FILLING CURVES ON CLOSED SURFACES OF GENUS 2

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Abstract. Let $F_g$ denote the closed oriented surface of genus $g$ and $\text{Mod}(F_g)$ denote the mapping class group of genus $g$. The group $\text{Mod}(F_g)$ acts on the set of fillings of $F_g$. The union of the curves in a filling form a graph on the surface which is so called decorated fat graph. We show that two fillings of $F_g$ are in the same $\text{Mod}(F_g)$-orbit if and only if the corresponding fat graphs are isomorphic. Next, we prove that any filling of $F_2$ with complement a single disc (i.e. a so called minimal filling) has either three or four closed curves. Moreover, we prove that there exist unique $\text{Mod}(F_2)$-orbit of minimal filling with three curves and unique $\text{Mod}(F_2)$-orbit of minimal filling with four curves.

We show that the minimum number of discs in the complement of a filling pair of $F_2$ is two and we construct filling pair of $F_2$ so that the complement is the union of two topological discs. Finally, for given any $k \geq 2$ we construct filling pair of $F_2$ such that the complement is the union of $k$ topological discs.

1. Introduction

Suppose $F_g$ is the closed surface of genus $g$ and $X = \{\gamma_i| i = 1, \ldots, n\}$ is a collection of simple closed curves on $F_g$ which are in pairwise intersect minimally, i.e., $i(\gamma_i, \gamma_j) = |\gamma_i \cap \gamma_j|$. Here, $i(\alpha, \beta)$ denotes the geometric intersection number of the closed curves $\alpha$ and $\beta$ (see [2]). The set $X$ is called a filling of the surface $F_g$ if the complement $F_g \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$ is a disjoint union of topological discs. If the number of curves in $X$ is $n = 2$ then we say that $X$ is a filling pair of $F_g$. If the complement $F_g \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$ is a single disc then we say that $X$ is a minimal filling of $F_g$. If $X$ is a filling on $F_g$, $g \geq 2$ and $T_1(X)$ is the number of simple closed curves on $F_g$ which intersect $\gamma_1 \cup \cdots \cup \gamma_n$ exactly once then $T_1(X) \leq 4g - 2$, with equality if and only if $X$ is minimal filling of $F_g$ [1].

Our motivation to study fillings of closed surfaces is the following. The set of all hyperbolic structure on the surface $F_g$ ($g \geq 2$) up to isometry is called the moduli space of genus $g$ and is denoted by $\mathcal{M}_g$. A systole of a hyperbolic surface is the length of a non-trivial closed geodesic of minimal length. A geodesic is called systolic curve if the systole is realized by its length. It is a well known and difficult problem of constructing a spine of $\mathcal{M}_g$, i.e., a deformation retraction in $\mathcal{M}_g$ of minimal dimension. The subset of $\mathcal{M}_g$ consisting of all hyperbolic surfaces whose systolic curves fill the surface is called Thurston set, denoted by $\mathcal{X}_g$. In [3] Thurston proposed $\mathcal{X}_g$ as a candidate spine of the moduli space $\mathcal{M}_g$. Thurston provided a sketch of a proof that $\mathcal{X}_g$ is a deformation retract, but it is difficult to complete the proof. Moreover the contractibility, connectivity, dimension of the set $\mathcal{X}_g$ remains open.

Let $\text{Mod}(F_g)$ denote the mapping class group of the closed surface of genus $g$ which is the group of all orientation preserving homeomorphism up to isotopy (see [1]).
It is a fact that if $X = \{\gamma_1, \ldots, \gamma_n\}$ is a filling of the surface $F_g$ then for every $[f] \in \text{Mod}(F_g)$ the set $f \cdot X = \{f \circ \gamma_1, \ldots, f \circ \gamma_n\}$ is also a filling of $F_g$.

Let $I(F_g)$ denote the set of all simple closed curves on $F_g$. There is a natural action of $\text{Mod}(F_g)$ on the quotient space $P_n(F_g) := I(F_g)^n/[(\gamma_1, \ldots, \gamma_n) \sim (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})]$ where $\sigma \in \Sigma_n$ is a permutation.

From another point of view one can think the union of the curves in $X$ as a connected graph on the surface associated with the filling $X$, denoted by $G_X(F_g)$. This is in fact a so called fat graph, with all vertices of valence 4. We show:

**Theorem 1.1.** Suppose $F_g$ is the closed surface of genus $g$ and $X, Y \in P_n(F_g)$ are two fillings of $F_g$. Then they are in the same $\text{Mod}(F_g)$-orbit if and only if $G_X(F_g)$ and $G_Y(F_g)$ are isomorphic as fat graphs.

In [1], authors shown that there does not exist minimal filling pair on $S_2$ and for all $g > 2$ there exist minimal filling pair on the closed surface $S_g$ of genus $g$. In this situation we have the following questions.

**Question.** What is the minimum number of curves is needed to fill $F_g$ such that the complement is a single disc?

The following theorem answers the question. Moreover, it says that there are exactly two $\text{Mod}(F_2)$-orbits of such fillings of $F_2$. We prove the following:

**Theorem 1.2.**
1. There exists a unique $\text{Mod}(F_2)$-orbit of minimal fillings triple $\{\alpha, \beta, \gamma\}$ in $F_2$.
2. There exists a unique $\text{Mod}(F_2)$-class of minimal fillings quadruple $\{\alpha_i\} i = 1, 2, 3, 4$ of $F_2$.

Now we are studying filling pair on $F_2$. Let $(\alpha, \beta)$ be a filling pair of $F_2$. The number of disjoint topological discs in the complement $F_2 \setminus (\alpha \cup \beta)$ is denoted by $K(\alpha, \beta)$. We define

$$K(F_2) = \min\{K(\alpha, \beta) | (\alpha, \beta) \text{ is a filling pair of } F_2\}.$$ 

We have the following question:

**Question.** What is the value of $K(F_2)$?

It is known that there does not exist minimal pair of $F_2$(see [1]). Hence we have a lower bound for $K(F_2)$, i.e.

$$2 \leq K(F_2).$$

There exists a filling pair $(\alpha, \beta)$ such that the complement is a union of four disjoint topological discs(see [1], [2]). Hence we have an upper bound

$$K(F_2) \leq 4.$$ 

Hence, combining the inequalities we have followed

$$2 \leq K(F_2) \leq 4.$$ 

The following theorem gives the exact value of $K(F_2)$.

**Theorem 1.3.** There exists a filling pair $(\alpha, \beta)$ of $F_2$ such that the complement is the union of two topological discs which follows that

$$K(F_2) = 2.$$
Let \( k \) be an integer such that \( k \geq 2 \).

**Question.** Does there exist a filling pair \((\alpha, \beta)\) of \( F_2 \) such that the complement \( F_2 \setminus (\alpha \cup \beta) \) is the union of \( k \) topological discs?

We prove:

**Theorem 1.4.** For every \( k(\geq 2) \in \mathbb{Z} \) there exist a filling pair \((\alpha_k, \beta_k)\) of \( F_2 \) such that the complement \( F_2 \setminus (\alpha_k \cup \beta_k) \) is the union of \( k \)-topological discs.

This paper is organized as follows. In section 2, we give an introduction to fat graphs. We describe the method of construction of the surface associated with a fat graph. We state a lemma which computes the number of boundary components of the surface associated with a given fat graph. The lemma in the section 2 is useful in the subsequent sections. In section 3, we study the action of mapping class group action on fillings of \( F_g \) and prove the Theorem 1.1. In section 4, we focus on the minimal fillings of \( F_2 \). First, we show that there does not exist minimal filling pairs on \( F_2 \). We conclude this section with a proof of Theorem 1.2. In section 5, we study filling pairs on \( F_2 \). In this section we prove Theorem 1.3 and Theorem 1.4.

## 2. Fat Graphs

A fat graph is a finite connected graph equipped with a cyclic ordering of the directed edges going out from each vertex. Before going to define a fat graph we define a graph. The following definition of graph is not the standard one which is used in ordinary graph theory. One can easily see that this definition is equivalent to the standard definition of a graph. We consider this definition as this is the convenient starting point to describe a fat graph.

**Definition 2.1.** A graph \( G \) is a triple \((E, \sim, \sigma_1)\) where

1. \( E \) is a finite set.
2. \( \sigma_1 : E \to E \) is a fixed point free involution, i.e. \( \sigma_1^2 = Id \) and \( \sigma_1(e) \neq \bar{e}, \forall e \in E \).
3. \( \sim \) is an equivalence relation on \( E \).

In ordinary language,

1. \( E \) is the set of all oriented edges of the graph.
2. The involution \( \sigma_1 \) maps an oriented edge to its reverse edge \( \sigma_1(e) = \bar{e}, \forall e \in E \).
3. The equivalence relation \( \sim \) is defined by following:

\[ e_1 \sim e_2 \iff e_1, e_2 \text{ has same initial vertex.} \]

**Remarks 2.2.**

1. An edge \( e = \{e, \bar{e}\} \in E_1 \) is called a loop if and only if \( e \sim \bar{e} \).
2. The set \( V = E/\sim \) of all equivalence classes of \( \sim \) is the set of vertices of the graph. For \( v \in V \) the degree of \( v \) is \( \deg(v) = |v| \).

Now we are ready to define a fat graph.

**Definition 2.3.** A fat graph \( G \) is a quadruple \((E, \sim, \sigma_1, \sigma_0)\) where
(1) \((E, \sim, \sigma_1)\) is a graph.
(2) \(\sigma_0\) is a permutation on \(E\) so that each cycle corresponds to an cyclic ordering
on the set of oriented edges going out from a vertex.

**Remark 2.4.** A fat graph is also known as ribbon graph.

**Convention.** When we draw a fat graph the cyclic ordering on the set of edges
incident at each vertex is the cyclic ordering induced from the positive orientation
of the plane.

We define
\[
\sigma_\infty := \sigma_1 \ast \sigma_0^{-1}
\]
and denote the set of orbits of \(\sigma_\infty\) by \(E_\infty\).

### 2.1. Surface associated to a fat graph.

We construct a surface corresponding to a fat graph \(G\) as described below. We take a closed disc corresponding to each vertex and a rectangle corresponding to each edge. Then we identify the sides of the rectangles with the segments of the boundary component of the discs according to the ordering of the edges incident to a vertex. The local picture at a vertex of degree four is given in the Figure 1.

**Figure 1.** Fat graph locally at a vertex of degree 4.

In this way, we obtain the oriented topological surface denoted by \(\Sigma(G)\) corresponding to a given fat graph \(G\). Thus we can talk about number of boundary components, genus of fat graph, Euler characteristic and many other topological notions. By the number of boundary components of a fat graph \(G\) we mean the number of boundary components of \(\Sigma(G)\).

**Lemma 2.5.** Given a fat graph \(G = (E, \sim, \sigma_1, \sigma_0)\), the number of boundary components of the surface \(\Sigma(G)\) is the number of orbits of \(\sigma_\infty\).

**Example 2.6.** We give an explicit example of a fat graph and count its number of boundary components.

The graph is given by \(G = (E, \sim, \sigma_1, \sigma_2)\) where

1. \(E = \{\vec{e}_i, \vec{e}_i\} \mid i = 1, 2, 3\) is the set of all oriented edges.
2. The involution \(\sigma_1 : E \rightarrow E\) is defined by
   \[
   \sigma(\vec{e}_i) = \vec{e}_i, \ \sigma_1(\vec{e}_i) = \vec{e}_i, \ i = 1, 2, 3.
   \]
3. We define
   \[
   v_1 := \{\vec{e}_1, \vec{e}_2, \vec{e}_3\},
   \]
   \[
   v_2 := \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}.
   \]
∼ is the equivalence relation on \( E \) such that \( v_1 \) and \( v_2 \) are the equivalence classes.

(4) The permutation \( \sigma_0 : E \to E \) is given by
\[
\sigma_0 = (\tilde{e}_1, \tilde{e}_3, \tilde{e}_2)(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3).
\]

It is easy to calculate that the permutation \( \sigma_1 \ast \sigma_0^{-1} \) is given by
\[
(\tilde{e}_1, \tilde{e}_2)(\tilde{e}_3, \tilde{e}_1)(\tilde{e}_2, \tilde{e}_3).
\]
Hence \( \sigma_1 \ast \sigma_0^{-1} \) a product of three disjoint cycles. So, it follows from the Lemma 2.5 that the number of boundary components of \( G \) is three (see Figure 2).

![Figure 2. The fat graph G.](image)

Remark 2.7. In Example 2.6 if we consider the following permutation \( \bar{\sigma}_0 : E \to E \) given by
\[
(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)
\]
instead of \( \sigma_0 \) then the number of boundary component is only one as \( \sigma_1 \ast \bar{\sigma}_0^{-1} \) is given by
\[
(\tilde{e}_1, \tilde{e}_3, \tilde{e}_2, \tilde{e}_1, \tilde{e}_3, \tilde{e}_2)
\]
is only one cycle. Therefore we conclude that as ordinary graphs they are the same but as fat graphs they are different.

Definition 2.8. A fat graph is called decorated if degree of each vertex is even and at least 4.

Definition 2.9. Let \( G = (E, \sim, \sigma_1, \sigma_0) \) and \( G' = (E', \sim', \sigma_1', \sigma_0') \) be two fat graphs. \( G \) and \( G' \) are said to be isomorphic if there exists a bijective function \( f : E \to E' \) such that:

(1) For \( x_1, x_2 \in E \) we have
\[
x_1 \sim x_2 \text{ if and only if } f(x_1) \sim' f(x_2).
\]

(2) The following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\sigma_1 \downarrow & \swarrow & \sigma_1' \\
E & \xrightarrow{f} & E
\end{array}
\]
(3) \((x_1, x_2, \ldots, x_n)\) is the cyclic ordering on the set of edges going out from the vertex \(v = \{x_1, x_2, \ldots, x_n\}\) of the graph \(G\) if and only if \((f(x_1), f(x_2), \ldots, f(x_n))\) is the cyclic ordering on the set of edges going out from the vertex \(v' = \{f(x_1), f(x_2), \ldots, f(x_n)\}\) of the graph \(G'\).

**Remark 2.10.** The equivalent condition of the condition (3) in Definition 2.9 is the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\sigma_0 \downarrow & & \downarrow \sigma'_0 \\
E & \xrightarrow{f} & E
\end{array}
\]

3. **Mapping class group orbits**

Let \(I(F_g)\) denote the set of all isotopy classes of simple closed curves on the surface \(F_g\). For \(n \in \mathbb{N}\), \(I(F_g)^n\) denotes the cartesian product \(I(F_g) \times \cdots \times I(F_g)\) of \(n\) copies of \(I(F_g)\). We define an equivalence relation \(\sim\) on \(I(F_g)^n\) by following: Let \(X = (\gamma_1, \gamma_2, \ldots, \gamma_n), Y = (\beta_1, \beta_2, \ldots, \beta_n) \in I(F_g)^n\) and \(\Sigma_n\) denote the permutation group on the set \(\{1, 2, \ldots, n\}\). Then \(X \sim Y\) if and only if \(Y = \sigma \cdot X\) for some \(\sigma \in \Sigma_n\) where \(\sigma \cdot X = (\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \ldots, \gamma_{\sigma(n)})\).

The mapping class group \(\text{Mod}(F_g)\) acts coordinate-wise on the set \((I(F_g))^n\):

\[
[f] : \text{Mod}(F_g) \times (I(F_g))^n \to (I(F_g))^n
\]

\[
[f] \cdot (\gamma_1, \gamma_2, \ldots, \gamma_n) = (f \circ \gamma_1, f \circ \gamma_2, \ldots, f \circ \gamma_n).
\]

This group action descends to an action on the quotient space

\[
\mathcal{P}_n(F_g) := I(F_g)^n / \sim.
\]

Let \(X = (\alpha_1, \ldots, \alpha_n) \in \mathcal{P}_n(F_g)\) be a filling of the surface \(F_g\) and \([f] \in \text{Mod}(F_g)\) then \(f \cdot X = (f \circ \alpha_1, \ldots, f \circ \alpha_n)\) is also a filling of \(F_g\) with the same number of topological discs in the complement.

Suppose \(X = \{\alpha_1, \ldots, \alpha_n\}\) is a filling of \(F_g\). Then we can think of the union of the curves in \(X\) as a 4-valent decorated fat graph which is described by following:

1. The intersection points \(\alpha_i \cap \alpha_j, i \neq j \in \{1, \ldots, n\}\) of the curves in \(X\) are the vertices.
2. The sub-arcs of \(\alpha_i\)’s joining vertices are the edges.
3. The cyclic order on the set of edges incident at each vertex is uniquely determined by the orientation of the surface.

We denote the decorated fat graph for a given filling \(X\) of the surface \(F_g\) by \(G_X(F_g)\). Conversely, suppose \(G\) is a given decorated 4-valent fat graph whose standard cycles are \(C_i; 1, \ldots, n\). We obtain the closed surface \(F(G)\) by capping each boundary component of \(G\) by topological a disc. Then \(X_G = \{C_i|i = 1, \ldots, n\}\) is a filling pair of \(F(G)\)

**Theorem 3.1.** Suppose \(F_g\) is the closed surface of genus \(g\) and \(X, Y \in \mathcal{P}_n(F_g)\) are two filling of \(F_g\). Then they are in the same \(\text{Mod}(F_g)\)-orbit if and only if \(G_X(F_g)\) and \(G_Y(F_g)\) are isomorphic as fat graphs.
Proof. Suppose two fillings $X$ and $Y$ of $F_g$ are in the same $\text{Mod}(F_g)$ class. Therefore we have an element $[f] \in \text{Mod}(F_g)$ such that $Y = f \cdot X$. The restriction $F = f|_{G_X(F_g)}$ on $G_X(F_g)$ of the homeomorphism $f : F_g \to F_g$ gives a fat graph isomorphism
\[ \tilde{f} : G_X(F_g) \to G_Y(F_g). \]
Conversely, if $\tilde{f} : G_X(F_g) \to G_Y(S_g)$ is an isomorphism then the isomorphism can be extended to a homeomorphism $f : F_g \to F_g$ such that $Y = f \cdot X$. So, $X$ and $Y$ are in the same $\text{Mod}(F_g)$-orbit. □

4. Minimal filling of $F_2$

Lemma 4.1. Let $G$ be a 4-valent decorated fat graph with 3 vertices and two standard cycles. The number of boundary components in $G$ is at least 2.

Proof. Let $C_i; i = 1, 2$ are the standard cycles of $G$. Then each $C_i$ is simple and consists of three edges. There are up to isomorphism three such 4-valent fat graphs with three vertices and two standard cycles which are denoted by $H_i; i = 1, 2, 3$(see Figure 3). For each $i \in \{1, 2, 3\}$ the graph $H_i$ has three boundary components.

![Figure 3](image)

**Figure 3.** The graphs $H_i; i = 1, 2, 3.$

Corollary 4.2. There does not exist any minimal filling pair of $F_2$.

Proof. Suppose there is a minimal filling pair $(\alpha, \beta)$ of $F_2$. Then, we define $G := \alpha \cup \beta$. We can think of $G$ as a decorated fat graph where the intersection points of $\alpha, \beta$ are the vertices, the sub-segments of $\alpha$ and $\beta$ joining two vertices are the edges. The cyclic ordering on the set of edges incident at each vertex is determined by the orientation of the surface.

In another way we can think $G$ as the 1-skeleton of a cellular decomposition of $F_2$. In the cell decomposition the number of 0-cell is $v = i(\alpha, \beta)$. The 4-valency
condition gives the number of 1-cell is $2i(\alpha, \beta)$ and the minimality condition gives the number of 2-cell is $f = 1$. Therefore we have

$$v - e + f = \chi(F_2)$$

$$\Rightarrow i(\alpha, \beta) = 3.$$ 

Hence, $G$ is a 4-valent decorated fat graph with three vertices, two standard cycles and single boundary component which contradicts the Lemma. □

**Theorem 4.3.**

1. There exists a unique $\text{Mod}(F_2)$-orbit of minimal filling triple $\{\alpha, \beta, \gamma\}$ on $F_2$.
2. There exists a unique $\text{Mod}(F_2)$-orbit of minimal filling quadruple $\{\alpha_i | i = 1, 2, 3, 4\}$ of $F_2$.

**Proof.** (1) Suppose $\{\alpha, \beta, \gamma\}$ is a minimal filling triple of $F_2$. We define, $G := \alpha \cup \beta \cup \gamma$. Then $G$ is a 4-valent decorated fat graph with three standard cycles $\alpha, \beta$ and $\gamma$. In an another way we can think $G$ is the 1-skeleton of a cellular decomposition of $F_2$. The number of 0-cell is $v$, the number of vertices in $G$. The number of 1-cells is $e$ which is the number of edges in $G$. Using the valency condition of $G$ we have $2e = 4v \Rightarrow e = 2v$ as the graph is 4-valent. The number of 2-cells is $f = 1$. Therefore we have $v - e + f = -\chi(F_2) \Rightarrow v = 3$.

Therefore, to prove the theorem it suffices to prove that there exists a unique 4-valent decorated fat graph $G$ satisfying the following:

1. The number of vertices is three.
2. The number of standard cycles is three.
3. The number of boundary components is one.

Let $C_i; i = 1, 2, 3$ are the standard cycles of $G$. There are two cases to be considered.

Case 1. In this case, we consider that each standard cycle $C_i$ consists of two edges. Up to isomorphism there are only two distinct 4-valent decorated fat graphs with three vertices, three standard cycles such that each standard cycle consists of two edges. The two fat graphs $H_1, H_2$ are given in Figure 4.

![Figure 4](image-url)

**Figure 4.** The graphs $H_i, i = 1, 2$.

For each $i = 1, 2$, the fat graph $H_i$ has three boundary components. Therefore, this case is not possible.
Case 2. The rest is the following: there is a cycle $C_i$ consists of three edges. Without loss of generality assume that $C_1$ consists of three edges. So other two standard cycles $C_2$ and $C_3$ consist of 2 edges and 1 edge respectively.

Again, up to isomorphism there are two distinct 4-valent decorated fat graphs $\Gamma_1, \Gamma_2$ with three vertices and standard cycles $C_i$, $i = 1, 2, 3$. The fat graphs $\Gamma_1$ and $\Gamma_2$ are given in the Figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.png}
\caption{The graphs $\Gamma_i, i = 1, 2$.}
\end{figure}

The graph $\Gamma_1$ has three boundary components and the graph $\Gamma_2$ has one boundary component. Therefore $G = \Gamma_2$ is the only 4-valent decorated fat graph which satisfy the theorem.

(2) To prove the part-(2), it suffices to show that there exist unique 4-valent decorated fat graph $G$ with 3 vertices and 4 standard cycles and single boundary component. Suppose $C_i$, $i = 1, 2, 3, 4$ are the standard cycles of $G$. As in the proof of the part-(1) there are two cases to be considered.

Case 1. Suppose there is a standard cycle consists of three edges. So, without loss of generality assume that $C_1$ has length 3. The number of edges in $G$ is six. Therefore it follows that length of each $C_i$ is one, for $i = 2, 3, 4$. On the standard cycle $C_1$ there are three vertices and $C_i$, $i = 2, 3, 4$ are the loops at the vertices on $C_1$. Such a graph $H$ is uniquely determined (up to isomorphism) and is given in the Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure6.png}
\caption{The graph $H$.}
\end{figure}

The number of boundary components in $H$ (Figure 6) is three. So this case is not possible.
Case 2. In this case, we assume that there are no standard cycle of length three. The only possibility is the following: there are two cycles of length 2 and two cycles of length 1. Such a 4-valent graph $K$ is uniquely (up to isomorphism) determined and is given in the Figure 7.

\[ \text{Figure 7. The graph } K. \]

By construction the graph $K$ has the following properties:

1. The number of vertices is 3.
2. The degree of each vertex is 4.
3. The number of standard cycles is 4.
4. The number of boundary components is 1.

Hence $G = K$ is the unique 4-valent decorated fat graph satisfies the theorem and the proof follows.

Remark 4.4. For $n \geq 5$ there does not exist a connected 4-valent decorated fat graph with $n$ standard cycles and three vertices which follows that there does not exist any minimal filling $X = \{ \gamma_i \mid i = 1, 2, \ldots n \}$ of $F_2$ if $n \geq 5$.

5. Filling pairs of $F_2$

The following theorem determine the exact value of $K(F_2)$.

Theorem 5.1. There exist a filling pair $(\alpha, \beta)$ of $F_2$ such that the complement is union of two topological discs which follows that

$$K(F_2) = 2.$$  

Proof. If $(\alpha, \beta)$ is a filling pair of $F_2$ such that the complement is a union of two disjoint topological discs then we have the following: $G := \alpha \cup \beta$ is a 4-valent decorated fat graph where the intersection points $\alpha \cap \beta$ of $\alpha$ and $\beta$ are the vertices and the sub-arcs of $\alpha, \beta$ between two vertices are the edges. The graph $G$ has the following properties.

1. The degree of each vertex is 4.
2. If the number of vertices is $v$ and the number of edges is $e$ then we have $4v = 2e \Rightarrow e = 2v$.

Therefore, comparing the Euler characteristic we have

\[
\begin{align*}
v - e + f &= -2 \\
\Rightarrow v - 2v + 2 &= -2 \\
\Rightarrow v &= 4 \text{ and} \\
e &= 8.
\end{align*}
\]
The number of standard cycles in $G$ is two which are correspond to the simple closed curves $\alpha$ and $\beta$.

Conversely, if we have a 4-valent decorated fat graph $G$ with the properties (1)-(4) then capping each boundary of the fat graph by a topological disc we obtain closed surface $F_2$ of genus 2. If $\alpha$ and $\beta$ are the standard cycles of $G$ then $(\alpha, \beta)$ is a filling pair of $F_2$ such that the complement is union of two topological discs.

So the theorem boils down once we prove that there exist a 4-valent decorated fat graph $G$ with the properties (1)-(4). Consider the graph $G = (E, \sim, \sigma_1, \sigma_0)$ given by following(Figure 8):

- $E = \{\vec{e}_i, \vec{f}_i, \bar{e}_i, \bar{f}_i \mid i = 1, 2, 3, 4\}$
- The equivalence classes of the relation $\sim$ on $E$ are given by
  
  $v_1 = \{\bar{e}_1, \bar{f}_1, \bar{e}_4, \bar{f}_4\}, \quad v_2 = \{\bar{e}_2, \bar{f}_2, \bar{e}_1, \bar{f}_1\}, \quad v_3 = \{\bar{e}_3, \bar{f}_3, \bar{e}_2, \bar{f}_4\}, \quad v_4 = \{\bar{e}_4, \bar{f}_2, \bar{e}_3, \bar{f}_3\}$.
- $\sigma_1 : E \rightarrow E$ is given by
  
  $\sigma_1(\vec{e}_i) = \bar{e}_i, \quad \sigma_1(\bar{e}_i) = \vec{e}_i, \quad \sigma_1(\vec{f}_i) = \bar{f}_i, \quad \sigma_1(\bar{f}_i) = \vec{f}_i, \quad i = 1, 2, 3, 4$.
- $\sigma_0 : E \rightarrow E$ is given by
  
  $(\vec{e}_1, \bar{f}_1, \bar{e}_4, \bar{f}_4)(\vec{e}_2, \bar{f}_2, \bar{e}_1, \bar{f}_1)(\vec{e}_3, \bar{f}_3, \bar{e}_2, \bar{f}_4)(\vec{e}_4, \bar{f}_2, \bar{e}_3, \bar{f}_3)$.

The graph $G$ is a 4-valent decorated fat graph with 4 vertices and has two standard cycles. Now we need to show that $G$ has two boundary components.

The permutation $\sigma_\infty = \sigma_1 \ast \sigma_0^{-1}$ is given by

$(\bar{e}_1, \bar{f}_2, \bar{e}_4, \bar{f}_1, \bar{e}_1, \bar{f}_4, \bar{e}_2, \bar{f}_1)(\bar{e}_3, \bar{f}_3, \bar{e}_2, \bar{f}_4)(\bar{e}_4, \bar{f}_2, \bar{e}_3, \bar{f}_3)$.

Hence, there are two orbits of $\sigma_\infty$ which is equal to the number of boundary components in the fat graph $G$ (see Lemma 4.1).

Let $k$ be an integer such that $k \geq 2$. Now we show that there exist a filling pair of $F_2$ such that the complement is the disjoint union of $k$ topological sides.

Let us consider some examples. First example considered the case when $k$ is an even integer.
Example 5.2. Here we consider $k = 4$. We construct a filling pair of $F_2$ such that the complement is union of 4 topological discs.

Suppose $(\alpha, \beta)$ is a filling pair of $F_2$ such that $F_2 \setminus (\alpha \cup \beta)$ is union of 4 topological discs. We can think of $G_4 = \alpha \cup \beta$ as a decorated fat graph on the surface $F_2$. The intersection points of $\alpha$ and $\beta$ are the vertices of $G_4$ and the arc segments of the curves $\alpha, \beta$ joining the vertices are the edges. The cyclic ordering on the set of edges incident at each vertex is induced from the orientation of the surface. The degree of each vertex is 4. The number of standard cycles is 2 which are corresponds to $\alpha$ and $\beta$. We can think the 4-valent graph $G$ as the 1-skeleton of a cellular decomposition of $F_2$. The number of 0-cell is $v = i(\alpha, \beta)$ and 4-valency of $G_2$ implies that the number of 1-cell(edges) is $e = 2i(\alpha, \beta)$. The number of 2-cells is $f = 4$. Therefore by Euler characteristic number argument we have

$$v - e + f = -2$$

$$\Rightarrow i(\alpha, \beta) - 2i(\alpha, \beta) + 4 = -2$$

$$\Rightarrow i(\alpha, \beta) = 6.$$  

To find a filling pair $(\alpha, \beta)$ of $S_2$ with the complement consists of four topological discs it is suffices to give an example of four valent decorated fat graph $G_4$ such that

1. The number of vertices is 6 and the number of edges is 12.
2. The number of standard cycles is 2.
3. The number of boundary components is 4.

If such a graph $G_4$ exists then we can cap each boundary component with a topological disc and obtain the surface $F_2$. Then if $\alpha$, $\beta$ are the standard cycles in $G_4$, $(\alpha, \beta)$ is a filling pair of $F_2$ such that the complement is union of four topological discs.

Let us consider the graph $G_4 = (E, \sim, \sigma_1, \sigma_0)$ given by following:

- $E = \{\tilde{e}_i, \tilde{e}_i, \tilde{f}_1, \tilde{f}_j | i = 1, \ldots, 6\}$.
- $\sim$ is the equivalence relation on $E$ such that the equivalence classes are given by,
  $$v_1 = \{\tilde{e}_1, \tilde{f}_1, \tilde{e}_6, \tilde{f}_6\}, \quad v_2 = \{\tilde{e}_2, \tilde{f}_2, \tilde{e}_1, \tilde{f}_1\}, \quad v_3 = \{\tilde{e}_3, \tilde{f}_3, \tilde{e}_2, \tilde{f}_2\}, \quad v_4 = \{\tilde{e}_4, \tilde{f}_5, \tilde{e}_3, \tilde{f}_6\}, \quad v_5 = \{\tilde{e}_5, \tilde{f}_4, \tilde{e}_4, \tilde{f}_5\}, \quad v_6 = \{\tilde{e}_6, \tilde{f}_3, \tilde{e}_5, \tilde{f}_4\}.$$  
- $\sigma_1 : E \to E$ is given by,
  $$\sigma_1(\tilde{e}_i) = \tilde{e}_i, \quad \sigma_1(\tilde{f}_i) = \tilde{f}_i, \quad \sigma_1(\tilde{f}_j) = \tilde{f}_j, \quad i = 1, 2, \ldots, 6.$$  
- The permutation $\sigma_0$ on $E$ is given by
  $$\sigma_0 := (\tilde{e}_1, \tilde{f}_1, \tilde{e}_6, \tilde{f}_6)(\tilde{e}_2, \tilde{f}_2, \tilde{e}_1, \tilde{f}_1)(\tilde{e}_3, \tilde{f}_3, \tilde{e}_2, \tilde{f}_2)(\tilde{e}_4, \tilde{f}_4, \tilde{e}_3, \tilde{f}_6)(\tilde{e}_5, \tilde{f}_5, \tilde{e}_4, \tilde{f}_5)(\tilde{e}_6, \tilde{f}_3, \tilde{e}_5, \tilde{f}_4).$$

Then the graph $G_4$ is a 4-valent decorated fat graph with six vertices and two standard cycles. Now we count the number of boundary components. The permutation $\sigma_1 \circ \sigma_0^{-1}$ is given by following

$$(\tilde{e}_1, \tilde{f}_2, \tilde{e}_2, \tilde{f}_1)(\tilde{e}_2, \tilde{f}_3, \tilde{e}_6, \tilde{f}_6, \tilde{e}_1, \tilde{f}_1, \tilde{e}_6, \tilde{f}_2, \tilde{e}_3, \tilde{f}_2), (\tilde{e}_3, \tilde{f}_5, \tilde{e}_4, \tilde{f}_6, \tilde{e}_6, \tilde{f}_4, \tilde{e}_5, \tilde{f}_5, \tilde{e}_3, \tilde{f}_3), (\tilde{e}_4, \tilde{f}_4, \tilde{e}_5, \tilde{f}_5)$$

which is a product of four disjoint cycles. Hence $G_4$ has four boundary components.

Now we construct examples for odd integer $k$.  


Example 5.3. Let $k = 3$. Then the number of vertices in $G_3$ is 5 and each standard cycle consists of 5 edges. The graph $G_3 = (E, \sim, \sigma_1, \sigma_0)$ is given by following,

- $E = \{\vec{e}_i, \vec{f}_i, \vec{e}_5, \vec{f}_5 | i = 1, 2, \ldots, 5\}$.
- The equivalence classes of the equivalence relation $\sim$ on $E$ is given by
  
  $v_1 = \{\vec{e}_1, \vec{f}_1, \vec{e}_5, \vec{f}_5\}$, $v_2 = \{\vec{e}_2, \vec{f}_3, \vec{e}_1, \vec{f}_4\}$,
  $v_3 = \{\vec{e}_3, \vec{f}_2, \vec{e}_2, \vec{f}_4\}$, $v_4 = \{\vec{e}_4, \vec{f}_1, \vec{e}_3, \vec{f}_2\}$,
  $v_5 = \{\vec{e}_5, \vec{f}_3, \vec{e}_4, \vec{f}_4\}$.

- $\sigma_1 : E \to E$ is given by
  
  $\sigma_1(\vec{e}_i) = \vec{e}_i$, $\sigma_1(\vec{f}_i) = \vec{f}_i$, $\sigma_1(\vec{f}_i) = \vec{f}_i, \ i = 1, 2, \ldots, 5$.

- $\sigma_0 : E \to E$ is given by,
  
  $(\vec{e}_1, \vec{f}_1, \vec{e}_5, \vec{f}_5)(\vec{e}_2, \vec{f}_3, \vec{e}_1, \vec{f}_4)(\vec{e}_3, \vec{f}_2, \vec{e}_2, \vec{f}_4)(\vec{e}_4, \vec{f}_1, \vec{e}_3, \vec{f}_2)(\vec{e}_5, \vec{f}_3, \vec{e}_4, \vec{f}_4)$.

The permutation $\sigma_\infty = \sigma_1 \ast \sigma_0^{-1}$ is given by

$(\vec{e}_1, \vec{f}_1, \vec{e}_5, \vec{f}_5)(\vec{e}_2, \vec{f}_3, \vec{e}_1, \vec{f}_4)(\vec{e}_3, \vec{f}_2, \vec{e}_2, \vec{f}_4)(\vec{e}_4, \vec{f}_1, \vec{e}_3, \vec{f}_2)(\vec{e}_5, \vec{f}_3, \vec{e}_4, \vec{f}_4)$.

So, the number of orbits of $\sigma_\infty$ is three which is the same as the number of boundary components of $G_3$.

Theorem 5.4. For every $k(\geq 2) \in \mathbb{Z}$ there exists a filling pair $(\alpha_k, \beta_k)$ of $F_2$ such that the complement $F_2 \setminus (\alpha_k \cup \beta_k)$ is the union of $k$-topological discs.

Proof. Suppose $(\alpha, \beta)$ is a filling pair of $F_2$ such that the complement is the union of $k$-topological discs. We can think $G_k := \alpha_k \cup \beta_k$ as the 1-skeleton of a cellular decomposition of the surface $F_2$. The number of 0-cell is $v := i(\alpha_k, \beta_k)$. As $G_k$ is a
4-valent graph the number of edges (1-cell) is \( e := 2i(\alpha_k, \beta_k) \). The number of 2-cell is \( f := k \). Therefore, by Euler characteristic argument as before we have

\[
\begin{align*}
\chi(F_2) &= v - e + f \\
&= 2i(\alpha_k, \beta_k) - 2i(\alpha_k, \beta_k) \\
&= i(\alpha_k, \beta_k) = 2 + k.
\end{align*}
\]

In another point of view we can think of \( G_k \) as a decorated fat graph embedded on the surface \( F_2 \). The intersection points \( \alpha \cap \beta \) of \( \alpha_k \) and \( \beta_k \) are the vertices. The sub-arcs of \( \alpha_k, \beta_k \) joining the vertices are the edges. The cyclic ordering on the set of edges incident at each vertex is determined by the orientation on the surface \( F_2 \). The number of boundary components of \( G_k \) is \( k \).

Conversely, if \( G_k \) is a 4-valent decorated fat graph such that

1. The number of vertices is \( k + 2 \).
2. The number of standard cycles is 2.
3. The number of boundary components is \( k \).

Then attaching topological discs along each boundary component we obtain closed surface \( F_2 \) of genus 2. If \( \alpha_k \) and \( \beta_k \) are the two standard cycles of \( G_k \) then \((\alpha_k, \beta_k)\) is a filling pair of \( F_2 \) such that the complement \( F_2 \setminus (\alpha_k \cup \beta_k) \) is union of \( k \)-topological discs. To prove the theorem it suffices to show that for every integer \( k \geq 2 \) there exist a 4-valent decorated fat graph \( G_k \). We consider two cases. In the first case we consider even integers and in the second case we construct \( G_k \) for odd integers \( k \).

Case 1. Let \( k \) be an even integer and \( k = 2n \) for some positive integer \( n \). Then the number of vertices is \( k + 2 = 2(n + 1) = 2m \) where \( m = n + 1 \) (say). Now consider the graph \( G_k = (E, \sim, \sigma_1, \sigma_0) \) described by following:

- \( E = \{ \bar{e}_i, \bar{e}_i, \bar{f}_i, \bar{f}_i | i = 1, 2, \ldots, 2m \} \) is the set of all directed edges.
\( \sim \) is an equivalence relation on \( E \) such that the equivalence classes are given by (Figure 11)

\[
\begin{align*}
v_1 &= \{ \tilde{e}_1, \tilde{f}_1, \tilde{e}_{2m}, \tilde{f}_{2m} \}, \\
v_i &= \{ \tilde{e}_i, \tilde{f}_i, \tilde{e}_{i-1}, \tilde{f}_{i-1} \}; \quad 1 < i \leq m, \\
v_j &= \{ \tilde{e}_j, \tilde{f}_{3m-j}, \tilde{e}_{j-1}, \tilde{f}_{3m-j+1} \}; \quad m + 1 \leq j \leq 2m.
\end{align*}
\]

- The permutation \( \sigma_1 : E \to E \) is given by

\[
\begin{align*}
\sigma_1(\tilde{e}_i) &= \tilde{e}_i, \quad \sigma_1(\tilde{f}_i) = \tilde{e}_i, \\
\sigma_1(\tilde{f}_i) &= \tilde{f}_i, \quad \sigma_1(\tilde{f}_i) = \tilde{f}_i; \quad i = 1, 2, \ldots, 2m.
\end{align*}
\]

- \( \sigma_0 : E \to E \) is given by

\[
\begin{align*}
(\tilde{e}_1, \tilde{f}_1, \tilde{e}_{2m}, \tilde{f}_{2m})&(\tilde{e}_2, \tilde{f}_2, \tilde{e}_1, \tilde{f}_1) \cdots (\tilde{e}_i, \tilde{f}_i, \tilde{e}_{i-1}, \tilde{f}_{i-1}) \cdots \\
(\tilde{e}_m, \tilde{f}_m, \tilde{e}_{m-1}, \tilde{f}_{m-1})&(\tilde{e}_{m+1}, \tilde{f}_{2m-1}, \tilde{e}_m, \tilde{f}_{2m}) \cdots \\
(\tilde{e}_j, \tilde{f}_{3m-j}, \tilde{e}_{j-1}, \tilde{f}_{3m-j+1})&(\tilde{e}_{2m}, \tilde{f}_m, \tilde{e}_{2m-1}, \tilde{f}_{m+1})
\end{align*}
\]

\[\text{Figure 11. The graph } G_k \text{.}(k \text{ is even)}\]

The graph \( G_k \) is 4-valent decorated fat graph with only two standard cycles. Now we count the orbits of \( \sigma_\infty = \sigma_0 \circ \sigma_1 \). The orbits of \( \sigma_\infty \) are given by following:

\[
\begin{align*}
\partial_i &= (\tilde{e}_i, \tilde{f}_{i+1}, \tilde{e}_{i+1}, \tilde{f}_i), \quad i = 1, 2, \ldots, m - 2; \\
\partial_{m-1} &= (\tilde{e}_{m-1}, \tilde{f}_m, \tilde{e}_{2m}, \tilde{f}_{2m}, \tilde{e}_{m}, \tilde{f}_{m-1}), \\
\partial_m &= (\tilde{e}_m, \tilde{f}_{2m-1}, \tilde{e}_{m+1}, \tilde{f}_{2m}, \tilde{e}_{m}, \tilde{f}_{m+1}, \tilde{e}_{2m}, \tilde{f}_{m+1}, \tilde{e}_{2m-1}, \tilde{f}_m), \\
\partial_j &= (\tilde{e}_j, \tilde{f}_{3m-j-1}, \tilde{e}_{j+1}, \tilde{f}_{3m-j}), \quad j = m + 1, m + 2, \ldots, 2m - 2.
\end{align*}
\]

So the total number of distinct orbits of \( \sigma_\infty \) is \( 2m - 2 = 2n = k \). Hence the theorem is proved for all even integer \( k \).
Case 2. In this case, we construct a 4-valent decorated fat graph $G_k$ for odd $k$. For some integer $n$ we have $k = 2n + 1$ as $k$ is an odd integer. The number of vertices in $G_k$ is $2n + 3$. The graph $G_k = (E, \sim, \sigma_1, \sigma_0)$ is described by following (Figure 12).

- $E = \{\bar{e}_i, \bar{f}_i, \bar{f}_i, \bar{f}_i | i = 1, 2, \ldots, 2n + 3\}$.
- $\sim$ is a equivalence relation on $E$ such that the equivalence classes are given by
  
  \[
  v_1 = \{\bar{e}_1, \bar{f}_1, \bar{e}_{2n+3}, \bar{f}_{2n+3}\},
  \]
  
  \[
  v_i = \{\bar{e}_i, \bar{f}_{2n+3-i}, \bar{e}_{i-1}, \bar{f}_{2n+4-i}\}, \quad i = 2, 3, \ldots, 2n + 2 \text{ and } \]
  
  \[
  v_{2n+3} = \{\bar{e}_{2n+3}, \bar{f}_{2n+3}, \bar{e}_{2n+2}, \bar{f}_{2n+2}\}.
  \]

- The permutation $\sigma_1 : E \to E$ is given by
  
  \[
  \sigma_1(\bar{e}_i) = \bar{e}_i, \quad \sigma_1(\bar{f}_i) = \bar{f}_i; \quad i = 1, 2, \ldots, 2n + 3.
  \]

- The permutation $\sigma_0 : E \to E$ is given by
  
  \[
  (\bar{e}_1, \bar{f}_1, \bar{e}_{2n+3}, \bar{f}_{2n+3})(\bar{e}_2, \bar{f}_{2n+1}, \bar{e}_{1}, \bar{f}_{2n+2}) \ldots (\bar{e}_i, \bar{f}_{2n+3-i}, \bar{e}_{i-1}, \bar{f}_{2n+4-i})
  \]
  
  \[
  \ldots (\bar{e}_{2n+2}, \bar{f}_{1}, \bar{e}_{2n+1}, \bar{f}_{2})(\bar{e}_{2n+3}, \bar{f}_{2n+3}, \bar{e}_{2n+2}, \bar{f}_{2n+2}).
  \]

![Figure 12. The graph $G_k$ (k is odd)](image-url)

Now we count the orbits of the permutation $\sigma_{\infty} = \sigma_1 \circ \sigma_0^{-1}$.

The orbit of $\sigma_{\infty}$ that contains $\bar{e}_1$ is consists of eight elements of $E$. The orbit is given by

\[
D_0 := \{\bar{e}_1, \bar{f}_{2n+1}, \bar{e}_2, \bar{f}_{2n+2}, \bar{e}_{2n+2}, \bar{f}_2, \bar{e}_{2n-1}, \bar{f}_1\}.
\]

Also the orbit of $\sigma_{\infty}$ containing $\bar{f}_1$ consists of eight elements of $E$. The orbit is given by

\[
D_1 := \{\bar{f}_1, \bar{e}_{2n+2}, \bar{f}_{2n+3}, \bar{e}_{2n+3}, \bar{f}_{2n+2}, \bar{e}_1, \bar{f}_{2n+3}, \bar{e}_{2n+3}\}.
\]
The other orbits each consists of 4 elements of $E$. The other orbits are given by the following:

$$D_i := \{ \vec{e}_i, \vec{f}_{2k+2-i}, \vec{e}_{i+1}, \vec{f}_{2k+3-i} \}, \ i = 2, 3, \ldots, 2k.$$  

Hence the total number of orbits of $\sigma_\infty$ is $2n + 1 = k$ which is equal to the number of boundary components of the fat graph $G_k$. \hfill \Box

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