A LIOUVILLE-TYPE THEOREMS FOR SOME CLASSES OF COMPLETE RIEMANNIAN ALMOST PRODUCT MANIFOLDS AND FOR SPECIAL MAPPINGS OF COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. In the present paper we prove Liouville-type theorems: non-existence theorems for some complete Riemannian almost product manifolds and submersions of complete Riemannian manifolds which generalize similar results for compact manifolds.

Keywords: complete Riemannian manifold, two complementary orthogonal distributions, submersion, non-existence theorems.

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1. Introduction

S. Bochner devised an analytic technique to obtain non-existence theorems for some geometric objects on a closed (compact, boundaryless) Riemannian manifold, under some curvature assumption (see [1]). Currently, there are two different points of view about classical Bochner technique; the first one uses the Green’s divergence theorem, and the second uses the Hopf’s theorem which were obtained from the Stokes’s theorem and classical maximum principle for compact Riemannian manifold, respectively. In particular, a good account of applications of the Bochner technique in differential geometry of Riemannian almost product manifolds and submersions may be found in [2]. We recall here that a Riemannian almost product manifold is a Riemannian manifold \((M, g)\) equipped with two complementary orthogonal distributions. For instance, the total space of any submersion of an arbitrary Riemannian manifold onto another Riemannian manifold admits such a structure.

In the present paper we will use a generalized Bochner technique: our proofs will be based on generalized divergence theorems and a generalized maximal principle for complete, noncompact Riemannian manifolds (see [3]). We will prove Liouville type non-existence theorems for some complete, noncompact Riemannian almost product manifolds, conformal and projective diffeomorphisms and submersions of complete, noncom-
pact Riemannian manifolds which generalize similar well known results for closed manifolds.

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2. Three global divergence theorems

Let \((M, g)\) an \(n\)-dimensional oriented Riemannian manifold \((M, g)\) with volume form \(dVol_g = \sqrt{\det g} \, dx^1 \wedge \ldots \wedge dx^n\) for positively oriented local coordinates \(x^1, \ldots, x^n\). Then we can define the divergence \(\text{div} X\) of the vector field \(X\) via the formula 
\[
d(i_X dVol_g) = (\text{div} X) dVol_g
\]
where \(i_X\) denotes contraction with \(X\) (see [4, p. 281-283]). Furthermore, if \(\omega\) is an \((n - 1)\)-form on \((M, g)\), then we can write \(\omega = i_X dVol_g\) where \(X = \ast_g \omega\) for the Hodge star operator relative to \(g\). Thus when \(\omega\) is an \((n - 1)\)-form with compact support in an orientable \(n\)-dimensional Riemannian manifold \((M, g)\) without boundary, the Stokes theorem \(\int_M d\omega = 0\) follows the classic Green divergence theorem 
\[
\int_M (\text{div} X) \, dVol_g = 0
\]
if the vector field \(X\) has compact support in a (not necessarily oriented) Riemannian manifold \((M, g)\) (see [5, p. 11]). On the other hand, there are some \(L^p(M, g)\)-extensions of the classical Green divergence theorem to complete, noncompact Riemannian manifolds without boundary. Firstly, we formulate the following (see [6])

**Theorem 1.** Let \((M, g)\) be geodesically complete Riemannian manifold and \(X\) be a smooth vector field on \((M, g)\) which satisfies the conditions \(\|X\| \in L^1(M, g)\) and \(\text{div} X \in L^1(M, g)\), then \(\int_M (\text{div} X) \, dVol_g = 0\) where \(\|X\|\) is a norm of the vector field \(X\) induced by the metric \(g\).

Later, L. Karp showed in [7] the generalized version of Theorem 1. Namely, he has provided the following theorem.

**Theorem 2.** Let \((M, g)\) be a complete, noncompact Riemannian manifold and \(X\) be a smooth vector field on \((M, g)\) which satisfies the condition
\[
\lim_{r \to \infty} \inf \frac{1}{r} \int_{B(r)/B(2r)} \| X \| \, dV_{g} = 0
\]  
(1.2)

for the geodesic ball \( B(r) \) of radius \( r \) with the center at some fixed \( x \in M \). If \( \text{div} \, X \) has an integral (i.e. if either \( (\text{div} \, X)^{+} \) or \( (\text{div} \, X)^{-} \) is integrable) then \( \int_{M} (\text{div} \, X) \, dV_{g} = 0 \).

In particular, from the above theorem we conclude that if outside some compact set \( \text{div} \, X \) is everywhere \( \geq 0 \) (or \( \leq 0 \)) then \( \int_{M} (\text{div} \, X) \, dV_{g} = 0 \).

In conclusion, we formulate the third generalized Green’s divergence theorem (see [8]; [9]), which can be regarded, as a consequence of the above two theorems and Yau lemma from [10].

**Theorem 3.** Let \( X \) be a smooth vector field on a connected complete, noncompact and oriented Riemannian manifold \((M, g)\), such that \( \text{div} \, X \geq 0 \) (or \( \text{div} \, X \leq 0 \)) everywhere on \((M, g)\). If the norm \( \| X \| \in L^{1}(M, g) \), then \( \text{div} \, X = 0 \).

The Laplace-Beltrami operator of any \( f \in C^{\infty}M \) is defined as \( \Delta f = \text{div}_{g} (\text{grad} \, f) \) where \( \text{grad} \, f \) is the unique vector field that satisfies \( g(X, \text{grad} \, f) = df(X) \) for all vector fields \( X \) on \( M \). The scalar function \( f \in C^{\infty}M \) is said to be harmonic if it satisfies \( \Delta f = 0 \). It is well known, that if \( f \in C^{\infty}M \) is a harmonic function on any complete Riemannian manifold satisfying \( f \in L^{p}(M, g) \) for some \( 1 < p < \infty \), then \( f \) is constant (see [10]).

In addition, we recall that the scalar function \( f \in C^{\infty}M \) is called subharmonic (resp. superharmonic) if \( \Delta f \geq 0 \) (resp. \( \Delta f \leq 0 \)). In particular, if \((M, g)\) is compact then every harmonic (subharmonic and superharmonic) function is constant by the Hopf’s theorem.

On the other hand, Yau has proved in [10] that any subharmonic function \( f \in C^{2}M \) defined on a complete, noncompact Riemannian manifold with \( \int_{M} \| df \| dV_{g} < \infty \) is harmonic. Then based on this statement (or on the Theorem 3) we conclude that the following lemma is true.

**Lemma.** If \((M, g)\) is a complete, noncompact Riemannian manifold, then any superharmonic function \( f \in C^{2}M \) with gradient in \( L^{1}(M, g) \) is harmonic.
**Proof.** Let \((M, g)\) be a complete Riemannian manifold and \(f\) be a scalar function such that 
\(f \in C^2 M, \Delta f \leq 0\) and \(\|\text{grad} \, f\| \in L^1(M, g)\). If we suppose that \(\phi = -f\) then the above conditions can be written in the form \(\phi \in C^2 M, \Delta \phi \geq 0\) and \(\|\text{grad} \, \phi\| \in L^1(M, g)\). In this case, from the Yau statement we conclude that \(\Delta \phi = 0\) and hence \(f = -\phi\) is a harmonic function.

3. Liouville-type theorems for some complete Riemannian almost product manifolds

Let \((M, g)\) be an \(n\)-dimensional \((n \geq 2)\) Riemannian manifold with the Levi-Civita connection \(\nabla\) and \(TM = \mathcal{V} \oplus \mathcal{H}\) be an orthogonal decomposition of the tangent bundle \(TM\) into vertical \(\mathcal{V}\) and horizontal \(\mathcal{H}\) distributions of dimensions \(n - m\) and \(m\), respectively. We shall use the symbols \(V\) and \(H\) to denote the orthogonal projections onto \(\mathcal{V}\) and \(\mathcal{H}\), respectively. In this case we can define a Riemannian almost product manifold (see [11]) as the triple \((M, g, P)\) for \(P = V - H\), where \((M, g)\) is a Riemannian manifold \(M\) and \(P\) is a \((1,1)\)-tensor field on \(M\) satisfying \(P^2 = \text{id}\) and \(g(P, P) = g\). In addition, the eigenspaces of \(P\) corresponding to the eigenvalues \(1\) and \(-1\), at each point, determine two orthogonal complementary distributions \(\mathcal{V}\) and \(\mathcal{H}\).

The second fundamental form \(Q_H\) and the integrability tensor \(F_H\) of \(\mathcal{H}\) are defined by (see [12, p. 148])

\[
Q_H = \frac{1}{2}V(\nabla_{hX}HY + \nabla_{hY}HX), \quad F_H = \frac{1}{2}V(\nabla_{hX}HY - \nabla_{hY}HX)
\]

for any smooth vector fields \(X\) and \(Y\) on \(M\). It is well known that \(Q_H\) vanishes if and only if \(\mathcal{H}\) is a totally geodesic distribution. We recall that a distribution on a Riemannian manifold is totally geodesic if each geodesic which is tangent to it at point remains for its entire length (see [12, p. 150]). On the other hand, \(F_H\) vanishes if and only if \(\mathcal{H}\) is an integrable distribution. A maximal connected integral manifold of \(\mathcal{H}\) is called a leaf of the foliation. The collection of leaves of \(\mathcal{H}\) is called a foliation of \(M\). By interchanging and we define the corresponding tensor fields \(Q_V\) and \(F_V\) for \(V = H^\perp\).

We define now the mixed scalar curvature of \((M, g)\) as the function
\[
    s_{\text{mix}} = \sum_{a=1}^{m} \sum_{\alpha=m+1}^{n} \sec(E_a, E_\alpha)
\]

where \( \sec(E_a, E_\alpha) \) is the sectional curvature of the mixed plane \( \pi = \text{span}\{E_a, E_\alpha\} \) for the local orthonormal frames \( \{E_1, \ldots, E_m\} \) and \( \{E_{m+1}, \ldots, E_n\} \) on \( TM \) adapted to \( \mathcal{V} \) and \( \mathcal{H} \), respectively (see [11]; [13, p. 23]). It is easy to see that this expression is independent of the chosen frames. With this in hand we can now state the formula which can be found in [14] and [15]. Namely, the following formula:

\[
    \text{div}(\xi_{\mathcal{V}} + \xi_{\mathcal{H}}) =
\]

\[
    = s_{\text{mix}} + \|Q_{\mathcal{V}}\|^2 + \|Q_{\mathcal{H}}\|^2 - \|F_{\mathcal{V}}\|^2 - \|F_{\mathcal{H}}\|^2 - \|\xi_{\mathcal{V}}\|^2 + \|\xi_{\mathcal{H}}\|^2 \quad (2.1)
\]

where \( \xi_{\mathcal{V}} = \text{trace}_g Q_{\mathcal{V}} \) and \( \xi_{\mathcal{H}} = \text{trace}_g Q_{\mathcal{H}} \) are the mean curvature vectors of \( \mathcal{V} \) and \( \mathcal{H} \), respectively (see [11]; [12, p. 149]).

Assume that \( \mathcal{V} \) and \( \mathcal{H} \) are totally umbilical distributions, i.e. \( Q_{\mathcal{V}} = (n - m)^{-1} g(V, V) \otimes \xi_{\mathcal{V}} \) and \( Q_{\mathcal{H}} = m^{-1} g(H, H) \otimes \xi_{\mathcal{H}} \) (see [11]; [12, p. 151]). In this case the formula (2.1) can be rewrite in the form (see [15]; [16]; [17])

\[
    \text{div}(\xi_{\mathcal{V}} + \xi_{\mathcal{H}}) = s_{\text{mix}} - \|F_{\mathcal{V}}\|^2 - \|F_{\mathcal{H}}\|^2 - \frac{n - m - 1}{n - m} \|\xi_{\mathcal{V}}\|^2 + \frac{m - 1}{m} \|\xi_{\mathcal{H}}\|^2 \quad (2.2)
\]

If in addition to the assumption above we now suppose that \((M, g)\) is a connected complete and oriented Riemannian manifold without boundary and with nonpositive mixed scalar curvature \( s_{\text{mix}} \), then from (3.2) we obtain the inequality \( \text{div} \left( \xi_{\mathcal{V}} + \xi_{\mathcal{H}} \right) \leq 0 \). If at the same time, \( \|\xi_{\mathcal{V}} + \xi_{\mathcal{H}}\| \in L^1(M, g) \) then by Theorem 4 we conclude that \( \text{div} \left( \xi_{\mathcal{V}} + \xi_{\mathcal{H}} \right) = 0 \).

In this case, from (3.2) we obtain \( \|\xi_{\mathcal{H}}\|^2 = \|\xi_{\mathcal{V}}\|^2 = \|N\|^2 = 0 \). It means that \( \mathcal{V} \) and \( \mathcal{H} \) are two integrable distributions with totally geodesic integral manifolds (totally geodesic foliations). We fix now a point \( x \in M \) and let \( M_1 \) and \( M_2 \) be the maximal integral manifolds of distributions through \( x \), respectively. Then by the de Rham decomposition theorem (see [4, p. 187]) we conclude that if \((M, g)\) is a simply connected Riemannian manifold then it is isometric to the direct product \((M_1 \times M_2, g_1 \oplus g_2)\) of some Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) for the Riemannian metric \( g_1 \) and \( g_2 \) which induced by \( g \) on \( M_1 \) and \( M_2 \), respectively. In addition, we recall that every simply connected mani-
fold $M$ is orientable. Summarizing, we formulate the statement which generalizes a theorem on two orthogonal complete totally umbilical distributions on compact Riemannian manifold with non positive mixed scalar curvature that has been proved in [11]; [15]; [16] and [18].

**Theorem 4.** Let $(M, g)$ be a complete, noncompact and simply connected Riemannian manifold with two orthogonal complementary totally umbilical distributions $\mathcal{V}$ and $\mathcal{H}$ such that their mean curvature vectors $\xi_\mathcal{V}$ and $\xi_\mathcal{H}$ satisfy the condition $\|\xi_\mathcal{V} + \xi_\mathcal{H}\| \in L^1(M, g)$. If the mixed scalar curvature $s_{\text{mix}}$ of $(M, g)$ is nonpositive then $\mathcal{V}$ and $\mathcal{H}$ are integrable and $(M, g)$ is isometric to a direct product $(M_1 \times M_2, \ g_1 \oplus g_2)$ of some Riemannian manifolds $(M_1, g_2)$ and $(M_1, g_2)$ such that integral manifolds of $\mathcal{V}$ and $\mathcal{H}$ correspond to the canonical foliations of the product $M_1 \times M_2$.

We consider now an $(n - 1)$-dimensional totally geodesic distribution $\mathcal{H}$ on $(M, g)$. In this case the formula (2.2) can be rewrite in the form (see [15])

$$\text{div} \xi_\mathcal{V} = s_{\text{mix}} - \|F_H\|^2$$

(2.2)

where $s_{\text{mix}} = \sum_{\alpha = 2}^{m} \sec(e_1, e_\alpha) = \text{Ric}(e_1, e_1)$ is the vertical Ricci curvature for an orthonormal frame $\{e_1, \ldots, e_n\}$ at a point $x \in M$ such that $\mathcal{H}_x = \text{span}\{e_1\}$ and $\mathcal{V}_x = \text{span}\{e_2, \ldots, e_n\}$.

Hence, an immediate consequence of (2.2) and Theorem 4 is following

**Corollary 1.** Let $(M, g)$ be an $n$-dimensional complete noncompact and simply connected Riemannian manifold with $(n - 1)$-dimensional totally geodesic horizontal distribution $\mathcal{H}$. If the vertical Ricci curvature of $(M, g)$ is nonpositive and $\|\xi_\mathcal{V}\| \in L^1(M, g)$ for the mean curvature vector of $\mathcal{V}$, then $\mathcal{H}$ is integrable and $(M, g)$ is isometric to a direct product $(M_1 \times M_2, \ g_1 \oplus g_2)$ of Riemannian manifolds $(M_1, g_2)$ and $(M_2, g_2)$ such that $\dim M_1 = 1$ and integral manifolds of $\mathcal{V}$ and $\mathcal{H}$ correspond to the canonical foliations of the product $M_1 \times M_2$.

The integral formula (2.1) can be reformulated as follows (see [11])

$$2\text{div}_\mathcal{V} \xi_\mathcal{H} + 2\text{div}_\mathcal{H} \xi_\mathcal{V} = 4s_{\text{mix}} + \frac{1}{2}\|\nabla P\|^2 - \|F_V\|^2 - \|F_H\|^2.$$  

(2.3)
where \( \text{div}_V \xi_H = \sum_{a=1}^{m} g(\nabla_{E_a} \xi_H, E_a) \) and \( \text{div}_H \xi_V = \sum_{a=m+1}^{n} g(\nabla_{E_a} \xi_V, E_a) \). If \( \mathcal{V} \) and \( \mathcal{H} \) are integrable distributions then (2.3) can be rewrite in the form

\[
2\text{div}_V \xi_H + 2\text{div}_H \xi_V = 4s_{\text{mix}} + \frac{1}{2} \| \nabla P \|^2. \tag{2.4}
\]

Suppose now that all integral manifolds of the vertical distribution \( \mathcal{V} \) are minimal submanifolds of the Riemannian manifold \((M, g)\) and \( s_{\text{mix}} \geq 0 \). Then from (2.4) we obtain

\[
\text{div}_V \xi_H = 2s_{\text{mix}} + \frac{1}{4} \| \nabla P \|^2 \geq 0. \tag{2.5}
\]

If at least one connected complete and oriented maximal integral manifold \( M' \) of \( \mathcal{V} \) exists. We assume that \( M' \) equipped with the Riemannian metric \( g' \) inherited from \((M, g)\) such that \( \| \xi_H \| \in L^1(M', g') \) for the mean curvature vector \( \xi_H \) of \( \mathcal{H} = \mathcal{V}^\perp \) which belongs to \( M' \) at each point \( x \in M' \). Then by applying Theorem 3 to \( \xi_H \), from (2.5) we get

\[
\text{div}_{g'} \xi_H = 0. \]

Therefore, if all integral manifolds of \( \mathcal{V} \) are connected complete and oriented minimal submanifolds of the Riemannian manifold \((M, g)\) and \( \xi_H \) is a \( L^1 \)-vector field for every of them, then \( \nabla P = 0 \). In this case \( \mathcal{V} \) and \( \mathcal{H} \) are two integrable distributions with totally geodesic integral manifolds (totally geodesic foliations) on \((M, g)\) (see [11]). If at the same time, \((M, g)\) is complete, noncompact and simply connected Riemannian manifold then by the de Rham decomposition theorem (see [4, p. 187]) it is isometric to the direct product \( (M_1 \times M_2, g_1 \oplus g_2) \) of some Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) for the Riemannian metric \( g_1 \) and \( g_2 \) which induced by \( g \) on \( M_1 \) and \( M_2 \), respectively. Summarizing, we formulate the statement which generalizes the main theorem of [12].

**Theorem 5.** Let \((M, g)\) be complete, noncompact and simply connected Riemannian manifold. If the following three conditions are satisfied:

1) \((M, g)\) admits an integrable distribution \( \mathcal{V} \) such that an arbitrary integral manifold \((M', g')\) of \( \mathcal{V} \) which equipped with the Riemannian metric \( g' \) inherited from \((M, g)\) is a connected complete and oriented minimal submanifold of \((M, g)\);
2) the orthogonal complementary distribution $\mathcal{H} = \mathcal{V}^\perp$ is also integrable and its mean curvature vectors $\xi_H$ satisfies the condition $\|\xi_H\| \in L^1(M', g')$;

3) the mixed scalar curvature $s_{mix} \geq 0$, then $(M, g)$ is isometric to a direct product $(M_1 \times M_2, g_1 \oplus g_2)$ of some Riemannian manifolds $(M_1, g_1)$ and $(M_1, g_2)$ such that integral manifolds of $\mathcal{V}$ and $\mathcal{H}$ correspond to the canonical foliations of the product $M_1 \times M_2$.

**Remark 1.** If in addition, at least one closed integral manifold $M'$ of $\mathcal{V}$ exists, then, by applying the classic Green divergence theorem to $\xi_H$, from (2.5) we get

$$\int_{M'} \left( 8s_{mix} + \|\nabla P\|^2 \right) dVol_{g'} = 0$$

where $dVol_{g'}$ is the volume form of $(M', g')$. If $(M', g')$ is non-oriented we can consider its orientable double cover. In this case the inequality $s_{mix} > 0$ is a condition of nonexistence of two orthogonal complementary foliations one of which consists of minimal submanifolds.

### 3. Applications to the theory of projective mappings of Riemannian manifolds

We recall here the definition of *pregeodesic* and *geodesic* curves. Namely, a *pregeodesic curve* is a smooth curve $\gamma: t \in J \subset \mathbb{R} \rightarrow \gamma(t) \in M$ on a Riemannian manifold $(M, g)$, which becomes a geodesic curve after a change of parameter. Let us change the parameter along $\gamma$ so that $t$ becomes an *affine parameter*. Then $\nabla_X X = 0$ for $X = d\gamma/dt$, and $\gamma$ is called a *geodesic curve*. By analyzing of the last equation, one can conclude that either $\gamma$ is an immersion, i.e., $d\gamma/dt \neq 0$ for all $t \in J$, or $\gamma(t)$ is a point of $M$.

Let $(M, g)$ and $(\bar{M}, \bar{g})$ be Riemannian manifolds of dimension $n \geq 2$. Then a smooth map $f: (M, g) \rightarrow (\bar{M}, \bar{g})$ of Riemannian manifolds is a *projective map* if $f(\gamma)$ is a pregeodesic in $(\bar{M}, \bar{g})$ for an arbitrary pregeodesic $\gamma$ in $(M, g)$ (see [19]). In particular, if a projective map $f: (M, g) \rightarrow (\bar{M}, \bar{g})$ is called *totally geodesic* if it maps linearly parametrized geodesics of $(M, g)$ to linearly parametrized geodesics of $(\bar{M}, \bar{g})$. An equiva-
lent definition is that \( f \) is connection-preserving, or affine. The global structure of these maps is investigated in the paper [20].

For a projective diffeomorphism \( f : (M, g) \rightarrow (\overline{M}, \overline{g}) \) we have (see [21, p. 135])

\[
\overline{\text{Ric}} = \text{Ric} + (n - 1)(\nabla d\psi - d\psi \otimes d\psi)
\]

(3.1)

where \( \text{Ric} \) and \( \overline{\text{Ric}} \) denote the Ricci tensors of \((M, g)\) and \((\overline{M}, \overline{g})\), respectively, and

\[
\psi = \frac{1}{2(n+1)} \log \left( \frac{\det \overline{g}}{\det g} \right) + C
\]

(3.2)

for some constant \( C \). Now we can formulate the following

**Theorem 6.** Let \((M, g)\) be a connected complete, noncompact Riemannian manifold and \( f : (M, g) \rightarrow (\overline{M}, \overline{g}) \) be a projective diffeomorphism onto another Riemannian manifold \((\overline{M}, \overline{g})\) such that \( \text{trace}_g \overline{\text{Ric}} \geq s \) for the Ricci tensor \( \overline{\text{Ric}} \) of \((\overline{M}, \overline{g})\) and the scalar curvature \( s \) of \((M, g)\). If the gradient of the function \( \log \left( \frac{\det \overline{g}}{\det g} \right) \) has integrable norm on \((M, g)\) then \( f \) is affine map.

**Proof.** We conclude immediately from (3.1) that

\[
\Delta \psi = \frac{1}{n-1} \left( \text{trace}_g \overline{\text{Ric}} - s \right) + \left\| \text{grad} \ \psi \right\|^2
\]

(3.3)

Let \( \text{trace}_g \overline{\text{Ric}} \geq s \) then (3.3) shows \( \Delta \psi \geq 0 \). If \((M, g)\) is a complete, noncompact Riemannian manifold and \( \left\| \text{grad} \ \psi \right\| \in L'(M, g) \) then by the Yau statement (see [10, p. 660]) we conclude that \( \Delta \psi = 0 \) and \( \psi \) must be harmonic on \((M, g)\). At the same time, we see from (3.3) that \( \psi \) is constant. Then according to the formula (40.8) from [21, p. 133] we obtain \( \nabla \overline{g} = 0 \). Hence by [20], \( f \) is affine map.

Let \((M, g)\) and \((\overline{M}, \overline{g})\) be Riemannian manifolds of dimension \( n \) and \( m \) such that \( n > m \). A surjective map \( f : (M, g) \rightarrow (\overline{M}, \overline{g}) \) is a submersion if it has maximal rank \( m \) at any point \( x \) of \( M \), that is, each differential map \( f_{*x} \) of \( f \) is surjective, hence, for \( y \in \overline{M} \). In this case, \( f^{-1}(y) \) for an arbitrary \( y \in \overline{M} \) is an \((n - m)\)-dimensional closed submanifold \( M' \) of \((M, g)\) (see [22, p.11]). We call the submanifolds \( f^{-1}(y) \) fibers.
Putting $V_x = Ker(f_*)_x$, for any $x \in M$, we obtain an integrable vertical distribution $V$ which corresponds to the foliation of $M$ determined by the fibres of $f$, since each $V_x = T_x f^{-1}(y)$ coincides with tangent space of $f^{-1}(y)$ at $x$ for $f(x) = y$.

Let $H$ be the complementary distribution of $V$ determined by the Riemannian metric $g$, i.e. $H_x = V_x^\perp$ at each $x \in M$. So, at any $x \in M$, one has the orthogonal decomposition $T_x(M) = V_x \oplus H_x$ where $H_x$ is called the horizontal space at $x$. Thus we have defined a Riemannian almost product structure on $(M, g)$.

Consider now an $n$-dimensional simple connected complete Riemannian manifold $(M, g)$, and suppose that a projective submersion $f : (M, g) \to (\overline{M}, \overline{g})$ onto an $m$-dimensional ($m < n$) Riemannian manifold $(\overline{M}, \overline{g})$ exists. Then each pregeodesic line $\gamma \subset M$ which is an integral curve of the distribution $Ker f_*$ is mapped into a point $f(\gamma)$ in $\overline{M}$. Note that this fact does not contradict the definition of projective submersion.

In addition, we have proved in [23] and [24] that $(M, g)$ is isometric to a twisted product $(M_1 \times M_2, g_1 + e^{2\alpha_2} g_2)$ of some Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$, and for smooth function $\alpha_2 : M_1 \times M_2 \to \mathbb{R}$ such that all fibres of submersion and their orthogonal complements correspond to the canonical foliations of $M_1 \times M_2$ (see [23] and [24]).

In this case, the following corollary of Theorem 4 is true.

**Corollary 2.** Let $(M, g)$ be an $n$-dimensional complete, noncompact and simply connected Riemannian manifold and $f : (M, g) \to (\overline{M}, \overline{g})$ be a projective submersion onto another $m$-dimensional ($m < n$) Riemannian manifold $(\overline{M}, \overline{g})$ such that the mean curvature vector $\xi_H$ of the horizontal distribution $(Ker f_*)^\perp$ satisfies the condition $\|\xi_H\| \in L^1(M, g)$. If the mixed scalar curvature $s_{\text{mix}}$ of $(M, g)$ is nonpositive then $(Ker f_*)^\perp$ is integrable and $(M, g)$ is isometric to a direct product $(M_1 \times M_2, g_1 \oplus g_2)$ of some Riemannian manifolds $(M_1, g_2)$ and $(M_1, g_2)$ such that integral manifolds of $Ker f_*$ and $(Ker f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$.

Moreover, we have proved in [24] that if a simple connected complete $n$-dimensional Riemannian manifold $(M, g)$ has a nonnegative sectional curvature and admits a projec-
tive submersion onto another $m$-dimensional ($m < n$) Riemannian manifold $(\overline{M}, \overline{g})$, then $(M, g)$ is isometric to a direct product $(M_1 \times M_2, g_1 \oplus g_2)$ of some Riemannian manifolds $(M_1, g_2)$ and $(M_2, g_2)$ such that the integral manifolds of $\text{Ker} f_*$ and $(\text{Ker} f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$. We can formulate now a statement which will supplement this theorem. The statement is a corollary of Theorem 1 and Theorem 6.

**Corollary 3.** Let $(M, g)$ be an $n$-dimensional complete, noncompact and simply connected Riemannian manifold and $f : (M, g) \to (\overline{M}, \overline{g})$ be a projective submersion onto another $m$-dimensional ($m < n$) Riemannian manifold $(\overline{M}, \overline{g})$ with connected fibres. If the mixed scalar curvature $s_{\text{mix}} \geq 0$ then $(M, g)$ is isometric to a direct product $(M_1 \times M_2, g_1 \oplus g_2)$ of some Riemannian manifolds $(M_1, g_2)$ and $(M_2, g_2)$ such that the integral manifolds of $\text{Ker} f_*$ and $(\text{Ker} f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$.

**Proof.** Let $(M, g)$ be an $n$-dimensional complete, noncompact and simply connected Riemannian manifold and $f : (M, g) \to (\overline{M}, \overline{g})$ be a projective submersion onto another $m$-dimensional ($m < n$) Riemannian manifold $(\overline{M}, \overline{g})$ with connected fibres. It follows from the above, the fibre $f^{-1}(y)$ for an arbitrary $y \in \overline{M}$ is an $(n - m)$-dimensional closed connected submanifold $M'$ of $(M, g)$ equipped with the Riemannian metric $g'$ inherited from $(M, g)$. The mean curvature vector $\xi_H$ of $\mathcal{H} = (\text{Ker} f_*)^\perp$ belongs to $T_xM'$ at each point $x \in M'$ then, by applying the classic Green divergence theorem

$$\int_{M'} (\text{div}_g \xi_H) \, d\text{Vol}_{g'} = 0 \quad \text{to} \quad \xi_H,$$

from (2.4) we get the following equation

$$\int_{M'} \left( 2s_{\text{mix}} + 1/4 \|\nabla P\|^2 \right) d\text{Vol}_{g'} = 0$$

(3.4)

where $d\text{Vol}_{g'}$ is the volume form of $(M', g')$. If $(M', g')$ is non-oriented we can consider its orientable double cover. If the mixed scalar curvature $s_{\text{mix}} \geq 0$ then from (3.4) we obtain that $\nabla P = 0$ at each point of $M'$. At the same time, we recall that $M'$ is an arbitrary fibre of the projective submersion $f : (M, g) \to (\overline{M}, \overline{g})$. Therefore, $\text{Ker} f_*$ and $(\text{Ker} f_*)^\perp$ are two integrable distributions with totally geodesic integral manifolds (totally geodesic
foliations) on the complete, noncompact and simply connected Riemannian manifold \((M, g)\). Then by the well known de Rham decomposition theorem it is isometric to the direct product \((M_1 \times M_2, g_1 \oplus g_2)\) of some Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) for the Riemannian metric \(g_1\) and \(g_2\) which induced by \(g\) on \(M_1\) and \(M_2\), respectively. The proof of our corollary is complete.

Using the equality \((3.4)\) once again we get the following

**Corollary 4.** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold and \(f : (M, g) \to (\overline{M}, \overline{g})\) be a submersion onto another \(m\)-dimensional \((m < n)\) Riemannian manifold \((\overline{M}, \overline{g})\) with connected fibres. If the mixed scalar curvature \(s_{\text{mix}} > 0\) then \(f : (M, g) \to (\overline{M}, \overline{g})\) is not a projective submersion.

**Proof.** For the case \(s_{\text{mix}} > 0\) we can rewrite \((3.4)\) in the form

\[
\int_{M'} \left(2s_{\text{mix}} + 1/4 \| \nabla P \|^2 \right) dVol_{g'} > 0.
\]

The contradiction just obtained with the classic Green divergence theorem completes the proof of our corollary.

4. Applications to the theory of conformal mappings of Riemannian manifolds

Let \(\varphi \in C^2 M\) then (see [21, p. 90])

\[
e^{2\sigma} s = s - 2(n-1)\Delta \sigma - (n-1)(n-2)\| \text{grad} \, \sigma \|^2
\]

where \(s\) denote the scalar curvature \((\overline{M}, \overline{g})\). Now we can formulate the following

**Theorem 6.** Let \((M, g)\) be an \(n\)-dimensional \((n \geq 3)\) complete, noncompact Riemannian manifold and \(f : (M, g) \to (\overline{M}, \overline{g})\) be a conformal diffeomorphism onto another Riemannian manifold \((\overline{M}, \overline{g})\) such that \(\overline{g} = e^{2\sigma} g\) and \(\overline{s} \geq e^{-2\sigma} s\) for some function \(\sigma \in C^2 M\).
and the scalar curvatures $s$ and $\overline{s}$ of $(M, g)$ and $(\overline{M}, \overline{g})$, respectively. If $\|\text{grad} \, \sigma\| \in L^1(M, g)$, then $f$ is a homothetic mapping.

**Proof.** If $f : (M, g) \to (\overline{M}, \overline{g})$ is a conformal diffeomorphism a connected complete non-compact and oriented Riemannian manifold $(M, g)$ onto another Riemannian manifold $(\overline{M}, \overline{g})$ such that $\overline{g} = e^{2\sigma} g$ for some function $\sigma \in C^2(M)$, then from (4.1) we obtain

$$2(n-1)\Delta \sigma = s - e^{2\sigma} \overline{s} - (n-1)(n-2)\|\text{grad} \, \sigma\|^2. \quad (4.2)$$

Let $s \leq e^{2\sigma} \overline{s}$ then (2) shows $\Delta \sigma \leq 0$. It means that $\sigma$ is a superharmonic function. By the condition of our theorem, the gradient of $\sigma$ has integrable norm on $(M, g)$ and we obtain from (4.2) that $\Delta \sigma = 0$ and $\sigma$ must be harmonic (see our Lemma). Since $n \geq 3$, we see from (4.2) that $\sigma$ is constant. The proof of the theorem is complete.

Let $(M, g)$ and $(\overline{M}, \overline{g})$ be Riemannian manifolds of dimension $n$ and $m$ for $n > m$. A submersion $f : (M, g) \to (\overline{M}, \overline{g})$ is called a horizontal conformal if $f_*$ restricted to the horizontal distribution $\mathcal{H} = (\text{Ker} \, f_*)^\perp$ is conformal mapping.

Next, we consider a horizontal conformal submersion $f : (M, g) \to (\overline{M}, \overline{g})$ for the case $m < n$. We note here that horizontal conformal mappings were introduced by Ishihara [25]. From the above discussion, one can conclude that the notion of horizontally conformal mappings is a generalization of concept of Riemannian submersions. In addition, we note that a natural projection onto any factor of a double-twisted product $(M_1 \times M_2, \lambda_1^2 g_1 + \lambda_2^2 g_2)$ of any Riemannian manifolds $(M_a, g_a)$ and smooth positive functions $\lambda_a : M_1 \times M_2 \to \mathbb{R}$ for an arbitrary $a = 1, 2$ is horizontal conformal submersion with umbilical fibres (see [20]).

Let $f : (M, g) \to (\overline{M}, \overline{g})$ be a horizontal conformal submersion and $\text{Ker} f_*$ be an umbilical distribution then (2.2) can be rewrite in the form

$$\text{div}(\xi_v + \xi_H) = s_{\text{mix}} - \|F_H\|^2 - \frac{n-m-1}{n-m} \|\xi_v\|^2 + \frac{m-1}{m} \|\xi_H\|^2. \quad (4.3)$$

In this case, we can formulate a corollary of Theorem 4 which generalizes our theorem on the horizontal conformal submersions of compact Riemannian manifolds with non positive mixed scalar curvature that has been proved in [26] (see also [27]).
Corollary 5. Let \((M, g)\) be an \(n\)-dimensional complete, noncompact and simply connected Riemannian manifold and \(f : (M, g) \to (\overline{M}, \overline{g})\) be a horizontal conformal submersion with umbilical fibres onto another \(m\)-dimensional \((m < n)\) Riemannian manifold \((\overline{M}, \overline{g})\). If the mean curvature vector \(\xi_V\) of \(\text{Ker} f_*\) and the mean curvature vector \(\xi_H\) of \((\text{Ker} f_*)_\perp\) satisfy the condition \(\|\xi_V + \xi_H\| \in L^1(M, g)\) and the mixed scalar curvature \(s_{mix}\) of \((M, g)\) is nonpositive then \((\text{Ker} f_*)_\perp\) is integrable and \((M, g)\) is isometric to a direct product \((M_1 \times M_2, g_1 \oplus g_2)\) of some Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) such that integral manifolds of \(\text{Ker} f_*\) and \((\text{Ker} f_*)_\perp\) correspond to the canonical foliations of the product \(M_1 \times M_2\).

5. Applications to the theory of Riemannian submersions

A submersion \(f : (M, g) \to (\overline{M}, \overline{g})\) is called Riemannian submersion if \((f_*)_x\) preserves the length of the horizontal vectors at each point \(x \in M\) (see [13, p. 3]). In this case, the horizontal distribution \(\mathcal{H} = (\text{Ker} f_*)_\perp\) is totally geodesic (see [17]). In the paper [28] and in the monograph [13, p. 235] was proved the following theorem. Let \(f : (M, g) \to (\overline{M}, \overline{g})\) be a Riemannian submersion with totally umbilical fibres. If \((M, g)\) is a closed and orientable manifold with nonpositive mixed sectional curvature (i.e. \(\text{sec}(X, Y) \leq 0\) for every horizontal vector field \(X\) and for every vertical vector field \(Y\)), then all fibres are totally geodesic and horizontal distribution \(\mathcal{H} = (\text{Ker} f_*)_\perp\) is integrable, and the mixed sectional curvature is equals to zero. We present a generalization of this theorem.

The following result is deduced immediately from Corollary 6.

Corollary 6. Let \((M, g)\) be an \(n\)-dimensional complete, noncompact and simply connected Riemannian manifold and \((\overline{M}, \overline{g})\) be another \(m\)-dimensional \((m < n)\) Riemannian manifold and \(f : (M, g) \to (\overline{M}, \overline{g})\) be a Riemannian submersion with totally umbilical fibres. If the mixed scalar curvature \(s_{mix}\) is nonpositive and the mean curvature vector \(\xi_V\) of fibres satisfies the condition \(\|\xi_V\| \in L^1(M, g)\), then the horizontal distribution \((\text{Ker} f_*)_\perp\) is integrable and the Riemannian manifold \((M, g)\) is isometric to a direct product \((M_1 \times M_2, g_1 \oplus g_2)\) of some Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) such that the
integral manifolds of $\text{Ker} f_*$ and $(\text{Ker} f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$.

We know from [29] that there are no Riemannian submersions from closed Riemannian manifolds with positive Ricci curvature to Riemannian manifolds with nonpositive Ricci curvature. The following statement is a direct consequence of Corollary 3 and complements the above vanishing theorem.

**Corollary 7.** Let $(M, g)$ be an $n$-dimensional complete, noncompact and simply connected Riemannian manifold and $f : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian submersion onto an $(n - 1)$-dimensional Riemannian manifold $(\overline{M}, \overline{g})$. If the vertical Ricci curvature of $(M, g)$ is nonpositive and the mean curvature vector $\xi_1$ of fibres satisfies the condition $\|\xi_1\| \in L^1(M, g)$, then the horizontal distribution $(\text{Ker} f_*)^\perp$ is integrable and the Riemannian manifold $(M, g)$ is isometric to direct product $(M_1 \times M_2, g_1 \oplus g_2)$ of some Riemannian manifolds $(M_1, g_2)$ and $(M_2, g_2)$ such that $\dim M_1 = 1$ and the integral manifolds of $\text{Ker} f_*$ and $(\text{Ker} f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$.

### 5. Applications to the theory of harmonic submersions of Riemannian manifolds

A smooth mapping $f : (M, g) \to (\overline{M}, \overline{g})$ is said to be harmonic if $f$ provides an extremum of the energy functional $E_\Omega(f) = \int_{\Omega} \|f_*\|^2 d\text{Vol}_g$ for each relatively closed open subset $\Omega \subset M$ with respect to the variations of $f$ that are compactly supported in $\Omega$.

If $(M, g)$ is an $n$-dimensional Riemannian manifold and $f : (M, g) \to (\overline{M}, \overline{g})$ is a harmonic submersion onto another $m$-dimensional ($m < n$) Riemannian manifold $(\overline{M}, \overline{g})$ then each its fibre $(M', g')$ is an $(n - m)$-dimensional closed imbedded minimal submanifold of $(M, g)$ (see [30]). Then from the above arguments and Theorem 4 we conclude that the following corollary is true.

**Corollary 9.** Let $(M, g)$ be an $n$-dimensional complete, noncompact and simply connected Riemannian manifold and $f : (M, g) \to (\overline{M}, \overline{g})$ be a harmonic submersion onto another $m$-dimensional ($m < n$) Riemannian manifold $(\overline{M}, \overline{g})$ with connected fibres. If the hori-
horizontal distribution $(\text{Ker } f_*)^\perp$ is integrable and the mixed scalar curvature $s_{\text{mix}}$ is non-negative, then $(M, g)$ is isometric to a direct product $(M_1 \times M_2, \, g_1 \oplus g_2)$ of some Riemannian manifolds $(M_1, g_2)$ and $(M_2, g_2)$ such that the integral manifolds of $\text{Ker } f_*$ and $(\text{Ker } f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$.

Using the Remark 1 we get the following

**Corollary 8.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $f : (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$ be a submersion onto another $m$-dimensional ($m < n$) Riemannian manifold $(\widetilde{M}, \widetilde{g})$ with connected fibres. If the horizontal distribution $(\text{Ker } f_*)^\perp$ is integrable and the mixed scalar curvature $s_{\text{mix}}$ is positive, then $f : (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$ is not harmonic.

**References**

[1] Wu H.H., The Bochner technique in differential geometry, Harwood Acad. Publ., Harwood (1987).

[2] Stepanov S.E., Riemannian almost product manifolds and submersions. Journal of Mathematical Sciences, 99:6 (2000), 1788-1831.

[3] Pigola S., Rigoli M., Setti A.G., Vanishing and Finiteness Results in Geometric Analysis. A Generalization of the Bochner Technique, Birkhäuser Verlag AG, Berlin (2008).

[4] Koboyashi S., Nomizu K., Foundations of differential geometry, Volume I, Interscience Publishers, New York, 1963.

[5] Pigola S., Setti A.G., Global divergence theorems in nonlinear PDEs and geometry, Ensaios Matemáticos, 26 (2014), 1-77.

[6] Gaffney M.P., A special Stokes’s theorem for complete Riemannian manifolds, Annals of Mathematics, Second Series, 60:1 (1954), 140-145.

[7] Karp L., On Stokes’ theorem for noncompact manifolds, Proceedings of the American Mathematical Society, 82:3 (1981), 487-490.

[8] Caminha A., Souza P., Camargo F., Complete foliations of space forms by hypersurfaces, Bull. Braz. Math. Soc., New Series, 41:3 (2010), 339-353.
[9] Caminha A., The geometry of closed conformal vector fields on Riemannian spaces, Bull. Braz. Math. Soc., New Series, 42:2 (2011), 277-300.
[10] Yau S.T., Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Univ. Math. J., 25 (1976), 659-670.
[11] Rocamora A.H., Some geometric consequences of the Weitzenböck formula on Riemannian almost-product manifolds; weak-harmonic distributions, Illinois Journal of Mathematics, 32:4 (1988), 654-671.
[12] Reinhart B.L., Differential Geometry of Foliations, Springer Verlag, Berlin – New York, 1983.
[13] Falcitelli M., Ianus S., Pastore A.M., Riemannian submersions and related topics, World Scientific Publishing, Singapore, 2004.
[14] Walczak P.G., An integral formula for a Riemannian manifold with two orthogonal complementary distributions, Colloquium Mathematicum, LVIII:2 (1990), 243-252.
[15] Stepanov S.E., An integral formula for a Riemannian almost-product manifold, Tensor, N. S., 55 (1994), 209-214.
[16] Stepanov S.E., Bochner’s technique in the theory of Riemannian almost product structures, Mathematical notes of the Academy of Sciences of the USSR, 1990, 48:2, 778-781.
[17] Stepanov S.E., A class of Riemannian almost-product structures, Soviet Mathematics (Izv. VUZ), 33:7 (1989), 51-59.
[18] Luzynczyk M., Walczak P., New integral formula for two complementary orthogonal distributions on Riemannian manifolds, Annals of Global Analysis and Geometry, 48 (2015), 195-209.
[19] Hebda J.J., Projective maps of rank ≥ 2 are strongly projective, Differential geometry and its applications, 12 (2000), 271-280.
[20] Fernández-López M., García-Rio E., Kupeli D.N., Ünal B., A curvature condition for a twisted product to be a warped product, Manuscripta Mat., 106:2 (2001), 213-217.
[21] Eisenhart L. P., Riemannian Geometry, Princeton University Press, Princeton, 1949.
[22] Giachetta G., Mangiartti I., Sardanashvily G., New Lagrangian and Hamiltonian Methods in Field Theory, Word Scientific Publishing, Singapore (1997).

[23] Stepanov S.E., On the global theory of projective mappings, Mathematical Notes, 58:1 (1995), 752-756.

[24] Stepanov S.E., Geometry of projective submersions of Riemannian manifolds, Russian Mathematics (Iz. VUZ), 43:9 (1999), 44-50.

[25] Ishihara T. A mapping of Riemannian manifolds which preserves harmonic functions, J. Math. Kyoto Univ., 19 (1979), 215-229.

[26] Stepanov S.E., Weyl submersions, Russian Mathematics (Izvestiya VUZ, Matematika), 36:5 (1992), 87-89.

[27] Zawadzki T., Existence conditions for conformal submersions with totally umbilical fibers, Differential Geometry and its Applications, 35 (2014), 69-85.

[28] Bădiţoiu G., Ianuş S., Semi-Riemannian submersions with totally umbilical fibres, Rendiconti del Circolo Matematico di Palermo, 51:2 (2002), 249-276.

[29] Pro C., Wilhelm F., Riemannian submersions need not preserve positive Ricci curvature, Proc. Amer. Math. Soc., 142:7 (2014), 2529-2535.

[30] Stepanov S.E., $O(n) \times O(m - n)$-structures on $m$-dimensional manifolds, and submersions of Riemannian manifolds, St. Petersburg Math. J., 7:6, 1005-1016 (1996).