CCS-normal spaces

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Abstract. A space $X$ is called CCS-normal space if there exist a normal space $Y$ and a bijection $f : X \to Y$ such that $f|_C : C \to f(C)$ is homeomorphism for any cellular-compact subset $C$ of $X$. We discuss about the relations between $C$-normal, $CC$-normal, $Ps$-normal spaces with CCS-normal.

1. Introduction

A topological space $X$ is called cellular-compact space if for any disjoint family $U$ of non-empty open subsets of $X$, there exists a compact subset $K$ of $X$ such that $K \cap U \neq \emptyset$, for any $U \in U$. The notion of the cellular-compact space was first introduced in [7].

A space $X$ is called CCS-normal (resp. $C$-normal, $CC$-normal, $Ps$-normal) space if there exist a normal space $Y$ and a bijection $f : X \to Y$ such that $f|_C : C \to f(C)$ is a homeomorphism for any cellular-compact (resp. compact, countably compact, pseudocompact) subspace of $X$. The notion of $C$-normality is defined in [2]. The authors of [2] mentioned that $C$-normality was first introduced by Arhangel’skii while presenting a talk in a seminar held at Mathematics Department, King Abdulaziz University at Jeddah, Saudi Arabia in 2012. $CC$-normal spaces and $Ps$-spaces are initiated in [5] and [3] respectively. CCS-normal is a generalization of normal space. Every $Ps$-normal space is CCS-normal and every CCS-normal space is $C$-normal. We give an example of a $C$-normal space which is not CCS-normal [Example 2.8]. Also we give an example of a space which is $CCS$-normal but not $Ps$-normal [ Example 2.9]. Finally we conclude this article by raising some open problems related to CCS-normal spaces.

2. CCS-normal spaces

**Definition 2.1.** A space $X$ is called CCS-normal if there exist a normal space $Y$ and a bijection $f : X \to Y$ such that $f|_C : C \to f(C)$ is a homeomorphism for every cellular-compact subset $C$ of $X$.

**Theorem 2.2.** Every CCS-normal space is a $C$-normal space.

**Proof.** Let $X$ be a CCS-normal space. Then there exist a normal space $Y$ and a bijection $f : X \to Y$ such that $f|_C : C \to f(C)$ is a homeomorphism for every cellular-compact subset $C$ of $X$. Let $C$ be a compact subset of $X$. Then $C$ is cellular-compact. Hence $f|_C : C \to f(C)$ is a homeomorphism. Therefore $X$ is a $C$-normal space. \qed

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In Proposition 3.1, the authors proved that every Tychonoff Cellular-Compact space is pseudocompact. But we prove that the result holds anyway. We do not need Tychonoffness property.

**Theorem 2.3.** Any cellular-compact space is pseudocompact.

**Proof.** Let $X$ be a Cellular-Compact space. Let $\mathcal{U}$ be an infinite family of non-empty disjoint open sets. There exists a compact set $K \subset X$ such that $K \cap U \neq \emptyset$ for $U \in \mathcal{U}$. Choose $x_U \in K \cap U, U \in \mathcal{U}$. The infinite set $\{x_U : U \in \mathcal{U}\}$ is contained in $K$, a compact set. Hence $\{x_U : U \in \mathcal{U}\}$ has an accumulation point, say $x \in K$. Then the family $\mathcal{U}$ has an accumulation point. Therefore $X$ is feebly compact which implies pseudocompact. \(\square\)

**Theorem 2.4.** Any $Ps$-normal space is $CCS$-normal.

**Proof.** Let $X$ be a $Ps$-normal space. Then there exist a normal space $Y$ and a bijection $f : X \to Y$ such that $f|_P : P \to f(P)$ is a homeomorphism for every pseudocompact subset $P$ of $X$. Let $C$ be a cellular-compact subclass of $X$. So, $C$ is a pseudocompact subset of $X$ by Theorem 2.3. Since $X$ is $Ps$-normal, $f|_C : C \to f(C)$ is a homeomorphism. Hence $X$ is a $CCS$-normal space. \(\square\)

In [2] the authors have shown that the Dieudonné Plank is $C$-normal. In what follows we can demonstrate that it is, indeed, $Ps$-normal. Let $X$ be the Dieudonné plank i.e., $X = [0, \omega_1] \times [0, \omega_0] - (\omega_1, \omega_0)$ with the topology described below:

Write $X = A \cup B \cup N$ where $A = \{\omega_1\} \times \mathbb{N}, B = [0, \omega_1] \times \{\omega_0\}, N = [0, \omega_1] \times \mathbb{N}$, where $N$ includes 0 also. All points of $N$ are isolated. Basic open neighbourhoods of $(\omega_1, n) \in A$ are of the form $U_\alpha(\alpha) = (\alpha, \omega_1] \times \{n\}, \alpha < \omega_1, n \in \mathbb{N}$. Basic open neighbourhoods of $\alpha, \omega_0 \in B$ are of the form $V_\alpha(n) = [\alpha, n) \times (n, \omega_0], n \in \mathbb{N}, \alpha < \omega_0$. $A$ and $B$ are closed subsets of $X$. Any horizontal line $B_n = [0, \omega_1] \times \{n\}$ is clopen set, $n \in \mathbb{N}$. No $B_n$ is pseudocompact: $B_n = [0, \omega_1] \times \{n\}$. Let $U_\alpha(\alpha) = (\alpha, \omega_1] \times \{n\}$ be an open neighbourhood of $(\omega_1, n) \in B_n$. Since $\alpha < \omega_1$ is an infinite countable ordinal, $[0, \alpha)$ is a countably infinite set. We can write $[0, \alpha) = \{0 = \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_m < \cdots\}$. Define $f : B_n \to \mathbb{R}$ by $f(\beta) = 0, \alpha \leq \beta \leq \omega_1 = m$ if $\beta = \alpha_m, m \in \mathbb{N}$. Clearly $f$ is continuous and unbounded. This shows that $X$ is not pseudocompact: for if $X$ is pseudocompact, each $B_n$, being clopen subset, would be pseudocompact—which is false. Suppose $C \subset X$ is pseudocompact. Then $C \cap B_n$, for each $n \in \mathbb{N}$ is clopen subset of $C$ and hence, pseudocompact. Now $C \cap B_n$ consists of either isolated points of $B_n$ or at most one non isolated point, viz, $(\omega_1, n)$. Hence $C \cap B_n$ cannot be pseudocompact if infinite. So $C \cap B_n$ is either empty or a non-empty finite set for each $n \in \mathbb{N}$. Now take $Y = X$ as set with a topology described by the following nbhood system: All points of $N \cup A = [0, \omega_1] \times \mathbb{N}$ are isolated. The neighbourhoods of the points $\alpha, \omega_0$ are unaltered. The topology of $Y$ is $T_4$, first-countable and is finer than the topology of $X$. Let $C$ be a pseudocompact subset of $X$. Let $(\alpha, v)$ be a typical point of $X, \alpha \in [0, \omega_1], v \in [0, \omega_0]$. Let $(\alpha, v) \in C$ and $1_X : X \to Y$ be the identity map. If $(\alpha, v) \in N$, say $(\alpha, n), (\alpha, n)$ is isolated both in $X$ and $Y$. Then $1_X$ is continuous at $(\alpha, n)$. If $(\alpha, v) \in B, (\alpha, \omega_0), (\alpha, \omega_0)$ has the same nbhood in $X$ and $Y$. $1_X$ is continuous at $(\alpha, \omega_0)$ as well.

If $(\alpha, v) \in A$ i.e. $(\omega_1, n) \in A$. Then $(\omega_1, n) \in C \cap B_n$. Now $C \cap B_n$ is a finite set containing $(\omega_1, n)$. Let $C \cap B_n = (\omega_1, n) = \{\omega_1, n), (\alpha_2, n), \ldots (\alpha_k, n)\}$, if non-empty. Choose $\beta < \omega_1$ such that $\alpha_i < \beta, 1 \leq i \leq k, (\beta, \omega_1] \times \{n\}$ is a neighbourhood of $(\omega_1, n)$ in $B_n$ and $C \cap (\beta, \omega_1] \times \{n\} = \{\omega_1, n))$ is an open nbhood of $(\omega_1, n)$ in $C$ as a subspace of $X$. Since $\{\omega_1, n)\}$ is an open nbhood of $(\omega_1, n)$ in $Y$, $1_X$ is continuous at $(\omega_1, n)$. So we come to the conclusion that $1_X : C \to 1_X(C) \subset Y$ is a continuous map. So $1_X(C)$ is pseudocompact in $Y$. Now the topology of $Y$ is finer than that of $X$, the identity map $1_Y : Y \to X$ is continuous. This implies
that $1_Y : 1_X(C) \to 1_Y(1_X(C)) = C$ is continuous. We can now conclude that $1_X:C \to 1_X(C)$ is a homeomorphism for each pseudocompact subset $C \subset X$. $X$ is then $P$s-normal and by Theorem 2.4 it is CCS-normal.

**Theorem 2.5.** If $X$ is CC-normal, first-countable, regular space. Then $X$ is a CCS-normal space.

**Proof.** Since $X$ is a CC-normal space, then there exist a normal space $Y$ and a bijection $f : X \to Y$ such that $f|C : C \to f(C)$ is a homeomorphism, for every countably-compact subset $D$ of $X$. Take any cellular-compact subset $D$ of $X$. Then $D$ is a cellular-compact, first countable and regular space. So $D$ is countably compact, [Corollary 4.2, [7]]. Hence $f|D : D \to f(D)$ is a homeomorphism. So, $X$ is a CCS-normal. 

**Theorem 2.6.** CCS-normality is a topological property.

**Proof.** Let $X$ be a CCS-normal space and $X$ is homeomorphic to $Z$. Let $Y$ be a normal space and $f : X \to Y$ be a bijective mapping such that $f|C : C \to f(C)$ is a homeomorphism for any cellular-compact subspace $C$ of $X$. Let $g : X \to Z$ be the homeomorphism. Then $f \circ g^{-1} : Z \to Y$ is the required map. 

**Theorem 2.7.** If $X$ is cellular-compact but not normal, then $X$ is not CCS-normal.

**Proof.** Let $X$ be cellular-compact but not normal. On the contrary assume that there exist a normal space $Y$ and a bijective mapping $f : X \to Y$ such that $f|C : C \to f(C)$ is a homeomorphism, for any cellular-compact subspace $C$ of $X$. In particular $f : X \to Y$ is a homeomorphism. This is a contradiction as $Y$ is normal and $X$ is not normal. Hence $X$ is not CCS-normal.

The following example is an example of a CC-normal space which is not CCS-normal.

**Example 2.8.** Consider $(\mathbb{R}, \tau_c)$, where $\tau_c$ is the co-countable topology on $\mathbb{R}$. Then $(\mathbb{R}, \tau_c)$ is CC-normal,[ Example 2.6, [5]]. Since there is no disjoint family of open sets in $(\mathbb{R}, \tau_c)$, so $(\mathbb{R}, \tau_c)$ is cellular-compact (vacuously). It is known that $(\mathbb{R}, \tau_c)$ is not normal. Hence by Theorem 2.7 $(\mathbb{R}, \tau_c)$ is not CCS-normal. Also every CC-normal space is C-normal. Therefore $(\mathbb{R}, \tau_c)$ is C-normal but not CCS-normal.

Now we give an example of a CCS-normal space which is not $P$s-normal.

**Example 2.9.** Let $X = [0, \omega_1] \times [0, \omega_1]$ be equipped with the product of the respective order topologies. Then

1) $X$ is countably compact, because $\Omega = [\alpha, \omega_1]$ is countably compact and $\Omega_0 = [0, \omega_1]$ is compact.

2) $X$ is locally compact, because $\Omega$ is locally compact and $\Omega_0$ is compact.

3) $X$ is not normal.

Following steps are needed to prove that $X$ is not normal.

a) Interlacing Lemma: Let $\{\alpha_n, n \in \mathbb{N}\}$ and $\{\beta_n, n \in \mathbb{N}\}$ be two sequences in $\Omega = [0, \omega_1]$ such that $\alpha_n \leq \beta_n \leq \alpha_{n+1}$ for each $n \in \mathbb{N}$ . Then both sequence converge and to the same point of $\Omega$.

b) If $f : \Omega \to \Omega$ is a function such that $\alpha \leq f(\alpha)$ for each $\alpha$, then for some $\alpha \in \Omega$, the point $(\alpha, f(\alpha))$ is an accumulation point of the graph $G(f) = \{(\alpha, f(\alpha)) : \alpha \in \Omega\}$ of $f$.

**Proof.** Let $\alpha$ be any element of $\Omega$. Let $\alpha_1 = \alpha, \alpha_2 = f(\alpha_1), \ldots, \alpha_{n+1} = f(\alpha_n), n \in \mathbb{N}$. Now $\alpha_1 \leq f(\alpha_1) = \alpha_2 \leq f(\alpha_2) \leq \alpha_3, \ldots$. Let $\beta_n = f(\alpha_n), n \in \mathbb{N}$. Applying Interlacing Lemma to sequences $\{\alpha_n, n \in \mathbb{N}\}, \{\beta_n, n \in \mathbb{N}\}$ we obtain a
point $a_0 \in \Omega$. Consider the point $(a_0, a_0) \in X$ and any open nbhd of $(a_0, a_0)$ of the form $(\gamma, a_0] \times (\gamma, a_0]$, where $0 \leq \gamma \leq a_0$. Since $a_0 = \lim a_n = \lim \beta_n$, there exists $n_0 \geq 1$ such that $\gamma < a_n \leq a_0$, $\gamma < \beta_n \leq a_0$ for $n \geq n_0$. Then $(a_n, f(a_n)) = (a_n, \beta_n) \in (\gamma, a_0] \times (\gamma, a_0]$ for $n \geq n_0$. This shows that $(a_0, a_0)$ is an accumulation point of $G(f) = \{(a, f(a)) : a \in \Omega\}$. Now in order to show that $X$ is not normal, let $A = \{(\alpha, \alpha) : \alpha \in \Omega\}$ and $B = [0, \omega_1] \times [\omega_1]$. Then $A$ is a compact subspace of $\Omega$ as $X$ is not normal. Because of (1) $X$ is not $CCS$-normal and because of (2) $X$ is $C$-normal. As $X$ is countably compact, $X$ is Pseudocompact but not normal. So $X$ is not $PS$-normal.

(4): $X$ is $CCS$-normal: $X$ is not cellular-compact. This is because $X$ has a dense subset of isolated points. Let $A \subset X$ be a cellular-compact subspace. Let $p_1 : X \to \Omega, p_2 : X \to \Omega$ be two projections. Then $A \subset p_1(A) \times p_2(A)$. Now $p_1(A)$ is a cellular-compact in $\Omega$. We claim that $p_1(A)$ is contained in a compact subset of $\Omega$.

Case(i): $p_1(A)$ is bounded set in $\Omega$. Then for $a < \omega_1, p_1(A) \subset [0, a]$ and $[0, a]$ is a compact subspace.

Case(ii): $p_1(A)$ is unbounded. Then $p_1(A)$ contains a dense subspace of isolated points of $p_1(A)$ and must be compact. But unbounded subset of $\Omega$ cannot be compact. Assume $p_1(A) \subset K$ a compact subset of $\Omega$. $A \subset p_1(A) \times p_2(A) \subset K \times \Omega_0$ and $K \times \Omega_0$ is compact in $X$. Since $X$ is $C$-normal and every cellular-compact subset is contained in a compact subspace, $X$ becomes $CCS$-normal.

Thus we obtain $X$ as an example of a $CCS$-normal space which is not $PS$-normal. \hfill $\Box$

**Theorem 2.10.** If $X$ is $CCS$-normal and Frechet, $f : X \to Y$ is witness of $CCS$-normality, Then $f$ is a continuous mapping.

**Proof.** It is sufficient to show that for any $A \subset X, \overline{f(A)} \subset f(\overline{A})$. Let $\emptyset \neq A \subset X$ and $y \in f(\overline{A})$. Let $x$ be the unique point in $\overline{A}$ such that $f(x) = y$. Since $X$ is Frechet, then there exists a sequence $\{x_n, n \geq 1\}$ in $A$ such that $x = \lim_{n \to \infty} x_n$. Let $K = \{x_n : n \geq 1\} \cup \{x\}$. Then $K$ is compact and therefore $K$ is cellular-compact. Then $f|K : K \to f(K)$ is a homeomorphism, as $X$ is $CCS$-Normal. Let $U$ be any open set in $Y$ containing $y$. Then $U \cap f(K)$ is open set in $f(K)$ containing $y$. Now $\{x_n : n \geq 1\}$ is dense in $K$ and is contained in $K \cap A \Rightarrow K \cap A$ is dense in $K \Rightarrow f(K \cap A)$ is dense in $f(K) \Rightarrow f(K \cap A) \cap f(K) \cap U \neq \emptyset \Rightarrow U \cap f(A) \neq \emptyset$. Therefore $y \in f(\overline{A})$. Hence $\overline{f(A)} \subset f(\overline{A})$. \hfill $\Box$

**Theorem 2.11.** Let $X = \bigoplus_{\alpha \in A} X_\alpha$ and $C \subseteq X$. Then $C$ is cellular-compact if and only if $\Lambda_\alpha = \{\alpha \in A : C \cap X_\alpha \neq \emptyset\}$ is finite and $C \cap X_\alpha$ is cellular compact in $X_\alpha$ for each $\alpha \in \Lambda_\alpha$.

**Proof.** Let $C$ be cellular-compact. If possible let $\Lambda_\alpha$ be infinite set. Then $\{C \cap X_\alpha : \alpha \in \Lambda_\alpha\}$ is a disjoint family of open sets in $C$. So there exists a compact set $K$ in $C$ such that $K \cap C \cap X_\alpha \neq \emptyset$ for all $\alpha \in \Lambda_\alpha$. Thus $\{C \cap X_\alpha : \alpha \in \Lambda_\alpha\}$ is
an open cover of $K$ which has no finite subcover, which is a contradiction as $K$ is compact. Therefore $\Lambda_\alpha$ is finite.

Let $\alpha \in \Lambda_\alpha$. We show $C \cap X_\alpha$ is cellular-compact. Let $\{U_\gamma : \gamma \in I\}$ be a disjoint family of open sets in $C \cap X_\alpha$. Now each $U_\gamma$ is open in $C$ as $C \cap X_\alpha$ is open in $C$. Then there exists a compact subset $K$ of $C$ such that $K \cap U_\gamma \neq \emptyset$ as $C$ is cellular-compact. $K$ is compact in $C$ then $K \cap X_\alpha$ is compact in $C \cap X_\alpha$ and also $(K \cap X_\alpha) \cap U_\gamma \neq \emptyset$ for all $\gamma \in I$. Therefore $C \cap X_\alpha$ is cellular compact for all $\alpha \in \Lambda_\alpha$.

Conversely, let the given condition hold. Let $\{U_\gamma : \gamma \in I\}$ be a disjoint family of open sets in $C$. For $\alpha \in \Lambda_\alpha$, $\{U_\gamma \cap X_\alpha : \gamma \in I\}$, if non-empty, is a disjoint family of open sets in $C \cap X_\alpha$. This implies there exists a compact set $K_\alpha$ in $C \cap X_\alpha$ such that $K_\alpha \cap (U_\gamma \cap X_\alpha) \neq \emptyset$ for all $\gamma \in I$. Let $K = \bigoplus_{\alpha \in \Lambda_\alpha} K_\alpha$. Then $K$ is a compact set in $C$ such that $K \cap U_\gamma \neq \emptyset$ for all $\gamma \in I$. Hence $C$ is cellular compact. □

**Theorem 2.12.** CCS-normality is an additive property.

**Proof.** Let $X = \bigoplus_{\alpha \in \Lambda} X_\alpha$ and each $X_\alpha$ is cellular-compact. Then for each $\alpha \in \Lambda$ there exists a normal space $Y_\alpha$ and a bijective mapping $f_\alpha : X_\alpha \to Y_\alpha$ such that $f_\alpha|C_\alpha : C_\alpha \to f_\alpha(C_\alpha)$ is a homeomorphism for each cellular-compact subspace $C_\alpha$ of $X_\alpha$. Then $Y = \bigoplus_{\alpha \in \Lambda} Y_\alpha$ is a normal space. Consider the function $f = \bigoplus_{\alpha \in \Lambda} f_\alpha : X \to Y$. Let $C$ be a cellular -compact subspace of $X$. Then by Theorem 2.11, $\Lambda_\alpha = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is finite and $C \cap X_\alpha$ is cellular -compact for each $\alpha \in \Lambda$. Then $f|C : C \to f(C)$ is a homeomorphism. Hence $X$ is CCS-normal. □

3. Some special results

**Definition 3.1.** A topological space $X$ is called submetrizable if the topology admits of a weaker metrizable topology.

It is proved in [2] that submetrizable space is C-normal. In [5], the authors raised a question. Is a submetrizable space CC-normal? We can claim that the answer is affirmative.

**Theorem 3.2.** Every submetrizable space is CC-normal.

In order to prove this we need the following Lemmas.

**Lemma 3.3.** Let $X$ be a first countable Hausdorff space. If $A \subset X$ is countably compact, then $A$ is closed in $X$.

**Proof.** Let $\emptyset \neq A \subset X$ be countably compact subspace. Let $\{U_n : n \geq 1\}$ be a countable local base at $x \notin A$. For each $y \in A$ there exist open sets $U_n(y), V_n(y)$ such that $x \in U_n(y), y \in V_n(y)$ and $U_n(y) \cap V_n(y) = \emptyset$. Now $\{V_n(y) : y \in A\}$ is , in fact , a countable open cover of $A$ and hence , admits of a finite subcover, say $\{V_{n(i)} : 1 \leq i \leq k\}$. Look at $U_x = \bigcap_{i=1}^k U_{n(i)}$. Then $U_x \cap A = \emptyset$ and $x \in U_x$ an open neighbourhood of $x$ . Thus $X - A$ is open $\Longrightarrow$ $A$ is closed.

**Lemma 3.4.** If $X$ be a countably compact space, $Y$ is first countable Hausdorff space and $f : X \to Y$ is a continuous bijection. Then $f$ is a homeomorphism.

**Proof.** Since $f$ is a continuous bijection, suffices to show that $f$ is a closed map. Let $A \subset X$ be closed. Then $A$ is countably compact and ipso facto, $f(A)$ is countably compact in $Y$. Since $Y$ is first countable , Hausdorff, $f(A)$ is closed in $Y$, i.e, $f$ is a closed map. Consequently $f$ is a homeomorphism. □

**Theorem 3.5.** Every submetrizable space is CC-normal.
countable, \( T \) and \( C \) countably compact subset of a metrizable space, is indeed compact which renders submetrizable space, compact and countably compact subsets coincide.

Claim: any countably compact subspace such that (a homeomorphism, so that (X, \( \tau \)) and 1 \( X \) : \( \tau \) \( C \) is a homeomorphism, so that (X, \( \tau \)) is CC-normal. Further note that 1 \( X \) being a countably compact subset of a metrizable space, is indeed compact which renders \( C \) into a compact subset of (X, \( \tau \)). This yields the further information that in a submetrizable space, compact and countably compact subsets coincide. \( \square \)

**Example 3.6.** Example \[3.3\] produces an example of a submetrizable space which is not cellular-compact. It is proved in \[7\] that a cellular-compact, first-countable \( T_3 \) space is countably-compact and ipso facto, closed in view of Lemma \[3.3\]. We can conclude that in a metrizable space all three concepts, viz, Compactness, countable-compactness and Cellular-compactness coincide.

**Definition 3.7.** Let \( M \) be a non-empty proper subset of a topological space (X, \( \tau \)). Define a new topology \( \tau(M) \) on X as follows: \( \tau(M) = \{ U \cup K : U \in \tau \) and \( K \subset X \setminus M \} \), (X, \( \tau(M) \)) is called a discrete extension of (X, \( \tau \)) and we denote it by \( X_M \).

Discrete extension of a cellular-compact space need not be cellular compact. This can be shown in the following example.

**Example 3.8.** Let \( X = [0, 1] \) with subspace topology of usual topology on \( \mathbb{R} \) and \( M = X \times \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{ 0 \} \). Then X is compact \( \Rightarrow \) X is cellular-compact.

Claim: \( X_M \) is not cellular-compact. Let \( \mathcal{U} = \{ (\frac{1}{n}, \frac{1}{m}) : n \in \mathbb{N} \} \cup \{ \{ \frac{1}{i} \} : n \in \mathbb{N} \} \cup \{ 0 \} \) is a disjoint family of open sets in \( X_M \). There does not exist any compact set \( K \) in \( X_M \) such that \( K \cap U \neq \emptyset \), for all \( U \in \mathcal{U} \). Therefore \( X_M \) is not cellular-compact.

Let X be any topological space. Let \( X' = X \times \{ 1 \} \). Let \( A(X) = X \cup X' \). For \( x \in X \) we denote \( < x, 1 > \) as \( x' \) and for \( B \subset X \) let \( B' = \{ x' : x \in B \} = B \times \{ 1 \} \). Let \( \beta(x) : x \in X \cup \{ x' : x \in X' \} \) be neighbourhood system for some topology \( \tau \) on A(X), where for \( x \in X, \beta(x) = \{ U \cup (U' \setminus \{ x' \}) : U \) is an open set in X containing \( x \} \) and for \( x' \in X', \beta(x') = \{ \{ x' \} \} \). (A(X), \( \tau \)) is called Alexandroff Duplicate of X.

**Observation:** Let X be a topological space and \( A(X) = X \cup X' \) is the Alexandroff duplicate space. Let \( C \subset A(X) \) be a compact subspace of A(X). Let \( C \cap X = D, C \cap X' = E \). If \( D = \emptyset \), then \( E = C \subset X' \Rightarrow C \) is finite. Suppose \( D \neq \emptyset \). For \( x \in D \) let \( U_x \cup U'_x \setminus \{ x \} \) be an open neighbourhood of \( x \). Consider the family \( \{ U_x \cup U'_x \setminus \{ x \} : x' \in E \} \cup \{ \{ x' \} : x' \in E \} \) of open subsets of A(X), which is an open cover of \( C \). Let \( \{ U_x \cup U'_x \setminus \{ x \} : 1 \leq i \leq n \} \cup \{ \{ y_j \} : y_j \in E, 1 \leq j \leq m \} \) be a finite subcover of \( C \cap X \cup D \cap E \). Then \( C \cap X = D \cup \cup_{i=1}^{n} U_x \cup E \cup \cup_{j=1}^{m} U'_x \cap \{ y_j \} \subset \{ x \} \) \( \cup_{i=1}^{n} U_x \cup \cup_{j=1}^{m} U'_x = U' \). Hence \( C \cap X = D \cup E \cup U' \cup \{ \{ y_j \} : y_j \in E, 1 \leq j \leq m \} \) \( \Rightarrow C \cap X = D \cup E \cup U' \). So, we conclude that: A subset \( C \subset A(X) \) is compact \( \Rightarrow \) for any open set \( U \supset \) compact set \( C \cap X \cap \cap X' \setminus U' \) is a finite subset of \( X' \).

Conversely let \( C \subset A(X) \) be a subset such that \( C \cap X \) is compact and for any open set \( U \subset X, C \cap X \subset U, (C \cap X') \setminus U' \) is a finite subset of \( X' \).

Claim: \( C \) is compact. It suffices to prove that any open cover of \( C \) by means of basic open sets of \( A(X) \) has a finite subcover. Let \( \{ U_x \cup U'_x \setminus \{ x \} : x \in C \cap X \cup \{ x' : x' \in C \cap X' \} \) be an open cover of \( C \). Since \( C \cap X \) is compact and \( \{ U_x : x \in C \cap X \} \) is an open cover \( C \cap X \) in X. There exists a finite subcover, say,
Let $D = \{ x_i : i \leq n \}$ be a finite subset of $X \subseteq \mathbb{R}$. By hypothesis $(C \cap X') \setminus U'$ is a finite open subset of $X'$, say, $\{y_1, y_2, \ldots, y_m\}$. Hence $C \cap X = U' \cup \{y_1, y_2, \ldots, y_m\}$. Let $D = \{x_i \cap C \cap X : 1 \leq i \leq n\}$. Now $\{U_{x_i} \cup U_{x_i} \setminus \{x_i\} : 1 \leq i \leq n\} \cup \{x_i : x_i \in C \cap X, 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq m\}$ is a finite open subcover for $C$ from $\{U_x \cup U_{x_i} \setminus \{x_i\} : x \in C \cap X\} \cup \{x_i : x_i \in C \cap X\}$. Hence $C$ is compact.

We come to the conclusion that $C \subseteq A(X)$ is compact if and only if every open set $U \subseteq X, C \cap X \subseteq U$ we must have $C \cap X' \setminus U'$ is a finite subset of $X'$.

Using this result we can cite an example for which $A(X)$ is compact in $X$, $C$ is cellular-compact but $X$ is cellular-compact.

**Example 3.9.** Suppose $A(X)$ is Cellular-Compact and $X$ is infinite. Now isolated points of $A(X) = X \cup X'$, namely the points of $X'$ form an infinite family of disjoint open sets. Hence $A(X)$ must contain a compact set $C$ such that $C \cap \{x\} \neq \emptyset$ for each $x \in X'$, i.e., $C \supset X$. Since $C = X'$ is impossible, $C \cap X = \emptyset$ and $C \cap X$ is a compact set. As $C$ is compact in $A(X)$, by the preceding observation, if $U \subseteq X$ is an open set containing $C \cap X$, then $X' \setminus U = C \cap X' \setminus U'$ is a finite subset of $X'$ vide Theorem 2.2, i.e., $X \setminus U$ is a finite subset of $X$. i.e., every open set of $X$ containing $C \cap X$ is cofinite in $X$.

Now take $X = \mathbb{R}$ with cocountable topology. Then $X$ is a $T_1$ space which is cellular-compact. Also any compact subset of $X$ is finite. Now look at $A(X) = X \cup X'$. If $A(X)$ has to be cellular-compact, from the preceding para it is clear that $X$ must have a compact set $K$ such that any open set containing $K$ must be cofinite. Since $K$ is anyhow finite, this is not true! Thus $A(X)$ cannot be cellular-compact.

**Theorem 3.10.** Let $X$ be a Hausdorff space and $A(X) = X \cup X'$ be the Alexandroff duplicate space. If $A(X)$ is cellular-compact, then there should exist a compact set $K \supset$ the dense set of isolated points of $X'$. Since $K$ is closed, as $A(X)$ is $T_2$, $K = A(X)$, hence $A(X)$ is compact. Consequently if $X$ is a non-compact $T_2$ space, then $A(X)$ cannot be cellular-compact. In presence of Hausdorffness $A(X)$ is cellular-compact if and only if $X$ is compact if and only if $A(X)$ is compact.

The following problems are open.

1) Is the discrete extension of a $CCS$-normal space $CCS$-normal?
2) Is $A(X)$ $CCS$-normal, when $X$ is $CCS$-normal?

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