SUPERELLIPTIC CURVES WITH MANY AUTOMORPHISMS
AND CM JACOBIANS

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Abstract. Let $C$ be a smooth, projective, genus $g \geq 2$ curve, defined over $\mathbb{C}$. Then $C$ has many automorphisms if its corresponding moduli point $p \in \mathcal{M}_g$ has a neighborhood $U$ in the complex topology, such that all curves corresponding to points in $U \setminus \{p\}$ have strictly fewer automorphisms than $C$. We compute completely the list of superelliptic curves having many automorphisms. For each of these curves, we determine whether its Jacobian has complex multiplication. As a consequence, we prove the converse of Streit’s complex multiplication criterion for these curves.

1. Introduction

An abelian variety $A$ has complex multiplication (or is of CM-type) over a field $k$ if $\text{End}^0_k(A)$ contains a commutative, semisimple $\mathbb{Q}$-algebra of dimension $2 \dim A$. They were first studied by M. Deuring [1, 2] for elliptic curves and generalized to Abelian varieties by Shimura and Tanyama in [3]. By abuse of terminology, a curve is said to have complex multiplication (or to be of CM-type) when its Jacobian is of CM-type. Since the CM property is a property of the Jacobian, it is an invariant of the curve. A natural question is whether there is anything special about the points in the moduli space $\mathcal{M}_g$ of genus $g \geq 2$ curves, for which the Jacobian is of CM-type; see [4]. F. Oort asked if curves with many automorphisms (cf. Section 2.3) are all of CM-type. The answer to this question is negative, as explained in [5], where a full history of the problem and recent developments are given.

Let $C$ be a smooth, projective, genus $g \geq 2$ curve defined over $k$, $p \in \mathcal{M}_g$ its corresponding moduli point, and $G := \text{Aut}(C)$ the automorphism group of $C$ over the algebraic closure of $k$. For our purposes we will assume $k = \mathbb{C}$. We say that $C$ has many automorphisms if its corresponding point $p \in \mathcal{M}_g$ has a neighborhood $U$ (in the complex topology) such that all curves corresponding to points in $U \setminus \{p\}$ have automorphism group strictly smaller than $G$. They shouldn’t be confused with curves with large automorphism group which are curves with automorphism group $|G| > 4(g - 1)$. Not all curves with large automorphism group are curves with many automorphisms.

As mentioned above, Oort asked if such curves are of CM-type, and this is not true in general. However it remains an interesting question to determine which curves with many automorphisms are of CM-type. In general, for a given $g \geq 2$ we can determine the full list of automorphism groups that occur; see [6] and [7] for a complete survey on automorphism groups of algebraic curves. It is difficult from

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the group Aut(\(C\)) alone to determine if \(C\) is of CM-type without knowing anything about the equation of the curve. However, there is only one class of curves for which we can determine the equation of the curves explicitly starting from the automorphism group, namely the superelliptic curves. Hence, it is a natural choice to try to determine which superelliptic curves with many automorphisms are of CM-type.

In [8], the authors solved this problem for hyperelliptic curves. Their main tool is a formula of Streit from [9], which gives conditions on the characters of the group of automorphisms of the curve. More precisely, let \(\chi_C\) be the character of \(\sigma\) on \(H^0(C, \omega_C)\), and let \(\text{Sym}^2 \chi_C\) be the character of \(\sigma\) on \(\text{Sym}^2 H^0(C, \omega_C)\). By \(\chi_{\text{triv}}\) we denote the character of the trivial representation on \(C\). Streit showed that if \(\langle \text{Sym}^2 \chi_C, \chi_{\text{triv}} \rangle = 0\) then \(\text{Jac}\, C\) has complex multiplication; see [9]. We say that \(C\) satisfies Streit’s criterion if this inner product is 0.

For \(C\) hyperelliptic, the authors in [8] determine a formula which computes \(\text{Sym}^2 \chi_C\) and through this formula are able to determine precisely if a hyperelliptic curve with many automorphisms is or is not of CM-type. They prove their formula using the fact that it is easy to write a monomial basis of holomorphic differentials for hyperelliptic function fields. As a consequence, the converse of Streit’s criterion holds for hyperelliptic curves with many automorphisms and reduced automorphism group isomorphic to \(A_4, S_4,\) or \(A_5\). In other words, no such curve that fails Streit’s criterion can have complex multiplication.

Using a similar approach we are able to prove a similar formula for superelliptic curves (cf. Prop. 4.7). Let \(C : y^n = f(x)\) be a smooth superelliptic curve defined over \(\mathbb{C}\) (in particular, \(f(x)\) is assumed to be a separable polynomial, see Section 3), and let \(\tau\) be the order \(n\) automorphism given by \(y \mapsto \zeta_n y\), where \(\zeta_n\) is a primitive \(n\)th root of unity. Let \(G\) be the normalizer of \(\tau\) in Aut(\(C\)), and let \(\overline{G} = G/\langle \tau \rangle\), which naturally lies in Aut(\(\mathbb{P}^1\)) \(\cong PGL_2(\mathbb{C})\). For each \(\sigma \in \overline{G}\), let \(m\) be its order, and let \(\zeta_{\sigma}\) be either ratio of the eigenvalues when \(\sigma\) is thought of as an element of \(PGL_2(\mathbb{C})\). Observe that \(\zeta_{\sigma}\) is a primitive \(m\)th root of unity. Let \(\zeta_{n,\sigma}\) be a primitive \(mn\)th root of unity such that \(\zeta_{n,\sigma}^n = \zeta_{\sigma}\). Define

\[
k_{\sigma} = \begin{cases} 
1 & \text{if } \sigma \text{ fixes a branch point of } C \to \mathbb{P}^1 \\
0 & \text{otherwise.}
\end{cases}
\]

Let \(A \subseteq \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\) be the set of ordered pairs defined below in (Eq. (4)). If

\[
\sum_{(a, b) \in A} \sum_{i=0}^{n-1} \left( \sum_{(a, b) \in A} \zeta_{n,\sigma}^{2(a-1) + 2(b+1)i} \right) \sum_{(a, b) \in A} \zeta_{n,\sigma}^{2(b-n-1) + 2(a + 1)i} + \sum_{(a, n,\sigma) \in A} \zeta_{n,\sigma}^{a + 1 + 2(b-n-1) + 2(a + 1)i} \right)^2 \]

vanishes, then \(C\) satisfies Streit’s criterion, and thus has CM. However, not all superelliptic curves with many automorphisms satisfy Streit’s criterion; see Table 1. We in fact prove the converse of Streit’s criterion for superelliptic curves with many automorphisms; any such curve not satisfying Streit’s criterion does not have CM Cor. 5.14. To do this we use stable reduction (cf. Section 5.2) and the theory of semistable models as in [10], as well as the so-called criterion of Müller-Pink; see Section 5.3. Our computations were done using Sage and GAP and are made available along with the arxiv posting of this paper.

It is still an open question whether these results can be generalized to larger families of curves, for example for generalized superelliptic curves as listed in [11].
In general, it is believed that curves with extra automorphisms give Jacobian varieties with a large number of endomorphisms (here “large” is used informally). Hence, perhaps an interesting family of curves to investigate would be all curves with large automorphism groups and many automorphisms; these were determined in [6]. It would also be interesting to obtain a theoretical explanation for Cor. 5.14.

2. Preliminaries

Throughout, we work over the field \( \mathbb{C} \). An Abelian variety is an absolutely irreducible projective variety which is a group scheme. A morphism of Abelian varieties \( A \to B \) is a homomorphism if and only if it maps the identity element of \( A \) to the identity element of \( B \).

Let \( A, B \) be abelian varieties. We denote the \( \mathbb{Z} \)-module of homomorphisms \( A \to B \) by \( \text{Hom}(A, B) \) and the ring of endomorphisms \( A \to A \) by \( \text{End} A \). It is more convenient to work with the \( \mathbb{Q} \)-vector spaces \( \text{Hom}_0(A, B) := \text{Hom}(A, B) \otimes \mathbb{Z} \mathbb{Q} \), and \( \text{End}_0 A := \text{End} A \otimes \mathbb{Z} \mathbb{Q} \). Determining \( \text{End} A \) or \( \text{End}_0 A \) is an interesting problem on its own; see [4].

The ring of endomorphisms of a generic Abelian variety \( A \) over \( \mathbb{C} \) is “as small as possible”, that is, \( \text{End}(A) = \mathbb{Z} \) in general. In general, \( \text{End}_0(A) \) is a \( \mathbb{Q} \)-algebra of dimension at most \( 4 \dim(A)^2 \). Indeed, \( \text{End}_0(A) \) is a semi-simple algebra, and by duality one can apply a complete classification due to Albert of possible algebra structures on \( \text{End}_0(A) \), see [12, pg. 202]. We say that an abelian variety \( A \) has complex multiplication if \( \text{End}_0(A) \) contains a commutative, semisimple \( \mathbb{Q} \)-algebra of dimension \( 2 \dim(A) \).

For the following elementary result, see [3].

**Lemma 2.1.** If \( A \) is an Abelian variety with CM, then every abelian subvariety of \( A \) also has CM.

**2.1. Curves and their Jacobians.** Throughout, \( C \) is a smooth, projective curve defined over \( \mathbb{C} \). We will denote its group of automorphisms by \( \text{Aut}(C) \). It is a well known fact that \( |\text{Aut}(C)| \leq 84(g-1) \). One gets a stratification of \( \mathcal{M}_g \) by strata of curves with the same automorphism group, and the generic curve of genus \( g > 2 \) has trivial automorphism group.

We state the following Corollary of Lem. 2.1, which is used repeatedly.

**Corollary 2.2.** If \( \psi : C \to Y \) is a finite morphism of curves and \( \text{Jac} C \) has CM, then so does \( \text{Jac} Y \).

**Proof.** Since \( \psi^* \) gives an embedding of \( \text{Jac} Y \) in \( \text{Jac} C \), the corollary follows from Lem. 2.1. \( \square \)

**2.2. Hurwitz spaces.** We consider finite covers \( \eta : C \to \mathbb{P}^1 \) of degree \( n \). Then \( \eta^* \) identifies \( \mathbb{C}(\mathbb{P}^1) =: \mathbb{C}(x) \) with a subfield of \( \mathbb{C}(C) \). First, we introduce the equivalence: \( \eta \sim \eta' \) if there are isomorphisms \( \alpha : C \to C' \) and \( \beta \in \text{Aut}(\mathbb{P}^1) \) with

\[ \beta \circ \eta = \eta' \circ \alpha. \]

The monodromy group of \( \eta \) is the Galois group of the Galois closure \( L \) of \( \mathbb{C}(C)/\mathbb{C}(x) \). We embed \( G \) into \( S_n \), the symmetric group with \( n \) letters, and fix the ramification type of the covers \( \eta \). We assume that exactly \( r \geq 3 \) points in \( \mathbb{P}^1(k) \) are ramified (i.e. their preimages contain fewer than \( n \) points). Note that the ramification groups are cyclic.
By the classical theory of covers of Riemann surfaces, which can be transferred to the algebraic setting by the results of Grothendieck, it follows that there is a tuple \((\sigma_1, \ldots, \sigma_r)\) in \(S_n\), such that \(\sigma_1 \cdots \sigma_r = 1\), \(\text{ord}(\sigma_i) = e_i\) is the ramification order of the \(i\)-th ramification point \(P_i\) in \(L\) and \(G := \langle \sigma_1, \ldots, \sigma_r \rangle\) is a transitive group in \(S_n\). We call such a tuple the signature \(\sigma\) of the covering \(\eta\) and remark that such tuples are determined up to conjugation in \(S_n\), and that the genus of \(C\) is determined by the signature because of the Hurwitz genus formula.

Let \(H_\sigma\) be the set of pairs \([(\eta), (p_1, \ldots, p_r)]\), where \([\eta]\) is an equivalence class of covers of type \(\sigma\), and \((p_1, \ldots, p_r)\) is an ordering of the branch points of \(\phi\) modulo automorphisms of \(\mathbb{P}^1\). The set \(H_\sigma\) carries the structure of a scheme; in fact it is a quasi-projective variety called the Hurwitz space. We have the forgetful morphism

\[
\Phi_\sigma : H_\sigma \to \mathcal{M}_g
\]

mapping \([(\eta), (p_1, \ldots, p_r)]\) to the isomorphism class \([C]\) in the moduli space \(\mathcal{M}_g\). Each component of \(H_\sigma\) has the same image in \(\mathcal{M}_g\).

Define the moduli dimension of \(\sigma\) (denoted by \(\text{dim}(\sigma)\)) as the dimension of \(\Phi_\sigma(H_\sigma)\); i.e., the dimension of the locus of genus \(g\) curves admitting a cover to \(\mathbb{P}^1\) of type \(\sigma\). We say \(\sigma\) has full moduli dimension if \(\text{dim}(\sigma) = \text{dim} \mathcal{M}_g\); see [6].

### 2.3. Curves with many automorphisms.

Let \(C\) be a genus \(g \geq 2\) curve defined over \(\mathbb{C}\), \(p \in \mathcal{M}_g\) its corresponding moduli point, and \(G := \text{Aut}(C)\). We say that \(C\) has many automorphisms if \(p \in \mathcal{M}_g\) has a neighborhood \(U\) (in the complex topology) such that all curves corresponding to points in \(U \setminus \{p\}\) have automorphism group strictly smaller than \(p\).

**Lemma 2.3** ([5, Lemma 4.4] or [8, Theorem 2.1]). Let \(C\) have genus \(\geq 2\) as above, let \(G := \text{Aut}(C)\), and let \(\eta : C \to \mathbb{P}^1\) the corresponding map with signature \(\sigma\). Then the following are equivalent:

1. \(C\) has many automorphisms.
2. There exists a subgroup \(H < G\) such that \(g(C/H) = 0\) and \(C \to C/H\) has exactly three branch points.
3. The quotient \(C/G\) has genus 0 and \(C \to C/G\) has exactly three branch points.
4. The signature \(\sigma\) has moduli dimension 0.

**Question 2.4** (F. Oort). If \(C\) has many automorphisms, does \(\text{Jac} C\) have complex multiplication?

Wolfart answered this question for all curves of genus \(g \leq 4\); see [13, §5].

### 3. Superelliptic curves with many automorphisms

The term superelliptic curve has been used differently by many authors. Most use it to mean a smooth projective curve \(C\) with affine equation of the form \(y^n = f(x)\), where \(f(x) \in \mathbb{C}[x]\) has discriminant \(\Delta(f) \neq 0\). If \(H := \langle \tau \rangle \subseteq \text{Aut}(C)\) is the subgroup generated by \(\tau(y) = \zeta_n y\), then it is sometimes further required that \(H\) be normal (or central) in \(\text{Aut}(C)\).

We will follow the definition in [11]. Specifically, a superelliptic curve is a smooth projective curve \(C\) of genus \(\geq 2\) with affine equation \(y^n = \prod_{i=1}^r (x - a_i)\), with the \(a_i\) distinct complex numbers such that

1. If \(H\) is as above, then \(H\) is normal in \(\text{Aut}(C)\).
(ii) Either \( n \mid r \) or \( \gcd(n, r) = 1 \) (this guarantees that all branch points have index \( n \)).

**Remark 3.1.** In fact, if \( C \) is a superelliptic curve with many automorphisms, we have that \( n \mid r \) or \( r \equiv -1 \mod n \), see

If \( C, H, \) and \( \tau \) are as above, we call \( \tau \) an *superelliptic automorphism (of level \( n \)) and \( H \) a *superelliptic group (of level \( n \)) of \( C \).

Suppose \( C \) is a superelliptic curve, with superelliptic group \( H \) and corresponding \( H \)-cover \( \pi : C \to \mathbb{P}^1 \). If \( G = \text{Aut}(C) \), then there is a short exact sequence \( 1 \to H \to G \to \overline{G} \to 1 \), where \( \overline{G} \) is a group of Möbius transformations keeping the set of branch points of \( \pi \) invariant. We call \( \overline{G} \) the *reduced automorphism group* of \( C \).

We also define a *pre-superelliptic curve* to be a curve satisfying all the requirements of a superelliptic curve except possibly for (i) above. In this case, if \( N \) is the normalizer of \( H \) in \( \text{Aut}(C) \), then we have a similar exact sequence \( 1 \to H \to N \to \overline{N} \to 1 \), and we call \( \overline{N} \) the reduced automorphism group of \( C \). In this case, \( H \) is called a *pre-superelliptic group* and \( \tau \) is called a *pre-superelliptic automorphism*.

Because verifying that a curve is pre-superelliptic does not depend on computing its entire automorphism group, it can be significantly easier than verifying that a curve is superelliptic.

As an immediate consequence of [11, Lem. 1], we obtain the following proposition.

**Proposition 3.2.** If \( C \) is a pre-superelliptic curve, then any pre-superelliptic group is in fact central in its normalizer.

Superelliptic curves over *finite* fields were described up to isomorphism in [14]. For more on arithmetic aspects of such curves we refer to [15].

### 3.1. Superelliptic curves with many automorphisms.

In the rest of this subsection, we construct a list containing all superelliptic curves with many automorphisms. As above, let \( C \) be a pre-superelliptic curve defined over \( \mathbb{C} \) with automorphism group \( G := \text{Aut}(C) \), and pre-superelliptic group \( H \) generated by \( \tau \) of order \( \geq 2 \). Let \( N \subseteq G \) be the normalizer of \( H \), and let \( \overline{N} := N / H \) be the reduced automorphism group. In fact, what we construct is the list of all pre-superelliptic curves \( C \) as above such that the quotient morphism \( C \to C / N \) is branched at exactly three points. Since \( N = G \) for a superelliptic curve, this list contains all superelliptic curves \( C \) such that \( C \to C / G \) is branched at exactly three points, thus, by Lem. 2.3, all superelliptic curves with many automorphisms.

**Proposition 3.3.** Let \( C \) be a pre-superelliptic curve, let \( N \subseteq \text{Aut}(C) \) be the normalizer of the a pre-superelliptic group \( H \) of level \( n \), and let \( \overline{N} = N / H \). Then \( \overline{N} \) is isomorphic to either \( C_m \), \( D_{2m} \), \( A_4 \), \( S_4 \), or \( A_5 \). If the quotient map \( \Phi : C \to C / N \) is branched at exactly three points, then furthermore:

(i) If \( \overline{N} \cong C_m \) then \( C \) has equation \( y^n = x^m + 1 \) or \( y^n = x (x^m + 1) \).

(ii) If \( \overline{N} \cong D_{2m} \) then \( C \) has equation \( y^n = x^{2m} - 1 \) or \( y^n = x (x^{2m} - 1) \).

(iii) If \( \overline{N} \cong A_4 \) then \( C \) has equation \( y^n = f(x) \) where \( f(x) \) is the following

\[
y^n = x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1).
\]

Furthermore, the \( A_4 \)-orbit of \( \infty \) consists of itself and the roots of \( x(x^4 - 1) \).
(iv) If $\mathcal{N} \cong S_4$ then $C$ has equation $y^n = f(x)$ where $f(x)$ is one of the following: $r_4(x), s_4(x), t_4(x), r_4(x)s_4(x), r_4(x)t_4(x), r_4(x)s_4(x)t_4(x)$, where $r_4, s_4, t_4$ are as in Eq. (2). Furthermore, the $S_4$-orbit of $\infty$ consists of itself and the roots of $t_4(x)$.

(v) If $\mathcal{N} \cong A_5$ then $C$ has equation $y^n = f(x)$ where $f(x)$ is one of the following: $r_5(x), s_5(x), t_5(x), r_5(x)s_5(x), s_5(x)t_5(x), r_5(x)s_5(x)t_5(x)$, where $r_5, s_5, t_5$ are as in Eq. (3). Furthermore, the $A_5$-orbit of $\infty$ consists of itself and the roots of $s_5(x)$.

Remark 3.4. The notation $r_4, s_4, t_4, r_5, s_5, t_5$ in Prop. 3.3 above is consistent with that used in [8].

Proof. The first statement holds because $\mathcal{N} \subseteq PGL_2(\mathbb{C})$, using the well-known classification of finite subgroups of $PGL_2(\mathbb{C})$.

We have the following diagram:

$$\Phi : C \rightarrow \mathbb{P}_x^1, \mathcal{N} \rightarrow \mathbb{P}_x^1 = \mathbb{C}/N.$$  

The group $\mathcal{N}$ is the monodromy group of the cover $\phi : \mathbb{P}_x^1 \rightarrow \mathbb{P}_x^1$. Let $y^n = f(x)$ be the equation of $C$, where $f(x)$ is a separable polynomial. Now, the map $\phi$ is given by the rational function $z$ in $x$, which has degree $|\mathcal{N}|$.

Let $S = \{q_1, q_2, q_3\}$ be the set of branch points of $\Phi : C \rightarrow \mathbb{P}_x^1$, let $W$ be the set of branch points of $\pi : C \rightarrow \mathbb{P}_x^1$ (that is, the roots of $f$ and possibly $\infty$). Since $C \rightarrow \mathbb{P}_x^1$ is Galois and $|H| \geq 2$, there exists a non-empty set $T \subseteq S$ such that $W = \phi^{-1}(T)$.

Write the rational function $z$ in lowest terms as a ratio of polynomials $\frac{\Psi(x)}{\Gamma(x)}$. We write $z = q_i = \frac{\Gamma_i(x)}{\Gamma(x)}$ in lowest terms, for each branch point $q_i$, $i = 1, 2, 3$, where $\Gamma_i(x) \in \mathbb{C}[x]$. Hence,

$$\Gamma_i(x) = \Psi(x) - q_i \cdot \Upsilon(x)$$

is a degree $|\mathcal{N}|$ polynomial and the multiplicity of each root of $\Gamma_i(x)$ corresponds to the ramification index for each $q_i$ (if $q_i = \infty$, then $\Gamma_i(x) := \Upsilon(x)$). The roots of $\Gamma_i(x)$ are the preimages of $q_i$ under $\phi$. So letting $\gamma_i(x) = \text{rad}(\Gamma_i(x))$, we conclude that the equation of $C$ is given by $y^n = f(x)$, where

$$f(x) = \prod_{q_i \in T} \gamma_i(x).$$

The rest of the proof proceeds similarly to [16, §4]. In particular, for $\mathcal{N} \in \{C_m, D_{2m}, A_4, S_4, A_5\}$, we can make a change of variables in $x$ and $z$ so that $z = \phi(x)$ is given by the appropriate entry in the first 5 rows of [16, Table 1]. We now go case by case.

i) $\mathcal{N} \cong C_m$: In this case, $\phi(x) = x^m$, which has branch points $q_1 = \infty$ and $q_2 = 0$. Hence $\gamma_1(x) = 1$ and $\gamma_2(x) = x$. After a change of variables in $z$, we may assume without loss of generality that $q_3 = -1$, which yields $\gamma_3(x) = x^m + 1$. Since the covering $C \rightarrow \mathbb{P}_x^1$ has three branch points, we must have $q_3 \in T$. From Eq. (1), we have $f(x) = x^m + 1$ or $f(x) = x(x^m + 1)$. This proves (i).

ii) $\mathcal{N} \cong D_{2m}$: In this case,

$$\phi(x) = x^m + \frac{1}{x^m} = \frac{x^{2m} + 1}{x^m},$$

we have $\phi(x) = x^m + 1$ or $\phi(x) = x(x^m + 1)$. This proves (ii).
which has branch points \( q_1 = \infty, q_2 = 2, \) and \( q_3 = -2. \) Hence \( \gamma_1(x) = x, \gamma_2(x) = x^m - 1, \) and \( \gamma_3(x) = x^m + 1. \) The involution in the dihedral group permutes the branch points \( q_2 \) and \( q_3, \) so \( q_2 \in T \) if and only if \( q_3 \in T. \) But if neither is in \( T, \) then Eq. (1) shows that \( C \) has equation \( y^n = x, \) contradicting the assumption that \( g(C) \geq 2. \) So \( T = \{q_1, q_2, q_3\} \) or \( T = \{q_2, q_3\}. \) From Eq. (1), we have the two possible equations

\[ y^n = x^{2m} - 1, \quad y^n = x(x^{2m} - 1). \]

This proves (ii).

iii) \( \mathcal{N} \cong A_4: \) In this case,

\[ \phi(x) = \frac{x^{12} - 33x^8 - 33x^4 + 1}{x^2(x^4 - 1)^2}, \]

which has branch points \( q_1 = \infty \) of index 2, \( q_2 = 6i\sqrt{3} \) of index 3, and \( q_3 = -6i\sqrt{3} \) of index 3, where \( i^2 = -1. \) Hence

\[ t_4 := \gamma_1 = x(x^4 - 1), \quad \gamma_2 = x^4 + 2i\sqrt{3}x^2 + 1, \quad \gamma_3 = x^4 - 2i\sqrt{3}x^2 + 1, \]

with \( \infty \) in the fiber of \( q_1 \) as well. The branch points \( q_1 = \infty, q_2 = 6i\sqrt{3}, \) and \( q_3 = -6i\sqrt{3} \) are the branch points of the covering \( \pi : C \to \mathbb{P}^1. \) Let \( s_4 \) and \( t_4 \) be as in Eq. (2) below. If neither \( q_2 \) nor \( q_3 \) is in \( T, \) then Eq. (1) shows that the equation of \( C \) is \( y^n = t_4. \) Observe that \( \gamma_2 \gamma_3 = s_4. \) So if both \( q_2 \) and \( q_3 \) are in \( T, \) then Eq. (1) shows that the equation of \( C \) is either \( y^n = s_4 t_4 \) or \( y^n = s_4. \) In all cases, the reduced automorphism group is actually \( S_4, \) so we may assume that exactly one of \( q_2 \) or \( q_3 \) is in \( T. \) Since the two choices are conjugate, we may assume \( q_2 \in T \) but \( q_3 \notin T. \)

So \( T = \{q_1, q_2\} \) or \( \{q_2\}. \) By Eq. (1), we have the two possible equations

\[ y^n = (x^4 + 2i\sqrt{3}x^2 + 1), \quad y^n = x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1). \]

The last assertion of (iii) is true because \( \infty \in \phi^{-1}(q_1). \)

iv) \( \mathcal{N} \cong S_4: \) In this case,

\[ \phi(x) = \frac{(x^8 + 14x^4 + 1)^3}{108 x^4(x^4 - 1)^4}, \]

which has branch points \( q_1 = 1, q_2 = 0 \) and \( q_3 = \infty. \) Also, \( \infty \in \phi^{-1}(q_3). \) Then

\[ r_4(x) : \gamma_1(x) = x^{12} - 33x^8 - 33x^4 + 1 \]

\[ s_4(x) := \gamma_2(x) = x^8 + 14x^4 + 1 \]

\[ t_4(x) := \gamma_3(x) = x(x^4 - 1), \]

Every possible case occurs here. So by Eq. (1), the equation of the curve \( C \) is \( y^n = f(x) \) for \( f(x) \) one of

\[ r_4(x), s_4(x), t_4(x), r_4(x)s_4(x), r_4(x)t_4(x), s_4(x)t_4(x), r_4(x)s_4(x)t_4(x). \]

The last assertion of (iv) is true because \( \infty \in \phi^{-1}(q_3). \)

v) \( \mathcal{N} \cong A_5: \) In this case,

\[ \phi(x) = \frac{(-x^{20} + 228x^{15} - 494x^{10} - 228x^5 - 1)^3}{(x(x^{10} + 11x^5 - 1))^5}, \]
which has branch points of $q_1 = 0$, $q_2 = 1728$, and $q_3 = \infty$. One computes
\[ r_5(x) : \gamma_1(x) = x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1 \]
\[ s_5(x) := \gamma_2(x) = x(x^{10} + 11x^5 - 1) \]
\[ t_5(x) := \gamma_3(x) = x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1, \]
and one notes that $\infty \in \phi^{-1}(q_2)$ as well. Every possible case occurs here. So by Eq. (1), the equation of the curve $C$ is $y^n = f(x)$ for $f(x)$ one of
\[ r_5(x), s_5(x), t_5(x), r_5(x)s_5(x), r_5(x)t_5(x), s_5(x)t_5(x), r_5(x)s_5(x)t_5(x). \]
The last assertion of (v) is true because $\infty \in \phi^{-1}(q_2)$. \hfill \Box

**Remark 3.5.** It is clear from the proof of Prop. 3.3 that in all cases, the group $\overline{N}$ permutes the branch locus of $\pi : C \to \mathbb{P}^1$.

**Proposition 3.6.** (i) Let $C$ be a pre-superelliptic curve satisfying the conditions of Prop. 3.3 given by an affine equation $y^n = f(x)$. Suppose $\overline{N} \in \{A_1, S_4, A_5\}$ as in Prop. 3.3(iii), (iv), or (v). Then $n \mid (\deg(f(x) + \delta)$, where
\[ \delta = \begin{cases} 1 & t_4(x) \mid f(x) \text{ or } s_5(x) \mid f(x) \\ 0 & \text{otherwise.} \end{cases} \]
Furthermore, $\deg(f) + \delta$ is the number of branch points of the map $C \to \mathbb{P}^1$ given by projection to the $x$-coordinate.

(ii) Conversely, if $C$ is a smooth projective curve given by an affine equation $y^n = f(x)$ as in Prop. 3.3(iii), (iv), or (v) with $n \mid (\deg(f(x) + \delta)$, then $C$ satisfies the assumptions of Prop. 3.3 (in particular, $C \to C/N$ is branched at three points, with $N$ as in Prop. 3.3).

**Proof.** We first note that by Prop. 3.3(iii), (iv), and (v), the orbit of $\infty$ under $\Aut(C)$ consists of the roots of $t_4(x)$ (in cases (iii) and (iv)) or of $s_5(x)$ (in case (v)). So $\infty$ is a branch point of $C \to \mathbb{P}^1$ if and only if $t_4(x) \mid f(x)$ or $s_5(x) \mid f(x)$; that is, if and only if $\delta = 1$.

Since $f(x)$ is separable, the monodromy action induced by a small counterclockwise loop around any root of $f(x)$ takes a point $(a, b)$ to $(a, e^{2\pi i/n}b)$. In particular, if $\tau \in \Aut_{\pi_0}(C)$ is the isomorphism $(a, b) \mapsto (a, e^{2\pi i/n}b)$, then $\tau$ has order $n$ and the signature of $C \to \mathbb{P}^1$ is
\[ \begin{cases} (\tau, \ldots, \tau) & \delta = 0 \\ (\tau, \ldots, \tau, \sigma) & \delta = 1, \end{cases} \]
for some $\sigma \in \mathbb{Z}/n$ corresponding to the branch point $\infty$. Now, the product of all entries in the signature must be the identity. In the first case, this implies that the number of branch points is divisible by $n$, so $\deg f(x)$ is divisible by $n$. In the second case, since $\infty$ is permuted with other branch points of the cover and $\mathbb{Z}/n$ is central in $\Aut(C)$, we have that $\sigma = \tau$. Thus the number of branch points, which is now $\deg f(x) + 1$, is also divisible by $n$. This completes the proof of part (i).

Let $\pi : C \to \mathbb{P}^1$ be the projection to the $x$-coordinate. To prove (ii), it suffices to show that if $\overline{\pi} \in \Aut(\mathbb{P}^1)$ preserves the branch locus of $\pi$, then $\overline{\pi}$ lifts to an element $\alpha \in \Aut(X)$. By the proof of part (i), the signature of $\pi$ is $(\tau, \ldots, \tau)$ for some $\tau \in \Aut(C/\mathbb{P}^1)$. After a change of variables, we may assume that $\infty$ is not a branch point, so the affine equation is $y^n = f(x)$ where $f(x)$ is separable. Now, $\overline{\pi}$
permutes the roots of $f(x)$, since $\infty$ is not a branch point. Thus it is clear that $\pi$ lifts to an automorphism of $\mathcal{C}$ given by acting trivially on $y$. This proves (ii). $\square$

The following corollary is immediate.

**Corollary 3.7.** If $\mathcal{C}$ is a superelliptic curve with many automorphisms and $H$ is a superelliptic group, then the cover $\mathcal{C} \to \mathcal{C}/H$ has signature $(\tau, \ldots, \tau)$ for some superelliptic automorphism $\tau$.

Let $\mathcal{C}$ be a pre-superelliptic curve with pre-superelliptic group $H$ with normalizer $N$ in $\text{Aut}(\mathcal{C})$. Suppose $X \to X/N$ is branched at exactly three points. As a consequence of Prop. 3.3 and Prop. 3.6, there are finitely many such curves with reduced automorphism group $A_4$, $S_4$, or $A_5$, and the list of such curves is exactly Table 1 below. In particular, all superelliptic curves with many automorphisms and reduced automorphism group not isomorphic to $C_m$ or $D_{2m}$ appear in Table 1.

For the rest of the paper, our goal is to determine which superelliptic curves with many automorphisms are CM-type.

Table 1: Curves with exceptional reduced automorphism groups.

| Nr. | $N$ | $n$ | $g$ | $f(x)$ | CM? | Justification |
|-----|-----|-----|-----|--------|-----|---------------|
| $\mathcal{C}_0$ | $A_4$ | 4 | 3 | $t_4$ | YES | Prop. 4.8 |
| $\mathcal{C}_1$ | $A_4$ | 2 | 4 | $p_4t_4$ | YES | Prop. 4.8 |
| $\mathcal{C}_2$ | $A_4$ | 5 | 16 | | NO | Prop. 5.11 |
| $\mathcal{C}_3$ | | 10 | 36 | | NO | Cor. 5.12 |
| $\mathcal{C}_4$ | $S_4$ | 2 | 5 | $r_4$ | YES | Prop. 4.8 |
| $\mathcal{C}_5$ | $S_4$ | 3 | 10 | | NO | Prop. 5.11 |
| $\mathcal{C}_6$ | $S_4$ | 4 | 15 | | NO | Prop. 5.11 |
| $\mathcal{C}_7$ | $S_4$ | 6 | 25 | | NO | Cor. 5.12 |
| $\mathcal{C}_8$ | $S_4$ | 12 | 55 | | NO | Cor. 5.12 |
| $\mathcal{C}_9$ | | 2 | 3 | $s_4$ | NO | Prop. 5.1 |
| $\mathcal{C}_{10}$ | $S_4$ | 4 | 9 | | NO | Prop. 5.1 |
| $\mathcal{C}_{11}$ | | 8 | 21 | | NO | Prop. 5.1 |
| $\mathcal{C}_{12}$ | $S_4$ | 2 | 2 | $t_4$ | YES | Prop. 4.8 |
| $\mathcal{C}_{13}$ | $S_4$ | 3 | 4 | | YES | Prop. 4.8 |
| $\mathcal{C}_{14}$ | | 6 | 10 | | YES | Prop. 4.8 |
| $\mathcal{C}_{15}$ | | 2 | 9 | | NO | Prop. 5.1 |
| $\mathcal{C}_{16}$ | | 4 | 27 | | NO | Prop. 5.1 |
| $\mathcal{C}_{17}$ | $S_4$ | 5 | 36 | $r_4s_4$ | NO | Prop. 5.5 |
| $\mathcal{C}_{18}$ | | 10 | 81 | | NO | Prop. 5.1 |
| $\mathcal{C}_{19}$ | | 20 | 171 | | NO | Prop. 5.1 |
| $\mathcal{C}_{20}$ | | 2 | 8 | | YES | Prop. 4.8 |
| $\mathcal{C}_{21}$ | | 3 | 16 | | NO | Prop. 5.5 |
| $\mathcal{C}_{22}$ | $S_4$ | 6 | 40 | $r_4t_4$ | NO | Cor. 5.6 |
| $\mathcal{C}_{23}$ | | 9 | 64 | | NO | Cor. 5.6 |
| $\mathcal{C}_{24}$ | | 18 | 136 | | NO | Cor. 5.6 |
| $\mathcal{C}_{25}$ | | 2 | 6 | | NO | Prop. 5.1 |
| $\mathcal{C}_{26}$ | $S_4$ | 7 | 36 | $s_4t_4$ | NO | Prop. 5.11 |
| $\mathcal{C}_{27}$ | | 14 | 78 | | NO | Prop. 5.1 |
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$C_{28}$ & $S_4$ & 2 & 12 & $r_4s_4t_4$ & NO & Prop. 5.1 \\
$C_{29}$ & 13 & 144 & NO & Prop. 5.11 \\
$C_{30}$ & 26 & 300 & NO & Prop. 5.1 \\
\hline
$C_{31}$ & 2 & 9 & NO & Prop. 5.1 \\
$C_{32}$ & 4 & 27 & NO & Prop. 5.1 \\
$C_{33}$ & $A_5$ & 5 & 36 & $r_5$ & NO & Prop. 5.11 \\
$C_{34}$ & 10 & 81 & NO & Prop. 5.1 \\
$C_{35}$ & 20 & 171 & NO & Prop. 5.1 \\
\hline
$C_{36}$ & 2 & 5 & NO & Prop. 5.1 \\
$C_{37}$ & 3 & 10 & YES & Prop. 4.8 \\
$C_{38}$ & $A_5$ & 4 & 15 & $s_5$ & NO & Prop. 5.1 \\
$C_{39}$ & 6 & 25 & NO & Prop. 5.1 \\
$C_{40}$ & 12 & 55 & NO & Prop. 5.1 \\
\hline
$C_{41}$ & 2 & 14 & YES & Prop. 4.8 \\
$C_{42}$ & 3 & 28 & NO & Prop. 5.5 \\
$C_{43}$ & 5 & 56 & NO & Prop. 5.11 \\
$C_{44}$ & $A_5$ & 6 & 70 & $t_5$ & NO & Cor. 5.6 \\
$C_{45}$ & 10 & 126 & NO & Cor. 5.12 \\
$C_{46}$ & 15 & 196 & NO & Cor. 5.6 \\
$C_{47}$ & 30 & 406 & NO & Cor. 5.6 \\
\hline
$C_{48}$ & 2 & 15 & NO & Prop. 5.1 \\
$C_{49}$ & 4 & 45 & NO & Prop. 5.1 \\
$C_{50}$ & $A_5$ & 8 & 105 & $r_5s_5$ & NO & Prop. 5.1 \\
$C_{51}$ & 16 & 225 & NO & Prop. 5.1 \\
$C_{52}$ & 32 & 465 & NO & Prop. 5.1 \\
\hline
$C_{53}$ & 2 & 24 & NO & Prop. 5.1 \\
$C_{54}$ & 5 & 96 & NO & Prop. 5.5 \\
$C_{55}$ & $A_5$ & 10 & 216 & $r_5t_5$ & NO & Prop. 5.1 \\
$C_{56}$ & 25 & 576 & NO & Cor. 5.6 \\
$C_{57}$ & 50 & 1176 & NO & Prop. 5.11 \\
\hline
$C_{58}$ & 2 & 20 & NO & Prop. 5.1 \\
$C_{59}$ & 3 & 40 & NO & Prop. 5.5 \\
$C_{60}$ & 6 & 100 & NO & Prop. 5.1 \\
$C_{61}$ & $A_5$ & 7 & 120 & $s_5t_5$ & NO & Prop. 5.5 \\
$C_{62}$ & 14 & 260 & NO & Prop. 5.1 \\
$C_{63}$ & 21 & 400 & NO & Cor. 5.6 \\
$C_{64}$ & 42 & 820 & NO & Prop. 5.1 \\
\hline
$C_{65}$ & 2 & 30 & NO & Prop. 5.1 \\
$C_{66}$ & $A_5$ & 31 & 900 & $r_5s_5t_5$ & NO & Prop. 5.11 \\
$C_{67}$ & 62 & 1830 & NO & Prop. 5.1 \\
\hline
\end{tabular}

4. Positive CM Results

In this section, we confirm that all superelliptic curves of separable type with many automorphisms and reduced automorphism group $C_m$ or $D_{2m}$ have CM, and we show that the curves in Table 1 marked “YES” have CM as well.

4.1. Quotients of Fermat curves. A Fermat curve is a projective curve with affine equation $x^a + y^a + z^a = 0$ for some $a \in \mathbb{N}$. It is easy to show directly that any
superelliptic curve $\mathcal{C}$ with many automorphisms and reduced automorphism group $\overline{\mathcal{C}} = \mathcal{C}_m$ or $\overline{\mathcal{C}} = D_{2m}$ is isomorphic to a quotient of a Fermat curve, and thus has CM:

**Lemma 4.1.** If an algebraic curve $\mathcal{C}$ is the quotient of a Fermat curve then $\text{Jac}(\mathcal{C})$ has CM.

*Proof.* It is well-known that Fermat curves have CM Jacobians, see, e.g., [17, Ch. VI.1]. The lemma follows since any quotient of a curve with CM Jacobian has CM Jacobian Cor. 2.2. □

**Theorem 4.2.** Let $\mathcal{C}$ be a superelliptic curve with many automorphisms.

i) If $\overline{\mathcal{N}}$ is cyclic then $\text{Jac}(\mathcal{C})$ has CM.

ii) If $\overline{\mathcal{N}}$ is dihedral then $\text{Jac}(\mathcal{C})$ has CM.

*Proof.* First note that for any $a \in \mathbb{N}$, the smooth proper curve with affine equation $x^a = y^a + 1$ is isomorphic to a Fermat curve over $\overline{\mathcal{C}}$.

If $\overline{\mathcal{C}} \cong \mathcal{C}_m$ or $\overline{\mathcal{C}} \cong D_{2m}$, then by Prop. 3.3(i) and (ii), $\mathcal{C}$ has (affine) equation $y^n = x^r + 1$ or $y^n = x(x^r + 1)$ for some $n$ and $r$. In the first case, $\mathcal{C}$ is clearly a quotient of the Fermat curve $\mathcal{Y}$ with affine equation $u^{rn} = v^{rn} + 1$ under the automorphism group generated by $u \mapsto \zeta_n u$ and $v \mapsto \zeta_n v$. In the second case, the quotient of the Fermat curve with affine equation $u^{rn} = v^{rn} + 1$ by the automorphism group generated by $(u, v) \mapsto (\zeta_n^{-1}u, \zeta_n v)$ and $(u, v) \mapsto (\zeta_n u, v)$ is $\mathcal{C}$, as we see by setting $x = v^n$ and $y = u^{rn}$. By Lem. 4.1, $\mathcal{C}$ has CM, proving part (i). □

**Remark 4.3.** In Prop. 4.10 below, we give another proof of Thm. 4.2.

4.2. Streit’s criterion. Let $\mathcal{C}/\mathbb{C}$ be a superelliptic curve , with $\sigma \in \text{Aut}(\mathcal{C})$ and $\bar{\sigma}$ its image in $\overline{\mathcal{N}}$. Let $\chi_{\mathcal{C}}(\sigma)$ be the character of $\sigma$ on $H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}/\mathbb{C})$, and let $\text{Sym}^2 \chi_{\mathcal{C}}(\sigma)$ be the character of $\sigma$ on the $\text{Aut}(\mathcal{C})$-representation $\text{Sym}^2 H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}/\mathbb{C})$.

**Lemma 4.4** (Streit [9]). $\text{Jac}(\mathcal{C})$ has CM if $\langle \text{Sym}^2 \chi_{\mathcal{C}}, \chi_{\mathcal{triv}} \rangle = 0$.

**Remark 4.5.** If $H$ is any subgroup of $\text{Aut}(\mathcal{C})$, it suffices to verify Streit’s criterion considering $\text{Sym}^2 H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}/\mathbb{C})$ as an $H$-representation. This is because $\langle \text{Sym}^2 \chi_{\mathcal{C}}, \chi_{\mathcal{triv}} \rangle_H = 0$ implies that $\langle \text{Sym}^2 \chi_{\mathcal{C}}, \chi_{\mathcal{triv}} \rangle = 0$, since the former means that $\text{Sym}^2 H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}/\mathbb{C})$ has no $H$-invariant vectors whereas the latter means it has no $G$-invariant vectors.

To calculate $\text{Sym}^2 \chi_{\mathcal{C}}$, we record the following lemma.

**Lemma 4.6.** We have

$$\sum_{\sigma \in \mathcal{T}} \text{Sym}^2 \chi_{\mathcal{C}}(\sigma) = \frac{1}{2} \left( \sum_{\sigma \in \mathcal{T}} \chi_{\mathcal{C}}(\sigma^2) + \sum_{\sigma \in \mathcal{T}} \chi_{\mathcal{C}}(\sigma^2) \right).$$

*Proof.* This is a basic result of representation theory. □

We now have the following:

**Proposition 4.7.** Let $\mathcal{C} : y^n = f(x)$ be a smooth superelliptic curve defined over $\overline{\mathcal{C}}$. For each $\mathcal{T} \in \text{Aut}(\mathcal{C})$, let $m$ be its order, and let $\mathcal{T}$ be either ratio of the eigenvalues when $\mathcal{T}$ is thought of as an element of $\text{PGL}_2(\mathbb{C})$ (if $\mathcal{T}$ is a primitive $m$th root of unity). Define

$$k_{\mathcal{T}} = \begin{cases} 1 & \text{if } \mathcal{T} \text{ fixes a branch point of } \mathcal{C} \to \mathbb{P}^1 \\ 0 & \text{otherwise.} \end{cases}$$
Let $\zeta_n$, be a primitive $n$th root of unity, and let $\zeta_{n,\sigma}$ be a primitive $mn$th root of unity such that $\zeta_{n,\sigma}^n = \zeta_\sigma$. Let $A$ be the set of ordered pairs defined below in Eq. (4).

If

$$\sum_{\sigma \in \text{Aut}(\mathcal{C})} \sum_{i=0}^{n-1} \left( \sum_{(a,b) \in A} \zeta_{\sigma}^{2(a+1)i}\zeta_{n,\sigma}^{2(b-1)k_{\sigma}} + \left( \sum_{(a,b) \in A} \zeta_{\sigma}^{a+1}\zeta_{n,\sigma}^{(b-1)k_{\sigma}} \right)^2 \right)$$

vanishes, then $\mathcal{C}$ has CM.

Proof. Suppose $\mathcal{C}$ has genus $g$ and $\deg(f) = d$. From [18], a basis for the space of holomorphic differentials on $\mathcal{C}$ is given by

$$x^a y^b \left( \frac{dx}{y^d} \right),$$

as $(a, b)$ ranges through the set

$$A := \{(a, b) \in \mathbb{Z}^2 \mid a \geq 0, 0 \leq b < n, 0 \leq an + bd \leq 2g - 2\}.$$ 

By the Hurwitz formula, $2g - 2 = -n - \gcd(n, d) + d(n - 1)$, so we can also write $A$ as

$$\{(a, b) \in \mathbb{Z}^2 \mid 0 \leq b < n, 0 \leq a \leq d - 1 - \frac{d(1 + b) + \gcd(n, d)}{n}\}.$$ 

Let $\sigma \in \text{Aut} \mathcal{C}$, $\sigma$ its image in $\overline{\text{Aut}}(\mathcal{C})$, and $m$ be the order of $\sigma$. If $\tau$ is the superelliptic automorphism of $\mathcal{C}$ and $H = \langle \tau \rangle$, then $\sigma H$ is a coset consisting of $n$ different automorphisms of $\mathcal{C}$, say $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$, all projecting to $\sigma \in \overline{\text{Aut}}(\mathcal{C}) \cong \text{Aut}(\mathbb{P}^1)$. Now, $\sigma$ acts on $\mathbb{P}^1$ with two fixed points, and after a change of coordinate we may assume that they are 0 and $\infty$. After possibly replacing $x$ by $1/x$, we may assume that $\sigma$ acts on the coordinate $x$ via $x \mapsto \zeta_\sigma \cdot x$. After this change of variables, there is a polynomial $h \in k[x]$ such that the equation for $\mathcal{C}$ is given by

$$y^n = x^k h(x^m),$$

where $k_{\sigma} = 1$ if the fixed point 0 of $\sigma$ is a ramification point of $X \to \mathbb{P}^1$, and $k_{\sigma} = 0$ otherwise, as in the statement of the proposition.

Fix a primitive $n$-th root of unity $\zeta_n$, as well as a primitive $mn$-th root $\zeta_{n,\sigma}$ as in the statement of the proposition, so $\zeta_{n,\sigma}^n = \zeta_n$. Since $\sigma \mathcal{C} = (\zeta_\sigma \cdot x)$ and $y^n = x^k h(x^m)$, we have that

$$\sigma(y) = \zeta_{n,\sigma}^i \cdot x^k y,$$

for some $i \in \{1, \ldots, n\}$. After reordering $\sigma_1, \ldots, \sigma_n$, we may assume that $\sigma_i(y) = \zeta_{n,\sigma}^i \cdot x^k y$ for each $i$. In particular, $x^a y^b \frac{dx}{y^d}$ is an eigenvector for every $\sigma_i$. Its eigenvalue is

$$\zeta_{n,\sigma}^{a+1}\zeta_{n,\sigma}^{i(b-1)+k_{\sigma}}$$

We thus have

$$\sum_{\sigma \to \tau} \chi_X(\sigma) = \sum_{i=0}^{n-1} \sum_{(a,b) \in A} \zeta_{n,\sigma}^{a+1}\zeta_{n,\sigma}^{i(b-1)+k_{\sigma}},$$

Likewise,

$$\sum_{\sigma \to \tau} \chi_X(\sigma^2) = \sum_{i=0}^{n-1} \sum_{(a,b) \in A} \zeta_{n,\sigma}^{2(a+1)+2i(b-1)+2k_{\sigma}}.$$
Also, 

\[ \sum_{\sigma \in \Psi} \chi(\sigma)^2 = \sum_{i=0}^{n-1} \left( \sum_{(a,b) \in A} \zeta_{\sigma}^{(a+1)i} \zeta_{n,\Psi}^{2(b+1)i + (b-n+1)k} \right)^2. \]

By Lem. 4.6, \( \text{Sym}^2 \chi_C(\sigma) = \frac{1}{2} (\chi_C(\sigma^2) + \chi_C(\sigma)^2) \). Combining this with Eq. (7) and Eq. (8), we have

\[
\sum_{\sigma \in \Psi} \text{Sym}^2 \chi_C(\sigma) = \frac{1}{2} \sum_{i=0}^{n-1} \left( \sum_{(a,b) \in A} \zeta_{\sigma}^{2(a+1)i} \zeta_{n,\Psi}^{2(b+1)i + (b-n+1)k} \right)^2 + \left( \sum_{(a,b) \in A} \zeta_{\sigma}^{a+1} \zeta_{n,\Psi}^{(b+1)i + (b-n+1)k} \right)^2. 
\]

Combining the above with Lem. 4.4 we claim the result.

\[
\text{Proof.} \] The curves \( C_0, C_1, C_4, C_{12}, C_{13}, C_{14}, C_{20}, C_{37}, C_{41} \) all have CM.

\[
\text{Proof.} \] The GAP program \texttt{streit\_program\_gap}\footnote{available as part of the arxiv posting for this paper} computes the sum in Prop. 4.7 for any superelliptic curve, presented as in Prop. 3.3, with \( n, \text{Aut}(C) \), and \( f(x) \) as inputs. The program is modelled on that of Pink and Müller used in \cite{8}. To calculate \( k_\Psi \), we use the embedding of \( G = \text{Aut}(C) \) into \( PGL_2(\mathbb{C}) \) from Prop. 3.3 and its proof. The rest of the calculation is straightforward. For all of the curves in the proposition, the sum in Prop. 4.7 comes to 0.

\[
\text{Remark 4.9.} \] The curves \( C_1, C_4, C_{12}, C_{20}, \) and \( C_{41} \) are all hyperelliptic, and Streit’s criterion was already verified for them in \cite{8}. These correspond to \( X_4, X_7, X_5, X_9 \), and \( X_{14} \) respectively in that paper.

\[
\text{Proposition 4.10.} \] Let \( C \) be a superelliptic curve with many automorphisms, and assume that \( \text{Aut}(C) = C_m \) or \( D_{2m} \). Then \( C \) satisfies Streit’s criterion. That is, \( \langle \text{Sym}^2 \chi_C, \chi_{\text{triv}} \rangle = 0 \).

\[
\text{Proof.} \] Let \( V = \text{Sym}^2 (H^0(C, \omega_C/C)) \), and let \( G = \text{Aut}(C) \). It suffices to show that no non-trivial points of \( V \) are fixed by \( \text{Aut}(C) \).

Let us first assume that \( \text{Aut}(C) = C_m \). By Prop. 3.3, the affine equation of \( C \) is \( y^n = x^k(x^m + 1), \) where \( k \in \{0,1\} \). The set \( \{v_{a,b} := x^a y^b dx/y^{n-1} \mid (a,b) \in A\} \) is a basis of simultaneous eigenvectors for the action of \( G \) on \( H^0(C, \omega_C/C) \), where \( A \) is as in Eq. (4). For each \( v_{a,b} \) in this basis, let \( \lambda_{a,b}(g) \) be its eigenvalue under the action of \( g \in G \). It suffices to prove that there is no set \( \{(a_1,b_1),(a_2,b_2)\} \) of indices such that \( \lambda_{a_1,b_1}(g) \lambda_{a_2,b_2}(g) \) takes the constant value 1 on \( G \).

Suppose \( \sigma(x,y) = (\zeta_m x, \zeta_k y) \), and \( \tau(x,y) = (x,\zeta_n y) \), where \( \zeta_m/n \) is a primitive \( m \)th root of unity with \( \zeta_m = \zeta_n \). The elements \( \sigma \) and \( \tau \) generate \( G \). Then

\[
\lambda_{a_1,b_1}(\sigma^i \tau^j) \lambda_{a_2,b_2}(\sigma^i \tau^j) = \zeta_{mn}^{i(a_1+a_2+2)j(b_1+b_2+2-2n)} \zeta_{mn}^{ik(b_1+b_2+2-2n)}. 
\]

This is independent of \( j \) only if \( b_1 + b_2 + 2 = n \), in which case it equals \( \zeta_{mn}^{i(a_1+a_2+2-k)} \).
By Eq. (5), we have
\[
(9) \quad a_1 + a_2 + 2 \leq 2d - \frac{d(b_1 + b_2 + 2) + \gcd(n, d)}{n} = d - \frac{\gcd(n, d)}{n},
\]
where \(d = m + k\). So \(0 < a_1 + a_2 + 2 - k < m\), which means that \(\zeta_m^{(a_1 + a_2 + 2 - k)}\) is not independent of \(i\). Thus the eigenvalue cannot take the constant value 1 on \(G\).

Now, assume \(\text{Aut}(\mathcal{C}) = D_{2m}\). By Prop. 3.3, the affine equation of \(\mathcal{C}\) is \(y^n = x^k(x^{2m} - 1)\), where \(k \in \{0, 1\}\). The automorphism group \(G\) of \(X\) is generated by
\[
\sigma(x, y) = (\zeta_m x, \zeta_m y), \quad \tau(x, y) = (x, \zeta_n y), \quad \rho(x, y) = (1/x, \zeta_{2n} y/x^{2(m+k)/n}),
\]
where \(\zeta_{2m}\) is any 2th root of unity and \(\zeta_m, \zeta_n\), and \(\zeta_{mn}\) are as before. Let \(H\) be the index two subgroup of \(G\) generated by \(\sigma\) and \(\tau\). The same \(v_{a,b}\) as in the \(C_m\) case form a basis of simultaneous eigenvectors for the action of \(H\) on \(H^0(\mathcal{C}, \omega_{\mathcal{C}/\mathcal{C}})\).

If we again set \(\lambda_{a,b}\) to be the respective eigenvalues, then exactly as in the \(C_m\) case, \(\lambda_{a_1, b_1}(h)\lambda_{a_2, b_2}(h)\) takes the constant value 1 as \(h\) ranges over \(H\) only if \(b_1 + b_2 + 2 = n\) and \(a_1 + a_2 + 2 - k\) is divisible by \(m\). Furthermore, since \(d = 2m + k\), we know from Eq. (9) that \(a_1 + a_2 + 2 - k\) is divisible by \(m\) only if \(a_1 + a_2 + 2 - k = m\).

The only eigenvector with eigenvalue 1 for the action of \(H\) on \(\text{Sym}^2(H^0(\mathcal{C}, \omega_{\mathcal{C}/\mathcal{C}}))\) is
\[
\omega := x^{a_1 + a_2} y^{b_1 + b_2 - 2n + 2} (dx)^2 = x^{m+k-2} y^{-n} (dx)^2 = \frac{x^{m+k-2}}{x^{2m} - 1} (dx)^2 = \frac{x^m}{x^{2m} - 1} \left(\frac{dx}{x}\right)^2.
\]
One sees immediately that \(\rho(\omega) = -\omega\), so \(\omega\) is not fixed under \(G\), which completes the proof. \(\square\)

Note that Prop. 4.10 gives another proof of Thm. 4.2.

5. Negative CM results

In this section, we show that the remaining curves in Table 1 do not have CM.

5.1. Bootstrapping the hyperelliptic case. The following proposition is a direct consequence of the main result of [8].

**Proposition 5.1.** None of the curves \(\mathcal{C}_i\) in Table 1 for
\[
i \in \{9, 10, 11, 15, 16, 18, 19, 25, 27, 28, 30, 31, 32, 34, 35, 36, 38, 39, 40, 48, 49, 50, 51, 52, 53, 55, 57, 58, 60, 62, 64, 65, 67\}
\]
has CM.

**Proof.** The curves \(\mathcal{C}_i\) for \(i \in S := \{9, 15, 25, 28, 31, 36, 48, 53, 58, 65\}\) are all hyperelliptic, and were shown not to have CM in [8, Table 1]. For each of the other curves \(\mathcal{C}_i\) in the proposition, there exists \(j \in S\) such that \(\mathcal{C}_i\) has \(\mathcal{C}_j\) as a quotient by an automorphism fixing \(x\) and multiplying \(y\) by an appropriate root of unity. Since \(\mathcal{C}_j\) does not have CM, neither does \(\mathcal{C}_i\). \(\square\)
5.2. Using stable reduction. It is well-known that if \( C \) is a CM curve defined over a number field \( K \), then its Jacobian \( \text{Jac} C \) has potentially good reduction modulo all primes of \( K \) ([19, Theorem 6]). For any such prime \( p \), let \( K_p \) be the corresponding completion of \( K \). Assume the genus of \( C \) is at least 2. The stable reduction theorem states that there exists a finite extension \( L/K_p \) for which \( C \times_K L \) has a stable model \( C^{ss} \) over Spec \( \mathcal{O}_L \), where \( \mathcal{O}_L \) is the ring of integers of \( L \) (see, e.g., [20, Corollary 2.7]). Specifically, \( C^{ss} \rightarrow \text{Spec} \mathcal{O}_L \) is a flat relative curve whose generic fiber is isomorphic to \( C \) and whose special fiber \( \mathcal{C} \) (called the stable reduction of \( C \) modulo \( p \)) is reduced, has smooth irreducible components, has only ordinary double points for singularities, and has the property that each irreducible component of genus zero contains at least three singular points of \( \mathcal{C} \). One forms the dual graph \( \Gamma_{\mathcal{C}} \) of \( \mathcal{C} \) by taking the vertices of \( \Gamma_{\mathcal{C}} \) to correspond to the irreducible components of \( \mathcal{C} \), with an edge between two vertices for each point where the two corresponding components intersect. A model \( C^{ss} \) of \( C \times_K L \) is called a semistable model if it satisfies all the properties of a stable model, except possibly the requirement on genus zero irreducible components. The dual graph of the special fiber of any \( C^{ss} \) is homeomorphic to \( \Gamma_{\mathcal{C}} \).

One has a similar construction for smooth curves \( Y/K \) with marked points. To wit, if \( \lambda_1, \ldots, \lambda_m \) are points of \( Y(K) \) and \( 2g + m \geq 3 \), there exists a unique stable model \( Y^{ss} \) of the marked curve \( (Y, \{\lambda_1, \ldots, \lambda_m\}) \), which is defined as above, except that we require only that each genus zero component of the special fiber \( \overline{Y} \) contain at least three points that are either singular points of \( \overline{Y} \) or specializations of marked points. We also require that the marked points specialize to distinct smooth points of \( \overline{Y} \). Note that if \( Y = \mathbb{P}^1 \), then \( \Gamma_{\mathcal{Y}} \) is always a tree.

By [21, Chapter 9, §2], \( \text{Jac} C \) has potentially good reduction if and only if \( \Gamma_{\mathcal{C}} \) is a tree (i.e., has trivial first homology). Thus, if we can find a prime \( p \) for which \( \Gamma_{\mathcal{C}} \) is not a tree, then \( \text{Jac} C \) does not have CM. We will use this criterion for several of the curves in Table 1, generalizing [22, §10] to the superelliptic case.

For the rest of Section 5.2, let \( K \) be a complete discrete valuation field with residue field \( k \), and let \( n \in \mathbb{N} \) with char\((k) \nmid n \). Let \( C \rightarrow \mathbb{P}^1_K \) be the (potentially) \( \mathbb{Z}/n \)-cover of \( \mathbb{P}^1_K \) given by the affine equation

\[
y^n = \prod_{i=1}^{m}(x - \alpha_i),
\]

where the \( \alpha_i \) are pairwise distinct elements of \( K \). Let \( B \) be the set of branch points of \( C \rightarrow \mathbb{P}^1_K \) (so \( B \) consists of the \( \alpha_i \), as well as \( \infty \)) if \( n \nmid \deg(\prod_{i=1}^{m}(x - \alpha_i)) \). Let \( \mathcal{Y}^{ss} \) be the stable model of the marked curve \((\mathbb{P}^1_K, B)\), and let \( \Gamma_{\mathcal{Y}} \) be the dual graph of its special fiber. Assume that \( g(C) \geq 2 \), which in turn implies \( |B| \geq 3 \).

**Lemma 5.2.** There exists a finite extension \( L/K \) with valuation ring \( \mathcal{O}_L \), such that the normalization of \( \mathcal{Y}^{ss} \times_{\mathcal{O}_K} \mathcal{O}_L \) in \( L(C) \) is a semistable model of \( C \) over \( L \).

**Proof.** This follows from [10, Corollary 3.6] and its proof, with \((\mathbb{P}^1_K, B)\) playing the role of \((X_{L_0}, D_{L_0})\). \( \square \)

The normalization from Lem. 5.2 induces a map

\[
\pi : \Gamma_{\mathcal{Y}} \rightarrow \Gamma_{\mathcal{C}}
\]

of graphs.

**Lemma 5.3.** Let \( \pi \) be as in Eq. (10).
(i) If $v$ is a leaf of $\Gamma_{\overline{C}}$ (i.e., a vertex incident to only one edge), then $|\pi^{-1}(v)| \leq n/2$.

(ii) Suppose there exists an edge $e$ of $\Gamma_{\overline{C}}$ such that removing $e$ splits $\Gamma_{\overline{C}}$ into two trees $T_1$ and $T_2$ where $n$ divides the number of elements of $B$ specializing to each of $T_1$ and $T_2$. Then $|\pi^{-1}(e)| = n$.

Proof. By the definition of marked stable model, the irreducible component $J$ of $\overline{C}$ corresponding to $v$ contains the specialization of at least one element $b$ of $B$. Since the cover $C \to \mathbb{P}^1_K$ is potentially Galois and ramification indices are at least 2, the preimage of $b$ in $\mathcal{C}$ contains at most $n/2$ points. Since every irreducible component of $\overline{C}$ lying above $J$ contains the specialization of one of these points, there are at most $n/2$ such irreducible components. By the construction of $\pi$, this proves part (i).

Now, assume $e$, $T_1$, and $T_2$ are as in part (ii). Let $v$ be the unique vertex of $T_2$ incident to $e$, and let $J$ be the corresponding irreducible component of $\overline{C}$. For $i \in \{1, 2\}$, write $B_i$ for the subset of $B$ consisting of branch points specializing to $T_i$. Choose three distinct elements $\alpha$, $\beta$, and $\gamma$ of $B$, such that $\alpha \in B_1$ and such that $(\alpha, \beta, \gamma)$ corresponds to $v$ via [10, Proposition 4.2(3)]. Note that in the construction of [10], $\alpha$ can be chosen freely, and then it is automatic that $\beta, \gamma \in B_2$. This is because if, say, $\beta \in B_1$, then in the language of [10], we would have $\overline{\lambda}_i(\beta) = \overline{\lambda}_i(\alpha)$, which contradicts the definition of $\lambda_i$ above [10, Proposition 4.2]). The same holds for $\gamma$.

As in [10, Notation 4.4], the triple $(\alpha, \beta, \gamma)$ gives rise to a coordinate $x_v$ on $\mathbb{P}^1_K$ whose reduction $\overline{x}_v$ is a coordinate on $J$ such that $\overline{x}_v = 0$ at the point of $J$ corresponding to the edge $e$. Since $\overline{x}_v$ is a coordinate on $J$, we in fact have that an element $b \in B$ satisfies $\overline{x}_v(b) = 0$ if and only if $b \in B_2$.

As in [10, §4.3], we can write $f$ in terms of the variable $x_v$, multiply by an appropriate element of $K$, and then reduce modulo a uniformizer of $K$ to obtain an element $\overline{f}_v \in k(\overline{x}_v)$. Since $\overline{x}_v(b) = 0$ if and only if $b \in B_2$, the order of $\overline{f}_v$ at $\overline{x}_v = 0$ is $|B_2|$, which is assumed to be divisible by $n$.

By [10, Proposition 4.5], the restriction of the cover $\overline{C} \to \overline{C}$ above $J$ is given bireationally by the equation $y^n_v = f_v$. Since $\text{ord}_{\overline{x}_v = 0}(\overline{f}_v)$ is divisible by $n$, the preimage of $\overline{x}_v = 0$ in $\overline{C}$ has cardinality $n$. This means that $|\pi^{-1}(e)| = n$, proving part (ii).

Proposition 5.4. In the situation of Lem. 5.3(ii), the graph $\Gamma_{\overline{C}}$ is not a tree.

Proof. Let $e$ be the edge from Lem. 5.3(ii). Consider the graph $\Gamma$ constructed by removing $\pi^{-1}(e)$ from $\Gamma_{\overline{C}}$. Then $\Gamma$ is the disjoint union of $\pi^{-1}(T_i)$ and $\pi^{-1}(T_2)$. Since $\pi$ is ultimately constructed from a normalization Lem. 5.2, every connected component of $\pi^{-1}(T_i)$ maps surjectively onto $T_i$ for $i \in \{1, 2\}$. If $v_i$ is a leaf of $\Gamma_{\overline{C}}$ in $T_i$, then Lem. 5.3(i) shows that $|\pi^{-1}(v_i)| \leq n/2$, so there are at least $n/2$ connected components in $\pi^{-1}(T_i)$. So if $V$ (resp. $E$) is the number of vertices (resp. edges) in $\Gamma$, then $V \leq E + n/2 + n/2 = E + n$. Since $V$ (resp. $E + n$) is the number of vertices (resp. edges) in $\Gamma_{\overline{C}}$, this shows that the first homology of $\Gamma_{\overline{C}}$ has dimension at least 1, so $\Gamma_{\overline{C}}$ is not a tree.

Proposition 5.5. None of the curves $C_{17}$, $C_{21}$, $C_{42}$, $C_{54}$, $C_{59}$, or $C_{61}$ in Table 1 has CM Jacobian.

2Specifically, we have $x_v = \frac{\beta - \gamma}{\beta - \alpha} \frac{\alpha - \alpha}{\alpha - \gamma}$. 
Proof. For $C_{17}$, the tree $\Gamma_{C_{17}}$ as in Prop. 5.4 for the reduction modulo 3 is shown in [22, Figure 3, p. 43]. In this case, $n = 5$, and there are four edges which split the tree up into subtrees with 5 and 15 marked points. By Prop. 5.4, $\Gamma_{C_{17}}$ is not a tree. So Jac$C_{17}$ has bad reduction, and thus does not have CM.

For $C_{61}$, the tree $\Gamma_{C_{61}}$ for reduction modulo 5 is shown in [22, Figure 11, p. 46]. In this case, $n = 7$, and there are four edges which split the tree up into subtrees with 7 and 35 marked points. Using Prop. 5.4 as before, Jac$C_{61}$ has bad reduction, and thus does not have CM.

For curves $C_{21}$, $C_{42}$, $C_{54}$, and $C_{59}$, the program \texttt{stable_reduction.sage}\(^3\) gives the tree $\Gamma_{C_i}$ for reduction modulo 2.

- For curve $C_{21}$, there is an edge splitting $\Gamma_{C_{21}}$ into two trees with 6 and 12 markings, and $n = 3$.
- For curve $C_{42}$, there is an edge splitting $\Gamma_{C_{42}}$ into two trees with 6 and 24 markings, and $n = 3$.
- For curve $C_{54}$, there is an edge splitting $\Gamma_{C_{54}}$ into two trees with 10 and 40 markings, and $n = 5$.
- For curve $C_{59}$, there is an edge splitting $\Gamma_{C_{59}}$ into two trees with 6 and 36 markings, and $n = 3$.

In all cases $i \in \{21, 42, 54, 59\}$, using Prop. 5.4 as before shows that $\Gamma_{C_i}$ is not a tree, which means that Jac$C_i$ has bad reduction, and thus does not have CM. □

Corollary 5.6. None of the curves $C_{22}$, $C_{23}$, $C_{24}$, $C_{44}$, $C_{46}$, $C_{47}$, $C_{56}$, or $C_{63}$ in Table 1 has CM Jacobian.

Proof. The curves $C_{22}$, $C_{23}$, and $C_{24}$ have $C_{21}$ as a quotient (via an automorphism multiplying $y$ by an appropriate root of unity). Likewise, the curves $C_{44}$, $C_{46}$, and $C_{47}$ have $C_{42}$ as a quotient. The curve $C_{56}$ has $C_{54}$ as a quotient. The curve $C_{63}$ has $C_{61}$ as a quotient.

Since all quotients of a CM curve must be CM curves, Prop. 5.5 shows that none of the curves in the corollary is a CM curve. □

5.3. Frobenius criterion. To show that the remaining curves in Table 1 do not have CM, we use a criterion of Müller–Pink. Let $A$ be an abelian variety defined over a number field $K$. For any prime $p$ of $K$ where $A$ has good reduction, let $f_p \in \mathbb{Q}[T]$ be the minimal polynomial of the Frobenius at $p$ acting on the Tate module of the reduction of $A$ modulo $p$. Let $E_{f_p}$ equal $\mathbb{Q}[T]/f_p$. Since the Frobenius action is semisimple, the polynomial $f_p$ has no multiple factors.

If $t$ is the image of $T$ in $E_{f_p}$, then let $E_{f_p}' \subseteq E_{f_p}$ be the subring given by intersecting the rings of $\mathbb{Q}[t^n]_{n \in \mathbb{N}}$ inside $E_{f_p}$. Observe that, if $f_p = g(T^m)$ for some polynomial $g$ and $m \in \mathbb{N}$, we can replace $f_p$ by $g(T)$ when computing $E_{f_p}'$.

The following criterion can be used to show that $A$ does not have CM.

Proposition 5.7 ([8, Theorem 6.2 (a) ⇒ (b)]). Maintain notation as above. If $\dim(A) = g$ and $A$ has CM, then there exists a product of number fields $E$ with $\dim \mathbb{Q}E \leq 2g$ such that for any good prime $p$, we have an embedding $E_{f_p}' \hookrightarrow E$.

Remark 5.8. The paper of Müller–Pink uses the simpler (but weaker) criterion of [8, Corollary 6.7]. However, it appears to be difficult to use this criterion to prove Prop. 5.11 below.

\(^3\)available as part of the arxiv posting for this paper
Before our main application of Prop. 5.7, we prove two lemmas.

**Lemma 5.9.** If $K_1, \ldots, K_n$ and $L_1, \ldots, L_m$, are characteristic 0 fields, then there exists a $\mathbb{Q}$-algebra embedding of $K_1 \times \cdots \times K_n$ into $L_1 \times \cdots \times L_m$ if and only if there exists a surjective map $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that for all $i \in \{1, \ldots, m\}$, there exists an embedding $\gamma_i : K_{\phi(i)} \hookrightarrow L_i$.

**Proof.** An embedding $\pi : K_1 \times \cdots \times K_n \hookrightarrow L_1 \times \cdots \times L_m$ gives rise to a $\mathbb{Q}$-algebra morphism $\pi_i : K_1 \times \cdots \times K_n \to L_i$ for each $i$. Since the kernel is a prime ideal, this morphism is projection onto some $K_j$ followed by an embedding $K_j \hookrightarrow L_i$. Set $j = \phi(i)$. The kernel of $\pi$ is the intersection of the kernels of the $\pi_i$, which is trivial only if each $j$ is equal to $\phi_i$ for some $i$, i.e., if $\phi$ is surjective. Thus the condition in the lemma is necessary for the existence of an embedding.

Conversely, if the condition in the lemma is satisfied, then we define the following $\mathbb{Q}$-algebra morphism, which is easily seen to be an embedding.

$$
\pi : K_1 \times \cdots \times K_n \hookrightarrow L_1 \times \cdots \times L_m
$$

$$(r_1, \ldots, r_n) \mapsto (\gamma_1(r_{\phi(1)}), \ldots, \gamma_i(r_{\phi(i)})).$$

□

**Lemma 5.10.** The curves $C_5, C_6, C_{26}, C_{29}, C_{33}, C_{43}$, and $C_{66}$ from Table 1 have quotients with the following affine equations, respectively, where $i$ is a square root of $-1$.

| Curve | Affine Birational Equation of Quotient Curve |
|-------|---------------------------------------------|
| $C_5$ | $y^3 = x^6 - 33x^4 - 33x^2 + 1$ |
| $C_6$ | $y^4 = x^6 - 33x^4 - 33x^2 + 1$ |
| $C_{26}$ | $y^7 = (x^4 - 4ix^2 + 12)(x^2 - 2i)$ |
| $C_{29}$ | $y^{13} = (x^2 - 4)(x + 14)(x - 34)$ |
| $C_{33}$ | $y^5 = x^{10} + 10x^8 + 35x^6 - 228x^5 + 50x^4 - 1140x^3 + 25x^2 - 1140x + 496$ |
| $C_{43}$ | $y^9 = x^6 + 522x^5 - 10005x^4 - 10005x^2 + 522x + 1$ |
| $C_{66}$ | $y^{31} = (x^2 - 228x + 496)^2(x^2 + 522x - 10004)(x + 11)^2(x^2 + 4)$ |

**Proof.** For $C_5$ and $C_6$, the affine equation given is the obvious quotient by the automorphism fixing $y$ and sending $x$ to $-x$. Likewise, for $C_{43}$, the affine equation given is the obvious quotient by the automorphism fixing $y$ and sending $x$ to $\zeta_5x$.

For $C_{26}$, consider the order 2 automorphism $\sigma$ of $\mathbb{C}(C_{26})$ given by $\sigma(x) = i/x$ and $\sigma(y) = iy/x^2$. The fixed subfield is generated by $z$ and $w$, where $z = x + i/x$ and $w = y/x$. The affine equation of $C_{26}$ is

$$y^7 = x(x^4 - 1)(x^8 + 14x^4 + 1).$$

In terms of $w$ and $z$, this becomes

$$w^7 = \left(x^2 - \frac{1}{x^2}\right)\left(x^4 + 14 + \frac{1}{x^4}\right) = (z^2 - 2i)(z^4 - 4iz^2 + 12),$$

as can be checked by hand or with a computer. Changing back to $x$ and $y$ gives the equation in the table.
For $C_{29}$, consider the automorphism group isomorphic to $D_8$ generated by $\sigma$ and $\tau$ where $\sigma(x,y) = (ix, iy)$ and $\tau(x,y) = (1/x, y/x^2)$. The fixed subfield is generated by $z$ and $w$, where $z = x^4 + 1/x^4$ and $w = y(x^8 - 1)/x^5$. The affine equation of $C_{29}$ is

$$y^{13} = x(x^8 - 1)(x^8 + 14x^4 + 1)(x^8 - 34x^4 + 1).$$

In terms of $w$ and $z$, this becomes

$$w^{13} = (z^2 - 4)^7(z + 14)(z - 34),$$

as can be checked by hand. Changing back to $x$ and $y$ gives the equation in the table.

For $C_{33}$, consider the order 2 automorphism $\sigma$ of $\mathbb{C}(C_{33})$ given by $\sigma(x) = -1/x$ and $\sigma(y) = y/x^4$. The fixed subfield is generated by $z$ and $w$, where $z = x - 1/x$ and $w = y/x^2$. The affine equation of $C_{33}$ is

$$y^5 = x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1.$$ 

In terms of $w$ and $z$, this becomes

$$w^5 = x^{10} - 228x^5 + 494 + \frac{228}{x^5} + \frac{1}{x^{10}}$$

$$= z^{10} + 10z^8 + 35z^6 - 228z^5 + 50z^4 - 1140z^3 + 25z^2 - 1140z + 496,$$

as can be checked by hand or with a computer. Changing back to $x$ and $y$ gives the equation in the table.

For $C_{66}$, consider the automorphism group isomorphic to $D_{10}$ generated by $\sigma$ and $\tau$ where $\sigma(x,y) = (\zeta_5 x, \zeta_5 y)$ and $\tau(x,y) = (-1/x, y/x^2))$. Here $\zeta_5$ is some primitive 5th root of unity. The fixed subfield is generated by $z$ and $w$, where $z = x^5 - 1/x^5$ and $w = y^2/x^2$. The affine equation of $C_{66}$ is

$$y^{31} = x(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1)(x^{10} + 11x^5 - 1).$$

In terms of $w$ and $z$, this becomes

$$w^{31} = (z^2 - 228z + 496)^2(z^2 + 4)(z^2 + 522z - 10004)^2(z + 11)^2$$

as can be checked tediously by hand or more easily with some basic computational assistance. Changing back to $x$ and $y$ gives the equation in the table.

\[\square\]

Proposition 5.11. None of the curves $C_2$, $C_5$, $C_6$, $C_{26}$, $C_{29}$, $C_{33}$, $C_{43}$, or $C_{66}$ in Table 1 has CM Jacobian.

Proof. For $i \in \{5, 6, 26, 29, 33, 43, 66\}$, let $C'_i$ be the quotient curve of $X_i$ from Lem. 5.10. It suffices to prove that neither $C_2$ nor any of these $C'_i$ has CM. For each curve, we use the the program \texttt{frobenius_polynomials.sage} to compute the algebras $E'_q$ for various $q$, and then we show that it is impossible for all the $E'_q$ to embed into a $\mathbb{Q}$-algebra of the correct dimension.

For all cases other than $C'_{29}$ and $C'_{66}$, the curve is $C'_i$ is itself a superelliptic curve, and we can compute the Frobenius minimal polynomial using the superelliptic curve package in Sage (which only works on these types of curves). For $C'_{29}$ and $C'_{66}$, we instead use the function \texttt{ZetaFunction} from Magma, and take the reciprocal polynomial of radical of the numerator. The results are in the following charts.

\footnote{available as part of the arxiv posting for this paper}
Table 2: Curve $C_2$, genus $g_2 := 16$

| p   | $E'_{f,p}$                      |
|-----|---------------------------------|
| 7   | $\mathbb{Q}(\sqrt{-3}) \times \mathbb{Q}(\sqrt{-5 \cdot 23})$ |
| 13  | $\mathbb{Q}(\sqrt{-3})$        |
| 31  | $K_2 \times L_2$               |
| 43  | $\mathbb{Q}(\sqrt{-3}) \times \mathbb{Q}(\sqrt{-5 \cdot 127})$ |
| 67  | $\mathbb{Q}(\sqrt{-3}) \times \mathbb{Q}(\sqrt{-5 \cdot 11 \cdot 13})$ |

Here, $[K_2 : \mathbb{Q}] = [L_2 : \mathbb{Q}] = 8$. Additionally, one verifies that the only quadratic field contained in $K_2$ is $\mathbb{Q}(\sqrt{5})$ and the only quadratic fields contained in $L_2$ are those contained in $\mathbb{Q}(\sqrt{-3}, \sqrt{5})$. Using Lem. 5.9, one sees that a minimum-dimensional product of number fields containing all these $E'_{f,p}$ is $K_2(\sqrt{-3}) \times L_2 \times \mathbb{Q}(\sqrt{-3}, \sqrt{-5 \cdot 23}) \times \mathbb{Q}(\sqrt{-3}, \sqrt{-5 \cdot 127}) \times \mathbb{Q}(\sqrt{-3}, \sqrt{-5 \cdot 11 \cdot 13})$, which has dimension $16 + 8 + 4 + 4 + 4 > 32 = 2g_2$. By Prop. 5.7, $C_2$ does not have CM.

Table 3: Curve $C'_5$, genus $g_5 := 4$

| p   | $E'_{f,p}$                      |
|-----|---------------------------------|
| 7   | $\mathbb{Q}(\sqrt{-3})$        |
| 17  | $\mathbb{Q} \times \mathbb{Q}(\sqrt{-2})$ |
| 23  | $\mathbb{Q} \times \mathbb{Q}(\sqrt{-33})$ |
| 29  | $\mathbb{Q} \times \mathbb{Q}(\sqrt{-39})$ |

Using Lem. 5.9, one sees that a minimum-dimensional product of number fields containing all of these $E'_{f,p}$ is $\mathbb{Q}(\sqrt{-2}, \sqrt{-3}) \times \mathbb{Q}(\sqrt{-3}, \sqrt{-33}) \times \mathbb{Q}(\sqrt{-3}, \sqrt{-39})$, which has dimension $12 > 8 = 2g_5$. By Prop. 5.7, $C'_5$ does not have CM.

Table 4: Curve $C'_6$, genus $g_6 := 7$

| p   | $E'_{f,p}$                      |
|-----|---------------------------------|
| 5   | $\mathbb{Q}(\sqrt{-1})$        |
| 7   | $\mathbb{Q} \times \mathbb{Q}(\sqrt{-6})$ |
| 17  | $\mathbb{Q}(\sqrt{-1}) \times \mathbb{Q}(\sqrt{-33})$ |
| 19  | $\mathbb{Q}(\sqrt{-1}) \times \mathbb{Q}(\sqrt{-21})$ |
| 29  | $\mathbb{Q} \times \mathbb{Q}(\sqrt{-13})$ |

Using Lem. 5.9, one shows that a minimum-dimensional product of number fields containing all of these $E'_{f,p}$ is $\mathbb{Q}(\sqrt{-1}, \sqrt{-6}) \times \mathbb{Q}(\sqrt{-1}, \sqrt{-13}) \times \mathbb{Q}(\sqrt{-1}, \sqrt{-21}) \times \mathbb{Q}(\sqrt{-1}, \sqrt{-33})$, which has dimension $16 > 14 = 2g_6$. By Prop. 5.7, $C'_6$ does not have CM.
Table 5: Curve $C_{26}'$, genus $g_{26} := 15$

| $p$ | $E_{f,p}'$ |
|-----|------------|
| 5   | $Q \times Q(\sqrt{-2 \cdot 7 \cdot 11})$ |
| 17  | $Q \times Q(\sqrt{-2 \cdot 5 \cdot 7 \cdot 13})$ |
| 29  | $Q(\sqrt{-7}) \times Q(\zeta_7) \times L_{26}$ |
| 37  | $Q(\sqrt{-7}) \times Q(\sqrt{-7})$ |
| 61  | $Q \times Q(\sqrt{-2 \cdot 193})$ |
| 73  | $Q \times Q(\sqrt{-2 \cdot 7 \cdot 23 \cdot 113 \cdot 211 \cdot 1571})$ |

Here $[L_{26} : Q] = 12$ and the only quadratic field contained in $L_{26}$ is $Q(\sqrt{-7})$. Using Lem. 5.9, one shows that a minimum-dimensional product of number fields containing all of these $E_{f,p}'$ is

$$Q(\sqrt{-7}, \sqrt{-2 \cdot 7 \cdot 11}) \times Q(\sqrt{-7}, \sqrt{-2 \cdot 5 \cdot 7 \cdot 13}) \times Q(\sqrt{-7}, \sqrt{-2 \cdot 193}) \times Q(\sqrt{-7}, \sqrt{-a}) \times Q(\zeta_7) \times L_{26},$$

where $a = 2 \cdot 7 \cdot 23 \cdot 113 \cdot 211 \cdot 1571$. This has dimension $34 > 30 = 2g_{26}$. By Prop. 5.7, $C_{26}'$ does not have CM.

Table 6: Curve $C_{29}'$, genus $g_{29} := 18$

| $p$ | $E_{f,p}'$ |
|-----|------------|
| 19  | $Q \times Q(\sqrt{-3 \cdot 7 \cdot 3847})$ |
| 53  | $K_{29} \times L_{29}$ |

Here $[K_{29} : Q] = 12$ and $[L_{29} : Q] = 24$, and the only quadratic field contained in $K_{29}$ or $L_{29}$ is $Q(\sqrt{13})$. Using Lem. 5.9, one shows that a minimum-dimensional product of number fields containing $E_{f,p}'$ for $p \in \{19, 53\}$ is

$$K_{29}(\sqrt{-3 \cdot 7 \cdot 3847}) \times L_{29},$$

which has dimension $48 > 36 = 2g_{29}$. By Prop. 5.7, $C_{29}'$ does not have CM.

Table 7: Curve $C_{33}'$, genus $g_{33} := 16$

| $p$ | $E_{f,p}'$ |
|-----|------------|
| 7   | $Q \times Q(\sqrt{-6})$ |
| 13  | $Q \times Q(\sqrt{-1})$ |
| 17  | $Q \times Q(\sqrt{-5 \cdot 41})$ |
| 37  | $Q \times Q(\sqrt{-3 \cdot 47})$ |
| 43  | $Q \times Q(\sqrt{-3 \cdot 83})$ |
| 61  | $Q(\zeta_5)$ |

Using Lem. 5.9, one shows that a minimum-dimensional product of number fields containing all of these $E_{f,p}'$ is

$$Q(\zeta_5, \sqrt{-6}) \times Q(\zeta_5, \sqrt{-1}) \times Q(\zeta_5, \sqrt{-5 \cdot 41}) \times Q(\zeta_5, \sqrt{-3 \cdot 47}) \times Q(\zeta_5, \sqrt{-3 \cdot 83}).$$
This has dimension $40 > 32 = 2g_{43}$. By Prop. 5.7, $C'_{43}$ does not have CM.

Table 8: Curve $C'_{43}$, genus $g_{43} := 10$

| $p$  | $E'_{f,p}$                |
|------|---------------------------|
| 7    | $\mathbb{Q} \times \mathbb{Q} (\sqrt{-10})$ |
| 11   | $K_{43} \times L_{43}$    |
| 13   | $\mathbb{Q} \times \mathbb{Q} (\sqrt{-3})$ |
| 17   | $\mathbb{Q} \times \mathbb{Q} (\sqrt{-2 \cdot 3 \cdot 7 \cdot 19})$ |

Here, $[K_{43} : \mathbb{Q}] = 12$ and $[L_{43} : \mathbb{Q}] = 4$. Additionally, one verifies that the only quadratic field contained in $K_{43}$ is $\mathbb{Q} (\sqrt{-5})$ and the only quadratic field contained in $L_{43}$ is $\mathbb{Q} (\sqrt{-43 \cdot 1361})$. Using Lem. 5.9, a minimum-dimensional product of number fields containing all these $E'_{f,p}$ is

$$K_{43} \times L_{43} \times \mathbb{Q} (\sqrt{-3}) \times \mathbb{Q} (\sqrt{-10}) \times \mathbb{Q} (\sqrt{-2 \cdot 3 \cdot 7 \cdot 19})$$

which has dimension $22 > 20 = 2g_{43}$. By Prop. 5.7, $C'_{43}$ does not have CM. This completes the proof.

Table 9: Curve $C'_{66}$, genus $g_{66} := 90$

| $p$  | $E'_{f,p}$                |
|------|---------------------------|
| 13   | $\mathbb{Q} \times L_{13}$ |
| 17   | $\mathbb{Q} \times L_{17}$ |
| 37   | $\mathbb{Q} \times L_{37}$ |
| 47   | $K_{147} \times L_{47}$   |

Here, $[K_{13} : \mathbb{Q}] = [L_{13} : \mathbb{Q}] = 4$, and $[L_{37} : \mathbb{Q}] = 20$, $[K_{47} : \mathbb{Q}] = 6$, and $[L_{47} : \mathbb{Q}] = 30$. Furthermore, there is an embedding $K_{47} \hookrightarrow L_{47}$, the fields $L_{13}$ and $L_{17}$ are linearly disjoint, and $K_{47}$ is linearly disjoint from $L_i$ for $i \in \{13, 17, 37\}$. All of this is verified in `frobenius_polynomials.sage`. Since $K_{47} \hookrightarrow L_{47}$, Lem. 5.9 shows that any product of number fields containing all these $E'_{f,p}$ must have an embedding of $K_{47}$ into each factor. Another application of Lem. 5.9 shows that a minimum-dimensional product of number fields containing all these $E'_{f,p}$ is

$$K_{147} L_{13} \times K_{147} L_{17} \times K_{47} L_{37} \times L_{47}$$

which has dimension $198 > 180 = 2g_{66}$. By Prop. 5.7, $C'_{43}$ does not have CM. This completes the proof.

\[\square\]

**Corollary 5.12.** None of the curves $C_3$, $C_7$, $C_8$, or $C_{45}$ in Table 1 has CM Jacobian.

*Proof.* The curve $C_3$ has $C_2$ as a quotient. The curves $C_7$ and $C_8$ have $C_5$ as a quotient, and the curve $C_{45}$ has $C_{43}$ as a quotient. Since all quotients of a CM curve must be CM curves, Prop. 5.7 shows that none of the curves in the corollary is a CM curve.

\[\square\]
Now we are ready to state the main theorem. Recall that if \( C \) is a smooth superelliptic curve with reduced automorphism group isomorphic to \( A_4, S_4, \) and \( A_5 \), then \( C \) is isomorphic to one of the curves in Table 1.

**Theorem 5.13.** For each case in Table 1, whether or not \( \text{Jac}C \) has CM is determined in the 6-th column of the Table.

**Proof.** This follows from Prop. 4.8, Prop. 5.1, Prop. 5.5, Cor. 5.6, Cor. 5.12, and Prop. 5.11. \( \square \)

**Corollary 5.14.** For superelliptic curves with many automorphisms, having complex multiplication is equivalent to satisfying Streit’s Criterion (Lem. 4.4).

**Proof.** One direction is immediate from Lem. 4.4. For the other, observe first that for each entry in Table 1 that is a CM curve, the fact that the Jacobian has CM is proven using Streit’s criterion in Prop. 4.8. Combining this with Prop. 4.10 finishes the proof. \( \square \)

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