Dependency Weighted Aggregation on Factorized Databases

Florent Capelli\textsuperscript{1}, Nicolas Crosetti\textsuperscript{2}, Joachim Niehren\textsuperscript{2}, and Jan Ramon\textsuperscript{2}

\textsuperscript{1}Universit\'e de Lille
\textsuperscript{2}Inria, Lille

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Abstract

We study a new class of aggregation problems, called dependency weighted aggregation. The underlying idea is to aggregate the answer tuples of a query while accounting for dependencies between them, where two tuples are considered dependent when they have the same value on some attribute. The main problem we are interested in is to compute the dependency weighted count of a conjunctive query. This aggregate can be seen as a form of weighted counting, where the weights of the answer tuples are computed by solving a linear program. This linear program enforces that dependent tuples are not over represented in the final weighted count. The dependency weighted count can be used to compute the s-measure, a measure that is used in data mining to estimate the frequency of a pattern in a graph database.

Computing the dependency weighted count of a conjunctive query is $\text{NP}$-hard in general. In this paper, we show that this problem is actually tractable for a large class of structurally restricted conjunctive queries such as acyclic or bounded hypertree width queries. Our algorithm works on a factorized representation of the answer set, in order to avoid enumerating it exhaustively. Our technique produces a succinct representation of the weighting of the answers. It can be used to solve other dependency weighted aggregation tasks, such as computing the (dependency) weighted average of the value of an attribute in the answers set.

1 Introduction

A central task in data mining and machine learning is to estimate the frequency of the matchings of a graph pattern in a data graph \cite{YH02,AIS93,BN08}. A graph pattern can be expressed by a conjunctive query on a graph database. The frequency of a graph pattern can be measured by the number of matchings in the data graph. From a database point of view, this amounts to counting the number of answers of a conjunctive query in a graph database. There exists a rich literature on the complexity of this problem: Without restrictions on the class of conjunctive queries, the counting problem is known to be $\#\text{P}$-complete \cite{BCC04}. In other words, the problem is as hard as counting the number of satisfying assignments of an instance of SAT, the generic $\text{NP}$-complete problem. It is thus very unlikely that the problem can be solved in polynomial time in the size of the database. Various tractable fragments were obtained with structural restrictions on the class of conjunctive queries \cite{PS13,GS14}.

Data mining and machine learning algorithms however often focus on patterns that fall in this class of tractable conjunctive queries, with patterns having a tree-like structure (acyclic or of small treewidth). This class of patterns has been observed to already contain a lot of the information one would like to mine \cite{NK05}. While it is possible that patterns with high treewidth may contain additional interesting information, due to the intractability of mining them they are almost never considered. Graph patterns can be matched to graph databases under homomorphism or subgraph isomorphism, i.e., conjunctive queries can be solved while allowing different variables to get the same value or while imposing (through an "all-diff" constraint) that all variables get different values. The first option is more tractable (e.g., for bounded treewidth graphs, in polynomial time), while the latter has a higher complexity but may be in some cases more appealing \cite{KR13}. While the results in the current paper can be applied to both types of matching, for simplicity of explanation we will consider mainly the first option.
Using the number of matchings of a pattern as a frequency measure, is questionable from the viewpoint of graph theory and data mining. The problematic fact is that different matchings of a given pattern may overlap, and as such they share some kind of dependencies that is relevant from a statistical point of view. More importantly, due to the overlaps, this measure fails to be anti-monotone, meaning that a subpattern may be counter-intuitively matched less frequently than the pattern itself. Therefore, the finding of better anti-monotonic frequency measures - also known as support measures - has received a lot of attention in the data mining community [BN08, CRVD11, FB07]. A first idea is to count the maximal number of non-overlapping patterns [VGS02]. However, finding such a maximal subset of patterns essentially boils down to finding a maximal independent set in a graph, a notorious NP-complete problem [GJ02]. A second alternative approach is to count the number of matchings with respect to some kind of weighting of the matching that takes dependencies into account. In this paper, we will refer to this task as the dependency weighted counting problem, a generalization to databases of the s-measure of Wang, Ramon, and Fannes [WRF13] introduced in the context of graph databases. The idea is to counter-balance the dependencies of matchings by assigning positive weight to each matching, and then to measure the frequency of the pattern by the sum of the weights of all matchings. The only constraint that applies to the weights is that for every vertex \( v \) of the graph, the sum of the weights of the matchings of the pattern containing \( v \) has to be at most one. Intuitively, it means that the overall contribution of all matchings overlapping on \( v \) should not be too large. We are interested in the maximal value that the total sum of the weights can take, considering all possible weightings. In other words, we choose the weights as an optimal solution of a linear program. This frequency measure is indeed anti-monotone and has been proven to have useful statistical properties.

Dependency weighted counting can be reformulated in the database setting as follows: given a conjunctive query, we want to assign a positive weight \( \omega(\tau) \) to each answer \( \tau \) of the query such that for every attribute \( x \) and for every value \( d \) of this attribute, the sum of the weights of the tuples taking value \( d \) on attribute \( x \) is at most 1. We choose a weighting \( \omega \) of the answers that maximizes the sum of all weights and are interested in the value of this optimal sum. This problem may be seen as a form of weighted counting of the set of answers with weights that are implicitly defined by an optimization task.

The naive way to perform dependency weighted counting is to first compute the answer set of the query on the database, and then to reduce dependency weighted counting for this answer set to solving a linear program, which is known to be feasible in polynomial time [Kha79]. However, this does not seem to be a feasible approach as the size of the answer set may be exponentially larger than the database itself. Even worse, we can show that the optimal value of the linear program is non-zero if and only if the query has at least one answer in the database. Thus, the dependency weighted counting problem is harder than deciding if the query has at least one answer, a well-known NP-hard problem.

The main contribution of this paper is to prove that the dependency weighted counting problem can be solved in polynomial time for a large class of conjunctive queries, namely, the class of all bounded fractional hypertreewidth conjunctive queries [Mar11], which includes acyclic and bounded hypertreewidth queries [CLS01, CLS02]. In contrast to the problem of counting the number of answers, our tractability result still holds for structured conjunctive queries in the presence of existential quantifiers. As a byproduct, we get that evaluating the frequency for the s-measure of a pattern of bounded treewidth in a graph can be done in polynomial time.

Our algorithm works on factorized representations of relations [BKOZ13], that can be used to represent the answer set of a conjunctive query in a succinct manner. More concretely, the factorized representations of [BKOZ13] are circuits performing only Cartesian products and disjoint unions of atomic relations. In this paper, we will refer to these representations as \( \{\cup, \times\}\)-circuits. Olteanu’s and Závodný [OZ13] proved that one can represent the answer set of a conjunctive query over a database as a \( \{\cup, \times\}\)-circuit of polynomial size, under the assumption that the query is free of existential quantifiers and that its fractional hypertreewidth is bounded. The existence of a \( \{\cup, \times\}\)-circuit representing the answer set of a conjunctive query with such structural restrictions in a factorized manner has made explicit some known mechanisms that were already used to solve various aggregation problems on quantifier-free conjunctive queries such as counting the number of answers [PS13, GSH14] or enumerating them [BDG07]. See Section 2 for a deeper introduction of related work. Our main result on quantifier-free conjunctive queries is a corollary of the fact that dependency weighted counting for \( \{\cup, \times\}\)-circuits can be performed in polynomial time. We handle existential projection separately, by showing that the dependency weighted count of the existential projection of a set of tuples is the same as computing the dependency weighted count while ignoring dependencies generated by projected variables, a task that
we can still solve it in polynomial time on \(\{\sqcup, \times\}\)-circuits.

To solve the dependency weighted counting problem on \(\{\sqcup, \times\}\)-circuits without explicitly computing the set of answers, we show that the optimization we need to perform on the weights associated to each answer may be done by associating weights to the edges of the \(\{\sqcup, \times\}\)-circuit and then by optimizing on these weights. This is reflected by our main technical theorem, Theorem 5.2, which shows a natural correspondence between weights on answers of query and weights on edges of the circuit used as factorized representation. This correspondence is later used in Section 6 to show that the possibly exponential size linear program on the weights of tuples has the same optimal value than a polynomial size linear program on the weights of edges, which can then be solved in overall polynomial time. We then show that we can solve the dependency weighted counting problem on \(\{\sqcup, \times\}\)-circuits with existential projections by solving a relaxed form of this linear program on the edges of the \(\{\sqcup, \times\}\)-circuit without projections.

Outline. The paper is organized as follows. We present an overview of related work in Section 2, presenting previous work in the estimation of pattern frequency in data graphs and giving a quick presentation of known results concerning the complexity of answering conjunctive queries. Section 3 introduces the main notations and concepts that we will use throughout the paper. Section 4 formally defines the dependency weighted counting problem, and states basic observation on its complexity in general. Section 5 contains the technical core of the paper with the proof of Theorem 5.2, which makes a correspondence between weights associated to the edges of a \(\{\sqcup, \times\}\)-circuit \(C\) and weights associated to the tuples in the relation computed by \(C\). Section 6 uses it to solve the dependency weighted counting problem and other related aggregation problems. Finally, we we give perspectives for future work in the conclusion.

2 Related work

Pattern frequencies and s-measure. Evaluating the frequency of a pattern in a graph database is a central task in data mining. A naive way of evaluating this frequency is to use the number of matchings of the pattern as a frequency measure. The most important drawback of this approach is the fact that this measure is not anti-monotone, meaning that a subpattern may be counter-intuitively less frequent than the pattern itself. A simple illustration of this phenomenon is when the data graph is a \(k\)-clique and we consider the triangle as a pattern and a single edge as a subpattern of the triangle. Using the number of occurrences as a frequency measure, we realize that the \(k\)-cliques contains \(\binom{k}{2}\) edges and \(\binom{k}{3} \geq \binom{k}{2}\) triangles. This is because many triangles share common edges. In many applications, it seems abusive to consider two triangles having a common edge as two distinct occurrences of the same pattern to estimate its frequency (in other cases, other notions of “overlap” can be considered such as edge overlap, see [CRVD11] for an overview). The number of occurrences also has drawbacks from a statistical point of view, in the sense that occurrences of patterns overlap and do not represent independent observations of the phenomenon described by the pattern.

Earlier work, e.g., [VGS02] and [CRVD11], have considered this problem from the point of view of pattern mining, and have considered several combinations of matching operators (e.g., homomorphism, isomorphism, homeomorphism) and overlap (e.g., vertex overlap, edge overlap). Calders et al. [CRVD11] have shown that for a large class of such settings, support measures exist which are anti-monotonic and satisfy a number of common counting intuitions. The maximal and minimal support measures can only be computed by solving a maximal independent set or minimum clique partition problem, which is NP hard in the number of occurrences. [CRVD11] also showed a polynomial time computable measure sandwiched between these two and based on the Lovasz \(\theta\) function. The frequency measure we consider in this paper, the dependency weighted count presented in Section 4, is a generalization to databases of the s-measure introduced in [WRF13], where it is shown that s-measure satisfies the nice properties introduced by [CRVD11] and can be more efficiently computed. They also argued that the measure makes more sense from a statistical point of view, see also [Wan15].

Conjunctive queries and factorized databases. Conjunctive queries are arguably one of the most studied class of queries, covering a large body of natural queries since they correspond to the SELECT-PROJECT-JOIN fragment of SQL and have strong relations with the Constraint Satisfaction Problem [Dec03] studied in AI. Given a conjunctive query \(Q = S_1(X_1) \land \cdots \land S_k(X_k)\) with free variables
We denote by $X \subseteq \bigcup_{i=1}^{k} X_i$ (the others are existentially quantified) and a database $\mathbb{D}$ on domain $D$, computing the answer set $Q(\mathbb{D}) \subseteq D^X$ of $Q$ on database $\mathbb{D}$ is well-known to be NP-complete when both query and database are given as part of the input. A successful line of research consists in finding classes of conjunctive queries that are tractable, that is, a class $C$ of conjunctive queries such that for every $Q \in C$ and database $\mathbb{D}$, one can decide whether $Q(\mathbb{D}) \neq \emptyset$ in polynomial time in the size of the $Q$ and $\mathbb{D}$. Yannakakis initiated this research by showing in [Yan81] that so-called $\alpha$-acyclic queries can be solved in linear time in the size of the database, opening the way to several generalisations [GLS01, CDL14, GM14, Mar11].

The idea underlying this body of work is that one can exploit the structure of the interaction between variables and atoms to evaluate a conjunctive query efficiently. This interaction is characterized by the hypergraph $H(Q)$ whose vertices are the variables $X$ of the query and edges are the variables of its atoms, that is, $\{X_i \mid i \leq k\}$ using above notations. This hypergraph is then decomposed following a tree, in the same spirit as what is done on graphs with tree decompositions. This decomposition is then used to evaluate the query. The complexity of this evaluation depends on a notion of width of the decomposition, that is, a rational number $k$ is associated to the decomposition to measure its complexity. The smaller the width, the better the algorithm performs. Several notions of width have been introduced, such as, from the least to the most general one, hypertree width [GLS02], generalized hypertree width [GLS01, CDM14, GM14, Mar11] or fractional hypertree width [GM14] (see [GGLS16] for a survey), all giving polynomial time algorithms for the query evaluation problem.

This approach has been shown to work to efficiently solve more involved aggregation tasks on conjunctive queries such as counting [PS13, DM15] or enumeration [BDG07] and has found applications in practical tools in order to design good query plans [ALT+17, AJR+14]. More recently, Olteanu et al. [OZ12] proposed the framework of factorized databases, unifying these approaches. They have shown that when $Q$ has bounded (fractional hypertree) width and is not existentially quantified, $Q(\mathbb{D})$ can be efficiently represented by a $\{\alpha, \times\}$-circuit, i.e. a circuit performing Cartesian product and disjoint union, starting with atomic relations. This representation is of size polynomial in the size of the database and the query, which can be exponentially more succinct than the explicit list of answers. This circuit representation allows to solve several aggregation problems such as counting or probabilistic evaluation in time polynomial in the size of the representation. This two steps approach of first changing the representation of the data before analysing it can be seen as a form of knowledge compilation [DM02, Olt16].

A technique used in AI to solve various reasoning problem on complex data. From this point of view, the factorized databases considered in [OZ12] exactly correspond to the class of circuits known as structured decision DNNF [PD08] in knowledge compilation. In this paper, we consider a slight generalization of these objects that we will call $\{\alpha, \times\}$-circuits in this paper. These circuits correspond to the so-called deterministic DNNF introduced by Darwiche in [Dar01].

Factorized databases has already been used in the context of solving statistical tasks on the data by Schleich et al. [SOC16] where they show that one can perform linear regression on data represented as a factorized databases much more efficiently than by first enumerating the complete answer set. We use a similar approach to solve our problem in this paper.

3 Preliminaries

3.1 General notations

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}_+$ the set of non-negative real numbers. For any positive natural number $n$, we denote by $[n]$ the set $\{1, \ldots, n\}$. For two sets $Y$ and $Z$, we denote by $Y^Z$ the set of (total) functions from $Z$ to $Y$. A finite function $R = \{(z_1, y_1), \ldots, (z_n, y_n)\}$ where all $z_i$ are pairwise distinct is denoted by $[z_1/y_1, \ldots, z_n/y_n]$. In the case $n = 1$ we will simply write $z_1/y_1$ instead of $[z_1/y_1]$. Given a function $f \in Y^Z$ and $Z' \subseteq Z$, we denote by $f_{|Z'}$ the projection of $f$ on variable $Z'$, that is, $f_{|Z'} \in Y^{Z'}$ and for every $z \in Z'$, $f_{|Z'}(z) = f(z)$.

A directed graph is pair $G = (V, E)$ with $E \subseteq V \times V$. Given a directed graph $G$ we write $\text{Nodes}(G) = V$ for the set of nodes and $\text{Edges}(G) = E$ for the set of edge. For any $u \in V$, we denote by $\text{In}(u)$ (resp. $\text{Out}(u)$) the set of ingoing (resp. outgoing) edges of $u$. 
3.2 Linear expressions

We define linear expressions that will denote real numbers. Let $Y$ be a finite set of variables. The set of linear expressions over $Y$ denoted by $\mathcal{T}_Y$ is the least set that contains all variables $y \in Y$, for all $a \in \mathbb{R}$ and $t, t' \in \mathcal{T}_Y$ the products $at$ and the sums $t + t'$. We assume that $+$ is associative and commutative, given that it will be interpreted as the addition of real numbers. Therefore we can write $t_1 + \ldots + t_n$ without any parenthesis or equivalently $\sum_{i=1}^n t_i$. For $n = 0$ we define $\sum_{i=1}^n t_i$ to be equal to 0. If all $t_i$ are pairwise distinct and $T = \{t_1, \ldots, t_n\}$ then we write $\sum T$ instead of $\sum_{i=1}^n t_i$.

Any variable assignment $\mu : Y \to \mathbb{R}$ can be lifted to an evaluator on linear expressions $[.]_\mu : \mathcal{T}_Y \to \mathbb{R}$ such that for all $t, t' \in \mathcal{T}_Y$, $[t + t']_\mu = [t]_\mu + [t']_\mu$, for all $a \in \mathbb{R}$, $[at]_\mu = a[t]_\mu$, and for all $y \in Y$, $[y]_\mu = \mu(y)$.

3.3 Linear programs

A linear equation over $Y$ is an equation of the form $t = a$ where $t \in \mathcal{T}_Y$ and $a \in \mathbb{R}$. A solution of a linear equation is a variable assignment $\mu : Y \to \mathbb{R}$ such that $[t]_\mu = a$. A linear inequality over $Y$ is an inequality of the form $t \leq a$. A solution of a linear inequality is a variable assignment $\mu : Y \to \mathbb{R}$ such that $[t]_\mu \leq a$. A set of linear constraints $c$ over $Y$ is a conjunction of linear equations and inequalities over $Y$:

$$t_1 = a_1 \wedge \ldots \wedge t_n = a_n \wedge t'_1 \leq a'_1 \wedge \ldots \wedge t'_m \leq a'_m$$

A solution of a set of linear constraints $c$ over $Y$ is a variable assignment $\mu : Y \to \mathbb{R}$ that jointly satisfies all equations and inequalities in $c$. The set of all solutions of $c$ is denoted by $\text{sol}(c)$.

A linear program over $Y$ is a pair $L = (t, c)$ where $t \in \mathcal{T}_Y$ is a linear expression that is called the objective function of $L$, and $c$ a set of linear constraints over $Y$. We denote $L$ by:

$$\text{max } t \text{ subject to } c$$

The set $\text{sol}(c)$ is called the feasible region of $L$. An optimal solution of $L$ is a solution $\mu$ in the feasible region that maximizes the objective function of $L$, i.e. $\mu \in \text{sol}(c)$ and for all $\mu' \in \text{sol}(c)$, $[t]_\mu \geq [t]_{\mu'}$. The optimal value of $L$ is defined as $[t]_\mu$ where $\mu$ is some optimal solution of $L$.

It is known that we can find an optimal solution of a linear program in polynomial time in the number of its variables, its constraints and the size of the coefficients appearing in the constraints. The first polynomial time algorithm for solving this problem is the ellipsoid algorithm [Kha79]. Other more efficient algorithms have been proposed later [Kar84, LS15] and there exists many tools using this techniques together with heuristics to solve this problem in practice such as lp_solve [2] or glpk. In this paper, we will use these tools as black boxes and will only use the fact that finding optimal solutions of linear programs can be done in polynomial time.

3.4 $\{\cup, \times\}$-Circuits

In this paper, we choose to directly work on a factorized representation of the answers set of a conjunctive query instead of working directly with a hypertree decomposition of the query’s hypergraph. This has mainly two advantages: it makes the algorithm easier to describe and it makes it more general in the sense that our result does not only apply to well structured conjunctive queries but to any query that has a small factorized representation.

We use a generalization of the framework of factorized databases introduced by Olteanu et al. [OZ12]. Our factorized representations are defined similarly but we keep only a subset of the syntactic restrictions on the circuits since our algorithm does not need all of them to work.

Let $X$ be a finite set of attributes and $D$ be a finite domain. A (database) tuple with attributes in $X$ and domain $D$ is a function $\tau : X \to D$ mapping attributes to database elements. A (database) relation with attributes $X$ and domain $D$ is a set of such tuples, that is, a subset of $D^X$.

A $\{\cup, \times\}$-circuit on attributes $X$ and domain $D$ is a a directed acyclic graph $C$ with edges directed from the leaves to a single root $r$ called the output of $C$ and whose nodes, called gates, are labeled as follows:

http://lp_solve.sourceforge.net/
https://www.gnu.org/software/glpk/
Due to these restrictions, any gate define inductively as follows. If $\uplus$-gates during the construction of the circuit, usually by using $\times$-gates of the form:

```
x × y × z
0 1 0
1 0 1
0 1 1
0 0 0
1 1 1
```

Figure 1 pictures a $\{\uplus, \times\}$-circuit representing the database relation on the right. This circuit is also a $\{\uplus, \times\}$-circuit.

- internal gates are labeled by either $\times$ or $\uplus$,
- inputs, i.e. nodes of in-degree 0, are labeled by atomic database relation of the form $x/d$ for an attribute $x$ and an element $d \in D$.

Figure 1 pictures a $\{\uplus, \times\}$-circuit on attributes $\{x, y, z\}$ and domain $\{0, 1\}$. We denote by $\text{Attr}(C)$ the attributes appearing in the database relations of the input nodes of $C$. We define the size of $C$ as the number of edges in the underlying graph of $C$.

Given a gate $u$ of $C$, we denote by $C_u$ the subcircuit of $C$ rooted in $u$ (in which dangling edge from above are removed). We denote by $\times$-gates($C$) the set of all the $\times$-gates and by $\uplus$-gates($C$) the set of all $\uplus$-gates of $C$.

We impose the any $\{\uplus, \times\}$-circuit must satisfy the following restrictions.

- If $u$ is an $\uplus$-gate with children $u_1, \ldots, u_k$, we impose that $\text{Attr}(C_{u_1}) = \cdots = \text{Attr}(C_{u_k})$.
- If $u$ is a $\times$-gate with children $u_1, \ldots, u_k$, we impose that $\text{Attr}(C_{u_i}) \cap \text{Attr}(C_{u_j}) = \emptyset$ for every $i < j \leq k$. Observe that it implies that $C_{u_i}$ and $C_{u_j}$ are disjoint graphs since otherwise they would share an input and, thus, an attribute of the relation of this input node.

Due to these restrictions, any gate $u$ of a $\{\uplus, \times\}$-circuit $C$ specifies a database relation $rel(C_u)$ that we define inductively as follows. If $u$ is an input then $rel(C_u)$ is the relation labelling $u$. If $u$ is a $\times$-gate with children $u_1, \ldots, u_k$, $rel(C_u)$ is defined as $rel(C_{u_1}) \times \cdots \times rel(C_{u_k})$. If $u$ is an $\uplus$-gate with children $u_1, \ldots, u_k$, $rel(C_u)$ is defined as $rel(C_{u_1}) \cup \cdots \cup rel(C_{u_k})$. We define $rel(C) = rel(C_r)$ where $r$ is the output of the circuit.

The way we have currently defined circuits is not yet enough to allow for efficient aggregation. Given a $\{\uplus, \times\}$-circuit $C$ and an $\uplus$-gate $u$ of $C$ with children $u_1, \ldots, u_k$, we say that $u$ is disjoint if $rel(C_{u_i}) \cap rel(C_{u_j}) = \emptyset$ for every $i < j \leq k$. In our figures and proofs, we will use the symbol $\uplus$ to indicate that a union is disjoint. A $\{\uplus, \times\}$-circuit is a $\{\uplus, \times\}$-circuit in which every $\uplus$-gate is an $\uplus$-gate.

Observe that given a $\{\uplus, \times\}$-circuit $C$, one can compute the cardinality of its relation $|rel(C)|$ in time polynomial in the size of $C$ by a simple inductive algorithm that adds the sizes of the children of a $\times$-gate and multiply the sizes of the children of the $\uplus$-gates.

Disjointness is a semantic condition and it is coNP-hard to check the disjointness of a gate given a $\{\uplus, \times\}$-circuit. In practice however, algorithms producing $\{\uplus, \times\}$-circuit ensure the disjointness of all $\uplus$-gates during the construction of the circuit, usually by using $\uplus$-gates of the form: $\biguplus_{d \in D}(x/d) \times C_d$, so that the different values $d$ assigned to $x$ ensure disjointness.

**Theorem 3.1.** [OZT13] Theorem 7.1] Given a quantifier free conjunctive query $Q$ and a database $D$, one can construct a $\{\uplus, \times\}$-circuit $C$ of size at most $|Q| \cdot |D|^k$ whose attributes are the variables of $Q$ such that $Q(D) = rel(C)$ where $k$ is the fractional hypertreewidth of $Q$.

For the special case of quantifier free acyclic conjunctive query the theorem follows from a variant of Yanakakis’ algorithm. We refer the reader to [CCS10] for the definition of fractional hypertreewidth as we will not work with this notion directly in this paper but only with $\{\uplus, \times\}$-circuits, using Theorem 3.1 to bridge both notions.

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3Observe that every possible input relation could be easily simulated with this restricted input using $\times$ and $\uplus$. For example, the relation $\{(x/1, y/1, z/1), (x/0, y/0, z/0)\}$ can be rewritten as $\{(x/1 \times y/1 \times z/1) \uplus (x/0 \times y/0 \times z/0)\}$.
4 Dependency Weighted Counting

In this section, we formalize the main problem we are solving in this paper, the dependency weighted counting problem and give basic observations on its complexity.

**Definition 4.1.** Let $S$ be a relation with attributes $X$ and domain $D$, that is, $S \subseteq D^X$. A weighting for $S$ is a function $\omega : S \rightarrow \mathbb{R}_+$. A dependency weighting for $S$ is a weighting for $S$ that satisfies $\sum_{\tau \in S} \omega(\tau) \leq 1$ for all attributes $x \in X$ and database elements $d \in D$. The dependency weighted count of $S$, denoted as $\text{DWC}(S)$, is the maximal value of $\sum_{\tau \in S} \omega(\tau)$ where the maximum is taken over every dependency weighting $\omega$ of $S$.

In the definition of dependency weightings and in rest of the paper, we will often refer to the subset of tuples of $S$ whose value on an attribute $x$ equals $d$. We will denote this set as $S_{x/d} := \{ \tau \in S \mid \tau(x) = d \}$.

Intuitively, $\text{DWC}(S)$ measures the statistical quality of the sample $S$, considering that any two database tuples in $S_{x/d}$ are not independent, given that they share the value $d$ on attribute $x \in X$. Therefore, the weights of the dependent tuples in $S_{x/d}$ are constrained to sum up to at most $1$.

**Example 4.2.** Consider the database domain $D = \{0, 1\}$, the set of attributes $\{x, y, z\}$ and the relation $S$:

|    | x | y | z | | tuple |
|----|---|---|---|---|--------|
| 1  | 1 | 0 | z₀ | |        |
| 1  | 0 | 1 | z₁ | |        |
| 0  | 1 | 1 | z₂ | |        |

The set of linear constraints that any dependency weighting $\omega : S \rightarrow \mathbb{R}_+$ must satisfy is the following:

\[
\begin{align*}
\omega(\tau_2) &\leq 1 & (S_{x/0}) \\
\omega(\tau_1) &\leq 1 & (S_{y/0}) \\
\omega(\tau_0) &\leq 1 & (S_{z/0}) \\
\omega(\tau_0) + \omega(\tau_1) &\leq 1 & (S_{x/1}) \\
\omega(\tau_0) + \omega(\tau_2) &\leq 1 & (S_{y/1}) \\
\omega(\tau_1) + \omega(\tau_2) &\leq 1 & (S_{z/1}) \\
\end{align*}
\]

For this example, $\text{DWC}(S) = 1.5$, obtained by taking the dependency weighting $\omega : S \rightarrow \mathbb{R}_+$ defined as $\omega(\tau) = 0.5$ for all $\tau \in S$.

We can define the $\text{DWC}(S)$ equivalently by the following linear program $\text{DWC-LP}(S)$ whose variables are the database tuples in $S$:

\[
\text{max } \sum_{S} S \text{ subject to } \bigwedge_{\tau \in S} 0 \leq \tau \land \bigwedge_{x \in X} \bigwedge_{d \in D} S_{x/d} \leq 1
\]

The positivity constraint $0 \leq \tau$ is syntactic sugar for the inequality $(-1)\tau \leq 0$. It should be obvious that the optimal value of $\text{DWC-LP}(S)$ is equal to $\text{DWC}(S)$.

**Example 4.3.** The linear program $\text{DWC-LP}(S)$ for the set of database tuples from Example 4 is:

\[
\text{max } \tau_0 + \tau_1 + \tau_2 \text{ subject to } \begin{pmatrix}
0 & \leq \tau_0 & \land & \tau_2 \leq 1 & \land & \tau_0 + \tau_1 \leq 1 \\
0 & \leq \tau_1 & \land & \tau_1 \leq 1 & \land & \tau_0 + \tau_2 \leq 1 \\
0 & \leq \tau_2 & \land & \tau_0 \leq 1 & \land & \tau_1 + \tau_2 \leq 1
\end{pmatrix}
\]

A set $S$ of database tuples may be specified in many different ways, for instance by the answer set of a database query, by an explicit list, or by a $\{\psi, \times\}$-circuit. The complexity of DWC thus depends on the way $S$ is given in the input.

If $S$ is given explicitly as a list, then it is easy to see that the problem can be solved in polynomial time, since then $\text{DWC-LP}(S)$ is of size $O(|X||D||S|)$. That is, $\text{DWC-LP}(S)$ is of size polynomial in the input and one just has to solve the linear program in polynomial time. However, if $S$ is specified as the answers set of a query $Q$ on a database $D$, it may be too costly to compute it explicitly and then solve $\text{DWC-LP}(S)$ as $S$ may be of exponential size. Unfortunately, it turns out that solving DWC when $S$ is given as a conjunctive query on a database is not tractable.
Lemma 4.4. The problem of deciding on input \((Q, D)\) whether \(DWC(Q(D)) > 0\) is NP-hard, where \(Q\) is a conjunctive query and \(D\) a database.

Proof. We show this lemma by showing that \(DWC(Q(D)) > 0 \iff Q(D) \neq \emptyset\). The NP-hardness then follows since it is known that deciding whether the answer set of a conjunctive query is non-empty is NP-hard for combined complexity [CM77].

We start by showing that if \(Q(D) \neq \emptyset\) then \(DWC(Q(D)) > 0\). Let \(\tau \in Q(D)\). Let \(\omega\) be the tuple weighting such that \(\omega(\tau) = 1\) and \(\omega(\tau') = 0\) for every \(\tau' \in Q(D)\) with \(\tau' \neq \tau\). It is easy to check \(\tau\) is a dependency weighting of \(Q(D)\), thus, \(DWC(Q(D)) \geq 1 > 0\).

Now assume that \(DWC(Q(D)) = s > 0\). That is, there exists a weighting \(\omega\) of \(Q(D)\) such that \(\sum_{\tau \in Q(D)} \omega(\tau) = s\). If \(Q(D) = \emptyset\), this would imply \(\sum_{\tau \in Q(D)} \omega(\tau) = 0\), a contradiction.

When \(S\) is given as a \(\{\wedge, \times\}\)-circuit however, the DWC problem can be solved in polynomial time. This is the main result of this paper:

Theorem 4.5. Given a \(\{\wedge, \times\}\)-circuit \(C\), one can compute \(DWC(\text{rel}(C))\) in time polynomial in the size of \(C\).

As a corollary of Theorem 4.5 and Theorem 3.1, we get that for every \(k \in \mathbb{N}\), computing \(DWC(Q(D))\) is tractable for the class of all conjunctive queries of fractional hypertree width \(k\).

We do not give complexity bounds in Theorem 4.5 as we will see that it mainly depends on a call to an external linear program solver. Our algorithms roughly works as follows: we show in Section 5 that any tuple-weighting of \(\text{rel}(C)\) can be naturally represented as a weighting of the edges of \(C\). We then show in Section 6 how to use this correspondence to rewrite DWC-LP into an equivalent polynomial size linear program on the edges of \(C\) that we can then solve by calling an external linear program solver.

5 Weighting correspondence

In order to solve the DWC problem on a relation given as a \(\{\wedge, \times\}\)-circuit, we have seen in Section 4 that we need to solve a linear program to optimize the weights we assign to the answer tuples. The number of such weights may be exponentially larger than the size of the circuit itself, making it impossible to directly manipulate them efficiently. In this section, we show in Theorem 5.2 that we can naturally represent any such weighting with a weighting on the edges of the circuit.

This section is organized as follows: we start by formalizing the notions we need to state Theorem 5.2. In Section 5.1, we make some observations and define useful notions on \(\{\wedge, \times\}\)-circuits. Section 5.2 and Section 5.3 are dedicated to the proof of Theorem 5.2.

Let \(C\) be a \(\{\wedge, \times\}\)-circuit. A tuple-weighting \(\omega\) of \(C\) is defined as a variable assignment of \(\text{rel}(C)\), e.g. \(\omega \in \mathbb{R}_+^{\text{rel}(C)}\). An edge-weighting \(W\) of \(C\) is similarly defined as a variable assignment of \(\text{Edges}(C)\), e.g. \(W \in \mathbb{R}_+^{\text{Edges}(C)}\).

Definition 5.1. An edge-weighting \(W\) is sound if for every gate \(u\) of \(C\) we have:

- if \(u\) is a \(\wedge\)-gate that is not the output of the circuit then \(\sum_{i \in \text{In}(u)} W(i) = \sum_{o \in \text{Out}(u)} W(o)\),
- if \(u\) is a \(\times\)-gate then \(\forall i, i' \in \text{In}(u): W(i) = W(i')\). If \(u\) is not the output of the circuit, we also have \(W(i) = \sum_{o \in \text{Out}(u)} W(o)\).

Throughout the paper, we will always use the symbol \(\omega\) for tuple-weightings and \(W\) for edge-weightings.

Given an edge \(e = \langle u, v \rangle\) of \(C\), we define the relation induced by \(e\), denoted by \(\text{rel}(C, e)\), as the subset of tuples of \(\text{rel}(C)\) that could not be computed by \(C\) without the presence of \(e\); more precisely, \(\tau \in \text{rel}(C, e)\) if \(\tau \in \text{rel}(C)\) and if it is not in the relation computed by the circuit where we replace the subcircuit rooted in \(u\) by the empty relation on \(\text{Attr}(C_u)\) (an alternative and more formal definition is given in Section 5.1).

Given a tuple-weighting \(\omega\) of a \(\{\wedge, \times\}\)-circuit \(C\), it naturally induces an edge-weighting \(W\) defined for every \(e \in \text{Edges}(C)\) as \(W(e) := \sum_{\tau \in \text{rel}(C, e)} \omega(\tau)\). We call \(W\) the edge-weighting induced by \(\omega\).
$W(e_1) := 1$ $W(e_2) := 1$
$\top$ $\top$
$W(e_3) = 0$ $W(e_4) = 0$
$x/1$ $z/2$
$y/0$
$x/0$ $x/2$ $z/1$ $z/0$

Figure 2: A $\{\top, \times\}$-circuit with a non-sound edge weighting since $W(e_1) + W(e_2) \neq W(e_4)$.

$W(e_1) := 0$ $W(e_2) := 0$
$\top$ $\top$
$W(e_3) = 0$ $W(e_4) = 0$
$x/1$ $z/2$
$y/0$
$x/0$ $x/2$ $z/1$ $z/0$

Figure 3: Proof-tree of the tuple $[x/1, y/1, z/1]$

**Theorem 5.2.** Given a $\{\top, \times\}$-circuit $C$.

(a) For every tuple-weighting $\omega$ of $C$, the edge-weighting $W$ induced by $\omega$ is sound.

(b) Given a sound edge-weighting $W$ of $C$, there exists a tuple-weighting $\omega$ of $C$ such that $W$ is the edge-weighting induced by $\omega$.

We provide the proof for this theorem in Sections 5.2 and 5.3.

**Example 5.3.** The notion of sound edge weighting is essential for Theorem 5.2 (a) to work. Consider the $\{\top, \times\}$-circuit of Figure 2 with a non-sound edge weighting $W$ drawn on the edges.

This circuit computes the relation $\{x/1, x/2, x/3\}$. Observe that removing the edge $e_1$ removes the tuple $x/1$ from the final relation. Thus $rel(C, e_1) = \{x/1\}$. Now, if we want to construct a tuple weighting $\omega$ as in Theorem 5.2 we will have $1 = W(e_1) = \sum_{\tau \in rel(C, e_1)} \omega(\tau) = \omega(x/1)$. Similarly, $1 = W(e_2) = \omega(x/2)$.

Now, removing $e_4$ from the circuit removes tuples $x/1$ and $x/2$ from the relation. Thus, $rel(C, e_4) = \{x/1, x/2\}$ and the construction of Theorem 5.2 would give $0 = W(e_4) = \sum_{\tau \in rel(C, e_4)} \omega(\tau) = \omega(x/1) + \omega(x/2) = 2$, a contradiction.

If $W(e_4) = 2$ however, we see that $W$ would be sound and we would not have the previous contradiction.

\[\square\]

### 5.1 Proof-trees

From now on, to simplify the proofs, we assume wlog that every internal gate of a $\{\top, \times\}$-circuit has fan-in two. It is easy to see that, by associativity, $\times$-gates and $\top$-gates of fan-in $k > 2$ can be rewritten with $k - 1$ similar gates which is a polynomial size transformation of the circuit. We also assume that for every $x \in \text{Attr}(C)$ and $d \in D$, we have at most one input labeled with $x/d$. This can be easily achieved by merging all such inputs.

Let $C$ be a $\{\top, \times\}$-circuit and let $\tau \in rel(C)$. The proof-tree of $\tau$, denoted $T_C(\tau)$, is a subcircuit of $C$ participating to the computation of $\tau$. More formally, $T_C(\tau)$ is defined inductively by starting from the output as follows: the output of $C$ is in $T_C(\tau)$. Now if $u$ is a gate in $T_C(\tau)$ and $v$ is a child of $u$, then we add $v$ in $T_C(\tau)$ if and only if $\tau_{\text{Attr}(C_u)} \in rel(C_v)$. A proof-tree is depicted in red in Figure 3.

**Proposition 5.4.** Given a $\{\top, \times\}$-circuit $C$ and $\tau \in rel(C)$, $T := T_C(\tau)$, the following holds:

- every $\times$-gate of $T$ has all its children in $T$,
- every $\top$-gate of $T$ has exactly one of its children in $T$,
• T is connected and every gate of T has out-degree at most 1 in T,
• for any \( x \in \text{Attr}(C) \), T contains exactly one input labeled with \( x/\tau(x) \).

Proof. By induction, it is clear that for every gate \( u \) of T, \( \tau_{\text{Attr}}(C_u) \in \text{rel}(C_u) \). Thus, if \( u \) is a \( \psi \)-gate of T, as \( u \) is disjoint, exactly one of it child \( v \) has \( \tau_{\text{Attr}}(C_v) \in \text{rel}(C_v) \). If \( u \) is a \( \times \)-gate with children \( u_1, u_2 \), then by definition, \( \tau_{\text{Attr}}(C_{u_1}) \in \text{rel}(C_{u_1}) \) if and only if \( \tau_{\text{Attr}}(C_{u_2}) \in \text{rel}(C_{u_2}) \). Thus both \( u_1 \) and \( u_2 \) are in T.

It is clear from definition that T is connected since T is constructed by inductively adding children of gates that are already in T. Now assume that T has a gate \( u \) of out-degree greater than 1 in T. Let \( u_1, u_2 \) be two of its parents and let \( v \) be their least common ancestor in T. By definition, \( v \) has in-degree 2, this it is necessarily a \( \times \)-gate with children \( v_1, v_2 \). Thus \( u \) is both in \( C_{v_1} \) and \( C_{v_2} \), which is impossible since they are disjoint subcircuits.

Finally, let \( x \in \text{Attr}(C) \). Observe that if T has an input labeled \( x \) with \( x/d \) then \( d = \tau(x) \) since \( \tau_{\text{Attr}}(C_u) \in \text{rel}(C_u) \). Thus, if T contains two inputs \( v_1 \) and \( v_2 \) on attribute \( x \), they are both labeled with \( x/\tau(x) \) and are thus the same input.

A proof-tree may be seen as the only witness that a tuple belongs to \( \text{rel}(C) \). We can now define the relation induced by an edge \( e \) more formally as the set of tuples of \( \text{rel}(C) \) such that their proof-tree contains the edge \( e \), that is \( \text{rel}(C,e) := \{ \tau \in \text{rel}(C) \mid e \in T_C(\tau) \} \).

In the following we show some fundamental properties of proof-trees that we will use to prove the correctness of our algorithm. For the rest of this section, we fix a \( \{\psi,\times\} \)-circuit \( C \) on attributes \( X \) and domain \( D \).

Proposition 5.5. Let \( u \) be a gate of \( C \) and \( e_1, e_2 \in \text{Out}(u) \) with \( e_1 \neq e_2 \). Then \( \text{rel}(C, e_1) \cap \text{rel}(C, e_2) = \emptyset \).

Proof. Assume that \( \tau \in \text{rel}(C, e_1) \) and \( \tau \in \text{rel}(C, e_2) \) for \( e_1 \neq e_2 \). By definition, it means that both \( e_1 \) and \( e_2 \) are in \( T_C(\tau) \) which is a contradiction as \( T_C(\tau) \) has out-degree 1 by Proposition 5.4.

Proposition 5.6. Let \( x \in X \) and \( d \in D \) and let \( u_{x/d} \) be the input of \( C \) labeled with \( x/d \).

\[
S_{x/d} = \bigcup_{e \in \text{Out}(u_{x/d})} \text{rel}(C, e).
\]

Proof. Let \( x \in \text{Attr}(C) \) and \( d \in D \). Let \( \tau \in S_{x/d} \). By Proposition 5.4, \( u_{x/d} \) belongs to \( T_C(\tau) \) so there is at least one \( e \in \text{Out}(u_{x/d}) \) that belongs to \( T_C(\tau) \). Similarly, if \( \tau \in \text{rel}(C, e) \) for some \( e \) then \( u_{x/d} \) is also in \( T_C(\tau) \), and then \( \tau(x) = d \). This proves the equality. The disjointness of the union directly follows from Proposition 5.5.

Next we show that the disjoint union of the relations induced by each ingoing edge of a \( \psi \)-gate is equal to the disjoint union of the relations induced by each outgoing edge of this gate.

Proposition 5.7. Let \( u \) be an internal \( \psi \)-gate of \( C \),

\[
\bigcup_{i \in \text{In}(u)} \text{rel}(C, i) = \bigcup_{o \in \text{Out}(u)} \text{rel}(C, o).
\]

Proof. Let \( S_u := \{ \tau \mid u \in T_C(\tau) \} \) and \( \tau \in S_u \). It is clear from Proposition 5.4 that a proof tree contains \( u \) if and only if it contains at least an edge of \( \text{In}(u) \) and at least an edge of \( \text{Out}(u) \). Thus, both unions are equal to \( S_u \).

The disjointness of the first union directly follows from the second item of Proposition 5.4 and the disjointness of the second union from Proposition 5.5.

Finally we show that the disjoint union of the relations induced by the outgoing edges of a \( \times \)-gate is equal to the relation induced by each ingoing edge of this gate.

Proposition 5.8. Let \( u \) be an internal \( \times \)-gate of \( C \),

\[
\forall i \in \text{In}(u) : \text{rel}(C, i) = \bigcup_{o \in \text{Out}(u)} \text{rel}(C, o)
\]
Proof. Let $S_u := \{ \tau \mid u \in T_{C}(\tau) \}$. By Proposition 5.4, it is clear that if $u$ is in a proof tree $T$, then every edge of $\text{In}(u)$ is also in $T$ and exactly one edge of $\text{Out}(u)$ is in $T$. Thus, both sets in the statement are equal to $S_u$. The disjointness of the union follows directly from Proposition 5.5. 

5.2 Proof of Theorem 5.2 (a)

This section is dedicated to the proof of Theorem 5.2 (a), namely that, given a \{⊎, ×\}-circuit $C$ and a tuple-weighting $ω$ of $C$, the edge-weighting $W$ of $C$ induces by $ω$ defined as $W(e) := \sum_{τ \in \text{rel}(C,e)} ω(τ)$ is sound.

We will see that the soundness of $W$ follows naturally from the properties of proof-trees of the previous section. We prove that $W$ is sound by checking the case of \{⊎\}-gates and \{×\}-gates separately.

- We first have to show that for every \{⊎\}-gate $u$ of $C$, it holds that $\sum_{e \in \text{Out}(u)} W(e) = \sum_{e \in \text{In}(u)} W(e)$. This is a consequence of Proposition 5.7. Let $u$ be a \{⊎\}-gate of $C$.

  $\sum_{e \in \text{Out}(u)} W(e) = \sum_{e \in \text{Out}(u)} \sum_{τ \in \text{rel}(C,e)} ω(τ)$ (by definition of $W$)
  $= \sum_{τ \in R} ω(τ)$ (where $R = \bigcup_{e \in \text{Out}(u)} \text{rel}(C,e)$)

  The disjointness of the union in $R$ has been proven in Proposition 5.7, which also states that $R = \bigcup_{e \in \text{In}(u)} \text{rel}(C,e)$. Thus, the last term in the sum can be rewritten as

  $\sum_{τ \in R} ω(τ) = \sum_{e \in \text{In}(u)} \sum_{τ \in \text{rel}(C,e)} ω(τ)$
  $= \sum_{e \in \text{In}(u)} W(e)$ (by definition of $W$)

- We now show that for every \{×\}-gate $u$ of $C$ and for every edge $i \in \text{In}(u)$ going in $u$, it holds that $\sum_{e \in \text{Out}(u)} W(e) = \sum_{e \in \text{In}(u)} W(e)$. The proof is very similar to the previous case but is now a consequence of Proposition 5.8. As before, we have $\sum_{τ \in \text{Out}(u)} W(e) = \sum_{τ \in R} ω(τ)$ where $R = \bigcup_{e \in \text{Out}(u)} \text{rel}(C,e)$. The disjointness of the union in $R$ has been proven in Proposition 5.8, which also implies that $R = \text{rel}(C,i)$. Thus, we have:

  $\sum_{e \in \text{Out}(u)} W(e) = \sum_{τ \in \text{rel}(C,i)} ω(τ)$
  $= W(i)$ (by definition of $W$)

5.3 Proof of Theorem 5.2 (b)

This section is dedicated to the proof of Theorem 5.2 (b), namely that, given a circuit $W$ and a sound edge-weighting $W$ of $C$, there exists a tuple-weighting $ω$ of $C$ such that, for every edge $e$ of $C$, $\sum_{τ \in \text{rel}(C,e)} ω(τ) = W(e)$.

In this section, we fix a \{⊎, ×\}-circuit $C$ and a sound edge-weighting $W$ of its edges. We assume wlog that the root $r$ of $C$ has a single ingoing edge which we call the output edge $α_r$. We construct $ω$ by induction. For every edge $e = (u,v)$ of $C$, we inductively construct a tuple weighting $ω_e$ of $\text{rel}(C_u)$. We will then choose $ω = ω_{α_r}$ and show that this tuple-weighting verifies $\sum_{τ \in \text{rel}(C,e)} ω(τ) = W(e)$ for every edge $e$ of $C$.

For $e = (u,v)$, we define $ω_e : \text{rel}(C_u) \to \mathbb{R}_+$ inductively as follows:

- **Case 1**: $u$ is an input labeled with $x/d$ then $\text{rel}(C_u)$ contains only the tuple $x/d$. We define $ω_e(x/d) := W(e)$.
• **Case 2**: $u$ is a $\psi$-gate with children $u_1, u_2$. Let $e_1 = \langle u_1, u \rangle$ and $e_2 = \langle u_2, u \rangle$. In this case, given $\tau \in rel(C_u)$, we have by definition that $\tau \in rel(C_{u_1})$ or $\tau \in rel(C_{u_2})$. Assume wlog that $\tau \in rel(C_{u_1})$. If $\sum_{f \in Out(u)} W(f) \neq 0$, we define: $\omega_e(\tau) := W(e)\frac{\omega_{e_1}(\tau_1)}{W(e_1)}$. Observe that since $W$ is sound, $W(e_1) + W(e_2) = \sum_{f \in Out(u)} W(f) \neq 0$. Otherwise $\omega_e(\tau) := 0$.

• **Case 3**: $u$ is a $\times$-gate with children $u_1, u_2$. Let $e_1 = \langle u_1, u \rangle$ and $e_2 = \langle u_2, u \rangle$. In this case, given $\tau \in rel(C_u)$, we have by definition that $\tau = \tau_1 \times \tau_2$ with $\tau_1 \in rel(C_{u_1})$ and $\tau_2 \in rel(C_{u_2})$. If $W(e_1) \neq 0$ and $W(e_2) \neq 0$, we define: $\omega_e(\tau) := W(e)\frac{\omega_{e_1}(\tau_1) \omega_{e_2}(\tau_2)}{W(e_1)W(e_2)}$. Otherwise $\omega_e(\tau) := 0$.

We begin our proof by showing that as we construct each $\omega_e$ through a bottom-up induction there is a relation between $W(e)$ and $\omega_e$ which resembles the property we aim to prove in this section.

**Lemma 5.9.** For every gate $u$ of $C$ and $e = \langle u, v \rangle \in Edges(C)$,

$$W(e) = \sum_{\tau \in rel(C_u)} \omega_e(\tau).$$

**Proof.** Let $e = \langle u, v \rangle$ be an edge of $C$. Observe that when $\sum_{o \in Out(u)} W(o) = 0$, then $W(e) = 0$ since $e \in Out(u)$ and $W$ has positive value. Moreover, by definition of $\omega_e$, for every $\tau \in rel(C_u)$, $\omega_e(\tau) = 0$. In particular, $\sum_{\tau \in rel(C_u)} \omega_e(\tau) = 0 = W(e)$. In this case then, the lemma holds.

In the rest of the proof, we now assume that $\sum_{o \in Out(u)} W(o) \neq 0$. We show the lemma by induction on the nodes of $C$ from the leaves to the root.

**Base case**: $u$ is a leaf labelled with $x/d$. Let $e$ be an outgoing edge of $u$. Observe that $rel(C_u)$ contains a single tuple $x/d$ and that, by definition of $\omega_e$, $W(e) = \omega_e(x/d) = \sum_{\tau \in rel(C_u)} \omega_e(\tau)$.

**Inductive case**: Now let $u$ be an internal gate of $C$ with children $u_1, u_2$ and let $e_1 = \langle u_1, u \rangle$ and $e_2 = \langle u_2, u \rangle$, as depicted in Figure 4.

![Figure 4](https://via.placeholder.com/150)

**Figure 4**: Inductive step notations.

**Case 1**: Assume that $u$ is a $\psi$-gate. Let $W = W(e_1) + W(e_2)$. Since $u$ is disjoint, given $\tau \in rel(C_u)$, either $\tau \in rel(C_{u_1})$ or $\tau \in rel(C_{u_2})$ but not both. It follows:

$$\sum_{\tau \in rel(C_u)} \omega_e(\tau) = \sum_{\tau \in rel(C_{u_1})} \omega_e(\tau) + \sum_{\tau \in rel(C_{u_2})} \omega_e(\tau) = \sum_{\tau \in rel(C_{u_1})} W(e)\frac{\omega_{e_1}(\tau_1)}{W(e_1)} + \sum_{\tau \in rel(C_{u_2})} W(e)\frac{\omega_{e_2}(\tau_2)}{W(e_2)}$$

by definition of $\omega_e$. Observe that by induction we have $W(e_1) = \sum_{\tau \in rel(C_{u_1})} \omega_{e_1}(\tau_1)$ and $W(e_2) = \sum_{\tau \in rel(C_{u_2})} \omega_{e_2}(\tau_2)$. Thus, by taking the constants out and using this identity, it follows:

$$\sum_{\tau \in rel(C_u)} \omega_e(\tau) = \frac{W(e)}{W} \sum_{\tau \in rel(C_{u_1})} \omega_{e_1}(\tau_1) + \frac{W(e)}{W} \sum_{\tau \in rel(C_{u_2})} \omega_{e_2}(\tau_2) = \frac{W(e)}{W} W(e_1) + \frac{W(e)}{W} W(e_2) = W(e).$$
Case 2: Assume that $u$ is a ×-gate. Applying the definition of $\omega_e$, we get:

$$\sum_{\tau \in \text{rel}(C_u)} \omega_e(\tau) = \sum_{\tau \in \text{rel}(C_u)} W(e) \frac{\omega_{e_1}(\tau_{\text{Attr}(C_{u_1})})}{W} \frac{\omega_{e_2}(\tau_{\text{Attr}(C_{u_2})})}{W}$$

Remember that by definition of $\{\omega, \times\}$-circuits, $\text{rel}(C_u) = \text{rel}(C_{u_1}) \times \text{rel}(C_{u_2})$. That is, $\tau \in \text{rel}(C_u)$ if and only if $\tau_1 := \tau_{\text{Attr}(C_{u_1})} \in \text{rel}(C_{u_1})$ and $\tau_2 := \tau_{\text{Attr}(C_{u_2})} \in \text{rel}(C_{u_2})$. Thus, we can rewrite the last sum as:

$$\sum_{\tau \in \text{rel}(C_u)} \omega_e(\tau) = \sum_{\tau_1 \in \text{rel}(C_{u_1})} \sum_{\tau_2 \in \text{rel}(C_{u_2})} W(e) \frac{\omega_{e_1}(\tau_1)}{W(e_1)} \frac{\omega_{e_2}(\tau_2)}{W(e_2)}$$

By taking the constant $W(e)$ out of the sum and observing that the sum is now separated into two independent terms, we have:

$$\sum_{\tau \in \text{rel}(C_u)} \omega_e(\tau) = W(e) \sum_{\tau_1 \in \text{rel}(C_{u_1})} \omega_{e_1}(\tau_1) \sum_{\tau_2 \in \text{rel}(C_{u_2})} \omega_{e_2}(\tau_2)$$

By induction, $W(e_i) = \sum_{\tau \in \text{rel}(C_{u_i})} \omega_e(\tau_i)$ for $i = 1, 2$. Hence both sums of the last term are equal to 1, which means that

$$\sum_{\tau \in \text{rel}(C_u)} \omega_e(\tau) = W(e).$$

We choose $\omega = \omega_o$, where $\omega_o$ is the output edge. Lemma 5.9 is however not enough to prove Theorem 5.2 as it only gives the equality $W(e) = \sum_{\tau \in \text{rel}(C,e)} \omega(\tau)$ for $e = o_r$. Fortunately we can show that it holds for every edge $e$ of the circuit. We actually prove a stronger property, that $\omega_e$ is, in some sense, a projection of $\omega$.

Given an edge $e = (u, v)$ of $C$ and $\tau' \in \text{rel}(C_u)$, we denote by $\text{rel}(C, e, \tau')$ the set of tuples $\tau$ of $\text{rel}(C, e)$ such that $\tau_{\text{Attr}(C_u)} = \tau'$. We prove the following:

**Lemma 5.10.** For every $e = (u, v) \in \text{Edges}(C)$, for every $\tau' \in \text{rel}(C_u)$,

$$\omega_e(\tau') = \sum_{\tau \in \text{rel}(C, e, \tau')} \omega(\tau).$$

**Proof.** The proof is by top-down induction on $C$.

**Base case:** We prove the result for $e = o_r = (u, v)$. Let $\tau' \in \text{rel}(C_u)$. Because $u$ is output gate, we have $\text{Attr}(C_u) = \text{Attr}(C)$ and hence $\text{rel}(C, e, \tau') = \{\tau'\}$. Recall that $\omega = \omega_o$, by definition. In other words, $\omega_e(\tau') = \omega(\tau') = \sum_{\tau \in \text{rel}(C, e, \tau')} \omega(\tau)$.

**Inductive case:** Now let $e = (u, v)$ be an internal edge of $C$. Let $o_1, \ldots, o_n$ be the outgoing edges of $v$, $u'$ be the only sibling of $u$ and let $e' = (u', v)$. See Figure 5 for a schema of these notations. We fix $\tau' \in \text{rel}(C_u)$ and prove the desired equality.

![Figure 5: Notations for the inductive step.](image)

**Case 1:** $v$ is a ⊕-gate. In this case, $\tau' \in \text{rel}(C_u)$. We claim that

$$\text{rel}(C, e, \tau') = \bigcup_{o \in \text{Out}(v)} \text{rel}(C, o, \tau').$$
For left-to-right inclusion, let \( \tau \in \text{rel}(C, e, \tau') \). By definition, its proof tree \( T_C(\tau) \) contains \( e \). Since \( T_C(\tau) \) is connected by Proposition 5.4, \( T_C(\tau) \) has to contain one edge of \( \Out(u) \). The disjointness of the right-side union is a direct consequence of Proposition 5.5. For the right-to-left inclusion, fix \( o \in \Out(v) \) and let \( \tau \in \text{rel}(C, o, \tau') \). By definition, its proof tree \( T_C(\tau) \) contains \( o \), thus it also contains the vertex \( v \). Now recall that \( \tau|_{\text{Attr}(C_u)} = \tau' \in \text{rel}(C_u) \). Thus, by definition of proof trees, \( u \) is also in \( T_C(\tau) \). In other words, \( \tau \in \text{rel}(C, e, \tau') \). Using this equality, we have:

\[
\sum_{\tau \in \text{rel}(C, e, \tau')} \omega(\tau) = \sum_{o \in \Out(v)} \sum_{\tau \in \text{rel}(C, o, \tau')} \omega(\tau) = \sum_{o \in \Out(v)} \omega_o(\tau').
\]

Since, by induction, we have that for every \( o \in \Out(v) \), \( \omega_o(\tau') = \sum_{\tau \in \text{rel}(C, o, \tau')} \omega(\tau) \).

Assume first that \( \sum_{o \in \Out(v)} W(o) = 0 \). In this case, by definition, for every \( o \), \( \omega_o(\tau') = 0 \). Since \( W \) is sound however, we also have \( W(e) = 0 \), which implies by Lemma 5.9 that \( \omega_e(\tau') = 0 \) as well. In this case, \( \sum_{\tau \in \text{rel}(C, e, \tau')} \omega(\tau) = 0 = \omega_e(\tau') \) which is the induction hypothesis.

Now assume that \( \sum_{o \in \Out(v)} W(o) \neq 0 \). We can thus apply the definition of \( \omega_o(\tau') = \frac{W(o)}{W(e)} \omega_e(\tau') \) in the last sum. It gives

\[
\sum_{o \in \Out(v)} \omega_o(\tau') = \sum_{o \in \Out(v)} \frac{W(o)}{W(e)} \omega_e(\tau') = \frac{1}{W(e)} \sum_{o \in \Out(v)} W(o) = \omega_e(\tau')
\]

where the last equality follows from the fact that \( W \) is sound and thus the ratio is 1.

**Case 2:** \( v \) is a \( \times \)-gate. Similarly as before, we have:

\[
\text{rel}(C, e, \tau') = \biguplus_{\tau'' \in \text{rel}(C_u)} \biguplus_{o \in \Out(v)} \text{rel}(C, o, \tau'' \times \tau''').
\]

For left-to-right inclusion, let \( \tau \in \text{rel}(C, e, \tau') \). By definition, its proof tree \( T_C(\tau) \) contains \( e \). Since \( T_C(\tau) \) is connected by Proposition 5.4, \( T_C(\tau) \) has to contain one edge of \( \Out(u) \). Thus, \( \tau \in \text{rel}(C, o, \tau'' \times \tau''') \) and by definition of proof trees, \( u \) is also in \( T_C(\tau) \). Using this equality, we have:

\[
\sum_{\tau \in \text{rel}(C, e, \tau')} \omega(\tau) = \sum_{o \in \Out(v)} \sum_{\tau'' \in \text{rel}(C_u)} \sum_{\tau''' \in \text{rel}(C, o, \tau'' \times \tau''')} \omega(\tau) = \sum_{o \in \Out(v)} \sum_{\tau'' \in \text{rel}(C_u)} \omega_o(\tau'' \times \tau''').
\]

Since, by induction, we have that for every \( o \in \Out(v) \), \( \omega_o(\tau'' \times \tau'''') = \sum_{\tau \in \text{rel}(C, o, \tau'' \times \tau''')} \omega(\tau) \).

Assume first that \( \sum_{o \in \Out(v)} W(o) = 0 \). In this case, by definition, for every \( o \) and \( \tau''' \), \( \omega_o(\tau'' \times \tau'''') = 0 \). Since \( W \) is sound however, we also have \( W(e) = 0 \), which implies by Lemma 5.9 that \( \omega_e(\tau') = 0 \) as well.

In this case, \( \sum_{\tau \in \text{rel}(C, e, \tau')} \omega(\tau) = 0 = \omega_e(\tau') \) which is the induction hypothesis.

Now assume that \( \sum_{o \in \Out(v)} W(o) \neq 0 \). We can thus apply the definition of \( \omega_o(\tau'' \times \tau''') = \frac{W(o)}{W(e)} \frac{W(e)}{W(e)} \omega_e(\tau') \)
Thus, the last sum equals to $\omega$ in the last sum. It gives

$$\sum_{o \in \text{Out}(v)} \sum_{\tau'' \in \text{rel}(C_u)} \omega_v(\tau' \times \tau'') = \sum_{o \in \text{Out}(v)} \sum_{\tau'' \in \text{rel}(C_u)} W(o) \frac{\omega_v(\tau')}{W(e)} \frac{\omega_v(\tau'')}{W(e')} = \frac{\omega_v(\tau')}{W(e)} \left( \sum_{o \in \text{Out}(v)} W(o) \sum_{\tau'' \in \text{rel}(C_u)} \omega_v(\tau'') \right) \frac{\omega_v(\tau'')}{W(e')}.$$

Since $W$ is sound, $\sum_{o \in \text{Out}(v)} W(o) = W(e)$. Moreover, by Lemma 5.9 $\sum_{\tau'' \in \text{rel}(C_u)} \omega_v(\tau'') = W(e')$. Thus, the last sum equals to $\omega_v(\tau')$ which concludes the proof.

Theorem 5.2 (b) is now an easy consequence of Lemma 5.9 and Lemma 5.10:

$$W(e) = \sum_{\tau' \in \text{rel}(C_u)} \omega_v(\tau') \quad \text{(by Lemma 5.9)}$$
$$= \sum_{\tau' \in \text{rel}(C_u)} \sum_{\tau \in \text{rel}(C,e,\tau')} \omega(\tau) \quad \text{(by Lemma 5.10)}$$
$$= \sum_{\tau \in \text{rel}(C,e)} \omega(\tau) \quad \text{(since rel}(C,e) = \bigcup_{\tau' \in \text{rel}(C_u)} \text{rel}(C,e,\tau')).$$

Indeed, $\text{rel}(C,e) = \bigcup_{\tau' \in \text{rel}(C_u)} \text{rel}(C,e,\tau')$. For the left-to-right inclusion, if $\tau \in \text{rel}(C,e)$ then by definition, $\tau' = r_{\text{Attr}(C_u)} \in \text{rel}(C_u)$ and thus $\tau \in \text{rel}(C,e,\tau')$. The other inclusion follows by definition since $\text{rel}(C,e,\tau') \subseteq \text{rel}(C,e)$ for every $\tau' \in \text{rel}(C_u)$.

6 Applications

6.1 Solving the DWC problem

We are now ready to explain the polynomial time algorithm to compute $\text{DWC}(\text{rel}(C))$ of a $\{\text{\#}, \times\}$-circuit $C$ given in the input. It actually boils down to solving a linear program whose size is linear in the size of $C$ and whose optimal value is $\text{DWC}(\text{rel}(C))$.

In this section, we fix a $\{\text{\#}, \times\}$-circuit $C$. For $x \in \text{Attr}(C)$ and $d \in \mathbb{D}$, we denote by $u_{x/d}$ the only input of $C$ labeled with $x/d$. We assume wlog that the root $r$ of $C$ has a single ingoing edge which we call the output edge $o_r$. We define $C\text{-DWC-LP}(C)$ to be the following linear program whose variables are the edges of $C$:

$$\begin{align*}
\text{maximize } & e_r \\
\text{subject to } & \bigwedge_{e \in \text{Edges}(C)} e \geq 0 \land \\
& \bigwedge_{i \in \text{gates}(C)} \text{Out}(u) = \text{ln}(u) \land \\
& \bigwedge_{x \in \text{gates}(C)} \bigwedge_{i \in \text{ln}(u)} \text{Out}(u) = i \land \\
& \bigwedge_{x \in \text{K}} \bigwedge_{d \in \mathbb{D}} \text{Out}(u_{x/d}) \leq 1
\end{align*}$$

The first three lines of $C\text{-DWC-LP}(C)$ ensure that every solution $W : \text{Edges}(C) \rightarrow \mathbb{R}$ is a sound edge weighting of $C$. The last line ensures that the corresponding tuple weighing $\omega$ given by Theorem 5.2 from $W$ is a dependency weighting, that is, it realizes the necessary linear program associated to the DWC problem.

We now prove these observations formally by proving that $C\text{-DWC-LP}(C)$ and $\text{DWC-LP}(\text{rel}(C))$ have the same optimal value. We denote the optimal value of $C\text{-DWC-LP}(C)$. 

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Lemma 6.1. DWC(rel(C)) \leq C-DWC(C)

Proof. Let S = rel(C) and let \( \omega \) be a tuple weighting that realizes the optimum of DWC-LP(S). Let W be the edge-weighting induced by \( \omega \). We claim that W is a solution of C-DWC-LP(C).

Since W is sound by Theorem 5.2, it is clear that it satisfies the first three lines of C-DWC-LP(C). Now let \( x \in \text{Attr}(C) \) and \( d \in D \). We check that the last constraints is satisfied, that is, \( \sum_{e \in \text{Out}(u_{x,d})} W(e) \leq 1 \). By Theorem 6.1 for \( e \in \text{Out}(u_{x,d}) \), \( W(e) = \sum_{\tau \in \text{rel}(C,e)} \omega(\tau) \).

Thus

\[
\sum_{e \in \text{Out}(u_{x,d})} W(e) = \sum_{e \in \text{Out}(u_{x,d})} \sum_{\tau \in \text{rel}(C,e)} \omega(\tau) = \sum_{\tau \in \text{rel}(C)} \omega(\tau) \text{ by Proposition 5.6} \leq 1
\]

since \( \omega \) is a solution of DWC(S). Thus W is a solution of C-DWC-LP(C). Moreover, its objective value is \( W(e_r) \) which is \( \sum_{\tau \in \text{rel}(C,e_r)} \omega(\tau) \) by Theorem 5.2. Since \( \text{rel}(C,e_r) = \text{rel}(C) \), we have that \( W(e_r) = \sum_{\tau \in \text{rel}(C)} \omega(\tau) = DWC(\text{rel}(C)) \). The optimal value of C-DWC-LP(C) is thus greater than \( W(e_r) \) as W is a solution of C-DWC-LP(C). In other words, C-DWC(C) \( \geq W(e_r) = DWC(\text{rel}(C)) \).

\[\square\]

Lemma 6.2. DWC(C) \( \geq C\text{-DWC}(C) \)

Proof. The proof follows the same schema as Lemma 6.1. Let W be an optimal solution of C-DWC-LP(C). As observed before, W is a sound edge weighting of C. Let \( \omega \) be the corresponding tuple weighting given by Theorem 5.2. We claim that \( \omega \) is a solution of DWC-LP(\text{rel}(C)). Indeed, for the same reasons as before, for every \( x \in \text{Attr}(C) \) and \( d \in D \), \( \sum_{e \in \text{Out}(u_{x,d})} W(e) = \sum_{\tau \in \text{rel}(C)} \omega(\tau) \), since W is a solution of C-DWC-LP(C). Thus, \( \omega \) is a solution of DWC-LP(\text{rel}(C)). As before again,

\[\text{C-DWC}(C) = W(e_r) = \sum_{\tau \in \text{rel}(C)} \omega(\tau) \leq \text{DWC(\text{rel}(C))}.\]

\[\square\]

It follows from Lemma 6.1 and Lemma 6.2 that DWC(C) = C-DWC(C) so DWC can be solved in polynomial time by running an LP-solver on C-DWC-LP.

6.2 Generalized DWC

Our definition of DWC assumes that two tuples are not independent whenever they have the same value on some attribute. This notion of dependency is too strong in practice since we may not consider every attribute to generate a dependency. For example, if one tries to compute statistics on the average height of people in a country, it is relevant to consider two people in the same family as a factor of dependency while the type of music they like to listen to seems less relevant.

Given a set of tuples S on attribute X and domain D and \( Y \subseteq X \), denote by DWC-LP(S,Y) the linear program defined as DWC-LP(S) but where we only keep the constraints \( \sum_{y/d} S_{y/d} \leq 1 \) for \( y \in Y \), that is, we only consider the variables of Y to generate dependencies between tuples. Let DWC(S,Y) be the optimal value of DWC-LP(S,Y). Observe that DWC-LP(S,X) = DWC-LP(S). Similarly, for a \( \{\omega, x\} \)-circuit \( C \), denote by C-DWC-LP(C,Y) the linear program defined as C-DWC-LP(C) but where we only keep the constraints Out\( (u_{y/d}) \leq 1 \) for \( y \in Y \) and let C-DWC(C,Y) be its optimal value. The same proof as the previous section gives:

Proposition 6.3. For every \( \{\omega, x\} \)-circuit C on attribute X and Y \( \subseteq X \), DWC(\text{rel}(C),Y) = C-DWC(C,Y).

In particular, one can compute DWC(\text{rel}(C),Y) in polynomial time in the size of C.

This observation actually allows us to solve DWC even when we have existentially projected variables. Given a set of tuples S on attributes X and domain D and \( Z \subseteq X \), denote by \( IIZ.S = \{\tau_{|X \setminus Z} \mid \tau \in S\} \). For the DWC problem, existentially projecting variables Z is the same as ignoring them in the dependencies relation:

Lemma 6.4. DWC(IIZ.S) = DWC(S,X \setminus Z).

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Proof. Given $\tau' \in \Sigma.S$, we denote by

$$\text{Ext}(\tau') = \{\tau'' : Z \to R_+ \mid \tau' \times \tau'' \in S\}.$$ 

Let $\omega' : \Sigma.S \to R_+$ be an optimal solution of DWC-LP$(\Sigma.S)$. We define $\omega : S \to R_+$ as for every $\tau \in S$, $\omega(\tau) := \frac{\omega'(\tau \times x) \# \text{Ext}(\tau \times x)}{\# \text{Ext}(\tau')}$. We claim that $\omega$ is a solution of DWC-LP$(S, X \setminus Z)$. Indeed, let $x \in X \setminus Z$ and $d \in D$, we have:

$$\sum_{\tau \in S_x \setminus d} \omega(\tau) = \sum_{\tau' \in \Sigma.S} \sum_{\tau'' \in \text{Ext}(\tau')} \omega(\tau' \times \tau'')$$

$$= \sum_{\tau' \in \Sigma.S} \sum_{\tau'' \in \text{Ext}(\tau')} \omega'(\tau') \frac{\# \text{Ext}(\tau')}{\# \text{Ext}(\tau')} \quad \text{(by definition of } \omega)$$

$$= \sum_{\tau' \in \Sigma.S} \omega'(\tau')$$

$$\leq 1 \quad \text{(since } \omega' \text{ is a solution of DWC(}\Sigma.S)) \).$$

Thus $\omega$ is a solution of DWC-LP$(S, X \setminus Z)$. We can also similarly prove that $\sum_{\tau \in S} \omega(\tau) = \sum_{\tau' \in \Sigma.S} \omega'(\tau)$. Thus, DWC$(S, X \setminus Z) \geq \text{DWC(}\Sigma.S)$. 

Now let $\omega : S \to R_+$ be an optimal solution of DWC-LP$(S, Z)$. We define $\omega' : \Sigma.S \to R_+$ as for every $\tau' \in \Sigma.S$, $\omega'(\tau') = \sum_{\tau'' \in \text{Ext}(\tau')} \omega(\tau' \times \tau'')$. A similar calculation as before gives that $\omega'$ is a solution of DWC-LP$(S, Z)$ and that DWC$(\Sigma.S) \geq \text{DWC}(S, X \setminus Z)$. \qed

As a corollary, we have that DWC is tractable on bounded fractional hypertree width conjunctive queries, even in the presence of existential quantification as we can compute the circuit for the conjunctive query without quantifiers and then solve the relaxed DWC problem on the unquantified attributes.

Another possible generalization is to consider DWC-like problems where the weighting is the optimal solution of a linear program whose variables are the $S_{x/d}$ (more formally this means that the constraints and objective function of these linear programs are in $T_S$ where $S = \{S_{x/d} \mid x \in X, d \in D\}$). Some interesting features of these programs are the ability to write expressions involving multiple attributes (e.g., $S_{age/42} + S_{city/Lille}$) or to multiply some parts of the expression by a real number (e.g., $1.5 \cdot S_{city/Amsterdam} + S_{city/Lille}$).

Remark that using Theorem 5.2 it is actually possible to handle expressions in $T_R$ where $R = \{\text{rel}(C,e) \mid e \in \text{Edges}(C)\}$. However it is difficult to give a meaning to most of the variables in $R$ in practice.

6.3 Dependency weighted average

Observe that Theorem 5.2 gives a way of encoding any tuple weighting as a sound edge weighting. It is interesting to observe that one can use this encoding to perform other aggregation tasks using this tuple weighting, that can then be applied to dependency weightings.

For example, we can imagine that the tuples of a database relation $S$ are ratings given by users to a movie. In this case, after having computed a dependency tuple weighting $\omega$ on $S$, one may also now be interested in the average rating of the movie but using a weighted average taking $\omega$ into account. In other words, we are interested in the dependency weighted average $\text{DWA}(S, \omega, p) := \frac{\sum_{\tau \in S} \omega(\tau)p(\tau)}{\sum_{\tau \in S} \omega(\tau)}$ where $p(\tau)$ is the rating contained in $\tau$. It turns out that if $\omega$ is a dependency tuple weighting, $\text{DWA}$ has interesting statistical properties \cite{WRF13} concerning the expected value of $p$.

In this section, we assume that $p$ is always given as follows. A value function $p$ on attributes $X$ and domain $D$ is a set of function $(p_x)_{x \in X}$ where for each attribute $x \in X$, $p_x : D \to R$. The value $p(\tau)$ of a tuple $\tau$ on attributes $X$ is then given as $p(\tau) := \prod_{x \in X} p_x(\tau(x))$. Depending on the context, $p$ may be some numerical value extracted from $\tau$, such as the ratings of a movie, or information on the provenance of $\tau$ such as the probability of $\tau$ being in the answer set in the context of probabilistic databases.

We prove the following:

**Proposition 6.5.** Let $C$ be a $\{\omega, \times\}$-circuit and $W$ a sound edge weighting of $C$. Let $\omega$ be the tuple weighting of $\text{rel}(C)$ so that $W$ is induced by $\omega$. One can compute $\sum_{\tau \in \text{rel}(C)} \omega(\tau)p(\tau)$ in time polynomial in the size of $C$.
The algorithm works by a bottom-up induction. For every edge \( e = \langle u, v \rangle \) of \( C \), we inductively compute a value \( M(e) \in \mathbb{R} \) as follows:

- If \( u \) is an input labeled with \( x/d \) then \( M(e) := p_x(d) \cdot W(e) \).
- If \( u \) is a \( \omega \)-gate with children \( u_1, u_2 \). Let \( e_1 = \langle u_1, u \rangle \) and \( e_2 = \langle u_2, u \rangle \).
  - If \( W(e_1) \neq 0 \) and \( W(e_2) \neq 0 \), we define \( M(e) := \frac{W(e)}{W(e_1) + W(e_2)} \cdot (M(e_1) + M(e_2)) \).
  - Otherwise \( M(e) := 0 \).
- If \( u \) is a \( \times \)-gate with children \( u_1, u_2 \). Let \( e_1 = \langle u_1, u \rangle \) and \( e_2 = \langle u_2, u \rangle \).
  - If \( \sum_{o \in \text{Out}(u)} W(o) \neq 0 \), then \( M(e) := \frac{W_o(e_1)}{W(e_1)} \cdot \frac{W_o(e_2)}{W(e_2)} \cdot W(e) \) (Observe that because \( W \) is sound, \( W(e_1) = W(e_2) = \sum_{o \in \text{Out}(u)} W(o) \)).
  - Otherwise \( M(e) := 0 \).

It turns out that if \( o_e \) is the output edge, \( M(o_e) = \sum_{\tau \in \text{rel}(C)} \omega(\tau) p(\tau) \).

We start by introducing another tuple-weighting \( \mu_e \) for each edge \( e = \langle u, v \rangle \in \text{Edges}(C) \). This weighting is defined similarly to \( \omega_e \) unless \( u \) is an input labelled with \( x/d \) in which case \( \mu_e(\tau) := W(e) \cdot p(x/d) \).

**Lemma 6.6.** For any edge \( e = \langle u, v \rangle \in \text{Edges}(C) \), for any \( \tau \in \text{rel}(C_u) \), \( \mu_e(\tau) = \omega_e(\tau) \cdot p(\tau) \).

*Proof.* Let \( e = \langle u, v \rangle \) be an edge of \( C \). Observe that when \( \sum_{o \in \text{Out}(u)} W(o) = 0 \), then by definition for every \( \tau \in \text{rel}(C_u) \), \( \omega_e(\tau) = 0 \) and \( \mu_e(\tau) \) thus \( \mu_e(\tau) = 0 = \omega_e(\tau) \cdot p(\tau) \). In this case then, the lemma holds.

In the rest of the proof, we now assume that \( \sum_{o \in \text{Out}(u)} W(o) \neq 0 \). We show the lemma by induction on the nodes of \( C \) from the leaves to the root.

**Base case:** \( u \) is a leaf labelled with \( x/d \). Let \( e \) be an outgoing edge of \( u \). Observe that \( \text{rel}(C_u) \) contains a single tuple \( x/d \) and that, by definition of \( \mu_e \) and \( \omega_e \), \( \mu_e(x/d) = W(e) p(\tau) = \omega_e(x/d) p(\tau) \).

**Inductive case:** Now let \( u \) be an internal gate of \( C \) with children \( u_1, u_2 \) and let \( e_1 = \langle u_1, u \rangle \) and \( e_2 = \langle u_2, u \rangle \), as depicted in Figure 4. We fix \( \tau \in \text{rel}(C_u) \).

**Case 1:** Assume that \( u \) is a \( \omega \)-gate. Let \( W := W(e_1) + W(e_2) \). Since \( u \) is disjoint, given \( \tau \in \text{rel}(C_u) \), either \( \tau \in \text{rel}(C_{u_1}) \) or \( \tau \in \text{rel}(C_{u_2}) \) but not both. We assume wlog that \( \tau \in \text{rel}(C_{u_1}) \).

\[
\mu_e(\tau) = \frac{W(e)}{W} \mu_{e_1}(\tau) \quad \text{(by definition of } \mu_e) \\
= \frac{W(e)}{W} \omega_{e_1}(\tau) p(\tau) \quad \text{(by induction)} \\
= \omega_e(\tau) p(\tau) \quad \text{(by definition of } \omega_e) 
\]

**Case 2:** Assume that \( u \) is a \( \times \)-gate. Let \( \tau_1 \in \text{rel}(C_{u_1}) \) and \( \tau_2 \in \text{rel}(C_{u_2}) \) such that \( \tau = \tau_1 \tau_2 \).

\[
\mu_e(\tau) = W(e) \frac{\mu_{\tau_1}(\tau_1) \mu_{\tau_2}(\tau_2)}{W} \quad \text{(by definition of } \mu_e) \\
= W(e) \frac{\omega_{\tau_1}(\tau_1) p(\tau_1) \omega_{\tau_2}(\tau_2) p(\tau_2)}{W} \quad \text{(by induction)} \\
= W(e) \frac{\omega_{\tau_1}(\tau_1) \omega_{\tau_2}(\tau_2) p(\tau_1) p(\tau_2)}{W} \quad \text{(by def. of } \omega_e \text{ and } p) 
\]

There is a natural relation between \( W_p \) and \( \mu \) that can be proven by a bottom up induction:
Lemma 6.7. For any edge $e = \langle u, v \rangle$ of $C$, $W_p(e) = \sum_{\tau \in rel(C)} \mu_e(\tau)$

Proof. Let $e = \langle u, v \rangle$ be an edge of $C$. Observe that when $\sum_{\tau \in rel(C)} W(e) = 0$, then by definition of $\omega_e$, for every $\tau \in rel(C)$, $\omega_e(\tau) = 0$ thus $\mu_e(\tau) = 0$ by Lemma 6.6. In particular, $\sum_{\tau \in rel(C)} \mu_e(\tau) = 0 = W_p(e)$. In this case then, the lemma holds.

In the rest of the proof, we now assume that $\sum_{\tau \in rel(C)} W(e) \neq 0$. We show the lemma by induction on the nodes of $C$ from the leaves to the root.

Base case: $u$ is a leaf labelled with $x/d$. Let $e$ be an outgoing edge of $u$. Observe that $rel(C_u)$ contains a single tuple $x/d$ and that, by definition of $\omega_e$, $W_p(e) = \omega_e(x/d)$, $p(\tau) = \mu_e(\tau) = \sum_{\tau \in rel(C_u)} \mu_e(\tau)$.

Inductive case: Now let $u$ be an internal gate of $C$ with children $u_1, u_2$ and let $e_1 = \langle u_1, u \rangle$ and $e_2 = \langle u_2, u \rangle$, as depicted in Figure 4.

Case 1: Assume that $u$ is a $\odot$-gate. Let $W = W(e_1) + W(e_2)$. Since $u$ is disjoint, given $\tau \in rel(C_u)$, either $\tau \in rel(C_{u_1})$ or $\tau \in rel(C_{u_2})$ but not both.

It follows:

\[
\sum_{\tau \in rel(C_u)} \mu_e(\tau) = \sum_{\tau \in rel(C_{u_1})} \mu_e(\tau) + \sum_{\tau \in rel(C_{u_2})} \mu_e(\tau)
= \sum_{\tau \in rel(C_{u_1})} \frac{W(e)}{W} \mu_e(\tau_1) + \sum_{\tau \in rel(C_{u_2})} \frac{W(e)}{W} \mu_e(\tau_2)
\]

by definition of $\mu_e$. Observe that by induction we have $W_p(e_1) = \sum_{\tau \in rel(C_{u_1})} \mu_e(\tau_1)$ and $W_p(e_2) = \sum_{\tau \in rel(C_{u_2})} \mu_e(\tau_2)$.

Thus, by taking the constants out and using this identity, it follows:

\[
\sum_{\tau \in rel(C_u)} \mu_e(\tau) = \frac{W(e)}{W} W_p(e_1) + \frac{W(e)}{W} W_p(e_2)
= \frac{W(e)}{W} (W_p(e_1) + W_p(e_2))
= W_p(e).
\]

Case 2: Assume that $u$ is a $\times$-gate. Applying the definition of $\mu_e$, we get:

\[
\sum_{\tau \in rel(C_u)} \mu_e(\tau) = \sum_{\tau \in rel(C_{u_1})} W(e) \frac{\mu_e(\tau_1)}{W} + \sum_{\tau \in rel(C_{u_2})} W(e) \frac{\mu_e(\tau_2)}{W}
\]

Remember that by definition of $\{\odot, \times\}$-circuits, $rel(C_u) = rel(C_{u_1}) \times rel(C_{u_2})$. That is, $\tau \in rel(C_u)$ if and only if $\tau_1 = \tau_{\odot}(C_{u_1}) \in rel(C_{u_1})$ and $\tau_2 := \tau_{\odot}(C_{u_2}) \in rel(C_{u_2})$. Thus, we can rewrite the last sum as:

\[
\sum_{\tau \in rel(C_u)} \mu_e(\tau) = \sum_{\tau_1 \in rel(C_{u_1})} \sum_{\tau_2 \in rel(C_{u_2})} W(e) \frac{\mu_e(\tau_1)}{W} \frac{\mu_e(\tau_2)}{W}
\]

By taking the constant $W(e)$ out of the sum and observing that the sum is now separated into two independent terms, we have:

\[
\sum_{\tau \in rel(C_u)} \mu_e(\tau) = W(e) \sum_{\tau_1 \in rel(C_{u_1})} \frac{\mu_e(\tau_1)}{W} \sum_{\tau_2 \in rel(C_{u_2})} \frac{\mu_e(\tau_2)}{W}
\]

By induction, $W(e_i) = \sum_{\tau \in rel(C_u)} \mu_e(\tau_i)$ for $i = 1, 2$. Hence

\[
\sum_{\tau \in rel(C_u)} \omega_e(\tau) = W(e) \frac{W_p(e_1)}{W} \frac{W_p(e_2)}{W} = W_p(e).
\]

\[\square\]
The correctness of the algorithm (and of Proposition 6.5) immediately follows from Lemma 6.7 and Lemma 6.6 applied to $e = o_r$, the output edge.

7 Conclusion

In this paper, we have studied aggregations problem taking into account dependencies between tuples. While these problems are intractable for conjunctive queries, we provide an algorithm that runs in polynomial time when the fractional hypertree width of the conjunctive query is bounded. Our technique relies on factorized representation of the answer set of the conjunctive query.

There are many possible future work concerning dependency weighted aggregation. A first interesting step would be to compute DWC in practice for some real life queries on which one wants to perform a statistical analysis. We would like to try to run the algorithm on benchmarks from data mining and see how they compare with different approaches. We already tried a basic implementation of our algorithm on syntactical data to see if the linear program solver scales but did not yet consider any real benchmark.

On a more theoretical side, we would like to explore other notions of dependencies. In this work, we consider two tuples to be dependent if they have the same value on some attribute. It may be interesting for example, to study the DWC problem where we consider two tuples to be dependent if they have the same value on two or more attributes, or if their value on some attribute are close, for some given distance. These generalizations are likely to be much harder than the case we study in this paper.

Another more concrete extension of this work could be to solve DWC without calling an external linear program solver or to see if one can generate a better linear program with a known structure that these solvers could exploit to solve the instance more quickly, or even in linear time depending on the size of the database, rather than polynomial time.

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