Continuity of weak solutions to an elliptic problem on $p$-fractional Laplacian

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1 | INTRODUCTION

We consider in this paper the following problem:

\[
(-\Delta)^s_p u + V(x)|u|^{p-2}u = \lambda a(x)|u|^{r-2}u - b(x)|u|^{q-2}u
\]

(1.1)

in $\mathbb{R}^N$, where up to a normalization constant one defines

\[
(-\Delta)^s_p u(x) := 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_{\epsilon}(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy
\]

for $x \in \mathbb{R}^N$ with $B_{\epsilon}(x) := \{ y \in \mathbb{R}^N : |x - y| < \epsilon \}$, under the assumptions as below.

1. $\lambda > 0$, $0 < s < 1$, and $1 < p < r < \min\{q, p^*_s\}$, with $N > ps$ and $p^*_s := \frac{Np}{N-ps}$.
2. $V : \mathbb{R}^N \to \mathbb{R}^+$ with a positive constant $V_0$ such that $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$.
3. $a, b : \mathbb{R}^N \to \mathbb{R}^+$ satisfying

\[
0 < \int_{\mathbb{R}^N} a(x) \frac{\beta(p^*_s - p)(p^*_s - q)}{\gamma(p^*_s - p) - \beta(p^*_s - q)} b(x)^{-\gamma} dx < \infty
\]

for some $\beta \in [0, \infty)$ and $\gamma \in \left(0, \frac{(r-p)(p^*_s + \beta(p^*_s - p))}{(q-r)(p^*_s - p)}\right)$ when $a, b \in L^1_{loc}(\mathbb{R}^N)$, or some $\beta \in [0, \infty)$ and $\gamma = \frac{(r-p)(p^*_s + \beta(p^*_s - p))}{(q-r)(p^*_s - p)}$ when $a \in L^k(\mathbb{R}^N)$ with $k > 1$ and $b \in L^1_{loc}(\mathbb{R}^N)$. 

In this paper, we study an elliptic variational problem regarding the $p$-fractional Laplacian in $\mathbb{R}^N$ on the basis of recent result which generalizes some nice published work, and then give some sufficient conditions under which some weak solutions to our studied elliptic variational problem are continuous in $\mathbb{R}^N$. In the final appendix, we correct the proofs of two published lemmas for $1 < p < 2$.

KEYWORDS
elliptic partial differential equation, $p$-fractional Laplacian, variational method

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It is closely related to the following Dirichlet problem with indefinite weights:

\[
\begin{cases}
-\Delta u - \lambda u = \omega(x)u^{q-1} - h(x)u^{r-1} & \text{in } \Omega, \\
u(x) > 0 & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial\Omega,
\end{cases}
\]  

(1.2)

where \(\lambda \in \mathbb{R}, \Omega \subset \mathbb{R}^N (N \geq 3)\) is a bounded domain with smooth boundary, the coefficients \(\omega, h \in L^\infty(\Omega)\) are nonnegative, and \(2 < q < r\). Alama and Tarantello [1] proved the existence, nonexistence and multiplicity of solutions to (1.2) depending on \(\lambda\) and according to the integrability of the ratio \(w^{-2}/h^{q/2}\).

In [2], Pucci and Rădulescu considered the following related problem in the whole space:

\[
\begin{cases}
-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) + |u|^{p-2}u = \lambda |u|^{q-2}u - h(x)|u|^{r-2}u & \text{in } \mathbb{R}^N, \\
u(x) \geq 0 & \text{in } \mathbb{R}^N,
\end{cases}
\]  

(1.3)

where \(h > 0\) satisfies \(0 < \int_{\mathbb{R}^N} h(x)^{q/(q-r)}dx < \infty\) and \(\lambda\) is a positive parameter and \(2 \leq p < q < r < p^*\), with \(p^* = Np/(N-p)\) if \(N > p\) and \(p^* = \infty\) if \(N \leq p\). They showed the nonexistence and existence of nontrivial solutions for the smallness and the largeness of \(\lambda\), respectively.

Later, Autuori and Pucci [3] extended above (1.3) to the following quasi-linear elliptic equation:

\[
-\text{div}A(x, \nabla u) + a(x)|u|^{p-2}u = \lambda \omega(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N,
\]  

(1.4)

where \(A(x, \nabla u)\) acts like the \(p\)-Laplacian, \(\max\{2, p\} < q < \min\{r, p^*\}\) with \(p^* = Np/(N-p)\) and \(1 < p < N\), the coefficients \(\omega\) and \(h\) are related by the integrability condition

\[
\int_{\mathbb{R}^N} \left[ \frac{\omega^p(x)}{h^q(x)} \right]^{1/(r-q)} dx \in \mathbb{R}^+.
\]

After that, Autuori and Pucci in [4] turned to the following elliptic equation involving the fractional Laplacian:

\[
(-\Delta)^s u + a(x)u = \lambda \omega(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N,
\]  

(1.5)

where \(\lambda \in \mathbb{R}, 0 < s < 1, 2s < N\) and \(2 < q < \min\{r, 2^*_s\}\) with \(2^*_s = 2N/(N-2s)\), \((-\Delta)^s\) is the fractional Laplacian operator. They studied the existence and multiplicity of entire solutions to (1.5) by using variational methods and the mountain pass theorem.

In [5], Pucci and Zhang considered the following quasilinear elliptic equations in the setting of variable exponents:

\[
-\text{div}A(x, \nabla u) + a(x)|u|^{p(x)-2}u = \lambda \omega(x)|u|^{q(x)-2}u - h(x)|u|^{r(x)-2}u \quad \text{in } \mathbb{R}^N.
\]  

(1.6)

In fact, they obtained the existence of entire solutions of (1.6), which generalized (1.4) from the case of constant exponents \(p, q\) and \(r\) to the case of variable exponents. They also extended the previous work of Alama and Tarantello [1] from Dirichlet Laplacian problems in bounded domains of \(\mathbb{R}^N\) to the case of general variable exponent differential equation in the \(\mathbb{R}^N\). Furthermore, they solved the two open problems proposed in [4]. Recently, Pucci et al. [6] also gave a positive answer to these open problems in the context of Kirchhoff problems involving the fractional \(p\)-Laplacian.

More recently, Xiang et al. [7] investigated the existence, nonexistence, and multiplicity of nontrivial weak solutions to (1.1) depending on \(\lambda\) and according to the integrability of the ratio \(a^{q/p}/b^{r/p}\) by using variational methods. In fact, they extended the results of Autuori and Pucci to the fractional \(p\)-Laplacian and weakened the conditions in their paper.

Motivated by [7], in this paper we will extend the well-known results on existence and multiplicity of weak solutions of (1.1). Furthermore, we would like to give some sufficient conditions under which the weak solutions of (1.1) are continuous. Finally, we correct the proofs of both [8, Lemma 10] and [6, Lemma A.6] for \(1 < p < 2\).

Now we give the definition of weak solutions of problem (1.1).
Definition 1.1. We say that \( u \in W \) (or \( u \in W_M \subset W \)) is a weak solution of problem (1.1) if

\[
\langle (-\Delta)^s u, \varphi \rangle + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x)dx = \lambda \int_{\mathbb{R}^N} a(x)|u(x)|^{r-2}u(x)\varphi(x)dx - \int_{\mathbb{R}^N} b(x)|u(x)|^{q-2}u(x)\varphi(x)dx
\]

(1.7)

for any \( \varphi \in W \), where

\[
\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dxdy,
\]

and the solution spaces \( W \) and \( W_M \) will be introduced in Section 2.

First, we shall generalize the condition and result of [7, Theorem 1.1] on problem (1.1) and prove the following theorem.

Theorem 1.2. Suppose that the three conditions (1)–(3) are satisfied. Then there exist a universal constant \( \lambda_* > 0 \) and two constants \( \lambda_M \) and \( \lambda_M^* \) depending on \( M \) with \( \lambda_M^* \geq \lambda_M > 0 \) for each fixed \( M > 0 \) such that problem (1.1) has

(i) only the trivial weak solution in \( W \) if \( \lambda < \lambda_* \) when \( a \in L^k (\mathbb{R}^N) \) with \( k > 1 \) and \( b \in L^1_{loc} (\mathbb{R}^N) \), and only the trivial weak solution in \( W_M \) if \( \lambda < \lambda_M \) when \( a, b \in L^1_{loc} (\mathbb{R}^N) \);

(ii) at least two nontrivial nonnegative weak solutions in \( W_M \) in which one has negative energy and another has positive energy if \( \lambda > \lambda_M^* \).

Remark 1.3. Since \( W_M \subset W \), our result generalizes Theorem 1.1 in [7]. And our condition (3) is new even in the regular Laplacian setting. More precisely, \( L^{\frac{p^*_s}{r^*_s}} (\mathbb{R}^N) \subset L^{\frac{p^*_s}{r^*_s}}_{loc}(\mathbb{R}^N) \) implies the condition (H3) in [7] is essentially \( a \in L^{\frac{p^*_s}{r^*_s}} (\mathbb{R}^N) \), which corresponds to the case when \( a \in L^k (\mathbb{R}^N) \) with \( k = \frac{p^*_s}{r^*_s} \) > 1 and \( b \in L^1_{loc} (\mathbb{R}^N) \) in our condition (3); in this case if \( \beta = 0 \) and \( \gamma = \frac{(r-p)(p^*_s+r-p^*_s)}{(q-r)(p^*_s-r)} \), then \( \frac{p^*_s+r-p^*_s}{p^*_s-r} > 1 \) and \( \gamma = \frac{r-p}{q-r} \cdot \frac{N}{ps} \), which implies the condition (H4) in [7] is a special case of our condition (3), so our condition (3) generalizes the conditions (H3) and (H4) in [7]. Moreover, our result also generalizes the nice work [1, 4] as well as the closely related ones [5, 6, 9–11] and can be extended to improve the work [12–14], among many others.

Second, applying Theorem 3.1 and Theorem 3.13 in [15], we give some sufficient conditions under which some weak solutions to the problem (1.1) are continuous in \( \mathbb{R}^N \).

Theorem 1.4. Suppose that the three conditions (1)–(3) are satisfied and that \( a(x), b(x), V(x) \in L^\infty (\Omega) \) for any bounded domain \( \Omega \subset \mathbb{R}^N \). If \( q < p^*_s \), then any weak solution in \( W \) to problem (1.1) is continuous in \( \mathbb{R}^N \). If \( q > p^*_s \), then any weak solution in \( W \cap L^{\frac{p^*_s}{r^*_s}}_{loc}(\mathbb{R}^N) \) to problem (1.1) is continuous in \( \mathbb{R}^N \).

Remark 1.5. When \( q \leq p^*_s \) and \( u \in W \) is a weak solution with bounded support denoted by \( \text{supp} u \), this theorem follows from applying the argument of [16, Theorem 3.3] with corresponding \( \Omega = \text{supp} u \). \( a = 0 \) and \( f(x, u) = \lambda a(x)|u|^{r-2}u - b(x)|u|^{q-2}u - V(x)|u|^{p-2}u \) since the condition (3.8) of [16, Theorem 3.3] holds a.e. in \( \Omega \). However, the condition (3.8) of [16, Theorem 3.3] is not applicable for our case when \( q > p^*_s \) since our condition is not included in (3.8) of [16, Theorem 3.3] in this case.

2 | PRELIMINARIES

In this section, we first give some basic results of fractional Sobolev spaces that will be used in the next section. Let \( 0 < s < 1 < p < \infty \) be real numbers and the fractional Sobolev space \( W^{s,p} (\mathbb{R}^N) \) be defined as follows:
equipped with the norm

\[ ||u||_{W^{s,p}(\mathbb{R}^N)} = \left( ||u||_{L^p(\mathbb{R}^N)}^p + ||u||_{L^p(\mathbb{R}^N)}^p \right)^{\frac{1}{p}} \]

where

\[ ||u||_{L^p(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |u(x)|^p \, dx \right)^{\frac{1}{p}}. \]

By [17, Theorem 6.7], we know that the embedding \( W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \) is continuous for any \( r \in [p, p^*_s] \) with a positive constant \( C = C(N, p, s) \) such that

\[ ||u||_{L^r(\mathbb{R}^N)} \leq C ||u||_{W^{s,p}(\mathbb{R}^N)} \quad \text{for all } u \in W^{s,p}(\mathbb{R}^N). \]  

Let \( E \) denote the completion of \( C_0^\infty(\mathbb{R}^N) \) endowed with the norm

\[ ||u||_E = \left( ||u||_{L^p(\mathbb{R}^N)}^p + ||u||_{L^p(\mathbb{R}^N)}^p \right)^{\frac{1}{p}}, \]

where \( ||u||_{L^p(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |u(x)|^p \, dx \right)^{\frac{1}{p}}. \)

Now we introduce the main space \( W \) which is the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm

\[ ||u||_W = ||u||_E + ||u||_{L^p(\mathbb{R}^N)}, \quad \text{where } ||u||_{L^p(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |u(x)|^p \, dx \right)^{\frac{1}{p}}. \]

It is proved in [7, Lemma 2.2] that \( W \) is a reflexive Banach space by using the fact that \( (E, || \cdot ||_E) \) is a uniformly convex Banach space, which is proved in [8, Lemma 10] and [6, Lemma A.6]. However, there are mistakes in the proof of both lemmas when \( 1 < p < 2 \). For instance, since \( p'/p = 1/(p-1) > 1 \) when \( 1 < p < 2 \), applying \( (a+b)^p \leq 2^{p-1}(a^p + b^p) \) for all \( a, b > 0 \) and \( p \geq 1 \) we have

\[
\left\| \frac{u+v}{2} \right\|_{p'}^p + \left\| \frac{u-v}{2} \right\|_{p'}^p \leq 2^{\frac{1}{p-1}} \left\{ \left\| \frac{u(x)+v(x)}{2} - \frac{u(y)+v(y)}{2} \right\|_{L^{p-1}(\mathbb{R}^N)}^{p'} \right. \\
+ \left. \left\| \left( V(x)^{\frac{1}{2}} \left| \frac{u(x)+v(x)}{2} \right| \right)^{p'} \right\|_{L^{p-1}(\mathbb{R}^N)} \right. \\
+ \left. \left( V(x)^{\frac{1}{2}} \left| \frac{u(x)-v(x)}{2} \right| x - y \right|^{\frac{N-p}{p}} \right)^{p'} \right\|_{L^{p-1}(\mathbb{R}^N)} \\
+ \left. \left( V(x)^{\frac{1}{2}} \left| \frac{u(x)-v(x)}{2} \right| x - y \right|^{\frac{N-p}{p}} \right)^{p'} \right\|_{L^{p-1}(\mathbb{R}^N)} \right\},
\]  

(2.2)
which shows the last inequality in (5.3) of [8, Lemma 10] lacks a factor $2^{1/p-1}$. Besides, in (A.4) of [6, Lemma A.6], the equality that $2^{1/(1-p)}(||u||^p_E + ||v||^p_E)^{1/(p-1)} = 2^{1/(1-p)}$ should be $2^{1/(1-p)}(||u||^p_E + ||v||^p_E)^{1/(p-1)} = 1$, so neither (5.3) of [8, Lemma 10] nor (A.4) of [6, Lemma A.6] can imply the uniform convexity of $E$. We shall give another proof by contradiction in Appendix A.

From now on, $B_R$ denotes the ball of radius $R$ in $\mathbb{R}^N$ that is centered at the origin, $D^{s,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $||u||_E$, and

$$W_M = \{u \in W : ||u||_E \leq M\}$$

for each fixed $M > 0$.

**Lemma 2.1.** (See [8, Lemma 1]) The embeddings $E \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ are continuous with

$$\min\{1, V_0\}||u||^p_{W^{s,p}(\mathbb{R}^N)} \leq ||u||^p_{L^r(\mathbb{R}^N)}$$

for all $u \in W$ and $v \in [p, p_0^*]$. Moreover, for any $R > 0$ and $v \in [1, p_0^*]$, the embeddings $E \hookrightarrow L^v(B_R)$ is compact for all $m \in [1, p_0^*)$.

The following lemma is [6, Lemma 2.1], which is an application of [17, Theorem 6.5] and [17, Corollary 7.2].

**Lemma 2.2.** The embeddings $W \hookrightarrow E \hookrightarrow D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{r'}(\mathbb{R}^N)$ are continuous with $||u||_{E} \leq ||u||_{W}$ for all $u \in W$, and

$$||u||_{L^{r'}(\mathbb{R}^N)} \leq C_{p_0^*}||u||_{k,p},$$

for all $u \in D^{s,p}(\mathbb{R}^N)$. Moreover, for any $R > 0$, the embeddings $E \hookrightarrow L^{m}(B_R)$ and $W \hookrightarrow L^{m}(B_R)$ are compact for all $m \in [1, p_0^*)$.

**Lemma 2.3.** If the three conditions (1)–(3) are satisfied, then $W \hookrightarrow L^{r}(\mathbb{R}^N, a)$ is compact.

**Proof.** Denote

$$s_1 = \frac{\beta(p_0^* - r)}{p_0^* + \beta (p_0^* - p) + \gamma (p_0^* - q)}, s_2 = \frac{\gamma(p_0^* - r)}{p_0^* + \beta (p_0^* - p) + \gamma (p_0^* - q)},$$

and

$$s_3 = \frac{r + \beta(r - p) + \gamma(r - q)}{p_0^* + \beta (p_0^* - p) + \gamma (p_0^* - q)}, s_4 = \frac{p_0^* - r}{p_0^* + \beta (p_0^* - p) + \gamma (p_0^* - q)}.$$

One can check that $\sum_{i=1}^{4} s_i = 1$.

If $p_0^* \geq q$, then by $\beta \geq 0$, $0 < \gamma \leq \frac{(r-p)\beta(p_0^*+p_0^*-p)}{q-r} \leq \frac{\beta(r-p)+r}{q-r}$ and condition (1), we have $s_1 \geq 0$ and $s_i > 0$ for each $i = 2, 3, 4$. So $\sum_{i=1}^{4} s_i = 1$ implies $s_1 \in (0, 1)$ and $s_i \in (0, 1)$ for each $i = 2, 3, 4$.

If $p_0^* < q$, then $0 < \gamma \leq \frac{(r-p)\beta(p_0^*+p_0^*-p)}{q-r} \leq \frac{\beta(r-p)+r}{q-r} < \frac{\beta(p_0^*+p_0^*-p)}{q-r}$ and so $p_0^* + \beta(p_0^* - p) + \gamma(p_0^* - q) > 0$. Combining this with $\beta \geq 0$, $0 < \gamma \leq \frac{\beta(r-p)+r}{q-r}$ and condition (1) yields $s_1 \geq 0$ and $s_i > 0$ for each $i = 2, 3, 4$, thus $\sum_{i=1}^{4} s_i = 1$ implies $s_1 \in (0, 1)$ and $s_i \in (0, 1)$ for each $i = 2, 3, 4$. Since $ps_1 + qs_2 + p_0^*s_3 = r$, $\beta = s_1/s_4$ and $\gamma = s_2/s_4$, applying H"{o}lder's inequality with Lemma 2.2 and condition (2), we obtain
\[||u||_{L^r(\mathbb{R}^N, a)}^r = \int_{\mathbb{R}^N} a(x)|u|^r \, dx \]
\[= \int_{\mathbb{R}^N} |u|^{p_1-q_2} \left\{ V(x)|u|^p \right\}^{s_1} \left\{ b(x)|u|^q \right\}^{s_2} \left\{ \frac{a(x)}{V(x)^{\frac{1}{r}} b(x)^{\frac{1}{q}}} \right\} \, dx \]
\[\leq \left( \int_{\mathbb{R}^N} |u|^{\alpha} \, dx \right)^{s_1} \left( \int_{\mathbb{R}^N} V(x)|u|^p \, dx \right)^{s_1} \left( \int_{\mathbb{R}^N} b(x)|u|^q \, dx \right)^{s_2} \left( \int_{\mathbb{R}^N} \frac{a(x)}{V(x)^{\frac{1}{r}} b(x)^{\frac{1}{q}}} \, dx \right)^{s_2} \tag{2.3} \]
\[\leq (C_{p_1})^{s_1} \left( \int_{\mathbb{R}^N} V(x)|u|^p \, dx \right)^{s_1} \left( \int_{\mathbb{R}^N} b(x)|u|^q \, dx \right)^{s_2} \left( \int_{\mathbb{R}^N} \frac{a(x)}{b(x)^{\frac{1}{q}}} \, dx \right)^{s_2} \leq C_1 \left( \int_{\mathbb{R}^N} \frac{a(x)}{b(x)^{\frac{1}{q}}} \, dx \right)^{s_2} \leq W \rightarrow L^r(\mathbb{R}^N, a) \]

with constant \( C_1 = C_{p_1}^{s_1} V_{0}^{-s_1} > 0 \).

By \( p s_1 + q s_2 + p_3 s_3 = r \) and (2.3), we have \( ||u||_{L^r(\mathbb{R}^N, a)} \leq C_1^{\frac{1}{s_1}} \left( \int_{\mathbb{R}^N} a(x) \, dx \right)^{\frac{s_1}{s_2}} ||u||_W \), and then the embedding \( W \hookrightarrow L^r(\mathbb{R}^N, a) \) is continuous. Next, we will show that \( W \hookrightarrow L^r(\mathbb{R}^N, a) \) is compact.

Indeed, by condition (3) for any \( \varepsilon > 0 \), there exists an \( R_1 > 0 \) such that \( \int_{\mathbb{R}^N \setminus B_{R_1}} \frac{a(x)}{b(x)^{\frac{1}{q}}} \, dx < \varepsilon^{1/s_1} \) for all \( R \geq R_1 \). Fix \( R_1 > 0 \) and let \( \{u_n\}_n \) be a bounded sequence in \( W \), then by Lemma 2.1, [17, Theorem 6.5], and [17, Corollary 7.2] as in [7, Theorem 2.1], we obtain a subsequence of \( \{u_n\}_n \), which is also denoted by \( \{u_n\}_n \) for convenience, satisfying \( u_n \rightharpoonup u \) weakly in \( W \cap L^r \) and \( a(x)|u_n - u|^r \rightarrow 0 \) a.e. in \( B_{R_1} \) as \( n \rightarrow \infty \). Since \( u_n \rightharpoonup u \) weakly in \( W \cap L^r \), applying condition (3) and the argument of (2.3) yields

\[\int_{\mathbb{R}^N \setminus B_{R_1}} a(x)|u_n(x) - u(x)|^r \, dx \leq C_1 \left( \int_{\mathbb{R}^N \setminus B_{R_1}} \frac{a(x)}{b(x)^{\frac{1}{q}}} \, dx \right)^{s_4} ||u_n - u||_W \leq C_2 \varepsilon \tag{2.4} \]

and

\[\int_U a(x)|u_n(x) - u(x)|^r \, dx \leq C_1 \left( \int_{\mathbb{R}^N} \frac{a(x)}{b(x)^{\frac{1}{q}}} \, dx \right)^{s_4} ||u_n - u||_W \leq C_3 < \infty \tag{2.5} \]

for each measurable subset \( U \subset B_{R_1} \) with constants \( C_2, C_3 > 0 \). Then (2.5) and the Vitali convergence theorem imply

\[\lim_{n \rightarrow \infty} \int_{B_{R_1}} a(x)|u_n(x) - u(x)|^r \, dx = 0.\]

Thus, for the above \( \varepsilon > 0 \), there exists an integer \( N_1 > 0 \) such that

\[\int_{B_{R_1}} a(x)|u_n(x) - u(x)|^r \, dx < \varepsilon \tag{2.6} \]

for all \( n \geq N_1 \). Hence, using (2.4) and (2.6) obtains

\[\int_{\mathbb{R}^N} a(x)|u_n - u|^r \, dx = \int_{\mathbb{R}^N \setminus B_{R_1}} a(x)|u_n(x) - u(x)|^r \, dx + \int_{B_{R_1}} a(x)|u_n(x) - u(x)|^r \, dx < (C_2 + 1)\varepsilon \]

for all \( n \geq N_1 \), which proves \( W \hookrightarrow L^r(\mathbb{R}^N, a) \) is compact. \( \square \)
3 | PROOF OF THEOREM 1.2

As in [7] for each \( u \in W \), we define \( I(u) = J(u) - H(u) \), whose critical points are weak solutions of problem (1.1), where

\[
J(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p}}{|x - y|^{N + ps}} \, dx \, dy + \frac{1}{p} \int_{\mathbb{R}^{N}} V(x)|u(x)|^{p} \, dx + \frac{1}{q} \int_{\mathbb{R}^{N}} b(x)|u(x)|^{q} \, dx,
\]

\[
H(u) = \frac{\lambda}{r} \int_{\mathbb{R}^{N}} a(x)|u(x)|^{r} \, dx.
\]

**Lemma 3.1.** Under the conditions (1)–(3), the function \( I \) is coercive and weakly lower semicontinuous in \( W_{M} \) for each constant \( M > 0 \).

**Proof.** Let \( M > 0 \) be a constant. By (2.3) for each \( u \in W_{M} \), we have

\[
H(u) = \frac{\lambda}{r} \int_{\mathbb{R}^{N}} a(x)|u(x)|^{r} \, dx \\
\leq \frac{\lambda C_{1}}{r} \left( |u|_{W_{M}}^{p} + \left( \int_{\mathbb{R}^{N}} V(x)|u|^{p} \, dx \right)^{\frac{s_{1}}{s_{1}}} \right) \left( \int_{\mathbb{R}^{N}} b(x)|u|^{q} \, dx \right)^{\frac{s_{2}}{s_{2}}} \left( \int_{\mathbb{R}^{N}} a(x)^{\frac{r}{q}} \, dx \right)^{S_{r}} \\
\leq C_{4} ||u||_{L_{q}(\mathbb{R}^{N}, b)}^{p_{2}^{s_{2}} + p_{1}^{s_{1}}} ||u||_{L_{q}(\mathbb{R}^{N}, b)}^{q_{2}}
\]

with constant \( C_{4} = \frac{\lambda C_{1}}{r} \left( \int_{\mathbb{R}^{N}} \frac{a(x)^{\frac{r}{r}}}{b(x)} \, dx \right)^{S_{r}} > 0 \). Then by \( ||u||_{L_{E}} \leq M \), we get

\[
I(u) \geq \frac{1}{p} ||u||_{E}^{p} + \frac{1}{q} ||u||_{L_{q}(\mathbb{R}^{N}, b)}^{q} - C_{4} ||u||_{L_{q}(\mathbb{R}^{N}, b)}^{p_{2}^{s_{2}} + p_{1}^{s_{1}}} ||u||_{L_{q}(\mathbb{R}^{N}, b)}^{q_{2}} \geq \frac{1}{q} ||u||_{L_{q}(\mathbb{R}^{N}, b)}^{q} - C_{4} M^{p_{2}^{s_{2}} + p_{1}^{s_{1}}} ||u||_{L_{q}(\mathbb{R}^{N}, b)}^{q_{2}} \geq \frac{1}{q} ||u||_{W} - M^{q} - C_{4} M^{p_{2}^{s_{2}} + p_{1}^{s_{1}}} ||u||_{W}^{q_{2}}.
\]

This, together with \( q > q_{2} \), implies \( I \) is coercive in \( W_{M} \) for each constant \( M > 0 \).

Note that \( W_{M} \subset W \), then [7, Lemma 3.2] implies the functional \( J \) is weakly lower semicontinuous in \( W_{M} \) with \( J \in C^{1}(W_{M}, \mathbb{R}) \). Also, Lemma 2.3 implies \( W_{M} \hookrightarrow L^{r}(\mathbb{R}^{N}, \alpha) \) is compact, and then by applying the similar argument as in [7, Lemma 3.3], one can get that functional \( H \) is weakly continuous in \( W_{M} \) with \( H \in C^{1}(W_{M}, \mathbb{R}) \). Hence, the functional \( I \) is weakly lower semicontinuous in \( W_{M} \) with \( I \in C^{1}(W_{M}, \mathbb{R}) \).

For each fixed \( M > 0 \), we define

\[
\bar{W}_{M} := \left\{ u \in W_{M} : ||u||_{L_{q}(\mathbb{R}^{N}, b)} = 1 \right\} \quad \text{and} \quad \lambda_{M}^{*} := \inf_{u \in \bar{W}_{M}} rJ(u).
\]

Applying Lemma 2.3 with a similar argument as in [7, Lemma 3.4], we have the following.

**Lemma 3.2.** Let \( M > 0 \) be fixed. Then \( \inf_{u \in \bar{W}_{M}} J(u) \) can be achieved at some \( u_{M} \in \bar{W}_{M} \) and \( \lambda_{M}^{*} = r \inf_{u \in \bar{W}_{M}} J(u) = rJ(u_{M}) > 0 \).

Moreover, \( |u_{M}| \) is also a minimizer of \( \inf_{u \in \bar{W}_{M}} J(u) \), which means that \( J(|u_{M}|) = J(u_{M}) \).

The following theorem is the modification of the mountain pass theorem of Ambrosetti-Rabinowitz (see [3, Theorem A.3]), which will be used to prove Theorem 1.2(ii).

**Theorem 3.3.** Let \( (X, || \cdot ||_{X}) \) and \( (Y, || \cdot ||_{Y}) \) be two Banach spaces. Suppose \( X \) can be continuously embedded into \( Y \). Let \( \Phi : X \rightarrow \mathbb{R} \) be a \( C^{1} \) functional with \( \Phi(0) = 0 \). Assume that there exist \( \rho, \alpha > 0 \) and \( e \in X \) such that \( ||e||_{Y} > \rho, \Phi(e) < \alpha \) and \( \Phi(u) \geq \alpha \) for all \( u \in X \) with \( ||u||_{Y} = \rho \). Then there exists a sequence \( \{u_{n}\} \subset X \) such that for all \( n \)

\[
c \leq \Phi(u_{n}) \leq c + \frac{1}{n} \quad \text{and} \quad ||\Phi'(u_{n})||_{X'} \leq \frac{2}{n},
\]
where
\[ c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \Phi(\gamma(t)) \quad \text{and} \quad \Gamma = \{ \gamma \in C([0, 1]; X) : \gamma(0) = 0, \gamma(1) = e \}. \]

Now for each \( e \in W \setminus \{0\} \), we find a lower bound of the functional \( I \) on the boundary of \( W_M \) with \( 0 < M < ||e||_E \) to establish the next lemma, which is similar to [7, Lemma 3.5], but its proof is different.

**Lemma 3.4.** Suppose the conditions (1)–(3) are satisfied. Then for each \( e \in W \setminus \{0\} \), there exist \( \rho \in (0, ||e||_E) \) and \( \alpha > 0 \) such that
\[ I(u) \geq \alpha > 0, \]
for all \( u \in W \) with \( ||u||_E = \rho \).

**Proof.** We divide the proof of this lemma into two cases as follows.

Case 1 If \( \beta \in [0, \infty) \) and \( \gamma \in \left(0, \frac{\alpha - p(p_{t+} + p_{q^-} - p)}{(q - r)(p_t - p)}\right) \) when \( a, b \in L^{1}_{\text{loc}}(\mathbb{R}^N) \), by (2.3) and (3.1) for each \( u \in W \), we have
\[ I(u) \geq \frac{1}{p} ||u||_{E}^{p} + \frac{1}{q} ||u||_{E}^{q} - C_3 (||u||_{E}^{p})^{\frac{p_{t+}}{p}} (||u||_{E}^{p})^{\frac{s_{t}}{p}} ||u||_{E}^{q_{t+}}. \]

Case 2 If \( \beta \in [0, \infty) \) and \( \gamma \in \left(0, \frac{\alpha - p(p_{t+} + p_{q^-} - p)}{(q - r)(p_t - p)}\right) \) when \( a, b \in L^{1}_{\text{loc}}(\mathbb{R}^N) \), we can take sufficiently small \( t_5 \in (0, 1/k) \), \( t_4 = s_4(1 - k t_5) \), \( t_3 = \frac{1}{p_t - p} [(p + p_{q^-} - q_{t+}) t_4 + p t_5 + (r - p)] \), \( t_2 = \gamma t_4 \), and \( t_1 = 1 - t_3 - (1 + \gamma) t_4 - t_5 \) satisfying \( p t_1 + q t_2 + p_{t+} t_3 = r \) and \( \sum_{i=1}^{5} t_i = 1 \) with \( t_i > 0 \), and use the same argument as in (2.3) to obtain
\[ ||u||_{E}^{r} \leq C_5 (||u||_{E}^{p})^{\frac{p_{t+}}{p}} \left( \int_{\mathbb{R}^N} V(x) ||u||^{p} dx \right)^{t_1} \left( \int_{\mathbb{R}^N} b(x) ||u||^{q} dx \right)^{t_2} \left( \int_{\mathbb{R}^N} \frac{a(x)}{b(x)}^{\frac{1}{q}} dx \right)^{t_3}. \]
with constant $C_8 = C_p \gamma 0 V_0^{-\gamma} |a|^{\gamma} L^t(\mathbb{R}^n) > 0$. In this case, it can be checked that

$$\frac{1}{1 - s_2} \left( \frac{p_s^{s_4} t_3}{p} + t_1 \right) > \frac{1}{1 - s_2} \left( \frac{p_s^{s_4} s_3}{p} + s_1 \right) = 1,$$

and then applying the argument of Case 1 yields that

$$I(u) \geq \rho^\alpha \left( \frac{1}{p} - C_9 \rho^{\frac{p}{1 - s_2}} \left( \frac{p_s^{s_4} + t_1}{p} \right)^{-p} \right) =: \alpha > 0$$

for all $u \in W$ with $||u||_E = \rho \in \left(0, \min \left\{ ||e||_E, (pC_9)^{-1} \left( \frac{1}{p} \left( \frac{p_s^{s_4} + t_1}{p} \right)^{-p} \right) \right\} \right)$, where $C_9 > 0$ is a constant. \qed

Now we first give the proof of Theorem 1.2.

**Proof of Theorem 1.2.**

(i) Suppose $u \neq 0$ is a nontrivial weak solution of problem (1.1). Since condition (1) implies $\gamma (r - q_2) \in [p, p_*]$ and condition (3) implies $r - q_2 \in (0, p)$, taking $\varphi = u$ in Definition 1.1 and applying Hölder’s inequality with inequality (2.1), Lemma 2.1 and [7, inequality(3.1)], we have

$$||u||^p_E = \lambda \int_{\mathbb{R}^n} a(x)|u(x)|^r dx - \int_{\mathbb{R}^n} b(x)|u(x)|^s dx.$$

$$= \int_{\mathbb{R}^n} \left( \lambda a(x)|u|^r \frac{q_2}{q_2 - q} - b(x)|u|^s \right) |u|^\frac{r - q_2}{s} dx$$

$$\leq \int_{\mathbb{R}^n} \frac{1}{\lambda^{\frac{1}{q_2 - q}}} \left( \frac{a(x)}{b(x)} \right)^\frac{1}{q_2 - q} |u|^\frac{r - q_2}{s} dx$$

$$\leq \lambda^{\frac{1}{q_2 - q}} \int_{\mathbb{R}^n} \left( \frac{a(x)}{b(x)} \right)^\frac{1}{q_2 - q} dx \left( \frac{\lambda}{q_2 - q} \right)^\frac{r - q_2}{s}$$

$$\leq \lambda^{\frac{1}{q_2 - q}} \int_{\mathbb{R}^n} \left( \frac{a(x)}{b(x)} \right)^\frac{1}{q_2 - q} dx \left( \min \{1, V_0\} \right)^{-\frac{1}{q}} ||u||_{L^p(\mathbb{R}^n)}^{r - q_2},$$

with the same constants $s_2$ and $s_4$ as in (2.3).

Case 1. If $u$ is a nontrivial weak solution in $W$ when $a \in L^k(\mathbb{R}^n)$ with $k > 1$ and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $\gamma = \frac{r - q_2}{q_2 - q} \in (0, 1)$ implies $\frac{r - q_2}{1 - s_2} = p$. By (3.6), we have

$$\lambda \geq \left( \int_{\mathbb{R}^n} \frac{a(x)}{b(x)} dx \right)^{-\frac{1}{q}} \left( \min \{1, V_0\} \right)^{-\frac{1}{q}} ||u||_{L^p(\mathbb{R}^n)}^{r - q_2} =: \lambda_s.$$

So problem (1.1) has only the trivial weak solution in $W$ if $\lambda < \lambda_s$ when $a \in L^k(\mathbb{R}^n)$ with $k > 1$ and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Case 2. Let $M > 0$ be fixed. If $u$ is a nontrivial weak solution in $W_M$ when $a, b \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $\gamma \in \left(0, \frac{r - p}{p_s^{s_4} + \beta (p_s^{s_4} - p)} \right)$ implies $\frac{r - q_2}{1 - s_2} > p$. By (3.6) and $||u||_E \leq M$, we have

$$\lambda \geq \left( \int_{\mathbb{R}^n} \frac{a(x)}{b(x)} dx \right)^{-\frac{1}{q}} \left( \min \{1, V_0\} \right)^{-\frac{1}{q}} ||u||_{L^p(\mathbb{R}^n)}^{r - q_2} M^{(p - q_2) - p} =: \lambda_M.$$

So problem (1.1) has only the trivial weak solution in $W_M$ if $\lambda < \lambda_M$ when $a, b \in L^1_{\text{loc}}(\mathbb{R}^n)$. 


(ii) Let $M > 0$ be an arbitrary constant and $\lambda_M^*$ be the same as in Lemma 3.2. Recall that $W$ is a reflexive Banach space; it can be checked that $W_M$ is a closed and convex subset of $W$, and then $W_M$ is a weakly closed subset of $W$ and also a reflexive Banach subspace of $W$. Since Lemma 3.1 and 3.4 imply the functional $I$ is weakly lower semicontinuous, bounded below and coercive in $W_M$ for all $\lambda > 0$, by [18, Theorem 1.2], there exists a $u_M^* \in W_M$ such that $I(u_M^*) = \inf_{u \in W_M} I(u)$. When $\lambda > \lambda_M^*$, applying Lemma 3.2 and similar argument of [7, Theorem 3.1] yields $I(u_M^*) < 0$.

Taking $e = u_M^*$ in Lemma 3.4, we can check that the functional $I$ satisfies the assumptions of Theorem 3.3. Then for all $\lambda > \lambda_M^*$ by Theorem 3.3, there is a sequence $\{u_{n,M}\}_{n \geq 1} \subset W_M$ such that

$$I(u_{n,M}) \to c_M \quad \text{and} \quad ||I(u_{n,M})||_{W_M} \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$c_M = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} I(\gamma(\tau))$$

So coerciveness of the functional $I$ in $W_M$ and reflexivity of $W_M$ imply $\{u_{n,M}\}_{n \geq 1}$ has a subsequence, which is also denoted by $\{u_{n,M}\}$ for convenience, such that $u_{n,M} \to u_M$ weakly in $W_M$ and thus $u_{n,M} \to u_M$ strongly in $L^r(\mathbb{R}^N, a)$ by Lemma 2.3. By the similar proof of [7, Theorem 3.3], we obtain $I(u_M) = \lim_{n \to \infty} I(u_{n,M}) = c_M > 0$.

Moreover, using the similar argument of [7, Corollary 3.4] with Theorem 3.3 and Lemma 3.4, we can show that problem (1.1) has at least two nontrivial nonnegative weak solutions in $W_M$ in which one has negative energy and another has positive energy if $\lambda > \lambda_M^*$. So Theorem 1.2 is proved.

4 PROOF OF THEOREM 1.4

Proof. Suppose that the three conditions (1)–(3) are satisfied and that $a(x), b(x), V(x) \in L^\infty(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}^N$. Let $u \in W$ be a weak solution of problem (1.1) and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Define $\tilde{u}(x) := u(x)$ for each $x \in \Omega$ and $\tilde{u}(x) := 0$ for each $x \in \mathbb{R}^N \setminus \Omega$. Then $(-\Delta)\tilde{u} = f(x, \tilde{u})$ for each $x \in \Omega$ with $f(x, \tilde{u}) = \lambda a(x)|\tilde{u}|^{r-2}\tilde{u} - b(x)|\tilde{u}|^{q-2}\tilde{u} - V(x)|\tilde{u}|^{p-2}\tilde{u}$ and $\|\tilde{u}\|_W \leq \|u\|_W < \infty$.

Given $k > 0$, $\mu \geq 1$ and $\nu > 0$, define $g_{\mu,\nu}(t) := t^{\mu}(t_k)^\nu$ with $t_k = \min\{t, k\}$ for all $t \geq 0$ and $g_{\mu,\nu}(t) := -g_{\mu,\nu}(-t)$ for all $t < 0$. We claim that

$$G_{\mu,\nu}(t) := \int_0^t \left( g_{\mu,\nu}(\tau) \right)^{\frac{\nu}{\mu + \nu}} d\tau \geq \frac{p(\mu + \nu)^{\frac{\nu}{\mu + \nu}}}{\mu + \nu + p - 1} g_{\nu+1,\nu}(t) \geq 0 \quad (4.1)$$

for all $t \geq 0$ and

$$G_{\mu,\nu}(t) := -G_{\mu,\nu}(-t) \leq \frac{p(\mu + \nu)^{\frac{\nu}{\mu + \nu}}}{\mu + \nu + p - 1} g_{\nu+1,\nu}(t) < 0 \quad (4.2)$$

for all $t < 0$.

Indeed, for each $t \geq 0$, if $t \leq k$, then $G_{\mu,\nu}(t) = t^{\mu+\nu}$ and so

$$G_{\mu,\nu}(t) = \int_0^t \left( (\mu + \nu) t^\mu \right)^{\frac{1}{\mu + \nu + 1}} d\tau = \frac{p(\mu + \nu)^{\frac{\nu}{\mu + \nu}}}{\mu + \nu + p - 1} t^{\frac{\nu+1}{\mu + \nu}} g_{\nu+1,\nu}(t).$$

If $t > k$, then

$$G_{\mu,\nu}(t) = \int_0^k \left( (\mu + \nu) t^\mu \right)^{\frac{1}{\mu + \nu + 1}} d\tau + \int_k^t k^\frac{1}{\mu + \nu + 1} d\tau$$

$$= \frac{p(\mu + \nu)^{\frac{\nu}{\mu + \nu}}}{\mu + \nu + p - 1} \left( k^{\nu+1,\nu} + k^{\frac{\nu}{\mu + \nu}} \right) + \frac{p\mu^\nu}{\mu + \nu + p - 1} \left( t^\nu - k^\nu \right).$$
Since \( \mu \geq 1, \nu > 0, \) and \( p \geq 1, \) one can check that \((\mu + \nu)(\mu + p - 1)^p \leq \mu(\mu + \nu + p - 1)^p, \) which implies \( \frac{\mu + p - 1}{\mu + \nu + p - 1} \leq \left( \frac{\mu}{\mu + \nu} \right)^\frac{1}{\nu}. \) Therefore,

\[
G_{\mu, \nu}(t) \geq \frac{p(\mu + \nu)^\frac{1}{\nu}}{\mu + \nu + p - 1} \cdot t^{\frac{\mu + p - 1}{\mu + \nu + p - 1}} = \frac{p(\mu + \nu)^\frac{1}{\nu}}{\mu + \nu + p - 1} g_{\mu, \nu}(t)
\]

if \( t > k. \) So (4.1) holds. Then (4.2) follows directly from (4.1) by definitions of \( g_{\mu, \nu}(t) \) and \( G_{\mu, \nu}(t). \)

Recall that \( \tilde{u} |_{\mathbb{R}^N \setminus \Omega} = 0 \) and \( |g_{\mu, \nu}(\tilde{u})| = |\tilde{u}|^\mu |\tilde{u}_k|^{\nu} \) with \( \tilde{u}_k = \min\{\tilde{u}, k\}. \) Now by [15, Lemma A.2] and Definition 1.1 with triangle inequality, we have

\[
\left[ G_{\mu, \nu}(\tilde{u}) \right]_{s, p}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|G_{\mu, \nu}(\tilde{u})(x) - G_{\mu, \nu}(\tilde{u})(y)|^p}{|x - y|^{N+p}} \, dx \, dy
\]

\[
\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+p}} \left( G_{\mu, \nu}(\tilde{u})(x) - g_{\mu, \nu}(\tilde{u})(y) \right) \, dx \, dy
\]

\[
= \langle (-\Delta)^{\frac{p}{2}}(\tilde{u}), g_{\mu, \nu}(\tilde{u}) \rangle
\]

\[
= \int_{\Omega} \lambda a(x)|\tilde{u}|^r \tilde{u} - V(x)|\tilde{u}|^{p-2} \tilde{u} - b(x)|\tilde{u}|^{q-2} \tilde{u} \rangle g_{\mu, \nu}(\tilde{u}) \, dx
\]

\[
\leq \int_{\Omega} \lambda a(x)|\tilde{u}|^r \tilde{u} \, dx + \int_{\Omega} V(x)|\tilde{u}|^{p-1} \tilde{u} \, dx + \int_{\Omega} b(x)|\tilde{u}|^{q-1} \tilde{u} \, dx.
\]

Applying [19, Theorem 1] or [17, Theorem 6.5] with (4.1)–(4.3) obtains

\[
\left( \int_{\Omega} \frac{p(\mu + \nu)^\frac{1}{\nu}}{\mu + \nu + p - 1} \left( g_{\mu, \nu}(\tilde{u}) \right)^{p'} \, dx \right)^\frac{p}{p'} \leq \left( \int_{\Omega} |G_{\mu, \nu}(\tilde{u})|^{p'} \, dx \right)^\frac{p}{p'} \leq \tilde{C} \left[ G_{\mu, \nu}(\tilde{u}) \right]_{s, p}^p \leq \tilde{C} \left( \int_{\Omega} \lambda a(x)|\tilde{u}|^r \tilde{u} \, dx + \int_{\Omega} V(x)|\tilde{u}|^{p-1} \tilde{u} \, dx + \int_{\Omega} b(x)|\tilde{u}|^{q-1} \tilde{u} \, dx \right),
\]

where \( \tilde{C} \) depends only on \( N, p, \) and \( s, \) which implies

\[
\left( \int_{\Omega} \frac{1}{|\tilde{u}|^{r \frac{p-1}{p}}} |\tilde{u}_k|^{\frac{p}{p'}} \, dx \right)^\frac{p}{p'} \leq C_0 \left( \int_{\Omega} \lambda a(x)|\tilde{u}|^r \tilde{u} \, dx + \int_{\Omega} V(x)|\tilde{u}|^{p-1} \tilde{u} \, dx + \int_{\Omega} b(x)|\tilde{u}|^{q-1} \tilde{u} \, dx \right)
\]

with \( C_0 = \frac{(\mu+p-1)^{p'}}{p'(\mu+p)} \tilde{C} > 0. \)

Since \( |\tilde{u}|_{s, p} \leq \| \tilde{u} \|_W < \infty, \) again by [19, Theorem 1] or [17, Theorem 6.5], we get \( \tilde{u} \in L^{p'}(\Omega), \) and then by \( 1 < p < r < \min\{q, p'_s\}, \) we have \( \tilde{u} \in L^{p+1}(\Omega), L^{p'}(\Omega), L^{p+1}(\Omega) \) if \( 1 \leq \mu \leq 1 + p'_s - \min\{q, p'_s\}. \) Thus, for each \( 1 \leq \mu \leq 1 + p'_s - \min\{q, p'_s\}, \) there must exist a positive number \( K_0 \geq 1 \) such that

\[
\int_{\Omega \cap \{|\tilde{u}| \geq K_0\}} |\tilde{u}|^{p+1} \, dx \leq \int_{\Omega \cap \{|\tilde{u}| \geq K_0\}} |\tilde{u}|^{r+1} \, dx
\]

\[
\leq \int_{\Omega \cap \{|\tilde{u}| \geq K_0\}} |\tilde{u}|^{q+1} \, dx
\]

\[
\leq \left( 4C_0 \left( \lambda \|a\|_{L^\infty(\Omega)} + \|V\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} \right) \right)^\frac{p}{p'},
\]
which, together with Hölder's inequality and (2.3), implies

\[
\begin{align*}
\int_{\Omega} \lambda a(x) \vert \tilde{u} \vert^{r+\mu-1} \vert \tilde{u}_k \vert^r \, dx & = \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \lambda a(x) \vert \tilde{u} \vert^{r+\mu-1} \vert \tilde{u}_k \vert^r \, dx + \int_{\Omega \cap \{ \vert \tilde{u} \vert < K_0 \}} \lambda a(x) \vert \tilde{u} \vert^{r+\mu-1} \vert \tilde{u}_k \vert^r \, dx \\
& \leq \lambda \| a \|_{L^\infty(\Omega)} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u} \vert^{r+\mu-1} \, dx \right)^{\frac{r}{\mu}} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u}_k \vert^r \, dx \right)^{\frac{r}{\mu}} \\
& \quad + K_0^{\mu+1} \lambda \int_{\Omega} a(x) \vert \tilde{u} \vert^r \, dx \\
& \leq \frac{1}{4C_0} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u} \vert^{r+\mu-1} \vert \tilde{u}_k \vert^r \, dx \right)^{\frac{r}{\mu}} + K_0^{\mu+1} \int_{\Omega} a(x) \frac{1}{b(x)^r} \, dx \left( \| u \|_{W^1}^r \right)^{\frac{r}{p}} \\
& \quad + K_0^{\mu+1} \int_{\Omega} a(x) \frac{1}{b(x)^r} \, dx \left( \| u \|_{W^1}^r \right)^{\frac{r}{p}} \tag{4.7}
\end{align*}
\]

with

\[
\begin{align*}
\int_{\Omega} V(x) \vert \tilde{u} \vert^{p+\mu-1} \vert \tilde{u}_k \vert^r \, dx & = \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} V(x) \vert \tilde{u} \vert^{p+\mu-1} \vert \tilde{u}_k \vert^r \, dx + \int_{\Omega \cap \{ \vert \tilde{u} \vert < K_0 \}} V(x) \vert \tilde{u} \vert^{p+\mu-1} \vert \tilde{u}_k \vert^r \, dx \\
& \leq \| V \|_{L^\infty(\Omega)} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u} \vert^{p+\mu-1} \, dx \right)^{\frac{r}{p+\mu-1}} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u}_k \vert^r \, dx \right)^{\frac{r}{p+\mu-1}} \\
& \quad + K_0^{\mu+1} \int_{\Omega} V(x) \vert \tilde{u} \vert^p \, dx \\
& \leq \frac{1}{4C_0} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u} \vert^{p+\mu-1} \vert \tilde{u}_k \vert^r \, dx \right)^{\frac{r}{p+\mu-1}} + K_0^{\mu+1} \| u \|_{W^1}^p \tag{4.8}
\end{align*}
\]

and

\[
\begin{align*}
\int_{\Omega} b(x) \vert \tilde{u} \vert^{q+\mu-1} \vert \tilde{u}_k \vert^r \, dx & = \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} b(x) \vert \tilde{u} \vert^{q+\mu-1} \vert \tilde{u}_k \vert^r \, dx + \int_{\Omega \cap \{ \vert \tilde{u} \vert < K_0 \}} b(x) \vert \tilde{u} \vert^{q+\mu-1} \vert \tilde{u}_k \vert^r \, dx \\
& \leq \| b \|_{L^\infty(\Omega)} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u} \vert^{q+\mu-1} \, dx \right)^{\frac{r}{q+\mu-1}} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u}_k \vert^r \, dx \right)^{\frac{r}{q+\mu-1}} \\
& \quad + K_0^{\mu+1} \int_{\Omega} b(x) \vert \tilde{u} \vert^q \, dx \\
& \leq \frac{1}{4C_0} \left( \int_{\Omega \cap \{ \vert \tilde{u} \vert \geq K_0 \}} \vert \tilde{u} \vert^{q+\mu-1} \vert \tilde{u}_k \vert^r \, dx \right)^{\frac{r}{q+\mu-1}} + K_0^{\mu+1} \| u \|_{W^1}^q \tag{4.9}
\end{align*}
\]

Next, we consider the following two cases to check that \( f(x, \tilde{u}) \in L^{\tilde{q}}(\Omega) \) for \( \tilde{q} > \frac{N}{p} \).

Case 1. If \( q \leq p_s^* \), we take \( \mu = 1 + p_s^* - q \) such that \( p + \mu - 1 < r + \mu - 1 < q + \mu - 1 \leq \frac{(\mu+1)p^*}{p} \). Combining (4.5) and (4.7)–(4.9) yields

\[
\left( \int_{\Omega} \frac{\lambda a(x) \vert \tilde{u} \vert^{(r+\mu-1)p^*}}{r \vert \tilde{u}_k \vert^r} \, dx \right)^{\frac{p}{p^*}} \leq 4C_0 K_0^{\mu+1} \left( \lambda C_1 \left( \int_{\mathbb{R}^N} \frac{a(x)^{\frac{1}{p}}}{b(x)^r} \, dx \right)^{\frac{r}{p}} \right) \left( \| u \|_{W^1}^r + \| u \|_{W^1}^p + \| u \|_{W^1}^q \right)
\]

and then letting \( k \to \infty \) obtains

\[
\left( \int_{\Omega} \frac{\lambda a(x) \vert \tilde{u} \vert^{r+\mu-1} \vert \tilde{u}_k \vert^r}{r} \, dx \right)^{\frac{p}{p^*}} < \infty \tag{4.10}
\]

for all \( \nu > 0 \). Recall that \( 1 < p < r < q \leq p_s^* \) and \( f(x, \tilde{u}) = \lambda a(x)(\tilde{u}^{r-2} \tilde{u}_k - \tilde{u}) \tilde{u} + b(x) \tilde{u}^{q-2} \tilde{u} - V(x) \tilde{u}^{p-2} \tilde{u} \). Since \( \mu + \nu + p - 1 = p_s^* - q + \nu + p > p \), it can be easily verified that \( f(x, \tilde{u}) \in L^{\tilde{q}}(\Omega) \) for \( \tilde{q} > \frac{N}{p} \) by substituting some proper values of \( \nu \) into (4.10).
Case 2. If \( q > p_s^* \), we take \( \mu = 1 + \frac{p(q - p)}{p - p} \) such that
\[
p + \mu - 1 < r + \mu - 1 < q + \mu - 1 = \frac{p_s^*(q - p)}{p_s^* - p} = \frac{(\mu + p - 1)p_s^*}{p}.
\]

Using the assumption that \( u \in W \cap L_{loc}^{p(q-p)}(\mathbb{R}^N) \) and repeating the argument of (4.6)–(4.9), we can also get the same result (4.10) and thus \( f(x, \tilde{u}) \in L^q(\Omega) \) for \( \tilde{q} > \frac{N}{p} \).

Therefore, by the arbitrariness of \( \Omega \), we can deduce from [15, Theorem 3.1] and [15, Theorem 3.13] that any weak solution in \( W \) to problem (1.1) is continuous in \( \mathbb{R}^N \) if \( q \leq p_s^* \) and that any weak solution in \( W \cap L_{loc}^{p(q-p)}(\mathbb{R}^N) \) to problem (1.1) is continuous in \( \mathbb{R}^N \) if \( q > p_s^* \), which proves Theorem 1.4.

\[ \Box \]

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APPENDIX A

Now we correct the proofs of both [8, Lemma 10] and [6, Lemma A.6] for $1 < p < 2$.

**Lemma 5.1.** The set $E = (E, \| \cdot \|_E)$ is a uniformly convex Banach space.

**Proof.** Since it has been proved in [8, Lemma 10] and [6, Lemma A.6] that $E$ is a Banach space for $p > 1$ and $E$ is uniformly convex when $p \geq 2$, we only need to show that $E$ is uniformly convex when $1 < p < 2$. Recall that $\|u\|_E = (\|u\|_{s,p}^p + \|u\|_{p,V}^p)^{\frac{1}{p}}$ and denote

$$W^{s,p} (\mathbb{R}^N) = \{ u \in L^p (\mathbb{R}^N) : [u]_{s,p} < \infty \},$$

$$L^p (\mathbb{R}^N, V) = \{ u \in L^p (\mathbb{R}^N) : \|u\|_{p,V} < \infty \}.$$

We first show that both $(W^{s,p} (\mathbb{R}^N), [\cdot]_{s,p})$ and $(L^p (\mathbb{R}^N, V), \| \cdot \|_{p,V})$ are uniformly convex.

Since $1 < p < 2$, by [20, Lemma 2.13] and [20, Lemma 2.37] for any $u, v \in W^{s,p} (\mathbb{R}^N)$, we have

$$\left[ \frac{u + v}{2} \right]_{s,p}^{p'} + \left[ \frac{u - v}{2} \right]_{s,p}^{p'} = \left\| \left( \frac{|u(x) + v(x)|}{2} - \frac{|u(y) + v(y)|}{2} \right) \frac{|x - y|^{-N-p}}{p'} \right\|_{L^{p'}(\mathbb{R}^N)}$$

$$+ \left\| \left( \frac{|u(x) - v(x)|}{2} - \frac{|u(y) - v(y)|}{2} \right) \frac{|x - y|^{-N-p}}{p'} \right\|_{L^{p'}(\mathbb{R}^N)}$$

$$\leq |x - y|^{-N-p} \left( \left( \frac{|u(x) + v(x)|}{2} - \frac{|u(y) + v(y)|}{2} \right) \frac{1}{p'} + \left( \frac{|u(x) - v(x)|}{2} - \frac{|u(y) - v(y)|}{2} \right) \frac{1}{p'} \right) $$

$$\leq |x - y|^{-N-p} \left( \frac{1}{2} |u(x) - u(y)|^p + \frac{1}{2} |v(x) - v(y)|^p \right) \frac{1}{p'}$$

$$= \left( \frac{1}{2} [u]_{s,p}^p + \frac{1}{2} [v]_{s,p}^p \right) \frac{1}{p'}.$$

So for each $\epsilon \in (0, 2]$, by (A1), there exists $\delta_1 = 1 - (1 - (\epsilon/2)^{p'})^{1/p'} > 0$ such that if $[u]_{s,p} = [v]_{s,p} = 1$ and $[u - v]_{s,p} \geq \epsilon$, then $[u]_{s,p}^p + \|u\|_{p,V}^p = \|u\|_{E}^p = 1$ and $[u - v]_{s,p}^p + \|u - v\|_{p,V}^p = \|u - v\|_{E}^p \geq \epsilon$. Similarly, it can be proved that $(L^p (\mathbb{R}^N, V), \| \cdot \|_{p,V})$ is also uniformly convex by repeating the above argument.

Now for any given $\epsilon \in (0, 2]$ we take $u, v \in E$ with $\|u\|_E = \|v\|_E = 1$ and $\|u - v\|_E \geq \epsilon$, then

$$\|u\|_{s,p}^p + \|u\|_{p,V}^p = \|u\|_E^p = 1,$$

$$\|v\|_{s,p}^p + \|v\|_{p,V}^p = \|v\|_E^p = 1,$$

$$[u - v]_{s,p}^p + \|u - v\|_{p,V}^p = \|u - v\|_E^p \geq \epsilon^p.$$
Without loss of generality, we may assume that $[u-v]_{k,p}^p \geq \epsilon^p/2$, that is, $[u-v]_{k,p} \geq \epsilon/2^{1/p}$. Next we show by contradiction that there exists $\delta_2 > 0$ depending on $\epsilon$ such that

$$\left[ \frac{u+v}{2} \right]_{s,p}^p \leq \frac{1-\delta_2}{2} \left( [u]_{k,p}^p + [v]_{k,p}^p \right). \quad (A2)$$

Note that $[u]_{k,p} \leq 1$ and $[v]_{k,p} \leq 1$, we have two cases for the proof of (A2) as follows.

Case 1. When $[u]_{k,p} = 1$ and $[v]_{k,p} \leq 1$, suppose that (A2) is false, and then there must exist an $\epsilon_0 > 0$ and two sequences $\{u_k\}$ and $\{v_k\}$ in $E$ with $[u_k]_{k,p} = 1, [v_k]_{k,p} \leq 1$ and $[u_k - v_k]_{k,p} \geq \epsilon_0/2^{1/p}$ such that

$$\left[ \frac{u_k + v_k}{2} \right]_{s,p}^p \geq \frac{1}{2} \left( 1 - \frac{1}{k} \right) \left( [u_k]_{k,p}^p + [v_k]_{k,p}^p \right). \quad (A3)$$

We claim that $\lim_{k \to \infty} [v_k]_{k,p} = 1$. Otherwise, one can take a subsequence $\{v_k\} \subset \{v_k\}$ such that $[v_k]_{k,p} \leq A < 1$ and apply triangle inequality to obtain

$$\left[ \frac{u_k + v_k}{2} \right]_{s,p}^p \leq \frac{1}{2p} \left( 1 + [v_k]_{k,p} \right)^p \leq \frac{[u_k]_{k,p}^p + [v_k]_{k,p}^p}{2} \cdot \left( \frac{1+A}{2} \right)^p / \left( \frac{1 + A^p}{2} \right), \quad (A4)$$

which contradicts (A3) since $\left( \frac{1+A}{2} \right)^p / \left( \frac{1 + A^p}{2} \right) < 1$, so the claim holds.

Denote $w_k = \frac{v_k}{[v_k]_{k,p}}$, then $\lim_{k \to \infty} [v_k - w_k]_{k,p} = 0$. This, together with (A3) and $\lim_{k \to \infty} [v_k]_{k,p} = 1$, yields

$$1 = \lim_{k \to \infty} \left[ \frac{u_k + v_k}{2} \right]_{s,p} \leq \lim_{k \to \infty} \left[ \frac{u_k + w_k}{2} \right]_{s,p} \leq 1,$$

and thus $\lim_{k \to \infty} \left[ \frac{u_k + w_k}{2} \right]_{k,p} = 1$. However, by $[u_k - v_k]_{k,p} \geq \epsilon_0/2^{1/p}$ for all $k \geq 1$, we can get an integer $k_0$ such that $[u_k - v_k]_{k,p} \geq \frac{\epsilon_0}{2^{1/p}}$ for each $k \geq k_0$, and thus $\left[ \frac{u_k + w_k}{2} \right]_{k,p} \leq 1 - \delta_3$ by definition of uniform convexity, where $\delta_3 > 0$ depends on $\epsilon_0$, which is a contradiction to $\lim_{k \to \infty} \left[ \frac{u_k + w_k}{2} \right]_{k,p} = 1$. Hence, (A2) follows.

Case 2. When $[u]_{k,p} \leq 1$ and $[v]_{k,p} \leq 1$, without loss of generality we may assume $[u]_{k,p} \geq [v]_{k,p} > 0$. Denote $\bar{u} = \frac{u}{[u]_{k,p}}, \bar{v} = \frac{v}{[v]_{k,p}}$, then $[\bar{u}]_{k,p} = 1, [\bar{v}]_{k,p} \leq 1$ and $[\bar{u} - \bar{v}]_{k,p} \geq \epsilon/2^{1/p}$ in that $[u-v]_{k,p} \geq \epsilon/2^{1/p}$ for the above $\epsilon$. By the result of Case 1, the inequality (A2) holds for $\bar{u}$ and $\bar{v}$, and so (A2) follows for $u$ and $v$.

Therefore, noting $\frac{[u]_{k,p}^p + [v]_{k,p}^p}{2} \geq \left( \frac{u-v}{2} \right)_{s,p}^p \geq \frac{\epsilon^p}{2^{2s+1}}$ and applying (A2) with $(a+b)^p \leq 2^p-1(a^p + b^p)$ for all $a, b > 0$ and $p \geq 1$ obtain

$$\left\| \frac{u+v}{2} \right\|_E = \left( \left\| \frac{u+v}{2} \right\|_{s,p}^p + \left\| \frac{u+v}{2} \right\|_{p,v}^p \right)^{\frac{1}{p}} \leq \left( 1 - \delta_2 \right) \left( \frac{[u]_{k,p}^p + [v]_{k,p}^p}{2} \right)^{\frac{1}{p}} \frac{1}{2} + \frac{\left\| u \right\|_{p,v}^p + \left\| v \right\|_{p,v}^p}{2} \leq \left( 1 - \delta_2 \right)^{\frac{1}{p}} \frac{\epsilon^p}{2^{2p+1}} = 1 - \delta$$

with $\delta = 1 - \left( 1 - \delta_2 \frac{\epsilon^p}{2^{2p+1}} \right)^{1/p} > 0$, which proves $E$ is uniformly convex when $1 < p < 2$. □