Minimax in Geodesic Metric Spaces: Sion’s Theorem and Algorithms

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Abstract

Determining whether saddle points exist or are approximable for nonconvex-nonconcave problems is usually intractable. We take a step towards understanding certain nonconvex-nonconcave minimax problems that do remain tractable. Specifically, we study minimax problems cast in geodesic metric spaces, which provide a vast generalization of the usual convex-concave saddle point problems. The first main result of the paper is a geodesic metric space version of Sion’s minimax theorem; we believe our proof is novel and transparent, as it relies on Helly’s theorem only. In our second main result, we specialize to geodesically complete Riemannian manifolds: we devise and analyze the complexity of first-order methods for smooth minimax problems.

1 Introduction

Minimax optimization refers to jointly optimizing two variables $x, y$: for $x$ we wish to minimize a bifunction $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, while for $y$ we wish to maximize it. The formulation of minimax optimization has drawn great attention recently in problems such as generative adversarial networks (Goodfellow et al., 2014a), robust learning (El Ghaoui and Lebret, 1997; Montanari and Richard, 2015), multi-agent reinforcement learning (Busoniu et al., 2008), adversarial training (Goodfellow et al., 2014b), among many others.

One common goal of solving minimax problems is to find saddle points. A pair $(x^*, y^*)$ is a saddle point if $x^*$ is a global minimum of $f(\cdot, y^*)$ and $y^*$ is a global maximum of $f(x^*, \cdot)$. In game theory, a saddle point is a special Nash equilibrium (Nash, 1950) for a two-player (zero-sum) game. When $f$ is convex-concave, existence of global saddle points is guaranteed by Sion’s minimax theorem (Sion, 1958), and computation is often tractable (e.g., Nemirovski (2004)). But without the simple convex-concave structure, global saddle points may fail to exist and computing them can be intractable (Daskalakis et al., 2009). Even computing local saddle points with linear constraints is PPAD-complete (Daskalakis et al., 2021). Therefore, a key question naturally arises in the subject:

Which nonconvex-nonconcave minimax problems admit saddle points, and can we compute them?

Existence. While in general this question is unlikely to admit satisfactory answers, it motivates a more nuanced study that seeks tractable subclasses or alternative criteria—e.g., (Jin et al., 2020; Mangoubi and Vishnoi, 2021; Fiez et al., 2021) explore this topic by establishing novel optimality criteria for nonconvex-nonconcave problems. We instead explore a rich subclass of nonconvex-nonconcave problems that admit saddle points: minimax problems over geodesic metric spaces (Burago et al., 2001). Therein, we provide sufficient conditions for the existence of (global) saddle points by establishing a metric space analog of Sion’s theorem. Informally, our result is as follows:

\textbf{Theorem 1.1} (Informal, see Theorem 3.1). Let $X, Y$ be geodesically convex subsets of geodesic metric spaces $\mathcal{M}$ and $\mathcal{N}$. If a bifunction $f : X \times Y \to \mathbb{R}$ is geodesically (quasi)-convex-concave and (semi)-continuous, then there exists a global saddle point $(x^*, y^*) \in X \times Y$.

Computation. Subsequently, we address computability of saddle points by focusing on special case of Riemannian manifolds. Here, we devise first-order algorithms for Riemannian minimax...
problem
\[
\min_{x \in \mathcal{M}} \max_{y \in \mathcal{N}} f(x, y),
\]
where \(\mathcal{M}, \mathcal{N}\) are finite-dimensional complete Riemannian manifolds, and \(f: \mathcal{M} \times \mathcal{N} \to \mathbb{R}\) is a smooth geodesically convex-concave bifunction. When the manifolds in \((P)\) are Euclidean, first-order methods such as optimistic gradient descent ascent and extragradient (ExtraG) can find saddle points efficiently. But in the Riemannian case, the extragradient steps do not succeed by merely translating Euclidean concepts into Riemannian ones. We introduce an additional correction to offset the distortion caused by nonlinear geometry, which helps us obtain a suitable Riemannian corrected extragradient (RCEG) algorithm. We provide non-asymptotic convergence rate guarantees, and informally state our result below.

**Theorem 1.2** (Informal, see Theorem 4.1). Under suitable conditions on the complete Riemannian manifolds \(\mathcal{M}, \mathcal{N}\), the proposed Riemannian corrected extragradient method admits a curvature-dependent \(O(\sqrt{\tau}/\epsilon)\) convergence to an \(\epsilon\)-approximate global saddle point for geodesically convex-concave problems, where \(\tau\) is a constant determined by the manifold curvature.

Our analysis enables us to study applications on nonlinear spaces; we note some examples below.

### 1.1 Motivating examples and applications

Minimix problems on geodesic metric spaces subsume Euclidean minimax problems. The more general structure from nonlinear geometry can offer more concise problem formulations, solutions, or even more efficient algorithms. We mention below several examples of minimax problems on geodesic spaces. Some of the examples possess a geodesically convex-concave structure, whereas others are more general and worth of further research on their theoretical properties.

**Constrained Riemannian optimization.** A first example arises in constrained minimization on Riemannian manifolds [Khuzani and Li, 2017; Liu and Boumal, 2020]; these works also note several applications of constrained Riemannian optimization: non-negative PCA, weighted MAX-CUT and etc. Here, we tackle the following problem:

\[
\min_{x \in \mathcal{M}} g(x), \quad \text{for } x \in \mathcal{M},
\]
\[
s.t. \quad h(x) = 0, \quad h := (h_1, \ldots, h_n): \mathcal{M} \to \mathbb{R}^n,
\]
where \(\mathcal{M}\) is a finite-dimensional Hadamard manifold. The basic idea is to convert (1) into an unconstrained Riemannian minimization problem via the augmented Lagrangian, which is:

\[
\max_{\lambda \in \mathbb{R}^n} \min_{x \in \mathcal{M}} f_\alpha(x, \lambda) := g(x) + \langle h(x), \lambda \rangle - \frac{\alpha}{2} \|\lambda\|^2.
\]

If \(g\) and all \(h_i\) are proper smooth and geodesically convex, then (2) resembles a geodesic-convex-Euclidean-concave minimax problem. A strong-duality condition hence can be established as the application of Theorem 1.1 which leads to the following important corollary:

**Corollary 1.3** (Informal, see Corollary 5.1). Lagrangian duality holds for geodesically convex Riemannian minimization problems with geodesically convex constraints in (1).

Hence, the minimizer can be efficiently found. A detailed statement is postponed to Section 5.

**Geometry-aware Robust PCA.** The second example is more specialized. It concerns efficient dimensionality reduction of symmetric positive definite (SPD) matrices. The geometry-aware Principal Component Analysis (PCA) in [Horev et al., 2016] better exploits geometric structure of SPD matrices, which is usually disregarded in the Euclidean view. Denote the \(n\)-dimensional SPD manifold \(\mathcal{P}(n) := \{A \in \mathbb{R}^{n \times n} : A \succeq 0 \text{ and } A = A^\top\}\) and the \(n\)-dimensional sphere \(\mathcal{S}(n) := \{A \in \mathbb{R}^n : A^\top A = I_n\}\). Let \(\{M_i \in \mathcal{P}(n)\}_{i=1}^k\) be a set of \(k\) observed instances. Then, robust PCA on the SPD manifold can be stated as:

\[
\max_{M \in \mathcal{P}(n) \times \mathcal{S}(n)} \min_{x \in \mathcal{S}(n)} f_\alpha(x, M) := -x^\top Ax - \alpha \mathbb{E}_i [d(M, M_i)],
\]
where \(\alpha > 0\) controls the penalty. Problem (3) formalizes a locally geodesic strongly-convex-strongly-concave structure. We verify the empirical performance of our proposed algorithm on this example in Section 5.
Robust optimization and further examples. We also hope to promote future study of geodesic minimax problems by noting below some applications that lack the convex-concave structure, yet where non-Euclidean geometry is still important. Although, our (global) theoretical results do not apply directly here, we believe local versions of our results should apply; nevertheless, we leave a deeper investigation of these problems to future work. Broadly construed, these applications arise via the lens of robust learning with underlying geometric structure (e.g., manifolds).

For instance, suppose that we seek to estimate the covariance matrix $S$ for Gaussian model on a set of $k$ data points $x_i \in M$, contaminated by errors $\Delta a_i \in M$. Assuming a constraint set $\Gamma$ for noise $\Delta a_i$, the robust maximal likelihood estimation (Bertsimas and Nohadani, 2019) formalizes the following problem

$$
\min_{\Delta a_i \in \Gamma} \max_{S \in \mathcal{P}(n)} -\frac{k n}{2} \log(2\pi) - \frac{k}{2} \log \det(S) - \frac{1}{2} \sum_{i=1}^{k} (a_i - \Delta a_i - \mu)^\top S^{-1} (a_i - \Delta a_i - \mu),
$$

which is geodesic concave in $S$ (Hosseini and Sra, 2015), and not necessarily convex. Other examples include robust computation of Wasserstein barycenters (Huang et al., 2021; Tiapkin et al., 2020) and computation of operator eigenvalues (Pesenson, 2004; Riddell, 1984). We expect that novel tools for geodesic nonconvex-nonconcave problems will prove valuable for these problems.

1.2 Related work on saddle points beyond Euclidean geometry

We provide a summary on the existence of saddle point in nonlinear geometry. Sion (1958) proved a general minimax theorem for quasi-convex-quasi-concave problems in the Euclidean space via both Knaster–Kuratowski–Mazurkiewicz (KKM) theorem and Helly’s theorem. Nevertheless, the proof relies on the linear geometry and can not be extended. Several recent works attempt to extend Sion’s minimax result into the non-Euclidean geometry. (Kristály, 2014; Colao et al., 2012; Bento et al., 2021) established results to guarantee the existence of Nash equilibrium or saddle point for geodesic convex games on Hadamard manifolds. Our analysis generalizes their result by removing the reliance on Riemannian differential structure along with other additional conditions.

| Differentiability | Convexity | Smoothness | Geometry |
|-------------------|-----------|------------|----------|
| Not required      | Quasi-conv. | Semi-cont. | Geodesic space |
| Not required      | Quasi-conv. | Semi-cont. | KKM space |
| Subdiff.          | Conv.     | Cont. subdiff. | Hadamard |
| Not required      | Conv.     | Cont.      | Hadamard |

Table 1: Results on global saddle point in nonlinear geometry. We compare our Theorem 3.1 with several similar existing results. These results are established for different geometry and relies on different continuity, differentiability and convexity conditions of objective $f$.

Our work is closest to Park (2019, 2010), who shows that Sion’s theorem can be established for the novel KKM space that subsumes Hadamard manifolds. Nevertheless, it remains difficult to verify whether a given geometry satisfies the KKM conditions. In contrast, our analysis is inspired by Komiya (1988), and applies more easily. However, we note that although Komiya (1988) claims to prove Sion’s minimax theorem in Euclidean space based on an elementary proof (without Helly’s theorem or KKM theorem), we found his proof to have a gap. Nevertheless, inspired by proof structure in Komiya (1988), we extend Sion’s theorem to nonlinear space by providing a new approach based on Helly’s theorem alone. In order to illustrate the strength of our result, we present a comparison with the existing works in Table 1.

2 Preliminaries on metric and Riemannian geometry

Before proceeding to the algorithm design and convergence analysis, we introduce our notation for geodesic metric spaces and Riemannian manifolds. For more details, we recommend textbooks (Burago et al., 2001; Lee, 2006; Do Carmo and Flaherty Francis, 1992) for reference.
Metric (geodesic) geometry. A metric space equipped with a geodesic structure resembles a generalization of Euclidean space and linear segments to nonlinear geometry. Such a metric space is called a geodesic metric space. Examples of geodesic metric space are CAT(0) spaces or Busemann convex spaces (Burago et al., 2001; Ivanov, 2014). Formally, a metric space is a pair \((\mathcal{M}, d_\mathcal{M})\) of a non-empty set \(\mathcal{M}\) and a distance function \(d_\mathcal{M} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}\) defined on \(\mathcal{M}\). We occasionally omit the subscript \(\mathcal{M}\) when it causes no confusion.

A map \(\gamma : [a, b] \subset \mathbb{R} \to \mathcal{M}\) is called a path on \(\mathcal{M}\). For any two points \(x, y \in \mathcal{M}\), a path \(\gamma : [0, 1] \to \mathcal{M}\) is referred as a geodesic joining \(x, y\) if

\[
\begin{align*}
(1) & \quad \gamma(0) = x, \quad (2) \quad \gamma(1) = y \quad \text{and} \quad (3) \quad d_\mathcal{M}(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1| \cdot d_\mathcal{M}(x, y)
\end{align*}
\]

for any \([t_1, t_2] \subseteq [0, 1]\). A metric space \((\mathcal{M}, d_\mathcal{M})\) is referred as a (unique) geodesic metric space if any two points \(x, y \in \mathcal{M}\) are joined by a (unique) geodesic. Using geodesics, the concept of convexity can be established in metric spaces. To be precise, a non-empty set \(X \subseteq \mathcal{M}\) is referred to as a geodesic convex set, if any geodesic connecting two points in \(X\) lies completely within \(X\). We can further define the concept of convex functions.

**Definition 2.1** (Geodesic (quasi)-convexity). A function \(f : \mathcal{M} \to \mathbb{R}\) is geodesically convex, if, for any \(x, y \in \mathcal{M}\) and \(t \in [0, 1]\), following inequality holds: \(f(\gamma(t)) \leq tf(x) + (1 - t)f(y)\), where \(\gamma\) is the geodesic connecting \(x, y\). Moreover, \(f\) is referred as geodesically quasi-convex if \(f(\gamma(t)) \leq \min\{f(x), f(y)\}\); (concavity and quasi-concavity are defined by considering \(-f\)).

Riemannian geometry. An \(n\)-dimensional manifold is a topological space that is locally Euclidean. A smooth manifold is referred as a Riemannian manifold if it is endowed with a Riemannian metric \(\langle \cdot, \cdot \rangle_x\) on the tangent space \(T_x\mathcal{M}\), where \(x\) is a point on \(\mathcal{M}\). The inner metric induces a natural norm on the tangent space as \(\|\cdot\|_x\). We usually omit \(x\) when it brings no confusion.

A curve \(\gamma : [0, 1] \to \mathcal{M}\) on Riemannian manifold is a geodesic if it is locally length-minimizing and of constant speed. An exponential map at point \(x \in \mathcal{M}\) defines a mapping from tangent space \(T_x\mathcal{M}\) to \(\mathcal{M}\) as \(\text{Exp}_x(v) = \gamma(1)\), where \(\gamma\) is the unique geodesic with \(\gamma(0) = x\) and \(\gamma'(0) = v\). The inverse map, if exists, is denoted as \(\text{Log}_x : \mathcal{M} \to T_x\mathcal{M}\). The exponential map also induces the Riemannian distance as \(d_\mathcal{M}(x, y) = \|\text{Log}_y(x)\|\). Parallel transport \(\Gamma^y_x : T_x\mathcal{M} \to T_y\mathcal{M}\) provides a way of comparing vectors between different tangent spaces. Parallel transport preserves inner product, i.e., \(\langle u, v \rangle_x = \langle \Gamma^y_x u, \Gamma^y_x v \rangle_y\) for points \(x, y \in \mathcal{M}\).

Unlike Euclidean space, a Riemannian manifold is not always flat. Sectional curvature \(\kappa\) (or simply “curvature”) provides a tool to characterize the distortion of geometry on the Riemannian manifold. In Riemannian optimization, an upper-bound \(\kappa_{\text{max}}\) on sectional curvature (e.g., \(\kappa_{\text{max}} \leq 0\)) is commonly employed to guarantee that the manifold is geodesically complete and unique. We will specify our condition on sectional curvature in the next section and also in Appendix B.1.

Lastly, we emphasize the (Riemannian) geodesics on complete manifolds are also geodesics in the sense of geodesic metric spaces and hence inherit the definition of geodesically convex sets and geodesically convex/concave function in unique geodesic spaces (see Lemma B.4 in Appendix B.2).

### 3 Existence of saddle points in nonlinear geometry

In Euclidean space, the famous Sion’s minimax theorem guarantees existence of saddle points for convex-concave minimax problems. In this section, we establish an analog of Sion’s theorem in the nonlinear geometry of unique geodesic metric spaces. The result automatically applies to complete manifolds, which are instances of geodesic metric spaces. This observation allows us to establish an optimality criterion for the geodesic convex-concave minimax problem (P).

#### 3.1 Sion’s Theorem based on geodesic structure

We consider the general form of (P) that is defined over unique geodesic metric spaces, where \(\mathcal{M}, \mathcal{N}\) are finite-dimensional geodesic metric spaces, \(f|_{X \times Y}\) is a geodesically (quasi-)convex-concave bifunction restricted to (compact) convex subsets \(X \subseteq \mathcal{M}, Y \subseteq \mathcal{N}\). We present below our core theorem that guarantees the existence of a saddle point for this general minimax problem.
Theorem 3.1 (Sion’s theorem in geodesic metric space). Let \((M, d_M)\) and \((N, d_N)\) be finite dimensional unique geodesic metric spaces. Suppose \(X \subseteq M\) is a compact and geodesically convex set, \(Y \subseteq N\) is a geodesically convex set. If following conditions holds for the bifunction \(f : X \times Y \rightarrow \mathbb{R}\):

1. \(f(\cdot, y)\) is geodesically-quasi-convex and lower semi-continuous;
2. \(f(x, \cdot)\) is geodesically-quasi-concave and upper semi-continuous.

Then we have an equality as

\[
\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).
\]

Proof sketch. The detailed proof is postponed to Appendix \(\[A\]\). We highlight the key steps here.

The major difficulty in the proof is to make sure, for any \(\alpha < \min_x \max_y f(x, y)\), there is always a \(y_0 \in Y\) such that condition \(\alpha < \min_x f(x, y_0)\) holds. This is done as following. We firstly specify the following claim.

Claim 1. For any \(\alpha < \min_x \max_y f(x, y)\), there exists for finite \(k\) points \(y_1, \ldots, y_k \in Y\) s.t. \(\alpha < \min_{i \leq k} f(x, y_i)\) holds.

If Claim[1] is true, then by following lemma, we can always guarantee the existence of the above \(y_0\).

Lemma 3.2. Under the condition of Theorem 3.1, for any finite \(k\) points \(y_1, \ldots, y_k \in Y\) and any real number \(\alpha < \min_{x \in X} \max_{i \leq k} f(x, y_i)\), there exists \(y_0 \in Y\) s.t. \(\alpha < \min_{x \in X} f(x, y_0)\).

Nevertheless, it is not trivial to find finite \(k\) points satisfying Claim[1]. To this end, we define level sets \(\phi_y(\alpha) := \{x \in X | f(x, y) \leq \alpha\}\) and consider the following version of Helly’s theorem.

Proposition 3.3 (Theorem 1 in [ivanov 2014]). Let \((M, d_M)\) be an \(m\)-dimensional unique geodesic metric space. Suppose a geodesically convex and compact set \(C\) in \(M\) and a collection of compact and convex subsets \(\{C_i\}_{i \in I} \subseteq C\). If arbitrary \(k + 1\) sets \(C_i_1, \ldots, C_i_{k+1}\) from \(\{C_i\}\) has nonempty intersection, i.e. \(\cap_{i=1}^{k+1} C_i \neq \emptyset\), then it follows that \(\cap_{i \in I} C_i \neq \emptyset\).

More precisely, for any \(\alpha, \alpha < \min_x f(x, y_0)\) is equivalent to \(\cap_{y \in Y} \phi_y(\alpha) = \emptyset\). Therefore, the contrapositive form of Proposition 3.3 guarantees that we can always find \(k\) points such that \(\cap_{i \in [k]} \phi_{\alpha_i}(y_i) = \emptyset\), this is immediately the content of Claim[1].

After we establish Lemma[A,5], we can complete the proof by proving the nontrivial direction of the minimax inequality as follows:

\[
\max_{y \in Y} \min_{x \in X} f(x, y) \geq \min_{x \in X} \max_{y \in Y} f(x, y).
\]

3.2 Saddle point and duality gap for Riemannian minimax

In the later part of our work, we are interested in the Riemannian minimax optimization problem (P). We state Sion’s theorem on Riemannian manifold as a corollary because of its importance in the paper’s layout.

Corollary 3.4 (Riemannian minimax). Suppose that \(M, \mathcal{N}\) are finite dimensional complete Riemannian manifolds. If subsets \(X, Y\) and \(f\) satisfies the same condition in Theorem 3.1, then it admits the identity as

\[
\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).
\]

Proof. This is a direct result from Theorem[3.1] since the whole Riemannian manifold defined in the corollary is an example of unique geodesic metric space.

From Corollary[3.4] we deduce that there is at least one saddle point \((x^*, y^*)\) such that the following equality holds:

\[
\min_{x \in X} f(x, y^*) = f(x^*, y^*) = \max_{y \in Y} f(x^*, y), \quad \forall x \in M, \quad \forall y \in N.
\]
As a consequence, the minimax problem $\mathcal{P}$ can be tackled by closing the duality gap, evaluated at a given pair $(\hat{x}, \hat{y})$, as

$$\max_y f(\hat{x}, y) - \min_x f(x, \hat{y}).$$

The duality gap then serves as an optimality criterion and a measure of convergence. Formally, we define $\epsilon$-saddle points.

**Definition 3.5.** A pair of points $(\hat{x}, \hat{y})$ is an $\epsilon$-saddle point of $f$ if $\max_y f(\hat{x}, y) - \min_x f(x, \hat{y}) \leq \epsilon$.

The above definition enables us to provide nonasymptotic convergence bounds for gradient methods on Riemannian manifolds.

### 4 Riemannian Minimax Algorithms and Complexity Analysis

In this section we present our algorithm for the minimax optimization of geodesic convex-concave bifunction $f$ on complete Riemannian manifolds. Furthermore, building upon the aforementioned optimality criterion, we establish the convergence properties by a non-asymptotic analysis. This is summarized in Table 2.

| Geometry       | Setting          | Algorithm | Complexity                | Theorem       |
|----------------|------------------|-----------|---------------------------|---------------|
| Riemannian     | convex-concave   | RCEG      | $O\left(\frac{\epsilon}{\mu}^2\right)$ | Theorem 4.1    |
| Euclidean      | convex-concave   | ExtraG    | $O\left(\frac{1}{\mu}\right)$ | Nemirovski (2004) |
| Euclidean      | strongly-convex  | ExtraG    | $O\left(\frac{1}{\mu \log(\frac{1}{\nu})}\right)$ | Mokhtari et al. (2020) |

Table 2: **Comparison of minimax algorithms.** The table summarizes the convergence properties of Riemannian minimax algorithms and presents a comparison with their Euclidean counterparts. We provide an explanation of each symbol. $L$: Lipschitz constant of $f$. $\mu$: strong-convexity/convexity constant of $f$. $\tau$: a constant parameterized by curvature and domain diameter (see below and Theorem 4.1).

In particular, we are interested in the smooth minimax optimization of $\mathcal{P}$ on Riemannian manifold. To this end, we assume the following regularity conditions.

**Assumption 1.** The gradient of $f$ is geodesically $L$-smooth, i.e. for any two pair $(x, y), (x', y') \in \mathcal{M} \times \mathcal{N}$, it satisfies that

$$\|\nabla_x f(x, y) - \Gamma^x_y \nabla_x f(x', y')\| \leq L \left( d_\mathcal{M}(x, x') + d_\mathcal{N}(y, y') \right),$$

$$\|\nabla_y f(x, y) - \Gamma^y_x \nabla_y f(x', y')\| \leq L \left( d_\mathcal{M}(x, x') + d_\mathcal{N}(y, y') \right).$$

We further require the curvature of manifolds $\mathcal{M}, \mathcal{N}$ to be bounded in range $[\kappa_{\min}, \kappa_{\max}]$. In order to guarantee the geodesic uniqueness and completeness, the diameter is also bounded when $\kappa_{\max} > 0$. This is also a regularity condition in Riemannian optimization literature (Alimisis et al. 2020; Zhang and Sra 2016). Formally, it is stated in the following assumption.

**Assumption 2.** The sectional curvature of manifold $\mathcal{M}$ and $\mathcal{N}$ lies in the range $[\kappa_{\min}, \kappa_{\max}]$ with $\kappa_{\min} \leq 0$. Moreover, if $\kappa_{\max} > 0$, the diameter of manifold satisfies $\text{diam} < \pi / \sqrt{\kappa}$.

For the non-asymptotic convergence analysis on the curved Riemannian manifold, a key ingredient is the comparison inequalities. These inequalities allow us to measure the distorted length by constants conditioned on the curvature bound and domain diameter. In particular, our upper and lower bound decides the minimal and maximal distortion rate (see Lemma B.1; Lemma B.2 in Appendix B.2). In our analysis, we use a ratio $\tau$ between the two rates to measure how non-flatness changes in the space, as:

$$\tau([\kappa_{\min}, \kappa_{\max}], \epsilon) = \begin{cases} \sqrt{\frac{\kappa_{\min}}{\kappa_{\max}}} \cdot \coth \left(\sqrt{\frac{\kappa_{\min}}{\kappa_{\max}}} \epsilon \right) / \cot \left(\sqrt{\frac{\kappa_{\min}}{\kappa_{\max}}} \epsilon \right), & \kappa_{\max} \leq 0, \\ \sqrt{\frac{\kappa_{\min}}{\kappa_{\max}}} \cdot \coth \left(\sqrt{\frac{\kappa_{\min}}{\kappa_{\max}}} \epsilon \right) / \cot \left(\sqrt{\frac{\kappa_{\min}}{\kappa_{\max}}} \epsilon \right), & \kappa_{\max} > 0. \end{cases} \tag{5}$$
Algorithm 1: Riemannian Corrected Extragradient (RCEG)

**Input:** objective $f$, initialization $(x_0, y_0)$, step-size $\eta$

1. Set $w_0 \leftarrow x_0$, $z_0 \leftarrow y_0$

2. for $t = 0, 1, 2, \ldots, T - 1$

3. \hspace{1em} $(w_t, z_t) \leftarrow \text{Exp}_{(x_t, y_t)}(-\nabla_x f(x_t, y_t), \nabla_y f(x_t, y_t))$

4. \hspace{1em} $(x_{t+1}, y_{t+1}) \leftarrow \text{Exp}_{w_t, z_t}(-\nabla_x f(w_t, z_t) + \text{Log}_{w_t}(x_t), \nabla_y f(w_t, z_t) + \text{Log}_{z_t}(y_t))$

**Output:** geodesic averaging scheme $(\overline{w}_T, \overline{z}_T)$ as in (7)

### 4.1 Convex-concave setting

We present a Riemannian version of extragradient method with an additional correction term (RCEG) for geodesically convex-concave $f$ (see Algorithm 1). We slightly abuse the manifold operation symbols to allow a more compact notation for the Riemannian gradient step of pair $(x, y) \in \mathcal{M} \times \mathcal{N}$:

$$\text{Exp}_{(x, y)}(u, v) := (\text{Exp}_x(u), \text{Exp}_y(v)).$$

We utilize an geodesic averaging scheme on Algorithm 1 in each iteration, we calculate

$$\overline{w}_{t+1} = \text{Exp}_{\overline{w}_t, \overline{z}_t} \left( \frac{1}{t+1} \cdot \text{Log}_{\overline{w}_t}(w_{t+1}), \frac{1}{t+1} \cdot \text{Log}_{\overline{z}_t}(z_{t+1}) \right).$$

The output produced by averaging is then $(\overline{w}_T, \overline{z}_T)$. The following theorem shows that the averaged output of RCEG achieves a curvature-dependent convergence rate for smooth convex-concave $f$ on Riemannian manifolds.

**Theorem 4.1.** Suppose Assumption 1-2 $f$ is geodesic convex-concave in $(x, y)$ and the iterations remain in the subdomain of bounded curvature $D_M$ and $D_N$. Let $(x_t, y_t, w_t, z_t)$ be the sequence obtained from the iteration of Algorithm 1 with initialization $x_0 = w_0$, $y_0 = z_0$. Then under constant step-size $\eta = \frac{1}{2T} \min \{ \sqrt{1/\tau_M}, \sqrt{1/\tau_N} \}$, the following inequality holds for $T$:

$$\min_{y \in \mathcal{N}} f(\overline{w}_T, y) - \min_{x \in \mathcal{M}} f(x, \overline{z}_T) \leq \frac{d_M^2(x_0, x^*) + d_N^2(y_0, y^*)}{\eta T}$$

with $(\overline{w}_T, \overline{z}_T)$ is the geodesic averaging scheme as in (7), and $\tau_M = \tau([\kappa_{\min}, \kappa_{\max}], D_M)$, $\tau_N = \tau([\kappa_{\min}, \kappa_{\max}], D_N)$ where $\tau$ is defined in (5).

Theorem 4.1 is a natural nonlinear extension of the known result achieved by extragradient method in the Euclidean setting. When the curvature $\kappa$ is set to zero, we retrieve its curvature-free Euclidean convergence rate.

**Proof sketch.** The translation of extragradient method to Riemannian manifold is non-trivial. We briefly elaborate on the proof technique and focus on the update of $x_t$ for simplicity. For any $x \in \mathcal{M}$, in each step, we need to bound the difference as

$$f(w_t, z_t) - f(x_t, z_t) \leq a_t \cdot d_M^2(x_{t+1}, x) - b_t \cdot d_M^2(x_t, x),$$

where $a_t, b_t$ are undetermined constants. We start with geodesic convexity, i.e. $f(w_t, z_t) - f(x_t, z_t) \leq -\langle \nabla_x f(w_t, z_t), \text{Log}_{w_t}(x) \rangle$. The correction term in RCEG allows a nice relationship

$$\text{Log}_{w_t}(x_{t+1}) = \text{Log}_{w_t}(x_t) - \eta \nabla_x f(w_t, z_t).$$

The above equality leads to a decomposition of cross terms in $\langle \nabla_x f(w_t, z_t), \text{Log}_{w_t}(x) \rangle$. As a result, we obtain

$$f(w_t, z_t) - f(x_t, z_t) \leq 1/\eta \cdot \langle \text{Log}_{w_t}(x_{t+1}), \text{Log}_{w_t}(x) \rangle - 1/\eta \cdot \langle \text{Log}_{w_t}(x_t), \text{Log}_{w_t}(x) \rangle.$$

Applying comparison inequalities on Eq. (10) leads to an efficient upper-bound in Eq. (8). By a telescoping on Eq. (8), we obtain the convergence result.

The condition is also regular in Riemannian optimization (Alimisis et al., 2020; Zhang and Sra, 2016).
It’s worth noting that the correction term is crucial to convergence in Riemannian setup. In the Euclidean case, the update of extragradient method is simply realized as \( x_{t+1} \leftarrow x_t - \eta \nabla_x f(w_t, z_t) \). However, a direct Riemannian counterpart update, i.e. \( x_{t+1} \leftarrow \text{Exp}_{x_t}(-\eta \Gamma f_{w_t} \nabla_x f(w_t, z_t)) \), does not make a convergent algorithm, since it does not allow a decomposition as in Eq. (9). In this case, we need to bound cross-term \( \Gamma f_{w_t} \nabla_x f(w_t, z_t) \), \( \Log_{w_t}(x) - \Gamma f_{w_t} \Log_{w_t}(x) \). This leads to intractable reliance on distortion caused by nonlinear geometry and cannot be upper-bounded.

### 4.2 Discussion on strongly-convex-strongly-concave setting

Unlike the Euclidean case, a simple single-loop algorithm like Riemannian-extragradient or our RCEG does not guarantee convergence at a linear rate\(^3\). The intractability can be intuitively explained as follows: in Euclidean space (Mokhtari et al., 2020), the key to convergence is to prove that \( d^2(x_t, x^*) \) forms a decreasing sequence. However, this is not true in nonlinear geometry. On Riemannian manifolds, a direct comparison between the distance \( d^2(x_{t+1}, x^*) \) and \( d^2(x_t, x^*) \) is infeasible. Hence the difference \( d^2(x_{t+1}, x^*) - d^2(x_t, x^*) \) generally cannot be bounded due to the existence of distance distortion in (Ahn and Sra, 2020). We expect the problem can be solved by considering more complicated double-looped algorithms like (Thekumparampil et al., 2019).

### 5 Applications and experiments

In this section, we illustrate the theoretical and algorithmic strength of our paper by two applications.

#### 5.1 Strong duality for constrained Hadamard optimization

From the theoretical aspect, we show the prowess of Theorem 4.1 and Corollary 3.2 by establishing a strong-duality result for the constrained Hadamard optimization problem. The setting is previously entailed in Section 1.1.

**Corollary 5.1.** Consider the constrained optimization problem (1) on a Hadamard manifold. If both \( g \) and each \( h_i, i = 1, \ldots, n \), are lower semi-continuous and geodesically convex, then the Lagrangian \( f(x, \lambda) = g(x) + \langle h(x), \lambda \rangle \) satisfies the following identity:

\[
\min_{x \in M} \max_{\lambda \in \mathbb{R}^n} f(x, \lambda) = \max_{\lambda \in \mathbb{R}^n} \min_{x \in X} f(x, \lambda).
\]

**Proof.** This claim is immediate from Theorem 4.1. \( \square \)

Similar to the Euclidean case, Corollary 5.1 guarantees that the minimizer for (1) can be efficiently found by maximizing the dual problem \( g^*_\alpha(\lambda) = \min_{x} f_\alpha(x, \lambda) \). We point out a similar result can be found in (Yang et al., 2014), which establishes a KKT theorem for constrained Hadamard optimization problem.

#### 5.2 Robust manifold PCA

We demonstrate the tractability of our RCEG by conducting experiments on the task of robust PCA on SPD matrix. While the problem in Eq. (3) is difficult in the Euclidean space, we show that it can be efficiently solved under our geodesic convex-concave setting. In precise, the SPD manifold is Hadamard. The sphere manifold is of positive curvature +1 and is a complete manifold.

**Data generation.** We run our test on a synthetic dataset \( \{M_i \in \mathcal{P}(n)\}_{i=1}^k \). For each \( M_i \), we first produce \( B \in \mathbb{R}^{n \times n} \) with i.i.d. random entries from standard Gaussian distribution and compute its QR decomposition as \( B = QR \), where \( Q \in \mathbb{R}^{n \times n} \) is orthonormal matrix and \( R \in \mathbb{R}^{n \times n} \) is upper-triangular matrix. We then generate random eigenvalue \( \sigma = (\sigma_1, \ldots, \sigma_n) \) in range \([\mu, L]\). Finally we obtain \( M_i = Q \text{diag}(\sigma) Q^\top \).

\(^3\)Nevertheless, an (asymptotic) linear convergence might be observed for concrete instances. We will elaborate on this in Section 5.
Figure 1: Convergence of RCEG for robust PCA on SPD manifold. We vary the penalty term $\alpha$ for different trajectories. In the left two subplots, we use configuration $n = 50, k = 40, \mu = 0.2$ and $L = 4.5$. In the right two subplots, we use configuration $n = 100, k = 40, \mu = 0.2$ and $L = 4.5$.

Results. The empirical performance of Riemannian minimax algorithms is illustrated in Figure 1. As predicted by Theorem 4.1, our RCEG is able to converge in an almost linear rate in late stages due to the gradient dominance and local geodesic strong-convexity and strong-concavity of our objective (Zhang and Sra, 2016). Though we are not able to prove the linear convergence of RCEG under the strongly-convex strongly-concave setting, we hope the empirical convergence of single-looped algorithm can be justified theoretically in future works.

Besides, we run a simple synthetic test to show the instability of Riemannian gradient descent-ascent method (RGDA). We set $\mathcal{M}, \mathcal{N}$ to be SPD manifold of same dimension and employ $f(x, y) = \text{tr}(\text{Log}_x(x_0) \cdot \text{Log}_y(y_0))$, where $x_0 \in \mathcal{M}, y_0 \in \mathcal{N}$. Then $f$ formalizes an analogy of Euclidean bilinear function. The result in Figure 2 illustrates that, similar to its Euclidean counterpart, the naive Riemannian gradient descent-ascent method is diverging for geodesic convex-concave setting.

Figure 2: Comparison between RGDA and RCEG for bilinear objective. While RCEG is convergent, the RGDA method is divergent for minimax problem $f(x, y) = \text{tr}(\text{Log}_x(x_0) \cdot \text{Log}_y(y_0))$, where $x, y$ are defined on a 50-dimensional SPD manifolds. We utilize a step-size $\eta = 0.2$.

6 Discussion of additional related work

Here we cover some topics and works relevant to our theme.

Convex-concave minimax algorithms. The majority of results on minimax optimization leverages the convex-concave setting. The optimal convergence rate for smooth convex-concave problems is $O(1/\epsilon)$ in terms of duality gap, achieved by mirror-prox method (Nemirovski, 2004), extragradient (Mokhtari et al., 2020) or proximal gradient descent (Tseng, 1995). The rate is matched by the lower-bound analysis in Ouyang and Xu (2021). Another line (Tseng, 1995; Mokhtari et al., 2020; Gidel et al., 2018) studied the strongly-convex-strongly-concave setting, establishing a several works (Thekumparampil et al., 2019; Lin et al., 2020; Alkousa et al., 2019) focused on the accelerated algorithms to improve the reliance on conditional number. Specifically, a recent work (Lin et al., 2020) established a near-optimal rate, matching the lower-bound (Ouyang and Xu, 2021).

Nonconvex-nonconcave minimax. Determining the existence of global saddle point in general nonconvex-nonconcave minimax problems is NP-hard. Hence a prominent task is to find a well-defined and tractable notion of stationarity. Along this line, works like (Lin et al., 2020).
Mangoubi and Vishnoi (2021); Fiez et al. (2021) investigated different notions of local optimality and their properties. Concurrently, several results (Daskalakis and Panageas, 2018; Adolfs et al., 2019; Mazumdar et al., 2019) focused on the relations between the stable fixed points of algorithm and local stationarity. Another line of research also considers problems with additional structure. For instance, Yang et al. (2020) tackled problem with Polyak-Lojasiewicz (PL) inequality; Diakonikolas et al. (2021); Malitsky (2020); Liu et al. (2019) explored Minty variational inequality condition.

**Geodesic convex optimization.** Geodesic convex optimization is a natural extension of convex optimization in Euclidean space onto Riemannian manifolds. The pioneer work of this field includes Udriste (1994); Absil et al. (2009). More recently, Zhang and Sra (2016) provided a first non-asymptotic analysis for Riemannian gradient methods. Subsequent works of the flourishing line explored topics such as acceleration (Zhang and Sra, 2018; Ahn and Sra, 2020; Hamilton and Moitra, 2021; Criscitiello and Boumal, 2021), variance reduction method (Zhang et al., 2016; Sato et al., 2019), and adaptive methods (Kasai et al., 2019). A parallel line of research tackled constrained Riemannian optimization by studying a hybrid minimax setting, in which \( M \) is the Riemannian manifold and \( N \) is Euclidean space. In particular, Khuzani and Li (2017); Liu and Boumal (2020) formalized the task of constrained geodesic-convex optimization on Riemannian manifold as a minimax problem by augmented Lagrangian method. Huang et al. (2020) considered a geodesic-convex-Euclidean-concave minimax problem and analyzed the convergence complexity of a novel Riemannian descent-ascent method.

7 Conclusions and Future Directions

In this work, we provide a new perspective into nonconvex-nonconcave minimax optimization and game theory by considering geodesic convex-concave problems in non-linear geometries. First, we provide an analog of Sion’s theorem on geodesic metric spaces. Second, we provide novel and efficient minimax algorithm for a different class of geodesic convex-concave games on geodesically complete Riemannian manifolds. We believe our work takes a significant step towards understanding the properties of minimax problems in non-linear geometry, and should help inform the study of many structured learning problems on manifolds.

**Future directions.** We would like to promote the future investigations and applications by raising several open questions. On the empirical side, several potential applications are discussed in Section 1.1. Here we propose some possible theoretical questions. First, while our RCEG tackles the unconstrained case, the tractability of projected single-looped algorithm for constrained setting deserves a future exploration: additional distortion caused by projection operation requires novel tools to bound. Second, it remains open to establish a linear convergence for the geodesic strongly-convex-strongly-concave setting by devising double-looped algorithm or incorporating the adaptive metric distortion in (Ahn and Sra, 2020). Third, a lower bound analysis like (Ouyang and Xu, 2021) is still lacking for the minimax problems in Riemannian geometry. Last, for Riemannian minimax problems that (partially) lack a geodesic convexity or concavity, additional structure like geodesic PL property or Minty variational condition may help overcome the intractability.

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Appendix: Proofs and Supplementaries

A Proof of Sion’s Theorem in nonlinear geometry

In this section we present a novel proof to Sion’s minimax theorem in the nonlinear geometry of metric spaces with geodesic structure. Our analysis indicates that, any nonlinear geometry that has a geodesic structure and satisfies the finite-intersection property (Helly’s theorem) admits Sion-style saddle point for convex-concave minimax problems. In particular, this also applies to the geodesic convex-concave minimax optimization on complete manifolds. We establish our proof by revising the technique in Komiya (1988). Nevertheless, we complete the proof and extend the result to geodesic spaces by resorting to a generalized Helly’s theorem.

Helly’s Theorem. In Euclidean space, Helly’s theorem is an important result on the intersection of convex bodies. Our proof to Sion’s theorem in general nonlinear geometry requires an analogy of Helly’s theorem in the geodesic metric space. This was established in Ivanov (2014).

Proposition A.1 (Theorem 1 in Ivanov (2014), restated Proposition 3.3). Let $(\mathcal{M}, d_{\mathcal{M}})$ be an $n$-dimensional unique geodesic metric space. Suppose a geodesic convex and compact set $C$ in $\mathcal{M}$ and a collection of compact and convex subsets $\{C_i\}_{i \in I} \subset C$. If arbitrary $k + 1$ sets $C_{i_1}, \ldots, C_{i_{k+1}}$ from $\{C_i\}$ has nonempty intersection, i.e. $\cap_{j=1}^{k+1} C_{i_j} \neq \emptyset$, then it follows that $\cap_{i \in I} C_i \neq \emptyset$.

We slightly modify the original statement in Ivanov (2014) by incorporating additional compact condition to contain the infinite collection case. We need in our proof a particular contrapositive form of the original Helly’s theorem. Due to its importance, we state it as in the next corollary.

Corollary A.2. Suppose the conditions of Proposition A.1. If the intersection of the whole family $\{C_i\}_{i \in I}$ is empty, i.e. $\cap_{i \in I} C_i = \emptyset$, then there exists at least one $(k \leq m)$ subsets of $k + 1$ elements $C_{i_1}, \ldots, C_{i_{k+1}}$, such that $\cap_{j=1}^{k+1} C_{i_j} = \emptyset$.

A.1 Proof of Sion’s Theorem.

Now we proceed to a proof to Sion’s theorem on the geodesic metric space. We restrict the domain to be geodesic convex set $X, Y$ on metric space $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$. Given a bifunction $f$ and a fixed $y \in Y$, we define its level set in the first variable as

$$\phi_y(\alpha) := \{x \in X | f(x, y) \leq \alpha\}.$$

Likewise in the Euclidean space, $\phi_y(\alpha)$ is a geodesic convex set if $f(\cdot, y)$ is a geodesic convex function, and closed if $f(\cdot, y)$ is lower semi-continuous. We restate Sion’s minimax theorem in the unique geodesic space and elaborate the proof.

Theorem A.3 (Sion’s theorem in geodesic space, restated Theorem 3.1). Let $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$ be finite dimensional unique geodesic metric spaces. Suppose $X \subseteq \mathcal{M}$ is a compact and geodesically convex set, $Y \subseteq \mathcal{N}$ is a geodesically convex set. If following conditions holds for the bifunction $f : X \times Y \rightarrow \mathbb{R}$:

1. $f(\cdot, y)$ is geodesically-quasi-convex and lower semi-continuous;
2. $f(x, \cdot)$ is geodesically-quasi-concave and upper semi-continuous.

Then we have an equality as

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).$$

Proof. It is obvious that the inequality $\max_{y \in Y} \min_{x \in X} f(x, y) \leq \min_{x \in X} \max_{y \in Y} f(x, y)$ holds. We now prove its reverse.

For any $\alpha < \min_{x \in X} \max_{y \in Y} f(x, y)$ we have the intersection of all the level sets over the whole $Y$, i.e. $\cap_{y \in Y} \phi_y(\alpha)$, to be empty. We also know each level set $\phi_y(\alpha)$ is a geodesic convex and

\[\text{(1)}\] by the Weierstrass minimum theorem, $\min_{x \in X} \max_{y \in Y} f(x, y)$ is bounded from $-\infty$ since (1) $X$ is compact and (2) $y(x) = \max_{y \in Y} f(x, y)$ is lower semi-continuous. The latter is due to the fact that the supremum of any collection of lower semi-continuous function is still lower semi-continuous.
We then argue the intersection is empty. In the meanwhile, since
\begin{equation}
\alpha < \min_{x \in X} \max_{j \in [k+1]} f(x, y_j).
\end{equation}
Applying Lemma [A.3] we conclude that there exists a \( y_0 \in Y \) such that \( \alpha < \min_{x \in X} f(x, y_0) \leq \max_{y \in Y} \min_{x \in X} f(x, y) \). Therefore, by considering a monotonic increasing sequence \( \alpha_k \rightarrow \min_{x \in X} \max_{y \in Y} f(x, y) \), we know that
\begin{equation}
\min_{x \in X} \max_{y \in Y} f(x, y) = \lim_{k \to \infty} \alpha_k \leq \max_{y \in Y} \min_{x \in X} f(x, y).
\end{equation}
This proves the theorem. \( \square \)

**Technical lemmas.** The proof for the Sion’s Theorem in geodesic space relies on the following lemmas.

**Lemma A.4.** Under the condition of Theorem [A.3] For any two points \( y_1, y_2 \in Y \) and any real number \( \alpha < \min_{x \in X} \max \{ f(x, y_1), f(x, y_2) \} \), there exists \( y_0 \in Y \) such that \( \alpha < \min_{x \in X} f(x, y_0) \).

**Proof.** We prove by contradiction and assume for such an \( \alpha \), the inequality \( \min_{x \in X} f(x, y) \leq \alpha \) for any \( y \in Y \). As a consequence, there is at least a constant \( \beta \) such that
\begin{equation}
\min_{x \in X} f(x, y) \leq \alpha < \beta < \min_{x \in X} \max \{ f(x, y_1), f(x, y_2) \}.
\end{equation}
Consider the geodesic \( \gamma : [0, d(y_1, y_2)] \rightarrow Y \) (this is possible since \( Y \) is a g-convex set) connecting \( y_0, y_1 \). For any \( t \in [d(y_1, y_2)] \) and corresponding \( z = \gamma(t) \) on the geodesic, the level sets \( \phi_z(\alpha), \phi_z(\beta) \) are nonempty due to Eq. (11), and closed due to lower semi-continuity of \( f \) regarding the first variable. In the meanwhile, since \( f \) is geodesic quasi-concave in the second variable, we obtain
\begin{equation}
f(x, z) \geq \min \{ f(x, y_1), f(x, y_2) \}, \quad \text{with } z = \gamma(t), \quad \forall x \in X, \quad \forall t \in [0, 1].
\end{equation}
This is equivalent to say \( \phi_z(\alpha) \subseteq \phi_z(\beta) \subseteq \phi_{y_1}(\beta) \cup \phi_{y_2}(\beta) \).

We then argue the intersection \( \phi_{y_1}(\beta) \cap \phi_{y_2}(\beta) \) should be empty. Otherwise, there exists \( x \in X \) such that \( \max \{ f(x, y_1), f(x, y_2) \} \leq \beta \), contradicting Eq. (11).

Next, by quasi-convexity, since level set \( \phi_z(\beta) \) is geodesic convex for any \( z \), it is also connected.

Consider the three facts:

- \( \phi_z(\alpha) \subseteq \phi_{y_1}(\beta) \cup \phi_{y_2}(\beta) \);
- \( \phi_{y_1}(\beta) \cap \phi_{y_2}(\beta) \) is empty;
- \( \phi_z(\alpha), \phi_{y_1}(\beta) \) and \( \phi_{y_2}(\beta) \) are closed (due to lower semi-continuity), connected and convex.

We claim that either \( \phi_z(\alpha) \subseteq \phi_{y_1}(\alpha) \) or \( \phi_z(\alpha) \subseteq \phi_{y_2}(\alpha) \) holds for any point \( z \) on the geodesic \( \gamma \). Suppose not, then we can always find two points \( w_1 \in \phi_{y_1}(\beta), w_2 \in \phi_{y_2}(\beta) \) such that \( w_1, w_2 \in \phi_z(\alpha) \). Since \( \phi_z(\alpha) \) is convex, then there is a geodesic \( \gamma : [0, 1] \rightarrow X \) in \( \phi_z(\alpha) \) connecting \( w_1, w_2 \).

Therefore \( \gamma \) also lies in \( \phi_{y_1}(\beta) \cup \phi_{y_2}(\beta) \). Because \( \phi_{y_1}(\beta) \cap \phi_{y_2}(\beta) \), \( \gamma^{-1} \) induces a partition on \( [0, 1] \) as \( J_1 \cap J_2 = \emptyset \) and \( J_1 \cup J_2 = [0, 1] \) where \( \gamma(J_1) \subseteq \phi_{y_1}(\beta), \gamma(J_2) \subseteq \phi_{y_2}(\beta) \). Therefore at least one of \( J_1, J_2 \) is not closed. Since \( \gamma \) is a continuous map, at least one of \( \phi_{y_1}(\beta) \) or \( \phi_{y_2}(\beta) \) is also not closed, contradicting known conditions.

Since either \( \phi_z(\alpha) \subseteq \phi_{y_1}(\alpha) \) or \( \phi_z(\alpha) \subseteq \phi_{y_2}(\alpha) \), the two sets below
\begin{align*}
I_1 & := \{ t \in [0, 1] | \phi_{\gamma(t)}(\alpha) \subseteq \phi_{y_1}(\beta) \}, \\
I_2 & := \{ t \in [0, 1] | \phi_{\gamma(t)}(\alpha) \subseteq \phi_{y_2}(\beta) \}.
\end{align*}
form a partition of the interval \([0, 1]\).
We prove $I_1$ is closed and nonempty. The latter is obvious since at least $\gamma^{-1}(y_1) \in I_1$. Now we turn to prove closedness. Let $t_k$ be an infinite sequence in $I_1$ with a limit point of $t$. We consider any $x \in \phi_{\gamma(t)}(\alpha)$. The upper semi-continuity of $f(x, \cdot)$ implies
\[
\limsup_{k \to \infty} f(x, \gamma(t_k)) \leq f(x, \gamma(t)) \leq \alpha < \beta.
\]
Therefore, there exists a large enough integer $l$ such that $f(x, \gamma(t_l)) < \beta$. This implies $x \in \phi_{\gamma(t_l)}(\beta) \subseteq \phi_{y_l}(\beta)$. Therefore for any $x \in \phi_{\gamma(l)}(\alpha)$, condition $x \in \phi_{y_l}(\beta)$ also holds. This is equivalent to $\phi_{\gamma(l)}(\alpha) \subseteq \phi_{y_l}(\beta)$. Hence by the definition of level set, we know the $t \in I_1$ and $I_1$ is then closed. By a similar argument, $I_2$ is also closed and nonempty. This contradicts the definition of partition and hence proves the lemma.

The next lemma extends the conclusion of Lemma A.4 to the minimum of any finite $k$ points and then provides a requisite for Helly’s theorem.

**Lemma A.5** (Restated Lemma A.4). Under the condition of Theorem A.3 For any finite $k$ points $y_1, \ldots, y_k \in Y$ and any real number $\alpha < \min_{x \in X} \max_{i \in [k]} f(x, y_i)$, there exists $y_0 \in Y$ s.t. $\alpha < \min_{x \in X} f(x, y_0)$.

**Proof.** This is proved by an induction on Lemma A.4. For $n = 1$, the result is trivial. We assume the lemma holds for $k - 1$. Now, for any $k$ points $y_1, \ldots, y_k \in N$, let us denote $X' = \phi_{y_k}(\alpha)$. For any $\alpha < \min_{x \in X} \max_{i \in [k]} f(x, y_i)$, we have
\[
\alpha < \min_{x \in X'} \max_{i \in [k-1]} f(x, y_i).
\]
By our assumption on $k - 1$ on set $X' \times Y$, there exists a $y'_0$ such that $\alpha < \min_{x \in X'} f(x, y_0)$. As a result, we have $\alpha < \min_{x \in X} \max\{f(x, y'_0), f(x, y_n)\}$. Then applying Lemma A.4 leads to the conclusion.

## B Riemannian optimization prerequisite

### B.1 Curvature and geodesic completeness

In Riemannian optimization literature, it is a common strategy to guarantee the completeness and uniqueness of geodesic by asserting an upper-bound $\kappa_{\text{max}}$ for the curvature. Specifically, Hadamard manifold (manifolds with non-positive curvature, $\kappa_{\text{max}} = 0$) admits unique geodesic between any two points. Moreover, a (sub)manifold of bounded diameter strictly less than $\pi/\sqrt{\kappa_{\text{max}}}$ is also geodesic complete and unique when it admits positive maximal curvature $\kappa_{\text{max}} > 0$. We employ such conditions in our algorithmic analysis.

### B.2 Comparison inequalities in non-flat geometry

The flat geometry of Euclidean space enables trigonometric equalities such as the law of cosine. Nevertheless, such results do not hold under non-flat geometry. Therefore, for the non-asymptotic convergence analysis on the curved Riemannian manifold, a key ingredient is the following trigonometric comparison inequalities. A first result is due to Zhang and Sra (2016), which is obtained when the sectional curvature is bound from below.

**Lemma B.1** (Lemma 5 in Zhang and Sra (2016)). Let $\mathcal{M}$ be a Riemannian manifold with sectional curvature lower bounded by $-\kappa \leq 0$. If $a, b, c$ are the sides length of a geodesic triangle in $\mathcal{M}$, and $A$ is the angle between $b$ and $c$, then
\[
a^2 \leq \zeta(\kappa, c)b^2 + c^2 - 2bc \cos A
\]
where $\zeta(\kappa, c) := \sqrt{-\kappa} \coth(\sqrt{-\kappa}c)$.

The second inequality characterizes the trigonometric length when sectional curvature is bounded from above. In particular, if the upper-bound $\kappa$ is positive, the diameter of manifold should be bounded.
Lemma B.2 (Corollary 2.1 in Alimisis et al. (2020)). Let \( M \) be a Riemannian manifold with sectional curvature bounded above by \( \kappa \) and diameter \( \text{diam}(M) < D(\kappa) \). If \( a, b, c \) are the sides length of a geodesic triangle in \( M \), and \( A \) is the angle between \( b \) and \( c \), then

\[
a^2 \geq \xi(\kappa, c)b^2 + c^2 - 2bc \cos A
\]

where

\[
D(\kappa) := \begin{cases} \infty, & \kappa \leq 0, \\ \pi/\sqrt{\kappa}, & \kappa > 0, \end{cases}
\]

and \( \xi(\kappa, c) := \left\{ \begin{array}{ll} \sqrt{-\kappa c \cot(\sqrt{-\kappa c})}, & \kappa \leq 0, \\ \sqrt{-\kappa c \cot(\sqrt{-\kappa c})}, & \kappa > 0. \end{array} \right. \)

Remark 1. When \( \kappa \) is set to 0, \( \xi(\kappa, \cdot) \) reduces to 1.

B.3 Function classes on Riemannian manifold

We first provide an additional definition of geodesically strongly-convex function, which we omit due to the lack of space and usage.

Definition B.3 (Strong convexity). A function \( f : M \to \mathbb{R} \) is geodesically strongly-convex modulus \( \mu \), if, for any \( x, y \in M \) and \( t \in [0, 1] \), following inequality holds: \( f(\gamma(t)) \leq tf(x) + (1-t)f(y) + \frac{\mu(1-t)}{2}d_M^2(x, y) \), where \( \gamma \) is the geodesic connecting \( x, y \).

Complete manifold inherits the definition of geodesically convex set and geodesically convex/concave function in unique geodesic space. We can readily verify the inherited convexity is consistent with regular definition of geodesic convexity in the Riemannian optimization literature.

Lemma B.4. The function \( f : M \to \mathbb{R} \) is geodesically convex if and only if for any two points \( x, y \in M \),

\[
f(y) \geq f(x) + \langle \nabla f(x), \log_y(x) \rangle.
\]

Besides, \( f \) is geodesically \( \mu \)-strongly convex if and only if for any two points \( x, y \in M \),

\[
f(y) \geq f(x) + \langle \nabla f(x), \log_y(x) \rangle + \frac{\mu}{2}d^2(x, y).
\]

We state also the regularity definition of smooth functions on Riemannian manifold using the aforementioned manifold operations.

Definition B.5. \( f \) is geodesically Lipschitz smooth modulus \( L \), if, there exists a constant \( L \), for any \( x, w \in M \), \( \| \nabla f(x) - \nabla f(w) \| \leq Ld(x, w) \).

C Proofs for Section 4.1

Theorem C.1 (Restated Theorem 4.1). Suppose Assumption 4.2 \( f \) is geodesically convex-concave in \( (x, y) \) and the iterations remain in the subdomain \( U \) of with bounded curvature \( D_M \) and \( D_N \). Let \( (x_t, y_t, w_t, z_t) \) be the sequence obtained from the iteration of Algorithm 4.1 with initialization \( x_0 = w_0, y_0 = z_0 \). Then under constant step-size \( \eta = \frac{1}{\tau M} \min\{ \sqrt{1/\tau_M}, \sqrt{1/\tau_N} \} \), the following inequality holds for \( T \):

\[
\min_{y \in N} f(\tilde{w}_T, y) - \min_{x \in M} f(x, \tilde{x}_T) \leq \frac{d_M^2(x_0, x^*) + d_N^2(y_0, y^*)}{\eta T}
\]

with \( (\tilde{w}_T, \tilde{x}_T) \) is the geodesic averaging scheme as in 4.1, and \( \tau_M = \tau([\kappa_{\min}, \kappa_{\max}], D_M), \tau_N = \tau([\kappa_{\min}, \kappa_{\max}], D_N) \) where \( \tau \) is defined in 4.3.

Proof. Since \( f \) is geodesically convex in \( x \) and geodesically concave in \( y \), for any two points \( x \in M, y \in N \), the following inequality holds

\[
f(w_t, y) - f(x, z_t) = f(w_t, z_t) - f(x, z_t) - (f(w_t, z_t) - f(w_t, y)) \\
\leq -\langle \nabla_x f(w_t, z_t), \log_{w_t}(x) \rangle + \langle \nabla_y f(w_t, z_t), \log_{w_t}(y) \rangle.
\]

The condition is also regular in Riemannian optimization (Alimisis et al., 2020; Zhang and Sra, 2016).
Plugging the iteration of Algorithm 1 into the above inequality, we obtain
\[
\begin{align*}
& f(w_t, y) - f(x, z_t) \\
& \leq \frac{1}{\eta} \cdot (\log_{w_1}(x_t) - \eta \nabla_z f(w_t, z_t)) - \frac{1}{\eta} \cdot \langle \log_{x_1}(x_t), \log_{w_1}(x) \rangle \\
& + \frac{1}{\eta} \cdot (\log_{z_1}(y_t) + \eta \nabla_y f(w_t, z_t)) - \frac{1}{\eta} \cdot \langle \log_{z_1}(y_t), \log_{z_1}(y) \rangle \\
& = \frac{1}{\eta} \cdot (\log_{w_1}(x_{t+1}), \log_{w_1}(x)) - \frac{1}{\eta} \cdot \langle \log_{w_1}(x_t), \log_{w_1}(x) \rangle \\
& + \frac{1}{\eta} \cdot (\log_{z_1}(y_{t+1}), \log_{z_1}(y)) - \frac{1}{\eta} \cdot \langle \log_{z_1}(y_t), \log_{z_1}(y) \rangle.
\end{align*}
\]
As the next step, we bound these terms by leveraging comparison inequalities on Riemannian manifold with bounded sectional curvature. Combined with the bounded domain condition, we define the following constants as
\[
\begin{align*}
\zeta_M &= \zeta(\kappa_{\min}, D_M), & \zeta_N &= \zeta(\kappa_{\min}, D_N), \\
\xi_M &= \xi(\kappa_{\max}, D_M), & \xi_N &= \xi(\kappa_{\max}, D_N)
\end{align*}
\]
where \( \xi \) and \( \zeta \) is defined as in Lemma B.1 and Lemma B.2. As a result, we have
\[
-2\langle \log_{w_1}(x_t), \log_{w_1}(x) \rangle \leq -\xi_M d_M^2(w_{t+1}, x_t) - d_M^2(w_{t+1}, x) + d_M^2(x_t, x)
\]
and
\[
-2\langle \log_{z_1}(y_t), \log_{z_1}(y) \rangle \leq -\xi_N d_N^2(z_t, y_t) - d_N^2(z_t, y) + d_N^2(y_t, y).
\]
Using Lemma B.1 results in
\[
\begin{align*}
2\langle \log_{w_1}(x_{t+1}), \log_{w_1}(x) \rangle & \leq \zeta_M \| \eta \nabla_x f(w_t, z_t) - \log_{w_1}(x_t) \|^2 + d_M^2(w_t, x) - d_M^2(x_{t+1}, x) \\
& = \zeta_M \eta^2 \| \nabla_x f(w_t, z_t) - \Gamma_{w_t} \nabla f(x_t, y_t) \|^2 + d_M^2(w_t, x) - d_M^2(x_{t+1}, x) \\
& \leq 2\zeta_M \eta^2 L^2 \cdot (d_M^2(x_t, w_t) + d_N^2(y_t, z_t)) + d_N^2(y_t, x) - d_N^2(x_{t+1}, x)
\end{align*}
\]
where the last inequality is due to smoothness and fact \((a-b)^2 \leq 2a^2 + 2b^2\). Similarly, we also have
\[
\begin{align*}
2\langle \log_{z_1}(y_{t+1}), \log_{z_1}(y) \rangle & \leq \zeta_N \| \eta \nabla_y f(w_t, z_t) - \log_{z_1}(y_t) \|^2 + d_N^2(z_t, y) - d_N^2(y_{t+1}, y) \\
& = \eta^2 \| \nabla_y f(w_t, z_t) - \Gamma_{z_t} \nabla f(x_t, y_t) \|^2 + d_N^2(z_t, y) - d_N^2(y_{t+1}, y) \\
& \leq 2\zeta_N \eta^2 L^2 \cdot (d_M^2(x_t, w_t) + d_N^2(y_t, z_t)) + d_N^2(z_t, y) - d_N^2(y_{t+1}, y)
\end{align*}
\]
Putting all the inequalities together, it yields
\[
\begin{align*}
f(w_t, y) - f(x, z_t) & \leq \frac{1}{\eta} \cdot \left( \xi_M - 2\zeta_M \eta^2 L^2 \right) d_M^2(w_t, x_t) + \frac{1}{\eta} \cdot (d_M^2(x_t, x) - d_M^2(x_{t+1}, x)) \\
& - \frac{1}{\eta} \cdot \left( \xi_N - 2\zeta_N \eta^2 L^2 \right) d_N^2(z_t, y_t) + \frac{1}{\eta} \cdot (d_N^2(y_t, y) - d_N^2(y_{t+1}, y)) \\
& \leq \frac{1}{\eta} \cdot (d_M^2(x_t, x) - d_M^2(x_{t+1}, x) + d_N^2(y_t, y) - d_N^2(y_{t+1}, y))
\end{align*}
\]
where the second inequality subjects to the parameter choice \( \eta = \frac{1}{2 L} \cdot \min \{ 1/\sqrt{\tau_M}, 1/\sqrt{\tau_N} \} \) and \( \tau_M = \zeta_M / \xi_M, \quad \tau_N = \zeta_N / \xi_N \). Lastly, by Lemma C.2, the averaging scheme satisfies
\[
f(\overline{w}_T, y) - f(x, \overline{z}_T) \leq \frac{1}{T} \sum_{i=1}^{T} (f(x_t, y_t) - f(x, y_t)) \leq \frac{d_M^2(x_0, x^*) + d_N^2(y_0, y^*)}{\eta T}
\]
where \((x^*, y^*)\) is the pair of global saddle point. Hence the result follows. \qed

The following lemma characterizes the behavior for geodesic averaging scheme in Eq. (7).

**Lemma C.2.** Suppose \( f \) is geodesically convex-concave in \((x, y)\). Then for any iteration \((w_t, z_t)\), the geodesic averaging scheme \((\overline{w}_T, \overline{z}_T)\) in Eq. (7) satisfies, for any positive integer \( T \),
\[
f(\overline{w}_T, y) - f(x, \overline{z}_T) \leq \frac{1}{T} \cdot \sum_{i=1}^{T} [f(\overline{w}_T, y) - f(x, \overline{z}_T)].
\]
Proof. For the case \( T = 1 \), the result trivially holds. Now suppose the condition holds for \( T - 1 \). Since \( \overline{w}_{t+1} \) and \( \overline{z}_{t+1} \) lie, respectively, on the geodesic connecting \( \overline{w}_{t+1}, w_{t+1} \) and \( \overline{z}_{t+1}, z_{t+1} \), it yields

\[
f(\overline{w}_T, y) - f(x, \overline{z}_T) \leq \frac{T - 1}{T} \cdot \left[ f(\overline{w}_T, y) - f(x, \overline{z}_T) \right] + \frac{1}{T} \cdot \left[ f(w_{T-1}, y) - f(x, z_T) \right]
\]

\[
\leq \frac{T - 1}{T} \cdot \frac{1}{T - 1} \cdot \sum_{i=1}^{T-1} \left[ f(\overline{w}_i, y) - f(x, \overline{z}_i) \right] + \frac{1}{T} \cdot \left[ f(w_{T-1}, y) - f(x, z_T) \right]
\]

\[
= \frac{1}{T} \cdot \sum_{i=1}^{T} \left[ f(\overline{w}_i, y) - f(x, \overline{z}_i) \right]
\]

where the first inequality comes from the fact the \( f \) is geodesically convex-concave and the second is due to induction. \( \square \)