) real, generalizing the notion of \( k \)-contracting systems with \( k \) an integer, recently analyzed in \([57]\). Thus, rather than a binary choice—contracting or not contracting in a given metric—one can place any system on a continuous axis of contraction levels.

Owing to space limitations, we focused here on theoretical results, but we believe that many applications are possible. First, there exist nonlinear systems, where the “level of contraction” naturally changes in a continuous way, for example, systems that involve a continuous-time dynamics and discrete-time switching (see, e.g., \([27]\)). Second, contraction theory (i.e., the theory of \( 1 \)-contracting systems \([26]\)) has found many applications in control synthesis (see, e.g., \([30]\), \([31]\), \([40]\), \([42]\), \([43]\), \([47]\), and \([59]\)). An interesting research direction is to apply the generalization described here to control synthesis in such contexts.

Finally, our results could be used to define the generalized notions of convexity in optimization and machine learning. Just as Riemannian convexity of a scalar function with respect to a metric is equivalent to contraction in that metric of natural gradient descent \([54]\), the notions of \( \alpha \) Riemannian convexity could similarly be defined through equivalent \( \alpha \)-contracting autonomous dynamical systems.

**APPENDIX A**

**Properties of Kronecker Multiplications and Sums**

**Lemma 3** (see e.g., \([14]\)): Consider matrices \( A, C \in \mathbb{C}^{n \times m}, B, D \in \mathbb{C}^{r \times q}, F \in \mathbb{C}^{m \times r}, G \in \mathbb{C}^{q \times r}, X \in \mathbb{C}^{m \times n}, \) and \( Y \in \mathbb{C}^{q \times n} \). Then, we have the following:

a) \( (cA) \otimes B = A \otimes (cB) = c(A \otimes B) \) for any \( c \in \mathbb{C} \).

b) \( (A + C) \otimes B = A \otimes B + C \otimes B \).

c) \( A \otimes (B + D) = A \otimes B + A \otimes D \).

d) \( (A \otimes B)(F \otimes G) = (AF) \otimes (BG) \).

e) \( (A \otimes B)^{T} = A^{T} \otimes B^{T} \).

f) If \( X \) and \( Y \) are nonsingular, then \( (X \otimes Y)^{-1} = X^{-1} \otimes Y^{-1} \).

g) Let \( \lambda_{i}(X) \), \( i = 1, \ldots, n \), and \( \lambda_{j}(Y) \), \( j = 1, \ldots, m \), denote the eigenvalues of \( X \) and \( Y \), respectively. Then,

\[ X \otimes Y \text{ has eigenvalues } \lambda_{i}(X)\lambda_{j}(Y), \ i = 1, \ldots, n, \ j = 1, \ldots, m. \]

h) \( X \oplus Y \) has eigenvalues \( \lambda_{i}(X) + \lambda_{j}(Y), \ i = 1, \ldots, n, \ j = 1, \ldots, m. \)

i) \( \exp(X) \otimes \exp(Y) = \exp(X \otimes Y) \).

Property (a) implies that we can write \( cA \otimes B := (cA) \otimes B \) or \( A \otimes (cB) \), without any ambiguity.

**APPENDIX B**

**Explicit Formulas for Matrix Measures of Additive Compounds**

Let \( A \in \mathbb{C}^{n \times n} \). We use \( \bar{A} \) to denote the complex conjugate of \( A \). Pick \( k \in \{1, \ldots, n\} \).

The matrix measures for \( A^{k} \) are \([35]\)

\[
\mu_{1}(A^{k}) = \max_{\{i_{1}, \ldots, i_{k}\} \subset \{1, \ldots, n\}} \left( \sum_{p=1}^{k} \Re(a_{i_{p}i_{p}}) \right)
\]

\[ + \sum_{j \notin \{i_{1}, \ldots, i_{k}\}} (|a_{j,i_{1}}| + \cdots + |a_{j,i_{k}}|) \]

\[
\mu_{2}(A^{k}) = \sum_{i=1}^{n} \lambda_{i} \left( (A + \bar{A})/2 \right)
\]

\[
\mu_{\infty}(A^{k}) = \max_{\{i_{1}, \ldots, i_{k}\} \subset \{1, \ldots, n\}} \left( \sum_{p=1}^{k} \Re(a_{i_{p}i_{p}}) \right)
\]

\[ + \sum_{j \notin \{i_{1}, \ldots, i_{k}\}} (|a_{i_{1}j}| + \cdots + |a_{i_{k}j}|) \quad (70) \]

For \( k = 1, (70) \) reduces to well-known formulas for \( \mu_{p}(A) \).

**APPENDIX C**

**Proofs of Auxiliary Results**

**Proof of Theorem 1:** Fix \( \varepsilon > 0 \). Properties (a) and (i) in Lemma 3 yield

\[ \| \exp(\varepsilon (X \otimes Y)) \| = \| \exp(\varepsilon X \otimes \varepsilon Y) \| \]

\[ = \| \exp(\varepsilon X) \otimes \exp(\varepsilon Y) \|. \]

By \((26), \| \exp(\varepsilon (X \otimes Y)) \| = \| \exp(\varepsilon X) \| \| \exp(\varepsilon Y) \|. \)

Let \( D^{+} \) denote the upper Dini derivative w.r.t. \( \varepsilon \) (see, e.g., \([2]\)). Then

\[ D^{+} \| \exp(\varepsilon (X \otimes Y)) \| = \left( D^{+} \| \exp(\varepsilon X) \| \| \exp(\varepsilon Y) \| \right) + \| \exp(\varepsilon X) \| \left( D^{+} \| \exp(\varepsilon Y) \| \right). \]

It follows from \((24)\) that

\[ \mu(A) = \left( D^{+} \| \exp(\varepsilon A) \| \right)_{\varepsilon=0} \]

for any \( A \in \mathbb{R}^{n \times n} \). Thus

\[ \mu(X \oplus Y) \]

\[ = \left( D^{+} \| \exp(\varepsilon (X \oplus Y)) \| \right)_{\varepsilon=0} \]

\[ = \left( D^{+} \| \exp(\varepsilon X) \| \| \exp(\varepsilon Y) \| \right)_{\varepsilon=0} \]

\[ = \mu(X) + \mu(Y) \]

and this completes the proof. ■
Proof of Lemma 2: The conditions in the Lemma guarantee that \( A(\varepsilon) \) is differentiable w.r.t. \( \varepsilon \in (a, b) \). By (20), we have
\[
\frac{d}{d\varepsilon} J^\alpha(\varepsilon) = \alpha J^{\alpha-1}(\varepsilon) \frac{d}{d\varepsilon} \ell(\varepsilon).
\]
Hence
\[
\frac{d}{d\varepsilon} A^\alpha(\varepsilon) = \alpha J^{\alpha-1}(\varepsilon) \text{diag} \left( \frac{d}{d\varepsilon} \ell_1(\varepsilon) I_{m_1}, \ldots, \frac{d}{d\varepsilon} \ell_p(\varepsilon) I_{m_p} \right)
\]
\[
= \alpha J^{\alpha-1}(\varepsilon) \frac{d}{d\varepsilon} J(\varepsilon).
\]
Thus
\[
\frac{d}{d\varepsilon} A^\alpha(\varepsilon) = T^T \frac{d}{d\varepsilon} J^\alpha(\varepsilon) T^{-1}
\]
\[
= \alpha T^T J^{\alpha-1}(\varepsilon) T^{-T} \frac{d}{d\varepsilon} J(\varepsilon) T^{-1}
\]
and using Definition 1 yields (40). □

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