On consistency of hydrodynamic approximation for chiral media

A. Avdoshkin,1, 2 V.P. Kirilin,1, 3 A.V. Sadofyev,1, 4 and V.I. Zakharov1, 2, 5

1ITEP, B. Cheremushkinskaya 25, Moscow, 117218 Russia
2Moscow Inst Phys & Technol, Dolgoprudny, Moscow Region, 141700 Russia.
3Department of Physics, Princeton University, Princeton, NJ 08544, USA
4Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA, 02139
5Max-Planck Institut für Physik, 80805 München, Germany;

We consider chiral liquids, consisting of massless fermions and right-left asymmetric. In such media, one expects existence of electromagnetic current in equilibrium flowing along external magnetic field. The current is predicted to be dissipation free. We argue that actually the chiral liquids in the hydrodynamic approximation should satisfy further constraints, like infinite classical conductivity. Inclusion of higher orders in electromagnetic interactions is crucial to reach the conclusions.

INTRODUCTION

Interest in theory of chiral liquids was boosted by discovery of the quark-gluon plasma since the light quarks are nearly massless. In theoretical studies one mostly concentrates on a generic plasma of massless fermions which interact in a chiral-invariant way and possess U(1) charges. A remarkable feature of the chiral materials is that the chiral anomaly, which is a loop, or quantum effect, is predicted to have macroscopic consequences and effectively modifies the Maxwell equations. In particular, in the equilibrium there is an electric current $j^e_\mu$ proportional to external magnetic field $B_\mu$:

$$j^e_\mu = \sigma MB_\mu . \quad (1)$$

Here $B_\mu$ is the magnetic field in the rest frame of the element of the liquid, or $B_\mu \equiv (1/2)\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$, where $u^\mu$ is the 4-velocity of an element of the liquid.

The constant $\sigma_M$ can be called magnetic conductivity and is related to the coefficient in front of the chiral anomaly $\bar{A}$. In particular, for a single (massless) fermion of charge $e$

$$\sigma_M = \frac{e^2 \mu_5}{2\pi^2} \quad (2)$$

where $\mu_5$ is the chiral chemical potential, $\mu_5 \equiv \mu_L - \mu_R$, so that $\mu_5 \neq 0$ implies that the medium is not invariant under parity transformation. Using the hydrodynamic approximation and equations of motion one can demonstrate that the magnetic conductivity is protected against corrections [5]. This non-renormalization of $\sigma_M$ goes back to the Adler-Bardeen theorem. Moreover, the current (1) is dissipation free. The simplest argument in favour of this is that the current (1) flows in the equilibrium. A more involved reasoning [6] is that both the r.h.s. and l.h.s. of (1) are odd under time reversal. This is a strong indication that the dynamics behind (1) is Hamiltonian and there is no dissipation [6]. Analogy to the superconducting current in the London limit, $j^i_\mu = m^2_\gamma \bar{A}$ where $m_\gamma$ is the photon mass and $\bar{A}$ is the vector potential, supports the conclusion on the dissipation-free nature of the current (1).

In short, one predicts a kind of novel superconductivity for chiral media: there exists dissipation-free current (1) in equilibrium. What seems puzzling, there is no apparent requirement that the chiral media are quantum in nature, like superconductors. Indeed, to derive (1) one uses only the standard hydrodynamic expansion in derivatives and the anomalous divergence of the axial current [6].

Next, one applies the argumentation with time-reversal invariance [6] and concludes that the current (1) is dissipation free.

We will argue in this note that chiral liquids in the hydrodynamic approximation have in fact to satisfy further constraints. The reasoning is as follows. Consider the anomalous axial charge conservation:

$$Q^A = Q^A_{\text{naive}} + \frac{e^2}{4\pi^2} \mathcal{H}, \quad \frac{d}{dt} Q^A = 0 , \quad (3)$$

where $Q^A_{\text{naive}}$ is the axial charge which is conserved according to the classical equations of motion (without account of the anomaly) and $\mathcal{H}$ is the so called helicity of the magnetic field:

$$\mathcal{H} = \int \bar{A} \cdot \mathbf{B} d^3 x . \quad (4)$$

Note that in the hydrodynamic approximation the axial current corresponding to $Q^A_{\text{naive}}$ takes the form $j^A_\mu = n^A u_\mu$ where $n^A \equiv n^L + n^R$ and $n^L, R$ are the densities of the left- and right-handed fermionic constituents, respectively.

Mostly, one considers the magnetic field in (1) as external and fixed. However, in case of plasma the motion of charged carriers creates electromagnetic fields and the conservation of the helicity (4) is not granted, generally speaking. Formally, we include higher orders in electromagnetic interactions. Our central point is that for the conservation of the axial charge (3) to be consistent in the hydrodynamic approximation the helicity (4) is to be conserved classically. Indeed, if, for example, the
helicity can be destroyed by the temperature then the axial charge $\mathcal{Q}$ could be changed in an uncontrollable way and the conservation $\mathcal{Q}$ could not hold.

It is amusing that the possibility of the helicity conservation in the ordinary magneto-hydrodynamics (without any reference to the chiral liquids) was studied in great detail, for review see, e.g., [8, 9]. The conclusion is that for the conservation of the helicity $\mathcal{Q}$ one needs electric conductivity tending to infinity:

$$\sigma_E \to \infty \quad \text{if} \quad \frac{dH}{dt} = 0 \quad (5)$$
or, equivalently, resistance of the plasma tending to zero. In other words, the helicity is conserved in case of ideal magnetodynamics which does admit for a field-theoretic relativistic description, see, e.g., [10]. In this perspective, it seems indeed illuminating that the anomalous equation $\mathcal{Q}$ allows for mixing between two pieces of the axial charge which conserve separately in the classical limit in case of chiral liquids. The physical meaning of $\mathcal{Q}$ seems obvious: it does not allow for the helicity, or the axial charge of the constituents to be transferred to heat.

We will discuss also in some detail definition of the axial charge in hydrodynamics. The point is that in field theoretic treatment of hydrodynamics one unifies the fields of vector potential $A_\mu$ entering the standard field theoretic expressions like $\mathcal{Q}$ and of the four-dimensional velocities $u_\mu$. Roughly speaking [11]:

$$eA_\mu \to eA_\mu + \mu u_\mu \quad (6)$$

where $\mu$ is the chemical potential conjugated to the charge $\mathcal{Q}$ whose matrix element over the constituents is equal to $e$. Possibility of this kind of unification was mentioned in early studies of ordinary (not chiral) magnetodynamics, (see, e.g. [4,5] and references therein). In modern approaches to the anomalous hydrodynamics, which utilize the geometric language of curved Euclidean space-time, there arises an extra symmetry which also prescribes an extension of electromagnetic potential $eA_\mu$ incorporating the chemical potential [12].

As a result of substitution $\mathcal{Q}$ the definition of the conserved axial charge $\mathcal{Q}$ is generalized in hydrodynamics as follows:

$$Q^A_{\text{hydro}} = Q^A_{\text{naive}} + Q^A_{mh} + Q^A_{mfh} + Q^A_{fh} \quad (7)$$

where indices “$mh$”, “$fh$” and “$mfh$” stand for “magnetic helicity”, “fluid helicity” and mixed “magnetic-fluid helicity”, respectively. Note that we are using here the standard terminology of magneto-hydrodynamics, see, e.g., [6, 7], where the fluid and magnetic helicities were considered phenomenologically, without reference to chiral liquids. In particular, $Q^A_{mh}$ stands for helicity $\mathcal{H}$ of Eq. $\mathcal{Q}$ while the two other terms in the r.h.s. of Eq. $\mathcal{Q}$ involve fluid vorticity, and explicit expressions will be given later. In the limit of ideal liquid all three types of the hydrodynamic helicities, entering $\mathcal{Q}$ conserve separately [4, 5]. Thus, the anomalous charge $\mathcal{Q}$ unifies in a well defined way all four terms which can be conserved classically.

Eq. (7) has important implications for the issue of stability of chiral media. Consider the case of a non-vanishing chiral chemical potential, $\mu_5 \neq 0$. Then one expects that in the equilibrium all the degrees of freedom with a non-vanishing axial charge are equally excited and all the helicities entering Eq. (7) are non-vanishing. This implies, in turn, that if one starts with the state where the whole axial charge is attributed to a single term in the r.h.s. of $\mathcal{Q}$, say to the charge of elementary constituents,

$$Q^A_{\text{naive}} \neq 0, \quad Q^A_{mh} = Q^A_{fh} = Q^A_{mfh} = 0 \quad ,$$

then this state is unstable with respect to generation of all types of helicities. Recently, this type of instability with respect to generation of the magnetic field with non-vanishing helicity $\mathcal{Q}$ was considered in detail and in various applications [14–16]. Eq. (7) implies that in fact one can expect that in the hydrodynamic approximation there are more general instabilities which would result in generation of all possible helicities:

$$Q^A_{\text{naive}} \sim Q^A_{mh} \sim Q^A_{mfh} \sim Q^A_{fh} \quad . \quad (8)$$

In other words, all types of helical motions are generated in the chiral plasmas on macroscopic scales.

It is worth emphasizing, that all the conclusions concerning chiral plasmas are subject to the reservation that, from the microscopical point of view, the underlying field theories are infrared unstable. As is emphasized, e.g., in [16] finite chemical potential provides only partial infrared cut off. As a result, the standard hydrodynamic expansion in derivatives is in fact not granted. First terms, including the chiral-anomaly effects can be described by an effective action, see, in particular, [12, 13]. There is no guarantee, however, that terms formally of higher order in derivatives are actually suppressed. We will mostly assume that there does exist an infrared cut-off which preserves chiral symmetries in the hydrodynamic approximation, as is (at least tacitly) assumed in the bulk of papers on the anomalous hydrodynamics, for review see, e.g., [17]. Note, however, that within holographic approach it turns possible in some cases to study dynamics of chiral liquids in infrared. There are indications that the physics in the infrared could be richer than is usually assumed. In particular, new scales can be generated, for a recent study and further references see [18].

EVALUATION OF AXIAL CHARGE

In this section we outline evaluation of the anomalous pieces in the conserved axial charge $\mathcal{Q}$. We emphasize
that the calculation is valid in the limit of exact symmetry. In particular, fermion masses are assigned to be exact zero. Considering this limit is common to the recent papers on the anomalous hydrodynamics, see, in particular, [12]. Only in this limit the effect of the anomaly is reduced to local terms in the effective action. Moreover, there is no explicit time dependence as if we are discussing static processes. Specific feature of such local, or polynomial terms is that the action is gauge invariant while the density of the action is not gauge invariant. The expression (11) for the magnetic helicity provides the best known example of such a term.

If one introduces explicit violation of the chiral symmetry, say, through the masses of the constituents, then effect of the anomaly does not reduce to local terms in the effective action. Nevertheless, in a certain kinematic limit the matrix element of the axial current becomes again the same polynomial as in (3). We emphasize that this kinematic limit actually assumes non-vanishing time-dependent fields. In particular, the expression (3) for the matrix element of the axial charge in the limit of electric fields much stronger than the fermionic masses:

\[ m_f^2 \ll E \ll H , \quad (9) \]

where \( E, H \) are electric and magnetic fields, respectively. The constraint (9) is mentioned in [10]. Here we present a more detailed derivation of (9).

Thus, our aim here is to evaluate the matrix element of the axial charge over a photonic state \( \langle \gamma | Q^A | \gamma \rangle \), where

\[ Q^A = \int d^3x_j x_j A(x,t) = \int d^3x \bar{\psi}_0 \gamma_0 \gamma_3 \psi \quad (10) \]

where \( \psi \) is a massless Dirac field of charge \( e \). Moreover, consider temperature-zero case and the photons on mass shell. Then, it is well known that the matrix element of the axial current \( j_\mu^A \) corresponding to the anomalous triangle graph has a pole. In the momentum space,

\[ \langle \sigma \rangle_{j_\mu^A} = \frac{e^2}{2\pi^2} \frac{i q_\mu}{q^2} \epsilon_{\sigma\alpha\beta} \epsilon^{(1)}_\rho k^{(1)}_\sigma \epsilon^{(2)}_\alpha k^{(2)}_\beta , \quad (11) \]

where \( q_\mu \) is the 4-momentum brought in by the axial current, \( \epsilon^{(1)}_\rho, k^{(1)}_\sigma \) and \( \epsilon^{(2)}_\alpha, k^{(2)}_\beta \) are the polarization vectors and momenta of the photons.

The matrix element (11) is clearly non-local in nature, by virtue of the Lorentz covariance and gauge invariance. Concentrate, however, on the matrix element of the axial charge (11). Since the charge is defined as \( Q^A = \int d^3x_j x_j A(x,t) \), evaluating the charge implies considering the kinematical limit

\[ \vec{q} \rightarrow 0, \quad q_0 \rightarrow 0 . \]

In this limit the matrix element (11) reduces to a polynomial:

\[ \langle \sigma | Q^A | \gamma \rangle = i \frac{e^2}{4\pi^2} \epsilon_{ij}^{(1)} \epsilon_j^{(2)} (k^{(1)} - k^{(2)}) , \quad (12) \]

In other words, we come to the standard expression (3).

Evaluating the charge (10) starting from the non-local expression (11) for the current has advantages, from the theoretical point of view. In particular, we avoided considering contribution of heavy regulator fields, and our derivation of (10) is given entirely in terms of physical, or light (massless) degrees of freedom. On the other hand, the now-standard way of evaluating the magnetic conductivity \( \sigma_M \) is to reduce it to the spatial correlator of two electromagnetic currents (for review see, e.g., [19]). In the momentum space:

\[ \sigma_M = \lim_{q_0 \rightarrow 0} \frac{\epsilon_{ik} e^{ik}}{2q^2} (j_i^{el}, j_j^{el}) \quad (13) \]

Although taking the limit of \( q_0 \rightarrow 0 \) implies, at first sight, that the correlator (13) is sensitive to large distances, \( r \sim 1/|q| \), in fact, it depends on the correct definition of the correlator at the coinciding points. Therefore, one has to consider carefully the ultraviolet regularization procedure, for details see [19].

Necessity of a careful treatment of the time-dependent fields looks counter-intuitive in view of the fact that Eq. (13) relates the magnetic conductivity to a pure spatial correlator. It might, therefore, worth reminding the reader that in the original derivation of the axial anomaly in terms of zero modes in magnetic field [20] one evaluates actually the work \( W \) produced by an external electric field \( E \):

\[ W \equiv \vec{E} \cdot \vec{r} \cdot \vec{e} = \vec{E} \cdot \vec{B} \frac{e^2}{2\pi^2} \mu_5 . \quad (14) \]

This work compensates the energy needed for massless pair production. And only after cancelling the electric field from the both sides of Eq. (13) one arrives at the current (1) which depends exclusively on the magnetic field.

If, on the other hand, one introduces finite fermionic masses then there is no pair production for \( E \ll m_f^2 \) and the role of the time-dependent electromagnetic potentials is made explicit. In particular, taking the limit \( q_0 \rightarrow 0 (|q| \equiv 0) \) now gives

\[ \langle \sigma | Q^A | \gamma \rangle_{m_f \neq 0} = 0 , \]

since there no singularity at \( q^2 = 0 \) in the matrix element corresponding to the triangle graph.

**AXIAL CHARGE IN HYDRODYNAMICS**

Recently, it has been realized that to derive chiral effects in hydrodynamics it is useful to consider motion in both electromagnetic and gravitational backgrounds, see, e.g., [12, 13]. This seems to be rooted in the very nature of the hydrodynamics which is entirely determined by conservation laws, of energy-momentum tensor and
of relevant currents. Technically, one of the ways to trace this kind of unification of electromagnetic and gravitational interactions is to start with the covariant action in higher dimensions. One can demonstrate then [13] that the mixed gauge-gravity anomaly in higher dimensions generates 4d action which is responsible for the chiral effects.

To be more specific, concentrate on the so called chiral vortical effect [17]. The effect itself is the flow of axial current along the fluid vorticity:

$$J^A_{\mu} = \frac{1}{2} \sigma_\omega \epsilon_{\mu \nu \rho \sigma} u^\nu \partial^\rho u^\sigma ,$$  \hspace{1cm} (15)

where $u^\alpha$ is the 4-velocity of an element of the liquid. One can demonstrate that the conductivity $\sigma_\omega$ is related to the correlator of components of the energy-momentum tensor and electric current:

$$\sigma_\omega = \lim_{q_0 \to 0} \frac{i}{q_k} \langle j^A_{\mu}, T_{0j} \rangle .$$  \hspace{1cm} (16)

The conductivity $\sigma_\omega$ was calculated explicitly in hydrodynamic approximation [6 21] in case of a single $U(1)$ symmetry:

$$\sigma_\omega = \frac{\mu^2}{2\pi^2} ,$$  \hspace{1cm} (17)

where we keep in Eq. (17) the leading in the chemical potential $\mu$ term. Terms of higher order depend in fact on the choice of coordinate frame to describe motion of the liquid.

Another geometric approach [12] starts with considering the static metric

$$ds^2 = -\exp(2\sigma(x))(dt + a_0(x)dx^0)^2 + g_{ij}(x)dx^i dx^j$$

There is also electromagnetic background $A_\mu(x)$. Then one can demonstrate that the symmetries of the problem imply that the partition function depends in fact on the combinations $A_0, A_1, A_2 = A_0 + \mu , A_i = A_i - A_0 a_i$ which are Kaluza-Klein gauge invariant and are replacing $A_\mu$ in the standard field theoretic expressions.

As is mentioned above, we will generalize the field theoretic definition of charge [3] to the hydrodynamic case replacing $eA_\mu$ entering the standard anomaly [3] by the combination [4]. In this way, electromagnetic and gravitational anomalies get unified as well.

The most straightforward way to justify [5] is to observe that chemical potential is introduced through an extension of the original Hamiltonian $\hat{H}$:

$$\hat{H} \to \hat{H} - \mu \hat{Q} - \mu_5 \hat{Q}^A .$$  \hspace{1cm} (19)

As far as the chemical potentials $\mu, \mu_5$ are considered to be small, the corresponding change in the Lagrangian $\delta L$ is given by

$$\delta L = -\delta H = \mu Q + \mu_5 Q^A = \mu \bar{\psi} \gamma_0 \psi + \mu_5 \bar{\psi} \gamma_0 \gamma_5 \psi .$$  \hspace{1cm} (20)

The next step is to generalize [20] to the case of hydrodynamics. The generalization assumes rewriting [20] in an explicitly Lorentz-covariant way:

$$\delta L = \mu u^\alpha \bar{\psi} \gamma_\alpha \psi + \mu_5 u^\alpha \bar{\psi} \gamma_\alpha \gamma_5 \psi .$$  \hspace{1cm} (21)

Now, the substitution [6] becomes obvious. The limitation is that in each particular case only the lowest orders in the chemical potential(s) can be evaluated in this way.

Moreover, it is quite obvious that the procedure we are using has much in common with approaches [12]. Indeed the field $a_i$ entering eq. [15] is also proportional to $u_i$. Note, however, that the approach of [12] applies only in equilibrium while the substitution [6] works in more general case. Connection of our procedure to that of [12] can be readily established as well. Indeed, modification of the naively conserved axial charge $Q^A_{\text{naive}}$ by the anomaly is in one-to-one correspondence with the non-vanishing correlator [13]. The hydrodynamic modification [19] of the field theoretic Hamiltonian implies modification of the $T^{0i}$ component of the energy-momentum tensor:

$$(\delta T^{0i})_{\text{hydro}} = \mu J^i .$$

There fore, there arises an anomalous piece in the correlator

$$\epsilon_{ijk} \frac{\langle T^{0i}, T^{0j} \rangle}{q^k} = \mu^2 \epsilon_{ijk} \frac{\langle J^i, J^j \rangle}{q^k} .$$  \hspace{1cm} (22)

Emergence of this anomaly modifies definition of axial charge, similar to the case of the correlator [13].

It is another question, whether the substitution [6] is unique in the relativistic case, $u_i \sim 1$. Probably, it is not. Moreover, it is not clear, how one could even fix the “correct” version theoretically. Indeed, it is well known that all such relativistic extensions of the standard hydrodynamics do not account, at least explicitly, for the finiteness of speed of light. Probably, the most rigorous approach is provided by the geometric language developed recently [12]. It refers to the Euclidean space-time and, for this reason, can be applied only in case of equilibrium, means stationary processes. Then there is no problem of upholding causality and there does exist a geometrically motivated extension of $eA_\mu$ which can replace [4]. For our purposes, we need, however, to consider non-equilibrium as well. Moreover, in the approximations we consider the geometric construct of [12] reduces to [4].

After these preliminary remarks, there is no difficulty to obtain explicit relations for the extra terms entering [7]. For convenience, we use notations of Ref. [3] which apply also in case of an external gravitational field. In case of the fluid helicity, the basic object to consider is the vorticity pseudotensor:

$$\omega_{\alpha \beta} \equiv (\mu u_\beta)_{,\alpha} - (\mu u_\alpha)_{,\beta} .$$  \hspace{1cm} (23)
The covariant current associated with the fluid helicity is defined as
\[ j^\alpha_{fh} = 1/2 \epsilon^{\alpha\beta\gamma\delta} \omega_{\gamma\delta}(\mu u_\beta) \ . \tag{24} \]
And, finally the fluid helicity itself is the volume integral from the temporal component of the current (23):
\[ Q_{fh}^A = \frac{1}{4\pi^2} \int d^4x j_0^{fh} \ . \tag{25} \]
In particular, axial charge associated with the fluid helicity in the non-relativistic limit is given by
\[ (Q_{fh}^A)_{\text{non-rel}} = \frac{\mu^2}{4\pi^2} \int d^3x u^0 \epsilon_{ijk} u^i \nabla^j u^k \ . \tag{26} \]
In other words, \( Q_{fh}^A \) reduces to the volume integral of the vorticity.

Similarly, the mixed magnetic-fluid helicity is defined in terms of the current \( j^\alpha_{mfh} \):
\[ j^\alpha_{mfh} = 1/2 \epsilon^{\alpha\beta\rho\sigma} \omega_{\rho\sigma} A_\beta \tag{27} \]
where \( \omega_{\alpha\beta} \) is defined in Eq. (23). The corresponding axial charge, \( Q_{mfh}^A \) entering Eq. (14) is defined as the volume integral from the temporal component of \( j^\alpha_{mfh} \) multiplied by a \( \frac{1}{\mu^2} \) factor. There is also an alternative form of \( j^\alpha_{mfh} \) defined as:
\[ j^\alpha_{mfh} = 1/2 \epsilon^{\alpha\beta\rho\sigma} F_{\rho\sigma}(\mu u_\beta) \ . \tag{28} \]
The corresponding charge, \( Q_{mfh}^A \) is the same in the both cases of (27) and (28).

This concludes the derivation of the expression of the axial charge (4) in the hydrodynamic limit. Let us emphasize again that so far we considered exact chiral limit. In this limit, one can consider stationary motion and concludes that, apart from the chiral magnetic effect there exists the chiral vortical effect, or the axial current proportional to the fluid vorticity:
\[ j^\alpha = \frac{\mu^2}{4\pi^2} \epsilon^{\alpha\beta\gamma\delta} u_\beta \nabla_\gamma u_\delta \ . \tag{29} \]
The reservation is that if we introduce small fermionic masses, as a measure of the chiral symmetry violation, then we need actually nontrivial time-dependence for Eqs (24), (27), (28) to effectively be valid. For example, in case of Eq. (27) we would require
\[ \frac{d(\mu_\alpha)}{dt} \gg m_f^2 \ . \tag{30} \]
In other words, the masses are to be small even on the hydrodynamic scale.

It should be noted that relative coefficients in front of classical helicities are fixed here from the gauge potential extension while the coefficient in front of microscopic chirality is fixed by the anomaly. However it is interesting to find whether there is other way to fix coefficients in the conserved axial charge.

**Classical Conservation of Magnetic and Fluid Helicities**

As is mentioned above, possibility of conservation of the magnetic and fluid helicities was intensely discussed in the context of ordinary magneto-hydrodynamics. Here we will reproduce the main results, following the paper in Ref. [5]. Let us begin with the conservation of the fluid helicity. The main tool to be used is the general relativistic version of the Euler equation:
\[ (\rho + p) u^\beta u_{\alpha,\beta} = -p_{,\alpha} - u_\alpha u^\beta p_{,\beta} \ , \tag{31} \]
where \( \rho \) and \( p \) stand for proper mass density and pressure, respectively and \( u^3 \) is the 4-velocity normalized as \( u_\beta u^\beta = -1 \). Moreover, the first law of thermodynamics can be represented as
\[ dp = nd\mu = nTds \ , \tag{32} \]
where \( n \) is the conserved-charge number density, \( \mu = (\rho + p)/n \) is the relativistic enthalpy per charged particle.

Then, after some algebra, one can demonstrate that the current \( j^\alpha_{fh} \) associated with the fluid helicity is conserved for isentropic flow:
\[ (j^\alpha_{fh})_{,\alpha} = -2\mu T s_{,\alpha}\omega^{\alpha} \ , \tag{33} \]
where \( j^\alpha_{fh},\omega^{\alpha} \) are defined in Eq. (23), (24), respectively. Thus, if \( s_{,\alpha}\omega^{\alpha} = 0 \) the axial charge (25) is conserved.

Turn now to the magnetic helicity (4). The corresponding 4d current is defined as
\[ j^\alpha_{mh} = 1/2 \epsilon^{\alpha\beta\gamma\delta} A_\beta F_{\gamma\delta} \ . \tag{34} \]
The divergence of this current is proportional to the product of magnetic and electric fields \( B_\mu \) and \( E_\mu \),
\[ (j^\alpha_{mh})_{,\alpha} = -2B^\mu E_\mu \ , \tag{35} \]
where \( B_\mu \) is defined in Eq. (11), \( E_\mu = F_{\mu\nu}u^\nu \) and one finds that \( Q_{mh} = \frac{e^2}{4\pi^2} H \).

Eq. (35) is pure kinematic in nature. The dynamic input that ensures conservation of the current \( j^\alpha_{mh} \) is that in case of infinitely conducting liquid, or perfect magneto-hydrodynamics \( E^\mu \) is to vanish:
\[ E_\mu \to 0 \ , \text{ if } \sigma_E \to \infty \ . \tag{36} \]
One can also evaluate the dissipation rate of the magnetic helicity in case of finite conductivity \( \sigma_E \) [9]:
\[ \frac{dH}{dt} = -\frac{2}{\sigma_E} \int d^3x B \cdot \text{curl } \vec{B} \ , \tag{37} \]
for details of the derivation see, e.g., [8].

Finally, consider the fluid-magneto helicity introduced above. The divergence of the corresponding current (28) is given by:
\[ (j^\alpha_{fh})_{,\alpha} = 1/2 \epsilon^{\alpha\beta\gamma\delta} \omega_{\alpha\beta} F_{\gamma\delta} \ . \tag{38} \]
The next step is to express \( F_{\alpha\beta} \) in terms of \( B_\mu \) when \( E_\nu = 0 \) \[ \text{[22]} \]:

\[
F_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} B^\gamma u^\delta, \quad (39)
\]

Using this as an input one comes to \[ \text{[3]} \]:

\[
(\jmath_{fh})_\alpha = -TB^\alpha s, \quad (40)
\]

and the current is conserved in case of isentropic flow.

Another feature which unifies various types of helicity is that the corresponding charges are related to linkage of magnetic and fluid vortices. In particular, the fluid helicity is a measure of linkage of vortex lines in the liquid \[ \text{[23]} \]. The fluid-magnetic helicity measures the linkage number of closed vortex lines and magnetic flux lines. Finally, the magnetic helicity can be interpreted in terms of the fluxes of linked flux tubes.

This relation of various types of helicities to topology is a source of non-renormalization theorems. In particular, the anomalous term in the charge \[ \text{[4]} \] in case of magnetostatics can be rewritten (by using Eq. \([20]\)) as a 3d topological photon mass, see, in particular, \[ \text{[16]} \] and references therein. Furthermore, consider currents of the form \( J_\mu(x) = I \int d\tau \delta^3(\bar{x} - \bar{x}(\tau)) \bar{x}_\mu(\tau) \). Then the interaction term of two current loops is given by:

\[
V = \frac{2I I'}{\sigma_M} \int_{C_l} \int_{C_r} dx^i dy^j \epsilon_{ijk} \frac{(x - y)^k}{4\pi |x - y|^3}, \quad (41)
\]

where \( \sigma_M \) is defined in Eq. \([11]\). The integral in \([11]\) is apparently proportional to the Gauss linking number of the two current loops. Moreover one can demonstrate \([10]\) that the interaction term \([11]\) is not renormalized to any order in electromagnetic interactions. As for the fluid helicity, it seems to be subject to renormalization, once higher orders in electromagnetic interactions are included, for a related discussion see, in particular, \[ \text{[24, 25]} \]

To summarize this section, consideration of the chiral anomaly in the hydrodynamic limit led us to include into the definition of the conserved axial charge fluid, magnetic and mixed helicities. All three helicities are conserved in the limit of perfect magnetohydrodynamics, as has been clarified many years ago. It is amusing that the chiral anomaly unifies all types of helicities which were considered separately so far.

**INSTABILITIES OF CHIRAL PLASMA**

As is mentioned in the Introduction, chiral plasma can well be unstable if one starts with a state where, say, \( Q_{\text{naive}}^A \neq 0 \) while all other terms in the conserved charge \([7]\) are vanishing. This kind of instabilities has been discussed recently \([14, 16]\).

We have a few, rather minor points to add:

- The state with \( Q_{\text{naive}}^A \neq 0 \), \( Q_{fh}^A = Q_{mh}^A = Q_{fh}^m = 0 \) can decay not only into the domains with non-vanishing magnetic field \([14, 16]\) but also into domains with helical motion of the plasma, so that \( Q_{fh}^A \neq 0 \).

- In particular, we expect that not only primodial magnetic field could be produced from an original right-left asymmetric state \([14]\), but primodial helical motion could be generated on a cosmological scale as well.

- We do not have any particular mechanism of chiral media instabilities in mind. The considerations presented above do not suffice to estimate, in particular, lifetime of unstable states.

- It is amusing to observe, again, that transitions among various kinds of helicities have been discussed in the literature, independent of the issue of chiral media. In particular, in paper \([26]\) there is rather detailed discussion of generation of magnetic field from the initial helical motion. In our language, this is about the instability:

\[
Q_{fh}^A \rightarrow Q_{mh}^A
\]

- Thus, the only novel point brought by consideration of chiral media is the transition of \( Q_{\text{naive}}^A \neq 0 \) to other components of the conserved axial charge \([7]\). Here, we have actually a word of caution. Realistically, one has charged massive particles. Then to apply the ideas on the instabilities one has to check whether the violations of the chiral symmetry are small on the scale of time dependent fields,

\[
|E| \gg m^2 \nu
\]

In each particular case one is to verify whether this condition is fulfilled.

**CONCLUSIONS.**

We have shown that conservation of the axial charge \([7]\) implies that classically chiral media are perfect liquids, with no dissipation:

\[
(\sigma_E)_{\text{classical}} \rightarrow \infty, \quad (\eta/s)_{\text{classical}} \rightarrow 0, \quad (42)
\]

where \( \eta \) is the viscosity and normalization of the \( \eta \) to the entropy density \( s \) is introduced on the dimensional grounds, as usual. The statement \([12]\) is the central point of the present notes. As is mentioned in the Introduction, it was demonstrated recently that the chiral anomaly modifies hydrodynamics of chiral media on the classical
level. What we are adding to this observation, is that for the consistency of the hydrodynamic approximation, the novel pieces in the axial charge should conserve on the classical level as well. This condition applies out-of-equilibrium as well and results in the constraints \[ (42) \]. If these constraints are satisfied, there is no surprise that in the equilibrium there can exist dissipation-free currents like \[ (1) \].

There is another question, whether such media actually exist as classical or quantum systems. The systems whose dynamics is understood explicitly represent rather examples of superconductivity, i.e. of quantum systems. We have in mind, in particular, the quantum wires and He II, see \[ [28] \]. The most interesting example—not fully understood dynamically though—is provided by holography. In particular, it seems to be a universal feature of the holographic systems is that viscosity is fixed to be the minimal value consistent with the uncertainty principle \[ [27] \]:

\[
\frac{\eta}{s} = \frac{1}{4\pi},
\]

where the presence of the Planck constant in the r.h.s. is in fact implicit. Such liquids could also be consistent with the constraints derived in the present notes.

ACKNOWLEDGMENTS

The work on this paper has been partly supported by RFBR grant 14-02-01185.

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