Fibonacci connection between Huffman codes
and Wythoff array

Alex Vinokur
Holon, Israel
alexvn@barak-online.net
alex.vinokur@gmail.com
Home Page: http://alexvn.freeservers.com/

Abstract. A non-decreasing sequence of positive integer weights \( P = \{ p_1, \ldots, p_2, p_n \} \) is called \( k \)-ordered if an intermediate sequence of weights produced by Huffman algorithm for initial sequence \( P \) on \( i \)-th step satisfies the following conditions: \( p_2^{(i)} = p_3^{(i)}, \) \( i = 0, k; \) \( p_2^{(i)} < p_3^{(i)}, \) \( i = k + 1, n - 3 \). Let \( T \) be a binary tree of size \( n \) and \( M = M(T) \) be a set of such sequences of positive integer weights that the tree \( T \) is the Huffman tree of \( P \) (\( |P| = n \)). A sequence \( P_{\text{min}} \) of \( n \) positive integer weights is called a minimizing sequence of the binary tree \( T \) in class \( M(P_{\text{min}} \in M) \) if \( P_{\text{min}} \) produces the minimal Huffman cost of the tree \( T \) over all sequences from \( M \), i.e., \( E(T, P_{\text{min}}) \leq E(T, P) \) \( \forall P \in M \).

Fibonacci related connection between minimizing \( k \)-ordered sequences of the maximum height Huffman tree and the Wythoff array \([\text{Sloane, A035513}]\) has been proved. Let \( M_{n,k} \) denote the set of all \( k \)-ordered sequences of size \( n \) for which the Huffman tree has maximum height. Let \( F(i) \) denote \( i \)-th Fibonacci number. **Theorem:** A minimizing \( k \)-ordered sequence of the maximum height Huffman tree in class \( M_{n,k} \) is \( P_{\text{min}}_{n,k} = \{ \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n \} \), where \( \bar{p}_1 = 1, \) \( \bar{p}_2 = F(1), \ldots, \) \( \bar{p}_{k+2} = F(k + 1), \) \( \bar{p}_{k+3} = F(k + 2), \) \( \bar{p}_{k+4} = w_{F(k+2),0}, \) \( \bar{p}_{k+5} = w_{F(k+2),2}, \ldots, \) \( \bar{p}_n = w_{F(k+2),n-k-3}; \) \( w_{i,j} \) is \((i,j)\)-th element of the Wythoff array.

The cost of Huffman trees for those sequences has been computed. Several examples of minimizing ordered sequences for Huffman codes are shown.

1 Main Conceptions and Terminology

1.1 Binary Trees

A (strictly) binary tree is an oriented ordered tree where each nonleaf node has exactly two children (siblings). A binary tree is called elongated if at least one of any two sibling nodes is a leaf. An elongated binary tree of size \( n \) has maximum height among all binary trees of size \( n \). An elongated binary tree is called left-sided if the right node in each pair of sibling nodes is a leaf.

A binary tree is called labeled if a certain positive integer (weight) is set in correspondence with each leaf.
1. Main Conceptions and Terminology

Size of a tree is the total number of leaves of this tree.

**Definition.** Let $T$ be a binary tree with positive weights $P = \{p_1, p_2, \ldots, p_n\}$ at its leaf nodes. The weighted external path length of $T$ is

$$E(T, P) = \sum_{i=1}^{n} l_i \cdot p_i,$$

where $l_i$ is the length of the path from the root to leaf $i$.

1.2 Huffman Algorithm

**Problem definition.** Given a sequence of $n$ positive weights $P = \{p_1, \ldots, p_n\}$. The problem is to find binary tree $T_{\min}$ with $n$ leaves labeled $p_1, p_2, \ldots, p_n$ that has minimum weighted external path length over all possible binary trees of size $n$ with the same sequence of leaf weights. $T_{\min}$ is called the Huffman tree of the sequence $P$; $E(T, T_{\min})$ is called the Huffman cost of the tree $T$.

The problem was solved by Huffman algorithm [1]. That algorithm builds $T_{\min}$ in which each leaf (weight) is associated with a (prefix free) codeword in alphabet $\{0, 1\}$.

**Note.** A code is called a prefix (free) code if no codeword is a prefix of another one.

**Algorithm description** (in the reference to the discussed issue).

**Algorithm input.** A non-decreasing sequence of positive weights $P = \{p_1, p_2, \ldots, p_n\}$ ($p_k \leq p_{k+1}$; $k = 1, n-1$).

**Algorithm output.** The sum of all the weights.

The algorithm is performed in $n-1$ steps. $i$-th step ($i = 1, n-1$) is as follows:

- **$i$-th step input.** A non-decreasing sequence of weights of size $n-i+1$.
  $$P^{(i-1)} = \{p_1^{(i-1)}, p_2^{(i-1)}, \ldots, p_{n-i+1}^{(i-1)}\} (p_k^{(i-1)} \leq p_{k+1}^{(i-1)}; k = 1, n-i).$$

- **$i$-th step method.** Build a sequence $\{p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, \ldots, p_{n-i+1}^{(i)}\}$ and sort its.

- **$i$-th step output.** A non-decreasing sequence of weights of size $n-i$.
  $$P^{(i)} = \{p_1^{(i)}, p_2^{(i)}, \ldots, p_{n-i}^{(i)}\} (p_k^{(i)} \leq p_{k+1}^{(i)}; k = 1, n-i-1).$$

**Note 1.** $P^{(0)}$ is an input of Huffman algorithm, i.e.,

$$p_k^{(0)} = p_k \ (k = 1, n). \quad (1)$$

**Note 2.** If an input sequence on $i$-th step(s) of the algorithm satisfies condition

$$p_2^{(i)} = p_3^{(i)} \ (0 \leq i \leq n-3).$$
then several Huffman trees can result from initial sequence $P$ of weights, but the weighted external path length is the same in all these trees.

Let $P = \{p_1, p_2, \ldots, p_n\}$ be a sequence of size $n$ for which the binary Huffman tree is elongated. Then according to Huffman algorithm

$$p^{(i)}_1 + p^{(i)}_2 \leq p^{(i)}_4 \quad (i = 0, n - 3).$$

### 1.3 Wythoff Array

The Wythoff array is shown below. It has many interesting properties \[2\], \[3\], \[4\].

| Row number | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | Note       |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|------------|
|            |     |     |     |     |     |     |     |     |     |     |     |     |            |
| Wythoff array |     |     |     |     |     |     |     |     |     |     |     |     |            |
| Fib[2]     | 1   | 3   | 4   | 7   | 11  | 18  | 29  | 47  | 76  | 123 | 199 | 322 | Fibonacci seq |
| Fib[3]     | 2   | 4   | 6   | 10  | 16  | 26  | 42  | 68  | 110 | 178 | 288 | 466 | Lucas seq   |
| Fib[4]     | 3   | 6   | 9   | 15  | 24  | 39  | 63  | 102 | 165 | 267 | 432 | 699 |            |
| Fib[5]     | 5   | 9   | 14  | 23  | 37  | 60  | 97  | 157 | 254 | 411 | 665 | 1076 | 1741       |
| Fib[6]     | 8   | 13  | 21  | 34  | 55  | 89  | 144 | 233 | 383 | 617 | 990 | 1607 | 2597       |
| Fib[7]     | 13  | 22  | 35  | 57  | 92  | 149 | 241 | 390 | 631 | 1021 | 1652 | 2673 |            |
|            |     |     |     |     |     |     |     |     |     |     |     |     |            |
| Generalized Wythoff array |     |     |     |     |     |     |     |     |     |     |     |     |            |

The two columns to the left of the Wythoff array consist respectively of the nonnegative integers $n$, and the lower Wythoff sequence whose $n$-th term is $\lfloor (n + 1) \cdot \phi \rfloor$, where $\phi = (1 + \sqrt{5})/2$ (Golden Ratio). The rows are then filled in by the Fibonacci rule that each term is the sum of the two previous terms. The entry $n$ in the first column is the index of that row.

**Note.** The Wythoff array description above has been taken from [2]. Let $w_{i,j}$ denote an $(i, j)$-th element of the generalized Wythoff array (row number $i \geq 0$, column number $j \geq 0$).

### 1.4 Fibonacci Numbers and Auxiliary Relations

Let $F(i)$ denote $i$-th Fibonacci number, i.e., $F(0) = 0, F(1) = 1, F(i) = F(i - 1) + F(i - 2)$ when $i > 1$, $L(i)$ denote $i$-th Lucas number, i.e. $L(1) = 1, L(2) = 3, L(i) = L(i - 1) + L(i - 1)$ when $i > 2$. Note some property of the Wythoff array that is related to the discussed issue:

$$w_{F(i),j} = F(i + j) + F(j), \quad i \geq 2, j \geq 0.$$  (3)
2. Main Results

Note also the following property of Fibonacci numbers

\[ 1 + \sum_{j=1}^{i} F(j) = F(i + 2). \]  \hspace{1cm} (4)

2 Main Results

Let \( T \) be a binary tree of size \( n \) and \( M = M(T) \) be a set of such sequences of positive integer weights that the tree \( T \) is the Huffman tree of \( P(|P| = n) \).

**Definition.** A sequence \( P_{\min} \) of \( n \) positive integer weights is called a minimizing sequence of the binary tree \( T \) in class \( M(P_{\min} \in M) \) if \( P_{\min} \) produces the minimal Huffman cost of the tree \( T \) over all sequences from \( M \), i.e.,

\[ E(T, P_{\min}) \leq E(T, P) \forall P \in M. \]

**Definition.** A non-decreasing sequence of positive integer weights \( P = \{p_1, p_2, \ldots, p_n\} \) is called absolutely ordered if the intermediate sequences of weights produced by Huffman algorithm for initial sequence \( P \) satisfy the following conditions

\[ p^{(i)}_2 < p^{(i)}_3, \quad i = 0, n-3. \]

**Theorem 1** ([5]). A minimizing absolutely ordered sequence of the elongated binary tree is

\[ P_{\min_{\text{abs}}} = \{F(1), F(2), \ldots, F(n)\}, \]

where \( F(i) \) is \( i \)-th Fibonacci number. The weighted external path length of elongated binary tree \( T \) of size \( n \) for the minimizing absolutely ordered sequence \( P_{\min_{\text{abs}}} \) is

\[ E(T, P_{\min_{\text{abs}}}) = F(n + 4) - (n + 4). \]

*Proof.* The proof of Theorem 1 of [5]. \( \square \)

**Definition.** A non-decreasing sequence of positive integer weights \( P^{(i)} = \{p^{(i)}_1, p^{(i)}_2, \ldots, p^{(i)}_{n-1}\} \) is called \( k \)-ordered if the intermediate sequences of weights produced by Huffman algorithm for initial sequence \( P \) satisfy the following conditions

\[ p^{(i)}_2 = p^{(i)}_3, \quad i = 0, k; \]  \hspace{1cm} (5)

\[ p^{(i)}_2 < p^{(i)}_3, \quad i = k + 1, n-3. \]  \hspace{1cm} (6)

Let \( M_{n,k}(k = 0, n-3) \) denote the set of all \( k \)-ordered sequences of size \( n \) for which the binary Huffman tree is elongated, i.e. an elongated binary tree of size \( n \) is the Huffman tree of \( P \).
Theorem 2. A minimizing $k$-ordered sequence of the elongated binary tree in class $M_{n,k}(k=0,n-3)$ is

$$
p_1 = 1,
$$

$$
p_i = F(i-1), \ i = 2,k+2,
$$

$$
p_{k+3} = F(k+2) = w_{F(k+2),0},
$$

$$
p_i = w_{F(k+2),i-k-3}, \ i = k+4,n,
$$

where $w_{i,j}$ is the $(i,j)$-th element of the Wythoff array.

Proof. Because $P_{\text{min}}_{n,k} = \{p_1,p_2,\ldots,p_n\}$ is minimizing sequence of positive integer values, $p_1$ and $p_2$ should have minimal positive integer values, i.e.,

$$
p_1 = p_2 = 1.
$$

(7)

$P_{\text{min}}_{n,k}$ is $k$-ordered ($k \geq 0$) sequence, so according to (5)

$$
p_2 = p_3 = 1,
$$

therefore according to (1) and (7)

$$
p_3 = 1.
$$

Thus

$$
p_1 = 1, \ p_2 = F(1), \ p_3 = F(2).
$$

(8)

Further, taking into account (2) and (6) we obtain the following Huffman algorithm steps for $k$-ordered ($k \geq 0$) sequence of the elongated (left-sided) binary tree.

| Step | Input sequence | Relation |
|------|----------------|----------|
| 0    | $p_1, p_2, p_3, p_4, p_5, p_6, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_{n-1}, p_n$; | $p_2 = p_3$ |
| 1    | $p_3, \sum_{i=1}^{2} p_i, p_4, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_{n-1}, p_n$; | $\sum_{i=1}^{2} p_i = p_4$ |
| 2    | $p_4, \sum_{i=1}^{3} p_i, p_5, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_{n-1}, p_n$; | $\sum_{i=1}^{3} p_i = p_5$ |
| 3    | $p_5, \sum_{i=1}^{4} p_i, p_6, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_{n-1}, p_n$; | $\sum_{i=1}^{4} p_i = p_6$ |
| ...  |                 |          |
| $k-1$| $p_{k+1}, \sum_{i=1}^{k} p_i, p_{k+2}, \ldots, p_{n-1}, p_n$; | $\sum_{i=1}^{k} p_i = p_{k+2}$ |
| $k$  | $p_{k+2}, \sum_{i=1}^{k+1} p_i, p_{k+3}, \ldots, p_{n-1}, p_n$; | $\sum_{i=1}^{k+1} p_i = p_{k+3}$ |
2. Main Results

| Steps \((k + 1) - (n - 3)\) | Input sequence | Relation |
|---|---|---|
| \(k+1\) | \(p_{k+3} + \sum_{i=1}^{k+2} p_i, p_{k+4}, \ldots, p_{n-1}, p_n;\) | \(\sum_{i=1}^{k+2} p_i < p_{k+4}\) |
| \(k+2\) | \(p_{k+4} + \sum_{i=1}^{k+3} p_i, p_{k+5}, \ldots, p_{n-1}, p_n;\) | \(\sum_{i=1}^{k+3} p_i < p_{k+5}\) |
| \(\ldots\) | \(\ldots\) | \(\ldots\) |
| \(n-4\) | \(p_{n-2} + \sum_{i=1}^{n-3} p_i, p_{n-1}, p_n;\) | \(\sum_{i=1}^{n-3} p_i < p_{n-1}\) |
| \(n-3\) | \(p_{n-1} + \sum_{i=1}^{n-2} p_i, p_n;\) | \(\sum_{i=1}^{n-2} p_i < p_n\) |

Consider two cases.

**Case 1. Steps** \(0 - k\).

It follows from relations for steps \(0 - k\) that

\[
p_i = \sum_{j=1}^{i-2} p_j, \quad i = 4, k + 3.
\]

Thus,

\[
p_i - p_{i-1} = \sum_{j=1}^{i-2} p_j - \sum_{j=1}^{i-3} p_j = p_{i-2}, \quad i = 4, k + 3.
\]

So, we have

\[
p_i = p_{i-1} + p_{i-2}, \quad i = 4, k + 3.
\]

Taking into account (8), we obtain

\[
p_i = F(i - 1).
\]  \(9\)

In particular,

\[
p_{k+3} = F(k + 2) = F(k + 2) + F(0).
\]  \(10\)

**Case 2. Steps** \((k + 1) - (n - 3)\).
Because $P_{\min,n,k} = \{p_1, p_2, \ldots, p_n\}$ is lineminimizing sequence of positive integer values, inequalities for steps $(k+1) - (n-3)$ are transformed to the following equalities:

| Step | Input sequence | Relation |
|------|----------------|----------|
| $k+1$ | $p_{k+3}, \sum_{i=1}^{k+2} p_i, p_{k+4}, p_{k+3}, \ldots, p_{n-1}, p_n,$; $\sum_{i=1}^{k+2} p_i = p_{k+4} + 1$ | $k+2$ |
| $k+2$ | $p_{k+4}, \sum_{i=1}^{k+3} p_i, p_{k+5}, \ldots, p_n,$; $\sum_{i=1}^{k+3} p_i = p_{k+5} + 1$ | $\ldots$ |
| $\ldots$ | $\ldots$ | $\ldots$ |
| $n-4$ | $p_{n-2}, \sum_{i=1}^{n-3} p_i, p_{n-1}, p_n,$; $\sum_{i=1}^{n-3} p_i = p_{n-1} + 1$ | $n-3$ |
| $n-3$ | $p_{n-1}, \sum_{i=1}^{n-2} p_i, p_n,$; $\sum_{i=1}^{n-2} p_i = p_n + 1$ | $\ldots$ |

From the equality for step $(k+1)$, (9), (7) and (4) results

$$p_{k+4} = F(k+3) + 1 = F(k+3) + F(1).$$  \hfill (11)

Further, it follows from relations with equalities for steps $(k+1) - (n-3)$ that

$$p_i = 1 + \sum_{j=1}^{i-2} p_j, \quad i = k + 5, n.$$  \hfill (12)

Thus,

$$p_i - p_{i-1} = (1 + \sum_{j=1}^{i-2} p_j) - (1 + \sum_{j=1}^{i-3} p_j) = p_{i-2}, \quad i = k + 5, n.$$  \hfill (13)

So, we have

$$p_i = p_{i-1} + p_{i-2}.$$  \hfill (14)

Therefore, taking into account (14) and (11), we have

$$p_i = F(i-1) + F(i-k-3).$$  \hfill (15)

From this and (15) it follows that

$$p_i = w_{F(k+2), i-k-3}.$$  \hfill (16)

where $w_{i,j}$ is $(i, j)$-th element of the Wythoff array.

The statement of the theorem follows from (7), (9) and (12).  \hfill □
2. Main Results

**Corollary 1.** A minimizing $0$-ordered sequence of size $n$ for the elongated binary tree in $M_{n,0}$ is the Lucas sequence shifted two places right, i.e.

$$\{1, 1, L(1), L(2), \ldots, L(n-2)\},$$

where $L(i)$ is $i$-th Lucas number.

**Corollary 2.** A minimizing $(n-3)$-ordered sequence of size $n$ for the elongated binary tree in $M_{n,n-3}$ is the Fibonacci sequence shifted one place right, i.e.

$$\{1, F(1), F(2), \ldots, F(n-1)\},$$

where $F(i)$ is $i$-th Fibonacci number.

Note that normalized $(n-3)$-ordered sequence of size $n$

$$\{1/F(n+1), F(1)/F(n+1), F(2)/F(n+1), \ldots, F(n-1)/F(n+1)\}$$

has maximum weighted external path length over all possible normalized sequences of size $n$ for which Huffman tree is elongated [6].

**Theorem 3.** The weighted external path length of the elongated binary tree $T$ of size $n$ for the minimizing $k$-ordered sequence $P_{\min_{n,k}}$ is

$$E(T, P_{\min_{n,k}}) = F(n + 3) + F(n - k + 1) - (n - k + 3).$$

**Proof.** Let $P_{\min_{n,k}} = \{p_1, p_2, \ldots, p_n\}$ be the minimizing $k$-ordered sequence of the elongated binary tree $T$ of size $n$.

According to Theorem 2 $P_{\min_{n,k}} = \{1, F(1), \ldots, F(k+2), F(k+3) + F(1), \ldots, F(n-1) + F(n-k-3)\}$.

Weighted external path length $E(T, P_{\min_{n,k}})$ is

$$E(T, P_{\min_{n,k}}) = \sum_{i=1}^{n} l_i \cdot p_i,$$

where $l_i$ is the length of the path from the root to leaf $i$.

$T$ is the elongated binary tree, therefore $l_1 = n - 1, l_i = n - i + 1 (i = 2, n)$.
Thus, taking into account (4), we obtain

\[ E(T, P_{\min_{n,k}}) = \sum_{i=1}^{n} i \cdot p_i = (n - 1) \cdot p_1 + \sum_{i=2}^{n} (n - i + 1) \cdot p_i = \]

\[ (n - 1) + \sum_{i=2}^{k+3} (n - i + 1) \cdot F(i - 1) + \]

\[ \sum_{i=k+4}^{n} (n - i + 1) \cdot (F(i - 1) + F(i - k - 3)) = \]

\[ (n - 1) + \sum_{i=1}^{n} (n - i) \cdot F(i) + \sum_{i=k+3}^{n-1} (n - i) \cdot (F(i) + F(i - k - 2)) = \]

\[ (n - 1) + \sum_{i=1}^{n-1} (n - i) \cdot F(i) + \sum_{i=k+3}^{n-3} (n - k - i - 2) \cdot F(i) = \]

\[ (n - 1) + \sum_{i=1}^{n-1} F(i) + \sum_{i=1}^{n-k-3} \sum_{j=1}^{n-k-3} F(i) = \]

Thus, taking into account (4), we obtain

\[ E(T, P_{\min_{n,k}}) = (n - 1) + \sum_{j=1}^{n-1} (F(j + 2) - 1) + \sum_{j=1}^{n-k-3} (F(j + 2) - 1) = \]

\[ \sum_{j=3}^{n+1} F(j) + \sum_{j=3}^{n-k-1} F(j) - (n - k - 3) = \]

\[ \sum_{j=1}^{n+1} F(j) - (F(2) - F(1)) + (\sum_{j=1}^{n-k-1} F(j) - (F(2) - F(1))) - (n - k - 3) = \]

\[ = (F(n + 3) - 1 - F(3)) + (F(n - k + 1) - 1 - F(3)) - (n - k - 3) = \]

\[ (F(n + 3) - 3) + (F(n - k + 1) - 3) - (n - k - 3) = \]

\[ F(n + 3) + F(n - k + 1) - (n - k + 3). \]

The statement of the theorem proved. \( \square \)

**Corollary 3.** The weighted external path length of the elongated binary tree \( T \) of size \( n \) for the minimizing 0-ordered sequence \( P_{\min_{n,0}} \) (the Lucas sequence shifted two places right) is

\[ E(T, P_{\min_{n,0}}) = F(n + 3) + F(n + 1) - (n + 3). \]
3. Examples

**Corollary 4.** The weighted external path length of the elongated binary tree $T$ of size $n$ for the minimizing $(n - 3)$-ordered sequence $P_{\text{min}_{n,n-3}}$ (the Fibonacci sequence shifted one place right) is

$$E(T, P_{\text{min}_{n,n-3}}) = F(n + 3) + F(n - (n - 3) + 1) - (n - (n - 3) + 3) = F(n + 3) + F(4) - 6 = F(n + 3) - 3.$$ 

3 Examples

Several examples of minimizing ordered sequences for Huffman codes are shown below. An underlined integer in the tables means a nonleaf node cost obtained as a result of merging two leaf nodes on the previous step of the Huffman algorithm.

**Example 1.** Absolutely minimizing ordered sequence $P_{\text{min}_{10}}$ of size 10

| Step | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| 0    | 0     | 1     | 1     | 2     | 3     | 5     | 8     | 13    | 21    | 34      |
| 1    | 1     | 1     | 2     | 3     | 5     | 8     | 13    | 21    | 34    | 55      |
| 2    | 2     | 2     | 3     | 5     | 8     | 13    | 21    | 34    | 55    |          |
| 3    | 3     | 4     | 5     | 8     | 13    | 21    | 34    | 55    |       |          |
| 4    | 5     | 7     | 8     | 13    | 21    | 34    | 55    |       |       |          |
| 5    | 8     | 12    | 13    | 21    | 34    | 55    |       |       |       |          |
| 6    | 13    | 20    | 21    | 34    | 55    |       |       |       |       |          |
| 7    | 21    | 33    | 34    | 55    |       |       |       |       |       |          |
| 8    | 34    | 54    | 55    |       |       |       |       |       |       |          |
| 9    | 55    | 88    |       |       |       |       |       |       |       |          |

$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55\}$ - Fibonacci sequence of size 10.

**Example 2.** Minimizing 0-ordered sequence $P_{\text{min}_{0,0}}$

| Step | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| 0    | 0     | 1     | 1     | 3     | 4     | 7     | 11    | 18    | 29    | 47      |
| 1    | 1     | 2     | 3     | 4     | 7     | 11    | 18    | 29    | 47    |          |
| 2    | 2     | 3     | 4     | 7     | 11    | 18    | 29    | 47    |       |          |
| 3    | 3     | 6     | 7     | 11    | 18    | 29    | 47    |       |       |          |
| 4    | 4     | 10    | 11    | 18    | 29    | 47    |       |       |       |          |
| 5    | 6     | 11    | 17    | 18    | 29    | 47    |       |       |       |          |
| 6    | 10    | 18    | 28    | 29    | 47    |       |       |       |       |          |
| 7    | 18    | 28    | 46    | 47    |       |       |       |       |       |          |
| 8    | 28    | 47    | 72    |       |       |       |       |       |       |          |
| 9    | 47    | 72    |       |       |       |       |       |       |       |          |

$\{p_2, p_3\} = \{1, 1\} = \{F(1), F(2)\}$ - Fibonacci sequence of size 2;

$\{p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} = \{1, 3, 4, 7, 11, 18, 29, 47\}$ -
### 3. Examples

Wythoff array row-1 (row-$F(2)$) sequence of size 8 (the Lucas sequence).

#### Example 3. Minimizing 1-ordered sequence $P_{\min_{10}}$

| Step | $P(0)$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ |
|------|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| 0    | $P(0)$ | 1     | 1     | 2     | 4     | 6     | 10    | 16    | 26    | 42    |         |
| 1    | $P(1)$ | 1     | 2     | 2     | 4     | 6     | 10    | 16    | 26    | 42    |         |
| 2    | $P(2)$ | 2     | 2     | 4     | 6     | 10    | 16    | 26    | 42    |       |         |
| 3    | $P(3)$ | 4     | 2     | 4     | 6     | 10    | 16    | 26    | 42    |       |         |
| 4    | $P(4)$ | 6     | 2     | 6     | 10    | 16    | 26    | 42    |       |       |         |
| 5    | $P(5)$ | 10    | 15    | 16    | 26    | 42    |       |       |       |       |         |
| 6    | $P(6)$ | 16    | 25    | 26    | 42    |       |       |       |       |       |         |
| 7    | $P(7)$ | 26    | 41    | 42    |       |       |       |       |       |       |         |
| 8    | $P(8)$ | 42    | 67    |       |       |       |       |       |       |       |         |
| 9    | $P(9)$ | 109   |       |       |       |       |       |       |       |       |         |

$\{p_2, p_3, p_4\} = \{1, 1, 2\} = \{F(1), F(2), F(3)\}$ \(- \text{Fibonacci sequence of size 3;}

$\{p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} = \{2, 4, 6, 10, 16, 26, 42\} \text{- Wythoff array row-2 (row-F(3)) sequence of size 7.}$

#### Example 4. Minimizing 4-ordered sequence $P_{\min_{10}}$

| Step | $P(0)$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ |
|------|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| 0    | $P(0)$ | 1     | 1     | 2     | 3     | 5     | 8     | 14    | 22    | 36    |         |
| 1    | $P(1)$ | 1     | 2     | 2     | 3     | 5     | 8     | 14    | 22    | 36    |         |
| 2    | $P(2)$ | 2     | 2     | 3     | 5     | 8     | 14    | 22    | 36    |       |         |
| 3    | $P(3)$ | 3     | 2     | 3     | 5     | 8     | 14    | 22    | 36    |       |         |
| 4    | $P(4)$ | 5     | 8     | 8     | 14    | 22    | 36    |       |       |       |         |
| 5    | $P(5)$ | 8     | 13    | 14    | 22    | 36    |       |       |       |       |         |
| 6    | $P(6)$ | 14    | 21    | 22    | 36    |       |       |       |       |       |         |
| 7    | $P(7)$ | 22    | 35    | 36    |       |       |       |       |       |       |         |
| 8    | $P(8)$ | 36    | 57    |       |       |       |       |       |       |       |         |
| 9    | $P(9)$ | 57    |       |       |       |       |       |       |       |       |         |

$\{p_2, p_3, p_4, p_5, p_6, p_7\} = \{1, 1, 2, 3, 5, 8\} = \{F(1), F(2), F(3), F(4), F(5), F(6)\}$ \(- \text{Fibonacci sequence of size 6;}

$\{p_7, p_8, p_9, p_{10}\} = \{8, 14, 22, 36\} \text{- Wythoff array row-8 (row-F(6)) sequence of size 4.}$
3. Examples

Example 5. Minimizing 7-ordered sequence $P_{\min_{10,7}}$

| Step | $P_{(0)}$ | $P_{(1)}$ | $P_{(2)}$ | $P_{(3)}$ | $P_{(4)}$ | $P_{(5)}$ | $P_{(6)}$ | $P_{(7)}$ | $P_{(8)}$ | $P_{(9)}$ | $P_{(10)}$ |
|------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0    | 1         | 1         | 1         | 2         | 3         | 5         | 8         | 13        | 21        | 34        |
| 1    | 1         | 2         | 2         | 3         | 5         | 8         | 13        | 21        | 34        |
| 2    | 2         | 3         | 3         | 5         | 8         | 13        | 21        | 34        |
| 3    | 3         | 5         | 5         | 8         | 13        | 21        | 34        |
| 4    | 5         | 8         | 8         | 13        | 21        | 34        |
| 5    | 8         | 13        | 13        | 21        | 34        |
| 6    | 13        | 21        | 21        | 34        |
| 7    | 21        | 34        | 34        |
| 8    | 34        | 55        |
| 9    | 55        | 89        |

$\{p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34\} = \{F(1), F(2), F(3), F(4), F(5), F(6), F(7), F(8), F(9)\}$ - Fibonacci sequence of size 9;

$\{p_{10}\} = \{34\}$ - Wythoff array row-34 (row-F(9)) sequence of size 1.

References

1. Huffman D., A method for the construction of minimum redundancy codes. Proc. of the IRE 40 (1952) 1098–1101
2. Sloane N.J.A., Classic Sequences In The On-Line Encyclopedia of Integer Sequences: The Wythoff Array and The Para-Fibonacci Sequence. Published electronically at http://wwww.research.att.com/~njas/sequences/classic.html
3. Sloane N.J.A., My Favorite Integer Sequences. Published electronically at http://wwww.research.att.com/~njas/doc/sg.pdf
4. Fraenkel A., Kimberling C., Generalized Wythoff arrays, shuffles and interspersions. Discrete Math. 126 (1994) 137–149
5. Vinokur A.B., Huffman trees and Fibonacci numbers. Kibernetika Issue 6 (1986) 9-12 (in Russian), English translation in Cybernetics 22 Issue 6 (1986) 692-696; http://springerlink.metapress.com/link.asp?ID=W32X70520K3J617
6. Vinokur A.B., Huffman codes and maximizing properties of Fibonacci numbers. Kibernetika i Systemnyi Analiz Issue 3 (1992) 10-15 (in Russian), English translation in Cybernetics and Systems Analysis 28 Issue 3 (1993) 329–334; http://springerlink.metapress.com/link.asp?ID=NJ5073781H237182