A Greedy Partition Lemma for Directed Domination

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Abstract

A directed dominating set in a directed graph $D$ is a set $S$ of vertices of $V$ such that every vertex $u \in V(D) \setminus S$ has an adjacent vertex $v$ in $S$ with $v$ directed to $u$. The directed domination number of $D$, denoted by $\gamma(D)$, is the minimum cardinality of a directed dominating set in $D$. The directed domination number of a graph $G$, denoted $\Gamma_d(G)$, which is the maximum directed domination number $\gamma(D)$ over all orientations $D$ of $G$. The directed domination number of a complete graph was first studied by Erdős [Math. Gaz. 47 (1963), 220–222], albeit in disguised form. In this paper we prove a Greedy Partition Lemma for directed domination in oriented graphs. Applying this lemma, we obtain bounds on the directed domination number. In particular, if $\alpha$ denotes the independence number of a graph $G$, we show that $\alpha \leq \Gamma_d(G) \leq \alpha(1 + 2\ln(n/\alpha))$.

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1 Introduction

An asymmetric digraph or oriented graph \( D \) is a digraph that can be obtained from a graph \( G \) by assigning a direction to (that is, orienting) each edge of \( G \). The resulting digraph \( D \) is called an orientation of \( G \). Thus if \( D \) is an oriented graph, then for every pair \( u \) and \( v \) of distinct vertices of \( D \), at most one of \((u, v)\) and \((v, u)\) is an arc of \( D \). A directed dominating set, abbreviated DDS, in a directed graph \( D = (V, A) \) is a set \( S \) of vertices of \( V \) such that every vertex in \( V \setminus S \) is dominated by some vertex of \( S \); that is, every vertex \( u \in V \setminus S \) has an adjacent vertex \( v \) in \( S \) with \( v \) directed to \( u \). Every digraph has a DDS since the entire vertex set of the digraph is such a set.

The directed domination number of a directed graph \( D \), denoted by \( \gamma(D) \), is the minimum cardinality of a DDS in \( D \). A DDS of \( D \) of cardinality \( \gamma(D) \) is called a \( \gamma(D) \)-set. Directed domination in digraphs is well studied (cf. [2, 3, 9, 10, 12, 16, 17, 19, 22, 24]).

The directed domination number of a graph \( G \), denoted \( \Gamma_d(G) \), is defined in [7] as the maximum directed domination number \( \gamma(D) \) over all orientations \( D \) of \( G \); that is,

\[
\Gamma_d(G) = \max \{ \gamma(D) \mid \text{over all orientations } D \text{ of } G \}.
\]

The directed domination number of a complete graph was first studied by Erdős [14] albeit in disguised form. In 1962, Schütte [14] raised the question of given any positive integer \( k > 0 \), does there exist a tournament \( T_{n(k)} \) on \( n(k) \) vertices in which for any set \( S \) of \( k \) vertices, there is a vertex \( u \) which dominates all vertices in \( S \). Erdős [14] showed, by probabilistic arguments, that such a tournament \( T_{n(k)} \) does exist, for every positive integer \( k \). The proof of the following bounds on the directed domination number of a complete graph are along identical lines to that presented by Erdős [14]. This result can also be found in [24]. Throughout this paper, \( \log \) is to the base 2 while \( \ln \) denotes the logarithm in the natural base \( e \).

**Theorem 1** (Erdős [14]) For \( n \geq 2 \), \( \log n - 2 \log(\log n) \leq \Gamma_d(K_n) \leq \log(n + 1) \).

In [7] this notion of directed domination in a complete graph is extended to directed domination of all graphs.

1.1 Notation

For notation and graph theory terminology we in general follow [18]. Specifically, let \( G = (V, E) \) be a graph with vertex set \( V \) of order \( n = |V| \) and edge set \( E \) of size \( m = |E| \), and let \( v \) be a vertex in \( V \). The open neighborhood of \( v \) is \( N_G(v) = \{ u \in V \mid uv \in E \} \) and the closed neighborhood of \( v \) is \( N_G[v] = \{ v \} \cup N_G(v) \). If the graph \( G \) is clear from context, we simply write \( N(v) \) and \( N[v] \) rather than \( N_G(v) \) and \( N_G[v] \), respectively. For a set \( S \subseteq V \), the subgraph induced by \( S \) is denoted by \( G[S] \). If \( A \) and \( B \) are subsets of \( V(G) \), we let \([A, B]\) denote the set of all edges between \( A \) and \( B \) in \( G \).
We denote the degree of \( v \) in \( G \) by \( d_G(v) \), or simply \( d(v) \) if the graph \( G \) is clear from context. The average degree in \( G \) is denoted by \( d_{av}(G) \). The minimum degree among the vertices of \( G \) is denoted by \( \delta(G) \), and the maximum degree by \( \Delta(G) \). The parameter \( \gamma(G) \) denotes the domination number of \( G \). The parameters \( \alpha(G) \) and \( \alpha'(G) \) denote the (vertex) independence number and the matching number, respectively, of \( G \), while the parameters \( \chi(G) \) and \( \chi'(G) \) denote the chromatic number and edge chromatic number, respectively, of \( G \). The covering number of \( G \), denoted by \( \beta(G) \), is the minimum number of vertices that covers all the edges of \( G \).

A vertex \( v \) in a digraph \( D \) out-dominates, or simply dominates, itself as well as all vertices \( u \) such that \( (v, u) \) is an arc of \( D \). The out-neighborhood of \( v \), denoted \( N^+(v) \), is the set of all vertices \( u \) adjacent from \( v \) in \( D \); that is, \( N^+(v) = \{ u \mid (v, u) \in A(D) \} \). The out-degree of \( v \) is given by \( d^+(v) = |N^+(v)| \), and the maximum out-degree among the vertices of \( D \) is denoted by \( \Delta^+(D) \). The in-neighborhood of \( v \), denoted \( N^-(v) \), is the set of all vertices \( u \) adjacent to \( v \) in \( D \); that is, \( N^-(v) = \{ u \mid (u, v) \in A(D) \} \). The in-degree of \( v \) is given by \( d^-(v) = |N^-(v)| \). The closed in-neighborhood of \( v \) is the set \( N^-[v] = N^-(v) \cup \{ v \} \). The maximum in-degree among the vertices of \( D \) is denoted by \( \Delta^-(D) \).

### 1.2 Known Results

We shall need the following inequality chain established in \([7]\).

**Theorem 2** \([7]\) For every graph \( G \) on \( n \) vertices, \( \gamma(G) \leq \alpha(G) \leq \Gamma_d(G) \leq n - \alpha'(G) \).

## 2 The Greedy Partition Lemma and its Applications

In this section we present our key lemma, which we call the Greedy Partition Lemma, and its applications. The Greedy Partition Lemma is a generalization of earlier results by Caro \([5, 6]\), Caro and Tuza \([8]\), and Jensen and Toft \([20]\).

First we introduce some additional terminology. Let \( G \) be a hypergraph and let \( P \) be a hypergraph property. Let \( P(G) = \max \{|V(H)| \mid H \text{ is an induced subhypergraph of } G \text{ that satisfies property } P \} \). Let \( \chi(G, P) \) be the minimum number \( q \) such that there exist a partition \( V(G) = (V_1, V_2, \ldots, V_q) \) such that \( V_i \) induces a subhypergraph having property \( P \) for all \( i = 1, 2, \ldots, q \). For example, if \( P \) is the property of independence, then \( P(G) = \alpha(G) \), while \( \chi(G, P) = \chi(G) \). If \( P \) is the property of edge independence, the \( P(G) = \alpha'(G) \), while \( \chi(G, P) = \chi'(G) \). If \( P \) is the property of being \( d \)-degenerate (recall that a \( d \)-degenerate graph is a graph \( G \) in which every induced subgraph of \( G \) has a vertex with degree at most \( d \)), then \( P(G) \) is the maximum cardinality of a \( d \)-degenerate subgraph and \( \chi(G, P) \) is the minimum partition of \( V(G) \) into induced \( d \)-degenerate graphs. For a subhypergraph \( H \) of a hypergraph \( G \), we let \( G - H \) be the subhypergraph of \( G \) with vertex set \( V(G) \setminus V(H) \).

We are now in a position to state the Greedy Partition Lemma.
Lemma 3 (Greedy Partition Lemma) Let $\mathcal{H}$ be a class of hypergraphs closed under induced subhypergraphs. Let $t \geq 2$ be an integer and let $f : [t, \infty) \to [1, \infty)$ be a positive nondecreasing continuous function. Let $P$ be a hypergraph property such that for every hypergraph $G \in \mathcal{H}$ the following holds.

(a) If $|V(G)| \leq t$, then $\chi(G, P) \leq |V(G)|$.

(b) If $|V(G)| \geq t$, then $|V(G)| \geq P(G) \geq f(|V(G)|)$.

Then for every hypergraph $G \in \mathcal{H}$ of order $n$,

$$\chi(G, P) \leq t + \int_t^{\max(n, t)} \frac{1}{f(x)} \, dx.$$ 

Proof. We proceed by induction on $n$. We first observe that the value of the given integral is always non-negative. If $n \leq t$, then by condition (a), $\chi(G, P) \leq n \leq t$, and the inequality holds trivially. This establishes the base case. For the inductive hypothesis, assume the inequality holds for every hypergraph in $\mathcal{H}$ with less than $n$ vertices and let $G \in \mathcal{H}$ of order $n$. As observed earlier, if $n \leq t$, then the inequality holds trivially. Hence we may assume that $n > t$. Let $P(G) = z = |V(H)|$ be the cardinality of the largest induced subhypergraph $H$ of $G$ that has property $P$. By condition (b), $z \geq f(n)$. If $z \geq n - t + 1$, then $n - z = |V(G) \setminus V(H)| \leq t - 1$, and so by condition (a), $\chi(G - H, P) \leq t - 1$. Hence, $\chi(G, P) \leq \chi(G - H, P) + 1 \leq t$ and the inequality holds trivially. Therefore we may assume that $z \leq n - t$, and so $|V(G) \setminus V(H)| \geq t$. Thus applying the inductive hypothesis to the induced subhypergraph $G - H \in \mathcal{H}$, and using condition (b), we have that

$$\int_t^n \frac{1}{f(x)} \, dx = \int_t^{n-z} \frac{1}{f(x)} \, dx + \int_{n-z}^n \frac{1}{f(x)} \, dx$$

$$\geq \chi(G - H, P) - t + \int_{n-z}^n \frac{1}{f(x)} \, dx$$

$$\geq \chi(G - H, P) - t + \int_{n-z}^n \frac{1}{f(n)} \, dx$$

$$= \chi(G - H, P) - t + z/f(n)$$

$$\geq \chi(G, P) - 1 - t + 1$$

$$\geq \chi(G, P) - t,$$

which completes the proof of the Greedy Partition Lemma. ☐

We next discuss several applications of the Greedy Partition Lemma. For this purpose, we shall need the following lemma. Recall that $d_{av}(G)$ denotes the average degree in a graph $G$. 

4
Lemma 4  For \( k \geq 1 \) an integer, let \( G \) be a graph with \( k \geq \alpha(G) \) and let \( D \) be an orientation of \( G \). Let \( H \) be an induced subgraph of \( G \) of order \( n_H \geq k \) and size \( m_H \), and let \( D_H \) be the orientation of \( H \) induced by \( D \). Then the following holds.

(a) \( m_H \geq n_H(n_H - k)/2k \).
(b) \( \Delta^+(D_H) \geq (n_H - k)/2k \).

Proof. Since \( H \) is an induced subgraph of \( G \), every independent set in \( H \) is an independent set in \( G \). In particular, \( k \geq \alpha(G) \geq \alpha(H) \). Thus applying the Caro-Wei Theorem (see [4, 25]), we have

\[
\sum_{v \in V(H)} \frac{1}{d_H(v) + 1} \geq \frac{n_H}{d_{av}(H) + 1} = \frac{n_H}{(2m_H/n_H) + 1} = \frac{n_H^2}{2m_H + n_H},
\]

or, equivalently, \( m_H \geq n_H(n_H - k)/2k \). This establishes part (a). Part (b) follows readily from Part (a) and the observation that

\[
n_H \cdot \Delta^+(D_H) \geq \sum_{v \in V(D_H)} d^+(v) = m_H. \quad \Box
\]

2.1 Independence Number

Using the Greedy Partition Lemma we present an upper bound on the directed domination number of a graph in terms of its independence number. First we introduce some additional notation. Let \( \alpha \geq 1 \) be an integer and let \( \mathcal{G}_\alpha \) be the class of all graphs \( G \) with \( \alpha \geq \alpha(G) \). Since every induced subgraph \( F \) of \( G \in \mathcal{G}_\alpha \) satisfies \( \alpha \geq \alpha(G) \geq \alpha(F) \), the class \( \mathcal{G}_\alpha \) of graphs is closed under induced subgraphs.

Theorem 5  For \( \alpha \geq 1 \) an integer, if \( G \in \mathcal{G}_\alpha \) has order \( n \geq \alpha \), then

\[
\Gamma_d(G) \leq \alpha(1 + 2 \ln(n/\alpha)).
\]

Proof. If \( \alpha = 1 \), then \( G = K_n \) and by Theorem 4 \( \Gamma_d(G) \leq \log(n + 1) \leq 1 + 2 \ln n = \alpha(1 + 2 \ln(n/\alpha)). \) Hence we may assume that \( \alpha \geq 2 \), for otherwise the desired bound holds. We now apply the Greedy Partition Lemma with \( t = \alpha \) and with \( f(x) \) the positive nondecreasing continuous function on \([\alpha, \infty)\) defined by \( f(x) = (x - \alpha)/2\alpha + 1 \) where \( x \geq [\alpha, \infty) \). Let \( P(G) = 1 + \min\{\Delta^+(D)\} \), where the minimum is taken over all orientations \( D \) of \( G \). Then, \( \Gamma_d(G) \leq \chi(G, P) \). To show that the conditions of the Greedy Partition Lemma are satisfied, we consider an arbitrary graph \( H \in \mathcal{G}_\alpha \), where \( H \) has order \( |V(H)| = n_H \). If \( |V(H)| \leq \alpha \), then \( \Gamma_d(H) \leq \chi(H, P) \leq \alpha \) since in this case \( H \) may be the empty graph on \( \alpha \) vertices. Thus condition (a) of Lemma 3 holds. If \( |V(H)| \geq \alpha \) and \( D \) is an arbitrary orientation of \( H \), then by Lemma 4 \( \Delta^+(D) \geq (n_H - \alpha)/2\alpha \), and so \( |V(H)| \geq P(H) \geq \)}
\[(n_H - \alpha)/2\alpha + 1 = f(n_H).\] Therefore condition (b) of Lemma 3 holds. Hence by the Greedy Partition Lemma,

\[
\Gamma_d(G) \leq \alpha + \int_{\alpha}^{n} \frac{1}{(x - \alpha)/2\alpha + 1} \, dx \\
= \alpha + 2\alpha \int_{\alpha}^{n} \frac{1}{x + \alpha} \, dx \\
= \alpha + 2\alpha \ln((n + \alpha)/2\alpha) \\
\leq \alpha + 2\alpha \ln(n/\alpha) \\
= \alpha(1 + 2\ln(n/\alpha)). \tag*{\Box}
\]

Observe that for every graph \(G\) of order \(n\), we have \(\chi(G) \geq n/\alpha(G)\) and \(d_{av}(G) + 1 \geq n/\alpha(G)\). Hence as an immediate consequence of Theorem 5, we have the following bounds on the directed domination number of a graph.

**Corollary 1** Let \(G\) be a graph of order \(n\). Then the following holds.

(a) \(\Gamma_d(G) \leq \alpha(G)(1 + 2\ln(\chi(G)))\).
(b) \(\Gamma_d(G) \leq \alpha(G)(1 + 2\ln(d_{av}(G) + 1))\).

### 2.2 Degenerate Graphs

A \(d\)-degenerate graph is a graph \(G\) in which every induced subgraph of \(G\) has a vertex with degree at most \(d\). The property of being \(d\)-degenerate is a hereditary property that is closed under induced subgraphs, as is the property of the complement of a graph being \(d\)-degenerate. For \(d \geq 1\) an integer, let \(\mathcal{F}_d\) be the class of all graphs \(G\) whose complement is a \(d\)-degenerate graph. Thus the class \(\mathcal{F}_d\) of graphs is closed under induced subgraphs. We shall need the following lemma.

**Lemma 6** For \(d \geq 1\) an integer, let \(G \in \mathcal{F}_d\) and let \(H\) be an induced subgraph of \(G\) of order \(n_H\). If \(D\) is an orientation of \(G\) and \(D_H\) is the orientation of \(H\) induced by \(D\), then \(\Delta^+(D_H) > (n_H - 1)/2 - d\).

**Proof.** Since \(G \in \mathcal{F}_d\), the graph \(G\) is the complement of a \(d\)-degenerate graph \(\overline{G}\). Let \(G\) have order \(n\) and size \(m\), and let \(\overline{G}\) have size \(\overline{m}\). It is a well-known fact that we can label the vertices of the \(d\)-degenerate graph \(\overline{G}\) with vertex labels \(1, 2, \ldots, n\) such that each vertex with label \(i\) is incident to at most \(d\) vertices with label greater than \(i\), implying that \(\overline{m} \leq dn - d(d + 1)/2\). Therefore, \(m \geq n(n - 1)/2 - dn + d(d + 1)/2\). This is true for every graph \(G\) whose complement is a \(d\)-degenerate graph. In particular, this is true for the induced subgraph \(H\) of \(G\). Therefore if \(H\) has size \(m_H\), we have \(\sum_{v \in V(H)} d^-_{D_H}(v) = m_H \geq n_H(n_H - 1)/2 - dn_H + d(d + 1)/2\). Hence, \(\Delta^+(D_H) > (n_H - 1)/2 - d\). \(\Box\)
Theorem 7  For \( d \geq 1 \) an integer, if \( G \in \mathcal{F}_d \) has order \( n \), then
\[
\Gamma_d(G) \leq 2d + 1 + 2\ln(n - 2d + 1)/2.
\]

Proof. We apply the Greedy Partition Lemma with \( t = 2d + 1 \) and with \( f(x) = (x - 1)/2 - d + 1 \) where \( x \geq [2d + 1, \infty) \). Let \( P(G) = 1 + \min\{\Delta^+(D)\} \), where the minimum is taken over all orientations \( D \) of \( G \). Then, \( \Gamma_d(G) \leq \chi(G, P) \). To show that the conditions of the Greedy Partition Lemma are satisfied, we consider an arbitrary graph \( H \in \mathcal{F}_d \), where \( H \) has order \( |V(H)| = n_H \). If \( |V(H)| \leq 2d + 1 \), then \( \Gamma_d(H) \leq \chi(H, P) \leq 2d + 1 \) since in this case \( H \) may be the empty graph on \( 2d + 1 \) vertices. Thus condition (a) of Lemma 3 holds. If \( |V(H)| \geq 2d + 1 \) and \( D \) is an arbitrary orientation of \( H \), then by Lemma 6, \( \Delta^+(D) \geq (n_H - 1)/2 - d \), and so \( |V(H)| \geq P(H) \geq (n_H - 1)/2 - d + 1 = f(n_H) \). Therefore condition (b) of Lemma 3 holds. Hence by the Greedy Partition Lemma,
\[
\Gamma_d(G) \leq 2d + 1 + \int_{2d+1}^{n} \frac{1}{(x - 1)/2 - d + 1} \, dx
\]
\[
= 2d + 1 + \int_{2d+1}^{n} \left( \frac{2}{x - 2d + 1} \right) \, dx
\]
\[
= 2d + 1 + 2 \int_{2}^{n-2d+1} \frac{1}{x} \, dx
\]
\[
\leq 2d + 1 + 2\ln(n - 2d + 1)/2. \square
\]

2.3 \( K_{1,m} \)-Free Graphs

In this section, we establish an upper bound on the directed domination number of a \( K_{1,m} \)-free graph. We first recall the well-known bound for the usual domination number \( \gamma \), which was proved independently by Arnautov in 1974 and in 1975 by Lovász and by Payan.

Theorem 8 (Arnautov [1], Lovász [21], Payan [23]) If \( G \) is a graph on \( n \) vertices with minimum degree \( \delta \), then \( \gamma(G) \leq n(\log(\delta + 1) + 1)/(\delta + 1) \).

We show that the above bound on \( \gamma \) is nearly preserved by the directed domination number \( \Gamma_d \) when we restrict our attention to \( K_{1,m} \)-free graphs. For this purpose, we shall need the following result due to Faudree et al. [15].

Theorem 9 ([15]) If \( G \) is a \( G \) is a \( K_{1,m} \)-free graph of order \( n \) with \( \delta(G) = \delta \) and \( \alpha(G) = \alpha \), then \( \alpha \leq (m - 1)n/(\delta + m - 1) \).

We shall prove the following result.
Theorem 10  For $m \geq 3$, if $G$ is a $K_{1,m}$-free graph of order $n$ with $\delta(G) = \delta$, then
\[ \Gamma_d(G) < (2(m - 1)n \ln(\delta + m - 1))/ (\delta + m - 1).\]

Proof. If $\delta < (\sqrt{e} - 1)(m - 1)$, where $e$ is the base of the natural logarithm, then $\delta < m - 1$ and so $(2(m - 1)n \ln(\delta + m - 1))/ (\delta + m - 1) > n \ln(\delta + m - 1) > n$. Hence we may assume that $\delta \geq (\sqrt{e} - 1)(m - 1)$, for otherwise the desired upper bound holds trivially. By Theorem 9, $\alpha \leq (m - 1)n/(\delta + m - 1)$. Substituting $\delta \geq (\sqrt{e} - 1)(m - 1)$ into this inequality, we get $\alpha \leq (m - 1)n / ((\sqrt{e} - 1)(m - 1) + m - 1) = (m - 1)n / (\sqrt{e}(m - 1) = n / \sqrt{e}$. Since the function $x(1 + 2 \ln(n / x))$ is monotone increasing in the interval $[1, n / \sqrt{e}]$, we get, by Theorem 5, that
\[\Gamma_d(G) \leq \alpha(1 + 2 \ln(n / \alpha)) \leq ((m - 1)n / (\delta + m - 1))(1 + 2 \ln(n(\delta + m - 1) / (m - 1)n)) = ((m - 1)n / (\delta + m - 1))(1 + 2 \ln((\delta + m - 1)/(m - 1))) = 2(m - 1)n(1/2 + \ln((\delta + m - 1)/(m - 1)))/ (\delta + m - 1) = 2(m - 1)n(\ln \sqrt{e} + \ln((\delta + m - 1)/(m - 1)))/ (\delta + m - 1) < (2(m - 1)n \ln(\delta + m - 1)/ (\delta + m - 1),\]
as $\sqrt{e} < m - 1$. □

We observe that as a special case of Theorem 10, we have that if $G$ is a claw-free graph of order $n$ with $\delta(G) = \delta$, then $\Gamma_d(G) \leq (4n(\log(\delta + 2)))(\delta + 2)$.

2.4 Nordhaus-Gaddum-Type Bounds

In this section we consider Nordhaus-Gaddum-type bounds for the directed domination of a graph. Let $\mathcal{G}_n$ denote the family of all graphs of order $n$. We define
\[\text{NG}_{\min}(n) = \min \left\{ \Gamma_d(G) + \Gamma_d(\overline{G}) \right\}\]
\[\text{NG}_{\max}(n) = \max \left\{ \Gamma_d(G) + \Gamma_d(\overline{G}) \right\}\]
where the minimum and maximum are taken over all graphs $G \in \mathcal{G}_n$. Chartrand and Schuster [11] established the following Nordhaus-Gaddum inequalities for the matching number: If $G$ is a graph on $n$ vertices, then $\lceil n / 2 \rceil \leq \alpha'(G) + \alpha'(\overline{G}) \leq 2 \lceil n / 2 \rceil$.

Theorem 11  The following holds.
(a) $c_1 \log n \leq \text{NG}_{\min}(n) \leq c_2 (\log n)^2$ for some constants $c_1$ and $c_2$.
(b) $n + \log n - 2 \log(\log n) \leq \text{NG}_{\max}(n) \leq n + \lceil n / 2 \rceil$.

Proof. (a) By Ramsey’s theory, for all graphs $G \in \mathcal{G}_n$ we have $\max \{ \alpha(G), \alpha(G) \} \geq c \log n$ for some constant $c$. Hence by Theorem 2(a), $\Gamma_d(G) + \Gamma_d(\overline{G}) \geq \alpha(G) + \alpha(\overline{G}) \geq c \log n$.
for some constant $c_1$. Further by Ramsey’s theory there exists a graph $G \in \mathcal{G}_n$ such that $\max\{\alpha(G), \alpha(\overline{G})\} \leq d \log n$ for some constant $d$. Hence by Theorem 3 $\Gamma_d(G) + \Gamma_d(\overline{G}) \leq 2d \log n(1 + 2 \log(n/d \log n)) \leq c_2(\log n)^2$ for some constant $c_2$. This establishes Part (a).

(b) By Theorem 1 $\Gamma_d(K_n) + \Gamma_d(\overline{K}_n) \leq n + \log n - 2 \log(\log n)$. Hence, $\text{NG}_{\text{max}}(n) \geq n + \log n - 2 \log(\log n)$. By Theorem 2(b) and by the Nordhaus–Gaddum inequalities, we have that $\Gamma_d(G) + \Gamma_d(\overline{G}) \leq 2n - (\alpha'(G) + \alpha'(\overline{G})) \leq 2n - \lfloor n/2 \rfloor = n + \lfloor n/2 \rfloor$. $\square$

3 Two Generalizations

In this section, we present two general frameworks of directed domination in graphs.

3.1 Directed Multiple Domination

For an integer $r \geq 1$, a directed $r$-dominating set, abbreviated DrDS, in a directed graph $D = (V, A)$ is a set $S$ of vertices of $V$ such that for every vertex $u \in V \setminus S$, there are at least $r$ vertices $v$ in $S$ with $v$ directed to $u$. The directed $r$-domination number of a directed graph $D$, denoted by $\gamma_r(D)$, is the minimum cardinality of a DrDS in $D$. An DrDS of $D$ of cardinality $\gamma_r(D)$ is called a $\gamma_r(D)$-set. The directed $r$-domination number of a graph $G$, denoted $\Gamma_{d,r}(G)$, is defined as the maximum directed $r$-domination number $\gamma_r(D)$ over all orientations $D$ of $G$; that is, $\Gamma_{d,r}(G) = \max\{\gamma_r(D) \mid \text{over all orientations } D \text{ of } G\}$. In particular, we note that $\Gamma_d(G) = \Gamma_{d,1}(G)$.

Theorem 12 Let $r \geq 1$ be an integer. Let $G$ be a graph of order $n$ with $\alpha(G) = \alpha$. Then the following holds.

(a) $\Gamma_{d,r}(K_n) \leq r \log(n + 1)$.

(b) $\Gamma_{d,r}(G) \leq r \alpha(1 + 2 \ln(n/\alpha))$.

Proof. (a) By Theorem 3 $\Gamma_d(K_n) \leq \log(n + 1)$. Let $D_1$ be an orientation of $K_n$ and let $S_1$ be a $\gamma(D_1)$-set. Then, $|S_1| \leq \log(n + 1)$. We now remove the vertices of the DDS $S_1$ from $D_1$ to produce an orientation $D_2$ of $K_{n_1}$ where $n_1 = n - |S_1|$. Let $S_2$ be a $\gamma(D_2)$-set. By Theorem 3 $|S_2| \leq \log(n_1 + 1) < \log(n + 1)$. We now remove the vertices of the DDS $S_2$ from $D_2$ to produce an orientation $D_3$ of $K_{n_2}$ where $n_2 = n - |S_1| - |S_2|$ and we let $S_3$ be a $\gamma(D_3)$-set. Continuing in this way, we produce a sequence $S_1, S_2, \ldots, S_r$ of sets whose union is a DrDS of $K_n$ of cardinality $\sum_{i=1}^{r} |S_i| \leq r \log(n + 1)$. This is true for every orientation $D$ of $K_n$. Hence, $\Gamma_{d,r}(K_n) \leq r \log(n + 1)$. This establishes Part (a).

(b) By Theorem 3 $\Gamma_d(G) \leq \alpha(1 + 2 \ln(n/\alpha))$. We first consider the case when $\alpha \geq n/\sqrt{e}$. Then, $r \alpha(1 + 2 \ln(n/\alpha)) > n$ for $r = 2$. However the function $x(1 + 2 \ln(n/x))$ is monotone increasing in the interval $[1, n/\sqrt{e}]$ and we may therefore assume that $\alpha \leq n/\sqrt{e}$, for otherwise the desired result holds trivially.
Let $D_1$ be an arbitrary orientation of $G$ and let $S_1$ be a DDS of $G$. We now remove the vertices of $S_1$ from $D_1$ to produce an orientation $D_2$ of the graph $G_1 = G - S_1$ where $G_1$ has order $n_1 = n - |S|$. Let $\alpha(G_1) = \alpha_1$. Since $G_1$ is an induced subgraph of $G$, we have $\alpha_1 \leq \alpha$. By Theorem 13, $\Gamma_d(G_1) \leq \alpha_1 \left(1 + 2\ln \left(n_1/\alpha_1\right)\right) < \alpha_1 \left(1 + 2\ln \left(n/\alpha_1\right)\right)$. Since $\alpha_1 \leq \alpha \leq n/\sqrt{e}$, the monotonicity of the function $x(1 + 2\ln(n/x))$ in the interval $[1, n/\sqrt{e}]$ implies that $\alpha_1 \left(1 + 2\ln \left(n/\alpha_1\right)\right) \leq \alpha \left(1 + 2\ln \left(n/\alpha\right)\right)$. Hence, $\Gamma_d(G_1) < \alpha \left(1 + 2\ln \left(n/\alpha\right)\right)$.

Let $S_2$ be a $\gamma(D_2)$-set, and so $|S_2| < \alpha \left(1 + 2\ln \left(n/\alpha\right)\right)$. We now remove the vertices of the DDS $S_2$ from $D_2$ to produce an orientation $D_3$ of $G_2 = G_1 - S_2$ where $n_2 = n - |S_1| - |S_2|$ and we let $S_3$ be a $\gamma(D_3)$-set. Continuing in this way, we produce a sequence $S_1, S_2, \ldots, S_r$ of sets whose union is a DrDS of $G$ of cardinality $\sum_{i=1}^r |S_i| \leq r\alpha \left(1 + 2\ln \left(n/\alpha\right)\right)$. This is true for every orientation $D$ of $G$. Hence, $\Gamma_{d,r}(G) \leq r\alpha \left(1 + 2\ln \left(n/\alpha\right)\right)$. This establishes Part (b). □

### 3.2 Directed Distance Domination

Let $D = (V, A)$ be a directed graph. The distance $d_D(u, v)$ from a vertex $u$ to a vertex $v$ in $D$ is the number of edges on a shortest directed path from $u$ to $v$. For an integer $d \geq 1$, a directed $d$-distance dominating set, abbreviated DdDDS, in $D$ is a set $U$ of vertices of $V$ such that for every vertex $v \in V \setminus U$, there is a vertex $u \in U$ with $d_D(u, v) \leq d$. The directed $d$-distance domination number of a directed graph $D$, denoted by $\gamma(D, d)$, is the minimum cardinality of a DdDDS in $D$. The directed $d$-distance domination number of a graph $G$, denoted $\Gamma_d(G, d)$, is defined as the maximum directed $d$-distance domination number $\gamma_d(D, d)$ over all orientations $D$ of $G$; that is, $\Gamma_d(G, d) = \max\{\gamma(D, d) \mid \text{ over all orientations } D \text{ of } G\}$.

In particular, we note that $\Gamma_d(G) = \Gamma_d(G, 1)$.

An independent set $U$ of vertices in $D$ is called a semi-kernel of $D$ if for every vertex $v \in V(D) \setminus U$, there is a vertex $u \in U$ such that $d_D(u, v) \leq 2$. For the proof of our next result we will use the following theorem due to Chvátal and Lovász [13].

**Theorem 13** (Chvátal, Lovász [13]) Every directed graph contains a semi-kernel.

**Theorem 14** For every integer $d \geq 2$, $\gamma_d(G, d) = \alpha(G)$.

**Proof.** Let $S$ be a maximum independent set in $G$ and let $D$ be an orientation obtained from $G$ by directing all edges in $[S, V \setminus S]$ from $S$ to $V \setminus S$ and directing all other edges arbitrarily. Every directed $d$-distance dominating set must contain $S$ since no vertex of $S$ is reachable in $D$ from any other vertex of $V(D)$. Hence, $\Gamma_d(G, d) \geq |S| = \alpha(G)$. However if $D^*$ is an arbitrary orientation of the graph $G$, then by Theorem 13 the oriented graph $D^*$ has a semi-kernel $S^*$. Thus, $\gamma(D, d) \leq |S^*| \leq \alpha(G)$. Since this is true for every orientation of $G$, we have that $\Gamma_d(G, d) \leq \alpha(G)$. Consequently, $\gamma_d(G, d) = \alpha(G)$. □
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