Prime Sums of Primes

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Abstract

We present a variety of prime-generating constructions that are based on sums of primes. The constructions come in all shapes and sizes, varying in the number of dimensions and number of generated primes. Our best result is a construction that produces 6 new primes for every starting prime.

1 Introduction

Constructions made from primes have fascinated mathematicians for many decades due to the beauty of their design. A number of such constructions have been proposed, such as: prime magic squares [4, 9], prime arrays [8] and primes in arithmetic progressions [1, 2].

In this paper we investigate some new prime-generating constructions that are based on sums of primes. Our constructions come in two flavours: standard and recursive. In standard constructions new primes are generated as the sum of primes used in the construction. Recursive constructions generate new primes, which in turn generate further primes. The recursion terminates when no more primes can be generated. Typically we only use odd primes (ignore 2), forcing our sums to contain an odd number of elements. Our overall aim is to generate constructions of the largest size (order). If two constructions have the same order then we typically prefer the one with smallest sum of elements (weight). To find all the constructions we use a variant of the randomised hill-climbing algorithm. For small constructions we were able to find the optimal solutions (smallest weight) by using a brute force method.

We describe the following standard constructions: prime vectors (Section 2), cyclic prime vectors (Section 2.1), Goldbach squares (Section 3) and prime matrices (Section 7). We describe the following recursive constructions: prime tuples (Section 3), prime stairs (Section 4), prime pyramids (Section 4.1) and prime cylinders (Section 5).

2 Prime Vectors

Definition 2.1. A prime vector of order n is an array of distinct primes \( P = (p_0, p_1, \ldots, p_{n-1}) \), such that every sum of an odd number of consecutive primes in the array results in a prime.
utive elements is also prime. In other words
\[ \sum_{0 \leq k \leq 2L} P(i + k) \] is prime for \[ \forall i \text{ such that } 0 \leq i \leq i + 2L < n. \] (1)

In the above definition, \( i \) is the index of the first prime in each sum, while \((2L + 1)\) is the number of terms in each sum. For a given \( n \) there are \([ (n - 1)^2 / 4 ] \) sums. Consider a prime vector of order 5: \((3, 11, 5, 7, 17)\). Its every element is prime, as well as, every sum of an odd number of consecutive elements:

\[
3 + 11 + 5 = 19, \quad 11 + 5 + 7 = 23, \\
5 + 7 + 17 = 29, \quad 3 + 11 + 5 + 7 + 17 = 43.
\] (2)

We used a variant of hill-climbing to find prime vectors (see Algorithm 1). We start with a random array of distinct primes and then perform various mutations, such as swapping two primes or replacing one prime with a new one. If the mutation improves the score then we keep it, otherwise we revert it. The score measures the number of “incorrect” (composite) sums that the array generates. Hence we want to minimise this score. Using this algorithm we were able to obtain a prime vector of order 23 that generates 121 primes:

\[
\begin{align*}
(239, 131, 109, 181, 83, 43, 41, 223, 53, 233, 271, 103, 269, 71, 19, 47, \\
241, 23, 277, 199, 281, 29, 37).
\end{align*}
\]

For small orders it is possible to obtain multiple solutions. In such cases we choose the solution with the smallest weight - sum of all elements. In fact, this allows us to define an optimal prime vector:

**Definition 2.2.** A prime vector is **optimal** if its weight is the lowest possible.

For \( n \leq 14 \) we were able to find the optimal prime vectors (see Table 1). To achieve this we used a brute force algorithm. This algorithm iterates through every permutation of \( n \) distinct odd primes whose weight is below the best known weight. If a permutation forms a prime vector then the best known weight is updated and the array is printed out. The algorithm terminates when there are no more permutations whose weight is less than the best known weight. Table 1 also shows the running time of this algorithm.

For \( n > 14 \) we used Algorithm 1 to find the upper bounds on the minimal weight (see Table 2). To obtain the lower bound we used sequences from the OEIS. For odd \( n \) the weight must be a prime, so we used sequence A068873 - smallest prime which is a sum of \( n \) distinct primes. For even \( n \) we used sequence A071148 - sum of the first \( n \) odd primes.

### 2.1 Cyclic Prime Vectors

We can also introduce a cyclic prime vector and define its optimality in a similar fashion:

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1. If \( L = 0 \) then we have a singleton rather than a sum.
2. Prime vectors of smaller orders are sub-arrays of this array.
Algorithm 1: Algorithm for finding prime vectors.

1. \( \text{bestScore} \leftarrow \infty \)
2. \( S^* \leftarrow \) random set of \( n \) distinct primes
3. 
4. while True
5. \( S \leftarrow S^* \)
6. mutate(\( S \))
7. \( \text{score} \leftarrow \text{score}(S) \)
8. if \( \text{score} < \text{bestScore} \)
9. \( \text{bestScore} \leftarrow \text{score} \)
10. \( S^* \leftarrow S \)
11. print(\( S^* \))
12. end
13. end

| \( n \) | Prime Vector | Weight | Time  |
|-------|--------------|--------|-------|
| 1     | (2)          | 2      |       |
| 2     | (3, 5)       | 8      |       |
| 3     | (3, 5, 11)   | 19     |       |
| 4     | (3, 5, 11, 7)| 26     |       |
| 5     | (3, 11, 5, 7, 17) | 43 |       |
| 6     | (3, 11, 5, 7, 17, 13) | 56 |       |
| 7     | (3, 17, 23, 7, 11, 13, 5) | 79 |       |
| 8     | (3, 11, 17, 13, 29, 19, 5, 7) | 104 |       |
| 9     | (7, 17, 13, 23, 11, 3, 29, 5, 19) | 127 |       |
| 10    | (3, 7, 19, 11, 13, 23, 31, 5, 37, 17) | 146 |       |
| 11    | (3, 23, 41, 19, 11, 13, 17, 7, 5, 31, 53) | 223 | 17s   |
| 12    | (7, 41, 19, 11, 23, 3, 5, 29, 13, 47, 43, 17) | 258 | 8m    |
| 13    | (13, 53, 7, 23, 11, 3, 29, 5, 19, 17, 43, 47, 37) | 307 | 73m   |
| 14    | (17, 43, 47, 13, 29, 5, 3, 23, 11, 19, 41, 7, 53, 37) | 348 | 14h   |

Table 1: Optimal prime vectors for \( n \leq 14 \), their weight and the time required to compute them. Computation times less than 1 second are not shown.

| \( n \) | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|-------|----|----|----|----|----|----|----|----|----|
| lower bound | 379 | 438 | 499 | 566 | 643 | 710 | 809 | 872 | 983 |
| upper bound  | 443 | 522 | 641 | 888 | 983 | 1430 | 1627 | 1824 | 3203 |

Table 2: Best bounds on the minimal weight of prime vectors for \( 15 \leq n \leq 23 \).
**Definition 2.3.** A *cyclic prime vector* of order $n$ is a prime vector $P$ of order $n$ with the additional property that prime sums can span from the end to the start of the array. In other words

$$\sum_{0 \leq k \leq 2L} P((i + k) \mod n) \text{ is prime for}$$

$$\forall \ i \text{ such that } 0 \leq i < n \text{ and}$$

$$\forall \ L \text{ such that } 0 \leq 2L < n.$$  \hspace{1cm} (3)

For a given $n$ there are $(n - 2)(2n - 1 + (-1)^n)/4$ sums. For example the cyclic prime vector $(5, 7, 17, 13, 11)$ generates the following 6 sums:

$$5 + 7 + 17 = 29, \quad 7 + 17 + 13 = 37,$$

$$17 + 13 + 11 = 41, \quad 13 + 11 + 5 = 29,$$

$$11 + 5 + 7 = 23, \quad 5 + 7 + 17 + 13 + 11 = 53.$$  \hspace{1cm} (4)

Cyclic prime vectors differ from normal prime vectors in a few key ways. Every cyclic prime vector is also a normal prime vector, but the opposite may not be the case. Unlike normal prime vectors, cyclic prime vectors can be permuted without affecting their prime sums. Also we cannot easily generate cyclic prime vectors as sub-arrays of larger cyclic prime vectors. Due to the cyclic requirement, cyclic prime vectors require more prime sums for the same order, making them significantly harder to find.

Using the brute force algorithm described above we were able to find the optimal cyclic prime vectors for $n \leq 10$ (see Table 3). The computation for the optimal cyclic prime vector of order 11 was still running after 4 days, so it is not shown. It is interesting to note that the weight for $n = 9$ is smaller than the weight for $n = 8$. Using an algorithm similar to Algorithm 1 we found cyclic prime vectors up to order 14 (see Table 4). The largest array generates 84 primes.

| $n$ | Cyclic Prime Vector | Weight | Time |
|-----|---------------------|--------|------|
| 1   | (2)                 | 2      |      |
| 2   | (3, 5)              | 8      |      |
| 3   | (3, 5, 11)          | 19     |      |
| 4   | (5, 7, 17, 19)      | 48     |      |
| 5   | (5, 7, 17, 13, 11)  | 53     |      |
| 6   | (5, 29, 7, 11, 19, 37) | 108 |      |
| 7   | (5, 7, 17, 13, 29, 31, 11) | 113 |      |
| 8   | (11, 17, 43, 47, 13, 19, 29, 31) | 210 |      |
| 9   | (7, 17, 13, 11, 19, 41, 29, 37, 23) | 197 | 9s   |
| 10  | (11, 19, 23, 47, 31, 53, 43, 67, 89, 127) | |      |

Table 3: Optimal cyclic prime vectors for $n \leq 10$, their weight and the time required to compute them. Computation times less than 1 second are not shown.

### 3 Prime Tuples

**Definition 3.1.** A *prime tuple* of order $n$ (odd) with length $k$ is an array of distinct odd primes $(p_0, p_1, \ldots, p_{k-1})$, such that every term after the
Table 4: Smallest (by weight) cyclic prime vectors found for $11 \leq n \leq 14$.

$n$-th term is the sum of the previous $n$ terms. In other words

$$p_i = \sum_{q=i-n}^{i-1} p_q, \quad \forall i \geq n.$$  \hspace{1cm} (5)

Note it is sufficient to use the first $n$ terms to represent a prime tuple, since the remaining terms can be generated via sums of previous terms. We seek to find prime tuples of order $n$ such that their length is greatest. For example, here is a prime tuple of order 7 with length 25 - the longest we have found:

$$(157, 379, 487, 109, 13, 7, 271, 1423, 2689, 4999, 9511, 18913, 37813, 75619, 150967, 3728733, 9419653, 18763687, 37376407, 74452303, 148306273).$$

The first 7 terms are shown in bold. The weight of a prime tuple of order $n$ is the sum of its first $n$ terms. When two tuples of the same order have the same length, then we prefer the one with the smaller weight.

Table 5 shows the best prime tuples that we found for $n \leq 19$. We have used a brute force approach to prove that the prime tuples for $n \in \{3, 5, 9, 11\}$ are optimal. We notice that for $n \mod 6 = 3$ and $n \mod 6 = 5$ the optimal prime tuples have length $2n + 1$ and must contain a 3.

| $n$ | Prime Tuple | Length | Weight |
|-----|-------------|--------|--------|
| 3   | (3, 13, 7)  | 7      | 23     |
| 5   | (17, 3, 19, 7, 13) | 11 | 59 |
| 7   | (157, 379, 487, 109, 13, 7, 271) | 25 | 1423 |
| 9   | (11, 47, 17, 23, 41, 5, 3, 13, 19) | 19 | 179 |
| 11  | (43, 7, 19, 13, 3, 17, 11, 5, 29, 41, 23) | 23 | 211 |
| 13  | (53, 137, 11, 17, 41, 227, 47, 101, 83, 5, 149, 263, 29) | 34 | 1163 |
| 15  | (29, 5, 23, 11, 41, 47, 89, 17, 71, 3, 7, 13, 37, 19, 79) | 31 | 491 |
| 17  | (5, 47, 53, 11, 17, 41, 89, 3, 61, 43, 97, 19, 13, 7, 37, 31, 73) | 35 | 647 |
| 19  | (89, 227, 29, 17, 5, 251, 269, 107, 101, 197, 41, 191, 173, 179, 47, 53, 71, 11, 23) | 43 | 2081 |

Table 5: Best prime tuples found for $n \leq 19$. 

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4 Prime Stairs

Definition 4.1. A prime stair of order \( n \geq 3 \) is a \( \lceil \frac{n}{2} \rceil \times n \) matrix \( P \) such that every element \( P(r, c) \) at row \( r > 0 \) and column \( c \) is a distinct prime and each new row is generated from the previous row as follows:

\[
P(r, c) := P(r - 1, c - 1) + P(r - 1, c) + P(r - 1, c + 1).
\] (6)

For a given \( r > 0 \) we must have \( c \in [r, n - r - 1] \). For a given \( n \) there are \( \lfloor (n - 1)^2/4 \rfloor \) sums. As a shorthand we can represent a prime stair of order \( n \) via its first (top) row only, i.e., using an array of length \( n \). For example, the prime stair \((13, 17, 7, 5, 11, 3, 23)\) looks like this:

\[
\begin{align*}
13, 17, 7, 5, 11, 3, 23 \\
37, 29, 23, 19, 37 \\
89, 71, 79 \\
239
\end{align*}
\]

The weight of a prime stair is defined as the sum of all elements in the first row. We were able to find the optimal prime stairs for \( n \leq 11 \) (see Table 6). The computation for the optimal prime stair of order 12 was still running after 4 days, so it is not shown. Using an algorithm similar to Algorithm 1, we found prime stairs up to order 15 (see Table 7). The largest stair generates 49 primes.

\[
\begin{array}{|c|c|c|}
\hline
n & Prime Stair & Weight \\
\hline
3 & (3, 5, 11) & 19 \\
4 & (7, 5, 11, 3) & 26 \\
5 & (7, 13, 11, 5, 3) & 39 \\
6 & (7, 17, 13, 11, 5, 3) & 56 \\
7 & (13, 17, 7, 5, 11, 3, 23) & 79 \\
8 & (5, 17, 31, 11, 19, 7, 3, 13) & 106 \\
9 & (29, 23, 37, 13, 11, 5, 7, 19, 17) & 161 \\
10 & (5, 29, 7, 17, 13, 11, 37, 23, 19, 41) & 202 \\
11 & (7, 17, 19, 5, 73, 11, 13, 23, 31, 29, 41) & 269 \\
\hline
\end{array}
\]

Table 6: Optimal prime stairs for \( n \leq 11 \), their weight and the time required to compute them. Computation times less than 1 second are not shown.

\[
\begin{array}{|c|c|c|}
\hline
n & Prime Stair & Weight \\
\hline
12 & (37, 13, 17, 23, 7, 67, 5, 59, 19, 61, 29, 11) & 348 \\
13 & (29, 19, 11, 127, 89, 7, 17, 37, 5, 31, 23, 43, 41) & 479 \\
14 & (53, 17, 67, 29, 13, 5, 19, 149, 31, 101, 79, 11, 7, 43) & 624 \\
15 & (433, 139, 491, 97, 89, 163, 29, 7, 5, 61, 17, 79, 263, 541, 83) & 2497 \\
\hline
\end{array}
\]

Table 7: Best (by weight) prime stairs found for \( 12 \leq n \leq 15 \).

4.1 Prime Pyramids

Similarly we can define a 3D version of the prime stair that we will call a prime pyramid:
Definition 4.2. A prime pyramid of order $n \geq 3$ is a $\lceil \frac{n^2}{4} \rceil \times n \times n$ matrix $P$ such that every element $P(k, r, c)$ at level $k > 0$, row $r$ and column $c$ is a distinct prime and each new level is generated from the previous level as follows:

$$P(k, r, c) := \sum_{-1 \leq dr \leq 1} \sum_{-1 \leq dc \leq 1} P(k - 1, r + dr, c + dc).$$ (7)

For a given $k > 0$ we must have $r, c \in [k, n - k - 1]$. For a given $n$ there are $n(n - 1)(n - 2)/6$ sums. As a shorthand we can represent a prime pyramid of order $n$ via its first (bottom) level only, i.e., using a $n \times n$ array. For example, Table 8 shows a prime pyramid of order 5:

|       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|
| Level 0 | Level 1 | Level 2 |
| 73 | 11 | 67 | 71 | 53 |       |
| 101 | 41 | 43 | 79 | 83 |       |
| 13 | 3 | 31 | 7 | 23 |       |
| 17 | 61 | 37 | 5 | 29 |       |
| 97 | 89 | 19 | 59 | 47 |       |
| 383 | 353 | 457 |       |       |       |
| 347 | 307 | 337 |       |       |       |
| 367 | 311 | 257 |       |       |       |
|       |       |       |       | 3119 |       |

Table 8: Prime pyramid of order 5.

The weight of a prime pyramid is the sum of all elements in its first level. We were able to find all the optimal prime pyramids up to order 8 (see Table 9). We also found an order 9 prime pyramid with a weight of 27325, but its optimality is not confirmed (see Table 10).
| n | Prime Pyramid | Weight |
|---|-------------|--------|
| 3 | 7 11 29  
19 17 23  
5 13 3 | 127 |
| 4 | 53 19 47 7  
37 3 41 13  
43 5 29 17  
11 79 23 31 | 458 |
| 5 | 73 11 67 71 53  
101 41 43 79 83  
13 3 31 7 23  
17 61 37 5 29  
97 89 19 59 47 | 1159 |
| 6 | 97 43 47 149 79 3  
113 103 61 151 11 37  
109 53 107 19 127 67  
31 23 101 29 13 7  
83 5 59 71 73 157  | 2582 |
| 7 | 199 47 223 79 61 107 157  
229 89 5 71 29 163 211  
167 109 83 3 23 137 151  
97 197 43 73 59 19 31  
191 103 139 179 41 7 11  
17 173 227 37 13 149 127  
181 53 67 113 131 193 101 | 5115 |
| 8 | 263 229 23 167 89 61 109 79  
97 149 173 67 281 211 59 47  
113 163 283 197 11 7 233 227  
277 19 293 223 181 251 43 241  
179 139 191 239 193 71 41 103  
101 107 13 83 73 137 269 37  
3 127 311 271 157 307 199 53  
5 17 313 131 257 151 29 31 | 9204 |

Table 9: Optimal prime pyramids for $3 \leq n \leq 8$. 

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Table 10: Prime pyramid of order 9 with weight 27325.

|    |     |     |     |     |     |     |     |     |
|----|-----|-----|-----|-----|-----|-----|-----|-----|
| 11 | 79  | 349 | 461 | 433 | 859 | 683 | 587 | 631 |
| 367| 31  | 593 | 167 | 331 | 307 | 277 | 577 | 743 |
| 311| 67  | 191 | 151 | 281 | 47  | 101 | 619 | 439 |
| 389| 761 | 613 | 229 | 173 | 607 | 13  | 43  | 271 |
| 421| 563 | 241 | 557 | 317 | 337 | 673 | 751 | 113 |
| 73 | 71  | 127 | 137 | 163 | 193 | 661 | 23  | 181 |
| 409| 571 | 691 | 61  | 83  | 251 | 179 | 233 | 877 |
| 467| 53  | 227 | 59  | 89  | 373 | 401 | 37  | 149 |
| 19 | 547 | 809 | 521 | 131 | 41  | 659 | 503 | 491 |

5 Prime Cylinders

Definition 5.1. A prime cylinder of order $n$ with $k$ layers is a $n \times k$ matrix $P$ of odd primes, such that for every $c$ and $r > 0$: $P(r, c) = P(r - 1, c - 1) + P(r - 1, c) + P(r - 1, c + 1)$.

Note that the columns wrap around and hence the term ‘cylinder’. For example here is a prime cylinder of order 4 and 6 layers - the best found so far:

1091, 3001, 271, 257
4349, 4363, 3529, 1619
10331, 12241, 9511, 9497
32069, 32083, 31249, 29339
93491, 95401, 92671, 92657
281549, 281563, 280729, 278819

Since all the values below the first layer can be generated from previous values, a prime cylinder can be described using its first layer only. So the above prime cylinder would be described as (1091, 3001, 271, 257). The weight of a prime cylinder is the sum of values in its first layer. When multiple prime cylinders have the same order and number of layers, then we prefer the one with the smaller weight. Prime cylinders were originally introduced in [5], but were limited to $n = 4$. Here we investigate other values of $n$. It turns out that prime cylinders of odd orders cannot have more than two layers, so we focus on prime cylinders of even orders. Table 11 shows the best prime cylinders found for $n \leq 12$.

6 Goldbach Squares

The famous Goldbach conjecture states that
Table 11: Best prime cylinders found for $n \leq 12$.

| $n$ | Prime Cylinder | Layers | Weight |
|-----|----------------|--------|--------|
| 4   | (1091, 3001, 257, 271) | 6      | 4620   |
| 6   | (163, 1109, 307, 1163, 109, 1307) | 6      | 4158   |
| 8   | (67, 541, 23, 137, 109, 193, 389, 431) | 5      | 1890   |
| 10  | (19, 17, 7, 107, 43, 23, 13, 71, 79, 101) | 4      | 480    |
| 12  | (11, 29, 31, 79, 53, 5, 109, 43, 47, 41, 61, 139) | 4      | 648    |

Every even integer greater than 2 can be expressed as the sum of two primes.

Although the conjecture has been verified up to $4 \times 10^{18}$ [3], a proof still remains elusive. Here we investigate a problem related to the Goldbach conjecture: can we place primes into a square such that every even number is generated as the sum of two adjacent cells? This puzzle has been explored in [6]. More formally we have:

**Definition 6.1.** A Goldbach square of order $n$ is a $n \times n$ matrix of odd primes (not necessarily unique) such that the sum of any two adjacent cells is one of the even numbers from 6 to $4 + 4n(n - 1)$ inclusive and every even number in this range appears exactly once.

For example, here is a Goldbach square of order 3:

$$
\begin{array}{ccc}
7 & 5 & 3 \\
17 & 11 & 3 \\
3 & 7 & 19 \\
\end{array}
$$

The sums across rows are:

$7 + 5 = 12, \quad 5 + 3 = 8, \\
17 + 11 = 28, \quad 11 + 3 = 14, \\
3 + 7 = 10, \quad 7 + 19 = 26.$

The sums down columns are:

$7 + 17 = 24, \quad 17 + 3 = 20, \\
5 + 11 = 16, \quad 11 + 7 = 18, \\
3 + 3 = 6, \quad 3 + 19 = 22.$

Notice that every even number from 6 to 28 appears exactly once. If there are multiple Goldbach squares for a given $n$ then we prefer the one with the smallest sum of cells (weight). Tables 12 and 13 show the best Goldbach squares that we found for $n \leq 10$. 

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| n   | Goldbach Square | Weight |
|-----|-----------------|--------|
| 2   | \( \begin{array}{c|c|c} 5 & 7 \\ \hline 3 & 3 \end{array} \) | 18     |
| 3   | \( \begin{array}{c|c|c|c} 3 & 5 & 7 \\ \hline 7 & 11 & 17 \\ \hline 19 & 3 & 3 \end{array} \) | 75     |
| 4   | \( \begin{array}{c|c|c|c|c} 5 & 11 & 13 & 5 \\ \hline 17 & 19 & 31 & 7 \\ \hline 3 & 29 & 11 & 3 \\ \hline 5 & 23 & 23 & 3 \end{array} \) | 208    |
| 5   | \( \begin{array}{c|c|c|c|c|c} 5 & 7 & 11 & 17 & 5 \\ \hline 31 & 31 & 23 & 53 & 3 \\ \hline 11 & 37 & 43 & 29 & 23 \end{array} \) | 499    |
| 6   | \( \begin{array}{c|c|c|c|c|c|c} 7 & 79 & 13 & 17 & 11 & 3 \\ \hline 41 & 17 & 71 & 43 & 31 & 19 \end{array} \) | 1078   |
| 7   | \( \begin{array}{c|c|c|c|c|c|c|c|c} 37 & 11 & 19 & 7 & 83 & 47 & 53 \\ \hline 5 & 17 & 109 & 61 & 19 & 17 & 53 \end{array} \) | 2077   |

Table 12: Goldbach squares for \( 2 \leq n \leq 7 \).
| n  | Goldbach Square | Weight |
|----|-----------------|--------|
| 8  | 17 23 157 13 43 37 31 67 17 7 47 71 79 79 97 127 3 71 131 31 61 127 37 73 23 5 7 11 149 19 89 139 41 53 113 41 3 3 127 17 149 59 11 131 83 13 47 179 71 29 3 5 31 59 3 7 11 37 181 41 151 47 101 31 3766 |
| 9  | 101 47 13 107 97 131 113 59 149 59 191 43 107 31 101 163 13 73 107 29 19 3 7 173 109 139 7 139 79 13 13 11 89 127 151 137 23 173 37 31 83 7 7 13 61 89 113 29 227 41 5 23 41 43 101 179 5 3 127 73 29 41 199 73 3 37 223 31 67 89 17 71 113 3 151 61 163 103 103 19 107 6187 |
| 10 | 89 11 101 71 271 13 5 89 59 197 107 227 23 251 67 19 3 211 29 47 13 83 47 79 43 313 3 139 127 131 3 223 11 163 7 13 181 181 37 61 89 79 73 151 3 173 73 131 227 97 199 149 179 97 41 7 127 233 71 47 97 17 11 31 5 7 109 53 37 31 257 89 251 101 131 19 37 241 67 7 79 109 31 191 29 11 43 19 163 53 127 163 181 89 179 23 59 5 47 19 9212 |

Table 13: Goldbach squares for $8 \leq n \leq 10$. 

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7 Prime Matrices

Definition 7.1. A prime matrix of order $n$ is a $n \times n$ matrix $P$ of odd primes, such that the sum of every odd number of elements in any straight line is prime. More formally, we have

\[
\sum_{0 \leq k \leq 2L} P(r + kd_r, c + kd_c) \quad \text{is prime for}
\]

\[
\forall r, c \text{ such that } 0 \leq r, c < n \quad \text{and}
\]

\[
\forall d_r, d_c \text{ such that } (d_r, d_c) \in \{(0, 1), (1, 0), (1, 1)\} \quad \text{and}
\]

\[
\forall L \text{ such that } 0 \leq 2L < n \quad \text{and}
\]

\[
r + 2Ld_r < n \text{ and } c + 2Ld_c < n.
\]

(10)

We were able to find prime matrices up to order 7. For $n \leq 4$ we found optimal (smallest weight) prime matrices. The results can be seen in Table 14. The lower bound on the optimal weight is the sum of the first $n^2$ odd primes.
| n  | Prime Matrix | Weight | Lower Bound |
|----|--------------|--------|-------------|
| 3  | 5 19 13     |        | 127         |
|    | 3 17 23     |        |             |
|    | 29 11 7     |        |             |
| 4  | 19 53 17 3  |        | 438         |
|    | 47 23 31 29 |        |             |
|    | 43 7 11 5   |        |             |
|    | 41 13 59 37 |        |             |
| 5  | 101 107 61 109 41 | | 1403 | 1159 |
|    | 127 43 11 29 31 | | | |
|    | 5 17 37 13 59 | | | |
|    | 79 97 23 131 19 | | | |
|    | 47 53 7 67 89 | | | |
| 6  | 73 131 113 109 61 227 | | 5796 | 2582 |
|    | 149 541 229 41 11 211 | | | |
|    | 379 491 419 349 139 53 | | | |
|    | 89 97 13 71 83 277 | | | |
|    | 193 59 137 307 127 29 | | | |
|    | 167 43 31 5 101 241 | | | |
| 7  | 547 719 1117 983 1201 29 397 | | 25891 | 5115 |
|    | 691 827 103 1307 373 131 37 | | | |
|    | 53 457 503 7 419 73 557 | | | |
|    | 347 463 683 307 647 337 1433 | | | |
|    | 163 167 313 1013 127 1217 643 | | | |
|    | 367 677 787 107 193 653 13 | | | |
|    | 1223 1087 23 769 227 487 887 | | | |

Table 14: Prime matrices for $3 \leq n \leq 7$. 
8 Conclusion and Future Work

We have investigated a number of constructions that generate primes via the sum of primes. Some constructions are more efficient than others at generating primes. We can define a construction’s efficiency as the number of primes it generates divided by the number of primes used to construct the construction. Table 8 shows the greatest efficiency achieved by each construction sorted from highest to lowest:

| Construction       | Order $n$ | Efficiency |
|--------------------|-----------|------------|
| Cyclic Prime Vector| 14        | 6          |
| Prime Vector       | 23        | 5.26       |
| Prime Cylinder     | 4 and 6   | 5          |
| Prime Matrix       | 7         | 4          |
| Prime Stair        | 15        | 3.27       |
| Prime Tuple        | 7         | 2.57       |
| Prime Pyramid      | 9         | 1.04       |

Many questions remain unresolved:
- What are the optimal prime vectors for $15 \leq n \leq 23$ ?
- Is there a prime vector of order 24 ?
- What are the optimal cyclic prime vectors for $11 \leq n \leq 14$ ?
- Is there a cyclic prime vector of order 15 ?
- What are the optimal prime tuples for $n = 7, 13, 19$ ?
- What are the optimal prime stairs for $12 \leq n \leq 15$ ?
- Is there a prime stair of order 16 ?
- What is the optimal prime pyramid of order 9 ?
- Is there a prime pyramid of order 10 ?
- What are the optimal prime cylinders for $n \leq 12$ ?
- Is there a Goldbach square of order 11 ?
- What are the optimal prime matrices for $5 \leq n \leq 7$ ?
- Is there a prime matrix of order 8 ?

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