Controlling the non-linear intracavity dynamics of large He–Ne laser gyroscopes

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Abstract
A model based on Lamb’s theory of gas lasers is applied to a He–Ne ring laser (RL) gyroscope to estimate and remove the laser dynamics contribution from the rotation measurements. The intensities of the counter-propagating laser beams exiting one cavity mirror are continuously observed together with a monitor of the laser population inversion. These observables, once properly calibrated with a dedicated procedure, allow us to estimate cold cavity and active medium parameters driving the main part of the non-linearities of the system. The quantitative estimation of intrinsic non-reciprocal effects due to cavity and active medium non-linear coupling plays a key role in testing fundamental symmetries of space–time with RLs. The parameter identification and noise subtraction procedure has been verified by means of a Monte Carlo study of the system, and experimentally tested on the G-PISA RL oriented with the normal to the ring plane almost parallel to the Earth’s rotation axis. In this configuration the Earth’s rotation rate provides the maximum Sagnac effect while the contribution of the orientation error is reduced to a minimum. After the subtraction of laser dynamics by a Kalman filter, the relative systematic errors of G-PISA reduce from 50 to 5 parts in 103 and can be attributed to the residual uncertainties on geometrical scale factor and orientation of the ring.

Keywords: laser gyroscopes, Kalman filter, gas laser theory, GINGER
(Some figures may appear in colour only in the online journal)

1. Introduction

Modern ring laser (RL) gyroscopes based on the Sagnac effect are standard inertial sensors of rotation rates with many applications, ranging from inertial guidance [1] to angle metrology [2], geodesy [3, 4] and geophysics [4, 5]. Their application to general relativity tests has also been recently considered [6] as in the GINGER proposal (Gyroscopes In General Relativity), an underground experiment on the detection of the dragging of the inertial frame (Lense–Thirring effect) due to the Earth’s rotation5. A RL ‘gyro’ consists of two counter-propagating electromagnetic waves along a polygonal closed path that acts as a resonant cavity. The theory of the Sagnac effect predicts that the resonance frequencies for the two counter-propagating modes inside a RL cavity, rotating with respect to an inertial frame, differ by

$$\omega_s = \frac{8\pi A}{\lambda L} \omega_r + \Delta \omega + \Delta \omega_0,$$

where \(\lambda\) is the wavelength of the laser beam, \(\omega_r = n \cdot \Omega\) is the projection of the angular velocity of the cavity \(\Omega\) (relative to an inertial frame) on the normal vector \(n\) of the plane of the ring cavity, and \(A\) and \(L\) are its area and perimeter, respectively. Laser non-linearities, cavity non-reciprocities and backscattering of light on cavity mirrors require the application of some corrections to the ideal case of equation (1) and we have to generalize it with the effective formula [7, 8]

$$\omega_s = \frac{8\pi A}{\lambda L} (1 + \delta_s) \omega_r + \Delta \omega + \Delta \omega_0,$$
where \( \Delta \omega \) gives the null shift error from the zero value of \( \Omega \), \( \Delta \omega \) accounts for the non-linear contributions from laser dynamics and two-beam coupling and \( \delta \) represents the scale factor modifications produced by fluctuations of temperature, pressure and gain of the plasma, as well as by fluctuations of losses and laser frequency. An exhaustive treatment of gas laser phenomena is given by Lamb in [9]. He described the coupling between the laser beams and the atomic polarization at the third-order expansion in the interaction field, introducing a system of differential equations ruled by a set of parameters known in the subsequent literature as ‘Lamb parameters’.

The accurate knowledge of scale factor and null shift is of paramount importance in tests of fundamental physics with RLs, e.g. breaking of local Lorentz invariance, axion detection, Lense–Thirring frame dragging, etc [10]. In fact, RLs have the potential to detect time reversal violating signals with extraordinarily high sensitivity [11]. In this paper, we demonstrate a considerable improvement of large RLs in sensing such signals by measuring laser induced non-reciprocities with an accuracy of 1 part in 10^7.

Another motivation for our work is to foresee a calibration procedure for large RLs that exploits the non-linearity of laser active medium without the need for rotating the apparatus through a calibrated angle. The possibility of performing a calibration without knowing the input signal is a typical feature of non-linear systems with non-homogeneous input–output relations. With the separation of the identification of cavity losses and active medium parameters we have, on the one hand, that we can increase the time stability of a RL with Kalman filtering by subtracting the backscattering. On the other hand, with the characterization of laser active medium (single pass gain and plasma dispersion function), we can achieve a good accuracy of the rotation estimate. In this respect, we already developed [12] an algorithm for the identification of the Lamb parameters for excess gain minus losses \( \alpha_{1,2} \), backscattering amplitude \( r_{1,2} \) and phases \( \epsilon_{1,2} \), that are associated with cavity dissipation. After providing a raw estimate of the laser medium gain, we run an extended Kalman filter that is able to remove a fraction of the backscattering induced drift from measurements of the Earth’s rotation rate. Thus we improved the long-term stability of the ring G-PISA [12–14] and demonstrated the effectiveness of non-linear Kalman filtering in removing backscattering effects. In this paper we extend our previous analysis by presenting a novel identification and calibration procedure for estimating the full set of Lamb parameters, i.e. \( \alpha_{1,2}, r_{1,2}, \epsilon_{1,2}, \) self- and cross-saturation coefficients \( \beta_{1,2} \) and \( \theta_{12,21} \), and scale factor and null shift errors \( \sigma_{1,2} \) and \( \tau_{12,21} \), which depend on their turn on cavity losses \( \mu_{1,2} \), plasma polarizability and single pass gain \( G \) of the laser medium.

The paper is organized as follows. In section 2 we report the basic equations of the RL dynamics. With the help of a Monte Carlo simulation, we show in section 3 the required bound on relative errors of Lamb parameters as a function of the accuracy goal in the estimation of rotation rate. The experimental apparatus of G-PISA and the calibration setup are presented in section 4. In section 5 we apply our calibration procedure to G-PISA data and show the results. In section 6 conclusions are drawn, and limitations of calibration and identification procedures, including calibration goals for application of RL to fundamental physics, are discussed.

2. RL dynamics and estimation of cold cavity parameters

To study the RL dynamics we refer to the self-consistent equations derived by Aronowitz [7] starting from Lamb’s third-order expansion of the polarization vector in powers of the electric field amplitude. The standard equations of a RL with backscattering can be conveniently written using the complex representation of the optical fields, expressed in Lamb units.

\[
\begin{align*}
\dot{E}_1(t) &= \left[ A_1 - B_1|E_1(t)|^2 - C_{11}|E_2(t)|^2 \right] E_1(t) + \mathcal{R}_1 E_2 \\
\dot{E}_2(t) &= \left[ A_2 - B_2|E_2(t)|^2 - C_{12}|E_1(t)|^2 \right] E_2(t) + \mathcal{R}_2 E_1,
\end{align*}
\]

where \( E_{1,2}(t) \) and \( \Omega_{1,2} \) are the complex amplitudes and frequencies of the clockwise (identified by subscript 1) and counter-clockwise (identified by subscript 2) waves propagating in the cavity, and the complex coefficients \( A_{1,2}, B_{1,2}, C_{11,12} \) and \( \mathcal{R}_{1,2} \) are related to the Lamb parameters \( \alpha_{1,2}, \sigma_{1,2}, \beta_{1,2}, \theta_{12,21}, \tau_{12,21}, r_{1,2} \) and \( \epsilon_{1,2} \) by

\[
\begin{align*}
A_{1,2} &= \frac{c}{L} \alpha_{1,2} + i \left( \Omega_{1,2} + \frac{c}{L} \sigma_{1,2} \right) \\
B_{1,2} &= \frac{c}{L} \beta_{1,2} \\
C_{21,12} &= \frac{c}{L} \left( \theta_{21,12} - i \tau_{21,12} \right) \\
\mathcal{R}_{1,2} &= \frac{c}{L} r_{1,2} e^{i \epsilon_{1,2}}.
\end{align*}
\]

By introducing the two-dimensional complex-valued column vector \( E(t) = (E_1, E_2) \), equations (3) can be written as

\[
\dot{E} = \left( A - \mathcal{D}(E)B \right) \mathcal{D}(E^*) E,
\]

where we have defined the complex-valued matrices

\[
A = \begin{pmatrix} A_1 & \mathcal{R}_2 \\ \mathcal{R}_1 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & C_{21} \\ C_{12} & B_2 \end{pmatrix},
\]

and

\[
\mathcal{D}(E) = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}.
\]

In standard RL operation the rotational frequency is much smaller than the cavity mode spacing and we have \( B_1 \approx B_2 = B, C_{21} \approx C_{12} = C \). We will use these approximations throughout the paper when we neglect the subscript 1, 2 in the corresponding Lamb parameters. For instance, the relative difference is of the order of \( 10^{-6} \) for large RLs biased by Earth’s rotation [15].

It is worth noticing that analytical solutions of equations (5) have been found in the case of \( |E_1| \ll |E_2| \) (Adler solution) [16], \( C_{21} = C_{12} \) and \( \mathcal{R}_1 = \mathcal{R}_2 \) [17] or \( \text{Re}[C_{21}] = \text{Re}[C_{12}] = B_1 = B_2 \) [18]. However, general analytic solutions of the RL equations system with non-reciprocal parameters are not known, and therefore perturbative solutions have to be used [12].

By introducing the new coordinates \( I = (E_1^* E_1, E_2^* E_2)^T \), and \( X = \log(E_1/E_2) \), equations (5) can be written as

\[
\begin{align*}
\frac{d}{dt} \log(I) &= 2\text{Re} \left[ \frac{A_1}{A_2} - BI + \frac{\mathcal{R}_2 e^{-X}}{\mathcal{R}_1 e^X} \right] \\
\frac{d}{dt} X &= A_1 - A_2 - (B - C)(I_1 - I_2) + \mathcal{R}_2 e^{-X} + \mathcal{R}_1 e^X,
\end{align*}
\]
which reduce to equations (6) of [12] after one identifies $I_1 = E_1^* E_1$, $I_2 = E_2^* E_2$, and further assumes that $C = 0$. In the new coordinates equations (8) are independent of the mean frequency of the electric field ($\Omega_1 + \Omega_2$)/2. In the reciprocal case (i.e., $A_1 = A_2$, $B_1 = B_2$, $C_1 = C_2$, and $R_1 = R_2$), equation (5) is clearly invariant under the transformation $(E_1, E_2, \Omega_1, \Omega_2) \rightarrow (E_2, E_1, \Omega_1, \Omega_2)$, and so the corresponding phase portrait is topologically equivalent to a torus [19]. The system exhibits two asymptotic time behaviours depending on whether or not the phases in the phase space can be continuously shrunk to a point. For behaviours of the first kind, equations (5) exhibit a fixed point, the beat frequency is equal to zero and the light intensities are constant (laser switched off or locked-in). Conversely, the behaviours of the second kind are limit cycle regimes, characterized by non-zero beat frequency of the counter-propagating waves (single mode laser operation). When the reciprocal conditions do not hold, the qualitative characteristics of asymptotic time behaviour are left unchanged, as a result of the invariance of phase portrait topology under continuous variation of RL parameters.

The RL equations depend on two sets of parameters with distinct physical origin: (i) cold cavity parameters associated with losses and (ii) active medium parameters associated with atomic polarizability. According to this classification, the RL matrices can be naturally written as

$$A \equiv \frac{c}{L} P^{(0)} - M,$$

$$B \equiv \frac{c}{L} P^{(2)},$$

where $P^{(0)}$ and $P^{(2)}$ are the zeroth and second order contributions to the gas mixture polarizability [8]:

$$P^{(0)} = \frac{G}{2} \begin{pmatrix} z^{(0)}(\xi_1) & 0 \\ 0 & z^{(0)}(\xi_2) \end{pmatrix},$$

$$P^{(2)} = \frac{G}{2} \begin{pmatrix} z^{(2)}(\xi_1) & z^{(2)}(\xi_1, \xi_2) \\ z^{(2)}(\xi_1, \xi_2) & z^{(2)}(\xi_2) \end{pmatrix},$$

and $M$ is the linear coupling matrix associated with cavity losses

$$M = \begin{pmatrix} \frac{c}{L} \mu_1 + i \omega_k & -\frac{c}{L} \tau_1 e^{i \omega_k} \\ \frac{c}{L} \tau_2 e^{i \omega_k} & \frac{c}{L} \mu_2 - i \omega_k \end{pmatrix},$$

where $G$ is the laser single pass gain, $\mu_{1,2}$ are the cavity losses and $z^{(0)}(\xi_{1,2}) = (a_{1,2} + \mu_{1,2} + i \sigma_{1,2})/G$, $z^{(2)}(\xi_{1,2}) = \beta_{1,2}/G$, $\xi_{1,2} = (\theta_{1,2} + i \tau_{1,2})/G$ are functions which depend on the normalized detuning $\xi_{1,2}$ of the optical frequencies to the cavity centre frequency, and will be explicitly calculated in the next section. Here we only recall that, in the case of large size RLSs sensing the Earth’s rotation, we have $\xi_{1} - \xi_{2} < 10^{-6}$, and so we can approximate the frequency detuning of each beam by its average $\xi = (\xi_1 + \xi_2)/2$. Consequently, cross- and self-saturation parameters can be assumed equal for the two beams, i.e., $\theta_{12} = \theta_{21} = \theta$, $\tau_{12} = \tau_{21} = \tau$ and $\beta_{1} = \beta_{2} = \beta$.

As mirror losses of an optical cavity are unpredictable, they must be identified in the RL outputs and tracked in time. To this aim, we assume for the moment that the first and second order polarizations $P^{(0)}$ and $P^{(2)}$ are given, and that the light intensities have already been calibrated in Lamb units (see section 3). The identification procedure of cold cavity parameters is based on the existence of a limit cycle that makes it possible to estimate the steady-state dynamical losses of a RL system, as asymptotic solutions are periodic with period $T = 2\pi/\omega$. In fact, from equation (8) we can construct the functional

$$J(I, X, M) \equiv \left\| \frac{d}{dt} \log(I) - 2\text{Re} \left[ \frac{A_1}{A_2} - BI + \left( \frac{R_2 e^{-X}}{R_1 e^X} \right) \right] \right\|_{L^2(T)},$$

where $\| \cdot \|_{L^2(T)} = \sqrt{\int_{0}^{T} (\cdot)^2 dt}$ is the norm in the Hilbert space of $L^2$ $T$-periodic signals, and search for its minimum value to derive statistics of parameter estimation. From the perturbative solutions in equations (9) of [12] approximated up to the first harmonic terms, we can write the steady-state intensities and complex phase difference of the electrical fields as

$$I_1(t) = I_1 + i_1 \sin(\omega t + \phi_1),$$

$$I_2(t) = I_2 + i_2 \sin(\omega t + \phi_2),$$

$$X(t) = \frac{1}{2} \log \left( \frac{I_1(t)}{I_2(t)} \right) + i\omega t,$$

where $I_{1,2}$, $i_{1,2}$ and $\phi_{1,2}$ are the intensity offsets, monobeam modulation amplitudes and phases, which can be readily measured from RL outputs [12]; it also follows that the backscattering phases can be estimated by $\hat{\xi}_1 = \phi_1$ and $\hat{\xi}_2 = \phi_2$ modulo a common initial phase. Thus, after substituting equations (14) into equations (13), we can estimate the cavity loss parameters $\mu_{1,2}$ and $\tau_{1,2}$ as

$$(\hat{\mu}_{1,2}, \hat{\tau}_{1,2}) = \arg\min_{\mu_{1,2}, \tau_{1,2}} \{ J(I, X, M) \}.$$
The accuracy of the Sagnac frequency estimation is dominated by systematic errors of the single pass gain. For instance, by biasing the relevant active medium parameters $\beta$, $\theta$ and $\tau$ of a relative error of $10^{-1}$, $10^{-2}$ and $10^{-3}$, the corresponding relative error in the Sagnac frequency turns out to be $10^{-4}$, $10^{-5}$ and $10^{-6}$, respectively.

3.2. Second-order calculation of plasma polarization

The complex-valued functions $z^{(0)}(\xi_1, \xi_2)$, $z^{(2)}(\xi_1, \xi_2)$ and $z^{(2)}(\xi_1, \xi_2)$ which allow us to estimate self- and cross-saturation parameters are rather common in plasma physics, and were calculated for the first time by Aronowitz with two counter-propagating laser beams [7, 20]. His model of plasma requires that the ratio $\eta$ between homogeneous $\gamma_{ab}$ and inhomogeneous $\Gamma$ broadening line width is $\eta \equiv \gamma_{ab}/\Gamma \ll 1$. For instance, in the experiment of [20], the typical value of $\eta$ is $\simeq 10^{-2}$.

For gas mixtures with pressure $4–8$ mbar and temperature $300–500$ K, as in G-PISA, we have instead $0.2 \leq \eta \leq 0.5$ [12], and so we must perform a more general calculation of the atomic polarization. However, we will follow the approach of Aronowitz in [7] for what concerns the series expansion in powers of the electric fields describing the interaction between radiation and atoms. Aronowitz showed that the complex polarization of the active medium in a gas He–Ne laser, expanded to the third order in the field amplitude, can be written in the following integral form:

$$P^{(3)}(E_{1,2}) = \frac{-2|\mu_{ab}|^2 E_{1,2}}{\gamma_{ab}} \int_0^\infty \chi_{1,2}^{(2)}(v) \rho^{(2)}(v, E_{1,2}) \, dv,$$

(16)

where $|\mu_{ab}|$ is the electric dipole moment between states $a$ and $b$, $\gamma_{ab}$ is the homogeneous line width, $v$ is the velocity of atoms, $\chi_{1,2}^{(2)}(v) = 1/(i\eta + \xi_1, \xi_2 \pm v/u)$ is the complex susceptibility, $u$ is the atomic mass constant, and

$$\rho^{(2)}(v, E_{1,2}) = \frac{N e^{-\frac{v}{\eta}}}{2\gamma_a \gamma_b \Gamma} \times \left(1 - I_1 \frac{1}{1 + (\xi_1 + v/u)^2} - \gamma_a + \gamma_b I_2 \frac{1}{1 + (\xi_2 - v/u)^2}\right).$$

(17)

3.1. Monte Carlo simulations

Firstly, we study the bias induced in the estimate of the Sagnac frequency by the approximations made in equation (15). We use the typical parameters of G-PISA [12] as given in table 1, and a step size of 0.2 ns to integrate equations (8). The parameters $\alpha_{1,2}$, $r_{1,2}$ and $\xi$ are simulated as independent random walks with starting value as in table 1, correlation time of half of the simulation length 300 s, and relative step size of $10^{-2}$. The initial value of the backscattering phase $\xi$ is assumed uniformly distributed in $[0, \pi/2]$. We find that the relative accuracy of the Sagnac frequency estimation is a few parts in $10^5$, with relative systematic errors of $10^{-4}$ and $10^{-2}$ on the identified parameters $\mu_{1,2}$ and $\tau_{1,2}$, respectively. The phase errors are $\sim 10^{-3}$ rad.

Moreover, the Monte Carlo simulations show that the precision and accuracy of the identification procedure increases with the dimension of the ring, as the values of Lamb parameters decrease linearly with the free spectral range; e.g. for a square ring with a side of $\simeq 1$ m, 5 m and 10 m we found that the relative frequency accuracy is a few parts in $10^5$, $10^3$ and $10^2$, respectively. Other Monte Carlo simulations have been run by appropriately biasing the intensity time series, to mimic a systematic error in their calibration in Lamb units or acquisition process. It results that the corresponding relative error in the Sagnac frequency estimation mainly depends on the values of $r_{1,2}$, $\tau$ and $\epsilon$. In particular, for the typical G-PISA parameters in table 1, the relative frequency error scales linearly with the intensity error, and it turns out to be $\simeq 1$ part in $10^4$ times the intensity relative error. Finally, we studied the effects of systematic errors of the laser active medium parameters. The end result of these Monte Carlo runs is that

| $c/L$ | $5.5 \times 10^3$ Hz |
| $\alpha_{1,2}$ | $\sim 10^{-6}$ |
| $\beta$ | $5 \times 10^{-5}$ |
| $\theta$ | $6.5 \times 10^{-6}$ |
| $r_{1,2}$ | $\sim 2 \times 10^{-7}$ |
| $\tau$ | $180$ rad s$^{-1}$ |
dispersion function, and are proportional to the spectral gain and dispersion profiles of the unsaturated active medium, respectively.

The expression for the atomic polarization up to the third-order approximation reads

\[
P^{(3)}(E_{1,2}) \equiv \chi(E_{1,2})e^{I_{1,2}}
\]

\[
= \sqrt{2\pi} AZ_1(0)
\]

\[
\gamma_{ab}y_{ab}y_{ab}
\]

\[
\left( z^{(0)}(\xi_{1,2}) + z^{(2)}(\xi_{1,2})I_{1,2} - z^{(2)}(\xi)I_{1,2} \right) E_{1,2},
\]

where \( \chi(E_{1,2}) \) is the cavity atomic polarizability, \( A = N|\mu_{ab}|^2/(6\Gamma) \).

\[
\begin{align*}
\left\{ z^{(0)}(\xi_{1,2}) &= \frac{Z_R(\xi_{1,2})}{Z_0(0)} \\
\left( 1 - 2\eta(\eta + i\xi_{1,2}) \right) \\
\frac{Z_R(\xi_{1,2})}{Z_0(0)} \\
\gamma_{1,2} &= \gamma_{ab} + \gamma_{ab} \eta \frac{Z_R(\xi) - Z_R(\xi)}{Z_R(\xi)} \\
\frac{Z_R(\xi) - Z_R(\xi)}{Z_R(\xi)}
\end{align*}
\]

(20)

and \( \xi_{1,2} = (\omega - \omega_{1,2})/\Gamma \) is the frequency detuning to the Doppler width ratio, \( Z_R = [Z_R(\xi_1) + Z_R(\xi_2)]/2, Z_I = [Z_I(\xi_1) + Z_I(\xi_2)]/2 \). Using equation (19), the expression of the right-hand sides of equations (11), (12) can be identified with the coefficients of \( z^{(0)}(\xi_{1,2}), z^{(2)}(\xi_{1,2}) \) and \( z^{(2)}(\xi_{1,2}) \) of equation (20), except for the constant laser single pass gain \( G \).

In our calculations we must take into account that usually a RL cavity is filled with a gas mixture of two neon isotopes. Thus the matrix elements of equation (11) must be substituted by

\[
\begin{align*}
\left\{ z^{(0)}(\xi_{1,2}) &= k'z^{(0)}(\xi_{1,2}) + k''z^{(0)}(\xi_{1,2}) \\
+ k''z^{(0)}(\xi_{1,2}) + k''z^{(0)}(\xi_{1,2}) \\
+ k''z^{(0)}(\xi_{1,2}) + k''z^{(0)}(\xi_{1,2}) \\
+ k''z^{(0)}(\xi_{1,2}) + k''z^{(0)}(\xi_{1,2}) \right. \\
\left. z^{(2)}(\xi_{1,2}) = k'z^{(2)}(\xi_{1,2}) + k''z^{(2)}(\xi_{1,2}) \right.
\end{align*}
\]

(21)

where the symbols ‘ and ” refer to the \(^{20}\)Ne and \(^{22}\)Ne isotopes, \( k' \) and \( k'' \) are the fractional amount of each isotope, and \( \xi_{1,2}, \xi'_{1,2} \) are the detuning to the centre frequency of each isotope. Practically, the values of \( \eta, \xi \) and \( k \) are rescaled by the square root of the ratio of the atomic mass of the two isotopes. As an example, we show in figure 1 the polarization of a 50–50 \(^{20}\)Ne–\(^{22}\)Ne gas mixture as the sum of contributions arising from each Ne isotope, according to equation (21).

### 3.3. Round trip losses

The spectroscopic technique known as ‘ring-down time’ (RDT) measurement allows us to estimate mirror losses from the impulse response of a linear system. In fact, from equation (5) with \( G = 0 \), we have

\[
E = ME,
\]

(22)

and so the system shows two exponential decays with a rate proportional to the total cavity losses. If we switch off the laser excitation at the time \( t = 0 \), the initial conditions are \( E(0) = \sqrt{U_I(0)e^{i\phi_i}}, \sqrt{U_J(0)e^{i\phi_j}} \), where \( I_{1,2}(0) \) are the initial intensities and \( \phi_{1,2} \) are the initial phases.

The solutions of equation (22) for the light intensities, expanded in a Taylor series of \( \omega \gg \omega_{1,2}, \mu_1, \mu_2 \) to the first order, read

\[
\begin{align*}
I_1(t) &= I_1(0)e^{-\mu_1 t} \\
I_2(t) &= I_2(0)e^{-\mu_2 t}
\end{align*}
\]

(23)

To measure light decay times, the experimental procedure consists in recording the RF discharge with a fast detector, (photomultiplier Hamamatsu H7827012) loaded on a 1 kΩ impedance, after a rapid switch-off of the RF discharge. The switching-off operation must be much faster than the laser decay time. In our setup we obtained a sufficiently rapid switch-off by grounding one of the two electrodes of the radio-frequency discharge by means of a mechanical switch. A validation of this technique is obtained by measuring the decay time of the plasma fluorescence, which is found to be of the order of a few microseconds. Finally, we performed an exponential fit of the collected data.

### 3.4. Calibration of intensities in Lamb units

To get accurate estimates of the Sagnac frequency, the light intensity input of the extended Kalman filter must be
calibrated in Lamb units [7]. To this aim, we propose an experimental method based on the observation of the birth of additional longitudinal modes of the laser while increasing the laser excitation power. This dynamical change is known as ‘multimode transition’, and has been widely studied in the literature, mainly for medium size and large size RL [15,21]. The value of the mean light intensity for the multimode transition expressed in Lamb units is commonly defined as the multimode threshold. Different calculations of the multimode threshold were proposed [15] which take into account only the plasma dispersion function, evaluated at frequencies of fundamental and higher order modes. To increase the accuracy of the multimode threshold it is convenient to account for cavity losses and backscattering in the balance of gained and lost photons for longitudinal lasing modes.

The threshold condition for multimode transition can be derived by solving equations (27): $G_{th} z_m^{(0)}(\xi) - \bar{\mu} = 2 I_{th} G_{th} z_m^{(2)}(\xi_m)$ \(\xi_m\) is the normalized detuning averaged over the beams $I_1$, $I_2$, $I_3$, $I_4$. The evolution of the system is still ruled by equations (24), provided that we substitute the $2 \times 2$ matrices $A$ and $B$ with the corresponding $4 \times 4$ matrices, calculated by means of the plasma dispersion function for four intensities (see the appendix).

We now consider the system in the initial configuration $(I_1, I_2, 0, 0)^T$ and look for the condition of the $m$th mode growth. The lower diagonal block of the Jacobian matrix of the four-mode dynamical system is given by

$$
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \alpha_3 - \theta_{31} I_1 - \theta_{32} I_2 & 0 \\
0 & 0 & 0 & \alpha_4 - \theta_{41} I_1 - \theta_{42} I_2
\end{pmatrix}
$$

where $\alpha_{3,4}$ are the gain minus losses of the clockwise and counter-clockwise $m$th mode, and $\theta_{31,41,32,42}$ are the cross-saturation coefficients between clockwise and counter-clockwise modes of the fundamental and $m$th modes. By the Lyapunov linearization theorem [22], if the eigenvalues of the above matrix lie in the strictly positive complex half-plane, the equilibrium point is unstable, and the new laser modes can start to grow. Therefore, a higher $m$th mode can be excited if $\alpha_3 - \theta_{31} I_1 - \theta_{32} I_2 > 0$ and $\alpha_4 - \theta_{41} I_1 - \theta_{42} I_2 > 0$. Since the frequency difference of the two counter-propagating beams is much smaller than the Doppler width (see the appendix), we have $\theta_{31} \sim \theta_{42} \sim \theta_{m1}, \theta_{32} \sim \theta_{41} \sim \theta_{mc}$. Thus, to derive a threshold condition for the multimode operation we can take the average of the two inequalities and write

$$\bar{\alpha}_m > \bar{\theta}_m \bar{T},$$

where $\bar{\alpha}_m = (\alpha_3 + \alpha_4)/2, \bar{T} = (I_1 + I_2)/2$ and $\bar{\theta}_m = \theta_{m1} + \theta_{mc}$, and so the threshold condition for the intensity $I_{th}$ is

$$\bar{\alpha}_m = \bar{\theta}_m I_{th}, \quad (26)$$

This condition is not sufficient to determine the multimode threshold because active laser parameters depend on the value of the single pass gain at the threshold $G_{th}$, which is also unknown. However, we can add to equation (26) a second equation representing the balance of the mean intensity for the fundamental modes, taking into account the average $(\bar{\mu}_1 + \bar{\mu}_2)/2$ in equation (15). Therefore, the system of equations in the variables $G_{th}$ and $I_{th}$ reads

$$
\begin{align*}
\left\{ \begin{array}{l}
G_{th} z_m^{(0)}(\xi) - \bar{\mu} = 2 I_{th} G_{th} z_m^{(2)}(\xi_m) \\
G_{th} z_m^{(0)}(\xi) - \bar{\mu} = I_{th} G_{th} \left[ z_m^{(0)}(\xi) + z_m^{(2)}(\xi) \left(1 + \frac{3 \delta_{l1}^2}{1 - \delta_{l1}^2}\right) \right]
\end{array} \right.
\end{align*}
$$

where $\xi_m$ is the normalized detuning averaged over the beams $I_3$ and $I_4$, $\delta_{l1} = (I_1 - I_2)/(I_1 + I_2)$, $\bar{\mu} = (\mu_1 + \mu_2)/2$, $z_m^{(0)}$, $z_m^{(2)}$ and $z_m^{(2)}$ are polarization contributions from the plasma dispersion function which are computed in the appendix. As $m$-modes are very close in frequency, their losses can be assumed with good approximation to be equal. The quantity $\delta_{l1}$ can be estimated from the acquired intensity channels $V_{1,2}$ as $\delta_{l1} = (V_1 - V_2)/(V_1 + V_2)$. It is worth noting that the measure of $\delta_{l1}$ is independent of multiplicative change of scale. The multimode threshold can be derived by solving equations (27): $I_{th} = \frac{\Delta z_m^{(0)}(\xi) - \bar{\mu}}{\Delta z_m^{(0)}(\xi) - \bar{\mu}}$. It is worth noting that the measure of $\delta_{l1}$ is independent of multiplicative change of scale. The multimode threshold can be derived by solving equations (27):

3.5. Gain monitor

The intensity of the plasma fluorescence line at 632.8 nm provides a good observable for monitoring the relative variations of the atomic population in the upper laser level. The calibration of the monitor signal is obtained by performing intensity steps in the neighbourhhod of the monomode working regime of the RL, exploiting the identification procedure described in equation (15), and a linear least squares fit. In fact, the second equation of the system in equation (27), representing the balance among gain, losses and mean intensities, holds for any value of $G$ and $\bar{T}$. By solving this equation for the variable $G$, and using $N$ measurements $\left\{ \bar{T}(n) \right\}$ and $\left\{ \delta_{l1}(n) \right\}$ $(n = 1, 2, 3, \ldots, N)$ in the monomode regime, we obtain $N$ estimates of the gain signal

$$G(n) = z_m^{(0)}(\xi) - \bar{T}(n) \left( \frac{\bar{\mu} + 3 \delta_{l1}(n)/2}{1 - \delta_{l1}^2} \right)$$

where the mean loss value $\bar{\mu}$ is supposed to be constant and equal to the mean of RDT estimation.
To account for the experimental setup, we can consider a simple linear measure model $G = aV_p + b$ for the gain monitor signal $V_p$, where the constants $a$ and $b$ have to be estimated by the linear least squares fit

$$\begin{align*}
\left(\hat{a} \quad \hat{b}\right) &= \arg\min_{a,b} \left\| \begin{pmatrix} V_p(1) & 1 \\ \vdots & \vdots \\ V_p(N) & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} - G \right\|_2^2,
\end{align*}$$

(30)

where $G = [G(1), \ldots, G(n)]$ is the vector of gain estimates. The estimated constants $\hat{a}$, $\hat{b}$ and $G_{th}$ allow us to monitor the laser single pass gain by acquiring the signal $V_p$ without affecting the continuous operation of a RL.

4. Experimental apparatus

4.1. Mechanical mounting

The RL G-PISA consists of a square optical cavity with a sidelength of 1.350 m. The four cavity mirrors are contained in a steel vacuum chamber entirely filled with a He–Ne gas mixture. Two optically transparent windows are mounted on each corner of the cavity and allow one to measure the eight beams emitted by the cavity. In this work the G-PISA cavity has been fixed to a granite table that, in its turn, is firmly attached to a special monument completed in 2012. The monument is made of reinforced concrete, is oriented towards the local North direction and allows one to hold the granite table with a tilt angle equal to the latitude of the laboratory. The concrete monument has two niches located on two opposite sides (one visible in figure 2) for the housing of seismic equipment (a broad-band seismometer and a tiltmeter). The positioning of the monument has been done using topographic references (angle with the north) and an inclinometer (angle with the local vertical). The error in the monument orientation was estimated to be less than $1^\circ$. In this configuration, the Sagnac frequency due to the Earth’s rotation is maximum and the rotation rate of the instrument is very stable in time. A picture of the G-PISA assembly, already oriented at the maximum signal, is shown in figure 2. This is the best condition for reducing the environmental disturbances, and for the systematic study of the laser dynamics and control.

4.2. Measurement setup

The optical setup is shown in figure 3. In the middle of one side of the cavity a Pyrex capillary is mounted, 4 mm in internal diameter and 150 mm in length. The capillary plays a double role: it acts as a diaphragm selecting the TEM$_{00}$ spatial mode and it gives the possibility to apply a radio-frequency electric field for the excitation of the He–Ne plasma. The RL G-PISA is operated with a gas mixture of 50% $^{20}$Ne and 50% $^{22}$Ne, at a total pressure of 7.5 mbar. The measurement of the Sagnac interferogram is obtained by combining the two output beams exiting one corner through an intensity beam splitter while the single beam intensities are directly detected at the output of the adjacent corner. The Sagnac signal and the single beams are detected with a large area (5.8 mm × 5.8 mm) Si photodiode (Hamamatsu S1227-66BR) followed by a transimpedance amplifier (FEMTO LCA-4K-1G) with a gain of $10^9 \Omega$ and bandwidth of 4 kHz. On the corner opposite to the one for the Sagnac detection an optical beat setup is mounted between the clockwise beam and the I$_2$ stabilized He–Ne reference laser. The beat is detected by an avalanche photodiode whose current is amplified by a transimpedance amplifier (FEMTO HCA-400M-5K-C) with a gain of 4 k$\Omega$ and a bandwidth of 400 MHz. During G-PISA operation, the detuning of the clockwise wave is kept constant by a perimeter stabilization loop [13, 14], acting on the position
of two opposite mirrors of the cavity. The RL frequency is locked in this way to the value where \( \text{Re}[z^{(0)}(\xi,\eta)] \) attains its maximum.

4.3. Diagnostic apparatus

To increase the level of precision and accuracy in the rotation-rate measurement, cold cavity and active medium parameters (neon atomic kinetic temperature, total homogeneous broadening and isotopic composition) should be directly measured on the experimental apparatus. In order to measure the ring-down time of both the clockwise and counterclockwise modes, the beams exiting the corner opposite to the one dedicated to the intensity monitor are detected by two fiber-coupled photomultipliers (Hamamatsu H7827012). Two diagnostics have been arranged for the estimation of the active medium parameters: a fluorescence monitor for the gain variations and a laser probe interrogating the plasma through the Pyrex capillary.

4.3.1. Population inversion monitor. In order to perform an on-line measurement of the laser gain, we coupled part of the plasma fluorescence to a multi-fibre bundle. The collected light containing all the spectral contribution of the He–Ne discharge is filtered by a line filter 1 nm wide around 632.8 nm and detected with a photodiode. The photocurrent is amplified with a transimpedance stage with a gain of 1 G. The voltage \( V_p \) of the photodiode is used as an optical monitor of the laser gain by recording the dependence of the output powers \( I_1 \) and \( I_2 \) on \( V_p \), after losses have been estimated.

4.3.2. Spectroscopic probe of the gain medium. Essential information about the gain medium can be extracted by observing the Doppler absorption of the plasma at 640.2 nm (the strongest closed optical transition of Ne). We set up a frequency tunable ECDL (extended cavity diode laser) crossing the He–Ne plasma through the Pyrex capillary [23]. From this measurement one can get a precise estimation of the Doppler broadening, as well as of the isotopic composition of the gas. An example of this measurement is given in figure 5, where a ‘standard’ He–Ne gas mixture has been used. The plasma temperature of G-PISA has been experimentally estimated as \( T_p = (360 \pm 12) \) K.

5. Results and discussion

We implement estimation and calibration routines for G-PISA with models and techniques described in sections 2 and 3, respectively. In addition, we also implemented a data quality criterion that discards large outliers due to electronic spikes.

In figure 4 we report the results of the RDT measurements; the losses for the two beams were estimated as \( \mu_1 = (1.136 \pm 0.02) \times 10^{-4} \), \( \mu_2 = (1.146 \pm 0.02) \times 10^{-4} \). Note that within the precision of the fit it is \( \mu_1 \sim \mu_2 \).

To get an estimation of the plasma temperature, pressure and isotopic concentration, we fit the normalized measures of a laser diode on the standard He–Ne gas mixture in figure 5 to the function \( z^{(0)}(\xi) \) in equation (20). To account for detuning uncertainties in the laser probe, we scaled the experimental abscissa so that \( \xi' = a'\xi + b' \), \( \xi'' = a''\xi + b'' \). The fit parameters \( a', b', \eta, k' \) and \( b', k'' \) are related to \(^{20}\text{Ne} \) and \(^{22}\text{Ne} \) isotopes, respectively; the remaining parameters of \(^{22}\text{Ne} \) isotope are calculated as \( a'' = \sqrt{22/20}a' \) and \( \eta'' = \sqrt{22/20}\eta \). From the fit results, we get \( T_{\text{Ne}} = (360 \pm 12) \) K.

Figure 6 shows the calibration line of the single pass gain as a function of \( V_p \), and the experimental data used for the linear fit.

The Lamb parameters for excess gain minus losses \( \alpha_{1,2} \) and self-saturation \( \beta \) can be calculated using the estimated laser single pass gain, the estimated losses, the plasma temperature, and a nominal value for the gas mixture pressure of 7.5 mbar. By equations (9) and (10) \( \alpha_{1,2} \sim 8.3 \times 10^{-7} \), \( \beta \sim 1.2 \times 10^{-4} \), in agreement with monomode laser operation.

Once the calibration of the experimental apparatus has been performed, we used the monobeam intensity offsets, modulation amplitudes and phases, and the gain monitor to estimate both cold cavity and active medium parameters.
Figure 5. Absorption profile of the closed optical transition in neon at 640.2 nm, allowing for the neon temperature estimation. The measurement is taken in typical operation conditions for a plasma of He–Ne standard mixture at 4.5 mbar. Using the standard Matlab fitting procedure and the function \(z(0)\), we get a reduced \(R\)-squared of 0.9947 and the fitting parameters \(a' = -4.9 \pm 0.1\), \(b' = -0.2 \pm 0.05\), \(\eta' = 0.27 \pm 0.02\), \(b'' = -2.44 \pm 0.05\) and \(k'/k'' = 0.11 \pm 0.03\).

Figure 6. Plot of the single pass gain \(G\) of G-PISA as a function of the gain monitor \(V_p\). Each point represents the average of 10 measurements and the corresponding error bar is their standard deviation. The linear fit gives \(\hat{a} = 9.87 \times 10^{-5}\), \(\hat{b} = 1.303 \times 10^{-4}\), \(\sigma_a = 1.57 \times 10^{-6}\) and \(\sigma_b = 2 \times 10^{-7}\).

The intensities \(I_{1,2}(t)\) and interferogram \(S(t) = |E_1 + E_2|^2\), sampled at 5 kHz, were collected in two days. We estimated the Sagnac frequency by means of the EKF [12], with deterministic dynamics given by equations (8), and with measure vector

\[
y = \begin{pmatrix} I_1(t) \\ I_2(t) \\ S(t) \end{pmatrix},
\]

where \(S(t) \simeq \sqrt{I_1(t)I_2(t)} \cos(\text{Im}(X(t)))\) neglecting the contribution \(I_1(t) + I_2(t)\). Finally, the performances of the EKF routine were compared with AR2, the standard frequency estimation algorithm for RL beat note.

Figure 7 shows the histograms of the AR2 and EKF estimates. Note that the EKF mean is shifted with respect to AR2 mean (effective removal of the frequency null shift), and that the EKF standard deviation is \(\sim 10\) times smaller than the standard deviation of the AR2 estimator. The G-PISA long-term stability and accuracy have thus been increased.

Moreover, in figure 8 we plot the Allan standard deviation of the AR2 and EKF estimates, and the expected Allan deviation curve of a frequency signal corrupted by shot noise.

Table 2 summarizes the systematic error contributions to rotation-rate accuracy of G-PISA. For middle size RL,
backscattering phenomena are the most important systematic error source [12]. The noise filtering algorithm developed in this work promises to be also effective in improving the long-term stability of larger RLs. For instance, we know from Monte Carlo simulations that to filter out a relative backscattering contribution of 1 part in 10^6 from the Sagnac frequency, the inaccuracy on the laser dynamics parameters should be of the order of 1 part in 10^5. Hardware improvements, on-line calibration procedures and Kalman filtering methods for the diagnostic apparatus could be implemented in order to reach this kind of precision and are currently under study.

6. Conclusions

We have thoroughly studied the RL dynamics in order to implement identification and calibration methods for the cold cavity and active medium parameters. The identification method is based on the first harmonic approximation of the steady-state solution of RL equations and the minimization of a quadratic functional over the Hilbert space of periodic signals. On the other hand, the dynamics of a laser with cavity detuning shows many monomode or multimode dynamical behaviours that can be exploited to get rid of systematic errors of Sagnac frequency estimation. The calibration method presented in this paper is based on the measure of the threshold of the multimode transition and the plasma dispersion function. The systematic errors associated with the latter measurements dominate over statistical errors, as plasma parameters depend also on RF discharge details, e.g. shape of the capillary discharge. However, the accuracy of the identified cold cavity parameters depends on the amount of losses and backscattering of light: higher quality mirrors lead to potentially higher accuracy of the estimated Sagnac frequency. Much work has still to be devoted to improving calibration procedures and estimation of non-reciprocities for testing fundamental physics with RLs. The problem of pushing calibration and identification methods to their intrinsic accuracy limit, with or without the addition of a calibrated rotation signal to the RL input, would deserve further investigation and it will be the topic of a forthcoming paper.

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Appendix. Calculation of cross-saturation

We address to the problem of calculating the cross-saturation coefficients \( z_{ms}^{(2)} (\xi) \) for two multimode counter-propagating waves \( E_{3,4} \), lasing at \( \xi_{3} = \xi_{1} + nc/L, n \in \mathbb{Z} \), that arise at multimode transition of a RL.

The polarization components for the waves \( E_{3,4} \) at multimode threshold are given by equation (16), provided that the subscript 1, 2 is substituted with 3, 4, and \( \rho^{(2)} (v, E_{1,2}) \) is replaced with the following expression:

\[
\rho^{(2)}_m (v, E_{1,2}) = \frac{N e^{-\frac{v^2}{2} \pi^2}}{2 \gamma_a \gamma_b \hbar^2} \left( 1 - \frac{\gamma_a + \gamma_b}{\gamma_{ab}} I_1 \right) \frac{1}{1 + (\xi_3 - f_m + v/u)^2} \left( \frac{1}{I_2} + \frac{1}{1 + (\xi_3 - f_m + v/u)^2} \right),
\]

where \( f_m = nc/(L \Gamma) \). The expression for the multimode atomic polarization reads

\[
\rho^{(3)}_m (E_{3,4}) = \frac{\sqrt{\pi}}{\gamma_a \gamma_b \gamma_{ab}} E_{3,4} z^{(0)} (\xi_{3,4})
\]

\[
- z_{ms}^{(2)} (\xi_{3,4}) I_{1,2} - z_{mc}^{(2)} (\xi) I_{2,1},
\]

where

\[
\begin{align*}
\xi_{ms}^{(2)} (\xi_{3,4}) &= 2 \eta^2 \gamma_a + \gamma_b \frac{Z (\xi_{3,4} + f_m - Z (\xi_{3,4})^*)}{\gamma_{ab}} \left( \frac{1}{f_m (f_m + \eta) Z} \right) Z (\xi_{3,4})^* \left( \frac{1}{Z} \right) Z (\xi_{3,4})^* \left( \frac{1}{Z} \right) \\
\xi_{mc}^{(2)} (\xi_{3,4}) &= - \eta \gamma_a + \gamma_b \frac{Z (\xi_{3,4} + f_m - Z (\xi_{3,4})^*)}{\gamma_{ab}} \left( \frac{1}{f_m (f_m + \eta) Z} \right) Z (\xi_{3,4})^* \left( \frac{1}{Z} \right) Z (\xi_{3,4})^* \left( \frac{1}{Z} \right)
\end{align*}
\]

In addition, for the cross-saturation coefficients between 1 and 2 modes of the fundamental and the mth mode, we have

\[
\begin{align*}
\theta_{31,42} &= \text{Re} \left[ z_{ms}^{(2)} (\xi_{3,4}) \right] \\
\theta_{32,41} &= \text{Re} \left[ z_{mc}^{(2)} (\xi_{3,4}) \right].
\end{align*}
\]

When more than one isotope is present in the gas mixture, one must modify equations (34) using the weighted average of each isotope contribution, as in equations (21). As a final remark, we note that, for large RL sensing the Earth’s rotation, the difference between the normalized detunings of each m-mode is very small, e.g. \(|z^{(0)} (\xi_3) - z^{(0)} (\xi_4)| \leq 10^{-6}\) for G-PISA. Consequently, the following approximation holds:

\[
\begin{align*}
z_{ms}^{(2)} (\xi_{3,4}) &\sim z_{ms}^{(2)} \left( \frac{\xi_3 + \xi_4}{2} \right) \\
z_{mc}^{(2)} (\xi_{3,4}) &\sim z_{mc}^{(2)} \left( \frac{\xi_3 + \xi_4}{2} \right).
\end{align*}
\]

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