Superlattice structure in the antiferromagnetically ordered state in the Hubbard model on the Ammann-Beenker tiling

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We study magnetic properties in the half-filled Hubbard model on the Ammann-Beenker tiling. First, we focus on the domain structure with locally eightfold rotational symmetry to examine the strictly localized confined states for the tightbinding model. We count the number of vertices and confined states in the larger domains generated by the deflation operations systematically. Then, the fraction of the confined states, which plays an important role for magnetic properties in the weak coupling limit, is obtained as \( p = 1/2\tau^2 \), where \( \tau = 1 + \sqrt{2} \) is the silver ratio. It is also found that the wave functions for confined states are densely distributed in the system and thereby the introduction of the Coulomb interactions immediately induces the finite staggered magnetizations. Increasing the Coulomb interactions, the spatial distribution of the magnetizations continuously changes to those of the Heisenberg model. We discuss crossover behavior in the perpendicular space representation and reveal the superlattice structure in the spatial distribution of the staggered magnetizations.

I. INTRODUCTION

Quasicrystal without translational symmetry has attracted much interest since its first observation in the Al-Mn array \[1\]. Among them, the Au-Al-Yb array with Tsai-type clusters \[2\] is one of the interesting examples with anomalous low temperature properties. The quasicrystal \( \text{Au}_{15}\text{Al}_{34}\text{Yb}_{15} \) shows quantum critical behavior with unusual exponents, while the approximant \( \text{Au}_{15}\text{Al}_{34}\text{Yb}_{15} \) shows heavy fermion behavior \[3\]. These experiments should clarify that electron correlations play an important role in the quasicrystals. Furthermore, the superconductivity has recently been observed in the Al-Zn-Mn quasicrystal \[4\], stimulating further investigations on electron correlations and induced ordered states in the quasiperiodic systems \[5\]-\[14\].

Up to now, no magnetically ordered states have been found in the quasicrystals although it has recently been observed in the approximants \( \text{Cd}_6\text{Nb} \[15\] and \( \text{Au-Al-Gd} \[16\] and \( \text{Au-Al-Tb} \[17\]. In contrast to the experiments, there are many theoretical works for the spontaneously symmetry breaking states on the two-dimensional quasiperiodic lattices. Among them, the system on the Penrose tiling \[18\]-\[24\] has been examined, where the magnetically ordered states \[12\]-\[25\]-\[29\], superconductivity \[11\]-\[13\]-\[14\], and excitonic insulator \[30\] have been discussed. The Ammann-Beenker tiling \[31\]-\[32\] (see Fig. 1) is another example for two dimensional quasiperiodic structures, where the superconducting \[33\] and higher order topological states \[34\] have recently been examined. The magnetic instability has been discussed in the Hubbard \[35\], Heisenberg \[36\]-\[37\], and Anderson lattice \[38\] models. However, the system size treated is not large enough to discuss magnetic properties inherent in the quasiperiodic lattice. In particular, the role of the strictly localized states, which should play a crucial role in the weak coupling limit, has not been discussed up to now. Therefore, it is instructive to examine the confined states and to clarify magnetic properties in the Hubbard model with larger clusters.

In the manuscript, we study the half-filled Hubbard model on the Ammann-Beenker tiling. First, we focus on the macroscopically degenerate states in the noninteracting case. By examining the domain structures generated by the deflation operations systematically, we obtain the fraction of the confined states in the thermodynamic limit. To clarify the effects of the Coulomb interactions, we apply the real-space Hartree approximation to the system and calculate the local magnetization at each site. We reveal that the superlattice structure appears in the weak coupling case. Mapping the spatial distribution of the magnetization to the perpendicular space, we also discuss the crossover in the antiferromagnetically ordered state.

The paper is organized as follows. In Sec. II we introduce the half-filled Hubbard model on the Ammann-Beenker tiling. In Sec. III we study the confined states with \( E = 0 \), which should play an important role for magnetic properties in the weak coupling limit. Counting the number of the confined states in the domains systematically, we exactly obtain their fraction. We discuss how the antiferromagnetically ordered state is realized in the Hubbard model in Sec. IV. The
crossover behavior in the ordered state is addressed, by mapping the spatial distribution of the magnetization to the perpendicular space. A summary is given in the last section.

II. MODEL AND HAMILTONIAN

We study the Hubbard model on the Ammann-Beenker tiling, which should be given by the following Hamiltonian,

\[ H = -t \sum_{\langle ij \rangle \sigma} \left( c_{i \sigma}^\dagger c_{j \sigma} + \text{h.c.} \right) + \sum_i U \left( n_{i \uparrow} - \frac{1}{2} \right) \left( n_{i \downarrow} - \frac{1}{2} \right), \]

where \( c_{i \sigma} (c_{i \sigma}^\dagger) \) annihilates (creates) an electron with spin \( \sigma (= \uparrow, \downarrow) \) at the \( i \)-th site and \( n_{i \sigma} = c_{i \sigma}^\dagger c_{i \sigma} \). \( t \) is the transfer integral and \( U \) is the onsite Coulomb interaction. Since the Hubbard model on the Ammann-Beenker tiling is bipartite, the chemical potential is always \( \mu = 0 \) when the electron density is fixed to be half filling.

The Ammann-Beenker tiling is composed of squares and rhombuses, which is schematically shown in Fig. 1. There exist six types of vertices. In the manuscript, the vertices are denoted as A, B, \( \cdots \), and F for the coordination number 3, 4, \( \cdots \), and 8, respectively. Since the vertex lattice is bipartite, it is naively expected that the introduction of the Coulomb potential is always \( \mu = 0 \) when the electron density is fixed to be half filling.

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According to the Lieb’s theorem \[ \text{39} \], the half-filled Hubbard model on the bipartite lattice has the total spin \( S_{\text{tot}} = \frac{1}{2}(N_A - N_B) \) in the ground state, where \( N_A \) and \( N_B \) are the numbers of sites in the A and B sublattices. Therefore, the imbalance in their numbers yields the ferrimagnetically ordered state e.g. Lieb lattice \[ \text{40} \]. In our model, one can prove that the antiferromagnetically ordered state is realized without uniform magnetizations, considering the deflation rule. Figure 2 shows the deflation rule for the directed squares and rhombuses, where the open and solid circles at the corners represent the distinct sublattices. By applying the deflation operations to the squares and rhombuses, their numbers are changed as

\[ S_\sigma \rightarrow S_\sigma + 2S_\bar{\sigma} + 2R_\sigma + 2R_\bar{\sigma}, \]

\[ R_\sigma \rightarrow 2S_\bar{\sigma} + 2R_\sigma + R_\bar{\sigma}, \]

where \( S_\sigma (R_\sigma) \) is the number of the squares (rhombuses) with spin \( \sigma \) where two spins connected by the arrows are \( \sigma \) and the other spins are \( \bar{\sigma} \). It is known that in the thermodynamic limit, the numbers of squares and rhombuses \( \tau^2 \) times increase for each deflation process and \( S/R = 1/\sqrt{2} \), where \( \tau = 1 + \sqrt{2} \) is the silver ratio \[ \text{31} \text{32} \]. From the above relations \[ \text{2} \] and \[ \text{3} \], we obtain that \( S_\sigma = S/2 \) and \( R_\sigma = R/2 \). Since the number of squares and rhombuses are independent of spins in the thermodynamic limit, the vertices are also independent. Its proof is explicitly shown in Appendix. Then, we can say that the antiferromagnetically ordered state without uniform magnetizations is realized in the thermodynamic limit.

On the other hand, the magnetization profile may not be trivial since in the quasicrystals, each lattice site is not equivalent, in contrast to the conventional lattice with translational symmetry. In particular, in the weak coupling limit, magnetic properties strongly depend on the noninteracting density of states (DOS) at the Fermi level. Figure 3 shows the DOS in the tightbinding model on the Ammann-Beenker tiling. We find the delta-function like peak at \( E = 0 \), meaning the existence of the confined states. When magnetic properties are studied at half filling, the confined states should play an essential role in understanding magnetic properties. In the following section, we focus on these macroscopically degenerate states with \( E = 0 \).
III. CONFINED STATES IN THE TIGHTBINDING MODEL ON THE AMMANN-BEENKER TILING

In the section, we focus on the confined states in the tightbinding model. As seen in Fig. 3, the eigenstates are macroscopically degenerate at \( E = 0 \), which means that the corresponding states are exactly localized in certain regions. This is similar to the model on the Penrose tiling [12, 13, 23]. The key of the confined states is the fact that the Ammann-Beenker tiling has the eightfold rotational symmetry. Here, we focus on the F vertex with locally eightfold rotational symmetry, which is closely related to the confined states, as discussed later. Due to the matching rule of the Ammann-Beenker tiling, there always appear eight squares and sixteen rhombuses around each F vertex, as shown in Fig. 4(a). For convenience, when the local eightfold rotational symmetry is satisfied in the domain shown in Fig. 4(a) and is not satisfied outside, we define this domain composed of seventeen sites (the boundary sites are excluded) as \( D_1 \). By applying the deflation operation to the domain \( D_1 \), a new domain is generated, as shown in Fig. 4(b). If one focuses on the F vertex at the center, the rotational symmetry is satisfied in the domain with larger lattice sites, which is bounded by the regular octagon shown as the dashed line in Fig. 4(b). This domain is denoted as \( D_2 \). Repeating the deflation operations, we obtain the \( D_i \) domains. Then, we can define the F vertex at the center of the domain \( D_i \) as \( F_i \). Figure 4 shows the domains \( D_1, D_2, \) and \( D_3 \), where \( F_1, F_2, \) and \( F_3 \) vertices are located at their centers, respectively. In the \( D_3 \) domain, we find sixteen \( D_1 \) domains with the \( F_1 \) vertices. Note that there does not exist the \( D_1 \) domain at the center because of its definition. It is known that, in each deflation operation, \( F_i (i > 1) \) vertices are generated from the \( F_{i-1} \) vertices and the \( F_1 \) vertices are generated from half of the C vertices, and D and E vertices (see Fig. 4). Then, in the thermodynamic limit, the fraction of the \( F_i \) vertices is obtained as

\[
\rho_i = 2\tau^{-2i+3},
\]

since \( \rho_i = \left( \frac{1}{2} \rho^C + \rho^D + \rho^E \right) / \tau^2 = 2\tau^{-5} \) and \( \rho_{i+1} = \rho_i / \tau^2 \), where \( \rho^A = \tau^{-1}, \rho^B = 2\tau^{-2}, \rho^C = 2\tau^{-3}, \rho^D = 2\tau^{-4}, \rho^E = \tau^{-4}, \rho^F = \tau^{-4} \). Since the \( F_i \) vertex is defined as the center vertex of the domain \( D_i \), the fraction of the domain \( D_i \) is given as \( \rho_i = \rho_i^F \).

By counting the numbers of all vertices up to the domain \( D_1 \) numerically, we obtain the domain profile, as shown in Table II. From these data, one finds relations between the number of vertices. For examples, \( N^C_{i+1} = 2N^A_i, N^D_{i+1} = N^B_i, N^E_{i+2} = N^A_i, \) and \( N^F_{i+1} = N^C_i / 2 + N^D_i + N^E_i + N^F_i \). Estimating the general terms for \( N^A_i, N^B_i, \) and \( N_i \) as,

\[
N^A_i = 2\sqrt{2}[(-\tau)^{i-1} - \tau^{-i-1}] + 4\tau^{2i-1} - \tau^{1-i}],
\]

\[
N^B_i = 8[\tau^{2i-2} + \tau^{2i-2i} - \delta_{i1}],
\]

\[
N_i = 1 + 2\sqrt{2}[-(-\tau)^{-i} - \tau^{-1}] + 4\tau^{2i} + \tau^{-2i}],
\]

we obtain the general terms for all vertices in each domain.
TABLE I. Profile of each domain $D_i$. $p_i$ is its fraction, $N_i$ is the number of vertices, and $N_i^\alpha$ is the number of $\alpha$ vertices in the $i$th domain, where the sites on the boundary are excluded. $N_i^{\text{tot}}$ is the total number of the confined states, $N_i^{\text{net}}$ is the net number of the confined states, and $p_i^{\text{conf}} (= N_i^{\text{net}}/N_i)$ is the fraction of the confined states in the $i$th domain (see text).

| $i$ | $p_i$ | $N_i$ | $N_i^A$ | $N_i^B$ | $N_i^C$ | $N_i^D$ | $N_i^{\text{tot}}$ | $N_i^{\text{net}}$ | $p_i^{\text{conf}}$ |
|-----|-------|-------|--------|--------|--------|--------|----------------|----------------|----------------|
| 1   | 2 $\tau^{-5}$ | 17    | 8      | 8      | 0      | 0      | 0              | 1              | 2              | 2 0.1176       |
| 2   | 2 $\tau^{-7}$  | 121   | 48     | 48     | 16     | 8      | 0              | 1              | 6              | 6 0.0496       |
| 3   | 2 $\tau^{-9}$  | 753   | 312    | 272    | 96     | 48     | 8              | 17             | 44             | 12 0.0584      |
| 4   | 2 $\tau^{-11}$ | 4521  | 1872   | 1584   | 624    | 272    | 48             | 121            | 324            | 20 0.0717      |
| 5   | 2 $\tau^{-13}$ | 26763 | 11048  | 9232   | 3744   | 1584   | 312            | 753            | 2110           | 30 0.0791      |
| 6   | 2 $\tau^{-15}$ | 156249| 64720  | 53808  | 22096  | 9232   | 1872           | 4521           | 12938          | 42 0.0828      |
| 7   | 2 $\tau^{-17}$ | 912593| 378008 | 313616 | 129440 | 53808  | 11048          | 26673          | 77112          | 56 0.0845      |
| 8   | 2 $\tau^{-19}$ | 532393| 2205104| 1827888| 756016 | 313616 | 64720          | 156249         |                |                |
| 9   | 2 $\tau^{-21}$ | 3103931| 12856904| 10653712| 4440208| 1827888 | 378008        | 912593         |                |                |
| 10  | 2 $\tau^{-23}$ | 180937273| 74946672| 62094384| 25713808| 10653712| 2205104       | 532393         |                |                |
| 11  | 2 $\tau^{-25}$ | 105464457| 436848120| 361912592| 149893344| 62094384 | 12856904      | 3103931        |                |                |

Now, we consider the confined states in each domain with the eightfold rotational symmetry. In the domain $D_1$, there are two confined states. Since the confined states satisfy the Schödinger equation $H\Psi = 0$ with $U = 0$, it is always possible to choose each eigenstate such that it can be described by the irreducible representation of the point group $D_8$. Table II shows a part of the irreducible characters of the dihedral group $D_8$, where there exist four one-dimensional irreducible representations. Namely, the confined states $\Psi_1$ and $\Psi_2$, which are schematically shown in Fig. 5, are described by the irreducible representation $B_1$ and $B_2$, and $\langle \Psi_1 | \Psi_2 \rangle = 0$. We wish to note that these confined states are always exact eigenstates even when the system does not have eightfold rotational symmetry outside of the domain $D_1$. We also find that the amplitudes of the wave function $\Psi_1$ are finite only in the sublattice $B$, and the others are in the sublattice $A$ when the sublattice for the center site is regarded as the sublattice $A$. This is contrast to the case in the vertex model on the Penrose tiling [15, 23].

TABLE II. A part of the irreducible characters of the dihedral group $D_8$. $E$ is an identity operator, $C_8$ is a rotation operator of $\pi/4$, and $I_y$ is a reflection operator about the $y$ axis.

| $E$ | $C_8$ | $I_y$ | $I_C8$ |
|-----|-------|-------|--------|
| $A_1$ | 1     | 1     | 1      |
| $A_2$ | 1     | -1    | -1     |
| $B_1$ | 1     | -1    | 1      |
| $B_2$ | 1     | -1    | 1      |

Namely, the domain $D_{\infty}$ can be regarded as the Ammann-Beenker tiling in the thermodynamic limit and we have confirmed that the fraction for each vertex $p_{i\infty} = \lim_{i \to \infty} N_i^\alpha/N_i$ is reduced to the well-known value $p = \frac{\tau_0}{\tau_1}$ [32]. Now, we consider the confined states in each domain with the eightfold rotational symmetry. In the domain $D_1$, there are two confined states. Since the confined states satisfy the Schödinger equation $H\Psi = 0$ with $U = 0$, it is always possible to choose each eigenstate such that it can be described by the irreducible representation of the point group $D_8$. Table II shows a part of the irreducible characters of the dihedral group $D_8$, where there exist four one-dimensional irreducible representations. Namely, the confined states $\Psi_1$ and $\Psi_2$, which are schematically shown in Fig. 5, are described by the irreducible representation $B_1$ and $B_2$, and $\langle \Psi_1 | \Psi_2 \rangle = 0$. We wish to note that these confined states are always exact eigenstates even when the system does not have eightfold rotational symmetry outside of the domain $D_1$. We also find that the amplitudes of the wave function $\Psi_1$ are finite only in the sublattice $B$, and the others are in the sublattice $A$ when the sublattice for the center site is regarded as the sublattice $A$. This is contrast to the case in the vertex model on the Penrose tiling [15, 23].

where finite amplitudes appears in one of the sublattices in the cluster defined in Ref [12]. This should induce distinct spatial distribution of the magnetization in the weak coupling limit, which will be discussed in the next section.

FIG. 5. Two confined states in the domain $D_1$ for the tight-binding model on the Ammann-Beenker tiling. The number at the vertices represent the amplitudes of confined state.

In the domain $D_2$, there is the structure of the domain $D_1$ around the center. Therefore, in the domain $D_2$, $\Psi_1$ and $\Psi_2$ located there are the confined states. Furthermore, we find four confined states $\Psi_3$, $\Psi_4$, $\Psi_5$, and $\Psi_6$, as shown in Fig. 6. It is found that these confined states are described by the irreducible representations $A_2$, $B_1$, $A_1$, and $B_2$. Namely, $\Psi_1$ and $\Psi_4$ ($\Psi_2$ and $\Psi_5$) are described by the same irreducible representation $B_1$ ($B_2$), but there are no overlap in their wave functions. In the domain $D_3$, in addition to the six confined states shown above, we find six confined states $\Psi_7$, $\Psi_8$, $\cdots$, and $\Psi_{12}$, which are explicitly shown in Fig. 7. These are described by the irreducible representations $A_1$, $B_1$, $B_2$, $A_1$, and $B_2$. We note that, in the domain $D_3$, there exist sixteen $D$ domains (shown as the shaded regions in Figs. 4 and 7), where two confined states $\Psi_1$ and $\Psi_2$ exist locally. Therefore, in the domain $D_3$, the net number of the confined states $N_{3\text{tot}}$ is 12, and the total number of the confined states $N_{3\text{tot}} = N_{3\text{tot}}^{\text{net}} + 16N_{1\text{tot}}^{\text{net}} = 44$, with $N_{1\text{tot}}^{\text{net}} = 2$.

To count the number of the confined states in larger domains systematically, we perform the exact diagonalization method for the tightbinding Hamiltonian. The results up to the domain $D_7$ are shown in Table I. The net number of the confined states is evaluated by taking into account the smaller domains, as $N_i^{\text{net}} = N_{i\text{tot}}^{\text{net}} - \sum_{j=1}^{i-1} N_jN_j^{\text{net}}$, where $N_i$ is the number of domain $D_j$ inside of the domain $D_i$. Namely, $N_j$ satis-
The relations as \( N_{i+1,j+1} = N_{i,j} \), \( N_{i,1} = \frac{N^C_{i-1} + N^D_{i-1} + N^E_{i-1}}{2} \), and \( N^F_i = \sum N_{i,j} \), where \( N^\alpha_i \) is the number of the \( \alpha \) vertex in the domain \( D_i \). Since the net number of confined states should be given as \( N^\text{net}_i = i(i + 1) \), we obtain the fraction of the confined states in the tight-binding model on the Ammann-Beenker tiling as,

\[
p = \frac{1}{2\tau^2} \sim 8.579 \times 10^{-2},
\]

where \( p_i \) is the fraction of the \( D_i \) domain. We have also confirmed that it corresponds to the ratio in the domain \( D_\infty \), \( p = \lim_{\tau \to \infty} p_i N^\text{net}_i/N_i \), where the general term for the total number of the confined states is given as

\[
N^\text{net}_i = 4 + 2 \sqrt{2(\tau^{-i} - \tau^i)} + 2(\tau^{2i-2} + \tau^{2-2i}) + i(i + 1). \tag{9}
\]

In the following, we consider electron correlations in the Hubbard model to discuss how the antiferromagnetically ordered state is realized in the Ammann-Beenker tiling. In the weak coupling limit, it is, in principle, possible to evaluate the magnetization by means of the Gram-Schmidt orthogonalization for the confined states at \( E = 0 \) since their degeneracy should be lifted by the introduction of the Coulomb interactions. However, the confined state are densely distributed in the lattice. Figures 5, 6, and 7 show that the confined states have amplitudes in almost whole of the domain. In addition, the amplitudes of confined states in a certain domain \( D \) sometimes appear on the smaller domains inside of \( D \), where some confined states exist locally. For example, in Fig. 7 the wave function \( \Psi_{11} \) has amplitudes in each domain \( D_1 \) (the shaded areas) with the local wave functions \( \Psi_1 \) and \( \Psi_2 \). Therefore, the wave functions for confined states multiply overlap in the space. This is contrast to the Penrose-Hubbard model, where there exist finite number of confined states in a certain region “cluster” and the seventy percents of magnetizations are exactly obtained in the thermodynamic limit [12]. Figure 8 shows the local magnetization for the \( D_7 \) domain, which is obtained from 77 112 confined states. It is found that the distribution of the magnetization is classified into some groups. The group with large magnetizations is mainly contributed from the A and B vertices around the F vertex, which originates from the confined states \( \Psi_1 \) and \( \Psi_2 \) (see Fig. 5). In the manuscript, we apply the simple mean-field theory to the Hubbard model to discuss magnetic properties inherent in the Ammann-Beenker tiling [15].
IV. ANTIFERROMAGNETICALLY ORDERED STATE

In the section, we consider the Hubbard model with finite $U$. To study the antiferromagnetically ordered state characteristic of the Ammann-Beenker tiling, we make use of the real-space Hartree approximation and the Hamiltonian $H_{MF}$ is reduced to

$$H_{MF} = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + h.c.) + U \sum_{\sigma} \langle n_{i\sigma} \rangle n_{i\sigma},$$

where $\langle n_{i\sigma} \rangle$ is the expectation value of the number of electron with spin $\sigma$ at the $i$th site. In our calculations, we use the open boundary condition and examine finite lattices with $N = 180\,329$ and $1049\,137$, where the largest domains are $D_6$ and $D_7$, respectively. The lattices are generated by the deflation operations to the $D_1$ domain [shown in Fig. 4(a)], and therefore have the global eightfold rotational symmetry. For given values of mean-fields, we numerically diagonalize the mean-field Hamiltonian $H_{MF}$ and update the mean-fields, and iterate this selfconsistent procedure until the result converges within numerical accuracy.

We show in Fig. 8 the spatial pattern of the magnetization $n_{i\sigma}(= \langle n_{i\uparrow} \rangle - \langle n_{i\downarrow} \rangle)/2$ when $U/t = 1.0 \times 10^{-7}$. It is found that finite staggered magnetizations are induced even in the limit. This is due to the existence of the confined states, as discussed above. We note that the F vertices are also magnetized except for the F vertex at the center of the system. This originates from the fact that the amplitude of the confined states at the F vertex is zero in the $D_6$ domain, while should be finite in the larger domains, discussed before. Therefore, it is naively expected that, in the thermodynamic limit, each lattice site have a finite magnetization even in the weak coupling limit. This is in contrast to the systems with delta-function peak in DOS such as the Lieb and Penrose lattices, where there exist a finite density of nonmagnetic sites. A remarkable point is that eight A and B vertices around the F vertex have large magnetizations with $m \sim 1/16$ and the other A and B vertices are less magnetized, as shown in Fig. 9. Then, the Ammann-Beenker tiling with the larger lattice constant $\tau^2$ is formed in the spatial distribution of the magnetizations if the F vertex and adjacent A and B vertices with large magnetizations are regarded as its “unit cell”. This may imply the superlattice structure (fractal behavior) in the magnetic profile, which will be discussed later.

Increasing the Coulomb interactions, the magnetizations monotonically increase and finally the system should be described by the Heisenberg model in the strong coupling limit. To clarify the crossover in the ordered state between weak and strong coupling regimes, we show in Fig. 10 the distribution of local magnetizations in the system with $N = 180\,329$. When $U/t \rightarrow 0$, a finite distribution appears in the magnetization, where the average of the staggered magnetization $\bar{m}_0 \sim 0.043$. This originates from the existence of the macroscopically degenerate states discussed above and the staggered magnetization should be given as $1/4\tau^2$ in the thermodynamic limit. The increase of the Coulomb interactions monotonically increases the absolute value of local magnetization $\bar{m}_i \sim \bar{m}_{0i} + c_i U$, where $\bar{m}_{0i}$ is the local magnetization at $U \rightarrow 0$ and $c_i$ is the...
constant. This $U$ dependence differs from that in the conventional bipartite system, where the staggered magnetization usually increases as $m \sim \exp(-a/U)$, with $a$ is constant. On the other hand, this behavior is common to that in the bipartite systems with the macroscopically degenerate states at the Fermi level such as the Lieb [40] and Penrose [12] lattices. Increasing the interaction strength, the distribution of local magnetizations gradually changes. At last, when $U/t \gtrsim 2$, the magnetizations are classified by some peaks.

This classification is closely related to the coordination number for each site, which is different from the weak coupling case. Therefore, the crossover occurs in the antiferromagnetically ordered state around $U/t \sim 1.5$. Namely, in the strong coupling regime, the larger magnetization appears in the A vertices with smaller coordinations. This should be consistent with the quantum Monte Carlo results for the Heisenberg model [36, 37] although the mean-field treatment cannot take into account quantum fluctuations originating from intersite correlations.

Finally, let us study the spatial profile of the magnetizations characteristic of the Ammann-Beenker tiling. To this end, we map it to the perpendicular space. The positions in the perpendicular space have one-to-one correspondence with the position in the physical space. Each vertex site in the Ammann-Beenker tiling is described by the four dimensional lattice points $\vec{n} = (n_0, n_1, n_2, n_3)$ labeled with integers $n_m$ (see Fig. 1). Their coordinates are the projections onto the two-dimensional space:

$$\vec{r} = (x, y) = (\vec{n} \cdot \vec{e}^x, \vec{n} \cdot \vec{e}^y),$$

where $e^e_m = \cos(m\pi/4)$ and $e^a_m = \sin(m\pi/4)$. The projection onto the two-dimensional perpendicular space has information specifying the local environment of each site,

$$\vec{r} = (\tilde{x}, \tilde{y}) = (\vec{n} \cdot \vec{e}^x, \vec{n} \cdot \vec{e}^y),$$

where $e^e_m = \cos(3m\pi/4)$ and $e^a_m = \sin(3m\pi/4)$. Namely, six kinds of vertices have the corresponding regions in the perpendicular space, as shown in Fig. 1(d). Since vertices in both sublattices are uniformly distributed in the corresponding regions of the perpendicular space, the absolute value of magnetization are shown in Fig. 11. In the weak coupling limit, we find the detailed structure in the perpendicular space, meaning that the magnetization is not classified by the kinds of vertices. Therefore, this magnetic profile is reflected by the spatial structure of the macroscopically degenerate confined states, where large magnetizations appear in the A and B vertices around the F vertices, as shown in Fig. 11(a). Increasing the Coulomb interaction, interesting detailed structures smear in the perpendicular space. When $U/t = 5$, the magnetizations are almost specified by the vertices, where large magnetization appears in the A vertices and small magnetization appears in the F vertices. This tendency should be consistent with the results obtained from the quantum Monte Carlo simulations [36], as mentioned above.

Before summary, we wish to comment on fractal behavior in the magnetic properties in the weak coupling case. In the spatial distribution, A and B vertices around the F vertex have large magnetizations and these units form the Ammann-Beenker tiling with the lattice constant $\tau^2$, as shown in Fig. 9. This superlattice structure in the magnetizations allows us to consider the perpendicular space for the F vertex lattice. Figure 12(a) shows the magnetization profile for the F vertices in the weak coupling limit, which is the same as that of the F vertex part in Fig. 11(a). The average of the staggered magnetizations for the F vertices $\bar{m}_F \sim 0.005$ are much smaller than its bulk average $\bar{m}_0 \sim 0.043$, and therefore the magnetic profile for the F vertices may be invisible in Fig. 11(a). Figure 12(a) clearly shows that the magnetizations are not classified by the kinds of the F vertices ($F_m$), which are octagonally distributed in the perpendicular space, as shown in Fig. 12(b). Instead, we find the detailed structure in the distribution, where “A” and
“B” vertices around “F” vertex have large magnetization in the Ammann-Beenker tiling with the lattice constant $\tau_1$. This is similar to that in the original lattice shown in Fig. 11(a). Therefore, we can say that a similar magnetic profile is found in this scale. This may expect a further nested structure in the perpendicular space. Considering the $F_i$ ($i \geq 3$) vertex lattice in the Ammann-Beenker tiling with the lattice constant $\tau_1$, we show the magnetic profile in their perpendicular space in Fig. 12(c). We find a similar detailed structure in the staggered magnetizations although the number of the corresponding vertices are not large enough and the absolute value of the magnetization is much smaller. Then, we can say that fractal behavior appears in the magnetization profile, in particular, in the weak coupling limit.

V. SUMMARY

We have investigated magnetic properties in the half-filled Hubbard model on the Ammann-Beenker tiling. Considering the domain structure with locally eightfold rotational symmetry, we have examined the strictly localized confined states. We have then obtained their exact fractions in the thermodynamic limit. In contrast to the vertex model on the Penrose tiling, the wave functions for confined states are densely distributed in the lattice and thereby the introduction of the Coulomb interactions should induce finite staggered magnetizations in each site. Increasing the interaction strength, the spatial distribution of the magnetizations continuously changes to those of the Heisenberg model. Mapping the magnetization profiles to the perpendicular space, we have clarified that the superlattice structure appears in the magnetization profiles.

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APPENDIX

Here, we prove that the number of the $\sigma$ vertex is independent of the spin. In the main text, we have proved that the numbers of squares and rhombuses are independent of the spin, $S_\sigma = S/2$ and $R_\sigma = R/2$. Now, we consider the inflation-deflation process for the vertices $\{1\}$ $\{2\}$. Each vertex with the spin $\sigma$ is transformed under the inflation process as,

$$
\begin{align*}
A_\sigma &\rightarrow 0 \\
B_\sigma &\rightarrow 0 \\
C_{1\sigma} &\rightarrow 0 \\
C_{2\sigma} &\rightarrow A_\sigma \\
D_\sigma &\rightarrow B_\sigma \\
E_\sigma &\rightarrow C_{1\sigma} \\
F_\sigma &\rightarrow D_\sigma \\
E_\sigma &\rightarrow E_{\sigma'}
\end{align*}
$$

(13)

where 0 means that the vertices vanish under the inflation process. Since there are two kinds of the C vertices in the tiling, we have introduced C₁ and C₂ vertices. Under the deflation process, a C₁ vertex is not changed from any vertex, but is generated inside of each square with spin $\vec{S}_\sigma$, $\vec{S}_\sigma$, as shown in Fig. 2. Therefore $p^\text{C₁} = p^\text{S₁}/\tau^2 = 1/2\tau^3$, where the fraction of the squares with spin $\sigma$ is $p^\text{S₁} = S_\sigma/\sum_\sigma(S_\sigma + R_\sigma) = 1/2\tau$. Another $C_{2\sigma}$ vertex is always generated from the $A_\sigma$ vertex, $p^\text{C₂} = p^\text{A₁}/\tau^2$. Note that C₁ and C₂ vertices always appear as the nearest-neighbor pair in the tiling, as shown in Fig. 2. Therefore, we can say that C₁ vertex is also independent of spin. Since C₂ vertex is always changed to the $A_\sigma$ ($E_\sigma$) vertex under the inflation (deflation) process, $p^\text{C₂} = p^\text{A₁}/2$ ($p^\text{C₂} = p^\text{S₁}/2$). Two Bₗ vertices are generated inside of each square $S_\sigma$ and each rhombus $R_\sigma$, as shown in Fig. 2. This implies that the fraction of the B vertex is independent of the spin, $p^\text{B₁} = p^\text{B}/2$. All Bₗ vertices are changed to the Dₗ vertices under the deflation process. Therefore, the D vertex is also independent and immediately we find that the F vertices are also independent. Then, we can say that all vertices are independent of spins. Namely, the F₁ vertices are also independent since the Fₗ vertices are generated from the F₁₋₁ vertices and the F₁ vertices are changed from the C₂ vertices and Eₗ vertices under the deflation process.
