Geometry of bifurcation sets of generic unfoldings of corank two functions

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Abstract
We study the geometry of bifurcation sets of generic unfoldings of $D_4^{±}$-functions. Taking blow-ups, we show each of the bifurcation sets of $D_4^{±}$-functions admits a parametrization as a surface in $R^3$. Using this parametrization, we investigate the behavior of the Gaussian curvature and the principal curvatures. Furthermore, we investigate the number of ridge curves and subparabolic curves near their singular point.

Keywords Bifurcation set · Caustics · Principal curvature · Parabolic curve

Mathematics Subject Classification 53A05 · 58K05

1 Introduction

In recent decades, the differential geometry of fronts (wave fronts) has been studied by many researchers. In the Euclidean space, the set consisting of the collection of singular values of a front and its parallel surfaces is called the caustic. Front and caustics are
both fundamental objects in Lagrangian and Legendrian singularity theory, and they are closely related (see [2,7,8,16], and also [6,9]). A front is a projection of the wave front set of an unfolding of a function, and a caustic is the bifurcation set of an unfolding of a function. Although sometimes a singular point of a bifurcation set is a singular point of a front, the bifurcation sets of the versal unfolding of $D_4^{\pm}$-functions do not appear as fronts. In this case, a parametrization of the bifurcation set has not been given in the literature to the best knowledge of the authors. In this paper, to see the geometry of the bifurcation set, we simplify a versal unfolding $\left(\mathbb{R}^2 \times \mathbb{R}^3, 0\right) \to \left(\mathbb{R}, 0\right)$ of a function $\left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}, 0\right)$ which is $\mathcal{R}$-equivalent to the $D_4^{\pm}$-function

$$f(u, v) = u^3/3! \pm uv^2/2,$$

by using a coordinate change on $\mathbb{R}^2$ and an isometry on $\mathbb{R}^3$. By using the blow-up method, we give a parametrization of a generic versal unfolding of such a function, and we show that the parametrization is a front, and investigate its geometry. For fronts, one can define classical differential geometric invariants (Gaussian, mean and principal curvatures) even though they diverge on the set of singular points. We show:

**Theorem A** One of the two principal curvatures of the parametrization of the bifurcation set of a versal unfolding of a $D_4^{\pm}$-singularity $C^\infty$-extends across the set of singular points, and the other is unbounded near the singular point of the bifurcation set. Moreover, the principal directions of these principal curvatures $C^\infty$-extend across the set of singular points.

By using the asymptotic behavior of the Gaussian curvature, we obtain the behavior of parabolic curves (the curves consisting of Gaussian curvature zero points) in Theorems 4.3 and 4.5. Moreover, by Theorem A, we can discuss the conditions for ridges and subparabolic curves near the singular point. Let $g$ be an umbilic free regular surface, and let $\kappa_i$ ($i = 1, 2$) be the principal curvatures, and $V_i$ the principal vector fields corresponding to $\kappa_i$. A point $p$ on $g$ is called a ridge point with respect to $\kappa_i$ if $V_i\kappa_i(p) = 0$. A point $p$ on $g$ is called a subparabolic point with respect to $\kappa_i$ if $V_j\kappa_i(p) = 0$, where $j = 1, 2$ and $j \neq i$. A curve on $g$ is called a ridge curve (respectively, subparabolic curve) if it consists of ridge points (respectively, subparabolic points). We show the following theorem under the assumption that the set of ridge points and the set of subparabolic points are curves. See Sect. 4.5 and Proposition 4.2 for the concrete conditions.

**Theorem B** We assume that the set of ridge points and the set of subparabolic points are curves. Then the number of ridge curves with respect to the unbounded principal curvature emanating from the singular point is at most 18; the number of ridge curves with respect to the bounded principal curvature emanating from the singular point is at most 18; there are no subparabolic points with respect to the unbounded principal curvature near the singular point; the number of subparabolic curves with respect to the bounded principal curvature emanating from the singular point is at most 10.
2 Preliminaries

2.1 Unfoldings and bifurcation sets

Let $f : (\mathbb{R}^m, 0) \to \mathbb{R}$ be a function. A function $F : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ is called an unfolding of $f$ if $F(u, 0) = f(u)$. The catastrophe set $C_F$ of the unfolding $F$ of $f$ is

$$C_F = \left\{ (u, x) \in (\mathbb{R}^m \times \mathbb{R}^r, 0) \mid \frac{\partial F}{\partial u_1}(u, x) = \cdots = \frac{\partial F}{\partial u_m}(u, x) = 0 \right\},$$

where $u = (u_1, \ldots, u_m)$. An unfolding $F : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ of $f : (\mathbb{R}^m, 0) \to \mathbb{R}$ is a Morse family of $f$ if $0 \in \mathbb{R}^m$ is a critical point of $f$ and

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^2 F}{\partial u_1^2} & \cdots & \frac{\partial^2 F}{\partial u_1 \partial u_m} & \cdots & \frac{\partial^2 F}{\partial u_1 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial u_1 \partial x_r} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 F}{\partial u_m \partial x_1} & \cdots & \frac{\partial^2 F}{\partial u_m \partial x_1} & \cdots & \frac{\partial^2 F}{\partial u_m \partial x_r} \end{pmatrix} = m, \quad (2.1)$$

holds at $0$, where $x = (x_1, \ldots, x_r)$. By the implicit function theorem, if $F$ is a Morse family, then $C_F$ is an $r$-dimensional submanifold of $(\mathbb{R}^m \times \mathbb{R}^r, 0)$. We set its parametrization $B_1 : C_F \to \mathbb{R}^m \times \mathbb{R}^r$ as an inclusion. Let $\pi : \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^r$ be the projection, and set $B_F = \pi \circ B_1$. The singular set of $B_F$ is $S(B_F) = \{(u, x) \in C_F \mid \operatorname{rank} H_F(u, x) < m\}$, where

$$H_F = \left( \frac{\partial^2 F}{\partial u_i \partial u_j} \right)_{1 \leq i, j \leq m}.$$

The image $B_F(S(B_F))$ is called the bifurcation set, and denoted by $\mathcal{B}_F$:

$$\mathcal{B}_F = \{ x \in \mathbb{R}^r \mid \text{there exists } (u, x) \in C_F \text{ such that } \operatorname{rank} H_F(u, x) < m \}.$$

The following $P$-$\mathcal{R}^+$ equivalence plays a fundamental role in investigating bifurcation sets (see [2, Chapter 8], [8, Chapter 5] for details).

**Definition 2.1** Let $F_i : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ be Morse families of functions $f_i : (\mathbb{R}^m, 0) \to \mathbb{R} (i = 1, 2)$. They are said to be $P$-$\mathcal{R}^+$ equivalent if there exists a triple $(g(u, x), G(x), h(x))$, where $g : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to (\mathbb{R}^m, 0), G : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$ is a diffeomorphism-germ, and $h : (\mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ such that

$$F_2(u, x) = F_1(\tilde{G}(u, x)) + h(x), \quad (\tilde{G}(u, x) = (g(u, x), G(x))). \quad (2.2)$$

The following lemma is well-known (see [8, Proposition 3.1] and its proof):

**Lemma 2.2** Let $F_i : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ be Morse families of $f_i : (\mathbb{R}^m, 0) \to \mathbb{R}$ ($i = 1, 2$). If $F_1, F_2$ are $P$-$\mathcal{R}^+$-equivalent as in (2.2), then $G(\mathcal{B}_{F_1}) = \mathcal{B}_{F_2}$ as set germs at 0.
By this lemma, to investigate the geometry of bifurcation sets, with respect to Euclidean geometry in $\mathbb{R}^3$, we introduce the following $P$-$\mathcal{R}^+$-isometricity:

**Definition 2.3** Let $F_i : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ be Morse families of functions $f_i : (\mathbb{R}^m, 0) \to \mathbb{R}$ ($i = 1, 2$). They are said to be $P$-$\mathcal{R}^+$-isometric if they are $P$-$\mathcal{R}^+$-equivalent, and the diffeomorphism-germ $G : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$ in the triple $(g(u, x), G(x), h(x))$ which gives $P$-$\mathcal{R}^+$-equivalence is an isotropy-germ.

We may simplify the functions $f$ and $F$ by $P$-$\mathcal{R}^+$-isometry. Two functions $f_i : (\mathbb{R}^m, 0) \to \mathbb{R}$ ($i = 1, 2$) are $\mathcal{R}$-equivalent if there exists a diffeomorphism-germ $\varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ such that $f_1 = f_2 \circ \varphi$. A function-germ $f$ at 0 of two variables is a $D^+_4$-germ if it is $\mathcal{R}$-equivalent to $f(u, v) = u^3/3! \pm uv^2/2$, where $(u_1, u_2)$ is denoted by $(u, v)$. Let $\varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ be a diffeomorphism-germ. Then a Morse family $F : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ of $f : (\mathbb{R}^m, 0) \to \mathbb{R}$ is $P$-$\mathcal{R}^+$-equivalent to the unfolding $F(\varphi(u, x), x)$. Since we study the geometry of the bifurcation set of a $D^+_4$-germ $f$, under the $P$-$\mathcal{R}^+$-isometricity, we may assume $f = u^3/3! \pm uv^2/2$ without loss of generality.

### 2.2 Simplification of an unfolding by $P$-$\mathcal{R}^+$-equivalence

**Definition 2.4** Let $F_i : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ ($i = 1, 2$) be two unfoldings of $f : (\mathbb{R}^m, 0) \to \mathbb{R}$. An $\mathcal{R}^+$-$f$-morphism from $F_2$ to $F_1$ is a triple $(g(u, x), G(x), h(x))$, where $g : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to (\mathbb{R}^m, 0)$, $G : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$, $h : (\mathbb{R}^r, 0) \to (\mathbb{R}, 0)$, and they satisfy $g(u, 0) = u$ and

$$F_2(u, x) = F_1(g(u, x), G(x)) + h(x). \quad (2.3)$$

**Definition 2.5** An unfolding $F_1 : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ of $f : (\mathbb{R}^m, 0) \to \mathbb{R}$ is an $\mathcal{R}^+$-$versal$ unfolding if for any unfolding $F_2 : (\mathbb{R}^m \times \mathbb{R}^r, 0) \to \mathbb{R}$ of $f$, there exists an $\mathcal{R}^+$-$f$-morphism from $F_2$ to $F_1$.

It is known that the function $F_{0, \varepsilon_1}(u, v, x, y, z) = u^3/3! + \varepsilon_1 uv^2/2 + xu + yv + z(u^2 - \varepsilon_1 v^2)/2$ is an $\mathcal{R}^+$-versal unfolding of $u^3/3! + \varepsilon_1 uv^2/2$, where $\varepsilon_1 = \pm 1$ ([2, Chapter 8], [8, Chapter 5]). We call the bifurcation set of an $\mathcal{R}^+$-versal unfolding of $u^3/3! \pm uv^2/2$ a $D^+_4$-singularity. The bifurcation set of $F_{0, \varepsilon_1}$ is the set

$$B_{0, \varepsilon_1} = \{(-u^2/2 - \varepsilon_1 v^2/2 - z u, -\varepsilon_1 uv + \varepsilon_1 z v, z) : \varepsilon_1 (u^2 - z^2) - v^2 = 0\}.$$

We can observe that $B_{0, 1}$ consists of two sheets and they have intersection curves $c(t) = \{(-t, \pm t, 0) \mid t > 0\}$. All $D^+_4$-singularities are locally diffeomorphic to $B_{0, 1}$, and we call the corresponding curves to $c(t)$ the intersection curve. See Fig. 1 for the bifurcation sets of $F_{0, \varepsilon_1}$.

We remark that $\mathcal{R}^+$-versal unfolding $(\mathbb{R}^2 \times \mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ of the $D^+_4$-germ is unique ([2, Chapter 8], [8, Chapter 5]).

Let $f : (\mathbb{R}^2, 0) \to \mathbb{R}$ be $f(u, v) = u^3/3! + \varepsilon_1 uv^2/2$ ($\varepsilon_1 = \pm 1$) and let $F : (\mathbb{R}^2 \times \mathbb{R}^3, 0) \to \mathbb{R}$ be an $\mathcal{R}^+$-versal unfolding of $f$. We have the following:
Proposition 2.6 The function $F$ is $P$-$R^+$-isometric to

$$F(u, x) = F_0(u, G(x)) = \frac{u^3}{3!} + \varepsilon_1 \frac{uv^2}{2} + P(x)u + Q(x)v + R(x)\frac{u^2 - \varepsilon_1 v^2}{2} \quad (2.4)$$

with the condition

$$g_{1,010} = g_{1,001} = g_{2,001} = 0 \text{ and } g_{1,100}, g_{2,010}, g_{3,001} > 0, \quad (2.5)$$

where

$$G_n(x) = \sum_{i+j+k \geq 1} \frac{g_{n,ijk}}{i!j!k!} x_i^j x_k^k + O(l+1) \quad (2.6)$$

where $l$ is sufficiently large, and $O(l+1)$ stands for the terms whose degrees are greater than $l$.

Proof Since $F_{0,\varepsilon_1}$ is a versal unfolding, there exist $g : (R^2 \times R^3, 0) \to (R^2, 0)$, $G : (R^3, 0) \to (R^3, 0)$, $h : (R^3, 0) \to (R, 0)$, and they satisfy $g(u, 0) = u$ and $F(u, x) = F_0(g(u, x), G(x)) + h(x)$. By the triple $(g(u, x), x, h(x))$, we see $F$ is $P$-$R^+$-isometric to $F_0(u, G(x))$ as in (2.4). Let us set

$$g_1 = \begin{pmatrix} g_{1,100} \\ g_{1,010} \\ g_{1,001} \end{pmatrix}, \quad g_2 = \begin{pmatrix} g_{2,100} \\ g_{2,010} \\ g_{2,001} \end{pmatrix}, \quad g_3 = \begin{pmatrix} g_{3,100} \\ g_{3,010} \\ g_{3,001} \end{pmatrix},$$

and let $\overline{g}_1, \overline{g}_2, \overline{g}_3$ be the Gram-Schmidt orthonormalized vectors, namely,

$$\overline{g}_1 = \frac{g_1}{|g_1|}, \quad \tilde{g}_2 = g_2 - (g_1 \cdot g_2)\overline{g}_1, \quad \overline{g}_2 = \frac{\tilde{g}_2}{|\tilde{g}_2|},$$

$$\overline{g}_3 = g_3 - (g_1 \cdot g_3)\overline{g}_1 - (g_2 \cdot g_3)\overline{g}_2, \quad \overline{g}_3 = \frac{g_3}{|g_3|}. $$
We set

\[ M = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \]

where '() stands for matrix transposition. Then \( M \) is an orthonormal matrix, and is identified with a linear map. By the triple \((\text{id}, M, 0)\), we see the first condition of (2.5) can be satisfied. By the versality, \( g_1, 0, g_2, 0, g_3, 0 \neq 0 \). The second condition of (2.5) can be satisfied by the linear map defined by the orthonormal matrix

\[ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}. \]

\[ \square \]

For any \( l \geq 3 \), the coefficients \( g_{n,ijk} \) are unique. Geometric meanings of coefficients of \( G_1, G_2, G_3 \) are discussed in Sect. 4.

### 2.3 Fronts

Since we shall show the bifurcation sets are fronts, we give a fundamental definition of fronts. Let \( f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be a map-germ. The map \( f \) is a frontal if there exists a unit normal vector field \( \nu \) along \( f \) such that \( [df_p(X), \nu(p)] = 0 \) for any \( X \in T_p(\mathbb{R}^2, 0) \). A frontal \( f \) with a unit normal vector is a front if the pair \((f, \nu)\) is an immersion. Let \( f \) be a frontal. A function \( \lambda: (\mathbb{R}^2, 0) \to \mathbb{R} \) is called an identifier of singularities if it is a non-zero multiple of the function \( \det(f_u, f_v, \nu) \). If a function is an identifier of singularities, then the set of singular points \( S(f) \) satisfies \( S(f) = \lambda^{-1}(0) \). Since the unit normal vector field is well-defined for frontals, one can define the Gaussian, mean and principal curvatures in a natural way. However, they may diverge on the set of singular points. See [10,11,13–15] for differential geometric study of these curvatures. Let \( f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be a front with a unit normal vector field \( \nu \), which is a cuspidal edge (by coordinate transformations on the source and the target space, it can be written as \((u, v) \mapsto (u, v^2, v^3)\)). Let \( \gamma: (J, 0) \to (\mathbb{R}^3, 0) \) be a parametrization of the set of singular points of \( f \), where \( J \) is an open interval containing 0. We set \( \hat{\gamma}(t) = f \circ \gamma(t) \) and \( \hat{\nu}(t) = \nu \circ \gamma(t) \). The singular curvature \( \kappa_s \) and the (limiting) normal curvature \( \kappa_n \) are defined by

\[ \kappa_s(t) = \pm \frac{\det(\hat{\gamma}', \hat{\gamma}'', \hat{\nu})}{|\hat{\gamma}'|^3}(t), \quad \kappa_n(t) = \frac{\hat{\gamma}'' \cdot \hat{\nu}}{|\hat{\gamma}'|^2}(t). \quad (2.7) \]

See [14] for details.
3 Description of bifurcation sets

We assume $f(u, v) = u^3/3! + \varepsilon_1 uv^2/2$ and $F$ is written as (2.4) with the conditions (2.5). Then by the implicit function theorem, there exist two functions $x(u, v, z), y(u, v, z)$ such that

$$F_u(u, v, x(u, v, z), y(u, v, z), z) = F_v(u, v, x(u, v, z), y(u, v, z), z) = 0.$$  

Thus $C_F$ can be parametrized by $(u, v, z) \mapsto (u, v, x(u, v, z), y(u, v, z), z)$, and the map $B_F$ is $B_F(u, v, z) = (x(u, v, z), y(u, v, z), z)$. Since $d_B F = \begin{pmatrix} x_u & x_v & x_z \\ y_u & y_v & y_z \\ 0 & 0 & 1 \end{pmatrix}$, (3.1)

it holds that $S(B_F) = \{x_u y_v - x_v y_u = 0\}$. By the implicit function theorem,

$$\begin{pmatrix} F_u & F_v \\ F_v & F_v \end{pmatrix} \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = -\begin{pmatrix} F_u & F_v \\ F_v & F_v \end{pmatrix}$$ (3.2)

and by (2.5), the bifurcation set can be written as

$$B_F = \{(x, y, z) | \text{there exists } (u, v) \in C_F \text{ such that } \det H_F(u, v, x, y, z) = 0\}. \quad (3.3)$$

3.1 $D^-_4$ singularity

We give a parametrization of the bifurcation set by using the blow-up method at a singular point [3] (see also [5, Example (a) in p. 221]). Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be a circle, and let $I = (-\varepsilon, \varepsilon)$ be an open interval. Two points $(\theta_i, r_i) \in S^1 \times I$ ($i = 1, 2$) are equivalent ($\sim$) if $(\theta_2, r_2) = (\theta_1 + \pi, r_1)$. The quotient space $\mathcal{M} = S^1 \times I / \sim$ is topologically a Möbius strip. There is a natural map $\pi : \mathcal{M} \to \mathbb{R}^2$, where $\pi([\theta, r]) = (r \cos \theta, r \sin \theta)$. This is usually called a blow-up. Furthermore, we take a double cover $\hat{\mathcal{M}}$ of $\mathcal{M}$, and consider a natural map $\hat{\pi} : \hat{\mathcal{M}} \to \mathbb{R}^2$, where $\hat{\pi}([\theta, r]) = (r \cos 2\theta, r \sin 2\theta)$. Then $\hat{\mathcal{M}}$ is topologically an annulus. We assume $\varepsilon_1 = -1$ in (2.4). Then det $F_H = -u^2 - v^2 + R(x, y, z)^2$. We set

$$u = r \cos 2\theta, \quad v = r \sin 2\theta, \quad (3.4)$$

where $(\theta, r) \in \hat{\mathcal{M}}$. Then $-r^2 + R^2 = 0$ can be solved by $r = \varepsilon_2 R(x, y, z)$, where $\varepsilon_2 = \pm 1$. Then the equation for the bifurcation set is

$$F_u(2\theta, r, x, y, z) = 0, \quad F_v(2\theta, r, x, y, z) = 0, \quad r = \varepsilon_2 R(x, y, z). \quad (3.5)$$
This is equivalent to
\[
X_{-1,\epsilon_2}(\theta, x, y, z) = 0, \quad Y_{-1,\epsilon_2}(\theta, x, y, z) = 0, \quad (3.6)
\]
where
\[
\begin{align*}
X_{-1,\epsilon_2}(\theta, x, y, z) &= F_u(\epsilon_2 R(x, y, z), 2\theta, x, y, z) \\
&= P(x, y, z) + R(x, y, z)^2 \alpha_{-1,\epsilon_2}(\theta), \\
Y_{-1,\epsilon_2}(\theta, x, y, z) &= F_v(\epsilon_2 R(x, y, z), 2\theta, x, y, z) \\
&= Q(x, y, z) + R(x, y, z)^2 \beta_{-1,\epsilon_2}(\theta), \quad (3.7)
\end{align*}
\]
\[
\alpha_{-1,\epsilon_2}(\theta) = \frac{\cos^2 2\theta - \sin^2 2\theta}{2} + \epsilon_2 \cos 2\theta,
\]
\[
\beta_{-1,\epsilon_2}(\theta) = -\cos 2\theta \sin 2\theta + \epsilon_2 \sin 2\theta.
\]

By the condition (2.5), there exist functions \(x_{-1,\epsilon_2}(\theta, z), y_{-1,\epsilon_2}(\theta, z)\) such that \(X_{-1,\epsilon_2}(\theta, x_{-1,\epsilon_2}(\theta, z), y_{-1,\epsilon_2}(\theta, z), z) = Y_{-1,\epsilon_2}(\theta, x_{-1,\epsilon_2}(\theta, z), y_{-1,\epsilon_2}(\theta, z), z) = 0\) holds identically. In this setting, we have:

**Lemma 3.1** It holds that
\[
x_{-1,\epsilon_2}(\theta, 0) = y_{-1,\epsilon_2}(\theta, 0) = (x_{-1,\epsilon_2})_z(\theta, 0) = (y_{-1,\epsilon_2})_z(\theta, 0) = 0 \quad (3.8)
\]
and
\[
R(x_{-1,\epsilon_2}(\theta, 0), y_{-1,\epsilon_2}(\theta, 0), 0) = 0. \quad (3.9)
\]

**Proof** We consider equations \(X_{-1,\epsilon_2}(\theta, x, y, 0) = Y_{-1,\epsilon_2}(\theta, x, y, 0) = 0\). Then since (2.5), we have functions \(\bar{x}_{-1,\epsilon_2}(\theta), \bar{y}_{-1,\epsilon_2}(\theta)\) such that
\[
X_{-1,\epsilon_2}(\theta, \bar{x}_{-1,\epsilon_2}(\theta), \bar{y}_{-1,\epsilon_2}(\theta), 0) = Y_{-1,\epsilon_2}(\theta, \bar{x}_{-1,\epsilon_2}(\theta), \bar{y}_{-1,\epsilon_2}(\theta), 0) = 0.
\]

We remark that by the implicit function theorem, each \(\bar{x}_{-1,\epsilon_2}(\theta), \bar{y}_{-1,\epsilon_2}(\theta)\) is unique. On the other hand, since \(P(0, 0, 0) = Q(0, 0, 0) = 0\), the equality \(X_{-1,\epsilon_2}(\theta, 0, 0, 0) = Y_{-1,\epsilon_2}(\theta, 0, 0, 0) = 0\) is satisfied. Thus \(\bar{x}_{-1,\epsilon_2}(\theta) = \bar{y}_{-1,\epsilon_2}(\theta) = 0\) is a solution of \(X_{-1,\epsilon_2}(\theta, x, y, 0) = Y_{-1,\epsilon_2}(\theta, x, y, 0) = 0\). By the uniqueness, \(\bar{x}_{-1,\epsilon_2}(\theta) = \bar{y}_{-1,\epsilon_2}(\theta) = 0\) holds. By the definition of the functions \(\bar{x}_{-1,\epsilon_2}(\theta), \bar{y}_{-1,\epsilon_2}(\theta)\), it holds that \(\bar{x}_{-1,\epsilon_2}(\theta) = x_{-1,\epsilon_2}(\theta, 0), \bar{y}_{-1,\epsilon_2}(\theta) = y_{-1,\epsilon_2}(\theta, 0)\). This shows \(x_{-1,\epsilon_2}(\theta, 0) = 0\) and \(y_{-1,\epsilon_2}(\theta, 0) = 0\). Moreover, by (3.7) and (2.5), it holds that \((x_{-1,\epsilon_2})_z(\theta, 0) = (y_{-1,\epsilon_2})_z(\theta, 0) = 0\). The Eq. (3.9) is obvious by (3.8). \(\Box\)

Thus a double cover of the bifurcation set can be parameterized by
\[
b_{-1,\epsilon_2}(\theta, z) = b(\theta, z) = (x_{-1,\epsilon_2}(\theta, z), y_{-1,\epsilon_2}(\theta, z), z) : \mathcal{M} \to \mathbb{R}^3 \quad (\epsilon_2 = \pm 1). \quad (3.10)
\]

We define the source space \(\mathcal{M}\) of \(b_{-1,\epsilon_2}\) by \(\mathcal{M}_{\epsilon_2}\). On the other hand, since \(R_z(0, 0, 0) = g_{3,001} \neq 0\), and
\[
\alpha_{-1,\epsilon_2}(\theta) = \alpha_{-1,\epsilon_2}(\theta + \pi/2), \quad \beta_{-1,\epsilon_2}(\theta) = \beta_{-1,\epsilon_2}(\theta + \pi/2),
\]

\(\Box \text{ Springer} \)
we can regard \((\theta, z) \in \mathcal{M}\) as a parameter of the bifurcation set. We have the following proposition.

**Proposition 3.2** The map \(b_{\theta, \varepsilon} : \tilde{\mathcal{M}}_{\varepsilon_2} \to \mathbb{R}^3\) is a front near \(\{z = 0\} \subset \tilde{\mathcal{M}}\).

**Proof** By (3.9), there exists a function \(\tilde{R}_{-1, \varepsilon_2}(\theta, z)\) such that \(R(\theta, x_{-1, \varepsilon_2}(\theta, z), y_{-1, \varepsilon_2}(\theta, z), z) = z\tilde{R}_{-1, \varepsilon_2}(\theta, z)\). By (3.8) and \(R_z(0, 0, 0) \neq 0\), it holds that \(\tilde{R}_{-1, \varepsilon_2}(\theta, 0) \neq 0\). By a direct calculation,

\[
\begin{align*}
\left( (X_{-1, \varepsilon_2})_\theta (x_{-1, \varepsilon_2}(\theta, z), y_{-1, \varepsilon_2}(\theta, z), z), \\
(Y_{-1, \varepsilon_2})_\theta (x_{-1, \varepsilon_2}(\theta, z), y_{-1, \varepsilon_2}(\theta, z), z), 0 \right)
\end{align*}
\]

(3.11)

\[
= \begin{cases} 4\varepsilon^2 \tilde{R}_{-1, \varepsilon_2}(\theta, z)^2 \sin 3\theta (-\cos \theta, \sin \theta, 0) (\varepsilon_2 = 1), \\
-4\varepsilon^2 \tilde{R}_{-1, \varepsilon_2}(\theta, z)^2 \cos 3\theta (\sin \theta, \cos \theta, 0) (\varepsilon_2 = -1)
\end{cases}
\]

holds. By the implicit function theorem,

\[
\begin{align*}
\left( (X_{-1, \varepsilon_2})_\theta (x_{-1, \varepsilon_2}(\theta, z), y_{-1, \varepsilon_2}(\theta, z), z), \\
(Y_{-1, \varepsilon_2})_\theta (x_{-1, \varepsilon_2}(\theta, z), y_{-1, \varepsilon_2}(\theta, z), z), 0 \right)
\end{align*}
\]

(3.12)

\[
\tilde{A}_{-1, \varepsilon_2} = \begin{pmatrix} (Y_{-1, \varepsilon_2})_y & -(X_{-1, \varepsilon_2})_y \\ -(Y_{-1, \varepsilon_2})_x & (X_{-1, \varepsilon_2})_x \end{pmatrix},
\]

(3.13)

\[
\tilde{V}_{-1, \varepsilon_2} = \begin{cases} t(-\cos \theta, \sin \theta) (\varepsilon_2 = 1), \\
t(\sin \theta, \cos \theta) (\varepsilon_2 = -1)
\end{cases},
\]

(3.14)

\[
\tilde{\lambda}_{-1, \varepsilon_2} = \begin{cases} \sin 3\theta (\varepsilon_2 = 1), \\
\cos 3\theta (\varepsilon_2 = -1),
\end{cases}
\]

(3.15)

\[
a_{-1, \varepsilon_2} = \frac{4\tilde{R}_{-1, \varepsilon_2}(\theta, z)^2}{|\det \tilde{A}_{-1, \varepsilon_2}|},
\]

(3.16)

hold. Thus \(b_{\theta} \times b_{\varepsilon} \) is proportional to

\[
\tilde{v}_{-1, \varepsilon_2}(\theta, z)
\]

(3.17)

where

\[
\begin{align*}
dX_{-1, \varepsilon_2} &= t((X_{-1, \varepsilon_2})_x, (X_{-1, \varepsilon_2})_y, (X_{-1, \varepsilon_2})_z), \\
dY_{-1, \varepsilon_2} &= t((Y_{-1, \varepsilon_2})_x, (Y_{-1, \varepsilon_2})_y, (Y_{-1, \varepsilon_2})_z).
\end{align*}
\]

Since \(\tilde{v}_{-1, \varepsilon_2}(\theta, 0) \neq 0\), the unit vector \(v_{-1, \varepsilon_2} = \tilde{v}_{-1, \varepsilon_2}/|\tilde{v}_{-1, \varepsilon_2}|\) is a unit normal vector of \(b\). Since \(b_{\theta}(\theta, 0) = 0\), to show that \(b\) is a front it is enough to see \(\tilde{v}_{-1, \varepsilon_2}(\theta, 0)\) and
$(\tilde{v}_{-1, \varepsilon_2})_\theta (\theta, 0)$ are linearly independent. It is easy to see by $dX$ and $dY$ are linearly independent at $(\theta, 0)$. This shows the assertion.

The singular set $S(b)$ is

$$S(b) = \begin{cases} \{(\theta, z) \mid \sin 3\theta = 0\} = \{(n\pi/3, z) \mid n = 0, 1, 2, 3, 4, 5\} & (\varepsilon_2 = 1), \\ \{(\theta, z) \mid \cos 3\theta = 0\} = \{(n\pi/3 + \pi/2, z) \mid n = -1, 0, 1, 2, 3, 4\} & (\varepsilon_2 = -1). \end{cases} \tag{3.18}$$

### 3.2 $D_4^+$ singularity

We will give a parametrization of the bifurcation set by a similar method to that of Sect. 3.1 in the case of $\varepsilon_1 = 1$. Let $I = (-\varepsilon, \varepsilon)$ be an open interval. We consider a map $\pi : \mathbb{R} \times I \to \mathbb{R}^2$ defined by $\pi_1((\theta, r)) = (r \cosh \theta, r \sinh \theta)$.

We assume $\varepsilon_1 = 1$ in $(2.4)$. Then $\det H_F = u^2 - v^2 + R(x, y, z)^2$. We set

$$u = r \cosh 2\theta, \quad v = r \sinh 2\theta. \tag{3.19}$$

Then $r^2 + R^2 = 0$ can be solved by $r = \varepsilon_2 R$, where $\varepsilon_2 = \pm 1$. We set

$$X_{1, \varepsilon_2} = P(x, y, z) + R(x, y, z)^2 \alpha_{1, \varepsilon_2}(\theta),$$

$$Y_{1, \varepsilon_2} = Q(x, y, z) + R(x, y, z)^2 \beta_{1, \varepsilon_2}(\theta),$$

$$\alpha_{1, \varepsilon_2}(\theta) = \frac{\cosh 2\theta^2 + \sinh 2\theta^2}{2} + \varepsilon_2 \cosh 2\theta,$$

$$\beta_{1, \varepsilon_2}(\theta) = \cosh 2\theta \sinh 2\theta - \varepsilon_2 \sinh 2\theta. \tag{3.20}$$

There exist $x_{1, \varepsilon_2}(\theta, z)$ and $y = y_{1, \varepsilon_2}(\theta, z)$ such that

$$X_{1, \varepsilon_2}(\theta, x_{1, \varepsilon_2}(\theta, z), y_{1, \varepsilon_2}(\theta, z), z) = Y_{1, \varepsilon_2}(\theta, x_{1, \varepsilon_2}(\theta, z), y_{1, \varepsilon_2}(\theta, z), z) = 0$$

holds identically. By the same proof as for Lemma 3.1, we have $x_{1, \varepsilon_2}(\theta, 0) = y_{1, \varepsilon_2}(\theta, 0) = 0, (x_{1, \varepsilon_2})_z(\theta, 0) = (y_{1, \varepsilon_2})_z(\theta, 0) = 0, R(x_{1, \varepsilon_2}(\theta, 0), y_{1, \varepsilon_2}(\theta, 0), 0) = 0$. Thus there exists a function $\tilde{R}_{1, \varepsilon_2}(\theta, z)$ such that

$$R(\theta, x_{1, \varepsilon_2}(\theta, z), y_{1, \varepsilon_2}(\theta, z), z) = z \tilde{R}_{1, \varepsilon_2}(\theta, z).$$

The bifurcation set, except for its intersection set, can be parameterized by $b(\theta, z) = b_{1, \varepsilon_2}(\theta, z) = (x_{1, \varepsilon_2}(\theta, z), y_{1, \varepsilon_2}(\theta, z), z)$. We define the source space $\mathbb{R} \times I$ of $b_{1, \varepsilon_2}$.
by $D_{\varepsilon_2}$. By a similar argument as in the case of a $D_4^-$ singularity, setting

$$V_{1,\varepsilon_2} = \begin{cases} t'(\cosh \theta, \sinh \theta) \ (\varepsilon_2 = 1) \\ t'(\sinh \theta, \cosh \theta) \ (\varepsilon_2 = -1), \end{cases} \quad (3.21)$$

$$\lambda_{1,\varepsilon_2} = \begin{cases} \sinh 3\theta \ (\varepsilon_2 = 1) \\ \cosh 3\theta \ (\varepsilon_2 = -1), \end{cases} \quad (3.22)$$

$$a_{1,\varepsilon_2} = \frac{4R^2_{1,\varepsilon_2}}{\det \tilde{A}_{1,\varepsilon_2}}, \quad \tilde{A}_{1,\varepsilon_2} = \begin{pmatrix} (Y_{1,\varepsilon_2})_y & -(X_{1,\varepsilon_2})_y \\ -(Y_{1,\varepsilon_2})_x & (X_{1,\varepsilon_2})_x \end{pmatrix}, \quad (3.23)$$

we have

$$((x_{1,\varepsilon_2})_\theta, (y_{1,\varepsilon_2})_\theta) = -z^2\lambda_{1,\varepsilon_2} a_{1,\varepsilon_2} \tilde{A}_{1,\varepsilon_2} V_{1,\varepsilon_2}. \quad (3.24)$$

We set

$$\tilde{v}_{1,\varepsilon_2}(\theta, z) = \begin{cases} (-\cosh \theta dY_{1,\varepsilon_2} + \sinh \theta dX_{1,\varepsilon_2})(x_{1,\varepsilon_2}(\theta, z), y_{1,\varepsilon_2}(\theta, z), z) \ (\varepsilon_2 = 1) \\ (-\sinh \theta dY_{1,\varepsilon_2} + \cosh \theta dX_{1,\varepsilon_2})(x_{1,\varepsilon_2}(\theta, z), y_{1,\varepsilon_2}(\theta, z), z) \ (\varepsilon_2 = -1). \end{cases} \quad (3.25)$$

Then $v_{1,\varepsilon_2} = \tilde{v}_{1,\varepsilon_2}/|\tilde{v}_{1,\varepsilon_2}|$ is a unit normal vector field for $b$. We can see $b_{1,\varepsilon_2} : D_{\varepsilon_2} \to \mathbb{R}^3$ are fronts by a similar method as in Sect. 3.1. The singular set $S(b)$ are

$$S(b) = \begin{cases} \{(\theta, z) \mid \sinh 3\theta = 0\} = \{(0, z)\} \ (\varepsilon_2 = 1), \\ \{(\theta, z) \mid \cosh 3\theta = 0\} = \{z = 0\} \ (\varepsilon_2 = -1). \end{cases} \quad (3.26)$$

### 4 Geometry of bifurcation sets

#### 4.1 Asymptotic behavior of parametrization near singular points

Let $b = b_{\varepsilon_1,\varepsilon_2}$ be the parametrization of the bifurcation set as in Sects. 3.1 and 3.2, and $v = v_{\varepsilon_1,\varepsilon_2}$ its unit normal vector field. We set the coefficients of the first and second fundamental forms as follows:

$$E = E_{\varepsilon_1,\varepsilon_2} = b_\theta \cdot b_\theta, \quad F = F_{\varepsilon_1,\varepsilon_2} = b_\theta \cdot b_z, \quad G = G_{\varepsilon_1,\varepsilon_2} = b_z \cdot b_z, \quad L = L_{\varepsilon_1,\varepsilon_2} = -b_\theta \cdot v_z, \quad M = M_{\varepsilon_1,\varepsilon_2} = -b_\theta \cdot v_z, \quad N = N_{\varepsilon_1,\varepsilon_2} = -b_z \cdot v_z.$$

We have the following lemma.
Lemma 4.1 The coefficients of the first and second fundamental forms satisfy

\[ E = z^4 \lambda(\theta)^2 a(\theta, z)^2 (E_0(\theta) + zE_1(\theta) + z^2 E(\theta, z)), \] (4.1)
\[ F = z^3 \lambda(\theta)a(\theta, z)(F_0(\theta) + zF_1(\theta, z)), \] (4.2)
\[ G = 1 + z^2 G_0(\theta, z) + z^3 G_1(\theta) + z^4 G_2(\theta, z), \] (4.3)
\[ L = z^2 \lambda(\theta)a(\theta, z)\tilde{v}^{-1}_0(L_0 + zL_1(\theta) + z^2 L_2(\theta) + z^3 L_3(\theta) + z^4 L_4(\theta, z)), \] (4.4)
\[ M = z^2 \lambda(\theta)a(\theta, z)|\tilde{v}^{-1}_0(M_0(\theta) + zM_1(\theta, z)), \] (4.5)
\[ N = |\tilde{v}^{-1}_0(N_0(\theta) + zN_1(\theta) + z^2 N_2(\theta, z)), \] (4.6)

where \( E_i, F_i, G_i, L_i, M_i, G_i \) \((i = 0, 1, \ldots, 4)\) are functions of variables \( \theta \) or \((\theta, z)\) as indicated, and \( a(\theta, z) = a_{\varepsilon_1, \varepsilon_2}(\theta, z), \lambda(\theta) = \lambda_{\varepsilon_1, \varepsilon_2}(\theta), \tilde{v}(\theta, z) = \tilde{v}_{\varepsilon_1, \varepsilon_2}(\theta, z)\) are defined in (3.14), (3.15), (3.16), (3.21), (3.22), (3.23) respectively. Moreover, \( E_0(\theta) > 0 \) and \( L_0 \) is a non-zero constant.

Proof By (3.12) and (3.24), it holds that the first two components of \( b_\theta \) are \( z^2 a \lambda A V \), and the third component of \( b_\theta \) is zero, where \( A = A_{\varepsilon_1, \varepsilon_2}, V = V_{\varepsilon_1, \varepsilon_2} \). Then we see (4.1), (4.4), (4.5) and \( E_0 > 0 \). By \( x_\varepsilon(\theta, 0) = y_\varepsilon(\theta, 0) = 0 \), it holds that \( b_\varepsilon(\theta, 0) = (0, 0, 1) \), and this shows (4.3). To show (4.2), since the third component of \( b_\theta \) is zero, and \( b_\varepsilon(\theta, 0) = (0, 0, 1) \), we see (4.2). We show \( L_0 \) is a non-zero constant in the case of \( \varepsilon_2 = 1 \). It is sufficient to show

\[ A^{-1} = \begin{pmatrix} -\cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \cdot (\tilde{v})_\theta \neq 0. \] (4.7)

Since \( X_\theta = Y_\theta = X_\varphi = Y_\varphi = 0 \) on \( z = 0 \), the right-hand side of (4.7) is

\[ \begin{pmatrix} -\cos \theta Y_y \\ \cos \theta Y_x + \sin \theta X_x \end{pmatrix} \cdot \begin{pmatrix} -\sin \theta X_y + \cos \theta X_x \\ -\sin \theta Y_x \end{pmatrix} = X_\varphi(\theta, 0)Y_\varphi(\theta, 0) = g_{1,100} g_{2,010} \]
on \( (z = 0) \). Thus this is a non-zero constant, and the other cases can be shown by the same calculation. Finally, (4.6) is clear.

4.2 Principal curvatures and principal directions

In this section, we first show Theorem A.

Proof of Theorem A Let \( K \) and \( H \) be the Gaussian curvature and the mean curvature. By Lemma 4.1, one can see \( z^2 \lambda(\theta) K(\theta, z) \) and \( z^2 \lambda(\theta) H(\theta, z) \) are \( C^\infty \) functions. We define \( \tilde{K}, \tilde{H}, \tilde{I} \) by

\[ K = \frac{\tilde{K}}{z^2 \lambda a|\tilde{v}|^2 \tilde{I}}, \quad H = \frac{\tilde{H}}{2z^2 \lambda a|\tilde{v}|\tilde{I}}, \quad \tilde{I} = \frac{EG - F^2}{z^4 \lambda^2 a^2}, \] (4.8)
where we exclude the variables \((\theta)\) and \((\theta, z)\) in the notation for functions, when it is clear. Here,

\[
\begin{align*}
\tilde{H} &= L_0 + L_1 z + (G_0 L_0 + L_2 + 2 \lambda a E_0 N_0) z^2 + H_3 z^3 + z^4 O(0), \\
H_3 &= G_0 L_1 + L_0 G_1 + L_3 + 6 \lambda (-2 a F_0 M_0 + a E_1 N_0 + a E_0 N_1 + (a)_z E_0 N_0), \\
\tilde{K} &= L_0 N_0 + (L_1 N_0 + L_0 N_1) z + z^2 O(0), \\
\bar{I} &= E_0 + z E_1 + z^2 O(0).
\end{align*}
\]

(4.9)

where the term \(z^i O(0)\) stands for a function of the form \(z^i h(\theta, z)\). Then

\[
\sqrt{H^2 - K} = \frac{1}{2 z^2 |\lambda| \bar{I} |\bar{\nu}|} \sqrt{\bar{H}^2 - 4 z^2 \lambda a \bar{I} \bar{K}}.
\]

We note that the function \(f(x) = \sqrt{b_1(\theta, z)^2 - 4 x b_2(\theta, z)}\) of the variable \(x (b_1 \neq 0)\) satisfies that \(f(0) = |b_1(\theta, z)|\). Thus there exists a function \(\tilde{f}(\theta, z, x)\) such that

\[
f(x) = |b_1(\theta, z)| - x \tilde{f}(\theta, z, x).
\]

(4.10)

Substituting \(x = z^2 \lambda(\theta)\) into (4.10), there exists a function \(\tilde{h}(\theta, z)\) such that

\[
\sqrt{\tilde{H}^2 - 4 z^2 \lambda a \bar{I} \bar{K}} = |\tilde{H}| - z^2 \lambda \tilde{h}.
\]

Thus

\[
H \pm \sqrt{H^2 - K} = \begin{cases}
\tilde{H} / z^2 \lambda a |\bar{\nu}| \bar{I} \pm \text{sgn} \lambda \tilde{h} / 2 |a| |\bar{\nu}| \bar{I} \\
\pm \text{sgn} \lambda \tilde{h} / 2 |a| |\bar{\nu}| \bar{I}
\end{cases}.
\]

We see \(\tilde{H}(\theta, 0) \neq 0\) and \(|a| |\bar{\nu}| \bar{I}(\theta, 0) \neq 0\). This implies that one of the principal curvatures of \(f\) is unbounded near \(z = 0\) and \(S(f)\), and the other is bounded. This proves the assertion.

By (4.9), we see that there are no vanishing mean curvature points, no umbilic points, and no zeros of unbounded principal curvature near the singular point. We refer to the unbounded principal curvature for the unbounded one, and to the bounded principal curvature for the other. We give evaluation formulas for these principal curvatures, such as \(\tilde{h}(\theta, z)\) when \(\tilde{H}(0) > 0\). Because the square root function \((b_0 \neq 0)\) is expanded as

\[
\sqrt{b_0^2 + b_1 z + b_2 z^2 + b_3(z) z^3} = |b_0| + z \frac{b_1}{2 |b_0|} + z^2 \frac{b_1^2 + 4 b_0 b_2}{8 |b_0|^3} + z^3 \frac{b_1^3 - 4 b_0^2 b_1 b_2 + 8 b_0^3 b_3(0)}{16 |b_0|^5} + z^4 O(0),
\]
we have

$$
\sqrt{H^2 - 4z^2 \lambda a \tilde{K}} = L_0 + L_1 z + (G_0 L_0 + L_2 - \lambda a E_0 N_0)z^2 \\
+ (G_0 L_1 + G_1 L_0 + L_3 - 2\lambda a F_0 M_0 - \lambda a E_0 N_1)
- a E_1 \lambda N_0 - N_0 E_0 \lambda (a)_z)z^3 + z^4 O(0).
$$

(4.11)

Hence

$$
\tilde{h}(\theta, z) = 2a E_0 N_0 + 2(a E_0 N_1 + a E_1 N_0 + E_0 N_0(a)_z)z + z^2 O(0).
$$

Thus the unbounded principal curvature $\kappa_1$ and the bounded principal curvature $\kappa_2$ can be written as

$$
\kappa_1 = \frac{1}{\lambda a |\tilde{v}|} (L_0 + L_1 z + z^2 O(0)),
$$

(4.12)

$$
\kappa_2 = \frac{1}{a |\tilde{v}|} (a E_0 N_0 + (a E_1 N_0 + a E_0 N_1 + (a)_z E_0 N_0)z + z^2 O(0)).
$$

(4.13)

By a direct calculation, the principal directions with respect to $\kappa_1$ and $\kappa_2$ are

$$
V_1 = a \lambda |\tilde{v}| \tilde{z}^2 (-N + \kappa_1 G, M - \kappa_1 F) = (L_0 + z O(0), z^2 O(0)),
$$

$$
V_2 = \frac{|\tilde{v}|}{z^2 \lambda a} (-M + \kappa_2 F, L - \kappa_2 E) = (-M_0 + z O(0), L_0 + z O(0)),
$$

(4.14)

respectively, under the identification $\partial_0 = (1, 0)$ and $\partial_z = (0, 1)$. Since the zero set of a function $\delta(\theta, z)$ is a regular curve and is transverse to the $\theta$-axis if $\delta(\theta, 0) \neq 0$ at $(\theta, 0)$ which satisfies $\delta(\theta, 0) = 0$, setting $' = \partial/\partial \theta$, the following proposition holds:

**Proposition 4.2** The ridge curve with respect to the unbounded principal curvature emanates from the direction $\theta$ satisfying $a(\theta, 0)(2 \lambda' E_0 + 3 \lambda E_0') + 2 \lambda E_0 a'(\theta, 0) = 0$ under the condition $(a(\theta, 0)(2 \lambda E_0' + 3 \lambda E_0') + 2 \lambda E_0 a'(\theta, 0))' \neq 0$. The ridge curve with respect to the bounded principal curvature emanates from the direction $\theta$ satisfying $-E_1 L_0 N_0 + E_0' M_0 N_0 + 2 E_0 (L_0 N_1 - M_0 N_0')' = 0$ under the condition $(-E_1 L_0 N_0 + E_0' M_0 N_0 + 2 E_0 (L_0 N_1 - M_0 N_0'))' \neq 0$. There are no subparabolic curves with respect to the unbounded principal curvature. The subparabolic curve with respect to the bounded principal curvature emanates from the direction $\theta$ satisfying $2 \lambda E_0 N_0' - E_0' N_0 = 0$ under the condition $(2 \lambda E_0' N_0' - E_0' N_0)' \neq 0$.

**Proof** Using (4.12), (4.13) and (4.14), we have the assertions by direct calculations. We note that $\tilde{v} \cdot \tilde{v} = E_0$, $(\tilde{v} \cdot \tilde{v})' = E_0'$, $(\tilde{v} \cdot \tilde{v})_z = E_1$ at $(\theta, 0)$. □

### 4.3 Singular curvature and limiting normal curvature

As the set of singular points near the singular point consists of a cuspidal edge, we calculate the singular curvature and limiting normal curvature. For simplicity, we just
give conditions for whether their limits vanish or not. By (2.7), \( \kappa_n \) vanishes if and only if \( N_0 \) vanishes. Thus if \( \varepsilon_1 = -1 \), then \( \kappa_n = 0 \) at \( (0, 0) \) (respectively, \( (\pi/3, 0) \), \( (2\pi/3, 0) \)) if and only if \( g_{2,002} = 0 \) (respectively, \( -\sqrt{3}g_{1,002} - g_{2,002} = 0 \), \( -\sqrt{3}g_{1,002} + g_{2,002} = 0 \)). These will appear as conditions for the configurations of parabolic curves in Sect. 4.4. Furthermore, if \( \varepsilon_1 = 1 \), then \( \kappa_n = 0 \) at \( (0, 0) \) if and only if \( g_{2,002} = 0 \). On the other hand, by (2.7), (3.17), (3.25) and \((x_{zz}, y_{zz}) = \det A_{\varepsilon_1,\varepsilon_2}(X_{zz}, X_{zz}) \) at \( z = 0 \), we obtain that \( \kappa_s = 0 \) at \( (0, 0) \) (respectively, \( (\pi/3, 0) \), \( (2\pi/3, 0) \)) if and only if \( K_1 = 0 \) (respectively, \( K_2 + 2\sqrt{3}K_3 = 0, -K_2 + 2\sqrt{3}K_3 = 0 \)), where

\[
K_1 = g_{1,100}g_{2,002}g_{2,100} - (g_{2,010}^2 + g_{2,100}^2)(g_{1,002} + 3g_{3,001}) \\
K_2 = (2g_{1,002} + 3g_{3,001})(g_{2,010}^2 + g_{2,100}^2) + g_{1,100}(2g_{2,002}g_{2,100} + 9g_{1,100}g_{3,001}) \\
K_3 = g_{1,100}(g_{1,100}g_{2,002} - g_{1,002}g_{2,100} + 3g_{2,002}g_{3,001}).
\]

Furthermore, if \( \varepsilon_1 = 1 \), then \( \kappa_n = 0 \) at \( (0, 0) \) if and only if \( K_1 = 0 \).

### 4.4 Configurations of parabolic curves emanating from \( D_4^{\pm} \)-singularities

In this section, we consider configurations of parabolic curves emanating from \( D_4^{\pm} \)-singularities. By (4.8) and (4.9), and the same reason just before Proposition 4.2, if \( N'_0(\theta) \neq 0 \) for any \( \theta \) satisfying \( N_0(\theta) = 0 \), then the parabolic curve emanating out at the singular point emanates in the direction \( \theta \) satisfying \( N_0(\theta) = 0 \) at \( (\theta, 0) \) in \( M_{\varepsilon_2} \) or \( D_{\varepsilon_2} \). We set \( \xi = g_{2,002}, \eta = g_{1,002} \) and \( \zeta = g_{3,001} \). We assume \( \xi \neq 0 \) and \( \eta - \zeta^2 \neq 0 \). Moreover, we assume \( N'_0(\theta) \neq 0 \) for any \( \theta \) such that \( N_0(\theta) = 0 \). By \( b_\varepsilon(\theta, 0) = (0, 0, 1) \), (3.13) and (3.17), we see that \( N_0(\theta) = 0 \) is equivalent to

\[
\xi \cos \theta + \eta \sin \theta + \zeta^2 \sin 3\theta = 0 \quad (\varepsilon_1 = -1, \varepsilon_2 = 1), \quad (4.15) \\
-\xi \sin \theta + \eta \cos \theta - \zeta^2 \cos 3\theta = 0 \quad (\varepsilon_1 = -1, \varepsilon_2 = -1), \quad (4.16) \\
-\xi \cosh \theta + \eta \sinh \theta + \zeta^2 \sinh 3\theta = 0 \quad (\varepsilon_1 = 1, \varepsilon_2 = 1), \quad (4.17) \\
-\xi \sinh \theta + \eta \cosh \theta - \zeta^2 \cosh 3\theta = 0 \quad (\varepsilon_1 = 1, \varepsilon_2 = -1), \quad (4.18)
\]

by the assumption \( \xi \neq 0 \) and \( \eta - \zeta^2 \neq 0 \), and we see \( \cos \theta = 0, \sin \theta = 0 \) are not the solutions of (4.15), (4.16), and also \( \sinh \theta = 0 \) is not the solution of (4.17), (4.18).

#### 4.4.1 The case of \( D_4^- \)-singularities

Setting \( \varepsilon_2 = 1 \) and \( t = \cot \theta \), the Eq. (4.15) is equivalent to

\[
p(t) = \xi t^3 + (\eta + 3\zeta^2)t^2 + \xi t + \eta - \zeta^2 = 0. \quad (4.19)
\]

Thus the number of asymptotic curves emanating from the singular point is generically 3 or 1. In this case, the singular set is \( t = \pm 1/\sqrt{3} \). We consider where the solutions \( t \) of (4.19) are in \( (-\infty, -1/\sqrt{3}), (-1/\sqrt{3}, 1/\sqrt{3}) \) or \( (1/\sqrt{3}, \infty) \). Let \( D_{-1} \) be the cubic
is the same as Theorem 4.3.

The discriminant of (4.19):

\[ D_{-1}/4 = -\xi^4 - 2\xi^2\eta^2 - \eta^4 + 24\xi^2\eta^2\zeta^2 - 8\eta^3\zeta^2 - 18\xi^2\zeta^4 - 18\eta^2\zeta^4 + 27\zeta^8. \]

We set the following notation for the direction of the parabolic curves. We divide \( \hat{\mathcal{M}}_1 \) into \( \Omega_1 = \{ (\theta, r) \in \hat{\mathcal{M}}_1 \mid \cot \theta < -1/\sqrt{3} \}, \Omega_2 = \{ (\theta, r) \in \hat{\mathcal{M}}_1 \mid -1/\sqrt{3} < \cot \theta < 1/\sqrt{3} \}, \Omega_3 = \{ (\theta, r) \in \hat{\mathcal{M}}_1 \mid \cot \theta > 1/\sqrt{3} \}. \) For the case of \( \varepsilon_2 = -1 \), setting \( t = \tan \theta \), the Eq. (4.16) is equivalent to \( p(-t) = 0 \). Thus the cubic discriminant is the same as \( D_{-1} \). We also divide \( \hat{\mathcal{M}}_{-1} \) into \( \tilde{\Omega}_1 = \{ (\theta, r) \in \hat{\mathcal{M}}_{-1} \mid \tan \theta < -1/\sqrt{3} \}, \tilde{\Omega}_2 = \{ (\theta, r) \in \hat{\mathcal{M}}_{-1} \mid -1/\sqrt{3} < \tan \theta < 1/\sqrt{3} \}, \tilde{\Omega}_3 = \{ (\theta, r) \in \hat{\mathcal{M}}_{-1} \mid \tan \theta > 1/\sqrt{3} \}. \) By (3.17) and (3.18), the directions defined by \( \tilde{v}_{-1,1}(0, z) \) and \( \tilde{v}_{-1,1}(\pi/2, z) \) (respectively, \( \tilde{v}_{-1,1}(\pi/3, z) \) and \( \tilde{v}_{-1,1}(\pi/3, z) \), \( \tilde{v}_{-1,1}(2\pi/3, z) \) and \( \tilde{v}_{-1,1}(2\pi/3, z) \)) are continuously connected across \( \{ z = 0 \} \). Thus, the region \( \Omega_1 \) corresponds to \( \tilde{\Omega}_3 \), \( \Omega_3 \) corresponds to \( \tilde{\Omega}_1 \), and \( \Omega_2 \) corresponds to \( \tilde{\Omega}_2 \). The notation \( (ijk|lmn) \), \( (i, j, k, l, m, n \in \{ 1, 2, 3 \}) \) stands for three parabolic curves of \( b_{1,1} \) emanating from the singular point, as they emanate into the regions \( \Omega_1, \Omega_2, \Omega_3 \), and three parabolic curves of \( b_{1,1} \) emanating from the singular point, as they emanate into the regions \( \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3 \), respectively. We set \( c_1 = \sqrt{3}\xi + \eta + 3\zeta^2 \), \( c_2 = \sqrt{3}\xi - \eta - 3\zeta^2 \). We have the following theorem:

**Theorem 4.3** The regions \( \Omega_1, \Omega_2, \Omega_3, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3 \) that the parabolic curves emanating from the singular point emanate into are summarized as in the “configurations” column of Table 1 according to the sign of \( D_{-1} \).

Needless to say, geometrically, all of the cases \( (iij) \) (\( i \neq j \)) are the same. Thus we can draw the pictures of the configurations of parabolic curves in Fig. 3. We draw “open” pictures instead of the usual pictures as in Fig. 2, with the parabolic curves drawn as thick lines, the set of singular points drawn as thin lines, and the intersection curves drawn as thin dotted lines.

To show the theorem, we use the following fact, known as Budan’s theorem or Descartes’ rule of signs (see [1], for example):

\[ \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3 \]
Fact 4.4 Let $p(t)$ be a polynomial in $t$. Then the number of roots of $p(t)$ that are greater than $\alpha$ is the same or less than the number of sign changes in the sequence of the coefficients of $p(t + \alpha)$, and their difference is an even number. Furthermore, the number of roots of $p(t)$ that are less than $\alpha$ is the same or less than the number of sign changes in the sequence of the coefficients of $p(-t + \alpha)$, and their difference is an even number.

Proof of Theorem 4.3 We assume $\varepsilon_2 = 1$. Using Fact 4.4, to study the numbers of roots of $p(t) = 0$ in each interval $(-\infty, -1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3})$, $(1/\sqrt{3}, \infty)$, we look
at the numbers of sign changes of
\[
\begin{align*}
p_1(t) &= 9p(t + 1/\sqrt{3}) = 9\xi t^3 + 9c_1 t^2 + 6\sqrt{3}c_1 t + 4\sqrt{3}(\xi + \sqrt{3}\eta), \\
p_2(t) &= 9p(t - 1/\sqrt{3}) = 9\xi t^3 - 9c_2 t^2 + 6\sqrt{3}c_2 t + 4\sqrt{3}(-\xi + \sqrt{3}\eta), \\
p_3(t) &= 9p(-t + 1/\sqrt{3}) = -9\xi t^3 + 9c_1 t^2 - 6\sqrt{3}c_1 t + 4\sqrt{3}(\xi + \sqrt{3}\eta), \\
p_4(t) &= 9p(-t - 1/\sqrt{3}) = -9\xi t^3 - 9c_2 t^2 - 6\sqrt{3}c_2 t + 4\sqrt{3}(-\xi + \sqrt{3}\eta).
\end{align*}
\]

We look at the number of sign changes of these polynomials under the conditions of each case. This is summarized in Table 2.

We first consider the case (I): \( \xi > 0 \). We assume (I-1): \(-\xi + \sqrt{3}\eta > 0 \). Then \( \eta > 0, \xi + \sqrt{3}\eta > 0 \) and \( c_1 > 0 \) hold. Since the number of sign changes in the sequence \( \xi, c_1, \xi + \sqrt{3}\eta \) of the coefficients of \( p_1 \) (the number of sign changes in \( p_1 \)) is zero, there are no roots in \((1/\sqrt{3}, \infty)\), since the number of sign changes in \( p_4 \) i.e., the number of sign changes in \(-\xi, -c_2, -c_2, -\xi + \sqrt{3}\eta \) is one for any of the cases of \( c_2 \), thus the configuration of the roots is \((223)\) or \((3)\) according to the sign of \( D_{-1} \). We assume (I-2): \(-\xi + \sqrt{3}\eta < 0 \). We consider the case (I-2-1): \( \xi + \sqrt{3}\eta > 0 \). Then \( \sqrt{3}c_1 = 2\xi + \xi + \sqrt{3}\eta + 3\sqrt{3}\xi^2 > 0 \). Since the number of sign changes in \( p_1 \) is zero, there are no roots in \((1/\sqrt{3}, \infty)\). If \( c_2 > 0 \), then the number of sign changes in \( p_4 \) is 0. So, in this case we have the configuration is \((222)\) or \((2)\). If \( c_2 < 0 \), then the number of sign changes in \( p_2 \) is 1. So, in this case we have \((233)\) or \((2)\). We consider the case (I-2-2): \( \xi + \sqrt{3}\eta < 0 \). Then the number of sign changes in \( p_1 \) is 1. If \( c_2 > 0 \), then the number of sign changes in \( p_4 \) is 0. So, in this case we have \((122)\) or \((1)\). If \( c_2 < 0 \), then the number of sign changes in \( p_2 \) is 1. So, in this case we have \((133)\) or \((1)\). We secondly consider the case (II): \( \xi < 0 \). We assume (II-1): \(-\xi + \sqrt{3}\eta > 0 \). We consider the case (II-1-1): \( \xi + \sqrt{3}\eta > 0 \). Then \( \eta > 0 \) and \( c_2 < 0 \). The number of sign changes in \( p_1 \) is 1 for any \( c_1 \), and there are no sign changes in \( p_4 \). So, in this case we have \((122)\) or \((1)\). We consider the case (II-1-2): \( \xi + \sqrt{3}\eta < 0 \). Then \( c_2 < 0 \) holds. If \( c_1 > 0 \) (II-1-2-1), then the number of sign changes in \( p_3 \) is 1, and that in \( p_4 \) is zero. So, in this case we have \((112)\) or \((2)\). If \( c_1 < 0 \) (II-1-2-2), then the number of sign changes in \( p_1 \) and \( p_3 \) are both 0. So, in this case we have \((222)\) or \((2)\). We assume (II-2): \(-\xi + \sqrt{3}\eta < 0 \). Then \( \eta > 0 \) and \( \xi + \sqrt{3}\eta < 0 \). If \( c_1 > 0 \), then the number of sign changes in \( p_3 \) and \( p_4 \) are both 1 for any \( c_2 \). So this case \((113), (3)\). If \( c_1 < 0 \), then there are no sign changes in \( p_1 \) and the number of sign changes in \( p_4 \) is 1. So this case \((223), (3)\). Since the above takes all the possibilities of signs of \( \xi \), \(-\xi + \sqrt{3}\eta, \xi + \sqrt{3}\eta, c_1 \) and \( c_2 \), we see the assertion. For the case of \( b_{-1}, -1 \), since the equation which we have to consider is \( p(-t) \), the configurations of this case can be obtained by interchanging \( \Omega_1 \) with \( \tilde{\Omega}_3 \), \( \Omega_2 \) with \( \tilde{\Omega}_2 \) and \( \Omega_3 \) with \( \tilde{\Omega}_1 \).

\[\square\]

4.4.2 The case of \( D_4^+ \)-singularity

Setting \( t = \coth \theta \), the Eq. (4.17) is equivalent to
\[q(t) = -\xi t^3 + (\eta + 3\xi^2)t^2 + \xi t - \eta + \xi^2 = 0, \quad (4.20)\]
and setting \( t = \tanh \theta \), the Eq. (4.18) is equivalent to \( q(t) = 0 \). Thus the number of asymptotic curves emanating from the singular point in \((\mathcal{D}_1 \cup \mathcal{D}_{-1}) \cap \{r > 0\}\) is generically 3 or 1 according to the cubic discriminant \( D_1 \) of \( q(t) = 0 \):

\[
D_1/4 = \xi^4 - 2\xi^2 \eta^2 + \eta^4 + 24\xi^2 \eta \xi^2 + 8\eta^3 \xi^2 - 18\xi^2 \xi^4 + 18\eta^2 \xi^4 - 27\xi^8.
\]

We divide \( \mathcal{D}_1 \) and \( \mathcal{D}_{-1} \) into the regions \( \Omega_1 = \{(\theta, r) \in \mathcal{D}_1 | r > 0, \ \coth \theta < -1\} \), \( \Omega_2 = \{(\theta, r) \in \mathcal{D}_{-1} | r > 0, \ \coth \theta > 1\} \), \( \Omega_3 = \{(\theta, r) \in \mathcal{D}_1 | r > 0, \ \coth \theta < 1\} \), \( \Omega_4 = \{(\theta, r) \in \mathcal{D}_{-1} | r < 0, \ \coth \theta > 1\} \), \( \Omega_5 = \{(\theta, r) \in \mathcal{D}_1 | r < 0, \ \coth \theta > 1\} \). The notation \((ijk|lmn), (i, j, k, l, m, n \in \{1, 2, 3\})\) stands for three parabolic curves of \( b_{1,1}\) \((\mathcal{D}_1 \cap \{r > 0\})\) (if \( i, j, k \) are 1 or 3) or \( b_{1,-1}\) \((\mathcal{D}_{-1} \cap \{r > 0\})\) (if \( i, j, k \) are 2) emanating from the singular point, as they emanate into the regions \( \Omega_i, \Omega_j, \Omega_k \), and three parabolic curves of \( b_{1,1}\) \((\mathcal{D}_1 \cap \{r < 0\})\) (if \( l, m, n \) are 1 or 3) or \( b_{1,-1}\) \((\mathcal{D}_{-1} \cap \{r < 0\})\) (if \( l, m, n \) are 2) emanating from the singular point, as they emanate into the regions \( \tilde{\Omega}_i, \tilde{\Omega}_j, \tilde{\Omega}_k \). We set \( c_3 = \eta + 3\xi^2, \ c_4 = -\eta + \xi^2 \). We have the following theorem:

**Theorem 4.5** For the regions \( \Omega_1, \Omega_2, \Omega_3, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3 \), the way how the parabolic curves emanate from the singular point to the regions are summarized as in the “configurations” columns in the Tables 3, 4, 5 and 6 according to the sign of \( D_1 \).

The pictures of the configurations of parabolic curves are in Fig. 4.

**Proof** We show the assertion by the same method as in the Proof of Theorem 4.3. Since the method is completely the same, we state here just the information about sign changes. We set \( q_1(t) = q(t) \) as in (4.20), \( q_2(t) = q_1(-t), \ q_3(t) = q_1(t + 1), \)

\[  \tag{5}  \]

\[  \tag{5}  \]
Table 4 Configurations of parabolic curves when $\xi < 0, c_3 > 0$

| Assumption II: $\xi < 0, c_3 > 0$ | $3\xi + c_3$ | $3\xi - c_3$ | $\xi + c_3$ | $\xi - c_3$ | $c_4$ | Configurations | $D_1 > 0$ | $D_1 < 0$ |
|----------------------------------|---------------|---------------|--------------|--------------|-------|----------------|----------|----------|
| (II-1-1)                         | +             | −             | +            | −            | +     | $(223|122)$    |          | $(3|1)$ |
| (II-2-1)                         | −             | −             | +            | −            | +     | $(223|122)$    |          | $(3|1)$ |
| (II-3-1)                         | −             | −             | −            | −            | +     | $(223|122)$    |          | $(3|1)$ |
| (II-1-2)                         | +             | −             | +            | −            | −     | $(223|122)$    |          | no       |
| (II-2-2)                         | −             | −             | any          | −            | −     | $(223|122)$    |          | $(3|1)$ |

Table 5 Configurations of parabolic curves when $\xi < 0, c_3 < 0$

| Assumption III: $\xi < 0, c_3 < 0$ | $3\xi + c_3$ | $3\xi - c_3$ | $\xi + c_3$ | $\xi - c_3$ | $c_4$ | Configurations | $D_1 > 0$ | $D_1 < 0$ |
|-----------------------------------|---------------|---------------|--------------|--------------|-------|----------------|----------|----------|
| (III-1-1)                         | −             | −             | −            | −            | +     | $(223|122)$    |          | $(3|1)$ |
| (III-2-1)                         | −             | any          | −            | +            | +     | $(113|133)$    |          | $(3|1)$ |

Table 6 Configurations of parabolic curves when $\xi > 0, c_3 < 0$

| Assumption IV: $\xi > 0, c_3 < 0$ | $3\xi + c_3$ | $3\xi - c_3$ | $\xi + c_3$ | $\xi - c_3$ | $c_4$ | Configurations | $D_1 > 0$ | $D_1 < 0$ |
|-----------------------------------|---------------|---------------|--------------|--------------|-------|----------------|----------|----------|
| (IV-1-1)                          | any          | +             | −            | +            | +     | $(133|133)$    |          | $(1|3)$ |
| (IV-2-1)                          | +             | +             | +            | +            | +     | $(122|223)$    |          | $(1|3)$ |

Fig. 4 The configurations (113), (122), (1), (2) (left to right) of parabolic curves in the case of $\varepsilon_1 = -1$

$q_4(t) = q_1(t - 1), q_5(t) = q_1(-t + 1)$ and $q_6(t) = q_1(-t - 1)$. Then each coefficient of these polynomials is one of $\xi, c_3, 3\xi + c_3, 3\xi - c_3, \xi + c_3, \xi - c_3$ and $c_4$. The necessary number of sign changes of these polynomials under the conditions of each case are given in Table 7, and it takes all the possibilities of signs of $\xi, c_3, 3\xi + c_3, 3\xi - c_3, \xi + c_3, \xi - c_3$ and $c_4$, so the assertion for the case of $r > 0$ is proven.

Since $(-r \cosh \theta, -r \sinh \theta)$ is a $\pi$-rotation of $(r \cosh \theta, r \sinh \theta)$, the configurations of the case $r < 0$ can be obtained by interchanging $\Omega_1$ with $\tilde{\Omega}_3$, $\Omega_2$ with $\tilde{\Omega}_2$ and $\Omega_3$ with $\tilde{\Omega}_1$. $\Box$
Lemma 4.6

Here we show Theorem B. A pair of two hyperbolic-trigonometric polynomials

$$g_n(s) = \sum_{1 \leq i+j \leq n} a_{i,j} \cosh^i s \sinh^j s$$

and

$$h_n(s) = \sum_{1 \leq i+j \leq n} b_{i,j} \cosh^i s \sinh^j s,$$

satisfying $a_{i,j} = b_{i,j} = 0$ for any odd $i + j$, or $a_{i,j} = b_{i,j} = 0$ for any even $i + j$, are said to be adapted if by setting $\cosh s = 1/(1-t^2)^{1/2}$, $\sinh s = t/(1-t^2)^{1/2}$, $(t = \tanh s)$, the polynomial $(1-t^2)^{n/2} g_n(s)$ with respect to $t$, and the polynomial $(1-t^2)^{n/2} h_n(s)/t^n$ with respect to $1/t$, are the same. For example, since $\cosh 3s = (1-t^2)^{-3/2}(1+3t^2)$, $\sinh 3s = (1-t^2)^{-3/2}t^3(1+3t^{-2})$, the pair $\cosh 3s$ and $\sinh 3s$ is adapted.

Lemma 4.6 (1) The number of roots of $f_n(s) = \sum_{i+j=1}^n a_{i,j} \cos^i s \sin^j s$ is at most $n$ for $s \in [0, \pi]$ if $d(f_n)/ds(s) \neq 0$ for all $s$ satisfying $f_n(s) = 0$. (2) We assume that the pair of two hyperbolic-trigonometric polynomials $g_n(s)$ and $h_n(s)$ is adapted. Then the sum of the numbers of roots $s \in \mathbb{R}$ of $g_n(s) = 0$ and $h_n(s) = 0$ is at most $n$ under the condition $d(g_n)/ds(s) \neq 0$ (respectively, $d(h_n)/ds(s) \neq 0$) for all $s$ satisfying $g_n(s) = 0$ (respectively, $h_n(s) = 0$).

Proof If $s = \pm \pi/2$ are solutions, factoring out $\cos s$ from $f_n(s)$, and we may assume $s = \pm \pi/2$ are not solutions of $f_n(s) = 0$. Setting $\cos s = \pm 1/(1 + \tan^2 s)^{1/2}$ and $\sin s = \pm \tan s/(1 + \tan^2 s)^{1/2}$, twice the number of roots of the equation $f_n(s) = 0$ can be reduced to a polynomial equation with respect to $\tan s$ with degree $2n$. This shows (1). See [4, Lemma 2] for a detailed proof. For a proof of (2), setting $\cosh s = 1/(1-t)^{1/2} \sinh s = t/(1-t)^{1/2}$, $(t = \tanh s)$, since $g_n$ and $h_n$ consist only of the terms where $i + j$ is odd or even, the equations $g_n(s) = 0$, $h_n(s) = 0$ are reduced to the same polynomial equations with respect to $t$ and $1/t$, respectively, with degree $n$. Since $\tanh s$ takes value in $(-1, 1)$, we see the assertion.

The number $n$ of the above $f_n$ (respectively, $g_n$, $h_n$), is called the degree, and it is denoted by $\text{deg}(f_n)$ (respectively, $\text{deg}(g_n)$, $\text{deg}(h_n)$). We give a Proof of Theorem B under the same assumption as in Proposition 4.2.
Proof of Theorem B  We see the degrees of
\[ C_1(\theta) = a(\theta, 0)(2\lambda' E_0 + 3\lambda E_0') + 2\lambda E_0 a z(\theta, 0), \]
\[ C_2(\theta) = -E_1 L_0 N_0 + E_1'M_0 N_0 + 2E_0(L_0 N_1 - M_0 N_0'), \]
\[ C_3(\theta) = 2E_0 N_0' - E_0' N_0. \]

These appear as the conditions of Proposition 4.2. We easily see \( \deg(L_0) = \deg(a) = 0, \deg(\lambda) = \deg(\lambda') = 3, \) and \( \lambda, \lambda' \) have only odd degree terms. Furthermore, we see \( \deg(E_0) = \deg(E_0') = 2, \deg(E_1) = 4 \) and these have only even degree terms. For the degree of \( a_z \), by (3.16) and (3.23), the degree of \( a_z \) is the same as that of \( a \) and \( \beta \). Thus \( \deg(a_z) = 4 \), and it has only even degree terms. We now consider the degrees of \( M_0, N_0, N_0', N_1 \). We assume \( \varepsilon_1 = \varepsilon_2 = 1 \). Since \( \varepsilon_1 = \varepsilon_2 = 1 \), we have \( \deg(M_0) = 4 \). Moreover, we see \( \varepsilon_1 = \varepsilon_2 = 1 \) has degree 3, together with \( \deg(V) = 1, \deg(\sin(V)) = \deg(\sin(V)) = 2 \), we have \( \deg(M_0) = 4 \). Moreover, we see \( \varepsilon_1 = \varepsilon_2 = 1 \) has only even degree terms. Similarly, we see \( \deg(N_0) = \deg(N_0') = 3 \) and they have only odd degree terms. Since \( \deg(\sin(\sin(V))) = 4 \) holds that \( \deg(x_{zz}) = \deg(y_{zz}) = \deg(X_{zz}) = \deg(Y_{zz}) = 4 \) and they have only even degree terms. Furthermore, since \( b_z(\theta, 0) = (0, 0, 1) \), \( b_{zz} = (x_{zz}, y_{zz}, 0) \),
\[ \tilde{v}_{zz} = -\cosh \theta (dY_x x_{zz} + dY_y y_{zz} + dY_{zz}) + \sinh \theta (dX_x x_{zz} + dX_y y_{zz} + dX_{zz}) \]
at \( \theta, 0 \) has degree 3, together with \( \deg(V) = 1, \deg(\sin(V)) = \deg(\sin(V)) = 2 \), we have \( \deg(M_0) = 4 \). Moreover, we see \( \varepsilon_1 = \varepsilon_2 = 1 \) has degree 3, together with \( \deg(V) = 1, \deg(\sin(V)) = \deg(\sin(V)) = 2 \), we have \( \deg(M_0) = 4 \). Moreover, we see \( \varepsilon_1 = \varepsilon_2 = 1 \) has only even degree terms. Similarly, we see \( \deg(N_0) = \deg(N_0') = 3 \) and they have only odd degree terms. Since \( \deg(\sin(\sin(V))) = 4 \) holds that \( \deg(x_{zz}) = \deg(y_{zz}) = \deg(X_{zz}) = \deg(Y_{zz}) = 4 \) and they have only even degree terms. Furthermore, since \( b_z(\theta, 0) = (0, 0, 1) \), \( b_{zz} = (x_{zz}, y_{zz}, 0) \),
\[ \tilde{v}_{zz} = -\cosh \theta (dY_x x_{zz} + dY_y y_{zz} + dY_{zz}) + \sinh \theta (dX_x x_{zz} + dX_y y_{zz} + dX_{zz}) \]
at \( \theta, 0 \) and \( 2(\alpha \sinh \theta - \beta \cosh \theta) = \sinh 3\theta \), we have \( \deg(N_1) = 7 \), and has only odd degree terms. Thus \( \deg(C_1) = \deg(C_2) = 9 \) and \( \deg(C_3) = 5 \). On the other hand, by (3.20), (3.21) and (3.22), the degrees of \( C_i (i = 1, 2, 3) \) in the case of \( \varepsilon_2 = -1 \) are the same as the case of \( \varepsilon_2 = 1 \). Moreover, for each \( i = 1, 2, 3 \), the pair \( C_i (\varepsilon_2 = \pm 1) \) of hyperbolic-trigonometric polynomial is adapted, so we have the assertion. For the case of \( \varepsilon_1 = -1 \), we can obtain the degrees by similar calculations. Summarizing up these degrees, Proposition 4.2 and the fact that each \( D^\pm_4 \) singularity consists of two sheets which correspond to \( \varepsilon_2 = \pm 1 \) respectively, we have the assertion.

See [12] for the ridge curves and the sub-parabolic curves on a regular surface near an umbilic point.

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