Research Article

On Discrete Fractional Integral Inequalities for a Class of Functions

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Discrete fractional calculus (\(\mathcal{FC}\)) is proposed to depict neural systems with memory impacts. This research article aims to investigate the consequences in the framework of the discrete proportional fractional operator. \(h\)-discrete exponential functions are assumed in the kernel of the novel generalized fractional sum defined on the time scale \(\mathbb{Z}_h\). The \(h\)-discrete sums are accounted in particular. The governing high discretization of problems is an advanced version of the existing forms that can be transformed into linear and nonlinear difference equations using appropriately adjusted transformations invoking property of observing the new chaotic behaviors of the logistic map. Based on the theory of discrete fractional calculus, explicit bounds for a class of positive functions \(n(n \in \mathbb{N})\) concerned are established. These variants can be utilized as a convenient apparatus in the qualitative analysis of solutions of discrete fractional difference equations. With respect to applications, we can apply the introduced outcomes to explore boundedness, uniqueness, and continuous reliance on the initial value problem for the solutions of certain underlying worth problems of fractional difference equations.

1. Introduction

Recently, \(\mathcal{FC}\) and its concrete utilities have increased a great deal of significance in light of the fact that fractional operators have become a useful asset with more precise and victories in demonstrating a few complex marvels in various apparently differing and broad fields of science and numerous areas, for example, fluid flow, optics, chaos, image processing, virology, and financial economics [1–3]. A few decades ago, the fractional differential equations and dynamical frameworks have been substantiated as being important devices in displaying several phenomena in various branches of pure and applied sciences. They attract incredible utility in research-oriented areas, for example, fluid mechanics, thermodynamics, vibration, groundwater flow with memory, and image processing (see the fundamental monograph and the fascinating paper [4–6]). An assortment of consequences that facilitated in emergence of the theory of discrete \(\mathcal{FC}\) is presented in [7–9]. Atici and Eloe prudentively provoked the enthusiasm for the theory of fractional difference operators [10]. Numerous researchers characterized fractional difference with various sorts of kernel having a discrete force law with discrete exponential and generalized Mittag–Leffler functions [11] and discrete exponential and Mittag–Leffler functions on generalized \(h\mathbb{Z}\) time scale [12] and kernel depending on the consequence of both power-law and exponential functions [13]. It is notable that the discretization cycle is one of the most requested devices for scientists who are intrigued in reproduction and computational examination. In arrangement with the reality that not all discrete operators acquire similar features as of the continuous ones, the exploration of the discrete alignment of \(\mathcal{FC}\) has become squeezing prerequisites (see [4, 7, 14–21]). Numerous authors devoted their attention to searching novel operators of arbitrary order. Definitely, the assortment of such tools provides analysts more opportunities to apply them to various models (see [22, 23]). In [24, 25], the authors presented...
and explored local-type derivatives and integrals with arbitrary order and without memory, called conformable derivatives and integrals, with the disadvantage that the function itself is not acquired in the restricting situation when the order of the derivative, \( \alpha \), approaches 0. Afterward, the researchers [26] proposed proportional-type derivatives that legitimately combine the function itself and its derivative as the parameter \( \rho \) tends to 0 and 1, respectively. We have comprehended that these local-type derivatives and integrals are valuable to produce new sorts of operators with memory through various kinds of kernels [27, 28]. In recent years, the study which is promptly increasing the extent of incredible intrigue both from the hypothetical and applied perspective is the study of fractional \( h \)-discrete calculus. Regarding utilities in various fields of mathematics, we refer to [29, 30]. Additionally, we notice that \( h \)-discrete \( FC \) is extremely significant in applied analysis, for example, financial mathematics, banking, and material sciences. Finally, the calculus emerging from the definition of discrete proportional fractional sums has become appealing to many authors and now it is a matter of strong research, in various directions: existence and uniqueness of solutions to discrete fractional equations modelling tumor growths [22], continuity of solutions with respect to initial conditions and also to the order \( \alpha \) of the derivative [31], and the Euler–Lagrange equation and Legendre’s optimality condition for the calculus of variations problems [32]. Discrete fractional variants have consistently been of excessive prominence for the advancement of remarkable momentous approaches midst investigators and accumulate productive purposeful demonstrations in several areas of science and technology. Certain distinctive examples are Ostrowski, Lyenger [33], Grownwall [34], Čebyšev [35], Hermite-Hadamard [36], and henceforth. Several researchers have devoted their concentrations for exploring the novel versions of fractional integral inequalities for a family of \( n ( n \in \mathbb{N} ) \)-positive increasing functions. In this flow, we observe that some variants have been concerned with the qualitative investigation of solutions of discrete fractional difference equations arising in the theory of discrete \( FC \). For current consequences on this trend, we refer the readers to [37–53]. Inspired by the discretization process, the principal purpose of this investigation is to present a new research area for mathematicians in the frame of \( h \)-discrete fractional and discrete \( h \)-proportional fractional sums. Taking into account the concepts of our noteworthy discrete fractional operators, we demonstrate the generalizations associating \( h \)-anologue for a class of family of continuous positive decreasing functions on \( \mathbb{N}_{a_1,b} \) by the proposed discrete \( h \)-proportional fractional sums. Additionally, it is accentuated that mingling these two methodologies, discrete \( FC \) and integral inequalities, might be the supreme proficient approach of combining inequities into both times and fractional operator theory. Finally, our findings can deliver a prevailing instrument to illustrate the dynamics of discrete complex frameworks all the more profoundly.

2. Preliminaries

Particular imperative characterizations and deductions in discrete \( FC \) are mentioned as follows [10]. For the accessibility, for \( a_1, b_1 \in \mathbb{R} \) and \( h > 0 \), we symbolize \( \mathbb{N}_{a_1,b} = \{ a_1, a_1 + h, a_1 + 2h, \ldots \} \) and \( \mathbb{N}_{b_1,b} = \{ b_1, b_1 - h, b_1 - 2h, \ldots \} \).

**Definition 1** (see [10]). The consequent equalities are valid:

(i) Let \( \eta \) be a natural number; then, the \( \eta \) rising factorial of \( t \) is expressed as

\[
\begin{align*}
\bar{t}^\eta &= \prod_{k=0}^{\eta-1} (t + k), \quad \bar{t}^0 = 1. \\
\end{align*}
\]

(ii) For any real number, the \( \beta \) rising function becomes

\[
\begin{align*}
t^\beta &= \frac{\Gamma(t + \beta)}{\Gamma(t)}, \quad \text{such that } t \in \mathbb{R} \setminus \{ \ldots, -2, -1, 0 \}, \quad \bar{t}^\beta = 0.
\end{align*}
\]

Additionally, we have

\[
\psi(t^\beta) = \beta t^{\beta-1}.
\]

Hence, \( t^\beta \) is increasing on \( \mathbb{N}_0 \).

(iii) Let \( \Gamma \) denote the usual special gamma function and recall the notation that is known as the falling factorial power:

\[
\begin{align*}
t^\beta &= \frac{\Gamma(t + 1)}{\Gamma(t + \beta - 1)}.
\end{align*}
\]

Throughout, we assume that if \( t + \beta - 1 \in \{ \ldots, -2, -1, 0 \} \), then \( t^\beta = 0 \).

Now, \( \nabla^\beta_h Y(t) = ((Y(t) - Y(t+h))/h) \) represents the backward difference operator on \( h\mathbb{Z} \) and \( \Delta^\beta_h Y(t) = ((Y(t + h) - Y(t))/h) \) is the backward difference operator on \( h\mathbb{Z} \).

Also, we can achieve the backward and forward difference operators \( \bar{\nabla}^\beta_h Y(t) = Y(t) - Y(t-1) \) and \( \Delta^\beta_h Y(t) = Y(t+1) - Y(t) \), respectively. Moreover, \( \sigma_h(t) = t + h \) and \( \rho_h(t) = t - h \) are the forward and backward jumping operator on time scale \( h\mathbb{Z} \), respectively.

**Definition 2.** For arbitrary \( t, \beta \in \mathbb{R} \) and \( h > 0 \), and the nabla \( h \)-factorial function is defined by

\[
\begin{align*}
\nabla^\beta_h t^\beta &= h^\beta \frac{\Gamma((t/h) + \beta)}{\Gamma(t/h)}.
\end{align*}
\]

Specifically, for \( h = 1 \), we obtain identity (2). A forthright confirmation prompts

\[
\begin{align*}
\nabla^\beta_h t^\beta &= \beta t^{\beta-1}.
\end{align*}
\]

**Definition 3** (see [54]). (Nabla \( h \)-fractional sums). For backward jump operator \( \rho(t) = t - h \) with \( h > 0 \), let a
function $Y$: $\mathcal{N}_{a_i, h} = \{a_1, a_1 + h, a_1 + 2h, \ldots\} \mapsto \mathbb{R}$ be the nabla left $h$-fractional sum of order $\beta > 0$, stated as follows:

$$
\begin{aligned}
\left( a_i \nabla^\beta_h Y \right) (t) &= \frac{1}{\Gamma(\beta)} \int_{a_i}^t (t - \rho_h (\phi)) \frac{\nabla^\beta_h Y (\phi) \nabla_h \phi}{\rho_h (\phi)} \, d\phi, \\
&= \frac{1}{\Gamma(\beta)} \sum_{k = (a_i/h+1)}^{(t/h)} (t - \rho_h (kh)) \frac{\nabla^\beta_h Y (kh) h}{\rho_h (kh)} \; t \in \mathbb{N}_{a_i + h, \infty}.
\end{aligned}
$$

(7)

Let a function $Y$: $\mathcal{N}_{b_i, h} = \{b_1, b_1 - h, b_1 - 2h, \ldots\} \mapsto \mathbb{R}$ be the nabla right $h$-fractional sum of order $\beta > 0$ (ending at $b_1$), stated as follows:

$$
\begin{aligned}
\left( b_i \nabla^\beta_h Y \right) (t) &= \frac{1}{\Gamma(\beta)} \int_t^{b_i} (\phi - \rho_h (t)) \frac{\nabla^\beta_h Y (\phi) \nabla_h \phi}{\rho_h (\phi)} \, d\phi, \\
&= \frac{1}{\Gamma(\beta)} \sum_{k = (t/h+1)}^{(b_i/h-1)} (kh - \rho_h (t)) \frac{\nabla^\beta_h Y (kh) h}{\rho_h (kh)} \; t \in \mathbb{N}_{a_i + h, a_i}.
\end{aligned}
$$

(8)

Definition 4 (see [54]). (Nabla $h$-Riemann–Liouville fractional differences). The nabla left- and right-sided difference of order $\beta > 0$ (starting from $a_1$) has the form

$$
\begin{cases}
\left( a_i \nabla^\beta_h Y \right) (t) = \frac{1}{\Gamma(\beta)} \int_{a_i}^t (t - \rho_h (\phi)) \frac{\nabla^\beta_h Y (\phi) \nabla_h \phi}{\rho_h (\phi)} \, d\phi, \\
&= \frac{1}{\Gamma(\beta)} \sum_{k = (a_i/h+1)}^{(t/h)} (t - \rho_h (kh)) \frac{\nabla^\beta_h Y (kh) h}{\rho_h (kh)} \; t \in \mathbb{N}_{a_i + h, \infty},
\end{cases}
$$

(9)

$$
\begin{cases}
\left( b_i \nabla^\beta_h Y \right) (t) = \frac{1}{\Gamma(\beta)} \int_t^{b_i} (\phi - \rho_h (t)) \frac{\nabla^\beta_h Y (\phi) \nabla_h \phi}{\rho_h (\phi)} \, d\phi, \\
&= \frac{1}{\Gamma(\beta)} \sum_{k = (t/h+1)}^{(b_i/h-1)} (kh - \rho_h (t)) \frac{\nabla^\beta_h Y (kh) h}{\rho_h (kh)} \; t \in \mathbb{N}_{b_i, b_i/h},
\end{cases}
$$

(10)

Now, we demonstrate the proportional fractional sum with memory depending on the proportional difference, which is mainly due to Abdeljawad et al. [54].

$$
\begin{aligned}
\left( a_i \nabla^\beta_h Y \right) (t) &= \frac{1}{\rho^{\beta \Gamma(\beta)}} \int_{a_i}^t \nabla_h \left( t - \phi + \beta h, 0 \right) (t - \rho_h (\phi)) \frac{\nabla^\beta_h Y (\phi) \nabla_h \phi}{\rho_h (\phi)} \, d\phi, \\
&= \frac{1}{\rho^{\beta \Gamma(\beta)}} \sum_{k = (a_i/h+1)}^{(t/h)} \nabla_h \left( t - \rho_h (kh) + \beta h, 0 \right) (t - \rho_h (kh)) \frac{\nabla^\beta_h Y (kh) h}{\rho_h (kh)} h,
\end{aligned}
$$

(11)

$$
\begin{aligned}
\left( b_i \nabla^\beta_h Y \right) (t) &= \frac{1}{\rho^{\beta \Gamma(\beta)}} \int_t^{b_i} \nabla_h \left( \phi - t + \beta h, 0 \right) (\phi - \rho_h (t)) \frac{\nabla^\beta_h Y (\phi) \nabla_h \phi}{\rho_h (\phi)} \, d\phi, \\
&= \frac{1}{\rho^{\beta \Gamma(\beta)}} \sum_{k = (t/h+1)}^{(b_i/h-1)} \nabla_h \left( kh - \rho_h (t) + \beta h, 0 \right) (kh - \rho_h (t)) \frac{\nabla^\beta_h Y (kh) h}{\rho_h (kh)} h,
\end{aligned}
$$

(12)

where $\gamma = (\rho - 1/\rho)$.

Remark 1. In view of Definition 5, if we choose $\rho \to 1$, then we acquire the left and right nabla $h$-fractional sums of order $\beta$, respectively.

$$
\begin{aligned}
\left( a_i \nabla^\alpha_h Y \right) (t) &= \nabla^\alpha_h \left( a_i \nabla_h \left( - (a - \alpha) h, 0 \right) Y \right) (t), \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k = (a_i/h+1)}^{(t/h)} \nabla_h \left( t - \rho_h (kh) + (n - \alpha) h, 0 \right) (t - \rho_h (kh)) \frac{\nabla^\alpha_h Y (kh) h}{\rho_h (kh)} h,
\end{aligned}
$$

(13)
and the right (proportional) fractional difference ending at $b_1$ is stated as follows:

\[
\begin{align*}
(\tilde{Y}_{b_1}^\beta)^{(n-\alpha)}(t) \\
= a_0 \tilde{Y}_{h}^\beta_h \tilde{Y}_{b_1}^{-(n-\alpha)}(t), \\
= \frac{a_0 \tilde{Y}_{h}^\beta_h}{\rho^{\alpha-\beta} \Gamma(n-\beta)} \sum_{n=h}^{b_1/h-1} \tilde{Y}_{n}(kh - t + (n - \beta)h,0) (kh - \rho_h(t))^{\alpha-1} \tilde{Y}(kh)h,
\end{align*}
\]  

where $n = [\Re(\beta) + 1]$ and $\nu = (\rho - 1/\rho)$.

### 3. New Estimations within Proportional Fractional Sums

This segment is dedicated to giving our fundamental consequences of this paper. We define new forms for a class of a family of $n(n \in \mathbb{N})$ continuous positive decreasing functions on $\mathbb{N}_{a_1,h}$ in the settings of discrete proportional fractional operator.

**Theorem 1.** For $\alpha > 0, \sigma \geq \varsigma > 0$, and let there be a continuous positive decreasing function $Y$ defined on $\mathbb{N}_{a_1,h}$. Then, the discrete $h$-proportional fractional sum satisfies the following inequality:

\[
\begin{align*}
\frac{a_1 \tilde{Y}_{h}^\beta_h [Y^\sigma(t)]}{a_1 \tilde{Y}_{h}^\beta_h [Y^\varsigma(t)]} & \geq \frac{a_1 \tilde{Y}_{h}^\beta_h [(t - a_1)^{\alpha}Y^\sigma(t)]}{a_1 \tilde{Y}_{h}^\beta_h [(t - a_1)^{\alpha}Y^\varsigma(t)]},
\end{align*}
\]  

Proof. By means of the given assumption in Theorem 1, we have

\[
((\theta - a_1)^{\alpha} - (\phi - a_1)^{\alpha})(Y^{\alpha-\varsigma}(\phi) - Y^{\alpha-\varsigma}(\theta)) \geq 0, \tag{16}
\]

where $\alpha > 0, \sigma \geq \varsigma > 0$, and $\phi, \theta \in \{a_1, a_1 + h, a_1 + 2h, \ldots\}$. By (16), we have

\[
(\theta - a_1)^{\alpha}Y^{\alpha-\varsigma}(\phi) - (\phi - a_1)^{\alpha}Y^{\alpha-\varsigma}(\theta) + (\phi - a_1)^{\alpha}Y^{\alpha-\varsigma}(\phi) \geq 0. \tag{17}
\]

Therefore, multiplying (17) by $((\theta - a_1)^{\alpha} - (\phi - a_1)^{\alpha})(t - \rho_h(\phi))h^{\alpha-1} / \rho^{\alpha-\beta} \Gamma(\beta) Y^{\alpha-\varsigma}(\phi)$, $t \in \mathbb{N}_{a_1,h}$, and $\phi \in \{a_1, a_1 + h, a_1 + 2h, \ldots\}$, we obtain
and again, summing both sides of (18) for 
\( \phi \in \{a_i, a_i + h, a_i + 2h, \ldots\} \), we obtain

\[
(\theta - a_i)^{\alpha} \frac{1}{\rho^{\Gamma}(\beta)} \sum_{k = (a_i/h)}^{(t/h)} \tilde{c}_\gamma(t - k\beta, 0) (t - \rho_h(k\beta))^{\beta - 1} (k\beta)h h Y^{\alpha - \zeta}(k\beta),
\]

\[
- \frac{1}{\rho^{\Gamma}(\beta)} \sum_{k = (a_i/h)}^{(t/h)} \tilde{c}_\gamma(t - k\beta, 0) (t - \rho_h(k\beta))^{\beta - 1} Y^\varsigma(k\beta) h Y^{\alpha - \zeta}(\theta)(k\beta - a_i)^{\alpha},
\]

\[
- (\theta - a_i)^{\alpha} \frac{1}{\rho^{\Gamma}(\beta)} \sum_{k = (a_i/h)}^{(t/h)} \tilde{c}_\gamma(t - k\beta, 0) (t - \rho_h(k\beta))^{\beta - 1} Y^\varsigma(k\beta) h Y^{\alpha - \zeta}(\theta),
\]

\[
+ \frac{1}{\rho^{\Gamma}(\beta)} \sum_{k = (a_i/h)}^{(t/h)} \tilde{c}_\gamma(t - k\beta, 0) (t - \rho_h(k\beta))^{\beta - 1} Y^\varsigma(k\beta) h Y^{\alpha - \zeta}(k\beta)(k\beta - a_i)^{\alpha} \geq 0.
\]

It follows that

\[
(\theta - a_i)^{\alpha} \left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) + \Psi^{\alpha - \zeta}(\theta) \left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) - (\theta - a_i)^{\alpha} Y^{\alpha - \zeta}(\theta) \left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) \geq 0.
\]

Multiplying (20) by \( (\tilde{c}_\gamma(t - \theta + \beta h, 0) (t - \rho_h(\theta))^{\beta - 1}) / \rho^{\Gamma}(\beta) Y^\varsigma(\theta) \) and summing both sides for \( \theta \in \{a_i, a_i + h, a_i + 2h, \ldots\} \) show

\[
\left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) \left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) \geq 0.
\]

Dividing the above inequality by \( (\tilde{\nabla}^{-\beta, \rho}(t - a_i)^{\alpha} Y^\varsigma(t)) (\tilde{\nabla}^{-\beta, \rho}(t - a_i)^{\alpha} Y^\varsigma(t)) \), we obtain desired inequality (15). \( \square \)

**Theorem 2.** For \( \alpha > 0, \sigma \geq \zeta > 0 \), and let there be a continuous positive decreasing function \( \Psi \) defined on \( \mathbb{R}^+ \). Then, the discrete \( h \)-proportional fractional sum satisfies the following inequality:

\[
\left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) \left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) \left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) \left( a_i \tilde{\nabla}^{-\beta, \rho}[Y^\varsigma(t)] \right) \geq 1.
\]
Proof. Multiplying both sides of (20) by 
\((\varepsilon_{t} - \theta + \lambda h, 0) - \rho_{h}(\theta) - \rho_{h}(\theta)\)) / \rho_{h}(\lambda)\) \((\lambda)\) and summing both sides for \(\theta \in \{a_{1}, a_{1} + h, a_{1} + 2h, \ldots\}\) show

\[
\begin{align*}
&\left( a_{i} \nabla_{h}^{-\beta-\rho} [Y^\varphi (t)] \right) \left( a_{i} \nabla_{h}^{-\beta-\rho} [t - a_{i})^{a}Y^\varphi (t)] \right) + \left( a_{i} \nabla_{h}^{-\beta-\rho} [t - a_{i})^{a}Y^\varphi (t)] \right) \\
&- \left( a_{i} \nabla_{h}^{-\beta-\rho} [Y^\varphi (t)] \right) \left( a_{i} \nabla_{h}^{-\beta-\rho} [t - a_{i})^{a}Y^\varphi (t)] \right) \geq 0.
\end{align*}
\]

Hence, dividing (23) by

\[
\begin{align*}
&\left( a_{i} \nabla_{h}^{-\beta-\rho} [Y^\varphi (t)] \right) \left( a_{i} \nabla_{h}^{-\beta-\rho} [t - a_{i})^{a}Y^\varphi (t)] \right) \\
&\left( a_{i} \nabla_{h}^{-\beta-\rho} [Y^\varphi (t)] \right) \left( a_{i} \nabla_{h}^{-\beta-\rho} [t - a_{i})^{a}Y^\varphi (t)] \right),
\end{align*}
\]

we obtain desired inequality (23).

Theorem 3. For \(\alpha > 0, \sigma \geq \varsigma > 0, \) and let there be an continuous positive decreasing function \(Y^\varphi\) defined on \(\mathbb{N}_{h}.\) Also, we have a continuous positive increasing function \(\Phi^\alpha\) defined on \(\mathbb{N}_{h}.\) Then, the discrete \(h\)-proportional fractional sum satisfies the following inequality:

\[
\begin{align*}
\left( a_{i} \nabla_{h}^{-\beta-\rho} [Y^\varphi (t)] \right) \left( a_{i} \nabla_{h}^{-\beta-\rho} [t - a_{i})^{a}Y^\varphi (t)] \right) \geq 1, \quad (25)
\end{align*}
\]

where \(v = (\rho - 1/\rho), \rho \in (0, 1), \beta \in \mathbb{R}, \) and \(\Re (\beta) > 0.

Proof. Using the hypothesis given in Theorem 3, we have

\[
\Phi^\alpha (\beta)Y^\varphi - \Phi^\alpha (\varphi)Y^\varphi (\beta) \geq 0, \quad (26)
\]

where \(\alpha > 0, \sigma \geq \varsigma > 0, \) and \(\phi, \beta \in \{a_{1}, a_{1} + h, a_{1} + 2h, \ldots\}.
\)

From (26), we have

\[
\Phi^\alpha (\beta)Y^\varphi (\beta) - \Phi^\alpha (\varphi)Y^\varphi (\beta) + \Phi^\alpha (\beta)Y^\varphi (\beta) - \Phi^\alpha (\beta)Y^\varphi (\beta) \geq 0.
\]

Taking product of (27) by \((\varepsilon_{t} - \phi + \beta h, 0)(t - \rho_{h}(\phi)) / \rho_{h}(\beta)) \Y^\varphi (t), \) \(t \in h_{a_{1}, a_{1} + h}, \) and \(\phi \in \{a_{1}, a_{1} + h, a_{1} + 2h, \ldots\},\) we have

\[
\frac{\Phi^\alpha (\beta)}{\rho_{h}} \sum_{k=(a_{1}/h+1)}^{(t/h)} \varepsilon_{t} (t - kh + \beta h, 0)(t - \rho_{h}(kh))^{\rho_{h}}Y^\varphi (kh),
\]

\[
- \frac{1}{\rho_{h}} \sum_{k=(a_{1}/h+1)}^{(t/h)} \varepsilon_{t} (t - kh + \beta h, 0)(t - \rho_{h}(kh))^{\rho_{h}}Y^\varphi (kh) \Phi^\alpha (kh),
\]

\[
\frac{\Phi^\alpha (\beta)}{\rho_{h}} \sum_{k=(a_{1}/h+1)}^{(t/h)} \varepsilon_{t} (t - kh + \beta h, 0)(t - \rho_{h}(kh))^{\rho_{h}}Y^\varphi (kh) \Phi^\alpha (kh) \geq 0.
\]
From (30), it follows that
\[
\Phi^\alpha(\theta)\left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a} + Y^{\alpha - \varsigma}(\theta)\left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right)\right)
- \Phi^\alpha(\theta)Y^{\alpha - \varsigma}(\theta)\left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) - \left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) \geq 0.
\]
(30)

Again, taking the product (20) by \((\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]/\rho^\alpha \Gamma(\beta))\) and summing both sides for \(\theta \in [a_1, a_1 + h, a_1 + 2h, \ldots] \) show
\[
\left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) + \left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) - \frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a} \geq 0,
\]
(31)

where \(\nu = (\rho - 1)/\rho, \rho \in (0, 1], \beta, \lambda \in \mathcal{C}, \) and \(\mathfrak{R}(\beta)\mathfrak{R}(\lambda) > 0\).

Proof. Multiplying both sides of (30) by \((\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]/\rho^\alpha \Gamma(\beta))\) and summing both sides for \(\theta \in [a_1, a_1 + h, a_1 + 2h, \ldots] \) show
\[
\left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) + \left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) - \frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a} \geq 0.
\]
(33)

It follows that
\[
\left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) + \left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) - \frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a} \geq 0.
\]
(34)

Dividing the above inequality by
\[
\left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) + \left(\frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a}\right) - \frac{\bar{\nabla}_h ^r \Phi^\alpha[Y^\alpha(t)]}{a},
\]
we obtain the desired inequality (32).

\[ \square \]

4. Certain Bounds for a Sequence of Decreasing Functions within Proportional Fractional Sums

Now, we demonstrate the discrete \(h\)-proportional fractional sum to derive some inequalities for a class of \(n\)-decreasing positive functions.

Theorem 4. For \(\alpha > 0, \sigma \geq \varsigma > 0\) and let there be a continuous positive decreasing function \(Y\) defined on \(\mathbb{N}_{a_1,h}\). Also, we have a continuous positive increasing function \(\Phi\) defined on \(\mathbb{N}_{a_1,h}\). Then, the discrete \(h\)-proportional fractional sum satisfies the following inequality:

\[ \square \]

Proof. Since \(\{Y_l, l = 1, 2, 3, \ldots, n\} \) is a sequence of continuous positive decreasing functions on \(\{a_1, a_1 + h, a_1 + 2h, \ldots\} \), we have

\[ \square \]
for any fixed $\theta \in \{1, 2, 3, \ldots, n\}, \alpha > 0, \sigma > \varsigma > 0$, and
\( \phi, \theta \in \{a_1, a_1 + h, a_1 + 2h, \ldots\} \). By (37), we have
\[
(\theta - a_1)^a a Y^c (\phi) + (\phi - a_1)^a a Y^c (\theta) \\
\geq (\theta - a_1)^a a Y^c (\theta) + (\phi - a_1)^a a Y^c (\phi).
\] (38)

Taking product of (38) by 
\[
(\bar{c}_v (t + \beta h, 0) (t - \rho_h (\phi))_h^{\beta - 1} / \rho^\beta (\beta))^n \prod_{l=1}^n Y^c (\phi),
\]
and summing both sides for $\theta \in \{a_1, a_1 + h, a_1 + 2h, \ldots\}$ shows
where $\nu = (\rho - 1/\rho), \rho \in (0, 1], \beta, \lambda \in \mathcal{C}$, and $\mathcal{R}(\beta), \mathcal{R}(\lambda) > 0$. 

Proof. Taking product on both sides of (41) by \((\zeta(t - \theta + \lambda h, 0) (t - \rho_h(\theta))^{-1})/\rho^\beta(\lambda))\prod_{i=1}^n Y_{i}^\rho(\theta)\) and summing both sides for $\theta \in \{a_1, a_1 + h, a_1 + 2h, \ldots\}$ show

\[
\begin{align*}
&\geq a_i \vec{\nu}^{-\rho_p} \left[ \prod_{i \neq 0}^{n} Y_{i}^{\rho}(t) \right] a_i \vec{\nu}^{-\rho_p} \left[ (t - a_i) \prod_{i = 1}^{n} Y_{i}^{\rho}(t) \right], \\
&\quad + \frac{a_i \vec{\nu}^{-\rho_p} \left[ \prod_{i \neq 0}^{n} Y_{i}^{\rho}(t) \right] a_i \vec{\nu}^{-\rho_p} \left[ (t - a_i) \prod_{i = 1}^{n} Y_{i}^{\rho}(t) \right]}{a_i \vec{\nu}^{-\rho_p} \left[ \prod_{i \neq 0}^{n} Y_{i}^{\rho}(t) \right] a_i \vec{\nu}^{-\rho_p} \left[ (t - a_i) \prod_{i = 1}^{n} Y_{i}^{\rho}(t) \right]} \geq 1.
\end{align*}
\]

Dividing the above inequality by

\[
\begin{align*}
&\geq a_i \vec{\nu}^{-\rho_p} \left[ \prod_{i \neq 0}^{n} Y_{i}^{\rho}(t) \right] a_i \vec{\nu}^{-\rho_p} \left[ (t - a_i) \prod_{i = 1}^{n} Y_{i}^{\rho}(t) \right], \\
&\quad + \frac{a_i \vec{\nu}^{-\rho_p} \left[ \prod_{i \neq 0}^{n} Y_{i}^{\rho}(t) \right] a_i \vec{\nu}^{-\rho_p} \left[ (t - a_i) \prod_{i = 1}^{n} Y_{i}^{\rho}(t) \right]}{a_i \vec{\nu}^{-\rho_p} \left[ \prod_{i \neq 0}^{n} Y_{i}^{\rho}(t) \right] a_i \vec{\nu}^{-\rho_p} \left[ (t - a_i) \prod_{i = 1}^{n} Y_{i}^{\rho}(t) \right]} \geq 1.
\end{align*}
\]

gives the desired inequality (43).

\[\Box\]

Theorem 6. For $\alpha > 0, \sigma \geq \varsigma_0 > 0$, for any $\theta \in \{1, 2, 3, \ldots, n\}$ and let there be a sequence of continuous positive decreasing functions $\{Y_{i}^\rho, l = 1, 2, 3, \ldots, n\}$ defined on $\mathbb{N}_{\varsigma_0, \alpha}$. Then, the discrete $\rho$-proportional fractional sum satisfies the following inequality:

\[\Phi^\alpha(\beta) \geq (t - \phi + \beta h, 0) (t - \rho_h(\phi))^{-1} (\rho^\beta(\lambda)) \prod_{i = 1}^{n} Y_{i}^{\rho}(\phi), \quad \phi \in \{a_1, a_1 + h, a_1 + 2h, \ldots\},
\]

which gives the desired inequality (36).

\[\Box\]

Theorem 7. For $\alpha > 0, \sigma \geq \varsigma_0 > 0$, for any $\theta \in \{1, 2, 3, \ldots, n\}$ and let there be a sequence of continuous positive decreasing functions $\{Y_{i}^\rho, l = 1, 2, 3, \ldots, n\}$ defined on $\mathbb{N}_{\varsigma_0, \alpha}$. Also, there is a continuous positive increasing function $\Phi$ defined on $\mathbb{N}_{\varsigma_0, \alpha}$. Then, the discrete $\rho$-proportional fractional sum satisfies the following inequality:

\[\Phi^\alpha(\beta) \geq (t - \phi + \beta h, 0) (t - \rho_h(\phi))^{-1} (\rho^\beta(\lambda)) \prod_{i = 1}^{n} Y_{i}^{\rho}(\phi), \quad \phi \in \{a_1, a_1 + h, a_1 + 2h, \ldots\},
\]

where $\nu = (\rho - 1/\rho), \rho \in (0, 1], \beta, \lambda \in \mathcal{C}$, and $\mathcal{R}(\beta), \mathcal{R}(\lambda) > 0$. 

Proof. Under the given hypothesis, we have

\[\Phi^\alpha(\theta)Y_{\phi}^{\sigma-\phi}(\beta) - \Phi^\alpha(\phi)Y_{\phi}^{\sigma-\phi}(\theta) \geq 0,
\]

for any fixed $\theta \in \{1, 2, 3, \ldots, n\}, \alpha > 0, \sigma \geq \varsigma_0 > 0$, and

\[\phi, \theta \in \{a_1, a_1 + h, a_1 + 2h, \ldots\}.
\]

From (47), we have

\[\Phi^\alpha(\theta)Y_{\phi}^{\sigma-\phi}(\beta) - \Phi^\alpha(\phi)Y_{\phi}^{\sigma-\phi}(\theta) \geq 0.
\]

Taking product on both sides of (50) by \((\zeta(t - \phi + \beta h, 0) (t - \rho_h(\phi))^{-1})/\rho^\beta(\lambda))\prod_{i = 1}^{n} Y_{i}^{\rho}(\phi), t \in \mathbb{N}_{\varsigma_0, \alpha}
\]

and $\phi \in \{a_1, a_1 + h, a_1 + 2h, \ldots\}$, we have

\[\Box\]
\[ \Phi^\alpha(\theta) \frac{1}{\rho^\beta \Gamma(\beta)} \sum_{\kappa=(a_1+h+1)}^{(t+h)} \tilde{c}_\kappa (t - k h + \beta h, 0) (t - \rho_h (k h))^{\kappa - 1} \prod_{l=1}^{n} Y^\sigma_{l} (k h) Y_{\rho}^{\sigma - \varphi} (k h), \]

\[ + \frac{1}{\rho^\beta \Gamma(\beta)} \sum_{\kappa=(a_1+h+1)}^{(t+h)} \tilde{c}_\kappa (t - k h + \beta h, 0) (t - \rho_h (k h))^{\kappa - 1} \prod_{l=1}^{n} Y^\sigma_{\rho} (k h) Y_{\rho}^{\sigma - \varphi} (k h), \]

\[ - \Phi^\alpha(\theta) \frac{1}{\rho^\beta \Gamma(\beta)} \sum_{\kappa=(a_1+h+1)}^{(t+h)} \tilde{c}_\kappa (t - k h + \beta h, 0) (t - \rho_h (k h))^{\kappa - 1} \prod_{l=1}^{n} Y^\sigma_{l} (k h) Y_{\rho}^{\sigma - \varphi} (k h) \Phi^\alpha (k h) \geq 0. \]  

(50)

From (50), it follows that

\[ \Phi^\alpha(\theta) \alpha \tilde{a}_h^{-\beta \rho} \left[ \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] + Y_{\rho}^{\sigma - \varphi} (t) \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] \]

\[ - \Phi^\alpha(\theta) Y_{\rho}^{\sigma - \varphi} (\theta) \alpha_h^{-\beta \rho} \left[ \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] - a_1 \alpha^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] \geq 0, \]

(51)

and we get the desired inequality (46).

\[ \left[ (\tilde{c}_\kappa (t - \theta + \beta h, 0) (t - \rho_h (k h))^{\kappa - 1} / \rho^\beta \Gamma (\beta)) \prod_{l=1}^{n} Y^\sigma_{l} (\theta), \right. \]

and summing both sides for \( \theta \in \{a_1, a_1 + h, a_1 + 2h, \ldots \} \) shows

\[ a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] \]

\[ - a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] \geq 0, \]

(52)

\[ \left( a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] \right) \left( a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] \right) \]

\[ = a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] \left( a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] \right) \]

(53)

\[ \geq 0. \]

Theorem 8. For \( a > 0, \sigma \geq \zeta > 0 \) for any \( \theta \in \{1, 2, 3, \ldots, n \} \) and let there be a sequence of continuous positive decreasing functions \{ \( Y^\sigma_l, l = 1, 2, 3, \ldots, n \} \) defined on \( N_{a_1, h} \). Also, there is a continuous positive increasing function \( \Phi \) defined on \( N_{a_1, h} \).

Then, the discrete h-proportional fractional sum satisfies the following inequality:

\[ \left( a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] \right) \left( a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] \right) \]

\[ = a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] \left( a_1 \alpha_h^{-\beta \rho} \left[ \prod_{l=1}^{n} Y^\sigma_{l} (t) \right] a_1 \alpha_h^{-\beta \rho} \left[ \Phi^\alpha (t) \prod_{l \neq \theta}^{n} Y^\sigma_{l} (t) \right] \right) \]

\[ \geq 0. \]
Complexity

Proof. Multiplying both sides of (51) by \((\xi_{\theta} - \theta + \lambda h, 0) (t - \rho_{h} (\theta) h^{-1}) / p^\gamma (\lambda)) \prod_{l=1}^{n} Y_{l}^{\theta} (t)\), and summing both sides for \(\theta \in \{a_{1}, a_{1} + h, a_{1} + 2h, \ldots\}\) show
\[
\begin{align*}
\al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg],
+ \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg],
- \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg],
- \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \geq 0.
\end{align*}
\]

The data follows that
\[
\begin{align*}
\al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg],
+ \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg],
\geq \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg],
+ \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg].
\end{align*}
\]

Dividing both sides by
\[
\begin{align*}
\al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg],
+ \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg] \al h \sum_{l=1}^{n} \bigg[ Y_{l}^{\theta} (t) \bigg],
\end{align*}
\]
we get the desired inequality (53). □

5. Conclusion

Discrete \(\mathcal{F}\) has gained a lot of achievement in real-world phenomena, such as fractional chaotic maps, image encryption, and another discrete-time modelling. One of the supreme fundamental problems in the study of difference equations is to investigate the qualitative characteristics of the solutions of these aforesaid fields. Discrete fractional variants are noteworthy mechanisms that expedite discovering such properties. In the present note, we have proposed a novel proportional fractional sum to derive innovative descriptions for the noted class of a family of continuous positive decreasing functions on time scale \(h\mathbb{Z}\). By considering the \(h\)-proportional fractional sums within the nabla \(h\)-fractional sum, we derived the extension of sequences of decreasing functions in the frame of time-scale domains. The noted consequences can also be extended to the weighted function case. One can straightforwardly make sense that the contemporary consequences simplify the ones formerly achieved in the literature. Certainly, the case \(p \rightarrow 1\) recaptures the outcomes of nabla \(h\)-fractional sums. For indicating the strength of the offered fallouts, we employ them to investigate numerous initial value problems of fractional difference equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

[1] A. Atangana, “Modelling the spread of COVID-19 with new fractal-fractional operators: can the lockdown save mankind before vaccination?” Chaos Solitons Fractals, vol. 136, Article ID 109860, 2020.
[2] J. Danane, K. Allali, and Z. Hammouch, “Mathematical analysis of a fractional differential model of HBV infection with antibody immune response,” Chaos, Solitons Fractals, vol. 136, Article ID 109787, 2020.
[3] J. Singh, D. Kumar, Z. Hammouch, and A. Atangana, “A fractional epidemiological model for computer viruses pertaining to a new fractional derivative,” Applied Mathematics and Computation, vol. 316, pp. 504–515, 2018.
[4] C. Goodrich and A. C. Peterson, Discrete Fractional Calculus, Springer, Berlin, Germany, 2015.
[5] L. Huang, J. H. Park, G.-C. Wu, and Z.-W. Mo, “Variable-order fractional discrete-time recurrent neural networks,” Journal of Computational and Applied Mathematics, vol. 370, Article ID 112633, 2019.
[6] G.-C. Wu, T. Abdeljawad, J. Liu, D. Baleanu, and K.-T. Wu, “Mittag-Leffler stability analysis of fractional discrete-time neural networks via fixed point technique,” Nonlinear Analysis: Modelling and Control, vol. 24, no. 6, pp. 919–936, 2019.
[7] V. T. Holm, “The Laplace transform in discrete fractional calculus,” Computers Mathematics with Applications, vol. 52, no. 9, pp. 1591–1601, 2006.
[8] G.-C. Wu and D. Baleanu, “Discrete fractional logistic map and its chaos,” Nonlinear Dynamics, vol. 75, no. 1-2, pp. 283–287, 2014.
[9] G.-C. Wu, Z. G. Deng, D. Baleanu, and D. Q. Zeng, “New variable-order fractional chaotic systems for fast image encryption,” Chaos, vol. 29, no. 8, 11 pages, Article ID 083103, 2019.
S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “Some new discrete fractional inequalities associated with Hermite-Hadamard type inequalities,” *Symmetry, Inference and Applications*, vol. 2, no. 1, pp. 1–12, 2009.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 226, no.16–18, pp. 3457–3471, 2018.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.

S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, “New multi-parametrized estimates having m-convex functions and applications,” *AIMS Mathematics*, vol. 5, no. 6, pp. 330, p.17, 2020.
[46] S. Rashid, Z. Hammouch, H. Kalsoom, R. Ashraf, and Y.-M. Chu, “New investigation on the generalized K fractional integral operators,” *Frontiers in Physics*, vol. 8, no. 25, 2020.

[47] S. Rashid, M. A. Noor, M. Aslam Noor, K. Noor, and Y.-M. Chu, “Ostrowski type inequalities in the sense of generalized (K)-fractional integral operator for exponentially convex functions,” *AIMS Mathematics*, vol. 5, no. 3, pp. 2629–2645, 2020.

[48] H.-H. Chu, S. Rashid, Z. Hammouch, and Y.-M. Chu, “New fractional estimates for Hermite-Hadamard-Mercer’s type inequalities,” *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3079–3089, 2020.

[49] J.-M. Shen, S. Rashid, M. A. Noor, R. Ashraf, and Y.-M. Chu, “Certain novel estimates within fractional calculus theory on time scales,” *AIMS Mathematics*, vol. 5, no. 6, pp. 6073–6086, 2020.

[50] L. Xu, Y.-M. Chu, S. Rashid, A. A. El-Deeb, and K. S. Nisar, “On new unified bounds for a family of functions via fractional q calculus theory,” *Journal of Function Spaces*, vol. 2020, Article ID 4984612, 9 pages, 2020.

[51] Y.-M. Chu, M. Adil Khan, T. Ali, and S. S. Dragomir, “Inequalities for α-fractional differentiable functions,” *Journal of Inequalities and Applications*, vol. 2017, Article ID 93, p. 12, 2017.

[52] S. Rashid, İ. İşcan, D. Baleanu, and Y.-M. Chu, “Generation of new fractional inequalities via n-polynomials s-type convexity with applications,” *Advances in Difference Equations*, vol. 2020, Article ID 264, p. 20, 2020.

[53] L. Xu, Y.-M. Chu, S. Rashid, A. A. El-Deeb, and K. S. Nisar, “On new unified bounds for a family of functions via fractional q-calculus theory,” *Journal of Function Spaces*, vol. 2020, Article ID 4984612, 9 pages, 2020.

[54] T. Chu, F. Jarad, and J. Alzabut, “Fractional proportional differences with memory,” *The European Physical Journal Special Topics*, vol. 226, no. 16–18, pp. 3333–3354, 2017.