Finite-time condensation in 1D Fokker–Planck model for bosons

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Abstract

We consider a one-dimensional analogue of the three-dimensional Fokker–Planck equation for bosons. The latter is still only partially understood, and, in particular, the physically relevant question of whether this equation has solutions which form a Bose–Einstein condensate has remained unanswered. After a change of variables, we establish global-in-time existence and uniqueness for our 1D model (and generalisations thereof) using the concept of viscosity solutions. We show that such solutions enjoy good regularity properties, which guarantee that in the original variables blow-up can only occur at the origin and with a fixed spatial profile, up to leading order, following a power law linked to the steady states of the equation. This enables us to extend entropy methods beyond the first blow-up time. As a consequence, in the mass-supercritical case, solutions will blow up in $L^\infty$ in finite time and—understood in an extended, measure-valued sense—they will eventually have a condensed part, i.e. a Dirac measure at the origin. In this case, the density of the absolutely continuous part of the solution is unbounded near the origin.

1 Introduction

The Kaniadakis–Quarati model or Bose–Einstein–Fokker–Planck equation (BEFP) has been proposed as a model for the dynamics of the velocity distribution of a spatially homogeneous Bose gas in [19]. It reads as

$$\partial_t f = \Delta f + \nabla \cdot (vf(f+1)), \quad t > 0, \quad v \in \mathbb{R}^d,$$

$$f(0, \cdot) = f_0 \geq 0.$$  \hspace{1cm} (1.1)$$

In physical terms $v$ represents the velocity variable while $f$ denotes the number density of particles, whose integral over $\mathbb{R}^d$—its mass—is formally preserved under the evolution. An important feature of equation (1.1) are its steady states, which coincide with the Bose–Einstein distributions and which in the $L^1$ supercritical case $d > 2$—which comprises the physically most relevant case $d = 3$—give rise to a finite critical mass $m_c$ (i.e. the least upper bound for the $L^1$-norm of all regular steady states of the equation). In the $L^1$ supercritical regime the problem of understanding the long-time dynamics has remained
largely open. Toscani [25] demonstrated that, for highly concentrated initial data or data with very large mass (above a threshold $\mathcal{m} \gg m_c$), solutions must blow up after finite time (in the sense that they cannot be extended to a global in time classical solution). The proof is indirect—based on a virial type argument—and does not provide any insights in the nature of blow-up. The $L^1$ critical case $d = 2$ has recently been investigated in the ref. [8]. Exploiting the fact that in this case the nonlinear equation (1.1) in isotropic form is closely related to a linear Fokker–Planck equation (by means of the Hopf–Cole transformation), the authors are able to prove global existence of classical solutions and relaxation to equilibrium for a large class of initial data. Besides the transformation, the main tools are comparison principles and entropy techniques. In the $L^1$ subcritical case $d = 1$ a formal study of the relaxation to equilibrium was performed in the ref. [5]. Actually, the global existence of regular solutions, which in this work was only guaranteed for initial data lying below one of the steady states in the pointwise sense, can be obtained for any sufficiently regular initial datum by means of a comparison argument for the distribution function. In summary, while in the $L^1$ subcritical and critical case solutions are globally regular and converge to the steady state of the same mass, in the supercritical regime there do exist solutions which become unbounded after finite time, but beyond this little is known in that case.

In this work we aim to shed some light on the problem of the formation of condensates in eq. (1.1) by studying in one dimension the $L^1$ supercritical case of the following generalisation of equation (1.1):

$$\frac{\partial f}{\partial t} = \Delta f + \nabla \cdot (vf(f^{\gamma} + 1)), \quad t > 0, \, v \in \mathbb{R}^d,$$

(1.2)

where $\gamma > 0$ is a fixed parameter. This equation appears as a special case in [4], where an associated (stationary) minimisation problem is studied. Here, the $L^1$ subcritical, critical resp. supercritical regimes are given by $\gamma < \frac{2}{d}, \gamma = \frac{2}{d}$ resp. $\gamma > \frac{2}{d}$. For $\gamma \leq \frac{2}{d}$ no (finite) critical mass exists so that condensation cannot be expected. In this case methods similar to those employed for the analysis of eq. (1.1) apply, and our focus will thus be on the $L^1$ supercritical case. Still, the theory we develop remains valid in the critical case, where global regularity of solutions to the Cauchy problem will be a simple corollary of our results.

We should mention that there exist several other models in the literature for Bose–Einstein condensation and the dynamics of a weakly interacting quantum gas of Bose particles. Many of these are based on spatially homogeneous Boltzmann type equations such as the quantum Boltzmann equation and the Boltzmann–Nordheim equation, which have been the subject of various studies (mainly within the framework of isotropic solutions), see, e.g., [12, 13, 14, 2, 23, 24] and references therein. Formally more similar to our problem is a model due to Kompaneets [20] for the relaxation to thermal equilibrium of the momentum distribution of photons in a homogeneous plasma under the assumption that interaction with matter occurs via Compton scattering. As a special case of this model one obtains a nonlinear Fokker–Planck type equation, which was thoroughly studied in the ref. [11]. Further models describing quantum effects in a gas of (weakly interacting) bosons at very low temperatures involve nonlinear equations of Schrödinger type and, in particular, the Gross–Pitaevskii equation, see, e.g., [10, 3] and references therein. Finally, the relaxation to the unique minimising measure of a natural entropy functional with possibly singular parts with respect to Lebesgue was derived in [16] for a one-dimensional Fokker–Planck type equation with sublinear diffusion and linear drift. They crucially used that the equation is the gradient flow of this entropy functional with
respect to the $L^2$-Wasserstein distance. We obtain a similar result in our problem, see Section 2.6 and Theorem 4.4 resp. Theorem 5.4. However, since the drift is linear, for bounded initial data finite-time condensation cannot occur in their case.

1.1 The equation in mass variables

From now on we will assume that $d = 1$ and $\gamma \geq 2$. In the following $R > 0$ denotes a fixed but arbitrarily large parameter. The core theory will first be established for the problem posed on a bounded domain, viz.\[
\begin{align*}
\partial_t f &= \partial_r^2 f + \partial_r(r f(\gamma + 1)), & t > 0, \ r \in (-R, R), \\
f(0, r) &= f_0(r), & r \in (-R, R), \\
0 &= \partial_r f + r f(\gamma + 1), & t > 0, \ r \in \{-R, R\}. \tag{1.3}
\end{align*}
\]
Notice that the boundary condition (1.4) formally ensures the conservation of mass. Also note that $r$ can be negative, and we stress that we do not assume radial symmetry. The results obtained for this problem will enable the passage to the limit $R \to \infty$ for initial data $f_0$ satisfying a suitable decay condition at infinity. Technically speaking, we do not directly investigate the formulation (1.3), but instead study the problem obtained upon a change of variables, which is motivated by the quest for a concept of solution capable of making sense of Dirac deltas at the origin and which was used before in the ref. [6] treating the non-diffusive case.

For a non-negative finite Borel measure $\nu$ on $[-R, R]$ we define the cumulative distribution function (cdf) $M$ associated with $\nu$ via
\[
M(r) = \nu([-R, r]), \quad r \in [-R, R].
\]
The cumulative distribution function of a function $f \in L^1(-R, R)$ is defined as the cdf associated with the measure $L^1 f$, where $L^1$ denotes the one-dimensional Lebesgue measure restricted to the interval $[-R, R]$.

**Definition 1.1.** Suppose $m > 0$. Given a strictly increasing, right continuous function $M : [-R, R] \to [0, m]$ with $M(R) = m$, we define its pseudo-inverse $u : [0, m] \to [-R, R]$ via
\[
u = \text{Def}(0, m) = \text{Def}(-R, R), \quad u(x) = \min\{r \in [-R, R] : M(r) \geq x\}, \quad x \in [0, m].
\]
The function $u$ is well-defined and continuous, satisfies $u(0) = -R$, $u(m) = R$ and $u(x) = r$ whenever $x \in [M(r-), M(r)]$, $r \in [-R, R]$. For simplicity, we will often omit the term “pseudo-”.

Assume for the moment that $f = f(t, r)$, $t > 0$, is a strictly positive classical solution of problem (1.3)–(1.4) of mass $m$. Then for fixed $t$ its cumulative distribution function $M(t, \cdot)$ satisfies the assumptions in Definition 1.1 and we can consider the pseudo-inverse $u(t, \cdot)$ of $M(t, \cdot)$, which satisfies $M(t, u(t, x)) = x$ for $x \in [0, m]$. Then, by a straightforward calculation—using in particular the relation
\[
\partial_x u = \frac{1}{f(u)},
\]
where we omitted the time argument—one finds that $u$ satisfies the equation
\[
\partial_t u - (\partial_x u)^{-2} \partial_x^2 u + u((\partial_x u)^{-\gamma} + 1) = 0.
\]
Following an idea in [6, Section 4], we multiply the last equation by \((\partial_x u)^\gamma\) to obtain
\[
(\partial_x u)^\gamma \partial_t u - (\partial_x u)^\gamma - 2 \partial_x^2 u + u(1 + (\partial_x u)^\gamma) = 0 \quad \text{in } \Omega,
\] (1.5)
where \(\Omega := (0, T) \times (0, m)\). Observing that the zero-flux boundary condition for the density \(f(t, r)\) formally converts into the constant-in-time Dirichlet conditions \(u(t, 0) = -R\) and \(u(t, m) = R\), equation (1.5) is to be complemented with the following conditions on the parabolic boundary:
\[
\begin{align*}
\quad & u(0, x) = u_0(x), \quad x \in (0, m), \quad (1.6) \\
\quad & u(t, 0) = -R, \quad u(t, m) = R, \quad t > 0. \quad (1.7)
\end{align*}
\]

1.2 Outline

The paper is structured as follows. In Section 2 we establish global-in-time existence, uniqueness and Lipschitz regularity of (monotonic) viscosity solutions for a class of equations generalising problem (1.5)–(1.7). In Section 3 we derive improved regularity and establish a result implying that, at the level of \(f\), blow-up can only happen at the origin, see Section 3.2. Here, for simplicity, we mostly focus on the 1D Fokker–Planck model for bosons. In Section 4 we explore the connections with the original equation. Back-transforming the generalised solution of equation (1.5), we obtain a finite measure consisting of an absolutely continuous part whose density is smooth away from the origin and satisfies the equation in the classical sense (away from \(r = 0\)), plus a Dirac measure centred at the origin. In Section 4.1 it is shown that to leading order the spatial behaviour near the singularity at the origin is determined by the so-called singular steady state associated with our equation, i.e. by the power \(c_\gamma r^{-\frac{2}{\gamma}}\), where \(c_\gamma = (2/\gamma)^{\frac{1}{2}}\). Exploiting this property, we then show that entropy methods can be extended globally in time (Sections 4.2 and 4.3). In the final part, Section 5, we transfer the techniques and results used in the analysis of problem (1.3) to the equation posed on the whole line. In particular, we establish global existence, uniqueness, relaxation to equilibrium (with possibly non-trivial singular part), and—in the mass-supercritical case—condensation in finite time.

1.3 Notations

- We let \(\Omega := I \times J := (0, T) \times (0, m)\), where \(0 < T \leq \infty\) and \(0 < m < \infty\).
- Unless stated otherwise, functions are to be understood as maps from \(\Omega\) to \(\mathbb{R}\).
- For an interval \(I \subset \mathbb{R}\), any measure on \(I\) is understood to be a non-negative Borel measure, and we denote by \(\mathcal{M}_b^+(I)\) the set of finite measures on \(I\).
- Test functions are \(C^1\) in time and \(C^2\) in space (meaning that the first time derivative and the second spatial derivative exist and are in \(C(\Omega)\)).
- In general, the expression \(\partial_x v\) denotes the weak derivative (in the distributional sense) of the function \(v\). The pointwise derivative of a function \(v\) will be denoted by \((p)\partial_x v\).
- For a function \(v : (a, b) \subset \mathbb{R} \to \mathbb{R}\) we denote by \(v'\) its (weak) derivative.
- For \(U \subset \mathbb{R}^d\) we denote by USC(\(U\)) (resp. LSC(\(U\))) the space of upper semicontinuous (resp. lower semicontinuous) functions on \(U\).
- For \(d \in \mathbb{N}\) the expression \(\text{Sym}(d)\) denotes the space of symmetric \(d \times d\) matrices with real components.
- For \(\alpha \in (0, 1]\) and \(U \subset \mathbb{R}^d\) we abbreviate \([u]_{C^{0, \alpha}(U)} := \sup_{x, y \in U, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\).
2 Monotonic viscosity solutions for 2nd order equations

In this section we introduce a weak notion of solution for a class of equations generalising eq. (1.5) and establish an associated wellposedness theory. The equations we consider take on the form

\[ G(u, \partial_t u, \partial_x u, \partial^2_x u) = 0 \quad \text{in } \Omega, \]  

with \( \Omega := (0, T) \times (0, m) \), where \( G : \mathbb{R}^4 \to \mathbb{R} \) is a continuous function satisfying:

(A0) The function \( q \mapsto G(z, \alpha, p, q) \) is non-increasing for all \( z, \alpha, p \in \mathbb{R} \).

Additional structural assumptions on \( G \) will be formulated when needed the first time. We will use the “curly font” to denote the corresponding operator, i.e. we let

\[ G(u) := G(u, \partial_t u, \partial_x u, \partial^2_x u) \]  

and similarly \( F(u) := F(u, \partial_t u, \partial_x u, \partial^2_x u) \), where the function \( F \) is to be specified.

In comparison to the existing literature \[18, 9, 17\], our approach has the following two main novelties: the first one consists in the fact that it can deal with parabolic equations which are not strictly monotonic in the time derivative, as long as \( G \) saturates a certain strict monotonicity condition in its first argument, the second one lies in the preservation of monotonicity in \( x \), provided the problem admits monotonic barriers.

2.1 Preliminary definitions and the notion of solution

Our concept of solution for equation (2.1) is the standard notion of a viscosity solution. In order to formulate it, we first need to introduce some additional notations.

We say that a test function \( \phi \) touches the function \( u \) from above (resp. from below) at the point \( \omega \in \Omega \) if \( \phi(\omega) = u(\omega) \) and if there exists a neighbourhood \( N \subseteq \Omega \) of \( \omega \) such that \( \phi \geq u \) (resp. \( \phi \leq u \)) in \( N \).

**Definition 2.1** (Parabolic super-/subdifferential). For a function \( u \) defined on \( \Omega \) and a point \( \omega \in \Omega \) we let

\[ \mathcal{P}^+ u(\omega) = \{ (\alpha, p, q) \in \mathbb{R}^3 : (\alpha, p, q) = (\partial_t \phi, \partial_x \phi, \partial^2_x \phi)|_{\omega} \text{ for some test function } \phi \text{ which touches } u \text{ from above at } \omega \}. \]

Analogously, we define

\[ \mathcal{P}^- u(\omega) = \{ (\alpha, p, q) \in \mathbb{R}^3 : (\alpha, p, q) = (\partial_t \phi, \partial_x \phi, \partial^2_x \phi)|_{\omega} \text{ for some test function } \phi \text{ which touches } u \text{ from below at } \omega \}. \]

We further let \( \mathcal{P} u(\omega) = \mathcal{P}^+ u(\omega) \cap \mathcal{P}^- u(\omega) \).

**Remark.** The set \( \mathcal{P} u(\omega) \) is non-empty if and only if the pointwise derivatives \( (p)\partial_t u(\omega), (p)\partial_x u(\omega), (p)\partial^2_x u(\omega) \) exist. In this case, \( \mathcal{P} u(\omega) = \{ (p)\partial_t u(\omega), (p)\partial_x u(\omega), (p)\partial^2_x u(\omega) \} \) is a singleton, which we will then identify with its unique element, i.e.

\[ \mathcal{P} u(\omega) = (p)\partial_t u(\omega), (p)\partial_x u(\omega), (p)\partial^2_x u(\omega)). \]

**Definition 2.2.** We let

\[ \mathcal{P}^\pm u(\omega) = \{ (\alpha, p, q) \in \mathbb{R}^3 : \exists \omega_n \in \Omega \text{ and } \exists (\alpha_n, p_n, q_n) \in \mathcal{P}^\pm u(\omega_n) \text{ such that } (\omega_n, u(\omega_n), \alpha_n, p_n, q_n) \to (\omega, u(\omega), \alpha, p, q) \}. \]
We will also need the elliptic analogues of $\mathcal{P}$ and its versions.

**Definition 2.3** (Second order sub-/superdifferential). Let $d \in \mathbb{N}^+$ and $U \subset \mathbb{R}^d$ be open. For a function $v : U \to \mathbb{R}$ and $x \in U$ we define

$$\mathcal{J}^{2,+} v(x) = \left\{(p,q) \in \mathbb{R}^d \times \text{Sym}(d) : \exists \phi \in C^2(U) \text{ with } v(y) - \phi(y) \leq v(x) - \phi(x) \text{ for all } y \in U \right\}.$$  

The sets $\mathcal{J}^{2,-} u(x), \mathcal{J}^{2} u(x), \mathcal{J}^{2,+} u(x)$ are then defined analogously as in the parabolic case and, if $\mathcal{J}^{2} u(x)$ is non-empty, this set will be identified with its unique element $((p) Du(x), (p)D^2 u(x)).$

We remark that $(\alpha, p, q) \in \mathcal{P}^+ u(t,x)$ resp. $(\alpha, p, q) \in \mathcal{P}^- u(t,x)$ if and only if there exists a neighbourhood $N$ of $(t,x)$ such that as $N \ni (s, y) \to (t,x):$

$$u(s,y) \leq u(t,x) + \alpha(s-t) + p(y-x) + \frac{q}{2}|y-x|^2 + o(|s-t| + |y-x|^2) \quad (2.3)$$

resp.

$$u(s,y) \geq u(t,x) + \alpha(s-t) + p(y-x) + \frac{q}{2}|y-x|^2 + o(|s-t| + |y-x|^2). \quad (2.4)$$

If $u(t,\cdot)$ is non-decreasing, letting $s = t$ in ineq. (2.3) resp. ineq. (2.4) and $y \to x^+$ resp. $y \to x^-$, it follows that $p \geq 0$. In particular, for functions $u$ which are non-decreasing in $x$, we have

$$\mathcal{P}^\pm u(\omega) \subseteq \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$$

for all $\omega \in \Omega$.

**Definition 2.4** (Semicontinuous envelopes). Given $u = u(\omega)$ we define the functions

$$u^*(\omega) = \limsup_{r \to 0} \left\{ u(\xi) : \xi \in \Omega, |\xi - \omega| \leq r \right\},$$

$$u_*(\omega) = \liminf_{r \to 0} \left\{ u(\xi) : \xi \in \Omega, |\xi - \omega| \leq r \right\}.$$  

The function $u$ is upper semicontinuous (usc) if $u = u^*$, and lower semicontinuous (lsc) if $u = u_*$. We call $u^*$ (resp. $u_*$) the usc (resp. lsc) envelope of $u$.

Notice that for any $\omega \in \Omega$ there exists a sequence $\xi_k \xrightarrow{k \to \infty} \omega$ such that $u(\xi_k) \xrightarrow{k \to \infty} u^*(\omega)$. Also note that the function $u$ is usc if and only if $u(\omega) \geq \limsup_{k \to \infty} u(\xi_k)$ for any sequence $\xi_k \xrightarrow{k \to \infty} \omega$. Furthermore, $v$ is lsc if and only if $-v$ is usc.

Now we are in a position to state the notion of solution we propose for eq. (2.1).

**Definition 2.5** (Viscosity (sub-/super-) solution). Suppose that the continuous function $G$ satisfies property \[\text{(A0)}\] and let $u$ be a function defined on $\Omega$. We call $u$ a

1. (viscosity) subsolution of equation (2.1) in $\Omega$ if it is upper semicontinuous and if for any $\omega \in \Omega$ and any $(\alpha, p, q) \in \mathcal{P}^+ u(\omega)$ we have

$$G(u(\omega), \alpha, p, q) \leq 0.$$
• \textit{(viscosity) supersolution} of equation \((2.1)\) in \(\Omega\) if it is lower semicontinuous and if for any \(\omega \in \Omega\) and any \((\alpha, p, q) \in \mathcal{P}^- u(\omega)\) we have
\[G(u(\omega), \alpha, p, q) \geq 0.\]

• \textit{viscosity solution} of equation \((2.1)\) in \(\Omega\) if it is both a subsolution and a supersolution of equation \((2.1)\) in \(\Omega\). (In this case \(u\) is necessarily continuous.)

In places we use the short phrase “\(u\) is a viscosity (sub-/super-\) solution of \(G = 0\)” if it is a viscosity (sub-/super-) solution of eq. \((2.1)\). Since we will only deal with sub- and supersolutions in the viscosity sense, we usually drop the word “viscosity” in these cases.

Notice that, by the continuity of \(G\), in Definition 2.5 one can replace \(\mathcal{P}^\pm u(\omega)\) with \(\mathcal{P}^\pm u(\omega)\).

\textbf{Remark.} Of course, the mere formulation of Definition 2.5 does not require the assumption \((A0)\). However, it is this property which ensures that the definition is meaningful in the sense that it generalises the notion of a classical solution.

2.2 Stability

One advantage of the notion of viscosity solutions lies in its good stability properties. In order to demonstrate this, we reformulate [9, Proposition 4.3] (for elliptic problems) in terms of our parabolic problem.

\textbf{Proposition 2.6.} Let \(v \in \text{USC}(\Omega)\), let \(\omega \in \Omega\) and assume that \((\alpha, p, q) \in \mathcal{P}^+ v(\omega)\). Suppose that \(u_n \in \text{USC}(\Omega)\) is a sequence of functions satisfying
\[
\begin{cases}
(i) \text{ there exist } \omega_n \in \Omega \text{ such that } (\omega_n, u_n(\omega_n)) \to (\omega, v(\omega)) \\
(ii) \text{ if } \xi_n \in \Omega \text{ and } \xi_n \to \xi, \text{ then } \limsup_{n \to \infty} u_n(\xi_n) \leq v(\xi).
\end{cases}
\]

Then there exist \(\hat{\omega}_n \in \Omega\), \((\alpha_n, p_n, q_n) \in \mathcal{P}^+ u_n(\hat{\omega}_n)\) such that
\[ (\hat{\omega}_n, u_n(\hat{\omega}_n), \alpha_n, p_n, q_n) \to (\omega, v(\omega), \alpha, p, q). \]

\textbf{Proof.} The proof is similar to the one of [9, Proposition 4.3]. Notice that this result does not involve the equation. \qed

\textbf{Remark 2.7 (Stability).} Observe that we have the following corollaries of Proposition 2.6.

(a) The notion of viscosity solutions is stable under locally uniform convergence: let \(G_n = G_n(z, \alpha, p, q)\), \(n \in \mathbb{N}\), be continuous and such that \(G_n \to G\) as \(n \to \infty\) locally uniformly. Furthermore assume that, for each \(n\), \(u_n\) is a viscosity solution of \(G_n = 0\) in \(\Omega\) and that the sequence \((u_n)\) converges locally uniformly in \(\Omega\) to some function \(u\). Then \(u\) is a viscosity solution of \(G = 0\) in \(\Omega\).

(b) If \(V\) is a family of subsolutions of equation \((2.1)\) and \(u := \sup_{v \in V} v\) is such that the usc envelope \(u^*\) of \(u\) satisfies \(u^*(\omega) < \infty\) for all \(\omega \in \Omega\), then \(u^*\) is a subsolution of equation \((2.1)\).
2.3 Comparison

Given that our notion of solution is a rather weak one, our first concern is the question of uniqueness subject to prescribed data.

**Proposition 2.8** (Comparison). Suppose that, in addition to (A0), the continuous function \( G \) has the following property:

\[(A1) \text{ For all } p, q \text{ the function } (z, \alpha) \mapsto G(z, \alpha, p, q) \text{ is weakly strictly increasing in the sense that for all } (z, \alpha), (z', \alpha') \in \mathbb{R}^2 \]

\[
\begin{cases}
[z \leq z' \text{ and } \alpha \leq \alpha'] & \Rightarrow G(z, \alpha, p, q) \leq G(z', \alpha', p, q), \\
[z < z' \text{ and } \alpha < \alpha'] & \Rightarrow G(z, \alpha, p, q) < G(z', \alpha', p, q).
\end{cases}
\]

Let \( 0 < T < \infty \) and assume that \( u \in \text{USC}(\Omega \cup \partial_p \Omega) \) is a subsolution bounded from above and \( v \in \text{LSC}(\Omega \cup \partial_p \Omega) \) a supersolution bounded from below of eq. (2.1) in \( \Omega \) satisfying \( u \leq v \) on \( \partial_p \Omega \). Then \( u \leq v \) in \( \Omega \).

**Remark.** Since in Proposition 2.8 the time \( T \) can be chosen arbitrarily large, the assertion remains valid if \( T = \infty \).

**Proof of Proposition 2.8.** Due to the lack of regularity we cannot argue as in the classical case since, for instance, the super- resp. subdifferentials of \( u \) resp. \( v \) are in general not non-empty. Also notice that while the superdifferential of \( u - v \) at a maximum point is not empty, we do not know whether \( u - v \) is the subsolution of a suitable parabolic equation.

Arguing by contradiction, we assume that

\[ \sup_{\Omega} (u - v) > 0. \]

This implies that for \( \eta > 0 \) sufficiently small

\[ K := \sup_{(t,x) \in \Omega} \left( u(t,x) - v(t,x) - \frac{\eta}{T-t} \right) > 0. \]

Notice that the function

\[ \tilde{u}(t,x) := u(t,x) - \frac{\eta}{T-t} \]

is a subsolution of eq. (2.1) which is bounded from above and satisfies \( \lim_{t \nearrow T} u(t, \cdot) = -\infty \) where the convergence is uniform in \( x \in J \).

To compensate for the lack of regularity, we use a well-known technique consisting in first doubling the independent variables and then penalising the deviation of corresponding variables. Concretely, for \( \varepsilon > 0 \) we consider the function

\[ h_\varepsilon(t,x,s,y) := \tilde{u}(t,x) - v(s,y) - \frac{|t-s|^2}{2\varepsilon} - \frac{|x-y|^2}{2\varepsilon}. \]

Now let

\[ K_\varepsilon := \sup_{(t,x),(s,y) \in \Omega} h_\varepsilon(t,x,s,y) \]

and notice that \( K_\varepsilon \geq K > 0 \). The fact that \( h_\varepsilon \) is usc and bounded from above combined with the behaviour of \( \tilde{u}(t, \cdot) \) as \( t \to T \) implies that for sufficiently small \( \varepsilon > 0 \) the supremum
is attained at some point \( \omega \varepsilon := (\omega_{1, \varepsilon}, \omega_{2, \varepsilon}) : (x_{\varepsilon}, y_{\varepsilon}) \in (\Omega \cup \partial_p \Omega) \times (\Omega \cup \partial_p \Omega) \).

Moreover, \( (\omega_{1, \varepsilon} - \omega_{2, \varepsilon}) \to 0 \) as \( \varepsilon \to 0 \) and, after passing to a subsequence, \( \omega_{i, \varepsilon} \to \bar{\omega}, \)
\( i = 1, 2, \) for some \( \bar{\omega} \in \Omega \cup \partial_p \Omega \). First assume \( \bar{\omega} \in \partial_p \Omega \).

Then we obtain
\[
0 < K \leq \limsup_{\varepsilon \to 0} h_{\varepsilon}(\omega_{1, \varepsilon}, \omega_{2, \varepsilon}) \leq \limsup_{\varepsilon \to 0}(\tilde{u}(\omega_{1, \varepsilon}) - v(\omega_{2, \varepsilon})) \leq \tilde{u}(\bar{\omega}) - v(\bar{\omega}) \leq 0,
\]
a contradiction. Hence, we must have \( \bar{\omega} \in \Omega \), so that for small enough \( \varepsilon \), we have \( \omega_{1, \varepsilon}, \omega_{2, \varepsilon} \in \Omega \). Now we can apply [9, Theorem 3.2], with \( k = 2, N_i = 2, O_i = \Omega, \)
\( u_1 = \tilde{u}, u_2 = -v \) (which is usc), \( \phi(t, x, s, y) = \frac{|t-s|^2}{2\varepsilon} \), and the maximiser \( \hat{x} = (\omega_{1, \varepsilon}, \omega_{2, \varepsilon}) \). Then, [9, Theorem 3.2] guarantees the existence of \( Q_{i, \varepsilon} \in \text{Sym}(2), i = 1, 2, \) such that
\[
(D_{\omega_i \phi}(\omega_{\varepsilon}), Q_{i, \varepsilon}) \in J^{2, +} u_i(\omega_{i, \varepsilon}), \quad i = 1, 2
\]
and
\[
Q_{\varepsilon} \coloneqq \begin{pmatrix} Q_{1, \varepsilon} & 0 \\ 0 & Q_{2, \varepsilon} \end{pmatrix} \leq A + A^2, \tag{2.5}
\]
where \( A = D^2 \phi(\omega_{\varepsilon}) \). Notice that
\[
D_{\omega_1 \phi}(\omega_{\varepsilon}) = \frac{1}{\varepsilon}(t_{\varepsilon} - s_{\varepsilon}, x_{\varepsilon} - y_{\varepsilon})^t =: (\tau_{\varepsilon}, p_{\varepsilon})^t,
\]
\[
D_{\omega_2 \phi}(\omega_{\varepsilon}) = - (\tau_{\varepsilon}, p_{\varepsilon})^t,
\]
and
\[
A = D^2 \phi(\omega_{\varepsilon}) = \frac{1}{\varepsilon} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.
\]

Writing
\[
Q_{i, \varepsilon} =: \begin{pmatrix} a_{i, \varepsilon} & b_{i, \varepsilon} \\ b_{i, \varepsilon} & q_{i, \varepsilon} \end{pmatrix}
\]
we have for
\[
\xi := (0, 1, 0, 1)^t
\]
the identity
\[
\xi^t Q_{\varepsilon} \xi = q_{1, \varepsilon} + q_{2, \varepsilon}.
\]
Hence, since \( \xi \in \ker(A) \), the matrix inequality [2, 5] implies
\[
q_{1, \varepsilon} + q_{2, \varepsilon} \leq 0.
\]
By definition, the fact that
\[
(D_{\omega_1 \phi}(\omega_{\varepsilon}), Q_{1, \varepsilon}) \in J^{2, +} u_1(\omega_{1, \varepsilon}), \quad u_1 = \tilde{u}
\]

means that there exist sequences \( \omega_{1,\varepsilon}^{(n)} := (t_{\varepsilon}^{(n)}, x_{\varepsilon}^{(n)}) \) and
\[
(t_{\varepsilon}^{(n)}, p_{\varepsilon}^{(n)}, Q_{1,\varepsilon}^{(n)}) \in J^{\tau_{\varepsilon}^{(n)}, P}\]
such that as \( n \to \infty \)
\[\omega_{1,\varepsilon}^{(n)} \to \omega_{1,\varepsilon}, \quad \tilde{u}(\omega_{1,\varepsilon}^{(n)}) \to \tilde{u}(\omega_{1,\varepsilon}) \quad \text{and} \quad (t_{\varepsilon}^{(n)}, p_{\varepsilon}^{(n)}, Q_{1,\varepsilon}^{(n)}) \to (t_{\varepsilon}, p_{\varepsilon}, Q_{1,\varepsilon}).\]
In particular, we have as \((t, x) \to \omega_{1,\varepsilon}^{(n)}:\)
\[
\tilde{u}(t, x) \leq \tilde{u}(\omega_{1,\varepsilon}^{(n)}) + (t - t_{\varepsilon}^{(n)})\tau_{\varepsilon}^{(n)} + (x - x_{\varepsilon}^{(n)})p_{\varepsilon}^{(n)} + \frac{1}{2}(t - t_{\varepsilon}^{(n)})^2a_{1,\varepsilon}^{(n)} + \frac{1}{2}(x - x_{\varepsilon}^{(n)})^2q_{1,\varepsilon}^{(n)}
\]
\[+ (t - t_{\varepsilon}^{(n)})(x - x_{\varepsilon}^{(n)})p_{1,\varepsilon}^{(n)} + o((t - t_{\varepsilon}^{(n)})^2 + (x - x_{\varepsilon}^{(n)})^2)
\]
\[\leq \tilde{u}(\omega_{1,\varepsilon}^{(n)}) + (t - t_{\varepsilon}^{(n)})\tau_{\varepsilon}^{(n)} + (x - x_{\varepsilon}^{(n)})p_{\varepsilon}^{(n)} + \frac{1}{2}(x - x_{\varepsilon}^{(n)})^2q_{1,\varepsilon}^{(n)} + \frac{\sigma}{2}(x - x_{\varepsilon}^{(n)})^2
\]
\[+ o((t - t_{\varepsilon}^{(n)}) + (x - x_{\varepsilon}^{(n)})^2),\]
where \( \sigma > 0 \) can be chosen arbitrarily small. This means that for all \( \sigma > 0 \)
\[\tau_{\varepsilon}^{(n)}, p_{\varepsilon}^{(n)}, q_{1,\varepsilon}^{(n)} + \sigma) \in \mathcal{P}^{+} \tilde{u}(\omega_{1,\varepsilon}^{(n)}),\]
which, upon choosing \( \sigma = \frac{1}{n} \) and letting \( n \to \infty \), yields
\[(t_{\varepsilon}, p_{\varepsilon}, q_{1,\varepsilon}) \in \mathcal{P}^{+} \tilde{u}(\omega_{1,\varepsilon})\]
or, equivalently,
\[
(t_{\varepsilon} + \frac{\eta}{(T - t_{\varepsilon})^2}, p_{\varepsilon}, q_{1,\varepsilon}) \in \mathcal{P}^{+} \tilde{u}(\omega_{1,\varepsilon}). \tag{2.6}
\]
Starting from
\[(-\tau_{\varepsilon}, -p_{\varepsilon}, Q_{2,\varepsilon}) \in J^{\tau_{\varepsilon}, u_2(\omega_{2,\varepsilon})},\]
we can argue analogously for \( u_2 \) to find
\[(-\tau_{\varepsilon}, -p_{\varepsilon}, Q_{2,\varepsilon}) \in \mathcal{P}^{+} u_2(\omega_{2,\varepsilon}),\]
or, equivalently,
\[(t_{\varepsilon}, p_{\varepsilon}, -q_{2,\varepsilon}) \in \mathcal{P}^{-} v(\omega_{2,\varepsilon}). \tag{2.7}\]
Thanks to the conclusions (2.6) and (2.7), we can make use of the fact that \( u \) (resp. \( v \)) is a subsolution (resp. a supersolution) of equation (2.1) and obtain the inequalities
\[G(u(\omega_{1,\varepsilon}), \tilde{\tau}_{\varepsilon}, p_{\varepsilon}, q_{1,\varepsilon}) \leq 0, \tag{2.8}\]
where \( \tilde{\tau}_{\varepsilon} = \tau_{\varepsilon} + \frac{\eta}{(T - t_{\varepsilon})^2} > \tau_{\varepsilon} \), and
\[G(v(\omega_{2,\varepsilon}), \tau_{\varepsilon}, p_{\varepsilon}, -q_{2,\varepsilon}) \geq 0. \tag{2.9}\]
Subtracting ineq. (2.8) from ineq. (2.9), we infer the following contradiction
\[0 \geq G(u(\omega_{1,\varepsilon}), \tilde{\tau}_{\varepsilon}, p_{\varepsilon}, q_{1,\varepsilon}) - G(v(\omega_{2,\varepsilon}), \tau_{\varepsilon}, p_{\varepsilon}, -q_{2,\varepsilon})
\]
\[\geq G(u(\omega_{1,\varepsilon}), \tilde{\tau}_{\varepsilon}, p_{\varepsilon}, q_{1,\varepsilon}) - G(v(\omega_{2,\varepsilon}), \tau_{\varepsilon}, p_{\varepsilon}, q_{1,\varepsilon}) > 0,\]
where we used hypothesis \([A0]\) and \([A1]\).
2.4 Perron method

As a preparative step towards existence we establish a Perron method for equation (2.1) for monotonic (and non-monotonic) functions, which roughly states that once a subsolution \( u^- \) and a supersolution \( u^+ \) satisfying \( u^- \leq u^+ \) are found, there exists an “almost” viscosity solution squeezed between these barriers. Since in our applications we are particularly interested in functions which are non-decreasing with respect to \( x \), we start with some preliminaries on monotonicity.

**Definition 2.9 (x-monotonicity).** We say that a function \( u = u(t, x) \) is \( x \)-monotonic, in short \( x \)-m, if the function \( x \mapsto u(t, x) \) is non-decreasing for any \( t \).

**Fact 1.** If \( u = u(t, x) \) is \( x \)-monotonic, so are \( u^- \) and \( u^+ \).

Let us sketch the elementary argument demonstrating the assertion for \( u^- \)—the claim for \( u^+ \) can be obtained by a similar reasoning. Fix \( t \geq 0 \) and \( x < y \). The definition of \( u^- \) implies that there exists a sequence \((t_j, x_j) \to (t, x)\) such that \( u(t_j, x_j) \to u^-(t, x) \). Then, for large enough \( j \), we have \( x_j < y \) and therefore \( u(t_j, x_j) \leq u(t_j, y) \). Hence

\[
u^-(t, x) \leq \limsup_{j \to \infty} u(t_j, y) \leq \limsup_{j \to \infty} u^+(t_j, y) \leq u^+(t, y),
\]

where the last inequality holds thanks to the semi-continuity of \( u^+ \).

**Fact 2.** If \( V \) is a set of functions such that all \( v \in V \) are \( x \)-m, then the function \( u \) defined via \( u(t, x) := \sup_{v \in V} v(t, x) \) is \( x \)-m.

While the idea of the Perron method is well-known in the literature, the assumption of monotonicity requires some non-trivial modifications. The version provided below is an adaptation of [17, Lemma 2.3.15].

**Proposition 2.10.** Suppose that hypothesis [A0] holds true and let \( 0 < T \leq \infty \). Assume that \( u^\pm \) are locally bounded \( x \)-m functions satisfying \( u^- \leq u^+ \) in \( \Omega \) and suppose that \( u^- \) is a subsolution and \( u^+ \) a supersolution of eq. (2.1) in \( \Omega \). Then there exists an \( x \)-m function \( u : \Omega \to \mathbb{R} \) such that \( u^- \) is a subsolution of eq. (2.1) in \( \Omega \), \( u^+ \) a supersolution and \( u^- \leq u \leq u^+ \).

The statement remains valid when the \( x \)-m property is dropped everywhere.

**Proof.** We confine ourselves to showing the (more interesting) assertion regarding the \( x \)-monotonic case. The proof of the second assertion is easier and can be carried out along similar lines (without the need of a distinction of cases). Consider the non-empty set

\[ V = \{ v : \Omega \to \mathbb{R} \mid u^- \leq v \leq u^+, \ v \text{ is } x \text{-monotonic}, \ v^* \text{ is a subsolution of eq. (2.1)} \} \]

and let

\[ u = \sup_{v \in V} v. \]

Then \( u \) is \( x \)-monotonic and, by Remark 2.7 (b), \( u^* \) is a subsolution of eq. (2.1) in \( \Omega \).

It remains to show that the \( x \)-m, lsc function \( u^* \) is a supersolution of eq. (2.1). We argue by contradiction and assume that there exists \( \omega \in \Omega \), \((\alpha, p, q) \in \mathcal{P}^- u^*(\omega)\) and \( \theta > 0 \) such that

\[ G(z, \alpha, p, q) \leq -\theta, \tag{2.10} \]
where \( z := u_*(\omega) \). Notice that, since \( u_* \leq u^+ \), if \( u_*(\omega) = u^+(\omega) \), then \((\alpha, p, q) \in \mathcal{P}^- u^+(\omega) \), and the fact that \( u^+ \) is a supersolution would then imply that \( G(z, \alpha, p, q) \geq 0 \), which contradicts (2.10). Therefore

\[
u_*(\omega) < u^+(\omega),
\]

and, after possibly decreasing \( \theta > 0 \), we can assume that

\[
u_*(\omega) - u^+(\omega) \leq -\theta < 0. \tag{2.11}
\]

By the translation invariance of the equation with respect to the independent variable \( \theta \), we can further assume that \((0, 0) \in \Omega \) and \( \omega = (0, 0) \). For small parameters \( \delta, \epsilon > 0 \) to be determined later, we define

\[
P(s, y) = z + \alpha s + py + \frac{1}{2} qy^2 + \delta - \epsilon \left( |s| + \frac{1}{2} |y|^2 \right).
\]

Note that for any \((s, y) \in \Omega \) and \((\alpha', p', q') \in \mathcal{P}^+ P(s, y) \) one has \( |\alpha' - \alpha| \leq \epsilon, p' = p + qy - \epsilon y \) and \( q' \geq q - \epsilon \). We further let \( N_r := \{ (s, y) : |s| + |y|^2/2 < r \} \).

We now have to distinguish between the case in which \( p > 0 \) and the one in which \( p \) vanishes.

\textbf{Case 1:} \( p > 0 \).

In this case, \( P \) is \( x \)-monotonic in \( N_r \) for \( r > 0 \) small enough, and after decreasing \( r \) again and choosing \( \epsilon, \delta > 0 \) sufficiently small, we have

\[
G(P(s, y), \alpha', p', q') \leq -\frac{\theta}{2}
\]

for any \((s, y) \in \Omega \) and any \((\alpha', p', q') \in \mathcal{P}^+ P(s, y) \). Thus, \( P \) is a subsolution of eq. (2.11) in \( N_r \).

Since, by inequality (2.11), we have \( P(\omega) \leq u^+(\omega) \) + \( \delta - \theta \), the fact that \( P \) is usc and \( u^+ \) lsc ensures that, after possibly decreasing \( \delta > 0 \),

\[
P(s, y) < u^+(s, y) \quad \text{for} \quad (s, y) \in N_r.
\tag{2.12}
\]

Since \((\alpha, p, q) \in \mathcal{P}^- u_*(\omega) \), by inequality (2.4),

\[
u_*(s, y) \geq z + \alpha s + py + \frac{1}{2} qy^2 + o(|s| + |y|^2)
\]

\[
\geq P(s, y) - \delta + \epsilon \left( |s| + \frac{1}{2} |y|^2 \right) + o(|s| + |y|^2).
\]

After possibly decreasing \( r \), we can choose \( \delta = \frac{\epsilon r}{4} \). Then for \((s, y) \in N_r \setminus N_r/2 \)

\[
u_*(s, y) \geq P(s, y) - \frac{\epsilon r}{4} + \frac{\epsilon r}{2} + o(r) = P(s, y) + \frac{\epsilon r}{4} + o(r)
\]

and hence, for \( r \) sufficiently small,

\[
u(s, y) - P(s, y) \geq \frac{\epsilon r}{8} > 0 \quad \text{for} \quad (s, y) \in N_r \setminus N_r/2.
\]

Let us now define

\[
U(s, y) = \begin{cases}
\max\{u(s, y), P(s, y)\} & \text{if} \ (s, y) \in N_r,
\end{cases}
\]

\[
u(s, y)
\]

else.
\tag{2.13}
Then $U$ is non-decreasing, $U^*$ is a subsolution of (2.14) in $\Omega$ and $u^- \leq U \leq u^+$, where the last bound follows from ineq. (2.12). Hence $U \in V$ and thus $U \leq u$. However, by definition there exists a sequence $\xi_n \to \omega$ such that $u(\xi_n) \to u_*(\omega) = z$ and therefore
\[
\liminf_{n \to \infty} (U(\xi_n) - u(\xi_n)) \geq \lim_{n \to \infty} (P(\xi_n) - u(\xi_n)) = \delta > 0.
\]
This contradicts $U \leq u$.

**Case 2:** $p = 0$.

In this case the $x$-monotonicity of $u_*$ implies that $q \leq 0$. Hence, hypothesis (A0) and inequality (2.10) imply that $G(z, \alpha, 0, q) \leq G(z, \alpha, 0, 0) \leq -\theta$.

The competitor $P = P(s, y)$ needs to be adapted since it is strictly decreasing in $y$ for $y > 0$. We define
\[
\hat{P}(s, y) = \begin{cases} 
P(s, y) & \text{if } y \leq 0, \\
P(s, 0) = z + \delta + s\alpha - \epsilon|s| & \text{if } y > 0.
\end{cases}
\]

Notice that we can choose $r, \delta, \epsilon$ sufficiently small such that for all $\sigma \in [-1, 1]$
\[
G(\hat{P}, \alpha + \sigma\epsilon, \partial_y \hat{P}, \partial^2_y \hat{P})_{(s,y)} = G(P, \alpha + \sigma\epsilon, \partial_y P, \partial^2_y P)_{(s,y)} \leq -\frac{\theta}{2} \forall (s, y) \in N_r \text{ s.t. } y < 0
\]
and
\[
G(\hat{P}, \alpha + \sigma\epsilon, \partial_y \hat{P}, \partial^2_y \hat{P})_{(s,y)} = G(P(s,0), \alpha + \sigma\epsilon, 0, 0) \leq -\frac{\theta}{2} \forall |s| < r, \ y > 0.
\]
Moreover, since $\partial_y \hat{P} \in C^0$ with $\partial_y \hat{P}(s,0) = 0$, whenever $(\hat{\alpha}, \hat{\rho}, \hat{\eta}) \in P^+ \hat{P}(s,0)$, we must have $\hat{\rho} = 0, \hat{\eta} \geq 0, \hat{\alpha} = \alpha + \sigma\epsilon$ for some $\sigma \in [-1, 1]$ and therefore $G(\hat{P}(s,0), \hat{\alpha}, \hat{\rho}, \hat{\eta}) \leq -\frac{\theta}{2}$ whenever $|s| < r$. Hence, $\hat{P}$ is a subsolution of $G = 0$ in the domain $\tilde{N}_r$ defined via
\[
\tilde{N}_r := N_r \cup \{(s,y) \in \Omega : |s| < r, y \geq 0\}.
\]
As in Case 1 we have $P(\omega) < u^+(\omega)$ for $\delta$ sufficiently small, so that after possibly decreasing $r$ once more, we obtain
\[
\hat{P} < u^+ \text{ in } \tilde{N}_r.
\]

For this conclusion we have used in particular the $x$-monotonicity of $u^+$.

Arguing as in Case 1 and letting in particular $\delta = \frac{\theta}{4}$, for $r, \epsilon$ sufficiently small, we can guarantee that
\[
u > \hat{P} \text{ in } \left(N_r \setminus \mathbb{N}_{\frac{\theta}{4}}\right) \cap \{(s,y) \in \Omega : y \leq 0\}. \tag{2.14}
\]

The inequality (2.14) implies that $u(s,0) > \hat{P}(s,0)$ for $\frac{\theta}{2} \leq |s| < r$, and thanks to the $x$-monotonicity of $u$ therefore
\[
u(s,y) > \hat{P}(s,y) \text{ for all } \frac{\theta}{2} \leq |s| < r, y \geq 0.
\]

We now define $U$ as in formula (2.13) with $P$ replaced by $\hat{P}$ and $N_r$ replaced by $\tilde{N}_r$. Then $U$ is $x$-monotonic, $U^*$ is a subsolution of $G = 0$ in $\Omega$, $u^- \leq u \leq U \leq u^+$ but $U \neq u$, which contradicts the maximality of $u$. \qed
2.5 Existence, uniqueness and Lipschitz regularity

We are now in a position to show existence and uniqueness for the Cauchy–Dirichlet problem associated with equation (2.1).

Theorem 2.11 (Existence and uniqueness). Suppose that the continuous function $G$ satisfies the conditions $[A0]$ and $[A1]$. Given $0 < T \leq \infty$ and locally bounded $x$-monotonic functions $u^\pm : \Omega \cup \partial_p \Omega \to \mathbb{R}$ such that $u^-$ is a subsolution and $u^+$ a supersolution of eq. (2.1) in $\Omega$ satisfying

(B1) $u^- \leq u^+$ in $\Omega$

(B2) $(u^-)_* = (u^+)_*$ on $\partial_p \Omega$,

there exists a unique $x$-monotonic viscosity solution $u \in C(\Omega \cup \partial_p \Omega)$ of eq. (2.1) in $\Omega$ with the property that $u = u^-(= u^+)$ on $\partial_p \Omega$. This solution satisfies $u^- \leq u \leq u^+$.

The assertion remains valid when dropping the $x$-monotonicity everywhere.

Remark. By replacing $u^\pm$ with $-u^\mp$ one obtains the same result for functions which are non-increasing in $x$.

Proof. We only consider the $x$-m case since the reasoning in the non-monotonic case is completely similar. From the assumptions we infer that

$$\lim_{\omega \in \Omega, \omega \to \bar{\omega} \in \partial_p \Omega} u^\pm(\omega) = u^-(\bar{\omega}) = u^+(\bar{\omega}) \in \mathbb{R}.$$  

Thus, Proposition 2.10 guarantees the existence of an $x$-m function $u : \Omega \cup \partial_p \Omega \to \mathbb{R}$ satisfying $u^- \leq u \leq u^+$ such that $u^*$ is a subsolution, $u_*$ a supersolution of eq. (2.1) and $u_* = u^* = u^\pm$ on $\partial_p \Omega$. Hence, Proposition 2.8 implies that $u^* \leq u_*$, and thus $u = u^* = u_* \in C(\Omega \cup \partial_p \Omega)$ is a viscosity solution of eq. (2.1). Uniqueness subject to prescribed values on $\partial_p \Omega$ is a consequence of Proposition 2.8.

Before providing concrete examples to Theorem 2.11 we show that if the barriers $u^\pm$ are Lipschitz continuous, the viscosity solution obtained in Theorem 2.11 inherits this regularity. The main ingredients in the proof are again versions of the so-called “Theorem on Sums” (a maximum type principle for semi-continuous functions), which already was the key to proving the comparison principle (Proposition 2.8). Related approaches can be found in [18] and [17].

Proposition 2.12 (Lipschitz continuity in time). Suppose that the conditions $[A0]$ and $[A1]$ hold true and assume that, in addition to the hypotheses in Theorem 2.11 the barriers $u^\pm$ are locally Lipschitz continuous with respect to $t$ in $\Omega \cup \partial_p \Omega$, i.e. for any $T' < T$ there exists $K_{T'} < \infty$ such that for all $s, t \in [0, T']$ and all $x \in \bar{J}$

$$|u^\pm(t, x) - u^\pm(s, x)| \leq K_{T'}|t - s|.$$  

Then for any $T' < T$ and the same constant $K_{T'}$ the associated viscosity solution $u$ saturates the estimate

$$|u(t, x) - u(s, x)| \leq K_{T'}|t - s|$$  

for all $s, t \in [0, T']$ and all $x \in \bar{J}$.
Proof. Assume that the assertion is false. Then there exists $T' < T$ such that for $K = K_{T'}$ 

$$\sup_{t,s \in [0,T'],x \in J} (u(t,x) - u(s,x) - K|t - s|) > 0$$

and thus for $\eta = \eta(K) > 0$ sufficiently small 

$$M := \sup_{t,s \in [0,T'],x \in J} \left( u(t,x) - \frac{\eta}{T'} - \left( u(s,x) + \frac{\eta}{T''} \right) - K|t - s| \right) > 0.$$ 

With the abbreviation $u_1(t,x) := u(t,x) - \frac{\eta}{T'}$ and $u_2(s,x) := -\left( u(s,x) + \frac{\eta}{T''} \right)$ it follows that for any $\varepsilon > 0$ 

$$M_\varepsilon := \sup_{t,s \in [0,T'],x \in J} \left( u_1(t,x) + u_2(s,x) - (K|t - s| + \frac{1}{2\varepsilon}|x - y|^2) \right) \geq M > 0.$$ 

Let now $\varphi(t,x, s, y) := (K|t - s| + \frac{1}{2\varepsilon}|x - y|^2)$ and define $w(t,x, s, y) := u_1(t,x) + u_2(s,x) - \varphi(t,x, s, y)$. Since $u \in C([0,T'] \times J)$, the function $w$ attains its maximum at some point $(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \in [0,T') \times J \times [0,T') \times J$. Notice that by the properties of $u^\pm$ one has $u^-(t,x) - u^-(0,x) \leq u(t,x) - u(0,x) \leq u^+(t,x) - u^+(0,x)$ and thus for all $x \in J$ and $t \in [0,T')$ 

$$|u(t,x) - u(0,x)| \leq Kt,$$

which implies that $\bar{t}, \bar{s} > 0$. whenever $\varepsilon = \varepsilon(u^\pm(0, \cdot), M) > 0$ is sufficiently small.

We next claim that $\bar{x}, \bar{y} \notin \partial J$ for small enough $\varepsilon = \varepsilon(K) > 0$. Indeed, assuming that this is not the case, we find a sequence $\varepsilon_n \to 0$ such that $\bar{x} \in \partial J$ for all $n$ or $\bar{y} \in \partial J$ for all $n$. By the boundedness of $u$, we must have $\bar{x} - \bar{y} \to 0$ as $n \to \infty$, and there exist $x_\infty \in \partial J$, $t_\infty, s_\infty \in [0,T')$ such that after passing to a subsequence $\bar{x}, \bar{y} \to x_\infty, \bar{t} \to t_\infty, \bar{s} \to s_\infty$ as $n \to \infty$. But then the continuity of $u$ and the fact that $u = u^\pm$ on $\partial_p \Omega$ lead to a contradiction to the assumption $M > 0$.

Hence $(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \in (0,T') \times J \times (0,T') \times J$. Notice also that $\bar{t} \neq \bar{s}$ for $\varepsilon$ sufficiently small since otherwise $M_\varepsilon \to 0$ along a subsequence. This guarantees that for small enough $\varepsilon$, the function $\varphi$ is $C^2$ in a neighbourhood of the maximiser of $w$.

We can now argue as in the proof of Proposition 2.2 by [9 Theorem 3.2] there exist $\tau, p \in \mathbb{R}$, where $p \geq 0$, and $Q_1, Q_2 \in \text{Sym}(2)$ satisfying $(\tau, p, Q_1) \in \mathcal{T}^{\pm} u_1(\bar{t}, \bar{x})$, $(\tau, -p, Q_2) \in \mathcal{T}^{\pm} u_2(\bar{s}, \bar{y})$ such that for $Q = \text{diag}(Q_1, Q_2)$ and $A = D^2 \varphi(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ the matrix inequality $Q \leq A + A^T$ holds true. Letting $q_i := (Q_1)_{i2}$ for $i = 1, 2$, it follows that $q_1 + q_2 \leq 0$ and, furthermore, $(\tau, p, q_1) \in \mathcal{T}^+ u_1(\bar{t}, \bar{x})$, $(\tau, -p, q_2) \in \mathcal{T}^+ u_2(\bar{s}, \bar{y})$. By the definition of $u_i$, $i = 1, 2$, this means that $(\tau + \frac{\eta}{|T'| - \varepsilon}, p, q_1) \in \mathcal{T}^+ u(\bar{t}, \bar{x})$, $(\tau - \frac{\eta}{|T''| - \varepsilon}, p, -q_2) \in \mathcal{T}^+ u(\bar{s}, \bar{y})$. A contradiction is now inferred in precisely the same way as in the proof of Proposition 2.2.

The Lipschitz bound (2.15) implies that for all $\omega = (t,x) \in \Omega$ with $t \leq T'$ we have the implication 

$$\left[ \exists p, q \in \mathbb{R} : (\tau, p, q) \in \mathcal{T}^+ u(\omega) \text{ or } (\tau, p, q) \in \mathcal{T}^+ u(\omega) \right] \implies |\tau| \leq K_{T'}.$$  (2.16)

Thanks to this observation, we easily obtain full Lipschitz regularity of viscosity solutions admitting barriers as in Theorem 2.11 which are Lipschitz continuous.
Proposition 2.13 (Lipschitz continuity). Suppose that the conditions (A0), (A1) hold true and assume that the barriers $w^\pm$ in Theorem 2.11 are in addition locally Lipschitz continuous in $\Omega \cup \partial_p \Omega$. Then for any $T' < T$ the associated viscosity solution $u$ saturates the estimate

$$|u(t,x) - u(t,y)| \leq \tilde{K}_{T'}|x - y|$$

for all $t \in [0, T']$ and all $x, y \in J$, where

$$\tilde{K}_{T'} := \max\{|u^-|_{L^\infty(0, T', C^{0,1}(J))}, |u^+|_{L^\infty(0, T', C^{0,1}(J))}\}.$$

Proof. Arguing by contradiction, we assume that there is $T' < T$ such that for $\tilde{K} := \tilde{K}_{T'}$

$$\sup_{t \in [0, T'), x, y \in J} \left( u(t,x) - u(t,y) - \tilde{K}|x - y| \right) > 0.$$ 

This implies that for $\eta > 0$ sufficiently small

$$M := \sup_{t \in [0, T'), x, y \in J} \left( u(t,x) - \frac{\eta}{T' - t} - u(t,y) - \tilde{K}|x - y| \right) > 0.$$ 

We now define $u_1(t,x) := u(t,x) - \frac{\eta}{T' - t}, u_2 := -u$ and $\varphi(x,y) := \tilde{K}|x - y|$, and then set $w(t,x,y) := u_1(t,x) + u_2(t,y) - \varphi(x,y)$. Since $u \in C([0, T') \times J)$, the function $w$ reaches its maximum $M$ at some point $(\bar{t}, \bar{x}, \bar{y}) \in [0, T) \times [0, m] \times [0, m]$. Arguing similarly as in the proof of Proposition 2.12 we find that the maximiser $(\bar{t}, \bar{x}, \bar{y})$ of $w$ is an interior point. Thus, in view of property (2.16) and the fact that $\bar{x} \neq \bar{y}$, the spatial version of the Theorem on Sums [9, Theorem 8.3] is applicable and yields the existence of $\tau, q_1, q_2 \in \mathbb{R}$ satisfying $q_1 + q_2 \leq 0$ and which are such that $(\tau, p, q_1) \in \overline{\mathcal{P}}^+ u_1(\bar{t}, \bar{x})$ and $(-\tau, -p, q_2) \in \overline{\mathcal{P}}^- u_2(\bar{t}, \bar{y})$, where $p = \partial_x \varphi(\bar{x}, \bar{y})$. Thus

$$\left( \tau + \frac{\eta}{(T' - \bar{t})^2}, p, q_1 \right) \in \overline{\mathcal{P}}^+ u(\bar{t}, \bar{x}) \quad \text{and} \quad (\tau, p, -q_2) \in \overline{\mathcal{P}}^- u(\bar{t}, \bar{y}).$$

Now the contradiction is obtained by using the fact that $u$ is a sub- and a supersolution of eq. 2.11. \qed

As an immediate consequence of Proposition 2.12 and Proposition 2.13 we obtain

Corollary 2.14 (Lipschitz continuity). Under the hypotheses in Proposition 2.13 the corresponding viscosity solution $u$ of equation (2.1) is locally Lipschitz continuous in $\Omega \cup \partial_p \Omega$ and satisfies the estimate

$$[u]_{C^{0,1}([0, T'] \times J)} \leq \sqrt{2}\max\{K_{T'}, \tilde{K}_{T'}\},$$

where $K_{T'}$ and $\tilde{K}_{T'}$ denote the constants defined in Proposition 2.12 and Proposition 2.13.

2.6 Applications

Here we demonstrate how Theorem 2.11 can be used to derive global-in-time existence and uniqueness for a class of equations generalising problem (1.5)–(1.7). While the approach is reminiscent of the setting in the ref. [3], our precise regularity assumptions are slightly different. For $R > 0$ define the functional

$$\mathcal{H}_R(f) = \int_{-R}^R \left( \frac{|v|^2}{2} + \Phi(f) \right) \, dv, \quad (2.17)$$

where $\Phi'(s) = -\int_s^\infty \frac{1}{h(z)} \, dz$ for a positive function $h \in C((0, \infty))$ satisfying $1/h \in L^1(1, \infty)$, and consider the equation

$$
\frac{\partial f}{\partial t} = \frac{d}{dr} \left( h(f) \frac{d\delta H_R}{d\delta f}(f) \right)
$$

(2.18)

for $t > 0$, $r \in (-R, R)$ subject to zero-flux boundary conditions $\frac{d}{dr} \frac{\delta H_R}{\delta f}(f) = 0$ on $\{-R, R\}$. Formally, solutions of eq. (2.18) satisfy the *entropy dissipation identity*

$$
\frac{d}{dt} \delta H_R(f) = -\int_{-R}^R h(f) \left| \frac{d\delta H_R}{d\delta f}(f) \right|^2 \, dr.
$$

We define the *steady states* of this conservative problem to be the solutions $f$ of

$$
h(f) \frac{d\delta H_R}{d\delta f}(f) = 0,
$$

i.e. the solutions $f_{\infty, \theta}$ of

$$
|\theta|^2 + \Phi'(f_{\infty, \theta}) = -\theta,
$$

where $\theta$ is a constant of integration.

In the following we assume that $1/h(s)$ is not integrable near $s = 0$, which implies that $\lim_{s \to 0^+} \Phi'(s) = -\infty$. Since $\Phi'$ is strictly increasing with $\lim_{s \to \infty} \Phi'(s) = 0$, we can then solve the last equation for $f_{\infty, \theta}$ to obtain

$$
f_{\infty, \theta}(r) = (\Phi')^{-1} \left(-(|\theta|^2/2 + \theta)\right), \quad r \in (-R, R),
$$

provided that $\theta \in (0, \infty)$. For $\theta = 0$ we define $f_{\infty, \theta}$ by the same formula if $r \neq 0$ and let $f_{\infty, 0}(0) := \infty$. Observe that $f_{\infty, 0}(r) \to \infty$ as $r \to 0$. Furthermore, notice that $f_{\infty, \theta} \to 0$ uniformly in $[-R, R]$ as $\theta \to \infty$ and that for any $\theta < \infty$ there exists $c_\theta > 0$ such that $f_{\infty, \theta} \geq c_\theta$ in $[-R, R]$. Thus, letting $m_\theta = \int_{-R}^R f_{\infty, \theta}(r) \, dr$ and denoting for given $m \in (0, \infty)$ and $\theta$ satisfying $m_\theta < m$ by $u_{\theta, -} : [0, m] \to [-R, R]$ (resp. $u_{\theta, +} : [0, m] \to [-R, R]$) the inverse cdf of the measure $(m - m_\theta)\delta_{-R} + C^1 L_\infty \delta_{-\infty}$ (resp. of $(m - m_\theta)\delta_R + C^1 L_\infty \delta_{\infty}$), we infer that $u_{\theta, -}$ are Lipschitz continuous in $[0, m]$ and that for any non-decreasing function $u_0 \in C^1([0, m])$ with $u_0(0) = -R, u_0(m) = R$ there exists $\theta < \infty$ such that

$$
uu_{\theta, -} \leq u_0 \leq u_{\theta, +}.
$$

(2.19)

We finally note that the map $(0, \infty) \ni \theta \mapsto m_\theta \in (0, m_0)$ is a bijection.

Formally, the equation for the inverse cdf $u(t, \cdot)$ of $f(t, \cdot)$ states

$$
uu_t - \frac{uxx}{ux} + ux h(1/ux) u = 0 \quad \text{in} \; \Omega := (0, \infty) \times (0, m),
$$

(2.20)

where $m$ denotes the mass of the initial datum $f_0$, i.e. $m = \int_{-R}^R f_0(r) \, dr$. In view of the no-flux boundary conditions for eq. (2.18), we complement eq. (2.20) with the Dirichlet conditions

$$
uu u(t, 0) = -R, \quad u(t, m) = R
$$

(2.21)

for all $t > 0$.

We henceforth suppose that $\lim_{s \to \infty} s^3/h(s)$ exists in $[0, \infty)$ and define

$$
G(z, \alpha, p, q) = (|p|^3 h(1/|p|))^{-1} (|p|^2 \alpha - q) + z,
$$

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with the understanding that for all \( z, \alpha, q \in \mathbb{R} \)

\[
G(z, \alpha, 0, q) := \lim_{p \to 0} G(z, \alpha, p, q),
\]

which, by assumption, exists in \( \mathbb{R} \). Then the function \( G \) is continuous on \( \mathbb{R}^4 \), saturates the conditions \([A0]\) and \([A1]\) and defining \( \mathcal{G} \) by formula \((2.2)\), equation \((2.20)\) can be reformulated as

\[
\mathcal{G}(u) = 0 \quad \text{in } \Omega. \tag{2.22}
\]

Notice that equations \((2.20)\) and \((2.22)\) are equivalent if \( 0 < u_x < \infty \).

**Definition 2.15** (Admissible initial datum for BEFP type). We call a non-decreasing function \( u_0 \in C^1(\bar{J}) \) an admissible initial datum for the BEFP type problem \((2.21)-(2.22)\) if it has the following properties:

- \( u_0(0) = -R, \; u_0(m) = R \),
- \( u'_0(x) > 0 \) for all \( x \in [0, m] \) with \( |u_0(x)| > 0 \),
- \( u_0 \in C^2(\{|u_0| > 0\}) \) and

\[
C := C(u_0) := \sup_{\{|u_0|>0\}} |p_0 h(p_0^{-1}) \mathcal{G}(u_0)| < \infty, \tag{2.23}
\]

with \( p_0 := u'_0 \) and where we have used the abbreviation \((2.2)\).

Notice that the choice of \( C \) in formula \((2.23)\) guarantees that \( u_0 \pm Ct, \; t \geq 0 \), is a sub- resp. supersolution of equation \((2.22)\) in \( \Omega := (0, \infty) \times (0, m) \). Of course, any initial datum \( u_0 \in C^2([0, m]) \) with \( \min_{[0,m]} u'_0 > 0 \) and \( u_0(0) = -R, \; u_0(m) = R \) is admissible in the sense of Definition \(2.15\) but, in general, Definition \(2.15\) also allows for functions which have a flat part at height zero, see Remark \(2.18\) for details and the meaning of the bound \((2.23)\).

We are now in a position to show wellposedness for the BEFP type problems introduced above.

**Theorem 2.16** (Global existence, uniqueness and Lipschitz continuity for BEFP type). Suppose that the function \( h \in C((0, \infty), \mathbb{R}^+) \) satisfies \( 1/h \notin L^1(0, 1) \) and that the limit \( \lim_{s \to \infty} s^3/h(s) \) exists in \([0, \infty)\). Given an admissible initial datum \( u_0 \) for the BEFP type problem \((2.21)-(2.22)\) (as in Definition \(2.15\)), there exists a unique, \( x \)-monotonic viscosity solution \( u \in C(\Omega \cup \partial_p \Omega) \) of problem \((2.21)-(2.22)\) such that \( u(0, \cdot) = u_0 \). This solution is globally Lipschitz continuous with constant \( K = \sqrt{2} \max\{C(u_0), [u_\infty, \theta]_{C^{0,1}}\} \).

**Proof.** Since \( u_0 \in C^1([0, m]) \), there exists \( \theta < \infty \) such that ineq. \((2.19)\) holds true. Notice that the function

\[
u^-(t, x) := \max\{u_0(x) - Ct, u_{\theta}^-(x)\}
\]

is a subsolution, while the function

\[
u^+(t, x) := \min\{u_0(x) + Ct, u_{\theta}^+(x)\}
\]

is a supersolution satisfying \( u^- \leq u^+ \). The functions \( u^\pm \) are of class \( C^{0,1}(\Omega \cup \partial_p \Omega) \) and have the desired behaviour on \( \partial_p \Omega \). Thus, Theorem \(2.11\) yields the first claim. The Lipschitz continuity is a consequence of Corollary \(2.14\). \( \square \)
Remark 2.17 (Critical mass). In general, the singularity of \( f_{\infty,0} \) near the origin may not be integrable and we may have \( m_0 = \infty \). Computing

\[
m_0 = \int_{-R}^{R} f_{\infty,0}(r) \, dr = 2 \int_{R_1}^{\infty} s \Phi''(s)(-2\Phi'(s))^{-\frac{1}{2}} \, ds,
\]

where \( R_1 = (\Phi')^{-1}(-R^2/2) > 0 \), we infer that \( m_0 < \infty \) if and only if

\[
\infty > \int_{1}^{\infty} s \Phi''(s)(-2\Phi'(s))^{-\frac{1}{2}} \, ds = \int_{1}^{\infty} \frac{s}{h(s)} \left( 2 \int_{s}^{\infty} \frac{1}{h(\sigma)} \, d\sigma \right)^{-\frac{1}{2}} \, ds. \tag{2.24}
\]

If inequality 2.24 is satisfied, the problem has a critical mass given by \( m_c := m_0 \). In this case, there does not exist a steady state of mass \( m > m_c \). Notice that for \( m \geq m_c \) the inverse cdf \( u_c : [0,m] \rightarrow [-R,R] \) of the measure \( L^1 \mathbb{L} f_{\infty,0} + (m - m_c) \delta_0 \) is of class \( C^1([0,m]) \) and is a viscosity solution of equation 2.22 while the inverse cdf \( u_\theta, \theta > 0 \), of the measure \( L^1 \mathbb{L} f_{\infty,\theta} + (m - m_\theta) \delta_0 \) is neither a sub- nor a supersolution of equation 2.22.

Since \( \lim_{s \to \infty} \Phi(s)/s = 0 \), we can proceed as in the reference [4] and extend the functional \( \mathcal{H}_R \) to the set of non-negative finite measures on \([-R,R]\) by ignoring the singular part (with respect to Lebesgue) in the nonlinear term of \( \mathcal{H}_R \) involving \( \Phi \). Following the proof of [4] Theorem 3.1 one finds that the unique minimiser of \( \mathcal{H}_R \) among non-negative measures of mass \( m \) is given by \( L^1 \mathbb{L} f_{\infty,0} + (m - m_c) \delta_0 \) if \( m \geq m_c \), and by \( L^1 \mathbb{L} f_{\infty,\theta} \) if \( m = m_\theta \).

Remark 2.18. If the BEFP type problem has a finite critical mass, Definition 2.15 allows for initial data \( u_0 \in C^1([0,m]) \) which have a flat part at height zero, so that there exist \( 0 < x_- \leq x_+ < m \) such that \( u_0(x) = 0, u'_0(x) = 0 \) for all \( x \in [x_-, x_+] \) and \( u_0(x) > 0 \) for \( x \not\in [x_-, x_+] \). In this case, condition 2.22 is non-trivial and enforces that, loosely speaking, the asymptotic behaviour of \( u_0(x) \) as \( x \to (x_\pm)^\pm \) agrees with the corresponding behaviour of \( u_c \). Its precise meaning at the level of the density \( f_0 \) associated with \( u_0 \) (for a specific choice of \( h \)) can be deduced by following the derivation in Section 4.

Remark 2.19. For \( h(s) = s(s^7 + 1) \), \( \gamma \geq 2 \), we obtain an equation equivalent to the Fokker–Planck model for bosons, which is given by \( \mathcal{F} = 0 \) with

\[
F(z, \alpha, p, q) := |p|^\gamma \alpha - |p|^\gamma - 2q + z(1 + |p|^\gamma). \tag{2.25}
\]

In this case we have \( \Phi'(f) = \frac{1}{\gamma} \log \left( \frac{f^\gamma}{1 + f^\gamma} \right) \), and the steady states take on the form

\[
f_{\infty,\theta}(r) = \frac{1}{(\exp(\gamma(\frac{|r|^2}{2} + \theta)) - 1)^{\frac{1}{\gamma}}} \cdot \theta > 0
\]

and saturate eq. 1.2 in the classical sense. As in the general case, we have \( f_{\infty,\theta_1} > f_{\infty,\theta_2} \) for \( 0 < \theta_1 < \theta_2 \) and \( f_{\infty,\theta} \nrightarrow f_c \) uniformly away from the origin as \( \theta \searrow 0 \), where the function

\[
f_c(v) := f_{\infty,0}(v) = \frac{1}{(\exp(\gamma(\frac{|r|^2}{2})) - 1)^{\frac{1}{\gamma}}}
\]

is unbounded near the origin. Besides, notice that condition 2.22 is equivalent to \( \gamma > 2 \).

Remark 2.20. For \( h(s) = s(s^7 + 1) \), \( \gamma \geq 2 \), the comments in Remarks 2.17 and 2.19 remain valid when replacing the bounded interval \([-R,R]\) by the whole line \( \mathbb{R} \), and analogous results hold true in higher dimensions. The extended entropy functional 2.17 for finite non-negative measures on the whole line will be denoted by \( \mathcal{H} \).
3 Refined regularity for bosonic Fokker–Planck model

For simplicity, we henceforth focus on the Fokker–Planck equation for bosons given by

\[ F(u, \partial_t u, \partial_x u, \partial^2_x u) = 0, \]

where \( F \) is defined by formula (2.25) with \( \gamma \geq 2 \). Later on it will be convenient to use the following notations: for \( \theta \geq 0 \) let \( \tilde{M}_\theta \) denote the cumulative distribution function of \( f_{\infty, \theta}\left[\left.-R,R\right\rangle \right] \), and then define \( \theta_{m,R} := \min\{\theta \geq 0 : \tilde{M}_\theta(R) \leq m\} \).

For \( \theta \geq \theta_{m,R} > 0 \) let \( \tilde{M}_\theta : \left[-R,R\right) \times \mathbb{R} \to [0,\bar{m}] \) denote the cumulative distribution function associated with \( \mathcal{L}_\infty \mathcal{L}_t f_{\infty, \theta} + (m - \tilde{M}_\theta(R))\delta_0 \), and denote by \( u_\theta \) the pseudo-inverse of \( \tilde{M}_\theta \).

If \( \theta_{m,R} = 0 \), we avoid this notation (in view of clashes with the notation of the initial value). In this case the notation \( u_c \) (introduced in Remark 2.17) will be used. Notice that \( u_c, u_{\theta_{m,R}} \in C^1([0,\bar{m}]) \) while for \( \theta > \theta_{m,R} \) the function \( \frac{du_\theta}{dx} \) is discontinuous at \( \tilde{M}_\theta(0) \) and at \( m - \tilde{M}_\theta(0) \).

In this section we aim to establish higher regularity of \( x \)-monotonic viscosity solutions of \( \mathcal{F} = 0 \). For this purpose, we first consider a regularised problem.

3.1 Approximation

From now on we assume that \( u_0 \in C^2(\bar{J}) \) with \( \min_J u_0' > 0 \) and—as before—\( u_0(0) = -R, u_0(m) = R \). We then consider a regularised problem in \( \Omega := (0,\infty) \times \bar{J} \), obtained by replacing the function \( F(z,\tau,p,q) \) with \( F_\sigma(z,\tau,p,q) := p^\gamma \tau - (p + \sigma)^{\gamma - 2} q + z(1 + p^\gamma) \), \( 0 < \sigma \ll 1 \), the lateral boundary conditions with \( u(t,0) = -R_\sigma \) and \( u(t,m) = R_\sigma \) for suitable \( 0 < R_\sigma \leq R \) with \( R_\sigma \to R \) as \( \sigma \to 0 \) and the initial value \( u_0 \) by suitable approximations \( u_{0,\sigma} \in C^2(\bar{J}) \) with \( \min_J u_{0,\sigma}'>0 \) satisfying \( u_{0,\sigma}(0) = -R_\sigma, u_{0,\sigma}(m) = R_\sigma, u_{0,\sigma} \not\nearrow u_0 \in C^2(\bar{J}) \). It is easy to see that such a sequence \( (u_{0,\sigma}) \) exists. Under these conditions the constants \( C_\sigma(u_{0,\sigma}) \), where

\[
C_\sigma(v) := \sup_{x \in \bar{J}} \left| \frac{(p(x) + \sigma)^{\gamma - 2}}{p^\gamma(x)} q(x) + v(x)(p(x)^{-\gamma} + 1) \right|, \quad p = v', q = v'',
\]

are uniformly bounded in \( 0 < \sigma \ll 1 \).

Existence and uniqueness of \( x \)-monotonic viscosity solutions are obtained by Theorem 2.11 provided appropriate barriers can be found. A possible construction of the barriers is as follows: we fix some \( \theta > 0 \) such that

\[
u_{\theta,-} \leq u_0 \leq u_{\theta,+}
\]

and define

\[
\kappa(\sigma) := \sup_{x \in \bar{J} ; u_0(x)>0} \left| u_\theta(x) - \frac{(p_\theta(x) + \sigma)^{\gamma - 2} q_\theta(x)}{1 + p_\theta(x)} \right|
\]

where we abbreviated \( p_\theta := u_\theta' \) and \( q_\theta := u_\theta'' \) (which are well-defined on \( \{|u_\theta| > 0\} \)). We note that \( \kappa \in C([0,1]) \) with \( \kappa(0) = 0 \), and let

\[
R_\sigma := R - \kappa(\sigma).
\]

By construction the function

\[
u_{\theta,-,\sigma} := \max\{-R_\sigma, u_{\theta,-} - \kappa(\sigma)\}
\]

is a subsolution of \( \mathcal{F}_\sigma = 0 \), while the function

\[
u_{\theta,+} := \min\{R_\sigma, u_{\theta,+} + \kappa(\sigma)\}
\]

is a supersolution of \( \mathcal{F}_\sigma = 0 \). This approach allows the construction of suitable barriers and the use of the maximum principle to control the height of the jump.
is a supersolution. Both functions are continuous on \( \bar{J} \) and they satisfy \( u_{\sigma}^{\pm}(0) = -R_{\sigma}, \)
\( u_{\theta,\sigma}(m) = R_{\sigma} \). It is also clear that after possibly slightly modifying the choice of \( u_{0,\sigma} \), we can assume that \( u_{\theta,\sigma}^{-} \leq u_{0,\sigma} \leq u_{\theta,\sigma}^{+} \).

Letting
\[
 u_{\sigma}^{-}(t, x) := \max\{u_{0,\sigma}(x) - C_{\sigma} t, u_{\theta,\sigma}^{-}(x)\}
\]
and
\[
 u_{\sigma}^{+}(t, x) := \min\{u_{0,\sigma}(x) + C_{\sigma} t, u_{\theta,\sigma}^{+}(x)\},
\]
where \( C_{\sigma} := C_{\sigma}(u_{0,\sigma}) \) (see formula (A.1)), defines bounded \( x \)-\( m \) functions \( u_{\sigma}^{\pm} \in C^{0}(\Omega \cup \partial_{p} \Omega) \) with the desired behaviour on \( \partial_{p} \Omega \) such that \( u_{\sigma}^{-} \) is a subsolution and \( u_{\sigma}^{+} \) a supersolution of \( \mathcal{F}_{\sigma} = 0 \). Thus, subject to the conditions on \( \partial_{p} \Omega \) specified above, there exists a unique viscosity solution \( u_{\sigma} \) of \( \mathcal{F}_{\sigma} = 0 \) in \( (0, \infty) \times J \), which, by Corollary 2.11, is such that the Lipschitz norm \( \|u_{\sigma}\|_{C^{0,1}(\Omega)} \) is uniformly bounded in \( 0 < \sigma < 1 \). The Arzelà–Ascoli theorem combined with Remark 2.7(a) and the uniqueness part of Theorem 2.11 now implies that, upon passing to a subsequence, we have \( u_{\sigma} \rightharpoonup u \) locally uniformly in \( \Omega \).

(Notice that the passage to a subsequence was not necessary.)

The approximate solutions \( u_{\sigma} \) are more regular: for any \( \omega \in \Omega \) and any \((\tau, p, q) \in \mathcal{P}^{-} u_{\sigma}(\omega)\) we have
\[
p^{\gamma} \tau - (p + \sigma)^{\gamma-2} q + u_{\sigma}(\omega)(1 + p^{\gamma}) \geq 0
\]
and therefore
\[
 q \leq p^{2} \tau + u_{\sigma}(\omega)((p + \sigma)^{2-\gamma} + p^{2})
\]
\[
 \leq C([u_{\sigma}]_{C^{0,1}(\bar{\Omega})}) + R \left( \sigma^{2-\gamma} + C([u_{\sigma}]_{C^{0,1}(\bar{\Omega})}) \right).
\]
Similarly, for any \( \omega \in \Omega \) and any \((\tau, p, q) \in \mathcal{P}^{+} u_{\sigma}(\omega)\) we have
\[
p^{\gamma} \tau - (p + \sigma)^{\gamma-2} q + u_{\sigma}(\omega)(1 + p^{\gamma}) \leq 0
\]
and therefore
\[
 q \geq p^{2} \tau + u_{\sigma}(\omega)((p + \sigma)^{2-\gamma} + p^{2})
\]
\[
 \geq -C([u_{\sigma}]_{C^{0,1}(\bar{\Omega})}) - R \left( \sigma^{2-\gamma} + C([u_{\sigma}]_{C^{0,1}(\bar{\Omega})}) \right).
\]

By Proposition A.2 (see also Definition A.1), we conclude that for all \( t > 0 \) (and uniformly in \( t \)) the function \( u_{\sigma}(t, \cdot, \cdot) \) is semi-concave as well as semi-convex, which implies (see Lemma A.3) the regularity \( u_{\sigma}(t, \cdot, \cdot) \in C^{1,1}(\bar{J}) \). Then, as demonstrated in Appendix A.2 the second pointwise derivative \((p) \partial_{x}^{2} u_{\sigma}\) of \( u_{\sigma} \) with respect to \( x \) exists \( \mathcal{L}^{2} \)-almost everywhere in \( \Omega \) and \( \partial_{x} u_{\sigma} \) has a weak derivative \( \partial_{x}^{2} u_{\sigma} = (p) \partial_{x}^{2} u_{\sigma} \in L^{\infty}(\Omega) \). Now we can relate the viscosity solution property to a more classical notion of solution. From the preceding observations and Rademacher’s theorem (see, e.g., [15]), it follows that \( \mathcal{P} u_{\sigma}(\omega) \) exists for \( \mathcal{L}^{2} \)-almost every \( \omega \in \Omega \) and that the function \( u_{\sigma} \) is a strong solution in the sense that the weak derivatives \( \partial_{t} u_{\sigma}, \partial_{\theta} u_{\sigma}, \partial_{x}^{2} u_{\sigma} \) exist in \( L^{\infty}(\Omega) \) and satisfy \( F_{\sigma}(u_{\sigma}, \partial_{t} u_{\sigma}, \partial_{\theta} u_{\sigma}, \partial_{x}^{2} u_{\sigma}) = 0 \) in \( L^{\infty}(\Omega) \). In particular, in view of the inequality
\[
\frac{1}{\gamma-1} |\partial_{x}((\partial_{x} u_{\sigma})^{\gamma-1})| \leq |(\partial_{x} u_{\sigma} + \sigma)^{\gamma-2} \partial_{x}^{2} u_{\sigma}|,
\]
the equation \( F_{\sigma}(u_{\sigma}) = 0 \) and the fact that \( [u_{\sigma}]_{C^{0,1}(\bar{\Omega})} \leq C([u]_{C^{0,1}(\bar{\Omega})}, R) \) yield the bound
\[
\|\partial_{x}((\partial_{x} u_{\sigma})^{\gamma-1})\|_{L^{\infty}(\Omega)} \leq C([u]_{C^{0,1}(\bar{\Omega})}, R).
\]  
(3.2)
Hence, switching to the Bochner function perspective via Fubini’s theorem, we have for any $T < \infty$

$$u_\sigma \in L^\infty(0, T; C^{1, \frac{1}{1-\gamma}}(J)), \quad \partial_t u_\sigma \in L^\infty(0, T; L^\infty(J)),$$

with norms uniformly bounded in $\sigma$ (and $T$). Thus, thanks to the Aubin–Lions lemma and the locally uniform convergence $u_\sigma \to u$, we can pass to a subsequence satisfying for $\beta \in (0, \frac{1}{1-\gamma})$ and any $T < \infty$

$$u_\sigma \to u \quad \text{in} \quad C([0, T]; C^{1, \beta}(J)).$$

In particular $\partial_x u_\sigma \to \partial_x u$ in $C_{\text{loc}}(\Omega)$, which implies that $(\partial_x u_\sigma)^{\gamma-1} \to (\partial_x u)^{\gamma-1}$ in $C_{\text{loc}}(\Omega)$. Now, the bound (3.2) yields

$$||\partial_x ((\partial_x u)^{\gamma-1})||_{L^\infty(\Omega)} \leq C([u]_{C^{0,1}(\Omega)}, R) \quad (3.3)$$

and $u \in C_b([0, \infty); C^{1,\beta}(J))$, with

$$\sup_{t \geq 0} ||u(t, \cdot)||_{C^{1,\beta}(J)} \leq C(\beta, [u]_{C^{0,1}(\Omega)}, R)$$

for $\beta \in (0, \frac{1}{1-\gamma})$.

### 3.2 The set $\Omega^+ \setminus \Omega^{++}$ is empty

Let us introduce the open sets

$$\Omega^+ := \{ \omega \in \Omega : |u(\omega)| > 0 \}$$

and

$$\Omega^{++} := \{ \omega \in \Omega : \partial_x u(\omega) > 0 \}.$$

From estimate (3.3) it follows that in any open subset $\Omega' \subset \subset \Omega^{++}$ we have $\partial^2_x u \in L^\infty(\Omega')$. Arguing as for $u_\sigma$ (see Section 3.1), it follows that $u|_{\Omega'}$ is a strong solution of a uniformly parabolic equation in $\Omega'$ (where the equality holds in $L^\infty(\Omega')$). Hence, classical regularity theory for quasilinear parabolic equations (see, e.g., [21]) implies that $u$ is smooth in $\Omega^{++}$.

Now define $\mathcal{N} := \Omega^+ \setminus \Omega^{++}$. Our goal is to show that $\mathcal{N}$ is empty. We proceed indirectly supposing that there exists a point $\omega = (t, x) \in \mathcal{N}$, where—by the symmetry of the equation—we may assume without loss of generality that $u(\omega) > 0$. From now on, we fix this particular time $t$, define $v(y) = u(t, y)$, $J' := \{x_0, x\}$, where $x_0 := \max\{y \in J : u(t, y) = 0\}$, and the non-empty set $A := J' \setminus (\Omega^{++})_t$, where $(\Omega^{++})_t := \{y \in J : (t, y) \in \Omega^{++}\}$ denotes the cross section of $\Omega^{++}$ at $t$. We call a point $y \in A$ a left-isolated point (of $A$) if there exists $\delta > 0$ such that $(y - \delta, y) \subset J' \setminus A$. Notice that in this case $(y - \delta, y) \subset (\Omega^{++})_t$, so that $v$ is smooth in $(y - \delta, y)$.

**Lemma 3.1.** Let $y \in A$. There cannot exist a sequence $x_n \to y$ with the property that for every $n$ there are $(p_n, q_n) \in J^{2,+}(u(t, \cdot))(x_n)$, where $p_n := \partial_x u(t, x_n)$, satisfying $q_n \leq 0$.

**Proof.** We argue by contradiction and assume that such a sequence $x_n \to y$ exists. Let $\delta := u(t, y) > 0$ and choose $\varepsilon > 0$ small enough such that

$$-\varepsilon^7 K + \delta/2 > 0, \quad (3.4)$$

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where $K := ||\partial_t u||_{L^\infty(\Omega)}$. Next, fix some sufficiently large $n$ such that $u(t, x_n) \geq \delta/2$, $\partial_x u(t, x_n) \leq -\varepsilon$ and choose $(p_n, q_n) \in J^{2+}(u(t, \cdot))(x_n)$ such that $q_n \leq 0$. Then there exists a function $\phi \in C^2(J)$ satisfying $u(t, \cdot) - \phi \leq u(t, x_n) - \phi(x_n) = 0$ and $\phi'(x_n) = p_n, \phi''(x_n) = q_n$. After possibly replacing $\phi$ with $\phi(y) := \phi(y) + |x_n - y|^4$, we can assume that the maximum of $u(t, \cdot) - \phi$ at $x_n$ is strict.

Now consider for some small $\delta > 0$ the function

$$w(s, y) := u(s, y) - \left(\phi(y) + \frac{1}{2\varepsilon}|s-t|^2\right) \quad \text{in} \quad Q_\delta := [t-\delta, t+\delta] \times [x_n - \delta, x_n + \delta],$$

which, by continuity, reaches its (non-negative) maximum at some point $(s_\varepsilon, y_\varepsilon)$. Notice that $s_\varepsilon \to t$ as $\varepsilon \to 0$ and, moreover, $y_\varepsilon \to x_n$. In particular, $(s_\varepsilon, y_\varepsilon) \in \text{Int}(Q_\delta)$ for small enough $\varepsilon > 0$, so that

$$(0, 0, 0) \in \mathcal{P}^+(s_\varepsilon, y_\varepsilon)$$

or, equivalently,

$$\left(\frac{s_\varepsilon - t}{\varepsilon}, \phi'(y_\varepsilon), \phi''(y_\varepsilon)\right) \in \mathcal{P}^+u(s_\varepsilon, y_\varepsilon).$$

Since $|\frac{s_\varepsilon - t}{\varepsilon}| \leq K$, there exists $\bar{\varepsilon} \in [-K, K]$ and a sequence $\varepsilon_i \to 0$ such that $\frac{s_{\varepsilon_i} - t}{\varepsilon_i} \to \bar{\varepsilon}$. Letting $i \to \infty$, we find

$$(\bar{\varepsilon}, p_n, q_n) \in \mathcal{P}^+u(t, x_n).$$

The subsolution property of $u$, the fact that $q_n \leq 0$ and the choice of $n$ now imply the inequality

$$-\varepsilon^2 K + \delta/2 \leq 0,$$

which contradicts $[3.4]$.

Thanks to Lemma $[3.1]$ we have

**Lemma 3.2.** There cannot be any left-isolated point in the set $A$.

**Proof.** We argue again by contradiction, assuming that there exists a point $y \in A$ and $\delta > 0$ such that $(y - \delta, y) \subset J' \setminus A$. Then $v'$ is strictly positive and smooth in $(y - \delta, y)$ and reaches its global minimum at the point $y$. Hence, there exists a strictly increasing sequence $(y - \delta, y) \ni \bar{x}_n \nearrow y$, $n \geq 0$, such that $(v'(\bar{x}_n))_n$ is strictly decreasing. Now for $n \geq 1$ let $y_n := \bar{x}_n$ and $h_n := \bar{x}_n - \bar{x}_{n-1} > 0$. We then have

$$v'(y_n) - v'(y_n - h_n) = v'(\bar{x}_n) - v'(\bar{x}_{n-1}) < 0$$

and thus

$$\frac{v'(y_n) - v'(y_n - h_n)}{h_n} < 0$$

for all $n \geq 1$. Since $v'$ is absolutely continuous in $(y - \delta, y)$, we then have

$$\frac{1}{h_n} \int_{y_n-h_n}^{y_n} v''(z) \, dz = \frac{v'(y_n) - v'(y_n - h_n)}{h_n} < 0.$$ 

Hence, there exists $x_n \in (y_n - h_n, y_n)$ such $q_n := v''(x_n) < 0$. In particular, letting $p_n := v'(x_n)$, we have $(p_n, q_n) \in J^2 v(x_n)$ and by construction $x_n \to y$ as $n \to \infty$. This contradicts Lemma $[3.1]$. \qed
Notice that the case $A = J'$ is impossible (for a trivial reason or as a simple consequence of Lemma 3.1). Therefore, there exists $y \in J' \setminus A$. Now let $y_1 := \min (A \cap [y, x])$, which exists since $x \in A$ and since, by the continuity of $v'$, $A$ is relatively closed in $J'$. Then $y_1 > y$, which implies that $y_1 \in A$ is left-isolated, contradicting Lemma 3.2.

We therefore conclude

$$\Omega^+ \setminus \Omega^{++} = \emptyset.$$ 

Let us round this section off with a summary of the results derived.

**Theorem 3.3.** Suppose that $\gamma \geq 2$. Given $R > 0$ and an initial datum $u_0 \in C^2([0, m])$ which is admissible for BEFP type (in the sense of Definition 2.13) and satisfies in addition $\min_{[0,m]} u_0' > 0$, the unique, $x$-monotonic viscosity solution $u$ of the bosonic Fokker–Planck equation $F(u) = 0$ (i.e. of eq. (1.5)) saturating the Cauchy–Dirichlet conditions (1.6)–(1.7) satisfies $[u]_{C^{0,1}(\Omega)} \leq C(u_0)$ (by Corollary 2.14),

$$u \in L^{\infty}(0, \infty; C^{1,\gamma - 1}(\bar{J})),$$  \hspace{1cm} (3.5)

with the estimate

$$\|\partial_x((\partial_x u)^{-1})\|_{L^{\infty}(\Omega)} \leq C([u]_{C^{0,1}(\Omega)}, R),$$  \hspace{1cm} (3.6)

and, thus, $u \in C_0([0, \infty); C^{1,\beta}(\bar{J}))$ with

$$\sup_{t \geq 0} \|u(t, \cdot)\|_{C^{1,\beta}(\bar{J})} \leq C(\beta, [u]_{C^{0,1}(\Omega)}, R)$$  \hspace{1cm} (3.7)

for $\beta \in (0, 1)$. Furthermore, $\partial_x u(t, x) > 0$ for all $(t, x) \in [0, \infty) \times [0, m] \setminus \{u = 0\}$, and away from $\{u = 0\}$ the solution $u$ is smooth and, hence, saturates the equation $F(u) = 0$ in the classical sense.

**Remark 3.4.** The specific form of the regularised equation in Section 3.1 is not important. For instance, we could have chosen $F_\sigma(z, \alpha, p, q) := F(z, \alpha, p + \sigma, q)$ instead.

**Remark 3.5.** Except for the specific regularity (3.5)–(3.7), the assertions in Theorem 3.3 are valid for the viscosity solution $u$ of the general BEFP type equation $G(u) = 0$ (subject to the same Cauchy–Dirichlet conditions) whenever $h$ satisfies the hypotheses in Theorem 2.10. Let us sketch how to argue in the general case. The family $(u_\sigma)$ of approximate solutions is constructed analogously, where one can choose, for instance, as regularised problem $G_\sigma(z, \alpha, p, q) := G(z, \alpha, p + \sigma, q)$. Of course, we cannot expect to obtain the uniform bound (3.2) (as $h$ may have rapid growth at infinity), but notice that in order to ensure compactness it is sufficient to deduce equicontinuity in $x$ of the family $(\partial_x u_\sigma)_{\sigma \in (0, 1)}$. To see the latter, define the continuous function $\kappa : [0, \infty) \rightarrow [0, \infty)$ via

$$\kappa(v) = (v^3 h(1/v))^{-1},$$

observe that $\kappa$ is strictly positive for $v > 0$, and then consider the strictly increasing function

$$K(v) = \int_0^v \kappa(s) \, ds, \quad v \geq 0,$$
given by the inverse of $\kappa \partial_x u_\sigma \partial_x^2 u_\sigma$ denoting for a uniformly continuous function $u_\sigma$, we infer that

$$\partial_{\partial_x u_\sigma}(t, \cdot)(\delta) \leq \partial_{K^{-1}}(C_2 \delta) \quad \text{for } \delta > 0.$$ 

Now compactness is obtained from the Arzelà–Ascoli theorem, so that the Aubin–Lions lemma applies as before and yields the bound

$$\left\| \frac{d}{dx} K(\partial_x u) \right\|_{L^\infty(\Omega)} \leq C([u]_{C^0(1)}(\Omega), R) =: C_2,$$

so that $K(\partial_x u_\sigma)$ is Lipschitz continuous with respect to $x$ uniformly in $\sigma$ with constant bounded from above by $C_2$. In the following we let $C_1 := C_1([u]_{C^0(1)}) + 1$ and denote the inverse of $K_{[0,C_1]} : [0, C_1] \to [0, K(C_1)]$ by $K^{-1}$. Then $\partial_x u_\sigma = K^{-1} \circ (K \circ \partial_x u_\sigma)$, and denoting for a uniformly continuous function $a$ by $\partial_a$ its modulus of continuity, we infer that

$$\partial_{\partial_x u_\sigma}(t, \cdot)(\delta) \leq \partial_{K^{-1}}(C_2 \delta) \quad \text{for } \delta > 0.$$ 

Now compactness is obtained from the Arzelà–Ascoli theorem, so that the Aubin–Lions lemma applies as before and yields the bound

$$\left\| \frac{d}{dx} K(\partial_x u) \right\|_{L^\infty(\Omega)} \leq C([u]_{C^0(1)}(\Omega), R) =: C_2$$

as well as the regularity $\partial_x u \in C(\Omega)$. Here $\frac{d}{dx} K(\partial_x u)$ denotes the weak derivative of $K(\partial_x u)$ with respect to $x$. The reasoning in Section 3.2 can then be carried out as before.

### 4 Relation to the original equation on a bounded domain

For $\gamma \geq 2$ and a fixed admissible initial datum $u_0$ satisfying $u_0 \in C^2(J)$ and $\min_J u'_0 > 0$, we henceforth denote by $u$ the unique global-in-time viscosity solution of the Cauchy–Dirichlet problem \[1.5]-[1.7] obtained in Theorem 3.3. Recall, in particular, the regularity $\partial_x u \in C([0, \infty) \times [0, m])$, and in this section we investigate the conclusions which can be drawn from our theory established at the level of $u$ for the problem in its original formulation \[1.3] on a bounded interval. Since the solution $u(t, \cdot)$ may form a flat part after some time, it can no longer be associated with a function $f(t, \cdot) \in L^1(-R, R)$ but must in general be associated with a finite measure. From Section 3.2 we know that for any $t > 0$ there exist unique $x_{+}(t) \in (0, m)$ such that $u(t, x) = 0 \iff x_{-}(t) \leq x \leq x_{+}(t)$ and $\partial_x u(t, x) > 0$ for $x \in (0, m) \setminus [x_{-}(t), x_{+}(t)]$. Let $M(t, \cdot)$ denote the generalised inverse of $u(t, \cdot)$ defined by

$$M(t, r) := \max \{x \in [0, m] : u(t, x) \leq r\}, \quad r \in [-R, R].$$

Then $M(t, \cdot)$ is right continuous and strictly increasing with $M(t, 0-) = x_{-}(t)$ and $M(t, 0) = x_{+}(t)$, and it is smooth away from $\{(t, 0) : t > 0\}$. Hence, $M(t, \cdot)$ is the cumulative distribution function of a (uniquely determined) measure $\mu(t) \in M_1^+(\{-R, R\})$ given by

$$\mu(t) = x_{+}(t) \delta_0 + L^1 \mathbf{1}_{f(t, \cdot)}, \quad t \in (0, \infty).$$

Here $x_{+}(t) := x_{+}(t) - x_{-}(t)$, and the density $f(t, \cdot)$ satisfies $f(t, u(t, x)) = 1/\partial_x u(t, x)$ for $x \in (0, m) \setminus [x_{-}(t), x_{+}(t)]$ or, equivalently, $f(t, r) = 1/\partial_x u(t, M(t, r))$ for $|r| > 0$. Observe that $\int_{-R}^R f(t, r) \, dr \leq m$ for all $t \geq 0$.

The above notation will be used for the rest of this manuscript.
4.1 Bound on blow-up profile and behaviour near singularity

Here we fix an arbitrary time \( t > 0 \). For \( x > x_+ (t) \) we let \( r = u(t, x), \tau = \partial_t u(t, M(t, r)), p = \partial_x u(t, M(t, r)) \) and \( q = \partial_x^2 u(t, M(t, r)) \). Notice that \( r, p > 0 \) and that \( \tau = \tau(r) \) defines a bounded function on \((0, R)\). We have

\[
p^{\gamma} \tau - p^{\gamma-2} q + r(1 + p^{\gamma}) = 0
\]

and thus

\[
\tau - p^{-2} q + r(p^{-\gamma} + 1) = 0. \tag{4.1}
\]

In the following the fixed time argument \( t \) will be dropped. From the identity \( f(u) = \frac{1}{\partial_x u} \), we deduce

\[
\frac{f'(u)}{f(u)} = -\frac{\partial_x^2 u}{(\partial_x u)^2},
\]

so that equation (4.1) can be rewritten as

\[
\frac{f'(r)}{f(r)} + rf(r) = -\tau(r) - r.
\]

Letting \( k(r) := f^{-\gamma}(r) \), which, by the regularity of \( u \), is well-defined, bounded and strictly positive for \( r \in (0, R) \), the last equation becomes

\[
-\frac{1}{\gamma} \frac{k'(r)}{k(r)} + rk^{-1}(r) = -\tau(r) - r,
\]

or, equivalently,

\[
k'(r) + a(r)k(r) = \gamma r, \tag{4.2}
\]

where we abbreviated \( a(r) := -\gamma(\tau(r) + r) \). Introducing

\[
q(r) = \exp \left( \int_0^r a(s) \, ds \right)
\]

the LHS of eq. (4.2) equals \( \frac{1}{q} (q \cdot k)' \), so that upon integration from \( \varepsilon > 0 \) to \( r \)

\[
(qk)(r) = (qk)(\varepsilon) + \gamma \int_{\varepsilon}^r sq(s) \, ds.
\]

Thus,

\[
k(r) = \frac{q(\varepsilon)k(\varepsilon)}{q(r)} + \frac{\gamma}{q(r)} \int_{\varepsilon}^r sq(s) \, ds,
\]

which in terms of \( f = k^{-\frac{1}{\gamma}} \) becomes

\[
f(r) = \left( \frac{q(\varepsilon)k(\varepsilon)}{q(r)} + \frac{\gamma}{q(r)} \int_{\varepsilon}^r sq(s) \, ds \right)^{-\frac{1}{\gamma}}. \tag{4.3}
\]
Spatial behaviour near singularity. Assume now that the function \( f(t, \cdot) \) is unbounded (from the right) near the origin, i.e. \( \limsup_{r \to 0} f(t, r) = \infty \). By the continuity of \( \partial_x u(t, \cdot) \), we infer \( \lim_{\varepsilon \to 0} k(\varepsilon) = 0 \) and thus

\[
f(t, r) = \left( \frac{\gamma}{q(t, r)} \int_0^r sq(t, s) \, ds \right)^{-\frac{1}{\gamma}},
\]

where \( q(t, r) = 1 - \gamma \tau(t, r) r + O(r^2) \) as \( r \to 0 \) with uniform control in \( t \). Hence,

\[
f(t, r) = \left( \frac{2}{\gamma} \right)^{\frac{1}{\gamma}} r^{-\frac{2}{\gamma}} \left( 1 + O(r) \right) \quad \text{as } r \to 0,
\]

which again holds true uniformly in \( t \) (provided \( f(t, \cdot) \) is unbounded at \( v = 0 \)).

Remark 4.1 (Bound on blow-up profile). The non-negativity of the first summand in identity (4.3) yields the estimate

\[
f(t, r) \leq \left( \frac{\gamma}{q(t, r)} \int_0^r sq(t, s) \, ds \right)^{-\frac{1}{\gamma}}.
\]

Recall that \( |\tau| \leq K := \|\partial_t u\|_{L^\infty(\Omega)} \) and \( q(r) = 1 - \gamma \tau(r) r + O(r^2) \) as \( r \to 0 \) (where \( O \) can be controlled independently of \( t \)). This implies that for any \( \delta > 0 \) there exists \( r^* = r^*(\delta, K) \) such that \( 1 - \delta \leq q(r) \leq 1 + \delta \) for all \( r \in (0, r^*) \). For \( r \in (0, r^*) \), we thus have

\[
f(t, r) \leq \left( \frac{2}{\gamma} \right)^{\frac{1}{\gamma}} r^{-\frac{2}{\gamma}} \left( \frac{1 + \delta}{1 - \delta} \right)^{\frac{1}{\gamma}}.
\]

Let us also note the rough global bound

\[
f(t, r) \leq \exp(2\gamma K) \left( \frac{2}{\gamma} \right)^{\frac{1}{\gamma}} r^{-\frac{2}{\gamma}}, \quad r \in (0, R), \ t > 0,
\]

which is a simple consequence of estimate (4.5).

Improved spatial regularity. From now on we assume that \( \gamma > 2 \). By the smoothness of \( u \) in \( \Omega^+ \), it is clear that the regularity of \( u(t, \cdot) \) in \( J \) is determined by the regularity of \( u(t, \cdot) \) at \( x = x_\pm(t) \). From identity (4.7) we observe

\[
\partial_x u = \left( \frac{\gamma}{q(u)} \int_0^u sq(s) \, ds \right)^{\frac{1}{\gamma}},
\]

from which we infer \( \partial_x u = \left( \frac{2}{\gamma} \right)^{\frac{1}{\gamma}} u^{\frac{2}{\gamma}} (1 + O(u)) \) as \( u \searrow 0 \) and, hence, as \( u \searrow x_+(t) \)

\[
u(t, x) \approx (x - x_+(t))^{\frac{\gamma}{\gamma - 2}}
\]

as well as

\[
\partial_x u(t, x) \approx (x - x_+(t))^{\frac{2}{\gamma - 2}}.
\]

Furthermore, differentiating identity (4.7) yields

\[
\partial_x^2 u = \left( \frac{\gamma}{q(u)} \int_0^u sq(s) \, ds \right)^{\frac{1}{\gamma} - 1} \partial_x u \left( u - \frac{q'(u)}{(q(u))^2} \int_0^u sq(s) \, ds \right),
\]
In particular, 

Proof.

in the classical sense.

(1.4)

fact that $p < 4$ for

identity (4.4) implies that

$T$

diction, that there exists a time

Note by

following properties hold true:

generalised inverse of $u$

means that there exists a constant $1 < C < \infty$ such that $C^{-1}A \leq B \leq CA$ holds.

In particular,

$$u(t, \cdot) \in W^{2,p}(J),$$

for $p < \frac{2}{4-\gamma}$ if $\gamma > 4$ and for $p = \infty$ if $\gamma \in (2, 4]$.

Remark 4.2. It is clear that the estimates and asymptotics established in this section have immediate analogues in the region where $0 < x < x_-(t)$. This is left as a simple exercise for the reader.

**Continuity of $x_p(t)$.** It is now easy to see that the mass concentrated at the origin depends continuously on time. Noticing that $x_p(t) = M(t, 0) - M(t, 0^-)$, we can estimate using the bound [4.0] (and its counterpart for $x < x_-(t)$)

$$|x_p(t) - x_p(s)| \leq |M(t, r) - M(t, 0)| + |M(s, r) - M(s, 0)| + |M(t, r) - M(s, r)|$$

$$+ |M(t, 0^-) - M(t, -r)| + |M(s, 0^-) - M(s, -r)| + |M(t, -r) - M(s, -r)|$$

$$\leq Cr^{1-\frac{\gamma}{2}} + |M(t, r) - M(s, r)| + |M(t, -r) - M(s, -r)|, \quad 0 < r \ll R.$$ 

Thus $\limsup_{s \to t} |x_p(t) - x_p(s)| \leq Cr^{1-\frac{\gamma}{2}}$. Since $r > 0$ can be chosen arbitrarily small, the continuity of $t \mapsto x_p(t)$ follows.

The arguments presented in this section yield the following

**Proposition 4.3.** Using the notations and assuming the hypothesis of Theorem 3.3, denote by $f$ the density of the absolutely continuous part of the measure associated with the generalised inverse of $u$ (as introduced on page 23). Then, if $\gamma > 2$, for any $t > 0$ the following properties hold true:

(i) If $f(t, \cdot)$ is unbounded near the origin, then

$$f(t, r) = \left(\frac{2}{\gamma}\right)^{\frac{1}{\gamma}}r^{-\frac{\gamma}{2}}(1 + O(r)) \quad \text{as } r \to 0.$$ 

(ii) As $x \to (x_+(t))^\pm$, $u(t, x) \approx \pm|x - x_+(t)|^{-\frac{\gamma}{2}}$, and we have the regularity $u(t, \cdot) \in W^{2,p}(J)$

for $p < \frac{2}{4-\gamma}$ if $\gamma > 4$ and for $p = \infty$ if $\gamma \in (2, 4]$.

(iii) The function $t \mapsto x_p(t) := L^1(\{u(t, \cdot) = 0\})$, denoting the size of the condensate, is continuous.

If $\gamma = 2$, the density $f(t, \cdot)$ is bounded and smooth in $(-R, R)$ for all $t \in (0, \infty)$. In particular, in this case $\min_{[0, t]} \partial_x u(t, \cdot) > 0$ for all $t > 0$, and $f$ satisfies problem (11.3) - (11.4) in the classical sense.

Proof. It remains to show the assertion concerning the case $\gamma = 2$. Assuming, by contradiction, that there exists a time $T \in (0, \infty)$ such that $f(T, \cdot)$ is unbounded near the origin, identity [1.4] implies that $f(T, r) \geq r^{-1}/2$ for small enough $r > 0$. This contradicts the fact that $\|f(T, \cdot)\|_{L^1(-R, R)} \leq m$. 

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4.2 Entropy dissipation identity

Here our goal is to establish an entropy technique which is valid globally in time. By the definition of the extended entropy, it is natural to consider the quantity

$$H(t) = \int_{-R}^{R} \left( \frac{\gamma^2}{2} f(t,r) + \Phi(f(t,r)) \right) dr.$$  

We also define a truncated version

$$H^{(\epsilon)}(t) = \int_{A_{\epsilon,R}} \left( \frac{\gamma^2}{2} f(t,r) + \Phi(f(t,r)) \right) dr,$$

where $A_{\epsilon,R} := (-R, -\epsilon) \cup (\epsilon, R)$. For fixed $\epsilon > 0$ the integrand is smooth in $A_{\epsilon,R}$ with bounded derivatives, and we may compute for $t > 0$

$$\frac{d}{dt} H^{(\epsilon)}(t) = \int_{A_{\epsilon,R}} \left( \frac{\gamma^2}{2} + \Phi'(f) \right) \partial_r f dr$$

$$= -\int_{A_{\epsilon,R}} (r + \Phi'(f) \partial_r f)(\partial_r f + rh(f)) dr + \left[ \left( \frac{\gamma^2}{2} + \Phi'(f) \right) (\partial_r f + rh(f)) \right]_{r=\epsilon}^{r=R} - \left[ \left( \frac{\gamma^2}{2} + \Phi'(f) \right) (\partial_r f + rh(f)) \right]_{r=-\epsilon}^{r=-R}$$

$$+ \left( \frac{\gamma^2}{2} + \Phi'(f(t,-\epsilon)) \right) (\partial_r f(t,-\epsilon) - \epsilon h(f(t,-\epsilon))). \quad (4.9)$$

Here, we have used the fact that at $r = \pm R$ the function $f(t, \cdot)$ satisfies the zero-flux boundary condition associated with our equation. The latter is a consequence of the fact that since $\partial_\xi u(t, m) > 0$, by parabolic regularity, the function $u(t, \cdot)$ is smooth near (and up to) the lateral boundary point $m$. Hence, the identity $\lim_{\omega \to (t,m)} F(u)|_{\omega} = 0$ combined with the fact that, by the constant Dirichlet b.c., $\partial_t u(t, m) = 0$ imply the equality $- (\partial_\xi u)^{-2} \partial_\xi^2 u + uh(1/\partial_\xi u) \partial_\xi u = 0$ at the point $(t, m)$. In terms of $f$ this means $\partial_r f + rh(f) = 0$ at $(t, R)$. The same reasoning applies to the left lateral boundary point.

Next, notice that if $f(t, \cdot)$ is unbounded near the origin, then, by identity (4.4),

$$-\Phi'(f(t, \pm \epsilon)) = \frac{1}{\gamma} \log \left( f^{-\gamma}(t, \pm \epsilon) + 1 \right) \approx f^{-\gamma}(t, \pm \epsilon) \approx \epsilon^2$$

as $\epsilon \to 0$ (where the hidden constants are independent of $t$). Hence the two summands in the last line of eq. (4.9) behave like $O(\epsilon^{1-\frac{2}{\gamma}})$ as $\epsilon \to 0$. We therefore have for $t \in (0, \infty)$

$$H^{(\epsilon)}(t) = H^{(\epsilon)}(0) - \int_{0}^{t} \int_{A_{\epsilon,R}} \frac{1}{h(f)} |\partial_r f + rh(f)|^2 dr ds + t O(\epsilon^{1-\frac{2}{\gamma}}). \quad (4.10)$$

Applying the monotone convergence theorem to the integral and the dominated convergence theorem to the terms involving $H^{(\epsilon)}$, we can pass to the limit $\epsilon \to 0$ in the identity (4.10) to obtain

$$H(t) = H(0) - \int_{0}^{t} \int_{-R}^{R} \frac{1}{h(f)} |\partial_r f + rh(f)|^2 dr ds. \quad (4.11)$$

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In particular,
\[ \int_0^\infty \int_{-R}^R \frac{1}{h(f)} |\partial_r f + r h(f)|^2 dr \, ds \leq H(0) - H_R(f_c), \tag{4.12} \]
where we have used the fact that \( H_R(f_c) > -\infty \) is the global minimum of the entropy functional \( H_R \) on \( \mathcal{M}^+_R([-R, R]) \).

4.3 Finite-time condensation and asymptotic behaviour

Thanks to identity (4.11), we can now show convergence in entropy to the minimiser of \( H_R \) among non-negative measures of the same mass. Parts of the arguments provided below are analogous to the reasoning in [8]. Let
\[ D(t) = \int_{-R}^R \frac{1}{h(f(t,r))} |\partial_r f(t,r) + r h(f(t,r))|^2 dr \]
and note that, by the bound (4.12), \( D \in L^1(0, \infty) \), which implies that there exists a sequence \( t_k \to \infty \) such that \( D(t_k) \to 0 \). By estimate (3.7), there exists \( u_\infty \) such that, after transition to a subsequence,
\[ u(t_k, \cdot) \to u_\infty \text{ in } C^{1,\beta} (\bar{J}) \]
for \( \beta \in (0, \frac{1}{\gamma-1}) \), and
\[ f(t_k, \cdot) \to f_\infty \text{ locally uniformly in } A_{0,R} \cup \{-R, R\}, \]
where \( f_\infty \) is defined via \( f_\infty(u_\infty) = \frac{1}{u_\infty} \). Letting \( f_k := f(t_k, \cdot) \) and \( h_k := \frac{1}{f_k(1+f_k^\gamma)} \), it follows that
\[ h_k \to h_\infty := \frac{1}{f_\infty^\gamma + 1} \tag{4.13} \]
locally uniformly in \( A_{0,R} \).

We next compute
\[ \gamma rh_k + \partial_r h_k = \gamma h_k \left[ r + \frac{\partial_r f_k}{f_k(1+f_k^\gamma)} \right], \]
so that, by the Cauchy–Schwarz inequality,
\[ \left( \int_{-R}^R |\gamma rh_k + \partial_r h_k| dr \right)^2 = \gamma^2 \left( \int_{-R}^R h_k \left[ r + \frac{\partial_r f_k}{f_k(1+f_k^\gamma)} \right] |dr |^2 \right) \]
\[ \leq \gamma^2 \|h_k\|_{L^1} \int_{-R}^R h_k \left[ r + \frac{\partial_r f_k}{f_k(1+f_k^\gamma)} \right] |dr |^2 \]
\[ \leq \gamma^2 \|h_k\|_{L^1} \int_{-R}^R f_k(1+f_k^\gamma) \left[ r + \frac{\partial_r f_k}{f_k(1+f_k^\gamma)} \right] |dr |^2 \]
\[ = \gamma^2 \|h_k\|_{L^1} D(f_k) \leq CD(f_k) \to 0 \quad \text{as } k \to \infty. \]

Thus, we deduce that
\[ \gamma rh_k + \partial_r h_k \to 0 \text{ in } L^1(-R, R) \quad \text{as } k \to \infty, \]

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which, thanks to (4.13), implies \( \gamma rh_\infty + \partial_r h_\infty = 0 \) in \( D'(A_{0,R}) \) and hence \( \gamma rh_\infty + \partial_r h_\infty = 0 \) almost everywhere in \( A_{0,R} \). This implies that

\[
f_\infty = f_\infty,\theta_+, \chi_{(-R<r<0)} + f_\infty,\theta_-, \chi_{(0<r<R)}
\]

for certain \( \theta_\pm \geq 0 \). Notice that the assumption \( \theta_+ \neq \theta_- \) contradicts the regularity \( u'_\infty \in C((0,m)) \). Hence \( \theta_+ = \theta_- \), and for the same reason, we then conclude that \( \theta_+ = \theta_- = \theta_{m,R} \) and thus

\[
f_\infty = f_\infty,\theta_{m,R}, u_\infty = u_{\theta_{m,R}},
\]

where \( u_{\theta_{m,R}} \) should be replaced by \( u_c \) if \( \theta_{m,R} = 0 \). By the dominated convergence theorem, we now have \( H(t_k) \rightarrow H_R(f_\infty) \), which, combined with the monotonicity of \( H = H(t) \), implies that

\[
\lim_{t \to \infty} H(t) = H_R(f_\infty).
\]

We also have convergence at the level of \( u \):

**Theorem 4.4** (Long-time asymptotics). Under the hypotheses of Theorem 3.3, the corresponding viscosity solution \( u \) satisfies

\[
\lim_{t \to \infty} \|u(t,\cdot) - u_{\theta_{m,R}}\|_{C(J)} = 0, \tag{4.14}
\]

while the associated point mass at the origin converges to the excess mass, i.e.

\[
x_p(t) \rightarrow m - \|f_{\theta_{m,R}}\|_{L^1(-R,R)} \quad \text{as } t \to \infty.
\]

**Corollary 4.5** (No Condensate after finite time). Under the hypotheses of Theorem 3.3, the following holds true:

- If \( m > m_c \), there exists \( T < \infty \) such that \( x_p(t) > 0 \) for all \( t > T \).
- If \( m < m_c \), there exists \( T < \infty \) such that \( \min_{[0,m]} \partial_x u(t,\cdot) > 0 \) for all \( t > T \). In particular, the condensed component is compactly supported, i.e. \( \text{supp } x \subset \subset (0,\infty) \), and the density \( f(t,\cdot) \) is smooth for all \( t > T \).

**Proof of Corollary 4.5** The assertion concerning the case \( m > m_c \) follows from the fact that \( x_p(t) \rightarrow m - m_c > 0 \) as \( t \to \infty \), which is implied by Theorem 4.4. Let us now assume that \( m < m_c \). Recall that identity (4.13) holds true uniformly in time whenever the density is unbounded near the origin. This implies that

\[
\|u(t,\cdot) - u_{\theta_{m,R}}\|_{C([0,m])} \geq c(\theta_{m,R}) > 0
\]

whenever \( \min_{[0,m]} \partial_x u(t,\cdot) = 0 \). (Here the constant \( c(\theta_{m,R}) \) depends, of course, also on the initial datum \( u_0 \).) Hence, the assertion follows from Theorem 4.4.

**Proof of Theorem 4.4** We first prove the convergence (4.14): for an arbitrary time sequence \( s_n \to \infty \) we want to show that \( \lim_{n \to \infty} \|u(s_n,\cdot) - u_{\theta_{m,R}}\|_{C(J)} = 0 \). By the global Lipschitz continuity of \( u \) (in time), we can assume without loss of generality that \( |s_n - s_{n+1}| \geq 2 \). We now let \( I_n = \{|t-s_n| \leq \frac{1}{n}\} \). Then, since \( D \in L^1(0,\infty) \), there exists a

\[\text{If } \theta_{m,R} = 0, u_{\theta_{m,R}} \text{ should be replaced by } u_c.\]
sequence \(n_k\) and \(t_k \in I_{n_k}\) such that \(D(t_k) \to 0\). Now the reasoning preceding Theorem 4.4 shows that after passing to a subsequence,

\[ u(t_k, \cdot) \to u_{\theta_{m,R}} \text{ uniformly in } \bar{J}. \]

Finally notice that for \(K := \|\partial_t u\|_{L^\infty(\Omega)}\) we have

\[ |u(s_{n_k}, x) - u_{\theta_{m,R}}(x)| \leq K |s_{n_k} - t_k| + |u(t_k, x) - u_{\theta_{m,R}}(x)|. \]

Thus the (arbitrary) sequence \((s_{n_k})\) has a subsequence \((s_{n_k})\) such that \(u(s_{n_k}, \cdot) \to u_{\theta_{m,R}}\) uniformly in \(\bar{J}\). This implies (4.11).

To show the convergence of the point mass, we argue by contradiction, assuming that there exists \(\varepsilon > 0\) and a sequence \(t_n \to \infty\) such that \(|m - m_{\theta_{m,R}} - x_p(t_n)| \geq \varepsilon\) for all \(n\) where \(m_{\theta_{m,R}} := \|f_{\theta_{m,R}}\|_{L^1(-R,R)}\). In view of the convergence (4.11) there cannot be a subsequence \((t_{n_i})\) satisfying \(m - m_{\theta_{m,R}} + \varepsilon \leq x_p(t_{n_i})\) for all \(i\). Thus \(x_p(t_n) \leq m - m_{\theta_{m,R}} - \varepsilon\) for all \(n\) large enough (of course, this is only possible if \(\theta_{m,R} = 1\)). But then, by formula (4.4) (see also (4.8)), there exists \(\delta > 0\) such that \(\max\{|u(t_n, m_{\theta_{m,R}}/2)|, |u(t_n, m - m_{\theta_{m,R}}/2)|\} \geq \delta\) for all such \(n\), thus contradicting the limit (4.11).

\[ \square \]

5 Extension to whole space

In this section we are concerned with the problem posed on the real line, i.e. equation (1.2) with \(d = 1\), where we assume that \(\gamma \geq 2\).

5.1 Uniqueness

Here we extend the comparison principle (Prop. 2.8) for \(x\)-monotonic functions to the case of unbounded (but continuous) functions.

**Proposition 5.1 (Comparison).** Let \(0 < T < \infty\) and assume that the continuous function \(G\) satisfies the hypothesis (A0), (A1). Suppose that \(u \in C([0,T] \times (0,m))\) is an \(x\)-monotonic subsolution, \(v \in C([0,T] \times (0,m))\) an \(x\)-monotonic supersolution of \(\mathcal{G} = 0\) in \(\Omega = (0,T) \times (0,m)\) with the boundary behaviour \(-\infty \leq \lim_{x \to 0} u(t,x) \leq \lim_{x \to 0} v(t,x), \lim_{x \to m} u(t,x) \leq \lim_{x \to m} v(t,x) \leq \infty\) for all \(t \in [0,T]\). If, in addition, \(u(0, \cdot) \leq v(0, \cdot)\) in \((0,m), then \(u \leq v\) in \(\Omega\).

**Proof.** For functions \(w = w(t,x)\) and \(0 < \delta \ll 1\) denote by \(w^{\pm\delta}(t,x)\) the spatially shifted function \(w(t,x \pm \delta)\). The same notation will be used for time-independent functions. We further abbreviate \(\Omega^\delta := (0,T) \times (\delta,m - \delta)\). Then \(u^\delta\) (resp. \(v^\delta\)) is a viscosity subsolution (resp. supersolution) of \(\mathcal{G} = 0\) in \(\Omega^\delta\), which is bounded from above (resp. from below), \(u^\delta(0, \cdot) \leq v^\delta(0, \cdot)\) in \((\delta,m - \delta)\) and \(\lim_{x \to \delta} u^\delta(t,x) \leq \lim_{x \to \delta} v^\delta(t,x)\) as well as \(\lim_{x \to m - \delta} u^\delta(t,x) \leq \lim_{x \to m - \delta} v^\delta(t,x)\). Thus, Proposition 2.8 implies that \(u^\delta \leq v^\delta\) in \(\Omega^\delta\). The assertion now follows from the continuity of \(u\) and \(v\). \(\square\)

Notice that Proposition 5.1 implies uniqueness for unbounded \(x\)-monotonic viscosity solutions of \(\mathcal{G} = 0\) in \((0, \infty) \times (0,m)\). The construction of such solutions in the bosonic Fokker–Planck case \(G := F\) is our next goal.
5.2 Existence, regularity and long-time behaviour

Here we will show that, under suitable assumptions on the initial datum $u_0$, a global-in-time viscosity solution to the equation corresponding to the problem for the density $f$ posed on the entire line $\mathbb{R}$ (i.e. eq. (1.2) with $\delta = 1$), can be constructed. This solution satisfies an entropy dissipation identity analogous to formula (4.11), and its asymptotic behaviour is similar to the one outlined in Section 4.3. In particular, in the mass-supercritical case, a condensate forms after finite time.

**Assumptions on initial value.** We assume that the initial datum $u_0 \in C^2((0, m))$ satisfies $\inf_{(0, m)} u_0' > 0$ and $\lim_{x \to 0^+} u_0(x) = -\infty$, $\lim_{x \to m^-} u_0(x) = \infty$. These hypotheses are not very restrictive in the sense that for small positive times the pseudo-inverse of the distribution function of a local-in-time mild solution of equation (1.2) has these properties. Furthermore, we assume that the density $f_0$ associated with the inverse of $u_0$ satisfies

$$f_0 \geq f_{\infty, \theta} \quad \text{in } \mathbb{R}$$

for some (possibly large) $\theta > 0$, and that there exists $\varepsilon_0 > 0$ such that the function $r \mapsto |r|^{1+\varepsilon_0} f_0(r)$ lies in $L^\infty(\mathbb{R})$ and such that

$$\int_{\mathbb{R}} |r|^{2+\varepsilon_0} f_0(r) \, dr < \infty.$$  \tag{5.2}

The moment bound (5.2) will be needed for entropy computations on the whole line, while the preceding hypotheses appear to be mostly technical. Under these assumptions, for any $R \geq 1$ there exist unique points $a_R$ and $b_R$ satisfying $u_0(a_R) = -R$ and $u_0(b_R) = R$. Abbreviating $J_R := (a_R, b_R)$ and $\Omega_R := (0, \infty) \times J_R$, we denote by $u(R)$ the unique $x$-monotonic viscosity solution of $F = 0$ in $\Omega_R$ subject to the conditions $u(R)(0, \cdot) = u_0|_{J_R}$, $u(R)(t, a_R) = -R$, $u(R)(t, b_R) = R$.

We will now show that, under the above assumptions, one can construct a locally Lipschitz continuous viscosity solution $u$ of eq. (1.3) in $\Omega := (0, \infty) \times (0, m)$ satisfying $\lim_{t \searrow 0} u(t, \cdot) = u_0$ in $C_{\text{loc}}((0, m))$. A fundamental ingredient in the construction is

**Lemma 5.2.** For any $R \geq 1$ there exists $c_R < \infty$ such that for all $\bar{R} \geq R$ and all $t \geq 0$

$$\sup_{x \in J_R} |u(R)(t, x)| \leq c_R.$$

Lemma 5.2 is an immediate consequence of

**Lemma 5.3.** For all $R \geq 1$

$$\sup_{t>0} \|u(R)(t, \cdot)\|_{L^2(J_R)} \leq 2 \max\{m, \|u_0\|_{L^2}\}.$$

**Proof of Lemma 5.3.** The idea of the proof is to exploit the fact that the (bosonic) Fokker–Planck type equations on $\mathbb{R}$ propagate moments. First, observe that since $u(R)(t, \cdot) \in C^1(J_R)$ with $\partial_x u(R)(t, x) > 0$ for $x \in \{u(R)(t, \cdot) > 0\}$ and $u(R)(t, a_R) = -R$, $u(R)(t, b_R) = R$, we have

$$\int_{J_R} |u(R)(t, x)|^2 \, dx = \int_{\{|u(R)(t, \cdot)| > 0\}} |u(R)(t, x)|^2 \, dx = \int_{-R}^R r^2 f(R)(t, r) \, dr,$$

where $f(R)$ is the density associated with the generalised inverse of $u(R)$. Thus, in order to prove the assertion, it suffices to show that the RHS is uniformly bounded in $R \geq 1$ and
\( t \in [0, \infty) \). Abbreviating \( E_R(t) := \int_{-R}^R r^2 f^{(R)}(t, r) \, dr \) and \( E^{(c)}_R(t) := \int_{A_{c, R}} r^2 f^{(R)}(t, r) \, dr \), we compute

\[
\frac{d}{dt} E^{(c)}_R(t) = \int_{A_{c, R}} r^2 \partial_r(\partial_r f^{(R)} + r f^{(R)}((f^{(R)})^\gamma + 1)) \, dr \\
= -2 \int_{A_{c, R}} r(\partial_r f^{(R)} + r f^{(R)}((f^{(R)})^\gamma + 1)) \, dr + O(\varepsilon^{1 - \frac{\gamma}{2}}) \\
\leq 2\|f^{(R)}(t, \cdot)\|_{L^1(A_{c, R})} - E^{(c)}_R(t) + O(\varepsilon^{1 - \frac{\gamma}{2}}).
\]

In the last step, we have used the fact that the boundary terms at \( r = \pm R \) which arise upon integration by parts are non-negative. Integration in time yields the estimate

\[
E^{(c)}_R(t) \leq E^{(c)}_R(0) + \int_0^t (2\|f^{(R)}(s, \cdot)\|_{L^1(A_{c, R})} - E^{(c)}_R(s) + O(\varepsilon^{1 - \frac{\gamma}{2}})) \, ds,
\]

from which we deduce

\[
E_R(t) \leq E_R(0) + \int_0^t (2m - E_R(s)) \, ds.
\]

Using the fact that \( f^{(R)}(0, \cdot) = f_0 \) on \((-R, R)\), we conclude

\[
E_R(t) \leq \max\{2m, E_0\},
\]

where \( E_0 := \int_{-R}^R r^2 f_0(r) \, dr < \infty \) by assumption \((5.2)\). \( \square \)

We next assert that for any \((\text{large enough})\) \( R \geq 1 \)

\[
K_R := \sup_{R \geq R} \|u^{(R)}\|_{C^{0,1}(\Omega_R)} < \infty. \tag{5.3}
\]

This is will be shown in the following. Defining \( \tilde{f}_0(r) = \max_{\sigma \in \{\pm 1\}} f_0(\sigma r) \) and recalling that, by assumption, \( \sup_{r \in \mathbb{R}} |r|^{1+\varepsilon} \tilde{f}_0(r) < \infty \), we find

\[
f_0(r) \leq \tilde{f}_0(r) \leq C(1 + |r|^2)^{-\frac{1+\varepsilon}{2}} =: \tilde{f}_0(r), \quad r \in \mathbb{R}.
\]

Notice that \( \tilde{f}_0 \) is even, non-increasing in \( |r| \), and, moreover, \( \tilde{f}_0 \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R}) \). Furthermore, an elementary argument shows that for large enough \( C \) and small enough \( \varepsilon_0 > 0 \) the function \( \tilde{f}_0 \) has exactly one intersection with \( f_\varepsilon \) in \( \mathbb{R}^+ \). For \( R \geq 1 \) consider the solutions \( \tilde{u}^{(R)} \) and \( u^{(R)} \) emanating from the inverse distribution functions of \( \tilde{f}_0|_{(-R, R)} \) and \( f_0|_{(-R, R)} \) and the corresponding densities \( \tilde{f}^{(R)} \) and \( f^{(R)} \) on \((0, \infty) \times (-R, R)\). Then, for any \( t \geq 0 \), the function \( \tilde{f}^{(R)}(t, \cdot) \) is non-increasing in \( |\cdot| \) and if there is a first time \( T < \infty \) such that \( \tilde{f}^{(R)} \) is singular at the origin \( r = 0 \), then there is a first time \( T < \infty \) such that \( \tilde{f}^{(R)}(t, \cdot) > f_\varepsilon \) in \((0, R)\) and by a comparison argument \( \tilde{f}^{(R)}(t, \cdot) > f_\varepsilon \) in \((0, R)\) for all \( t \geq T \). (In order to deal with the singularity, one can compare \( \tilde{u}^{(R)} \) and a suitable shift of \( u_\varepsilon \) near \( x = \frac{C}{\varepsilon} \). For the non-standard boundary conditions one can use a boundary point lemma such as [22, Lemma 2.6].) Then, again by comparison, it follows that \( f^{(R)}(t, \cdot) \leq \tilde{f}^{(R)}(t, \cdot) \) for any \( t \geq 0 \). Hence

\[
f^{(R)}(t, r) \leq \tilde{f}^{(R)}(t, r) \leq \frac{\tilde{m}}{|r|^{1/2}} \quad \text{for } t \geq 0, \ r \in (-R, R) \setminus \{0\}, \quad \text{where } \tilde{m} := \|\tilde{f}_0\|_{L^1(\mathbb{R})}, \quad \text{and thus}
\]

\[
\partial_r u^{(R)} \geq \frac{2|u^{(R)}|}{\tilde{m}}. \tag{5.4}
\]

\footnote{Comparison remains true even if \( \tilde{f}^{(R)}(t, \cdot) \) and \( f^{(R)}(t, \cdot) \) are singular at the origin. This follows from formula \((4.4)\) and the fact that \( \tilde{f}^{(R)}(t, \cdot) > f_\varepsilon \) in \((0, R)\) for \( t \geq T \), which precludes a “blow-down” of \( \tilde{f}^{(R)} \).}
As a side note, we remark that, in view of the bound \( f_0 \geq f_{\infty, \theta} \), a similar comparison argument combined with Lemma 5.3 yields the estimate
\[
\| \partial_x u(\tilde{R})(t, \cdot) \|_{L^\infty(a_\tilde{R}, b_\tilde{R})} \leq (f_{\infty, \theta}(c_\tilde{R}))^{-1} > 0
\]
for any \( t \geq 0 \) and any \( \tilde{R} \geq R \).

Using the steady states, it is easy to see that there exist functions \( u_\pm : (0, m) \to [-\infty, \infty] \) which act as upper resp. lower barrier, such that \( u_-(x) \leq u_0(x) \leq u_+(x) \) for all \( x \in (0, m) \) and such that there exists \( \delta > 0 \) such that \( \partial_x u_\pm < 0 \) in \( (0, \delta) \) and \( u_- > 0 \) in \( (m-\delta, m) \). Owing to bound (5.4), we infer the existence of \( \tilde{R} > 0 \) (large) and \( c_1 > 0 \) (small) such that for any \( R \geq \tilde{R} \) the inequality \( \partial_x u(R) \geq c_1 > 0 \) holds true in \( (a_\tilde{R}, a_R) \cup (b_R, b_\tilde{R}) \).

Now, for \( R \) sufficiently large we can apply classical parabolic estimates [see Theorem V.5.1] to the equation for \( u(\tilde{R}) \), \( \tilde{R} \geq R + 1 \), in \( (0, \infty) \times I_{\eta,R} \), where for \( 0 < \eta \ll 1 \) we denote \( I_{\eta,R} := (a_\tilde{R}, a_\tilde{R} + \eta) \cup (b_\tilde{R} - \eta, b_\tilde{R}) \) and for \( \varepsilon > 0 \) small \( I_{\eta,R,\varepsilon} := \{ x \in (0,m) : \text{dist}(x, I_{\eta,R}) < \varepsilon \} \). In particular one has the bound
\[
\| \partial_t u(\tilde{R}) \|_{L^\infty((0,\infty) \times I_{\eta,R})} \leq C \left( \epsilon, R, \| u(\tilde{R}) \|_{L^\infty((0,\infty) \times I_{\eta,R,\varepsilon})}, \| u_0 \|_{C^2(I_{\eta,R,\varepsilon})}, c_1, \theta \right)
\]
for any \( \tilde{R} > R + 1 \). From Lemma 5.2 we thus infer \( \| \partial_t u(\tilde{R}) \|_{L^\infty((0,\infty) \times I_{\eta,R,\varepsilon})} \leq C(R) \), and, arguing as in Propositions 2.12 and 2.13 we deduce estimate (5.3).

The bound (5.3) and the fact that each \( u(\tilde{R}) \) satisfies the equation imply the estimate
\[
\sup_{R > R_0} \| \partial_x ((\partial_x u(\tilde{R}))^{\gamma-1}) \|_{L^\infty(\Omega_R)} \leq C(R).
\]

Now we argue as in Section 3.1 to find \( \gamma_0 > 0 \), a function
\[
u \in C([0, \infty); C^{1,\beta_0}_{\text{loc}}((0, m)) \cap C^{0,1}_{\text{loc}}(\Omega) \times (0, m))
\]
and a sequence \( \tilde{R} \to \infty \) such that
\[
u(\tilde{R}) \tilde{R} \to \infty \nu \in C([0, T]; C^{1,\beta_0}(J_\tilde{R}))
\]
for any \( T > 0 \). By Remark 2.7(a) the limit \( u \) is itself a viscosity solution of equation (2.1), and, by construction, \( u(t, 0) = u_0 \). Owing to Footnote we have \( \lim_{x \to 0^+} u(t, x) = -\infty \), \( \lim_{x \to 0^-} u(t, x) = \infty \) for all \( t \geq 0 \).

Furthermore, results analogous to those derived in Sections 3.2 and 4.1 hold true. In particular, the density \( f \) corresponding to \( u \) is smooth away from the origin. Finally, the entropy dissipation identity is a consequence of the following uniform control of moments combined with assumption (5.2).

**Remark** (Higher moment bounds). Lemma 5.3 can easily be generalised to \( L^p \) for \( p \in [2, \infty) \) as long as
\[
\| u_0 \|_{L^p(0, m)}^p = \int_R |r|^p f_0(r) \, dr < \infty.
\]

Below we provide the formal argument, which can be made rigorous by following the proof of Lemma 5.3. The main reasoning given below is similar as in [7]. We let \( E_{p,R}(t) :=\)

\[\text{(5.5)}\]
\[ f_{(-R,R)} |r|^p f^{(R)}(t, r) \, dr, \quad p \geq 0, \] and compute for \( p \geq 2 \)

\[
\frac{d}{dt} E_{p,R}(t) = \int_{(-R,R)} |r|^p \partial_r (\partial_r f(t, r) + rf(1 + f^\gamma)) \, dr \\
= -p \int_{(-R,R)} |r|^{p-2} \partial_r f(t, r) + |r|^p f(1 + f^\gamma) \, dr \\
\leq p(p-1) \int_{(-R,R)} |r|^{p-2} f(t, r) \, dr - p \int_{(-R,R)} |r|^p f(1 + f^\gamma) \, dr;
\]

from which we deduce

\[
\frac{1}{p} \frac{d}{dt} E_{p,R}(t) \leq (p-1)E_{p-2,R}(t) - E_{p,R}(t). \quad (5.6)
\]

Defining \( K_0 = m \) and then for \( p \in 2\mathbb{N}^+ \) recursively \( K_p = \max \left\{ (p-1)K_{p-2}, \|u_0\|_{L^p[0,m)} \right\} \), we inductively obtain for all \( p \in 2\mathbb{N} \) satisfying condition (5.5)

\[
\sup_{R \geq 1} E_{p,R}(t) \leq K_p \quad \text{for all } t \geq 0.
\]

For general real \( q \geq 2 \) we use interpolation: if condition (5.5) is satisfied for \( p = q \) then

\[
\sup_{R \geq 1} E_{q-2,R}(t) \leq C(K_{2[q/2]}, m) \quad \text{for all } t \geq 0,
\]

and using once more (5.6) with \( p = q \), we find that

\[
\sup_{R \geq 1} E_{q,R}(t) \leq C(K_{2[q/2]}, m, q, \|u_0\|_{L^q[0,m)}) \quad \text{for all } t \geq 0.
\]

The long-time asymptotics and finite-time condensation for \( m > m_c \) can now be deduced similarly as in Section 4.3 where the uniform convergence (4.14) holds true locally, i.e. for any \( J' \subset J \)

\[
\lim_{t \to \infty} \|u(t, \cdot) - u_{\theta_{m,R}}\|_{C(J')} = 0. \quad (5.7)
\]

Moreover, in view of the uniform integrability of \( \{|u(t, \cdot)|^p\}_t \) for \( p < 2 + \varepsilon_0 \), we also have

\[
\lim_{t \to \infty} \|u(t, \cdot) - u_{\theta_{m,R}}\|_{L^p(J')} = 0 \quad \text{for } p < 2 + \varepsilon_0. \quad (5.8)
\]

If \( \theta_{m,R} = 0 \), \( u_{\theta_{m,R}} \) should be replaced by \( u_c \) in the identities (5.7) and (5.8).

We conclude by summarising the main results of this section in the following

**Theorem 5.4.** Let \( \gamma \geq 2 \) and suppose that the initial value \( u_0 \in C^2((0, m)) \) satisfies the assumptions formulated in the paragraph on page 33. Then there exists a unique \( x \)-monotonic viscosity solution \( u \) of \( \mathcal{F}(u) = 0 \) in \( (0, \infty) \times (0, m) \) such that \( \lim_{x \to 0^+} u(t, x) = -\infty \), \( \lim_{x \to m^-} u(t, x) = \infty \) for all \( t \geq 0 \). This solution has the following properties:

(i) Regularity: \( \partial_x u(t, x) > 0 \) for all \( (t, x) \in [0, \infty) \times (0, m) \setminus \{u = 0\} \), and away from \( \{u = 0\} \) the solution \( u \) is smooth and, thus, saturates the equation \( \mathcal{F}(u) = 0 \) in the classical sense.
(ii) Relation to original formulation and spatial blow-up profile: the integrable, strictly positive function \( f = 1/(\partial_x u \circ M) \) is smooth away from \( r = 0 \), and for all \( t \) satisfying \( \limsup_{|r| \to 0^+} f(t, r) = \infty \), we have
\[
f(t, r) = \left( \frac{2}{\gamma} \right)^{\frac{1}{2}} r^{-\frac{2}{\gamma}} (1 + O(r)) \quad \text{as } r \to 0,
\]
uniformly in \( t \). As a consequence, if \( \gamma = 2 \), the density \( f(t, \cdot) \) is bounded and smooth for all \( t \in (0, \infty) \), and \( f \) satisfies problem \([13] - [14]\) in the classical sense.

(iii) Entropy dissipation: for all \( t > 0 \)
\[
\mathcal{H}(f(t, \cdot)) = \mathcal{H}(f(0, \cdot)) - \int_0^t \int_{\mathbb{R}} \frac{1}{h(f)} |\partial_x f + rh(f)|^2 dr ds.
\]
(See Remark \([2.20]\) for the definition and properties of \( \mathcal{H} \).)

(iv) Long-time asymptotics and finite-time condensation: we have
\[
\lim_{t \to \infty} \mathcal{H}(f(t, \cdot)) = \inf \mathcal{H}(g),
\]
where the infimum is taken over all non-negative, integrable functions \( g \) on \( \mathbb{R} \) satisfying \( \int g = m. \) (Recall that the infimum is finite and coincides with \( \mathcal{H}(L^1 U_\gamma f_c + (m - m_c)\delta_0) \) (which equals \( \mathcal{H}(f_c) \)) if \( m \geq m_c \) resp. with \( \mathcal{H}(f_{\infty, \theta}) \) if \( m = m_{\theta} := \int f_{\infty, \theta} \).
Furthermore, the solution \( u \) satisfies the identities \([5.7]\) and \([5.8]\), while the function \( t \mapsto x_p(t) := L^1 \{ u(t, \cdot) = 0 \} \) is continuous and converges to \( m - m_c \) if \( m \geq m_c \) resp. to \( m - m_{\theta} \) if \( m = m_{\theta} \).

\[\text{A} \quad \text{Appendix}\]

A.1 Semi-convexity

Definition A.1 (Semi-convexity and -concavity). Let \( U \subset \mathbb{R}^d \) be convex. A function \( v : U \to \mathbb{R} \) is called semi-convex (resp. semi-concave) if there exists a constant \( C \in \mathbb{R} \) such that the function \( x \mapsto v(x) + \frac{C}{2} |x|^2 \) is convex (resp. such that \( v(x) - \frac{C}{2} |x|^2 \) is concave).

Remark. By Aleksandrov’s theorem on the twofold differentiability almost everywhere of convex functions, if \( v : J \to \mathbb{R} \) is semi-convex (or semi-concave), then it is twice differentiable \( L^1 \)-almost everywhere. See [15] Theorem 6.9] for a measure-theoretic proof of Aleksandrov’s theorem or [9] Appendix] for a proof based on convex analysis.

Proposition A.2. Let \( u : \Omega \to \mathbb{R} \) be continuous. Suppose that there exists a constant \( C < \infty \) such that for all \( \omega \in \Omega \) for all \( (\tau, p, q) \in \mathcal{P}^u u(\omega) \) the bound \( q \geq -C \) (resp. \( q \leq C \)) holds true. Then, for all \( t > 0 \) the function \( u(t, \cdot) \) is semi-convex (semi-concave) in \( J \) with constant bounded from above by \( C \).

Proof. By symmetry, it suffices to prove the statement asserting semi-convexity. Thanks to [11] Lemma 1], it is enough to show that for all \( t \in (0, \infty) \) and all \( x \in J \)
\[
(p, q) \in \mathcal{J}^{2,+}(u(t, \cdot))(x) \Rightarrow q \geq -C. \tag{A.1}
\]
The implication \( \text{(A.1)} \) is a consequence of the following general argument. A similar reasoning can be found in [17].
In order to see implication \( A.1 \), we fix \( t \in (0, \infty) \) and \( x \in J \) and assume that \((p, q) \in \mathcal{F}^{2,+}(u(t, \cdot))(x)\). By definition (and the local boundedness of \( u \)), there exists \( \phi \in C^2(J) \) such that \( 0 \geq u(t, y) - \phi(y), 0 = u(t, x) - \phi(x) \) and \( p = \phi'(x), q = \phi''(x) \). In particular, \( u(t, \cdot) - \phi \) reaches a maximum at \( x \). After possibly replacing \( \phi \) with \( \phi(y) + |x - y|^4 \), we may assume that the maximum is strict. Now consider for suitably small \( 0 < \delta \ll 1 \) the function

\[
 w(s, y) := u(s, y) - \left( \phi(y) + \frac{1}{2\varepsilon}|s-t|^2 \right) \quad \text{in } Q_\delta := [t-\delta, t+\delta] \times [x-\delta, x+\delta].
\]

By continuity, \( w \) reaches its (non-negative) maximum at some point \( (s_\varepsilon, y_\varepsilon) \in Q_\delta \) and as \( \varepsilon \to 0 \) we must have \( s_\varepsilon \to t \). Moreover, \( y_\varepsilon \to x \) since if this was not the case, then along a subsequence \( (s_\varepsilon, y_\varepsilon) \to (t, \bar{x}) \) for some \( \bar{x} \neq x \) and therefore \( 0 \leq w(s_\varepsilon, y_\varepsilon) \leq u(s_\varepsilon, y_\varepsilon) - \phi(y_\varepsilon) \to u(t, \bar{x}) - \phi(\bar{x}) < 0 \) by the strictness of the maximum, a contradiction.

Hence for small enough \( \varepsilon > 0 \)

\[
(0, 0, 0) \in \mathcal{P}^+ w(s_\varepsilon, y_\varepsilon)
\]
or, equivalently,

\[
\left( \frac{s_\varepsilon - t}{\varepsilon}, \phi'(y_\varepsilon), \phi''(y_\varepsilon) \right) \in \mathcal{P}^+ u(s_\varepsilon, y_\varepsilon).
\]

Hence \( \phi''(y_\varepsilon) \geq -C \) and, letting \( \varepsilon \to 0 \), we conclude

\[
q = \phi''(x) \geq -C.
\]

\[ \square \]

**Lemma A.3.** Suppose the function \( v : J \to \mathbb{R} \) is semi-convex and semi-concave with constant \( C < \infty \). Then \( v \in C^{1,1}(\bar{J}) \) and \( |v'|_{C^{0,1}(\bar{J})} \leq C \).

**Proof.** The fact that \( v \) is semi-convex and semi-concave implies that \( v \) is differentiable at every point (since the first order sub- and superdifferential exist everywhere). Thus, since \( v(x) + \frac{C}{2}|x|^2 \) is convex and \( v(x) - \frac{C'}{2}|x|^2 \) concave, we deduce \( v'(x) + Cx \leq v'(y) + Cy \) and \( v'(x) - Cy \geq v'(y) - Cx \) whenever \( x \leq y \). In combination, this yields

\[
|v'(x) - v'(y)| \leq C|x - y|.
\]

\[ \square \]

### A.2 \( L^2 \)-measurability

**Lemma A.4.** Using the notation from Section \( \S 1 \), the second order pointwise derivative \( \partial_x^2 u_\sigma \) of \( u_\sigma \) with respect to \( x \) exists \( L^2 \)-almost everywhere in \( \Omega \) and the function \( \partial_x u_\sigma \) has a weak derivative in \( x \)-direction satisfying

\[
\partial_x^2 u_\sigma = (p)\partial_x^2 u_\sigma \text{ in } L^\infty(\Omega).
\]

**Proof.** Throughout the proof we abbreviate \( u := u_\sigma \). Recall that for fixed time this function is semi-convex, semi-concave (uniformly in \( t \)) and, thus, by Lemma \( A.3 \) of the class \( C^{1,1}(\bar{J}) \) (uniformly in \( t \)). For any \( t > 0 \) we denote by \( N_t \) the subset of points in \( J \) where the second pointwise derivative of \( u(t, \cdot) \) does not exist. Then the set \( N_t \) is an \( L^1 \)-null set, and our goal is to show that the set \( \bigcup_t \{t\} \times N_t \subset \Omega \) is \( L^2 \)-measurable.

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We choose \( C \) large enough such that the function \( \tilde{u}(t, x) = u(t, x) + \frac{C}{1} |x|^2 \) is convex for all \( t \) and define \( v(t, x) := \partial_x \tilde{u}(t, x) \). Then \( v(t, \cdot) \) is non-decreasing and \( v(t, \cdot) \in C^{0,1}(\mathcal{J}) \). Moreover, \( v \) lies in \( L^\infty(\Omega) \) and is thus \( \mathcal{L}^2 \)-measurable. Now define

\[
\overline{\partial v} := \limsup_{h \to 0} \partial^h v
\]

and

\[
\underline{\partial v} := \liminf_{h \to 0} \partial^h v,
\]

where the function \( \partial^h v(t, x) := \frac{v(t,x+h) - v(t,x)}{h} \) is bounded. In view of the monotonicity and the continuity of \( v(t, \cdot) \), it is clear that in taking the \( \limsup \) resp. the \( \liminf \) one can restrict to \( h = \frac{1}{n}, \ n \in \mathbb{Z} \). Since \( w_n := \partial^1_{\infty} v \) is \( \mathcal{L}^2 \)-measurable, the pointwise \( \limsup \) resp. \( \liminf \) of this countable family \( \{w_n\} \) must itself be \( \mathcal{L}^2 \)-measurable. Therefore the set

\[
G := \{ \omega \in \Omega : \overline{\partial v}(\omega) - \underline{\partial v}(\omega) = 0 \},
\]

which is exactly the set where \( (p)\partial^2_x u \) exists, is \( \mathcal{L}^2 \)-measurable. Hence its complement \( \Omega \setminus G = \bigcup_i \{(t) \times N_i \} \) is \( \mathcal{L}^2 \)-measurable and thus, by Fubini’s theorem, an \( \mathcal{L}^2 \)-null set. Extending the function \( (p)\partial^2_x u \) defined on \( G \) to \( \Omega \), e.g., by setting \( (p)\partial^2_x u(\omega) = 0 \) for all \( \omega \in \Omega \setminus G \), the fact that \( (p)\partial^2_x u(\omega) = \overline{\partial v}(\omega) \) for any \( \omega \in G \) implies that \( (p)\partial^2_x u \) is \( \mathcal{L}^2 \)-measurable, so that, thanks to the boundedness of \( \overline{\partial v} \), \( (p)\partial^2_x u \in L^\infty(\Omega) \). Fubini’s theorem finally yields that the identity \( (p)\partial^2_x u = \partial^2_x \overline{\partial v} \) holds true \( \mathcal{L}^2 \)-almost everywhere in \( \Omega \).

\[\square\]

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