A direct kinematical derivation of the relativistic Sagnac effect for light or matter beams

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Abstract

The Sagnac time delay and the corresponding Sagnac phase shift, for relativistic matter and electromagnetic beams counter-propagating in a rotating interferometer, are deduced on the ground of relativistic kinematics. This purely kinematical approach allows to explain the “universality” of the effect, namely the fact that the Sagnac time difference does not depend on the physical nature of the interfering beams. The only prime requirement is that the counter-propagating beams have the same velocity with respect to any Einstein synchronized local co-moving inertial frame.

1 Introduction

The phase shift due to the interference of two coherent light beams, propagating in the two opposite directions along the rim of a rotating ring interferometer, was observed for the first time by Sagnac[1] in 1913. Indeed, some years before[2], he had predicted the following fringe shift (with respect to the interference pattern when the device is at rest), for monochromatic light waves in vacuum:

\[ \Delta z = \frac{4\Omega \cdot S}{\lambda c} \]  

(1)

where \( \Omega \) is the (constant) angular velocity vector of the turntable, \( S \) is the vector associated to the area enclosed by the light path, and \( \lambda \) is the
wavelength of light, as seen by an observer at rest on the rotating platform. The time difference associated to the fringe shift (1) turns out to be

$$\Delta t = \frac{\lambda}{c} \Delta z = \frac{4 \Omega \cdot S}{c^2}$$  \hspace{1cm} (2)

His interpretation of these results was entirely in the framework of the classical (non Lorentz!) ether theory; however, Sagnac was the first scientist who reported an experimental observation of the effect of rotation on space-time, which, after him, was named "Sagnac effect". It is interesting to notice that the Sagnac effect was interpreted as a disproof of the Special Theory of Relativity (SRT) not only during the early years of relativity (in particular by Sagnac himself), but, also, more recently, in the 1990's by Selleri[3,4], Croca-Selleri[5], Goy-Selleri[6], Vigier[7], Anastasovski et al.[8], Klauber[9]. However, this claim is incorrect: as a matter of fact, the Sagnac effect for counter-propagating light beams (in vacuum) can be explained completely in the framework of SRT, see for instance Weber[10], Dieks[11], Anandan[12], Rizzi-Tartaglia[13], Bergia-Guidone[14], Rodrigues-Sharif[15], Henriksen[16], Rizzi-Ruggiero[17]. According to SRT, eq. (2) turns out to be just a first order approximation of the relativistic proper time difference between counter-propagating light beams.

The experimental data show that the Sagnac fringe shift (1) does not depend either on the light wavelength nor on the presence of a co-moving optical medium. This is a first important clue for the so called "universality of the Sagnac effect". However, the most compelling claim for the universal character of the Sagnac effect comes from the validity of eq. (1) not only for light beams, but also for any kind of "entities" (such as electromagnetic and acoustic waves, classical particles and electron Cooper pairs, neutron beams and De Broglie waves and so on...) travelling in opposite directions along a closed path in a rotating interferometer, with the same (in absolute value) velocity with respect to the turntable. Of course the entities take different times for a complete round-trip, depending on their velocity relative to the turntable; but the difference between these times is always given by eq. (2). So, the amount of the time difference is always the same, both for matter and light waves, independently of the physical nature of the interfering beams.

There have been many tests of the effect that prove its universality. For instance, the Sagnac effect with matter waves has been verified experimentally using Cooper pair[18] in 1965, using neutrons[19] in 1984, using $^{40}$Ca atoms beams[20] in 1991 and using electrons, by Hasselbach-Nicklaus[21], in 1993. The effect of the terrestrial rotation on neutrons phase was demonstrated in 1979 by Werner et al.[22] in a series of famous experiments.
However, as far as we know, a clear - and universally shared - derivation of the Sagnac effect for matter waves, in the full framework of SRT, seems to be lacking\(^1\) - or it is at least hard to find it in the literature.

In this paper we are going to provide a direct and simple derivation of the Sagnac effect, using the relativistic law of addition of velocities. Our derivation applies to any kind of light or matter beams, counter-propagating in a rotating interferometer. More explicitly, we shall show that, if a very simple and sound requirement is fulfilled, the Sagnac time delay does not depend on the physical nature of the interfering beams. The "simple and sound requirement" is the following: the counter-propagating beams must have the same velocity with respect to any local co-moving inertial frame (LCIF), provided that it is Einstein synchronized. Of course an alternative synchronization is allowed\(^2\), but this statement explicitly requires local Einstein’s synchronization on the platform.

## 2 The Sagnac Effect for material and light beams

Two light or matter beams are constrained to follow a circular path along the rim of a rotating disk, with constant angular velocity, in opposite directions. Let us suppose that a beam source and an interferometric detector are lodged on a point \(\Sigma\) of the rim of the disk. Let \(K\) be the central inertial frame, parameterized by a set of cylindrical coordinates \(\{x^\mu\} = (ct, r, \theta, z)\), with line element given by\(^3\)

\[
\text{ds}^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + dr^2 + r^2d\theta^2 + dz^2
\]

In particular, if we confine ourselves to a disk (\(z = \text{const}\)), the metric which we have to deal with is

\[
\text{ds}^2 = -c^2dt^2 + dr^2 + r^2d\theta^2
\]

With respect to \(K\), the disk (whose radius is \(R\)) rotates with angular velocity \(\Omega\), and the world line \(\gamma_{\Sigma}\) of \(\Sigma\) is

\[
\gamma_{\Sigma} \equiv \left\{
\begin{array}{l}
x^0 = ct \\
x^1 = r = R \\
x^2 = \theta = \Omega t
\end{array}
\right.
\]

\(^1\)As we pointed out before, the Sagnac effect has been derived by many Authors, in the full framework of SRT, only for electromagnetic waves in vacuum.

\(^2\)Let us remind that an alternative synchronization is actually needed \textit{globally} on the platform.

\(^3\)The signature is (-1,1,1,1), Greek indices run from 0 to 3, while Latin indices run from 1 to 3.
Figure 1: The world-line of $\Sigma$, a point on the disk where a beam source and interferometric detector are lodged, is $\gamma_{\Sigma}$; $\gamma_+$ and $\gamma_-$ are the world-lines of the co-propagating (+) and counter-propagating (-) beams. The first intersection of $\gamma_+$ ($\gamma_-$) with $\gamma_{\Sigma}$ takes place at the time $\tau_+$ ($\tau_-$), as measured by a clock at rest in $\Sigma$.

or, eliminating $t$

$$\gamma_{\Sigma} \equiv \begin{cases} 
    x^0 &= \frac{c}{\Omega} \theta \\
    x^1 &= R \\
    x^2 &= \theta 
\end{cases} \quad (6)$$

The world-lines of the co-propagating (+) and counter-propagating (-) beams emitted by the source at time $t = 0$ (when $\theta = 0$) are, respectively:

$$\gamma_+ \equiv \begin{cases} 
    x^0 &= \frac{c}{\omega_+} \theta \\
    x^1 &= R \\
    x^2 &= \theta 
\end{cases} \quad (7)$$

$$\gamma_- \equiv \begin{cases} 
    x^0 &= \frac{c}{\omega_-} \theta \\
    x^1 &= R \\
    x^2 &= \theta 
\end{cases} \quad (8)$$
where \( \omega_+, \omega_- \) are their angular velocities, as seen in the central inertial frame\(^4\). The first intersection of \( \gamma_+ (\gamma_-) \) with \( \gamma_{\Sigma} \) is the event "absorption of the co-propagating (counter-propagating) beam after a complete round trip" (see figure\(^1\)). This event takes place when

\[
\frac{1}{\Omega} \theta_\pm = \frac{1}{\omega_\pm} (\theta_\pm \pm 2\pi)
\]  

(9)

where the \(+\) (\(-\)) sign holds for the co-propagating (counter-propagating) beam. The solution of eq. (9) is:

\[
\theta_\pm = \pm \frac{2\pi \Omega}{\omega_\pm - \Omega} 
\]  

(10)

If we introduce the dimensionless velocities \( \beta = \Omega R/c \), \( \beta_\pm = \omega_\pm R/c \), the \( \theta \)-coordinate of the absorption event can be written as follows:

\[
\theta_\pm = \pm \frac{2\pi \beta}{\beta_\pm - \beta} 
\]  

(11)

The proper time read by a clock at rest in \( \Sigma \) is given by

\[
\tau = \frac{1}{c} \int_{\gamma_{\Sigma}} ds = \frac{1}{c} \int_{\gamma_{\Sigma}} \sqrt{c^2 dt^2 - R^2 d\theta^2} = \frac{1}{\Omega} \sqrt{1 - \beta^2} \int_{\gamma_{\Sigma}} d\theta
\]

(12)

Taking into account eq. (11), the proper time \( \tau_+ (\tau_-) \) elapsed between the emission and the absorption of the co-propagating (counter-propagating) beam, read by a clock at rest in \( \Sigma \), is given by

\[
\tau_\pm = \pm \frac{2\pi \beta \sqrt{1 - \beta^2}}{\Omega (\beta_\pm - \beta)} 
\]

(13)

and the proper time difference \( \Delta \tau \equiv \tau_+ - \tau_- \) turns out to be

\[
\Delta \tau = \frac{2\pi \beta}{\Omega} \sqrt{1 - \beta^2} \frac{\beta_- - 2\beta + \beta_+}{(\beta_+ - \beta)(\beta_- - \beta)}
\]

(14)

Without specifying any further conditions, the proper time difference (14) appears to depend upon \( \beta, \beta_+, \beta_- \): this means that it does depend, in general, both on the velocity of rotation of the disk and on the velocities of the beams.

\(^4\)Notice that \( \omega_- \) is positive if \( |\omega'_-| < \Omega \), null if \( |\omega'_-| = \Omega \), and negative if \( |\omega'_-| > \Omega \), see eq. (15) below.
Let $\beta'_\pm$ be the dimensionless velocities of the beams as measured in any Minkowski inertial frame, locally co-moving with the rim of the disk, or briefly speaking in any locally co-moving inertial frame (LCIF). Provided that each LCIF is Einstein synchronized (see Rizzi-Serafini [23]), the Lorentz law of velocity addition gives the following relations between $\beta'_\pm$ and $\beta_\pm$:

$$\beta_\pm = \frac{\beta'_\pm + \beta}{1 + \beta'_\pm \beta}$$ (15)

By substituting (15) in (14) we easily obtain

$$\Delta \tau = \frac{4\pi \beta^2}{\Omega} \frac{1}{\sqrt{1-\beta^2}} + \frac{2\pi \beta}{\Omega} \frac{1}{\sqrt{1-\beta^2}} \left( \frac{1}{\beta'_+} + \frac{1}{\beta'_-} \right)$$ (16)

Now, let us impose the condition "equal relative velocity in opposite directions":

$$\beta'_+ = -\beta'_-$$ (17)

Such condition means that the beams are required to have the same velocity (in absolute value) in every LCIF, provided that every LCIF is Einstein synchronized. If condition (17) is imposed, the proper time difference (16) reduces to

$$\Delta \tau = \frac{4\pi R^2 \Omega}{c^2} \left( 1 - \frac{\Omega^2 R^2}{c^2} \right)^{-1/2}$$ (18)

which is the relativistic Sagnac time difference.

A very relevant conclusion follows. According to eq. (13), the beams take different times - as measured by the clock at rest on the starting-ending point $\Sigma$ on the platform - for a complete round trip, depending on their velocities $\beta'_\pm$ relative to the turnable. However, when condition (17) is imposed, the difference $\Delta \tau$ between these times does depend only on the angular $\Omega$ of the disk, and it does not depend on the velocities of propagation of the beams with respect the turnable.

This is a very general result, which has been obtained on the ground of a purely kinematical approach. The Sagnac time difference (18) applies to any couple of (physical or even mathematical) entities, as long as a velocity, with respect the turnable, can be consistently defined. In particular, this result applies as well to photons (for which $|\beta'_\pm| = 1$), and to any kind of classical or quantum particle under the given conditions (or electromagnetic/acoustic waves in presence of an homogeneous co-moving medium)\(^5\).

\(^5\)Provided that a group velocity can be defined.
This fact evidences, in a clear and straightforward way, the universality of the Sagnac effect.

3 A remark on the synchronization

As it is well known (see for instance Rizzi-Serafini or Minguzzi), in a local or global inertial frame (IF) the synchronization can be arbitrarily chosen within the synchronization gauge

\[
\begin{align*}
    t' &= t' (t, x^1, x^2, x^3) \\
    x'_i &= x_i
\end{align*}
\]

(with the additional condition \(\partial t'/\partial t > 0\), which ensures that the change of time parameterization does not change the arrow of time). In eq. (19) the coordinates \((t, x_i)\) are Einstein coordinates, and \((t', x'_i)\) are re-synchronized coordinates of the IF under consideration. Of course, the IF turns out to be optically isotropic if and only if it is parameterized by Einstein coordinates \((t, x_i)\).

According to the previous section, the central inertial frame \(K\) is Einstein synchronized; let us call \(F(K)\) the "simultaneity foliation" of space-time with respect to \(K\). However, on the rotating platform many synchronization choices can be done, depending on the aims and circumstances. In particular, exploiting the gauge freedom, two different synchronization choices turn out to be specially useful.

If we look for a global synchronization on the rotating platform, any LCIF must share the synchronization of \(K\), that is the "simultaneity foliation" \(F(K)\) of space-time.

On the other hand, if we look for a plain kinematical relationship between local velocities, in order to explain the universality of the Sagnac effect, any LCIF must be Einstein synchronized. In fact, only Einstein’s synchronization allows the clear and meaningful requirement: "equal relative velocity in opposite directions".

\footnote{Eq. (19) is a subset of the set of all the possible parameterizations of the given physical IF, see for instance Cattaneo, Møller, Nikolić.}

\footnote{We want to point out that the local isotropy or anisotropy of the velocity of light in an IF is not a fact, with a well defined ontological meaning, but a convention which depends on the synchronization chosen in the IF. In particular, the velocity of light has the invariant value \(c\) in every LCIF, both in co-rotating and counter-rotating direction, if and only if the LCIF are Einstein-synchronized.}

\footnote{Formally expressed by condition (17).}
4 A remark on the interferometric detectability of the Sagnac effect

The Sagnac time difference also applies to the Fourier components of the wave packets associated to a couple of matter beams counter-propagating, with the same relative velocity, along the rim. Of course only matter beams are physical entities, while Fourier components are just mathematical entities, which no energy transport is associated to.

With regard to the interferometric detection of the Sagnac effect, the crucial point is the following. Despite the lack of a direct physical meaning and energy transfer, the phase velocity of these Fourier components (which is the same for both the co-rotating and counter-rotating ones) complies with the Lorentz law of velocity composition.

Moreover, the interferometric detection of the Sagnac effect requires that the wave packet associated to the matter beam should be sharp enough in the frequency space to allow the appearance, in the interferometric region, of an observable fringe shift. It may be worth recalling that:

(i) the observable fringe shift \( \Delta z \) depends on to the phase velocity of the Fourier components of the packet wave;

(ii) with respect to an Einstein synchronized LCIF, the velocity of every Fourier component of the wave packet associated to the matter beam, moving with the velocity (in absolute value) \( v \equiv c|\beta'| \), is given by the De Broglie expression \( v_f = c^2 / v \).

The consequent Sagnac phase shift, due to the relativistic time difference, is

\[
\Delta \Phi = 2\pi \Delta z = 2\pi \left( \frac{v_f}{\lambda} \Delta \tau \right) = \frac{8\pi^2 R^2 \Omega}{\lambda v} \left( 1 - \frac{\Omega^2 R^2}{c^2} \right)^{-1/2}
\]

5 Conclusions

We have given a direct derivation of the Sagnac effect on the bases of the relativistic kinematics. In particular, only the law of velocities addition, together with the condition that the counter-propagating beams have the same velocity with respect to any Einstein synchronized LCIF, have been used to obtain the Sagnac time difference. In this way, we have shown, in a straightforward way, the independence of the Sagnac time difference

\(^9\)That is, the Fourier components of the packet wave should have slightly different wavelengths.
from the physical nature and the velocities (relative to the turntable) of the interfering beams.

The simple derivation that we have outlined proves, in a clear and understandable way, the universal features of the Sagnac effect, which can be clearly understood as a purely geometrical effect in the Minkowski spacetime of SRT, while it would be hard to grasp in the context of classical physics.

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