A THEOREM ON PLACES OF FUNCTION FIELDS, WITH APPLICATIONS TO REAL HOLOMORPHY RINGS

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ABSTRACT. Given an algebraic function field $F|K$, we prove that the places that are composite with a fixed place of $K$ lie dense in the space of all places of $F$, in a strong sense. We apply the result to the case of $K = R$ any real closed field and the fixed place on $R$ being its natural (finest) real place. This leads to a new description of the real holomorphy ring of $F$ which can be seen as an analogue to a certain refinement of Artin’s solution of Hilbert’s 17th problem. We also determine the relation between the topological space $M(F)$ of all $R$-places of $F$ (places with residue field contained in $R$), its subspace of all $R$-places of $F$ that are composite with the natural $R$-place of $R$, and the topological space of all $R$-rational places. Further results about these spaces as well as various classes of relative real holomorphy rings are proven. At the conclusion of the paper the theory of real spectra of rings will be applied to interpret basic concepts from that angle and to show that the space $M(F)$ has only finitely many topological components.

1. Introduction and main theorems on places

The Main Theorem of [23] showed the density (in a very strong sense) of certain types of places in the space of all places of a function field of characteristic 0 (by “function field” we will always mean an algebraic function field of transcendence degree at least 1). A modification of the Main Theorem was then applied to various classes of holomorphy rings, including the real and the $p$-adic. In a subsequent paper [19], the Main Theorem was generalized to arbitrary characteristic. The density
of several important sets of places was shown, such as prime divisors, as well as the Abhyankar places which play a crucial role e.g. in [14, 21]. While the paper [23] only considered the space
\[ S(F|K) = \{ \xi \text{ place of } F \mid \xi|_K = \text{id}_K \} \]
of all places of an algebraic function field \( F|K \) that are trivial on \( K \), the scope was widened in [19] to the spaces
\[ S(F|K; \wp) = \{ \xi \in S(F) \mid \xi|_K = \wp \} \]
of places of \( F \) that extend a fixed place \( \wp \) of \( K \). We note that every \( S(F|K; \wp) \) is a subset of the Zariski space \( S(F) \) of all places of \( F \), and that \( S(F|K) = S(F|K; \text{id}_K) \).

However, one interesting subset of these spaces was entirely missed: the set consisting of those places that factor over \( S(F|K) \) (see below for the precise definition). In this paper we will adapt the proofs of the density theorems from [23, 19] so as to prove the density of this subset and show how this is used to obtain ample information on real holomorphy rings and the topologies of various spaces of real places.

In order to present our central Theorem 1.1, we need some preparations. In contrast to the usage in [14, 19, 20, 21, 22] we will treat places as usual functions and apply them to elements from the left, that is, the image of \( a \) under a place \( \xi \) will be denoted by \( \xi(a) \). However, we will keep one convention: the residue field of \( F \) under \( \xi \) will be denoted by \( F\xi \). Further, the valuation, valuation ring and valuation ideal associated with the place \( \xi \) will be denoted by \( v_\xi, O_\xi \) and \( M_\xi \), respectively. The value group of \( v_\xi \) on \( F \) will be denoted by \( v_\xi F \).

If \( < \) is an ordering on the field \( F \), then we will say that \( \xi \) (or its associated valuation \( v_\xi \)) is **compatible with** \( < \) if \( O_\xi \) is convex relative to this given ordering. That a place \( \xi \) on \( F \) is compatible with some of the orderings on \( F \) is equivalent to the statement that \( F\xi \) is a formally real field. This is one essential part of the so-called Baer-Krull Theorem (cf. [2, Theorem 10.1.10]).

Further, recall that a field \( K \) is **existentially closed in** an extension field \( F \) if every existential sentence in the language of rings with parameters from \( K \) which holds in \( F \) will also hold in \( K \). For further explanations, see [23, Section 1]. Similarly, a valued field \( K \) (or an ordered field \( K \) with a valuation) is existentially closed in an extension field \( F \) with ordering and valuation extending those of \( K \) if every existential sentence in the language of rings with a relation symbol for a valuation (or with relation symbols for an ordering and a valuation, respectively) and with parameters from \( K \) which holds in \( F \) will also hold in \( K \).
Now we have all definitions in place to state our central theorem.

**Theorem 1.1.** Take an arbitrary field $K$ with a place $\varphi$, a function field $F$ over $K$, a place $\xi \in S(F|K; \varphi)$, and nonzero elements $a_1, \ldots, a_m \in F$. Choose $r \in \mathbb{N}$ such that $1 \leq r \leq s = \mathrm{trdeg} F|K$ and an arbitrary ordering on $\mathbb{Z}^r$; denote by $\Gamma$ the so obtained ordered abelian group. If $\mathrm{trdeg} F|K > 1$ and $\varphi$ is trivial while $\xi$ is not, then we assume in addition that $\Gamma$ is the lexicographic product $\Gamma' \times \mathbb{Z}$, where $\Gamma' = \mathbb{Z}^{r-1}$ endowed with an arbitrary ordering.

Then there is a place $\lambda \in S(F|K)$ and an extension $\varphi'$ of $\varphi$ from $K$ to $F\lambda$ such that, with $\xi' := \varphi' \circ \lambda \in S(F|K; \varphi)$,

(a) $F\lambda$ is a finite extension of $K$,

(b) $v_{\lambda}F \subseteq \Gamma$ with $(\Gamma : v_{\lambda}F)$ finite,

(c) if $a_i \in \mathcal{O}_\xi$, then $\lambda(a_i) \in \mathcal{O}_{\varphi'}$ and $a_i \in \mathcal{O}_\xi$.

The following assertions can also be realized, unless $\varphi$ is trivial while $\xi$ is not:

(d) if $a_i \in \mathcal{M}_\xi$, then $\lambda(a_i) \in \mathcal{M}_{\varphi'}$ and $a_i \in \mathcal{M}_\xi$,

(e) if $\xi(a_i) \in K\varphi$, then $\xi'(a_i) = \xi(a_i)$ for $1 \leq i \leq m$.

(f) $\lambda(a_i) \neq 0, \infty$ for $1 \leq i \leq m$.

If $\varphi$ is trivial while $\xi$ is not, then we have: if $\mathrm{trdeg} F|K > 1$, then either (d) and (e) or (f) can also be realized, and if $\mathrm{trdeg} F|K = 1$, then (f) can also be realized.

In addition:

A) If $F\xi = K\varphi$, and if $\mathrm{trdeg} F|K > 1$ in case $\varphi$ is trivial, then we can also obtain that $F\xi'|K\varphi$ is a finite purely inseparable extension.

B) If $(K, \varphi)$ is existentially closed in $(F, \xi)$, then in addition, we can obtain that $F\lambda = K$, $v_{\lambda}F = \Gamma$, $F\xi' = K\varphi$ and $\varphi' = \varphi$.

C) If $K = R$ is a real closed field, $\varphi$ is a place compatible with its ordering and $\xi$ is compatible with an ordering $< \xi$ of $F$, then we can also obtain that $F\lambda = R$, $v_{\lambda}F = \Gamma$, $F\xi' = R\varphi$, $\varphi' = \varphi$, and

(g) if $a_i > 0$, then $\lambda(a_i) > 0$, unless $\varphi$ is trivial while $\xi$ is not and we want $\lambda$ and $\xi'$ to satisfy assertions (d) and (e),

(h) if $a_i > 0$ and $\xi(a_i) \neq 0, \infty$, then $\xi'(a_i) > 0$.

The latter implies that if $\infty \neq \xi(a_i) > 0$, then $\xi'(a_i) > 0$.

Moreover, there are infinitely many nonequivalent places $\lambda$ and $\xi'$ with the above properties. If $\mathrm{trdeg} F|K = 1$ and $\varphi$ is trivial while $\xi$ is not, then $\xi$ itself satisfies assertions (a)-(e), and also (h) under the conditions of C), but it may be the only such place.
Remark 1.2. In the case where \( \wp \) is trivial while \( \xi \) is not, assertion (d) is incompatible with (f) and (g) because in this case, \( \mathcal{O}_\xi = \mathcal{O}_\lambda \). Hence if \( 0 \neq a_i \in \mathcal{M}_\xi \) and \( \lambda \) satisfies assertion (d), then \( a_i \in \mathcal{M}_\lambda \), hence \( \lambda(a_i) = 0 \) so that assertion (f) is not satisfied by \( \lambda \); if in addition \( a_i > 0 \), then also assertion (g) is not satisfied by \( \lambda \).

Theorem 1.1 will be proven in Section 4. The proof of assertion C) for trivial \( \wp \) uses the fact that a real closed field is existentially closed in every formally real extension field (cf. Theorem 4.2), which allows us to apply Theorem 23 of [19].

Given two places \( \xi' \) and \( \pi' \), we will say that \( \xi' \) **factors over** \( \pi' \) (or in other words, is **composite with** \( \pi' \)) if there is a place \( \lambda \) such that \( \xi' = \pi' \circ \lambda \). Theorem 1.1 shows the strong density of the subset of \( S(F|K; \wp) \) of all places \( \xi' \) that factor over \( \wp' \) for a suitable finite extension \( (K', \wp') \) of \( (K, \wp) \). Let us describe one consequence of the strong density. Every set \( S(F|K; \wp) \) carries the **Zariski topology**, for which the basic open sets are the sets of the form

\[
\{ \xi \in S(F|K; \wp) \mid a_1, \ldots, a_k \in \mathcal{O}_\xi \},
\]

where \( k \in \mathbb{N} \cup \{0\} \) and \( a_1, \ldots, a_k \in F \). With this topology, \( S(F|K; \wp) \) is a spectral space (see [19, Appendix] for a proof, and [12] for details on spectral spaces); in particular, it is quasi-compact. Its associated **patch topology** (or **constructible topology**) is the finer topology whose basic open sets are the sets of the form

\[
\{ \xi \in S(F|K; \wp) \mid a_1, \ldots, a_k \in \mathcal{O}_\xi; a_{k+1}, \ldots, a_{k+\ell} \in \mathcal{M}_\xi \},
\]

where \( k, \ell \in \mathbb{N} \cup \{0\} \) and \( a_1, \ldots, a_{k+\ell} \in F \). With the patch topology, \( S(F|K; \wp) \) is a totally disconnected compact Hausdorff space. From the previous theorem, in particular assertions (c) and (d), we obtain:

**Corollary 1.3.** Take a function field \( F|K \) and a place \( \wp \) on \( K \). Then every nonempty open set in the Zariski topology of \( S(F|K; \wp) \) contains infinitely many places that factor over \( \wp' \) for a suitable finite extension \( (K', \wp') \) of \( (K, \wp) \). The same holds in the Zariski patch topology, unless \( \text{trdeg} F|K = 1 \) and \( \wp \) is trivial.

In view of our later applications to real holomorphy rings, we will need a stronger result in the case where \( K = R \) is a real closed field. The additional information is provided by statement C). Let us give the definitions necessary to deal with formally real function fields \( F \) over real closed fields \( R \). By

\[
M(F)
\]
we will denote the set of all \( \mathbb{R} \)-places of \( F \), that is, places \( \xi \) of \( F \) with residue field \( F\xi \subseteq \mathbb{R} \). These are exactly (up to equivalence of places) the places associated with the natural valuations of the orderings on \( F \), where the natural valuations of an ordered field \((F, <)\) is the finest valuation compatible with that ordering. In particular, every real closed field \( R \) has a unique \( \mathbb{R} \)-place \( \xi_R \), which we will call its natural \( \mathbb{R} \)-place.

Instead of the set \( S(F|R) \) of all places of \( F \) that are trivial on \( R \), we are rather interested in the set \( M(F|R) = \{ \lambda \in S(F|R) : F\lambda = R \} \).

of \( R \)-rational places. The new object we study in this paper is the set

\[ M_R(F) := \{ \xi_R \circ \lambda : \lambda \in M(F|R) \} \subseteq S(F|R ; \xi_R) \]

of all \( \mathbb{R} \)-places of \( F \) that factor over \( \xi_R \). Statement C) of Theorem 1.1 implies that for every \( \mathbb{R} \)-place \( \xi \) of \( F \) there is an \( \mathbb{R} \)-place \( \xi' \) of \( F \) that factors over \( \xi_R \) and is “very close to \( \xi \”).

Remark 1.4. Note that we usually do not identify equivalent real places. However, here any two equivalent places in \( M_R(F) \) are equal since their residue fields are equal to the archimedean real closed field \( R\xi_R \subseteq \mathbb{R} \) which does not allow any nontrivial isomorphism into \( \mathbb{R} \). Also in \( M(F|R) \), by its definition equivalent places are equal. As we are interested in the compositions of \( R \)-rational places with the natural \( \mathbb{R} \)-place \( \xi_R \) of \( R \), this constitutes no loss of information. Indeed, assume that \( \lambda_1 \) and \( \lambda_2 \) are equivalent \( R \)-rational places, and write \( \lambda_2 = \sigma \circ \lambda_1 \) for some isomorphism \( \sigma \). As \( \lambda_2 \) is assumed to be \( R \)-rational, \( \sigma \) must be an automorphism of \( R \). Since \( R \) is real closed, it is also order preserving. As \( \mathcal{O}_R := \mathcal{O}_{\xi_R} \) is the convex hull of \( \mathbb{Q} \) in \( R \) and \( \mathbb{Q} \) is left elementwise fixed by \( \sigma \), it follows that \( \sigma \mathcal{O}_R = \mathcal{O}_R \). This implies that \( \xi_R \) and \( \xi_R \circ \sigma \) are equivalent, and with the same argument as before, we find that they are equal. Thus, \( \xi_R \circ \lambda_1 \) and \( \xi_R \circ \lambda_2 = \xi_R \circ \sigma \circ \lambda_1 \) are equal.

Theorem 1.1 is essential for the description of the relation between the sets \( M(F|R) \), \( M_R(F) \) and \( M(F) \). It will be applied to formally real function fields \( F \) over arbitrary real closed fields \( R \) where we address the following issues.

The set \( M(F) \), its subset \( M_R(F) \) and the set \( M(F|R) \) carry natural topologies. The topology of \( M(F) \) as described by Dubois in [9] is compact and Hausdorff; we will denote it by \( \text{Top} \ M(F) \). It is a quotient
topology of the space of orderings with the Harrison topology. Its basic open sets are
\[ U(f_1, \ldots, f_m) := \{ \xi \in M(F) \mid \xi(f_i) > 0 \text{ for } 1 \leq i \leq m \} \]
where \( f_1, \ldots, f_m \) lie in the real holomorphy ring \( H(F) \) of the field \( F \), which is defined to be the intersection of the valuation rings of all real places of \( F \); it is equal to the intersection of the valuation rings of all \( \mathbb{R} \)-places of \( F \). Note that if \( R \) is a real closed field, then \( H(R) = \mathcal{O}_R \).

When we speak of the topological space \( M(F) \), we will always refer to \( \text{Top} M(F) \), and the subset \( M_R(F) \subseteq M(F) \) will always carry the subspace topology. So far, the topological space \( M_R(F) \) has not found any attention in the literature. Yet, for our present study it is highly relevant. In particular, Proposition 2.1 will show that \( M_R(F) \) is dense in \( M(F) \), which is a very important fact.

In an analogous way, a topology \( \text{Top} M(F|R) \) on \( M(F|R) \) will be introduced. It is then shown in Theorem 2.4 that the mapping
\[
(3) \quad \iota_{F|R} : M(F|R) \to M(F), \lambda \mapsto \xi_R \circ \lambda
\]
is a topological embedding with image \( M_R(F) \). In the same theorem, it is shown that all three topological spaces have no isolated points.

Similar to the space \( M_R(F) \), the space \( M(F|R) \) has found little, if any, attention in real algebraic geometry. It was passed by in favour of stronger topological spaces, see e.g. [26]. In the concluding section of this paper we re-address these three topological spaces by invoking the theory of real spectra of rings, a cornerstone of modern real algebraic and semi-algebraic geometry, see [2]. As a surprising application we derive that the space \( M(F) \), where \( F \) a formally real function field over any real closed field, admits only finitely many connected components.

So far, various authors have already studied the relative real holomorphy ring
\[ H(F|R) := \{ a \in F \mid \xi(a) \neq \infty \text{ for all real places } \xi \in S(F|R) \} \]
for function fields \( F \) over real closed fields \( R \), and its extensions
\[ H(F|R)D \]
(the smallest subring of \( F \) containing \( H(F|R) \) and \( D \)), where \( D \) is a finitely generated \( R \)-algebra inside \( F \), cf. [2, 4, 5, 13, 15, 17, 18, 23, 26, 27, 28]. Model theory or algebraic geometry or a combination of both theories have been used. Common to all of these approaches is that they use the fact that \( F \) admits many smooth models (projective or real complete affine ones), which in turn allows to study the behaviour of the elements in \( F \) as functions on the set \( M(F|R) \).
In the case of a non-archimedean real closed base field $R$, this relationship seems to get lost once one turns to the absolute real holomorphy ring $H(F)$ in place of $H(F|R)$. However, using the set $M_R(F)$ of all $\mathbb{R}$-places of $F$ that factor over the natural $\mathbb{R}$-place of $R$, we are able to prove representations for $H(F)$ and related rings that still retain the geometric flavour; cf. Section 2.2.

Theorem 1.1 allows much wider application to all composite places which factor over places in $M(F|R)$. It is this strength that allows to broadly extend previous results on the relative real holomorphy rings. In fact we can include the class of rings $H(F|D)$ where $D$ is a general finitely generated ring extension over any real valuation ring $B$ of the base field $R$.

2. Applications to topologies and holomorphy rings

2.1. Sets of real places and their topologies. From Theorem 1.1 we will deduce:

**Proposition 2.1.** Take a function field $F$ over a real closed field $R$. Then the set $M_R(F)$ is dense in $M(F)$ with respect to $\text{Top}M(F)$. If in addition $R$ is non-archimedean, then every nonempty intersection of an open set in the Zariski patch topology of $M(F)|R$ with an open set in $\text{Top}M(F)$ contains infinitely many places from $M_R(F)$.

*Proof.* Take a function field $F$ over a real closed field $R$. Assume that there is an $\mathbb{R}$-place $\xi \in U(f_1, \ldots, f_m)$ and choose a compatible ordering $\prec_\xi$ on $F$. Then there are positive rational numbers $q_1$ and $q_2$ such that

$q_1 \prec_\xi f_i \prec_\xi q_2 \quad \text{for } 1 \leq i \leq m$.

Using assertion C) of Theorem 1.1, we obtain a place $\lambda \in M(F|R)$ such that

$q_1 < \lambda(f_i) < q_2$ in $R$. Composing $\lambda$ with $\xi_R$, we obtain

$q_1 \leq \xi_R \circ \lambda(f_i) \leq q_2$,

which shows that the $\mathbb{R}$-place $\xi_R \circ \lambda$ is in $U(f_1, \ldots, f_m)$. This proves the first assertion of Proposition 2.1.

In order to prove the second assertion, assume in addition that $R$ is nonarchimedean. Then $\xi_R$ is a non-trivial place. Further, consider elements $a_1, \ldots, a_{k+\ell+m} \in F$ and an $\mathbb{R}$-place $\xi$ of $F$ such that $a_1, \ldots, a_k \in \mathcal{O}_\xi$, $a_{k+1}, \ldots, a_{k+\ell} \in \mathcal{M}_\xi$ and $\xi(a_{k+\ell+1}) > 0, \ldots, \xi(a_{k+\ell+m}) > 0$. Note
that $\xi|_R = \xi_R$. Hence by assertion C) of Theorem 1.1 there are infinitely many $R$-rational places $\lambda$ of $F$ and places $\xi' = \xi_R \circ \lambda$ such that:

1. $a_1, \ldots, a_k \in \mathcal{O}_{\xi'}$ and $a_{k+1}, \ldots, a_{k+\ell} \in \mathcal{M}_{\xi'}$;
2. $\xi'(a_{k+\ell+1}) > 0, \ldots, \xi'(a_{k+\ell+m}) > 0$.

We observe the following equivalences that hold for all $a \in F$:

3. $\lambda(a) > 0 \iff \lambda\left(\frac{a}{1+a^2}\right) > 0$,
4. $\lambda(a) \neq 0, \infty \iff \lambda\left(\frac{a^2}{1+a^2}\right) > 0$.

We introduce a topology $\text{Top} M(F|R)$ on $M(F|R)$ through the basic open sets

$$V(f_1, \ldots, f_m) := \{\lambda \in M(F|R) \mid \lambda(f_i) > 0 \text{ for } 1 \leq i \leq m\}$$

where $f_1, \ldots, f_m \in H(F|R)$. Note that $\frac{f_i}{1+f_i^2} \in H(F) \subseteq H(F|R)$. Using the equivalence (4) we can thus replace the condition “$f_1, \ldots, f_m \in H(F|R)$” by “$f_1, \ldots, f_m \in H(F)$” without changing the collection of basic sets.

**Proposition 2.2.** Take a function field $F$ over a real closed field $R$, and choose elements $f_1, \ldots, f_k \in H(F)$, and nonzero elements $a_1, \ldots, a_k \in F$. If the basic set $V(f_1, \ldots, f_k)$ of $\text{Top} M(F|R)$ is non-empty, then there are infinitely many places in

$$\{\lambda \in V(f_1, \ldots, f_k) \mid \lambda(a_j) \neq 0, \infty \text{ for } 1 \leq j \leq \ell\}.$$

**Proof.** Take $\lambda_0 \in V(f_1, \ldots, f_k)$. Then $\lambda_0$ is an $R$-rational place and therefore $F$ admits an ordering which is compatible with $\lambda_0$ and under which $f_1, \ldots, f_k$ are positive. This ordering together with $\lambda_0$ can be extended to a function field $F'$, with $F'|F$ finite, and an $R$-rational place $\lambda'_0$ of $F'$ in which there are elements $a_{\ell+1}, \ldots, a_{\ell+k}$ such that $f_i = a_{\ell+i}^2$ for $1 \leq i \leq k$.

Since $R$ is real closed and $F'$ is formally real, we know that the field $R$ is existentially closed in $F'$. By Theorem 4.3 below there are infinitely many $R$-rational places of $F'$ which do not take the value $\infty$ on $a_1, \ldots, a_{\ell+k}, a_1^{-1}, \ldots, a_{\ell+k}^{-1}$. Hence they do not take the values $0, \infty$ on $a_1, \ldots, a_{\ell+k}$. In particular, they send $f_1, \ldots, f_k$ to squares $\neq 0, \infty$ which consequently are positive elements of $R$. The restrictions of these places yield the desired places in the set (6). Indeed, since $F'|F$
Theorem 2.4. Take a function field \( \xi \in M(F) \) over a real closed field \( F \) which contains all elements \( f_1, \ldots, f_k, \frac{a_i^2}{1+a_i^2}, \ldots, \frac{a_i^2}{1+a_i^2} \). Then there are positive rationals \( c, d \) such that \( c < \xi(f) < d \). Thus \( V(f) \) is an open set.

Alternatively, one may prove the last proposition by passing to an regular affine \( R \)-algebra \( A \) with quotient field \( F \) which contains all elements \( f_1, \ldots, \frac{a_i^2}{1+a_i^2} \). One then applies the so-called Artin-Lang Homomorphism Theorem (see [2, 4.1.2]) and uses the fact that every regular \( R \)-point is the center of a rational \( R \)-place.

Remark 2.3. By the equivalence (5), the set in (6) is equal to

\[
V \left( f_1, \ldots, f_k, \frac{a_i^2}{1+a_i^2}, \ldots, \frac{a_i^2}{1+a_i^2} \right).
\]

A point \( x \) in a topological space is called isolated if the singleton \( \{x\} \) is an open set.

Theorem 2.4. Take a function field \( F \) over a real closed field \( R \) and let \( \xi_R \) be the natural \( \mathbb{R} \)-place of \( R \).

1) The mapping \( \iota_{F|R} : M(F|R) \to M_R(F) \) defined in (3) is a bijection.
2) \( \iota_{F|R} \) is a topological embedding of \( M(F|R) \) into \( M(F) \).
3) All nonempty open sets in \( M(F|R), M(F) \) and \( M_R(F) \) are infinite.
4) In particular, none of the spaces \( M(F|R), M(F) \) and \( M_R(F) \) admit any isolated points.

Proof. 1): This is a special instance of Proposition 2.7 in the next section.

2): We first prove that \( \iota_{F|R} : M(F|R) \to M(F) \) is continuous. Take \( \lambda \in M(F|R) \) and \( f \in H(F) \) such that \( \xi := \iota_{F|R}(\lambda) = \xi_R \circ \lambda \in U(f) \). Then there are positive rationals \( c, d \) such that \( c < \xi(f) < d \). Then also \( c < \lambda(f) < d \), so \( \lambda \in V(f-c) \cap V(d-f) =: V \) and \( V \) is an open neighbourhood of \( \lambda \). We will show that \( \iota_{F|R}(V) \subseteq U(f) \). If \( \lambda' \in V \), then \( c < \lambda'(f) < d \), whence \( c \leq \xi_R \circ \lambda'(f) \leq d \). Thus \( \xi_R \circ \lambda' \in U(f) \).

Hence, \( \iota_{F|R} \) is shown to be continuous.

Next, we prove that \( \iota_{F|R} : M(F|R) \to M_R(F) \) is an open map. To this end, take an arbitrary subbasic set \( V(f) = \{ \lambda \in M(F|R) \mid \lambda(f) > 0 \} \) where we may take \( f \in H(F) \). We have to show that \( \iota_{F|R}(V(f)) \) is open in the subspace topology on \( M_R(F) \). Take any \( \lambda \in V(f) \) and set \( \xi = \xi_R \circ \lambda \). Then \( a := \lambda(f) \in H(R) \), \( a > 0 \). Set \( g := \frac{af}{a^2+f^2} \). One sees that \( g \in H(F) \). We obtain that \( \lambda(g) = \frac{1}{2} = \xi(g) \) and therefore \( \xi \in U(g) \cap M_R(F) \). We want to show that the whole neighbourhood \( U(g) \cap M_R(F) \) of \( \xi \) is contained in \( \iota_{F|R}(V(f)) \).
If $\xi' \in U(g) \cap M_R(F)$, then $\xi' = \xi_R \circ \lambda'$ with $\lambda' \in M(F|R)$, and $\xi_R(\lambda'(g)) = \xi'(g) > 0$ implies that $\lambda'(g) > 0$, whence $\lambda(f) > 0$. This yields that $\lambda' \in V(f)$, and the inclusion $U(g) \cap M_R(F) \subseteq \iota_{F|R}(V(f))$ is proven.

3): The assertion about $M(F|R)$ follows from Proposition 2.2. From this the assertion about $M_R(F)$ follows by part 2) of our theorem, which together with the density of $M_R(F)$ in $M(F)$ (cf. Proposition 2.1) implies the assertion for $M(F)$.

4): The assertions follow directly from part 3). □

Remark 2.5. Here is an even simpler proof of the fact that $M(F)$ has no isolated points (from which the same follows for $M_R(F)$ and $M(F|R)$ via the density of $M_R(F)$ in $M(F)$ and part 2) of the above theorem). We have that $F$ is a finite extension of some rational function field $R(x_1, \ldots, x_n)$. Assume that $\xi$ is an isolated point in $F$, i.e., $U := \{\xi\}$ is an open subset of $M(F)$. Take the inverse image $V$ of $U$ in the space of orderings of $F$. It is open since $M(F)$ is a quotient space of the space of orderings of $F$, and it has only finitely many elements by the Baer-Krull Theorem (note that the value group of the restriction of any place $\xi \in M(F)$ to the real closed field $R$ has divisible value group and as $F|R$ has finite transcendence degree, it follows that $v_\xi(F)/2v_\xi(F)$ is finite). Consider the set of all orderings on $R(x_1, \ldots, x_n)$ induced by the orderings in $V$. By the openness of the restriction function for orderings (cf. [10, Theorem 4.4]), this set is open in the space of orderings of $R(x_1, \ldots, x_n)$. As it contains a finite number of elements, this is impossible, as [6, Theorem 10] shows that the space of orderings of $R(x_1, \ldots, x_n)$ does not have isolated points. □

2.2. Holomorphy rings. In the introduction we alluded to the rings $H(F)B[x_1, \ldots, x_n]$, $B$ any real valuation ring of $R$. We will show that these rings admit a description as an intersection of valuation rings of a family $\mathcal{F}$ of composite places, or in other words: they are the holomorphy ring of this family. This section begins with a general study of rings which are intersections of families of valuation rings of composite places. It turns out that this property is closely related to a certain type of Nullstellensatz, a fact which was first observed by H.-W. Schülting, cf. [27, Section 2].

Two further issues will be discussed in this section. We look at the existence of minimal representations as an intersection of valuation rings of composite places. Secondly, a new description of the real holomorphy ring $H(F)$ will be presented which can be seen as an analogue to a certain refinement of Artin’s solution of Hilbert’s 17th problem.
Given any subring $D$ of $F$, the relative real holomorphy ring $H(F|D)$ is defined as follows:

$$H(F|D) := \bigcap\{O \mid O \text{ a real valuation ring of } F \text{ and } D \subseteq O\}.$$  

We find that

$$H(F|D) = H(F)D,$$

$$H(F|B[x_1, \ldots, x_n]) = H(F|B)[x_1, \ldots, x_n] = H(F)B[x_1, \ldots, x_n],$$

since all rings are Prüfer rings, hence intersections of their valuation overrings, and one checks that the rings to be compared in each of the two cases admit the same set of valuation overrings.

Note: if a subring $A \subseteq F$ is the intersection of real valuation rings of $F$, then it must contain $H(F)$. Hence, for a general discussion we will impose the condition

$$H(F) \subseteq A$$

throughout, if not stated otherwise. Under this condition, the ring $A$ is a Prüfer ring. Hence it is the intersection of all valuation rings which are real valuation rings as they contain $H(F)$. However, we are not interested in this sort of presentation of $A$ as an intersection of valuation rings. As mentioned before, we want to study rings $A$ which admit an intersection presentation by valuation rings of composite places.

At this point, let us note:

**Lemma 2.6.** Assume that $H(F) \subseteq A$, $x_1, \ldots, x_n \in F$, and $a$ is a finitely generated ideal of $A$. Then:

1) there is $x \in F$ with $A[x_1, \ldots, x_n] = A[x] = A[1 + x^2],$

2) $\sqrt{a}$ is the radical of a principal ideal.

**Proof.** 1): We show that $A[x_1, \ldots, x_n] = A[1 + \sum_1^n x_i^2]$. The inclusion “$\supseteq$” is clear, while the inclusion “$\subseteq$” follows from the fact that $\frac{x_i}{1+\sum_1^n x_i^2} \in H(F)$ for all $i$, and $H(F) \subseteq A$. We set $x = 1 + \sum_1^n x_i^2$ and observe that a similar argument shows that $A[x] = A[1 + x^2]$ because $\frac{x}{1+\sum_1^n x_i^2} \in H(F)$.

2): Let $a = (f_1, \ldots, f_n)$. Then $a^2 = (\sum_1^n f_i^2)$ as $\frac{f_i f_j}{\sum_1^n f_k^2} \in H(F) \subseteq A$. Now our assertion follows since $\sqrt{a} = \sqrt{a^2}$.  

In view of part 1) of this lemma, whenever we will consider a finitely generated ring extension of $A$, we may always assume it to be of the form $A[x]$ for some $x \in F$. 

The real valuation rings of the base field \( R \) are just the overrings of \( \mathcal{O}_R \). They will be denoted by \( B \) and \( C \), and their canonical places by \( \pi_B \) and \( \pi_C \). The valuation ring of \( \pi_B \circ \lambda \) equals \( \lambda^{-1}(B) \), its maximal ideal is \( \lambda^{-1}(\mathcal{M}_{\pi_B}) \) and the residue field of \( \pi_B \circ \lambda \) is the real closed residue field of \( \pi_B \). Consequently, the specific choice of the real place with valuation ring \( B \) will not affect the concepts and results below, as one may check. In case of \( B = H(R) = O_R \) the only \( \mathbb{R} \)-place is the canonical real place \( \xi_R \) of \( R \).

We will be dealing with the set
\[ C(F) := \{ \pi_B \circ \lambda \mid B \text{ real valuation ring of } R, \lambda \in M(F|R) \} \]
of composite places, which we will abbreviate as \( C \), and for a given ring subring \( A \) of \( F \) with the set
\[ C_A := \{ \xi \in C \mid \xi \text{ finite on } A \} = \{ \pi_C \circ \lambda \mid \lambda(A) \subseteq C \}. \]
In particular,
\[ C_{H(F|B)} = \{ \pi_C \circ \lambda \mid B \subseteq C \}, \]
since for each \( \lambda \in M(F|R) \) we have that \( \lambda(H(F)) = H(R) = \mathcal{O}_R \) and therefore \( \lambda(H(F|B)) = \lambda(H(F)B) = B \). Since
\[ C_{A[x]} = \{ \xi \in C_A \mid \xi(x) \neq \infty \} \]
and \( \lambda(H(F|B)[x]) = B[\lambda(x)] \) if \( \lambda \) is finite on \( H(F|B)[x] \), we obtain that
\[ (7) \quad C_{H(F|B)[x]} = \{ \pi_C \circ \lambda \mid B \subseteq C \text{ and } \lambda(x) \in C \}. \]

We note:

**Proposition 2.7.** The place \( \pi_B \circ \lambda \) determines \( B \) and \( \lambda \) uniquely.

**Proof.** In fact, the restriction of \( \pi_B \circ \lambda \) to \( R \) equals \( \pi_B \). Therefore, the valuation ring \( B \) is determined uniquely. If \( \pi_B \circ \lambda = \pi_C \circ \mu \), then \( B = C \), and \( \mathcal{O}_\lambda \) and \( \mathcal{O}_\mu \) are both overrings of the valuation ring of the composite place. Hence, these two valuation rings are comparable, say \( \mathcal{O}_\lambda \subseteq \mathcal{O}_\mu \). Pick any \( a \in \mathcal{O}_\mu \) then \( \mu(a - \mu(a)) = 0 \). As the maximal ideal of the larger valuation ring \( \mathcal{O}_\mu \) is contained in the maximal ideal of \( \mathcal{O}_\lambda \), we find that \( \lambda(a - \mu(a)) = 0 \), whence \( \lambda = \mu \).

We say that the ring \( A \) satisfies the **intersection property** if
\[ A = \bigcap_{\xi \in C_A} \mathcal{O}_\xi, \]
or in other words, if \( A \) is the **holomorphy ring of the family** \( C_A \).

Next consider an ideal \( a \) of \( A \), from which we obtain the zero set
\[ V(a) := \{ \xi \in C_A \mid \xi = 0 \text{ on } a \}. \]
Likewise, from a subset $V \subseteq \mathcal{C}_A$ we obtain the vanishing ideal

$$I(V) := \{a \in A \mid \xi(a) = 0 \text{ for all } \xi \in V\}.$$ 

Clearly, $\sqrt{a} \subseteq I(V(a))$. Following the usual terminology, we say that the ideal $a$ satisfies the **Nullstellensatz** if $I(V(a)) = \sqrt{a}$.

The following proposition extends Schütting’s result [27, 2.6].

**Proposition 2.8.** Assume that $H(F) \subseteq A$. Then the following statements are equivalent:

1) $A$ satisfies the Nullstellensatz for finitely generated ideals,
2) every finite ring extension $A[x]$, where $x \in F$, satisfies the Nullstellensatz for finitely generated ideals,
3) every finite ring extension $A[x]$, where $x \in F$, has the intersection property.

**Proof.** The implication 2) $\Rightarrow$ 1) is trivial, as $A$ is its own finite ring extension. Once the equivalence of 1) and 3) is proven for all overrings of $H(F)$, the implication 1) $\Rightarrow$ 2) also follows: if 1) holds, then by 3), every finite ring extension of $A[x]$, being also a finite ring extension of $A$, has the intersection property, which implies that 1) holds for $A[x]$.

In view of the previous lemma we can restrict our attention in 3) to extensions of the form $A[x]$.

1) $\Rightarrow$ 3): Set $A' = A[x] = A[1 + x^2]$ and consider $f \in \bigcap_{\xi \in \mathcal{C}_A} \mathcal{O}_\xi$. Using the previous lemma we obtain that for all $\xi \in \mathcal{C}_A$, the implication $\xi(1+x^2) \neq \infty \Rightarrow \xi(1+f^2) \neq \infty$, holds, and hence also its contraposition

$$\xi \left( \frac{1}{1+f^2} \right) = 0 \Rightarrow \xi \left( \frac{1}{1+x^2} \right) = 0. \quad (8)$$

We observe that $\frac{1}{1+x^2}, \frac{1}{1+f^2} \in H(F) \subseteq A$. Thus we may set $a = \left( \frac{1}{1+f^2} \right)$, and from (8) we obtain that $\frac{1}{1+f^2} \in I(V(a))$. From 1) we infer that

$$\left( \frac{1}{1+x^2} \right)^k = a \frac{1}{1+f^2} \quad \text{for some } k \in \mathbb{N}, a \in A \text{ and } 1 + f^2 \in A[x].$$

The Prüfer ring $A[x]$ is integrally closed, so $f \in A[x]$.

3) $\Rightarrow$ 1): Take any finitely generated ideal of $A$; by the previous lemma we may assume that it is a principal ideal $(f)$. Consider $g \in I(V(f))$, so $\xi(f) = 0 \Rightarrow \xi(g) = 0$ holds for all $\xi \in \mathcal{C}_A$. The composite places which are finite on the extension $A[\frac{1}{g}]$ are just the composite ones which are finite on $A$ and satisfy $\xi(\frac{1}{g}) \neq \infty$. By the contrapositive of the above implication, these places also satisfy $\xi(\frac{1}{f}) \neq \infty$. By 3), the extension $A[\frac{1}{g}]$ has the intersection property, so we obtain that $\frac{1}{f} \in A[\frac{1}{g}]$. From this, $g^k \in (f)$ follows for some $k \in \mathbb{N}$. $\square$
Remark 2.9. 1) That a ring \( A \) admits the intersection property does not imply that the Nullstellensatz holds for finitely generated ideals of \( A \). To obtain this implication one really needs the hypothesis for all finitely generated extensions as above. For an example, take \( A = \mathcal{O}_\lambda \) for some \( \lambda \in M(F|R) \). Then \( C_A = \{ \lambda \} \) and the intersection property trivially holds. But if the rank of \( \lambda \) is greater than 1, then there is \( f \in \mathcal{O}_\lambda \) with \( \sqrt{\langle f \rangle} \neq \mathcal{M}_\lambda \), while \( I(V(f)) = \mathcal{M}_\lambda \).

Pick any \( x \in F \setminus \mathcal{O}_\lambda \). Then there is no composite place which is finite on the extension \( \mathcal{O}_\lambda[x] \). Hence this ring does not have the intersection property.

2) Assume that \( A \) has the intersection property. Then \( \mathcal{O}_R \) is contained in \( A \) since it is contained in \( \mathcal{O}_\xi \) for all \( \xi \in C_A \). It follows that for every \( \lambda \in M(F|R) \) that is finite on \( A \), \( \lambda(A) \) is a ring containing \( \mathcal{O}_R \), hence a real valuation ring of \( R \). This leads to the representation

\[
A = \bigcap \{ \mathcal{O}_{\pi\lambda(A)} \mid \lambda \in M(F|R) \text{ finite on } A \},
\]

since the valuation rings on the right hand side are the minimal ones among all valuation rings \( \mathcal{O}_\xi \) with \( \xi \in C_A \).

\[\square\]

Theorem 2.10. For every real valuation ring \( B \subseteq R \), each finite ring extension of \( H(F|B) \) within \( F \) satisfies the Nullstellensatz for finitely generated ideals and has the intersection property.

Proof. We may write \( A = H(F|B)[x] \) for some \( x \in F \). By Proposition 2.8, it suffices to prove that \( A \) has the intersection property. Suppose that there exists \( f \in \bigcap_{\xi \in C_A} \mathcal{O}_\xi \) with \( f \notin A \). As said above, \( A \) is the intersection of all real valuation rings in which it is contained. Hence we find a real place \( \xi_0 \) with \( A \subseteq \mathcal{O}_{\xi_0} \), \( x \in \mathcal{O}_{\xi_0} \) and \( f \notin \mathcal{O}_{\xi_0} \). Applying Theorem 1.1 with \( \xi_0 \) in place of \( \xi \) and \( \varphi \) the restriction of \( \xi_0 \) to \( R \), we obtain \( \lambda \in M(F|R) \) such that \( \lambda(x) \in \mathcal{O}_\varphi \) and \( \lambda(\frac{1}{x}) \in \mathcal{M}_\varphi \), whence \( f \notin \mathcal{M}_\varphi \).

We set \( C := \mathcal{O}_\varphi = \mathcal{O}_{\xi_0} \cap R \), so \( \lambda(x) \in C \). Since \( H(F|B) \subseteq A \subseteq \mathcal{O}_{\xi_0} \), we also have that \( B \subseteq C \). Hence by (7), \( \pi_C \circ \lambda \in C_A \). But \( f \notin \mathcal{M}_\varphi \) implies that \( f \notin \mathcal{O}_{\pi_C \circ \lambda} \), a contradiction to our choice of \( f \). \[\square\]

The three distinguished cases of \( A = H(F|B) \), \( A = H(F) = H(F|\mathcal{O}_R) \) and \( A = H(F|R) \) deserve special attention:

\[
(9) \quad H(F|B) = \bigcap \{ \mathcal{O}_{\pi_{\varphi\circ\lambda}} \mid \lambda \in M(F|R) \},
\]
\[
(10) \quad H(F) = \bigcap \{ \mathcal{O}_{\xi\circ\lambda} \mid \lambda \in M(F|R) \}
\]
\[
= \bigcap \{ \mathcal{O}_\xi \mid \xi \in M_R(F) \},
\]
\[
(11) \quad H(F|R) = \bigcap \{ \mathcal{O}_\lambda \mid \lambda \in M(F|R) \}.
\]
In particular, $H(F)$ is the intersection of the family of valuation rings of the real places in $M_R(F)$. This is a straightforward and appealing geometric generalization of the situation in case of $R = \mathbb{R}$.

In what follows we address the question whether there are minimal representations for the relative real holomorphy rings $H(F|B)$ of the type above. More precisely, we will study subfamilies $\mathcal{F} \subseteq M(F|R)$ such that $H(F|B) = \bigcap_{\lambda \in \mathcal{F}} \mathcal{O}_{\pi_B \circ \lambda}$ and look at the existence of minimal families $\mathcal{F}$. This is a topic dealt with by Schütling in [4, 3.13] and [28, 1.3 ff] for the case $B = R$. Here, we allow $B$ to range over all real valuation rings of the base field $R$.

**Theorem 2.11.** Let $B, C$ be real valuation rings of $R$ such that $B \subseteq C$. Then we have:

1) $H(F|B) \subsetneq H(F|C)$;
2) the following statements are equivalent for each subset $\mathcal{F}$ of $M(F|R)$:
   
   a) $H(F|B) = \bigcap_{\lambda \in \mathcal{F}} \mathcal{O}_{\pi_B \circ \lambda}$,
   b) $\mathcal{F}$ is dense in $M(F|R)$;
3) There is no representation of the form (a) with minimal $\mathcal{F}$,
4) $H(F|C)$ admits a representation of the form (a) with a minimal $\mathcal{F}$ if and only if $C = R$ and trdeg $F|R = 1$. In the case of a minimal representation we necessarily have that $\mathcal{F} = M(F|R)$.

**Proof.** 1): Clearly $H(F|B) = H(F)B \subseteq H(F)C = H(F|C)$. If we had $H(F|B) = H(F|C)$ then $B = H(F|B) \cap R = H(F|C) \cap R = C$ would follow.

2): Assume that $\mathcal{F}$ is not dense in $M(F|R)$. Hence by part 2) of Theorem 2.4, $\iota_{F|R}(\mathcal{F})$ is not dense in $M_R(F)$ and thus also not in $M(F)$. Let $N$ be the closure of $\iota_{F|R}(\mathcal{F})$ in $M(F)$ and take $\eta \in M(F) \setminus N$. By the Separation Criterion given in [24, Proposition 9.13], there is $f \in H(F)$ such that $N \subseteq U(-f)$ and $\eta \in U(f)$. Since $M_R(F) = \iota_{F|R}(M(F|R))$ is dense in $M(F)$ by Proposition 2.1, there is $\lambda_0 \in M(F|R)$ such that $\xi_R \circ \lambda_0 \in U(f)$ and thus $a := \lambda_0(f)$ is an element of the set $E^+(R)$ of positive units of $\mathcal{O}_R$. For $\lambda \in \mathcal{F}$ we have that $-\lambda(f) \in E^+(R)$. Define $g := \frac{1}{a \cdot f}$. We have that $\lambda_0(g) = \infty$ and therefore $g \notin \mathcal{O}_{\pi_B \circ \lambda_0}$, whence $g \notin H(F|B)$. But for $\lambda \in \mathcal{F}$ we have $\lambda(g) \in E^+(R) \subseteq \mathcal{O}_R \subseteq B$ and therefore $g \in \mathcal{O}_{\pi_B \circ \lambda}$. Hence (a) cannot be true. This proves that (a) implies (b).

Now we prove that (b) implies (a). We have that $H(F|B) \subseteq \bigcap_{\lambda \in \mathcal{F}} \mathcal{O}_{\pi_B \circ \lambda}$. Suppose that equality does not hold. Then there is some $f \in F$ and $\lambda_0 \in M(F|R)$ such that $\lambda_0(f) \notin B$ but $\lambda(f) \in B$ for all $\lambda \in \mathcal{F}$. Then either $\lambda_0(f) = \infty$ or $\lambda_0(f) \in R \setminus B$. In the first case
we choose $a \in R \setminus B$ with $a > 0$ (note that $B \subsetneq R$ since $B \subsetneq C$). In the second case we choose $a$ such that $2a = \lambda_0(f)$; switching $f$ to $-f$ if necessary, we may again assume that $a > 0$. In both cases we see that $-a < b < a$ for all $b \in B$ as $B$ is convex under the ordering $< \subseteq R$. We consider the set

$$S := \{ \lambda \in M(F|R) \mid \lambda(f) = \infty \text{ or } |\lambda(f)| > a \}$$

and note that it contains $\lambda_0$. With $g := \frac{f}{1+\varepsilon a} \in H(F)$, the condition defining $S$ holds if and only if $\lambda(g) = 0$ or $|\lambda(g)| < \frac{1}{1+\varepsilon a} =: b$, which means that $-b < \lambda(g) < b$. This shows that $S$ is an open subset of $M(F|R)$. By the density of $F$ in $M(F|R)$ there is some $\lambda \in F \cap S$. Consequently, $\lambda(f) = \infty$ or $|\lambda(f)| > a$ so that $\lambda(f) \notin B$, a contradiction to our assumption on $\lambda(f)$. Thus equality, and hence (a), must hold.

3:) Assume that $F$ is a dense subset of $M(F|R)$ and that $\eta \in F$. We wish to show that $F \setminus \{\eta\}$ is still dense in $M(F|R)$. Suppose not. Then there is an open subset $V$ of $M(F|R)$ such that $F \cap (V \setminus \{\eta\}) = (F \setminus \{\eta\}) \cap V = \emptyset$. But by part 3) of Theorem 2.4, $V$ is infinite and hence $V \setminus \{\eta\}$ is a nonempty open set, so we obtain a contradiction to the density of $F$.

4:) Assume first that $H(F|C)$ admits a minimal representation. Then part 3) of our theorem cannot hold, which implies that there is no real valuation ring of $R$ of which $C$ is a proper subring. In particular, $C$ is not a proper subring of $R$, so $C = R$. We are therefore dealing with the case that $H(F|R)$ admits a minimal representation. Then necessarily $\text{trdeg} (F|R) = 1$; this is proven by Schüttling in [28, Section 1] for $R = \mathbb{R}$. Transferring the arguments, one can deduce it from [4, 3.13] also for the case of any real closed base field.

For the converse, consider $H(F|R)$ in the case of $\text{trdeg} (F|R) = 1$ and choose an arbitrary $\eta \in M(F|R)$. We use that $H(F|R)$ contains an affine algebra $A$ with quotient field $F$. Take the integral closure $B$ of $A$ in $F$. Due to the Noether Normalization Theorem we have that $B$ is an affine $R$-algebra, say $B = R[x_1, \ldots, x_n]$. We observe that $B$ is a noetherian, integrally closed ring of dimension 1, in other words, a smooth affine model with $B$ a Dedekind domain. As it is contained in $H(F|R)$, we get that the places in $M(F|R)$ bijectively correspond to the $R$-epimorhisms $B \to R$, the local rings of which are exactly the valuation rings of the places in $M(F|R)$. Now fix a place $\eta$ and set $a_i = x_i - \eta(x_i), 1 \leq i \leq n$. The $a_i$'s also generate the algebra $B$. Thus if $\lambda \neq \eta$, then $\lambda(a_i) \neq \eta(a_i) = 0$ for some $i$. It follows that for $a = \sum_{1 \leq i \leq n} a_i^2$, the element $1/a$ lies in the valuation ring of $\lambda$ but not
in the valuation ring of \( \eta \). This shows that \( \bigcap_{\lambda \in F \setminus \{ \eta \}} \mathcal{O}_{\pi \circ \lambda} \) is strictly larger than \( H(F|R) \).

For a function field \( F \) over a nonarchimedean real closed field \( R \) we will give yet another, geometric representation of \( H(F) \). Choose \( X \) to be any smooth projective model of \( F \). For \( x \in X \) we denote by \( \mathcal{O}_x \) the local ring in \( x \) and by \( X(R) \) the set of rational points of \( X \). For \( f \in F \), by \( X_f \) we denote the set of those rational points for which \( f \in \mathcal{O}_x \), i.e., \( f \) is defined in \( x \). Note that \( X_f \) is open in the Zariski topology. As \( X \) is smooth, every point in \( X(R) \) is the center of some \( \lambda \in M(F|R) \).

Using this fact, Artin’s solution to Hilbert’s 17th Problem can be rephrased as follows:

\[
f \in \sum \hat{F}^2 \iff f(x) \geq 0 \text{ for every } x \in X_f .
\]

The question arises whether this characterization can be suitably reformulated to obtain a geometric characterization of \( H(F) \).

Take a function \( f \in H(F) \) and take \( x \in X(R) \) such that \( f \in \mathcal{O}_x \). Since \( x \) is the center of some \( \lambda \in M(F|R) \), we obtain that \( f(x) = \lambda(f) \). Since \( \xi_R \circ \lambda \) is an \( \mathbb{R} \)-place and \( f \in H(F) \), we have that \( \xi_R \circ \lambda(f) \neq \infty \). Therefore \( \lambda(f) \in \mathcal{O}_R = H(R) \). We have shown:

\[
f \in H(F) \implies f(x) \in H(R) \text{ for every } x \in X_f .
\]

The converse is in general not true. If \( R \) is an archimedean real closed field, then \( H(R) = R \) and the right hand side of the implication is always true, while the left hand side is not.

To understand what is going on, we have to turn again to the relative real holomorphy ring \( H(F|R) \) and its geometrical description given by Schüting in [26]:

\[
(12) \quad H(F|R) = \{ f \in F \mid f \text{ is bounded on } X_f \text{ by elements of } R \} .
\]

For a smooth real projective variety \( X \), define

\[
H_X := \{ f \in F \mid f(x) \in H(R) \text{ for every } x \in X_f \} .
\]

Then \( H(F) \subseteq H_X \). But functions in \( H_X \) are not necessarily bounded on \( X_f \) in the case of an archimedean ordered base field \( R \). But in the nonarchimedean case every function in \( H_X \) is bounded by the elements with negative values under \( v_R \). Therefore, for \( R \) nonarchimedean we have:

\[
(13) \quad H(F) \subseteq H_X \subseteq H(F|R) ,
\]

where the latter inclusion follows from (12).
Proposition 2.12. Take a function field $F$ over a nonarchimedean real closed field $R$. Then $H(F)$ is the intersection of the sets $H_X$ where $X$ runs through all smooth projective models of $F$.

Proof. As we observed before, $H(F) \subseteq H_X$ for any smooth model of $F$. Therefore, $H(F) \subseteq \bigcap\{H_X \mid X$ smooth projective model of $F\}$.

Assume that $f$ is in the intersection of the sets $H_X$. Since $R$ is nonarchimedean, (13) shows that $f$ is in $H(F|R)$. By a theorem of Schütling (see [26, page 437]) there is a smooth projective model $X_0$ such that $f$ is regular in every point of $X_0(R)$. Take any $R$-place $\xi$ such that $\xi = \xi_R \circ \lambda$, $\lambda \in M(F|R)$. Since $f \in H(F|R)$, we have $f \in \mathcal{O}_\lambda$. The place $\lambda$ has a center $c(\lambda)$ on the projective model, so $c(\lambda) \in X_0(R)$. Then $\lambda(f) = f(c(\lambda)) \in H(F)$ by our assumption. This means that $\xi(f) \neq \infty$, so $f \in \mathcal{O}_\xi$ for every $\xi \in M_R(F) = \{\xi_R \circ \lambda \mid \lambda \in M(F|R)\}$. Thus $f \in \bigcap\{\mathcal{O}_{\xi_R \circ \lambda} \mid \lambda \in M(F|R)\}$, which by (10) is equal to $H(F)$. \qed

Note that this theorem is not true for an archimedean real closed field $R$ since in this case $H_X = F$ for every smooth projective model $X$ of $F$.

3. The Real Spectrum of $H(F|R)$ and $H(F)$

As before, $F$ denotes a formally real function field over a real closed base field $R$.

The topologies on $M(F|R)$ and $M(F)$ find natural interpretations via the theory of the real spectrum $\text{Spec}_r(A)$ of a commutative ring $A$. Regarding general concepts and results we refer to [2, Chapter 7] and [15, Kapitel III]; however, note that the authors of the latter reference are using the notation $\text{Sper}A$ for the real spectrum of $A$. The real spectrum $\text{Spec}_r(A)$ is a quasi-compact space; we reserve the term “compact”, in contrast to the use in [2], for quasi-compact Hausdorff spaces. It is its compact subspace of closed points $\text{MaxSpec}_r(A)$ that we are mainly interested in.

In our situation, we will prove:

Proposition 3.1. There is a commutative diagram

\[
\begin{array}{ccc}
M(F|R) & \xrightarrow{i} & \text{MaxSpec}_r(H(F|R)) \\
\downarrow \iota_{F|R} & \xrightarrow{\text{///}} & \downarrow \tau \\
M(F) & \xrightarrow{j} & \text{MaxSpec}_r(H(F))
\end{array}
\]

where

(1) the maps $i, \iota_{F|R}$ are topological embeddings with dense images,
(2) the map \( j \) is a homeomorphism,
(3) the map \( \tau \) is continuous and surjective.

Using this proposition and results from real algebraic geometry over arbitrary real closed fields (cf. [2, 7]), we can prove:

**Proposition 3.2.** \( M(F) \) has only finitely many connected components.

It was already known that the space \( M(F) \) of a rational function field \( F = R(X_1, \ldots, X_n) \) is connected, cf. [11, Theorem 2.12].

Let \( A \) denote any commutative ring. By definition, the real spectrum \( \text{Spec}_r(A) \), as a set, is the collection of all so-called **prime cones** \( \alpha \subseteq A \) satisfying the conditions

\[
\alpha + \alpha \subseteq \alpha, \alpha \cdot \alpha \subseteq \alpha, \alpha \cup \alpha = A, \alpha \cap -\alpha \text{ is a prime ideal of } A.
\]

Let a prime cone \( \alpha \) be given. We set \( \text{supp}(\alpha) = \alpha \cap -\alpha \). This prime ideal is called the **support** of \( \alpha \). By the **residue field** of \( \alpha \) we will mean the quotient field of \( A/\text{supp}(\alpha) \). Given any \( a \in A \) and \( \alpha \in \text{Spec}_r(A) \), we write \( a(\alpha) := a + \text{supp}(\alpha) \). An ordering \( \bar{\alpha} \) with order relation \( \leq_\alpha \) (or in short, \( \leq \)) is induced by requiring, for all \( a \in A \),

\[
0 \leq a(\alpha) \iff a \in \alpha \text{ and } -a /\in \alpha.
\]

The topology on \( \text{Spec}_r(A) \) is defined by the following family of basic open sets:

\[
\tilde{U}(a_1, \ldots, a_n) = \{ \alpha \mid a_1(\alpha) > 0, \ldots, a_n(\alpha) > 0 \}
\]

for all \( n \in \mathbb{N}, a_1, \ldots, a_n \in A \). As ring homomorphisms \( \phi : A \to B \) are well behaved with respect to the assignment \( A \mapsto \text{Spec}_r(A) \), we are dealing with a contravariant functor \( \text{Spec}_r \) from the category of rings to the category of quasi-compact spaces. Here we will only be using the simplest case, where \( A \) is a subring of the ring \( B \). It is readily seen that we obtain a continuous map, the restriction

\[
\text{res} = \text{res}_{A,B} : \text{Spec}_r(B) \to \text{Spec}_r(A), \ \alpha \mapsto \alpha \cap A.
\]

If \( \alpha, \beta \in \text{Spec}_r(A) \) satisfy \( \alpha \subseteq \beta \), then \( \beta \) is called a **specialization** of \( \alpha \) and \( \alpha \) a **generalization** of \( \beta \). The specializations of a given prime cone \( \alpha \) form a totally ordered set with respect to inclusion, and there is a unique maximal specialization of \( \alpha \), denoted by \( \rho(\alpha) \). The maximal prime cones are exactly the closed points in \( \text{Spec}_r(A) \). For example, a prime cone whose support is a maximal ideal is a maximal prime cone. We set

\[
\text{MaxSpec}_r(A) = \{ \alpha \in \text{Spec}_r(A) \mid \alpha \text{ maximal} \}.
\]
It turns out that the subspace $\text{MaxSpec}_r(A)$ is compact and that the specialization map

$$
\rho = \rho_A : \text{Spec}_r(A) \rightarrow \text{MaxSpec}_r(A), \quad \alpha \mapsto \rho(\alpha)
$$

is continuous and a closed retraction, cf. [2, 7.1.25] and [15, p.128, Satz 5]. By composing the assignment $A \mapsto \text{Spec}_r(A)$ with the specialization map, we obtain a functor $A \mapsto \text{MaxSpec}_r(A)$ into the category of compact spaces. In the case where $A$ is a subring of $B$ we obtain the continuous map

$$
\tau = \tau_{A,B} := \rho_A \circ \text{res}_{A,B} : \text{MaxSpec}_r(B) \rightarrow \text{MaxSpec}_r(A).
$$

In what follows we will use the following, easily proven observation: if $\beta$ is a specialization of $\alpha$ and $\text{supp}(\alpha) = \text{supp}(\beta)$, then $\alpha = \beta$.

As already stated above, if $\text{supp}(\alpha)$ is a maximal ideal, then $\alpha$ is a maximal prime cone.

We now turn to the proof of Proposition 3.1.

**Proof.** The map $\iota_{F|R}$ has already been introduced and shown in Theorem 2.4 to be a topological embedding of $M(F|R)$ into $M(F)$; by Proposition 2.1, its image $M_R(F)$ is dense in $M(F)$. The map $\tau$ on the right hand side equals $\tau_{A,B}$ for $A = H(F), B = H(F|R)$. So it is continuous. Surjectivity follows once the statements on the maps $i, j$ and the commutativity of the diagram have been shown; this is seen as follows. As we are dealing with compact spaces the image of $\tau$ is closed, and furthermore, it contains the image of the dense subspace $M_R(F)$ under the homeomorphism $j$. All this implies that $\tau$ is surjective.

To define the map $i : M(F|R) \rightarrow \text{MaxSpec}_r(H(F|R))$ and study its properties we need the following facts. A place $\lambda \in M(F|R)$ induces an epimorphism $H(F|R) \rightarrow R$ whose kernel $p_\lambda$ is a maximal ideal of $H(F|R)$. As this ring is a Prüfer ring we see that the valuation ring of $\lambda$ is just the localization $H(F|R)_{p_\lambda}$. Altogether we obtain that the places in $M(F|R)$ are determined by their restriction to $H(F|R)$. Using the unique ordering on $R$ we now define the natural map

$$
i : M(F|R) \rightarrow \text{MaxSpec}_r(H(F|R)),
\lambda \mapsto \alpha_\lambda := \{a \in H(F|R) \mid \lambda(a) \geq 0\}.
$$

We observe that $\text{supp}(\alpha_\lambda) = p_\lambda$, so indeed, $\alpha_\lambda$ is a maximal prime cone, as its support is a maximal ideal. As each $\lambda \in M(F|R)$ is the identity on $R$ we find that for each $a \in H(F|R)$ we have $a - \lambda(a) \in p_\lambda$. From this the injectivity of $i$ follows: indeed, if $\alpha_\lambda = \alpha_\mu$, then $p_\lambda = p_\mu$, so $\mu(a - \lambda(a)) = 0$ and therefore $\mu(a) = \lambda(a)$ for every $a \in H(F|R)$, whence $\mu = \lambda$. In addition we obtain that $a(\alpha_\lambda) = \lambda(a)$ for any
$a \in H(F|R)$, from which we deduce:
\[
\left.i^{-1}\left(\overline{U}(a_1, \ldots, a_n) \cap \text{MaxSpec}_r(H(F|R))\right)\right) = V(a_1, \ldots, a_n).
\]

This means that the map $i$ is a topological embedding. To prove that the image is dense, consider a nonempty basic open subset $\overline{U} = \overline{U}(a_1, \ldots, a_n) \cap \text{MaxSpec}_r(H(F|R))$ and pick one of its elements $\alpha$. Set $p = \text{supp}(\alpha)$. The residue field of the valuation ring $H(F|R)_p$ equals the residue field of $\alpha$. Therefore we can pull back the ordering $\bar{\alpha}$ to construct an ordering $>$ on $F$ which satisfies $a_i > 0$ for $1 \leq i \leq n$. Now the arguments presented in the proof of Proposition 2.2 yield the existence of $\lambda \in M(F|R)$ with $\lambda(a_i) > 0$ for $1 \leq i \leq n$. We see that $\alpha \lambda \in \overline{U}$.

In the case of the map $j$ we follow a similar route. A place $\xi \in M(F)$ induces a homomorphism $H(F) \to \mathbb{R}$. We define
\[
j : M(F) \to \text{Spec}_r(H(F)), \xi \mapsto \alpha_\xi := \{a \in H(F) \mid \xi(a) \geq 0\}.
\]

This time however, the kernel $p_\xi = \text{supp}(\alpha_\xi)$ need not be a maximal ideal. Nevertheless, $\alpha_\xi \in \text{MaxSpec}_r(H(F))$. To see this, first note that the residue field of $\xi$ equals the residue field of $\alpha_\xi$, which embeds into $\mathbb{R}$. Hence the induced ordering $\overline{\alpha_\xi}$ is nothing but the pullback of the natural ordering on $\mathbb{R}$. Thus it is an archimedean ordering of the residue field.

Now assume that $\alpha_\xi \subsetneq \beta$ for some $\beta \in \text{Spec}_r(H(F))$; we wish to deduce a contradiction. Then, due to the above observation, we obtain that $p := \text{supp}(\alpha_\xi) \subsetneq q := \text{supp}(\beta)$. Then we can choose $a \in q \setminus p$, and we can assume that $a \in \alpha_\xi$ since otherwise, we can replace $a$ by $-a$. For each rational number $r > 0$ we have that $r + a \in \alpha_\xi$ but also $r - a \in \alpha_\xi$: if not, then we would obtain that $r - a \in -\alpha_\xi \subseteq -\beta$ and $r - a \in \beta$ as $a \in \pm \beta$. This would imply that $r - a \in q$, which leads to the contradiction $r \in q$. Passing to the residue field we see that the non-zero element $\bar{a}$ is infinitesimally small relative to the archimedean ordering $\overline{\alpha_\xi}$: a contradiction to our assumption. Thus the image of $j$ is contained in $\text{MaxSpec}_r(H(F))$.

To prove the injectivity of $j$ assume that $\alpha_\xi = \alpha_\zeta$. Then both places have the same valuation ring and the same residue field on which they induce embeddings $\bar{\xi}, \bar{\zeta}$ into $\mathbb{R}$, subject to the condition $\xi(\bar{a}) > 0 \iff \zeta(\bar{a}) > 0$ for every $a \in H(F)$. As $\mathbb{Q}$ is dense in $\mathbb{R}$ we find that $\bar{\xi} = \bar{\zeta}$, whence $\xi = \zeta$.

From the equivalence $a(\alpha_\xi) > 0 \iff \xi(a) > 0$ we find that $j$ is a topological embedding of $M(F)$ into $\text{MaxSpec}_r(H(F))$. 

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Now we show that \( j \) is surjective. Consider any \( \alpha \in \text{Spec}_r(H(F)) \). We want to show that \( \alpha \subseteq \alpha_\xi \) for some \( \xi \in M(F) \). This, of course, will settle our claim. Set \( p = \text{supp}(\alpha) \). Then \( H(F)_p \) is the valuation ring of a place \( \zeta : F \to k(p) \cup \infty \), where \( k(p) \) is the quotient field of \( H(F)/p \). It is known that \( H(F)/p = H(k(p)) \). The ordering \( \bar{\alpha} \) induces a real place \( \lambda_{\bar{\alpha}} \) with a valuation ring which contains \( H(k(p)) = H(F)/p \). Using the residue map \( \pi : H(F) \to H(k(p)) \), we find \( \xi \in M(F) \), determined by the condition \( \xi|_{H(F)} = \lambda_{\bar{\alpha}} \circ \pi \). One readily checks that \( \alpha \subseteq \alpha_\xi \).

It remains to address the commutativity of the diagram. Starting with \( \lambda \in M(F|R) \) we have to show that \( \rho(\alpha_\lambda \cap H(F)) = \alpha_\xi \) with \( \xi = \xi_R \circ \lambda \).

As \( \alpha_\xi \) is a maximal prime cone it is sufficient to prove that \( \alpha_\lambda \cap H(F) \subseteq \alpha_\xi \). Pick any \( a \in H(F) \) with \( \lambda(a) \geq 0 \). Then \( \lambda(a) \in H(R) \) and consequently, \( \xi_R(\lambda(a)) \geq 0 \), i.e., \( a \in \alpha_\xi \).

Next, the proof of Proposition 3.2 will be sketched.

**Proof.** We know that \( \tau \) is continuous and surjective. Therefore, once we know that \( \text{MaxSpec}_r(H(F|R)) \) has only finitely many connected components, we can derive the same for \( \text{MaxSpec}_r(H(F)) \). We list the arguments needed to show that \( \text{MaxSpec}_r(H(F|R)) \) decomposes into finitely many connected components. First of all, for any given ring \( A \) the specialization map \( \rho : \text{Spec}_r(A) \to \text{MaxSpec}_r(A) \) induces a bijection between the set of connected components of \( \text{Spec}_r(A) \) and that of \( \text{MaxSpec}_r(H(A)) \), see for instance [15, p.129, Satz 6]. Consequently, we are facing the problem to show that \( \text{Spec}_r(H(F|R)) \) admits only finitely many connected components. This follows from Schültig's result [26, p. 436, Theorem] as it is known that algebraic sets over real closed fields decompose into finitely many semi-algebraically connected components. By the way, they are exactly the semi-algebraic path connected components, see [2, Sections 2.4.,2.5] and [7, Theorem 4.1].

Note that the surjectivity of \( \tau \) can be obtained in a more direct way by appealing to the Baer-Krull Theorem. But we preferred to convey the present argument for the sake of a coherent presentation.

**Remark 3.3.** Without providing any further details, we want to conclude by another observation. The number of connected components \( s_F \) of \( \text{Spec}_r(H(F|R)) \), which is a geometric invariant of \( F \), is an upper bound for the number of connected components \( t_F \) of \( M(F) \). This is a consequence of the last proof. However, it may happen that \( s_F > t_F \), as we will show now.
Take a nonarchimedean real closed field $R$, and denote by $R^+$ the set of its positive elements and by $I^+$ the set of its positive infinitesimals. Take $a \in I^+$. Let $F$ be the function field of the real complete affine curve $C$ given by
\[ y^2 = (x^2 - a^2)(1 - x^2). \]
The relative real holomorphy $H(F|R)$ equals the coordinate ring $A := R[C]$ and is a Dedekind ring. The curve $C$ has two semialgebraic connected components separated by the function $x$.

The real spectrum $\text{Spec}_r(A)$ consists of the prime cones
\[ P(\alpha, \beta) := \{ f \in A \mid f(\alpha, \beta) \geq 0 \} \]
attached to the points $(\alpha, \beta) \in C$ and the prime cones $P \cap A$, where $P$ runs through the orderings of $F$. The first ones are maximal prime cones. A prime cone of the second type is maximal if and only if $(F, P)$ is archimedean over $A$, and this holds if and only if $(F, P)$ is archimedean over $R$ (see [15, Corollary 5, p. 134]).

Take the ordering
\[ P := \{ f \mid \exists d \in I^+ \exists e \in R^+ \setminus I^+ : f(c) > 0 \text{ for all } c \in (d, e) \} \]
\[ \text{of } R(x). \]
The ordering $P$ has exactly one extension $P'$ to $F$ in which $y$ is positive. Take the automorphism $\sigma$ of $F$ such that $\sigma(x) = -x$ and $\sigma(y) = y$. Then $Q' = \sigma(P')$ is an ordering of $F$ such that $\lambda_{P'} = \lambda_{Q'}$. Since $P$ is archimedean over $R$, the same is true for $P'$ and $Q'$. The function $x$ is positive in $P'$ and negative in $Q'$, therefore $P' \cap A$ and $Q' \cap A$ belong to different components of $\text{Spec}_r(H(F|R))$. But the map $\tau$ from Proposition 3.1 sends the maximal prime cones $P' \cap A$ and $Q' \cap A$ to prime cones related with the real place $\lambda_{P'} = \lambda_{Q'}$, which shows that the number of components drops.

The example above was also studied in the paper [16], where the relation between cuts on the real curve and the orderings of its function field was described. In general, the study of $t_F$ and its comparison to $s_F$ seem to be an interesting task.

4. Proof of Theorem 1.1

We will need the following fact, which has been shown in [1, Theorem 1.1]:

**Proposition 4.1.** Let $L|K$ be an extension of finite transcendence degree, and $v_\xi$ a nontrivial valuation on $L$ with associated place $\xi$. If $v_\xi L/v_\xi K$ is not a torsion group or $L\xi|K\xi$ is transcendental, then $(L, v_\xi)$ admits an immediate extension of infinite transcendence degree.
The proof of Theorem 1.1 is an adaptation of the proof of the Main Theorem in [23], but instead of the Ax-Kochen-Ershov Theorem used there we will have to use other transfer principles. To prove assertion A) of Theorem 1.1, we will make use of a version of the Ax-Kochen-Ershov Theorem for the theory of tame fields as presented in [22]. These are henselian valued fields \((K,v)\) whose absolute ramification field is algebraically closed. Here, the absolute ramification field of \((K,v)\) with respect to an extension of the valuation \(v\) to the separable algebraic closure \(K^{\text{sep}}\) of \(K\) is the ramification field of the extension \((K^{\text{sep}}|K,v)\).

To prove assertion C) and other special cases of our theorem, we will need analogues for algebraically closed fields, algebraically closed fields with valuation, ordered real closed fields, and ordered real closed fields with compatible valuation.

**Theorem 4.2.** 1) In the language of rings, an algebraically closed field is existentially closed in every extension field \(F\).

2) In the language of rings with a relation symbol for a valuation, an algebraically closed nontrivially valued field is existentially closed in every valued extension field \(F\).

3) In the language of rings with a relation symbol for an ordering, a real closed field \(R\) is existentially closed in every ordered extension field \(F\).

4) In the language of rings with relation symbols for an ordering and a valuation, a real closed field \(R\) with nontrivial compatible valuation is existentially closed in every ordered extension field \(F\) equipped with a compatible valuation which extends the valuation of \(R\).

**Proof.** 1): Take an algebraic closure \(F^{\text{ac}}\) of \(F\). By the model completeness of the theory of algebraically closed valued fields (cf. [RO]), \(F^{\text{ac}}\) is an elementary extension of \(K\) in the language of rings. Every existential sentence in this language with parameters from \(K\) that holds in \(F\) also holds in \(F^{\text{ac}}\), and by what we just have stated, it then also holds in \(K\). This proves that \(K\) is existentially closed in \(F\) in this language.

2): Take an algebraic closure \(F^{\text{ac}}\) of \(F\) together with some extension of the valuation. By Abraham Robinson’s theorem on the model completeness of the theory of algebraically closed valued fields (cf. [RO]), \(F^{\text{ac}}\) is an elementary extension of \(K\) in the language of rings with a relation symbol for a valuation. The remainder of the argument is as in 1).

3): Take a real closure \(F^{\text{rc}}\) of \(F\) together with the corresponding extension of the ordering. Then the ordering on \(F^{\text{rc}}\) extends the unique
ordering of the real closed field $R$. By [8, Theorem 4.5.1], $F^{\text{rc}}$ is an elementary extension of $R$ in the language of rings with a relation symbol for an ordering. Now our assertion follows as in the proof of part 1), with $F^{\text{rc}}$ and $R$ in place of $F^{\text{ac}}$ and $K$, respectively.

4): Take a real closure $F^{\text{rc}}$ of $F$ together with the corresponding extensions of the ordering and the compatible valuation of $F$. Again, the ordering on $F^{\text{rc}}$ extends the unique ordering of the real closed field $R$. As the compatible valuation on $F^{\text{rc}}$ extends the one of $F$, which in turn extends the one of $R$, it also extends the one of $R$. By [8, Corollary 4.5.4] and the fact that the ordering is definable in a real closed field in the language of rings, $F^{\text{rc}}$ is an elementary extension of $R$ in the language of rings with relation symbols for an ordering and a valuation. Now our assertion follows as in the proof of part 3). □

Further, we will need a generalization of [19, Theorem 23].

**Theorem 4.3.** Let $F|K$ be an algebraic function field and choose $\Gamma$ as in Theorem 1.1. Take any nonzero elements $a_1, \ldots, a_m \in F$. Then there are infinitely many (nonequivalent) places $\lambda \in S(F|K)$ such that $F_\lambda|K$ is finite, $v_\lambda F \subseteq \Gamma$, and $\lambda(a_i) \neq 0, \infty$ for $1 \leq i \leq m$.

If in addition $K$ is existentially closed in $F$, then these places can be chosen to be $K$-rational with $v_\lambda F = \Gamma$.

**Proof.** We adapt the proof of the lemma on p. 190 of [K–P]. In some algebraic closure $F^{\text{ac}}$ of $F$ we find an algebraic closure $K_0$ of $K$ and let $F' := K_0.F$ be the field compositum of $K_0$ and $F$ inside of $F^{\text{ac}}$. By part 1) of Theorem 4.2, $K_0$ is existentially closed in $F'$.

Since $K_0|K$ is algebraic, trdeg $F'|K_0 = \text{trdeg } F|K = s$. The extension $F'|K_0$ is separable and finitely generated, so we can pick in $F'$ a separating transcendence basis $t_1, \ldots, t_s$ together with an element $y$ separable algebraic over $K_0(t_1, \ldots, t_s)$ such that $F' = K_0(t_1, \ldots, t_s, y)$. Take $f \in K_0[t_1, \ldots, t_s, Y]$ to be an irreducible polynomial of $y$ over $K_0[t_1, \ldots, t_s]$. We write $t = (t_1, \ldots, t_s)$ and

\begin{equation}
(14) \quad a_i = \frac{g_i(t, y)}{h_i(t)} \quad \text{for} \quad 1 \leq i \leq m,
\end{equation}

where $g_i$ and $h_i$ are polynomials over $K_0$, with $h_i(t) \neq 0$. Since the elements $t_1, \ldots, t_s, y$ satisfy

\begin{equation}
(15) \quad f(t, y) = 0, \quad \frac{\partial f}{\partial Y}(t, y) \neq 0 \quad \text{and} \quad h_i(t) \neq 0 \quad \text{for} \quad 1 \leq i \leq m
\end{equation}
in $F'$, we infer from $K_0$ being existentially closed in $F'$ that there are $t'_1, \ldots, t'_s, y'$ in $K_0$ such that
\[
f(t'_i, y') = 0, \quad \frac{\partial f}{\partial y}(t'_i, y') \neq 0 \quad \text{and} \quad h_i(t'_i) \neq 0 \quad \text{for} \quad 1 \leq i \leq m.
\]

Now let $K_1$ be the subfield of $K_0$ which is generated over $K$ by the following elements:

- $t'_1, \ldots, t'_s, y'$,
- the coefficients of $f$, $g$, and $h_i$ for $1 \leq i \leq m$.

We note that $K_1$ is a finite extension of $K$. We will now construct an extension $K_4$ of $K_1$ with $K_1$-rational place $\lambda_4$, which will contain an isomorphic copy of $K_1.F$. The construction will be done in such a way that the place $\lambda$ induced on $F$ through the resulting embedding of $F$ in $K_4$ and the place $\lambda_4$ will satisfy the assertions of our theorem.

We write $\Gamma = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \alpha_i$ with $\alpha_i > 0$. We adjoin $r$ many algebraically independent elements $x_1, \ldots, x_r$ to $K_1$ and denote the resulting field by $K_2$. By [3, Chapter VI, §10.3, Theorem 1] (see also [19, Lemma 25]), there is a place $\lambda_2$ of $K_2$ whose restriction to $K_1$ is the identity, such that $K_2 \lambda_2 = K_1$ and $v_{\lambda_2} x_i = \alpha_i$ for $1 \leq i \leq r$, whence $v_{\lambda_2} K_2 = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \alpha_i = \Gamma$.

Since $r \geq 1$, Proposition 4.1 shows that $(K_2, \lambda_2)$ admits an immediate extension of transcendence degree $s - r$. We pick a transcendence basis $x_{r+1}, \ldots, x_s$ of this extension and take $(K_3, \lambda_3)$ to be the immediate subextension which it generates over $(K_2, \lambda_2)$. It follows that $\lambda_3|_{K_1} = \lambda_2|_{K_1} = \text{id}_{K_1}$. We may choose the elements $x_i$ such that $v_{\lambda_3} x_i > 0$, $r + 1 \leq i \leq s$. We have the same for $1 \leq i \leq r$ since all $\alpha_i$ are positive.

Now we take $(K_4, \lambda_4)$ to be the henselization of $(K_3, \lambda_3)$. Since it is an immediate extension of $(K_3, \lambda_3)$, in turn is an immediate extension of $(K_2, \lambda_2)$, we have that $v_{\lambda_4} K_4 = v_{\lambda_2} K_2 = \Gamma$ and $K_4 \lambda_4 = K_2 \lambda_2 = K_1$, as well as $\lambda_4|_{K_1} = \text{id}_{K_1}$.

We wish to show that $F$ can be embedded in $K_4$ over $K$. In fact, we find an embedding $\iota$ of $K_1.F$ over $K_1$ in $K_4$ as follows. We set $t_i^* := t_i + x_i \in K_4$, $1 \leq i \leq s$; since $v_{\lambda_4} x_i = v_{\lambda_2} x_i = \alpha_i > 0$ we have that $\lambda_4(x_i) = 0$ and obtain that $\lambda_4(t_i^*) = t_i^*$. Using Hensel’s Lemma, we lift the simple root $y'$ of $f(t'_i, Y)$ to an element $y^* \in K_4$ which satisfies $f(t_i^*, y^*) = 0$ and $\lambda_4(y^*) = y'$. By construction, $t_1^*, \ldots, t_s^*$ are algebraically independent over $K_1$, so we obtain the desired embedding by setting $\iota(t_i) = t_i^*$ and $\iota(y) = y^*$. Then we take $\lambda$ to be the restriction of $\lambda_4 \circ \iota$ to $F$. As $x_i = t_i^* -


Now suppose that we have already constructed places \( \lambda_1, \ldots, \lambda_k \in S(F|K) \) which are finite on \( a_1, \ldots, a_m \) and satisfy all additional assertions. Since \( \text{trdeg}_F K \geq 1 \) by our general assumption, but \( F\lambda_j|K \) is algebraic, the places \( \lambda_j \) are nontrivial. Hence there are elements \( a_{m+j} \in F \) such that \( \lambda_j(a_{m+j}) = \infty \) for \( 1 \leq j \leq k \). As shown above, there exists a place \( \lambda \) which is finite on \( a_1, \ldots, a_{m+k} \) and satisfies all additional assertions. It follows that \( \lambda(a_{m+j}) \neq \infty = \lambda_j(a_{m+j}) \) and hence \( \lambda \) is not equivalent to \( \lambda_j \) for \( 1 \leq j \leq k \). This shows that there are infinitely many nonequivalent places which satisfy all assertions of the first part of our theorem.

If \( K \) is existentially closed in \( F \), then \( F|K \) is separable (cf. [22, Lemma 5.3]). In this case, the proof proceeds as above with \( K \) in place of \( K_0 \) and \( F \) in place of \( F' \). We then have that \( K_1 = K \), which implies that \( F\lambda = K \). We also have that \( t_i' \in K \) for all \( i \), which yields that \( x_i \in K(t_i') \subseteq \iota(F) \). As a consequence, \( \Gamma \subseteq v_\lambda F \), so that \( v_\lambda F = \Gamma \).
From part 2) of Theorem 4.2 we obtain that \((K_0, \xi_0)\) is existentially closed in \((F', \xi)\). Hence there exist elements
\[
t_1', \ldots, t_s', y' \in K_0
\]
such that for \(1 \leq i \leq m\),
\[
(i) \quad f(t', y') = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'}(t', y') \neq 0,
(ii) \quad g_i(t', y') \neq 0, \quad h_i(t') \neq 0,
(iii) \quad v_{\xi_0}g_i(t', y') \geq v_{\xi_0}h_i(t') \quad \text{if} \quad a_i \in \mathcal{O}_\xi,
(iv) \quad v_{\xi_0}g_i(t', y') > v_{\xi_0}h_i(t') \quad \text{if} \quad a_i \in \mathcal{M}_\xi,
(v) \quad v_{\xi_0} \left( \frac{g_i(t', y')}{h_i(t')} - a'_i \right) > 0 \quad \text{if} \quad \xi(a_i) \in K\varphi,
\]
since these assertions are true in \(F'\) for \(t, y\) in place of \(t', y'\) and \(v_{\xi}\) in place of \(v_{\xi_0}\).

Now let \(K_1\) be as in the proof of Theorem 4.3, and let \(\varphi_1\) denote the restriction of \(\xi_0\) to \(K_1\). As before, \(K_1\) is a finite extension of \(K\) and \(\varphi_1\) is an extension of \(\varphi\). The extension \(K_4\) of \(K_1\) with the \(K_1\)-rational place \(\lambda_4\) and the embedding \(\iota\) of \(F'\) over \(K_1\) in \(K_4\) are constructed as in the proof of Theorem 4.3. As before, we obtain \(\lambda \in S(F[K])\) with \(v_{\lambda}\) finite. We take \(\varphi'\) to be the restriction of \(\varphi_1\) to \(F\lambda\). Then assertions \((a)\) and \((b)\) of our theorem are satisfied.

We still have to check assertions \((c)\), \((d)\), \((e)\) and \((f)\) on the elements \(a_i\). Since \(\lambda_4(t_i^*) = t'_i\) and \(\lambda_4(y^*) = y'\), we have that
\[
\lambda(g(t, y)) = \lambda_4(\iota(g(t, y))) = \lambda_4(g(t^*, y^*)) = g(t', y')
\]
for every polynomial \(g \in K_1[X_1, \ldots, X_s, Y]\). Consequently, using that \(h_i(t') \neq 0\) by \((ii)\),
\[
\lambda(a_i) = \lambda \left( \frac{g_i(t^*, y^*)}{h_i(t^*)} \right) = \frac{g_i(t', y')}{h_i(t')}.
\]
Hence \((iii)\) and \((iv)\) imply that \((c)\), \((d)\) and \((f)\) hold (note that \(\lambda(a_i) \in \mathcal{O}_{\varphi'}\) implies that \(a_i \in \mathcal{O}_{\varphi'\circ\lambda} = \mathcal{O}_{\varphi'}\)). If \(\xi(a_i) \in K\varphi\), then by \((v)\),
\[
\xi'(a_i) = \varphi'(\lambda(a_i)) = \varphi' \left( \frac{g_i(t', y')}{h_i(t')} \right) = \xi_0 \left( \frac{g_i(t', y')}{h_i(t')} \right)
\]
\[
= \xi_0(a'_i) = \xi(a'_i) = \xi(a_i),
\]
which shows that also assertion \((e)\) holds.
To prove assertion A), we modify our above proof as follows. If $E$ is any field, we will denote by $E^{1/p^\infty}$ its perfect hull (which is equal to $E$ if char $E = 0$).

Now we take $(L, \xi)$ to be a maximal algebraic extension of $(F, \xi)$ with the property of having $(F \xi)^{1/p^\infty} = (K \wp)^{1/p^\infty}$ as its residue field. Then $(L, \xi)$ will have a divisible value group. For the construction of such an extension, cf. Section 2.3 of [20]. Further, $(L, \xi)$ is algebraically maximal (i.e., does not admit nontrivial immediate algebraic extensions) and therefore, it is a tame field by [22, Theorem 3.2].

This time we take $K_0$ to be the relative algebraic closure of $K$ in $L$, and $\xi_0$ the restriction of $\xi$ to $K_0$; as before, $\xi_0$ is an extension of $\wp$. Since $L \xi = (K \wp)^{1/p^\infty}$ is algebraic over $K \wp$, [22, Lemma 3.7] shows that $(K_0, \xi_0)$ is a tame field with $K_0 \xi_0 = L \xi = (K \wp)^{1/p^\infty}$ and $v_{\xi_0} K_0$ equal to the divisible hull of $v_\wp K$.

Since a divisible ordered abelian group is existentially closed in every ordered abelian group extension, and since $\xi_0$ is nontrivial, we can apply [22, Theorem 1.4] to obtain that $(K_0, \xi_0)$ is existentially closed in $(L, \xi)$. By [22, Lemma 3.1], the tame field $K_0$ is perfect, hence again $K_0, F|K_0$ is separably generated.

From here, the construction proceeds as before. Since $K_1|K$ is finite and $K_1 \wp$ is contained in the purely inseparable extension $L \xi$ of $K \wp$, we conclude that $K_1 \wp$ is a finite purely inseparable extension of $K \wp$. Since $\iota(F) \subseteq K_4$, we have that $F \xi' \subseteq (K_4 \lambda_4) \wp = K_1 \wp$. Therefore, $F \xi'|K \wp$ is a finite purely inseparable extension.

In order to prove assertion B), we just take $K_0 = K$; then also $K_1 = K$. Again, the construction proceeds as before. Now we have that $F \lambda = K$, $F \xi' = K \wp$ and $\wp' = \wp$. Further, one shows as in the proof of Theorem 4.3 that $v_\lambda F = \Gamma$.

Finally, we prove assertion C). Under the assumptions of C) we can take $K_0 = K$ and obtain from part 4) of Theorem 4.2 that in the language of rings with relation symbols for an ordering and a valuation, $K$ is existentially closed in $F$. Hence from assertion B), we obtain a place $\lambda$ of $F$ such that the assertions (a) – (f) are satisfied and in addition, $F \lambda = R$, $v_\lambda F = \Gamma$, $F \xi' = R \wp$, $\wp' = \wp$.

To obtain assertion (g), we can choose the elements $t'_i$ and $y'$ such that in addition to (i) – (v), also the following holds:

\[ \frac{g_i(t', y')}{h_i(t')} > 0 \quad \text{if} \quad a_i > 0. \]

To prove assertion (h), we assume in addition that $\xi(a_i) \neq 0, \infty$. This means that $a_i, a_i^{-1} \in \mathcal{O}_\xi$. Hence, in view of assertion (c), we can
choose $\lambda$ such that $\lambda(a_i), \lambda(a_i^{-1}) \in O_\varphi$, so $\xi'(a_i) = \varphi(\lambda(a_i)) \neq 0, \infty$. Since $\lambda(a_i) > 0$ and $\varphi$ is compatible with the ordering on $K$, this implies that $\xi'(a_i) = \varphi(\lambda(a_i)) > 0$. This proves assertion (h).

Since $\lambda(a_i) > 0$ and $\varphi$ is compatible with the ordering on $K$, this implies that $\xi'(a_i) = \varphi(\lambda(a_i)) > 0$. This proves assertion (h).

Part II: We will now assume that $\varphi$ is trivial. In this case we can assume that $\varphi = \text{id}_K$ since otherwise we replace $\xi$ by $\xi \circ \sigma$ where $\sigma$ is any monomorphism on $F$ which extends $\varphi^{-1}$. We then also choose every extension $\varphi'$ of $\varphi$ to be the identity. Further, we have that $O_\varphi = K$ and $M_\varphi = \{0\}.$

Part II.1: First we discuss the case where the place $\xi$ is trivial. Then $\xi$ is a monomorphism and we may assume that $\xi|_F = \text{id}_F$ since otherwise, we apply the following proof to $F\xi$ and $\xi(a_i)$ in place of $F$ and $a_i$ and then replace the places $\lambda$ of $F\xi$ that we obtain by the places $\lambda \circ \xi$ of $F$.

Since $\xi$ is trivial, we have that $O_\xi = F$ and $M_\xi = \{0\}$. Hence assertion (d) of our theorem is satisfied for every $\lambda \in S(F|K)$ because $a_i \in M_\xi$ would imply that $a_i = 0$, contrary to our choice of the elements $a_i$. Also assertion (e) is always satisfied, as the condition $\xi(a_i) \in K_\varphi$ means that $a_i \in K$, whence $\xi'(a_i) = \varphi(a_i) = a_i = \xi(a_i)$ as $\lambda$ and $\varphi$ are trivial on $K$.

For any choice of finitely many elements $a_1, \ldots, a_m \in F$, Theorem 4.3 shows the existence of infinitely many places $\lambda$ which satisfy assertions (a) and (b) as well as $\lambda(a_i) \neq 0, \infty$ for $1 \leq i \leq m$. The latter implies that they also satisfy assertions (c) and (f).

By our general assumption on function fields $F|K$, we have that $F \neq K$. Hence in our present setting ($\varphi$ and $\xi$ trivial), the condition $F\xi = K_\varphi$ is never satisfied, and therefore, assertion A) of our Theorem is trivially true.

In the present setting, assertion B) follows from the above together with the second statement of Theorem 4.3.

In order to prove assertion C) in the present setting, we apply the already proven assertion B). In addition, we just have to show that $\lambda$ can be chosen such that if $a_i > 0$, then $\infty \neq \lambda(a_i) > 0$. To this end, we may replace $F$ by a larger ordered function field in which every positive $a_i$ is a square. As we already have that $\lambda(a_i) \neq 0, \infty$, it then follows that also $\xi'(a_i) = \lambda(a_i)$ is a nonzero square, hence positive.

Part II.2: Now we deal with the case of $\xi$ being nontrivial.

Part II.2a: We wish to satisfy assertions (d) and (e), but not necessarily assertion (f).
Assume first that $\text{trdeg } F/K = 1$. We claim that $\lambda = \xi$ satisfies assertions (a)-(e). Indeed, (a) and (b) are satisfied since $\xi$ is a nontrivial place of the function field $F|K$ of transcendence degree 1 which is trivial on $K$. As indicated before, we choose $\varphi'$ on $F\xi$ to be the identity, so we have that $\xi' = \xi$, $O_{\varphi'} = F\xi$ and $O_{\xi'} = O_\xi$. Hence if $a_i \in O_\xi$, then $\lambda(a_i) = \xi(a_i) \in O_{\varphi'}$ and $a_i \in O_{\xi'}$, so that $\lambda$ also satisfies assertion (c). Likewise, we have that $M_{\varphi'} = \{0\}$ and $M_{\xi'} = M_\xi$. Hence if $a_i \in M_\xi$, then $\lambda(a_i) = \xi(a_i) = 0 \in M_{\xi'}$ and $a_i \in M_{\xi'}$, so that $\lambda$ also satisfies assertion (d). Further, $\xi'(a_i) = \xi(a_i)$, so also assertion (d) is satisfied.

Assume now that $\text{trdeg } F|K > 1$. Since $\xi$ is nontrivial, there is some $x \in F$ such that $\xi(x) = 0$. We denote the $x$-adic place of $K(x)$ by $\xi_x$. We apply the already proven part of our theorem to the function field $F|K(x)$, with $\varphi$ replaced by $\xi_x$, to obtain a place $\lambda' \in S(F|K(x))$ such that, with $\xi_x$ extended to $F\lambda'$ and $\lambda := \xi_x \circ \lambda' \in S(F|K)$, (a') $F\lambda'$ is a finite extension of $K(x)$,
(b') $v_{\lambda'} F \subseteq \Gamma'$,
(c') if $a_i \in O_\xi$, then $\lambda'(a_i) \in O_{\xi_x}$ and, consequently, $a_i \in O_\lambda$,
(d') if $a_i \in M_\xi$, then $\lambda'(a_i) \in M_{\xi_x}$ and, consequently, $a_i \in M_\lambda$,
(e') if $\xi(a_i) \in K(x)\xi_x = K$, then $\lambda(a_i) = \xi(a_i)$ for $1 \leq i \leq m$,
(f') $\lambda'(a_i) \neq 0, \infty$.

Now (a') implies that $F\lambda = (F\lambda')\xi_x$ is a finite extension of $K(x)\xi_x = K$, so assertion (a) of our theorem is satisfied. Since $\text{trdeg } F\lambda'|K = \text{trdeg } K(x)|K = 1$, we obtain that $v_{\xi_x}(F\lambda') = \mathbb{Z}$, so that $v_{\xi_x} F$ is the lexicographic product of $v_{\lambda'} F$ with $\mathbb{Z}$, which by (b') is a subgroup of $\Gamma$.

To see that assertions (c) and (d) of our theorem are satisfied, we recall that we take the extension $\varphi'$ of the trivial place $\varphi$ of $K$ to $F\lambda$ to be the identity. Consequently, $O_{\lambda} = O_{\varphi' \circ \lambda}$ and $M_{\lambda} = M_{\varphi' \circ \lambda}$. To see that assertion (e) of our theorem is satisfied, we use statement (e') above and observe that $\xi' = \varphi' \circ \lambda = \lambda$.

**Part II.2b:** We wish to satisfy assertion (f), but not necessarily assertions (d) and (e). Note that in this setting, assertion (c) follows directly from assertion (f). Hence in fact, the given place $\xi$ does not play any role. Therefore we obtain infinitely many places $\lambda$ with the required properties by just applying Theorem 4.3.

**Part II.2c:** We prove the additional assertions in the present setting ($\varphi$ trivial while $\xi$ nontrivial).

To prove assertion A), we assume that $F\xi = K\varphi = K(x)\xi_x$. Then the condition of assertion A) is satisfied for $(K(x), \xi_x)$ in place of $(K, \varphi)$,
and from the already proven part of our theorem we infer that in addition to the above we can choose \( \lambda' \) such that \( F(\xi_x \circ \lambda')|K(x)x \) is a finite purely inseparable extension. As \( \xi' = \varphi' \circ \xi_x \circ \lambda = \xi_x \circ \lambda' \) and \( K(x)x = K = K\varphi \), this yields that \( F\xi'|K\varphi \) is a finite purely inseparable extension.

We do not have to deal with assertion B) since its condition is never satisfied in the present setting.

Next, we turn to the proof of assertion C) in the present setting, so we assume that \( K = R \) is a real closed field and that \( \xi \) is compatible with an ordering \( < \) of \( F \). First, we deal with the case where we want \( \lambda \) and \( \xi' \) to satisfy assertions (f), (g) and (h), but not necessarily assertions (d) and (e). As we have done before, we replace \( F \) by a larger ordered function field in which every positive \( a_i \) is a square. As we already have obtained a place \( \lambda \) such that \( \lambda(a_i) \neq 0, \infty \), it then follows that also \( \xi'(a_i) = \lambda(a_i) \) is a nonzero square, hence positive.

Now we deal with the case where we want \( \lambda \) and \( \xi' \) to satisfy assertions (d), (e) and (h), but not necessarily assertions (f) and (g). We choose a real closure \( F^{\text{rc}} \) and take \( K' := R(x)^{\text{rc}} \) to be the corresponding real closure of \( R(x) \) within \( F^{\text{rc}} \). Further, we extend the ordering of \( F \) and the place \( \xi \) to an ordering and a compatible place on the compositum \( F' := K'.F \). We endow \( K' \) with the restriction of the place of \( F' \); as it is an extension of \( \xi \), we also denote it by \( \xi_x \). Note that \( K'\xi_x = R \). Now we can apply the already proven part of assertion C) of our theorem with \((R,\varphi)\) replaced by \((K',\xi_x)\) to obtain a \( K'\)-rational place \( \lambda' \) of \( F' \) such that the above assertions \((c')-(f')\) are satisfied and in addition,

\[(h') \text{ if } a_i > 0 \text{ and } \xi(a_i) \neq 0, \infty, \text{ then } \xi_x \circ \lambda'(a_i) > 0.\]

We take \( \lambda \) to be the restriction of \( \xi_x \circ \lambda' \) to \( F \). Since \( F'\lambda' = K' \) and \( K'\xi_x = R \), we have that \( F\lambda = R \). By assumption, \( \varphi = \text{id}_R \), so \( \xi' = \varphi \circ \lambda = \lambda \). Hence assertions \((c'), (d'), (e') \) and \((h')\) show that assertions \((c), (d), (e) \) and \((h)\) of our theorem hold. In the present setting, \( \lambda \) does not have to satisfy assertions \((f)\) and \((g)\).

In order to prove that \( \lambda' \) can be taken such that \( v_{\lambda'}F = \Gamma \), we have a closer look at our construction method that is taken from the proof of Theorem 4.3. In our present setting, the elements \( t \) will form a separating transcendence basis of \( F'|K' \). But since \( \text{char} K' = 0 \), every transcendence basis is separating, and since \( F'|F \) and \( K'|R(x) \) are algebraic, we can choose it inside of \( F \). As a consequence, \( x_i \in \iota(F) \) for
all $i$, showing that $v_{\lambda}^i F = \Gamma'$. Further, the field $K_1$ is a finite extension of $R(x)$, hence $v_{\xi_x} K_1 = \mathbb{Z}$. It follows that $v_{\lambda} F = v_{\xi_x \circ \lambda} F$ is the lexicographic product of $\Gamma'$ and $\mathbb{Z}$, which is $\Gamma$.

**Part III:** We prove the last assertion of our theorem, except in the case where $\text{trdeg} F|K = 1$ and $\wp$ is trivial while $\xi$ is not.

It suffices to show the assertion for the places $\lambda$. This is seen as follows. The valuation ring of $\lambda$ is an overring of $\wp \circ \lambda$ and the overrings of a valuation ring in a field are linearly ordered by inclusion. Hence if $\lambda_1$ and $\lambda_2$ are such that $\wp \circ \lambda_1$ and $\wp \circ \lambda_2$ have the same valuation ring, then the valuation rings of $\lambda_1$ and $\lambda_2$ must be comparable by inclusion. But since $F \lambda_1$ and $F \lambda_2$ are algebraic over $K$, this implies that the two valuation rings are equal.

In all cases where the constructed places $\lambda$ satisfy assertion $(f)$ of our theorem, the argument given in the proof of Theorem 4.3 shows the existence of infinitely many nonequivalent places $\lambda$.

It remains to prove our assertion in the case where $\wp$ is trivial while $\xi$ is not and we showed the existence of places that satisfy assertions $(d)$ and $(e)$. In this case we constructed places $\lambda'$ satisfying assertion $(f')$, hence by what we just said, there are infinitely many nonequivalent such places $\lambda'$. By our above argument (with $\lambda$ replaced by $\lambda'$ and $\wp$ replaced by $\xi_x$), also the resulting places $\lambda = \xi_x \circ \lambda'$ are nonequivalent.

\[\square\]

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