MEMBERSHIP IN MOMENT CONES, QUIVER SEMI-INVAR IANTS, AND GENERIC SEMI-STABILITY FOR BIPARTITE QUIVERS

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ABSTRACT. Let \( Q \) be a bipartite quiver with vertex set \( Q_0 \) such that the number of arrows between any two source and sink vertices is constant. Let \( \beta = (\beta(x))_{x \in Q_0} \) be a dimension vector of \( Q \) with positive integer coordinates, and let \( \Delta(Q, \beta) \) be the moment cone associated to \((Q, \beta)\). We show that the membership problem for \( \Delta(Q, \beta) \) can be solved in strongly polynomial time.

As a key step in our approach, we first solve the polytopal problem for semi-invariants of \( Q \) and its flag-extensions. Specifically, let \( Q_\beta \) be the flag-extension of \( Q \) obtained by attaching a flag \( F(x) \) of length \( \beta(x) - 1 \) at every vertex \( x \) of \( Q \), and let \( \overline{\beta} \) be the extension of \( \beta \) to \( Q_\beta \) that takes values \( 1, \ldots, \beta(x) \) along the vertices of the flag \( F(x) \) for every vertex \( x \) of \( Q \). For an integral weight \( \overline{\sigma} \) of \( Q_\beta \), let \( K_{\overline{\sigma}} \) be the dimension of the space of semi-invariants of weight \( \overline{\sigma} \) on the representation space of \( \overline{\beta} \)-dimensional complex representations of \( Q_\beta \).

We show that \( K_{\overline{\sigma}} \) can be expressed as the number of lattice points of a certain hive-type polytope. This polytopal description together with Derksen-Weyman’s Saturation Theorem for quiver semi-invariants allows us to use Tardos’s algorithm to solve the membership problem for \( \Delta(Q, \beta) \) in strongly polynomial time. In particular, this yields a strongly polynomial time algorithm for solving the generic semi-stability problem for representations of \( Q \) and \( Q_\beta \).

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1. INTRODUCTION

Let \( Q \) be a general quiver with set of vertices \( Q_0 \) and set of arrows \( Q_1 \). For an arrow \( a \in Q_1 \), we denote its tail and head by \( ta \) and \( ha \), respectively. Let \( \beta = (\beta(x))_{x \in Q_0} \in \mathbb{N}^{Q_0} \).
be a sincere dimension vector of \( Q \) and let us consider the representation space of \( \beta \)-dimensional representations of \( Q \),

\[
\text{rep}(Q, \beta) := \prod_{a \in Q_1} \mathbb{C}^{\beta(\text{ha}) \times \beta(\text{ta})}.
\]

The base change group \( \text{GL}(\beta) := \prod_{x \in Q_0} \text{GL}(\beta(x)) \) acts on \( \text{rep}(Q, \beta) \) by simultaneous conjugation. This action gives rise to a rational convex polyhedral cone (see [Sja98]), which we refer to as the moment cone associated to \( (Q, \beta) \) (see also [Chi06]). It is defined as follows:

\[
\Delta(Q, \beta) := \left\{ (\lambda(x))_{x \in Q_0} \middle| \lambda(x) \text{ is a weakly decreasing sequence of } \beta(x) \text{ real numbers such that there exists } W \in \text{rep}(Q, \beta) \text{ with } \lambda(x) \text{ the spectrum of } \sum_{a \in Q_1} W(a)^* \cdot W(a) - \sum_{a \in Q_1} W(a) \cdot W(a)^* \text{ for all } x \in Q_0 \right\},
\]

where \( W(a)^* \in \mathbb{C}^{\beta(\text{ta}) \times \beta(\text{ha})} \) denotes the adjoint of the complex matrix \( W(a) \), i.e., \( W(a)^* \) is the transpose of the conjugate of \( W(a) \) for every \( a \in Q_1 \).

For example, when \( Q = \bullet \rightarrow \bullet \leftarrow \bullet \) and \( \beta = (n, n, n) \), the moment cone \( \Delta(Q, \beta) \) is essentially the Klyachko cone; see Example 5.1 for more details. In general, it can be realized as the cone over the moment polytope for the projectivization of \( \text{rep}(Q, \beta) \) (see [Sja98]). In the context of finite-dimensional unitary representations of compact Lie groups, it has been proved in [BCMW17] that the problem of deciding membership in the associated moment polytopes is in the complexity classes NP and co-NP.

In this paper, we address the membership problem in moment cones of the form \( \Delta(Q, \beta) \).

We show that for a large class of bipartite quivers \( Q \), there exists a strongly polynomial time algorithm to decide membership in \( \Delta(Q, \beta) \). Specifically, let \( m, l, \) and \( n \) be positive integers and let \( Q \) be the bipartite quiver with \( m \) source vertices, \( l \) sink vertices, and \( n \) arrows between any two source and sink vertices.

We will refer to \( Q \) as an \( n \)-complete bipartite quiver. It is well-known that the membership problem for the moment cone \( \Delta(Q, \beta) \) can be reduced to the membership problem for the so-called cone of effective weights associated to the flag-extension \( (Q_\beta, \tilde{\beta}) \) of \( (Q, \beta) \) (see (2) and Proposition 5.3 for more details). Applying King’s [Kin94] and Schofield’s [Sch92] results to \( (Q_\beta, \tilde{\beta}) \), one can describe \( \Delta(Q, \beta) \) by means of a (finite) list of Horn-type linear homogeneous inequalities. A more refined description of \( \Delta(Q, \beta) \) can be found in [BR22] and [BVW19]. Nonetheless, these characterizations do not seem to lead to a strongly polynomial time algorithm to check membership in \( \Delta(Q, \beta) \).
In this paper, we follow a strategy similar to the one used in [MNS12] and [DLM06] (see also [ARY19]). More precisely, our main goal is to provide a polytopal description for weight spaces of semi-invariants of $Q_\beta$. This description combined with Derksen-Weyman’s Saturation Theorem (see [DW00]) allows us to use Tardos’s strongly polynomial time algorithm (see [Tar86]) in our context.

**Theorem 1.1.** Let $Q$ be an $n$-complete bipartite quiver with source vertices $x_1, \ldots, x_m$ and sink vertices $y_1, \ldots, y_\ell$. Let $\beta = (\beta(x))_{x \in Q_0}$ be a sincere dimension vector of $Q$ and let $\tilde{\beta}$ be the extension of $\beta$ to the flag-extended quiver $Q_\beta$.

1. Let $\tilde{\sigma}$ be an integral weight of $Q_\beta$ with $\tilde{\sigma} \cdot \tilde{\beta} = 0$ and such that $\tilde{\sigma}$ is non-negative/non-positive along the vertices of the flag $F(x)$ if $x$ is a source/sink. Then there exists a hive-type polytope $P_{\tilde{\sigma}}$ such that the number of lattice points of $P_{\tilde{\sigma}}$ is equal to $K_{\tilde{\sigma}}$.

2. There exists a strongly polynomial time algorithm to check membership in the moment cone $\Delta(Q, \beta)$.

In particular, there exists a strongly polynomial time algorithm for solving the generic semi-stability problem for representations of $Q$ and $Q_\beta$.

The problem of finding a polytopal description for dimensions of weight spaces of quiver semi-invariants is very difficult in general; see Problem 1 for more details. Our $n$-complete bipartite quivers and their flag-extensions are a case in point. Directly computing their weight spaces of semi-invariants leads to very complicated formulas involving large sums of large products of multiple Littlewood-Richardson coefficients (see Remark 4.8(2)). To prove the first part of Theorem 1.1, we first use quiver exceptional sequences and Derksen-Weyman’s Embedding Theorem to embed $Q_\beta$ into another quiver $T$, introduced in Section 3, without changing the dimensions of the weight spaces of semi-invariants for $Q_\beta$. After we perform this embedding, we show in Section 4 that the dimensions of weight spaces of semi-invariants for $T$, and thus $K_{\tilde{\sigma}}$, can be expressed as the number of lattice points of certain polytopes. Furthermore, these polytopes can be described as combinatorial linear programs, and the positivity of $K_{\tilde{\sigma}}$ is equivalent to the feasibility of the corresponding combinatorial linear program (see Proposition 4.15). In Section 5, we first show that a tuple $\underline{\lambda}$ of sequences of integers lies in $\Delta(Q, \beta)$ if and only if $K_{\tilde{\sigma}_{\underline{\lambda}}}$ is positive, where $\tilde{\sigma}_{\underline{\lambda}}$ is the weight of $Q_\beta$ induced by $\underline{\lambda}$. Thus, checking membership in $\Delta(Q, \beta)$ is equivalent to checking the feasibility of a combinatorial linear program of the form $A \cdot \mathbf{x} \leq \mathbf{b}$, where the entries of $A$ are 0, 1, or −1, while the entries of the vector $\mathbf{b}$ are homogeneous linear integral forms in the parts of the sequences of the tuple $\underline{\lambda}$.

### 2. Background on Quiver Invariant Theory

#### 2.1. Quivers and their representations.
Throughout, we work over the field $\mathbb{C}$ of complex numbers and denote by $N = \{0, 1, \ldots\}$. For a positive integer $L$, we denote by $[L] = \{1, \ldots, L\}$.

A quiver $Q = (Q_0, Q_1, t, h)$ consists of two finite sets $Q_0$ (vertices) and $Q_1$ (arrows) together with two maps $t : Q_1 \to Q_0$ (tail) and $h : Q_1 \to Q_0$ (head). We represent $Q$ as a directed graph with set of vertices $Q_0$ and directed edges $a : ta \to ha$ for every $a \in Q_1$. A quiver is said to be acyclic if it has no oriented cycles. We call a quiver connected if its underlying graph is connected.
A representation of $Q$ is a family $V = (V(x), V(a))_{x \in Q_0, a \in Q_1}$, where $V(x)$ is a finite-dimensional $\mathbb{C}$-vector space for every $x \in Q_0$, and $V(a) : V(ta) \to V(ha)$ is a $\mathbb{C}$-linear map for every $a \in Q_1$. After fixing bases for the vector spaces $V(x)$, $x \in Q_0$, we often think of the linear maps $V(a)$, $a \in Q_1$, as matrices of appropriate size. A subrepresentation $W$ of $V$, written as $W \subseteq V$, is a representation of $Q$ such that $W(x) \subseteq V(x)$ for every $x \in Q_0$, and moreover $V(a)(W(ta)) \subseteq W(ha)$ and $W(a) = V(a)|_{W(ta)}$ for every arrow $a \in Q_1$.

A morphism $\varphi : V \to W$ between two representations is a collection $(\varphi(x))_{x \in Q_0}$ of $\mathbb{C}$-linear maps with $\varphi(x) \in \text{Hom}_\mathbb{C}(V(x), W(x))$ for every $x \in Q_0$, and such that $\varphi(ha) \circ V(a) = W(a) \circ \varphi(ta)$ for every $a \in Q_1$. The $\mathbb{C}$-vector space of all morphisms from $V$ to $W$ is denoted by $\text{Hom}_Q(V, W)$.

The dimension vector $\dim V \in \mathbb{N}^{Q_0}$ of a representation $V$ is defined by $\dim V(x) = \dim_\mathbb{C} V(x)$ for all $x \in Q_0$. By a dimension vector of $Q$, we simply mean an $\mathbb{N}$-valued function on the set of vertices $Q_0$. We say a dimension vector $\beta$ is sincere if $\beta(x) > 0$ for every $x \in Q_0$. For every vertex $x \in Q_0$, the simple dimension vector at $x$, denoted by $e_x$, is defined by $e_x(y) = \delta_{x,y} \forall y \in Q_0$, where $\delta_{x,y}$ is the Kronecker symbol. We point out that $e_x$ is the dimension vector of the simple representation $S_x$ defined by assigning a copy of $\mathbb{C}$ to vertex $x$, the zero vector space at all other vertices, and the zero linear map along all arrows.

The Euler form (also known as the Ringel form) of $Q$ is the bilinear form on $\mathbb{Z}^{Q_0}$ defined by

$$\langle \alpha, \beta \rangle := \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha), \forall \alpha, \beta \in \mathbb{Z}^{Q_0}.$$ 

From now on, we assume that all of our quivers are connected and acyclic. Then, for any integral weight $\sigma \in \mathbb{Z}^{Q_0}$, there exists a unique $\alpha \in \mathbb{Z}^{Q_0}$ such that $\sigma(x) = \langle \alpha, e_x \rangle, \forall x \in Q_0$.

2.2. Weight spaces of semi-invariants and quiver semi-stability. Let $\beta$ be a sincere dimension vector of a quiver $Q$. As mentioned in Section 1, there is a natural action via simultaneous conjugation of $\text{GL}(\beta)$ on $\text{rep}(Q, \beta)$, i.e., for $g = (g(x))_{x \in Q_0}$ and $W = (W(a))_{a \in Q_1}$, we define $g \cdot W \in \text{rep}(Q, \beta)$ by

$$(g \cdot W)(a) := g(ha) \cdot W(a) \cdot g(ta)^{-1}, \forall a \in Q_1.$$ 

This action descends to that of the subgroup

$$\text{SL}(\beta) := \prod_{x \in Q_0} \text{SL}(\beta(x)),$$

giving rise to a highly non-trivial ring of semi-invariants $\text{SI}(Q, \beta) := \mathbb{C}[\text{rep}(Q, \beta)]^{\text{SL}(\beta)}$. (We point out that since $Q$ is assumed to be acyclic, the invariant ring $\mathbb{C}[\text{rep}(Q, \beta)]^{\text{GL}(\beta)}$ is precisely $\mathbb{C}$.) Since $\text{GL}(\beta)$ is linearly reductive and $\text{SL}(\beta)$ is its commutator subgroup, we have the weight space decomposition

$$\text{SI}(Q, \beta) = \bigoplus_{\chi \in X^*(\text{GL}(\beta))} \text{SI}(Q, \beta)_\chi,$$

where $X^*(\text{GL}(\beta))$ is the group of rational characters of $\text{GL}(\beta)$ and

$$\text{SI}(Q, \beta)_\chi := \{ f \in \mathbb{C}[\text{rep}(Q, \beta)] \mid g \cdot f = \chi(g)f, \forall g \in \text{GL}(\beta)\}$$
is the space of semi-invariants of weight $\chi$. Every integral weight $\sigma \in \mathbb{Z}^{Q_0}$ defines a character $\chi_\sigma$ of $\text{GL}(\beta)$ by $\chi_\sigma(g) := \prod_{x \in Q_0} (\det g(x))^{\sigma(x)}$, $\forall g = (g(x))_{x \in Q_0} \in \text{GL}(\beta)$. Moreover, since $\beta$ is sincere, any character of $\text{GL}(\beta)$ is of the form $\chi_\sigma$ for a unique $\sigma \in \mathbb{Z}^{Q_0}$, allowing us to identify the character group with $\mathbb{Z}^{Q_0}$. In what follows, we write $\text{SI}(Q, \beta)_\sigma$ for $\text{SI}(Q, \beta)_{\chi_\sigma}$, and $K_\sigma$ for the dimension of $\text{SI}(Q, \beta)_\sigma$.

In [Kin94], King used weight spaces of semi-invariants and tools from Geometric Invariant Theory to construct moduli spaces of quiver representations. Our focus in this paper is on combinatorial/computational aspects of weight spaces of semi-invariants.

**Problem 1 (The polytopal problem for quiver semi-invariants).** Let $Q$ be a quiver, $\beta$ a sincere dimension vector of $Q$, and $\sigma$ an integral weight of $Q$ such that $\sigma \cdot \beta = 0$. Find a rational polytope $P_\sigma$ such that

1. $\dim \text{SI}(Q, \beta)_\sigma = \text{the number of lattice points of } P_\sigma$;
2. $P_\sigma$ can be described by a combinatorial linear program $Ax \leq b$, where $A$ does not depend on $\sigma$, and the coordinates of $b$ are homogeneous linear forms in the coordinates of $\sigma$. (This latter condition implies that $rP_\sigma = P_{r\sigma}$ for any positive integer $r$.)

The polytopal problem for quiver semi-invariants seems to be very difficult in general. There are only a few examples of quivers in the literature where Problem 1 has been solved; see [CDW07], [Chi08], [Chi09], [Col20], and [DW00]. All these examples rely on Knutson-Tao’s hive model [KT99] for Littlewood-Richardson coefficients. In this paper, we solve Problem 1 for $n$-complete bipartite quivers and their flag-extensions by using quiver exceptional sequences to embed these quivers into other quivers and then compute the dimensions of the weight spaces of semi-invariants for those quivers (see the quiver $T$ defined in Section 3). Directly computing dimensions of weight spaces of semi-invariants for these quivers without embedding is, in general, very difficult to do and leads to very complicated formulas (see Remark 4.8).

The notion of a semi-stable quiver representation, introduced by King [Kin94] in the context of moduli spaces of quiver representations, plays a key role in understanding the positivity of the dimensions of weight spaces of semi-invariants.

Let $\sigma \in \mathbb{Z}^{Q_0}$ be an integral weight of $Q$. A representation $W$ of $Q$ is $\sigma$-semi-stable if and only if the following conditions hold:

(1) $\sigma \cdot \dim W = 0$ and $\sigma \cdot \dim(W') \leq 0$, $\forall W' \subseteq W$.

**Remark 2.1.** We point out that in order to check whether a given representation $W$ is $\sigma$-semi-stable one needs to check a finite number of linear homogeneous inequalities where this number can grow exponentially with the dimension of $W$. Nonetheless, as shown in [CD21, CK21, CK23], given $W$ and $\sigma$, there is a polynomial time algorithm that can check in deterministic polynomial time if $W$ is $\sigma$-semi-stable.

Let $\beta'$ be a dimension vector of $Q$ with $\beta' \leq \beta$, i.e., $\beta'(x) \leq \beta(x)$, $\forall x \in Q_0$. In what follows, we write $\beta' \leftrightarrow \beta$ to mean that a generic (equivalently, every) $\beta$-dimensional representation has a subrepresentation of dimension vector $\beta'$.

**Example 2.2.** If $x$ is a source vertex of $Q$, it is immediate to see that any $\beta$-dimensional representation has the simple representation $S_x$ as a subrepresentation, and thus $e_x \leftrightarrow \beta$. On the other hand, if $x$ is a source vertex of $Q$, one can also easily see that $\beta - e_x \leftrightarrow \beta$. □
A fundamental result due to King [Kin94], as well as Derksen and Weyman [DW00] (see also [CBG02]), gives necessary and sufficient conditions for the positivity of \( \dim SI(Q, \beta)_\sigma \).

**Theorem 2.3.** For an integral weight \( \sigma \in \mathbb{Z}^{Q_0} \) of \( Q \), the following statements are equivalent:
1. \( \dim SI(Q, \beta)_\sigma > 0 \);
2. \( \sigma \cdot \beta = 0 \) and \( \sigma \cdot \beta' \leq 0 \) for all \( \beta' \hookrightarrow \beta \);
3. there exists a \( \sigma \)-semi-stable \( \beta \)-dimensional representation of \( Q \);
4. there exists \( W \in \text{rep}(Q, \beta) \) such that
   \[
   \sum_{\begin{array}{c}
   aa \in Q_1 \\
   ta = x
   \end{array}} W(a)^* \cdot W(a) - \sum_{\begin{array}{c}
   a \in Q_1 \\
   ha = x
   \end{array}} W(a) \cdot W(a)^* = \sigma(x) \cdot \text{Id}_{\beta(x)} \ \forall x \in Q_0.
   \]

Consequently, weight spaces of quiver semi-invariants have the following **Saturation Property**:
\( \dim SI(Q, \beta)_\sigma > 0 \) for some positive integer \( r \geq 1 \) implies that \( \dim SI(Q, \beta)_\sigma > 0 \).

**Remark 2.4.**

1. We point out that if \( \dim SI(Q, \beta)_\sigma > 0 \), then \( \dim SI(Q, \beta)_{r \sigma} > 0 \) for any positive integer \( r \). Indeed, if \( f \in SI(Q, \beta)_\sigma \) is a non-zero semi-invariant then \( f^r \) is a non-zero semi-invariant of weight \( r \sigma \).
2. Assume that \( \dim SI(Q, \beta)_\sigma > 0 \). Then it follows from Theorem 2.3 and Remark 2.2 that
   \( \sigma(x) \geq 0 \) for any source vertex \( x \), and \( \sigma(y) \leq 0 \) for any sink vertex \( y \).

We recall another important result [IOTW09, Lemma 6.5.7] (see also [CG19, Lemma 3]) that gives necessary conditions for the positivity of \( \dim SI(Q, \beta)_\sigma \). It comes in handy in the proof of our main result, Theorem 1.1.

**Proposition 2.5.** Let \( \sigma \in \mathbb{Z}^{Q_0} \) be an integral weight of \( Q \) with \( \sigma = \langle \alpha, \cdot \rangle \) for a unique \( \alpha \in \mathbb{Z}^{Q_0} \). If \( \dim SI(Q, \beta)_\sigma > 0 \) then \( \alpha \) must be a dimension vector of \( Q \), i.e., \( \alpha(x) \geq 0, \forall x \in Q_0 \).

**Remark 2.6.** If \( \beta \) is not sincere, then the positivity of \( \dim SI(Q, \beta)_\sigma \) does not necessarily imply that all the coordinates of \( \alpha \) are non-negative.

2.3. **The cone of effective weights.** Let \( Q \) be a quiver and \( \beta \) a sincere dimension vector of \( Q \). The cone of effective weights associated to \( (Q, \beta) \) is the rational convex polyhedral cone defined by
   \[
   \text{Eff}(Q, \beta) := \{ \sigma \in \mathbb{R}^{Q_0} \mid \sigma \cdot \beta = 0, \text{ and } \sigma \cdot \beta' \leq 0 \ \forall \beta' \hookrightarrow \beta \}.
   \]

It follows from Theorem 2.3 that the lattice points of \( \text{Eff}(Q, \beta) \) is the affine semi-group of all integral weights \( \sigma \in \mathbb{Z}^{Q_0} \) for which \( \dim SI(Q, \beta)_\sigma > 0 \). This is further equivalent to saying that there exists a \( \beta \)-dimensional \( \sigma \)-semi-stable representation.

**Problem 2** (The generic quiver semi-stability problem). Let \( Q \) be a quiver, \( \beta \) a sincere dimension vector of \( Q \), and \( \sigma \) an integral weight of \( Q \) such that \( \sigma \cdot \beta = 0 \). Decide whether \( \sigma \) belongs to \( \text{Eff}(Q, \beta) \).

**Remark 2.7.** We point out that a positive answer to Problem 1 combined with the Saturation Property for quiver semi-invariants would prove that for a given \( \sigma \in \mathbb{Z}^{Q_0} \),

\[
\sigma \in \text{Eff}(Q, \beta) \iff \mathcal{P}_\sigma \neq \emptyset,
\]

at which point one could use Tardos’s strongly polynomial time algorithm to check whether \( \sigma \) belongs to \( \text{Eff}(Q, \beta) \).
For the remainder of this section we assume that $Q$ is a bipartite quiver (not necessarily $n$-complete) with source vertices $x_1, \ldots, x_m$, and sink vertices $y_1, \ldots, y_l$. For a sincere dimension vector $\beta$, let $Q_\beta$ be the flag extension of $Q$ defined as below, where the flag $\mathcal{F}(x)$ has $\beta(x) - 1$ vertices for each $x \in Q_0$. We use $\longrightarrow$ to indicate that multiple arrows are allowed between vertices but $Q$ need not be $n$-complete.

(2)

Recall that we define $\tilde{\beta}$ to be the extension of $\beta$ to $Q_\beta$ that takes values $1, \ldots, \beta(x_i)$ along the vertices (from left to right) of the flag $\mathcal{F}(x_i)$, $i \in [m]$, and $\beta(y_j), \ldots, 1$ along the vertices (from left to right) of the flag $\mathcal{F}(y_j)$, $j \in [l]$.

**Lemma 2.8.** Let $Q$ be a bipartite quiver with source vertices $x_1, \ldots, x_m$, and sink vertices $y_1, \ldots, y_l$, and let $\beta$ be a sincere dimension vector of $Q$. If $\tilde{\sigma} \in \text{Eff}(Q_\beta, \tilde{\beta})$ is an effective weight, then

$$\tilde{\sigma}|_{\mathcal{F}(x_i)} \geq 0, \forall i \in [m], \text{ and } \tilde{\sigma}|_{\mathcal{F}(y_j)} \leq 0, \forall j \in [l].$$

**Proof.** We already know that $\tilde{\sigma}$ is non-negative at the $m$ source vertices of $Q_\beta$ and non-positive at the $l$ sink vertices of $Q_\beta$ by Remark 2.4.

Now let $W \in \text{rep}(Q_\beta, \tilde{\beta})$ be a generic representation such that $W(a)$ is injective along any given arrow $a$ of a flag $\mathcal{F}(x_i)$ and $W(b)$ is surjective along any given arrow $b$ of a flag $\mathcal{F}(y_j)$. Since $\tilde{\beta}(ha) = \tilde{\beta}(ta) + 1$ and $\tilde{\beta}(tb) = \tilde{\beta}(hb) + 1$, it is immediate to see that $W$ has subrepresentations $W'_1$ and $W'_2$ of dimension vector $\tilde{\beta} - e_{ha}$ and $e_{tb}$, respectively, where $W'_1$ is the same as $W$ except that at vertex $ha$ where $W'_1$ is the $(\beta(ha) - 1)$-dimensional image of $W(a)$, and $W'_2$ is zero everywhere except at vertex $tb$ where $W'_2$ is the one-dimensional kernel of $W(b)$.

The argument above shows that if $z$ is a non-source vertex of $Q_\beta$ lying along one of the flags $\mathcal{F}(x_i)$, then $\tilde{\beta} - e_z \hookrightarrow \tilde{\beta}$ and thus $\tilde{\sigma}(z) \geq 0$. Furthermore, if $z$ is a non-sink vertex of $Q_\beta$ lying along one of the flags $\mathcal{F}(y_j)$, then $e_z \hookrightarrow \tilde{\beta}$ and thus $\tilde{\sigma}(z) \geq 0$. This now completes the proof. \qed

**Remark 2.9.**

1. As hinted in Theorem 2.3, there is a tight relationship between the moment cone $\Delta(Q, \beta)$ and the cone of effective weights $\text{Eff}(Q_\beta, \tilde{\beta})$; see Proposition 5.3 for full details.

2. Let $\sigma$ be an integral weight of $Q$ and let $\sigma'$ be its trivial extension to $Q_\beta$ defined to be zero at all other vertices of $Q_\beta$. Then one can check that

$$\text{SI}(Q, \beta)_\sigma = \text{SI}(Q_\beta, \tilde{\beta})_{\sigma'}.$$
3. QUIVER EXCEPTIONAL SEQUENCES AND THE EMBEDDING THEOREM FOR QUIVER SEMI-INVARIANTS

In this section, we first review Derksen-Weyman’s Embedding Theorem for quiver semi-invariants. This result allows us to embed the quiver $Q_\beta$ into a new quiver, denoted below by $\mathcal{T}$, without changing the dimensions of the weight spaces of semi-invariants for $Q_\beta$. The advantage of working with $\mathcal{T}$ is that it is easier to find a polytopal description for the dimensions of its spaces of semi-invariants than for those of $Q_\beta$ (see Sections 4.3 and 4.4).

In what follows, for two dimension vectors $\alpha$ and $\beta$ of a quiver $Q$, we define $(\alpha \circ \beta)_{Q} := \dim \text{SI}(Q,\beta)_{\langle \alpha,\cdot \rangle}$. (Whenever the quiver is understood from the context, we drop the subscript $Q$ and simply write $\alpha \circ \beta$ for the dimension of $\text{SI}(Q,\beta)_{\langle \alpha,\cdot \rangle}$.)

**Definition 3.1 (Quiver Exceptional Sequences).** Let $Q = (Q_0, Q_1, t, h)$ be a quiver. A sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ of dimension vectors is said to be a quiver exceptional sequence if:

1. each $\epsilon_i$ is a real Schur root, i.e., $\langle \epsilon_i, \epsilon_i \rangle = 1$ and $\epsilon_i$ is the dimension vector of a Schur representation for all $i \in [N]$;
2. $\langle \epsilon_i, \epsilon_j \rangle \leq 0$ and $\epsilon_j \circ \epsilon_i \neq 0$ for all $1 \leq i < j \leq N$.

**Remark 3.2.** To check the second condition in the definition above, we will use the following fact which is a consequence of Derksen-Weyman’s First Fundamental Theorem for quiver semi-invariants [DW00] (see also [CBG02]). For two dimension vectors $\alpha$ and $\beta$ of $Q$, we have that $\alpha \circ \beta \neq 0$ if and only if $\langle \alpha, \beta \rangle = 0$ and $\text{Hom}_Q(V,W) = 0$ for some representations $V$ and $W$ of dimension vectors $\alpha$ and $\beta$, respectively. □

To any quiver exceptional sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$, we associate the quiver $Q(\epsilon)$ with vertices $1, \ldots, N$ and $-\langle \epsilon_i, \epsilon_j \rangle$ arrows from vertices $i$ to $j$ for all $1 \leq i \neq j \leq N$. Let

$$\mathcal{I}: \mathbb{R}^N \longrightarrow \mathbb{R}^{Q_0}$$

be the map defined by

$$\mathcal{I}(\gamma(1), \ldots, \gamma(N)) : = \sum_{i=1}^{N} \gamma(i)\epsilon_i \quad \text{for all} \quad \gamma = (\gamma(1), \ldots, \gamma(N)) \in \mathbb{R}^N.$$

We are now ready to state Derksen-Weyman’s Embedding Theorem which plays a key role in our approach to computing the dimensions of weight spaces of quiver semi-invariants.

**Theorem 3.3 (The Embedding Theorem for Quiver Semi-Invariants).** [DW11] Let $Q = (Q_0, Q_1, t, h)$ be a quiver and $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ a quiver exceptional sequence. If $\alpha$ and $\beta$ are two dimension vectors of $Q(\epsilon)$, then

$$(\alpha \circ \beta)_{Q(\epsilon)} = (\mathcal{I}(\alpha) \circ \mathcal{I}(\beta))_{Q}.$$ 

We end this section with an important example. Let $Q$ be the $n$-complete bipartite quiver with source vertices $x_1, \ldots, x_m$, sink vertices $y_1, \ldots, y_\ell$, and $n$ arrows from $x_i$ to $y_j$ for every $i \in [m]$ and $j \in [\ell]$. Let $\beta$ be a sincere dimension vector of $Q$ and let $Q_\beta$ be the
corresponding flag-extension of $Q$. In what follows, we show how to realize $Q_{\beta}$ as $T(\epsilon)$ for a suitable quiver $T$ and quiver exceptional sequence $\epsilon$.

Let us consider the quiver

with the flag $F(x_i)$ going in vertex $x_i$ of length $\beta(x_i) - 1$, $\forall i \in [m]$, and the flag $F(y_j)$ going out of vertex $y_j$ of length $\beta(y_j) - 1$, $\forall j \in [\ell]$. Note that there are no flags attached to the $n$ vertices $x_{m+1}, \ldots, x_{m+n}$. Also, if $\beta$ takes value one at a vertex of $Q$, then no flag is attached to that vertex in $T$.

Next, consider the dimension vectors $\delta_1, \ldots, \delta_m$ of $T$ defined by

$$\delta_i(x_0) = n + 1, \delta_i(y_0) = n, \delta_i(x_i) = \delta_i(x_{m+1}) = \ldots = \delta_i(x_{m+n}) = 1,$$

and $\delta_i$ is zero at all other vertices of $T$. To build the desired quiver exceptional sequence, we will work with the following dimension vectors:

- the simple roots at the vertices of the flag $F(x_i) \setminus \{x_i\}, i \in [m]$;
- $\delta_1, \ldots, \delta_m$;
- the simple roots at the vertices of the flag $F(y_j), j \in [\ell]$.

**Proposition 3.4.** The dimension vectors above can be ordered to form a quiver exceptional sequence $\epsilon$ for $T$ such that $T(\epsilon) = Q_{\beta}$.

**Proof.** To obtain the sequence $\epsilon$, we list the simple roots at the vertices of the flags $F(x_1) \setminus \{x_1\}, \ldots, F(x_m) \setminus \{x_m\}$ by going through the vertices of each flag from left to right starting with the flag $F(x_1)$. Next, we list the dimension vectors $\delta_1, \ldots, \delta_m$. Finally, we list the simple roots at the vertices of the flags $F(y_1), \ldots, F(y_\ell)$ by going through the vertices of each flag from left to right starting with the flag $F(y_1)$.

It is clear that any simple root is a real Schur root. Next, we show that the $\delta_i$ are real Schur roots and $\delta_i \perp \delta_j = 0$ for all $i, j \in [m]$. For each $i \in [m]$, consider the representation $V_i$ of $T$ defined by

- $V_i(x_0) = \mathbb{C}^{n+1}, V_i(y_0) = \mathbb{C}^n, V_i(x_i) = V_i(x_{m+1}) = \ldots = V_i(x_{m+n}) = \mathbb{C}$, and $V$ is zero at the remaining vertices;
- $V_i(a) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ sends $(\ell t_1, \ldots, tn+1)$ to $(t_1 + t_{n+1}, \ldots, t_n + t_{n+1})$;
- $V_i(a_i) : \mathbb{C} \rightarrow \mathbb{C}^{n+1}$ is the $(n + 1)^{th}$ canonical inclusion of $\mathbb{C}$ into $\mathbb{C}^{n+1}$;
- $V_i(a_{m+k}) : \mathbb{C} \rightarrow \mathbb{C}^{n+1}$ is the $k^{th}$ canonical inclusion of $\mathbb{C}$ into $\mathbb{C}^{n+1}$ for every $k \in [n]$.

It is immediate to check that $\text{End}_T(V_i) = \{ \lambda \text{Id}_{V_i} \mid \lambda \in \mathbb{C} \}$, $i.e., V_i$ is a Schur representation of dimension vector $\delta_i$ which together with the fact that $\langle \delta_i, \delta_i \rangle = 1$ proves that $\delta_i$
is a real Schur root for all \( i \in [m] \). Also, we have that \( \text{Hom}_T(V_i, V_j) = 0 \) and \( \langle \delta_i, \delta_j \rangle = 0 \), which imply that \( \delta_i \circ \delta_j \neq 0 \) for all \( 1 \leq i \neq j \leq m \). Thus, it is now clear that \( \epsilon \) is a quiver exceptional sequence with \( T(\epsilon) = Q_\beta \). □

**Example 3.5.** In what follows, for two quivers \( Q' \) and \( Q \), we write \( Q' \hookrightarrow Q \) to mean that \( Q' = Q(\epsilon) \) for an explicit quiver exceptional sequence \( \epsilon \).

1. \( (n\text{-Kronecker quivers}) \)

   \[
   \begin{array}{ccc}
   n \text{ arrows} & \rightarrow & n + 1 \text{ sources}
   \end{array}
   \]

2. \( (\text{complete bipartite quivers}) \)

   \[
   \begin{array}{ccc}
   m \text{ sources} & \rightarrow & m + 1 \text{ sources}
   \end{array}
   \]

**Remark 3.6.** If \( \gamma \in \mathbb{Z}^{(Q_\beta)_0} \) is an integral vector, then \( I(\gamma) \in \mathbb{Z}^{T_0} \) is the same as \( \gamma \) at the vertices of the flags \( F(x_i), i \in [m] \), and \( F(y_j), j \in [\ell] \). Furthermore, we have that

\[
I(\gamma)(x_0) = (n + 1)C, \quad I(\gamma)(y_0) = nC, \quad \text{and} \quad I(\gamma)(x_{m+1}) = \ldots = I(\gamma)(x_{m+n}) = C,
\]

where \( C := \sum_{i=1}^{m} \gamma(x_i) \). Moreover, if \( \tilde{\sigma} \) is a weight of \( T \) of the form \( \langle I(\gamma), \cdot \rangle_T \), then

1. \( \tilde{\sigma}(x_0) = 0 \) and \( \tilde{\sigma}(y_0) = -C \), and
2. \( \tilde{\sigma} \) is equal to \( \tilde{\sigma} = \langle \gamma, \cdot \rangle_{Q_\beta} \) at the vertices of the flags \( F(x_i), i \in [m] \), and \( F(y_j), j \in [\ell] \).

4. HIVE-TYPE POLYTOPES FOR QUIVER SEMI-INVARIANTS

4.1. The irreducible representations of the general linear group. In this section we review the basics of the representation theory of the general linear group, which can be found in [Ful97]. A partition is a sequence \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of integers with \( \lambda_1 \geq \ldots \geq \lambda_r \geq 0 \). The length of a partition is denoted \( \ell(\lambda) \) and is defined to be the number of its nonzero parts. If \( \lambda \) is a partition, we define \( |\lambda| \) to be the sum of its parts. The Young diagram of a partition \( \lambda \) is a collection of boxes, arranged in left-justified rows with \( \lambda_i \) boxes in row \( i \). If \( a \) and \( b \) are two positive integers, \( (b^a) \) denotes the partition that has \( a \) parts, all equal to \( b \). We say that the diagram of \( (b^a) \) is the \( a \times b \) rectangle.

Now let \( N \) be a fixed integer. Denote the set of partitions with length at most \( N \) by \( P_N \). For a partition \( \lambda \in P_N \), \( S^\lambda V \) denotes the irreducible (polynomial) representation of \( \text{GL}(V) \) with highest weight \( \lambda \), called a Schur module, where \( V \) is any fixed \( N \)-dimensional
complex vector space. Given partitions $\lambda, \mu, \nu \in P_N$, we define the Littlewood-Richardson coefficient $c^\nu_{\lambda,\mu}$ to be the multiplicity of $S^\nu V$ in $S^\lambda V \otimes S^\mu V$, that is,

$$c^\nu_{\lambda,\mu} = \dim_{\mathbb{C}} \left( S^\nu V^* \otimes S^\lambda V \otimes S^\mu V \right)^{\text{GL}(V)}.$$

More generally, if $\nu, \lambda^{(1)}, \ldots, \lambda^{(r)} \in P_N$, we define

$$c^\nu_{\lambda^{(1)}, \ldots, \lambda^{(r)}} = \dim_{\mathbb{C}} \left( S^\nu V^* \otimes S^\lambda^{(1)} V \otimes \cdots \otimes S^\lambda^{(r)} V \right)^{\text{GL}(V)}.$$

Following [Zel99], given partitions $\lambda^{(1)}, \ldots, \lambda^{(r)} \in P_N$, we define partitions $\tilde{\lambda}, \tilde{\mu} \in P_{rN}$ by

$$\tilde{\mu}_{(j-1)N+i} := \sum_{k=j+1}^r \lambda_1(k) \text{ and } \tilde{\lambda}_{(j-1)N+i} = \lambda_i(j) + \tilde{\mu}_{(j-1)N+i}, \forall j \in [r], i \in [N].$$

Diagrammatically, these partitions are defined as

![Diagram of partitions](image)

**Remark 4.1.** (1) We emphasize that if the partitions $\lambda^{(1)}, \ldots, \lambda^{(r)}$ have different lengths, we first choose an integer $N \geq 1$ such that $\ell(\lambda^{(1)}), \ldots, \ell(\lambda^{(r)}) \leq N$ and extend each $\lambda^{(i)}$ by adding $N - \ell(\lambda^{(i)})$ zero parts. Then we construct the partitions $\tilde{\lambda}$ and $\tilde{\mu}$ according to Equation (3). This is emphasized in the above diagram by using red vertical lines to indicate that zeros may have been added to the end of the partitions.

(2) The last $N$ parts of the partition $\tilde{\mu}$ are zero. Furthermore, $\tilde{\lambda} - \tilde{\mu}$ is a skew diagram whose connected components are translates of the diagrams of $\lambda^{(1)}, \ldots, \lambda^{(r)}$.

**Proposition 4.2.** [Zel99, Proposition 9] Keep the same notations as above. If $\nu, \lambda^{(1)}, \ldots, \lambda^{(r)} \in P_N$ are partitions, then

$$c^\nu_{\lambda^{(1)}, \ldots, \lambda^{(r)}} = c^\nu_{\tilde{\lambda}, \tilde{\mu}}.$$

We end this subsection by listing some very useful properties of the irreducible representations of $\text{GL}(V)$.

**Proposition 4.3.** (1) Let $\lambda \in P_N$. Then $(S^\lambda(V))^{\text{SL}(V)} \neq 0$ if and only if $\dim S^\lambda(V) = 1$ if and only if $\lambda = (w^N)$. In this case, $(S^\lambda V)^{\text{SL}(V)}$ is spanned by one semi-invariant of weight $w$. 


Let \( \lambda = (\lambda_1, \ldots, \lambda_N) \) and \( \mu = (\mu_1, \ldots, \mu_N) \) be two partitions. Then \( (S^\lambda V^* \otimes S^\mu V)^{SL(V)} \neq 0 \) if and only if \( \mu_i - \lambda_i = w \) for all \( i \in [N] \) for some integer \( w \). If this is the case, \( (S^\lambda V^* \otimes S^\mu V)^{SL(V)} \) is a one-dimensional vector space spanned by a semi-invariant of weight \( w \).

Let \( U \) be a rational representation of \( GL(V) \). Then \( U^{SL(V)} = \bigoplus_{\theta \in \mathbb{Z}} U_\theta \), where

\[
U_\theta = \{ u \in U \mid g \cdot u = \det(g)^\theta \cdot u, \; g \in GL(V) \}
\]

is the space of semi-invariants of weight \( \theta \). Moreover, \( U_\theta = (U \otimes \det^{-\theta})^{GL(V)} \), where \( \det^{-\theta} : GL(V) \to \mathbb{C}^* \) is the one-dimensional representation of \( GL(V) \) that sends \( g \in GL(V) \) to \( \det^{-\theta}(g) \in \mathbb{C}^* \).

4.2. Knutson-Tao’s hive polytopes for Littlewood-Richardson coefficients. In this subsection we review a combinatorial model for computing Littlewood-Richardson coefficients that was introduced by A. Knutson and T. Tao in [KT99] and [KT01]. Further details about this combinatorial description and its consequences can be found in, for instance, [KTT04], [KTT07a], [KTT07b], and [KTT09].

To define the polytope whose number of lattice points is the Littlewood-Richardson coefficient \( c^\nu_{\lambda,\mu} \) for a specific choice of partitions \( \nu, \lambda, \) and \( \mu \) with at most \( N \) parts, we start by considering a triangular graph obtained by dividing an equilateral triangle into \( N^2 \) smaller equilateral triangles of the same size by plotting \( N + 1 \) vertices along each edge of the large triangle.

An \( N \)-hive is a tuple of numbers \( (e_{i,j}, f_{i,j}, g_{i,j}) \) with \( 0 \leq i, j, i+j \leq N-1 \) where the entries \( e_{i,j} \) label the edges parallel to the left boundary of the large triangle, the entries \( f_{i,j} \) label the edges parallel to the right boundary of the large triangle, and the entries \( g_{i,j} \) label the horizontal edges. Furthermore, these numbers must satisfy the hive conditions (4) – (6) described below. A hive is said to be an integral hive if all of its entries are non-negative integers. A 3-hive is depicted in Figure 1 below.

The hive conditions are a set of constraints on the edge labels of each of the following two elementary triangles and three elementary rhombi:

\[
\begin{align*}
T_1 & \quad \alpha \quad \beta \quad \gamma \\
R_1 & \quad \delta \quad \alpha \quad \beta \quad \gamma \\
R_2 & \quad \delta \quad \alpha \quad \beta \quad \gamma \\
R_3 & \quad \alpha \quad \beta \quad \gamma \quad \delta \\
\end{align*}
\]

In each of the two triangles \( T_1 \) and \( T_2 \), we want

\[
\alpha + \beta = \gamma.
\]
In particular, this implies that in the three rhombi with our labeling, we must have
\[(5) \quad \alpha + \delta = \beta + \gamma.\]
Furthermore, we want the elementary rhombi to satisfy the rhombus inequalities, i.e., for each of \(R_1, R_2,\) and \(R_3,\) we want
\[(6) \quad \alpha \geq \gamma \quad \text{and} \quad \beta \geq \delta,\]
where it is clear that either one of the two inequalities in (6) implies the other one. Moreover, note that inequalities (4) – (6) define a convex polyhedral cone in \(\mathbb{R}^{3N(N+1)/2}.\)

**Definition 4.4.** An LR-hive is an integer \(N\)-hive whose border labels are determined by three partitions \(\lambda, \mu,\) and \(\nu\) with at most \(N\) non-zero parts such that \(|\nu| = |\lambda| + |\mu|\) and
\[e_{i,0} = \lambda_{i+1}, \quad f_{j,N-1-j} = \mu_{N-j}, \quad \text{and} \quad g_{0,k} = \nu_{k+1}, \quad \forall 0 \leq i, j, k \leq N - 1.\]

**Theorem 4.5 ([KT99], Theorem 4).** Let \(\lambda, \mu,\) and \(\nu\) be three partitions with at most \(N\) nonzero parts such that \(|\nu| = |\lambda| + |\mu|\). Then the Littlewood-Richardson coefficient \(c_{\lambda, \mu}^\nu\) is the number of LR-hives with boundary labels determined by \(\lambda, \mu,\) and \(\nu.\)

**4.3. Computing weight spaces of semi-invariants via Littlewood-Richardson coefficients.** Let \(m, n,\) and \(\ell\) be positive integers and let \(Q\) be the \(n\)-complete bipartite quiver with source vertices \(x_1, \ldots, x_m,\) and sink vertices \(y_1, \ldots, y_\ell.\) Let \(\beta\) be a sincere dimension vector of \(Q.\)

Let \(\mathcal{T}\) be the quiver introduced in Section 3. Our goal in this section is to find a hive-type polytopal description for the weight spaces of semi-invariants for the quiver set-up \((\mathcal{T}, \hat{\beta}),\) where \(\hat{\beta} = \mathcal{I}(\tilde{\beta})\) with \(\beta\) being a sincere dimension vector of \(Q\) and \(\tilde{\beta}\) its extension to \(Q_{\beta}.\) More precisely, we have that \(\hat{\beta}\) is the dimension vector of \(\mathcal{T}\) given by:
\( \hat{\beta}(x_0) = (n + 1)d, \hat{\beta}(y_0) = nd, \) and \( \hat{\beta}(x_{m+k}) = d \) for \( k \in [n], \) where \( d := \sum_{i=1}^{m} \beta(x_i); \)

- traversing the flag \( F(x_i) \) going into the vertex \( x_i \) from left to right, the values of \( \hat{\beta} \) at the vertices of this flag are 1, \ldots, \beta(x_i) \) for every \( i \in [m]; \)

- traversing the flag \( F(y_j) \) going out of the vertex \( y_j \) from left to right, the values of \( \hat{\beta} \) at the vertices of this flag are \( \beta(y_j), \ldots, 1 \) for every \( j \in [\ell]. \)

Next, let \( \hat{\sigma} \) be a weight of \( T \) such that \( \hat{\sigma} \cdot \hat{\beta} = 0. \) Furthermore, we assume that:

\[
\hat{\sigma} \big|_{F(x_i)} \geq 0, \; i \in [m], \quad \text{and} \quad \hat{\sigma} \big|_{F(y_j)} \leq 0, \; j \in [\ell],
\]

and

\[
\hat{\sigma}(x_0) = 0, \; \hat{\sigma}(y_0) = -f, \; \text{and} \; \hat{\sigma}(x_{m+k}) = f, \; \forall k \in [n], \; \text{for some non-negative integer} \; f.
\]

**Remark 4.6.** Let \( \alpha \) be a dimension vector of \( Q_\beta \) and let us assume that the weight \( \sigma := \langle \alpha, \cdot \rangle_{Q_\beta} \) of \( Q_\beta \) satisfies \( \sigma \big|_{F(x_i)} \geq 0, \; \forall i \in [m], \) and \( \sigma \big|_{F(y_j)} \leq 0, \; \forall j \in [\ell]. \) Note that according to Lemma 2.8, these constraints are just some necessary conditions for \( \sigma \) to belong to \( \text{Eff}(Q_\beta, \hat{\beta}). \)

It then follows from Remark 3.6 that the weight \( \hat{\sigma} := \langle I(\alpha), \cdot \rangle_T \) of \( T \) satisfies (7) and (8), where \( f := \sum_{i=1}^{m} \alpha(x_i). \)

For each \( i \in [m], \) let us label the vertices of the flag \( F(x_i) \) of the quiver \( T \) as follows

\[
F(x_i) : \; \bullet \to \bullet \to \cdots \to \bullet \to \bullet
\]

\[i_1 \quad i_2 \quad i_{\beta(x_i)-1} \quad i_{\beta(x_i)}\]

and define the partition \( \lambda(i) = \left( \sum_{k \leq r \leq \beta(x_i)} \hat{\sigma}(i_r) \right)_{k \in [\beta(x_i)]}. \) For each \( j \in [\ell], \) let us label the vertices of the flag \( F(y_j) \) of the quiver \( T \) as follows

\[
F(y_j) : \; \bullet \to \bullet \to \cdots \to \bullet \to \bullet
\]

\[j_{\beta(y_j)} \quad j_{\beta(y_j)-1} \quad j_2 \quad j_1\]

and define the partition \( \nu(j) = \left( -\sum_{k \leq r \leq \beta(y_j)} \hat{\sigma}(j_r) \right)_{k \in [\beta(y_j)]}. \)

We point out that since \( \hat{\sigma} \cdot \hat{\beta} = 0 \) we have that

\[
\sum_{i=1}^{m} |\lambda(i)| = \sum_{j=1}^{\ell} |\nu(j)|.
\]

**Proposition 4.7.** Keeping the same notation as above, the following formula holds:

\[
\dim \text{SI}(T, \hat{\beta} \hat{\sigma}) = \sum_{\ell(\mu) \leq nd} \epsilon^d \lambda(1), \ldots, \lambda(m), (f^d)^{0}, \ldots, (f^d)^{n \text{ times}} \cdot \epsilon^d \nu(1), \ldots, \nu(\ell).(f^{nd}),
\]

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Proof. To find the desired formula for \( \dim \text{SI}(\mathcal{T}, \mathcal{\hat{\beta}}) \), we proceed as follows. First, we use Cauchy’s formula to decompose \( \mathbb{C}[\text{rep}(\mathcal{T}, \mathcal{\hat{\beta}})] \) into a direct sum of irreducible representations of \( \text{GL}(\mathcal{\hat{\beta}}) \). Then, we consider the ring of semi-invariants \( \text{SI}(\mathcal{T}, \mathcal{\hat{\beta}}) = \mathbb{C}[\text{rep}(\mathcal{T}, \mathcal{\hat{\beta}})]^{\text{SL}(\mathcal{\hat{\beta}})} \) and sort out those semi-invariants that have weight \( \mathcal{\hat{\sigma}} \).

For each \( i \in [m] \), let us focus on the following subquiver of \( \mathcal{T} \):

\[
\begin{array}{c}
\bullet & \rightarrow & \bullet \\
& \text{\ldots} & \\
& \rightarrow & \\
\mathcal{F}(x_i) & \rightarrow & a_i \\
& \rightarrow & \bullet \\
& & \mathcal{F}(x_0)
\end{array}
\]

For convenience, let us denote \( \beta(x_i) = r \) and write \( V_k = \mathbb{C}^k \), \( \forall 1 \leq k \leq r \), and \( V = \mathbb{C}^\beta(x_0) = \mathbb{C}^{(n+1)d} \). Then the contribution of the subquiver above to \( \mathbb{C}[\text{rep}(\mathcal{T}, \mathcal{\hat{\beta}})] \) is:

\[
\mathbb{C} \left[ \prod_{k=1}^{r-1} \text{Hom}(V_k, V_{k+1}) \times \text{Hom}(V_r, V) \right] = \bigotimes_{k=1}^{r-1} S(V_k \otimes V_{k+1}^*) \otimes S(V_r \otimes V^*)
\]

\[
= \bigoplus_{\gamma(1), \ldots, \gamma(r-1), \gamma(i)} S^{\gamma(1)}(V_1) \otimes \bigotimes_{k=2}^{r-1} \left( S^{\gamma(k-1)}V_k^* \otimes S^{\gamma(k)}V_k \right) \otimes \left( S^{\gamma(r-1)}V_r^* \otimes S^{\gamma(i)}V_r \right)
\]

This yields the following contribution of the vertices of the flag \( \mathcal{F}(x_i) \) to \( \text{SI}(\mathcal{T}, \mathcal{\hat{\beta}}) \):

\[
\bigoplus_{\gamma(1), \ldots, \gamma(r-1), \gamma(i)} \left( S^{\gamma(1)}V_1 \right)^{\text{SL}(V_1)} \otimes \bigotimes_{k=2}^{r-1} \left( S^{\gamma(k-1)}V_k^* \otimes S^{\gamma(k)}V_k \right)^{\text{SL}(V_k)} \otimes \left( S^{\gamma(r-1)}V_r^* \otimes S^{\gamma(i)}V_r \right)^{\text{SL}(V_r)}
\]

Sorting out those semi-invariants of weight \( \mathcal{\hat{\sigma}} \) completely determines the partitions \( \gamma(1), \ldots, \gamma(r-1) \), and \( \gamma(i) \). By Proposition 4.3(1), we have that \( \left( S^{\gamma(1)}V_1 \right)^{\text{SL}(V_1)} \neq 0 \) if and only if it is one-dimensional. If this is the case, then \( \gamma(1) \) is a \( 1 \times w \) rectangle with \( w \in \mathbb{N} \) and \( \left( S^{\gamma(1)}V_1 \right)^{\text{SL}(V_1)} \) is spanned by a semi-invariant of weight \( w \). Thus, \( \left( S^{\gamma(1)}V_1 \right)^{\text{SL}(V_1)} \) contains a semi-invariant of weight \( \mathcal{\hat{\sigma}}(i_1) \) if and only if \( \gamma(1) = (\mathcal{\hat{\sigma}}(i_1)) \).

Next, using Proposition 4.3(2), we have that the space \( \left( S^{\gamma(1)}V_2^* \otimes S^{\gamma(2)}_2 \right)^{\text{SL}(V_2)} \) is nonzero if and only if it is one-dimensional. If that is the case, then \( \gamma(2) \) is \( \gamma(1) \) plus some extra columns of height 2, with the number of these extra columns equaling the weight of the semi-invariant spanning \( \left( S^{\gamma(1)}V_2^* \otimes S^{\gamma(2)}_2 \right)^{\text{SL}(V_2)} \). Thus, this space contains a nonzero semi-invariant of weight \( \mathcal{\hat{\sigma}}(i_2) \) if and only if \( \gamma(2) = (\mathcal{\hat{\sigma}}(i_2) + \mathcal{\hat{\sigma}}(i_1), \mathcal{\hat{\sigma}}(i_2)) \). Continuing with this reasoning, we see that \( \gamma(1), \ldots, \gamma(r-1) \), and \( \gamma(i) \) are completely determined by \( \mathcal{\hat{\sigma}} \) with

\[
\gamma(i) = (\mathcal{\hat{\sigma}}(x_i) + \mathcal{\hat{\sigma}}(i_{r-1}) + \ldots + \mathcal{\hat{\sigma}}(i_1), \ldots, \mathcal{\hat{\sigma}}(x_i))
\]

which is precisely \( \lambda(i) \). Now, let us focus on vertex \( x_0 \) and its neighbors:
We write $W = C^\beta(y_0) = C^{nd}$. The contribution of this subquiver to $C[\text{rep}(\mathcal{T}, \hat{\beta})]$ is

$$C[\text{Hom}(V_1, V) \times \cdots \times \text{Hom}(V_{m+n}, V) \times \text{Hom}(V, W)] = S(V_1 \otimes V^*) \otimes \cdots \otimes S(V_{m+n} \otimes V^*) \otimes S(V \otimes W^*).$$

Using Cauchy’s Formula again, we can write

$$(10) \quad S(V \otimes W^*) = \bigoplus S^\mu(V) \otimes S^\mu(W^*),$$

where the sum is over all partitions $\mu$ of length at most $\min\{\dim V, \dim W\} = nd$. Since the weight $\hat{\sigma}$ is zero at vertex $x_0$, the calculations above together with Proposition 4.3(3) show that the contribution of $x_0$ to $\text{SI}(\mathcal{T}, \hat{\beta})_{\hat{\sigma}}$ is made of spaces of the form

$$\left( S^{\lambda(1)}V^* \otimes \cdots \otimes S^{\lambda(m)}V^* \otimes S^{(f_1^d)} V^* \otimes \cdots \otimes S^{(f_1^d)} V^* \otimes S^\mu V \right)^{\text{GL}(V)} \text{n times},$$

with $\mu$ a partition of length at most $nd$.

Taking into account the contributions of all the other vertices of $\mathcal{T}$, we get that $\text{SI}(\mathcal{T}, \hat{\beta})_{\hat{\sigma}}$ is isomorphic to

$$\bigoplus_{\ell(\mu) \leq nd} \left( S^{\lambda(1)}V^* \otimes \cdots \otimes S^{\lambda(m)}V^* \otimes S^{(f_1^d)} V^* \otimes \cdots \otimes S^{(f_1^d)} V^* \otimes S^\mu V \right)^{\text{GL}(V)} \otimes \left( S^{\nu(1)} W \otimes \cdots \otimes S^{\nu(\ell)} W \otimes S^\mu W^* \otimes \text{det}_W \right)^{\text{GL}(W)}.$$

Thus, we conclude that

$$\dim \text{SI}(\mathcal{T}, \hat{\beta})_{\hat{\sigma}} = \sum_{\ell(\mu) \leq nd} \ell(\mu) \cdot c^\mu_{\lambda(1), \ldots, \lambda(m), (f_1^d), \ldots, (f_1^d)} \cdot c^\mu_{\nu(1), \ldots, \nu(\ell), (f_1^d)}. $$
Remark 4.8. (1) We will mainly use Proposition 4.7 with \( \hat{\sigma} \) a weight of \( \mathcal{T} \) of the form 
\[ \langle I(\alpha), \cdot \rangle_{\mathcal{T}}, \] 
where \( \alpha \) a dimension vector of \( Q_\beta \) such that the weight \( \sigma := \langle \alpha, \cdot \rangle_{Q_\beta} \) of \( Q_\beta \) satisfies the conditions in Remark 4.6. Then, it follows from Theorem 3.3, Proposition 3.4, and Proposition 4.7 that

\[ \dim \text{SI}(Q_\beta, \hat{\beta})_{\hat{\sigma}} = \dim \text{SI}(\mathcal{T}, \hat{\beta})_{\hat{\sigma}} = \sum_{\mu \colon \ell(\mu) \leq nd} c^{\mu}_{\lambda(1), \ldots, \lambda(m)(f^d), \ldots, (f^d)} \cdot c^{\mu}_{\nu(1), \ldots, \nu(t),(f^{nd})}, \]

where \( \hat{\beta} = I(\hat{\beta}) \). Working with this formula allows us to express \( \dim \text{SI}(Q_\beta, \hat{\beta})_{\hat{\sigma}} \) as the number of lattice points of a certain hive-type polytope; see Proposition 4.15 in Section 4.4.

(2) One can also compute \( \dim \text{SI}(Q_\beta, \hat{\beta})_{\hat{\sigma}} \) directly (without embedding \( Q_\beta \) into \( \mathcal{T} \)) in terms of Littlewood-Richardson coefficients. The problem with this direct approach is that it computes \( \dim \text{SI}(Q_\beta, \hat{\beta})_{\hat{\sigma}} \) as a sum over \( lmn \) variable partitions \( \mu^{(r)}_{i,j}, i \in [m], j \in [l], r \in [n] \), where each term of the sum is a product of \( ml \) multiple Littlewood-Richardson coefficients. Specifically, one obtains the following formula for \( \dim \text{SI}(Q_\beta, \hat{\beta})_{\hat{\sigma}} \):

\[ \sum_{\mu^{(r)}_{i,j}} \prod_{i=1}^{m} \prod_{j=1}^{l} \prod_{r=1}^{n} c^{\lambda(x_i)}_{\mu^{(1)}_{i,1}, \ldots, \mu^{(n)}_{i,1}} \cdot \prod_{j=1}^{l} \prod_{r=1}^{n} c^{\lambda(y_j)}_{\mu^{(1)}_{1,j}, \ldots, \mu^{(n)}_{1,j}}, \]

where \( \lambda(x_i), i \in [m], \) and \( \lambda(y_j), j \in [l], \) are partitions determined by \( \hat{\sigma} \). The result is very difficult to work with, making our approach based on quiver exceptional sequences and the quiver \( \mathcal{T} \) essential for our purposes.

\[ \square \]

Remark 4.9. We point out that when \( \ell = 1 \), i.e., \( Q \) has only one sink vertex and thus \( \mathcal{T} \) is a star quiver, the right hand side of (9) can be simplified down to one multiple Littlewood-Richardson coefficient. Indeed, for a partition \( \mu \) with \( \ell(\mu) \leq nd \), the Littlewood-Richardson coefficient \( c^{\mu}_{\nu(1),(f^{nd})} \neq 0 \) if and only if \((S^{\mu}(W)^* \otimes S^{\nu(1)}(W) \otimes \det f^f_W)^{\text{GL}(W)} \neq 0\), where \( W = \mathbb{C}^{nd} \). This is further equivalent to saying that the weight space of weight \( f \) that occurs in the weight space decomposition of \((S^{\mu}(W)^* \otimes S^{\nu(1)}(W))^{\text{SL}(W)} \) is not zero. Finally, using Proposition 4.3(3), we see that this is equivalent to \( \mu \) being equal to \( \nu(1) \) plus \( f \) columns of length \( nd \) and \( c^{\mu}_{\nu(1),(f^{nd})} = 1 \). Thus, we get that

\[ \dim \text{SI}(\mathcal{T}, \hat{\beta})_{\hat{\sigma}} = c^{\nu(1) + (f^{nd})}_{\lambda(1), \ldots, \lambda(m)(f^d), \ldots, (f^d)}. \]

This can be further expressed as a single Littlewood-Richardson coefficient via Proposition 4.2.

\[ \square \]

As a consequence of Proposition 3.4 and Proposition 4.7, we obtain the following interesting combinatorial identity.
Corollary 4.10. Let \( d \) and \( n \) be two positive integers and let \( \lambda = (\lambda_1, \ldots, \lambda_d) \) and \( \nu = (\nu_1, \ldots, \nu_d) \) be two partitions of length at most \( d \) such that \( n\lambda_1 \geq \nu_1 \). Then
\[
\sum_{\mu(1), \ldots, \mu(n)} c^\lambda_{\mu(1), \ldots, \mu(n)} c^\nu_{\mu(1), \ldots, \mu(n)} = c^{\nu + (\lambda_1^{nd})}_{\lambda_1^{d}, \ldots, (\lambda_d^{d})},
\]
where the sum on the left hand side is over all partitions \( \mu(1), \ldots, \mu(n) \) of length at most \( n \).

Proof. Let \( Q \) be the \( n \)-Kronecker quiver
\[Q = \begin{array}{c} n \text{ arrows} \\
\vdots \\
x_1 \end{array} \rightarrow \begin{array}{c} \vdots \\
y_1 \end{array}\]
and let \( \beta = (d, d) \). Let \( \alpha \) be the dimension vector of the flag-extension quiver \( Q_\beta \) that takes the following values at the vertices of \( F(x_1) \):
\[
\lambda_1 - \lambda_2, \lambda_1 - \lambda_3, \ldots, \lambda_1 - \lambda_d, \lambda_1 = \alpha(x_1).
\]
At the vertices of the flag \( F(y_1) \), \( \alpha \) takes the following values:
\[
\alpha(y_1) = n\lambda_1 - \nu_d, n\lambda_1 - \nu_{d-1}, \ldots, n\lambda_1 - \nu_1.
\]
Then, using the same strategy as in the proof of Proposition 4.7, we get that
\[
\dim \text{SI}(Q_\beta, \beta)_{\tilde{\sigma}} = \sum_{\mu(1), \ldots, \mu(n)} c^\lambda_{\mu(1), \ldots, \mu(n)} c^\nu_{\mu(1), \ldots, \mu(n)},
\]
where \( \tilde{\sigma} = \langle \alpha, \cdot \rangle_{Q_\beta} \). Furthermore, it follows from Theorem 3.3 and Proposition 3.4 that
\[
\dim \text{SI}(Q_\beta, \beta)_{\tilde{\sigma}} = \dim \text{SI}(\mathcal{T}, \beta)_{\tilde{\sigma}},
\]
where \( \tilde{\beta} = I(\beta) \) and \( \tilde{\sigma} = \langle I(\alpha), \cdot \rangle_{\mathcal{T}} \). Finally, using Proposition 4.7 and Remark 4.9, we can write
\[
\dim \text{SI}(\mathcal{T}, \beta)_{\tilde{\sigma}} = c^{\nu + (\lambda_1^{nd})}_{\lambda_1^{d}, \ldots, (\lambda_d^{d})},
\]
which together with (12) and (13) yields the desired identity. \( \square \)

Remark 4.11. If we let \( \lambda = \nu = (x^d) \) for some non-negative integer \( x \), then the left hand side of (11) is precisely \( \dim \text{SI}(Q, (d, d))_{(x, -x)} \) where \( Q \) is the \( n \)-Kronecker quiver. Note that in this case \( \dim \text{SI}(Q, (d, d))_{(x, -x)} \) is a parabolic Kostka coefficient.

4.4. Hive-type polytopes for quiver semi-invariants. Our goal in this subsection is to find a polytopal description for constants of the form
\[
K_{\lambda_{\mu}, f(d, n)} := \sum_{\ell(\mu) \leq nd} c^\mu_{\lambda(1), \ldots, \lambda(m), (f^d), \ldots, (f^d)} c^\nu_{\nu(1), \ldots, \nu(\ell), (f^{nd})},
\]
where \( f, d, \ell, m, n \) are fixed positive integers and \( \lambda(1), \ldots, \lambda(m), \nu(1), \ldots, \nu(\ell) \) are fixed partitions such that \( \sum_{i=1}^m |\lambda(i)| = \sum_{j=1}^\ell |\nu(j)| \). As we have seen in Section 4.2, these types of structure constants occur as dimensions of weight spaces of semi-invariants for \( Q_\beta \) and \( \mathcal{T} \).
We begin by applying Proposition 4.2 to the terms of the sum in the definition of $K_{\Delta \Psi, f}(d, n)$. To this end, we first extend each of the partitions $\lambda(i), \nu(j), (f^d)$, and $(f^{nd})$ by adding zero parts so that their length is at most $\sum_{i=1}^{l} \ell(\lambda(i)) + \sum_{j=1}^{\ell} \ell(\nu(j)) + nd$. Using (3), we next construct the partitions $\gamma(1) \subset \gamma(2)$ and $\gamma(3) \subset \gamma(4)$ such that

$$c_{\gamma(1), \mu} = c_{\gamma(2)}^{(1)} \text{ and } c_{\nu(1), \ell} = c_{\mu, \gamma(3)}^{(4)}.$$ 

Note that $\gamma(1), \gamma(2), \gamma(3)$, and $\gamma(4)$ have at most $N$ parts where

$$N := (m + n + l + 1) \left( \sum_{i=1}^{m} \ell(\lambda(i)) + \sum_{j=1}^{\ell} \ell(\nu(j)) + nd \right).$$

It now follows from Proposition 4.2 that

$$K_{\Delta \Psi, f}(d, n) = \sum_{\ell(\mu) \leq nd} c_{\gamma(1), \mu} \cdot c_{\mu, \gamma(3)}^{(4)}.$$ 

Let us now consider the polytope obtained by gluing two hive polytopes as follows:

In other words, the resulting polytope, denoted by $P$, consists of tuples of non-negative numbers $(x_{i,j}, y_{i,j}, t_{i,j}, \tilde{x}_{i,j}, \tilde{y}_{i,j}, \tilde{t}_{i,j})$ such that

1. $x_{i,0} = \gamma_{i+1}(1), t_{0,k} = \gamma_{k+1}(2), \forall i, k \in \{0, \ldots, N - 1\}$;
2. $y_{j,N-1-j} = \gamma_{j,N-1-j}, \forall j \in \{0, \ldots, N - 1\}$;
3. $\tilde{x}_{i,0} = \gamma_{i+1}(3), \tilde{t}_{0,k} = \gamma_{k+1}(4), \forall i, k \in \{0, \ldots, N - 1\}$;
4. $\sum_{j=0}^{N-1} y_{j,N-1-j} = |\gamma(2)| - |\gamma(1)| = |\gamma(4)| - |\gamma(3)|$;
5. $(x_{i,j}, y_{i,j}, t_{i,j})$ and $(\tilde{x}_{i,j}, \tilde{y}_{i,j}, \tilde{t}_{i,j})$ are $N$-hives.

**Remark 4.12.** Note that the number of lattice points of $P$ is precisely

$$\sum_{\mu} c_{\gamma(1), \mu} \cdot c_{\mu, \gamma(3)}^{(4)};$$

where the sum is over all partitions $\mu$ of length at most $N$. Furthermore, these numbers turn out to be the tensor product multiplicities for extremal weight crystals of type $A_\infty$ [Kwo09]. They have been also studied in [Chi08] in the context of Horn-type eigenvalue problems and long exact sequences of finite abelian $p$-groups.

In what follows, if $a = (x_{i,j}, y_{i,j}, t_{i,j}, \tilde{x}_{i,j}, \tilde{y}_{i,j}, \tilde{t}_{i,j}) \in P$, we denote $y_{nd+j,N-1-(nd+j)}$ by $a_j, \forall j \in [N - nd]$. 

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Remark 4.14. The linear inequalities defining the polytope \( P \) can be written in the form of an integer linear program
\[
A \cdot x \leq b,
\]
where the entries of \( A \) are 0, 1, and \(-1\), and the entries of \( b \) are homogeneous linear integral forms in the parts of the partitions \( \lambda(i), \nu(j), \) and \( f \). This is a combinatorial linear program in the sense of Tardos [Tar86].

Proposition 4.15. (1) With the same notation as above, the number of lattice points of \( P \) is precisely \( K \).

(2) With the same notation as in Proposition 4.7, \( \text{dim} \, \text{SI}(\mathcal{T}, \hat{\beta})_{\overline{a}} \) is equal to the number of lattice points of \( P \). Furthermore,
\[
\text{dim} \, \text{SI}(\mathcal{T}, \hat{\beta})_{\overline{a}} = 0 \iff P \neq \emptyset.
\]

Proof. (1) It follows from the considerations above that the number of lattice points of \( P \) is
\[
\sum_{\mu} c_{\gamma(2)}^{(2)} \cdot c_{\mu, \gamma(3)}^{(4)},
\]
where the sum is over all partitions \( \mu \) with \( \ell(\mu) \leq N \) whose last \( N-nd \) parts are zero. But this is precisely \( K \).

(2) The first claim of this part follows from Proposition 4.7 and part (1). Finally let us prove that (15) holds. The implication “\( \implies \)” is obvious. For the other implication, assume that \( P \neq \emptyset \) and let \( v \) be one of its vertices. Then \( v \) must have rational coefficients, and therefore \( r \cdot v \) is a lattice point of \( P \) for some positive integer \( r \), and thus \( \text{dim} \, \text{SI}(\mathcal{T}, \hat{\beta})_{r, \overline{a}} = 0 \) by parts (1) and (2). It now follows from the Saturation Property stated in Theorem 2.3 that \( \text{dim} \, \text{SI}(\mathcal{T}, \hat{\beta})_{\overline{a}} = 0 \). \( \square \)

5. Moment cones for quivers and the proof of Theorem 1.1

Let \( Q = (Q_0, Q_1, t, h) \) be a connected acyclic quiver and \( \beta \in \mathbb{Z}^Q_0 \) be a sincere dimension vector of \( Q \). If \( U(\beta(x)) \) is the group of \( \beta(x) \times \beta(x) \) unitary matrices for every \( x \in Q_0 \), then
\[
U(\beta) := \prod_{x \in Q_0} U(\beta(x))
\]
is a maximal compact subgroup of \( \text{GL}(\beta) \). The conjugation action of \( U(\beta) \) on \( \text{rep}(Q, \beta) \) is Hamiltonian with the moment map given by
\[
\phi : \text{rep}(Q, \beta) \to \text{Herm}(\beta)
\]
\[
W \mapsto \phi(W) := \left( \sum_{a \in Q_1} W(a)^* \cdot W(a) - \sum_{a \in Q_1} W(a) \cdot W(a)^* \right)_{x \in Q_0}
\]
where \( \text{Herm}(\beta) := \prod_{x \in Q_0} \text{Herm}(\beta(x)) \) with \( \text{Herm}(\beta(x)) \) being the space of \( \beta(x) \times \beta(x) \) Hermitian matrices for every \( x \in Q_0 \) and \( W(a)^* \) denotes the adjoint of the complex matrix \( W(a) \), i.e., \( W(a)^* \) is the transpose of the conjugate of \( W(a) \). The moment cone corresponding to this moment map is \( \Delta(Q, \beta) \), which is a rational convex polyhedral cone (see [Sja98, Theorem 4.9]) and can be viewed as the cone over the moment polytope of the projectivization of \( \text{rep}(Q, \beta) \) (see [Sja98, Corollary 4.11]).

If \( \lambda = (\lambda_1, \ldots, \lambda_N) \) is a weakly decreasing sequence of real numbers, then the weakly decreasing sequence \((-\lambda_N, \ldots, -\lambda_1)\) will be denoted by \(-\lambda\).

**Example 5.1.** Let \( Q = \bullet \rightarrow \bullet \leftarrow \bullet \) and \( \beta = (n, n, n) \). Then \( \Delta(Q, \beta) \) consists of all triples \((\lambda(1), \lambda(2), -\lambda(3))\) with each \( \lambda(i) \) a weakly decreasing sequence of \( n \) (non-negative) real numbers for which there are positive semi-definite \( n \times n \) Hermitian matrices \( H(1), H(2), \) and \( H(3) \) with spectra \( \lambda(1), \lambda(2), \) and \( \lambda(3), \) respectively, and \( H(3) = H(1) + H(2) \).

Recall that the Klyachko cone, denoted by \( \mathcal{K}(n) \), consists of all triples \((\lambda(1), \lambda(2), \lambda(3))\) of weakly decreasing sequences of \( n \) real numbers for which there are \( n \times n \) Hermitian matrices \( H(1), H(2), \) and \( H(3) \) with spectra \( \lambda(1), \lambda(2), \) and \( \lambda(3), \) respectively, and \( H(3) = H(1) + H(2) \).

Now, let \((\lambda(1), \lambda(2), \lambda(3))\) be a triple of weakly decreasing sequences of \( n \) real numbers, and consider the following sequences of non-negative real numbers:

\[
\begin{align*}
\tilde{\lambda}(1) & := (\lambda_1(1) - \lambda_n(1), \ldots, \lambda_{n-1}(1) - \lambda_n(1), 0), \\
\tilde{\lambda}(2) & := (\lambda_1(2) - \lambda_n(2), \ldots, \lambda_{n-1}(2) - \lambda_n(2), 0), \\
\tilde{\lambda}(3) & := (\lambda_1(3) - (\lambda_n(1) + \lambda_n(2)), \ldots, \lambda_n(3) - (\lambda_n(1) + \lambda_n(2))).
\end{align*}
\]

It is now immediate to see that

\[(\lambda(1), \lambda(2), \lambda(3)) \in \mathcal{K}(n) \iff (\tilde{\lambda}(1), \tilde{\lambda}(2), -\tilde{\lambda}(3)) \in \Delta(Q, \beta). \]

We next explain how to view \( \Delta(Q, \beta) \) as the cone of effective weights associated to a different quiver. While this result holds for general quivers (see [BR87]), we will focus in what follows on bipartite quivers since this suffices for our purposes.

Assume that \( Q \) is a bipartite quiver (not necessarily \( n \)-complete) with source vertices \( x_1, \ldots, x_m \) and sink vertices \( y_1, \ldots, y_\ell \), and let \((Q_\beta, \tilde{\beta})\) be its flag-extension. (Note that we orient our flags slightly differently than in [BR87].)

Let \( \bar{\lambda} = (\lambda(x_i), -\lambda(y_j))_{i \in [m], j \in [\ell]} \) be a tuple of sequences with \( \lambda(x) \) a weakly decreasing sequence of \( \beta(x) \) real numbers for every vertex \( x \in Q_0 \).

**Remark 5.2.** It is immediate to see that \( \bar{\lambda} \) belongs to \( \Delta(Q, \beta) \) if and only if there is a representation \( W \in \text{rep}(Q, \beta) \) such that

1. the spectrum of the Hermitian matrix \( \sum_{i=1}^m W(a)^* \cdot W(a) \) is \( \lambda(x_i) \) for every \( i \in [m] \);

2. the spectrum of Hermitian matrix \( \sum_{j=1}^\ell W(a)^* \cdot W(a) \) is \( \lambda(y_j) \) for every \( j \in [\ell] \).

This shows that a necessary condition for \( \bar{\lambda} \) to belong to \( \Delta(Q, \beta) \) is that \( \lambda(x_i), i \in [m], \) and \( \lambda(y_j), j \in [\ell], \) are non-negative sequences. \( \square \)
Now let $\tilde{\sigma}_\lambda \in \mathbb{R}^{(Q_\beta)_0}$ be the real weight defined as follows: If $x$ is a source vertex of $Q$, the values of $\tilde{\sigma}_\lambda$ along the $\beta(x)$ vertices of the flag

\begin{equation}
\mathcal{F}(x) : \bullet \to \bullet \to \cdots \to \bullet \to x
\end{equation}

are

\begin{equation}
\lambda_1(x) - \lambda_2(x), \ldots, \lambda_{\beta(x)-1}(x) - \lambda_{\beta(x)}(x), \lambda_{\beta(x)}(x).
\end{equation}

If, instead, $y$ is a sink vertex of $Q$, the values of $\tilde{\sigma}_\lambda$ along the $\beta(y)$ vertices of the flag

\begin{equation}
\mathcal{F}(x) : \bullet \to \bullet \to \cdots \to \bullet \to y
\end{equation}

are

\begin{equation}
-\lambda_{\beta(y)}(y), \lambda_{\beta(y)}(y) - \lambda_{\beta(y)-1}(y), \ldots, \lambda_2(y) - \lambda_1(y).
\end{equation}

**Proposition 5.3** (compare to [BR87]). Let $Q$ be a bipartite quiver and $\beta$ a sincere dimension vector of $Q$. Let $T$ be the function from the set of all tuples $\Delta = (\lambda(x_i), -\lambda(y_j))_{i \in [m], j \in [r]}$ as above to $\mathbb{R}^{(Q_\beta)_0}$ defined by $T(\lambda) = \tilde{\sigma}_\lambda$. Then

\begin{equation}
T(\Delta(Q, \beta)) = \text{Eff}(Q_\beta, \tilde{\beta}),
\end{equation}

and $T$ is an isomorphism of rational convex polyhedral cones.

To prove this result, we require the following very useful lemma.

**Lemma 5.4** (see [CBG02, Sec. 3.4]). (1) Let $\sigma(1), \ldots, \sigma(N - 1)$ be non-positive real numbers. Then the following are equivalent:

(a) There exist matrices $W_i \in \mathbb{C}^{i \times (i+1)}$, $1 \leq i \leq N - 1$, such that

\begin{align*}
W_i \cdot W_i^* - W_{i-1}^* \cdot W_{i-1} &= -\sigma(i) \cdot \text{Id}_{\mathbb{C}^i} & \text{for } 2 \leq i \leq N - 1, \\
W_1 \cdot W_1^* &= -\sigma(1).
\end{align*}

(b) There exists an $N \times N$ Hermitian matrix $H$ (= $W_{N-1}^* \cdot W_{N-1}$) with eigenvalues

\begin{equation}
\gamma(i) = -\sum_{i \leq j \leq N-1} \sigma(j), \; \forall \; 1 \leq i \leq N - 1
\end{equation}

and $\gamma(N) = 0$.

(2) Let $\sigma(1), \ldots, \sigma(N - 1)$ be non-negative real numbers. Then the following are equivalent:

(c) There exist matrices $W_i \in \mathbb{C}^{(i+1) \times i}$, $1 \leq i \leq N - 1$, such that

\begin{align*}
W_i^* \cdot W_i - W_{i-1} \cdot W_{i-1}^* &= \sigma(i) \cdot \text{Id}_{\mathbb{C}^i} & \text{for } 2 \leq i \leq N - 1, \\
W_1^* \cdot W_1 &= \sigma(1).
\end{align*}

(d) There exists an $N \times N$ Hermitian matrix $H$ (= $W_{N-1} \cdot W_{N-1}^*$) with eigenvalues

\begin{equation}
\gamma(i) = \sum_{i \leq j \leq N-1} \sigma(j), \; \forall \; 1 \leq i \leq N - 1,
\end{equation}

and $\gamma(N) = 0$. 

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We are now ready to prove Proposition 5.3.

**Proof of Proposition 5.3.** Let \( \Delta = (\lambda(x_i), -\lambda(y_j))_{i \in [m], j \in [\ell]} \) be a tuple of sequences with \( \lambda(x) \) a weakly decreasing sequence of \( \beta(x) \) real numbers for every vertex \( x \) of \( Q \). From Remark 5.2 and Lemma 5.4 we obtain that \( \Delta \in \Delta(Q, \beta) \) if and only if there exists \( \tilde{W} \in \text{rep}(Q_\beta, \tilde{\beta}) \) such that

\[
\sum_{a \in Q_1} \tilde{W}(a)^* \cdot \tilde{W}(a) - \sum_{a \in Q_1 \backslash a = x} \tilde{W}(a) \cdot \tilde{W}(a)^* = \tilde{\sigma}_\Delta(x) \cdot \text{Id}_{\tilde{\beta}(x)} \quad \forall x \in (Q_\beta)_0.
\]

It now follows from Theorem 2.3 that \( T(\Delta(Q, \beta) \cap Z(Q_\beta)_0) \subseteq \text{Eff}(Q_\beta, \tilde{\beta}) \cap Z(Q_\beta)_0 \). To prove the other inclusion, let \( \tilde{\sigma} \in \text{Eff}(Q_\beta, \tilde{\beta}) \) be any effective weight. Then \( \tilde{\sigma} \) is non-negative/non-positive along the vertices of the flag \( F(x) \) if \( x \) is a source/sink of \( Q \) by Lemma 2.8. For any such \( \tilde{\sigma} \), consider the partitions

\[
\lambda_{\tilde{\sigma}}(x) := \begin{cases} 
\left( \sum_{i \leq j \leq \beta(x) \cap \tilde{\beta}(x)} \tilde{\sigma}(j) \right)_{i \in [\beta(x)]} & \text{if } x \text{ is a source}, \\
\left( - \sum_{i \leq j \leq \beta(x)} \tilde{\sigma}(j) \right)_{i \in [\beta(x)]} & \text{if } x \text{ is a sink},
\end{cases}
\]

where \( \tilde{\sigma}(k) \) denotes the value of \( \tilde{\sigma} \) at the \( k^{th} \) vertex of the flag \( F(x) \) as we traverse the flag from left/right to right/left for any source/sink vertex \( x \in Q_0 \) and \( k \in [\beta(x)] \). Then, using Lemma 5.4 once again, we get that \( \lambda_{\tilde{\sigma}} := (\lambda_{\tilde{\sigma}}(x_i), -\lambda_{\tilde{\sigma}}(y_j))_{i \in [m], j \in [\ell]} \) belongs to \( \Delta(Q, \beta) \) and \( T(\Delta_{\tilde{\sigma}}) = \tilde{\sigma} \).

This shows that \( T(\Delta(Q, \beta) \cap Z(Q_\beta)_0) = \text{Eff}(Q_\beta, \tilde{\beta}) \cap Z(Q_\beta)_0 \), which implies the claim of the proposition since \( \Delta(Q, \beta) \) and \( \text{Eff}(Q_\beta, \tilde{\beta}) \) are both rational convex polyhedral cones. \( \square \)

Finally, we are ready to prove our main result.

**Proof of Theorem 1.1.** Let \( \tilde{\sigma} \in Z(Q_\beta)_0 \) be an integral weight of \( Q_\beta \). For each \( i \in [m] \), let us label the vertices of the flag \( F(x_i) \) of \( Q_\beta \) as follows

\[
F(x_i) : \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow x_i \rightarrow \bullet
\]

and define the sequence

\[
\lambda(i) = \left( \sum_{k \leq r \leq \beta(x_i) \cap \tilde{\beta}(x_i)} \tilde{\sigma}(i_r) \right)_{k \in [\beta(x_i)]}.
\]

For each \( j \in [\ell] \), let us label the vertices of the flag \( F(y_j) \) of \( Q_\beta \) as follows

\[
F(y_j) : \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet
\]

and define the sequence

(18) \[
\lambda(i) = \left( \sum_{k \leq r \leq \beta(y_j) \cap \tilde{\beta}(y_j)} \tilde{\sigma}(i_r) \right)_{k \in [\beta(y_j)]}.
\]
and define the sequence

\[
(19) \quad \nu(j) = \left( - \sum_{k \leq r \leq \beta(y_j)} \tilde{\sigma}(j_r) \right)_{k \in [\beta(y_j)]}.
\]

(1) Let us assume now that \(\tilde{\sigma}\) is a non-trivial integral weight of \(Q_\beta\) with \(\tilde{\sigma} \cdot \tilde{\beta} = 0\) and such that \(\tilde{\sigma}\) is non-negative/non-positive along the vertices of the flag \(\mathcal{F}(x)\) if \(x\) is a source/sink. Then \(\lambda(1), \ldots, \lambda(m)\), and \(\nu(1), \ldots, \nu(\ell)\) are partitions.

Since \(Q\) is acyclic, we can write

\[
\tilde{\sigma} = \langle \alpha, \cdot \rangle_{Q_\beta}
\]

for a unique \(\alpha \in \mathbb{Z}^{(Q_\beta)^\circ}\). As \(\tilde{\sigma}\) is assumed to be non-negative at the vertices of \(\mathcal{F}(x_i)\), it is immediate to see that \(\alpha\) is non-negative at the vertices of the flags \(\mathcal{F}(x_i), i \in [m]\), and thus

\[
f := \sum_{i=1}^m \alpha(x_i) \geq 0.
\]

Let us also consider the weight \(\tilde{\sigma} := \langle \mathcal{I}(\alpha), \cdot \rangle_{T}\) of \(T\), and note that it satisfies conditions (7) and (8). If \(\alpha\) is a dimension vector, then \(K_{\tilde{\sigma}} = \dim \text{SI}(T, \tilde{\beta})_{\tilde{\sigma}}\) by Theorem 3.3, where \(\tilde{\beta} = \mathcal{I}(\beta)\) and \(\tilde{\sigma} = \langle \mathcal{I}(\alpha), \cdot \rangle_{T}\). If \(\alpha\) is not a dimension vector of \(Q_\beta\), then \(\mathcal{I}(\alpha)\) is not a dimension vector of \(T\) by Remark 3.6, and thus \(K_{\tilde{\sigma}} = \dim \text{SI}(T, \tilde{\beta})_{\tilde{\sigma}} = 0\) by Proposition 2.5. Either way, we obtain that

\[
(20) \quad K_{\tilde{\sigma}} = \dim \text{SI}(T, \tilde{\beta})_{\tilde{\sigma}}.
\]

Now let us consider the polytope

\[
\mathcal{P}_{\tilde{\sigma}} := \mathcal{P}_{\Delta_{\tilde{\sigma}}, f}(d, n),
\]

where \(f, n\), the partitions \(\lambda(i)\) and \(\nu(j)\) are as above, and \(d := \sum_{i=1}^m \beta(x_i)\). Then it follows from (20) and Propositions 4.7 and 4.15 that

\[
(21) \quad K_{\tilde{\sigma}} = \text{the number of lattice points of } \mathcal{P}_{\tilde{\sigma}},
\]

and this completes the proof of the first part of the theorem.

(2) Let \(\lambda = (\lambda(x_i), -\lambda(y_j))_{i \in [m], j \in [\ell]}\) be a tuple of sequences with \(\lambda(x)\) a weakly decreasing sequence of \(\beta(x)\) integers for every \(x \in Q_0\), and let \(\tilde{\sigma}_\lambda\) be the weight of \(Q_\beta\) corresponding to \(\lambda\) defined by (16) and (17).

We assume that the \(\lambda(x_i)\) and \(\lambda(y_j)\) are partitions since otherwise we know that \(\lambda \notin \Delta(Q, \beta)\) by Remark 5.2. Then \(\tilde{\sigma}_{\Delta_{\mathcal{F}(x)}} \geq 0\) for all \(i \in [m]\), and \(\tilde{\sigma}_{\Delta_{\mathcal{F}(y)}} \leq 0\) for all \(j \in [\ell]\). Furthermore, writing \(\tilde{\sigma}_\lambda = \langle \alpha, \cdot \rangle_{Q_\beta}\), we get that \(\alpha(x_i) = \lambda_1(x_i)\) for all \(i \in [m]\), and thus

\[
f = \sum_{i=1}^m \lambda_1(x_i),
\]

where \(\lambda_1(x_i)\) is the largest part of the partition \(\lambda(x_i), \forall i \in [m]\). Also the partitions corresponding to \(\tilde{\sigma}_\lambda\) defined by (18) and (19) are precisely the partitions \(\lambda(x_i)\) and \(\lambda(y_j)\). It
now follows from part (1) and Propositions 5.3 and 4.15 that
\[ \lambda \in \Delta(Q, \beta) \iff \sum_{\ell(\mu) \leq n d} c^\mu_{\lambda(x_1), \ldots, \lambda(x_m)}(f^{d_1}), \ldots, (f^{d_n}) \cdot c^\mu_{\lambda(y_1), \ldots, \lambda(y_l)}(f^{d_1}) \neq 0 \iff \mathcal{P}_{\bar{\lambda}} \neq \emptyset. \]

Since $\mathcal{P}_{\bar{\lambda}}$ can be described as a combinatorial linear program (see Remark 4.14), deciding whether $\lambda$ belongs to $\Delta(Q, \beta)$ can be done in strongly polynomial time using Tardos’s [Tar86] combinatorial linear programming algorithm.

Finally, the generic semi-stability problem for $Q_{\beta}$, in particular for $Q$, follows immediately from part (1), Proposition 4.15, and Remark 2.7. □

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