A NOTE ON THE NON-HOMOGENEOUS INITIAL BOUNDARY PROBLEM FOR AN OSTROVSKY-HUNTER TYPE EQUATION

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Abstract. We consider an Ostrovsky-Hunter type equation, which also includes the short pulse equation, or the Kozlov-Sazonov equation. We prove the well-posedness of the entropy solution for the non-homogeneous initial boundary value problem. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method.

1. Introduction. In this paper, we investigate the well-posedness in classes of discontinuous functions for the equation
\[ \partial_x (\partial_t u + \kappa \partial_x f(u)) = \gamma u, \quad t > 0, \quad x > 0, \quad f(u) = u^2, u^3, \quad \kappa, \gamma \in \mathbb{R}. \] (1)
We are interested in the initial-boundary value problem for this equation, so we augment (1) with the boundary condition
\[ u(t, 0) = g(t), \quad t > 0, \] (2)
and the initial datum
\[ u(0, x) = u_0(x), \quad x > 0, \] (3)
on which we assume that
\[ u_0 \in L^\infty(0, \infty) \cap L^1(0, \infty), \quad \int_0^\infty u_0(x) dx = 0. \] (4)
On the function
\[ P_0(x) = \int_0^x u_0(y) dy, \] (5)
we assume that
\[ \|P_0\|_{L^2(0, \infty)}^2 = \int_0^\infty \left( \int_0^x u_0(y) dy \right)^2 dx < \infty. \] (6)

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On the boundary datum $g(t)$, we assume that
\[ g \in L^\infty(0, \infty), \] (7)
when
\[ f(u) = u^2, \quad \kappa \in \mathbb{R}, \quad \gamma > 0, \quad \text{or,} \quad f(u) = u^3, \quad \kappa > 0, \quad \gamma > 0. \] (8)
If
\[ f(u) = u^2, \quad \kappa \in \mathbb{R}, \quad \gamma < 0, \] (9)
we assume
\[ g \in W^{1,\infty}(0, \infty). \] (10)
Instead, if
\[ f(u) = u^3, \quad \kappa < 0, \quad \gamma > 0, \] (11)
we assume
\[ g \in W^{1,\infty}(0, \infty), \quad 0 < \alpha \leq |g(\cdot)|. \] (12)
When $f(u) = u^2$, (1) reads
\[ \partial_x (\partial_t u + \kappa \partial_x u^2) = \gamma u. \] (13)
(13) with $\gamma > 0$ is known under different names such as the reduced Ostrovsky equation [31, 32], the Ostrovsky-Hunter equation [3] and the short-wave equation [21].

It was deduced considering two asymptotic expansions of the shallow water equations, first with respect to the rotation frequency and then with respect to the amplitude of the waves [20, 22].

Instead, if $\gamma < 0$, (13), known as the Vakhnenko equation [27, 42], describes the short-wave perturbations in a relaxing medium when the equations of motion are closed by the dynamic equation of state (see [4, 43]).

Moreover, (1) with $f(u) = u^2$ can be seen as the limit of no high-frequency dispersion ($\beta = 0$) of the nonlinear evolution equation
\[ \partial_x (\partial_t u + \kappa \partial_x u^2 - \beta \partial^3_{xxx} u) = \gamma u, \] (14)
known as the Ostrovsky equation [30]. This equation generalizes the Korteweg-deVries equation that corresponds to $\gamma = 0$.

In [8, 15], the authors proved the wellposedness of the entropy solution of the Cauchy problem for (13), while, following [7, 38], in [6], the convergence of the solutions of (14) to the entropy solutions of (13) is proven.

In [9, 16, 20], the authors study the well-posedness for the initial boundary value problem for (13). In particular, in [9], they prove the well-posedness of the entropy solution of non-homogeneous initial boundary value problem for (13), assuming on the initial datum (4) and (6), while, on the boundary datum they assume (10).

The fundamental assumption, which is made in [9], is the following one:
\[ \gamma > 0. \] (15)
Observe that, if we have (13) with $\gamma < 0$, [9, Theorem 1.2] is not true, because, from a mathematical point of view, [9, Lemma 2.6] does not hold (see also Lemmas 2.4, 3.7.)

In this paper, we improve the result of [9], when $\gamma > 0$, assuming on the boundary datum (7). Moreover, we extend the results of [9] in the case $\gamma < 0$, assuming on the boundary datum (10).
When \( f(u) = u^3 \), (1) reads

\[
\partial_x \left( \partial_t u + \kappa \partial_x u^3 \right) = \gamma u. \tag{16}
\]

It was introduced both Kozlov and Sazonov [23] as a model equation describing the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media, and Schäfer and Wayne [37] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers. Moreover, [2, 5, 36, 34] show that (16) is a particular Rabelo equation which describes pseudospherical surfaces.

It also is interesting to remind that equation (16) was proposed earlier in [29] in the context of plasma physic and that the similar equations describe the dynamics of radiating gases [25, 39].

In [41], the authors deduce (16) to describe the short pulse propagation in nonlinear metamaterials characterized by a weak Kerr-type nonlinearity in their dielectric response. In particular, they proved that

\[
\frac{\gamma}{\kappa} < 0, \quad \text{or} \quad \frac{\gamma}{\kappa} > 0. \tag{17}
\]

Finally, an interpretation of (16) in the context of Maxwell equations is given in [33].

On the other hand, (16) is the limit of no high-frequency dispersion (\( \beta = 0 \)) of non-linear equation

\[
\partial_x \left( \partial_t u + \kappa \partial_x u^3 - \beta \partial^{3}_{xxx} u \right) = \gamma u. \tag{18}
\]

It was derived by Costanzino, Manukian and Jones [19] in the context of the non-linear Maxwell equations with high-frequency dispersion. Kozlov and Sazonov [23] show that (18) is an more general equation than (16) to describe the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media.

In [10, 15], The wellposedness of the Cauchy problem for (16) is proven, while, in [11, 12], the authors proved the convergence of the solutions of (18) to the the entropy ones of (16). In [18], the authors prove the convergence of a finite difference scheme to the unique entropy solution of (13) and (16) on a bounded domain with periodic boundary conditions. That result also provides an existence proof for periodic entropy solutions for (13) and (16).

In [13], the authors prove the well-posedness of the entropy solution of non-homogeneous initial boundary value problem for (16), under the assumption

\[
\kappa = \gamma = 1, \tag{19}
\]

and, assuming on the boundary datum (7). In this paper, under the assumption (7), we extend the result of [13] in the case

\[
\kappa > 0, \quad \gamma > 0. \tag{20}
\]

Moreover, we prove the well-posedness of the entropy solution of non-homogeneous initial boundary value problem for (16) in the case:

\[
\kappa < 0, \quad \gamma > 0, \tag{21}
\]

under the assumption (12).
Observe that, integrating (1) in $(0, x)$, we gain the integro-differential formulation of (1) (see [26, 35]):

\[
\begin{cases}
\partial_t u + \kappa \partial_x f(u) = \gamma \int_0^x u(t, y) dy, & t > 0, x > 0, \\
u(t, 0) = g(t), & t > 0, \\
u(0, x) = u_0(x), & x > 0,
\end{cases}
\]

that is equivalent to

\[
\begin{cases}
\partial_t u + \kappa \partial_x f(u) = \gamma P, & t > 0, x > 0, \\
\partial_x P = u, & t > 0, x > 0, \\
u(t, 0) = g(t), & t > 0, \\
P(t, 0) = 0, & t > 0, \\
u(0, x) = u_0(x), & x > 0,
\end{cases}
\]

where

\[f(u) = u^2, u^3, \quad \kappa, \gamma \in \mathbb{R}.
\]

One of the main issues in the analysis of (23) is that the equation is not preserving the $L^1$ norm, as a consequence the nonlocal source term $P$ and the solution $u$ are a priori only locally bounded. Indeed, from (22) and (23) is clear that we cannot have any $L^\infty$ bound without an $L^1$ bound. Since we are interested in the bounded solutions of (1), some assumptions on the decay at infinity of the initial condition $u_0$ are needed.

The unique useful conserved quantities are

\[t \mapsto -\int u(t, x) dx = 0, \quad t \mapsto -\int u^2(t, x) dx.
\]

In the sense that if $u(t, \cdot)$ has zero mean at time $t = 0$, then it will have zero mean at any time $t > 0$. In addition, the $L^2$ norm of $u(t, \cdot)$ is constant with respect to $t$. Therefore, we require that initial condition $u_0$ belongs to $L^2 \cap L^\infty$ and has zero mean.

Due to the regularizing effect of the $P$ equation in (23) we have that

\[u \in L^\infty((0, T) \times (0, \infty)) \implies P \in L^\infty(0, T; W^{1,\infty}(0, \infty)), \quad T > 0.
\]

Therefore, if a map $u \in L^\infty((0, T) \times (0, \infty)), T > 0$, satisfies, for every convex map $\eta \in C^2(\mathbb{R})$,

\[\partial_t \eta(u) + \partial_x q(u) - \eta'(u) P \leq 0, \quad q(u) = \int_u^u \kappa f'(\xi) \eta'(\xi) d\xi,
\]

in the sense of distributions, then [17, Theorem 1.1] provides the existence of strong trace $u_0^\tau$ on the boundary $x = 0$.

Following [1, 9, 13, 15, 20], we give the following definition of solution

**Definition 1.1.** Assume (24). We say that $u \in L^\infty((0, T) \times (0, \infty)), T > 0$, is an entropy solution of the initial-boundary value problem (1), (2) and (3) if for every
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nonnegative test function \( \phi \in C^2(\mathbb{R}^2) \) with compact support, and \( c \in \mathbb{R} \)

\[
\int_0^\infty \int_0^\infty \left( |u - c| \partial_t \phi + \kappa \text{sign} (u - c) (f(u) - f(c)) \partial_x \phi \right) dt dx \\
+ \gamma \int_0^\infty \int_0^\infty \text{sign} (u - c) P \phi dt dx \\
+ \kappa \int_0^\infty \text{sign} (g(t) - c) \left( f(u_0^\varepsilon(t)) - f(c) \right) \phi(0, \cdot) dt \\
+ \int_0^\infty |u_0(x) - c| \phi(0, x) dx \geq 0,
\]

(28)

where \( u_0^\varepsilon(t) \) is the trace of \( u \) on the boundary \( x = 0 \).

Following [24], the main result of this paper is the following theorem.

**Theorem 1.2.** Assume (4), (6) and (24). If one of the following holds

(i) (7) and (8);
(ii) (9) and (10);
(iii) (11) and (12);

the initial-boundary value problem (1), (2) and (3) possesses an unique entropy solution \( u \) in the sense of Definition 1.1. Moreover, if \( u \) and \( v \) are two entropy solutions of (1), (2), (3) in the sense of Definition 1.1, the following inequality holds

\[
\|u(t, \cdot) - v(t, \cdot)\|_{L^1(0, R)} \leq e^{C(T)t} \|u(0, \cdot) - v(0, \cdot)\|_{L^1(0, R + C(T)t)},
\]

(29)

for almost every \( 0 < t < T, R > 0 \), and some suitable constant \( C(T) > 0 \).

The special case \( \kappa = \gamma = 1 \) has been treated in [13].

The paper is organized as follows. In Sections 2 and 4 we prove Theorem 1.2 under the assumption (7) and (8). Sections 3 and 5 are devoted to the proof of Theorem 1.2 when (9)-(10), or (11)-(12) holds.

2. The case \( f(u) = u^2, \kappa \in \mathbb{R}, \gamma > 0 \). In this section, we prove Theorem 1.2 in the case

\[
f(u) = u^2, \quad \kappa \in \mathbb{R}, \quad \gamma = a^2.
\]

(30)

Therefore, we consider (22), or (23), with \( f(u), \kappa, \gamma \) as in (30)

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (23).

Following [13], fix a small number \( \varepsilon > 0 \) and let \( u_\varepsilon = u_\varepsilon(t,x) \) be the unique classical solution of the following mixed problem

\[
\begin{align*}
\partial_t u_\varepsilon + \kappa \partial_x u_\varepsilon^2 &= a^2 P_\varepsilon + \varepsilon \partial_{xx} u_\varepsilon, & t > 0, \quad x > 0, \\
\partial_x P_\varepsilon &= u_\varepsilon, & t > 0, \quad x > 0, \\
u_\varepsilon(t,0) &= g_\varepsilon(t), & t > 0, \\
P_\varepsilon(t,0) &= 0, & t > 0, \\
u_\varepsilon(0,x) &= u_{\varepsilon,0}(x), & x > 0,
\end{align*}
\]

(31)
where \( u_{\varepsilon,0} \) and \( g_\varepsilon \) are \( C^\infty(0, \infty) \) approximations of \( u_0 \) and \( g \) such that

\[
\begin{align*}
&u_{0,\varepsilon} \to u_0, \quad \text{a.e. and in } L^p(0, \infty), \quad 1 \leq p < \infty, \\
P_{0,\varepsilon} \to P_0, \quad \text{in } L^2(0, \infty), \\
g_\varepsilon \to g, \quad \text{a.e. and in } L^p_{\text{loc}}(0, \infty), \quad 1 \leq p < \infty,
\end{align*}
\]

\[
\|u_{\varepsilon,0}\|_{L^\infty(0, \infty)} \leq \|u_0\|_{L^\infty(0, \infty)}, \quad \|u_{\varepsilon,0}\|_{L^2(0, \infty)} \leq \|u_0\|_{L^2(0, \infty)},
\]

\[
\int_0^\infty u_{\varepsilon,0}(x)dx = 0, \quad \|P_{\varepsilon,0}\|_{L^2(0, \infty)} \leq \|P_0\|_{L^2(0, \infty)},
\]

\[
\|g_\varepsilon\|_{L^\infty(0, \infty)} \leq C_0,
\]

and \( C_0 \) is a constant independent on \( \varepsilon \).

Let us prove some a priori estimates on \( u_\varepsilon \) denoting with \( C_0 \) the constants which depend only on the initial data, and \( C(T) \) the constants which depend also on \( T \).

Arguing as in [13, Lemma 2.1], or [15, Lemma 2.1], or [20, Lemma 2.2.1], we have the following result

**Lemma 2.1.** The following statements are equivalent

\[
\int_0^\infty u_\varepsilon(t, x)dx = 0, \quad t \geq 0,
\]

\[
\frac{d}{dt}\|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 
\]

\[
= \frac{2\kappa}{3}\varepsilon^3(t) - 2\varepsilon g_\varepsilon(t)\partial_x u_\varepsilon(t, 0), \quad t \geq 0.
\]

**Lemma 2.2.** For each \( t \geq 0 \), (33) holds. In particular we get

\[
\|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds 
\]

\[
\leq C_0(t + 1) + 2\varepsilon \int_0^t |g_\varepsilon(s)|\|\partial_x u_\varepsilon(s, 0)\|ds.
\]

Moreover,

\[
\|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds 
\]

\[
\leq C(T) \left( 1 + \varepsilon \|\partial_x u_\varepsilon(\cdot, 0)\|_{L^2(0, T)} \right),
\]

for every \( 0 \leq t \leq T \).

**Proof.** Arguing as in [13, Lemma 2.2], we have (33) and (35).

We prove (36). Fix \( T > 0 \). From (32), (35) and the Hölder inequality, we get

\[
\|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds
\]

\[
\leq C(T) + 2\varepsilon \int_0^T |g_\varepsilon(t)|\|\partial_x u_\varepsilon(t, 0)\|dt
\]

\[
\leq C(T) + 2 \|g_\varepsilon\|_{L^\infty(0, \infty)} \varepsilon \int_0^T |\partial_x u_\varepsilon(t, 0)| dt
\]

\[
\leq C(T) + C_0 \varepsilon \int_0^T |\partial_x u_\varepsilon(t, 0)| dt
\]
\[ \leq C(T) + C_0 \sqrt{T} \varepsilon \left\| \partial_x u_x(\cdot, 0) \right\|_{L^2(0,T)} \]
\[ \leq C(T) \left( 1 + \varepsilon \left\| \partial_x u_x(\cdot, 0) \right\|_{L^2(0,T)} \right), \]

which gives (36).

\[ \square \]

**Lemma 2.3.** Let us consider the following function
\[ F_{\varepsilon}(t, x) = \int_0^x P_{\varepsilon}(t, y) dy, \quad t > 0, \ x > 0, \]  
(37)

We have that \[ a^2 F_{\varepsilon}(t, \infty) = a^2 \int_0^\infty P_{\varepsilon}(t, x) dx = \varepsilon \partial_x u_x(t, 0) - \kappa g_{\varepsilon}^2(t). \]  
(38)

**Proof.** We begin by observing that, integrating on \((0, x)\) the second equation of (31), we get \[ P_{\varepsilon}(t, x) = \int_0^x u_x(t, y) dy. \]  
(39)

Differentiating (39) with respect to \(t\), we have \[ \partial_t P_{\varepsilon}(t, x) = \frac{d}{dt} \int_0^x u_x(t, y) dy = \int_0^x P_{\varepsilon} u_x(t, y) dy. \]  
(40)

It follows from (33) and (40) that \[ \lim_{x \to \infty} \partial_t P_{\varepsilon}(t, x) = \frac{d}{dt} \int_0^\infty u_x(t, x) dx = 0. \]  
(41)

Integrating on \((0, x)\) the first equation of (31), thanks to (37) and (40), we have \[ \partial_t P_{\varepsilon}(t, x) + \kappa u_{\varepsilon}^2(t, x) - \kappa g_{\varepsilon}^2(t) - \varepsilon \partial_x u_x(t, x) = \varepsilon \partial_x u_x(t, x) + \varepsilon \partial_x u_x(t, 0) = a^2 F_{\varepsilon}(t, x). \]  
(42)

It follows from the regularity of \(u_x\) that \[ \lim_{x \to \infty} \left( \kappa u_{\varepsilon}^2(t, x) - \varepsilon \partial_x u_x(t, x) \right) = 0. \]  
(43)

(38) follows from (41) and (43). \[ \square \]

Following [11, Lemma 2.4], or [14, Lemma 2.2], we prove the following result.

**Lemma 2.4.** Fix \(T > 0\). There exists a constant \(C(T) > 0\), independent on \(\varepsilon\), such that 
\[ \varepsilon \left\| \partial_x u_x(\cdot, 0) \right\|_{L^2(0,T)} \leq C(T). \]  
(44)

In particular, we have 
\[ \| u_x(t, \cdot) \|^2_{L^2(0,\infty)} + 2 \varepsilon \int_0^t \left\| \partial_x u_x(s, \cdot) \right\|^2_{L^2(0,\infty)} ds \leq C(T). \]  
(45)

for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). Multiplying (42) by \(2a^2 P_{\varepsilon}\), we have that
\[ 2a^2 P_{\varepsilon} \partial_t P_{\varepsilon} + 2a^2 \kappa u_{\varepsilon}^2 P_{\varepsilon} - 2a^2 \kappa g_{\varepsilon}^2(t) P_{\varepsilon} - 2a^2 \varepsilon \partial_x u_x P_{\varepsilon} + 2a^2 \varepsilon \partial_x u_x(t, 0) P_{\varepsilon} = 2a^4 P_{\varepsilon} F_{\varepsilon}. \]  
(46)

Observe that, by (37) and (38),
\[ -2a^2 \kappa g_{\varepsilon}^2(t) \int_0^\infty P_{\varepsilon}(t, x) dx = -2 \kappa \varepsilon \partial_x u_x(t, 0) g_{\varepsilon}^2(t) + 2 \kappa^2 g_{\varepsilon}^4(t), \]
Therefore, by (32) and (49),
\[
\int_0^\infty P_\varepsilon(t,x)dx = 2\varepsilon^2(\partial_x u_\varepsilon(t,0))^2 - 2\kappa\varepsilon\partial_x u_\varepsilon(t,0)g_\varepsilon^2(t). \tag{47}
\]
Moreover, by (31) and (37),
\[
-2a^2\varepsilon \int_0^\infty \partial_x u_\varepsilon(t,x)P_\varepsilon(t,x)dx = 2a^2\varepsilon \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2,
\]
\[
2a^4 \int_0^\infty P_\varepsilon(t,x)F_\varepsilon(t,x)dx = a^4 F_\varepsilon^2(t,\infty)
\]
\[
\leq 2a^2\kappa \int_0^\infty u_\varepsilon^2P_\varepsilon dx + 2\kappa\varepsilon\partial_x u_\varepsilon(t,0)g_\varepsilon^2(t) - \kappa^2 g_\varepsilon^4(t). \tag{48}
\]
Multiplying (42) by $2a^2P_\varepsilon$, an integration on $(0,\infty)$, (47) and (48) give
\[
a^2 \frac{d}{dt}\|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + 2a^2\varepsilon \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \varepsilon^2(\partial_x u_\varepsilon(t,0))^2
\]
\[
= -2a^2\kappa \int_0^\infty u_\varepsilon^2P_\varepsilon dx + 2\kappa\varepsilon\partial_x u_\varepsilon(t,0)g_\varepsilon^2(t) - \kappa^2 g_\varepsilon^4(t). \tag{49}
\]
Due to the Young inequality,
\[
2\kappa\varepsilon\partial_x u_\varepsilon(t,0)g_\varepsilon^2(t) \leq \varepsilon(\partial_x u_\varepsilon(t,0))2\kappa g_\varepsilon^4(t) \leq \frac{\varepsilon^2}{2}(\partial_x u_\varepsilon(t,0))^2 + 2\kappa^2 g_\varepsilon^4(t).
\]
Therefore, by (32) and (49),
\[
a^2 \frac{d}{dt}\|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 \leq 2a^2\varepsilon \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \varepsilon^2(\partial_x u_\varepsilon(t,0))^2 \leq a^2\kappa \int_0^\infty |P_\varepsilon|u_\varepsilon^2 dx + 2\kappa^2 g_\varepsilon^4(t) \leq a^2\kappa \int_0^\infty |P_\varepsilon|u_\varepsilon^2 dx + C_0. \tag{50}
\]
Due to the Young inequality,
\[
2a^2\kappa \int_0^\infty |P_\varepsilon|u_\varepsilon^2 dx \leq 2a^2\kappa \|P_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)} \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2
\]
\[
\leq \frac{a^2\kappa}{D} \|P_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)}^2 + \frac{a^2\kappa}{D} |u_\varepsilon(t,\cdot)|_{L^2(0,\infty)}^4 + C_0,
\]
where $D$ is a positive constant which will be specified later. Consequently, by (50),
\[
a^2 \frac{d}{dt}\|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 \leq 2a^2\varepsilon \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \varepsilon^2(\partial_x u_\varepsilon(t,0))^2 \leq \frac{a^2\kappa}{D} \|P_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)}^2 + \frac{a^2\kappa}{D} |u_\varepsilon(t,\cdot)|_{L^2(0,\infty)}^4 + C_0. \tag{51}
\]
Observe that, by (31),
\[
P_\varepsilon^2(t,x) = 2 \int_0^x P_\varepsilon(t,y)\partial_x P_\varepsilon(t,y)dy = 2 \int_0^x P_\varepsilon(t,y)\partial_x u_\varepsilon(t,y)dy. \tag{52}
\]
Therefore, by the Young inequality,
\[
\|P_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)}^2 \leq 2 \int_0^\infty |P_\varepsilon|u_\varepsilon dx \leq \frac{1}{E} \|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + E \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2, \tag{53}
\]
where \( E \) is a positive constant which be specified later. It follows from (51) and (53) that

\[
a^2 \frac{d}{dt} \| P_e(t, \cdot) \|_{L^2(0, \infty)}^2 + 2a^2 \varepsilon \| u_e(t, \cdot) \|_{L^2(0, \infty)}^2 + \frac{\varepsilon^2}{2} (\partial_x u_e(t, 0))^2 \\
\leq \frac{a^2 |\kappa|}{DE} \| P_e(t, \cdot) \|_{L^2(0, \infty)}^2 + \frac{a^2 |\kappa| E}{D} \| u_e(t, \cdot) \|_{L^2(0, \infty)}^2 \\
+ a^2 |\kappa| D \| u_e(t, \cdot) \|_{L^2(0, \infty)}^2 + C_0.
\]

(54)

The Gronwall Lemma, (32) and an integration on \((0, T)\) give

\[
a^2 \| P_e(T, \cdot) \|_{L^2(0, \infty)}^2 + 2a^2 e^{\frac{|\kappa| T}{2}} \varepsilon \int_0^T e^{-\frac{|\kappa| t}{2}} \| u_e(t, \cdot) \|_{L^2(0, \infty)}^2 dt \\
+ e^{\frac{|\kappa| T}{2}} \varepsilon \int_0^T e^{-\frac{|\kappa| t}{2}} (\partial_x u_e(t, 0))^2 dt \\
\leq C_0 e^{\frac{|\kappa| T}{2}} + a^2 |\kappa| E e^{\frac{|\kappa| T}{2}} \int_0^T e^{-\frac{|\kappa| t}{2}} \| u_e(t, \cdot) \|_{L^2(0, \infty)}^2 dt \\
+ a^2 |\kappa| De^{\frac{|\kappa| T}{2}} \int_0^T \| u_e(t, \cdot) \|_{L^2(0, \infty)}^4 dt \\
+ C_0 e^{\frac{|\kappa| T}{2}} \int_0^T \| u_e(t, \cdot) \|_{L^2(0, \infty)}^4 dt.
\]

(55)

Thanks to (36) and the Young inequality,

\[
\| u_e(t, \cdot) \|_{L^2(0, \infty)}^4 \leq C(T) \left( 1 + \varepsilon \| \partial_x u_e(\cdot, 0) \|_{L^2(0, T)} \right)^2 \\
\leq C(T) \left( 1 + \varepsilon^2 \| \partial_x u_e(\cdot, 0) \|_{L^2(0, T)}^2 \right).
\]

(56)

Consequently, by (36), (55) and (56), we get

\[
a^2 \| P_e(T, \cdot) \|_{L^2(0, \infty)}^2 + 2a^2 e^{\frac{|\kappa| T}{2}} \varepsilon \int_0^T e^{-\frac{|\kappa| t}{2}} \| u_e(t, \cdot) \|_{L^2(0, \infty)}^2 dt \\
+ e^{\frac{|\kappa| T}{2}} \varepsilon \int_0^T e^{-\frac{|\kappa| t}{2}} (\partial_x u_e(t, 0))^2 dt \\
\leq C(T) e^{\frac{|\kappa| T}{2}} + C(T) e^{\frac{|\kappa| T}{2}} \left( 1 + \varepsilon \| \partial_x u_e(\cdot, 0) \|_{L^2(0, T)} \right) \\
+ De^{\frac{|\kappa| T}{2}} C(T) \left( 1 + \varepsilon^2 \| \partial_x u_e(\cdot, 0) \|_{L^2(0, T)}^2 \right) \\
\leq C(T) e^{\frac{|\kappa| T}{2}} + C(T) e^{\frac{|\kappa| T}{2}} \left( 1 + \varepsilon \| \partial_x u_e(\cdot, 0) \|_{L^2(0, T)} \right) \\
+ De^{\frac{|\kappa| T}{2}} C(T) \left( 1 + \varepsilon^2 \| \partial_x u_e(\cdot, 0) \|_{L^2(0, T)} \right) \\
+ De^{\frac{|\kappa| T}{2}} C(T) e^{\frac{|\kappa| T}{2}} \| \partial_x u_e(\cdot, 0) \|_{L^2(0, T)}^2.
\]

(57)
Observe that
\[
\frac{\varepsilon^2}{2} \| \partial_x u_\varepsilon (\cdot, 0) \|^2_{L^2(0,T)} = \frac{\varepsilon^2}{2} \int_0^T (\partial_x u_\varepsilon (t, 0))^2 dt \\ \leq e^{\frac{|\kappa| T}{2}} \varepsilon^2 \int_0^T e^{-\frac{|\kappa| t}{2}} (\partial_x u_\varepsilon (t, 0))^2 dt.
\]
(58)

Therefore, (57) and (58) give
\[
\frac{\varepsilon^2}{2} \| \partial_x u_\varepsilon (\cdot, 0) \|^2_{L^2(0,T)} \leq C(T) e^{\frac{|\kappa| T}{2}} + \frac{C(T) E}{D} e^{\frac{|\kappa| T}{2}} \\
+ D e^{\frac{|\kappa| T}{2}} C(T) + \frac{C(T) E}{D} e^{\frac{|\kappa| T}{2}} \varepsilon \| \partial_x u_\varepsilon (\cdot, 0) \|_{L^2(0,T)} \\
+ D e^{\frac{|\kappa| T}{2}} C(T) \varepsilon^2 \| \partial_x u_\varepsilon (\cdot, 0) \|^2_{L^2(0,T)},
\]
that is
\[
\left( \frac{1}{2} - C(T) D e^{\frac{|\kappa| T}{2}} \right) \varepsilon^2 \| \partial_x u_\varepsilon (\cdot, 0) \|^2_{L^2(0,T)} - \frac{C(T) E}{D} e^{\frac{|\kappa| T}{2}} \varepsilon \| \partial_x u_\varepsilon (\cdot, 0) \|_{L^2(0,T)} \\
- C(T) e^{\frac{|\kappa| T}{2}} - \frac{C(T) E}{D} e^{\frac{|\kappa| T}{2}} - D e^{\frac{|\kappa| T}{2}} C(T) \leq 0.
\]
(59)

We search \( D, E \) such that
\[
C(T) D e^{\frac{|\kappa| T}{2}} = \frac{1}{3},
\]
that is
\[
e^{\frac{|\kappa| T}{2}} = \frac{1}{3C(T) D}.
\]
(60)

(60) admits a solution if and only if
\[
\frac{1}{3C(T) D} > 1 \quad \Rightarrow \quad D < \frac{1}{3C(T)}.
\]

Taking
\[
D = \frac{1}{6C(T)},
\]
(61)
by (60), we have
\[
e^{\frac{6|\kappa| C(T) T}{D}} = 2.
\]
(62)

Hence,
\[
E = \frac{6|\kappa| C(T) T}{\log(2)}.
\]
(62)

It follows from (59), (61) and (62) that
\[
\frac{\varepsilon^2}{6} \| \partial_x u_\varepsilon (\cdot, 0) \|^2_{L^2(0,T)} - C(T) \varepsilon \| \partial_x u_\varepsilon (\cdot, 0) \|_{L^2(0,T)} - C(T) \leq 0,
\]
which gives (44).

Finally, (44) follows from (36) and (44).

\[
\text{Lemma 2.5. Fix } T > 0. \text{ There exists a constant } C(T) > 0, \text{ independent on } \varepsilon \text{ such that }
\]
\[
a \| P_\varepsilon (t, \cdot) \|^2_{L^2(0,\infty)} + 2a^2 \varepsilon e^t \int_0^t e^{-s} \| u_\varepsilon (s, \cdot) \|^2_{L^2(0,\infty)} ds \\
+ \frac{\varepsilon^2 e^t}{2} \int_0^t e^{-s} (\partial_x u_\varepsilon (s, 0))^2 ds \leq C(T).
\]
(63)
In particular, we have that
\[ \|P_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)} \leq C(T), \] (64)
for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). Arguing as in Lemma 2.4, we have (54) with \(D = 1\) and \(E = |\kappa|\). Therefore, by (45), we get
\[ a^2 \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2a^2 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \frac{\varepsilon^2}{2} (\partial_x u_\varepsilon(t, 0))^2 \]
\[ \leq a^2 \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C(T). \] (65)
The Gronwall Lemma and (32) give (63).

Finally, (64) follows from (45), (53) with \(E = |\kappa|\) and (63).

**Lemma 2.6.** Let \(T > 0\). There exists a constant \(C(T) > 0\), independent on \(\varepsilon\), such that
\[ \|u_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))} \leq C(T). \] (66)

**Proof.** Due to (31) and (64),
\[ \partial_t u_\varepsilon + 2\kappa u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon \leq a^2 \|P_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)} \leq C(T). \]
Since the map
\[ F(t) := \|u_0\|_{L^\infty(0, \infty)} + C(T)t, \]
solves the equation
\[ \frac{dF}{dt} = C(T) \]
and
\[ \max\{u_\varepsilon(0, x), 0\} \leq F(t), \quad (t, x) \in (0, T) \times (0, \infty), \]
the comparison principle for parabolic equations implies that
\[ u_\varepsilon(t, x) \leq F(t), \quad (t, x) \in (0, T) \times (0, \infty). \]
In a similar way we can prove that
\[ u_\varepsilon(t, x) \geq -F(t), \quad (t, x) \in (0, T) \times (0, \infty). \]
Therefore,
\[ |u_\varepsilon(t, x)| \leq \|u_0\|_{L^\infty(0, \infty)} + C(T)t \leq \|u_0\|_{L^\infty(0, \infty)} + C(T)T \leq C(T), \]
which gives (66). \[ \square \]

Let us continue by proving the existence of a distributional solution to (13), (2) and (3) satisfying (27) with \(f(u) = u^2\).

**Lemma 2.7.** Fix \(T > 0\). There exists a function \(u \in L^\infty((0, T) \times \mathbb{R})\) that is a distributional solution of (23) with \(f(u) = u^2\) and satisfies (28).

We construct a solution by passing to the limit in a sequence \(\{u_\varepsilon\}_{\varepsilon > 0}\) of viscosity approximations (31). We use the compensated compactness method [40].
Lemma 2.8. Let $T > 0$. There exists a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ of $\{u_{\varepsilon}\}_{\varepsilon > 0}$ and a limit function $u \in L^\infty((0, T) \times (0, \infty))$ such that

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0, T) \times (0, \infty)), \quad 1 \leq p < \infty.$$  \hspace{1cm} (67)

In particular, we have

$$P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{loc}(0, T; W^{1,p}_{loc}(0, \infty)), \quad 1 \leq p < \infty,$$

where

$$P(t, x) = \int_0^x u(t, y)dy, \quad t \geq 0, \quad x \geq 0,$$  \hspace{1cm} (69)

and (28) holds.

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex $C^2$ entropy function, and $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q'(u) = -\kappa u \eta'(u)$. By multiplying the first equation in (31) with $\eta'(u_{\varepsilon_k})$ and using the chain rule, we get

$$\partial_t \eta(u_{\varepsilon_k}) + \partial_x q(u_{\varepsilon_k}) = \varepsilon \partial_x^2 \eta(u_{\varepsilon_k}) - \varepsilon \eta''(u_{\varepsilon_k})(\partial_x u_{\varepsilon_k})^2 + \eta'(u_{\varepsilon_k}) P_{\varepsilon_k},$$

where $\mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon}, \mathcal{L}_{3,\varepsilon}$ are distributions. Fix $T > 0$. Arguing as in [13, Lemma 3.2], we have that

$$\mathcal{L}_{1,\varepsilon} \to 0 \text{ in } H^{-1}((0, T) \times (0, \infty)) \text{ as } \varepsilon \to 0,$$

$$\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon > 0} \text{ is uniformly bounded in } L^1((0, T) \times (0, \infty)),$$

$$\{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon > 0} \text{ is uniformly bounded in } L^1_{loc}((0, T) \times (0, \infty)).$$

Therefore, Murat’s lemma [28] implies that

$$\{\partial_t \eta(u_{\varepsilon_k}) + \partial_x q(u_{\varepsilon_k})\}_{\varepsilon > 0} \text{ lies in a compact subset of } H^{-1}_{loc}((0, T) \times (0, \infty)).$$  \hspace{1cm} (70)

The $L^\infty$ bound stated in Lemma 2.6, (70), and the Tartar’s compensated compactness method [40] give the existence of a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ and a limit function $u \in L^\infty((0, T) \times (0, \infty)), \quad T > 0$, such that (67) holds.

(68) follows from (67), the Hölder inequality and the identity

$$P_{\varepsilon_k} = \int_0^x u_{\varepsilon_k}dy, \quad \partial_x P_{\varepsilon_k} = u_{\varepsilon_k}.$$

Finally, arguing as in [9, Lemma 3.2], we have (28). \hspace{1cm} \Box

Proof of Theorem 1.2. Lemma (2.8) gives the existence of an entropy solution $u$ for (22), or equivalently (23), with $f(u), \kappa, \gamma$ defined in (30).

We observe that, fixed $T > 0$, the solutions of (22), or equivalently (23), are bounded in $(0, T) \times \mathbb{R}$. Therefore, using [9, Theorem 1.1], $u$ is unique and (29) holds. \hspace{1cm} \Box

3. The case $f(u) = u^2, \kappa \in \mathbb{R}, \gamma < 0$. In this section, we prove Theorem 1.2 under the assumption (9). Therefore, we consider (22), or (23), with $f(u)$, $\kappa$, $\gamma$ as in (9).

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (23).
Following [13], defined $\gamma = -a^2$, fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of the following mixed problem

$$
\begin{align*}
\partial_t u_\varepsilon + \kappa \partial_x u_\varepsilon^2 &= -a^2 P_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon, \quad t > 0, \ x > 0, \\
\partial_x P_\varepsilon &= u_\varepsilon, \\
u_\varepsilon(t, 0) &= g_\varepsilon(t) \quad t > 0, \\
P_\varepsilon(t, 0) &= 0 \\
u_\varepsilon(0, x) &= u_{\varepsilon,0}(x),
\end{align*}
$$

(71)

where $u_{\varepsilon,0}$ and $g_\varepsilon$ are $C^\infty(0, \infty)$ approximations of $u_0$ and $g$ such that

$$
\|u_{\varepsilon,0}\|_{L^\infty(0, \infty)} \leq \|u_0\|_{L^\infty(0, \infty)}; \quad \|u_{\varepsilon,0}\|_{L^2(0, \infty)} \leq \|u_0\|_{L^2(0, \infty)}, 
$$

(72)

$$
\begin{align*}
\int_0^\infty u_{\varepsilon,0}(x) dx &= 0, \\
\|P_{\varepsilon,0}\|_{L^2(0, \infty)} &\leq \|P_0\|_{L^2(0, \infty)}, \\
\varepsilon^2 \|\partial_x u_{\varepsilon,0}\|_{L^2(0, \infty)} &\leq \varepsilon^2 \|\partial_x u_0\|_{L^2(0, \infty)}, \\
\|g_\varepsilon\|_{L^\infty(0, \infty)} + \|g_\varepsilon\|_{L^\infty(0, \infty)} &\leq C_0, \quad g_\varepsilon(0) = u_{\varepsilon,0}(0),
\end{align*}
$$

and $C_0$ is a constant independent on $\varepsilon$. Clearly, (71) is equivalent to the integro-differential problem

$$
\begin{align*}
\partial_t u_\varepsilon + \kappa \partial_x u_\varepsilon^2 &= -a^2 \int_0^x u_\varepsilon y dy + \varepsilon \partial_{xx}^2 u_\varepsilon, \quad t > 0, \ x > 0, \\
u_\varepsilon(t, 0) &= g_\varepsilon(t) \quad t > 0, \\
u_\varepsilon(0, x) &= u_{\varepsilon,0}(x), \quad x > 0,
\end{align*}
$$

(73)

Let us prove some a priori estimates on $u_\varepsilon$ denoting with $C_0$ the constants which depend only on the initial data, and $C(T)$ the constants which depends also on $T$.

**Lemma 3.1.** For each $t \geq 0$, (33) holds.

**Proof.** We begin by observing that $\partial_t u_\varepsilon(t, 0) = g_\varepsilon'(t)$, being $u_\varepsilon(t, 0) = g_\varepsilon(t)$. It follows from (73) that

$$
\varepsilon \partial_{xx}^2 u_\varepsilon(t, 0) = \partial_t u_\varepsilon(t, 0) + 2\kappa g_\varepsilon(t) \partial_x u_\varepsilon(t, 0) + a^2 \int_0^0 u_\varepsilon dx
$$

(74)

Differentiating (73) with respect to $x$, we have

$$
\partial_x (\partial_t u_\varepsilon + 2\kappa u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon) = -a^2 u_\varepsilon.
$$

(75)

From (74) and being $u_\varepsilon$ a smooth solution of (73), an integration over $(0, \infty)$ gives (33)

Let us consider the following function

$$
u_\varepsilon(t, x) = u_\varepsilon(t, x) - g_\varepsilon(t) e^{-x}.
$$

(76)

Observe that, by (71), (72) and (76),

$$
\nu_\varepsilon(t, 0) = 0, \quad \|\nu_{\varepsilon,0}\|_{L^2(0, \infty)} \leq C_0.
$$

(77)

Moreover,

$$
\begin{align*}
\partial_t \nu_\varepsilon(t, x) &= \partial_t u_\varepsilon(t, x) - g_\varepsilon'(t) e^{-x}, \\
\partial_x \nu_\varepsilon(t, x) &= \partial_x u_\varepsilon(t, x) + g_\varepsilon(t) e^{-x}, \\
\partial_{xx}^2 \nu_\varepsilon(t, x) &= \varepsilon \partial_{xx}^2 u_\varepsilon(t, x) - g_\varepsilon(t) e^{-x}.
\end{align*}
$$

(78)
Lemma 3.2. For each \( t > 0 \), we have that
\[
\| u_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 \leq 2 \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + C_0,
\] (79)
\[
\varepsilon \int_0^t \| \partial_x u_\varepsilon(s, \cdot) \|_{L^2(0, \infty)}^2 ds \leq 2\varepsilon \int_0^t \| \partial_x v_\varepsilon(s, \cdot) \|_{L^2(0, \infty)}^2 ds + C_0,
\] (80)
\[
-a^2 \int_0^t \partial_x v_\varepsilon dx \leq 2 \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + C_0.
\] (81)

Proof. We begin by observing that (79) and (80) are proven in [9, Lemmas 2.3 and 2.4]. We prove (81). From (76), we have that
\[
-a^2 \int_0^\infty P_\varepsilon v_\varepsilon dx = -a^2 \int_0^\infty P_\varepsilon u_\varepsilon dx + a^2 g_\varepsilon(t) \int_0^\infty P_\varepsilon e^{-x} dx.
\] (82)
Observe that, by (33), (71) and (73),
\[
-a^2 \int_0^\infty P_\varepsilon u_\varepsilon dx = -a^2 \int_0^\infty P_\varepsilon \partial_x P_\varepsilon = -a^2 \frac{1}{2} (\int_0^\infty u_\varepsilon dx)^2 = 0.
\] (83)
Since
\[
\int_0^\infty e^{-y} dy = 1 - e^{-x},
\] (84)
from (33) and (71), we have that
\[
a^2 g_\varepsilon(t) \int_0^\infty P_\varepsilon e^{-x} dx = -a^2 g_\varepsilon(t) \int_0^\infty \partial_x P_\varepsilon (1 - e^{-x}) dx
\]
\[
=-a^2 g_\varepsilon(t) \int_0^\infty u_\varepsilon dx + a^2 g_\varepsilon(t) \int_0^\infty u_\varepsilon e^{-x} dx
\] (85)
\[
=a^2 g_\varepsilon(t) \int_0^\infty u_\varepsilon e^{-x} dx.
\] (86)
(82), (83), (85) give
\[
-a^2 \int_0^\infty P_\varepsilon v_\varepsilon dx = a^2 g_\varepsilon(t) \int_0^\infty u_\varepsilon e^{-x} dx.
\] (86)
Since
\[
\int_0^{-\infty} e^{-2x} dx = \frac{1}{2},
\] (87)
due to (72) and the Young inequality,
\[
a^2 |g_\varepsilon(t)| \int_0^\infty |u_\varepsilon| e^{-x} dx \leq a^2 \| g_\varepsilon(t) \|_{L^\infty(0, \infty)} \int_0^\infty |u_\varepsilon| e^{-x} dx
\]
\[
\leq C_0 \| u_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + C_0 \int_0^\infty e^{-2x} dx
\] (88)
(81) follows from (79), (86) and (88).

Following [9, Lemma 2.4], we prove the following result.
Lemma 3.3. Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\varepsilon$, such that
\[
\|v'_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + 2\varepsilon e^{C_{\text{const}}} \int_0^\infty e^{-C_{\text{const}}} \|\partial_x v'_\varepsilon(s, \cdot)\|^2_{L^2(0, \infty)} \, ds \leq C(T). \tag{89}
\]
for every $0 \leq t \leq T$. In particular, we have that
\[
\|u\|^2_{L^\infty(0, T; L^2(0, \infty))} \leq C(T), \quad \varepsilon \int_0^T \|\partial_x u\|_{L^2(0, \infty)}^2 \, ds \leq C(T). \tag{90}
\]
Proof. We begin by observing that from the first equation of (71), (76) and (78), we have
\[
\partial_t v'_\varepsilon + g'_\varepsilon(t)e^{-x} + 2\kappa (v'_\varepsilon + g_\varepsilon(t)e^{-x}) (\partial_x v'_\varepsilon - g_\varepsilon(t)e^{-x}) = -a^2 P_x + \varepsilon \partial_{xx}^2 v'_\varepsilon + g_\varepsilon(t)e^{-x},
\]
that is
\[
\partial_t v'_\varepsilon + 2\kappa v'_\varepsilon \partial_x v'_\varepsilon = -a^2 P_x + \varepsilon \partial_{xx}^2 v'_\varepsilon + g_\varepsilon(t)e^{-x} - g'_\varepsilon(t)e^{-x} + 2\kappa g_\varepsilon(t) v'_\varepsilon e^{-x} - 2\kappa g_\varepsilon(t) \partial_x v'_\varepsilon e^{-x} + 2\kappa g''_\varepsilon(t) e^{-2x}. \tag{91}
\]
Since, from (77),
\[
2 \int_0^\infty v'_\varepsilon \partial_t v'_\varepsilon \, dx = \frac{d}{dt} \|v'_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)},
\]
\[
4\kappa \int_0^\infty v''_\varepsilon \partial_x v'_\varepsilon \, dx = 0,
\]
\[
2 \varepsilon \int_0^\infty v'_\varepsilon \partial_{xx}^2 v'_\varepsilon \, dx = -2\varepsilon \|\partial_x v'_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)},
\]
\[
-4\kappa g'_\varepsilon(t) \int_0^\infty v'_\varepsilon \partial_x v'_\varepsilon e^{-x} \, dx = 2\kappa g_\varepsilon(t) \int_0^\infty v''_\varepsilon e^{-x} \, dx.
\]
Multiplying (91) by $2v'_\varepsilon$, (72), (81) and an integration on $(0, \infty)$ give
\[
\frac{d}{dt} \|v'_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + 2\varepsilon \|\partial_x v'_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)}
\]
\[
= -a^2 \int_0^\infty P_x v'_\varepsilon \, dx + 2g_\varepsilon(t) \int_0^\infty v'_\varepsilon e^{-x} \, dx
\]
\[
- 2g'_\varepsilon(t) \int_0^\infty v'_\varepsilon e^{-x} \, dx + 2\kappa g_\varepsilon(t) \int_0^\infty v''_\varepsilon e^{-x} \, dx
\]
\[
+ 2\kappa g'_\varepsilon(t) \int_0^\infty v''_\varepsilon e^{-x} \, dx + 4\kappa g''_\varepsilon(t) \int_0^\infty v'_\varepsilon e^{-2x} \, dx
\]
\[
\leq -a^2 \int_0^\infty P_x v'_\varepsilon \, dx + 2|g_\varepsilon(t)| \int_0^\infty |v'_\varepsilon| e^{-x} \, dy
\]
\[
+ 2|g'_\varepsilon(t)| \int_0^\infty |v'_\varepsilon| e^{-x} \, dx + 6\kappa |g_\varepsilon(t)| \int_0^\infty v''_\varepsilon e^{-x} \, dx
\]
\[
+ 4\kappa |g''_\varepsilon(t)| \int_0^\infty |v'_\varepsilon| e^{-2x} \, dx
\]
\[
\leq C_0 \|v'_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + C_0 + 2 \|g_\varepsilon\|_{L^\infty(0, \infty)} \int_0^\infty |v'_\varepsilon| e^{-x} \, dy
\]
Since
\[ \int_0^\infty |v_\varepsilon| e^{-2x} \, dx \leq C_0 \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0 \int_0^\infty |v_\varepsilon| e^{-2x} \, dx \]

it follows from (87), (93) and the Young inequality that
\[ C_0 \int_0^\infty |v_\varepsilon| e^{-2x} \, dx \leq C_0 \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0, \]

Therefore, by (92),
\[ \frac{d}{dt} \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon \|\partial_x v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \leq C_0 \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0. \]

The Gronwall Lemma and (77) give
\[ \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon e^{C_0 t} \int_0^t e^{-C_0 s} \|\partial_x v_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 \, ds \leq C_0 + C_0 e^{C_0 t} \int_0^t e^{-C_0 s} \, ds \leq C_0 + C_0 T e^{C_0 T} \leq C(T), \]

which gives (89).

Finally, (90) follows from (79), (80) and (89).

\[ \square \]

**Lemma 3.4.** Let \( T > 0 \). We have that
\[ \|u_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))} \leq C(T) \left( 1 + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))} \right). \]

In particular, we obtain that
\[ \|u_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 \leq C(T) \left( 1 + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 \right). \]

**Proof.** Let \( 0 \leq t \leq T \). Arguing as in Lemma 2.6, we have (94). (94) and the Young inequality give (95).

\[ \square \]

**Lemma 3.5.** Fix \( T > 0 \). There exists a constant \( C(T) > 0 \), independent on \( \varepsilon \), such that
\[ \varepsilon^2 \int_0^T (\partial_x u_\varepsilon(t, 0))^2 \, dx \leq C(T) \left( 1 + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 \right). \]

In particular, we have that
\[ \varepsilon^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \varepsilon^3 \int_0^t \|\partial_{xx} u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 \, ds \]
\[ \leq C(T) \left( 1 + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 \right), \]

for every \( 0 \leq t \leq T \).
Proof. Let $0 \leq t \leq T$. We begin by observing that, by (75), we have
\[
\partial_{xx}^2 u_x = -2\kappa (\partial_x u_x)^2 - 2\kappa u_x \partial_{xx} u_x + \varepsilon \partial_{xxx}^3 u_x - \alpha^2 u_x.
\] (98)

Multiplying (98) by $2\varepsilon^2 \partial_x u_x$, we get
\[
2\varepsilon^2 \partial_x u_x \partial_{xx}^2 u_x = -4\kappa \varepsilon^2 (\partial_x u_x)^3 + 4\kappa \varepsilon^2 u_x \partial_x u_x \partial_{xx}^2 u_x
+ 2\varepsilon^3 \partial_x u_x \partial_{xxx}^3 u_x - 2\varepsilon^2 u_x \partial_x u_x.
\] (99)

Observe that, by (71),
\[
-4\kappa \varepsilon^2 \int_0^\infty (\partial_x u_x)^3 dx = 4\kappa \varepsilon^2 \int_0^\infty g_{t,e}(t) \partial_x u_x(t,0)^2 + 8\kappa \varepsilon^2 \int_0^\infty u_x \partial_x u_x \partial_{xx}^2 u_x dx,
\]
\[
2\varepsilon^3 \int_0^\infty \partial_x u_x \partial_{xxx}^3 u_x dx = -2\varepsilon^3 \int_0^\infty \partial_x u_x(t,0) \partial_{xx}^3 u_x(t,0)
- 2\varepsilon^3 \int_0^\infty \partial_x u_x(t,\cdot) \partial_{xx}^2 u_x(t,\cdot) \| \partial_{xx}^2 u_x(t,\cdot) \|^2_{L^2(0,\infty)},
\] (100)
\[
-2\varepsilon^2 \int_0^\infty u_x \partial_x u_x dx = \varepsilon^2 \int_0^\infty g_{t,e}(t).
\]

Moreover, by (74),
\[
-2\varepsilon^3 \partial_x u_x(t,0) \partial_{xx}^2 u_x(t,0) = -2\varepsilon^2 g_{t,e}(t) \partial_x u_x(t,0) - 2\kappa \varepsilon^2 g_{t,e}(t) (\partial_x u_x(t,0))^2.
\] (101)

Therefore, since $0 < \varepsilon < 1$, (73), (100), (101) and an integration on $(0, \infty)$ of (98) give
\[
\varepsilon^2 \frac{d}{dt} \| \partial_x u_x(t, \cdot) \|^2_{L^2(0,\infty)} + 2\varepsilon^3 \| \partial_{xx}^2 u_x(t, \cdot) \|^2_{L^2(0,\infty)}
= 12\kappa \varepsilon^2 \int_0^\infty u_x \partial_x u_x \partial_{xx}^2 u_x dx + 2\kappa \varepsilon^2 \int_0^\infty g_{t,e}(t) (\partial_x u_x(t,0))^2
\]
\[
+ \varepsilon^2 \int_0^\infty g_{t,e}(t) \partial_x u_x(t,0)
\]
\[
\leq 12|\kappa| \varepsilon^2 \int_0^\infty |u_x \partial_x u_x| |\partial_{xx}^2 u_x| dx + 2|\kappa| |g_{t,e}| \varepsilon^2 (\partial_x u_x(t,0))^2
\]
\[
+ \int_0^\infty g_{t,e}(t) |\partial_x u_x(t,0)|
\]
\[
\leq 12|\kappa| \varepsilon^2 \int_0^\infty |u_x \partial_x u_x| |\partial_{xx}^2 u_x| dx
\]
\[
+ 2|\kappa| \| g_{t,e} \|_{L^\infty(0,\infty)} \varepsilon^2 (\partial_x u_x(t,0))^2
\]
\[
+ 2\varepsilon^2 |g_{t,e}(t)| |\partial_x u_x(t,0)|
\]
\[
\leq 12|\kappa| \varepsilon^2 \int_0^\infty |u_x \partial_x u_x| |\partial_{xx}^2 u_x| dx + C_0 \varepsilon^2 (\partial_x u_x(t,0))^2
\]
\[
+ C_0 + 2\varepsilon^2 |g_{t,e}(t)| |\partial_x u_x(t,0)|.
\] (102)

Due to the Young inequality,
\[
12|\kappa| \varepsilon^2 \int_0^\infty |u_x \partial_x u_x| |\partial_{xx}^2 u_x| dx
\]
\[
= 2 \int_0^\infty |6\kappa \varepsilon^2 u_x \partial_x u_x| \varepsilon^2 |\partial_{xx}^2 u_x| dx
\]
\[
\leq 36|\kappa|^2 \varepsilon \int_0^\infty u_x^2 (\partial_x u_x)^2 dx + \varepsilon^3 \| \partial_{xx}^2 u_x(t, \cdot) \|^2_{L^2(0,\infty)}
\]
\[
\leq 36|\kappa|^2 \varepsilon \int_0^\infty (u_x^2)^2 dx + \varepsilon^3 \| \partial_{xx}^2 u_x(t, \cdot) \|^2_{L^2(0,\infty)}.
\]
Since $0 < \varepsilon < 1$, due to (72) and the Young inequality,
\[
2\varepsilon^2 g'_\varepsilon(t)|\partial_x u_\varepsilon(t, 0)| \leq \varepsilon^2 (g'_\varepsilon(t))^2 + \varepsilon^2 (\partial_x u_\varepsilon(t, 0))^2
\leq \|g'_\varepsilon\|^2_{L^2(0, \infty)} + \varepsilon^2 (\partial_x u_\varepsilon(t, 0))^2
\]
\[
\leq C_0 + \varepsilon^2 (\partial_x u_\varepsilon(t, 0))^2.
\]
It follows from (102) that
\[
\varepsilon^2 \frac{d}{dt}\|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + \varepsilon^3 \|\partial_x^2 u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)}
\leq 36\kappa^2 \varepsilon \|u_\varepsilon\|^2_{L^\infty((0,T) \times (0,\infty))} \|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)}
+ C_0 \varepsilon^2 (\partial_x u_\varepsilon(t, 0))^2 + C_0.
\]
An integration on $(0, t)$, (72) and (90) give
\[
\varepsilon^2 \|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + \varepsilon^3 \int_0^t \|\partial_x^2 u_\varepsilon(s, \cdot)\|^2_{L^2(0, \infty)} ds
\leq C_0 + 36\kappa^2 \varepsilon \|u_\varepsilon\|^2_{L^\infty((0,T) \times (0,\infty))} \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|^2_{L^2(0, \infty)} ds
+ C_0 \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, 0))^2 ds + C_0 t
\leq C(T) \left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T) \times (0,\infty))}\right) + C_0 \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, 0))^2 ds.
\]
We prove (96). Due to the Young inequality,
\[
\varepsilon^2 (\partial_x u_\varepsilon(t, 0))^2 = -2\varepsilon^2 \int_0^\infty \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx
\leq 2\varepsilon^2 \int_0^\infty |\partial_x u_\varepsilon| |\partial_x^2 u_\varepsilon| dx
= 2 \int_0^\infty \left| \varepsilon \sqrt{D} \partial_x u_\varepsilon \right| |\partial_x^2 u_\varepsilon| dx
\leq \frac{\varepsilon}{D} \|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + D\varepsilon^3 \|\partial_x^2 u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)},
\]
where $D$ is a positive constant which will be specified later. Therefore, an integration on $(0, t)$ gives
\[
\varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, 0))^2 ds \leq \frac{\varepsilon}{D} \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|^2_{L^2(0, \infty)} ds
+ D\varepsilon^3 \int_0^t \|\partial_x^2 u_\varepsilon(s, \cdot)\|^2_{L^2(0, \infty)} ds.
\]
It follows from (90), (103) and (104) that
\[
\varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, 0))^2 ds \leq \frac{C(T)}{D} + DC(T) \left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T) \times (0,\infty))}\right)
+ DC_0 \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, 0))^2 ds,
\]
that is

$$(1 - DC_0) \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s,0))^2 ds \leq \frac{C(T)}{D} + DC(T) \left(1 + \|u_\varepsilon\|_{L^\infty((0,T) \times (0,\infty))}^2\right).$$

Taking $D = \frac{1}{2C_0}$, we have that

$$\frac{\varepsilon^2}{2} \int_0^t (\partial_x u_\varepsilon(s,0))^2 ds \leq C(T) \left(1 + \|u_\varepsilon\|_{L^\infty((0,T) \times (0,\infty))}^2\right),$$

which gives (96).

Finally, (97) follows from (96) and (103).

**Lemma 3.6.** Consider the function defined in (37). We have that

$$F_\varepsilon(t,\infty) = -\frac{\varepsilon}{a^2} \partial_x u_\varepsilon(t,0) + \frac{\kappa}{a^2} g_\varepsilon(t).$$

In particular, we get

$$\int_0^t F_\varepsilon^2(s,\infty) ds \leq C(T) \left(1 + \|P_\varepsilon\|_{L^\infty((0,T) \times (0,\infty))}^2\right),$$

for every $0 \leq t \leq T$.

**Proof.** Arguing as in Lemma 2.3, we have (106).

We prove (107). Fix $T > 0$. Thanks to (72), (106) and the Young inequality,

$$F_\varepsilon^2(t,\infty) = \left(-\frac{\varepsilon}{a^2} \partial_x u_\varepsilon(t,0) + \frac{\kappa}{a^2} g_\varepsilon(t)\right)^2$$

$$\leq \frac{2\varepsilon^2}{a^4} (\partial_x u_\varepsilon(t,0))^2 + \frac{2\kappa^2}{a^4} g_\varepsilon^4(t)$$

$$\leq \frac{2\varepsilon^2}{a^4} (\partial_x u_\varepsilon(t,0))^2 + \frac{2\kappa^2}{a^4} \|g_\varepsilon\|_{L^\infty((0,\infty))}^4$$

$$\leq \frac{2\varepsilon^2}{a^4} (\partial_x u_\varepsilon(t,0))^2 + C_0.$$

An integration on $(0, t)$ gives

$$\int_0^t F_\varepsilon^2(s,\infty) ds \leq \frac{2\varepsilon^2}{a^4} \int_0^t (\partial_x u_\varepsilon(s,0))^2 ds + C_0t$$

$$\leq \frac{2\varepsilon^2}{a^4} \int_0^t (\partial_x u_\varepsilon(s,0))^2 ds + C(T).$$

(107) follows from (96) and (108).

**Lemma 3.7.** Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\varepsilon$, such that

$$\|P_\varepsilon\|_{L^\infty((0,T) \times (0,\infty))} \leq C(T).$$

In particular, we have

$$\|u_\varepsilon\|_{L^\infty((0,T) \times (0,\infty))} \leq C(T),$$

$$\|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \leq C(T),$$

$$\varepsilon \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \leq C(T),$$

$$\varepsilon^3 \int_0^t \|\partial_{xx} u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \leq C(T),$$

(110)
Due to the Young inequality, 
\[ a^2 \int_0^t (\partial_x u_\varepsilon(s,0))^2 ds \leq C(T) , \]
\[ \int_0^t F_\varepsilon^2(s,\infty) ds \leq C(T) . \]

Proof. Let \( 0 \leq t \leq T \). Arguing as in Lemma 2.3, we have (42) with \(-a^2\) instead of \(a^2\), that is
\[ \partial_t P_\varepsilon(t,x) + \kappa u_\varepsilon^2(t,x) - \kappa g_\varepsilon^2(t) - \varepsilon \partial_x u_\varepsilon(t,x) + \varepsilon \partial_x u_\varepsilon(t,0) = -a^2 F_\varepsilon(t,x) . \] (111)

Observe that, by (37) and (71),
\[ -2\varepsilon \int_0^\infty P_\varepsilon \partial_x u_\varepsilon dx = 2\varepsilon \| u_\varepsilon(t,\cdot) \|_{L^2(0,\infty)}^2 , \]
\[-2a^2 \int_0^\infty F_\varepsilon P_\varepsilon dx = -a^2 F_\varepsilon^2(t,\infty) . \] (112)

Consequently, multiplying (111) by \(2P_\varepsilon\), it follows by (37), (112) and an integration on \((0,\infty)\) that
\[ \frac{d}{dt} \| P_\varepsilon(t,\cdot) \|_{L^2(0,\infty)}^2 + 2\varepsilon \| u_\varepsilon(t,\cdot) \|_{L^2(0,\infty)}^2 \]
\[ = -2\kappa \int_0^\infty P_\varepsilon u_\varepsilon^2 dx + 2\kappa g_\varepsilon^2(t) F_\varepsilon(t,\infty) -2\varepsilon \partial_x u_\varepsilon(t,0) F_\varepsilon(t,\infty) - 2a^2 F_\varepsilon^2(t,\infty) . \] (113)

Due to the Young inequality,
\[ 2|\kappa| g_\varepsilon^2(t) F_\varepsilon(t,\infty) | \leq \kappa^2 g_\varepsilon^4(t) + F_\varepsilon^2(t,\infty) , \]
\[ 2\varepsilon |\partial_x u_\varepsilon(t,0)| F_\varepsilon(t,\infty) | \leq \varepsilon^2 (\partial_x u_\varepsilon(t,0))^2 + F_\varepsilon^2(t,\infty) . \]

Therefore, by (72) and (113), we have
\[ \frac{d}{dt} \| P_\varepsilon(t,\cdot) \|_{L^2(0,\infty)}^2 + 2\varepsilon \| u_\varepsilon(t,\cdot) \|_{L^2(0,\infty)}^2 \]
\[ \leq 2|\kappa| \int_0^\infty | P_\varepsilon u_\varepsilon^2 dx + 2|\kappa| g_\varepsilon^2(t) F_\varepsilon(t,\infty) |
+ 2\varepsilon |\partial_x u_\varepsilon(t,0)| F_\varepsilon(t,\infty) | + a^2 F_\varepsilon^2(t,\infty) \]
\[ \leq 2|\kappa| \int_0^\infty | P_\varepsilon u_\varepsilon^2 dx + \kappa^2 u_\varepsilon^4(t) + \varepsilon^2 (\partial_x u_\varepsilon(t,0))^2 + (a^2 + 2) F_\varepsilon^2(t,\infty) \] (114)
\[ \leq 2|\kappa| \int_0^\infty | P_\varepsilon u_\varepsilon^2 dx + \kappa^2 \| g_\varepsilon \|_{L^\infty(0,\infty)}^4
+ \varepsilon^2 (\partial_x u_\varepsilon(t,0))^2 + (a^2 + 2) F_\varepsilon^2(t,\infty) \]
\[ \leq 2|\kappa| \int_0^\infty | P_\varepsilon u_\varepsilon^2 dx + \varepsilon^2 (\partial_x u_\varepsilon(t,0))^2 + (a^2 + 2) F_\varepsilon^2(t,\infty) + C_0 . \]

Thanks to (90) and the Young inequality,
\[ 2|\kappa| \int_0^\infty | P_\varepsilon u_\varepsilon^2 dx \leq 2|\kappa| \| P_\varepsilon \|_{L^\infty((0,T) \times (0,\infty))} \| u_\varepsilon(t,\cdot) \|_{L^2(0,\infty)}^2 \]
\[ \leq \| P_\varepsilon \|_{L^\infty((0,T) \times (0,\infty))}^2 + \kappa^2 \| u_\varepsilon(t,\cdot) \|_{L^2(0,\infty)}^2 \]
\[ \leq \| P_\varepsilon \|_{L^\infty((0,T) \times (0,\infty))}^2 + C(T) . \]
It follows from (114) that

\[
\frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \\
\leq \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 + \varepsilon^2(\partial_x u_\varepsilon(t, 0))^2 + (a^2 + 2)F_\varepsilon^2(t, \infty) + C(T).
\]

An integration on \((0, t)\), (96) and (107) give

\[
\|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon \int_0^t \|u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 \, ds \\
\leq C_0 + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 t + \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, 0))^2 \, ds \\
+ (a^2 + 2) \int_0^t F_\varepsilon^2(s, \infty) \, ds + C(T)t \\
\leq C(T) + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 T + C(T) \left(1 + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2\right) \\
\leq C(T) \left(1 + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2\right).
\]

We prove (109). Due to (71), (90), (115) and the Hölder inequality,

\[
P_\varepsilon^2(t, x) = 2 \int_0^x P_\varepsilon \partial_x P_\varepsilon \, dy \\
= \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)} \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)} \\
\leq C(T) \sqrt{1 + \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2}.
\]

Hence,

\[
\|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 - C(T) \|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))}^2 - C(T) \leq 0,
\]

which gives (109).

Finally, (110) follows from (94), (96), (97), (107), (109) and (115).

\(\square\)

**Proof of Theorem 1.2.** Arguing as in [9, Theorem 1.2], the proof is concluded. \(\square\)

4. **The case** \(f(u) = u^3, \kappa > 0, \gamma > 0\). In this section, we prove Theorem 1.2 in the case

\[
f(u) = u^3, \quad \kappa = b^2, \quad \gamma = a^2.
\]

Therefore, we consider (22), or (23), with \(f(u), \kappa, \gamma\) as in (117).

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (23).

Following [13], fix a small number \(\varepsilon > 0\) and let \(u_\varepsilon = u_\varepsilon(t, x)\) be the unique classical solution of the following mixed problem

\[
\begin{cases}
\partial_t u_\varepsilon + b^2 \partial_x u_\varepsilon^3 = a^2 P_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon, \quad t > 0, \quad x > 0, \\
\partial_x P_\varepsilon = u_\varepsilon, \quad t > 0, \quad x > 0, \\
u_\varepsilon(t, 0) = g_\varepsilon(t) \quad t > 0, \\
P_\varepsilon(t, 0) = 0 \quad t > 0, \\
u_\varepsilon(0, x) = u_{\varepsilon,0}(x), \quad x > 0,
\end{cases}
\]

(118)
where \( u_{\varepsilon,0} \) and \( g_{\varepsilon} \) are \( C^\infty(0, \infty) \) approximations of \( u_0 \) and \( g \) such that
\[
u_0, \varepsilon, \rightarrow u_0, \quad \text{a.e. and in } L^p(0, \infty), \quad 1 \leq p < \infty,
\]
\( P_{0, \varepsilon} \rightarrow P_0 \), in \( L^2(0, \infty) \),
\( g_\varepsilon \rightarrow g \), a.e. and in \( L^p_{\text{loc}}(0, \infty), \quad 1 \leq p < \infty,
\]
\[
\|u_{\varepsilon,0}\|_{L^\infty(0, \infty)} \leq \|u_0\|_{L^\infty(0, \infty)}, \quad \|u_{\varepsilon,0}\|_{L^2(0, \infty)} \leq \|u_0\|_{L^2(0, \infty)},
\]
\[
\|u_{\varepsilon,0}\|_{L^4(0, \infty)} \leq \|u_0\|_{L^4(0, \infty)}, \quad \int_0^\infty u_{\varepsilon,0}(x)dx = 0,
\]
\[
\|P_{\varepsilon,0}\|_{L^2(0, \infty)} \leq \|P_0\|_{L^2(0, \infty)}, \quad \|g_\varepsilon\|_{L^\infty(0, \infty)} \leq C_0,
\]
and \( C_0 \) is a constant independent on \( \varepsilon \).

Let us prove some a priori estimates on \( u_\varepsilon \), denoting with \( C_0 \) the constants which depend only on the initial data, and \( C(T) \) the constants which depend also on \( T \).

Arguing as in Section 2, we have the following result.

**Lemma 4.1.** (33) is equivalent to (34) with \( \kappa = b^2 \) and \( \frac{3g_\varepsilon^2(t)}{2} \) instead of \( \frac{2g_\varepsilon^2(t)}{3} \).

**Lemma 4.2.** For each \( t > 0 \), (33) and (35) hold. Moreover, fixed \( T > 0 \),
\[
\|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds \leq C(T) + \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, \cdot))^2 ds,
\]
for every \( 0 \leq t \leq T \).

**Proof.** Arguing as in Lemma 2.2 we have (33) and (35).

We prove (120). Let \( 0 \leq t \leq T \). We begin by observing that, from (119) and the Young inequality,
\[
2\varepsilon \int_0^t |g_\varepsilon(s)| \|\partial_x u_\varepsilon(s, 0)\| ds \leq \int_0^t g_\varepsilon^2(s) ds + \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, x))^2 ds \leq ||g_\varepsilon||_{L^\infty(0, \infty)}^2 t + \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, x))^2 ds \leq C(T) + \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, x))^2 ds.
\]
(35) and (121) give (120). \( \square \)

Arguing as in Lemma 2.3, we have the following result.

**Lemma 4.3.** Consider the function defined in (37). For each \( t > 0 \), we have that
\[
a^2 F_\varepsilon(t, \infty) = \varepsilon \partial_x u_\varepsilon(t, 0) - b^2 g_\varepsilon^3(t).
\]

**Lemma 4.4.** Fix \( T > 0 \). There exists a constant \( C(T) > 0 \), independent on \( \varepsilon \), such that
\[
\frac{b^2}{2} \|u_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)}^2 + a^2 \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 6b^2 \varepsilon \int_0^t \|u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)} ds \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds \leq C(T),
\]
\[
+ 2a^2 \varepsilon \int_0^t \|u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds + \varepsilon^2 \int_0^t (\partial_x u_\varepsilon(s, 0))^2 ds \leq C(T),
\]
for every 0 ≤ t ≤ T.

Proof. Let 0 ≤ t ≤ T. We take A is a positive constant which will be specified later. Multiplying (118) by 4Auτ, an integration on (0, 8) gives

\[
\frac{d}{dt} \left\| u_\tau(t, \cdot) \right\|_{L^4(0, \infty)}^4 = 4A \int_0^\infty u_\tau^3 \partial_x u_\tau \, dx
\]

\[
= -12b^2 A \int_0^\infty u_\tau^5 \partial_x u_\tau \, dx + 4Aa^2 \int_0^\infty u_\tau^3 P_\tau \, dx
\]

\[
+ 4A\varepsilon \int_0^\infty u_\tau^3 \partial_x^2 u_\tau \, dx
\]

\[
= -2b^2 A g_\tau^6(t) + 4Aa^2 \int_0^\infty u_\tau^3 P_\tau \, dx + 4A\varepsilon g_\tau^3(t) \partial_x u_\tau(t, 0)
\]

\[
- 12A\varepsilon \left\| u_\tau(t, \cdot) \partial_x u_\tau(t, \cdot) \right\|_{L^2(0, \infty)}^2,
\]

that is,

\[
\frac{d}{dt} \left\| u_\tau(t, \cdot) \right\|_{L^4(0, \infty)}^4 + 12A\varepsilon \left\| u_\tau(t, \cdot) \partial_x u_\tau(t, \cdot) \right\|_{L^2(0, \infty)}^2 = 2b^2 A g_\tau^6(t) + 4Aa^2 \int_0^\infty u_\tau^3 P_\tau \, dx + 4A\varepsilon g_\tau^3(t) \partial_x u_\tau(t, 0).
\]

Arguing as in Lemma 2.3, we get

\[
\partial_t P_\tau = a^2 F_\tau(t, x) + \varepsilon \partial_x u_\tau(t, \cdot) - \varepsilon \partial_x P_\tau(t, 0) - b^2 u_\tau^3 + b^2 g_\tau^3(t).
\]

Observe that, by (37) and (122),

\[
-2a^2 \varepsilon \partial_x u_\tau(t, 0) \int_0^\infty P_\tau(t, x) \, dx = -2\varepsilon^2 (\partial_x u_\tau(t, 0))^2 - 2b^2 \varepsilon \partial_x u_\tau(t, 0) g_\tau^3(t),
\]

\[
2a^2 b^2 g_\tau^3(t) \int_0^\infty P_\tau(t, x) \, dx = 2b^2 \varepsilon \partial_x u_\tau(t, 0) g_\tau^3(t) - 2b^4 g_\tau^6(t).
\]

Moreover, by (119) and (37),

\[
2a^2 \varepsilon \int_0^\infty \partial_x u_\tau(t, x) P_\tau(t, x) \, dx = -2a^2 \varepsilon \left\| u_\tau(t, \cdot) \right\|_{L^2(0, \infty)}^2,
\]

\[
2a^4 \int_0^\infty P_\tau(t, x) F_\tau(t, x) = a^4 F_\tau^2(t, \infty)
\]

\[
= \varepsilon^2 (\partial_x u_\tau(t, 0))^2 - 2b^2 \varepsilon \partial_x u_\tau(t, 0) g_\tau^3(t) + b^4 g_\tau^6(t).
\]

Multiplying (125) by 2a^2 P_\tau, an integration on (0, \infty), (127) and (128) give

\[
a^2 \frac{d}{dt} \left\| P_\tau(t, \cdot) \right\|_{L^2(0, \infty)}^2 + 2a^2 \varepsilon \left\| u_\tau(t, \cdot) \right\|_{L^2(0, \infty)}^2 + \varepsilon^2 (\partial_x u_\tau(t, 0))^2
\]

\[
= -2a^2 b^2 \int_0^\infty u_\tau^3 P_\tau \, dx - b^4 g_\tau^6(t) + 2b^2 \varepsilon \partial_x u_\tau(t, 0) g_\tau^3(t).
\]

Adding (124) and (129), we have

\[
\frac{d}{dt} \left( A \left\| u_\tau(t, \cdot) \right\|_{L^4(0, \infty)}^4 + a^2 \left\| P_\tau(t, \cdot) \right\|_{L^2(0, \infty)}^2 \right)
\]

\[
+ 12A\varepsilon \left\| u_\tau(t, \cdot) \partial_x u_\tau(t, \cdot) \right\|_{L^2(0, \infty)}^2.
\]
+ 2a^2 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \leq (\varepsilon^2 \partial_x u_\varepsilon(t, 0))^2 \\
= 2a^2 (2A - b^2) \int_0^\infty u_\varepsilon^3 P_\varepsilon \ dx + b^2 (2A - 1) g_\varepsilon^6(t) + 2 (2A + b^2) \varepsilon \partial_x u_\varepsilon(t, 0) g_\varepsilon^6(t). \\
(130)

We search \( A \) such that
\[ 2A - b^2 = 0, \]
that is
\[ A = \frac{b^2}{2}. \] (131)

It follows from (130) and (131) that
\[
\frac{d}{dt} \left( \frac{b^2}{2} \|u_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)}^4 + a^2 \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \right) \\
+ 6b^2 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \partial_x u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \leq b^2 (b^2 - 1) g_\varepsilon^6(t) + 4b^2 \varepsilon \partial_x u_\varepsilon(t, 0) g_\varepsilon^6(t). \\
(132)

Thanks to (119) the Young inequality,
\[
4b^2 \varepsilon \|\partial_x u_\varepsilon(t, 0)\|_{L^2(0, \infty)} \|g_\varepsilon(t)\|^6 \leq \frac{\varepsilon^2}{2} (\partial_x u_\varepsilon(t, 0))^2 + 8b^4 g_\varepsilon^6(t) \\
\leq \frac{\varepsilon^2}{2} (\partial_x u_\varepsilon(t, 0))^2 + 8b^4 \|g_\varepsilon\|^6_{L^\infty(0, \infty)} \\
\leq \frac{\varepsilon^2}{2} (\partial_x u_\varepsilon(t, 0))^2 + C_0.
\]

Therefore, by (119) and (132),
\[
\frac{d}{dt} \left( \frac{b^2}{2} \|u_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)}^4 + a^2 \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \right) \\
+ 6b^2 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \partial_x u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2a^2 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \frac{\varepsilon^2}{2} (\partial_x u_\varepsilon(t, 0))^2 \\
\leq b^2 (b^2 - 1) \|g_\varepsilon(t)\|^6 + C_0 \leq b^2 (b^2 - 1) \|g_\varepsilon\|^6_{L^\infty(0, \infty)} + C_0 \leq C_0.
\]

An integration on \( (0, t) \) and (119) give
\[
b^2 \|u_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)}^4 + a^2 \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 6b^2 \varepsilon \int_0^t \|u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 \partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds \\
+ 2a^2 \varepsilon \int_0^t \|u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds + \frac{\varepsilon^2}{2} \int_0^t (\partial_x u_\varepsilon(s, 0))^2 ds \leq C_0(1 + t) \leq C(T),
\]
which gives (123). \( \square \)

**Corollary 1.** Assume
\[ a^2 = b^2 = 1. \] (133)

**Estimate (2.17) of [13, Lemma 2.4] holds.**

**Proof.** We begin by observing that, thanks to (131) and (133)
\[ A = \frac{1}{2}. \] (134)
Consequently, (130) reads
\[
\frac{d}{dt} \left( \frac{1}{2} \| u_\varepsilon(t, \cdot) \|_{L^\infty(0,\infty)}^4 + \| P_\varepsilon(t, \cdot) \|_{L^2(0,\infty)}^2 \right) + 6\varepsilon \| u_\varepsilon(t, \cdot) \|_{L^2(0,\infty)}^4 + 2a^2 \varepsilon \| u_\varepsilon(t, \cdot) \|_{L^2(0,\infty)}^2 + \varepsilon^2 (\partial_x u_\varepsilon(t,0))^2 \\
= 4\varepsilon \partial_x u_\varepsilon(t,0) g_\varepsilon^2(t),
\]
which coincides with the equality (2.25) of [13, Lemma 2.4].

\[\tag{130}\]

**Lemma 4.5.** Fix \(T > 0\). There exists a constant \(C(T) > 0\), independent on \(\varepsilon\), such that (45) holds for every \(0 \leq t \leq T\). In particular, (109) and (66) hold.

**Proof.** We begin by observing that (45) follows from (120) and (123).

We prove (109). Due to (118) and the Hölder inequality,
\[
P_\varepsilon^2(t,x) = 2 \int_0^x P_\varepsilon \partial_x P_\varepsilon dy \leq \int_0^\infty |P_\varepsilon| u_\varepsilon dx \\
= \|P_\varepsilon(t, \cdot)\|_{L^2(0,\infty)} \|u_\varepsilon(t, \cdot)\|_{L^2(0,\infty)}.
\]

Hence, by (45) and (123),
\[
\|P_\varepsilon\|_{L^\infty((0,T) \times (0,\infty))} \leq C(T),
\]
which gives (109).

Finally, arguing as in Lemma 2.6, we (66). \(\square\)

**Proof of Theorem 1.2.** Arguing as in [9, Theorem 1.1], the proof is concluded. \(\square\)

5. **The case** \(f(u) = u^3\), \(\kappa < 0\), \(\gamma > 0\). In this section, we prove Theorem 1.2 in the case (11). Therefore, we consider (22), or (23), with \(f(u), \kappa, \gamma\) as in (11).

Observe that (23) with \(f(u), \kappa, \gamma\) defined in (9) is equivalent to the following one (see [16]):
\[
\begin{aligned}
&\partial_t u + \kappa \partial_x u^3 = \gamma P, \quad t > 0, \ x > 0, \\
&\partial_x P = u, \quad t > 0, \ x > 0, \\
&u(t,0) = g(t) \quad t > 0, \\
&\partial_x u^3(t,0) = \frac{g'(t)}{\kappa} \quad t > 0, \\
&u(0,x) = u_0(x), \quad x > 0.
\end{aligned}
\]

Unlike Sections 2, 3, 4, our existence argument is based on passing to the limit in a vanishing viscosity approximation of (135).

Following [16], defined \(\kappa = -b^2\), \(\gamma = a^2\), fixed \(0 < \varepsilon < 1\), let \(u_\varepsilon = u_\varepsilon(t,x)\) be the unique classical solution of the following mixed problem
\[
\begin{aligned}
&\partial_t u_\varepsilon - b^2 \partial_x u_\varepsilon^3 = a^2 P_\varepsilon + \varepsilon \partial_{xx} u_\varepsilon, \quad t > 0, \ x > 0, \\
&\partial_x P_\varepsilon = u_\varepsilon, \quad t > 0, \ x > 0, \\
&u_\varepsilon(t,0) = g_\varepsilon(t) \quad t > 0, \\
&\partial_x u_\varepsilon^3(t,0) = -\frac{g'_\varepsilon(t)}{b^2} \quad t > 0, \\
&u_\varepsilon(0,x) = u_{\varepsilon,0}(x), \quad x > 0.
\end{aligned}
\]

(136)
where \( u_{\varepsilon,0} \) and \( g_\varepsilon \) are \( C^\infty(0, \infty) \) approximations of \( u_0 \) and \( g \) such that
\[
\|u_{\varepsilon,0}\|_{L^\infty(0, \infty)} \leq \|u_0\|_{L^\infty(0, \infty)}, \quad \|u_{\varepsilon,0}\|_{L^2(0, \infty)} \leq \|u_0\|_{L^2(0, \infty)}; \\
\|P_{\varepsilon,0}\|_{L^2(0, \infty)} \leq \|P_0\|_{L^2(0, \infty)}; \\
\|g_\varepsilon\|_{L^\infty(0, \infty)} + \|g'_\varepsilon\|_{L^\infty(0, \infty)} \leq C_0, \quad 0 < \alpha \leq |g_\varepsilon(t)|,
\]
and \( C_0 \) is a constant independent on \( \varepsilon \).

Let us prove some a priori estimates on \( u_\varepsilon \) denoting with \( C_0 \) the constants which depend only on the initial data, and \( C(T) \) the constants which depend also on \( T \).

**Lemma 5.1.** We have that
\[
P_\varepsilon(t, \infty) = 0, \quad (138)
\]
\[
\varepsilon \partial_{xx}^2 u_\varepsilon(t, 0) = -a^2 P_\varepsilon(t, 0), \quad (139)
\]
\[
\int_0^\infty u_\varepsilon dx = -P_\varepsilon(t, 0). \quad (140)
\]

Moreover,
\[
\|\partial_x u_\varepsilon(\cdot, 0)\|_{L^\infty(0, \infty)} \leq C_0. \quad (141)
\]

**Proof.** We begin by observing that, arguing as in [9, Lemma 2.1], we have (138).

We prove (139). Observe that, by (136), we have
\[
\partial_t u_\varepsilon(t, 0) - b^2 \partial_{xx}^2 u_\varepsilon(t, 0) = a^2 P_\varepsilon(t, 0) + \varepsilon \partial_{xx}^2 u_\varepsilon(t, 0).
\]

Since \( \partial_t u_\varepsilon(t, 0) = g'_\varepsilon(t) \), again by (136), we get
\[
b^2 P_\varepsilon(t, 0) + \varepsilon \partial_{xx}^2 u_\varepsilon(t, 0) = 0,
\]
which gives (139).

Now, we prove (140). Integrating the second equation of (136) on \( (0, x) \), we have
\[
P_\varepsilon(t, x) - P_\varepsilon(t, 0) = \int_0^x u_\varepsilon(t, y) dy. \quad (143)
\]
Consequently,
\[
\int_0^\infty u_\varepsilon dx = \lim_{x \to \infty} (P_\varepsilon(t, x) - P_\varepsilon(t, 0)) = P_\varepsilon(t, \infty) - P_\varepsilon(t, 0).
\]
(140) follows from (138) and (144).

Finally, we prove (141). By (136), we obtain that
\[
3b^2 g_\varepsilon^2(t) \partial_x u_\varepsilon(t, 0) = -g'_\varepsilon(t), \quad (145)
\]
Squaring equation (145), we have
\[
9b^4 g_\varepsilon^4(t) (\partial_x u_\varepsilon(t, 0))^2 = (g'_\varepsilon(t))^2. \quad (146)
\]
Due to (137), we get
\[
9b^4 \alpha^4 (\partial_x u_\varepsilon(t, 0))^2 = (g'_\varepsilon(t))^2 \leq \|g'_\varepsilon\|_{L^2(0, \infty)} \leq C_0.
\]
Consequently,
\[
(\partial_x u_\varepsilon(t, 0))^2 \leq C_0 \quad \Rightarrow \quad |\partial_x u_\varepsilon(t, 0)| \leq C_0,
\]
which gives (141).
Lemma 5.2. Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\varepsilon$, such that
\[
\|u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|^2_{L^2(0, \infty)} ds + a^2 \int_0^t P_\varepsilon^2(s, 0) ds \leq C(T),
\]
for every $0 \leq t \leq T$. In particular, we have that
\[
\varepsilon^2 \int_0^t (\partial_{xx}^2 u_\varepsilon(s, 0))^2 ds \leq C(T).
\]

Proof. Let $0 \leq t \leq T$. Multiplying the first equation of (136) by $2u_\varepsilon$, an integration on $(0, \infty)$ and (136) give
\[
\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} = 2 \int_0^\infty u_\varepsilon \partial_x u_\varepsilon dx
\]
\[
= 6b^2 \int_0^\infty u_\varepsilon^3 \partial_x u_\varepsilon dx + 2a^2 \int_0^\infty P_\varepsilon u_\varepsilon dx + 2\varepsilon \int_0^\infty u_\varepsilon \partial_{xx} u_\varepsilon dx
\]
\[
= - \frac{3}{2} g_\varepsilon^2(t) + 2a^2 \int_0^\infty P_\varepsilon u_\varepsilon dx
\]
\[
- 2\varepsilon g_\varepsilon(t) \partial_x u_\varepsilon(t, 0) - 2\varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)}.
\]
Therefore, we have
\[
\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + 2\varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)}
\]
\[
= - \frac{3}{2} g_\varepsilon^2(t) + 2a^2 \int_0^\infty P_\varepsilon u_\varepsilon dx - 2\varepsilon g_\varepsilon(t) \partial_x u_\varepsilon(t, 0).
\]

Thanks to the second equation of (136) and (138),
\[
+ 2a^2 \int_0^\infty P_\varepsilon u_\varepsilon dx = 2a^2 \int_0^\infty P_\varepsilon \partial_x P_\varepsilon dx = a^2 P_\varepsilon^2(t, \infty) - a^2 P_\varepsilon^2(t, 0) = -a^2 P_\varepsilon^2(t, 0).
\]
Therefore, since $0 < \varepsilon < 1$, by (141),
\[
\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + 2\varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + a^2 P_\varepsilon^2(t, 0)
\]
\[
= - \frac{3}{2} g_\varepsilon^2(t) - 2\varepsilon g_\varepsilon(t) \partial_x u_\varepsilon(t, 0)
\]
\[
\leq \frac{3}{2} \|g_\varepsilon\|^2_{L^\infty(0, \infty)} + 2|g_\varepsilon(t)||\partial_x u_\varepsilon(t, 0)|
\]
\[
\leq C_0 + 2 \|g_\varepsilon\|_{L^\infty(0, \infty)} \|\partial_x u_\varepsilon(\cdot, 0)\|_{L^\infty(0, \infty)} \leq C_0.
\]

An integration on $(0, t)$ and (137) give
\[
\|u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|^2_{L^2(0, \infty)} ds + a^2 \int_0^t P_\varepsilon^2(s, 0) ds \leq C_0(1+t) \leq C(T),
\]
which gives (148).

Finally, we prove (149). Squaring equation (139), (148) and an integration on $(0, t)$ give
\[
\varepsilon^2 \int_0^t (\partial_{xx}^2 u_\varepsilon(s, 0))^2 ds = a^4 \int_0^t P_\varepsilon^2(s, 0) ds \leq C(T).
\]
Therefore, the proof is concluded. \(\square\)
Lemma 5.3. Consider the function defined in (37). For each $t \geq 0$,
\[ a^2 F_\varepsilon(t, \infty) = a^2 \int_0^\infty P_\varepsilon(t, x) dx = -\partial_t P_\varepsilon(t, 0) + \varepsilon \partial_x u_\varepsilon(t, 0) + b^2 g_\varepsilon^3(t). \tag{151} \]

**Proof.** Integrating the first equation of (136) on $(0, x)$, we have
\[ \int_0^x \partial_t u_\varepsilon(t, y) dy - b^2 u_\varepsilon^3(t, x) + b^2 g_\varepsilon^3(t) - \varepsilon \partial_x u_\varepsilon(t, x) + \partial_x u_\varepsilon(t, 0) = a^2 F_\varepsilon(t, x). \tag{152} \]
Observe that, by (140), we get
\[ \lim_{x \to \infty} \int_0^x \partial_t u_\varepsilon(t, y) dy = \int_0^\infty \partial_t u_\varepsilon(t, x) dx = \frac{d}{dt} \int_0^\infty u_\varepsilon(t, x) dx = -\partial_t P_\varepsilon(t, 0). \tag{153} \]
It follows from the regularity of $u_\varepsilon$ that
\[ \lim_{x \to \infty} (-b^2 u_\varepsilon^3(t, x) - \varepsilon \partial_x u_\varepsilon(t, x)) = 0. \tag{154} \]

(151) follows from (152), (153) and (154). \qed

**Lemma 5.4.** Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\varepsilon$, such that
\[ \| u_\varepsilon(t, \cdot) \|_{L^4((0, \infty), \mathbb{R}^3)}^4 + a^2 \| P_\varepsilon(t, \cdot) \|_{L^2((0, \infty), \mathbb{R}^3)}^2 \]
\[ + 12 \varepsilon C(T) t \int_0^t e^{-C(T)s} \| u_\varepsilon(s, \cdot) \|_{L^2((0, \infty), \mathbb{R}^3)}^2 ds \]
\[ + 2a^2 \varepsilon C(T) t \int_0^t e^{-C(T)s} \| u_\varepsilon(s, \cdot) \|_{L^2((0, \infty), \mathbb{R}^3)}^2 ds \]
\[ + \frac{1}{2} C(T) t \int_0^t e^{-C(T)s} (\partial_t P_\varepsilon(s, \cdot))^2 ds \leq C(T), \tag{155} \]
for every $0 \leq t \leq T$.

**Proof.** Let $0 \leq t \leq T$. Arguing as in Lemma 4.4, we have
\[ \frac{d}{dt} \| u_\varepsilon(t, \cdot) \|_{L^4((0, \infty), \mathbb{R}^3)}^4 + 12 \varepsilon \| u_\varepsilon(t, \cdot) \|_{L^2((0, \infty), \mathbb{R}^3)}^2 \]
\[ = -2b^2 g_\varepsilon^3(t) + 4a^2 \int_0^\infty u_\varepsilon^3 P_\varepsilon dx + 4\varepsilon g_\varepsilon^3(t) \partial_x u_\varepsilon(t, 0). \tag{156} \]
Observe that, differentiating (143) with respect to $t$, we get
\[ \frac{d}{dt} \int_0^x u_\varepsilon dy = \int_0^y \partial_t u_\varepsilon dy = \partial_t P_\varepsilon - \partial_t P_\varepsilon(t, 0). \tag{157} \]
It follows from (152) and (157) that
\[ \partial_t P_\varepsilon - \partial_t P_\varepsilon(t, 0) - b^2 u_\varepsilon^3 + b^2 g_\varepsilon^3(t) - \varepsilon \partial_x u_\varepsilon + \partial_x u_\varepsilon(t, 0) = a^2 F_\varepsilon. \tag{158} \]
Observe that, by (151),
\[ -2a^2 \partial_t P_\varepsilon(t, 0) \int_0^\infty P_\varepsilon dx = (2(\partial_t P_\varepsilon(t, 0))^2 \]
\[ - 2 \varepsilon \partial_t P_\varepsilon(t, 0) \partial_x u_\varepsilon(t, 0) - 2b^2 \partial_t P_\varepsilon(t, 0) g_\varepsilon^3(t), \]
\[ 2a^2 b^2 g_\varepsilon^3(t) \int_0^\infty P_\varepsilon dx = - 2b^2 \partial_t P_\varepsilon(t, 0) g_\varepsilon^3(t) \]
Moreover, being $0 < \varepsilon < 1$, thanks to the Young inequality,

$$2a^2 \varepsilon |P_\varepsilon(t, 0)|g_\varepsilon(t) \leq 2a^2 |P_\varepsilon(t, 0)|g_\varepsilon(t) \leq P_\varepsilon^2(t, 0) + a^4 g_\varepsilon^2(t).$$

Consequently, since $0 < \varepsilon < 1$, by (137), (141) and (162), we get

$$\frac{d}{dt} \left( \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^4 + a^2 \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \right)$$

$$+ 12\varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + 2a^2 \varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \frac{1}{2} (\partial_t P_\varepsilon(t, 0))^2$$

$$\leq 2a^2 (2 + b^2) \int_0^\infty |u_\varepsilon|^3 |P_\varepsilon| \, dx + P_\varepsilon^2(t, 0) + a^4 g_\varepsilon^2(t) + b^2 (5b^2 + 2) g_\varepsilon^6(t).$$

(163)
\[ + 2|(2 - b^2)|g_\varepsilon(t)|^3|\partial_x u_\varepsilon(t, 0)| + 4(\partial_x u_\varepsilon(t, 0))^2 \\]
\[ \leq 2a^2(2 + b^2) \int_0^\infty |u_\varepsilon|^3|P_\varepsilon|dx + P_\varepsilon^2(t, 0) + a^4 \|g_\varepsilon\|_{L^\infty_{0, \infty}}^2 \]
\[ + b^2(5b^2 + 2) \|g_\varepsilon\|_{L^\infty_{0, \infty}}^6 \]
\[ + 2|(2 - b^2)||g_\varepsilon|_{L^\infty_{0, \infty}}^3 \|\partial_x u_\varepsilon(\cdot, 0)\|_{L^\infty_{0, \infty}} + 4 \|\partial_x u_\varepsilon(\cdot, 0)\|_{L^\infty_{0, \infty}}^2 \\]
\[ \leq 2a^2(2 + b^2) \int_0^\infty |u_\varepsilon|^3|P_\varepsilon|dx + P_\varepsilon^2(t, 0) + C_0. \]

Thanks to (148) and the Young inequality,
\[ 2a^2(2 + b^2) \int_0^\infty |u_\varepsilon|^3|P_\varepsilon|dx \]
\[ \leq a^4(2 + b^2)^2 \int_0^\infty P_\varepsilon^2 u_\varepsilon^2 dx + \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 \]
\[ \leq a^4(2 + b^2)^2 \|P_\varepsilon(t, \cdot)\|_{L^\infty_{0, \infty}}^2 \|u_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 \]
\[ \leq a^2 C(T) \|P_\varepsilon(t, \cdot)\|_{L^\infty_{0, \infty}}^2 + \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 + P_\varepsilon^2(t, 0) + C_0. \]

It follows from (163) that
\[ \frac{d}{dt} \left( \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 + a^2 \|P_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 \right) \]
\[ + 12\varepsilon \|u_\varepsilon(t, \cdot)\partial_x u_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 \]
\[ + 2a^2\varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + \frac{1}{2}(\partial_t P_\varepsilon(t, 0))^2 \]
\[ \leq a^2 C(T) \|P_\varepsilon(t, \cdot)\|_{L^\infty_{0, \infty}}^2 + \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 + P_\varepsilon^2(t, 0) + C_0. \] (164)

Observe that, by (135),
\[ P_\varepsilon^2(t, x) = 2 \int_0^x P_\varepsilon \partial_x P_\varepsilon dy + P_\varepsilon^2(t, 0) = 2 \int_0^x P_\varepsilon \partial_x u_\varepsilon dy + P_\varepsilon^2(t, 0). \]

Consequently, by (148) and the Young inequality,
\[ \|P_\varepsilon(t, \cdot)\|_{L^\infty_{0, \infty}}^2 \leq 2 \int_0^\infty |P_\varepsilon|\partial_x u_\varepsilon|dx + P_\varepsilon^2(t, 0) \]
\[ \leq \|P_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + \|u_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + P_\varepsilon^2(t, 0) \]
\[ \leq \|P_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + P_\varepsilon^2(t, 0) + C(T). \] (165)

Therefore, by (164), we have that
\[ \frac{d}{dt} \left( \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 + a^2 \|P_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 \right) \]
\[ + 12\varepsilon \|u_\varepsilon(t, \cdot)\partial_x u_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + 2a^2\varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + \frac{1}{2}(\partial_t P_\varepsilon(t, 0))^2 \]
\[ \leq a^2 C(T) \|P_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 + C(T) P_\varepsilon^2(t, 0) + C(T) \]
\[ \leq C(T) \left( a^2 \|P_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 + \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 \right) + C(T) P_\varepsilon^2(t, 0) + C(T). \]

The Gronwall Lemma (137) and (148) give
\[ \|u_\varepsilon(t, \cdot)\|_{L^4_{0, \infty}}^4 + a^2 \|P_\varepsilon(t, \cdot)\|_{L^2_{0, \infty}}^2 \]
Lemma 5.5. Fix $T > 0$. There exists a constant $C(T) > 0$, independent on $\varepsilon > 0$, such that
\[
\|P_\varepsilon(\cdot,0)\|_{L^\infty(0,T)} \leq C(T).
\] In particular, we have
\[
\varepsilon \|\partial_x^2 u_\varepsilon(\cdot,0)\|_{L^\infty(0,T)} \leq C(T).
\] Moreover, (66) and (109) hold.

Proof. Let $0 \leq t \leq T$. We begin by proving (166). Multiplying the second equation of (136) by $-2P_\varepsilon$, from (138) and an integration on $(0,\infty)$, we have
\[
P_\varepsilon^2(t,0) = -2 \int_0^\infty P_\varepsilon \partial_x P_\varepsilon dx = -2 \int_0^\infty P_\varepsilon u_\varepsilon dx.
\] Thanks to (148), (155) and the Hölder inequality,
\[
P_\varepsilon^2(t,0) \leq 2 \int_0^\infty |P_\varepsilon| u_\varepsilon dx \leq 2 \|P_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)} \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \leq C(T).
\] Hence,
\[
\|P_\varepsilon(\cdot,0)\|_{L^\infty(0,T)}^2 \leq C(T),
\] which gives (166).

(167) follows from (139) and (166), while (155), (165) and (166) give (109).

Finally, arguing as in Lemma 2.6 we have (66).

Therefore, the proof is concluded.

Let us continue by proving the existence of a distributional solution to (16), (2) and (3) satisfying (27) with $f(u) = u^3$.

Lemma 5.6. Fix $T > 0$. There exists a function $u \in L^\infty((0,T) \times (0,\infty))$ that is a distributional solution of (23) with $f(u) = u^3$ and satisfies (28).

We construct a solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon \geq 0}$ of viscosity approximations (136). We use the compensated compactness method [40].

Lemma 5.7. Let $T > 0$. We have that (67) and (28) hold. In particular, we get
\[
\int_0^T u_\varepsilon \, dy \to \int_0^T u \, dy \text{ a.e. and in } L^p_{loc}(0,T;W^{1,p}_{loc}(0,\infty)), 1 \leq p < \infty,
\]
\[
P_\varepsilon(\cdot,0) \to P(\cdot,0) \text{ in } L^p(0,T), 2 \leq p < \infty.
\]
Moreover,
\[ P_{te}(t, x) \to \int_0^x u(t, y)dy + P(t, 0), \quad t \geq 0, \quad x \geq 0. \tag{171} \]

\textbf{Proof.} Arguing as in [9, Lemma 3.2], we have (67), (28) and (169). Lemmas 5.2 and 5.5 give (170).

Finally, (171) follows from (143), (169) and (170).

\textbf{Proof of Theorem 1.2.} Lemma 5.7 give the existence of entropy solution of (135). Since it is equivalent to (23) with \( f(u), \kappa, \gamma \) defined in (11), we have that
\[ P(t, 0) = 0. \]

Arguing as in [9, Theorem 1.2], the proof is concluded.

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