Dark solitons dynamics and snake instability in superfluid Fermi gases trapped by an anisotropic harmonic potential

Wen Wen and Changqing Zhao
Department of Mathematics and Physics, Hohai University,
Changzhou Campus, Changzhou 213022, China

Xiaodong Ma
College of Physics and Electronic Engineering,
Xinjiang Normal University, Urumchi 830054, China

(Dated: February 26, 2022)

Abstract

We present an investigation of generation, dynamics and stability of dark solitons in anisotropic Fermi gases for a range of particle numbers and trap aspect ratios within the framework of the order-parameter equation. We calculate the periods of dark solitons oscillating in a trap, and find a good agreement with the results based on the Bogoliubov-de Gennes equations. By studying the stability of initially off-center dark solitons under various tight transverse confinements in the unitarity limit, we not only give the criterion of dynamical stability, but also find that the soliton and a hybrid of solitons and vortex rings can be characterized by different oscillation period. The stability criterion is not fulfilled by the parameters of the recent experiment [Nature 499, 426 (2013)]. Therefore, instead of a very slow oscillation as observed experimentally, we find that the created dark soliton undergoes a transverse snake instability with collapsing into vortex rings, which propagate in soliton-like manner with a nearly two times larger period.

PACS numbers: 03.75.Ss, 03.75.Lm
I. INTRODUCTION

Dark solitons, namely localized density dips with a phase-jump across their density minimum, are the most fundamental nonlinear excitations in nonlinear dispersive media. They appear in many areas of science, such as water waves, nonlinear optics, biophysics and plasma and particle physics[1], and more recently in Bose-Einstein condensate[2–4]. Since the first observation of the crossover from a Bardeen-Cooper-Schrieffer (BCS) superfluid to a Bose-Einstein condensation (BEC) in ultracold fermionic atomic gases[5–9], understanding formation, dynamics and stability of solitons in a strongly interacting fermionic system has been attracted great attention[10, 11], and explored theoretically[12–19].

Based on the BCS mean-field theory[20], the existence and properties of black solitons in the BCS-BEC crossover were demonstrated[12] by the real solutions of the Bogoliubov-de Gennes (BdG) equations. The more general case of the complex solutions corresponding to grey solitons was also considered[13, 16]. Furthermore, the periodic dynamics of dark solitons in a harmonic trap[14, 15] and two solitons collision[16] were studied by numerically solving the time-dependent BdG equations. It was predicted that the oscillation period of a soliton in a harmonic trap increases as one moves from the BEC to BCS regimes[14, 15]. From the computational side, calculating solutions to both the time-independent and time-dependent BdG equations is numerically intensive, since they require a self-consistent calculations of single-particle states whose number increases linearly with the number of particles. For this reason, these investigations have essentially been restricted to Fermi gases confined in a box[12, 13] or one dimensional (1D) trapping potential[15, 16] and for a small number of atoms, that is essentially quasi-1D. However, it is not particularly relevant to current experimental setting.

In the very recent MIT experiment performed by Yefsah et al. with a fermionic superfluid of $^6$Li near a Feshbach resonance the long-lived solitons were observed[11]. These authors created dark solitons by phase imprinting in the cigar-shaped superfluid. Instead of in situ imaging solitons at the Feshbach resonance, the visualization of solitons has been relied on the method of time-of-flight. It was obtained after releasing the superfluid cloud from its trap and letting it expand with the rapid ramp to the weakly interacting BEC regime. The oscillation period of dark solitons in the unitary regime was measured, which is ten times larger than one predicted by the BCS mean-field theory[14, 15]. Note that Yefsah et al.
essentially have prepared the superfluid Fermi gases containing about $2 \times 10^5$ atom pairs, and confined in an external anisotropic harmonic trap, which is not satisfied by the quasi-1D condition. Hence it is of great interest in generation, dynamics and stability of dark solitons in a genuinely three-dimensional (3D) superfluid Fermi gas.

Different from previous investigations by the extend BdG equations [12–16], our theoretical investigations are based on the time dependent order-parameter equation [21, 22]. The order-parameter equation only can describe superfluidity features macroscopically, but its mathematical framework is simple that involves a single function of the coordinate, i.e. the superfluid density. Thus we encounter no limitation in the number of particles and external potentials, and analyze easily and clearly. Our calculation is carried out for a wide range of the number of atoms and trap aspect ratios. We not only present the oscillation periods of dark soliton in the trapped Fermi gases containing small number of atoms, but also find that as the number of atoms is increased by two orders, the period of stable solitons increases 8%. By examining the effects of transverse confinements on the stability of initially off-center solitons through their phase profiles, we give the criterion of dynamical stability. Finally we find that the dark soliton created in the MIT experiment is subject to snake instability, splitting into two vortex rings, and eventually reduces to one vortex ring, which performs a soliton-like oscillatory motion.

This paper is organized as follows: after a brief description of the order-parameter equation and numerical method in Sec.II, the dynamics of dark solitons in the quasi-1D regime is studied in Sec.III, and the soliton periods along the BCS-BEC crossover are compared with ones obtained from the BdG equations. In Sec.IV, the snake instability of dark soliton in the unitary limit is studied, and the dynamic stability criterion is given. The results of our calculation on the dark soliton dynamics in the parameters of the MIT experiment are presented in Sec.V. Finally a conclusion is given in Sec.VI.

II. MODEL AND METHOD

We consider an ultracold Fermi gas at zero temperature, in which fermionic atoms have two spin states with equal number. In a ground state all atoms are paired and in the superfluid state, which can be described by the following time dependent order-parameter
(or macroscopic wavefunction) equation

\[ i\hbar \frac{\partial \Psi_s}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2M} + V_s(\mathbf{r}) + \mu_s(n_s) \right] \Psi_s, \]  

where \( \Psi_s \) is the order parameter of fermionic atomic pairs in the superfluid state, with the superfluid density \( n_s = |\Psi_s|^2 \) and the normalized condition \( \int d\mathbf{r} |\Psi_s|^2 = N \) (\( N \) is the total atomic pair number of the superfluid Fermi gas, also equal to the number in each spin state). \( M \) is the mass of atom pair (i.e. \( M = 2m \) with \( m \) being atomic mass), and \( V_s(\mathbf{r}) \) is the external potential. We write the order parameter \( \Psi_s = \sqrt{n_s} e^{i\Phi_s} \) in terms of its amplitude \( \sqrt{n_s} \) and the phase \( \Phi_s \), which can be understood by the superfluid velocity \( \mathbf{v}_s(\mathbf{r}, t) = \hbar \nabla \Phi_s(\mathbf{r}, t)/M \).

Defining a dimensionless interaction parameter \( \eta \equiv 1/(k_F a_s) \), where \( k_F \) is the Fermi wavenumber and \( a_s \) is \( s \)-wave scattering length, one can distinguish several different superfluid regimes: BCS regime (\( \eta \leq -1 \)), BEC regime (\( \eta \geq 1 \)), and BCS-BEC crossover regime (\( -1 < \eta < 1 \)). A special case \( \eta = 0 \) is called unitarity limit where the scattering length is infinity. The BCS-BEC crossover regime is a strongly interacting regime, and BCS (\( \eta \ll -1 \)) and BEC (\( \eta \gg 1 \)) limits are actually weakly interacting. In general, the expression for the equation of state \( \mu_s(n_s) = 2\mu(n) \) with \( n = 2n_s \) being atomic density is very complicated, but it can be fitted by the analytical formula based on the Monte Carlo data\[30\], and approximated by the polytropic approximation\[31–33\]

\[ \mu_s(n_s) = 2\mu^0 \left( \frac{n_s}{n^0} \right)^\gamma, \]  

\[ \gamma = \gamma(\eta) = n \frac{\partial \mu}{\partial n} = \frac{\frac{2}{3} \sigma(\eta) - \frac{2n}{5} \sigma'(\eta) + \frac{(n^0)^2}{15} \sigma''(\eta)}{\sigma(\eta) - \frac{2n}{5} \sigma'(\eta)}, \]

where \( \mu^0 \) and \( n^0 \) are respectively reference chemical potential and particle number density\[33\]. Usually we take reference particle number density \( n^0 = (2mE_F)^{3/2}/(6\pi^2\hbar^3) \) to be the per spin density of the non-interacting Fermi gas at the trapping center, and reference chemical is thus \( \mu^0 = E_F(\sigma(\eta) - \eta\sigma'(\eta)/5) \) proportional to the Fermi energy \( E_F = (\hbar k_F)^2/(2m) \). The order-parameter equation incorporated with the equation of state allows one to investigate the smooth crossover from the BEC limit to BCS regime\[34\] in a unified way.

It is noticed that in the unitarity limit (\( \eta = 0, \gamma = 2/3 \)), the order-parameter equation is exactly equivalent to one derived by Salasnich \textit{et al}. from an extended Thomas-Fermi density functional theory\[21\]. In the BEC limit (\( \eta \gg 1, \gamma = 1 \)), the order-parameter equation coincides exactly with one derived by Pieri and Strinati based on BdG equations\[35\].
It has been demonstrated that the order-parameter equation is very reliable to capture ground-states properties \cite{27, 32} and low-energy collective dynamics \cite{27, 33} in the BCS-BEC crossover. Furthermore, the results given by the order-parameter equation in the BEC side of the crossover are found to be in good agreement with the zero-temperature BdG equations \cite{36, 37}. However, the order-parameter equation can not completely capture dynamical properties in the BCS regime ($\eta < 0$) for the reason that dynamical behaviors can easily result in pair breaking due to very small gap energy \cite{38}, while the order-parameter equation ignores single atom excitation.

We consider a cylindrically symmetric harmonic trap

$$V_s(r) = \frac{1}{2}M\omega_z^2(\lambda^2r^2 + z^2),$$

(3)

where ($r, z$) are cylindrical coordinates with $r = \sqrt{x^2 + y^2}$. The aspect ratio (anisotropy) of the trap is defined by $\lambda = \omega_\perp/\omega_z$, with the trapping frequencies $\omega_\perp$ and $\omega_z$. So the Fermi energy for the fermions trapped by a 3D harmonic potential is given by $E_F = \hbar(6N\omega^2_\perp\omega_z)^{1/3}$. The axial trapping frequency is $\omega_z = 2\pi \times 10.66$Hz in the MIT experiment \cite{11}, and the transversal frequency $\omega_\perp = \lambda \omega_z$ is determined by the fixing axial frequency and different aspect ratios. We choose total number of atom pairs in a wide range $N = 2 \times 10^6 \sim 2 \times 10^2$. In the following, the length is in units of $a_z \equiv \sqrt{\hbar/(M\omega_z)} = 8.89$µm and time is in units of axial trapping period $T_z = 93.76$ms \cite{11}. Density profiles are presented by normalized cross-sectional densities at the $y = 0$ plane, i.e. $n_s(x, z) = \frac{a_z^3}{2\sqrt{2N}}|\Psi_s(x, y = 0, z)|^2$.

Recently we have presented the dark soliton solutions of quasi-1D order-parameter equation by the multiple scale method in the small-amplitude limit \cite{18}. Later bell solitons along the BCS-BEC crossover as exact soliton solutions of the order-parameter equation in arbitrary amplitudes were found analytically \cite{19}. Dark solitons have been created in Fermi gases by using phase imprinting technique \cite{11}, which originally has been proposed to generate vortices and solitons in weakly interacting atomic BECs \cite{39}, and experimentally implemented \cite{40, 41}. The main ideal of this technique is described as shining an off-resonance laser on a condensate in order to create phase steps between its different parts. Instead of exposing the analytic 1D soliton solution to 3D, we simulate the phase imprinting method to generate initial solitons in anisotropic Fermi gases along the BCS-BEC crossover.

We solve the order-parameter equation by discretizing with the split-step Crank-Nicolson algorithm \cite{42}. The initial dark soliton is created by employing imaginary time propagation.
FIG. 1. (Color online) Dark solitons dynamics in anisotropic superfluid Fermi gases in the unitarity limit, with the total atom pairs \( N = 2 \times 10^2 \) and aspect ratio \( \lambda = 6.5 \). Plane(a) is the initial density profiles of dark soliton generated at the axial position \( z_0 = 3R_z/4 \) with \( R_z \) being the axial half length. Plane (b), (c) and (d) are snapshots of dark soliton dynamics at times \( t = 0.4T_z \), \( t = 0.8T_z \), and \( t = 3.6T_z \), respectively. Light/dark regions indicate high/low densities. Plane (e) is the spatiotemporal contour plot of the axial density at \( r = 0 \) plane. The dark soliton oscillates in the trap with the first period of 1.75\( T_z \) and the reduced second of 1.57\( T_z \). Subject to an enforced axially-symmetric \( \pi \) phase step, and then its dynamics is calculated by using real time propagation.

III. THE SOLITON PERIODS ACROSS THE BCS-BEC Crossover

The beginning of our calculation is to apply the parameters \( N = 2 \times 10^2 \) and \( \lambda = 6.5 \) for the superfluid Fermi gas in the unitarity limit\( (\eta = 0) \). By imposing \( \pi \) phase step at the axial position \( z_0 = 3R_z/4 \) with \( R_z \) being the axial half length, the density profile evolves into that of an axially-symmetric dark soliton shown in Fig.1(a). This off-center dark soliton under a harmonic confinement is expected to oscillate back and forth along the trap as a quasiparticle[43-45]. Fig.1(b)-(d) depict the density profiles of the dark soliton at different evolution times. As shown in Fig.1(b), the initial black soliton moves towards the trap center at \( t = 0.4T_z \) from the right side, in which it becomes a shallower gray soliton due to
FIG. 2. (Color online) The spatiotemporal contour plots of dark soliton dynamics in superfluid Fermi gases containing $N = 2 \times 10^2$ atomic pairs. (a) corresponds to $\eta = 0$ (the unitarity limit) with the aspect ratio $\lambda = 15$, and (b) for $\eta = 0$ with $\lambda = 30$; (c) and (d) are for $\eta = 1.0$ (the BEC side) and $\eta = 6.0$ (the BEC limit), respectively, both with $\lambda = 30$.

the higher density near the center and the faster it gets. For the timescale of axial trapping period $t = 0.8T_z$ in Fig.1(c), the soliton is prone to be instable, emitting radiation in the form of sound waves. After a long live oscillations accompanied by the sound waves at $t = 3.6T_z$, the dark soliton completely decays into a train of sound waves (see Fig.1(d)).

In order to monitor the soliton trajectory, we give the spatiotemporal evolution of density along a cross section at $r = 0$ in Fig.1(e), where the soliton and sound waves are indicated by the light regions. Fig.1(e) provides a clearer indication of dynamical instability in the form of sound radiation, and gives the first period of the dark soliton oscillating in the trap to be $T_s = 1.75T_z$. It is shown that the dissipation of energy by the sound waves from the soliton is associated with increase of the amplitude of oscillation (anti-damping), and soliton becomes shallower and, as a result, it accelerates with the second oscillation period of $1.57T_z$.

Such dynamical decay of a moving soliton via the emission of sound waves can be accounted for two instability mechanisms in the framework of the order-parameter equation: (i) axial background inhomogeneity due to the trapping potential [44, 46, 48], and (ii) the effects of transverse degree of freedom coupled with the axial degree by the atomic
FIG. 3. (Color online) Periods of dark solitons in the quasi-1D regime along the BCS-BEC crossover. Our results(□) based on the order-parameter equation are compared with ones from the the BdG equations by Scott et al.(△)[14] and Liao et al.(▽)[15]. The horizontal dashed line indicates the well-known period $\sqrt{2}T_z$ for atomic BECs. The inset shows the periods of the soliton oscillating in the unitarity limit as a function of the aspect ratio.

The relevant size of solitons in the crossover from the BEC limit up to the unitary limit can be characterized by the healing length $\xi = \hbar/\sqrt{2M\mu_s}$, with local chemical potential $\mu_s(r)$ depending on the spatially inhomogeneous density $n_s(r)$. The local chemical potential is determined by the ground state solution of the order-parameter equation Eq.(1), that is $\mu_s(r) + V_s(r) = \mu_G$. By using the normalized condition, one can obtain the bulk chemical potential $\mu_G$[33]. In the presence of the harmonic trap, the axial size of the superfluid $R_z = \sqrt{2\mu_G/M\omega_z^2}$ is set by the trapping frequency. For the case of Fig.1, we find that $R_z/\xi = 2\sqrt{\mu_s\mu_G}/\hbar\omega_z = 63$, which means that the change in the axial background is very weak over the size of soliton.

On the other hand, the characteristic size of the soliton is the order of the healing length, thus the corresponding (axial) kinetic energy is the order of $\mu_s$. The dimensionality parameter defined by $\alpha = \mu_s/\hbar\omega_\perp = 3.22$ implies that the kinetic energy is larger than the transverse energy $\hbar\omega_\perp$, and the atomic interaction can induce transfer of axial energy to the transversal degrees of freedom. Therefore, we conclude that the dominant decay mechanism is due to the coupling to transverse modes.
To suppress the dynamical instability, we perform a stronger transverse confinement of their motion, that is $\lambda = 15(\alpha = 2.44$ correspondingly). Therefore, the growth of transverse energy $\hbar \omega_\perp$ suppresses the transfer of kinetic energy of the soliton to transverse modes and resulting in the decay of the soliton(see Fig.2(a)). For a tight enough trapping potential($\lambda = 30, \alpha = 1.93$) in Fig.2(b), the instability does not occur as a consequence of possibility of separation of the axial and transversal degrees of the freedom. It is seen that the period of the stable soliton oscillation is $T_s = 1.7T_z$, which agrees well with one calculated by the BdG equations[14, 15] for the case in the unitarity limit. As shown in the inset of Fig.3, the soliton periods have a relatively weak dependence on the anisotropy, that is larger values of $\lambda$ yield smaller periods. Notice that under the conditions of tight transverse confinement $\lambda = 30$ and small pairs number $N = 2 \times 10^2$, the system is in the quasi-1D regime that corresponds to the cases discussed by the mean-field theory[14, 15]. We also present the spatiotemporal evolutions of dark solitons in the quasi-1D regime for the cases of the BEC side in Fig.2(c) and BEC limit in Fig.2(d), respectively.

Fig.3 shows our results (□) on the periods $T_s$ of dark soliton in the quasi-1D regime as a function of the dimensionless interaction parameter $\eta \equiv 1/k_Fa_s$. The decrease of $T_s$ as one moves from the BCS regime to the BEC limit is consistent with the mean-field theory computations by Scott et al.($\triangle$)[14] and Liao et al.($\triangledown$)[15]. In the BEC limit($\eta = 6$), our result is very close to the well-known value $\sqrt{2}T_z$ for atomic BECs[43–45], which is indicated by the horizontal dashed line in Fig.3. We find that in the BEC regime($\eta > 0$), our results are slightly larger than ones based on the extended BdG equations. It is because that the BdG equations can not obtain the beyond mean-field term of the equation of state correctly[9]. Notice that the order-parameter equation fails to give correct results in the BCS side. For comparison purposes, we also present our results in the BCS side($\eta < 0$). Interestingly, different from the small discrepancy in the BEC regime, in the BCS side the BdG results are significantly larger than our results, which may be interpreted by the coupling to fermionic quasiparticle excitations near the soliton[14] that are completely disregarded in the order-parameter equation.
FIG. 4. (Color online) The snapshots of the stationary dark soliton dynamics in the unitarity limit with \( \lambda = 6.5 \). Planes (a), (b) and (c) correspond to the total numbers of atom pairs \( N = 2 \times 10^2 \), \( N = 2 \times 10^3 \) and \( N = 2 \times 10^5 \), respectively.

IV. SNAKE INSTABILITY OF UNITARY FERMI GASES

Dark solitons have 1D character, which are stable in the quasi-1D regime, but feature a long-wavelength transverse instability known as the "snake instability" [52–56], when extended into higher dimensions. The snake instability originates from the transfer of the soliton kinetic energy to the transverse modes parallel to the soliton nodal plane. Generally dark solitons undergo a snake deformation, causing the nodal plane collapse into vortex rings [54–56] in 3D (or vortex-antivortex pairs in two dimensions [53]). For atomic BECs, it has been shown that this instability leads to a strong bending of the nodal plane, which breaks down into vortex rings and sound waves, as experimentally observed [57, 58].

The snake stability of dark solitons of superfluid Fermi gases in the unitarity limit can be studied by monitoring the evolution of a standing dark soliton created at the trap center [52–56]. The development of the snake instability and the concomitant vortex rings for a wide range of the numbers of atom pairs with \( \lambda = 6.5 \) are displayed in Fig.4(a) for \( N = 2 \times 10^2 \), Fig.4(b) for \( 2 \times 10^3 \), and Fig.4(c) for \( 2 \times 10^5 \), respectively. Vortex rings resemble toroids where the superfluid density is depleted, and so the slice of the vortex ring appears as two dark spots separated vertically. As shown in Fig.4(a), the stationary dark soliton is
subject to the snake instability, bending into one vortex ring, in strongly contrast to the moving soliton slowly decaying into sound radiation (see Fig.1). This is due to the larger dimensionality parameter $\alpha = \mu_G / \hbar \omega_\perp = 7.38$ at the trap center. As the number of atom pairs rises, the increased chemical potential opens more decay channels, which result in the formation of more vortex rings, that is $3 \alpha = 15.90$ in Fig.4(b) and $14 \alpha = 73.80$ in Fig.4(c). For comparison purposes, the results for the three cases are presented on the same spatial grid. From plane (a) to (c), the chemical potential increases tenfold and inversely the size of solitons and vortex rings decreases threefold. We can estimate the size of the nonlinear excitations for the experimental relevant case of $N = 2 \times 10^5$ in Fig.4(c). The size scale is given by $\xi = R_\perp / 959 = 0.29 \mu m$, which is too small to be resolved by optical means directly, but by the time of flight expansion acting as a magnifying glass [11]. In addition, we find the time for the start of snake instability reduces significantly from $0.32 T_\perp$ in Fig.4(a) to $0.05 T_\perp$ in Fig.4(c).

Now we show that how the instability mechanism can be suppressed under tight transverse confinements [59] and determine the criterion for stability against the transverse decay [49]. We consider the superfluid Fermi gases containing $N = 2 \times 10^4$ atom pairs in the unitarity limit, and examine the evolution of dark solitons generated at an off-center position of $3R_\perp / 4$. Fig.5 shows the density (left planes) and corresponding phase (right planes) of the evolutions of the solitons at the time when the solitons reach the trap center. Plane (a), (b), (c), (d), and (e) correspond to $\lambda = 6.5, 50, 100, 180$ and 250, and the evolution time $t = 0.56 T_\perp, 0.53 T_\perp, 0.47 T_\perp, 0.45 T_\perp,$ and $0.42 T_\perp$, respectively.

With a weak transverse trapping $\lambda = 6.5$ in Fig.5(a), the created soliton subjecting snake instability is decay to vortex rings, one of which reaches the trap center at $t = 0.56 T_\perp$. Vortex rings can be evidenced by the $2\pi$ phase change at any point of the circle, as shown in the right plane of Fig.5(a). Increasing the transverse frequency $\lambda = 50$ in Fig.5(b) leads to a decrease in the bending of the soliton, and hence the production of a single vortex ring. It is seen that only by observing dips in the density profile from the left plane of Fig.5(b), it is very hard to discriminate between soliton and vortex rings. We find that the vortex ring evolves back into a soliton, when moving near the ends of the trap due to the decrease of the dimensionality parameter. It is seen that such periodic soliton/vortex ring is stable with a oscillation period of $2.2 T_\perp$, which was observed firstly in atomic BECs [60]. In the geometries ($\lambda = 100$ in Fig.5(c) and $\lambda = 180$ in Fig.5(d)) where the soliton is transversely unstable but the
FIG. 5. (Color online) Close-up snapshots of density (left) and phase (right) profiles of the evolutions of the off-centered dark solitons initially generated at $3R_z/4$, when they evolve near the center of the unitary Fermi gases with $N = 2 \times 10^4$ for different transverse confinement. Plane (a), (b), (c), (d), and (e) correspond to $\lambda = 6.5, 50, 100, 180$ and 250, and evolution time $t = 0.56T_z, 0.53T_z, 0.47T_z, 0.45T_z$, and $0.42T_z$, respectively.

transverse width of the system is too small to support vortex rings, an excitation with soliton and vortex properties, known as a hybrid of soliton and vortex rings[60, 61], is predicted to occur. The hybrid of solitons and vortex rings can be evidenced in the right planes, by the emergence of not only phase azimuthal dependence, but also phase jump that is the character of soliton. We find that the oscillation period of the hybrid of soliton and vortex rings is $2.0T_z$ for the case of Fig.5(c). Finally, very tight transverse confinement ($\lambda = 250$) results in a highly elongated quasi-1D geometry, evidencing by a stable soliton with its step phase profile of Fig.5(e). The period of the stable soliton in such highly elongated geometry is found to be $1.83T_z$, only 8% larger than one in the quasi-1D regime.

Therefore, the criterion of dynamical stability of dark solitons in trapped unitary Fermi gases can be estimated[49] by the case of Fig.5(e), that is $\alpha_c = \mu_s/h\omega_\perp = 4.2$. We find that the stability criterion, that is $\alpha < \alpha_c$, is very strict for the conditions of current experiments. For the system with total number of atom pairs of $10^5$ in most Fermi experiment, it requires at least $\lambda = 1500$ in the unitarity limit when solitons are generated in the off-center position of $3R_z/4$, while only $\lambda = 350$ for $\alpha_c = 2.4$ of weakly interacting atomic BECs[52].

12
FIG. 6. (Color online) Close-up snapshots of the evolution of density (top) and corresponding phase profiles (bottom) for the dark dynamics in the unitary Fermi gas with the MIT experimental parameters of $N = 2 \times 10^5$ and $\lambda = 6.5$ at (a) $t = 0$, (b) $t = 0.07T_z$, (c) $t = 0.12T_z$ and (d) $t = 0.24T_z$, showing the onset of the snake instability and the decay of the soliton into two vortex rings as evidenced by the corresponding phase profiles.

V. COMPARISON WITH EXPERIMENT

In the MIT experiment[11], the superfluid Fermi gas containing $2 \times 10^5$ atom pairs was prepared in a cylindrically symmetric trap with $\omega_z = 2\pi \times 10.66$Hz and $\lambda = 6.5$. In order to observe dark solitons dynamics in the Fermi gas, they optically applied a step-function potential to advance a $\pi$ phase shift of the superfluid order parameter, thereby imprinting a moving soliton at the off-center position.

We perform numerical simulations using the experimental parameters in the unitary limit. The results of the close-up snapshots of density profiles (top row) of dark soliton dynamics and corresponding phase profiles (bottom row) are presented in Fig.6. In Fig.6(a), the initial soliton has a node of zero density at $3R_z/4$ (top) and a $\pi$ phase step (bottom). As the soliton starts to move at $t = 0.07T_z (6.56 \text{ms})$, the soliton plane is dynamically unstable subjecting to a gradual bending shown in Fig.6(b), which is resulted from the inhomogeneous transverse density. Subsequently in Fig.6(c), the soliton plane tears into pieces, creating two vortex rings and radiating sound waves. Vortex ring is evidenced by a $2\pi$ phase dependence...
azimuthally in the bottom plane. At 11.2 ms (Fig. 6(c)), the produced two vortex rings propagate in opposing directions, that is one propagates to the left and another to the right. Finally the vortex ring propagating rightly is absorbed by the boundary, and only the left vortex ring survives. Note that the dimensionality parameter at $3R_z/4$ position is $\alpha = 32$, much larger than the stability criterion $\alpha_c = 4.2$ estimated by us, so such decay channel can be anticipated.

It is interesting to investigate the evolution of the survival vortex ring in the trapped Fermi gas. As shown in Fig. 7, we find that the vortex ring is stable and presents a oscillation in a soliton-like manner, which is highlighted by red lines. Interestingly, after reflecting by the left end, it is also disappear when moves to the right boundary, thus only a single oscillation period can be observed. This period is given by about $2.8T_z$ that is nearly two times than the soliton period of $1.7T_z$, and also different from the period of $2.0T_z$ for a hybrid of solitons and vortex rings. The results suggest that in a strongly interacting superfluidity to probe 1D character of stable solitons needs very small number of atoms or very large harmonic trapping frequency against the snake instability, which is hard to be fulfilled by current experimental situation. But it may be an ideal system to experimentally study the dynamics of vortex rings or hybrids of solitons and vortex rings resulting from the snake

FIG. 7. (Color online) Snapshots of the oscillation of a vortex ring created by the snake instability of the unitary Fermi gases from Fig. 6 at (a) $t = 0.4T_z$ as well as (d) corresponding phase, (b) $t = 2.0T_z$, and (c) $t = 2.8T_z$. 


instability, which can be distinguished by measuring different oscillation periods or their phases profiles as we discussed.

VI. CONCLUSIONS

We perform the calculations for the dynamics and stability of dark solitons in anisotropic superfluid Fermi gases for a wide range of atomic particle numbers and ratio aspects within the framework of the order-parameter equation. We study the dynamics of solitons in the trapped superfluid Fermi gases with small number of atoms, and the computed soliton period of $1.7T_z$ in the unitary limit is in a good agreement with one by the BdG equations. The snake instability of unitary Fermi gases is studied. By examining the evolutions of an initially off-center dark soliton under various aspect ratios, and a transition from hybrids of solitons and vortex rings to stable soltion discriminated by their phase profiles, we give the criterion for stability against transverse decay, that is $\alpha_c = 4.2$. In addition, it is found that the soliton period increases 8% as the number of atom pairs is increased two orders, and the hybrid of solitons and vortex rings has a larger period of $2.0T_z$. We simulate the recent MIT experiment on the dark soliton dynamics in the unitary Fermi gas. Instead of performing a very slow oscillation as observed experimentally, the imprinted soliton is found to evolve into vortex rings, which propagate in soliton-like manner with a period of $2.8T_z$. Such disagreement between theory and experiment may be accounted for the time-of-flight method with the Feshbach resonance, which will be considered in the future.

After finishing this manuscript, we became aware of Ref. [62] and Ref. [63] addressing the snake instability of the unitary Fermi gases using different theoretical frameworks, which reached the same conclusions.

ACKNOWLEDGMENTS

We gratefully acknowledge Renyuan Liao and Hui Zhai for enlightening discussions and helpful comments. W.W. is supported by the NSFC under Grant No. 11105039, the Fundamental Research Funds for Central Universities of China (program No. 2012B05714 and 2010B23414), and Doctoral Foundation of Hohai university 2010. X.D.M. is supported by the NSFC Grant No. 11264039 and the Key Research Project of Xinjiang Higher Education,
China under Grant No. XJED2010141.

[1] T. Dauxois and M. Peyrard, *Physics of Solitons* (Cambridge University Press, Cambridge, 2006).

[2] C. J. Pethick and H. Smith, *Bose-Einstein condensation in Dilute Gases*, 2nd edn. (Cambridge University Press, Cambridge, 2008).

[3] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, ed. *Emergent Nonlinear Phenomena in Bose-Einstein Condensates* (Springer-Verlag, Berlin, 2008).

[4] D. J. Frantzeskakis, J. Phys. A:Math. Theor. **43**, 213001 (2010).

[5] K. M. O'Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, and J. E. Thomas, Science **298**, 2197 (2002).

[6] C. A. Regal, M. Greiner and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004).

[7] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. **92**, 120401 (2004).

[8] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, Phys. Rev. Lett. **92**, 120403 (2004).

[9] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008), and references therein.

[10] A. Adams, L. D. Carr, T. Schäfer, P. Steinberg and J. E. Thomas, New J. Phys. **14**, 115009 (2012).

[11] T. Yefsah, A. T. Sommer, M. J. H. Ku, L. W. Cheuk, W. Ji, W. S. Bakr, and M. W. Zwierlein, Nature **499**, 426 (2013).

[12] M. Antezza, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A **76**, 043610 (2007).

[13] A. Spuntarelli, L. D. Carr, P. Pieri, and G. C. Strinati, New J. Phys. **13**, 035010 (2011).

[14] R. G. Scott, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **106**, 185301 (2011).

[15] R. Liao and J. Brand, Phys. Rev. A **83**, 041604(R) (2011).

[16] R. G. Scott, F. Dalfovo, L. P. Pitaevskii, S. Stringari, O. Fialko, R. Liao and J. Brand, New. J. Phys. **14**, 023044 (2012).

[17] A. Bulgac, Y. L. Luo, and K. J. Roche, Phys. Rev. Lett. **108**, 150401 (2012).
[18] W. Wen and G. X. Huang, Phys. Rev. A 79, 023605 (2009).
[19] A. Khan and P. K. Panigrahi, J. Phys. B: At. Mol. Opt. Phys. 46, 115302 (2013).
[20] P. G. de Gennes, Superconductivity of Metals and Alloys (Addison-Wesley, New York, 1989).
[21] L. Salasnich, N. Manini, and F. Toigo, Phys. Rev. A 77, 043609 (2008); L. Salasnich and F. Toigo, ibid. 78, 053626 (2008).
[22] W. Wen, Y. Zhou, and G. X. Huang, Phys. Rev. A 77, 033623 (2008).
[23] Y. E. Kim and A. L. Zubarev, Phys. Rev. A 70, 033612 (2004).
[24] A. L. Zubarev, J. Phys. B: At. Mol. Opt. Phys. 42, 011001 (2009).
[25] S. K. Adhikari, Phys. Rev. A 77, 045602 (2008).
[26] F. Ancilotto, L. Salasnich and F. Toigo, Phys. Rev. A 85, 063612 (2012).
[27] S. K. Adhikari, J. Phys. B: At. Mol. Opt. Phys. 43, 085304 (2010).
[28] G. Rupak and T. Schäfer, Nucl. Phys. A 816, 52 (2009).
[29] H. W. Xiong, S. J. Liu and M. S. Zhan, Phys. Rev. A 74, 033602 (2006).
[30] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. 93, 200404 (2004).
[31] N. Manini and L. Salasnich, Phys. Rev. A 71, 033625 (2005).
[32] G. Diana, N. Manini, and L. Salasnich, Phys. Rev. A 73, 065601 (2006).
[33] W. Wen, S.-Q. Shen, and G. X. Huang, Phys. Rev. B 81, 014528 (2010).
[34] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. 59, 195 (1985).
[35] P. Pieri and G. C. Strinati, Phys. Rev. Lett. 91, 030401 (2003).
[36] L. Salasnich, F. Ancilotto, N. Manini, and F. Toigo, Laser Phys. 19, 636 (2009).
[37] F. Ancilotto, L. Salasnich and F. Toigo, Phys. Rev. A 79, 033627 (2009).
[38] H. Zhai and T.-L. Ho, Phys. Rev. Lett. 97, 180414 (2006).
[39] L. Dobrek, M. Gajda, M. Lewenstein, K. Sengstock, G. Birkl and W. Ertmer, Phys. Rev. A 60, R3381 (1999).
[40] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 83, 5198 (1999).
[41] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, Science 287, 97 (2000).
[42] P. Muruganandam and S. K. Adhikari, Comput. Phys. Commun. 180, 1888 (2009).
[43] Th. Busch and J. R. Anglin, Phys. Rev. Lett. 84, 2298 (2000).
[44] G. X. Huang, J. Szeftel and S. H. Zhu, Phys. Rev. A 65, 053605 (2002).
[45] V. A. Brazhnyi, V. V. Konotop, and L. P. Pitaevskii, Phys. Rev. A 73, 053601 (2006).
[46] N. G. Parker, N. P. Proukakis, M. Leadbeater, and C. S. Adams, Phys. Rev. Lett. 90, 220401 (2003); N. G. Parker, N. P. Proukakis, M. Leadbeater and C. S. Adams, J. Phys. B: At. Mol. Opt. Phys. 36, 2891 (2003).
[47] V. V. Konotop and L. Pitaevskii, Phys. Rev. Lett. 93, 240403 (2004); V. A. Brazhnyi and V. V. Konotop, Phys. Rev. A 68, 043613 (2003).
[48] D. E. Pelinovsky, D. J. Frantzeskakis, and P. G. Kevrekidis, Phys. Rev. E 72, 016615 (2005).
[49] A. Muryshev, G. V. Shlyapnikov, W. Ertmer, K. Sengstock, and M. Lewenstein, Phys. Rev. Lett. 89, 110401 (2002).
[50] G. Theocharis, P. G. Kevrekidis, M. K. Oberthaler, and D. J. Frantzeskakis, Phys. Rev. A 76, 045601 (2007).
[51] A. Weller, J. P. Ronzheimer, C. Gross, J. Esteve, M. K. Oberthaler, D. J. Frantzeskakis, G. Theocharis and P. G. Kevrekidis, Phys. Rev. Lett. 101, 130401 (2008).
[52] A. E. Muryshev, H. B. van Linden van den Heuvell, and G. V. Shlyapnikov, Phys. Rev. A 60, R2665 (1999).
[53] G. X. Huang, V. A. Makarov and M. G. Velarde, Phys. Rev. A 67, 023604 (2003).
[54] D. L. Feder, M. S. Pindzola, L. A. Collins, B. I. Schneider, and C. W. Clark, Phys. Rev. A 62, 053606 (2000).
[55] J. Brand and W. P. Reinhardt, Phys. Rev. A 65, 043612 (2002).
[56] P. G. Kevrekidis, G. Theocharis, D. J. Frantzeskakis, and A. Trombettoni, Phys. Rev. A 70, 023602 (2004).
[57] Z. Dutton, M. Budde, C. Slowe, and L. V. Hau, Science 293, 663 (2001).
[58] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Phys. Rev. Lett. 86, 2926 (2001).
[59] N. P. Proukakis, N. G. Parker, D. J. Frantzeskakis and C. S. Adams, J. Opt. B: Quantum Semiclass. Opt. 6, S380 (2004).
[60] S. Komineas and N. Papanicolaou, Phys. Rev. Lett. 89, 070402 (2002); S. Komineas and N. Papanicolaou, Phys. Rev. A 67, 023615 (2003).
[61] I. Shomroni, E. Lahoud, S. Levy and J. Steinhauer, Nature Phys. 5, 193 (2009).
[62] A. Bulgac, M. M. Forbes, M. M. Kelley, K. J. Roche, and G. Wlazłowski, e-print arXiv:1306.4266.

[63] A. Cetoli, J. Brand, R. G. Scott, F. Dalfovo, and L. P. Pitaevskii, e-print arXiv:1307.3717.