Properties and Applications of Truncated Exponential Marshall Olkin Weibull Distribution

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Abstract. A new flexible compound distribution with four parameters called Truncated Exponential Marshall Olkin Weibull (TEMOW) is proposed as a sub-model of a new generator of continuous distributions named Truncated Exponential Marshall Olkin-G family. Reliability characteristics with several main statistical properties are presented. The maximum likelihood estimation method is adopted to estimate the unknown parameters. Furthermore, to assess the usefulness and flexibility, the TEMOW distribution applied upon simulation study besides real application with two real data set. The simulation results clearly shown the flexible performance of the maximum likelihood estimators for the parameters. Also, the real application results clearly shown that the proposed distribution has outstanding performance than other considered distributions for all information criteria.

1. Introduction

Over the last decades, various statistical distributions have been used to model data in a variety of areas. However, in some areas of use, there is a strong need to construct expanded versions of these distributions to adapt to particular real-life data. Several new generalized distributions have recently been developed and studied. Moreover, attempts continue to identify new families of probability distributions that expand well-known distribution families and at the same time providing greater flexibility in the practice of modeling results. One of the most important distributions is the Weibull distribution. It has been applied in various fields, especially to fit lifetime data. However, some of these applications are limited partly and this limitation undoubtedly inspired researchers to develop different generalizations and many modifications of this distribution in order to improve its flexibility. So, in this paper, based on the existing Weibull distribution, we introduce a new family of probability distributions arising from composing the work of Marshall and Olkin[1] with the work of Eugene et al. [2].

Marshall and Olkin [1] used tilting parameter $\alpha$ ($\alpha \geq 0$) to proposed an elastic family of probability distributions named Marshall Olkin-G (MO-G) with the following cumulative distribution function (cdf) and probability density function (pdf)

$$M(x)_{MO} = \frac{G(x)}{\alpha + \bar{\alpha} G(x)} ; \alpha \geq 0 , \bar{\alpha} = 1 - \alpha$$

(1)

$$m(x)_{MO} = \frac{\alpha g(x)}{(\alpha + \bar{\alpha} G(x))^2}$$

(2)

where $G(x)$ and $g(x)$ are the cdf and pdf of any baseline distribution(see [3][4]).
Eugene et al. [2] introduced a new family built on the interval [0,1] beta distribution with the following cdf and pdf
\[
F(x) = \frac{1}{\beta(a,b)} \int_0^x (1 - z)^{a-1} (1 - (1 - z)^b) dz ; \quad 0 < a, b < \infty
\]
\[
f(x) = \frac{1}{\beta(a,b)} (1 - M(x))^{a-1} (1 - M(x))^{b-1} m(x)
\]
where \(\beta(a,b)\) is the Beta function, \(M(x)\) and \(m(x)\) are the cdf and pdf of any distribution. Then by taking \(M(x)\) to be the cdf of the normal distribution, Eugene et al. defined the beta normal distribution and also derived some properties of this new distribution (see [2][5]).

Based on the two works above, our new family called truncated exponential Marshall Olkin – G has been built (for truncated distribution see [6]). The rest of this paper is structured as follows: In Section 2, the truncated exponential Marshall Olkin – G family is introduced with some general expressions. Then, in Section 3, attention is given to the truncated exponential Marshall Olkin Weibull as a particular member of this new family. In Sections 4, and 5 reliability measures and various properties of new distribution along with its entropies, and reliability stress strength model are introduced. In Section 6, the maximum likelihood estimators (MLEs) of the unknown parameters are presented. In Section 7, a simulation study is carried out to exhibit the performances of the MLEs, as well as, real applications are adopted to illustrate the behavior of the new distribution with some existing distributions. Finally, conclusions are addressed in Section 8.

2. Truncated exponential Marshall Olkin-G family

Consider the cdf and pdf of the [0,1] truncated exponential (TE) distribution as
\[
W(x)_{TE} = \frac{1}{1 - e^{-\theta x}} ; \quad 0 < x < 1, \theta > 0
\]
\[
w(x)_{TE} = \frac{\theta e^{-\theta x}}{1 - e^{-\theta}}
\]
Now, composing the two continuous cdfs, i.e., \(W\) and \(M\), so that the cdf \(F(x) = W(M(x))\) and pdf \(f(x) = \frac{\partial}{\partial x} F(x)\) will be
\[
F(x) = \int_0^{M(x)} \frac{\theta e^{-\theta u}}{1 - e^{-\theta}} du = \frac{1 - e^{-\theta M(x)}}{1 - e^{-\theta}}
\]
\[
f(x) = \frac{\theta m(x)e^{-\theta M(x)}}{1 - e^{-\theta}}
\]
Substituting (1) and (2) in (7) and (8), a new family of probability distributions called Truncated Exponential Marshall Olkin-G (TEMO-G) can be proposed with the following cdf and pdf
\[
F(x)_{TEMO-G} = \frac{1 - e^{-\theta \frac{G(x)}{\alpha + \bar{G}(x)}}}{1 - e^{-\theta}}
\]
\[
f(x)_{TEMO-G} = \frac{\theta \alpha g(x)e^{-\theta \frac{G(x)}{\alpha + \bar{G}(x)}}}{(1 - e^{-\theta})(\alpha + \bar{G}(x))^2}
\]
The formula of the pdf in (10) can be expansion by using the expressions \(e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^i\) and \((1 - x)^{-a} = \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{\Gamma(a)} \frac{x^i}{i!}, |x| < 1, a > 0\) with performing some simple mathematical steps as
\[
f^E(x)_{TEMO-G} = \delta g(x) (G(x))^{1+j}
\]
where
\[
\delta = \frac{1}{1 - e^{-\theta}} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} \Gamma(i+j+2) \theta^{i+1} (\bar{G})^j}{\Gamma(i+2) \alpha^{i+j+1}}
\]
3- Truncated exponential Marshall Olkin Weibull distribution

Let \( G(x) \) and \( g(x) \) in (9), (10), and (11) be the cdf and pdf of the Weibull distribution [7] with two non-negative parameters \( \lambda \) and \( \beta \), a new proposed distribution named Truncated Exponential Marshall Olkin Weibull (TEMOW) distribution is attained as a member of TEMO-G family with the cdf, pdf and expanded pdf respectively given by

\[
F(x)_{TEMOW} = 1 - e^{-\frac{1 - e^{-\lambda x^\beta}}{\alpha + \tilde{\alpha}(1 - e^{-\lambda x^\beta})}}
\]

\[
f(x)_{TEMOW} = \frac{\theta \alpha \beta x^{\beta-1} e^{-\lambda x^\beta} e^{-\frac{1 - e^{-\lambda x^\beta}}{\alpha + \tilde{\alpha}(1 - e^{-\lambda x^\beta})}}}{(1 - e^{-\beta})(\alpha + \tilde{\alpha}(1 - e^{-\lambda x^\beta}))^2}
\]

and

\[
f^E(x)_{TEMOW} = \delta \lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \left(1 - e^{-\lambda x^\beta}\right)^{1+j}
\]

with \( \delta \) as in (12).

The plots of some possible shapes of the cdf and pdf of the TEMOW distribution for specific values of four parameters are presented in figures 1 and 2. Figure 1 clearly shows the characteristics of the cdf that are \( 0 \leq F(x)_{TEMOW} \leq 1 \), strictly increasing, and continuous. Figure 2 shows some of the possible shapes of TEMOW pdf such as decreasing, right-skewed, symmetric, and semi-symmetric. So, it is very flexible to model positive data.

4- Reliability measures and properties of TEMOW distribution

The reliability measures (reliability, hazard, reverse hazard, and cumulative hazard functions) (see [8]) of the proposed TEMOW distribution easily can be found respectively as follows

\[
R(x)_{TEMOW} = 1 - F(x)_{TEMOW} = 1 - \frac{1 - e^{-\frac{1 - e^{-\lambda x^\beta}}{\alpha + \tilde{\alpha}(1 - e^{-\lambda x^\beta})}}}{1 - e^{-\beta}}
\]
The plots of some possible shapes of the reliability and hazard functions of the TEMOW distribution for specific values of four parameters are presented in figures 3 and 4. Figure 3 clearly shows the characteristics of the reliability that are $0 \leq R(x)_{TEMOW} \leq 1$, strictly decreasing, and continuous. Figure 4 shows some of the possible shapes of TEMOW hazard function such as monotonically increasing, monotonically decreasing, bathtub, right-skewed, left-skewed, and semi-constant.

Furthermore, the most essential statistical properties of TEMOW distribution can be found as follows: The $r^{th}$ moment can be found regarding to (15) as follows

$$E(X^r)_{TEMOW} = \int_0^\infty x^r f(x)_{TEMOW} dx = \delta \int_0^\infty x^r \lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \left(1 - e^{-\lambda x^{\beta}}\right)^i e^{-\theta} dx$$

Using the expression $(1 - x)^a = \sum_{i=0}^\infty (-1)^i \Gamma(a+1) \Gamma(a-i+1) x^i$, $|x| < 1, a > 0$, the $E(X^r)_{TEMOW}$ will be
\[ E(X^r)_{TEMOW} = \delta \int_0^\infty x^r \lambda^\beta \ x^{\beta-1} e^{-\lambda x^\beta} \frac{\Gamma(i + j + 1)}{m!} \frac{\sum_{m=0}^\infty \frac{(-1)^m \Gamma(i + j + 1)}{m!} \Gamma(i + j - m + 1)}{m!} \int_0^\infty x^r \lambda^\beta \ x^{\beta-1} e^{-\lambda x^{\beta+1}} \ dx \]

\[ = \delta \sum_{m=0}^\infty \frac{(-1)^m \Gamma(i + j + 1)}{m!} \Gamma(i + j - m + 1) \frac{1}{m!} \frac{\Gamma(i + j - m + 1)}{\lambda^{i+1}} \Gamma\left(\frac{r}{\beta}\right) (m + 1)^\beta \]

But \( \int_0^\infty x^r \lambda m \ x^{\beta-1} e^{-\lambda x^\beta} \ dx \) represent the \( r^{th} \) moment of Weibull distribution with parameters \( \lambda(m + 1) \) and \( \beta \), i.e. \( \int_0^\infty x^r \lambda m \ x^{\beta-1} e^{-\lambda x^\beta} \ dx \) is defined by Moors \[10\], as

\[ E(X^r)_{TEMOW} = \delta \sum_{m=0}^\infty \frac{(-1)^m \Gamma(i + j + 1)}{m!} \Gamma(i + j - m + 1) \frac{1}{m!} \frac{\Gamma(i + j - m + 1)}{\lambda^{i+1}} \Gamma\left(\frac{r}{\beta}\right) (m + 1)^\beta \]

5. The entropies, reliability stress strength and order statistics of TEMOW distribution

Shannon entropy: Regarding to (14), the TEMOW Shannon entropy can be achieved as follows

\[ SE_{TEMOW} = -\int_0^\infty \ln(f(x)_{TEMOW}) f(x)_{TEMOW} \ dx \]

\[ \ln(f(x)_{TEMOW}) = \ln\left(\frac{\theta \alpha \lambda^\beta}{1 - e^{-\theta}}\right) + (\beta - 1) \ln(x) - \lambda x^\beta - \theta \frac{1 - e^{-\lambda x^\beta}}{\alpha + \bar{\alpha} (1 - e^{-\lambda x^\beta})} \]

\[ - 2 \ln\left(\alpha + \bar{\alpha} (1 - e^{-\lambda x^\beta})\right) \]

Then

\[ SE_{TEMOW} = \ln\left(\frac{1 - e^{-\theta}}{\theta \alpha \lambda^\beta}\right) - (\beta - 1) E(\ln(X)) + \lambda E(X^\beta) + \theta E\left(\frac{1 - e^{-\lambda x^\beta}}{\alpha + \bar{\alpha} (1 - e^{-\lambda x^\beta})}\right) + 2E\left(\ln\left(\alpha + \bar{\alpha} (1 - e^{-\lambda x^\beta})\right)\right) \]

where \( E(X^\beta) \) as in (20) with \( r = \beta \), and
\[ E(\ln(X)) = \int_0^\infty \ln(x) f(x)_{\text{TEMOW}} \, dx = \delta \int_0^\infty \ln(x) \lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \left( 1 - e^{-\lambda x^\beta} \right)^{i+j} \, dx \]

Since \( \left( 1 - e^{-\lambda x^\beta} \right)^{i+j} = \sum_{m=0}^\infty \frac{(-1)^m \Gamma(i+j+1)}{m! \Gamma(i+j-m+1)} e^{-\lambda m x^\beta} \), then

\[ E(\ln(X)) = \delta \sum_{m=0}^\infty \frac{(-1)^m \Gamma(i+j+1)}{\beta(m+1)! \Gamma(i+j-m+1)} \left[ \psi(1) - \ln(\lambda(m+1)) \right] \] (25)

and

\[ E \left( \frac{1 - e^{-\lambda x^\beta}}{\alpha + \tilde{a} (1 - e^{-\lambda x^\beta})} \right) = \int_0^\infty \frac{1 - e^{-\lambda x^\beta}}{\alpha + \tilde{a} (1 - e^{-\lambda x^\beta})} f(x)_{\text{TEMOW}} \, dx \]

Using the expressions \((1-x)^{-\alpha}, (1-x)^{\alpha}, \) and \(e^{-x}\) that mentioned previously, we obtain

\[ E \left( \frac{1 - e^{-\lambda x^\beta}}{\alpha + \tilde{a} (1 - e^{-\lambda x^\beta})} \right) = \sum_{m,l,t=0}^\infty \frac{(-1)^{m+l+t} \Gamma(m+2) \tilde{a}^m}{(m+2)!! \Gamma(m-l+2)} (\lambda l)^t E(X^t) \beta \] (26)

and

\[ E \left( \ln \left( \alpha + \tilde{a} (1 - e^{-\lambda x^\beta}) \right) \right) = \int_0^\infty \ln \left( \alpha + \tilde{a} (1 - e^{-\lambda x^\beta}) \right) f(x)_{\text{TEMOW}} \, dx \]

Using the expressions \(\ln(1-x) = -\sum_{i=0}^\infty \frac{x^{i+1}}{i+1}; |x| < 1, (1-x)^{\alpha}, \) and \(e^{-x}\) we obtain

\[ E \left( \ln \left( \alpha + \tilde{a} (1 - e^{-\lambda x^\beta}) \right) \right) = \ln(\alpha) + \sum_{m,l,t=0}^\infty \frac{(-1)^{m+l+t+2} \Gamma(m+2) \tilde{a}^m}{(m+1)!! \Gamma(m-l+2)} (\lambda l)^t E(X^t) \beta \] (27)

where \(E(X^t)\beta\) as in (20) with \(r = t \beta\).

Relative entropy: The relative entropy of the TEMOW distribution can be obtained by taking the mathematical expectation of \(\ln \left( \frac{f(x)_{\text{TEMOW}}}{f_1(x)_{\text{TEMOW}}} \right)\) where \(f(x)_{\text{TEMOW}}\) is the pdf with parameters \((\theta, \alpha, \lambda, \beta)\) as in (14) and \(f_1(x)_{\text{TEMOW}}\) is the pdf with parameters \((\theta_1, \alpha_1, \lambda_1, \beta_1)\) as

\[ \ln \left( \frac{f(x)_{\text{TEMOW}}}{f_1(x)_{\text{TEMOW}}} \right) = \ln \left( \frac{\theta \alpha \beta (1 - e^{-\theta_1})}{\theta_1 \alpha_1 \beta_1 (1 - e^{-\theta_1})} + (\beta - \beta_1) \ln(x) - \lambda x^\beta + \lambda_1 x^{\beta_1} - \theta \frac{1 - e^{-\lambda x^\beta}}{\alpha + \tilde{a} (1 - e^{-\lambda x^\beta})} \right) \]

\[ \quad + \frac{1 - e^{-\lambda x^\beta}}{\alpha_1 + \tilde{a_1} (1 - e^{-\lambda_1 x^{\beta_1}})} \ln \left( \alpha + \tilde{a} (1 - e^{-\lambda x^\beta}) \right) \]

\[ \quad + 2 \ln \left( \alpha_1 + \tilde{a_1} (1 - e^{-\lambda_1 x^{\beta_1}}) \right) \]

Thus, the TEMOW relative entropy is given by

\[ RE_{\text{TEMOW}} = E \left( \ln \left( \frac{f(x)_{\text{TEMOW}}}{f_1(x)_{\text{TEMOW}}} \right) \right) = \ln \left( \frac{\theta \alpha \beta (1 - e^{-\theta_1})}{\theta_1 \alpha_1 \beta_1 (1 - e^{-\theta_1})} + (\beta - \beta_1) E(\ln(X)) \right) \]

\[ - \lambda E(X^\beta) + \lambda_1 E(X^{\beta_1}) - \theta E \left( \frac{1 - e^{-\lambda x^\beta}}{\alpha + \tilde{a} (1 - e^{-\lambda x^\beta})} \right) + \theta_1 E \left( \frac{1 - e^{-\lambda_1 x^{\beta_1}}}{\alpha_1 + \tilde{a_1} (1 - e^{-\lambda_1 x^{\beta_1}})} \right) \]

\[ - 2E \left( \ln \left( \alpha + \tilde{a} (1 - e^{-\lambda x^\beta}) \right) \right) + 2E \left( \ln \left( \alpha_1 + \tilde{a_1} (1 - e^{-\lambda_1 x^{\beta_1}}) \right) \right) \] (28)

where \(E(X^\beta)\) and \(E(X^{\beta_1})\) as in (20) with \(r = \beta\) and \(\beta_1\). The other mathematical expectations as in (25), (26) and (27) with indicated parameters.

Reliability stress strength model: Consider \(f_X(x)\) the pdf of the strength random variable \(X\) as in (14) and \(F_Y(x)\) the cdf of the stress random variable \(Y\) as in (13) with parameters \((\theta_1, \alpha_1, \lambda_1, \beta_1)\). The
reliability stress strength model associated with two independent TEMOW random variables can be achieved as follows

$$SS_{TEMOW} = P(Y < X) = \int_0^{\infty} f_X(x) F_Y(x) \, dx = \frac{1}{1 - e^{-\theta_1}} \left( 1 - e^{-\frac{1-e^{-\lambda x \beta_1}}{\alpha_1 + \bar{\alpha}(1-e^{-\lambda x \beta_1})}} \right) f_X(x) \, dx$$

After using the expansion formulas of $e^{-x} \cdot (1-x)^{-a}$ and $(1-x)^a$ along with some simplification steps we get

$$SS_{TEMOW} = \frac{1}{1 - e^{-\theta_1}} \left[ 1 - \sum_{i,j,l,t=0}^{\infty} (-1)^{i+j+i+t} \frac{\Gamma(i+j)\Gamma(i+j+1)}{i!j!l!t!} \frac{\Theta(i)}{\Gamma(i+j-l+1)} \frac{\Theta_1^j}{\alpha_1^i} (\lambda l)^t x^t \beta_1 \right] f_X(x) \, dx$$

Then

$$SS_{TEMOW} = \frac{1}{1 - e^{-\theta_1}} \left[ 1 - \sum_{i,j,l,t=0}^{\infty} (-1)^{i+j+i+t} \frac{\Gamma(i+j)\Gamma(i+j+1)}{i!j!l!t!} \frac{\Theta(i)}{\Gamma(i+j-l+1)} \frac{\Theta_1^j}{\alpha_1^i} (\lambda l)^t E(X^t \beta_1) \right]$$

where $E(X^t \beta_1)$ as in (20) with $r = t \beta_1$.

Order statistics: Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ be the corresponding order statistics of a random sample $X_1, X_2, \ldots, X_n$ of size $n$ taken independently from TEMOW distribution. The pdf and the joint pdf of order statistics can be found based on (13) and (14) via the following standard formulas (see [11])

$$f_{k:n}(x) = \frac{n!}{(k-1)! (n-k)!} (F(x))^{k-1} (1 - F(x))^{n-k} f(x) ; 0 \leq x_{k:n} < \infty, k \leq n$$

$$f_{j,k:n}(x, y) = \frac{n!}{(j-1)! (k-j-1)! (n-k)!} (F(x))^{j-1} (F(y) - F(x))^{k-j-1} (1 - F(y))^{n-k} f(x) f(y) ; 1 \leq j \leq k, 0 \leq y < \infty$$

As follows

$$f_{k:n}(x) = \frac{n!}{(k-1)! (n-k)!} \left( 1 - e^{-\frac{1-e^{-\lambda x \beta_1}}{\alpha + \bar{\alpha}(1-e^{-\lambda x \beta_1})}} \right)^{k-1} \left( -\frac{1-e^{-\lambda x \beta_1}}{\alpha + \bar{\alpha}(1-e^{-\lambda x \beta_1})} e^{-\theta} - e^{-\theta} \right)^{n-k}$$

$$\theta \alpha \lambda \beta x_{\beta-1} e^{-\lambda x \beta} \frac{\Theta(1-e^{-\lambda x \beta})}{(1 - e^{-\theta})^n (\alpha + \bar{\alpha}(1-e^{-\lambda x \beta}))^2} ; 0 \leq x_{k:n} < \infty, k \leq n$$

$$f_{j,k:n}(x, y) = \frac{n!}{(j-1)! (k-j-1)! (n-k)!} \left( 1 - e^{-\frac{1-e^{-\lambda x \beta_1}}{\alpha + \bar{\alpha}(1-e^{-\lambda x \beta_1})}} \right)^{j-1}$$

$$\left( -\frac{1-e^{-\lambda y \beta_1}}{\alpha + \bar{\alpha}(1-e^{-\lambda x \beta_1})} e^{-\theta} - e^{-\theta} \right)^{n-k}$$

$$\Theta^2 \alpha^2 \lambda^2 \beta^2 x_{\beta-1} y_{\beta-1} e^{-\lambda(x+y) \beta} \frac{\Theta(1-e^{-\lambda x \beta} + 1-e^{-\lambda y \beta})}{(1 - e^{-\theta})^n (\alpha + \bar{\alpha}(1-e^{-\lambda x \beta}))^2 (\alpha + \bar{\alpha}(1-e^{-\lambda y \beta}))^2} ; 1 \leq j \leq k, 0 \leq x \leq y < \infty$$
6. Maximum likelihood estimators of TEMOW parameters

Consider \(x_1, x_2, \ldots, x_n\) a complete random sample of size \(n\) follow TEMOW distribution with the vector of parameters \(\Delta = (\theta, \alpha, \lambda, \beta)^T\). Then natural logarithm likelihood function based on (14) is

\[
\ell(\Delta | x) = n \ln \left( \frac{\partial a \lambda \beta}{1 - e^{-\beta + \lambda x_i}} \right) + (\beta - 1) \sum_{i=1}^{n} \ln(x_i) - \lambda \sum_{i=1}^{n} x_i^\beta - \theta \sum_{i=1}^{n} \frac{1 - e^{-\lambda x_i^\beta}}{\alpha + \lambda (1 - e^{-x_i^\beta})} - 2 \sum_{i=1}^{n} \ln \left( \frac{1 - e^{-\lambda x_i^\beta}}{\alpha + \lambda (1 - e^{-x_i^\beta})} \right)
\]

(32)

The maximum likelihood estimators of four parameters can be attained either directly by using the package (AdeqeuityModel) in R software or by solving the nonlinear four likelihood equations

\[
\frac{\partial \ell(x | \delta)}{\partial \delta} = \left( \frac{\partial \ell(x | \delta)}{\partial \alpha}, \frac{\partial \ell(x | \delta)}{\partial \lambda}, \frac{\partial \ell(x | \delta)}{\partial \beta}, \frac{\partial \ell(x | \delta)}{\partial \theta} \right)^T = 0
\]

through computational iterative techniques.

In AdeqeuityModel package, there exists various maximization algorithms such as Newton-Raphson (NR), Broyden-Fletcher-Goldfarb-Shanno (BFGS), Berndt-Hall-Hall-Hausman (BHHH), Limited-Memory quasi-Newton code for Bound-constrained optimization (L-BFGS-B), Nelder-Mead (NM), and Simulated-Annealing (SANN). Here, the MLEs are computed directly by using the package (AdeqeuityModel) in R software with BFGS method.

7. Simulation and real application

In this section, numerical illustrations (simulation and real application) are presented to exhibit the abilities of the proposed distribution.

7.1. Simulation process

The first step of the simulation process is to generate i.i.d. random samples (1000 times) follow TEMOW distribution each with size \(n = 25,50,100,200\) and 300 where the true or initial values of parameters are chosen to be as in table 1 (also see figure 2). Then, for each parameter, calculate the Bias and root mean squared error (RMSE) as

\[
\text{Bias}(\hat{\eta}) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\eta}_i - \eta)
\]

\[
\text{RMSE}(\hat{\eta}) = \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} (\hat{\eta}_i - \eta)^2}
\]

where \(\eta\) can be \(\theta, \alpha, \lambda,\) or \(\beta\).

It clearly appears from the simulation results (table 1) that RMSE values decrease as the sample size increases.

| Table 1. The Bias and RMSE of the TEMOW parameters estimation using MLE. |
|----------------|----------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \(n\) | Par. | Init. | Bias   | RMSE  | Init. | Bias   | RMSE  | Init. | Bias   | RMSE  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 25  | \(\theta\) | 3.3  | -0.213 | 0.871 | 9.1 | 0.405 | 1.520 | 3.5 | -0.096 | 0.640 |
|     | \(\alpha\) | 1.2  | 0.312 | 1.098 | 2.8 | 0.448 | 1.690 | 1 | 0.048 | 0.689 |
|     | \(\lambda\) | 0.8  | 0.209 | 0.360 | 2.1 | 0.567 | 1.305 | 0.5 | 0.348 | 0.381 |
|     | \(\beta\) | 2.1  | 0.136 | 0.502 | 0.9 | 0.056 | 0.193 | 0.39 | 0.221 | 0.771 |
| 50  | \(\theta\) | 3.3  | -0.103 | 0.649 | 9.1 | 0.395 | 1.388 | 3.5 | -0.032 | 0.493 |
|     | \(\alpha\) | 1.2  | 0.173 | 0.781 | 2.8 | 0.302 | 1.277 | 1 | -0.022 | 0.502 |
|     | \(\lambda\) | 0.8  | 0.157 | 0.271 | 2.1 | 0.379 | 0.948 | 0.5 | 0.335 | 0.356 |
|     | \(\beta\) | 2.1  | 0.065 | 0.338 | 0.9 | 0.023 | 0.128 | 3.9 | 0.134 | 0.538 |
| 100 | \(\theta\) | 3.3  | -0.007 | 0.534 | 9.1 | 0.288 | 1.152 | 3.5 | 0.034 | 0.347 |
|     | \(\alpha\) | 1.2  | 0.094 | 0.565 | 2.8 | 0.144 | 0.793 | 1 | -0.097 | 0.378 |
|     | \(\lambda\) | 0.8  | 0.123 | 0.202 | 2.1 | 0.255 | 0.617 | 0.5 | 0.318 | 0.330 |
|     | \(\beta\) | 2.1  | 0.034 | 0.242 | 0.9 | 0.013 | 0.087 | 3.9 | 0.116 | 0.383 |
| 200 | \(\theta\) | 3.3  | 0.051 | 0.410 | 9.1 | 0.284 | 0.989 | 3.5 | 0.054 | 0.253 |
7.2. Real Application

In this subsection, we include applications for two real data sets (right-skewed and semi-symmetric) to exhibit the abilities and flexibility of the TEMOW distribution.

The first real data (Data-1) collection reflects the body fat percentage of 202 Australian athletes [12].

"19.75, 21.30, 19.88, 23.66, 17.64, 15.58, 19.99, 22.43, 17.95, 15.07, 28.83, 18.08, 23.30, 17.71, 18.77, 19.83, 25.16, 18.04, 21.29, 22.15, 16.15, 16.38, 19.35, 19.20, 17.89, 12.20, 23.70, 24.69, 16.58, 21.47, 20.12, 17.51, 23.70, 22.39, 20.43, 11.29, 25.16, 19.39, 19.63, 23.11, 16.86, 21.32, 26.57, 17.93, 24.97, 22.62, 15.01, 18.14, 26.78, 17.22, 26.50, 23.01, 30.10, 13.93, 26.65, 35.52, 15.59, 19.61, 14.52, 11.47, 17.71, 18.48, 11.22, 13.61, 12.87, 11.85, 13.35, 11.77, 11.07, 21.30, 20.10, 24.88, 19.26, 19.51, 23.01, 8.07, 11.05, 12.39, 15.95, 9.91, 16.20, 9.02, 14.26, 10.48, 11.29, 19.94, 13.91, 6.10, 7.52, 9.56, 6.06, 7.90, 6.46, 9.00, 12.61, 9.03, 9.66, 10.05, 9.56, 9.36, 10.81, 8.61, 9.53, 7.42, 9.79, 8.97, 7.49, 11.95, 7.35, 7.16, 8.77, 9.56, 14.53, 8.51, 10.64, 7.06, 8.87, 7.88, 9.20, 7.19, 6.06, 5.63, 6.59, 9.50, 13.97, 11.66, 6.43, 6.99, 6.00, 6.56, 6.03, 6.33, 6.82, 6.20, 5.93, 5.80, 6.56, 6.76, 7.22, 8.51, 7.72, 19.94, 13.91, 6.10, 7.52, 9.56, 6.06, 7.35, 6.00, 6.92, 6.33, 5.90, 8.84, 8.94, 6.53, 9.40, 8.18, 17.41, 18.08, 9.86, 7.29, 18.72, 10.12, 19.17, 17.24, 9.89, 13.06, 8.84, 8.87, 14.69, 8.64, 14.98, 7.82, 8.97, 11.63, 13.49, 10.25, 11.79, 10.05, 8.51, 11.50, 6.26".

The second real data (Data-2) collection reflects the strengths of 1.5 cm glass fibers of 63 observations [13].

"0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24".

The fitting of TEMOW distribution is compared with the Beta Weibull (BW), Kumaraswamy Weibull (KuW), Exponentiated Generalized Weibull (EGW), Weibull Weibull (WW), Gompertz Weibull (GoW), and Weibull (W) distributions (see [14]-[18] for more details). The software R was used to calculate their negative log-likelihood (-LL), Consistent Akaike Information Criteria (CAIC), Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Hanan and Quinn Information Criteria (HQIC), besides MLEs of the parameters.

The findings are shown in tables 2 - 5. In addition to the plots of empirical cdfs, the histogram plot and the estimated densities are shown in figures 5 - 8. Based on the results, TEMOW has the lowest values of information criteria, making it the most fitting to represent two data sets compared to other distributions. In addition, this best fitting can be seen via the plots.

| Table 2. The information criteria to fitting Data-1. |
|-----------------------------------------------|
| Dist. | -LL   | CAIC  | AIC   | BIC   | HQIC  |
| TEMOW | 625.8387 | 1259.880 | 1259.677 | 1272.910 | 1265.031 |
| BW    | 635.6115 | 1279.504 | 1279.301 | 1292.534 | 1284.656 |
| KuW   | 626.1613 | 1260.526 | 1260.323 | 1273.556 | 1265.767 |
| EGW   | 630.3769 | 1268.957 | 1268.754 | 1281.987 | 1274.108 |
| WW    | 642.4163 | 1293.036 | 1292.833 | 1306.066 | 1298.187 |
| GoW   | 641.4215 | 1291.046 | 1290.843 | 1304.076 | 1296.197 |
| W     | 642.7304 | 1289.524 | 1289.464 | 1296.080 | 1292.141 |
Table 3. The MLE values to Data-1.

| Dist. | $\hat{\theta}_{ML}$ | $\hat{\alpha}_{ML}$ | $\hat{\lambda}_{ML}$ | $\hat{\beta}_{ML}$ |
|-------|----------------------|----------------------|----------------------|----------------------|
| TEMOW | -17.5717             | 0.0208               | 0.0557               | 2.1454               |
| BW    | 1.7612               | 0.2322               | 0.1540               | 1.9505               |
| KuW   | 7.9067               | 0.1593               | 0.2580               | 1.6154               |
| EGW   | 1.3058               | 97.7112              | 1.1087               | 0.5177               |
| WW    | 3.3540               | 1.7800               | 0.1485               | 0.7018               |
| GoW   | 1.2484               | -0.1433              | 0.0623               | 2.5427               |
| W     | ---                  | ---                  | 0.0658               | 2.2554               |

Table 4. The information criteria to fitting Data-2.

| Dist. | -LL  | CAIC | AIC  | BIC  | HQIC |
|-------|------|------|------|------|------|
| TEMOW | 12.0290 | 32.748 | 32.058 | 40.631 | 35.430 |
| BW    | 14.5926 | 37.875 | 37.185 | 45.758 | 40.557 |
| KuW   | 13.3785 | 35.447 | 34.757 | 43.329 | 38.129 |
| EGW   | 20.8999 | 50.489 | 49.800 | 58.372 | 53.171 |
| WW    | 15.2068 | 39.103 | 38.414 | 46.986 | 41.785 |
| GoW   | 14.7764 | 38.242 | 37.553 | 46.125 | 40.924 |
| W     | 18.0609 | 40.355 | 40.155 | 44.441 | 41.840 |

Table 5. The MLE values to Data-2.

| Dist. | $\hat{\theta}_{ML}$ | $\hat{\alpha}_{ML}$ | $\hat{\lambda}_{ML}$ | $\hat{\beta}_{ML}$ |
|-------|----------------------|----------------------|----------------------|----------------------|
| TEMOW | -0.5754              | 12.2078              | 0.8896               | 3.2084               |
| BW    | 0.6194               | 6.2381               | 0.4480               | 7.7638               |
| KuW   | 0.4934               | 0.2121               | 0.7400               | 7.0644               |
| EGW   | 0.1377               | 3.2010               | 1.048               | 2.6592               |
| WW    | 3.4845               | 3.9363               | 1.4029               | 1.6589               |
| GoW   | 0.0080               | 3.5273               | 1.0602               | 0.9752               |
| W     | ---                  | ---                  | 0.6240               | 4.4768               |
8. Conclusions
In this paper, a newly generated family of continuous distributions with Marshall Olkin is introduced. Then a truncated distribution as a sub-model with four parameters called Truncated Exponential Marshall Olkin Weibull is proposed. Reliability analysis with several main statistical properties such as moments, characteristic function, quantile function, median, skewness, kurtosis, simulated data, Shannon entropy, relative entropy, reliability stress-strength model, and order statistics are presented. Furthermore, the maximum likelihood estimation method is used to estimate the unknown four parameters. To assess the usefulness and flexibility, the TEMOW distribution applied upon simulation study besides real application by the implementation of two real data sets (right-skewed and semi-symmetric) with different information criteria. The simulation results clearly shown the flexible and consistent performance of the maximum likelihood estimators for the parameters. Also, the real application results clearly shown that the proposed distribution has outstanding performance than other considered distributions for all information criteria. This flexibility allows using the TEMOW distribution in various application areas.

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