Finite–dimensional global attractor for a system modeling the 2D nematic liquid crystal flow

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Abstract

We consider a 2D system that models the nematic liquid crystal flow through the Navier–Stokes equations suitably coupled with a transport-reaction-diffusion equation for the averaged molecular orientations. This system has been proposed as a reasonable approximation of the well-known Ericksen–Leslie system. Taking advantage of previous well-posedness results and proving suitable dissipative estimates, here we show that the system endowed with periodic boundary conditions is a dissipative dynamical system with a smooth global attractor of finite fractal dimension.

Keywords: Liquid crystal flow, kinematic transport, global attractors, finite fractal dimension.

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1 Introduction

We consider the following hydrodynamical system that models the nematic liquid crystal flows (cf. e.g., [6,7,16,22])

\[ \begin{align*}
  v_t + v \cdot \nabla v - \nu \Delta v + \nabla P &= -\lambda \nabla \cdot [\nabla d \odot \nabla d + \alpha (\Delta d - f(d)) \odot d - (1 - \alpha) d \otimes (\Delta d - f(d))], \\
  \nabla \cdot v &= 0, \\
  d_t + (v \cdot \nabla)d - \alpha (\nabla v)d + (1 - \alpha) (\nabla^T v)d &= \gamma (\Delta d - f(d)),
\end{align*} \]

(1.1) (1.2) (1.3)
in $Q \times (0, \infty)$. Here, $Q$ is a unit square in $\mathbb{R}^2$ (the more general case $Q = \Pi_{i=1}^2(0, L_i)$ with different periods $L_i$ in different directions can be treated in a similar way). The state variables $v, d$ and $P$ represent, respectively, the velocity field of the flow, the averaged macroscopic/continuum molecular orientations in $\mathbb{R}^2$ and the hydrodynamic pressure. The positive constants $\nu, \lambda$ and $\gamma$ stand for viscosity, the competition between kinetic energy and potential energy, and macroscopic elastic relaxation time (Deborah number) for the molecular orientation field, respectively. The parameter $\alpha \in [0, 1]$ is related to the shape of the liquid crystal molecule. The symbol $\nabla d \otimes \nabla d$ denotes the $2 \times 2$ matrix whose $(i, j)$-th entry is given by $\nabla_i d \cdot \nabla_j d$, for $1 \leq i, j \leq 2$. $\otimes$ is the usual Kronecker product, e.g., $(a \otimes b)_{ij} = a_i b_j$ for $a, b \in \mathbb{R}^2$. $f(d) = \frac{1}{\eta^2}(|d|^2 - 1)d : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\eta \in (0, 1]$ can be seen as a penalty function to approximate the strict unit-length constraint $|d| = 1$, which is due to liquid crystal molecules being of similar size (cf. [14]). This approximation fits well with the general theory of Landau’s order parameter (cf. [13]) and the Ginzburg–Landau type energy is also consistent with the model on variable degree of orientation (cf. [8]). It is obvious that $f(d)$ is the gradient of the scalar valued function $F(d) = \frac{1}{\eta^2}(|d|^2 - 1)^2 : \mathbb{R}^2 \rightarrow \mathbb{R}$.

In the present paper, we consider system (1.1)–(1.3) subject to the periodic boundary conditions

$$v(x + e_i) = v(x), \quad d(x + e_i) = d(x), \quad \text{for } x \in \mathbb{R}^2,$$

where unit vectors $e_i$ ($i = 1, 2$) are the canonical basis of $\mathbb{R}^2$. Namely, $v, d$ are well defined in the 2-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}$. Besides, we have the initial conditions

$$v|_{t=0} = v_0(x) \text{ with } \nabla \cdot v_0 = 0, \quad d|_{t=0} = d_0(x), \quad \text{for } x \in Q.$$  

Well-posedness issues for problem (1.1)–(1.5) for $\alpha \in [0, 1]$ have been studied in [23] (see also [22] for the case $\alpha = 1$ and [15] for the general Ericksen–Leslie model). More recently, the existence of a global weak solution has been proven in [2] with boundary conditions which are not necessarily periodic. Prior to these results, a number of papers (see, e.g., [3, 4, 9–11, 17–19]) have been devoted to the theoretical and numerical analysis of the highly simplified system studied first in [14]. In this case, the liquid crystal molecules are assumed to be "small" enough so that kinematic transport is neglected. However, such an assumption is physically questionable. On the contrary, system (1.1)–(1.3) accounts for the kinematic transport and also preserves dissipative properties expressed by a basic energy law similar to [14] (compare with [5]). Indeed, letting $(v, d)$ be a classical solution to problem (1.1)–(1.5). Multiplying equation (1.1) with $v$, equation (1.3) with $-\lambda(\Delta d - f(d))$, adding them together and integrating over $Q$, we get (cf. also [10])

$$\frac{1}{2} \frac{d}{dt} \int_Q (|v|^2 + \lambda |\nabla d|^2 + 2\lambda F(d)) \, dx = -\int_Q (\nu |\nabla v|^2 + \lambda \gamma |\Delta d - f(d)|^2) \, dx.$$

Taking advantage of this dissipative feature, in [23] it has also been proven that a given solution converges to a single stationary state and an estimate of the convergence rate has been obtained (cf. [24] for the simplified model). Here we want to show that the dissipative dynamical system associated with problem (1.1)–(1.5) possesses a global attractor with finite fractal dimension (see [12] for the simplified model). Our argument is slightly nonstandard since we do not know whether the semigroup defined through the global solution of problem (1.1)–(1.5) is strongly
continuous on the phase space. Thus, we achieve our goal by observing that the semigroup is closed in the sense of [20].

The plan of the paper goes as follows. In the next section we introduce the functional setup, we recall the well-posedness results established in [23] and we state the main theorem. Section 3 is devoted to prove a number of dissipative estimates that entail the existence of smooth compact absorbing sets in the phase space. This will yield the existence of the global attractor. Finally, in Section 4 we prove the finite dimensionality of the global attractor.

2 Preliminaries and Main Result

We recall the well-established functional setting for periodic boundary value problems (cf. e.g., [23, Chapter 2], also [22]):

\[ H^m_p(Q) = \{ v \in H^m(\mathbb{R}^2, \mathbb{R}^2) \mid v(x + e_i) = v(x) \}, \]
\[ \dot{H}^m_p(Q) = H^m_p(Q) \cap \left\{ v : \int_Q v(x) dx = 0 \right\}, \]
\[ H = \{ v \in L^2_p(Q, \mathbb{R}^2), \nabla \cdot v = 0 \}, \text{ where } L^2_p(Q, \mathbb{R}^2) = H^0_p(Q), \]
\[ V = \{ v \in H^1_p(Q), \nabla \cdot v = 0 \}, \]
\[ V' = \text{ the dual space of } V. \]

For the sake of simplicity, we denote the inner product on \( L^2_p(Q, \mathbb{R}^2) \) as well as \( H \) by \( (\cdot, \cdot) \) and the associated norm by \( \| \cdot \| \). The space \( H^m(Q, \mathbb{R}^2) \) will be shorthanded by \( H^m \) and the \( H^m \)-inner product \((\cdot, \cdot)_{H^m} \) can be given by
\[ (v, u)_{H^m} = \sum_{|\kappa|=0}^m (D^\kappa v, D^\kappa u), \]
where \( \kappa = (\kappa_1, \ldots, \kappa_n) \) is a multi-index of length \(|\kappa| = \sum_{i=1}^n \kappa_i \) and \( D^\kappa = \partial_{x_1}^{\kappa_1} \cdots \partial_{x_n}^{\kappa_n} \). For any \( m \in \mathbb{N}, m \geq 2 \), we recall the interior elliptic estimate, which states that for any \( U_1 \subset U_2 \) there is a constant \( C > 0 \) depending only on \( U_1 \) and \( U_2 \) such that
\[ \|d\|_{H^m(U_1)} \leq C (\|\Delta d\|_{H^{m-2}(U_2)} + \|d\|_{L^2(U_2)}). \]

In our case, we can choose \( Q' \) to be the union of \( Q \) and its neighborhood copies. Then we have
\[ \|d\|_{H^m(Q)} \leq C (\|\Delta d\|_{H^{m-2}(Q')} + \|d\|_{L^2(Q')}) = 9C (\|\Delta d\|_{H^{m-2}(Q')} + \|d\|_{L^2(Q')}), \tag{2.1} \]

Following [23], one can define mapping \( S \) (Stokes operator in the periodic case)
\[ Su = -\Delta u, \quad \forall u \in D(S) := \{ u \in H, \Delta u \in H \} = \dot{H}^2_p \cap H. \tag{2.2} \]

The operator \( S \) can be seen as an unbounded positive linear self-adjoint operator on \( H \). If \( D(S) \) is endowed with the norm induced by \( \dot{H}^0_p(Q) \), then \( S \) becomes an isomorphism from \( D(S) \) onto \( H \). More detailed properties of operator \( S \) can be found in [23].

We shall denote by \( C \) the genetic constants depending on \( \lambda, \gamma, \nu, Q, f \) and the initial data. Special dependence will be pointed out explicitly in the text if necessary. Since the parameters \( \nu, \lambda \) and \( \gamma \) do not play important roles in the proof when \( n = 2 \), we just set \( \nu = \lambda = \gamma = 1 \) in the remaining part of the paper.

We now report the global existence of strong/classical solutions to problem (1.1)–(1.5) for \( \alpha \in [0, 1] \) proven in [25] (see [22] for the special case \( \alpha = 1 \)).
Remark 2.1. We are not able to prove that functional. However, thanks to Lemma 2.1.

\[ \text{Lemma 2.1. Suppose that } v_i, d_i \text{ are global solutions to the problem \( (1.1)-(1.5) \) corresponding to the initial data } (v_0, d_0) \in V \times H^2_p(Q), i = 1, 2, \text{ respectively. Moreover, assume that the following estimate holds for any } T > 0 \]

\[ \|v_i(t)\|_V + \|d_i(t)\|_{H^2} \leq M, \quad \forall t \in [0, T]. \] (2.4)

Then for any } t \in [0, T], we have

\[ \|(v_1 - v_2)(t)\|^2 + \|(d_1 - d_2)(t)\|^2_{H^1} + \int_0^t (\|\nabla(v_1 - v_2)(\tau)\|^2 + \|\Delta(d_1 - d_2)(\tau)\|^2) d\tau \]

\[ \leq 2e^{C \tau} \left( \|v_0 - v_0\|^2 + \|d_0 - d_0\|^2_{H^1} \right), \] (2.5)

where } C \text{ is a constant depending on } M \text{ but not on } t. \]

Therefore, problem \( (1.1)-(1.5) \) has a unique strong solution. On account of the stated results, we are able to define a semigroup } \Sigma(t) \text{ by setting } (v(t), d(t)) = \Sigma(t)(v_0, d_0), \text{ for all } t \geq 0 \text{ and for any } (v_0, d_0) \in V \times H^2_p, \text{ where } (v, d) \text{ is the solution to } (1.1)-(1.5).

Remark 2.1. We are not able to prove that } \Sigma(t) \text{ is Lipschitz continuous from } V \times H^2_p \text{ to } V \times H^2_p. \text{ However, thanks to (2.5), } \Sigma(t) \text{ turns out to be a closed semigroup in the sense of [22].}

The dynamical system } (V \times H^2_p, \Sigma(t)) \text{ is a gradient system since it has a global Lyapunov functional

\[ \mathcal{E}(t) = \frac{1}{2}\|v(t)\|^2 + \frac{1}{2}\|\nabla d(t)\|^2 + \int_Q F(d(t)) dx, \] (2.6)

which satisfies the following basic energy law (cf. [18][22])

\[ \frac{d}{dt} \mathcal{E}(t) = -\|\nabla v(t)\|^2 - \|\Delta d(t) - f(d(t))\|^2, \quad \forall t \geq 0. \] (2.7)

The main result of this paper is as follows:

**Theorem 2.1.** \( (V \times H^2_p, \Sigma(t)) \) possesses a connected global attractor } \mathcal{A} \text{ with finite fractal dimension that is bounded in } (V \cap H^s_p) \times H^{s+1}_p \text{ for any given } s \in \mathbb{N}, s \geq 2. \]

**Remark 2.2.** Due to the existence of global Lyapunov functional } \mathcal{E}, \text{ a well-known result entails that } \mathcal{A} \text{ coincides with the unstable manifold of the set of equilibria (cf. e.g., [28]) . Moreover, on account of Proposition 2.7 (see also Proposition 3.3), we have that } \mathcal{A} \subset (C^\infty(Q))^2. \]
3 Dissipative Estimates

We begin to prove the first basic dissipative inequality that is a direct consequence of the basic energy law (2.7).

Lemma 3.1. There exist constants $C_0 > 0, \kappa > 0$ independent of initial data $(v_0, d_0)$ such that
\[
\frac{d}{dt} E(t) + \kappa E(t) \leq C_0, \quad \forall t \geq 0.
\] (3.1)

Proof. Set\[
g = -\Delta d + f(d).
\] (3.2)

Multiplying (3.2) by $d$, integrating over $Q$ and using the periodic boundary condition, we get
\[
\frac{1}{2} \|\nabla d\|^2 + \int_Q (|d|^4 - |d|^2) dx = \int_Q g \cdot d dx.
\]

By the Young inequality, we have
\[
\int_Q |d|^2 dx \leq \frac{1}{3} \int_Q |d|^4 dx + \frac{3}{4}|Q|.
\] (3.3)

Then it follows that
\[
\frac{1}{2} \|\nabla d\|^2 + \int_Q |d|^4 dx \leq \frac{1}{2} \|g\|^2 + \frac{3}{2} \|d\|^2 \leq \frac{1}{2} \int_Q |d|^4 dx + \frac{1}{2} \|g\|^2 + \frac{9}{8}|Q|.
\] (3.4)

Thus, we obtain
\[
\frac{1}{2} \|\nabla d\|^2 + \int_Q F(d) dx \leq \|\nabla d\|^2 + \int_Q |d|^4 dx \leq \frac{9}{4}|Q| + \| - \Delta d + f(d)\|^2.
\]

On the other hand, we have the Poincaré inequality for $v \in V$ that $\|v\| \leq C_P \|\nabla v\|$.
As a result, we can see that there exist constants $C_0 > 0, \kappa > 0$ independent of initial data $(v_0, d_0)$ such that
\[
\kappa E(t) \leq \|\nabla v\| + \| - \Delta d + f(d)\|^2 + C_0,
\]

which together with (2.7) yields (3.1). \hfill \Box

For any $R > 0$ and $(v_0, d_0) \in V \times H^2_p$ satisfying
\[
\|v_0\|^2_{H^1} + \|d_0\|^2_{H^2} \leq R,
\]
it is easy to infer from Lemma 3.1 the following

Proposition 3.1. There exists a time $t_0 = t_0(R)$ and positive constants $M_1, M_2$ independent of $R$ such that
\[
\|v(t)\|^2 + \|d(t)\|^2_{H^1} \leq M_1, \quad \forall t \geq t_0,
\] (3.5)

and
\[
\int_t^{t+1} (\|v(\tau)\|^2_{H^1} + \|d(\tau)\|^2_{H^2}) d\tau \leq M_2, \quad \forall t \geq t_0.
\] (3.6)
Proof. It easily follows from (3.1) that
\[ \mathcal{E}(t) \leq \mathcal{E}(0)e^{-\kappa t} + \frac{C_0}{\kappa}, \quad \forall \ t \geq t_0. \]
Taking
\[ t_0 := \frac{1}{\kappa} \ln \mathcal{E}(0) + \ln \kappa - \ln C_0, \]
we have for all \( t \geq t_0 \),
\[ \mathcal{E}(t) \leq \frac{2C_0}{\kappa}. \]
This and (3.3), (3.4) implies that there exists \( M_1 \) independent of \( R \) such that (3.5) holds. Next, from (2.7) we see that
\[ \int_t^{t+1} \left( \| \nabla v(\tau) \|^2 + \| - \Delta d(\tau) + f(d(\tau)) \|^2 \right) d\tau \leq \mathcal{E}(t) \leq \frac{2C_0}{\kappa}, \quad \forall \ t \geq t_0. \]
(3.7)
Using (3.5), we can immediately deduce
\[ \text{Lemma 3.3.} \]
The inequality (3.10) holds for the classical solution \((v,d)\) to problem (1.1)–(1.5) for \( t \geq t_0 \), with \( C \) being a constant that only depends on \( M_1 \).

As a result, we have
\[ \text{Proposition 3.2.} \]
There exists positive constants \( M_3, M_4 \) independent of \( R \) such that for all \( t_1 := t_0 + 1 \), the following uniform estimates hold
\[ \| v(t) \|^2_{H^1} + \| d(t) \|^2_{H^2} \leq M_3, \quad \forall \ t \geq t_1, \quad (3.11) \]
\[ \int_t^{t+1} \left( \| v(\tau) \|^2_{H^2} + \| d(\tau) \|^2_{H^1} \right) d\tau \leq M_4, \quad \forall \ t \geq t_1. \quad (3.12) \]
Proof. It follows from Lemma 3.3 (3.7) and the uniform Gronwall inequality that

\[ A(t + 1) \leq \frac{4C_0}{\kappa} e^{2C_0 \kappa}, \quad \forall \ t \geq t_0. \]  

Then by (3.5) and (3.8), there exists \( M_3 \) independent of \( R \) such that (3.11) holds. Besides, we infer from Lemma 3.3 and (3.13) that

\[
\int_t^{t+1} (\|\Delta v(\tau)\|^2 + \|\nabla(\Delta d(\tau) - f(d(\tau)))\|^2) d\tau \\
\leq A(t) + C \sup_{s \in [t,t+1]} (\|A(s)\|^2 + \|A(s)\|) \\
\leq M_5, \quad \forall \ t \geq t_1 := t_0 + 1,
\]

where \( M_5 \) is independent of \( R \). Since

\[
\|d\|_{H^3} \leq C (\|\nabla \Delta d\| + \|d\|_{H^2}) \\
\leq C (\|\nabla(\Delta d - f(d))\| + \|\nabla f(d)\| + \|d\|_{H^2}) \\
\leq C (\|\nabla(\Delta d - f(d))\| + \|d\|_{H^2}^5 + \|d\|_{H^2}),
\]

we can infer from (3.14) and (3.11) that (5.12) holds. The proof is complete.

Hence we have

**Theorem 3.1.** The dynamical system \((V \times H^2_p, \Sigma(t))\) has a bounded absorbing set in the phase space \( V \times H^2_p \). Namely, for any given \( R > 0 \) and each initial data \((v_0, d_0)\) in the ball \( B_R := \{(v_0, d_0) \in V \times H^2_p : \|v_0\|_V^2 + \|d_0\|_{H^2}^2 \leq R\}\), there exists \( B_0 \subset V \times H^2_p \) whose radius is independent of \( R \) such that \((v(t), d(t)) = \Sigma(t)(v_0, d_0) \in B_0 \) for all \( t \geq t_1(R) \).

We now prove some uniform higher-order estimates for the global solution \((v, d)\). For this purpose, we will take advantage of the following (cf., e.g., [12]):

**Lemma 3.4.** When \( s \geq 2 \), \( H^s \) is a Banach algebra. Assume that \( f, g \in H^s \). Then we have

\[
\|fg\|_{H^s} \leq C (\|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}),
\]

where the constant \( C \) is independent of \( f, g \).

**Lemma 3.5.** For any \( s \in \mathbb{N}, s \geq 2 \), the solution \((v, d)\) satisfies the inequality

\[
\frac{d}{dt} \left( \|v\|_{H^s}^2 + \|d\|_{H^{s+1}}^2 \right) + \|v\|_{H^{s+1}}^2 + \|d\|_{H^{s+2}}^2 \leq J(t), \quad \forall \ t > 0,
\]

where \( J \) is a positive function only depending on the norms \( \|d(t)\|_{H^2} \) and \( \|v(t)\|_{H^1} \) as well as on the parameter \( s \).
Proof. Taking the $H^s$ inner-product of (1.1) with $v$ and adding the $H^{s+1}$ inner-product of (1.3) with $d$, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|v\|_{H^s}^2 + \|d\|_{H^{s+1}}^2) + \|\nabla v\|_{H^s}^2 + \|\nabla d\|_{H^{s+1}}^2$$

$$= -\langle v, \nabla \cdot (\nabla v \cdot \nabla \cdot (\nabla d \circ \nabla d)) \rangle_{H^s} - \alpha \langle v, \nabla \cdot (\Delta d - f(d)) \nabla d \rangle_{H^s}$$

$$+ (1 - \alpha) \langle v, \nabla \cdot [\nabla \cdot (\Delta d - f(d))] \rangle_{H^s} - \langle v \cdot \nabla d, d \rangle_{H^{s+1}}$$

$$+ \alpha \langle d \cdot \nabla v, d \rangle_{H^{s+1}} - (1 - \alpha) \langle d \cdot \nabla v, v \rangle_{H^{s+1}} - \langle f(d), d \rangle_{H^{s+1}}.$$  

(3.17)

Arguing as in Proposition 3.2, we can easily show that if $v_0 \in V$ and $d_0 \in H^2$, then the uniform estimates holds

$$\|v(t)\|_{H^1} + \|d(t)\|_{H^2} \leq C, \quad \forall \, t \geq 0,$$

where $C > 0$ depends on $\|v_0\|_{H^1}$ and $\|d_0\|_{H^2}$.

Using Agmon’s inequality and suitable interpolation inequalities, we have

$$\|d\|_{L^\infty} \leq C \|d\|_{H^2}^\frac{1}{2} \|d\|^{\frac{1}{2}}$$

$$\|\nabla d\|_{L^\infty} \leq C \|d\|_{H^2}^\frac{1}{2} \|d\|_{H^1} \leq C \|d\|_{H^{s+2}} \|d\|_{H^1}^\frac{1}{2} \|d\|_{H^1}^\frac{1}{2}$$

$$\|\Delta d\|_{L^\infty} \leq C \|d\|_{H^2} \|d\|_{H^1} \leq C \|d\|_{H^{s+2}} \|d\|_{H^1}^\frac{1}{2} \|d\|_{H^1}^\frac{1}{2}$$

$$\|v\|_{L^\infty} \leq C \|v\|_{H^2} \|v\|_{H^1} \leq C \|v\|_{H^{s+1}} \|v\|_{H^1}$$

$$\|\nabla v\|_{L^\infty} \leq C \|v\|_{H^2} \|v\|_{H^1} \leq C \|v\|_{H^{s+1}} \|v\|_{H^1}$$

$$\|\nabla v\|_{H^s} \leq C \|v\|_{H^{s+1}} \|v\|_{H^1} \leq C \|v\|_{H^{s+1}} \|v\|_{H^1}$$

$$\|v\|_{H^s} \leq C \|v\|_{H^{s+1}} \|v\|_{H^1} \leq C \|v\|_{H^{s+1}} \|v\|_{H^1}$$

$$\|d\|_{H^{s+k}} \leq C \|d\|_{H^{s+2}} \|d\|_{H^2}^{-k}, \quad k = 1, 0, -1.$$

Then we can apply Lemma 3.4 to deduce

$$|\langle v, \nabla \cdot (\nabla d \circ \nabla v) \rangle_{H^s}| = |\langle \nabla v, \nabla v \cdot \nabla d \rangle_{H^s}|$$

$$\leq C \left( \|v\|_{L^\infty}^2 \|\nabla v\|_{H^s} + \|v\|_{L^\infty} \|\nabla v\|_{L^\infty} \|v\|_{H^s} \right)$$

$$\leq C \|v\|_{H^{s+1}} \|v\|_{H^1} \|v\| + C \|v\|_{H^{s+2}} \|v\|_{H^1} \|v\|_{H^1}^\frac{1}{2}$$

$$\leq \varepsilon \|v\|_{H^{s+1}}^2 + C \varepsilon \left( \|v\|_{H^1}^2 \|v\|_{H^1}^{2-k} \right).$$

$$|\langle v, \nabla \cdot (\nabla d \circ \nabla d) \rangle_{H^s}| = |\langle \nabla v, \nabla d \circ \nabla d \rangle_{H^s}|$$

$$\leq C \|\nabla v\|_{L^\infty} \|\nabla d\|_{L^\infty} \|\nabla d\|_{H^s} + C \|\nabla v\|_{H^s} \|\nabla d\|_{L^\infty}$$

$$\leq C \|v\|_{H^{s+1}} \|d\|_{H^{s+2}} \|v\|_{H^1} \|d\|_{H^1} \|d\|_{H^1} \|d\|_{H^1}$$

$$\leq \varepsilon \|v\|_{H^{s+1}}^2 + \varepsilon \|d\|_{H^{s+2}}^2 + C \varepsilon \left( \|v\|_{H^{s+2}}^2 \|d\|_{H^1} + \|v\|_{H^{s+1}} \|d\|_{H^1}^2 \|d\|_{H^1}^2 \right).$$

$$|\langle v, \nabla d, d \rangle_{H^{s+1}}|$$

$$\leq C \|v\|_{L^\infty} \|\nabla d\|_{L^\infty} \|d\|_{H^{s+1}} + C \|v\|_{L^\infty} \|\nabla d\|_{H^{s+1}} \|d\|_{L^\infty}$$

$$+ C \|v\|_{H^{s+1}} \|\nabla d\|_{L^\infty} \|d\|_{L^\infty}$$
\[
\begin{align*}
\leq C\|v\|_H^{\frac{1}{2}} \|d\|_H^{\frac{2+1}{2}} \|v\|_H^{\frac{2+1}{2}} \|v\|_H^{\frac{1}{2}} \|\nabla \varepsilon\|_H^{\frac{1}{2}} \|\nabla \alpha\|_H^\frac{1}{2} \|d\|_H^\frac{1}{2} \\
+ C\|v\|_H \|d\|_H^{\frac{2+1}{2}} \|v\|_H^{\frac{2+1}{2}} \|v\|_H^{\frac{1}{2}} \|\nabla \alpha\|_H^{\frac{1}{2}} \|d\|_H^{\frac{1}{2}} \\
+ C\|v\|_H \|d\|_H \|d\|_H^{\frac{2+1}{2}} \|\nabla \alpha\|_H^{\frac{1}{2}} \|d\|_H^{\frac{1}{2}} \\
\leq \varepsilon \|d\|_H^{\frac{2+2}{2}} + \varepsilon \|v\|_H^{\frac{2+1}{2}} + C\|v\|_H \|d\|_H \|d\|_H \|d\|_H \|d\|_H^\frac{2+1}{2} \|d\|_H^\frac{2+1}{2} \\
+ C\|v\|_H \|d\|_H \|v\|_H \|d\|_H \|d\|_H \|d\|_H \|d\|_H^\frac{2+1}{2} \|d\|_H^\frac{2+1}{2}. 
\end{align*}
\]

It remains to estimate the other four terms involving the parameter \(\alpha\) on the right-hand side of (3.11). We notice that

\[
\begin{align*}
-\alpha \langle \nabla v, (\Delta d - f(d)) \otimes d \rangle_{H^s} + \alpha \langle d \cdot \nabla v, d \rangle_{H^{s+1}} \\
= \alpha \langle \nabla v, (\Delta d - f(d)) \otimes d \rangle_{H^s} + \alpha \langle d \cdot \nabla v, d \rangle_{H^{s+1}} \\
= -\alpha \langle \nabla v, f(d) \otimes d \rangle_{H^s} + \alpha \langle \nabla v, \Delta d \otimes d \rangle_{H^s} + \alpha \langle d \cdot \nabla v, d \rangle_{H^{s+1}}. 
\end{align*}
\]

The first term is estimated as follows

\[
\begin{align*}
\alpha \langle \nabla v, f(d) \otimes d \rangle_{H^s} \\
\leq C \|\nabla v\|_{L^\infty} (\|d \otimes d\|_{H^s} + \|d^2 \otimes d\|_{H^s}) + C \|\nabla v\|_{H^s} \|f(d) \otimes d\|_{L^\infty} \\
\leq C \|\nabla v\|_{L^\infty} \|\nabla v\|_{H^s} + C \|d\|_{H^s} \|\nabla v\|_{L^\infty} \|d\|_{H^s} + C \|d\|_{H^s} (\|d\|_{L^\infty}^2 + \|d\|_{L^\infty}^4) \\
:= I_1 + I_2 + I_3,
\end{align*}
\]

where

\[
\begin{align*}
I_1 & \leq C \|\nabla v\|_H^\frac{1}{2} \|d\|_H^\frac{1}{2} \|v\|_H^\frac{1}{2} \|d\|_H^\frac{1}{2} \|d\|_H^\frac{1}{2} \\
& \leq \varepsilon \|d\|_H^{2+2} + \varepsilon \|v\|_H^{2+1} + C\|v\|_H \|d\|_H \|d\|_H \|d\|_H^\frac{2+1}{2} \|d\|_H^\frac{2+1}{2}. \\
I_2 & \leq C \|\nabla v\|_H^\frac{1}{2} \|d\|_H^\frac{1}{2} \|v\|_H^\frac{1}{2} \|d\|_H^\frac{1}{2} \|d\|_H^\frac{1}{2} \\
& \leq \varepsilon \|d\|_H^{2+2} + \varepsilon \|v\|_H^{2+1} + C\|v\|_H \|d\|_H \|d\|_H \|d\|_H^\frac{2+1}{2} \|d\|_H^\frac{2+1}{2}, \\
I_3 & \leq C \|\nabla v\|_{H^{s+1}} (\|d\|_{H^s} \|d\| + \|d\|_{H^s} \|d\|^2) \\
& \leq \varepsilon \|v\|_{H^{s+1}} + C\|d\|_{H^s} \|d\|^2 + \|d\|_{H^s} \|d\|^4. 
\end{align*}
\]

For the remaining two terms in (3.18), we have

\[
\alpha \langle \nabla v, f(d) \otimes d \rangle_{H^s} + \alpha \langle d \cdot \nabla v, d \rangle_{H^{s+1}}
\]

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\[
\begin{align*}
&= \alpha(\nabla v, \Delta d \otimes d)_{H^{-1}} + \alpha \sum_{|\kappa|=s} (D^\kappa \nabla v, D^\kappa (\Delta d \otimes d)) \\
&\quad + \alpha(d \cdot \nabla v, d)_{H^s} + \alpha \sum_{|\kappa|=s+1} (D^\kappa (d \cdot \nabla v), D^\kappa d) \\
&= \alpha(\nabla v, \Delta d \otimes d)_{H^{-1}} + \alpha(d \cdot \nabla v, d)_{H^s} + \alpha \sum_{|\kappa|=s} (D^\kappa \nabla v, D^\kappa \Delta d \otimes d) \\
&\quad + \alpha \sum_{|\kappa|=s} \sum_{|\xi| \leq s-1, \kappa \geq \xi} C_{\kappa, \xi} (D^\kappa \nabla v, D^\xi \Delta d \otimes D^{\kappa-\xi} d) \\
&\quad - \alpha \sum_{|\kappa|=s} (D^\kappa (d \cdot \nabla v), D^\kappa \Delta d) \\
&= \alpha(\nabla v, \Delta d \otimes d)_{H^{-1}} + \alpha(d \cdot \nabla v, d)_{H^s} + \alpha \sum_{|\kappa|=s} (D^\kappa \nabla v, D^\kappa \Delta d \otimes d) \\
&\quad + \alpha \sum_{|\kappa|=s} \sum_{|\xi| \leq s-1, \kappa \geq \xi} C_{\kappa, \xi} (D^\kappa \nabla v, D^\xi \Delta d \otimes D^{\kappa-\xi} d) \\
&\quad - \alpha \sum_{|\kappa|=s} \sum_{|\xi| \leq s-1, \kappa \geq \xi} C_{\kappa, \xi} (D^{\kappa-\xi} d \cdot D^\xi \nabla v, D^\kappa \Delta d) \\
&= \alpha(\nabla v, \Delta d \otimes d)_{H^{-1}} + \alpha(d \cdot \nabla v, d)_{H^s} \\
&\quad + \alpha \sum_{|\kappa|=s} \sum_{|\xi| \leq s-1, \kappa \geq \xi} C_{\kappa, \xi} (D^\kappa \nabla v, D^\xi \Delta d \otimes D^{\kappa-\xi} d) \\
&\quad - \alpha \sum_{|\kappa|=s} \sum_{|\xi| \leq s-1, \kappa \geq \xi} C_{\kappa, \xi} (D^{\kappa-\xi} d \cdot D^\xi \nabla v, D^\kappa \Delta d) \\
&:= \underbrace{I_4 + I_5 + I_6 + I_7}_{(3.19)}.
\end{align*}
\]

\begin{align*}
I_4 &\leq C\|\nabla v\|_{L^\infty} \|\Delta d \otimes d\|_{H^{s-1}} + \|\nabla v\|_{H^{s-1}} \|\Delta d \otimes d\|_{L^\infty} \\
&\leq C\|\nabla v\|_{L^\infty} \|\Delta d\|_{L^\infty} \|d\|_{H^{s-1}} + C\|\nabla v\|_{L^\infty} \|d\|_{H^{s+1}} \|d\|_{L^\infty} \\
&\quad + C\|\nabla v\|_{H^s} \|\Delta d\|_{L^\infty} \|d\|_{L^\infty} \\
&\leq C\|\nabla v\|_{H^{s+1}} \|d\|_{H^{s+2}}^{\frac{1}{2}} \|\Delta d\|_{H^1}^{\frac{1}{2}} \|d\|_{H^2}^{\frac{1}{2}} + C\|\nabla v\|_{H^{s+1}} \|d\|_{H^{s+2}}^{\frac{1}{2}} \|\Delta d\|_{H^1}^{\frac{1}{2}} \|d\|_{H^2}^{\frac{1}{2}} \|d\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\nabla v\|_{H^{s+1}} \|d||d||_{H^{s+2}}^{\frac{1}{2}} \|\Delta d\|_{H^1}^{\frac{1}{2}} \|d||d||_{H^2}^{\frac{1}{2}} \|d||d||_{L^2}^{\frac{1}{2}} \\
&\leq \varepsilon \|\nabla v\|_{H^{s+2}}^{\frac{2(s+1)}{s+2}} + \varepsilon \|d\|_{H^{s+2}}^{\frac{2(s+1)}{s+2}} \\
&\quad + C\varepsilon \left( \|\nabla v\|_{H^{s+1}} \|d||d||_{H^{s+2}}^{\frac{1}{2}} \|d||d||_{H^1}^{\frac{1}{2}} \|d||d||_{H^2}^{\frac{1}{2}} \|d||d||_{H^2}^{\frac{1}{2}} \|d||d||_{L^2}^{\frac{1}{2}} \right),
\end{align*}

\begin{align*}
I_5 &\leq C\|d\|_{L^\infty} \|\nabla v\|_{H^s} + C\|d\|_{L^\infty} \|\nabla v\|_{L^\infty} \|d\|_{H^s} \\
&\leq C\|\nabla v\|_{H^{s+1}} \|d||d||_{H^2} \|d||d||_{H^{s+1}} \|d||d||_{H^2} \|d||d||_{H^{s+1}} \|d||d||_{H^2} \|d||d||_{L^2} \|d||d||_{L^2} \\
&\leq \varepsilon \|\nabla v\|_{H^{s+2}}^{\frac{2(s+1)}{s+2}} + \varepsilon \|d\|_{H^{s+2}}^{\frac{2(s+1)}{s+2}} + C\varepsilon \left( \|d||d||_{H^{s+1}} \|d||d||_{H^2} \|d||d||_{H^{s+1}} \|d||d||_{H^2} \|d||d||_{H^2} \|d||d||_{L^2} \right),
\end{align*}

\begin{align*}
I_6 &\leq C \sum_{|\kappa|=s} \sum_{|\xi| \leq s-1, \kappa \geq \xi} \|D^\kappa \nabla v\| \|D^\xi \Delta d\|_{L^4} \|D^{\kappa-\xi} d\|_{L^4}
\end{align*}
Collecting all the above estimates and taking can be treated exactly as in (3.18) and (3.19). On the other hand, the interpolation inequalities since for all \( \alpha = 2 \), it follows that \( t(\|d\|_{H^2}^2 + \|d\|_{H^2}^2) \) depends only on \( \|H^s\|_{H^s} \leq \|H^s\|_{H^s} + C_\varepsilon (\|d\|_{H^2}^2 + \|d\|_{H^2}^2) \).

The term 
\[
(1 - \alpha) \langle v, \nabla \cdot (d \otimes (\Delta d - f(d))) \rangle - (1 - \alpha) \langle d \cdot \nabla^T v, d \rangle_{H^s+1}
\]
can be treated exactly as in (3.21) and (3.22). On the other hand, the interpolation inequalities and Young inequality yield that 
\[
\|v\|_{H^s} \leq \varepsilon \|v\|_{H^{s+1}} + C_\varepsilon \|v\|_{H^s}, \quad \|d\|_{H^{s+1}} \leq \varepsilon \|d\|_{H^{s+2}} + C_\varepsilon \|d\|_{H^2}.
\]
Collecting all the above estimates and taking \( \varepsilon \) sufficiently small, we obtain 
\[
\frac{d}{dt} (\|v\|_{H^s}^2 + \|d\|_{H^{s+1}}^2 + \|v\|_{H^{s+1}}^2 + \|d\|_{H^{s+2}}^2) \leq J(t),
\]
where \( J(t) \) depends only on \( \|v(t)\|_{H^1}, \|d(t)\|_{H^2} \) and \( s \).

**Remark 3.1.** Since \( \alpha \in [0, 1] \), it is easy to realize that the higher-order differential inequality (3.16) is uniform in \( \alpha \).

**Proposition 3.3.** For each \( s \in \mathbb{N}, s \geq 2 \), there exist positive constants \( M_6, M_7 \) independent of \( R \) such that for all \( t_s := t_0 + s \), the following uniform estimates hold 
\[
\|v(t)\|_{H^s}^2 + \|d(t)\|_{H^{s+1}}^2 \leq M_6, \quad \forall t \geq t_s,
\]
and 
\[
\int_t^{t+1} (\|v(\tau)\|_{H^{s+1}}^2 + \|d(\tau)\|_{H^{s+2}}^2) \, d\tau \leq M_7, \quad \forall t \geq t_s.
\]

**Proof.** By (3.11), we can see that inequality (3.16) holds for \( t \geq t_1 \), with \( J(t) \) being uniformly bounded by a constant \( J_0 = J_0(M_1, s) \) depending only on \( M_1 \) and \( s \). We argue by induction on \( s \). For \( s = 2 \), it follows that 
\[
\int_t^{t+1} J(\tau) d\tau \leq J_0(M_1, 2), \quad \forall t \geq t_1.
\]
We conclude from (3.12) and the uniform Gronwall inequality that
\[ \|v(t+1)\|^2_{H^2} + \|d(t+1)\|^2_{H^3} \leq M_4 + J_0(M_1, 2), \quad \forall t \geq t_1. \]

For \( t \geq t_2 := t_1 + 1 \), integrating (3.16) from \( t \) to \( t+1 \), we obtain
\[ \int_t^{t+1} (\|v(\tau)\|^2_{H^3} + \|d(\tau)\|^2_{H^4})d\tau \leq M_4 + 2J_0(M_1, 2), \quad \forall t \geq t_2 := t_1 + 1. \]

Assume that for \( s = k \), we have the following uniform estimates for \( t_k := t_1 + k - 1 \):
\[ \|v(t)\|^2_{H^k} + \|d(t)\|^2_{H^{k+1}} \leq K_1, \quad \forall t \geq t_k, \quad (3.23) \]
\[ \int_t^{t+1} (\|v(\tau)\|^2_{H^{k+1}} + \|d(\tau)\|^2_{H^{k+2}})d\tau \leq K_2, \quad \forall t \geq t_k, \quad (3.24) \]
where \( K_1 \) and \( K_2 \) are constants independent of \( R \). Then repeating the above argument, we can see that for \( t \geq t_{k+1} := t_1 + k \),
\[ \|v(t)\|^2_{H^{k+1}} + \|d(t)\|^2_{H^{k+2}} \leq K_2 + J_0(M_1, k + 1), \quad \forall t \geq t_{k+1}, \]
\[ \int_t^{t+1} (\|v(\tau)\|^2_{H^{k+2}} + \|d(\tau)\|^2_{H^{k+3}})d\tau \leq K_2 + 2J_0(M_1, k + 1), \quad \forall t \geq t_{k+1}. \]
The proof is complete. \( \square \)

Proposition 3.3 implies the existence of a compact absorbing set for our dynamical system:

**Theorem 3.2.** For each \( s \in \mathbb{N}, s \geq 2 \), the dynamical system \((V \times H^2_p, \Sigma(t))\) has a compact absorbing set bounded in the space \((V \cap H^s_p) \times H^{s+1}_p\). Namely, for any given \( R > 0 \) and each initial data \((v_0, d_0)\) in the ball
\[ B_R := \left\{ (v_0, d_0) \in V \times H^2_p : \|v_0\|^2_V + \|d_0\|^2_{H^2} \leq R \right\}, \]
there exists \( B_1 \subset (V \cap H^s_p) \times H^{s+1}_p \) whose radius is independent of \( R \) such that
\[ (v(t), d(t)) = \Sigma(t)(v_0, d_0) \in B_1, \quad \text{for all } t \geq t_s(R). \]

## 4 Finite-dimensional Global Attractor

To prove the existence of the global attractor, we make use of the following abstract result (cf. [20 Corollary 6])

**Lemma 4.1.** Let the closed semigroup \( \Sigma(t) \) has a connect compact attracting set \( \mathcal{K} \). Assume also that \( \Sigma(t)\mathcal{K} \subset \mathcal{K} \) for every \( t \) sufficient large. Then \( \Sigma(t) \) has a connected global attractor \( \mathcal{A} \).

Owing to the above lemma, we can take \( \mathcal{K} = B_1 \) as in Theorem 3.2 and obtain the existence of the global attractor \( \mathcal{A} \). Its boundedness properties follow from Proposition 3.3.

It remains to prove that the attractor \( \mathcal{A} \) has finite fractal dimension. First, we recall the definition of box-counting dimension.
Definition 4.1. Let $X$ be a (relatively) compact subset of a metric space $E$. For a given $\epsilon > 0$, let $N_\epsilon(X)$ be the minimal number of balls of radius $\epsilon$ that are necessary to cover $X$. Denote the Kolmogorov $\epsilon$-entropy of $X$ in $E$ by $\mathcal{H}_\epsilon(X) = \log_2 N_\epsilon(X)$. Then the fractal dimension of $X$ is the quantity

$$\dim_F X := \limsup_{\epsilon \to 0} \frac{\mathcal{H}_\epsilon(X)}{\log_2 \epsilon}.$$  

Also, we report a general result that ensures the finite fractal dimensionality of a compact set, namely (cf., e.g., [27, Theorem 4.1]),

Lemma 4.2. Let $X$ be a compact subset of Banach space $E$. We assume that there exist a Banach space $E_1$ such that $E_1$ is compactly embedded into $E$ and a mapping $L : X \to X$ such that $L(X) = X$ and

$$\|L(x_1) - L(x_2)\|_{E_1} \leq c\|x_1 - x_2\|_E, \quad \forall \, x_1, x_2 \in X.$$  

Then the fractal dimension of $X$ is finite and satisfies

$$\dim_F X \leq \mathcal{H}_{\frac{1}{4\epsilon}}(B_{E_1}(0, 1)),$$

where $B_{E_1}(0, 1)$ is the unit ball at origin in $E_1$.

We now prove the following smoothing property:

Lemma 4.3. For any given $(v_0, d_0) \in A$, $i = 1, 2$, the corresponding complete bounded trajectories $(v_i, d_i)$ satisfy the estimate

$$\|(v_1 - v_2)(1)\|_{H^2}^2 + \|(d_1 - d_2)(1)\|_{H^3}^2 \leq C \left(\|v_0 - v_2\|_{H^3}^2 + \|d_0 - d_0\|_{H^2}^2\right),$$

where $C$ is a constant depending on $\|v_0\|_{H^3}$ and $\|d_0\|_{H^4}$.

Proof. Denote

$$\bar{v} = v_1 - v_2, \quad \bar{d} = d_1 - d_2, \quad \bar{v}_0 = v_0 - v_2, \quad \bar{d}_0 = d_0 - d_0.$$  

(4.1)

Since $(v_i, d_i)$ are solutions to problem (1.1)–(1.5), we have

\begin{align*}
\bar{v}_{1t} + v_1 \cdot \nabla \bar{v}_1 - \nu \Delta \bar{v}_1 + \nabla P_1 & = -\nabla \cdot [\nabla d_1 \otimes \nabla d_1 + \alpha(\Delta d_1 - f(d_1)) \otimes d_1 - (1 - \alpha)d_1 \otimes (\Delta d_1 - f(d_1))], \quad (4.2) \\
\nabla \cdot \bar{v}_1 & = 0, \quad (4.3) \\
\bar{d}_{1t} + v_1 \cdot \nabla \bar{d}_1 - \alpha(\nabla v_1) & = \Delta d_1 - f(d_1), \quad (4.4) \\
v_{2t} + v_2 \cdot \nabla v_2 - \nu \Delta v_2 + \nabla P_2 & = -\nabla \cdot [\nabla d_2 \otimes \nabla d_2 + \alpha(\Delta d_2 - f(d_2)) \otimes d_2 - (1 - \alpha)d_2 \otimes (\Delta d_2 - f(d_2))], \quad (4.5) \\
\nabla \cdot v_2 & = 0, \quad (4.6) \\
\bar{d}_{2t} + v_2 \cdot \nabla \bar{d}_2 - \alpha(\nabla v_2) & = \Delta d_2 - f(d_2), \quad (4.7)
\end{align*}

in $Q \times \mathbb{R}$. For $s \geq 1$, taking the $H^s$-inner product of $\bar{v}$ with the equation obtained by subtracting (4.3) from (4.2) and the $H^{s+1}$-inner product of $\bar{d}$ with the equation obtained by subtracting (4.7) from (4.4), respectively, adding the two resulting equations together, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|\bar{v}\|_{H^s}^2 + \|\bar{d}\|_{H^{s+1}}^2) + \|\nabla \bar{v}\|_{H^s}^2 + \|\nabla \bar{d}\|_{H^{s+1}}^2.$$  

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The proof is complete.

which implies

Since $\mathcal{A} \subset (C^\infty(Q))^2$, then $(v_0, d_0) \in H^s \times H^{s+1}$ for any $s \geq 2$. We infer from (3.10) that

$$\|v_i(t)\|^2_{H^s} + \|d_i(t)\|^2_{H^{s+1}} \leq \left(\|v_0\|^2_{H^s} + \|d_0\|^2_{H^{s+1}}\right) e^{-t} + J, \quad \forall t \geq 0,$$

where $J$ depends only on $\|v_0\|_{H^1}, \|d_i\|_{H^2}$ that are uniformly bounded by Lemma [3.2]. As a result, we have uniform-in-time estimates of the Sobolev norms of $(v_i, d_i)$ of any order $s \in \mathbb{N}$. Using these higher-order estimates, it is not difficult to bound the right-hand side of (4.8) as in Lemma [3.5] (actually much simpler). We have

For the r.h.s of (4.8) $\leq \varepsilon \|\tilde{v}\|^2_{H^{s+1}} + \varepsilon \|\tilde{d}\|^2_{H^{s+2}} + C\varepsilon \left(\|\tilde{v}\|^2_{H^s} + \|\tilde{d}\|^2_{H^{s+1}}\right)$.

Choosing $\varepsilon = \frac{1}{2}$, we obtain that

$$\frac{d}{dt} \left(\|\tilde{v}\|^2_{H^s} + \|\tilde{d}\|^2_{H^{s+1}} + \|\tilde{v}\|^2_{H^{s+1}} + \|\tilde{d}\|^2_{H^{s+2}} \leq K(s) \left(\|\tilde{v}\|^2_{H^s} + \|\tilde{d}\|^2_{H^{s+1}}\right)\right),$$

where $K(s)$ is a constant depending on $\|v_0\|_{H^{s+1}}$ and $\|d_0\|_{H^{s+2}}$ at most.

Taking $s = 1$ in (4.9), it follows from the Gronwall inequality that

$$\|\tilde{v}(t)\|^2_{H^1} + \|\tilde{d}(t)\|^2_{H^2} \leq e^{K(1)t} \left(\|v_0\|^2_{H^1} + \|d_0\|^2_{H^2}\right), \quad \forall t \geq 0,$$

which implies

$$\int_0^1 (\|\tilde{v}(t)\|^2_{H^2} + \|\tilde{d}(t)\|^2_{H^3}) dt \leq \left(1 + e^{K(1)}\right) \left(\|v_0\|^2_{H^1} + \|d_0\|^2_{H^2}\right).$$

Next, taking $s = 2$, multiplying (4.9) by $t$ and integrating in time from 0 to 1, we obtain

$$\|\tilde{v}(1)\|^2_{H^2} + \|\tilde{d}(1)\|^2_{H^3} \leq \int_0^1 (\|\tilde{v}(t)\|^2_{H^2} + \|\tilde{d}(t)\|^2_{H^3}) dt + K(2) \int_0^1 t \left(\|\tilde{v}(t)\|^2_{H^2} + \|\tilde{d}(t)\|^2_{H^3}\right) dt \leq \left(1 + K(2)\right) \left(1 + e^{K(1)}\right) \left(\|v_0\|^2_{H^1} + \|d_0\|^2_{H^2}\right).$$

The proof is complete.

Taking

$$X = \mathcal{A}, \quad E = V \times H^2_p, \quad E_1 = (V \cap H^2_p) \times H^3_p, \quad L = \Sigma(1),$$

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we can apply Lemma 4.2 to conclude that the global attractor $\mathcal{A}$ has finite fractal dimension in the $H^1 \times H^2$-metric. The proof of Theorem 2.1 is now complete.

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