Conformal Invariance, Universality, and the Dimension of the Brownian Frontier

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Abstract

This paper describes joint work with Oded Schramm and Wendelin Werner establishing the values of the planar Brownian intersection exponents from which one derives the Hausdorff dimension of certain exceptional sets of planar Brownian motion. In particular, we prove a conjecture of Mandelbrot that the dimension of the frontier is $4/3$. The proof uses a universality principle for conformally invariant measures and a new process, the stochastic Loewner evolution (SLE), introduced by Schramm. These ideas can be used to study other planar lattice models from statistical physics at criticality. I discuss applications to critical percolation on the triangular lattice, loop-erased random walk, and self-avoiding walk.

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1. Exceptional sets for planar Brownian motion

Let $B_t$ be a standard Brownian motion taking values in $\mathbb{R}^2 = \mathbb{C}$ and let $B[s, t]$ denote the random set $B[s, t] = \{ B_r : s \leq r \leq t \}$. For $0 \leq t \leq 1$, we say that $B_t$ is a

- cut point for $B[0, 1]$ if $B[0, t] \cap B(t, 1] = \emptyset$;
- frontier point for $B[0, 1]$ if $B_t$ is on the boundary of the unbounded component of $\mathbb{C} \setminus B[0, 1]$;
- pioneer point for $B[0, 1]$ if $B_t$ is on the boundary of the unbounded component of $\mathbb{C} \setminus B[0, t]$, i.e., if $B_t$ is a frontier point for $B[0, t]$.

I will discuss the following result proved by Oded Schramm, Wendelin Werner, and myself.
**Theorem 1.** [17] [18] [20] If $B_t$ is a standard Brownian motion in $\mathbb{R}^2 = \mathbb{C}$, then with probability one,

$$
\dim_{h} \{ \text{cut points for } B[0,1] \} = 3/4,
$$

$$
\dim_{h} \{ \text{frontier points for } B[0,1] \} = 4/3,
$$

$$
\dim_{h} \{ \text{pioneer points for } B[0,1] \} = 7/4,
$$

where $\dim_{h}$ denotes Hausdorff dimension.

Mandelbrot [27] first gave the conjecture for the Brownian frontier, basing his conjecture on numerical simulation and then noting that simulations of the frontier resembled simulations of self-avoiding walks. It is conjectured that the scaling limit of planar self-avoiding walks has paths of dimension $4/3$. Duplantier and Kwon [5] used nonrigorous conformal field theory techniques to make the above conjectures for the cut points and pioneer points. More precisely, they made conjectures about certain exponents called the Brownian or simple random walk intersection exponents. More recently, Duplantier [6] has given other nonrigorous arguments for the conjectures using quantum gravity.

To prove Theorem 1, it suffices to find the values of the Brownian intersection exponents. In fact, before Theorem 1 had been proved, it had been established [11] [12] [13] that the Hausdorff dimensions of the set of cut points, frontier points, and pioneer points were $2 - \eta_1$, $2 - \eta_2$, and $2 - \eta_3$, respectively, where $\eta_1, \eta_2, \eta_3$ are defined by saying that as $\epsilon \to 0+$,

$$
P\{ B[0, \frac{1}{2} - \epsilon^2] \cap B[\frac{1}{2} + \epsilon^2, 1] = \emptyset \} \approx \epsilon^{\eta_1},
$$

$$
P\{ B[0, \frac{1}{2} - \epsilon^2] \cup B[\frac{1}{2} + \epsilon^2, 1] \text{ does not disconnect } B_{1/2} \text{ from infinity} \} \approx \epsilon^{\eta_2},
$$

$$
P\{ B[\epsilon^2, 1] \text{ does not disconnect } 0 \text{ from infinity} \} \approx \epsilon^{\eta_3}.
$$

It had also been established [3] [16] that the analogous exponents for simple random walk are the same as for Brownian motion.

There are two main ideas in the proof. The first is a one parameter family of conformally invariant processes developed by Oded Schramm [30] which he named the Stochastic Loewner evolution ($SLE$). The second is the idea of “universality” which states roughly that all conformally invariant measures that satisfy a certain “locality” or “restriction” property must have the same exponents as Brownian motion (see [26]). In this paper, I will define $SLE$ and give some of its properties; describe how analysis of $SLE$ leads to finding the Brownian intersection exponents; and finally describe some other planar lattice models in statistical physics at criticality that can be understood using $SLE$.

### 2. Stochastic Loewner evolution

I will give a brief introduction to the stochastic Loewner evolution ($SLE$); for more details, see [29] [17] [18] [15] [28]. Let $W_t$ denote a standard one dimensional...
Brownian motion. If \( \kappa \geq 0 \) and \( z \) is in the upper half plane \( \mathbb{H} = \{ w \in \mathbb{C} : \Im(w) > 0 \} \), let \( g_t(z) \) be the solution to the Loewner differential equation

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} W_t}, \quad g_0(z) = z. \tag{2.1}
\]

For each \( z \in \mathbb{H} \), the solution \( g_t(z) \) is defined up to a time \( T_z \in (0, \infty) \). Let \( H_t = \{ z : T_z > t \} \). Then \( g_t \) is the unique conformal transformation of \( H_t \) onto \( \mathbb{H} \) with \( g_t(z) - z = o(1) \) as \( z \to \infty \). In fact,

\[
g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.
\]

It is easy to show that the maps \( g_t \) are well defined. It has been shown \(^{28},^{22}\) that there is a (random) continuous path \( \gamma : [0, \infty) \to \mathbb{H} \) such that \( H_t \) is the unbounded component of \( \mathbb{H} \setminus \gamma[0,t] \) and \( g_t(\gamma(t)) = \sqrt{\kappa} W_t \). The conformal maps \( g_t \) or the corresponding paths \( \gamma(t) \) are called the \textit{chordal stochastic Loewner evolution with parameter} \( \kappa \) (chordal \( \text{SLE}_\kappa \)). It is easy to check that the distribution of \( \text{SLE}_\kappa \) is invariant (modulo time change) under dilations \( z \mapsto rz \). Using this, we can use conformal transformations to define chordal \( \text{SLE}_\kappa \) connecting two distinct boundary points of any simply connected domain. This gives a family of probability measures on curves (modulo reparametrization) on such domains that is invariant under conformal transformation.

Chordal \( \text{SLE}_\kappa \) can also be considered as the only probability distributions on continuous curves (modulo reparametrization) \( \gamma : [0, \infty) \to \mathbb{H} \) with the following properties.

- \( \gamma(0) = 0, \gamma(t) \to \infty \) as \( t \to \infty \), and \( \gamma(t) \in \partial H_t \) for all \( t \in [0, \infty) \), where \( H_t \) is the unbounded component of \( \mathbb{H} \setminus \gamma[0,t] \).
- Let \( h_t : H_t \to \mathbb{H} \) be the unique conformal transformation with \( h_t(\gamma(t)) = 0, h_t(\infty) = \infty, h_t'(\infty) = 1 \). Then the conditional distribution of \( \hat{\gamma}(s) := h_t \circ \gamma(s+t), 0 \leq s < \infty \), given \( \gamma[0,t] \) is the same as the original distribution.
- The measure is invariant under \( x + iy \mapsto -x + iy \).

There is a similar process called \textit{radial \( \text{SLE}_\kappa \)} on the unit disk. Let \( W_t \) be as above, and for \( z \) in the unit disk \( \mathbb{D} \), consider the equation

\[
\partial_t g_t(z) = g_t(z) \frac{e^{i\sqrt{\kappa} W_t} + g_t(z)}{e^{i\sqrt{\kappa} W_t} - g_t(z)}, \quad g_0(z) = z.
\]

Let \( U_t \) be the set of \( z \in \mathbb{D} \) for which \( g_t(z) \) is defined. It can be shown that there is a random path \( \gamma : [0, \infty) \to \mathbb{H} \), such that \( U_t \) is the component of \( \mathbb{D} \setminus \gamma[0,t] \) containing the origin; \( g_t(\gamma(t)) = e^{i\sqrt{\kappa} W_t} \); and \( g_t \) is a conformal transformation of \( U_t \) onto \( \mathbb{D} \) with \( g_t(0) = 0, g_t'(0) = e^i \). We can define radial \( \text{SLE}_\kappa \) connecting any boundary point to any interior point of a simply connected domain by conformal transformation.

The qualitative behavior of the paths \( \gamma \) varies considerably as \( \kappa \) varies, although chordal and radial \( \text{SLE}_\kappa \) for the same \( \kappa \) are qualitatively similar. The
Hausdorff dimension of $\gamma[0,t]$ for chordal or radial $\text{SLE}_\kappa$ is conjectured to be $\min\{1 + (\kappa/8), 2\}$. This has been proved for $\kappa = 8/3, 6$, see [4], and for other $\kappa$ it is a rigorous upper bound [23]. For $0 < \kappa \leq 4$, the paths $\gamma$ are simple (no self-intersections) and $\gamma(0, \infty)$ is a subset of $\mathbb{H} \cup \mathbb{D}$. For $\kappa > 4$, the paths have double points and hit $\partial \mathbb{H}$ or $\partial \mathbb{D}$ infinitely often. If $\kappa \geq 8$, the paths are space filling.

Investigation of $\text{SLE}_\kappa$ requires studying the behavior of $\text{SLE}_\kappa$ under conformal maps. Suppose $A$ is a compact subset of $\overline{\mathbb{H}}$ not containing the origin such that $\overline{A \cap \mathbb{H}} = A$ and $\mathbb{H} \setminus A$ is simply connected. Let $\Phi$ denote the conformal transformation of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with $\Phi(0) = 0, \Phi(\infty) = \infty, \Phi'(\infty) = 1$. Let $\gamma$ denote a chordal $\text{SLE}_\kappa$ starting at the origin, and let $T$ be the first time $t$ that $A \cap \mathbb{H} \notin H_t$. For $t < T$, let $\tilde{\gamma}(t) = \Phi \circ \gamma(t)$. Let $\tilde{g}_t$ be the conformal transformation of the unbounded component of $\mathbb{H} \setminus \tilde{\gamma}[0,t]$ onto $\mathbb{H}$ with $\tilde{g}_t(z) - z = o(1)$ as $z \to \infty$; define $a(t)$ by $\tilde{g}(z) - z \sim a(t) z^{-1}$. Then $\tilde{g}(t)$ satisfies the modified Loewner equation

$$\partial_t \tilde{g}(t) = \frac{\partial a}{\tilde{g}(z) - \tilde{W}_t}, \quad \tilde{g}_0(z) = z,$$

for some $\tilde{W}_t$. In fact $\tilde{W}_t = \tilde{g}_t \circ \Phi \circ \tilde{g}_t^{-1}(\sqrt{\kappa} W_t)$. Using the Loewner differential equation and Itô’s formula, we can write $\tilde{W}_t$ as a local semimartingale, $d\tilde{W}_t = b(t) dt + \sqrt{\kappa} \partial_t a/2 dW_t$; here $b(t)$ and $a(t)$ are random depending on $W_s, 0 \leq s < t$. For $\kappa = 6$, and only $\kappa = 6$, the drift term $b(t)$ disappears and hence $\tilde{W}_t$ is a time change of Brownian motion.

**Locality property for $\text{SLE}_6$.** [17] If $\kappa = 6$, $\tilde{\gamma}(t), 0 \leq t < T$, has the same distribution as a time change of $\text{SLE}_6$.

For other values of $\kappa$, the image $\tilde{\gamma}(t), t < T$, has a distribution that is absolutely continuous with respect to that of (a time change of) $\text{SLE}_\kappa$. This follows from Girsanov’s theorem (see, e.g. [1] Theorem I.6.4) that states roughly that Brownian motions with the same variance but different drifts give rise to absolutely continuous measures on paths. Similarly, radial $\text{SLE}_\kappa$ can be obtained from chordal $\text{SLE}_\kappa$ by considering its image under a map taking $\mathbb{H}$ to $\mathbb{D}$. For all values of $\kappa$ we get absolutely continuous measures (which is why radial $\text{SLE}_\kappa$ is qualitatively the same as chordal $\text{SLE}_\kappa$), but for $\kappa = 6$ we get a special relationship [13] Theorem 4.1.

One of the reasons that $\text{SLE}_\kappa$ is useful is that “crossing probabilities” and “critical exponents” for the process can be calculated. The basic idea is to relate an event about the planar path $\gamma$ to an event about the driving process $\sqrt{\kappa} W_t$ and then to use standard methods of stochastic calculus to relate this to solutions of partial differential equations. As an example, consider chordal $\text{SLE}_6$ in the upper half plane $\mathbb{H}$ going from $x \in (0,1)$ to infinity. Let $T$ be the first time $t$ that $\gamma(t) \in (-\infty,0] \cap [1,\infty)$; since $\kappa > 4$, $T < \infty$ with probability one. Let $E$ be the event that $\gamma(T) \in (-\infty,0]$, and let $H_T$ be the unbounded component of $\mathbb{H} \setminus \gamma[0,T]$. Let $y_1, y_2$ be the minimum and maximum of $\gamma[0,T] \cap \mathbb{R}$; on the event $E^c$, $y_1 \leq 0$ and $x \leq y_2 < 1$. Let $L$ denote the $\pi$-extremal distance between $(-\infty, y_1)$ and $[y_2, 1]$ in $H_T$, i.e., the number $L$ such that $H_T$ can be mapped conformally onto $[0,L] \times [0,\pi]$ in a way that $(-\infty, y_1]$ and $[y_2, 1]$ are mapped onto the vertical boundaries. In order to relate $\text{SLE}_6$ to intersection exponents for Brownian motion one needs to
understand the behavior of \( E^x \{ 1 \exp \{ -\lambda L \} \} \) as \( x \to 1^- \) for \( \lambda \geq 0 \). It is not hard to show that this quantity is closely related to \( E^x \{ 1 \exp \{ \lambda T \} \} \). If we differentiate (2.1) with respect to \( z \) we get an equation for \( \partial_t g'(1) \), and standard techniques of stochastic calculus can be applied to give a differential equation for the function \( r(x, \lambda) = E^x \{ 1 \exp \{ \lambda T \} \} \). We get an exact solution in terms of hypergeometric functions [17, Theorem 3.2]. If \( \lambda = 0 \), so that \( r(x) = P^x \{ E \} \), we get the formula given by Cardy [4] for crossing probabilities of percolation clusters (see §3.2).

3. Applications

3.1. Brownian motion

As already mentioned, computation of dimensions for many exceptional sets for Brownian motion reduces to finding the Brownian intersection exponents. These exponents, which can be defined in terms of crossing probabilities for non-intersecting paths, were studied in [25, 26]. In these papers, relations were given between different exponents and a “universality” principle was shown for conformally invariant processes satisfying an additional hypothesis (the term completely conformally invariant was used there). Heuristic arguments indicated that self-avoiding walks and percolation should also satisfy this hypothesis. Unfortunately, from a rigorous standpoint, we had only reduced a hard problem, computing the Brownian intersection exponents, to the even harder problem of showing conformal invariance and computing the exponents for self-avoiding walks or critical percolation.

At the same time Schramm [29] was completing his beautiful construction of \( SLE_\kappa \) and conjecturing that \( SLE_6 \) gave the boundaries of critical percolation clusters. While he was unable to prove that critical percolation has a conformally invariant limit, he was able to conclude that if the limit was conformally invariant then it must be \( SLE_6 \). The identification \( \kappa = 6 \) was determined from rigorous “crossing probabilities” for \( SLE_\kappa \): only \( \kappa = 6 \) was consistent with Cardy’s formula (see §3.2) or even the simple fact that a square should have crossing probability 1/2.

Since both Brownian motion and \( SLE_6 \) were conjectured to be related to the scaling limit of critical percolation, it was natural to try to use \( SLE_6 \) to prove results about Brownian motion (and, as mentioned before, the Hausdorff dimension of exceptional sets on the path); see [17, 18, 20, 19]. There were two major parts of the proof. First, the locality property for \( SLE_6 \) was formulated and proved; this allowed ideas as in [26] to show that the exponents of \( SLE_6 \) can be used to find the exponents for Brownian motion. Second, the exponents for \( SLE_6 \) had to be computed. The basic idea is discussed at the end of the last section. What makes \( SLE \) so powerful is that it reduces problems about a two-dimensional process to analysis of a one-dimensional stochastic differential equation (and hence a partial differential equation in one space variable).

The universality in these papers was in terms of exponents. We now know that the paths of planar Brownian motion and \( SLE_6 \) are even more closely related.
The “hull” generated by an $SLE_6$ is the same as the hull generated by a Brownian motion with oblique reflection (see [33]). In particular, the frontiers (outer boundaries) of the two processes have the same dimension. There are now direct proofs that the Hausdorff dimension of the frontier of $SLE_6$ is $4/3$ ([2]) and this stronger universality principle implies the same holds for Brownian paths.

3.2. Critical percolation

Suppose each vertex of the planar triangular lattice is colored independently white or black, with the probability of a white being $1/2$. This is called critical percolation (on the triangular lattice). Let $D$ be a simply connected domain in $\mathbb{C} = \mathbb{R}^2$ and let $A_1, A_2$ be disjoint nontrivial connected arcs on $\partial D$. Consider the limit as $\delta \to 0$ of the probability that in critical percolation on a lattice with mesh size $\delta$ that there is a connected set of white vertices in $D$ connecting $A_1, A_2$. It has long been believed that this limit, $p(A_1, A_2; D)$, exists and is strictly between 0 and 1. (Note: if the probability of a white vertex is $p$, then $p(A_1, A_2; D)$ is 0 for $p < 1/2$ and 1 for $p > 1/2$. One of the features of critical percolation is the fact that this quantity is strictly between 0 and 1.) Moreover, it has been conjectured that $p(A_1, A_2; D)$ is a conformal invariant [4, 10]. It is also believed that this limit does not depend on the nature of the lattice; for example, critical bond percolation in $\mathbb{Z}^2$ (each bond is colored white or black independently with probability $1/2$) should give the same limit.

Cardy [4] used nonrigorous methods from conformal field theory to find an exact formula for $p(A_1, A_2; D)$; his calculations were done for $D = \mathbb{H}$ and the formula involves hypergeometric functions. Carleson noted that the formula was much nicer if one chooses $D$ to be an equilateral triangle of side length 1; $A_1$, one of the sides; and $A_2$, a line segment of length $x$ with one endpoint on the vertex opposite $A_1$. In this case, Cardy’s formula is $p(A_1, A_2; D) = x$. Schramm [29] went further and, assuming existence and conformal invariance of the limit, showed that the limiting boundary between black and white clusters can be given in terms of $SLE_6$. If $A_3$ denotes the third side of the triangle (so that $A_3 \cap A_2$ is a single point), we can consider the limiting cluster formed by taking all the white vertices that are connected by a path of white vertices to $A_3$. In the limit, the outer boundary of this “hull” has the same distribution as the outer boundary of the hull of chordal $SLE_6$ going from the vertex $A_3 \cap A_1$ to the vertex $A_3 \cap A_2$. The identification with $SLE$ comes from the conformal invariance assumption; Schramm determined the value $\kappa = 6$ from a particular crossing probability, but we now understand this in terms of the locality property which scaling limits of these boundary curves can be seen to satisfy. Cardy’s formula (and generalizations) were computed for $SLE_\kappa$ in [17].

Recently Smirnov [31] made a major breakthrough by proving conformal invariance and Cardy’s formula for the limit of critical percolation in the triangular lattice. As a corollary, the identification of the limit with $SLE_6$ has become a theorem. This has also led to rigorous proofs of a number of critical exponents for the lattice model [21, 30, 32]. The basic strategy is to compute the exponent for $SLE_6$ and to then to use Smirnov’s result to relate this exponent to lattice percolation.
It is an open problem to show that critical percolation on other planar lattices, e.g., bond percolation on the square lattice, has the same limiting behavior.

### 3.3. Loop-erased random walk

Loop-erased random walk (LERW) in a finite set $A \subset \mathbb{Z}^2$ starting at $0 \in A$ is the measure on self-avoiding paths obtained from starting a simple random walk at the origin, stopping at the first time that it leaves $A$, and erasing loops chronologically from the path. It can also be defined as a non-Markov chain which at each time $n$ chooses a new step using probabilities weighted by the probability that simple random walk starting at the new point avoids the path up to that point (see, e.g., [14]). It is also related to uniform spanning trees; if one choose a spanning tree uniformly among all spanning trees of $A$, considered as a graph with appropriate boundary conditions, then the distribution of the unique self-avoiding path from the origin to the boundary is the same as LERW. Wilson gave a beautiful algorithm to generate uniform spanning trees using LERW [34].

One can hope to define a scaling limit of planar LERW on a domain connecting an interior point to a boundary point by taking LERW on finer and finer grids and taking the limit. There are a number of reasons to believe that this limit is conformally invariant. For example, the limit of simple random walk (Brownian motion) is conformally invariant and the ordering of points used in the loop-erasing procedure is not changed under conformal maps. Also, certain crossing probabilities for LERW can be given by determinants of probabilities for simple random walk (see [7]), and hence these quantities are conformally invariant. Kenyon [9] used a conformal invariance argument (using a determinant relation from a related domino tiling model) to prove that the growth exponent for LERW is $5/4$; roughly, this says it takes about $r^{5/4}$ steps for a LERW to travel distance $r$.

Schramm [29] showed that under the assumption of conformal invariance, the scaling limit of LERW must be radial $SLE_2$. He used conformal invariance and a natural Markovian-type property of LERW to conclude that it must be an $SLE_{\kappa}$, and then he used Kenyon’s result to determine $\kappa$. Recently, Schramm, Werner, and I [22] proved that the scaling limit of loop-erased random walk is $SLE_2$.

There is another path obtained from the uniform spanning tree that has been called the uniform spanning tree Peano curve. This path, which lies on the dual lattice, encodes the entire tree (not just the path from the origin to the boundary). A similar, although somewhat more involved, argument can be used to show that this process converges to the space-filling curve $SLE_8$ [22].

### 3.4. Self-avoiding walk

A self-avoiding walk (SAW) in the lattice $\mathbb{Z}^2$ is a nearest neighbor walk with no self-intersections. The problem of the SAW is to understand the uniform measure on all such walks of a given length (or sometimes the measure that assigns weight $a^n$ to all walks of length $n$). It is still an open problem to prove there is a limiting distribution; it is believed that such a limit in conformally invariant (see [23] for
precise statements). However, if the conjectures hold there is only one possible limit, $SLE_{8/3}$.

The conformal invariance property leads one to conclude that the limit must be an $SLE_\kappa$ and $\kappa \leq 4$ is needed in order to have a measure on simple paths. The property that $SLE_{8/3}$ has that is not held by $SLE_\kappa$ for other $\kappa \leq 4$ is the restriction property. The restriction property is similar to, but not the same, as the locality property. Let $A$ be a compact subset and $\Phi$ the transformation as in §2 and let $\gamma$ be an $SLE_{8/3}$ path from 0 to $\infty$. Then the distribution of $\Phi \circ \gamma$ given the event $\{\gamma[0, \infty) \cap A = \emptyset\}$ is the same as (a time change of) $SLE_{8/3}$. In fact, the probability that $\{\gamma[0, \infty) \cap A = \emptyset\}$ is $\Phi'(0)^{5/8}$.

If the scaling limit of SAW has a conformally invariant limit then one can show easily that the limit satisfies the restriction property. Hence, the only candidate for the limit (assuming a conformally invariant scaling limit) is $SLE_{8/3}$. The conjectures for critical exponents for SAW can be interpreted in terms of rigorous properties of $SLE_{8/3}$ (see [24]). For example, the Hausdorff dimension of $SLE_{8/3}$ paths is 4/3 [24]; this gives strong evidence that the limit of SAWs should give paths of dimension 4/3. Monte Carlo simulations [8] support the conjecture that the limit of SAW is $SLE_{8/3}$.

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