SIMPLICIAL AND CONICAL DECOMPOSITION OF
POSITIVELY SPANNING SETS

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Abstract. We investigate the decomposition of a set $X$, which positively spans the Euclidean space $\mathbb{R}^d$ into a set of minimal positive bases, we call simplices, and into maximal sets positively spanning pointed cones, i.e. cones with exactly one apex. For any set $X$, let $\mathcal{S}(X)$ denote the set of simplex subsets of $X$, and let $\ell(X)$ denotes the linear hull of $X$. The set $X$ is said to fulfill the factorisation condition if and only if for each subset $Y \subset X$ and each simplex $S \in \mathcal{S}(X)$, $\ell(Y) \cap \ell(S) = \ell(Y \cap S)$.

We demonstrate that $X$ is a positive basis if and only if it is the union of most $d$ simplices, and $X$ satisfies the factorization condition. In this case, $X$ contains a linear basis $B$ such that each simplex in $\mathcal{S}(X)$ has with $B$, all but one exactly one element in common. We show that for sets positively spanning $\mathbb{R}^d$, the set of subbases of $X$ forms a boolean lattice, which can be embedded into the set $2^{\mathcal{S}(X)}$, with isomorphy for positive bases.

Our second main result depending on the former is as follows. A finite set $X \subset \mathbb{R}^d \setminus \{0\}$ can be written as the union of at most $2^d$ maximal sets spanning pointed cones, which, if $X$ is a positive basis, are tantamount to frames of the cones. The inequality holds sharply if and only if $X$ is a cross, that is, a union of 1-simplices derived from a linear basis of $\mathbb{R}^d$.

We also show that there can be at the most $2^d$ maximal subsets of $X$ spanning pointed cones, when intersections of two of them do not span a set of full dimension.

1. Introduction

Since Chandler Davis’s seminal paper [2], analysis of the structure of sets of vectors positively generating solid cones (i.e. linear spaces) has gained attention. The attempts to understand the structure of such sets has been focused on positive bases, which are minimal positive generating sets. These approaches involved the decomposition into disjointed subsets [10], analysis through the Gale transformation and related techniques [3, 4], and generation through certain kind of matrices [2]. The latter results include the only characterizations of positive bases; however technical, and beyond the obvious equivalence of the minimal generating sets being positively independent.

Our approach is novel as far as it focuses on positive linear relations, equations yielding zero, instead of linear relations. We identified a simple and intuitive characterization of positive bases (theorem 3). Building on this,
we show that each positive spanning set (not only bases) can be decomposed into not greater than $2^d$ sets that positively span pointed cones.

**Notation.** All spaces $\mathbb{R}^d$ are Euclidean. By $\ell(X)$ and $\varphi(X)$, we denote the linear and positive span of $X$, respectively, i.e. the set of linear combinations of the following form:

$$a_1x_1 + \cdots + a_kx_k, \ x_1, \ldots, x_k \in X$$

with real and non-negative coefficients $a_1, \ldots, a_n$, respectively. By convention, the empty set spans, positively and linearly, the null space $\{0\}$. For general properties of $\varphi(\cdot)$ see [7]. A positive spanning set (PSS) is any set $X$ that positively spans a linear space [4]. A positive basis is a PSS with no proper subset that spans the same space. $|X|$ denotes the cardinality of $X$.

For $A \subset \mathbb{R}^d$, $\text{rint}(A)$ is the interior relative to the affine space spanned by $A$.

**Definition 1.** A set $X \subset \mathbb{R}^d$ is called linearly / positively / negatively dependent, if and only if for some $x \in X$ we have the relation $x \in \ell(X \setminus \{x\}) / x \in \varphi(X \setminus \{x\}) / -x \in \varphi(X \setminus \{x\})$, respectively. It is called linearly / positively / negatively independent, if it is not dependent.

The upper definition of linear dependence is obviously equivalent to the standard definition. Linear independence implies positive and negative independent, but the converse does not hold save in dimension 2 (lemma 9 and example 10). Negative independence of a set means that the positive cone generated by this set is pointed with 0 as the only apex; another equivalent term is strictly one-sided [1, p. 112] (see lemma 7).

**Definition 2.** A set $S \subset \mathbb{R}^d$ is called a simplex basis (or simplex, for short) if and only if $0 \in \varphi(S)$, and no proper subset has this property. For a set $X$, $S(X)$ denote the set of all simplices contained in $X$.

In the literature, the term minimal basis refers to the simplex basis [2]; However, this notation may lead yo confusion, because the positive basis itself is a minimal positively spanning set. The term is justified as a simplex basis and can be understood as the vertices of a geometrical simplex with its origin in its relative interior. For example, a 1-simplex consists of two opposite vectors $x, -\alpha x$, $x \neq 0$, $\alpha > 0$.

The opposite extreme of a basis is the cross. This is generated from a linear basis $B$ of the space by adding a negative multiple, forming a 1-simplex, to each element of the basis. The convex hull of these simplices all intersect at the origin. While the cardinality of a simplex spanning $\mathbb{R}^d$ is always $d + 1$, the cardinality of a cross spanning $\mathbb{R}^d$ is $2d$ [2]. It is well know that these are unique instances that yield sharp boundaries for the cardinality of positive bases [2, 5, 7].

It is easy to see that each positively spanning set is the union of its simplices (theorem 13). For a positive basis, these simplices overlap in common linear subspaces, as the following theorem shows.

**Theorem 3.** The following statements are equivalent for a set $X$ positively spanning $\mathbb{R}^d$:

(i) $X$ is positively independent.
(ii) For all subsets $Y \subset X$ and all simplices $S \in \mathcal{S}(X)$

$$\ell(Y) \cap \ell(S) = \ell(Y \cap S).$$

(iii) For all positively spanning subsets $Y \subset X$ and all simplices $S \in \mathcal{S}(X)$

$$\ell(Y) \cap \ell(S) = \ell(Y \cap S).$$

(iv) We can write $X = B \cup \{x_1, \ldots, x_n\}$, $1 \leq n \leq d$, with a basis $B$ of $\ell(X)$, such that each $x_i \in -\operatorname{rint}_\varphi(A_i)$ for some $A_i \subset B$, and $A_i \not\subseteq A_j$ for $i \neq j$. Moreover, each simplex $S \in \mathcal{S}(X)$ has all but one element in common with $B$.

Following Reay [6], the Bonnice-Klee theorem can be derived from a corollary to condition (iv) (corollary 22). Our first main result is the previous theorem along with the following: We show that for any positively spanning set $X$, the set of subsets positively generating linear subspaces form a boolean lattice isomorphic to a sublattice of $2^{\mathcal{S}(X)}$, with isomorphy if $X$ is a positive basis (theorem 14 and corollary 23).

The second main result emanates from the following theorem.

**Definition 4.** By $\mathcal{M}(X)$, we denote the set of all maximal negatively independent subsets of $X$.

**Theorem 5.** Let $X$ be a positive basis of $\mathbb{R}^d$. Then $X$ can be written as the union of $n$ simplices such that the following inequalities hold

$$1 \leq n \leq d,$$

$$d + 1 \leq |X| \leq 2d,$$

$$d + 1 \leq |\mathcal{M}(X)| \leq 2^d,$$

with equality in each of the upper inequalities (and therefore in all) if and only if $X$ is a cross, and equality in one or all of the lower equations if and only if $X$ is a simplex.

There are two extensions for the last inequality. We show that each positively spanning set can be decomposed into not more than $2^d$ sets that positively span pointed cones (theorem 29). Our last theorem (and second main result) extends the last inequality to the case when $X$ is not necessarily a positive basis. Another way to state this is that there can be no more than $2^d$ maximally pointed $d$-cones in $\mathbb{R}^d$ (maximally pointed in the sense that no element of the frame of another cone can be added to it and keep it pointed), such that two cones overlap only at the boundary.

**Theorem 6.** Let $X$ positively span $\mathbb{R}^d$, and $\mathcal{A} \subset \mathcal{M}(X)$ such that for all $A, B \in \mathcal{A}$, $A \neq B$, $\varphi(A \cap B)$ does not have full dimension. Then $|\mathcal{A}| \leq 2^d$, with equality if and only if $X$ is a union of at least $d$ 1-simplices, some $d$ of which form a cross.

2. Preliminaries

More notation. For any set $A$ we write $A[x \rightarrow y]$ for $(A \setminus \{x\}) \cup \{y\}$. Let $\operatorname{conv}(X)$ denote the convex hull of $X$.

**Lemma 7.** For a set $X \subset \mathbb{R}^d$ the following is equivalent.
(i): $X$ is negatively independent,
(ii): $S(X) = \emptyset$,
(iii): There is a $z$ with $z \cdot x > 0$ for all $x \in X$.

Proof. “(i)$\Rightarrow$(ii)”: Let $S \in S(X)$ with $S = \{x_1, \ldots, x_n\}$. Then there are $\alpha_i \geq 0$, not all zero, with $\sum_{i=1}^n \alpha_i x_i = 0$. Hence $x_k = -\sum_{i=1}^n (\alpha_i/\alpha_k) x_k$ for one $k$. Thus $X$ is negatively dependent.

“(ii)$\Rightarrow$(iii)”: Assume that for all $z$ there is an $x \in X$ with $z \cdot x \leq 0$. Let $C = \varphi(X)$. If $C$ and $-C$ intersected only in 0, then $C \setminus \{0\}$ and $-C \setminus \{0\}$ could be strictly separated by a hyperplane through the origin. Let $z$ be the normal vector orthogonal to the hyperplane lying in the half space containing $C$, then $z \cdot x > 0$ for all $x \in C \setminus \{0\}$, a contradiction to the assumption. Thus $C$ and $-C$ contain a common nonzero vector $x$. There are $x_1, \ldots, x_m \in X$ and $\alpha_1, \ldots, \alpha_m > 0$ such that

$$x = \sum_{i=1}^k \alpha_i x_i, \text{ and } -x = \sum_{i=k+1}^m \alpha_i x_i,$$

and $\sum_{i=1}^m \alpha_i x_i = 0$. Thus $\{x_1, \ldots, x_m\}$ contain a minimal set with this property, which forms a simplex $S \in S(X)$.

“(iii)$\Rightarrow$(i)”: Assume $z \cdot x > 0$ for all $x \in X$ for some $z \in \mathbb{R}^d \setminus \{0\}$. Assume further $X$ is negatively dependent. Then there is $x_0, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n > 0$ with $x_0 = -\sum_{i=1}^n \alpha_i x_i$. But then, $z \cdot x_0 > 0$ and $z \cdot \sum_{i=1}^n \alpha_i x_i > 0$, a contradiction. Hence $X$ is negatively independent. \[\square\]

The following result is trivial.

Remark 8. Let $X = \{x_1, \ldots, x_m\}$ with $m \leq d = \dim \varphi(X)$. Then $X$ is a linear basis of $\ell(X)$.

Proof. We choose a maximal linear independent subset $B \subset X$. Then $\ell(B) = \ell(X)$, and $\dim \ell(B) \geq d$. But this requires $|B| \geq d$, hence $B = X$ and $m = d$. \[\square\]

Linearly independent sets are positively and negatively independent. The converse holds only in two dimensions, as the following two results show.

Lemma 9. Let $X \subset \mathbb{R}^2 \setminus \{0\}$. Then $X$ is linearly independent if and only if it is both positively and negatively independent.

Proof. Only sufficiency has to be shown. Let us first assume that $X = \{x_1, \ldots, x_n\}$ is positively and negatively independent. We have to show that $X$ is linearly independent. For $n = 1$ this is trivial. For $n \geq 2$, this means $x_1 \neq ax_2$ and $x_1 \neq -ax_2$ for all $\alpha > 0$, as $x_1, x_2 \neq 0$, hence $x_1 \neq ax_2$ for any $a \in \mathbb{R}$, and $\{x_1, x_2\}$ are linearly independent and span $\mathbb{R}^2$. For $n \geq 3$ we therefore have $x_3 = ax_1 + bx_2$ for $a, b$ not both zero. If $a, b \geq 0$, then $X$ positively dependent. If $a, b \leq 0$, then $X$ is negatively dependent. If one coefficient is positive, and one negative, say $a > 0$ and $b < 0$, then $x_1 = \frac{a}{b} x_3 + \frac{b}{a} x_2$, and $X$ is positively dependent. Hence $n \leq 2$ and $X$ is linearly independent. This also covers the infinite case. \[\square\]

We only need a counterexample in three dimensions.
Example 10. Let $e_1, e_2, e_3$ be the standard basis in $\mathbb{R}^3$ and $z = e_1 + e_2 - e_3$. Then $X = \{e_1, e_2, e_3, z\}$ is both positively and negatively independent, but linearly dependent.

Proof. We observe that for $i \neq j$, $\{e_i, e_j, z\}$ is a linear basis of $\mathbb{R}^3$. As all elements of $X$ are linear combinations of the others,

$$
e_1 = z - e_2 + e_3, \\
e_2 = z - e_1 + e_2, \\
e_3 = z - e_1 - e_2, \\
z = e_1 + e_2 - e_3,$$

with both positive and negative coefficients, $X$ is both negatively and positively independent. \(\square\)

We show Caratheodory’s theorem in a version for positive linear combinations in cones.

Lemma 11. Let $X \subset \mathbb{R}^d$ and $x \in \wp(X)$. Then there are $x_1, \ldots, x_n \in X$ with $n \leq d$ and $\alpha_i > 0$ such that

$$x = \sum_{i=1}^n \alpha_i x_i$$

Proof. Let $x \in \wp(X)$, such that $x = \sum_{i=1}^n \alpha_i x_i$ for $x_i \in X$ and, without loss of generality, $\alpha_i > 0$. The set $B = \{x_1, \ldots, x_n\}$ can be chosen negatively independent: If $x_k = - \sum_{i \neq k} \beta_i x_i$, letting $\beta_k = 1$ and $I = \{i \mid \beta_i > 0\}$, select an $l \in J$ with $\gamma := \alpha_l / \beta_l \leq \alpha_i / \beta_i$ for all $i \in I$. Then $\sum_{i \in I} \beta_i x_i = 0$ and letting $\beta_i = 0$ for $i \notin I$, we obtain

$$x = x - \sum_{i \notin I} \beta_i x_i = \sum_{i=1}^n \left( \alpha_i - \frac{\alpha_l}{\beta_l} \beta_i \right) x_i = \sum_{i \neq l} \left( \alpha_i - \frac{\alpha_l}{\beta_l} \beta_i \right) x_i,$$

where the coefficients are $\alpha_i - (\alpha_l / \beta_l) \beta_i \geq \alpha_i - (\alpha_i / \beta_i) \beta_i = 0$.

By lemma there is a $z \neq 0$ with $z \cdot x_i > 0$. By $H = \{y \in \mathbb{R}^d \mid z \cdot y = z \cdot x\}$, we define a hyperplane such that for each $i = 1, \ldots, n$ there is a $y_i > 0$ with $y_i = \gamma_i x_i \in H$. Since $x \in H$, there are now $\beta_i = \alpha_i / \gamma_i > 0$ with $x = \sum_{i=1}^n \beta_i y_i$. Since $x \in \wp(B) \cap H = \text{conv}(\{y_1, \ldots, y_n\})$, by Caratheodory’s theorem, a subset of $d$ elements of $\{y_1, \ldots, y_n\}$ suffice to have $x$ in its convex hull. Hence $n \leq d$ can be chosen. \(\square\)

3. Simplicial Decompositions and Gale Diagrams

Lemma 12. The following statements are equivalent.

(i): $S$ is a simplex.

(ii): For all $z \in S$, $B_z = S \setminus \{z\}$ is linearly independent with $z \in -\text{rint}(\wp(B_z))$.

(iii): There is a $z \in S$ such that $B_z = S \setminus \{z\}$ is linearly independent, and $z \in -\text{rint}(\wp(B_z))$.

(iv): $S$ is a minimal set with $\wp(S) = \ell(S)$.
Proof. “(i)⇒(ii)”: Let $S = \{x_0, \ldots, x_m\}$ be a simplex. Then, by definition, there are $\alpha_i > 0$ with $\sum_{i=0}^m \alpha_i x_i = 0$. Fix any $x_k$ and set $B_k = S \setminus \{x_k\}$. Then

$$-x_k = \sum_{i \neq k} \frac{\alpha_i}{\alpha_k} x_i \in \text{rint} (\varphi(B_k)).$$

We show that $B_k$ is linearly independent by induction over $m = |B_k|$. For $m = 1$, which is trivial. We assume now the proposition has been shown for $m-1$. Choose $x_i \in B_k$ and set $B_{kl} = B_k \setminus \{x_i\}$. By induction hypothesis, $B_{kl}$ is linearly independent. To show that $B_k$ is linearly independent, it is sufficient to demonstrate that $x_i \notin \ell(B_{kl})$. Assume the contrary, that is, $x_i \in \ell(B_{kl}) = \ell(B_k)$. With $-x_k \in \varphi(B_k) \subset \ell(B_k) = \ell(B_{kl})$, we find $S \subset \ell(B_{kl})$. Since $0 \notin \varphi(S)$ with $m-1 = |B_{kl}| = \dim \ell(B_{kl})$, by Caratheodory’s theorem for cones (lemma 11), either $0 \notin \varphi(B_{kl} \cup \{x_i\})$, or $0 \in \varphi(B_{kl} \cup \{x_i\})$. In either case, $S$ contains a proper subset with zero in its positive span, a contradiction to $S$ being a simplex.

“(ii)⇒(iii)”: Trivial.

“(iii)⇒(ii)”: Assume, without loss, that $x_0 \in S = \{x_0, \ldots, x_m\}$ with $x_0 \in -\text{rint} (\varphi(B))$, and $B = \{x_1, \ldots, x_m\}$ is linearly independent. Thus,

$$-x_0 = \sum_{i=1}^m \alpha_i x_i$$

is a unique linear combination and $\alpha_i > 0$. Fix any $x_k \in B$ and set $B' = B \cup \{x_k \rightarrow x_0\}$. If $B'$ is linearly dependent, then $x_0 \in \ell(B' \setminus \{x_0\})$, and further $x_0 = \sum_{i \neq k} \alpha_i x_i$, which implies $\alpha_k = 0$, a contradiction. Moreover,

$$x_0 + \alpha_k x_k + \sum_{1 \leq i \neq k} \alpha_i x_i = 0.$$

Setting $\alpha_0 = 1$ we find that

$$-x_k = \sum_{0 \leq i \neq k} \alpha_i x_i \in \text{rint} \varphi(B'),$$

which proves the assertion.

“(ii)⇒(iv)”: Assume that for any $z \in S$, $B_z = S \setminus \{z\}$ is linearly independent with $z \in -\text{rint} (\varphi(B_z))$. 

The main proposition of the following theorem with positively spanning sets are the sums of simplices is not new; however, can be obtained from Blaschke’s Theorem on decomposition of polytopes into simplices [3, Ch 15.3]. The additional requirement by Minkowski’s Theorem of equilibration of the vectors in a positively spanning set (summing up to zero) that indicates no loss of generality for our result.

**Theorem 13.** For a finite set $X \subset \mathbb{R}^d$, the following statements are equivalent.

(i): $X$ is a positively spanning set ($\varphi(X) = \ell(X)$).

(ii): $\varphi(X) = -\varphi(X)$.

(iii): $X = \bigcup S(X)$. 

Proof. “(i)⇒(ii)”: Immediate, as \( \varphi (X) = \ell (X) = -x = -\varphi (X) \).

“(ii)⇒(iii)”: Choose \( x \in X \). As \( x \in \varphi (X) \), by assumption, also \( -x \in \varphi (X) \). Hence \( 0 \in \varphi (X) \). Then there is a minimal subset \( S \subset X \) with \( x \in S \) and \( 0 \in \varphi (S) \). By definition of the simplex, \( S \in \mathcal{S}(X) \).

“(iii)⇒(i)”: Let \( x \in \ell (X) \), \( x = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \sgn (a_i) x_i \) with \( x_i \in X \). If \( \sgn (a_i) = -1 \), it is sufficient to show that \( -x_i \in \varphi (X) \) to conclude that \( x \in \varphi (X) \). By assumption, each \( x_i \) is contained in a simplex \( S_i \). Thus lemma 12 assures that \( -x_i \in \ell (S_i) = \varphi (S_i) \subset \varphi (X) \). □

Further characterizations can be found in [7]. Examples of decomposition of positive bases into simplices can be found in [8, Ch 2]. The simplices induce a lattice structure given by the next theorem.

**Theorem 14.** For any positively spanning set \( X \) of a linear space \( \mathbb{R}^d \), the set \( \mathcal{L}(X) \) of subsets positively spanning linear spaces form a boolean lattice by set inclusion. If, moreover, \( X \) is a positive basis, the lattice \( \mathcal{L}(X) \) is isomorphic to \( 2^{\mathcal{S}(X)} \).

**Proof.** Let \( X \) be a set positively spanning \( \mathbb{R}^d \), and, denoted by \( \mathcal{L} = \mathcal{L}(X) \), the set of subsets positively spanning a linear subspace of \( \mathbb{R}^d \). By convention, the empty set spanning the null space is included. We introduce the following order on \( \mathcal{L} \).

\[
Y \sqsubset Z \iff \mathcal{S}(Y) \subset \mathcal{S}(Z).
\]

Our goal is to demonstrate that \( \mathcal{S} : \mathcal{L} \to 2^{2^{\mathcal{S}(X)}} \) is a injective homomorphism into a sublattice of \( 2^{2^{\mathcal{S}(X)}} \). First, we show that for any \( \mathcal{S}^* \subset \mathcal{S}(X) \), \( Y = \bigcup \mathcal{S}^* \) is a positively spanning set. By theorem 13, \( Y \) is a positively spanning set if and only if \( Y = \bigcup \mathcal{S}(Y) \). Since \( \mathcal{S}^* \subset \mathcal{S}(Y) \), we find that \( Y = \bigcup \mathcal{S}^* \subset \bigcup \mathcal{S}(Y) \subset Y \), i.e. what needed to be shown.

Second, we show that \( \mathcal{S} \) is injective. Take any \( Y, Z \in \mathcal{L} \) with \( \mathcal{S}(Y) = \mathcal{S}(Z) \). It is immediate that \( Y \sqsubset X \) implies \( \mathcal{S}(Y) \subset \mathcal{S}(X) \); hence, \( \mathcal{S}(Y) \) and \( \mathcal{S}(Z) \) are subsets of \( \mathcal{S}(X) \). From the previous result, we obtain

\[
Y = \bigcup \mathcal{S}(Y) = \bigcup \mathcal{S}(Z) = Z.
\]

It is easy to show that \( \mathcal{S}(Y \cap Z) = \mathcal{S}(Y) \cap \mathcal{S}(Z) \) and \( \mathcal{S}(Y) \cup \mathcal{S}(Z) \subset \mathcal{S}(Y \cup Z) \). From these relations and the monotonicity of \( \mathcal{S} \), we find

\[
Y \sqsubset Z \iff Y \subset Z,
\]

\[
Y \cap Z = \bigcup \mathcal{S}(Y \cap Z),
\]

\[
Y \cup Z = \bigcup \mathcal{S}(Y \cup Z),
\]

where, as usual, \( \cap \) and \( \cup \) denote infimum and supremum, respectively. The last equation is derived as follows:

\[
Y \cup Z = \bigcup (\mathcal{S}(Y) \cup \mathcal{S}(Z)) \subset \bigcup \mathcal{S}(Y \cup Z) \subset Y \cup Z
\]

\[
= \bigcup \mathcal{S}(Y) \cup \bigcup \mathcal{S}(Z) = \bigcup (\mathcal{S}(Y) \cup \mathcal{S}(Z)).
\]

The complement is given by

\[
Y' = \bigcup (\mathcal{S}(X) \setminus \mathcal{S}(Y)).
\]
From the above result, \( Y' \) is well-defined as a positively spanning set and element of \( \mathcal{L} \). Altogether, we have shown that \( \mathcal{S} \) is a homomorphism of \( \mathcal{L} \) ordered by \( \sqsubseteq \) with complement \( \cdot' \) into a boolean sublattice of \( 2^{\mathcal{S}(X)} \).

The last statement will be proven later. Corollary 23 will assure that for a positive basis \( X \), the lattice \( \mathcal{L}(X) \) is actually isomorphic to \( 2^{\mathcal{S}(X)} \). 

In the remaining part we study the relation between the simplicial decomposition of a positive spanning set and its Gale Diagram following the literature \([4, 8]\). The definition of a Gale Diagram differs slightly in the literature up to a convention, but all definitions are essentially interdefinable. The following definition is compatible with all conventions. We show that the structure given by the simplices \( \mathcal{S}(X) \) and their intersections on one side, and the Gale Diagrams on the other side, coincide in a very special case, which has a simple characterisation in terms of the simplices. All examples in \([8, \text{Ch 2}]\) are of this type.

**Definition 15.** Let \( X \subset \mathbb{R}^d \setminus \{0\} \) be a finite set of vectors. A dependency for \( X \) is a function \( v : X \to \mathbb{R} \) such that

\[
\sum_{x \in X} v(x) \cdot x = 0.
\]

Let \( \mathcal{D}(X) \) denote the linear space of dependencies of \( X \), and \( \mathcal{P}(X) \) the convex cone of non-negative dependencies \( v \geq 0 \). \( X \) is said to be equilibrated if and only if it has a non-zero constant function as a dependency. \( X \) is called locally equilibrated if and only if every simplex \( S \in \mathcal{S}(X) \) is equilibrated.

The Gale Diagram \( \hat{x} \) for \( x \in X \) is defined as follows. Select a linear basis \( v_1, \ldots, v_n \) of \( \mathcal{D}(X) \). The Gale Transform of \( x \) is the ray \( \alpha \cdot (v_1(x), \ldots, v_n(x)) \) for \( \alpha \geq 0 \) in \( \mathbb{R}^n \). The Gale Diagram is the unique point of the Gale Transform intersecting some given \( n-1 \)-dimensional hypersurface in \( \mathbb{R}^n \). If the intersection is empty, then the Gale Diagram of \( x \) is 0.

Observe that for positive spanning sets, the Gale Transform of a non-zero element can not be zero, as by theorem 16 every element is contained in a simplex, which corresponds to a non-negative dependency.

**Theorem 16.** Let \( X \subset \mathbb{R}^d \setminus \{0\} \) be a finite set of vectors positively spanning \( \mathbb{R}^d \). Then the dependency space \( \mathcal{D}(X) \) has a linear basis in \( \mathcal{P}(X) \). Moreover, \( X \) is locally equilibrated if and only if the following holds: Any \( x, y \in X \) have the same Gale Diagram if and only if they are contained in the same simplices from \( \mathcal{S}(X) \).

**Proof.** For the first proposition it is sufficient to show that \( \mathcal{D}(X) \subset \ell(\mathcal{P}(X)) \), then there is a linear basis of \( \mathcal{D}(X) \) contained in \( \mathcal{P}(X) \). Let \( v \in \mathcal{D}(X) \). If \( v(x) \geq 0 \) for all \( x \in X \), then the proof is complete. Otherwise, let \( x_k \in X \) be with \( v(x_k) < 0 \). By theorem ..., there is an \( S \in \mathcal{S}(X) \) with \( x_k \in S = \{x_1, \ldots, x_m\} \). Hence there are \( \alpha_1, \ldots, \alpha_m > 0 \) with \( \sum_{i=1}^m \alpha_i \cdot x_i = 0 \). Without loss of generality, we chose \( \alpha_k = -v(x_k) \). Then there is a \( v_1 \in \mathcal{P}(X) \) with \( v_1(x_i) = \alpha_i \) and \( v_1(x) = 0 \) otherwise. Then \( v' = v + v_1 \in \mathcal{D}(X) \) has \( v'(x) < 0 \) only if \( v(x) < 0 \), and \( v'(x_k) = 0 \). We repeat this procedure until we end with \( v' \in \mathcal{P}(X) \). We have shown that \( v + \sum_{i=1}^n v_i \in \mathcal{P}(X) \), hence \( v \in \ell(\mathcal{P}(X)) \).
If $X$ is locally equilibrated, then for each simplex $S \in \mathcal{S}(X)$ the cone $\mathcal{P}(X)$ contains the characteristic function $\chi_S$ of $S$. From the result above and theorem 13 we find that the characteristic functions of simplices positively span $\mathcal{P}(X)$. Thus, there is a linear basis of $\mathcal{D}(X)$ consisting only of characteristic functions $\chi_{S_1}, \ldots, \chi_{S_n}$ of simplices $S_1, \ldots, S_n \in \mathcal{S}(X)$, which we choose to construct the Gale Transforms. Then, for any $S \in \mathcal{S}(X)$, $\chi_S$ is a positive linear combination of the elements of the basis. Then, clearly, whenever $x \in S \iff y \in S$ for all $S \in \mathcal{S}(X)$ for $x, y \in X$, then also $\chi_{S_i}(x) = \chi_{S_i}(y)$ for $i = 1, \ldots, n$, and the Gale Transform of $x$ and $y$ coincides. Conversely, if $x$ and $y$ have the same Gale transform, then $\chi_{S_i}(x) = \chi_{S_i}(y)$ for $i = 1, \ldots, n$, and $x \in S \iff y \in S$ for all $S \in \mathcal{S}(X)$, which proofs the assertion.

In order to show the converse, first note that whenever $x, y \in X$ have identical Gale Transform, then there is an $\alpha > 0$ such that $v(x) = \alpha \cdot v(y)$ for each $v \in \mathcal{P}(X)$. For each simplex $S \in \mathcal{S}(X)$ there is a dependency $v_S \in \mathcal{P}(X)$ with $v_S(z) > 0$ if and only if $z \in S$. This yields $x \in S \iff y \in S$ for all $S \in \mathcal{S}(X)$. Now, assume the latter equivalence holds for some $x, y \in X$, but their Gale Transform differs. Then there are simplices $S_1, S_2 \in \mathcal{S}(X)$ such that for all $\alpha > 0$

$$\begin{pmatrix} v_{S_1}(x) \\ v_{S_2}(x) \end{pmatrix} \neq \alpha \cdot \begin{pmatrix} v_{S_1}(y) \\ v_{S_2}(y) \end{pmatrix}.$$  

But this requires that at least one of the vectors has nonzero components. In any case, by assumption, $x$ and $y$ are both contained in $S_1$ and $S_2$, and both vectors have nonzero components. As a consequence, not both dependencies can be characteristic functions or positive multiples thereof. Thus, either $S_1$ or $S_2$ is not equilibrated.  

$\square$

4. Positive Dependence

We now define the skeleton of a set with subsets linearly spanning a proper subset. This definition is consistent with the usual one on simplicial complexes.

**Definition 17.** For any set $X$ we let

$$\text{skel}(X) = \bigcup \{ \wp(A) \mid A \subset X, \ell(A) \subsetneq \ell(X) \},$$

$$\text{core}(X) = \wp(X) \setminus \text{skel}(X).$$

The skeleton is monotoneous.

**Lemma 18.** If $X \subset Y$, then $\text{skel}(X) \subset \text{skel}(Y)$.

**Proof.** Assume $X \subset Y$ and $z \in \text{skel}(X)$. Then there is an $A \subset X$ such that $\ell(A) \subsetneq \ell(X)$ with $z \in \wp(A)$. But then a fortiori, $A \subset Y$ and $\ell(A) \subsetneq \ell(Y)$, so $z \in \text{skel}(Y)$.  

If $X$ is finite, then the skeleton of $X$ is a closed subset of $X$; thus, the core is open. Its components are characterized by the following lemma, which is not essential for the main results, but for illustration purposes only.

**Lemma 19.** For a finite set $X$ linearly spanning $\mathbb{R}^d$, each component of the core of $X$ is contained in a set of the form $\text{rint}\wp(B)$ for some linear
basis $B \subset X$ of $\mathbb{R}^d$. If, moreover, $X$ is positively independent, then each component is of this form.

Proof. Let $X$ as above and $O$ be a component of core $(X)$. Then $O$ is open. For $x \in O$, by lemma 11 there are $x_1, \ldots, x_m \in X$, $m \leq d$, with $x = \sum_{i=1}^{m} \alpha_i x_i$, $\alpha_i > 0$. Set $B = \{x_1, \ldots, x_m\}$. If the dimension of $\varphi (B)$ is less than $d$, then $\ell (B) \subseteq \ell (X)$ and $x \in \varphi (B) \subset \text{skel} (X)$, a contradiction. So, the dimension of $\varphi (B)$ is $d$, which (by remark 5) requires $m = d$ and $B$ to be a basis of $\mathbb{R}^d$. Hence $x \in \text{rint} \varphi (B)$.

For any $y \in O \setminus \text{rint} \varphi (B)$, there must be a path from $x$ to $y$ contained in $O$. As the boundary of $\varphi (B)$ is contained in the skeleton of $X$, the arc must intersect the boundary at some $z \in \text{skel} (X)$, a contradiction to $O$ being contained in the core of $X$. Thus $O \subset \text{rint} \varphi (B)$.

Now, assume further that $X$ is positively independent. If $y \in \text{rint} \varphi (B) \cap \text{skel} (X)$, then there would be a set $A \subset X$ with $y \in \varphi (A)$ and $\ell (A) \subseteq \ell (X) = \ell (B)$. Since $X$ is positively independent, $A \cap \varphi (B) \subset B$. Hence $y$ lies in the boundary of $\varphi (B)$, a contradiction. Thus $\text{rint} \varphi (B) \subset O$. □

The following lemma is crucial for the proof of theorem 3. It states two necessary and sufficient conditions for the existence of a positive dependency within a simplex and one added element.

Lemma 20. Let $S$ be a simplex, and $y \in \ell (S) \setminus S$. The following three propositions are equivalent:

(i): There is an $x \in S$ with $x \in \varphi (S[x \to y])$.
(ii): For all $R \in S (S \cup \{y\})$, $\ell (R) = \ell (S)$.
(iii): $-y \notin \text{skel} (S)$.

Proof. “(i)⇒(ii)”: Assume there is an $R \in S (S \cup \{y\})$ with $\ell (R) \neq \ell (S)$. Since $y \in \ell (S)$, this means $\ell (R) \subset \ell (S)$, and further $R \neq S$. By definition of a simplex, $R \not\subseteq S$. Hence $y \in R$. For the same reason, $S \not\subseteq R$, thus, by lemma 12, $C = R \cap S$ is a basis of a proper subspace of $\ell (S)$.

Extend $C$ to a basis $B \subset S$ of $\ell (S)$, so that $B = \{x_1, \ldots, x_n\}$, $C = \{x_1, \ldots, x_m\}$, $m < n$, $S = B \cup \{x_0\}$, and $R = C \cup \{y\}$. By lemma 12 we can write

\[
x_0 = -\sum_{i=1}^{n} \delta_i x_i,
\]
\[
y = -\sum_{i=1}^{m} \gamma_i x_i,
\]

with $\delta_i, \gamma_i > 0$. 

We have to show that no \( x_k \in \varphi(S[x \to y]) \) for \( k = 0, \ldots, n \). Assume the contrary. For \( k > 0 \) we can write with \( \alpha_i \geq 0, \alpha_k = 0, \beta \geq 0 \),
\[
x_k = \sum_{i=0}^{n} \alpha_i x_i + \beta y
\]
\[
= \sum_{i=1}^{n} (\alpha_i - \beta \gamma_i - \alpha_0 \delta_i) x_i, \text{ or}
\]
\[
x_k (1 + \beta \gamma_k + \alpha_0 \delta_k) = \sum_{i=1}^{n} (\alpha_i - \beta \gamma_i - \alpha_0 \delta_i) x_i.
\]
As \( \beta \gamma_k + \alpha_0 \delta_k \geq 0 \), this means \( x_k \in \varphi(S \setminus \{x_k\}) \), contradicting the assumption that \( S \) is a simplex. For the case \( k = 0 \), since \( C \subseteq B \), by lemma 12 one chooses a different basis \( B' \supseteq C \) containing \( x_0 \) and proceeds as above. Thus \( x_k \notin \varphi(S[x \to y]) \) for \( k = 0, \ldots, n \).

“(ii)⇒(i)”: Assume that for all \( R \in S(S \cup \{y\}) \) we have \( \ell(R) = \ell(S) \).
Since \( \ell(S) = \varphi(S) \) is symmetric, and \( y \in \ell(S) \), by theorem 13 there is an \( R \in S(S \cup \{y\}) \) with \( y \in R \). By assumption, \( \ell(R) = \ell(S) \). By definition of a simplex, we can not have \( S \subseteq R \). So, there is an \( x \in S \setminus R \) with
\[
x \in \ell(S) = \varphi(R) \subset \varphi(S[x \to y]).
\]

“(ii)⇒(iii)”: Let \( R \subset S \cup \{y\} \) be a simplex with \( \ell(R) \subseteq \ell(S) \). If \( y \notin R \), then \( R \subset S \) and, by definition, \( R = S \), a contradiction. Thus \( y \in R \). By lemma 12 \( A = R \setminus \{y\} \) is linearly independent, and \( -y \in \text{rint} \varphi(A) \). Since \( \ell(A) \subset \ell(S) \), \( \varphi(A) \subset \text{skel}(S) \). Hence \( -y \in \text{skel}(S) \).

“(iii)⇒(ii)”: Assume \( -y \in \text{skel}(S) \). Then there is an \( A \subset S \) with \( -y \in \varphi(A \subseteq \ell(A) \subseteq \ell(S) \). Let \( B \subset A \) be a minimal set with \( -y \in \varphi(B) \). Then \( -y \in \text{rint} \varphi(B) \), and by lemma 12 \( R = B \cup \{y\} \) is a simplex with \( \ell(R) \subset \ell(A) \subseteq \ell(S) \). \( \square \)

**Example 21.** Let \( S = \{a, b, c\} \subset \mathbb{R}^2 \setminus \{0\} \) be a simplex. Then \( \text{skel}(S) = \varphi(\{a\}) \cup \varphi(\{b\}) \cup \varphi(\{c\}) \). For \( y \in -S \) we have \( -y \in \text{skel}(S) \) and \( R = \{y, -y\} \) is simplex in \( S \cup \{y\} \). For \( y \in \ell(S) \setminus -S \) we find an \( x \in S \) with \( x \in \varphi(S[x \to y]) \).

5. Positive Bases

We are now in a position to prove theorem 3

**Proof.** “(i)⇒(ii)”: We assume that condition (ii) does not hold for some subset \( Y \subseteq X \) and a simplex \( S \in S(X) \). Let \( L = \ell(Y) \cap \ell(S) \) and \( M = \ell(Y \cap S) \). Since, obviously, \( M \subseteq L, M \neq L \) spells out as \( M \not\subseteq L \) (\( M \) may be null). Then we indicate that \( L = M^\perp \oplus M \) with some non-null linear space \( M^\perp \) orthogonal to \( M \). Pick \( y \in M^\perp \setminus (S \cup -S \cup \{0\}) \), then also \( -y \in M^\perp \setminus (S \cup \{0\}) \).

Assuming both \( y, -y \in \text{skel}(S) \). Then, by definition, there are sets \( C, D \subsetneq S \) with \( y \in \varphi(C) \subset \ell(C) \) and \( \ell(C) \subseteq \ell(S) \), and \( -y \in \varphi(D) \subset \ell(D) \) and \( \ell(D) \subseteq \ell(S) \). Set \( R = C \cap D \). Since both \( C \) and \( D \) have at least two elements less than \( S \), \( C \cup D \) is linearly independent (lemma 12), and so is \( R \). Moreover, \( \ell(C) \cap \ell(D) = \ell(R) \), hence \( y, -y \in \ell(R) \). By comparison
of the coefficients we conclude that both $y$ and $-y$ are a linear combination of elements of $R$ with non-negative coefficients, as the coefficients of $y$ in $C$ and $-y$ in $D$ are non-negative. Hence $y, -y \in \varphi (R)$. It follows that $0 \in \varphi (R) \subseteq \varphi (S)$, a contradiction to the fact that $S$ is a simplex (lemma 12).

Therefore, either $y \notin \text{skel} (S)$, and, by lemma 20, there is an $x \in S$ with $x \in \varphi (S [x \rightarrow y])$, or $-y \notin \text{skel} (S)$, and, by the same lemma, there is an $x \in S$ with $x \in \varphi (S [x \rightarrow -y])$, as $\{y, -y\} \cap S = \emptyset$ but $y, -y \in \ell (Y) \subset \varphi (X)$. In either case, $X$ is positively dependent.

“(iii)⇒(i)”: Trivial.

“(iii)⇒(ii)”: The proof is by induction over the number of simplices contained in $X$. Let $n = |S (X)|$. For $n = 1$ we find that $X$ is a single simplex, and there is nothing left to prove. Now assume that the implication has been proven for all proper subsets $Y$ of $X$ positively spanning a linear subspace. Assume that (i) is false and there is an $x \in X$ with $x \in \varphi (X \setminus \{x\})$. Since $X$ is a positively spanning set, we can choose an $S \in S (X)$ with $x \in S$. Let $Y = S'$ be the lattice complement of $S$ in $\mathcal{L} (X)$ from theorem 14, then $Y$ is a proper positively spanning subset, which can be written as a union of $n-1$ elements. As $X = Y \cup S$ we can write $x = y + z$ with $y \in \varphi (Y \setminus \{x\}) \subset \ell (Y)$ and $z \in \varphi (S \setminus (Y \cup \{x\})) \subset \ell (S)$. So, $y = x - z \in \ell (S) \cap \ell (Y)$.

Assume furthermore, that $y \in \ell (S \cap Y)$. This requires $z = 0$, and thus $x = y$. But then we have $x \in \varphi (Y \setminus \{x\})$ and $Y$ is positively dependent, in contradiction to the induction hypothesis which states (i) for $Y$. We conclude that $x \in \varphi (X \setminus \{x\})$ implies that $\ell (Y) \cap \ell (S) \neq \ell (Y \cap S)$ for some positively spanning subset $Y \subset X$ and a simplex $S \in S (X)$.

“(i)\cap(ii)⇒(iv)”: As $X$ positively spans $\mathbb{R}^d$, by theorem 13 $X$ is a sum of simplices. Let $S_1, \ldots, S_n$ be a minimal set of simplices from $S (X)$ with $\bigcup_{i=1}^n S_i = X$, where $X$ is a positive basis of $\mathbb{R}^d$. Then each $S_i$ contains an element $x_i$ not contained in any other simplex. By lemma 12 $A_i = S_i \setminus \{x_i\}$ is linearly independent for $i = 1, \ldots, n$.

We set $B = A_1 \cup \cdots \cup A_n$ and show that $B$ is a linear basis of $\mathbb{R}^d$. Assume, in contrast, that $B$ is linearly dependent. Then there is an $x \in B$ with $x \in \ell (B \setminus \{x\})$. Choose an $i$ with $x \in A_i \subset S_i$. As $x_i \notin B$, we obtain by (ii)

$$x \in \ell (B \setminus \{x\}) \cap \ell (S_i) = \ell ((B \setminus \{x\}) \cap S_i) = \ell (A_i \setminus \{x\}).$$

But this is impossible, since $A_i$ is linearly independent. Hence, $B$ is linearly independent. Moreover,

$$\ell (B) = \sum_{i=1}^n \ell (A_i) = \sum_{i=1}^n \ell (S_i) = \ell (X) = \mathbb{R}^d,$$

thus $B$ is a linear basis of $\mathbb{R}^d$. Therefore, $|B| = d$ and, further, $1 \leq n \leq d$.

We show that each $S \in S (X)$ has with $B$ all but one element in common. Indeed, by (ii)

$$\ell (S) = \ell (S) \cap \ell (B) = \ell (S \cap B).$$

The assertion then follows from lemma 12. Lemma 12 also asserts that $x_i \in -\text{rint} \varphi (A_i)$.
It remains to show that \( A_j \not\subseteq A_i \) for \( i \neq j \). Assuming the contrary, then \( x_j \in -\varphi(A_i) = \varphi(S_i) \), and \( x_j \not\in S_i \). This constitutes a positive dependency within \( X \), contradicting (i). This completes the proof of (iv).

“(iv)⇒(iii)”: Assume that \( X = B \cup \{x_1, \ldots, x_n\} \), with a basis \( B \) of \( \ell(X) \), such that each \( x_i \in -\text{rint} \varphi(A_i) \) for a minimal subset \( A_i \subseteq B \), and \( A_i \not\subseteq A_j \) for \( i \neq j \). Lemma \( \ref{lemma:positively_spanning_sets} \) assures that \( S_i = A_i \cup \{x_i\} \) is a simplex. This implies that \( x_i \neq x_j \) and \( A_i \not\subseteq A_j \) for \( i \neq j \). Since, by assumption, every simplex from \( S(X) \) has exactly all but one element with \( B \) in common, \( S_1, \ldots, S_n \) are exactly the simplices in \( X \).

Let \( Y \subseteq X \) be a positively spanning set. By theorem \( \ref{thm:simplicial_conical_decomposition} \), \( Y = \bigcup_{j \in J} S_j \) with \( S_j \in S(Y) \). If \( i \in J \), then \( S_i \subseteq Y \), and \( \ell(Y) \cap \ell(S_j) = \ell(Y \cap S_j) \). Therefore, \( i \notin J \). For \( i \neq j \), we find that \( S_i \cap S_j = A_i \cap A_j \subseteq B \) and \( x_i \notin \ell(A_j) \) as well as \( x_j \notin \ell(A_i) \). Hence,

\[
\ell(Y) \cap \ell(S_i) = \left( \sum_{j \in J} \ell(A_j) \right) \cap \ell(A_i) = \sum_{j \in J} \ell(A_j \cap A_i) \subseteq \ell(Y \cap S_i) \subseteq \ell(Y) \cap \ell(S_i).
\]

□

The following result is from Reay \( \cite{Reay1987} \) Th 2, which was used to prove the Bonnice-Klee Theorem (see also \( \cite{mccomas2002} \) Th 10).

**Corollary 22.** Let \( X \) be any positive basis of \( \mathbb{R}^d \). Then \( X \) admits a decomposition into pairwise disjoint subsets \( X = X_1 \cup \cdots \cup X_n \), \( n \leq d \), such that \( |X_i| \geq |X_{i+1}| \geq 2 \) for \( i = 1, \ldots, n-1 \), and \( \varphi(X_1 \cup \cdots \cup X_k) \) is a linear subspace of \( \mathbb{R}^d \) of dimension \( \sum_{i=1}^k |X_i| - k \).

**Proof.** Let \( B \) be the linear basis of \( \mathbb{R}^d \) and \( A_i \) the sets from condition (iv). In order to remove the overlap between the sets \( A_i \), we set \( B_0 = \emptyset \) and \( B_i = A_{\pi(i)} \setminus \bigcup_{j=0}^{i-1} B_j \) with a permutation \( \pi : [1, \ldots, n] \to [1, \ldots, n] \), for all \( i = 1, \ldots, n \). \( \pi \) will be chosen to ensure that the sets \( B_i \) are ordered in non-increasing cardinality. Hence \( B_i \cap B_j = \emptyset \) for \( i \neq j \) and \( |B_i| \geq |B_{i+1}| \).

Since \( \bigcup_{i=1}^n A_i = B \), we find \( B = B_1 \cup \cdots \cup B_n \). Since \( |X_i| = |B_i| + 1 \), we obtain the theorem \( \ref{thm:simplicial_conical_decomposition} \)

\[
\varphi(X_1 \cup \cdots \cup X_k) = \varphi(S_1 \cup \cdots \cup S_k) = \ell(B_1 \cup \cdots \cup B_k),
\]

which implies the assertion. □

The special case when the simplices span disjoint subspaces is covered by \( \cite{mccomas2002} \) Th 8.

Although this statement could be used to construct the basis \( B \) in (iv), our result proves more, by, namely, showing that each simplex has all but one element with \( B \) in common (see counterexample \( \ref{counterexample:positive_dependency} \)). This insight immediately yields the missing part of the proof of theorem \( \ref{thm:simplicial_conical_decomposition} \).
Corollary 23. Let $X$ be a positive basis of $\mathbb{R}^d$. Then $\mathcal{S}(X)$ is a minimal set of simplices spanning $\mathbb{R}^d$. Furthermore, the lattice $\mathcal{L}(X)$ is isomorphic to $2\mathcal{S}(X)$.

Example 24. The following set $X = \{x_1, \ldots, x_9\}$ positively spans $\mathbb{R}^6$ and satisfies the condition of corollary 23 with simplices $X_1 = \{x_1, x_2, x_3\}$, $X_2 = \{x_4, x_5, x_6\}$, and $X_3 = \{x_7, x_8, x_9\}$, but is positively dependent, as $x_6 = x_1 + x_2 + x_8 + x_9$. We observe that $X_1 \cup X_2$ spans a 4-dimensional subset, and $S = \{x_3, x_6, x_7\}$ is another simplex, violating the last condition in theorem 4 (iv) for any basis $B \subset X$ (as not all four simplices can have one element not in $B$). Hence, this last condition is unable to be removed.

\[ x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, x_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \]

\[ x_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, x_7 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, x_8 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, x_9 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}. \]

6. Conical Decompositions

This chapter contains the proof of the main theorem 4.

Lemma 25. Let $X$ be a set positively spanning $\mathbb{R}^d$. Then each $A \in \mathcal{M}(X)$ linearly spans $\mathbb{R}^d$.

Proof. Let $A \in \mathcal{M}(X)$ and set $M = \ell(A) \subset \mathbb{R}^d = \ell(X)$. If $M \subset \mathbb{R}^d$, then there is a hyperplane $H$ of $\mathbb{R}^d$ containing $M$. Let $z$ be a normal vector of $H$. Since $X$ positively generates $L$, there must be an $x \in X$ lying in the open half-space in direction $z$. Clearly, $x$ is not in $A$. Since $A$ is negatively independent, lemma 7 warrants the existence of a vector $w$ with $w \cdot y > 0$ for all $y \in A$. By construction, $y \cdot z = 0$ for all $y \in A$, and $x \cdot z > 0$. Then there is an $\varepsilon > 0$ such that $\varepsilon \cdot w \cdot x < z \cdot x$. For $u = z + \varepsilon \cdot w$ we have $u \cdot x < z \cdot x + \varepsilon \cdot w \cdot x > 0$, and for $y \in A$ we obtain $u \cdot y = \varepsilon \cdot w \cdot y > 0$. Thus, $A \cup \{x\}$ is negatively independent, a contradiction to the assumption that $A$ is a maximally negatively independent subset of $X$. Hence $M = \mathbb{R}^d$, or $\ell(A) = \ell(X)$. \hfill \Box

The following lemma constitutes the crucial combinatorical insight for our main theorem.

Lemma 26. Let $X$ be a positively spanning set. If $A$ contains all but one element of each simplex $S \in \mathcal{S}(X)$, then $A \in \mathcal{M}(X)$. If $X$ is positively independent, then the converse holds.
Proof. Assume first, $|S \cap A| = |S| - 1$ for all simplices $S \in \mathcal{S}(X)$. If $A$ is not negatively independent, then by lemma 7 there would be a simplex $S \subset A \subset X$, contradicting the assumption. Now let $B$ be a negatively independent set with $A \subset B \subset X$. For $x \in B \setminus A$, since $X = \bigcup \mathcal{S}(X)$, there is a simplex $S \in \mathcal{S}(X)$ with $x \in S$ and $|S \cap A| = |S| - 1$, hence $(S \cap A) \cup \{x\} = S$. But this implies $S \subset B$, in contradiction to the assumption that $B$ is negatively independent. We conclude $A = B$. So, $A \in \mathcal{M}(X)$.

Conversely, assume that $X$ is positively independent, let $A \in \mathcal{M}(X)$ be a maximally negatively independent subset of $X$ and consider any simplex $S \in \mathcal{S}(X)$. So, $S$ cannot be a subset of $A$. Select $x \in S \setminus A$. By lemma 12 $B = S \setminus \{x\}$ is a basis spanning $\ell(S)$. By lemma 25 $A$ linearly spans $\ell(X)$. Theorem 3 yields

$$\ell(B) = \ell(S) \cap \ell(X) = \ell(S) \cap \ell(A) = \ell(S \cap A) = \ell(B \cap A),$$

so $B \subset A$, as $B$ is linearly independent. This gives us $|S \cap A| = |S| - 1$. \hfill $\blacksquare$

The converse proposition does not hold if $X$ is positively dependent, as the following example shows.

Example 27. Let $S$ be a simplex of any dimension greater one and set $X = S \cup -S$. Then for any $x \in S$, $S[x \mapsto -x]$ is in $\mathcal{M}(X)$, and the intersection with the only other simplex $-S$ consists only of the element $-x$.

Proof. Let $S$ and $X$ be as above. Then also $-S$ is a simplex. For $x \in S$, by lemma 12 $-x \notin \varphi(S)$, and $S \setminus \{x\}$ is negatively independent, and so is $A = S[x \mapsto -x]$. For any $y \in X \setminus A$ either $y = x$ or $y \in -S \setminus \{-x\}$. In any case, $-y \notin A$, thus $A \cup \{y\}$ is negatively dependent. Hence $A \in \mathcal{M}(X)$, and $A \cap -S = \{-x\}$. \hfill $\blacksquare$

Next is a simple inequality.

Lemma 28. For natural numbers $k_1, \ldots, k_n \geq 1$ with $\sum_{i=1}^n k_i = d$ we have

$$\prod_{i=1}^n k_i \leq 2^{d-n}.$$  

The equality is strict whenever one $k_i$ is greater than one.

Proof. The case $n = 1$ follows from $k \leq 2^{k-1}$ for $k \geq 1$ with equality for $k = 1$ only. Assume now that the above formula has been proven for $n-1$. Then

$$\prod_{i=1}^n k_i \leq 2^{d-k_n-(n-1)} \cdot k_n \leq 2^{d-n}.$$  

Again, the inequality is strict, if $k_n > 1$. \hfill $\blacksquare$

Now to the proof of the main theorem 3

Proof. Let $X$ be a positive basis. By theorem 3 (iv), we can write $X = B \cup \{x_1, \ldots, x_n\}$, $1 \leq n \leq d$, with a basis $B$ of $\ell(X) = \mathbb{R}^d$, such that each $x_i \in -\text{rint} \varphi(A_i)$ for $A_i \subset B$, and $A_i \neq A_j$ for $i \neq j$. Therefore, $|B| = d$. Moreover, the theorem assures that each simplex $S \in \mathcal{S}(X)$ has all but one element in common with $B$. Hence $S_i = A_i \cup \{x_i\}$, $i = 1, \ldots, n$, are exactly the simplices in $\mathcal{S}(X)$.
As in the proof of theorem 3 (iv), in order to remove the overlap between those sets, we set $B_0 = \emptyset$ and $B_i = A_i \setminus \bigcup_{j=0}^{i-1} B_i$. Hence $B_i \cap B_j = \emptyset$ for $i \neq j$.

First, we show that each $B_k$ is non-empty for $k \geq 1$. Assume $B_k = \emptyset$. Then $A_k \subset \bigcup_{i=1}^{k-1} A_i$, and

$$x_k \in \ell (A_k) \subset \ell \left( \bigcup_{i=1}^{k-1} A_i \right) = \varnothing \left( \bigcup_{i=1}^{k-1} S_i \right).$$

The last equation holds because for $x \in A_i$, $-x \in \emptyset (S_i)$. As $x_k \notin \bigcup_{i=1}^{k-1} S_i$, which indicates a positive dependence within $X$, a contradiction. Hence $|B_i| \geq 1$ for $i = 1, \ldots, n$.

Since $X = B \cup \{x_1, \ldots, x_n\}$ with $B \cap \{x_1, \ldots, x_n\} = \emptyset$ and $1 \leq n \leq d$, we find well-known inequalities [2, Th. 3.8 and 6.7]

$$d + 1 \leq |X| = d + n \leq 2d.$$

As $B$ is a basis of $\mathbb{R}^d$, we must have $|B| = d$. For $|X| = 2d$, we find $n = d$, which requires $|B_i| = 1$ for all $i = 1, \ldots, n$, such that $X$ is a union of $d$ 1-simplices forming a cross. The lower bound $|X| = d + 1$ yields $n = 1$, and $X = S_1$ is a single simplex.

We now proceed to give the upper bound of the number of maximally negatively independent subsets of $X$. Let $X_i = B_i \cup \{x_i\}$, $i = 1, \ldots, n$. Since $x_i \notin B_j$ we find that the $X_i$ are a pairwise disjoint decomposition of $X$. Let $A \in \mathcal{M}(X)$. By lemma [23] each set has with each simplex $S_i$ all but one element in common. We show that $A$ can be constructed by picking one element from each $X_i$, which is to be excluded from $A$. At the same time, we establish the upper bound for a number of those constructions.

Since $X_i \subset S_i$, $A$ contains either all or all but one elements of $X_i$.

As $X_1 = S_1$, we can pick any $y_1 \in X_1$, and there are exactly $|X_1|$ choices. Thus, $A$ must contain $X_1$ except for one $y_1 \in X_1$. Upon constructing $A$, we start with $A = X_1 \setminus \{y_1\}$. Assume now we have picked $y_1, \ldots, y_j$ from $X_1, \ldots, X_{k-1}$, $j \leq k - 1 < n$, and established that $A$ contains all of $S_i$, $1 \leq i < k$, except for one element from the list $y_1, \ldots, y_j$. Now, going from $k - 1$ to $k$, three cases have to be distinguished.

(i) If $S_k$ contains more than one element from the list $y_1, \ldots, y_j$, then the list is incompatible with the goal of constructing an element of $\mathcal{M}(X)$. There are zero choices. Since we are looking for the upper bound of possible choices, there is no problem with this case (which can not occur if $A$ is already given as a maximal negatively independent set). (ii) If $S_k$ contains exactly one element $y_j$ from the list, then $A$ does not have this element or should not have in common with $S_k$, and $A$ contains all the other elements from $S_k$ (upon constructing $A$, we are adding all elements of $X_k \setminus \{y_j\}$ to it). (iii) If $X_k$ does not contain an element from the list, then any element $y_{j+1} \in X_k$ can be chosen and added to the list, and there are $|X_k|$ choices. In the latter two cases, since $S_k \subset X_1 \cup \cdots \cup X_k$, we are assured that $S_k$ has all but one element in common with $A$. At each step, there are at most $|X_k|$ choices, which remains true also in case (i).
As \( d = |B| = \sum_{i=1}^{n} |B_i| \) and \( |B_i| \geq 1 \), we find with lemma 28

\[
|M(X)| \leq \prod_{i=1}^{n} |X_i| = \prod_{i=1}^{n} (|B_i| + 1) \leq 2^n \prod_{i=1}^{n} |B_i| \leq 2^n \cdot 2^{d-n} = 2^d.
\]

By the same lemma, the inequality is strict save for \( |B_i| = 1 \) for \( i = 1, \ldots, n \). This requires all \( S_i \) to be 1-simplices. But this also yields \( n = d \). Hence, equality holds in the above relation if and only if \( X \) is a cross.

If \( X \) consists of one simplex, then \( |M(X)| = d + 1 \). Otherwise, if \( n > 1 \) it is easy to see that \( |M(X)| > d + 1 \), hence the lower bound of the last inequality, which holds with equality if and only if \( X \) is a simplex. \( \square \)

7. Consequences of the main theorem

The last inequality of the main theorem 3 can be extended to arbitrary positively spanning sets.

**Theorem 29.** Let \( X \) be a set positively spanning \( \mathbb{R}^d \). Then the elements of \( X \) can be subdivided into at most \( 2^d \) negatively independent sets. The boundary is met exactly in case that \( X \) is a sum of 1-simplices.

**Proof.** Assume \( X \) is positively spanning \( \mathbb{R}^d \). Pick a positive basis \( Y \subset X \) [7, Th. 4.3]. Consider \( x \in X \). Since \( Y \) positively spans \( L \), there is a negatively independent set \( A \subset Y \) with \( x \in \partial(A) \). But \( A \) can be expanded to a maximally negatively independent set \( B \in M(Y) \). So, every \( x \in X \) lies in the positive span of some element of \( M(Y) \), and there are at most \( 2^d \) of them.

Now assume that \( X \) can not be subdivided into less then \( 2^d \) negatively independent sets. Then \( S(X) \) does not contain any simplex, which is not a 1-simplex. For if there is a simplex \( S_0 \in S(X) \), which is not a 1-simplex, we can construct this basis by extending \( S_0 \) to a minimal set that positively spans \( \mathbb{R}^d \), and has \( |M(Y)| < 2^d \), a contradiction to the assumption. Hence \( X \) can be written as a sum of 1-simplices, of which \( d \) must form a cross. \( \square \)

The next lemma prepares for the last major result.

**Lemma 30.** Let \( X, Y \) be sets positively spanning \( \mathbb{R}^d \) with \( Y \subset X \). Then \( A \in M(Y) \) if and only if there is a \( B \in M(X) \) with \( A = B \cap Y \). Moreover, for \( A, B \in M(X) \), \( A \cap Y = B \cap Y \) implies \( \ell(A \cap B) = \mathbb{R}^d \).

**Proof.** Let \( B \in M(X) \) and set \( A = B \cap Y \). By lemma 26, for every \( S \in S(Y) \), \( S \cap B \) contains all but one element of \( S \). For \( S \in S(Y) \), by lemma , \( S \in S(X) \), and \( S \cap A = S \cap B \cap Y = S \cap B \) contain all but one element from \( S \). Hence \( A \in M(Y) \).

Conversely, let \( A \in M(Y) \). Since \( A \) is a negatively independent subset of \( X \), there is a maximal extension \( B \in M(X) \) with \( A \subset B \). Clearly, \( A \subset B \cap Y \). If there existed an \( x \in (B \cap Y) \setminus A \), then \( A \cup \{x\} \) would be negatively independent, a contradiction to \( A \) being a maximally independent subset of \( Y \). Hence \( A = B \cap Y \).

To show the second statement, assume \( A, B \in M(X) \), \( A \cap Y = B \cap Y \), and \( \ell(X) = \ell(Y) = \mathbb{R}^d \). Then \( \ell(S \cap A) = \ell(S) \) for every \( S \in S(Y) \subset S(X) \),
and further
\[ \ell(A \cap Y) = \ell\left( \bigcup_{S \in \mathcal{S}(Y)} A \cap S \right) = \ell\left( \bigcup_{S \in \mathcal{S}(Y)} S \right) = \ell(Y). \]
So,
\[ \ell(A \cap B) \subset \ell(Y) = \ell(A \cap Y) = \ell(A \cap B \cap Y) \subset \ell(A \cap B), \]
which assures \( \ell(A \cap B) = \ell(Y) \).
\[
\square
\]
In general, even in two dimensions, \( \mathcal{M}(X) \) can be arbitrarily large.

Example 31. For any \( n \geq 1 \) there is a set \( X_n \subset \mathbb{R}^2 \) with \( |\mathcal{M}(X_n)| = |X_n| = 2n \).

To see this, divide the unit circle into \( 2n \) segments by taking \( n \) antipodal pairs of points with equal distances to the neighbour. Then the maximal negatively independent sets are exactly those consisting of \( n \) consecutive points, and there are exactly \( 2n \) of them. Thus, the intersection condition in theorem 6 is necessary for the inequality.

We are now in a position to prove theorem 6.

Proof. Let \( A \subset \mathcal{M}(X) \), where \( X \) positively spans \( \mathbb{R}^d \), such that for all \( A, B \in A, A \neq B, \varphi(A \cap B) \) does not have full dimension. Choose a positive basis \( Y \subset X \). By lemma 30, for each \( B \in A \), \( B \cap Y \in \mathcal{M}(Y) \). Moreover, for \( A, B \in A \) we do not have \( A \cap Y = B \cap Y \), because this would imply \( \ell(A \cap B) = \mathbb{R}^d \), contradicting the assumption that \( \varphi(A \cap B) \) does not have full dimension. Hence \( |A| \leq |\mathcal{M}(Y)| \leq 2^d \).

Now consider the case \( |A| = 2^d \). If \( Y = \bigcup A \) does not span the whole space, then by the separating hyperplane theorem, \( Y \) is contained in a half-space with the boundary containing the origin. Then there must be another element of \( \mathcal{M}(X) \) contained in the opposite half space, which could be added to \( A \), contradicting the last inequality. Moreover, we must have \( \bigcup A = X \), otherwise any \( x \in X \setminus Y \) must be contained in the positive span of one set in \( A \), contradicting its maximality. If \( X \) contained a positive basis, which is not a cross, then by theorem 29, \( |A| < 2^d \). Thus, \( X \) does not contain any simplices apart from 1-simplices. So, \( X \) is a union of at least \( d \) 1-simplices, of which some \( d \) of them form a cross.

\[
\square
\]

References

[1] Boltyanski, V., Martini, H., and Soltan, P.S., *Excursions into Combinatorial Geometry*, Springer Berlin, 1997.
[2] Davis, C., “Theory of Positive Linear Dependence,” *American Journal of Mathematics*, 76 (4), 1954, 733-746.
[3] Grünbaum, B., *Convex Polytopes*, Second edition, Springer, 2003.
[4] Marcus, D., “Gale Diagrams of Convex Polytopes and Positive Spanning Sets of Vectors,” *Discrete Applied Mathematics* 9, 1984, 47-67.
[5] Hare W., Song H., “On the Cardinality of Positively Linearly Independent Sets,” *arXiv:1509.07496v1*, 2015.
[6] Reay, J. R., “A new Proof of the Bonnice-Klee Theorem,” *Proc. Amer. Math. Soc.*, 16, 1965, 585-586.
[7] Regis, R.G., “On the Properties of Positive Spanning Sets and Positive Bases,” *Optimization and Engineering* 17(1), 2016, 229-262.
[8] Shepard, G.C., “Diagrams for Positive Bases,” *Journal of the London Mathematical Society*, 2 (4), 1971, 165-175.

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