On the Vacua of Mass-deformed Gaiotto-Tomasiello Theories

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Abstract

We write explicit Lagrangian and supersymmetry transformation rules using the component fields in the $\mathcal{N} = 2,3$ GT theories. In the component field expansion, the manifestation of an additional $\mathcal{N} = 1$ supersymmetry is verified in the $\mathcal{N} = 3$ GT theory. We find maximal supersymmetry preserving mass-deformation of the GT theories and their classical supersymmetric discrete vacua. Some interesting aspects of the set of discrete vacua are discussed in comparison with the ABJM case.
1 Introduction

Massive type IIA supergravity \cite{1} has many supersymmetric and nonsupersymmetric solutions of the form $\text{AdS}_4 \times \mathcal{M}_6$, where $\mathcal{M}_6$ is a six dimensional manifold. Some of the nonsupersymmetric solutions were already found in \cite{1}, while the supersymmetric ones were not known until recently. The first such solution was $\mathcal{N} = 1$ solution constructed in \cite{2} and later generalized in \cite{3, 4, 5}. Based on these works, $\mathcal{N} = 2$ solutions were found by compactifying $\text{AdS}_4 \times M^{(1,1,1)}$ solutions and introducing mass deformation \cite{6} (see \cite{7, 8} for more general consideration). Other family of $\mathcal{N} = 2$ solutions, including massive deformation of the compactified $\text{AdS}_4 \times Q^{(1,1,1)}$ solution, were also constructed \cite{9}.

In relation with the dual three dimensional superconformal field theories for the above solutions, Gaiotto and Tomasiello \cite{10} considered some deformations in the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory \cite{12} such that the sum of Chern-Simons (CS) levels for the two gauge fields is not zero \cite{11}. The deformations yield in different superconformal CS-matter theories with $\mathcal{N} = 0, 1, 2, 3$ supersymmetries and $\text{SO}(6), \text{SO}(5), \text{SO}(2)_R \times \text{SO}(4)$, and $\text{SO}(3)_R \times \text{SO}(3)$ global symmetries, respectively. We refer to these theories as GT theories in the sequel. The authors argued that the deformed theories are dual to massive type IIA supergravity with the Roman mass parameter, which is understood as the RR zero-form flux $F_0$, is identified as the sum of CS

\footnote{The shift in Chern-Simons levels in the presence of D8-branes was also discussed in \cite{11}.}
levels of the two gauge fields \( F_0 = k_1 + k_2 \) \[10\] \[13\]. See also \[11\]. In particular, the gravity dual for the \( \mathcal{N} = 0 \) GT theory was identified as \( \text{AdS}_4 \times \mathbb{C}P^3 \) solution in \[1\] and the solutions in \[2\] \[4\] \[5\] are conjectured to be dual to \( \mathcal{N} = 1 \) GT theory. As a further evidence, a brane configuration for the \( \mathcal{N} = 0 \) theory was proposed in Type IIB string theory \[12\], by introducing D7 branes to the brane configuration of the ABJM theory \[12\]. Here \( F_0 \) is identified with the number of D7-branes. The gravity duals for the \( \mathcal{N} = 2 \) and \( \mathcal{N} = 3 \) GT theories with small \( F_0 \) were obtained in \[13\] by using first order perturbation for the \( \text{AdS}_4 \times \mathbb{C}P^3 \) solution, which is dual to the \( \mathcal{N} = 6 \) ABJM theory. Some nonperturbative aspects of \( \mathcal{N} = 2, 3 \) GT theories were also discussed by calculating partition functions \[15\] \[16\] \[17\].

An important missing point in this subject is that we have no known M-theory interpretation and dual eleven dimensional gravity for the GT theories. The reason is that the existence of the M-theory limit of the massive type IIA string theory is unclear. Recently, there was a conjecture of nonexistence of that limit \[8\], though the authors only considered weakly-curved solutions in massive type IIA gravity. The validity of this conjecture in the strongly-curved theory is unclear and needs further investigation.

On the other hand maximal supersymmetry preserving nonconformal deformations of the ABJM theory were found in \[18\] \[19\] and interpreted as the worldvolume theory of multiple M2-branes with a constant transverse four-form field strength \[20\] \[21\]. An important feature of this theory is that it has many classical discrete supersymmetric vacua \[19\] with their number much larger than the number expected from the dual gravity solution which is the Lin-Lunin-Maldacena (LLM) geometry \[22\] \[3\] for the CS level \( k = 1 \). The latter problem was recently resolved by realizing that many of the classical supersymmetric vacua dynamically breaks supersymmetry and results in the expected number \[24\]. This gauge/gravity duality relation between the mass-deformed ABJM theory and the LLM geometry was extended to generic \( k \) in \[25\] and the role of \( k \) was identified as the \( \mathbb{Z}_k \) quotients of the LLM geometry.

In this paper, we consider the mass deformation of the \( \mathcal{N} = 2, 3 \) GT theories. We verify that like the ABJM theory the GT theories allow maximal supersymmetry preserving mass deformations. We solve the vacuum equation and find discrete supersymmetric vacua which are similar in structure and property to the discrete solutions of the mass-deformed ABJM theory \[19\] \[24\] \[25\]. More precisely, the solutions are represented in terms of \( n \times (n + 1) \) or \( (n + 1) \times n \) irreducible blocks. One basic difference is that there are overall coefficients which depend on the ratio of the CS levels \( t = -\frac{k_2}{k_1} \) and the size of the irreducible block \( n \). For some special values of \( t \), the coefficients of some blocks are singular and those blocks are not allowed. This fact reduces the total number of vacua as compared to the case of ABJM theory.

\(^2\)See also \[23\].
This paper is organized as follows. In section 2 we write the $\mathcal{N} = 2, 3$ GT Lagrangian in terms of component fields and the corresponding supersymmetry transformation rules. In section 3 we obtain the maximal supersymmetry preserving mass deformation of these theories. In section 4 we solve the vacuum equations for the mass-deformed theories and find sets of discrete vacua. Section 5 includes conclusion and feature directions.

2 $\mathcal{N} = 2$ and $\mathcal{N} = 3$ GT Theories

The actions for these theories were written in [10] using the $\mathcal{N} = 2$ superfield formulation. In this section, we will give the Lagrangians in terms of component fields and find the corresponding supersymmetric transformation rules.

For convenience we review the $\mathcal{N} = 2$ superfield formulation of the GT theories with gauge group $U(N)_{k_1} \times U(N)_{k_2}$. The action is given by

$$S_{\mathcal{N}=2} = -\frac{ik_1}{8\pi}S_{\text{CS}}(\mathcal{V}_1) - \frac{ik_2}{8\pi}S_{\text{CS}}(\mathcal{V}_2) + S_{\text{mat}} + S_{\text{pot}},$$

where

$$S_{\text{CS}}(\mathcal{V}) = \int d^3xd^4\theta \int_0^1 dt \text{tr}[\mathcal{V}D^\alpha(e^{\mathcal{V}}D_\alpha e^{-\mathcal{V}})],$$

$$S_{\text{mat}} = \int d^3xd^4\theta \text{tr}[-\bar{Z}_A e^{-\mathcal{V}_1} Z^A e^{\mathcal{V}_2} - \bar{W}_A e^{-\mathcal{V}_2} W_A e^{\mathcal{V}_1}],$$

$$S_{\text{pot}} = c_1 \int d^3xd^2\theta \text{tr}[Z^A W_A Z^B W_B] + c_1 \int d^3xd^2\theta \text{tr}[W^A Z_A Z^B \bar{W}^B \bar{Z}_B] + c_2 \int d^3xd^2\theta \text{tr}[W_A Z^A W_B Z^B] + c_2 \int d^3xd^2\theta \text{tr}[ar{Z}_A \bar{W}^A \bar{Z}_B \bar{W}^B].$$

with $A, B = 1, 2$ and arbitrary real numbers $c_1, c_2$. The component field expansions for the chiral superfields, $Z^A, W_A$, and anti-chiral ones, $\bar{Z}_A, \bar{W}^A$, are given by

$$Z^A = Z^A(y) + \sqrt{2}\theta^i \xi^A_i(y) + \theta^2 F^A(y), \quad \bar{Z}_A = \bar{Z}^A_i(y^\dagger) - \sqrt{2}\bar{\theta}^i \xi_A^i(y^\dagger) - \bar{\theta}^2 F^A_i(y^\dagger),$$

$$W_A = W_A(y) + \sqrt{2}\bar{\theta} \omega_A(y) + \theta^2 G_A(y), \quad \bar{W}^A = W^A(y^\dagger) - \sqrt{2} \bar{\theta} \omega^A(y^\dagger) - \bar{\theta}^2 G^A(y^\dagger),$$

where the supercoordinate $y$ is defined as

$$y^\mu = x^\mu - i\theta \gamma^\mu \bar{\theta}, \quad y^{\dagger}_\mu = x^\mu + i\bar{\theta} \gamma^\mu \theta. \quad (2.4)$$

---

3We mainly follow the notations of [26].
$Z^A$ and $\mathcal{W}^A$ are in the bifundamental representations, while $\bar{Z}_A$ and $\mathcal{W}_A$ are in the anti-bifundamental representations of the gauge group. The component field expansions of the vector superfields $V_1$ and $V_2$ in Wess-Zumino gauge are

$$
V_1 = 2\bar{\theta} \theta \sigma_1 (x) - 2\theta \gamma^\mu \bar{\theta} A_\mu (x) + \sqrt{2} i \theta^2 \bar{\theta} \chi_1 (x) - \sqrt{2} i \bar{\theta}^2 \theta \chi_1 + \theta^2 \bar{\theta}^2 D_1 (x),
$$

$$
V_2 = 2\bar{\theta} \theta \sigma_2 (x) - 2\theta \gamma^\mu \bar{\theta} \hat{A}_\mu (x) + \sqrt{2} i \theta^2 \bar{\theta} \chi_2 (x) - \sqrt{2} i \bar{\theta}^2 \theta \chi_2 + \theta^2 \bar{\theta}^2 D_2 (x).
$$

(2.5)

Some conventions of the $\mathcal{N} = 2$ superspace are given in appendix A.

For generic $c_1$, the theory (2.1) has $\mathcal{N} = 2$ supersymmetry (SO(2) R-symmetry) and SU(2) flavor symmetry. For a particular case of $c_1 = -c_2 = c$, the superpotential in (2.2) can be rewritten as

$$
S_{\text{pot}} = c \int d^3 x d^2 \theta \epsilon_{AB} \epsilon^{CD} \text{tr} [Z^A \mathcal{W}_C Z^B \mathcal{W}_D] + c \int d^3 x d^2 \bar{\theta} \epsilon_{AB} \epsilon^{CD} \text{tr} [\mathcal{W}^C \bar{Z}_B \mathcal{W}^D \bar{Z}_B].
$$

(2.6)

The supersymmetry of this theory remains $\mathcal{N} = 2$, however, the flavor symmetry is enhanced to SU(2)×SU(2). On the other hand, if we choose $c_1 = \frac{2m}{k_1}$, the supersymmetry is enhanced to $\mathcal{N} = 3$, while the flavor symmetry remains SU(2) [10] [13]. In addition, if $F_0 = 0$, the supersymmetry is enhanced to $\mathcal{N} = 6$ [12] [27] and to $\mathcal{N} = 8$ for $k = 1, 2$ [28] [29].

2.1 $\mathcal{N} = 2$

In component field notation the $\mathcal{N} = 2$ Lagrangian can be written as

$$
\mathcal{L}_{\mathcal{N}=2} = \mathcal{L}_0 + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{form}} + \mathcal{L}_{\text{bos}} + \mathcal{L}_{\text{bos}}
$$

(2.7)

where

$$
\mathcal{L}_0 = \text{tr} \left[ -D_\mu Z_A^A D^\mu Z_A - D_\mu W^A D^\mu W_A + i \xi_A^A \gamma_\mu D_\mu \xi^A + i \omega^A \gamma_\mu D_\mu \omega_A \right],
$$

$$
\mathcal{L}_{\text{CS}} = \frac{k_1}{4\pi} \epsilon^{\mu \nu \rho} \text{tr} \left( A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) + \frac{k_2}{4\pi} \epsilon^{\mu \nu \rho} \text{tr} \left( \hat{A}_\mu \partial_\nu \hat{A}_\rho + \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho \right),
$$

(2.8)

$$
\mathcal{L}_{\text{form}}^D = -\frac{2\pi i}{k_1} \text{tr} \left[ (\xi_A^A - \omega^A A_\omega) (Z_B^B Z_B^B - W^B W_B) + 2(Z_A^A \xi_B^B - \omega^A W^B) \right],
$$

$$
\mathcal{L}_{\text{form}}^F = -c_1 \text{tr} \left( Z_A^A W_B^B Z_B^B + \xi_A^A W_B^A \xi_B^B W_B^B + 2Z_A^A W_B^A \xi_B^B \omega_B^B + 2Z_A^A \omega_A^A \xi_B^B W_B^B \right)
$$

$$
- \omega^A Z_A^A \omega_B^B W_B^B + W_A^A \xi_B^B W_B^B \xi_B^B \omega_B^B - 2\omega^A \xi_A^A \omega_B^B \xi_B^B \omega_B^B - W_A^A \omega_B^B 
$$

$$
- c_2 \text{tr} \left( \omega_A^A Z_B^B + \xi_A^A W_B^A \xi_B^B + 2\omega_A^A Z_B^B \xi_B^B + 2W_A^A \omega_B^B \xi_B^B \right)
$$

$$
- Z_A^A \omega_B^B \xi_B^B + 2\omega_A^A \xi_A^A \omega_B^B - 2c_1^2 W_A^A Z_B^B \omega_B^B - 2c_1^2 W_A^A Z_B^B \omega_B^B - 2c_1^2 \omega_A^A Z_B^B W_B^B)
$$

(2.9)
and

\[
\mathcal{L}^D_{\text{bos}} = -\frac{4\pi^2}{k_1^2} \text{tr} \left[ (Z^A Z_A^\dagger + W^A W_A^\dagger) (Z^B Z_B^\dagger - W^B W_B^\dagger) (Z^C Z_C^\dagger - W^C W_C^\dagger) \right] \\
- \frac{8\pi^2}{k_1 k_2} \text{tr} \left[ (Z^A Z_A^\dagger - W^A W_A^\dagger) Z^B (Z_C^\dagger Z^C - W^C W_C^\dagger) Z_B^\dagger \\
+ (Z^A Z_A^\dagger - W^A W_A^\dagger) W^B (Z_C^\dagger Z^C - W^C W_C^\dagger) W_B^\dagger \right] \\
- \frac{4\pi^2}{k_2^2} \text{tr} \left[ (Z^A_A Z^A_A + W_A W_A^\dagger) (Z_B^\dagger Z^B - W_B W_B^\dagger) (Z_C^\dagger Z^C - W_C W_C^\dagger) \right],
\]

\[
\mathcal{L}^F_{\text{bos}} = -4\text{tr} \left[ (c_1 W_A Z^B W_B + c_2 W_B Z^B W_A) (c_1 W^C Z_C^\dagger W^A + c_2 W^A Z_C^\dagger W^C) \\
+ (c_1 Z^B W_B Z^A + c_2 Z^A W_B Z^B) (c_1 Z_A^\dagger W^C Z_C^\dagger + c_2 Z_C^\dagger W^C Z_A^\dagger) \right].
\] (2.10)

In the generic case of \( c_1 \), the \( \mathcal{N} = 2 \) supersymmetric transformation rules for the component fields are given by

\[
\delta_\epsilon Z^A = i\epsilon \xi^A, \quad \delta_\epsilon Z_A^\dagger = i\xi_A^\dagger \epsilon, \\
\delta_\epsilon W_A = i\epsilon \omega_A, \quad \delta_\epsilon W^A = i\epsilon W^A, \\
\delta_\epsilon \xi^A = -D_\mu Z^A \gamma^\mu \epsilon - \sigma_1 Z^A \epsilon + Z^A \sigma_2 \epsilon - 2i\epsilon (c_1 W^B Z_B^\dagger W^A + c_2 W^A Z_B^\dagger W^B), \\
\delta_\epsilon \xi_A^\dagger = \epsilon \gamma^\mu D_\mu Z_A^\dagger - \epsilon \sigma_1 Z_A^\dagger + \epsilon \sigma_2 Z_A^\dagger + 2i (c_1 W_A Z^B W_B + c_2 W_B Z^B W_A) \epsilon, \\
\delta_\epsilon \omega_A = -D_\mu W_A \gamma^\mu \epsilon + W_A \sigma_1 \epsilon - \sigma_2 W_A \epsilon - 2i\epsilon (c_1 Z_A^\dagger W^B Z_B^\dagger + c_2 Z_B^\dagger W^B Z_A^\dagger), \\
\delta_\epsilon W^A = \epsilon \gamma^\mu D_\mu W^A + \epsilon \sigma_1 W^A \epsilon - \epsilon W^A \sigma_2 + 2i (c_1 Z^B W_B Z^A + c_2 Z^A W_B Z^B) \epsilon, \\
\delta_\epsilon A_\mu = \frac{1}{2} (\epsilon \gamma_\mu \chi_1 + \chi_1 \gamma_\mu \epsilon), \quad \delta_\epsilon \dot{A}_\mu = \frac{1}{2} (\epsilon \gamma_\mu \chi_2 + \chi_2 \gamma_\mu \epsilon),
\] (2.11)

where the supersymmetry parameters \( \epsilon \) and \( \bar{\epsilon} \) are complex two component spinor and its complex conjugate, respectively. Here we also defined

\[
\sigma_1 \equiv \frac{2\pi}{k_1} (Z^B Z_B^\dagger - W^B W_B^\dagger), \quad \sigma_2 \equiv -\frac{2\pi}{k_2} (Z_B^\dagger Z^B - W_B W^B), \\
\chi_1 \equiv -\frac{4\pi}{k_1} (Z^A \xi_A^\dagger - \omega^A W_A), \quad \chi_2 \equiv \frac{4\pi}{k_2} (\xi_A^\dagger Z^A - W_A \omega^A). \] (2.12)

The \( \mathcal{N} = 2 \) supersymmetric parameter \( \epsilon \) is inherited from the original \( \mathcal{N} = 6 \) supersymmetry in the ABJM theory. The \( \mathcal{N} = 6 \) supersymmetric parameters \( \omega^{AB} \), \( (A = 1, 2, 3, 4) \), can be grouped as the parameters of \( \mathcal{N} = 2 \) and those of \( \mathcal{N} = 4 \). The parameters of \( \mathcal{N} = 2 \) are \( \omega^{ab} \) \( (a, b = 1, 2) \) and they are related to \( \omega_{\hat{a}\hat{b}} \) \( (\hat{a}, \hat{b} = 3, 4) \) by reality condition, while those of \( \mathcal{N} = 4 \) are \( \omega^{ab} \). The parameter \( \epsilon \) in \( (2.11) \) is identified with \( \omega^{12} = \omega_{34} \). Therefore, the \( \mathcal{N} = 4 \) part in the ABJM theory are broken by introducing nonvanishing \( F_0 \) with generic \( c_i \) in the \( F \)-terms of the \( \mathcal{N} = 2 \) GT theory.
In this subsection, we find the additional $\mathcal{N} = 1$ supersymmetry in the action (2.7) when $c_i = \frac{2\pi}{k_i}$. This additional supersymmetry is slightly different from the $\mathcal{N} = 2$ of the previous section. For this reason we will briefly summarize the invariance of the action under this supersymmetry.

We start by noting that under this supersymmetry we have

$$\delta_\eta L_0 + \delta_\eta^A L_{CS} = 0,$$

(2.13)

where the supersymmetric variations are

$$\begin{align*}
\delta_\eta Z^A &= -\eta \omega^A, & \delta_\eta Z^A_\nu &= -i \omega_\nu \eta, \\
\delta_\eta W_A &= \eta \xi^A, & \delta_\eta W^A_\nu &= i \xi^A \eta, \\
\delta_\eta \xi^A &= i D_{\mu} W^A_\gamma \eta \gamma^{\mu} \eta, & \delta_\eta ^A_\nu &= \eta \gamma^{\mu} D_\mu W_A, \\
\delta_\eta \omega_A &= -i D_{\mu} Z^A_\gamma \eta \gamma^{\mu} \eta, & \delta_\eta ^A_\nu &= -\eta \gamma^{\mu} D_\mu Z^A, \\
\delta_\eta A_\mu &= -\frac{1}{2} \left( \eta \gamma_\mu \zeta_1 + \bar{\zeta}_1 \gamma_\mu \bar{\eta} \right), & \delta_\eta ^A_\nu &= \frac{1}{2} \left( \eta \gamma_\mu \zeta_2 + \bar{\zeta}_2 \gamma_\mu \bar{\eta} \right).
\end{align*}$$

(2.14)

Here we defined

$$\zeta_1 \equiv \frac{4\pi}{k_1} \left( \xi^A W_A + Z^A \omega_A \right), \quad \zeta_2 \equiv \frac{4\pi}{k_2} \left( W_A \xi^A + \omega_A Z^A \right),$$

(2.15)

and the supersymmetric parameter $\eta$ is a two component complex spinor. Later, this parameter will be constrained to give the $\mathcal{N} = 1$ parameter.

In order to complete the invariance of the action, we introduce an additional transformation for the fermions as

$$\begin{align*}
\delta_\eta^\prime \xi^A &= i \eta \sigma_1 W^A - i \eta W^A_\sigma_2 + \frac{4\pi i}{k_1} \eta W^B_\nu Z^A_\nu Z^A + \frac{4\pi i}{k_2} \eta Z^A_\nu Z^B_\nu W^{\nu B}, \\
\delta_\eta^\prime \xi^A_\nu &= -\eta W_A \sigma_1 + \eta \sigma_2 W_A - \frac{4\pi i}{k_1} \eta Z^A W^B Z^B_\nu W^{\nu B} - \frac{4\pi i}{k_2} \eta W^B Z^B_\nu W^{\nu B}, \\
\delta_\eta^\prime \omega_A &= i \eta Z^A_\sigma_1 - i \eta \sigma_2 Z^A_\nu - \frac{4\pi i}{k_1} \eta W_A W^B Z^B_\nu Z^A_\nu W^{\nu B} - \frac{4\pi i}{k_2} \eta W^B W^{\nu B} W_A, \\
\delta_\eta^\prime \omega^A_\nu &= -\eta \sigma_1 Z^A + \eta Z^A \sigma_2 + \frac{4\pi i}{k_1} \eta Z^A W^B W^{\nu B} + \frac{4\pi i}{k_2} \eta W^{\nu A} W_B Z^B.
\end{align*}$$

(2.16)

Then we note that

$$\delta_\eta L_0 + \delta_\eta^\prime L_0 + \delta_\eta L_{\text{ferm}}^D + \delta_\eta L_{\text{ferm}}^F = 0,$$

(2.17)

if the complex two component spinor $\eta$ satisfies the relation

$$\eta = i \eta^*.$$

(2.18)
The remaining variations of the Lagrangian satisfy
\[ \delta'_{\eta} \mathcal{L}_{\text{term}}^D + \delta'_{\eta} \mathcal{L}_{\text{term}}^F + \delta_{\eta} \mathcal{L}_{\text{bos}}^D + \delta_{\eta} \mathcal{L}_{\text{bos}}^F = 0 \] (2.19)
without further constraint. This completes the verification of the invariance of the action under the additional $\mathcal{N} = 1$ supersymmetry.

We have relations similar to (2.17) and (2.19) in the case of the $\mathcal{N} = 2$ supersymmetric theory discussed in the previous subsection. In that case, the variations of the $F$-term Lagrangians cancel with some part of variations of $\mathcal{L}_0$ in (2.17) and each other in (2.19), independent of the variations of the $D$-term Lagrangians. However, in the case of the additional $\mathcal{N} = 1$, the supersymmetric variations of the $D$-term and $F$-term Lagrangians are mixed. This indicates that the additional $\mathcal{N} = 1$ supersymmetry is inherited from the $\mathcal{N} = 4$ part of the original ABJM theory, which mixes the component fields of the superfields $Z^A$ and $W_A$. For this reason, the supersymmetric parameter $\eta$ is inherited from the $\mathcal{N} = 4$ supersymmetric parameter $\omega^{ab}$. In our conventions $\eta = i\omega^{12}$.

3 Mass-deformed GT Theories

It is well-known that the maximal supersymmetry preserving mass deformation is possible in the ABJM theory [18, 19]. There are several methods to obtain such mass-deformed theory, for instance, $\mathcal{N} = 1$ superfield formalism [18], $D$-term and $F$-term deformations in $\mathcal{N} = 2$ superfield formalism [19]. These different versions of mass-deformed theories are actually equivalent since they are connected by field redefinitions [30]. Based on the close relation between the ABJM and GT Lagrangians, we expect that such maximal supersymmetry preserving mass deformation can exist for the supersymmetric GT theories as well. In this section, we will find such deformation for both $\mathcal{N} = 2$ and $\mathcal{N} = 3$ GT theories.

3.1 $D$-term deformation

As in the ABJM theory the $D$-term deformation (FI deformation) is one way of obtaining the supersymmetry preserving mass-deformation of the GT theories. In the $\mathcal{N} = 2$ superfield formulation, the $D$-term deformation is given by
\[
S_D = -\frac{\mu}{4\pi} \int d^3x d^4\theta \text{tr}(k_1 \mathcal{V} - k_2 \mathcal{V}) = -\frac{\mu}{4\pi} \int d^3x \text{tr}(k_1 D_1 - k_2 D_2),
\] (3.20)
where the vector superfield $\mathcal{V}$ is as defined in section 2 and $\mu$ is the mass parameter. In this action as well as in the component field expansion of the original GT action (2.1), the auxiliary fields $D_1$ and $D_2$ are mixed with $F$-term Lagrangians. This indicates that the additional $\mathcal{N} = 1$ supersymmetry is inherited from the $\mathcal{N} = 4$ part of the original ABJM theory, which mixes the component fields of the superfields $Z^A$ and $W_A$. For this reason, the supersymmetric parameter $\eta$ is inherited from the $\mathcal{N} = 4$ supersymmetric parameter $\omega^{ab}$. In our conventions $\eta = i\omega^{12}$.
and $D_2$ appear linearly. Their equations of motion determine the auxiliary scalar fields $\sigma_1$ and $\sigma_2$, respectively. The effect of the above $D$-term deformation is then, shifting the values of $\sigma_1$ and $\sigma_2$ as follows

$$
\sigma_1 \rightarrow \sigma_1 - \frac{\mu}{2}, \quad \sigma_2 \rightarrow \sigma_2 + \frac{\mu}{2}.
$$

(3.21)

After integrating out $D_1$ and $D_2$ using their equations of motion, $\sigma_1$ and $\sigma_2$ appear only in the $L_D^{\text{ferm}}$ and $L_D^{\text{bos}}$ terms of the GT Lagrangian. Therefore, the shifting in (3.21) affects only the $D$-term fermionic and bosonic potentials in (2.7). Explicitly, we will obtain

$$
L_D^{\text{ferm}} = L_D^{\text{ferm}}(\text{GT}) + \mu \text{ tr}(i\xi_A^\dagger \xi^A - i\omega_A^\dagger \omega^A)
$$

(3.22)

$$
L_D^{\text{bos}} = L_D^{\text{bos}}(\text{GT}) - \text{ tr}\left[\mu^2 (Z_A^A Z^A_A + W_A^A W^A_A) - \frac{4\pi\mu}{k_1} (Z_A^A Z_B^B Z_B^A - W_A^A W_B^B W_B^A) - \frac{4\pi\mu}{k_2} (Z_A^A Z_B^B Z_B^A - W_A^A W_B^B W_B^A)\right].
$$

(3.23)

In summary, the Lagrangian of the deformed theory in terms of the component fields is written as

$$
L_{\text{tot}} = L_{\text{GT}} + L^\mu_{\text{ferm}} + L^\mu_{\text{bos}},
$$

(3.24)

where the first term in the right hand side is the original GT Lagrangian and the last two are the $D$-term deformations in (3.22) and (3.23). It is important to note that the $D$-term deformation does not affect the $F$-term potentials where we have the crucial difference between the $\mathcal{N} = 2$ and $\mathcal{N} = 3$ theories. As a result, the supersymmetry preserving mass deformation, which is derived from the $D$-term deformation, has the same form for these two theories.

It is straightforward to show that the Lagrangian (3.24) is invariant under the $\mathcal{N} = 2$ supersymmetry (2.11) if we include the following additional variations for the fermionic fields,

$$
\delta_\epsilon^\mu \xi^A = \mu \epsilon Z^A, \quad \delta_\epsilon^\mu \xi_A^\dagger = \mu \epsilon Z_A^\dagger, \quad \delta_\epsilon^\mu \omega_A = -\mu \epsilon W_A, \quad \delta_\epsilon^\mu \omega_A^\dagger = -\mu \epsilon W_A^\dagger.
$$

(3.25)

Furthermore, in the special case of $c_i = \frac{2\pi}{k_i}$ the action (3.24) is invariant under the additional $\mathcal{N} = 1$ supersymmetry (2.14) and (2.16) with additional supersymmetry variations,

$$
\delta_\eta^\mu \xi^A = -i\mu \eta W_A^\dagger, \quad \delta_\eta^\mu \xi_A^\dagger = \mu \eta W_A, \quad \delta_\eta^\mu \omega_A = -i\mu \eta Z_A^\dagger, \quad \delta_\eta^\mu \omega_A^\dagger = \mu \eta Z_A.
$$

(3.26)
3.2 F-term deformation

An alternative realization of the mass deformation is in terms of F-term superpotential deformation. In this section we will show that such realization of the supersymmetry preserving mass deformation of the GT theories is possible for \( N = 2 \) and \( N = 3 \) cases. In the case of \( N = 3 \), the F-term deformation is equivalent to the D-term deformation of the previous subsection up to field redefinition, while for the \( N = 2 \) theory these deformations cannot be related by field redefinition.

In the \( N = 2 \) superfield language the F-term deformation is given by

\[
S_F = -\mu \int d^3 x d^2 \theta \, \text{tr}(Z^A W_A) - \mu \int d^3 x d^2 \bar{\theta} \, \text{tr}(\bar{Z_A} \bar{W}^A).
\] (3.27)

Carrying out the component field expansion of the GT Lagrangian including this F-term deformation, we obtain

\[
L_{\text{tot}} = L_{\text{GT}} + L_{F}^\mu,
\] (3.28)

where

\[
L_{F}^\mu = \mu \text{tr}(\xi^A \omega_A - \omega^\dagger A \xi_A^\dagger) - \mu^2 (Z^A Z_A^\dagger + W_A W_A^\dagger)
\]
\[+ 2\mu \text{tr} \left[ (c_1 W_A Z^B W_B + c_2 W_B Z^B W_A) W_A^\dagger + W_A (c_1 W_A Z^B Z_B^\dagger + c_2 W_A^\dagger Z_B^\dagger W_B^\dagger) \right.
\]
\[+ (c_1 Z^B W_B Z_A^\dagger + c_2 Z_A^\dagger W_B Z_B^\dagger) Z_A^\dagger + Z^A (c_1 Z_A^\dagger W_A Z_B^\dagger + c_2 Z_B^\dagger W_A^\dagger Z_A^\dagger) \right].
\] (3.29)

In order to cast this term into the form of the D-term deformation in (3.24), we introduce the following field redefinitions

\[
Z^A = \frac{1}{\sqrt{2}} (P^A - Q^A), \quad W_A = \frac{1}{\sqrt{2}} (P_A^\dagger + Q_A),
\]
\[
\xi^A = \frac{1}{\sqrt{2}} (\chi^A - i\eta^A), \quad \omega_A = \frac{1}{\sqrt{2}} (i\chi_A^\dagger + \eta_A).
\] (3.30)

With this field redefinition we obtain

\[
L_{F}^\mu = \mu \text{tr}(i\chi_A^\dagger \chi^A - i\eta_A \eta^A) - \mu^2 \text{tr}(P_A^\dagger P_A + Q_A^\dagger Q_A)
\]
\[- 2\mu \text{tr} \left[ c_1 (P_A^\dagger P_B^\dagger P_B^\dagger P^B - Q_A^\dagger Q_A Q_B^\dagger Q_B) + c_2 (P_A^\dagger P_A P_B^\dagger P_B - Q_A Q_B^\dagger Q_B Q_B) \right].
\] (3.31)

In the special case of \( c_i = \frac{2 \pi}{k_i} \), this Lagrangian is equivalent to the D-term deformation (3.24). In addition, one can show that the form of the original GT Lagrangian is invariant under our field redefinition (3.30). Therefore, we realize that in the case of \( N = 3 \) GT theory the F-term deformation is equivalent to the supersymmetry preserving mass deformation derived from the D-term deformation. However, in the case of \( N = 2 \), the D-term and F-term deformations give two different supersymmetry preserving mass deformations.
4 Vacua of the Mass-deformed GT Theories

The classical supersymmetric discrete vacua of the mass-deformed ABJM theory were obtained in [19] and refined in [24]. In this section, we will follow a similar procedure with [19, 24] to obtain the classical discrete vacua of the mass-deformed GT theory. The structures of the vacua are the same as those of the mass-deformed ABJM theory, except for overall coefficients which depend on the CS levels \( k_1, k_2 \), and the size of the irreducible blocks inside matrix representations of the vacua.

In the \( D \)-term deformed \( \mathcal{N} = 2, 3 \) GT theories the bosonic potential can be written as

\[
V^\mu_{\text{bos}} = |\sigma_1 Z^A - Z^A \sigma_2 - \mu Z^A|^2 + |\sigma_2 W_A - W_A \sigma_1 + \mu W_A|^2 + |F_A|^2 + |G_A|^2, \tag{4.32}
\]

where \( \sigma_1 \) and \( \sigma_2 \) were defined in (2.12), \( F_A \) and \( G_A \) are

\[
F_A = -2 (c_1 W^A Z_B^B W^A_B + c_2 W^A_B Z_B^A W^A_B), \quad G_A = -2 (c_1 Z^A_B W^A_B Z_B^A + c_2 Z^A_B W^A_B Z_B^A). \tag{4.33}
\]

Here we have introduced the notation \( |O|^2 \equiv \text{tr} O \dagger O \), for convenience. At the vacuum, \( V^\mu_{\text{bos}} = 0 \). This means each of the summand in (4.32) is vanishing separately. Vanishing of the first two terms in the right hand side of (4.32) is rewritten as

\[
k_1 Z^A Z_B^B + k_2 Z^A_B Z_B^A = \frac{k_1 k_2}{2\pi} \mu Z^A, \quad k_1 W^A W^A_B W^A_B + k_2 W^A_B W^A_B = -\frac{k_1 k_2}{2\pi} \mu W^A. \tag{4.34}
\]

In order to solve the vacuum equation we assume that \( \mu > 0, k_1 > 0, \) and \( k_2 < 0, \) without loss of generality. Then the equations in (4.34) are simplified as

\[
\tilde{Z}^A \tilde{Z}_B^B \tilde{Z}^B - t \tilde{Z}^B \tilde{Z}_B^A \tilde{Z}^A + \tilde{Z}^A = 0, \\
\tilde{W}^A \tilde{W}_B^B \tilde{W}^B - t \tilde{W}^B \tilde{W}_B^A \tilde{W}^A - \tilde{W}^A = 0, \tag{4.35}
\]

where \( t = -\frac{k_2}{k_1} \) and we rescaled the fields as

\[
Z^A = \left( \frac{|k_2| \mu}{2\pi} \right)^{\frac{1}{2}} \tilde{Z}^A, \quad W^A = \left( \frac{|k_2| \mu}{2\pi} \right)^{\frac{1}{2}} \tilde{W}^A. \tag{4.36}
\]

As pointed out earlier the F-term deformation is equivalent to D-term deformation in \( \mathcal{N} = 3 \) theory. However, for \( \mathcal{N} = 2 \) theory the two are not the same and the vacuum equations for F-term deformed theory are different from what we are considering here.
In the mass-deformed ABJM theory with $U(1) \times U(1)$ gauge group, there is only trivial vacuum solution $Z^A = W_A = 0$. However, the vacuum equation of the mass-deformed GT theory with $U(1) \times U(1)$ gauge group is nontrivial and can be written as

$$|\tilde{Z}^1|^2 + |\tilde{Z}^2|^2 = \frac{1}{t-1}, \quad \tilde{W}_A = 0, \quad \text{or} \quad |	ilde{W}_1|^2 + |	ilde{W}_2|^2 = \frac{1}{1-t}, \quad \tilde{Z}^A = 0. \quad (4.37)$$

Expanding the complex fields in terms of real fields as $\tilde{Z}^A = \tilde{X}_A + i\tilde{X}_{A+4}$, $\tilde{W}_A = \tilde{X}_{A+2} - i\tilde{X}_{A+6}$, the vacuum equations in (4.37) are given by

$$\tilde{X}_1^2 + \tilde{X}_2^2 + \tilde{X}_3^2 + \tilde{X}_6^2 = \frac{1}{t-1}, \quad \tilde{W}_A = 0, \quad \text{or} \quad \tilde{X}_3^2 + \tilde{X}_4^2 + \tilde{X}_7^2 + \tilde{X}_8^2 = \frac{1}{1-t}, \quad \tilde{Z}^A = 0. \quad (4.38)$$

For $t > 1$ the first equation defines a $S^3$ while for $t < 1$ the second equation defines a $S^3$. We note that the vacuum equations (4.37) are singular for $t = 1$, as a result the $S^3$ vacuum modulus does not exist in the mass-deformed abelian ABJM theory.

For non-abelian GT theory, the solutions of (4.35) are represented by a direct sum of two sets of irreducible rectangular matrices\(^5\). The first set is composed of two $n \times (n + 1)$ matrices,

$$M^{(n)}_1 = \frac{1}{\sqrt{(n+1)t-n}} \begin{pmatrix} \sqrt{n} & 0 & 0 & \cdots & \cdots & \sqrt{n} \\ 0 & \sqrt{n-1} & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{n} \end{pmatrix},$$

$$M^{(n)}_2 = \frac{1}{\sqrt{(n+1)t-n}} \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \sqrt{2} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \sqrt{2} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \sqrt{n} \end{pmatrix}. \quad (4.39)$$

\(^5\)We follow the notation introduced in [24, 25].
and the second one is composed of two \((n+1) \times n\) matrices,

\[
\mathcal{M}_{1}^{(n)} = \frac{1}{\sqrt{n+1} - nt} \begin{pmatrix}
\sqrt{n} & 0 & \cdots & 0 \\
0 & \sqrt{n-1} & \cdots & \sqrt{2} \\
\cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & 0
\end{pmatrix},
\]

\[
\mathcal{M}_{2}^{(n)} = \frac{1}{\sqrt{n+1} - nt} \begin{pmatrix}
0 & 1 & \cdots & 0 \\
1 & 0 & \cdots & \sqrt{2} \\
\cdots & \cdots & \cdots & \cdots \\
0 & \sqrt{n-1} & \cdots & 0
\end{pmatrix}.
\]

(4.40)

\(\mathcal{M}_{a}^{(n)}\) and \(\mathcal{M}_{a}^{(n)}\) can be used to construct the solutions of the first and the second equations in (4.35), respectively. These irreducible blocks are similar to those in [24, 25], except for the overall coefficients\(^6\). As a result of these non trivial overall coefficients, there is an interesting difference between the vacuum solution here and those of [24, 25]. For given \(t < 1\) the blocks \(\mathcal{M}_{1,2}^{(n)}\) with \(n = \frac{t}{1-t}\) are not allowed while for \(t > 1\) the blocks \(\mathcal{M}_{1,2}^{(n)}\) with \(n = \frac{1}{t-1}\) are not allowed. Therefore, for GT theories with CS levels satisfying any of these two restrictions on the ratio \(t\) of the CS levels, the total number of classical supersymmetric vacua is reduced as compared to the ABJM case.

The general solutions satisfying the equations (4.34) and the F-term equations \(|F^A| = 0, \ |G_A| = 0\), are represented in terms of the irreducible blocks as

\[
Z^A = \left(\frac{|k_2| \mu}{2\pi}\right)^{\frac{1}{2}} \begin{pmatrix}
\mathcal{M}_{A}^{(n_1)} \\
\cdots \\
\mathcal{M}_{A}^{(n_i)} \\
0_{(n_{i+1}+1) \times n_{i+1}} \\
\cdots \\
0_{(n_f+1) \times n_f}
\end{pmatrix},
\]

---

\(^6\)These blocks are obtained following the method of [19], where the uniqueness of the irreducible blocks with \(t = 1\) was confirmed. Therefore, the same argument of uniqueness can be applied to the irreducible blocks in (4.39) and (4.40).
\[ W^{\dagger A} = \left( \frac{|k_2|/\mu}{2\pi} \right)^{1/2} \begin{pmatrix} 0_{n_1 \times (n_1+1)} & \cdots & 0_{n_1 \times (n_1+1)} \\ \vdots & \ddots & \vdots \\ 0_{n_1 \times (n_1+1)} & \cdots & \tilde{M}_A^{(n_1+1)} \\ \vdots & \ddots & \vdots \\ \tilde{M}_A^{(n_f)} \end{pmatrix}, \quad (4.41) \]

where \( n_i = 0, 1, 2, \cdots \) and \( 0_{m \times n} \) denotes \( m \times n \) zero matrix. Since \( Z^A \) and \( W^{\dagger A} \) are \( N \times N \) matrices, there are two constraints

\[
\sum_{n=0}^{\infty} [n\tilde{N}_n + (n + 1)\hat{N}_n] = N, \quad \sum_{n=0}^{\infty} [(n + 1)\tilde{N}_n + n\hat{N}_n] = N, \quad (4.42)
\]

where \( \tilde{N}_n \) denotes the number of block of \( M_A^{(n)} \)-type and \( \hat{N}_n \) is the number of block of \( \bar{M}_A^{(n)} \)-type. Here \( \tilde{N}_0 \) and \( \hat{N}_0 \) represent the numbers of empty columns and empty rows, respectively.

Next we discuss one more interesting feature of the classical supersymmetric discrete vacua we found here. It is understood that at the discrete vacua, the U(\( N \)) \times U(\( N \)) gauge symmetry is partially broken. The unbroken gauge symmetry corresponds to the reshuffling of irreducible blocks of the same type. More precisely, the gauge fields for the symmetry that reshuffles the block \( M_A^{(n)} \) is given by \[24\]

\[
A_\mu = 1_n \otimes [a_\mu]_{\tilde{N}_n \times \tilde{N}_n}, \quad \hat{A}_\mu = 1_{n+1} \otimes [a_\mu]_{\hat{N}_n \times \hat{N}_n}, \quad (4.43)
\]

where \( n = 0, 1, 2, \cdots \) and \( a_\mu \) denotes the unbroken gauge field generating the reshuffling of \( \tilde{N}_n \) blocks. Inserting (4.43) into the CS action (2.8) we obtain

\[
\frac{1}{4\pi} \epsilon^{\mu\nu\rho} (k_1 \text{tr} 1_n + k_2 \text{tr} 1_{n+1}) \text{tr} \tilde{N}_n \left[ a_\mu \partial_\nu a_\rho + \frac{2i}{3} a_\mu a_\nu a_\rho \right] = \frac{k_1 n + k_2 (n + 1)}{4\pi} \epsilon^{\mu\nu\rho} \text{tr} \tilde{N}_n \left[ a_\mu \partial_\nu a_\rho + \frac{2i}{3} a_\mu a_\nu a_\rho \right]. \quad (4.44)
\]

Similarly for the blocks \( \bar{M}_A^{(n)} \) the unbroken gauge fields are

\[
A_\mu = 1_{n+1} \otimes [a_\mu]_{\hat{N}_n \times \hat{N}_n}, \quad \hat{A}_\mu = 1_n \otimes [a_\mu]_{\tilde{N}_n \times \tilde{N}_n}, \quad (4.45)
\]

and the corresponding CS term is given by

\[
\frac{k_1 (n + 1) + k_2 n}{4\pi} \epsilon^{\mu\nu\rho} \text{tr} \tilde{N}_n \left[ a_\mu \partial_\nu a_\rho + \frac{2i}{3} a_\mu a_\nu a_\rho \right]. \quad (4.46)
\]
In summary, for a given vacuum there exist CS theories with gauge group

$$\prod_{n=0}^{\infty} U(\tilde{N}_n)_{\tilde{k}_n} \times U(\hat{N}_n)_{\hat{k}_n},$$

where CS levels are

$$\tilde{k}_n = k_1 n + k_2 (n+1) = nF_0 + k_2, \quad \hat{k}_n = k_1 (n+1) + k_2 n = nF_0 + k_1.$$  \hspace{0.5cm} (4.48)

Here we would like to point out two interesting facts about the unbroken gauge group. The first point is that unlike the mass-deformed ABJM theory, where the CS levels of the unbroken gauge fields are unshifted i.e., $\tilde{k}_n = -k$ and $\hat{k}_n = k$ \cite{24}, in the present case the CS levels are shifted by $nF_0$. The other point is that when $n = \frac{t}{t-1}$ or $n = \frac{1}{t-1}$ the CS levels, $\tilde{k}_n$ or $\hat{k}_n$ are vanishing, respectively. This is consistent with what we explained previously about the coefficients of the irreducible blocks.

5 Conclusion

In this paper we found the maximal supersymmetry preserving mass deformation of the $\mathcal{N} = 2, 3$ GT theories. Since the original GT Lagrangian was written in the $\mathcal{N} = 2$ superfield formalism, the additional $\mathcal{N} = 1$ supersymmetry is not manifest in the case of the $\mathcal{N} = 3$ GT theory. To clarify this point we started with the component field expansion of the GT Lagrangians. Then we wrote the $\mathcal{N} = 2$ supersymmetry transformation rules for the component fields. Gaiotto and Tomasiello pointed out that when the coefficients of the superpotential $c_i = \frac{2\pi}{k_i}$, the supersymmetry is enhanced to $\mathcal{N} = 3$ \cite{10}. We found the explicit supersymmetry transformation rules for the component fields under the additional $\mathcal{N} = 1$ supersymmetry.

Following the line of the original ABJM theory we found the mass deformations of the GT theories which preserve the maximal supersymmetry. The mass deformations can be realized either as D-term or F-term deformations. The D-term deformation does not affect the F-term potentials. Since the $\mathcal{N} = 2, 3$ theories differ by the F-term potential, the mass deformation which is derived from the D-term deformation for the two theories are equivalent. On the other hand, for the $\mathcal{N} = 3$ GT theory the F-term deformation is equivalent to the mass deformation obtained from the D-term deformation, up to field redefinition, while the $\mathcal{N} = 2$ D-term and F-term deformations give two distinct supersymmetry preserving mass deformations.

Using the mass-deformed GT theories we found set of discrete classical supersymmetric vacua. The classical vacuum solutions are expressed in terms irreducible blocks of size $n \times (n+1)$ or

\footnote{We are indebted to Seok Kim for clarifying this point.}
\((n+1) \times n\). An important feature of the current situation is that when the ratio of the CS levels \(t = \frac{n+1}{n}\) or \(t = \frac{n}{n+1}\), those irreducible blocks have singular coefficients and are not allowed. Therefore, in mass-deformed GT theories with the ratio of CS levels satisfying the above conditions the total number of supersymmetric vacua is reduced.

The purpose of this work is mainly to clear the way for future perspective in this subject. There are many unanswered questions in the GT theory. The important ones are the facts that M-theory interpretation and M-theory limit of dual gravity are not understood. Recently, Cheon et al \([25]\) found a one-to-one correspondence between the supersymmetric vacua in gauge theory side and \(\mathbb{Z}_k\) quotients of the LLM geometry. This correspondence is valid for weakly-curved as well as strongly-curved background in dual gravity. Based on the close relation between the discrete vacua of the mass-deformed ABJM and GT theories, there is a possibility that the investigation of \([25]\) (see also \([31]\)) can be carried out for mass-deformed GT theories as well. The result can shed some light on the physics of strongly-curved and strongly-coupled massive type IIA string theory in relation with the conjecture of \([8]\).

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**A Conventions and Fierz Identity**

We choose (2+1)-dimensional gamma matrices as \(\gamma^0 = i\sigma^2, \gamma^1 = \sigma^1\), and \(\gamma^2 = \sigma^3\), which satisfy \(\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu\rho} \gamma_\rho\). The conventions for spinor indices are

\[
\theta^\alpha = \epsilon^{\alpha\beta} \theta_\beta, \quad \theta_\alpha = \epsilon_{\alpha\beta} \theta^\beta, \quad \epsilon^{12} = -\epsilon_{12} = 1,
\]

\[
\theta^\alpha \theta_\alpha \equiv \bar{\theta}^2, \quad \theta^\alpha \theta_\alpha \equiv \theta \bar{\theta}, \quad \theta^\alpha \gamma_\alpha \gamma_\beta \theta_\beta \equiv \theta \gamma^\mu \bar{\theta}.
\]  

(A.49)

In terms of these conventions for spinors, we obtain

\[
\theta_\alpha \theta_\beta = \frac{1}{2} \epsilon_{\alpha\beta} \theta^2, \quad \theta^\alpha \theta^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \theta^2.
\]  

(A.50)

\footnote{We use the spinor convention of \([26]\). There is one difference in the convention of the suppressed spinor indices, i.e., in our case \(\xi \gamma^\mu \chi = \xi^\alpha \gamma_\alpha^{\mu\beta} \chi_\beta\), where \(\xi\) and \(\chi\) are two component spinors.}
Useful Fierz identities inside the trace are
\[
\text{tr}\left[(\varepsilon^\mu \psi_A)(\psi_B \gamma_\mu \psi_C)\right] = -\text{tr}\left[2(\psi_C)(\psi_A \psi_B) + (\psi_A)(\psi_B \psi_C)\right]
\]
\[
= -\text{tr}\left[(\psi_C)(\psi_A \psi_B) - (\psi_B)(\psi_C \psi_A)\right]
\]
\[
= \text{tr}\left[(\psi_A)(\psi_B \psi_C) + 2(\psi_B)(\psi_C \psi_A)\right],
\]
\[
\text{tr}\left[(\psi_A \gamma_\mu \psi_B)(\psi_C \gamma_\mu \psi)\right] = -\text{tr}\left[2(\psi_B \psi_C)(\psi_A \epsilon) + (\psi_A \psi_B)(\psi_C \epsilon)\right]
\]
\[
= -\text{tr}\left[(\psi_B \psi_C)(\psi_A \epsilon) - (\psi_C \psi_A)(\psi_B \epsilon)\right]
\]
\[
= \text{tr}\left[(\psi_A \psi_B)(\psi_C \epsilon) + 2(\psi_B \psi_C)(\psi_A \epsilon)\right],
\]
\[
\text{tr}\left[(\varepsilon^\mu \psi_A)(\psi_B \psi_C)\right] = -\text{tr}\left[(\psi_A \psi_B)(\psi_C \epsilon) + (\psi_C \psi_A)(\psi_B \epsilon)\right],
\]
where \(\epsilon\) is a spinor without gauge indices. We also have the relations,
\[
(\theta \bar{\theta})^2 = -\frac{1}{2} \theta^2 \bar{\theta}^2, \quad \theta \bar{\theta}(\theta \gamma^\mu \bar{\theta}) = 0, \quad (\theta \gamma^\mu \bar{\theta})(\theta \gamma^\nu \bar{\theta}) = \frac{1}{2} \eta^{\mu \nu} \theta^2 \bar{\theta}^2.
\]
We adapt the convention for integrations,
\[
d^2 \theta \equiv -\frac{1}{4} d\theta^\alpha d\bar{\theta}_\alpha, \quad d^2 \bar{\theta} \equiv -\frac{1}{4} d\bar{\theta}^\alpha d\theta_\alpha, \quad d^4 \theta \equiv d^2 \theta d^2 \bar{\theta},
\]
\[
\int d^2 \theta d^2 \bar{\theta} = 1, \quad \int d^2 \theta d^2 \bar{\theta} = 1, \quad \int d^4 \theta d^2 \bar{\theta} = 1,
\]
and have supercovariant derivatives and supersymmetry generators,
\[
D_\alpha = \partial_\alpha - i \gamma^\mu \beta \bar{\theta}_\beta \partial_\mu, \quad \bar{D}_\alpha = -\bar{\partial}_\alpha - i \theta^\beta \gamma^\mu \bar{\beta}_\alpha \partial_\mu,
\]
\[
Q_\alpha = \partial_\alpha + i \gamma^\mu \beta \bar{\theta}_\beta \partial_\mu, \quad \bar{Q}_\alpha = -\bar{\partial}_\alpha + i \theta^\beta \gamma^\mu \bar{\beta}_\alpha \partial_\mu
\]
with anti-commutation relations,
\[
\{D_\alpha, \bar{D}_\beta\} = -2i \gamma^\mu_\alpha \partial_\mu, \quad \{Q_\alpha, \bar{Q}_\beta\} = 2i \gamma^\mu_\alpha \partial_\mu.
\]

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