QUANTUM LAKSHMIBAI-SESHADRI PATHS AND THE SPECIALIZATION OF MACDONALD POLYNOMIALS AT $t = 0$ IN TYPE $A_{2n}^{(2)}$.

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Abstract. In this paper, we give a combinatorial realization of the crystal basis of a quantum Weyl module over a quantum affine algebra of type $A_{2n}^{(2)}$, and a representation-theoretic interpretation of the specialization $P_{\lambda}^{A_{2n}^{(2)}}(q, 0)$ of the symmetric Macdonald polynomial $P_{\lambda}(q, t)$ at $t = 0$, where $\lambda$ is a dominant weight and $P_{\lambda}^{A_{2n}^{(2)}}(q, t)$ denotes the specific specialization of the symmetric Macdonald-Koornwinder polynomial $P_{\lambda}(q, t_1, t_2, t_3, t_4, t_5)$. More precisely, as some results for untwisted affine types, the set of all $A_{2n}^{(2)}$-type quantum Lakshmibai-Seshadri paths of shape $\lambda$, which is described in terms of the finite Weyl group $W$, realizes the crystal basis of a quantum Weyl module over a quantum affine algebra of type $A_{2n}^{(2)}$ and its graded character is equal to the specialization $P_{\lambda}^{A_{2n}^{(2)}}(q, 0)$ of the symmetric Macdonald-Koornwinder polynomial.

1. Introduction

Symmetric Macdonald polynomials with two parameters $q$ and $t$ were introduced by Macdonald [M1] as a family of orthogonal symmetric polynomials, which include as special or limiting cases almost all the classical families of orthogonal symmetric polynomials. This family of polynomials are characterized in terms of the double affine Hecke algebra (DAHA) introduced by Cherednik ([Ch1], [Ch2]). In fact, there exists another family of orthogonal polynomials, called nonsymmetric Macdonald polynomials, which are simultaneous eigenfunctions of $Y$-operators acting on the polynomial representation of the DAHA; by “symmetrizing” nonsymmetric Macdonald polynomials, we obtain symmetric Macdonald polynomials (see [M]).

Based on the characterization above of nonsymmetric Macdonald polynomials, Ram-Yip [RY] obtained a combinatorial formula expressing symmetric or nonsymmetric Macdonald polynomials associated to an arbitrary untwisted affine root system; this formula is described in terms of alcove walks, which are certain strictly combinatorial objects. In addition, Orr-Shimozono [OS] refined the Ram-Yip formula above, and generalized it to an arbitrary affine root system (including the twisted case); also, they specialized their formula at $t = 0$, $t = \infty$, $q = 0$, and $q = \infty$.

As for representation-theoretic interpretations of the specialization of symmetric or nonsymmetric Macdonald polynomials at $t = 0$, we know the following. Ion [I] proved that for a dominant integral weight $\lambda$ and an element $x$ of the finite Weyl group $W$, the specialization $E_{x\lambda}(q, 0)$ of the nonsymmetric Macdonald polynomial
$E_{x\lambda}(q,t)$ at $t = 0$ is equal to the graded character of a certain Demazure submodule of an irreducible highest weight module over an affine Lie algebra of a dual untwisted type. Afterward, Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS2] proved that for a dominant integral weight $\lambda$, the set $\text{QLS}(\lambda)$ of all quantum Lakshmibai-Seshadri (QLS) paths of shape $\lambda$ provides a realization of the crystal basis of a special quantum Weyl module over a quantum affine algebra $U_q'(\mathfrak{g}_{\text{aff}})$ (without degree operator) of an arbitrary untwisted type, and that its graded character equals the specialization $E_{w_0\lambda}(q,0)$ at $t = 0$, which equals the specialization $P_\lambda(q,0)$ of the symmetric Macdonald polynomial $P_\lambda(q,t)$ at $t = 0$, where $w_0$ denotes the longest element of $W$. Here a QLS path is described in terms of (the parabolic version of) the quantum Bruhat graph, introduced by Brenti-Fomin-Postnikov [BFP]; the set of QLS paths is endowed with an affine crystal structure in a way similar to the one for the set of ordinary LS paths introduced by Littelmann [L]. Moreover, Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS3] obtained a formula for the specialization $E_{x\lambda}(q,0)$, $x \in W$, at $t = 0$ in an arbitrary untwisted affine type, which is described in terms of QLS paths of shape $\lambda$, and proved that the specialization $E_{x\lambda}(q,0)$ is just the graded character of a certain Demazure-type submodule of the special quantum Weyl module.

Also, Ishii-Naito-Sagaki proved that for a level-zero dominant weight $\Lambda$, the set of semi-infinite Lakshmibai-Seshadri (SiLS) paths of shape $\Lambda$ provides a realization of the crystal basis of the level-zero extremal weight module of extremal weight $\Lambda$ over a quantum affine algebra $U_q'(\mathfrak{g}_{\text{aff}})$ of an untwisted affine type, and by factoring the null root $\delta$ of the affine Lie algebra $\mathfrak{g}_{\text{aff}}$, we obtain a surjective strict morphism of crystals from the crystal of all SiLS paths of shape $\Lambda$ onto the set $\text{QLS}(\lambda)$; here $\lambda$ denotes the image of $\Lambda$ via the projection $P_{\text{aff}} \to P$ (see §2).

This paper is mainly a $A_{2n}^{(2)}$-type-analog of the Lenart-Naito-Sagaki-Schilling-Shimozono’s work [LNSSS2]. More precisely, the purpose of this paper is to define a QLS path of type $A_{2n}^{(2)}$, and to give a representation-theoretic interpretation of the specialization $P_{\lambda}^{A_{2n}^{(2)}}(q,0) = E_{w_0\lambda}(q,0)$ at $t = 0$ of the symmetric Macdonald polynomial $P_{\lambda}^{A_{2n}^{(2)}}(q,t)$ of type $A_{2n}^{(2)}$ in terms of the set $\text{QLS}^{A_{2n}^{(2)}}(\lambda)$ of all $A_{2n}^{(2)}$-type QLS paths of shape $\lambda$; here $P_{\lambda}^{A_{2n}^{(2)}}(q,t)$ (resp., $E_{\mu}^{A_{2n}^{(2)}}(q,t)$) is a specific specialization of the symmetric Macdonald-Koornwinder polynomial $P_{\lambda}(q,t_1,t_2,t_3,t_4,t_5)$ (resp., nonsymmetric Macdonald-Koornwinder polynomial $E_{\mu}(q,t_1,t_2,t_3,t_4,t_5)$) (see §5.1 and [OS]). We prove the following 2 theorems:

**Theorem A** (= Theorem 1.2.9). Let $\lambda$ be a dominant weight. Then, the set $\text{QLS}^{A_{2n}^{(2)}}(\lambda)$ of all QLS path of shape $\lambda$ provides a realization of the crystal basis of a quantum Weyl module $W_q(\Xi(\lambda))$ over a quantum affine algebra $U_q'(\mathfrak{g}(A_{2n}^{(2)}))$ of type $A_{2n}^{(2)}$, where $\Xi$ denotes the bijection $\Xi : P \to P_{\text{aff}}^0/(P_{\text{aff}}^0 \cap \mathbb{Q}\delta)$, and $P_{\text{aff}}^0$ the set of all level-zero weight (see §2).

**Theorem B** (= Theorem 5.1.2). Let $\lambda$ be a dominant weight. Then,

$$P_{\lambda}^{A_{2n}^{(2)}}(q,0) = \sum_{\eta \in \text{QLS}^{A_{2n}^{(2)}}(\lambda)} q^{\text{Deg}(\eta)} e^{\text{wt}(\eta)}.$$
where for \( \eta \in \text{QLS}(\lambda) \), \( \text{Deg}(\eta) \) is a certain nonnegative integer, which is explicitly described in terms of the quantum Bruhat graph; see §5.1 for details.

Theorem A implies that the definition of a QLS path of type \( A_{2n}^{(2)} \) is reasonable. By Theorem B, \( P_{A_{2n}^{(2)}}^{(2)}(q, 0) \) is equal to the graded character of the set \( \text{QLS}^{A_{2n}^{(2)}}(\lambda) \). Moreover, by Theorem A, \( P_{A_{2n}^{(2)}}^{(2)}(q, 0) \) is equal to the graded character of the crystal basis of the quantum Weyl module \( W_q(\xi(\lambda)) \) over a quantum affine algebra \( U_q'((\mathfrak{g}(A_{2n}^{(2)})). \) Also, we state an explicit formula for the specialization \( E_{\mu}^{A_{2n}^{(2)}}(q, t) \) of nonsymmetric Macdonald polynomial \( E_{\mu}^{A_{2n}^{(2)}}(q, t) \) at \( t = 0 \) in terms of specific subset of \( \text{QLS}^{A_{2n}^{(2)}}(\lambda) \). Furthermore, we establish the relation between QLS paths and SiLS paths of type \( A_{2n}^{(2)} \), which is \( A_{2n}^{(2)} \)-type analog of the Ishii-Naito-Sagaki's work [NS].

This paper is organized as follows. In Section 2, we fix our notation. In Section 3, we define a QLS path of dual untwisted types and type \( A_{2n}^{(2)} \). Also, we prove Theorem A by using root operators acting on the set \( \text{QLS}^{A_{2n}^{(2)}}(\lambda) \). In Section 4, we prove Theorem B; this theorem gives the description of the specialization \( P_{A_{2n}^{(2)}}^{(2)}(q, 0) \) at \( t = 0 \) in terms of QLS paths of shape \( \lambda \). In Appendix, we define a SiLS path of type \( A_{2n}^{(2)} \). Also, we prove that the set of QLS paths of shape \( \lambda \) is obtained from the set of SiLS paths of shape \( \Lambda \) through the projection \( \mathbb{R} \otimes_{\mathbb{Z}} P_{\text{aff}} \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} P \).

2. Notation

Let \( A = (a_{ij})_{i,j \in I} \) be a Cartan matrix, and \( A_{\text{aff}} = (a_{ij})_{i,j \in I_{\text{aff}}} \) be a generalized Cartan matrix of (twisted) affine type; in this paper, we consider the following pairs \((A, A_{\text{aff}})\).

|      |      |      |
|------|------|------|
|      |      |      |
|      |      |      |

Let \( \mathfrak{g}(A) \) be a finite dimensional simple Lie algebra associated with the Cartan matrix \( A \), and \( \mathfrak{g}(A_{\text{aff}}) \) be a twisted affine Lie algebra associated with the generalized Cartan matrix \( A_{\text{aff}} \). We denote by \( I \) and \( I_{\text{aff}} = I \sqcup \{0\} \) the vertex sets for the Dynkin diagrams of \( A \) and \( A_{\text{aff}} \), respectively.

First, we consider the cases when \( \mathfrak{g}(A_{\text{aff}}) \) is a dual untwisted affine Lie algebra; that is, \( A_{\text{aff}} = A_{2n-1}^{2}, D_{n+1}^{(2)}, E_6^{(2)}, \) or \( D_4^{(3)} \). Let \( \{\alpha_i\}_{i \in I} \) (resp., \( \{\alpha_i^\vee\}_{i \in I} \)) be the set of all simple roots (resp., coroots) of \( \mathfrak{g}(A) \); \( \mathfrak{h} = \bigoplus_{i \in I} \mathbb{C} \alpha_i \) a Cartan subalgebra of \( \mathfrak{g}(A) \), \( \mathfrak{h}^* = \bigoplus_{i \in I} \mathbb{C} \alpha_i \) the dual space of \( \mathfrak{h} \), and \( \mathfrak{h}_{\mathbb{R}} = \bigoplus_{i \in I} \mathbb{R} \alpha_i \) (resp., \( \mathfrak{h}_{\mathbb{R}}^* = \bigoplus_{i \in I} \mathbb{R} \alpha_i \)) the real form of \( \mathfrak{h} \) (resp., \( \mathfrak{h}^* \)); the duality pairing between \( \mathfrak{h} \) and \( \mathfrak{h}^* \) is denoted by \( \langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C} \). Let \( Q = \sum_{i \in I} \mathbb{Z} \alpha_i \subset \mathfrak{h}_{\mathbb{R}}^* \) (resp., \( Q' = \sum_{i \in I} \mathbb{Z} \alpha_i^\vee \subset \mathfrak{h}_{\mathbb{R}} \)) denote the root (resp., coroot) lattice of \( \mathfrak{g}(A) \), and \( P = \sum_{i \in I} \mathbb{Z} \varpi_i \subset \mathfrak{h}_{\mathbb{R}}^* \) the weight lattice of \( \mathfrak{g}(A) \); here \( \varpi_i, i \in I \), are the fundamental weights for \( \mathfrak{g}(A) \), that is, \( \langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij} \) for \( j \in I \). Let us denote by \( \Delta \) the set of all roots, and by \( \Delta^+ \) (resp., \( \Delta^- \)) the set of
all positive (resp., negative) roots. For a subset \( \Gamma \subset \Delta \), we set \( \Gamma^\vee \defeq \{ \alpha^\vee \mid \alpha \in \Gamma \} \).

Also, let \( W \defeq \langle s_i \mid i \in I \rangle \) be the Weyl group of \( g(A) \), where \( s_i, i \in I \), are the simple reflections acting on \( \mathfrak{h}^* \) and on \( \mathfrak{h}^* \):

\[
\begin{align*}
s_i x &= s_{\alpha_i} x = x - \langle x, \alpha_i^\vee \rangle \alpha_i, \quad x \in \mathfrak{h}^*, \\
s_i y &= s_{\alpha_i^\vee} y = y - \langle \alpha_i, y \rangle \alpha_i^\vee, \quad y \in \mathfrak{h};
\end{align*}
\]

we denote the identity element and the longest element of \( g \) by \( w_0 \) and the longest element of \( \Delta \). For a subset \( \Gamma \) of all positive (resp., negative) roots. For a subset \( \Gamma \) of all positive (resp., negative) roots.

For a subset \( S \subset I \), we set \( W_S \defeq \langle s_i \mid i \in S \rangle \). We denote the length coset representative of the coset \( wW_S \) by \( \ell(w) \), which equals \( \#(\Delta^+ \cap u^{-1} \Delta^-) \).

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Let \( \{ \alpha_i \}_{i \in I_{aff}} \) (resp., \( \{ \alpha_i^\vee \}_{i \in I_{aff}} \)) be the set of all simple roots (resp., coroots), and \( d \) the degree operator; here, \( \alpha_0 = \delta - \theta \) and \( \alpha_0^\vee = c - \theta^\vee \), where \( \delta \) denotes the null root of \( g(A_{aff}) \), \( c \) denotes the canonical central element of \( g(A_{aff}) \), and \( \theta \) denotes the highest short root of \( g(A) \). Then the Cartan subalgebra of \( g(A_{aff}) \) is \( \mathfrak{h}_{aff} = \left( \bigoplus_{i \in I_{aff}} \mathbb{C} \alpha_i^\vee \right) \oplus \mathbb{C} d \). We denote by \( \Delta_{aff} \) the set of all real roots of \( g(A_{aff}) \), and by \( \Delta^+_{aff} \) (resp., \( \Delta^-_{aff} \)) the set of all positive (resp., negative) roots.

Also, let \( \Lambda_i \in \mathfrak{h}^*_{aff}, i \in I_{aff} \), be the (affine) fundamental weights. We have

\[
\begin{align*}
\langle \alpha_j, \alpha_i^\vee \rangle_{aff} &= a_{ij}, \\
\langle \alpha_j, d \rangle &= \delta_{j,0}, \\
\langle \Lambda_j, \alpha_i^\vee \rangle_{aff} &= \delta_{i,j}, \\
\langle \Lambda_j, d \rangle_{aff} &= 0,
\end{align*}
\]

for \( i, j \in I_{aff} \), where \( \langle \cdot, \cdot \rangle_{aff} : \mathfrak{h}^*_{aff} \times \mathfrak{h}_{aff} \to \mathbb{C} \) denotes the duality pairing.

The weight lattice and the coweight lattice of \( g(A_{aff}) \) are \( P_{aff} \defeq \left( \bigoplus_{i \in I_{aff}} \mathbb{Z} \Lambda_i \right) \oplus \mathbb{Z} \delta \subset \mathfrak{h}^*_{aff} \) and \( P_{aff}^\vee \defeq \text{Hom}_{\mathbb{Z}}(P_{aff}, \mathbb{Z}) = \left( \bigoplus_{i \in I_{aff}} \mathbb{Z} \alpha_i^\vee \right) \oplus \mathbb{Z} d \subset \mathfrak{h}_{aff} \), respectively. We also consider the following lattices:

\[
\overline{P}_{aff} = P_{aff}/(P_{aff} \cap \mathbb{Q} \delta) \cong \bigoplus_{i \in I_{aff}} \mathbb{Z} \Lambda_i,
\]

\[
\overline{P}_{aff}^\vee = \bigoplus_{i \in I_{aff}} \mathbb{Z} \alpha_i^\vee \subset P_{aff}^\vee.
\]

We denote the canonical projection \( P_{aff} \to \overline{P}_{aff} \) by \( \text{cl} \), and take an injective \( \mathbb{Z} \)-linear map \( \xi : P \to P_{aff} \) given by \( \xi(x_i) = \Lambda_i - \langle \Lambda_i, c \rangle_{aff} \Lambda_0, i \in I \).

**Remark 2.0.1.** For \( x \in P_{aff} \), \( x \) is a level-zero weight if \( \langle x, c \rangle_{aff} = 0 \). We denote the set of all level-zero weights by

\[
P_{aff}^0 \defeq \{ x \in P_{aff} \mid \langle x, c \rangle_{aff} = 0 \},
\]
and set

$$I^\dagger_{\text{aff}} \overset{\text{def}}{=} \{ \xi(x) \in I^\dagger_{\text{aff}} \mid x \in P^0_{\text{aff}} \} = \bigoplus_{i \in I} \mathbb{Z}(\Lambda_i - \langle \Lambda_i, e \rangle_{\text{aff}} \Lambda_0) \left( = \bigoplus_{i \in I} \mathbb{Z}(\varpi_i) \right).$$

Then $\xi(P) \subset P^0_{\text{aff}}$, and $\xi \overset{\text{def}}{=} \xi \circ \xi : P \xrightarrow{\xi} P^0_{\text{aff}} \xrightarrow{\xi} P^0_{\text{aff}}$ is a linear isomorphism of $\mathbb{Z}$-modules. Also, $x \in P^0_{\text{aff}}$ is a level-zero dominant weight if $\xi^{-1} \circ \xi(x) \in P$ is a dominant weight.

We consider the case $(A, A_{\text{aff}}) = (C_n, A^{(2)}_{2n})$. In this case, we use all the notations which we set above, putting the dagger for $g(A^{(2)}_{2n})$ and $g(C_n)$ except $\theta^\dagger$, $\alpha_0^\dagger$ and $P^\dagger_{\text{aff}}$; here we denote by $\theta^\dagger$ the highest root, and set $\alpha_0^\dagger \overset{\text{def}}{=} \frac{1}{2}(-\theta^\dagger + \delta^\dagger)$, and $P^\dagger_{\text{aff}} \overset{\text{def}}{=} (\bigoplus_{i \in I^\dagger_{\text{aff}}} \mathbb{Z} \Lambda_i^\dagger) \oplus \frac{1}{2} \mathbb{Z} \delta^\dagger \subset (h_{\text{aff}}^\dagger)^*$. 

Remark 2.0.2. Let $I_{\text{aff}} = \{0, \ldots, n\}$ and $I = \{1, \ldots, n\}$ denote the vertex sets for the Dynkin diagrams of $g(D^{(2)}_{n+1})$ and $g(B_n)$, respectively, indexed as follows

$$0 \quad 1 \quad 2 \quad \ldots n-1 \quad n.$$

Also, let $I^\dagger_{\text{aff}} = \{0, \ldots, n\} = I_{\text{aff}}$ and $I^\dagger = \{1, \ldots, n\} = I$ denote the vertex sets for the Dynkin diagrams of $g(A^{(2)}_{2n})$ and $g(C_n)$, respectively, indexed as follows

$$0 \quad 1 \quad 2 \quad \ldots n-1 \quad n.$$

Let $h_{\mathbb{R}}$ and $h_{\mathbb{R}}^\dagger$ be a Cartan subalgebra of $g(B_n)$ and $g(C_n)$, respectively. We define linear isomorphisms $\iota : h_{\mathbb{R}}^\dagger \rightarrow h_{\mathbb{R}}$ and $\iota^* : (h_{\mathbb{R}}^\dagger)^* \rightarrow h_{\mathbb{R}}^\dagger$ by

$$\iota^*(\alpha_i^\dagger) \overset{\text{def}}{=} \left\{ \begin{array}{ll} \alpha_i, & \text{if } i \neq n, \\ 2 \alpha_i, & \text{if } i = n, \end{array} \right. \quad \text{and} \quad \iota((\alpha_i^\dagger)^\vee) = \left\{ \begin{array}{ll} \alpha_i^\vee, & \text{if } i \neq n, \\ \frac{1}{2} \alpha_i^\vee, & \text{if } i = n; \end{array} \right.$$ for $i \in I$; here we notice that

$$\langle x, y \rangle^\dagger = \langle \iota^*(x), \iota(y) \rangle,$$

for $x \in (h_{\mathbb{R}}^\dagger)^*$ and $y \in h_{\mathbb{R}}^\dagger$, where the pairing $\langle \cdot, \cdot \rangle^\dagger$ in the left-hand side is the duality pairing $(h_{\mathbb{R}}^\dagger)^* \times h_{\mathbb{R}}^\dagger \rightarrow \mathbb{R}$ and the pairing $\langle \cdot, \cdot \rangle$ in the right-hand side is the duality pairing $h_{\mathbb{R}}^\dagger \times h_{\mathbb{R}} \rightarrow \mathbb{R}$. Then, for $\alpha^\dagger \in \Delta^\dagger$, $\iota^*(\alpha^\dagger)$ is a long root in $\Delta$ if $\alpha^\dagger$ is a short root in $\Delta^\dagger$, or twice as a short root in $\Delta$ if $\alpha^\dagger$ is a half-long root in $\Delta^\dagger$. Also, for $\alpha^\dagger \in \Delta^\dagger$, $\iota((\alpha^\dagger)^\vee)$ is a short coroot in $\Delta^\vee$ if $\alpha^\dagger$ is a short root in $\Delta^\dagger$, or half as a long coroot in $\Delta$ if $\alpha^\dagger$ is a long root in $\Delta^\dagger$. If we identify $h_{\mathbb{R}}^\dagger$ and $h_{\mathbb{R}}^\dagger$ with $h_{\mathbb{R}}$ and $h_{\mathbb{R}}^\dagger$, respectively, then, the set of all roots $\Delta^\dagger$, the set of all coroots $(\Delta^\dagger)^\vee$, the highest root $\theta^\dagger$, the weight lattice $P^\dagger$, and the Weyl group $W^\dagger$ of $g(C_n)$ can be described in terms of those of $g(B_n)$. More precisely, $\iota^*(\Delta^\dagger) = \{ \alpha \mid \alpha \in \Delta \text{ is a long root} \} \cup \{ 2 \alpha \mid \alpha \in \Delta \text{ is a short root} \}$ and $\iota^*((\Delta^\dagger)^\vee) = \{ \alpha^\vee \mid \alpha \in \Delta \text{ is a long root} \} \cup \{ \frac{1}{2} \alpha^\vee \mid \alpha \in \Delta \text{ is a short root} \}$; in particular, the highest root $\theta^\dagger$ of $g(C_n)$ is identified with $2 \theta$, where $\theta$ is the highest root of $g(B_n)$. The weight lattice $P^\dagger$ of $g(C_n)$ is identified with the root lattice $Q$.
of \( \mathfrak{g}(B_2) \); indeed,

\[
\iota^*(P^\dagger) = \left\{ x \in \mathfrak{h}_R^* \bigg| \langle x, \alpha_i^\vee \rangle \in \mathbb{Z}, 1 \leq i \leq n-1, \text{and } \langle x, \frac{1}{2} \alpha_n^\vee \rangle \in \mathbb{Z} \right\}
\]

\[
= \left( \bigoplus_{i \neq n} \mathbb{Z} \omega_i \right) \oplus 2\mathbb{Z} \omega_i = Q;
\]

notice that \( \iota^*(P^\dagger) = Q \subset P \). There exists a group isomorphism \( \iota^*: W^\dagger \to W \) which satisfies \( \iota^*(s_i) = s_i, \ i \in I \). Furthermore, if we identify the null root \( \delta^\dagger \) of \( \mathfrak{g}(A_{2n}^{(2)}) \) with \( 2\delta \), where \( \delta \) is the null root of \( \mathfrak{g}(B_{n+1}^{(2)}) \), then \( \alpha_i^\dagger \) is identified with \( \alpha_0 \).

We fix a pair \((A, A_{aff})\) with \( \mathfrak{g}(A_{aff}) \) a dual untwisted affine Lie algebra. For \( \alpha \in \Delta \) of \( \mathfrak{g}(A) \), we set

\[
c_\alpha \overset{\text{def}}{=} \left\{ \begin{array}{cl}
1 & \text{if } \alpha \text{ is a short root}, \\
2 & \text{if } \alpha \text{ is a long root and } A \neq G_2, \\
3 & \text{if } \alpha \text{ is a long root and } A = G_2.
\end{array} \right.
\]

Then \( \Delta_{aff} = \{ \alpha + c_\alpha a \delta \mid \alpha \in \Delta, a \in \mathbb{Z} \} \).

**Remark 2.0.3.** Keep the notation and setting in Remark 2.0.2. Then, for every \( \alpha^\dagger \in \Delta^\dagger \), there exists \( \alpha \in \Delta \) such that \( \iota^*(\alpha^\dagger) = \frac{2}{c_\alpha} \alpha \) and \( \iota((\alpha^\dagger)^\vee) = \frac{2}{c_\alpha} \alpha^\vee \).

**Remark 2.0.4.** For \( A = B_n \), we define the standard bilinear form \((\cdot, \cdot): \mathfrak{h}_R^* \times \mathfrak{h}_R^* \to \mathbb{R}\) by

\[
(x, \alpha_i) \overset{\text{def}}{=} \frac{c_\alpha}{2} \langle x, \alpha_i^\vee \rangle, \ x \in \mathfrak{h}_R^*.
\]

We can identify \( \mathfrak{h}_R \) with \( \mathfrak{h}_R^* \) by \((\cdot, \cdot): \mathfrak{h}_R^* \times \mathfrak{h}_R^* \to \mathbb{R}; \alpha^\vee \) is identified with \( \frac{2}{c_\alpha} \alpha \). Then, by Remark 2.0.3 we have \( \iota^*(\Delta^\dagger) = \Delta^\vee \) and \( \iota((\Delta^\dagger)^\vee) = \Delta \). We claim that

\[
\langle \alpha^\dagger, (\beta^\dagger)^\vee \rangle^\dagger = \langle \iota((\beta^\dagger)^\vee), \iota((\alpha^\dagger)^\vee) \rangle,
\]

for \( \alpha^\dagger, \beta^\dagger \in \Delta^\dagger \); here, the pairing \((\cdot, \cdot)^\dagger \) in the left-hand side is the duality pairing \((\mathfrak{h})^\dagger \times \mathfrak{h}^\dagger \to \mathbb{R}\) and the pairing \((\cdot, \cdot)\) in the right-hand side is the duality pairing \(\mathfrak{h}^\dagger \times \mathfrak{h} \to \mathbb{R}\). Indeed, since there exist \( \alpha, \beta \in \Delta \) such that \( \iota^*(\alpha^\dagger) = \frac{2}{c_\alpha} \alpha \) and \( \iota((\beta^\dagger)^\vee) = \frac{c_\beta}{2} \beta^\vee \), then

\[
\langle \alpha^\dagger, (\beta^\dagger)^\vee \rangle^\dagger = \langle \iota^*(\alpha^\dagger), \iota((\beta^\dagger)^\vee) \rangle = \langle \frac{2}{c_\alpha} \alpha, \frac{c_\beta}{2} \beta^\vee \rangle = \frac{2}{c_\alpha} \langle \alpha, \beta \rangle = \frac{2}{c_\alpha} (\beta, \alpha) = \langle \beta, \alpha^\vee \rangle.
\]

3. **The (parabolic) quantum Bruhat graphs of type \( A_{2n}^{(2)} \)**

3.1. **Definition of the (parabolic) quantum Bruhat graphs.** First, we fix \((A, A_{aff})\) with \( \mathfrak{g}(A_{aff}) \) a dual untwisted affine Lie algebra.

**Definition 3.1.1** ([BFPP Definition 6.1]). The quantum Bruhat graph, denoted by QBG, is the directed graph with vertex set \( W \), and directed edges labeled by positive coroots; for \( u, v \in W \), and \( \beta \in \Delta^+ \), an arrow \( u \xrightarrow{\beta^\dagger} v \) is an edge of QBG if the following conditions hold:

1. \( v = us_\beta \), and
2. either \((2a)\): \( \ell(v) = \ell(u) + 1 \), or \((2b)\): \( \ell(v) = \ell(u) - 2\langle \beta, \rho^\vee \rangle + 1 \),

where \( \rho^\dagger \overset{\text{def}}{=} \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha^\vee \). An edge satisfying condition \((2a)\) (resp., \((2b)\)) is called a Bruhat (resp., quantum) edge.
**Definition 3.1.2** (see [LNSSS1 §4.3] for untwisted types). The parabolic quantum Bruhat graph, denoted by \( \text{QBG}^S \), is the directed graph with vertex set \( W^S \), and directed edges labeled by positive coroots in \( (\Delta^+ \setminus \Delta^+_S)^\vee \); for \( u, v \in W^S \), and \( \beta \in \Delta^+ \setminus \Delta^+_S \), an arrow \( u \xrightarrow{\beta} v \) is an edge of \( \text{QBG}^S \) if the following conditions hold:

1. \( v = [us_\beta] \), and
2. either (2a): \( \ell(v) = \ell(u) + 1 \), or (2b): \( \ell(v) = \ell(u) - 2(\rho, \beta) + 1 \),

where \( \rho_S = \frac{1}{2} \sum_{\alpha \in \Delta^+_S} \alpha \vee \). An edge satisfying condition (2a) (resp., (2b)) is called a Bruhat (resp., quantum) edge.

For a dominant weight \( \lambda \in P \), we set \( S = S_\lambda \).

**Definition 3.1.3** (see [LNSSS2 §3.2] for untwisted types). Let \( \lambda \in P \) be a dominant weight and \( b \in \mathbb{Q} \cap [0, 1] \). We denote by \( \text{QBG}_{b\lambda} \) (resp., \( \text{QBG}_{b\lambda}^S \)) the subgraph of \( \text{QBG} \) (resp., \( \text{QBG}^S \)) with the same vertex set but having only the edges: \( u \xrightarrow{\beta^\vee} v \) with \( b(\lambda, \beta) \in \mathbb{Z} \).

Let us fix \((A, A_{\text{aff}}) = (B_n, D_n^{(2)})\). Let us define \( \text{QBG}_{b\lambda}^{A_n^{(2)}} \) and \((\text{QBG}_{b\lambda}^{A_n^{(2)}})^S\) in terms of \( \text{QBG}_{b\lambda} \) and \( \text{QBG}_{b\lambda}^S \).

**Definition 3.1.4.** Let \( \lambda \in Q \) be a dominant weight and \( b \in \mathbb{Q} \cap [0, 1] \). We denote by \( \text{QBG}_{b\lambda}^{A_n^{(2)}} \) (resp., \((\text{QBG}_{b\lambda}^{A_n^{(2)}})^S\)) the subgraph of \( \text{QBG}_{b\lambda} \) (resp., \( \text{QBG}_{b\lambda}^S \)) with the same vertex set but having only the edges:

\[
\begin{align*}
\text{If } \beta \text{ is a long root of } \Delta, & \quad u \xrightarrow{\beta^\vee} v \quad \text{with} \quad b(\lambda, \beta^\vee) \in \mathbb{Z} \\
\text{If the edge is a quantum edge with } \beta \text{ a short root of } \Delta, & \quad u \xrightarrow{\beta^\vee} v \quad \text{with} \quad b(\lambda, \beta^\vee) \in 2\mathbb{Z} \\
\text{If the edge is a Bruhat edge with } \beta \text{ a short root of } \Delta, & \quad u \xrightarrow{\beta^\vee} v \quad \text{with} \quad b(\lambda, \beta^\vee) \in \mathbb{Z}
\end{align*}
\]

3.2. **QBGA_{2n}^{(2)} in terms of the root system of type C_n.** Let \((A, A_{\text{aff}}) = (C_n, A_{2n}^{(2)})\). In this subsection, we use the notation in Remark 2.0.2.

**Definition 3.2.1** ([BPP Definition 6.1]). The quantum Bruhat graph (of type \( C_n \)), denoted by \( \text{QBG}^\dagger \), is the directed graph with vertex set \( W^\dagger \), and directed edges labeled by positive roots; for \( u, v \in W^\dagger \), and \( \beta \in (\Delta^\dagger)^+ \), an arrow \( u \xrightarrow{\beta} v \) is an edge of \( \text{QBG}^\dagger \) if the following conditions hold:

1. \( v = us_\beta \), and
2. either (2a): \( \ell(v) = \ell(u) + 1 \), or (2b): \( \ell(v) = \ell(u) - 2(\rho^\dagger, \beta^\dagger) + 1 \),

where \( \rho^\dagger \) is a short root of \( \Delta^\dagger \). An edge satisfying condition (2a) (resp., (2b)) is called a Bruhat (resp., quantum) edge.

We remark that \( \text{QBG}^\dagger \) is identical to the quantum Bruhat graph for the untwisted affine Lie algebra of type \( C_n^{(1)} \) (see [LNSSS1]).

**Definition 3.2.2** ([LNSSS1 §4.3]). The parabolic quantum Bruhat graph, denoted by \( (\text{QBG}^\dagger)^S \), is the directed graph with vertex set \( (W^\dagger)^S \), and directed edges labeled by positive coroots in \((\Delta^\dagger)^+ \setminus (\Delta^\dagger)_S^\vee \); for \( u, v \in (W^\dagger)^S \), and \( \beta \in (\Delta^\dagger)^+ \setminus (\Delta^\dagger)_S^\vee \), an arrow \( u \xrightarrow{\beta} v \) is an edge of \( (\text{QBG}^\dagger)^S \) if the following conditions hold:

1. \( v = [us_\beta] \), and
Lemma 3.3.2. Remark 3.2.4

Some properties on QBG.

Definition 3.2.3. Proposition 3.3.1 (BFP, Theorem 6.4)

where \( \rho \) following condition: if \( \alpha \), \( \beta \)

We take and fix an arbitrary dominant weight \( \lambda \in P^\dagger \), i.e., \( \langle \lambda, (\alpha_i^\dagger)\rangle^\dagger \geq 0 \) for all \( i \in I \). We set

\[
S = S_\lambda \overset{\text{def}}{=} \{ i \in I \mid \langle \lambda, (\alpha_i^\dagger)\rangle^\dagger = 0 \} \subset I.
\]

Definition 3.2.3. Let \( \lambda \in P^\dagger \) be a dominant weight and \( b \in \mathbb{Q} \cap [0, 1] \). We denote by \((QBG^\dagger_{b\lambda})^{A^{(2)}_{2n}} \) (resp., \((QBG^\dagger_{b\lambda})^{A^{(2)}_{2n}}\) ) the subgraph of \( QBG^\dagger \) (resp., \( QBG^\dagger\) ) with the same vertex set but having only the edges:

\[
\begin{aligned}
\text{if } \beta \text{ is a short root of } \Delta^\dagger, \\
\text{if the edge is a quantum edge with } \beta \text{ a long root of } \Delta^\dagger, \\
\text{if the edge is a Bruhat edge with } \beta \text{ a long root of } \Delta^\dagger.
\end{aligned}
\]

Remark 3.2.4. It follows from Lemmas 2.0.2, 2.0.3, and 2.0.4 that

(1) \( W^\dagger \overset{\iota^*}{\cong} W \),
(2) \( \iota^* (\rho^\dagger) = \rho^\dagger \) and \( \iota^* (\rho^\dagger)_S = \rho_S^\dagger \) since \( \iota^* (\Delta^\dagger) = \Delta^\dagger \), and \( \iota^* ((\Delta^\dagger)^\dagger) = \Delta \),
(3) \( \langle \alpha, \beta^\dagger \rangle = \langle \iota (\beta^\dagger), \iota^* (\alpha) \rangle \) for \( \alpha, \beta \in \Delta^\dagger \),
(4) we can rewrite integrality condition by using \( \iota^*(\Delta^\dagger) = \{ \alpha \mid \alpha \in \Delta \text{ is a long root} \} \cup \{ 2\alpha \mid \alpha \in \Delta \text{ is a short root} \} \).

Therefore, by [2.1], for \( \lambda \in P^\dagger \) a dominant weight, and \( b \in \mathbb{Q} \cap [0, 1] \), there exist graph isomorphisms \((QBG^\dagger_{b\lambda})^{A^{(2)}_{2n}} \cong QBG^\dagger_{b\iota^*(\lambda)}^{A^{(2)}_{2n}} \text{ and } ((QBG^\dagger_{b\lambda})^{A^{(2)}_{2n}})^S \cong (QBG^\dagger_{b\iota^*(\lambda)})^S \)
induced by the group isomorphism \( \iota^*: W^\dagger \to W \).

3.3. Some properties on QBG. Let \( \prec \) be a total order on \( (\Delta^\dagger)^\dagger \) satisfying the following condition: if \( \alpha, \beta, \gamma \in \Delta^\dagger \) with \( \gamma = \alpha + \beta \), then \( \gamma^\dagger \prec \gamma^\dagger \prec \beta^\dagger \) or \( \beta^\dagger \prec \gamma^\dagger \prec \alpha^\dagger \) (see §2.2).

Proposition 3.3.1 (BFP Theorem 6.4]). Let \( u \) and \( v \) be elements in \( W \).

(1) There exists a unique directed path from \( u \) to \( v \) in QBG for which the edge labels are strictly increasing (resp., strictly decreasing) in the total order \( \prec \) above.
(2) The unique label-increasing (resp., label-decreasing) path

\[
u = u_0 \xrightarrow{\gamma^\dagger} u_1 \xrightarrow{\gamma^\dagger} \cdots \xrightarrow{\gamma^\dagger} u_r = v
\]

from \( u \) to \( v \) in QBG is a shortest directed path from \( u \) to \( v \). Moreover, it is lexicographically minimal (resp., lexicographically maximal) among all shortest directed paths from \( u \) to \( v \); that is, for an arbitrary shortest directed path

\[
u = u_0' \xrightarrow{\gamma^\dagger} u_1' \xrightarrow{\gamma^\dagger} \cdots \xrightarrow{\gamma^\dagger} u_r' = v
\]

from \( u \) to \( v \) in QBG, there exists \( 1 \leq j \leq r \) such that \( \gamma_j^\dagger \prec \gamma_j^\dagger \) (resp., \( \gamma_j^\dagger \succ \gamma_j^\dagger \)), and \( \gamma_k^\dagger = \gamma_k^\dagger \) for \( 1 \leq k \leq j - 1 \).

Lemma 3.3.2. If \( x \xrightarrow{\beta^\dagger} y \) is a Bruhat (resp., quantum) edge of QBG, then \( yw_0 \xrightarrow{w_0 \beta^\dagger} \)
\( xw_0 \) is also a Bruhat (resp., quantum) edge of QBG.
Proof. This follows easily from equalities \( \ell(y) - \ell(x) = \ell(xw_0) - \ell(yw_0) \) and \( \langle -w_0 \beta, \rho^\vee \rangle = \langle \beta, \rho^\vee \rangle \). \(\square\)

Lemma 3.3.3 (see [LNSS1] Proposition 5.10 and Proposition 5.11) for untwisted types. Let \( \lambda \in P \) be a dominant weight. Let \( w \in W^S \), \( i \in I \), and \( \beta \in \Delta^+ \).

1. If \( \langle w\lambda, \alpha_i^\vee \rangle > 0 \) (resp., \( \lambda_i < 0 \)), then \( w \leftarrow w^{-1} \alpha_i^\vee \) is a Bruhat edge.
2. If \( \langle w\lambda, -\theta^\vee \rangle < 0 \) (resp., \( \lambda_i > 0 \)), then \( w \to w^{-1} \theta^\vee \) is a quantum edge where \( z \in W_S \) is defined by \( s_\theta w \) is a dominant weight. Let \( \gamma \in P \) for a dual untwisted type, or \( \gamma \in Q \) for type \( A_2^{(2)} \);
   that is, if we consider \( \text{QB}_{b\lambda} \) and \( \text{QB}_{b\lambda}^S \), then \( \lambda \in P \), and if we consider \( \text{QB}_{b\lambda}^{A_2^{(2)}} \) and \( (\text{QB}_{b\lambda}^{A_2^{(2)}})^S \), then \( \lambda \in Q \).

In diagrams of the following lemma, a plain (resp., dotted) edge represents a Bruhat (resp., quantum) edge.

Lemma 3.3.4 (see [LNSS1] Proposition 4.1.4 (3), (4) and Proposition 4.1.5 (3), (4) for untwisted types). Let \( \lambda \in P \) be a dominant weight. Let \( w \in W^S \), and \( i \in I \).

1. If \( \langle w\lambda, \alpha_i^\vee \rangle \geq 0 \) and \( \langle ws_\beta \lambda, \alpha_i^\vee \rangle < 0 \), then \( w\beta = \pm \alpha_i \).
2. If \( \langle w\lambda, \alpha_i^\vee \rangle > 0 \) and \( \langle ws_\beta \lambda, \alpha_i^\vee \rangle \leq 0 \), then \( w\beta = \pm \alpha_i \).
3. If \( \langle w\lambda, -\theta^\vee \rangle \geq 0 \) and \( \langle ws_\beta \lambda, -\theta^\vee \rangle < 0 \), then \( w\beta = \pm \theta \).
4. If \( \langle w\lambda, -\theta^\vee \rangle > 0 \) and \( \langle ws_\beta \lambda, -\theta^\vee \rangle \leq 0 \), then \( w\beta = \pm \theta \).

In what follows, we fix a dominant weight \( \lambda \in P \) for a dual untwisted type, or a dominant weight \( \lambda \in Q \) for type \( A_2^{(2)} \); that is, if we consider \( \text{QB}_{b\lambda} \) and \( \text{QB}_{b\lambda}^S \), then \( \lambda \in P \), and if we consider \( \text{QB}_{b\lambda}^{A_2^{(2)}} \) and \( (\text{QB}_{b\lambda}^{A_2^{(2)}})^S \), then \( \lambda \in Q \).

In diagrams of the following lemma, a plain (resp., dotted) edge represents a Bruhat (resp., quantum) edge.

Lemma 3.3.5 (see [LNSS2] Lemma 5.14) for untwisted types. Let \( G \) be \( \text{QB}_{b\lambda}^S \) or \( (\text{QB}_{b\lambda}^{A_2^{(2)}})^S \). Let \( i \in I \), \( \gamma \in \Delta^+ \setminus \Delta^+_+ \), and \( w \in W^S \). Then we have following cases in which the bottom two edges in \( G \) imply the top two edges in \( G \) in the left diagram, and the top two edges in \( G \) imply the bottom two edges in \( G \) in the right diagram.

1. Here we assume that \( \gamma \neq w^{-1} \alpha_i \) and have \( s_i [ws_\gamma] = s_i ws_\gamma = [s_i ws_\gamma] \) in both cases.

   \[
   s_i [ws_\gamma] \quad \gamma^\vee \quad [ws_\gamma]^{-1} \quad w^{-1} \alpha_i^\vee \quad w \quad \gamma^\vee
   \]
   \[
   s_i (s_i [ws_\gamma]) \quad \gamma^\vee \quad [ws_\gamma]^{-1} \quad w^{-1} \alpha_i^\vee \quad w \quad \gamma^\vee
   \]
(2) Here we have \( s_i \{ w s_\gamma \} = \{ s_i w s_\gamma \} \) in both cases.

\[
\begin{align*}
\gamma^\vee & \quad s_i \{ w s_\gamma \} \\
& \xrightarrow{\gamma^\vee} [w s_\gamma]^{-1} \alpha_i^\vee \\
\gamma^\vee & \quad s_i \vert \end{align*}
\]

\[
\begin{align*}
\gamma^\vee & \quad [w s_\gamma] \\
& \xrightarrow{\gamma^\vee} s_i \{ w s_\gamma \} \\
\gamma^\vee & \quad w \\
& \xrightarrow{-w^{-1} \alpha_i^\vee} s_i \vert
\end{align*}
\]

(3) Here \( z, z' \in W_S \) are defined by \( s_\theta w = [s_\theta w] z, s_\theta \{ w s_\gamma \} = [s_\theta \{ w s_\gamma \}] z' = [s_\theta w s_\gamma] z' \). In subcase (3.3) (resp., (3.4)) we assume that \( \langle w^{-1} \theta, \gamma^\vee \rangle \) is nonzero (resp., zero). In both cases, we have \( w s_\gamma = [w s_\gamma] \).

\[
\begin{align*}
\gamma^\vee & \quad [s_\theta w s_\gamma] \\
& \xrightarrow{\gamma^\vee} [w s_\gamma]^{-1} \theta^\vee \\
\gamma^\vee & \quad [s_\theta w] \\
& \xrightarrow{-w^{-1} \theta^\vee} w \\
\gamma^\vee & \quad \gamma \\
& \xrightarrow{zw^{-1} \theta^\vee} s_\theta \gamma \\
\gamma^\vee & \quad [s_\theta w s_\gamma] \\
& \xrightarrow{\gamma^\vee} [s_\theta w] \quad z\gamma \\
\gamma^\vee & \quad w \\
& \xrightarrow{zw^{-1} \theta^\vee} [s_\theta w] \\
\gamma^\vee & \quad z\gamma \\
& \xrightarrow{\gamma^\vee} [s_\theta w s_\gamma] \quad z'
\end{align*}
\]

(4) Here we assume \( \gamma \neq -w^{-1} \theta \) in all cases, and \( z, z' \in W_S \) are defined as in (3). In subcase (3.5) (resp., (3.6)) we assume that \( \langle w^{-1} \theta, \gamma^\vee \rangle \) is nonzero (resp., zero).

\[
\begin{align*}
\gamma^\vee & \quad [s_\theta w s_\gamma] \\
& \xrightarrow{\gamma^\vee} [w s_\gamma]^{-1} \theta^\vee \\
\gamma^\vee & \quad [s_\theta w] \\
& \xrightarrow{-w^{-1} \theta^\vee} w \\
\gamma^\vee & \quad \gamma \\
& \xrightarrow{zw^{-1} \theta^\vee} s_\theta z \gamma \\
\gamma^\vee & \quad [s_\theta w s_\gamma] \\
& \xrightarrow{\gamma^\vee} [s_\theta w] \\
\gamma^\vee & \quad w \\
& \xrightarrow{zw^{-1} \theta^\vee} [s_\theta w] \\
\gamma^\vee & \quad s_\theta \gamma \\
\gamma^\vee & \quad [s_\theta w s_\gamma] \\
& \xrightarrow{\gamma^\vee} [s_\theta w] \\
\gamma^\vee & \quad w \\
& \xrightarrow{zw^{-1} \theta^\vee} [s_\theta w] \\
\gamma^\vee & \quad z\gamma \\
& \xrightarrow{\gamma^\vee} [s_\theta w s_\gamma] \quad z'
\end{align*}
\]
Proof. If \( G = \text{QBG}^S_{b\lambda} \), then the proof is similar to the proof of Lemma 5.14. Hence we assume that \( G = (\text{QBG}^{A_{2n}^{(2)}})_{b\lambda} \). Then in all the cases except \( 3.3 \) and \( 3.5 \), the assertion is obvious by Lemma 3.3.3 for \( G = (\text{QBG}^{S_{b\lambda}})_{b\lambda} \) of type \( D_{n+1}^{(2)} \).

In cases \( 3.3 \) and \( 3.5 \), the assertion for \( b = 1 \) is obvious by Lemma 3.3.3 for \( G = (\text{QBG}^{S_{b\lambda}})_{b\lambda} \) of type \( D_{n+1}^{(2)} \). Suppose that \( w \gamma \rightarrow w s_{\lambda} \) is a Bruhat edge of \( (\text{QBG}^{A_{2n}^{(2)}})_{b\lambda} \). It suffices to show that \( s_{gw} \xrightarrow{z\gamma} s_{gws_{\lambda}} \) is a Bruhat edge of \( (\text{QBG}^{A_{2n}^{(2)}})_{b\lambda} \). By Lemma 3.3.3 for \( G = (\text{QBG}^{S_{b\lambda}})_{b\lambda} \) of type \( D_{n+1}^{(2)} \), we know that \( s_{gw} \xrightarrow{z\gamma} s_{gws_{\lambda}} \) is a Bruhat edge of \( \text{QBG}^{S_{b\lambda}} \). Recall that \( \theta \) is the highest short root. Therefore, \( \langle \lambda, z\gamma \rangle = b(\lambda, \gamma) \in \mathbb{Z} \), and hence \( s_{gw} \xrightarrow{z\gamma} s_{gws_{\lambda}} \) is a Bruhat edge of \( (\text{QBG}^{A_{2n}^{(2)}})_{b\lambda} \).

For an edge \( u \xrightarrow{\beta} v \) of QBG, we set

\[
\text{wt}(u \rightarrow v) \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } u \xrightarrow{\beta} v \text{ is a Bruhat edge}, \\
\beta^\gamma & \text{if } u \xrightarrow{\beta} v \text{ is a quantum edge}.
\end{cases}
\]

Also, for \( u, v \in W \), we take a shortest directed path \( u = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = v \) in QBG, and set

\[
\text{wt}(u \Rightarrow v) = \text{wt}(x_0 \rightarrow x_1) + \cdots + \text{wt}(x_{r-1} \rightarrow x_r) \in Q^\gamma;
\]

we know from [Pa, Lemma 1 (2), (3)] that this definition does not depend on the choice of a shortest directed path from \( u \) to \( v \) in QBG. For a dominant weight \( \lambda \in P \), we set \( \text{wt}_\lambda(u \Rightarrow v) \overset{\text{def}}{=} \langle \lambda, \text{wt}(u \Rightarrow v) \rangle \), and call it the \( \lambda \)-weight of a directed path from \( u \) to \( v \) in QBG.

For an edge \( u \xrightarrow{\beta} v \) in \( \text{QBG}^S \), we set

\[
\text{wt}^S(u \rightarrow v) \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } u \xrightarrow{\beta} v \text{ is a Bruhat edge}, \\
\beta^\gamma & \text{if } u \xrightarrow{\beta} v \text{ is a quantum edge}.
\end{cases}
\]

Also, for \( u, v \in W^S \), we take a shortest directed path \( p : u = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = v \) in \( \text{QBG}^S \) (such a path always exists by [LNSSSI, Lemma 6.12]), and set

\[
\text{wt}^S(p) \overset{\text{def}}{=} \text{wt}^S(x_0 \rightarrow x_1) + \cdots + \text{wt}^S(x_{r-1} \rightarrow x_r) \in Q^\gamma;
\]
we know from [LNSSS2] Proposition 8.1 that if \( q \) is another shortest directed path from \( u \) to \( v \) in QBG\( ^S \), then \( \text{wt}^S(p) - \text{wt}^S(q) \in Q^X_S = \sum_{i \in S} \mathbb{Z}_{\geq 0} \alpha^X_i \).

Now, we take and fix an arbitrary dominant weight \( \lambda \in P^* \), and set
\[
S = S_\lambda \overset{\text{def}}{=} \{ i \in I \mid \langle \lambda, \alpha^\vee_i \rangle = 0 \}.
\]

By the remark just above, for \( u, v \in W^S \), the value \( \langle \lambda, \text{wt}^S(p) \rangle \) does not depend on the choice of a shortest directed path \( p \) from \( u \) to \( v \) in QBG\( ^S \); this value is called the \( \lambda \)-weight of a directed path from \( u \) to \( v \) in QBG\( ^S \). Moreover, we know from [LNSSS2, Lemma 7.2] that the value \( \langle \lambda, \text{wt}^S(p) \rangle \) is equal to the value \( \text{wt}_\lambda(x \Rightarrow y) = \langle \lambda, \text{wt}(x \Rightarrow y) \rangle \) for all \( x \in uW_S \) and \( y \in vW_S \). In view of this, for \( u, v \in W^S \), we write \( \text{wt}_\lambda(u \Rightarrow v) \) also for the value \( \langle \lambda, \text{wt}^S(p) \rangle \) by abuse of notation.

The proof of the following lemma is similar to that of [LNSSS2] Lemma 6.1.

**Lemma 3.3.6** (see [LNSSS2] Lemma 6.1] for untwisted types). Let \( b \in \mathbb{Q} \cap [0, 1] \); notice that \( b \) may be 1. If \( u \rightarrow v \) is an edge of QBG\( _{b\lambda}^{A(2)_{2n}} \), then there exists a directed path from \( [u] \) to \( [v] \) in (QBG\( _{b\lambda}^{A(2)_{2n}} \))\( ^S \).

For \( u, v \in W \) (resp., \( \in W^S \)), let \( \ell(u, v) \) denote the length of a shortest path in QBG (resp., QBG\( ^S \)) from \( u \) to \( v \). For \( w \in W \), we define the \( w \)-tilted (dual) Bruhat order \( <_w \) on \( W \) as follows: for \( u, v \in W \),
\[
\ell(u, v) \overset{\text{def}}{=} \ell(v, u) + \ell(u, w).
\]

The proof of the following lemma is similar to those of [LNSSS1] Theorem 7.1] and [LNSSS2] Lemma 6.5].

**Lemma 3.3.7** (see [LNSSS1] Theorem 7.1], [LNSSS2] Lemma 6.5] for untwisted types). Let \( u, v, w \in W^S \), and \( w \in W_S \).

1. There exists a unique minimal element in the coset \( vW_S \) in the uw-tilted Bruhat order \( <_{uw} \). We denote it by \( \text{min}(vW_S, <_{uw}) \).
2. There exists a unique directed path from some \( x \in vW_S \) to some \( uw \) in QBG whose edge labels are increasing in the total order \( < \) on \( (\Delta^+)^\vee \), defined at the beginning of §2.3, and lie in \( (\Delta^+ \setminus \Delta_0^+)\)\( ^\vee \). This path begins with \( \text{min}(vW_S, <_{uw}) \).
3. Let \( b \in \mathbb{Q} \cap [0, 1] \). If there exists a directed path from \( v \) to \( u \) in QBG\( _{b\lambda}^S \) (resp., QBG\( _{b\lambda}^{A(2)_{2n}} \))\( ^S \), then the directed path in (2) is in QBG\( _{b\lambda} \) (resp., QBG\( _{b\lambda}^{A(2)_{2n}} \)).

4. **Quantum Lakshmibai-Seshadri paths and root operators**

4.1. **Quantum Lakshmibai-Seshadri paths.** First of all, we define quantum Lakshmibai-Seshadri paths for dual untwisted type.

**Definition 4.1.1** (see [LNSSS2] Definition 3.1] for untwisted types). We fix \((A, A_{\text{aff}})\) with \( g(A_{\text{aff}}) \) a dual untwisted affine Lie algebra. Let \( \lambda \in P \) be a dominant weight and set \( S = S^\lambda = \{ i \in I \mid \langle \lambda, \alpha^\vee_i \rangle = 0 \} \). A pair \( \eta = (w_1, w_2, \ldots, w_s; \tau_0, \tau_1, \ldots, \tau_s) \) of a sequence \( w_1, \ldots, w_s \) of elements in \( W \) such that \( w_k \neq w_{k+1} \) for \( 1 \leq k \leq s - 1 \) and a increasing sequence \( 0 = \tau_0, \ldots, \tau_s = 1 \) of rational numbers is called a quantum Lakshmibai-Seshadri (QLS) path of shape \( \lambda \) if

(C) for every \( 1 \leq i \leq s - 1 \), there exists a directed path from \( w_{i+1} \) to \( w_i \) in QBG\( ^S_{\tau_i \lambda} \).
Let $QLS(\lambda)$ denote the set of all QLS paths of shape $\lambda$.

**Remark 4.1.2.** As in [LNSSS3, Definition 3.2.2 and Theorem 4.1.1], condition (C) can be replaced by

$(C')$ for every $1 \leq i \leq s - 1$, there exists a shortest directed path in $\text{QBG}^S$ from $w_{i+1}$ to $w_i$ which is also a directed path in $\text{QBG}_{r_{i\lambda}}^S$.

Let $(A, A_{\text{aff}}) = (B_n, D_{n+1}^{(2)})$. We define a $A_{2n}^{(2)}$-type quantum Lakshmibai-Seshadri (QLS) path (or a QLS path of type $A_{2n}^{(2)}$) as a quantum Lakshmibai-Seshadri paths for type $D_{n+1}^{(2)}$ with a specific condition.

**Definition 4.1.3.** Let $\lambda \in Q$ be a dominant weight and set $S = S_{\lambda} = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. A pair $\eta = (w_1, w_2, \ldots, w_s; \tau_0, \tau_1, \ldots, \tau_s)$ of a sequence $w_1, \ldots, w_s$ of elements in $W^S$ such that $w_k \neq w_{k+1}$ for $1 \leq k \leq s - 1$ and a increasing sequence $0 = \tau_0, \ldots, \tau_s = 1$ of rational numbers is called a $A_{2n}^{(2)}$-type quantum Lakshmibai-Seshadri (QLS) path (or a QLS path of type $A_{2n}^{(2)}$) of shape $\lambda$ if

$(C)$ for every $1 \leq i \leq s - 1$, there exists a directed path from $w_{i+1}$ to $w_i$ in $(\text{QBG}_{r_{i\lambda}}^S)^S$.

Let $QLS_{A_{2n}^{(2)}}(\lambda) \subset QLS(\lambda)$ denote the set of all $A_{2n}^{(2)}$-type QLS paths of shape $\lambda$.

**Definition 4.1.4.** We define a subset $\overline{QLS}_{A_{2n}^{(2)}}(\lambda) \subset QLS_{A_{2n}^{(2)}}(\lambda)$ by

$\overline{QLS}_{A_{2n}^{(2)}}(\lambda) \overset{\text{def}}{=} \{\eta = (w_1, w_2, \ldots, w_s; \tau_0, \tau_1, \ldots, \tau_s) \in QLS_{A_{2n}^{(2)}}(\lambda) \mid \eta \text{ satisfies condition } (C')\}$,

where

$(C')$ for every $1 \leq i \leq s - 1$, there exists a shortest directed path in $\text{QBG}^S$ from $w_{i+1}$ to $w_i$ which is also a directed path in $(\text{QBG}_{r_{i\lambda}}^S)^S$.

Indeed, the set $\overline{QLS}_{A_{2n}^{(2)}}(\lambda)$ is identical to $QLS_{A_{2n}^{(2)}}(\lambda)$ (see §3.4).

Let $(A, A_{\text{aff}}) = (C_n, A_{2n}^{(2)})$.

**Definition 4.1.5.** Let $\lambda \in P^\dagger$ be a dominant weight and set $S = S_{\lambda} = \{i \in I^\dagger \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. A pair $\eta = (w_1, w_2, \ldots, w_s; \tau_0, \tau_1, \ldots, \tau_s)$ of a sequence $w_1, \ldots, w_s$ of elements in $(W^\dagger)^S$ and a increasing sequence $0 = \tau_0, \ldots, \tau_s = 1$ of rational numbers is called a $A_{2n}^{(2)}$-type quantum Lakshmibai-Seshadri (QLS$^\dagger$) path of shape $\lambda$ if for every $1 \leq i \leq s - 1$, there exists a directed path from $w_{i+1}$ to $w_i$ in $(\text{QBG}_{r_{i\lambda}}^S)^S$.

Let $QLS^\dagger(\lambda)$ denote the set of all $A_{2n}^{(2)}$-type QLS$^\dagger$ paths of shape $\lambda$.

**Remark 4.1.6.** It follows from Remark 3.2.4 that for $\lambda \in P^\dagger$, there exists a bijective $t^*: QLS^\dagger(\lambda) \to QLS_{A_{2n}^{(2)}}(t^*(\lambda))$ which satisfies $t^*((w_1, \ldots, w_s; \tau_0, \ldots, \tau_s)) = (t^*(w_1), \ldots, t^*(w_s); \tau_0, \ldots, \tau_s)$, where $t^*: W^\dagger \to W$ denotes the group isomorphism defined in Remark 2.0.2. Since in dual untwisted types, a QLS path of type $A_{\text{aff}}$ is defined in terms of the root system of type $A$, and in untwisted types, a QLS path of type $A_{\text{aff}}^{(1)}$ is defined in terms of the root system of type $A$ (see [LNSSS2]), QLS path of type $A_{2n}^{(2)}$ should be defined in terms of the root system of type $C_n$. However, in this paper, we mainly study $QLS_{A_{2n}^{(2)}}(\lambda)$, because the proofs of all properties on
QLS paths of dual untwisted types are similar to that of untwisted types, and Orr-Shimozono formula (see §4.2) of type $A_{2n}^{(2)}$ is described in terms of the root system of types $D_{n+1}^{(2)}$ and $C_n$.

4.2. Root operators. We fix a pair $(A,A_{\text{aff}})$ with $\mathfrak{g}(A_{\text{aff}})$ a dual untwisted affine Lie algebra. We mainly consider the case $(A,A_{\text{aff}}) = (B_n,D_{n+1}^{(2)})$. In what follows, we use the following notation: for $i \in I_{\text{aff}},$

$$\bar{\alpha}_i \overset{\text{def}}{=} \left\{ \begin{array}{ll} -\theta & \text{for } i = 0, \\ \alpha_i & \text{for } i \in I, \end{array} \right. \quad \text{and} \quad r_i \overset{\text{def}}{=} \left\{ \begin{array}{ll} s_{i} & \text{for } i = 0, \\ s_i \bar{h}_i & \text{for } i \in I. \end{array} \right.$$

For a piecewise-linear, continuous (PLC for short) map $\pi : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$, we define a function $H(t) = H_i^\pi(t)$ on $[0,1]$ by $H(t) = H_i^\eta(t) = \langle \eta(t), (\bar{\alpha}_i)^\vee \rangle, t \in [0,1]$, and set $m = m_i^\eta = \min \{ H_i^\eta(t) \mid t \in [0,1] \}$. Let $\mathfrak{B}_{\text{int}}$ be the set of all PLC maps $[0,1] \to \mathfrak{h}_{\mathbb{R}}^*$ whose all local minima of the function $H_i^\pi(t), t \in [0,1]$ are integers.

We fix a dominant weight $\lambda \in P$.

**Remark 4.2.1.** We identify an element $\eta = (v_1, \ldots, v_s; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \text{QLS}(\lambda)$ with the following PLC map $\eta : [0,1] \to \mathfrak{h}_{\mathbb{R}}^* = \bigoplus_{i \in I} \mathbb{R} \bar{\alpha}_i$:

$$\eta(t) \overset{\text{def}}{=} \sum_{k=1}^{p-1} (\sigma_k - \sigma_{k-1})v_k \lambda + (t - \sigma_{p-1})v_p \lambda \quad \text{for } \sigma_{p-1} \leq t \leq \sigma_p, 1 \leq p \leq s.$$

Then, $\eta \in \mathfrak{B}_{\text{int}}$ (see Lemma 4.3.7). We set $\text{wt}(\eta) \overset{\text{def}}{=} \eta(1)$. In fact, $\text{wt}(\eta) \in P$ (see Lemma 4.3.2).

Let $i \in I_{\text{aff}}$. We define the root operators $e_i, f_i : \mathfrak{B}_{\text{int}} \to \mathfrak{B}_{\text{int}} \sqcup \{0\}$ as follows.

If $m = m_i^\eta = 0$, then $e_i \eta \overset{\text{def}}{=} 0$. If $m \leq -1$, then we set

$$t_1 \overset{\text{def}}{=} \min \{ t \in [0,1] \mid H(t) = m \},$$

$$t_0 \overset{\text{def}}{=} \max \{ t \in [0,t_1] \mid H(t) = m+1 \},$$

and define $e_i \eta$ by

$$e_i \eta(t) \overset{\text{def}}{=} \left\{ \begin{array}{ll} \eta(t) & \text{for } t \in [0,t_0], \\ \eta(t_0) + r_i(\eta(t) - \eta(t_0)) & \text{for } t \in [t_0,t_1], \\ \eta(t) + \bar{\pi}_i & \text{for } t \in [t_1,1]. \end{array} \right.$$

Similarly, we define $f_i$ as follows. If $m - H(1) = 0$, then $f_i \eta \overset{\text{def}}{=} 0$. Otherwise, we set

$$t_2 \overset{\text{def}}{=} \max \{ t \in [0,1] \mid H(t) = m \},$$

$$t_3 \overset{\text{def}}{=} \min \{ t \in [t_2,1] \mid H(t) = m+1 \},$$

and define $f_i \eta$ by

$$f_i \eta(t) \overset{\text{def}}{=} \left\{ \begin{array}{ll} \eta(t) & \text{for } t \in [0,t_2], \\ \eta(t_2) + r_i(\eta(t) - \eta(t_2)) & \text{for } t \in [t_2,t_3], \\ \eta(t) - \bar{\pi}_i & \text{for } t \in [t_3,1]. \end{array} \right.$$

Then, it is a well-known fact that $e_i \eta, f_i \eta \in \mathfrak{B}_{\text{int}} \sqcup \{0\}$ for $\eta \in \text{QLS}(\lambda)$. Moreover,
Proposition 4.2.2 (see [LNSSS4, Theorem 4.1.2] for untwisted types). The set 
$\text{QLS}(\lambda) \sqcup \{0\}$ is stable under the action of the root operators $e_i$ and $f_i$ for all $i \in I_{\text{aff}}$.

Also, for $i \in I_{\text{aff}}$, we define $\varepsilon_i, \varphi_i : \text{QLS}(\lambda) \to \mathbb{Z}$ by $\varepsilon_i(\eta) \overset{\text{def}}{=} \max\{k \in \mathbb{Z}_{\geq 0} \mid \varepsilon_i^k \eta \neq 0\}$, $\varphi_i(\eta) \overset{\text{def}}{=} \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k \eta \neq 0\}$. Then we see that

(4.1) $\varepsilon_i(\eta) = m^\eta_i$ and $\varphi_i(\eta) = H_i^\eta(1) - m_i^\eta$.

The proof of the following proposition is similar to that of [LNSSS4, Theorem 4.1.1].

Proposition 4.2.3 (see [LNSSS4, Theorem 4.1.1] for untwisted types). The tuple 
$(\text{QLS}(\lambda), \xi \circ \text{wt}, e_i, f_i, \varepsilon_i, \varphi_i \ (i \in I_{\text{aff}}))$ is a $U_q^t(\mathfrak{g}(D^{(2)}_{n+1}))$-crystal. Moreover, it is a realization of the crystal basis of a particular quantum Weyl module $W_q(\xi(\lambda))$ over a quantum affine algebra $U_q^t(\mathfrak{g}(D^{(2)}_{n+1}))$.

Remark 4.2.4. Proposition 4.2.3 also holds for every dual untwisted types.

Remark 4.2.5. We set $\eta_a \overset{\text{def}}{=} (\varepsilon; 0, 1) \in \text{QLS}(\lambda)$. By [LNSSS4, Remark 2.3.6 (2)], for $\eta \in \text{QLS}(\lambda)$, there exist $i_1, \ldots, i_k \in I_{\text{aff}}$ such that $e_{i_1}^\text{max} \cdots e_{i_k}^\text{max} \eta = \eta_a$. Also, there exist $i_1, \ldots, i_k \in I_{\text{aff}}$ such that $f_{i_1}^\text{max} \cdots f_{i_k}^\text{max} \eta = \eta_a$.

We consider the case $(A, A_{\text{aff}}) = (C_n, A^{(2)}_{2n})$. In this case, we use all the notation which we set above, putting the dagger for $(C_n, A^{(2)}_{2n})$ except $\overline{\eta}_a$; here we set $\overline{\eta}_a \overset{\text{def}}{=} -\frac{1}{2} \theta^\dagger (=-\theta)$. Thus we obtain the root operators $\tilde{e}_i^\dagger$ and $\tilde{f}_i^\dagger$, $i \in I_{\text{aff}}$, acting on the set $\mathbb{B}_{\text{int}}^\dagger$.

We fix a dominant weight $\lambda^\dagger \in P^\dagger$. We set

(4.2) $\chi_i \overset{\text{def}}{=} \begin{cases} 1 & \text{if } i \in I_{\text{aff}} \setminus \{n\}, \\ 2 & \text{if } i = n. \end{cases}$

We remark that $\iota^*(\overline{\eta}_a^\dagger) = \chi_i \overline{\eta}_a$ and $\iota((\overline{\eta}_a^\dagger)^\vee) = \frac{1}{\chi_i} \overline{\eta}_a^\vee$ for $i \in I_{\text{aff}}$ by Remark 2.0.2.

Lemma 4.2.6. When we see an element $\eta \in \text{QLS}^\dagger(\lambda^\dagger)$ as a PLC map $\eta(t) : [0, 1] \to (\mathfrak{h}^{(i)}_{\text{aff}})^*$, all local minima of $H_i^\eta(\lambda)(t) = \langle \eta(t), \overline{\eta}_a^\dagger \rangle^\vee$, $t \in [0, 1]$, are integers; that is, 
$\text{QLS}^\dagger(\lambda^\dagger) \subset \mathbb{B}_{\text{int}}^\dagger$.

Proof. In the proof, we assume that Lemma 4.3.7 holds; we will prove the Lemma in §3.3. Let $\lambda = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \text{QLS}^\dagger(\lambda^\dagger)$. Then we have

$\iota^*(\eta) = (\iota^*(w_1), \ldots, \iota^*(w_s); \sigma_0, \ldots, \sigma_s) \in \text{QLS}^\dagger_{\text{aff}}(\iota^*(\lambda^\dagger))$,

where $\iota^*(\eta)$ is defined in Remark 4.1.6. By Lemma 4.3.7, all local minima of $H_i^\iota^*(\eta)(t) = \langle \iota^*(\eta)(t), \overline{\eta}_a^\dagger \rangle = \langle \iota^*(\eta)(t), \overline{\eta}_a^\dagger \rangle^\vee$ lie in $\chi_i \mathbb{Z}$. Also, it follows from Remark 2.0.2 that $\iota((\overline{\eta}_a^\dagger)^\vee) = \frac{1}{\chi_i} \overline{\eta}_a^\vee$ for $i \in I_{\text{aff}}$. Therefore, all local minima of $H_i^\eta(\lambda)(t) = \langle \eta(t), \overline{\eta}_a^\dagger \rangle^\vee = \frac{1}{\chi_i} \langle \iota^*(\eta), \overline{\eta}_a^\dagger \rangle$, $t \in [0, 1]$, are integers. $\square$

We define an injective map $\iota^* : \mathbb{B}_{\text{int}}^\dagger \to \mathbb{B}_{\text{int}}$ by $\eta(t) \mapsto \iota^*(\eta(t))$, $t \in [0, 1]$. The bijection $\iota^* : \text{QLS}^\dagger(\lambda^\dagger) \to \text{QLS}^\dagger_{\text{aff}}(\iota^*(\lambda^\dagger))$ defined in Remark 4.1.6 is identical to the restriction of the injective map $\iota^* : \mathbb{B}_{\text{int}}^\dagger \to \mathbb{B}_{\text{int}}$ to $\text{QLS}^\dagger(\lambda)$.
Since the root operators $e_i^\dagger$ and $f_i^\dagger$, $i \in I_{\text{aff}}$, act on $\mathbb{B}_{\text{int}}^\dagger$, let us define $\varepsilon_i^\dagger$ and $\mathfrak{f}_i^\dagger$, $i \in I_{\text{aff}}$, acting on $\mathbb{B}_{\text{int}}$ which are compatible with $\iota^* : \mathbb{B}_{\text{int}}^\dagger \to \mathbb{B}_{\text{int}}$.

If $m = m_i^0 > -\chi_i$, then $\varepsilon_i^\dagger \eta = 0$. If $m \leq -\chi_i$, then we set
\[
t_1 \overset{\text{def}}{=} \min \{ t \in [0, 1] \mid H(t) = m \}, \quad t_0^\chi \overset{\text{def}}{=} \max \{ t \in [0, t_1] \mid H(t) = m + \chi_i \},
\]
and define $\varepsilon_i^\dagger \eta$ by
\[
\varepsilon_i^\dagger \eta(t) \overset{\text{def}}{=} \begin{cases} 
\eta(t) & \text{for } t \in [0, t_0], \\
\eta(t_0) + r_i(\eta(t) - \eta(t_0)) & \text{for } t \in [t_0, t_1], \\
\eta(t) + \chi_i \bar{x}_i & \text{for } t \in [t_1^\chi, 1].
\end{cases}
\]

Similarly, we define $\mathfrak{f}_i^\dagger$ as follows. If $m - H(1) > -\chi_i$, then $\mathfrak{f}_i^\dagger \eta = 0$. Otherwise, we set
\[
t_2 \overset{\text{def}}{=} \max \{ t \in [0, 1] \mid H(t) = m \}, \quad t_3^\chi \overset{\text{def}}{=} \min \{ t \in [t_2, 1] \mid H(t) = m + \chi_i \},
\]
and define $\mathfrak{f}_i^\dagger \eta$ by
\[
\mathfrak{f}_i^\dagger \eta(t) \overset{\text{def}}{=} \begin{cases} 
\eta(t) & \text{for } t \in [0, t_2], \\
\eta(t_2) + r_i(\eta(t) - \eta(t_2)) & \text{for } t \in [t_2, t_3], \\
\eta(t) - \chi_i \bar{x}_i & \text{for } t \in [t_3^\chi, 1].
\end{cases}
\]

Then we see that
\[
(4.3) \quad \iota^* \circ e_i^\dagger \eta = e_i^\dagger \circ \iota^* \eta, \text{ and } \iota^* \circ f_i^\dagger \eta = \mathfrak{f}_i^\dagger \circ \iota^* \eta
\]
for $i \in I_{\text{aff}}$, $\eta \in \mathbb{B}_{\text{int}}^\dagger$.

Remark 4.2.7. Let $\eta \in \mathbb{B}_{\text{int}}$ and $i \in I_{\text{aff}}$ be such that $\varepsilon_i^\dagger \eta \neq 0$ (resp., $\mathfrak{f}_i^\dagger \eta \neq 0$). By the definition of $\varepsilon_i^\dagger$ (resp., $\mathfrak{f}_i^\dagger$), we have $\varepsilon_i^\dagger \eta = e_i^\dagger \eta$ (resp., $\mathfrak{f}_i^\dagger \eta = f_i^\dagger \eta$); here $e_i$ and $f_i$ are root operators on $\mathbb{B}_{\text{int}}$.

For $i \in I_{\text{aff}}$, we define $\varepsilon_i^\dagger$, $\mathfrak{f}_i^\dagger : \text{QLS}(\iota^*(\lambda_i^\dagger)) \to \mathbb{Z}$ by $\varepsilon_i^\dagger(\eta) \overset{\text{def}}{=} \max \{ k \in \mathbb{Z}_{\geq 0} \mid (\varepsilon_i^\dagger)^k \eta \neq 0 \}$, $\mathfrak{f}_i^\dagger(\eta) \overset{\text{def}}{=} \max \{ k \in \mathbb{Z}_{\geq 0} \mid (\mathfrak{f}_i^\dagger)^k \eta \neq 0 \}$, and set $(\varepsilon_i^\dagger)^{\max \eta} \overset{\text{def}}{=} (\varepsilon_i^\dagger)^{\varepsilon_i^\dagger(\eta)} \eta$ and $(\mathfrak{f}_i^\dagger)^{\max \eta} \overset{\text{def}}{=} (\mathfrak{f}_i^\dagger)^{\mathfrak{f}_i^\dagger(\eta)} \eta$. Then we see that
\[
(4.4) \quad \varepsilon_i^\dagger(\eta) = \begin{cases} 
\frac{m_i^0 - m_i^0}{2} & \text{if } i \neq n, \\
\frac{m_i^0 - m_i^0}{2} & \text{if } i = n,
\end{cases} \quad \mathfrak{f}_i^\dagger(\eta) = \begin{cases} 
H_i^0(1) - m_i^0 & \text{if } i \neq n, \\
\frac{H_i^0(1) - m_i^0}{2} & \text{if } i = n.
\end{cases}
\]

The following Theorem will be proved in §3.4.

Theorem 4.2.8. For a dominant weight $\lambda_i^\dagger \in P^\dagger$, the sets $\text{QLS}^{A_{\bar{\text{int}}}^{(2)}}(\iota^*(\lambda_i^\dagger)) \sqcup \{ 0 \}$ and $\text{QLS}^{A_{\bar{\text{int}}}^{(2)}}(\iota^*(\lambda_i^\dagger)) \sqcup \{ 0 \}$ are stable under the action of the root operators $\varepsilon_i^\dagger$ and $\mathfrak{f}_i^\dagger$ for all $i \in I_{\text{aff}}$.

We define a subset $\mathbb{B}^\dagger(\lambda_i^\dagger)_{\text{cl}}$ of $\mathbb{B}_{\text{int}}^\dagger$ by
\[
\eta \in \mathbb{B}^\dagger(\lambda_i^\dagger)_{\text{cl}} \iff \text{there exists a sequence } (i_1, \ldots, i_k) \text{ such that } (f_{i_1}^\dagger)^{\max} \ldots (f_{i_k}^\dagger)^{\max} \eta = \eta_{\lambda_i^\dagger}.
\]
By [LNSSS4] Theorem 2.4.1, the set $B^\dagger(\lambda^\dagger)_{cl}$ is a realization of the crystal basis $B(\xi^\dagger(\lambda^\dagger))$ of a particular quantum Weyl module $W_q(\xi^\dagger(\lambda^\dagger))$ over a quantum affine algebra $U'_q(\mathfrak{g}(A_n^{(2)}))$. It follows from [4.3] that $\iota^\ast(B^\dagger(\lambda^\dagger)_{cl}) = \widetilde{B}(\iota^\ast(\lambda^\dagger))_{cl}$ is also a realization of the $U'_q(\mathfrak{g}(A_n^{(2)}))-\text{crystal } B(\xi^\dagger(\lambda^\dagger))$, where for a dominant weight $\lambda \in Q$, $\widetilde{B}(\lambda)_{cl} \subset B_{\text{int}}$ is defined by

$$\eta \in \widetilde{B}(\lambda)_{cl} \iff \text{there exists a sequence } (i_1, \ldots, i_k) \text{ such that } (f^\dagger_{i_1})_{\text{max}} \cdots (f^\dagger_{i_k})_{\text{max}} \eta = \eta_\lambda.$$  

We will prove the following theorem in \S 3.4.

**Theorem 4.2.9.** Let $\lambda^\dagger \in P^\dagger$ be a dominant weight. QLS$^{A_n^{(2)}}(\iota^\ast(\lambda^\dagger)) = \overline{\text{QLS}}A_n^{(2)}(\iota^\ast(\lambda^\dagger)) = \widetilde{B}(\iota^\ast(\lambda^\dagger))_{cl}$; that is, the set QLS$^{A_n^{(2)}}(\iota^\ast(\lambda^\dagger)) = \overline{\text{QLS}}A_n^{(2)}(\iota^\ast(\lambda^\dagger))$ is a realization of the $U'_q(\mathfrak{g}(A_n^{(2)}))-\text{crystal } B(\xi^\dagger(\lambda^\dagger))$.

4.3. **Some technical lemmas.** In this subsection, we state some lemmas for dual untwisted types and type $A_n^{(2)}$, but prove them only for $A_n^{(2)}$, the proofs of them for dual untwisted types are similar to those of lemmas in [LNSSS4] \S 4.1. We fix a dominant weight $\lambda \in P$ for a dual untwisted type (especially for type $D_n^{(2)}$), or a dominant weight $\lambda \in Q$ for type $A_n^{(2)}$; that is, if we consider QBG$_{b\lambda}$, QBG$^S_{b\lambda}$, and QLS($\lambda$), then $\lambda \in P$, and if we consider QBG$_{b\lambda}^{A_n^{(2)}}$, (QBG$_{b\lambda}^{A_n^{(2)}}$)$^S$, and QLS$^{A_n^{(2)}}(\lambda)$, then $\lambda \in Q$, where $P$ denote the weight lattice of $\mathfrak{g}(B_n)$, and $Q$ the root lattice of $\mathfrak{g}(B_n)$. Also, we set $S = S_\lambda = \{i \in I \mid (\lambda, \alpha^\vee_i) = 0\}$.

**Lemma 4.3.1** (see [LNSSS4] Lemma 4.1.6) for untwisted types. Let $u, v \in W^S$, $b \in Q \cap [0, 1]$, and $G = \text{QBG}_{b\lambda}^S$ or $(\text{QBG}_{b\lambda}^{A_n^{(2)}})^S$. Let

$$u = x_0 \xleftarrow{\gamma^\vee_1} x_1 \xleftarrow{\gamma^\vee_2} \cdots \xleftarrow{\gamma^\vee_r} x_r = v$$

be a directed path from $v$ to $u$ in $G$. Then $b(x\lambda - y\lambda) \in Q$.  

**Proof.** This is obvious even for $G = \text{QBG}_{b\lambda}^{A_n^{(2)}}$ since $(\text{QBG}_{b\lambda}^{A_n^{(2)}})^S \subset \text{QBG}_{b\lambda}^S$. \hfill $\Box$

**Lemma 4.3.2** (see [LNSSS4] Lemma 4.1.7) for untwisted types. If $\eta \in \text{QLS}^{A_n^{(2)}}(\lambda)$ (resp., $\in \text{QLS}^{A_n^{(2)}}(\lambda)$), then $\eta(1)$ is contained in $\lambda + Q$.  

**Proof.** Even for $\eta \in \text{QLS}^{A_n^{(2)}}(\lambda)$, this is obvious since $\text{QLS}^{A_n^{(2)}}(\lambda) \subset \text{QLS}(\lambda)$. \hfill $\Box$

**Lemma 4.3.3** (see [LNSSS4] Lemma 4.1.8) for untwisted types. Let $u, v \in W^S$, and

$$u = x_0 \xleftarrow{\gamma^\vee_1} x_1 \xleftarrow{\gamma^\vee_2} \cdots \xleftarrow{\gamma^\vee_r} x_r = v$$

be a directed path from $v$ to $u$ in QBG$^S$. Let $i \in I_{\text{aff}}$.

1. If there exists $1 \leq p \leq r$ such that $\langle x_k \lambda, \alpha^\vee_i \rangle < 0$ for all $0 \leq k \leq p - 1$ and $\langle x_p \lambda, \alpha^\vee_i \rangle \geq 0$, then $[r_i x_{p-1}] = x_p$, and there exists a directed path from $v$ to $r_i u$ of the form:

$$[r_i u] = [r_i x_0] \xleftarrow{\gamma^{r_1}_1} \cdots \xleftarrow{\gamma^{r_{p-1}}_{p-1}} [r_i x_{p-1}] = x_p \xleftarrow{\gamma^{r_{p+1}}_{p+1}} \cdots \xleftarrow{\gamma^{r_{r}}_{r}} x_r = v.$$  

Here, if $i \in I$, then we define $z_k = \epsilon$ for all $1 \leq k \leq p - 1$; if $i = 0$, then we define $z_k \in W_S$ by $s_0 x_k = [s_0 x_k] z_k$ for all $1 \leq k \leq p - 1$.  

}\end{document}
(2) If the directed path \( (4.5) \) from \( v \) to \( u \) is shortest, i.e., \( \ell(v, u) = r \), then the directed path \( (4.6) \) from \( v \) to \( [r_iu] \) is also shortest, i.e., \( \ell(v, [r_iu]) = r - 1 \).

(3) Let \( b \in \mathbb{Q} \cap [0, 1] \), and suppose that the directed path \( (4.3) \) is a path in \( \text{QBG}^S_{\mathcal{B}_b} \). Then the directed path \( (4.4) \) is also a path in \( \text{QBG}^S_{\mathcal{B}_b} \).

(4) Let \( b \in \mathbb{Q} \cap [0, 1] \), and suppose that the directed path \( (4.3) \) is a path in \( \text{QBG}^{A(2)}_{\mathcal{B}_b} \). Then the directed path \( (4.4) \) is also a path in \( \text{QBG}^{A(2)}_{\mathcal{B}_b} \), and \( [r_iu] \to u \) for \( i \in I \) (resp., \( i = 0 \)) is a Bruhat (resp., quantum) edge of \( \text{QBG}^{A(2)}_{\mathcal{B}_b} \).

Proof. (1) The proof of (1), (2), and (3) is similar to that of [LNSSS4, Lemma 4.1.8]; here we introduce their proof of (1) in order to prove (4). Assume that \( i \in I \). The proof for the case \( i = 0 \) is similar; replace \( \alpha_i \) and \( \alpha_i^\vee \) by \( -\theta \) and \( -\theta^\vee \), respectively, and use Lemmas 3.3.3 (3), (4) and 3.3.4 (3) instead of Lemmas 3.3.5 (1), (2) and 3.3.4 (1). First, we show that \( x_k\gamma_k \neq \pm \alpha_i \) for all \( 1 \leq k \leq p - 1 \). Suppose that \( x_k\gamma_k = \pm \alpha_i \) for some \( 1 \leq k \leq p - 1 \). Then, \( x_{k-1}\lambda = x_k s_\gamma \lambda = s_\gamma x_k \lambda = r_i x_k \lambda \), and hence \( \langle x_{k-1}\lambda, \alpha_i^\vee \rangle = \langle x_k \lambda, \gamma^\vee \rangle = -\gamma_k^\vee \alpha_i^\vee > 0 \), which contradicts our assumption. Thus \( x_k \gamma_k \neq \pm \alpha_i \) for \( 1 \leq k \leq p - 1 \). By Lemma 3.3.3 (1), (2) for \( b = 1 \), we see that \( [r_i x_{k-1}] \xrightarrow{\gamma_k^\vee} [x_k x_i] \) is an edge of \( \text{QBG}^S \) for all \( 1 \leq k \leq p - 1 \).

Also, since \( \langle x_{p-1}\lambda, \gamma_i^\vee \rangle < 0 \), and \( \langle x_p \lambda, \alpha_i^\vee \rangle \geq 0 \), it follows from Lemma 3.3.4 (1) that \( x_p \gamma_p = \pm \alpha_i \), and hence \( r_i x_p \lambda = r_i x_p \gamma_p \lambda = r_i x_p \alpha_i \gamma_i \lambda = r_i x_p \lambda = x_p \lambda \), which implies \( [x_p x_i] = x_p \). Therefore, we obtain a directed path of the form \( (4.6) \) from \( v \) to \( [r_iu] \).

Let us show (4). Since \( [x_p x_i] = x_p \) and \( x_{p-1} \xrightarrow{\gamma_i^\vee} x_p \) is an edge of \( (\text{QBG}^{A(2)}_{\mathcal{B}_b})^S \), we see that \( x_{p-1} \xrightarrow{\gamma_i^\vee} [s_i x_{p-1}] \) is an edge of \( (\text{QBG}^{A(2)}_{\mathcal{B}_b})^S \); note that \( x_{p-1} \xrightarrow{\gamma_i^\vee} [r_i x_{p-1}] \) is a Bruhat (resp., quantum) edge for \( i \in I \) (resp., \( i = 0 \)) by Lemma 3.3.3 (1) (resp., (2)). Hence by Lemma 3.3.5 for \( 1 \leq k \leq p - 1 \), \( [x_k x_{k-1}] \xrightarrow{\gamma_k^\vee} [x_k x_i] \) is an edge of \( (\text{QBG}^{A(2)}_{\mathcal{B}_b})^S \), and \( x_{k-1} \xrightarrow{r_i x_{k-1}} [x_k x_{k-1}] \) is a Bruhat (resp., quantum) edge of \( (\text{QBG}^{A(2)}_{\mathcal{B}_b})^S \) for \( i \in I \) (resp., \( i = 0 \)). Thus the directed path \( (4.6) \) is a path in \( (\text{QBG}^{A(2)}_{\mathcal{B}_b})^S \), and \( [r_i u] \to u \) is a Bruhat (resp., quantum) edge of \( (\text{QBG}^{A(2)}_{\mathcal{B}_b})^S \) for \( i \in I \) (resp., \( i = 0 \)).

The following lemma can be shown in the same way as Lemma 4.3.3.

Lemma 4.3.4 (see [LNSSS4, Lemma 4.1.9] for untwisted types). Keep the notation and setting in Lemma 4.3.3.

(1) If there exists \( 1 \leq p \leq r \) such that \( \langle x_k \lambda, \alpha_i^\vee \rangle > 0 \) for all \( p \leq k \leq r \) and \( \langle x_{p-1}\lambda, \gamma_i^\vee \rangle \leq 0 \), then \( x_{p-1} = [r_i x_p] \), and there exists a directed path from \( [r_i x] \) to \( u \) of the form:

\[
\begin{align*}
  u &= x_0 \gamma_1^\vee \cdots \gamma_{p-1}^\vee x_{p-1} = [r_i x_p] \xleftarrow{\gamma_{p+1}^\vee} \cdots \xleftarrow{\gamma_r^\vee} [r_i x_r] = [r_i u].
\end{align*}
\]

Here, if \( i \in I \), then we define \( z_k = e \) for all \( p + 1 \leq k \leq r \); if \( i = 0 \), then we define \( z_k \in \mathcal{S}_\lambda \) by \( s_\lambda x_k = [s_\lambda x_k] z_k \) for all \( p + 1 \leq k \leq r \).

(2) If the directed path \( (4.5) \) from \( v \) to \( u \) is shortest, i.e., \( \ell(v, u) = r \), then the directed path \( (4.7) \) from \( [r_i v] \) to \( u \) is also shortest, i.e., \( \ell([r_i v], u) = r - 1 \).
(3) Let \( b \in \mathbb{Q} \cap [0, 1] \), and suppose that the directed path \((4.5)\) is a path in \( \mathrm{QB}_{\ell b}^S \). Then the directed path \((4.7)\) is also a path in \( \mathrm{QB}_{\ell b}^S \).

(4) Let \( b \in \mathbb{Q} \cap [0, 1] \), and suppose that the directed path \((4.5)\) is a path in \((\mathrm{QB}_{\ell b}^{A^{(2)}})^{S})\). Then the directed path \((4.7)\) is also a path in \((\mathrm{QB}_{\ell b}^{A^{(2)}})^{S})\), and \( v \rightarrow \{r_i v\} \) for \( i \in I \) (resp., \( i = 0 \)) is a Bruhat (resp., quantum) edge of \((\mathrm{QB}_{\ell b}^{A^{(2)}})^{S})\).

Lemma 4.3.5 (see [LNSSS4] Lemma 4.1.10) for untwisted types.

Let \( \eta = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda) \) (resp., \( \in \mathrm{QLS}^{A^{(2)}}(\lambda) \)). Let \( i \in I_{\text{aff}} \) and \( 1 \leq u \leq s - 1 \) be such that \( \langle w_{u+1} \lambda, c_i^\vee \rangle > 0 \). Let

\[
 w_u = x_0 \xleftarrow{\gamma_i^\vee} x_1 \xleftarrow{\gamma_i^\vee} \cdots \xleftarrow{\gamma_i^\vee} x_r = w_{u+1}
\]

be a directed path from \( w_{u+1} \) to \( w_u \) in \( \mathrm{QB}_{\sigma_{u+1}}^S \) (resp., \( (\mathrm{QB}_{\sigma_{u+1}}^{A^{(2)}})^S) \). If there exists \( 0 \leq k < r \) such that \( \langle x_k \lambda, \pi_i^\vee \rangle \leq 0 \), then \( H_i^{\eta}(\sigma_u) \in \mathbb{Z} \) (resp., \( \in \chi_i \mathbb{Z} \)). In particular, if \( \langle w_u \lambda, c_i^\vee \rangle \leq 0 \), then \( H_i^{\eta}(\sigma_u) \in \mathbb{Z} \) (resp., \( \in \chi_i \mathbb{Z} \)).

Proof. Since \( \eta = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \mathrm{QLS}^{A^{(2)}}(\lambda) \), we see from the definition of \( A^{(2)} \)-type QLS path that \( \eta' = (w_1, \ldots, w_u, w_{u+1}; \sigma_0, \ldots, \sigma_u, \sigma_s) \in \mathrm{QLS}(\lambda)^{A^{(2)}} \).

Observe that \( \eta'(t) = \eta(t) \) for \( 0 \leq t \leq \sigma_{u+1} \), and hence \( H_i^{\eta'}(t) = H_i^{\eta}(t) \) for \( 0 \leq t \leq \sigma_{u+1} \). Also, it follows that \( H_i^{\eta'}(\sigma_u) = H_i^{\eta'}(\sigma_u) = H_i^{\eta'}(1 - (1 - \sigma_u)\langle w_{u+1} \lambda, \pi_i^\vee \rangle) \). Since \( \eta'(1) \in Q \) by Lemma 4.3.2 and since \( w_{u+1} \lambda \in Q \), we have \( H_i^{\eta'}(1) = H_i^{\eta'}(\sigma_u) \in \chi_i \mathbb{Z} \). Hence it suffices to show that \( \sigma_u \langle w_{u+1} \lambda, c_i^\vee \rangle \in \chi_i \mathbb{Z} \).

We deduce from Lemma 4.3.3 (4) that for \( i \in I \) (resp., \( i = 0 \)), \( w_{u+1} \rightarrow \{r_i w_{u+1}\} \) is a Bruhat (resp., quantum) edge of \((\mathrm{QB}_{\sigma_{u+1}}^{A^{(2)}})^S) \). Hence if \( i \in I \), then \( \sigma_u \langle \lambda, w_{u+1}^{-1} c_i^\vee \rangle \in \chi_i \mathbb{Z} \); if \( i = 0 \), then \( \sigma_u \langle \lambda, w_{u+1}^{-1} (-\theta)^\vee \rangle \in \mathbb{Z} = \chi_0 \mathbb{Z} \). Therefore, in both cases, we have \( (1 - \sigma_u) \langle w_{u+1} \lambda, c_i^\vee \rangle \in \chi_i \mathbb{Z} \).

The following Lemma can be shown in the same way as Lemma 4.3.5.

Lemma 4.3.6 (see [LNSSS4] Lemma 4.1.11) for untwisted types.

Let \( \eta = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda) \) (resp., \( \in \mathrm{QLS}^{A^{(2)}}(\lambda) \)). Let \( i \in I_{\text{aff}} \) and \( 1 \leq u \leq s - 1 \) be such that \( \langle w_u \lambda, c_i^\vee \rangle < 0 \). Let

\[
 w_u = x_0 \xleftarrow{\gamma_i^\vee} x_1 \xleftarrow{\gamma_i^\vee} \cdots \xleftarrow{\gamma_i^\vee} x_r = w_{u+1}
\]

be a directed path from \( w_{u+1} \) to \( w_u \) in \( \mathrm{QB}_{\sigma_{u+1}}^S \) (resp., \( (\mathrm{QB}_{\sigma_{u+1}}^{A^{(2)}})^S) \). If there exists \( 0 < k < r \) such that \( \langle x_k \lambda, \pi_i^\vee \rangle > 0 \), then \( H_i^{\eta}(\sigma_u) \in \mathbb{Z} \) (resp., \( \in \chi_i \mathbb{Z} \)). In particular, if \( \langle w_{u+1} \lambda, c_i^\vee \rangle > 0 \), then \( H_i^{\eta}(\sigma_u) \in \mathbb{Z} \) (resp., \( \in \chi_i \mathbb{Z} \)).

Proposition 4.3.7 (see [LNSSS4] Proposition 4.1.12) for untwisted types. Let \( \eta \in \mathrm{QLS}(\lambda) \) (resp., \( \in \mathrm{QLS}^{A^{(2)}}(\lambda) \)), and \( i \in I_{\text{aff}} \). Then all local minima of \( H_i^{\eta}(t) \), \( t \in [0, 1] \), are elements in \( \mathbb{Z} \) (resp., \( \chi_i \mathbb{Z} \)).

Proof. Assume that the function \( H_i^{\eta}(t) \) attains a local minimum at \( t' \in [0, 1] \); we may assume \( t' = \sigma_u \) for some \( 0 \leq u \leq s \). If \( u = 0 \), then \( H_i^{\eta}(0) = 0 \in 2 \mathbb{Z} \subset \mathbb{Z} \); if \( u = s \), then \( H_i^{\eta}(1) \in \chi_i \mathbb{Z} \) since \( \eta(1) \in Q \). Otherwise, we have either \( \langle w_u \lambda, c_i^\vee \rangle \leq 0 \) and \( \langle w_{u+1} \lambda, c_i^\vee \rangle > 0 \), or \( \langle w_u \lambda, c_i^\vee \rangle > 0 \) and \( \langle w_{u+1} \lambda, c_i^\vee \rangle > 0 \). Therefore, it follows from Lemma 4.3.5 or 4.3.6 that \( H_i^{\eta}(\sigma_u) \in \chi_i \mathbb{Z} \). □
Lemma 4.3.8 (see [LNSSS4] Lemma 4.1.13 for untwisted types).

Let \( \eta = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \text{QLS}(\lambda) \) (resp., \( \in \text{QLS}(\lambda)^{(2)}_{2n} \)). Let \( i \in I_{\text{aff}} \) and \( 1 \leq u \leq s - 1 \) be such that \( \langle w_{u+1} \lambda, \overline{\alpha}_i \rangle > 0 \) and \( H_i^u(\sigma_u) \notin \mathbb{Z} \) (resp., \( \notin \mathbb{Z} \)). Let

\[
(4.8) \quad w_u = x_0 \leftarrow \cdots \leftarrow x_r = w_{u+1}
\]

be a directed path from \( w_{u+1} \) to \( w_u \) in \( \text{QBG}^S_{\sigma_\lambda, \lambda} \) (resp., \( \text{QBG}^S_{\sigma_\lambda, \lambda}^{(2)} \)). Then, \( \langle x_k \lambda, \overline{\alpha}_i \rangle > 0 \) for all \( 0 \leq k \leq r \), and there exists a directed path from \( [r_i w_{u+1}] \) to \( [r_i w_u] \) in \( \text{QBG}^S_{\sigma_\lambda, \lambda} \) (resp., \( \text{QBG}^S_{\sigma_\lambda, \lambda}^{(2)} \)) of the form:

\[
(4.9) \quad [w_u] = [x_0] \leftarrow \cdots \leftarrow [x_r] = [w_{u+1}].
\]

Here, if \( i \in I \), then we define \( z_k = e \) for all \( 1 \leq k \leq r \); if \( i = 0 \), then we define \( z_k \in W_S \) by \( s_k z_k = (s_{k+1}^u z_k) \) for all \( 1 \leq k \leq r \). Moreover, if the directed path \( 4.8 \) is a shortest one from \( w_{u+1} \) to \( w_u \), i.e., \( \ell(w_{u+1}, w_u) = r \), then the directed path \( 4.9 \) is a shortest one from \( [r_i w_{u+1}] \) to \( [r_i w_u] \), i.e., \( \ell([r_i w_{u+1}], [r_i w_u]) = r \).

**Proof.** It follows from Lemma 4.3.5 that if \( H_i^u(\sigma_u) \notin \mathbb{Z} \), then \( \langle x_k \lambda, \overline{\alpha}_i \rangle > 0 \) for all \( 0 \leq k \leq r \) (in particular, \( \langle w_u \lambda, \overline{\alpha}_i \rangle > 0 \)). Assume that \( i \in I \) (resp., \( i = 0 \)), and for a contradiction, that \( x_k \gamma_i = \alpha_i \) (resp., \( = \pm \theta \)) for some \( 1 \leq k \leq r \). Then, \( x_{k-1} \lambda = x_k s_k \gamma_i \lambda = s_k \gamma_i x_k \lambda = r_i x_k \lambda \), and hence \( \langle x_{k-1} \lambda, \overline{\alpha}_i \rangle = \langle r_i x_k \lambda, \overline{\alpha}_i \rangle = -\langle x_k \lambda, \overline{\alpha}_i \rangle \), which contradicts the fact that \( \langle x_k \lambda, \overline{\alpha}_i \rangle > 0 \) and \( \langle x_{k-1} \lambda, \overline{\alpha}_i \rangle > 0 \). Thus, we conclude that \( x_k \gamma_i \neq \pm \alpha_i \) (resp., \( \neq \pm \theta \)) for all \( 1 \leq k \leq r \). Therefore, we deduce from Lemma 4.3.5 (1), (2) (resp., (3), (4)) that there exists a directed path of the form \( 4.9 \) from \( [r_i w_{u+1}] \) to \( [r_i w_u] \). Because the directed path \( 4.8 \) lies in \( \text{QBG}^S_{\sigma_\lambda, \lambda}^{(2)} \) and \( \sigma_u \lambda, z \gamma_i \gamma_i = \sigma_u \lambda, \gamma_i \gamma_i \), the directed path \( 4.9 \) is a path from \( [r_i w_{u+1}] \) to \( [r_i w_u] \) in \( \text{QBG}^S_{\sigma_\lambda, \lambda} \). The proof of being the shortest is similar to that of [LNSSS4] Lemma 4.1.13. \( \square \)

4.4. Proof of Theorems 4.2.8 and 4.2.9. We set \( \lambda = e^*(\lambda^\dagger) \). We recall that \( \lambda \) is a dominant weight in \( Q \).

**Proof of Theorem 4.2.8.** The outline of the proof is similar to that of [LNSSS4] Proposition 4.2.1. It suffices to show that the set \( \text{QLS}^{(2)}_{2n}(\lambda) \) is stable under the action of the root operators \( \overline{J}_i^u \), \( i \in I_{\text{aff}} \). Let \( \eta = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \text{QLS}^{(2)}_{2n}(\lambda) \), and assume that \( \overline{J}_i^u \eta \neq 0 \). Then there exists \( 0 \leq u < s \) such that \( \sigma_u = t_2 \). Let \( u \leq m < s \) be such that \( \sigma_m < t_3 \leq \sigma_{m+1} \); notice that \( H_i^u(t) \) is strictly increasing on \( [t_2, t_3^u] \), which implies that \( \langle w_p \lambda, \alpha_i \rangle > 0 \) for all \( u + 1 \leq p \leq m + 1 \). We need to consider the following four cases:

- **Case 1:** \( w_u \neq [s_j w_{u+1}] \) or \( u = 0 \), and \( \sigma_m < t_3^u < \sigma_{m+1} \),
- **Case 2:** \( w_u \neq [s_j w_{u+1}] \) or \( u = 0 \), and \( t_3^u = \sigma_{m+1} \),
- **Case 3:** \( w_u = [s_j w_{u+1}] \), and \( \sigma_m < t_3^u < \sigma_{m+1} \),
- **Case 4:** \( w_u = [s_j w_{u+1}] \), and \( t_3^u = \sigma_{m+1} \).

In what follows, we assume Case 2; the proofs of other cases are easier. Then it follows from the definition of the root operator \( \overline{J}_i^u \) that

\[
\overline{J}_i^u \eta = (w_1, \ldots, w_u, [s_j w_{u+1}], \ldots, [s_j w_{m+1}], w_{m+2}, \ldots, w_s; \\
\sigma_0, \ldots, \sigma_u, \ldots, \sigma_m, t_3^u = \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_s)
\]
It suffices to show that
(i) there exists a directed path from $[s_i w_{u+1}]$ to $w_u$ in $(\text{QBG}_{\sigma_{u\lambda}}^{A_{2n}^{(2)}})^S$ (when $u > 0$);
(ii) there exists a directed path from $[s_i w_{p+1}]$ to $[s_i w_p]$ in $(\text{QBG}_{\sigma_{\lambda}}^{A_{2n}^{(2)}})^S$ for each $u + 1 \leq p \leq m$;
(iii) there exists a directed path from $w_{m+2}$ to $[s_i w_{m+1}]$ in $(\text{QBG}_{\sigma_{m+1\lambda}}^{A_{2n}^{(2)}})^S$.

Also, we will show that if $\eta \in \widetilde{\text{QLS}}_{A_{2n}^{(2)}}(\lambda)$, then the directed paths in (i)-(iii) above can be chosen from the shortest ones, which implies that $f_i \eta \in \widetilde{\text{QLS}}_{A_{2n}^{(2)}}(\lambda)$.

(i) We deduce that from the definition of $\eta$, $w_{u+1} \lambda, \pi_{i}^{\gamma} > 0$. Since $\eta \in \text{QLS}_{A_{2n}^{(2)}}(\lambda)$, there exists a directed path from $w_{u+1}$ to $w_u$ in $(\text{QBG}_{\sigma_{u\lambda}}^{A_{2n}^{(2)}})^S$. Hence it follows from Lemma 4.3.4 (1), (4) that there exists a directed path from $[r_i w_{u+1}]$ to $w_u$ in $(\text{QBG}_{\sigma_{u\lambda}}^{A_{2n}^{(2)}})^S$. Moreover, we see from the definition of $\widetilde{\text{QLS}}_{A_{2n}^{(2)}}(\lambda)$ and Lemma 4.3.4 (2) that if $\eta \in \widetilde{\text{QLS}}_{A_{2n}^{(2)}}(\lambda)$, then there exists a directed path from $[r_i w_{u+1}]$ to $w_u$ in $(\text{QBG}_{\sigma_{u\lambda}}^{A_{2n}^{(2)}})^S$ whose length is equal to $\ell([r_i w_{u+1}], w_u)$.

(ii) Recall that $\lambda_i^\eta(t) = m_i^\eta \in \chi_i \mathbb{Z}$ and $H_i^\eta(t_0) = m_i^\eta + \chi_i$. Hence it follows that $H_i^\eta(\sigma_p) \notin \chi_i \mathbb{Z}$ for any $u + 1 \leq p \leq m$. Therefore, we deduce from Lemma 4.3.3 that there exists a directed path from $[r_i w_{p+1}]$ to $[r_i w_p]$ in $(\text{QBG}_{\sigma_p}^{A_{2n}^{(2)}})^S$ for each $u + 1 \leq p \leq m$. Moreover, we see from the definition of $\widetilde{\text{QLS}}_{A_{2n}^{(2)}}(\lambda)$ and Lemma 4.3.8 that if $\eta \in \widetilde{\text{QLS}}_{A_{2n}^{(2)}}(\lambda)$, then for each $u + 1 \leq p \leq m$, there exists a directed path $[r_i w_{p+1}]$ in $(\text{QBG}_{\sigma_p}^{A_{2n}^{(2)}})^S$ for each $u + 1 \leq p \leq m$ whose length is equal to $\ell([r_i w_{p+1}], [r_i w_p])$.

(iii) Since $\langle w_{m+1} \lambda, \pi_{i}^{\gamma} \rangle > 0$, it follows from Lemma 5.3.3 (1), (2) that if $i \in I$ (resp., $i = 0$), then $[r_i w_{m+1}]$ is a Bruhat (resp., quantum) edge; here if $i \in I$ (resp., $i = 0$), then $\gamma = w_{m+1}^{-1} \alpha_i$ (resp., $\gamma = -w_{m+1}^{-1} \theta$). Note that $\langle \lambda, \gamma^\vee \rangle = \langle x_{m+1} \lambda, \pi_{i}^{\gamma} \rangle$.

Claim. $[r_i w_{m+1}]$ is an edge of $(\text{QBG}_{t_3 \lambda}^{A_{2n}^{(2)}})^S$; that is, $t_3^{\gamma} \langle w_{m+1} \lambda, \pi_{i}^{\gamma} \rangle \in \chi_i \mathbb{Z}$.

Proof of Claim. Since

$$\eta(t_3^{\gamma}) = \sum_{k=1}^{m+1} (\sigma_k - \sigma_{k-1})w_k \lambda = t_3^{\gamma} w_{m+1} \lambda + \sum_{k=1}^{m} \sigma_k (w_k \lambda - w_{k+1} \lambda),$$

we have

$$H_i^\eta(t_3^{\gamma}) = t_3^{\gamma} \langle w_{m+1} \lambda, \pi_{i}^{\gamma} \rangle + \sum_{k=1}^{m} \langle \sigma_k (w_k \lambda - w_{k+1} \lambda), \pi_{i}^{\gamma} \rangle.$$

Here, $\sigma_k (w_k \lambda - w_{k+1} \lambda) \in Q$ by Lemma 4.3.1 and hence $\langle \sigma_k (w_k \lambda - w_{k+1} \lambda), \pi_{i}^{\gamma} \rangle \in \chi_i \mathbb{Z}$. Also, it follows from Lemma 4.3.7 that $H_i^\eta(t_3^{\gamma}) = m_i^\eta + \chi_i \in \chi_i \mathbb{Z}$. Therefore, we deduce that $t_3^{\gamma} \langle w_{m+1} \lambda, \pi_{i}^{\gamma} \rangle \in \chi_i \mathbb{Z}$. ■
By the definition of $\text{QLS}^{(2)}_{(2n)}(\lambda)$, there exists a directed path from $w_{m+2}$ to $w_{m+1}$ in $(\text{QBGS}_{\sigma_{m+1}}^{(2n)}(\lambda))^S$. Concatenating this directed path and the edge $[r_i w_{m+1}] \leftrightarrow w_{m+1}$, we obtain a directed path from $w_{m+2}$ to $[r_i w_{m+1}]$ in $(\text{QBGS}_{\sigma_{m+1}}^{(2n)}(\lambda))^S$. Thus, we have proved that $f_i \eta \in \text{QLS}^{(2)}_{(2n)}(\lambda)$.

Assume that $\eta \in \widehat{\text{QLS}}^{(2)}_{(2n)}(\lambda)$, and set $r \overset{\text{def}}{=} \ell(w_{m+2}, w_{m+1})$. We see from the argument above that there exists a directed path from $w_{m+2}$ to $[r_i w_{m+1}]$ in $(\text{QBGS}_{\sigma_{m+1}}^{(2n)}(\lambda))^S$ whose length is equal to $r + 1$. Suppose, for a contradiction, that there exists a directed path to $[r_i w_{m+1}]$ in $\text{QBGS}^S$ whose length $l$ is less than $r + 1$. Since $\langle r_i w_{m+1}, \pi_i \rangle < 0$ and $\langle w_{m+2}, \pi_i \rangle \geq 0$, we deduce from Lemma \ref{lem:4.3.3} that there exists a directed path from $w_{m+2}$ to $[r_i w_{m+1}] = [w_{m+1}] = w_{m+1}$ in $\text{QBGS}^S$ whose length is equal to $l - 1 = \ell(w_{m+2}, w_{m+1})$, a contradiction. Thus, we have proved that if $\eta \in \widehat{\text{QLS}}^{(2)}_{(2n)}(\lambda)$, then $f_i \eta \in \widehat{\text{QLS}}^{(2)}_{(2n)}(\lambda)$. \hfill $\Box$

**Proof of Theorem \ref{thm:4.2.9}** Since $\eta \lambda \in \widehat{\text{QLS}}^{(2)}_{(2n)}(\lambda) \subset \text{QLS}^{(2)}_{(2n)}(\lambda)$, it follows from the definition of $\widehat{B}(\lambda)_{\text{cl}}$ and Theorem \ref{thm:4.2.8} that $\widehat{B}(\lambda)_{\text{cl}} \subset \widehat{\text{QLS}}^{(2)}_{(2n)}(\lambda) \subset \text{QLS}^{(2)}_{(2n)}(\lambda)$. Hence it suffices to show that $\text{QLS}^{(2)}_{(2n)}(\lambda) \subset \widehat{B}(\lambda)_{\text{cl}}$.

**Claim.** $(f_i^\dagger)^{\max \pi} = f_i^{\max \pi}$ for each $\pi \in \text{QLS}^{(2)}_{(2n)}(\lambda)$ and $i \in I_{\text{aff}}$.

**Proof of Claim.** It follows from Proposition \ref{prop:4.3.7} and \ref{prop:4.1} that $\varphi_i(\pi) \in \chi_i Z$. Therefore, by \ref{prop:4.4}, $\varphi_i^\dagger(\pi) = \varphi_i^!(\pi) \in Z$. Hence by Remark \ref{rem:4.2.7} $(f_i^\dagger)^{\pi} = f_i^{\varphi_i(\pi)}$, which implies $(f_i^\dagger)^{\max \pi} = f_i^{\max \pi}$. \hfill $\blacksquare$

Since $\text{QLS}^{(2)}_{(2n)}(\lambda) \subset \text{QLS}(\lambda)$, for each element $\eta \in \text{QLS}^{(2)}_{(2n)}(\lambda)$, there exist $i_1, \ldots, i_k \in I_{\text{aff}}$ such that

$$f_{i_1}^{\max} \ldots f_{i_k}^{\max} \eta = \eta \lambda,$$

by Remark \ref{rem:4.2.5}. It follows from the claim above that

$$(f_{i_1}^\dagger)^{\max} \ldots (f_{i_k}^\dagger)^{\max} \eta = \eta \lambda,$$

which implies that $\eta \in \widehat{B}(\lambda)_{\text{cl}}$. \hfill $\square$

### 5. The Graded Character of $\text{QLS}(\lambda)$

In this section, we fix $(A, A_{\text{aff}}) = (B_n, D_{n+1})^{(2)}$. For $x \in \mathfrak{h}_R^* \oplus \mathbb{R} \delta$, we define $\pi \in \mathfrak{h}_R^*$ by $\deg(x) \in \mathbb{R}$ by

$$x = \pi + \deg(x) \delta. \quad (5.1)$$

For $x \in Q$, let $t(x)$ denote the linear transformation on $\mathfrak{h}_R^* \oplus \mathbb{R} \delta$: $t(x)(y + r \delta) \overset{\text{def}}{=} y + (r - 2(x, y)) \delta$ for $y \in \mathfrak{h}_R^*$, $r \in \mathbb{R}$, where $\delta$ denotes the null root of $g(D_{n+1}^{(2)})$ and $(\cdot, \cdot) : \mathfrak{h}_R^* \times \mathfrak{h}_R^* \to \mathbb{R}$ the bilinear form defined in Remark \ref{rem:2.0.4}. The affine Weyl group of $g(D_{n+1}^{(2)})$ are defined by $W_{\text{aff}} \overset{\text{def}}{=} t(Q) \times W$. Also, we define $s_0 : \mathfrak{h}^* \to \mathfrak{h}^*$ by $y + r \delta \mapsto s_0 y - (r - 2(x, \theta)) \delta = s_0 x - (r - (x, \theta^\vee)) \delta$. Then $W_{\text{aff}} = \langle s_i \mid i \in I_{\text{aff}} \rangle$; note that $s_0 = t(\theta)s_0$. 


An element \( u \in W_{aff} \) can be written as
\[
(5.2) \quad u = t(\text{wt}(u)) \text{dir}(u),
\]
where \( \text{wt}(u) \in Q \) and \( \text{dir}(u) \in W \), according to the decomposition \( W_{aff} = t(Q) \times W \).

For \( w \in W_{aff} \), we denote the length of \( w \) by \( \ell(w) \), which equals \( \# (\Delta_{aff}^+ \cap w^{-1} \Delta_{aff}^-) \).

### 5.1. Nonsymmetric and symmetric Macdonald polynomials of type \( A_{2n}^{(2)} \)

In this subsection, we introduce the notion of nonsymmetric and symmetric Macdonald polynomials of type \( A_{2n}^{(2)} \); see [OS §3.6] for more detail. We fix \( (A, A_{aff}) = (B_n, D_{n+1}^{(2)}) \). Set \( \tilde{\Delta}_{aff} \) as the nonsymmetric Macdonald-Koornwinder polynomial. Then we define the nonsymmetric and symmetric Macdonald polynomials of type \( A_{2n}^{(2)} \).

Let \( t_i \) be an indeterminate with respect to \( O_i, i = 1, \ldots, 5 \). For \( \mu \in Q \), let \( E_{\mu}(q, t_1, t_2, t_3, t_4, t_5) \) denote the nonsymmetric Macdonald-Koornwinder polynomial. Also, for a dominant weight \( \lambda \in Q \), let \( P_{\lambda}(q, t_1, t_2, t_3, t_4, t_5) \) denote the symmetric Macdonald-Koornwinder polynomial. Then we define the nonsymmetric and symmetric Macdonald polynomials of type \( A_{2n}^{(2)} \) by \( E_{\mu}^{A_{2n}^{(2)}}(q, t) \) and \( P_{\lambda}^{A_{2n}^{(2)}}(q, t) \), respectively. We set \( E_{\mu}^{A_{2n}^{(2)}}(q, 0) \) and \( P_{\lambda}^{A_{2n}^{(2)}}(q, 0) \), which are well-defined.

**Remark 5.1.1.** As in [LNSSS2 Lemma 7.7], for a dominant weight \( \lambda \in Q \), we have
\[
P_{\lambda}^{A_{2n}^{(2)}}(q, 0) = E_{w_0 \lambda}^{A_{2n}^{(2)}}(q, 0),
\]
where \( w_0 \) denotes the longest element of \( W \).

Let \( \lambda \in Q \) be a dominant weight. For \( \eta = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \text{QLS}^{A_{2n}^{(2)}}(\lambda) \), we set
\[
\text{Deg}(\eta) = \sum_{i=0}^{s-1} (1 - \sigma_i) \text{wt}_{\lambda}(w_i \Rightarrow w_{i+1}),
\]
where \( \text{wt}_{\lambda}(w_i \Rightarrow w_{i+1}) \) is the \( \lambda \)-weight of the path from \( w_{i+1} \) to \( w_i \) in QBG (or QBG\(^5 \)) as defined in §2.3. We define
\[
\text{gch}(\text{QLS}^{A_{2n}^{(2)}}(\lambda)) \overset{\text{def}}{=} \sum_{\eta \in \text{QLS}^{A_{2n}^{(2)}}(\lambda)} q^{\text{Deg}(\eta)} e^{\text{wt}(\eta)}.
\]

We will show the following theorem in §4.4.

**Theorem 5.1.2.** Let \( \lambda \in Q \) be a dominant weight. Then, \( P_{\lambda}^{A_{2n}^{(2)}}(q, 0) = \text{gch}(\text{QLS}^{A_{2n}^{(2)}}(\lambda)) \).
5.2. Orr-Shimozono formula. For $\mu \in Q$, we denote the shortest element in the coset $t(\mu)W$ by $m_\mu \in W_{\text{aff}}$. In the following, we fix $\mu \in Q$, and take a reduced expression $m_\mu = s_{\ell_1} \cdots s_{\ell_L}$, where $\ell_1, \ldots, \ell_L \in I_{\text{aff}}$.

For each $J = \{j_1 < j_2 < j_3 < \cdots < j_r\} \subset \{1, \ldots, L\}$, we define an alcove path

$$p_J^{\text{OS}} = (m_\mu = z_0^{\text{OS}}, z_1^{\text{OS}}, \ldots, z_r^{\text{OS}}, \beta_{j_1}^{\text{OS}}, \ldots, \beta_{j_r}^{\text{OS}})$$

as follows: we set $\beta_k^{\text{OS}} = s_{t_L} \cdots s_{t_{k+1}} \alpha_{t_k} \in \Delta_+^{\text{aff}}$ for $1 \leq k \leq L$, and set $z_0^{\text{OS}} = m_\mu$ and $z_k^{\text{OS}} = z_{k-1}^{\text{OS}} s_{\beta_k^{\text{OS}}}$, $1 \leq k \leq s$. Also, following [OS §3.3], we set $B(e; m_\mu) = \{p_J^{\text{OS}} \mid J \subset \{1, \ldots, L\}\}$ and $\text{end}(p_J^{\text{OS}}) = z_r^{\text{OS}} \in W_{\text{aff}}$. Then we define $\text{QB}^{(2)}_{A_{2n}}(e; m_\mu)$ to be the following subset of $B(e; m_\mu)$:

$$\left\{ p_J^{\text{OS}} \in B(e; m_\mu) \right\} \text{dir}(z_i^{\text{OS}}) = \frac{\text{wt}(\text{end}(p_J^{\text{OS}}))}{\text{wt}(\text{end}(p_J^{\text{OS}}))} \text{dir}(z_{i-1}^{\text{OS}}) \text{ is an edge of QBG, } 0 \leq i \leq r - 1, \text{ if this edge is a Bruhat edge, then } \beta_{j_i}^{\text{OS}} \eta_{\Delta} \in \Delta \oplus 2\mathbb{Z}$$

Remark 5.2.1 ([M (2.4.7)]). If $j \in J$, then $-\beta_j^{\text{OS}} \in \Delta^+$. For $p_J^{\text{OS}} \in \text{QB}^{(2)}_{A_{2n}}(e; m_\mu)$, we define $\text{qwt}(p_J^{\text{OS}})$ as follows. Let $J_0$ denote the set of all indices $j_i$ such that $\ell_{j_i} = 0$, and $J^- = J \setminus J_0$ the set of all indices $j_i \in J \setminus J_0$ such that $\text{dir}(z_{i-1}^{\text{OS}}) = \frac{\text{wt}(\text{end}(p_J^{\text{OS}}))}{\text{wt}(\text{end}(p_J^{\text{OS}}))} \text{dir}(z_i^{\text{OS}})$ is a quantum edge; we remark that for $j_i \in J_0$, $\text{dir}(z_{i-1}^{\text{OS}}) = \frac{\text{wt}(\text{end}(p_J^{\text{OS}}))}{\text{wt}(\text{end}(p_J^{\text{OS}}))} \text{dir}(z_i^{\text{OS}})$ is a quantum edge. Then we set $\text{qwt}(p_J^{\text{OS}}) = \sum_{j \in J_0 \cup J^-} \beta_j^{\text{OS}}$. We know the following formula for the specialization of nonsymmetric Macdonald polynomials at $t = 0$ of type $A_{2n}^{(2)}$.

Proposition 5.2.2 ([OS Proposition 5.5]). Let $\mu \in Q = P^+ \subset P$. Then,

$$E_\mu^{(2)} = \sum_{p_J^{\text{OS}} \in \text{QB}^{(2)}_{A_{2n}}(e; m_\mu)} q^{\text{deg}(\text{qwt}(p_J^{\text{OS}}))} e^{\text{wt}(\text{end}(p_J^{\text{OS}}))}.$$

By Remark 5.1.1 we obtain the Orr-Shimozono formula for the specialization of symmetric Macdonald polynomials at $t = 0$ of type $A_{2n}^{(2)}$.

Corollary 5.2.3. Let $\lambda \in Q$ be a dominant weight. Then,

$$P_\lambda^{(2)}(e; t(w_0^\lambda)) = \sum_{p_J^{\text{OS}} \in \text{QB}^{(2)}_{A_{2n}}(e; t(w_0^\lambda))} q^{\text{deg}(\text{qwt}(p_J^{\text{OS}}))} e^{\text{wt}(\text{end}(p_J^{\text{OS}}))}.$$

Here we remark that

$$m_{w_0^\lambda} = t(w_0^\lambda),$$

and

$$\text{dir}(t(w_0^\lambda)) = e$$

(see [M §2.4]).
5.3. Reduced expression for $t(w_0 \lambda)$ and total order on $\Delta^+_{\text{aff}} \cap t(\lambda_-)^{-1} \Delta^-_{\text{aff}}$. Let $\lambda \in Q \subset P$ be a dominant weight, and set $\lambda_- = w_0 \lambda$ and $S = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0\}$. For $\mu \in W \lambda$, we denote by $v(\mu) \in W^S$ the minimal-coset representative for the coset $\{w \in W \mid w \lambda = \mu\}$. Since $w_0$ is the longest element of $W$, we have $w_0 = v(\lambda_-)w_0(S)$ and $\ell(w_0) = \ell(v(\lambda_-)) + \ell(w_0(S))$.

We fix reduced expressions

$$v(\lambda_-) = s_{i_1} \cdots s_{i_M},$$

$$w_0(S) = s_{i_{M+1}} \cdots s_{i_N}$$

for $v(\lambda_-)$ and $w_0(S)$, respectively. Then $w_0 = s_{i_1} \cdots s_{i_N}$ is a reduced expression for $w_0$. We set $\beta_j = s_{i_N} \cdots s_{i_{j+1}} \alpha_{i_j}$, $1 \leq j \leq N$. Then we have $\Delta^+ \setminus \Delta^+_S = \{\beta_1, \ldots, \beta_M\}$, and $\Delta^+_S = \{\beta_{M+1}, \ldots, \beta_N\}$. We fix a total order on $(\Delta^+)^\vee$ by

$$\beta_1^\vee \succ \beta_2^\vee \succ \cdots \succ \beta_M^\vee \succ \beta_{M+1}^\vee \succ \beta_N^\vee \in (\Delta^+ \setminus \Delta^+_S)^\vee.$$ (5.5)

Remark 5.3.1. This total order is a (weak) reflection order on $(\Delta^+)^\vee$; that is, if $\alpha, \beta, \gamma \in \Delta^+$ with $\gamma = \alpha + \beta$, then $\alpha^\vee \prec \gamma^\vee \prec \beta^\vee$ or $\beta^\vee \prec \gamma^\vee \prec \alpha^\vee$.

We define an injective map $\Phi$ by:

$$\Phi : \Delta^+_{\text{aff}} \cap t(\lambda_-)^{-1} \Delta^-_{\text{aff}} \rightarrow \mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)^\vee,$$

$$\beta = \overline{\beta} + \deg(\beta) \delta \mapsto \left( \frac{c_{\overline{\beta}}(\lambda_-, \overline{\beta}^\vee) - \deg(\beta)}{c_{\overline{\beta}}(\lambda_-, \overline{\beta}^\vee)}, w_0 \overline{\beta}^\vee \right);$$

note that $\langle \lambda_-, \overline{\beta}^\vee \rangle > 0$, $c_{\overline{\beta}}(\lambda_-, \overline{\beta}^\vee) - \deg(\beta) \geq 0$, and $w_0 \overline{\beta} \in \Delta^+ \setminus \Delta^+_S$ since $\langle \lambda_-, \overline{\beta}^\vee \rangle = \langle \lambda, w_0 \overline{\beta}^\vee \rangle > 0$, since we know from [M] (2.4.7) (i) that

$$(\Delta^+_{\text{aff}} \cap t(\lambda_-)^{-1} \Delta^-_{\text{aff}} = \{\alpha + a c_\alpha \delta \mid \alpha \in \Delta^-, a \in \mathbb{Z}, \text{ and } 0 < a \leq \langle \lambda_-, \alpha^\vee \rangle\}.$$ (5.6)

We now consider the lexicographic order $\prec$ on $\mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)^\vee$ induced by the usual total order on $\mathbb{Q}_{\geq 0}$ and the inverse of the restriction to $(\Delta^+ \setminus \Delta^+_S)^\vee$ of the total order $\prec$ on $(\Delta^+)^\vee$ defined above; that is, for $(a, a^\vee), (b, b^\vee) \in \mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)^\vee$,

$$(a, a^\vee) \prec (b, b^\vee) \text{ if and only if } a < b \text{ or } a = b \text{ and } a^\vee \prec b^\vee.$$ Then we denote by $\prec'$ the total order on $\Delta^+_{\text{aff}} \cap t(\lambda_-)^{-1} \Delta^-_{\text{aff}}$ induced by the lexicographic order on $\mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)^\vee$ through the map $\Phi$, and write $\Delta^+_{\text{aff}} \cap t(\lambda_-)^{-1} \Delta^-_{\text{aff}}$ as $\{\gamma_1 \prec' \cdots \prec' \gamma_L\}$; we call this total order a weak reflection order.

The proof of the following proposition is similar to that of [NNS] Proposition 3.1.8.

Proposition 5.3.2 (see [NNS] Proposition 3.1.8 for untwisted types). Keep the notation and setting above. Then, there exists a unique reduced expression $t(\lambda_-) = s_{\ell_1} \cdots s_{\ell_L}$ for $t(\lambda_-)$, $\{\ell_1, \ldots, \ell_L\} \subset I_{\text{aff}}$, such that $\beta_{\ell_j}^\text{OS} = \gamma_j$ for $1 \leq j \leq L$.

In what following, we fix the reduced expression $t(\lambda_-) = s_{\ell_1} \cdots s_{\ell_L}$ for $t(\lambda_-)$ as above. Recall that $\beta_{\ell_j}^\text{OS} = s_{\ell_1} \cdots s_{\ell_{j-1}} \alpha_{\ell_j}$, $1 \leq j \leq L$.

The proof of the following lemma is similar to that of [NNS] Lemma 3.1.9,
Lemma 5.3.3 ([NNS Lemma 3.1.10]). Keep the notation and setting above. Then, $s_1 \cdots s_{\ell_M}$ is a reduced expression for $v(\lambda)$, and $s_{\ell_M+1} \cdots s_{\ell_L}$ is a reduced expression for $m_\lambda$. Moreover, $i_k = \ell_k$ for $1 \leq k \leq M$.

We set $a_k = \deg(\vartheta_k) / \deg(\vartheta_k^\vee) \in \mathbb{Z}_{\geq 0}$; since $\Delta_+^{\text{aff}} \cap t(\lambda_-)^{-1}\Delta_-^{\text{aff}} = \{\vartheta_1^\text{OS}, \ldots, \vartheta_L^\text{OS}\}$, we see by (5.6) that $0 < a_k \leq \langle \lambda_-, \vartheta_k^\text{OS}^\vee \rangle$.

Corollary 5.3.4. For $1 \leq k \leq M$, $w_0\vartheta_k^\text{OS} = \vartheta_k$, where $\vartheta_k = s_{i_1} \cdots s_{i_{k+1}}\alpha_{i_k}$.

Proof. We set $\vartheta_k^L = s_{\ell_1} \cdots s_{\ell_{k-1}}\alpha_{\ell_k}, 1 \leq k \leq M$. Then we have

$$-t(\lambda_-)\vartheta_k^\text{OS} = -(s_{\ell_1} \cdots s_{\ell_{\ell_k+1}}\alpha_{\ell_k}) = -s_{\ell_1} \cdots s_{\ell_{k-1}}s_{\ell_k}\alpha_{\ell_k} = -s_{\ell_1} \cdots s_{\ell_{k-1}}(-\alpha_{\ell_k}) = us_{\ell_1} \cdots s_{\ell_{k-1}}\alpha_{\ell_k} = \vartheta_k^L.$$ 

From this, together with $-t(\lambda_-)\vartheta_k^\text{OS} = w_0\vartheta_k^\text{OS} - c_{\vartheta_k^\text{OS}}(a_k - \langle \lambda_-, \vartheta_k^\text{OS}^\vee \rangle)\delta$, we obtain $\vartheta_k = w_0\vartheta_k^\text{OS} = w_0(\vartheta_k^L) = w_0(-s_{i_1} \cdots s_{i_{k+1}}\alpha_{i_k})$ Lemma 5.3.3 $w_0(-s_{i_1} \cdots s_{i_{k+1}}\alpha_{i_k}) = (s_{i_1} \cdots s_{i_1})(-s_{i_1} \cdots s_{i_{k+1}}\alpha_{i_k}) = (s_{i_1} \cdots s_{i_{k+1}}\alpha_{i_k}) = \vartheta_k$.

Remark 5.3.5 ([LNSS2 Theorem 8.3], [NNS proof of Lemma 3.1.10]). For $1 \leq k \leq L$, we set

$$d_k = \frac{\langle \lambda_-, \vartheta_k^\text{OS}^\vee \rangle - a_k}{\langle \lambda_-, \vartheta_k^\text{OS}^\vee \rangle} = \frac{c_{\vartheta_k^\text{OS}}(\lambda_-, \vartheta_k^\text{OS}^\vee)}{\langle \lambda_-, \vartheta_k^\text{OS}^\vee \rangle} - \deg(\vartheta_k^\text{OS})$$

Here $d_k$ is just the first component of $\Phi(\vartheta_k^\text{OS}) \in \mathbb{Q}_{\geq 0} \times (\Delta_+^{\text{aff}} \setminus \Delta_+^\text{aff})^\vee$. For $1 \leq k, j \leq L$, $\Phi(\vartheta_k^\text{OS}) < \Phi(\vartheta_j^\text{OS})$ if and only if $k < j$, and hence we have

$$0 \leq d_1 \leq \cdots \leq d_L \leq 1.$$  

Moreover,

$$d_i = 0 \iff i \leq M.$$  

Lemma 5.3.6 ([NNS Lemma 3.1.112]). If $1 \leq k < j \leq L$ and $d_k = d_j$, then $w_0\vartheta_k^\text{OS}^\vee > w_0\vartheta_j^\text{OS}^\vee$.

5.4. Proof of Theorem 5.1.2. We continue to use all notations in §4.3. In this subsection, we give a bijection

$$\Xi : \mathcal{Q}B_1^{(2)}(e; t(\lambda_-)) \rightarrow \mathcal{Q}L_1^{(2)}(\lambda)$$

that preserves weights and degrees.

Remark 5.4.1. Let $\gamma_1, \gamma_2, \ldots, \gamma_r \in \Delta_+^{\text{aff}} \cap t(\lambda_-)^{-1}\Delta_-^{\text{aff}}$, and consider the sequence $(y_0, y_1, \ldots, y_r; \gamma_1, \gamma_2, \ldots, \gamma_r)$ defined by $y_0 = t(\lambda_-)$, and $y_i = y_{i-1}^+=\gamma_i$ for $1 \leq i \leq r$. Then, the sequence $(y_0, y_1, \ldots, y_r; \gamma_1, \gamma_2, \ldots, \gamma_r)$ is an element of $\mathcal{Q}B_1^{(2)}(e; t(\lambda_-))$ if and only if the following conditions hold:

(1) $\gamma_1 < \gamma_2 < \cdots < \gamma_r$, where the order $<$ is the weak reflection order on $\Delta_+^{\text{aff}} \cap t(\lambda_-)^{-1}\Delta_-^{\text{aff}}$ introduced in §4.3;
(2) For $1 \leq i \leq r$, $\text{dir}(y_i - 1) \xleftarrow{\text{dir}(y_i)}$ is an edge of QBG, and if this edge is a Bruhat one, then $\deg(y_i) \in 2\mathbb{Z}$.

In the following, we define a map $\Xi : \mathcal{QB}^{A_{2n}^{(2)}}(e; t(\lambda_-)) \to \mathcal{QLS}^{A_{2n}^{(2)}}(\lambda)$. Let $p_j^{\Omega S}$ be an arbitrary element of $\mathcal{QB}^{A_{2n}^{(2)}}(e; t(\lambda_-))$ of the form

$$p_j^{\Omega S} = (t(\lambda_-) = z_0^{\Omega S}, z_1^{\Omega S}, \ldots, z_r^{\Omega S}; \beta_j^{\Omega S}, \beta_{j_1}^{\Omega S}, \beta_{j_2}^{\Omega S}, \ldots, \beta_{j_r}^{\Omega S}) \in \mathcal{QB}^{A_{2n}^{(2)}}(e; t(\lambda_-)),$$

with $J = \{j_1 < \cdots < j_r\} \subset \{1, \ldots, L\}$. We set $x_k \overset{\text{def}}{=} \text{dir}(z_k^{\Omega S}), 0 \leq k \leq r$. Then, by the definition of $\mathcal{QB}^{A_{2n}^{(2)}}(e; t(\lambda_-))$,

$$e = x_0 \xleftarrow{-\frac{\beta_j^{\Omega S}}{\beta_{j_1}^{\Omega S}}} x_1 \xleftarrow{-\frac{\beta_j^{\Omega S}}{\beta_{j_2}^{\Omega S}}} \cdots \xleftarrow{-\frac{\beta_j^{\Omega S}}{\beta_{j_r}^{\Omega S}}} x_r$$

is a directed path in QBG; the first equality follows from (5.4). We take $0 = 0 = \sigma_0 \leq \sigma_1 < \cdots < \sigma_{s-1} < 1 = \sigma_s$ in such a way that (see (5.7))

$$0 = d_{j_1} = \cdots = d_{j_{u_1}} < d_{j_{u_1+1}} = \cdots = d_{j_{u_2}} < \cdots < d_{j_{u_{s-1}+1}} = \cdots = d_{j_r} < 1 = \sigma_s;$$

note that $d_{j_1} > 0$ if and only if $u_1 = 0$, and $\sigma_0 = 0$ even if $d_{j_1} = 0$. We set $w_p \overset{\text{def}}{=} x_{u_p}$ for $0 \leq p \leq s - 1$, and $w'_p \overset{\text{def}}{=} x_r$. Then, by taking a subsequence of (5.9), we obtain the following directed path in QBG for each $0 \leq p \leq s - 1$:

$$w'_p = x_{u_p} \xleftarrow{-\frac{\beta_j^{\Omega S}}{\beta_{j_{u_p+1}}^{\Omega S}}} x_{u_{p+1}} \xleftarrow{-\frac{\beta_j^{\Omega S}}{\beta_{j_{u_{p+2}}^{\Omega S}}} \cdot \cdots \xleftarrow{-\frac{\beta_j^{\Omega S}}{\beta_{j_{u_{p+1}}^{\Omega S}}}} x_{u_{p+1}} = w'_{p+1}.$$ 

Multiplying this directed path on the right by $w_0$, we obtain the following directed path in QBG for each $0 \leq p \leq s - 1$ (see Lemma 3.3.2):

$$w_p \overset{\text{def}}{=} w'_p w_0 = x_{u_p} w_0 \xleftarrow{w_0} \cdots \xleftarrow{w_0} x_{u_{p+1}} w_0 = w'_{p+1} w_0 \overset{\text{def}}{=} w_{p+1}.$$ 

Note that $w_0 \overset{\text{def}}{=} w_0 w_0 = x_0 w_0 = w_0$, and the edge labels of this directed path are increasing in the weak reflection order $\prec$ on $(\Delta^+)\vee$ introduced at the beginning of §4.3 (see Lemma 5.3.6) and lie in $(\Delta^+ \setminus \Delta^+_S)\vee$; this property will be used to give the inverse to $\Xi$. Because

$$\sigma_p(\lambda, w_0 \beta_{j_u}^{\Omega S}) = d_{j_u}(\lambda, w_0 \beta_{j_u}^{\Omega S}) = \frac{\langle \lambda, \beta_{j_u}^{\Omega S} \rangle - a_{j_u}}{\langle \lambda, \beta_{j_u}^{\Omega S} \rangle} = \frac{\langle \lambda, \beta_{j_u}^{\Omega S} \rangle - a_{j_u}}{\langle \lambda, \beta_{j_u}^{\Omega S} \rangle} - a_{j_u} \in \mathbb{Z}$$

for $u_{p+1} \leq u \leq u_{p+1}$, $0 \leq p \leq s - 1$, we find that (5.11) is a directed path in $\mathcal{QB}^{A_{2n}^{(2)}}_{\sigma_p}$ for $0 \leq p \leq s - 1$; indeed, if $x_{u-1} w_0 \xleftarrow{w_0} x_{u} w_0$ is a Bruhat edge with $w_0 \beta_{j_u}^{\Omega S}$ a short root, that is, $\text{dir}(z_{u-1}^{\Omega S}) = x_{u-1} \xleftarrow{w_0} x_u = \text{dir}(z_u^{\Omega S})$ is a Bruhat edge with
\(\overline{\beta}^{\text{OS}}_{j_u}\) a short root, then \(\deg(\overline{\beta}^{\text{OS}}_{j_u}) = \alpha_{\overline{\beta}^{\text{OS}}_{j_u}} a_{j_u} \in 2\mathbb{Z}\) by the definition of \(Q B^{A_2}_w(e; t(\lambda_-))\), and hence \(a_{j_u} \in 2\mathbb{Z}\) since \(\alpha_{\overline{\beta}^{\text{OS}}_{j_u}} = 1\). Also, \(\langle \lambda_, \overline{\beta}^{\text{OS}}_{j_u}\rangle \in 2\mathbb{Z}\) since \(\overline{\beta}^{\text{OS}}_{j_u}\) is a short root. Hence we have \(\sigma_{p}(\lambda, w_0\overline{\beta}^{\text{OS}}_{j_u}) \in 2\mathbb{Z}\) by (5.12). Therefore, by Lemma 3.3.6 there exists a directed path in \((Q B^{A_2}_w)^S\) from \([w_{p+1}]\) to \([w_p]\), where \(S = \{i \in I | \langle \lambda, \alpha_{j}^{-1}\rangle = 0\}\). Also, we claim that \(\langle \lambda_-, \overline{\beta}^{\text{OS}}_{j_u}\rangle \neq [w_{p+1}]\) for \(1 \leq p \leq s - 1\). Suppose, for contradiction, that \(\langle \lambda, \alpha_{j}^{-1}\rangle = 0\) for some \(p\). Then, \(w_p W_S = w_{p+1} W_S\), and hence \(\min(w_p W_S, w_{p+1}) = \min(w_{p+1} W_S, w_{p-1}) = w_{p-1}\). Recall that the directed path (5.11) is a path in \(Q B\) from \(w_{p+1}\) to \(w_p\) whose labels are increasing and lie in \(\Delta^+ \setminus \Delta^-\). By Lemma 3.3.7 (1), (2), the directed path (5.11) is a shortest path in \(Q B\) from \(\min(w_{p+1} W_S, w_p) = \min(w_p W_S, w_p)\) to \(w_p\), which implies that the length of the directed path (5.11) is equal to 0. Therefore, \(\{j_{u_{p+1}}, \ldots, j_{u_p}\} = \emptyset\), and hence \(u_p = u_{p+1}\), which contradicts the fact that \(u_p < u_{p+1}\).

Thus we obtain

\[
\eta \overset{\text{def}}{=} ([w_1], \ldots, [w_s]; \sigma_0, \ldots, \sigma_s) \in Q L S^{A_2}_w(\lambda).
\]

We now define \(\Xi(\eta_{j_{OS}}) \overset{\text{def}}{=} \eta\).

**Lemma 5.4.2.** The map \(\Xi : Q B^{A_2}_w(e; t(\lambda_-)) \to Q L S^{A_2}_w(\lambda)\) is injective.

**Proof.** We first show that the map \(\Xi\) is injective. Let \(J = \{j_1, \ldots, j_r\}\) and \(K = \{k_1, \ldots, k_{r'}\}\) be subsets of \(\{1, \ldots, L\}\) such that \(\Xi(p_{j_{OS}}) = \Xi(p_{K_{OS}}) = (v_1, \ldots, v_s; \sigma_0, \ldots, \sigma_s) \in Q L S^{A_2}_w(\lambda)\). As in (5.11), we set \(0 = u_0 \leq u_1 < \cdots < u_{s-1} = r\) and \(0 = u_0' \leq u_1' < \cdots < u_{s-1}' = r'\) in such a way that

\[
0 = d_{j_1} = \cdots = d_{j_{u_1}} < d_{j_{u_1+1}} = \cdots = d_{j_{u_2}} < \cdots < d_{j_{u_{s-1}+1}} = \cdots = d_{j_r} < 1 = \sigma_s,
\]

\[
0 = d_{k_1} = \cdots = d_{k_{u_1'}} < d_{k_{u_1'+1}} = \cdots = d_{k_{u_2'}} < \cdots < d_{k_{u_{s-1}'+1}} = \cdots = d_{k_{r'}} < 1 = \sigma_{s}.
\]

As in (5.11), we consider the directed paths in \(Q B\)

\[
\begin{align*}
\eta_p &\overset{\text{def}}{=} (\overline{\beta}^{\text{OS}}_{j_{u_1}+1}, \ldots, \overline{\beta}^{\text{OS}}_{j_{u_2}+1})^w, \quad \text{for } 0 \leq p \leq s - 1; \\
y_p &\overset{\text{def}}{=} (\overline{\beta}^{\text{OS}}_{k_{u_1}'+1}, \ldots, \overline{\beta}^{\text{OS}}_{k_{u_2}'+1})^w, \quad \text{for } 0 \leq p \leq s - 1,
\end{align*}
\]

here we note that \(w_0 = y_0 = e, \) and \([w_p] = [y_p] = v_p, 0 \leq p \leq s - 1\).

Suppose that \(u_p = y_p\) and \(p = y_p\). Then the paths (5.14) are both directed paths from some element in \(v_{p+1} W_S\) to \(w_p\) in \(Q B\) whose labels are increasing and lie in \(\Delta^+ \setminus \Delta^-\)^\vee. Hence \(w_{p+1} = y_{p+1} \in v_{p+1} W_S\) and \(w_0 \overline{\beta}^{\text{OS}}_{j_{i_j}} = w_0 \overline{\beta}^{\text{OS}}_{j_{k_i}}\) for \(u_p + 1 \leq i \leq u_{p+1}\), and \(u_{p+1} = u'_{p+1}\) by Lemma 3.3.7 (2). Also, \(d_{j_1} = d_{k_1} = \sigma_{p}\) for \(u_p + 1 \leq i \leq u_{p+1}\). It follows from \(\overline{\beta}^{\text{OS}}_{j_{i_j}} = \overline{\beta}^{\text{OS}}_{k_{i_k}}\) that \(\beta^{\text{OS}}_{j_{i_j}} = \beta^{\text{OS}}_{k_{i_k}}, u_p + 1 \leq i \leq u_{p+1}\). Thus, by induction on \(p\), we deduce that \(u_p = u'_p\) for \(0 \leq p \leq s - 1\), and \(\beta^{\text{OS}}_{j_{i_j}} = \beta^{\text{OS}}_{k_{i_k}}, u_0 + 1 \leq i \leq u_{s-1}\). Therefore, \(r = u_{s-1} = u'_{s-1} = r'\), and hence \(J = \{j_1, \ldots, j_r\} = \{k_1, \ldots, k_{r'}\} = K\).
**Lemma 5.4.3.** The map \( \Xi : \text{QB}^{A_{2n}^{(2)}}(e; t(\lambda_-)) \rightarrow \text{QLS}^{A_{2n}^{(2)}}(\lambda) \) is surjective.

**Proof.** Take an arbitrary \( \eta = (y_1, \ldots, y_s; \tau_0, \ldots, \tau_s) \in \text{QLS}^{A_{2n}^{(2)}}(\lambda) \). By convention, we set \( y_0 = v(\lambda_-) \in W^S \). We define the elements \( v_p, 0 \leq p \leq s \), by: \( v_0 = w_0 \), and \( v_p = \min(y_pW_s; \leq v_{p-1}) \) for \( 1 \leq p \leq s \).

Because there exists a directed path in \( (\text{QB}^{A_{2n}^{(2)}}(\tau_p, \lambda)) \) from \( y_{p+1} \) to \( y_p \) for \( 1 \leq p \leq s - 1 \), we see from Lemma 3.3.3.2 (2), (3) that there exists a unique directed path

\[
(5.15) \quad v_p \leftarrow w_0 \gamma_{p,1} \cdots \leftarrow w_0 \gamma_{p,p} \rightarrow v_{p+1}
\]

in \( \text{QB}^{A_{2n}^{(2)}}(\tau_p, \lambda) \) from \( v_{p+1} \) to \( v_p \) whose edge labels \( -w_0 \gamma_{p,1}, \ldots, -w_0 \gamma_{p,p} \) are increasing in the weak reflection order \( \preceq \) and lie in \( (\Delta^+ \setminus \Delta_{\lambda}^{\tau_p})^\vee \) for \( 1 \leq p \leq s - 1 \); we remark that this is also true for \( p = 0 \), since \( \tau_0 = 0 \). Multiplying this directed path on the right by \( w_0 \), we obtain by Lemma 3.3.2 the following directed paths

\[
(5.16) \quad v_{p,0} = v_0 w_0 \stackrel{\gamma_{0,1}}{\rightarrow} v_{0,1} \stackrel{\gamma_{0,2}}{\rightarrow} \cdots \rightarrow v_{p+1,0} \stackrel{w_0}{\rightarrow} v_{p,t_p}, \quad 0 \leq p \leq s - 1.
\]

Concatenating these paths for \( 0 \leq p \leq s - 1 \), we obtain the following directed path

\[
e = v_{0,0} \gamma_{0,1} \cdots \gamma_{0,t_0} \rightarrow v_{0,t_0} = v_{1,0} \gamma_{1,1} \cdots \gamma_{1,t_1} \rightarrow v_{1,t_1} = v_{2,0} \gamma_{2,1} \cdots \gamma_{s-1,t_{s-1}} \rightarrow v_{s-1,t_{s-1}}
\]

in \( \text{QB}^G \). Now, for \( 0 \leq p \leq s - 1 \) and \( 1 \leq m \leq t_p \), we set \( d_{p,m} \overset{\text{def}}{=} \tau_p \in \mathbb{Q} \cap [0, 1) \), \( a_{p,m} \overset{\text{def}}{=} (1 - d_{p,m})/(\lambda_-, -\gamma_{p,m}^\vee) \), and \( \tilde{\gamma}_{p,m} \overset{\text{def}}{=} c_{p,m}a_{p,m} \delta - \gamma_{p,m} \). It follows from (5.6) that \( \tilde{\gamma}_{p,m} \in \Delta_{\text{aff}}^+ \cap t(\lambda_-)^{-1} \Delta_{\text{aff}}^+ \).

**Claim 1.**

(1) We have

\[
\tilde{\gamma}_{0,1} \prec \cdots \prec \tilde{\gamma}_{0,t_0} \prec \tilde{\gamma}_{1,1} \prec \cdots \prec \tilde{\gamma}_{s-1,t_{s-1}},
\]

where \( \prec \) denotes the weak reflection order on \( \Delta_{\text{aff}}^+ \cap m_{\lambda,1}^{-1} \Delta_{\text{aff}} \) introduced in §4.3; we choose \( J' = \{j_1, \ldots, j_{r'}\} \subset \{1, \ldots, L\} \) in such way that

\[
\left( \beta_{j_1}^{OS}, \ldots, \beta_{j_{r'}}^{OS} \right) = (\gamma_{0,1}, \ldots, \tilde{\gamma}_{0,t_0}, \tilde{\gamma}_{1,1}, \ldots, \tilde{\gamma}_{s-1,t_{s-1}}).
\]

(2) Let \( 1 \leq k \leq r' \), and take \( 1 \leq p \leq s \) such that

\[
\left( \beta_{j_1}^{OS} \prec \cdots \prec \beta_{j_k}^{OS} \right) = (\gamma_{0,1} \prec \cdots \prec \gamma_{p,m}).
\]

Then, we have \( \text{dir}(z_k^{OS}) = v_{p,m-1} \). Moreover, \( \text{dir}(z_{k-1}^{OS}) \rightarrow \text{dir}(z_k^{OS}) \) is an edge of \( \text{QB}^G \).

**Proof of Claim 1.** (1) It suffices to show the following:

(i) for \( 0 \leq p \leq s - 1 \) and \( 1 \leq m < t_p \), we have \( \tilde{\gamma}_{p,m} \prec \tilde{\gamma}_{p,m+1} \);

(ii) for \( 1 \leq p \leq s - 1 \), we have \( \tilde{\gamma}_{p,t_p} \prec \tilde{\gamma}_{p+1,1} \);

(i) Because \( \frac{(\lambda_-, -\gamma_{p,m}) - a_{p,m}}{(\lambda_-, -\gamma_{p,m+1})} = d_{p,m} \) and \( \frac{(\lambda_-, -\gamma_{p,m+1}) - a_{p,m+1}}{(\lambda_-, -\gamma_{p,m+1})} = d_{p,m+1} \), we have

\[
\Phi(\tilde{\gamma}_{p,m}) = (d_{p,m}, -w_0 \gamma_{p,m}), \quad \Phi(\tilde{\gamma}_{p,m+1}) = (d_{p,m+1}, -w_0 \gamma_{p,m+1}).
\]
Therefore, the first component of $\Phi(\tilde{\eta})$ is equal to that of $\Phi(\tilde{\eta}_{p,m+1})$ since $d_{p,m} = 1 - \tau_p = d_{p,m+1}$. Moreover, since $-w_0\gamma \geq -w_0\gamma_{p,m+1}$, we have $\Phi(\tilde{\eta}) < \Phi(\tilde{\eta}_{p,m+1})$. This implies that $\tilde{\eta}_{p,m} < \tilde{\eta}_{p,m+1}$.

(ii) The proof of (ii) is similar to that of (i). The first components of $\Phi(\tilde{\eta}_{p,t_p})$ and $\Phi(\tilde{\eta}_{p+1,1})$ are $d_{p,t_p}$ and $d_{p+1,1}$, respectively. Since $d_{p,t_p} = \tau_p < \tau_{p+1} = d_{p+1,1}$, we have $\Phi(\tilde{\eta}_{p,t_p}) < \Phi(\tilde{\eta}_{p+1,1})$. This implies that $\tilde{\eta}_{p,t_p} < \tilde{\eta}_{p+1,1}$.

(2) We proceed by induction on $k$. If $\beta_{j_1}^{OS} = \gamma_{0,1}$, i.e., $y_1 \neq v(\lambda_\gamma)$, then, we have

$$\text{dir}(z_1^{OS}) = \text{dir}(z_0^{OS})s_{-\beta_{j_1}^{OS}} = v_{0,0}s_{\gamma_{0,1}} = v_{0,1}$$

Thus, $\text{dir}(z_0^{OS}) = \text{dir}(t(\lambda_\gamma)) = e = v_{0,0}$. Therefore, the first component of $\Phi(\tilde{\eta})$ is $\tilde{\eta}_{p,0} = \tilde{\eta}$.

Assume that $\text{dir}(z_{k-1}^{OS}) = v_{p,m-1}$ for $0 \leq m \leq t_p$; here we remark that $v_{p,m}$ is the successor of $v_{p,m-1}$ in the directed path (5.16). Hence we have

$$\text{dir}(z_k^{OS}) = \text{dir}(z_{k-1}^{OS})s_{-\beta_{j_k}^{OS}} = v_{p,m-1}s_{\gamma_{p,m}} = v_{p,m}.$$

Also, since (5.16) is a directed path in $\text{QB}$, we have $v_{p,m} = \text{dir}(z_{k-1}^{OS})^{-1} \text{dir}(z_k^{OS}) = v_{p,m-1}$ is an edge of $\text{QB}$. □

Since $J' = \{j_1, \ldots, j_{r'}\} \subset \{K + 1, \ldots, L\}$, we can define an element $p_{J'}^{OS}$ to be

$$(m_0 = z_0^{OS}, z_1^{OS}, \ldots, z_{r'}^{OS}; \beta_{j_1}^{OS}, \beta_{j_2}^{OS}, \ldots, \beta_{j_{r'}}^{OS})$$

where $z_0^{OS} = m_0$, $z_k^{OS} = z_{k-1}^{OS}\beta_{j_k}^{OS}$ for $1 \leq k \leq r'$; it follows from Remark 5.4.1 and Claim 1 that $p_{J'}^{OS} \in \text{QB}^{A^{(2)}_{r'}(e; t(\lambda_\gamma))}$.

Claim 2. $\Xi(p_{J'}^{OS}) = \eta$.

Proof of Claim 2. In the following description of $p_{J'}^{OS}$, we employ the notation $u_p$, $v_p$, $w_p$, and $w_p$ used in the definition of $\Xi(p_{J'}^{OS})$.

For $1 \leq k \leq r'$, if we set $\beta_{j_k}^{OS} = \tilde{\eta}_{p,m}$, then we have

$$d_{j_k'} = 1 + \frac{\deg(\beta_{j_k}^{OS})}{c_{j_k}^{OS}(\lambda_\gamma, -\beta_{j_k}^{OS})} = 1 + \frac{\deg(\tilde{\eta}_{p,m})}{c_{\tilde{\eta}_{p,m}}(\lambda_\gamma, -\tilde{\eta}_{\tilde{\gamma}_{p,m}})} = 1 + \frac{c_{\tilde{\eta}_{p,m}}\alpha_{p,m}}{c_{\tilde{\eta}_{p,m}}(\lambda_\gamma, \tilde{\gamma}_{p,m})} = d_{p,m}.$$

Therefore, the sequence (5.10) determined by $p_{J'}^{OS}$ is (5.17)

$$0 = d_{0,1} = \cdots < d_{0,t_0} < d_{1,1} = \cdots < d_{1,t_1} < \cdots < d_{s-1,1} = \cdots = d_{s-1,t_{s-1}} < 1 = \tau_s = \sigma_s.$$

Because the sequences (5.17) of rational numbers is just the sequence (5.10) for $\Theta(\eta) = p_{J'}^{OS}$, we deduce that $u_{p+1} - u_p = t_p$ for $0 \leq p \leq s - 1$, $\beta_{j_{p+k}}^{OS} = \tilde{\eta}_{p,k}$ for $0 \leq p \leq s - 1, 1 \leq k \leq s$, and $\sigma_p = \tau_p$ for $0 \leq p \leq s$. Therefore, we have $w_p = \text{dir}(z_p^{OS}) = v_{p,0}$ and $w_p = v_{p,0}w_0 = v_p$. Since $|w_p| = |v_p| = y_p$, we conclude that $\Xi(p_{J'}^{OS}) = (\{w_1\}, \ldots, \{w_s\}; \sigma_0, \ldots, \sigma_s) = (y_1, \ldots, y_s; \tau_0, \ldots, \tau_s) = \eta$. □

This completes the proof of Lemma 5.4.3.

By Lemmas 5.4.2 and 5.4.3, we have the following Proposition.
Proposition 5.4.4. The map \( \Xi \) is bijective.

We recall from (5.1) and (5.2) that \( \deg(x) \) is defined by: \( x = x + \deg(x)\delta \) for \( x \in \mathbb{R} \), and \( \wt(u) \in Q \) and \( \dir(u) \in W \) are defined by: \( u = t(\wt(u))\dir(u) \) for \( u \in W_{aff} = t(Q) \times W \).

Proposition 5.4.5. The bijection \( \Xi : \mathcal{Q} \mathcal{B}^{A_{2n}}(c; t(\lambda_-)) \to \mathcal{Q} \mathcal{L} S^{A_{2n}}(\lambda) \) satisfies the following:

1. \( \wt(\Xi(p_{j_{0}}^{Q})) = \wt(\Xi(p_{j_{0}}^{Q})) \);
2. \( \deg(qwt(p_{j_{0}}^{Q})) = \Deg(\Xi(p_{j_{0}}^{Q})) \).

Proof. We proceed by induction on \( \#J \).

If \( J = 0 \), it is obvious that \( \deg(qwt(p_{j_{0}}^{Q})) = \Deg(\Xi(p_{j_{0}}^{Q})) = 0 \) and \( \wt(\end(p_{j_{0}}^{Q})) = \wt(\Xi(p_{j_{0}}^{Q})) = \lambda_- \), since \( \Xi(p_{j_{0}}^{Q}) = (v(\lambda_-); 0, 1) \).

Let \( J = \{ j_{1} < j_{2} < \cdots < j_{s} \} \), and set \( K \) of the form: \( \Xi(p_{j_{0}}^{Q}) = ([w_{1}], \ldots, [w_{s}]; \sigma_{0}, \ldots, \sigma_{s}) \). In the following, we employ the notation \( w_{0} \), \( 0 \leq p \leq s \), used in the definition of the map \( \Xi \). Note that \( \dir(p_{j_{0}}^{Q}) = w_{s}w_{0} \) by the definition of \( \Xi \). Also, observe that if \( d_{j_{r}} = d_{j_{r-1}} = \sigma_{s-1} \), then \( \{ d_{j_{1}} \leq \cdots \leq d_{j_{r-1}} \leq d_{j_{r}} \} = \{ d_{j_{1}} \leq \cdots \leq d_{j_{r-1}} \} \), and if \( d_{j_{r}} > d_{j_{r-1}} = \sigma_{s-1} \), then \( \{ d_{j_{1}} \leq \cdots \leq d_{j_{r-1}} \leq d_{j_{r}} \} = \{ d_{j_{1}} \leq \cdots \leq d_{j_{r-1}} \leq d_{j_{r}} \} \). From these, we deduce that

\[
\Xi(p_{j_{0}}^{Q}) = \begin{cases} ([w_{1}], \ldots, [w_{s-1}], [w_{s}^{s_{0}}w_{0}^{j_{r}}]; \sigma_{0}, \ldots, \sigma_{s-1}, \sigma_{s}) & \text{if } d_{j_{r}} = d_{j_{r-1}} = \sigma_{s-1}, \\ ([w_{1}], \ldots, [w_{s-1}], [w_{s}], [w_{s}^{s_{0}}w_{0}^{j_{r}}]; \sigma_{0}, \ldots, \sigma_{s-1}, d_{j_{r}}, \sigma_{s}) & \text{if } d_{j_{r}} > d_{j_{r-1}} = \sigma_{s-1}. \end{cases}
\]

For the induction step, it suffices to show the following claims.

Claim 1.

1. We have
   \[
   \wt(\Xi(p_{j_{0}}^{Q})) = \wt(\Xi(p_{j_{0}}^{Q})) + a_{j_{r}}w_{s}w_{0}(-j_{r}),
   \]
2. We have
   \[
   \Deg(\Xi(p_{j_{0}}^{Q})) = \Deg(\Xi(p_{j_{0}}^{Q})) + \zeta \deg(j_{r}),
   \]
   where \( \zeta \) is a Bruhat (resp., quantum) edge.

Claim 2.

1. We have
   \[
   \wt(\end(p_{j_{0}}^{Q})) = \wt(\end(p_{j_{0}}^{Q})) + a_{j_{r}}w_{s}w_{0}(-j_{r}),
   \]
2. We have
   \[
   \deg(qwt(p_{j_{0}}^{Q})) = \deg(qwt(p_{j_{0}}^{Q})) + \zeta \deg(j_{r}).
   \]

The proofs of Claims 1 and 2 are similar to those of Claims 1 and 2 in [NNS Proposition 3.2.6].
Hence we conclude that $P_{\lambda}^{A_{2n}}(q, 0) = \sum_{p_{j}^{QS} \in QS_{2n}^{A_{2n}}(c_{t}(\lambda_{-}))} q^{\deg(qwt(p_{j}^{QS}))} e^{\wt(\text{end}(p_{j}^{QS}))}$.

Therefore, it follows from Propositions 5.4.3 and 5.4.5 that

$$P_{\lambda}^{A_{2n}}(q, 0) = \sum_{\eta \in \text{QLS}_{\leq v(\mu)}^{A_{2n}}(2)} q^{\deg(\eta)} e^{\wt(\eta)}.$$  

Hence we conclude that $P_{\lambda}^{A_{2n}}(q, 0) = \text{gchQLS}_{\leq v(\mu)}^{A_{2n}}(2)$, as desired. \hfill \Box

5.5. Nonsymmetric Macdonald polynomials $E_{\mu}^{A_{2n}}(q, 0)$ at $t = 0$ with respect to arbitrary weights. Let $\lambda \in Q$ be a dominant weight. By Remark 5.1.4 and Theorem 5.1.2, we have

$$E_{w_{0}\lambda}^{A_{2n}}(q, 0) = \sum_{\eta \in \text{QLS}_{\leq v(\mu)}^{A_{2n}}(2)} q^{\deg(\eta)} e^{\wt(\eta)}.$$  

In general, we have the following theorem. The proof is similar to that of [LNSSS4, Theorem 1.1].

For $\eta = (w_{1}, \ldots, w_{s}; \sigma_{0}, \ldots, \sigma_{s}) \in \text{QLS}_{\leq v(\mu)}^{A_{2n}}(2)$, we set $i(\eta) \equiv w_{1}$. Then for $\mu \in W\lambda$, we define $\text{QLS}_{\leq v(\mu)}^{A_{2n}}(2) \equiv \{ \eta \in \text{QLS}_{\leq v(\mu)}^{A_{2n}}(2) \mid i(\eta) \leq v(\mu) \}$, where $\leq$ denotes the Bruhat order on $W$.

**Theorem 5.5.1.** Let $\lambda \in Q$ be a dominant weight and $\mu \in W\lambda$. Then

$$E_{\mu}^{A_{2n}}(q, 0) = \sum_{\eta \in \text{QLS}_{\leq v(\mu)}^{A_{2n}}(2)} q^{\deg(\eta)} e^{\wt(\eta)}.$$  

**Appendix A. Semi-infinite Bruhat graph of type $A_{2n}^{(2)}$**

We fix $(A, A_{\text{aff}}) = (C_{n}, A_{2n}^{(2)})$. Throughout Appendix, we omit the dagger for the notation because we do not consider other cases than $(A, A_{\text{aff}}) = (C_{n}, A_{2n}^{(2)})$. We remark that, in this section QBG, QBG$^{S}$, QBG$^{\lambda \lambda_{\lambda}}$, $(\text{QBG}_{\lambda \lambda_{\lambda}}^{A_{2n}^{(2)}})^{S}$, QLS($\lambda$) mean QBG$^{\lambda}$, $(\text{QBG}_{\lambda \lambda_{\lambda}}^{A_{2n}^{(2)}})^{S}$, (QBG$^{\lambda \lambda_{\lambda}}$)$^{S}$, ((QBG$^{\lambda \lambda_{\lambda}}$)$^{A_{2n}^{(2)}}$)$^{S}$, QLS$^{(\lambda)}$, respectively, defined in Definitions 3.2.1, 3.2.3, and 4.1.5.

We set $c_{\alpha} \equiv \begin{cases} 1 & \text{if } \alpha \text{ is a intermediate root of } \Delta, \\ 2 & \text{if } \alpha \text{ is a long root of } \Delta, \end{cases}$

and recall that the set of all real root of $g(A_{2n}^{(2)})$ is

$$\Delta_{\text{aff}} = \{ \alpha + c_{\alpha}a\delta \mid \alpha \in \Delta, a \in \mathbb{Z} \} \cup \{ \frac{1}{2}(\alpha + (2a - 1)\delta) \mid \alpha \in \Delta_{\ell}, a \in \mathbb{Z} \},$$

where $\Delta_{\ell}$ denotes the set of all long roots in $\Delta$.

For $x \in Q^{\vee}$, let $t(x)$ denote the linear transformation on $P_{\text{aff}}^{0}: t(x)(y + a\delta) = y + (a - \langle x, \cdot \rangle_{\text{aff}})\delta$ for $y \in \oplus_{i \in I} \mathbb{Z}(\Lambda_{i} - \langle \Lambda_{i}, c_{\text{aff}}\Lambda_{0} \rangle), a \in \mathbb{Z}$. The affine Weyl group of $g(A_{2n}^{(2)})$ is defined by $W_{\text{aff}} \equiv t(Q^{\vee}) \rtimes W$. Also, we define $s_{0}: P_{\text{aff}}^{0} \to P_{\text{aff}}^{0}$ by $x \mapsto s_{0}x - \langle x, \theta^{\vee} \rangle_{\text{aff}}\delta$. Then $W_{\text{aff}} = \{ s_{i} \mid i \in I_{\text{aff}} \}$. 

The affine Weyl group also acts on $P_{\text{aff}}^\vee$ by

$$s_i y = y - (\alpha_i, y)_{\text{aff}} \alpha_i^\vee$$

for $i \in I_{\text{aff}}$ and $\mu \in P_{\text{aff}}^\vee$.

For $\beta \in \Delta_{\text{aff}}^+$ let $w \in W_{\text{aff}}$ and $i \in I_{\text{aff}}$ be such that $\beta = w\alpha_i$. We define the associated reflection $s_\beta \in W_{\text{aff}}$ and the associated coroot $\beta^\vee \in P_{\text{aff}}^\vee$ by

$$s_\beta = ws_iw^{-1},$$

$$\beta^\vee = w\alpha_i^\vee.$$  

Note that for $\Lambda \in P_{\text{aff}}^0$,

(A.1) \hspace{1cm} \langle \Lambda, (\alpha + c_\alpha a\delta)^\vee \rangle_{\text{aff}} = (\zeta^{-1} \circ \text{cl}(\Lambda), \alpha^\vee), \hspace{0.5cm} \alpha \in \Delta, a \in \mathbb{Z},

(A.2) \hspace{1cm} \langle \Lambda, \left(\frac{1}{2}(\alpha + (2a - 1)\delta)\right)^\vee \rangle_{\text{aff}} = 2(\zeta^{-1} \circ \text{cl}(\Lambda), \alpha^\vee), \hspace{0.5cm} \alpha \in \Delta^\ell, a \in \mathbb{Z}.

The proof of the following lemma is straightforward.

**Lemma A.0.2.** (1) Let $\beta \in \Delta_{\text{aff}}^+$ be a positive root of the form $\beta = \alpha + c_\alpha a\delta$ with $\alpha \in \Delta$ and $a \in \mathbb{Z}$, and let $x \in P_{\text{aff}}^0$. Then, $s_\beta x = s_\alpha x - ac_\alpha \langle x, \alpha^\vee \rangle \delta = s_{\alpha t}(ac_\alpha \alpha^\vee) x$.

(2) Let $\beta \in \Delta_{\text{aff}}^+$ be a positive root of the form $\beta = \frac{1}{2}(\alpha + (2a - 1)\delta)$ with $\alpha \in \Delta$ and $a \in \mathbb{Z}$, and let $x \in P_{\text{aff}}^0$. Then, $s_\beta x = s_\alpha x - (2a - 1)\langle x, \alpha^\vee \rangle \delta = s_{\alpha t}(2a - 1)\alpha^\vee x$.

**A.1. Peterson’s coset representatives $W_{\text{aff}}^S$.** Let $S$ be a subset of $I$. We define

$$\Delta_{S,\text{aff}} \overset{\text{def}}{=} \{\alpha + c_\alpha a\delta \mid \alpha \in \Delta_S, a \in \mathbb{Z}\} \cup \{\frac{1}{2}(\alpha + (2a - 1)\delta) \mid \alpha \in \Delta_S \cap \Delta^\ell, a \in \mathbb{Z}\},$$

$$\Delta^+_S,\text{aff} \overset{\text{def}}{=} \Delta_{S,\text{aff}} \cap \Delta_{\text{aff}}^+,$$

$$W_{S,\text{aff}} \overset{\text{def}}{=} W \ltimes \{t(\mu) \mid \mu \in Q_{S}^\vee\},$$

$$W_{S,\text{aff}}^S \overset{\text{def}}{=} \{x \in W_{\text{aff}} \mid x\beta \in \Delta_{\text{aff}}^+ \text{ for all } \beta \in \Delta^+_S,\text{aff} \}.$$  

**Lemma A.1.1** (Pe, see also [LS, Lemma 10.6]). For every $x \in W_{\text{aff}}$, there exists a unique factorization $x = x_1 x_2$ with $x_1 \in W_{S,\text{aff}}^S$, and $x_2 \in W_{S,\text{aff}}$.

We define a surjective map $\Pi^S : W_{\text{aff}} \to W_{S,\text{aff}}^S$ by $\Pi^S(x) \overset{\text{def}}{=} x_1$ if $x = x_1 x_2$ with $x_1 \in W_{S,\text{aff}}^S$ and $x_2 \in W_{S,\text{aff}}$.

**Definition A.1.2** (see [LNS, Lemma 3.8]). An element $\mu \in Q^\vee$ is said to be $S$-adjusted if $(\mu, \gamma) \in \{0, -1\}$ for all $\gamma \in \Delta_{S}^\vee$. Let $Q_{S-\text{adj}}^\vee$ denote the set of $S$-adjusted elements.

Since $W_{\text{aff}}$ is isomorphic to the affine Weyl group of $\mathfrak{g}(\mathfrak{c}_{n}^{(1)})$, the proof of the following Lemma is the same as the proof in [LNS, Lemma 2.3.5].

**Lemma A.1.3** ([LNS, Lemma 2.3.5], see [LNS, (3.6), (3.7), and Lemma 3.7]). Let $S$ be a subset of $I$.

(1) For each $\mu \in Q^\vee$, there exists a unique $\phi_S(\mu) \in Q_{S-adj}^\vee$ such that $\mu + \phi_S(\mu) \in Q_{S-\text{adj}}^\vee$. 
(2) For each \( \mu \in Q^\vee \), there exists a unique \( z_\mu \in W_\mathsf{aff} \) such that \( \Pi^S(t(\mu)) = z_\mu t(\mu + \phi_S(\mu)) \).
(3) For \( w \in W \) and \( \mu \in Q^\vee \), we have \( \Pi^S(wt(\mu)) = \lfloor w \rfloor z_\mu t(\mu + \phi_S(\mu)) \). In particular,
\[
W_\mathsf{aff}^S = \{ wz_\mu t(\mu) : w \in W^S, \mu \in Q^\vee_{S-\mathsf{adj}} \}.
\]

**Lemma A.1.4** ([INS Lemma 2.2.7]). Let \( x \in W_\mathsf{aff} \) and \( \mu \in Q^\vee_{S-\mathsf{adj}} \). Then, \( xz_\mu t(\mu) \in W_\mathsf{aff}^S \) if and only if \( x \in W_\mathsf{aff}^S \).

### A.2. Semi-infinite Bruhat graph of type \( A_{2n}^{(2)} \)

**Definition A.2.1** ([Pe]). Let \( x = wt(\mu) \in W_\mathsf{aff} \) with \( w \in W \) and \( \mu \in Q^\vee \). Then we define \( \ell^\mathsf{aff}(x) \defeq \ell(x) + 2\langle \rho, \mu \rangle \), where \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \).

**Definition A.2.2** (see [INS Definition 2.4.2] for untwisted types). Let \( S \) be a subset of \( I \). The semi-infinite Bruhat graph \( \mathsf{SiB}^S \) is the directed graph with vertex set \( W_\mathsf{aff}^S \) and directed edges labeled by elements in \( \Delta^+_{\mathsf{aff}} \); for \( x \in W_\mathsf{aff}^S \) and \( \beta \in \Delta^+_{\mathsf{aff}} \), \( x \xrightarrow{\beta} s_\beta x \) is an edge of \( \mathsf{SiB}^S \) if the following hold:

1. \( s_\beta x \in W_\mathsf{aff}^S \),
2. \( \ell^\mathsf{aff}(s_\beta x) \overline{=} \ell^\mathsf{aff}(x) + 1 \).

**Definition A.2.3** (see [INS Definition 3.1.1] for untwisted types). Let \( \Lambda \in P_\mathsf{aff}^0 \) be a level-zero dominant weight, and set \( \lambda = \frac{1}{s} \circ \mathsf{cl}(\Lambda) \in P \) and \( S = S_\lambda \). Let \( b \in \mathbb{Q} \cap [0, 1] \). We denote by \( \mathsf{SiB}^S_{b\Lambda} \) the subgraph of \( \mathsf{SiB}^S \) with the same vertex set but having only the edges:

\[
x \xrightarrow{\beta} s_\beta x \text{ with } b\langle x, \Lambda, \beta^\vee \rangle_{\mathsf{aff}} \in \mathbb{Z}.
\]

**Definition A.2.4** (see [INS Definition 3.1.2] for untwisted types). Let \( \Lambda \in P_\mathsf{aff}^0 \) be a level-zero dominant weight, and set \( \lambda = \frac{1}{s} \circ \mathsf{cl}(\Lambda) \) and \( S = S_\lambda = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} \). A pair \( \pi = (x_1, x_2, \ldots, x_s; \tau_0, \tau_1, \ldots, \tau_s) \) of a sequence \( w_1, \ldots, w_s \) of elements in \( W_\mathsf{aff}^S \) and an increasing sequence \( 0 = \tau_0, \ldots, \tau_s = 1 \) of rational numbers is called a semi-infinite Lakshmibai-Seshadri (SiLS) path of shape \( \Lambda \) if

(C) for every \( 1 \leq i \leq s - 1 \), there exists a directed path from \( w_{i+1} \) to \( w_i \) in \( \mathsf{SiB}^S_{b\Lambda} \). Let \( \mathsf{SiLS}(\Lambda) \) denote the set of all SiLS paths of shape \( \Lambda \).

Let \( \Lambda \in P_\mathsf{aff}^0 \) be a level-zero dominant weight, and set \( \lambda = \frac{1}{s} \circ \mathsf{cl}(\Lambda) \). We take \( \pi = (x_1, \ldots, x_s; \sigma_0, \ldots, \sigma_s) \in \mathsf{SiLS}(\Lambda) \). It follows from Lemma A.1.3 that for each \( 1 \leq i \leq s \), there exists a unique decomposition \( x_i = w_i z_{\mu_i} t(\mu_i) \) with \( w_i \in W^S \) and \( \mu_i \in Q^\vee_{S-\mathsf{adj}} \). Then we set \( \mathsf{cl}(\pi) = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \).

We will show the following theorem in §A.3.

**Theorem A.2.5.** For each \( \eta \in \mathsf{SiLS}(\Lambda) \), \( \mathsf{cl}(\eta) \in \mathsf{QLS}(\lambda) \). Hence we obtain the map \( \mathsf{cl} : \mathsf{SiLS}(\Lambda) \to \mathsf{QLS}(\lambda) \); moreover, the map \( \mathsf{cl} : \mathsf{SiLS}(\Lambda) \to \mathsf{QLS}(\lambda) \) is surjective.

Let \( \Lambda \in P_\mathsf{aff}^0 \) be a level-zero dominant weight. Let \( B(\Lambda) \) be the crystal basis of the extremal weight module \( V(\Lambda) \) over \( U_q(g(A_2^{(2)}) \) (see [K] §3.1]). The proof of the following theorem is the same as [INS Theorem 3.2.1].

**Theorem A.2.6** (see [INS Theorem 3.2.1] for untwisted types). Let \( \Lambda \in P \) be a level-zero dominant weight. Then, \( \mathsf{SiLS}(\Lambda) \) has a \( U_q(g(A_{2n}^{(2)}) \)-crystal structure.
Moreover, $B(\Lambda)$ is isomorphic to the set $\text{SiLS}(\Lambda)$ as a $U_q(\mathfrak{g}(\text{aff}))$-crystal, and the surjective map $\text{cl} : \text{SiLS}(\Lambda) \to \text{QLS}(\mathbb{Z}^{-1} \circ \text{cl}(\Lambda))$ satisfies the following: for $\eta \in \text{SiLS}(\Lambda)$ and $i \in I_{\text{aff}}$,

\[
\begin{align*}
\text{cl}(e_i(\eta)) &= e_i(\text{cl}(\eta)), \\
\text{cl}(f_i(\eta)) &= f_i(\text{cl}(\eta)),
\end{align*}
\]

where $\text{cl}(0) = 0$.

**A.3. Proof of Theorem [A.2.5]**

**Lemma A.3.1** ([BFP] Lemma 4.3). We have $\ell(s_\alpha) \leq 2\langle \rho, \alpha^\vee \rangle - 1$ for all $\alpha \in \Delta^+$. 

**Lemma A.3.2** (see [INS] Corollary 4.2.2 for untwisted types). Let $S$ be a subset of $I$. Let $x = wz_\mu t(\mu) \in W^\text{aff}_S$ with $w \in W^S$ and $\mu \in Q^+_S$, and let $\beta \in \Delta^+_\text{aff}$ be such that $x \mapsto s_\beta x$ in $\text{SiB}^S$. If $\beta$ is a long or intermediate root, then we write $\beta$ as $\beta = \gamma + ac_\alpha\delta$ with $\gamma \in \Delta$ and $a \in \mathbb{Z}_{\geq 0}$; if $\beta$ is a short root, then we write $\beta$ as $\beta = \frac{1}{2}(2a + 1)(\alpha + (2\alpha + 1)\delta)$ with $\gamma \in \Delta'$ and $a \in \mathbb{Z}_{\geq 0}$. Set $\alpha \overset{\text{def}}{=} w^{-1}\gamma \in \Delta$. Then the following hold.

1. $\alpha \in \Delta^+ \setminus \Delta^+_S$.
2. If $\beta$ is a long or intermediate root, then $\ell(ws_\alpha z_\mu) = \ell(wz_\mu) + 1 - 2\langle \alpha, z_\mu^{-1}\alpha^\vee \rangle$; if $\beta$ is a short root, then $\ell(ws_\alpha z_\mu) = \ell(wz_\mu) + 2 - 2\langle \alpha, z_\mu^{-1}\alpha^\vee \rangle$.
3. $a \in \{0, 1\}$. Moreover, $\beta$ is a long or short root, then $a = 0$.

**Proof.** (1) We first show that $\alpha \notin \Delta_S$. Suppose that $\alpha \in \Delta_S$. If $\beta$ is a long or intermediate root, then we see that

$$x^{-1}\beta = t(-\mu)z_\mu^{-1}w^{-1}(w\alpha + ac_\alpha\delta) = t(-\mu)(z_\mu^{-1}\alpha + ac_\alpha\delta) = z_\mu^{-1}\alpha + \zeta c_\alpha\delta$$

for some $\zeta \in \mathbb{Z}$; if $\beta$ is a short root, then we see that $x^{-1}\beta = \frac{1}{2}(z_\mu^{-1}\alpha + \zeta\delta)$ for some odd integer $\zeta$.

Because $\alpha \in \Delta_S$ and $z_\mu \in W_S$, it follows that $x^{-1}\beta \in \Delta_S$, and hence $s_{x^{-1}\beta} \in W_{S,\text{aff}}$. Therefore, we deduce from Lemma A.1.1 that $s_{x}\beta \neq x$s_{x^{-1}\beta}$ is not contained in $W^S_{\text{aff}}$, which contradicts the assumption that $x \mapsto s_\beta x$ is an edge of $\text{SiB}^S$. Thus we obtain $\alpha \notin \Delta_S$.

We set

$$a' \overset{\text{def}}{=} \begin{cases} ac_\alpha & \text{if } \beta \text{ is a long or intermediate root}, \\ 2a + 1 & \text{if } \beta \text{ is a short root}. \end{cases}$$

It follows from Lemma A.0.2 that $s_\beta = s_\gamma t(a'\gamma^\vee)$. Hence we have

$$s_\beta x = s_\gamma t(a'\gamma^\vee)wz_\mu t(\mu) = ws_{-1}z_\mu t(a'z_\mu^{-1}w^{-1}\gamma^\vee + \mu) = ws_\alpha z_\mu t(a'z_\mu^{-1}\alpha^\vee + \mu).$$
Suppose, for a contradiction, that $-\alpha \in \Delta^+$. Then, $-z_{\mu}^{-1} \alpha \in \Delta^+$ since $\alpha \notin \Delta_S$. We have

$$\ell^\infty(s_\beta x) = \ell^\infty(ws_\alpha z_\mu, t(a'z_{\mu}^{-1}\alpha^\vee + \mu))$$

$$= \ell(ws_\alpha z_\mu) + 2\langle \rho, a'z_{\mu}^{-1}\alpha^\vee + \mu \rangle$$

$$= \ell(wz_\mu s_{z_\mu^{-1}}\alpha) + 2\langle \rho, a'z_{\mu}^{-1}\alpha^\vee + \mu \rangle$$

$$\leq \ell(wz_\mu) + \ell(s_{z_\mu^{-1}}\alpha) + 2\langle \rho, a'z_{\mu}^{-1}\alpha^\vee + \mu \rangle$$

$$\leq \ell(wz_\mu) + (2\langle \rho, -z_{\mu}^{-1}\alpha^\vee \rangle - 1) + 2\langle \rho, a'z_{\mu}^{-1}\alpha^\vee + \mu \rangle \quad \text{(by Lemma A.3.1)}$$

$$= \ell^\infty(x) - 1 + 2(a' - 1)\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle.$$ 

Since $\ell^\infty(s_\beta x) = \ell^\infty(x) + 1$, we deduce that $(a' - 1)\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle \geq 1$. Because $\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle < 0$, we obtain $a' = 0$, and hence $\beta = w\alpha$. Recall that $w\alpha \in \Delta^+$ by assumption. Because $-\alpha \in \Delta^+$ and $w\alpha \in \Delta^+$, we see that $\ell(ws_\alpha) < \ell(w)$. However, we have

$$\ell^\infty(s_\beta x) = \ell^\infty(ws_\alpha z_\mu, t(\mu)) \quad \text{(since $a' = 0$)}$$

$$= \ell(ws_\alpha z_\mu) + 2\langle \rho, \mu \rangle$$

$$\leq \ell(ws_\alpha) + \ell(z_\mu) + 2\langle \rho, \mu \rangle$$

$$< \ell(w) + \ell(z_\mu) + 2\langle \rho, \mu \rangle = \ell^\infty(x) \quad \text{(since $\ell(ws_\alpha) < \ell(w)$)},$$

which contradicts $\ell^\infty(s_\beta x) = \ell^\infty(x) + 1$. Thus we obtain $\alpha \in \Delta^+$. This proves (1).

(2) As in (1), we see that

$$1 = \ell^\infty(s_\beta x) - \ell^\infty(x)$$

$$= \ell(ws_\alpha z_\mu) + 2\langle \rho, a'z_{\mu}^{-1}\alpha^\vee + \mu \rangle - \ell(wz_\mu) - 2\langle \rho, \mu \rangle \quad \text{(by Definition A.2.1)}$$

$$= \ell(ws_\alpha z_\mu) - \ell(wz_\mu) + 2a'\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle,$$

which proves (2).

(3) It follows that

$$1 = \ell(ws_\alpha z_\mu) - \ell(wz_\mu) + 2a'\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle \quad \text{(by (2))}$$

$$= \ell(wz_\mu s_{z_\mu^{-1}}\alpha) - \ell(wz_\mu) + 2a'\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle$$

$$\geq \ell(s_{z_\mu^{-1}}\alpha) + 2a'\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle$$

$$\geq 1 - 2\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle + 2a'\langle \rho, z_{\mu}^{-1}\alpha^\vee \rangle \quad \text{(by Lemma A.3.1)}$$

which implies that $a' \in \{0, 1\}$, since $z_{\mu}^{-1}\alpha \in \Delta^+$. If $\beta$ is an intermediate root, then $c_\alpha = 1$ and hence $a \in \{0, 1\}$. If $\beta$ is a long root, then $c_\alpha = 2$ and hence $a = 0$. If $\beta$ is a short root, then $2a + 1 \in \{0, 1\}$ and hence $a = 0$. □

**Lemma A.3.3** ([LNSSS2, Lemma 3.10]). For $\mu \in Q_{S^-adj}^\vee$, we have $\ell(z_\mu) = -2\langle \rho_S, \mu \rangle$.

**Remark A.3.4.**

(1) If $w \xrightarrow{\Delta^+} [ws_\alpha]$ is a Bruhat edge of QBG$^S$, then we have $ws_\alpha = [ws_\alpha] \in W^S$ ([LNSSS4, Remark 3.1.2]).

(2) Let $w \xrightarrow{\Delta^+} [ws_\alpha]$ be a quantum edge of QBG$^S$. Then we have $ws_\alpha t(\alpha^\vee) \in W^S_{aff}$, and $\ell(ws_\alpha) = \ell(w) + 1 - \langle \rho, \alpha^\vee \rangle$ ([LNSSS1, §4.3]; see also [LS, §10]).
Proposition A.3.5 (see INS Proposition A.1.2 for untwisted types). Let $0 < b \leq 1$ be a rational number. Let $x = wz_{\mu} t(\mu) \in W_{aff}^S$ with $w \in W^S$ and $\mu \in Q^\vee_{S-adj}$.

(1) Assume that $x \overset{\beta}{\rightarrow} s_{\beta}x \in \text{SiB}^S_{b\Lambda}$ for $\beta \in \Delta^+_\text{aff}$. We set $\alpha$ as in Lemma A.3.2.

Then $w \overset{\alpha}{\rightarrow} [w_{\alpha}]$ is an edge of $(QBG_{A(2)_{bt}}^S)^S$.

(2) Assume that $w \overset{\alpha}{\rightarrow} [w_{\alpha}]$ is a Bruhat edge of $(QBG_{A(2)_{bt}}^S)^S$. We set $\beta \overset{\defeq}{=} w\alpha \in \Delta^+ \subset \Delta^+_{aff}$. Then $s_{\beta}x \in W_{aff}^S$. Moreover, for each $\mu \in Q^\vee_{S-adj}$, $wz_{\mu} t(\mu) \overset{\beta}{\rightarrow} s_{\beta}wz_{\mu} t(\mu)$ is an edge of $\text{SiB}^S_{b\Lambda}$.

(3) Assume that $w \overset{\alpha}{\rightarrow} [w_{\alpha}]$ is a quantum edge of $(QBG_{A(2)_{bt}}^S)^S$ with $\alpha \in \Delta^+$ a short root. We set $\beta \overset{\defeq}{=} w\alpha + \delta \in \Delta^+_\text{aff}$. Then $s_{\beta}x \in W_{aff}^S$. Moreover, for each $\mu \in Q^\vee_{S-adj}$, $wz_{\mu} t(\mu) \overset{\beta}{\rightarrow} s_{\beta}wz_{\mu} t(\mu)$ is an edge of $\text{SiB}^S_{b\Lambda}$.

(4) Assume that $w \overset{\alpha}{\rightarrow} [w_{\alpha}]$ is a quantum edge of $(QBG_{A(2)_{bt}}^S)^S$ with $\alpha \in \Delta^+$ a long root. We set $\beta \overset{\defeq}{=} \frac{1}{2}(w\alpha + \delta) \in \Delta^+_\text{aff}$. Then $s_{\beta}x \in W_{aff}^S$. Moreover, for each $\mu \in Q^\vee_{S-adj}$, $wz_{\mu} t(\mu) \overset{\beta}{\rightarrow} s_{\beta}wz_{\mu} t(\mu)$ is an edge of $\text{SiB}^S_{b\Lambda}$.

Proof. (1) By Lemma A.3.2 (1), (3), the affine root $\beta$ is of the form:

(i) (long or intermediate root) $\beta = w\alpha$ with $\alpha \in \Delta^+$, or

(ii) (intermediate root) $\beta = w\alpha + \delta$ with $\alpha \in \Delta^+$ a short root, or

(iii) (short root) $\beta = \frac{1}{2}(w\alpha + \delta)$ with $\alpha \in \Delta^+$ a long root.

Consider case (i). Since $W_{aff}^S \ni s_{\beta}x = s_{\beta}wz_{\mu} t(\mu) = w_{\alpha}z_{\mu} t(\mu)$, we have $w_{\alpha} \in W^S$, and hence $\ell(w_{\alpha}z_{\mu}) = \ell(w_{\alpha}) + \ell(z_{\mu})$. Also, since $\ell(wz_{\mu}) = \ell(w) + \ell(z_{\mu})$, $\ell(w_{\alpha}) = \ell(w) + 1$ because $\ell(wz_{\mu}) = \ell(w) + 1$. Hence $w \overset{\alpha}{\rightarrow} [w_{\alpha}] = w_{\alpha}$ is a Bruhat edge of $QBG$. Moreover, $w \overset{\alpha}{\rightarrow} [w_{\alpha}]$ is an edge of $(QBG_{A(2)_{bt}}^S)^S$ since $\langle \lambda, \alpha^\vee \rangle = \langle z_{\mu} t(\mu) \Lambda, w^{-1} \beta^\vee \rangle_{aff} = \langle x\Lambda, \beta^\vee \rangle_{aff} \in \mathbb{Z}$ by A.1.

Consider cases (ii) and (iii). Since $W_{aff}^S \ni s_{\beta}x = w_{\alpha}z_{\mu} t(z_{\mu}^{-1} \alpha^\vee + \mu)$, we obtain $z_{\mu}^{-1} \alpha^\vee + \mu \in Q^\vee_{S-adj}$ by Lemma A.1.3 (3). We have

$$w_{\alpha}z_{\mu} t(z_{\mu}^{-1} \alpha^\vee + \mu) = \Pi^S(w_{\alpha}z_{\mu} t(z_{\mu}^{-1} \alpha^\vee + \mu)) = [w_{\alpha}] z_{z_{\mu}^{-1} \alpha^\vee + \mu} t(z_{\mu}^{-1} \alpha^\vee + \mu) \quad \text{(by Lemma A.1.3 (3))}$$

which implies that $w_{\alpha}z_{\mu} = [w_{\alpha}] z_{z_{\mu}^{-1} \alpha^\vee + \mu}$, and hence $\ell(w_{\alpha}z_{\mu}) = \ell([w_{\alpha}]) + \ell(z_{z_{\mu}^{-1} \alpha^\vee + \mu})$. Therefore,

$$\ell([w_{\alpha}]) = \ell(w_{\alpha}z_{\mu}) - \ell(z_{z_{\mu}^{-1} \alpha^\vee + \mu}) = \ell(wz_{\mu}) + 1 - 2\langle \rho, \alpha^\vee \rangle - \ell(z_{z_{\mu}^{-1} \alpha^\vee + \mu}) \quad \text{(by Lemma A.3.2 (2))}$$

$$= \ell(w) + \ell(z_{\mu}) + 1 - 2\langle \rho, \alpha^\vee \rangle - \ell(z_{z_{\mu}^{-1} \alpha^\vee + \mu})$$

$$= \ell(w) - 2\langle \rho_S, \mu \rangle + 1 - 2\langle \rho, \alpha^\vee \rangle + 2\langle \rho_S, z_{\mu}^{-1} \alpha^\vee + \mu \rangle \quad \text{(by Lemma A.3.3)}$$

$$= \ell(w) + 1 - 2\langle \rho, \alpha^\vee \rangle + 2\langle \rho_S, z_{\mu}^{-1} \alpha^\vee \rangle \quad \text{(since $z_{\mu} \in W^S$)}$$

$$= \ell(w) + 1 - 2\langle \rho - \rho_S, \alpha^\vee \rangle,$$
which implies that $w \xrightarrow{\alpha} [ws_\alpha]$ is a quantum edge of QBG. Moreover, if $\beta = \gamma + \delta$, then $\langle \lambda, \alpha' \rangle = \langle x\Lambda, \beta' \rangle_{\text{aff}} \in \mathbb{Z}$ by (A.1); if $\beta = \frac{1}{2}(\gamma + \delta)$, then $\langle \lambda, \alpha' \rangle = \frac{1}{2}\langle x\Lambda, \beta' \rangle_{\text{aff}} \in \frac{1}{2}\mathbb{Z}$ by (A.2). Therefore, in both cases, we have $w \xrightarrow{\alpha} [ws_\alpha]$ is a quantum edge of $(\text{QBG}_{b_\Lambda}^{\alpha_{2n}})^S$.

(2) Since $s_\beta x = ws_\alpha z_\mu t(\mu)$ and $ws_\alpha = [ws_\alpha] \in W^S$ by Remark A.3.3 (1), it follows from Lemma A.1.3 (3) that $s_\beta x \in W^S_{\text{aff}}$. Also, we have

$$\ell^\infty(s_\beta x) = \ell(ws_\alpha z_\mu) + 2(\rho, \mu)$$

since $ws_\alpha \in W^S$ and $z_\mu \in W_S$

$$= \ell(ws_\alpha) + \ell(z_\mu) + 2(\rho, \mu)$$

(4) Since $w_\alpha t(\alpha') \in W^S_{\text{aff}}$ by Remark A.3.4 (2), it follows from Lemma A.1.3 that $s_\beta x = ws_\alpha t(\alpha') z_\mu t(\mu) \in W^S_{\text{aff}}$. Because

$$s_\beta x = \Pi^S(s_\beta x)$$

since $s_\beta x \in W^S_{\text{aff}}$

$$= \Pi^S(ws_\alpha z_\mu t(\mu + z_\mu^{-1} \alpha'))$$

$$= [ws_\alpha] z_\mu^{-1} \alpha' t(\mu + z_\mu^{-1} \alpha')$$

(by Lemma A.1.3 (3)),

we deduce that

$$\ell^\infty(s_\beta x) = \ell^\infty([ws_\alpha] z_\mu^{-1} \alpha' t(\mu + z_\mu^{-1} \alpha'))$$

$$= \ell([ws_\alpha] z_\mu^{-1} \alpha') + 2(\rho, \mu + z_\mu^{-1} \alpha')$$

$$= \ell([ws_\alpha] z_\mu^{-1} \alpha') + 2(\rho, \mu + z_\mu^{-1} \alpha')$$

(3) Moreover, $b(\lambda, \alpha') = b(\lambda, \alpha')_{\text{aff}} \in \mathbb{Z}$ by (A.1). Therefore, $x \xrightarrow{\beta} s_\beta x$ is an edge of $\text{SiB}^S_{b_\Lambda}$.

Thus, $x \xrightarrow{\beta} s_\beta x$ is an edge of $\text{SiB}^S_{b_\Lambda}$. Moreover, in case (3), we have $b(x\Lambda, \beta'_{\text{aff}}) = b(x\Lambda, \beta') \in \mathbb{Z}$ by (A.1); in case (4), we have $b(x\Lambda, \beta')_{\text{aff}} = 2b(x\Lambda, \alpha') \in 2 \cdot \frac{1}{2}\mathbb{Z} = \mathbb{Z}$ by (A.2). Therefore, in both cases, $x \xrightarrow{\beta} s_\beta x$ is an edge of $\text{SiB}^S_{b_\Lambda}$.

Proof of Theorem A.2.5 Let $\pi = (x_1, \ldots, x_s; \sigma_0, \ldots, \sigma_s) \in \text{SiLS}(A)$. Let $w_i \in W^S$ and $\mu_i \in Q^S_{S_{-\text{adj}}}$ be such that $x_i = w_i z_{\mu_i} t(\mu_i)$ for $1 \leq i \leq s$. If there exists a directed path from $x_{i+1}$ to $x_i$ in $\text{SiB}^S_{b_\Lambda}$, then there exists a directed path from $w_{i+1}$ to $w_i$
in \((\text{QBG}_{la}^{A(2)})^S)\) by Lemma \(\text{A.3.5}(1)\). This implies \(\text{cl}(\pi) = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \text{QLS}(\lambda)\).

Let \(\eta = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s) \in \text{QLS}(\lambda)\), and set \(\mu_s = 0\). We proceed by descending induction on \(1 \leq p \leq s\). Assume that for every \(i + 1 \leq p \leq s\) there exist an element \(\mu_p \in Q^{S, \text{adj}}_S\) and a directed path from \(w_{p+1}z_{\mu_{p+1}}t(\mu_{p+1})\) to \(w_pt(\mu_p)\) in \(\text{SiB}_{\lambda}^S\). By the definition of \(\text{QLS}(\lambda)\), there exists a directed path from \(w_{i+1}\) to \(w_i\) in \((\text{QBG}_{la}^{A(2)})^S)\). It follows from Lemmas \(\text{A.1.3}\) and \(\text{A.3.5}\) that there exist an element \(\mu_i \in Q^{S, \text{adj}}_S\) and a directed path from \(w_{i+1}z_{\mu_{i+1}}t(\mu_{i+1})\) to \(w_iz_{\mu_i}t(\mu_i)\) in \(\text{SiB}_{\lambda}^S\). Thus we obtain \(\pi \overset{\text{def}}{=} (w_1z_{\mu_1}t(\mu_1), \ldots, w_sz_{\mu_s}t(\mu_s); \sigma_0, \ldots, \sigma_s) \in \text{SiLS}(\Lambda)\). It is obvious that \(\text{cl}(\pi) = (w_1, \ldots, w_s; \sigma_0, \ldots, \sigma_s)\), which implies that the map \(\text{cl} : \text{SiLS}(\Lambda) \to \text{QLS}(\lambda)\) is surjective. \(\square\)

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