Pair Creation of Black Holes

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Abstract

This article is based on a talk given at the IIInd International Colloquium on Modern Quantum Field Theory, Bombay 1994. The Ernst solution of dilaton gravity describes charged black holes undergoing uniform acceleration in a background magnetic field. By analytically continuing the Ernst solution one obtains instantons that describe the pair production of black holes in the background field. We review various aspects of these solutions paying special attention to the Einstein-Maxwell, low-energy string and $d = 5$ Kaluza-Klein theories. It is based on work done in collaboration with F. Dowker, S. Giddings, G. Horowitz, D. Kastor and J. Traschen [1,2].

1. Introduction

It was first demonstrated by Schwinger [3] that a uniform electric field is unstable to the formation of electron-positron pairs. This was generalised by Affleck and Manton [4] who showed that a background magnetic field will spontaneously produce monopole-antimonopole pairs. In general relativity, the analogue of a monopole is a magnetically charged black hole, and the question naturally arises as to whether black holes can be pair produced by a background magnetic field.

Since the configuration of two black holes has a different spatial topology than the vacuum, unlike the monopole case, one cannot continuously deform one into the other: pair production of black holes necessarily involves topology change. The most natural
framework for quantum gravity in which to investigate such processes is consequently the sum-over-histories formulation. We assume that the path-integrals can be evaluated semiclassically using instanton techniques and thus we look for finite action solutions to the euclidean equations of motion that interpolate between the appropriate initial and final 3-geometries.

Affleck and Manton used an approximate instanton to estimate the pair creation rate for monopoles (see also [5]). As Gibbons first realized [6], an exact instanton for the Einstein-Maxwell theory can be obtained by analytically continuing a solution found by Ernst almost twenty years ago [7]. The Ernst solution describes two oppositely charged black holes undergoing uniform acceleration in a background magnetic field. (Ernst actually considered electric fields, but by duality, that is equivalent to the magnetic fields we will consider here.) The Ernst solution describes the evolution of the black holes after their creation. Regularity of the euclidean instanton turns out to restrict the charge to mass ratio of the black holes. Gibbons believed that only extremal black holes could be created. But Garfinkle and Strominger [8] found a regular instanton for which the black holes were slightly non-extremal and the horizons of the two black holes were identified to form a wormhole in space. The action of the instanton was evaluated in [8] to give the rate of production of the wormhole instantons and it was shown that the leading term was simply the Schwinger rate.

In more recent work these investigations have been extended in the context of “dilaton gravity” describing the interaction between a dilaton, a $U(1)$ gauge field and gravity via the action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2(\nabla\phi)^2 - e^{-2a\phi}F^2 \right]. \quad (1.1)$$

For $a = 0$ it is always consistent with the equations of motion to set $\phi = 0$ and we recover the standard Einstein-Maxwell theory. For $a = 1$, $S$ is part of the action describing the low energy dynamics of string theory. Note that for some physical questions it is more appropriate to use the conformally rescaled string metric, $\tilde{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu}$. The value $a = \sqrt{3}$ is also of special interest since this corresponds to standard Kaluza-Klein theory, i.e. (1.1) is equivalent to the five-dimensional vacuum Einstein action for geometries with a spacelike symmetry if we define the five dimensional metric via

$$ds^2 = e^{-4\phi/\sqrt{3}}(dx_5 + 2A_\mu dx^\mu)^2 + e^{2\phi/\sqrt{3}}g_{\mu\nu}dx^\mu dx^\nu. \quad (1.2)$$

For all values of $a$ there exist static charged black hole solutions which we shall briefly review below. In [1] the generalisation of the Ernst solution was constructed for all values of
\( a \) and it was shown that regular wormhole type instantons exist only for \( a < 1 \). However, it was shown in [3] that extremal instantons exist for all values of \( a \). Furthermore the action for both type of instantons was calculated in [2] and again it was shown that the rate of pair production, to leading order, is the Schwinger rate. This article is essentially a summary of the results of [1] and [2].

The plan of the rest of the article is as follows. In section 2 we review the black hole solution and discuss the analogue of a uniform magnetic field in General Relativity, the Melvin solution. The dilaton Ernst solution is discussed in section 3 and in particular we examine the extremal limit. In section 4 we discuss the euclidean instantons obtained by analytic continuation of the Ernst solution and calculate their action. In section 5 we make some comments on quantum corrections and section 6 contains some conclusions.

2. Background

2.1. Black Hole Solutions

The static, spherically symmetric magnetically charged black hole solution is given by [9,10]

\[
\begin{align*}
 ds^2 &= -\lambda^2 dt^2 + \lambda^{-2} dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \\
 e^{-2a\phi} &= \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{(1+a^2)}}, \\
 A_{\varphi} &= q(1 - \cos \theta) \\
 \lambda^2 &= \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{(1+a^2)}}, \\
 R^2 &= r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{(1+a^2)}}.
\end{align*}
\]

If \( r_+ > r_- \), the surface \( r = r_+ \) is the event horizon. For \( a = 0 \), the surface \( r = r_- \) is the inner Cauchy horizon of the Reissner-Nordstrom black hole, however for \( a > 0 \) this surface is singular. The parameters \( r_+ \) and \( r_- \) are related to the ADM mass \( m \) and total charge \( q \) by

\[
 m = \frac{r_+ + r_-}{2} + \left(1 - \frac{a^2}{1+a^2}\right) \frac{r_-}{2}, \quad q = \left(\frac{r_+ r_-}{1+a^2}\right)^{\frac{1}{2}}.
\]

The extremal limit occurs when \( r_+ = r_- \). The spatial geometry of the extreme Reissner-Nordstrom black hole resembles an infinite throat connected onto an asymptotically flat region. The situation in low-energy string theory is similar: the extremal spatial geometry is the same as long as one uses the string metric. In addition the entire extremal string 4-geometry is non-singular. In the Kaluza-Klein case the extremal geometry is just the Kaluza-Klein monopole [11,12]: the five dimensional metric (1.2) is the product of euclidean Taub-Nut with a real line and is non-singular.
2.2. Dilaton Melvin Spacetimes

The concept of a uniform magnetic field in flat spacetime must be modified when gravity is included since the field has non-zero stress-energy. The natural generalisation in Einstein-Maxwell theory is the Melvin solution \([13]\) which represents an infinitely long straight flux-tube. The solution is axisymmetric and static, the gravitational attraction being balanced by the transverse magnetic pressure.

The generalisation of the Melvin solution to dilaton gravity is given by \([9]\)

\[
\begin{align*}
ds^2 &= \Lambda^{2 \frac{2}{1+a^2}} [-dt^2 + d\rho^2 + dz^2] + \Lambda^{-2 \frac{2}{1+a^2}} \rho^2 d\varphi^2 \\
e^{-2a\phi} &= \Lambda^{2 \frac{2}{1+a^2}}, \\
A_\varphi &= -\frac{2}{(1 + a^2)BA} \\
\Lambda &= 1 + \frac{(1 + a^2)}{4} B^2 \rho^2
\end{align*}
\]

and now both the gravitational and scalar attraction balance the magnetic pressure. The square of the Maxwell field is \(F^2 = 2B^2/\Lambda^4\), which is a maximum on the axis \(\rho = 0\) and decreases to zero at infinity. The parameter \(B\) labels the strength of the magnetic field.

3. Dilaton Ernst

3.1. General Properties

The generalisation of the Ernst metric of Einstein-Maxwell theory to dilaton gravity was constructed in \([1]\) and is given by

\[
\begin{align*}
ds^2 &= (x - y)^{-2} A^{-2} \Lambda^{\frac{2}{1+a^2}} \left[ F(x) \left\{ G(y) dt^2 - G^{-1}(y) dy^2 \right\} + F(y) G^{-1}(x) dx^2 \right] \\
&\quad + (x - y)^{-2} A^{-2} \Lambda^{-\frac{2}{1+a^2}} F(y) G(x) d\varphi^2 \\
e^{-2a\phi} &= e^{-2a\phi_0} \Lambda^{\frac{2a^2}{1+a^2}} \frac{F(y)}{F(x)}, \\
A_\varphi &= -\frac{2e^{a\phi_0}}{(1 + a^2)BA} \left[ 1 + \frac{(1 + a^2)}{2} Bqx \right]
\end{align*}
\]

where the functions \(\Lambda \equiv \Lambda(x, y), F(\xi)\) and \(G(\xi)\) are given by

\[
\begin{align*}
\Lambda &= \left[ 1 + \frac{(1 + a^2)}{2} Bqx \right]^2 + \frac{(1 + a^2)B^2}{4A^2(x - y)^2} G(x) F(x) \\
F(\xi) &= (1 + r_- A\xi)^{\frac{2a^2}{(1+a^2)}} \\
G(\xi) &= (1 - \xi^2 - r_+ A\xi^3) (1 + r_- A\xi)^{\frac{(1-a^2)}{(1+a^2)}}.
\end{align*}
\]
and \( q \) is related to \( r_+ \) and \( r_- \) by (2.2). As we will discuss, this solution possesses both an inner and outer black hole horizon in addition to an acceleration horizon and asymptotically approaches the Melvin solution. Consequently, we can interpret the solution as describing charged black holes being uniformly accelerated in a background magnetic field.

The constant \( \phi_0 \) in the solution for the dilaton determines the value of the dilaton at infinity. Although one could keep this as a free parameter, we will fix it so that the dilaton vanishes on the axis at infinity in agreement with (2.3). The solution (3.1) depends on four other parameters, \( r_\pm, A, B \). Defining \( m \) and \( q \) via (2.2) we can loosely think of these parameters together with \( A, B \) as denoting the mass, charge and acceleration of the black holes and the strength of the magnetic field which is accelerating them, respectively. We emphasize, however, that this is heuristic since, for example, the mass and acceleration are not in general precisely defined and, further, we will see that \( q \) and \( B \) only approximate the physical charge \( \hat{q} \) and magnetic field \( \hat{B} \) in the limit \( r_\pm A \ll 1 \).

Before discussing the causal structure of the solution, it is convenient to introduce the following notation. Define \( \xi_1 \equiv -\frac{1}{r_- A} \) and let \( \xi_2 \leq \xi_3 < \xi_4 \) be the three roots of the cubic in \( G \). We restrict the range of the parameters \( r_+ \) and \( A \) so that \( r_+ A \leq 2/(3\sqrt{3}) \), so that the \( \xi_i \) are all real; the limit \( r_+ A = 2/(3\sqrt{3}) \) corresponds to \( \xi_2 = \xi_3 \). We also restrict the parameter \( r_- \) so that \( \xi_1 \leq \xi_2 \).

The metric (3.1) has two Killing vectors, \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \varphi} \). The surface \( y = \xi_1 \) is singular for \( a > 0 \), as can be seen from the square of the field strength. This surface is analogous to the singular surface (the “would be” inner horizon) of the dilaton black holes. The surface \( y = \xi_2 \) is the black hole horizon and the surface \( y = \xi_3 \) is the acceleration horizon; they are both Killing horizons for \( \frac{\partial}{\partial t} \).

The coordinates \( (x, \varphi) \) in (3.1) are angular coordinates. To keep the signature of the metric fixed, the coordinate \( x \) is restricted to the range \( \xi_3 \leq x \leq \xi_4 \) in which \( G(x) \) is positive. Due to the conformal factor \( (x - y)^{-2} \) in the metric, spatial infinity is reached by fixing \( t \) and letting both \( y \) and \( x \) approach \( \xi_3 \). Letting \( y \to x \) for \( x \neq \xi_3 \) gives null or timelike infinity [14]. Since \( y \to x \) is infinity, the range of the coordinate \( y \) is \(-\infty < y < x \) for \( a = 0 \), \( \xi_1 < y < x \) for \( a > 0 \).

The norm of the Killing vector \( \frac{\partial}{\partial \varphi} \) vanishes at \( x = \xi_3 \) and \( x = \xi_4 \), which correspond to the poles of the spheres surrounding the black holes. The axis \( x = \xi_3 \) points along the symmetry axis towards spatial infinity. The axis \( x = \xi_4 \) points towards the other black hole. Note that the coordinates we are using only cover one region of spacetime containing one of the black holes. As discussed in [1], to ensure that the metric is free of conical
singularities at both poles, \( x = \xi_3, \xi_4 \), for a single choice of period for \( \varphi \), we must impose the condition
\[
G'(\xi_3)\Lambda(\xi_4)\sqrt{1+a^2} = -G'(\xi_4)\Lambda(\xi_3)\sqrt{1+a^2}.
\] (3.3)
where \( \Lambda(\xi_i) \equiv \Lambda(x = \xi_i) \). When (3.3) is satisfied, the spheres are regular as long as \( \varphi \) has period
\[
\Delta \varphi = \frac{4\pi\Lambda^{\frac{2}{1+a^2}}(\xi_3)}{G'(\xi_3)}.
\] (3.4)
The condition (3.3) can be readily understood in the limit \( r_\pm A \ll 1 \) where it becomes Newton’s law,
\[
mA \approx qB,
\] (3.5)
where we have used (2.2) to replace \( r_\pm \) with \( m, q \). This is true for all \( a \). More generally, the condition (3.3) reduces the number of free parameters for the solution to three by relating the acceleration to the magnetic field, mass, and charge.

To show that the Ernst solution approaches the Melvin solution at large spacelike distances, \( x, y \to \xi_3 \), we change coordinates from \((x, y, t, \varphi)\) to \((\rho, \zeta, \eta, \tilde{\varphi})\) using
\[
x - \xi_3 = \frac{4F(\xi_3)\Lambda^{\frac{2}{1+a^2}}(\xi_3)}{G'(\xi_3)A^2} \frac{\rho^2}{(\rho^2 + \zeta^2)^2}, \quad \xi_3 - y = \frac{4F(\xi_3)\Lambda^{\frac{2}{1+a^2}}(\xi_3)}{G'(\xi_3)A^2} \frac{\zeta^2}{(\rho^2 + \zeta^2)^2}
\] (3.6)
\[
t = \frac{2\eta}{G'(\xi_3)}, \quad \varphi = \frac{2\Lambda^{\frac{2}{1+a^2}}(\xi_3)}{G'(\xi_3)} \tilde{\varphi}
\]
Note that \( \eta, \tilde{\varphi} \) are related to \( t, \varphi \) by a simple rescaling and that \( \tilde{\varphi} \) has period \( 2\pi \) due to (3.4). Choosing the constant \( \phi_0 \) appropriately, for large \( \rho^2 + \zeta^2 \), the dilaton Ernst metric reduces to
\[
ds^2 \to \tilde{\Lambda}^{\frac{2}{1+a^2}} \left( -\zeta^2 d\eta^2 + d\zeta^2 + d\rho^2 \right) + \tilde{\Lambda}^{\frac{2}{1+a^2}} \rho^2 d\tilde{\varphi}^2
\] (3.7)
where
\[
\tilde{\Lambda} = \left( 1 + \frac{1+a^2}{4}\tilde{B}^2 \rho^2 \right),
\]
\[
\tilde{B}^2 = \frac{B^2G'(\xi_3)^2}{4\Lambda^{\frac{2}{1+a^2}}(\xi_3)}.
\] (3.8)
which we identify as the dilaton Melvin metric in Rindler-type coordinates. In particular we note the coordinate \( t \) in the dilaton Ernst solution is the analogue of Rindler time (and

\footnote{It follows from (3.2) that when \( x \) is equal to a root of \( G(x) \), \( \Lambda(x, y) \) is independent of \( y \). So \( \Lambda(\xi_i) \) are constants.}
hence, since $\frac{\partial}{\partial t}$ is a Killing vector the spacetime is boost invariant, not static). It is clear that the physical magnetic charge is $\hat{B}$. The physical charge of the black hole is defined by $\hat{q} = \frac{1}{4\pi} \int F$ where the integral is over any two sphere surrounding the black hole. In the weak field limit $r_+ a \ll 1$, $\hat{B} \approx B$ and $\hat{q} \approx q$.

3.2. The Limit $\xi_1 = \xi_2$: Accelerating Extremal Black Holes

Since $y = \xi_2$ is the event horizon and $y = \xi_1$ is an inner horizon ($a = 0$) or singularity ($a > 0$), it follows that the extremal limit of the dilaton Ernst solution is given by choosing the parameter $r_-$ so that $\xi_1 = \xi_2$. Recalling the regularity condition (3.3), it follows that the extremal solution is described by two parameters which we can take to be the physical charge $\hat{q}$ and magnetic field $\hat{B}$. It was shown in detail in \cite{2} that that as one approaches the horizon $y \to \xi_2$ the extremal solution becomes spherically symmetric, and approaches the static black hole solution (2.1) with $r_- = r_+$. This surprising result has a number of consequences which we now discuss.

First, all the geometric properties of the extremal static solution near the horizon carry over immediately to the accelerated case. In particular, for $a = 0$, a constant-$t$ slice of the solution has an infinitely long throat. For $a = 1$, the string metric $ds^2 = e^{2\phi} ds^2$ also has an infinite throat in which the solution takes the form of the linear dilaton vacuum. For $a = \sqrt{3}$, the five dimensional metric (1.2) approaches that of the Kaluza-Klein monopole.

A second consequence is that there is a sense in which the extremal black holes are not accelerating. For $a = 0$, this is suggested by the fact that the event horizon is exactly spherical. But a more convincing argument comes from examining the acceleration of a family of observers near the horizon whose four velocities are proportional to $\partial/\partial t$. For the static black hole, the acceleration of these observers approaches the finite limit $1/q$ as they approach the horizon. (This is related to the fact that the surface gravity vanishes for extremal black holes and is in contrast to the non-extremal case in which the acceleration diverges.) If one computes the acceleration of these observers for the Ernst solution, one again finds that it approaches $1/\hat{q}$ as $y \to \xi_2$ independent of direction. Although this particular argument cannot be extended to $a > 0$ since the acceleration (in the Einstein metric) now diverges for the static extremal solution near the horizon, other arguments can be made. For example, when $a = 1$, in the string metric the acceleration of these observers tends to zero down the throat. In addition, when $a = \sqrt{3}$, $y = \xi_2$ is a regular origin in the five dimensional Kaluza-Klein solution, and one can show that its worldline is a geodesic!
Even though the black hole itself is not accelerating, the region around the black hole is. This is clear from the relation between the solution and the dilaton Melvin solution in accelerating coordinates discussed in the previous subsection. In terms of the infinite throats, one might say that the mouth of the throat is accelerating while the region down the throat is not.

4. Dilaton Ernst Instantons

To calculate tunnelling effects using instanton methods one looks for a classical solution to the euclidean equations of motion that interpolates between the appropriate initial and final 3-geometries. In addition, the classical solution must have zero momentum (zero extrinsic curvature) at the initial and final configurations. Let us call such a solution a “tunneling geometry” [15]. By analytically continuing the Ernst solution we obtain instantons or more precisely “bounces” that have a moment of time symmetry. The corresponding tunneling geometries are obtained by slicing the manifold along the moment of time symmetry. By construction it is clear that the final geometry has vanishing extrinsic curvature and furthermore that the final geometry provides good initial data for the subsequent lorentzian evolution given by the Ernst solution. To leading order in the semi-classical expansion, the tunnelling rate is given by $e^{-S_E}$ where $S_E$ is the euclidean action of the instanton after subtracting off the action of the background, in our case the background magnetic field.

Euclideanizing (3.1) by setting $\tau = it$, we find that another condition must be imposed on the parameters in order to obtain a regular solution. Two distinct ways that this may be achieved were discussed in [1] and are reviewed in the first subsection below. These include the wormhole instantons. There is a third way [2] which leads to the extremal instantons and is described in the second subsection. The calculation of the action for the wormhole and extremal instantons is given in the third subsection.

4.1. Wormhole Instantons

In the lorentzian solution, the vector $\partial/\partial t$ is timelike only for $\xi_2 < y < \xi_3$. If we make the restriction $\xi_1 < \xi_2$ then the Einstein metric has a regular horizon for all values of $a$. In this case, one must impose a condition on the parameters in order to eliminate conical singularities in the euclidean solution at both the black hole ($y = \xi_2$) and acceleration ($y = \xi_3$) horizons with a single choice of the period of $\tau$. This is equivalent to demanding
that the Hawking temperature of the black hole horizon equal the Unruh temperature of the acceleration horizon.

In terms of the metric function $G(y)$ appearing in (3.1), the period of $\tau$ is taken to be

$$\Delta \tau = \frac{4\pi}{G'(\xi_3)} \quad (4.1)$$

and the constraint is

$$G'(\xi_2) = -G'(\xi_3), \quad (4.2)$$

yielding

$$\left( \frac{\xi_2 - \xi_1}{\xi_3 - \xi_1} \right)^{\frac{1-a^2}{1+a^2}} (\xi_4 - \xi_2)(\xi_3 - \xi_2) = (\xi_4 - \xi_3)(\xi_3 - \xi_2). \quad (4.3)$$

With $\xi_1 < \xi_2$ there are two ways to satisfy this condition and correspondingly two types of instantons. The first one exists when $\xi_2 \neq \xi_3$ and only for $0 \leq a < 1$. It has topology $S^2 \times S^2 - \{pt\}$ where the removed point is $x = y = \xi_3$. This instanton is readily interpreted as a bounce: the surface defined by $\tau = 0, \Delta \tau/2$ has topology $S^2 \times S^1 - \{pt\}$, which is that of a wormhole attached to a spatial slice of Melvin and is the zero momentum initial data for the lorentzian Ernst solution. The tunneling geometry describes the pair creation of a pair of oppositely charged dilaton black holes in a magnetic field which subsequently uniformly accelerate away from each other. From the metric we deduce that there is a horizon sitting inside the wormhole throat, located at a finite proper distance from the mouth.

These “wormhole” instantons generalize the Einstein-Maxwell instanton discussed in $[8]$. The reason these instantons only exist for $0 \leq a < 1$ can be understood (for weak fields) by recalling the thermodynamic behavior of the dilaton black holes as extremality is approached: the Hawking temperature, as defined from the period of $\tau$ in the euclidean section, goes to zero for $0 \leq a < 1$, approaches a constant for $a = 1$ and diverges for $a > 1$. Thus, for small magnetic fields and hence accelerations, we expect to be able to match the resultant Unruh temperature and the black hole temperature by a small perturbation of the black hole away from extremality only for $0 \leq a < 1$.

The second class of instantons we mention only for completeness since their interpretation is obscure. They are defined by $\xi_2 = \xi_3$ which is equivalent to $r_+ A = 2/(3\sqrt{3})$, and have topology $S^2 \times R^2$. Note that for these instantons one does not have to impose the condition (3.3) for regularity.
4.2. Extremal Instantons

The wormhole type instantons discussed above were made regular by the condition that the temperatures of the black hole and acceleration horizons should be equal. Gibbons [6] pointed out (for \(a = 0\)) that there is another way that the temperatures of the black hole and acceleration horizons can be equal: that is if the black hole is extremal. This might seem strange since the extremal Reissner-Nordstrom black hole has zero temperature in the sense that the euclidean time coordinate need not be periodically identified to obtain a regular geometry. But, of course we can periodically identify the euclidean time and with any period we like (just as for flat space). In particular we choose \(\tau\) to have period (4.1) to eliminate the conical singularity at the acceleration horizon, \(y = \xi_3\).

For \(a = 0\) the extremal condition \(\xi_1 = \xi_2\) does indeed lead to a smooth instanton. The range of the coordinate \(y\) is \(\xi_2 < y \leq \xi_3\) in the euclidean section. We noted in Section 3.2 that the lorentzian solution near the back hole is just that of an extremal black hole. The same holds for the euclidean solution. The horizon \(y = \xi_2\) is infinitely far away (in every direction since every direction is now spacelike) and gives no restriction on the period of \(\tau\). Thus we have obtained a regular geometry with internal infinities down the throats of the extremal black holes. The length of the \(y = \) constant circles in the \((y, \tau)\) section tend to zero as \(y \to \xi_2\) but the curvature remains bounded and the radii of the two spheres approach a constant. The topology of this instanton is \(R^2 \times S^2 - \{ pt\}\) with the removed point again being \(x = y = \xi_3\). The 3-geometry created by the tunneling geometry, the \(\tau = 0, \Delta \tau/2\) zero momentum slice, is a spatial slice of a Melvin universe with two infinite tubes attached.

The extremal case \(\xi_1 = \xi_2\) also gives well defined instantons for \(0 < a \leq 1\). Although the Einstein metric has a singularity, the so called “total” metric [5], \(ds_T = e^{2\phi} ds^2\), which is the same as the string metric for \(a = 1\), is perfectly regular. We noted in Section 3.2 that the metric close to the singularity is that of the extremal black hole. In the total metric this looks like

\[
 ds_T^2 \propto -dt^2 + \left(1 - \frac{r_+}{r}\right)\frac{4}{1+a^2} dr^2 + r_+^2 \left(1 - \frac{r_+}{r}\right)^{\frac{2(a^2-1)}{1+a^2}} d\Omega_2^2 \tag{4.4}
\]

For \(0 < a < 1\), the total metric is geodesically complete and the spatial sections have the form of two asymptotic regions joined by a wormhole, one region being flat, the other having a deficit solid angle. Hence the corresponding extremal instantons are regular as long as we again choose the period of \(\tau\) to be (1.1) to ensure regularity as \(y \to \xi_3\). For
The geometry of the string metric is that of an infinitely long throat of constant radius and thus the $a = 1$ extremal instanton looks very much like that of the $a = 0$ extremal instanton described above: the topology is the same, $R^2 \times S^2 - \{pt\}$, and the major difference is that the proper radius of $y = \text{constant}$ circles in the $(y, \tau)$ section tends to a finite limit as $y \to \xi_2$. The $\tau = 0, \frac{\Delta \tau}{2}$ slice resembles the $a = 0$ case and hence the tunneling geometry describes the pair production of extreme $a = 1$ black holes with their infinite throats (in the string metric). It is perhaps worth pointing out that as $a \to 1$, the wormhole instanton approaches the $a = 1$ extremal instanton.

For $a > 1$, both the Einstein metric and the total metric have a naked singularity in the extremal limit. It has been argued in [16], however, that these “black holes” should be interpreted as elementary particles. The extremal instantons can then be interpreted as pair creating such objects. For $a = \sqrt{3}$, the five dimensional metric of the extremal instanton is already regular at $y = \xi_2$ and hence is regular if $\tau$ has period $(4.1)$. It was shown in [2] that the topology of the instanton is $S^5 - S^1$ and that the topology of the zero momentum slice is $S^4 - S^1$. It describes the creation of a Kaluza-Klein monopole-anti-monopole pair.

### 4.3. The action

To leading semiclassical order, the pair production rate of non-extremal or extremal black holes is given by $e^{-S_E}$ where $S_E$ is the euclidean action of the corresponding instanton solutions. The euclidean action including boundary terms is given by

$$S_E = \frac{1}{16\pi} \int_V d^4x \sqrt{g} \left[-R + 2(\nabla \phi)^2 + e^{-2a\phi}F^2\right] - \frac{1}{8\pi} \int_{\partial V} d^3x \sqrt{h}K$$  \hspace{1cm} (4.5)

where $h$ is the induced three metric and $K$ is the trace of the extrinsic curvature of the boundary.

For both the wormhole and extremal instantons there is a boundary at infinity, $x = y = \xi_3$ which contributes an infinite amount to the action. However, the action of the background magnetic field solution is itself infinite. It was shown in [2] how to subtract the infinite background contribution to obtain the physical result. For the extremal instantons there is also an additional boundary down the throats of the black holes i.e. at $y = \xi_2$. The contribution to the action from this boundary vanishes.

The result of the calculations of [2] is that the action is finite for both types of instantons and is given by

$$S_E = 2\pi d^2 \frac{\Lambda(\xi_4)(\xi_3 - \xi_2)}{\Lambda(\xi_3)(\xi_4 - \xi_3)}.$$  \hspace{1cm} (4.6)
For the $a = 0$ wormhole instantons this agrees with the result first obtained (using a different method) by Garfinkle and Strominger [8]. Notice that the result is finite for the extremal instantons despite the infinite throats for $0 \leq a \leq 1$ and despite the fact that there are singularities in both the Einstein and the total metric for $a > 1$. The action can be expressed in terms of the physical charge $\hat{q}$ and magnetic field $\hat{B}$ by expanding out in the parameter $\hat{q}\hat{B}$. The action for the wormhole type instantons is

$$S_E = \pi \hat{q}^2 \left[ \frac{1}{\hat{q}\hat{B}} - \frac{1}{2} + \cdots \right] \quad a = 0$$

$$S_E = \pi \hat{q}^2 \left[ \frac{1}{(1 + a^2)\hat{q}\hat{B}} + \frac{1}{2} + \cdots \right] \quad 0 < a < 1$$

while the action for the extremal type instantons for all $a$ is given by

$$S_E = \pi \hat{q}^2 \left[ \frac{1}{(1 + a^2)\hat{q}\hat{B}} + \frac{1}{2} + \cdots \right]$$

where dots denote higher order terms which may be fractional powers of $\hat{q}\hat{B}$. To leading order these all give the Schwinger result, $\pi m^2 / \hat{q}\hat{B}$ after using the relation between the mass and charge of extremal black holes, $(1 + a^2)m^2 = \hat{q}^2$.

To next-to-leading order, for $a = 0$ the action of the extremal instanton is greater than the action of the wormhole instanton by $\pi \hat{q}^2 = \frac{1}{4}A$ where $A$ is the area of the horizon of an extremal black hole of charge $\hat{q}$. In fact, to this order it could also be the area of the horizon of the wormhole instanton. This difference is precisely the Bekenstein-Hawking entropy. For $0 < a < 1$ the difference is zero to this order, which is consistent with the difference being the area of the horizon of the extremal instanton since that vanishes for $a > 0$. The area of the horizon in the wormhole instanton is non-zero, but higher order in $\hat{q}\hat{B}$.

In [17] a comparison was made between the wormhole action for $a = 0$ and the action of an instanton describing the creation of a monopole-anti-monopole pair. It was found that the action of the monopole instanton was greater than that of the wormhole instanton by the black hole entropy. Our result thus suggests that, at least for $a = 0$, the extremal black holes behave more like elementary particles than non-extremal ones. However, these conclusions neglect quantum corrections, to which we now turn.
5. Quantum Considerations

Until now we have been discussing the solutions purely at the classical level. In this section we will make some general comments about quantum corrections referring the reader to [3] for more details.

Let's first consider the lorentzian solution with general $m$ and $q$. An observer travelling on a trajectory at a fixed distance from the black hole will be accelerated\(^2\) and therefore would observe acceleration radiation if carrying a detector. This suggests that we should describe the black hole as being in contact with this approximately thermal radiation. If so, then the black hole would be expected to absorb energy, thus perturbing the solution. However, the black hole can also emit Hawking radiation, and therefore achieve a time-independent equilibrium state where the emission and absorption rates match. We have seen evidence that this might be possible (at least for $0 \leq a < 1$) in the construction of the euclidean wormhole instantons: it is possible for $0 \leq a < 1$ to constrain the parameters via (4.2) to match the Hawking and Unruh temperatures.

To address the issue of quantum corrections to the extremal geometries it is crucial to know the forms of the effective potentials for fluctuations about the extremal black holes. This was studied in detail in [16,18] and was shown to depend critically on the value of $a$. For $a > 1$ the potentials grow arbitrarily large as the horizon (singularity) is approached suggesting that the quantum corrections do not move the solution away from extremality. However, for $0 \leq a \leq 1$ there are potential barriers outside the black hole, but these vanish at the horizon (for the string theory case this requires the addition of a particular kind of matter to the action (1.1)). This suggests that the acceleration radiation can perturb the solution away from extremality. As noted above, there is a potential equilibrium solution for $0 \leq a < 1$ into which the extreme solution could evolve. However, it is not at all clear what happens to the $a = 1$ extremal geometry. It is perhaps worth pointing out that the string coupling constant $e^{2\phi}$ is getting large down the infinite throat and we expect string loop effects to substantially modify the solution in this region.

Similar comments apply to the instanton solutions. Since the Hawking and Unruh temperatures are matched for the wormhole instantons we do not expect significant quantum corrections. For the extremal instantons with $a > 1$ we don’t expect quantum corrections because of the infinite potential barriers around the singularity. On the other hand we

\(^2\) This is of course in addition to the usual acceleration needed to avoid falling into the black hole were it static.
expect significant quantum corrections to the extremal instantons with $0 \leq a \leq 1$. Are the extremal instanton solutions with $0 \leq a < 1$ somehow perturbed into the wormhole solutions? This is not clear since the two types of instantons pair produce black holes in a different topological class. It is even less clear what happens to the extremal instantons in string theory.

### 6. Conclusions

We have reviewed the Ernst solution of dilaton gravity which describes two dilaton black holes being uniformly accelerated in a uniform magnetic field. In the extremal limit, as one approaches the black hole horizon the solution reduces exactly to the static dilaton black hole solution. In this limit there is thus a sense in which although the surrounding spacetime is accelerating the black holes themselves are not.

By analytically continuing the Ernst solution one obtains two types of finite action instantons. The extremal instantons exist for all values of $a$ and describe the pair creation of extremal black holes. For $a = \sqrt{3}$ the extremal instanton describes the pair creation of Kaluza-Klein monopoles. The wormhole instantons exist for $0 \leq a < 1$ and describe the pair creation of two non-extremal black holes connected at their horizons to form a wormhole. Although we do not expect quantum corrections to significantly change the geometry of the wormhole instantons or the extremal instantons with $a > 1$, we do expect significant corrections to the extremal instantons with $a \leq 1$.

There is some evidence that $a = 1$ wormhole instantons exist if one allows the black holes to rotate. One piece of evidence comes from [19], where an approximate wormhole instanton was constructed which includes rotation. Another comes from the fact that for the rotating black hole with $a = 1$, the Hawking temperature goes to zero in the extremal limit whenever the angular momentum is non-zero [20]. Thus one could match the Unruh temperature at the acceleration horizon by a slightly non-extremal rotating black hole. Note that a wormhole solution to the $a = 1$ theory with 2 $U(1)$‘s has recently been constructed in [21].

One of the most important issues is to develop a better understanding of quantum corrections to the instanton approximation and their effects on the geometry and pair creation rate. It is particularly important to understand the calculation of the rate, as a finite answer may indicate that such black holes serve as a model for black hole remnants [22,23,18]. A better understanding of these corrections will also help to resolve the question
of whether an infinite volume of space can really be created in a finite amount of time. If so, there would appear to be problems with causality, unless the state down the throats were fixed uniquely. Hopefully I will have the opppurtunity of reporting on these and other topics at a future colloquium.

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