ON THE INTRINSIC TORSION OF SPACETIME STRUCTURES

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Dedicated to Dmitri Vladimirovich Alekseevsky on his eightieth birthday

Abstract. We briefly review the notion of the intrinsic torsion of a G-structure and then go on to classify the intrinsic torsion of the G-structures associated with spacetimes: namely, galilean (or Newton–Cartan), carrollian, aristotelian and bargmannian. In the case of galilean structures, the intrinsic torsion classification agrees with the well-known classification into torsionless, twistless torsional and torsional Newton–Cartan geometries. In the case of carrollian structures, we find that intrinsic torsion allows us to classify Carroll manifolds into four classes, depending on the action of the Carroll vector field on the spatial metric, or equivalently in terms of the nature of the null hypersurfaces of a lorentzian manifold into which a carrollian geometry may embed. By a small refinement of the results for galilean and carrollian structures, we show that there are sixteen classes of aristotelian structures, which we characterise geometrically. Finally, the bulk of the paper is devoted to the case of bargmannian structures, where we find twenty-seven classes which we also characterise geometrically while simultaneously relating some of them to the galilean and carrollian structures.

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1. Introduction

What are the possible geometries of space and time? An answer to this question was given (subject to some assumptions) by Bacry and Lévy-Leblond [1], who pioneered the classification of kinematical symmetries. Later work of Bacry and Nuysts [2] relaxed some of the assumptions in the earlier work and classified kinematical Lie algebras in four space-time dimensions. Taking these ideas to their logical conclusion, Stefan Prohazka and I classified (simply-connected, spatially isotropic) homogeneous kinematical spacetimes in arbitrary dimension [3]. We found that such spacetimes are of one of several classes: lorentzian, galilean (a.k.a. Newton–Cartan), carrollian and aristotelian.\(^1\) The geometry of such homogeneous kinematical spacetimes was further studied in [4], together with Ross Grassie.

Being homogeneous, these spacetimes serve as Klein models for more realistic spacetime geometries, in the same way that Minkowski spacetime serves as a model for the lorentzian spacetimes of General Relativity. Technically, the realistic spacetimes are Cartan geometries modelled on the kinematical Klein geometries. A closer analysis of the Klein geometries reveals that they fall into far fewer classes than their number might suggest: all galilean homogeneous spacetimes, for example, are Klein models for the same Cartan geometry and the same is true for aristotelian and carrollian (with the exception of the lightcone) spacetimes. Hence it makes sense to study galilean, carrollian, aristotelian structures without reference to a particular homogeneous model.

One way to do this is to re-interpret the relevant structure as a G-structure; that is, as a principal G-subbundle of the frame bundle or, more prosaically, as a consistent way to restrict to moving frames which are related by local G-transformations; such as orthonormal frames in a riemannian manifold. There is a notion of affine connection adapted to a G-structure. Typically these connections have torsion and there exists a component of the torsion which is independent of the adapted connection. This is the intrinsic torsion of the G-structure and it is the subject of the present paper. The intrinsic torsion is the first obstruction to the integrability of the G-structure, which roughly speaking says that there exists a coordinate atlas to the manifold whose transition functions take values in G.

Studying the intrinsic torsion might seem a strange approach coming from the direction of General Relativity because in lorentzian geometry and in the absence of any additional structure, the intrinsic torsion of a metric connection vanishes – that being essentially the Fundamental Theorem of riemannian geometry. That this is not the end of the story can be gleaned from the emergence of natural connections other than the Levi-Civita connection in the context of 1/c expansions of General Relativity [5, 6] and in particular from the emergence of (torsional) Newton–Cartan geometry in that limit. Indeed, we will see that for the non-lorentzian G-structures, the intrinsic torsion will give us some information. For example, we will see that the classification of galilean G-structures [7] by intrinsic torsion coincides with the classification of Newton–Cartan geometries into torsionless (NC), twistless torsional (TTNC) and torsional (TN) geometries [8]. For carrollian, aristotelian and indeed bargmannian structures, their classifications via intrinsic torsion seem to be novel. We will see that there are four classes of carrollian G-structures, sixteen classes of aristotelian G-structures and twenty-seven classes of bargmannian G-structures, which we will classify and characterise geometrically in terms of the tensor fields which characterise the G-structure.

This paper is organised as follows. In Section 2 we review the very basic notions about G-structures and their intrinsic torsion. In Section 2.1 we review the useful language of associated vector bundles, which we use implicitly in much of the paper, discuss adapted connections in Section 2.2 and the intrinsic torsion of a G-structure in Section 2.3. The rest of the paper consists of four worked out examples of increasing complexity of the classification of the intrinsic torsions of a G-structure. For each one of the spacetime G-structures (galilean, carrollian, aristotelian and bargmannian) we first work out the group G and identify the characteristic tensor fields which define and are defined by the G-structure, work out the lattice of G-submodules where the intrinsic torsion lives and hence classify the distinct classes of G-structures, and then we characterise them geometrically in terms of the characteristic tensor fields of the G-structure.

Section 3 is devoted to galilean structures. A galilean structure is defined by a nowhere-vanishing “clock” one-form \(\tau\) spanning the kernel of a positive-semidefinite cometric \(\gamma\). Proposition 5 shows that the intrinsic torsion of a galilean structure is captured by \(d\tau\). Theorem 6 then shows that there are three types of galilean structures, depending on whether or not \(\tau \wedge d\tau\) is zero and, if so, whether or not \(d\tau = 0\).

Section 4 is devoted to carrollian structures, which are defined by a nowhere-vanishing vector field \(\xi\) spanning the kernel of a positive-semidefinite metric \(h\). Proposition 8 shows that the intrinsic torsion is...\(^3\)The classification also gives some riemannian spaces and in two dimensions also some spacetimes without any discernable structure.
captured by $\mathcal{L}_h$ and in Theorem 10 we show that there are four types of carrollian structures: depending on whether or not $\mathcal{L}_h = 0$, and if not, whether the symmetric tensor $\mathcal{L}_h$ is traceless or pure trace or neither. Recently a different approach to the study of carrollian geometry has been proposed [9], exhibiting the carrollian geometry as a principal line bundle (with structure group the one-dimensional group generated by $\xi$) over a riemannian manifold with metric $h$. It would be interesting to relate our two approaches.

Section 5 is devoted to aristotelian structures. An aristotelian geometry admits simultaneously a galilean structure and a carrollian structure and hence we can re-use and refine the results in the previous two sections to arrive at Theorem 12, which lists the sixteen types of aristotelian structures.

Section 6 is the longest and is devoted to bargmannian structures. A bargmannian structure consists of a lorentzian manifold $(M, g)$ and a nowhere-vanishing null vector field $\xi$. As advocated in [10], bargmannian structures serve as a bridge between galilean and carrollian structures and one can recover the results of Sections 3 and 4 as special cases. Proposition 15 shows that the intrinsic torsion of a bargmannian structure is captured by $\nabla^g \xi$, the covariant derivative of the null vector field relative to the Levi-Civita connection of $g$. We find that, perhaps surprisingly, there are twenty-seven types of bargmannian structures, as described in Theorem 21. These structures defined a partially ordered set which is depicted in Figure 2.

In deriving the results on bargmannian structures we found the need to extend the theory of null hypersurfaces (e.g., [11, 12]) to non-involutive null distributions. In Section 6.4 we relate them to galilean and carrollian structures. We will find that all three classes of galilean structures can arise as null reductions of bargmannian manifolds, whereas all four classes of carrollian structures can arise as embedded null hypersurfaces in bargmannian manifolds. This then allows us to rephrase the carrollian classification in terms of the classification of null hypersurfaces in a lorentzian manifold. The role of null hypersurfaces in carrollian geometry was already emphasised in [13].

The paper ends with some conclusions and two appendices. Appendix A, included for completeness, contains a proof of a result concerning hypersurface orthogonality which is used often in Section 6. The result is often quoted, but hardly ever proved. Finally, Appendix B treats some special dimensions. In the bulk of the paper we work with generic $n$-dimensional galilean, carrollian and aristotelian structures and $(n+1)$-dimensional bargmannian structures and the results hold for $n > 2$ and $n \neq 5$. When $n = 2$ there is no distinction between carrollian and galilean structures and hence we will need to look again at the classifications. This is done in Appendix B.1, which also treats the two-dimensional aristotelian structures. We find that there are now two galilean, two carrollian and four aristotelian structures in two dimensions. When $n = 2$ the classification of bargmannian structures also simplifies and this is described in Appendix B.2. There are now only eleven three-dimensional bargmannian structures. When $n = 5$ we find in the galilean, aristotelian and bargmannian cases, $so(4)$-submodules of type $\Lambda^2 \mathbb{R}^4$, which are not irreducible, leading to a refinement of the classifications. This is described briefly in Appendices B.3 for galilean structures, B.4 for aristotelian structures and B.5 for bargmannian structures. We find that there are 5 galilean structures, 32 aristotelian structures and 47 bargmannian structures in these dimensions.

2. The intrinsic torsion of a $G$-structure

In this section we briefly review the language associated to $G$-structures, adapted connections and their intrinsic torsion. It sets the stage for the calculations in the remaining sections. Readers familiar with this language may simply skim for notation and go directly to the calculations starting in the next section. I do not include any proofs, which can be found in, say, [14, 15].

2.1. $G$-structures. Let $M$ be an $n$-dimensional smooth manifold and let $p \in M$. By a frame at $p$ we mean a vector space isomorphism $u : \mathbb{R}^n \to T_p M$. Since $\mathbb{R}^n$ has a distinguished basis (the elementary vectors $e_i$), its image under $u$ is a basis $(u(e_1), u(e_2), \ldots, u(e_n))$ for $T_p M$. If $u, u'$ are two frames at $p$ then $g := u^{-1} \circ u \in GL(n, \mathbb{R})$. Rewriting this as $u' = u \circ g$ defines a right action of $GL(n, \mathbb{R})$ on the set $F_p(M)$ of frames at $p$. This action is transitive and free, making $F_p(M)$ into a torsor (a.k.a. principal homogeneous space) of $GL(n, \mathbb{R})$.

The disjoint union $F(M) = \bigsqcup_{p \in M} F_p(M)$ can be made into the total space of a principal $GL(n, \mathbb{R})$-bundle called the frame bundle of $M$. In particular, we have a smooth free right-action of $GL(n, \mathbb{R})$; that is, a diffeomorphism $R_g : F(M) \to F(M)$ for every $g \in GL(n, \mathbb{R})$, where $R_g u = u \circ g$ for every frame $u \in F(M)$. Let $\pi : F(M) \to M$ be the smooth map sending a frame $u \in F_p(M)$ to $p \in M$. It follows that $\pi \circ R_g = \pi$ for all $g \in GL(n, \mathbb{R})$, since $GL(n, \mathbb{R})$ acts on the frames at $p$. A local section $s : U \to F(M)$, where $U \subset M$, defines a moving frame (or vielbein) $(X_1, \ldots, X_n)$ in $U$, where $(X_i)_p = s(p)(e_i)$ for all $p \in U$. 
Moving frames exist on $M$ by virtue of it being a smooth manifold. Indeed, if $\{U, x^1, \ldots, x^n\}$ is a local coordinate chart, then $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ is a moving frame in $U$. If $\{V, y^1, \ldots, y^n\}$ is an overlapping coordinate chart, then in the overlap $U \cap V$, the moving frames are related by a local $\text{GL}(n, \mathbb{R})$ transformation $g_{UV} : U \cap V \to \text{GL}(n, \mathbb{R})$: namely, the Jacobian matrix of the change of coordinates.

It may happen, though, that we can restrict ourselves to distinguished moving frames which are related on overlaps by local $G$-transformations, for some subgroup $G < \text{GL}(n, \mathbb{R})$. For example, we may endow $M$ with a riemannian metric and restrict ourselves to orthonormal moving frames, which are related on overlaps by local $\text{O}(n)$ transformations. For every $p \in M$ let $P_p \subset P(M)$ denote the set of orthonormal frames at $p$. Then $G \subset \text{O}(n)$ acts on $P_p$ by sending an orthonormal frame $u$ to $u' := u \circ \gamma$, which is also an orthonormal frame. The disjoint union $P = \bigsqcup_{p \in M} P_p$ defines a principal $\text{O}(n)$-subbundle of $F(M)$. We call $P \subset F(M)$ an $\text{O}(n)$-structure on $M$.

More generally, a $G$-structure on $M$ is a principal $G$-subbundle $P \subset F(M)$. As in the riemannian example just considered, a $G$-structure on $M$ can be defined in terms of certain characteristic tensor fields on $M$. In order to explain this, we have to briefly recall the concepts of an associated vector bundle to a $G$-structure and of the soldering form.

Let $P \subset F(M)$ be a $G$-structure on $M$. Then $P \to M$ is a principal $G$-bundle. Let $\rho : G \to \text{GL}(\mathbb{V})$ be a representation of $G$ on some finite-dimensional vector space $\mathbb{V}$. The group $G$ acts on $P \times \mathbb{V}$ on the right:

$$(u, v) \cdot g := (u \circ g, \rho(g^{-1})v).$$

Since $G$ acts freely on $P$, this action is free and the quotient $(P \times \mathbb{V})/G$ is the total space of an associated vector bundle $P \times_G \mathbb{V} \to M$. Sections of $P \times_G \mathbb{V}$ may be identified with $G$-equivariant functions $P \to \mathbb{V}$. More precisely, there is an isomorphism of $C^\infty(M)$-modules

$$\Gamma(P \times_G \mathbb{V}) \cong C^\infty_G(P, \mathbb{V}) := \{ \sigma : P \to \mathbb{V} \mid R^*_G \sigma = \rho(g^{-1}) \circ \sigma \}.$$  
(2.2)

(We observe that $\pi : P \to M$ allows us to pull-back smooth functions on $M$ to $P$ and view $C^\infty(M)$ as the $G$-invariant functions $C^\infty_G(P) = \{ f \in C^\infty(P) \mid R^*_G f = f \quad \forall g \in G \}$. Hence any $C^\infty(P)$-module becomes a $C^\infty(M)$-module by restricting scalars.) If $\mathcal{W}$ is another representation, a $G$-equivariant linear map $\Phi : \mathbb{V} \to \mathcal{W}$ defines a bundle map $\Phi : P \times_G \mathbb{V} \to P \times_G \mathcal{W}$, whose corresponding map on sections sends $\sigma \in C^\infty_G(P, \mathbb{V})$ to $\Phi \circ \sigma \in C^\infty_G(P, \mathcal{W})$.

The preceding discussion holds for any principal $G$-bundle, but in the case of a $G$-structure we have an additional structure not present in a general principal bundle: namely, an $\mathbb{R}^n$-valued one-form $\theta$ on $P$. To define it, suppose that $X_u \in T_u P$ is tangent to $P$ at $u \in P_p$. Then $\theta_u(X_u) := u^{-1}(\pi_u X_u)$, where $\pi : P \to M$ is the restriction of $P$ to the map sending a frame $u$ at $p \in M$. In words, $\theta_u(X_u)$ is the coordinate vector of $\pi_u X_u \in T_u M$ relative to the frame $u : \mathbb{R}^n \to T_u M$. The components of $\theta$ relative to the standard basis $(e_1, \ldots, e_n)$ for $\mathbb{R}^n$ are one-forms $\theta^i \in \Omega^1(P)$. If $s = (X_1, \ldots, X_n) : U \to P$ is a local moving frame, then the pull-backs $(s^*\theta^1, \ldots, s^*\theta^n)$ make up the local coframe on $U$ canonically dual to $s$: that is, $(s^*\theta^i)(X_j) = \delta_{ij}$. We call $\theta \in \Omega^1(P, \mathbb{R}^n)$ the soldering form of the $G$-structure.

The soldering form defines an isomorphism $\mathcal{T}M \cong P \times_G \mathbb{R}^n$, where $G$ acts on $\mathbb{R}^n$ via the defining representation $G < \text{GL}(n, \mathbb{R})$. In general, the soldering form allows us to identify tensor bundles over $M$ with the corresponding associated vector bundles $P \times_G \mathbb{V}$. We will use this often and tacitly in this paper.

Let $\rho : G \to \text{GL}(\mathbb{V})$ be a representation and let $0 \neq v \in \mathbb{V}$ be $G$-invariant: namely, $\rho(g)v = v$ for all $g \in G$. Then the constant function $\sigma_v : P \to \mathbb{V}$ sending $u \mapsto v$ obeys $\sigma_v(u \circ g) = \rho(g^{-1})\sigma_v(u)$ and therefore gives a (nowhere-vanishing) section of the associated vector bundle $P \times_G \mathbb{V}$. If $\mathbb{V}$ is a tensor representation of $\mathbb{R}^n$, then the soldering form allows us to view $\sigma_v$ as a (nowhere-vanishing) tensor field on $M$.

For example, if $\mathbb{V} = \otimes^2(\mathbb{R}^n)^*$ is the space of symmetric bilinear forms on $\mathbb{R}^n$, then $\delta \in \mathbb{V}$ defined by $\delta(e_i, e_j) = \delta_{ij}$ is $O(n)$-invariant. In fact, $O(n)$ is precisely the subgroup of $\text{GL}(n, \mathbb{R})$ which leaves $\delta$ invariant. If $P \to M$ is an $O(n)$-structure, the constant function $\delta : P \to \mathbb{V}$ sending $u \mapsto \delta$ defines a section of $P \times_G \mathbb{V}$. The soldering form induces an isomorphism $P \times_G \mathbb{V} \cong \otimes^2 T^*M$ and hence $\delta$ defines a section $g \in \Gamma(\otimes^2 T^*M)$, which relative to a local moving frame $s = (X_1, \ldots, X_n) : U \to P$ satisfies $g(X_i, X_j) = \delta_{ij}$. Equivalently, $g = \delta_{ij} s^*\theta^i s^*\theta^j$ (using Einstein summation convention here and from now on). In other words, $g$ is the riemannian metric which defines the $O(n)$-structure. The group $O(n)$ is not connected and it may happen that a $O(n)$-structure further reduces to an $SO(n)$-structure. In that case, there is an additional invariant tensor: namely the volume form of the riemannian metric.

In this paper we shall be interested in several different types of $G$-structures on an $n$-dimensional smooth manifold. Each such group $G$ can be defined as the subgroup of $\text{GL}(n, \mathbb{R})$ which leaves invariant one or more tensors of the defining representation. These $G$-invariant tensors will then give rise to a set of characteristic tensor fields on $M$ in the manner illustrated above in the case of a riemannian structure.
2.2. Adapted connections. From now on we shall write \( V \) for \( \mathbb{R}^n \). In other words, \( V \) is not an abstract vector space but simply our notation for \( \mathbb{R}^n \). We shall also write \( \text{GL}(V) \) for \( \text{GL}(n, \mathbb{R}) \) and \( \mathfrak{gl}(V) \) for its Lie algebra. If \( G \subset \text{GL}(V) \) we shall let \( g \subset \mathfrak{gl}(V) \) denote its Lie algebra. Let \( \pi : P \to M \) be a \( G \)-structure and let \( \theta \in \Omega^1(P, V) \) be the soldering form.

If \( u \in P \) is a frame at \( p = \pi(u) \), then \( (\pi)_u : T_u P \to T_p M \) is a surjective linear map, whose kernel \( \mathcal{V}_u = \ker(\pi)_u \) is called the vertical subspace of \( T_u P \). The rank theorem says that \( \dim \mathcal{V}_u = \dim g \). The disjoint union \( \mathcal{V} = \bigsqcup_{u \in P} \mathcal{V}_u \) defines a \( G \)-invariant distribution \( \mathcal{V} \subset TP \). Indeed \( \pi \circ R_g = \pi \) implies that \((R_g)_\mathcal{V}
\) preserves the kernel of \( \pi_u \). The distribution \( \mathcal{V} \) is also involutive and the leaves of the corresponding foliation of \( P \) are the fibres \( \pi^{-1}(p) \).

By an Ehresmann connection on \( P \) we mean a \( G \)-invariant distribution \( \mathcal{H} \subset TP \) complementary to \( \mathcal{V} \). At every frame \( u \in P \), \( \mathcal{H}_u \) restricts to an isomorphism \( \mathcal{H}_u \cong T_p M \). We will let \( h_u : T_u P \to \mathcal{H}_u \) denote the horizontal projector along \( \mathcal{V}_u \). Equivalently, we may define an Ehresmann connection via a connection one-form \( \omega \in \Omega^1(P, g) \) defined uniquely by the properties:

\[
\ker \omega_u = \mathcal{H}_u \quad \text{and} \quad \omega(\xi_X) = X \quad \forall X \in g,
\]

where \( \xi_X \in \mathfrak{X}(P) \) is the fundamental vector field corresponding to \( X \in g \) and defined by \( (\xi_X)_u = \frac{d}{dt} (u \circ e^{tX})|_{t=0} \). It follows that

\[
R^g_u \omega = Ad(g^{-1}) \circ \omega,
\]

where \( Ad : G \to \text{GL}(g) \) is the adjoint representation.

An Ehresmann connection allows us to extend the \( C^\infty(M) \)-module isomorphism (2.2) to differential forms. Let us define

\[
\Omega^P_G(P, V) := \{ \varphi \in \Omega^P(P, V) \mid R^g_u \varphi = \rho(g^{-1}) \circ \varphi \quad \text{and} \quad h^* \varphi = \varphi \},
\]

where \( h^* \varphi(Y_1, \ldots, Y_p) = \varphi(hY_1, \ldots, hY_p) \), with \( h \) the horizontal projector. The condition \( R^g_u \varphi = \rho(g^{-1}) \circ \varphi \) says that \( \varphi \) is invariant, whereas the condition \( h^* \varphi = \varphi \) says that it is horizontal. A form \( \varphi \in \Omega^P_G(P, V) \) is said to be basic because it defines a \( p \)-form on \( M \) with values in the associated bundle \( P \times_G V \). Indeed, we have a \( C^\infty(M) \)-module isomorphism

\[
\Omega^G_P(P, V) \cong \Omega^P(M, P \times_G V).
\]

An Ehresmann connection on \( P \) defines a Koszul connection on any associated vector bundle. Its expression is particularly transparent in terms of the equivariant functions \( C^\infty_G(P, V) \), where the Koszul connection defines a covariant derivative operator:

\[
\nabla : C^\infty_G(P, V) \to \Omega^1_G(P, V) \quad \text{with} \quad \nabla \sigma := h^* d\sigma.
\]

In calculations, it is more convenient to use the equivalent expression \( \nabla \sigma = d\sigma + \rho_*(\omega) \circ \sigma \), where \( \rho : g \to \text{gl}(V) \) is the representation of \( g \) induced by \( \rho : G \to \text{GL}(V) \).

It is easy to see that the soldering form is actually basic: \( \theta \in \Omega^1_P(P, V) \) and hence it defines a one-form on \( M \) with values in \( P \times_G V \); that is, a section of \( \text{Hom}(TM, P \times_G V) \). This is none other but the isomorphism \( \Phi \cong P \times_G V \). Functorially, it induces isomorphisms between the bundle of \((r, s)\)-tensors on \( M \) and \( P \times_G T^s(M) \), with \( T^s(M) = (V^r \otimes V^s)^* \).

The Koszul connection on \( P \times_G V \) induces an affine connection (also denoted \( \nabla \)) on \( TM \), which is said to be adapted to the \( G \)-structure \( P \):
where the second term in the RHS involves also the action of $g$ on $V$ via the embedding $g < gl(V)$; that is, for all $X, Y ∈ X(P)$, we have
\[ Θ(X, Y) = dθ(X, Y) + ω(X)θ(Y) - ω(Y)θ(X). \] (2.11)

Let us now investigate how the torsion changes when we change the connection. Let $Χ' \subset TP$ be a second Ehresmann connection on $P$ with connection one-form $ω' ∈ Ω^1(P, g)$. Let $κ = ω' - ω ∈ Ω^1(P, g)$. Since $ω$ and $ω'$ are invariant, so is $κ$; but since $ω$ and $ω'$ agree on vertical vectors, $κ$ is now also horizontal. Therefore $κ ∈ Ω^1_G(P, g)$ and hence it descends to a one-form with values in $Ad P := P × G g$.

In general, the difference $∇' - ∇$ between two affine connections belongs to $Ω^1(M, End TM)$, but if the connections are adapted to the $G$-structure, then $∇' - ∇$ is a one-form with values in the sub-bundle of $End TM$ corresponding to $Ad P$ via the soldering form.

Let $Θ'$ be the torsion two-form of $Χ'$. From the first structure equation (2.10), we see that
\[ Θ' - Θ = κ ∧ θ \] (2.12)
or, equivalently, for all $X, Y ∈ X(P)$,
\[ (Θ' - Θ)(X, Y) = κ(X)θ(Y) - κ(Y)θ(X) \] (2.13)

The passage from $κ$ to $Θ' - Θ$ defines a $C^∞(M)$-linear map
\[ Ω^1(M, P × G g) → Ω^2(M, P × G V) \] (2.14)
which is induced from a bundle map
\[ P × G (g ⊗ V*) → P × G (V ⊗ ^2V*), \] (2.15)
which is in turn induced from a $G$-equivariant linear map, a special instance of a **Spencer differential**, \[ Hom(V, g) \xrightarrow{∂} Hom(^2V, V) \text{ defined by } (∂κ)(v, w) = κ_v w - κ_w v, \] (2.16)
for all $v, w ∈ V$ and where $κ : V → g$ sends $v → κ_v$.

We may summarise this discussion as follows.

**Proposition 1.** Let $P \xrightarrow{π} M$ be a $G$-structure and $ω ∈ Ω^1(P, g)$ the connection one-form of an Ehresmann connection with torsion two-form $Θ ∈ Ω^2_G(P, V)$. If $ω' = ω + κ$ is another Ehresmann connection, then its torsion two-form $Θ' = Θ + ∂κ$, where $∂ : Ω^2_G(P, g) → Ω^3_G(P, V)$ is induced from the Spencer differential
\[ ∂ : Hom(V, g) → Hom(^2V, V) \] (2.17)
defined by $∂κ(v, w) = κ_v w - κ_w v$ for all $v, w ∈ V$.

Under the isomorphisms $Hom(V, g) ≅ g ⊗ V^*$ and $Hom(^2V, V) ≅ V ⊗ ^2V^*$, the Spencer differential is the composition
\[ g ⊗ V^* \xrightarrow{id ⊗ id} V ⊗ V^* ⊗ V^* \xrightarrow{id ⊗ ^2Λ} V ⊗ ^2V^* \] (2.18)
where $i : g → V ⊗ V^*$ is the embedding $g < gl(V)$ composed with the isomorphism $gl(V) ≅ V ⊗ V^*$, and $Λ : V^* ⊗ V^* → ^2V^*$ is skew-symmetrisation.

To the linear map $∂ : g ⊗ V^* → V ⊗ ^2V^*$ there is associated an exact sequence:
\[ 0 → ker ∂ → g ⊗ V^* → ^2Λ → coker ∂ → 0, \] (2.19)
where $coker ∂ = (V ⊗ ^2V^*)/im ∂$. Since these maps are $G$-equivariant, we obtain an exact sequence of associated vector bundles:
\[ 0 → P × G ker ∂ → P × G (g ⊗ V^*) → P × G (V ⊗ ^2V^*) → P × G coker ∂ → 0. \] (2.20)

These bundles have the following interpretation:
- the torsion of (adapted) affine connections are sections of $P × G (V ⊗ ^2V^*) ≅ TM ⊗ ^2T^*M$;
- the contorsions (i.e., the differences between adapted affine connections) are sections of $Ad P ⊗ T^*M$;
- the contorsions which do not alter the torsion are sections of $P × G ker (g ⊗ V^*)$; and
- the "intrinsic torsion" (see below) of an adapted connection is a section of $P × G coker (g ⊗ V^*)$. 
Since \( T^\nabla - T^\nabla = \delta(\nabla^\prime - \nabla) \), we see that the image \([T^\nabla] \in \Gamma(P \times_G \ker \delta)\) of the torsion is independent of the connection and is an intrinsic property of the G-structure. We say \([T^\nabla] \in \Gamma(P \times_G \ker \delta)\) is the intrinsic torsion of the G-structure.

As an example, consider a lorentzian G-structure. It is customary here to label the standard basis of \( V = \mathbb{R}^n \) as \( \{e_0, e_1, \ldots, e_{n-1}\} \) with canonical dual basis \( \{\alpha^0, \alpha^1, \ldots, \alpha^{n-1}\} \) for \( V^* \). Then \( G < \text{GL}(V) \) is the subgroup leaving invariant the lorentzian inner product

\[
\eta = -(\alpha^0)^2 + \sum_{i=1}^{n-1} (\alpha^i)^2.
\]

The Lie algebra \( g = \mathfrak{so}(V) \) is the space of \( \eta\)-skew-symmetric endomorphisms of \( V \). As we now show, the Spencer differential is an isomorphism in this case.

**Lemma 2.** The Spencer differential

\[
\delta : \mathfrak{so}(V) \otimes V^* \rightarrow V \otimes \wedge^2 V^*
\]

is an isomorphism.

**Proof.** Notice that \( \dim(\mathfrak{so}(V) \otimes V^*) = \dim(V \otimes \wedge^2 V^*) \), so the result will follow if we show that \( \ker \delta = 0 \).

Let \( \kappa \in \mathfrak{so}(V) \otimes V^* \) so that \( \delta \kappa(v, w) = \kappa(w)v - \kappa(v)w \). Introduce the notation \( T(v, w, z) := \eta(\kappa(w)v, z) \). Since \( \kappa_w \in \mathfrak{so}(V), T(v, w, z) = -T(v, z, w) \) and if \( \delta \kappa = 0 \) then also \( T(v, w, z) = T(w, v, z) \), so that for all \( v, w, z \in V \),

\[
T(v, w, z) = T(w, v, z) = -T(z, w, v) = -T(z, v, w) = T(z, v, w) = T(v, z, w) = -T(v, w, z) \implies T = 0.
\]

Since \( \eta \) is non-degenerate, it follows that \( \kappa = 0 \) and hence \( \ker \delta = 0 \). \( \square \)

It follows therefore that \( \ker \delta = 0 \) and hence any adapted connection (here any metric connection) can be modified to be torsionless, and since \( \ker \delta = 0 \), there is a unique such modification. In other words, we have rederived the Fundamental Theorem of riemannian geometry: the existence of a unique torsionless metric connection; namely, the Levi-Civita connection.

We close this short review with two observations. Firstly, if \( g < \mathfrak{so}(V) \), then since \( \delta \) is the restriction to \( g \) of the map in Lemma 2, it is still the case that \( \ker \delta = 0 \) and hence the exact sequence (2.19) becomes short exact:

\[
0 \rightarrow g \otimes V^* \xrightarrow{\delta} V \otimes \wedge^2 V^* \rightarrow \ker \delta \rightarrow 0,
\]

and in particular \( \dim \ker \delta = n \left( \binom{n}{2} - \dim g \right) \).

The second observation is that \( P \times_G (V \otimes \wedge^2 V^*) \) is the bundle of which the torsion of any connection is a section. It is not clear that any section of that bundle can be identified with the torsion tensor of an adapted connection. This is not unrelated to the fact that the classification of G-structures by their intrinsic torsion may result in classes which may not actually be realised geometrically. For example, it is well-known that in the case of \( G_2 < \text{SO}(7) \) structures in a 7-manifold, only 15 of the possible 16 structures are realised \([16, 17]\).

The rest of the paper consists in the calculation of \( \ker \delta \) for four types of G-structures relevant to spacetime geometries: galilean, carrollian, aristotelian and bargmannian. Our strategy will be the following. For each such type of geometry we will first determine the corresponding subgroup \( G < \text{GL}(V) \) and determine \( \ker \delta \) as a G-module. This usually allows us to interpret \( \ker \delta \), which is a quotient module, as a certain tensor module and hence will allow us to determine which expression in terms of the characteristic tensors of the G-structure captures the intrinsic torsion. We will then classify the G-submodules of \( \ker \delta \) and in this way characterise them geometrically in terms of properties of the characteristic tensors of the G-structure.

### 3. Galilean G-structures

Galilean G-structures were first discussed by Hans-Peter Künzle \([7]\), who proved, among other things, that they are of infinite type. Some of the results in this section can already be found in \([7]\): the determination of the group \( G \) and of the characteristic tensors and the identification of the intrinsic torsion (which is termed the “first structure function”) with the exterior derivative \( d\tau \) of the clock one-form. The main deviation from \([7]\) is that we claim that there is an additional “distinguished condition” other than “flatness” which can be imposed on the torsion of an adapted connection, which follows from our more detailed analysis of the G-module structure of \( \ker \delta \). Later papers on the subject of adapted connections to a galilean structure are \([18, 19, 20]\).
3.1. The group $G$ of a galilean structure. Let $V = \mathbb{R}^n$. We will use a suggestive notation for the standard basis for $V$: namely, $(H, p_1, \ldots, p_{n-1})$ with canonical dual basis $(\eta, \pi^1, \ldots, \pi^{n-1})$ for $V^*$. Indices $a, b, \ldots$ will run from 1 to $n - 1$ and we will write $P_a$ and $\pi^a$. Let $G < GL(V)$ be the subgroup which leaves invariant $\eta \in V^*$ and $\delta^{ab}P_aP_b \in \otimes^2 V$. It is not hard to show that
\[
G = \left\{ \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \middle| v \in \mathbb{R}^{n-1}, A \in O(n-1) \right\} < GL(n, \mathbb{R}),
\]
with Lie algebra
\[
g = \left\{ \begin{pmatrix} 0 & 0 \\ v & A \end{pmatrix} \middle| v \in \mathbb{R}^{n-1}, A \in so(n-1) \right\} < gl(n, \mathbb{R}).
\]

The characteristic tensor fields of a galilean $G$-structure are a nowhere-vanishing one-form $\tau \in \Omega^1(M)$, typically called the clock one-form and a corank-one positive-semidefinite $\gamma \in \Gamma(\otimes^2 TM)$ with $\gamma(\tau, -) = 0$, typically called the spatial cometric.

We will choose a basis $J_{ab} = -J_{ba}, B_a$ for $g$, with Lie brackets
\[
\begin{align*}
[J_{ab}, J_{cd}] &= \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac} + \delta_{ad}J_{bc} \\
[J_{ab}, B_c] &= \delta_{bc}B_a - \delta_{ac}B_b \\
[B_a, B_b] &= 0.
\end{align*}
\]

The actions of $g$ on $V$ and $V^*$ are given by
\[
\begin{align*}
J_{ab} \cdot P_c &= \delta_{bc}P_a - \delta_{ac}P_b \\
J_{ab} \cdot H &= 0 \\
B_a \cdot P_b &= 0 \\
B_a \cdot H &= P_a.
\end{align*}
\]

Letting $\langle \cdot \cdot \cdot \rangle$ denote the real span, we see that $\langle P_a \rangle \subset V$ and $\langle \eta \rangle \subset V^*$ are $g$-submodules. Hence neither $V$ nor $V^*$ are irreducible. The absence of complementary submodules says that they are nevertheless indecomposable.

3.2. The intrinsic torsion of a galilean structure. The Spencer differential $\partial : g \otimes V^* \to V \otimes \wedge^2 V^*$ is given by
\[
\begin{align*}
\partial(J_{ab} \otimes \pi^c) &= (\delta_{bd}P_a - \delta_{ad}P_b) \otimes \pi^d \wedge \pi^c \\
\partial(J_{ab} \otimes \eta) &= (\delta_{bd}P_a - \delta_{ad}P_b) \otimes \pi^c \wedge \eta \\
\partial(B_a \otimes \pi^b) &= P_a \otimes \eta \wedge \pi^b \\
\partial(B_a \otimes \eta) &= 0.
\end{align*}
\]

Therefore we see that its kernel is given by
\[
\ker \partial = \langle B_a \otimes \eta, J_{ab} \otimes \eta + (\delta_{bc}B_a - \delta_{ac}B_b) \otimes \pi^c \rangle.
\]

**Lemma 3.** As $g$-modules, $\ker \partial \cong \wedge^2 V^*$.

**Proof.** The $g$ action on $\wedge^2 V^*$ is given by the obvious action of $so(n-1)$ and then
\[
B_c \cdot \pi^a \wedge \pi^b = -\delta^a_c \eta \wedge \pi^b + \delta^b_c \eta \wedge \pi^a
\]
\[
B_c \cdot \pi^a \wedge \eta = 0.
\]

The action of $g$ on $\ker \partial$ is again given by the obvious action of $so(n-1)$ and then
\[
B_c \cdot (J_{ab} \otimes \eta + (\delta_{bd}B_a - \delta_{ad}B_b) \otimes \pi^d) = 2(\delta_{ca}B_c - \delta_{cb}B_a) \otimes \eta
\]
\[
B_c \cdot (B_a \otimes \eta) = 0.
\]

This suggests defining a linear map $\varphi : \ker \partial \to \wedge^2 V^*$ by
\[
\varphi(B_a \otimes \eta) = \delta_{ab} \pi^b \wedge \eta
\]
\[
\varphi(J_{ab} \otimes \eta + (\delta_{bc}B_a - \delta_{ac}B_b) \otimes \pi^c) = 2(\delta_{ca} \delta_{bd}) \tau^c \wedge \pi^d.
\]

This map is clearly an $so(n-1)$-equivariant isomorphism and one can easily check that it is also equivariant under the action of $B_a$.

The cokernel of the Spencer differential is spanned by the image in $\coker \partial$ of $\langle H \otimes \pi^a \wedge \pi^b, H \otimes \eta \wedge \pi^a \rangle$.

**Lemma 4.** As $g$-modules, $\coker \partial \cong \wedge^2 V^*$. 
Proof. We consider the $g$-equivariant linear map $\eta \otimes \text{id}_{\Lambda^2 V^*} : V \otimes \Lambda^2 V^* \to \Lambda^2 V^*$, which simply applies $\eta \in V^*$ to the $V$-component. From equation (3.5), we see that the image of the Spencer differential is contained in its kernel and hence it induces a $g$-equivariant linear map $\text{coker} \, \delta \to \Lambda^2 V^*$. Explicitly, it is given on the basis for $\text{coker} \, \delta$ by

$$
[\mathcal{H} \otimes \pi^a \wedge \pi^b] \mapsto \pi^a \wedge \pi^b
$$

which is clearly seen to be an isomorphism. $\square$

As a $g$-module, $\Lambda^2 V^*$ is indecomposable but not irreducible. Indeed, we have the following chain of submodules:

$$
0 \subset (\eta \wedge \pi^a) \subset \Lambda^2 V^*.
$$

(3.7)

By Lemma 4, this is also the case for $\text{coker} \, \delta$ and hence we see that there are three classes of galilean structures depending on whether the intrinsic torsion vanishes, lands in the submodule $(\eta \wedge \pi^a)$ or is generic.

Note that the short exact sequence of $g$-modules does not split; although it does split as vector spaces. This means that whereas it is possible to find a vector subspace of $V \otimes \Lambda^2 V^*$ complementary to $\text{im} \, \delta$, it is not possible to demand in addition that it should be stable under $g$. In this case we have chosen $(\mathcal{H} \otimes \pi^a \wedge \pi^b, \mathcal{H} \otimes \eta \wedge \pi^a)$ as the vector space complement of $\text{im} \, \delta$ in $V \otimes \Lambda^2 V^*$. This subspace is not preserved under $g$, but only modulo $\text{im} \, \delta$. This has the following geometrical consequence. Having intrinsic torsion in the submodule $\mathcal{G} := (\mathcal{H} \otimes \pi^a \wedge \eta) \subset \text{coker} \, \delta$ does not mean that there exists an adapted connection $\nabla$ whose torsion $\nabla^\tau$ is a section of $P \times_G \mathcal{G}$, where $\mathcal{G} = (\mathcal{H} \otimes \pi^a \wedge \eta) \subset V \otimes \Lambda^2 V^*$. What it does mean is that relative to some local moving frame (in P), the torsion will be represented by a function $U \to \mathcal{G}$, but if we change the frame (while still in P), this might not persist. However one can modify the connection such that relative to the new adapted connection, the torsion is again represented by a function $U \to \mathcal{G}$. This is why it is important to derive consequences of the fact that the intrinsic torsion lands in $\mathcal{G}$ which are independent of the choice of the adapted connection.

This is something one seldom sees in riemannian $G$-structures where $G < O(n)$, since $G$, being compact, is reductive: sequences of $G$-modules split and modules are fully reducible into irreducibles. This is why results of the kind reported in this paper are typically simpler to state in that situation.

3.3. Geometric characterisation. It follows from the isomorphism in Lemma 4, that there is bundle isomorphism $P \times_G \text{coker} \, \delta \cong \Lambda^2 T^*M$ and therefore the intrinsic torsion of an adapted connection is captured by a two-form. To identify this two-form, we notice that the $g$-equivariant linear map $\eta \otimes \text{id}_{\Lambda^2 V^*} : V \otimes \Lambda^2 V^* \to \Lambda^2 V^*$ in the proof of Lemma 4, induces a bundle map $TM \otimes \Lambda^2 T^*M \to \Lambda^2 T^*M$ and hence a $\text{C}^{\infty}(M)$-linear map $\Phi : \Omega^2(M, TM) \to \Omega^2(M)$ which is given by composing with the clock form $\tau$. In other words, $\Phi(T) = \tau \circ T$ for any $T \in \Omega^2(M, TM)$.

Proposition 5. Let $\nabla$ be an adapted affine connection with torsion $\nabla^\tau \in \Omega^2(M, TM)$. Its image under $\Phi : \Omega^2(M, TM) \to \Omega^2(M)$ is given by $\Phi(\nabla^\tau) = d\tau$, where $\tau \in \Omega^1(M)$ is the clock one-form.

Proof. Since the clock one-form $\tau$ is parallel relative to any adapted affine connection, we have that for all $X, Y \in \mathcal{X}(M)$,

$$
X\tau(Y) = \tau(\nabla_X Y).
$$

Skew-symmetrising,

$$
X\tau(Y) - Y\tau(X) = \tau(\nabla_X Y - \nabla_Y X) = \tau([X, Y]) + T^\tau([X, Y]),
$$

(by definition of $T^\tau$)

so that

$$
d\tau(X, Y) = X\tau(Y) - Y\tau(X) - \tau([X, Y]) = \tau(T^\tau(X, Y)).
$$

In other words, $d\tau = \tau \circ T^\tau = \Phi(T^\tau)$, as desired. $\square$

---

2This is for $n \neq 5$. The case $n = 5$ is treated in Appendix B.3.
If the intrinsic torsion vanishes, then $\mathrm{d}\tau = 0$. If the intrinsic torsion lands in the subbundle $P \times_G \mathfrak{g}$, then $\mathrm{d}\tau$ is represented locally by a function $U \to \langle \eta \wedge \pi^a \rangle$, which says that $\mathrm{d}\tau = \tau \wedge \alpha$ for some $\alpha \in \Omega^1(U)$. This implies that $\mathrm{d}\tau \wedge \tau = 0$ which, as shown in Appendix A, implies in turn that $\mathrm{d}\tau = \tau \wedge \alpha$ for a global one-form $\alpha \in \Omega^1(M)$. Finally, the generic case is where $\mathrm{d}\tau \wedge \tau \neq 0$.

We summarise this discussion as follows, which is to be compared with [8, Table I].

**Theorem 6.** Let $n > 2$ and $n \neq 5$. A galilean $G$-structure on an $n$-dimensional manifold $M$ may be of one of three classes, according to its intrinsic torsion. If $\tau \in \Omega^1(M)$ is the clock one-form, then three cases can exist:

1. $(S_0)$ $\mathrm{d}\tau = 0$, corresponding to a torsionless Newton–Cartan geometry (NC);
2. $(S_1)$ $\mathrm{d}\tau \neq 0$ and $\mathrm{d}\tau \wedge \tau = 0$, corresponding to a twistless torsional Newton–Cartan geometry (TTNC); and
3. $(S_2)$ $\mathrm{d}\tau \wedge \tau \neq 0$, corresponding to a torsional Newton–Cartan geometry (TNC).

All spatially isotropic homogeneous galilean spacetimes in $[3, 4]$ have $\mathrm{d}\tau = 0$, but there exist homogeneous examples of all three kinds $[21]$.

4. **Carrollian G-structures**

4.1. **The group $G$ of a carrollian structure.** We use the same notation as in the previous section: with $V = \langle H, P \rangle$ and $V^* = \langle \eta, \pi^a \rangle$. Let $G < \text{GL}(V)$ be the subgroup leaving invariant $H \in V$ and $\delta_{ab} \pi^a \pi^b \in \odot^2 V^*$. Explicitly,

$$G = \left\{ \begin{pmatrix} 1 & v^T \\ 0 & A \end{pmatrix} \bigg| v \in \mathbb{R}^{n-1}, A \in \text{O}(n-1) \right\} < \text{GL}(n, \mathbb{R}), \quad (4.1)$$

with Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & v^T \\ 0 & A \end{pmatrix} \bigg| v \in \mathbb{R}^{n-1}, A \in \text{so}(n-1) \right\} < \text{gl}(n, \mathbb{R}). \quad (4.2)$$

We remark that the groups for the galilean and carrollian structures are abstractly isomorphic, being isomorphic to the semi-direct product $\text{O}(n-1) \rtimes \mathbb{R}^{n-1}$, but crucially they are not conjugate inside $\text{GL}(V)$. Indeed, if they were conjugate, they would have the invariants in the same representations. To see this let $\rho : \text{GL}(V) \to \text{GL}(V)$ be a representation and suppose that $G, G' < \text{GL}(V)$ are conjugate subgroups. This means that there exists $\gamma \in \text{GL}(V)$ such that $G' = \gamma G \gamma^{-1}$. Suppose now that $v \in V$ is $G$-invariant, so that $\rho(g)v = v$ for all $g \in G$. Then $v' = \rho(\gamma)v \in V$ is $G'$-invariant. To show this let $g' \in G'$ be arbitrary. Then $g' = \gamma g \gamma^{-1}$ for some $g \in G$ and calculate

$$\rho(g'v') = \rho(g')\rho(\gamma)v = \rho(g'\gamma)v = \rho(\gamma g)v = \rho(\gamma)\rho(g)v = \rho(\gamma)v = v'.$$

Since the galilean structure group has an invariant in the representation $V^*$ and the carrollian structure group does not, they cannot be conjugate subgroups of $\text{GL}(V)$.

The characteristic tensor fields of a carrollian $G$-structure are a nowhere-vanishing vector field $\xi \in \mathfrak{X}(M)$, typically called the **carrollian vector field** and a corank-one positive-semidefinite $h \in \Gamma(\odot^2 T^*M)$ with $h(\xi, -) = 0$, typically called the **spatial metric**.

The group $G$ has two connected components, corresponding to the value of the determinant of the matrix $A \in \text{O}(n-1)$. If the $G$-structure reduces further to a $G_0$-structure, where $G_0$ is the identity component of $G$, there is an additional characteristic tensor: namely a **volume form** $\mu \in \Omega^n(M)$, corresponding to the $G_0$-invariant tensor $\eta \wedge \pi^1 \wedge \cdots \wedge \pi^{n-1} \in \wedge^\circ V^*$. Even if the $G$-structure does not reduce to $G_0$, the volume form exists locally, but it may change by a sign on overlaps.

As before, let $g = (J_{ab}, B_a)$ with the same Lie brackets as in (3.3). The action of $g$ on $V$ and $V^*$ is given by

$$J_{ab} \cdot P_c = \delta_{bc} P_a - \delta_{ac} P_b \quad \text{and} \quad J_{ab} \cdot \pi^c = (\delta^c_d \delta_{ad} + \delta^c_d \delta_{bd}) \pi^d$$

$$J_{ab} \cdot H = 0 \quad \text{and} \quad B_a \cdot \pi^b = 0$$

$$B_a \cdot H = 0 \quad \text{and} \quad B_a \cdot \eta = -\delta_{ab} \pi^b. \quad (4.3)$$

As in the galilean case, the $g$-modules $V$ and $V^*$ are indecomposable but not irreducible, since $\langle H \rangle \subset V$ and its annihilator $\text{Ann} H := \langle \pi^a \rangle \subset V^*$ are submodules without complementary submodules.

---

5See Appendices B.1 for $n = 2$ and B.3 for $n = 5$. 
4.2. The intrinsic torsion of a carrollian structure. The Spencer differential $\partial : \mathfrak{g} \otimes V^* \to V \otimes \wedge^2 V^*$ is given relative to our choice of basis by

\[
\partial(J_{ab} \otimes \eta) = (\delta_{bc} P_a - \delta_{ac} P_b) \otimes \pi^c \wedge \eta \\
\partial(J_{ab} \otimes \pi^c) = (\delta_{ba} P_a - \delta_{ad} P_b) \otimes \pi^d \wedge \pi^c \\
\partial(B_a \otimes \eta) = \delta_{ab} H \otimes \pi^b \wedge \eta \\
\partial(B_a \otimes \pi^b) = \delta_{ac} H \otimes \pi^c \wedge \pi^b.
\]

(4.4)

Lemma 7. As $\mathfrak{g}$-modules, $\ker \partial \cong \text{coker } \partial \cong \otimes^2 \text{Ann } H$, where $\text{Ann } H \subset V^*$ is the annihilator of $H$.

Proof. The kernel of the Spencer differential is easily seen to be

\[
\ker \partial = \langle (\delta_{bc} B_a + \delta_{ac} B_b) \otimes \pi^c \rangle
\]

which suggests defining $\ker \partial \to \otimes^2 \text{Ann } H$ by

\[
(\delta_{bc} B_a + \delta_{ac} B_b) \otimes \pi^c \mapsto \delta_{bd} \pi^c \pi^d.
\]

This is clearly $\mathfrak{g}$-equivariant, since it is manifestly $so(n-1)$-equivariant and $B_a$ acts trivially on both sides. It is also clearly an isomorphism. Similarly the cokernel of the Spencer differential is the image in $\text{coker } \partial$ of

\[
\langle (\delta_{bc} P_a + \delta_{ac} P_b) \otimes \eta \wedge \pi^c \rangle \subset V \otimes \wedge^2 V^*.
\]

and we define $\text{coker } \partial \to \otimes^2 \text{Ann } H$ by

\[
(\delta_{bc} P_a + \delta_{ac} P_b) \otimes \eta \wedge \pi^c \mapsto \delta_{bd} \pi^c \pi^d,
\]

which can be easily checked to be a $\mathfrak{g}$-equivariant isomorphism. □

Since $B_a$ acts trivially on $\otimes^2 \text{Ann } H$, we may think of it simply as an $so(n-1)$-module. It is therefore fully reducible into a direct sum of two irreducible submodules:

\[
\otimes^2 \text{Ann } H = \mathcal{O}_0^2 \text{Ann } H \oplus \mathcal{O}_1^2 \text{Ann } H \oplus \mathcal{O}_2^2 \text{Ann } H
\]

where $\mathcal{O}_0^2 \text{Ann } H$ are the traceless symmetric bilinear forms and $\mathcal{O}_1^2 \text{Ann } H$ decomposes $\text{coker } \partial$ into a direct sum of two irreducible submodules

\[
\text{coker } \partial = \mathcal{C}_1 \oplus \mathcal{C}_2,
\]

where the submodule $\mathcal{C}_1$ is of type $\mathcal{O}_0^2 \text{Ann } H$ and is the image in $\text{coker } \partial$ of the subspace

\[
\langle (\delta_{bc} P_a + \delta_{ac} P_b - \frac{2}{n-1} \delta_{ab} P_c) \otimes \eta \wedge \pi^c \rangle \subset V \otimes \wedge^2 V^*.
\]

(4.7)

whereas the trivial submodule $\mathcal{C}_2$ is the image of $\partial$ of the one-dimensional subspace

\[
\langle P_a \otimes \eta \wedge \pi^c \rangle \subset V \otimes \wedge^2 V^*.
\]

(4.8)

Thus we see that there are four submodules of $\text{coker } \partial$: $0, \mathcal{C}_1, \mathcal{C}_2$ and $\text{coker } \partial = \mathcal{C}_1 \oplus \mathcal{C}_2$ and hence we conclude that there are four classes of carrollian $G$-structures according to which submodule of $\text{coker } \partial$ the intrinsic torsion lands in.

4.3. Geometric characterisation of carrollian structures. The isomorphism $\text{coker } \partial \cong \otimes^2 \text{Ann } H$ of $\mathfrak{g}$-modules in Lemma 7 is induced (up to an inconsequential factor of 2) by the $\mathfrak{g}$-equivariant linear map

\[
\phi : \text{Hom}(\wedge^2 V, V) \to \otimes^2 \text{Ann } H
\]

defined for $T \in \text{Hom}(\wedge^2 V, V)$ by

\[
\phi(T)(v, w) := \delta^+(T(H, v), w) + \delta^+(T(H, w), v) \quad \forall v, w \in V.
\]

(4.10)

We check that $\phi(T)$ does land in $\otimes^2 \text{Ann } H$:

\[
\phi(T)(H, v) = \delta^+(T(H, H), v) + \delta^+(T(H, v), H) = 0,
\]

(4.11)

where the first term vanishes because of skew-symmetry of $T$ and the second because $\delta^+(H, -) = 0$. Explicitly,

\[
\phi(P_a \otimes \pi^b \wedge \pi^c) = 0 \\
\phi(P_a \otimes \eta \wedge \pi^b) = \delta_{ac} \pi^b \pi^c \\
\phi(H \otimes \pi^a \wedge \pi^b) = 0 \\
\phi(H \otimes \eta \wedge \pi^a) = 0.
\]

(4.12)
and we check that \( \text{im} \, \partial \subset \text{ker} \, \phi \), so that \( \phi \) does induce a map \( \text{coker} \, \partial \to \mathcal{O}^2 \text{Ann} \, \mathcal{H} \) which coincides with the one in Lemma 7, up to an overall factor of 2.

The map \( \phi \) induces a bundle map of the relevant associated vector bundles and hence a \( C^\infty(M) \)-linear map

\[
\Phi : \mathcal{O}^2(M, TM) \to \Gamma(\mathcal{O}^2 \text{Ann} \, \xi)
\]

where, for \( T \in \mathcal{O}^2(M, TM) \),

\[
\Phi(T)(X, Y) = h(T(\xi, X), Y) + h(T(\xi, Y), X) \quad \forall \, X, Y \in \mathfrak{X}(M).
\]

**Proposition 8.** Let \( \nabla \) be an affine connection adapted to a carrollian \( G \)-structure on \( M \) with torsion \( T^\nabla \). Then under the map \( \Phi \) in equation (4.13),

\[
\Phi(T^\nabla) = \mathcal{L}_\xi h.
\]

**Proof.** Since \( \nabla \) is adapted, both the carrollian vector field \( \xi \in \mathfrak{X}(M) \) and the spatial metric \( h \in \Gamma(\mathcal{O}^2 \text{Ann} \, \xi) \) are parallel. From \( \nabla \xi = 0 \) we have that

\[
T^\nabla(\xi, X) = \nabla_\xi X - [\xi, X], \quad \forall \, X \in \mathfrak{X}(M),
\]

and from \( \nabla_\xi h = 0 \) we have that for all \( X, Y \in \mathfrak{X}(M) \),

\[
\xi h(X, Y) = h(\nabla_\xi X, Y) = h(X, \nabla_\xi Y) = 0.
\]

We may expand the first term using the Lie derivative and arrive at

\[
(\mathcal{L}_\xi h)(X, Y) + h([\xi, X], Y) + h(X, [\xi, Y]) + h(\nabla_\xi X, Y) = 0,
\]

which, using equation (4.16), becomes

\[
(\mathcal{L}_\xi h)(X, Y) + h(T^\nabla(\xi, X), Y) + h(X, T^\nabla(\xi, Y)) = 0
\]

or, equivalently,

\[
(\mathcal{L}_\xi h)(X, Y) = \Phi(T^\nabla)(X, Y).
\]

**Proposition 9.** Let \( \mu \) denote the (perhaps only locally defined) volume form on \( M \). Then

\[
\mathcal{L}_\xi \mu = \text{tr}(S) \mu,
\]

where \( S(X) := T^\nabla(\xi, X) \) for all \( X \in \mathfrak{X}(M) \).

**Proof.** Let \( s = (X_0, \xi, X_1, \ldots, X_{n-1}) : U \to P \) be a local moving frame with \( h(X_a, X_b) = \delta_{ab} \) and, of course \( h(X_0, -) = 0 \). Let \((0^0, 0^1, \ldots, 0^{n-1})\) be the canonically dual coframe, so that \( h = \delta_{ab} \theta^a \theta^b \). Then the local expression for the volume form is \( \mu = \theta^0 \wedge \theta^1 \wedge \cdots \wedge \theta^{n-1} \) and hence by the Cartan formula

\[
\mathcal{L}_\xi \mu = d \mu = d(\theta^0 \wedge \cdots \wedge \theta^{n-1}).
\]

Pulling the first structure equation (2.10) back to \( M \) via \( s \), we have

\[
d \theta^a = T^a - \omega^a_0 \wedge \theta^0 - \omega^a_\theta \wedge \theta^b,
\]

where \( T^a = \theta^a \circ T^\nabla \). Since \( \nabla \xi = 0 \), we have that, for all \( Y \in \mathfrak{X}(M) \),

\[
0 = \nabla_Y X_0 = X_0 \omega(Y)^0_0 + X_\theta \omega(Y)^\theta_0 \Rightarrow \omega^0_0 = \omega^a_0 = 0,
\]

so that

\[
d \theta^a = T^a - \omega^a_\theta \wedge \theta^b,
\]

and hence

\[
d(\theta^0 \wedge \cdots \wedge \theta^{n-1}) = (T^1 - \omega^1_\theta \wedge \theta^0) \wedge \theta^2 \wedge \cdots \wedge \theta^{n-1} - \theta^1 \wedge (T^2 - \omega^2_\theta \wedge \theta^0) \wedge \cdots \wedge \theta^{n-1} + \cdots
\]

The only terms which contribute to this sum are \( T^a(X_0, X_a) \theta^0 \wedge \theta^a \) and \( \omega(X_0)^a_0 \theta^0 \) with no summation implied in either term. In summary,

\[
d(\theta^0 \wedge \cdots \wedge \theta^{n-1}) = (\theta^a \circ T^\nabla(\xi, X_a) - \omega(\xi)^a_0) \mu.
\]

We claim that \( \omega(\xi)^a_0 = 0 \) since \( \nabla h = 0 \). Indeed,

\[
0 = (\nabla_\xi h)(X_a, X_b) = \xi h(X_a, X_b) - h(\nabla_\xi X_a, X_b) - h(X_a, \nabla_\xi X_b) = -h(X_c \omega(\xi)^c_a, X_b) - h(X_a, X_c \omega(\xi)^c_b) = -\omega(\xi)^{ba}_a - \omega(\xi)^{ab}_a,
\]

(used that \( h(\xi, -) = 0 \)).
which implies that \( \omega(\xi)^a_a = \delta^{ab} \omega(\xi)_{ab} = 0 \). In summary,
\[
d(\Theta^1 \wedge \cdots \wedge \Theta^{n-1}) = (\Theta^a \circ T^\nu(\xi, X_a)) \mu = \Theta^a S(X_a) \mu = \text{tr}(S) \mu.
\]

We can now recognise the geometrical significance of the different intrinsic torsion conditions. If \( \Phi(T^\nu) = 0 \), then \( \mathcal{L}_\xi h = 0 \) and hence \( \xi \) is h-Killing. If \( \Phi(T^\nu) = fh \), for some \( f \in C^\infty(M) \), then \( \mathcal{L}_\xi h = fh \) and hence \( \xi \) is h-conformal Killing. Finally, if \( \Phi(T^\nu) \) is traceless, then \( \mathcal{L}_\xi \mu = 0 \), so that \( \xi \) is volume-preserving. Otherwise, we have a generic carrollian structure.

We may summarise this discussion as follows.

**Theorem 10.** Let \( n > 2 \). A carrollian G-structure on an \( n \)-dimensional manifold \( M \) can be of one of four classes depending on the Lie derivative \( \mathcal{L}_\xi h \) of the spatial metric \( h \) along the carrollian vector field \( \xi \):

- (\( E_0 \)) \( \mathcal{L}_\xi h = 0 \) (\( \xi \) is h-Killing);
- (\( E_1 \)) \( \mathcal{L}_\xi \mu = 0 \) (\( \xi \) is volume-preserving);
- (\( E_2 \)) \( \mathcal{L}_\xi h = fh \) (\( \exists \neq f \in C^\infty(M) \)) (\( \xi \) is h-conformal Killing);
- (\( E_3 \)) none of the above.

The symmetric carrollian spaces in \([3, 4]\) all have \( \mathcal{L}_\xi h = 0 \), but the formulae in \([4, \text{Section 7.3}] \) show that the carrollian lightcone has \( \mathcal{L}_\xi h = 2h \), so that \( \xi \) is h-homothetic. I am not aware of any explicit homogeneous carrollian manifolds in the other two classes; although it should not be hard to construct them as null hypersurfaces of lorentzian manifolds using as a hint the relationship with bargmannian structures in Section 6.4, where we will reformulate the conditions in Theorem 10 in terms of the different types of null hypersurfaces in a lorentzian manifold.

## 5. Aristotelian G-structures

### 5.1. The group \( G \) of an aristotelian structure.

An aristotelian space admits simultaneously a galilean and a carrollian structure, so the group \( G \triangleleft \text{GL}(V) \) corresponding to an aristotelian G-structure is the intersection of the groups in equations (3.1) and (4.1), namely

\[
G = \left\{ \begin{pmatrix} 1 & 0^T \\ 0 & A \end{pmatrix} \Bigg| A \in O(n-1) \right\} \triangleleft \text{GL}(n, \mathbb{R}),
\]

with Lie algebra

\[
g = \left\{ \begin{pmatrix} 0 & 0^T \\ 0 & A \end{pmatrix} \Bigg| A \in so(n-1) \right\} \triangleleft \text{so}(n, \mathbb{R}).
\]

In other words \( G \triangleleft O(n-1) \) and \( g \triangleleft so(n-1) \) is spanned by \( I_{ab} \), consistent with the fact that there are no boosts in an aristotelian spacetime.

Under the action of \( G \), both \( H \in V \) and \( \eta \in V^* \) are invariant, as are \( \delta_{ab} P_a P_b \in \otimes^2 V \) and \( \delta_{ab} \pi^a \pi^b \in \otimes^2 V^* \). Therefore, as G-modules, we have decompositions into irreducible submodules:

\[
V = \langle H \rangle \oplus \text{Ann} \eta \quad \text{and} \quad V^* = \langle \eta \rangle \oplus \text{Ann} H.
\]

Moreover, \( V \) and \( V^* \) are isomorphic G-modules. For example, the map \( \phi : V \to V^* \) defined by

\[
\phi(H) = \eta \quad \text{and} \quad \phi(P_a) = \delta_{ab} \pi^b
\]

is a G-equivariant isomorphism.

This means that an aristotelian spacetime has the following characteristic tensor fields: a nowhere vanishing vector field \( \xi \) and a nowhere-vanishing one-form \( \tau \) which can be normalised to \( \tau(\xi) = 1 \), and corank-one positive-semidefinite \( \gamma \in \Gamma(\otimes^2 TM) \) and \( h \in \Gamma(\otimes^2 T^* M) \) with \( \gamma(\tau, -) = 0 \) and \( h(\xi, -) = 0 \).

### 5.2. The intrinsic torsion of an aristotelian structure.

Since \( g \triangleleft so(V) \), for either a lorentzian or euclidean inner product on \( V \), Lemma 2 says that the Spencer differential \( \partial : g \otimes V^* \to V \otimes \wedge^2 V^* \) is injective. It is given explicitly by

\[
\partial(J_{ab} \otimes \pi^c) = (\delta_{aa} P_b - \delta_{bb} P_a) \otimes \pi^c \wedge \pi^d,
\]

\[
\partial(J_{ab} \otimes \eta) = (\delta_{bc} P_a - \delta_{ac} P_b) \otimes \pi^c \wedge \eta.
\]

It then follows that the image of the Spencer differential is given by

\[
\text{im} \partial = \langle P_a \otimes \pi^b \wedge \pi^c, (\delta_{bc} P_a - \delta_{ac} P_b) \otimes \pi^c \wedge \eta \rangle
\]

\[^4\text{See Appendix B.1 for } n = 2.\]
and hence the cokernel is the image in coker $\partial$ of

$$
\langle H \otimes \pi^a \land \pi^b, H \otimes \pi^a \land \eta, (\delta_{bc}P_a + \delta_{ac}P_b) \otimes \pi^c \land \eta \rangle.
$$

(5.7)

The cokernel of the Spencer differential is fully reducible into irreducible $G$-submodules.\(^5\)

$$
coker \partial \cong A_1 \oplus A_2 \oplus A_3 \oplus A_4,
$$

(5.8)

where, letting $W$ stand for the vector representation of $g \cong so(n-1)$,

- $A_1 \cong \land^3 W$ consists of the image in coker $\partial$ of $\langle H \otimes \pi^a \land \pi^b \rangle$;
- $A_2 \cong W$ consists of the image in coker $\partial$ of $\langle H \otimes \pi^a \land \eta \rangle$;
- $A_3 \cong \ominus^3 W$ (symmetric traceless) consists of the image in coker $\partial$ of $\langle (\delta_{bc}P_a + \delta_{ac}P_b - \frac{2}{n-1}\delta_{ab}P_c) \otimes \pi^c \land \eta \rangle$;
- and $A_4 \cong \mathbb{R}$ consists of the image in coker $\partial$ of $\langle P_a \otimes \pi^a \land \eta \rangle$.

We conclude that there are sixteen $G$-submodules of coker $\partial$ and therefore sixteen classes of aristotelian $G$-structures.

5.3. Geometric characterisation of aristotelian structures. Since an aristotelian $G$-structure is a simultaneous reduction of galilean and carrollian $G$-structures, we may reuse the results in Sections 3.3 and 4.3 in order to characterise the sixteen classes of aristotelian $G$-structures geometrically. This seems to give only twelve aristotelian classes: four carrollian structures for each of the three galilean structures. There is, however, a new ingredient in the aristotelian case: namely, the Lie derivative along the vector field $\xi$ of the one-form $\tau$.

**Proposition 11.** With the above notation, $\mathcal{L}_\xi \tau = \tau \circ S$, where $S(X) = T^\nabla(\xi, X)$.

**Proof.** First of all notice that if $\nabla$ is an adapted connection then both $\xi$ and $\tau$ are parallel and hence the function $\tau(\xi)$ is constant. (Being nonzero, we can assume that it is equal to 1 without loss of generality, simply by rescaling either $\tau$ or $\xi$.) Using this and the Cartan formula, we have that

$$
\mathcal{L}_\xi \tau = \tau \xi d\tau.
$$

(5.10)

But from Proposition 5, $d\tau = \tau \circ T^\nabla$, so that

$$
\mathcal{L}_\xi \tau = \tau(\xi \circ T^\nabla) = \tau \circ (\xi T^\nabla).
$$

(5.11)

$\square$

It follows that if $d\tau = 0$ then $\mathcal{L}_\xi \tau = 0$, whereas if $d\tau \neq 0$ but $\tau \land d\tau = 0$, then $\mathcal{L}_\xi \tau \neq 0$. If $\tau \land d\tau \neq 0$, then it may or may not happen that $\mathcal{L}_\xi \tau = 0$.

It is now simply a matter of inspecting the sixteen classes and determine whether $d\tau = 0$ or $\tau \land d\tau = 0$ or $d\tau$ is unconstrained, and then whether $\xi$ leaves invariant $\tau$, $\mu$ and the volume form $\nu$ or whether it rescales $h$. The results are summarised as follows.

**Theorem 12.** Let\(^6\) $n > 2$ and $n \neq 5$. An aristotelian $G$-structure on an $n$-dimensional manifold $M$ can be of sixteen different classes depending on its intrinsic torsion. These classes are summarised in the table below. Each class is labelled by the submodule of coker $\partial$ where the intrinsic torsion lands and is characterised geometrically as

\(^5\)If $n = 5$ and assuming that the structure group $O(4)$ reduces further to $SO(4)$, then the module $A_4$ is not irreducible but decomposes into selfdual and antiselfdual pieces. This is discussed briefly in Appendix B.4.

\(^6\)See Appendices B.1 for $n = 2$ and B.4 for $n = 5$. 
Bargmannian structures were introduced in [22], where the relation between bargmannian and galilean structures was initially explored. In particular, it was shown that Newton–Cartan gravity could be obtained as a null-reduction of a pp-wave: a lorentzian manifold with a nonzero parallel null vector field. The relation between bargmannian, galilean and carrollian structures was further explored in [10] and in [23]. Although both papers concentrate on pp-waves, the latter paper announces some work where the bargmannian structure is allowed to be more general. Indeed, below we will see that pp-waves are precisely the bargmannian manifolds with vanishing intrinsic torsion, which are one of (generically) twenty-seven different classes of bargmannian structures.

6.1. The group G of a bargmannian structure. In this section, we will assume that the dimension of the manifold M is n + 1. Therefore in this section V = \mathbb{R}^{n+1}.

An (n + 1)-dimensional bargmannian structure on M is a pair \((g, \xi)\) consisting of a lorentzian metric g and a nowhere-vanishing null vector field \(\xi\). g(\xi, \xi) = 0. Since \(\xi\) is nowhere-vanishing, around every point in M we may construct local Witt frames \((e_\alpha = \xi, e_1, \ldots, e_n)\), with \(g(e_\alpha, e_{\alpha'}) = 0\), \(g(e_\alpha, e_{-}) = 1\) and \(g(e_{\alpha'}, e_\beta) = \delta_{ab}\). On overlaps, such frames are related by local G-transformations, where G is the subgroup of the Lorentz group of V which preserves \(e_\alpha\). Explicitly,

\[
G = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}v^T \nu & \nu^T \\ 0 & 1 & 0 \\ 0 & \nu & A \end{pmatrix} \middle| v \in \mathbb{R}^{n-1}, A \in O(n-1) \right\} < GL(n+1, \mathbb{R}),
\]

with Lie algebra

\[
g = \left\{ \begin{pmatrix} 0 & 0 & \nu^T \\ 0 & 0 & 0 \\ 0 & \nu & A \end{pmatrix} \middle| v \in \mathbb{R}^{n-1}, A \in so(n-1) \right\} < gl(n+1, \mathbb{R}).
\]

Let us choose a Witt basis \((Z, H, P_a)\) for V with lorentzian inner product \(\gamma \in \otimes^2 V^*\) given by \(\gamma(Z, H) = 1\) and \(\gamma(P_a, P_{a'}) = \delta_{ab}\), and all other inner products not related to these by symmetry vanishing. The canonical dual basis for \(V^*\) will be denoted \((\zeta, \eta, \pi^a)\). We may choose basis \(J_{ab}, B_a\) for \(g\) with brackets given by equation (3.3). Indeed, the Lie algebras \(g\) in the galilean (and carrollian) and bargmannian cases are abstractly isomorphic, but whereas the galilean algebra \(g < gl(n+1, \mathbb{R})\), the bargmannian algebra \(g < gl(n+1, \mathbb{R})\).

The G-modules V and \(V^*\) are isomorphic: they are indecomposable, but not irreducible.

**Lemma 13.** There are \(G\)-invariant filtrations

\[
0 \subset \langle Z \rangle \subset Z^\perp \subset V \quad \text{and} \quad 0 \subset \langle \eta \rangle \subset \text{Ann} \, Z \subset V^*,
\]

where \(Z^\perp = \langle Z, P_a \rangle\) and \(\text{Ann} \, Z = \langle \eta, \pi^a \rangle\).
Proof. This follows from the explicit actions of $g$ on $V$ and $V^*$:

$$
\begin{align*}
J_{ab} \cdot P_c &= \delta_{bc} P_a - \delta_{ac} P_b \\
J_{ab} \cdot H &= 0 \\
J_{ab} \cdot Z &= 0 \\
B_a \cdot P_b &= \delta_{ab} Z \\
B_a \cdot H &= P_a \\
B_a \cdot Z &= 0
\end{align*}
$$

and

$$
\begin{align*}
J_{ab} \cdot \pi^c &= (-\delta^c_a \delta_{bd} + \delta^c_b \delta_{ad}) \pi^d \\
J_{ab} \cdot \eta &= 0 \\
J_{ab} \cdot \zeta &= 0 \\
B_a \cdot \pi^b &= -\delta^b_a \eta \\
B_a \cdot \eta &= 0 \\
B_a \cdot \zeta &= -\delta^b_{ab} \pi^b.
\end{align*}
$$

$\square$

6.2. The intrinsic torsion of a bargmannian structure. Since $g \subset so(V)$, Lemma 2 says that the Spencer differential $\partial : g \otimes V^* \to V \otimes \wedge^2 V^*$ is injective.

**Proposition 14.** As $G$-modules, $\ker \partial \cong Z^+ \otimes V^*$.

**Proof.** Since $\ker \partial = 0$, it is enough to exhibit a short exact sequence of $G$-modules

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{Hom}(V, g) & \longrightarrow \\
& & \delta & \longrightarrow \\
& & \text{Hom}(\wedge^2 V, V) & \longrightarrow \\
& & \lambda & \longrightarrow \\
0 & \longrightarrow & \text{Hom}(V, Z^+) & \longrightarrow 0,
\end{array}
$$

(6.3)

for some $G$-equivariant map $\lambda : \text{Hom}(\wedge^2 V, V) \to \text{Hom}(V, Z^+)$, sending $\varphi \mapsto \lambda_\varphi$. Let us define $\lambda_\varphi$ by

$$
2\gamma(\lambda_\varphi(v), w) := \gamma(\varphi(v, w), v) + \gamma(\varphi(v, w), v) + \gamma(\varphi(v), w), Z).
$$

(6.4)

Putting $w = Z$ and using the skew-symmetry of $\varphi$, we see that $\gamma(\lambda_\varphi(v), Z) = 0$ for all $v$. Therefore $\lambda_\varphi : V \to Z^+$, as desired. The map $\lambda$ is $G$-equivariant since it is constructed out of $\gamma$ and $Z$, which are $G$-invariant. It remains to show that $\ker \varphi = \ker \lambda$.

Let us first show that $\ker \varphi \subset \ker \lambda$. Suppose that $\varphi = \varphi_\varphi$, so that $\varphi(u, v) = \kappa(u) - \kappa(v)$. Then

$$
2\gamma(\lambda_\varphi(v), w) = \gamma(\kappa_Z v - \kappa_Z Z, w) + \gamma(\kappa_Z w - \kappa_w Z, v) + \gamma(\kappa_v w - \kappa_v v, Z).
$$

(6.5)

Since $\kappa_v \in so(V)$ for all $v \in V$, we have that $\gamma(\kappa_v u, w) = -\gamma(\kappa_v w, u)$ for all $v, w \in V$, and using this we can show that the terms in the RHS cancel pairwise.

Conversely, let $\varphi \in \ker \lambda$, so that

$$
\gamma(\varphi(v), w) + \gamma(\varphi(v, w), v) + \gamma(\varphi(v), \text{w}, Z) = 0 \quad \forall v, w \in V.
$$

Notice that the first two terms are symmetric in $(v, w)$, whereas the third term is skew-symmetric, so that both terms are zero separately. From Lemma 2, we know that there exists a unique $\kappa : V \to so(V)$ such that $\varphi = \partial \kappa$. We claim that if $\lambda_\varphi = 0$, then $\kappa$ actually maps to $g$. Write $\varphi(v, w) = \kappa(v) w - \kappa(w) v$ with $\kappa : V \to so(V)$ and insert this expression into the symmetric and skew-symmetric components of equation (6.5). The skew-symmetric component gives

$$
\gamma(\kappa w - \kappa_v v, Z) = 0
$$

and the symmetric component gives

$$
\gamma(\kappa_Z v - \kappa_Z Z, w) + \gamma(\kappa_Z w - \kappa_w Z, v) = 0 \iff \gamma(\kappa w + \kappa_v v, Z) = 0.
$$

Adding the two equations we see that $\gamma(\kappa, Z) = 0$ for all $v, w \in V$, which is equivalent to $\gamma(\kappa, Z) = 0$ for all $v, w \in V$. Since $\gamma$ is nondegenerate, this says $\kappa, Z = 0$ for all $v \in V$ and hence $\kappa : V \to g$ as desired.

We must now determine the $G$-submodules of $Z^+ \otimes V^*$ and hence the $G$-submodules of $\ker \partial$. Our strategy is to first decompose $Z^+ \otimes V^*$ into irreducible $so(n-1)$-modules and then to see how the bargmannian boosts $B_a$ act on them. First of all, $Z^+ \otimes V^* = \langle Z, P_a \rangle \otimes \langle \zeta, \eta, \pi^b \rangle$, resulting in the following $so(n-1)$-submodules:

$$
\langle Z \otimes \zeta \rangle, \quad \langle Z \otimes \eta \rangle, \quad \langle Z \otimes \pi^b \rangle, \quad \langle P_a \otimes \zeta \rangle, \quad \langle P_a \otimes \eta \rangle \quad \text{and} \quad \langle P_a \otimes \pi^b \rangle.
$$

(6.6)

All submodules but the last are irreducible. The last submodule breaks up into three$^7$ irreducible submodules:

$$
\langle P_a \otimes \pi^b \rangle = \langle P_a \otimes \pi^b \rangle_{\lambda_2} \oplus \langle P_a \otimes \pi^b \rangle_{\lambda_0} \oplus \langle P_a \otimes \pi^b \rangle_{\lambda_1},
$$

(6.7)

$^7$except for $n = 5$, in which case $\langle P_a \otimes \pi^b \rangle_{\lambda_2}$ breaks up further into selfdual and antiselfdual pieces. That case will be treated separately in Appendix B.5.
where
\[
\begin{align*}
\langle P_a \otimes \pi^b \rangle_{\wedge 2} &= \langle (\delta_{bc} P_a - \delta_{ac} P_b) \otimes \pi^c \rangle \\
\langle P_a \otimes \pi^b \rangle_{\otimes \delta} &= \langle (\delta_{bc} P_a + \delta_{ac} P_b - \frac{2}{n-1} \delta_{ab} P_c) \otimes \pi^c \rangle \\
\langle P_a \otimes \pi^b \rangle_{\text{tr}} &= \langle P_a \otimes \pi^a \rangle.
\end{align*}
\]

(6.8)

The action of the boosts are given by
\[
\begin{align*}
B_c \cdot (Z \otimes \pi^a) &= -\delta_c^a Z \otimes \eta \\
B_c \cdot (Z \otimes \eta) &= 0 \\
B_c \cdot (Z \otimes \zeta) &= -\delta_{ac} Z \otimes \pi^a \\
B_c \cdot (P_a \otimes \eta) &= \delta_{ac} Z \otimes \eta \\
B_c \cdot (P_a \otimes \zeta) &= \delta_{ac} Z \otimes \zeta - \delta_{cd} P_a \otimes \pi^d \\
B_c \cdot (P_a \otimes \pi^b) &= \delta_{ac} Z \otimes \pi^b - \delta_b^a P_a \otimes \eta.
\end{align*}
\]

(6.9)

Projecting the last term into its three irreducible components,
\[
B_c \cdot (P_a \otimes \pi^a) = \delta_{ac} Z \otimes \pi^a - P_c \otimes \eta
\]
\[
B_c \cdot (\delta_{bd} P_a + \delta_{ad} P_b - \frac{2}{n-1} \delta_{ab} P_d) \otimes \pi^d = (\delta_{bc} \delta_{bd} + \delta_{bc} \delta_{ad} - \frac{2}{n-1} \delta_{ab} \delta_{cd}) Z \otimes \pi^d \\
- (\delta_{bc} P_a + \delta_{ac} P_b - \frac{2}{n-1} \delta_{ab} P_c) \otimes \eta
\]
\[
B_c \cdot (\delta_{bd} P_a - \delta_{ad} P_b) \otimes \pi^d = (\delta_{bc} \delta_{bd} - \delta_{bc} \delta_{ad}) Z \otimes \pi^d - (\delta_{bc} P_a - \delta_{ac} P_b) \otimes \eta.
\]

(6.10)

We observe that \( Z \otimes \pi^a + \delta_{ab} P_b \otimes \eta \) is boost-invariant, and hence we introduce
\[
\Xi^a_{\pm} := Z \otimes \pi^a \pm \delta_{ab} P_b \otimes \eta,
\]
so that
\[
B_c \cdot \Xi^a_{\pm} = 0 \quad \text{and} \quad B_c \cdot \Xi^a = -2 \delta_c^a Z \otimes \eta.
\]

(6.11)

(6.12)

We rewrite
\[
Z \otimes \pi^a = \frac{1}{2}(\Xi^a_+ + \Xi^a_-) \quad \text{and} \quad P_a \otimes \eta = \frac{1}{2} \delta_{ab}(\Xi^b_+ - \Xi^b_-),
\]

(6.13)

in terms of which
\[
\begin{align*}
B_c \cdot (P_a \otimes \pi^a) &= \delta_{cd} \Xi^d_+ \\
B_c \cdot (\delta_{bd} P_a - \delta_{ad} P_b) \otimes \pi^d &= (\delta_{bc} \delta_{bd} - \delta_{bc} \delta_{ad}) \Xi^d_+ \\
B_c \cdot (\delta_{bd} P_a + \delta_{ad} P_b - \frac{2}{n-1} \delta_{ab} P_d) \otimes \pi^d &= (\delta_{bd} \delta_{ac} + \delta_{ad} \delta_{bc} - \frac{2}{n-1} \delta_{ab} \delta_{cd}) \Xi^d_+.
\end{align*}
\]

(6.14)

We may summarise this graphically as in Figure 1, where we have used self-explanatory abbreviations for the irreducible \( so(n-1) \)-submodules and where the arrows represent the action of the boosts \( B_a \).

![Figure 1. Action of boosts on \( so(n-1) \)-submodules of \( Z^+ \otimes V^* \)](image)
It is possible now to list the \( g \)-submodules of \( Z^\perp \otimes V^* \) from Figure 1: if a certain \( \mathfrak{so}(n-1) \)-submodule appears, then all other \( \mathfrak{so}(n-1) \)-submodules which can be reached from it following the arrows in the diagram must appear as well. This process yields twenty-seven \( g \)-submodules, which we process to list below in abbreviated form. The reason for the primes is that these are the submodules of \( Z^\perp \otimes V^* \) and we are eventually interested in the submodules of \( \text{coker } \partial \).

- \( B'_0 = 0 \)
- \( B'_1 = \langle Z \otimes \eta \rangle \)
- \( B'_2 = \langle \Xi_+ \rangle \)
- \( B'_3 = \langle Z \otimes \eta, \Xi_+ \rangle \)
- \( B'_4 = \langle Z \otimes \eta, \Xi_- \rangle \)
- \( B'_5 = \langle \Xi_+, \langle P \oplus \pi \rangle_\Lambda^2 \rangle \)
- \( B'_6 = \langle Z \otimes \eta, \Xi_+, \langle P \oplus \pi \rangle_\Theta \rangle \)
- \( B'_7 = \langle Z \otimes \eta, \Xi_-, \langle P \oplus \pi \rangle_\Theta \rangle \)
- \( B'_8 = \langle Z \otimes \eta, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_9 = \langle Z \otimes \eta, \Xi_+, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{10} = \langle Z \otimes \eta, \Xi_-, \langle P \oplus \pi \rangle_\Theta \rangle \)
- \( B'_{11} = \langle Z \otimes \eta, \Xi_+, \langle P \oplus \pi \rangle_\Theta \rangle \)
- \( B'_{12} = \langle Z \otimes \eta, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{13} = \langle Z \otimes \eta, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{14} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{15} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{16} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_\Theta \rangle \)
- \( B'_{17} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_\Theta \rangle \)
- \( B'_{18} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{19} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{20} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{21} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{22} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{23} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{24} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{25} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)
- \( B'_{26} = \langle Z \otimes \eta, \Xi_+, \Xi_-, \langle P \oplus \pi \rangle_{\Lambda^2} \rangle \)

Next, we exhibit the dictionary between the \( \mathfrak{so}(n-1) \)-submodules of \( Z^\perp \otimes V^* \) and those of \( \text{coker } \partial \), which we list in abbreviated form:

| \( Z^\perp \otimes V^* \) | \( \text{coker } \partial \) |
|-----------------|-----------------|
| \( \langle Z \otimes \eta \rangle \) | \( \langle Z \otimes \eta \wedge \zeta \rangle \) |
| \( \langle \Xi_+ \rangle \) | \( \langle \eta \otimes \pi \wedge \eta \rangle \) |
| \( \langle \Xi_- \rangle \) | \( \langle Z \otimes \pi - P \otimes \eta \rangle \wedge \zeta \) |
| \( \langle P \otimes \pi \rangle_{\Lambda^2} \) | \( \langle H \otimes \pi \wedge \pi \rangle \) |
| \( \langle Z \otimes \zeta \rangle \) | \( \langle H \otimes \eta \wedge \zeta \rangle \) |
| \( \langle P \otimes \pi \rangle_{\Theta} \) | \( \langle P \otimes \pi \rangle_{\Theta} \wedge \zeta \) |
| \( \langle P \otimes \pi \rangle_{\Theta} \) | \( \langle P \otimes \pi \rangle_{\Theta} \wedge \zeta \) |
| \( \langle P \otimes \pi \rangle_{\Theta} \) | \( \langle P \otimes \pi \rangle_{\Theta} \wedge \zeta \) |
| \( \langle P \otimes \pi \rangle_{\Theta} \) | \( \langle P \otimes \pi \rangle_{\Theta} \wedge \zeta \) |

One can use this dictionary to read off the \( g \)-submodules of \( \text{coker } \partial \) from the ones of \( Z^\perp \otimes V^* \) listed above and in this way set up a correspondence between \( B'_i \subset Z^\perp \otimes V^* \) and \( \mathcal{B}_i \subset \text{coker } \partial \) for \( i = 0, 1, \ldots, 26 \). The set of twenty-seven \( g \)-submodules of \( \text{coker } \partial \) is partially ordered by inclusion. Figure 2 illustrates the Hasse diagram of this poset. The node labelled \( i \) corresponds to the submodule \( \mathcal{B}_i \subset \text{coker } \partial \) (or \( B'_i \subset Z^\perp \otimes V^* \)) and an arrow indicates inclusion. The meaning of the labels is explained in Section 6.3.

In summary, we see that there are twenty-seven classes of bargmannian \( G \)-structures, which we will characterise geometrically in the next section.
induces a bundle map \(X\) with the covariant derivative \(\nabla\). The Levi-Civita connection is not adapted unless \(\nabla g = 0\) so that \(g\) is a Brinkmann metric (a.k.a. a generalised pp-wave). Nevertheless it is an intrinsic object in \((M, g)\) and therefore \(\nabla g \in \Omega^1(M, TM)\) is intrinsic to the G-structure. Since \(g(\xi, \xi) = 0\) and \(\nabla g\) is metric, we have that \(g(\nabla^g_\xi \xi, \xi) = 0\) for all \(X \in \mathfrak{X}(M)\). Therefore, \(\nabla g \xi : TM \to \xi^\perp\) or, equivalently, \(\nabla g \xi\) is a section of the associated vector bundle \(P \times_G (\xi^\perp \otimes V^*)\), which as we saw before is isomorphic to \(P \times_G \coker \partial\). The next Proposition shows that this is not a coincidence.

The map \(\lambda : V \otimes \wedge^2 V^* \to \xi^\perp \otimes V^*\) in the proof of Proposition 14 induces a bundle map \(TM \otimes \wedge^2 T^*M \to \xi^\perp \otimes T^*M\) and hence a \(C^\infty(M)\)-linear map
\[
\Lambda : \Omega^2(M, TM) \to \Omega^1(M, \xi^\perp), \quad T \mapsto \Lambda_T, \tag{6.15}
\]
defined, for all \(T \in \Omega^2(M, TM)\) and \(X, Y \in \mathfrak{X}(M)\), by
\[
2g(\Lambda_T(X, Y)) := g(T(\xi, X), Y) + g(T(\xi, Y), X) + g(T(X, Y), \xi). \tag{6.16}
\]

**Proposition 15.** Let \(\nabla\) be an adapted connection to a bargmannian G-structure with torsion \(T^\nabla\). Then \(\Lambda_{T^\nabla} = \nabla^g \xi\).

**Proof.** Let us denote by \(\kappa \in \Omega^1(M, so(TM))\) the contorsion \(\kappa := \nabla - \nabla^g\). It takes values in \(so(TM)\), the bundle of skew-symmetric endomorphisms of \(TM\), because both connections are metric-compatible. Since \(\nabla^g\) has zero torsion, \(T^\nabla = \partial \kappa\) and since \(\nabla\) is adapted,
\[
0 = \nabla_X \xi = \nabla^g_X \xi + \kappa_X \xi, \quad \Rightarrow \quad \kappa_X \xi = -\nabla^g_X \xi.
\]

We calculate for \(X, Y \in \mathfrak{X}(M)\),
\[
2g(\Lambda_{T^\nabla}(X, Y)) = g(T^\nabla(\xi, X), Y) + g(T^\nabla(\xi, Y), X) + g(T^\nabla(X, Y), \xi)
= g(\kappa \xi X - \kappa \xi Y, Y) + g(\kappa \xi Y - \kappa \xi X, X) + g(\kappa \xi Y - \kappa \xi X, \xi)
= -g(\kappa \xi Y, Y) - g(\kappa \xi Y, X) + g(\kappa \xi Y, \xi) + g(\kappa \xi X, Y) + g(\kappa \xi X, Y) \quad \text{(using} \kappa_X \in so(TM))
= -2g(\kappa \xi X, Y).
\]

Hence \(\Lambda_{T^\nabla}(X) = -\kappa \xi X = \nabla^g_X \xi\), as claimed. \(\square\)
6.3. Geometric characterisation of bargmannian structures. Let \((M^{n+1}, g, \xi)\) be a bargmannian manifold; that is, \((M, g)\) is an \((n+1)\)-dimensional lorentzian manifold and \(\xi \in \mathfrak{X}(M)\) is a nowhere-vanishing null vector. Let \(\xi^\perp \in \Omega^1(M)\) denote the one-form dual to \(\xi\); \(\xi^\perp(X) = g(\xi, X)\) for all \(X \in \mathfrak{X}(M)\). Let \(\nu\) denote the (possibly only locally defined) volume form. If we assume that \(M\) is orientable (e.g., if it is simply-connected) then \(\nu \in \Omega^{n+1}(M)\) defines an orientation; that is, a nowhere-vanishing top form.

Let \(\xi^\perp = \ker \xi^\ast\) denote the characteristic distribution consisting of tangent vectors perpendicular to \(\xi\). Since \(\xi\) is null, \(\xi\) belongs to the distribution and hence the restriction of the metric to \(\xi^\perp\) is degenerate. If the distribution is involutive \((\xi^\perp, \xi^\perp) \subset \xi^\perp\), which is equivalent to \(\xi^\perp \wedge d\xi^\perp = 0\), then \(M\) is foliated by null hypersurfaces \(N\) whose tangent space \(T_pN\) at \(p\) coincides with \(\xi^\perp_p \subset T_pM\). There is a well established theory of null hypersurfaces (see, e.g., [11, 12]) from which we will borrow in this section. However since not all bargmannian structures are such that \(\xi^\perp\) is involutive, we will have to extend this theory slightly to non-involutive distributions.

It is not hard to see that in all bargmannian structures but the generic one \((B_{20})\), the distribution is \(\xi\)-invariant; that is, for all \(X \in \Gamma(\xi^\perp)\), \([\xi, X] \in \Gamma(\xi^\perp)\), which we abbreviate by \([\xi, \xi^\perp] \subset \xi^\perp\), with some abuse of notation. We may also write \(X \perp \xi\) for \(X \in \Gamma(\xi^\perp)\).

We recall that \(\xi\) is said to be geometric if \(\nabla^g_\xi \in \mathfrak{X}\), for some \(f \in C^\infty(M)\). The name is apt, because integral curves of \(\xi\) can be parametrised in such a way that they satisfy the geodesic equation.

**Lemma 16.** Let \(\xi^\perp = \ker \xi^\ast\). Then \([\xi, \xi^\perp] \subset \xi^\perp\) if and only if \(\xi\) is geometric.

**Proof.** We have that \(\xi\) is geometric if and only if \(g(\nabla^g_\xi \xi, X) = 0\) for all \(X \perp \xi\), and then

\[
g(\nabla^g_\xi \xi, X) = 0 \iff g(\xi, \nabla^g_\xi X) = 0
\]

\[
\iff g(\xi, [\xi, X] + \nabla^\xi_\xi X) = 0 \quad \text{(since } \nabla^g \text{ has zero torsion)}
\]

\[
\iff g(\xi, [\xi, X]) = 0 \quad \text{(since } g(\xi, \nabla^\xi_\xi X) = 0 \text{ for any } X)
\]

\[
\iff [\xi, X] \perp \xi.
\]

\[\Box\]

From now on we will assume that \([\xi, \xi^\perp] \subset \xi^\perp\); that is, we are dealing with any one but the generic bargmannian structure.

Let \(L \subset \xi^\perp\) denote the line sub-bundle spanned by \(\xi\) and let \(E := \xi^\perp / L\) denote the quotient vector bundle: it is a corank-2 vector bundle over \(M\). If \(X \perp \xi\), we let \(X \in \Gamma(E)\) denote its equivalence class modulo \(L\); that is, \(X, Y \perp \xi\) satisfy \(X = Y\) if and only if \(X = fY\) for some \(f \in C^\infty(M)\). On \(E\) we have a positive-definite metric \(h\) defined by

\[
h(\overline{X}, \overline{Y}) := g(X(Y), Y) \quad \forall X, Y \perp \xi. \tag{6.17}
\]

This is well-defined on equivalence classes precisely because \(X, Y \perp \xi\) and it is positive-definite because \((M, g)\) is lorentzian. This makes \((E, h)\) into a corank-2 riemannian vector bundle over \(M\).

Following [12] we define the null Weingarten map \(W : \Gamma(E) \to \Gamma(E)\) by

\[
W(\overline{X}) := \overline{\nabla^h_X \xi}. \tag{6.18}
\]

Although in [12] this map is shown to be well-defined for the case of involutive \(\xi^\perp\), it turns out that it is well-defined under the weaker hypothesis that \([\xi, \xi^\perp] \subset \xi^\perp\). Indeed, let \(\overline{X} = \overline{\nabla_\xi} Y\) and calculate

\[
W(\overline{X}) - W(\overline{Y}) = \overline{\nabla_X \xi} - \overline{\nabla_Y \xi} = \overline{\nabla_X - Y} \xi = \overline{\nabla_{[X, Y]} \xi} = \overline{\nabla_{\xi} \xi} = 0,
\]

where we have used that \(\xi\) is geodetic, which as shown in Lemma 16 follows by virtue of \([\xi, \xi^\perp] \subset \xi^\perp\).

We define the null second fundamental form \(B \in \Gamma(E^* \otimes E^*)\) by

\[
B(\overline{X}, \overline{Y}) := h(W(\overline{X}), \overline{Y}) = g(\nabla^g_\xi \xi, Y). \tag{6.19}
\]

We see that this is well-defined because both \(h\) and \(W\) are well-defined. In contrast to the case of a null hypersurface, the second fundamental form of \(\xi^\perp\) need not be symmetric.

**Lemma 17.** The null second fundamental form \(B\) is symmetric if and only if \(\xi^\perp\) is involutive.
Proof. We calculate
\[ B(\tilde{X}, \tilde{Y}) - B(\tilde{Y}, \tilde{X}) = g(\nabla_Y^g \xi, Y) - g(\nabla_X^g \xi, X) \]
\[ = -g(\xi, \nabla_Y^g Y) + g(\xi, \nabla_X^g X) \quad \text{(since } \nabla^g g = 0 \text{ and } X, Y \perp \xi) \]
\[ = -g(\xi, [X, Y]) \quad \text{(since } \nabla^g \text{ has zero torsion)} \]
and hence \( B \) is symmetric if and only if \([X, Y] \perp \xi\) for all \( X, Y \perp \xi \); that is, if and only if \([\xi^\perp, \xi^\perp] \subset \xi^\perp \). \( \square \)

Let \( B_{\text{sym}} \) denote (twice) the symmetric part of \( B \):
\[ B_{\text{sym}}(\tilde{X}, \tilde{Y}) := B(\tilde{X}, \tilde{Y}) + B(\tilde{Y}, \tilde{X}). \] (6.20)

Lemma 18. As sections of \( \odot^2 \mathbb{E}^* \), \( B_{\text{sym}} = \mathcal{L}_\xi h \).

Proof. Let \( X, Y \perp \xi \). Then
\[ (\mathcal{L}_\xi h)(\tilde{X}, \tilde{Y}) = (\mathcal{L}_\xi g)(X, Y) \]
\[ = g(\nabla_Y^g \xi, Y) + g(\nabla_X^g \xi, X) \]
\[ = B(\tilde{X}, \tilde{Y}) + B(\tilde{Y}, \tilde{X}). \]

\( \square \)

Definition 19. Let \( \xi^\perp = \ker \xi \) be \( \xi \)-invariant, so that \([\xi, \xi^\perp] \subset \xi^\perp \). We say that \( \xi^\perp \) is
\begin{itemize}
  \item totally geodesic if \( B_{\text{sym}} = 0 \);
  \item minimal if \( \text{tr } B_{\text{sym}} = 0 \); and
  \item totally umbilical if \( B_{\text{sym}} = fh \) for some \( f \in C^\infty(\mathcal{M}) \).
\end{itemize}

Of course, if \( \xi^\perp \) is involutive, then \( B_{\text{sym}} = B \) and hence these are the natural extension to null hypersurfaces of the well-known concepts for hypersurfaces of riemannian manifolds. Even if \( \xi^\perp \) is not involutive, the condition of being totally geodesic simply says that any lorentzian geodesic whose initial velocity belongs to \( \xi^\perp \) is such that its velocity remains in \( \xi^\perp \). Indeed, suppose that \( \gamma \) is a geodesic with \( \gamma(0) = p \) and \( \dot{\gamma}(0) \in \xi^\perp_p \). Consider the function of \( t \) defined by \( t \mapsto g(\dot{\gamma}(t), \xi(\gamma(t))) \). This function vanishes at \( t = 0 \) because \( \gamma(0) \in \xi^\perp_p \). Differentiating with respect to \( t \), we obtain
\[ \frac{d}{dt} g(\dot{\gamma}, \xi) = g(\frac{D}{dt} \dot{\gamma}, \xi) + g(\dot{\gamma}, \nabla^g_\dot{\gamma} \xi) \]
\[ = g(\dot{\gamma}, \nabla^g_\dot{\gamma} \xi) \quad \text{(since } \dot{\gamma} \text{ is a geodesic)} \]
\[ = B(\dot{\gamma}, \xi). \]

By polarisation, this vanishes for all \( \dot{\gamma} \) if and only if \( B \) is skew-symmetric. If (and only if) that is the case, then \( g(\dot{\gamma}, \xi) = 0 \) for all \( t \).

Before we go on to characterise geometerically the different bargmannian structures, let us observe that many of the calculations already done in the galilean, carrollian and aristotelian sections imply some results also for bargmannian structures.

If \( \nabla \) is an adapted connection, then \( \nabla g = 0, \nabla \xi = 0, \nabla \xi^\perp = 0 \) and \( \nabla \sigma = 0 \). Let \( T^\xi \in \Omega^2(\mathcal{M}, \mathcal{T}\mathcal{M}) \) denote its torsion. We define \( S \in \Omega^1(\mathcal{M}, \mathcal{T}\mathcal{M}) \) by \( S(X) := T^\xi(\tilde{X}, \xi) \) for all \( X \in \mathcal{X}(\mathcal{M}) \) and \( \Sigma \in \Gamma(\odot^2 \mathcal{T}\mathcal{M}) \) by \( \Sigma(X, Y) := g(S(X, Y) + g(S(Y), X) \). The following result follows from the calculations already done in the galilean, carrollian and aristotelian sections.

Corollary 20. With the notation of the previous paragraph, the following identities hold:
\[ d\xi^\perp = \xi^\perp \circ T^\xi, \quad \mathcal{L}_\xi g = \Sigma, \quad \mathcal{L}_\xi \sigma = \text{tr}(S) \nu \quad \text{and} \quad \mathcal{L}_\xi \xi^\perp = \xi^\perp \circ S. \] (6.21)

Proof. The proof of Proposition 5 shows that \( d\xi^\perp = \xi^\perp \circ T^\xi \), whereas the proof of Proposition 8 shows that \( \mathcal{L}_\xi g = \Sigma \) and that of Proposition 9 shows that \( \mathcal{L}_\xi \sigma = \text{tr}(S) \nu \). Finally, the proof of Proposition 11 shows that \( \mathcal{L}_\xi \xi^\perp = \xi^\perp \circ S \). \( \square \)

It follows from this corollary, from Proposition 15 and from the explicit form of the map \( \Lambda \) in equation (6.16) that
\[ 2g(\nabla^g_X \xi, Y) = \Sigma(X, Y) + d\xi^\perp(X, Y), \quad \forall X, Y \in \mathcal{X}(\mathcal{M}). \] (6.22)

In other words, \( \Sigma \) and \( d\xi^\perp \) are the extensions to TM of the symmetric and skew-symmetric parts of the second fundamental form of the distribution \( \xi^\perp \), respectively. In particular, it should be emphasised that the skew-symmetric component of \( S \) is not related to the skew-symmetric component of the second fundamental form.

In order to make full use of these results in the geometric characterisation of bargmannian structures, we should first determine which \( \sigma(n - 1) \)-submodules of \( \text{coker } \delta \) (or of \( \mathcal{Z}^\perp \odot V^* \)) contribute to which
geometric data. This can be read off by inspection and the results are collected in Table 1, which turns out to be quite handy. In that table, a • indicates that the submodule does contribute and ○ indicates that it does not.

| \( \mathfrak{so}(n-1) \)-submodule of \( Z^\perp \otimes V^* \) | \( \text{coker } \partial \) | \( \Sigma \) | \( S_{\Lambda^2} \) | \( S_{\otimes^3} \) | \( \Sigma \circ S \) | \( \xi^3 \otimes \xi^2 \) | \( \xi^2 \wedge \xi^3 \) | \( B_{\Lambda^2} \) | \( B_{\otimes^3} \) | \( \text{tr}(B) \) |
|----------------|----------------|-------|------|--------|--------|--------|--------|--------|--------|-------|
| \( \langle Z \otimes \eta \rangle \) | \( Z \otimes \eta \wedge \zeta \) | ○     | •     | ○      | ○      | ○      | ○      | ○      | ○      | ○     |
| \( \langle Z \otimes \pi + P \otimes \eta \rangle \) | \( (H \otimes \pi \wedge \zeta) \) | ●     | ●     | ○      | ○      | ●      | ○      | ○      | ○      | ○     |
| \( \langle Z \otimes \pi - P \otimes \eta \rangle \) | \( (Z \otimes \pi - P \otimes \eta) \wedge \zeta \) | ●     | ●     | ○      | ○      | ○      | ○      | ○      | ○      | ○     |
| \( \langle (P \otimes \pi)_{\otimes^2} \rangle \) | \( (H \otimes \pi \wedge \zeta) \) | ○     | ○     | ○      | ○      | ○      | ○      | ○      | ○      | ○     |
| \( \langle (P \otimes \pi)_{\otimes^2} \rangle \) | \( (H \otimes \pi \wedge \zeta) \) | ○     | ○     | ○      | ○      | ○      | ○      | ○      | ○      | ○     |
| \( \langle (P \otimes \pi)_{\otimes^2} \rangle \) | \( (H \otimes \pi \wedge \zeta) \) | ○     | ○     | ○      | ○      | ○      | ○      | ○      | ○      | ○     |

Table 1. Contributions of each \( \mathfrak{so}(n-1) \)-submodule of \( Z^\perp \otimes V^* \cong \text{coker } \partial \) (• contributes and ○ does not)

6.3.1. Totally geodesic bargmannian structures. We start with those bargmannian structures which are totally geodesic; that is, for which the symmetric part of the second fundamental form vanishes. From Table 1 we see that only the \( \mathfrak{so}(n-1) \)-submodules \( \langle (P \otimes \pi)_{\otimes^2} \rangle \cong \langle (P \otimes \pi)_{\otimes^2} \wedge \zeta \rangle \) and \( \langle P \otimes \pi \rangle \cong \langle P \otimes \pi \rangle_{\otimes^2} \) contribute to \( \mathcal{B}_{\text{sym}} \). Therefore these modules cannot be present in the \( \mathcal{B}_1 \). There are precisely eleven bargmannian structures not containing any of these \( \mathfrak{so}(n-1) \)-submodules: \( \mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_9, \mathcal{B}_{13}, \mathcal{B}_{14} \) and \( \mathcal{B}_{15} \), which are depicted in Figure 3.

They can be distinguished by the properties in Table 2. Notice that \( \nabla^g \xi \) is a one-form on \( M \) with values in the distribution \( \xi^\perp \). We can restrict \( \nabla^g \xi \) to \( \xi^\perp \) (resulting in the column \( \nabla^g \xi \) in the table). Alternatively we can quotient by (the line bundle associated to) \( \xi \) in order to define \( \nabla^g \xi \), which is a one-form on \( M \) with values in the quotient vector bundle \( \xi^\perp / \xi \). Structures \( \mathcal{B}_9 \) and \( \mathcal{B}_{13} \) can be distinguished by the fact that for \( \mathcal{B}_9, \nabla \xi = 0 \) for all \( X \perp \xi \), whereas this is not the case for \( \mathcal{B}_{13} \); but it would be better to distinguish them in a way which is manifestly independent of \( \nabla \).

| Structure | \( \nabla^g \xi \) | \( \nabla^g \xi \) | \( \mathcal{L}_\xi g \) | \( \xi^3 \wedge d\xi^3 \) | \( \nabla^g \xi \wedge \xi \) | \( \mathcal{L}_\xi^2 \xi^3 \) | \( \mathcal{L}_\xi^2 \) |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( \mathcal{B}_0 \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_1 \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_2 \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_3 \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_4 \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_5 \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_6 \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_9 \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_{13} \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_{14} \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |
| \( \mathcal{B}_{15} \) | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              | ✓              |

Table 2. Totally geodesic bargmannian structures (✓ means the expression vanishes)
6.3.2. Minimal bargmannian structures. We continue with those bargmannian structures which are minimal. From Table 1 it follows that such $B_i$ cannot contain the $\mathfrak{so}(n-1)$-modules $\langle (P \otimes \pi)_{\text{tr}} \rangle \cong \langle (P \otimes \pi)_{\text{tr}} \wedge \zeta \rangle$ nor $\langle (P \otimes \zeta) \rangle \cong \langle (H \otimes \pi \wedge \zeta) \rangle$ and must contain $\langle (P \otimes \pi)_{\text{哲}} \rangle \cong \langle (P \otimes \pi)_{\text{哲}} \wedge \zeta \rangle$. There are precisely five bargmannian structures satisfying these conditions: $B_6$, $B_{11}$, $B_{16}$, $B_{18}$ and $B_{21}$, which are depicted in Figure 4. They can be distinguished by the properties listed in Table 3.

| Structure | $d\xi^0$ | $\xi^0 \wedge d\xi^0$ | $\nabla^b_\zeta \xi^0$ | $\mathcal{L}_\xi \xi^0$ |
|-----------|----------|-------------------------|--------------------------|--------------------------|
| $B_6$     | ✓        | ✓                       | ✓                        | ✓                        |
| $B_{11}$  | ✓        | ✓                       | ✓                        | ✓                        |
| $B_{16}$  | ✓        | ✓                       | ✓                        | ✓                        |
| $B_{18}$  | ✓        | ✓                       | ✓                        | ✓                        |
| $B_{21}$  | ✓        | ✓                       | ✓                        | ✓                        |

Table 3. Minimal bargmannian structures ($\checkmark$ means the expression vanishes)

6.3.3. Totally umbilical bargmannian structures. We continue with those bargmannian structures which are totally umbilical. From Table 1 it follows that such $B_i$ cannot contain the $\mathfrak{so}(n-1)$-modules $\langle (P \otimes \pi)_{\text{哲}} \rangle \cong \langle (P \otimes \pi)_{\text{哲}} \wedge \zeta \rangle$ nor $\langle (P \otimes \zeta) \rangle \cong \langle (H \otimes \pi \wedge \zeta) \rangle$ and must contain $\langle (P \otimes \pi)_{\text{tr}} \rangle \cong \langle (P \otimes \pi)_{\text{tr}} \wedge \zeta \rangle$. There are precisely five bargmannian structures satisfying these conditions: $B_7$, $B_{12}$, $B_{17}$, $B_{19}$ and $B_{22}$, which are depicted in Figure 5. They can be distinguished by the properties listed in Table 4.

![Hasse diagram of totally geodesic bargmannian structures](image-url)
6.3.4. Other Bargmannian structures. We end with those Bargmannian structures which are neither totally geodesic, totally umbilical nor minimal, but still not generic. From Table 1 it follows that such $B_1$ cannot contain the $\mathfrak{so}(n-1)$-module $\langle P \otimes \zeta \rangle \cong (H \otimes \pi \wedge \zeta)$ and must contain both $\langle (P \otimes \pi)_{0g} \rangle \cong \langle (P \otimes \pi)_{0g} \wedge \zeta \rangle$ and $\langle (P \otimes \pi)_{1g} \rangle \cong \langle (P \otimes \pi)_{1g} \wedge \zeta \rangle$. There are precisely five Bargmannian structures satisfying these conditions: $B_{10}, B_{20}, B_{23}, B_{24}$ and $B_{25}$, which are depicted in Figure 6. They can be distinguished by the properties listed in Table 5.

We may summarise the preceding discussion as follows.

**Theorem 21.** Let $n > 2$ and $n \neq 5$. A Bargmannian $G$-structure on an $(n+1)$-dimensional manifold $(M, g, \xi)$ can be of twenty-seven different classes depending on its intrinsic torsion. These classes are summarised in Table 6, where each class is labelled by the smallest $G$-submodule of $\text{coker} \, \partial$ containing the intrinsic torsion and is characterised geometrically as indicated in the table.

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See Appendices B.2 for $n = 2$ and B.5 for $n = 5$. 

---

**Table 4.** Totally umbilical Bargmannian structures

| Structure | $d\xi^p$ | $\xi^p \wedge d\xi^p$ | $\nabla^g_\xi \xi$ | $\mathcal{L}_\xi \xi^p$ |
|-----------|-----------|-----------------------|------------------|------------------|
| $B_7$     | ✓         | ✓                     | ✓                | ✓                |
| $B_{12}$  | ✓         | ✓                     | ✓                | ✓                |
| $B_{17}$  | ✓         | ✓                     | ✓                | ✓                |
| $B_{19}$  | ✓         | ✓                     | ✓                | ✓                |
| $B_{22}$  | ✓         | ✓                     | ✓                | ✓                |

**Figure 4.** Hasse diagram of minimal Bargmannian structures
6.4. Correspondences between bargmannian, galilean and carrollian structures. As pioneered in [10], bargmannian structures may be related to galilean and carrollian structures and the interplay between these structures can prove to be very useful.

6.4.1. Bargmannian structures reducing to galilean structures. These are the bargmannian structures where $\xi$ is a Killing vector: $L_\xi g = 0$. Let us assume for the purposes of exposition that $\xi$ generates the action of a one-dimensional Lie group $\Gamma$ and we can perform the null reduction of the bargmannian structure as in [22, 24]. Indeed, we may view $M$ as the total space of a principal $\Gamma$-bundle $\pi: M \to N$ over an $n$-dimensional manifold $N = M/\Gamma$. The one-form $\xi^\flat$ is both horizontal (since $\xi$ is null) and invariant (since $\xi$ is Killing). Then $\xi^\flat = \pi^*\tau$ for a nowhere-vanishing one-form $\tau \in \Omega^1(N)$. If $\alpha, \beta \in \Omega^1(N)$, then $g((\pi^*\alpha)^\flat, (\pi^*\beta)^\flat)$, where $\pi: \Omega^1(M) \to \mathcal{X}(M)$ is one of the musical isomorphisms associated to $g$, is a $\Gamma$-invariant function on $M$ since so are $g$, $\pi^*\alpha$ and $\pi^*\beta$. We can define $\gamma \in \Gamma(\otimes^2TN)$ by $\gamma(\xi, \alpha) = g((\pi^*\alpha)^\flat, (\pi^*\beta)^\flat)$. Notice that $\gamma(\xi, \alpha) = 0$ since $(\pi^*\tau)^\flat = \xi$ and hence for all $\alpha \in \Omega^1(N)$,

$$g((\pi^*\tau)^\flat, (\pi^*\alpha)^\flat) = g(\xi, (\pi^*\alpha)^\flat) = (\pi^*\alpha)(\xi) = \pi^*(\alpha(\pi_*\xi)) = 0.$$  

(6.23)

It follows that $(N, \tau, \gamma)$ is a galilean structure and we may distinguish these bargmannian structures by which of the three galilean structures they induce.

---

**Figure 5.** Hasse diagram of totally umbilical bargmannian structures

| Structure | $d\xi^\flat$ | $\xi^\flat \wedge d\xi^\flat$ | $\nabla^g_\xi \xi$ | $L_\xi \xi^\flat$ |
|-----------|--------------|-----------------|-----------------|-----------------|
| $B_{10}$  | ✓            | ✓               | ✓               | ✓               |
| $B_{20}$  | ✓            | ✓               | ✓               | ✓               |
| $B_{23}$  | ✓            | ✓               | ✓               | ✓               |
| $B_{24}$  | ✓            | ✓               | ✓               | ✓               |
| $B_{25}$  | ✓            | ✓               | ✓               | ✓               |

**Table 5.** Other bargmannian structures

(✓ means the expression vanishes)
It turns out that there are precisely three bargmannian structures where \( \xi \) is Killing: \( B_0, B_2 \) and \( B_5 \), and they can be distinguished by the galilean structure induced on their null reductions.

(\( B_0 \)) Here \( \nabla^g \xi = 0 \) and hence \( g \) is a Brinkmann metric (i.e., a generalised pp-wave). Since \( \nabla^g \xi^\flat = 0 \), it follows that \( d\xi^\flat = 0 \) and hence the null reduction gives rise to a torsionless Newton–Cartan structure.

(\( B_2 \)) Here \( d\xi^\flat \neq 0 \) but \( \xi^\flat \wedge d\xi^\flat = 0 \), so that the null reduction gives a twistless torsionless Newton–Cartan structure.

(\( B_5 \)) Here \( \xi^\flat \wedge d\xi^\flat \neq 0 \), so that the null reduction gives a torsionless Newton–Cartan structure.

6.4.2. Bargmannian structures with embedded carrollian structures. As shown in [10] (see also [13]), a null hypersurface in a lorentzian manifold admits a carrollian structure. A bargmannian manifold \((M, g, \xi)\) where \( \xi^\flat \wedge d\xi^\flat = 0 \), is foliated by null hypersurfaces and we can relate the carrollian structure on the null hypersurfaces to the ambient bargmannian structure.

Lemma 22. If \( d\xi^\flat = 0 \), the vector field \( \xi \) is self-parallel relative to the Levi-Civita connection: \( \nabla^g_\xi \xi = 0 \), whereas if \( d\xi^\flat \neq 0 \) but \( \xi^\flat \wedge d\xi^\flat = 0 \), then \( \nabla^g_\xi \xi = f\xi \) for some nonzero function \( f \in C^\infty(M) \).

Proof. Let \( \xi^\flat \wedge d\xi^\flat = 0 \). Then by Proposition A.1, there exists some \( \alpha \in \Omega^1(M) \) such that \( d\xi^\flat = \alpha \wedge \xi^\flat \). The one-form \( \alpha \) is defined up to the addition of a one-form \( f\xi^\flat \) for some \( f \in C^\infty(M) \). If \( d\xi^\flat = 0 \) we can choose \( \alpha = 0 \).

For all \( X, Y \in \mathfrak{X}(M) \), the equation \( d\xi^\flat = \xi^\flat \wedge \alpha \) becomes

\[
Xg(\xi, Y) - Yg(\xi, X) - g(\xi, [X, Y]) = \alpha(X)g(\xi, Y) - g(\xi, X)\alpha(Y).
\]

Putting \( Y = \xi \) and using that \( g(\xi, \xi) = 0 \), we have that

\[
\xi g(\xi, X) + g(\xi, [X, \xi]) - g(\xi, X)\alpha(\xi) = 0.
\]

We use that \( \nabla^g \) is metric to expand the first term as

\[
\xi g(\xi, X) = g(\nabla^g_\xi \xi, X) + g(\xi, \nabla^g_X \xi),
\]

resulting in

\[
g(\nabla^g_\xi \xi - \alpha(\xi)\xi, X) + g(\xi, \nabla^g_X \xi + [X, \xi]) = 0.
\]
Using that $\nabla^g$ has zero torsion, $\nabla^g \xi + [X, \xi] =\nabla^g_X \xi$ and hence the second term becomes $g(\xi, \nabla^g_X \xi)$, which vanishes since this is half the derivative of $g(\xi, \xi)$ along $X$ and $\xi$ is null. This leaves the first term: since $g$ is nondegenerate and $X \in \mathfrak{X}(M)$ is arbitrary, we conclude that $\nabla^g_X \xi = \varphi(\xi) \xi$. It follows that if $d\xi^\perp = 0$ then $\nabla^g_X \xi = 0$, otherwise $\xi := \varphi(\xi)$ is not identically zero and hence $\nabla^g_X \xi = f\xi$.

It bears repeating that there is no converse to the above result: there are bargmannian structures with $\nabla^g_X \xi = 0$ for which $d\xi^\perp \neq 0$ and bargmannian structures with $\nabla^g_X \xi = f\xi$ for which $d\xi^\perp \neq 0$.

If $M$ is orientable, then since $\xi^\perp$ is null, we have that $\xi^\perp \wedge d\xi^\perp = 0$, where $\ast$ is the Hodge star. This says that $\ast \xi^\perp = \xi^\perp \wedge \mu$, for some $\mu \in \Omega^{n-1}(M)$ which is defined up to the addition of a term $\xi^\perp \wedge \varphi$ for some $\varphi \in \Omega^{n-2}(M)$. In particular, $\mu$ is well defined on the distribution $\xi^\perp$ and gives a “volume form” on the associated null hypersurfaces, which is precisely the volume form of the carrollian structure, when it exists. Even if $M$ is not orientable, $\mu$ exists locally.

**Proposition 23.** Let $(M, g, \xi)$ be a bargmannian structure with $\xi^\perp = \ker \xi^\perp$ involutive. Then any affine connection $\nabla$ on $M$ adapted to the bargmannian structure induces a connection on every leaf $N$ of $\xi^\perp$ which is adapted to the carrollian structure on $N$ and whose torsion is the restriction of $T^\xi$ to $N$.

**Proof.** Since $\xi$ and $g$ are parallel, it follows that so is $\xi^\perp$:

\[ \xi^\perp(YX) = X\xi^\perp(Y) \]

for all $X, Y \in \mathfrak{X}(M)$. In particular, if $Y \in \Gamma(\xi^\perp)$, so that $\nabla_X Y = 0$, then $X \in \mathfrak{X}(M)$. In other words, $\nabla$ induces a connection on the distribution $\xi^\perp$ or, equivalently, an affine connection on every leaf of the associated foliation. Since $\xi$ and $g$ are parallel, so are their restriction to the leaves of the foliation and hence the induced connection is adapted to the carrollian structure. Finally, notice that if $X, Y \in \Gamma(\xi^\perp)$, then

\[ T^\xi(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \in \Gamma(\xi^\perp), \]

(6.24)

| Structure | Type of $\xi^\perp$ | Geometrical characterisation |
|-----------|---------------------|-----------------------------|
| $B_0$     | totally geodesic    | $\nabla^g \xi^\perp = 0$    |
| $B_1$     | totally geodesic    | $\nabla^g \xi^\perp = 0$    |
| $B_2$     | totally geodesic    | $\nabla^g \xi^\perp = 0$    |
| $B_3$     | totally geodesic    | $\nabla^g \xi^\perp = 0$    |
| $B_4$     | totally geodesic    | $\nabla^g \xi^\perp = 0$    |
| $B_5$     | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_6$     | totally umbilical   | $\nabla^g \xi^\perp = 0$    |
| $B_7$     | totally umbilical   | $\nabla^g \xi^\perp = 0$    |
| $B_8$     | totally geodesic    | $\nabla^g \xi^\perp = 0$    |
| $B_9$     | totally geodesic    | $\nabla^g \xi^\perp = 0$    |
| $B_{10}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{11}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{12}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{13}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{14}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{15}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{16}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{17}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{18}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{19}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{20}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{21}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{22}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{23}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{24}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{25}$  | minimal             | $\nabla^g \xi^\perp = 0$    |
| $B_{26}$  | minimal             | $\nabla^g \xi^\perp = 0$    |

Table 6. Summary of bargmannian structures
where we have used that $\xi^\perp$ is involutive to show that $T^\nabla(X, Y) \in \nabla(\xi^\perp)$. Finally we notice that by definition, $T^\nabla$ given by equation (6.24) is the torsion of the induced connection. □

The different classes of null hypersurfaces can be distinguished by their second fundamental form. Explicitly, the condition $B = 0$ is equivalent to $L_\xi h = 0$:

$$g(\nabla^g_\xi Y, X) + g(\nabla^g_\xi X, Y) = 0, \quad \forall X, Y \perp \xi, \quad (6.25)$$

whereas the condition $B = fh$ is equivalent to $L_\xi h = fh$:

$$g(\nabla^g_\xi Y, X) + g(\nabla^g_\xi X, Y) = \frac{2}{n-1} g(X, Y) \div \xi \neq 0, \quad \forall X, Y \perp \xi, \quad (6.26)$$

where $\div \xi$ is the Levi-Civita divergence of $\xi$. Finally, the condition that $\text{tr} \ B = 0$ is equivalent to $L_\xi \mu = 0$.

Therefore we see that the type of carrollian structure induced on the null hypersurfaces corresponds with the type of the distribution $\xi^\perp$. This suggests that we rename the four types of carrollian structures in Theorem 10 as totally geodesic (if $L_\xi h = 0$), minimal (if $L_\xi \mu = 0$), totally umbilical (if $L_\xi h = fh$) and otherwise generic.

![Hasse diagram of bargmannian structures with involutive $\xi^\perp$](image)

**Figure 7.** Hasse diagram of bargmannian structures with involutive $\xi^\perp$

7. Conclusions

In this paper we have studied spacetime structures from the point of view of G-structures and studied their intrinsic torsion. Whereas this provides no information for the case of lorentzian spacetimes, the situation for non-lorentzian spacetimes is very different. As Theorem 6 shows, the classification of galilean structures by intrinsic torsion coincides with the classification of Newton–Cartan geometries in [8] into what those authors call torsionless, twistless torsional and torsional Newton–Cartan geometries. As Theorem 10 shows there are 4 types of carrollian structures, which as discussed in Section 6.4.2, may be distinguished by the geometrical properties of the null hypersurfaces of bargmannian manifolds into which they embed: totally geodesic, totally umbilical, minimal or generic. The intersection of the galilean and carrollian structures consists of the aristotelian structures, and as Theorem 12 shows, there are 16 classes depending on their intrinsic torsion. As advocated in [10], bargmannian structures are a subclass of lorentzian structures which are intimately linked with both galilean and carrollian structures. The study of the intrinsic torsion of bargmannian structures is surprisingly rich and as Theorem 21 shows
there are 27 bargmannian structures, many of which can be related to galilean and carrollian structures in a way made explicit in Section 6.4. We find that all three classes of galilean structures can arise as null reductions of bargmannian structures, whereas all four classes of carrollian structures can be induced from suitable bargmannian structures by restriction to null hypersurfaces integrating the distribution $\xi^\perp$.

The above results hold in generic dimension, which means that $n \neq 2, 5$. As shown in Appendix B, there are only 2 galilean and carrollian $G$-structures in two dimensions, and hence 4 aristotelian struc-
tures, whereas there are 11 three-dimensional bargmannian structures. Similarly, there are 5 five-dimensional
galilean structures, 32 five-dimensional aristotelian structures and 47 six-dimensional bargmannian struc-
tures.

It remains to understand whether all the different classes of (five-dimensional) galilean, aristotelian and
bargmannian structures can be realised geometrically or whether, as is the case with $G_2$-structures
on 7-manifolds [16, 17], some of the inclusions between the different classes (e.g., those in Figure 2 for
bargmannian structures) are not strict.

The classification of $G$-structures via intrinsic torsion is still somewhat coarse – after all, the intrinsic
torsion is the first of a sequence of obstructions to the integrability of the $G$-structure – but the results in
this paper may help to add some structure to the zoo of non-orentzian geometries.

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like to record my gratitude to him, especially for the careful reading of a previous version of this paper.
Last, but by no means least, I would like to dedicate this paper to Dmitri Alekseevsky on his eightieth
birthday, in hopes that he might derive some pleasure in seeing some familiar structures in a possibly
novel context.

Appendix A. Hypersurface orthogonality

It is of course a well-known fact that if a nowhere-vanishing one-form $\tau \in \Omega^1(M)$ satisfies $d\tau \wedge \tau = 0$
then there exists a one-form $\omega \in \Omega^1(M)$ such that $d\tau = \tau \wedge \omega$. The statement is ubiquitous in the
literature, but the proof is not. In this appendix I record a proof of this fact. Of course the condition simply
says that the characteristic distribution $\ker \tau \subset TM$ is Frobenius integrable and hence $M$ is foliated by
hypersurfaces whose tangent spaces agree with $\ker \tau$. By abuse of language one says that $\tau$ is hypersurface
orthogonal, a concept taken from riemannian geometry where the vector field dual to $\tau$ would indeed
be orthogonal to the hypersurfaces integrating $\ker \tau$.

Proposition A.1. Let $\tau \in \Omega^1(M)$ be nowhere vanishing. Then the following are equivalent

1. $d\tau \wedge \tau = 0$
2. $d\tau = \tau \wedge \omega$, for some $\omega \in \Omega^1(M)$.

Proof. It is clear that (2) implies (1), so we need to prove that (1) implies (2). The idea is to show this
locally and then to show that the local $\omega$'s glue to a global one-form.

Since $\tau$ is nowhere-vanishing, we may complete to a local coframe $(\theta^1 = \tau, \theta^2, \ldots, \theta^n)$ defined on
some chart $(U, \phi)$ for $M$. Then

$$d\tau = \sum_{i<j} f_{ij} \theta^i \wedge \theta^j$$

for some $f_{ij} \in C^\infty(U)$. Then

$$\tau \wedge d\tau = \sum_{i<j} f_{ij} \theta^i \wedge \theta^j \wedge \theta^i = \sum_{1<i<j} f_{ij} \theta^i \wedge \theta^j \wedge \theta^i,$$

so that $\tau \wedge d\tau = 0$ says that $f_{ij} = 0$ for $1 < i < j$, and hence

$$d\tau = \sum_{1<i<j} f_{ij} \theta^i \wedge \theta^j = 0^1 \wedge \sum_{1<i<j} f_{ij} \theta^i = \tau \wedge \omega,$$

for $\omega = \sum_{1<i<j} f_{ij} \theta^i \in \Omega^1(U)$. Notice that $\omega$ is not unique, since we could always add a component along $\tau$. We will exploit this ambiguity when we glue the local $\omega$'s.
Let \( (((U_\alpha, \varphi_\alpha))_{\alpha \in A}) \) be an atlas for \( M \). Then we have just shown that there exists \( \omega_\alpha \in \Omega^1(U_\alpha) \), where \( d\tau = \tau \wedge \omega_\alpha \) on \( U_\alpha \). Since \( \tau \) and \( d\tau \) are global forms, on a non-empty overlap \( U_{\alpha\beta} \),

\[
\tau \wedge (\omega_\alpha - \omega_\beta) = 0.
\]

We claim that \( \omega_\alpha - \omega_\beta = f_{\alpha\beta} \tau \) for some \( f_{\alpha\beta} \in \mathcal{C}^\infty(U_{\alpha\beta}) \). To see this, write

\[
\omega_\alpha - \omega_\beta = \sum_{i=1}^{n} g_i \theta^i,
\]

for some \( g_i \in \mathcal{C}^\infty(U_{\alpha\beta}) \), so that

\[
\tau \wedge (\omega_\alpha - \omega_\beta) = \tau \wedge \sum_{i} g_i \theta^i = \sum_{i} g_i \theta^i \wedge \theta^i = \sum_{i>1} g_i \theta^i \wedge \theta^i.
\]

If \( \tau \wedge (\omega_\alpha - \omega_\beta) = 0 \), we see that \( g_i = 0 \) for \( i > 1 \) and hence

\[
\omega_\alpha - \omega_\beta = f_{\alpha\beta} \tau,
\]

where \( f_{\alpha\beta} = g_1 \).

On a triple overlap \( U_{\alpha\beta\gamma} \), we have that

\[
(f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}) \tau = \omega_\alpha - \omega_\beta + \omega_\beta - \omega_\gamma + \omega_\gamma - \omega_\alpha = 0,
\]

and since \( \tau \) is nowhere-vanishing,

\[
f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0. \tag{A.1}
\]

Let \( \{\rho_\alpha\} \) denote a partition of unity subordinate to the atlas, with \( \rho_\alpha \) supported in \( U_\alpha \). Define \( g_\beta = \sum_\alpha \rho_\alpha f_{\alpha\beta} \in \mathcal{C}^\infty(M) \). Then

\[
g_\alpha - g_\beta = \sum_\gamma (\rho_\gamma f_{\gamma\alpha} - \rho_\gamma f_{\gamma\beta})
\]

\[
= \sum_\gamma (\rho_\gamma f_{\gamma\alpha} + \rho_\gamma f_{\beta\gamma}) \quad \text{(since \( f_{\beta\gamma} = -f_{\gamma\beta} \))}
\]

\[
= -\sum_\gamma \rho_\gamma f_{\alpha\beta} \quad \text{(by (A.1))}
\]

\[
= -f_{\alpha\beta}. \quad \text{(since \( \sum_\gamma \rho_\gamma = 1 \))}
\]

Therefore \( \omega_\alpha - \omega_\beta = (g_\beta - g_\alpha) \tau \), so that on \( U_{\alpha\beta} \),

\[
\omega_\alpha + g_\alpha \tau = \omega_\beta + g_\beta \tau.
\]

Let \( \omega_\alpha = \omega_\alpha + g_\alpha \tau \in \Omega^1(U_\alpha) \). Then \( \omega_\alpha = \omega_\beta \) on \( U_{\alpha\beta} \) and hence it glues to a global form \( \omega \in \Omega^1(M) \). Notice that on \( U_\alpha \),

\[
\tau \wedge \omega = \tau \wedge (\omega_\alpha + g_\alpha \tau) = \tau \wedge \omega_\alpha = d\tau,
\]

as desired. \( \square \)

**Appendix B. Some special dimensions**

In the bulk of the paper we have taken the dimension \( n \) to be generic, but for some special values of \( n \) (i.e., \( n = 2 \) and \( n = 5 \)) the discussion needs to be refined. We will briefly comment on how the results in the bulk of paper are modified for such values of \( n \).

**B.1. Two-dimensional galilean, carrollian and aristotelian structures.** If we think of galilean and carrollian structures as arising from non- and ultra-relativistic limits of lorentzian geometry, it is visually clear that in two dimensions the limits are equivalent simply by re-interpreting what we call time and space, which are geometrically equivalent in this dimension. This would seem to contradict the results of Sections 3 and 4, which therefore require modification.

When it comes to galilean structures, \( \text{coker } \mathcal{V} \) is now one-dimensional and the intrinsic torsion is still determined by \( d\tau \). The main difference is that now \( \tau \wedge d\tau = 0 \) by dimension, so we only have two (and not three) galilean structures, depending on whether or not \( d\tau \) vanishes.

Similarly, in the case of carrollian structures \( \text{coker } \mathcal{V} \) is again one-dimensional and the intrinsic torsion is still determined by \( \mathcal{L}_h \), except that since \( h \) is rank-one, there are no non-zero traceless symmetric tensors. Hence here too we have only two (and not four) carrollian structures, depending on whether or not \( \mathcal{L}_h \) vanishes.

Therefore the seeming discrepancy between galilean and carrollian structures is not there in two dimensions after all.
The classes of aristotelian structures for \( n = 2 \) also simplifies as a result. Now the structure group is \( O(1) \cong \mathbb{Z}_2 \) and the submodules \( A_1 \) and \( A_3 \) are absent. The submodules \( A_2 \) and \( A_4 \) are one-dimensional: \( G \) acts trivially on \( A_1 \) and via the "determinant" on \( A_2 \). All said, there are four aristotelian structures for \( n = 2 \), depending on whether either of \( d\tau \) and \( \mathcal{L}_\xi h \) vanishes or not.

**B.2. Three-dimensional bargmannian structures.** The classification of bargmannian structures also changes when \( n = 2 \). Now two submodules are absent: \( (P \otimes \pi) \wedge^2 \cong H \otimes \pi \wedge \pi \) and \( (P \otimes \pi) \wedge^3 \cong (P \otimes \pi) \wedge^3 \wedge \mathcal{L} \). This results in a somewhat simplified version of Figure 1, which we omit. There are some coincidences between the twenty-six bargmannian structures: \( B_2 = B_5 \), \( B_3 = B_9 \), \( B_4 = B_6 \), \( B_7 = B_{10} \), \( B_8 = B_{11} = B_{13} = B_{18} \), \( B_{14} = B_{15} = B_{16} = B_{21} \), \( B_{12} = B_{19} = B_{20} \), \( B_{17} = B_{22} = B_{23} = B_{24} = B_{25} \). The minimal structures coincide with the totally geodesic structures and the "none of the above" structures (except for the generic structure \( B_{26} \)) are now totally umbilical. In summary, there are eleven three-dimensional bargmannian structures, whose Hasse diagram is depicted in Figure 8.

![Hasse diagram of three-dimensional bargmannian structures](image)

**Figure 8.** Hasse diagram of three-dimensional bargmannian structures

**B.3. Five-dimensional galilean structures.** When \( n = 5 \), the \( \mathfrak{so}(4) \)-submodule \( H \otimes \pi \wedge \pi \) in \( \text{coker} \ \partial \) described in Section 3.2 is not irreducible, breaking up into selfdual and antiselfdual summands. This means that if the galilean structure reduces further to a \( G_0 \)-structure, with \( G_0 \cong SO(4) \ltimes \mathbb{R}^4 \), we have five \( G_0 \)-submodules of \( \text{coker} \ \partial \) and hence five galilean structures instead of three. The torsional Newton–Cartan geometries, where \( d\tau \wedge \tau \neq 0 \), now come in three flavours: selfdual, antiselfdual and neither, according to whether the restriction of \( d\tau \) to the four-dimensional oriented sub-bundle \( \ker \tau \) is selfdual, antiselfdual or neither.

**B.4. Five-dimensional aristotelian structures.** When \( n = 5 \), the \( \mathfrak{so}(4) \)-submodule \( A_1 \cong \wedge^2 W \) defined in Section 5.2, with \( W \) the four-dimensional vector representation of \( \mathfrak{so}(4) \), is no longer irreducible. Indeed, it decomposes into selfdual and antiselfdual summands:

\[
A_1 = A_1^+ \oplus A_1^- = \wedge^2_+ W \oplus \wedge^2_- W. \tag{B.1}
\]

If the aristotelian structure reduces further to \( G_0 = SO(4) \), then \( A_1^\pm \) are \( G_0 \)-submodules and we must refine the classification of aristotelian structures. Theorem 12 gets modified: there are not sixteen, but thirty-two aristotelian structures. Each of the eight structures in Theorem 12 whose intrinsic torsion have a nonzero component in \( A_1 \) – namely, those for which \( \tau \wedge d\tau \neq 0 \) – now can be of three distinct types, depending on whether \( d\tau \) is selfdual, antiselfdual or neither when restricted to the four-dimensional distribution \( \ker \tau \).
If $n = 5$, and if the group of the Bargmannian structure reduces to the identity component $G_0 \cong SO(4) \times R^4$, then $SO(4)$-submodule $(P \otimes \pi)_{\Lambda^2}$ is no longer irreducible and decomposes into selfdual and antselfdual parts. Under the action of the boosts, it is still the case that the $SO(4)$-submodule $P \otimes \Lambda^3 \subset (P \otimes \pi)_{\Lambda^2}$ and each of $(P \otimes \pi)_{\Lambda^2}$ maps into $\Xi$. Therefore all that happens is that every Bargmannian structure $B_{1}$ (except for $B_{26}$) which contains $(P \otimes \pi)_{\Lambda^2}$ now comes in two more flavours: $B_{1}^+$ and $B_{1}^-$, where $(P \otimes \pi)_{\Lambda^2}$ is replaced by the submodules $(P \otimes \pi)_{\Lambda^2}^+$ or $(P \otimes \pi)_{\Lambda^2}^-$, respectively. So now we have twenty additional structures: $B_{1}^+_{5}$, $B_{1}^+_{12}$, $B_{1}^+_{13}$, $B_{1}^+_{15}$, $B_{1}^+_{18}$, $B_{1}^-_{14}$, $B_{1}^-_{16}$, $B_{1}^-_{19}$, $B_{1}^-_{23}$, and $B_{1}^-_{20}$. I omit the rather more involved Hasse diagram of the 47 six-dimensional Bargmannian structures, as I do their geometric characterisation.