NonSTOP: A NonSTationary Online Prediction Method for Time Series

Christopher Xie
University of Washington
chrisxie@cs.washington.edu

Avleen Bijral
Microsoft
avbijral@microsoft.com

Juan Lavista Ferres
Microsoft
jlavista@microsoft.com

October 4, 2017

Abstract

We present online prediction methods for univariate and multivariate time series that allow us to handle nonstationary artifacts present in most real time series. Specifically, we show that applying appropriate transformations to such time series can lead to improved theoretical and empirical prediction performance. Moreover, since these transformations are usually unknown, we employ the learning with experts setting to develop a fully online method (NonSTOP) for predicting nonstationary time series. This framework allows for seasonality and/or other trends in univariate time series and cointegration in multivariate time series. Our algorithms and regret analysis subsumes recent related work while significantly expanding the applicability of such methods. For all the methods, we provide sub-linear regret bounds using relaxed assumptions. We note that the theoretical guarantees do not fully capture the benefits of the nonstationary transformations, thus we provide a data-dependent analysis of the follow-the-leader algorithm for least squares loss that provides insight into the success of using nonstationary transformations. We support all of our results with experiments on simulated and real data.

1 Introduction

In the analysis of time series, AutoRegressive Moving Average (ARMA) models [Box et al., 2008; Brockwell and Davis, 2009; Hamilton, 1994] are simple and powerful descriptors of weakly stationary processes. As such, they have found tremendous application in many domains including linear dynamical systems, econometrics and forecasting resource consumption [Hamilton, 1994]. It is imminent that with the advent of internet of things (IoT), connected devices will generate large quantities of time series data and thus efficient estimation and prediction with such models will become much more relevant.

Despite a large amount of literature on estimation and prediction for these models, most of it remains within the confines of the statistical assumption of Gaussianity. Such assumptions are often unrealistic [Thompson, 1994] and lead to poor prediction performance. Moreover, since the noise sequence is not known beforehand, standard methods of ARMA estimation rely on conditional
likelihood estimation. These methods usually lead to nonlinear estimation problems and only hold for Gaussian residual sequences and the least squares loss.

In the setting of streaming or high-frequency time series, one would ideally like to have methods that update the model, predict sequentially, and do not rely on any restrictive assumptions on the noise sequence or the loss function. This brings attention to the paradigm of online learning [Cesa-Bianchi and Lugosi, 2006]. In that vein, Anava et al. [2013] recently presented online gradient and online Newton methods (ARMA-OGD and ARMA-ONS) for ARMA prediction. Using a truncated auto-regressive (AR) representation of an ARMA process, the authors provide online ARMA prediction algorithms with sublinear regret, where the regret is with respect to the best conditionally expected one-step ARMA prediction loss in hindsight (See Anava et al. [2013] for more details).

While Anava et al. [2013] make no assumption about the stationarity of the generating ARMA process, in the absence of appropriate modifications, the empirical performance of ARMA-OGD suffers in the presence of seasonality and/or trends which are very common in real time series [Box et al., 2008]. To handle a trend in the time series, Liu et al. [2016] recently presented ARIMA-OGD, a straightforward extension of ARMA-OGD also with sublinear regret. This still doesn’t account for seasonality in the time series or the presence of both a trend and seasonality. More importantly, the trend transform and its parameters are assumed to be known. This is rarely the case in most online settings as one typically needs adequate data to test for such nonstationarities. Also, a fixed transform may not adapt well to the changes in the incoming data. Moreover, existing methods (Liu et al. [2016] or Anava et al. [2013]) do not carry over to the multivariate domain. These shortcomings of existing work necessitate the development of broader methods that take into account different types of nonstationarities (with unknown parameters) with extensions to multivariate time series.

In this paper, we present general methods for online time series prediction that account for possible nonstationarities in the data. When the form of these nonstationarities are known, this leads to transformation of the data (depending on the type of nonstationarity) before prediction in the univariate case. For the general multivariate case with cointegration, we propose a novel algorithm for prediction in potentially nonstationary vector time series generated by error corrected VARMA (EC-VARMA) [Tsay, 2013; Lütkepohl, 2005] processes. Estimating EC-VARMA models are non-trivial in general and the algorithm we propose simultaneously estimates both the cointegrating rank and the VARMA matrix parameters. Please see Tsay [2013] for more details on error corrected models. In the realistic unknown transform setting, we unify the above methods into a meta-algorithm called NonSTOP that is essentially the weighted majority method (Cesa-Bianchi and Lugosi [2006]) wherein each expert corresponds to different parameter settings of the nonstationary transformation (e.g. trend only, trend and seasonality, no trend/seasonality, etc). The weighted majority paradigm allows us to quickly hone in on the correct transformation and make improved predictions. As expected, NonSTOP also allows for flexibility in adapting to changes in the data. Relaxing the assumptions compared to previous work, we provide sublinear regret guarantees for all methods.

Our regret guarantees and that of [Anava et al., 2013; Liu et al., 2016] do not completely explain why the convergence for these online prediction methods is faster for seasonal/trend adjusted data. We conjecture that these bounds are missing data dependent terms that capture correlations inherent in many real nonstationary time series. To give a flavor of what a satisfactory data dependent regret bound might look like, we analyze the regret for the Follow-The-Leader (FTL) algorithm in the case of least squares loss and show that these bounds depend on a data dependent term and can be compared across the different spectrum of real time series (stationary/trend/seasonal etc).
1.1 Contributions

Our contributions in this paper can be highlighted as follows:

1. We provide general methods for time series prediction using Online Gradient Descent (OGD) [Zinkevich, 2003] that allow for appropriate modifications to nonstationary time series before making a prediction. Our approach, being more general, includes existing work while expanding the applicability of such online methods to realistic time series settings.

   (a) For the univariate case, we propose simple modifications to ARMA-OGD that lead to improved empirical and theoretical performance.

   (b) In the multivariate setting, we provide a novel algorithm that learns the error correction term alongside the VARMA matrix coefficients.

2. Since the appropriate transformations are usually unknown in the online setting, we propose NonSTOP, an algorithm that allows us to model each transformation as an expert in the weighted majority setting. This lets us deal with the uncertainty in the transform and changes in the incoming data in case the nonstationarity changes over time.

3. Our regret analysis only requires invertibility of the moving average polynomial (See [Box et al., 2008] for a discussion of invertibility), while the assumptions in [Anava et al., 2013] and [Liu et al., 2016] are less natural. Moreover, we don’t require an upper bound on the data as nonstationary data can be unbounded.

4. To emphasize the effect of the these nonstationary transformations, we prove a data dependent regret guarantee for FTL (for least squares loss) that gives insights into why adjusting for nonstationarities can give faster convergence.

Note that we can easily extend our algorithms and analysis for the online Newton step method of [Anava et al., 2013]. But for simplicity and efficiency, we limit our analysis to OGD. All proofs of theorems can be found in the Supplement.

2 Preliminaries: Time Series Modeling

In this section, we provide a summary of seasonal and/or integrated extensions to standard ARMA processes. We also provide a brief introduction to VARMA and EC-VARMA processes. For an introduction to online convex optimization and online gradient descent, please see [Shalev-Shwartz, 2011] [Zinkevich, 2003] [Hazan et al., 2007]. We introduce standard notation for time series in Table 1.

2.1 SARIMA Processes

Time series exhibiting seasonal patterns can be modeled by Seasonal AutoRegressive Integrated Moving Average (SARIMA) Processes. SARIMA\((p,d,q) \times (P,D,Q)_s\) processes are described by the following equation:

\[
\phi(L)\Phi(L^s)\Delta^d\Delta_s^D x_t = \theta(L)\Theta(L^s)\epsilon_t
\] (1)

\[
\phi(L)\Phi(L^s)\Delta^d\Delta_s^D x_t = \theta(L)\Theta(L^s)\epsilon_t
\]
### Table 1: Notation

| Symbol | Description |
|--------|-------------|
| $x_t$ | Denotes the time series |
| $L$ | Lag operator: $L x_t = x_{t-1}$ |
| $\Delta$ | Differencing operator: $\Delta x_t = x_t - x_{t-1}$ |
| $\Delta_s$ | Seasonal differencing operator: $\Delta_s x_t = x_t - x_{t-s}$ |
| $\varepsilon_t$ | Noise |
| $d$ | Differencing order |
| $s$ | Seasonal period |
| $D$ | Seasonal differencing order |

where $\phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i$, $\theta(L) = 1 + \sum_{i=1}^{q} \theta_i L^i$, $\Phi(L^s) = 1 - \sum_{i=1}^{P} \Phi_i L^{is}$, $\Theta(L^s) = 1 + \sum_{i=1}^{Q} \Theta_i L^{is}$ and $\phi, \Phi, \theta, \Theta \in \mathbb{R}$. Note that

1. $\tilde{D} = 0$ implies a ARIMA($p, d, q$) process.

2. $\tilde{D} = d = 0$ implies a ARMA($p, q$) process.

SARIMA processes explicitly model trend and seasonal nonstationarities by assuming that the differenced process $\Delta^d \Delta_s^s x_t$ is an ARMA process with AR lag polynomial $\phi(L) \Phi(L^s)$ and MA lag polynomial $\theta(L) \Theta(L^s)$. We denote the order of the underlying AR and MA lag polynomials as $l_a$ and $l_m$, respectively. For SARIMA($p, d, q$) $\times$ ($P, \tilde{D}, Q)_s$ processes, Eq. 1 gives us that $l_a = p + Ps$ and $l_m = q + Qs$.

If the MA lag polynomial has all of its roots outside of the complex unit circle, then the SARIMA process is defined as invertible. Let $\beta_i$ be the scalar coefficients of the MA lag polynomial. Invertibility is equivalent to saying that the companion matrix

$$
F = \begin{bmatrix}
-\beta_1 & -\beta_2 & \cdots & \cdots & -\beta_{l_m} \\
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & \vdots \\
\vdots & 0 & \ddots & \ddots & \vdots \\
0 & \vdots & \vdots & 1 & 0 
\end{bmatrix}
$$

has eigenvalues less than 1 in magnitude. If this is the case, then the underlying ARMA process $\Delta^d \Delta_s^s x_t$ can be written as an AR($\infty$) process and can be approximated by a finite truncated AR process.

#### 2.2 VARMA Processes

Vector AutoRegressive Moving Average (VARMA) processes provide a parsimonious description of modeling linear multivariate time series. Let $x_t, \varepsilon_t \in \mathbb{R}^k$, $\Phi_i \in \mathbb{R}^{k \times k}$, $\Theta_i \in \mathbb{R}^{k \times k}$. A VARMA($p, q$) process is described by:

$$
x_t = \sum_{i=1}^{p} \Phi_i x_{t-i} + \sum_{i=1}^{q} \Theta_i \varepsilon_{t-i} + \varepsilon_t
$$
which can also be written in lag polynomial form:

$$\Phi(L)x_t = \Theta(L)\varepsilon_t$$

(4)

with $$\Phi(L) = I - \sum_{i=1}^{p} \Phi_i L^i$$, $$\Theta(L) = I + \sum_{i=1}^{q} \Theta_i L^i$$. The requirements for invertibility are very similar to the univariate case. We require that $$\det(\Theta(L))$$ must have all of its roots outside of the complex unit circle. Again, this is equivalent to saying that the companion matrix has eigenvalues less than 1 in magnitude [Lütkepohl, 2005, Tsay, 2013]. If the process is invertible, then it can be rewritten as a VAR(\infty) process.

2.3 EC-VARMA Model

In many cases, a collection of time series may follow a common trend. This phenomenon, known as cointegration, is ubiquitous in economic times series [Tsay, 2013]. Let $$x_t$$ be described by Eq. 3. Formally, $$x_t$$ is cointegrated if $$\Delta x_t$$ is stationary and there exists a vector $$\mu \in \mathbb{R}^k$$ such that $$\mu^\top x_t$$ is a stationary process. If $$x_t$$ is cointegrated, then we can rewrite the original VARMA representation of $$x_t$$ as

$$\Delta x_t = \Pi x_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} + \sum_{i=1}^{q} \Theta_i \varepsilon_{t-i} + \varepsilon_t$$

(5)

where $$\Pi = -\Phi(1)$$ is low rank, and $$\Gamma_j = -(\Phi_{j+1} + \ldots + \Phi_p)$$ for $$j = 1, \ldots, p-1$$. Eq. 5 is known as an Error-Corrected VARMA (EC-VARMA) model. Note that this looks like a VARMA($p - 1$, $q$) process in the differenced series $$\Delta x_t$$, except that there is the error-correction term $$\Pi x_{t-1}$$. See [Lütkepohl, 2005, Tsay, 2013] for more details.

This family of models is the multivariate analogue of the ARIMA model. However, differencing in multivariate time series may lead to a phenomenon called over-differencing and thus we must consider the notion of error correction models. We refer the reader to [Tsay, 2013] for a discussion of this issue.

3 Univariate Methods

In this section, we present algorithmic methods for online univariate time series prediction which subsumes recent works such as ARMA-OGD as presented in Anava et al. [2013] and its extension to trend nonstationarities ARIMA-OGD as presented in Liu et al. [2016].

Precisely, we show that time series with certain characteristics (such as a trend and/or seasonality) can be appropriately transformed before prediction to give better theoretical and empirical results. To achieve this goal, we present a unified template for time series prediction using OGD, denoted TSP-OGD, that allows for prediction of transformed time series. We will show that the choice of the transformation, dependent on the underlying data generation process (DGP), can lead to improved regret guarantees, partially explaining why these transformations lead to better empirical performance.

This framework includes some of the commonly used transformations of seasonal and non-seasonal differencing [Box et al., 2008]. Table 2 shows the explicit form of such transformations. In practice, the order of differencing is usually determined by statistical tests (e.g. Elliot et al., 1996) on a given dataset, which is not realistic when considering the online setting.
Algorithm 1 TSP-OGD Framework

**Require:** DGP parameters $l_a, l_m$. Horizon $T$. Learning rate $\eta$. Data: $\{x_t\}$. Transformation $\tau$.

Inverse Transformation $\zeta$.

1: Set $M = \log_{\lambda_{\max}} \left( (2\kappa TL_{\max} \sqrt{T_{\max}})^{-1} \right) + l_a$
2: Transform $x_t$ to get $\tau(x_t)$.
3: Choose $\gamma(1) \in E$ arbitrarily.
4: for $t = 1$ to $T$ do
5: $\tau(\tilde{x}_t) = \sum_{i=1}^{M} \gamma(t,i) \cdot \tau(x_{t-i})$
6: Predict $\tilde{x}_t = \zeta(\tau(\tilde{x}_t))$
7: Observe $x_t$ and receive loss $\ell_t^M(\gamma(t))$
8: Set $\gamma(t+1) = \Pi_E (\gamma(t) - \eta \nabla \ell_t^M(\gamma(t)))$
9: end for

| DGP   | $\tau(x_t)$ | $\zeta(y_t)$ |
|-------|-------------|--------------|
| ARMA  | $x_t$       | $y_t$        |
| ARIMA | $\Delta^d x_t$ | $y_t + \sum_{i=0}^{d-1} \Delta^i x_{t-1}$ |
| SARIMA| $\Delta^d \Delta^p x_t$ | $y_t + \sum_{i=0}^{d-1} \Delta^i \Delta^p x_{t-1}$ + $\sum_{i=0}^{p-1} \Delta^i x_{t-p}$ |

### 3.1 TSP-OGD

We assume the following for the remainder of this section:

U1) $x_t$ is generated by a DGP such that there exists a transformation $\tau(x_t)$ which results in an invertible ARMA process. Moreover, there corresponds an inverse transformation $\zeta$ that satisfies $\zeta(\tau(x_t)) = x_t$. Examples of such processes are ARMA, ARIMA, and SARIMA processes.

U2) The noise sequence $\epsilon_t$ of the process is independent. Also, it satisfies that $\mathbb{E}[|\epsilon_t|] < M_{\max} < \infty$.

U3) $\ell_t : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a convex loss function with Lipschitz constant $L > 0$.

U4) We assume the companion matrix $F$ (as defined in Eq. (2)) of the MA lag polynomial is diagonalizable, i.e. $F = T \Lambda T^{-1}$ where $\Lambda$ is a diagonal matrix of eigenvalues. Denote $\lambda_{\max}$ as the magnitude of the largest eigenvalue ($\lambda_{\max} < 1$ by definition of invertibility), and $\kappa \in \mathbb{R}$ such that $\left( \sigma_{\max}(T)/\sigma_{\min}(T) \right) \leq \kappa$.

In Algorithm 1 the model parameters of the stochastic process are fixed by an adversary. At time $t$, $\epsilon_t$ and $x_t$ are generated by the DGP. Before $x_t$ is revealed to us, the learner (see Algorithm 1) makes a prediction $\tilde{x}_t$ which incurs a prediction loss of $\ell_t(x_t, \tilde{x}_t)$. This prediction is preceded by a transform $\tau$ (See Table 2) that may require data points from previous rounds (we suppress that dependence in the notation for convenience). The prediction $\tau(\tilde{x}_t)$ is computed using an AR model
of order $M$ to approximate the underlying invertible ARMA process. Then it is inverted with $\zeta$ and incurs a loss

$$\ell_t^M(\gamma) := \ell_t(x_t, \zeta(\tau(\tilde{x}_t))) = \ell_t\left(x_t, \zeta\left(\sum_{i=1}^{M} \gamma_i \tau(x_t-i)\right)\right)$$

(6)

where $\gamma$ is the vector of parameters of the approximating AR model. The prediction performance is evaluated using an “extended” notion of regret that looks at the prediction loss of the best process in hindsight. Precisely, let $\alpha, \beta$ denote the set of AR and MA parameters, respectively, of the underlying ARMA process $\tau(x_t)$. Define

$$f_t(\alpha, \beta) = \ell_t\left(x_t, \zeta\left(\mathbb{E}\left[\tau(x_t)\right|\{\tau(x_t)\}_{t=1}^{t-1}; \alpha, \beta\right]\right)$$

(7)

Note that $f_t$ depends on the transformations $\tau, \zeta$ in $U_1$. The extended regret is defined as comparing the accumulated loss in Eq. (6) to the loss of the best process in hindsight:

$$\text{Regret} = \sum_{t=1}^{T} \ell_t^M(\gamma(t)) - \min_{\alpha, \beta \in K} \sum_{t=1}^{T} \mathbb{E}[f_t(\alpha, \beta)]$$

(8)

where $K$ is the set of invertible ARMA processes. Note that the randomness in the expectation is w.r.t. the noise sequence $\varepsilon_t$ while the data $x_t$ is fixed.

Furthermore, let $E \subseteq \mathbb{R}^M$ be a convex set of approximating AR models, i.e. $\gamma \in E$. $E$ should be chosen to be large enough to include a valid approximation to the DGP described in $U_1$. However, since the DGP is unknown in practice, one usually chooses something simple such as $E = \{\gamma : \|\gamma\|_\infty \leq 1\}$. Let $D = \sup_{\gamma_1, \gamma_2 \in E} \|\gamma_1 - \gamma_2\|_2$, and $\|\nabla \ell_t^M(\gamma)\|_2 \leq G(t)$ for some monotonically increasing $G(t)$. This assumption stems from the fact that we allow the time series to be potentially unbounded. As an example, the norm of the gradient for the squared loss depends on the bound on the data. Let $\Pi_E$ denote the projection operator onto the set $E$.

We present a general regret bound for Algorithm 1:

**Theorem 3.1.** Let $\eta = \frac{D}{G(T)\sqrt{T}}$. Then for any data sequence $\{x_t\}_{t=1}^{T}$ that satisfies assumptions $U1-U4$, Algorithm 1 generates a sequence $\{\gamma(t)\}$ in which

$$\sum_{t=1}^{T} \ell_t^M(\gamma(t)) - \min_{\alpha, \beta \in K} \sum_{t=1}^{T} \mathbb{E}[f_t(\alpha, \beta)] = O\left(DG(T)\sqrt{T}\right)$$

**Remark 1:** Note that plugging in the ARMA transformation and ARIMA transformation in Table 2 to Algorithm 1 recovers ARMA-OGD as presented in Anava et al. [2013] and ARIMA-OGD as presented in Liu et al. [2016], respectively. Plugging in the SARIMA transformation results in a variation which we will denote as SARIMA-OGD.

For the following remarks, assume that $\ell_t$ is squared loss, the DGP is a SARIMA process, and $|x_t| < C(t) = O(\log t)$ (note that the log transformation is commonly employed as a variance stabilizer in many time series domains).

**Remark 2:** With these assumptions, Table 3 shows the regret bounds obtained by using different transformations/algorithms. The differencing transforms remove any growth trends in the data; as a consequence the transformed time series is bounded by a constant. In our case, this implies
Table 3: Regret Bounds for Different Transformations

| Algorithm      | $\tau(x_t)$ | Regret Bound                          |
|----------------|-------------|---------------------------------------|
| ARMA-OGD       | $x_t$       | $O\left(M^2 \log^2 (T) \sqrt{T}\right)$ |
| ARIMA-OGD      | $\Delta^d x_t$ | $O\left(M^2 \sqrt{T}\right)$          |
| SARIMA-OGD     | $\Delta^d \Delta^D x_t$ | $O\left(M^2 \sqrt{T}\right)$          |

$|\Delta^d x_t|, |\Delta^d \Delta^D x_t| < C_\Delta$ (a constant), which leads to an improvement over the regret bound obtained from ARMA-OGD. This improvement can be seen in the empirical results section of Liu et al. [2016].

Remark 3: When the DGP is assumed to be SARIMA, we require that $l_a = p + P s, l_m = q + Q s$ as mentioned in Section 2 i.e. $l_a, l_m$ both need to essentially be a multiplicative factor larger than $s$. This affects the length of the required AR approximation $M$ as described in line 1 of Algorithm 1.

3.2 Data Transformation Dependent Regret

The transformations discussed in the previous sections essentially diminish the effect of serial correlation in the data due to any existing nonstationary trends. However, our regret bounds (shown in Table 3) do not accurately reflect this. We conjecture that these bounds are missing data-dependent terms that capture correlations inherent in many nonstationary time series. To give a flavor of what a satisfactory data dependent regret bound might look like, we analyze the regret for the FTL algorithm for the case of least squares loss

$$\ell_t(\gamma) = \frac{1}{2}(x_t - \gamma^\top \psi_t)^2$$  \hspace{1cm} (9)

and show that these bounds depend on a data dependent term. We look at the standard notion of regret and hence the result in this section is much more general than time series prediction and is also relevant to general regression problems.

The FTL algorithm follows a simple update [Shalev-Shwartz 2011]:

$$\gamma_{t+1} \in \arg \min_{\gamma} \sum_{i=1}^{t} \ell_t(\gamma)$$  \hspace{1cm} (10)

Plugging Eq. 9 in Eq. 10 reveals that the FTL algorithm for least squares loss is just recursive least squares (RLS). Using the relevant RLS update equations [Ljung 1998, Lai and Wei 1982], we present a data dependent regret bound for FTL with least squares loss.

Theorem 3.2. Let $\ell_t(\gamma)$ be defined in Eq. 9 with Lipschitz constant $L > 0$. Then FTL generates a sequence $\{\gamma_t\}$ in which

$$\sum_{t=1}^{T} \ell_t(\gamma_t) - \min_{\gamma} \sum_{t=1}^{T} \ell_t(\gamma) = O\left(\sum_{t=1}^{T} \frac{1}{\lambda_{\min}(t)}\right)$$
where

\[ \lambda_{\min}(t) = \lambda_{\min}\left(\frac{1}{t} \sum_{i=1}^{t} \psi_i \psi_i^T\right). \]

At the heart of our framework in Algorithm 1, we are approximating an ARMA process with an AR model. In order to apply Theorem 3.2 to our time series prediction setting for DGPs as described in assumption U1, we use FTL and least squares loss to predict the underlying ARMA process \( \tau(x_t) \) with an AR model \( \gamma^T \tau(\xi_t) \), where \( \xi_t = [x_{t-1} \ldots x_{t-M}]^T \) and \( \tau(\xi_t) = [\tau(x_{t-1}) \ldots \tau(x_{t-M})]^T \). This results in \( \lambda_{\min}(t) = (\frac{1}{t} \sum_{i=1}^{t} \tau(\xi_i) \tau(\xi_i)^T) \), which is the empirical non-centered autocovariance of the transformed data. Ideally, we want this quantity to be large, meaning that the individual samples \( \tau(x_t) \) are not highly correlated.

To empirically assess the regret bound when accounting for the appropriate nonstationarities, we calculate the bound \( \sum_{i=1}^{T} 1/(t \lambda_{\min}(t)) \) for the three transforms in Table 2. We simulated a SARIMA process 50 times with \( T = 10,000 \). We then averaged the calculated regret bound across the 50 simulated datasets using each transformation. The result is shown in Figure 1. The appropriate transformations essentially decrease correlations making the data more like realizations of a stationary ARMA process; we can see that accounting for the appropriate nonstationarities results in tighter regret bounds.

Missing from the analysis present in this section and other related works is the case of multivariate time series. The issue of nonstationarity in such series is complicated by the fact that differencing transforms do not preserve the conintegration structure prevalent in many real time series. Thus, online methods must additionally infer this relationship from the data.

### 4 Multivariate Methods

Online prediction using multivariate nonstationary models present an additional difficulty due to the notion of cointegration (Section 2). Estimating EC-VARMA models in the static setting is non-trivial in general since the cointegrating rank is unknown and is typically determined by statistical tests (e.g. trace statistic of Johansen [1988]), which again is not realistic in the online setting. We propose a novel online method for cointegrated vector valued time series that simultaneously
updates both the cointegrating matrix (including its rank) and the approximating VAR matrix parameters in order to accurately adapt to the underlying DGP and make predictions.

### 4.1 Approximating an EC-VARMA Process

Given that an EC-VARMA process starts at some fixed time \( t = 0 \) with fixed initial values, we can write Eq. 5 in a pure EC-VAR form [Lütkepohl] [2006]:

\[
\Delta x_t = \Pi^* x_{t-1} + \sum_{i=1}^{t-1} \Gamma_i^* \Delta x_{t-i} + \varepsilon_t, \quad t \in \mathbb{N}
\]  

(11)

This allows us to approximate an EC-VARMA process with an EC-VAR model. To use EC-VARMA as a DGP in Algorithm 1, we edit line 5 to be:

\[
\Delta \tilde{x}_t = \hat{\Pi} x_{t-1} + \sum_{i=1}^{M} \hat{\Gamma}_i \Delta x_{t-i}
\]

where \( \gamma = \{ \hat{\Pi}, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_M \} \) are the approximating EC-VAR parameters.

### 4.2 Online Prediction for EC-VARMA Models

We generalize the assumptions U1-U4 to the multivariate setting:

M1) \( x_t \) is generated by an EC-VARMA process. The noise sequence \( \varepsilon_t \) of the underlying VARMA process is independent. Also, it satisfies that \( E[\|\varepsilon_t\|^2] < M_{\text{max}} < \infty \).

M2) We overload notation for the vector case and let \( \ell_t : \mathbb{R}^{2k} \to \mathbb{R} \) be a convex loss function with Lipschitz with constant \( L > 0 \).

M3) We assume the companion matrix \( F \) of the MA lag polynomial is diagonalizable. \( \lambda_{\text{max}} \) and \( \kappa \) are the same as in assumption U4.

The resulting algorithm is summarized in Algorithm 2 denoted EC-VARMA-OGD. The setup of this algorithm is the same as described in Section 3. We overload more notation to generalize Equations 6 and 7:

\[
\ell^M_t(\gamma) = \ell_t \left( x_t, x_{t-1} + \hat{\Pi} x_{t-1} + \sum_{i=1}^{M} \hat{\Gamma}_i \Delta x_{t-i} \right)
\]  

(12)

\[
f_t(\Pi, \Gamma, \Theta) = \ell_t \left( x_t, x_{t-1} + \Pi x_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} + \sum_{i=1}^{q} \Theta_i \varepsilon_{t-i} \right)
\]

The regret as defined in Eq. 8 can be generalized to

\[
\text{Regret} = \sum_{t=1}^{T} \ell^M_t(\gamma_t) - \min_{\Pi, \Gamma, \Theta \in \mathcal{K}} \sum_{t=1}^{T} E[f_t(\Pi, \Gamma, \Theta)]
\]  

(13)

where \( \mathcal{K} \) is the set of invertible EC-VARMA processes.
Algorithm 2 EC-VARMA-OGD

Require: DGP parameters $p, q$. Horizon $T$. Learning rate $\eta$. Data: $\{x_t\}$.

1: Set $M = \log_{\lambda_{\text{max}}} \left( \left( 2\kappa TLM_{\text{max}} \sqrt{q} \right)^{-1} \right) + p$
2: Choose $\gamma^{(1)} \in \mathcal{E}$ arbitrarily.
3: for $t = 1$ to $T$ do
4: Predict $\tilde{x}_t = x_{t-1} + \hat{\Pi}x_{t-1} + \sum_{i=1}^{M} \hat{\Gamma}_i \Delta x_{t-i}$
5: Observe $x_t$ and receive loss $\ell_t(\gamma^{(t)})$
6: Set $\hat{\Gamma}^{(t+1)} = \Pi_{\mathcal{E}} \left( \hat{\Gamma}^{(t)} - \eta \nabla \ell_t(\gamma^{(t)}) \right)$, for all $i$
7: Set $\hat{\Pi}^{(t+1)} = \Pi_{\mathcal{B}^*(\rho)} \left( \hat{\Pi}^{(t)} - \eta t \nabla \ell_t(\gamma^{(t)}) \right)$
8: end for

To encourage $\hat{\Pi}$ to be low rank, we project it onto $\mathcal{B}(\cdot, \rho)$, which is the nuclear norm ball of radius $\rho$. This involves projecting the singular values of $\hat{\Pi}$ onto an $\ell_1$-ball and can be efficiently done [Duchi et al., 2008]. In our framework, this is handled by letting the convex set $\mathcal{E}$ be described as $\{\gamma : ||\hat{\Pi}||_* \leq \rho, ||\hat{\Gamma}_i||_{\text{max}} \leq 1, i = 1, \ldots, M\}$ and plugging it into OGD where projections are made at each iteration. For convenience of notation, let $\mathcal{E}_{\Gamma} = \{\hat{\Gamma} : ||\hat{\Gamma}_i||_{\text{max}} \leq 1, i = 1, \ldots, M\}$.

As in Section 3, $\mathcal{E}$ should be chosen to be large enough to encompass a valid approximation to the true DGP.

We present the following regret bound:

**Theorem 4.1.** Let $\eta = \frac{D}{G(T)^{\sqrt{T}}}$. Then for any data sequence $\{x_t\}_{t=1}^T$ that satisfies assumptions M1-M3, Algorithm 2 generates a sequence $\{\gamma_t\}$ in which

$$\sum_{t=1}^{T} \ell_t(\gamma_t) - \min_{\Pi, \Gamma, \Theta \in \mathcal{K}} \sum_{t=1}^{T} \mathbb{E}[f_t(\Pi, \Gamma, \Theta)] = O \left( DG(T) \sqrt{T} \right)$$

(14)

For the remainder of the section, we assume that $\ell_t$ is the squared loss and $||x_t||_2 < C(t) = O(\log t)$.

**Remark 1:** With the above assumptions, the resulting regret bound of EC-VARMA-OGD is $O \left( k^2 M^2 \log^2(T) \sqrt{T} \right)$.

**Remark 2:** By setting $\rho = 0$ and using $x_t$ in place of $\Delta x_t$ (i.e. not differencing) in Algorithm 2 we use a VARMA process as the DGP and achieve an equivalent regret bound as in the previous remark. Denote this adaptation as VARMA-OGD. However, if the DGP is EC-VARMA, we expect this to empirically perform worse than EC-VARMA-OGD since the latter exploits a valid transformation of the data.

**Remark 3:** Assume that the DGP is an EC-VARMA process and $\rho = o(1/\log^2(T))$. Then the regret bound obtained is $O \left( k^2 M^2 \sqrt{T} \right)$. In Section 6, we find that this choice of $\rho$ works well empirically.

5 NonSTOP

Algorithms 1 and 2 assume that the appropriate transformation is known apriori. Typically, statistical tests are used to determine the degree of differencing on a fixed dataset (e.g. Elliot et al. [1996]).
Algorithm 3 NonSTOP

**Require:** DGP parameters $l_a, l_m$. Horizon $T$. Data: $\{x_t\}$. Models $M$. Window size $k$.

1: Set $\eta = \sqrt{\frac{\log |M|}{T}}$
2: Initialize $w_1 = [1 \ldots 1]$
3: for $t = 1$ to $T$ do
4: Set $b_t = \max_{\tau \in \{t-k, \ldots, t\}, h \in M} \ell_{\tau}(h), W_t = \sum_h w_t(h)$
5: Predict using $h_t \in M$, where $h_t$ is chosen using probability distribution $\frac{w_t(h)}{W_t}$
6: For each model in $M$, run the update according to Algorithm 1.
7: Update $w_{t+1}[h] = w_t[h](1 - \eta)^{\ell_t(h)}$ for all $h \in M$
8: end for

and these usually come with assumptions and sample size requirements. In the online setting, these requirements are not realistic and an ideal method must adapt to the incoming data leading to a possibly time dependent sequence of transformations. We approach this problem by using the online learning with experts (OLE) setup wherein each expert corresponds to a specific transformation (including none at all). Specifically, we adapt the (randomized) weighted majority algorithm (WM, for details see [Shalev-Shwartz, 2011]) as a meta-algorithm to select a transformation at each time step and allow for potentially unbounded loss functions.

More precisely, let $M$ be the set of experts we consider. The set of experts can either be instantiations of Algorithm 1 or 2. For example, in the univariate setting, we could have $M = \{ARMA-OGD, ARIMA-OGD, SARIMA-OGD\}$ with $d = D = 1$, and in the multivariate setting we can have $M = \{VARMA-OGD, EC-VARMA-OGD\}$. We assume that the seasonal period $s$ is known.

The resulting algorithm is summarized in Algorithm 3, denoted NonSTOP (NonSTationary Online Prediction). In each round, the online meta-algorithm randomly selects a prediction from one of its experts. After receiving the loss, it then updates its view about its experts, while the experts themselves are adapting to the data. We scale the loss function with a sliding window maximum such that the losses stay bounded. Since $D, G(T)$, and $\ell_t(h)$ as shown in Algorithm 1 and 2 are now dependent on the specific transformation, we denote this as $D_h, G_h(T), \ell_{t,h}(\gamma_h(t))$ for a model $h \in M$. With these definitions in hand, we give the following theorem:

**Theorem 5.1.** Let $\ell_t(h) := \ell_{t,h}(\gamma_h(t))$. Define $B_T := \max_{\tau \in \{1, \ldots, T\}, h \in M} \ell_{\tau}(h)$. Then Algorithm 3 plays a sequence of predictions that satisfies

$$\sum_{t=1}^{T} \mathbb{E}[\ell_t(h_t)] - \min_{\alpha, \beta \in K} \sum_{t=1}^{T} \mathbb{E}[f_t(\alpha, \beta)] = O \left( \max \{B_T, D_s G_s(T)\} \sqrt{T} \right)$$

(15)

where $D_s = \max_h D_h$, $G_s(T) = \max_h G_h(T)$. Note that this bound is the same as if you used $B_T$ in place of $b_t$.

**Remark 5.1.1.** When using least squares loss, $B_T = O(G_s(T))$ and the regret bound defaults to $O \left( D_s G_s(T) \sqrt{T} \right)$.
In this section, we show empirically the effectiveness of methods described in Sections 3, 4, and 5 on synthetic and real datasets. In each scenario, we use squared loss and plot the log average squared loss vs. iteration. For all experiments, we set $E = \{ \gamma : \| \gamma \|_{\text{max}} \leq 1 \}$, initialize all parameters to 0, and set the sliding window length $k = 10$. For all real world datasets, we log transform the time series. Plots of these datasets can be found in the Supplement.

### 6.1 Univariate Setting

We first simulate 20 synthetic time series with $T = 20000$ from the following SARIMA model:

$$\Delta^2_1 x_t = (1 - 0.95L)(1 - 0.4L^{12}) \epsilon_t$$  \hspace{1cm} (16)

We compare NonSTOP to each algorithm in Table 3 on the generated data. We run the algorithms on each generated time series and average the log average loss in Figure 2a. As expected, SARIMA-OGD outperforms ARMA-OGD and ARIMA-OGD, converging quickly as it accounts for the appropriate nonstationarities. This behaviour is consistent with our hypothesis that in the absence of an appropriate transformation, existing methods will underperform. NonSTOP gradually adapts and learns to heavily weight the correct transformation (expert) and outperforms ARMA-OGD and ARIMA-OGD. Note that the initial bump in the convergence of ARMA-OGD is due to the fact that we are plotting (log) average loss and there is an initial period where the average loss grows faster than $T$, and then it grows slower.
To showcase the adaptability of NonSTOP, we simulate 20 synthetic time series from Eq. (16) for $T = 4000$ timesteps, and then simulate data from an ARIMA model for another 16000 timesteps. Results are shown in Figure 2b. NonSTOP learns to weight SARIMA-OGD, but quickly adapts at $t = 4000$ to weight ARIMA-OGD. In fact, at the end of the run, it actually outperforms all experts, showing the power of this fully online adaptable algorithm.

Next, we consider a dataset that contains daily electricity demand in Turkey from January 1, 2000, to December 31, 2008. The seasonality in this dataset is biannual (rounded down to $s = 182$ days). The results of running the algorithms are shown in Figure 2c. Again, SARIMA-OGD accounts for the appropriate nonstationarities and performs the best. ARMA-OGD suffers severely due to not accounting for nonstationarity. As such, NonSTOP quickly finds that ARMA-OGD is not a reliable expert and performs well in comparison to the other experts. For the daily recorded births in Quebec from Jan. 01, 1977 to Dec. 31, 1990 there is a weekly seasonality pattern with $s = 7$. Figure 2d reveals that the results here are similar to previous datasets. Because NonSTOP starts with an equal weight for each expert, it pays a large penalty for selecting ARMA-OGD in initial iterations. However, it approaches the performance of the other algorithms as it learns to optimally weight the correct transformation.

Lastly, we consider a dataset consisting of daily river flow values from the Saugeen River from Jan 01, 1915 to Dec 31, 1979. This data exhibits a yearly ($s = 365$) seasonality pattern. The results are plotted in Figure 2e. While accounting for any non-stationarity improves convergence, accounting for the seasonality actually hurts the performance compared to accounting for the trend only as SARIMA-OGD is outperformed by ARIMA-OGD. In our experience, sometimes ARIMA can outperform SARIMA even on seasonal data. Yet NonSTOP learns to weight ARIMA-OGD and quickly approaches the best performance. This showcases the efficacy of the NonSTOP algorithm in a fully online setting.

### 6.2 Multivariate Setting

In the multivariate setting we show empirically that accounting for cointegration results in faster convergence. We look at the results of running EC-VARMA-OGD as described in Algorithm 2 vs. VARMA-OGD on two real datasets.

We collected 7 time series of stock prices from Yahoo Finance (http://finance.yahoo.com/) of large technology companies, and also includes the S&P500 index. By including the
S&P500, which is essentially an weighted average of 500 company stock prices, we have partially introduced cointegration into the time series. We set $M = 10$, $\rho = 0.5$ and ran all algorithms with the resulting plots in Figure 3a. Accounting for cointegration results in considerably stronger performance. There is a bump in the convergence plot due to a spike in the data (see Supplement). We also evaluated the algorithms on the Google Flu dataset (https://www.google.com/publicdata/explore/) which contains influenza rates of 28 countries. There are two distinct seasonality patterns: the northern hemisphere countries have flu incidents that peak in one part of the year while the southern hemisphere countries have flu incidents that peak in the other part of the year. Thus, it makes sense to believe that the time series exhibit cointegration. This dataset exhibits yearly seasonality, thus we set $M = 60$ to be larger than one seasonal period. We choose $\rho = 0.5$ and plot the results are given in Figure 3b. Again, adjusting for the cointegration dramatically increases predictive performance.

On both datasets, NonSTOP pays a penalty for selecting VARMA-OGD in the initial iterations before learning to heavily weight EC-VARMA-OGD. Note that NonSTOP outperforms VARMA-OGD by at least a factor of 3 on the original scale for both datasets.

7 Conclusions and Future Work

We presented general online algorithms that account for time series to account for nonstationary artifacts in both univariate and multivariate data. If known in advance, we demonstrate that these transformations lead to superior theoretical and empirical performance. In the case that these are unknown, we incorporate a finite set of possible transformations into a OLE framework called NonSTOP that can learn to appropriately weight the correct transformation. Speculating that accounting for nonstationary artifacts reduces correlation in the data, we presented a data dependent bound for FTL in the case of squared loss. In future work, we plan to explore extensions that hold for more complicated models like ARFIMA.

References

Oren Anava, Elad Hazan, Shie Mannor, and Ohad Shamir. Online learning for time series prediction. In JMLR: Workshop and Conference Proceedings of Conference on Learning Theory, volume 13, 2013.

Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-algorithm and applications. Theory of Computing, 8(1):121–164, 2012.

George Box, Gwilym Jenkins, and Gregory Reinsel. Time Series Analysis: Forecasting and Control. Wiley, 2008.

Peter Brockwell and Richard Davis. Time Series: Theory and Methods. Springer, 2009.

Nicolo Cesa-Bianchi and GÁbor Lugosi. Prediction, learning, and games. Cambridge university press, 2006.

John Duchi, Shai Shalev-Shwartz, Yoram Singer, and Tushar Chandra. Efficient projections onto the 1 1-ball for learning in high dimensions. In Proceedings of the 25th international conference on Machine learning, pages 272–279. ACM, 2008.
Graham Elliot, Thomas J. Rothenberg, and James H. Stock. Efficient tests for an autoregressive unit root. *Econometrica*, 64:813–836, 1996.

James D Hamilton. Time series analysis princeton university press. *Princeton, NJ*, 1994.

Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2-3):169–192, 2007.

Soren Johansen. Statistical analysis of cointegration vectors. *Journal of Economic Dynamics and Control*, 12:1–47, 1988.

Tze Leung Lai and Ching Zong Wei. Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems. *The Annals of Statistics*, pages 154–166, 1982.

Percy Liang. Cs229t/stat231: Statistical learning theory (winter 2014).

Chenghao Liu, Steven CH Hoi, Peilin Zhao, and Jianling Sun. Online arima algorithms for time series prediction. In *Thirtieth AAAI Conference on Artificial Intelligence*, 2016.

Lennart Ljung. System identification. In *Signal Analysis and Prediction*, pages 163–173. Springer, 1998.

Helmut Lütkepohl. *New introduction to multiple time series analysis*. Springer Science & Business Media, 2005.

Helmut Lütkepohl. Forecasting with varma models. *Handbook of economic forecasting*, 1:287–325, 2006.

Shai Shalev-Shwartz. Online learning and online convex optimization. *Foundations and Trends in Machine Learning*, 4(2):107–194, 2011.

David Thompson. Jackknifing multiple-window spectra. In *Proceedings of the 6th ICASSP*, pages 73–76, 1994.

Ruey Tsay. *Multivariate Time Series Analysis: With R and Financial Applications*. Wiley, 2013.

Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *ICML*, 2003.
A Proof of Theorem 3.1

We give a proof similar to [Anava et al., 2013] and [Liu et al., 2016] using our transformation notation, and with the more natural and relaxed assumption of invertibility of the MA process.

Proof. Step 1: Assume that \( \zeta(\tau) \) is a linear function such as the ones given in Table 2. Then \( \{\ell^M_t\} \) are convex loss functions, and we may invoke [Zinkevich, 2003] with a fixed step size \( \eta = \frac{D}{G(T)\sqrt{T}} \):

\[
\sum_{t=1}^{T} \ell^M_t(\gamma_t) - \min_{\gamma} \sum_{t=1}^{T} \ell^M_t(\gamma) = O\left(DG(T)\sqrt{T}\right)
\]

Note that the proof in [Zinkevich, 2003] uses a constant upper bound \( G \) on the gradients. Since we assume \( G(T) \) is a monotonically increasing function, the proof in [Zinkevich, 2003] follows through straightforwardly.

Step 2: Let \( \alpha, \beta \) denote the parameters of the underlying ARMA\((l_a, l_m)\) process. We define a few things:

\[
\tau(x_t^\infty(\alpha, \beta)) = \sum_{i=1}^{l_a} \alpha_i \tau(x_{t-i}) + \sum_{i=1}^{l_m} \beta_i \left( \tau(x_{t-i}) - \tau(x_{t-i}^\infty(\alpha, \beta)) \right)
\]

\[
x_t^\infty(\alpha, \beta) = \zeta(\tau(x_t^\infty(\alpha, \beta)))
\]

with initial condition \( \tau(x_t^\infty(\alpha, \beta)) = \tau(x_t) \) for \( t < 0 \). For convenience, assume that we have fixed data \( x_0, \ldots, x_{-h} \) so that \( \tau(x_0), \ldots, \tau(x_{-l_a}) \) exists. Denote

\[
f_t^\infty(\alpha, \beta) = \ell_t(x_t, x_t^\infty(\alpha, \beta))
\]

With this definition, we can write \( \tau(x_t^\infty(\alpha, \beta)) = \sum_{i=1}^{l_a+l_m} c_i(\alpha, \beta) \tau(x_{t-i}) \), i.e. as a growing AR process. Next, we define

\[
\tau(x_t^m(\alpha, \beta)) = \sum_{i=1}^{l_a} \alpha_i \tau(x_{t-i}) + \sum_{i=1}^{l_m} \beta_i \left( \tau(x_{t-i}) - \tau(x_{t-i}^m(\alpha, \beta)) \right)
\]

\[
x_t^m(\alpha, \beta) = \zeta(\tau(x_t^m(\alpha, \beta)))
\]

with initial condition \( \tau(x_t^m(\alpha, \beta)) = \tau(x_t) \) for \( m < 0 \). We relate \( M \) and \( m \) with this relation: \( M = m + l_a \). With this definition, we can write \( \tau(x_t^m(\alpha, \beta)) = \sum_{i=1}^{M} \tilde{c}_i(\alpha, \beta) \tau(x_{t-i}) \), i.e. as a fixed length AR process. Denote

\[
f_t^m(\alpha, \beta) = \ell_t(x_t, x_t^m(\alpha, \beta))
\]

Let \( (\alpha^*, \beta^*) = \arg\min_{\alpha, \beta \in K} \sum_{t=1}^{T} E[f_t(\alpha, \beta)] \). Recall that the only random part of the expectation is \( \varepsilon_t \). \( x_t \) is fixed in this quantity.

Lemma A.1 gives us that

\[
\min_{\gamma} \sum_{t=1}^{T} \ell^M_t(\gamma) \leq \sum_{t=1}^{T} f_t^m(\alpha^*, \beta^*)
\]
Lemma A.3 says that choosing \( m = \log_{\lambda_{\text{max}}} \left( (2\kappa TLM_{\text{max}} \sqrt{T}) ^{-1} \right) \) results in
\[
\left| \sum_{t=1}^{T} E[f^m_t(\alpha^*, \beta^*)] - \sum_{t=1}^{T} E[f^\infty_t(\alpha^*, \beta^*)] \right| = O(1)
\]

Lemma A.2 gives us that
\[
\left| \sum_{t=1}^{T} E[f^\infty_t(\alpha^*, \beta^*)] - \sum_{t=1}^{T} E[f_t(\alpha^*, \beta^*)] \right| = O(1)
\]

Chaining all of these gives us the final result:
\[
\sum_{t=1}^{T} \ell^m_t(\gamma_t) - \min_{\alpha, \beta \in K} \sum_{t=1}^{T} E[f_t(\alpha, \beta)] = O\left( DG(T) \sqrt{T} \right)
\]

Lemma A.1. For all \( m \) and \( \{x_t\} \) that satisfies the assumptions U1-U4, we have that
\[
\min_{\gamma} \sum_{t=1}^{T} \ell^m_t(\gamma) \leq \sum_{t=1}^{T} f^m_t(\alpha^*, \beta^*)
\]

Proof. We simply set \( \gamma^*_t = \tilde{c}_t(\alpha^*, \beta^*) \) and get \( \sum_{t=1}^{T} \ell^m_t(\gamma^*) = \sum_{t=1}^{T} f^m_t(\alpha^*, \beta^*) \). Thus, the minimum holds trivially. Note that we assume \( \gamma^* \in \mathcal{E} \). \( \square \)

Lemma A.2. For any data sequence \( \{x_t\} \) that satisfies the assumptions U1-U4, it holds that
\[
\left| \sum_{t=1}^{T} E[f^\infty_t(\alpha^*, \beta^*)] - \sum_{t=1}^{T} E[f_t(\alpha^*, \beta^*)] \right| = O(1)
\]

Proof. Let \( (\alpha', \beta') \) denote the parameters that generated the signal. Thus,
\[
f_t(\alpha', \beta') = \ell_t(x_t, x_t - \varepsilon_t)
\]
for all \( t \). Since \( \varepsilon_t \) is independent of \( \varepsilon_1, \ldots, \varepsilon_{t-1} \), the best prediction at time \( t \) will cause a loss of at least \( E[\ell_t(x_t, x_t - \varepsilon_t)] \). Since \( E[\varepsilon_t] = 0 \) and \( \ell_t \) is convex, it follows that \( (\alpha^*, \beta^*) = (\alpha', \beta') \) and that
\[
f_t(\alpha^*, \beta^*) = \ell_t(x_t, x_t - \varepsilon_t)
\]

We define a few things first. Let
\[
y_t = \tau(x_t) - \tau(x^\infty_t(\alpha^*, \beta^*)) - \varepsilon_t, \quad y_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-q+1} \end{bmatrix}
\]
Next, we note that $F$ as defined in Eq. 2. Thus, which shows that $y = \sum_{t=1}^{\infty} F y_{t-1}$, where $WLOG$ (and by assumption), we can assume that $E[\|y_0\|_2] \leq \rho$, where $\rho$ is some positive constant. 

Next we show that
\[
E[|y_t|] = E[|\tau(x_t) - \tau(x_t^\infty(\alpha^*, \beta^*)) - \varepsilon_t|] \leq \kappa \lambda_{\max}^t \rho
\]

We have that
\[
\tau(x_t) - \tau(x_t^\infty(\alpha^*, \beta^*)) - \varepsilon_t = \sum_{i=1}^{l_0} \alpha_i^* \tau(x_{t-i}) + \sum_{i=1}^{l_m} \beta_i^* \varepsilon_{t-i} + \varepsilon_t
\]
\[
- \sum_{i=1}^{l_0} \alpha_i^* \tau(x_{t-i}) - \sum_{i=1}^{l_m} \beta_i^* (\tau(x_{t-i}) - \tau(x_{t-i}^\infty(\alpha^*, \beta^*))) - \varepsilon_t
\]
\[
= - \sum_{i=1}^{l_m} \beta_i^* (\tau(x_{t-i}) - \tau(x_{t-i}^\infty(\alpha^*, \beta^*)) - \varepsilon_{t-i})
\]

which shows that $y_t = - \sum_{i=1}^{l_m} \beta_i^* y_{t-i}$. The companion matrix to this difference equation is exactly $F$ as defined in Eq. 2. Thus,
\[
y_t = F y_{t-1}
\]

Next, we note that
\[
|y_t| \leq \|y_t\|_2 = \|F y_{t-1}\|_2
\]
\[
= \|F^2 y_{t-2}\|_2
\]
\[
= \|F^3 y_0\|_2
\]
\[
= \|T \Lambda^t \Lambda^{-1} y_0\|_2
\]
\[
\leq \|T\|_2 \|T^{-1}\|_2 \|\Lambda^t\|_2 \|y_0\|_2
\]
\[
= \sigma_{\max}(T) \sigma_{\min}(T) \lambda_{\max}^t \|y_0\|_2
\]
\[
\leq \kappa \lambda_{\max}^t \|y_0\|_2
\]

Taking the expectation gives us $E[|y_t|] \leq \kappa \lambda_{\max}^t E[\|y_0\|_2] \leq \kappa \lambda_{\max}^t \rho$.

Now we combine this with the Lipschitz continuity of $\ell_t$ to get
\[
E[f_t^\infty(\alpha^*, \beta^*)] - E[f_t(\alpha^*, \beta^*)] = E[\ell_t(x_t, x_t^\infty(\alpha^*, \beta^*)) - \ell_t(x_t, x_t - \varepsilon_t)]]
\]
\[
\leq E[|\ell_t(x_t, x_t^\infty(\alpha^*, \beta^*)) - \ell_t(x_t, x_t - \varepsilon_t)|]
\]
\[
\leq L \cdot E[|x_t - x_t^\infty(\alpha^*, \beta^*) - \varepsilon_t|]
\]
\[
= L \cdot E[|\tau(x_t) - \tau(x_t^\infty(\alpha^*, \beta^*)) - \varepsilon_t|]
\]
\[
\leq \kappa L \rho \lambda_{\max}^t
\]

where we used Jensen’s inequality in the first inequality. Note that we also assume $x_t - \tilde{x}_t = \zeta(\tau(x_t)) - \zeta(\tau(\tilde{x}_t)) = (x_t) - \tau(\tilde{x}_t)$. This holds true for the transformations given in Table 2. Summing this from $t = 1$ to $T$ gives us the result.
Lemma A.3. For any data sequence \( \{x_t\} \) that satisfies the assumptions U1-U4, it holds that

\[
\left| \sum_{t=1}^{T} \mathbb{E} \left[ f_t^m (\alpha^*, \beta^*) \right] - \sum_{t=1}^{T} \mathbb{E} \left[ f_t^\infty (\alpha^*, \beta^*) \right] \right| = O(1)
\]

if we choose \( m = \log_{\lambda_{\text{max}}} \left( (2\kappa TL M_{\text{max}} \sqrt{t_m})^{-1} \right) \).

Proof. Fix \( t \). Note that for \( m < 0 \),

\[
|\tau (x_t^m (\alpha^*, \beta^*)) - \tau (x_t^\infty (\alpha^*, \beta^*))| = |\tau (x_t) - \tau (x_t^\infty (\alpha^*, \beta^*))| \\
\leq |\tau (x_t) - \tau (x_t^\infty (\alpha^*, \beta^*))| - \varepsilon_t|
\]

The right hand side of the inequality is simply \( |y_t| + |\varepsilon_t| \), where \( y_t \) is as defined in Lemma A.2. By assumption, \( \mathbb{E}[|\varepsilon_t|] < M_{\text{max}} \). Assume that \( M_{\text{max}} \) is large enough such that \( \mathbb{E}[|y_t|] \leq M_{\text{max}} \). This is a valid assumption since it is decaying exponentially as proved in Lemma A.2. It is important to note that \( \tau (x_t^m (\alpha, \beta)) \) and \( \tau (x_t^\infty (\alpha, \beta)) \) have no randomness in them since \( \tau \) is deterministic. Thus, for \( m < 0 \),

\[
|\tau (x_t^m (\alpha^*, \beta^*)) - \tau (x_t^\infty (\alpha^*, \beta^*))| = \mathbb{E} [|\tau (x_t^m (\alpha^*, \beta^*)) - \tau (x_t^\infty (\alpha^*, \beta^*))|] \\
\leq \mathbb{E}[|y_t| + |\varepsilon_t|] \\
\leq 2M_{\text{max}}
\]

Squaring both sides of the inequality results in

\[
(\tau (x_t^m (\alpha^*, \beta^*)) - \tau (x_t^\infty (\alpha^*, \beta^*))))^2 \leq 4M_{\text{max}}^2
\]

Next, we define

\[
z_t^m = \tau (x_t^m (\alpha^*, \beta^*)) - \tau (x_t^\infty (\alpha^*, \beta^*)) \quad \text{and} \quad z_t^m = \begin{bmatrix} z_t^m \\ z_{t-1}^m \\ \vdots \\ z_{t-q+1}^m \end{bmatrix}
\]

We have that

\[
\tau (x_t^m (\alpha^*, \beta^*)) - \tau (x_t^\infty (\alpha^*, \beta^*)) = \sum_{i=1}^{l_a} \alpha_i^* \tau (x_{t-i}) + \sum_{i=1}^{l_m} \beta_i^* (\tau (x_{t-i}) - \tau (x_{t-i}^\infty (\alpha^*, \beta^*))) \\
- \sum_{i=1}^{l_a} \alpha_i^* \tau (x_{t-i}) - \sum_{i=1}^{l_m} \beta_i^* (\tau (x_{t-i}) - \tau (x_{t-i}^\infty (\alpha^*, \beta^*))) \\
= - \sum_{i=1}^{l_m} \beta_i^* (\tau (x_{t-i}^m (\alpha^*, \beta^*)) - \tau (x_{t-i}^\infty (\alpha^*, \beta^*)))
\]

Thus, \( z_t^m = -\sum_{i=1}^{l_m} \beta_i^* z_{t-i}^m \). The companion matrix to this difference equation is exactly \( F \) as defined in Eq. [2]. Thus,

\[
z_t^m = F z_{t-1}^m
\]
We have that
\[ |z_t^m| \leq \|z_t^m\|_2 = \|Pz_{t-1}^m\|_2 \]
\[ = \|P^2z_{t-2}^m\|_2 \]
\[ = \|FM_0z_{t-m}\|_2 \]
\[ \leq \|T\|_2\|T^{-1}\|_2\|LM\|_2\|x_{t-m}\|_2 \]
\[ = \frac{\sigma_{\text{max}}(T)\lambda_{\text{max}}}{\sigma_{\text{min}}(T)}\sum_{i=0}^{t-m-1}(-i)^2 \]
\[ \leq \kappa\lambda_{\text{max}}\sqrt{q4M^2_{\text{max}}} \]
\[ = \kappa\lambda_{\text{max}}2M_{\text{max}}\sqrt{I_m} \]

Now we combine this with the Lipschitz continuity of $\ell_t$ to get
\[ |\mathbb{E}[f_t^m(\alpha^*, \beta^*)] - \mathbb{E}[f_t^\infty(\alpha^*, \beta^*)]| = |\mathbb{E}[\ell_t(x_t, x_t^m(\alpha^*, \beta^*))] - \mathbb{E}[\ell_t(x_t, x_t^\infty(\alpha^*, \beta^*))]| \]
\[ \leq |\mathbb{E}[\ell_t(x_t, x_t^m(\alpha^*, \beta^*)) - \ell_t(x_t, x_t^\infty(\alpha^*, \beta^*))]| \]
\[ \leq L \cdot |\mathbb{E}[x_t^m(\alpha^*, \beta^*) - x_t^\infty(\alpha^*, \beta^*)]| \]
\[ = L \cdot |\tau(x_t^m(\alpha^*, \beta^*)) - \tau(x_t^\infty(\alpha^*, \beta^*))| \]
\[ \leq 2\kappa LM_{\text{max}}\sqrt{I_m}\lambda_{\text{max}}^m \]

where in the first inequality we used Jensen’s inequality and we again used the assumption that $x_t - \tilde{x}_t = \tau(x_t) - \tau(\tilde{x}_t)$.

Summing this quantity from $t = 1$ to $T$ gives us the result:
\[ \left| \sum_{t=1}^T \mathbb{E}[f_t^\infty(\alpha^*, \beta^*)] - \sum_{t=1}^T \mathbb{E}[f_t^m(\alpha^*, \beta^*)] \right| \leq 2\kappa LM_{\text{max}}\sqrt{I_m}\lambda_{\text{max}}^m \]

Choosing $m = \log\lambda_{\text{max}}((2\kappa LM_{\text{max}}\sqrt{I_m})^{-1})$ gives us the desired $O(1)$ property. \qed

**B Proof of Theorem 4.1**

**Proof.** We again produce a proof of very similar structure to Anava et al. [2013] and Liu et al. [2016]. We first need to redefine a few things for the vector case. Let $D = \sup_{\gamma_1, \gamma_2 \in K} \|\gamma_1 - \gamma_2\|_F$, and $\|\nabla_\gamma \ell_t^m(\gamma)\|_F \leq G(T)$.

**Step 1:** Since $\{\ell_t^M\}$ are convex loss functions, we may invoke Zinkevich [2003] with a fixed step size $\eta = \frac{D}{G(T)\sqrt{T}}$:
\[ \sum_{t=1}^T \ell_t^M(\gamma_t) - \min_{\gamma} \sum_{t=1}^T \ell_t^M(\gamma) = O\left(DG(T)\sqrt{T}\right) \]
Furthermore, we define with initial condition
\[ x_t = \Delta x_t^\infty (\Pi, \Gamma, \Theta) = \Pi x_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} + \sum_{i=1}^{q} \Theta_i \left( \Delta x_{t-i} - \Delta x_{t-i}^\infty (\Pi, \Gamma, \Theta) \right) \]
\[ x_t^\infty (\Pi, \Gamma, \Theta) = \Delta x_t^\infty (\Pi, \Gamma, \Theta) + x_{t-1} \]
\[ f_t^\infty (\Pi, \Gamma, \Theta) = \ell_t (x_t, x_t^\infty (\Pi, \Gamma, \Theta)) \]

with initial condition \( \Delta x_t^\infty (\Pi, \Gamma, \Theta) = \Delta x_t \) for all \( t < 0 \). Note that we are assuming that we have fixed data \( x_0, \ldots, x_{-p} \). With this definition, we can write \( \Delta x_t^\infty (\Pi, \Gamma, \Theta) = c_0(\Pi, \Gamma, \Theta)x_{t-1} + \sum_{i=1}^{t+p-1} c_i(\Pi, \Gamma, \Theta)\Delta x_{t-i} \), i.e. as a growing AR process. This is because we can undo the reparameterization and write \( \Delta x_t \) in its original VARMA process form
\[ x_t^\infty (\Pi, \Gamma, \Theta) = \sum_{i=1}^{p} A_i x_{t-i} + \sum_{i=1}^{q} \Theta_i \left( x_{t-i} - x_{t-i}^\infty (\Pi, \Gamma, \Theta) \right) \]
\[ = \sum_{i=1}^{t+p} c_i(A, \Theta) x_{t-i} \]

as shown in the proof of Algorithm 1. Using the error corrected reparameterization here results in
\[ \Delta x_t^\infty (\Pi, \Gamma, \Theta) = c_0(\Pi, \Gamma, \Theta)x_{t-1} + \sum_{i=1}^{t+p-1} c_i(\Pi, \Gamma, \Theta)\Delta x_{t-i} \]

Furthermore, we define
\[ \Delta x_t^m (\Pi, \Gamma, \Theta) = \Pi x_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} + \sum_{i=1}^{q} \Theta_i \left( \Delta x_{t-i} - \Delta x_{t-i}^m (\Pi, \Gamma, \Theta) \right) \]
\[ x_t^m (\Pi, \Gamma, \Theta) = \Delta x_t^m (\Pi, \Gamma, \Theta) + x_{t-1} \]
\[ f_t^m (\Pi, \Gamma, \Theta) = \ell_t (x_t, x_t^m (\Pi, \Gamma, \Theta)) \]

with initial condition \( \Delta x_t^m (\Pi, \Gamma, \Theta) = \Delta x_t \) for all \( m < 0 \). We relate \( M = m + p - 1 \). With this definition, we can write \( \Delta x_t^m (\Pi, \Gamma, \Theta) = c_0(\Pi, \Gamma, \Theta)x_{t-1} + \sum_{i=1}^{M} c_i(\Pi, \Gamma, \Theta)\Delta x_{t-i} \) by using similar rearrangement arguments as shown above.

Lastly, we define
\[ (\Pi^*, \Gamma^*, \Theta^*) = \arg\min_{\Pi, \Gamma, \Theta} \sum_{t=1}^{T} \mathbb{E}[f_t(\Pi, \Gamma, \Theta)] \]

Recall that \( x_t \) is fixed in the expectation.

Lemma [B.1] gives us that
\[ \min_{\gamma} \sum_{t=1}^{T} \ell_t^M (\gamma) \leq \sum_{t=1}^{T} f_t^m (\Pi^*, \Gamma^*, \Theta^*) \]
Lemma B.3 says that choosing $m = \log_{\lambda_{\max}} \left( \left(2\kappa TLM_{\max}\sqrt{q} \right)^{-1} \right)$ results in

$$\left| \sum_{t=1}^{T} E[f_t^m(\Pi^*, \Gamma^*, \Theta^*)] - \sum_{t=1}^{T} E[f_t^\infty(\Pi^*, \Gamma^*, \Theta^*)] \right| = O(1)$$

Lemma B.2 gives us that

$$\left| \sum_{t=1}^{T} E[f_t^\infty(\Pi^*, \Gamma^*, \Theta^*)] - \sum_{t=1}^{T} E[f_t(\Pi^*, \Gamma^*, \Theta^*)] \right| = O(1)$$

Chaining all of these gives us the final result:

$$\sum_{t=1}^{T} \ell_t^M(\gamma_t) - \min_{\Pi, \Gamma, \Theta} \sum_{t=1}^{T} E[f_t(\Pi, \Gamma, \Theta)] = O \left( DG(T) \sqrt{T} \right)$$

Lemma B.1. For all $m$ and $\{x_t\}$ that satisfies assumptions M1-M3, we have that

$$\min_{\gamma} \sum_{t=1}^{T} \ell_t^m(\gamma) \leq \sum_{t=1}^{T} f_t^m(\Pi^*, \Gamma^*, \Theta^*)$$

Proof. Recall that $\gamma = \{\Pi, \Gamma, \Theta\}$ We simply set $\Pi^* = \tilde{c}_0(\Pi^*, \Gamma^*, \Theta^*)$, $\Gamma^* = \Gamma^*$ and let that be denoted by $\gamma^*$. Thus, we get $\sum_{t=1}^{T} \ell_t^M(\gamma^*) = \sum_{t=1}^{T} f_t^m(\Pi^*, \Gamma^*, \Theta^*)$. Thus, the minimum holds trivially. Note that we assume $\gamma^* \in \mathcal{E}$.

Lemma B.2. For any data sequence $\{x_t\}_{t=1}^{T}$ that satisfies assumptions M1-M5, it holds that

$$\left| \sum_{t=1}^{T} E[f_t^\infty(\Pi^*, \Gamma^*, \Theta^*)] - \sum_{t=1}^{T} E[f_t(\Pi^*, \Gamma^*, \Theta^*)] \right| = O(1)$$

Proof. We start the proof in the same exact way that Anava does. Let $(\Pi', \Gamma', \Theta')$ denote the parameters that generated the signal. Thus,

$$f_t(\Pi', \Gamma', \Theta') = \ell_t(x_t, x_t - \varepsilon_t)$$

for all $t$. Since $\varepsilon_t$ is independent of $\varepsilon_1, \ldots, \varepsilon_{t-1}$, the best prediction at time $t$ will cause a loss of at least $E[\ell_t(x_t, x_t - \varepsilon_t)]$. Since $E[\varepsilon_t] = 0$ and $\ell_t$ is convex, it follows that $(\Pi^*, \Gamma^*, \Theta^*) = (\Pi', \Gamma', \Theta')$ and that

$$f_t(\Pi^*, \Gamma^*, \Theta^*) = \ell_t(x_t, x_t - \varepsilon_t)$$

We define a few things first. Let

$$y_t = \Delta x_t - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*) - \varepsilon_t, \quad Y_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-q+1} \end{bmatrix}$$
Thus, \( y \) which shows that

\[
E[\|y_t\|_2] = E[\|\Delta x_t - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*) - \epsilon_t\|_2] \leq \kappa \lambda^t_{\text{max}} \rho
\]

We have that

\[
\Delta x_t - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*) - \epsilon_t = \Pi^* x_{t-1} + \sum_{i=1}^{p-1} \Gamma_i^* \Delta x_{t-i} + \sum_{i=1}^{q} \Theta_i^* \epsilon_{t-i} + \epsilon_t
\]

\[- \Pi^* x_{t-1} - \sum_{i=1}^{p-1} \Gamma_i^* \Delta x_{t-i} - \sum_{i=1}^{q} \Theta_i^* (\Delta x_{t-i} - \Delta x_{t-i}^\infty(\Pi^*, \Gamma^*, \Theta^*)) - \epsilon_t
\]

\[= - \sum_{i=1}^{q} \Theta_i^* (\Delta x_{t-i} - \Delta x_{t-i}^\infty(\Pi^*, \Gamma^*, \Theta^*)) - \epsilon_{t-i})
\]

which shows that \( y_t = - \sum_{i=1}^{q} \Theta_i^* y_{t-i} \). The companion matrix to this difference equation is \( F \). Thus,

\[
Y_t = FY_{t-1}
\]

Next, we note that

\[
\|y_t\|_2 \leq \|Y_t\|_2 = \|FY_{t-1}\|_2
\]

\[
= \|F^2Y_{t-2}\|_2
\]

\[
= \|F^2Y_0\|_2
\]

\[
= \|TA^tT^{-1}Y_0\|_2
\]

\[
\leq \|T\|_2 \|A^t\|_2 \|L\|_2 \|Y_0\|_2
\]

\[
= \frac{\sigma_{\text{max}}(T)}{\sigma_{\text{min}}(T)} \lambda^t_{\text{max}} \|Y_0\|_2
\]

\[
\leq \kappa \lambda^t_{\text{max}} \|Y_0\|_2
\]

Taking the expectation gives us

\[
E[\|y_t\|_2] \leq \kappa (1 - \epsilon)^t E[\|Y_0\|_2] \leq \kappa \lambda^t_{\text{max}} \rho.
\]

Now we combine this with the Lipschitz continuity of \( \ell_t \) to get

\[
|E[f_t^\infty(\Pi^*, \Gamma^*, \Theta^*)] - E[f_t(\Pi^*, \Gamma^*, \Theta^*)]| = |E[\ell_t(x_t, x_t^\infty(\Pi^*, \Gamma^*, \Theta^*))] - E[\ell_t(x_t, x_t - \epsilon_t)]|
\]

\[
\leq E[|\ell_t(x_t, x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)) - \ell_t(x_t, x_t - \epsilon_t)|]
\]

\[
\leq L \cdot E[\|x_t - x_t^\infty(\Pi^*, \Gamma^*, \Theta^*) - \epsilon_t\|_2]
\]

\[
= L \cdot E[\|\Delta x_t - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*) - \epsilon_t\|_2]
\]

\[
\leq \kappa L \rho \lambda^t_{\text{max}}
\]

where we used Jensen’s inequality in the first inequality. Summing this from \( t = 1 \) to \( T \) gives us the result.
Lemma B.3. For any data sequence \( \{x_t\}_{t=1}^T \) that satisfies assumptions M1-M3, it holds that
\[
\left| \sum_{t=1}^T \mathbb{E} [f_t^m(\Pi^*, \Gamma^*, \Theta^*)] - \sum_{t=1}^T \mathbb{E} [f_t^\infty(\Pi^*, \Gamma^*, \Theta^*)] \right| = O(1)
\]
if we choose \( m = \log_{\lambda_{\max}} \left( (2\kappa T LM_{\max}\sqrt{q})^{-1} \right) \).

Proof. Fix \( t \). Note that for \( m < 0 \),
\[
|\Delta x_t^m(\Pi^*, \Gamma^*, \Theta^*) - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)| = |\Delta x_t - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)| \\
\leq |\Delta x_t - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)| - \varepsilon_t + |\varepsilon_t|
\]
The right hand side of the inequality is simply \( \|y_t\|_2 + \|\varepsilon_t\|_2 \), where \( y_t \) is as defined in Lemma B.2. By assumption, \( \mathbb{E}[\|\varepsilon_t\|_2] < M_{\max} \). Assume that \( M_{\max} \) is large enough such that \( \mathbb{E}[\|y_t\|_2] \leq M_{\max} \). This is a valid assumption since it is decaying exponentially as proved in Lemma B.2. It is important to note that \( \Delta x_t^m(\Pi, \Gamma, \Theta) \) and \( \Delta x_t^\infty(\Pi, \Gamma, \Theta) \) have no randomness in them (recall that they can be written as a linear combination of past values of the realized data sequence \( \Delta x_t \)). Thus, for \( m < 0 \),
\[
\|\Delta x_t^m(\Pi^*, \Gamma^*, \Theta^*) - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)\|_2 = \mathbb{E}[\|\Delta x_t^m(\Pi^*, \Gamma^*, \Theta^*) - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)\|_2] \\
\leq \mathbb{E}[\|y_t\|_2 + \|\varepsilon_t\|_2] \\
\leq 2M_{\max}
\]

Squaring both sides of the inequality results in
\[
\|\Delta x_t^m(\Pi^*, \Gamma^*, \Theta^*) - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)\|_2^2 \leq 4M_{\max}^2
\]

Next, we define
\[
z_t^m = \Delta x_t^m(\Pi^*, \Gamma^*, \Theta^*) - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*), \quad Z_t^m = \begin{bmatrix} z_t^m \\ z_{t-1}^{m-1} \\ \vdots \\ z_{t-q+1}^{m-q+1} \end{bmatrix}
\]

We have that
\[
\Delta x_t^m(\Pi^*, \Gamma^*, \Theta^*) - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*) = \Pi^* x_{t-1} + \sum_{i=1}^k \Gamma_i^* \Delta x_{t-i} + \sum_{i=1}^q \Theta_i^* (\Delta x_{t-i} - \Delta x_{t-i}^m(\Pi^*, \Gamma^*, \Theta^*)) \\
- \Pi^* x_{t-1} - \sum_{i=1}^k \Gamma_i^* \Delta x_{t-i} - \sum_{i=1}^q \Theta_i^* (\Delta x_{t-i} - \Delta x_{t-i}^\infty(\Pi^*, \Gamma^*, \Theta^*)) \\
= - \sum_{i=1}^q \Theta_i^* (\Delta x_{t-i}^m - \Delta x_{t-i}^\infty) (\Pi^*, \Gamma^*, \Theta^*)
\]

Thus, \( z_t^m = - \sum_{i=1}^q \Theta_i^* z_{t-i}^{m-1} \). The companion matrix to this difference equation is exactly \( F \) as defined above. Thus,
\[
Z_t^m = FZ_{t-1}^{m-1}
\]
We have that
\[
\|z_t^m\|_2 \leq \|z_t^m\|_2 = \|Fz_{t-1}^m\|_2 \\
= \|F^2z_{t-2}^m\|_2 \\
= \|Fa_{2}\|_2 \\
= \|Ta^mT^{-1}z_{t-m}\|_2 \\
\leq \|T\|_2\|T^{-1}\|_2\|a^m\|_2\|z_{t-m}\|_2 \\
= \sigma_{\max}(T)\lambda_{\max}\left[\sum_{i=0}^{q-1}z_{t-m-i}\right]^2 \\
\leq k\lambda_{\max}^m\sqrt{q}\lambda_{\max}^m \\
= k\lambda_{\max}^m2M_{\max}\sqrt{q}
\]

Now we combine this with the Lipschitz continuity of \(\ell_t\) to get
\[
|E[f_t^m(\Pi^*, \Gamma^*, \Theta^*)] - E[f_t^\infty(\Pi^*, \Gamma^*, \Theta^*)]| = |E[\ell_t(x_t, x_t^m(\Pi^*, \Gamma^*, \Theta^*))] - E[\ell_t(x_t, x_t^\infty(\Pi^*, \Gamma^*, \Theta^*))]| \\
\leq L\cdot E[\|x_t^m(\Pi^*, \Gamma^*, \Theta^*) - x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)\|_2] \\
= L\cdot \|\Delta x_t^m(\Pi^*, \Gamma^*, \Theta^*) - \Delta x_t^\infty(\Pi^*, \Gamma^*, \Theta^*)\|_2 \\
\leq 2\kappa L\sigma_{\max}\sqrt{q}\lambda_{\max}^m
\]

where in the first inequality we used Jensen’s inequality.

Summing this quantity from \(t = 1\) to \(T\) gives us the result:
\[
\left|\sum_{t=1}^{T} E[f_t^m(\Pi^*, \Gamma^*, \Theta^*)] - \sum_{t=1}^{T} E[f_t^\infty(\Pi^*, \Gamma^*, \Theta^*)]\right| \leq 2\kappa TL\sigma_{\max}\sqrt{q}\lambda_{\max}^m
\]

Choosing \(m = \log_{\lambda_{\max}}\left((2\kappa TL\sigma_{\max}\sqrt{q})^{-1}\right)\) gives us the desired \(O(1)\) property. \(\square\)

C Proof of Theorem 3.2

Proof. Recall that for FTL, we have that
\[
\gamma_t \in \arg\min_{\gamma} \sum_{i=1}^{t-1} \ell_t(\gamma) = \arg\min_{\gamma} \frac{1}{2} \sum_{i=1}^{t-1} (x_t - \gamma^T \psi_t)^2 = \arg\min_{\gamma} \frac{1}{2} \|X_t - \Psi_t\gamma\|^2_2
\]

where \(X_t = [x_t \ldots x_1]^T, \Psi_t = [\psi_t \ldots \psi_1]^T\). Note that this is simply a recursive least squares procedure. This procedure can be computed in a recursive manner using the update equations:
\[
\gamma_{t+1} = \gamma_t + \frac{x_t - \psi_{t+1}^T \gamma_t}{1 + \psi_{t+1}^T \psi_{t+1}} \psi_{t+1} \psi_{t+1}^T V_{t-1} \psi_{t+1} \\
V_{t+1} = V_t - \frac{V_t \psi_{t+1} \psi_{t+1}^T V_t}{1 + \psi_{t+1}^T V_t \psi_{t+1}}
\]
where $V_t = \left(\sum_{i=1}^{t} \psi_i \psi_i^T\right)^{-1}$. Using the fact that $\ell_t$ is Lipschitz, we have that

$$
|\ell_t(\gamma_t) - \ell_t(\gamma_{t+1})| \leq L \|\gamma_{t+1} - \gamma_t\|_2
$$

$$
= L \left\| \frac{x_t - \gamma_t^T \psi_t}{1 + \psi_t^T V_{t-1} \psi_t} \psi_{t+1} \right\|_2
$$

$$
\leq L \left\| \frac{x_t - \gamma_t^T \psi_t}{1 + \psi_t^T V_{t-1} \psi_t} \right\|_2 \|\psi_t\|_2
$$

$$
\leq L^2 \|\psi_{t+1}\|_2
$$

$$
= L^2 \lambda_{\max}(V_{t-1})
$$

$$
= \frac{L^2}{(t-1)\lambda_{\min}(t-1)}
$$

where we used the fact that $\|\nabla_{\gamma} \ell_t(\gamma)\|_2 = |x_t - \gamma^T \psi_t| \|\psi_t\|_2 \leq L$, with $\frac{1}{1 + \psi_t^T V_{t-1} \psi_t} \leq 1$.

To complete the proof, we sum this quantity up and invoke Lemma C.1. To avoid the divide-by-zero, simply start the indexing at $t = 2$.

**Lemma C.1.** Let $\ell_1, \ldots, \ell_T$ be a sequence of loss functions. Let $\gamma_1, \ldots, \gamma_t$ be produced by FTL. Then

$$
\sum_{t=1}^{T} \ell_t(\gamma_t) - \min_{\gamma} \sum_{t=1}^{T} \ell_t(\gamma) \leq \sum_{t=1}^{T} \left[ \ell_t(\gamma_t) - \ell_t(\gamma_{t+1}) \right]
$$

This is fairly standard material. For reference to a proof, see [Liang].

## D Proof of Theorem 5.1

We first give an extension of the (randomized) Weighted Majority Algorithm to handle unbounded loss. We include the proof for completeness.

**Lemma D.1.** Assume we run the weighted majority algorithm (see [Shalev-Shwartz, 2011]) with the modified update rule $w_{t+1}(h) = w_t(h)(1 - \eta) \frac{\ell_t(h)}{b_t}$, where $b_t = \max_{t \in \{t-\delta, \ldots, t\}, h \in M} \ell_t(h)$, and $\ell_t(h) \geq 0$ for all $t, h$. Define the expected loss of the algorithm to be $\ell_t(ALG) := \mathbb{E}_{w_t} [\ell_t(h_t)] = \sum_h w_t(h) \ell_t(h)$, where $W_t := \sum_h w_t(h)$. Then the resulting regret bound is

$$
\text{Regret}_T := \sum_{t=1}^{T} \ell_t(ALG) - \min_{h \in M} \sum_{t=1}^{T} \ell_t(h) \leq 2B_T \sqrt{T \log n}
$$

where $B_T = \max_{t \in \{1, \ldots, T\}} b_t = \max_{t \in \{1, \ldots, T\}} \max_{h \in M} \ell_t(h)$.

**Proof.** Using the update

$$
w_{t+1}(h) = w_t(h)(1 - \eta) \frac{\ell_t(h)}{b_t} \implies w_{t+1}(h) = (1 - \eta) \sum_{r=1}^{t} \frac{\ell_r(h)}{b_r}
$$
Note that $\ell_{t_i}(h) / b_t \in [0, 1]$ by definition of $b_t$. Using this, we have

$$W_{T+1} = \sum_{h \in H} w_{T+1}(h) = \sum_{h \in H} w_{T}(h) (1 - \eta) \frac{\ell_{T}(h)}{b_T}$$

$$\leq \sum_{h \in H} w_{T}(h) \left( 1 - \eta \left( \frac{\ell_{T}(h)}{b_T} \right) \right)$$

$$= \sum_{h \in H} w_{T}(h) - \eta \sum_{h \in H} w_{T}(h) \left( \frac{\ell_{T}(h)}{b_T} \right)$$

$$= W_{T} - \frac{\eta}{b_T} \left( \sum_{h \in H} w_{T}(h) \ell_{T}(h) \right)$$

$$= W_{T} - \frac{\eta}{b_T} W_{T} \ell_T(\text{ALG})$$

$$= W_{T} \left( 1 - \frac{\eta}{b_T} \ell_T(\text{ALG}) \right)$$

$$\leq \frac{W_{T} e^{-\eta \ell_T(\text{ALG})}}{b_T}$$

$$\leq n e^{-\eta \sum_{i=1}^{T} \ell_{T}(\text{ALG})}$$

We also note that

$$W_{T+1} \geq \max_{h \in M} w_{T+1}(h) = \max_{h \in M} (1 - \eta) \sum_{i=1}^{T} \frac{\ell_{i}(h)}{b_t} = (1 - \eta) \min_{h \in M} \sum_{i=1}^{T} \frac{\ell_{i}(h)}{b_t}$$

And thus,

$$(1 - \eta) \frac{1}{b_T} \min_{h \in M} \sum_{i=1}^{T} \ell_{i}(h) \leq (1 - \eta) \frac{\min_{h \in M} \sum_{i=1}^{T} \ell_{i}(h)}{b_T} \leq n e^{-\eta \sum_{i=1}^{T} \ell_{i}(\text{ALG})} \leq n e^{-\eta \sum_{i=1}^{T} \ell_{i}(\text{ALG})}$$

The first and third inequalities are due to the fact that $B_T \geq b_t$, $\forall t \in \{1, \ldots, T\}$. For concise notation, denote $\ell_{1...T}(h) := \sum_{i=1}^{T} \ell_{i}(h)$.

Taking logs and using the fact that $x \leq -\log(1 - x) \leq x(1 + x)$, we have

$$\log(1 - \eta) \frac{1}{B_T} \left( \min_{h \in M} \ell_{1..T}(h) \right) \leq \log n - \frac{\eta}{B_T} \ell_{1..T}(\text{ALG})$$

$$\log(1 - \eta) \min_{h \in M} \ell_{1..T}(h) \leq B_T \log n - \eta \ell_{1..T}(\text{ALG})$$

$$-\eta(1 + \eta) \min_{h \in M} \ell_{1..T}(h) \leq B_T \log n - \eta \ell_{1..T}(\text{ALG})$$

$$-(1 + \eta) \min_{h \in M} \ell_{1..T}(h) \leq B_T \log n - \ell_{1..T}(\text{ALG})$$

$$\text{Regret}_T \leq \frac{B_T}{\eta} \log n + \eta \min_{h \in M} \ell_{1..T}(h)$$

Since $\ell_{1..T}(h) \leq B_T T$,

$$\text{Regret}_T \leq \frac{B_T}{\eta} \log n + \eta B_T T$$
Choosing $\eta = \sqrt{\frac{\log n}{T}}$, we get

$$\text{Regret}_T \leq 2B_T \sqrt{T \log n}$$

Our result from Theorem 3.1 gives us

$$\sum_{t=1}^{T} \ell_{t, h} \left( \gamma_{h}^{(t)} \right) - \min_{\alpha, \beta \in K} \sum_{t=1}^{T} \mathbb{E} \left[ f_t(\alpha, \beta) \right] = O \left( D_h G_h(T) \sqrt{T} \right)$$

for all $h \in M$. Adding these together gives

$$\sum_{t=1}^{T} \mathbb{E} \left[ \ell_t(h_t) \right] - \min_h \sum_{t=1}^{T} \ell_t(h) + \sum_{t=1}^{T} \ell_{t, h^*} \left( \gamma_{h^*}^{(t)} \right) - \min_{\alpha, \beta \in K} \sum_{t=1}^{T} \mathbb{E} \left[ f_t(\alpha, \beta) \right] = O \left( B_T \sqrt{T \log n} \right) + O \left( D_{h^*} G_{h^*}(T) \sqrt{T} \right)$$

where we define $h^* := \arg\min_h \sum_{t=1}^{T} \ell_t(h)$ for brevity of notation. The middle two terms cancel, and since $n$ is typically very small, we can treat it as a small constant and absorb it into the big $O$ notation. Combined with the definitions of $D_s, G_s(T)$, this leaves us with

$$\sum_{t=1}^{T} \mathbb{E} \left[ \ell_t(h_t) \right] - \min_{\alpha, \beta \in K} \sum_{t=1}^{T} \mathbb{E} \left[ f_t(\alpha, \beta) \right] \leq O \left( \max_{h} \{ B_T, D_s G_s(T) \} \sqrt{T} \right)$$

Regarding Remark 5.1.1 we show that $\ell_t(h) = \ell_{t, h} \left( \gamma_{h}^{(t)} \right) = O \left( G_h(t) \right)$. In the setting of least squares (which is one of the most widely used loss function in time series):

$$\ell_t^M \left( \gamma \right) = \frac{1}{2} (x_t - \gamma^T \psi_t)^2$$

The norm of the gradient of this loss is

$$\|x_t - \gamma^T \psi_t\| \|\psi_t\|_2 \leq G(t)$$

where the bound is by definition of $G(t)$. It’s easy to see that $(x_t - \gamma^T \psi_t) \leq O \left( \|\psi_t\|_2 \right)$ when $\mathcal{E}$ is a norm ball. Thus, $\ell_t(h) = O \left( G_h(t) \right)$. Because $B_T, G_h(T)$ are nondecreasing in $T$, it follows that $B_T = O(G_s(T))$.
E Data for Experiments

In this section, we display the data we used in Section 6.

Figure 4: Data plots. The top line has plots for univariate data, and the bottom line has plots for multivariate data.