Stability of impulsive switched systems with sampled-data control

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Abstract
This paper investigates the sampled-data control of impulsive switched systems with asynchronous switching. The impulses are not required to synchronise the switching. Since it is possible that switches may occur in sampling intervals, the mismatched problem may happen between controllers and system modes. Aiming for this, a functional consisting of multiple impulse-dependent Lyapunov functions and looped functionals is constructed, which does not increase at impulsive times. By using ADT method, some sufficient conditions for exponential stability are proposed in terms of linear matrix inequalities. Furthermore, sampled-data controllers are presented to stabilise the impulsive switched systems. The efficiency of the proposed results is verified by an F-18 aircraft.

1 INTRODUCTION

Hybrid systems are composed of the continuous dynamic behaviour, which is governed by differential equations, and the discrete dynamic behaviour [1]. Switched systems [2–5] and impulsive systems [6, 7] are two important classes of hybrid systems. Switched systems consist of some subsystems described by a collection of continuous dynamics and a switching rule deciding the mode transition among these subsystems. Impulsive systems exhibit continuous evolutions described by differential equations and instantaneous state jumps. Such systems have a lot of practical applications in a broad range of areas, for example, networked control systems [8–10], robot control systems [11], flight control systems [12]. As is known, stability analysis and stabilisation are the fundamental problems. Many methods such as the piecewise Lyapunov function method and the average dwell time (ADT) method are proposed for switched or impulsive systems and some marked results can be found in [13–20].

In practice, impulses and switches exist synchronously in many physical systems, which are usually called impulsive switched systems and have been intensely studied. Up to now, many results have been obtained. In [21], a new adaptive control scheme for switched systems is proposed. Some sufficient conditions for input/output-to-state stability (IOSS) are addressed by using Lyapunov and ADT methods in [22]. [23] studies robust stabilisation for a class of uncertain impulsive switched systems. Input-to-state stability (ISS) and integral input-to-state stability (iISS) for a class of impulsive switched systems under asynchronous switching is studied in [24].

However, these results only focus on the case where switches and impulses are synchronous. In fact, some practical systems cannot be simply modelled as dynamic systems mentioned above. This is because a switching rule that defines the switching signal may be unknown. Switching may be caused by unpredictable environmental factors or component failures thus the switching times are more likely not at the same time with impulses. For example, a given chemical process prone to component failures, an automobile running in a harsh environment, a robotic manipulator moving different specific loads or an aircraft flying in different flight conditions is better to be modelled as impulsive switched systems, where the switching and impulse times are different, for example, [25–27]. To be specific, a simplified F-18 aircraft is provided in [27] with a family of modes
determined by Mach number and flight altitude. As the aircraft enters different flight altitudes and flies at different Mach numbers, the subsystem describing the dynamic of the flight changes accordingly. On top of that, since it is possible that the aircraft is subjected to shock effects, the system state including the angle of attack and the pitch rate suffers jumps. Therefore, the simplified F-18 aircraft system can be modeled as an impulsive switched system. Hence, this paper considers the scenarios where switching and impulsive time instants are not necessary synchronous. This would inevitably bring the difficulty of analysis because two kinds of discontinuities in the state have to be taken into account. There are only a few results published, see [28–30]. In [28, 29], the stability for impulsive switched time-delay systems with time and state-dependent impulses is investigated. The IOSS and integral IOSS for non-linear impulsive switched delay systems are studied in [30]. In addition, because of the developments of hard-ware technology and digital technologies, sampled-data controllers are more favorable in practical applications. Some insightful results can be seen in [31–33]. In [31], discontinuous Lyapunov functions are introduced for impulsive systems under the variable and bounded sampling. [32] improves the results on sampled-data systems by using new discontinuous Lyapunov functions. Based on the discrete-time Lyapunov theorem, [33] provides stability criteria with looped functionals for the continuous-time model. Nevertheless, when applying sampled-data controllers to impulsive switched systems, the approaches proposed in [28–30] are difficult to handle the potential mismatched problem. The mismatched problem results from the unknown switching times. Specifically, it is because it will take a while to identify the system mode and switching time and then apply the matched controller, which means that there inevitably exists time delay between the system mode and the controller, resulting in the asynchronous switching. Still, the methods used in [31–33] can also not be applied to impulsive switched systems because the effect of switchings are not involved in these works and thus the mismatched problem cannot be addressed.

Based on the above discussions, in this paper, we study the stability and stabilisation problem of impulsive switched systems with sampled-data control. A functional consisting of multiple impulse-dependent Lyapunov functions and looped functionals is constructed, which does not increase at impulsive time instants. By using the ADT method, some sufficient conditions for exponential stability are given in terms of linear matrix inequalities. Furthermore, exponential stabilisation conditions are derived to design sampled-data controllers. As an application, these results are applied to ensure the stability of the simplified F-18 aircraft given in [27].

The remainder of this paper is arranged as follows: Section 2 describes the model of impulsive switched system via sampled-data control and some relevant notations. The multiple impulse-dependent Lyapunov functions and looped functionals are introduced in Section 3. Stability and stabilisation conditions are provided in Sections 4 and 5, respectively. An example is used to demonstrate the efficiency of the obtained results in Section 6. Finally, conclusions are listed in Section 7.

Notation: Throughout this paper, \( \mathbb{R} \) and \( \mathbb{N} \) are sets of real numbers and natural numbers, respectively, \( \mathbb{Z}_+ \) is the set of positive integers. \( \| \cdot \| \) is a vector norm defined in \( \mathbb{R}^n \). The notation \( M \succ (\succeq) 0 \) denotes a symmetric positive (semi-) definite matrix. Given two sets \( C_1 \) and \( C_2 \), we denote by \( C_2 \setminus C_1 \) the relative complement of \( C_1 \) in \( C_2 \), that is, the set of all elements belonging to \( C_2 \), but not to \( C_1 \). For two integers \( n_1 \) and \( n_2 \) with \( n_1 < n_2 \), the notation \( \overrightarrow{n_1, n_2} \) represents the set of \( \{n_1, n_1+1, \ldots, n_2\} \). Asterisk * in a symmetric matrix denotes the entry implied by symmetry. For any square matrix \( A \in \mathbb{R}^{n \times n} \), we define \( \text{He}(A) = A + A^T \). For any non-singular matrix \( A \in \mathbb{R}^{n \times n} \), define \( A^{-T} = (A^T)^{-1} \). For any symmetric matrix \( A \), \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote the maximum and minimum eigenvalues, respectively. The notation \( I \) stands for the identity matrix. The block matrix \( \bar{I}_i \) is defined as \( \bar{I}_i = \{0_{a \times (i-1)a}, I_{a \times a}, 0_{a \times (4-i)a}\} \in \mathbb{R}^{a \times 4a} \), where \( i \in \{1, 2, 3, 4\} \).

## 2 | PRELIMINARIES AND PROBLEM FORMULATION

In this paper, we consider the following impulsive switched systems:

\[
\begin{align*}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \neq t_k, \\
\Delta x(t_k) &= J_{i_k} x(t_k^-), \quad t = t_k,
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the state and \( u \in \mathbb{R}^m \) is the control input. \( \sigma : [0, \infty) \to \mathbb{M} \) is the switching signal taking value from the finite index set \( \mathbb{M} = \{1, 2, \ldots, L\} \). \( \mathcal{S} = \{s, s \in \mathbb{Z}_+\} \) is the switching time sequence. For instance, \( \sigma(s^-) = i \) is switched to \( \sigma(s) = j \neq i \), where \( \sigma(s^-) = \lim_{\Delta s \to 0^+} \sigma(s) - \Delta) \). \( A_i, B_i \) and \( J_i \) are known constant matrices with appropriate dimensions. Moreover, we assume that for each \( i \in \mathbb{M} \), the pair \((A_i, B_i)\) is stabilisable and \( A_i \) is non-singular matrix. \( x(t_k^-) = J_{i_k} x(t_k^-) \) describes the state jump, where the matrix \( J_{i_k} \) is dependent on the active subsystem, and \( x(t^-) = \lim_{\Delta t \to 0^+} x(t - \Delta) \). In this paper, for given \( \varepsilon_0 \) and \( \varepsilon_1 \) with \( 0 < \varepsilon_0 \leq \varepsilon_1 \), \( I(\varepsilon_0, \varepsilon_1) \) denotes a class of impulsive time sequences satisfying \( \varepsilon_0 \leq t_{k+1} - t_k \leq \varepsilon_1 \), \( k \in \mathbb{Z}_+ \).

Denote the sampling sequence \( \{\bar{t}_r, r \in \mathbb{Z}_+\} \), where \( \bar{t}_0 = 0 \). The sampled-data controller is

\[
\nu(t) = K_{\sigma(t)} x(\bar{t}_r), \quad t \in [\bar{t}_r, \bar{t}_{r+1}),
\]

where \( K_{\sigma(t)} \) is the controller gain. The sampling and switching sequences satisfy the following assumptions:

**Assumption 1.** The lengths of sampling intervals \( b_r = \bar{t}_{r+1} - \bar{t}_r \) are bounded by

\[
0 < b_{\min} \leq b_r \leq b_{\max} \leq \tau_d, \quad \forall r \in \mathbb{N},
\]

**Assumption 2.** (Slow switching [3])

1) There exists a positive number \( \tau_d \) such that any two switches are separated by at least \( \tau_d \), that is, \( \bar{t}_{r+1} - \bar{t}_r \geq \tau_d \).
2) There exist numbers \( \tau_d > \tau_d \) (\( \tau_d \) called an average dwell time) and \( N_0 \geq 1 \) such that
\[
N_d(T,t) \leq N_0 + \frac{T - t}{\tau_d}, \quad \forall T > t \geq t_0,
\]
where \( N_d(T,t) \) stands for the number of switches on time interval \([t, T]\). For simplicity, we denote such kind of switching signal \( \sigma(t) \in S[\tau_d, N_0] \).

Remark 1. Observing from Assumption 1, the lengths of sampling interval are variable in \([b_{\min}, b_{\max}]\). By taking \( b_{\max} \leq \tau_d \), one can guarantee that there is at most one switching in each sampling interval, that is, there exists at least one sampling within each switching interval. In this way, the controller is updated in each switching interval, which is helpful for the system stabilisation.

Combining (1) with (2) yields the following closed-loop system for \( t \in [t, \bar{t}+1) \):
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + K_x(t) \dot{x}(\bar{t}), \quad t \neq t_k \\
\dot{x}(t) &= Jx(t) - \dot{x}(\bar{t}), \quad t = t_k.
\end{align*}
\]

The objective of this paper is to exponentially stabilise system (4) through designing sampled-data controllers. The exponential stability definition is presented as follows.

Definition 1. System (1) under a class of impulses \( I(\epsilon_0, \epsilon_1) \) is said to be exponentially stable (ES) with rate of convergence \( \gamma > 0 \), if there exists \( \chi > 0 \) such that, for any impulse \( t_k \in I(\epsilon_0, \epsilon_1) \) and any corresponding solution \( x \), we have
\[
\|x(t)\| \leq e^{-\gamma t}\|x(0)\|, \quad \forall t \geq 0.
\]

Next, we introduce the merging switching signal technique, which mainly comes from [4], to deal with mismatched switching. Rewrite \( \sigma(t) \) as \( \sigma'(t) = \sigma(t - d(t)) \), in which, \( d(t) = t - \bar{t}, \) for \( t \in [\bar{t}, \bar{t}+1) \). \( \sigma'(t) \) determines which controller is active at current time. Denote \( \delta(t) = (\sigma(t), \sigma'(t)) \) as the augmented switching signal. Given \( t > \tau \geq 0 \), let \( \Xi_1(t, \tau) \) denote the matched interval and \( \Theta_1(t, \tau) := [t, \bar{t}] \setminus \Xi_1(t, \tau) \) denote the mismatched interval in \([t, \bar{t}]\). The notations \( \Xi_1(t, \tau) \) and \( \Theta_1(t, \tau) \) are used to denote the lengths of the interval \( \Xi_1(t, \tau) \) and \( \Theta_1(t, \tau) \), respectively. For example, in Figure 1, \( \Xi_1(\bar{t}_1, \bar{t}_2) = [\bar{t}_1, \bar{t}_1], \Theta_1(\bar{t}_1, \bar{t}_2) = [\bar{t}_1, \bar{t}_2], \Xi_1(\bar{t}_1, \bar{t}_2) = [\bar{t}_1, \bar{t}_1] \cup [\bar{t}_2, \bar{t}_2], \Theta_1(\bar{t}_1, \bar{t}_2) = [\bar{t}_1, \bar{t}_2] \cup [\bar{t}_2, \bar{t}_2], \Xi_1(\bar{t}_1, \bar{t}_2) = [\bar{t}_1, \bar{t}_1], [\bar{t}_2, \bar{t}_2], \Theta_1(\bar{t}_1, \bar{t}_2) = [\bar{t}_2, \bar{t}_2]. \]

Some lemmas needed for our main results are given in the following.

Lemma 1. ([4]). Consider the sampling condition (3) and switching signal \( \sigma(t) \in S[\tau_d, N_0] \), then \( \delta(t) \in S[\frac{1}{2}, 2N_0, \tau_d / \tau_d]. \)

It follows from (3) that \( d(t) \in [0, \tau_d) \). Then based on Lemma 1, one has the following Lemma 2.

Lemma 2. ([19]). For some positive scalars \( \alpha, \beta \) and \( \lambda \in (0, \alpha) \), if \( (\alpha + \beta)\tau_d \leq (\alpha - \lambda)\tau_d \), then for an interval \([\tau, t)\), the following inequality holds,
\[
-\alpha|\Xi(\tau, t)| + \beta|\Theta(\tau, t)| \leq \eta - \lambda(t - \tau),\]
where \( \eta = (\alpha + \beta)N_0\tau_d \).

### 3 | MULTIPLE IMPULSE-DEPENDENT LYAPUNOV FUNCTIONS AND LOOPED FUNCTIONALS

In this section, multiple impulse-dependent Lyapunov functions and looped functionals are introduced. The multiple impulse-dependent Lyapunov functions \( V'(\dot{x}(t)) \) are non-increasing at impulsive time instants and satisfy certain growth condition at the augment switching time instants. The looped functionals \( v(\bar{t}_1, \bar{t}_2, \dot{x}(t)) \) \( (t \in [\bar{t}_1, \bar{t}_2+1)) \), which are similar to the form in [33], are used to enhance the flexibility of Lyapunov functions.

Based on the fact that there is at most one switching in a sampling interval, two cases should be considered. One is that there is no switching happening in the sampling time interval \([\bar{t}_1, \bar{t}_2+1)\), then the total time interval \([\bar{t}_1, \bar{t}_2+1)\) is matched. The other is that there is one switching \( \bar{t} \), occurring in the sampling time interval \([\bar{t}_1, \bar{t}_2+1)\). Thus, we can discuss separately for the matched interval \([\bar{t}_1, \bar{t}_2)\) and mismatched interval \([\bar{t}_2, \bar{t}_2+1)\).

#### 3.1 | Matched Intervals

In this subsection, the impulse-dependent Lyapunov function and the time-dependent looped functional assigned to the matched interval \( \Xi(\bar{t}_1, \bar{t}_2+1) \) \( (\epsilon \in \mathbb{N}) \) are introduced. Without loss of generality, for the matched interval \( \Xi(\bar{t}_1, \bar{t}_2+1) \) and assuming...
where $\sigma(t) = i \in \mathbb{M}$, the looped subsystem is
\begin{equation}
\dot{x}(t) = A_i x(t) + A_{ij} x(t), \quad t \neq t_k
\end{equation}
\begin{equation}
\dot{x}(t_k) = f_i(x(t_k')), \quad t = t_k
\end{equation}
where $A_{ij} = B_j K_i$.

For system (5), we choose the impulse-dependent Lyapunov function $V_j(x(t)) = \varphi_j(t)x^T(t)\varphi_j(t)x(t)$. In order to find $P_i(t)$ and $\varphi_j(t)$, the impulse time interval $[t_{k-1}, t_k]$ is divided into $N$ subintervals $[t_{k-1}, t_k)$, where $k \in \mathbb{Z}_+$, $l \in \mathbb{N}$ and $t_{k,0} = t_{k-1}$, $t_{k,N} = t_k$, $t_0 = 0$. Define $\rho(t)$ and $\varphi(t)$ as follows:
\begin{equation}
\rho(t) = \frac{1}{t - t_{k-1}}, \quad t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+
\end{equation}
\begin{equation}
\varphi(t) = \frac{t - t_{k-1}}{t_{k,j} - t_{k,j-1}}, \quad t \in [t_{k,j-1}, t_{k,j}), k \in \mathbb{Z}_+, l \in \mathbb{N}.
\end{equation}
For $t \in \Xi(t_k, t_{k+1})$, $t$ must also be in some subinterval $[t_{k,j-1}, t_{k,j})$ ($k \in \mathbb{Z}_+, l \in \mathbb{N}$), then for $t \in \Xi(t_k, t_{k+1}) \cap [t_{k,j-1}, t_{k,j})$ ($k \in \mathbb{Z}_+, l \in \mathbb{N}$), $P(t)$ and $\varphi(t)$ take the following form:
\begin{equation}
\varphi(t) = \left( \prod_{q=1}^{l-1} \mu_{ij} \right) \mu_{ij}^{(l)} P_{ij}(t), \quad P(t) = \rho_{10}(t) P_{ij} + \rho_{11}(t) P_{ij-1},
\end{equation}
where $\mu_{ij} \geq 1$ ($l \in \mathbb{N}$) are scalars, $\rho_{10}(t), P_{ij} \in \mathbb{R}_{+}^{n \times n}$ ($l \in \mathbb{N}$) are positive definite matrices. Moreover, for $t_k \in \Xi(t_k, t_{k+1})$ ($k \in \mathbb{Z}_+$),
\begin{equation}
P(t_k) = P_{ij0}, \quad P(t_k^-) = P_{ijN}, \quad \varphi(t_k) = 1, \quad \varphi(t_k^-) = \prod_{q=1}^{N} \mu_{ijq}.
\end{equation}

Remark 2. $\rho_{10}(t)$ is used to define the matrices $P(t)$ which change piecewise. $\varphi(t)$ can be viewed as a weighting factor. $\varphi(t)$ and $P(t)$ are used to satisfy that $V(t)$ are non-increasing at impulsive instants under appropriate condition.

According to (6), it is obtained that for $t_k \in I(\varepsilon_0, \varepsilon_1)$, \begin{equation}
\frac{1}{t - t_{k-1}} \leq \frac{1}{t_k}.
\end{equation}
Hence we can express $\rho(t)$ in the convex combination form, $\rho(t) = \sum_{\varepsilon \in \mathbb{E}_0} \rho_{ij}(t) - \varepsilon \varepsilon_0$, where $\rho_{30}(t) = \frac{\rho_{ij}(t) - \varepsilon_0}{1 - \varepsilon_0}$ if $\varepsilon_0 < \varepsilon_1$ and $\rho_{30}(t) = \frac{\varepsilon_1}{1 - \varepsilon_0}$ if $\varepsilon_0 = \varepsilon_1$. Besides, from (7), one has
\begin{equation}
\prod_{q=1}^{l-1} \mu_{ij} \leq \varphi(t) \leq \prod_{q=1}^{l} \mu_{ij}.
\end{equation}
For simplicity, let $\tilde{\mu}_{ij} = 1$, $\tilde{\mu}_{ij} = \prod \mu_{ijq}$, then $\tilde{\mu}_{ij} \leq \varphi(t) \leq \tilde{\mu}_{ij}$. In order to let $\varphi(t)$ have the same form with $\rho(t)$, define $\tilde{\mu}_{ij0} = \tilde{\mu}_{ij-1} \mu_{ij0}^{-1} = \tilde{\mu}_{ij}$ and $\tilde{\mu}_{ij1} = \tilde{\mu}_{ij-1} \mu_{ij1}^{-1} = \tilde{\mu}_{ij}$, hence
\begin{equation}
\varphi(t) = \sum_{\varepsilon \in 0} \rho_{ij}(t) \tilde{\mu}_{ij}, \quad \rho_{ij}(t) = \frac{\varphi(t) - \tilde{\mu}_{ij}}{\tilde{\mu}_{ij} - \tilde{\mu}_{ij-1}} 
\end{equation}
if $\mu_{ij} \neq 1$ and $\rho_{ij}(t) = \tilde{\mu}_{ij-1}$ if $\mu_{ij} = 1$.

Next the time-dependent looped functional for the matched interval $\Xi(t_k, t_{k+1})$ is introduced
\begin{equation}
\varphi(t_k, t_{k+1}, t) = \left( t_{k+1} - t \right) \int_{t}^{t_{k+1}} e^{2\beta(t-t)} x^T(t) R_j x(t) dt
\end{equation}
where $\alpha > 0, R_j \in \mathbb{R}^{n \times n}$ is positive definite matrix.

3.2 Mismatched intervals

In this subsection, for the mismatched interval $\Theta(t_k, t_{k+1})$ and assuming that mode $i$ is switched to mode $j$ at $t_k$ ($t_k \leq t' \leq t_{k+1}$), the looped dynamics is
\begin{equation}
\dot{x}(t) = A_i x(t) + A_{ij} x(t), \quad t \neq t_k
\end{equation}
\begin{equation}
\dot{x}(t_k) = f_i(x(t_k')), \quad t = t_k
\end{equation}
where $A_{ij} = B_j K_i$.

We choose the impulse-dependent Lyapunov function $V_j(x(t)) = \varphi_j(t)x^T(t)\varphi_j(t)x(t)$ for system (10). For $t \in \Theta(t_k, t_{k+1}) \cap [t_{k,j-1}, t_{k,j})$ ($k \in \mathbb{Z}_+, l \in \mathbb{N}$), one has
\begin{equation}
\varphi(t) = \prod_{q=1}^{l} \mu_{ijq} \mu_{ij}^{(l)} P_{ij}(t), \quad P(t) = \rho_{10}(t) P_{ij} + \rho_{11}(t) P_{ij-1},
\end{equation}
where $\mu_{ij} \geq 1$ ($l \in \mathbb{N}$) are scalars, $P_{ij} \in \mathbb{R}_{+}^{n \times n}$ ($l \in \mathbb{N}$) are positive definite matrices. Moreover, for $t_k \in \Theta(t_k, t_{k+1})$ ($k \in \mathbb{Z}_+$),
\begin{equation}
P(t_k) = P_{ij0}, \quad P(t_k^-) = P_{ijN}, \quad \varphi(t_k) = 1, \quad \varphi(t_k^-) = \prod_{q=1}^{N} \mu_{ijq}.
\end{equation}

Similar to analysis of subsection 3.1, set $\tilde{\mu}_{ij0} = 1$, $\tilde{\mu}_{ij1} = \prod_{q=1}^{l-1} \mu_{ijq}$, then $\tilde{\mu}_{ij} \leq \varphi(t) \leq \tilde{\mu}_{ij}$. In order to let $\varphi(t)$ have the same form with $\rho(t)$, define $\tilde{\mu}_{ij0} = \tilde{\mu}_{ij-1} \mu_{ij0}^{-1} = \tilde{\mu}_{ij}$ and $\tilde{\mu}_{ij1} = \tilde{\mu}_{ij-1} \mu_{ij1}^{-1} = \tilde{\mu}_{ij}$, hence
\begin{equation}
\varphi(t) = \sum_{\varepsilon \in 0} \rho_{ij}(t) \tilde{\mu}_{ij}, \quad \rho_{ij}(t) = \frac{\varphi(t) - \tilde{\mu}_{ij}}{\tilde{\mu}_{ij} - \tilde{\mu}_{ij-1}} 
\end{equation}
if $\mu_{ij} \neq 1$ and $\rho_{ij}(t) = \tilde{\mu}_{ij-1}$ if $\mu_{ij} = 1$. Denote $\rho_{ij}(t) = \rho_{ij}(t)$ to $\varphi(t)$.

In what follows, we construct the time-dependent looped functional for the matched interval $\Theta(t_k, t_{k+1})$.
\begin{equation}
\varphi(t_k, t_{k+1}, t) = \left( t_{k+1} - t \right) \int_{t}^{t_{k+1}} e^{2\beta(t-t)} x^T(t) R_j x(t) dt
\end{equation}
where $\beta > 0, R_j \in \mathbb{R}^{n \times n}$ is positive definite matrix.
Define functional $W_i(t) := V_i(x(t)) + v_i(t)$ that is assigned to the matched interval $\Xi(l_i, l_{i+1})$, and $W_i(t) := V_i(x(t)) + v_i(t)$ that is assigned to the mismatched interval $\Theta(l_i, l_{i+1})$. Set $W^*(t) = W_i(t)$ when $t \in \Xi(l_i, l_{i+1})$, and $W^*(t) = W_i(t)$ when $t \in \Theta(l_i, l_{i+1})$.

Remark 3. The functional $W^*(t)$ is characterised by the following properties. (i) $W^*(t)$ keeps the value of $V(t)$ at the sampling time instants, that is $W_i(l_i) = V_i(l_i)$, $W_i(l_{i+1}) = V_i(l_{i+1})$; (ii) $W^*(t)$ is non-increasing at the impulsive time instants under appropriate conditions and satisfies certain growth condition at the augment switching time instants; (iii) The derivative of $V(t)$ is not required to be negative, while we only require the derivative of $W^*(t)$ is negative.

4 STABILITY

The theorem below provides conditions of exponential stability of system (4) based on the functional $W^*(t)$ discussed in Section 3.

Theorem 1. Consider system (4) with variable samplings satisfying Assumption 1. Given $\alpha > 0$, $\beta > 0$, $\gamma > 1$, $\varepsilon_0$ and $\varepsilon_1$ with $0 < \varepsilon_0 \leq \varepsilon_1$, if for prescribed positive integer $N_l$, and scalars $\mu_{ij}, \mu_{ij}, \mu_{ij}, (l = 1, N)$, there exist positive definite matrices $P_{ij}, P_{ij}, P_{ij}, R_{ij}, R_{ij}, R_{ij} \in \mathbb{R}^{n \times n}$ and arbitrary matrices $M_{ij}, M_{ij}, M_{ij}, N_{ij}, N_{ij} \in \mathbb{R}^{m \times n}$ such that the following LMIs hold:

$$
\Pi_{i,\text{langf}} + b \Pi_{i,\text{langf}} < 0,
$$

$$
\Pi_{ij,\text{langf}} + b \Pi_{ij,\text{langf}} < 0,
$$

$$
P_{ij} < \gamma P_{ij}, P_{ij} < \gamma P_{ij},
$$

$$
R_{ij} < \gamma e^{-(\alpha+\beta)\varepsilon_{\text{loc}}} R_{ij},
$$

$$
\mu_{ij} < \gamma \mu_{ij}, \mu_{ij} < \gamma \mu_{ij},
$$

$$
J^T_l P_{0,l} J_l \leq \sum_{q=1}^{N_l} \mu_{ij} P_{ij,N},
$$

$$
J^T_l P_{0,l} J_l \leq \sum_{q=1}^{N_l} \mu_{ij} P_{ij,N},
$$

where $b \in \{b_{\text{min}}, b_{\text{max}}\}$, $l = 1, N$ in (14) (15), $l = 0, N$ in (16), $m, g, \ell = 0, 1$ and

$$
\Pi_{i,\text{langf}} = \text{He}\left\{ \bar{\mu}_{ij} \left( \frac{N \ln \mu_{ij}}{2\varepsilon_{\ell}} + \alpha \bar{1} + \bar{1}_2 \right) P_{ij,\text{langf}} \bar{1} + \frac{N}{2\varepsilon_{\ell}} \bar{1}_1 \left( P_{ij} - P_{ij,\text{langf}} \right) + M_{ij} \left( \bar{1}_1 - \bar{1}_3 - \bar{1}_4 \right) \right\},
$$

$$
\Pi_{ij} = \bar{1}_2 R_{ij} \bar{1}_2,
$$

$$
\Pi_{ij,\text{langf}} = \text{He}\left\{ \bar{\mu}_{ij} \left( \frac{N \ln \mu_{ij}}{2\varepsilon_{\ell}} + \alpha \bar{1} + \bar{1}_2 \right) P_{ij,\text{langf}} \bar{1} + \frac{N}{2\varepsilon_{\ell}} \bar{1}_1 \left( P_{ij} - P_{ij,\text{langf}} \right) + M_{ij} \left( \bar{1}_1 - \bar{1}_3 - \bar{1}_4 \right) \right\},
$$

then system (1) is exponentially stable over $I(\varepsilon_0, \varepsilon_1)$ under the ADT

$$
\tau_x > \frac{\ln(\gamma + \tau_x (\alpha + \beta))}{\alpha}.
$$

Proof: Since the sampling intervals are divided into matched intervals and mismatched intervals, we shall prove that $W^*(t) < e^{-2\varepsilon_{\text{loc}}} W^*(t)$ in the matched interval $\Xi(l_i, l_{i+1})$ and $W^*(t) < e^{2\varepsilon_{\text{loc}}} W^*(t)$ in the mismatched interval $\Theta(l_i, l_{i+1})$, where $l_i \leq t < l_{i+1}$, then establish the exponential decay of the Lyapunov function on sampling intervals under the ADT constraint.

Part 1: In the matched interval $\Xi(l_i, l_{i+1})$.

When the $i$-th system is activated in the matched interval $\Xi(l_i, l_{i+1})$, system (5) is considered. As described in Section 3, we choose the functional $W_i(t) := V_i(x(t)) + v_i(t)$. Then for $t \in \Xi(l_i, l_{i+1}) \cap [\xi_k, l_{k+1}, l_{k+1}) (k \in \mathbb{Z}_+, l \in 1, N)$, one has

$$
\dot{V}_i(t) + 2\alpha V_i(t)
$$

$$
= \varphi_i(t) x^T(t) P_{ij} x(t) N \rho(t) \ln \mu_{ij} + 2 \varphi_i(t) x^T(t) P_{ij} x(t)
$$

$$
+ 2 \alpha \varphi_i(t) x^T(t) P_{ij} x(t) + (\varphi_i(t) x(t))(P_{ij} - P_{ij,\text{langf}}) N \rho(t) x(t)
$$

$$
= \sum_{q=0}^{N_l} \rho_{ij,q} \bar{\mu}_{ij} \bar{1} \bar{1}_1 \bar{1}_2 \bar{1}_1 \bar{1}_3 \bar{1}_4
$$

$$
+ M_{ij} \left( \bar{1}_1 - \bar{1}_3 - \bar{1}_4 \right)
$$

$$
= \sum_{q=0}^{N_l} \rho_{ij-q} \bar{\mu}_{ij} \bar{1} \bar{1}_1 \bar{1}_2 \bar{1}_1 \bar{1}_3 \bar{1}_4
$$

$$
+ M_{ij} \left( \bar{1}_1 - \bar{1}_3 - \bar{1}_4 \right).
$$
\[+2\alpha x^T(t)P_{ij-l-m}(t)\dot{x}(t)+2\alpha x^T(t)P_{ij-m}x(t)\]
\[+\frac{N}{\epsilon_\ell}x^T(t)(P_{ij}-P_{ij-l-m})x(t).\]  

(20)

Since \(R_i\) is positive definite, it is obtained that

\[\int_{\tau}^{t'} \dot{x}^T(s)R_i\dot{x}(s)ds - 2\dot{x}^T(t)NJ_i\dot{x}(t)\]
\[+\tau\delta^T(t)NJ_iR_i^{-1}N_i^T\delta(t) \geq 0,\]  

(21)

where \(\tau = t - \bar{t},\ \delta(t) = \delta_1\{x(t), \dot{x}(t), x(\bar{t}), \dot{x}(\bar{t})\}\) and \(\delta(t) = x(t) - x(\bar{t}).\)

Using (21), we have

\[\dot{V}_i^i(t) + 2\alpha\nu_i(t)\]
\[= (b_i - \tau)\dot{x}^T(t)R_i\dot{x}(t) - \int_{\tau}^{t'} 2\alpha x^T(t)R_i\dot{x}(s)ds\]
\[< (b_i - \tau)\dot{x}^T(t)R_i\dot{x}(t) - e^{-2\alpha h_{\text{max}}} \left(2\delta^T(t)NJ_i\dot{x}(t) - \tau\delta^T(t)NJ_iR_i^{-1}N_i^T\delta(t)\right).\]  

(22)

In addition, the following two slack variables \(M_{i1}\) and \(M_{i2}\) are introduced:

\[2\delta^T(t)M_{i1}[x(t) - x(\bar{t})] = 0,\]  

(23)

\[2\delta^T(t)M_{i2}[A_i x(t) + A_{i1} x(\bar{t})] = 0.\]  

(24)

Combining (20–24), for \(t \in \Sigma(\bar{t}, \bar{t} + 1) \cap \{t_{k-l-1}, t_{k+1}\} (k \in \mathbb{Z}_+,\ l \in 1, N),\) one has

\[W_i^i(t) + 2\alpha W_i^i(t) < \sum_{a_{ij}=0}^{1} \sum_{b_{ij}=0}^{1} \rho_{ij-m\ell} \delta^T(t)\Pi(a, b)\delta(t),\]  

(25)

where \(\Pi(a, b) = \Pi_{ij, m\ell} + (b - \tau)\Pi_{ij} + \tau\Pi_{ij-l},\) and \(\Pi_{ij} = e^{-2\alpha h_{\text{max}}} N_i R_i^{-1} N_i^T.\)

Note that \(\Pi(a, b)\) is affine, and thus convex, with respect to \(a \in [0, 1].\) Condition (14a) means \(\Pi(0, b) < 0,\) and condition (14b) means \(\Pi(b, b) < 0.\) Hence, by the convexity of \(\Pi(a, b),\) we can obtain for \(t \in \Sigma(\bar{t}, \bar{t} + 1) \cap \{t_{k-l-1}, t_{k+1}\} (k \in \mathbb{Z}_+,\ l \in 1, N),\)

\[W_i^i(t) + 2\alpha W_i^i(t) < 0.\]  

(26)

For the impulsive time sequence \(I \in I(\varepsilon_0, \varepsilon_1),\) one should consider two cases as follows:

**Case 1.** There are no impulses happening in the matched interval \(\Sigma(\bar{t}, \bar{t} + 1).\) From the form of \(W_i^i(t),\) we can obtain that \(W_i^i(t)\) varies continuously in \(\Sigma(\bar{t}, \bar{t} + 1),\) hence it is obtained that \(W_i^i(t) < e^{-2\alpha \varepsilon_1} W_i^i(t)\) in the matched interval \(\Sigma(\bar{t}, \bar{t} + 1)\) from (26).

**Case 2.** There is at least one impulse occurring in the matched interval \(\Sigma(\bar{t}, \bar{t} + 1).\) According to (17), for \(k \in \mathbb{Z}_+,\) we have

\[V_i(t_k) = x^T(t_k^+) P_{ij} x(t_k^-)\]
\[\leq \sum_{j=1}^{N} \mu_{ij} x^T(t_k^-)P_{ij} x(t_k^-) = V_i(t_k^-),\]  

(27)

Along with \(V_i(t_k) = v_i(t_k),\) we can obtain \(W_i^i(t_k) \leq W_i^i(t_k^-).\) Then by (26), we have \(W_i^i(t) < e^{-2\alpha \varepsilon_1} W_i^i(t_k^-)\) in the matched interval \(\Sigma(\bar{t}, \bar{t} + 1),\) \(\bar{t} \leq \tau < t < \bar{t} + 1.\)

**Part 2:** In the mismatched interval \(\Theta(\bar{t}, \bar{t} + 1)\)

When a switching happens at \(\bar{t} \in (\bar{t}, \bar{t} + 1),\) system (10) is considered. As discussed in Section 3, we choose the functional \(V_i^i(t) := V_i(x(t)) + \nu_i(t),\) then for \(t \in \Theta(\bar{t}, \bar{t} + 1) \cap \{t_{k-l-1}, t_{k+1}\} (k \in \mathbb{Z}_+,\ l \in 1, N),\) we have

\[V_i^i(t) - 2\beta V_i^i(t)\]
\[= \sum_{a_{ij}=0}^{1} \sum_{b_{ij}=0}^{1} \sum_{q=0}^{N} P_{ij-qlm} \mu_{ij-lq} \left(\frac{N \ln \mu_{ij-lq}}{\epsilon_\ell} x^T(t)P_{ij-l-q} x(t)\right)\]
\[+ 2\alpha x^T(t)P_{ij-l-q}(t)\dot{x}(t) - 2\beta x^T(t)P_{ij-l-q}(t) x(t)\]
\[+ \frac{N}{\epsilon_\ell} x^T(t)(P_{ij-l-q} - P_{ij-l-q-1}) x(t),\]
\[\dot{V}_i^i(t) + 2\alpha \nu_i(t) < (b_i - \tau)\dot{x}^T(t)R_i\dot{x}(t) - \left(2\delta^T(t)NJ_i\dot{x}(t)\right)\]
\[+ \tau\dot{x}^T(t)NJ_iR_i^{-1}N_i^T\delta(t).\]  

(28)

In addition, we make use of the following two slack variables \(M_{i1, j}, M_{i2, j}\)

\[2\delta^T(t)M_{i1, j}[x(t) - x(\bar{t})] = 0,\]  

(30)

\[2\delta^T(t)M_{i2, j}[A_i x(t) + A_{i1} x(\bar{t})] = 0.\]  

(31)

Combining (28–31) and using (15), for \(t \in \Theta(\bar{t}, \bar{t} + 1) \cap \{t_{k-l-1}, t_{k+1}\} (k \in \mathbb{Z}_+,\ l \in 1, N),\) we get

\[W_i^i(t) - 2\beta W_i^i(t) < 0.\]  

(32)

For the impulsive time sequence \(I \in I(\varepsilon_0, \varepsilon_1),\) one should consider two cases as follows:

**Case 1.** There are no impulses happening in the mismatched interval \(\Theta(\bar{t}, \bar{t} + 1).\) From the form of \(W_i^i(t),\) we can obtain that \(W_i^i(t)\) varies continuously in \(\Theta(\bar{t}, \bar{t} + 1);\) hence, it is obtained that
$W_i(t) < e^{2\|\Theta(t)\|_{\infty}} W_i(t)$ in the mismatched interval $\Theta(\tilde{t}, \tilde{t}+1)$ form (32).

**Case 2.** There is at least one impulse occurring in the mismatched interval $\Theta(\tilde{t}, \tilde{t}+1)$. According to (18), for $k \in \mathbb{Z}_+$, we have

$$V_{ij}(t_k) = x^T(t_k) \frac{\partial}{\partial t} f_i(t_j(t_k)) x(t_k) \
\leq \prod_{q=1}^{N} \mu_{ij, q} x^T(t_k) P_{ij, N} x(t_k) = V_{ij}(t_{k-1}).$$

Along with $v_j(t_k) = v_j(t_k)$, we can obtain $W_{ij}(t_k) \leq W_{ij}(t_{k-1})$. Then by (32), we have $W_{ij}(t) < e^{2\|\Theta(t)\|_{\infty}} W_{ij}(t)$. Considering the mismatched interval $\Theta(\tilde{t}, \tilde{t}+1)$, where $\tilde{t} \leq \tau < \tilde{t}+1$. Part 3. Synthesize both parts and derive the ADT condition.

Next consider the augment switching signal $\delta(t)$, for any $\tau > \tilde{t}_0$, let $0 = \tilde{t}_0 < \tau_1, \tau_2, \ldots, \tau_{N_i} < \tau$. Let $\sigma$ be the augment switching time sequence for $\delta(t)$ in interval $[\tilde{t}_0, \tau)$. Without loss of generality, let $\tilde{t}_0 = \tilde{t}_0$, $\delta_{i,j} = 1$, from (26), (32), (34), (35) and by using of recursive calculation, one has

$$W_{ij}(\tilde{t}_0) < vW_{ij}(\tilde{t}_0),$$

and at the next sampling time $\tilde{t}+1$ after switching $\tilde{t}$, we can obtain the relationship between $W_{ij}(\tilde{t}+1)$ and $W_{ij}(\tilde{t}+1)$

$$W_{ij}(\tilde{t}+1) < vW_{ij}(\tilde{t}+1).$$

Remark 4. It is not hard to get solutions of the LMIs in Theorem 1 for the following reasons: (1) Since $\mu_{ij, q}$, $\mu_{ij}$ are adjustable, (17) and (18) are not hard to be satisfied. (2) Inequalities in (16) are not hard to verify with $\nu > 1$; (3) $P_{ij, q}$, $P_{ij}$, $R_{ij}$, $R_{ij}$ are required to be positive definite while $M_{ij, q}$, $M_{ij}$, $M_{ij, q}$, $N_{ij}$, and $N_{ij}$ are arbitrary matrices. Hence, (14a) and (15a) are not hard to verify; (4) In LMIs (14b) and (15b), $h_{R_{ij}}$, $h_{R_{ij}}$ are both negative definite, which make (14b) and (15b) easy to be satisfied.

Remark 5. It is worth noting that the impulse-dependent Lyapunov functional method proposed here allows the functional not to increase at destabilising impulses. As a tradeoff, the number of linear matrix inequalities needed to be solved is $16m^2(N + 1) + 3m(m-1) + m^2$, where $m$ is the number of subsystems, and $N$ is the number of subintervals between every two impulses. So a large partition number $N$ and a system mode number $m$ would increase the computational cost and complexity.

Remark 6. In [15, 17, 18], a linear time-varying Lyapunov function $V(x(t)) = x^T(t) P(t) x(t)$ is used, which is, however, infeasible in this paper because the sampled state is used for feedback in our method. In addition, it is too strict to find such a Lyapunov function, which is required to be independent of the sampled state. In this paper, a new functional $W(t)$ consisting of Lyapunov function $V(x(t))$ and looped-functional $v(t)$ is constructed to overcome this difficulty. Furthermore, in order to handle impulses and the mismatched problem, multiple impulse-time dependent weighting factors are introduced.
ALGORITHM 1 Solution search algorithm
1: Fixed $\varepsilon_0$ and $\varepsilon_1$;
2: Select a partition number $N$, and a growth bound $\nu$;
3: Choose the exponential decay rate $\alpha$ and growth rate $\beta$;
4: Select a smaller number $\varepsilon$;
5: Initialise $h_{\text{min}}$ and $h_{\text{max}}$;
6: for $i := 1:1000$
7: \[ \mu_i, \mu_j \text{ take value from } \mu_{\text{low}} \text{ to } \mu_{\text{high}} \text{ randomly}; \]
8: Solve LMIs (14)-(18);
9: if the LMIs are feasible
10: finish;
11: end if
12: end for
13: Update $h_{\text{min}} = 0.8h_{\text{min}} - h_{\text{max}} = 0.5h_{\text{max}}$;
14: if $h_{\text{min}} \leq h_{\text{max}} \leq \varepsilon \| h_{\text{min}} \geq h_{\text{max}}$;
15: goto 3;
16: else
17: goto 6;
18: end if

\[
\begin{bmatrix}
P_{i0} - J_i^{-1} Q_i - Q_i^T J_i^{-T} \\
- \left( \prod_{j=1}^N \mu_{ij,q} \right) P_{i,N}
\end{bmatrix} \leq 0, \quad (41)
\]

\[
\begin{bmatrix}
P_{j0} - J_j^{-1} Q_j - Q_j^T J_j^{-T} \\
- \left( \prod_{l=1}^N \mu_{lj,q} \right) P_{j,N}
\end{bmatrix} \leq 0, \quad (42)
\]

where $b \in \{h_{\text{min}}, h_{\text{max}}\}$, $l = 1, N$ in (37) (38), $l = 0, N$ in (39) (40),
$m, q, \ell = 0, 1$ and

\[
\Pi_{i1,\text{loop}} = H_{e} \left\{ \hat{\mu}_{i,l} \left( \frac{N \ln \mu_{ij,l}}{2 \varepsilon} \hat{I}_1 + \hat{\alpha} \hat{I}_1 + \hat{I}_2 \right) T \Pi_{j,\text{loop}} \right\}
\]

\[
\Pi_{i2} = T_i^T R_i \hat{I}_1,
\]

\[
\Pi_{i1,\text{loop}} = H_{e} \left\{ \hat{\mu}_{i,l} \left( \frac{N \ln \mu_{ij,l}}{2 \varepsilon} \hat{I}_1 - \hat{\beta} \hat{I}_1 + \hat{I}_2 \right) T \Pi_{j,\text{loop}} \right\}
\]

\[
\Pi_{i2} = T_i^T R_i \hat{I}_1,
\]

\[
\Pi_{i2} = T_i^T R_i \hat{I}_1,
\]

then the system (1) with sampled-data controller (2), where $K_i = L_i Q_i^{-1}$, is exponentially stabilisable over $I(\varepsilon_0, \varepsilon_1)$ under the ADT condition (19).

Proof: Based on Theorem 1, the proof is also proceeded in three parts.

Part 1: By matched interval $E(\bar{t}_l, \bar{t}_l+1)$.

Without loss of generality, we analyse the matched interval $E(\bar{t}_l, \bar{t}_l+1)$ and assume $\sigma(\bar{t}_l) = l \in \mathcal{M}$. Let the slack matrix variable \[ M_{ij} = \sum_{l=0}^4 s_{lj} R_{lj}, \quad Q_i = \Gamma_i^{-1}, \quad \Theta_i = \text{diag}(Q_i, Q_i, Q_i, Q_i), \quad \hat{\xi}(t) = \Theta_i^{-1} \hat{\xi}(t), \quad \hat{I}_1 = \Theta_i^{\dagger} \hat{I}_1, \] following above variables transformation, substitute $\hat{\xi}(t) = \Theta_i \hat{\xi}(t)$ into (25), then according to (37), for $t \in E(\bar{t}_l, \bar{t}_l+1) \cap \left\{ t_{k,l-1}, t_{k,l} \right\}$ ($k \in \mathbb{Z}^+, \ell = 1, N$), we can obtain

\[
\mathcal{W}_i(t) + 2\alpha \mathcal{W}_i(t) < \sum_{m=0}^1 \sum_{\ell=0}^1 \mathcal{P}_i,m,\ell \hat{\xi}^T(t) \Pi(t, \ell) \hat{\xi}(t) < 0,
\]

An algorithm above is given to search for feasible solutions of inequalities (19)–(27) in Theorem 1.

5 STABILISATION

In this section, a sampled-data controller is designed, and the exponential stabilisation conditions are established to calculate controller gains $K_i (i \in \mathcal{M})$.

Theorem 2. Consider system (4) with variable samplings satisfying Assumption 1. Given $\alpha > 0, \beta > 0, \nu > 1, \varepsilon_0$ and $\varepsilon_1$ with $0 < \varepsilon_0 \leq \varepsilon_1$, for prescribed positive integer $N$, scalars $s_{ij}, s_{ijp} (p \in \{1, 4\})$, and $\mu_{ij}, \mu_{ij} (l = 1, N)$, there exist positive definite matrices $P_{j0}, P_{j,l} (l \neq 0, N), R_{ij}, R_{ij} \in \mathbb{R}^{n \times n}$, non-singular matrix $Q_i \in \mathbb{R}^{n \times n}$ and arbitrary matrices $L_{ij} \in \mathbb{R}^{4 \times n}$, $M_{ij}, N_{ij} \in \mathbb{R}^{1 \times n}$ such that the following LMIs hold:

\[
\begin{bmatrix}
P_{i0} - J_i^{-1} Q_i - Q_i^T J_i^{-T} \\
- \left( \prod_{j=1}^N \mu_{ij,q} \right) P_{i,N}
\end{bmatrix} \leq 0, \quad (41)
\]

\[
\begin{bmatrix}
P_{j0} - J_j^{-1} Q_j - Q_j^T J_j^{-T} \\
- \left( \prod_{l=1}^N \mu_{lj,q} \right) P_{j,N}
\end{bmatrix} \leq 0, \quad (42)
\]

\[
\Pi_{i1,\text{loop}} = H_{e} \left\{ \hat{\mu}_{i,l} \left( \frac{N \ln \mu_{ij,l}}{2 \varepsilon} \hat{I}_1 + \hat{\alpha} \hat{I}_1 + \hat{I}_2 \right) T \Pi_{j,\text{loop}} \right\}
\]

\[
\Pi_{i2} = T_i^T R_i \hat{I}_1,
\]

\[
\Pi_{i1,\text{loop}} = H_{e} \left\{ \hat{\mu}_{i,l} \left( \frac{N \ln \mu_{ij,l}}{2 \varepsilon} \hat{I}_1 - \hat{\beta} \hat{I}_1 + \hat{I}_2 \right) T \Pi_{j,\text{loop}} \right\}
\]

\[
\Pi_{i2} = T_i^T R_i \hat{I}_1,
\]

\[
\Pi_{i2} = T_i^T R_i \hat{I}_1,
\]

then system (1) with sampled-data controller (2), where $K_i = L_i Q_i^{-1}$, is exponentially stabilisable over $I(\varepsilon_0, \varepsilon_1)$ under the ADT condition (19).

Proof: Based on Theorem 1, the proof is also proceeded in three parts.

Part 1: By matched interval $E(\bar{t}_l, \bar{t}_l+1)$.

Without loss of generality, we analyse the matched interval $E(\bar{t}_l, \bar{t}_l+1)$ and assume $\sigma(\bar{t}_l) = l \in \mathcal{M}$. Let the slack matrix variable \[ M_{ij} = \sum_{l=0}^4 s_{lj} R_{lj}, \quad Q_i = \Gamma_i^{-1}, \quad \Theta_i = \text{diag}(Q_i, Q_i, Q_i, Q_i), \quad \hat{\xi}(t) = \Theta_i^{-1} \hat{\xi}(t), \quad \hat{I}_1 = \Theta_i^{\dagger} \hat{I}_1, \] following above variables transformation, substitute $\hat{\xi}(t) = \Theta_i \hat{\xi}(t)$ into (25), then according to (37), for $t \in E(\bar{t}_l, \bar{t}_l+1) \cap \left\{ t_{k,l-1}, t_{k,l} \right\}$ ($k \in \mathbb{Z}^+, \ell = 1, N$), we can obtain

\[
\mathcal{W}_i(t) + 2\alpha \mathcal{W}_i(t) < \sum_{m=0}^1 \sum_{\ell=0}^1 \mathcal{P}_i,m,\ell \hat{\xi}^T(t) \Pi(t, \ell) \hat{\xi}(t) < 0,
\]

An algorithm above is given to search for feasible solutions of inequalities (19)–(27) in Theorem 1.
where \( \hat{\Pi}(\tau, h) = \hat{\Pi}_{1,\text{long}} + (h - \tau)\hat{\Pi}_{2,\text{long}} + \tau\hat{\Pi}_{3} \) and \( \hat{\Pi}_{3} = -e^{-2\alpha T}N_{\tau}R_{\beta}^{-1}N_{\beta} \). In view of (41), by leveraging Schur complement, one has

\[
\left( \sum_{\mu_{ij} \neq 0} \frac{1}{\mu_{ij}} \right)^{-1} \leq j_{\mu_{ij}}^{-1} Q_{i} + Q_{i}^{T} J_{j_{\mu_{ij}}} - P_{\mu_{ij}},
\]

then we have \( \left( \sum_{\mu_{ij} \neq 0} \frac{1}{\mu_{ij}} \right)^{-1} \leq j_{\mu_{ij}}^{-1} Q_{i} + Q_{i}^{T} J_{j_{\mu_{ij}}} - P_{\mu_{ij}} \), which is equivalent to \( J_{j_{\mu_{ij}}}^{T} P_{\mu_{ij}} Q_{i} - j_{\mu_{ij}}^{-1} \leq \left( \sum_{\mu_{ij} \neq 0} \frac{1}{\mu_{ij}} \right)^{-1} P_{\mu_{ij}} Q_{i} - j_{\mu_{ij}}^{-1} \), i.e.,

\[
J_{j_{\mu_{ij}}}^{T} P_{\mu_{ij}} Q_{i} - j_{\mu_{ij}}^{-1} \leq \left( \sum_{\mu_{ij} \neq 0} \frac{1}{\mu_{ij}} \right)^{-1} P_{\mu_{ij}} Q_{i} - j_{\mu_{ij}}^{-1} \leq P_{\mu_{ij}} Q_{i} - j_{\mu_{ij}}^{-1},
\]

as that of Part 1 in Theorem 1, we have \( W_{\tau}(\tau) < e^{-2\alpha T}w_{\tau}(\tau) W_{\tau}(\tau) \) in the mismatched interval \( \Xi(\tilde{t}_{i}, \tilde{t}_{i+1}) \), where \( \tilde{t}_{i} \leq \tau < \tilde{t}_{i+1} \).

Part 2: In the mismatched interval \( \Theta(\tilde{t}_{i}, \tilde{t}_{i+1}) \)

Without loss of generality, we analyse the mismatched interval \( \Theta(\tilde{t}_{i}, \tilde{t}_{i+1}) \) and assume that mode \( i \) is switched to mode \( j \) at \( \tilde{t}_{i} \), \( \tilde{t}_{i} < \tilde{t}_{i+1} \). (32) can also be ensured for \( i \in \Theta(\tilde{t}_{i}, \tilde{t}_{i+1}) \cap \{k_{i}, i = 1, \ldots, N_{i}\} \) by defining \( M_{ij} = \sum_{k_{i} = 1}^{N_{i}} \Gamma_{j_{k_{i}}}^{T} \Gamma_{j_{k_{i}}} \), and let \( P_{j_{k_{i}}} = Q_{j_{k_{i}}}^{T} Q_{j_{k_{i}}} \), \( M_{i} = \sum_{k_{i} = 1}^{N_{i}} M_{i_{k_{i}}} \), \( M_{j_{k_{i}}} = \sum_{k_{i} = 1}^{N_{i}} M_{j_{k_{i}}} \), and \( N_{j_{k_{i}}} = \sum_{k_{i} = 1}^{N_{i}} N_{j_{k_{i}}} \). Besides, (42) implies \( \tilde{t}_{i} \leq \tilde{t}_{i} < \tilde{t}_{i+1} \).

Part 3: Synthesize both parts

The analysis is the same as that of Part 3 in Theorem 1, by using (39) and (40), system (1) with sampled-data controller (2), where \( K_{f} = I_{2}Q_{j_{k_{i}}}^{-1} \), is exponentially stabilisable over \( I_{2}(\varepsilon_{0}, \varepsilon_{1}) \) under the ADT condition (19).

6  SIMULATION EXAMPLE

In this section, we take an aircraft as an example to show the effectiveness of our proposed approach. The longitudinal linear equations of motion of the aircraft come from [27]. The simplified longitudinal linear equations are

\[
\begin{align*}
\dot{\alpha} & = \begin{bmatrix} Z_{\alpha} & Z_{q} \\ M_{\alpha} & M_{q} \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_{\alpha E} & Z_{\alpha \text{PTV}} \\ M_{\alpha E} & M_{\alpha \text{PTV}} \end{bmatrix} \begin{bmatrix} \delta_{E} \\ \delta_{\text{PTV}} \end{bmatrix} \\
\dot{q} & = A_{\text{long}} \begin{bmatrix} \alpha \\ q \end{bmatrix} + B_{\text{long}} \begin{bmatrix} \delta_{E} \\ \delta_{\text{PTV}} \end{bmatrix},
\end{align*}
\]

(43)

A set of baseline aerodynamic data are obtained from wind tunnel and flight test data [27]. Considering two flight conditions of Mach 0.8, altitude 10 kft and Mach 0.8 altitude 12 kft, we obtain the following two subsystems:

\[
\begin{align*}
\sigma & = \begin{cases} 1, & \text{if altitude } \in [9, 11) \text{ kft}, \\
2, & \text{if altitude } \in [11, 13) \text{ kft},
\end{cases} \\
A_{1} & = A_{\text{long}}^{0.8\text{10}} \begin{bmatrix} -1.675 & 9.853 \\ -16.16 & -1.212 \end{bmatrix}, \\
B_{1} & = B_{\text{long}}^{0.8\text{10}} \begin{bmatrix} -0.2449 & -0.04649 \\ -28.34 & -5.742 \end{bmatrix}, \\
A_{2} & = A_{\text{long}}^{0.8\text{12}} \begin{bmatrix} -1.562 & 9.862 \\ -14.96 & -1.132 \end{bmatrix}, \\
B_{2} & = B_{\text{long}}^{0.8\text{12}} \begin{bmatrix} -0.2316 & -0.04349 \\ -26.48 & -5.323 \end{bmatrix},
\end{align*}
\]

where \( A_{\text{long}}^{0.8\text{10}} \) means the longitudinal state matrix at Mach 0.8 and altitude 10 kft. We assume that at least \( \tau_{\alpha} \) is needed before the altitude changed from one interval, say (9,11) kft, to the other. Further, at some moments let \( \tau_{\alpha} \), we suppose the system is subjected to shock effects because of which the angle of attack \( \alpha \) and the pitch rate \( q \) suffer an instantaneous increment. For simplicity, we model the impulse time sequence as \( I \in I(\varepsilon_{0}, \varepsilon_{1}) \) and the increment as

\[
\begin{align*}
\Delta \alpha(t_{k}) & = \left[ \begin{array}{c} \alpha(t_{k}) - \alpha(t_{k}^{-}) \\ \alpha(t_{k}) - \alpha(t_{k}^{-}) \end{array} \right] = D_{\alpha} \left[ \begin{array}{c} \alpha(t_{k}) \\ q(t_{k}) \end{array} \right], \\
\Delta q(t_{k}) & = \left[ \begin{array}{c} q(t_{k}) - q(t_{k}^{-}) \\ q(t_{k}) - q(t_{k}^{-}) \end{array} \right] = D_{q} \left[ \begin{array}{c} \alpha(t_{k}) \\ q(t_{k}) \end{array} \right],
\end{align*}
\]

where

\[
D_{\alpha} = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0.3 \end{bmatrix}.
\]

Then applying sampled-data controller \( u(t) = K_{G}(\tau, x(t)) \) for \( t \in I(t_{k}, t_{k+1}) \), the dynamic equation (43) becomes

\[
\begin{align*}
x(t) & = X_{\alpha}(t) + B_{\alpha} \left( K_{G}(x(t)) \right), \\
x(t) & = X_{\alpha}(t) + B_{\alpha} \left( K_{G}(x(t)) \right),
\end{align*}
\]

(44)

Then Theorem 2 is applied to this situation. A class of impulsive time sequences \( I(0.2, 3) \) is considered. Comparing with (14), an apparent distinction in (37) is \( \sum_{j=1}^{4} \delta_{j} T_{j}^{T} \left( A_{\alpha} Q_{\alpha} - Q_{\alpha} T_{j} + B_{\alpha} I_{\alpha} \right) \) where \( Q_{\alpha} \) and \( I_{\alpha} \) are slack variables which the role of is more like as that of \( M_{ij} \) in (14), and \( \delta_{j} \) parameters provide more flexibility herein. So, it is sufficient for finding feasible solution using Algorithm 1 to select the slack variables \( \delta_{j} = s_{ij} \delta_{j} = 0.5 \).
for all $i \neq j \in \{1, 2\}$ and $p \in [1, 4]$ as a starting point. Then by utilising Algorithm 1, we choose $N = 2$ due to the computational cost and choose a modest exponential decay rate $\alpha = 0.4$ and growth $\beta = 1.6$. Note that $\nu$ affects the overshoot bound for Lyapunov stability. A large $\nu$ implies a large overshoot bound, which is not desirable system performance thus we let $\nu = 1.2$. Notice also that we use sampled-data controllers to save the cost of calculation and avoid the frequent computation because only the states on sampling times are fed back, hence the sampling interval cannot be too small and we choose $\varepsilon = 0.05$. Initialise $b_{\text{min}} = 0.5$, $b_{\text{max}} = 1$, $\mu_{\text{min}} = 1$, $\mu_{\text{max}} = 3$. As a result, we figure out that when $\mu_{1,1} = \mu_{1,2} = \mu_{2,1} = \mu_{2,2} = 1.2$, $\mu_{12,1} = \mu_{12,2} = \mu_{21,1} = \mu_{21,2} = 1.3$, $b_{\text{min}} = b_{\text{max}} = 0.1$, a feasible solution is as follows:

$$Q_1 = \begin{bmatrix} 0.2167 & 0.0143 \\ -0.0419 & 0.4001 \end{bmatrix} \times 10^{-3},$$

$$L_1 = \begin{bmatrix} -0.0012 & 0.0020 \\ 0.0056 & -0.0098 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.2310 & -0.0021 \\ -0.0361 & 0.3939 \end{bmatrix} \times 10^{-3},$$

$$L_2 = \begin{bmatrix} 0.0016 & -0.0009 \\ -0.0084 & 0.0043 \end{bmatrix}.$$ 

Thus, the controller gains are obtained by $K_i = L_i Q_i^{-1}$ as follows:

$$K_1 = \begin{bmatrix} -4.7384 & 5.1268 \\ 20.8990 & -25.1200 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 6.4789 & -2.1392 \\ -34.6325 & 10.8415 \end{bmatrix},$$

and the ADT should satisfy $\tau_a > 1.0058 \xi$.

The impulsive time sequence is illustrated in Figure 2. Choose $N_0 = [2, 2]^T$, $N_0 = 2$, $\tau_a = 1.01 \xi$, $\tau_d = 0.11 \xi > 0.1 \xi$, which satisfies Assumption 1. From Figure 3, one can see that the states of the switched systems converge to the origin, meaning the system (44) is stable and our method is effective. In addition, the trajectories of functionals $W(t)$, $v(t)$ and $V(t)$ are illustrated in Figure 4.

7 | CONCLUSIONS

In this paper, we have investigated the stability and stabilisation of impulsive switched system with sampled-data controller. The functional consisting of multiple impulse-dependent Lyapunov functions and looped functionals has been proposed. By solving a set of LMIs, we could check the stability of impulsive switched system with sampled-data controllers. Moreover, exponential stabilisation conditions have been derived for the
design of sampled-data controller. At last, the example has been given to show the effectiveness of the proposed approach.

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