General decay for a viscoelastic wave equation with dynamic boundary conditions and a time-varying delay

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Abstract

The goal of this paper is to study a nonlinear viscoelastic wave equation with strong damping, time-varying delay and dynamical boundary condition. By introducing suitable energy and Lyapunov functionals, under suitable assumptions, we then prove a general decay result of the energy, from which the usual exponential and polynomial decay rates are only special cases.

Keywords: viscoelastic equation, strong damping, time-varying delay, dynamic boundary conditions.

AMS Subject Classification (2000): 35L05, 93D15.

1 Introduction

In this paper, we consider the following problem:

\[
\begin{cases}
  u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \alpha \Delta u_t = 0, & x \in \Omega, t > 0, \\
  u(x,t) = 0, & x \in \Gamma_0, t > 0, \\
  u_t(x,t) = -\frac{\partial u}{\partial \nu}(x,t) + \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(x,s)ds - \alpha \frac{\partial u_t}{\partial \nu}(x,t) \\
  -\mu_1 u_t(x,t) - \mu_2 u_t(x,t - \tau(t)), & x \in \Gamma_1, t > 0, \\
  u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\
  u_t(x,t - \tau(0)) = f_0(x,t - \tau(0)), & x \in \Gamma_1, t \in (0, \tau(0)),
\end{cases}
\]

(1.1)

where \( \Omega \) is a regular and bounded domain of \( \mathbb{R}^N \), \( (N \geq 1) \), \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), \( mes(\Gamma_0) > 0 \), \( \Gamma_0 \cap \Gamma_1 = \emptyset \) and \( \frac{\partial}{\partial \nu} \) denotes the unit outer normal derivative. Moreover, \( \tau(t) > 0 \) is the time-varying delay term, \( \alpha, \mu_1 \) and \( \mu_2 \) are positive constants. The initial datum \( u_0, u_1 \) and \( f_0 \) are given functions belonging to suitable spaces.

From the mathematical point of view, these problems like (1.1) take into account acceleration terms on the boundary. Such type of boundary conditions are usually called dynamic boundary conditions (see [1], [2], [3], [4], [5] for more details). The above model without delay term (i.e., \( \mu_2 = 0 \)), has been studied by many authors in recent years. For example, Gerbi and Said-Houair in [7] studied problem (1.1) with source term \( |u|^{p-2}u \) and nonlinear damping on the boundary but without the relaxation function \( g \). They showed that if the initial data are large enough then the energy and the \( L^p \) norm of the solution of the problem is unbounded,
grows up exponentially as time goes to infinity. Later in [8], they established the global existence and asymptotic stability of solutions starting in a stable set by combining the potential well method and the energy method. A blow-up result for the case \(m = 2\) with initial data in the unstable set was also obtained. Recently, when the relaxation function \(g \neq 0\), they in [10] got the existence and exponential growth results. For the other works, we refer the readers to ([11, 12, 17, 19]) and the references therein.

On the other hand, the above model with delay term (i.e., \(\mu_2 \neq 0\)) has become an active area of research. The delay term may be a source of instability, we refer the readers to ([3, 13, 14, 18]) and the references therein. For example, in [9], Stephane Gerbi and Belkacem Said-Houari considered the following linear damped wave equation with dynamic boundary conditions and a delay boundary term:

\[
\begin{aligned}
  u_{tt} - \Delta u - \alpha \Delta u_t &= 0, \\
  u(x, t) &= 0, \\
  u_{tt}(x, t) &= -\left( \frac{\partial u}{\partial \nu}(x, t) + \alpha \frac{\partial u}{\partial \nu}(x, t) + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) \right), \\
  u(x, 0) &= u_0(x), \\
  u_t(x, 0) &= u_1(x), \\
  u_t(x, t - \tau) &= f_0(x, t - \tau)
\end{aligned}
\]

under the condition that if the weight of the delay term in the feedback is less than the weight of the term without delay or if it is greater under an assumption between the damping factor and the difference of two weights, they proved the global existence of the solutions and the exponential stability of the system. Later, Mohamed Ferhat and Ali Hakem in [6] considered the following wave equation with dynamic boundary conditions:

\[
\begin{aligned}
  u_{tt} - \Delta u - \alpha \Delta u_t - \int_0^t g(t - s) \Delta u(x, s) ds &= |u|^{p-1}u, & \text{in } \Omega \times (0, +\infty), \\
  u(x, t) &= 0, & \text{on } \Gamma_0 \times (0, +\infty), \\
  u_{tt}(x, t) &= -a \left[ \frac{\partial u}{\partial \nu}(x, t) - \alpha \frac{\partial u_t}{\partial \nu}(x, t) - \int_0^t g(t - s) \frac{\partial u}{\partial \nu}(x, s) ds \\
  &\quad + \mu_1 \psi(u_t(x, t)) + \mu_2 \psi(u_t(x, t - \tau)) \right], & \text{on } \Gamma_1 \times (0, +\infty), \\
  u(x, 0) &= u_0(x), \\
  u_t(x, 0) &= u_1(x), & x \in \Omega, \\
  u_t(x, t - \tau) &= f_0(x, t - \tau), & x \in \Gamma_1 \times (0, +\infty).
\end{aligned}
\]

By using the potential well method and introducing suitable Lyapunov function, they proved the global existence and established general decay estimates for the energy.

Recently, the case of time-varying delay has been studied by ([4, 15, 16]). For example, Nicaise, Valein and Fridman [16] in one space dimension. They proved the exponential stability result under the condition

\[
\mu_2 < \sqrt{1 - d \mu_1}
\]  

(1.2)
where $d$ is a constant such that
\[ \tau'(t) \leq d < 1, \quad \forall t > 0. \quad (1.3) \]

Later, Serge Nicaise, Cristina Pignotti and Julie Valein considered the following problem
\[
\begin{aligned}
    & u_{tt} - \Delta u = 0, \quad \text{in } \Omega \times (0, \infty), \\
    & u(x, t) = 0, \quad \text{on } \Gamma_D \times (0, \infty), \\
    & \frac{\partial u}{\partial \nu}(x, t) = -\mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau(t)), \quad \text{on } \Gamma_N \times (0, \infty), \\
    & u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad \text{in } \Omega, \\
    & u_t(x, t - \tau(0)) = f_0(x, t - \tau(0)), \quad \text{on } \Gamma_N \times (0, \tau(0)),
\end{aligned}
\quad (1.4)
\]

they extend the last result to general space dimension under the hypothesis
\[ \tau(t) \geq \tau_0 > 0, \quad \forall t > 0, \]

assumed in [16], that is the delay may degenerate. They also gave a well-posedness result and an exponential stability estimate for problem (1.4) under a suitable relation between the coefficients.

Motivated by these results, in this paper, we intend to study the general decay result to problem (1.1). Our main contribution is an extension of previous result from [9] to relaxation function $g$ and time-varying delays with $\tau(t) \geq 0$. By introducing new energy and Lyapunov functionals, we show in this article that the decay rates of the solution energy is similar to the relaxation function, which are not necessarily decaying like polynomial or exponential functions.

The paper is organized as follows. In Section 2, we present some assumptions needed for our work and state the main result. The general decay result is given in Section 3.

## 2 Preliminaries

In this section we present some assumptions and state the main result. For the relaxation function $g$, we assume the following

(G1) $g: \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function satisfying
\[ g(0) > 0, \quad 1 - \int_{0}^{\infty} g(s) ds = l > 0. \]

(G2) There exists a nonincreasing differentiable function $\xi: \mathbb{R}_+ \to \mathbb{R}_+$ such that
\[ g'(s) \leq -\xi(s) g(s), \quad \forall s \in \mathbb{R}_+ \]

and
\[ \int_{0}^{+\infty} \xi(t) dt = \infty. \]
We denote $H^1_{\Gamma_0} = \{ u \in H^1(\Omega) | u|_{\Gamma_0} = 0 \}$, $\mathcal{V} = H^1_{\Gamma_0}(\Omega) \cap L^2(\Gamma_1)$ and by $(\cdot, \cdot)$ we denote the scalar product in $L^2(\Omega)$; i.e.,

$$(u, v)(t) = \int_{\Omega} u(x, t)v(x, t)dx.$$ 

As in [13], let us introduce the new variable

$$z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad x \in \Gamma_1, \rho \in (0, 1), t > 0.$$ 

Then, we have

$$\tau(t)z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad \text{in } \Gamma_1 \times (0, 1) \times (0, +\infty).$$

Therefore, problem (1.1) is equivalent to

$$
\begin{cases}
  u_{tt} - \Delta u + \int_0^t g(t - s)\Delta u(x, s)ds - \alpha\Delta u_t = 0, & x \in \Omega, t > 0, \\
  \tau(t)z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, & x \in \Gamma_1, \rho \in (0, 1), t > 0, \\
  u(x, t) = 0, & x \in \Gamma_0, t > 0, \\
  u_t(x, t) = -\frac{\partial u}{\partial \nu}(x, t) + \int_0^t g(t - s)\frac{\partial u}{\partial \nu}(x, s)ds - \alpha\frac{\partial u_t}{\partial \nu}(x, t) - \mu_1u_t(x, t) - \mu_2z(x, 1, t), & x \in \Gamma_1, t > 0, \\
  z(x, 0, t) = u_t(x, t), & x \in \Gamma_1, t > 0, \\
  u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega, \\
  z(x, \rho, 0) = f_0(x, -\rho(0)), & x \in \Gamma_1, \rho \in (0, 1).
\end{cases}
$$

(2.1)

We now state, without a proof, local existence result, which can be established by using the Fadeo-Galerkin approximation method (see [15], [16] for more details).

**Lemma 2.1** Suppose that (G1) and (G2) hold. Then given $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$ and $f_0 \in L^2(\Omega \times (0, 1))$, then there exist $T > 0$ and a unique weak solution $(u, z)$ of problem (2.1) on $(0, T)$ satisfying

$$u \in C\left([0, T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1 \left([0, T], L^2(\Omega)\right),$$

$$u_t \in L^2(0, T; H^1_{\Gamma_0}(\Omega)) \cap L^2((0, T) \times \Gamma_1).$$

We define the new energy of system (1.1) as

$$E(t) := \frac{1}{2} \left| |u_t(t)||_2^2 + \left( 1 - \int_0^t g(s)ds \right) \| \nabla u(t) \|_2^2 + \langle g \circ \nabla u(t) \rangle + \| u_t(t) \|_{2, \Gamma_1}^2 \right]$$

$$+ \frac{\xi}{2} \tau(t)\int_{\Gamma_1} u_t^2(x, t - \tau(t)\rho)d\rho d\sigma,$$

(2.2)
where $\zeta$ is a positive constant such that
\[
2\mu_1 - \frac{\mu_2}{\sqrt{1-d}} - \zeta > 0 \quad \text{and} \quad \zeta - \frac{\mu_2}{\sqrt{1-d}} > 0
\]  
and
\[
(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2_2 ds \geq 0.
\]

Then, we state the main result as follows

**Theorem 2.2** Let $(u_0, u_1) \in H^1_{L^2} \times L^2(\Omega)$ be given. Assume that $g$ and $\xi$ satisfy (G1) and (G2). Then, for each $t_0 > 0$, there exist two positive constants $K$ and $k$ such that, for any solution of the problem (1.1), the energy satisfies
\[
E(t) \leq Ke^{-k\int_{t_0}^t \xi(s)ds}.
\]  

3 Decay of solutions

As mentioned earlier, in this section, we prove the general decay result for problem (1.1) under the assumption (1.2).

**Proposition 3.1** For any regular solution of problem (1.1) we have
\[
E'(t) = -\alpha\|\nabla u_t(t)\|^2_2 - \mu_1\|u_t(t)\|^2_{2,\Gamma_1} - \mu_2 \int_{\Gamma_1} u_t(x,t)u_t(x,t-\tau(t))d\sigma + \frac{1}{2}(g' \circ \nabla u)(t)
\]
\[
- \frac{1}{2}g(t)\|\nabla u(t)\|^2_2 - \zeta \int_{\Gamma_1} u^2_t(x,t-\tau(t))(1-\tau'(t))d\sigma + \frac{\zeta}{2}\|u_t(t)\|^2_{2,\Gamma_1}.
\]

**Proof.** Differentiating (2.2) we get
\[
E'(t) = \int_{\Omega} u_{tt}u_t dx + \left(1 - \int_0^t g(s)ds\right) \int_{\Omega} \nabla u \nabla u_t dx - \frac{1}{2}g(t)\|\nabla u\|^2_2 + \frac{1}{2}(g' \circ \nabla u)(t)
\]
\[
+ \int_{\Gamma_1} u_{tt}u_t d\sigma + \frac{\zeta}{2}\tau'(t) \int_0^t \int_{\Gamma_1} u^2_t(x,t-\tau(t)\rho)d\rho d\sigma
\]
\[
+ \zeta \tau(t) \int_0^t \int_{\Gamma_1} u_{tt}(x,t-\tau(t)\rho)u_t(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\sigma.
\]  

**Case 1.** If $\tau(t) \neq 0$, then
\[
u_t(x,t-\tau(t)\rho) = -\tau^{-1}(t)u_{\rho}(x,t-\tau(t)\rho)
\]
and
\[
u_{tt}(x,t-\tau(t)\rho) = \tau^{-2}(t)u_{\rho\rho}(x,t-\tau(t)\rho).
\]

So we get
\[
\int_0^1 u_{tt}(x,t-\tau(t)\rho)u_t(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho.
\]
\[-\tau^{-3}(t) \int_0^1 u_{\rho\rho}(x, t - \tau(t)\rho)u_\rho(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) \, d\rho \]
\[-\tau^{-3}(t) \left[u_\rho^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)\right]_0^1 + \tau^{-3}(t) \int_0^1 u_{\rho\rho}(x, t - \tau(t)\rho)u_\rho(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) \, d\rho \]
\[-\tau'(t)\tau^{-3}(t) \int_0^1 u_\rho(x, t - \tau(t)\rho)u_\rho(x, t - \tau(t)\rho) \, d\rho \]
\[-\frac{1}{2} \tau'(t)\tau^{-3}(t) \int_0^t u_\rho^2(t - \tau(t)\rho) \, d\rho \]
\[-\frac{\tau^{-1}(t)}{2} u_\rho^2(x, t - \tau(t))(1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2} u_\rho^2(x, t) \]
\[-\frac{\tau^{-1}(t)}{2} u_\rho^2(x, t - \tau(t))(1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2} u_\rho^2(x, t). \tag{3.3} \]

By using (3.2), (3.3) and the boundary condition on \(\Gamma_1\), we obtain (3.1).

**Case 2.** If \(\tau(t) = 0\), then from (3.2), we get

\[E'(t) = -\alpha \|\nabla u_t(t)\|_2^2 - (\mu_1 + \mu_2) \|u_t(t)\|_{2,\Gamma_1}^2 + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 \]
\[+ \frac{\zeta}{2} \|u_t(t)\|_{2,\Gamma_1}^2. \tag{3.4} \]

Therefore, (3.1) is proved for all times \(t > 0\).

**Lemma 3.2** For any regular solution of problem (1.1) the energy decays and there exists a positive constant \(C\) such that

\[E'(t) \leq -\alpha \|\nabla u_t(t)\|_2^2 + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 - \mu_1 \|u_t(t)\|_{2,\Gamma_1}^2 + \frac{\mu_2}{2\sqrt{1 - d}} \|u_t(t)\|_{2,\Gamma_1}^2 \]
\[-C (\|u_t(t)\|_{2,\Gamma_1}^2 + \|u_t(t - \tau(t))\|_{2,\Gamma_1}^2). \tag{3.5} \]

**Proof.** In the case of \(\tau(t) \neq 0\), by Cauchy-Schwarz’s inequality, we have

\[E'(t) \leq -\alpha \|\nabla u_t(t)\|_2^2 + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 - \mu_1 \|u_t(t)\|_{2,\Gamma_1}^2 + \frac{\mu_2}{2\sqrt{1 - d}} \|u_t(t)\|_{2,\Gamma_1}^2 \]
\[+ \frac{\mu_2\sqrt{1 - d}}{2} \|u_t(x, t - \tau(t))\|_{2,\Gamma_1}^2 - \frac{\zeta}{2} (1 - \tau'(t)) \|u_t(x, t - \tau(t))\|_{2,\Gamma_1}^2 + \zeta \|u_t(t)\|_{2,\Gamma_1}^2 \]
\[\leq -\alpha \|\nabla u_t(t)\|_2^2 + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 - C (\|u_t(t)\|_{2,\Gamma_1}^2 + \|u_t(t - \tau(t))\|_{2,\Gamma_1}^2), \]

by (3.3) we easily get (3.5). In the case of \(\tau(t) = 0\), when \(\zeta < 2\mu_1 < \frac{2(\mu_1 + \mu_2)}{d}\), by (3.4) we obtain (3.5).

Now, we use the following modified functional, for positive constants \(\varepsilon_1, \varepsilon_2\) and \(\varepsilon_3\), we have

\[L(t) = E(t) + \varepsilon_1 \psi(t) + \varepsilon_2 \phi(t) + \varepsilon_3 I(t), \tag{3.6} \]
where
\[
\psi(t) = \int_{\Omega} u_t u dx + \int_{\Gamma_1} u_t u d\sigma + \frac{\alpha}{2} \|\nabla u\|^2_2,
\] (3.7)
\[
\dot{\phi}(t) = -\int_{\Omega} u_t \int_{0}^{t} g(t-s) (u(t) - u(s)) \, ds \, dx
\] (3.8)
and
\[
I(t) = -\zeta(t) \int_{\Gamma_1} \int_{0}^{1} e^{-2\tau(t)\rho} u_1^2(x, t - \tau(t)\rho) \, d\rho \, d\sigma.
\] (3.9)

It is easy to check that, by using Poincaré’s inequality, trace inequality, (2.3) and for \(\varepsilon_1, \varepsilon_2, \varepsilon_3\) small enough, there exists two constants \(\alpha_1\) and \(\alpha_2\) such that
\[
\alpha_1 L(t) \leq E(t) \leq \alpha_2 L(t).
\] (3.10)

Next, we estimate the derivative of \(L(t)\) according to the following lemmas.

**Lemma 3.3** Under the conditions of Theorem 2.2, the functional \(\psi(t)\) defined in (3.7) satisfies
\[
\psi'(t) \leq \|u_t(t)\|^2_2 + \left(1 + \frac{\mu_1^2}{4\delta}\right) \|u_t(t)\|^2_{2, \Gamma_1} + (\delta - l + 2c\delta) \|\nabla u(t)\|^2_2 + \frac{\mu_2^2}{4\delta} \|u_t(t - \tau(t))\|^2_{2, \Gamma_1} + \frac{1 - l}{4\delta} (g \circ \nabla u)(t),
\] (3.11)
for some \(\delta > 0\).

**Proof.** By using the differential equation in (1.1), we get
\[
\psi'(t) = \|u_t(t)\|^2_2 + \int_{\Omega} u_t(t) u(t) \, dx + \int_{\Gamma_1} u_t(t) u(t) \, d\sigma + \|u_t(t)\|^2_{2, \Gamma_1} + \alpha \int_{\Omega} \nabla u_t(t) \cdot \nabla u(t) \, dx
\]
\[
= \|u_t(t)\|^2_2 + \|u_t(t)\|^2_{2, \Gamma_1} + \int_{\Gamma_1} \frac{\partial u_t(t)}{\partial \nu} u(t) \, d\sigma - \|\nabla u(t)\|^2_2 + \int_{0}^{t} g(t-s) \int_{\Gamma_1} \frac{\partial u(s)}{\partial \nu} u(t) \, d\sigma \, ds
\]
\[
+ \int_{0}^{t} g(t-s) \int_{\Omega} \nabla u(s) \cdot \nabla u(t) \, dx \, ds + \alpha \int_{\Gamma_1} \frac{\partial u_t(t)}{\partial \nu} u(t) \, d\sigma - \mu_1 \int_{\Gamma_1} u_t(t) u(t) \, d\sigma
\]
\[
- \mu_2 \int_{\Gamma_1} u_t(t - \tau(t)) u(t) \, d\sigma
\]
\[
= \|u_t(t)\|^2_2 + \|u_t(t)\|^2_{2, \Gamma_1} - \|\nabla u(t)\|^2_2 + \int_{\Omega} \nabla u(t) \cdot \int_{0}^{t} g(t-s) \nabla u(s) \, ds \, dx
\]
\[
- \mu_1 \int_{\Gamma_1} u_t(t) u(t) \, d\sigma - \mu_2 \int_{\Gamma_1} u_t(t - \tau(t)) u(t) \, d\sigma.
\] (3.12)

We now estimate the right hand side of (3.12). For a positive constant \(\delta\), we have the estimates as follows
\[
\int_{\Omega} \nabla u(t) \cdot \int_{0}^{t} g(t-s) \nabla u(s) \, ds \, dx \leq (\delta + 1 - l) \|\nabla u(t)\|^2_2 + \frac{1 - l}{4\delta} (g \circ \nabla u)(t).
\] (3.13)
By Young’s inequality and trace inequality, we have
\[
-\mu_1 \int_{\Gamma_1} u_t(t)u(t)d\sigma - \mu_2 \int_{\Gamma_1} u(t - \tau(t))u(t)d\sigma \leq 2c\delta \|\nabla u(t)\|_2^2 + \frac{\mu_1^2}{4\delta} \|u_t\|_{2,\Gamma_1}^2 + \frac{\mu_2^2}{4\delta} \|u(t - \tau(t))\|_{2,\Gamma_1}^2.
\]
Combining (3.12)-(3.14), we arrive at (3.11).

**Lemma 3.4** Under the conditions of Theorem 2.2, the functional \(\phi(t)\) defined in (3.3) satisfies
\[
\phi'(t) \leq (\delta - g(0)) \|u_t(t)\|_2^2 + \mu_1 \|u_t(t)\|_{2,\Gamma_1}^2 + \left[\delta + 2\delta(1 - l)^2\right] \|\nabla u(t)\|_2^2 + \frac{\alpha(1 - l)}{2} \|\nabla u(t)\|_2^2 \frac{1 - l}{2\delta\lambda_1} (g \circ \nabla u)(t) + \frac{1 - l}{4\delta\lambda_1} (g' \circ \nabla u)(t)
\]
\[
+ \mu_2 \|u(t - \tau(t))\|_{2,\Gamma_1}^2,
\]
for some \(\delta > 0\).

**Proof.** By using the differential equation in (1.1), we get
\[
\phi'(t) = -\int_{\Omega} u_t(t) \int_0^t g(t - s) (u(t) - u(s)) dsdx - \int_{\Omega} u_t(t) \int_0^t g'(t - s) (u(t) - u(s)) dsdx
\]
\[
- \left(\int_0^t g(s)ds\right) \int_{\Omega} |u_t(t)|^2 dx
\]
\[
= \int_{\Omega} \nabla u(t) \cdot \int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) dsdx - \left(\int_0^t g(s)ds\right) \int_{\Omega} u_t^2(t) dx
\]
\[
- \int_{\Omega} \left(\int_0^t g(t - s) \nabla u(s)ds\right) \left(\int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) ds\right) dx
\]
\[
+ \alpha \int_{\Omega} \nabla u_t(t) \cdot \int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) dsdx
\]
\[
- \int_{\Omega} u_t(t) \int_0^t g'(t - s) (u(t) - u(s)) dsdx
\]
\[
+ \mu_1 \int_{\Gamma_1} u_t(t) \int_0^t g(t - s) (u(t) - u(s)) dsd\sigma
\]
\[
+ \mu_2 \int_{\Gamma_1} u_t(t - \tau(t)) \int_0^t g(t - s) (u(t) - u(s)) dsd\sigma.
\]
We now estimate the right side of (3.16), using Young’s inequality, Hőlder’s inequality and Cauchy-Schwarz’s inequality, we get
\[
\int_{\Omega} \nabla u(t) \cdot \int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) dsdx \leq \delta \|\nabla u(t)\|_2^2 + \frac{1 - l}{4\delta} (g \circ \nabla u)(t),
\]
\[
- \int_{\Omega} \left(\int_0^t g(t - s) \nabla u(s)ds\right) \left(\int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) ds\right) dx
\]
\[
\leq \frac{1}{\delta} \int_\Omega \left( \int_0^t g(t-s) \nabla u(s) ds \right)^2 dx + \frac{1}{\delta} \int_\Omega \left( \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) ds \right)^2 dx \\
\leq 2\delta (1 - \frac{1}{4\delta})^2 \| \nabla u(t) \|_2^2 + \frac{4}{\delta} (1 - \frac{1}{4\delta})^2 (g \circ \nabla u)(t),
\]

(3.18)

\[
\alpha \int_\Omega \nabla u(t) \cdot \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) ds dx \\
\leq \frac{\alpha}{2} \| \nabla u(t) \|_2^2 + \frac{\alpha(1 - \frac{1}{4\delta})}{2} (g \circ \nabla u)(t),
\]

(3.19)

\[
\mu_1 \int_{\Gamma_1} t_{\tau}(t) \int_0^t g(t-s) (u(t) - u(s)) ds d\sigma \\
\leq \mu_1 \| u_{\tau}(t) \|_{2,\Gamma_1}^2 + \frac{1}{4\delta \lambda_1} (g \circ \nabla u)(t),
\]

(3.20)

\[
\mu_2 \int_{\Gamma_1} \tau(t) \int_0^t g(t-s) (u(t) - u(s)) ds d\sigma \\
\leq \mu_2 \| u_{\tau}(t) \|_{2,\Gamma_1}^2 + \frac{1}{4\delta \lambda_1} (g \circ \nabla u)(t),
\]

(3.21)

since \( g \) is continuous and \( g(0) > 0 \), then for any \( t_0 > 0 \), we have

\[
\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds = g_0, \quad \forall t \geq t_0,
\]

(3.22)

then we use (3.22) to get

\[
- \int_\Omega u_{\tau}(t) \int_0^t g'(t-s) (u(t) - u(s)) ds dx - \left( \int_0^t g(s) ds \right) \int_\Omega u_{\tau}^2(t) dx \\
\leq \delta \| u_{\tau}(t) \|_2^2 + \frac{1}{4\delta \lambda_1} (-g' \circ \nabla u)(t) - g_0 \| u_{\tau}(t) \|_2^2.
\]

(3.23)

A combination of (3.16)-(3.23) yields (3.15).

Lemma 3.5 Under the conditions of Theorem 2.2, the functional \( I(t) \) defined in (3.9) satisfies

\[
I'(t) \leq -2I(t) + \zeta \| u_{\tau}(t) \|_{2,\Gamma_1}^2
\]

(3.24)

for some \( \delta > 0 \).

Proof. Differentiating (3.9) we have

\[
I'(t) = \zeta \tau'(t) \int_{\Gamma_1} \int_0^1 e^{-2\tau(t)\rho} u_{\tau}^2(x, t - \tau(t)\rho) d\rho d\sigma \\
- 2\zeta \tau(t) \tau(t) \int_{\Gamma_1} \int_0^1 e^{-2\tau(t)\rho} u_{\tau}^2(x, t - \tau(t)\rho) d\rho d\sigma
\]
\begin{equation}
+ 2\zeta \tau(t) \int_{\Gamma_1} \int_0^1 e^{-2r(t)\rho} u_t(x, t - \tau(t)\rho)u_{tt}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho d\sigma. \tag{3.25}
\end{equation}

Now, let us suppose \( \tau(t) \neq 0 \) and integrate by parts the last term in (3.25). We get

\begin{align*}
\int_0^1 e^{-2r(t)\rho} u_t(x, t - \tau(t)\rho)u_{tt}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho \\
= -\tau^{-3}(t) \int_0^1 e^{-2r(t)\rho} u_{\rho}(x, t - \tau(t)\rho)u_{\rho\rho}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho \\
=\tau^{-3}(t) \int_0^1 e^{-2r(t)\rho} u_{\rho}(x, t - \tau(t)\rho)u_{\rho\rho}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho \\
- \tau'(t)\tau^{-3}(t) \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)d\rho \\
- 2\tau^{-2}(t) \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho \\
- \tau^{-3}(t) \left[ e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) \right]_0^1, \tag{3.26}
\end{align*}

then we have

\begin{align*}
\int_0^1 e^{-2r(t)\rho} u_t(x, t - \tau(t)\rho)u_{tt}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho \\
= -\frac{1}{2} \tau'(t)\tau^{-1}(t) \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)d\rho \\
- \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho \\
- \frac{\tau^{-1}(t)}{2} e^{-2r(t)} u_{\rho}^2(x, t - \tau(t))(1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2} u_{\rho}^2(x, t). \tag{3.27}
\end{align*}

Inserting (3.27) in (3.25), we obtain

\begin{align*}
I'(t) &= \zeta \tau'(t) \int_{\Gamma_1} \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)d\rho d\sigma \\
&- 2\zeta \tau'(t)\tau(t) \int_{\Gamma_1} \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)d\rho d\sigma \\
&- \zeta \tau'(t) \int_{\Gamma_1} \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)d\rho d\sigma \\
&- 2\zeta \tau(t) \int_{\Gamma_1} \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho d\sigma \\
&- \zeta e^{-2r(t)} \int_{\Gamma_1} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\sigma + \zeta \int_{\Gamma_1} u_{\rho}^2(x, t)d\sigma \\
&= -2\zeta \tau(t) \int_{\Gamma_1} \int_0^1 e^{-2r(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)d\rho d\sigma \\
&- \zeta e^{-2r(t)} \int_{\Gamma_1} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\sigma + \zeta \int_{\Gamma_1} u_{\rho}^2(x, t)d\sigma, \tag{3.28}
\end{align*}
from which immediately follows estimate (3.24) for $t$ such that $\tau(t) \neq 0$. However, if $\tau(t) = 0$, from (3.11) we get

$$I'(t) = \zeta' \tau'(t) \int_{\Gamma_1} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho d\sigma$$

$$\leq \zeta d \int_{\Gamma_1} \int_0^1 u_t^2(x, t) d\rho d\sigma$$

$$= \zeta d \int_{\Gamma_1} u_t^2(x, t) d\sigma$$

$$= \zeta d \int_{\Gamma_1} u_t^2(x, t) d\sigma - 2I(t).$$

Then, we obtain (3.24).

**Proof of Theorem 2.2.** From (3.1), (3.11), (3.15) and (3.24), then from (3.6), we get

$$L'(t) = E'(t) + \varepsilon_1 G'(t) + \varepsilon_2 H'(t) + \varepsilon_3 I'(t)$$

$$\leq - [\varepsilon_1 - \varepsilon_2(\delta - g_0)] \|u_t(t)\|_2^2 - \left[ C - \varepsilon_1 \left( 1 + \frac{\mu_1^2}{4\delta} \right) - \varepsilon_2 \mu_1 - \varepsilon_3 \zeta \right] \|u_t(t)\|_{2, \Gamma_1}^2$$

$$- \left[ \frac{1}{2} g(t) - \varepsilon_1 (\delta - l + 2c\delta) - \varepsilon_2 (\delta + 2\delta(1 - l)^2) \right] \|\nabla u(t)\|_2^2$$

$$- \left( \alpha - \frac{\alpha\varepsilon_2}{2} \right) \|\nabla u_t(t)\|_2^2 + \left[ \frac{1}{2} - \varepsilon_2(1 - l) \right] \left( g' \circ \nabla u \right)(t)$$

$$+ \left[ \frac{\varepsilon_1(1 - l)}{4\delta\lambda_1} + \varepsilon_2 \left( \frac{1 - l}{4\delta} + \left( 2\delta + \frac{1}{4\delta} \right)(1 - l)^2 + \frac{\alpha(1 - l)}{2} + \frac{1 - l}{2\delta\lambda_1} \right) \right] \left( g \circ \nabla u \right)(t)$$

$$- \left[ C - \frac{\varepsilon_1\mu_2^2}{4\delta} - \varepsilon_2\mu_2 \right] \|u_t(t - \tau(t))\|_{2, \Gamma_1}^2 - 2\varepsilon_3 I(t),$$

(3.29)

By the trace inequality

$$\|u_t\|_{2, \Gamma_1}^2 \leq C \|u_t\|_{W^{1, 2}(\Omega)} \leq C_1 \|u_t\|_2^2,$$

then we have

$$L'(t) = E'(t) + \varepsilon_1 G'(t) + \varepsilon_2 H'(t) + \varepsilon_3 I'(t)$$

$$\leq - [\varepsilon_1 - \varepsilon_2(\delta - g_0) + C_1 \left( \alpha - \frac{\alpha\varepsilon_2}{2} \right)] \|u_t(t)\|_2^2$$

$$- \left[ \frac{1}{2} g(t) - \varepsilon_1 (\delta - l + 2c\delta) - \varepsilon_2 (\delta + 2\delta(1 - l)^2) \right] \|\nabla u(t)\|_2^2$$

$$- \left[ C - \varepsilon_1 \left( 1 + \frac{\mu_1^2}{4\delta} \right) - \varepsilon_2\mu_1 - \varepsilon_3 \zeta \right] \|u_t(t)\|_{2, \Gamma_1}^2 + \left[ \frac{1}{2} - \varepsilon_2(1 - l) \right] \left( g' \circ \nabla u \right)(t)$$

$$+ \left[ \frac{\varepsilon_1(1 - l)}{4\delta\lambda_1} + \varepsilon_2 \left( \frac{1 - l}{4\delta} + \left( 2\delta + \frac{1}{4\delta} \right)(1 - l)^2 + \frac{\alpha(1 - l)}{2} + \frac{1 - l}{2\delta\lambda_1} \right) \right] \left( g \circ \nabla u \right)(t)$$

$$- \left[ C - \frac{\varepsilon_1\mu_2^2}{4\delta} - \varepsilon_2\mu_2 \right] \|u_t(t - \tau(t))\|_{2, \Gamma_1}^2 - 2\varepsilon_3 I(t).$$

(3.30)
At this point, we choose $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ so small that (3.10) remain valid and

\[
k_1 = -\varepsilon_1 - \varepsilon_2(\delta - g_0) + C_1 \left(\alpha - \frac{\alpha \varepsilon_2}{2}\right) > 0,
\]

\[
k_2 = \frac{1}{2} g(t) - \varepsilon_1(\delta - l + 2c\delta) - \varepsilon_2 \left(\delta + 2\delta(1 - l)^2\right) > 0,
\]

\[
k_3 = C - \varepsilon_1 \left(1 + \frac{\mu_1^2}{4\delta}\right) - \varepsilon_2\mu_1 - \varepsilon_3\xi > 0,
\]

\[
k_4 = \frac{1}{2} - \frac{\varepsilon_2(1 - l)}{4\delta\lambda_1} > 0,
\]

\[
k_5 = \frac{\varepsilon_1(1 - l)}{4\delta\lambda_1} + \varepsilon_2 \left(\frac{1 - l}{4\delta} + \left(2\delta + \frac{1}{4\delta}\right)(1 - l)^2 + \frac{\alpha(1 - l)}{2} + \frac{1 - l}{2\delta\lambda_1}\right),
\]

\[
k_6 = C - \frac{\varepsilon_1\mu_2}{4\delta} - \varepsilon_2\mu_2 > 0.
\]

Therefore, (3.30) takes the form

\[
L'(t) \leq -k_1\|u_t(t)\|_2^2 - k_2\|\nabla u(t)\|_2^2 - k_3\|u_t(t)\|_{2,\Gamma_1}^2 + k_1(g' \circ \nabla u)(t) + k_5 \left(g \circ \nabla u\right)(t) - k_6\|u_t(t - r(t))\|_{2,\Gamma_1}^2 - 2\varepsilon_3I(t).
\]

(3.31)

Since $I(t) \geq 0$ and by (2.2), (G2) there exists a positive constant $M$ such that

\[
L'(t) \leq -ME(t) + k_5\left(g \circ \nabla u\right)(t), \forall t \geq t_0.
\]

(3.32)

Multiplying (3.32) by $\xi(t)$, we have

\[
\xi(t)L'(t) \leq -M\xi(t)E(t) + k_5\xi(t)\left(g \circ \nabla u\right)(t), \forall t \geq t_0.
\]

(3.33)

Because $\xi$ and $g$ are nonincreasing, we get

\[
\xi(t)\int_0^t g(t - s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds \leq -\int_0^t g'(t - s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds \leq -2E'(t)
\]

Inserting the last inequality in (3.33), we obtain

\[
\xi(t)L'(t) + 2k_5E'(t) \leq -M\xi(t)E(t), \forall t \geq t_0.
\]

(3.34)

Now, we define

\[
H(t) = \xi L(t) + 2k_5E(t).
\]
Since $\xi(t)$ is nonincreasing positive function, we can easily get that $H \sim E$. Thus (3.34) implies that

$$H'(t) \leq -k\xi(t)H(t), \forall t \geq t_0,$$

for some $k > 0$. Then, by direct integration over $(t_0, t)$, we have

$$H(t) \leq H(t_0)e^{-k \int_{t_0}^{t} \xi(s)ds}, \forall t \geq t_0.$$

Consequently, using the equivalent relations of $H(t)$ and $E(t)$, we can conclude

$$E(t) \leq k_5 \Phi(t_0)e^{-k \int_{t_0}^{t} \xi(s)ds} = Ke^{-k \int_{t_0}^{t} \xi(s)ds}, \forall t \geq t_0,$$

where $k_5$ is a positive constant and $K = k_5 \Phi(t_0)$. This completes the proof.

References

[1] K. T. Andrews, K. L. Kuttler and M. Shillor, Second order evolution equations with dynamic boundary conditions, J. Math. Anal. Appl. 197 (1996), no. 3, 781–795.

[2] J. T. Beale, Spectral properties of an acoustic boundary condition, Indiana Univ. Math. J. 25 (1976), no. 9, 895–917.

[3] A. Benaissa and M. Bahlil, Global existence and energy decay of solutions to a nonlinear Timoshenko beam system with a delay term, Taiwanese J. Math. 18 (2014), no. 5, 1411–1437.

[4] A. Benaissa and S. A. Messaoudi, Global existence and energy decay of solutions for a nondissipative wave equation with a time-varying delay term, in Progress in partial differential equations, 1–26, Springer Proc. Math. Stat., 44, Springer, Cham.

[5] B. M. Budak, A. A. Samarskii and A. N. Tikhonov, A collection of problems on mathematical physics, Translated by A. R. M. Robson; translation edited by D. M. Brink. A Pergamon Press Book, Macmillan, New York, 1964.

[6] M. Ferhat and A. Hakem, On convexity for energy decay rates of a viscoelastic wave equation with a dynamic boundary and nonlinear delay term, Facta Univ. Ser. Math. Inform. 30 (2015), no. 1, 67–87.

[7] S. Gerbi and B. Said-Houari, Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions, Adv. Differential Equations 13 (2008), no. 11-12, 1051–1074.

[8] S. Gerbi and B. Said-Houari, Asymptotic stability and blow up for a semilinear damped wave equation with dynamic boundary conditions, Nonlinear Anal. 74 (2011), no. 18, 7137–7150.
[9] S. Gerbi and B. Said-Houari, Existence and exponential stability of a damped wave equation with dynamic boundary conditions and a delay term, Appl. Math. Comput. 218 (2012), no. 24, 11900–11910.

[10] S. Gerbi and B. Said-Houari, Global existence and exponential growth for a viscoelastic wave equation with dynamic boundary conditions, Adv. Nonlinear Anal. 2 (2013), no. 2, 163–193.

[11] G. R. Goldstein, Derivation and physical interpretation of general boundary conditions, Adv. Differential Equations 11 (2006), no. 4, 457–480.

[12] P. J. Graber and B. Said-Houari, Existence and asymptotic behavior of the wave equation with dynamic boundary conditions, Appl. Math. Optim. 66 (2012), no. 1, 81–122.

[13] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim. 45 (2006), no. 5, 1561–1585 (electronic).

[14] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, Differential Integral Equations 21 (2008), no. 9-10, 935–958.

[15] S. Nicaise and C. Pignotti, Interior feedback stabilization of wave equations with time dependent delay, Electron. J. Differential Equations 2011, No. 41, 20 pp.

[16] S. Nicaise, J. Valein and E. Fridman, Stability of the heat and of the wave equations with boundary time-varying delays, Discrete Contin. Dyn. Syst. Ser. S 2 (2009), no. 3, 559–581.

[17] F. Sun and M. Wang, Non-existence of global solutions for nonlinear strongly damped hyperbolic systems, Discrete Contin. Dyn. Syst. 12 (2005), no. 5, 949–958.

[18] S.-T. Wu, Asymptotic behavior for a viscoelastic wave equation with a delay term, Taiwanese J. Math.

[19] R. Z. Xu, Global existence, blow up and asymptotic behaviour of solutions for nonlinear Klein-Gordon equation with dissipative term, Math. Methods Appl. Sci. 33 (2010), no. 7, 831–844.