On a class of PDEs with nonlinear distributed in space and time state-dependent delay term

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Abstract. A new class of nonlinear partial differential equations with distributed in space and time state-dependent delay is investigated. We find appropriate assumptions on the kernel function which represents the state-dependent delay and discuss advantages of this class. Local and long-time asymptotic properties, including the existence of global attractor, are studied.

Key words: Partial functional differential equation, state-dependent delay, delay selection, global attractor.

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1. Introduction

Theory of delay differential equations is one of the oldest and simultaneously, intensively developing branches of the theory of infinite-dimensional dynamical systems. This theory covers ordinary and partial delay differential equations, includes studies of discrete and distributed, finite and infinite delays. Classical methods of differential equations, theory of distributions and functional analysis allow one to study wide classes of ordinary and partial differential equations with delay. We mention only several monographs which are classical references for delay equations [9, 7, 1, 14, 31] and also works which are close to this investigation [28, 5, 6, 18, 3, 20, 21]. Nevertheless, each (nonlinear) equation requires a separate and careful studying.

Recently, a new class of delay equations attracts attention of many researchers. These equations have delay (delay term) which may change, according to the state of the system i.e. state-dependent (state-selective) delay. The study of such equations was started in the case of ordinary equations [15, 16, 30] and it was recently continued for P.D.E.s in [20, 21]. For more detailed discussion and references on delay equations see e.g. introduction in [20]. We continue our previous research [20, 21] and present a wider class of nonlinear equations with distributed in space and time state-dependent delay terms.

Let us illustrate the main question studied in this article on the simplified object which is a local in space delay term. Consider the following simple distributed in time delay term $\int_{-r}^{0} b(u(t+\theta,x))\xi(\theta)d\theta$. Here function $\xi$ belongs to some space of real valued functions defined on the delay interval $(-r,0)$. This (kernel) function represents the rule how the information on the previous stages of the system (function $u$) is used to model the process. As discussed (see e.g. [20, 21]), this rule may change according to the state of the system. Let us denote by $v \in H$ the state coordinate, where $H$ represents the phase space. With this notations the state-dependent (state-selective) delay rule reads $\xi(\theta,v) : (-r,0) \times H \rightarrow \mathbb{R}$ and the corresponding delay term
becomes \( f^0_r, b(u(t + \theta, x))\xi(\theta, v)d\theta \). This is a simplified (local) example of delay terms studied in \[20\] [21]. As we will see (section 3), studying some questions (e.g. stationary solutions), there is a need to use a wider class of functions \( \xi \) (delay rules) which are space-dependent i.e. \( \xi(\theta, x, v) \). For example, considering a biological system, where \( u(t, x) \) represents the density of a population at time moment \( t \) at point \( x \in \Omega \), the delay rule \( \xi(\theta, v) \) is the same for all points \( x \) in the domain \( \Omega \), while the delay rule \( \xi(\theta, x, v) \) depends on the point \( x \in \Omega \) (e.g. due to the dependence of food resources on points in \( \Omega \)). This interpretation shows that the delay rule \( \xi(\theta, x, v) \) is more realistic biologically and as we will see (section 3) mathematically. Taking into account the above motivation we need to find an appropriate class of functions \( \xi \), concentrating on the character of dependence of \( \xi \) on the coordinate \( x \in \Omega \). This is the main goal of the article. It is interesting to mention that in spite of the fact that for any time moment \( t \geq 0 \) solutions \( u(t) \) belong to the space \( L^2(\Omega) \) and the phase coordinate \( (u(t); u(t+\theta)) \) belongs to \( L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \), the values of function \( \xi \) (as functions of \( x \)) do not necessary belong to \( L^2(\Omega) \). They belong to a wider space \( D(A^{-1/2}) \supset L^2(\Omega) \) (for more details see theorems 1 and 2 below).

The proposed model has an essential advantage in comparison with the previous ones (see \[20\] [21]) to cover the case of finite and even infinite sequences of isolated stationary solutions. We also present an algorithm to construct such state-dependent delay terms.

The article is organized as follows. In section 2 we present the model, prove the existence and uniqueness of weak solutions, construct the dynamical system and prove the existence of a global attractor. Section 3 is devoted to stationary solutions and the possibility to use our system to construct a dynamical system with an a-priory given set of isolated stationary solutions. The results may be applied to the diffusive Nicholson’s blowflies equation.

### 2. Formulation of the model with distributed delay

Consider the following non-local partial differential equation with state-dependent distributed in space and time delay

\[
\begin{aligned}
\frac{\partial}{\partial t}u(t, x) + Au(t, x) + du(t, x) &= \int_{-r}^{0} \{ \int_{\Omega} b(u(t + \theta, y))f(x - y)dy \} \xi(\theta, x, u(t), u_t)d\theta \\
&\equiv (F(u_t))(x), \quad x \in \Omega,
\end{aligned}
\]  

(1)

where \( A \) is a densely-defined self-adjoint positive linear operator with domain \( D(A) \subset L^2(\Omega) \) and with compact resolvent, \( \Omega \) is a smooth bounded domain in \( R^m \), \( f : \Omega \rightarrow R \) is a bounded function to be specified later, \( b : R \rightarrow R \) is a locally Lipschitz bounded map \( |b(w)| \leq C_b \) with \( C_b \geq 0 \), \( d \) is a positive constant. As usually for delay systems (see \[9\]) for any function \( u(t), t \in [a, b], b > a + r \) with values in a Banach space \( X \), we denote by \( u_t \equiv u_t(\theta) \equiv u(t + \theta) \), which is a function of \( \theta \in [-r, 0] \) with parameter \( t \in [a + r, b] \). Constant \( r > 0 \) is the (maximal) delay of the system.

The function \( \xi(\cdot, \cdot, \cdot) : [-r, 0] \times \Omega \times H \rightarrow R \) represents the state-dependent distributed delay. We denote for short \( H \equiv L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \) and also use \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) to denote the norm and scalar product in \( L^2(\Omega) \).
We consider equations (1) with the following initial conditions

\[ u(0+) = u^0 \in L^2(\Omega), \quad u|(−r,0) = \varphi \in L^2(−r,0;L^2(\Omega)). \tag{2} \]

So we write \((u^0, \varphi) \in H\).

Now we study the existence and properties of solutions for distributed delay problem (1, 2).

**Definition 1.** A function \(u\) is a weak solution of problem (1) subject to the initial conditions (2) on an interval \([0,T]\) if \(u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(−r,T; L^2(\Omega)) \cap L^2(0,T; D(A^{\frac{1}{2}}))\), \(u(\theta) = \varphi(\theta)\) for \(\theta \in (−r,0)\) and

\[-\int_0^T \langle u, \dot{v} \rangle dt + \int_0^T \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v \rangle dt + \int_0^T \langle du - F(u_t), v \rangle dt = −\langle u^0, v(0) \rangle \tag{3} \]

for any function \(v \in L^2(0,T; D(A^{\frac{1}{2}}))\) with \(\dot{v} \in L^2(0,T; D(A^{-\frac{1}{2}}))\) and \(v(T) = 0\).

**Theorem 1.** Assume that

(i) \(b : \Omega → R\) is locally Lipschitz and bounded i.e., there exists a constant \(C_b\) so that \(|b(w)| \leq C_b\) for all \(w \in R\);

(ii) \(f : \Omega → R\) is bounded \(|f(\cdot)| \leq M_f\);

(iii) \(ξ : [−r,0] × Ω × L^2(Ω) × L^2(−r,0;L^2(\Omega)) → R\) satisfies the following conditions:

a) for any \(M > 0\) there exists \(L_{ξ,M}\) so that for all \((v^i, ψ^i) \in H\) satisfying

\[ ||v^i||^2 + \int_{−r}^0 ||ψ^i(s)||^2 ds \leq M^2, i = 1, 2 \]

one has

\[ \int_{−r}^0 ||ξ(θ,⋅,v^1,ψ^1) - ξ(θ,⋅,v^2,ψ^2)||_{D(A^{−1/2})} dθ \]

\[ \leq L_{ξ,M} \left[ ||v^1 - v^2||^2 + \int_{−r}^0 ||ψ^1(s) - ψ^2(s)||^2 ds \right]^{1/2}, \tag{4} \]

b) there exists \(C_{(ξ,−1/2)} > 0\) so that

\[ \int_{−r}^0 ||ξ(θ,⋅,v,ψ)||_{D(A^{−1/2})} dθ \leq C_{(ξ,−1/2)} \text{ for all } (v, ψ) \in H. \tag{5} \]

Then for any \((u^0, \varphi) \in H \equiv L^2(Ω) × L^2(−r,0;L^2(\Omega))\) the problem (1) subject to the initial conditions (2) has a weak solution \(u(t)\) on every given time interval \([0,T]\) and this solution satisfies

\[ u(t) \in C([0,T]; L^2(\Omega)). \tag{6} \]

**Remark.** Properties (iii)-a) and (iii)-b) mean that \(ξ\) as a function of the third and fourth coordinate \((v, ψ) \in H\) is a (nonlinear) locally Lipschitz and globally bounded mapping \(ξ : H → L^1(−r,0; D(A^{−\frac{1}{2}}))\).
Proof of Theorem 1. Let us denote by $\{e_k\}_{k=1}^{\infty}$ an orthonormal basis of $L^2(\Omega)$ such that $Ae_k = \lambda_k e_k$, $0 < \lambda_1 < \ldots < \lambda_k \to +\infty$. We say that function $u^m(t, x) = \sum_{k=1}^{m} g_{k,m}(t) e_k(x)$ is a Galerkin approximate solution of order $m$ for the problem \( (1), (2) \) if

\[
\begin{aligned}
\langle \dot{u}^m + Au^m + du^m - F(u^m_t), e_k \rangle &= 0, \\
\langle u^m(0), e_k \rangle &= \langle u^0, e_k \rangle, \\
\langle u^m(\theta), e_k \rangle &= \langle \varphi(\theta), e_k \rangle, \quad \forall \theta \in (-r, 0)
\end{aligned}
\]

\forall k = 1, \ldots, m. Here $g_{k,m} \in C^1(0, T; R) \cap L^2(-r, T; R)$ with $\dot{g}_{k,m}$ being absolutely continuous.

Equations $(7)$ for fixed $m$ can be rewritten as a system for the $m$-dimensional vector-function $v(t) = v^m(t) = (g_{1,m}(t), \ldots, g_{m,m}(t))^T$. We notice that

$$
\|u^m(t, \cdot)\|^2_{L^2(\Omega)} = \sum_{k=1}^{m} g_{k,m}(t)^2 = \|v(t)\|^2_{R^m}.
$$

The standard technique (see e.g. \cite{9}) gives that for any initial data $\varphi \in L^2(-r, 0; R^m)$, $a \in R^m$ there exist $\alpha > 0$ and a unique solution of \( (7) \) $v \in L^2(-r, \alpha; R^m)$ such that $v_0 = \varphi$ and $v(0) = a$, and $v|_{(0,\alpha]} \in C([0,\alpha]; R^m)$ (for more details see Theorem 6 and Remark 9 from \cite{19} and also Lemma from \cite{21}).

It is easy to get from \( (1) \) and the boundedness of $b$ and $f$ that

$$
|\langle F(u_t), v \rangle_{L^2(\Omega)}| = \left| \int_{\Omega} \left\{ \int_{-r}^{0} \left[ \int_{\Omega} b(u(t + \theta, y)) f(x - y) dy \right] \xi(\theta, x, u(t), u_t) d\theta \right\} v(x) dx \right|
$$

$$
= \left| \int_{-r}^{0} \left[ \int_{\Omega} b(u(t + \theta, y)) \left\{ \int_{\Omega} f(x - y) \xi(\theta, x, u(t), u_t) dx \right\} dy \right] d\theta \right|
$$

$$
\leq C_b M_f |\Omega| \int_{-r}^{0} \|\xi(\theta, \cdot, u(t), u_t)\|_{D(A^{1/2})} d\theta \cdot \|v\|_{D(A^{1/2})}.
$$

Using \( (5) \), one has

$$
|\langle F(u_t), v \rangle_{L^2(\Omega)}| \leq C_b M_f |\Omega| C(\xi, -1/2) \cdot A^{1/2} \cdot \|v\|.
$$

Now, we will get an $a$-priori estimate for the Galerkin approximate solutions for the problem \( (1), (2) \). We multiply \( (7) \) by $g_{k,m}$ and sum over $k = 1, \ldots, m$. Hence for $u(t) = u^m(t)$ and $t \in (0, \alpha] \equiv (0, \alpha(m)]$, the local existence interval for $u^m(t)$, we get

$$
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \|A^{1/2} u(t)\|^2 + d \|u(t)\|^2 \leq |\langle F(u_t), u(t) \rangle|.
$$

Using \( (8) \), \( (9) \) we obtain

$$
\frac{d}{dt} \|u(t)\|^2 + \|A^{1/2} u(t)\|^2 + 2d \|u(t)\|^2 \leq C_b^2 M_f^2 |\Omega|^2 C^2(\xi, -1/2) \equiv \tilde{k}_1.
$$

Since

$$
\frac{d}{dt} \|u(t)\|^2 + \|A^{1/2} u(t)\|^2 + 2d \|u(t)\|^2 = \frac{d}{dt} \left( \|u(t)\|^2 + \int_0^t \|A^{1/2} u(\tau)\|^2 d\tau + 2d \int_0^t \|u(\tau)\|^2 d\tau \right),
$$

we denote by $\chi(t) \equiv \|u(0)\|^2 + \int_0^t \|A^{1/2} u(\tau)\|^2 d\tau + 2d \int_0^t \|u(\tau)\|^2 d\tau$ and rewrite the last estimate as follows $\frac{d}{dt} \chi(t) \leq \tilde{k}_1 t$. We obtain $\chi(t) \leq \chi(0) + \tilde{k}_1 t = \|u(0)\|^2 + \tilde{k}_1 t$. So, we have the $a$-priori estimate

$$
\|u(t)\|^2 + \int_0^t \|A^{1/2} u(\tau)\|^2 d\tau + 2d \int_0^t \|u(\tau)\|^2 d\tau \leq \|u(0)\|^2 + \tilde{k}_1 t.
$$
Estimate (11) gives that, for \( u^0 \in L^2(\Omega) \) the family of approximate solutions \( \{u^m(t)\}_{m=1}^{\infty} \) is uniformly (with respect to \( m \in \mathbb{N} \)) bounded in the space \( L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;D(A^{1/2})) \), where \( D(A^{1/2}) \) is the domain of the operator \( A^{1/2} \) and \([0,T]\) is the local existence interval. From (11) we also get the continuation of \( u^m(t) \) on any interval, so (11) holds for all \( t > 0 \).

Using the definition of Galerkin approximate solutions (7) and their property (11), we can integrate over \([0,T]\) to obtain \( \int_0^T \|A^{-1/2}\dot{u}^m(\tau)\|^2 \, d\tau \leq C_T \) for any \( T \). These properties of the family \( \{u^m(t)\}_{m=1}^{\infty} \) give that \( \{(u^m(t) ; \dot{u}^m(t))\}_{m=1}^{\infty} \) is a bounded sequence in the space

\[
X_T \equiv L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;D(A^{1/2})) \times L^2(0,T;D(A^{-1/2})).
\]

Then there exist a function \((u(t) ; \dot{u}(t))\) and a subsequence \( \{u^{m_k}\} \subset \{u^m\} \) such that

\[
(u^{m_k} ; \dot{u}^{m_k}) \quad \ast\text{-weakly converges to} \quad (u ; \dot{u}) \quad \text{in the space} \quad X_T.
\]

By a standard argument (using the strong convergence \( u^{m_k} \to u \) in the space \( L^2(0,T;L^2(\Omega)) \) which follows from (13) and the Doubinskii’s theorem, one can show (see e.g. Lions (1969), Chueshov (1999) and Rezounenko (1997)) that any \( \ast\)-weak limit is a solution of (1) subject to the initial conditions (2). To prove the continuity of weak solutions we use the well-known (see also [13 thm. 1.3.1])

**Proposition 1** (Proposition 1.2 in [23]). Let the Banach space \( V \) be dense and continuously embedded in the Hilbert space \( X \); identify \( X = X^* \) so that \( V \hookrightarrow X \hookrightarrow V^* \). Then the Banach space \( W_p(0,T) \equiv \{u \in L^p(0,T;V) : \dot{u} \in L^q(0,T;V^*)\} \) (here \( p^{-1} + q^{-1} = 1 \)) is contained in \( C([0,T];X) \).

In our case \( X = L^2(\Omega), V = D(A^{1/2}), V^* = D(A^{-1/2}), p = q = 1/2 \) (see (12), (13)). Hence Proposition 1 gives (3). The proof of Theorem 1 is complete.

Now we describe a sufficient condition for the uniqueness of weak solutions.

**Theorem 2.** Assume that functions \( b \) and \( f \) are as in Theorem 1 (satisfy properties (i),(ii)), function \( \xi \) satisfies property (iii)-a) and

\[
\xi(\cdot,\cdot,v,\psi) \in L^\infty(-r,0;D(A^{-1/2})) \quad \text{for all} \quad (v,\psi) \in H.
\]

Then solution of (1), (2) given by Theorem 1 is unique.

**Proof of Theorem 2.** Let \( u^1 \) and \( u^2 \) be two solutions of (1), (2). Below we denote for short \( w(t) = w^m(t) = u^{1,m}(t) - u^{2,m}(t) \) - the difference of corresponding Galerkin approximate solutions. Hence

\[
\frac{d}{dt} \|w(t)\|^2 + 2\|A^{1/2}w(t)\|^2 + 2d\|w(t)\|^2 = \langle F(u^1_t) - F(u^2_t), w(t) \rangle.
\]

Let us consider the difference \( \langle F(u^1_t) - F(u^2_t), w(t) \rangle \) in details (see (11)).

\[
\langle F(u^1_t) - F(u^2_t), w(t) \rangle \equiv \int_{-r}^{r} \left[ \int_0^T \left\{ \int_\Omega b(u^1(t+\theta,y))f(x-y)dy \right\} \xi(\theta,x,u^1(t),u^1_t) \, d\theta -
\right.
\]

\[
\left. \int_0^T \left\{ \int_\Omega b(u^2(t+\theta,y))f(x-y)dy \right\} \xi(\theta,x,u^2(t),u^2_t) \, d\theta \right] \, dt.
\]
\[- \int_0^r \left\{ \int_\Omega b(u^2(t + \theta, y)) f(x - y) dy \right\} \xi(\theta, x, u^2(t), u_t^2) d\theta \cdot w(t, x) dx \]
\[= \int_\Omega \left[ \int_0^r \left\{ \int_\Omega b(u^1(t + \theta, y)) f(x - y) dy \right\} \xi(\theta, x, u^1(t), u_t^2) d\theta - \right. \]
\[- \int_0^r \left\{ \int_\Omega b(u^2(t + \theta, y)) f(x - y) dy \right\} \xi(\theta, x, u^1(t), u_t^2) d\theta - \right. \]
\[\left. + \int_\Omega \left[ \int_0^r \left\{ \int_\Omega b(u^2(t + \theta, y)) f(x - y) dy \right\} \xi(\theta, x, u^1(t), u_t^2) d\theta - \right. \]
\[- \int_0^r \left\{ \int_\Omega b(u^2(t + \theta, y)) f(x - y) dy \right\} \xi(\theta, x, u^2(t), u_t^2) d\theta \right] \cdot w(t, x) dx, \]

Using the local Lipschitz property of \( b \), (11) and (1), we deduce

\[|\langle F(u_t^1) - F(u_t^2), w(t) \rangle| \leq L_{hf} \int_0^r \left\{ \int_\Omega |w(t + \theta, y)| dy \cdot \int_\Omega |\xi(\theta, x, u^1(t), u_t^2)| \cdot |w(t, x)| dx \right\} d\theta \]
\[+ C_{bf} M_f |\Omega| \int_0^r ||\xi(\theta, \cdot, u^1(t), u_t^2) - \xi(\theta, \cdot, u^2(t), u_t^2)||_{D(A^{-1/2})} d\theta \cdot ||A^{1/2}w(t)|| \]
\[\leq L_{hf} \sqrt{|\Omega|} \sup_{\theta \in (-r, 0)} ||\xi(\theta, \cdot, u^1(t), u_t^2)||_{D(A^{-1/2})} \cdot \int_0^r ||w(t + \theta, \cdot)|| \cdot ||A^{1/2}w(t)|| d\theta \]
\[+ C_{bf} M_f |\Omega| L_{f,M} \left[ ||w(t)||^2 + \int_0^r ||w(t + s)||^2 ds \right]^{1/2} \cdot ||A^{1/2}w(t)|| \]
\[\leq L_{hf} \sqrt{|\Omega|} \sup_{\theta \in (-r, 0)} ||\xi(\theta, \cdot, u^1(t), u_t^2)||_{D(A^{-1/2})} \cdot \int_0^r ||w(t + \theta, \cdot)|| \cdot ||A^{1/2}w(t)|| d\theta \]
\[+ \frac{1}{2} ||A^{1/2}w(t)||^2 + \frac{1}{2} C_{bf}^2 M_f^2 |\Omega| L_{f,M}^2 \left[ ||w(t)||^2 + \int_0^r ||w(t + s)||^2 ds \right] \]

Finally, we get the existence of positive constants \( C_1, C_2 \) such that

\[|\langle F(u_t^1) - F(u_t^2), w(t) \rangle| \leq ||A^{1/2}w(t)||^2 + C_1 \int_0^r ||w(t + \theta)||^2 d\theta + C_2 ||w(t)||^2 \]

The last estimate and (15) give

\[ \frac{d}{dt} ||w(t)||^2 + 2 ||A^{1/2}w(t)||^2 + 2d ||w(t)||^2 \leq ||A^{1/2}w(t)||^2 + C_1 \int_0^r ||w(t + \theta)||^2 d\theta + C_2 ||w(t)||^2 \]
\[\leq ||A^{1/2}w(t)||^2 + C_1 \left( \int_0^r ||w(\theta)||^2 d\theta + \int_0^t ||w(s)||^2 ds \right) + C_2 ||w(t)||^2. \]

Hence

\[ \frac{d}{dt} ||w(t)||^2 + ||A^{1/2}w(t)||^2 + 2d ||w(t)||^2 \leq C_1 \left( \int_0^r ||w(\theta)||^2 d\theta + \int_0^t ||w(s)||^2 ds \right) + C_2 ||w(t)||^2. \]
and property \( \|A^{1/2}v\|^2 \geq \lambda_1 \|v\|^2 \) gives
\[
\frac{d}{dt} \left[ \|w(t)\|^2 + (\lambda_1 + 2d) \int_0^t \|w(s)\|^2 ds \right] \leq C_1 \left( \int_{-r}^0 \|w(\theta)\|^2 d\theta + \int_0^t \|w(s)\|^2 ds \right) + C_2 \|w(t)\|^2.
\]

It implies that there exists \( C_3 > 0 \), such that for \( Z(t) \equiv \|w(t)\|^2 + (\lambda_1 + 2d) \int_0^t \|w(s)\|^2 ds \), we have
\[
\frac{d}{dt} Z(t) \leq C_3 Z(t) + C_1 \int_{-r}^0 \|w(\theta)\|^2 d\theta.
\]

Gronwall lemma implies
\[
Z(t) \leq \left( \|w(0)\|^2 + C_1 C_3^{-1} \int_{-r}^0 \|w(\theta)\|^2 d\theta \right) \cdot e^{C_3 t}.
\] (16)

The last estimate allows one to apply the well-known

**Proposition 2.** [32] Theorem 9] Let \( X \) be a Banach space. Then any \(*\)-weak convergent sequence \( \{w_k\}_{n=1}^\infty \in X^* \) \(*\)-weakly converges to an element \( w_\infty \in X^* \) and \( \|w_\infty\|_X \leq \liminf_{n \to \infty} \|w_n\|_X \).

Hence, for the difference \( u^1(t) - u^2(t) \) of two solutions we have
\[
\|u^1(t) - u^2(t)\|^2 + 2(\lambda_1 + d) \int_0^t \|u^1(s) - u^2(s)\|^2 ds
\leq \left( \|u^1(0) - u^2(0)\|^2 + C_1 C_3^{-1} \int_{-r}^0 \|\varphi^1(\theta) - \varphi^2(\theta)\|^2 d\theta \right) \cdot e^{C_3 t}.
\] (17)

We notice that by (6) the difference \( \|u^1(t) - u^2(t)\| \) makes sense for all \( t \in [0,T], \forall T > 0 \). The last estimate gives the uniqueness of solutions and completes the proof of Theorem 2.

Theorems 1 and 2 allow us to define the evolution semigroup \( S_t : H \to H \), with \( H \equiv L^2(\Omega) \times L^2((-r,0); L^2(\Omega)) \), by the formula \( S_t(u^0; \varphi) \equiv (u(t); u(t+\theta)), \theta \in (-r,0) \), where \( u(t) \) is the weak solution of (11), (2). The continuity of the semigroup with respect to time follows from (6), and with respect to initial conditions from (17).

For the study of long-time asymptotic properties of the above evolution semigroup we recall (see e.g. [2] [27])

**Definition 2.** A global attractor of the semigroup \( S_t \) is a closed bounded set \( U \) in \( H \), strictly invariant \( (S_t U = U \) for any \( t \geq 0 \)), such that for any bounded set \( B \subset H \) we have \( \lim_{t \to +\infty} \sup \{ \text{dist}_H(S_t v, U) ; y \in B \} = 0 \).

**Theorem 3.** Assume functions \( b \) and \( f \) satisfy properties (i), (ii) of Theorem 1. Let function \( \xi \) satisfy properties (iii)-a) of Theorem 1, (14) and also there exists \( C_{\xi,0} > 0 \) such that (c.f. (3))
\[
\int_{-r}^0 \|\xi(\theta, \cdot, v, \psi)\| d\theta \leq C_{(\xi,0)} \text{ for all } (v, \psi) \in H.
\] (18)
Then the dynamical system \((S_t; H)\) has a compact global attractor \(U\) which is a bounded set in the space \(H_1 \equiv D(A^α) \times W\), where \(W = \{φ : φ ∈ L^∞(−r, 0; D(A^α)), ϕ ∈ L^∞(−r, 0; D(A^{α−1}))\}, α ≤ \frac{1}{2}\).

Proof of Theorem 3. To prove the existence of the global attractor we use classical theorem saying that it is sufficient for the dynamical system \((S_t, H)\) to be dissipative and asymptotically compact (see [2, 27, 4]).

The property (18) gives the estimate stronger than (8):

\[ \|⟨F(u_t), v⟩_{L^2(Ω)}\| ≤ C_b M_f|Ω|C(ξ,0) \cdot \|v\| \] (19)

which is necessary for the property of dissipativeness of \((S_t; H)\). The rest of the proof, including the property of asymptotic compactness, is standard (see e.g. [2, 4, 18] and also [20, 21]).

3. Stationary solutions

For simplicity of presentation, in this section we consider operator \(A = (−Δ_D) > 0\), where \(Δ_D\) is the Laplace operator in \(L^2(Ω)\) with the Dirichlet boundary conditions. In this case (which is sufficient for the application to the Nicholson’s blowfly equation), we have \(D(A) = H^2(Ω) ∩ H^1_0(Ω), D(A^{1/2}) = H^1_0(Ω), D(A^{−1/2}) = H^−1(Ω)\). For more details on this classical Sobolev spaces see e.g. [13].

In this section we concentrate on the stationary solutions. First of all, by definition 1 (of a weak solution), \(u(t, x) ∈ L^2(0, T; D(A^{1/2})) = L^2(0, T; H^1_0(Ω))\), so for the stationary solution \(u(t, x) ≡ u^{st}(x)\), one has \(u^{st} ∈ H^1_0(Ω)\).

Let us consider an arbitrary function \(u^{st} ∈ H^1_0(Ω) ⊂ L^2(Ω)\). Our goal is to find conditions on a function \(ξ(\cdot, \cdot, \cdot, \cdot)\) such that the system (11) has stationary solution \(u(t) ≡ u^{st} ∈ H^1_0(Ω)\) for all \(t ∈ R\). Let us denote by \(\overline{u^{st}} ≡ u^{st}(θ) ≡ u^{st}, θ ∈ [−r, 0]\).

Since for \(u^{st} = 0 ∈ H^1_0(Ω)\) we can choose \(ξ(\cdot, \cdot, 0, 0) ≡ 0\), we concentrate below on the case \(u^{st} ≠ 0 ∈ H^1_0(Ω)\).

From (11) and \(\frac{∂}{∂t} u(t, x) ≡ 0\), we have

\[ Au^{st}(x) + d \cdot u^{st}(x) = f_Ω b(u^{st}(y))f(x − y)dy \cdot f_0^r ξ(θ, x, u^{st}, \overline{u^{st}})dθ, \quad x ∈ Ω. \] (20)

As we will show, it is sufficient to define in a proper way the value of \(ξ\) for the second and third coordinates equal \((u^{st}, \overline{u^{st}}) ∈ H \equiv L^2(Ω) × L^2(−r, 0; L^2(Ω))\) only. We propose to look for this value, decomposing it on the time and space coordinates i.e.

\[ ξ(θ, x, u^{st}, \overline{u^{st}}) = χ(θ) \cdot v(x), \quad θ ∈ [−r, 0], \quad x ∈ Ω. \] (21)

Now equation (20) reads

\[ Au^{st}(x) + d \cdot u^{st}(x) = f_Ω b(u^{st}(y))f(x − y)dy \cdot v(x) \cdot f_0^r χ(θ)dθ, \quad x ∈ Ω. \] (22)

We need the following elementary

**Lemma.** Assume \(u^{st} ≠ 0 ∈ L^2(Ω)\). Let the function \(f\) be strictly positive and \(f ∈ C^∞(Ω − Ω)\). Let function \(b\) be bounded and satisfy \(b(w) > 0\) for all \(w ≠ 0\).
Then the function
\[ p(x) \equiv \int_{\Omega} b(u^s(y))f(x - y)dy \]  \hspace{1cm} (23)

satisfies properties: \( p \in C(\overline{\Omega}) \), \( \inf \{p(x) : x \in \overline{\Omega}\} \equiv p_{\min} > 0 \) and \( \sup \{\frac{\partial}{\partial x_i}p(x) : x \in \overline{\Omega}, i = 1, \ldots, n_0\} \equiv p'_{\max} < \infty. \)

The continuity of \( p \) on \( \overline{\Omega} \) follows immediately from the continuity of \( f \) and the Cauchy-Schwartz inequality
\[ |p(x^1) - p(x^2)| = \left| \int_{\Omega} b(u^s(y)) \left[ f(x^1 - y) - f(x^2 - y) \right] dy \right| \leq \|b(u^s(\cdot))\| \cdot \left\| \int_{\Omega} \left| f(x^1 - y) - f(x^2 - y) \right|^2 dy \right\|^{1/2}. \]  \hspace{1cm} (24)

We also use that for all \( y \in \Omega \) one has \( |(x^1 - y) - (x^2 - y)| = |x^1 - x^2|. \)

Properties \( b(w) > 0 \) for all \( w \neq 0 \), \( u^s \neq 0 \in L^2(\Omega) \) and strict positivity of \( f \) imply that \( p(x) > 0 \) for all \( x \in \overline{\Omega} \). Hence, the continuity of \( p \) and the Weierstrass theorem give \( \inf \{p(x) : x \in \overline{\Omega}\} \equiv p_{\min} > 0. \) The boundedness of partial derivatives of \( p \) is due to \( f \in C^\infty(\overline{\Omega} - \Omega). \)

We also assume
\[ \int_{-r}^0 \chi(\theta)d\theta \neq 0. \]  \hspace{1cm} (25)

Under the assumptions of Lemma and (25) we have \( p(x) \cdot \int_{-r}^0 \chi(\theta)d\theta \neq 0 \) for all \( x \in \Omega \). So we can write (see (22))
\[ \hat{v}(x) = \frac{Au^s(x) + d \cdot u^s(x)}{\int_{\Omega} b(u^s(y))f(x - y)dy \cdot \int_{-r}^0 \chi(\theta)d\theta}. \]  \hspace{1cm} (26)

We notice that (26) is the equality in \( D(A^{-1/2}) = H^{-1}(\Omega) \) (in the sense of distributions). As we saw, by definition 1 (of a weak solution), \( u(t, x) \in L^2(0, T; D(A^{1/2})) = L^2(0, T; H_0^1(\Omega)) \), so for the stationary solution \( u(t, x) \equiv u^s(x) \), one has \( u^s \in H^1_0(\Omega). \) This implies \( Au^s \in H^{-1}(\Omega) = D(A^{-1/2}) \) and \( Au^s + d \cdot u^s \in H^{-1}(\Omega). \) To show that \( \hat{v} \in H^{-1}(\Omega) \) we remind the following

**Proposition 3.** [13] Theorem 12.1 Let \( m \) be positive integer. Then any element \( h \in H^{-m}(\Omega) \) may be represented (in the non-unique way) in the form
\[ h = \sum_{|\alpha| \leq m} D^\alpha h_j, \quad h_j \in L^2(\Omega). \]

Here \( D^\alpha \equiv \frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \quad \alpha = \{\alpha_1, \ldots, \alpha_n\}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n. \)

In our case \( m = 1 \) and if we denote by \( h = Au^s + d \cdot u^s \in H^{-1}(\Omega) \) and by \( q(x) \equiv p^{-1}(x) \), then \( \hat{v} \in H^{-1}(\Omega) \) reads as \( qh \in H^{-1}(\Omega). \) Using proposition 3, we write \( h = h_0 + \sum_{i=1}^{n_0} \frac{\partial}{\partial x_i} h_i, \quad h_i \in L^2(\Omega). \) Hence,
\[ \hat{v} = qh = qh_0 + \sum_{i=1}^{n_0} q \frac{\partial}{\partial x_i} h_i = qh_0 + \sum_{i=1}^{n_0} \left[ \frac{\partial}{\partial x_i}(qh_i) - h_i \frac{\partial}{\partial x_i} q \right]. \]  \hspace{1cm} (27)
Remark. We notice that all the derivatives are understood in the sense of distributions (see [22, 10]). The term \( q \cdot h \) is understood as the distribution which is obtained by multiplication of the distribution \( h \) by the infinitely differentiable function \( q \) (by definition, \( (q \cdot h, \varphi) \equiv (h, q \cdot \varphi), \forall \varphi \in D \) as in [22, 10]), since the operation of multiplication is not defined for two distributions. Using this definition, it is easy to check that \( \frac{\partial}{\partial x_i}(q \cdot h) = h \cdot \frac{\partial}{\partial x_i} q + q \cdot \frac{\partial}{\partial x_i} h \).

By proposition 3, to get \( \hat{v} \in H^{-1}(\Omega) \) it is enough to show (see (27)) that

\[
q h_i \in L^2(\Omega), \quad h_i \frac{\partial}{\partial x_i} q \in L^2(\Omega). \tag{28}
\]

The first inclusion in (28) follows from Lemma and

\[
\int_\Omega |q(x) h_i(x)|^2 dx \leq \left[ \sup \{ |q(x)| : x \in \Omega \} \right]^2 \cdot ||h_i||^2 = p_{\min}^{-2} \cdot ||h_i||^2 < +\infty.
\]

The second inclusion in (28) holds due to

\[
\int_\Omega |h_i(x) \frac{\partial}{\partial x_i} q(x)|^2 dx \leq \left[ \sup \left\{ \left| \frac{\partial}{\partial x_i} q(x) \right| : x \in \Omega, i = 1, \ldots, n_0 \right\} \right]^2 \cdot ||h_i||^2
\]

\[
\leq \left[ \frac{p_{\max}'}{p_{\min}^2} \right]^2 \cdot ||h_i||^2 < +\infty.
\]

Here we use (see Lemma)

\[
\left| \frac{\partial}{\partial x_i} q(x) \right| = \left| \frac{\partial}{\partial x_i} (p^{-1}(x)) \right| = \left| - \frac{\partial p(x)}{\partial x_i} \cdot p^{-2}(x) \right|
\]

\[
\leq \sup \left\{ \left| \frac{\partial}{\partial x_i} q(x) \right| : x \in \Omega, i = 1, \ldots, n_0 \right\} p_{\min}^{-2} \leq p_{\max}' \cdot p_{\min}^{-2}.
\]

So we get the property \( \hat{v} \in H^{-1}(\Omega) = \mathcal{D}(A^{-1/2}) \) which is very important for us to justify the choice of assumptions on the state-dependent function \( \xi \) (the choice of a class of functions \( \xi \)) in this article. Now we see that, assuming (in addition to (25)) that \( \int_0^r |\chi(\theta)| d\theta < \infty \), the function \( \xi \) defined by (21) with \( \hat{v} \) defined by (26) possesses the property (5).

As a result, we may conclude that for any (finite or infinite) sequence of isolated points \( \{u^{st,k}\} \subset H^1_0(\Omega) \subset L^2(\Omega) \) we can define a state-dependent function \( \xi \), which satisfies assumptions of Theorem 1 and such that system (11) with this \( \xi \) will have all the points \( \{u^{st,k}\} \subset H^1_0(\Omega) \) as stationary solutions \( u^k(t) \equiv u^{st,k}, t \in \mathbb{R} \). The last property means that our model with distributed in space and time state-dependent delay term may be successfully used having information (say from experiments) on an arbitrary set of isolated stationary solutions.

We notice that the definition of values of \( \xi(\cdot, \cdot, u^{st,k}, u^{st,k}) \) by (21), (26) on a set of isolated points does not contradict property (11) since the last one deals with the case of convergent sequence of points in \( H \).

To conclude this section we collect all the assumptions on functions \( \xi, b, f \) used in our considerations:
Ab) Function $b : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, bounded and satisfies $b(w) > 0$ for all $w \neq 0$.

Af1) Function $f : \overline{\Omega} - \Omega \to \mathbb{R}$ is bounded.

Af2) Function $f$ is strictly positive and $f \in C^\infty(\overline{\Omega} - \Omega)$.

Aξ1) Function $\xi : \{-r, 0\} \times \Omega \times L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \to R$ satisfies the following condition:

for any $M > 0$ there exists $L_{\xi,M}$ so that for all $(v^i, \psi^i) \in H$ satisfying $\|v^i\|^2 + \int_{-r}^0 \|\psi^i(s)\|^2 ds \leq M^2, i = 1, 2$ one has

$$
\int_{-r}^0 \|\xi(\theta, \cdot, v^1, \psi^1) - \xi(\theta, \cdot, v^2, \psi^2)\|_{D(A^{-1/2})} d\theta
\leq L_{\xi,M} \cdot \left[\|v^1 - v^2\|^2 + \int_{-r}^0 \|\psi^1(s) - \psi^2(s)\|^2 ds\right]^{1/2}.
$$

Aξ2) There exists $C_{(\xi, -, 1/2)} > 0$ so that $\int_{-r}^0 \|\xi(\theta, \cdot, v, \psi)\|_{D(A^{-1/2})} d\theta \leq C_{(\xi, -, 1/2)}$ for all $(v, \psi) \in H$.

Aξ3) Function $\xi$ satisfies $\xi(\cdot, \cdot, v, \psi) \in L^\infty(-r, 0; D(A^{-1/2}))$ for all $(v, \psi) \in H$.

Aξ4) There exists $C_{(\xi, 0)} > 0$ so that $\int_{-r}^0 \|\xi(\theta, \cdot, v, \psi)\| d\theta \leq C_{(\xi, 0)}$ for all $(v, \psi) \in H$.

Aχ) Function $\chi$ satisfies $\int_{-r}^0 \chi(\theta) d\theta \neq 0$ and $\int_{-r}^0 |\chi(\theta)| d\theta < \infty$.

As an application we can consider the diffusive Nicholson’s blowflies equation (see e.g. [26, 24]) with state-dependent delays. More precisely, we consider equation (11) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary, the function $f$ can be a constant as in [26, 24] which leads to the local in space coordinate term or, for example, $f(s) = \frac{1}{\sqrt{4\pi\alpha}}e^{-s^2/4\alpha}$, as in [25] which corresponds to the non-local term, the nonlinear function $b$ is given by $b(w) = p \cdot we^{-w}$. Function $b$ is bounded and $b(w) > 0$ for all $w \neq 0$. As a result, we conclude that for any functions $\xi$ satisfying conditions of Theorems 2 and 3 the dynamical system $(S_t, H)$ has a global attractor (Theorem 3).

So, our system (11) with distributed in space and time state-dependent delay term may be successfully used to study Nicholson’s blowflies equation with an arbitrary set of isolated stationary solutions.

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References

[1] N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina, Introduction to the theory of functional differential equations, Moscow, Nauka, 1991.

[2] A. V. Babin, and M. I. Vishik, Attractors of Evolutionary Equations, Amsterdam, North-Holland, 1992.

[3] L. Boutet de Monvel, I. D. Chueshov and A. V. Rezounenko, Inertial manifolds for retarded semilinear parabolic equations, *Nonlinear Analysis*, 34 (1998), 907-925.

[4] I. D. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, Acta, Kharkov (1999), (in Russian). English transl. Acta, Kharkov (2002).

[5] I. D. Chueshov, On a certain system of equations with delay, occurring in aeroelasticity, *J. Soviet Math.* 58, 1992, p.385-390.

[6] I. D. Chueshov, A. V. Rezounenko, Global attractors for a class of retarded quasilinear partial differential equations, *C.R.Acad.Sci.Paris*, Ser.I 321 (1995), 607-612, (detailed version: *Math.Physics, Analysis, Geometry*, Vol.2, N.3 (1995), 363-383).

[7] O. Diekmann, S. van Gils, S. Verduyn Lunel, H.-O. Walther, Delay Equations: Functional, Complex, and Nonlinear Analysis, Springer-Verlag, New York, 1995.

[8] J. K. Hale, Theory of Functional Differential Equations, Springer, Berlin-Heidelberg- New York, 1977.

[9] J. K. Hale and S. M. Verduyn Lunel, Theory of Functional Differential Equations, Springer-Verlag, New York, 1993.

[10] A. Kolmogorov, S. Fomine, Elements of the theory of functions and functional analysis. 3rd ed. Moscow: MIR, 536p. 1994.

[11] T. Krisztin, H.-O. Walther and J. Wu, Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback, *Fields Institute Monographs*, 11, AMS, Providence, RI, 1999.

[12] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.

[13] J. L. Lions and E. Magenes, Problèmes aux Limites Non Homogénes et applications, Dunod, Paris, 1968.

[14] A.D. Mishkis, Linear differential equations with retarded argument. 2nd edition, Nauka, Moscow, 1972.
[15] J. Mallet-Paret and R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags I, Archive for Rational Mechanics and Analysis 120 (1992), 99-146.

[16] J. Mallet-Paret and R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags II, J. Reine Angew. Math., 477 (1996), 129-197.

[17] J. Mallet-Paret, R. D. Nussbaum, P. Paraskevopoulos, Periodic solutions for functional-differential equations with multiple state-dependent time lags, Topol. Methods Nonlinear Anal. 3 (1994), no. 1, 101–162.

[18] A. V. Rezounenko, On singular limit dynamics for a class of retarded nonlinear partial differential equations, Matematicheskaya fizika, analiz, geometriya. -1997. N.4 (1/2), 193-211.

[19] A.V. Rezounenko, A short introduction to the theory of ordinary delay differential equations. Lecture Notes. Kharkov University Press, Kharkov, 2004.

[20] A.V. Rezounenko, J. Wu, A non-local PDE model for population dynamics with state-selective delay: local theory and global attractors // Journal of Computational and Applied Mathematics. -2006. Vol. 190, Issues 1-2, P.99-113.

[21] A.V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays // Journal of Mathematical Analysis and Applications. -2007. Vol. 326, Issue 2, (15 February 2007), 1031-1045. ( see preprint version: "A.V. Rezounenko, Two models of partial differential equations with discrete and distributed state-dependent delays", preprint. March 22, 2005, http://arxiv.org/abs/math.DS/0503470 ).

[22] L. Schwartz, Theorie des distributions. I et II, Hermann, Paris, 1950-1951.

[23] R.E.Showalter, Monotone operators in Banach space and nonlinear partial differential equations, AMS, Mathematical Surveys and Monographs, vol. 49, 1997.

[24] J. W. -H. So, J. Wu and Y. Yang, Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson’s blowflies equation. Appl. Math. Comput. 111 (2000), no. 1, 33–51.

[25] J. W. -H. So, J. Wu and X.Zou, A reaction diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains, Proc. Royal. Soc. Lond. A (2001) 457, 1841-1853.

[26] J. W.- H. So and Y. Yang, Dirichlet problem for the diffusive Nicholson’s blowflies equation, J. Differential Equations 150 (1998), no. 2, 317–348.

[27] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer, Berlin-Heidelberg-New York, 1988.
[28] C. C. Travis and G. F. Webb, Existence and stability for partial functional
differential equations, *Transactions of AMS* **200**, (1974), 395-418.

[29] H. -O. Walther, Stable periodic motion of a system with state dependent delay,
*Differential and Integral Equations* **15** (2002), 923-944.

[30] H.-O. Walther, The solution manifold and $C^1$-smoothness for differential equa-
tions with state-dependent delay, *J. Differential Equations* **195** (2003), no. 1,
46–65.

[31] J. Wu, Theory and Applications of Partial Functional Differential Equations,
Springer-Verlag, New York, 1996.

[32] K. Yosida, Functional analysis, Springer-Verlag, New York, 1965.

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