Verification of Nonblockingness in Bounded Petri Nets With Minimax Basis Reachability Graphs

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Abstract—This paper proposes a semi-structural approach to verify the nonblockingness of a Petri net. We construct a structure, called minimax basis reachability graph (minimax-BRG): it provides an abstract description of the reachability set of a net while preserving all information needed to test if the net is blocking. We prove that a bounded deadlock-free Petri net is nonblocking if and only if its minimax-BRG is unobstructed, which can be verified by solving a set of integer constraints and then examining the minimax-BRG. For Petri nets that are not deadlock-free, one needs to determine the set of deadlock markings. This can be done with an approach based on the computation of maximal implicit firing sequences enabled by the markings in the minimax-BRG. The approach we developed does not require the construction of the reachability graph and has wide applicability.

Index Terms—Petri net, basis reachability graph, nonblocking-ness.

I. INTRODUCTION

As discrete event models, Petri nets are commonly used in the framework of supervisory control theory (SCT) [4], [18], [22], [28]. From the point of view of computational efficiency, Petri nets have several advantages over simpler models such as automata [6], [22], [23]: since states in Petri nets are not explicitly represented in the model in many cases, and structural analysis and linear algebraic approaches can be used without exhaustively enumerating the state space of a system.

A suite of supervisory control approaches in discrete event systems focuses on an essential property, namely nonblockingness [7], [24]. As defined in [24], nonblockingness is a property prescribing that all reachable states should be co-reachable to a set of final states representing the completions of pre-specified tasks. Consequently, to verify and ensure the nonblockingness of a system is a problem of primary importance in many applications and should be addressed with state-of-the-art techniques.

The nonblockingness verification (NB-V) problem in automata can be solved in a relatively straightforward manner. The authors in [16] address several sufficient conditions for nonblockingness; however, they are not very suitable for systems that contain complex feedback paths. In [13], [14], a method called hierarchical interface-based supervisory control, which consists in breaking up a plant into two subsystems and restricting the interaction between them, is developed to verify if a system is nonblocking. Based on the state tree structure, [19] studies an efficient algorithm for nonblocking supervisory control design in reasonable time and memory cost. To mitigate the state explosion problem, in the framework of compositional verification an abstraction approach is proposed in [21] to verify discrete event systems modelled by extended finite-state machines (EFSMs) and such verification approach is typically designed for large models consisting of several EFSMs that interact both via shared events and variables. Based on the automaton abstraction technique, the work in [25] presents an aggregative synthesis approach to obtain nonblocking supervisors in a distributed way.

Using Petri net models, the works in [6], [7] study NB-V and enforcement from the aspect of Petri net languages; however, these methods rely on the construction and analysis of the reachability graph, which is practically inefficient. A different approach based on the theory of regions [27] was used in [5] to design a maximally permissive controller ensuring the nonblockingness of a system; however, it still requires an exhaustive enumeration of the state space. For a class of Petri nets called G-systems, [30] reports a deadlock prevention policy that can usually lead to a nonblocking supervisor with high computational efficiency but cannot guarantee maximally permissive behavior. A nonblocking and maximally permissive supervisor with a distributed architecture is designed in [12] and modelled by a class of Petri nets namely BS4R.

As is known, the difficulty of enforcing nonblockingness lies in the fact that the optimal nonblocking supervisory control problem is NP-hard [9]. Moreover, the problem of efficiently verifying nonblockingness of a Petri net without
constructing its reachability graph remains open to date. By this motivation, in this paper, we aim to develop a method to cope with the NB-V problem in Petri nets.

A state-space abstraction technique in Petri nets, called basis reachability graph (BRG) approaches, was recently proposed in [2, 3]. In these approaches, only a subset of the reachable markings, called basis markings, are enumerated. This method can be used to solve marking reachability [20] and diagnosis [2]. Preliminary results are presented in [10] to show how it is in the classical BRG of the system. As a countermeasure, a Petri net is blocking if a livelock that contains no final markings cannot further advance [15].

We propose a structure called minimax-BRG. In minimax-BRGs, only part (minimax-BRG) of the reachability space. Thus, deadlock control problems can be solved efficiently. The BRG-based methods are semi-structural since only basis markings are explicitly enumerated in the BRG while all other reachable markings are abstracted by linear algebraic equations.

On the other hand, in our previous work [10] we show that the standard BRG cannot be directly used to solve the NB-V problem due to the possible presence of livelocks and deadlocks. In particular, livelocks describe an undesirable non-dead repetitive behavior such that the system is bound to evolve along a particular subset of its reachability space. Thus, a Petri net is blocking if a livelock that contains no final markings is reachable. However, the set of markings that form a livelock is usually hard to characterize and is not encoded in the classical BRG of the system. As a countermeasure, preliminary results are presented in [10] to show how it is possible to tailor the BRG to detect livelocks. In more detail, a structure named the expanded BRG is proposed, which expands the BRG so that all markings in $R(N,M_0)$ reached by firing a sequence of transitions ending with an explicit transition are included. The set of markings in an expanded BRG is denoted as the expanded basis marking set $M_{\text{BE}}$. However, this approach presents two major drawbacks. First, it only applies to deadlock-free nets, which is an undesirable restriction considering that dead non-final markings are one of the causes of blockingness. Second, while the expanded BRG can abstract part of the reachability set, its size can still be very large and its practical efficiency needs to be improved.

When a system is not deadlock-free, a dead marking in the state space characterizes a condition from which the system cannot further advance [15]. If there exists a dead marking that is not final (we call such a state a non-final deadlock), the system is blocking.

Inspired by the classical BRG-based methodology, in this paper, we develop a semi-structural approach to tackle the NB-V problem. The contribution consists of three aspects:

- We propose a structure called minimax basis reachability graph (minimax-BRG). In minimax-BRGs, only part of the state space, namely minimax basis markings, is encoded and all other markings can be characterized as the integer solutions of a linear constraint set.

- Owing to properties of the minimax-BRG, when a bounded Petri net is known to be deadlock-free, we prove that it is nonblocking if and only if its minimax-BRG consists of all nonblocking nodes (such a minimax-BRG is said to be unobstructed), which can be verified by solving a set of integer constraints and then examining the minimax-BRG.

- We generalize the results to arbitrary bounded Petri nets (not necessarily be deadlock-free) and propose a necessary and sufficient condition for NB-V. Numerical results demonstrate the proposed approach.

The rest of the paper is organized as follows. Some basic concepts and formalisms used in the paper are recalled in Section II. Section III dissects the NB-V problem. Section IV introduces the minimax-BRG. Section V investigates how minimax-BRGs can be applied to solving the NB-V problem. Numerical analyses are given in Section VI, while discussions are reported in Section VII. Conclusions and future work are given in Section VIII.

II. Preliminaries

A. Automata and Petri nets

An automaton [29] is a five-tuple $A = (X, \Sigma, \eta, x_0, X_m)$, where $X$ is a set of states, $\Sigma$ is an alphabet of events, $\eta : X \times \Sigma \rightarrow X$ is a state transition function, $x_0 \in X$ is an initial state and $X_m \subseteq X$ is a set of final states (also called marker states in [24]). $\eta$ can be extended to a function $\eta : X \times \Sigma^* \rightarrow X$.

A state $x \in X$ is reachable if $x = \eta(x_0, s)$ for some $s \in \Sigma^*$; it is co-reachable if there exists $s' \in \Sigma^*$ such that $\eta(x, s') \in X_m$. An automaton is said to be nonblocking if every reachable state is co-reachable.

A Petri net [23] is a four-tuple $N = (P,T,\text{Pre},\text{Post})$, where $P$ is a set of $m$ places (graphically represented by circles) and $T$ is a set of $n$ transitions (graphically represented by bars). $\text{Pre} : P \times T \rightarrow N$ and $\text{Post} : P \times T \rightarrow N (N = \{0,1,2,\cdots\})$ are the pre- and post- incidence functions that specify the arcs directed from places to transitions, and vice versa in the net, respectively. The incidence matrix of $N$ is defined by $C = \text{Post} - \text{Pre}$. A Petri net is acyclic if there are no directed cycles in its underlying digraph.

Given a Petri net $N = (P,T,\text{Pre},\text{Post})$ and a set of transitions $T_x \subseteq T$, the $T_x$-induced sub-net of $N$ is a net resulting by removing all transitions in $T \setminus T_x$ and corresponding arcs from $N$, denoted as $N_x = (P,T_x,\text{Pre}_x,\text{Post}_x)$ where $T_x \subseteq T$ and $\text{Pre}_x(\text{Post}_x)$ is the restriction of $\text{Pre}(\text{Post})$ to $P$ and $T_x$. The incidence matrix of $N_x$ is denoted by $C_x = \text{Post}_x - \text{Pre}_x$.

A marking $M$ of a Petri net $N$ is a mapping: $P \rightarrow N$ that assigns to each place of a Petri net a non-negative integer number of tokens. The number of tokens in a place $p$ at a marking $M$ is denoted by $M(p)$. A Petri net $N$ with an initial marking $M_0$ is called a marked net, denoted by $(N,M_0)$.

For a place $p \in P$, the set of its input transitions is defined by $\cdot p = \{ t \in T \mid \text{Post}(p,t) > 0 \}$ and the set of its output transitions is defined by $\cdot t = \{ t \in T \mid \text{Pre}(p,t) > 0 \}$. The notions for $\cdot t$ and $\cdot p$ are analogously defined.

A transition $t \in T$ is enabled at a marking $M$ if $M \geq \text{Pre}(t)$, denoted by $M[t]$. If $t$ is enabled at $M$, the firing of $t$ yields marking $M' = M + C(\cdot,t)$, which is denoted as $M[t]M'$. A marking $M$ is dead if for all $t \in T$, $M \not\geq \text{Pre}(t)$.

1 We use $A(\cdot,x)$ ($A(x,\cdot)$) to denote the column (row) vector corresponding to the element $x$ in matrix $A$. 
Marking \( M' \) is reachable from \( M_1 \) if there exist a sequence of transitions \( \sigma = t_1t_2 \cdots t_n \), and markings \( M_2, \ldots, M_n \) such that \( M_1(t_1)M_2(t_2) \cdots M_n(t_n)M' \) holds. When \( \sigma = \epsilon \), where \( \epsilon \) denotes the empty sequence, then it holds that \( M(\epsilon)M \). We denote by \( T^* \) the set of all finite sequences of transitions over \( T \). Given a transition sequence \( \sigma \in T^* \), \( \varphi : T^* \rightarrow \mathbb{N}^n \) is a function that associates to \( \sigma \) a vector \( y = \varphi(\sigma) \in \mathbb{N}^n \), called the firing vector of \( \sigma \), i.e., \( y(t) = k \) if transition \( t \in T \) appears \( k \) times in \( \sigma \). In particular, it holds that \( \varphi(\epsilon) = 0 \).

\[
M \text{ is said to be } T \text{-disjoint sets if the reachability set of } \varphi(\sigma) \text{ for all } \sigma \in \mathbb{N}^n \text{ is reachable from it, i.e., }
M(\sigma)M,'
\]

characterize \( M \).

Hereinafter, we adopt the GMEC-based representation to the incidence matrix of the \( T \)-induced sub-net. A GMEC is a pair \((GMECs) [8]\). A GMEC is a pair \((w, k)\), where \( w \in \mathbb{Z}^m \) and \( k \in \mathbb{Z} (\mathbb{Z} \text{ is the set of integers}) \), that defines a set of markings \( L_{(w, k)} = \{ M \in \mathbb{N}^m | w^T \cdot M \leq k \} \).

Definition 1: A marking \( M \in R(N, M_0) \) of a plant \( G = (N, M_0, \mathcal{M}) \) is said to be blocking if no final marking is reachable from it, i.e., \( R(N, M) \cap M(\sigma) = \emptyset \); otherwise \( M \) is said to be nonblocking. System \( G \) is nonblocking if no reachable marking is blocking; otherwise \( G \) is blocking.

B. Basis Reachability Graph (BRG) [2, 3, 20]

Definition 2: Given a Petri net \( N = (P, T, \text{Pre}, \text{Post}) \), transition set \( T \) can be partitioned into \( T = T_E \cup T_I \), where the disjoint sets \( T_E \) and \( T_I \) are called the explicit transition set and the implicit transition set, respectively. A pair \( \pi = (T_E, T_I) \) is called a basis partition of \( T \) if the \( T_I \)-induced sub-net of \( N \) is acyclic. We denote \( |T_E| = n_E \) and \( |T_I| = n_I \). Let \( C_I \) be the incidence matrix of the \( T_I \)-induced sub-net of \( N \).

Note that in a BRG with respect to a basis partition \((T_E, T_I)\), the firing information of explicit transitions in \( T_E \) is explicitly encoded in the BRG, while the firing information of implicit transitions in \( T_I \) is abstracted as firing vectors. Note that the selection of \( T_E \) and \( T_I \) does not related to the physical meaning of the transitions: the only restriction is that the \( T_I \)-induced sub-net is acyclic.

Definition 3: Given a Petri net \( N = (P, T, \text{Pre}, \text{Post}) \), a basis partition \( \pi = (T_E, T_I) \), a marking \( M \), and a transition \( t \in T_E \), we define

\[
\Sigma(M, t) = \{ \sigma \in T^*_E | M[\sigma]M', M' \geq \text{Pre}(\epsilon, t) \}
\]

as the set of explanations of \( t \) at \( M \), and we define

\[
Y(M, t) = \{ \varphi(\sigma) \in \mathbb{N}^m | \sigma \in \Sigma(M, t) \}
\]

as the set of explanation vectors; meanwhile we define \( \Sigma_{\min}(M, t) = \{ \sigma \in \Sigma(M, t) | 2\sigma' \in \Sigma(M, t) \parallel \varphi(\sigma') \leq \varphi(\sigma) \} \) as the set of minimal explanations of \( t \) at \( M \), and we define

\[
Y_{\min}(M, t) = \{ \varphi(\sigma) \in \mathbb{N}^m | \sigma \in \Sigma_{\min}(M, t) \}
\]

as the corresponding set of minimal explanation vectors.

Definition 4: Given a marked net \((N, M_0)\) and a basis partition \( \pi = (T_E, T_I) \), its basis marking set \( M_{\pi} \) is the smallest subset of reachable markings such that:

- \( M_0 \in M_{\pi} \);
- If \( M \in M_{\pi} \), then for all \( t \in T_E \), for all \( y \in Y_{\min}(M, t) \), \( M' = M + C_I \cdot y + C(\cdot, t) \Rightarrow M' \in M_{\pi} \).

A marking \( M \in M_{\pi} \) is called a basis marking of \((N, M_0)\) with respect to \( \pi = (T_E, T_I) \).

Definition 5: Given a bounded marked net \((N, M_0)\) and a basis partition \( \pi = (T_E, T_I) \), its basis reachability graph is a deterministic finite state automaton \( B = (M_{\pi}, Tr, \Delta, M_0) \), where the state set \( M_{\pi} \) is the set of basis markings, the event set \( Tr \) is the finite set of pairs \( (t, y) \in T_E \times \mathbb{N}^n \), the transition relation \( \Delta = \{(M_1, t, y), M_2 | t \in T_E \in \Sigma(M_1, t), M_2 = M_1 + C_I \cdot y + C(\cdot, t) \} \) and the initial state is the initial marking \( M_0 \).

We extend in the usual way the definition of transition relation to consider a sequence of pairs \( \sigma \in T^* \) and write \((M_1, \sigma, M_2) \in \Delta \) to denote that from \( M_1 \) sequence \( \sigma \) yields \( M_2 \).

Definition 6: Given a marked net \((N, M_0)\), a basis partition \( \pi = (T_E, T_I) \), and a basis marking \( M \in M_{\pi} \), we define \( R_I(M) = \{ M \in \mathbb{N}^m | (\exists \sigma \in T_I) M[\sigma]M \} \) as the implicit reach of \( M \).

The implicit reach of a basis marking \( M_0 \) is the set of all markings that can be reached from \( M_0 \) by firing only implicit transitions. Since the \( T_I \)-induced sub-net is acyclic, by Proposition 1 it holds that:

\[
R_I(M_0) = \{ M \in \mathbb{N}^m | (\exists y \in \mathbb{N}^n) M = M_0 + C_I \cdot y \}.
\]

III. BRG and Nonblockingness Verification

To efficiently solve the NB-V problem in Petri nets without constructing the reachability graph, we attempted to use the BRG-based approach in [10]. However, as observed in [10], the classical BRG does not necessarily encode all information needed to test nonblockingness. To help clarify, an example is provided in the following.

Example 1: Consider a parameterized plant \( G = (N, M_0, \mathcal{M}) \) in Fig. [1] with \( M_0 = [2 0 1]^T \) and \( M_\mathcal{F} = \{ M_0 \} \). In this net, \( \text{Pre}(p_2, t_3) = \alpha \) is set to be a parameter (\( \alpha \in \mathbb{N} \)). Assuming that \( T_E = \{ t_2 \} \), the BRG of this net (regardless of the value of \( \alpha \)) is also shown in the same figure, where \( M_{I_0} = M_0 \) and \( Y_{\min}(M_0, t_2) = \{ y_1 \} \).

It can be computed that \( y_1 = [1 0]^T \), which implies that the firing of sequence \( \sigma = t_1 \) is the prerequisite (the minimal one) of the firing of explicit transition \( t_2 \) at marking \( M_{I_0} \). The reachability graphs for \( \alpha = 1 \) and \( \alpha = 2 \) are shown in Fig. [2]
By inspection of the two reachability graphs, one can verify that
$G$ is deadlock-free if $\alpha = 1$ and not deadlock-free if $\alpha = 2$. When $\alpha = 1$ $G$ is blocking due to the livelock composed
by two markings $[1 \ 0 \ 0]^{T}$ and $[0 \ 1 \ 0]^{T}$. When $\alpha = 2$ $G$ is also
blocking because of the non-final deadlock $[0 \ 0 \ 0]^{T}$. However,
these blocking conditions are not captured in the BRG which,
in both cases, consists of a unique node $M_{b0} = M_{0}$ which is
also final.

Example 1 shows that when all basis markings in the
BRG are nonblocking, this does not necessarily imply that all
reachable markings in the corresponding plant are nonblock-
ing. Specifically, as we mentioned in Section I, two types of
reachable markings in the corresponding plant are nonblock-
ing because of the non-final deadlock $[1 \ 0 \ 0]^{T}$. When
$\alpha = 1$ $M$ is blocking due to the livelock composed
by two markings $[1 \ 0 \ 0]^{T}$ and $[0 \ 1 \ 0]^{T}$, i.e., they are dead
but non-final, and those are included in livelocks (ergodic strongly-
connected components of the reachability graph containing
non-final non-dead markings).

Notice that when tackling the NB-V problem by using the
basis marking approach, there may exist some (i) dead and
non-final markings, and/or (ii) livelock markings that are not
basis. Since such markings do not belong to set $M_{B}$, they are
not shown in the corresponding BRG. Therefore, the classical
structure of BRGs needs to be revised to encode additional
information for checking nonblockingness. To this end, in the
following, we propose a structure namely minimax-BRG and
show how it can be leveraged on solving the NB-V problem.

IV. MINIMAX BASIS MARKINGS AND MINIMAX-BRGs

A. Minimax Basis Markings

To define the minimax-BRG, we first introduce the set of
minimax basis markings. As two prerequisite concepts, we
define maximal explanations and maximal explanation vectors
as follows.

Definition 7: Given a Petri net $N = (P, T, Pre, Post)$, a
basis partition $\pi = (T_{E}, T_{I})$, a marking $M$, and a transition
t $t \in T_{E}$, we define

$\Sigma_{\text{max}}(M, t) = \{ \sigma \in \Sigma(M, t) | \exists \sigma' \in \Sigma(M, t): \phi(\sigma') \geq \phi(\sigma) \}$

as the set of maximal explanations of $t$ at $M$, and

$Y_{\text{max}}(M, t) = \{ \phi(\sigma) \in \mathbb{N}^{n} | \sigma \in \Sigma_{\text{max}}(M, t) \}$
as the corresponding set of maximal explanation vectors. □

From the standpoint of partial order set (poset), the set of
maximal explanation vectors $Y_{\text{max}}(M, t)$ is the set of maximal
elements in the corresponding poset $Y(M, t)$. Note that, as the
case for the set of minimal explanation vectors $Y_{\text{min}}(M, t)$
[2, 3, 20], $Y_{\text{max}}(M, t)$ may not be a singleton. In fact, there
may exist multiple maximal firing sequences $\sigma_{i} \in T_{E}$ that
enable an explicit transition $t$. Next, we define minimax basis
markings in an iterative way as follows.

Definition 8: Given a marked net $(\mathcal{N}, M_{0})$ with a basis
partition $\pi = (T_{E}, T_{I})$, its minimax basis marking set $M_{BM}$ is
recursively defined as follows

(a) $M_{0} \in M_{BM}$;
(b) $M \in M_{BM}$, $t \in T_{E}$, $y \in Y_{\text{min}}(M, t) \cup Y_{\text{max}}(M, t)$,
$M' = M + C_{t} \cdot y + C(\cdot, t) \Rightarrow M' \in M_{BM}$.

A marking in $M_{BM}$ is called a minimax basis marking of the
marked net with $\pi = (T_{E}, T_{I})$. □

In practice, the set of minimax basis markings is a smaller
subset of reachable markings that contains the initial marking
and is closed by reachability through a sequence that contains
an explicit transition and one of its maximal or minimal
explanations. Meanwhile, note that for a bounded marked net,
$M_{B} \subseteq M_{BM}$ holds. To compute $Y_{\text{min}}(M, t)$, one may refer
to Algorithm 1 in [20].

We introduce in Algorithm 1 how to calculate $Y_{\text{max}}(M, t)$ for a given
marking $M$ and an explicit transition $t$. The basic idea is first to
iteratively enumerate all explanation vectors in $Y(M, t)$ (not necessarily
stored), and then collect the set of maximal elements in $Y(M, t)$.

The computation as to $Y(M, t)$ is presented through lines
1–16. To put it simply, as a breadth-first-search technique,
all possible firing vectors $y \in \mathbb{N}^{n}$ such that $\sigma \in \varphi^{-1}(y)$ is an
explanation of $t$ at $M$ (i.e., $M[\sigma]M'[t]$) are iteratively
searched and enumerated. A detailed description is shown as
follows.

Initially, at line 1, the row $A = (M - Pre(\cdot, t))^{T}$ is
either nonnegative or contains at least a negative element.
The former implies that $t$ is sufficiently enabled at $M$ (thus
$0_{n_{s}} \in Y(M, t)$). The latter suggests that the number of
tokens in the corresponding place(s) as to $M$ is insufficient.
Then, at line 2, we call a subroutine that consists of lines
2–12 in Algorithm 1 of [20] to process the matrix $\Gamma$.

Precisely, this procedure enumerates part of the explanation
vectors (not all) by iteratively updating $\Gamma$, i.e., adding some
considered rows in $[ C_{t}^{T} | I_{n_{i} \times n_{i}} ]$ to rows in $[ A | B ]$
that contain negative elements to neutralize them eventually.
The physical interpretation of this manipulation is to test
whether the firing of an implicit transition $t_{i}, t_{i} \in T_{I}$ (i.e.,
corresponds to the $i$'th row of the matrix $C_{t_{i}}^{T}$) at marking
$M$ such that $M' = M + C_{t_{i}} \cdot y + C(\cdot, t) \Rightarrow M' \in M_{BM}$.
As a result, if $Y(M, t) \neq \emptyset$, the sub-matrix $[ A | B ]$ contains all
nonnegative rows and the corresponding explanation vectors are
stored individually in the form of row vectors in sub-matrix
$B$.

To complete $Y(M, t)$, analogously, from lines 6–12, we add
each of the rows in $[ C_{t}^{T} | I_{n_{i} \times n_{i}} ]$ to rows in $[ A | B ]$ in
the updated $\Gamma$. If the obtained new row, e.g., $R = [ C_{t_{i}}^{T} (i^{*}, \cdot) +
A(j^{*}, \cdot) | I_{n_{i} \times n_{i}} (i^{*}, \cdot) + B(j^{*}, \cdot) ]$, is nonnegative and does
Algorithm 1 Calculation of $Y_{\text{max}}(M,t)$

Input: A marked net $(N, M_0)$, a basis partition $\pi = (T_E, T_I)$, a marking $M \in R(N, M_0)$, and $t \in T_E$

Output: $Y_{\text{max}}(M,t)$

1: $\Gamma := \begin{bmatrix} C^T I_{n_I \times n_I} \\ A \end{bmatrix}$ where $A := (M - \text{Pre}(\cdot, t))^T$

and $B := 0_{n_I}$;

2: Subroutine: Update $\Gamma$ through lines 2–12 in Algorithm 1 of [20];

3: $\alpha := \text{row size}(\Gamma)$, $\alpha_{\text{old}} := 0$, and $\alpha'_{\text{old}} := n_I$;

4: while $\alpha_{\text{old}} - \alpha \neq 0$ do

5: $\alpha_{\text{old}} := \alpha$;

6: for $k = 1 : n_I$, do

7: for $l = (\alpha'_{\text{old}} + 1) : \alpha_{\text{old}}$, do

8: $R := \Gamma(l, \cdot) + \Gamma(k, \cdot)$;

9: if $R \geq 0$ and $\beta \Gamma(i, \cdot) = \Gamma$ where $i \in \{(n_I + 1), \ldots, \alpha_{\text{old}}\}$, then

10: $\Gamma_{\text{new}} := \begin{bmatrix} \Gamma \\ R \end{bmatrix}$;

11: end if

12: end for

13: end for

14: $\alpha := \text{row size}(\Gamma_{\text{new}})$, $\alpha'_{\text{old}} := \alpha_{\text{old}}$, and $\Gamma := \Gamma_{\text{new}}$;

15: end while

16: Let $Y(M, t)$ be the set of row vectors in the updated sub-matrix $B := \Gamma((n_I + 1) : \alpha, (m + 1) : (m + n_I))$;

17: Let $Y_{\text{max}}(M,t)$ be the set of maximal elements in $Y(M, t)$.

not equal to any of the rows in $\begin{bmatrix} A & B \end{bmatrix}$, it is then recorded in $\begin{bmatrix} A & B \end{bmatrix}$ as a new extended row and matrix $\Gamma$ will be updated. This act implies that $M + C_I \cdot (I_{n_I \times n_I}(i, \cdot) + B(j, \cdot))^T - \text{Pre}(\cdot, t) \geq 0$. Thus, it is deduced that the vector $(I_{n_I \times n_I}(i, \cdot) + B(j, \cdot))^T$ is another explanation vector of $t$ at $M$ and it will be recorded in the sub-matrix $B$.

Iteratively, represented in the sub-matrix $B$ of the updated matrix $\Gamma_{\text{new}}$, all explanations of $M$ at $t$ can be collected. The computation of $Y(M, t)$ ends when the sub-matrix $\begin{bmatrix} A & B \end{bmatrix}$ of $\Gamma_{\text{new}}$ reaches a fixed point. Note that due to the boundness of the net and the acyclicity of the $T_I$-induced sub-net, Algorithm 1 will not run endlessly, since $Y(M, t)$ is not infinite. Finally, at line 17, the set of maximal explanations is obtained by collecting all the maximal rows in $Y(M, t)$.

Remark 1: We analyze the complexity of Algorithm 7. The complexity of the while loop (lines 3–14) can be estimated as follows. We have $|Y(M, t)|$ iterations from lines 3, 6 and 8 (determine if $R$ is unique) and $n_I$ iterations from lines 5. Thus, the worst-case time complexity of this loop is $O(|Y(M, t)|^3 n_I)$. On the other hand, the complexity of line 17 is less than the above-mentioned loop. Thus, the overall complexity is cubic in $|Y(M, t)|$.

B. Minimax Basis Reachability Graph

Definition 9: Given a bounded marked net $(N, M_0)$ and a basis partition $\pi = (T_E, T_I)$, its minimax-BRG is a deterministic finite state automaton $B_M = (\mathcal{M}_{B_M}, \text{Tr}_M, \Delta_M, M_0)$, where $\mathcal{M}_{B_M}$ is the set of minimax basis markings, $\text{Tr}_M$ is a finite set of pairs $(t, y) \in T_E \times \mathbb{N}^n$, $\Delta_M$ is the transition relation $\{(M_1, (t, y), M_2) \mid t \in T_E; y \in (Y_{\text{min}}(M_1, t) \cup Y_{\text{max}}(M_1, t)) \cup M_2 = M_1 + C_I \cdot y + C(\cdot, t)) \}$ and $M_0$ is the initial marking.

We extend the definition of transition relation $\Delta_M$ for sequences of pairs $\sigma^+ = ((t_1, y_1), (t_2, y_2), \ldots, (t_k, y_k)) \in \text{Tr}_M$ and write $(M_1, \sigma^+, M_2) \in \Delta_M$ to denote that from $M_1$ sequence $\sigma^+$ yields $M_2$ in $B_M$.

According to Definitions 8 and 9, to build a minimax-BRG (e.g., see Algorithm 2 in [20]), the difference is that the construction of minimax-BRGs requires taking both minimal and maximal explanation vectors into consideration. We briefly explain the construction procedure as follows. First, the set $\mathcal{M}_{B_M}$ is initialized as $\{M_0\}$. Then, for all untested markings $M \in \mathcal{M}_{B_M}$ and for all explicit transitions $t \in T_E$, it is required to check whether there exist explanation vectors $y \in Y_{\text{min}}(M, t) \cup Y_{\text{max}}(M,t)$; if exist, the corresponding minimax basis marking (i.e., $M' = M + C_I \cdot y + C(\cdot, t)$) is computed and stored in $\mathcal{M}_{B_M}$ (on the condition that $M'$ is not included in the set $\mathcal{M}_{B_M}$ before). Moreover, the set of pairs $(t, y)$ and transition relations between $M$ and $M'$ are stored in $\text{Tr}_M$ and $\Delta_M$, respectively. Iteratively, the minimax-BRG $B_M$ can be constructed. We exemplify this procedure in Example 2.

As for the complexity of constructing the minimax-BRG, in common with the BRG, the upper bound of states in a minimax-BRG is the size of the reachable space of a net (consider $T_E = \varnothing$ and $T_I = \emptyset$). Nonetheless, first, the building of a minimax-BRG does not require constructing the reachability graph. Then, our numerical results (e.g., see Section VII and VIII) show that the minimax-BRG can often be more compact in size than that of the reachability graph in the considered cases.

Remark 2: Similar to the BRG, note that the selection of the basis partition may change the computational efficiency of constructing the minimax-BRG. In general, the larger is the set $T_I$ of implicit transitions in a basis partition $\pi = (T_E, T_I)$, the smaller is the number of nodes in the minimax-BRG and the time required for its construction. See Section IV in [20] for a discussion on how to choose basis partitions.

Example 2: Consider again the parameterized plant $G = (N, M_0, M_E)$ in Fig. 1 (left) with $\alpha = 1$ and $T_E = \{t_2\}$. We briefly introduce how to construct its minimax-BRG $B_M$. According to Definition 8, $M_{b_0} = M_0$ is a minimax basis marking. Next, we compute the minimal and maximal explanation vectors of the explicit transition $t_2$ at $M_{b_0}$ respectively and derive the other potential minimax basis markings. For instance, for $t_2$, the only minimal explanation vector $y_{\text{min}} = y_1 = [1 0]^T$ while the only maximal explanation vector $y_{\text{max}} = y_2 = [2 1]^T$. Since $M_{b_0} + C_I \cdot y_{\text{min}} + C(\cdot, t_2) = [2 0 1]^T = M_{b_0}$, no new minimax basis marking is generated. However, the pair $(t_2, y_1)$ is stored in $\text{Tr}_M$ and the transition relation $(M_{b_0}, (t_2, y_1), M_{b_0})$ is stored in $\Delta_M$. On the other hand, since $M_{b_1} = M_{b_0} + C_I \cdot y_{\text{max}} + C(\cdot, t_2) = [1 0 0]^T \neq M_{b_0}$, let $M_{b_1}$ be another minimax basis marking. Corresponding pair and transition relation are also collected.
Analogously, $y_3 = [1 0]^T$ and $B_M$ can be constructed which is graphically shown in Fig. 3.

In the following, we show that the minimax-BRG preserves the reachability information and other non-minimax-basis markings can be algebraically characterized by linear equations.

**Proposition 2:** Given a marked net $(N, M_0)$ with a basis partition $\pi = (T_E, T_I)$ and a marking $M \in \mathbb{N}^m$, $M \in R(N, M_0)$ if and only if there exists a minimax basis marking $M_b \in M_{BM}$ such that $M \in R_I(M_b)$, where $M_{BM}$ is the set of the minimax basis markings in minimax-BRG of $(N, M_0)$.

**Proof:** (only if) It is shown in [3] that such a property holds for the set of basis markings $M_B$. Since $M_{BM} \supseteq M_B$, the result follows.

(ii) since $M \in R_I(M_b)$, according to Definition 6 there exists a firing sequence $\sigma \in T^*$ such that $M_0[\sigma]M_b$. On the other hand, there exists another firing sequence $\sigma' \in T^*$ such that $M_0[\sigma']M_b$, which implies that $M_0[\sigma'\sigma]M$ and concludes the proof.

In summary, a marking $M$ is reachable from $M_0$ if and only if it belongs to the implicit reach of a minimax basis marking $M_b$ and thus $M$ can be characterized by a linear equation, i.e., $M = M_b + C_I \cdot y_I$, where $y_I = \varphi(\sigma_I)$, $\sigma_I \in T^*$ and $M_0[\sigma_I]M_B$.

V. VERIFYING NONBLOCKINGNESS OF BOUNDED PLANTS USING MINIMAX-BRGs

In this section, we investigate how minimax-BRGs can be applied to solving the NB-V problem.

A. Unobstructiveness of Minimax-BRGs

This subsection generalizes the notion of unobstructiveness that is given in Fig. 4 for a BRG to a minimax-BRG. Such a property is essential to establish our method since it is strongly related to the nonblockingness of a Petri net. First, we define the set of i-coreachable minimax basis markings, denoted by $M_{i\text{co}}$, which means all of the final markings in $M_F$ is reachable by firing implicit transitions only.

**Definition 10:** Consider a bounded plant $G = (N, M_0, M_F)$ with the set of minimax basis markings $M_{BM}$ in its minimax-BRG. The set of i-coreachable minimax basis markings of $M_{BM}$ is defined as $M_{i\text{co}} = \{M_b \in M_{BM} | R_I(M_b) \cap M_F \neq \emptyset\}$.

**Proposition 3:** Given a set of final markings defined by a single GMEC $L_{(w,k)}$ and a minimax basis marking $M_b$, $M_b$ belongs to $M_{i\text{co}}$ if and only if the following set of integer constraints is feasible.

$$
\begin{align*}
M_b + C_I \cdot y_I &= M; \\
w^T J'M_b &\leq k;
\end{align*}
$$

**Proof:** (only if) Since $M_b \in M_{i\text{co}}$, according to Definition 10 $R_I(M_b) \cap M_F \neq \emptyset$. Therefore, integer constraints [1] meets feasible solution $y_I$.

(ii) The state equation $M_b + C_I \cdot y_I = M$ provides necessary and sufficient conditions for reachability since the implicit subnet is acyclic (see Proposition [1]). Moreover, $M \in L_{(w,k)}$ is a final marking. Therefore, the statement holds.

The notion of unobstructiveness in a minimax-BRG is given in Definition 11. In the following, we show how the unobstructiveness of a minimax-BRG is related to the nonblockingness of the corresponding Petri net.

**Definition 11:** Given a minimax-BRG $M_{BM} = (M_{BM}, T_{BM}, \Delta M, M_0)$ and a set of i-coreachable minimax basis markings $M_{i\text{co}} \subseteq M_{BM}$, $B_M$ is said to be unobstructed if for all $M_b \in M_{BM}$, there exist a marking $M_b' \in M_{i\text{co}}$, in $B_M$, and a firing sequence $\sigma^* \in T_{BM}$ such that $(M_b, \sigma^*, M_b') \in \Delta M$. Otherwise it is obstructed.

**Proposition 4:** Given a plant $(N, M_0, M_F)$, its minimax-BRG is unobstructed if and only if all minimax basis markings are nonblocking.

**Proof:** (only if) If a minimax-BRG $B_M$ is unobstructed, then for all $M_b \in M_{BM}$, there exist a marking $M_b' \in M_{i\text{co}}$ in $B_M$, and a sequence of pairs $\sigma^* = (t_1, y_1), (t_2, y_2), \ldots, (t_k, y_k)$ $(\sigma^* \in T_{BM}^*)$ such that $(M_b, \sigma^*, M_b') \in \Delta M$. By Definition 8 this means that the net admits an evolution: $M_b[\sigma_1 t_1 \sigma_2 t_2 \ldots \sigma_k t_k]M_b'$, where $\sigma_i = \varphi^{-1}(y_i)$ $(i \in \{1, 2, \ldots, k\})$. Since $M_b' \in M_{i\text{co}}$, there exists an implicit firing sequence $\sigma_j$ such that $M_b[\sigma_j]M_j$, where $M_j \in M_F$. Thus it holds that $M_b[\sigma_1 t_1 \sigma_2 t_2 \ldots \sigma_k t_k]M_b'[\sigma_1]M_f$, implying that $M_b$ is nonblocking.

(ii) By prove this part by contradiction. Assume $B_M$ is obstructed. Then, there exists $M_b \in M_{BM}$ such that $M_b$ is not accessible to any of the i-coreachable marking in $M_{i\text{co}}$, through $\sigma^* \in T_{BM}^*$. However, since $M_b$ is nonblocking, there exists a firing sequence $\sigma \in T^*$ and a final marking $M_f \in M_F$ such that $M_b[\sigma]M_f$. We write $\sigma = \sigma_1 t_1 \ldots \sigma_k t_k$ $\sigma_{k+1}$ where all $\sigma_i \in T_{BM}^*, t_j \in T_E, j = 1, \ldots, k$. Following the procedure in the proof of Theorem 3.8 in [3], we can repeatedly move transitions in each $\sigma_j$ $(j \in \{1, \ldots, k\})$ to somewhere after $t_j$ to obtain a new sequence $\sigma_{\text{min},1 t_1 \sigma_{\text{min},2 t_2} \ldots \sigma_{\text{min},k t_k} \sigma_{k+1}}$ such that $M_b[\sigma_{\text{min},1 t_1} M_b[\sigma_{\text{min},2 t_2}] \ldots [\sigma_{\text{min},k t_k}] M_b[\sigma_{k+1}] M_f$, where each $\sigma_{\text{min},j} \in T_{BM}^*$ is a minimal explanation of $t_j$ at $M_{i\text{co}} \in M_{BM}$ for $j = 1, \ldots, k$. Thus, $M_b \in M_{i\text{co}}$, contradiction.

According to Proposition 4 to determine the unobstructiveness of minimax-BRG $B_M$, it is only required to check if all minimax basis markings are co-reachable to some i-coreachable minimax basis markings in $B_M$. This can be done by using some search algorithm (e.g., Dijkstra) in the underlying digraph of the minimax-BRG, whose complexity is
polynomial in $B_M$. An example is illustrated in the following to help clarify Proposition iv.

**Example 3:** Consider again the parameterized plant $(N, M_0, M_F)$ in Fig. 1 (left) with $\alpha = 1$, $T_E = \{t_2\}$ and $M_F = L(w, k)$ where $w = [1 \ 1 \ 0 \ 0]^T$ and $k = 1$. We explain how to verify the unobstructiveness of its minimax-BRG $B_M$ shown in Fig. 3. By solving the linear constraint in Proposition iv we conclude that $M_{i_{10}} = \{[2 \ 0 \ 1 \ 1]^T\}$. Since there is no directed path from $M_{i_{30}}$ to $M_{i_{40}}$, $M_{i_{10}}$ is not co-reachable to the only marking in $M_{i_{10}}$. Thus, the minimax-BRG $B_M$ in Fig. 5 is obstructed. □

B. Verifying Nonblockingness of Deadlock-Free Plants

In this subsection, we focus on deadlock-free plants. An intermediate result is shown in Proposition v.

**Proposition 5:** Given a bounded marked net $(N, M_0)$ with basis partition $\pi = (T_E, T_I)$, for all $M \in R(N, M_0)$, for all $t \in T_E$, for all $\sigma \in \Sigma(M, t)$ with $M[\sigma]t$, the following implication holds:

$$\forall \sigma' \in \Sigma(M, t) \quad \varphi(\sigma) - \varphi(\sigma') = \vec{y} \geq 0 \Rightarrow \quad (\exists \sigma'' \in \varphi^{-1}(\vec{y})) \quad M[\sigma't\sigma'']M'$$

**Proof:** Let $M'' \in N^M$ such that $M[\sigma't]M''$. Then it holds that:

$$M' = M + C_I \cdot \varphi(\sigma) + C(\cdot, t)$$
$$M'' = M + C_I \cdot \varphi(\sigma') + C(\cdot, t)$$

From Equation 3 we conclude that $M' - M'' = C_I(\varphi(\sigma) - \varphi(\sigma'))$, which implies $M' = M'' + C_I \cdot \vec{y}$ and $\vec{y} \in N^M$. This indicates:

$$\exists \sigma'' \in \varphi^{-1}(\vec{y}) : M''[\sigma'']M'$$

and thus $M[\sigma't]M''[\sigma'']M'$ that concludes the proof. □

**Proposition 5** shows the connection between two markings $M'$ and $M''$ reachable from $M \in R(N, M_0)$, i.e., $M[\sigma]tM'$ and $M[\sigma't]M''$, where $t \in T_E$, $\sigma \in \Sigma(M, t)$, $\sigma' \in \Sigma(M, t)$, and $\varphi(\sigma) - \varphi(\sigma') = \vec{y} \geq 0$. If $M'$ is nonblocking, then $M''$ is nonblocking as well, since there exists a firing sequence $\sigma'' \in \varphi^{-1}(\vec{y})$ such that $M''[\sigma'']M'$. According to this proposition, we next show that the unobstructiveness of the minimax-BRG is a necessary and sufficient condition for nonblockingness of a net in the considered class.

**Lemma 1:** Consider a bounded deadlock-free marked net $(N, M_0)$ with a basis partition $\pi = (T_E, T_I)$. For all markings $M \in R(N, M_0)$, there exists a firing sequence $\sigma t$, where $\sigma \in T_I$ and $t \in T_E$, such that $M[\sigma t]$ holds.

**Proof:** We prove this statement by contradiction. Assume the system is deadlock-free and there exists a marking $M$ from which all explicit transitions are not enabled. Since the implicit sub-net of the system is bounded and acyclic, the maximal length of sequences enabled at $M$ and composed by only implicit transitions is finite. Hence, from $M$, after the firing of such maximal sequences of implicit transitions, the net reaches a deadlock, which is a contradiction.

The result in Lemma 1 can be applied to both BRG and minimax-BRG. However, it does not imply that the marking reached after the firing of the explicit transition is a basis marking, as we have shown in Example 1. Hence, it does not rule out the presence of livelocks in the BRG.

**Lemma 2:** Consider a bounded deadlock-free marked net $(N, M_0)$ with a basis partition $\pi = (T_E, T_I)$. For all markings $M' \in R(N, M_0)$, for all explicit transition $t \in T_E$, the following holds:

$$\sigma \in \Sigma(M, t) \Rightarrow (\exists \sigma' \in \Sigma_{\max}(M, t)) \quad \varphi(\sigma') \geq \varphi(\sigma)$$

**Proof:** If $\sigma \notin \Sigma_{\max}(M, t)$, according to Definition iv, there exists an explanation $\sigma'' \in \Sigma_{\max}(M, t)$ such that $\varphi(\sigma'') > \varphi(\sigma)$; otherwise $\varphi(\sigma') = \varphi(\sigma)$, hence the result holds. □

**Lemma 3:** Given a bounded deadlock-free marked net $(N, M_0)$ with a basis partition $\pi = (T_E, T_I)$, the set of minimax basis markings of the system is $M_{B_{M_0}}$. For all markings $M \in R(N, M_0)$, there exists $M_b \in M_{B_{M_0}}$ such that $M_b \in R(N, M)$.

**Proof:** Due to Lemma 1 there exists a firing sequence $\sigma t$, where $\sigma \in T_I$ and $t \in T_E$, such that $M[\sigma t]$, which implies that $\sigma \in \Sigma(M, t)$. By Lemma 2 there exists a maximal explanation $\sigma' \in \Sigma_{\max}(M, t)$ such that $\varphi(\sigma') \geq \varphi(\sigma)$. Let $\varphi(\sigma') - \varphi(\sigma) = \vec{y}$ and $M[\sigma't]M'$, by Definition v $M' \in M_{B_{M_0}}$. According to Proposition 5 there exists a firing sequence $\sigma'' \in \varphi^{-1}(\vec{y})$ such that $M[\sigma\sigma''t]M'$, which implies that $M' \in R(N, M)$.

**Theorem 1:** A bounded deadlock-free plant $G = (N, M_0, M_F)$ is nonblocking if and only if its minimax-BRG $B_M$ is obstructed.

**Proof:** (only if) Since the net is nonblocking, all reachable markings, including all minimax basis markings, are nonblocking. By Proposition 4 its minimax-BRG $B_M$ is unobstructed.

(ii) Consider an arbitrary marking $M \in R(N, M_0)$. By Lemma 3 there exists a minimax basis marking $M_b \in M_{B_{M_0}}$ such that $M_b \in R(N, M)$, i.e., there exists a firing sequence $\sigma \in T_I$ such that $M[\sigma]M_b$. Since the minimax BRG $B_M$ is unobstructed, according to Proposition 4 all minimax basis markings including $M_b$ are nonblocking, which implies that marking $M$ is co-reachable to a nonblocking marking. Hence, $G$ is nonblocking.

By Theorem 1 for a deadlock-free net, one can use an arbitrary basis partition to construct the minimax-BRG to verify its nonblockingness. Since the existence of a livelock component that contains all blocking markings implies the existence of at least a blocking minimax basis marking $M_b$ in $B_M$, the potential livelock problem mentioned in Section III is avoided.

C. Verifying Nonblockingness of Plants with Deadlocks

In this subsection, we generalize the results in Section V-B to systems that are not deadlock-free. Notice that a dead marking $M \in R(N, M_0)$ can either be non-final (i.e., $M \notin M_F$) or final (i.e., $M \in M_F$).

**Theorem 2:** A bounded plant $G = (N, M_0, M_F)$ is nonblocking if and only if its minimax-BRG $B_M$ is unobstructed and all its dead markings are final.

**Proof:** (only if) When all reachable markings are nonblocking, all dead markings (if any exists) and all minimax basis markings are also nonblocking. Hence, all dead markings are final and by Proposition 4 the minimax-BRG $B_M$ is unobstructed.

(ii) If the minimax-BRG $B_M$ is unobstructed, all minimax basis markings are nonblocking, by Proposition 4.
an arbitrary marking \( M \in R(N, M_0) \). By Proposition 2 there exist a minimax basis marking \( M_b \in \mathcal{M}_{B,N} \) in the minimax-BRG of the system and an implicit firing sequence \( \sigma_t \in T_t^* \) such that \( M_b(\sigma_t)M \).

We prove that marking \( M \) is nonblocking by contradiction. In fact, if we assume that \( M \) is blocking, since all dead markings are final, \( M \) is neither dead nor co-reachable to a deadlock in the system. Suppose that from \( M \) no explicit transition can eventually fire: following the argument of the proof of Lemma 1 a dead marking will be reached, leading to a contradiction. Therefore, there exist \( \sigma'_t \in T_t^* \) and \( t \in T_E \) such that \( M[\sigma'_t,t] \) and thus \( M_b[\sigma'_t,t] \). Also, there exists a maximal explanation \( \sigma' \in \Sigma_{max}(M_b,t) \) such that \( \varphi(\sigma') \geq \varphi(\sigma_t \sigma'_t) \). According to Proposition 5 it follows that \( M \) is co-reachable to a minimax basis marking, which implies that \( M \) is nonblocking, another contradiction, which proves the proof.

According to Theorem 2 determining the nonblockingness of a plant \( G \) can be addressed by two steps: (1) determine if there exists a reachable non-final dead marking: if not, then (2) determine the unobstructiveness of a minimax-BRG of it.

Since step (2) has already been discussed in the previous section, we only need to study step (1). Next, we show how to determine the existence of non-final dead markings by using the minimax-BRG. Denote the set of non-final dead markings as \( \mathcal{D}_{nf} \). Then, we define the set of maximal implicit firing sequences and the corresponding set of vectors as follows.

**Definition 12:** Given a bounded marked net \( \langle N, M_0 \rangle \) with basis partition \( \pi = (T_E, T_I) \) and a marking \( M \in R(N, M_0) \), we define

\[
\Sigma_{max}(M) = \{ \sigma \in T_I^* | (M[\sigma]) \land (\exists \sigma' \in T_I^* : M[\sigma',t] \geq \varphi(\sigma')) \}
\]

as the set of maximal implicit firing sequences at \( M \), and

\[
Y_{max}(M) = \{ \varphi(\sigma) \in Y^{|t|} \ | \sigma \in \Sigma_{max}(M) \}
\]

as the corresponding set of maximal implicit firing vectors.

**Proposition 6:** Given a bounded marked net \( \langle N, M_0 \rangle \) with basis partition \( \pi = (T_E, T_I) \), let \( \mathcal{M}_{B,N} \) be its minimax basis marking set. Marking \( M \in R(N, M_0) \) is dead if and only if there exist \( M_b \in \mathcal{M}_{B,N} \) and \( \sigma \in \Sigma_{max}(M_b) \) such that for all \( t \in T_E, \sigma \notin \Sigma_{max}(M_b, t) \) and \( M_b[\sigma,t]M \).

**Proof:** (if) Since \( \sigma \in \Sigma_{max}(M_b) \), there does not exist an implicit transition \( t_I \in T_I \) such that \( M[t_I] \). On the other hand, since for all \( t \in T_E, \sigma \notin \Sigma_{max}(M_b, t) \), i.e., there does not exist an explicit transition \( t' \in T_E \) such that \( M[t'] \), which implies that \( M \) is dead.

(only if) Since \( M \) is dead, there does not exist \( t \in T \) such that \( M[t] \). Therefore, there exists \( \sigma \in \Sigma_{max}(M_b) \) such that for all \( t \in T_E, \sigma \notin \Sigma_{max}(M_b, t) \) and \( M_b[\sigma,t]M \).

**Proposition 6** shows the relation between dead markings and minimax basis markings in a bounded system, i.e., all reachable dead markings can be obtained by firing a maximal implicit firing sequence \( \sigma \) from a minimax basis marking \( M_b \) where for all \( t \in T_E, \sigma \) is not a maximal explanation of \( t \). Next, we introduce Algorithm 2 to verify if there exist non-final dead markings in a plant.

In Algorithm 2, first, from lines 1–4, we add an explicit transition \( t_0 \) to \( N \) with \( Pre(\cdot,t_0) = Post(\cdot,t_0) = \emptyset \) and derive a new plant \( \langle N', M_0, M_F \rangle \). Obviously, \( t_0 \) is enabled from any reachable marking and, since its firing does not modify the marking, it holds that \( R(N, M_0) = R(N', M_0) \). Hence, for all \( M_b \in \mathcal{M}_{B,N} \), we conclude that \( Y_{max}(M_b) = Y_{max}(M_b, t_0) \), i.e., the set of maximal implicit firing vectors at \( M_b \) can be determined by computing maximal explanation of \( t_0 \) at \( M_b \) based on Algorithm 1.

Then, we determine if, for all \( t \in T_E \), the obtained firing vector \( y \in Y_{max}(M_b) \) is not an explanation of \( t \) at \( M_b \). Implemented in lines 5–16, this consists in checking if, for all \( t \in T_E \), \( t \) is disabled at marking \( M' = M_b + C_I \cdot y \); since no implicit transition can fire at \( M' \), the only transitions that can possibly fire are those explicit ones. If no explicit transition is enabled at \( M' \), according to Proposition 6, marking \( M' \) is dead. Further, \( M' \) will be added into the set \( \mathcal{D}_{nf} \) if it is dead and not final. Note that Algorithm 2 also tests if a minimax basis marking \( M'_b \in \mathcal{M}_{B,N} \) is dead. Since \( Y_{max}(M'_b, t_0) = \emptyset \) for all \( t \in (T_E \setminus \{t_0\}) \), \( Y_{max}(M'_b, t_0) = \emptyset \); \( M'_b \) will be added to \( \mathcal{D}_{nf} \) if it is not final. When the algorithm terminates, if \( \mathcal{D}_{nf} = \emptyset \), we conclude that the plant is blocking; otherwise, the unobstructiveness verification procedure (mentioned in Section 4-A) of the minimax-BRG should be further executed.

The complexity of Algorithm 2 depends on the two for loops (lines 5–13). First, there are \( |\mathcal{M}_{B,N}| \) and \( |Y_{max}(M,t_0)| \) iterations in lines 5 and 6, respectively. In line 8, to verify \( M' \) is dead, one may need to test if \( M' \neq Pre'(\cdot,t) \) for all \( t \in T_E \) (no need to test transitions in \( T_I \) since no implicit transition is enabled at \( M' \)), which requires \( |T_E| \) iterations. In summary, the worst-case time complexity of Algorithm 2 is \( O(|\mathcal{M}_{B,N}| \cdot |Y_{max}(M,t_0)| \cdot |T_E|) \).

**VI. CASE STUDY**

We use a parameterized plant (chosen from [20]) depicted in Fig. 4 to test the efficacy and efficiency of our method in this section. Let
Table I: Analysis of the reachability graph, expanded BRG from \([10]\) and minimax-BRG for the plant in Fig. 4 with
\[T_E = \{t_3, t_6, t_{11}, t_{13}\}\].

| Run | \(\lambda\) | \(\mu\) | \(|R(N, M_0)|\) | Time (s) | \(|M_{BE}\)| | Time (s) | \(|M_{BE_{\mu}}|\) | Time (s) | \(P_{BE_{\mu}} = 0^\ast\) | Time (s) | Unobstructed? | Time (s) | NB? | \(|M_{BE_{\mu}}|/|M_{BE}|\) | \(|M_{BE_{\mu}}|/|R(N, M_0)|\) |
|-----|-----|-----|-------------|--------|-------------|--------|-------------|--------|----------------|--------|--------------|--------|-----|----------------|----------------|
| 1   | 5   | 1   | 102         | <1     | 31          | 0.2    | 11          | 0.04   | Yes            | 0.03   | Yes          | 1.9    | Yes | 35.5%         | 10.8%         |
| 2   | 5   | 2   | 384         | 1      | 191         | 0.7    | 37          | 0.2    | Yes            | 0.1    | Yes          | 6      | Yes | 19.3%         | 9.6%          |
| 3   | 5   | 3   | 688         | 2      | 405         | 1      | 68          | 0.4    | Yes            | 0.4    | Yes          | 12     | Yes | 10.8%         | 9.9%          |
| 4   | 5   | 4   | 840         | 4      | 449         | 2      | 81          | 0.5    | Yes            | 0.5    | Yes          | 15     | Yes | 18.0%         | 9.6%          |
| 5   | 6   | 2   | 12066       | 431    | 9117        | 302    | 1171        | 23     | Yes            | 16     | No           | 251    | No  | 12.8%         | 9.7%          |
| 6   | 6   | 3   | 88681       | 24354  | 75378       | 20944  | 9985        | 833    | No             | 58     | No           | 155    | No  | 13.2%         | 11.3%         |
| 7   | 6   | 4   | -           | -t.o.  | -           | -t.o.  | 31147       | 10062  | No             | 955    | No           | -t.o.  | -   | -             | -             |
| 8   | 6   | 5   | -           | -t.o.  | -           | -t.o.  | 41817       | 18295  | No             | 1618   | -            | -t.o.  | -   | -             | -             |
| 9   | 6   | 6   | -           | -t.o.  | -           | -t.o.  | 45458       | 21229  | No             | 1754   | -            | -t.o.  | -   | -             | -             |

* The computing time is denoted by *overtime* (o.t.) if the program does not terminate within 28,800 seconds (8 hours).

Figure 4: A parameterized manufacturing example.

\(M_0 = [\lambda 0 0 0 0 0 0 \mu \mu \mu 0 0 0 0 0 0 0 \mu \mu \mu \mu \mu \mu]^T\).

Consider \(T_E = \{t_3, t_6, t_{11}, t_{13}\}\) (marked as shadow bars). Also, we set \(M_E = L(w, k) = \{M \in \mathbb{N}^{|w|}\ | M \leq k\}\),
where \(w = [0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0]^T\)
and \(k = 3\) (for run 4) or \(k = 4\) (for runs 8–10) or \(k = 5\) (for runs 1–3) or \(k = 7\) (for runs 5–6) or \(k = 15\) (for run 7), to test nonblockingness of this plant for all cases.

We run several simulations on a laptop with Intel i7-5500U 2.40 GHz processor and 8 GB RAM. Table I shows, for different values of the parameters \(\lambda\) and \(\mu\), the sizes of the reachability graph \(|R(N, M_0)|\), of the expanded BRG \(|M_{BE}|\) \([10]\) and of minimax-BRG \(|M_{BE_{\mu}}|\) as well as the time required to compute them. We also show the ratios of \(|M_{BE_{\mu}}|\) to \(|M_{BE}|\) and \(|M_{BE_{\mu}}|\) to \(|R(N, M_0)|\). It can be verified that \(|M_{BE_{\mu}}| \ll |M_{BE}|\) and \(|M_{BE_{\mu}}| \ll |R(N, M_0)|\) in all cases. Note that the size of minimax-BRG depends on the net structure, initial resource distribution and choice of basis partition \(\pi = (T_E, T_I)\). Also in Table I we show the simulation results of determining if there exist non-final dead markings based on Algorithm \(2\) (columns 10–11), and verifying unobstructiveness (the set of i-coreachable markings \(M_{ic}\) of a minimax-BRG can be obtained by using two free MATLAB integer linear programming problems solver toolboxes namely YALMIP \([17]\) and lpsolve \([1]\) for all cases if necessary (columns 12–13)). Moreover, the nonblockingness of the system for all cases are listed in column 14. The test cases show that minimax-BRG-based technique achieves practical efficiency when coping with the NB-V problem in this considered case. Additional case studies are also considered in \([11]\), which consists of three Petri net benchmarks taken from the literature.

VII. DISCUSSIONS

We propose the minimax-BRG to ensure that the essential features of a system, from which a blocking condition may originate, are captured in the abstracted model. As a non-trivial task, it is necessary to formally characterize and validate the proposed approach with a series of theoretical results. When tackling the NB-V problem, the minimax-BRG-based approach is general and can be directly applied to arbitrary bounded plants (the only restriction is that the \(T_I\)-induced subnet is acyclic). This is a major practical advantage with respect to other abstraction approaches that are based on particular structures or symmetries, and require significant analysis of the model in a preliminary stage before they can be applied.

Further, our numerical results (i.e., Section VI and \([11]\)) show that the minimax-BRG can often be more compact in size than that of the reachability graph in the considered cases. Accordingly, as a potential advantage, when it comes to a related problem of NB-V, i.e., nonblocking enforcement, which consists of designing a supervisor (an online control agent) to ensure that the controlled plant does not reach a blocking marking, a supervisor designed based on the minimax-BRG can also be more compact than that of a reachability-graph-based one.

VIII. CONCLUSIONS AND FUTURE WORK

In this paper, we studied the problem of nonblockingness verification of a plant. A semi-structural method using minimax-BRG is developed, which can be used to determine the nonblockingness of a system modelled by bounded Petri nets by first determining the existence of non-final deadlocks and later checking the unobstructiveness of the corresponding minimax-BRG. The proposed approach does not require the construction of the reachability graph and has wide applicability. As for future work, we will investigate necessary
and sufficient conditions for verifying nonblockingness in unbounded nets. Second, if a system is blocking, we plan to study the nonblockingness enforcement problem and develop a supervisor to guarantee the closed-loop system to be non-blocking.

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