Accelerating Growth and Size-dependent Distribution of Human Online Activities

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Research on human online activities usually assumes that total activity $T$ increases linearly with active population $P$, that is, $T \propto P^\gamma (\gamma = 1)$. However, we find examples of systems where total activity grows faster than active population. Our study shows that the power law relationship $T \propto P^\gamma (\gamma > 1)$ is in fact ubiquitous in online activities such as micro-blogging, news voting and photo tagging. We call the pattern “accelerating growth” and find it relates to a type of distribution that changes with system size. We show both analytically and empirically how the growth rate $\gamma$ associates with a scaling parameter $b$ in the size-dependent distribution. As most previous studies explain accelerating growth by power law distribution, the model of size-dependent distribution is novel and worth further exploration.

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I. INTRODUCTION

There are two ways of describing the growth of human online activities: linear and nonlinear. Linear models assume that the average level of individual activities is a constant. Hence, the total amount of activity $T$ is a linear function of $P$, namely, $T \propto P^\gamma (\gamma = 1)$. For instance, the average degree is a constant parameter in the Barabasi - Albert (BA) model [1]. However, in nonlinear models the average level of individual activities increase with system size $P$, leading to a power law relationship $T \propto P^\gamma (\gamma > 1)$. This relationship is supported by empirical studies on different online activities including game playing [3], resource recommendation [4], collaborative programming [5] and tagging [6, 7], as well as off-line activities such as energy consumption [8–11] and wealth creation [12, 13]. Although there is plenty of evidence for nonlinear growth (referred to hereafter as “accelerating growth”), how such growth arises is still an open question. In the current study, we find a power law relationship $T \propto P^\gamma (\gamma > 1)$ in several types of online activities ranging from micro-blogging, news voting to photo tagging. While previous authors have explained it with a long-tail distribution $P \propto 1/T$ (which determines the activities of highly active users) in the DGBD, we are able to fit the empirical curves with $R^2 > 0.9$. Furthermore, it is observed that the rank curves of individual activities change with population size, and such a correlation can be controlled by adding a scaling factor $P^b$ in the DGBD function. We call the modified DGBD function the “size-dependent distribution” and derive the relationship $T \propto P^\gamma (\gamma > 1)$ from it analytically. We therefore find that the accelerating growth rate $\gamma$ is not determined by the exponent in Zipf’s law (or a power law distribution), as claimed by previous studies [5, 7, 18, 19], but is in fact related with the size dependent exponent $b$.

II. ACCELERATING GROWTH IN HUMAN ONLINE ACTIVITIES

Our data cover several types of typical human online activities, including the micro-blogging activities of 6,426 users on Jiwai, the news voting activities of 139,409 users on Digg, the photo tagging activities of 195,575 users on Flickr, and the book tagging activities of 13,988 users on Delicious. All the four data sets are publicly available. The Jiwai data set is published in [20] and is available at [http://www.fanpq.com/](http://www.fanpq.com/). The Digg data set is published in [18] and can be downloaded from [http://www.isi.edu/integration/people/lerman/downloads.html](http://www.isi.edu/integration/people/lerman/downloads.html). The Flickr and Delicious data sets are published in [21] and can be downloaded from [http://www.uni-koblenz-landau.de/koblenz/fb4/AGStaab/Research](http://www.uni-koblenz-landau.de/koblenz/fb4/AGStaab/Research). In the four systems, we define $P$ as the number of active users in a day, and $T$ the total activity generated by these users. Note that the unit of $T$ is different across systems: $T$ are micro-blogs in Jiwai, news votes in

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that greater $\gamma$ values of $T$ to overstate the effect of outliers [19]. The estimated values of $\gamma$ shown in Table I are all greater than 1. By noting that greater $\gamma$ means more activities would be generated by a given population, we can regard $\gamma$ as an indicator of productivity and compare the productivity between online and off-line systems. Off-line activities, including wealth creation and patent invention, have been found to scale linearly with population, $\gamma$ estimated to be between 1 and 1.35 [8]. As one of the $\gamma$ in our study exceeds 1.40 (Flickr), this is evidence that online systems could be more productive than off-line systems.

Our analysis of four online systems provides varying estimates of $\gamma$. But what determines the $\gamma$? In exploring the underlying distribution of individual activities, we discover a new type of distribution that changes with system size. In the following section, we introduce the size-dependent distribution, which we propose relates to accelerating growth in human online activities. In particular, we show how the exponent $b$ of the distribution can be used to predict the value of $\gamma$.

III. FROM SIZE DEPENDENT DISTRIBUTION TO ACCELERATING GROWTH

In this section, we show that the DGBD, which has been extensively studied for various data sets in [17], fits the daily distributions of individual activities in the four online systems. We then show that the distribution changes with system size and this correlation can be captured by a scaling parameter $P^b$. In other words, by adding $P^b$ to the distribution function, we can use it to fit the empirical distributions in different days. We call the modified distribution the size-dependent distribution and show that accelerating growth in total activity can be derived from this distribution.

A. The DGBD model

We use $t(r)$ to denote the activity of a user in one day, in which $r$ is the decreasing rank of the activity among all individual activities in the day. Thus the maximum value of $r$, $r_{\max}$, equals population $P$. The DGBD model [17] of individual activities is then

$$t(r) = A(P + 1 - r)^b r^{-a} \quad (a > 0, b > 0),$$

where $A$, $b$, and $a$ are parameters to be estimated. In Fig. 2, we show three examples of daily distributions for each system. Note that if we set $b = 0$ then Eq. (2) becomes Zipf’s law $t(r) \propto r^{-a}$. Therefore, the DGBD can be viewed as a generalized Zipf’s law (or power law distribution). The reason for using the DGBD instead of Zipf’s law, which has been widely used to fit human behavioral

TABLE I: Estimates of accelerating growth.

| Activity            | Dataset | $\gamma$   | 95% CI       | Adjusted $R^2$ | N of Days | URL          |
|---------------------|---------|------------|--------------|----------------|-----------|--------------|
| Photo tagging       | Flickr  | 1.48       | 1.43, 1.54   | 0.93           | 193       | flickr.com  |
| Micro-blogging      | Jiwai   | 1.19       | 1.03, 1.48   | 0.98           | 21        | m.jiwai.de  |
| News voting         | Digg    | 1.18       | 1.06, 1.22   | 1.00           | 31        | digg.com/news|
| Book tagging        | Delicious| 1.17       | 1.15, 1.19   | 0.93           | 663       | delicious.com|

FIG. 1: (Color online) Accelerating growth in human online activities. Different datasets are marked in points of different colors and shapes (green squares for Flickr, orange diamonds for Delicious, red circles for Jiwai, and blue triangles for Digg). The x axis shows the active population in a day and the y axis shows the total activity in the day. Both axes have a logarithmic scale. The orthogonal regression lines are also shown.

Digg, and tags in Delicious and Flickr. The systems are summarized in Table I.

We plot $T$ and $P$ in a log-log scale plot (Fig. 1) and find a power law relationship

$$T \propto P^\gamma. \quad (1)$$

The values of $\gamma$ estimated by orthogonal regression are summarized in Table I. We use orthogonal regression instead of ordinary least squares regression because the latter tends to overstate the effect of outliers [19]. The estimated values of $\gamma$ shown in Table I are all greater than 1. By noting that greater $\gamma$ means more activities would be generated by a given population, we can regard $\gamma$ as an indicator of productivity and compare the productivity between online and off-line systems. Off-line activities, including wealth creation and patent invention, have been found to scale linearly with population, $\gamma$ estimated to be be-

FIG. 1: (Color online) Accelerating growth in human online activities. Different datasets are marked in points of different colors and shapes (green squares for Flickr, orange diamonds for Delicious, red circles for Jiwai, and blue triangles for Digg). The x axis shows the active population in a day and the y axis shows the total activity in the day. Both axes have a logarithmic scale. The orthogonal regression lines are also shown.
B. The size-dependent DGBD model

In fitting the daily distributions of individual activities with the DGBD, we find that the parameter $A$ changes with population size. To confirm this empirical finding, we analyze the DGBD function and find that the relation-

data exhibiting a long tail [13, 16, 17, 22], is that in our data, the rank curves in the rank-ordered plots (individual activity vs. rank) are not perfectly straight lines. The empirical curves deviate from the straight line predicted by Zipf’s law at the right tails (Fig.2), and hence estimations based on Zipf’s law will be biased.

FIG. 2: (Color online) Three examples of daily distribution of individual activities: Flickr (a), Digg (b), Delicious (c), and Jiwai (d). Different colors and shapes of the data points indicate different days. The y axis shows the individual activities and the x axis shows the decreasing ranks of the activities. Both axes have a logarithmic scale. The rescaled form of example distributions and the theoretical curves predicted by the size-dependent DGBD model are shown in semi-log plots (in which the y axis has a logarithmic scale) in the insets.
TABLE II: Estimations of the size-dependent DGBD models.

| Dataset | b  | a  | Adjusted R² | N of Days |
|---------|----|----|-------------|-----------|
| Flickr  | 0.54 | 0.97 | 0.97 | 193 |
| Jiwai  | 0.04 | 0.90 | 0.96 | 21  |
| Digg   | 0.06 | 0.85 | 0.94 | 31  |
| Delicious   | 0.15 | 0.91 | 0.94 | 663 |

ship between A and P can be derived as follows. As \( r_{\text{max}} \) equals \( P \) and the minimum value of individual activities \( t(P) \) is 1 by construction (because we only consider active users), we derive the boundary condition

\[
t(r_{\text{max}}) = t(P) = 1. \tag{3}
\]

Substituting Eq. (3) into Eq. (2) gives

\[
A = P^a. \tag{4}
\]

Let \( k = r/(P+1) \). Obviously \( k \in (0, 1) \) and \( 1-k \in (0, 1) \). By replacing \( r \) in Eq. (2) with \( k \) we get

\[
t(k) = P^a(P+1)^{b-a}(1-k)^b k^{-a}. \tag{5}
\]

As in the data \( P \gg 1 \), namely, \( P+1 \approx P \), we can rewrite Eq. (5) as

\[
t(k) \approx P^b(1-k)^b k^{-a}, \tag{6}
\]

or

\[
\ln(t(k)) \approx b \ln(P(1-k)) - a \ln(k). \tag{7}
\]

Eq. (6) controls the variance of system size by replacing \( A \) with a scaling factor \( P^b \), as well as normalizing rank \( r \) into \( k \). We refer to Eq. (6) as the size-dependent DGBD model and estimate its parameters \( a \) and \( b \) by ordinary least squares regression (Table II). The large adjusted \( R^2 \) is evidence that the size-dependent DGBD model captures the dynamic properties of human online activities very well. Another way to validate the size-dependent DGBD model is to plot \( t(k)/P^b \) vs. \( k = r/(P+1) \) in different days and check whether the relationship Eq. (6) rises from data. The insets in Figure 2 shows that empirical distributions in different days collapse to the same theoretical curves predicted by Eq. (6), thus our deduction is empirically supported.

We can derive the probability density function \( f(x) \) of individual activity from the size-dependent function Eq. (6) as follows. We know that the cumulative function \( C(x) = P r\{X > x\} \) of the activity is the inverse function of the rank-activity curve \( t(k) \), namely,

\[
C(x) = t^{-1}(x) = h\left(\frac{x}{P^b}\right), \tag{8}
\]

where function \( h(x) \) is the inverse function of \((1-k)^b k^{-a}, \)

that is:

\[
h^{-1}(k) = (1-k)^b k^{-a}. \tag{9}
\]

The probability density function \( f(x) \) can be derived from Eq. (3) as

\[
f(x) = -\partial C(x)/\partial x = \frac{1}{P^b h'(x/P^b)}. \tag{10}
\]

Setting \( g(x) = -h'(x) \) in Eq. (10) gives a generalized form of the probability density function:

\[
f(x) = \frac{1}{P^b g(x/P^b)}, \tag{11}
\]

which has already been found in off-line systems such as the stock market and the income distribution [13, 23, 24].

C. From the size-dependent DGBD model to accelerating growth

Accelerating growth can be derived from the size-dependent DGBD model as follows. The integration of all user activities \( t(r) \) is total activity \( T \), that is,

\[
T = \int_{r_{\text{min}}}^{r_{\text{max}}} t(r) \, dr
= (P+1) \int_0^1 t(k) \, dk
\approx P^{b+1} \int_0^1 (1-k)^b k^{-a} \, dk. \tag{12}
\]

Using Euler integration, we can rewrite Eq. (12) as

\[
T \approx P^{b+1} \frac{\Gamma(1-a)\Gamma(1+b)}{\Gamma(2-a+b)} \tag{13}
\]

where \( \Gamma \) is the gamma function. As \( b \) and \( a \) are constants, according to the definition of the gamma function, we can replace \( \frac{\Gamma(1-a)\Gamma(1+b)}{\Gamma(2-a+b)} \) with a constant \( C \) and further rewrite Eq. (13) as

\[
T \approx CP^{b+1}. \tag{14}
\]

Eq. (14) is the accelerating growth relationship. By comparing Eq. (11) with Eq. (14), it is apparent that

\[
\gamma \approx b + 1. \tag{15}
\]

Note that the value of \( \gamma \) only relates to \( b \). As mentioned above, Eq. (2) becomes Zipf’s law when \( b = 0 \). Therefore Zipf’s law leads to \( \gamma = 1 \), namely, linear increase of total activity with the growth of population. Moreover, as it is the size-dependent parameter \( P^b \) in Eq. (5) that leads to accelerating growth, any distribution independent of system size, including power law distribution predicted by the BA model [1], can not result in accelerating growth.
TABLE III: The comparison between theoretical and empirical values of $\gamma$.

| Dataset | $\gamma$ within 95% CI | $\gamma'$ | $b$ | $a$ |
|---------|------------------------|----------|-----|-----|
| Flickr  | [1.43, 1.54]           | 1.64     | 0.54| 0.97|
| Jiwai   | [1.03, 1.48]           | 1.04     | 0.04| 0.90|
| Digg    | [1.06, 1.22]           | 1.06     | 0.06| 0.85|
| Delicious| [1.15, 1.19]          | 1.15     | 0.15| 0.91|

As [17] reports the finding of the DGBD with a parameter $b > 0$ in various empirical data sets, it is reasonable to conjecture the wide existence of accelerating growth, as we have shown that $\gamma = b + 1 > 1$.

To validate Eq. (15), we can compare the theoretical and empirical values of $\gamma$. Table III shows $\gamma$ estimated from empirical data (from Table II) and $\gamma'$ that is the theoretical value of $\gamma$ predicted by $b$ (from Table II). It is observed that the values of $\gamma$ and $\gamma'$ are consistent with each other: all $\gamma'$ fall into the 95% CI of $\gamma$. Therefore our analytical deduction of the relationship between size-dependent distribution and accelerating growth is justified.

It should be noted that the generalized form of the size-dependent probability density function Eq. (13) is a sufficient condition of accelerating growth, meaning that if we replace $g(x)$ with other functions, we can still obtain the power law relationship between system size $P$ and total activity $T$ [12]. This finding is consistent with our previous study on income distributions of countries [13].

IV. CONCLUSIONS

In this paper, we discuss accelerating growth in human online activities, that is, a power law relationship between total activity and active population with an exponent greater than 1. The power law relationship is found to be ubiquitous across different types of human online behaviors. We show analytically how size-dependent distribution relates to accelerating growth, and validate our deduction using several large data sets containing millions of human online activity records.

The major theoretical contribution of this paper is the finding that size-dependent distribution relates to accelerating growth quantitatively. Although our study is based on human online activities, this quantitative relationship is not necessarily confined to an online context. The model may also be used to explain accelerating growth patterns in off-line social systems such as cities [9] and countries [12, 13].

Beside the theoretical contribution, the model of size-dependent distribution has potential applications, e.g., in web crawling and website management. For example, with historical data on individual activities, webmasters can estimate the value of $b$ and predict the accelerating growth rate $\gamma$ of total activity, which may help webmasters plan the capacity of web server accordingly. Webmasters can also compare the values of $\gamma$ among websites with equivalent functions, leading to an innovative theoretically informed approach of benchmarking.

It should be noted that our findings appear to contradict conclusions of previous studies. For example, [3, 7, 18, 19] suggest that the exponent $\gamma$ in accelerating growth is determined by the exponent $\alpha$ in Zipf’s law, but our analysis suggests that a power law distribution that is independent of system size will not lead to accelerating growth. We conjecture this contradiction may be due to an unknown relationship between the parameters $\alpha$ (which, as mentioned, corresponds to the exponent $\alpha$ in Zipf’s law) and $b$ in the DGBD model, since we have shown that $\gamma = b + 1$. The unknown relationship between $\alpha$ and $b$, together with other unsolved problems such as the behavioral origins of size-dependent distributions, call for further exploration.

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