Interplay between Non-Hermitian Skin Effect and Magnetic Field: Skin Modes Suppression, Onsager Quantization and $MT$ Phase Transition

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The non-Hermitian skin effect (NHSE) refers to the exponential localization of the bulk wave functions to the system boundary, which corresponds to a directional current flow under the periodic boundary condition. A magnetic field, on the contrary, pins charged particles in space as cyclotron motion. Here, we investigate the interplay between the two seemingly incompatible effects by the nonreciprocal Harper-Hofstadter model. Our main findings are as follows. First, the magnetic field can drive the skin modes into the bulk so as to suppress the NHSE. Second, the magnetic energy spectra are entirely real and partially complex under open and periodic boundary conditions, respectively. Interestingly, real spectra are preserved in the long-wavelength limit for both boundary conditions, indicating that the Onsager-Lifshitz quantization rule persists against the NHSE. Third, a real-to-complex spectral transition can be induced by the boundary parameter and the magnetic field, which stems from the spontaneous breaking of the underlying mirror-time reversal ($MT$) symmetry. An order parameter is introduced to quantify the symmetry breaking which is formulated by the average quantum distance induced by the $MT$ operation. Our work uncovers several intriguing effects induced by the fascinating interplay between the NHSE and the magnetic field, which can be implemented in a variety of physical systems.

Introduction—Non-Hermitian physics has attracted growing research interest recently for its intriguing properties and potential applications that can be implemented in various physical systems, including photonic systems [1–8], open quantum systems coupled to the environment [9–20], quasiparticles with finite lifetimes in condensed matter physics [21–24] and the engineered Laplacian in electrical circuits [25–27]. Moreover, the interplay between the non-Hermiticity and the band topology has been extensively studied which yields plentiful interesting results [28], such as the anomalous edge modes [29–33], the enriched classification of the topological insulating and superconducting phases [34,35], exotic semimetal phases [36–38] and the topological lasing [39–43]. It is now accepted that the conventional bulk-boundary correspondence principle for the Hermitian system fails [29,30] due to the so-called non-Hermitian skin effect (NHSE) [41,45]. Instead, the band topology should be described by the non-Bloch band theory [44].

The NHSE is a unique phenomenon due to the non-Hermiticity [44,59], which refers to that all the bulk states are driven to the edge of the system under the open boundary condition (OBC). As its counterpart under the periodic boundary condition (PBC), a persistent current flows inside of the system as shown in Fig. 1(a) [51]. Therefore, the NHSE can be considered as a particular type of delocalization effect which can even induce the localization phase transition [60–72]. The NHSE has been confirmed in recent experiments in a variety of physical systems [73–78].

In contrast to the physical picture of the NHSE, a magnetic field gives rise to the opposite effect. The motion of free charged particles in the magnetic field forms cyclotron orbits with the guiding centers localized in space; see Fig. 1(b). The quantization of these cyclotron orbits results in dispersionless Landau levels with vanishing mobility. The high degeneracy of the Landau levels also indicates that the corresponding eigenstates distribute uniformly in space. Therefore, the physical effects of the magnetic field seem completely incompatible with that due to the NHSE. Given that the magnetic field and the NHSE may coexist in a variety of real and artificial systems [79,80], it is of importance to study the fascinating interplay and competition between these two effects.

In this work, we investigate the combined effects of the NHSE and the magnetic field on the nonreciprocal Harper-Hofstadter model with unequal hopping strengths ($t \pm \delta_x$) in the $x$-direction and equal hopping ($t$) in the $y$-direction. (a) The semiclassic cyclotron motion of charged particles in the magnetic field.

![FIG. 1.](image-url)
Other parameters are set as 0 the flux quantum $\Phi$ modulated by the nonreciprocal hopping $h_{\perp}$, we set $\alpha$ PBC in the role in the presence of NHSE. We denote the OBC and $a$ through a lattice cell (lattice constant $a$) 58th eigenstates numbered by ascending Re($E$) and (c3,c4) 58th eigenstates numbered by ascending Re($E$) modulated by the nonreciprocal hopping $\delta_x$, with $B = 0.02$. Other parameters are set as $t = 0.5$ and $M = N = 50$.

yielding a suppression of the NHSE. The inverse action of the NHSE on the whole magnetic energy spectra strongly relies on the boundary condition in the x-direction, which remain entirely real and become partially complex under the OBC and PBC, respectively. Interestingly, the low-energy spectra are always real, regardless of the boundary conditions, and the Onsager-Lifshitz quantization rule holds true in this regime despite the NHSE. Finally, we show that the real-to-complex phase transition of the energy spectra can be described by the spontaneous breaking of the inherent mirror-time reversal ($MT$) symmetry. We propose an order parameter to accurately describe the $MT$-symmetry-breaking defined by the average quantum distance between the eigenstates before and after the $MT$ operation.

Model.-To be concrete, we study the nonreciprocal Harper-Hofstadter model [81, 82] on a square lattice under a magnetic field $B$ in the $x$-direction as

$$H = \sum_{m,n} t^+_x c^\dagger_{m+1,n} c_{m,n} + t^+_x c^\dagger_{m,n} c_{m+1,n} + \cos (k_y a + 2 \pi m \phi) c^\dagger_{m,k_y} c_{m,k_y} + t^+_x c^\dagger_{m,n} c_{m+1,n} + t^+_x c^\dagger_{m,n} c_{m,n+1},$$

where $c^\dagger_{m,n}$ ($c_{m,n}$) are the creation (annihilation) operator on the site $(m, n)$, $t^+_x = t \pm \delta_x$ describe the nonreciprocal hopping in the $x$-direction with $\delta_x$ the strength of nonreciprocity or non-Hermiticity. The phase factor $\phi = \Phi/\Phi_0$ is defined by the magnetic flux $\Phi = B a^2$ through a lattice cell (lattice constant $a$) divided by the flux quantum $\Phi_0 = h/e$. Here, the Landau gauge $A = (0, Bx)$ has been adopted. In the rest of the paper, we set $h = e = a = 1$ in all numerical calculations for simplicity. The boundary condition plays an essential role in the presence of NHSE. We denote the OBC and PBC in the $x$-direction ($x = x, y$) as $\alpha$-OBC and $\alpha$-PBC for brevity.

For $B = 0$, the system exhibits NHSE in the $x$-direction under the OBC in both directions [44] with the eigenfunction expressed as $\psi(x, y) = \psi^\dagger_{m,n} = (t^+_x/t^-_x)^{m/2} \sin^2(k_x x) \sin^2(k_y y)$ where $k_x = \pi t_x/(M + 1)$, $l_x = 1, ..., M$ and $k_y = \pi l_y/(N + 1)$, $l_y = 1, ..., N$, and $M(N)$ is the system size in the $x(y)$-direction, respectively. For $\delta_x = 0$, the wave functions reduce to the standing waves; The nonreciprocal hopping with $\delta_x > 0$ results in an envelope function modulation $(t^+_x/t^-_x)^{m/2}$ on top of the standing waves, which is nothing but the NHSE. As a result, all the wave functions are localized at the right boundary [cf. Fig. 2(a)]. The opposite limit with $\delta_x = 0$ and finite $B$, Eq. (1) reduces to the conventional Harper-Hofstadter model [81, 82], that yields the familiar butterfly energy spectra; see Fig. 3. The Landau fan structure near the band edges in Fig. 3(b) for small $B$ represents the highly degenerate Landau levels which have vanishing band widths and zero mobility so that the wave packets are localized through cyclotron motion, in contrast to the NHSE picture in Fig. 2(a).

**Suppression of the NHSE.** It is of particular interest to investigate the interplay between the NHSE and the magnetic field. Under $y$-PBC, the Hamiltonian can be Fourier transformed into $H = \sum_{m,k_y} (t^+_x c^\dagger_{m+1,k_y} c_{m,k_y} + t^+_x c^\dagger_{m,k_y} c_{m+1,k_y})$ with $k_y$ the wave vector in the $y$-direction. Taking $x$-OBC, we plot in Fig. 2(a) the spatial distribution function $W(x, k_y) = \sum_i |\psi^\dagger_{i} (m, k_y)|^2/M$ defined by all the right eigenstates $\psi^\dagger_{i} (m, k_y)$ (labeled by $i$) of $H$ for a given $k_y$. One can see that a small $B$ is sufficient to drive the the skin modes penetrating deeply into the bulk, showing a considerable suppression of the NHSE, that generally holds for all $k_y$.

The results can be understood in the following ways. As we start with $B = 0$, the energy spectra $E_{k_y}(k_x)$ for a given $k_y$ forms a closed loop in the complex plane under the $x$-PBC; see Fig. 2(b), which predicts the presence of the NHSE under the $x$-OBC [51,52]. As $B$ increases, real energy spectra start to form from the band edges towards its center, accompanied by a shrinkage of the complex loop; see Fig. 2(b). According to the correspondence between the spectra configuration under the $x$-PBC and the occurrence of the NHSE under the $x$-OBC, this indicates a tendency to suppress the NHSE. One can also analyze the results from the real space perspective and start with the opposite limit $\delta_x = 0$. The wave functions under magnetic field are modulated by the exponential envelope function introduced by the NHSE as $\delta_x$ raises from zero; see Fig. 2(c). Then the results in Fig. 2(a) can be understood as the superposition of all the tilted loops in real space.

**NHSE on the magnetic energy spectra.** The non-Hermitian systems can have completely different energy spectra under the PBC and OBC [29,30] Next, we study the NHSE on the magnetic energy spectra for both the $x$-OBC and the $x$-PBC, while the $y$-PBC is adopted for both cases. The energy spectra under the $x$-OBC are shown in Fig. 3(a) [The in-gap streaks are edge states]. It turns out that the butterfly diagram depends on $\delta_x$
rather weakly as $\delta_x < 2t/5$ \cite{S}. Moreover, the energy spectra are entirely real. In fact, under the $x$-OBC, a similarity transformation $S^{-1}c_m,n = e^{-i_m}c_m,n, S^{-1}c_m,nS = r^{-m}c_m,n$ can be performed to Eq. (1) that transforms the original Hamiltonian into a Hermitian one as $H = H^\dagger = S^{-1}HS$ with real energy spectra \cite{60}.

Under the $x$-PBC, the complex energy spectra start to form from the band center due to the nonreciprocal hopping $\delta_x$; see Figs. 3(d)-3(f). As a result, the self-similar fractal patterns merge into continuous pieces along with multiple gap closing. The associated energy levels coalesce in pairs to create exceptional points \cite{83, 84}, indicating a non-Hermitian phase transition. At the same time, the energies near the band top and bottom remain real for $k \ll 1/l_B$ with $l_B = \sqrt{\hbar/(eB)}$ the magnetic length, immune to the NHSE; see Figs. 3(c) and 3(f). It means that the magnetic field prevents the system from the real-to-complex spectral transition induced by the NHSE, which again reflects their incompatible nature. One can infer from this observation that the combined $\mathcal{MT}$-symmetry as

$$\mathcal{M} \mathcal{T} H (\mathcal{M} \mathcal{T})^{-1} = H,$$

with the mirror reflection ($\mathcal{M}$) about the $x$-axis and time reversal ($\mathcal{T}$) operation defined by

$$\mathcal{M} c_m,n \mathcal{M}^{-1} = c_{-m,n}, \mathcal{T} c_m,n \mathcal{T}^{-1} = c_{m,n}, \mathcal{T} \mathcal{T}^{-1} = -i.$$  \label{1}

The constraint by the $\mathcal{MT}$ symmetry can be rewritten in another standard form of $\mathcal{MOH}^\dagger (\mathcal{MO})^{-1} = H$, with $O$ the transpose operation, and then $H$ is said to be $\mathcal{MO}$-pseudo-Hermitian \cite{81}. As a result, the energy spectra can be entirely real or be composed of complex conjugate pairs. For a specific state, the real and complex nature of the energy corresponds to its wave function with and without the $\mathcal{MT}$ symmetry, respectively \cite{88}.

The breaking of the $\mathcal{MT}$ symmetry for the $i$th eigenstate $\psi_i$ can be measured by the Hilbert-Schmidt quan-
For the state that respects the symmetry, $\gamma = 0$; In contrast, if the state breaks the symmetry, $\gamma \neq 0$. One judicious choice should be the average quantum distance $d_{\text{HS}}$ defined by the hopping between the boundary sites $(1, n)$. The parameter $\gamma_B \in [0, 1]$ and the two limits $\gamma_B = 0$ and $\gamma_B = 1$ correspond to the $x$-OBC and $x$-PBC, respectively. We plot in Fig. 4 the average quantum distance $d_{\text{HS}}$ as a function of $\gamma_B$ and $\delta_x$ for zero and finite $B$ [The states number $N$ in Eq. (5) counts all states labeled by $i$ and $k_{ij}$]. One can see that a clear phase boundary naturally forms between the $MT$-symmetric ($d_{\text{HS}} = 0$) and $MT$-broken ($d_{\text{HS}} > 0$) region. Moreover, such a phase boundary can also be obtained as the critical points of the real-to-complex spectral transition, which shows good coincidence with that given by the $d_{\text{HS}}$ contour. This is assured by the theorem associated with the $MT$ antunitary symmetry. Another interesting observation is that the critical phase boundary can be well fitted by the exponential function.

In the absence of the magnetic field, the system resides in the $MT$-symmetric and $MT$-broken phase for the $x$-OBC and $x$-PBC, respectively; see Fig. 4(a). By tuning the boundary parameter $\gamma_B$, a continuous $MT$ transition can be implemented that shows a crossover between two limiting cases. However, no phase transition takes place in either the $x$-OBC ($\gamma_B = 0$) or the $x$-PBC ($\gamma_B = 1$) as $\delta_x$ varies. Remarkably, a finite magnetic field can effectively suppress the $MT$ symmetry breaking; see Fig. 4(b), which is reflected in two aspects: first, as $B$ increases from zero, the $MT$-symmetric area expands, meaning that a stronger nonreciprocal hopping $\delta_x$ is required to break the $MT$ symmetry compared with that for $B = 0$; second, the order parameter $d_{\text{HS}}$ diminishes in the $MT$-broken area so that the symmetry breaking becomes weaker. This fact reflects again the incompatible nature between the NHSE and the magnetic field. A direct result of such magnetic suppression is that a $MT$ transition can be driven by either $\delta_x$ or $B$ under the $x$-PBC for a finite system.

Summary and outlook.—We have shown that the interplay between the NHSE and the magnetic field can result in several novel effects. It can be hopefully realized in various physical systems, such as cold atoms [79, 80], photonic [92, 93] and acoustic [94] systems, condensed matter physics [21, 24] and electric circuits [73, in which the two ingredients can be implemented by state-of-the-art techniques. There are a variety of open problems associated with the coexistence of the NHSE and the magnetic field, such as generalization of the theory to other systems, the wave packet dynamics, time dependent $MT$ transition, etc. Exploring potential applications of the $MT$ transition is also of great interest.

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Note added.—During the preparation of the manuscript,
we became aware of a related work posted on arXiv [93]. In addition to the magnetic suppression of the NHSE studied in both works from different perspectives, here, we also discuss the NHSE on the magnetic spectra, the Onsager quantization rule and uncover the $\mathcal{M}T$ phase transition.

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