The inverse problem for Hamilton-Jacobi equations and semiconcave envelopes

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This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 694126-DYCON).
We consider the following initial-value problem

\[
\begin{aligned}
\partial_t u + H(D_x u) &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

where \( u_0 \in \text{Lip}(\mathbb{R}^n) \) is the initial condition and

\[H : \mathbb{R}^n \rightarrow \mathbb{R}\]

is a \( C^2 \) Hamiltonian satisfying

\[
D^2 H(x) > 0, \quad \forall x \in \mathbb{R}^n \text{ and } \frac{H(|x|)}{|x|} \rightarrow +\infty \quad |x| \rightarrow \infty.
\]
A problem in calculus of variations

We are given \( T > 0 \) and two cost functions:

**Running cost:** \( L : \mathbb{R}^n \rightarrow \mathbb{R} \)

**Initial cost:** \( u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \)

For any \((t, x) \in ]0, T[ \times \mathbb{R}^n\), we introduce the set of **admissible arcs**

\[ \mathcal{A}(t, x) := \{ \alpha \in C^1([0, t]; \mathbb{R}^n) ; \alpha(t) = x \}, \]

and consider the following **minimization problem**:

\[
\text{minimize } \int_0^t L(\alpha'(s)) \, ds + u_0(\alpha(0)) \text{ over all arcs } \alpha \in \mathcal{A}(t, x).
\]

We define the **value function**:

\[
u(t, x) = \inf_{\alpha(\cdot) \in \mathcal{A}(t, x)} \left\{ \int_0^t L(\alpha'(s)) \, ds + u_0(\alpha(0)) \right\}.
\]

This function satisfies the equation

\[
\partial_t u + H(D_x u) = 0
\]

at all points of differentiability of \( u \). Here, \( H \) is given by

\[
H(v) = \sup_{z \in \mathbb{R}^n} \{ z \cdot v - L(z) \}.
\]

We remark that \( u \) is Lipschitz in \([0, T] \times \mathbb{R}\) provided \( u_0 \in \text{Lip}(\mathbb{R}) \) and \( H \) satisfies (H).
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We remark that \( u \) is Lipschitz in \([0, T] \times \mathbb{R}\) provided \( u_0 \in \text{Lip}(\mathbb{R}) \) and \( H \) satisfies (H).
Viscosity solutions

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t u + H(D_x u) &= 0, \\
u(0, x) &= u_0(x),
\end{array} \right.
\quad (t, x) \in (0, T) \times \mathbb{R}^n
\]

(HJ)

In general we cannot expect to have $C^1$ solutions. Therefore, we need to consider generalized solutions:

\[
u \in W^{1,\infty}_{\text{loc}}, \text{ satisfying } (HJ) \text{ a.e.}
\]

We have no uniqueness of generalized solutions.
Viscosity solutions

\[
\begin{aligned}
\begin{cases}
\partial_t u + H(D_x u) &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^n,
\end{cases}
\end{aligned}
\]  

(HJ)

**Definition**

We say \( u \in C([0, T] \times \mathbb{R}^n) \) is a *viscosity solution* if

\[
u(0, x) = u_0(x)
\]

and for any \((t, x) \in (0, T) \times \mathbb{R}^n\) we have

\[
\begin{aligned}
p_t + H(p_x) &\leq 0 \quad \text{for all } (p_t, p_x) \in D^+ u(t, x) \\
p_t + H(p_x) &\geq 0 \quad \text{for all } (p_t, p_x) \in D^- u(t, x)
\end{aligned}
\]

where the super- and sub-differentials are defined by

\[
\begin{aligned}
D^+ u(t, x) &= \{(p_t, p_x) : p_t = \varphi_t(t, x), \ p_x = D\varphi(t, x), \ \exists \varphi \in C^1, \\
&\quad u - \varphi \leq 0, \ (u - \varphi)(t, x) = 0\}, \\
D^- u(t, x) &= \{(p_t, p_x) : p_t = \varphi_t(t, x), \ p_x = D\varphi(t, x), \ \exists \varphi \in C^1, \\
&\quad u - \varphi \geq 0, \ (u - \varphi)(t, x) = 0\}.
\end{aligned}
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Viscosity solutions

\[
\begin{align*}
\begin{cases}
\frac{\partial t}{\partial t} u + H(D_x u) &= 0, \\
u(0, x) &= u_0(x),
\end{cases} \\
(t, x) &\in (0, T) \times \mathbb{R}^n, \\
x &\in \mathbb{R}^n,
\end{align*}
\]

(HJ)

Theorem: Crandall-P.L. Lions, 1980’s

Let $T > 0$, $u_0 \in \text{Lip}(\mathbb{R}^n)$ and $H$ satisfy (H). The problem (HJ) admits a unique viscosity solution and coincides with the value function of the problem in calculus of variations.

We define the following nonlinear operator:

$$
S_T^+ : \text{Lip}(\mathbb{R}) \longrightarrow \text{Lip}(\mathbb{R}), \\
u_0 \longmapsto S_T^+ u_0 := u(T, \cdot)
$$
Goals

Consider the nonlinear operator:

\[ S_T^+ : \text{Lip}(\mathbb{R}) \rightarrow \text{Lip}(\mathbb{R}) \]

\[ u_0 \mapsto S_T^+ u_0 := u(T, \cdot) \]

where \( u \) is the viscosity solution of (HJ).

**The inverse problem:** for a given target \( u_T \in \text{Lip}(\mathbb{R}^n) \) and \( T > 0 \) fixed,

- Study the reachability of \( u_T \), i.e. determine if the set

\[ I_T(u_T) := \{ u_0 \in \text{Lip}(\mathbb{R}) ; S_T^+ u_0 = u_T \} \]

is empty or not.

- If \( u_T \) is reachable, construct all the initial conditions in \( I_T(u_T) \).

- If \( u_T \) is not reachable, define a projection of \( u_T \) on the set of reachable targets and study its geometrical properties.

**Long-time behavior:** for a given initial condition \( u_0 \in \text{Lip}(\mathbb{R}^n) \), set the target

\[ u_T := S_T^+ u_0 \]

and the set of initial conditions \( I_T(u_T) \neq \emptyset \), for each \( T > 0 \).

- Describe the evolution of \( I_T(u_T) \) as \( T \) increases.

- Study the behavior of \( I_T(u_T) \) as \( T \) goes to infinity.
The backward operator

Definition

A uniformly continuous function \( w : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \) is called a **backward viscosity solution** of (HJ) if the function \( v(t, x) := w(T - t, x) \) is a viscosity solution of

\[
\partial_t v - H(D_x v) = 0, \quad \text{in} \ [0, T] \times \mathbb{R}^n.
\]

Lemma

We say \( w \in C([0, T] \times \mathbb{R}^n) \) is a backward viscosity solution if and only if for any \((t, x) \in (0, T) \times \mathbb{R}^n\) we have

\[
\begin{align*}
p_t + H(p_x) &\geq 0 \quad \text{for all} \ (p_t, p_x) \in D^+ w(t, x) \\
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Definition

A uniformly continuous function $w : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is called a **backward viscosity solution** of (HJ) if the function $v(t, x) := w(T - t, x)$ is a viscosity solution of

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Lemma

We say $w \in C([0, T] \times \mathbb{R}^n)$ is a backward viscosity solution if and only if for any $(t, x) \in (0, T) \times \mathbb{R}^n$ we have

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The backward operator

Using similar arguments as for (forward) viscosity solutions, for any terminal condition $u_T \in \text{Lip}(\mathbb{R}^n)$, the problem

$$\begin{cases} 
\partial_t w + H(\partial_x w) = 0, & \text{in } [0, T] \times \mathbb{R}^n, \\
w(T, x) = u_T(x), & \text{in } \mathbb{R}
\end{cases}$$  \hspace{1cm} \text{(BHJ)}$$

admits a unique backward viscosity solution.

We define the following nonlinear operator:

$$S_T^- : \text{Lip}(\mathbb{R}) \rightarrow \text{Lip}(\mathbb{R})$$

$$u_T \mapsto S_T^- u_T := w(0, \cdot)$$

where $w$ is the backward viscosity solution of (BHJ).

Hopf formula

$$S_T^+ u_0(x) = \min_{y \in \mathbb{R}^n} \left[ u_0(y) + T H \left( \frac{x - y}{T} \right) \right]$$

$$S_T^- u_T(x) = \max_{y \in \mathbb{R}^n} \left[ u_T(y) - T H \left( \frac{y - x}{T} \right) \right]$$
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S^-_T : \text{Lip}(\mathbb{R}) \rightarrow \text{Lip}(\mathbb{R})
\]

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u_T \mapsto S^-_T u_T := w(0, \cdot)
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where \( w \) is the backward viscosity solution of (BHJ).

Hopf formula

\[
S^+_T u_0(x) = \min_{y \in \mathbb{R}^n} \left[ u_0(y) + T \left( \frac{x - y}{T} \right) \right]
\]

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S^-_T u_T(x) = \max_{y \in \mathbb{R}^n} \left[ u_T(y) - T \left( \frac{y - x}{T} \right) \right]
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Hamilton-Jacobi equations
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\]
Semiconcavity and semiconvexity

**Definition**

1. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is semiconcave with linear modulus if it is continuous and there exists $C \geq 0$ such that

$$f(x + h) + f(x - h) - 2f(x) \leq C h^2,$$

for all $x, h \in \mathbb{R}^n$. The constant $C$ above is called a semiconcavity constant of $f$.

2. We say that $f$ is semiconvex with linear modulus and constant $C > 0$ if the function $g = -f$ is semiconcave with linear modulus and constant $C$.

**Lemma**

Let $T > 0$ and $u_0, u_T \in \text{Lip}(\mathbb{R})$. Then,

1. the function $S_T^+ u_0$ is semiconcave with linear modulus;
2. the function $S_T^- u_T$ is semiconvex with linear modulus.
Semiconcavity and semiconvexity

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1. the function \( S_T^+ u_0 \) is semiconcave with linear modulus;

2. the function \( S_T^- u_T \) is semiconvex with linear modulus.
Examples

\[ u_1(x) := \begin{cases} 
|x + 1| - 1 & \text{if } -2 < x \leq 0 \\
|x - 1| - 1 & \text{if } 0 < x < 2 \\
0 & \text{else.}
\end{cases} \]

For \( T = 1 \), the function \( S_T^+ u_1 \) at the left and the function \( S_T^- u_1 \) at the right.
Examples

\[ u_2(x) := \begin{cases} 
1 - 2|x + 1| & \text{if } -1.5 < x \leq 0 \\
1 - 2|x - 1| & \text{if } 0 < x < 1.5 \\
0 & \text{else.}
\end{cases} \]

For \( T = 0.5 \), the function \( S_T^+ u_2 \) at the left and the function \( S_T^- u_2 \) at the right.

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Hamilton-Jacobi equations
**Lemma**

Let $T > 0$ and $u_0 \in \text{Lip}(\mathbb{R}^n)$. Set the function

$$\tilde{u}_0(x) := S_T^- (S_T^+ u_0)(x), \quad \text{for all } x \in \mathbb{R}^n.$$  

Then it holds

$$S_T^+ u_0 = S_T^+ \tilde{u}_0, \quad \text{and} \quad u_0(x) \geq \tilde{u}_0(x), \quad \text{for all } x \in \mathbb{R}^n.$$
Reachability condition

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Theorem (reachability condition)
Let $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$. Then, the set $I_T(u_T)$ is nonempty if and only if

$$S_T^+ (S_T^- u_T) = u_T.$$
Theorem (initial data construction)

Let $T > 0$ and consider $u_T \in \text{Lip}(\mathbb{R}^n)$ such that $I_T(u_T) \neq \emptyset$. Define the function

$$\tilde{u}_0 := S_T^{-1} u_T.$$

For any $u_0 \in \text{Lip}(\mathbb{R}^n)$, the two following statements are equivalent:

1. $u_0 \in I_T(u_T)$;

2. $u_0(x) \geq \tilde{u}_0(x), \ \forall x \in \mathbb{R}^n$ and $u_0(x) = \tilde{u}_0(x), \ \forall x \in X_T(u_T)$,

where $X_T(u_T)$ is the subset of $\mathbb{R}$ defined by

$$X_T(u_T) := \{z - T \nabla_x u_T(z); \forall z \in \mathbb{R}^n \text{ such that } u_T(\cdot) \text{ is differentiable at } z\}.$$ 

Remark: Observe that, by the reachability condition,

$$I_T(u_T) \neq \emptyset, \ \text{implies} \ \tilde{u}_0 \in I_T(u_T).$$

In view of this theorem, we can write

$$I_T(u_T) = \{\tilde{u}_0 + \varphi; \ \varphi \in \text{Lip}(\mathbb{R}) \text{ such that } \varphi \geq 0 \text{ and } \text{supp}(\varphi) \subset \mathbb{R} \setminus X_T(u_T)\}.$$
Theorem (initial data construction)

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In view of this theorem, we can write

$$I_T(u_T) = \{\tilde{u}_0 + \varphi; \varphi \in \text{Lip}(\mathbb{R}) \text{ such that } \varphi \geq 0 \text{ and } \text{supp} \varphi \subset \mathbb{R} \setminus X_T(u_T)\}.$$
Consider $T = 0.5$ and the target function

$$u_T(x) := S_T^+ u_3(x), \quad \text{where} \quad u_3(x) := \begin{cases} 1 - |x + 1| & \text{if } -2 < x \leq 0 \\ 1 - |x - 1| & \text{if } 0 < x < 2 \\ 0 & \text{else.} \end{cases}$$

$$X_T(u_T) = \mathbb{R} \setminus \left([-1,5, -0.5] \cup [0.5, 1, 5]\right).$$

The function $u_T$ is represented at the left. The function $\tilde{u}_0$ is represented at the right. The restriction of $\tilde{u}_0$ to the set $X_T(u_T)$ is marked by a red line.
Consider $T = 0.5$ and the target function

$$u_T(x) := S^+_T u_4(x), \text{ where } u_4(x) := \begin{cases} -(1 - 4|x|^2) & \text{if } |x| < \frac{1}{2} \\ 1 - 4|x - (3, 0)|^2 & \text{if } |x - (3, 0)| < \frac{1}{2} \\ 0 & \text{else.} \end{cases}$$

From left to right we have: the function $u_T$, the function $\tilde{u}_0$ and the set $X_T(u_T)$ in blue.
For a given function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the **concave envelope** $f^*$ is the smallest concave function which stays above $f$.

$$f^*(x) := \inf \{ v(x) ; \ v \text{ is concave and } v(x) \geq f(x), \ \forall x \in \mathbb{R}^n \}.$$ 

**Theorem**: (Oberman in 2007) Let $f \in \text{Lip}(\mathbb{R}^n)$, then $f^*$ is the viscosity solution of the following fully nonlinear obstacle problem:

$$\min \{ v(x) - f(x), -\lambda_n[D^2v(x)] \} = 0.$$ 

Here, $\lambda_n[D^2v(x)]$ denotes the biggest eigenvalue of the Hessian matrix $D^2v(x)$.
The concave envelope

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For a given function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), the **concave envelope** \( f^* \) is the smallest concave function which stays above \( f \).

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This kind of operators, and its connection with geometry and game theory, have been largely studied during the past 10 year by many authors: A.M Oberman, L. Silvestre, I. Birindelli, F.R. Harvey, H.B. Lawson, H. Ishii, M. Parviainen, P. Blanc, J.D. Rossi...
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Hamilton-Jacobi equations
What if the target $u_T$ is not reachable?

Consider the composition operator

$$S^+_T \circ S^-_T : \text{Lip}(\mathbb{R}^n) \rightarrow \text{Lip}(\mathbb{R}^n)$$

$$u_T \mapsto S^+_T (S^-_T u_T)$$

Note the the function $u^*_T := S^+_T (S^-_T u_T)$ satisfies $I_T(u_T) \neq \emptyset$.

The operator $S^+_T \circ S^-_T$ can be viewed as a projection of Lip($\mathbb{R}^n$) onto the set of reachable targets.

**Theorem**

Let $H(p) = \frac{|p|^2}{2}$ and $u_T \in \text{Lip}(\mathbb{R}^n)$. Then, the function $u^*_T := S^+_T (S^-_T u_T)$ is the viscosity solution of the obstacle problem

$$\min\{v(x) - u_T(x), -\lambda_n [D^2 v(x)] + \frac{1}{\gamma} \} = 0.$$
What if the target $u_T$ is not reachable?

Consider the composition operator

$$S^+_T \circ S^-_T : \text{Lip}(\mathbb{R}^n) \longrightarrow \text{Lip}(\mathbb{R}^n)$$

$$u_T \longmapsto S^+_T(S^-_T u_T)$$

Note the the function $u^*_T := S^+_T(S^-_T u_T)$ satisfies $I_T(u_T) \neq \emptyset$.

The operator $S^+_T \circ S^-_T$ can be viewed as a projection of $\text{Lip}(\mathbb{R}^n)$ onto the set of reachable targets.

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Let $H(p) = \frac{|p|^2}{2}$ and $u_T \in \text{Lip}(\mathbb{R}^n)$. Then, the function $u^*_T := S^+_T(S^-_T u_T)$ is the viscosity solution of the obstacle problem

$$\min\{v(x) - u_T(x), -\lambda_n[D^2v(x)] + \frac{1}{T}\} = 0.$$
Semiconcave envelopes

What if the target $u_T$ is not reachable?

Consider the composition operator

$$S_T^+ \circ S_T^- : \text{Lip}(\mathbb{R}^n) \longrightarrow \text{Lip}(\mathbb{R}^n)$$

$$u_T \longmapsto S_T^+(S_T^- u_T)$$

Note the the function $u_T^* := S_T^+(S_T^- u_T)$ satisfies $I_T(u_T) \neq \emptyset$.

The operator $S_T^+ \circ S_T^-$ can be viewed as a projection of $\text{Lip}(\mathbb{R}^n)$ onto the set of reachable targets.

Theorem

Let $H(p) = \frac{|p|^2}{p}$ and $u_T \in \text{Lip}(\mathbb{R}^n)$. Then, the function $u_T^* := S_T^+(S_T^- u_T)$ is the viscosity solution of the obstacle problem

$$\min \{ v(x) - u_T(x), -\lambda_n[D^2 v(x)] + \frac{1}{T} \} = 0.$$  

Observe that, the inequality $\lambda_n[D^2 u_T^*(x)] \leq \frac{1}{T}$ implies that the function $u_T^*$ is semiconcave with linear modulus and constant $C = \frac{1}{T}$.
What if the target \( u_T \) is not reachable?

Consider the composition operator

\[
S^+_T \circ S^-_T : \text{Lip}(\mathbb{R}^n) \rightarrow \text{Lip}(\mathbb{R}^n)
\]

\[
u_T \rightarrow S^+_T(S^-_T u_T)
\]

Note the the function \( u^*_T := S^+_T(S^-_T u_T) \) satisfies \( I_T(u_T) \neq \emptyset \).

The operator \( S^+_T \circ S^-_T \) can be viewed as a projection of \( \text{Lip}(\mathbb{R}^n) \) onto the set of reachable targets.

Theorem

Let \( H(p) = \frac{|p|^2}{p} \) and \( u_T \in \text{Lip}(\mathbb{R}^n) \). Then, the function \( u^*_T := S^+_T(S^-_T u_T) \) is the viscosity solution of the obstacle problem

\[
\min\{v(x) - u_T(x), -\lambda n[D^2v(x)] + \frac{1}{T}\} = 0.
\]

In analogy with the concave envelope, we refer to the function \( u^*_T \) as the \( \frac{1}{T} \)-semiconcave envelope of \( u_T \) in \( \mathbb{R}^n \).
What if the target $u_T$ is not reachable?

Consider the composition operator

$$S_T^+ \circ S_T^- : \text{Lip}(\mathbb{R}^n) \rightarrow \text{Lip}(\mathbb{R}^n) \quad u_T \mapsto S_T^+(S_T^- u_T)$$

Note the the function $u_T^* := S_T^+(S_T^- u_T)$ satisfies $I_T(u_T) \neq \emptyset$.

The operator $S_T^+ \circ S_T^-$ can be viewed as a projection of $\text{Lip}(\mathbb{R}^n)$ onto the set of reachable targets.

**Theorem**

Let $H(p) = \frac{|p|^2}{p}$ and $u_T \in \text{Lip}(\mathbb{R}^n)$. Then, the function $u_T^* := S_T^+(S_T^- u_T)$ is the viscosity solution of the obstacle problem

$$\min\{v(x) - u_T(x), -\lambda_n[D^2 v(x)] + \frac{1}{T}\} = 0.$$ 

**Corollary (reachability condition)**

Let $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$, then the set $I_T(u_T)$ is nonempty if and only if $u_T$ satisfies the inequality $\lambda_n[D^2 u_T(x)] \leq \frac{1}{T}$ in a viscosity sense.
Examples

\[ u_1(x) := \begin{cases} 
|x + 1| - 1 & \text{if } -2 < x \leq 0 \\
|x - 1| - 1 & \text{if } 0 < x < 2 \\
0 & \text{else.}
\end{cases} \]

\[ u_2(x) := \begin{cases} 
1 - 2|x + 1| & \text{if } -1,5 < x \leq 0 \\
1 - 2|x - 1| & \text{if } 0 < x < 1,5 \\
0 & \text{else.}
\end{cases} \]

Here can see the \( \frac{1}{T} \)-semiconcave envelopes of \( u_1 \) and \( u_2 \) respectively and the functions \( u_1 \) and \( u_2 \) represented by dotted lines.
Thanks for the attention!

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