ON SOURCE-TYPE SOLUTIONS AND THE CAUCHY PROBLEM FOR A DOUBLY DEGENERATE SIXTH-ORDER THIN FILM EQUATION I. LOCAL OSCILLATORY PROPERTIES

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Abstract. As a key example, the sixth-order doubly degenerate parabolic equation from thin film theory,

$$u_t = (|u|^m |u_{xxxxx}|^n u_{xxxxx})_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

with two parameters, $n \geq 0$ and $m \in (-n, n + 2)$, is considered. In this first part of the research, various local properties of its particular travelling wave and source type solutions are studied. Most complete analytic results on oscillatory structures of these solutions of changing sign are obtained for $m = 1$ by an algebraic-geometric approach, with extension by continuity for $m \approx 1$.

1. Introduction: basic nonlinear model with complicated local and global properties of solutions

1.1. Higher-order degenerate parabolic PDEs: no potential, monotone, order-preserving properties, not of divergence form, and no weak solutions. We consider the sixth-order parabolic equation from thin film theory,

$$u_t = (|u|^m |u_{xxxxx}|^n u_{xxxxx})_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

with two parameters, $n \geq 0$ and $m \in (-n, n + 2)$. This equation is doubly degenerate and contains the higher-order $p$-Laplacian nonlinearity $|u_{xxxxx}|^n$ (then $p = 2 + n$), and another one $|u|^m$ of the porous medium type. The equation is written for solutions of changing sign, which is an intrinsic feature of the Cauchy problem (the CP) with bounded compactly supported initial data $u_0(x)$ to be studied.

The PDEs such as (1.1), which are called sixth-order thin film equations (the TFEs–6), were introduced by King [16] in 2001 among others for modelling of power-law fluids spreading on a horizontal substrate. Equation (1.1) is quasilinear, where the diffusion-like operator includes two parameters $m$ and $n$ and is not potential (variational) and/or monotone in any functional setting and topology. It is not also an operator of fully divergence form, since the PDE admits just a single integration by parts, so that a standard definition of weak solutions is entirely illusive. Of course, as a higher-order parabolic equation, (1.1) does not exhibit any order-preserving (via the Maximum Principle) features.

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Overall, the TFE–6 \((1.1)\), as a typical example of a variety of complicated nonlinear thin film models arising from modern applications, represents a serious challenge to general PDE theory of the twenty-first century, concerning principles and concepts of understanding the common local and global features and properties of its solutions, which also need proper definitions. For other sixth-order TFEs including their derivation and mathematical properties, see a survey in [8], where further key references are traced out.

It is well-known that nonnegative solutions of a wide class of higher-order TFEs can be obtained by special non-analytic and often “singular” \(\varepsilon\)-regularizations and passing to the limit \(\varepsilon \to 0^+\), that, in general, lead to free-boundary problems (FBPs). We refer to the pioneering work by Bernis and Friedman [3] and to the monograph on nonlinear parabolic PDEs [22, Ch. 4], where further references and results can be found. Actually, such singular \(\varepsilon\)-regularizations, as \(\varepsilon \to 0^+\), pose a kind of an “obstacle FBP”, where the solutions are obliged to be non-negative by special free-boundary conditions that, in general, are not easy to detect rigorously. Solutions of the CP cannot be obtained by such techniques and require more involved and different analysis. In particular, analytic \(\varepsilon\)-regularizations, with \(\varepsilon \to 0^+\), can be key for the CP, [7, 8].

There is a large amount of pure, applied, and numerical mathematical literature devoted to existence, uniqueness, and various local and asymptotic properties of TFEs, especially, for the standard TFE–4:

\[
(1.2) \quad u_t = -(|u|^nu_{xxx})_x, \quad \text{where } n > 0.
\]

Necessary key references on various results of modern TFE theory that are important for justifying principal regularity and other assumptions on solutions will be presented below and, in particular, are available in [14,15] and in a more recent paper [11]. See also [13, Ch. 3], where further references are given and several evolution properties of TFEs (with absorption, included) are discussed. However, even for simpler pure TFEs such as [12], questions of local and global properties of solutions of the CP, their oscillatory, and asymptotic behaviour are not completely well understood or proved in view of growing complexity of mathematics corresponding to higher-order degenerate parabolic flows.

Using this, rather complicated, and even exotic, model equation \((1.1)\), we plan to explain typical and unavoidable difficulties that appear even in the study of local properties of compactly supported solutions and their interfaces for higher-order degenerate nonlinear PDEs. We then intent to give insight and develop some general approaches, notions, and techniques, that are adequate and can be applied to a wide class of difficult degenerate parabolic (and not only parabolic) PDEs. Since even the related ODEs of the fifth order for particular solutions get very complicated with a higher-dimensional phase space, we cannot rely on traditional ODE methods, which were very successful in the twentieth century for second-order ODEs, occurred for many particular self-similar and other solutions, with clear phase-planes.

Instead, as a general idea, we propose to use parameter homotopy-continuity approaches using the fact that for some values of \(m, n\) (e.g., for \(m = 1\) or \(m = n = 0\), etc.), the ODEs can be solved by some algebraic-geometric methods, or leads to easier linear equations. Then, we use a stable “transversality-geometric” structure of the obtained solutions to
extend those into some surrounding parameter ranges. However, global extension of those solutions are not straightforward at all and often we are obliged to apply careful numerical methods to trace out some solution properties and their actual existence.

Therefore, we are not restricted to quasilinear equations with semi-divergent operators. Without essential changes and hesitation, we may consider other fully nonlinear models such as the following formal parabolic PDE:

\[(1.3) \quad |u_x|^\sigma u_t = (|u|^m |u_{xxxx}|^n u_{xxxx})_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \quad (\sigma > 0),\]

where \(\sigma = 0\) leads to the quasilinear counterpart (1.1). It is easy to propose other more artificial (and often awkward) versions of such PDEs without traces or remnants of monotone, potential, divergence form, etc., operators, which however can be used for applying our general mathematical concepts of analysis.

1.2. On other models, results, and extensions. In the present first part, in Sections 2–7 we present a detailed study of some, mainly local oscillatory and sign-changing, properties of travelling wave (TW) and source-type solutions. In this connection, let us mention the first important pioneering results on oscillatory source-type solutions of the fourth-order quasilinear parabolic equation of porous medium type (the PME–4) in the fully divergence form with the monotone operator in \(H^{-2}\):

\[(1.4) \quad u_t = -(|u|^{m-1} u)_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \quad (m > 1),\]

which were obtained by Bernis [1] and in Bernis–McLeod [4]; see also [22, § 4.2] for further details, and [9] for construction of a countable family of similarity solutions of (1.4).

Notice that many key features of a local oscillatory structure of solutions of quasilinear degenerate PDEs with such operators do not essentially change not only for equations of higher, \(2m\)th-order, but also for similar odd-order nonlinear dispersion equations (NDEs). In [5], as an illustration, we briefly review our approaches for the corresponding fifth-order counterpart of (1.1), which has the form (the NDE–5)

\[(1.5) \quad u_t = (|u|^m |u_{xxxx}|^n u_{xxxx})_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,\]

Various odd-order PDEs occur in nonlinear dispersion theory. As a key feature, they exhibit finite interfaces, compacton behaviour, and shock/rarefaction waves; see first results in Rosenau–Hyman [19], a survey in [13, Ch. 4], and [10], as a more recent reference for shock wave behaviour for NDEs–5. The counterpart of (1.5) for \(m = n = 0\) is the linear dispersion PDE

\[(1.6) \quad u_t = u_{xxxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,\]

which, by natural continuity-homotopy issues, gives some clues about oscillatory properties of solutions for the nonlinear one (1.5).

It is curious that the performed local analysis of oscillatory solutions close to finite interfaces remains similar for a number of other PDEs. For instance, in this sense, the parabolic equation (1.1) has the counterpart which is a rather unusual PDE also belonging to the class of nonlinear dispersion equations

\[(1.7) \quad u_{tt} = (|u|^m |u_{xxxx}|^n u_{xxxx})_{xx}.\]
In its turn, a “local counterpart” of the nonlinear dispersion PDE (1.5) is the quasilinear sixth-order hyperbolic equation

\[
\frac{\partial^2 u}{\partial t^2} = \left( |u|^m |u_{xxxx}|^n u_{xxxx} \right)_{xx}.
\]

The second part of our paper [5] is devoted to a global construction of source-type solutions and some general aspects concerning the Cauchy problem for (1.1). As a first general rule therein, we show that, for correct understanding the CP and some principal properties of its solutions, one should study the limit \(m, n \to 0^+\), where the linear triharmonic equation occurs

\[
\frac{\partial u}{\partial t} = u_{xxxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+.
\]

One can easily construct for (1.9) the necessary TW and source-type (ZKB) solutions, to be compared with solutions of its nonlinear counterpart, the TFE (1.1), at least for sufficiently small \(m\) and \(n\). As a principal feature, we observe that, for the CP, the solutions of both linear and nonlinear PDEs are oscillatory near finite interfaces (for (1.9), by implication, interfaces are at infinity, \(x = \pm \infty\)).

In higher-order quasilinear degenerate evolution PDEs of parabolic, nonlinear dispersion, or hyperbolic types, the identification of optimal local regularity properties of the FBP and the Cauchy problem becomes very difficult, to say nothing about general existence-uniqueness theory. Actually, we were surprised to observe that many classic techniques of PDE theory developed in the last fifty or so years not only cannot be applied but also cannot provide us with necessary results in principle, since these are strictly oriented to lower-order PDEs.

Evidently, related challenging difficulties appear even when constructing standard particular self-similar solutions leading to higher-order ODEs. These cannot be studied by phase-plane analysis which is usual for the first or second-order equations. Moreover, several results, which have been obtained for given third or fourth-order PDEs by a complicated study of the topology of orbits, are often hardly applied to similar ODEs of the order that is higher by one. Many mathematical approaches are strictly attached to ODEs of the given and sufficiently low order.

In this paper, we have specially chosen sixth- or fifth-order models (1.1) and (1.5) with complicated non-monotone and non-potential operators to describe a certain general scheme to study local and global properties of solutions including:

(I) existence and uniqueness of travelling wave solutions, and
(II) local oscillatory behaviour of solutions of changing sign near interfaces.

In a forthcoming paper [5], we continue this study of (1.1) and will concentrate upon:

(III) existence and uniqueness of source-type (ZKB-type) solutions, and
(IV) general properties of solutions and proper setting of the Cauchy problem.

2. TRAVELLING WAVE AND SOURCE-TYPE SIMILARITY SOLUTIONS

Let us now introduce two classes of particular solutions to be studied in detail.
2.1. **Travelling waves.** These are simplest solutions of nonlinear PDEs of the form

\[(2.1) \quad u_{TW}(x,t) = f(y), \quad y = x - \lambda t \quad (\lambda \in \mathbb{R}),\]

where, on substitution into (1.1), \(f\) satisfies the autonomous ODE

\[(2.2) \quad A(f) \equiv |f|^m |f^{(5)}|^n f^{(5)} = -\lambda f.\]

This is obtained on integration once with a zero constant via the required flux continuity at the interfaces, where \(f = 0\).

For \(n = m = 0\), the ODE (2.2) is easy, with a linear bundle of exponential solutions:

\[(2.3) \quad f^{(5)} = -\lambda f \quad \Rightarrow \quad f(y) = e^{\mu y}, \quad \text{where} \quad \mu^5 = -\lambda.\]

2.2. **Source-type solutions.** Equation (1.1) admits the standard self-similar solutions

\[(2.4) \quad u_S(x,t) = t^{-\beta} F(y), \quad y = x/t^\beta, \quad \text{where} \quad \beta = \frac{1}{6+(m+n)},\]

so (2.4) preserves the total mass of the solution:

\[\int_{\mathbb{R}} u_S(x,t) \, dx = \text{const.} \quad \text{for all} \quad t \geq 0.\]

Then \(F\) solves a slightly different ODE again obtained on integration once with the zero constant:

\[(2.5) \quad A(F) + \beta F y \equiv |F|^m |F^{(5)}|^n F^{(5)} + \beta F y = 0 \quad \text{in} \quad \mathbb{R}.\]

In view of the symmetry group of scalings, if \(F_1(y)\) is a solution of (2.5), then

\[(2.6) \quad F_a(y) = a^\gamma F_1(\frac{y}{a}), \quad \text{with} \quad \gamma = \frac{6}{m+n},\]

is a solutions for any \(a > 0\). Therefore, we are looking for a unit mass profile satisfying

\[(2.7) \quad \int F(y) \, dy = 1.\]

As another normalization, one can take the condition \(F(0) = 1\).

**Remark on finite propagation.** Finite propagation in the ODEs such as (2.2) and (2.5), written in a semilinear form (see below), i.e., existence of finite interface \(y_0\) of arbitrary sufficiently small solutions, was well-known for a long time. We refer to the first ODE proofs in [1, 4], [22, p. 392], and more involved energy estimates for general related higher-order elliptic and parabolic PDEs in [2, 20] and survey in [12]. However, we must admit that some extensions of energy methods to odd-order ODEs such as (2.2) or (2.5), i.e., for nonlinear dispersion operators involved, are not straightforward and can be technically rather involved.

Thus, the principal question remained open is the oscillatory and non-oscillatory behaviour of solutions close to interfaces.
2.3. **Fundamental solution for** \( n = m = 0 \). Then (2.4) is the fundamental solution

\[
(2.8) \quad b(x, t) = t^{-\frac{1}{6}} F_0(y), \quad y = x/t^{\frac{1}{6}},
\]

of the tri-harmonic equation (1.9). The corresponding linear problem for the rescaled kernel denoted by \( F_0 \) reads

\[
(2.9) \quad F_0^{(5)} + \frac{1}{6} F_0 y = 0 \quad \text{in} \quad \mathbb{R}, \quad \int F_0 = 1.
\]

This has the unique solution \( F_0(y) \) by classic linear theory, [6]; see [5] for extra properties of the kernel \( F_0(y) \). Recall that we intend to use the linear rescaled kernel \( F_0 \) given by (2.9) in trying to understand the nonlinear one (2.5), at least, for \( m, n \approx 0 \).

3. **Local existence of positive solutions and maximal regularity**

We now begin to study the behaviour of solutions of both ODEs (2.2) and (2.5) near the interface point. For (2.2), we just assume that the interface is at \( y = 0 \) and set \( f(y) \equiv 0 \) for \( y < 0 \). For (2.5), assuming that \( y_0 < 0 \) is the left-hand interface point of \( F \), we perform the change \( y - y_0 \mapsto y \) and obtain equation (2.2) with

\[
(3.1) \quad \lambda = \beta y_0 < 0,
\]

up to an exponentially small perturbation as \( y \to y_0^- \).

Thus, in both cases, we consider (2.2) in a neighbourhood of the interface at \( y = 0 \), where we scale out the parameter \( |\lambda| \neq 0 \), so that now \( \lambda = \pm 1 \):

\[
(3.2) \quad f^{(5)} = \mp |f|^{\alpha - 1} f \quad \text{for} \quad y > 0, \quad f(0) = 0 \quad (\alpha = \frac{1 - m}{1 + n} \in (-1, 1), \quad \lambda = \pm 1).
\]

Actually, the condition \( \alpha \in (-1, 1) \) holds in a wider parameter range

\[
(3.3) \quad n > -1, \quad m \in (-n, n + 2).
\]

The assumption \( \alpha > -1 \) \((m < n + 2)\) is purely technical that simplifies local analysis of \( f(y) \) close to “transversal” zeros and makes it quite standard. For \( \alpha \leq -1 \), some technicalities occur that are often not in the focus of the present study.

Recall that, in (3.2), both \( \lambda = \pm 1 \) can occur for the TWs, and, always, \( \lambda = -1 \) for the source-type solutions.

We should clarify what kind of solutions we are looking for, and, namely, which extra conditions are supposed to be posed at the interface \( y = 0 \) to create proper solutions of the CP. For instance, one can look for solutions \( f \in C^4((-\delta, \delta)) \), with a constant \( \delta > 0 \), so this demands

\[
(3.4) \quad f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 0,
\]

but it is not clear whether these correspond to the actual regularity associated with the CP. Bearing in mind more sophisticated ODEs that occur from models like (1.3), we would like to avoid any weak definitions of solutions that are based on integration by parts (for instance, for fully nonlinear operators, this makes no sense).
As a hint, we can compare the desired regularity for the CP with that for the standard zero height, zero-contact angle, zero curvature, and zero-flux FBP, for which the free-boundary conditions take the form

\[(3.5) \quad f(0) = f'(0) = f''(0) = 0 \quad \text{and} \quad (|f|^m|f^{(5)}|n f^{(5)})(0) = 0.\]

Therefore, at the interface \( y = 0 \), solutions of the FBP satisfy for any \( C > 0 \) and some \( C_1 = C_1(C) \in \mathbb{R} \),

\[(3.6) \quad f(y) = Cy^3 + C_1y^\gamma + \ldots \quad \text{for} \quad y \geq 0 \implies f \in C^2((−\delta, \delta)) \quad \text{(FBP)},
\]

which is true provided that

\[(3.7) \quad \gamma = \frac{8 + 5n - 3m - 1}{n+1} > 3 \implies m < \frac{5+2n}{3}.\]

Of course, (3.6) exhibits less regularity at \( y = 0 \) than (3.4). In fact, the solutions of the FBP do not need and/or admit the zero extension for \( y < 0 \) (by the definition, this makes no sense for the FBP). If they do, this already corresponds to the CP.

For the CP, we need to use the concept of the maximal regularity, which for the ODE (3.2) simply means that setting \( f(y) = 0 \) for \( y < 0 \) yields

\[(3.8) \quad \text{solutions that are maximally smooth at} \quad y = 0 \quad \text{admitted by the ODE} \quad \text{(CP)}.
\]

The actual maximal regularity associated with the ODE under consideration needs special local analysis close to \( y = 0^+ \). It is worth mentioning now that (3.4) are not conditions of maximal regularity in general, for arbitrary values of \( m \) and \( n \).

3.1. Existence, uniqueness, and nonexistence of positive TW solutions. We begin by noting that, for \( \lambda = −1 \), equation (3.2) admits the positive solution

\[(3.9) \quad f_0(y) = \varphi_0 y^\mu, \quad \text{where} \quad \mu = \frac{5}{1-\alpha} = \frac{5(n+1)}{m+n} \quad \text{and}
\]

\[(3.10) \quad \varphi_0 = \left[ \frac{1}{\mu(\mu-1)(\mu-2)(\mu-3)(\mu-4)} \right]^{\frac{1}{m+n+1}} (\lambda = -1).
\]

This formula makes sense in the intervals \( \mu \in (4, +\infty) \), (2, 3), and (0, 1), which give some relations between parameters \( m \) and \( n \). For \( \lambda = +1 \), two intervals \( \mu \in (1, 2) \) and (3, 4) are accepted. Note that the regularity of the solution (3.9) at \( y = 0 \) (with the trivial extension \( f(y) \equiv 0 \) for \( y < 0 \)) increases without bound as \( \alpha \to 1^- \), since

\[(3.11) \quad \mu = \frac{5}{1-\alpha} \to +\infty \quad \text{as} \quad \alpha \to 1^-.
\]

For instance, it follows that \( f_0 \in C^4 \) at the interface provided that

\[(3.12) \quad m < \frac{3+5}{4} \quad (\mu \in (4, +\infty)).\]

Below, we will explain the meaning of these positive solutions in the Cauchy problem.

Notice that, for \( \lambda = +1 \), in the most convenient interval (3.12), the ODE (3.2) does not admit solutions that are strictly positive in an arbitrarily small neighbourhood of the interface at \( y = 0 \).

We first consider the natural “smooth” version of solutions of the maximal regularity, where the conditions (3.4) hold, so that \( f_0(y) \) is such a solution in the interval (3.12). The following classification of possible solutions holds. By oscillatory solutions of (3.2) at
the origin, we mean those that have infinitely many sign changes (for instance, isolated transversal zeros) in any arbitrarily small neighbourhood \((0, \delta)\) of the interface at \(y = 0\).

**Proposition 3.1.** Let (3.12) hold. Then:

(i) for \(\lambda = +1\), any \(f(y) \neq 0\) for \(y \approx 0^+\) of the problem (3.2), (3.4) is oscillatory at \(y = 0^+\), and

(ii) for \(\lambda = -1\), there exists a unique local strictly monotone and positive solution of (3.2), which is given by (3.9), and all other solutions are oscillatory as \(y \to 0^+\).

**Remark: first comparison with the linear ODE.** The existence of a 1D manifold of positive solutions (with the interface parameter \(y_0\)) is in good agreement with the same conclusion for the linear equation (2.3), where the 1D manifold of positive exponentially decaying functions at the “infinite interface” as \(y \to -\infty\) is given by

\[ f(y) = C e^y, \quad C > 0. \]

The rest of decaying orbits are oscillatory as \(y \to -\infty\). We thus prove that a similar property persists for some \(m, n > 0\) in the interval (3.12).

**Proof of Proposition 3.1.** (i) We argue by contradiction. Assume that \(f(y)\) is a nontrivial solution of (3.2) with \(\lambda = +1\). Then, if \(f(y)\) is positive in an interval \((0, y^*)\), it follows from the equation (3.2) that \(f^{(5)} < 0\) for \(0 < y < y^*\). After integration and taking into account the conditions (3.4), we obtain that \(f'(y) < 0\) and \(f(y) < 0\) on \((0, y_*)\), whence comes the contradiction. Similarly, there is no a negative solution on \((0, y_*)\).

(ii) The key idea of the proof relies on the translation invariance property of the equation. For \(\lambda = -1\) and positive solutions, consider equation (3.2) in the form,

\[ f^{(5)} = |f|^{\alpha-1}f \quad \text{for} \quad y > 0 \quad (\alpha = \frac{1-m}{1+n} < 1). \]

As a first consequence, one can see after integrating this equation that any positive solution \(f(y)\) of the ODE problem (3.2) is also convex and strictly increasing.

We next prove that, for \(\lambda = +1\), the function \(f_0 = f_0(y)\) defined in (3.2)–(3.9), is the unique positive solution of the problem (3.2), (3.4). We begin by showing that if \(f_0(y)\) and \(f(y)\) are different positive solutions of (3.2), then either they are ordered, or they have mutually oscillatory behavior close to the origin (in the sense that they intersect each other infinitely many times at different values of \(y\) approaching the origin, i.e., the difference \(f_0(y) = f(y)\) is oscillatory at \(y = 0\)). Notice that, in any domain of analyticity of \(f(y) > 0\) (where \(f'(y) \neq 0\), all intersections with the analytic \(f_0(y)\) are isolated points.

We begin with the case \(\alpha > 0\). Assume that the assertion is false and that there exists \(y^* > 0\) such that \(f_0(y^*) = f(y^*)\). If \(f_0(y) < f(y)\) in this interval, it readily follows from the equation in the new form (3.13) and the regularity properties of the solutions that \(f_0^{(5)}(y) \leq f^{(5)}(y)\) for \(0 < y \leq y^*\). Hence, we obtain by integration in \((0, y^*)\) that \(f_0(y^*) < f(y^*)\), a contradiction with the assumption. The same argument applies to the opposite inequality and therefore, the assertion holds.
In order to establish the uniqueness result in both cases (ordered solutions or with mutually oscillatory behavior), we apply similar arguments to auxiliary solutions constructed by using the invariance translation properties of the equation. Assume for contradiction that there exists \( y_0 \) such that \( f_0(y) \neq f(y) \) with \( f_0(y_0) < f(y_0) \). Consider the auxiliary solution of (3.2), (3.4) defined as follows:

\[
g(y) = \begin{cases} 
  f(y - \varepsilon) & \text{for } y \geq \varepsilon, \\
  0 & \text{for } y < \varepsilon,
\end{cases}
\]

where \( \varepsilon > 0 \) is chosen small enough such that \( f_0(y_0) < g(y_0) \). So defined, we have that \( g(y) \) is a non-negative solution of (3.2) and, clearly, it satisfies by construction that \( g(y) < f(y) \) for \( y > \varepsilon \) close enough to \( \varepsilon \). Therefore we have by continuity and the previous arguments that there exists a value \( y^* \in (\varepsilon, y_0) \) such that \( g(y) \leq f(y) \) in the interval \((0, y^*)\) and \( g(y^*) = f(y^*) \). Hence, it follows from equation (3.13) that \( f_0^{(5)}(y) \leq g^{(5)}(y) \) for \( 0 < y \leq y^* \) and a contradiction at \( y^* \) follows after integrating this inequality as above. A similar contradiction argument applies if we assume that \( f_0(y_0) > f(y_0) \) by considering the auxiliary function \( g(y) = f_0(y - \varepsilon) \).

The uniqueness result for \( \alpha < 0 \) is obtained by using similar arguments but taking into account that in this case \( h(s) = |s|^{\alpha-1}s \) is a decreasing function for \( s > 0 \). This fact gives after subtracting and integrating five times that

\[
f(y) - f_0(y) = \int_0^y \cdots \int_0^y (f^{(\alpha)} - f_0^{(\alpha)})(r) \, dr \, dz,
\]

whence if the solutions are ordered close to the origin, the signs of the left- and right-hand sides of this expression are different, so the contradiction readily follows. The same argument applies to \( g(y) \) constructed as above, so mutually oscillatory behavior is also disregarded in this case. This completes the proof. \( \square \)

3.2. Positive solutions by fixed point theorem. Indeed, we were lucky to have the explicit positive solution (3.9), which is connected with the invariant scaling group of transformations of the ODE (3.2). We now sketch another approach to detecting a unique positive solution for more arbitrary nonlinearities without using and relying on explicit calculus via a scaling group.

We again consider the ODE (3.13) with \( \lambda = -1 \), i.e., look for positive solutions of (3.13), where \( f^\alpha \) can be replaced by more general functions \( q(f) > 0 \) for \( f > 0 \). The main non-uniqueness difficulty for (3.13) is that it always admits the trivial solution \( f(y) \equiv 0 \) if \( \alpha > 0 \). However, this disappears for the inverse function \( y = y(f) \), for which

\[
f' = \frac{1}{y}, \quad f'' = \left(\frac{1}{y}\right)'\frac{1}{y}, \quad f''' = \left(\left(\frac{1}{y}\right)'\frac{1}{y}\right)'\frac{1}{y}, \quad \ldots.
\]

Then we obtain the following ODE for \( y(f) \):

\[
f^{(5)} \equiv \left(\left(\left(\left(\left(\frac{1}{y}\right)'\frac{1}{y}\right)'\frac{1}{y}\right)'\frac{1}{y}\right)'\frac{1}{y}\right)'\frac{1}{y} = f^\alpha \quad \text{for } f > 0.
\]
Multiplying by \( y' \) and integrating over \((0, f)\) with the zero boundary condition yields
\[
(((\frac{1}{y'}))^4 \frac{1}{y'}) = \int_0^f f^\alpha y' \, df = f^\alpha y - \alpha \int_0^f f^{\alpha-1} y \, df.
\]

Multiplying again by \( y' \) and integrating gives on the right-hand side a smooth quadratic integral operator depending on \( y^2 \), etc.

After five integrations like that, we obtain an integral equation for \( y(f) \) of the form:
\[
(3.15) \quad y(f) = M(y)(f) \quad \text{for} \quad f > 0,
\]
with a smooth operator \( M \) of \( y \) containing the maximum fifth degree \( y^5 \) polynomial dependence on \( y \). Therefore, this operator is a contraction in a space of continuous functions provided that the typical integrals such as \( \int_0^f f^{\alpha-1}(\cdot) \, df \) converge, i.e.,
\[
(3.16) \quad \alpha > 0 \implies m \in (0, 1).
\]

This restriction can be weakened by choosing special classes of functions \( y(f) \) with prescribed envelopes as \( y \to 0 \).

Finally, Banach’s Contraction Principle (see e.g. [17, p. 206]) guarantees existence and uniqueness of a positive solution of \((3.15)\), and hence of \((3.13)\). For non-monotone changing sign solutions of \((3.13)\), this inverse function approach obviously fails. We will study such a behaviour in Section 5.1 by an extra scaling.

3.3. Maximal regularity. It is easy to see that \((3.9)\) describes the best (maximal) regularity at the interface that is provided by the ODE \((2.2)\). Indeed, \((3.9)\) established the only possible balance between two terms in \((2.2)\). In other words, loosely speaking, if such a balance is violated, the solution must behave along the kernel (the null-manifold) of the nonlinear or linear operator on both sides of \((2.2)\), which is obviously trivial. Recall that this maximal regularity reflects the intrinsic properties of the CP. For the FBP, the regularity is different; cf. \((3.6)\).

**Proposition 3.2.** Let \( \alpha > 0 \) and let the solution of \((2.2), (3.4)\) be written as
\[
(3.17) \quad f(y) = y^\mu \varphi(y) \quad \text{for} \quad y \geq 0 \quad (\mu = \frac{5}{1-\alpha} = \frac{5(n+1)}{m+n}).
\]

Then \( \varphi(y) \) is uniformly bounded for \( y > 0 \).

**Proof.** Let us prove that \( \varphi(y) \) is globally bounded, i.e., there exists a positive constant \( C \) such that
\[
(3.18) \quad |\varphi(y)| \leq C \quad \text{for all} \quad y > 0.
\]

Assume that \( f \neq 0 \), and, for every \( y > 0 \), define
\[
(3.19) \quad \bar{y} = \sup_{[0,y]} \{ z \in [0, y] : |f(s)| \leq |f(z)| \, \forall s \in (0, y) \}.
\]

Using the ODE such as \((3.13)\) and integrating five times over \((0, \bar{y})\), we obtain by the definition in \((3.19)\) that
\[
|f(\bar{y})| \leq C|f(\bar{y})|^\alpha \bar{y}^5,
\]
so the desired inequality \((3.18)\) readily follows for \( \bar{y} \) and, by \((3.19)\), for every \( y > 0 \). \( \square \)
3.4. **Nonexistence of positive source-type similarity profile.** We finish this analysis by dealing with the question of nonexistence of positive source-type solutions. We prove that, in the basic interval \([3.12]\), the ODE problem \((2.5)\) has no positive solutions with the symmetry conditions at the origin.

**Proposition 3.3.** There are no symmetric positive solutions with compact support of the problem \((2.5)\), with conditions \((3.4)\) for \(F(y)\) at the interfaces.

**Proof.** Let \(F(y)\) be a positive solution of \((2.5)\) with compact support \([-y_0, y_0]\). It is clear from the equation \((2.5)\) and the positivity assumption that \(F^{(5)}(y) > 0\) in \((-y_0, 0)\). By integrating this inequality over \((0, y_0)\) and assuming the required regularity \((3.4)\) of \(F(y)\) at \(y = y_0\), we infer that \(F'(0) > 0\). Hence the symmetry condition fails that completes the proof. □

Thus, the sufficiently smooth source-type profile in the CP must be oscillatory at the interfaces. We now begin to study the character of such oscillations.

4. **Existence and uniqueness for the initial value problem**

In order to analyze the existence and uniqueness problem, we introduce the following (cf. [II]):

**Definition 4.1.** A function \(f\) is said to be a weak \(C^4\)-solution of the equation \((2.2)\) on an interval \(I\), if \(f \in C^4\) and for any \(y_0, y \in I\),

\[
    f^{(4)}(y) - f^{(4)}(y_0) = \pm \int_{y_0}^{y} |f(s)|^{\alpha-1} f(s) \, ds. \tag{4.1}
\]

One can see that, for \(\alpha \geq 0\), solutions are \(C^5\) and hence classical. For \(\alpha < 0\), if we use the asymptotics \((3.17)\) with uniformly bounded (say periodic) oscillatory components \(\varphi\), weak \(C^4\)-solutions exist for \(\alpha > -\frac{1}{4}\) (i.e., \(\frac{5\alpha}{1-\alpha} > -1\)).

Consider the Cauchy problem for the equation \((2.2)\) with the initial conditions

\[
    f^{(j)}(y_0) = \alpha_j, \quad j = 0, 1, 2, 3, 4. \tag{4.3}
\]

It is clear that this initial value problem can also be written, by introducing the function \(h(y) = |f(y)|^{\alpha-1} f(y)\), in the equivalent form,

\[
    \begin{aligned}
    f^{(4)}(y) &= \alpha_4 \pm h(y), \\
    h'(y) &= |f(y)|^{\alpha-1} f(y), \\
    f^{(j)}(y_0) &= \alpha_j, \quad j = 0, 1, 2, 3, \\
    h(y_0) &= 0.
    \end{aligned} \tag{4.4}
\]

From the standard theory of ordinary differential equations, for every \(\alpha > 0\), problem \((4.4)\) is solvable in some neighborhood of \(y = y_0\). Let \((f, h)\) be a solution. On the one hand, it is clear that, for \(\alpha \geq 1\), such solution is unique since the Lipschitz condition on the nonlinearity is satisfied. On the other, for \(\alpha < 1\), the only continuity is guaranteed
and in fact uniqueness does not hold, for instance, if $\alpha_j = 0$ for every $j = 0, \ldots, 4$, as it is shown by means of the construction of positive solutions in the previous section. A partial answer to uniqueness is established in the following:

**Proposition 4.2.** Let $0 < \alpha < 1$. If in (3.4)

$$
\sum_{j=0}^{4} |\alpha_j| > 0,
$$

then the initial value problem (3.4) has at most one solution in a neighbourhood of $y = y_0$.

**Proof.** The proof follows the ideas in [1, 4]. We prove uniqueness in a small right-hand neighbourhood of $y = y_0$. Uniqueness on the left is obtained in a similar way. Denote by $j^*$ the smallest value of $j$ such that $\alpha_j$ is non-trivial, and, for such $j = j^*$, introduce the function

$$
g_i(y) = \begin{cases} 
\frac{f_i(y)}{(y-y_0)^j}, & \text{if } y > y_0, \\
g_i(y_0) = 0, & \text{if } y = y_0.
\end{cases}
$$

Assume for contradiction that there exist two different solutions $f_1$ and $f_2$ for small $y - y_0 > 0$. Due to the invariance properties of the equation, we may assume without loose of generality that $f_1 \geq f_2 \geq 0$. It is clear by integrating the equation and taking into account the initial conditions that

$$
|f_1(y) - f_2(y)| \leq \int_{y_0}^{y} \cdots \int_{y_0}^{s} |\alpha_1^{a-1} f_1(r) - |f_2|^{\alpha-1} f_2(r)| \, dr \cdots dz.
$$

Hence, after dividing by $(y - y_0)^j$, taking into account that $h(s)$ is Lipschitz continuous away from $s = 0$ and integrating five times, we get

$$
|g_1(y) - g_2(y)| \leq C(y - y_0)^{4+(-1-\alpha)j} \max_{[y_0,y]} |g_1(s) - g_2(s)|.
$$

Since $4 + (1 - \alpha)j > 0$ for any $j \leq 4$, the contradiction follows for $y - y_0 > 0$ sufficiently small. The proof is complete. □

Next we introduce a comparison principle which is valid for $\alpha > 0$ with $\lambda = -1$ (the positive sign on the right-hand side in (3.2)) and $\alpha < 0$ with $\lambda = 1$.

**Proposition 4.3.** Let $f_1$ and $f_2$ be two solutions of equation (4.1) on $(y_0, \infty)$ satisfying $f_1^{(j)}(y_0) \geq f_2^{(j)}(y_0)$ for $j = 0, 1, 2, 3, 4$, with strict inequality for at least one of them. Then, $f_1(y) > f_2(y)$ for every $y \geq y_0$.

**Proof.** The inequality is obvious by continuity and the assumptions on the initial conditions on an small interval $(y_0, y_0 + \delta)$. We see that in fact the inequality holds for every $y > y_0$. If the assertion is false, we denote by $y^* = \sup\{y \leq y_0 : f_1(s) > f_2(s), s \in (y_0, y)\}$. It is clear that, at this point, $f_1(y^*) = f_2(y^*)$. However, for $\alpha > 0$ and $\lambda = -1$, we obtain after subtracting and integrating five times,

$$
f_1(y^*) - f_2(y^*) = \sum_{j=0}^{4} (\alpha_1^{j} - \alpha_2^{j}) + \int_{y_0}^{y^*} \cdots \int_{y_0}^{s} (|f_1|^{\alpha-1} f_1(r) - |f_2|^{\alpha-1} f_2(r)) \, dr \cdots dy > 0,
$$

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leading to a contradiction. In a similar way, a contradiction is obtained for $\alpha < 0$ and $\lambda = 1$. \[\Box\]

5. Oscillatory component: dissipative systems, periodic behaviour, and heteroclinic bifurcations

5.1. Dissipative dynamical system for the oscillatory component. We now study solutions of (3.2) of changing sign. Bearing in mind (3.17), we will be looking for solutions in the form

\[(5.1) \quad f(y) = y^\mu \varphi(s), \quad s = \ln y, \quad \text{where} \quad \mu = \frac{5}{1-\alpha} = \frac{5(n+1)}{m+n}.\]

Here $\varphi$ is called the oscillatory component of the solution, and is a sufficiently smooth function; see below. The positive solution (3.9) corresponds to the particular case, where $\varphi(s) \equiv \varphi_0$ (see (3.10)), and this exists in the corresponding parameter ranges. Besides (3.9), there exist many other changing sign solutions.

According to Proposition 3.2, we know that, at least for $\alpha \geq 0$, the oscillatory component satisfies

\[(5.2) \quad \varphi(s) \text{ is bounded and } \not\to 0 \text{ as } s \to -\infty.\]

Therefore, (5.1) gives a clear picture of typical regularity of solutions at the interfaces,

\[(5.3) \quad f \in C^4((-\delta, \delta)) \quad \text{for} \quad m < \frac{n+5}{4}, \]

\[f \in C^5((-\delta, \delta)) \quad \text{for} \quad \frac{n+5}{4} \leq m < \frac{2n+5}{3}, \]

\[f \in C^2((-\delta, \delta)) \quad \text{for} \quad \frac{2n+5}{3} \leq m < \frac{3n+5}{2}.\]

According to (3.7), in the first two ranges the CP always has a better regularity than the FBP.

Substituting the representation (5.1) into the ODE (3.2), after simple manipulations via scaling properties, yields the following autonomous equation for $\varphi(s)$:

\[(5.4) \quad P_5(\varphi) = \mp |\varphi|^{\alpha-1} \varphi \quad \text{in} \quad \mathbb{R} \quad (\lambda = \pm 1),\]

where $\alpha = \frac{1-m}{1+n}$. Here $P_5$ is an easily derived linear differential operator of the form

\[(5.5) \quad P_5(\varphi) = e^{(-\alpha\mu-1)s}(e^{-s}(e^{-s}(e^{-s}(e^{\mu s}\varphi)''))'')' + 5(\mu - 2)\varphi^{(4)} + 5(2\mu^2 - 8\mu + 7)\varphi'' + 5(\mu - 2)(2\mu^2 - 8\mu + 5)\varphi' + (5\mu^4 - 40\mu^3 + 105\mu^2 - 100\mu + 24)\varphi' + \mu(\mu - 1)(\mu - 2)(\mu - 3)(\mu - 4)\varphi.\]

The best and simplest connection satisfying (5.2) with the interface at $s = -\infty$ is

\[(5.6) \quad \varphi = \varphi_*(s) \text{ is a (non-constant) periodic solution of (5.4)}.\]

On the other hand, any uniformly bounded solution of (5.4) for $y \ll -1$ will fit.

We now concentrate on periodic connections. We list the following properties that approach us to existence of a periodic orbit of changing sign:
Proposition 5.1. Let $\alpha \in [0,1)$ and (cf. (3.12))
\begin{equation}
\mu = \frac{5}{1-\alpha} = \frac{5(\alpha+1)}{\alpha+1} \geq 4, \quad \text{i.e.,} \quad m \leq \frac{n+5}{4}.
\end{equation}

Then the fifth-order dynamical system (5.4) satisfies:
(i) no orbits are attracted to infinity as $s \to +\infty$, and
(ii) it is a dissipative system with bounded absorbing sets defined for any $\delta > 0$ by
\begin{equation}
\limsup_{s \to +\infty} |\varphi(s)| \in B^*_\delta = [-C_* - \delta, C_* + \delta], \quad \text{where} \quad C_* = (5!)^{-\frac{1}{1-\alpha}}.
\end{equation}

The absorbing set in $\mathbb{R}^5$ includes all vectors $(\varphi, \varphi', \varphi'', \varphi^{(4)})^T$, where all derivatives are uniformly bounded by a certain constant.

Proof. (i) The operator in (5.4) is asymptotically linear [17, p. 77] with the derivative at the point at infinity $P_5(\varphi)$ that has the characteristic equation obtained by setting
\begin{equation}
\varphi = e^{\rho s} \implies p_5(\rho) \equiv (\mu + \rho)(\mu + \rho - 1)(\mu + \rho - 2)(\mu + \rho - 3)(\mu + \rho - 4) = 0.
\end{equation}
Therefore, all eigenvalues are real negative or non-positive:
\begin{equation}
\rho_k = k - \mu \leq 0 \quad \text{for} \quad k = 0, 1, 2, 3, 4, \quad \text{provided that} \quad 4 - \mu \leq 0, \quad \text{or} \quad m \leq \frac{n+5}{4}.
\end{equation}
Thus, $\varphi = \infty$ cannot attract orbits.

(ii) Actually, this follows from (i) by using an extra scaling. On the other hand, this is easy to see from the original equation such as (3.13) for $\lambda = -1$ (for $\lambda = +1$ the proof is precisely the same). Taking an arbitrary solution $f(y)$ defined for all $y > 0$, we integrate (3.13) five times to get that
\begin{equation}
\bar{f}(y) = \sup_{z \in (0,y)} |f(z)| > 0
\end{equation}
satisfies the inequality
\begin{equation}
\bar{f}(y) = C_0 + C_1 y + C_2 y^2 + C_3 y^3 + C_4 y^4 + \int \int \int \int z |f(z)|^{\alpha-1} f(z) \, dz \leq C_0 + C_1 y + C_2 y^2 + C_3 y^3 + C_4 y^4 + \frac{1}{\alpha} y^5 \tilde{f}^\alpha(y),
\end{equation}
where $C_i > 0$ are some constants. It follows that
\begin{equation}
\bar{f}(y) \leq C_0 + C_* y^{\frac{5}{1-\alpha}} \quad \text{for all} \quad y \geq 0,
\end{equation}
provided that $\frac{5}{1-\alpha} \geq 4$. In the variable (5.11), this yields (5.8). □

5.2. Periodic oscillatory component. In connection with Proposition 5.1, existence of a periodic orbit for dissipative dynamical systems is a standard result of degree theory; see [17] p. 235]. Nevertheless, classic theory deals with non-autonomous periodic systems, with the given fixed period. Therefore, these results do not directly apply to the fifth-order dissipative autonomous dynamical system (5.4), so we need some extra arguments to justify existence of periodic solutions for some values of $\alpha$ and how this disappears when $\alpha$ gets negative.

It turns out that it is easier to establish first where such periodic orbits are nonexistent. The existence evidence will be presented later on.
Figure 1. Coefficients of the polynomial operator $P_5(\varphi)$ in (5.5) and (5.15).

Proposition 5.2. (i) The ODE (5.4) with $\lambda = -1$ does not admit a nontrivial $T_\ast$-periodic solution $\varphi_\ast$ for

$$\mu \in \left(\frac{5}{2}, \mu_1^+\right) \text{ or } \alpha \in (-1, -0.9655...), \quad \text{where } \mu_1^- \approx 1.45608..., \mu_1^+ \approx 2.5439... .$$

(ii) Analogously, the ODE (5.4) with $\lambda = +1$ does not admit a nontrivial $T_\ast$-periodic solution $\varphi_\ast$ for

$$\frac{5}{2} < \mu \leq \mu_\ast^+ = 3.22474..., \quad \text{or } \alpha \in (-1, \alpha_\ast^+), \quad \text{where } \alpha_\ast^+ = \frac{\mu_\ast^+ - 5}{\mu_\ast^+} = -0.5505... .$$

Proof. We write down (5.4) as follows:

$$P_5(\varphi) = \varphi^{(5)} + a_4 \varphi^{(4)} + a_3 \varphi''' + a_2 \varphi'' + a_1 \varphi' + a_0 \varphi = \mp |\varphi|^{\alpha - 1} \varphi \quad (\lambda = \pm 1),$$

where the coefficients $a_i = a_i(\mu)$ are as in (5.5). For convenience, we present graphs of these coefficients in Figure 1.

Multiplying (5.15) by $\varphi$ and $\varphi'$ in $L^2(0, T_\ast)$ and integrating by parts yields,

$$a_4 \int (\varphi_\ast^{(5)})^2 - a_2 \int (\varphi_\ast^{(4)})^2 + a_0 \int \varphi_\ast^2 = \mp \int |\varphi_\ast|^\alpha \varphi, \quad (\lambda = \pm 1),$$

$$\int (\varphi_\ast^{(5)})^2 - a_3 \int (\varphi_\ast''')^2 + a_1 \int (\varphi_\ast'')^2 = 0,$$

respectively. It is not difficult to see that those identities make sense for all $\alpha > -1$, so this restriction enters all our conditions.

Note that multiplication by $\varphi''$ gives a new formal identity

$$-a_4 \int (\varphi_\ast'')^2 + a_2 \int (\varphi_\ast''')^2 - a_0 \int (\varphi_\ast')^2 = \pm \alpha \int |\varphi_\ast|^{\alpha - 1} (\varphi_\ast')^2.$$

However, on the right-hand side, there appears, after integration by parts, an integral with unknown converging properties, so using it could lead to wrong conclusions. More
Thus, here and later on various integrals of $\varphi_*$ and its derivatives appear. The convergence of these integrals for $\alpha < 0$ is checked by using local properties of solutions of (5.4). In particular, these are easy for $\alpha > -1$, which justifies identities (5.16), (5.17) in the range of parameters under consideration, (3.2)–(3.3). Such convergence can be quite tricky for negative $\alpha < -1$ that are not under scrutiny in the present study.

(i) To begin with our nonexistence purposes, it suffices to use just the single second identity (5.17). This gives that $\varphi_* \neq \text{const.}$ is nonexistent if

$$a_1 \geq 0 \quad \text{and} \quad a_3 \leq 0.$$  

It is easy to prove that there exist values $\mu_1^-$ and $\mu_1^+$ roots of $a_1$ such that $a_1 > 0$ and $a_3 < 0$ on $(\mu_1^-, \mu_1^+)$, and hence, $\varphi_*$ does not exist, at least, on such interval. Numerical approximation of these roots shows that $\mu_1^- \approx 1.45608...$ and $\mu_1^+ \approx 2.5439...$. The results are illustrated in Figure 1.

(ii) In a similar way, the nonexistence result for $\lambda = 1$ is extended to (5.14), by using identity (5.16). This directly gives that $\varphi_* \neq \text{const.}$ does not exist if

$$a_4 \geq 0 \quad \text{and} \quad a_2 \leq 0,$$

which is true if $\mu \in (\frac{5}{2}, \mu_2^+)$, where $\mu_2^+ = 3.22474...$ is the largest root of $a_2$. □

We continue with establishing the principal fact that the periodic solution $\varphi_*$ is hyperbolic in some parameter ranges of interest. This establishes an important corollary concerning the type of heteroclinic bifurcation, and where it appears from; see below. This also reflects the stability and instability properties of periodic solutions for $\lambda = +1$ and $\lambda = -1$. We concentrate on the more important case $\lambda = -1$, which in (5.4) corresponds to oscillatory behaviour of source-type solutions.

**Proposition 5.3.** Let $\varphi_*(s)$ be a non-constant $T_*$-periodic solution of (5.4), $\lambda = -1$, for

$$\alpha \in (-0.9655..., -\frac{2}{3}).$$

Then $\varphi_*$ is hyperbolic.

**Proof.** For convenience, we write down (5.4), $\lambda = -1$ as

$$\varphi_* : \quad A_-(\varphi) \equiv -P_5(\varphi) + |\varphi|^{\alpha - 1}\varphi = 0.$$  

Consider the eigenvalue problem for the corresponding linearized operator

$$A'_-(\varphi_*)\psi \equiv -P_5(\psi) - |\alpha| |\varphi_*|^{\alpha - 1}\psi = \lambda_k \psi,$$

where we use that $\alpha < 0$. Note that the potential here can be rather singular at zeros of $\varphi_*(y)$ that can essentially affect the setting of the eigenvalue problem and hence the actual regularity of eigenfunctions. In particular, we always assume that these zeros are transversal, so that the potential $|\varphi_*(y)|^{\alpha - 1}$ at zero, say, at $y = 0$ is not singular as $\sim \frac{c}{|y|^\alpha}$ as $y \to 0$. Then by Hardy–Rellich-type inequalities, we have got necessary embeddings that guarantee compactness of the resolvent and hence discreteness of the spectrum, which
allows to deal with the hyperbolicity issue for the periodic solutions. In further simple manipulations, we naturally assume that all these are justified.

Assume first, for simplicity, that an eigenvalue $\lambda_k$ is real. Multiplying (5.23) by $\psi$ and integrating over the period $(0, T_*)$ by parts yields

$$-a_4 \int (\psi')^2 + a_2 \int (\psi')^2 - a_0 \int \psi^2 - |\alpha| \int |\varphi|^2 = \lambda_k \int \psi^2.$$  

It follows that

$$\lambda_k < 0 \quad \text{(Im} \lambda_k = 0), \quad \text{provided that} \quad a_4 \geq 0, \quad a_2 \leq 0, \quad a_0 \geq 0.$$  

This gives the $\mu$-interval in (5.21). For $\lambda_k \in \mathbb{C}$, similarly, first multiplying in $L^2$ by the complex conjugate $\bar{\psi}$, and next by $\psi$ the complex conjugate ODE, summing up yields

$$-a_4 \int |\psi''|^2 + a_2 \int |\psi'|^2 - a_0 \int |\psi|^2 - |\alpha| \int |\varphi|^2 = \lambda_k \int |\psi|^2 <$ 0,  

i.e., $\text{Re} \lambda_k < 0$, under the same inequalities for the parameters as in (5.25).

The identities (5.24) and (5.26) can be used to detect other hyperbolicity (and stability) parameter ranges of $\varphi_*$ and also for $\lambda = +1$.

As an important corollary, we obtain that, by classic bifurcation-branching theory and implicit function theorem for periodic solutions [21, Ch. 6], for $\lambda = -1$, a hyperbolic periodic solution $\varphi_*(s)$, existing at some parameter value $\mu_0 \in (\mu_1^+, 3)$, can be extended into a small open neighbourhood of $(\mu_0 - \delta, \mu_0 + \delta)$. This implies the following:

**Corollary 5.4.** The $\mu$-parameter domain of existence of a periodic solution $\varphi_*(s)$ of (5.4), $\lambda = -1$ for $\mu \in (2, 3)$, if it is not empty, contains a connected interval $(\mu_1^+, \mu_2)$, where $\mu_1^+ \geq \mu_1^+ + 1$ and $\mu_2 \leq 3$.

Finally, we expect that the periodic connection (5.6) together with its 1D stable manifold as $s \to -\infty$ is the only transition to the interface point at $s = -\infty \ (y = 0^+)$, though this is difficult to justify completely rigorously. Also, we cannot prove uniqueness (see below) of a periodic solution of (5.4). Before, the only result on existence of a periodic orbit is obtained in [7, § 7.2] for the third-order ODE like (5.4) with $n = 0$ and replacing the linear operator,

$$P_3(\varphi) \mapsto P_3(\varphi) = \varphi''' + 3(\mu - 1)\varphi'' + (3\mu^2 - 6\mu + 2)\varphi' + \mu(\mu - 1)(\mu - 2)\varphi,$$

by using the fact that this dynamical system is dissipative. Then classic theory [17, § 39.3] applies together with a shooting-type argument. This results was improved in similar lines in [11, § 5] with a sharper estimate on the periodic solution existence interval:

$$0 < n < n_h \in \left(\frac{2}{3}, n_+\right), \quad \text{where} \quad n_+ = \frac{9}{3 + \sqrt{3}} = 1.9019238\ldots.$$  

The actual heteroclinic bifurcation occurs at

$$n_h = 1.7599\ldots \quad \text{(the TFE–4)},$$

and was calculated numerically, [7].
5.3. Heteroclinic bifurcations of periodic solutions. The ODE (5.4) is fifth-order quasilinear, is rather non-standard, and has a non-Lipschitz nonlinearity. Moreover, periodic solutions exist not for all \( m, n > 0 \), namely, on some open (cf. Corollary 5.4) interval \( m \in (0, m_h(n)) \), with the bifurcation exponent \( m_h \) to be discussed. The classic results on existence of nontrivial periodic solutions based on rotation vector field theory [17, p. 50] or branching theory [21, Ch. 6] do not apply to such ODEs. Solutions that are bounded on the whole axis (which is fine for (5.2)) also cannot be detected along the lines of classic theory in [17, p. 56]. Moreover, we are interested in solutions that are bounded (with non-zero limits) as \( s \to -\infty \) only. Several other techniques for existence of periodic solutions also fail; see references and comments in [13, p. 140]. Therefore, we will rely on careful numerical evidence, especially when talking about the uniqueness of periodic orbits and their stability.

The TW case \( \lambda = +1 \). In Figure 2, we show a stable periodic behaviour for the ODE (5.4) for \( m = 1 \) and \( n = 0 \) (so this is the standard thin film case) for \( \lambda = +1 \). It turns out that such a periodic solution persists until a heteroclinic-like bifurcation which occurs at (5.28) \( m_h = 1.337968147... \) \( (n = 0, \lambda = +1) \).

To compare with Proposition 5.2, we present the corresponding universal critical values (5.29) \[ \mu_h = \frac{5}{\mu} = 3.7370... \in (3, 4), \quad \alpha_h = \frac{\mu - 5}{\mu} = -0.3380... \]

In terms of the original parameters \( m \) and arbitrary \( n \), we have that the heteroclinic bifurcation occurs at (5.30) \[ m_h(n) = \frac{5}{\mu} + \left( \frac{5}{\mu} - 1 \right)n \approx 3.7370 + 2.3737n. \]

Note that, for \( \lambda = +1 \), (5.30) is essentially far from the predicted nonexistence value (5.13). It is interesting to check whether the system (5.16), (5.17) (plus other identities if any) can supply us in this case with a better estimate of \( \mu_h \).

This scenario of a heteroclinic bifurcation of stable and hyperbolic periodic solutions is typical for dynamical systems; see [18, Ch. 4]. Indeed, the hyperbolicity property as in Proposition 5.3 excludes saddle-node bifurcations of periodic solutions (these demand existence of a \( \lambda \in \mathbb{R} \)), at which \( \varphi_* \) can disappear. Therefore, the appearance of a (stable) heteroclinic orbit as \( \mu \to \mu_h \) in such dynamical systems is most plausible.

Figure 3 shows this typical formation of a heteroclinic orbit \( -\varphi_0 \to \varphi_0 \) as \( m \to m_h^- \). Here \( \pm \varphi_0 \) are constant equilibria of (5.4) given as in (3.10) by (5.31) \[ \varphi_0 = \left[ -\frac{1}{\mu(\mu-1)(\mu-2)(\mu-3)(\mu-4)} \right]^{\frac{\alpha+1}{m+n}} (\lambda = +1). \]

Since by (5.29), for \( n = 0 \) and \( m = m_h \), we have \( \mu_h \in (3, 4) \), both equilibria \( \pm \varphi_0 \) exist. These results have been obtained by the MatLab (the ode45 solver) with the enhanced accuracy parameters Tols = \( 10^{-11} \) and the same parameter of regularization in both degenerate and singular terms in (5.4).

Notice that, in view of (3.10), solutions (5.1) for \( m \in (\frac{5}{3}, m_h) \) do not satisfy the last condition in (3.4), i.e., \( f^{(4)}(y) \) is not bounded as \( y \to 0^+ \).
Nevertheless, the structure (5.1) corresponds to the maximal regularity for the ODE (3.2) and correctly establishes the balance of its nonlinear and linear terms. In Figure 4, we show a stable periodic behaviour for $m = n = 1$ and $\lambda = +1$.

The source-type case $\lambda = -1$. The case $\lambda = -1$ that includes the source-type solutions, is shown in Figure 5 for $m = n = 1$. In this case, the periodic behaviour is unstable as $s \to +\infty$ (as well as $s \to -\infty$), though is clearly visible. For $n = 0$, a “heteroclinic” bifurcation occurs at (see details in [13, p. 142])

$$m_h = 1.909... \quad (n = 0, \quad \lambda = -1).$$

The corresponding other critical values are

$$\mu_h = \frac{5}{1 - \alpha_h} = 2.619... \in (2, 3) \quad \text{and} \quad \alpha_h = \frac{\mu_h - 5}{\mu_h} = -0.909...,$$

Observe that this $\alpha_h$ is sufficiently close to the nonexistence one $\alpha_* = -0.9655...$ in (5.13) in Proposition 5.2. The corresponding critical values of $m_h(n)$ are then given by

$$m_h(n) = \frac{5}{\mu_h} + \left(\frac{\mu_h}{5} - 1\right)n \approx 1.909 + 0.909n.$$

Let us state the following conjecture on periodic solutions and their stable manifolds.

**Conjecture 5.2.** For all $n \geq 0$, there exists $m_h(n)$ given by (5.30) for $\lambda = +1$ (resp. by (5.34) for $\lambda = -1$), such that for all $m \in (0, m_h(n))$, the ODE (5.4) with $\lambda = +1$ (resp. $\lambda = -1$):

(i) has a unique stable (unstable) as $s \to +\infty$ periodic solution $\varphi_s(s)$ of changing sign;

(ii) as $s \to -\infty$, the periodic solution $\varphi_s(s)$ is unstable and has a 1D stable manifold; and

(iii) there exists a heteroclinic bifurcation at $m_h(n)$, so $\varphi_s(s)$ is nonexistent for $m \geq m_h(n)$.

It is worth mentioning the obvious instability of $\varphi_s(s)$ as $s \to -\infty$ (while approaching the interface point at some $y = y_0$) is associated with the translational invariance of the ODEs such as (3.2) admitting shifting in $y_0$. The precise meaning and the significance of
the conclusion (ii) will be explained in the second part of the paper [5], which will be key for a well-posed shooting of a global source-type similarity profile.

5.4. Oscillatory component: nonlinear dispersion model. Substituting TW solutions (2.1) into (1.7) and integrating twice with zero constants of integration yields the same ODE (2.2) with the only change on the right-hand side:

$$-\lambda \mapsto \lambda^2 > 0.$$ 

Therefore, for the oscillatory component we obtain the same equation (5.4), and in examples and figures presented there, we always take $\lambda = -1$, so that $-\lambda = 1 > 0$. In
Figure 5. The trace of an unstable periodic orbit of (5.4) for $m = n = 1, \lambda = -1$.

In particular, the oscillatory behaviour near interfaces persists until the bifurcation exponent (5.32) for $n = 0$.

6. Existence and uniqueness of oscillatory TW solutions for $m = 1$

In this section, we analyze the problem of existence and uniqueness of the oscillatory travelling waves solutions, as well as the character of their oscillations. We begin by proving existence of the oscillatory TW solutions via a different approach.

6.1. Existence and uniqueness: an algebraic-geometric approach to periodic orbits. Here, we develop an alternative approach to existence of periodic orbits of (5.4) for both $\lambda$, positive and negative. We focus our attention on the construction of solutions with right-hand side interfaces, $f^+(y)$ and $f^-(y)$ corresponding to $\lambda > 0$ and $\lambda < 0$ respectively. By reflection, we obtain solutions with left-hand side interfaces, namely $f^+( -y)$ for $\lambda < 0$ and $f^-( -y)$ for $\lambda > 0$.

We begin with the original equation for $f(y)$, (3.13), which, for the case $m = 1$ (and putting for convenience $|\lambda|$ to make calculations easier), takes especially simple “weakly nonlinear” form

\begin{equation}
(6.1) \quad f^{(5)} = 5! \text{sign } f.
\end{equation}

Let us explain the main ingredients of the strategy of our construction.

By using the properties of the solutions, we may consider for convenience, that the positivity domain of $f_0$ is given by $(-1, 0)$, so $f(0) = f(-1) = 0$ and $f(y) > 0$ on $(-1, 0)$. Hence, such a function denoted now by $f_0(y)$ satisfies on $(-1, 0)$,

\begin{equation}
(6.2) \quad f^{(5)} = 5! \quad \Longrightarrow \quad f_0(y) = y(y+1)[a + by + cy^2 + y^3],
\end{equation}

where the constants $a, b, c$ are chosen so that $f_0(y) > 0$ on $(-1, 0)$.

We next extend $f_0(y)$ to $y > 0$ as follows in three steps:
(i) take $-f_0(y)$, 
(ii) shift it to the right by $y_1 = 1$ to get $-f_0(y - 1)$, and 
(iii) rescale it by the invariant scaling group for (6.1) to get for $y > 0,$ 

$$f_0(y) \equiv -G^5 f_0(\frac{y}{G} - 1), \quad \text{with a parameter } G > 0.$$ 

In order to have a smooth solutions on $[-\delta, \delta]$, we need four matching conditions on zero jumps of four consecutive derivatives, 

$$[f_0'](0) = [f_0''](0) = [f_0'''](0) = [f_0^{(4)}](0) = 0.$$ 

Thus, we obtain four algebraic equations with four parameters $a, b, c$, and $G$. They can be written as:

$$
\begin{align*}
(1 - G^4)a + G^4b - G^4c &= -G^4, \\
(1 + G^3)a + (1 - 2G^3)b + 3G^3c &= 4G^3, \\
(1 + G^2)b + (1 - 3G^2)c &= -6G^2, \\
(1 + G)c &= 4G - 1.
\end{align*}
$$

We remark that by the argument of continuation of $f_0$ to a solution of the problem with a finite right-hand interface, the condition 

$$0 < G < 1$$

is required. Therefore, we restrict ourselves to analyzing system (6.5) in this range of parameters.

We begin by noting that, for every fixed value of $G \in (0, 1)$, (6.5) is a linear system in $a, b, c$ of four equations. Considering the three last equations, one can easily check that the determinant of the associated matrix 

$$d = (1 + G^3)(1 + G^2)(1 + G)$$

is not zero, and, hence, for every fixed $G \in (0, 1)$, the system of the last three equations has a unique solution. Therefore, it will be the solution of the complete system (6.5) if the determinant of the matrix, 

$$
\begin{pmatrix}
1 - G^4 & G^4 & -G^4 & -G^4 \\
1 + G^3 & 1 - 2G^3 & 3G^3 & 4G^3 \\
0 & 1 + G^2 & 1 - 3G^2 & -6G^2 \\
0 & 0 & 1 + G & 4G - 1
\end{pmatrix}
$$

is zero. Denoting by $D(G)$ the determinant of this matrix, we have that $D(0)$ and $D(1)$ are both negative. Hence, it readily follows that either no roots of $D(G)$ are available, or at least two roots exist. After analyzing in more detail $D(G)$, one can see that there exist two roots, that can be obtained numerically:

$$G_1 = 0.178318... \quad \text{and} \quad G_2 = 0.7060378....$$
However, the positivity condition of \( f_0 \) in \((-1, 0)\) implies \( a < 0 \) or, equivalently, by using the first equation in (6.5), gives \( 1 + b - c > 0 \). Manipulating the two last equations in (6.5), one obtains that this restriction holds, if only if,
\[
F(G) = 3G^2 - 10G + 3 > 0.
\]
One can check that this is satisfied by \( G_1 \) and allows to disregard \( G_2 \).

It is key that the existence of the value \( G_2 \) also plays an important role in the analysis, since it provides, by means of the construction explained above, solutions of the equation with the left-hand interface for \( \lambda < 0 \), i.e.,
\[
(6.7) \quad f^{(5)} = -5! \text{ sign } f.
\]

These arguments allow to state the following:

**Theorem 6.1.** (i) The above algebraic system (6.5) has a unique solution \( G_1 \in (0, 1) \) satisfying the positivity condition of \( f_0 = f_0(y; G_1) \) in \((-1, 0)\) and a unique solution \( G_2 \in (0, 1) \) satisfying the negativity condition of \( f_0 = f_0(y; G_2) \) in \((-1, 0)\).

(ii) Equation (5.4), with the sign “+” (respectively “−”) and \( \alpha = 0 \), has a nontrivial periodic solution \( f^+(y) \) (respectively \( f^−(y) \)), with the period
\[
(6.8) \quad T_* = 2 \ln G_1 \quad (\text{resp. } T_* = 2 \ln G_2).
\]

**Proof.** (i) We prove the result for \( \lambda \) positive and the solution corresponding to \( G_1 \). For \( \lambda < 0 \) the same arguments apply.

Once a solution of the algebraic system (6.5) has been found, we can extend the solution \( f_0(y) \) similarly to the next interval of oscillations, etc. indefinitely. Then the sequence of positive and negative humps will converge according to some geometric series. For instance, the right-hand interface of the interval of every extension, given by the standard geometric series,
\[
b_n = G + G^2 + G^3 + \ldots + G^n \quad (G = G_1 \text{ or } G_2),
\]
converges as \( n \to \infty \) to the right-hand interface of the solution, \( \frac{G}{e-1} \).

It is not hard to see that, after reflecting and moving the interface to \( y = 0^+ \), this gives precisely the behavior (5.1), i.e.,
\[
(6.9) \quad f_0(y) = (y_0 - y)^5 \varphi_*(s), \quad s = \ln(y_0 - y),
\]
where \( y_0 = \frac{G}{e-1} \) denotes the right-hand interface of \( f_0 \). We next prove that the oscillatory component \( \varphi_* \) is periodic with the period (6.8). In fact, we show that \( \varphi_* \) satisfies a stronger property,
\[
\varphi_*(s + \ln G) = -\varphi_*(s) \quad \text{for every } s \in \mathbb{R}.
\]

Hence,
\[
\varphi_*(s + 2 \ln G) = -\varphi_*(s + \ln G) = \varphi_*(s) \quad \text{for every } s \in \mathbb{R},
\]
and the periodicity stated for \( \varphi_* \) follows. According to the definitions in (6.9) and the expression of the right-hand interface \( y_0 \), one obtains that
\[
\varphi_*(s + \ln G) = G^{-5} e^{-5s} f_0(y_0 - Ge^s) \quad \text{and} \quad \varphi_*(s) = e^{-5s} f_0(y_0 - e^s).
\]
Assume, for instance, that $s$ is such that $y_0 - e^s \in (0, 1)$. Then, $y_0 - Ge^s \in (0, G)$ and, using the definition of $f_0$ in such an interval, we have that
\[ f_0(y_0 - Ge^s) = -G^5 f_0 \left( \frac{y_0 - Ge^s}{G} - 1 \right) = -G^5 f_0(y_0 - Ge^s). \]
The same argument applies in $(0, G)$ and in all the intervals obtained in every step of this construction. This completes the proof. □

As we mentioned above, by reflection of the solutions constructed above, we obtain analogous results for solutions with the left-hand interface. We note that, in both cases (the left- and right-hand interfaces), the behavior of the self-similar profile $F$ of (2.5) close to the interfaces is described by means of $f_0(y; G_2)$. The behavior corresponds to TW “travelling” in the opposite directions with $\lambda = +1$ is given by the profile $f_0(y; G_1)$.

### 6.2. Uniqueness of oscillatory TW solutions.

In Section 4, we have proved uniqueness of the solution to the Cauchy problem for the equation (2.2) with given initial conditions (4.3), when $\alpha_j \neq 0$ at least for some values of $j$. It is clear, due to the regularity conditions at the interface $y_0$, that this analysis of uniqueness does not apply to the oscillatory travelling wave solution at this point. In fact, it is not difficult to check that uniqueness fails and, for instance, the solutions $f \equiv 0$, the positive solution constructed above, and its negative counterpart are three different solutions of the same problem with analogous conditions at the interface $y = y_0$.

In order to understand the structure of the set of solutions of this type, it would be interesting to fix an additional condition that provides uniqueness of the TW. We next prove that uniqueness holds by adding to the previous conditions at the interface $y_0$, a fixed transversal zero $a_0$ of the solution and a sign to the function close to the fixed zero. Without loss of generality, we next assume $a_0 = 0$ and positivity of the solution to the left-hand side of $a_0$. The result is stated as follows:

\textbf{Theorem 6.2.} There exists a unique solution of the equation (6.1) with the interface $y_0$ with $f^{(j)}(y_0) = 0$ for $j = 0, 1, 2, 3, 4$ and satisfying $f(0) = 0$ and $f$ positive in $(-\varepsilon, 0)$. Moreover, the solution is given by $f_0(y)$, with the algebraic construction above.

\textbf{Proof.} Let $f(y)$ be a solution of (6.1) satisfying the stated assumptions. Define the auxiliary function $g(y) = f(y) + p(y)$, with $p(y)$ a polynomial of the fourth degree,
\[ p(y) = \sum_{i=1}^{4} \frac{m_j y^j}{j!}, \]
with $m_j = (f_0 - f)^{(j)}(0)$. By the definition of $g$ and taking into account the properties of $f$ and $f_0$ around $a_0 = 0$, it is not difficult to check that $g(y)$ satisfies the same equation as $f_0$ on $(-\varepsilon, \varepsilon)$ and that $g^{(j)}(0) = f^{(j)}(0)$ for $j = 0, 1, ..., 4$.

Hence, it follows from the uniqueness result of the solutions to the initial value problem stated in Section 4 that $g \equiv f_0$ at least in this interval, and, by a continuation argument (applied for instance to the integral version of the equation), we conclude the equivalence of both functions and its derivatives for every $y \leq y_0$. 

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In particular, we have at the interface \( y_0 \) that \( g^{(j)}(y_0) = 0 \) for \( j = 0, 1, 2, 3, 4 \). Taking into account the definition of \( g \) and that \( f_0 \) and \( f \) satisfy also these regularity conditions at \( y_0 \), it follows that \( p^{(j)}(y_0) = 0 \) for \( j = 0, 1, 2, 3, 4 \), whence \( p(y) \equiv 0 \) and the uniqueness result follows. □

This allows us to establish a result concerning the structure of the set of oscillatory TW solutions to equation (6.1) with finite interfaces.

**Theorem 6.3.** Assume that \( f(y) \) is an oscillatory TW solution of (6.1) with a finite interface and necessary regularity assumptions at the interface. Then, there exist real values \( \gamma > 0 \) and \( D \) such that

\[
f(y) = \pm \gamma^5 f_0(y + D).
\]

**Proof.** By using the rescaling properties of the solutions, it is enough to define an auxiliary function

\[
g(y) = \pm \lambda^5 f(y + C)
\]

with \( \lambda > 0 \) and \( C \) such that \( y_0 \) is the interface of \( g(y) \) and \( g(0) = 0 \). We also choose the sign in this definition in order to have all the hypotheses in Theorem 6.2 to hold, whence we deduce that \( g(y) \equiv f_0(y) \). It is not difficult to check that the result follows for \( \gamma = \frac{1}{\lambda} \) and \( D = -\frac{C}{\lambda} \). □

Both results have strong implications concerning the properties and the character of the oscillations of the solutions and therefore, the behaviour of solutions close to the interface. In particular, we prove the following:

**Proposition 6.4.** Assume that \( f(y) \) is an oscillatory TW solution with the interface \( y_0 \). Then there exists a finite limit:

\[
\limsup_{y \to y_0} \frac{f(y)}{(y_0 - y)^5} = C.
\]

**Proof.** We prove the result for \( f_0(y) \). Denote by \( y_1 \in (-1, 0) \), the absolute maximum point of \( f_0(y) \) in the interval \( (-1, 0) \) and define

\[
y_{n+1} = (y_n + 1)G, \quad \text{with} \quad G = \frac{y_0}{1+y_0}.
\]

From the properties of \( f_0 \) and its construction, it is not difficult to see that, so defined, \( y_n \) are the absolute maximum point of \( |f_0(y)| \) in every interval \( I_n \) of the extension defined in the algebraic construction of \( f_0 \). Taking into account the definition of \( f_0 \) and \( y_n \) yields, as \( n \to \infty \),

\[
y_n = \frac{G(1-y_1)G^{n+1}}{1-G} \quad \text{and} \quad \frac{f(y_n)}{(y_0 - y_n)^5} = \frac{G^{n+1}f(y_1)}{(y_0 - y_n)^5} \to C,
\]

so the result follows. □

As a straightforward consequence of the construction above, we also obtain the structure of the zeros of the solution \( f_0(y) \) constructed:

**Corollary 6.5.** Every zero of the solution \( f_0(y) \), except the one corresponding to the interface, is transversal, i.e., \( f' \neq 0 \).
7. Continuity geometric extension to $m \approx 1$

As the next natural step, we consider equation (3.13) with $\alpha \approx 0$. We then are going to perform the same geometric (but not algebraic!) construction explained in the steps (i)–(iii) in Section 6.1, where the only difference is that the basic profile $f_0(y)$ is not given explicitly as in (6.2). We can use the standard continuous dependence of local solutions of the ODE (3.13) on the parameter $\alpha \approx 0$. Here, we mean a class of good locally and uniquely extensible solutions without strong singularities and finite interface points. Of course, we crucially need the zero transversality property fixed in Corollary 6.5. Good local properties of such solutions are checked by standard contractivity techniques (as in Section 3.2), so we arrive at a local continuity (“homotopy $\alpha$-deformation”) result:

**Theorem 7.1.** Equation (5.4) with both signs “±” (i.e., for $\lambda = \mp 1$) has a nontrivial periodic solution $\varphi_*(s)$ of changing sign for all $\alpha \approx 0$.

It is worth recalling here that, in a global sense, for $\alpha$ sufficiently far away from 0, periodic solutions do not exist; cf. Proposition 5.2 and the heteroclinic bifurcation phenomena in (5.33).

Notice that the existence result in Theorem 7.1 is not associated with the “hyperbolicity” of the periodic orbit $\varphi_*$ in this range. Anyway, Theorem 7.1 altogether with Propositions 5.2 and 5.3, though not covering the whole $\alpha$-range, provide us with rather solid mathematical evidence on existence of periodic solutions and also on the type of heteroclinic bifurcations, at which these disappear.

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