Classical stability of supercurrent in one dimension: a numerical study

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We report results of a classical simulation of thermal phase slips, and the associated relaxation of supercurrent, in a ring-shaped one-dimensional superfluid. We find that the classical relaxation rate vanishes in the uniform limit. This leaves the quantum relaxation, with momentum transfer to phonons, the only mechanism of supercurrent decay in the uniform system. In the presence of a smooth periodic potential, classical decay becomes possible, and we identify a family of moving critical droplets that can mediate it.

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I. INTRODUCTION

Decay of supercurrent in one-dimensional (1D) superfluids and superconductors is a problem of longstanding interest. In 1967, Little [1] discussed the role of “large” thermal fluctuations in thin superconducting wires and observed that if, as a result of such a fluctuation, the order parameter $\Psi$ vanishes at some point, the phase of $\Psi$ can unwind, leading to a nonzero resistance. Such processes became known as thermally-activated phase slips (TAPS). Using thermodynamic arguments, Little had pointed out that at low temperatures the rate of these processes is exponentially small.

A more detailed theory of TAPS, based on the Ginzburg-Landau (GL) equations, was constructed in subsequent works [2] and became known as the LAMH theory. At first sight, it may appear that the LAMH saddle point solution to the classical equations of motion (in our case, the GL equation). Dynamics along the negative mode of this saddle point is supposed to mediate a transition between states that differ by one unit of the winding number. On a closer inspection, however, one notices that the LAMH saddle point actually mediates a transition between states with different values of superfluid velocity and so supercurrent state must be an integer.

Here we want to consider transitions between states with different values of superfluid velocity and so use periodic boundary conditions (the ring geometry). In this case, the winding number in the vicinity of a uniform supercurrent state must be an integer.

In a nearly uniform Bose fluid, a natural candidate for the critical fluctuation is one of the many time-dependent solutions (solitons [3]) of the perfectly uniform problem. According to the preceding discussion, we need a field for which the order parameter is completely extinguished at some point. For a large enough length $L$ of the ring, a solution having this property is

$$\Psi(x,t) = \Psi_1 e^{-i\mu t + iv_1 x} \tanh \frac{x - v_1 t}{2\xi_1}.$$  \hspace{1cm} (1)

Here $\Psi_1$ and $\mu$ are suitably chosen constants ($\mu$ is real, $\Psi_1$ is complex), $v_1 = (2\pi/mL)(W + \frac{1}{2})$, $W$ is the winding number of a nearby uniform state (an integer), $m$ is the mass of the particle, and $\xi_1$ is the “healing” length, $\xi_1^{-2} = 4gm|\Psi_1|^2$, $g$ is the coupling constant.

The field (1) solves the time-dependent Gross-Pitaevskii (GP) equation at zero external potential, and represents a depletion of density moving at velocity $v_1$. The classical energy of this state is computed in the Appendix. The momentum of this state is $mv_1 N$, where $N$ is the total number of particles. Relative to the uniform state with winding $W$ and superfluid velocity $v_0 = 2\pi W/mL$, the momentum is

$$\Delta P = mv_1 N - mv_0 N = \pi n,$$  \hspace{1cm} (2)

where $n = N/L$ is the linear density. In the limit $L \to \infty$ with $n$ and $W$ fixed, the field (1) becomes the classical counterpart of the extremal point of the Lieb-Liniger spectrum [4] of the corresponding quantum problem—a Bose gas with a delta-function repulsion.

Note that, while in the limit of small $v_1$ the solution (1) is in a certain sense close to the corresponding LAMH saddle point, for larger $v_1$ the two solutions are substantially different.

We want to identify (1) as a critical droplet mediating transitions between supercurrent states with winding...
numbers $W$ and $W+1$, i.e., superfluid velocities $v_0$ and $v_2 = 2\pi(W+1)/mL$. This identification shows immediately that in a nearly uniform superfluid TAPS may be strongly suppressed. Indeed, if the droplet is to decay into a state with winding $W$, the momentum $\Delta P$ needs to be transferred to another excitation branch, i.e., phonons, or to some external system. A look at the Lieb-Liniger spectrum tells us that any phonon state with momentum $\Delta P$ has energy larger than the energy of the droplet state.

The nearly uniform limit can be realized in a variety of physical systems. An interesting and perhaps somewhat unexpected realization is a thin superconducting wire. In this case, the analog of Bogoliubov’s phonon is the gapless plasmon mode $\xi$, describing fluctuations of the superconducting density. The large-scale (“hydrodynamic”) effective theory of these plasmons is a GP theory, with the coupling constant $g = 4e^2/C$, where $C$ is the capacitance of the wire per unit length (see, e.g., Ref. [8]). This description of superconductors is specific to one dimension, and the reason why it holds is that in 1D the screening of charges is weak, so the large-scale dynamics is dominated by the charging energy, rather than the energy of condensation (as in the GL theory). The order parameter $\Psi$ of this description is proportional (but not necessarily equal) to the GL order parameter. It can be interpreted as the field of Cooper pairs, and the entire description applies only at length scales larger than the “size” of a pair, i.e., the GL coherence length $\xi_{GL}$. Since in practice $\xi_{GL}$ is much larger than the “healing” length $\xi$ obtained from the GP description, the size of a critical droplet will now be determined by $\xi_{GL}$, rather than $\xi$. Nevertheless, as we discuss further in the concluding section, it is possible to adapt some of our results to this case. In particular, we expect that at scales of order $\xi_{GL}$ disorder, typically present in superconducting wires, self-averages, so that as an initial approximation we can use the uniform limit with suitably renormalized parameters. At the next level of accuracy, we will need to include momentum transfer to disorder via various mechanisms.

Another possible realization of the nearly uniform limit is atomic superfluids—specifically, trapped Bose gases—which can in principle be prepared is such a way that there are no significant sources of disorder on scales shorter than the healing length. In this case, the above considerations suggest that the requirement of momentum transfer can present a major bottleneck for TAPS.

We want to stress that, as shown in Refs. [8, 9], in quantum theory phase slips occur even in a perfectly uniform superfluid, by thermally-assisted tunneling with momentum transfer to phonons. The question we address in the present paper is whether phase slips are possible by a classical mechanism, such as thermal activation. This question is particularly relevant at temperatures $T \gtrsim gn$ ($g$ is the coupling constant), where the approximate methods used in Refs. [8, 9] do not apply.

Here, we attempt to simulate TAPS numerically, using classical equations of motion. Our simulations are microcanonical, i.e., there is no coupling to any external heat bath, and are similar in spirit to simulations of sphaleron transitions in the (1+1)-dimensional Abelian Higgs model [10]. In our case, however, because of the exact solvability of the theory at zero potential [11], we take special measures to ensure appropriate population of the phase space (see below).

Our main result is the complete absence, within the classical mechanics, of TAPS in the uniform Bose gas. This leaves quantum tunneling as the only mechanism of supercurrent decay in this case.

We have found that TAPS appear in the presence of an external potential $V(x)$, but at a rate that depends sensitively on the magnitude and scale of variation of $V$. When $V$ is smooth and not too large, the rate is much smaller than that derived from the LAMH theory. Fluctuations that mediate TAPS in such a smooth potential are similar to the field [11].

The paper is organized as follows. Details of the numerical procedure are described in Sects. [12] (initial conditions) and [13] (the evolution algorithm). Numerical results are presented in Sect. [14]. Our conclusions are summarized in Sect. [15], where we also discuss some applications of our results.

## II. INITIAL CONDITIONS

For the evolution equation, we have used the GP equation

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2m} \frac{\partial^2 \Psi}{\partial x^2} + g|\Psi|^2 \Psi + V(x)\Psi$$

with periodic boundary conditions

$$\Psi(x+L) = \Psi(x).$$

Here $\Psi(x,t)$ is the complex order parameter, $m$ is the mass of the particles, $g > 0$ is the coupling constant, and $V(x)$ is an external potential. For the uniform case, $V(x) = 0$. We have set $\hbar = 1$.

Important parameters that we will often use in what follows are the zero-temperature speed of Bogoliubov’s phonons

$$c_0 = (gn_{ave}/m)^{1/2}$$

and the zero-temperature “healing” length

$$\xi = (4gmn_{ave})^{-1/2};$$

$n_{ave}$ is the average density of the gas. In a classical simulation, $n_{ave}$ is obtained as the average of the density $n = \Psi\dagger\Psi$ over the entire lattice:

$$n_{ave} = \frac{1}{L} \int dx \Psi\dagger\Psi.$$

A precise characterization of temperature was not a goal of the present work. This allowed us to initialize the
field directly from the phonon power spectrum. Namely, suppose we separate the initial field $\Psi$ into the modulus (square root of the density) and the phase:

$$\Psi(x, t_i) = \sqrt{n(x)} e^{i\theta(x)} .$$

For the phase, we write

$$\theta(x) = \frac{2\pi W_i x}{L} + \theta_0 + \theta_1(x) ,$$

where $W_i$ is the initial winding number, $\theta_0$ is an uninteresting constant, which will be set to zero, and $\theta_1$ is a fluctuation. For the density, we write

$$n(x) = n_{\text{ave}} + \delta n(x) ,$$

where $n_{\text{ave}}$ is the uniform average density, which we specify, and $\delta n$ is a fluctuation.

Next, the fluctuations $\theta_1$ and $\delta n$ are expressed using Bogoliubov’s transformation

$$\theta_1(x) = \frac{1}{\sqrt{L}} \sum_{k \neq 0} \frac{\sqrt{Z_k}}{2\omega_k} \left[ b_k + b_k^\dagger \right] e^{ikx} ,$$

$$\delta n(x) = \frac{i}{\sqrt{L}} \sum_{k \neq 0} \frac{\omega_k}{2Z_k} \left[ b_k - b_k^\dagger \right] e^{ikx} ,$$

where

$$\omega_k = c_0 \left[ k^2 + k^4 \xi^2 \right]^{1/2} ,$$

$$Z_k = g(1 + k^2 \xi^2) .$$

In our classical simulation, we take $b_k$ and $b_k^\dagger$ to be random classical variables, whose moduli are given by the Bose distribution

$$|b_k|^2 = (\xi_0^2/T - 1)^{-1} ,$$

where

$$\Omega_k = \omega_k + v k .$$

is the quasiparticle dispersion law in the presence of a superfluid velocity $v$. The phase of $b_k$ is a (pseudo)random number uniformly distributed between 0 and 2$\pi$.

In numerical work, instead of (5) we use the linearized version

$$\Psi = \mathcal{N} \sqrt{n_{\text{ave}}} \left( 1 + \frac{\delta n}{2n_{\text{ave}}} \right) e^{i\theta} ,$$

where $\mathcal{N}$ is a normalization coefficient enforcing the condition (7).

The use of the linearized Eq. (17) is consistent with the assumption $\delta n \ll n_{\text{ave}}$ of Bogoliubov’s theory and with using the ideal-gas distribution (15) for phonons. But how far is the result from the true equilibrium of the interacting system and how much should we worry about the difference?

To begin with, notice that for a classical field the notion of a “true” equilibrium applies only to the low-frequency component, $\Omega_k \ll T$, which is more or less classical. For a TAPS, we expect $k \sim 1/\xi$, i.e., $\Omega_k \sim g n_{\text{ave}}$, so that at $T \gg g n_{\text{ave}}$ a TAPS is essentially a classical fluctuation. For high-frequency modes, the classical approximation is not parametrically justified, but under the above condition on the temperature these only provide a “heat bath”, and we do not expect the results to depend very sensitively on the model we choose for them.

By the same token, it seems extremely unlikely that modest deviations from thermal equilibrium can jeopardize the qualitative results of this work (such as vanishing of the TAPS rate in the uniform system).

One may, however, raise the objection that, in the uniform case or for a weak potential, our way of generating the initial conditions may not populate adequately the Lieb-Liniger branch, to which for instance the state (11) belongs. To counter this objection, we subject the field, after it has been initialized as described above, to a “premixing” stage, wherein it evolves for a while in a relatively strong potential. After that, the potential is switched to the desired strength, the winding number is reset to the original value, and the actual evolution begins. Premixing causes further deviations from thermal equilibrium but these are again deemed harmless for the reason stated above.

Two further comments are in order:

(i) Bogoliubov’s theory [12], when formulated in terms of density and phase, as in (11) and (12), requires smallness of $\delta n$ but does not require either smallness of $\theta_1$ or a non-vanishing expectation of the order parameter. It is therefore applicable even in 1D, cf. Ref. [13].

(ii) Within that theory, the properly subtracted variance of the density is

$$\langle \delta n^2 \rangle_T - \langle \delta n^2 \rangle_{T=0} = \frac{c_0}{2\pi g} \int_{-\infty}^{\infty} \frac{dk |k|}{\sqrt{1 + k^2 \xi^2}} e^{\Omega_k/T} - 1 .$$

Of main interest to us here is the temperature region $T \gtrsim g n_{\text{ave}}$. For an estimate of $\delta n$, we will assume the limit

$$T \gg \frac{c_0}{\xi} = 2g n_{\text{ave}} .$$

In this limit, the integral in (18) is saturated at small $k$: $k \sim \xi^{-1}(1 - v^2/c_0^2)^{1/2}$ (this infrared sensitivity is specific to one dimension). At these $k$, the Bose distribution can be replaced by the classical $T/\Omega_k$, and upon that replacement the integral can be easily evaluated:

$$\langle \delta n^2 \rangle_T - \langle \delta n^2 \rangle_{T=0} \approx \frac{n_{\text{ave}} T}{\sqrt{c_0^2 - v^2}} .$$

We conclude that (unless $v$ is very close to $c_0$) the smallness of a typical density fluctuation, $\delta n/n_{\text{ave}} \ll 1$, requires that

$$T \ll c_0 n_{\text{ave}} .$$
This condition is compatible with \[19\], provided the dimensionless coupling $g/c_0$ is chosen small.

### III. EVOLUTION ALGORITHM

In the code, it is convenient to work with rescaled, dimensionless variables, since this reduces the number of parameters that need to be input. A natural unit of length is the healing length $\xi$, and of time—the ratio

$$\frac{\xi}{c_0} = \frac{1}{2gn_{\text{ave}}}.$$  \hspace{1cm} (22)

So, we rescale variables in the GP equation \[4\] as follows:

\begin{align*}
 x & \rightarrow x/\xi, \\
 t & \rightarrow 2gn_{\text{ave}}t, \\
 V & \rightarrow V/2gn_{\text{ave}}.
\end{align*}  \hspace{1cm} (23-25)

In this section, and in the plots, we use the same letter for a rescaled quantity as in the rest of the paper for the original one. We also rescale the field $\Psi$ into

$$\psi = \Psi/\sqrt{n_{\text{ave}}}$$  \hspace{1cm} (26)

so that the average rescaled density is equal to 1: \[(1/L) \int \psi^\dagger \psi dx = 1.\]

In the rescaled variables, the GP equation becomes

$$i\frac{\partial \psi}{\partial t} = -\partial_x^2 \psi + \frac{1}{2} |\psi|^2 \psi + V(x) \psi,$$  \hspace{1cm} (27)

For the purpose of numerical evolution, we associate to the right-hand side of \[27\] an operator $U$ that updates the field $\psi_n(x)$ at the n-th time step:

$$\psi_{n+1}(x) = U[\psi_n(x), \Delta t];$$  \hspace{1cm} (28)

\[\Delta t = t_{n+1} - t_n.\] For this work, we have used an operator-splitting algorithm, in which the update is made in three steps:

$$U[\psi(x), \Delta t] = U_1[U_2[U_1[\psi(x), \frac{1}{2} \Delta t], \frac{1}{2} \Delta t], \frac{1}{2} \Delta t].$$  \hspace{1cm} (29)

where $U_1$ corresponds to the equation

$$i\frac{\partial \psi}{\partial t} = \frac{1}{2} |\psi|^2 \psi + V(x) \psi,$$  \hspace{1cm} (30)

and $U_2$ to the equation

$$i\frac{\partial \psi}{\partial t} = -\partial_x^2 \psi.$$  \hspace{1cm} (31)

To solve these equations, the field was discretized on a spatial lattice with a uniform step $\Delta x$ and periodic boundary conditions. Eq. \[30\] is local in space, so $U_1$ can be defined site-by-site:

$$U_1[\psi(x), \Delta t] = \prod_{j=0}^{M-1} u_j(\psi_j, \Delta t).$$  \hspace{1cm} (32)

We have used

$$u_j(\psi, \Delta t) = \frac{1 - \sigma_j}{1 + \sigma_j} \psi,$$  \hspace{1cm} (33)

with $\sigma_j = \frac{i}{2} [|\psi|^2 + V(x_j)] \Delta t$. The operator $U_2$ is non-local in $x$-space but local in $k$-space; we have used

$$U_2[\psi(x), \Delta t] = \prod_{k=0}^{M-1} v_k(\psi^F_k, \Delta t),$$  \hspace{1cm} (34)

where $\psi^F$ is the fast Fourier transform (FFT) of $\psi$, and

$$v_k(\psi^F, \Delta t) = \frac{1 - \rho_k}{1 + \rho_k} \psi^F$$  \hspace{1cm} (35)

with $\rho_k = \frac{1}{2} ik^2$. So, there are two FFTs (one direct and one inverse) at each time step.

The resulting algorithm has second-order accuracy in time. It conserves the number of particles exactly, while the energy non-conservation is controlled by the time-step $\Delta t$.

### IV. NUMERICAL RESULTS

We present results from simulations on an $M = 1024$ lattice with the following values of the dimensionless parameters: $L/\xi = 79$, $T/2gn_{\text{ave}} = 1.2$, and $g/c_0 = 0.1$ (a weak coupling). We have used the periodic potential

$$V(x) = V_0 \sin(2\pi x/l).$$  \hspace{1cm} (36)

For periodicity, $m$ must have \[l = L/q\], where $q$ is an integer. We have used $q = 15$, so that $l/\xi \approx 5.3$. “Premixing” was done with $V_0/2gn_{\text{ave}} = 0.3$ and lasted for $\Delta t = 200$ (in our dimensionless units). The premixing stage is not shown in the plots, i.e., $t = 0$ corresponds to the beginning of the actual evolution with a smaller potential. The winding number changes during premixing, but at $t = 0$ it is reset to the original value $W_i = 3$. In Fig. \[11\] we show the evolution of the winding number with and without a potential. In the uniform case ($V_0 = 0$), the winding number fluctuates but always quickly returns back to the initial value $W = 3$. There is no overall relaxation. This behavior persisted as we went to larger values of the temperature. Based on these and similar other results, we conclude that, in the uniform 1D superfluid, supercurrent is classically stable for any value of superfluid velocity satisfying Landau’s criterion. Thus, the only mechanism of supercurrent decay in this case is a quantum effect—the phonon-assisted tunneling considered in Refs. \[8,9\]. In the presence of a (weak) potential, while there are still many unsuccessful phase-slip attempts, the winding number eventually relaxes towards zero.

These results may be compared to those of numerical simulations of sphaleron transition in the (1+1)-dimensional Abelian Higgs model \[10\]. Sphalerons are
very similar to TAPS except that they connect states with zero current, so there is no issue of momentum transfer. In the simulations of Ref. [10], they readily occur in the absence of any external potential.

In Fig. 2 we show the power spectrum of $\psi$ in the case $V_0 = 0$ at the beginning and near the end of the simulation (only positive $k$ are shown; results for $k \leq 0$ are similar). We see that the spectrum exhibits remarkable stability. Although one expects that eventually fluctuations will propagate to the ultraviolet (a manifestation of the Rayleigh-Jeans problem of classical statistics), this clearly does not happen on the timescale of our simulation.

In Fig. 3 we plot the density profiles of the fluctuation mediating one of the phase slips in Fig. 1 for $V_0 \neq 0$, at three different moments of time. Comparing these profiles with Eq. (1), we find that the spatial size of the droplet is about 2.5 times smaller than predicted by that equation. This is not surprising given that, in the present simulation, fluctuations of density are relatively large: $\delta n$ is about half of $n_{\text{ave}}$. Since the typical wavelength of these fluctuations is of the same order as the size of the droplet [cf. comment (ii) in Sect. II], at $\delta n \sim n_{\text{ave}}$ we loose the notion of an effective long-range theory to which (1) could be a solution.

This interpretation is confirmed by going to smaller $T$ (and stronger potentials), for which $\delta n/n_{\text{ave}}$ is smaller, and the droplet size is expected to be closer to the value predicted by Eq. (1). For example, for $T/2gn_{\text{ave}} = 0.9$ and $V_0/2gn_{\text{ave}} = 0.1$, the difference reduces to a factor of about 1.6.

V. DISCUSSION

The main result of this paper is that in a uniform 1D Bose gas classical phase slips are completely blocked out. We find this result nontrivial and even surprising, given that momentum conservation, while characteristic of the uniform system, by itself does not prohibit phase slips: momentum released from the supercurrent can be absorbed by phonons. Indeed, quantum-mechanically, phase slips are possible even in the perfectly uniform system [8, 9]. What we have shown here, then, is that quantum effects remain the only source of supercurrent decay in the uniform case.

Our results, both for the uniform system and in the presence of a potential, are consistent with identifying the moving droplet (1) as the fluctuation mediating thermal phase slips. As we have already noted, at nonzero $v_1$ it is distinct from the LAMH saddle point.

The nearly uniform limit can presumably be realized in trapped Bose gases. Another system where, as already mentioned in the Introduction, the droplet (1) may play a role is a thin superconducting wire. In this case, the field $\Psi$ describes a fluid of Cooper pairs, and for the reasons...
already indicated—the weak screening in 1D and the resulting special role of the charging energy—the GP equation is well justified for description of long-wavelength plasma oscillations—the gapless plasmon mode \[ \xi_{\text{GL}} \]. However, this description breaks down at scales of order of the “size” of a pair, i.e., the Ginzburg-Landau coherence length \( \xi \). In the following discussion, we will need only the energy of droplet, which can be calculated using the Ginzburg-Landau free energy. The latter differs from the GP Hamiltonian \[ H_{\text{GP}} \] only by the value of the quartic coupling \( g \) and by the presence of a \( |\Psi|^2 \) term, which in the GP case would correspond to a chemical potential. Although the activation energies \[ E_{\text{A7}}, E_{\text{A8}} \] are computed in the Appendix at a fixed number of particles, the same expressions also apply at a fixed chemical potential. For as the coupling \( g \), it will be sufficient to consider it as a phenomenological parameter. This allows us to apply the expressions obtained in the Appendix to the case of a superconducting wire.

Since a moving droplet can now transfer momentum to normal electrons and, through them, to the disorder potential, we expect that for sufficiently strong disorder the TAPS rate is determined simply by the droplet’s activation energy. The activation energies for transitions decreasing and increasing the winding number by one unit are given by Eqs. \[ E_{\text{A7}}, E_{\text{A8}} \]. Unfortunately, the form of the \( I-V \) curve obtained from these expressions differs from that obtained in the LAMH theory only for currents comparable to the GL critical current. This makes it difficult to draw an experimental distinction between the activation energies \[ E_{\text{A7}}, E_{\text{A8}} \] and their LAMH counterparts. If, at smaller currents, we regard the last term in Eqs. \[ E_{\text{A7}}, E_{\text{A8}} \] as current-independent, the voltage drop found from these equations is proportional to \( \sinh(I_s/I_0) \) where \( I_s \) is the supercurrent, and \( I_0 = 2eT/\pi \). This form of the nonlinear \( I-V \) curve, the same as in the LAMH theory in the equivalent limit \[ \xi_{\text{GL}} \ll L \], has been recently found to be in good agreement with experiment \[ \xi_{\text{GL}} \].

Finally, we mention one technical result of our work, namely, the extent to which it turned out possible to simulate thermal field theory classically. We refer here to the remarkable stability of the classical power spectrum against spreading towards the ultraviolet, cf. Fig. 2. This is similar to behavior observed in three dimensions, in simulations of highly nonthermal states produced by parametric amplification \[ \xi_{\text{GL}} \]. One may wonder if this property holds more generally, so that the classical model of a thermal state we have used here can be applied to other (weakly-coupled) systems where the high-frequency modes provide an internal “heat bath” for the low-frequency, classical component.

After this work was completed, we have learned of a paper by Polkovnikov et al. \[ \xi_{\text{GL}} \] who studied numerically classical decay of supercurrent in an optical lattice. Their calculation corresponds to a strong periodic potential—the limit opposite to ours.

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**APPENDIX A: ENERGY OF A MOVING DROPLET**

Here we compute the energy of the field \[ \Psi \] relative to the energy of the neighboring uniform states: one with winding number \( W \) and superfluid velocity \( v_0 = 2\pi W/mL \), and the other with winding \( W + 1 \) and superfluid velocity \( v_2 = 2\pi(W + 1)/mL \).

We start with the Hamiltonian corresponding to Eq. \[ H_{\text{GP}} \] with \( V = 0 \),

\[
H_0 = \int dx \left( \frac{1}{2m} |\partial_x \Psi|^2 + \frac{g}{2} |\Psi|^4 \right),
\]
and compute the energy at a fixed total number of particles \( N \). So, there is no need to include a chemical potential.

Substituting \[ \Psi \] into \[ H_{\text{GP}} \] and integrating over \( x \), we obtain (to exponential accuracy in \( \xi_{\text{GL}} \))

\[
E_1 = -m \frac{v_1^2 N + \frac{g}{2} |\Psi_0|^4 (L - \frac{8}{3} \xi_1)}{L},
\]
where \( \xi_1 \) is the healing length defined after Eq. \[ \xi_{\text{GL}} \]. The particle number is

\[
N = \int dx \Psi^\dagger \Psi = (L - 4\xi_1)|\Psi_1|^2.
\]

The energies of the neighboring uniform states are

\[
E_{0,2} = \frac{m}{2} v_{0,2}^2 N + \frac{g}{2} |\Psi_0|^4 L,
\]
where \( |\Psi_0|^2 = N/L \). To the leading order in \( 1/L, \xi_1 \approx \xi \),

\[
v_1^2 - v_2^2 \approx -\frac{2\pi v_1 L}{mL},
\]

\[
|\Psi_1|^2 - |\Psi_0|^2 \approx \frac{4\xi}{L} |\Psi_0|^2
\]

(we allow for the possibility that \( W \propto L \)). Then,

\[
E_1 - E_0 \approx -\pi v_1 |\Psi_0|^2 + \frac{8}{3} g \xi |\Psi_0|^4,
\]

\[
E_1 - E_2 \approx -\pi v_1 |\Psi_0|^2 + \frac{8}{3} g \xi |\Psi_0|^4.
\]

These expressions coincide with the LAMH activation energies \[ \xi_{\text{GL}} \] expanded to the first order in supercurrent.
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