Schrodinger theory of black holes

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Abstract

The Schrodinger equation of the Schwarzschild black hole (SBH) is derived via Feynman’s path integral approach by re-obtaining the same results found by the Author and collaborators in two recent research papers. In this two-particle system approach to BH quantum physics the traditional classical singularity in the core of the SBH is replaced by a non-singular two-particle system where the two components, the “nucleus” and the “electron”, strongly interact with each other through a quantum gravitational interaction. In other words, the SBH is the gravitational analog of the hydrogen atom and this could, in principle, drive to a space-time quantization based on a quantum mechanical particle approach. By following with caution the analogy between this SBH Schrodinger equation and the traditional Schrodinger equation of the states \( l = 0 \) of the hydrogen atom, the SBH Schrodinger equation can be solved and discussed. The approach also permits us to find the quantum gravitational quantities which are the gravitational analogous of the fine structure constant and of the Rydberg constant. Remarkably, such quantities are not constants. Instead, they are dynamical quantities having well defined discrete spectra. In particular, the spectrum of the “gravitational fine structure constant” is exactly the set of non-zero natural numbers \( \mathbb{N} - \{0\} \). Therefore, one argues the interesting consequence that the SBH results in a well defined quantum gravitational system, which obeys Schrodinger’s theory: the “gravitational hydrogen atom”.

1 Introduction

It is often emphasized by theoretical physicists that the two main pillars of modern physics are Einstein’s general relativity and quantum theory. On the one
hand general relativity well-describes gravitational phenomena at large scales, starting from observations from cosmological distances to millimeter scales. On the other hand, quantum mechanics and quantum field theory well-describe phenomena at small scales from a fraction of a millimeter down to $10^{-19}$ meters. Such scales are dominated by strong and electroweak interactions. Although general relativity achieved great success (see for example the opinion of Landau who says that general relativity is, together with quantum field theory, the best scientific theory of all [2]) and withstood many experimental tests, it also displayed many shortcomings and flaws which today make theoreticians question whether it is the definitive theory of gravity [3]. As distinct from other field theories, like the electromagnetic theory, general relativity is very difficult to quantize. This fact rules out the possibility of treating gravitation like other quantum theories and precludes the unification of gravity with other interactions. At the present time, it is not possible to realize a consistent quantum gravity theory which leads to the unification of gravitation with the other forces.

From a historical point of view, Einstein believed that, in the path to unification of theories, quantum mechanics had to be subjected to a more general deterministic theory, which he called generalized theory of gravitation, but he did not obtain the final equations of such a theory (see for example the biography of Einstein which has been written by Pais [4]). At present, this point of view is partially retrieved by some theorists [5]. During the last 30 years, a strong, critical discussion about both general relativity and quantum mechanics has been undertaken by theoreticians in the Scientific Community [5]. The first motivation for this historical discussion arises from the fact that one of the most important goals of modern physics is to obtain an unified theory which could, in principle, show the fundamental interactions as different forms of the same symmetry. Considering this point of view, today one observes and tests the results of one or more breaks of symmetry. In this way, it is possible to say that we live in an asymmetrical world [5]. In the last 60 years, the dominant idea has been that a fundamental description of physical interactions arises from quantum field theory [6]. In this approach, different states of a physical system are represented by vectors in a Hilbert space defined in a space-time, while physical fields are represented by operators (i.e. linear transformations) on such a Hilbert space. The greatest problem is that this quantum mechanical framework is not consistent with gravitation, because this particular field, i.e. the metric $g_{mn}$, describes both the dynamical aspects of gravity and the space-time background [2, 5, 7]. In other words, one says that the quantization of dynamical degrees of freedom of the gravitational field is meant to give a quantum-mechanical description of space-time. This is an unequalled problem in the context of quantum field theories, because the other theories are founded on a fixed space-time background, which is treated like a classical continuum [5]. Thus, at the present time, an absolute quantum gravity theory, which implies a total unification of various interactions has not been obtained. In addition, general relativity assumes a classical description of the matter which is totally inappropriate at subatomic scales, which are the scales of the early Universe [5].

The necessity to produce a correct quantum gravity theory came into exis-
ence at the end of the 50’s of last century, when scientists tried to analyse the four interactions at a fundamental level in the sense of quantum field theory [5, 6]. The starting point was to follow the same type of analysis performed considering the other interactions: for example, the electromagnetic theory was quantized following both of the canonical and covariant approaches. In the first case, one considers magnetic and electric fields which satisfy the uncertainty principle and quantum states which are functions of invariant gauges generated by potential vectors on 3-surfaces [5]. Instead, in the covariant approach, one isolates and quantizes the two degrees of freedom of the Maxwellian field, without the 3+1 metric decomposition, and the quantum states are given by elements of the Fock space of photons [5, 8].

The two cited approaches are equivalent in the case of electromagnetic theory, but, when scientists tried to apply the same analysis to gravitation, they obtained deep differences [5]. The biggest difficulty is the fact that general relativity cannot be formulated like a quantum field theory on Minkowskian space-time, because in general relativity a geometry a priori is not present in the space-time background [2, 5, 7]. In fact, space-time is the final product of evolution, i.e. the dynamic variable [2, 5, 7]. Then, if one wants to introduce fundamental notions like causality, time and evolution of the system, one has to solve the field equations obtaining a particular space-time as a solution. Let us consider the classical example of a BH [2, 5, 7]. To understand if particular boundary constraints generate a BH, one has to solve the Einstein field equations. After this, by using the causal structure induced by the solution, one has to study the asymptotic future metric and connect it to the past initial data. It is very difficult to discuss the problem from a quantum point of view [5]. The uncertainty principle prevents particles having definite trajectories even in non-relativistic quantum mechanics; the time evolution gives only an amplitude probability rather than a precise trajectory [5]. In the same way, in quantum gravity, the evolution of the initial state cannot give a specific space-time. Then, it is not possible to introduce fundamental concepts like causality, time and matrix elements.

The two cited approaches, i.e. the covariant and canonical approaches, give different solutions to these problems. Substantially, the quantum gravity problem is represented exactly by this inconsistency [5].

What is the BHs’ role in the framework of quantum gravity? It is a general conviction, which arises from the famous, pioneering works of Bekenstein [9] and Hawking [10], that the role and the importance of BHs are fundamental. BHs are indeed considered as being theoretical laboratories for testing different models of quantum gravity. Bekenstein was the first physicist who observed that, in some respects, BHs play the same role in gravitation that atoms played in the nascent quantum mechanics [11]. This analogy implies that BH energy could have a discrete spectrum [11]. Therefore, BHs combine in some sense both the “hydrogen atom” and the “quasi-thermal” emission in quantum gravity [12].

1In next Sections this problem will be analysed in the simplest case of the historical Oppenheimer and Snyder gravitational collapse and by setting the constraints for the formation of the SBH.
As a consequence, BH quantization could be the key to a quantum theory of gravity and, for that reason BH quantization became, and currently remains, one of the most important research fields in theoretical physics of the last 50 years. Various Authors proposed and still propose various different approaches. Hence, the current literature is very rich, see for example [13–26] and references within.

2 The quantum black hole as two-particle system

In the framework of BH quantization, the Author developed a semiclassical approach to BH quantization [27–36], also with some international collaborations [37–42], which is somewhat similar to the historical semi-classical approach to the structure of a hydrogen atom introduced by Bohr in 1913 [43, 44]. Recently, the Author and Collaborators [45, 46] improved the analysis by adapting an approach to quantization of the famous collaborator of Einstein, N. Rosen [47] to the historical Oppenheimer and Snyder gravitational collapse [48]. The approach in [45, 46] has shown that the traditional classical singularity in the core of the SBH is replaced by a high symmetric nonsingular two-particle system where the two components, the “nucleus” and the “electron”, strongly interact with each other through a quantum gravitational interaction. In other words, the SBH is the gravitational analogous of the hydrogen atom, and this could, in principle, drive to a space-time quantization based on a quantum mechanical two-particle system approach. One asks, what is the physical meaning of the SBH “electron”? Following the analogy with the hydrogen atom, one can consider the de Broglie hypothesis [49] and the wave nature of the BH “electron”. This means that such a particle does not orbit the nucleus in the same way as a planet orbits the Sun, but instead exists as a standing wave. The correct analogy is that of a large and often oddly shaped “atmosphere” (the BH “electron”), distributed around a relatively tiny planet (the BH “nucleus”). The correct physical interpretation of such an “atmosphere” is nothing else than the BH horizon modes. In fact, the idea that the radius of the event horizon undergoes quantum oscillations has a longstanding history. Such horizon modes were introduced in a semi-classical framework about 50 years ago [50] in terms of BH quasi-normal modes (QNMs) which represent the BH back reaction to perturbations. Both of the absorptions of external particles and the emissions of Hawking quanta are BH perturbations and this allowed the Author and Collaborators to develop the Bohr-like approach to BH quantum physics [27–42] starting from a very interesting paper written by Hod [12] and then improved by Maggiore [51]. On one hand, the QNMs approach is a semi-classical approach similar to the approach that Bohr developed in 1913 [43, 44] concerning the structure of the hydrogen atom. On the other hand, the importance of horizon modes in a quantum gravity framework has been recently emphasized in [52], by considering them as being described by the periodic motion of their particle-like analogue, in full accordance with the de Broglie hypothesis. The Authors of [52] found an energy spectrum which scales as $\sim \sqrt{n}$, consistent with the results ob-
tained in [27–42] and in [46, 47]. The key point is that during both the processes of absorptions of external particles, including the original BH formation, and of emissions of Hawking quanta, the BH horizon is not fixed at a constant distance from the BH “nucleus” [32]. In fact, because of energy conservation, the BH contracts during the emission of a particle and expands during an absorption [32]. Such quantum contractions/expansions are not “one shot processes” [32]. They generate oscillations of the horizon instead [32].

The approach in [46, 47] seems also consistent with a similar one of Hajicek and Kiefer [19, 20], and permits to write down, explicitly, the gravitational potential and the Schrodinger equation for the SBH as (hereafter Planck units will be used, i.e. $G = c = k_B = \hbar = \frac{1}{4\pi\varepsilon_0} = 1$) [46, 47]

$$V(r) = -\frac{M_E^2}{r}, \quad (1)$$

$$-\frac{1}{2M_E} \left( \frac{\partial^2 X}{\partial r^2} + 2 \frac{\partial X}{r \partial r} \right) + V X = E X. \quad (2)$$

Here $E = -\frac{M_E^2}{r}$ is the BH total energy and $M_E$ is the BH effective mass introduced by the Author and Collaborators in [27–42]. $M_E$ is the average of the initial and final masses in a BH quantum transition. It represents the BH mass during the BH expansion (contraction), due to an absorption (emission) of a particle [27–42]. Its rigorous definition is [27–42]

$$M_E \equiv M \pm \frac{\omega}{2}. \quad (3)$$

where $\omega$ represents the mass-energy of the absorbed (emitted) particle. Thus, on one hand, the introduction of the BH effective mass in the BH dynamical equations is very intuitive. On the other hand, such an introduction is rigorously justified through Hawking periodicity argument [32, 33]. Thus, at the quantum level the SBH is interpreted as a two-particle system where the two components strongly interact with each other through the quantum gravitational interaction of Eq. (1) [46, 47]. The two-particle system BH seems to be nonsingular from the quantum point of view and is analogous to the hydrogen atom because it consists of a “nucleus” and an “electron” [46, 47].

Thus, the Schrodinger equation for the SBH of Eq. (2) is formally identical to the traditional Schrodinger equation of the $s$ states ($l = 0$) of the hydrogen atom with the traditional Coulombian potential [53]

$$V(r) = -\frac{e^2}{r}. \quad (4)$$

One observes that, on one hand, Eq. (2) seems simpler than the corresponding Schrodinger equation for the hydrogen atom because there is no angular dependence; on the other hand it seems also more complicated, because, differently from the Schrodinger equation for the hydrogen atom, where the electron’s charge is constant, in Eq. (2) the BH mass variates, and this is exactly the reason because the BH effective mass, that is indeed a dynamical quantity, has been introduced, see [27–42] for details.
3 Black hole Shrodinger equation via Feynman’s path integral approach

The historical Oppenheimer and Snyder gravitational collapse [48], is the simple case of a pressureless “star of dust”. The classical framework of this kind of gravitational collapse is well known [7, 48, 48, 54]. From the historical point of view, it was originally analysed in the famous paper of Oppenheimer and Snyder [48]. A different approach has been instead developed by Beckerhoff and Misner [54]. More recently, a non-linear electrodynamics Lagrangian has been added in this collapse’s framework by the Author and a collaborator in [55]. This different approach permitted a way to remove the BH singularity at the classical level [55].

For the interior of the collapsing star, one uses the well-known Friedmann-Lemaitre-Robertson-Walker (FLRW) line-element with comoving hyper-spherical coordinates $\chi, \theta, \varphi$ [7]. Thus, one writes down (hereafter Planck units will be used, i.e. $G = c = k_B = \hbar = \frac{1}{4\pi\alpha} = 1$) [7]

$$ds^2 = d\tau^2 + a(\tau)(-d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)),$$

where the origin of coordinates is set at the centre of the star, and $a(\tau)$ is the scale factor given by the familiar cycloidal relation [7]

$$a = \frac{1}{2}a_m (1 + \cos\eta),$$

$$\tau = \frac{1}{2}a_m (\eta + \sin\eta)$$

The density is given by [7]

$$\rho = \left(\frac{3a_m}{8\pi}\right)a^{-3} = \left(\frac{3}{8\pi a_m^2}\right)\left[\frac{1}{2} (1 + \cos\eta)\right]^{-3}.$$  

Setting $\sin^2\chi$ one chooses the case of positive curvature, which corresponds to a gas sphere whose dynamics begins at rest with a finite radius, and, in turn, it is the only one of interest [7]. Thus, the choice $k = 1$ is made for dynamical reasons (the initial rate of change of density is null, that means “momentum of maximum expansion” [7]), but the dynamics also depends on the field equations.

In order to discuss the simplest model of a “star of dust”, that is, the case of zero pressure, one sets the stress-energy tensor as [7]

$$T = \rho u \otimes u,$$

where $\rho$ is the density of the collapsing star and $u$ the four-vector velocity of the matter. On the other hand, the external geometry is given by the Schwarzschild line-element [7]

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - r^2 (\sin^2\theta d\varphi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{2M}{r}}.$$  

where $M$ is the total mass of the collapsing star. As there are no pressure gradients, which can deflect the motion of the particles, the particles on the surface of any ball of dust move along radial geodesics in the exterior Schwarzschild spacetime [7]. Considering a ball which begins at rest with finite radius (in terms of the Schwarzschild radial coordinate) $r = r_i$ at the (Schwarzschild) time $t = 0$, the geodesics motion of its surface is given by the following equations [7]:

$$r = \frac{1}{2} r_i \left(1 + \cos \eta \right), \quad (10)$$

$$t = 2 M \ln \left[ \frac{\sqrt{r_i^2 - 1 + \tan \left(\frac{\eta}{2}\right)}}{\sqrt{r_i^2 - 1 - \tan \left(\frac{\eta}{2}\right)}} \right] + 2 M \sqrt{\frac{r_i^2}{2 M} - 1} \left[ \eta + \left(\frac{r_i}{4 M}\right)(\eta + \sin \eta) \right]. \quad (11)$$

The proper time measured by a clock put on the surface of the collapsing star is [7]

$$\tau = \sqrt{\frac{r_i^3}{8 M}} (\eta + \sin \eta). \quad (12)$$

The collapse begins for $r = r_i$, $\eta = \tau = t = 0$, and terminates at the singularity $r = 0$, $\eta = \pi$ after a duration of proper time measured by the falling particles [7]

$$\Delta \tau = \pi \sqrt{\frac{r_i^2}{8 M}}. \quad (13)$$

which coincidentally corresponds, as it is well known, to the interval of Newtonian time for free-fall collapse in Newtonian theory. Differently from the cosmological case, where the solution is homogeneous and isotropic everywhere, here the internal homogeneity and isotropy of the FLRW line-element are broken at the star’s surface, that is, a some radius $\chi = \chi_0$ [7]. At that surface, which is a 3-dimensional world tube enclosing the star’s fluid [7], the interior FLRW geometry must match smoothly the exterior Schwarzschild geometry [7]. One considers a range of $\chi$ given by $0 \leq \chi \leq \chi_0$, with $\chi_0 < \frac{\pi}{2}$ during the collapse [7]. For the pressureless case the match is possible [7]. The external Schwarzschild solution predicts indeed a cycloidal relation for the star’s circumference [7]

$$C = 2 \pi r = 2 \pi \left[ \frac{1}{2} r_i \left(1 + \cos \eta \right) \right], \quad (14)$$

$$\tau = \sqrt{\frac{r_i^3}{8 M}} (\eta + \sin \eta).$$

The interior FLRW predicts a similar cycloidal relation [7]

$$C = 2 \pi r = 2 \pi a \sin \chi_0 = \pi \sin \chi_0 a_m \left(1 + \cos \eta \right), \quad (15)$$

$$\tau = \frac{1}{2} a_m (\eta + \sin \eta).$$

Therefore, the two predictions agree perfectly for all time if and only if [7]
\[ r_i = a_0 \sin \chi_0, \quad (16) \]

\[ M = \frac{1}{2} a_0 \sin^3 \chi_0, \]

where \( r_i \) and \( a_0 \) are the values of the Schwarzschild radial coordinate in Eq. (9) and of the scale factor in Eq. (5) at the beginning of the collapse, respectively. Thus, Eqs. (16) represent the requested match, while the Schwarzschild radial coordinate, in the case of the matching between the internal and external geometries, is \[ r = a \sin \chi_0. \quad (17) \]

The attentive reader notes that the initial conditions on the matching are the simplest possible that could be relaxed, still having a continuous matching without extra surface terms. In fact, taking the interior solution to be homogeneous requires very fine tuned initial conditions for the collapse and the dynamics of the edge. Thus, on one hand, further analyses for a better characterization of the initial conditions on the matching between the internal and external geometries could be the object of future works. On the other hand, although the analysis of this paper is not the most general possible, one recollects that, as it has been previously stressed in the Introduction of this Book Chapter, the problem of the BH quantization is one of the most important of modern theoretical physics. Thus, in order to solve such a fundamental problem, one must start from the simplest case rather than from more complicated ones. This is in complete analogy with the history of general relativity. The first solution of Einstein field equations was indeed the Schwartzschild solution, but it was not a general, rotating solution which included cosmological term or other sources of dark energy, as well the corresponding gravitational collapse developed by Oppenheimer and Snyder [48] did not include a class of non-homogeneous models. Thus, in this novel approach to the BH quantization, here one starts from the simplest conditions rather than from more complicated ones. The initial conditions on the matching that are applied here are exactly the ones proposed by Oppenheimer and Snyder in their historical paper on the gravitational collapse. It is well known that the final result of the gravitational collapse studied by Oppenheimer and Snyder is the Schwarzschild BH [7]. Hence, one also sees what happens when the star is completely collapsed, i.e. when the star is a BH. By inserting \( r_i = 2M = r_g \), where \( r_g \) is the gravitational radius (the Schwarzschild radius), in Eqs. (16), one gets \( \sin^2 \chi_0 = 1 \). Thus, as the range \( \chi > \frac{\pi}{2} \) must be discarded [7], one concludes that it is \( \chi_0 = \frac{\pi}{2} \) for a BH.

Now, the Schrödinger equation (2) will be obtained via Feynman’s path integral approach. One starts by rewriting the FLRW line-element (5) in spherical coordinates and comoving time as [7] [45] [46]

\[ ds^2 = d\tau^2 - a^2(\tau) \left( \frac{dr^2}{1 - r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right). \quad (18) \]
The Einstein field equation \[45, 47\]

\[ G_{\mu\nu} = -8\pi T_{\mu\nu} \]  

(19)
gives the relations (one assumes zero pressure) \[45, 47\]

\[ \dot{a}^2 = \frac{8\pi a^2 \rho}{3} - 1 \]  

(20)
\[ \ddot{a} = -\frac{4\pi a \rho}{3} \]

with \( \dot{a} = \frac{da}{d\tau} \). For consistency, one gets \[45, 47\]

\[ \frac{d\rho}{da} = -\frac{3\rho}{a} , \]  

(21)
which, when integrated gives \[45, 47\]

\[ \rho = \frac{C}{a^3} . \]  

(22)
In the collapse case, \( C \) is determined by the initial conditions as \[7\]

\[ C = \frac{3a_0}{8\pi} . \]  

(23)
Now, one rewrites Eq. \[22\] as

\[ \rho = \frac{3a_0}{8\pi a^3} . \]  

(24)
By multiplying the first of \[20\] for \( M/2 \) one gets \[45, 47\]

\[ \frac{Ma^2}{2} - \frac{4}{3} \pi Ma^2 \rho = -\frac{M}{2} , \]  

(25)
One considers the standard Einstein - Hilbert Lagrangian \[2\]

\[ L_{EH} = \sqrt{-gR} \frac{1}{16\pi} . \]  

(26)
By using the FLRW line-element \[18\] one gets

\[ L_{FLRW} = \dot{a}^2 + \frac{8\pi a^2 \rho}{3} . \]  

(27)
In order to have consistence with Feynman’s PhD thesis \[56\], one rescales the Lagrangian \[27\] as

\[ L = \frac{Ma^2}{2} + \frac{4}{3} M \pi a^2 \rho . \]  

(28)
In this way, the Lagrangian is rewritten in Feynman’s form \[56\]

\[ L = \frac{Ma^2}{2} - V(a) , \]  

(29)
where
\[ V(a) \equiv -\frac{4}{3} M \pi a^2 \rho. \] (30)

Inserting Eq. (24) into Eq. (30) one obtains
\[ V(a) = -\frac{M a_0}{2a}. \] (31)

The energy function associated to the Lagrangian is
\[ E = \frac{\partial L}{\partial \dot{a}} \dot{a} - L. \] (32)

Then, by inserting Eq. (28) in Eq. (32) and by using the first of Eqs. (20) one gets
\[ E = -\frac{M^2}{2}. \] (33)

By taking an infinitesimal time interval \( \tau \) to \( \tau + \delta \tau \), the wave function \( \Psi \) for the system at the time \( \tau + \delta \tau \) in terms of its value at time \( \tau \) is determined by
\[
\Psi (Q, \tau + \delta \tau) = \int \exp \left[ i \delta \tau L \left( \frac{Q - q}{\delta \tau}, Q \right) \right] \Psi (q, \tau) \frac{\sqrt{g(q)} dq}{A(\delta \tau)},
\] (34)

where, the transformation function \( (q', q' = Q | q, q' = q) \) which connects the representations referring to the two different times \( \tau \) and \( \tau + \delta \tau \), corresponds in the classical theory to \( \exp (i \delta \tau L) \). \( \sqrt{g(q)} dq \) is the volume element in q-space and \( A(\delta \tau) \) is a suitable normalization constant see [56] for details. In accordance with Eq. (34), then the wave function for the system must satisfy (where one writes \( \varepsilon \) for \( \delta \tau \)) for infinitesimal \( \varepsilon \), the equation [56]
\[
\Psi (a, \tau + \varepsilon) = \int \exp \left[ i M^2 \frac{(a - y)^2}{2 \varepsilon} - \varepsilon V(a) \right] \Psi (y, \tau) \frac{dy}{A},
\] (35)

Now, one replaces \( y = \eta + a \) in the integral. Thus, one writes [56]
\[
\Psi (a, \tau + \varepsilon) = \int \exp \left[ i \frac{M \eta^2}{2 \varepsilon} - \varepsilon V(a) \right] \Psi (\eta + a, \tau) \frac{d\eta}{A},
\] (36)

Only values of \( \eta \) close to zero will contribute to the integral [56]. In fact, for small \( \varepsilon \), other values of \( \eta \) make the exponential oscillate so rapidly that there will be little contribution to the integral [56]. Then, one expands \( \Psi (\eta + a, \tau) \) in a Taylor series around \( \eta = 0 \) [56]. After rearranging, one gets the integral [56]
\[
\Psi (a, \tau + \varepsilon) = \frac{\exp (i \varepsilon V(a))}{A} \int \exp \left[ \frac{i M^2 \eta^2}{2 \varepsilon} \right] \left[ \Psi (a, \tau) + \eta \frac{\partial \Psi (a, \tau)}{\partial a} + \frac{\eta^2}{2} \frac{\partial^2 \Psi (a, \tau)}{\partial \tau^2} + \ldots \right] d\eta.
\] (37)
Now, from Pierces integral tables in Feynman’s PhD thesis \[56\], one gets
\[
\int_{-\infty}^{\infty} \exp \left( \frac{i M \eta^2}{2 \varepsilon} \right) d\eta = \sqrt{\frac{2\pi i \varepsilon}{M}},
\]
(38)
and, by differentiating both sides with respect to \(M\), one finds \[56\]
\[
\int_{-\infty}^{\infty} \exp \left( \frac{i M \eta^2}{2 \varepsilon} \right) \eta^2 d\eta = \sqrt{\frac{2\pi i \varepsilon}{M}}.
\]
(39)
The integral with \(\eta\) in the integrand is the integral of an odd function. Then, it is zero \[56\]. Hence,
\[
\Psi (a, \tau + \varepsilon) = \frac{\sqrt{2\pi i \varepsilon}}{A} \int_{-\infty}^{\infty} \exp -[i\varepsilon V(a)] \left[ \Psi (a, \tau) + \frac{\varepsilon i}{2M} \frac{\partial^2 \Psi (a, \tau)}{\partial a^2} + \ldots \right].
\]
(40)
The left hand side of Eq. (40) approaches \(\Psi (a, \tau)\) for very small \(\varepsilon\). Thus, if one wants the equality to hold one must choose \[56\]
\[
A (\varepsilon) = \sqrt{\frac{2\pi i \varepsilon}{M}}.
\]
(41)
If one expands both sides Eq. (40) in powers of \(\varepsilon\) up to the first order, one finds \[56\]
\[
\Psi (a, \tau) + \varepsilon \frac{\partial \Psi (a, \tau)}{\partial t} = \Psi (a, \tau) - i\varepsilon V(a)\Psi (a, \tau) \frac{\varepsilon i}{2M} \frac{\partial^2 \Psi (a, \tau)}{\partial a^2},
\]
(42)
and therefore \[56\],
\[
i \frac{\partial \Psi}{\partial \tau} = -\frac{1}{2M} \frac{\partial^2 \Psi}{\partial a^2} + V\Psi,
\]
(43)
which is exactly Eq. (2.30) in \[45\]. On the other hand, in \[45\] Eq. (43) has been obtained via Rosen’s quantization approach \[47\]. In this paper, it has been obtained via Feynman’s path integrals approach. One can state that, perhaps, Rosen’s approach is more intuitive, while Feynman’s approach is more rigorous from the mathematical point of view. But what is really important from the physical point of view is that both of the approaches generate the same results for the quantization of the Oppenheimer and Snyder gravitational collapse. Now, one can closely follow the analysis in \[45\] in order to get the SBH Schrodinger equation from Eq. (43). Then, one considers the case of a completely collapsed star, i.e. a BH, which means \(\chi_0 = \tilde{r}, r = a\) and \(r_i = a_0 = 2M = r_g\), in Eqs. (16), see the discussion below Eq. (17). Then, setting
\[
\Psi = rX,
\]
(44)
Eq. (43) becomes
\[
-\frac{1}{2M} \left( \frac{\partial^2 X}{\partial r^2} + \frac{2}{\tilde{r}} \frac{\partial X}{\partial \tilde{r}} \right) + VX = EX,
\]
(45)
with the associated potential

\[ V(r) = -\frac{M^2}{r}. \]  

(46)

Equations (46) and (45) should be the exact gravitational potential energy, and Schrodinger equation, for the SBH interpreted as “gravitational hydrogen atom”, respectively. Actually, a further final correction is needed. To clarify this point, let us compare Eq. (46) with the analogous potential energy of an hydrogen atom which is Eq. (4). Equations (46) and (4) are formally identical, but there is an important difference. In the case of Eq. (4) the electron’s charge is constant for all the energy levels of the hydrogen atom. Instead, in the case of Eq. (46), based on the emissions of Hawking quanta or on the absorptions of external particles, the BH mass changes during the jumps from an energy level to another. In fact, such a BH mass decreases for emissions and increases for absorptions. Thus, one must also consider this dynamical behavior of quantum BHs. A good way to take into account this dynamical behavior is by using the BH effective mass of Eq. (3). One has to choose the positive sign in that equation because from the quantum point of view the BH formation is represented as absorptions of external particles. Thus, Eq. (3) becomes

\[ M_E \equiv M + \omega, \]  

(47)

and one associates this effective mass with the effective horizon [27–42]

\[ r_E \equiv 2M_E. \]  

(48)

Now the two effective quantities represent the BH mass and the BH horizon during the BH expansion, i.e. during the absorption of the particle. Thus, in order to take the BH dynamical behavior into account, one must replace the BH mass \( M \) with the BH effective mass \( M_E \) in Eqs. (46) and (45). This point is rigorously justified via Hawking periodicity argument [57], see [32, 33] for details. In that way, one obtains exactly Eqs. (1) and (2), while Eq. (33) becomes

\[ E = -\frac{M_E}{2}, \]  

(49)

in complete consistency with the results in [45, 46].

4 Rigorous solution of the black hole Shrodinger equation: the “gravitational fine structure constant”

One also recalls that \( e^2 = \alpha \) is the fine structure constant, which, in standard units, combines the constants \( \frac{e^2}{4\pi\varepsilon_0} \) from electromagnetism, \( \hbar \) from quantum mechanics, and the speed of light \( c \) from relativity, into the dimensionless irrational
number $\alpha \simeq 1/137.036$, which is one of the most important numbers in Nature. Thus, from the present approach one argues that the gravitational analogous of the fine structure constant is not a constant. It is a dynamical quantity instead. In fact, if one labels the “gravitational fine structure constant” as $\alpha_G$, by confronting Eqs. (1) and (4) one gets

$$\alpha_G = \left( \frac{M_E}{m_p} \right)^2,$$

in standard units, where $m_p$ is the Planck mass. It will be indeed shown that $\alpha_G$ has a spectrum of values which coincides with the set of natural numbers $\mathbb{N}$. In analogous way, the Rydberg constant, $R_\infty = \frac{1}{2}m_e\alpha^2$, is defined in terms of the electron mass $m_e$ and on the fine structure constant. In the current gravitational approach the effective mass and the “charge” are the same. Hence, the quantum gravitational analogous of the Rydberg constant is

$$(R_\infty)_G = \frac{M_E^5}{2m_p^5 l_p^5}.$$

By introducing the variable $y$

$$y(r) \equiv rX$$

Eq. (2) becomes

$$-\left( \frac{1}{2M_E} \frac{d^2}{dr^2} + \frac{M_E^2}{r} + E \right) y = 0.$$

Setting $y' \equiv \frac{d}{dr}$ and using $E = -\frac{M_E}{2}$, Eq. (53) can be rewritten as

$$y'' + M_E^2 \left( \frac{2M_E}{r} - 1 \right) y = 0.$$

As it is $E < 0$, the asymptotic form of the solution, which is regular at the origin, is a linear combination of exponentials $\exp(M_E r), \exp(-M_E r)$. If one wants this solution to be an acceptable eigensolution, the coefficient in front of $\exp(M_E r)$ must vanish. This happens only for certains discrete values of $E$. Such values will be the energies of the discrete spectrum of the SBH and the corresponding wave function represents one of the possible SBH bound states.

If one makes the change of variable

$$x = 2M_E r$$

Eq. (55) results equivalent to

$$\left[ \frac{d^2}{dx^2} + \frac{M_E^2}{x} - \frac{1}{4} \right] y = 0,$$
and } y \text{ is the solution which goes as } x \text{ at the origin. For } x \text{ very large, it increases exponentially, except for certain particular values of the SBH effective mass where it behaves as } \exp \left( -\frac{x}{2} \right). \text{ One wants to determine such special values and their corresponding eigenfunctions. One starts to perform the change of function}

\[ y = x \exp \left( -\frac{x}{2} \right) z(x), \tag{57} \]

which changes Eq. (56) to

\[ \left[ \frac{d^2}{dx^2} + (2 - x) \frac{d}{dx} - (1 - M_E^2) \right] z = 0. \tag{58} \]

This last equation is a Laplace-like equation. Within a constant, one finds only a solution which is positive at the origin. All the other solutions have a singularity in } x^{-1}. \text{ One can show that this solution is the confluent hypergeometric series}

\[ Z = \sum_{i=1}^{\infty} \frac{\Gamma (1 + i - M_E^2)}{\Gamma (1 - M_E^2)} \frac{1}{(1 + i)!} \frac{x^i}{i!}, \tag{59} \]

In fact, one expands the solution of Eq. (58) in Maclaurin series at the origin as

\[ z = 1 + \alpha_1 x + \alpha_2 x^2 + ... \alpha_i x^i + ... \tag{60} \]

By inserting Eq. (60) in Eq. (58) one writes the LHS in terms of a power series of } x. \text{ All the coefficients of this expansion must be null. Thus,}

\[ 2\alpha_1 = 1 - M_E^2 \]
\[ 2 \ast 3\alpha_2 = (2 - M_E^2) \alpha_1 \]
\[ \ldots \]
\[ i (1 + i) \alpha_i = (i - M_E^2) \alpha_{i-1}. \tag{61} \]

Then, one gets

\[ \alpha_i = \frac{\left( i - M_E^2 \right) \left( i - 1 - M_E^2 \right)}{\left( 1 + i \right) \left( 1 + i - 1 \right)} \ldots \frac{1 - M_E^2}{1 + 1} \frac{1}{i!}, \tag{62} \]

which is exactly the coefficient of } x^i \text{ in the expansion (60).}

From a mathematical point of view the series (59) is infinite and behaves as

\[ \frac{\exp x}{x^{(1+M_E^2)}} \]

for large } x. \text{ Consequently, } y \text{ behaves in the asymptotic region as}

\[ \frac{\exp x}{x^{M_E^2}} \]

for large } x. \text{ Consequently, } y \text{ behaves in the asymptotic region as}
and cannot be, in general, an eigensolution. On the other hand, for particular values of $M_E^2$ all the coefficients will vanish from a certain order on. In that case, the series (59) reduces to a polynomial. The requested condition is

$$1 - M_E^2 \leq 0, \text{ with } |1 - M_E^2| \in \mathbb{N}, \quad (63)$$

which implies the quantization condition

$$M_E^2 = n = n' + 1, \text{ with } n' = 0, 1, 2, 3, \ldots + \infty. \quad (64)$$

The hypergeometric series becomes a polynomial of degree $n'$ and $y$ behaves in the asymptotic region as

$$y \sim x^{n'} \exp \left( \frac{1}{2} \right).$$

Then, the regular solution of the SBH Schrödinger equation becomes an acceptable eigensolution. The quantization condition (64) permits us to obtain the spectrum of the effective mass as

$$(M_E)_n = \sqrt{n}, \text{ with } n = 1, 2, 3, \ldots + \infty. \quad (65)$$

The quantization condition (64) gives also the spectrum of the “gravitational fine structure constant” that one writes as

$$( \alpha_G)_n = \left( \frac{(M_E)_n}{m_p} \right)^2 = n, \text{ with } n = 1, 2, 3, \ldots + \infty. \quad (66)$$

Thus, the gravitational analogous of the fine structure constant is not a constant. It is a dynamical quantity instead, which has the spectrum of values (66) which coincides with the set of non-zero natural numbers $\mathbb{N} - \{0\}$. In the same way, one finds the spectrum of the “gravitational Rydberg constant” as

$$[(R_\infty)_G]_n = \frac{(M_E)_n^5}{2m_\text{pl}^5} = \frac{n^5}{2}, \text{ with } n = 1, 2, 3, \ldots + \infty. \quad (67)$$

Now, from the quantum point of view, one wants to obtain the mass eigenvalues as being absorptions starting from the BH formation, that is from the BH having null mass, where with “the BH having null mass” one means the situation of the gravitational collapse before the formation of the first event horizon. This implies that one must replace $M \to 0$ and $\omega \to M$ in Eqs. (47) and (48). Thus, one gets

$$M_E = \frac{M}{2}, \quad r_E = 2M_E = M. \quad (68)$$

By combining Eqs. (65) and (68) one immediately gets the SBH mass spectrum as

$$M_n = 2\sqrt{n}, \quad (69)$$

which is the same result of [45, 46]. Eq. (69) is consistent with the SBH mass spectrum found by Bekenstein in 1974 [9]. Bekenstein indeed obtained
\[ M_n = \sqrt{n^2} \] by using the Bohr-Sommerfeld quantization condition because he argued that the SBH behaves as an adiabatic invariant. It is also consistent with other BH energy spectra in the literature, see for example [12–16, 26]. Eq. \((49)\) permits also to find the negative energies of the SBH bound states as

\[ E_n = -\sqrt{n^4} \] (70)

The (unnormalized) wave function corresponding to each energy level is

\[ X(r) = \frac{1}{r} \left[ \frac{2M_Er}{2\pi} \left( \sum_{i=0}^{n-1} (-)^i \frac{(n-1)!}{(n-1-i)!(i+1)!} (2M_Er)^i \right) \right]. \] (71)

One observes that the number of nodes of the wave function is exactly \(n-1\) and that the energy spectrum \((70)\) contains a denumerably infinite number of levels because \(n\) can take all the infinite values \(n \in \mathbb{N} - \{0\}\). One can calculate the energy jump between two neighboring level as

\[ \Delta E = E_{n+1} - E_n = \sqrt{n^4} - \sqrt{\frac{(n+1)^4}{4}} = -\frac{1}{4} \left( \frac{\sqrt{n^4}}{4} + \frac{n+1}{4} \right). \] (72)

For large \(n\) one obtains

\[ \Delta E \simeq -\frac{1}{4\sqrt{n}} \]

when \(n \to +\infty\) one gets \(\Delta E \to 0\). Thus, one finds that the energy levels become more and more closely spaced and their difference tends to \(\Delta E = 0\) at the limit, at which point the continuous spectrum of the SBH begins.

One also observes that a two particle Hamiltonian

\[ H(\vec{p}, r) = \frac{\vec{p}^2}{2M_E} - \frac{M_E^2}{r}, \] (73)

which governs the SBH quantum mechanics, must exist in correspondence of Eqs. \(1\) and \(2\). Therefore, the square of the wave function \((71)\) must be interpreted as the probability density of a single particle in a finite volume. Thus, the integral over the entire volume must be normalized to unity as

\[ \int dx^3 |X|^2 = 1. \] (74)

For stable particles, this normalization must remain the same at all times of the SBH evolution. This issue has deep implications for the BH information paradox \[58\] because it guarantees preservation of quantum information. As the wave function \((71)\) obeys the SBH Schrödinger equation \((2)\), this is assured if and only if the Hamiltonian operator \((73)\) is Hermitian \[59\]. In other words, the Hamiltonian operator \((73)\) must satisfy for arbitrary wave functions \(X_1\) and \(X_2\) the equality \[59\]

\[ \int dx^3 [HX_2]^* X_1 = \int dx^3 X_2^* HX_1. \] (75)
One notes that both $\vec{p}$ and $r$ are Hermitian operators. Thus, the Hamiltonian \((73)\) will automatically be a Hermitian operator if it is a sum of a kinetic and a potential energy \([59]\)

\[
    H = T + V.
\]

This is always the case for non-relativistic particles in Cartesian coordinates \([59]\) and works also for SBHs.

Thus, it has been shown that the SBH is a well defined quantum system, which obeys Schrödinger’s theory. In a certain sense it is a “gravitational hydrogen atom”. One also observes that studying the SBH in terms of a well defined quantum mechanical system, having an ordered, discrete quantum spectrum, looks consistent with the unitarity of the underlying quantum gravity theory and with the idea that information should come out in BH evaporation.

5 Conclusion remarks

The Author and collaborators recently found the Schrödinger equation of the SBH \([45,46]\). In that approach, the traditional classical singularity in the core of the SBH has been replaced by a nonsingular two-particle system where the two components, the “nucleus” and the “electron”, strongly interact with each other through a quantum gravitational interaction. Thus, the SBH is nothing else than the gravitational analog of the hydrogen atom. In this paper the Schrödinger equation of the SBH has been rigorously obtained via Feynman’s path integral approach by re-obtaining the same result in \([45,46]\).

After this, by following with caution the analogy between this SBH Schrödinger equation and the traditional Schrödinger equation of the $s$ states \((l = 0)\) of the hydrogen atom, the SBH Schrödinger equation has been solved and discussed. The approach also permitted us to find the quantum gravitational quantities which are the gravitational analogous of the fine structure constant and of the Rydberg constant. Remarkably, it has been shown that such quantities are not constants. Instead, they are dynamical quantities having well defined discrete spectra. In particular, the spectrum of the “gravitational fine structure constant” is exactly the set of non-zero natural numbers $\mathbb{N} - \{0\}$.

Therefore, the interesting consequence of the results in this paper is that the SBH results in a well defined quantum gravitational system, which obeys Schrödinger’s theory: the “gravitational hydrogen atom”. This should lead to a space-time quantization based on a quantum mechanical particle approach.

References

[1] A. Einstein, Preuss. Akad. Wiss. Berlin, Sitzber., 778 (1915).

[2] L. Landau and E. Lifshits, Classical Theory of Fields (3rd ed.), London: Pergamon (1971).

[3] C. Corda, Int. Journ. Mod. Phys. D, 18, 2275 (2009).
[4] A. Pais, “Subtle Is the Lord: The Science and the Life of Albert Einstein”, Oxford University Press (2005).

[5] C. Corda, New Adv. Phys. 7, 67 (2013).

[6] L. H. Ryder, “Quantum Field Theory”, 2nd Edition, Cambridge University Press (1996).

[7] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (W. H. Feeman and Co., 1973).

[8] K. O. Friedrichs, Mathematical aspects of the Quantum Theory of Fields. Interscience Publishers (1953).

[9] J. Bekenstein, Lett. Nuovo Cimento, 11, 467 (1974).

[10] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).

[11] J. D. Bekenstein, in Proceedings of the Eight Marcel Grossmann Meeting, T. Piran and R. Ruffini, eds., pp. 92-111 (World Scientific Singapore 1999).

[12] S. Hod, Phys. Rev. Lett. 81, 4293 (1998).

[13] V. Mukhanov, JETP Letters 44, 63 (1986).

[14] J. D. Bekenstein and V. Mukhanov, Phys. Lett. B 360, 7 (1995).

[15] S. Das, P. Ramadevi and U. A. Yajnik, Mod. Phys. Lett. A 17, 993 (2002).

[16] A. Barvinski, S. Das, G. Kunstatter, Phys. Lett. B 517, 415 (2001).

[17] L. Modesto, J. W. Moffat, P. Nicolini, Phys. Lett. B 695, 397 (2011).

[18] C. Kiefer and T. Schmitz, Phys. Rev. D 99, 126010 (2019).

[19] P. Hajicek and C. Kiefer, Nucl. Phys. B 603, 531 (2001).

[20] P. Hajicek and C. Kiefer, Int. J. Mod. Phys. D 10, 775 (2001).

[21] A. Saini and D. Stojkovic, Phys. Rev. D 89, 044003 (2014).

[22] E. Greenwood and D. Stojkovic, JHEP 0806, 042 (2008).

[23] J. E. Wang, E. Greenwood and D. Stojkovic, Phys. Rev. D 80, 124027 (2009).

[24] A. Davidson, Phys. Rev. D 100, 081502 (2019).

[25] A. Davidson, Phys. Lett. B 780, 29 (2018).

[26] M. Maggiore, Nucl. Phys. B 429, 205 (1994).

[27] C. Corda, J. High Energ. Phys. 2011, 101 (2011).
[28] C. Corda, Int. Jour. Mod. Phys. D 21, 1242023 (2012).
[29] C. Corda, Eur. Phys. J. C 73, 2665 (2013).
[30] C. Corda, EJTP 11, 27 (2014).
[31] C. Corda, Ann. Phys. 353, 71 (2015).
[32] C. Corda, Class. Quantum Grav. 32, 195007 (2015).
[33] C. Corda, Adv. High En. Phys. 2015, 867601 (2015).
[34] C. Corda, AIP Conf. Proc. 1648, 020004 (2015).
[35] C. Corda, Int. Journ. Theor. Phys. 54, 3841 (2015).
[36] C. Corda, Mod. Phys. Lett. A, 33, 1850069 (2018).
[37] C. Corda, S. H. Hendi, R. Katebi, N. O. Schmidt, J. High Energ. Phys. 2013, 8 (2013).
[38] C. Corda, S. H. Hendi, R. Katebi, N. O. Schmidt, Adv. High En. Phys. 2014, 527874 (2014).
[39] C. Corda, S. H. Hendi, R. Katebi, N. O. Schmidt, Adv. High En. Phys. 2014, 530547 (2014).
[40] S. Haldar, C. Corda and S. Chakraborty, Adv. High En. Phys. 2018, 9851598 (2018).
[41] Y. Heydarzade, H. Hadi, C. Corda, F. Darabi, Phys. Lett. B 776, 457 (2018).
[42] L. Crowell and C. Corda, Entropy 22(3), 301 (2020).
[43] N. Bohr, Philos. Mag. 26, 1 (1913).
[44] N. Bohr, Philos. Mag. 26, 476 (1913).
[45] C. Corda and F. Feleppa, Adv. Theor. Math. Phys. (2022), pre-print in arXiv:1912.06478.
[46] C. Corda, F. Feleppa and F. Tamburini, EPL 132, 30001 (2020).
[47] N. Rosen, Int. Journ. Theor. Phys. 32, 8, (1993).
[48] J. R. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939).
[49] L. de Broglie, Ann. de Physique (10)3, 22 (1925).
[50] W. H. Press, Astrophys. J. 170, L105 (1971).
[51] M. Maggiore, Phys. Rev. Lett. 100, 141301 (2008).
[52] E. Spallucci, A. Smailagic, Phys. Lett B 816, 136180 (2021).

[53] A. Messiah, Quantum Mechanics, Vol. 1, North-Holland, Amsterdam (1961).

[54] D. L. Beckerhoff and C. W. Misner, D. L. Beckerhoff’s A. B. Senior Thesis, Princeton Univeristy (1962).

[55] C. Corda and H. J. Mosquera Cuesta, Mod. Phys. Lett. A 25, 2423 (2010).

[56] R. Feynman, “Feynman’s Thesis – A New Approach to Quantum Theory”, Edited by Laurie M Brown, World Scientific (2005).

[57] S. W. Hawking, “The Path Integral Approach to Quantum Gravity”, in General Relativity: An Einstein Centenary Survey, eds. S. W. Hawking and W. Israel, (Cambridge University Press, 1979).

[58] S. W. Hawking, Phys. Rev. D 14, 2460 (1976).

[59] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, 5th edition, World Scientific, Singapore (2009).