A Time Dependent Spacetime in $f(R,T)$ Gravity: Gravitational Collapse

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ABSTRACT: In this note a time dependent spacetime is explored in the background of $f(R,T)$ gravity via the gravitational collapse of a massive star. The star is modelled by the Vaidya spacetime which is time dependent in nature. The coupling of matter with curvature is the key feature of $f(R,T)$ theory and here we have investigated its effects on a collapsing scenario. Two different types of models, one involving minimal and the other involving non-minimal coupling between matter and curvature are considered for our study. Power law and exponential functionalities are considered as examples to check the outcome of the gravitational collapse. Our prime objective is to explore the nature of singularities (black hole or naked singularity) that form as an end state of the collapse. Existence of outgoing radial null geodesics from the central singularity was probed and such existence implied the formation of naked singularities thus defying the cosmic censorship hypothesis. The absence of such outgoing null geodesics would imply the formation of an event horizon and the singularity formed becomes a black hole. Conditions under which such possibilities occur are derived for all the models and sub-models. Gravitational strength of the singularity is also investigated and the conditions under which we can get a strong or a weak singularity is derived. The results obtained are very interesting and may be attributed to the coupling between curvature and matter. It is seen that for non-minimal coupling there is a possibility of a globally naked singularity, whereas for a minimal coupling scenario local nakedness is the only option. It is also found that the singularity formed can be sufficiently weak in nature, which is cosmologically desirable.

KEYWORDS: Modified gravity, gravitational collapse, black hole, naked singularity, Vaidya.
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1 Introduction

For the last two decades we have been aware of the fact that our universe has entered into a phase of accelerated expansion [1, 2]. Although this came as a total surprise to the scientific community, extensive research have been able to put some meaning to this observed phenomenon over the years. It is a widely known fact that this cosmic acceleration can be explained via two different theoretical frameworks. One is the theory of dark energy (DE) that recasts the matter content of the universe to some exotic substance possessing negative pressure. The other way is to modify Einstein’s theory of gravity leading to modified gravity theories. The reader may refer to Refs.[3–5] for extensive reviews on modified gravity theories and to the Ref.[6] for a detailed review on DE.

One of the most popular way to modify Einstein’s gravity is by replacing the Ricci scalar $R$ in the gravity Lagrangian of the Einstein-Hilbert action of general relativity (GR) by an analytic function of $R$, i.e. $f(R)$, which gives rise to $f(R)$ gravity theory. Extensive reviews on $f(R)$ gravity can be found in Refs.[7, 8]. In Ref.[9] the authors proposed an even more generic class of models by considering the gravitational lagrangian as an analytic function of Ricci scalar $R$ and matter Lagrangian $L_m$, paving the path for $f(R, L_m)$ theories. Further developments in $f(R, L_m)$ theories can be found in Refs.[10–12]. In Ref.[13] Harko et al. proposed the $f(R, T)$ theory, where the matter Lagrangian is given by the trace $T$ of the energy-momentum tensor $T_{\mu\nu}$. So here the gravitational Lagrangian is an analytic function of two scalar invariants, namely the Ricci scalar $R$ and the trace of the energy-momentum tensor $T$. Here the contributions of $T$ will come from the matter content of the universe. As a result it is found that the field equations of $f(R, T)$ theory depends on a source term, which is given by the variation of the energy-momentum tensor with respect to the metric. This will in turn depend on the matter Lagrangian or the nature of matter content of the universe. So it is obvious that for different types of matter, such as scalar fields, perfect fluid, electromagnetic field, etc. we will get different set of field equations. From the form of the function, it is obvious that this theory involves coupling between matter and geometry. So by studying this theory one can probe such coupling effects and their consequences on various astrophysical and cosmological phenomenon. It is seen that the covariant divergence of the energy-momentum tensor is non-zero for this model, which leads to non-geodesic motion of the massive test particles. This is because the coupling effects between matter and geometry induces an extra acceleration on the particles. Thermodynamics in $f(R, T)$ gravity was studied by Sharif and Zubair in [14]. Cosmological Evolution in $f(R, T)$ theory with collisional matter was studied in Ref.[15]. In Ref.[16] cosmic coincidence problem was studied in the background of $f(R, T)$ gravity. Cosmological models in $f(R, T)$ theories as phase space was explored in [17]. Dynamics of scalar perturbations in $f(R, T)$ gravity was studied by Alvarenga et al. in [18]. Gravastars in $f(R, T)$ gravity was studied in Ref.[19]. Dark matter from $f(R, T)$ gravity was investigated by the authors in [20]. Propagation of polar gravitational waves in $f(R, T)$ scenario was explored in [21]. Dynamical behavior of the Tolman metrics in $f(R, T)$ gravity was studied by Hansraj and Banerjee in [22].

Gravitational collapse is a key astrophysical phenomenon that helps us to understand various aspects of the universe such as structure formation, properties of stars, formation of black holes, white dwarfs, neutron stars, etc. A star undergoes a gravitational collapse due to its own mass at the end of its life cycle, when it has exhausted all its nuclear fuel. During its collapse journey there are various stages at which the collapse may stop, depending on the initial mass of the collapsing star. If the star is massive i.e. mass $> 20M_\odot$ ($M_\odot$ represents solar mass), then the collapse does not come to a halt at any of the intermediate stages (such as white dwarf or neutron star), but directly proceeds to form a singularity such as a black hole (BH). The study of gravitational collapse started with Oppenheimer and Snyder [23] in 1939 when they explored the gravitational collapse of a dust cloud modelled by a static Schwarzschild exterior and Friedmann interior. Following this, Tolman [24] and Bondi [25] studied the collapse of spherically symmetric inhomogeneous distribution of
dust. Subsequently a lot of interest was generated in this subject and numerous work related to this can be found in literature. Some reviews in gravitational collapse can be found in Refs.[26, 27]. Roger Penrose in 1969 proposed cosmic censorship hypothesis (CCH) [28], where he stated that any cosmological singularity will always be covered by an event horizon, thus censoring the singularity from an external observer. Such a singularity (popularly called a black hole) is associated with permanent loss of physical information allowing multiple physical states to devolve into a single state. This is known as the black hole information loss paradox [29–32]. Over the years, in the absence of a formidable proof of CCH, scientists started questioning its validity. As a consequence, a search was initiated that will culminate in the discovery of a singularity that will be free from any event horizon. This type of singularity will not only disprove CCH but also in the absence of information loss it will enhance our knowledge about gravity. Such a singularity is named as a naked singularity (NS) [33–44] which is considered to be a crucial tool in the formulation of an effective theory of quantum gravity.

The first effective relativistic line element representing the spacetime of a realistic star was given by P. C. Vaidya [45] in 1951. It represented the radiation for a non-static mass, thus generalizing the static solution of Schwarzschild. Schwarzschild’s solution basically represented the spacetime around a spherically symmetric cold dark body with a constant mass. So it is obvious that it could never model the spacetime outside a star. This is the problem that was addressed by Vaidya in his phenomenal paper [45] of 1951. The solution proposed by Vaidya was termed as Vaidya spacetime and is often referred to as the shining or radiating Schwarzschild metric. It should be noted that the basic difference between the two metrics is that the constant mass parameter in the Schwarzschild metric is replaced by a time dependent mass parameter in the Vaidya metric, which consequently becomes a time dependent spacetime. Notable studies in Vaidya metric can be found in the Refs.[46–54].

Here we are interested in exploring the gravitational collapse of a massive star modelled by the Vaidya metric in the background of $f(R, T)$ gravity. Collapsing scenario in the presence of coupling between matter and curvature is expected to be an interesting proposition. Moreover the behaviour of Vaidya spacetime has never been explored in the background of $f(R, T)$ gravity. So there is more than enough motivation for attempting this work. We will basically focus on the nature of the singularity formed (BH or NS) as the end state of the collapse. We will report the conditions under which these singularities can form in a comparative manner. We hope to obtain interesting and new results in our collapsing scheme in the background of curvature-matter coupling. In the next section we will report the basic equations of Vaidya spacetime in $f(R, T)$ gravity and find solutions for the system. In section III we will explore the collapsing scenario of a massive star. Section IV will deal with the strength of the singularity formed and finally the paper will end with a detailed discussion and conclusion in section V.

2 Vaidya spacetime in $f(R, T)$ gravity

The Einstein-Hilbert action for general relativity is given by,

$$ S_{EH} = \frac{1}{2\kappa} \int R \sqrt{-g} d^4x $$

(2.1)

where $\kappa \equiv 8\pi$, $g$ is the determinant of the metric and $R$ is the Ricci scalar (we have considered $G=c=1$). We replace the Ricci scalar, $R$ in the above action by a generalized function of $R$ to get the action for $f(R)$ gravity [7, 8],

$$ S = \frac{1}{2\kappa} \int f(R) \sqrt{-g} d^4x $$

(2.2)
Taking the action (2.2) and adding a matter term $S_M$, the total action for $f(R)$ gravity takes the form,

$$S_{f(R)} = \frac{1}{2\kappa} \int f(R)\sqrt{-g}d^4x + \int L_m \sqrt{-g}d^4x$$

(2.3)

where $L_m$ is the matter Lagrangian and the second integral on the R.H.S is $S_M$ representing the matter fields. To obtain the action for $f(R, T)$ gravity we further modify the action for $f(R)$ gravity by introducing the trace of the energy-momentum tensor $T_{\mu\nu}$ in the gravity Lagrangian as follows [13],

$$S_{f(R, T)} = \frac{1}{2\kappa} \int f(R, T)\sqrt{-g}d^4x + \int L_m \sqrt{-g}d^4x$$

(2.4)

Here $f(R, T)$ is an arbitrary function of the Ricci scalar $R$ and the trace $T$ of the energy-momentum tensor $T_{\mu\nu}$. The energy-momentum tensor is defined as [55],

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(-gL_m)}{\delta g^{\mu\nu}}$$

(2.5)

The trace of this tensor can be given as $T = g_{\mu\nu}T_{\mu\nu}$. Taking variation with respect to the metric we get the field equations for $f(R, T)$ gravity as,

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2} f(R, T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f(R, T) = \kappa T_{\mu\nu} - f_T(R, T)T_{\mu\nu} - f_T(R, T)\Theta_{\mu\nu}$$

(2.6)

where $\Theta_{\mu\nu}$ is given by,

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta} \frac{\delta^2 L_m}{\delta g^{\mu\nu} \delta g^{\alpha\beta}}$$

(2.7)

In the field equations $\nabla_\mu$ denotes covariant derivative associated with the Levi-Civita connection of the metric and $\Box \equiv \nabla^\mu \nabla_\mu$ is the D’Alembertian operator. Moreover we have denoted $f_R(R, T) = \partial f(R, T)/\partial R$ and $f_T(R, T) = \partial f(R, T)/\partial T$. The tensor $\Theta_{\mu\nu}$ can be calculated as,

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}L_m - 2g^{\alpha\beta} \frac{\delta^2 L_m}{\delta g^{\mu\nu} \delta g^{\alpha\beta}}$$

(2.8)

It is seen that the above tensor depends on the matter lagrangian. For perfect fluid the above tensor becomes,

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + pg_{\mu\nu}$$

(2.9)

The Vaidya metric in the advanced time coordinate system is given by,

$$ds^2 = f(t, r)dt^2 + 2dtdr + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

(2.10)

where $f(t, r) = -\left(1 - \frac{m(t, r)}{r}\right)$ and using the units $G = c = 1$. The total energy momentum tensor of the field equation (2.6) is given by the following sum,

$$T_{\mu\nu} = T_{\mu\nu}^{(n)} + T_{\mu\nu}^{(m)}$$

(2.11)

where $T_{\mu\nu}^{(n)}$ and $T_{\mu\nu}^{(m)}$ are the contributions from the Vaidya null radiation and perfect fluid respectively defined as,

$$T_{\mu\nu}^{(n)} = \sigma l_\mu l_\nu$$

(2.12)

and

$$T_{\mu\nu}^{(m)} = (\rho' + p')(l_\mu \eta_\nu + l_\nu \eta_\mu) + pg_{\mu\nu}$$

(2.13)

where '$\rho'$ and '$p'$ are the energy density and pressure for the perfect fluid and '$\sigma'$ is the energy density corresponding to Vaidya null radiation. In the co-moving co-ordinates $(t, r, \theta_1, \theta_2, ..., \theta_n)$,
the two eigen vectors of energy-momentum tensor namely \( l_\mu \) and \( \eta_\mu \) are linearly independent future pointing null vectors having components

\[
l_\mu = (1, 0, 0, 0) \text{ } \text{and} \text{ } \eta_\mu = \left( \frac{1}{2} \left( 1 - \frac{m}{r} \right), -1, 0, 0 \right)
\]

(2.14)

and they satisfy the relations

\[
l_\lambda l^\lambda = \eta_\lambda \eta^\lambda = 0, \text{ } l_\lambda \eta^\lambda = -1
\]

(2.15)

Therefore, the non-vanishing components of the total energy-momentum tensor will be as follows

\[
T_{00} = \sigma + \rho \left( 1 - \frac{m(t,r)}{r} \right), \text{ } T_{01} = -\rho \\
T_{22} = pr^2, \text{ } T_{33} = pr^2 \sin^2 \theta
\]

(2.16)

Here we consider matter in the form of perfect barotropic fluid given by the equation of state

\[
p = \omega \rho
\]

(2.17)

where \( \omega \) is the barotropic parameter.

The non-vanishing components of the Ricci tensors are given by,

\[
R_{00} = \frac{(m - r) m'' + 2\dot{m}}{2r^2}, \text{ } R_{01} = R_{10} = \frac{m''}{2r} \\
R_{22} = m', \text{ } R_{33} = m' \sin^2 \theta
\]

(2.18)

where \( . \) and \( ' \) represents the derivatives with respect to time coordinate \( 't' \) and radial coordinate \( 'r' \) respectively. For this system the Ricci scalar becomes,

\[
R = \frac{2m' + rm''}{r^2}
\]

(2.19)

The trace of the energy momentum tensor is calculated as,

\[
T = g'^{\mu \nu} T_{\mu \nu} = 2 (\omega - 1) \rho
\]

(2.20)

The relation between density and mass is considered as [56],

\[
\rho = n \times m(t,r)
\]

(2.21)

where \( n > 0 \) is the particle number density.

2.1 Field equations

Now we consider some particular classes of \( f(R, T) \) modified gravity models, which are obtained by some explicit functional forms of \( f(R, T) \). Since the field equations depend on the nature of matter through the tensor \( \Theta_{\mu \nu} \), here we will consider the field equations for a perfect fluid source, which will be our field of interest in this study, as discussed in the previous section. On a broad sense we are going to discuss two types of models.
2.1.1 Model-1: \( f(R, T) = f_1(R) + f_2(T) \)

Here we consider models of the form \( f(R, T) = f_1(R) + f_2(T) \), where \( f_1(R) \) and \( f_2(T) \) are arbitrary functions of \( R \) and \( T \) respectively. It is straightforward to see that for \( f_1(R) = R \) and \( f_2(T) = 0 \), we can retrieve GR from this model. Using Eq.(2.6), the gravitational field equations for this model is given by,

\[
f_1'(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_1'(R) = \kappa T_{\mu\nu} + f_2'(T)T_{\mu\nu} + \left[ f_2'(T)p + \frac{1}{2} f_2(T) \right] g_{\mu\nu}
\]

(2.22)

where ' represents derivative with respect to the argument. Now using the Vaidya metric given in Eq.(2.10) and using Eqs.(2.16), (2.17), (2.18) and (2.19) in Eq.(2.25), we compute all the components of the Einstein’s field equations for this model (taking \( \kappa = 1 \)). Here we report the (01), (22) and (33) components of the field equations which will be used in our analysis. The rest of the components are reported in the appendix section of the paper.

1. **The (01)-component of field equations is given by,**

\[
-r \{ f_1(R) + f_2(T) - 2\rho (1 + f_2'(T) - \omega f_2'(T)) \} + f_1'(R)m'' = 0
\]

(2.23)

2. **The (22) and (33) components of field equations are given by,**

\[
r^2 [ f_1(R) + f_2(T) + 2\omega \rho + 4f_2'(T)\omega \rho ] - 2f_1'(R)m' = 0
\]

(2.24)

The above equations along with the ones reported in the appendix are the Einstein’s field equations for \( f(R, T) \) gravity in the time dependent Vaidya spacetime for the first model.

2.1.2 Model-2: \( f(R, T) = f_1(R) + f_2(R)f_3(T) \)

Now we consider a second model given by \( f(R, T) = f_1(R) + f_2(R)f_3(T) \), where \( f_i(R) \), \( i = 1, 2 \) are arbitrary functions of \( R \) and \( f_3(T) \) is an arbitrary function of \( T \). Here the scalar invariants \( R \) and \( T \) are non-minimally coupled to each other via the second term. In order to realize GR from this model, we should have \( f_1(R) = R \) and either or both of \( f_2(R) \) and \( f_3(T) \) equal to zero. We may also take \( f_1(R) = 0, f_2(R) = R \) and \( f_3(T) = 1 \) to get GR from this model. Using Eq.(2.6), the gravitational field equations for this model is given by,

\[
R_{\mu\nu} [ f_1'(R) + f_2'(R)f_3(T) ] - \frac{1}{2} f_1(R)g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) [ f_1'(R) + f_2'(R)f_3(T) ] = \kappa T_{\mu\nu} + f_2(R)f_3'(T)T_{\mu\nu} + f_2(R) \left[ f_3'(T)p + \frac{1}{2} f_3(T) \right] g_{\mu\nu}
\]

(2.25)

Like the previous model here also we report the necessary components of the field equations for the second model (taking \( \kappa = 1 \)):

1. **The (01)-component of field equations is given by,**

\[
-r [ f_1(R) - 2\rho + f_2(R) \{ f_3(T) + 2f_3'(T)\rho (\omega - 1) \} ] + (f_1'(R) + f_2'(R)f_3(T))m'' = 0
\]

(2.26)

2. **The (22) and (33) components of field equations are given by,**

\[
r^2 [ f_1(R) + f_2(R)f_3(T) + 2\omega \rho + 4f_2(R)f_3'(T)\omega \rho ] - 2(f_1'(R) + f_2'(R)f_3(T))m' = 0
\]

(2.27)

The rest of the components are reported in the appendix section. The above equations along with the ones reported in the appendix are the Einstein’s field equations for \( f(R, T) \) gravity in the time dependent Vaidya spacetime for the second model.
2.2 Solution of the system

In this section we will proceed to find solutions of the above systems. In order to do that, we will have to consider special forms for the arbitrary functions of $R$ and $T$ as examples. We will do this for both the models studied in the previous section.

2.2.1 Model-1

Here we have consider four different toy models as examples in order to solve the field equations. The model involves arbitrary functions of $R$ and $T$ coupled in a minimal way. The functional forms used in the toy models are basically power law and exponential forms, which are observationally the most favoured models with constraints imposed on their free parameters.

Case-1: $f_1(R) = g_1 R^{\beta_1}$, $f_2(T) = g_2 T^{\beta_2}$, where $g_1, \beta_1, g_2, \beta_2$ are constants.

Here we have chosen power law forms for both $f_1(R)$ and $f_2(T)$. For our convenience we call this the double-power (DP) model. For $g_2 = 0$ and $g_1 = \beta_1 = 1$, we get back GR from this model.

For this case, the 22 or 33 component of the field equations give the following differential equation,

\[
\frac{1}{r^2} \left[ g_1 \left( 2 (\beta_1 - 1) m' - r m'' \right) \left\{ \frac{2 m' + r m''}{r^2} \right\}^{\beta_1 - 1} \right] + \frac{1}{1 - \omega} \left[ 2 \beta_2 g_2 (2 \beta_2 \omega + 1) \{ n (\omega - 1) m \}^{\beta_2} \right] - 2 n \omega m = 0 \quad (2.28)
\]

Solving the above differential equation for $\beta_1 = \beta_2 = 1$ we get,

\[
m(t, r) = h_1(t) \text{AiryAi} \left[ 2^{1/3} r \left\{ \frac{n (g_2 - \omega - 3g_2 \omega)}{g_1} \right\}^{1/3} \right] + h_2(t) \text{AiryBi} \left[ 2^{1/3} r \left\{ \frac{n (g_2 - \omega - 3g_2 \omega)}{g_1} \right\}^{1/3} \right] \quad (2.29)
\]

where AiryAi and AiryBi are the two Airy functions (see appendix) and $h_1(t), h_2(t)$ are arbitrary functions of time which arises from integration. We would like to mention here that the imposed conditions $\beta_1 = \beta_2 = 1$ are necessary to get a solution of this system by the known mathematical methods.

Now the 01 component of the field equations gives the differential equation,

\[
2 n r^2 m - 2 \beta_2 g_2 r^2 (1 + \beta_2) \{ n (\omega - 1) m \}^{\beta_2} + g_1 \left( \frac{2 m' + r m''}{r^2} \right)^{\beta_1 - 1} \{ r (\beta_1 - 1) m'' - 2 m' \} = 0 \quad (2.30)
\]

Solving this equation for $\beta_1 = \beta_2 = 1$ we get,

\[
m(t, r) = h_3(t) e^{\frac{2n r^2 (1 - 2g_2 (\omega - 1))}{g_1}} \quad (2.31)
\]

where $h_3(t)$ is an arbitrary function of time. For some values of the arbitrary functions it is expected that the solutions given by Eqs.(2.29) and (2.31) will match. In fact it has been checked that both these solutions give rise to similar scenarios in the gravitational collapse scheme which we are going to introduce in the next section. So we are going adopt one of these solutions depending upon the nature of genericity of the solution. Since the solution given by Eq.(2.29) has two arbitrary functions we will use this for our collapse study simply because it is more general in nature and can easily generate the other solution for some well chosen initial conditions. From here on, we will only concentrate on the differential equation and the solution obtained from the 22 or 33 component of the field equations for the reason discussed above. Now that we have obtained the mass parameter,
using it in the Vaidya metric given in Eq. (2.10), we can easily get the Vaidya spacetime for the corresponding model in \(f(R, T)\) gravity.

**Case-2:** \(f_1(R) = g_1 e^{\beta_1 R}, \quad f_2(T) = g_2 e^{\beta_2 T},\) where \(g_1, \beta_1, g_2, \beta_2\) are constants.

Here we have chosen exponential forms for both \(f_1(R)\) and \(f_2(T)\). For our convenience we call this the double-exponential (DE) model. Realizing GR from this model is difficult. Nevertheless an approximation will help us realize the scenario. Expanding \(e^{\beta_1 R}\) in Taylor’s series and keeping the linear terms in \(R\) only, will help us realize GR for \(g_2 = 0, g_1 = 1\) and \(\beta_1 = \frac{\beta_1 - 1}{\pi^2}.\) For this case, the 22 or 33 component of the field equations gives us the differential equation,

\[
2n^2\omega \left(1 + 2g_2\beta_2 e^{2n\beta_2(\omega-1)m}\right) m + \frac{g_2}{g_1} \left(1 - 2\beta_1 m^2 + g_2 r^2 e^{2n\beta_2(\omega-1)m}\right) = 0 \quad (2.32)
\]

This equation has got the unknown function \(m\) and its derivatives in exponential form. It is not possible to find a general solution of this equation. So we search for approximate solutions. We expand the exponentials in the first and the third term in Taylor series and take the linear terms only to get the following solution for \(\beta_1 = 0,\)

\[
m(t, r) = \frac{g_2 n \beta_2 (1 - 3\omega) - n\omega + \sqrt{n^2 \left(\omega^2 + g_2^2 \beta_2^2 (1 + \omega)^2 + 2g_2 \beta_2 \omega \left(3\omega - 4g_1 \beta_2 (\omega - 1) - 1\right)\right)}}{8g_2 n^2 \beta_2^2 \omega (\omega - 1)} \quad (2.33)
\]

We see that this a constant solution for the mass parameter. Using this in Eq. (2.10) we will get the Vaidya spacetime in \(f(R, T)\) gravity for this case.

**Case-3:** \(f_1(R) = g_1 R^{\beta_1}, \quad f_2(T) = g_2 e^{\beta_2 T},\) where \(g_1, \beta_1, g_2, \beta_2\) are constants.

Here we have chosen power law for \(f_1(R)\) and exponential form for \(f_2(T).\) For our convenience we call this the power-exponential (PE) model. For \(g_2 = 0\) and \(g_1 = \beta_1 = 1,\) we get back GR from this model. The 22 or 33 component of the field equations gives us the differential equation,

\[
2nr^2 \left(1 + 2g_2\beta_2 e^{2n\beta_2(\omega-1)m}\right) \omega m + e^{2n\beta_2(\omega-1)m} g_2^{2} - g_1 \left(2(\beta_1 - 1)m' - rm''\right) \left(\frac{2m' + rm''}{r^2}\right)^{\beta_1 - 1} = 0 \quad (2.34)
\]

For \(\beta_1 = 1\) and \(\beta_2 = 0\) we get the following solution for the above differential equation,

\[
m(t, r) = \frac{1}{2n\omega} \left[g_2 \pi AiryAi\left[2^{1/3}r \left(-\frac{n\omega}{g_1}\right)^{1/3}\right]\right. - \frac{2^{1/3}nr\omega}{g_1 \left(-\frac{n\omega}{g_1}\right)^{2/3}} - g_2 \pi AiryAi\left[\frac{2^{1/3}nr\omega}{g_1 \left(-\frac{n\omega}{g_1}\right)^{2/3}}\right] \times \left.ight.

\[
AiryBi'\left[2^{1/3}r \left(-\frac{n\omega}{g_1}\right)^{1/3}\right] + h_4(t) AiryAi \left[-\frac{2^{1/3}nr\omega}{g_1 \left(-\frac{n\omega}{g_1}\right)^{2/3}}\right] + h_5(t) AiryBi \left[-\frac{2^{1/3}nr\omega}{g_1 \left(-\frac{n\omega}{g_1}\right)^{2/3}}\right] \quad (2.35)
\]

where \(AiryAi', \ AiryBi'\) are derivatives of the Airy functions with respect to the argument and \(h_4(t), h_5(t)\) are arbitrary functions of time.
Case-4: \( f_1(R) = g_1 e^{\beta_1 R}, \quad f_2(T) = g_2 T^{\beta_2}, \) where \( g_1, \beta_1, g_2, \beta_2 \) are constants.

Here we have chosen exponential form for both \( f_1(R) \) and power law for \( f_2(T) \). For our convenience we call this the exponential-power (EP) model. A similar scenario as discussed in case-2, will help us realize GR from this model. The 22 or 33 component of the field equations gives us the differential equation,

\[
r^2 \left[ -2n\omega m - \frac{2^{\beta_2} g_2 (\omega + 2\beta_2 \omega - 1) \{ n (\omega - 1) m \}^{\beta_2}}{\omega - 1} \right] - g_1 \left( r^2 - 2\beta_1 m' \right) e^{\frac{\beta_1 m (2m' + rm'')}{r^2}} = 0 \quad (2.36)
\]

For \( \beta_1 = 0 \) and \( \beta_2 = 1 \) we get the following solution of the above equation,

\[
m(t, r) = -\frac{g_1}{2n(\omega + 3g_2\omega - g_2)} \quad (2.37)
\]

2.2.2 Model-2

Now again we consider some special models as sub-cases in order to solve the field equations. The basic difference between this model with the previous one is that here the functions of \( R \) and \( T \) will be minimally coupled to each other which is observationally the more favoured model.

Case-1: \( f_1(R) = g_1 R^{\beta_1}, \quad f_2(R) = g_2 R^{\beta_2}, \quad f_3(T) = g_3 T^{\beta_3}, \) \( (g_1, \beta_1, g_2, \beta_2, g_3, \beta_3 \) are constants)

Here we have considered power law forms for all the three functions. We call this the triple-power (TP) model. For \( g_1 = \beta_1 = 1 \) and \( g_2 = 0 \) or \( g_3 = 0 \), we can realize GR from this model. We may also realize GR for \( g_1 = \beta_3 = 0 \) and \( g_2 = g_3 = \beta_2 = 1 \). The 22 or 33 component of the field equations gives us the differential equation,

\[
2n\omega m (2m' + rm'') + g_1 \left( \frac{2m' + rm''}{r^2} \right)^{\beta_1} \{ 2(1 - \beta_1) m' + rm'' \}
+ 2^{\beta_3} g_2 g_3 \left\{ n (\omega - 1) m \right\}^{\beta_3} \left( \frac{2m' + rm''}{r^2} \right)^{\beta_2} \{ 2(\beta_2 + \omega - 2\beta_2 \omega + 2\beta_3 \omega - 1) m' + r (\omega + 2\beta_3 \omega - 1) m'' \}
+ \frac{2^{\beta_3} g_2 g_3 \left\{ n (\omega - 1) m \right\}^{\beta_3}}{\omega - 1} = 0 \quad (2.38)
\]

A solution for the above differential equation can be obtained for \( \beta_1 = 1, \beta_3 = 2 \) and \( \omega = 1 \) which is given below,

\[
m(t, r) = h_6(t) \text{AiryAi} \left[ -\frac{2^{1/3} n r}{g_1 (-n/g_1)^{2/3}} \right] + h_7(t) \text{AiryBi} \left[ -\frac{2^{1/3} n r}{g_1 (-n/g_1)^{2/3}} \right] \quad (2.39)
\]

where \( h_6(t) \) and \( h_7(t) \) are arbitrary functions of time. This solution corresponds to early universe \( (\omega = 1) \) representing stiff perfect fluid.

Case-2: \( f_1(R) = g_1 e^{\beta_1 R}, \quad f_2(R) = g_2 e^{\beta_2 R}, \quad f_3(T) = g_3 T^{\beta_3}, \) \( (g_1, \beta_1, g_2, \beta_2, g_3, \beta_3 \) are constants)

This is the double-exponential-power (DEP) model. For this model the 22 or 33 component of the field equations yields the following differential equation,

\[
2nr^2 (\omega - 1) \omega m + g_1 (\omega - 1) \left( r^2 e^{\frac{\beta_1 (2m' + rm'')}{r^2}} - 2\beta_1 R m' \right) + 2^{\beta_3} g_2 g_3 \left\{ n (\omega - 1) m \right\}^{\beta_3} \times
\]
\[
\left\{ \frac{r^2 (\omega + 3\beta \omega - 1) e^{\frac{\beta_2 (2m' + rm'')}{r^2}} - 2\beta_2 (\omega - 1) e^{\beta_2 R m'}}{2n} \right\} = 0
\]

For \( \beta_1 = 0, \beta_3 = 2 \) and \( \omega = 1 \) we get the following constant solution of the above equation,

\[
m(t, r) = \frac{g_1}{2n} \quad \text{(2.41)}
\]

We see that this is a constant solution. Moreover this solution is valid in the early universe for a stiff perfect fluid (\( \omega = 1 \)).

**Case-3:** \( f_1(R) = g_1 R^{\beta_1}, \) \( f_2(R) = g_2 R^{\beta_2}, \) \( f_3(T) = g_3 e^{\beta_3 T} \) (\( g_1, \beta_1, g_2, \beta_2, g_3, \beta_3 \) are constants)

From the choice of the functions we can see that this is a double-power-exponential (DPE) model. For \( g_2 = 0 \) or \( g_3 = 0 \) and \( g_1 = \beta_1 = 1 \) we get back GR from this model. In this case the (22) or (33) components of the field equations gives the differential equation,

\[
r^2 \left[ 2n' e^{\beta_1} + g_1 \left( \frac{2m' + rm''}{r^2} \right)^{\beta_1} + g_2 g_3 (1 + 4n' \beta_3 \omega m) e^{2n \beta_3 (\omega - 1)m} \left( \frac{2m' + rm''}{r^2} \right)^{\beta_1} \right] = 0
\]

Case-4: \( f_1(R) = g_1 e^{\beta_1 R}, \) \( f_2(R) = g_2 e^{\beta_2 R}, \) \( f_3(T) = g_3 e^{\beta_3 T} \) (\( g_1, \beta_1, g_2, \beta_2, g_3, \beta_3 \) are constants)

This is the triple-exponential (TE) model formed by three exponential functions. For this model the (22) or (33) components of the field equations give the differential equation,

\[
g_1 r^2 e^{\beta_1 (2m' + rm'')} + 2n' \omega m - 2g_1 \beta_1 e^{\beta_1 (2m' + rm'')} m' + g_2 g_3 \left( r^2 + 4n' \omega m^3 - 2\beta_2 m' \right) e^{2n \beta_3 (\omega - 1)m} \left( \frac{2m' + rm''}{r^2} \right)^{\beta_1} = 0
\]

A solution to the above equation is obtained for \( \beta_1 = \beta_2 = 1 \) and \( \omega = 0 \) which is given below,

\[
m(t, r) = \frac{r^3}{6} + h_{10}(t) \quad \text{OR} \quad m(t, r) = \frac{\log \left( -\frac{2g_{23}}{g_1} \right)}{2n \beta_3}
\]

where \( h_{10}(t) \) is an arbitrary function of time. Just like the previous model, here also we will use the first expression for the mass parameter, for reasons similar to the ones discussed in the previous model. It should be noted that this solution corresponds to dust (\( \omega = 0 \)) as far as the matter content of the universe is concerned and cosmologically this corresponds to early universe.
3 Gravitational Collapse

In this section, we will devise a mechanism in order to study the gravitational collapse of a massive star in this system. As mentioned before, we will consider that the parent star is a massive one so that collapse smoothly continues until a singularity (BH or NS) is formed. Our idea is to develop a set-up, via which the nature of the singularity (BH or NS) can be comprehensively identified. At least our aim is to derive a condition that will govern the nature of singularity (BH or NS) resulting out of the gravitational collapse.

Let us consider a spherical collapsing system, where the physical radius of the $r$-th shell of the star at time $t$ is $R(t,r)$. A suitable initial condition would be that in the epoch $t = 0$, we have $R(0,r) = r$. It is obvious that if the collapse is inhomogeneous, then different collapsing shells may become singular at different times. We are concerned with the light photons emerging from the singularity and travelling along the geodesics and reaching an external observer. An event horizon will be an obstruction for these photons and will resist them from reaching the observer. So here we will probe the existence of such outgoing non-spacelike geodesics. Theoretically if such geodesics possess well defined tangent at the singularity, the quantity $dR/dr$ will definitely tend towards a finite limit with the geodesics approaching the singularity in the past following the trajectories. When these trajectories reach the points $(t_0, r) = (t_0, 0)$, there is a complete breakdown of mathematical and physical concepts and a singularity occurs at $R(t_0, 0) = 0$. At these points ideally the collapsing matter shells are crushed to zero radius, which results in the formation of the central singularity. This is a highly compact object, since a huge amount of mass is packed inside an almost negligible volume. Now if we follow back the path of the outgoing non-spacelike geodesics that are emerging from the central singularity, it is highly probable that they will terminate in the past at the singularity $(r = 0, t = t_0)$ where $R(t_0, 0) = 0$. Therefore from our set-up, mathematically we should have $R \rightarrow 0$ as $r \rightarrow 0$.

We obtain the equation for outgoing radial null geodesics from the Vaidya metric (2.10) by putting $ds^2 = 0$ and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 = 0$ as furnished below

$$\frac{dt}{dr} = \frac{2}{1 - \frac{m(t,r)}{r}}.$$  \hspace{2cm} (3.1)

The above differential equation has a singularity at $r = 0$, $t = 0$. Mathematically this means that any solution to the above equation is not analytic at the point $r = 0$, $t = 0$. Since there is a mathematical breakdown at the singularity we are forced to study the limiting behaviour as one approaches the singularity. To facilitate this, we consider a parameter $X = t/r$. The idea is to study the limiting behaviour of the function $X$ as we approach the singularity at $r = 0$, $t = 0$ following the radial null geodesic. If we denote the limiting value of $X$ by $X_0$ then using L’Hospital’s rule we have

$$X_0 = \lim_{t \to 0} X = \lim_{r \to 0} \frac{t}{r} = \lim_{t \to 0} \frac{dt}{dr} = \lim_{t \to 0} \frac{2}{1 - \frac{m(t,r)}{r}} \hspace{2cm} (3.2)$$

This will actually generate an algebraic equation in terms of $X_0$. The roots of this equation will be our prime concern because they actually represent the slopes (direction) of the tangents to the geodesics. Here we are only interested in the real roots because we are dealing with a realistic collapsing scenario with no connection to the complex domain. For our set-up, any positive real root of this algebraic equation will give the direction of the tangent to an outgoing null geodesic at the singularity. Therefore the existence of positive real roots of this equation corresponds to a necessary and sufficient condition for the singularity to be naked in nature. Now as we have
discussed earlier, if a single null geodesic in the \((t,r)\) plane escapes the singularity, it would mean that a single wavefront emitted from the singularity reaches the external observer. In such a scenario the singularity would be visible only instantaneously to a distant observer and become a \textit{locally naked singularity}. Physically this will correspond to a situation where the event horizon was eliminated from the picture, but only temporarily. But this might not be enough for a complete exchange of information between the singularity and the observer. So for a formidable exchange of information, the singularity is to be seen for a finite period of time. This requires a family of null geodesics escaping from the singularity thus making it \textit{globally naked}. In our mathematical set-up this can be investigated very easily from the number of real positive roots obtained from the above algebraic equation. The above explained comprehensive mathematical set-up for identifying the nature of singularity formed as the end state of a gravitational collapse was first used by Joshi, Singh and Dwivedi in several of their papers \[37-39, 57\]. With the mathematical tools ready, we proceed to study the models one by one.

3.1 Model-1

3.1.1 Case-1

Using Eq.(2.29) in Eq.(3.2) we get,

\[
\frac{2}{X_0} = \lim_{t \to 0, r \to 0} \left[ 1 - \frac{h_1(t)}{r} \text{AiryAi} \left( 2^{1/3} \frac{n \left( g_2 - \omega - 3g_2\omega \right)}{g_1} \right)^{1/3} \right] - \frac{h_2(t)}{r} \text{AiryBi} \left( 2^{1/3} \frac{n \left( g_2 - \omega - 3g_2\omega \right)}{g_1} \right)^{1/3} \] \tag{3.3}

Here we will consider self-similar collapsing scenario. So we consider the following self-similar expressions for the arbitrary functions \(h_i(t), \ i = 1, 2\)

\[h_1(t) = \xi_1 t, \quad h_2(t) = \xi_2 t \quad \text{where } \xi_1 \text{ and } \xi_2 \text{ are arbitrary constants.}\]

Using the above chosen functions in Eq.(3.3) we get the following algebraic equation in \(X_0\),

\[
\frac{1}{\Gamma(2/3)} \left( \frac{\xi_1}{3^{2/3}} + \frac{\xi_2}{3^{1/3}} \right) X_0^2 - X_0 + 2 = 0 \tag{3.4}
\]

To evaluate the above limit, we have used the values of the Airy functions given by Eq.(6.8) in the appendix. Solving the above equation we get two values of \(X_0\) which are,

\[
X_{0,1,2}^{\text{case1}} = \frac{\sqrt{3} \Gamma(2/3) \left( \xi_1 \left( \frac{1}{3^{2/3}} \right) + \xi_2 \left( \frac{1}{3^{1/3}} \right) \right)}{2 \Gamma(2/3) \left( \frac{1}{3^{2/3}} + \frac{1}{3^{1/3}} \right)} \left[ \sqrt{3} \Gamma(2/3) \pm \sqrt{3} \Gamma(2/3) - 8 \times 3^{1/3} \xi_1 - 8 \times 3^{5/6} \xi_2 \right] \tag{3.5}
\]

Here we have considered and henceforth we will consider positive sign for root1 and negative sign for root2. Since we are dealing with a realistic situation we should have \(3^{1/3} \xi_1 + 3^{5/6} \xi_2 \leq \frac{3 \Gamma(2/3)}{8}\).

Now in order to get a NS we should have \(X_0 > 0\). We list below the respective conditions in detail.

**Conditions for a local NS:** \(X_{0,1}^{\text{case1}} > 0 \quad \text{OR} \quad X_{0,2}^{\text{case1}} < 0\)

**Conditions for global NS:** \(X_{0,1}^{\text{case1}} > 0 \quad \text{AND} \quad X_{0,2}^{\text{case1}} > 0\)
Figs. 1 and 2 show the variation of the collapse parameter $X_0$ for different values of $\xi_1$ and $\xi_2$ for Case-1 of Model-1. Fig. 1 shows the variation for the first root $X_{01}^{\text{case1}}$, whereas Fig. 2 shows the variation for the second root $X_{02}^{\text{case1}}$.

Condition for BH: $X_{01}^{\text{case1}} < 0 \& X_{02}^{\text{case1}} < 0$

We see that the above conditions put constraints on $\xi_1$ and $\xi_2$. So by clubbing this theory with observations of collapsing massive stars, we can get bounds on the model parameters. The roots $X_{01,2}^{\text{case1}}$ have been plotted against the parameters $\xi_1$ and $\xi_2$ in Figs. (1) and (2).

3.1.2 Case-2

Using Eq. (2.33) in Eq. (3.2) we get,

$$\frac{2}{X_0} = \lim_{t \to 0, r \to 0} \left[ 1 - \frac{g_2 n \beta_2 (1 - 3\omega) - n \omega + \sqrt{n^2 \left( \omega^2 + g_2^2 \beta_2^2 (1 + \omega)^2 + 2g_2 \beta_2 \omega \left( 3\omega - 4g_1 \beta_2 (\omega - 1) - 1 \right) \right)}{8g_2 n \beta_2 \omega (\omega - 1) r} \right].$$

(3.6)

Evaluating the above limit we get $\frac{2}{X_0} \to \infty$, which implies $X_0 \to 0$. Since the mass function in this case is not a function of $t$ and $r$, we do not get a realistic collapsing scenario for this particular model according to our scheme of study.

3.1.3 Case-3

Using Eq. (2.35) in Eq. (3.2) we get,

$$\frac{2}{X_0} = \lim_{t \to 0, r \to 0} \left[ 1 - \frac{1}{2n\omega r} \left\{ g_2 \pi \text{AiryAi}' \left[ 2^{1/3} r \left( \frac{-n \omega}{g_1} \right) \right]^{1/3} \times \text{AiryBi} \left[ -\frac{2^{1/3} r \omega}{g_1 \left( -\frac{n \omega}{g_1} \right)^{2/3}} \right] \right\} - \frac{h_1(t)}{r} \text{AiryAi} \left[ -\frac{2^{1/3} r \omega}{g_1 \left( -\frac{n \omega}{g_1} \right)^{2/3}} \right] \right].$$
We consider the following functions: \( h_4(t) = \xi_4 t, \quad h_5(t) = \xi_5 t, \) where \( \xi_4 \) and \( \xi_5 \) are arbitrary constants. Using the above chosen functions in Eq. (3.7) we get the following algebraic equation in \( X_0, \)

\[
\frac{1}{\Gamma(2/3)} \left( \frac{\xi_4}{3^{2/3}} + \frac{\xi_5}{3^{1/6}} \right) X_0^2 + (\xi_3 - 1) X_0 + 2 = 0 \tag{3.8}
\]

where \( \xi_3 \) is a constant that arises as a limiting value of the second term of the expression in Eq. (3.6). Solving the above quadratic we get,

\[
X_{01,2}^{\text{case} 3} = \frac{\Gamma(2/3)}{2} \left[ 1 - \xi_3 \pm \sqrt{(\xi_3 - 1)^2 - \frac{8}{\Gamma(2/3)} \left( \frac{\xi_4}{3^{2/3}} + \frac{\xi_5}{3^{1/6}} \right)} \right] \tag{3.9}
\]

where we should have \( (\xi_3 - 1)^2 \geq \frac{8}{\Gamma(2/3)} \left( \frac{\xi_4}{3^{2/3}} + \frac{\xi_5}{3^{1/6}} \right) \). The conditions for NS or BH will be similar as discussed in Case-1.

Conditions for a local NS: \( X_{01}^{\text{case} 3} > 0 \) \& \( X_{02}^{\text{case} 3} < 0 \) OR \( X_{01}^{\text{case} 3} < 0 \) \& \( X_{02}^{\text{case} 3} > 0 \)

Conditions for global NS: \( X_{01}^{\text{case} 3} > 0 \) \& \( X_{02}^{\text{case} 3} > 0 \)

Condition for BH: \( X_{01}^{\text{case} 3} < 0 \) \& \( X_{02}^{\text{case} 3} < 0 \)

We see that the above conditions put numerical bounds on \( \xi_3, \xi_4 \) and \( \xi_5 \) from the perspective of a collapsing scenario. The roots \( X_{01,2}^{\text{case} 3} \) have been plotted against the parameters \( \xi_4 \) and \( \xi_5 \) in Figs. (3) and (4).

3.1.4 Case-4

Using Eq. (2.37) in Eq. (3.2) we get,

\[
\frac{2}{X_0} = \lim_{t \rightarrow 0} \left[ 1 + \frac{g_1}{2nr(\omega + 3g_2 \omega - g_2)} \right] \tag{3.10}
\]

Just like Case-2, here also we get \( \frac{2}{X_0} \rightarrow \infty \), which implies \( X_0 \rightarrow 0 \). The mass function being independent of \( t \) and \( r \) does not generate a realistic collapsing scenario for this particular case according to our scheme of study.

3.2 Model-2

3.2.1 Case-1

In this model we will replace \( X_0 \) by \( Y_0 \), just to differentiate the results from those obtained for model-1. Moreover this is just a representational issue. The definition remains same as given in
Figs. 3 and 4 show the variation of the collapse parameter $X_0$ for different values of $\xi_4$ and $\xi_5$ for Case-3 of Model-1. Fig.3 shows the variation for the first root $X_{01}^{\text{case3}}$, whereas Fig.4 shows the variation for the second root $X_{02}^{\text{case3}}$. Here we have taken $\xi_3 = 0.5$.

Eq. (3.2). Using Eq. (2.39) in Eq. (3.2) we get,

$$2 \frac{Y}{Y_0} = t \to 0 \left[ 1 - \frac{h_6(t)}{r} \text{AiryAi} \left[ -2^{1/3}nr \frac{g_1}{g_1} (-n/g_1)^{2/3} \right] - \frac{h_7(t)}{r} \text{AiryBi} \left[ -2^{1/3}nr \frac{g_1}{g_1} (-n/g_1)^{2/3} \right] \right]$$

(3.11)

Here we consider: $h_6(t) = \xi_6 t$, $h_7(t) = \xi_7 t$, where $\xi_6$ and $\xi_7$ arbitrary constants. Using these functional forms in Eq. (3.11) we get the following algebraic equation for this case,

$$\frac{1}{\Gamma(2/3)} \left( \frac{\xi_6}{3^{2/3}} + \frac{\xi_7}{3^{1/6}} \right) Y_0^2 - Y_0 + 2 = 0$$

(3.12)

It should be noted that this equation is similar to the one obtained for Case-1 in model-1. This is due to the fact that, though the mass functions have different forms in the two cases, yet their limiting values coincide with other and hence generate similar collapsing scenario. The solution for the above equation is obtained as,

$$Y_{01,2}^{\text{case1}} = \frac{\sqrt{2} \Gamma(2/3)}{2^{3/2} \left( \frac{\xi_6}{3^{2/3}} + \frac{\xi_7}{3^{1/6}} \right)} \left[ \sqrt{3} \Gamma(2/3) \pm \sqrt{3} \Gamma(2/3) - 8 \times 3^{1/3} \xi_6 - 8 \times 3^{5/6} \xi_7 \right]$$

(3.13)

where $3 \Gamma(2/3) \geq 8 \times 3^{1/3} \xi_6 + 8 \times 3^{5/6} \xi_7$. The collapsing outcomes may be discussed as below,

- Conditions for a local NS: $Y_{01}^{\text{case1}} > 0 \ & \ Y_{02}^{\text{case1}} < 0 \ OR \ Y_{01}^{\text{case1}} < 0 \ & \ Y_{02}^{\text{case1}} > 0$

- Conditions for global NS: $Y_{01}^{\text{case1}} > 0 \ & \ Y_{02}^{\text{case1}} > 0$

- Condition for BH: $Y_{01}^{\text{case1}} < 0 \ & \ Y_{02}^{\text{case1}} < 0$

The above conditions put numerical bounds on $\xi_6$ and $\xi_7$ from the perspective of a collapsing scenario. The roots $Y_{01,2}^{\text{case1}}$ have been plotted against the parameters $\xi_6$ and $\xi_7$ in Figs. (5) and (6).
Figs. 5 and 6 show the variation of the collapse parameter $Y_0$ for different values of $\xi_6$ and $\xi_7$ for Case-1 of Model-2. Fig.5 shows the variation for the first root $Y_{01}^{\text{case1}}$, whereas Fig.6 shows the variation for the second root $Y_{02}^{\text{case1}}$.

### 3.2.2 Case-2

Using Eq. (2.41) in Eq. (3.2) we get,

$$
\lim_{t \to 0} \frac{2}{Y_0} = t \to 0 \left[ 1 + \frac{q_1}{2nr} \right]
$$

Here we get $\frac{2}{Y_0} \to \infty$, which implies $Y_0 \to 0$. The mass function being independent of $t$ and $r$ does not generate a realistic collapsing scenario for this particular case according to our scheme of study. This is equivalent to the scenarios in Case-2 and Case-4 in model-1.

### 3.2.3 Case-3

Using Eq. (2.43) in Eq. (3.2) we get,

$$
\lim_{t \to 0} \frac{2}{Y_0} = t \to 0 \left[ 1 - \frac{h_8(t)}{r} - h_9(t) \right]
$$

Here we consider $h_8(t) = \xi_8 t$ (where $\xi_8$ is an arbitrary constant). We do not need to consider any particular functional form for $h_9(t)$. This is because irrespective of the form of $h_9(t)$, it will always yield a constant value in the limit $t \to 0$. This gives an additional degree of freedom to the collapsing system. The above equation yields,

$$
\xi_8 Y_0^2 + (\xi_9 - 1) Y_0 + 2 = 0
$$

where $\xi_9$ is the limiting value of $h_9(t)$ as $t \to 0$. The above equations yields the solution,

$$
Y_{01,2}^{\text{case3}} = \frac{1 - \xi_9 \pm \sqrt{(\xi_9 - 1)^2 - 8\xi_8}}{2\xi_8}
$$

where $(\xi_9 - 1)^2 \geq 8\xi_8$. The conditions for different collapse outcomes are given below,

Conditions for a local NS: $Y_{01}^{\text{case3}} > 0 \& Y_{02}^{\text{case3}} < 0$ OR $Y_{01}^{\text{case3}} < 0 \& Y_{02}^{\text{case3}} > 0$
Figs. 7 and 8 show the variation of the collapse parameter $Y_0$ for different values of $\xi_8$ and $\xi_9$ for Case-3 of Model-2. Fig.7 shows the variation for the first root $Y^{\text{case 3}}_{0_1}$, whereas Fig.8 shows the variation for the second root $Y^{\text{case 3}}_{0_2}$.

Conditions for global NS: $Y^{\text{case 3}}_{0_1} > 0 \& Y^{\text{case 3}}_{0_2} > 0$

Condition for BH: $Y^{\text{case 3}}_{0_1} < 0 \& Y^{\text{case 3}}_{0_2} < 0$

The above conditions put numerical bounds on $\xi_8$ and $\xi_9$ from the perspective of gravitational collapse of a massive star. The roots $Y^{\text{case 3}}_{0_{1,2}}$ have been plotted against the parameters $\xi_8$ and $\xi_9$ in Figs.(7) and (8).

3.2.4 Case-4

Using Eq.(2.45) in Eq.(3.2) we get,

$$\lim_{t \to 0} \frac{2}{Y_0} = \lim_{r \to 0} \left[ 1 - \frac{r^2}{6} - \frac{h_{10}(t)}{r} \right]$$

(3.18)

Here we consider the functional form $h_{10}(t) = \xi_{10} t$, where $\xi_{10}$ is an arbitrary constant. From the above equation we get the algebraic equation,

$$\xi_{10} Y^2_{0_2} - Y_0 + 2 = 0$$

(3.19)

Solving the above equation we get,

$$Y^{\text{case 4}}_{0_{1,2}} = \frac{1 \pm \sqrt{1 - 8\xi_{10}}}{2\xi_{10}}$$

(3.20)

where $\xi_{10} \leq 1/8$. Here the conditions for NS and BH can be discussed as below,

Conditions for a local NS: $Y^{\text{case 4}}_{0_1} > 0 \& Y^{\text{case 4}}_{0_2} < 0$ OR $Y^{\text{case 4}}_{0_1} < 0 \& Y^{\text{case 4}}_{0_2} > 0$

Conditions for global NS: $Y^{\text{case 4}}_{0_1} > 0 \& Y^{\text{case 4}}_{0_2} > 0$

Condition for BH: $Y^{\text{case 4}}_{0_1} < 0 \& Y^{\text{case 4}}_{0_2} < 0$
Fig. 9 shows the variation of the collapse parameter $Y_0$ for different values of $\xi_{10}$ for Case-4 of Model-2.

The above conditions put numerical bounds on $\xi_{10}$ from the perspective of gravitational collapse of a massive star. The roots $Y_{01,2}^{\text{case4}}$ have been plotted against the parameter $\xi_{10}$ in Fig. (9).

### 3.3 Numerical Analysis

In order to get greater insights about the nature of the singularity formed as an end state of the gravitational collapse for our models, we have generated plots for the collapsing parameter ($X_0$ or $Y_0$) against the other free parameters. According to our scheme of study, we are interested only in the signature of the collapse parameter ($X_0$ or $Y_0$) and not in the actual value of the parameter.

In figs. (1) and (2) we have generated plots for the Case-1 (DP model) of Model-1. We see from the figures that the first root lies in the negative region, whereas the second root lies in the positive region. So it can be predicted that for this case, we will have a local NS. In figs. (3) and (4) plots have been obtained for Case-3 (PE model) of Model-1. We see that for the considered initial conditions the first root is again negative and the second root is positive. So here also the collapse will end in a local NS. Figures (5) and (6) show the corresponding plots for Case-1 (TP model) of Model-2. In fig. (5) although the major portion of the surface lies in the negative region, yet there is an array of points represented by a straight line lying in the positive region around $\xi_7 = 0$. In fig. (6) the entire surface lies in the positive region. So in this case, there is a possibility to get more than a local NS. By properly adjusting the initial conditions, it is quite possible that more than one null geodesic originating in the singularity reach a distant observer. In case of such an event, the NS will become global in nature. In figs. (7) and (8) we have obtained plots for Case-3 (DPE model) of Model-2. Here we see that by properly adjusting the parameters $\xi_8$ and $\xi_9$ both positive and negative values can be realized for both the roots. So in this case we can have BH, local NS and global NS depending on the chosen initial conditions. Finally in fig. (9) we have obtained plots for Case-4 (TE model) of Model-2. Here we see that irrespective of the initial conditions one root is always positive and the other is always negative. So here the singularity is destined to be a local NS.

### 4 Strength of the singularity (Curvature growth near the singularity)

The gravitational strength of a singularity is defined as the estimate of its destructive capacity. We know that most theories of gravity till date have been plagued by the existence of singularities. Though theoretical methods of removal of such singularities have been proposed in literature, yet
they are highly exotic in nature and far from being comprehensible. It is known that singularities are holes in the fabric of the otherwise continuous and smooth spacetime. Now for a weak singularity the hole is shallow and an extension of space-time is possible through the singularity. This is equivalent to a removable discontinuity mathematically and can be a cure for the discontinuity of spacetime at a singularity. From the above discussion it is quite clear that one should be highly interested in finding out whether a singularity is strong or weak in nature. According to Tipler [58] a curvature singularity is said to be strong if any object hitting it is squeezed to zero volume. In Ref.[58] the condition for a strong singularity has been given as,

\[ S = \lim_{\tau \to 0} \tau^2 \psi = \lim_{\tau \to 0} \tau^2 R_{\mu \nu} K^\mu K^\nu > 0 \] (4.1)

where \( R_{\mu \nu} \) is the Ricci tensor, \( \psi \) is a scalar given by the relation \( \psi = R_{\mu \nu} K^\mu K^\nu \), where \( K^\mu = dx^\mu / d\tau \) represents the tangent to the non spacelike geodesics at the singularity and \( \tau \) is the affine parameter.

In Ref.[59] Mkeneyeleye et al. have shown that,

\[ S = \lim_{\tau \to 0} \tau^2 \psi = \frac{1}{4} X_0^2 (2\dot{m}_0) \] (4.2)

where

\[ m_0 = \lim_{t \to 0, r \to 0} m(t,r) \] (4.3)

and

\[ \dot{m}_0 = \lim_{t \to 0, r \to 0} \frac{\partial}{\partial t} (m(t,r)) \] (4.4)

In ref. [59] it has also been shown that the relation between \( X_0 \) and the limiting values of mass is given by,

\[ X_0 = \frac{2}{1 - 2m'_0 - 2m_0 X_0} \] (4.5)

where

\[ m'_0 = \lim_{t \to 0, r \to 0} \frac{\partial}{\partial r} (m(t,r)) \] (4.6)

and \( \dot{m}_0 \) is given by the eqn.(4.4).

Studies by Dwivedi and Joshi in Refs.[38, 60] showed that any classical singularity in Vaidya spacetime in Einstein gravity is supposed to be a strong curvature singularity in a very strong sense. Additionally they have also shown that the conjecture [61] that the strong curvature singularities are never naked is not always true. It is speculated that in the background of \( f(R, T) \) gravity the strength of the singularity may weaken due to the exotic component arising from the modified gravity. Moreover the structure of such a NS was studied in detail in Ref.[62] and it was shown that the singularity admits a directional behaviour in terms of curvature growth along the geodesics terminating in the singularity. On the contrary it was found that in a quantum regime the singularity formed is supposed to be gravitationally weak, thus allowing a continuous extension of the spacetime beyond the singularity [63]. Below we study the strength of the singularities for the different models.
4.1 Model-1

4.1.1 Case-1

Using Eqs. (2.29), (4.2) and (4.4) we have,

\[ S = \lim_{\tau \to 0} \tau^2 \psi = \frac{X_0^2}{2\Gamma(2/3)} \left( \frac{\xi_1}{3^{2/3}} + \frac{\xi_2}{3^{1/3}} \right) \]  

(4.7)

It is obvious that the signature of the above expression is independent of the collapsing parameter $X_0$ since $X_0^2 > 0$. So the strength of the singularity ultimately depends on the values of the parameters $\xi_1$ and $\xi_2$. The condition for a strong singularity is $\frac{1}{\Gamma(2/3)} \left( \frac{\xi_1}{3^{2/3}} + \frac{\xi_2}{3^{1/3}} \right) > 0$ and that for a weak singularity is $\frac{1}{\Gamma(2/3)} \left( \frac{\xi_1}{3^{2/3}} + \frac{\xi_2}{3^{1/3}} \right) \leq 0$.

4.1.2 Case-2

Using Eqs. (2.33), (4.2) and (4.4) we have,

\[ S = \lim_{\tau \to 0} \tau^2 \psi = 0 \]  

(4.8)

The above value shows that the singularity formed is weak in nature. In the previous section it was seen that for this model, the constancy of mass parameter did not assist in studying the nature of the singularity. But whatever be the nature of the singularity formed, it should always be gravitationally weak in nature.

4.1.3 Case-3

Using Eqs. (2.35), (4.2) and (4.4) we have,

\[ S = \lim_{\tau \to 0} \tau^2 \psi = \frac{X_0^2}{2\Gamma(2/3)} \left( \frac{\xi_4}{3^{2/3}} + \frac{\xi_5}{3^{1/6}} \right) \]  

(4.9)

Similar to case-1, here the strength of singularity is independent of $X_0$. The condition for a strong singularity is $\frac{1}{\Gamma(2/3)} \left( \frac{\xi_4}{3^{2/3}} + \frac{\xi_5}{3^{1/6}} \right) > 0$ and that for a weak singularity is $\frac{1}{\Gamma(2/3)} \left( \frac{\xi_4}{3^{2/3}} + \frac{\xi_5}{3^{1/6}} \right) \leq 0$.

4.1.4 Case-4

Using Eqs. (2.37), (4.2) and (4.4) we have,

\[ S = \lim_{\tau \to 0} \tau^2 \psi = 0 \]  

(4.10)

This is a situation similar to case-2 where the singularity is always gravitationally weak.

4.2 Model-2

4.2.1 Case-1

Using Eqs. (2.39), (4.2) and (4.4) we have,

\[ S = \lim_{\tau \to 0} \tau^2 \psi = \frac{Y_0^2}{2\Gamma(2/3)} \left( \frac{\xi_6}{3^{2/3}} + \frac{\xi_7}{3^{1/6}} \right) \]  

(4.11)

Here the signature of the above expression and hence the strength of the singularity depends on the values of the parameters $\xi_6$ and $\xi_7$. We get a strong singularity if $\frac{1}{\Gamma(2/3)} \left( \frac{\xi_6}{3^{2/3}} + \frac{\xi_7}{3^{1/6}} \right) > 0$, and a weak singularity if $\frac{1}{\Gamma(2/3)} \left( \frac{\xi_6}{3^{2/3}} + \frac{\xi_7}{3^{1/6}} \right) \leq 0$. 

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4.2.2 Case-2
Using Eqs. (2.41), (4.2) and (4.4) we have,
\[
S = \lim_{\tau \to 0} \tau^2 \psi = 0 \quad (4.12)
\]
Hence the singularity is gravitationally weak in nature.

4.2.3 Case-3
Using Eqs. (2.43), (4.2) and (4.4) we have,
\[
S = \lim_{\tau \to 0} \tau^2 \psi = \frac{1}{2} Y_0^2 \xi_8 \quad (4.13)
\]
Here we have considered no special form for the function \( h_9(t) \) as was done in the previous section. It is clear from the above expression that the strength of the singularity basically depends on the signature of \( \xi_8 \). If \( \xi_8 > 0 \), then the singularity is strong and if \( \xi_8 \leq 0 \), then the singularity is weak in nature. However if we do consider a special form for the function \( h_9(t) \), we can have a different result. If we consider \( h_9(t) = \gamma_9 \log(t) \), then we have from the Eqs. (2.43), (4.2) and (4.4),
\[
S = \lim_{\tau \to 0} \tau^2 \psi = \frac{1}{2} Y_0^2 \left( \xi_8 + \frac{\gamma_9}{Y_0} \right) \quad (4.14)
\]
Now using Eqs. (2.43), (4.4), (4.5) and (4.6) we get a relation from where the values of \( X_0 \) may be extracted. Using these values of \( X_0 \) in the above equation we may have a different scenario for the strength of the singularity.

4.2.4 Case-4
Using Eqs. (2.45), (4.2) and (4.4) we have,
\[
S = \lim_{\tau \to 0} \tau^2 \psi = \frac{1}{2} Y_0^2 \xi_{10} \quad (4.15)
\]
Here the strength of the singularity depends on the signature of \( \xi_{10} \). If \( \xi_{10} > 0 \), then the singularity is strong, and if \( \xi_{10} \leq 0 \), the singularity is weak.

5 Conclusion and Discussion
In this work, we have explored a gravitational collapse mechanism of a massive star in \( f(R,T) \) gravity. A time dependent Vaidya spacetime is used to model the collapsing phenomenon. The Einstein’s field equations for \( f(R,T) \) gravity in the Vaidya spacetime are calculated and the corresponding solutions for the mass parameter \( m(t,r) \) are obtained. We have considered two different category of \( f(R,T) \) models, each consisting of four sub-models. The two models are considered on the basis of the nature of coupling between the scalar invariants \( R \) and \( T \). The sub-models for each model basically involve various combinations of power and exponential functional forms. Here we considered the collapse of a massive star (\( > 20M_\odot \)), which will invariably continue its collapse until the formation of a singularity. The huge mass of the parent star will always keep the collapsing mass beyond the Chandrasekhar limit (1.3\( M_\odot \)), and hence neither the electron nor the neutron degeneracy pressure will be able to counterbalance the inward collapsing force. Hence the collapse will not terminate in any middle stage like a white dwarf or a neutron star, but will continue all the way to a singularity (BH or NS). The scheme followed for the gravitational collapse study involved the quest for outgoing radial null geodesics from the central singularity formed as an end state of the collapse. If such outgoing geodesics exists then the singularity becomes a
naked singularity and the formation of the event horizon is hindered. Such a situation will definitely defy the cosmic censorship hypothesis. Moreover depending on the number of such escaping geodesics, we can have a locally or globally naked singularity. More number of escaping geodesics will mean greater exposure time of the singularity to an external observer, and hence result in a globally naked singularity. However if no such geodesic escape from the singularity, the collapse is destined to end in a black hole and thus favour the censorship hypothesis. Our study predicts that in almost all the cases of model-1 we get a locally naked singularity. Model-2 seems to be a mixed bag, predicting the formation of black holes, local and global naked singularities depending on the initial conditions. However in the Case-4 of Model-2 (TE model), the collapse always results in a local naked singularity. So here it should be noted that the nature of coupling between the scalar invariants $R$ and $T$ does play a very important role in the nature of singularity formed as an end state of the collapse. For minimal coupling (Model-1), we see that the collapse generally ends in a local naked singularity. But for non-minimal coupling (Model-2), all the options (BH, local and global NS) are possible except the TE model (Case-4). Hence these models resulting from the minimal and non-minimal coupling between curvature and matter, can be considered as significant counterexamples of the cosmic censorship hypothesis. But as we know that non-minimal coupling is observationally the favoured model, the result derived for this model-2 will be cosmologically more relevant. Moreover we see that for minimal coupling we generally do not get the global nature of the naked singularity, but in case of non-minimal coupling this can be a reality.

One thing which may be worrying for the reader is that the final limiting forms ($X_0$ or $Y_0$) in the collapsing scheme does not involve the model parameters $g_i$ or $\beta_i$, $i = 1,2,3$. So how does one differentiate the collapse outcomes between the models? We see that here the solutions are in terms of Airy functions which are relatively complicated mathematical forms. In the limiting scenario the argument of these functions vanish giving constant values, which is reason we do not see any model parameters in the limiting forms. However it should be mentioned here that the functional forms of $X_0$ or $Y_0$ are different for different models, which is testimony of the fact they arise from different functional forms. Moreover the imprints of such functional forms are carried by the functions $h_i(t)$, $i = 1,2,3...10$, and thus the parameters $\xi_i$, $i = 1,2,3...10$ which are present in the analysis. One more thing that the reader needs to note is that some of the solutions derived for the special cases in model-2 are valid for early universe. So in such cases we are actually studying the collapsing scenarios of primordial black holes that existed at the beginning of the universe. However it should also be kept in mind that these solutions are just specific examples to get greater insights into the bigger picture and in no sense represent the entire story.

To complement the collapsing scheme we have studied the strength of the singularity formed for all our models. We see that for the DP (case-1) and PE (case-3) models of model-1, the singularity can be both gravitationally weak or strong depending on the model parameters. However for the DE (case-2) and EP (case-4) models, the singularity formed is always gravitationally weak. A weak singularity will obviously be pathologically favoured because the spacetime can be extended beyond such a singularity and we get a sense of continuity. For model-2, we see that for the TP (case1), DPE (case-3) and TE (case-4) models the strength of the singularity depends on the initial conditions but for the DEP (case-2) model the singularity is always gravitationally weak. So it is understandable that for all models, by suitably adjusting the initial conditions, we can have a sufficiently weak singularity, which will be cosmologically desirable, since an extension of the spacetime beyond the singularity becomes a possibility. In principle we can create a scenario where the singularity may be completely avoided. This is a direct consequence of the coupling of matter with geometry and hence an intrinsic property of $f(R,T)$ models and their exotic nature.
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6 Appendix

Here we report the other components of the field equations for this model which have not been used in our analysis.

Model-1

1. The (00)-component of field equations is given by,

\[ r^4 \{ f_1(R) + f_2(T) - 2(f_2'(T) + 1)(\rho + \sigma) + 2f_2(T)\omega\rho \} - 8f_1''(R)\dot{m}' + r \{ 8f_1''(R)\ddot{m}' \dot{m}'' + r \} \]

2. The (11)-component of field equations is given by,

\[ f_1''(R) \left[-4m' + r \left( m'' + rm^{(3)} \right) \right]^2 + r^2 f_1''(R) \left( 12m' - 6rm'' + r^3 m^{(4)} \right) = 0 \]

where (3) and (4) in the power represents the third and fourth order derivative with respect to \( r \) respectively.

Model-2

1. The (00)-component of field equations is given by,

\[ 2r^4\sigma + 2r^3\rho (r - m) - f_2(R)r^3 (f_3(T) + 2f_3'(T)\omega\rho) (r - m) - f_1(R)r^3 (r - m) + 2f_2(R)f_3(T)r^3 \{ r(\rho + \sigma) - \rho m \}

\]

\[ + (f_1'(R) + f_2'(R)f_3(T)) r^2 \{ m'' (r - m) - 2\dot{m}' \} + 2f_2'(R)f_3'(T)r^4\lambda^2 \dot{\rho}^2 + 4f_2''(R)f_3'(T)r^4\lambda \dot{\rho} (2\dot{m}' + rm'') \]

\[ + 2f_1''(R) (2\dot{m}' + rm'')^2 + 2f_2''(R)f_3(T) (2\dot{m}' + rm'')^2 + 2f_2'(R)f_3'(T)r^4\lambda \ddot{\rho} + 4f_2''(R)r^2\dot{m}' + 4f_2''(R)f_3(T)r^2\dot{m}' 

\]

\[ + 2 (f_1'(R) + f_2'(R)f_3(T)) r^2 \ddot{m}'' = 0 \]

2. The (11)-component of field equations is given by,

\[ r^6 \left( -f_2'(R)f_3''(T)\lambda^2 (\rho')^2 - f_2'(R)f_3'(T)\lambda \dot{\rho}'' \right) - r^3 \left[ 2f_2''(R)f_3'(T)\lambda \dot{\rho} \left( -4m' + r \left( m'' + rm''' \right) \right) \right] 

\]

\[ - f_1''(R) \left[ -4m' + r \left( m'' + rm''' \right) \right]^2 - f_2''(R)f_3(T) \left[ -4m' + r \left( m'' + rm''' \right) \right]^2 - r^2 f_1''(R) \left( 12m' - 6rm'' + r^3 m^{(4)} \right) 

\]

\[ - r^2 f_2''(R)f_3(T) \left( 12m' - 6rm'' + r^3 m^{(4)} \right) = 0 \]
Here we would like to present a short description of Airy functions for the reader’s convenience. Airy function is a special function named after the British astronomer George Biddell Airy (1801-1892). There are in fact, two Airy functions $Ai(x)$ (Airy function of the first kind) and $Bi(x)$ (Airy function of the second kind), which are linearly independent solutions of the Airy differential equation given by,

$$\frac{d^2y}{dx^2} - xy = 0$$  \hspace{1cm} (6.5)

For real values of $x$ the Airy function of the first kind is defined by the improper integral,

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{t^3}{3} + xt \right) dt \equiv \frac{1}{\pi} \lim_{b \to \infty} \int_0^b \cos \left( \frac{t^3}{3} + xt \right) dt$$  \hspace{1cm} (6.6)

which is convergent. This solution is subject to the condition $y \to 0$ as $x \to \infty$. The Airy function of the second kind is defined as,

$$Bi(x) = \frac{1}{\pi} \int_0^\infty \left[ \exp \left( -\frac{t^3}{3} + xt \right) + \sin \left( \frac{t^3}{3} + xt \right) \right] dt$$  \hspace{1cm} (6.7)

This solution has the same amplitude of oscillation as $Ai(x)$ as $x \to -\infty$ differing in phase by $\pi/2$.

The values of Airy function ($Ai(x), Bi(x)$) and its derivatives ($Ai'(x), Bi'(x)$) at $x = 0$ are given by,

$$Ai(0) = \frac{1}{3^{2/3} \Gamma(2/3)}, \quad Bi(0) = \frac{1}{3^{1/6} \Gamma(2/3)}, \quad Ai'(0) = -\frac{1}{3^{1/3} \Gamma(1/3)}, \quad Bi'(0) = \frac{3^{1/6}}{\Gamma(1/3)}$$  \hspace{1cm} (6.8)

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