DACOROGNA-MOSER THEOREM ON THE JACOBIAN DETERMINANT EQUATION WITH CONTROL OF SUPPORT

PEDRO TEIXEIRA

Abstract. The original proof of Dacorogna-Moser theorem on the prescribed Jacobian PDE, det $\nabla \varphi = f$, can be modified in order to obtain control of support of the solutions from that of the initial data, while keeping optimal regularity. Briefly, under the usual conditions, a solution diffeomorphism $\varphi$ satisfying

$$\text{supp}(f - 1) \subset \Omega \implies \text{supp}(\varphi - \text{id}) \subset \Omega$$

can be found and $\varphi$ is still of class $C^{r+1,\alpha}$ if $f$ is $C^{r,\alpha}$, the domain of $f$ being a bounded connected open $C^{r+2,\alpha}$ set $\Omega \subset \mathbb{R}^n$.

In memoriam Jürgen Moser

In [DM, p.4], Dacorogna and Moser formulated a celebrated result on the solutions to the Jacobian determinant PDE with pointwise fixed boundary condition, which found many applications across several fields of research. It is one of the main tools for the correction of volume distortion (in relation to the standard volume), in Hölder spaces. Its main advantage over similar results lies in its optimal regularity, the solution diffeomorphism $\varphi$ is $C^{r+1,\alpha}$ if the initial data $f$ is $C^{r,\alpha}$. Nevertheless, from the point of view of applications, Dacorogna-Moser theorem has, perhaps, one main drawback: even if $\text{supp}(f - 1) \subset \Omega$, the solution obtained does not, in general, extend by the identity to the whole $\mathbb{R}^n$ (in the $C^{r+1,\alpha}$ class). This is a serious limitation, for it is often necessary to guarantee that the volume correcting diffeomorphism acts (by composition) only inside the region $\Omega$ where the volume distortion takes place, while keeping the original diffeomorphism (or map) unchanged outside that domain (see the Example below). This limitation comes from the elliptic regularity solutions to Neumann problems arising in the proof of the auxiliary linearized problem. Other approaches (e.g., the flow method of Moser) permit to take control of support

$$\text{supp}(f - 1) \subset \Omega \implies \text{supp}(\varphi - \text{id}) \subset \Omega \quad (0.1)$$

but fail to achieve the desired gain of regularity. Notwithstanding, it is possible to modify Dacorogna and Moser original proof (in its improved form given in [CDK, p.192]) in order to guarantee that condition (0.1) holds, keeping nevertheless optimal regularity, which significantly enlarges the scope of applicability of the original result.

Theorem 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2,\alpha}$ set, $r \geq 0$ an integer and $0 < \alpha < 1$. Given $f \in C^{r,\alpha} (\overline{\Omega})$ such that $f > 0$ in $\Omega$ and $\int_{\Omega} f = \text{meas} \, \Omega$, there exists $\varphi \in \text{Diff}^{r+1,\alpha} (\overline{\Omega}, \overline{\Omega})$ satisfying:

$$\begin{cases}
\det \nabla \varphi = f & \text{in } \Omega \\
\varphi = \text{id} & \text{on } \partial \Omega \\
\text{supp}(f - 1) \subset \Omega \implies \text{supp}(\varphi - \text{id}) \subset \Omega
\end{cases} \quad (0.2)$$

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Example. We give an application to a situation that arises in conservative dynamics. It corresponds to the natural improvement of the example given in [DM] p.3, made possible by the additional control of support of condition (0.1). Let $r, \alpha$ be as in Theorem 1. Suppose that $\psi \in \text{Diff}^{r+1,\alpha}(\mathbb{R}^n)$ and $\Omega$ is a bounded connected open set such that (a) $\psi(\partial \Omega) = \partial \Omega$ and (b) $\psi$ is volume preserving in a neighbourhood of $\mathbb{R}^n \setminus \Omega$ (always in relation to the standard volume; the diffeomorphisms are of $\mathbb{R}^n$ onto itself and orientation preserving). Then setting $f = \det \nabla \psi^{-1}|_{\overline{\Omega}}$ in Theorem 1 (noting that $\text{supp}(f - 1) \subset \Omega$), and extending the solution obtained by the identity to the whole $\mathbb{R}^n$, we find $\varphi \in \text{Diff}^{r+1,\alpha}(\mathbb{R}^n)$ such that

- $\Psi := \varphi \circ \psi \in \text{Diff}^{r+1,\alpha}(\mathbb{R}^n)$ i.e. $\Psi$ is volume preserving on $\mathbb{R}^n$
- $\Psi = \psi$ in a neighbourhood of $\mathbb{R}^n \setminus \Omega$

i.e. $\varphi$ corrects the volume distortion of $\psi$ inside $\Omega$, while keeping $\psi$ unchanged in $\mathbb{R}^n \setminus \Omega$ (actually, Theorem 1 requires $\Omega$ to be $C^{r+2,\alpha}$, but this condition is only necessary when $\text{supp}(f - 1) \not\subset \Omega$, which is not the case, see Remark 6, Section 6). Observe that under the above conditions (a) and (b) plus additional regularity imposed on $\Omega$, Dacorogna-Moser theorem only guarantees the existence of $\varphi_0 \in \text{Diff}^{r+1,\alpha}(\Omega, \overline{\Omega})$ such that $\Psi_0 = \varphi_0 \circ \psi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}, \overline{\Omega})$ and $\Psi_0 = \psi$ on $\partial \Omega$ (prescribed boundary data), but nothing guarantees that $\Psi_0$ extends by $\psi$ to the whole $\mathbb{R}^n$ (in the $C^{r+1,\alpha}$ class). Needless to say, the above reasoning immediately applies to precompact connected open subsets $\Omega$ of orientable $n$-manifolds (second countable, Hausdorff and boundaryless), provided $\Omega$ embeds in $\mathbb{R}^n$.

The key new ingredient in the proof of Theorem 1 is a quite recent global Poincaré lemma with optimal regularity obtained in Csatò, Dacorogna, Kneuss [CDK] p.148. It permits to construct an universal bounded linear operator $h \to u$ solving, with optimal regularity, the linearized problem

$$
\begin{cases}
\text{div} u = h & \text{in } \Omega \\
u = 0 & \text{in } U
\end{cases} \tag{0.3}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded connected open $C^\infty$ set (here briefly called a smooth domain), $h \in C^{r,\alpha}(\overline{\Omega})$ satisfies $\int_\Omega h = 0$ and $h = 0$ in a smooth collar $U$ of $\overline{\Omega}$ (see Definition 2). Granted this, the strategy for modifying Dacorogna-Moser original proof in order to include condition (0.1) is the obvious one and quite straightforward: if $\Omega$ is smooth and $f \in C^{r,\alpha}(\overline{\Omega})$ satisfies (1) $f > 0$ in $\overline{\Omega}$, (2) $\int_\Omega f = \text{meas } \Omega$ and (3) $f = 1$ in a neighbourhood of a compact collar $U$, then $\varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}, \overline{\Omega})$ solving

$$
\begin{cases}
\det \nabla \varphi = f & \text{in } \Omega \\
\varphi = \text{id} & \text{in } U
\end{cases} \tag{0.4}
$$

can be found using the above result (instead of Theorem 2 in [DM]). The case of general domains (with no regularity imposed on $\partial \Omega$), which includes those that are $C^{r+2,\alpha}$, easily reduces to the previous one, $\Omega$ having an exhaustion by smooth domains (Appendix, Lemma 2). A single difficulty emerges at last step (Proof of Theorem 6), where the solution of (0.4) is reduced to the case of $\|f - 1\|_{C^{r,\alpha}}$ small enough (for some fixed $0 < \gamma < \alpha$). The correction of the measure of an auxiliary function needed in this process can no longer be achieved multiplying it by a suitable constant, as all functions involved must now equal 1 in a neighbourhood of $\partial \Omega$. This difficulty is easily overcome multiplying the function in question by a suitable measure correcting smooth function found via the intermediate value theorem.

\[1\text{See Remark 3.}\]
0.1. Solution to the linearized problem \((0.3)\). Under the hypothesis of \((0.3)\), Theorem 2 in \([DM]\) provides a first solution \(u_0 \in C^{r+1,\alpha}(\Omega)\) to \(\text{div}u_0 = h\) in \(\Omega\) which, however, only guarantees that \(u_0 = 0\) on \(\partial\Omega\). But \(u_0\) can be modified to satisfy \(u = 0\) in \(U\) while still verifying the estimate

\[
\|u\|_{C^{r+1,\alpha}} \leq C\|h\|_{C^{r,\alpha}}
\]  

(0.5) for some constant \(C = C(r,\alpha,U,\Omega) > 0\) and furthermore, the correspondence \(h \mapsto u_0\) can be made linear and universal. It is enough to find an universal bounded linear operator extending \(u_0|_U\) to a divergence-free \(C^{r+1,\alpha}\) vector field \(\tilde{u}_0\) on \(\Omega\) and then set \(u := u_0 - \tilde{u}_0\). In the \(C^\infty\) class, there is simple procedure to achieve this, using the isomorphism between divergence-free vector fields and closed \((n-1)\)-forms together with the relative Poincaré lemma (see for instance [AV, Theorem 3] and [AMR, p.447]), but this approach no longer works in the \(C^{r+1,\alpha}\) class due to the loss of regularity under exterior derivation. To compensate for this, a relative Poincaré lemma with optimal regularity and control of the norm is needed (see Theorem 2), and this result actually becomes easily available combining the standard relative Poincaré lemma (with no gain of regularity, see the Appendix, Lemma 1) with the optimal regularity global Poincaré lemma mentioned above \([CDK, p.148]\). This simple but crucial remark was essentially made by Carlos Matheus in \([MA]\), where an approach along similar lines to those of this note is roughly sketched.

0.2. Limitations of the present solution to the main problem \((0.2)\). As it will be seen ahead, when \(\text{supp}(f - 1) \subset \Omega\), the construction provided here of diffeomorphism \(\varphi\) solving \((0.2)\) depends, on an essentially way, on the distance \(d\) from \(\text{supp}(f - 1)\) to \(\partial\Omega\). As a consequence, global estimates of the kind

\[
\|\varphi - \text{id}\|_{C^{r+1,\alpha}} \leq C\|f - 1\|_{C^{r,\alpha}}
\]  

(0.6) obtained in \([CDK, p.192]\), with \(C\) is independent of \(d\), which are valid if condition \((0.1)\) is dropped, are actually impossible to attain by the present method (see Section 7; c.f. Theorem 4). For this reason, a more uniform method of construction of the solutions to problem \((0.2)\), permitting useful estimates as \((0.6)\) with \(C\) independent of \(d\), would be desirable.

On the other hand, while the existence of optimal regularity solutions for the more general pullback equation between prescribed volume forms \(f, g\) (with equal total volume) follows immediately from Dacorogna-Moser result mentioned above (see \([DM, p.4]\)), presently we are unable to derive the corresponding control of support condition

\[
\text{supp}(f - g) \subset \Omega \implies \text{supp}(\varphi - \text{id}) \subset \Omega
\]  

(0.7) from the results here obtained.

Finally, from the technical point of view, it should be recognized that the present note adds little to the deepness and beauty of Dacorogna and Moser’s original proof, its main advantage being, perhaps, the complete transparency and the low deductive effort from previously known results. It is also worth mentioning that the key results involved in the simple deduction chain that follows are still due to the original authors, Bernard Dacorogna and Jürgen Moser, working together, alone or with other authors. This note is dedicated to the memory of the later.

1. Dimension one, notation and conventions

1.1. The one-dimensional case. When \(n = 1\) i.e. \(\mathbb{R}^n = \mathbb{R}\), Dacorogna-Moser Theorem 1’ \([DM, p.4]\) with additional control of support is trivially true: let \(\Omega = \)
(a, b), where \(-\infty < a < b < \infty\); let \(r \geq 0\) be an integer and \(0 \leq \alpha \leq 1\). If \(f \in C^{r,\alpha}(\overline{\Omega})\) satisfies \(f > 0\) in \(\overline{\Omega}\) and \(\int_a^b f = b - a\), then
\[
\varphi(x) = a + \int_a^x f(t) \, dt
\]
begins to \(\text{Diff}^{r+1,\alpha}(\overline{\Omega}, \overline{\Omega})\), \(\varphi = \text{id}\) on \(\partial \Omega = \{a, b\}\) and for any \(0 < \eta \leq (b - a)/2\), letting \(U = [a, a + \eta] \cup [b - \eta, b]\), it is immediate to verify that
\[
f = 1 \text{ in } U \implies \varphi = \text{id} \text{ in } U
\]
For this reason we shall concentrate on the case \(n \geq 2\). Note, for instance, that while Theorem 3 is trivially true for \(n = 1\), its proof fails in that dimension.

1.2. Notation and conventions. For brevity of expression, we introduce the following definition of domain, which is narrower than the usual one (as boundedness is imposed).

**Definition 1.** • Domain. A bounded, connected open set \(\Omega \subset \mathbb{R}^n\) is called here a domain. Let \(r \geq 0\) be an integer and \(0 \leq \alpha \leq 1\). Domains with \(C^{r,\alpha}\) boundary (briefly \(C^{r,\alpha}\) domains) are defined in the usual way [CDK] p.338. A domain is smooth if it is \(C^\infty\).

• Banach space \(C^{r,\alpha}(\overline{\Omega})\). Let \(\Omega \subset \mathbb{R}^n\) be a domain, \(r \geq 0\) an integer and \(0 \leq \alpha \leq 1\). The space \(C^{r,\alpha}(\Omega)\) is defined in the usual way [CDK] p.336-337]. \(C^{r,\alpha}(\overline{\Omega})\) is the space of continuous functions on \(\overline{\Omega}\) and we define \(C^{r,\alpha}(\overline{\Omega})\) as the subspace of all functions \(f \in C^0(\overline{\Omega})\) such that (1) \(f|_\Omega\) is \(C^r\), (2) all its partial up to order \(r\) extend continuously to \(\overline{\Omega}\) and (3) for every multiindex \(n\) of order \(|n| = r\), \(\partial^nf|_{C^{r,\alpha}(\overline{\Omega})} < \infty\), where for \(D \subset \mathbb{R}^n\),
\[
[g]_{C^{r,\alpha}(D)} := \sup_{x,y \in D; x \neq y} \left\{ \frac{|g(x) - g(y)|}{|x - y|^\alpha} \right\}
\]
As usual, \(u \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)\) if all its components belong to \(C^{r,\alpha}(\overline{\Omega})\). If \(\Omega, \Omega' \subset \mathbb{R}^n\) are domains and \(r \geq 1\), then \(\varphi \in \text{Diff}^{r,\alpha}(\overline{\Omega}, \overline{\Omega})\) if \(\varphi\) is a bijection from \(\overline{\Omega}\) to \(\overline{\Omega}'\) and \(\varphi \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)\), \(\varphi^{-1} \in C^{r,\alpha}(\overline{\Omega}'; \mathbb{R}^n)\).

**Remark on notation.** Our definition of \(C^{r,\alpha}(\overline{\Omega})\) is slightly different from the more common one, which usually defines \(C^{r,\alpha}(\overline{\Omega})\) as consisting of all restrictions \(f|_\Omega\), where \(f \in C^0(\overline{\Omega})\) satisfies conditions (1) to (3) above. The present definition is more convenient in the following sense: if \(f \in C^{r,\alpha}(\overline{\Omega})\) and \(\Omega\) is Lipschitz (i.e. \(C^{0,1}\)), then \(f\) has a \(C^{r,\alpha}\)-extension \(\tilde{f}\) to the whole \(\mathbb{R}^n\) (see [CDK] p.342)[2]. By \(C^r\) continuity, all the partial derivatives of \(\tilde{f}\) up to order \(r\) at the points of \(\partial \Omega\) (i.e. the \(r\)-jets of \(f|_{\partial \Omega}\)) are uniquely determined by \(f|_{\partial \Omega}\), and therefore they are common to all possible \(C^{r,\alpha}\)-extensions \(\tilde{f}\). Hence, it makes sense to evaluate all these partial derivatives of \(f\) on \(\partial \Omega\) and \(f\) should be regarded as the restriction to \(\overline{\Omega}\) of all possible \(C^{r,\alpha}\) extensions of \(f\) to the whole \(\mathbb{R}^n\). If \(u \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)\) and \(r \geq 1\), then the existence of these extensions for \(u\) also implies, for instance, that div \(u\) and det \(\nabla u\), which are defined in \(\Omega\), have \(C^{r-1,\alpha}\)-extensions to the whole \(\mathbb{R}^n\), which are uniquely determined on \(\partial \Omega\) by \(u\). Therefore, div \(u\) and \(\nabla u\) should also be seen as belonging to \(C^{r-1,\alpha}(\overline{\Omega})\). In this context, we see that in Dacorogna-Moser theorem [DM] Theorem 1], where \(\Omega\) is \(C^{r+3,\alpha}\) and thus Lipschitz, the solution \(\varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}, \overline{\Omega})\) actually satisfies
\[
\text{det} \nabla \varphi = f \quad \text{in} \ \overline{\Omega}\]

---

[2] All domains we will encounter are at least \(C^{2,\alpha}\).
or simply
\[ \det \nabla \varphi = f \]
as both \( \det \nabla \varphi \) and \( f \) are in \( C^0(\overline{\Omega}) \) and \( \det \nabla \varphi = f \) in \( \Omega \). The above identity actually means that \( \det \nabla \varphi \) and \( f \) have \( r \)-jet coincidence all over \( \overline{\Omega} \) (by \( C^r \) continuity, these \( r \)-jets still coincide on \( \partial \Omega \)). Analogously, in [DM] Theorem 2 we actually have
\[ \text{div} u = f \quad (\text{in } \overline{\Omega}) \]
For brevity, we shall adopt from now on this natural convention: \( f = g \) means that these two functions have the same domain and agree all over it.

The present definition of \( C^{r, \alpha}(\overline{\Omega}) \) is more consistent than the usual one, since adopting the latter is still often necessary to evaluate \( f \in C^{r, \alpha}(\overline{\Omega}) \) (and functions depending continuously on its \( r \)-jet) at points of \( \partial \Omega \), while the domain of \( f \) is actually \( \Omega \), by definition. Moreover, for \( u \) as above, \( \partial \Omega \subset \overline{\Omega} \) immediately implies
\[ \| u \|_{C^{r, \alpha}} = \| u \|_{C^{r, \alpha}(\Omega)} := \| u \|_{C^{r, \alpha}} \text{, where } \| u \|_{C^{r, \alpha}} = \| u \|_{C^r} + \max_{|\beta|=r} [\partial^\beta u]_{C^{r, \alpha}}(\overline{\Omega}) \]
(see [CDK] p.336).

2. Optimal regularity relative Poincaré lemma

**Definition 2.** (Collar of \( \overline{\Omega} \)). If \( \Omega \subset \mathbb{R}^n \) is a smooth domain, there is a smooth embedding \( \zeta : \partial \Omega \times [0, \infty) \hookrightarrow \overline{\Omega} \) such that \( \zeta(x,0) = x \) (collar embedding [HI, Chapter 4]). For each \( \epsilon > 0 \) we call \( U_\epsilon := \zeta(\partial \Omega \times [0, \epsilon]) \) a (compact) collar of \( \overline{\Omega} \). Every neighbourhood of \( \partial \Omega \) contains a collar and every collar is contained in the relative interior of another collar.

The following result is the key lemma in this note and it is interesting on its own.

**Theorem 2.** (Optimal regularity relative Poincaré lemma). Let \( r \geq 1 \) and \( 1 \leq k \leq n \) be integers and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^n, n \geq 2, \) be a bounded connected open smooth set and \( U \) a collar of \( \overline{\Omega} \). Then there is a constant \( C = C(r, \alpha, U) > 0 \) such that: given a closed form \( \beta \in C^{r, \alpha}(U; A^k) \) that vanishes when pulled back to \( \partial \Omega \), there exists \( \omega \in C^{r+1, \alpha}(\overline{\Omega}; A^{k-1}) \) satisfying:

1. \( d \omega = \beta \) in \( U \)
2. \( \| \omega \|_{C^{r+1, \alpha}} \leq C \| \beta \|_{C^{r, \alpha}(U)} \)

Furthermore, the correspondence \( \beta \rightarrow \omega \) can be chosen linear and universal.

**Remark 1.** (Universality). The above correspondence is universal in the sense that \( \omega \) depends only on \( \beta \) and \( \Omega \), but not on \( r, \alpha \). More precisely, if \( \beta \) also belongs to class \( C^{s, \delta}, s \in \mathbb{Z}^+, \) and \( 0 < \delta < 1 \), then the same solution \( \omega \) is of class \( C^{s+1, \delta} \) and the corresponding estimate (2) holds for some constant \( C = C(s, \delta, U) > 0 \). In particular, \( \omega \) is smooth if \( \beta \) is smooth. The universality of \( \beta \rightarrow \omega \) will be used in the proof of Theorem 3.

**Proof.** (A) \( \beta \) has a \( C^1 \) primitive \( \omega_0 \) in \( U \) (\( d\omega_0 = \beta \)).

Since \( U \) is a collar, \( \beta \) is \( C^r, r \in \mathbb{Z}^+ \), \( d\beta = 0 \) and \( \beta \) vanishes when pulled back to \( \partial \Omega \). \( \beta \) has a primitive \( \omega_0 \) of class \( C^r \) on \( U \). This is immediate to verify following the standard proof of the relative Poincaré lemma, which uses the homotopy formula with integration along times fibres. For the convenience of the reader, we include a complete proof that \( \beta \) has a \( C^r \) primitive \( \omega_0 \) in \( U \), see Lemma 1 in the Appendix.

(B) Finding a \( C^{r+1, \alpha} \) primitive \( \hat{\omega} \) of \( \beta \) in \( U \) with control of the norm.

Now observe that \( \Omega' := \text{int} U \) (in \( \mathbb{R}^n \)) and \( \beta \) satisfy the hypothesis of [CDK] Theorem 8.3] since \( d\beta = 0 \) and for every \( \psi \in \mathscr{H}_N(\text{int} U, A^k) \) (noting that each such
ψ extends to \( \tilde{\psi} \in C^\infty(U, A^k) \) \([CDK, p.122]\), we have, integrating by parts \([CDK, p.88]\),

\[
\int_{\text{int } U} \langle \beta, \psi \rangle = \int_{\text{int } U} \langle d\omega_0, \psi \rangle = -\int_{\text{int } U} \langle \omega_0, \delta \psi \rangle + \int_{\partial U} \langle \omega_0, \nu, \psi \rangle = 0
\]
as \( \delta \psi = 0 = \nu, \psi \) by definition of \( \mathcal{H} \). Therefore \( \beta \) is a \( C^{r+1,\alpha} \) primitive \( \tilde{\omega} \) on \( U \) satisfying

\[
\|\tilde{\omega}\|_{C^{r+1,\alpha}(U)} \leq C_1 \|\beta\|_{C^{r,\alpha}(U)}
\]
where \( C_1 = C_1(r, \alpha, U) > 0 \) is the constant given in \([CDK, Theorem 8.3]\).

(C) Universal extension of \( \tilde{\omega} \) to the whole \( \overline{U} \) with control of the \( C^{r+1,\alpha} \) norm. As a \( k \)-form is completely determined by its \( \binom{n}{2} \) components and \( U \) is smooth, there is a universal linear operator (see \([CDK, p.342]\))

\[
E : C^{r+1,\alpha}(U; A^{k-1}) \to C^{r+1,\alpha}(\overline{U}; A^{k-1})
\]
and a constant \( C_2 = C_2(r, U) \geq 1 \) such that

\[
E(\gamma)|_{U} = \gamma \quad \text{and} \quad \|E(\gamma)\|_{C^{r+1,\alpha}} \leq C_2 \|\gamma\|_{C^{r+1,\alpha}(U)}
\]
hence the extension \( \omega = E(\tilde{\omega}) \in C^{r+1,\alpha}(\overline{U}; A^{k-1}) \) satisfies

\[
\|\omega\|_{C^{r+1,\alpha}} \leq C_1 C_2 \|\beta\|_{C^{r,\alpha}(U)}
\]
Therefore, \( d\omega = \beta \) in \( U \) and there is \( C = C(r, \alpha, U) > 0 \) as claimed. Since \( \beta \to \tilde{\omega} \) and \( \tilde{\omega} \to \omega \) are both linear and universal (see \([CDK, p.148-149]\)) so is \( \beta \to \omega \).

**Remark 2.** Actually, in the above situation, a much simpler extension operator could be used. If \( f \in C^{r,\alpha}(\overline{U}) \) and \( \Omega \) is \( C^{r,\alpha} \cap C^1 \), then \( f \) can be \( C^{r,\alpha} \)-extended by a bounded linear operator as in \([GT, p.136]\). This extension operator is actually not universal, as \( E(\gamma) \) depends on \( r \). To render it universal, instead of balls and half-cubes we use (open) cubes \((-2, 2)^n\) and half-cubes \([0, 2) \times (-2, 2)^{n-1}\) for the boundary rectification, and use Seeley’s operator \( SE \) (c.f. also \([BI]\)) to extend functions from the right halfcube to the cube, noting that, by construction, Seeley’s extension of a function \( f \) to the left halfline \((-\infty, 0] \times y\) depends only on the values taken by \( f \) on the interval \([0, 2) \times y\). This operator provides a simultaneous bounded linear extension in all classes of differentiability.

### 3. The Linearized Problem when \( f = 1 \) in a Collar

**Theorem 3.** Let \( r \geq 0 \) be an integer and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a bounded connected open smooth set and \( U \) a collar of \( \overline{\Omega} \). Given \( f \in C^{r,\alpha}(\overline{\Omega}) \) satisfying

\[
\begin{cases}
\int_{\Omega} f = 0 \\
f = 0 \quad \text{in } U
\end{cases}
\]
there exists \( u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) \) satisfying

\[
\begin{cases}
\text{div } u = f \\
u = 0 \quad \text{in } U
\end{cases}
\]
Furthermore, the correspondence \( f \to u \) can be chosen linear and universal and there exists \( C = C(r, \alpha, U, \Omega) > 0 \) such that

\[
\|u\|_{C^{r+1,\alpha}} \leq C \|f\|_{C^{r,\alpha}}
\]
Applying Theorem 2 to closed (Remark 3), the isomorphism between $C$ and the opposite direction:

Moreover, the correspondence $X$ where $\omega$ being a composition of universal linear operators $d\gamma$ for some constant $C = C(s, \delta, U) > 0$. In particular, $u$ is smooth if $f$ is smooth. The universality of $f \mapsto u$ will be used in the proof of Theorem 4.

**Proof.** (A) Reduction to the existence of a divergence-free extension universal bounded linear operator. Theorem 9.2 in [CIDK, p.180] provides a first solution to the problem, $u_0 \in C^{r+1,\alpha}(\overline{U}; \mathbb{R}^n)$, which however only guarantees that $u_0 = 0$ on $\partial \Omega$. Moreover, the correspondence $f \mapsto u_0$ is linear and universal and there is a constant $C_1 = C_1(r, \alpha, \Omega) > 0$ such that

$$\|u_0\|_{C^{r+1,\alpha}} \leq C_1 \|f\|_{C^r,\alpha}$$

Since $\text{div} u_0 = f = 0$ in $U$ (see Remark on notation, Section 1.2), in order to find $u$ it is enough to construct an universal (bounded) linear operator $H(\cdot)$ extending each divergence-free $X \in C^{r+1,\alpha}(U; \mathbb{R}^n)$ vanishing on $\partial \Omega$, to a divergence-free $H(X) \in C^{r+1,\alpha}(\overline{U}; \mathbb{R}^n)$ such that

$$\|H(X)\|_{C^{r+1,\alpha}} \leq C_2 \|X\|_{C^{r+1,\alpha}(U)}$$

for some constant $C_2 = C_2(r, \alpha, U) > 0$, for it is then immediate to verify that $u := u_0 - H(u_0|_U)$ is the desired solution and $C = C(r, \alpha, U, \Omega) = C_1(1 + C_2)$. As $f \mapsto u_0$ and $u_0 \mapsto u$ are both linear and universal so is $f \mapsto u$.

(B) Construction of operator $H(\cdot)$. Suppose that $X \in C^{r+1,\alpha}(U; \mathbb{R}^n)$ is divergence-free and vanishes on $\partial \Omega$. In order to construct $\tilde{X} := H(X)$ we use, as in [AV, Theorem 3], the isomorphism between $C^{r+1,\alpha}$ divergence-free vector fields and $C^{r+1,\alpha}$ closed $(n-1)$-forms given by

$$X \longleftrightarrow X^* = X \omega$$

where $\omega$ is the standard volume form on $\mathbb{R}^n$, which immediately gives

- $X^* \in C^{r+1,\alpha}(U; A^{n-1})$
- $dX^* = (\text{div} X)\omega = 0$
- $X^* = 0$ in $\partial \Omega$ (as $X = 0$ there)
- $\|X\|_{C^{r+1,\alpha}(U)} = \|X^*\|_{C^{r+1,\alpha}(U)}$

Applying Theorem 2 to $X^*$ we find $\gamma \in C^{r+2,\alpha}(\overline{U}; A^{n-2})$ and a constant $C_3 = C_3(r, \alpha, U) > 0$ such that $d\gamma = X^*$ in $U$ and

$$\|\gamma\|_{C^{r+2,\alpha}} \leq C_3 \|X^*\|_{C^{r+1,\alpha}(U)}$$

Moreover, the correspondence $X^* \mapsto \gamma$ is both linear and universal. We now go in the opposite direction:

- $d\gamma \in C^{r+1,\alpha}(\overline{U}; A^{n-1})$ is a closed form
- $d\gamma = X^*$ in $U$
- $\|d\gamma\|_{C^{r+1,\alpha}} \leq (n-1) \|\gamma\|_{C^{r+2,\alpha}}$

and $d\gamma$ corresponds, under the isomorphism described above, to a divergence-free $\tilde{X} \in C^{r+1,\alpha}(\overline{U}; \mathbb{R}^n)$, $\tilde{X}|_U = X$. The correspondence $X \mapsto \tilde{X}$ is linear and universal, being a composition of universal linear operators

$$X \mapsto X^* \mapsto \gamma \mapsto d\gamma \mapsto \tilde{X} = H(X)$$

Following the above chain we readily get $C_2 = (n-1)C_3$ ($n = \dim \Omega$) and $H(\cdot)$ is as claimed. □
4. Solution when \( f = 1 \) in a collar and \( \|f - 1\|_{C^0,\gamma} \) is small

We now state a variant of Theorem 10.9 in [CDK] p.198 (which improves Lemma 4 in [DM] p.10), the original proof being adapted in the obvious way, with all the necessary changes highlighted. All references below are to that proof [CDK] p.198-201.

**Theorem 4.** Let \( r \geq 0 \) be an integer and \( 0 < \alpha, \gamma < 1 \) with \( \gamma \leq r + \alpha \). Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a bounded connected open smooth set and \( U \) a collar of \( \overline{\Omega} \). Then, there exists \( \varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}, \overline{\Omega}) \) satisfying

\[
\begin{cases}
\varphi = \text{id} & \text{in } U \\
\det \nabla \varphi = f & (\text{in } \overline{\Omega})
\end{cases}
\]

there exists \( \varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}, \overline{\Omega}) \) satisfying

\[
\begin{cases}
\det \nabla \varphi = f & (\text{in } \overline{\Omega}) \\
\varphi = \text{id} & \text{in } U
\end{cases}
\]

\( \| \varphi - \text{id} \|_{C^{r+1,\alpha}} \leq c \| f - 1 \|_{C^{r,\alpha}} \) and \( \| \varphi - \text{id} \|_{C^{1,\gamma}} \leq c \| f - 1 \|_{C^0,\gamma} \)

**Remark 4.** See Remark on notation (Section 1) for the identity \( \det \nabla \varphi = f \) in \( \overline{\Omega} \) above.

**Remark 5.** The following fundamental result on the inclusion of Hölder spaces will be often implicitly used without mention: if the domain \( \Omega \subset \mathbb{R}^n \) is Lipschitz, \( 0 \leq \bar{s} \leq s \) are integers and \( 0 \leq \beta, \bar{\beta} \leq 1 \), with \( \bar{s} + \bar{\beta} \leq s + \beta \), then \( C^{s,\beta}(\overline{\Omega}) \subset C^{\bar{s},\bar{\beta}}(\overline{\Omega}) \) [CDK] p.342.

**Proof.** Step 1. Let

\[ X = \{ a \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) : a = 0 \text{ in } U \} \]

\[ Y = \left\{ b \in C^{r,\alpha}(\overline{\Omega}) : \int_{\Omega} b = 0 \text{ and } b = 0 \text{ in } U \right\} \]

By the diverge theorem, \( L(a) = \text{div} a \) is a well defined bounded linear operator \( L : X \to Y \). By Theorem 3 there is a bounded linear operator \( L^{-1} : Y \to X \) such that \( LL^{-1} = \text{id} \) on \( Y \) and for which the correspondence \( b \to a \) is universal (i.e. \( a \) only depends on \( b \) but not on \( r, \alpha \), see Remark 3), therefore 10.16 and 10.17 in the original proof are simultaneously satisfied for \( K_1 = \max(C(r, \alpha, U, \Omega), C(0, \gamma, U, \Omega)) \), where the constants are provided by Theorem 3 (this fact is implicit but not mentioned in [CDK]).

**Step 2.** For any real \( n \times n \) matrix let

\[ Q(\xi) = \text{det}(I + \xi) - 1 - \text{trace}(\xi) \]

where \( I \) is the identity matrix. Observe that a solution to (4.1) is given by \( \varphi := v + \text{id} \) provided \( v \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) \) satisfies

\[
\begin{cases}
\text{div} \, v = f - 1 - Q(\nabla v) & \text{in } U \\
v = 0 & \text{in } U
\end{cases}
\]

(4.2) Letting \( N(v) = f - 1 - Q(\nabla v) \), it is immediate to verify that (4.2) is satisfied by any fixed point of the linear operator \( L^{-1} N : X \to X \). By Banach’s theorem, it remains to find a subset \( B \) of \( X \), complete in relation to an adequate norm and such that \( L^{-1} N \) maps \( B \) into itself and acts there as a contraction (\( N \) is well defined since
there exists $(1)$ and $(2)$ we have just used that $C_\alpha$ (CDK, Theorem 10.7). Note that (The proof is a trivial adaptation of that of Lemma 2 in [DM, p.9-10]; c.f. Proof. $\Omega$)

Let $\alpha$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded connected open smooth set and $U$ a collar of $\Omega$. Given $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$ in $\Omega$, satisfying
\[
\begin{align*}
\int_{\Omega} f &= \text{meas } \Omega \\
f &= 1 & \text{in } U 
\end{align*}
\]
there exists $\varphi \in \text{Diff}^{r,\alpha}(\overline{\Omega}, \overline{\Omega})$ satisfying
\[
\begin{align*}
\det \nabla \varphi &= f \\
\varphi &= \text{id} & \text{in } U
\end{align*}
\]

Proof. (The proof is a trivial adaptation of that of Lemma 2 in [DM] p.9-10); c.f. [CDK, Theorem 10.7]). Note that $\Omega$ being smooth, we have $(1) f \in C^{r-1.1/2}(\overline{\Omega})$ if $\alpha = 0$ and $(2) f \in C^{r.1/2}(\overline{\Omega})$ if $\alpha = 1$. By Theorem 3 there is a solution $v$ to
\[
\begin{align*}
div v &= f - 1 \\
v &= 0 & \text{in } U
\end{align*}
\]
which is $(1)$ in $C^{r,0}(\overline{\Omega}, \mathbb{R}^n) \supset C^{r.1/2}(\overline{\Omega}, \mathbb{R}^n)$ if $\alpha = 0$, $(2) in C^{r.1}(\overline{\Omega}, \mathbb{R}^n) \supset C^{r+1.1/2}(\overline{\Omega}, \mathbb{R}^n)$ if $\alpha = 1$, $(3) in C^{r,\alpha}(\overline{\Omega}, \mathbb{R}^n) \supset C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ if $0 < \alpha < 1$ (for $(1)$ and $(2)$ we have just used that $C^{r,0} \subset C^{r-1.1/2}$ and $C^{r.1} \subset C^{r.1/2}$, see Remark 5). Thus $v$ and $v_t$ are always of class $C^{r,\alpha}$ and so are the solution diffeomorphisms $\Phi_t, t \in [0,1]$ (see [DM]). With the above $v$, the proof is the same as the original one, noting that $v = 0$ in $U$ implies $v_t = 0$ in $U$ for all $t \in [0,1]$, which by its turn implies $\Phi_t = \text{id}$ in $U$, for all such $t$.

\[\square\]

Theorem 6. Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded connected open smooth set and $U$ a collar of $\Omega$. Given $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$ in $\Omega$, satisfying
\[
\begin{align*}
\int_{\Omega} f &= \text{meas } \Omega \\
f &= 1 & \text{in a neighbourhood of } U
\end{align*}
\]
there exists $\varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}, \overline{\Omega})$ satisfying
\[
\begin{align*}
\det \nabla \varphi &= f \\
\varphi &= \text{id} & \text{in a neighbourhood of } U
\end{align*}
\]

Proof. The adaptation of Step 4 in the proof of Theorem 1’ [DM] p.12] requires special attention since it is not straightforward (see the Introduction). Here the general case of arbitrary $f$ is reduced to the case where $\|f - 1\|_{C^{0,\gamma}}$ is small enough (Theorem 4), for some fixed $0 < \gamma < \alpha < 1$. Let $\epsilon = \epsilon(r, \alpha, \gamma, U, \Omega) > 0$ be the constant given by Theorem 4. We shall find $F \in C^{r,\alpha}(\overline{\Omega})$, $F > 0$ in $\Omega$, satisfying
\[
\begin{align*}
\int_{\Omega} F &= \text{meas } \Omega \\
F &= 1 & \text{in a neighbourhood of } U \\
\|F - 1\|_{C^{0,\gamma}} &\leq \epsilon
\end{align*}
\]
\( F \) being the product of \( f \) and \( h/\tilde{f} \), where \( \tilde{f} \) is a convolution of \( f \) and \( h \) is a measure correcting smooth function \( C^{0,\gamma} \)-close to 1.

Fix \( 0 < \gamma < \alpha \). Note that \( f \in C^{0,\gamma}(\overline{I}) \) since \( \Omega \) is smooth (and thus Lipschitz), see Remark 5. By continuity of the multiplication and reciprocal \((1/\cdot)\) operations in relation to the \( C^{0,\gamma} \) norm, there is \( \delta > 0 \) such that for any \( \tilde{f}, h \in C^{0,\gamma}(\overline{I}) \),

\[
\|\tilde{f} - f\|_{C^{0,\gamma}}, \|h - 1\|_{C^{0,\gamma}} \leq \delta \implies \left\| \frac{hf}{\tilde{f}} - 1 \right\|_{C^{0,\gamma}} \leq \epsilon
\]

Reparametrizing the 2nd factor of \( \partial\Omega \times [0,\infty) \) we may assume that \( U = U_1 = \zeta(\partial\Omega \times [0,1]) \) for some collar embedding \( \zeta: \partial\Omega \times [0,1] \to \overline{I} \) (see Definition 2), and that \( f = 1 \) in \( U_3 \), where \( U_t := \zeta(\partial\Omega \times [0,t]) \), for each \( t > 0 \).

(A) **Claim.** There is a sequence \( f_k \in C^{\infty}(\overline{I}) \), \( k \in \mathbb{Z}^+ \), satisfying

1. \( f_k > 0 \) in \( \overline{I} \)
2. \( f_k = 1 \) in \( U_2 \)
3. \( \|f_k - f\|_{C^{0,\gamma}} \leq \delta \)
4. \( \|f_k - f\|_{C^{0,\gamma}} \xrightarrow{k \to \infty} 0 \)

Fix a mollifier \( \rho \in C^\infty(\mathbb{R}^n) \) such that \( \rho > 0 \) in \( \mathbb{B}^n \), \( \rho = 0 \) elsewhere and \( \int_{\mathbb{R}^n} \rho = 1 \). Since \( f \) extends by 1 to the whole \( \mathbb{R}^n \) (in the \( C^{\alpha,\alpha} \) class), for \( k \in \mathbb{Z}^+ \)

\[
f_k := \rho_k * f \in C^\infty(\overline{I})
\]

where \( \rho_k(x) = k^n \rho(kx) \), is well defined and \( f_k > 0 \) in \( \overline{I} \) (\( * \) is the convolution operator), thus (1) holds; for \( k \) large enough, say \( k > k_0 \), (2) holds since \( \text{supp}(f_k - 1) \subset \text{supp}(f - 1) + \text{supp} \rho_k \subset \text{supp} \rho_k \to \{O\} \) (in the Hausdorff metric, \( O \) the origin of \( \mathbb{R}^n \)), \( \text{supp}(f - 1) \subset \overline{\Omega} \setminus U_3 \) and \( U_2 \subset \text{int} U_3 \) in \( \overline{I} \). To see that (4) holds, first note that \( f, f_k \in C^{0,\alpha}(\overline{I}) \) satisfy

(a) \( \|f_k - f\|_{C^{0,\alpha}} \xrightarrow{k \to \infty} 0 \)

(b) \( [f_k]_{C^{0,\alpha}} \leq [f]_{C^{0,\alpha}} \) (see Definition 1)

For this last assertion see for instance [GT] p.148, noting that \( f = 1 \) in a neighbourhood of \( \partial\Omega \), thus \( f \) extends by 1 to \( f \in C^{0,\alpha}(\mathbb{R}^n) \) and \( \|f\|_{C^{0,\alpha}} = [f]_{C^{0,\alpha}} \) (the norms being taken on the respective domains of definition). Now, (4) easily follows from (a) and (b), see for instance Step 1.1 in the proof of [CDK, Theorem 16.22]. Therefore, for \( k \) large enough, say \( k > k_1 \geq k_0 \), (3) holds and reindexing \( f_k \) as \( f_k \to f_{k+k_1} \), both (2) and (3) hold for all \( k \in \mathbb{Z}^+ \) and the Claim is proved. Note that by the continuity of the reciprocal operation in relation to the \( C^{0,\gamma} \) norm, (4) also implies

(5) \( \|f/f_k - 1\|_{C^{0,\gamma}} \xrightarrow{k \to \infty} 0 \)

(B) **Finding a measure correcting smooth function \( h \).**

Fix \( \phi \in C^\infty(\overline{I}) \) such that \( \phi = 0 \) in \( U_2 \) and \( 0 < \phi < 1 \) elsewhere. Take \( \eta > 0 \) small enough so that \( H := \eta \phi \) satisfies \( \|H\|_{C^{0,\gamma}} \leq \delta \). For \( t \in [-1,1] \) let \( h_t := 1 + tH \). Note that

- \( h_t > 0 \) in \( \overline{I} \)
- \( h_t = 1 \) in \( U_2 \)
- \( \|h_t - 1\|_{C^{0,\gamma}} \leq \|H\|_{C^{0,\gamma}} \leq \delta \)
- \( \int_{\overline{I}} h_1 = \text{meas } \Omega + \int_{\overline{I}} H \geq \text{meas } \Omega \)
- \( \int_{\overline{I}} h_{-1} = \text{meas } \Omega - \int_{\overline{I}} H < \text{meas } \Omega \)

Now, by (5) above, fixing \( k \in \mathbb{Z}^+ \) large enough and letting \( \tilde{f} := f_k \) we have

\[
\int_{\overline{I}} (f/\tilde{f})h_1 > \text{meas } \Omega \quad \text{and} \quad \int_{\overline{I}} (f/\tilde{f})h_{-1} < \text{meas } \Omega
\]
Figure 5.1. Finding $h_{\tilde{t}}$ satisfying $\int_{\Omega} (f/\tilde{f}) h_{\tilde{t}} = \text{meas} \Omega$. The functions $h_{\tilde{t}}$ are seen in the background (bell shaped).

(we can see $f/\tilde{f}$ acting (by multiplication) as a small $C^0$ perturbation on $h_1$ and $h_{-1}$, see Fig. 5.1). As $\int_{\Omega} (f/\tilde{f}) h_{1}$ varies continuously with $t$, by the intermediate value theorem

$$\int_{\Omega} (f/\tilde{f}) h = \text{meas} \Omega$$

where $h := h_{\tilde{t}}$ for some $-1 < \tilde{t} < 1$. Summing up, we now have

- $\tilde{f}, h \in C^\infty(\overline{\Omega})$, $\tilde{f}, h > 0$ in $\overline{\Omega}$
- $\tilde{f} = 1 = h$ in $U_2$
- $\|\tilde{f} - f\|_{C^{0,\gamma}}, \|h - 1\|_{C^{0,\gamma}} \leq \delta$

Therefore, $F := (h/\tilde{f}) f \in C^{r,\alpha}(\overline{\Omega})$, $F > 0$ in $\overline{\Omega}$ and (5.1) above holds, the neighbourhood of $U$ in question being $U_2$. Now (as in the original proof [DM, p.13]), use Theorem 4 to find a solution $\varphi_1 \in \text{Diff}^{r,1,\alpha}(\overline{\Omega}, \overline{\Omega})$ of

$$\begin{cases}
\det \nabla \varphi_1 = F \\
\varphi_1 = \text{id} \quad \text{in } U_2
\end{cases}$$

and Theorem 5 to find $\varphi_2 \in \text{Diff}^{r,1,\alpha}(\overline{\Omega}, \overline{\Omega})$ solving

$$\begin{cases}
\det \nabla \varphi_2 = (\tilde{f}/h) \circ \varphi_1^{-1} \\
\varphi_2 = \text{id} \quad \text{in } U_2
\end{cases}$$

noting that $G := (\tilde{f}/h) \circ \varphi_1^{-1} \in C^{r+1,\alpha}(\overline{\Omega})$, $G > 0$ in $\overline{\Omega}$, $\int_{\Omega} G = \text{meas} \Omega$ (by the change of variables theorem) and $G = 1$ in $U_2$. It is immediate to verify that $\varphi = \varphi_2 \circ \varphi_1$ has all claimed properties ($U_2$ being the neighbourhood of $U$ where $\varphi = \text{id}$). \qed

6. The main result

The next result is Theorem 10.3 in [CDK, 192], with control of support but without the estimate (which is impossible to obtain by the present method, see however Theorem 4). It shows, in particular, that we can make $\text{supp}(\varphi - \text{id})$ depend only on the distance from $\text{supp}(f - 1)$ to $\partial \Omega$. Note that besides the estimate, the result in [CDK] improves Dacorogna-Moser Theorem 1’ [DM, p.4], noting that the required regularity of the boundary can be lowered from $C^{r+3,\alpha}$ to $C^{r+2,\alpha}$.
Before stating the main result we need to establish a convention, whose reason is explained below.

**Convention.** If $\Omega \subset \mathbb{R}^n$ is a domain (see Definition 1), then $d(\emptyset, \partial \Omega) := \text{inradius } \Omega$, where $d(\cdot, \cdot)$ is the euclidean distance between two subsets of $\mathbb{R}^n$ and

$$\text{inradius } \Omega := \sup \{ \epsilon > 0 : \Omega \text{ contains an open ball of radius } \epsilon \}$$

The Hausdorff metric in the space of nonvoid closed subset of $\mathbb{R}^n$ is denoted by $d_H(\cdot, \cdot)$.

**Theorem 7.** Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2, \alpha}$ set, $r \geq 0$ an integer and $0 < \alpha < 1$. For each $0 < c \leq R := \text{inradius } \Omega$ there is a neighbourhood $V_c$ of $\partial \Omega$ in $\overline{\Omega}$ such that: given $f \in C^{r, \alpha}(\overline{\Omega})$, $f > 0$ in $\overline{\Omega}$, and $0 \leq d \leq R$ satisfying:

$$\left\{ \begin{array}{l}
  f \mid_{\Omega} = \text{meas } \Omega \\
  d(\text{supp}(f - 1), \partial \Omega) \geq d
\end{array} \right.$$ 

there exists $\varphi \in \text{Diff}^{r+1, \alpha}(\overline{\Omega}, \overline{\Omega})$ satisfying:

$$\left\{ \begin{array}{l}
  \det \nabla \varphi = f \\
  d = 0 \implies \varphi = \text{id} \quad \text{on } \partial \Omega \\
  d > 0 \implies \varphi = \text{id} \quad \text{in } \Omega_d
\end{array} \right.$$

Furthermore,

$$\left\{ \begin{array}{l}
  0 < \hat{c} < c \leq R \implies V_{\hat{c}} \subset V_c \\
  V_c \xrightarrow{d_H} \partial \Omega \\
  V_R = \overline{\Omega}
\end{array} \right.$$ 

**Remark 6.** As it is evident from the proof, the only reason for requiring that $\Omega$ is $C^{r+2, \alpha}$ is to guarantee the existence of a solution when $\text{supp}(f - 1) \not\subset \Omega$ (i.e. $d(\text{supp}(f - 1), \partial \Omega) = 0$). If $\text{supp}(f - 1) \subset \Omega$ is assumed, then Theorem 7 holds if $\Omega \subset \mathbb{R}^n$ is any bounded connected open set.

**Remark 7.** If $\text{supp}(f - 1) \neq \emptyset$ then this set has nonvoid interior, thus its distance to $\partial \Omega$ is smaller than $R := \text{inradius } \Omega$. The compactness of $\overline{\Omega}$ actually implies that $B_R(x) \subset \Omega$ for some $x \in \Omega$, and it is easily seen that for each $0 < \epsilon \leq 1$ there are functions $f_\epsilon$, as above for which $\text{supp}(f_\epsilon - 1) = B_\epsilon(x)$, thus implying that $d(\text{supp}(f_\epsilon - 1), \partial \Omega) = (1 - \epsilon)R$ may actually attain any value $0 \leq d < R$. That is why we have adopted the convention $d(\emptyset, \partial \Omega) := R$ thus covering the limit case $\text{supp}(f - 1) = \emptyset$ in a continuous way (roughly speaking, $\text{supp}(f_\epsilon - 1)$ vanishes in the limit, as $\epsilon \to 0^+$, since it cannot be reduced to a single point).

**Proof.** (of Theorem 7). If $n = 1$ then the solution is trivial, see Section 1.1. Suppose that $n \geq 2$. If $d = 0$ let $\varphi$ be the solution provided by Theorem 10.3 in [CDK], p.192. If $d = R := \text{inradius } \Omega$, then $\text{supp}(f - 1) = 0$ i.e. $f = 1$ in $\overline{\Omega}$, in which case we have the natural solution $\varphi = \text{id} \in \overline{\Omega} =: V_R$. Otherwise we proceed as follows.

(A) **Construction of a suitable smooth exhaustion of $\Omega$.** For each $0 < c < R$, let $[c]$ be the unique $k \in \mathbb{Z}^+$ such that

$$\frac{R}{k + 1} \leq c < \frac{R}{k}$$

By Lemma 2 (see 8.2 in the Appendix), we can find a subsequence of $\Omega_{k \in \mathbb{Z}^+}$, still labeled $\Omega_{k \in \mathbb{Z}^+}$, and a sequence $U_{k \in \mathbb{Z}^+}$, $U_k$ a small collar of $\partial \Omega_k$, such that

1. $d_H(U_1 \cup \overline{\Omega} \setminus \Omega_1, \partial \Omega) < R/2$
2. $d_H(U_{k+1} \cup \overline{\Omega} \setminus \Omega_{k+1}, \partial \Omega) < \min \left( \frac{R}{k+2}, d(\partial \Omega_k, \partial \Omega) \right)$, for each $k \geq 1$. 
Define

\[ \begin{aligned}
V^*_k &= U_k \cup \overline{\Omega} \setminus \Omega_k \quad \text{for } k \in \mathbb{Z}^+ \\
V_c &= V^*[c] \quad \text{for } 0 < c < R \\
V_R &= \overline{\Omega}
\end{aligned} \]

Clearly, by construction \( V^*_{k+1} \subset \text{int} V^*_k \) and 0 < \( \tilde{c} < c \leq R \) implies \( V_c \subset V^*_c \subset \overline{\Omega} \), as 0 < \( \tilde{c} < c \leq R \) implies \( [\tilde{c}] \geq [c] \). Also, by construction, \( d_H(V^*_c, \partial \Omega) \) tends to zero as \( k \to \infty \), hence \( d_H(V_c, \partial \Omega) \) tends to zero as \( c \to 0^+ \).

(B) Solution for 0 < \( d < R = \) inradius \( \Omega \). Given \( f \) and \( d \) as in the statement, note that, by construction of \( V^*_d \), \( f = 1 \) in a neighbourhood of \( V^*_d \), hence \( \int_{\Omega \setminus \Omega_d} f = \int_{\Omega \setminus \Omega_d} 1 = \text{meas } \Omega \setminus \Omega_d \), thus

\[ \int_{\Omega_d} f = \int_{\Omega} f - \int_{\Omega \setminus \Omega_d} f = \text{meas } \Omega - \text{meas } \Omega \setminus \Omega_d = \text{meas } \Omega_d \]

and \( f = 1 \) in a neighbourhood of collar \( U_{\{d\}} \) (in \( \overline{\Omega_{\{d\}}} \)). Apply Theorem 6 to get \( \hat{\varphi} \in \text{Diff}^{r+1,\alpha}(\overline{\Omega_d}, \overline{\Omega}) \) satisfying

\[ \begin{aligned}
\det \nabla \hat{\varphi} &= f|_{\Omega_{\{d\}}} \\
\hat{\varphi} &= \text{id} \quad \text{in } U_{\{d\}}
\end{aligned} \]

Finally, extend this solution to \( \varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}, \overline{\Omega}) \), letting \( \varphi = \text{id} \) in \( \overline{\Omega \setminus \Omega_d} \). It is immediate to check that \( \varphi \) has all claimed properties, in particular \( \varphi = \text{id} \) in \( V_{\{d\}} = V^*_{\{d\}} = U_{\{d\}} \cup \overline{\Omega} \setminus \Omega_d \).

7. Impossibility of the estimate \( \| \varphi - \text{id} \|_{C^{r+1,\alpha}} \leq C \| f - 1 \|_{C^{r,\alpha}} \)

By the present method, which guarantees control of the support of solutions, an estimate in Theorem 7 as that in Theorem 10.3 of [CDK, p.192] cannot be attained. Even restricting to functions \( f \) that are \( C^{0,\alpha} \)-close enough to 1 (c.f. Theorem 4 and [CDK, Theorem 10.9]), we shall exhibit strong evidence pointing to the fact that the estimate

\[ \| \varphi - \text{id} \|_{C^{r+1,\alpha}} \leq C \| f - 1 \|_{C^{r,\alpha}} \quad (7.1) \]

for some constant \( C = C(r, \alpha, \Omega) > 0 \), is impossible to obtain simultaneously with control of support

\( \text{supp}(f - 1) \subset \Omega \implies \text{supp}(\varphi - \text{id}) \subset \Omega \)

if the present method (or variants of it) for the construction of the solutions is employed. To simplify the explanation, suppose that \( \Omega \) is smooth. Obviously, any solution to \( \det \nabla \varphi = f \) in \( \Omega \) satisfies

\( \text{supp}(\varphi - \text{id}) \supset \text{supp}(f - 1) \)

Actually, if \( \text{supp}(f - 1) \subset \Omega \), then to carry out the present method for the construction of a solution \( \varphi \) satisfying \( \text{supp}(\varphi - \text{id}) \subset \Omega \), we need in first place to fix a collar \( \Upsilon = \zeta(\partial \Omega \times [0, \epsilon]) \) of \( \overline{\Omega} \) such that \( \Upsilon \subset \overline{\Omega} \setminus \text{supp}(f - 1) \), in order to be able to apply Theorem 4. Clearly,

\[ \text{thick } U < d(\text{supp}(f - 1), \partial \Omega) \]

where \( \text{thick } U \) (the thickness of \( U \)) is the distance between \( \partial \Omega \) and the “internal” boundary \( \partial_0 U \) of \( U \) i.e.

\[ \text{thick } U := d(\partial \Omega, \partial_0 U) \quad \text{where } \partial_0 U := \zeta(\partial \Omega \times \epsilon) = \partial U \setminus \partial \Omega \]

Let \( U_{k \in \mathbb{Z}^+} \) be a sequence of collars of \( \overline{\Omega} \) such that

\[ \text{thick } U_k \xrightarrow{k \to \infty} 0 \]
We shall now produce enough evidence that in Theorem 4, fixing $\gamma = \alpha$, the undesirable facts

$$c = c(r, \alpha, U_k, \Omega) \xrightarrow{k \to \infty} \infty \quad \text{and} \quad \epsilon = \epsilon(r, \alpha, U_k, \Omega) \xrightarrow{k \to \infty} 0$$

cannot be avoided, thus dissipating any hope of establishing the estimate (7.1) with $C$ independent of $d(\text{supp}(f-1), \partial \Omega)$. The proof of Theorem 4 is closely modeled on that of [CDK, Theorem 10.9], all the estimates being the same. In particular, a brief inspection reveals that

$$c = c(r, \alpha, U_k, \Omega) = 2K_1 \quad \text{and} \quad \epsilon = \epsilon(r, \alpha, U_k, \Omega) \leq 1/2K_1$$

where $K_1 \geq C$, $C = C(r, \alpha, U_k, \Omega) > 0$ being the constant given by Theorem 3 (see Step 1 in the Proof of Theorem 4). Actually, in this $C$ it enters as a multiplicative factor a constant $\hat{C} = C_2(r, U_k) \geq 1$, controlling (in the case under question) the $C^{r+2, \alpha}$ norm of an extension operator introduced in the Proof of Theorem 2. We now show evidence that, when $r \geq 1$, there is no way to keep $C_2(r, U_k)$ bounded as $k \to \infty$ i.e. when $k \to \infty$ the thickness of $U_k$ tends to zero and

$$C_2(r, U_k) \xrightarrow{k \to \infty} \infty$$

cannot be avoided. All extension methods (applicable in the context of (C) in the Proof of Theorem 2) that are known to us are variants of the same global strategy and are consequently subject to problem (7.2), see below. Let $U$ be any of the $U_k$’s. Briefly, the extension of a function $g \in C^1(U)$ to the whole $\overline{\Omega}$ is obtained gluing together finitely many local extensions from the interior to the exterior of $U$, performed on small balls (or cubes) centred at points of $\partial_0 U$. To simplify the explanation, we adopt the extension method described in [GT, p.136]
with the modification of Remark 2. Since we wish only to extend \( g \) to the whole \( \overline{\Omega} \) (and not to the whole \( \mathbb{R}^n \)), we start by covering \( \partial_0 U \) with finitely many open “cubes” \( V_i \), \( 1 \leq i \leq j \), each having one of its halves \( V_i^+ \) contained in \( U \) (see Fig. 7.1). As explained in [GT] (see also Remark 2), we then use the boundary rectifying diffeomorphisms (see Footnote 3) and Seeley’s extension operator to get, for each \( 1 \leq i \leq j \), a \( C^1 \) extension \( g_i \) of \( g|_{V_i^+} \) to the whole \( V_i \). Note that if we set \( V_0 := U \setminus \partial_0 U \), then \( \{ V_i \}_{0 \leq i \leq j} \) is an open covering of \( U \) in \( \overline{\Omega} \). Fix a partition of unity \( \{ \eta_i \}_{0 \leq i \leq j} \) subordinate to this covering. Finally, let \( g_0 = g|_{U \setminus \partial_0 U} \) and define the desired extension \( \tilde{g} \in C^1(\overline{\Omega}) \) of \( g \) as

\[
\tilde{g} = \sum_{i=0}^{j} \eta_i g_i
\]

(with the convention that \( g_i = 0 \) in \( \overline{\Omega} \setminus V_i \)). Now, in order to find a constant \( \tilde{C} = \tilde{C}(1, U) \geq 1 \) such that

\[
\| \tilde{g} \|_{C^1(\overline{\Omega})} \leq \tilde{C} \| g \|_{C^1(U)}
\]

we are naturally lead to the estimates

\[
\| \tilde{g} \|_{C^1(\overline{\Omega})} \leq \sum_{i=0}^{j} \| \eta_i g_i \|_{C^1}
\]

\[
\leq \sum_{i=0}^{j} \| \eta_i \|_{C^1} \| g_i \|_{C^0} + \| \eta_i \|_{C^0} \| g_i \|_{C^1}
\]

\[
\leq 2 \sum_{i=0}^{j} \| \eta_i \|_{C^1} \| g_i \|_{C^1}
\]

It is easily seen that there is a constant \( K \geq 1 \) depending only on the boundary rectifying diffeomorphisms \( \psi_i \) (see Footnote 3) and on the extension operator from the right halfcube \( \mathcal{C}^+ \) to the cube \( \mathcal{C} = (-2, 2)^n \) (in our case Seeley’s operator) such that

\[
\| g_i \|_{C^1} \leq K \| g \|_{C^1(U)}
\]

and therefore the above estimate leads to

\[
\| \tilde{g} \|_{C^1(\overline{\Omega})} \leq \tilde{C} \| g \|_{C^1(U)} \quad \text{where} \quad \tilde{C} = 2K \sum_{i=0}^{j} \| \eta_i \|_{C^1}
\]

But the problem now lies in the factor \( \sum_{i=0}^{j} \| \eta_i \|_{C^1} \), for one can easily see that as \( k \to \infty \), the thickness of \( U = U_k \) tends to zero and simultaneously this quantity diverges to \( \infty \), independently of the particular partition of unity employed for each \( U_k \). In fact let \( \delta = \text{thick } U \) and fix \( x \in \partial_0 U \) and \( y \in \partial \Omega \) such that \( d(x, y) = |x - y| = \delta \) (see Fig. 7.1). For \( 1 \leq i \leq j \), \( \psi_i \) since \( V_i \cap \partial_\Omega = \emptyset \) (see Footnote 4), hence for all such indices \( i \), \( \eta_i(y) = 0 \). Therefore, by the mean value theorem, for \( 1 \leq i \leq j \)

\[
\| \eta_i \|_{C^1} \geq \frac{\eta_i(x) - \eta_i(y)}{|x - y|} = \frac{\eta_i(x)}{\delta}
\]

and since \( \sum_{i=1}^{j} \eta_i(x) = 1 \) (as \( x \in \partial_0 U \) implies \( \eta_0(x) = 0 \)) we finally have

\[
\sum_{i=0}^{j} \| \eta_i \|_{C^1} \geq \delta^{-1} \sum_{i=1}^{j} \eta_i(x) = \delta^{-1}
\]

therefore, when \( k \to \infty \), \( \tilde{C} = \tilde{C}(1, U_k) \to \infty \) as \( \delta_k := \text{thick } U_k \to 0 \). It remains to mention that if instead of the above extension operator, that of [CDK] p.342 is

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3 More precisely, \( \cup_{1 \leq i \leq j} V_i \supset \partial_0 U \), where each \( V_i \) is an open set intersecting \( \partial_0 U \) for which there is a (boundary rectifying) diffeomorphism \( \psi_i \in \text{Diff}^\infty (V_i, (-2, 2)^n) \) such that \( \psi_i(V_i^+) = [0, 2^n \times (-2, 2)^{n-1}, \text{ where } V_i^+ = V_i \cap U \), thus implying \( V_i \cap \partial \Omega = \emptyset \).

4 Note that necessarily \( \eta_0 = 0 \) in a neighbourhood of \( \partial_0 U \) and \( \eta_0 = 1 \) in a neighbourhood of \( \partial \Omega \) (in \( \overline{\Omega} \)).
employed, then we run into the very same problem (see in particular [CDK] p.353-355) where an explicit formula for this extension operator is obtained, noting that the auxiliary functions $\lambda_i$ play the analogue role to the above $\eta_i$).

8. Appendix

8.1. Standard regularity relative Poincaré lemma for collars. The following result is invoked in the Proof of Theorem 2, step (A).

**Lemma 1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open smooth set, $U$ a collar of $\overline{\Omega}$ and $k \in \mathbb{Z}^+$. Given a closed form $\beta \in C^r(U; \Lambda^k)$ that vanishes when pulled back to $\partial \Omega$, there exists $\omega_0 \in C^r(U; \Lambda^{k-1})$ satisfying $d\omega_0 = \beta$.

**Proof.** Since $U = \zeta(\partial \Omega \times [0,\epsilon])$ is a collar and $\partial \Omega$ is a smooth $(n-1)$-manifold, we reason in product charts as if $\beta \in C^r(\mathbb{B}^{n-1} \times [0,\epsilon]; \Lambda)$. Let $P := \mathbb{B}^{n-1} \times [0,\epsilon]$ with points $p = (x, x_n) = (x_1, \ldots, x_{n-1}, x_n)$. For $t \in [0,1]$ let $j_t$ be the embedding $P \hookrightarrow P \times [0,1]: p \mapsto (p, t)$. Define the smooth homotopy

$$h : P \times [0,1] \rightarrow P \quad (x, x_n, t) \mapsto (x, tx_n)$$

Then, $h \circ j_1$ is the identity on $P$ and $h \circ j_0$ is the canonical projection $P \rightarrow \mathbb{B}^{n-1} \times 0$, thus $(h \circ j_0)^*\beta = 0$ as by hypothesis $\beta$ vanishes when pulled back to $\mathbb{B}^{n-1} \times 0$. Hence

$$\beta = (h \circ j_1)^*\beta - (h \circ j_0)^*\beta = \int_0^1 \frac{d}{dt} \bigg|_{t=s} (h \circ j_t)^*\beta \, ds$$

As $d\beta = 0$, denoting by $\partial_t$ the vertical vector field $\frac{\partial}{\partial t}$ on $P \times [0,1]$, by Cartan’s formula the right side expression equals

$$\int_0^1 j_s^* (\partial_t \wedge h^*\beta) + j_s^* d(\partial_t \wedge h^*\beta) \, ds = d \int_0^1 j_s^* (\partial_t \wedge h^*\beta) \, ds$$

and a primitive $\omega_0$ of $\beta$ is found. $\omega_0$ is $C^r$ since $\partial_t \wedge h^*\beta$ is a $C^r$ $(k-1)$-form on $P \times [0,1]$ (whose components involving $\wedge dt$ vanish), hence, by Leibniz rule, for any multiindex $\gamma$ of order $\leq r$ with directions among those of $P$,

$$\partial^\gamma \int_0^1 j_s^* (\partial_t \wedge h^*\beta) \, ds = \int_0^1 \partial^\gamma j_s^* (\partial_t \wedge h^*\beta) \, ds$$

and the right side expression is continuous on $P$. \hfill \Box

8.2. Existence of an exhaustion by smooth domains. For the sake of completeness we include a brief proof that any domain has an exhaustion by smooth domains (which we could not locate in the literature).

**Definition 3.** For each $\mathfrak{M} \subset 2^{\mathbb{R}^n}$ ($2^{\mathbb{R}^n}$ being the set of all subsets of $\mathbb{R}^n$), we define $\cup\mathfrak{M} = \cup_{S \in \mathfrak{M}} S \subset \mathbb{R}^n$.

**Lemma 2.** (Exhaustion by smooth domains). Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. Then there is a sequence $\Omega_k$ of bounded connected open smooth sets satisfying:

1. $\overline{\Omega_k} \subset \Omega_{k+1} \subset \Omega$ for all $k \in \mathbb{Z}^+$
2. $\cup_{k \in \mathbb{Z}^+} \Omega_k = \Omega$
3. $d_H(\overline{\Omega} \setminus \Omega_k) \rightarrow 0$ as $k \rightarrow \infty$

**Proof.** Let $x_k \in \mathbb{Z}^+$ be a dense sequence in $\Omega$ and define $B_k = B(x_k, \delta_k)$, where $\delta_k = d(x_k, \partial \Omega)/2$. Let $\mathcal{B} = \{B_k\}_{k \in \mathbb{Z}^+}$. Clearly

$$\cup \mathcal{B} = \Omega$$

(8.1)
Let $\Omega_1 = B_1$. We proceed by induction over $k \in \mathbb{Z}^+$. Supposing that $\Omega_k$ has already been found, we now find $\Omega_{k+1}$. Define

$$\xi(k) = \min \{ j \in \mathbb{Z}^+ : \cup_{i=1}^{k} B_i \supset \overline{\Omega_k} \}$$

$$\Theta_k = \{ B_i : 1 \leq i \leq \xi(k) \}$$

Eventually, $\cup \Theta_k$ is disconnected. In this case let $\gamma_k : [0, 1] \to \Omega$ be an injective path that intersects all the components of $\cup \Theta_k$. Since $\gamma_k = \mathrm{im} \gamma_k$ is compact, we can find a finite collection $\Upsilon_k \subset \mathcal{B}$ such that each ball in $\Upsilon_k$ intersects $\overline{\gamma_k}$ and $\overline{\gamma_k} \subset \cup \Theta_k$. If $\cup \Theta_k$ is connected, simply let $\Upsilon_k = \emptyset$. Let $\gamma_k = \Theta_k \cup \Upsilon_k$. Note that $\cup \Theta_k$ is connected. Slightly increasing the radius of each ball $B_i \in \Theta_k$ (always to less than the double of the original radius, in order to keep its closure within $\Omega$), we make all the boundary spheres $\partial B_i$ intersect transversely, so that the union of these enlarged balls is a connected open set $\Omega_{k+1}$ with piecewise smooth boundary containing $\overline{\Theta_k}$, whose closure $\overline{\Omega_{k+1}}$ is contained in $\Omega$. Smooth out the edges of $\overline{\Omega_{k+1}}$ to get $\overline{\Omega_{k+1}}$, the closure of a connected open smooth set $\Omega_{k+1}$ satisfying

1. $\cup \Theta_k \subset \Omega_{k+1}$
2. $\overline{\Theta_k} \subset \Omega$

This is clearly possible since the smoothing of $\overline{\Theta_k}$ can be performed arbitrarily near $\partial \overline{\Theta_k}$. Since $\overline{\Theta_k} \subset \cup \Theta_k$, it follows from (a) and (b) that (1) holds. Observe that the inductively defined function $\xi : \mathbb{Z}^+ \to \mathbb{Z}^+$ is strictly increasing, as for all $k \geq 1$, $\overline{\Theta_k} \subset \Theta_{k+1} \subset \Omega_{k+1}$, therefore by (8.1), (2) holds. To see that (3) holds we proceed by contradiction. First note that $\partial \Omega \subset \overline{\Theta_k}$ for all $k \geq 1$, hence if (3) fails then there is $\epsilon > 0$, a subsequence of $\Omega_k$, still labeled $\Omega_k$, and a sequence of points $z_k \in \overline{\Omega_k} \setminus \Omega_k$ such that $d(z_k, \partial \Omega) \geq \epsilon$. Since $\overline{\Omega}$ is compact, $z_k$ has a subsequence (still labeled $z_k$) converging to some point $z \in \Omega$. Clearly $z$ can belong to no $\Omega_k$, otherwise, by (1), there is a neighbourhood of $z_k$ contained in $\cup_{k \geq 1} \Omega_k$, which contradicts the existence of sequence $z_k$ as defined above. But $z \notin \cup_{k \in \mathbb{Z}^+} \Omega_k$ contradicts $\cup_{k \in \mathbb{Z}^+} \Omega_k = \Omega$.

\[ \square \]

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Centro de Matemática da Universidade do Porto
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
E-mail: pteixeira.ir@gmail.com
E-mail2: pedro.teixeira@fc.up.pt