Comment on “Greybody radiation and quasinormal modes of Kerr-like black hole in Bumblebee gravity model”

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Abstract It is shown that the paper “Greybody radiation and quasinormal modes of Kerr-like black hole in Bumblebee gravity model” (Kanzi and Sakalli in Eur Phys J C 81:501, 2021) recently published in this journal is based on an incorrect result obtained by Ding et al. (Eur Phys J C 80:178, 2020) for a Kerr-like black hole solution.

1 Comment

The paper “Greybody radiation and quasinormal modes of Kerr-like black hole in Bumblebee gravity model” [1] is an interesting work that seeks to establish a significant relationship between two important realms of the current research on astrophysics and high energy physics, namely, semi-classical gravitation associated to rotating black holes and Lorentz symmetry breakdown. This latter embodies features that are beyond the standard model of elementary particles and fundamental interactions. However, the aforementioned work exhibits inconsistencies that need to be clarified and corrected, and the present comment seeks to do this.

Firstly, the complete rotational Kerr-like black hole solution used in [1] does not satisfy the equation of motion for the bumblebee field, as it already was discussed in Ref. [2], although that has not been shown explicitly. Such a solution exists only in the slow rotation approximation, i.e., when we consider only linear terms in the rotational parameter, $a$. On the other hand, it is possible to show that neither the modified Einstein equations are satisfied. In fact, after performing a series of manipulations of the gravitational field equations, the authors in [3] claim to have found an exact Kerr-like solution for the Einstein-bumblebee theory, which in Boyer–Lindquist coordinates can be written as

$$ds^2 = -(1 - \frac{2Mr}{\Sigma})dt^2 - \frac{4Mra\sqrt{1 + \ell}\sin^2 \theta}{\Sigma} dt d\phi$$

$$+ \sum_{\Delta} d\rho^2 + \sum_{\Sigma} d\theta^2 + \frac{A}{\Sigma} \sin^2 \theta d\phi^2,$$ (1.1)

where $\ell = \rho b^2$ encodes contribution from the Lorentz-violating effects, and the functions $\Sigma$, $\Delta$ and $A$ are defined as

$$\Sigma(\rho, \theta) = r^2 + (1 + \ell)a^2 \cos^2 \theta,$$

$$\Delta(r) = \frac{r^2 - 2Mr}{1 + \ell} + a^2,$$

$$A(r, \theta) = [r^2 + (1 + \ell)a^2]^2 - \Delta(1 + \ell)a^2 \sin^2 \theta.$$ (1.2)

On the other hand, the equation of motion for the bumblebee field is

$$\nabla^\mu b_{\mu\nu} + \frac{q}{\kappa} b^{\mu} R_{\mu\nu} = 0,$$ (1.3)

where the bumblebee field strength is $b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$, with $q$ being a coupling constant, and $\kappa = 8\pi G_N$. Notice that the original bumblebee field $B_\mu$ was frozen at its vacuum expectation value (vev) i.e., $B_\mu = b_\mu$ and that its vev has a constant norm $b_\mu b^\mu = \mp b^2$. These vacuum conditions ensure that the bumblebee field preserves the minimum of the potential, i.e., $V' = 0 = V$.

The assumed condition for the bumblebee field is

$$b_\mu = \left(0, b_\ell, 0, 0\right),$$ (1.4)

such that $b_\mu b^{\mu\nu} = b^2$ is a positive constant. Thus, the nonzero components for the bumblebee field strength are $b_{\tau\phi} = -b_{\phi\tau} = -b_{\phi\tau} \Sigma/2\sqrt{\Delta} \Sigma$, and the relevant components obtained from Eq. (1.3) are $\nabla^\mu b_{\mu\nu} + \frac{q}{\kappa} b^\mu R_{\mu\nu}$ and $\nabla^\mu b_{\mu\phi} + \frac{q}{\kappa} b^\mu R_{\mu\phi}$.
For the metric (1.1), the bumblebee equations of motion can be explicitly written as

\[
\nabla^\theta b_{\theta r} + \frac{\rho}{\kappa} b' R_{r r} = \frac{a^2}{2 \kappa (r^2 + a^2(1 + \ell) \cos^2 \theta)} \\
\times \sqrt{(1 + \ell) \left( \frac{r^2 + a^2(1 + \ell) \cos^2 \theta}{r^2 - 2Mr + a^2(1 + \ell)} \right)} \\
\times \left[ a^2(1 + \ell) \cos^2 \theta \left( 3\ell^2 - 4b^2\kappa(1 + \ell) + \ell^2 \cos 2\theta \right) \\
+ r^2 \left( \ell^2 - b^2\kappa(1 + \ell) \right) (1 + 3 \cos 2\theta) \right] \\
\neq 0,
\]

(1.5)

and

\[
\nabla^r b_{\theta \theta} + \frac{\rho}{\kappa} b' R_{r \theta} = \frac{a^2 b \sin 2\theta}{4 \left( r^2 + a^2(1 + \ell) \cos^2 \theta \right)^3} \\
\times \sqrt{(1 + \ell) \left( \frac{r^2 + a^2(1 + \ell) \cos^2 \theta}{r^2 - 2Mr + a^2(1 + \ell)} \right)} \\
\times \left[ r \left( a^2(1 + \ell)(-5 + \cos 2\theta) + 10Mr - 4r^2 \right) \\
- 2a^2(1 + \ell)Mr \cos^2 \theta \right] \\
\neq 0,
\]

(1.6)

where we replace \( \rho = \ell/b^2 \). In this way, the solution presented in [3] fails to solve exactly the equation of motion for the bumblebee field. However, in the slowly rotating limit, i.e., \( a^2 \to 0 \), these equations are fulfilled. They become

\[
\nabla^\theta b_{\theta r} + \frac{\rho}{\kappa} b' R_{r r} = \frac{a^2(\ell^2 - b^2(1 + \ell)\kappa)(1 + 3 \cos 2\theta)}{2bkr^4} \sqrt{\frac{(1 + \ell)r}{r^2 - 2M}} \\
+ \mathcal{O}(a^3) \equiv 0 \quad \text{when} \quad a^2 \to 0,
\]

(1.7)

and

\[
\nabla^r b_{\theta \theta} + \frac{\rho}{\kappa} b' R_{r \theta} = \frac{-a^2b(2r - 5M) \sin 2\theta}{2r^4} \sqrt{\frac{(1 + \ell)r}{r^2 - 2M}} + \mathcal{O}(a^3) \\
\equiv 0 \quad \text{when} \quad a^2 \to 0,
\]

(1.8)

where \( \mathcal{O}(a^3) \) denotes terms of order greater than or equal to \( a^3 \). It is worth mentioning that for a small \( b \) limit, the Eqs. (1.5) and (1.6) should at least be proportional to \( b \). Indeed, these quantities take the form

\[
\nabla^\theta b_{\theta r} + \frac{\rho}{\kappa} b' R_{r r} = -\frac{ba^2}{2 \kappa} \frac{[4a^2 \cos^2 \theta + r^2 (1 + 3 \cos 2\theta)]}{\left( r^2 + a^2 \cos^2 \theta \right)^3} \\
\times \sqrt{\frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2Mr + a^2}} + \mathcal{O}(b^2),
\]

(1.9)

and

\[
\nabla^r b_{\theta \theta} + \frac{\rho}{\kappa} b' R_{r \theta} = -\frac{ba^2}{4 \kappa} \frac{[4r^3 - 10Mr^2 + 2a^2M \cos^2 \theta + a^2r(5 - \cos 2\theta)] \sin 2\theta}{\left( r^2 + a^2 \cos^2 \theta \right)^3} \\
\times \sqrt{\frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2Mr + a^2}} + \mathcal{O}(b^2),
\]

(1.10)

and essentially, the equation of motion (1.3) is not satisfied at the small \( b \) limit either. Such results are in agreement with those presented in [2], and similarly, it can be shown explicitly (using the computer algebra program [4]) that the Einstein-bumblebee gravitational equations are also not satisfied by the solution (1.1). Thus, the existence of a full rotating black hole solution for the Einstein-bumblebee theory remains an open issue.

From the foregoing, the authors of reference [1] spoiled their work from the beginning, and we fear it is irremediably. However, the physical implications of the studied model might be considered if the original results are fitted to the slow rotating solution approximation.

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Appendix: Some quantities

In this appendix, we list out the nonzero components of the Ricci tensor and the covariant derivative of $b_{\mu\nu}$ to make checking Eqs. (1.5) and (1.6) easier for the readers. For the metric (1.1), they take the explicit form:

$$R_{rr} = \frac{\ell a^2(1+\ell) [a^2(1+\ell) \cos^2 \theta (3 + \cos 2\theta) + r^2(1 + 3 \cos 2\theta)]}{2(r^2 - 2Mr + a^2(1 + \ell)(r^2 + a^2(1 + \ell) \cos^2 \theta))^2},$$

(A.1)

which at the limit $\ell \to 0$ it vanishes, and additionally, we verify that the component $R_{r\theta}$ is identically null. These results agree that the Kerr metric is Ricci flat in the absence of the Lorentz violation since it is the solution of the vacuum Einstein equations. Besides, the relevant covariant derivatives are given by

$$\nabla^\theta b_{r\ell} = -\frac{a^2b(1 + \ell)}{2(r^2 + a^2(1 + \ell) \cos^2 \theta)^3} \times \left[ \frac{(1 + \ell)(r^2 + a^2(1 + \ell) \cos^2 \theta)}{r^2 - 2Mr + a^2(1 + \ell)} \times \left[ 4a^2(1 + \ell) \cos^2 \theta + r^2(1 + 3 \cos 2\theta) \right] \right],$$

(A.2)

and

$$\nabla^r b_{r\theta} = \frac{a^2 b \sin 2\theta}{4(r^2 + a^2(1 + \ell) \cos^2 \theta)^3} \times \sqrt{\frac{(1 + \ell)(r^2 + a^2(1 + \ell) \cos^2 \theta)}{r^2 - 2Mr + a^2(1 + \ell)}} \times \left[ r \left( a^2(1 + \ell)(-5 + \cos 2\theta) + 10Mr - 4r^2 \right) - 2a^2(1 + \ell)M \cos^2 \theta \right].$$

(A.3)

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