Wave delocalization and transfer efficiency enhancement in locally symmetric disordered chains

C. V. Morfonios, M. Röntgen, F. K. Diakonos, and P. Schmelcher

1Zentrum für Optische Quantentechnologien, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
2Department of Physics, University of Athens, Panepistimiopolis, 15771 Athens, Greece
3The Hamburg Centre for Ultrafast Imaging, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
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The impact of local reflection symmetry on wave localization and transport within finite disordered chains is investigated. Local symmetries thereby play the role of a spatial correlation of variable range. We find that a transition to more delocalized and fragmented eigenstates occurs for intermediate average symmetry domain sizes, depending on the degree of disorder. This is caused by the partial formation of states with approximate local parity confined within fictitious, disorder-induced double wells and perturbed by the coupling to adjacent domains. The dynamical evolution of wave-packets shows that the average site-resolved transfer efficiency is enhanced between regions connected by local symmetry. The transfer may further be drastically amplified in the presence of spatial overlap between the symmetry domains, and in particular when global and local symmetry coexist. Applicable to generic discrete models for matter and light waves, our work provides a perspective to understand and exploit the impact of local order at multiple scales in complex systems.

I. INTRODUCTION

Since the pioneering theoretical work by Anderson it has been shown that, under many circumstances, spatial disorder in a medium leads to the exponential localization of waves due to multiple destructive interference. This behavior in turn suppresses the transport of an initial wave excitation through a disordered sample between remote sites. Initially explored for electrons in models of solids, disorder-induced localization has more recently also been demonstrated and intensively studied for light waves as well as for matter waves in optical lattices. Typically manifest in one-dimensional (1d) discrete lattice models with random onsite potential (diagonal disorder) or inter-site hoppings (off-diagonal disorder), localization also occurs for structural disorder in systems with binary constituents and may further be induced in the bulk of setups with disordered boundary.

The presence of spatial correlations between the constituents of an otherwise disordered medium generally causes a delocalization of wave excitations and enhanced transport, with the detailed system response depending on the type of correlated disorder. Correlation can be short-ranged, in the form of ordered clustered elements such as dimers, trimers, or polymers, it can have long-range character, while also mixed short- and long-range correlations as well as subsystem disorder have been explored. Further, delocalization may be facilitated by correlations between onsite elements alone, between hoppings, or between onsite and hopping elements.

In the meanwhile vast literature on wave localization, the impact of spatial symmetry has largely been used on the small scale of lattice constituents in resonant conditions of transport. On the other hand, a series of recent studies shows that the presence of global centrosymmetry—equivalent to reflection symmetry for 1d systems—in an otherwise disordered system may lead to a prevalence of delocalized states and thereby enhance wave transfer between symmetry related sites. This principle applies to general multi-dimensional networks of interconnected nodes, even in the presence of many-body interactions. The effect is closely connected to the definite parity of the system eigenmodes which may result in a corresponding delocalization on configuration average. Exploiting this property, global reflection symmetry has also been proposed as a generator of tunneling in 1d disordered potentials with applications to secure communication in classical circuits. The behavior of such globally symmetric systems in terms of optimal transfer efficiency is subject to further design conditions, but generally demonstrates the crucial role of symmetry coexisting with disorder.

In fact, global symmetry is seldom exactly fulfilled. Continuous symmetry measures and symmetry operation measures have been proposed to describe deviation from exact symmetry. A different paradigm of global symmetry breaking is the case of exact but local symmetries, that is, symmetries which are fulfilled in a restricted subdomain of a composite system. A recently developed theoretical framework addresses such local symmetries in terms of symmetry-adapted non-local currents governed by generalized non-local continuity equations. Their stationary versions reveal the presence of 1d local symmetries in generic wave-mechanical systems including non-Hermitian or even driver setups. In particular, they enable amplitude mappings which generalize the parity and Bloch theorems to the case of local symmetry and can be used to classify perfectly transmitting states. A well-known class of systems featuring abundant local symmetries is that of 1d binary deterministic aperiodic structures (see e.g. Ref. and references therein), where their combinatorial properties have been studied in terms of the so-called “palindrome complexity”, where their presence may affect order-disorder transitions, or exist “concurrently” in interplay with global symmetries within molecules. Ultimately, any system with global symmetry which is coupled to an environment can be considered locally symmetric. In view of the manifest role of global symmetry, this abundance of local symmetries raises the question of their impact on localization and transfer when multiply present at different locations and scales in disordered systems.
We here view local reflection symmetry, defined within different spatial subparts of a system, as a particular type of correlation of fixed or variable range in an otherwise disordered system. To study the effect solely of local symmetry in a simple setting, we consider finite 1d tight-binding chains with disordered onsite potentials which are mirror-symmetrized within adjacent or overlapping spatial domains of random or uniform size. The localization properties of the eigenstates of such locally reflection-symmetric disordered (LRD) chains are then studied numerically for varying disorder strength and symmetry domain sizes (for brevity, from now on “symmetry” will refer exclusively to reflection symmetry, unless otherwise stated). Apart from the widely used inverse participation ratio \( IPR \), we utilize a recently proposed \( \nu \) alternative localization measure which reflects the fragmentation of states induced by the local symmetries. An intricate interplay between the short- and long-range localization and fragmentation properties is observed. It indicates an overall transition to delocalized states, within the ensemble average, for increasing degree of local symmetry, with the uncorrelated case recovered in the limit of small symmetry domains. This behavior is analyzed by combining the notion of fictitious disorder-induced tunneling barriers with the concept of symmetrization of eigenstates into symmetry domains, in turn explained within a local resonant scattering picture. A crucial ingredient is here the concept of approximate local parity \( w, w \) of localized eigenstates within symmetry domains perturbed by the coupling to adjacent domains. Further, we explore the impact of local symmetry on the diffusion of time-evolved wave-packets, by computing statistical distributions of the site-resolved transfer efficiency upon a single site excitation in LRD chains with few domains. We here show that local symmetry may significantly enhance the transfer depending on the symmetry domain positions. A drastic increase in transfer is shown to occur when symmetry domains overlap with each other. In particular, the transfer enhancement induced by global symmetry can be further increased considerably when local symmetry is present simultaneously at smaller scales.

The paper is organized as follows. In Sec. II we first define the considered LRD chain setups and provide the analysis tools used to distinguish localization from fragmentation (Sec. II A). We then classify the types of eigenstate profiles present in the LRD chains (Sec. II B) which are employed to explain the distribution of the computed localization measures for varying disorder and symmetry (Sec. II C). In Sec. III we investigate wave-packet dynamics in LRD chains, demonstrating the enhancement of transfer efficiency via local symmetry (Sec. III A) and its further increase in the presence of symmetry domain overlaps (Sec. III B). Section IV concludes our investigations. Appendix A explains the typical “symmetrization” of eigenstates into symmetry domains and Appendix B provides a mapping of such eigenstates to “fictitious” double wells.

**FIG. 1.** Top: Locally reflection-symmetric disordered (LRD) chain of \( N = 500 \) sites for disorder strength \( w = 3h \), with initially random onsite elements \( v_n \in [-w, w] \) symmetrized around local symmetry centers \( c_d=1,2,\ldots,D \) (vertical dashed lines) of \( D = 10 \) domains attached at random interfaces (vertical solid lines), and corresponding Hamiltonian eigenvalues \( \epsilon_n \), for uniform hopping \( h = 0.01 \). Bottom: Inverse participation ratio (IPR) \( r_\nu \) and cumulative Friedel phase (CFS) \( \nu \) for increasing eigenstate index \( \nu \in [1, N] \). The purple (blue) circle indicates the eigenmode of minimal \( r_\nu \) \( f_\nu \), with correspondingly colored modulus \( |\phi_\nu^c| \) plotted in the top panel.
In the following we investigate the localization properties of the eigenvectors $|\phi^\nu\rangle = \sum_n \phi^\nu_m |n\rangle$ of $H$, given by

$$ H |\phi^\nu\rangle = \epsilon^\nu |\phi^\nu\rangle, \tag{3} $$

with eigenvalues $\epsilon^\nu$. The spatial profiles of the squared eigenmode norms $\rho^\nu_n = |\phi^\nu_n|^2$ are unaffected by the sign of the hopping $h$ which induces a relative $\pi/2$ phase flip between adjacent sites. We have here chosen $h > 0$, modeling e.g. the evanescent coupling between photonic waveguides, while the choice $h < 0$ would correspond to e.g. the kinetic energy of non-interacting electrons on a tight-binding lattice. We set $h = 0.01$ as the energy unit throughout.

### A. Localization versus fragmentation of states

A convenient and widely used single-parameter indicator of the grade of localization of a wavefunction is the inverse participation ratio (IPR) defined by \(^79,80\)

$$ r = \sum_{n=1}^{N} \rho^2_n \in [N^{-1}, 1] \tag{4} $$

for a normalized state $|\psi\rangle$ of squared modulus $\rho_n = |\psi_n|^2$ (with $\sum_{n=1}^{N} \rho_n = 1$). The IPR takes on its maximal value $r = 1$ in the limit of a state localized on a single site $m$, $\rho_m = \delta_{m,n}$, and its minimal value $r = 1/N$ for a state uniformly extended over the chain, $\rho_n = 1/N$. As desired for a localization measure, the IPR does not depend on the position at which a state is localized within a disordered system. At the same time, however, it is also largely insensitive to the spatial state profiles \(^81\), which in general do affect the static properties and dynamical response of the system. An alternative localization measure, proposed very recently \(^81\) and defined here as

$$ f = \frac{1}{N} \left| \sum_{n=1}^{N} e^{2 \pi i P_n} \right| \in [0, 1] \tag{5} $$

with

$$ P_n = \sum_{m=1}^{n} \rho_m, \tag{6} $$

reflects more details of the spatial profile $\rho_m$ via its cumulative sum $P_n$ up to site $n$ within Eq. (5). As noted in Ref. 81, Eq. (5) is inspired by the Friedel sum rule \(^83-85\), and we will here refer to $f$ as the “cumulative Friedel sum” (CFS) of a given state. Again, larger (smaller) CFS indicates a more (less) localized state, though now taking into account its total spatial extent instead of only its site participation, as described in the following.

The IPR and CFS distributions among the eigenmodes of a single LRD chain configuration are shown in the bottom panel of Fig. 1. An impression of the difference between IPR and CFS in indicating localization properties is provided by the eigenstates $\phi_{\nu: \text{min}\ r}$ and $\phi_{\nu: \text{min}\ f}$ in Fig. 1 having minimal $r$ and $f$, respectively, for the example setup. With a similar density contribution (comparable amplitudes at similar number of sites), the states have almost the same IPR, which thus does not distinguish them. In contrast, the drastically smaller CFS of $\phi_{\nu: \text{min}\ f}$ indicates its delocalized profile: The envelope consists of two individual maxima which are more peaked than in $\phi_{\nu: \text{min}\ r}$, but lie further apart, thus yielding an increased total extent. For (normalized) states with local maxima, the CFS can thus be seen to indicate the degree of spatial “fragmentation”, that is, how remote from each other the amplitude maxima are located. As an example, if we consider a (virtual) normalized state consisting of two single-site peaks at spacing $s$ and zero elsewhere in an $N$-site chain, then the CFS decreases monotonously from $f = 1$ at $s = 0$ (one single-site peak) to $f = 0$ at $s = N/2$ \(^86\). If each of the two peaks has a symmetric profile of common finite width (in the form of, e.g., a Gaussian or a rectangular step), then $f$ is independent of this width. Thus, the CFS complements the IPR, as a localization measure which is sensitive to the spacing of peaks in a wavefunction but relatively insensitive to the width of the peaks themselves (except for single peaks, that is, of non-frAGMENTED states). Other cases of states characterized by the CFS are studied in Ref. 81.

Before we present the statistical behavior of the IPR and CFS in Sec. II C, we next provide an intuitive interpretational tool where the symmetry domains effectively behave like double wells perturbed by the coupling to adjacent domains.

### B. Eigenstate symmetrization

The qualitative distinction between the IPR and CFS in the present context of LRD chains is closely linked to the fact that, as explained in Appendix A, the eigenstates generally tend to “symmetrize” into the symmetric chain domains. By this we mean that, for a sufficiently localized eigenvector $|\phi^\nu\rangle$ in a LRD finite chain, the density will have the approximate symmetry $\rho_n \approx \rho_{N-n}$ about the center $c_D$ of some domain $D_D \ni n$, while approximately vanishing outside of it, $\rho_{n \notin D_D} \approx 0$. Examples of this are states $\phi_{\nu: \text{min}\ r}$ and $\phi_{\nu: \text{min}\ f}$ already seen in Fig. 1. In other words, the states tend to become approximate local parity eigenstates \(^65\) of local reflections $\mathcal{P}_{D_D}$ as defined in Eq. (2) \(^87\).

Further, as explained in Appendix B, each such locally symmetrized eigenstate can be mapped to a “fictitious double well” with constant inter-well barrier strength $\tilde{\nu} = \vec{v}''$ and of width $\xi = \xi_D$ (corresponding to state $|\phi^\nu\rangle$ symmetrized into domain $D_D$) given by the spacing between the state’s maxima; see Eqs. (B1) and (B2), respectively. This mapping is visualized in Fig. 2 (a) for a selected state localized in domain $D_7$ of the setup in Fig. 1. Notably, for the eigenstate symmetrization to occur in a domain $D_D$, the short-range localization length $\ell$ of the state (see Appendix B) should be significantly smaller than that domain’s size $N_D$.

In terms of local symmetry, the eigenstates of a finite LRD chain will in general be of one of the following types, with examples shown in Fig. 2 (b):

(i) “Even-odd” (eo) pair: two quasidegenerate states of ap-
proximate even and odd local parity, resembling the energy-split states of an isolated symmetric double well, with approximately the same density profile (see φeo, φeo);

(ii) “Left-right” (lr) pair: two quasidegenerate states localized in the left and right half of a symmetry domain each, resembling the states of an isolated well and its mirror image, each with spatial profile being approximately the mirror image—under PD —of the other (see φ303, φ97);

(iii) Single states of approximate local (even or odd) parity (see φ433), in cases where the above-mentioned fictitious barrier width ξ is of the order of the short-range localization length ℓ (see Appendix B);

(iv) Single asymmetric states localized around the boundary between two symmetry domains for sufficiently strong disorder (see φ303) or extended over multiple domains for very weak disorder, sharing none of the above properties.

We emphasize here that the coupling of the symmetry domain boundaries to the surroundings (adjacent domains) acts as a perturbation on the local parity of domain-localized eigenstates. This perturbation increases with the overlap of those states with the domain boundaries, and depending on the fictitious inter-well barrier (see Appendix B), they may (like eo pairs) or may not (like lr pairs) have approximate local parity with respect to PD. Indeed, lr pairs can be seen as originating from eo pair states which are practically degenerate due to vanishingly small inter-well coupling (large ξeo and/or βeo of the fictitious barrier) and combine linearly into left- and right-localized states under the boundary perturbation. In other words, a stronger disorder-induced fictitious double-well barrier assists the local parity breaking caused by the coupling of the domain to its environment.

The IPR and CFS distribution among the eigenstates of a given LRD chain will highly depend on the occurrence of eo and lr pairs. For eo pair states, a localization peak at some position, denoted nvr, within a symmetry domain D is imposes the same localization peak at PD (nvr), which yields relatively small IPR and CFS values, each with a double multiplicity (since the pair states have almost the same density)—as evident, e. g., from pairs of equal consecutive rν or fν-bars in Fig. 1. The CFS will additionally decrease with the distance ξν between the two density peaks which represents the degree of the state’s fragmentation mentioned in Sec.II A. In contrast, lr state pairs contribute with relatively high IPR and CFS values (now without fragmentation present), again with double multiplicity.

The relative frequency of eo and lr pairs will depend on the average fictitious double-well barriers emerging among the different domains. For a given moderate disorder strength, the key analysis tool is here the average of the fictitious barrier widths ξν, which naturally follows the mean size of symmetry domains. Indeed, larger Nd allows for larger ξν corresponding to states which stochastically localize further from the domain center. This in turn increases the formation of lr pairs from boundary-perturbed combinations of eo pairs, as described above.

C. Statistical eigenstate localization properties

With the above insight into individual eigenstate profile characteristics, we now analyze the statistical behavior of eigenstate localization in LRD chains for varying disorder and setup symmetrization. To this end, we compute the probability distribution function (PDF) of the mean IPR ĵ and CFS ĵ over the eigenstates of a given configuration, where ĵ = 1 N ∑ xν/N with x = rνfν. The result is shown in Fig. 3 for different disorder strengths w and number of symmetry domains D. As we see, for each (w, D)-combination the IPR and CFS distributions have well-defined single maxima. Note that D = 0 represents a random chain without any symmetrization and D = 1 a globally symmetric disordered chain, while the maximal value D = N (not shown) is equivalent to D = 0.

As expected, we see in Fig. 3 (a) that the mean IPR is peaked at higher ĵ—that is, eigenstates are more localized—at stronger disorder, for any number of symmetry domains. Indeed, the Anderson localization mechanism will govern the spatial decay on the single-site length scale within the region where a state is concentrated, independently of the presence of symmetries on larger scales. Larger w then leads to faster decay and larger ĵ on average. At the same time, the fluctuations around the peak ĵ value (width of each PDF hump) increases with w, since individual rν-values—being quadratic in rν—are more sensitive to detailed differences between spatial configurations for more localized (larger rν-values) states.
On top of those short-range statistical characteristics, Fig. 3 (a) shows a systematic impact of the long-range spatial correlations of the chain on the IPR distributions induced by local symmetry: For a given (moderate) disorder strength \( w \), the PDFs are shifted to lower \( \bar{r} \) with increasing number of symmetry domains \( D \) up to some intermediate value \( (D \approx 50) \), and then back to higher \( \bar{r} \) for larger \( D (\gtrsim 100) \) approaching the random limit \( (D = N) \). This overall behavior is in accordance with viewing local symmetry as a type of long-range correlation, of variable correlation length, imposed on the disordered potential. With increasing \( D \) more symmetries are added and the long-range correlation is increased, resulting in an enhanced average delocalization of eigenstates. At the same time the symmetry domains become smaller, so that the correlation length also decreases, counteracting in general the delocalization. Approaching the limit \( D \to N \), the average size of the domains gets closer to the order of single sites and the correlation gradually vanishes, leading to usual Anderson-localized states again.

Given this general correspondence between correlation and delocalization, let us now analyze the PDF evolution with varying \( D \) more specifically as a result of the local symmetry-induced state profiles described in Sec. II B. Since increasing \( D \) yields smaller domains on average, it thereby also leads to smaller average widths \( \xi^w_d \) of the fictitious double-well barriers (see Fig. 2 (a) and Eq. (B2)) induced by the disorder. This in turn enhances the occurrence of \( oe \) state pairs at the expense of \( lr \) pairs (which have relatively larger IPR), as concluded in Sec. II B. Thus, the mean IPRs decrease with \( D \) until an average domain size is reached where the predominance of \( oe \) pairs is completed. At the same time, however, the occurrence of domain-interface-localized asymmetric states (of type (iv) in Sec. II B), which have relatively large IPR, increases for larger number \((D - 1) \) of domain interfaces. Also, \( oe \) pairs become less supported again for domains so small that the states do not have enough available space to localize away from the parity-breaking domain boundaries. Together, these effects lead to an increase of the mean IPRs for large \( D \) towards the limit \( D \to N \).

As alluded to in Secs. II A and II B, further localization aspects related to the spatial state profiles for different \( D \) and \( w \) are captured by the CFS distribution, shown in Fig. 3 (b). The short-range localization behavior of the mean CFS \( \bar{f} \) is similar to that of the IPR discussed above: For given \( D \), the peak \( \bar{f} \) values increase with disorder strength \( w \), indicating stronger localization. Interestingly, however, the fluctuations around the peaks show a behavior opposite to the IPRs, now being larger for weaker disorder. This indicates that the degree of spatial fragmentation (as measured by \( \bar{f} \)) is more homogeneous for large \( w \) with stronger short-range localization, while smaller \( w \) favors fragmentation variability for the more smeared out states.

The evolution of the CFSs with varying domain number \( D \) generally follows the same scheme as the IPRs, with an initial shift to smaller \( \bar{f} \) as \( D \) increases and a gradual recovering of the random limit as \( D \to N \). Again, this reflects the overall correspondence between the (symmetry-induced) long-range correlation and delocalization, as discussed above. However, unlike the similar \( \bar{r}(D) \)-evolution for the IPR at different \( w \) (see Fig. 3 (a)), here the shifts of PDF peaks to lower \( \bar{f} \) are inhomogeneous among different \( w \); namely, they occur around smaller \( D \) for weaker disorder. As a consequence, a striking feature of the CFS distribution for varying domain number is manifest along the increment steps \( D \approx 2 \to 10 \): The mean CFS decreases for \( w > 3h \) and increases for \( w < 3h \), while displaying both behaviors for the dividing case \( w = 3h \) (referring to the \( w \)-values considered). From this we anticipate that, with decreasing mean domain size, spatial eigenstate frag-
mentation becomes more (less) pronounced at strong (weak) disorder, thus revealing the role of oe and lr state pairs in a more resolved manner. Specifically, smaller domains (smaller fictitious intra-domain barrier widths $\xi_{\nu}^\rho$) on average favor the occurrence of oe pairs while depleting lr pairs at stronger disorder, thus lowering the average CFS. At weak disorder, the already occurring oe pair states as well as single local parity states (type (iii) in Sec. II B) simply become smaller in extent, contributing to larger mean CFSs.

Finally, we notice that an extreme manifestation of symmetry-induced localization in the CFS distribution of Fig. 3 (b) naturally occurs for global symmetry, $D = 1$, and weak disorder. Here, with a fixed domain size $N_1 = N$, the vast majority of eigenstates are divided into oe pairs, inducing a dramatic jump of the PDFs to lower $f$, especially for the smallest disorder strength $w = h$ (where also non-fragmented parity eigenstates are more extended). The IPR distribution of Fig. 3 (a), being relatively insensitive to individual eigenstate profiles, shows only a very slight corresponding jump to lower $f$, practically discernible only for $w = h$.

III. TRANSFER EFFICIENCY IN LRD CHAINS WITH ADJACENT AND OVERLAPPING SYMMETRY

Having investigated and explained the static eigenstate properties of LRD chains, let us now explore the impact of local symmetry on the dynamics of evolving wave-packets. Since any wave-packet will evolve according to its projection coefficients on the chain’s eigenstates, the question of interest here will be whether the occurrence of approximate local parity eigenstates may systematically affect the dynamics.

In the study of correlation-induced effects on wave diffusion during evolution, a widely used measure is the root-mean-square (standard) deviation, defined as $\langle m(t) \rangle = \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} (n - n_0)^2 |\psi_n(t)|^2}$.

for a discrete chain, where the wave-packet $|\psi(t)\rangle$ evolves here according to the Schrödinger equation (with $\hbar = 1$)

$$i\partial_t |\psi(t)\rangle = H |\psi(t)\rangle$$

under an initial unit excitation at the single site $n_0$, $|\psi(t = 0)\rangle = |n_0\rangle$. Such an evolution is shown in Fig. 4 for initial excitation at the left end $n_0 = 1$ of an LRD chain. For simplicity, we have chosen a relatively short chain with a small number $D = 4$ of symmetry domains. In this example, however, we also introduce a spatial overlap between consecutive domains which, as we will see below, may drastically affect the diffusion. Specifically, in Fig. 4 the overlap is created by extending each domain except for the first $D_1$ and last $D_N$ by a fixed number of sites $L$, keeping the domain centers fixed.

The mean displacement $m(t)$ for the configuration in Fig. 4 increases with time, as expected, until $\psi_n(t)$ has spread enough to reach the right end where it is back-reflected. From this point on $m(t)$ simply fluctuates around a constant mean value. This is, nevertheless, the generic behavior also for uniformly random chains without local symmetry correlations, with the saturation mean value for $m(t)$ decreasing with disorder strength. For other types of (short- or long-range) correlation, the effect on the rate of increase of $m(t)$ is usually studied before reflection at the end of the chain sets in.

Local-symmetry-induced correlations, however, do not affect the overall displacement behavior on ensemble average: In analogy to the IPR used in Sec. II, the mean displacement does not resolve details of the time-dependent spatial profile of $|\psi_n(t)\rangle$ (like, e.g., the faint but visible slight enhancement of $|\psi_n(t)\rangle$ close to the overlap between domains $D_2$ and $D_3$ seen in the color-plot of Fig. 4).

To probe possible symmetry-induced dynamical effects in the LRD chain in a site-resolved manner, we will utilize the so-called “transfer efficiency” of the initial excitation to site $n$, defined here as the maximum amplitude at $n$ over a fixed reference time $t_c$.

$$p_n = \max_{t \in [0, t_c]} |\langle n | e^{-iHt} | n_0 \rangle| = \max_{t \in [0, t_c]} |\psi_n(t)\rangle = 1$$

where we set the input site to $n_0 = 1$. To visualize an example, $p_{n=N}$ is indicated in Fig. 4 (right panel) for that setup. The transfer efficiency has been used to demonstrate that global symmetry in discrete disordered networks may generally lead to an enhanced signal transmission between diametrically located input and output end-sites. This effect relies on the commutation of the Hamiltonian with the global reflection operation, and was shown to be subject to further conditions and optimizations when promoted for efficient quantum transport.
A. Transfer enhancement by local symmetry

What we aim to investigate here is whether a statistical enhancement of signal transfer (compared to uncorrelated disorder) can be manifest if more than one local symmetries are present in the system, each of which now does not commute with \( H \) (as explained in Sec. II B). To this end, we first consider the case of adjacent, that is, non-overlapping symmetry domains. We compute the PDFs of the site-resolved transfer efficiency \( p_n \) over an ensemble of \( 10^5 \) disordered configurations of an \( N = 144 \)-site chain with \( D = 1, 2, 3 \) symmetry domains, and compare it to the non-symmetric, uniformly random case (\( D = 0 \)). The results are shown in Fig. 5 (a0)–(a3), where \( \sqrt{p_n} \) is plotted to increase detail visibility. We use here a disorder strength \( w = 1.2 \) and evolution time \( t_e = 2/\hbar \) in Eq. (9) such that the wave-packet has explored the whole chain. As we see, in all cases the PDF for any given site \( n \) is rather peaked (with peaks becoming narrower towards the chain ends), and the peak \( p_n \)-values fall monotonously with \( n \), as can be anticipated for a disordered chain. Notably, one can clearly distinguish a relatively small but statistically systematic enhancement of \( p_n \) when local symmetry is imposed, approximately in the right halves of the symmetry domains; see local humps of PDF peaks along \( n \) in Fig. 5 (a1),(a2),(a3).

For a clearer comparison with the non-symmetric case (a0), Fig. 5 (b) shows (scaled) the enhancement quotient

\[
g_n^D = \frac{\hat{p}_n^D}{\hat{p}_n^0}
\]

of the peak transfer efficiencies \( \hat{p}_n^D \) (corresponding to \( D = 1, 2, 3 \) domains) to that of the uncorrelated random chain, \( \hat{p}_n^0 \).

The PDF peak values (white dots in Fig. 5 (a0)–a3) have been estimated as the maxima of smoothed versions of the PDFs using local regression\(^{40}\). The \( g_n^D \) show a degree of fluctuation increasing along \( n \), which stems from the strong interference-induced \( p_n \)-fluctuations among individual configurations. They clearly demonstrate, though, an enhancement in transfer efficiency when local symmetry is added (\( g_n^D > 1 \)), to chain parts dependent on the symmetry domains.

For \( D = 1 \) (global symmetry), the enhancement practically starts when crossing the symmetry center, and is then steadily increased in the right chain half. In similarity to the network case of Ref. 54, this is a consequence of the definite parity of the eigenstates under global reflection: When those parity eigenstates have a finite projection onto the initial input state (on the left chain half), they will enable tunneling to their mirror-related part (on the right half). The total transfer is determined by the combination of such effective “double-well” tunnelings\(^{34}\). For \( D = 2 \), the two local reflection operations \( \mathcal{P}_{\mathcal{D}_1,2} \) do not commute with \( H \), but still there is a multitude of approximate local parity eigenstates (see Sec. II B) which can assist in tunneling between the two halves of each domain. Indeed, we observe a drastically enhanced transfer to the right half (\( n \in [36, 72] \)) of \( \mathcal{D}_1 \) which stops roughly at the interface to \( \mathcal{D}_2 \). Then \( g_n \) slightly drops, and increases again in the right half (\( n \in [108, 144] \)) of \( \mathcal{D}_2 \). The scheme of increased enhancement in right domain halves is similar for \( D = 3 \). Note that the \( g_n \)-curves are scaled by the domain size: Their slopes are approximately the same in corresponding domain parts, with the slopes overall decreasing towards the right chain end.

The main difference to the global symmetry case is a generally significant portion of \( lr \) pair states as well as asymmetric domain-interface-localized states, depending on the fictitious intra-domain barriers (see Sec. II B). These states do not contribute to the intra-domain tunneling and therefore lower the
transfer enhancement compared to the global symmetry case. For the weak disorder chosen in Fig. 5, the occurrence of such states is overall reduced, but at the same time extended asymmetric states are favored. Those may generally contribute to transfer, though also in the non-symmetric chain, and are therefore not expected to increase the $g_n^D$.

B. Overlap-induced transfer enhancement

Finally, an intriguing variation on the above LRD setups is to introduce spatial overlap between the domains $D_d$ (as in the explicit example of Fig. 4). We remark that such domain overlap is a unique characteristic accessible with local symmetry as opposed to global symmetry. The key feature here is that symmetry-adapted LRD chain eigenstates (that is, $oe$ and $lr$ pairs as well as single approximate local parity eigenstates; see Sec. II B) of one domain may have substantial spatial overlap with those of a consecutive domain within the overlap region. Since an evolving wave-packet generally has contributions from all available eigenstates, it may be transferred across domains via this spatial overlap of different eigenstates.

Two such scenarios are realized in Fig. 6 for (a) $D = 2$ and (b) $D = 3$ symmetry domains, in two different ways (see horizontal bars indicating domains): In Fig. 6 (a) both domains are extended by $L$ sites across the middle of the chain (with equal domain sizes $N_1 = N_2 = N/2$ for $L = 0$), with their centers shifted by $L/2$, while in Fig. 6 (b) only the middle one of three equally sized domains is extended symmetrically by $L$, such that the domain centers remain fixed. In both cases, we observe a clear enhancement of transfer to the right half of the whole chain in the presence of domain overlap compared to adjacent domains ($L = 0$, gray curves).

Note that for $D = 2$, the overlap leads to a so-called gapped translation symmetry. The chain along the first $2L$ sites is repeated in (i.e., finitely “translated” to) the last $2L$ sites, but not in the region between which constitutes a symmetry “gap”; note though, that the mirror image of the translated part appears within the domain overlap. For $D = 3$, the overlap yields a gapped reflection symmetry: the chain is reflection-symmetric about its center, with the exception of the $N_d - 2L$ sites around the centers of $D_1$ and $D_3$ forming a (locally symmetric) gap. As it appears, those long-range correlations induced via overlap-induced gapped symmetries may play a substantial role in enhancing signal transfer through LRD systems. An interesting prospect would be to explore their impact for larger number of overlapping domains (of same or different sizes) featuring multiple symmetry gaps.

In the special case of $L = N/6$, for both ($D = 2, 3$) of the two considered LRD chain setups in Fig. 6 there is a dramatic enhancement of the ensemble-average transfer efficiency, with maximal $g_n$-factors exceeding $g_n \approx 9$ (roughly double the average maximal $g_n$ for the globally symmetric setup in Fig. 5 (b)). Now, the $D = 2$ chain consists of a single part (first $2L$ sites) which is successively reflected two times at its right end, while the $D = 3$ chain becomes globally symmetric but additionally composed of two different symmetric units of size $2L$ (one in the middle and one repeated at the two ends). In particular, the latter case indicates that the possible transfer efficiency enhancement by global symmetry may be even further increased drastically if local symmetry is present simultaneously at smaller scales within a composite system.

IV. CONCLUSIONS

We have investigated the localization and signal transfer properties of finite, locally reflection-symmetric disordered (LRD) tight-binding chains, treating local symmetry as a spatial correlation of variable range. To reveal the localization behavior, we used the ensemble distributions of the inverse participation ratio (IPR) and a recently proposed measure of
confinement here coined ‘cumulative Friedel sum’ (CFS). It was shown that the spatial participation and fragmentation of eigenstates initially increases with the number of randomly sized symmetry domains and then decreases again towards the limit of uncorrelated disorder, though in an inhomogeneous manner with respect to disorder strength. The localization behavior is induced by the disordered symmetry domains acting as fictitious double wells in which eigenstates acquire approximate local parity. This type of symmetrized localization is explained within a local resonant scattering picture combined with the recent theory of effective confinement potentials. Further, the dynamics of a wave-packet upon excitation of the leftmost site in LRD chains was investigated in terms of the site-resolved transfer efficiency. Here, a systematic enhancement of transfer to the right halves of one, two, or three symmetry domains was shown to take place compared to the non-symmetric random chain. This enhancement can be drastically increased in the presence of overlap between symmetry domains; especially in the case of repeated extended constituents in the chain, or in the simultaneous presence of global and local reflection symmetry. In particular, the possibility to amplify signal transfer by the coexistence of global and local symmetry in composite systems is thus demonstrated.

We stress that the aim of the present work is to study the generic impact of the presence of local symmetry on localization and state transfer efficiency in a minimalistic setting. Disordered 1d chains with uniformly random potential were thus chosen as a platform to isolate the effect solely of the imposed symmetry—that is, without the influence of other structural characteristics or assumptions. Certainly, many alternative routes could be employed to optimize the parameters of LRD setups for efficiency or to probe the effect of local symmetries with improved symmetry-adapted measures. The insight provided here may then also be leveraged to design devices with (overlapping) local symmetries, in order to achieve controllable localization or signal transfer at desired locations. As an example, we mention the perspective to combine the constellation of spatially and spectrally quasidegenerate pair states resulting from combined quasidegenerate or pair states perturbed by domain boundaries. We now give an intuitive argument for the symmetrized eigenstate localization in LRD chains, based on the combination of a recent unifying theory of wave localization66 with a scattering picture of perfectly transmitted85,71 local resonant states. We split the argument by answering three questions, as follows.

1. Where in a disordered chain can an eigenstate localize?

To begin with, computing the eigenvectors of Eq. (3) for a generic disordered medium raises the question: What determines the positions and ranges of localization corresponding to given eigenvalues? The answer is provided in the fairly recently developed framework of “effective confining potentials”97 and “localization landscapes”96,98, formulated also for discrete models82. We now briefly outline this framework, and provide an example for an LRD chain in Fig. 7 (a) (see below).

For our discrete chains, the so-called effective confining potential $u_n$ is defined as the inverse of the “landscape function” $\tau_n$, in turn given as the site amplitudes of the response $|\tau\rangle$ (solving $H_s \tau = |\epsilon\rangle$) of the system to a spatially uniform excitation (source term) $|\epsilon\rangle$:

$$u_n = \frac{1}{\tau_n} = \frac{1}{\langle u | \tau \rangle}, \quad |\tau\rangle = H_s^{-1} |\epsilon\rangle \quad \text{(A1)}$$

with $\epsilon_n = 1 \forall n$, where $H_s = H + V_s$ with a constant offset diagonal $V_s$ added such that $u_n > 0$. As shown in Ref. 97, an eigenstate $|\phi_n\rangle$ of $H$ with eigenenergy $\epsilon_n$ decays exponentially within regions $n$ where $\epsilon_n < u_n$, and can thus have substantial amplitude only in the remaining regions—that is, within local minima of $u_n$ below the threshold $\epsilon_n$. In other words, $u_n$ defines the locations to which $\phi_n$ can be spatially confined according to its energy, namely between “effective barriers” where $\epsilon_n < \epsilon_n$. At larger $\epsilon_n$, the eigenstate will be more delocalized, since such barriers between local $u_n$ minima are exceeded and larger regions are available.

For discrete (tight-binding) models, eigenstates localize again for higher energies (like, e.g., state $|\phi^{489}\rangle$ in Fig. 2 (a)), though now confined by a so-called “dual” effective potential $u'_n$. It is obtained by using $H'_s = V'_s - H$ (with eigenenergies $\epsilon'_n$) instead of $H_s$ in Eq. (A1) where, again, the constant offset $V'_s$ is added to have $u'_n > 0$. This is shown in Fig. 7 (a) for a relatively high-energy eigenstate localizing into domain $D_9$ of a LRD chain—here an example with $N = 10$ equally sized domains. The state is indeed confined between two thick effective barriers of the corresponding $u'_n$ (where $\epsilon'_n < u'_n$) close to the borders of the domain. Smaller barriers lead to amplitude minima in the domain interior. Note here that $u_n$
eigenenergy $\epsilon_r$ of sub-Hamiltonians $H_C$ of (possible) domains of localization $C$:

The smaller $\delta \epsilon$ is for a given region $C$, the larger is the allowed norm of $|\phi^\nu(\epsilon)|$ (eigenvector of $H$) within $C$ for given boundary data (in particular, in the limiting case of zero boundary data, $\phi^\nu_{n\in C} = 0$ only if $\epsilon_r = \epsilon_{\mu}^C$ for some $H_C$-eigenstate $|\phi^\nu_{\mu}^C\rangle$ under Dirichlet boundary conditions). This essentially means that $|\phi^\nu(\epsilon)|$ will confine into the localization domain $C$ supporting a local $H_C$-eigenstate which best matches $|\phi^\nu(\epsilon)|$ in eigenenergy (i.e., with smallest $\delta \epsilon$). It will then also match it in spatial profile, that is, with (approximately) locally symmetric $|\phi^\nu_{\mu}^C\rangle$ for symmetric $v^\nu_{n\in C}$. An example of this is given in Fig. 7(b), showing the local eigenstate $|\phi^\nu_{\mu}^C\rangle$ (where we have chosen $C = D_5$) matching $|\phi^\nu(\epsilon = 360)|$ above of the full system which localizes in $D_5$.

As a side note, if the system contains repeated subdomains (not occurring in the present random potentials) such that corresponding repeated confining domains C occur, then also the localization of an eigenstate will be repeated in those C’s (since their $|\epsilon_r - \epsilon_{\mu}^C|$ values will be equally small) with factors depending on the detailed configuration at those domains’ boundaries. This intuitively explains, e.g., the repeated amplitude patterns occurring in eigenstates of deterministic aperiodic structures with correspondingly repeating sub-Hamiltonians, which feature abundant local symmetries at different scales.

3. Why is the chosen region of localization in the LRD chain symmetric?

Even within the above localization framework, however, the ultimate question of eigenstate symmetrization remains: Why does it happen that, for sufficiently strong localization, the domains C with smallest $|\epsilon_r - \epsilon_{\mu}^C|$, where the full eigenstates are confined, coincide with symmetry domains of the LRD chains?

To give an intuitive answer, let us view a subdomain D as a local scaterer within a generic chain, and consider the scattering of a monochromatic wave of energy $E$ incident from the left on the isolated $D$ connected to perfect semi-infinite chains (or “leads”). The transmission function $T(E) \in [0, 1]$, which is independent of the side of incidence of the wave, gives the portion of the (unit) wave amplitude that transmits through $D$ and leaves the scatterer on the right, while the reflected part is given by $R = 1 - T$. $T(E)$ naturally shows variations depending on the internal structure of $D$ and may, in particular, feature resonant peaks corresponding to quasi-bound states of the scatterer—with resonant widths proportional to the couplings of such states to the leads. Crucially, now, an energetically isolated resonance always has perfect transmission, $T(E) = 1$ at the resonance position $E = E_r$ if the scatterer $D$ is symmetric, with resonant state amplitude $|\phi^\nu_{\mu}^C|$. The position where the eigenstate will actually localize is then determined by the minimal spectral distance

$$\delta \epsilon = \min_{\epsilon_{\mu}, \epsilon_{\nu}} |\epsilon_{\nu} - \epsilon_{\mu}^C|$$

of the eigenenergy $\epsilon_{\nu}$ to the eigenspectra $\{\epsilon_{\mu}^C\}$ of sub-Hamiltonians $H_C$ of (possible) domains of localization $C$:
profile being symmetric too\textsuperscript{64,71,102}. This is shown in Fig.7 (c) for scattering off the isolated domain $\mathbb{D}_5$, which features a scattering resonance extremely close to the eigenenergy $\epsilon_{360}$ of the localized eigenstate in Fig.7 (a), with practically identical amplitude profile $|\psi_{\nu}(E_{\nu})|$ (note the relative factor between $|\psi_{n\epsilon\mathbb{D}_5}|$ within the scatterer and $|\psi_{n\epsilon\mathbb{D}_5}| = 1$ within the leads). This domain will thus be transparent at $E_r \approx \epsilon_{360}$ when embedded into the considered LRD chain, where eigenstates can be viewed as forming upon multiple scattering (and interference) of waves off local scatterers (as done also originally in, e.g., Anderson’s work\textsuperscript{1}). In other words, waves impinging from the left and right onto $\mathbb{D}_5$ are let inside without reflection, while being reflected back into $\mathbb{D}_5$ by adjacent domains when reaching its border from inside, as indicated by arrows in Fig.7 (a). Thus, there will be an accumulation of amplitude in $\mathbb{D}_5$ forming the localized eigenstate at $\epsilon_{360} \approx E_{\nu}$, while this eigenstate is expelled from other localization regions due to larger $|\epsilon_{360} - \epsilon|$, as discussed above.

The deviation from the above picture, that is, deviations of LRD chain eigenstate energies and profiles from local (perfectly transmitting) scattering resonance energies and profiles, increases with the leakage of the eigenstates through the symmetry domain boundaries. This naturally occurs for smaller disorder strength $\nu$ relative to given eigenenergies, where disorder-induced spatial decay is weaker and, equivalently, more maxima in the effective confining potential $u_n$ are exceeded by the eigenenergies.

Summarizing, eigenstate symmetrization into symmetry domains will occur for strong enough disorder (yielding short-range decay at the scale of the domain sizes), at eigenenergies matching perfect transmission resonance energies of the corresponding isolated domains. The link to the fictitious double wells defined in Appendix B can be viewed as follows. The effective confining potential of Eq.\textsuperscript{(A1)} governs the details of localization of an eigenstate. In the case of its symmetrization into a domain, its double-peak profile is represented by the simple picture of a fictitious double-well with corresponding strength and width.

**Appendix B: Fictitious eigenstate-specific double wells**

To provide a simple analysis tool relating the locally symmetrized eigenstates (see Appendix A) to the LRD chain characteristics, we here introduce an effective mapping of such eigenstates to corresponding local double wells.

Any finite piece of the disordered medium can be seen to act \textit{effectively} as a homogeneous potential barrier, in the sense that both may lead to a spatially exponential decay of an eigenstate. In the uncorrelated disordered chain, an eigenstate $|\phi^{\nu}\rangle$ will typically be localized with an exponential decay of modulus envelope $\chi^{\nu}_n$,\textsuperscript{103} that is, $|\phi^{\nu}_n| \leq \chi^{\nu}_n \propto e^{-|n-n_c|/\ell}$, in both directions outwards from its maximum position denoted $n_{\nu}$. Here, $\ell = \ell(\epsilon_{\nu}; \nu) \equiv 1/\gamma$ is the “localization length”, defined as the inverse of the so-called Lyapunov exponent $\gamma$, which generally depends on $\epsilon_{\nu}$ and $\nu$. On the other hand, for a homogeneous periodic chain with onsite potential $V_0 = v$ and dispersion relation $E = 2v \cos k + v$ of $H$ in Eq.\textsuperscript{(1)}, there are solutions exponentially decaying as $e^{-\kappa n}$ at imaginary momenta $k$ (with $i k \equiv \kappa \in \mathbb{R}$) for energies $E$ outside the band, $|E - v| > 2v$. We thereby associate an exponentially localized state $|\phi^{\nu}_n\rangle$ in the uncorrelated disordered chain with a constant \textit{fictitious potential barrier} of strength

$$\hat{\nu'} \equiv \nu - 2h \cos\gamma(\epsilon_{\nu}; w), \quad (B1)$$

that is, supporting decaying states with the same exponent $\kappa = \gamma(\epsilon_{\nu}; w)$ at $E = \epsilon_{\nu}$.

Spatially, this fictitious barrier starts roughly at the sites adjacent to the single site, denoted $n_{\nu}$, where $|\phi^{\nu}_n|$ is maximal. In the LRD chain, however, if an eigenstate is symmetrized in a domain $\mathbb{D}_{d}$, as described above, it has a second (local) maximum at the symmetry-related position $P_{\mathbb{D}_{d}}(n_{\nu})$; see e.g. $\phi_{\nu \text{min}} f$ in Fig.1. In this case, the fictitious barrier acquires a \textit{finite width}, which we simply take to be the number of sites

$$\epsilon_{\nu} = |n_{\nu} - P_{\mathbb{D}_{d}}(n_{\nu})| - 1 = |n_{\nu} - c_d| - 1 \quad (B2)$$

between the positions of the two symmetry-related local maxima of state $|\phi^{\nu}_n\rangle$ localized in $\mathbb{D}_{d}$. This is visualized in the example of Fig.2 (a), showing also $\hat{\nu}$ (superscripts dropped) for the selected state with an estimated $\ell = 1/\gamma = 0.8$ in Eq.\textsuperscript{(B1)}. As a comparison, also the corresponding (fictitious) localized state denoted $\varphi_n$ is shown, here produced by choosing a potential at $n_{\nu}$ and $P_{\mathbb{D}_{d}}(n_{\nu})$ (with $\hat{\nu}$ along the remaining chain) such that this state’s energy matches $\epsilon_{\nu}$\textsuperscript{106}.

In the above situation, the interior of the symmetry domain $\mathbb{D}_{d}$ effectively plays the role of a symmetric double well (like the globally symmetric setup of Ref.\textsuperscript{61}, but here coupled to adjacent sites), with a constant tunneling barrier of width $\xi_{d}^{\nu}$ and strength $\hat{\nu'}$. The outer “walls” of this fictitious double well are represented by the constant potential $\hat{\nu'}$ on the left and right of $n_{\nu}$ and $P_{\mathbb{D}_{d}}(n_{\nu})$ (see Fig.2 (a)). Note that, even for a locally symmetrized state, the maximum position $n_{\nu}$ is still a stochastic variable, determined by the details of the random potential in (one half of) $\mathbb{D}_{d} \ni n_{\nu}$ for a given disorder configuration. In fact, the possible localization positions for given potential and eigenenergy can be found via the chain’s “localization landscape”, as outlined in Appendix A. Distinct peaks at $n_{\nu}$ and $P_{\mathbb{D}_{d}}(n_{\nu})$ occur when $\xi_{d}^{\nu} \gg \ell(\epsilon_{\nu}; w)$, that is, when the fictitious tunneling barrier is sufficiently strong and/or wide. It may also often happen, however, that $\xi_{d}^{\nu} \sim \ell$, in which case the two fictitious wells practically merge into one, supporting a state peaked around the center $c_d$ of $\mathbb{D}_{d}$.

Note that the localization length $\ell = \ell(\epsilon_{\nu}; w)$, indirectly determining each barrier strength via Eq.\textsuperscript{(B1)}, generally depends in an involved manner on the energy and the disorder strength; see e.g. Refs.\textsuperscript{105,107,108}. For the analysis carried out in the present work, it suffices to say that $\ell(\epsilon_{\nu}; w)$ overall decreases with increasing $w$ and $|\epsilon_{\nu}|$ (for fixed $w$).
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