The potential function and ladder variables of a recurrent random walk on $\mathbb{Z}$ with infinite variance

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Abstract

We consider a recurrent random walk of i.i.d. increments on the one dimensional integer lattice and obtain a certain relation between the hitting distribution to a half line and the potential function, $a(x)$, of the walk. Applying it we derive an asymptotic estimate of $a(x)$ and thereby a criterion for $a$ to be bounded on a half line. We also apply it to a classical two-sided exit problem and show that if the expectation of the ladder height is finite, then Spitzer’s condition is necessary and sufficient for the probabilities of exiting a long interval $[-M, N]$ through the upper boundary to converge whenever $M/N$ tends to a positive constant.

1 Introduction and Statements of Results

In this paper we study properties of the potential function $a(x)$ of a recurrent random walk on the integer lattice $\mathbb{Z}$ with infinite variance and apply them to the two-sided exit problem for the walk. Our first result gives some relations of $a$ to the renewal function, $f_r$ say, of ladder height processes and the distributions of overshoots of the first entrances into the negative half line. Applying it we derive an asymptotic estimate of $a(x)$ in case when the ladder height variable, denoted by $Z$, has a finite expectation $EZ < \infty$ and thereby an analytic criterion for $a(x)$ to be bounded on $x < 0$ (Theorem 2). As for the exit problem we show that if $EZ < \infty$, then an asymptotic form of the probability of exiting an interval $[-x, N-x]$, $0 < x < N$ through the upper boundary is given by the ratio $f_r(x)/f_r(N)$ for large $N$, and relate this probability to Spitzer’s condition (Theorems 3 and 4). The results obtained complement those of [21] where similar matters are treated when one tail of the distribution of the increment of the walk is negligible with respect to the other. There are many investigations on the ladder variables and associated renewal function for the walks with infinite variance [1], [6], [14], [22] etc., but few of them relate these objects to the property of $a(x)$. Our results exhibit the significant relevance of $a$ to them at least as treated under the condition $EZ < \infty$. As an intelligible manifestation of this condition in the sample path behavior of the walk one may mention that $EZ < \infty$ if and only if the walk conditioned to avoid the origin to approach the positive infinity with probability one (Section 6).

1 key words: recurrent random walk; ladder height; potential function; Spitzer’s condition; two-sided exit problem

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sake of comparison we include the case of finite variance when all the results are known or easily derived from known ones.

Let \( S_n = S_0 + X_1 + \ldots + X_n \) be a random walk on \( \mathbb{Z} \) where the starting position \( S_0 \) is an unspecified (non-random) integer and the increments \( X_1, X_2, \ldots \) are independent and identically distributed random variables defined on some probability space \((\Omega, \mathcal{F}, P)\) and taking values in \( \mathbb{Z} \). Let \( X \) be a random variable having the same law as \( X_1 \). We suppose throughout the paper that the walk \( S_n \) is \textit{irreducible and recurrent}. For a non-empty subset \( B \subset \mathbb{R} \), put \( \sigma_B = \inf\{n \geq 1 : S_n \in B\} \), the first entrance time of the walk into \( B \) and define the random variable \( Z \) by

\[
Z = S_{\sigma(S_0 + 1, \infty)} - S_0
\]

(cf. [10] Section XII.1). We denote the dual variable by \( -\hat{Z} \) which is given by \( \hat{Z} = S_{\sigma(-\infty, S_0 - 1)} - S_0 \). Because of recurrence of the walk \( Z \) is a proper random variable whose distribution is concentrated on positive integers \( x = 1, 2, \ldots \) and similarly for \( -\hat{Z} \). Let \( E \) indicate the integration by \( P \) as usual. If \( \sigma^2 := EX^2 < \infty \), then \( EZ < \infty \), whereas if \( \sigma^2 = \infty \), either \( EZ = \infty \) or \( E|\hat{Z}| = -\infty \) (cf. [16], [4], Theorem 8.4.7).

Let \( P_x \) denote the probability of the random walk with \( S_0 = x \) and \( E_x \) the expectation by \( P_x \). Put \( p^n(x) = P_0[S_n = x] \), \( p(x) = p^1(x) \) and define the potential function

\[
a(x) = \sum_{n=0}^{\infty} [p^n(0) - p^n(-x)];
\]

the series on the right side is convergent and \( a(x)/|x| \to 1/\sigma^2 \) as \( |x| \to \infty \) (cf. Spitzer [16], P28.8, P29.2). Here and in the sequel \( 1/\infty \) is understood to be zero.

Put

\[
T = \sigma(-\infty, 0) = \inf\{n \geq 1 : S_n \leq 0\}
\]

(where \( \alpha, \beta \) denotes the interval \( \alpha < x \leq \beta \) as usual) and define

\[
H^x_{(-\infty, 0]}(y) = P_x[S_T = y],
\]

the hitting distribution of \((-\infty, 0]\) for the walk starting at \( x \in \mathbb{Z} \). Likewise let \( H^1_{[0, \infty)} \) be the hitting distribution of \([0, \infty)\). (Thus \( H^1_{(-\infty, 0]}(y) = P[\hat{Z} = y - 1], y \leq 0 \) and \( H^{-1}_{[0, \infty)}(y) = P[Z = y + 1], y \geq 0 \).) There exists \( \lim_{y \to \infty} H^x_{(-\infty, 0]}(y) \), which we denote by \( H^+_{(-\infty, 0]}(y) \) and similarly for \( H^x_{[0, \infty)} \). \( H^\infty_{[0, \infty)} \) is a probability if \( EZ < \infty \) and vanishes identically otherwise. Let \( V_{ds}(x) \), \( x = 0, 1, 2, \ldots \), be the renewal function of the weak descending ladder-height process (see [25]) or Appendix A). For our present purpose it is convenient to bring in the function \( f_r \), the shift of \( V_{ds} \) to the right by 1, namely

\[
f_r(x) = V_{ds}(x - 1) \quad (x \geq 1).
\]

According to [16] \( f_r \) is a positive harmonic function on \([1, \infty)\), i.e., a solution of the equation \( f_r(x) = E_x[f_r(S_1); S_1 \geq 1] = E[f_r(X + x); X > -x] \), which may be written as

\[
f_r(x) = \sum_{y=1}^{\infty} f_r(y)p(y - x), \quad x \geq 1, \quad (1.1)
\]
and the solution is unique apart from a constant factor; it turns out that the distribution of $Z$ is expressed as

$$P[Z > -x] = \sum_{y=1}^{\infty} f_r(y)p(y - x) \quad (x \leq 0), \tag{1.2}$$

(see Theorems A and B and (2.10) in Section 2 for more details). Define for any non-negative function $\varphi(y), y \leq 0$,

$$H_{(-\infty,0]}^x(\varphi) = E_x[\varphi(S_T)] = \sum_{y \leq 0} H_{(-\infty,0]}^x(y)\varphi(y).$$

The first result relates the summability of $Z$ with some properties of $a(x)$.

For a non-empty subset $B \subset \mathbb{Z}$ let $g_B(x,y)$ denote the Green function of the walk killed on $B$:

$$g_B(x,y) = E_x\left[\sum_{0 \leq n < \sigma_B} \delta(S_n, y)\right] \quad (x, y \in \mathbb{Z}),$$

where $\delta(x,y) = 1$ if $x = y$ and $= 0$ otherwise. This definition is different from one in [16], where the corresponding one agrees with our $g_B(x,y)$ if $x \in B$, but vanishes if $x \notin B$, whereas

$$g_B(x,y) = \sum_{z \notin B} p(z-x)g_B(z,y) + \delta(x,y) \quad \text{for} \quad x \in B$$

(valid also for $x \notin B$). This relation shows that $g_B(x,y)$ equals the hitting distribution of $B$ by the dual (or time-reversed) walk started at $y$ which fact is expressed as

$$g_B(x,y) = P_{-y}[S_{\sigma_B} = -x] \quad \text{for} \quad x \in B,$$

where $-B = \{-z : z \in B\}$ and $\sigma_B = \sigma$ if $S_0 \notin B$ and $= 0$ otherwise.

In case $B = (-\infty,0]$ $g_B(x,y), x,y \in B$ is expressed explicitly by means of the renewal functions of ascending and descending ladder height processes (cf. Theorem A in Section 2), by which it immediately follows that there exists $\lim_{y \to \infty} g_{(-\infty,0]}(x,y)$ which is denoted by $g_{(-\infty,0]}(x,\infty)$ and is given by

$$g_{(-\infty,0]}(x,\infty) = \begin{cases} f_r(x)/EZ & x > 0, \\ \lim_{y \to \infty} P_{-y}[S_\sigma|0,\infty) = -x] & x \leq 0; \end{cases} \tag{1.3}$$

if $EZ = \infty$, the limit on the right side vanishes so that $g_{(-\infty,0]}(x,\infty) = 0$ for all $x$ (cf. [21]).

**Theorem 1.** (i) For all $x, y \in \mathbb{Z}$,

$$a(x - y) - H_{(-\infty,0]}^x(a(-y)) + g_{(-\infty,0]}(x,y) = Ag_{(-\infty,0]}(x,\infty), \tag{1.4}$$

where $A$ is a positive constant such that $A = 1/2$ or $1$ according as $\sigma^2 < \infty$ or $\sigma^2 = \infty$.

(ii) If $EZ < \infty$, then as $x \to \infty$, $a(x)/f_r(x) \to A/EZ$ and $a(-x)/a(x) \to 0$, and

$$\sum_{x=0}^{\infty} a(-x)P[|X| > x] < \infty.$$

(iii) If $EZ = \infty$, then $\liminf_{x\to\infty} a(x)/f_r(x) = 0.$
It is natural to extend $f_r(x)$ to a function on $\mathbb{Z}$ by means of \((1.1)\) (so as to make \((1.1)\) valid for all $x \in \mathbb{Z}$), or what amounts to the same thing in view of \((1.2)\),
\[
 f_r(x) = P[Z > -x] \quad \text{for} \quad x \leq 0.
\]

Since $f_r(0) = 1 < c^{-1} = f_r(1)$, $f_r$ is increasing. According to this extension of $f_r$ relation \((1.3)\) is simply written as $g_{(-\infty,0)}(x,\infty) = f_r(x)/EZ$. \((1.3)\) entails that the left side of \((1.4)\) is independent of $y \in \mathbb{Z}$ and the special case $y = 0$ yields the next corollary. For brevity of expression we write
\[
a^\dagger(x) = a(x) + \delta(x,0).
\]

**Corollary 1.**

(i) If $EZ < \infty$, then $H_{(-\infty,0)}^x\{a\}/a(x) \to 0$ as $x \to \infty$ and
\[
a^\dagger(x) - H_{(-\infty,0)}^x\{a\} = Af_r(x)/EZ \quad \text{for} \quad x \in \mathbb{Z}.
\]

(ii) If $EZ = \infty$, then $H_{(-\infty,0)}^x\{a\} = a^\dagger(x)$ for $x \in \mathbb{Z}$.

**Remark 1.1.**

(a) By \((1.2)\) it follows that
\[
\sum_{y=1}^{\infty} f_r(y)P[X \geq y] = EZ,
\]
which together with Theorem \(1(i)\) implies that if $EZ < \infty$, then
\[
\sum_{x=1}^{\infty} |a(x) + a(-x)|P[X > x] < \infty. \tag{1.7}
\]

The converse also holds (cf. \[18\]), so that in view of Theorem \(1(ii)\) \((1.7)\) implies $a(-x)/a(x) \to 0$ but $\sum_{x=1}^{\infty} a(x)P[X > x] < \infty$ is not enough for \((1.7)\) to be true. (See also Remark \(2.2)\)

(b) The process $M_n := a(S_{n\wedge T})$ is a non-negative martingale under $P_x$, $x \neq 0$, in particular $a(x) = E_xM_n$. Clearly $M_\infty = a(S_T)$ a.s., so that $H_{(-\infty,0)}^x\{a\} = E_xM_\infty$. Corollary \(1\) implies that $(M_n)$ is uniformly integrable (so that $a(x) = E_xM_\infty$) if and only if $EZ = \infty$.

(c) As another application of Theorem \(1\) we shall consider the walk killed at the origin and observe that the killed walk distinguishes $+\infty$ and $-\infty$ if (and only if in a sense) either $EZ$ or $E\hat{Z}$ is finite, although its Martin compactification does not whenever $\sigma^2 = \infty$ (see Section 6).

(d) Let $EZ = \infty$ and consider asymptotic behaviour of $a(x)/f_r(x)$ as $x \to \infty$. Theorem \(1(iii)\) tells merely $\lim \inf a(x)/f_r(x) = 0$. It however seems to be true quite generally that $\lim a(x)/f_r(x) = 0$. Actually if $E|X| < \infty$ in addition, with $m_-$ and $m$ defined right after this remark, we have $f_r(x) \gg x/m_-(x)$ (cf. Lemma \(2.2)\) while $a(x) + a(-x) \asymp x/m(x)$ under a mild side condition that is satisfied if e.g., $\limsup x m'(x)/m(x) < 1$ or $\lim[m(x) \wedge m_+(x)]/m(x) = 0$ (\[21\] Theorem 1(ii))]: these two relations obviously entail that
\[
[a(x) + a(-x)]/f_r(x) \to 0 \quad (x \to \infty).
\]
Put
\[ m_-(x) = \int_0^x dy \int_y^\infty P[X < -u]du, \quad m_+(x) = \int_0^x dy \int_y^\infty P[X > u]du \quad (1.8) \]
and \( m(x) = m_-(x) + m_+(x) \), provided that \( E|X| < \infty \) which is valid if \( EZ \) or \( E\hat{Z} \) is finite since \( EX_+ \) and \( EX_- \) are simultaneously finite or infinite because of the assumed recurrence. Here \( X_\pm = \max\{\pm X, 0\} \). The first part of the following theorem provides asymptotic estimates of \( a(x) \) as \( |x| \to \infty \). The third part of it solves an open question mentioned at the very end of Spitzer’s book [16] (see Remark 1.2(e) below).

**Theorem 2.** (i) If \( EZ < \infty \) and \( \sigma^2 = \infty \), then
\[ 1 \leq \liminf_{x \to \infty} \frac{a(x)m_-(x)}{x} \leq \limsup_{x \to \infty} \frac{a(x)m_-(x)}{x} \leq 2; \quad (1.9) \]
and
\[ \lim_{x \to \infty} \frac{1}{a(-x)} \sum_{z=1}^{\infty} P[z < Z \leq z + x]a(z) = EZ. \quad (1.10) \]
(ii) If \( EZ = \infty \), then \( \lim_{x \to \infty} a(-x) = \infty \).
(iii) Suppose \( \sigma^2 = \infty \) and \( EZ < \infty \). Then
\[ a(-x) \asymp \frac{x}{2} \sum_{w=1}^{\infty} \sum_{z=1}^{\infty} p(w + z) \left( \frac{z}{m_-(z)} \right)^2, \quad (1.11) \]
there exists \( \lim_{x \to \infty} a(-x) \leq \infty \) and this limit is finite if and only if
\[ \int_1^{\infty} \frac{t^2}{m_-(t)} P[X > t]dt < \infty. \quad (1.12) \]
If this is the case, then \( \lim_{x \to \infty} a(-x) = H_{[0,\infty)} \{a\} \).

**Remark 1.2.**
(a) In [21] it is shown that if \( m_+(x)/m_-(x) \to 0 \) \( (x \to \infty) \), then \( a(-x)/a(x) \to 0 \) and \( a(x) \) is asymptotically monotone for large positive values of \( x \). By (1.10) \( a(x) \) is asymptotically monotone also for large negative values of \( x \) if \( EZ < \infty \).

(b) Under the assumption of (iii), \( m_-(t) \to \infty \) \( (t \to \infty) \) and \( E[X_+^3] < \infty \) is sufficient for (1.12) to hold, while the upper order of \( t/m_-(t) \) can be zero (i.e., \( \log[t/m_-(t)]/\log t \to 0 \)) and accordingly (1.12) is possibly true even if \( E[X_+^{1+\delta}] = \infty \) for every \( \delta > 0 \).

(c) Condition (1.12) implies \( EZ < \infty \), the latter being equivalent to the integrability condition \( \int_1^{\infty} tP[X > t]dt/m_-(t) < \infty \) (see [5], [18, Section 2.4]).

(d) Let \( EZ < \infty \). Then bound (1.9) (or rather Lemma 2.2(ii)) gives an estimate of \( P[Z > y] \) in view of identity (1.2) and Theorem 1(ii) (or rather Lemma 2.2(ii)), while
\[ P[\hat{Z} = -x] = v^-(0) \sum_{y=0}^{\infty} v(y)p(-x-y) \asymp P[X \leq -x] \quad (x \to \infty) \quad (1.13) \]
due to the dual of (1.2) (see (2.1), (2.2) for \( v, v^- \); also (2.3)), where \( a_n \sim b_n \) means that \( a_n/b_n \) is bounded away from zero and infinity.

(e) If \( E|X| = \infty \), then \( EZ = -E\hat{Z} = \infty \) (because of recurrence) so that \( \lim_{|x| \to \infty} a(x) = \infty \). Theorem 2(ii) therefore gives an exact criterion for the trichotomy into \( \lim_{|x| \to \infty} a(x) = \infty \) or \( \sup_{x<0} a(x) < \infty \) or \( \sup_{x>0} a(x) < \infty \).

(f) Let \( EZ < \infty \). Then (1.10) together with the equality stated last in Theorem 2 shows that \( H_{[0,\infty)}^{-\infty}(\{a\}) = \sum_{y>0} a(y) P[Z > y]/EZ \), but this follows from the identity \( H_{[0,\infty)}^{\infty}(y) = P[Z > y]/EZ \) (cf. (2.5) or Appendix A).

The following corollary is obtained by combining Theorems 1 and 2 in view of (1.6).

**Corollary 2.** For \( a(x) \) to be bounded for \( x < 0 \) each of the following conditions are necessary and sufficient.

(i) \( \sum_{z=1}^{\infty} P[X > z][f_r(z)]^2 < \infty \). 
(ii) \( \sum_{z=1}^{\infty} P[Z > z]f_r(z) < \infty \).
(iii) \( \sum_{z=1}^{\infty} H_{[0,\infty)}^{\infty}(z)f_r(z) \) is bounded for \( x < 0 \).

Theorem 1 also entails a probabilistically significant consequence. For \( y \in \mathbb{Z} \) write \( \sigma_y \) for \( \sigma_{\{y\}} \) (the first hitting time of \( y \)).

**Theorem 3.** (i) Suppose that either \( EZ < \infty \) or \( E\hat{Z} > -\infty \). Then, uniformly for \( 0 < x < N \), as \( N \to \infty \)

\[
P_x[\sigma_{[N,\infty)} < T] = P_x[\sigma_N < T](1 + o(1)),
\]

and

\[
P_x[\sigma_{[N,\infty)} < \sigma_0] = P_x[\sigma_N < \sigma_0](1 + o(1)).
\]

(ii) If the walk is not left-continuous (i.e. \( P[X \leq -2] > 0 \)), then for \( x > 0 \), as \( N \to \infty \)

\[
\frac{P_x[\sigma_N < T]}{P_x[\sigma_N < \sigma_0]} = \begin{cases} 1 - H_{[0,\infty)}^{\infty}(\{a\})/a(x) + o(1) & \text{if } EZ < \infty, \\ o(1) & \text{if } EZ = \infty. \end{cases}
\]

Here \( o(1) \) is uniform for \( 0 < x < N \) if \( EZ < \infty \).

The result of (ii) in case \( EZ = \infty \) says that \( P_x[T < \sigma_N \mid \sigma_N < \sigma_0] \to 1 \) as \( N \to \infty \), namely, for \( N \) large enough the walk conditioned on \( \sigma_N < \sigma_0 \) reaches \( N \) only after entering the negative half line with dominant probability as far as its starting position \( x \) is fixed. If \( N-x \) remain bounded, the same conditional probability approaches unity; the behaviour of it as \( x \land (N-x) \to \infty \) depends on the behaviour of tails of the distribution of \( X \) at both \(+\infty \) and \(-\infty \) and to find a fashion of the dependence would pose a serious problem.

The first relation of Theorem 3(i) provides a solution to the classical two-sided exit problem in case when \( EZ < \infty \) or \( E\hat{Z} > -\infty \). Let \( f_1(x), x \geq 1 \) denote the dual of \( f_r \), in other words \( f_1(x) = V_{as}(x - 1) \), where \( V_{as} \) is the renewal function of the weak ascending ladder height; \( f_1 \) is dual harmonic on \( x \geq 1 \) (cf. (2.5) or Appendix A).
Corollary 3. Suppose $EZ < \infty$. Then, as $N \to \infty$

$$P_x[\sigma_{[N,\infty)} < T] = \frac{f_r(x)}{f_r(N)} (1 + o(1)) \quad \text{uniformly for } 0 < x < N \quad (1.14)$$

and for $x > 0$, as $N - x \to \infty$,

$$P_x[\sigma_{[N,\infty)} > T] = \frac{f_r(N) - f_r(x)}{f_r(N)} (1 + o(1)). \quad (1.15)$$

In the case $E\hat{Z} > -\infty$ analogous formulae hold: they are obtained by interchanging the right sides of (1.14) and (1.15) and simultaneously replacing $f_r(N)$ and $f_r(x)$ by $f_l(N)$ and $f_l(N - x)$, respectively (see (5.8)).

The second formula (1.15) follows from the first if $x$ ranges over a set depending on $N$ in which $f_r(x) = O(f_r(N) - f_r(x))$ but does not otherwise. Since $f_r$ is sub-additive, so that $f_r([N/2])/f(N) \geq 1/2$ (cf. (7.1)), Corollary 3 says that recurrent random walks started at the origin leave a symmetric interval on its right side more likely than or at least equally likely as on its left side if $EZ < \infty$.

From (1.14) one derives the following overshoot estimate.

**Lemma 1.1.** If (1.14) holds, then, uniformly for $0 < x < N$ and $N' > N$, as $N \to \infty$

$$P_x[S_{\sigma_{[N,\infty)}} > N', \sigma_{[N,\infty)} < T] = \frac{f_r(x)}{f_r(N') - f_r(N)} \times o(1). \quad (1.16)$$

**Proof.** By Markov’s inequality $P_x[S_{\sigma_{[N,\infty)}} > N'; \sigma_{[N,\infty)} < T]$ is not larger than

$$E_x[f_r(S_{\sigma_{[N,\infty)}}) - f_r(N); \sigma_{[N,\infty)} < T] \leq \frac{f_r(x) \left(1 - \left\{\frac{f_r(N)}{f_r(x)}\right\}P_x[\sigma_{[N,\infty)} < T]\right)}{f_r(N') - f_r(N)},$$

which shows (1.16), the numerator of the last ratio approaching zero due to (1.14). 

**Remark 1.3.** Without the condition $EZ \wedge E|\hat{Z}| < \infty$ there are some results concerning the two sided exit problem. According to [21] if $m_+(x)/m_-(x) \to 0 \ (x \to \infty)$, then for each $\varepsilon > 0$, $P_x[\sigma_{[N,\infty)} < T] \sim a(x)/a(N)$ as $N \to \infty$ uniformly for $\varepsilon N < x < N$. If the distribution of $X$ is symmetric and belongs to the domain of attraction of a stable law, the problem is investigated by Kesten [12]: he identifies the limit of $P_x[\sigma_{[N,\infty)} < T] \text{ as } x \wedge N \to \infty$ so that $x/N \to \lambda \in (0, 1)$. For Lévy processes with no positive jumps there are certain definite results (cf. [2] Section 7.1-2, [7] Section 9.4)). It is also noted that if the walk $S$ is attracted to a stable process of index $1 < \alpha \leq 2$ with no positive jumps, then (1.14) holds (see Appendix (C) for the proof), and hence the overshoot estimate of Lemma 1.1.

If the walk is right-continuous, then $Z \equiv 1$ and $P_x[\sigma_{[N,\infty)} < T] = P_x[\sigma_N < T] = f_r(x)/f_r(N)$, showing that the existence of $\lim_{x/N \to \lambda} P_x[\sigma_{[N,\infty)} < T]$ is equivalent to regular variation of $f_r$ (see [2], [7] for the corresponding results for Lévy processes with no positive jumps). In [16] p. 227 there is given a criterion for Spitzer’s condition to hold in case the walk is left continuous (cf. [3] Section 8.9 for related results). Based on Corollary 3 above these are extended to the case $EZ < \infty$. 


Theorem 4. Suppose $EZ < \infty$ and $\sigma^2 = \infty$. Then, (i) the following are equivalent

1. for each $0 < \lambda \leq 1$, $P_{x}[\sigma_{(N,\infty)} < T]$ converges as $x/N \to \lambda$ (along with $N \to \infty$),
2. $f_r(x)$ is regularly varying at infinity,
3. $m_-(x)$ is regularly varying at infinity,
4. Spitzer’s condition holds, i.e.,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P[S_k > 0] ; \]  \hspace{1cm} (1.17)

and (ii) if any of these conditions holds, then

\[ m_-(x) \sim x^{2-\alpha}/L(x) \quad \text{and} \quad f_r(x)/EZ \sim [\Gamma(3-\alpha)\Gamma(\alpha)]^{-1}x^{\alpha-1}L(x) \]  \hspace{1cm} (1.18)

with some $1 \leq \alpha \leq 2$ and $L$ slowly varying at infinity, uniformly for $\lambda \in [\varepsilon, 1]$,

\[ P_x[\sigma_{(N,\infty)} < T] \to \lambda^{\alpha-1} \quad \text{as} \quad x/N \to \lambda \]  \hspace{1cm} (1.19)

for each $\varepsilon > 0$, and the limit in Spitzer’s condition (1.17) equals $1/\alpha$.

Remark 1.4. Suppose $EZ < \infty$ and $\sigma^2 = \infty$. Then the asymptotic forms of $f_r(x)$ and $m_-(x)$ given in (1.18), implies that as $x \to \infty$

1. $P[X < -x] \sim (2-\alpha)(\alpha-1)x^{-\alpha}/L(x)$;
2. $P[-\hat{Z} > x] \sim (cEZ)^{-1}(2-\alpha)x^{1-\alpha}/L(x)$,

with the obvious interpretation of $a_n \sim Cb_n$ in case $C = 0$ (see (5.13) and (1.13)); and

3. $v^-(x)/EZ \sim \left\{ \begin{array}{ll} [\Gamma(3-\alpha)\Gamma(\alpha-1)]^{-1}x^{\alpha-2}L(x) & \text{if} \ 3/2 < \alpha < 2, \\ L(x) & \text{if} \ \alpha = 2. \end{array} \right.$

(3) follows from (2) if $3/2 < \alpha < 2$ (cf. [6], [8], [11] etc.) and from Lemma 2.2(i), i.e.,

the relation $\int_0^\infty P[\hat{Z} < -t]dt \sim m_-(x)/cEZ$ if $\alpha = 2$ (cf. Appendix (B)). If the factor $(\alpha-1)(2-\alpha)$ does not vanish in (1), then the converse implication holds.In case $1 < \alpha \leq 2$ Spitzer’s condition implies that $X$ is in the domain of attraction of the spectrally negative $\alpha$-stable law [18, Theorem 1.2].

In the case $\sigma^2 < \infty$ the formulae of Corollaries 1 through 3 are given in [17], where they are applied to evaluate the transition probabilities of the walk killed at the origin. Some of results obtained in this paper will be used for a similar purpose for walks that are in the normal domain of attraction to stable laws in [20].

In Section 2 we collect fundamental facts used in this paper about $f_r$, $f_l$, $a$ and $g_{(-\infty,0]}$ given in Spitzer [16] and advance several lemmas that are directly derived from them. The proofs of Theorems 1 and 2 are given in Sections 3 and 4, respectively. In Section 5 Theorems 3 and 4 and its corollaries are proved after showing a sequence of lemmas. In Section 6 we briefly study large time behaviour of the walk conditioned to avoid the origin. In Section 7 we give Appendix A) stating a remark on the relations between strictly and weakly ascending ladder variables, Appendix B) providing a lemma concerning a renewal sequence in a critical case and Appendix C) providing a proof of the statement at the end of Remark 1.3.
## Preliminary lemmas

In this section we collect fundamental results of the recurrent random walks on \( \mathbb{Z} \) given in Spitzer’s book \[16\] and then derive some consequences of them that are used throughout the paper.

For \( B \subset \mathbb{Z} \) we have defined the first hitting time by \( \sigma_B = \inf \{ n \geq 1 : S_n \in B \} \). For a point \( x \in \mathbb{Z} \) write \( \sigma_x \) for \( \sigma \{ x \} \). For typographical reason we sometimes write \( \sigma(B) \) for \( \sigma_B \).

Let \( v(x), x = 0, 1, 2, \ldots \) be the renewal sequence of the ascending ladder variables, namely \( v(0) = 1 \) and

\[
v(x) = \sum_{n=1}^{\infty} P[Z_1 + \cdots + Z_n = x], \quad x \geq 1; \tag{2.1}
\]

and similarly \( v^-(x), x = 0, 1, 2, \ldots \) denotes the renewal sequence of the weak descending ladder variables, which may be given by

\[
v^-(0) = \frac{1}{c} \quad \text{and} \quad v^-(x) = \frac{1}{c} \sum_{n=1}^{\infty} P[\hat{Z}_1 + \cdots + \hat{Z}_n = -x], \quad x \geq 1, \tag{2.2}
\]

where

\[
c = \exp \left( -\sum_{k=1}^{\infty} \frac{1}{k} p^k(0) \right) = \exp \left( \frac{1}{\pi} \int_0^{\pi} \log |1 - E[e^{itX}]| dt \right).
\]

(See Appendix (A) for (2.2) as well as the probabilistic meaning of the constant \( c \).) Owing to the renewal theorem, there exist limits

\[
v_\infty := \lim_{x \to \infty} v(x) = 1/EZ \quad \text{and} \quad v^-_\infty := \lim_{x \to \infty} v^-(x) = 1/c E[-\hat{Z}]. \tag{2.3}
\]

Let \( g_B(x, y) \ (x, y > 0) \) denote the Green function of the walk \( S_n \) killed on \( B \neq \emptyset \):

\[
g_B(x, y) = \sum_{n=0}^{\infty} P_x[S_n = y, n < \sigma_B].
\]

The following theorem follows from the propositions P18.8, P19.3, P19.5 of \[16\]. For two real numbers \( s \) and \( t \) write \( s \wedge t = \min \{ s, t \} \) and \( s \vee t = \max \{ s, t \} \).

**Theorem A**

\[
g_{(-\infty,0]}(x, y) = \sum_{z=1}^{x \wedge y} v^-(x - z)v(y - z) \quad (x, y > 0)
\]

and \( g_{[0,\infty)}(x, y) = g_{(-\infty,0]}(-y, -x) = \sum_{z=1}^{[x] \wedge [y]} \sum_{z=1}^{[y] \wedge [x]} v(|x| - z)v^-(|y| - z) \quad (x, y < 0). \)

The formulae in Theorem A will often be used in combination with the following representation of the hitting distribution \( H^x_{(-\infty,0]}(y) \) of \( (-\infty,0] \):

\[
H^x_{(-\infty,0]}(y) = \sum_{z=1}^{\infty} g_{(-\infty,0]}(x, z)p(y - z) \quad (x > 0, y \leq 0), \tag{2.4}
\]

and analogous one for \( H^x_{[0,\infty)} \). The function \( f_r \) may be written as

\[
f_r(x) = v^-(0) + \cdots + v^-(x - 1) \quad (x \geq 1), \tag{2.5}
\]
and its dual as \( f_t(x) = c^{-1} [v(0) + \cdots + v(x-1)] \) \((x \geq 1)\). As stated in Section 1 the following theorem follows from [16] P19.5, P24.7, P18.8.

**Theorem B** (i) \( f_r \) is harmonic on \( x \geq 1 \) in the sense that \( f_r(x) = \sum_{y=1}^{\infty} p(y-x) f_r(y) \); any non-negative harmonic function on \( x \geq 1 \) is a constant multiple of \( f_r \).
(ii) \( 2E_0[-S_T]EZ = \sigma^2 \); in particular either \( EZ = \infty \) or \( E\hat{Z} = -\infty \) if \( \sigma^2 = \infty \).

By Theorem A and \( v(y) \leq 1 \) it follows that
\[
g_{(-\infty,0]}(x,y) \leq \begin{cases} f_r(x) & \text{if } x \leq y, \\ f_r(x) - f_r(x-y) & \text{if } x > y. \end{cases}
\] (2.6)

Noting \( \sum_y p(y-z)|z| < \infty \) we let \( x \to -\infty \) in \( H^x_{[0,\infty)}(y) = \sum g_{[0,\infty)}(x,z)p(y-z) \), which leads to
\[
H^{-x}_{[0,\infty)}(y) := \lim_{x \to -\infty} H^x_{[0,\infty)}(y) = \frac{1}{EZ} \sum_{w=1}^{\infty} f_r(w)p(y+w)
\] (2.7)
and
\[
H^x_{[0,\infty)}(y) \leq (EZ)H_{[0,\infty)}^{-x}(y) \text{ for all } x \leq 0 < y \text{ if } EZ < \infty.
\] (2.8)

In particular the three conditions (a) \( EZ < \infty \); (b) \( v_\infty > 0 \); (c) \( H_{[0,\infty)}^{-x}(0) > 0 \) are equivalent to one another. Since \( g_{[1,\infty)}(0,-y) = g_{(-\infty,0]}(y+1,1) = v(0)v^-(y) = v^-(y) \) we have for \( k > 0 \)
\[
P[Z = k] = \sum_{y=0}^{\infty} g_{[1,\infty)}(0,-y)p(k+y) = \sum_{y=0}^{\infty} v^-(y)p(k+y),
\] (2.9)
and, by summation by parts,
\[
P[Z > x] = \sum_{y=0}^{\infty} v^-(y)P[X > x+y] = \sum_{y=1}^{\infty} f_r(y)p(x+y) \quad (x \geq 0).
\] (2.10)

(Cf. e.g., Eq (3.7a) in Chapter XII of [10] for a similar identity.) Recalling \( f_r(x) \) is extended to \( x \leq 0 \) by (1.5) and employing (2.10) with \( x \) replacing \( -x \) for \( x \leq 0 \) we obtain
\[
\lim_{y \to +\infty} g_{(-\infty,0]}(x,y) = f_r(x)/EZ \quad \text{for } x \in \mathbb{Z}.
\] (2.11)

Similarly, by \( g_{[0,\infty)}(x,z) = g_{(-\infty,0]}(-z,-x), \lim_{x \to -\infty} g_{[0,\infty)}(x,z) = f_r(-z)/EZ \quad (z \leq -1) \).

The next theorem also is taken from Spitzer [16] T28.1, T29.1, P30.2, P30.3.

**Theorem C.** The series \( \sum_{n=0}^{\infty} [p^n(0) - p^n(-x)] \) converges for each \( x \in \mathbb{Z} \) and if \( a(x) \) denotes the sum, then the following relations hold.
\[
g_{[0]}(x,y) = \delta(0,x) + a(x) + a(-y) - a(x-y) \quad (x,y \in \mathbb{Z}),
\] (2.12)
\[
a(x+y) \leq a(x) + a(y),
\]
\[
\sum_{y=-\infty}^{\infty} p(y-x)a(y) = a(x) + \delta(0,x),
\] (2.13)
\[
\lim_{x \to \pm \infty} [a(x + 1) - a(x)] = \pm 1/\sigma^2 \quad \text{and} \quad \lim_{x \to \infty} [a(x) + a(-x)] = \infty. \tag{2.14}
\]

If the walk is left-continuous (i.e. \(P[X \leq -2] = 0\)), then \(a(x) = x/\sigma^2 \) for \(x > 0\); analogously \(a(x) = -x/\sigma^2 \) for \(x < 0\) for right-continuous walks; except for left- or right-continuous walks with infinite variance \(a(x) > 0 \) for all \(x \neq 0\).

We put \(a(x) = \frac{1}{2}[a(x) + a(-x)]\).

By (2.12) it follows that \(2\bar{a}(y) = \delta(0, y) = g_{(0)}(y, y)\) and that
\[
P_x[\sigma_y < \sigma_0] = \frac{a(x) + a(-y) - a(x - y)}{2\bar{a}(y)} \quad (x, y \neq 0). \tag{2.15}
\]

The equation (2.13) states that \(a\) is harmonic on \(x \neq 0\), which together with \(a(0) = 0\) entails that the process \(M_n := a(S_{\sigma_0^n} - \xi)\) is a martingale, provided that \(S_0 \neq \xi \in \mathbb{Z}\) a.s. Using the optional sampling theorem and Fatou’s lemma we obtain first the inequality
\[
a(x - \xi) = \lim_{n \to \infty} E_x[a(S_{n\sigma_B} - \xi)] \geq E_x[a(S_{\sigma_B} - \xi)] \quad \text{valid whenever} \quad \xi \in B \subset \mathbb{Z} \quad \text{and} \quad x \neq \xi,
\]
and then by using (2.13) again
\[
E_x[a(S_{\sigma_B} - \xi)] \leq a^+(x - \xi) \quad \text{for} \quad \xi \in B, x \in \mathbb{Z}, \tag{2.16}
\]
in particular
\[
a(y)P_x[\sigma_y < \sigma_0] = E_x[a(S_{\sigma_0^n\sigma_y})] \leq a^+(x) \quad (x, y \in \mathbb{Z}). \tag{2.17}
\]

In the rest of this section we prove several lemmas that are derived more or less directly from the results presented above and independently of one another except for Lemma 2.6 that is used for Lemmas 2.7 and 2.8.

**Lemma 2.1.** There exists \(\lim_{x \to \infty} a(x) \leq \infty\). This limit is zero if and only if the walk is right-continuous and \(\sigma^2 = \infty\).

**Proof.** We have only to consider the case \(\sigma^2 = \infty\). The relations (2.17) and (2.15) yield
\[
a(y) \geq \frac{a(x)}{a(x) + a(-x)}[a(y) + a(-y) - a(y - x)] \quad (x \neq 0).
\]

On using (2.14) it then follows that
\[
\liminf_{y \to \infty} a(y) \geq \frac{a(x)a(-x)}{a(x) + a(-x)} \quad \text{for all} \quad x \neq 0. \tag{2.18}
\]

If \(\limsup_{x \to \infty} a(x) < \infty\), then \(\limsup_{x \to \infty} a(-x) = \infty\) in view of (2.14) and the inequality (2.18) gives \(\liminf a(x) \geq \limsup a(x)\) so that \(\lim a(x)\) exists. If this limit is zero, then the right side of (2.18) is zero and \(\lim_{x \to \infty} a(-x) = \infty\), which in view of (2.17) is possible only if the walk is right-continuous.

Now suppose \(\limsup_{x \to \infty} a(x) = \infty\) and put \(M = \liminf_{x \to \infty} a(x) \leq \infty\). Contrary to what is to be shown let \(M < \infty\). Then one can choose \(N\) such that \(a(x) + a(-x) > 4M + 6\).
for $x > N$. In view of \((2.14)\) there must exists $x_0 > N$ such that $2M + 2 \leq a(x_0) < 2M + 3$, which entails $a(-x_0) > 2M + 3$. Combined with \((2.18)\) these lead to the absurdity

$$M \geq \frac{a(x_0)a(-x_0)}{a(x_0) + a(-x_0)} \geq \frac{a(x_0)}{2} \geq M + 1.$$ 

Hence $\liminf_{x \to \infty} a(x)$ must be infinite. \hfill \Box

**Lemma 2.2.** If $\sigma^2_\sim := E[X^2; X < 0] = \infty > -E[X; X < 0]$, then

(i) $\frac{1}{m_-(x)} \int_0^x P[\hat{Z} < -t] dt \to \frac{1}{cEZ}$ as $x \to \infty$.

(ii) $EZ \leq \liminf_{x \to \infty} \frac{\int_0^x f_r(x)m_-(x)}{x} \leq \limsup_{x \to \infty} \frac{\int_0^x f_r(x)m_-(x)}{x} \leq 2EZ$,

where $m_-(x)$ is the function defined in \((1.8)\). If $\sigma^2_\sim < \infty$, then the two limits in (ii) coincide and equal $m_-(\infty)/cE[\hat{Z}] \in (0, \infty)$.

**Proof.** As a dual relation of \((2.10)\) we have for $t \geq 0$

$$P[\hat{Z} < -t] = v^-(0) \sum_{y=0}^{\infty} v(y)P[X < -t - y]. \quad (2.19)$$

Let $\sigma^2_\sim = \infty > -E[X; X < 0]$, which entails that $\int_0^x P[X < -t - y] dt/m_-(x)$ tends to zero as $x \to \infty$ for each $y \geq 0$. Replacing $v(y)$ by $v_\infty + o(1)$ in \((2.19)\) and recalling $v^-(0)v_\infty = 1/cEZ \geq 0$ we then infer that

$$\frac{1}{m_-(x)} \sum_{t=0}^{x} P[\hat{Z} < -t] = \frac{v^-(0)}{m_-(x)} \sum_{t=0}^{x} \sum_{y=t}^{\infty} v(y - t)P[X < -y] = \frac{1}{cEZ} + o(1).$$

Thus (i) is verified. Noting that $cf_r(x + 1)$ is the renewal function for the variable $-\hat{Z}$ we use the first inequality of Lemma 1 of \cite{9} which may read

$$1 \leq \frac{cf_r(x + 1)}{x} \int_0^x P[\hat{Z} < -t] dt \leq 2;$$

combining this with (i) we can readily deduce (ii). The last assertion is obvious, for if $\sigma^2_\sim < \infty$, then $m_-(\infty) < \infty$, $E\hat{Z} > -\infty$ and $f_r(x) \sim x/cE[\hat{Z}]$. \hfill \Box

**Remark 2.1.** Lemma 2.2(ii) together with \((2.10)\) shows that $\int_0^\infty \{tP[X > t]\} dt/m_-(t) < \infty$ if $EZ < \infty$—the necessity half of the Chow’s criterion for $EZ < \infty$.

**Lemma 2.3.** If $EZ < \infty$, then $\lim_{x \to \infty} H_{[0,\infty)}^{-x}\{f_r\}/f_r(x) = 0$.

**Proof.** In the same way as for \((2.6)\) observe that that for $x, y > 0$ $g_{[0,\infty)}(-x, -y) \leq g_{[0,\infty)}(-x, -x) \wedge g_{[0,\infty)}(-y, -y) \leq f_r(x) \wedge f_r(y)$. Employing \((2.10)\) we then infer that

$$H_{[0,\infty)}^{-x}\{f_r\} \leq \sum_{y=1}^{x} f_r(y)P[Z > y] + f_r(x) \sum_{y>x} P[Z > y].$$

Hence the assertion of the lemma follows by dominated convergence. \hfill \Box
Lemma 2.4. If either $EZ < \infty$ or $E|\hat{Z}| < \infty$, then
\[
\lim_{x \to \infty} \frac{g(-\infty,0)(x,x)}{g_{(0)}(x,x)} = 1. \tag{2.20}
\]
If $EZ < \infty$, then
\[
\lim_{x \to \infty} \frac{f_r(x)}{2\bar{a}(x)} = EZ,
\]
and, if $E|\hat{Z}| < \infty$, then $\lim_{x \to \infty} f_l(x)/2\bar{a}(x) = -EZ$.

Proof. We prove the first half only, the second half being equivalent to it in view of Theorem A(i), (ii) and (2.12). For the proof consider the difference
\[
0 \leq g_{(0)}(x,x) - g_{(0)}(x,x) = \sum_{y < 0} P_x [T < \sigma_0, S_T = y] P_y [\sigma_x < \sigma_0] g_{(0)}(x,x).
\]
The first probability of the summand being equal to $H^x_{(-\infty,0]}(y)$, this may be written as
\[
0 \leq 1 - \frac{g_{(0)}(x,x)}{g_{(0)}(x,x)} = \sum_{y < 0} H^x_{(-\infty,0]}(y) P_y [\sigma_x < \sigma_0].
\]
The last sum is dominated by
\[
\sum_{y < 0} \sum_{z = 1}^K H^x_{(-\infty,0]}(y) H^y_{[0,\infty]}(z) P_z [\sigma_x < \sigma_0] + \sum_{y < 0} H^x_{(-\infty,0]}(y) H^y_{[0,\infty]} \{1_{[K,\infty]}\}
\]
for any integer $K > 0$ ($1_A$ denotes the indicator function of a set $A$), and this upper bound tends to zero as $x \to \infty$ and $K \to \infty$ in this order if $EZ < \infty$, since then the family $(H^y_{[0,\infty]}; y < 0)$, is tight. In the case $EZ > -\infty$ the same proof may be applied, with the last step being skipped. \hfill \Box

Lemma 2.5. Suppose $EZ < \infty$ and $\sigma^2 = \infty$. Then $\lim_{x \to \infty} a(-x)/a(x) = 0$.

Proof. Since $P_z [\sigma_0 < \sigma_x] = [a(z - x) + a(x) - a(z)]/2\bar{a}(x)$ ($z < 0 < x$), applications of (2.14), (2.8) and Theorem A show that under the assumption of the lemma
\[
\frac{a(x)}{a(x) + a(-x)} = \lim_{z \to -\infty} P_z [\sigma_0 < \sigma_x] = \sum_{y = 0}^\infty H^\infty_{[0,\infty]}(y) P_y [\sigma_0^o < \sigma_x^o], \tag{2.21}
\]
where $\sigma_x^o = \inf \{n \geq 0 : S_n = y\}$. As $x \to \infty$ the last sum approaches 1 and hence $a(-x)/a(x) \to 0$. \hfill \Box

Lemma 2.6. Let $B$ be a non-empty subset of $\mathbb{Z}$. Then for all $x, y \in \mathbb{Z}$,
\[
H^y_B \{a(\cdot - y)\} - a(x - y) \leq H^x_B \{a(\cdot - x)\} \tag{2.22}
\]
and for each $y \in B$ fixed, the difference $u(x) := a^\dagger (x - y) - H^y_B \{a(\cdot - y)\}$ is harmonic on $\mathbb{Z} \setminus B$ in the sense that
\[
\sum_{z \notin B} p(z - x) u(z) = u(x) \quad \text{for} \quad x \in \mathbb{Z}. \tag{2.23}
\]
Proof. \((2.22)\) is immediate from the subadditivity inequality \(a(S_{\sigma_B} - y) \leq a(S_{\sigma_B} - x) + a(x - y)\). Noting that \(\sum_{w \notin B} p(w - x)H_B^y\{a(\cdot - y)\} = H_B^x\{a(\cdot - y)\} - \sum_{z \in B} p(z - x)\{a(z - y)\}\) one deduces
\[
\sum_{w \notin B} p(w - x)[a(w - y) - H_B^y\{a(\cdot - y)\}] = a^*(x - y) - H_B^x\{a(\cdot - y)\},
\]
showing \((2.23)\), for \(\sum_{w \notin B} p(w - x)\delta(w, y) = 0\) if \(y \in B\).

For any non-empty subset \(B\) of \(\mathbb{Z}\) we can define a function \(u_B(x), x \in \mathbb{Z}\) by
\[
 u_B(x) = a(x - y) - H_B^x\{a(\cdot - y)\} + g_B(x, y)
\]
(2.25)
according to the following lemma.

**Lemma 2.7.** For each \(x \in \mathbb{Z}\) the right side of \((2.23)\) is independent of \(y \in \mathbb{Z}\) and \(u_B\) defined by it is non-negative and harmonic on \(\mathbb{Z} \setminus B\) in the same sense as \((2.23)\).

The same assertion holds for the two-dimensional recurrent random walks to which the same proof applies.

**Proof.** In the proof of [19, Lemma 2.9] it is shown that for each \(x \in \mathbb{Z}\) fixed, the right side of \((2.25)\) is a dual harmonic function of \(y \in \mathbb{Z}\). The first assertion of the lemma therefore follows from \((2.22)\) which implies that it is bounded below. Taking \(y\) from \(B\) in \((2.25)\) the inequality \((2.16)\) implies \(u_B \geq 0\). The identity \(\sum_{z \notin B} p(z - x)g_B(z, y) = g_B(x, y) - \delta(x, y)\) (valid for all \(x, y \in \mathbb{Z}\)) together with \((2.24)\) shows that \(u_B\) is harmonic as asserted.

**Remark 2.2.** For a positive integer \(N\) let \(\tau_N = \sigma_{\mathbb{Z} \setminus (-N,N)}\). The function \(u_B\) defined by \((2.25)\) is then given by
\[
u_B(x) = \lim_{N \to \infty} E_x[a(S_{\sigma_B} - y); \tau_N \leq \sigma_B],
\]
(2.26)
where the limit on the right side is independent of \(y\) as is readily ascertained. This equality is derived by taking \(\xi \in B\) for \(y \neq \xi\), \(M_n = a(S_{n \wedge \sigma_B} - \xi)\) being a martingale under \(P_x\), one verifies without difficulty that \(a(x - \xi) = E_x M_{\tau_N}\) and decomposing \(E_x M_{\tau_N}\) to see
\[
a(x - \xi) = E_x[a(S_{\tau_N} - \xi); \tau_N \leq \sigma_B] + E_x[a(S_{\sigma_B} - \xi); \tau_N > \sigma_B],
\]
making the corresponding decomposition of \(H_B^x\{a(\cdot - \xi)\}\) and passing to the limit as \(N \to \infty\) lead to \((2.26)\). When \(X\) is of finite range, the identity \((2.26)\) restricted to \(x \notin B\) is shown in the proof of [13, Proposition 4.6.3], which (somewhat different from ours) can be adapted to the setting of Lemma 2.7.

**Lemma 2.8.** If \(EZ < \infty\), then \(\sum_{y=-\infty}^0 a(y)P[X < y] < \infty\) and \(\lim_{x \to -\infty} H_{(-\infty,0]}^x\{a\}/f_r(x) = 0\).

**Proof.** By Theorem A \(g_{(-\infty,0]}(1, z) = v^- (0) v(z - 1)\) for \(z \geq 1\). Suppose \(EZ < \infty\). Then \(v_\infty > 0\) and for \(y \leq 0\),
\[
 H_{(-\infty,0]}^1(y) = v^- (0) \sum_{z=1}^\infty v(z - 1) p(y - z) \asymp P[X < y],
\]
and hence the first assertion follows, for $H_{(-\infty,0)}^1\{a\} = \sum_{y<0} a(y)H_{(-\infty,0)}^1(y) < \infty$ by virtue of Lemma 2.6. Since $H_{(-\infty,0)}^x\{y\} \leq f_r(x)P[X < y]$ and $H_{(-\infty,0)}^x\{y\}/f_r(x) \to 0$ as $x \to \infty$ for each $y \leq 0$, by dominated convergence $H_{(-\infty,0)}^x\{a\}/f_r(x) \to 0$, as desired.

**Lemma 2.9.** For all integers $x, y \geq 0$, $v(x + y) \geq v(x)v(y)$.

**Proof.** The ratio $v(x + y)/v(x)$ is not less than the conditional probability that $x + y$ is an ascending ladder point given so is $x$, but this conditional probability equals $v(y)$, showing the inequality of the lemma. 

\[\]  

\section{Proof of Theorem 1}  

**Lemma 3.1.** Suppose $EZ = \infty$. Then $a(x) = H_{(-\infty,0)}^x\{a\}$ ($x > 0$).

**Proof.** The proof rests on the fact that the function $h(x) := a(x) - H_{(-\infty,0)}^x\{a\}$ is non-negative and harmonic on $x > 0$ (according to Lemma 2.6). In view of the uniqueness of harmonic function it suffices to show

\[\lim_{x \to \infty} \inf \frac{a(x)}{f_r(x)} = 0. \tag{3.1}\]

We have extended $f_r$ to a function on $\mathbb{Z}$, denoted also by $f_r$, by (1.5), namely $f_r(x) = P[Z > -x]$ ($x \leq 0$). Accordingly, by (2.10) we have $f_r(x) = \sum_{y=1}^\infty p(y-x)f_r(y)$ for all $x \in \mathbb{Z}$.

By the assumption of the lemma the walk is not right-continuous. Hence by Lemma 2.1 $\inf_{x<0} a(x) > 0$, so that for some constant $C$

\[f_r(x) \leq Ca(x) \quad \text{for} \quad x < 0. \tag{3.2}\]

Define the operators $P$ and $P^-$ by

\[Pf(x) = \sum_{y \in \mathbb{Z}} p(y-x)f(y) \quad \text{and} \quad P^-f(x) = \sum_{y \leq 0} p(y-x)f(y) \quad (x \in \mathbb{Z}),\]

respectively. Put $G_n(x,y) = p^n(y-x) + \cdots + p(y-x) + \delta(x,y)$ and let $G_n$ also denote the corresponding operator. We may suppose $\inf_{x>0} a(x) > 0$. Owing to (3.2) relation (3.1) then follows if we have

\[\lim_{n \to \infty} \frac{P^n a^\dagger(0)}{P^n f_r(0)} = 0, \tag{3.3}\]

for if (3.1) does not hold, $a^\dagger = a + \delta(\cdot,0)$ must dominate a positive multiple of $f_r$ so that (3.3) is impossible. From the identity $Pa = a + \delta(\cdot,0)$ one deduces by induction that

\[P^n a^\dagger(x) = a(x) + G_n(x,0). \tag{3.4}\]

On the other hand one obtains that $Pf_r = f_r + P^-f_r$ and by induction again

\[P^n f_r(x) = f_r(x) + G_{n-1}P^-f_r(x). \tag{3.5}\]
Observe that \( \sum_{x \in \mathbb{Z}} P^{-} f_r(x) = \sum_{z < 0} f_r(z) = \sum_{y \geq 0} P[Z > y] = \infty \), where the last equality is due to the assumption of the lemma. For any \( K > 0 \) one can then choose a positive integer \( M \) so that \( \sum_{|z| \leq M} P^{-} f_r(z) \geq K \); and hence

\[
G_{n-1} P^{-} f_r(0) \geq K \min_{|z| \leq M} G_{n-1}(z, 0) \geq 2^{-1} K G_{n-1}(0, 0)
\]

if \( n \) is large enough, for the recurrence of the walk implies \( \lim_{n \to \infty} G_n(z, 0)/G_n(0, 0) = 1 \) (cf.\[16, P2.6\]). Combined with (3.4) and (3.5) the inequality derived above implies that

\[
EZ < \sup_{z \in \mathbb{Z}} G_n(z, 0) = 0.
\]

Observe that (3.3) since \( P^{-} f_r(0) \geq 0.5 K P^{n-1} a^\dagger(0) \) for all sufficiently large \( n \) and from this we can conclude the required relation (3.3) since \( P^{n-1} a^\dagger(0) = P^n a^\dagger(0) - p^n(0, 0) \) and \( \lim P^n a^\dagger(0) = \infty \).

**Proof of Theorem 7.** By Lemma 2.7 the formula of (i) follows if we verify its special case \( y = 0 \). Note that \( g_{(-\infty, 0)}(x, 0) = \delta(x, 0) \). It is then obvious that if \( EZ = \infty \), (i) and (iii) follows from Lemma 3.1 and (3.1), respectively. Let \( EZ < \infty \). Then by Lemma 2.6 the difference \( a^\dagger(x) - H_x^{(-\infty, 0)} \{a\} \) is non-negative and harmonic on \( x > 0 \), so that it is a constant multiple of \( f_r(x) \). The constant factor is determined by using Lemmas 2.4, 2.5, and 2.8. (Note that \( a(x) \sim a(x) \) if \( \sigma^2 < \infty \).) By these lemma (ii) also follows.

**4 Proof of Theorem 2**

*Proof of (i).* The first half of (i) of Theorem 2 follows from Lemma 2.2 and Corollary 1. Suppose that \( EZ < \infty \) and \( \sigma^2 = \infty \). The formula (1.10), what is asserted in the second half, may be written as

\[
a(-x) \sim \frac{1}{EZ} \sum_{k=1}^{\infty} P[k < Z \leq x + k] a(k) \quad (x \to \infty).
\]

We compute \( H_{-x}^{(-\infty, \infty)} \{a\} \), which equals \( a(-x) \) in view of Corollary 1(ii). We write

\[
H_{-x}^{(-\infty, \infty)} \{a\} = \sum_{z=1}^{\infty} a(z) \sum_{y=1}^{\infty} g_{(-\infty, \infty)}(-x, -y) p(z + y) = \sum_{y=1}^{x-1} I_y(x) + \sum_{k=1}^{x} J_k(x),
\]

where

\[
I_y(x) = \sum_{z=1}^{\infty} a(z) \sum_{k=1}^{y} v(x - k) v^{-}(y - k) p(z + y),
\]

\[
J_k(x) = \sum_{z=1}^{\infty} a(z) \sum_{y=x}^{\infty} v(x - k) v^{-}(y - k) p(z + y).
\]

First we verify that \( v(x - k) \) in these expressions may be replaced by \( v_\infty \), which is positive since \( EZ < \infty \). To this end it suffices to show that for each \( j = 0, 1, \ldots, \), \( I_{x-j}(x) \to 0 \) and \( J_{x-j}(x) \to 0 \) as \( x \to \infty \). Since \( a(z) \) is dominated by a constant multiple of \( f_r \), by (2.10)

\[
I_{x-j} \leq C \sum_{k=1}^{x-j} v^{-}(x - j - k) P[Z > x - j] = C f_r(x - j) P[Z > x - j].
\]
Since $Ef_r(Z) < \infty$, the rightmost member tends to zero. We see $J_{x-j}(x) \to 0$ in a similar way.

As a consequence of the replacement mentioned above we have two asymptotic equivalences

$$\sum_{y=1}^{x-1} I_y(x) \sim v_\infty \sum_{z=1}^{\infty} a(z) \sum_{y=1}^{x-1} f_r(y)p(z+y) \quad \text{and}$$

$$\sum_{k=1}^{x} J_k(x) \sim v_\infty \sum_{z=1}^{\infty} a(z) \sum_{y=x}^{\infty} [f_r(y) - f_r(y - x + 1)]p(z+y).$$

Finally summing them and applying the identity $\sum_{y=1}^{\infty} f_r(y)p(z+y) = P[Z > z]$ yield the required formula. \hfill \Box

**Proof of (ii).** If $\limsup_{x \to \infty} a(x) < \infty$, then $H_f^r_{\mathcal{P}_{-\infty,0}} \{a\}$ is bounded, so that $EZ$ cannot be infinite in view of Corollary 2(ii) and (2.14). This shows (ii).

**Proof of (iii).** Let $\sigma^2 = \infty$ and $EZ < \infty$. Then it follows from (4.1) that

$$a(-x) \sim \frac{1}{EZ} \sum_{w=1}^{x} \sum_{k=1}^{\infty} P[Z = k + w]a(k) = \frac{1}{EZ} \sum_{w=1}^{x} b(w)$$

where $b(w) = \sum_{k=1}^{\infty} P[Z = w + k]a(k)$. By (2.9)

$$b(w) = \sum_{k=1}^{\infty} \sum_{y=1}^{\infty} v^-(y)p(w + k + y)a(k)$$

After a change of variable this double series becomes

$$\sum_{j=1}^{\infty} p(w + j) \sum_{-j < z < j}^* v^\left(-\left(\frac{j - z}{2}\right)\right)a\left(\frac{j + z}{2}\right),$$

where $\sum^*$ indicates that the summation is restricted to $z$ such that $j + z$ is even. By sub-additivity of $a$ and the fact that $a(x)EZ \sim f_r(x) = v^-(1) + \cdots + v^-(x)$, we have $\frac{1}{2}f_r(j) \leq f_r\left(\frac{j+z}{2}\right) \leq f_r(j)$ for $0 \leq z \leq j$ with $j$ large enough, which shows

$$C^{-1}[f_r(j)]^2 \leq \sum_{0 \leq z < j}^* v^\left(-\left(\frac{j - z}{2}\right)\right)a\left(\frac{j + z}{2}\right) < \sum_{-j < z < j}^* v^\left(-\left(\frac{j - z}{2}\right)\right)a\left(\frac{j + z}{2}\right) \leq C[f_r(j)]^2,$$

so that $b(w) \asymp \sum_{j=1}^{\infty} p(w + j)[f_r(j)]^2$, the symbol $\asymp$ indicating that the ratio of the two sides of it is bounded away from zero and infinity. Now substitution into (4.2) yields

$$a(-x) \asymp \sum_{w=1}^{x} \sum_{j=1}^{\infty} p(w + j)[f_r(j)]^2.$$

In view of (1.9) (or Lemma 2.2(ii)) we can replace $f_r(j)$ by $j/m_-(j)$, showing (1.11), the desired asymptotics of $a(-x)$. The rest of (iii) is readily ascertained by (1.11), Lemma 2.1 and (2.8). \hfill \Box
Remark 4.1. The second result of (i) of Theorem 2 is effectively used for the proof of (iii). It would be desirable that we may dispense with it for the second assertion of (iii) giving the condition for the boundedness of \( a(-x), x > 0 \). By \((2.7), (2.8)\) and the dual assertion of Corollary 1(ii) if \( EZ < \infty \) and \( \sigma^2 = \infty \),

\[
\lim_{x \to \infty} a(-x) = H_{[0,\infty)}^{-\infty}\{a\} = (EZ)^{-1} \sum_{w=1}^{\infty} \sum_{z=1}^{\infty} f_r(w)p(z+w)a(z) \leq \infty,
\]

in particular for \( a(-x), x > 0 \) to be bounded it is necessary and sufficient that \( H_{[0,\infty)}^{-\infty}\{a\} < \infty \). As in the proof above we apply Corollary 1(i) and the sub-additivity of \( a \) to see that this is equivalent to

\[
\sum_{j=1}^{\infty} p(j) a(j) \sum_{z=1}^{j} a(z) < \infty. \tag{4.3}
\]

Now we replace \( a(x) \) by \( x/m_-(x) \). By an elementary computation we check that

\[
\left( \frac{1}{m_-(t)} \int_0^t \frac{s}{m_-(s)} ds \right)' = \frac{t^2 + \int_0^t P[X \leq -s]ds \int_0^t \frac{1}{m_-(s)} ds}{[m_-(t)]^2} \propto \frac{t^2}{[m_-(t)]^2},
\]

and then perform summation by parts to find that condition \((4.3)\) in turn is equivalent to \((1.12)\) as desired.

5 Proofs of Theorems 3 and 4 and Corollary 3

The proof of Theorem 3 consists of several lemmas that are given below. If \( \sigma^2 < \infty \) the results given in these lemmas are either trivial or easily obtained from those of Section 2, so the arguments for their proofs are given only in the case \( \sigma^2 = \infty \) in most places.

Lemma 5.1. If \( a(-z)/a(z) \to 0 \) as \( z \to \infty \), then \( a(-N) - a(x - N) = \sigma^{-2} x + o(a(x)) \) \((N \to \infty)\) uniformly for \( 0 < x < N \); in particular uniformly for \( 0 < x < N \)

\[
2\tilde{a}(N)P_x[\sigma_N < \sigma_0] = a(x)(1 + o(1)) + \sigma^{-2} x \quad (N \to \infty).
\]

Proof. This follows from Lemma 5.2(ii) of [21].

Lemma 5.2. (i) If the walk is not left-continuous, then as \( N \to \infty \)

\[
\frac{P_x[\sigma_N < T]}{P_x[\sigma_N < \sigma_0]} = \begin{cases} \frac{1 - H_{(-\infty,0)}^x\{a\}}{a(x)} + o(1) & \text{if } EZ < \infty; \\ o(1) & \text{if } EZ = \infty \end{cases} \quad (x > 0).
\]

In case \( EZ < \infty \), \( o(1) \) is uniform for \( 0 < x < N \).

(ii) If \( \sigma^2 = \infty \), then \( \lim_{N \to \infty} 2\tilde{a}(N)P_x[T < \sigma_N < \sigma_0] = H_{(-\infty,0)}^x\{a\} \quad (x > 0) \).

Proof. We employ the following identities:

\[
P_x[\sigma_N < T] = \frac{g(-\infty,0)(x,N)}{g(-\infty,0)(N,N)}; \quad P_x[\sigma_N < \sigma_0] = \frac{a(x) + a(-N) - a(x - N)}{2\tilde{a}(N)}. \tag{5.1}
\]
and  \( P_x[T < \sigma_N < \sigma_0] = P_x[\sigma_N < \sigma_0] - P_x[\sigma_N < T] \) \,(0 < x < N). \quad (5.2)

The case \( EZ < \infty \) of (i) follows from (5.1), Lemma 2.4 and Corollary (i). The uniformity of the convergence is verified by combining Lemma 5.1 and the uniformity of the convergence of \( \lim_{N \to \infty} g_{(-\infty,0]}(x, N)/f_r(x) = v_{\infty} \).

For the rest of proof observe that on the one hand (5.1) and (5.2) together yield
\[
2\bar{a}(N)P_x[T < \sigma_N < \sigma_0] = [a(x) + a(-N) - a(x - N)] \left(1 - \frac{P_x[\sigma_N < T]}{P_x[\sigma_N < \sigma_0]}\right) \quad (5.3)
\]
and on the other hand since \( P_x[T < \sigma_N < \sigma_0] = \sum_{y < 0} P_x[S_T = y, T < \sigma_N]P_y[\sigma_N < \sigma_0], \)
\[
2\bar{a}(N)P_x[T < \sigma_N < \sigma_0] = \sum_{y < 0} P_x[S_T = y, T < \sigma_N][a(y) + a(-N) - a(y - N)].
\]

In case \( \sigma^2 = \infty \) first applying Fatou’s lemma to the last infinite series and then using these two identities we obtain
\[
H^x_{(-\infty,0]}\{a\} \leq \liminf_{N \to \infty} 2\bar{a}(N)P_x[T < \sigma_N < \sigma_0]
= a(x) - a(x) \limsup_{N \to \infty} \frac{P_x[\sigma_N < T]}{P_x[\sigma_N < \sigma_0]} \leq a(x). \quad (5.4)
\]

Let \( EZ = \infty \). Then the two extreme members in (5.4) must coincide owing to Lemma 3.1 entailing that the two inequalities above are the equality, of which the latter means that the \( \limsup \) vanishes, showing the relation (i) of the lemma. We can interchange the \( \liminf \) and the \( \limsup \) in the above, which gives the equality of (ii).

If \( EZ < \infty \) and \( \sigma^2 = \infty \), on going back to (5.3) substitution of the first relation of (i) into it immediately concludes (ii). \( \square \)

**Lemma 5.3.** Suppose that either \( EZ < \infty \) or \( E\hat{Z} > -\infty \). Then as \( N \to \infty \)
\[
P_x[\sigma_{[N,\infty)} < T] = P_x[\sigma_N < T](1 + o(1)),
\]
where \( o(1) \) is uniform for \( 0 < x < N \).

**Proof.** First suppose \( EZ < \infty \). Since \( f_r \) is harmonic and increasing on \([1, \infty), \)
\[
f_r(x) \geq E_x[f_r(S_T \wedge \sigma_{[N,\infty)})] \geq f_r(N)P_x[\sigma_{[N,\infty)} < T] \quad (x > 0). \quad (5.5)
\]
Hence
\[
\frac{f_r(x)}{f_r(N)} \geq P_x[\sigma_{[N,\infty)} < T] \geq P_x[\sigma_N < T] = \frac{f_r(x)}{f_r(N)}(1 + o(1)), \quad (5.6)
\]
where for the last equality we have used the first relation of (5.1) as well as Theorem A.

In case \( E\hat{Z} > -\infty \), the family \( \{H^y_{(-\infty,0]} : y > 0\} \) is tight, implying that
\[
\sup_{y > 0} P_y[T < \sigma_N] = \sup_{y > 0} \sum_{z > 0} H^y_{(-\infty,0]}(-z)P_N_{-z}[T' < \sigma_N] \to 0 \quad \text{as} \quad N \to \infty,
\]
where \( T' = \inf\{n > 0 : S_n \leq 0\} \). Hence the ratio
\[
\frac{P_x[\sigma_{[N,\infty)} < T] - P_x[\sigma_N < T]}{P_x[\sigma_{[N,\infty)} < T]} = \sum_{y > N} P_x[S_{\sigma_{[N,\infty)}} = y | \sigma_{[N,\infty)} < T]P_y[T < \sigma_N] \quad (5.7)
\]
tends to zero uniformly for \( 0 < x < N \), which is the same as the relation of the lemma. \( \square \)
Remark 5.1. For contrast to (5.6) which holds only if $EZ < \infty$, in case $E\hat{Z} > -\infty$, or what amounts to the same, $v^-_\infty > 0$, by substituting the expression of $P_x[\sigma_N < T]$ given in (5.1) one obtains from Lemma 5.3

$$P_x[\sigma_{[N,\infty)} < T] = \frac{\sum_{k=1}^{\infty} v^-(x-k)v(N-k)}{f_1(N)} (E|\hat{Z}| + o(1)).$$

Lemma 5.4. Suppose that $a(-z)/a(z) \to 0$ or $\infty$ as $z \to +\infty$. Then as $N \to \infty$

$$P_x[\sigma_{[N,\infty)} < \sigma_0] = P_x[\sigma_N < \sigma_0](1 + o(1)),$$

where $o(1)$ is uniform for $0 < x < N$.

Proof. This is implied by Proposition 5.2 of [21].

Proof of Theorem 3. The first formula of (i) follows from Lemma 5.3 and the second one from Lemma 5.4. (ii) is the same as Lemma 5.2(i).

Proof of Corollary 3. The first half of Corollary 3 follows immediately from (5.6). For the second half, we use the relation of Remark 5.1 above to verify that if $E|\hat{Z}| < \infty$, then uniformly for $N > x$, as $x \to \infty$,

$$P_x[\sigma_{[N,\infty)} < T] = \frac{f_1(N) - f_1(N-x)}{f_1(N)} (1 + o(1)), \tag{5.8}$$

the dual of the relation to be shown. For verification of (5.8), on recalling $v^-_\infty = -E\hat{Z}$, it suffices to see that for each $K > 0$, $\sum_{k=1}^{\infty} v(N-k)$ divided by $\sum_{k=1}^{\infty} v(N-k)$ tends to zero as $N \to \infty$, but this follows from Lemma 2.9. Indeed, Lemma 2.9 says that for each $m \leq K$, $\sum_{k=1}^{\infty} v(N-k) \geq v(N-x+m)[v(0) + \cdots + v(x-m-1)]$, so that the ratio in question is dominated by $K/f_1(x-K)$, which tends to zero.

Proof of Theorem 4. Let $EZ < \infty$ and $\sigma^2 = \infty$. The proof rests on the first formula of Corollary 3(i) that entails that in order for $P_x[\sigma_{[N,\infty)} < T]$ to converge as $x/N \to \lambda$ for each $0 < \lambda < 1$, it is necessary and sufficient that

$$f_r(x) \sim x^\gamma L(x)/\beta \tag{5.9}$$

with some constants $\beta > 0$ and $0 \leq \gamma \leq 1$ and $L$ slowly varying.

Recall that $cf_r(x) = c[v^-(0) + \cdots + v^-(x-1)]$ is the renewal function associated with the variable $-\hat{Z}$. Taking the Abel transform coverts the corresponding renewal equation

$$v^-(x) = \frac{1}{c} \delta(0, x) + \sum_{y=0}^{x} P[-\hat{Z} = y]v^-(x-y), \quad x \geq 0 \tag{5.10}$$

into

$$c\hat{v}^-(s) = 1/(1 - \omega(s)), \quad 0 < s < 1, \tag{5.11}$$

where $\hat{v}^-(s) = \sum_{x=0}^{\infty} s^x \hat{v}^-(x)$ and $\omega(s) = \sum_{x=0}^{\infty} s^x P[-\hat{Z} = x]$. By Karamata’s Tauberian theorem (cf. [10, Section XIII.5]), on writing $\alpha = \gamma + 1$, (5.9) is equivalent that $\hat{v}^-(s) \sim \Gamma(\alpha)(1-s)^{-\alpha+1}L(1/(1-s))/\beta$, or by (5.11)

$$\sum_{x=0}^{\infty} s^x P[-\hat{Z} > x] = \frac{1-\omega(s)}{1-s} \sim \frac{\beta/c}{\Gamma(\alpha)(1-s)^{2-\alpha}L(1/(1-s))} \quad (s \uparrow 1),$$

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which, in view of Lemma 2.2(i), in turn is equivalent that

$$m_-(x) \sim \frac{\beta EZ}{\Gamma(\alpha)\Gamma(3-\alpha)} \cdot \frac{x^{2-\alpha}}{L(x)} \quad (x \to \infty),$$

(5.12)

showing the equivalence of (1), (2) and (3) of Theorem 4 and (1.18) as well as (1.19).

It remains to show the assertion concerning Spitzer’s condition (1.17). To this end we may let \(\frac{\beta EZ}{\Gamma(\alpha)\Gamma(3-\alpha)} = 1\). It suffices to show that under \(EZ < \infty\), (5.12) is equivalent to

\[
\begin{align*}
  a) & \quad P[X < -x] \sim -(2-\alpha)(\alpha-1)x^{-\alpha}/L(x) \quad \text{if } 1 < \alpha < 2, \\
  b) & \quad E[X^2; -x \leq X < 0] \sim 2/L(x) \quad \text{if } \alpha = 2, \\
  c) & \quad E[-X; X < -x] \sim 1/L(x) \quad \text{if } \alpha = 1,
\end{align*}
\]

(5.13)

the latter being true if and only if Spitzer’s condition holds [18]. For \(1 < \alpha < 2\), this equivalence is immediate from a monotone density theorem (cf. [10, Lemma XIII.5], [3]).

Let \(\alpha = 2\) and put \(G(x) = E[X^2; -x \leq X < 0]\). Write \(\ell_-(x)\) for \(P[-X > x]\) so that \(m_-(x) = \int_0^x dt \int_{-\infty}^x \ell_-(u)du\) and \(G(x) = -\int_0^x t^2 d\ell_-(t)\). If \(m_-(x) \sim 1/L(x)\), then in view of the monotone density theorem \(x \int_{-\infty}^\infty \ell_-(t)dt = o(m_-(x))\), hence \(x^2 \ell_-(x) = 2x \int_{x/2}^x \ell_-(t)dt = o(m_-(x))\) and by the identity

\[
G(x) = -x^2 \ell_-(x) - 2x \int_{-\infty}^\infty \ell_-(t)dt + 2m_-(x)
\]

(5.14)

one concludes that \(G(x) \sim 2m_-(x)\). For the converse, express \(\int_{-\infty}^\infty t^{-1}dG(t) > 0\) in two ways as follows

\[
\int_{-\infty}^\infty t^{-1}dG(t) = \begin{cases} 
-x^{-1}G(x) + \int_x^\infty t^{-2}G(t)dt, \\
x\ell_-(x) + \int_x^\infty \ell_-(t)dt.
\end{cases}
\]

If \(G(x) \sim 2/L(x)\), then the first expression is \(o(1/xL(x))\), hence so is the second and the latter bound together with (5.14) shows \(m_-(x) \sim 1/L(x)\).

The case \(\alpha = 1\) is similarly dealt with on putting \(G(x) = E[-X; X < -x]\) and using \(G(x) = -\int_x^\infty t \ell_-(t)dt = x\ell_-(x) + \int_x^\infty \ell_-(t)dt\) instead of (5.14). (Note that for a recurrent walk \(E[X; X > 0] < \infty\) implies \(E[-X; X < 0] < \infty\).) The details are omitted.

\section{The walk conditioned on \(\sigma_0 = \infty\)}

Write \(\bar{P}_x\) for \(P_x[\cdot | \sigma_0 = \infty] (x \neq 0)\), the probability law of the conditional process \(S_n\) given that it never visiting the origin. It is defined as a limit law of \(P_x[\cdot | \sigma_0 > k]\) as \(k \to \infty\). If \(\sigma = \infty\), suppose that \(P[X \leq -2]P[X \geq 2] > 0\) so that \(a(x) > 0\) for all \(x \neq 0\). The conditional process is Markovian with state space \(\mathbb{Z} \setminus \{0\}\) and the \(n\)-step transition law given by

\[
\frac{1}{a(x)} q^n(x, y)a(y) \quad (x, y \neq 0) \quad \text{where } q^n(x, y) = P_x[S_n = y, \sigma_0 > n],
\]

(6.1)

Indeed

\[
P_x[S_n = y | \sigma_0 > k] = q^n(x, y) \frac{P_y[\sigma_0 > k - n]}{P_x[\sigma_0 > k]},
\]

(6.2)
and as \( k \to \infty \), \( P_{x[\sigma_0>k]} \to a(y)/a(x) \) while \( P_{x[\sigma_0=k-j]} \to 0 \) for each \( j \) (see [16, T32.1, T32.2]) so that the ratio in the right side of (6.2) converges to \( a(y)/a(x) \). It follows from (6.1) that
\[
\tilde{H}^x_{(-\infty,0)}(y) := \tilde{P}_x[S_T = y] = \frac{1}{a(x)}H^x_{(-\infty,0)}(y)a(y) \quad (x > 0, y < 0).
\]
Therefore Corollary 1 gives that
\[
\tilde{P}_x[T < \infty] = 1 - \frac{f_r(x)}{E[Z]a^\dagger(x)} \quad (x \in \mathbb{Z}).
\]
By (6.1) \( \sum_n \tilde{P}_x[S_n = y] < \infty \). Hence
\[
\tilde{P}_x[|S_n| \to \infty \text{ as } n \to \infty] = 1.
\]
In fact we have that if \( \sigma^2 = \infty \), for every \( x \in \mathbb{Z} \),
\[
\tilde{P}_x[\lim S_n = +\infty] = 1 \quad \text{if } EZ < \infty;
\]
\[
\tilde{P}_x[\lim sup S_n = +\infty \text{ and } \lim inf S_n = -\infty] = 1 \quad \text{if } EZ = -E\hat{Z} = \infty;
\]
and if \( \sigma^2 < \infty \), either \( \lim S_n = +\infty \) or \( \lim S_n = -\infty \) with \( P_x \)-probability one and
\[
\tilde{P}_x[\lim S_n = +\infty] = \frac{a^\dagger(x) + \sigma^{-2}x}{2a(x)} \quad (x \in \mathbb{Z}).
\]
The first two identities above are readily deduced from (6.4) and its dual relation as well as (6.5). The last one is obtained by applying a theorem from the theory of Martin boundary (see [13, Theorem III29.2]): the Martin kernel \( \kappa(\cdot, \pm) \) relative to a reference point \( \xi \in \mathbb{Z}\setminus\{0\} \) is given by \([a(\cdot) \pm \sigma^{-2}\cdot]/[a(\xi) + \sigma^{-2}\xi])\). The conditional process \((\tilde{P}_x)_{x \neq 0}\) is a harmonic transform of the walk with absorption at the origin whose Martin boundary contains exactly two extremal harmonic functions \( h_+ \) and \( h_- \) given by \( h_\pm = \lim_{y \to \pm\infty} g_{0}(y)/\sum_z p(z)g_{0}(z, y) = a(x) \pm \sigma^{-2}x \). It is noticed that if \( \sigma^2 = \infty \), there is only one harmonic function, hence a unique Martin boundary point: \( \lim_{|y| \to \infty} g_{0}(\cdot, y)/g_{0}(\cdot, \xi) = a(\cdot)/a(\xi) \), so that two geometric boundary points \( +\infty \) and \( -\infty \) are not distinguished in the Martin boundary whereas the walk itself discerns them provided that either \( EZ \) or \( E\hat{Z} \) is finite.

Here we provide a direct proof of (6.7). To this end another characterization of \( \tilde{P}_x \) is convenient for the present purpose. Suppose \( \sigma^2 < \infty \) and put \( B(N) = (-\infty, -N) \cup [N, \infty) \). Using (2.15) one then infers first \( P_x[\sigma_{[N,\infty]} \vee \sigma_{(-\infty,-N]} < \sigma_0] = o(1/N) \) and then as \( N \to \infty \)
\[
P_x[\sigma_{B(N)} < \sigma_0] \sim \frac{a^\dagger(x)}{\alpha(N)}.
\]
This shows that \( \tilde{P}_x \) equals the limit as \( N \to \infty \) of another conditional law \( P_x[\cdot | \sigma_{B(N)} < \sigma_0] \), the walk conditioned on the event that \( B(N) \) is reached before the first visit of 0. In view of (6.5) \( \tilde{P}_x[\lim S_n = \infty] = \lim_{M \to \infty} \tilde{P}_x[\sigma_{(-\infty,-M]} = \infty] \), but
\[
\tilde{P}_x[\sigma_{(-\infty,-M]} = \infty] = \lim_{N \to \infty} \frac{P_x[\sigma_{[N,\infty]} \vee \sigma_{(-\infty,-M]} < \sigma_0]}{P_x[\sigma_{B(N)} < \sigma_0]} \quad (x \in \mathbb{Z}).
\]
where \( o_M(1) \to 0 \) as \( M \to \infty \) uniformly for \( N > M \) (since \( \sup_{y \geq 1} P_y[\sigma_{(-\infty,-M]} < \sigma_0] \to 0 \)). Since \( P_x[\sigma_{[N,\infty]} < \sigma_0] \sim [a^\dagger(x) + \sigma^{-2}x]/2\alpha(N) \), with the help of (6.8) this yields (6.7).
7 Appendix

A) Put $Z' = S_{\alpha}[0,\infty) - S_0$, the weak ascending ladder height. The renewal functions for the strictly and weakly ascending ladder height processes are defined by $U_{as}(x) = 1 + \sum_{k=1}^{\infty} P[Z_1 + \cdots + Z_k \leq x]$ and $V_{as}(x) = 1 + \sum_{k=1}^{\infty} P[Z'_1 + \cdots + Z'_k \leq x]$ $(x = 0, 1, 2, \ldots)$. Here $(Z_n)$ and $(Z'_n)$ are i.i.d. copies of $Z$ and $Z'$, respectively. It follows [10, Section XII.1] that $P[Z' \leq x] = P[Z' = 0] + P[Z' > 0] \Pr[Z \leq x]$ and

$$V_{as}(x) = U_{as}(x)/P[Z' > 0].$$

Let $\tau = \sigma_{[1,\infty)}$, $\tau' = \sigma_{[0,\infty)}$ and $c(t) = e^{-\sum_{k=1}^{\infty} k^{-1} t^k P(0)} (t \geq 0)$. Then $S_{\tau'} \overset{\text{law}}{=} Z'$ and $S_{\tau} \overset{\text{law}}{=} Z$ under $P_0$ and $1 - E_0[t^{\tau'} z^{S_{\tau'}}] = c(t)(1 - E_0[t^{\tau} z^{S_{\tau}}])$ for $0 \leq t < 1, 0 < |z| < 1$ ([16 Proposition 17.5]), so that on letting $z \downarrow 0$ and $t \uparrow 1$ in this order

$$P[Z' > 0] = c(1) = c.$$ 

Put $\tau(x) = \inf\{n \geq 1 : Z_1 + \cdots + Z_n > x\}$, the first epoch when the ladder height process enters $[x, \infty)$. Then from the identity $P[\tau(x) > n] = P[Z_1 + \cdots + Z_n \leq x]$ one finds that

$$U_{as}(x) = E[\tau(x)] \quad (x = 1, 2, \ldots),$$

which especially shows that $U_{as}(x)$ is sub-additive.

Analogous relations hold for $U_{ds}$ and $V_{ds}$, the renewal functions of strictly and weakly descending ladder height processes, respectively.

B) Let $T_0 = 0$ and $T_n$ be a random walk on $\{0, 1, 2, \ldots\}$ with i.i.d. increments. Put

$$u_x = \sum_{n=0}^{\infty} P[T_n = x] \quad (x = 1, 2, \ldots), \quad G(t) = \int_{0}^{t} P[T_1 > s]ds \quad (t \geq 0)$$

and suppose that $u_x$ is positive for all sufficiently large $x$. Erickson [8, §2(ii)] shows that $\lim u_x G(x) = 1$ if $tP[T_1 > t]$ is slowly varying at infinity. This restriction on $T_1$ is relaxed in the following lemma, which is used to obtain the relation (3) in case $\alpha = 2$ in Remark 1.4.

Lemma 7.1. If $G$ is slowly varying at infinity, then $u_x \sim 1/G(x)$ as $x \to \infty$.

Proof. We follow the argument made by Erickson [8]. Let $\phi(\theta) = E \exp\{i\theta T_1\}$. Unlike [8] we take up the sine series of coefficients $u_x$ that represents the imaginary part of $1/(1 - \phi(\theta))$. Fourier inversion yields

$$u_x = \frac{2}{\pi} \int_{0}^{\pi} S(\theta) \sin x\theta d\theta, \quad \text{where} \quad S(\theta) = \Im\left(\frac{1}{1 - \phi(\theta)}\right),$$

as is ensured shortly. Note that under the assumption of the lemma

$$tP[T_1 > t]/G(t) \to 0 \quad (t \to \infty).$$
(cf. [10, Theorem VIII.9.2]). On using this we observe that as \( \theta \downarrow 0 \), \( \int_0^{\varepsilon/\theta} t\,dP[T_1 \leq t] \sim G(1/\theta) \) for each \( \varepsilon > 0 \) and hence

\[
1 - \phi(\theta) = \left[ \int_0^{1/\theta} + \int_{1/\theta}^{\infty} \right] (1 - e^{i\theta \cdot})dP[T_1 \leq t] = -i\theta G(1/\theta)\{1 + o(1)\} + O(P[T_1 > 1/\theta])
\]

\[
= -i\theta G(1/\theta)\{1 + o(1)\}. \tag{7.3}
\]

Since for \( 0 < \theta \leq \pi \), \( 1 - \phi(\theta) \neq 0 \) and \( |\phi(\theta)| < 1 \) except for a finite number of points, it follows that \( \int_0^\pi [(1 - \phi^n)/(1 - \phi)](\theta) \sin x \, d\theta \to 0 \) \( (n \to \infty) \) for each \( x \), which entails \( (7.2) \).

Decomposing \( u_x = \frac{2}{\pi} \int_0^B \frac{\sin u \, du}{uG(x/u)} \{1 + o(1)\} \)

with \( o(1) \to 0 \) as \( x \to \infty \) for each \( B > 1 \). On the other hand

\[
\pi J_2 = \int_{B/x}^{\pi} \left[ S(\theta) - S\left( \frac{\theta + \pi}{x} \right) \right] \sin x \, d\theta + \left( \int_{\pi - B/x}^{\pi} - \int_{B-\pi/x}^{B/x} \right) S\left( \frac{\theta + \pi}{x} \right) \sin x \, d\theta.
\]

By \( (7.3) \) \( |S(\theta)| \leq C/\theta G(1/\theta) \) \( (\theta > 0) \) and it is easy to see that the second term on the right side above is bounded in absolute value by a constant multiple of \( B/x + B/xG(x/B) \). With the help of

\[
|\phi(\theta) - \phi(\theta')| \leq 2|\theta - \theta'|G(1/|\theta - \theta'|) \quad (\theta \neq \theta')
\]

(Lemma 5 of [3]) the same proof as given in [3] (5.15)) yields the bound \( C'/BG(x) \) for the first term. Thus \( \lim_{x \to \infty} G(x)J_2 \leq C'/B \). Combining this with \( (7.4) \) and letting \( x \to \infty \) and \( B \to \infty \) in this order we conclude \( G(x)u_x \to 1 \) as desired.

\[\square\]

C) Here we prove what is mentioned at the end of Remark 1.3: if \( X \) is in a domain of attraction of a stable law of index \( 1 < \alpha \leq 2 \) with skewness parameter \( \beta = -1 \) (i.e., Lévy measure concentrate on the negative half line), then as \( N \to \infty \)

\[
(*) \quad P_x[\sigma_{(N,\infty)} < \sigma_{(-\infty,0]}] \sim f_r(x)/f_r(N) \quad \text{uniformly for} \quad 1 \leq x < N,
\]

in particular the overshoots estimate \( (1.16) \) holds according to Lemma 1.1.

On recalling the proof of Lemma 5.3 the inequalities in \( (5.5) \) hold generally and this reduces our task to verification of the last inequality in \( (5.6) \), in other words, for the proof of \( (*) \) it suffices to obtain the lower bound

\[
P_x[\sigma_N < \sigma_{(-\infty,0]}] \geq [f_r(x)/f_r(N)](1 + o(1)). \tag{7.5}
\]

The probability on the left equals \( g_{(-\infty,0]}(x,N)/g_{(-\infty,0]}(N,N) \), which may be expressed by means of \( v \) and \( v_- \) (see Theorem A). Under the assumption of the assertion it follows that \( \int_0^x P[Z > t]dt \sim 1/L(x) \),

\[
\int_0^x P[Z > t]dt \sim 1/L(x), \quad f_i(x) \sim xL(x) \quad \text{and} \quad f_r(x) \sim Cx^{\alpha-1}/L(x) \tag{7.6}
\]

for some constant \( C > 0 \) and slowly varying function \( L \) (cf. [3], [14, Theorems 2, 6 and 9]), and by Lemma 7.1 the first one implies

\[
v(x) \sim L(x).
\]
We claim that
\[ g_{(-\infty,0]}(x, N) \sim f_r(x)L(N) \quad \text{uniformly for } 1 \leq x \leq N \]
as \( N \to \infty \), which plainly entails (7.5). On taking a smooth \( \tilde{L} \) such that \( \tilde{L}(t) \sim L(t) \) and \( \tilde{L}'(t)/L(t) = o(1/t) \) \( (t \to \infty) \), summing by parts leads to
\[ g_{(-\infty,0]}(x, N) \sim \sum_{n=1}^{x} v_-(n)\tilde{L}(N - x + n) \sim f_r(x)\tilde{L}(N) - \sum_{n=1}^{x} f_r(n)\tilde{L}'(N - x + n). \quad (7.7) \]

If \( x < N/2 \), then \( \tilde{L}(N - x + n) \sim L(N) \) for \( n \leq x \) so that the middle term above is asymptotically equivalent to \( f_r(x)L(N) \) and we obtain the claimed formula. If \( x \geq N/2 \), then noting that \( \tilde{L}(t)/t \) is decreasing for \( t \) large enough, we see that
\[ \tilde{L}'(N - x + n) = [\tilde{L}(N - x + n)/(N - x + n)] \times o(1) \leq [L(n)/n] \times o(1) \quad (n \to \infty), \]
and since \( \sum_{n=1}^{x} f_r(n)[L(n)/n] \sim C \sum_{n=1}^{x} n^{\alpha-2} \leq C'N^{\alpha-1} \), the last sum in (7.7) is a smaller order of \( f_r(x)L(N) \) \( (\asymp N^{\alpha-1}) \), showing the claim. The proof of (\ast) is complete.

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