Charged particle motion around a quasi-Kerr compact object immersed in an external magnetic field

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(Dated: February 12, 2019)

We explore the electromagnetic fields around a quasi-Kerr compact object assuming it is immersed in an external asymptotically uniform magnetic field. Using Wald method, components of the electromagnetic field in orthonormal basis have been obtained. We explore the charged particle motion around deformed Kerr compact objects in the presence of external asymptotically uniform magnetic fields. Using the Hamilton-Jacobi equation, we obtain the effective potential expression for the charged particle surrounding a quasi-Kerr compact object immersed in an external magnetic field. It is also derived the dependence of innermost stable circular orbits (ISCOs) from the magnetic and deformation parameters for charged particles motion around a rotating quasi-Kerr compact object. Comparison with ISCO radius measurements has provided the constraint to the deformation parameter as \( \epsilon \gtrsim -0.012 \). The center of mass (CM) energy of the colliding particles in several physically interesting cases has been studied.

I. INTRODUCTION

Astrophysical black holes are believed to be Kerr one, which has two parameters: its mass \( M \) and rotation parameter \( a \). However, there are many attempts to construct extension of Kerr black holes introducing extra parameters and parametrizations, see, for example, [1–10]. The presence of nonvanishing electric charge in the spacetime metric has been tested by different scenarios in [11–17]. The properties of the black holes with brane charge have been studied in [18–22]. Other extension of the Kerr solution is the solution with gravitomagnetic charge [2, 23–28]. Various works are dedicated to test the axial symmetric metrics with the deformation parameters [29–36].

The authors of the Ref. [3] proposed the deviations from the Kerr metric considering an approximate solution of Einstein vacuum equations and introducing the leading order deviation coming from spacetime quadrupole moment. Different physical properties of these quasi-Kerr black holes have been studied in [37, 38]. Our recent work has been devoted to study the weak lensing near the compact object with nonzero quadrupole momentum [39].

The magnetic field is very important in many astrophysical scenarios related to compact objects. Rotating neutron stars having the own magnetic field can be observed as pulsars [40–43]. However, the black holes, according to no-hair theorem, may not create their own magnetic field [41, 44–46]. The accretion disc around the rotating black holes can provide the magnetic field in the vicinity of the latters. The properties of the electromagnetic field structure around rotating black holes immersed in the external magnetic field first initiated by Wald [47]. The dipolar magnetic field configuration around black hole created by circular electric current has been studied by Petterson [48]. After that the properties of the electromagnetic field structure around rotating black holes have been considered by various authors. The role of the magnetic field through magnetic Penrose process has been studied by [49–55]. The similar scenarios in alternative/modified theories of gravity have been studied in [28, 56–61]. The charged particle motion around black holes immersed in external magnetic field has been studied by various authors [62–69].

In this work our main purpose is to study the electromagnetic field structure around rotating quasi-Kerr black holes with nonvanishing quadrupole momentum. The paper is organized as follows: Sect. II is devoted to study the electromagnetic field components around quasi-Kerr compact objects immersed in external asymptotically uniform magnetic field. The charged particle motion around the quasi-Kerr compact object is studied in Sect. III in the presence of magnetic field. In Sect. IV we analyze the particle acceleration process around a compact object with nonzero quadrupole moment and in the presence of magnetic field. In Sect. V, we summarize the obtained results.

II. COMPACT OBJECT IMMERSED IN MAGNETIC FIELD

The metric for quasi-Kerr compact object is given by (for \( G = c = 1 \), and with metric signature \((-+++)) \) [3]

\[
ds^2 = g_{00} dt^2 + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\phi^2 + 2 g_{03} dt d\phi , \tag{1}
\]
where

\[
g_{00} = - \left(1 - \frac{2Mr}{\Sigma} \right) + \epsilon \left(1 - 3 \cos^2 \theta \right) \times \left(\frac{F_1 (1 - \frac{2Mr}{r})^2}{\Sigma^2} + \frac{4a^2 F_2 M^2 \sin^2 \theta}{\Sigma^2} \right),
\]

\[
g_{11} = \frac{\Sigma}{\Delta} + \epsilon \frac{F_1 (1 - \frac{2Mr}{r}) \Sigma^2 (1 - 3 \cos^2 \theta)}{\Delta^2},
\]

\[
g_{22} = - \epsilon \frac{F_2 \Sigma^2 (1 - 3 \cos^2 \theta)}{r^2},
\]

\[
g_{33} = \left( r^2 + a^2 + \frac{2a^2 Mr \sin^2 \theta}{\Sigma} \right) \sin^2 \theta + \epsilon (1 - 3 \cos^2 \theta) \sin^2 \theta \left[ \frac{4a^2 F_1 M^2 r^2}{\left(1 - \frac{2Mr}{r}\right) \Sigma^2} - \frac{F_2}{r^2} \left( a^2 + r^2 + \frac{2a^2 Mr \sin^2 \theta}{\Sigma} \right)^2 \right],
\]

\[
g_{03} = - \frac{2aMr}{\Sigma} \sin^2 \theta + \epsilon (1 - 3 \cos^2 \theta) \sin^2 \theta \times \left[ \frac{2aF_1 M (1 - \frac{2Mr}{r})}{\left(1 - \frac{2Mr}{r}\right) \Sigma} + \frac{2aF_2 M}{r \Sigma} (a^2 + r^2 + \frac{2a^2 Mr \sin^2 \theta}{\Sigma}) \right],
\]

with \( \Sigma = r^2 + a^2 \cos^2 \theta, \Delta = r^2 - 2Mr + a^2 \) and

\[
F_1 = \frac{5(M - r) (2M^2 + 6Mr - 3r^2)}{8Mr(r - 2M)} - \frac{15r(r - 2M)}{16M^2} \ln \left( \frac{r}{r - 2M} \right),
\]

\[
F_2 = \frac{5(2M^2 - 3Mr - 3r^2)}{8Mr} + \frac{15(r^2 - 2M^2)}{16M^2} \ln \left( \frac{r}{r - 2M} \right),
\]

the constant \( \epsilon \) in the expression (2) indicates the small contribution to the quadrupole moment \( Q \) of the compact object with the total mass \( M \) as

\[
Q = -M(a^2 + \epsilon M^2),
\]

here the deformation parameter \( \epsilon \) might take either positive \( \epsilon > 0 \) or negative \( \epsilon < 0 \) values [3, 39]. One can easily see that the case when \( \epsilon = 0 \) the spacetime metric (1)–(4) describes the spacetime of a Kerr black hole.

In order to find the vector potential of the electromagnetic field in the vicinity of the compact object we will follow the Wald method that assumes the black hole is immersed in a uniform magnetic field [28, 47, 56, 61, 62]. Here we use the existence in this spacetime of a time-like Killing vector, \( \xi^\alpha(t) = \partial x^\alpha / \partial t \), and a spacelike one, \( \xi^\alpha(r) = \partial x^\alpha / \partial \phi \), which are responsible for the stationarity and axial symmetry of the spacetime geometry (1)–(4) which satisfies the Killing equations

\[
\xi_{\alpha\beta} + \xi_{\beta\alpha} = 0,
\]

and according to the Wald method [47] the solution of the vacuum Maxwell’s equations \( \Box A^\alpha = 0 \) for the vector potential \( A_{\mu} \) of the electromagnetic field in the Lorentz gauge can be written as

\[
A^\alpha = C_1 \xi^\alpha(t) + C_2 \xi^\alpha(r).
\]

The constant \( C_2 = B/2 \), where the compact object is immersed in the uniform magnetic field \( B \) that is aligned along its rotating axis. The remaining constant \( C_1 \) can be found from the asymptotic properties of spacetime (1)–(4) at infinity. Indeed in order to find the remaining constant one can use the electrical neutrality of the compact object \( Q^* = 0 \) evaluating the value of the integral through the spherical surface at the asymptotic infinity. Then one can easily get the value of constant \( C_1 = aB \).

The contravariant components of the vector potential \( A^\mu \) of the electromagnetic field will take the following form

\[
A^\mu = aB, A^1 = A^2 = 0, A^3 = \frac{1}{2} B.
\]

Now one can easily find the covariant components of the vector potential using \( A_{\mu} = g_{\mu\nu} A^\nu \) for metric (1)–(4).
where \( \mathcal{R} = 2a^2Mr \sin^2 \theta + (a^2 + r^2) \Sigma \).

The components of the electromagnetic fields can be found using following expressions in curved spacetime.

\[
E_\alpha = F_{\alpha \beta} u^\beta, \quad (10)
\]

\[
B^\alpha = \frac{1}{2} \epsilon^{\alpha \beta \gamma \delta} F_{\beta \gamma} u_\delta, \quad (11)
\]

with

\[
F_{\alpha \beta} = A_{\beta ; \alpha} - A_{\alpha ; \beta}, \quad (12)
\]

where \( F_{\alpha \beta}, u^\beta \) and \( \epsilon^{\alpha \beta \gamma \delta} \) are the electromagnetic field tensor, the four velocity of the observer and the Levi-Civita tensor, respectively.

In the case of the rotating quasi-Kerr compact object one can obtain simplified expressions for the fields in the linear approximation of \( \epsilon \).

\[
E^i = -\frac{aBM \rho_3}{2\Sigma^2 \sqrt{\mathcal{R}}} + \frac{aB \sqrt{\mathcal{R}}}{4 \Delta \Sigma^2} \epsilon \left\{ 2 \Delta \Sigma \left[ N_2 + \frac{2N_3 (1 - 3 \cos^2 \theta)}{(r - 2M)^2} \right] - \frac{Mr \sin^2 \theta (N_4 + 2N_5)}{\mathcal{R}} \right\} + N_1 \quad (16)
\]

\[
E^\theta = \frac{aBM \rho_3 \sin 2\theta + aB \sqrt{\mathcal{R}}}{4 \rho \Sigma \epsilon} \left\{ K_1 + \frac{K_2 \sin 2\theta}{\Sigma^2 (2M - r)} - \frac{2Mr^2}{\mathcal{R}} \left[ K_3 + \frac{2K_4 \sin 2\theta + K_5}{r^2} \right] \right\} + R_1 \quad (17)
\]

\[
B^i = \frac{BP_3 \sin 2\theta + \epsilon}{2 \sqrt{\Sigma \Delta}} \left\{ \frac{\Delta}{\Delta} \left[ \frac{1}{\sqrt{\mathcal{Q}}} \left( \frac{R_4 \sin 2\theta}{r} + R_5 \right) \right] + \frac{P_3 \sin 2\theta}{\Sigma} \left( \frac{R_3 \sqrt{2\Sigma (2M - r)} + R_2}{\mathcal{R}} \right) \right\} + R_1 \quad (18)
\]

\[
B^\theta = \frac{BP_4 \Delta \sin^2 \theta}{\Sigma \sqrt{\mathcal{Q}}} + \epsilon \left\{ \frac{1}{2} B \sqrt{\Sigma \sin^2 \theta} \left[ \frac{2D_2 + D_3}{\Sigma^2} \left( \frac{8a^2 Mr^2 - P_4}{\mathcal{R}} \right) + \frac{D_4 - D_5}{\sqrt{\mathcal{Q}}} - D_1 \right] \right\} \quad (19)
\]

where \( \mathcal{Q} = (\mathcal{R} - 2a^2 Mr - 2Mr^3) \mathcal{R} \sin^2 \theta, \) and explicit forms of functions \( P_i, i = 1, 4 \) with \( N_j, K_j, R_j \) and \( D_j, \) \( j = 1, 5 \) are given in Appendix A.

In the case of the rotating quasi-Kerr compact object one can obtain simplified expressions for the fields in the linear or quadratic approximation \( O(a^2/r^2, \epsilon) \).

\[
E^i = \frac{aBM}{r^2} \left[ \cos^2 \theta - 3 + \frac{3M^2 \epsilon (1 - 3 \cos^2 \theta)}{r^2} \right], \quad (20)
\]

\[
E^\theta = \frac{aBM}{r^2} \sin 2\theta \left[ 2 \frac{2M}{r} - \frac{3M^2 \epsilon}{r^2} \right. \\
\left. + \frac{3(\alpha^2 \cos^2 \theta - M^2)}{r^2} \right], \quad (21)
\]

\[
B^i = B \cos \theta \left[ 1 + \alpha^2 \frac{2}{2r^2} \sin^2 \theta - \frac{a^2 M}{2r^3} (5 + 3 \cos 2\theta) - \frac{3M^2 \epsilon}{r^2} \sin^2 \theta \right], \quad (22)
\]

\[
B^\theta = B \sin \theta \left[ 1 - \frac{M}{r} - \frac{M^2 + a^2 \cos^2 \theta}{2r^2} + \frac{M}{4r^3} \right. \\
\left. \times \left\{ a^2 (9 + 5 \cos 2\theta) - 3M^2 \epsilon (1 + 3 \cos 2\theta) - 2M^2 \right\} \right], \quad (23)
\]

It can be seen from the equations (20) and (21) that in the linear approximation in \( \epsilon \) and \( a \), electric field does not have any contribution from \( \epsilon \). In the limit of flat spacetime, i.e., for \( M/r \to 0 \), expressions (16) - (19) give

\[
E^i = E^\theta = 0, \quad B^i = B \cos \theta, \quad B^\theta = B \sin \theta. \quad (24)
\]

As expected, expressions (24) coincide with the solutions for the homogeneous magnetic field in the Newtonian spacetime. One can ensure that providing plots of the expressions (16) - (19) as in Fig.1. Indeed, we can see that for the large distances the absolute values of the components of the electric field tend to zero while the components of the magnetic field tend to the corresponding values of \( \sin \theta \) and \( \cos \theta \) for chosen angles \( \theta \) as in (24), when \( B=1 \).

### III. CHARGED PARTICLE MOTION AROUND MAGNETIZED COMPACT OBJECT

In this section we will consider the equation of motion of charged particles in the background spacetime of rotating compact object with the metric given in (1)-(4). We are aimed at investigating the particle motion
in the spacetime of a quasi-Kerr compact object immersed in a uniform magnetic field. In order to describe the charged particle motion we use the Hamilton-Jacobi equation which can be expressed as

$$g^{\alpha\beta} \left( \frac{\partial S}{\partial x^\alpha} - qA^\alpha \right) \left( \frac{\partial S}{\partial x^\beta} - qA^\beta \right) = -m^2,$$  \hspace{1cm} (25)

where \(m\) and \(q\) are the mass and charge of the test particle, respectively.

Due to existence of two Killing vectors \(\xi_{(t)}\) and \(\xi_{(\phi)}\) the action of charged particle around the compact object can be described as follows

$$S = -Et + L\phi + S_r(r) + S_\theta(\theta),$$  \hspace{1cm} (26)

where \(E\) and \(L\) are the energy and the angular momentum of the charged particle, respectively. Inserting (26) to (25) and after making calculations in equatorial plane \((\theta = \pi/2)\) one can easily find the equation for the radial part of motion which corresponds to the radial component of covariant 4-momentum of the charged particle \(p_r = \partial S_r/\partial r\). Radial contravariant component of the momentum can be obtained multiplying the metric (1)
with covariant momentum. On the other hand,
\[
\left( \frac{dr}{d\tau} \right)^2 = f(r) = \mathcal{E}^2 - 1 - 2V_{\text{eff}}^2, \tag{27}
\]
where \( \tau \) is the proper time of the test particle and \( V_{\text{eff}}^2 \) is the square of the effective potential that can be written as
\[
V_{\text{eff}}^2 = V_{(K)}^2 + V_{(q)}^2. \tag{28}
\]

The first term corresponds to Kerr one and the second indicates small deviation taking place in quasi-Kerr spacetime. Solving equations (25) and (27) we can obtain the following expressions for each term

\[
V_{(K)}^2 = \frac{1}{8\epsilon r^3} \left\{ a^4b^2(2M - 3r) + 8a^3b\mathcal{E}r - 2a^2 \left[ r \left( b(2M^2 - 4Mr + r^2) + 2\mathcal{L} \right) - 2 \right] + 2\mathcal{E}^2(2M + r) \\
+ 8a\mathcal{E} \left[ b(r^2 - 2M) + 2\mathcal{L}M \right] - 2b^2Mr^4 + b^2r^5 - 4b\mathcal{E}r^2(r - 2M) + 4\mathcal{E}^2(r - 2M) - 8Mr^2 \right\}, \tag{29}
\]

\[
V_{(q)}^2 = \frac{5\epsilon}{128M^2r^7(\alpha^2 + r(r - 2M))} \left\{ r^4 \left[ 2M(M - r) \left( 2M^2 + 6Mr - 3r^2 \right) - 3r^2(r - 2M)^2 \ln \frac{r}{r - 2M} \right] \right.
- 8a\mathcal{E} \left[ b(r^3 - 2b^2M + 2\mathcal{L}M) + r^2 \left( (b^2r^4 + 4)(2M - r) + 4\mathcal{E}^2r \right) + 4b\mathcal{E}r^2(r - 2M) + \mathcal{L}^2(8M - 4r) \right] \left. \right.
- \frac{1}{(r - 2M)^2} \left[ a^2 + r^2 - 2Mr \right]^2 \left[ (r - 2M)^2 \left( 2M(2M^2 - 3Mr - 3r^2) + 3r(r^2 - 2M^2) \ln \frac{r}{r - 2M} \right) \right.
\times \left[ a^2b(r - 2M) + br^3 + 2\mathcal{L}r \right] \left[ b(a^2r - 2a^2M + r^3) - 2\mathcal{L}r \right] + 4r^3 \left( 3r^2(r - 2M)^2 \ln \frac{r}{r - 2M} \right.
\left. \right.
- 2M(M - r) \left( 2M^2 + 6Mr - 3r^2 \right) \right\} (abr - abM + \mathcal{E}r)(abr - abM - \mathcal{E}r) \right\} ,
\]

where we have introduced new notations \( \mathcal{E} = E/m \) and \( \mathcal{L} = L/m \) and dimensionless magnetic parameter \( b = qB/m \) which characterizes the cyclotron frequency of the charged particle.

In Fig. 2 the radial dependence of the effective potential for different values of the magnetic and deformation parameters is shown. It is worth to note that, points where the lines turn correspond to ISCO of the charged test particle.

Now we consider circular orbits, especially the innermost stable circular orbit for the charged test particle in the spacetime of a quasi-Kerr compact object using the following conditions

\[
f(r) = 0, \tag{30}
\]

\[
f'(r) = 0, \tag{31}
\]

\[
f''(r) = 0. \tag{32}
\]

By solving equations (30), (31), and (32) all together one can find numerical values of the magnitudes for the energy, the angular momentum and the radius of ISCO of the charged particle orbiting around a quasi-Kerr compact object immersed in an asymptotically uniform external magnetic field. Obtained results are shown in Table I and Table II. It is apparent from the tables that, increasing the value of the deformation parameter \( \epsilon \) and the external magnetic field strength will reduce the ISCO radius. It is worth to note that as \( \epsilon = 0 \), the results coincide with the results for the Kerr compact object as expected.

TABLE I. ISCO radius of the particles moving around the rotating quasi-Kerr compact object (case of \( b = 0 \)).

| \( \epsilon \) | -0.006 | -0.003 | 0 | 0.003 | 0.006 |
|------------------|--------|--------|---|--------|--------|
| \( a = 0 \)      | 6.0063 | 6.0032 | 6.0 | 5.9968 | 5.9936 |
| \( a = 0.5 \)    | 4.2515 | 4.2423 | 4.233 | 4.224 | 4.214 |
| \( a = 0.7 \)    | 3.4430 | 3.4189 | 3.3931 | 3.3655 | 3.3355 |
| \( a = 0.8 \)    | 3.0287 | 2.9754 | 2.9066 | 2.7964 | 2.7551 |
TABLE II. ISCO radius of the particles moving around the rotating quasi-Kerr compact object (case of $a = 0.5$).

| $\epsilon$ | -0.006 | -0.003 | 0   | 0.003 | 0.006 |
|------------|--------|--------|-----|-------|------|
| $b = 0$    | 4.2515 | 4.2423 | 4.233 | 4.2236 | 4.2141 |
| $b = 0.05$ | 4.0576 | 4.0483 | 4.0399 | 4.0313 | 4.0226 |
| $b = 0.2$  | 3.3061 | 3.2979 | 3.2897 | 3.2812 | 3.2725 |
| $b = 0.5$  | 2.7418 | 2.7294 | 2.7162 | 2.7018 | 2.6860 |

IV. PARTICLE COLLISIONS IN THE VICINITY OF A COMPACT OBJECT

Here we study some simple cases of the collisional processes of test particles that could well represent the role of the deviation parameter and the external magnetic field added to the gravitational field of the quasi-Kerr compact objects.

\[
\frac{E^2_{CM}}{2m^2} = 2a^2(M + r) + L^2(r - 2M) + 2r^2(r - 2M) - Z \quad \frac{a^2r + r^2(r - 2M)}{2a^2M} \left[ F_1 r^3 + F_2 L^2(2M - r) + F_1 r^3 \left[ L^2(r - 2M) - 2Mr^2 + Z \right] + F_2 L^2(2M - r) \left[ L^2(2M - r) + 2Mr^2 + Z \right] \right]
\]

\[
 Z = \sqrt{2M ((a - L)^2 + r^2) - L^2 r} \left[ 2M ((a + L)^2 + r^2) - L^2 r \right].
\]

A. Collisions of neutral particles with opposite angular momentum

In this subsection we calculate the central mass energy of two colliding neutral particles falling from infinity with zero initial velocity and opposite angular momentum $L_1 = -L_2 = L$. For simplicity we assume that the particles are moving on the equatorial plane ($\theta = \pi/2$) and have the same initial rest energy ($E_{CM} = m_1 + m_2$) at infinity. Under such assumptions the 4-velocities of the particles read

\[
u^i = -g^{tt} + g^{t\phi} \frac{L_i}{m_i}.
\]

and $u^i$ can be found from the condition $g_{\alpha\beta} u^i_1 u^i_2 = -1$ with $i = 1, 2$.

The CM energy of the system of test particles can be found from the relation $E^2_{CM} = -g_{\alpha\beta} p^\alpha_{CM} p^\beta_{CM}$. The square of the CM energy then reads

\[
E^2_{CM} = -m_1^2 g_{\alpha\beta} (u^\alpha_1 + u^\alpha_2) (u^\beta_1 + u^\beta_2) = 2m^2 (1 - g_{\alpha\beta} u^\alpha_1 u^\beta_2),
\]

or

\[
E^2_{CM} = \frac{2a^2(M + r) + L^2(r - 2M) + 2r^2(r - 2M) - Z}{2a^2M} \left[ F_1 r^3 + F_2 L^2(2M - r) + F_1 r^3 \left[ L^2(r - 2M) - 2Mr^2 + Z \right] + F_2 L^2(2M - r) \left[ L^2(2M - r) + 2Mr^2 + Z \right] \right].
\]

or

\[
E^2_{CM} = \frac{2a^2(M + r) + L^2(r - 2M) + 2r^2(r - 2M) - Z}{2a^2M} \left[ F_1 r^3 + F_2 L^2(2M - r) + F_1 r^3 \left[ L^2(r - 2M) - 2Mr^2 + Z \right] + F_2 L^2(2M - r) \left[ L^2(2M - r) + 2Mr^2 + Z \right] \right],
\]

or

\[
E^2_{CM} = \frac{2a^2(M + r) + L^2(r - 2M) + 2r^2(r - 2M) - Z}{2a^2M} \left[ F_1 r^3 + F_2 L^2(2M - r) + F_1 r^3 \left[ L^2(r - 2M) - 2Mr^2 + Z \right] + F_2 L^2(2M - r) \left[ L^2(2M - r) + 2Mr^2 + Z \right] \right].
\]
When one assumes such particles to collide at the turning point which is given by the following condition

\[ u^r(r_t, \mathcal{L}) = 0 , \]  

the specific angular momentum of colliding particles as a function of the turning point radius reads

\[
\mathcal{L}(r_t) = \frac{\sqrt{2G} - 2aMr_t^2}{r_t^2(r_t - 2M)} + \frac{G\epsilon}{4M^2r_t^2(r_t - 2M)^2} \left( -2\sqrt{2a^2F_2Mr_t^3(2M + r_t)} + 8aF_2GM + \sqrt{2r_t^2(2M - r_t)}(2F_2M - F_1r_t) \right),
\]

with \( G = \sqrt{Mr_t^2[a^2 + r_t(r - 2M)]} \).

The behavior of the CM energy \( E_{CM}^2/2m^2 \) as a function of the radius \( r_t \) is illustrated in Fig. 3. It is apparent from the graph that in the absence of the deviation parameter, \( \epsilon = 0 \), the CM energy of the particles goes up exponentially near to the compact object as in the case of Kerr one, whereas, in the presence of some deviation one can see significant differences in the shape of lines and can conclude that increasing of the deviation parameter reduces the CM energy of the particles.

**B. Collision of charged particles with opposite angular momentum**

In this part we assume two charged particles falling from infinity which have the same masses, \( (E_1, E_2 = m) \), but with opposite electric charges \( (q_1 = -q_2 = q) \), and opposite angular momenta as in the previous subsection \( (L_1 = -L_2 = L) \). All steps and assumptions here are the same as in the previous subsection but calculations become slightly complicated as there are additional terms in the four velocities that involve the four potential components (8).

The components of the four velocities read

\[
\begin{align*}
    u^i_t &= -1 - b^i(q_{\phi}a + \frac{1}{2}g_{\phi\phi}) , \\
    u^i_\phi &= \frac{L_i}{m} - b^i(q_\phi a + \frac{1}{2}g_{\phi\phi}) , \\
    u^i_\theta &= 0 ,
\end{align*}
\]

where \( i = 1, 2 \) and \( b^i = q^iB/m \) are the cyclotron frequencies of the particles. Note that these are covariant components of the four velocities but not contravariant ones as in (33). As in the previous section radial components \( u^r_t \) can be found using the normalization of the four velocity \( g^{\alpha\beta}u_\alpha u_\beta = -1 \). Knowing the dependence between CM energy and the velocities of the particles

\[
E_{CM}^2 = 2m^2(1 - g^{\alpha\beta}u^i_\alpha u^i_\beta) ,
\]

one can easily plot the graph for the CM energy of the two colliding charged particles at turning point \( r_t \) that satisfies the condition (36) in the presence of the external magnetic field (see Fig. 4). The graph shows that the external magnetic field increases the CM energy of particles which plays a role of the accelerator of the particles while deviation parameter decreases it.
C. Collision of charged particles on circular orbits with radially falling neutral ones

In this subsection we focus on the collision of two particles where the first particle is charged and orbiting on a circular orbit around the compact object and the second one is electrically neutral and falling from infinity. To make our calculations easy to solve we set the following assumptions:

(i) both particles have the same rest mass \( m_1 = m_2 = m \);
(ii) rotation is absent \( \alpha = 0 \);
(iii) particles are moving on the same plane;
(iv) the neutral particle is falling radially from infinity.

The four velocity of the neutral particle that is falling from infinity reads

\[
\begin{align*}
    u^t_2 &= -g^{tt}, \\
    u^r_2 &= \sqrt{g^{rr}(-1 - g^{tt})}, \\
    \theta_2 &= \phi_2 = 0.
\end{align*}
\]

The four velocity of the charged particle moving on circular orbit can be found using conditions \((30)\) and \((31)\)

\[
\begin{align*}
    u^t_1 &= -g^{tt}E(r, b, \epsilon), \\
    u^r_1 &= u^\theta_1 = 0, \\
    u^\phi_1 &= g^{\phi\phi}L(r, b, \epsilon),
\end{align*}
\]

where \(E(r, b, \epsilon)\) and \(L(r, b, \epsilon)\) represent the energy and the angular momentum of the charged particle and they are derived solving equations \((30)\) and \((31)\).

The radial dependence of the CM energy can be obtained using the expression \((34)\). In Fig. 5 it is shown how the CM energy of the two colliding particles behaves in the presence of an external uniform magnetic field and deviation parameter.

From this section one can conclude that increase of the deviation parameter does not accelerate particles but slow them down which has an opposite character to the magnetic field. The effects of the quasi-Kerr term of the metric become stronger in the near regions of the compact object while in the higher distances it behaves as traditional Kerr black hole. This might play an important role for example when one attempts to identify if the object is a Kerr black hole or not.

V. CONCLUSION

In the present paper, we have investigated the explicit forms of the components of electromagnetic fields around a quasi-Kerr compact object immersed in an asymptotically uniform magnetic field. We have also investigated the motion of a charged test particle orbiting around a quasi-Kerr compact object immersed in an asymptotically uniform magnetic field. The main results obtained in this paper can be summarized as follows.

- We have obtained the exact analytic expressions for the electromagnetic field components around a quasi-Kerr compact object immersed in an external magnetic field. It was shown that at large distances the absolute values of the components of the electric field tend to zero while the components of the magnetic field tend to the corresponding values of \(\sin \theta\) and \(\cos \theta\) for chosen angles \(\theta\).
- We analyzed the equation of motion of the charged particle motion around a quasi-Kerr compact object in a magnetic field. The analysis of the circular orbits of charged particles showed that, increasing the value of the deformation parameter \(\epsilon\) and the external magnetic field \(B\) will reduce the ISCO radius. It is worth to note that as \(\epsilon = 0\), the results coincide with the results for a Kerr black hole as expected.
- The measurements of the ISCO radius in accretion disks around compact objects can be used to obtain the constraint on the values of the deformation parameter \(\epsilon\). Observable properties of the accretion disc around black hole can be modeled using the spacetime metric and X-ray observation could give information about spacetime parameters, particularly if one compares with Kerr spacetime, one can get estimation of spin parameter \([70–73]\). Comparison of the X-ray observations with the spacetime metric of alternative/modified theories gravity has been used to get constraint parameters of the spacetime metric \([36, 74]\). In this paper we have obtained numerical results on ISCO around quasi-Kerr compact object which can be used to get rough constraint on \(\epsilon\) parameter: using the error bar in the observation we can get rough constraint on the \(\epsilon\) parameter. Since the ISCO radius decreases with an increase of the \(\epsilon\) parameter, one may get rough estimation as \(\epsilon \gtrsim -0.012\).
- The results of the study the particles collision show that increase of the deviation parameter does not accel-
erate particles but slow them down which has an opposite character to the magnetic field. The effects of the quasi-Kerr term of the metric become stronger in the near regions of the compact object while in the higher distances it behaves as traditional Kerr black hole.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Grant No. U1531117) and Fudan University (Grant No. IDH1512060). B.N. also acknowledges support from the China Scholarship Council (CSC), grant No. 2018DFH009013. The research is supported in part by Grant No. VA-FA-F-2-008 and No.YFA-Ftech-2018-8 of the Uzbekistan Ministry for Innovation Development, by the Abdus Salam International Centre for Theoretical Physics through Grant No. OEA-NT-01 and by Erasmus+ exchange grant between Silesian University in Opava and National University of Uzbekistan. A.A. thanks Nazarbayev University for hospitality.

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The functions introduced in (16)–(19) have the following form:
\[ \mathcal{P}_1 = 2r \Sigma (\cos 2 \theta + 3) \left( -2a^2 M \sin^2 \theta + a^2 r + r^3 \right) + 8a^2 M r^3 \sin^2 \theta (\cos 2 \theta + 3) - \Sigma^2 \left( (a^2 + 3r^2) \cos 2 \theta + 3a^2 + r^2 \right) \]  
\quad \text{(A1)}

\[ \mathcal{P}_2 = a^2 M r (3 \cos 2 \theta + 1) - 2 (a^2 + r^2) \Sigma \]  
\quad \text{(A2)}

\[ \mathcal{P}_3 = 2a^2 M r \sin^2 \theta - 4a^2 M r + \mathcal{R} \]  
\quad \text{(A3)}

\[ \mathcal{P}_4 = a^2 M \Sigma \sin^2 \theta - 2a^2 M r^2 \sin^2 \theta - 4a^2 M r^2 - 2a^2 M \Sigma + r \Sigma^2 \]  
\quad \text{(A4)}

\[ N_1 = \frac{MF_1 (3 \cos^2 \theta - 1) (1 - 2M/r)}{\mathcal{R}} (-2r (\cos 2 \theta + 3) \Sigma (-2a^2 M \sin^2 \theta + a^2 r + r^3)) \]  
\quad \text{(A5)}

\[ N_2 = \frac{\Delta F_1 \Sigma M^2 (\cos 2 \theta + 3) (3 \cos 2 \theta + 1) (2r^2 - \Sigma)}{4 \mathcal{R} (1 - 2M/r)} \]  
\quad \text{(A6)}

\[ N_3 = \frac{F_1 M}{2} (2r^2 \Sigma (-18M^2 - 8Mr + 7r^2) + r \cos 2 \theta (2r \Sigma (r^2 - 6M^2) + 8Mr^2 (2M - r) + \Sigma^2 (4M - r))) \]  
\quad + 24Mr^4 (2M - r) + 7r \Sigma^2 (4M - r) - 4 \Sigma^3) + \frac{(1 - 2M/r)}{r} F_2 M \sin^2 \theta (2M - r) (8a^2 M r^3 \sin^2 \theta - 16a^2 M r^3 + 2r^2 \Sigma (a^2 + r^2) + \Sigma^2 (a - r) (a + r) + Mr \Sigma F_2 \sin^2 \theta (2M - r) (4a^2 M r - \mathcal{R}) + r^3 F_1' (Mr \Sigma (\sin^2 \theta (\Sigma - 2Mr) + 4Mr - 4 \Sigma) + \Sigma (r^3)) \]  
\quad \text{(A7)}

\[ N_4 = \frac{\Delta F_1 (\cos 2 \theta + 3) (3 \cos 2 \theta + 1) (a^2 M (2r^2 - \Sigma) + 2r \Sigma^2)}{2 \Sigma \mathcal{R} (1 - \frac{2M}{r})} \]  
\quad \text{(A8)}

\[ N_5 = \frac{(3 \cos 2 \theta + 1)}{\Sigma^3} \left( \frac{2a^2 F_1 M r^4}{(r - 2M)^2} \right) (2r \Sigma (-9M^2 - Mr + 2r^2) + 6Mr^3 (2M - r)) \]  
\quad + r \Sigma F_2 (r - 2M) (4a^2 Mr - \mathcal{R}) + 2F_2 (2M - r) (2a^2 Mr (4Mr^3 + \Sigma (2r^2 + \Sigma))) \]  
\quad + a^2 Mr (2r^2 - \Sigma) (2r \Sigma - \mathcal{R}) - a^2 Mr \cos 2 \theta (8a^2 Mr^3 + \mathcal{R} (\Sigma - 2r^2)) - \Sigma \mathcal{R} (\mathcal{R} - 2r^2 \Sigma)) \]  
\quad \text{(A9)}

\[ K_1 = \frac{-F_2 M \sin^2 \theta (3 \cos 2 \theta + 1) \left[ a^2 M r (3 \cos 2 \theta + 1) - 2 \Sigma (a^2 + r^2) \right]}{\mathcal{R}} \]  
\quad \text{(A10)}

\[ K_2 = F_2 M (2M - r) \left[ a^2 M r (28 \cos 2 \theta + 9 \cos 4 \theta - 5) - 4 \Sigma (a^2 + r^2) (3 \cos 2 \theta - 1) \right] \]  
\quad - \frac{F_1 \Delta Mr^3 \Sigma^2 (3 \cos 2 \theta + 1)}{\mathcal{R}} - 4F_1 r^3 (2Mr - \Sigma) (Mr (3 \cos 2 \theta + 5) - 3 \Sigma) \]  
\quad \text{(A11)}

\[ K_3 = \frac{F_1 \sin 2 \theta (3 \cos 2 \theta + 1) (\Sigma (a^2 + r^2) - \Delta Mr) (\Sigma (a^2 + r^2) - 4a^2 M r \cos^2 \theta)}{2 \Sigma \mathcal{R} (1 - \frac{2M}{r})} \]  
\quad \text{(A12)}

\[ K_4 = \frac{4F_1 a^2 M^2 r^5 \sin^2 \theta (9 \cos 2 \theta - 1)}{2M - r} + F_2 \mathcal{R} \left[ -3a^2 Mr \cos 2 \theta + \cos 2 \theta (4a^2 Mr + 6 \mathcal{R}) + a^2 (-M)r - 2 \mathcal{R} \right] \]  
\quad \text{(A13)}

\[ K_5 = \frac{a^2 M r \sin^2 \theta}{\Sigma^2 (2M - r)} \left[ 4F_2 (r - 2M) \left[ a^2 M r \sin^2 \theta (9 \cos 2 \theta - 1) + \Sigma (a^2 + r^2) (3 \cos 2 \theta - 1) \right] - 4F_1 r^3 (3 \cos 2 \theta - 1) (2Mr - \Sigma) \right] \]  
\quad \text{(A14)}
\[ R_1 = \frac{BF_1 \sin 2\theta (3 \cos 2\theta + 1)(1 - 2M/r) \sqrt{\frac{\Sigma}{\Sigma}} (a^2Mr \cos 2\theta + 3a^2Mr - R)}{8\sqrt{Q}} \]  
(A15)

\[ R_2 = \frac{\Delta^2 F_1 \Sigma (3 \cos^2 \theta - 1)}{2\sqrt{QR} (1 - 2M/r)} \]  
(A16)

\[ R_3 = (1 - 3 \cos^2 \theta) \left[ F_1 r [2\Delta M^2 r^3 + \Sigma^2 (r - 2M)^2 - \Delta r^2 \Sigma] - F_2 \Delta (r - 2M) [-a^2 Mr \cos 2\theta + a^2 (Mr + \Sigma) + \Sigma (r^2 + \Sigma)] \right] \]  
(A17)

\[ R_4 = \frac{2F_1 a^2 M^2 r^4 \Sigma^2 \sin^2 \theta (1 - 9 \cos 2\theta)}{\Sigma^4 (1 - 2M)} - \frac{F_2 R}{\Sigma^4} \left[ \Sigma^2 (R - 3R \cos 2\theta) - 2a^2 Mr \Sigma^2 \sin^2 \theta (3 \cos 2\theta + 1) \right] \]  
(A18)

\[ R_5 = -\frac{4F_2 a^2 M \sin 2\theta}{r \Sigma^2} (a^2 Mr \sin^2 \theta (9 \cos 2\theta - 1) + \Sigma (a^2 + r^2) (3 \cos 2\theta - 1)) \]  
(A19)

\[ D_1 = \frac{BF_2 \Delta \sin^2 \theta (1 - 3 \cos^2 \theta) \left[ -a^2 M \Sigma (\cos 2\theta + 3) + 2a^2 Mr^2 (\cos 2\theta + 3) + 2r \Sigma^2 \right]}{4r^2 \sqrt{Q} \Sigma} \]  
(A20)

\[ D_2 = \frac{F_1 \Delta^2 r \Sigma (1 - 3 \cos^2 \theta)}{2R (2M - r) \sqrt{Q}} \]  
(A21)

\[ D_3 = \frac{F_2 \Delta (1 - 3 \cos^2 \theta)}{r^2 \sqrt{Q} \Sigma} \left[ 2a^2 Mr \sin^2 \theta + \Sigma (a^2 + r^2 + \Sigma) \right] \]  
(A22)

\[ D_4 = \frac{\Delta (3 \cos 2\theta + 1)}{\Sigma^3} \times \left[ \frac{r \Sigma (4a^2 M^2 r^5 \sin^2 \theta F_1' + R^2 (2M - r) F_1')}{4M - 2r} - \frac{4a^2 F_1 M^2 r^5 \sin^2 \theta (2r^2 (2M - r) + \Sigma (r - 3M))}{(r - 2M)^2} \right] \]  
(A23)

\[ D_5 = \frac{2a^2 Mr (3 \cos 2\theta + 1)}{\Sigma^2 (r - 2M)^2} \left[ F_1 r^3 \left( \Sigma (12M^2 r + \Sigma (r - 4M) - 2r^3) + 8Mr^3 (r - 2M) \right) - (2M - r) F_2 (2M - r) (8a^2 Mr^3 \sin^2 \theta + 2r^2 \Sigma (a^2 + r^2) + \Sigma^2 (a - r)(a + r)) + r \Sigma (r^3 F_1' (\Sigma - 2Mr) + RF_2' (r - 2M)) \right] \]  
(A24)

where \( F_1' \) with \( F_2' \) are derivatives of the functions over \( r \).