RCF 2
Evaluation and Consistency
\[ \varepsilon \& \mathcal{C} * \pi_{O} \mathcal{R} \]
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Abstract: We construct here an iterative evaluation of all PR map codes: progress of this iteration is measured by descending complexity within “Ordinal” \( O := \mathbb{N}[\omega] \) of polynomials in one indeterminate, ordered lexicographically. Non-infinit descent of such iterations is added as a mild additional axiom schema \((\pi_{O})\) to Theory \( \text{PR}_{A} = \text{PR} + \text{(abstr)} \) of Primitive Recursion with predicate abstraction, out of forgoing part RCF 1. This then gives (correct) on-termination of iterative evaluation of argumented deduction trees as well, for theories \( \pi_{O} \mathcal{R} = \text{PR}_{A} + (\pi_{O}) \). By means of this constructive evaluation the Main Theorem is proved, on Termination-conditioned (Inner) Soundness for Theories \( \pi_{O} \mathcal{R} \), Ordinal \( O \) extending \( \mathbb{N}[\omega] \). As a consequence we get Self-Consistency for these theories \( \pi_{O} \mathcal{R} \), namely \( \pi_{O} \mathcal{R} \)-derivation of \( \pi_{O} \mathcal{R} \)'s own free-variable Consistency formula

\[ \text{Con}_{\pi_{O} \mathcal{R}} = \text{Con}_{\pi_{O} \mathcal{R}}(k) =_{\text{def}} \neg \text{Prov}_{\pi_{O} \mathcal{R}}(k, 'false') : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free}. \]
Here PR predicate \( \text{Prov}_{T}(k, u) \) says, for an arithmetical theory \( T : \text{number } k \in \mathbb{N} \text{ is a } T\text{-Proof code proving internally } T\text{-formula code } u : k \text{ is an arithmetised Proof for } u \text{ in Gödel’s sense}. \)

As to expect from classical setting, Self-Consistency of \( \pi_{O} \mathcal{R} \) gives (unconditioned) Objective Soundness. Termination-Conditioned Soundness holds “already” for \( \text{PR}_{A} \), but it turns out that at least present derivation of Consistency from this conditioned Soundness depends on schema \((\pi_{O})\) of non-infinit descent in Ordinal \( O := \mathbb{N}[\omega] \).

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1 Summary

Gödel’s first Incompleteness Theorem for *Principia Mathematica* and “*verwandte Systeme*”, on which in particular is based the second one, on non-provability of PM’s own *Consistency formula* $\text{Con}_{\text{PM}}$, exhibits a (closed) PM formula $\varphi$ with property that

$$\text{PM} \vdash [\varphi \iff \neg (\exists k \in \mathbb{N}) \text{Prov}_{\text{PM}}(k, \langle \varphi \rangle^\ast)],$$

in words:

Theory PM derives $\varphi$ to be equivalent to its “own” coded, arithmetised non-Provability.

Since this equivalence needs already for its statement “full” formal, “not testable” quantification, the Consistency Provability issue is not settled for Free-Variables Primitive Recursive Arithmetic and its strengthenings – Theories $T$ which express (formalised, “internal”) Consistency as free-variable formula

$$\text{Con}_T = \text{Con}_T(k) = \neg \text{Prov}_T(k, \langle \text{false} \rangle^\ast) : \mathbb{N} \to 2$$

“No $k \in \mathbb{N}$ is a Proof code proving $\langle \text{false} \rangle$.”

This is the point of depart for investigation of “suitable” strengthenings $\pi_O \mathbf{R} = \mathbf{PR}_A + (\pi_O)$ of categorical Theory $\mathbf{PR}_A$ of Primitive Recursion, enriched with *predicate abstraction Objects* $\{ A | \chi \} = \{ a \in A | \chi(a) \}$: Plausibel axiom schema $(\pi_O)$, more presisely: its contraposition $\tilde{\pi}_O$, states “weak” impossibility of infinite descending chains in any *Ordinal* $O$ extending polynomial semiring $\mathbb{N}[\omega]$, with its canonical, lexicographical order.

**Central Non-Infinist Descent Schema:** We need an axiom-schema for expressing – in free variables – finite descent of (endo-driven) chains, descending in complexity value out of Ordinal $O \geq \mathbb{N}[\omega]$, a schema called $(\pi_O)$, which gives the “name” to $\pi_O \mathbf{R} = \mathbf{PR}_A + (\pi_O)$: This theory is a pure strengthening of $\mathbf{PR}_A$, it has the same language.

Easier to interprete logically is $(\pi_O)$’s equivalent, Free-Variables contrapos-
tion, on “absurdity” of infinitely descending chains, namely:

\[ c = c(a) : A \rightarrow O \text{ PR (complexity)}, \]
\[ p = p(a) : A \rightarrow A \text{ PR (predecessor endo)}, \]
\[ \text{PR}_A \vdash c(a) > 0_O \implies c p(a) < c(a) \text{ (descent)}, \]
\[ \text{PR}_A \vdash c(a) = 0_O \implies p(a) = a \text{ (stationarity at zero)} \]
\[ \psi = \psi(a) : A \rightarrow 2 \text{ absurdity test predicate,} \]
\[ \text{PR}_A \vdash \psi(a) \implies c p^n(a) > 0_O, \]
\[ \text{with quantifier decoration:} \]
\[ \text{PR}_A \vdash \forall a [\psi(a) \implies \forall n c p^n(a) > 0_O] \]
\[ \text{the latter statement: “infinit descent”, is felt absurd,} \]
\[ \text{and “therefore” so “must be”, by axiom,} \]
\[ \text{condition } \psi \text{ implying this “absurdity”:} \]
\[ (\tilde{\pi}_O) \]
\[ \pi_O R \vdash \psi(a) = \text{false} : A \rightarrow 2, \text{ intuitively:} \]
\[ \pi_O R \vdash \forall a \neg \psi(a). \]

[The first four lines of the antecedent constitute \((c, p)\) as (the data of) a \(\text{CCI}_O\) : of a Complexity Controlled Iteration, with (stepwise) descending order values in Ordinal \(O\). Central example: General Recursive, Ackermann type \(PR\)-code evaluation \(\varepsilon\) will be resolved into such a \(\text{CCI}_O\), for \(O := \mathbb{N}[\omega].\)]

My Thesis then is that these theories \(\pi_O R\), weaker than PM, set theories and even Peano Arithmetic \(\text{PA}\) (when given its quantified form), derive their own internal (Free-Variable) Consistency formula \(\text{Con}_{\pi_O R}(k) : \mathbb{N} \rightarrow 2\), see above.

Notions and Arguments for Self-Consistency of \(\pi_O R\) : In order to obtain constructive Theories – candidates for self-Consistency – we introduce first, into fundamental Theory \(\text{PR}\) of (categorical) Free-Variables Primitive Recursion, predicate abstraction of \(\text{PR}\) maps \(\chi = \chi(a) : A \rightarrow 2\) (\(A\) a finite power of NNO \(\mathbb{N}\)), into defined Objects \(\{A \mid \chi\}\), and then strengthen Theory \(\text{PR}_A\) obtained this way, by a free-variables, (inferential) schema \((\pi_O)\) of “on”-terminating descent, into Theory(s) \(\pi_O R\), on-terminating descent of Complexity Controlled Iterations (CCI\(_O\)’s, see above), with (descending) complexity values in Ordinal \(O \geq \mathbb{N}[\omega]\).

Strengthened Theory \(\pi_O R = \text{PR}_A + (\pi_O)\), with its language equal to that of \(\text{PR}_A\), is asserted to derive the (Free-Variable) formula \(\text{Con}_{\pi_O R}(k)\) which expresses internally: within \(\pi_O R\) itself, Consistency of Theory \(\pi_O R\), see above.

Proof is by \(\text{CCI}_{\mathbb{N}[\omega]}\) (descent) property of a suitable, atomic \(\text{PR}\) evaluation step \(e\) applied to \(\text{PR-map-code/argument}\) pairs \((u, x) \in \text{PR}_A \times X\).

[Here \(X \subseteq \mathbb{N}\) denotes the Universal Object of all (codes of) singletons and (nested) pairs of natural numbers. Objects \(A\) of \(\text{PR}_A\), \(\pi_O R\) admit a natural embedding \(A \subseteq X\) into this universal Object.]

Iteration \(\varepsilon\), of step \(e\), is in fact controlled by a syntactic complexity \(c_{\text{PR}_A}(u) \in \mathbb{N}[\omega]\), descending with each application of \(e\) as long as minimum complexity
0 = c_{PR_A}(\forall \mathrm{id}^{-}) is not “yet” reached.

*Strengthening* of $PR_A$ by schema $(\pi_O)$ – cf. its free-variables contraposition $(\tilde{\pi}_O)$ above – into Theory $\pi_O R = PR + (\pi_O)$, is “just” to allow for a *sound*, canonical evaluation “algorithm” for $\pi_O R$:

On one hand it is proved straightforward that evaluation $\varepsilon$ above has the expected recursive properties of an *evaluation*, this within (categorical, Free-Variables) Theory $\mu R$ of $\mu$-Recursion.

On the other hand, $\pi_O R$ has the same *Language* as $PR_A$, so that this $\varepsilon$ is a natural candidate for likewise – *sound* – evaluation of internal version of theory $\pi_O R$, and for being *totally defined* in a suitable Free-Variables sense, technically: to *on-terminate*, this just by its property to be a *Complexity Controlled Iteration*, with order values in $\mathbb{N}[\omega]$. In fact, by schema $(\pi_O)$ itself, $\varepsilon$ will *preserve* the *extra* equation instances inserted by internalisation of $(\pi_O)$, argument see below.

**Dangerous bound:** is there a good reason that this evaluation is not a *self-evaluation* for Theory $\pi_O R$?

Answer: $\varepsilon$ is – by definition – *not* PR: If you take the diagonal

$$\text{diag}(n) =_{\text{def}} \varepsilon(\text{enum}_{PR}(n), \text{cantor}_X(n)) : \mathbb{N} \to \mathbb{N},$$

$\text{enum}_{PR}$ an internal PR *count* of all PR map codes, and $\text{cantor}_X : \mathbb{N} \overset{\cong}{\to} X$ “the” Cantor’s *count* of $X \subset \mathbb{N}$, then you get ACKERMANN’s original diagonal function which grows faster than any PR function: but $\pi_O R$ has only PR maps as its *maps*, it is a (pure) *strengthening* of $PR_A$.

On the other hand, $\varepsilon$ is *intuitively* total, since, intuitively, complexity $ce^m(u, x)$ “must” reach 0 in *finitely many* $e$-steps. The latter intuition can be, in free variables (!), expressed *formally* by $\pi_O R$’s *schema* $(\tilde{\pi}_O)$: Free-Variables contraposition of $(\pi_O)$. Schema $(\tilde{\pi}_O)$ says that a condition which implies *infinite descent* of such a chain (on all $x$), must be *false* (on all $x$), “absurd”.

**Complexity Controlled Iteration** $\varepsilon$ of $e$ extends canonically into a Complexity Controlled evaluation $\varepsilon_d$, of *argumented deduction trees*, $\varepsilon_d$ again defined as a CCI$[\omega]$ : this time as an iteration of a *tree evaluation step* $e_d$ suitably extending basic evaluation step $e$ to argumented deduction trees.

Deduction-tree evaluation starts on trees of form $dtree_k/x$, obtained as follows from $k$ and $x$ : Call $dtree_k$ the (first) *deduction tree* which (internally) *proves* $k$th internal equation $u \centcolon k \overset{\cong}{\Rightarrow} v$ of theory $\pi_O R$, enumeration of *proved* equations being (lexicographically) by code of (first) *Proof*. This argument-free deduction tree $dtree_k$ then is provided – node-wise top down from given $x \in X$ – with its *spread down* arguments in $X_\square =_{\text{def}} X \cup \{\square\} \subset \mathbb{N}$; (empty list $\square$ refers to a not yet known argument, not “yet” at a given time of stepwise evaluation $e_d$.)

*Spreading down* arguments this way eventually converts argument-free $k$th deduction tree $dtree_k$ into (partially non-dummy) *argumented deduction tree* $dtree_k/x$.

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1 for a two-parameter, simple genuine ACKERMANN function cf. Eilenberg/Elgot 1970
Iteration \( \varepsilon_d \), of tree evaluation step \( e_d \), again is Complexity Controlled descending in Ordinal \( \mathbb{N}[\omega] \), when controlled by deduction tree complexity \( c_d \). This complexity is defined essentially as the (polynomial) sum of all (syntactical) complexities \( c_{PR_A}(u) \) of map codes appearing in the deduction tree.

So, as it does to basic evaluation \( \varepsilon \), schema \( (\bar{\pi} = (\bar{\pi}\|\omega]) \) applies to complexity controlled evaluation \( \varepsilon_d \) of argumented deduction-trees as well, and gives

**Deduction-Tree Evaluation non-infinit Descent:** Infinit strict descent of endo map \( e_d \) – with respect to complexity \( c_d \) – is absurd.

This deduction-tree evaluation \( \varepsilon_d \) externalises, as far as terminating, \( k \) th internal equation \( u \rightleftharpoons_k v \) of theory \( \pi_0 R \) into complete evaluation \( \varepsilon(u, x) \rightleftharpoons \varepsilon(v, x) \):

**Termination-Conditioned Inner Soundness**, our Main Theorem.

For a given \( PR_A \)-predicate \( \chi = \chi(a) : A \to 2 \), the Main Theorem reads, again in \( \pi_0 R \)-terms alone:

Theory \( \pi_0 R \) derives: If for \( k \in \mathbb{N} \) and for \( a \in A \sqsubseteq X \) given, \( \text{Prov}_{\pi_0 R}(k, \lnot \chi^\top) \) “holds”, and if argumented \( \pi_0 R \) deduction tree \( \text{dtree}_k/a \) admits complete evaluation by \( m \) (“say”) deduction-tree evaluation-steps \( e_d \),

then the pair \( (k, a) \) is a Soundness-Instance, i.e. then \( k \) th given (internal) \( \pi_0 R \)-Provability \( \text{Prov}_{\pi_0 R}(k, \lnot \chi^\top) \) implies \( \chi(a) \), for the given argument \( a \in A \). All this within Theory \( \pi_0 R \) itself.

**Corollary:** Self-Consistency Derivability for Theory \( \pi_0 R \) :

\[
\pi_0 R \vdash \text{Con}_{\pi_0 R}, \text{i.e. “necessarily” in Free-Variables form:}
\]

\[
\pi_0 R \vdash \lnot \text{Prov}_{\pi_0 R}(k, \lnot \chi^\top) : \mathbb{N} \to 2, \text{i.e. equationally:}
\]

\[
\pi_0 R \vdash \lnot[\lnot \chi^\top \rightleftharpoons_k \chi^\top] : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free :}
\]

Theory \( \pi_0 R \) derives that no \( k \in \mathbb{N} \) is the internal \( \pi_0 R \)-Proof for \( \lnot \chi^\top \).

**Proof** of this Corollary to Termination-Conditioned Soundness:

By the last assertion of the Theorem we get, for \( \chi := \text{false} : 1 \to 2 \), and \( a := 0 \in 1 \):

Evaluation-effective internal inconsistency of \( \pi_0 R \), i.e.: availability of an evaluation-terminating internal deduction tree for \( \lnot \chi^\top \), implies false:

\[
\pi_0 R \vdash \lnot \chi^\top \rightleftharpoons_k \chi^\top \land c_d e_d^m(\text{dtree}_k/0) \not\geq 0 \implies \text{false.}
\]

Contraposition to this, still with \( k, m \in \mathbb{N} \) free:

\[
\pi_0 R \vdash \text{true} \implies \lnot[\lnot \chi^\top \rightleftharpoons_k \chi^\top] \lor c_d e_d^m(\text{dtree}_k/0) > 0,
\]
i.e. by Free-Variables (Boolean) tautology:

\[
\pi_0 R \vdash \lnot \chi^\top \rightleftharpoons_k \chi^\top \implies c_d e_d^m(\text{dtree}_k/0) > 0, \ k, m \text{ free.}
\]

This \( \pi_0 R \) derivative invites to apply schema \( (\bar{\pi}) \) of \( \pi_0 R \) :
“infinit endo-driven descent with order values in $\mathbb{N}[\omega]$ is absurd.”

We apply this schema to deduction tree evaluation $\varepsilon_d$ given by $\text{step } e_d$ and complexity $c_d$ which descends – this is $Argumented$-$Tree$ $Evaluation$ $Descent$ – with each application of $e_d$, as long as complexity 0 is not (“yet”) reached. We combine this with choice of “overall” absurdity condition

$$\psi = \psi(k) := \left[ \sim_k \text{false} \right] : \mathbb{N} \to 2, \ k \in \mathbb{N \ free \ (!)}$$

and get, by schema $(\tilde{\pi})$, overall negation of this (overall) “absurd” predicate $\psi$, namely

$$\pi_{OR} \vdash \sim_k \left[ \sim_k \text{false} \right] : \mathbb{N} \to 2, \ k \in \mathbb{N \ free}.$$

This is $\pi_{OR}$-derivation of the free-variable Consistency Formula of $\pi_{OR}$ itself.

From this Self-Consistency of Theorie(s) $\pi_{OR}$, which is equivalent to injectivity of (special) internal numeralisation $\nu_2 : 2 \mapsto [1, 2]_{\pi_{OR}}$, we get immediately injectivity of all these numeralisations $\nu_A = \nu_A(a) : A \mapsto [1, A] = [1, A]/\sim$, and from this, with naturality of this family, “full” objective Soundness of Theory $\pi_{OR}$ which reads:

Formalised $\pi_{OR}$-Provability of (code of) PR predicate $\chi : X \to 2$ implies – within Theory $\pi_{OR}$ – “validity” $\chi(x)$ of $\chi$ at “each” of $\chi$’s arguments $x \in X$.

In a Coda, we state a schema $(\rho_{OR})$, of predicate truth backpropagation along a Complexity Controlled Iteration (CCIO) – admitted by Theory $\text{PR}_A$ – which gives Termination-conditioned Soundness “already” for basic Theory $\text{PR}_A$.

But for derivation of Self-Consistency from Termination-conditioned Soundness, a suitable strengthening of $\text{PR}_A$, here by schema $(\tilde{\pi})$, stating absurdity of infinite descent in Ordinal $\mathbb{N}[\omega]$, seems to be necessary: my guess is that Theories such as classical Free-Variables Theory $\text{PRA}$ of Primitive Recursive Arithmetic, as well as $\text{PR}$ and hence $\text{PR}_A$, are not strong enough to derive their own (internal) Consistency.

On the other hand, we know from Gödel’s work that Principia Mathematica “und verwandte Systeme” are too strong for being self-consistent – if $\omega$-consistent. This is true for any (formally) quantified Arithmetical Theory $Q$ (classical properties of quantifiers), in particular for the (classically quantified) version $\text{PA}$ of Peano Arithmetic. Such theory $Q$ has all ingredients for Gödel’s Proof of his two Incompleteness Theorems. The latter at least when – explicitly or implicitly – Countable Choice comes into the game: see Discussion at end “of paper”.

6
Present basic section is to give an – iterative – Evaluation “map” – a formally partial PR map, not (“total”) one, evaluating map codes on suitable arguments.

Codes of (Object-) and map terms of all our theories are coded straight ahead, in particular since formally we have no (individual) variables on the Object Language level in our context of Cartesian (categorical) theories: Here free variables are seen – interpreted – as (possibly nested) projections.

So we code all our terms just as prime power products “over” the \( \LaTeX \) source codes describing these terms, this externally in naive numbers, out of \( \mathbb{N} \) as well as internally into the NNO \( \mathbb{N} \) of the (categorical) arithmetical theory itself.

**Equality Enumeration:** As “any” theories, fundamental Theory PR of Primitive Recursion as well as basic Theory \( \text{PR}_A = \text{PR} + (\text{abstr}) \), definitional enrichment of PR by the schema of predicate abstraction: \( \langle \chi : A \rightarrow 2 \rangle \mapsto \{ A | \chi \} \), a “virtual”, abstracted Object in \( \text{PR}_A \), admit an (external) primitive recursive enumeration of their respective theorems, ordered by length (more precisely: by lexicographical order) of the first proofs of these (equational) Theorems, here:

\[
\begin{align*}
\varphi^{\text{PR}}_k : \mathbb{N} \rightarrow \text{PR} \times \text{PR} \subseteq \mathbb{N}^2 \quad \text{and} \quad \varphi^{\text{PR}_A}_k : \mathbb{N} \rightarrow (\text{PR}_A)^2 \subseteq \mathbb{N}^2
\end{align*}
\]

respectively.

By the PR Representation Theorem 5.3 of ROMÀN 1989, these enumerations are represented by their internal versions

\[
\begin{align*}
\varphi_k^{\text{PR}} : \mathbb{N} \rightarrow \text{PR} \subset \mathbb{N}^2 \quad \text{and} \quad \varphi_k^{\text{PR}_A} : \mathbb{N} \rightarrow (\text{PR}_A)^2 \subset \mathbb{N}^2
\end{align*}
\]

PR = \( \{ \mathbb{N} | \text{PR} \} \) is the predicative, PR decidable subset of \( \mathbb{N} \) “of all PR codes” (a \( \text{PR}_A \)-Object), internalisation of PR \( \subset \mathbb{N} \) of all PR-terms on Object Language level. Analogous meaning for internalisation \( \text{PR}_A \subset \mathbb{N} \) of \( \text{PR}_A \subset \mathbb{N} \).

For discussion of “constructive” evaluation, we use representation of all \( \text{PR}_A \) maps within one PR endo map monoid, namely within \( \text{PR}(X\bot, X\bot) \), where \( X \subset \mathbb{N} \), \( X = \{ \mathbb{N} | X : \mathbb{N} \rightarrow 2 \} \) is the (predicative) Universal Object of \( \mathbb{N} \)-singletons \( \{ \langle n \rangle | n \in \mathbb{N} \} \), possibly nested \( \mathbb{N} \)-pairs \( \{ \langle a; b \rangle | a, b \in X \} \).

\[
X\bot = \text{def} \quad X \cup \{ \bot \} : \mathbb{N} \rightarrow 2
\]

is augmentation by symbol (code) \( \bot \) for taking care of defined undefined values of defined partial maps.

We define, within endo map set \( \text{PR}(\mathbb{N}, \mathbb{N}) \) a subTheory \( \text{PR}X \) externally PR, by mimikry of schema (abstr) for the special case of predicate \( X = X(a) : \mathbb{N} \rightarrow \mathbb{N} \), but without introduction of a coarser notion of equality, as in case of schema of abstraction constituting Theory \( \text{PR}_A = \text{PR} + (\text{abstr}) \), see RCF 1:

Objects of \( \text{PR}X \) are predicates \( \chi \subset X : \mathbb{N} \rightarrow 2 \), \( \text{PR}X \)-maps in \( \text{PR}X(\chi, \psi) \) are PR-maps \( f : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \chi(a) \iff \psi \circ f(a) : \mathbb{N} \rightarrow 2 \), cf. schema
(abstr) constituting basic Theory $\text{PR}_A$ over fundamental Theory $\text{PR}$ of Primitive Recursion, but here with $\nabla X(a) \Rightarrow f(a) \triangleq \perp$ : irregular elements go to $\perp \in X \perp \subset N$, so $\text{PR}_X \subset \text{PR}(N,N)$.

This gives a canonical embedding (Functor)

$$ I : \text{PR} \xrightarrow{\cong} I[\text{PR}] \subset \text{PR}_X \subset \text{PR}(N,N) $$

which extends into an embedding Functor

$$ I : \text{PR}_A = \text{PR}_A/\perp \xrightarrow{\cong} \text{PR}_A^x = \text{def} I[\text{PR}] \subset \text{PR}_X \subset \text{PR}(N,N), $$

converting $\text{PR}_A$’s notion of equality into that of $\text{PR}$. In this condensed version, we identify Theory $\text{PR}_A$ with the above Theory $\text{PR}_A^x$ which inherits the “one” notion of equality $\equiv_{\text{PR}}$ from Theory $\text{PR}$. Idem for the (predicative) internal versions $\text{PR}$, $\text{PR}_A$, $\text{PR}_A^x \subset N$. So this way, we don’t have to take care on the formally coarser notion of $\text{PR}_A$-equality between $\text{PR}_A$-maps $f,g : \{A | \chi \} \rightarrow \{B | \psi \}$.$^2$

With these formal ingredients we state (wanted) Double Recursive Characterisation of evaluation algorithm to be constructed later as “on-terminating” iteration of a suitable evaluation step. (In a set-theoretical framework, on-termination would be equivalent to termination = total definedness.)

Double Recursive Characterisation of Evaluation Algorithm

$$ \varepsilon = \varepsilon(u,a) : \text{PR}_A^x \times X \perp \rightarrow X \perp $$

to evaluate all map codes in $\text{PR}_A$ on all arguments of – free variable on – Universal Object $X$.

The (wanted) characterisation is the following:

- case of basic map constants $\text{bas} : A \rightarrow B$, namely $\text{bas}$ one of $0 : 1 \rightarrow N$,
  $s : N \rightarrow N$, $\text{id}_A : A \rightarrow A$, $\Delta_A : A \rightarrow A \times A$, $\Theta_{A,B} : A \times B \rightarrow B \times A$,
  $\ell_{A,B} : A \times B \rightarrow A$, and $r_{A,B} : A \times B \rightarrow B$, $A,B$ Objects of $\text{PR}_A$, identified
  with the corresponding ones of $\text{PR}_A^x$:
  $$ \varepsilon(\text{bas}, a) = \text{bas}(a) : X \sqsubseteq A \rightarrow B \sqsubseteq X, $$

- (recursive) case of evaluation of internally composed
  $$ \langle v \circ u \rangle = \text{by def} \langle v \circ_{\text{PR}_A} u \rangle = \langle v \circ_{\text{PR}_A^x} u \rangle , \text{ for} $$
  $$ u \in \{A,B\}, \ v \in \{B,C\} \subset \text{PR}_A^x \cong \text{PR}_A, \ a \in A \subset X : $$

  Characterisation in this composition case is (is wanted):
  $$ \varepsilon((v \circ u), a) = \varepsilon(v, \varepsilon(u,a)) = \varepsilon \circ (v, \varepsilon (u,a)) \rightarrow C \subset X. \quad (\odot) $$

$^2$see RCF1X for the details
- cylindrified \( (A \times v), v \in [B, B'], (a; b) \in (A \times B) : \)

\[
\varepsilon((A \times v), (a; b)) = (a; \varepsilon(v, b)) \rightarrow (A \times B') \subset X. \quad (\times)
\]

**evaluation in the cylindrified component.**

- internally iterated \( u^\gamma \), for \( u \in [A, A], (a; n) \in (A \times \mathbb{N}) \subset X : \)

\[
\begin{align*}
\varepsilon(u^\gamma, (a; 0)) &= a \in A \quad \text{(iteration anchor)} \\
\varepsilon(u^\gamma, (a; s n)) &= \varepsilon(u, \varepsilon(u^\gamma, (a; n))) \\
&= \varepsilon(\varepsilon(u, \varepsilon(\varepsilon(u^\gamma, (a; n)))) : \quad \text{(iteration step)} \\
&= \varepsilon(u^\gamma, \varepsilon(u, a)) \quad \text{(bottom up iteration step)} \\
&\rightarrow A \subset X.
\end{align*}
\]

**Recursive Legitimacy** for “definition” above of evaluation \( \varepsilon \) is obvious for all cases above, except for second subcase of case of iterated, since in the other cases recursive reference is made (only) to map terms of lesser depth. In case of an iterated, reference is made to a term with equal depth, but with decreased iteration counter.

This shows double recursive, (intuitive) legitimacy of our “definition”, more precisely: (double recursive) description of formally partial evaluation

\[
\begin{array}{ccc}
PR_A \times X & \xrightarrow{\varepsilon} & X \\
\downarrow \cong \times \subset & \cong & \subset \\
PR^X_A \times X_\bot & \xrightarrow{\varepsilon} & X_\bot
\end{array}
\]

A possible such (formally partial) map(-pair) \( \varepsilon \) is characterised by the above general recursive equation system. This system constitutes a definition by a (nested) double recursion à la ACKERMANN, legitimate e.g. in set theory.

We now resolve basic evaluation \( \varepsilon \), to be characterised – in case of termination – by the above double recursion, into a definition as an iteration of a suitable evaluation step

\[
e = \varepsilon(u, x) : PR_A \times X_\bot \rightarrow PR_A \times X_\bot.
\]

In fact resolution into a Complexity Controlled Iteration, CCI, which is to give, upon reaching complexity 0, evaluation result \( \varepsilon(u, a) \in X \) in its right component.

For discussion of termination of this (content driven) iteration, we consider

**Complexity Controlled Iterations** in general: Such a CCI is given – in Theory \( PR_A \) by data a (“predecessor”) step \( p : A \rightarrow A \) coming with a
complexity \( c : A \rightarrow O \), such that \( \text{PR}_A \vdash \text{DeSta}[p \mid c](a) : A \rightarrow 2 \), where

\[
\text{DeSta}[p \mid c](a) = \text{def } [c(a) > 0 \implies p(c(a) < c(a))]
\]

\( (\text{strict Descent above complexity zero}) \)

\( \land [c(a) \doteq 0 \implies p(a) \doteq_A a] \)  

\( (\text{Stationarity at complexity zero}). \)

\( O \) is an \emph{Ordinal}, here an extension of the semiring \( \mathbb{N}[\omega] \) of polynomials in one indeterminate, with lexicographical order.

Abbreviating predicate \( \text{DeSta}[p \mid c](a) : A \rightarrow 2 \) given, “positive” \textbf{axiom} schema \( (\pi_O) \), of all CCI\( _O \)'s to \emph{on-terminate} – whose equivalent \emph{contraposition} is schema \( (\bar{\pi}_O) \) of \emph{non-infinit descent} of the CCI\( _O \)'s –, reads:

\[
c : A \rightarrow O, \ p : A \rightarrow A \ \text{\textbf{PR}_A \ maps}
\]

\( \text{PR}_A \vdash \text{DeSta}[p \mid c](a) : A \rightarrow 2 \) (see above);

furthermore: for \( \chi : A \rightarrow 2 \) “test” predicate, in \( \text{PR}_A \):

\[\text{test on reaching } 0_O \text{ by chain } p^n(a) : \]

\( \text{PR}_A \vdash \text{TerC}[p, c, \chi] = \text{TerC}[p, c, \chi](a, n) : A \times \mathbb{N} \rightarrow 2, \]

\( = \text{def } [c p^n(a) \doteq 0 \implies \chi(a)] : A \times \mathbb{N} \rightarrow 2 \)

\( (\text{Termination Comparison condition}), \)

\( \text{with quantifier decoration:} \)

\[\frac{(\pi_O)}{\text{PR}_A \vdash (\forall a) [(\exists n) c p^n(a) \doteq 0_O \implies \chi(a)]} \]

\( \pi_O \vdash \chi : A \rightarrow 2, \ \text{i.e. } \chi = \pi_O \text{ true}_A : A \rightarrow 2. \)

It is important to note in context of \emph{evaluation} – that “emerging” Theory \( \pi_O \text{R} \) has same \textit{language} as basic PR Theory \( \text{PR}_A \). It just adds equations \textit{forced} by the additional schema. \textit{Axis case} is \( O := \mathbb{N}[\omega], \ (\pi) = \text{def } (\pi_{\mathbb{N}[\omega]}), \ \pi \text{R} = \text{def } \text{PR}_A + (\pi). \) Theory \( \pi \text{R}_O \) would be just Theory \( \text{PR}_A \). Isomorphic translation, with start \( \text{PR}_A^\times \cong \text{PR}_A : \pi_O^\times \text{R} = \text{PR}_A^\times + (\pi_O^\times) \cong \pi_O \text{R}. \)

\textbf{Characterisation Theorem} for CCI\( _O \)'s: Let \emph{complexity} \( c = c(a) : A \rightarrow O \) and \emph{predecessor} \( p = p(a) : A \rightarrow A \) be given, as in the antecedent of \( (\pi_O) \) above.

Then (formally partial) \( \bar{\text{PR}}_A \) map

\[f(a) = p^\bar{\circ} (a, \mu [c \mid p] \bar{\circ} a) : A \rightarrow A \times \mathbb{N} \rightarrow A \]

is nothing else then the \( \bar{\text{PR}}_A \) map \( (\text{while loop}) f = \text{wh}[c > 0_O \mid p] : A \rightarrow A, \)

and we “name” it \( \text{wh}_O[c \mid p] : A \rightarrow A. \)

[In terms of these \( \text{while} \) loops, equivalently: \emph{formally partial} PR maps, schema \( (\pi_O \text{R}) \) says map theoretically: \emph{Defined-arguments} enumeration of the CCI\( _O \)'s \emph{have image predicates}, and these predicative images equal \emph{true}, on the common \emph{Domain}, \( A, \) of the given step and complexity. By \textbf{definition}, this means that these enumerations are \emph{onto}, become so by axiomatic; and by this, all]
CCI\textsubscript{O}'s on-terminate. In our context – use equality definability – this is equivalent with epi property of the defined-arguments enumerations of the CCI\textsubscript{O}'s – but not with these enumerations to be retractions.]

“Following” the characterisation goal for PR\textsubscript{A}-map – formally partial PR\textsubscript{A}-map
\[ \varepsilon = \varepsilon(u, a) : PR \times X \rightarrow X \]
we can construct\[ the following PR\textsubscript{A} maps: fundamental evaluation step
\[ e = e^{PR}(u, x) : PR \times X \rightarrow PR \times X, \]
with fundamental complexity map \[ c = c^{PR} : PR \rightarrow N[\omega], \]
and their extension into basic evaluation step and complexity
\[ e = e^{PR} : PR \times X \rightarrow PR \times X \]

Note that this complexity \[ c = c^{PR} \] bears only on the map code part of pair \( (u, a) \in PR \times X \).
Partial evaluation map \( \varepsilon \) then is defined by iteration of these respective evaluation steps, descending in their respective complexities.

The (endo) evaluation step
\[ e = e(u, x) = (e_{map}(u, x), e_{arg}(u, x)) : PR \times X \rightarrow PR \times X \]
has left component \( e_{map}(u, x) : PR \times X \rightarrow PR \) designating the by-one-step evaluated, reduced map code, and right component \( e_{arg}(u, x) : PR \times X \rightarrow X \) which designates the by-one-step (in part) evaluated argument.

Central, (recursive) composition case for definition of step \( e \) is – essentially, for \( u \neq \text{id} \)
\[ e(\langle v \circ u \rangle, a) = (e_{map}(\langle v \circ u \rangle, a), e_{arg}(\langle v \circ u \rangle, a)), \]
with \[ e_{map}(\langle v \circ u \rangle, a) = e_{map}(u, a), \]
\[ e_{arg}(\langle v \circ u \rangle, a) = e_{arg}(u, a) : \]
begin with evaluation of first factor \( u \) on argument \( a \).

The only “tricky” case in definition of step \( e \) and complexity \( c \) is the one of an iterated \( (u \upharpoonright \subseteq, \langle a; s n \rangle) \in ([A \times N]_1, A) \times (A \times N) : \) Here we define:
\[ e(u \upharpoonright \subseteq, \langle a; n \rangle) = (u^{[n]}, a), \]
where, by PR definition \( u^{[0]} = \text{id}_A \in PR, \)
\[ u^{[n]} = (u^{[n]} \circ u) \in PR, \] is code expansion “at run time”.\[ ^3 \]explicitely in RCF2d
This latter case of definition by code expansion, is not very "effective", but logically simple.

The only "tricky" case for complexity – in the other case definition is (recursively) using structural depth of map code \( u \), – equally comes in with the case of an iterated code \( u^{\mathfrak{g}^{-}} \), as follows:

\[
c(u^{\mathfrak{g}^{-}}) = \text{def} \ c(u) \cdot \omega^1 + 1 \cdot \omega^1 = (c(u) + 1) \cdot \omega^1.
\]

(Polynomial) multiplication with \( \omega^1 \) in this iteration case is to take into account the a priori unrestricted, perhaps "big" iteration exponent \( n \) to be "substituted" at a later evaluation step for \( u^{\mathfrak{g}^{-}} \). It guarantees descent of order value with evaluation step \( e \) in case \( (u^{\mathfrak{g}^{-}}, \langle a; n+1 \rangle) \), and so gives the

**Basic Descent Lemma:** For \((u, a) \in \text{PR}_A \times X_\bot\) free with \( c(u) > 0 \), i.e. with \( u \neq \mathfrak{id}^{-} \):

\[
c_{\varepsilon}(u, x) < c_{\varepsilon}(u, x) = \text{by def} \ c(u) \in \mathbb{N}[\omega],
\]

with respect to the canonical, “lexicographic”, and – intuitively – finite-descent order of the polynomial semiring \( \mathbb{N}[\omega] \).

The Descent Lemma makes plausible global termination of the (\( \mu \)-recursive) version of evaluation \( \varepsilon = \varepsilon(u, x) : \text{PR}_A^X \times X_\bot \rightarrow X_\bot \), in a suitable framework, here: it proves that this basic (formally partial evaluation map out of \( \text{PR}_A^\mathfrak{X} : \varepsilon = \varepsilon(u, x) : \text{PR}_A^X \times X_\bot \rightarrow \text{PR}_A \times X_\bot \rightarrow X_\bot \)

on-terminates within Theory \( \pi_O \mathcal{R} = \text{PR}_A + (\pi_O \mathcal{R}) \), for Ordinal \( O \geq \mathbb{N}[\omega] \). This means that evaluation \( \varepsilon \) has an onto, epi defined arguments enumeration

\[
d_{\varepsilon} = d_{\varepsilon}(n, (u, x)) = \text{def} \ (u, x) :
\]

\[
D_{\varepsilon} = \{ (m, (u, x)) | c \in \varepsilon^n(u, a) = 0 \} \rightarrow \text{PR}_A^\mathfrak{X} \times X_\bot
\]

within \( \pi \mathcal{R} \), and a fortiori in \( \pi_O \mathcal{R} \), Ordinal \( O \geq \mathbb{N}[\omega] \), such choice of \( O \) taken always here.

**Remark:** Even if intuitively terminating, and derivably on-terminating, partial map \( \varepsilon \) does not constitute a self-evaluation of Theory \( \pi \mathcal{R} = \text{PR}_A + (\pi) \), “Dangerous bound” in Summary above. Nothing is said (above) on evaluation of Theory \( \pi_O \widehat{\mathcal{R}} = \widehat{\pi_O \mathcal{R}} \).

In present context, we need an “explicit”

Free-Variable Termination Condition, in particular for our basic evaluation \( \varepsilon \), and later for its extension, \( \varepsilon_d \), into an evaluation for argumented deduction trees.

We define dominated termination of \( \varepsilon \) by \( m \in \mathbb{N} \), as

\[
\text{PR}_A \vdash [ m \text{ def } \varepsilon(u, x) ] = [ c \in \varepsilon^m(u, x) = 0 \land \varepsilon(u, x) = r \varepsilon^m(u, x) ] : \mathbb{N} \times (\text{PR}_A^\mathfrak{X} \times X_\bot) \rightarrow 2.
\]
We will use this given termination counter “\( m \text{ def } \ldots \)” only as a (termination) condition (!), in implications of form \( m \text{ def } \varphi_\Omega(a) \implies \chi(a) \), \( \chi = \chi(a) \) a termination conditioned predicate. And we will make assertions on formally partial maps such as evaluation \( \varepsilon \) and argumented deduction-tree evaluation \( \varepsilon_d \) below, mainly in this termination-conditioned form.

So the main stream of our story takes place in theory \( \text{PR}_A \) (not “really” within \( \text{PR}_A \)): we go back usually to the \( \text{PR}_A \)-building blocks of formally partial maps occurring, in particular to those of basic evaluation \( \varepsilon \) as well as those of tree evaluation \( \varepsilon_d \) to come.

**Iteration Domination** above, applied to the Double Recursive equations for \( \varepsilon = \varepsilon^{\text{PR}_A} \equiv \varepsilon^{\text{PR}_A} / \text{CG}_A \) makes out of these a \( \text{PR}_A \) Characterisation Theorem for evaluation \( \varepsilon \), the non-trivial statements being the following \( m \)-dominated, “truncated” ones:

\[
\text{PR}_A \vdash [ m \text{ def } \varepsilon(v \odot u, a) ] \implies \varepsilon(\langle v \odot u \rangle, a) = \varepsilon(v, \varepsilon(u, a)) \\
\wedge \varepsilon(u^\text{§}, \langle a; 0 \rangle) = e^1(u^\text{§}, \langle a; 0 \rangle) = a \\
\wedge [ m \text{ def } \varepsilon(u^\text{§}, \langle a; s n \rangle) ] \implies : \\
m \text{ defines all } \varepsilon \text{ instances below, and : } \\
\varepsilon(u^\text{§}, \langle a; s n \rangle) = \varepsilon(u^\text{§}, \langle \varepsilon(u, a); n \rangle) = \varepsilon(u, \varepsilon(u^\text{§}, \langle a; n \rangle)) : \\
\mathbb{N} \times ((\text{PR}_A \times \text{PR}_A \times \text{CG}_A^2) \times \mathbb{N}) \rightarrow 2, \\
m \in \mathbb{N}, u, v \in \text{PR}_A^x \subset \mathbb{N}, a, b \in \text{CG}_A \subset \mathbb{N}, n \in \mathbb{N} \text{ all free.}
\]

**Proof** of this Theorem by Primitive Recursion (Peano Induction) on \( m \in \mathbb{N} \) free, via (recursive) case distinction on the “structure” of code \( w \in \text{PR}_A^x \cong \text{PR}_A \), and arguments \( a \in X \) appearing in the different cases of the asserted conjunction.

**Objectivity Theorem:** “Basic” evaluation \( \varepsilon \) is Objective: for each single, (meta free) \( f : A \rightarrow B \) in Theory \( \text{PR}_A \) itself, we have

\[
\text{PR}_A, \pi_{\text{OR}} \vdash \varepsilon(\langle f^\text{§}, a \rangle) = f(a) : A \rightarrow B.
\]

**Proof** by first: External structural recursion on the nesting depth \( \text{depth}[f] \) (“bracket depth”) of \( \text{PR}_A \)-map \( f : A \rightarrow B \) in question, seen as external code: \( f \in \mathbb{N} \), and

second: in case of an iterated, \( g^s = g^h(a, n) : A \times \mathbb{N} \rightarrow A \), by PR-recursion on iteration count \( n \in \mathbb{N} \). This uses (termination dominated) Double Recursive Characterisation of evaluation \( \varepsilon \).

“Uniform” Evaluation above splits into an (externally) indexed evaluation family \( \varepsilon_{A,B} = \varepsilon_{A,B}(u, a) : [A, B] \times A \rightarrow B \) in \( \text{PR}_A \) with

– in particular – Objectivity property above.
Central for all what follows is the (Inner) Soundness Problem

for evaluation \(\varepsilon = \varepsilon(u, a) : \text{PR}_A^X \times X \rightarrow X\), namely:

Is there a “suitable” Condition \(\Gamma = \Gamma(k, (u, v)) : N \times (\text{PR}_A^X)^2 \rightarrow 2\), under which Theory \(\text{PR}_A\) exports internal equality \(u \equiv_k v\) into Objective, predicative equality \(\varepsilon(u, a) \equiv \varepsilon(v, a)\)?

Such (“suitably conditioned”) evaluation Soundness is strongly expected, and derivable \textit{without condition} in classical Recursion Theory (and \textit{set} theory) – the latter two in the rôle of frame theory \(\text{PR}_A\) above.

The formal problem here lies in \textit{termination}. Problem and proposed Solution are “isomorphic” for theories \(\text{PR}_A, \pi_R = \text{PR}_A + (\pi_O)\) on one hand, and for theories \(\text{PR}_A^X, \pi_R^X = \text{PR}_A^X + (\pi_O^X)\) on the other.

Here is a list of headers of subsequent sections, all ready for distribution by email, except the last one, “Discussion”: I would be happy to get ERROR messages, ideas to correct these possible errors as well as comments in general, and last but not least hints to the “work of other people” (Jim Łambek): I am not aware of the contemporary litterature, as you can see in my – momentaneous – list of References.

3 Deduction Trees and Their Top Down \textit{Argumentation}

4 Evaluation Step on Map-Code/Argument Trees

5 Termination-Conditioned Soundness

6 (Unconditioned) Objective Soundness

7 Discussion

Coda: Termination Conditioned Soundness for Theory \(\text{PR}_A\)
References

J. Barwise ed. 1977: Handbook of Mathematical Logic. North Holland.

H.-B. Brinkmann, D. Puppe 1969: Abelsche und exakte Kategorien, Korrespondenzen. L.N. in Math. 96. Springer.

S. Eilenberg, C. C. Elgot 1970: Recursiveness. Academic Press.

S. Eilenberg, G. M. Kelly 1966: Closed Categories. Proc. Conf. on Categorical Algebra, La Jolla 1965, pp. 421-562. Springer.

G. Frege 1879: Begriffsschrift. Reprint in “Begriffsschrift und andere Aufsätze”, Zweite Auflage 1971, I. Angelelli editor. Georg Olms Verlag Hildesheim, New York.

P. J. Freyd 1972: Aspects of Topoi. Bull. Australian Math. Soc. 7, 1-76.

K. Gödel 1931: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. Monatsh. der Mathematik und Physik 38, 173-198.

R. L. Goodstein 1971: Development of Mathematical Logic, ch. 7: Free-Variable Arithmetics. Logos Press.

F. Hausdorff 1908: Grundzüge einer Theorie der geordneten Mengen. Math. Ann. 65, 435-505.

D. Hilbert: Mathematische Probleme. Vortrag Paris 1900. Gesammelte Abhandlungen. Springer 1970.

P. T. Johnstone 1977: Topos Theory. Academic Press

A. Joyal 1973: Arithmetical Universes. Talk at Oberwolfach.

J. Lambek, P. J. Scott 1986: Introduction to Higher order categorical logic. Cambridge University Press.

F. W. Lawvere 1964: An Elementary Theory of the Category of Sets. Proc. Nat. Acad. Sc. USA 51, 1506-1510.

S. Mac Lane 1972: Categories for the working mathematician. Springer.

B. Pareigis 1969: Kategorien und Funktoren. Teubner.

R. Péter 1967: Recursive Functions. Academic Press.

M. Pfender 1974: Universal Algebra in S-Monoidal Categories. Algebra-Berichte Nr. 20, Mathematisches Institut der Universität München. Verlag Uni-Druck München.

M. Pfender 2008 RCF1, RCF1d: Recursive Categorical Foundations part 1: Theories of PR Maps and Partial PR Maps. detailed as well as condensed, pdf files, TU Berlin.

M. Pfender 2008 RCF1X: Universal Object and Theory Embedding. pdf file. TU Berlin.

M. Pfender, M. Kröplin, D. Pape 1994: Primitive Recursion, Equality, and a Universal Set. Math. Struct. in Comp. Sc. 4, 295-313.

W. Rautenberg 1995/2006: A Concise Introduction to Mathematical Logic. Universitext Springer 2006.
R. Reiter 1980: Mengentheoretische Konstruktionen in arithmetischen Universen. Diploma Thesis. Techn. Univ. Berlin.

L. Román 1989: Cartesian categories with natural numbers object. J. Pure and Appl. Alg. 58, 267-278.

C. Smoryński 1977: The Incompleteness Theorems. Part D.1 in Barwise ed. 1977.

A. Tarski, S. Givant 1987: A formalization of set theory without variables. AMS Coll. Publ. vol. 41.

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