Extending a Chebyshev Subspace to a Weak Chebyshev Subspace of Higher Dimension and Related Results

Mansour Alyazidi-Asiry*
Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia

Abstract

Let \( G=\{g_1, \ldots, g_n\} \) be an \( n \)-dimensional Chebyshev sub-space of \( C[a, b] \) such that 1 \( \not\in \) \( G \) and \( U=(u_0, u_1, \ldots, u_n) \) be an \( (n+1) \)-dimensional subspace of \( C[a, b] \) where \( u_0=1, u_i=g_i, i=1, \ldots, n \). Under certain restriction on \( G \), we proved that \( U \) is a Chebyshev subspace if and only if it is a Weak Chebyshev subspace. In addition, some other related results are established.

Keywords: Chebyshev system; Weak Chebyshev system

Introduction

The finite set of functions \( \{g_1, \ldots, g_n\} \) and \( C[a, b] \) is called a Chebyshev system on \([a, b]\) if it is linearly independent and \( D \left( \begin{array}{c} g_1(x) \\ \vdots \\ g_n(x) \end{array} \right) \geq 0, j=1, \ldots, n \) for all \( \{x_i\}_{i=1}^n \) such that \( a \leq x_1 < x_2 < \ldots < x_n \leq b \), and the \( n \)-dimensional subspace \( G=\{g_1, \ldots, g_n\} \) of \( C[a, b] \) will be called a Chebyshev subspace \([1-4]\). Using the continuity of the determinant, it can be shown that the sign of the determinant is always positive through this paper (replace \( g_1 \) by \(-g_1\) if necessary). And the finite set of functions \( \{g_1, \ldots, g_n\} \) and \( C[a, b] \) is called a Weak Chebyshev subspace of \( C[a, b] \) if and only if it is a Weak Chebyshev system on \([a, b]\). An example illustrating that the preceding assertion is not true in general is presented and some related results are given.

The Main Result

We start this section with the following lemma.

Lemma

Let \( G=\{g_1, \ldots, g_n\} \) be an \( n \)-dimensional Chebyshev subspace of...
C[a, b] such that 1 ∈ U and U={u0, u1, ...., uν} be an (n+1)-dimensional
subspace of C[a,b] where u0=1, u1=g1, I=1,......,n. If there are two non-
triivial functions h, k ∈ U and a set of n points \{x_i\}_{i=1}^{n+1} with
\[ a ≤ x_1 ≤ x_2 ≤ ... ≤ x_n ≤ b \]
such that
\[ h(x_i)=k(x_i)=0, \quad i=1, ..., n, \]
then there is a nonzero constant \( λ \) such that \( h(x)=λ_k(x) \) for every \( x \) ∈ [a, b].

**Proof:** Write \( h=a_0+\sum_{i=1}^{n}a_ig_i \), if \( a_0=0 \) then \( h \) ∈ G and from theorem (1)
\[ h(x)=0 \]
for every \( x \) ∈ [a, b], so \( a_0 \neq 0 \) then \( h(x_i)=0, i=1, ..., n \), where
\[ \frac{-\bar{h}}{h} = \frac{1}{a_0} \]
Similarly, if \( K=b_0+\sum_{i=1}^{n}b_ig_i \), then \( b_0 \neq 0 \) and \( K(x_i)=0, i=1, ..., n \), where
\[ K\frac{1}{b_0}k = 1+\sum_{i=1}^{n}b_ig_i \]
Now let \( f=h-K \), then \( f \) is an element of the n-dimensional Chebyshev subspace G with \( f(x_i)=0, i=1, ..., n \), so \( f \) and \( h-K \) taking \( \frac{a}{b} \), we have \( \lambda \neq 0 \) and \( h(x)=λk(x) \) for every \( x \) ∈ [a, b].

**Assumption A:** We say that the subspace G of C[a, b] satisfies Assumption A if for each \( f \) ∈ G such that \( f(x)=f(y) \) for some \( x, y \) ∈ [a, b] with \( x \neq y \) there is an element \( f \) such that \( f(z) \neq f(x) \).

**Lemma:**
Let \( G=\{g_1, ..., g_n \} \) be an n-dimensional Chebyshev subspace of C[a, b]
such that \( \exists \)G and \( U=\{u_0, u_1, ..., u_n \} \) be an (n+1)-dimensional subspace of C[a, b] where \( u_0=1, u_1=g_1, i=1, ..., n \). If G satisfies Assumption A, then the zeros of each nontrivial function \( h \) ∈ U are separated and essential.

**Proof:** Let \( h \) be a nontrivial element of U such that \( h(x)=h(y)=0 \) for some \( x, y \) with \( a ≤ x ≤ y ≤ b \). If \( h \) ∈ G, then \( n = 3 \), for otherwise \( h \equiv 0 \), and since G is an n-dimensional Chebyshev subspace of C[a, b], there is a point \( z \) ∈ (x, y) such that \( h(z)=0 \). If \( h \) ∈ U, then \( h=αg \), where \( α \neq 0 \) and G being a Chebyshev subspace of C[a, b] satisfies Assumption A, so there is a point \( z \) ∈ (x, y) such that \( g(z)=αg \) for \( z \) ∈ (x, y). This shows that the zeros of \( h \) are separated. For the second part of the assertion of the lemma, it is clear that each zero of any nontrivial element of U is an essential zero, that is because \( \exists \)G.

**Remark 1:** Note that if \( f \) ∈ U and \( f(x)=f(y)=0 \), then since G is a Chebyshev space, \( x \) is an essential zero for \( f \). Indeed, there is an element \( g \) ∈ G such that \( g(x) \neq 0 \).

**Theorem:**
Let \( G=\{g_1, ..., g_n \} \) be an n-dimensional Chebyshev subspace of C[a, b] such that \( \exists \)G and \( U=\{u_0, u_1, ..., u_n \} \) be an (n+1)-dimensional subspace of C[a, b] where \( u_0=1, u_1=g_1, i=1, ..., n \). If G satisfies Assumption A, then U is a Chebyshev subspace of C[a, b] if and only if it is a Weak Chebyshev subspace of C[a, b].

**Proof:** One direction is trivial.

For the other direction, suppose \( U=\{u_0, u_1, ..., u_n \} \) is an (n+1) – dimensional Weak Chebyshev subspace of C[a, b] where \( u_0=1, u_1=g_1, i=1, n \) and G=(g1,..., gn) is an n-dimensional Chebyshev subspace of C[a, b] satisfying Assumption A. Let \( u \) be a nontrivial element of U such that \( \bar{u}(x_i)=0, i=1, ..., d, \)
\[ a ≤ x_i ≤ x_d, b, \]
If \( d > n+1 \), then by lemma (2) together with theorem (2) we must have \( \bar{v} \equiv 0 \), so \( d ≤ n+1 \), if \( d ≤ n \), then there is nothing to prove, so to this end, we will assume that \( d= n+1 \)
\[ \bar{u}(x_d)=0, i=1, ..., n+1, \]
\[ a ≤ x_1 < ... < x_{n+1} ≤ b, \]
again from law (2) and theorem (2) we must have \( a=x_i \) or \( x_{n+1} \) and \( u(x) \neq 0 \) for all \( x \) ∈ [a, b]. Writing \( \bar{u} = α+g, \)
then \( α \neq 0 \) that is because G is an n-dimensional Chebyshev subspace of C[a, b].

Taking \( u=\frac{1}{α} \bar{u} \), then
\[ u=1+\sum_{i=1}^{n}α_i g_i, \]
\[ u(x_i)=0, i=1, ..., n+1 \]
and \( u(x) \neq 0 \) for all \( x \) ∈ [a, b].

The rest of the proof is divided into several cases.

**Case A:** \( a=x_i \) and \( x_{n+1}=b \).

Since G is an n-dimensional Chebyshev subspace of C[a, b], then for any point \( q \) ∈ (x_i, b) there is a function \( g=\sum_{i=1}^{n}β_ig_i \) ∈ G such that
\[ g(y_i)=1, i=1, ..., n, \]
\[ g(y_n)=q, \]
Taking
\[ v=1-g \]
\[ v(x) \neq 0 \]
and by lemma (1) there is a non-zero constant \( λ \) such that \( u=λv \), this implies that
\[ u(x_i)=v(x_i)=0, i=1, ..., n \]
And \( v(t) \neq 0 \), \( t=1, ..., n+1 \)
This means that \( u \) has at least \( n+2 \) separated zeros in [a, b] which implies that \( u=v \equiv 0 \) contradicting the fact \( u \) and \( v \) are nontrivial elements of U, hence \( v \neq 0 \), \( x \) ∈ [a, b]. It is clear that
u(x) ≠ 0, x∈(x₁, b), u(x₁)=u(b)=0
and
v(yᵢ)=0, yᵢ∈(x₁, b), v(t)=0 for all t ∈ [x₁, b]/\{yᵢ\},
and if x∈[xᵢ, yᵢ], y∈(yᵢ, b) then sign v(x)=−sign v(y), subsequently,
We treat four different subcases.

**Case A1:** u(x) < 0 for all x ∈ (x₁, b) and v(x) > 0 for all x ∈ [xᵢ, yᵢ],
then v(x) <0 for all x ∈ (yᵢ, b), taking w=u−v, we have
w(xᵢ)=−v(xᵢ) <0 and w(yᵢ)=u(yᵢ) >0
by the continuity of w, there is a point s ∈ (xᵢ, yᵢ) such that w(s)=0,
hence we have:
w(zᵢ)=0, i=1,…,n, where zᵢ=xᵢ, I=1,…,n-1 and z_n=s.
But w belongs to the n-dimensional Chebyshev subspace G of C [a,b].
Hence w ≡ 0 and it follows that u=v and
u(t_i)=0, I=1,….n+2
Where
t_i=x_i, I=1,…,n,
t_n+1=y_n and t_{n+2}=x_{n+1}=b
So u must be identically zero.

**Case A2:** u(x) > 0 for all x ∈ (x₁, b) and v(x) < b0 for all x ∈ [xᵢ, yᵢ],
then v(x) >0 for all x ∈ (yᵢ, b), again taking w=u−v, we have
w(yᵢ)=u(yᵢ) >0 w(b)=−v(b)<0,
and there is a point s ∈ (yᵢ, b], such that w(s)=0, so w has at least n distinct zeros in [a,b].
A similar argument as in case A1 shows that u must be identically zero.

**Case A3:** u(x) < 0 for all x ∈ (x₁, b) and v(x)<b0 for all x ∈ [xᵢ, yᵢ],
then v(x) >0 for all x∈ (yᵢ, b), taking w=u−v, we have
w(yᵢ)=−v(yᵢ) >0 and w (yᵢ)=u (yᵢ)<0,
and continuing exactly as in case A1, we conclude that u must be identically zero.

**Case A4:** u(x) < 0 for all x ∈ (x₁, b) and v(x) > 0 for all x∈ [xᵢ, yᵢ],
then v(x) > 0 for all x∈ [xᵢ, yᵢ), taking w=u−v, we have
w(yᵢ)=−v(yᵢ) >0 and w (yᵢ)=u (yᵢ)<0,
and if there is a point t ∈ [a,b]/\{yᵢ\}, such that v(t)=0, then by theorem (2) we must have t=b or t=a.

If t=b, then u and v are two nontrivial elements of U such that
U(xᵢ)=v(xᵢ)=0,
u(x)=v(x)=0, I=1, …,n-1
and by lemma 1 there is a nonzero constant λ such that u=λ v, this implies that
u(t_i)=0, I=1,….n+2
where t_i=x_i, I=1,…,n,
t_{n+1}=y_n
and t_{n+2}=x_{n+1}=b
so u has at least n+2 separated zeros in[a, b] which implies that u=v
≡ 0 and this is a contradiction.

so t ≠ b and the situation becomes exactly as in case A, proceedings as in case A we conclude that u must be identically zero.

**Case C:** a=x₁ and xₙ₊₁<b
The proof of this case requires that n ≥ 2 and the proof for n=1 will be given in remark (2).

Now, for any point p ∈ (a,x₁) there is a function G = ∑_{i=1}^{∞} β_i g_i ∈ G
such that
G(yᵢ)=1, i=1,…,n,
Where y₁=p and yᵢ=xᵢ, i=3,…,n+1.
Taking
v = 1 − g = 1 − ∑_{i=1}^{∞} β_i g_i,
Then v is a nontrivial element of U with
v(yᵢ)=0, i=1,….n,
a < y₁ < ……< yᵦ < b
and if there is a point t ∈ [a,b]/\{yᵦ\}, such that v(t)=0, then by thorem (2) we must have t=a or t=b.

If t=a, then u and v are two nontrivial elements of U such that
U(xᵢ)=v(xᵢ)=0, u(x₁)=0, i=3,…,n+1.
A similar argument to that of the other cases leads to a contradiction.
So t ≠ a and on the interval [a, x₁] we have
u(a)=u(x₁)=0, u(x)≠0, x ∈ (a,x₁)
and
v(yᵦ)=0, yᵦ ∈ (a,x₁),v(t) ≠ 0 for every t ∈ [a,x₁]\{yᵦ\}.
If x∈[a,yᵦ], y∈(yᵦ, x₁] then sign v(x)=−sign v(y), and as in the other cases we are presented with four different subcases. In each case, a similar argument to that of the cases in A can be used to show that the function u-v in G has at least n zeros which leads to the conclusion that u must be identically zero. Hence U is a Chebyshev subspace of C [a,b].

**Remark 2:** The following is the proof for theorem (3) when n=1 which is somewhat more direct:
Suppose $g$ is a non constant continuous function on $[a,b]$ such that $G=[g]$ is a Chebyshev subspace of $C[a,b]$ of dimension 1 satisfying Assumption A and $U=[1,g]$ is a subspace of $C[a,b]$ of dimension 2. If $U$ is not a Chebyshev subspace, then there is a nontrivial element $u=α + βg$ of $U$ such that $u(x_1)=u(x_2)=0$ where $a ≤ x_1 < x_2 ≤ b$, clearly $α ≠ 0$ and $β ≠ 0$.

By lemma (2) there is a point $y_1$, $x_1< y_1< x_2$ such that $g(y_1)=d≠ c$. Taking $x_1=z_1$, $y_1=z_2$ and $x_2=z_3$, then

$$D\left(\begin{array}{c} 1 \\ z_1 \\ z_2 \end{array}\right) f = Det\left(\begin{array}{cc} 1 & 1 \\ c & d \end{array}\right) = d-c ≠ 0$$

And

$$D\left(\begin{array}{c} 1 \\ z_2 \\ z_3 \end{array}\right) = c-d ≠ 0,$$

Hence

$$\frac{g(z_1)}{g(z_2)} = \frac{-\text{sign} D\left(\begin{array}{c} 1 \\ z_1 \\ z_2 \end{array}\right)}{\text{sign} D\left(\begin{array}{c} 1 \\ z_2 \\ z_3 \end{array}\right)}$$

This shows that $U$ is not a weak Chebyshev subspace and the theorem is proved.

The following example illustrates that theorem (3) is not true in general is proved.

**Example 1**

Let $g(x)=\begin{cases} 1 & 0 ≤ x ≤ 1 \\ x & 1 < x ≤ 2 \end{cases}$

$G=[g]$ is a Chebyshev Subspace of $C[0,2]$ of dimension 1, if $U=[1,g]$ and $x_1 < x_2 ≤ 2$, then

$$D\left(\begin{array}{c} 1 \\ x_1 \\ x_2 \end{array}\right) = 0 \text{ if } x_2 \in [0,1]$$

And

$$D\left(\begin{array}{c} 1 \\ x_1 \\ x_2 \end{array}\right) > 0 \text{ if } x_2 \in (1,2].$$

That is $U$ is a 2-dimensional weak Chebyshev Subspace of $C[0,2]$ but not a Chebyshev Subspace.

If $H$ is $n$-dimensional subspace of $C[a,b]$, then it is possible that $H$ is a Chebyshev subspace on one of the intervals $(a,b]$ or $[a,b)$ but not on the closed interval $[a,b]$ as illustrated in the following example.

**Example 2**

Let $H=(\sin x, \cos x)$, it can be easily checked that $H$ is a Chebyshev subspace of dimension 2 on each of the intervals $(0,\pi]$ of dimension 2.

In next result we give a necessary and sufficient condition for an $n$-dimensional Chebyshev $H$ on $(a,b]$ or $[a,b)$ to be a Chebyshev subspace on the closed interval $[a,b]$.

**Theorem**

Let $H$ be an $n$-dimensional subspace of $C[a,b]$ such that $H$ is a Chebyshev subspace on $(a,b]$ or on $[a,b)$, then $H$ is a Chebyshev subspace on $[a,b]$ if and only if each function $h_i$, $i=1,...,n$ can have at most $n-1$ distinct zeros on $[a,b]$ whenever $H=[h_1,...,h_n]$.

**Proof:** If $(h_1,...,h_n)$ is a basis of $H$ such that for some $s \in \{1,...,n\}$, $h_s$ has at least $n$ zeros on $[a,b]$, then clearly $H$ is not a Chebyshev Subspace on $[a,b]$. For the other direction, suppose $H$ is Chebyshev subspace on $I=(a,b]$ but not on $[a,b)$, there is a non-trivial element $u=\alpha + \beta h_s$ of $U$ such that $u(x_1)=u(x_2)=0$, $i=2,...,n$.

Since $H$ is a Chebyshev subspace on $(a,b)$, there is a subset $E=[h_1,...,h_n]$ and $H$ such that

$$h_i(z_j) = \delta_{ij} \quad \text{if } i \neq j, \delta_{ii} = 0 \text{ otherwise}$$

The elements of $E$ are linearly independent and $H=[h_1,...,h_n]$, write

$$u = \sum_{i=1}^{n} a_i h_i, a_i \in \mathbb{R}, i=1,...,n,$$

Then

$$0 = u(z_1)=a_1 h_1(z_1)=a_1,$$

$$0 = u(z_i)=a_i h_i(z_i)=a_i, i=2,...,n,$$

Hence $h_i=\frac{1}{a_i}$ which has no zeros on $[a,b]$ and this is a contradiction.

Using a similar argument when $I=[a,b)$ leads to a contradiction and the theorem is proved.

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