WEAVING K-FRAMES IN HILBERT SPACES

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Abstract. Gávruţa introduced K-frames for Hilbert spaces to study atomic systems with respect to a bounded linear operator. There are many differences between K-frames and standard frames, so we study weaving properties of K-frames. Two frames \( \{ \phi_i \}_{i \in I} \) and \( \{ \psi_i \}_{i \in I} \) for a separable Hilbert space \( \mathcal{H} \) are woven if there are positive constants \( A \leq B \) such that for every subset \( \sigma \subset I \), the family \( \{ \phi_i \}_{i \in \sigma} \cup \{ \psi_i \}_{i \in \sigma^c} \) is a frame for \( \mathcal{H} \) with frame bounds \( A, B \). In this paper, we present necessary and sufficient conditions for weaving K-frames in Hilbert spaces. It is shown that woven K-frames and weakly woven K-frames are equivalent. Finally, sufficient conditions for Paley-Wiener type perturbation of weaving K-frames are given.

1. Introduction and Preliminaries

Let \( \mathcal{H} \) be a complex separable Hilbert space with an inner product \( \langle .., .. \rangle \). A countable sequence \( \{ f_k \}_{k=1}^\infty \subset \mathcal{H} \) is a frame for \( \mathcal{H} \) if there exist positive scalars \( A_o \leq B_o < \infty \) such that

\[
A_o \| f \|^2 \leq \sum_{k=1}^\infty | \langle f, f_k \rangle |^2 \leq B_o \| f \|^2 \text{ for all } f \in \mathcal{H}.
\]

The scalars \( A_o \) and \( B_o \) are called lower and upper frame bounds, respectively. The sequence \( \{ f_k \}_{k=1}^\infty \) is called a Bessel sequence with Bessel bound \( B_o \) if the upper inequality in (1.1) holds for all \( f \in \mathcal{H} \).

Following three operators are associated with a frame \( \{ f_k \}_{k=1}^\infty \) for \( \mathcal{H} \):

- **pre-frame operator** \( T : \ell^2(\mathbb{N}) \to \mathcal{H}, \quad T \{ c_k \}_{k=1}^\infty = \sum_{k=1}^\infty c_k f_k, \ \{ c_k \}_{k=1}^\infty \in \ell^2(\mathbb{N}) \),
- **analysis operator** (adjoint of \( T \)) \( T^* : \mathcal{H} \to \ell^2(\mathbb{N}), \quad T^* f = \{ \langle f, f_k \rangle \}_{k=1}^\infty, \ f \in \mathcal{H} \),
- **frame operator** \( S = T T^* : \mathcal{H} \to \mathcal{H}, \quad S f = \sum_{k=1}^\infty \langle f, f_k \rangle f_k, \ f \in \mathcal{H} \).

The frame operator \( S \) is a bounded, linear and invertible operator on \( \mathcal{H} \). This gives the reconstruction of each vector \( f \in \mathcal{H} \),

\[
f = SS^{-1} f = \sum_{k=1}^\infty \langle S^{-1} f, f_k \rangle f_k.
\]

Thus, a frame for \( \mathcal{H} \) allows each vector in \( \mathcal{H} \) to be written as a linear combination of the elements in the frame, but the linear independence between the frame elements is not required. Concerning the evolution of the notion of Hilbert frames and their applications in different directions in science and engineering, it is necessary to mention the noble books by Casazza and Kutyniok [4], Christensen [7] and beautiful research tutorials by Casazza [5] and Casazza and Lynch [6].

Next we give some basic notations. The family of all bounded linear operators from a Banach space \( \mathcal{X} \) into a Banach space \( \mathcal{Y} \) is denoted by \( \mathcal{B}(\mathcal{X}, \mathcal{Y}) \). If \( \mathcal{X} = \mathcal{Y} \), then we write \( \mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X}) \).

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The range and the kernel of $K \in B(\mathcal{X}, \mathcal{Y})$ are denoted by $R(K)$ and $N(K)$, respectively. The pseudo-inverse of $K \in B(\mathcal{H})$ is denoted by $K^\dagger$. Note that $KK^\dagger f = f$ for all $f \in R(K)$. Throughout the paper $R(K)$ is closed. By $\mathbb{N}$ we denote the set of all positive integers. The canonical orthonormal basis for $\ell^2(\mathbb{N})$ is the sequence $\{e_n\}_{n=1}^\infty$, where $e_n = \{0, 0, 0, \cdots, 1_{\text{nth place}}, 0, 0, \cdots\}$ for all $n \in \mathbb{N}$. For a sequence of vectors $\{f_k\}_{k \in I} \subset \mathcal{H}$, the closure of the span$\{f_k\}_{k \in I}$ is denoted by $[f_k]_{k \in I}$.

The following key-theorem can be found in [13]

**Theorem 1.1.** [13] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces and let $L_1 \in B(\mathcal{H}_1, \mathcal{H})$, $L_2 \in B(\mathcal{H}_2, \mathcal{H})$. The following statements are equivalent:

(i) $R(L_1) \subset R(L_2)$
(ii) $L_1L_1^* \leq \lambda^2 L_2 L_2^*$ for some $\lambda \geq 0$ and
(iii) there exists a $C \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $L_1 = L_2 C$ and $\|C\|^2 = \inf \{\mu : L_1 L_1^* \leq \mu L_2 L_2^*\}$.

1.1. **K-frames in Hilbert spaces.** Feichtinger and Werther [14] introduced a family of analysis and synthesis systems with frame-like properties for closed subspaces of $\mathcal{H}$ and call it an atomic system (or local atoms). The motivation for the atomic system is based on examples arising in sampling theory, see [15]. One of the important properties of the atomic system is that it can generate a proper subspace even though they do not belong to them.

**Definition 1.2.** [14] Let $\mathcal{H}_o$ be a closed subspace of $\mathcal{H}$. A sequence $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ is called a family of local atoms (or atomic system) for $\mathcal{H}_o$, if

(i) $\{f_k\}_{k=1}^\infty$ is Bessel sequence in $\mathcal{H}$,
(ii) there exists a sequence of linear functionals $\{c_k\}$ and a real number $C > 0$ such that

$$\sum_{k=1}^\infty |c_k(f)|^2 \leq C\|f\|^2 \text{ for all } f \in \mathcal{H}_o,$$

(iii) $f = \sum_{k=1}^\infty c_k(f)f_k$ for all $f \in \mathcal{H}_o$.

**Remark 1.3.** Note that the linear functionals $\{c_k\}$ need only to be defined on the subspace $\mathcal{H}_o$. We say that $c_k$ are associated functionals of the local atoms $\{f_k\}$. The constant $C$ is called the atomic bound. Furthermore, the partial sum $\sum_{k=1}^N c_k(f)f_k$ of the series in (iii) can be converges to $f$ from “outside” of $\mathcal{H}_o$. The family of local atoms for $\mathcal{H}_o$ is a so-called pseudo-frame as proposed by Li and Ogawa in [19].

Găvruţa in [16] introduced and studied K-frames in Hilbert spaces to study atomic systems with respect to a bounded linear operator $K$ on Hilbert spaces.

**Definition 1.4.** [16] Let $K \in B(\mathcal{H})$. A sequence $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ is called a K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{k=1}^\infty |\langle f, f_k \rangle|^2 \leq B\|f\|^2 \text{ for all } f \in \mathcal{H}. \quad (1.2)$$

The numbers $A$ and $B$ are called lower and upper K-frame bounds, respectively. If $I$ is the identity operator on $\mathcal{H}$, then $K$-frames are the standard frames. $K$-frames are more general than standard frames in the sense that the lower frame bound only holds for the elements in the range of $K^*$. Since a $K$-frame $\{f_k\}_{k=1}^\infty$ for $\mathcal{H}$ is a Bessel sequence, we can define the pre-frame operator, analysis operator and frame operator associated with $\{f_k\}_{k=1}^\infty$. The frame operator of a $K$-frame is not invertible on $\mathcal{H}$ in general, but it is invertible on a subspace $R(K)$, where the range $R(K) \subset \mathcal{H}$ is closed. Furthermore, there are many differences between $K$-frames and standard frames, see [16]. Găvruţa [16] characterize $K$-frames in Hilbert spaces by using bounded linear operators. In [8, 17, 18, 23] some new results about $K$-frames were obtained.
1.2. Background on weaving frames. The concept of “weaving frames” for complex separable Hilbert spaces introduced by Bemrose, Casazza, Gröchenig, Lammers and Lynch in [1]. For a fixed \( m \in \mathbb{N} \), we write \([m] = \{1, 2, ..., m\}\) and \([m]^c = \mathbb{N} \setminus [m] = \{m+1, m+2, ....\}\).

**Definition 1.5.** [1] Let \( I \) be a countable indexing set. A family of frames \( \{\phi_{ij} \}_{j \in I} : i \in [m]\) for \( \mathcal{H} \) is said to be \( \text{woven} \), if there are universal constants \( A \) and \( B \) such that for every partition \( \{\sigma_i\}_{i \in [m]} \) of \( I \), the family \( \bigcup_{i \in [m]} \{\phi_{ij} \}_{j \in \sigma_i} \) is a frame for \( \mathcal{H} \) with frame bounds \( A \) and \( B \).

This new notion of weaving frames is motivated by a problem in distributed signal processing. Weaving frames has potential applications in wireless sensor networks that require distributed processing under different frames, as well as pre-processing of signals using Gabor frames. Bemrose et al. [1] gave a characterization of weaving frames (that does not require universal frame bounds) and weaving Riesz bases. They proved a geometric characterization of woven Riesz bases in terms of distance between subspaces of a Hilbert space. Sufficient conditions for weaving frames by means of perturbation theory and diagonal dominance can be found in [1]. The fundamental properties of weaving frames reviewed by Casazza and Lynch in [2]. Casazza and Lynch [2] proved some basic properties in the theory of weaving frames. They proved that an invertible operator applied to woven frames leaves them woven. Casazza and Lynch [2] considered a “weaving equivalent” of an unconditional basis for weaving Riesz basis. Casazza, Freeman and Lynch [3] extended the concept of weaving Hilbert space frames to the Banach space setting. They introduced and studied weaving Schauder frames in Banach spaces. It is proved in [3] that for any two approximate Schauder frames for a Banach space \( X \), every weaving is an approximate Schauder frame if and only if there is a uniform constant \( C \geq 1 \) such that every weaving has a \( C \)-approximate Schauder frame. Some perturbation theorems for woven approximate Schauder frames can be found in [3]. Deepshikha and Vashisht studied weaving properties of an infinite family of frames in separable Hilbert spaces in [11]. In [22, 21], authors introduced and studied weaving frames with respect to measure spaces. Weaving properties of generalized frames and fusion frames can be found in [9, 10, 20].

1.3. Outline of the paper. In this paper we give necessary and sufficient conditions for weaving \( K \)-frames in Hilbert spaces. A characterization of weaving \( K \)-frames in terms of a bounded linear operator is given, see Theorem 2.3. In Theorem 2.5, a sufficient condition for \( K \)-frames not to be weakly woven in terms of lower \( K \)-frame bounds is given. Theorem 2.7 shows that woven \( K \)-frames and weakly woven \( K \)-frames are equivalent. A characterization of weaving \( K \)-frames in terms of action of operators on \( K \)-frames is presented, see Proposition 2.9. To be precise, it is shown that an operator \( U \in \mathcal{B}(\mathcal{H}) \) applied to woven \( K \)-frames leaves them \( UK \)-woven. A Paley-Wiener type perturbation result for weaving \( K \)-frames is given in Theorem 2.12. Several examples and counter-examples are given to illustrate the results.

2. Main Results

**Definition 2.1.** A family of \( K \)-frames \( \{\{\phi_{ij} \}_{j=1}^{\infty} : i \in [m]\} \) for \( \mathcal{H} \) is said to be \( K \)-\text{woven} if there exist universal positive constants \( A \) and \( B \) such that for any partition \( \{\sigma_i\}_{i \in [m]} \) of \( \mathbb{N} \), the family \( \bigcup_{i \in [m]} \{\phi_{ij} \}_{j \in \sigma_i} \) is a \( K \)-frame for \( \mathcal{H} \) with lower and upper \( K \)-frame bounds \( A \) and \( B \), respectively.

As in the case of ordinary frames, see [1, Proposition 3.1], every weaving has an universal upper frame bound.

**Proposition 2.2.** For each \( i \in [m] \), let \( \{\phi_{ij} \}_{j=1}^{\infty} \) be a \( K \)-frame for \( \mathcal{H} \) with \( K \)-frame bounds \( A_i \) and \( B_i \). Then, for any partition \( \{\sigma_i\}_{i \in [m]} \) of \( \mathbb{N} \), the family \( \bigcup_{i \in [m]} \{\phi_{ij} \}_{j \in \sigma_i} \) is a Bessel sequence with Bessel bound \( \sum_{i \in [m]} B_i \). That is, \( \sum_{i \in [m]} B_i \) is one of the choice for an universal upper \( K \)-frame bound.
Proof. Let $\{\sigma_i\}_{i \in [m]}$ be any partition of $\mathbb{N}$. Then
\[
\sum_{i \in [m]} \sum_{j \in \sigma_i} |(f, \phi_{ij})|^2 \leq \sum_{i \in [m]} \sum_{j \in \mathbb{N}} |(f, \phi_{ij})|^2 \leq \left( \sum_{i \in [m]} B_i \right) \|f\|^2 \text{ for all } f \in \mathcal{H}.
\]
This gives the required universal upper $K$-frame bound for the family $\bigcup_{i \in [m]} \{\phi_{ij}\}_{j \in \sigma_i}$. □

The following theorem gives a necessary and sufficient condition for weaving $K$-frames in terms of an operator.

Theorem 2.3. For each $i \in [m]$, suppose $\{\phi_{ij}\}_{j=1}^\infty$ is a $K$-frame for $\mathcal{H}$ with bounds $A_i$ and $B_i$. The following conditions are equivalent.

(i) The family $\{\{\phi_{ij}\}_{j=1}^\infty : i \in [m]\}$ is $K$-woven.

(ii) There exists $A > 0$ such that for any partition $\sigma = \{\sigma_i\}_{i \in [m]}$ of $\mathbb{N}$ there exists a bounded linear operator $M_\sigma : \ell^2(\mathbb{N}) \to \mathcal{H}$ such that
\[
M_\sigma(e_j) = \begin{cases} 
\phi_{1j}, & j \in \sigma_1 \\
\phi_{2j}, & j \in \sigma_2 \\
\vdots \\
\phi_{mj}, & j \in \sigma_m
\end{cases},
\]
and $AKK^* \leq M_\sigma M_\sigma^*$, where $\{e_j\}_{j=1}^\infty$ is the canonical orthonormal basis for $\ell^2(\mathbb{N})$.

Proof. (i) $\Rightarrow$ (ii) : Suppose $A$ is an universal lower $K$-frame bound for the family $\{\{\phi_{ij}\}_{j=1}^\infty : i \in [m]\}$. For any partition $\sigma = \{\sigma_i\}_{i \in [m]}$ of $\mathbb{N}$, let $T_\sigma$ be the pre-frame operator associated with the Bessel sequence $\bigcup_{i \in [m]} \{\phi_{ij}\}_{j \in \sigma_i}$.

Choose $M_\sigma = T_\sigma$. Then, $M_\sigma(e_j) = T_\sigma(e_j) = \phi_{ij}$ for all $j \in \sigma_i$ ($i \in [m]$).

Next, we compute
\[
A(KK^* f, f) = A\|K^* f\|^2 \\
\leq \sum_{i \in [m]} \sum_{j \in \sigma_i} |(f, \phi_{ij})|^2 \\
= \sum_{j \in \mathbb{N}} |(f, M_\sigma(e_j))|^2 \\
= \sum_{j \in \mathbb{N}} |(M_\sigma^* f, e_j)|^2 \\
= \|M_\sigma^* f\|^2 \\
= \langle M_\sigma M_\sigma^* f, f \rangle \text{ for all } f \in \mathcal{H}.
\]

This gives $AKK^* \leq M_\sigma M_\sigma^*$.

(ii) $\Rightarrow$ (i) Let $\{\sigma_i\}_{i \in [m]}$ be any partition of $\mathbb{N}$. Then, by using (ii), for all $f \in \mathcal{H}$ we have
\[
A\|K^* f\|^2 = A(KK^* f, f) \leq \langle M_\sigma M_\sigma^* f, f \rangle = \|M_\sigma^* f\|^2 = \sum_{j \in \mathbb{N}} |(M_\sigma^* f, e_j)|^2 = \sum_{i \in [m]} \sum_{j \in \sigma_i} |(f, \phi_{ij})|^2.
\]

This gives the lower $K$-frame inequality. On the other hand, by Proposition 2.2 the positive number $\sum_{i \in [m]} B_i$ is one of the choice of an universal upper $K$-frame bound. Hence the family $\{\{\phi_{ij}\}_{j=1}^\infty : i \in [m]\}$ is $K$-woven. □

Next we give an applicative example of Theorem 2.3.

Example 2.4. Let $\mathcal{H} = \ell^2(\mathbb{N})$, $m = 2$ and $\{e_j\}_{j=1}^\infty$ be the canonical orthonormal basis for $\mathcal{H}$. 
(a) Define families \( \Phi \equiv \{ \phi_{1j} \}_{j=1}^{\infty} \) and \( \Psi \equiv \{ \phi_{2j} \}_{j=1}^{\infty} \) in \( \mathcal{H} \) as follows:
\[
\{ \phi_{1j} \}_{j=1}^{\infty} = \{ 0 , e_2 , 0 , e_3 , 0 , e_4 , 0 , \ldots \} \\
\{ \phi_{2j} \}_{j=1}^{\infty} = \{ 0 , e_2 , e_3 , e_4 , e_4 , \ldots \}.
\]

Let \( K \) be the orthogonal projection of \( \mathcal{H} \) onto \( \{ e_j \}_{j=2}^{\infty} \). For any \( \sigma \subset \mathbb{N} \), define a bounded linear operator \( M_{\sigma} : \ell^2(\mathbb{N}) \to \mathcal{H} \) as follows:
\[
M_{\sigma}(e_j) = \begin{cases} 
\phi_{1j}, & j \in \sigma \\
\phi_{2j}, & j \in \sigma^c.
\end{cases}
\]

Then, for a given \( f = \{ \alpha_1 , \alpha_2 , \cdots \} \in \mathcal{H} \) we have \( M_{\sigma}^* f = (0, \alpha_2, \alpha_3, \alpha_3, \ldots) \), where
\[
\tilde{\alpha}_n = \begin{cases} 
0, & 2n - 1 \in \sigma \\
\alpha_n, & 2n - 1 \in \sigma^c.
\end{cases}
\]

Thus, for any \( f = \{ \alpha_1 , \alpha_2 , \cdots \} \in \mathcal{H} \), we have
\[
\langle M_{\sigma} M_{\sigma}^* f, f \rangle = \| M_{\sigma} f \|^2 \geq \sum_{i \geq 2} |\tilde{\alpha}_i|^2 = \| K^* f \|^2 = \langle KK^* f, f \rangle.
\]

Hence by Theorem 2.3 \( \Phi \) and \( \Psi \) are \( K \)-woven.

(b) Define families \( \Phi_1 \equiv \{ \phi_{1j} \}_{j=1}^{\infty} \) and \( \Psi_1 \equiv \{ \phi_{2j} \}_{j=1}^{\infty} \) in \( \mathcal{H} \) as follows:
\[
\{ \phi_{1j} \}_{j=1}^{\infty} = \{ e_1 , e_2 , 0 , e_3 , e_4 , \cdots \} \\
\{ \phi_{2j} \}_{j=1}^{\infty} = \{ e_1 , 0 , e_2 , e_3 , e_5 , \cdots \}.
\]

Let \( K \) be the orthogonal projection of \( \mathcal{H} \) onto \( \{ e_j \}_{j=2}^{\infty} \). Then, \( \Phi_1 \) and \( \Psi_1 \) are \( K \)-frames for \( \mathcal{H} \).

Next we show that \( \Phi_1 \) and \( \Psi_1 \) are not \( K \)-woven. Choose \( \sigma = \mathbb{N} \setminus \{ 2 \} \). Assume that there exists a bounded linear operator \( M_{\sigma} : \ell^2(\mathbb{N}) \to \mathcal{H} \) such that
\[
M_{\sigma}(e_j) = \begin{cases} 
\phi_{1j}, & j \in \sigma \\
\phi_{2j}, & j \in \sigma^c.
\end{cases}
\]

Then, for any \( A > 0 \), we have
\[
A \langle KK^* e_2, e_2 \rangle = A \langle K^* e_2, K^* e_2 \rangle = A \| e_2 \|^2 > 0 = \| M_{\sigma} e_2 \|^2 = \langle M_{\sigma} M_{\sigma}^* e_2, e_2 \rangle.
\]

Hence, by Theorem 2.3, \( \Phi_1 \) and \( \Psi_1 \) are not \( K \)-woven.

Recall that a family of \( K \)-frames \( \left\{ \{ \phi_{ij} \}_{j=1}^{\infty} : i \in [m] \right\} \) for \( \mathcal{H} \) is said to be weakly \( K \)-woven if for any partition \( \{ \sigma_i \}_{i \in [m]} \) of \( \mathbb{N} \), the family \( \bigcup_{i \in [m]} \{ \phi_{ij} \}_{j \in \sigma_i} \) is a \( K \)-frame for \( \mathcal{H} \). The following theorem provides a sufficient condition for \( K \)-frames not to be weakly woven in terms of a lower \( K \)-frame bound. This is inspired by [1, Lemma 4.3].

**Theorem 2.5.** Suppose \( \{ \phi_{ij} \}_{j=1}^{\infty} \) is a \( K \)-frame for \( \mathcal{H} \) with bounds \( A_1 \) and \( B_1 \) (i \( \in [m] \)). Assume for any partition \( \{ \tau_i \}_{i \in [m]} \) of a finite subset of \( \mathbb{N} \) and for every \( A > 0 \) there exists a partition \( \{ \sigma_i \}_{i \in [m]} \) of \( \mathbb{N} \) \( \setminus \{ \tau_i \}_{i \in [m]} \) such that \( \bigcup_{i \in [m]} \{ \phi_{ij} \}_{j \in \sigma_i} \cup \tau_i \) has a lower \( K \)-frame bound less than \( A \). Then, there exists a partition \( \{ \tau_i \}_{i \in [m]} \) of \( \mathbb{N} \) such that \( \bigcup_{i \in [m]} \{ \phi_{ij} \}_{j \in \tau_i} \) is not a \( K \)-frame for \( \mathcal{H} \).

**Proof.** Let \( \tau_{i1} = 0 \) for all \( i \in [m] \). Then, for \( A_1 = 1 \), there exists a partition \( \{ \sigma_{i1} \}_{i \in [m]} \) of \( \mathbb{N} \) such that \( \bigcup_{i \in [m]} \{ \phi_{ij} \}_{j \in \sigma_{i1} } \cup \tau_{i1} \) has a lower \( K \)-frame bound less than 1. Therefore, there exists a vector \( h_1 \in \mathcal{H} \) such that
\[
\sum_{i \in [m]} \sum_{j \in \sigma_{i1} \cup \tau_{i1} } |\langle h_1, \phi_{ij} \rangle|^2 < \| K^* h_1 \|^2.
\]
Since $\sum_{i \in [m]} \sum_{j \in \mathbb{N}} |\langle h_1, \phi_{ij} \rangle|^2 < \infty$, there exists $k_1 \in \mathbb{N}$ such that

$$\sum_{i \in [m]} \sum_{j > k_1} |\langle h_1, \phi_{ij} \rangle|^2 < \|K^*h_1\|^2.$$ 

Choose a partition $\{\tau_i\}_{i \in [m]}$ of $[k_1]$ such that $\tau_1 = \tau_1 \cup (\sigma_i \cap [k_1])$ for all $i \in [m]$ and $A_2 = \frac{1}{2}$. Then, there exists a partition $\{\sigma_i\}_{i \in [m]}$ of $\mathbb{N}$ such that for any $i \in [m]$ that $\sum_{i \in [m]} \sum_{j \in \sigma_i \cup \tau_i} |\langle h_2, \phi_{ij} \rangle|^2 < \frac{1}{2} \|K^*h_2\|^2$.

Continuing in this way, for $A_n = \frac{1}{n}$ and for a partition $\{\tau_{ni}\}_{i \in [m]}$ of $[k_n-1]$ such that $\tau_{ni} = \tau_{ni-1} \cup (\sigma_{n-1} \cap [k_n-1])$ ($i \in [m]$), there exists a partition $\{\sigma_{ni}\}_{i \in [m]}$ of $\mathbb{N} \setminus \{\tau_{ni}\}_{i \in [m]}$ such that $\sum_{i \in [m]} \sum_{j \in \sigma_{ni} \cup \tau_{ni}} |\langle h_n, \phi_{ij} \rangle|^2 < \frac{1}{n} \|K^*h_n\|^2$, (2.1)

and we can find a positive integer $k_n > k_{n-1}$ such that

$$\sum_{i \in [m]} \sum_{j > k_n} |\langle h_n, \phi_{ij} \rangle|^2 < \frac{1}{n} \|K^*h_n\|^2.$$ (2.2)

Choose a partition $\{\pi_i\}_{i \in [m]}$ of $\mathbb{N}$, where $\pi_i = \bigcup_{n \in \mathbb{N}} \{\tau_{ni}\}$. We show that the family $\bigcup_{i \in [m]} \{\phi_{ij}\}_{j \in \pi_i}$ is not a $K$-frame for $\mathcal{H}$. Assume that $\bigcup_{i \in [m]} \{\phi_{ij}\}_{j \in \pi_i}$ is a $K$-frame for $\mathcal{H}$ with bounds $\alpha$ and $\beta$.

By the Archimedean property there exists a $\eta \in \mathbb{N}$ such that $\eta > \frac{2}{\alpha}$.

By using (2.1), (2.2), we compute

$$\sum_{i \in [m]} \sum_{j \in \pi_i} |\langle h_\eta, \phi_{ij} \rangle|^2 \leq \sum_{i \in [m]} \sum_{j \in \tau_{n+1}} |\langle h_\eta, \phi_{ij} \rangle|^2 + \sum_{i \in [m]} \sum_{j \geq k_n} |\langle h_\eta, \phi_{ij} \rangle|^2$$

$$\leq \sum_{i \in [m]} \sum_{j \in \tau_{n+1}} |\langle h_\eta, \phi_{ij} \rangle|^2 + \sum_{i \in [m]} \sum_{j \geq k_n} |\langle h_\eta, \phi_{ij} \rangle|^2$$

$$\leq \frac{1}{\eta} \|K^*h_\eta\|^2 + \frac{1}{\eta} \|K^*h_\eta\|^2$$

$$= \frac{2}{\eta} \|K^*h_\eta\|^2$$

$$< \alpha \|K^*h_\eta\|^2,$$ a contradiction.

This completes the proof.

Theorem 2.5 gives a necessary condition for weakly woven $K$-frames.

**Proposition 2.6.** Suppose the family of $K$-frames $\{\{\phi_{ij}\}_{i \in [m]} \}_{i \in [m]}$ for $\mathcal{H}$ is weakly $K$-woven. Then, there exists a partition $\{\tau_i\}_{i \in [m]}$ of some finite subset of $\mathbb{N}$ and $A > 0$ such that for any partition $\{\sigma_i\}_{i \in [m]}$ of $\mathbb{N} \setminus \{\tau_i\}_{i \in [m]}$, the family $\bigcup_{i \in [m]} \{\phi_{ij}\}_{j \in \sigma_i \cup \tau_i}$ has a lower $K$-frame bound $A$. 


Next we show that woven $K$-frames are equivalent to weakly woven $K$-frames.

**Theorem 2.7.** Suppose $\Phi \equiv \{\phi_j\}_{j=1}^{\infty}$ and $\Psi \equiv \{\psi_j\}_{j=1}^{\infty}$ are $K$-frames for $\mathcal{H}$. The following are equivalent:

(i) $\Phi$ and $\Psi$ are $K$-woven.

(ii) $\Phi$ and $\Psi$ are weakly $K$-woven.

**Proof.** $(i) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (i)$: First we note that an universal upper $K$-frame bound for $\Phi$ and $\Psi$ can be obtained from Proposition 2.2. So it is sufficient to compute an universal lower $K$-frame bound for $\Phi$ and $\Psi$.

By Proposition 2.6, there exist disjoint finite sets $I$ and $J$ of $\mathbb{N}$ and $A > 0$ satisfying:

($\Phi$) For any partition $\{\sigma, \delta\}$ of $\mathbb{N} \setminus (I \cup J)$, the family $\{\phi_j\}_{j \in I \cup \sigma} \cup \{\psi_j\}_{j \in J \cup \delta}$ has a lower $K$-frame bound $A$.

We can permute both the $K$-frames $\Phi$ and $\Psi$ (if necessary), so that $I \cup J = [m]$.

**Claim:** For any partition $\{I_0, J_0\}$ of $[m]$, there exists $A_0 > 0$ such that for any partition $\{\sigma, \delta\}$ of $\mathbb{N} \setminus [m]$ the family $\{\phi_j\}_{j \in I_0 \cup \epsilon} \cup \{\psi_j\}_{j \in J_0 \cup \delta}$ has a lower $K$-frame bound $A_0$. So that $\Phi$ and $\Psi$ are $K$-woven with an universal lower $K$-frame bound $\min \{A_0 : \{I_0, J_0\} \text{ is a partition of } [m]\} > 0$.

Suppose our claim is not true. Then, there exists a partition $\{I_1, J_1\}$ of $[m]$ such that for each positive $\alpha$, there exists a partition $\{\sigma_\alpha, \delta_\alpha\}$ of $\mathbb{N} \setminus [m]$ such that $\{\phi_j\}_{j \in I_1 \cup \sigma_\alpha} \cup \{\psi_j\}_{j \in J_1 \cup \delta_\alpha}$ has a lower $K$-frame bound less than $\alpha$. Therefore, for any $n \in \mathbb{N}$, there exists a partition $\{\sigma_n, \delta_n\}$ of $\mathbb{N} \setminus [m]$ such that $\{\phi_j\}_{j \in I_1 \cup \sigma_n} \cup \{\psi_j\}_{j \in J_1 \cup \delta_n}$ has a lower $K$-frame bound less than $n$. Thus, there exists $h_n \in \mathcal{H}$ such that

$$
\sum_{j \in I_1 \cup \sigma_n} |\langle h_n, \phi_j \rangle|^2 + \sum_{j \in J_1 \cup \delta_n} |\langle h_n, \psi_j \rangle|^2 < \frac{1}{n} \|K^*(h_n)\|^2 \leq \frac{1}{n} \lambda^2 \|h_n\|^2, \text{ where } \lambda^2 = \|K^*\|^2.
$$

That is

$$
\sum_{j \in I_1 \cup \sigma_n} |\langle \zeta g_n, \phi_j \rangle|^2 + \sum_{j \in J_1 \cup \delta_n} |\langle \zeta g_n, \psi_j \rangle|^2 < \frac{1}{n}, \text{ where } \zeta = \frac{1}{\lambda} \text{ and } g_n = \frac{h_n}{\|h_n\|} \quad (n \in \mathbb{N}).
$$

Note that $K^*(\zeta g_n) \neq 0$ ($n \in \mathbb{N}$).

Now we proceed in the following steps:

**Step 1:** By hypothesis, for each $n \in \mathbb{N}$ there exist a partition $\{\sigma_n, \delta_n\}$ of $\mathbb{N} \setminus [m]$ and a unit vector $g_n \in \mathcal{H}$ such that

$$
\sum_{j \in I_1 \cup \sigma_n} |\langle \zeta g_n, \phi_j \rangle|^2 + \sum_{j \in J_1 \cup \delta_n} |\langle \zeta g_n, \psi_j \rangle|^2 < \frac{1}{n} \quad (2.3)
$$

and the sets $\sigma_n$ and $\delta_n$ satisfy the following properties:

1. For every $k = 1, 2, \ldots$, either $m + k \in \sigma_n$ or $m + k \in \delta_n$ for all $n \geq k$.
2. There is a partition $\{\sigma, \delta\}$ of $\mathbb{N} \setminus [m]$ such that $m + k \in \sigma$ implies that $m + k \in \sigma_n$ for all $n \geq k$ or if $m + k \in \delta$ implies that $m + k \in \delta_n$ for all $n \geq k$.

Furthermore, the sequence $\{\zeta g_n\}_{n=1}^{\infty}$ is bounded, so there exists a subsequence $\{\zeta g_{n_i}\}_{i=1}^{\infty}$ of $\{\zeta g_n\}_{n=1}^{\infty}$ such that $\{\zeta g_{n_i}\}_{i=1}^{\infty}$ converges weakly to $h \in \mathcal{H}$. We reindex, $\zeta g_{n_i} \to \zeta g_l$ and $\sigma_{n_i} \to \sigma_l$, $\delta_{n_i} \to \delta_l$. Note that (2.3), (1) and (2) are satisfied by this constructed sequence.

**Step 2:** In this step we show that $K^*(h) \neq 0$. Let $\tilde{K}^* : N(K^*)^1 \to R(K^*)$ be the restriction of $K^*$ on $N(K^*)^1$. Note that $R(K^*)$ is closed, $\tilde{K}^*$ is invertible and $\tilde{K}^*^{-1}$ is bounded.

For any non-zero $f \in N(K^*)^1$, $f = \tilde{K}^*^{-1}(g)$ for some $g \in R(K^*)$. 


We compute
\[
\frac{\|K^*f\|}{\|f\|} = \frac{\|\tilde{K}^*f\|}{\|f\|} = \frac{\|\tilde{K}^*(\tilde{K}^{-1}g)\|}{\|\tilde{K}^{-1}g\|} = \frac{\|g\|}{\|\tilde{K}^{-1}g\|} \geq \frac{1}{\|\tilde{K}^{-1}\|}. \tag{2.4}
\]

Fix \( p \in \mathbb{N} \) such that \( p > \frac{2\lambda^2\|\tilde{K}^{-1}\|^2}{A} \). So, we can find \( n_p \in \mathbb{N} \) such that for all \( n \geq n_p > p \), we have
\[
\sum_{j \in [m+p]} |\langle \zeta g_n - h, \phi_j \rangle|^2 + \sum_{j \in [m+p]} |\langle \zeta g_n - h, \psi_j \rangle|^2 < \frac{1}{2p}. \tag{2.5}
\]

From (\( \mathcal{S} \)), for each \( n \in \mathbb{N} \) we have
\[
\sum_{j \in I \cup \sigma_n} |\langle \zeta g_n, \phi_j \rangle|^2 + \sum_{j \in I \cup \delta} |\langle \zeta g_n, \psi_j \rangle|^2 \geq A \|K^*(\zeta g_n)\|^2. \tag{2.6}
\]

For \( m, p \in \mathbb{N} \), we write \([m, p] = \{m + 1, m + 2, \cdots, m + p\} \). By using (2.3), (2.4), (2.5), (2.6) and \( \sigma \cap [m, p] = \sigma_{n_p} \cap [m, p], \delta \cap [m, p] = \delta_{n_p} \cap [m, p] \), we compute
\[
\sum_{j \in I \cup \sigma} |\langle h, \phi_j \rangle|^2 + \sum_{j \in I \cup \delta} |\langle h, \psi_j \rangle|^2 \geq \sum_{j \in I \cup (\sigma \cap [m, p])} |\langle h, \phi_j \rangle|^2 + \sum_{j \in I \cup (\delta \cap [m, p])} |\langle h, \psi_j \rangle|^2 \geq \frac{1}{2} \left( \sum_{j \in I \cup (\sigma_{n_p} \cap [m, p])} |\langle \zeta g_{n_p}, \phi_j \rangle|^2 + \sum_{j \in I \cup (\delta_{n_p} \cap [m, p])} |\langle \zeta g_{n_p}, \psi_j \rangle|^2 \right) - \left( \sum_{j \in I \cup (\sigma_{n_p} \cap [m, p])} |\langle h - \zeta g_{n_p}, \phi_j \rangle|^2 + \sum_{j \in I \cup (\delta_{n_p} \cap [m, p])} |\langle h - \zeta g_{n_p}, \psi_j \rangle|^2 \right) \geq \frac{1}{2} \left( \sum_{j \in I \cup (\sigma_{n_p} \cap [m, p])} |\langle \zeta g_{n_p}, \phi_j \rangle|^2 + \sum_{j \in I \cup (\delta_{n_p} \cap [m, p])} |\langle \zeta g_{n_p}, \psi_j \rangle|^2 \right) - \frac{1}{2} \left( \sum_{j \in (\sigma_{n_p} \cap [m, p])} |\langle \zeta g_{n_p}, \phi_j \rangle|^2 + \sum_{j \in (\delta_{n_p} \cap [m, p])} |\langle \zeta g_{n_p}, \psi_j \rangle|^2 \right) - \left( \sum_{j \in [m+p]} |\langle h - \zeta g_{n_p}, \phi_j \rangle|^2 + \sum_{j \in [m+p]} |\langle h - \zeta g_{n_p}, \psi_j \rangle|^2 \right) \geq \frac{A}{2} \|K^*(\zeta g_{n_p})\|^2 - \frac{1}{2} n_p - \frac{1}{2p} \geq \frac{A}{2} \|K^*(g_{n_p})\|^2 - \frac{1}{2} n_p - \frac{1}{2p} \geq \frac{A}{2\lambda^2\|\tilde{K}^{-1}\|^2} - \frac{1}{p} > 0.
\]

This gives \( h \neq 0 \). Since \( \{\zeta g_n\}_{n=1}^\infty \subset N(K^*)^\perp \) and \( N(K^*)^\perp \) is closed, so \( K^*(h) \neq 0 \).
Step 3: In this step we will show that \( \{ \phi_j \}_{j \in I_1 \cup \sigma} \cup \{ \psi_j \}_{j \in J_1 \cup \delta} \) is not a K-frame for \( \mathcal{H} \).

By using (2.3) and (2.5), we compute
\[
\sum_{j \in I_1 \cup \sigma} |\langle h, \phi_j \rangle|^2 + \sum_{j \in J_1 \cup \delta} |\langle h, \psi_j \rangle|^2
\]
\[
= \lim_{p \to \infty} \left( \sum_{j \in I_1 \cup (\sigma \cap \{m, p\})} |\langle h, \phi_j \rangle|^2 + \sum_{j \in J_1 \cup (\delta \cap \{m, p\})} |\langle h, \psi_j \rangle|^2 \right)
\]
\[
= \lim_{p \to \infty} \left( \sum_{j \in I_1 \cup (\sigma \cap \{m, p\})} |\langle h, \phi_j \rangle|^2 + \sum_{j \in J_1 \cup (\delta \cap \{m, p\})} |\langle h, \psi_j \rangle|^2 \right)
\]
\[
\leq 2 \lim_{p \to \infty} \left( \sum_{j \in I_1 \cup (\sigma \cap \{m, p\})} |\langle \zeta g_{n_p}, \phi_j \rangle|^2 + \sum_{j \in J_1 \cup (\delta \cap \{m, p\})} |\langle \zeta g_{n_p}, \psi_j \rangle|^2 \right)
\]
\[
+ 2 \lim_{p \to \infty} \left( \sum_{j \in I_1 \cup (\sigma \cap \{m, p\})} |\langle h - \zeta g_{n_p}, \phi_j \rangle|^2 + \sum_{j \in J_1 \cup (\delta \cap \{m, p\})} |\langle h - \zeta g_{n_p}, \psi_j \rangle|^2 \right)
\]
\[
\leq 2 \lim_{p \to \infty} \frac{1}{n_p} + 2 \lim_{p \to \infty} \frac{1}{2p}
\]
\[
= 0
\]
\[
< \alpha \| K^* h \|^2 \text{ for any } \alpha > 0.
\]

Therefore, \( \{ \phi_j \}_{j \in I_1 \cup \sigma} \cup \{ \psi_j \}_{j \in J_1 \cup \delta} \) is not a K-frame for \( \mathcal{H} \). This contradicts the fact that \( \Phi \) and \( \Psi \) are weakly K-woven. The proof is complete.

\[ \square \]

Remark 2.8. If \( K = I \), the identity operator on \( \mathcal{H} \), then by Theorem 2.7 we can obtain Theorem 4.5 of [1].

Recall that the image of a standard frame under a bounded linear operator need not be a standard frame for the underlying space. But, the situation is different for K-frames. More precisely, if \( \{ f_k \}_{k=1}^\infty \) is a K-frame for \( \mathcal{H} \) then for any \( U \in B(\mathcal{H}) \), \( \{ Uf_k \}_{k=1}^\infty \) is a UK-frame for \( \mathcal{H} \). The following proposition characterizes weaving K-frames in terms of action of operators on K-frames.

**Proposition 2.9.** Let \( \{ \phi_{ij} \}_{j=1}^\infty : i \in [m] \) be a family of K-frames for \( \mathcal{H} \). The following are equivalent:

(i) \( \{ \phi_{ij} \}_{j=1}^\infty : i \in [m] \) is K-woven.

(ii) \( \{ U(\phi_{ij}) \}_{j=1}^\infty : i \in [m] \) is UK-woven for all \( U \in B(\mathcal{H}) \).

**Proof.** (i) \( \Rightarrow \) (ii) : Let \( A \) and \( B \) be universal K-frame bounds for the family \( \{ \phi_{ij} \}_{j=1}^\infty : i \in [m] \).

Let \( \sigma_i \) be any partition of \( \mathbb{N} \). Then, for any \( f \in \mathcal{H} \), we have
\[
\sum_{i \in [m]} \sum_{j \in \sigma_i} |\langle f, U(\phi_{ij}) \rangle|^2 = \sum_{i \in [m]} \sum_{j \in \sigma_i} |\langle U^* f, \phi_{ij} \rangle|^2 \leq B\| U^* f \|^2 \leq B\| U^* \|^2 \| f \|^2.
\]

Similarly
\[
\sum_{i \in [m]} \sum_{j \in \sigma_i} |\langle f, U(\phi_{ij}) \rangle|^2 = \sum_{i \in [m]} \sum_{j \in \sigma_i} |\langle U^* f, \phi_{ij} \rangle|^2 \geq A\| K^* U^* f \|^2 = A\| (UK)^* f \|^2 \text{ for all } f \in \mathcal{H}.
\]

Hence the family \( \{ U(\phi_{ij}) \}_{j=1}^\infty : i \in [m] \) is UK-woven with universal K-frame bounds \( A, B\| U^* \|^2 \).
(ii) ⇒ (i): Choose $U = I$, the identity operator on $\mathcal{H}$. Then, the family $\{\phi_{ij}\}_{j=1}^{\infty}$ is $K$-woven.

**Remark 2.10.** Let $\Phi$ and $\Psi$ be $K$-frames for $\mathcal{H}$ such that $U\Phi$ and $U\Psi$ are $UK$-woven for some $U \in B(\mathcal{H})$. Then, in general, $\Phi$ and $\Psi$ are not $K$-woven. This is justified in the following example.

**Example 2.11.** Let $\mathcal{H} = \ell^2(\mathbb{N})$ and let $K$ be the orthogonal projection of $\mathcal{H}$ onto $[e_j]_{j=2}^{\infty}$. Define $\Phi, \Psi \subset \mathcal{H}$ by

$$
\Phi = \{ \phi_{ij} \}_{j=1}^{\infty} = \{ e_1, 0, e_2, 0, e_3, 0, e_4, 0, e_5, 0, \cdots \},
$$

$$
\Psi = \{ \phi_{2j} \}_{j=1}^{\infty} = \{ 0, e_1, 0, e_2, e_3, e_4, e_5, e_6, \cdots \},
$$

where $\{e_j\}_{j=1}^{\infty}$ is the canonical orthonormal basis for $\mathcal{H}$. Then, $\Phi$ and $\Psi$ are $K$-frames for $\mathcal{H}$.

To show $\Phi$ and $\Psi$ are not $K$-woven. Choose $\sigma = \mathbb{N} \setminus \{1, 3\}$. Then, the family $\{\phi_{lj}\}_{j \in \sigma} \cup \{\phi_{2j}\}_{j \in \sigma}^c = \{0, 0, 0, e_3, 0, e_4, 0, e_5, 0, e_6, \cdots \}$ is not a $K$-frame for $\mathcal{H}$, since for any $A > 0$, we have

$$
\sum_{j \in \sigma} |\langle e_2, \phi_{lj}\rangle|^2 + \sum_{j \in \sigma^c} |\langle e_2, \phi_{2j}\rangle|^2 = \sum_{j \geq 3} |\langle e_2, e_j\rangle|^2 = 0 < A\|e_2\|^2 = A\|K^*e_2\|^2.
$$

Hence $\Phi$ and $\Psi$ are not $K$-woven.

Let $U$ be the orthogonal projection of $\mathcal{H}$ onto $[e_j]_{j=3}^{\infty}$. Then, $U\Phi \equiv \{ U(\phi_{lj}) \}_{j=1}^{\infty}$ and $U\Psi \equiv \{ U(\phi_{2j}) \}_{j=1}^{\infty}$ are $UK$-woven frames for $\mathcal{H}$. To show this, first we note that

$$
\{ U(\phi_{lj}) \}_{j=1}^{\infty} = \{ 0, 0, 0, e_3, 0, e_4, 0, e_5, 0, e_6, \cdots \}
$$

$$
\{ U(\phi_{2j}) \}_{j=1}^{\infty} = \{ 0, 0, 0, e_3, e_4, e_5, e_6, \cdots \}.
$$

For any subset $\sigma$ of $\mathbb{N}$, we have

$$
\sum_{j \in \sigma} |\langle f, U(\phi_{lj})\rangle|^2 + \sum_{j \in \sigma^c} |\langle f, U(\phi_{2j})\rangle|^2 \leq 2 \sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2 = 2\|f\|^2 \text{ for all } f \in \mathcal{H}.
$$

On the other hand, let $f \in \mathcal{H}$. Then, $f = \sum_{j=1}^{\infty} \alpha_j e_j$. Thus, we have

$$
\| (UK)^*f \|^2 = \| U^*f \|^2
$$

$$
= \| \sum_{j \in \mathbb{N}} \alpha_j U^*e_j \|^2
$$

$$
= \| \sum_{j=3}^{\infty} \alpha_j e_j \|^2
$$

$$
= \sum_{j=3}^{\infty} |\langle f, e_j\rangle|^2
$$

$$
\leq \sum_{j \in \sigma} |\langle f, U(\phi_{lj})\rangle|^2 + \sum_{j \in \sigma^c} |\langle f, U(\phi_{2j})\rangle|^2.
$$

Hence $U\Phi$ and $U\Psi$ are $UK$-woven with universal $UK$-frame bounds $1, 2$.

We end the paper with sufficient conditions for perturbation of weaving $K$-frames.

**Theorem 2.12.** Suppose $\Phi_1 \equiv \{ \phi_{lj} \}_{j=1}^{\infty}$ and $\Phi_2 \equiv \{ \phi_{2j} \}_{j=1}^{\infty}$ are $K$-frames for $\mathcal{H}$ with bounds $A_1, B_1$ (respectively, $A_2, B_2$) such that $\Phi_1$ is an orthogonal set and $\| \phi_{lj} \|^2 > \alpha > 0$ for all $j \in \mathbb{N}$. Assume that there exists $\lambda, \mu, \nu \geq 0$ such that

$$
\left( \sqrt{B_1} + \sqrt{B_2} \right) \left( \lambda + \mu \sqrt{B_1} + \nu \sqrt{B_2} \right) \| \tilde{K}^{-1} \| < \sqrt{\alpha A_1}
$$
and

$$\left\| \sum_{j=1}^{\infty} a_j (\phi_{1j} - \phi_{2j}) \right\| \leq \lambda \left\| \{a_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mathbb{N})} + \mu \left\| \sum_{j=1}^{\infty} a_j \phi_{1j} \right\| + \nu \left\| \sum_{j=1}^{\infty} a_j \phi_{2j} \right\|$$

(2.7)

for all sequence of scalars \( \{a_j\}_{j=1}^{\infty} \in \ell^2(\mathbb{N}) \), where \( \tilde{K}^*: N(K^*)^\perp \to R(K^*) \) is the restriction of \( K^* \) on \( N(K^*)^\perp \). Then, \( \Phi_1 \) and \( \Phi_2 \) are \( K \)-woven with universal bounds

$$\left( \sqrt{\alpha A_1} - \sqrt{\beta_1 + \beta_2} (\lambda + \mu \sqrt{\beta_1 + \nu \sqrt{\beta_2}}) \right)^2$$

and \( B_1 + B_2 \).

**Proof.** Let \( T^i \) be the pre-frame operator associated with \( \Phi_i, \ i = 1, 2 \). Then, \( \|T^i\| \leq \sqrt{B_i} \).

Let \( \sigma \) be any subset of \( \mathbb{N} \). Define \( \tau^i_\sigma: \ell^2(\mathbb{N}) \to \mathcal{H} \) by

$$\tau^i_\sigma(\{a_j\}_{j=1}^{\infty}) = \sum_{j \in \sigma^c} a_j \phi_{ij}, \ \{a_j\}_{j=1}^{\infty} \in \ell^2(\mathbb{N}) \quad (i = 1, 2).$$

Then, each \( \tau^i_\sigma \) is a bounded operator with norm at most \( \sqrt{B_i} \). Indeed, for all \( \{a_j\}_{j=1}^{\infty} \in \ell^2(\mathbb{N}) \), we have

$$\|\tau^i_\sigma(\{a_j\}_{j \in \mathbb{N}})\| = \left\| \sum_{j \in \sigma^c} a_j (\phi_{1j} - \phi_{2j}) \right\|$$

$$= \left\| \sum_{j \in \mathbb{N}} \gamma_j (\phi_{1j} - \phi_{2j}) \right\| \quad \text{(where } \gamma_j = a_j \text{ for } j \in \sigma^c \text{ and } \gamma_j = 0 \text{ otherwise})$$

$$\leq \lambda \left\| \{\gamma_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mathbb{N})} + \mu \left\| \sum_{j=1}^{\infty} \gamma_j \phi_{1j} \right\| + \nu \left\| \sum_{j=1}^{\infty} \gamma_j \phi_{2j} \right\|$$

$$\leq \lambda \left\| \{\gamma_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mathbb{N})} + \mu \|T^1\| \left\| \{\gamma_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mathbb{N})} + \nu \|T^2\| \left\| \{\gamma_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mathbb{N})}$$

$$\leq \left( \lambda + \mu \sqrt{B_1 + \nu \sqrt{B_2}} \right) \left\| \{a_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mathbb{N})} \text{ for all } \{a_j\}_{j=1}^{\infty} \in \ell^2(\mathbb{N}).$$

(2.8)

For any \( f \in N(K^*)^\perp \), by using (2.8) and orthogonality of \( \Phi_1 \), we compute

$$\left\| \sum_{j \in \sigma} \langle f, \phi_{1j} \rangle \phi_{1j} + \sum_{j \in \sigma^c} \langle f, \phi_{2j} \rangle \phi_{2j} \right\|$$

$$= \left\| \sum_{j \in \sigma} \langle f, \phi_{1j} \rangle \phi_{1j} + \sum_{j \in \sigma^c} \langle f, \phi_{1j} \rangle \phi_{1j} - \sum_{j \in \sigma} \langle f, \phi_{1j} \rangle \phi_{1j} + \sum_{j \in \sigma^c} \langle f, \phi_{2j} \rangle \phi_{2j} \right\|$$

$$= \left\| \sum_{j \in \mathbb{N}} \langle f, \phi_{1j} \rangle \phi_{1j} - \sum_{j \in \sigma^c} \langle f, \phi_{1j} \rangle \phi_{1j} + \sum_{j \in \sigma^c} \langle f, \phi_{2j} \rangle \phi_{2j} \right\|$$

$$\geq \left( \sum_{j \in \mathbb{N}} |\langle f, \phi_{1j} \rangle|^2 \|\phi_{1j}\|^2 \right)^{1/2} - \left\| \sum_{j \in \sigma^c} \langle f, \phi_{1j} \rangle \phi_{1j} - \sum_{j \in \sigma^c} \langle f, \phi_{2j} \rangle \phi_{2j} \right\|$$

$$\geq \sqrt{\alpha A_1} \|K^*(f)\| - \|T^1_{\sigma^c}T^{1*}(f) - T^2_{\sigma^c}T^{1*}(f)\| - \|T^2_{\sigma^c}T^{1*}(f) - T^2_{\sigma^c}T^{2*}(f)\|$$

$$= \sqrt{\alpha A_1} \|K^*(f)\| - \|T^1_{\sigma^c}T^{1*}(\tilde{K}^{*\perp}(f)) - T^2_{\sigma^c}T^{1*}(\tilde{K}^{*\perp}(f))\| - \|T^2_{\sigma^c}T^{2*}(\tilde{K}^{*\perp}(f)) - T^2_{\sigma^c}T^{2*}(\tilde{K}^{*\perp}(f))\|$$
\[
\geq \sqrt{\alpha A_1} \|K^*(f)\| - \|T^{1*}\| \|T_\sigma^* - T_\sigma^2\| \|K^{-1}\| - \|T_\sigma^2\| \|T^{1*}\| \|\tilde{K}^{*-1}\| \|\tilde{K}^*(f)\|
\]
\[
\geq \sqrt{\alpha A_1} \|K^*(f)\| - \sqrt{B_1} \left( \lambda + \mu \sqrt{B_1} + \nu \sqrt{B_2} \right) \|K^{-1}\| - \sqrt{B_2} \left( \lambda + \mu \sqrt{B_1} + \nu \sqrt{B_2} \right) \|K^{*-1}\| - \|K^*(f)\|
\]
\[
= \left[ \sqrt{\alpha A_1} - \left( \sqrt{B_1} + \sqrt{B_2} \right) \left( \lambda + \mu \sqrt{B_1} + \nu \sqrt{B_2} \right) \|K^{-1}\| \right]^2 \|K^*(f)\|^2
\]

Now by Proposition 2.2, the family \( \{\phi_{1j}\}_{j \in \sigma} \cup \{\phi_{2j}\}_{j \in \sigma^e} \) is a Bessel sequence in \( \mathcal{H} \) with universal Bessel bound \( B_1 + B_2 \). Let \( T \) be its associated pre-frame operator. Then, by using (2.9), we have

\[
\leq \left\| \sum_{j \in \sigma} \langle f, \phi_{1j} \rangle \phi_{1j} + \sum_{j \in \sigma^e} \langle f, \phi_{2j} \rangle \phi_{2j} \right\|^2
\]
\[
= \|TT^* f\|^2
\]
\[
\leq \|T\| \|T^* f\|^2
\]
\[
\leq (B_1 + B_2) \left( \sum_{j \in \sigma} |\langle f, \phi_{1j} \rangle|^2 + \sum_{j \in \sigma^e} |\langle f, \phi_{2j} \rangle|^2 \right) \text{ for all } f \in \mathcal{H}.
\]

Hence \( \Phi_1 \) and \( \Phi_2 \) are \( K \)-woven with the required universal bounds. \( \square \)

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