Nearly Bounded Regret of Re-solving Heuristics in Price-based Revenue Management

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Abstract

Price-based revenue management is a class of important questions in operations management. In its simplest form, a retailer sells a single product over $T$ consecutive time periods and is subject to constraints on the initial inventory levels. While the optimal pricing policy over $T$ periods could be obtained via dynamic programming, such an approach is sometimes undesirable because of its enormous computational costs. Approximately optimal policies, such as the re-solving heuristic, is often applied as a computationally tractable alternative. In this paper, we prove the following results:

1. We prove that a popular and commonly used re-solving heuristic attains an $O(\ln \ln T)$ regret compared to the value of the optimal DP pricing policy. This improves the $O(\ln T)$ regret upper bound established in the prior work of Jasin (2014).

2. We prove that there is an $\Omega(\ln T)$ gap between the value of the optimal DP pricing policy and that of a static LP relaxation. This complements our upper bound results in showing that the static LP relaxation is not an adequate information-relaxed benchmark when analyzing price-based revenue management algorithms.

Keywords: re-solving, self-adjusting controls, price-based revenue management, dynamic pricing

1 Introduction

We consider the simplest example of price-based revenue management, in which the retailer sells a single product repeatedly over $T$ consecutive time periods, subject to initial inventory level constraints. More specifically, let $f : [0, 1] \rightarrow [\underline{d}, \overline{d}]$ be a fixed demand rate function that is monotonically decreasing, and $x_T \in (\underline{d}, \overline{d})$ be an inventory ratio parameter. The price-based revenue management model consists of $T$ consecutive selling periods, with an initial inventory level of $y_0 = x_T T$. At time $t$, the retailer sets a price $p_t \in [0, 1]$. The realized demand $d_t$, instantaneous
revenue \( r_t \), and remaining inventory level \( y_t \) are governed by the following model:

\[
d_t = f(p_t) + \xi_t, \quad r_t = p_t \min\{d_t, y_{t-1}\}, \quad y_t = \max\{0, y_{t-1} - d_t\},
\]

where \( \xi_1, \cdots, \xi_T \) i.i.d. \( \sim Q \) are i.i.d. centered additive noise variables.

The retailer’s objective is to design an admissible pricing policy \( \pi \) to maximize his/her expected revenue over \( T \) periods. A pricing policy \( \pi \) is admissible if the advertised price \( p_t \) at time \( t \) is decided based only on the inventory level at the beginning of the \( t \)th time period. Mathematically, an admissible policy \( \pi \) can be parameterized as \( \pi = (\pi_1, \cdots, \pi_T) \), where \( \pi_t \) is a certain random function that maps from \( y_{t-1} \) to \( p_t \in [0,1] \). The expected revenue of an admissible policy \( \pi \) can then be written as

\[
R^\pi(T, y_0) := \mathbb{E}^\pi \left[ \sum_{t=1}^T r_t \bigg| p_t \sim \pi_t(y_{t-1}) \right].
\]

1.1 Existing results on self-adjusting controls

An optimal policy \( \pi^* \) maximizing \( R^\pi(T, x_T) \) defined in Eq. (2) can be in principle obtained via dynamic programming. Such an approach, however, is computationally very expensive because there are an infinite number of states (inventory levels) and actions (advertised prices). Although discretization is possible, it is not an exact solution and soon becomes intractable when the discretization grid becomes too dense. Furthermore, with multiple products for sale (e.g., network revenue management) the number of states and prices grow exponentially and the approach is therefore intractable.

The seminal work of Gallego and Van Ryzin (1994) proposed a useful and easy-to-compute benchmark for understanding and developing approximately optimal dynamic pricing control protocols. Suppose the inverse function of \( f \) exists and let \( x^* = \arg \max_{x \in [d, y_T]} r(x) \), where \( r(x) = xf^{-1}(x) \).

The following results are established in (Gallego and Van Ryzin 1994):

**Theorem 1 (Gallego and Van Ryzin (1994)).** For any admissible policy \( \pi \) and \( y_0 = x_TT \), \( R^\pi(T, y_0) \leq Tr(\min\{x_T, x^*\}) \). Furthermore, for the static pricing policy \( \pi^* : p_t = f^{-1}(\min\{x_T, x^*\}) \), \( R^{\pi^*}(T, y_0) \geq Tr(\min\{x_T, x^*\}) - O(\sqrt{T}) \).

It has been an interesting question to further improve the \( O(\sqrt{T}) \) gap in Theorem 1 by considering more sophisticated yet still computationally efficient dynamic pricing strategies. In the work of Jasin (2014) the gap is reduced from \( O(\sqrt{T}) \) to \( O(\ln T) \), as shown by the following result:

**Theorem 2 (Jasin (2014)).** Let \( \pi^r = (\pi^r_1, \cdots, \pi^r_T) \) be a re-optimizing pricing strategy defined as \( p_t = f^{-1}(\max(d, \min\{y_{t-1}/(T-t+1), x^*\})) \). Then \( R^{\pi^r}(T, y_0) \geq Tr(\min\{x_T, x^*\}) - O(\ln T) \), where \( y_0 = x_TT \).

Although Theorems 1 and 2 are often studied in the context of network revenue management in
which more than one products are present, for even the simplest single-product case it is still open whether the $O(\ln T)$ gap in Theorem 2 can be further reduced. This question is answered from both upper bound and lower bound sides in this paper, as we present in the next section.

1.2 Our results: nearly bounded regret and logarithmic gaps

In this paper we establish two main results: nearly bounded regret for the re-solving heuristic, and logarithmic regret lower bounds on the gap between the value of the optimal pricing policy and the static LP relaxation. Figure 1 summarizes results established in this paper (in red) and compares them with existing results in the prior literature (in blue).

Our first main result, as stated in Theorem 3 later in this paper, asserts that for any fixed $x_T \in (d, x^*)$ the cumulative regret of the re-solving heuristic $\pi^r$ is upper bounded by an iterated logarithmic term $O(\ln \ln T)$, compared against the expected reward of the optimal dynamic pricing policy $\pi^*$. Apart from the obvious improvement from $O(\ln T)$ to $O(\ln \ln T)$ in asymptotic regret upper bounds, our proof technique is different from existing works which compares the expected reward of the re-solving policy to a certain information relaxed benchmark, such as the static LP solution or the hindsight optimum benchmark. In contrast, because most benchmarks in the price-based revenue management setting are likely to be loose, we compare the value of $\pi^r$ directly with the value of the optimal DP policy $\pi^*$ by carefully analyzing the demand correction structures in $\pi^*$.

Our second main result, as stated in Theorem 4 later in this paper, shows that there is an $\Omega(\ln T)$ lower bound on the gap between the expected revenue of the re-solving heuristics $\pi^r$ and the static LP relaxation benchmark $Tr(\min\{x^*, x_T\})$. Coupled with the $O(\ln \ln T)$ regret upper bound established in Theorem 3, this shows that there is an $\Omega(\ln T)$ lower bound on the gap between the value of the optimal policy $\pi^*$ and the static LP relaxation as well. This demonstrates the fundamental limitation of analysis conducted using the static LP relaxation or other similar criteria because these information relaxed benchmarks give the pricing policy too much information ahead
of time and are therefore too loose for price-based revenue management problems.

2 Related work

The most relevant prior research to our paper is the work by Jasin (2014), who studied the network revenue management problem and showed that a re-optimization heuristic attains an $O(\ln T)$ asymptotic regret upper bound under mild conditions. Jasin (2014) also shows that infrequent re-solving has similar theoretical performance guarantees and is much more computationally efficient. In this paper, we improve the regret of frequent re-solving to $O(\ln \ln T)$, which is an iterated logarithmic term in time horizon $T$ and is very close to bounded. Our analysis is different from the one in (Jasin 2014) in the sense that we directly compare the expected revenue of re-solving with the value of the optimal DP policy, instead of a static LP relaxation. Additionally, we complement our results with an $\Omega(\ln T)$ lower bound between the expected revenue of the optimal DP policy and the static LP relaxation.

The idea of using simple, easy-to-compute pricing policies to approximate the optimal dynamic pricing strategy originates from the works of Gallego and Van Ryzin (1994, 1997), who studied static price policies. Maglaras and Meissner (2006) showed that frequent resolves does not diminish revenue asymptotically. Chen and Farias (2013) studied a single-product pricing problem under a specific class of demand models, and showed that re-optimization strictly improves the asymptotic performance compared to static price strategies. Due to the modeling difference the results in (Chen and Farias 2013) are not directly comparable with our setting. More specifically, Chen and Farias (2013) studied a market-size stochastic process that models inter-temporal correlations and non-stationarity in demand. As a result, the model in Chen and Farias (2013) is harder and therefore weaker performance guarantees are derived. In the model of (Chen and Farias 2013) the competitive ratio of the static price strategy is $O(1/\ln T)$ while the competitive ratio of re-optimization is around 0.5 when properly tuned; in contrast in the model considered in our paper (independent and stationary demands) the static price strategy has a $1 – O(\sqrt{T}/T)$ competitive ratio while re-optimization has a $1 – O(\ln \ln T/T)$ competitive ratio.

Re-solving has also been studied in several other settings such as quantity-based revenue management, for example in (Reiman and Wang 2008, Cooper 2002, Secomandi 2008, Jasin 2014, Jasin and Kumar 2013, Bumpensanti and Wang 2020, Wu et al. 2015). The quantity-based revenue management model exhibits some quite different structures from the price-based model we study, such as the fact that re-optimization having the potential of lowering the expected revenue, and the possibility of achieving bounded regret (i.e., $O(1)$ regret) by using hindsight-optimum (HO) benchmarks. Vera et al. (2019) studied a price-based revenue management model with a finite set of candidate prices.
A related yet significantly different problem is dynamic pricing with demand learning, in which the underlying demand rate function is unknown and needs to be learnt in the pricing process. Some representative recent works include (Besbes and Zeevi 2009, Keskin and Zeevi 2014, Wang et al. 2014, Besbes and Zeevi 2015, Broder and Rusmevichientong 2012, Cheung et al. 2017, Lei et al. 2014), and many more. In contrast, in this paper the retailer is assumed to have full information about the underlying demand distributions. Because of the retailer’s full information about the demand function, lower bounds/negative results are proved using completely different techniques from the lower bounds in (Broder and Rusmevichientong 2012, Wang et al. 2019), which rely on the customers’ lack of knowledge about the underlying demand function.

3 Main results

We make the following standard assumptions throughout this paper.

1. The demand rate function $f: [0, 1] \rightarrow [\bar{d}, \underline{d}]$ with $f(0) = \bar{d}$, $f(1) = \underline{d} > 0$ is strictly decreasing and admits a unique inverse function $g = f^{-1}$. Furthermore, there exists constant $C < \infty$ such that $|f(p) - f(p')| \leq C|p - p'|$ for all $p, p' \in [0, 1]$, and $|g(d) - g(d')| \leq C|d - d'|$ for all $d, d' \in [\bar{d}, \underline{d}]$.

2. The expected revenue $r(d) = df^{-1}(d)$ as a function of the demand rate $d$ is concave and three times continuously differentiable, with $\sup_d |r'(d)| + |r''(d)| + |r'''(d)| < \infty$. Furthermore, there exist constants $0 < m \leq M < \infty$ such that $m^2 \leq -r''(d) \leq M^2$ for all $d \in [\bar{d}, \underline{d}]$.

3. The noise variables $\{\xi_t\}_{t=1}^T$ are i.i.d. sampled from an underlying distribution $Q$. Furthermore, $\mathbb{E}_Q[\xi_t] = 0$, $|\xi_t| \leq B\xi$ almost surely for some constant $0 < B\xi \leq \bar{d}$, and $\mathbb{E}_Q[\xi_t^2] > 0$. Note this also ensures that the realized demands are non-negative almost surely.

3.1 Nearly bounded regret of re-solving heuristics

When the (normalized) initial inventory level $x_T$ exceeds the optimal demand rate $x^* = \arg \max_{x \in [\bar{d}, \underline{d}]} r(x)$, it is easy to verify that the stationary policy $\pi^s : p_t \equiv f^{-1}(x^*)$ has constant regret for sufficiently large $T$ in this case.

**Proposition 1.** Suppose $x_T \in (x^*, \bar{d})$ and let $\pi^s : p_t \equiv f^{-1}(x^*)$ be the stationary pricing policy defined in Theorem 1. Then for sufficiently large $T$, $R^s(T, x_T T) \geq Tr(x^*) - O(1) \geq R^s(T, x_T T) - O(1)$.

**Proof of Proposition 1.** Define $\mathcal{F} := \{\forall t, \sum_{\tau=1}^t \xi_T \leq T(x_T - x^*)\}$ be the event that the initial inventory is not completely depleted throughout the $T$ selling periods. By Hoeffding’s inequality, for every $t$ it holds that $\Pr[|\sum_{\tau=1}^t \xi_T \leq T(x_T - x^*)|] \leq O(\exp\{-\frac{T^2(x_T - x^*)^2}{t}\}) \leq O(\exp\{-T(x_T - x^*)^2\})$. 

5
With a union bound over all \( t = 1, 2, \cdots, T \), we have \( \Pr[\mathcal{F}] \geq 1 - O(T e^{-\Omega(T)}) \). Subsequently, with the definition of \( \tilde{\xi} = \frac{1}{T} \sum_{t=1}^{T} \xi_t \) and \( \mathbb{E}[\tilde{\xi}] = 0 \), \( |\tilde{\xi}| \leq B_\xi = O(1) \) a.s. we have

\[
R^{\pi^*}(T, x_T T) \geq \mathbb{E}[\sum_{t=1}^{T} r_t \mathbbm{1}\{\mathcal{F}\}] \geq Tr(x^*) \mathbb{P}[\mathcal{F}] - T \mathbb{E}[\tilde{\xi}\mathbbm{1}\{\mathcal{F}\}] = Tr(x^*) \mathbb{P}[\mathcal{F}] - T \mathbb{E}[\tilde{\xi}\mathbbm{1}\{\mathcal{F}^c\}]
\]

\[
\geq Tr(x^*)(1 - O(T e^{-\Omega(T)})) - O(T^2 e^{-\Omega(T)}) = Tr(x^*) - O(1),
\]

which is to be proved. \( \square \)

The case of insufficient inventory \( x_T \in (d, x^*) \), on the other hand, is much more complicated. The stationary policy \( \pi^* \equiv f^{-1}(x_T) \) typically suffers \( \Omega(\sqrt{T}) \) regret. On the other hand, the work of Jasin (2014) established that the regret of \( \pi^* \) when measured against \( Tr(x_T) \) is at most \( O(\ln T) \).

Our next theorem improves the regret to the iterated logarithm by switching from the static LP benchmark \( Tr(x_T) \) directly to the expected revenue of the optimal DP policy.

**Theorem 3.** Suppose \( x_T \in (d, x^*) \) and let \( \pi^r \) be the re-solving policy defined in Theorem 2. Let also \( \pi^* \) be the optimal DP pricing policy. For sufficiently large \( T \), it holds that \( R^{\pi^r}(T, x_T T) \geq R^{\pi^*}(T, x_T T) - O(\ln \ln T) \).

Theorem 3 is the main result of this section and its proof is given in Sec. 4. Note that instead of comparing with the static LP benchmark \( Tr(x_T) \), Theorem 3 compares the value of \( \pi^r \) directly with the optimal DP pricing policy \( \pi^* \), allowing for tighter regret bounds. On the other hand, the \( O(\ln \ln T) \) regret gap does not hold when compared against the \( Tr(x_T) \) benchmark, as we shall establish in the next section.

### 3.2 Logarithmic regret of the static LP benchmark

In this section, we show that the regret of the re-solving policy \( \pi^r \) measured against the static LP benchmark \( Tr(x_T) \) (in the insufficient inventory case) is at least logarithmic \( \Omega(\ln T) \).

**Theorem 4.** Suppose \( x_T \in (d, x^*) \) and let \( \pi^r \) be the re-solving policy defined in Theorem 2. For sufficiently large \( T \), it holds that \( R^{\pi^r}(T, x_T T) \leq Tr(x_T) - \Omega(\ln T) \).

Theorem 4 is the main result of this section and is proved later in Sec. 4. Because \( R^{\pi^*}(T, x_T T) \) is naturally an upper bound on \( R^{\pi^r}(T, x_T T) \), Theorem 4 shows that there is a logarithmic lower bound \( \Omega(\ln T) \) between the value of the optimal DP pricing policy and the static LP relaxation.

In the prior works of (Bumpensanti and Wang 2020, Vera et al. 2019) benchmarks weaker than the static LP relaxation are considered too, such as the “hindsight optimum” benchmark which assumes the pricing policy has knowledge of the average realized demands in later time periods. In the appendix of this paper we show that a popular version of the hindsight optimum benchmark has \( O(1) \) regret when measured against the static LP benchmark \( Tr(x_T) \), and is therefore also \( \Omega(\ln T) \) away from the value of the optimal DP policy.
Table 1: Notations used in the proof.

| Notation | Definition | Meaning |
|-----------|------------|---------|
| $\phi^*_t(x)$ | $\phi^*_t(x) = R^*_t(t, xt)$ | reward of $\pi^*$ with $t$ periods and $xt$ inventory |
| $\phi_t^r(x)$ | $\phi_t^r(x) = R_t^r(t, xt)$ | reward of re-solving with $t$ periods and $xt$ inventory |
| $z_\tau$ | $z_\tau = f(p_\tau)$ | the expected demand at time $\tau$ |
| $\xi_\tau$ | $\xi_\tau \sim Q$, $|\xi_\tau| \leq B_\xi$ a.s., $E[\xi_\tau] = 0$ | the stochastic demand noise at time $\tau$ |
| $x^*_\tau, x^r_\tau$ | remaining inventory divide by $\tau$ | normalized inventory levels under policy $\pi^*$ and $\pi^r$ |
| $\Delta_\tau$ | See Eq. (3) | the optimal demand correction with $\tau$ periods remaining |
| $\Delta_{\rightarrow t}$ | $\frac{\Delta_t}{t} + \frac{\Delta_{t-1}}{t-1} + \cdots + \frac{\Delta_1}{1}$ | harmonic series of demand corrections up to $t$ |
| $\xi_{\rightarrow t}$ | $\frac{\xi_t}{t} + \frac{\xi_{t-1}}{t-1} + \cdots + \frac{\xi_1}{1}$ | harmonic series of demand noises up to $t$ |
| $T^\pi$ | See Eq. (4) | stopping time that $\{x^*_\tau, x^r_\tau\}_{\tau \leq T^\pi}$ are well-behaved |

4 Proofs

To present our proof we first define some notations. Let $\phi^*_t(x) = R^*_t(t, xt)$ and $\phi_t^r(x) = R_t^r(t, xt)$ be the expected cumulative revenue of the optimal DP pricing policy $\pi^*$ and the re-solving policy $\pi^r$, respectively. For $\tau \leq t$, let $x^*_\tau$ and $x^r_\tau$ be the random variables of the normalized inventory levels under policy $\pi^*$ and $\pi^r$ when there are $\tau$ time periods remaining. Let also $p_\tau, z_\tau, \xi_\tau$ be the price, expected demand and stochastic demand noises at time $\tau$. These notations are summarized in Table 1, with some additional notations being defined later as the proof proceeds.

The rest of this section of proof is organized as follows. In the first two sub-sections we establish some properties of the optimal policy $\pi^*$ and the re-solving policy $\pi^r$. More specifically, we establish upper and lower bounds of the expected rewards $\phi^*_t(\cdot), \phi_t^r(\cdot)$ using the key quantities of $\{\Delta_{\rightarrow t}\}$ (harmonic series of optimal demand corrections), $\{\xi_{\rightarrow t}\}$ (harmonic series of stochastic noise variables) and $T^\pi$ (a carefully defined stopping time to ensure that the process is well-behaved before $T^\pi$). We then proceed with the proofs of Theorems 3, 4 by carefully analyzing the differences in the expansions of $\phi^*_t(\cdot), \phi_t^r(\cdot)$.

4.1 Properties of the optimal policy $\pi^*$

For any $\tau \geq 1$ and $x^*_\tau \geq \underline{d}/t$, the value of the optimal policy $\pi^*$ is defined by the following value iteration formula:

$$
\phi^*_{\tau}(x^*_\tau) = \max_{\Delta} r(x^*_\tau + \Delta_\tau) + \mathbb{E}
\phi^*_{\tau-1}\left(x^*_{\tau-1} - \frac{\Delta + \xi_\tau}{\tau - 1}\right) = r(x^*_\tau + \Delta_\tau) + \mathbb{E}\phi^*_{\tau-1}(x^*_{\tau-1}),
$$

(3)

where $x^*_{\tau-1} = x^*_\tau - \frac{\Delta + \xi_\tau}{\tau - 1}$, and the maximization of $\Delta$ is subject to the constraint that $x^*_\tau + \Delta \in [\underline{d}, \overline{d}]$. The random variable $\Delta_\tau$ is thus defined as the maximizing parameter of $\Delta_\tau$ which in turn depends on the random variable of $x^*_\tau$.

For any $t < T$ let $\overline{\Delta}_{\rightarrow t} := \frac{\Delta_t}{t} + \frac{\Delta_{t-1}}{t-1} + \cdots + \frac{\Delta_1}{1}$ and $\overline{\xi}_{\rightarrow t} := \frac{\xi_t}{t} + \frac{\xi_{t-1}}{t-1} + \cdots + \frac{\xi_1}{1}$. (For $t = T$
Define $\Delta_{\rightarrow t} = \xi_{\rightarrow t} = 0$. Define stopping time $T^\sharp$ as

$$T^\sharp := \max \left( \{ [\ln^2 T] \} \cup \{ t : \max(|\Delta_{\rightarrow t} + \xi_{\rightarrow t}|, |\xi_{\rightarrow t}|) > \min(x_T - d, x^* - x_T)/2 \} \right),$$

where $x_T \in (d, x^*)$ is the normalized initial inventory level (i.e., the initial inventory level is $x_T T$) and $x^* = \arg\max_{x \in [d, T]} r(x)$ is the optimal price without inventory considerations. Intuitively, $T^\sharp$ is the first time mark at which the remaining inventory level is either too low or too high. It is easy to verify that, as $t = T, T - 1, \cdots, 1$, the random variable $T^\sharp$ is a stopping time since it only depends on $\Delta_{\rightarrow t}$ and $\xi_{\rightarrow t}$, both of which are available at time $t$. It then holds that

$$x^*_\tau = x_T - \Delta_{\rightarrow \tau} - \xi_{\rightarrow \tau}, \quad \forall \tau \geq T^\sharp,$$

where $x^*_\tau$ is the random variable of the total inventory level when there are $\tau$ time periods remaining under policy $\pi^*$.

**Lemma 1.** Let $x_T \in (0, x^*)$ and $T^\sharp, \{x^*_\tau, \Delta_{\tau} \}_{\tau \leq T^\sharp}$ be defined in Eqs. (3,4,5). Then

$$\phi^*_T(x_T) \leq \mathbb{E} \left[ \sum_{\tau = T^\sharp + 1}^T r(x^*_\tau + \Delta_{\tau}) + T^\sharp r(x^*_T) \right].$$

**Proof of Lemma 1.** The reward collected in time periods $T, T - 1, \cdots, T^\sharp + 1$ are $\sum_{\tau = T^\sharp + 1}^{\tau = T^\sharp + 1} [r(x^*_\tau + \Delta_{\tau}) + f^{-1}(x^*_\tau + \Delta_{\tau})]\xi_{\tau}$. When there are $T^\sharp$ periods remaining the random variable of total remaining inventory level is $x^*_T$. By definition of $T^\sharp$ and the fact that $T^\sharp \geq [\ln^2 T]$, it is clear that $x^*_T \in [d, x^*]$ for sufficiently large $T$. It is a well-known result that (see Theorem 1 in this paper, or (Gallego and Van Ryzin 1994)) $\phi^*_T(x) = T^\sharp r(x)$ for all $x \in [d, x^*]$. Subsequently,

$$\phi^*_T(x_T) \leq \mathbb{E} \left[ \sum_{\tau = T^\sharp + 1}^T [r(x^*_\tau + \Delta_{\tau}) + f^{-1}(x^*_\tau + \Delta_{\tau})]\xi_{\tau}] + T^\sharp r(x^*_T) \right] = \mathbb{E} \left[ \sum_{\tau = T^\sharp + 1}^T r(x^*_\tau + \Delta_{\tau}) + T^\sharp r(x^*_T) \right],$$

where the second equality holds because $\mathbb{E} [\sum_{\tau = T^\sharp + 1}^T f^{-1}(x^*_\tau + \Delta_{\tau})\xi_{\tau}] = 0$ thanks to the Doob’s optimal stopping theorem.

**Lemma 2.** For $x_T \in (0, x^*)$, it holds that $\mathbb{E}[T^\sharp] = O(\ln^2 T)$.

**Remark 1.** In the $O(\cdot)$ notation in Lemma 2 we omit constants depending on $x_T, x^*$ and $r(\cdot)$.

**Proof of Lemma 2.** Fix any $t \leq T$, note that $\xi_{\rightarrow t}$ is the sum of $(T-t)$ centered independent random variables with variance $\mathbb{E}[|\xi_{\rightarrow t}|^2] = \sum_{\tau = t+1}^{\tau = T} O(1/(\tau - 1)^2) = O(1/t)$. Note also that each $|\xi_{\tau}|/(\tau - 1)$ are bounded by $B\xi/t$ almost surely. By Bernstein’s inequality, with probability $1 - \delta$ it holds that

$$|\xi_{\rightarrow t}| \leq O(t^{-1}\ln(1/\delta)) + O(t^{-1/2}\sqrt{\ln(1/\delta)}).$$
Let $T_0 = |\ln^2 T|$. With the above inequality and the union bound, we have for sufficiently large $T$ that

$$\Pr \left[ \forall t \geq T_0, \; |\xi_{-t}| \leq \min(x_T - d, x^* - x_T)/4 \right] \geq 1 - O(T^{-2}). \tag{6}$$

Let $\mathcal{E}$ be the event that $\forall t \geq T_0 = |\ln^2 T|$, $|\xi_{-t}| \leq \min(x_T - d, x^* - x_T)/4$. Eq. (6) shows that $\Pr[\mathcal{E}^c] = O(T^{-2})$. Lemma 1 and the law of total expectation imply that

$$\phi^*_T(x_T) \leq \mathbb{E} \left[ \left( \sum_{\tau=T^2+1}^{T^2} r(x^*_\tau + \Delta_\tau) + T^2 r(x^*_T) \right) 1{\{\mathcal{E}\}} \right] + O(1). \tag{7}$$

Next, consider arbitrary $\tau \geq T^2 + 1$. Using the smoothness and concavity of $r(\cdot)$, it holds that

$$r(x^*_\tau + \Delta_\tau) = r(x_T - \Delta_{\tau} - \xi_{-\tau} + \Delta_\tau) \leq r(x_T) + r'(x_T)(-\Delta_{\tau} - \xi_{-\tau} + \Delta_\tau). \tag{8}$$

Similarly, for $x^*_T = x_T - \Delta_{T^2} - \xi_{T^2}$ it holds that

$$r(x^*_T) \leq r(x_T) + r'(x_T)(-\Delta_{T^2} - \xi_{T^2}) - \frac{m^2}{2} |\Delta_{T^2} + \xi_{T^2}|^2. \tag{9}$$

Combining Eqs. (8,9) we have that

$$\sum_{\tau=T^2+1}^{T^2+1} r(x^*_\tau + \Delta_\tau) + T^2 r(x^*_T) \leq Tr(x_T) + r'(x_T) \left[ \sum_{\tau=T^2+1}^{T^2} (-\Delta_{\tau} - \xi_{-\tau} + \Delta_\tau) - T^2 \Delta_{T^2} - T^2 \xi_{T^2} \right] - \frac{m^2}{2} T^2 |\Delta_{T^2} + \xi_{T^2}|^2$$

$$= Tr(x_T) - r'(x_T) \left[ \sum_{\tau=T^2+1}^{T^2} \xi_{\tau} \right] - \frac{m^2}{2} T^2 |\Delta_{T^2} + \xi_{T^2}|^2. \tag{10}$$

By the law of total expectation, for every $\tau$ it holds that $|\mathbb{E}[\xi_T 1\{\mathcal{E}\}]| = |\mathbb{E}[\xi_T 1\{\mathcal{E}^c\}]| = O(T^{-2})$. Combining Eqs. (10) and (7), we have

$$\phi^*_T(x_T) \leq Tr(x_T) + O(1) - \frac{m^2}{2} \mathbb{E}[T^2 |\Delta_{T^2} + \xi_{T^2}|^2 1\{\mathcal{E}\}]$$

$$\leq Tr(x_T) + O(1) - \frac{m^2}{2} \times \frac{\min(x_T - d, x^* - x_T)^2}{16} \mathbb{E}[T^2 1\{(T^2 > |\ln^2 T|) \cap \mathcal{E}\}] \tag{11}$$

$$= Tr(x_T) + O(1) - \Omega(1) \times \mathbb{E}[T^2 1\{(T^2 > |\ln^2 T|) \cap \mathcal{E}\}].$$

Here, Eq. (11) holds because $T^2 > |\ln^2 T|$ implies that $|\xi_{T^2} + \Delta_{T^2}| \geq \min(x_T - d, x^* - x_T)/2$, which further implies $|\Delta_{T^2}| \geq \min(x_T - d, x^* - x_T)/4$ because $|\xi_{T^2}| \leq \min(x_T - d, x^* - x_T)/4$ conditioned on $\mathcal{E}$. On the other hand, the results of (Jasin 2014) shows that $\phi^*_T(x_T) \geq Tr(x_T) - O(\ln T)$. Subsequently,

$$\mathbb{E}[T^2 1\{(T^2 > |\ln^2 T|) \cap \mathcal{E}\}] = O(\ln T).$$
Consequently,
\[
\mathbb{E}[T^\sharp] = \mathbb{E}[T^\sharp \mathbf{1}\{(T^\sharp > \lceil \ln^2 T \rceil) \cap \mathcal{E}\}] + \mathbb{E}[T^\sharp \mathbf{1}\{(T^\sharp = \lceil \ln^2 T \rceil) \cap \mathcal{E}\}] + \mathbb{E}[T^\sharp \mathbf{1}\{\mathcal{E}^c\}] \\
\leq O(\ln T) + O(\ln^2 T) + O(1) = O(\ln^2 T),
\]
which is to be demonstrated. \qed

4.2 Properties of the re-solving heuristics \(\pi^r\)

For any \(\tau \geq 1\) and \(x^r_\tau \in [d/T, x^*]\), the value of the re-solving policy \(\pi^r\) can be written as
\[
\phi^r_t(x^r_\tau) = r(x^r_\tau) + \mathbb{E}\left[\phi^r_{t-1}\left(x^r_\tau - \frac{\xi^r_\tau}{\tau - 1}\right)\right]. \tag{12}
\]

Note that Eq. (12) does not hold for \(x^r_\tau > x^*\), in which case the re-solving policy \(\pi^r\) would commit to \(z_\tau = x^*\) instead of \(z_\tau = x^r_\tau\). Comparing Eq. (12) with Eq. (3), we remark that the re-solving heuristics \(\pi^r\) is the special case of the dynamic programming with decision rule \(\Delta_\tau \equiv 0\) for all \(x^r_\tau \leq x^*\).

Recall the definition of the stopping time \(T^\sharp\) in Eq. (4). Because of the upper bound \(|\xi^r_{t+1}| \leq \min\{x_T - d, x^* - x_T\}/2\) for all \(t > T^\sharp\), we have that
\[
x^r_\tau = x_T - \xi^r_\tau, \quad \forall \tau \geq T^\sharp. \tag{13}
\]

**Lemma 3.** Let \(x_T \in (0, x^*)\) and \(T^\sharp, \{x^r_\tau\}_{\tau \geq T^\sharp}\) be defined in Eqs. (4,13). Then
\[
\mathbb{E}\left[\sum_{\tau=T^\sharp+1}^{T} r(x^r_\tau) + T^\sharp r(x^r_{T^\sharp}) - O(\ln T^\sharp)\right] \leq \phi^r_T(x_T) \leq \mathbb{E}\left[\sum_{\tau=T^\sharp+1}^{T} r(x^r_\tau) + T^\sharp r(x^r_{T^\sharp})\right].
\]

**Proof of Lemma 3.** Suppose when there are \(T^\sharp\) time periods left the remaining inventory level is \(x^r_{T^\sharp} T^\sharp\) for some \(x^r_{T^\sharp} \in (0, x^*)\). The static LP relaxation claims that \(\phi^r_{T^\sharp}(x^r_{T^\sharp}) \leq T^\sharp r(x^r_{T^\sharp})\). On the other hand, the results of Jasin (2014) asserts that the re-solving heuristics has logarithmic regret compared against the static LP benchmark, or more specifically \(\phi^r_{T^\sharp}(x^r_{T^\sharp}) \geq T^\sharp r(x^r_{T^\sharp}) - O(\ln T^\sharp)\). The rest of the proof is identical to the proof of Lemma 1. \qed
4.3 Proof of Theorem 3

In this section we prove Theorem 3. By Lemmas 1 and 1, for any fixed \(x_T \in (d, x^*)\) and sufficiently large \(T\) it holds that

\[
\phi_T^*(x_T) \leq E\left[\sum_{\tau=T+1}^{T} r(x^*_\tau + \Delta_\tau) + T^2 r(x^*_\tau)\right]; \tag{14}
\]

\[
\phi_T^*(x_T) \geq E\left[\sum_{\tau=T+1}^{T} r(x^*_\tau) + T^2 r(x^*_\tau) - O(\ln T^2)\right], \tag{15}
\]

where \(x^*_\tau = x_T - \overline{\Delta}_{\tau} - \overline{\xi}_{\tau}, x^*_r = x_T - \overline{\xi}_{\tau}\) for all \(\tau \geq T^2, \overline{\Delta}_{\tau} = \frac{\Delta T}{T-1} + \cdots + \frac{\Delta T}{\tau}, \overline{\xi}_{\tau} = \frac{\xi_T}{T-1} + \cdots + \frac{\xi_T}{\tau},\) and \(T^2\) is the stopping time defined in Eq. (4).

For any \(\tau \geq T^2\), by the smoothness and concavity of \(r(\cdot)\) it holds that

\[
r(x^*_\tau + \Delta_\tau) - r(x^*_\tau) \leq r'(x_T)(\Delta_\tau - \overline{\Delta}_{\tau}) r''(x_T)\overline{\xi}_{\tau} \overline{\Delta}_{\tau} - - - \overline{\Delta}_{\tau} - \overline{\Delta}_{\tau}^2
\]

\[
\leq r'(x_T)[\Delta_\tau - \overline{\Delta}_{\tau}] - r''(x_T)\overline{\xi}_{\tau} \overline{\Delta}_{\tau} + O(\overline{\xi}_{\tau}^2)\overline{\Delta}_{\tau} - \overline{\Delta}_{\tau}^2
\]

\[
\leq r'(x_T)[\Delta_\tau - \overline{\Delta}_{\tau}] - r''(x_T)\overline{\xi}_{\tau} \overline{\Delta}_{\tau} + \frac{1}{2m} O(\overline{\xi}_{\tau}^4). \tag{16}
\]

Similarly, for \(\tau = T^2\), we have

\[
r(x^*_T) - r(x^*_T) \leq -r'(x_T)\overline{\Delta}_{T^2} + r''(x_T)\overline{\xi}_{T^2} \overline{\Delta}_{T^2} + \frac{1}{2m} O(\overline{\xi}_{T^2}^4). \tag{17}
\]

Combining Eqs. (16,17) we obtain

\[
\phi_T^*(x_T) - \phi_T^*(x_T) \leq E\left[r'(x_T)A - r''(x_T)B + O(1) \times C - O(\ln T^2)\right], \tag{18}
\]

where random variables \(A, B, C\) are defined as

\[
A = \sum_{\tau=T+1}^{T} [\Delta_\tau - \overline{\Delta}_{\tau}] - T^2 \overline{\Delta}_{T^2},
\]

\[
B = \sum_{\tau=T+1}^{T} \overline{\xi}_{\tau} [\Delta_\tau - \overline{\Delta}_{\tau}] - T^2 \overline{\xi}_{T^2} \overline{\Delta}_{T^2},
\]

\[
C = \sum_{\tau=T+1}^{T} |\overline{\xi}_{\tau}|^4 + T^2 |\overline{\xi}_{T^2}|^4.
\]

We next analyze the three terms \(A, B, C\) separately. Recall the definition that \(\overline{\Delta}_{\tau} = \frac{\Delta T}{T-1} + \cdots + \frac{\Delta T}{\tau}\). With elementary algebra it is easy to verify that

\[
A = \sum_{\tau=T+1}^{T} [\Delta_\tau - \overline{\Delta}_{\tau}] - T^2 \overline{\Delta}_{T^2} = 0. \tag{19}
\]
For the $B$ term, re-organizing all terms for each $\xi_t$, $t > T^\sharp$, we obtain

$$
B = \sum_{t=T^\sharp+1}^T \frac{\xi_t}{t-1} \left[ \sum_{\tau=T^\sharp+1}^{t-1} (\Delta_{\tau} - \overline{\Delta}_{\tau}) - T^\sharp \overline{\Delta}_{\tau} \right] = \sum_{t=T^\sharp+1}^T \frac{\xi_t}{t-1} \left[ -(t-1)\overline{\Delta}_{t-1} \right] = -\sum_{t=T^\sharp+1}^T \xi_t \overline{\Delta}_{t-1}.
$$

Note that, because $\overline{\Delta} = \frac{\Delta_T}{T} + \cdots + \frac{\Delta_t}{t}$ involve demand corrections $\Delta_T, \Delta_{T-1}, \cdots, \Delta_t$ when there are at least $t$ periods remaining, it holds that $E[\xi_t \overline{\Delta}_{t-1}] = E[\xi_t] E[\overline{\Delta}_{t-1}] = E[\xi_t] E[Q] = 0$ since the DP policy must be non-anticipating. Therefore, by Doob’s optimal stopping theorem we have

$$
E[B] = E[\sum_{t=T^\sharp+1}^T \xi_t] = 0.
$$

Finally we upper bound (the expectation) of term $C$. Recall the definition that $\overline{\xi}_{t} = \frac{\xi_T}{T-1} + \cdots + \frac{\xi_t}{t-1}$. Clearly, $\overline{\xi}_t$ is the sum of centered, independently distributed random variables with $E[\overline{\xi}_t] = 0$ and $E[\overline{\xi}_t] = O(1/(T - 1)^2 + \cdots + 1/t^2) = O(1/t)$. Note also that each $|\xi_t/(\tau - 1)|$ term is upper bounded by $B_{\xi}/t$ almost surely. By Bernstein’s inequality, with probability $1 - \delta$ it holds that

$$
|\overline{\xi}_t| \leq O(t^{-1}\ln(1/\delta)) + O(t^{-1/2}\sqrt{\ln(1/\delta)}).
$$

Setting $\delta = 1/T^3$ and taking the union bound over all $t \geq T^\sharp$, it holds with probability $1 - O(T^{-2})$ that

$$
|\overline{\xi}_{t}| \leq O(t^{-1}\ln T + t^{-1/2}\sqrt{\ln T}), \quad \forall t \geq T^\sharp.
$$

Consequently, with probability $1 - O(T^{-2})$ we have

$$
C = \sum_{t=T^\sharp+1}^T |\overline{\xi}_t|^4 + T^\sharp |\overline{\xi}_{T^\sharp}|^4 \leq \sum_{t=T^\sharp+1}^T O(t^{-4}\ln^4 T + t^{-2}\ln^2 T) + T^3 \times O([T^\sharp]^{-4}\ln^4 T + [T^\sharp]^{-2}\ln^2 T)
$$

$$
\leq O \left( \frac{\ln^4 T}{[T^\sharp]^3} + \frac{\ln^2 T}{T^\sharp} \right) \leq O(1),
$$

where the last inequality holds because $T^\sharp \geq \ln^2 T$ almost surely. On the other hand, because $|\overline{\xi}_{t}| \leq B_{\xi} \ln T$ almost surely, we have that $C \leq O(T \ln T)$ almost surely. Therefore,

$$
E[C] \leq O(1) + O(T \ln T) \times O(T^{-2}) = O(1).
$$

Combining Eqs. (19,20,21) with Eq. (18), we have

$$
\phi_T^r(x_T) - \phi_T^r(x_T) \leq O(1) + O(E[\ln T^\sharp]) \leq O(\ln(E[T^\sharp])) \leq O(\ln \ln T),
$$

where the last inequality holds by applying Lemma 2. This completes the proof of Theorem 3.
4.4 Proof of Theorem 4

Recall that \( x_T^r = x_T - \xi \rightarrow r \) for all \( r \geq T^\sharp \), where \( T^\sharp \) is the stopping time defined in Eq. (4). Expanding the difference \( r(x_T^r) - r(x_T) \) at \( x_T \) and using the smoothness and concavity of \( r(\cdot) \), we have

\[
r(x_T^r) - r(x_T) \leq -r'(x_T)\xi \rightarrow r - \frac{m^2}{2} |\xi \rightarrow r|^2.
\]

Invoking Lemma 3, we have

\[
\phi_T^r(x_T) - Tr(x_T) \leq \mathbb{E}\left[ \sum_{\tau=Tr+1}^T (r(x_T^r) - r(x_T)) + T^\sharp (r(x_T^r) - r(x_T)) \right] \\
\leq -r'(x_T)\mathbb{E}\left[ \sum_{\tau=Tr+1}^T \xi \rightarrow r + T^\sharp \xi \rightarrow T^\sharp \right] - \frac{m^2}{2} \mathbb{E}\left[ \sum_{\tau=Tr+1}^T |\xi \rightarrow r|^2 + T^\sharp |\xi \rightarrow T^\sharp|^2 \right].
\]

For the first term in Eq. (22), we have

\[
\mathbb{E}\left[ \sum_{\tau=Tr+1}^T \xi \rightarrow r + T^\sharp \xi \rightarrow T^\sharp \right] = \mathbb{E}\left[ \sum_{\tau=Tr+1}^T \xi \right] = 0,
\]

where the last equality holds thanks to the Doob’s optimal stopping theorem. For the second term in Eq. (22), we have

\[
\mathbb{E}\left[ \sum_{\tau=Tr+1}^T |\xi \rightarrow r|^2 + T^\sharp |\xi \rightarrow T^\sharp|^2 \right] \geq \mathbb{E}\left[ \sum_{t=Tr+1}^T |\xi \rightarrow t|^2 \right] \geq \mathbb{E}\left[ \sum_{t=Tr+1}^T \Omega(1/t) \right] \\
\geq \Omega(\ln T - \mathbb{E}[\ln T^\sharp]) \geq \Omega(\ln T - \mathbb{E}[\ln(T^\sharp)]) = \Omega(\ln T),
\]

where Eq. (24) holds by the Doob’s optimal stopping theorem (since \( \mathbb{E}[|\xi \rightarrow t|^2] \) is a deterministic quantity) and Eq. (25) holds by applying Lemma 2 and Jensen’s inequality. Combining Eqs. (22,23,25) we complete the proof of Theorem 4.

5 Numerical results

We corroborate the theoretical findings in this paper with a simple numerical experiment. In the simulation we adopt a Bernoulli demand model \( \Pr[d_t = 1|p_t] = \alpha - \beta p_t, \Pr[d_t = 0|p_t] = 1 - \Pr[d_t = 1|p_t] \) with \( p \in [0, 1] \), \( \alpha = 3/4 \) and \( \beta = 1/2 \). The (normalized) initial inventory level is \( x_T = 5/16 \), meaning that for problem instances with \( T \) time periods the initial inventory level is \( x_T T = 5T/16 \). It is easy to verify that the optimal demand rate \( x^* \) without inventory constraints is \( x^* = 3/8 > x_T \), and the static LP relaxation suggests a \( Tr(x_T) = (19/32)T = .59375T \) expected revenue. We select the Bernoulli demand model because the states of inventory levels are discrete and therefore the optimal dynamic programming pricing policy can be exactly obtained.

In Table 2 we report the regret of the static LP relaxation, the optimal stationary policy \( \pi^* : p_t \equiv f^{-1}(x_T) = 7/8 \) and the re-solving heuristics \( \pi_r \). All regret is defined with respect to the value
Table 2: Regret for the static LP relaxation, the optimal stationary policy $\pi^s$ and the re-solving heuristics $\pi^r$ compared against the value of the optimal DP pricing policy.

| $\log_2 T$ | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|------------|----|----|----|----|----|----|----|----|----|----|
| Static LP relaxation | -0.90 | -1.13 | -1.37 | -1.63 | -1.91 | -2.19 | -2.48 | -2.78 | -3.08 | -3.37 |
| Stationary policy $\pi^s$ | 0.38 | 0.70 | 1.22 | 2.03 | 3.27 | 5.13 | 7.84 | 11.81 | 17.55 | 25.84 |
| Resolving heuristics $\pi^r$ | 0.11 | 0.15 | 0.18 | 0.21 | 0.23 | 0.23 | 0.24 | 0.24 | 0.24 | 0.25 |

Figure 2: Plots of regret of the static LP relaxation, the optimal stationary policy $\pi^s$ and the re-solving heuristics $\pi^r$ compared against the value of the optimal DP pricing policy.

We also plot the regret in Figure 2 to make the regret growth of each policy more intuitive. As we can see from Table 2, the gap between the value of the optimal policy and the value of the static LP relaxation grows nearly linearly as the number of time periods $T$ grows geometrically, which verifies the $\Omega(\ln T)$ growth rate established in Theorem 4. On the other hand, the growth of regret of the re-solving heuristics $\pi^r$ stagnated at $T \geq 2^{10}$ and is nearly the same for $T$ ranging from $2^{10} = 1024$ to $2^{15} = 32768$. This shows the asymptotic growth of regret of $\pi^r$ is far slower than $O(\ln T)$ and is compatible with the $O(\ln \ln T)$ regret upper bound we proved in Theorem 3.
6 Conclusion

In this paper, we analyze the re-solving heuristics in single-product price-based revenue management and establish two complementary theoretical results: that the re-solving heuristic attains $O(\ln \ln T)$ regret compared against the value of the optimal dynamic programming pricing policy, and there exists an $\Omega(\ln T)$ lower bound on the gap between the re-solving heuristics (as well as the expected revenue of the optimal policy) and the static LP relaxations.

Going forward, one obvious question is whether it is possible to further sharpen the regret upper bound from the iterated logarithm $O(\ln \ln T)$ to bounded regret $O(1)$, which is suggested to hold by the numerical results presented in the previous section. Technically speaking, the $O(\ln \ln T)$ term in our analysis arises from the expectation of the stopping time $T^\sharp$, which characterizes how well-behaved the normalized inventory levels are before $T^\sharp$. To further reduce the impact of $T^\sharp$ one needs to carefully analyze the behavior of both the optimal pricing policy and the re-solving heuristics in the cases when inventories run out too fast or not sufficiently fast so that the normalized inventory levels near the end of the $T$ selling periods fall outside of their typical ranges.

Appendix: the Hindsight-Optimum (HO) benchmark

The HO benchmark was adopted in (Bumpensanti and Wang 2020) to develop constant-regret re-optimizing algorithms for item-based network revenue management. Since in item-based network revenue management the demand rates are not affected by the (adaptively chosen) prices, the formulation in (Bumpensanti and Wang 2020) is not directly applicable to our setting. Instead, we formulate an HO benchmark following the strategy in (Vera et al. 2019) which also considered price-based revenue management with a finite subset of prices.

**Definition 1** (The HO benchmark). For any $p$ define random variable $D_T(p) := \sum_{t=1}^{T} d_t$ as the total realized demand with fixed price $p_t \equiv p$. A policy $\pi$ is HO-admissible if at time $t$, the price decision $p_t$ depends only on $\{p_{t'}, x_{t'}, d_{t'}\}_{t' < t}$ and $\{D_T(p)\}_{p \in [0, 1]}$. The HO-benchmark $R^{\text{HO}}(T, x_0)$ is defined as the expected revenue of the optimal HO-admissible policy $\pi$.

At a higher level, the HO-benchmark equips a policy with the knowledge of the total realized demand for each hypothetical fixed price $p \in [0, 1]$ in hindsight. Clearly, such policies are more powerful than an ordinary admissible policy which only knows the expected demand but not the realized demand for a specific price $p$.

Our next proposition shows that the HO-benchmark $R^{\text{HO}}(T, x_0)$ has a constant gap compared against the $T p^* f(p^*)$ oracle. Hence, it also has an $\Omega(\ln T)$ gap from the re-solving heuristic and the optimal DP solution.
Proposition 2. For any $x_T \in (d, x^*)$, it holds that $R^{HO}(T, y_0) \geq Tr(x_T) - O(1)$ where $y_0 = x_T T$.

The proof of Proposition 2 is straightforward and presented later. The conclusion in Theorem 4 then holds with $Tr(x_T)$ replaced by $R^{HO}(T, x_T T)$.

Proof of Proposition 2. It is clear that, in our setting of $\xi_1, \cdots, \xi_T$ being i.i.d., knowing $\{D_T(p)\}_{p \in [0, 1]}$ is equivalent to knowing $\bar{\xi} = \frac{1}{T} \sum_{t=1}^{T} \xi_t$, since $D_T(p) = T(f(p) + \bar{\xi})$ for all $p$. Now consider the policy of fixed prices $p_t \equiv g(x_T + \bar{\xi})$. The expected regret of such a policy can be bounded as

$$T\mathbb{E}_{\bar{\xi}}[(x_T + \bar{\xi}) f^{-1}(x_T + \bar{\xi})] - T x_T f^{-1}(x_T) = T\mathbb{E}_{\bar{\xi}}[r(x_T + \bar{\xi}) - r(x_T)] \geq T\mathbb{E}_{\bar{\xi}}[r'(x_T) \bar{\xi} - \frac{M^2}{2} \bar{\xi}^2]$$

$$= \frac{M^2}{2} T\mathbb{E}[^2] = \frac{M^2}{2} \times O(1) = O(1),$$

which is to be demonstrated.

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