DARK ENERGY EFFECTS OF TWO MEASURES FIELD THEORY

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The dilaton-gravity sector of the Two Measures Field Theory (TMT) is explored in detail in the context of cosmology. The model possesses scale invariance which is spontaneously broken due to the intrinsic features of the TMT dynamics. The dilaton $\phi$ dependence of the effective Lagrangian appears only as a result of the spontaneous breakdown of the scale invariance. If no fine tuning is made, the effective $\phi$-Lagrangian $p(\phi, X)$ depends quadratically upon the kinetic energy $X$. Hence TMT may represent an explicit example of the effective $k$-essence resulting from first principles without any exotic term in the fundamental action intended for obtaining this result. Depending of the choice of regions in the parameter space, TMT exhibits different possible outputs for cosmological dynamics: a) Possibility of a power law inflation driven by the field $\phi$ which is followed by the late time evolution driven both by a small cosmological constant and the field $\phi$ with a quintessence-like potential. TMT enables two ways for achieving small cosmological constant without fine tuning of dimensionfull parameters: either by a seesaw type mechanism or due to a correspondence principle between TMT and conventional field theories (i.e theories with only the measure of integration $\sqrt{-g}$ in the action). b) Possibility of resolution of the old cosmological constant problem. From the point of view of TMT, it becomes clear why the old cosmological constant problem cannot be solved (without fine tuning) in conventional field theories. c) The power law inflation without any fine tuning can end with damped oscillations of $\phi$ around the state with zero cosmological constant. d) There is a broad range of the parameters such that: the equation-of-state in the late time universe $w = p/\rho < -1$; $w$ asymptotically (as $t \to \infty$) approaches $-1$ from below; $\rho$ approaches a cosmological constant, the smallness of which does not require fine tuning of dimensionfull parameters.

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I. INTRODUCTION

The cosmological constant (CC) problem [1-3], the accelerated expansion of the late time universe [4], the cosmic coincidence [5] are challenges for the foundations of modern physics (see also reviews on dark energy [6-8], dark matter [9] and references therein). Numerous models have been proposed with the aim to find answer to these puzzles, for example: the quintessence [10], coupled quintessence [11], $k$-essence [12], variable mass particles [13], interacting quintessence [14], Chaplygin gas [15], phantom field [16], tachyon matter cosmology [17], brane world scenarios [18], etc. These puzzles have also motivated an interest in modifications and even radical revisions of the standard gravitational theory (General Relativity (GR)) [19, 20]. Although motivations for most of these models can be found in fundamental theories like for example in bran world [21], the questions concerning the Einstein GR limit and relation to the regular particle physics, like standard model, still remain unclear.

It is very hard to imagine that it is possible to propose ideas which are able to solve the above mentioned problems keeping at the same time unchanged the basis of fundamental physics, i.e. gravity and particle field theory. This paper may be regarded as an attempt to find a way for satisfactory answers at least to a part of the fundamental questions on the basis of first principles, i.e. without using semi-empirical models. In this paper we explore a toy model including gravity and a single scalar field $\phi$ in the framework of the so called Two Measures Field Theory (TMT) [22-24]. In TMT, gravity (or more exactly, geometry) and particle field theory intertwined in a very non trivial manner, but the Einstein’s GR is restored when the fermion matter energy density is much larger than the vacuum energy density [25, 26].

Here we have no purpose of constructing a complete realistic cosmological scenario. Instead, we concentrate our attention on the possible role TMT may play in resolution of a number of the above mentioned problems. The field theory model we will study here is invariant under global scale transformation of the metric accompanied with an appropriate shift of the dilaton field $\phi$. This scale symmetry is spontaneously broken due to intrinsic features of the TMT dynamics. Except for a peculiar structure of the TMT action, the latter does not contain any exotic terms. The obtained dynamics represents an explicit example of $k$-essence resulting from first principles.

The parameter space of the model allows to find different regions where the following different effects can take place without fine tuning of the parameters and initial conditions:

a) Possibility of a power law inflation driven by a scalar field $\phi$ which is followed by the late time evolution...
driven both by a small cosmological constant and a scalar field $\phi$ with a quintessence-like potential; smallness of the cosmological constant can be achieved without fine tuning of dimensionfull parameters.

b) In another region of the parameters, there is a possibility of the power law inflation ended with damped oscillations of $\phi$ around the state with zero cosmological constant (we want to emphasize again that this is realized without fine tuning of the parameters and initial conditions). Thus this scenario includes an exit from inflation together with resolution of the old CC problem. This becomes possible because the basic assumptions of the Weinberg’s no-go theorem\[1\] are generically violated in TMT models.

c) There is a region of the parameters where the model exhibits the possibility for a superacceleration phase of the universe: in the late time universe the dark energy density $\rho$ increases asymptotically (as $t \rightarrow \infty$) approaching from below to a constant value; the equation-of-state $p/\rho = w < -1$ and $w$ asymptotically approaches $-1$ from below.

The organization of the paper is the following: In Sec.II we present a review of the basic ideas of TMT. Sec.III starts from formulation of a simple scale invariant model containing all the terms respecting the symmetry of the model but without any exotic terms. In Appendix A we present equations of motion in the original frame. Using results of Appendix A, the complete set of equations of motion in the Einstein frame is given in Sec.III as well. It is shown there that if no fine tuning of the parameters is made, the effective action of our model in the Einstein frame turns out to be a k-essence type action. We start in Sec.IV from a simple case with fine tuned parameters where the non-linear dependence of the kinetic term disappears. Then different shapes of the effective potential are possible. For the spatially flat FRW universe we study some features of the cosmological dynamics for each of the shapes of the effective potential. For one of the shapes of the effective potential, the zero vacuum energy is achieved without any fine tuning. In Sec.V we study the cosmological dynamics without tuning parameters. It is shown that there is the possibility of a superacceleration phase of the universe, and some details of the dynamics are explored. Sec.VI is devoted to studying in more details the CC problems: a) we discuss two mechanisms for resolution of the new CC problem; b) we analyze the difference between TMT and conventional field theories (where only the measure of integration $\sqrt{-g}$ is used in the fundamental action) which allows to understand why from the point of view of TMT the conventional field theories failed to solve the old CC problem. In Appendix B we shortly discuss what kind of model one would obtain when choosing fine tuned couplings to the measures of integration in the action. Some additional remarks concerning the relation between the structure of TMT action and the CC problem are given in Appendix C.

II. BASIS OF TWO MEASURES FIELD THEORY

TMT is a generally coordinate invariant theory where all the difference from the standard field theory in curved space-time consists only of the following three additional assumptions:

1. The main supposition is that for describing the effective action for 'gravity + matter' at energies below the Planck scale, the usual form of the action $S = \int L\sqrt{-g} d^4x$ is not complete. Our positive hypothesis is that the effective action has to be of the form\[24]-\[32]

$$S = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x$$

(1)

including two Lagrangians $L_1$ and $L_2$ and two measures of integration $\sqrt{-g}$ and $\Phi$ or, equivalently, two volume elements $\Phi d^4x$ and $\sqrt{-g} d^4x$ respectively. One is the usual measure of integration $\sqrt{-g}$ in the 4-dimensional space-time manifold equipped with the metric $g_{\mu\nu}$. Another is the new measure of integration $\Phi$ in the same 4-dimensional space-time manifold. The measure $\Phi$ being a scalar density and a total derivative\[46] may be defined

- either by means of four scalar fields $\varphi_a$ ($a = 1, 2, 3, 4$), (compare with the approach by Wilczek\[46]),

$$\Phi = \varepsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \partial_\mu \varphi_a \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d.$$  

(2)

- or by means of a totally antisymmetric three index field $A_{\alpha\beta\gamma}$

$$\Phi = \varepsilon^{\mu\nu\alpha\beta} \partial_\mu A_{\nu\alpha\beta}.$$  

(3)

To provide parity conservation in the case given by Eq.\(2\) one can choose for example one of $\varphi_a$’s to be a pseudoscalar; in the case given by Eq.\(3\) we must choose $A_{\alpha\beta\gamma}$ to have negative parity. Some ideas concerning the nature of the measure fields $\varphi_a$ are discussed in Ref.\[32\]. A special case of the structure \[11\] with definition \[13\] has been recently discussed in Ref.\[33\] in applications to supersymmetric theory and the CC problem.
2. It is assumed that the Lagrangian densities $L_1$ and $L_2$ are functions of all matter fields, the dilaton field, the metric, the connection but not of the "measure fields" ($\varphi_a$ or $A_{a\beta\gamma}$). In such a case, i.e. when the measure fields enter the theory only via the measure $\Phi$, the action possesses an infinite dimensional symmetry. In the case given by Eq. (2) these symmetry transformations have the form $\varphi_a \to \varphi_a + f_a(L_1)$, where $f_a(L_1)$ are arbitrary functions of $L_1$ (see details in Ref. [24]); in the case given by Eq. (3) they read $A_{a\beta\gamma} \to A_{a\beta\gamma} + \varepsilon_{\mu\alpha\beta\gamma} f^\mu(L_1)$ where $f^\mu(L_1)$ are four arbitrary functions of $L_1$ and $\varepsilon_{\mu\alpha\beta\gamma}$ is numerically the same as $\varepsilon^{\mu\alpha\beta\gamma}$. One can hope that this symmetry should prevent emergence of a measure fields dependence in $L_1$ and $L_2$ after quantum effects are taken into account.

3. Important feature of TMT that is responsible for many interesting and desirable results of the field theory models studied so far consists of the assumption that all fields, including also metric, connection and the measure fields ($\varphi_a$ or $A_{a\beta\gamma}$) are independent dynamical variables. All the relations between them are results of equations of motion. In particular, the independence of the metric and the connection means that we proceed in the first order formalism and the relation between connection and metric is not necessarily according to Riemannian geometry.

We want to stress again that except for the listed three assumptions we do not make any changes as compared with principles of the standard field theory in curved space-time. In other words, all the freedom in constructing different models in the framework of TMT consists of the choice of the concrete matter content and the Lagrangians $L_1$ and $L_2$ that is quite similar to the standard field theory.

Since $\Phi$ is a total derivative, a shift of $L_1$ by a constant, $L_1 \to L_1 + const$, has no effect on the equations of motion. Similar shift of $L_2$ would lead to the change of the constant part of the Lagrangian coupled to the volume element $\sqrt{-g}dx$. In the standard GR, this constant term is the cosmological constant. However in TMT the relation between the constant term of $L_2$ and the physical cosmological constant is very non trivial (see [24]-[31]).

In the case of the definition of $\Phi$ by means of Eq. (2), varying the measure fields $\varphi_a$, we get

$$B_a^\mu \partial_\mu L_1 = 0 \quad \text{where} \quad B_a^\mu = \varepsilon^{\mu\alpha\beta\gamma} \varepsilon_{abcd} \partial_\alpha \varphi_b \partial_\beta \varphi_c \partial_\gamma \varphi_d.$$

(4)

Since $\text{Det}(B_a^\mu) = \frac{4 \pi^4}{h} \Phi^3$ it follows that if $\Phi \neq 0$,

$$L_1 = sM^4 = const$$

(5)

where $s = \pm1$ and $M$ is a constant of integration with the dimension of mass. In what follows we make the choice $s = 1$.

In the case of the definition (3), variation of $A_{a\beta\gamma}$ yields

$$\varepsilon^{\mu\alpha\beta\gamma} \partial_\mu L_1 = 0,$$

(6)

that implies Eq. (6) without the condition $\Phi \neq 0$ needed in the model with four scalar fields $\varphi_a$.

One should notice the very important differences of TMT from scalar-tensor theories with nonminimal coupling:

a) In general, the Lagrangian density $L_1$ (coupled to the measure $\Phi$) may contain not only the scalar curvature term (or more general gravity term) but also all possible matter fields terms. This means that TMT modifies in general both the gravitational sector and the matter sector; b) If the field $\Phi$ were the fundamental (non composite) one then instead of (4), the variation of $\Phi$ would result in the equation $L_1 = 0$ and therefore the dimensionfull integration constant $M^4$ would not appear in the theory.

Applying the Palatini formalism in TMT one can show (see for example [24] and Appendix A of the present paper) that in addition to the Christoffel’s connection coefficients, the resulting relation between connection and metric includes also the gradient of the ratio of the two measures

$$\zeta \equiv \frac{\Phi}{\sqrt{-g}}$$

(7)

which is a scalar field. Consequently geometry of the space-time with the metric $g_{\mu\nu}$ is non-Riemannian. The gravity and matter field equations obtained by means of the first order formalism contain both $\zeta$ and its gradient as well. It turns out that at least at the classical level, the measure fields affect the theory only through the scalar field $\zeta$.

The consistency condition of equations of motion has the form of a constraint which determines $\zeta(x)$ as a function of matter fields. The surprising feature of the theory is that neither Newton constant nor curvature appear in this constraint which means that the geometrical scalar field $\zeta(x)$ is determined by the matter fields configuration locally and straightforward (that is without gravitational interaction).

By an appropriate change of the dynamical variables which includes a conformal transformation of the metric, one can formulate the theory in a Riemannian space-time. The corresponding conformal frame we call "the Einstein frame". The big advantage of TMT is that in the very wide class of models, the gravity and all matter fields equations of motion take canonical GR form in the Einstein frame. All the novelty of TMT in the Einstein frame as compared
with the standard GR is revealed only in an unusual structure of the scalar fields effective potential (produced in the Einstein frame), masses of fermions and their interactions with scalar fields as well as in the unusual structure of fermion contributions to the energy-momentum tensor: all these quantities appear to be $\zeta$ dependent [47–51]. This is why the scalar field $\zeta(x)$ determined by the constraint as a function of matter fields, has a key role in dynamics of TMT models.

### III. SCALE IN Variant MODEL

#### A. Symmetries and Action

The TMT models possessing a global scale invariance [25–27, 29–31] are of significant interest because they demonstrate the possibility of spontaneous breakdown of the scale symmetry [47]. In fact, if the action (1) is scale invariant then this classical field theory effect results from Eq.(5), namely from solving the equation of motion (4) or (6). One of the interesting applications of the scale invariant TMT models [29] is a possibility to generate the exponential potential for the scalar field $\phi$ by means of the mentioned spontaneous symmetry breaking even without introducing any potentials for $\phi$ in the Lagrangians $L_1$ and $L_2$ of the action (1). Some cosmological applications of this effect have been also studied in Ref. [29].

A dilaton field $\phi$ allows to realize a spontaneously broken global scale invariance [25] and together with this it governs the evolution of the universe on different stages: in the early universe $\phi$ plays the role of inflaton and in the late time universe it is transformed into a part of the dark energy.

According to the general prescriptions of TMT, we have to start from studying the self-consistent system of gravity (metric $g_{\mu\nu}$ and connection $\Gamma^\alpha_{\mu\nu}$), the measure $\Phi$ degrees of freedom (i.e. measure fields $\varphi_a$ or $A_{\alpha\beta\gamma}$) and the dilaton field $\phi$ proceeding in the first order formalism. We postulate that the theory is invariant under the global scale transformations:

$$g_{\mu\nu} \to e^\theta g_{\mu\nu}, \quad \Gamma^\mu_{\alpha\beta} \to \Gamma^\mu_{\alpha\beta} + \varphi_a \lambda_{ab}\varphi_b$$

where $\det(\lambda_{ab}) = e^{2\theta}, \quad \phi \to \phi - \frac{M_p}{\alpha} \theta$. \hfill (8)

If the definition (5) is used for the measure $\Phi$ then the transformations of $\varphi_a$ in (8) should be changed by $A_{\alpha\beta\gamma} \to e^{2\theta}A_{\alpha\beta\gamma}$. This global scale invariance includes the shift symmetry of the dilaton $\phi$ and this is the main factor why it is important for cosmological applications of the theory [25–27, 29–31].

We choose an action which, except for the modification of the general structure caused by the basic assumptions of TMT, does not contain any exotic terms and fields as like in the conventional formulation of the minimally coupled scalar-gravity system. Keeping the general structure (1), it is convenient to represent the action in the following form:

$$S = \int d^4x e^{\alpha\phi/M_p} \left[ -\frac{1}{\kappa} R(\Gamma, g)(\Phi + b_y \sqrt{-g}) + (\Phi + b_y \sqrt{-g}) \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - e^{\alpha\phi/M_p} (\Phi V_1 + \sqrt{-g}V_2) \right]$$ \hfill (9)

In the action (9) there are two types of the gravitational terms and of the "kinetic-like terms" which respect the scale invariance: the terms of the one type coupled to the measure $\Phi$ and those of the other type coupled to the measure $\sqrt{-g}$. Using the freedom in normalization of the measure fields ($\varphi_a$ in the case of using Eq (2) or $A_{\alpha\beta\gamma}$ when using Eq (5)), we set the coupling constant of the scalar curvature to the measure $\Phi$ to be $\frac{1}{\kappa}$. Normalizing all the fields such that their couplings to the measure $\Phi$ have no additional factors, we are not able in general to provide the same in terms describing the appropriate couplings to the measure $\sqrt{-g}$. This fact explains the need to introduce the dimensionless real parameters $b_y$ and $b_\phi$ and we will only assume that they have close orders of magnitudes. Note that in the case of the choice $b_y = b_\phi$ we would proceed with the fine tuned model. The real positive parameter $\alpha$ is assumed to be of the order of unity; in all solutions presented in this paper we set $\alpha = 0.2$. For Newton constant we use $\kappa = 16\pi G$, $M_p = (8\pi G)^{-1/2}$.

One should also point out the possibility of introducing two different pre-potentials which are exponential functions of the dilaton $\phi$ coupled to the measures $\Phi$ and $\sqrt{-g}$ with factors $V_1$ and $V_2$. Such $\phi$-dependence provides the scale symmetry (8). However $V_1$ and $V_2$ might be Higgs-dependent and then they play the role of the Higgs pre-potentials, but this will be done in the future publication.

#### B. Equations of motion in the Einstein frame.

In Appendix A we present the equations of motion resulting from the action (9) when using the original set of variables. The common feature of all the equations in the original frame is that the measure $\Phi$ degrees of freedom
appear only through dependence upon the scalar field $\zeta$, Eq. (7). In particular, all the equations of motion and the solution for the connection coefficients include terms proportional to $\partial_\mu \zeta$, that makes space-time non Riemannian and all equations of motion - noncanonical.

It turns out that when working with the new metric ($\phi$ remains the same)

$$\tilde{g}_{\mu\nu} = e^{\alpha\phi/M_p}(\zeta + b_\phi)g_{\mu\nu}, \quad (10)$$

which we call the Einstein frame, the connection becomes Riemannian. Since $\tilde{g}_{\mu\nu}$ is invariant under the scale transformations, spontaneous breaking of the scale symmetry (by means of Eq. (5) which for our model (4) takes the form (A1)) is reduced in the Einstein frame to the spontaneous breakdown of the shift symmetry

$$\phi \to \phi + \text{const.} \quad (11).$$

Notice that the Goldstone theorem generically is not applicable in this model[25]. The reason is the following. In fact, the scale symmetry (5) leads to a conserved dilatation current $j^\mu$. However, for example in the spatially flat FRW universe the spatial components of the current $j^j$ behave as $j^j \propto M^4 x^i$ as $|x^i| \to \infty$. Due to this anomalous behavior at infinity, there is a flux of the current leaking to infinity, which causes the non conservation of the dilatation charge. The absence of the latter implies that one of the conditions necessary for the Goldstone theorem is missing. The non conservation of the dilatation charge is similar to the well known effect of instantons in QCD where singular behavior at infinity, there is a flux of the current leaking to infinity, which causes the non conservation of the dilatation charge.

After the change of variables to the Einstein frame (10) and some simple algebra, Eq.(A4) takes the standard GR form

$$G_{\mu\nu}(\tilde{g}_{\alpha\beta}) = \frac{\kappa}{2} T^{eff}_{\mu\nu} \quad (12)$$

where $G_{\mu\nu}(\tilde{g}_{\alpha\beta})$ is the Einstein tensor in the Riemannian space-time with the metric $\tilde{g}_{\mu\nu}$; the energy-momentum tensor $T^{eff}_{\mu\nu}$ is now

$$T^{eff}_{\mu\nu} = \frac{\zeta + b_\phi}{\zeta + b_g}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} \delta_{\mu\nu} \tilde{g}_{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}) - \tilde{g}_{\mu\nu} \frac{b_g - b_\phi}{2(\zeta + b_g)} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \tilde{g}_{\mu\nu} V_{eff}(\phi; \zeta, M) \quad (13)$$

where the function $V_{eff}(\phi; \zeta, M)$ is defined as following:

$$V_{eff}(\phi; \zeta, M) = \frac{b_g [M^4e^{-2\phi/M_p} + V_1] - V_2}{(\zeta + b_g)^2} \quad (14).$$

Putting $M$ in the arguments of $V_{eff}$ we indicate explicitly that $V_{eff}$ incorporates our choice for the integration constant $M$ that appears as a result of the spontaneous breakdown of the scale symmetry. We will see in the next sections that $\zeta$-dependence of $V_{eff}(\phi; \zeta, M)$ in the form of square of $(\zeta + b_g)^{-1}$ has a key role in the resolution of the old CC problem in TMT. The reason that only such $\zeta$-dependence emerges in $V_{eff}(\phi; \zeta, M)$, and a $\zeta$-dependence is absent for example in the numerator of $V_{eff}(\phi; \zeta, M)$, is a direct result of our basic assumption that $L_1$ and $L_2$ are independent of the measure fields (see item 2 in Sec.II).

The dilaton $\phi$ field equation (A3) in the Einstein frame reads

$$\frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left[ \frac{\zeta + b_\phi}{\zeta + b_g} \sqrt{-\tilde{g}} \partial^\mu \phi \right] - \frac{\alpha}{M_p(\zeta + b_g)^2} \left[ (\zeta + b_g)M^4e^{-2\phi/M_p} - (\zeta - b_g)V_1 - 2V_2 - \delta b_g (\zeta + b_g)^2 \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right] = 0 \quad (15)$$

The scalar field $\zeta$ in Eqs.(13)-15 is determined by means of the constraint (A3) which in the Einstein frame (11) takes the form

$$(b_g - \zeta) \left[ M^4e^{-2\phi/M_p} + V_1 \right] - 2V_2 - \delta b_g (\zeta + b_g)X = 0 \quad (16)$$

where

$$X = \frac{1}{2} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \quad \text{and} \quad \delta = \frac{b_g - b_\phi}{b_g} \quad (17)$$
Applying the constraint (16) to Eq. (15) one can reduce the latter to the form
\[
\frac{1}{\sqrt{-g}} \partial_\mu \left[ \frac{\zeta + b_\phi}{\zeta + b_g} \sqrt{-g} g^{\mu \nu} \partial_\nu \phi \right] - \frac{2\alpha \zeta}{(\zeta + b_g)^2 M_p} M^4 e^{-2\alpha \phi/M_p} = 0,
\] (18)
where \(\zeta\) is a solution of the constraint (16).

The effective energy-momentum constraint (13) can be represented in a form of that of a perfect fluid
\[
T_{\mu \nu}^{\text{eff}} = (\rho + p)u_\mu u_\nu + p\tilde{g}_{\mu \nu}, \quad \text{where} \quad u_\mu = \frac{\phi_\mu}{(2X)^{1/2}}
\] (19)
with the following energy and pressure densities resulting from Eqs. (13) and (14) after inserting the solution \(\zeta = \zeta(\phi, X; M)\) of Eq. (16):
\[
\rho(\phi, X; M) = X + \frac{(M^4 e^{-2\alpha \phi/M_p} + V_1)^2 - 2\delta b_g (M^4 e^{-2\alpha \phi/M_p} + V_1)X - 3\delta^2 b_g^2 X^2}{4[b_g (M^4 e^{-2\alpha \phi/M_p} + V_1) - V_2]},
\] (20)
\[
p(\phi, X; M) = X - \frac{(M^4 e^{-2\alpha \phi/M_p} + V_1 + \delta b_g X)^2}{4[b_g (M^4 e^{-2\alpha \phi/M_p} + V_1) - V_2]}.
\] (21)

In a spatially flat FRW universe with the metric \(\tilde{g}_{\mu \nu} = \text{diag}(1, -a^2, -a^2, -a^2)\) filled with the homogeneous scalar sector field \(\phi\), the field equation of motion takes the form
\[
Q_1 \ddot{\phi} + 3HQ_2 \dot{\phi} - \frac{\alpha}{M_p} Q_3 M^4 e^{-2\alpha \phi/M_p} = 0
\] (22)
where \(H\) is the Hubble parameter and we have used the following notations
\[
\dot{\phi} \equiv \frac{d\phi}{dt}.
\] (23)
\[
Q_1 = 2[b_g (M^4 e^{-2\alpha \phi/M_p} + V_1) - V_2] \rho, \quad X = (b_g + b_\phi)(M^4 e^{-2\alpha \phi/M_p} + V_1) - 2V_2 - 3\delta^2 b_g^2 X
\] (24)
\[
Q_2 = 2[b_g (M^4 e^{-2\alpha \phi/M_p} + V_1) - V_2] p, \quad X = (b_g + b_\phi)(M^4 e^{-2\alpha \phi/M_p} + V_1) - 2V_2 - \delta^2 b_g^2 X
\] (25)
\[
- \frac{\alpha}{M_p} Q_3 M^4 e^{-2\alpha \phi/M_p} = 2[b_g (M^4 e^{-2\alpha \phi/M_p} + V_1) - V_2] \rho, \quad X = \frac{1}{[b_g (M^4 e^{-2\alpha \phi/M_p} + V_1) - V_2]}
\]
\[
\times \left[ (M^4 e^{-2\alpha \phi/M_p} + V_1)[b_g (M^4 e^{-2\alpha \phi/M_p} + V_1) - 2V_2] + 2\delta b_g V_2 X + 3\delta^2 b_g^2 X^2 \right].
\] (26)

It is interesting that the non-linear \(X\)-dependence appears here in the framework of the fundamental theory without exotic terms in the Lagrangians \(L_1\) and \(L_2\), see Eqs. (11) and (12). This effect follows just from the fact that there are no reasons to choose the parameters \(b_g\) and \(b_\phi\) in the action (10) to be equal in general; on the contrary, the choice \(b_g = b_\phi\) would be a fine tuning. Besides one should stress that the \(\phi\) dependence in \(\rho, p\) and in equations of motion emerges only in the form \(M^4 e^{-2\alpha \phi/M_p}\), where \(M\) is the integration constant (see Eq. (A1)), i.e. due to the spontaneous breakdown of the scale symmetry (8) (or the shift symmetry (11) in the Einstein frame). Thus the above equations represent an explicit example of k-essence [12] resulting from first principles. The system of equations (22), (24)-(26) accompanied with the functions (24)-(26) and written in the metric \(\tilde{g}_{\mu \nu} = \text{diag}(1, -a^2, -a^2, -a^2)\) can be obtained from the k-essence type effective action
\[
S_{\text{eff}} = \int \sqrt{-\tilde{g}} d^4 x \left[ -\frac{1}{\kappa} R(\tilde{g}) + p(\phi, X; M) \right],
\] (27)
where \(p(\phi, X; M)\) is given by Eq. (21). In contrast to the simplified models studied in literature [12], it is impossible here to represent \(p(\phi, X; M)\) in a factorizable form like \(K(\phi)p(X)\).
IV. COSMOLOGICAL DYNAMICS IN FINE TUNED $\delta = 0$ MODELS.

A. Equations of motion

The qualitative analysis of equations is significantly simplified if $\delta = 0$. This is what we will assume in this section. Although it looks like a fine tuning of the parameters (i.e. $b_g = b_\phi$), it allows us to understand qualitatively the basic features of the model. In fact, only in the case $\delta \neq 0$ the effective action \[ V_{eff}(0) \] takes the form of that of the scalar field without higher powers of derivatives. Role of $\delta \neq 0$ in a possibility to produce an effect of a super-accelerated expansion of the late time universe will be studied in Sec.V.

![Three possible shapes of the effective potential $V_{eff}(0)$ in the models](image)

FIG. 1: Three possible shapes of the effective potential $V_{eff}(0)$ in the models with $b_g V_1 > V_2$: Fig.(a) $b_g V_1 \geq 2V_2$, $V_1 > 0$; Fig.(b) $b_g V_1 < 2V_2$, $V_1 > 0$; Fig.(c) $V_1 < 0$, $V_2 < 0$.

So let us study spatially flat FRW cosmological models governed by the system of equations

\[ \frac{\dot{a}^2}{a^2} = \frac{1}{3M_p^2} \rho \]  

and \[(20)-(22)\] where one should set $\delta = 0$.

In the fine tuned case under consideration, the constraint \[16\] yields

\[ \zeta = \zeta(\phi, X; M)|_{\delta=0} \equiv b_g - \frac{2V_2}{V_1 + M_4 e^{-2\alpha \phi/M_p}}, \]  

(29)
The energy density and pressure take then the canonical form,
\[
\rho|_{\delta=0} = \frac{1}{2}\dot{\phi}^2 + V_{\text{eff}}^{(0)}(\phi); \quad p|_{\delta=0} = \frac{1}{2}\dot{\phi}^2 - V_{\text{eff}}^{(0)}(\phi),
\]
where the effective potential of the scalar field \(\phi\) results from Eq. (13)
\[
V_{\text{eff}}^{(0)}(\phi) = V_{\text{eff}}(\phi; \zeta, M)|_{\delta=0} = \frac{[V_1 + M^4 e^{-2\alpha\phi/M_p}]^2}{4[b_g (V_1 + M^4 e^{-2\alpha\phi/M_p}) - V_2]}
\]
and the \(\phi\)-equation is reduced to
\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dV_{\text{eff}}^{(0)}}{d\phi} = 0.
\]
Notice that \(V_{\text{eff}}^{(0)}(\phi)\) is non-negative for any \(\phi\) provided
\[
b_g V_1 \geq V_2,
\]
that we will assume in this paper. We assume also that \(b_g > 0\).

In the following three subsections we consider three different dilaton-gravity cosmological models determined by different choice of the parameters \(V_1\) and \(V_2\): one model with \(V_1 < 0\) and two models with \(V_1 > 0\). The appropriate three possible shapes of the effective potential \(V_{\text{eff}}^{(0)}(\phi)\) are presented in Fig.1. A special case with the fine tuned condition \(b_g V_1 = V_2\) is discussed in Appendix B where we show that equality of the couplings to measures \(\Phi\) and \(\sqrt{-g}\) in the action (equality \(b_g V_1 = V_2\) is one of the conditions for this to happen) gives rise to a symmetric form of the effective potential.

### B. Model with \(V_1 < 0\) and \(V_2 < 0\)

1. Resolution of the Old Cosmological Constant Problem in TMT

The most remarkable feature of the effective potential is that it is proportional to the square of \(V_1 + M^4 e^{-2\alpha\phi/M_p}\) which is a straightforward consequence of our basic assumption that \(L_1\) and \(L_2\) are independent of the measure fields (see item 2 in Sec.II, Eq. (14) and the discussion after it). Due to this, as \(V_1 < 0\) and \(V_2 < 0\), the effective potential has a minimum where it equals zero automatically, without any further tuning of the parameters \(V_1\) and \(V_2\) (see also Fig.1c). This occurs in the process of evolution of the field \(\phi\) at the value of \(\phi = \phi_0\) where
\[
V_1 + M^4 e^{-2\alpha\phi_0/M_p} = 0.
\]
This means that the universe evolves into the state with zero cosmological constant without any additional tuning of the parameters and initial conditions.

To provide the global scale invariance, the prepotentials \(V_1\) and \(V_2\) enter in the action with factor \(e^{2\alpha\phi/M_p}\). However, if quantum effects (considered in the original frame) break the scale invariance of the action via modification of existing prepotentials or by means of generation of other prepotentials with arbitrary \(\phi\) dependence (and in particular a "normal" cosmological constant term \(\int \Lambda\sqrt{-g}d^4x\)), this cannot change the result of TMT that the effective potential generated in the Einstein frame is proportional to a perfect square. Note that the assumption of scale invariance is not necessary for the effect of appearance of the perfect square in the effective potential in the Einstein frame and therefore for the described mechanism of disappearance of the cosmological constant, see Refs. [22-25].

If such type of the structure for the scalar field potential in a conventional (non TMT) model would be chosen "by hand" it would be a sort of fine tuning. But in our TMT model it is not the starting point, it is rather a result obtained in the Einstein frame of TMT models with spontaneously broken global scale symmetry including the shift symmetry \(\phi \rightarrow \phi + \text{const}\).

On the first glance this effect contradicts the Weinberg’s no-go theorem which states that there cannot exist a field theory model where the cosmological constant is zero without fine tuning. In Sec.VIB we will study in detail the manner our TMT model avoids this theorem.
FIG. 2: Phase portrait (plot of $\frac{d\phi}{dt}$ versus $\phi$) for the model with $V_1 < 0$ and $V_2 < 0$. All phase curves started with $|\phi| \gg M_p$ quickly approach the attractor long before entering the oscillatory regime. The region of the oscillatory regime is marked by point $B$. The oscillation spiral is not visible here because of the chosen scales along the axes.

FIG. 3: (a) Typical dependence of the field $\phi$ (fig. (a)) and the energy density $\rho$ (fig. (b)) upon $\ln(a/a_0)$. Here and in all the graphs of this paper describing scale factor $a$ dependences, $a(t)$ is normalized such that at the end point of the described process $a(t_{end}) = a_0$. In the model with $V_1 < 0$ the power law inflation ends with damped oscillations of $\phi$ around $\phi_0$ determined by Eq.(34). For the choice $V_1 = -30 M^4$ Eq.(34) gives $\phi_0 = -8.5 M_p$. (b) The exit from the early inflation is accompanied with approaching zero of the energy density $\rho$. The graphs correspond to the evolution which starts from the initial values $\phi_{in} = -85 M_p$, $\dot{\phi}_{in} = -8 \cdot 10^3 M^2/\sqrt{b_g}$.

2. Cosmological Dynamics

As $M^4 e^{-2\alpha\phi/M_p} \gg Max(|V_1|, |V_2|/b_g)$, the effective potential behaves as the exponential potential $V_{eff}(0) \approx \frac{1}{a_{in}} M^4 e^{-2\alpha\phi/M_p}$. So, as $\phi \ll -M_p$ the model describes the well studied power law inflation of the early universe.
if $\alpha < 1/\sqrt{2}$:

$$a(t) = a_{in} t^{1/2\alpha^2}, \quad \phi = \phi_{in} + \frac{M_p}{\alpha} \ln \frac{t}{t_{in}}, \quad (35)$$

Further behavior of the solutions in the model of Sec.IVA with $V_1 < 0$ and $V_2 < 0$, i.e. for the potential of Fig.1c, is qualitatively evident enough. With the choice $\alpha = 0.2$, $V_1 = -30 M^4$ and $V_2 = -50 b_g M^4$, the results of numerical solutions are presented in Figs. 2,3,4. The phase curves in Fig.2 demonstrate the well known attractor behavior of the solutions describing the power law inflation.

Exit from the inflation starts as $\phi$ becomes close to $\phi_0$ determined by Eq.13. Then the energy density starts to tend to zero very fast, Fig.3b. The exit from the inflation occurs when all the phase trajectories corresponding to different initial conditions with $\phi_{in} \ll -M_p$ practically coincide. The process ends with oscillatory regime, Fig.4a, where $\phi$ performs damped oscillations around the minimum of the effective potential (see also Fig.1c).

C. Model with $V_1 > 0$ and $b_g V_1 \geq 2V_2$: Early Power Law Inflation Ending With Small $\Lambda$ Driven Expansion

In this model the effective potential is a monotonically decreasing function of $\phi$ (see Fig.1b). As $\phi \ll -M_p$ the model describes the power law inflation, similar to what we discussed in the model of previous subsection.

Applying this model to the cosmology of the late time universe and assuming that the scalar field $\phi \rightarrow \infty$ as $t \rightarrow \infty$, it is convenient to represent the effective potential in the form

$$V^{(0)}_{eff}(\phi) = \Lambda_1 + V_{q-1}(\phi) \quad \text{where} \quad \Lambda_1 = \Lambda|_{b_g V_1 > 2V_2}, \quad (36)$$

with the definition

$$\Lambda = \frac{V_2}{4(b_g V_1 - V_2)}, \quad (37)$$

Here $\Lambda$ is the positive cosmological constant and

$$V^{(0)}_{q-1}(\phi) = \frac{(b_g V_1 - 2V_2) V_2 M^4 e^{-2\alpha \phi/M_p} + (b_g V_1 - V_2) M^8 e^{-4\alpha \phi/M_p}}{4(b_g V_1 - V_2)[b_g (V_1 + M^4 e^{-2\alpha \phi/M_p}) - V_2]}, \quad (38)$$
The early universe evolution is governed by the almost exponential potential (see Fig. 1a) providing the power law inflation ($w \approx -0.95$ interval in fig.(c)). After transition to the late time universe the scalar $\phi$ increases with the rate typical for a quintessence scenario. Later on the cosmological constant $\Lambda_1$ becomes a dominated component of the dark energy that is displayed by the infinite region where $w \approx -1$ in fig.(c).

Thus the effective potential $V_q^{(0)}(\phi)$ provides a possibility for a cosmological scenario which starts with a power law inflation and ends with a cosmological constant $\Lambda_1$. The smallness of $\Lambda_1$ may be achieved without fine tuning of dimensionful parameters, that will be discussed in Sec.VI. Such scenario may be treated as a generalized quintessential inflation type of scenario. Recall that the $\phi$-dependence of the effective potential $V_q^{(0)}(\phi)$ appears here only as the result of the spontaneous breakdown of the global scale symmetry $[54]$.

Results of numerical solutions for such type of scenario are presented in Figs.5 and 6 ($V_1 = 10M^4$, $V_2 = 4b_gM^4$)

The early universe evolution is governed by the almost exponential potential (see Fig.1a) providing the power low
inflation ($w \approx -0.95$ interval in fig. (c)) with the attractor behavior of the solutions, see Ref. \[37\]. After transition to the late time universe the scalar $\phi$ increases with the rate typical for a quintessence scenario. Later on the cosmological constant $\Lambda_1$ becomes a dominated component of the dark energy that is displayed by the infinite region where $w \approx -1$ in fig. (c). The phase portrait in Fig.3 shows that all the trajectories started with $|\phi| \gg M_p$ quickly approach the attractor which asymptotically (as $\phi \to \infty$) takes the form of the straight line $\dot{\phi} = 0$. Qualitatively similar results are obtained also when $V_1$ is positive but $V_2$ is negative.

D. Model with $V_1 > 0$ and $V_2 < b_g V_1 < 2V_2$

In this case the effective potential has the minimum (see Fig.1b)

$$V_{\text{eff}}(\phi_{\text{min}}) = \frac{V_2}{b_g^2} \quad \text{at} \quad \phi = \phi_{\text{min}} = -\frac{M_p}{2\alpha} \ln \left(\frac{2V_2 - b_g V_1}{b_g}\right). \quad (39)$$

For the choice of the parameters as in Fig.1b, i.e. $V_1 = 10M^4$ and $V_2 = 9.9b_g M^4$, the minimum is located at $\phi_{\text{min}} = -5.7M_p$. The character of the phase portrait one can see in Fig.7.

![Phase portrait](image)

**FIG. 7:** Phase portrait (plot of $\frac{d\phi}{dt}$ versus $\phi$) for the model with $b_g V_1 < 2V_2$ and $V_1 > 0$ (the parameters are chosen here as in Fig.1b). Trajectories started anywhere in the phase plane in a finite time end up at the same point $A(-5.7,0)$ which is a node sink. There exist two attractors ending up at $A$, one from the left and other from the right. All phase curves starting with $|\phi| \gg M_p$ quickly approach these attractors.

For the early universe as $\phi \ll -M_p$, similar to what we have seen in the models of the previous two subsections, the model implies the power low inflation. However, the phase portrait Fig.7 shows that now all solutions end up without oscillations at the minimum $\phi_{\text{min}} = -5.7M_p$ with $\frac{d\phi}{dt} = 0$. In this final state of the scalar field $\phi$, the evolution of the universe is governed by the cosmological constant $V_{\text{eff}}(\phi_{\text{min}})$ determined by Eq. (39). For some details of the cosmological dynamics see Fig.8. The desirable smallness of $V_{\text{eff}}(\phi_{\text{min}})$ can be provided again without fine tuning of the dimensionful parameters that will be discussed in Sec.VI. The absence of appreciable oscillations in the minimum is explained by the following two reasons: a) the non-zero friction at the minimum determined by the cosmological constant $V_{\text{eff}}(\phi_{\text{min}})$; b) the shape of the potential near to minimum is too flat.

The described properties of the model are evident enough. Nevertheless we have presented them here because this model is a particular (fine tuned) case of an appropriate model with $\delta \neq 0$ studied in the next subsection where we will demonstrate a possibility of states with $w < -1$ without any exotic contributions, like a phantom term, in the original action.
FIG. 8: Cosmological dynamics in the model with $b_1 V_1 < 2V_2$ and $V_1 > 0$: typical dependence of $\phi$ (Fig.(a)), the energy density $\rho$ (Fig.(b)) and the equation-of-state $w$ (Fig.(c)) upon $\ln(a/a_0)$ where the scale factor $a(t)$ normalized as in Fig.2. The graphs correspond to the initial conditions $\phi_{in} = -35M_p$, $\dot{\phi}_{in} = -10b^{-1/2}M^2$. The early universe evolution is governed by an almost exponential potential (see Fig.1b) providing the power low inflation ($w \approx -0.95$ interval in Fig.(c)). After arriving the minimum of the potential at $\phi_{min} = -5.7M_p$ (see Fig.1b and the point $A(-5.7M_p,0)$ of the phase plane in Fig.4) the scalar $\phi$ remains constant. At this stage the dynamics of the universe is governed by the constant energy density $\rho = V_{eff}(\phi_{min})$ (see the appropriate intervals $\rho = \text{const}$ in Fig.(b) and $w = -1$ in Fig.(c)).

V. DYNAMICS IN THE CASE $\delta \neq 0$: A SUPERACCELERATION PHASE OF THE UNIVERSE.

We return now to the general case of our model (see Sec.III) with no fine tuning of the parameters $b_1$ and $b_2$, i.e. the parameter $\delta$, defined by Eq.(17), is non zero. We have checked that the models of Secs.IVB and IVC do not suffer qualitative changes at all and quantitative modifications are not strong, at least in the range of $-1 < \delta < 1$. However the model of Sec.IVD changes drastically as $\delta \neq 0$: the model enables k-essence type solutions for the late time universe with equation-of-state $w < -1$.

So, we consider now the dynamics of the FRW cosmology described by Eqs. (28)-(26). Before choosing the appropriate parameters for numerical studies, let us start from the analysis of Eq.(22). The interesting feature of this equation is that for certain range of the parameters, each of the factors $Q_i(\phi,X)$ $(i = 1, 2, 3)$ can get to zero. Equation $Q_i(\phi,X) = 0$ determines a line in the phase plane $(\phi, \dot{\phi})$. In terms of a mechanical interpretation of Eq.(22), the change of the sign of $Q_1$ can be treated as the change of the mass of "the particle". Therefore one can think of situation where "the particle" climbs up in the potential with acceleration. It turns out that when the scalar field is behaving in this way, the flat FRW universe may undergo a super-acceleration.

For $Q_1 \neq 0$, Eqs. (22), (25) result in the well known equation [35]

$$\ddot{\phi} + \sqrt{3}\frac{\rho}{M_p} c_s^2 \dot{\phi} + \frac{\rho,\phi}{\rho,X} = 0,$$

(40)

where $c_s$ is the effective sound speed of perturbations [12]

$$c_s^2 = \frac{p,X}{\rho,X} = \frac{Q_2}{Q_1},$$

(41)

$$\frac{\rho,\phi}{\rho,X} = -\frac{\alpha}{M_p} \frac{Q_3}{Q_1} M^4 e^{-2\alpha\phi/M_p}.$$  \hspace{1cm} (42)

$\rho$ and $p$ are defined by Eqs. (20), (21) and $Q_i$ $(i = 1, 2, 3)$ - by Eqs. (24)-(26).

With simple algebra one can see that the following "sign rule" is fulfilled for the equation-of-state $w = p/\rho$:

$$\text{sign}(w + 1) = \text{sign}(Q_2)$$

(43)

and the line $Q_2(\phi,X) = 0$ divides the phase plane $\phi, \dot{\phi}$ into two regions: one with $w > -1$ and other with $w < -1$. 

FIG. 9: The phase portrait for the model with $\alpha = 0.2$, $\delta = 0.1$, $V_1 = 10M^4$ and $V_2 = 9.9b_gM^4$. The region with $\rho > 0$ is divided into two dynamically disconnected regions by the line $Q_1(\phi, \dot{\phi}) = 0$. To the left of this line $Q_1 > 0$ (the appropriate zone we call zone 1) and to the right $Q_1 < 0$. The phase portrait in zone 1 corresponds to processes similar to those of Sec.IVD. The $\rho > 0$ region to the right of the line $Q_1(\phi, \dot{\phi}) = 0$ is divided into two zones (zone 2 and zone 3) by the line $Q_2 = 0$ (the latter coincides with the line where $w = -1$). In zone 2 $w > -1$ but $c_s^2 < 0$. In zone 3 $w < -1$ and $c_s^2 > 0$. Phase curves can start in zone 2 in points very close to the line $Q_1 = 0$. After they cross the line $w = -1$, i.e. in zone 3, they exhibit processes with super-accelerating expansion of the universe. The phase curves in zone 3 demonstrate dynamical attractor behavior. As $\phi \to \infty$ the phase curves approach the straight line $\dot{\phi} = 0$.

There are a lot of sets of parameters providing the $w < -1$ phase in the late universe. For example we are demonstrating here this effect with the following set of the parameters of the original action (8): $\alpha = 0.2$, $V_1 = 10M^4$ and $V_2 = 9.9b_gM^4$ used in Sec.IVD but now we choose $\delta = 0.1$. The results of the numerical solution are presented in Figs.9-11.

The phase plane, Fig.9, is divided into two regions by the line $\rho = 0$. The region $\rho > 0$ is divided into two dynamically disconnected regions by the line $Q_1(\phi, X) = 0$.

To the left of the line $Q_1(\phi, X) = 0$ - zone 1 where $Q_1 > 0$. Comparing carefully the phase portrait in the zone 1 with that in Fig.4 of the previous subsection, one can see an effect of $\delta \neq 0$ on the shape of phase trajectories. However the general structure of these two phase portraits is very similar. In particular, they have the same node sink $A(-5.7M_p, 0)$. At this point "the force" equals zero since $Q_3|A = 0$. The value $\phi = -5.7M_p$ coincides with the position of the minimum of $V_{\text{eff}}^{(0)}(\phi)$ because in the limit $\dot{\phi} \to 0$ the role of the terms proportional to $\delta$ is negligible. Among trajectories converging to node $A$ there are also trajectories corresponding to a power low inflation of the early universe, which is just a generalization to the case $\delta \neq 0$ of the similar result discussed in the previous subsection.

In the region to the right of the line $Q_1(\phi, X) = 0$, all phase curves approach the attractor which in its turn asymptotically (as $\phi \to \infty$) takes the form of the straight line $\phi = 0$. This region is divided into two zones by the line $Q_2(\phi, X) = 0$. In all points of this line $w = -1$. In zone 2, i.e. between the lines $Q_1 = 0$ and $w = -1$, the equation-of-state $w > -1$ and the sound speed $c_s^2 < 0$. In zone 3, i.e. between the line $w = -1$ and the line $\rho = 0$,
FIG. 10: Typical scalar factor dependence of $\phi$ (Fig.(a)) and of the energy density $\rho$, defined by Eq.(20), (Fig.(b)) in the regime corresponding to the phase curves started in zone 2. Both graphs correspond to the initial conditions $\phi_{in} = M_p$, $\dot{\phi}_{in} = 5.7M^4/\sqrt{b_g}$. $\rho$ increases approaching asymptotically $\Lambda_2 = \frac{M^4}{b_g}e^{5.52}$, where $\Lambda_2$ is the value of $\Lambda$ determined by Eq.(37) as $V_1 = 10M^4$ and $V_2 = 9.9b_gM^4$; see also Fig.1b.

FIG. 11: The scale factor dependence of the equation-of-state $w$ (Fig.(a)) and effective speed of sound for perturbations $c_s^2$ (Fig.(b)) for the initial conditions $\phi_{in} = M_p$, $\dot{\phi}_{in} = 5.7M^4/\sqrt{b_g}$.

the equation-of-state $w < -1$ and the sound speed $c_s^2 > 0$.

For a particular choice of the initial data $\phi_{in} = M_p$, $\dot{\phi}_{in} = 9M^4/\sqrt{b_g}$, the features of the solution of the equations of motion are presented in Figs.10 and 11. The main features of the solution as we observe from the figures are the following: 1) $\phi$ slowly increases in time; 2) the energy density $\rho$ slowly increases approaching the constant $\Lambda = \Lambda_2$ defined by the same formula as in Eq.(37), see also Fig.1b; for the chosen parameters $\Lambda_2 \approx \frac{M^4}{b_g}e^{5.52}$; 3) $w \equiv p/\rho$ is less than $-1$ and asymptotically approaches $-1$ from below.

Using the classification of Ref.38 of conditions for the dark energy to evolve from the state with $w > -1$ to the phantom state, we see that transition of the phase curves from zone 2 (where $w > -1$) to the phantom zone 3 (where $w < -1$) occurs under the conditions $p,X = 0$, $\rho,X \neq 0$, $X \neq 0$. Qualitatively the same behavior one observes for all initial conditions $(\phi_{in}, \dot{\phi}_{in})$ disposed in the zone 2. The question of constructing a realistic scenario where the dark energy can evolve from the power low inflation state disposed in zone 1 to the phantom zone 3 is beyond the goal of this paper.
VI. RESOLUTIONS OF THE COSMOLOGICAL CONSTANT PROBLEMS AND CONNECTION BETWEEN TMT AND CONVENTIONAL FIELD THEORIES WITH INTEGRATION MEASURE $\sqrt{-g}$

A. The new cosmological constant problem

The smallness of the observable cosmological constant $\Lambda$ is known as the new cosmological constant problem. In TMT, there are two ways to provide the observable order of magnitude of $\Lambda \sim (10^{-3}eV)^4$ by an appropriate choice of the parameters of the theory (see Eqs. (37) and (33)) but without fine tuning of the dimensionfull parameters.

1. Seesaw mechanism

If $V_2 < 0$ then there is no need for $V_1$ and $V_2$ to be small: it is enough that $b_2V_1 < |V_2|$ and $V_1/|V_2| \ll 1$. This possibility is a kind of "seesaw mechanism" ([2] and [39]). For instance, if $V_1 \sim (10^5 GeV)^4$ and $V_2$ is determined by the Planck scale $V_2 \sim (10^{18} GeV)^4$ then $\Lambda_1 \sim (10^{-3}eV)^4$. The range of the possible scale of the dimensionless parameter $b_g$ remains very broad.

2. The TMT correspondence principle and the smallness of $\Lambda$

Let us start from the notion that if $V_2 > 0$ or alternatively $V_2 < 0$ and $b_2V_1 > |V_2|$ then $\Lambda \sim \frac{V_1^2}{V_2}$. Hence the second possibility to ensure the needed smallness of $\Lambda$ is to choose the dimensionless parameter $b_g > 0$ to be a huge number. In this case the order of magnitudes of $V_1$ and $V_2$ could be either as in the above case of the seesaw mechanism or to be not too much different from each other (or even of the same order). For example, if $V_1 \sim (10^5 GeV)^4$ then for getting $\Lambda_1 \sim (10^{-3}eV)^4$ one should assume that $b_g \sim 10^{60}$. Below we will use this value just for illustrative purposes.

Note that $b_g$ is the ratio of the coupling constants of the scalar curvature to the measures $\sqrt{-g}$ and $\Phi$ respectively in the fundamental action. The Lagrangians $L_1$ and $L_2$ have the same structure: both of them contain the scalar curvature, kinetic and pre-potential terms. It is natural to assume that the ratio of couplings of all the appropriate terms in $L_1$ and $L_2$ to the measures $\sqrt{-g}$ and $\Phi$ have the same or close orders of magnitude. This is why in Sec.III we have made an assumption that the dimensionless parameters $b_g$ and $b_2$ have close orders of magnitude. For the same reason we will also assume that $V_2/V_1 \sim b_2 \sim 10^{60}$. If this is the case then the huge value of $b_g$ can be treated as an indication that TMT implies a certain sort of "correspondence principle" between TMT and conventional field theories (i.e. theories with only the measure of integration $\sqrt{-g}$ in the action). In fact, using the notations of the general form of the TMT action in the case of the action, one can conclude that the relation between the "usual" (i.e. entering in the action with the usual measure $\sqrt{-g}$) Lagrangian density $L_2$ and the new one $L_1$ (entering in the action with the new measure $\Phi$) is roughly speaking $L_2 \sim 10^{60}L_1$. In the case $\int L_1 \Phi dx$ becomes negligible, the remaining term of the action $\int L_2 \sqrt{-g} dx$ would describe GR instead of TMT. It seems to be very interesting that such a correspondence principle for the TMT action may have a certain relation to the extreme smallness of the cosmological constant.

Appearance of a big dimensionless constant in particle field theory is usually associated with hierarchy of masses and/or interactions describing by different terms in the Lagrangian. The way big numbers can appear in the TMT action is absolutely different. It is easier to see this difference in the case of a fine tuned model, where $b_g = b_\phi$ and $b_2V_1 = V_2$, see Appendix B. In such a case the Lagrangians $L_1$ and $L_2$ not only have the same type of terms but they are just proportional: $L_2 = b_gL_1$. Therefore the nature of the huge value of $b_g$ differs here very much from the conventional hierarchy issue.

If such the ratio between $L_1$ and $L_2$ is actually realized, then taking into account the fact that $L_1$ and $L_2$ describe the same matter and gravity degrees of freedom in a very similar manner, the question arises why $L_1$ is not dynamically negligible in comparison with $L_2$. To answer this question we have to turn to the fundamental action that it is convenient to rewrite in the following form

$$S = \int \sqrt{-g} d^4x e^{\alpha\Phi/M_p} \left[ -\frac{b_\phi}{\kappa} R(\Gamma, g) \left( \frac{\zeta}{b_\phi} + 1 \right) + \frac{\zeta}{b_\phi} + \frac{b_\phi}{b_g} \right] \frac{b_2}{2} g^{\mu\nu} \frac{\phi^\mu \phi^\nu}{\kappa} - e^{\alpha\Phi/M_p} \left( \frac{\zeta V_1}{V_2} + 1 \right) V_2 \right]$$

where one can see that the ratio $\zeta/b_\phi$ has an important dynamical role. Analyzing the constraint and cosmological dynamics studied in Secs.IV and V it is easy to see that the order of magnitude of the scalar field $\zeta \equiv \Phi/\sqrt{-g}$ is generically close to that of $b_g$. In other words, when analyzing the action on the mass shell, it turns out that the ratio of the measures $\Phi/\sqrt{-g}$ generically compensates the smallness of $L_1/L_2$. So, similar terms (R-terms, kinetic terms
and pre-potential terms) appearing in the action \( \mathcal{L} \) with measures \( \Phi \) and \( \sqrt{-g} \) respectively, are both dynamically important in general.

In the light of this understanding of the general picture it is interesting to check the TMT dynamics in situations where \( \frac{\zeta}{b_g} \) becomes very small or very large. Let us start from the fine tuned model where \( b_g V_1 = 2V_2 \) and \( b_g = b_\phi \) (recall that \( V_1 > 0 \) and \( b_g > 0 \) in all our models). In this case it follows from the constraint (29) that

\[
\frac{\zeta}{b_g} = \frac{M^4 e^{-2\alpha \phi/M_p}}{V_1 + M^4 e^{-2\alpha \phi/M_p}},
\]

Then the effective potential (41) (see also Eqs.(36)-(38)) reads

\[
V_{eff}^{(0)}(\phi) = \frac{V_2}{b_g^2} + \frac{M^8 e^{-4\alpha \phi/M_p}}{2b_g(V_1 + 2M^4 e^{-2\alpha \phi/M_p})}.
\]

In such a fine tuned model, \( \zeta/b_g \) approaches zero asymptotically as \( \phi \to \infty \) where the effective potential becomes flat. However when looking into the TMT action written in the form (44) we see that the asymptotic disappearance of \( \zeta/b_g \) means that we deal with an asymptotic transition from TMT to a conventional field theory model with only measure of integration \( \sqrt{-g} \) and only one Lagrangian density. In the limit \( \zeta/b_g \to 0 \), the conformal transformation to the Einstein frame (10) takes the form \( \tilde{g}_{\mu\nu} = b_g e^{\alpha \phi/M_p} g_{\mu\nu} \). Therefore in the Einstein frame, the limit of the action (44) as \( \zeta/b_g \to 0 \) is reduced to the following model:

\[
S|_{\zeta=0, \text{Einstein frame}} = \int \sqrt{-\tilde{g}} d^4x \left[ -\frac{1}{\kappa} R(\tilde{g}) + \frac{1}{2} \tilde{g}^{\mu\nu} \phi,_{\mu} \phi,_{\nu} - \frac{1}{b_g^2} V_2 \right].
\]

It is easy to see that for example in the FRW universe, the asymptotic (as the scale factor \( a(t) \to \infty \)) behavior of the universe in the model (47) coincides with the appropriate asymptotic result of TMT model under consideration: both of them asymptotically describe the universe governed by the cosmological constant \( V_2/b_g^2 = V_1/2b_g \).

Similar conclusion is obtained in a model where \( V_2 < b_g V_1 < 2V_2 \), i.e. with no fine tuning of the prepotentials, which have been studied in Sec.IVD. The only difference is that now \( \frac{\zeta}{b_g} \to 0 \) and \( V_{cf} \to V_2/b_g^2 \sim V_1/b_g \) as \( \phi \to \phi_{min} \), see Eq.(43); in a small neighborhood of \( \phi_{min} \), the TMT action presented in the Einstein frame looks like (47).

Note however that in the context of the k-essence model studied in Sec.V where \( \delta \neq 0 \) and \( V_2 < b_g V_1 < 2V_2 \), the asymptotic value of \( \zeta \) in the late time universe (i.e. as \( \phi \to \infty \) and \( X \to 0 \)) is \( |\zeta| \sim b_g \) but nevertheless the energy density tends to \( \Lambda = \Lambda_2 = \frac{V_2^2}{4(4b_g V_1 - V_2)} \sim V_1/b_g \) as well (see Fig.1b).

Similar situation takes place in the model where \( b_g V_1 > 2V_2 \), studied in Sec.IVC. The asymptotic value of \( \zeta \) in the late time universe is again \( \zeta \sim b_g \) and the energy density tends to \( \Lambda = \Lambda_1 = \frac{V_2^2}{4(4b_g V_1 - V_2)} \sim V_1/b_g \) as well (see Fig.1a).

Thus in all the models, the huge value of \( b_g \) can ensure the needed smallness of the dark energy density in the late time universe but it is not always realized due to the limit \( \zeta/b_g \to 0 \).

\section*{B. The Old Cosmological Constant Problem Is Solved in the Dynamical Regime where the Fundamental TMT Action Tends to a Limit Opposite to Conventional Field Theory (with only measure \( \sqrt{-g} \)).}

As we have seen in Sec.IVB, in the model with \( V_1 < 0 \) and \( V_2 < 0 \), the old cosmological constant problem is resolved without fine tuning: the effective potential (31) is proportional to the square of \( V_1 + M^4 e^{-2\alpha \phi/M_p} \), and \( \phi = \phi_0 \) where \( V_1 + M^4 e^{-2\alpha \phi_0/M_p} = 0 \), is the minimum of the effective potential without any further tuning of the parameters and initial conditions. Now we want to analyze some of the essential differences we have in TMT as compared with the conditions of the Weinberg’s no-go theorem and show what are the reasons providing solution of the old CC problem in TMT. This has to be done when TMT is considered in the original frame since in the Einstein frame we observe only the results in the effective picture after some of the symmetries are broken.

\begin{itemize}
  \item The basic assumption of the Weinberg’s theorem is that in the vacuum all the fields (metric tensor \( g_{\mu\nu} \) and matter fields \( \psi_n \)) are constant. As it was pointed out by S.Weinberg in the review [1], the Euler-Lagrange equations for such constant fields (with the action \( \int L(g_{\mu\nu}, \psi_n) \, d^4x \)) have the form
    \[
    \frac{\partial L}{\partial g_{\mu\nu}} = 0,
    \]
    \[
    \frac{\partial L}{\partial \psi_n} = 0
    \]
\end{itemize}
and these equations constitute the basis for further Weinberg’s arguments. In particular, if $GL(4)$ symmetry

$$g_{\mu\nu} \rightarrow A_{\mu}^\alpha A_{\nu}^\beta g_{\alpha\beta}, \quad \psi_i \rightarrow D_i(A)\psi_j$$

(50)
survives as a vestige of general covariance when all the fields are constrained to be constant, the Lagrangian $\mathcal{L}$ transforms as a density:

$$\mathcal{L} \rightarrow \det A \cdot \mathcal{L}.$$  

(51)

Weinberg concludes that when Eq.(48) is satisfied then the unique form of $\mathcal{L}$ is

$$\mathcal{L} = c\sqrt{-g},$$

(52)

where $c$ is independent of $g_{\mu\nu}$. As a matter of fact this means that for example in the case of a scalar matter field $\phi$ model considered by Weinberg in Sec.VI of the review [1], $c$ is determined by the value of the scalar field $\phi$ potential as $\phi$ is a constant determined by Eq.(49).

However, if $\mathcal{L}$ contains a term linear in space-time derivatives then Eq.(49) may not be valid even for constant field. This is what happens in TMT where the first term in the action (1) is linear in space-time derivatives of $A_{\alpha\beta\gamma}$ (when using the definition (3))(see also [53]). Then instead of Eq.(49) which appears to be an identity in this case, the Euler-Lagrange equations for $A_{\alpha\beta\gamma}$ look

$$\partial_{\mu} \frac{\partial(\Phi L_1)}{\partial A_{\alpha\beta\gamma,\mu}} = 0,$$

(53)

which are nontrivial even for constant $A_{\alpha\beta\gamma}$ and resulting in Eq.(6). Note that $\Phi L_1$ is a scalar density and transforms exactly according to Eq.(51). Therefore generically (i.e. if $A_{\alpha\beta\gamma}$ are not constant while other fields are constant), the Lagrangian $\mathcal{L}$ satisfying (51) can have the following form

$$\mathcal{L} = c_1 \Phi + c_2 \sqrt{-g},$$

(54)

where $c_1$ and $c_2$ are independent of $g_{\mu\nu}$ and $\Phi$. This is why the equation

$$\partial \mathcal{L}/\partial \phi = T_\mu^\mu \sqrt{-g},$$

(55)

where $T_\mu^\mu$ is the trace of the energy-momentum tensor, obtained by Weinberg[1] for all constant $g_{\mu\nu}$ and matter fields, is generically no longer valid.

- Let us now note that $\zeta = \zeta_0(\phi)$, Eq.(29), becomes singular

$$|\zeta| \approx \frac{2|V_2|}{|V_1 + M^4 e^{-2\alpha\phi/M_p}|} \rightarrow \infty \quad \text{as} \quad \phi \rightarrow \phi_0.$$  

(56)

In this limit the effective potential (14) (see also Eq.(31)) behaves as

$$V_{eff} \approx \frac{|V_2|}{\zeta^2}.$$  

(57)

Thus, disappearance of the cosmological constant occurs in the regime where $|\zeta| \rightarrow \infty$. In this limit, the dynamical role of the terms of the Lagrangian $L_2$ (coupled with the measure $\sqrt{-g}$) in the action (1) becomes negligible in comparison with the terms of the Lagrangian $L_1$ (see also the general form of the action (11)). A particular realization of this we observe in the behavior of $V_{eff}$, Eq.(57). It is evident that the limit of the TMT action (1) as $|\zeta| \rightarrow \infty$ is opposite to the conventional field theory (with only measure $\sqrt{-g}$) limit of the TMT action discussed in the previous subsection. From the point of view of TMT, this is the answer to the question why the old cosmological constant problem cannot be solved (without fine tuning) in theories with only the measure of integration $\sqrt{-g}$ in the action.

- Recall that one of the basic assumptions of the Weinberg’s no-go theorem is that all fields in the vacuum must be constant. This is also assumed for the metric tensor components of which in the vacuum must be nonzero constants. However, this is not the case in the fundamental TMT action (1) defined in the original (non Einstein) frame if we ask what is the metric tensor $g_{\mu\nu}$ in the $\Lambda = 0$ vacuum. To see this let us note that in the Einstein frame all the terms in the cosmological equations are regular. This means that the metric tensor in the Einstein
frame $g_{\mu\nu}$ is always well defined, including the $\Lambda = 0$ vacuum state $\phi = \phi_0$ where $\zeta$ is infinite. Taking this into account and using the transformation to the Einstein frame (10) we see that all components of the metric in the original frame $g_{\mu\nu}$ go to zero overall in space-time as $\phi$ approaches the $\Lambda = 0$ vacuum state:

$$g_{\mu\nu} \sim \frac{1}{\zeta} \sim V_1 + M^4 e^{-2\alpha\phi/M_p} \to 0 \quad (\mu, \nu = 0, 1, 2, 3) \quad \text{as} \quad \phi \to \phi_0. \quad (58)$$

This result shows that the Weinberg’s analysis based on the study of the trace of the energy-momentum tensor misses any sense in the case $g_{\mu\nu} = 0$.

The metric is an attribute of the space-time term. Hence disappearance of the metric $g_{\mu\nu}$ in the limit $\phi \to \phi_0$ means that the strict formulation of the TMT model (9) with $V_1 < 0$ and $V_2 < 0$ requires a new mathematical basis. A manifold which is not equipped with the metric (corresponding to the $\Lambda = 0$ vacuum state) emerges as a certain limit of a sequence of space-times. Thus the model under consideration is in fact formulated not in a space-time manifold but rather by means of a set of space-time manifolds. A limiting point of a sequence of space-times is a ”vacuum space-time manifold” (VSTM) one of the differences of which from a regular space-time is the absence of the metric $g_{\mu\nu}$.

It follows immediately from (58) that $\sqrt{-g}$ tends to zero like

$$\sqrt{-g} \sim \frac{1}{\zeta^2} \sim \left(V_1 + M^4 e^{-2\alpha\phi/M_p}\right)^2 \to 0 \quad \text{as} \quad \phi \to \phi_0. \quad (59)$$

Then the definition $\zeta = \Phi/\sqrt{-g}$ implies that the integration measure $\Phi$ also tends to zero but rather like

$$\Phi \sim \frac{1}{\zeta} \sim V_1 + M^4 e^{-2\alpha\phi/M_p} \to 0 \quad \text{as} \quad \phi \to \phi_0. \quad (60)$$

Thus both the measure $\Phi$ and the measure $\sqrt{-g}$ become degenerate in the $\Lambda = 0$ vacuum state $\phi = \phi_0$. However $\sqrt{-g}$ tends to zero more rapidly than $\Phi$.

- As we have discussed in detail (see Secs.II, IIIA and Refs.[23],[24]), with the original set of variables used in the fundamental TMT action it is very hard or may be even impossible to display the physical meaning of TMT models. One of the reasons is that in the framework of the postulated need to use the Palatini formalism, the original metric $g_{\mu\nu}$ and connection $\Gamma^\alpha_{\mu\nu}$ appearing in the fundamental TMT action describe a non-Riemannian space-time. The transformation to the Einstein frame (10) enables to see the physical meaning of TMT because the space-time becomes Riemannian in the Einstein frame. Now we see that the transformation to the Einstein frame (10) plays also the role of a regularization of the space-time metric: the singular behavior of the $\Lambda = 0$ vacuum state.

As a result of this the metric in the Einstein frame $\tilde{g}_{\mu\nu}$ turns out to be well defined in all physical states including the $\Lambda = 0$ vacuum state.

VII. DISCUSSION AND CONCLUSION

A. Differences of TMT from the standard field theory in curved space-time

The main idea of TMT is that the general form of the action $\int L \sqrt{-\tilde{g}} d^4x$ is not enough in order to account for some of the fundamental problems of particle physics and cosmology. The key difference of TMT from the conventional field theory in curved space-time consists in the hypothesis [23]–[27] that in addition to the term in the action with the volume element $\sqrt{-\tilde{g}} d^4x$ there should be one more term where the volume element is metric independent but rather it is determined either by four (in the 4-dimensional space-time) scalar fields $\varphi_\alpha$ or by a three index potential $A_{\alpha\beta\gamma}$, see Eqs.(11–13). We would like to emphasize that including in the action of TMT the coupling of the Lagrangian density $L_3$ with the measure $\Phi$, we modify in general both the gravitational and matter sectors as compared with the standard field theory in curved space-time. Besides we made two more assumptions: the measure fields ($\varphi_\alpha$ or $A_{\alpha\beta\gamma}$) appear only in the volume element; one should proceed in the first order formalism. These assumptions constitute all the modifications of the general structure of the theory we have made as compared with the conventional field theory where only the measure of integration $\sqrt{-g}$ is used in the action principle. In fact, the Lagrangian densities $L_1$ and $L_2$ studied in the present paper, contain only such terms which should be present in a conventional model with minimally coupled to gravity scalar field. In particular there is no need for the non-linear in kinetic energy terms as well as in
the phantom type terms in the fundamental Lagrangian densities $L_1$ and $L_2$ in order to obtain a super-acceleration
phase at the late time universe.

After making use of the variational principle and formulating the resulting equations in the Einstein frame, we have seen that the effective action \[27\] represents a concrete realization of the $k$-essence \[12\] obtained from first principles of TMT without any exotic terms in the Lagrangian densities.

**B. Short summary of results**

1. **The early universe inflation**

As $\delta = 0$, the dynamics of $\phi$ can be analyzed by means of its effective potential \[31\]. As $\phi \ll -M_p$ the effective $\phi$ potential has the exponential form and it is proportional to the integration constant $M^4$. In other words, the effective potential governing the dynamics of the early universe results from the spontaneous breakdown of the global scale invariance \[5\] caused by the intrinsic feature of TMT (see Eqs. \[5\] and \[13\]). We have seen that independently of the values of the parameters $V_1$, $V_2$ and under very general initial conditions, solutions rapidly approach a regime characterized by a power law inflation. If $\delta \neq 0$, we deal with the intrinsically $k$-essence dynamics. The numerical solutions in this case have showed that there is no qualitative difference from the power law inflation obtained in the case with $\delta = 0$.

2. **End of inflation**

In our toy model there are three regions of the parameters $V_1$ and $V_2$ and appropriate three shapes of the effective potentials, Fig.1. Therefore three different types of scenarios for exit from inflation can be realized:

a) $V_1 < 0$ and $V_2 < 0$, Sec.IVB. In this case the power law inflation ends with damped oscillations of $\phi$ approaching the point of the phase plane ($\phi = \phi_0$, $\dot{\phi} = 0$) where the vacuum energy $V_{\text{eff}}(\phi_0) = 0$. This occurs without fine tuning of the parameters $V_1$, $V_2$ and the initial conditions.

b) $V_1 > 0$ and $b_2V_1 > 2V_2$, Sec.IVC. In this case the power law inflation monotonically transforms to the late time inflation asymptotically governed by the cosmological constant $\Lambda_1$.

Qualitatively the same results are also obtained in the cases a) and b) if $\delta \neq 0$.

c) $V_2 < b_2V_1 < 2V_2$, Sec.IVD. In this case the power law inflation ends without oscillations at the final value $\phi_{\text{min}}$, corresponding to the (non zero) minimum of the effective potential.

The model we have studied in this paper may be extended by including the Higgs field, as well as gauge fields and fermions. It turns out that the scalar sector of such an extended model enables a scenario which resembles a hybrid inflation \[11\]. These results will be presented in a future publication.

3. **Cosmological constant problems**

1) **The old cosmological constant problem.** In Sec.IVA we have seen in details that if $V_1 < 0$ then, for a broad range of other parameters, the vacuum energy turns out to be zero without fine tuning. This effect is a direct consequence of the TMT structure which yields the following results: a) the effective scalar sector potential generated in the Einstein frame is proportional to a perfect square of two terms; b) one of those terms is proportional to the integration constant $M^4$ the appearance of which is also the intrinsic feature of TMT. Note that the spontaneously broken global scale invariance is not necessary to achieve this effect \[24\]. If such type of the structure for the scalar field potential in a conventional (non TMT) model would be chosen "by hand" it would be a sort of fine tuning.

In Sec.VI we have explained in details how this result avoids the well known no-go theorem by Weinber,\[1\] stating that generically in field theory one cannot achieve zero value of the potential in the minimum without fine tuning. It is interesting that the resolution of the old CC problem in the context of TMT happens in the regime where $\zeta \rightarrow \infty$. From the point of view of TMT, the latter is the answer to the question why the old cosmological constant problem cannot be solved (without fine tuning) in theories with only the measure of integration $\sqrt{-g}$ in the action.

2) **The new cosmological constant problem.** Interesting result following from the general structure of the scale invariant TMT model with $V_1 > 0$ is that the cosmological constant $\Lambda$, Eq.\[37\], is a ratio of quantities constructed from pre-potentials $V_1$, $V_2$ and the dimensionless parameter $b_2$. Such structure of $\Lambda$ allows to propose two ways (see Sec.VI) for resolution of the problem of the smallness of $\Lambda$ that should be $\Lambda \sim (10^{-3} eV)^4$.
a) The first way is a kind of a *seesaw mechanism* [39]. For instance, if \( V_1 \sim (10^3 \text{GeV})^4 \) and \( V_2 \sim (10^{18} \text{GeV})^4 \) then \( \Lambda_1 \sim (10^{-3} \text{eV})^4 \).

b) The second way is realized if the dimensionless parameters \( b_a, b_\phi \) and \( V_2/V_1 \) of the action [9] are huge numbers of the close orders of magnitude. For example, if \( V_1 \sim (10^3 \text{GeV})^4 \) then for getting \( \Lambda \sim (10^{-3} \text{eV})^4 \) one should assume that \( b_2 \sim 10^{60} \). Possibility of this idea means that the resolution of the new cosmological constant problem may have a certain relation to the *correspondence principle* between TMT and conventional field theories (see details in Sec.VIA2).

4. Super-acceleration phase of the Universe.

If no fine tuning of the parameters is made in the fundamental action, namely if \( b_2 \neq b_\phi \), then our TMT model has big enough regions in the parameter space where the super-acceleration phase in the late time universe becomes possible. The appropriate phantom dark energy asymptotically approaches a cosmological constant. However it is impossible to obtain a pure classical solution which connects the early universe power law inflation with the late time super-acceleration. This problem is apparently related with the toy character of the scenario where the role of the matter creation has been ignored: in TMT the fermionic matter generically contributes to the constraint equation for the scalar field \( \zeta \) and so can effect the field \( \phi \) dynamics as well.

C. What can we expect from quantization

In this paper we have studied only classical TMT and its possible effects in the context of cosmology. However quantization of TMT as well as influence of quantum effects on the processes explored in this paper may have a crucial role. We summarize here some ideas and speculations which gives us a hope that quantum effects can keep the main results of this paper.

Recall first two fundamental facts of TMT as a classical field theory: (a) The measure degrees of freedom appear in the equations of motion only via the scalar \( \zeta \), Eq. (7); (b) The scalar \( \zeta \) is determined (as a function of matter fields, in our toy model - as a function of \( \phi \)) by the constraint which is nothing but a consistency condition of the equations of motion (see Eqs. (A1)-(A3) in Appendix A and Eq. (10)). Therefore the constraint plays a key role in TMT. Note however that if we were ignore the gravity from the very beginning in the action \( \Phi \), then instead of the constraint [A3] we would obtain Eq. (A1) (where one has to put zero the scalar curvature). In such a case we would deal with a different theory. This notion shows that the gravity and matter intertwined in TMT in a much more complicated manner than in GR. Hence introducing the new measure of integration \( \Phi \) we have to expect that the quantization of TMT may be a complicated enough problem. Nevertheless we would like here to point out that in the light of the recently proposed idea of Ref. [11], the incorporation of four scalar fields \( \varphi_a \) together with the scalar density \( \Phi \), Eq. (12), (which in our case are the measure fields and the new measure of integration respectively), is a possible way to define local observables in the local quantum field theory approach to quantum gravity. We regard this result as an indication that the effective gravity + matter field theory has to contain the new measure of integration \( \Phi \) as it is in TMT.

The assumption formulated in item 2 in Sec.IIA, that the measure fields \( \varphi_a \) (or \( A_{\alpha\beta\gamma} \)) appear in the action only via the measure of integration \( \Phi \), has a key role in the TMT results and in particular for the resolution of the old cosmological constant problem. In principle one can think of breakdown of such a structure by quantum corrections. However, TMT possesses an infinite dimensional symmetry mentioned in item 2 of Sec.II which, as we hope, is able to protect the postulated structure of the action from a deformation caused by quantum corrections. Another effect of quantum corrections is the possible appearance of a nonminimal coupling of the dilaton field \( \phi \) to gravity in the form like for example \( \xi R \phi^2 \). Proceeding in the first order formalism of TMT one can show that the nonminimal coupling can affect the k-essence dynamics but the mechanism for resolution of the old CC problem exhibited in this paper remains unchanged. This conclusion together with expected effect of quantum corrections on the scale invariance (see our discussion in the paragraph after Eq. (44)) allows us to hope that the exhibited resolution of the old CC problem holds in the quantized TMT as well.

Quantization of TMT being a constrained system requires developing the Hamiltonian formulation of TMT. Preliminary consideration shows that the Einstein frame appears in the canonical formalism in a very natural manner. A systematic exploration of TMT in the canonical formalism will be a subject of forthcoming research.
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APPENDIX A: EQUATIONS OF MOTION IN THE ORIGINAL FRAME

Variation of the measure fields $\varphi_{\alpha}$ with the condition $\Phi \neq 0$ leads, as we have already seen in Sec.II, to the equation $L_1 = s M^4$ where $L_1$ is now defined, according to Eq. (1), as the part of the integrand of the action (9) coupled to the measure $\Phi$. Equation (5) in the context of the model (9) reads (with the choice $s = +1$):

$$\left( -\frac{1}{\kappa} R(\Gamma, g) + \frac{1}{2} g_{\mu\nu} \phi,_{\mu} \phi,_{\nu} \right) e^{\alpha\phi/M_\nu} - V_1 e^{2\alpha\phi/M_\nu} = M^4,$$

(A1)

It can be noticed that the appearance of a nonzero integration constant $M^4$ spontaneously breaks the scale invariance (8).

Variation of the action (9) with respect to $g_{\mu\nu}$ yields

$$-\frac{1}{\kappa} (\zeta + b_\phi) R_{\mu\nu}(\Gamma) + (\zeta + b_\phi) \frac{1}{2} g_{\mu\nu} \phi,_{\mu} \phi,_{\nu} + \frac{b_\phi}{2} g_{\mu\nu} \left[ b_\mu R(\Gamma, g) - \frac{b_\phi}{2} g^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta} + V_2 e^{\alpha\phi/M_\nu} \right] = 0.$$  

(A2)

We see that in contrast to field theory models with only the measure $\sqrt{-g}$, in TMT there are two independent equations containing curvature. Contracting Eq. (A2) with $g_{\mu\nu}$ and solving Eq. (A1) for $R(\Gamma, g)$ we obtain the following consistency condition of these two equations:

$$(\zeta - b_\phi) \left( M^4 e^{-\alpha\phi/M_\nu} + V_1 e^{\alpha\phi/M_\nu} \right) + 2V_2 e^{\alpha\phi/M_\nu} + (b_\phi - b_\phi) \frac{1}{2} g_{\mu\nu} \phi,_{\mu} \phi,_{\nu} = 0,$$

(A3)

that we will call the constraint in the original frame.

It follows from Eqs. (A1) and (A2) that

$$\frac{1}{\kappa} R_{\mu\nu}(\Gamma) = \frac{\zeta + b_\phi}{\zeta + b_\phi} \frac{1}{2} g_{\mu\nu} \phi,_{\mu} \phi,_{\nu} - \frac{g_{\mu\nu}}{2(\zeta + b_\phi)} \left[ b_\mu M^4 e^{-\alpha\phi/M_\nu} + (b_\phi) V_1 e^{\alpha\phi/M_\nu} - (b_\phi - b_\phi) \frac{1}{2} g^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta} \right].$$

(A4)

The scalar field $\phi$ equation of motion in the original frame can be written in the form

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[ e^{\alpha\phi/M_\nu} \left( \zeta + b_\phi \right) \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right] - \frac{\alpha}{M_\nu} e^{\alpha\phi/M_\nu} \left[ (\zeta + b_\phi) M^4 e^{-\alpha\phi/M_\nu} + [(b_\phi - \zeta) V_1 - 2V_2] e^{\alpha\phi/M_\nu} - (b_\phi - b_\phi) \frac{1}{2} g^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta} \right] = 0$$

(A5)

where Eq. (A1) has been used.

Variation of the action (9) with respect to the connection degrees of freedom leads to the equations we have solved earlier[24]. The result is

$$\Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu} \} + \frac{1}{2} (\delta^\mu_\rho \sigma_{\nu\rho} + \delta^\nu_\rho \sigma_{\mu\rho} - \sigma_{\nu\rho} g_{\mu\nu} g^{\rho\beta})$$

(A6)

where $\{^\lambda_{\mu\nu} \}$ are the Christoffel’s connection coefficients of the metric $g_{\mu\nu}$ and $\sigma \equiv \ln \zeta$. 
APPENDIX B: ASYMMETRY BETWEEN EARLY AND LATE TIME DYNAMICS OF THE UNIVERSE AS RESULT OF ASYMMETRY IN THE COUPLINGS TO MEASURES $\Phi$ AND $\sqrt{-g}$ IN THE ACTION.

The results obtained in Secs.IV and V depend very much on the choice of the parameters $V_1$, $V_2$ and $\delta$ in the action $\mathcal{S}$. Let us recall that the curvature term in the action $\mathcal{S}$ couples to the measure $\Phi + b_\gamma \sqrt{-g}$ where the $\phi$ kinetic term couples to the measure $\Phi + b_\sigma \sqrt{-g}$. This is the reason of $\delta \neq 0$. If we were choose the fine tuned condition $\delta = 0$ then both the curvature term and the $\phi$ kinetic term would be coupled to the same measure $\Phi + b_\gamma \sqrt{-g}$. One can also pay attention that depending on the choice of one of the alternative conditions $b_\gamma V_1 > 2V_2$ or $b_\gamma V_1 < 2V_2$ we realize different shapes of the effective potential if $b_\gamma V_1 > 2V_2$ (see Fig.1). And again, if instead we were choose the fine tuned condition $b_\gamma V_1 = V_2$ then the action would contain only one prepotential coupled to the measure $\Phi + b_\gamma \sqrt{-g}$.

So, in order to avoid fine tunings we have introduced asymmetries in the couplings in the different terms in the Lagrangian densities $L_1$ and $L_2$ to measures $\Phi$ and $\sqrt{-g}$. In order to display the role of these asymmetries it is useful to consider what happens if such asymmetries are absent in the action at all. In other words we want to explore here the gravity+dilaton model where both $\delta = 0$ and $b_\gamma V_1 = V_2$. In such a case the action contains only one Lagrangian density coupled to the measure $\Phi + b_\gamma \sqrt{-g}$:

$$S = \int (\Phi + b_\gamma \sqrt{-g})d^4x e^{\alpha\phi/M_p} \left(-\frac{1}{\kappa}R + \frac{1}{2}g_{\mu\nu}\partial_\mu\phi \partial_\nu\phi - V e^{\alpha\phi/M_p}\right),$$  \hspace{1cm} (B1)

where $V = V_1 = V_2/b_\gamma$. An equivalent statement is that $L_1 = b_\gamma L_2$; it is an example of the very special class of the TMT models where $L_1$ is proportional to $L_2$.

To see the cosmological dynamics in this model one can use the results of Sec.IIIB. If we assume in addition $b_\gamma > 0$ and $V_1 > 0$, then after the shift $\phi \rightarrow \phi + \Delta\phi$ where $\Delta\phi = -M_p^2(2\alpha \ln(V/M^4))$ (which is not a shift symmetry in this case), the effective potential $V_{\phi\phi\sigma}$ takes the form

$$V_{\phi\phi\sigma}^{(\text{symm})}(\phi) = \frac{V^2}{b_\gamma M^4}\cosh^2(\alpha\phi/M_p).$$  \hspace{1cm} (B2)

In contrast to general cases ($b_\gamma V_1 \neq V_2$) this potential has no flat regions and it is symmetric around a certain point in the $\phi$-axis. This form of the potential (with an additional constant) has been used in a model of the early inflation.

APPENDIX C: SOME REMARKS ON THE MEASURE FIELDS INDEPENDENCE OF $L_1$ AND $L_2$

Although we have assumed in the main text that $L_1$ and $L_2$ are $\phi_n$ independent, a contribution equivalent to the term $\int f(\Phi/\sqrt{-g})d^4x$ can be effectively reproduced in the action $\mathcal{S}$ if a nondynamical field (Lagrange multipliers) is allowed in the action. For this purpose let us consider the contribution to the action of the form

$$S_{\text{auxiliary}} = \int [\sigma\Phi + l(\sigma)\sqrt{-g}]d^4x$$  \hspace{1cm} (C1)

where $\sigma$ is an auxiliary nondynamical field and $l(\sigma)$ is an analytic function. Varying $\sigma$ we obtain $dl/d\sigma \equiv l'(\sigma) = -\Phi/\sqrt{-g}$ that can be solved for $\sigma$: $\sigma = l^{-1}(-\Phi/\sqrt{-g})$ where $l^{-1}$ is the inverse function of $l'$. Inserting this solution for $\sigma$ back into the action $S_{\text{aux}}$ we obtain

$$S_{\text{aux.inTEGRATED}} = \int f(\Phi/\sqrt{-g})\Phi d^4x$$  \hspace{1cm} (C2)

where the auxiliary field has disappeared and

$$f(\Phi/\sqrt{-g}) \equiv l^{-1}(-\Phi/\sqrt{-g}) + l(l^{-1}(-\Phi/\sqrt{-g}))\sqrt{-g}/\Phi.$$  \hspace{1cm} (C3)

To see the difference between effect of this type of auxiliary fields as compared with a model where the $\sigma$ field is equipped with a kinetic term, let us consider two toy models including gravity and $\sigma$ field: one - without kinetic term

$$S_{\text{toy}} = \int \left[-\frac{1}{\kappa}R + \sigma\right] \Phi + bo\sigma^2\sqrt{-g} d^4x$$  \hspace{1cm} (C4)
and the other - with a kinetic term

\[ S_{\text{toy}, k} = \int \left[ \left( \frac{1}{\kappa} R + \sigma + \frac{1}{2} g^{\alpha \beta} \frac{\partial_\alpha \sigma \partial_\beta \sigma}{\sigma^2} \right) \Phi + b \sigma^2 \sqrt{-g} \right] d^4 x \]  

(C5)

where \( b \) is a real constant. For both of them it is assumed the use of the first order formalism. The first model is invariant under local transformations \( \Phi \to J \Phi, \ g_{\mu \nu} \to J g_{\mu \nu}, \ \sigma \to J^{-1} \sigma \) where \( J \) is an arbitrary space-time function while in the second model the same symmetry transformations hold only if \( J \) is constant.

Variation of the measure fields \( \varphi_a \) in the model (C5) leads (if \( \Phi \neq 0 \)) to

\[ -\frac{1}{\kappa} R + \sigma + \frac{1}{2} g^{\alpha \beta} \frac{\partial_\alpha \sigma \partial_\beta \sigma}{\sigma^2} = M^4, \]  

(C6)

where \( M^4 \) is the integration constant. On the other hand varying the action (C5) with respect to \( g^{\mu \nu} \) gives

\[ \chi \left( -\frac{1}{\kappa} R_{\mu \nu} + \frac{1}{2} \partial_\mu \sigma \partial_\nu \sigma \right) - \frac{1}{2} b \sigma^2 g_{\mu \nu} = 0, \]  

(C7)

where \( \chi \equiv \frac{\phi}{\sqrt{-g}} \). The corresponding equations in the model (C4) are obtained from (C6) and (C7) by omitting the terms with gradients of \( \sigma \). It follows from Eqs. (C6) and (C7) that

\[ \frac{1}{\chi} = \frac{M^4 - \sigma}{2b\sigma^2} \]  

(C8)

This result holds in both models.

In the model (C4), variation of \( \sigma \) results in \( \frac{1}{\chi} = -\frac{1}{2b\sigma} \) which is consistent with Eq. (C8) only if the integration constant \( M = 0 \). This means that through the classical mechanism displayed in TMT, it is impossible to achieve spontaneous breakdown of the local scale invariance. This appears consistent with arguments by Elitzur[13] concerning impossibility of a spontaneous breaking of a local symmetry without gauge fixing.

Transition to the Einstein frame where the space-time becomes Riemannian is implemented by means of the conformal transformation \( \tilde{g}_{\mu \nu} = \chi g_{\mu \nu} \). For the model (C4) the gravitational equations in the Einstein frame read

\[ \frac{1}{\kappa} G_{\mu \nu}(\tilde{g}_{\alpha \beta}) = \frac{M^4 - \sigma}{8b\sigma^2} \tilde{g}_{\mu \nu} + \frac{1}{2} \left( \frac{\partial_\mu \sigma \partial_\nu \sigma}{\sigma^2} - \frac{1}{2} \tilde{g}_{\mu \nu} \tilde{g}^{\alpha \beta} \partial_\alpha \sigma \partial_\beta \sigma \right). \]  

(C9)

This means that the model (C4) with auxiliary (nondynamical) field \( \sigma \) intrinsically contains a constant vacuum energy.

In the model (C5), where \( \sigma \) appears as a dynamical field, the gravitational equations in the Einstein frame results from Eq. (C7)

\[ \frac{1}{\kappa} G_{\mu \nu}(\tilde{g}_{\alpha \beta}) = \frac{(M^4 - \sigma)^2}{8b\sigma^2} \tilde{g}_{\mu \nu} + \frac{1}{2} \left( \frac{\partial_\mu \sigma \partial_\nu \sigma}{\sigma^2} + \frac{1}{2} \tilde{g}_{\mu \nu} \tilde{g}^{\alpha \beta} \partial_\alpha \sigma \partial_\beta \sigma \right). \]  

(C10)

It is convenient to rewrite this equation in terms of the scalar field \( \ln \sigma \equiv \phi \):

\[ \frac{2}{\kappa} G_{\mu \nu}(\tilde{g}_{\alpha \beta}) = V_{\text{eff}}(\phi) \tilde{g}_{\mu \nu} + \frac{1}{2} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \tilde{g}_{\mu \nu} \tilde{g}^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi \right), \]  

where \( V_{\text{eff}}(\phi) = \frac{1}{4b\sigma^2} (M^4 e^{-\phi} - 1)^2 \).  

(C11)

The \( \phi \)-equation reads \( \Box \phi + V'_{\text{eff}}(\phi) = 0 \). Similar to the general discussion in the main text we see that if the \( \sigma \) field is dynamical then TMT provides the vacuum with zero energy without fine tuning.

Hence the main difference between the TMT models with auxiliary and dynamical scalar fields consists in radically different results concerning the cosmological constant problem.

However it is very unlikely that a nondynamical scalar field will not acquire a kinetic term after quantum corrections[44]. Then it becomes dynamical which restores the above results for the model (C5). This is why we have ignored the rather formal possibility of introducing the nondynamical scalars into the fundamental action of the models studied in this paper.

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[46] See: F. Wilczek, Phys.Rev.Lett. 80, 4851 (1998). Wilczek’s goal was to avoid the use of a fundamental metric, and for this purpose he needs five scalar fields. In our case we keep the standard role of the metric from the beginning, but enrich the theory with a new metric independent density.
[47] The field theory models with explicitly broken scale symmetry and their application to the quintessential inflation type cosmological scenarios have been studied in Ref. [26]. Inflation and transition to slowly accelerated phase from higher curvature terms was studied in Ref. [26].
[48] Compare the way the shift symmetry is realized here with that of Ref. [35].
[49] Note that the solution \( \Phi \) may be nonapplicable for a relatively short period of time from the very beginning where \( \frac{d\Phi}{dt} \) may be negative that we observe in Fig.2.
[50] The particular case of this model with \( b_0 = 0 \) and \( V_2 < 0 \) was studied in Ref. [26]. The application of the TMT model with explicitly broken global scale symmetry to the quintessential inflation scenario was discussed in Ref. [26].
[51] Note that if \( V_2 < 0 \) then the choice \( |V_2|/V_1 \sim b_0 \) means that in this case the second way of resolution of the new CC problem is a particular case of the seesaw mechanism. However the second way is applicable also if \( V_2 > 0 \).
[52] Note that instead of using factors \( b_0 \) and \( b_0 \) one can define the fundamental TMT action such that the appropriate factors in \( L_2 \) equal unity. Then in front of the appropriate terms in \( L_1 \) one should add factors like \( b_0^{-1} \) and \( b_0^{-1} \). With such a definition of the Lagrangians \( L_1 \) and \( L_2 \), instead of a huge factor in \( L_2 \), a very small factor will appear in \( L_1 \).
[53] A possibility of a vacuum with non constant 3-form gauge field has been discussed in Footnote 8 of the Weiberg’s review.