SINGULAR INSTANTONS WITH SO(3) SYMMETRY

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ABSTRACT. The purpose of this article is to provide an explicit construction for a family of singular instantons on $S^4 \setminus S^2$ with arbitrary real holonomy parameter $\alpha$. This family includes the original $\alpha = 1/4$, $c_2 = 3/2$ solution discovered by P. Forgács, Z. Horváth, and L. Palla, and our approach is modeled on that of their 1981 paper. Our primary tool is the ansatz due to Corrigan, Fairlie, Wilczek, and 't Hooft that constructs a self-dual Yang-Mills connection using a positive real-valued harmonic super-potential. Here we reformulate this harmonic function ansatz in terms of quaternionic notation, and we show that it arises naturally from the Levi-Civita connection of a conformally Euclidean metric.

To simplify the construction, we introduce an SO(3)-action on $S^4$, and we show by dimensional reduction that the symmetric self-duality equation on $S^4$ is equivalent to the vortex equations over hyperbolic space $H^2$. We thus obtain a similar harmonic function ansatz for hyperbolic vortices, which we also derive using conformal transformations of $H^2$. Using this ansatz, we construct the vortex equivalents of the symmetric 't Hooft instantons, and we prove using the equivariant ADHM construction that they provide a complete description of all hyperbolic vortices. We also analyze when two vortices constructed by this ansatz are gauge equivalent, obtaining the surprising result that two such vortices are completely determined by the gauge transformation between them.

INTRODUCTION

In recent years, the study of singular Yang-Mills fields has been an extremely active area of research. Considering SU(2) instantons on four manifolds with codimension two singularities, it was found that these connections can admit non-trivial holonomy around arbitrarily small circles linking the embedded singular surface. An analytical theory for such instantons with holonomy singularity has been developed in [15, 16]. Although we currently have an understanding of the moduli space for such singular instantons, the literature in this field has a conspicuous dearth of explicit examples. A singular solution on $T^2 \times D^2$ is given in the appendix to [15], although of greater interest are solutions on the standard model $S^4 \setminus S^2$.

The first example of a singular instanton was discovered by P. Forgács, Z. Horváth, and L. Palla. In their 1981 paper [10], they describe a self-dual Yang-Mills field on $S^4 \setminus S^2$ with the fractional Chern class $c_2 = 3/2$. At first their result was not readily accepted, due to the resistance to the new idea of a fractional charge. They later published a second paper [11] defending their result, and since then over a decade of successful research into the field has eliminated any initial skepticism. Nevertheless, their construction itself remains poorly understood.

The goal of this article is to elucidate and extend the work of Forgács et al., writing their construction using simpler notation, explaining the motivation behind their formulae, and generalizing to obtain a family of singular instantons with varying holonomy parameter. To this work we shall contribute a mathematical perspective, exchanging indices and Pauli matrices for more invariant complex, quaternionic, and spinor notation, and offering geometric interpretations for the equations involved.

Section 1 is devoted to the construction of instantons on $S^4$ employing the ansatz proposed by the physicists Corrigan, Fairlie, and Wilczek in 1976 and described in [8, 12]. Starting with a
positive real-valued function $\rho$ on $\mathbb{R}^4$, known as the super-potential, we consider the Yang-Mills connection

$$A = \sum_{\mu, \nu} i\tilde{\sigma}_{\mu\nu} \partial' \log \rho \, dx^\mu.$$  

Here the anti-symmetric matrix $\tilde{\sigma}_{\mu\nu}$ is defined as

$$\tilde{\sigma}_{\mu\nu} = \begin{cases} \sigma_{ij} = \frac{1}{4!} [\sigma_i, \sigma_j] \\ \sigma_{i0} = -\frac{i}{2} \sigma_i \end{cases}$$

where the $\sigma_i$ are the standard Pauli matrices generating the Lie algebra $\mathfrak{su}(2)$. For such a connection, the self-duality equation $\ast F_A = F_A$ is equivalent to the condition $\Delta \rho = 0$. By reversing orientation, this construction can also be used to generate anti-self-dual connections from a harmonic super-potential.

This harmonic function ansatz was used by 't Hooft to construct a class of instantons with $5n$ parameters, corresponding to the centers and scales of $n$ superimposed basic instantons. Since then, this ansatz has been shown to be the simplest case of a more general algebraic-geometric construction involving twistors discussed in [4]. More recently, both constructions have been eclipsed by the ADHM description of instantons given in [2], which provides a complete construction for all ASD connections on $S^4$ up to gauge equivalence. In Section 1 we recast the harmonic function ansatz in terms of quaternionic notation. Not only does this greatly simplify the required calculations, but also it better exhibits the underlying structure. We then show how these connections arise naturally via conformal transformations.

In Section 2 we introduce an $\text{SO}(3)$-action on $S^4$. Taking advantage of the conformal equivalence $S^4 \setminus S^1 \cong H^2 \times S^2$, we show that the symmetric SD and ASD equations over $S^4$ are equivalent to the vortex and anti-vortex equations over hyperbolic space $H^2$. This technique is known as dimensional reduction. The harmonic function ansatz for instantons then reduces to a similar ansatz for hyperbolic vortices, which we also derive using conformal transformations of hyperbolic space. After computing the vortex equivalents of the symmetric 't Hooft instantons, we use an equivariant version of the ADHM construction to provide a classification for all hyperbolic vortices. Examining gauge transformations, we obtain the surprising result that if two hyperbolic vortices constructed by the harmonic function ansatz are gauge equivalent, then they are both completely determined by the gauge transformation between them.

We return to our primary task of constructing singular instantons in Section 3. Restricting our attention to $\text{SO}(3)$-invariant connections on $S^4$, we can work instead with hyperbolic vortices. Using the unit disc model of $H^2$, singular instantons correspond to vortices with a holonomy singularity at the origin. We then proceed to construct solutions on the cut disc using the harmonic function ansatz, patching them together with gauge transformations to form global solutions on the punctured disc. In §3.1 we essentially rewrite the paper [10] in this context, and in the following section we construct our desired family of singular vortices.

1. The Harmonic Function Ansatz

1.1. Quaternionic Notation. For the duration of this section, we adopt the quaternionic notation as used in [1]. Writing $x \in \mathbb{H}$ in the form $x = x^0 + ix^1 + jx^2 + kx^3$, its conjugate is $\bar{x} = x^0 - ix^1 - jx^2 - kx^3$, and the corresponding differentials are

$$dx = dx^0 + i \, dx^1 + j \, dx^2 + k \, dx^3 \quad d\bar{x} = dx^0 - i \, dx^1 - j \, dx^2 - k \, dx^3.$$
Proof of Theorem 1. For the purposes of this proof, we restrict our attention to the potentially self-dual connection $A$. Writing $\partial A = \partial x^0 + i \partial x^1 + j \partial x^2 + k \partial x^3$, we see that $dx$ is anti-self-dual and the connection $A$ is self-dual if and only if the super-potential $\rho$ is harmonic. As expected, the exterior derivative $d$ may be written as the sum of $\partial$ and $\bar{\partial}$ components, although there are now two distinct splittings

\[
d = dx \frac{\partial}{\partial x} + \frac{\partial}{\partial \bar{x}} d\bar{x} = \frac{\partial}{\partial x} dx + d\bar{x} \frac{\partial}{\partial \bar{x}} \bar{x}
\]
due to the non-abelian nature of the operators involved. Expanding the 2-form $dx \wedge d\bar{x}$ in terms of coordinates as

\[
dx \wedge d\bar{x} = -2 \left[ i (dx^0 \wedge dx^1 + dx^2 \wedge dx^3) + j (dx^0 \wedge dx^2 + dx^3 \wedge dx^1) + k (dx^0 \wedge dx^3 + dx^1 \wedge dx^2) \right],
\]
we see that $dx \wedge d\bar{x}$ is self-dual and likewise that $d\bar{x} \wedge dx$ is anti-self-dual.

Rewriting the connection (2) in terms of this new quaternionic notation, the harmonic function ansatz now takes the surprisingly familiar form

**Theorem 1.** Given a positive real-valued super-potential $\rho$ on $\mathbb{R}^4$, the Yang-Mills connection $A^+$ defined by

\[
A^+ = -\text{Im} \left( \frac{\partial}{\partial \bar{x}} \log \rho \, d\bar{x} \right) = -\frac{1}{2} \left( \frac{\partial}{\partial \bar{x}} \log \rho \, d\bar{x} - dx \frac{\partial}{\partial x} \log \rho \right)
\]
is anti-self-dual and the connection $A^-$ defined by the conjugate expression

\[
A^- = -\text{Im} \left( \frac{\partial}{\partial x} \log \rho \, dx \right) = -\frac{1}{2} \left( \frac{\partial}{\partial x} \log \rho \, dx - d\bar{x} \frac{\partial}{\partial \bar{x}} \log \rho \right)
\]
is self-dual if and only if the super-potential $\rho$ is harmonic.

Before proceeding with the proof of this theorem the reader may want to verify that (2) and (3) both yield the same self-dual connection. Expanding (3) using coordinates, we obtain the expression

\[
A^- = \frac{1}{2} \left( \left( +i \partial_1 + j \partial_2 + k \partial_3 \right) dx^0 + \left( -i \partial_0 - k \partial_2 + j \partial_3 \right) dx^1 \right.
\]
\[
\left. + \left( -j \partial_0 + k \partial_1 - i \partial_2 \right) dx^2 + \left( -k \partial_0 - j \partial_1 + i \partial_2 \right) dx^3 \right)
\]
writing $\partial_i$ as an abbreviation for $\partial_i \log \rho$.

**Proof of Theorem 1.** For the purposes of this proof, we restrict our attention to the potentially self-dual connection $A^-$ given in (3), calling it $A$. Explicitly computing the two components of the curvature $F_A = dA + A \wedge A$, we obtain

\[
A \wedge A = -\frac{1}{2} \left( \frac{\partial}{\partial x} \log \rho \, dx \wedge d\bar{x} \frac{\partial}{\partial \bar{x}} \log \rho + d\bar{x} \frac{\partial}{\partial \bar{x}} \log \rho \wedge \frac{\partial}{\partial x} \log \rho \, dx \right)
\]
\[
dA = -\frac{1}{2} \left( \frac{\partial}{\partial x} \, dx + d\bar{x} \frac{\partial}{\partial \bar{x}} \right) \left( \frac{\partial}{\partial x} \log \rho \, dx - d\bar{x} \frac{\partial}{\partial \bar{x}} \log \rho \right)
\]
\[
= -\frac{1}{2} \left( -\frac{\partial}{\partial x} \, dx \wedge d\bar{x} \frac{\partial}{\partial \bar{x}} \log \rho + d\bar{x} \frac{\partial}{\partial \bar{x}} \wedge \frac{\partial}{\partial x} \log \rho \, dx \right).
\]
Recalling that the 2-forms $dx \wedge d\bar{x}$ and $d\bar{x} \wedge dx$ are self-dual and anti-self-dual respectively, the curvature of $A$ then splits as $F_A = F_A^+ + F_A^-$ with

$$F_A^+ = \frac{1}{2} \left( \frac{\partial}{\partial x} dx \wedge d\bar{x} \frac{\partial}{\partial x} \log \rho - \frac{\partial}{\partial \bar{x}} \log \rho \right) + \frac{\partial}{\partial \bar{x}} \log \rho \frac{\partial}{\partial \bar{x}} \log \rho
$$

$$F_A^- = -\frac{1}{2} \left( \frac{\partial}{\partial x} \log \rho + \frac{\partial}{\partial \bar{x}} \log \rho \right) \left( \frac{\partial}{\partial x} \log \rho \right) d\bar{x} \wedge dx.$$

Noting the identity

$$\frac{\partial}{\partial x} \log \rho + \frac{\partial}{\partial \bar{x}} \log \rho = \frac{1}{\rho} \frac{\partial}{\partial \bar{x}} \log \rho = -\frac{1}{4\rho} \Delta \rho,$$

we see that the self-duality equation $F_A^- = 0$ is equivalent to the condition $\Delta \rho = 0$ that the super-potential $\rho$ be harmonic.

We now calculate the curvature density $|F_A|^2$ of the self-dual connection $F_A = F_A^+ + F_A^-$, from which we can construct the Yang-Mills functional $\|F_A\|^2$ and the Chern class $c_2(A)$. Using the decomposition $F_A = F_A^+ + F_A^-$ given above, we first compute the anti-self-dual component $|F_A^-|^2$. From equation (2) we observe that $(d\bar{x} \wedge dx) - (dx \wedge d\bar{x}) = 24 \, d\mu$, where $d\mu$ is the volume form, and we immediately obtain

$$|F_A^-|^2 = \frac{3}{8} \left( \frac{1}{\rho} \Delta \rho \right)^2,$$

which clearly vanishes if the super-potential $\rho$ is harmonic.

On the other hand, the self-dual component $|F_A^+|^2$ of the curvature density is significantly more difficult to compute. Again using the expansion (2) for $dx \wedge d\bar{x}$, we have

$$|F_A^+|^2 = 2 \left( \left| \frac{\partial}{\partial x} i \frac{\partial}{\partial x} \log \rho - \left( \frac{\partial}{\partial \bar{x}} \log \rho \right) i \left( \frac{\partial}{\partial \bar{x}} \log \rho \right) \right|^2 + \left| \frac{\partial}{\partial \bar{x}} j \frac{\partial}{\partial \bar{x}} \log \rho - \left( \frac{\partial}{\partial x} \log \rho \right) j \left( \frac{\partial}{\partial x} \log \rho \right) \right|^2 + \left| \frac{\partial}{\partial x} k \frac{\partial}{\partial x} \log \rho - \left( \frac{\partial}{\partial \bar{x}} \log \rho \right) k \left( \frac{\partial}{\partial \bar{x}} \log \rho \right) \right|^2 \right),$$

which when fully expanded in terms of coordinates becomes

$$|F_A^+|^2 = \frac{1}{8} \sum_{i,j} \left[ 4 \left( \partial_i (\partial_j \log \rho) \right)^2 + 3 \left( \partial_i \log \rho \right)^2 \left( \partial_j \log \rho \right)^2 - 8 \left( \partial_i \partial_j \log \rho \right) \left( \partial_i \log \rho \right) \left( \partial_j \log \rho \right) - \left( \partial_i^2 \log \rho \right) \left( \partial_j \log \rho \right) \left( \partial_j \log \rho \right) \right].$$

If the super-potential $\rho$ is harmonic, then we can take advantage of the identity $\sum_i \partial_i^2 \log \rho = -\sum_i (\partial_i \log \rho)^2$ to simply this expression for $|F_A^+|^2$ to

$$|F_A^+|^2 = \frac{1}{2} \sum_{i,j} \left[ (\partial_i (\partial_j \log \rho))^2 - 2 \left( \partial_i \partial_j \log \rho \right) \left( \partial_i \log \rho \right) \left( \partial_j \log \rho \right) \right].$$
On the other hand, expanding the expression $\Delta \Delta \log \rho$, we obtain
\[
\Delta \Delta \log \rho = \sum_{i,j} \partial_i^2 \partial_j^2 \log \rho = - \sum_{i,j} \partial_i^2 (\partial_i \log \rho)^2 \\
= -2 \sum_{i,j} \partial_j [ (\partial_i \log \rho)(\partial_i \partial_j \log \rho) ] \\
= -2 \sum_{i,j} [(\partial_i \partial_j \log \rho)^2 + (\partial_i \log \rho)(\partial_i \partial_j \partial_j \log \rho) ] \\
= -2 \sum_{i,j} [(\partial_i \partial_j \log \rho)^2 - (\partial_i \log \rho)\partial_i (\partial_j \log \rho)^2 ] \\
= -2 \sum_{i,j} [(\partial_i \partial_j \log \rho)^2 - 2(\partial_i \log \rho)(\partial_j \log \rho) (\partial_i \partial_j \log \rho) ],
\]
again assuming that $\rho$ is harmonic and using the identity $\sum_i \partial_i^2 \log \rho = - \sum_i (\partial_i \log \rho)^2$ repeatedly.

Hence if the super-potential $\rho$ is harmonic, then the components of the curvature density $|F_A|^2$ for the self-dual connection (3) are
\[
|F_A^+|^2 = -\frac{1}{4} \Delta \Delta \log \rho, \quad |F_A^-|^2 = 0,
\]
and the Chern class $c_2(A)$ and $L^2$ norm $\|F_A\|^2$ are given by
\[
c_2(A) = \frac{1}{4\pi^2} \|F_A\|^2 = -\frac{1}{16\pi^2} \int_{\mathbb{R}^4} \Delta \Delta \log \rho \, d\mu.
\]
Here we have a factor of $4\pi^2$ instead of the customary $8\pi^2$ because the function $\xi \mapsto \text{Tr}(\xi^2)$ on the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ corresponds to the map $\xi \mapsto 2\xi^2$ in our quaternionic notation. Similarly, if we take the anti-self-dual connection (2) then the two components $|F_A^+|^2$ and $|F_A^-|^2$ are interchanged and the Chern class $c_2(A)$ switches sign. It is important to note that the scalar curvature density is gauge invariant. In other words, if two harmonic super-potentials $\rho_1$ and $\rho_2$ yield gauge equivalent connections via the ansatz of Theorem 1, then they must satisfy the equation $\Delta \Delta \log \rho_1 = \Delta \Delta \log \rho_2$.

1.2. The 't Hooft Construction. As an example of the harmonic function ansatz, we take for our super-potential the Green’s functions of the Laplacian. Although these functions have $O(1/r^2)$ poles, the corresponding singularities can be removed from the resulting connections by a gauge transformation. In the simplest case, consider the spherically symmetric harmonic function
\[
\rho = 1 + \frac{1}{|x|^2} = 1 + \frac{1}{x\bar{x}},
\]
the sum of the Green’s functions centered at the origin and infinity. Applying formula (2), this super-potential generates the anti-self-dual connection
\[
A = \text{Im} \left( \frac{x^{-1} dx}{1 + x\bar{x}} \right) = -\text{Im} \left( \frac{dx \, x^{-1}}{1 + x\bar{x}} \right),
\]
which is simply the basic instanton with $c_2 = 1$ expressed in the “singular gauge”. Applying the gauge transformation $g = x^{-1}$, we can remove the $O(1/r)$ pole at the origin to obtain this instanton’s customary form
\[
g(A) = \text{Im} \left( -x^{-1} \frac{dx \, x^{-1}}{1 + x\bar{x}} x - dx^{-1} x \right) = \text{Im} \left( \frac{\bar{x} \, dx}{1 + x\bar{x}} \right).
\]
Note that if we switch to coordinates around infinity by putting $x = y^{-1}$, then we simply interchange these two gauges (5) and (6). This connection therefore takes the same form about infinity as it does about the origin.
More generally, we can modify the basic instanton \([7]\) by applying a dilation and translation 
\[ x \mapsto \lambda^{-1}(x - a) \] with \( \lambda > 0 \) real and \( a \in \mathbb{H} \). The super-potential and associated connection then
become
\[ \rho = 1 + \frac{\lambda^2}{|x - a|^2}, \quad g_\rho(A) = \text{Im} \left( \frac{(\bar{x} - \bar{a}) \, dx}{\lambda^2 + |x - a|^2} \right). \]
Here we have again used a gauge transformation \( g_\rho = (x - a)^{-1} \) in order to remove the singularity
at the point \( x = a \).

One of the interesting features of this ansatz is that it allows us to take the superposition of several
such instantons simply by adding their super-potentials. For instance, the ’t Hooft instantons with
\[ \lambda \]
can be constructed using the harmonic function
\[ \rho = 1 + \frac{\lambda_1^2}{|x - a_1|^2} + \cdots + \frac{\lambda_k^2}{|x - a_k|^2}, \]
combining \( k \) basic instantons of the form \([7]\) with scales \( \lambda_1, \ldots, \lambda_k \) and distinct centers \( a_1, \ldots, a_k \).

1.3. Conformal Transformations. In this section, we discuss a differential geometric interpretation
of the harmonic function ansatz introduced in \([11]\). Treating the super-potential as a conformal
transformation of flat Euclidean space, the connections \([22]\) and \([23]\) arise naturally from the action
of the Levi-Civita connection on the half-spin bundles. We can then express the curvatures of these
two connections in terms of the decomposition of the Riemann curvature into its scalar, trace-
free Ricci, and conformally invariant Weyl curvature components, thereby providing an alternative
proof of Theorem \([11]\).

Starting with the flat Euclidean metric \( g_{ij} = \delta_{ij} \) on \( \mathbb{R}^4 \), we consider the conformally equivalent
metric \( g' = \rho^2 g \), given a smooth, positive, real-valued super-potential \( \rho \). The condition that \( \rho \) be
harmonic enters when calculating the scalar curvature of this new metric as in the following lemma.

**Lemma 2.** The scalar curvature \( R' \) of the conformally Euclidean metric \( g' \) given by \( g' = \rho^2 \delta_{ij} \)
vanishes if and only if the super-potential \( \rho \) is harmonic.

**Proof.** Using the expression for \( R' \) computed in \([5]\, \text{p. 125} \), in dimension \( n = 4 \) we have
\[
R' = -\rho^{-2} \left[ (n-1) \sum_\nu \partial_\nu^2 \log \rho^2 + \frac{(n-1)(n-2)}{4} \sum_\nu (\partial_\nu \log \rho^2)^2 \right]
= -6\rho^{-2} \sum_\nu \left[ \partial_\nu^2 \log \rho + (\partial_\nu \log \rho)^2 \right] = 6\rho^{-3} \Delta \rho.
\]
Hence \( R' = 0 \) if and only if \( \Delta \rho = 0 \). \( \square \)

Let \( \{e_0, \ldots, e_3\} \) be an orthonormal tangent frame for the original metric \( g \). After applying
the conformal transformation, the Levi-Civita connection for the metric \( g' \) is given with respect to this
frame by Christoffel’s formula
\[
\Gamma'^{ij}_{ik} = \partial_i \log \rho \, \delta^j_k + \partial_k \log \rho \, \delta^j_i - \partial^j \log \rho \, \delta_{ik}.
\]
In order to express this as an \( so(4) \) connection, we must rescale the tangent frame so that it is
again orthonormal with respect to the new metric \( g' \). Switching to the frame \( e'_i = \rho^{-1} e_i \) introduces
a factor of \(-\partial_i \log \rho \, \delta^j_k \) into the connection, cancelling the diagonal term and leaving us with an
expression skew-symmetric in the indices \( j \) and \( k \).

Taking the double cover \( \text{Spin}(4) \) of \( \text{SO}(4) \), we recall that the Lie algebra isomorphism \( so(4) \cong \text{spin}(4) \)
associates to a skew-symmetric matrix \( a_{ij} \) the Clifford algebra element\(^2\)
\[
-\frac{1}{4} \sum_{i \neq j} a_{ij} e_i \cdot e_j = e_i \cdot e_j \quad \text{for} \quad i \neq j.
\]
\[^2\]The Clifford algebra \( \text{Cl}(4) \) is the algebra generated by \( \mathbb{R}^4 \) subject to the relation \( v \cdot w + w \cdot v = -2(v, w) \), and the
Lie algebra \( \text{spin}(4) \) is the subspace spanned by \( \{e_i \cdot e_j\}_{i \neq j} \).
We may thus write the Levi-Civita connection in this \(\text{spin}(4)\) notation as
\[
A = \frac{1}{\rho} \sum_{i \neq j} \left( e'_j \partial^i \log \rho \right) \cdot (e'_i \, dx^j).
\]
From the decomposition \(\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1)\), we see that the complex 4-dimensional spin space splits as the direct sum \(S = S^+ \oplus S^-\) of two half-spin spaces, each of which is isomorphic to the quaternions \(\mathbb{H}\). These spaces \(S^+\) and \(S^-\) are called the spaces of self-dual and anti-self-dual spinors respectively. The two half-spin representations \(\gamma^\pm\) of the Lie algebra \(\text{spin}(4)\) on \(\mathbb{H}^\pm\) are then given by
\[
\gamma^+: v \cdot w \mapsto -\gamma(v) \gamma^+(w) \quad \gamma^- : v \cdot w \mapsto -\gamma^+(v) \gamma(w),
\]
where the Clifford action \(\gamma(\cdot)\) is simply quaternion multiplication
\[
\gamma(e'_0) = 1, \quad \gamma(e'_1) = i, \quad \gamma(e'_2) = j, \quad \gamma(e'_3) = k,
\]
and \(\gamma^+(\cdot)\) is its adjoint. Hence, the Levi-Civita connection for the conformally transformed metric \(g' = \rho^2 g\) splits into the two \(\mathfrak{sp}(1)\) components
\[
A^+ = -\text{Im} \left( \frac{\partial}{\partial x} \log \rho \, dx \right) \quad A^- = -\text{Im} \left( \frac{\partial}{\partial x} \log \rho \, dx \right)
\]
acting on the positive and negative half-spin spaces respectively. Note that these two connections agree with the connections \(A^+\) and \(A^-\) given in equations (2) and (3).

By definition, the Riemann curvature tensor \(\mathcal{R}\) is an \(\mathfrak{so}(4)\)-valued 2-form. However, using the identification \(\Lambda^2 \cong \mathfrak{so}(4)\), we may view it as a self-adjoint linear map \(\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2\) given in coordinates by
\[
\mathcal{R} (dx^i \wedge dx^j) = \frac{1}{2} \sum_{k,l} R_{ijkl} \, dx^k \wedge dx^l.
\]
Relative to the familiar decomposition \(\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-\) of the space of two-forms into its self-dual and anti-self-dual subspaces, the Riemann curvature can be written in the block matrix form
\[
\mathcal{R} = \left( \begin{array}{cc} \mathcal{W}^+ - \frac{1}{12} R & 0 \\ 0 & \mathcal{W}^- - \frac{1}{12} R \end{array} \right).
\]
Here \(R\) denotes the scalar curvature multiplied by the identity matrix, while \(R_0 : \Lambda^2_+ \rightarrow \Lambda^2_+\) is the trace-free Ricci curvature tensor, \(R^*_0 : \Lambda^2_- \rightarrow \Lambda^2_-\) is its adjoint, and \(\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-\) is the conformally invariant Weyl tensor. A standard reference for this material is [3].

We now consider the Riemann curvature \(\mathcal{R}'\) of the metric \(g'\) discussed above. Since \(g'\) is by definition conformally flat, we see that the Weyl tensor \(\mathcal{W}'\) vanishes. We also recall from Lemma 2 that if our super-potential \(\rho\) is harmonic, then the scalar curvature \(R'\) vanishes as well. All that remains is the trace-free Ricci tensor \(R'_0\), and so the Riemann curvature is simply
\[
\mathcal{R}' = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).
\]
Note that the splitting \(\text{spin}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)\) which we used to construct the connections \(A^+\) and \(A^-\) is isomorphic to the decomposition \(\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-\). We can therefore read off the curvatures \(F_{A^+}\) and \(F_{A^-}\) of these connections directly from the block form of the Riemann curvature, giving us
\[
F^+_{A^+} = 0 \quad F^+_{A^-} = R'_0 \quad F^-_{A^+} = R^*_0 \quad F^-_{A^-} = 0.
\]
Hence the connections \(A^+\) and \(A^-\) are anti-self-dual and self-dual respectively as claimed in Theorem [1].

\[\text{[1]}\] Here the various signs are determined by the Clifford algebra relation \(\gamma^\pm(v \cdot w + w \cdot v) = -2(v, w)\) and also by the convention that \(\gamma^+(e'_0 \cdot e'_1 - e'_2 \cdot e'_3) = 0\) and \(\gamma^- (e'_0 \cdot e'_1 + e'_2 \cdot e'_3) = 0\).
2. Hyperbolic Vortices

2.1. Dimensional Reduction. In this section, we examine SO(3)-invariant instantons, showing that the SD and ASD equations for Yang-Mills connections over $S^4$ with SO(3) symmetry are equivalent to the U(1) vortex equations over the hyperbolic plane $\mathcal{H}^2$. This is an example of dimensional reduction, whereby the Yang-Mills or (A)SD equations for a symmetric connection reduce to differential equations for a connection and Higgs fields (sections of the Lie algebra bundle) over a lower dimensional space.

Viewing $S^4$ as the standard conformal compactification $\mathbb{R}^4 \cup \{\infty\}$, we let SO(3) act via its fundamental representation on a three-dimensional subspace of $\mathbb{R}^4$. Expressing this using quaternionic notation, we see that an element $g \in \text{Sp}(1)$ acts on $\mathbb{H}$ according to $g : x \mapsto g x g^{-1}$, fixing the real part of $x$ and acting by the adjoint representation on its imaginary part. Regarding $S^4$ as the quaternionic projective space $P(\mathbb{H}^2)$ with homogeneous coordinates $(x : y) = (x_\alpha : y_\alpha)$, the embedding of $\mathbb{H}$ is simply the map $x \mapsto (1 : x)$. The Sp(1)-action given by

$$g : (x : y) \mapsto (g x : g y) = (g x g^{-1} : g y g^{-1})$$

then provides an extension of the above action on $\mathbb{R}^4$ to all of $S^4$.

In order to discuss SO(3)-invariant connections, we must lift this action on $S^4$ to an action on the Lie algebra bundle with fibres $\mathfrak{so}(3)$. There are two possible lifts: either SO(3) acts trivially on each fibre or it acts via the adjoint representation. For our purposes, we will consider this second, more interesting, action. Note that for any $g \in \text{SO}(3)$, the adjoint action leaves fixed an $\mathbb{R} = \mathfrak{u}(1)$ subalgebra.

Again adopting quaternionic notation, any $x \in \mathbb{H}$ can be written in the form $x = t + r Q$, with $t, r$ real, $r \geq 0$, and $Q$ pure imaginary with $Q^2 = -1$. Note that $t, r$ coordinatize the upper half-plane, which we will later regard as hyperbolic space $\mathcal{H}^2$. If $A$ is an Sp(1)-invariant connection, then its connection one-form satisfies $g A(t, r, Q) g^{-1} A(t, r, q Q g^{-1})$. The most general connection exhibiting this symmetry is of the form

$$A = \frac{1}{2} (Q a + \Phi_1 dQ + \Phi_2 Q dQ),$$

where $a = a_t dt + a_r dr$, and the $a_t, a_r, \Phi_1, \Phi_2$ are all real functions of $t, r$. The curvature $F_A$ of this connection $A$ is then

$$F_A = \frac{1}{2} \left( Q da + \frac{1}{2} (\Phi_1^2 + \Phi_2^2 + 2 \Phi_2) dQ \wedge dQ + [d\Phi_1 - a(\Phi_2 + 1)] \wedge dQ + (d\Phi_2 + a \Phi_1) \wedge Q dQ \right).$$

Putting $\Phi = \Phi_1 + i (\Phi_2 + 1)$ and writing $d_a \Phi = d\Phi + i a \Phi$, the curvature can be written much more simply as

$$F_A = \frac{1}{2} \left( Q da - \frac{1}{2} (1 - |\Phi|^2) dQ \wedge dQ + \text{Re} (d_a \Phi) \wedge dQ + \text{Im} (d_a \Phi) \wedge Q dQ \right).$$

Note that multiplication by $Q$ here behaves like multiplication by $i$.

From the above discussion, we see that an SO(3)-invariant connection $A$ on $S^4$ gives rise in a natural way to a U(1) connection $ia$ and a complex scalar field $\Phi$ on the upper half-plane. The next step is to analyze the SD and ASD equations in terms of this dimensional reduction. To determine the action of the Hodge star operator, we consider the 2-form $dx \wedge d\bar{x}$ which we already know to be self-dual. In coordinates $t, r, Q$, we have

$$dx \wedge d\bar{x} = (dt + Q dr + r dQ) \wedge (dt - Q dr - r dQ) = 2Q dt \wedge dr + r^2 dQ \wedge dQ + 2r (dt \wedge dQ + dr \wedge Q dQ),$$

where $\text{Im} (\Phi) = a_r dr + a_t dt$.
and so the Hodge star operator acts according to
\[ * Q \, dt \wedge dr = \frac{r^2}{2} \, dQ \wedge dQ \]
\[ * dt \wedge dQ = dr \wedge Q \, dQ \]
\[ * dr \wedge dQ = -dt \wedge Q \, dQ. \]
Furthermore, using the hyperbolic metric
\[ h = \frac{1}{r^2} (dt^2 + dr^2) \]
on the upper half-plane, the corresponding Hodge star operator \( *_h \) satisfies
\[ *_h dt \wedge dr = r^2 dQ \wedge dQ \]
\[ *_h dt = dr \wedge Q \, dQ \]
\[ *_h dr = -dt \wedge Q \, dQ. \]
Using complex notation with \( z = t + ir \) and noting the identity \( *_h dz = -i \, dz \), we observe that
\[ 2 \, \overline{\partial_a \Phi} = d_a \Phi - i *_h d_a \Phi. \]
We therefore conclude that the SO(3)-symmetric self-duality equation
\[ F_A = *F_A \]
on \( S^4 \) is equivalent to the following two equations on hyperbolic space \( \mathcal{H}^2 \): (9)
\[ \overline{\partial}_a \Phi = 0 \]
(10)
\[ *_h iF_a = 1 - |\Phi|^2, \]
where \( F_a = i \, da \) is the curvature of the connection \( ia \). These equations are known as the vortex equations. Similarly, the SO(3)-symmetric anti-self-dual equation
\[ F_A = -*F_A \]
is equivalent to the anti-vortex equations: (11)
\[ \partial_a \Phi = 0 \]
(12)
\[ *_h iF_a = |\Phi|^2 - 1. \]
The first equation in each pair is simply the condition that \( \Phi \) be holomorphic (or anti-holomorphic) with respect to the holomorphic structure compatible with the connection \( ia \). The second equation then expresses a form of duality between the connection and Higgs field. These vortex equations are discussed in great detail in [13]. Note that if we consider these vortex and anti-vortex equations over the plane \( \mathbb{R}^2 \) with the flat metric \( h = \frac{1}{2} (dx^2 + dy^2) \), then we obtain the Euclidean vortex and anti-vortex equations in their customary form as given by (1.7) and (1.8) on p. 55 of [13].
We now compute the \( L^2 \) norm of the curvature \( F_A \) using the standard metric \( |\xi|^2 = \xi \bar{\xi} = -\xi^2 \) on the Lie algebra \( \mathfrak{sp}(1) \) of imaginary quaternions. The Yang-Mills action, or energy, of this SO(3)-invariant connection is thus
\[ \|F_A\|^2 = \int_{S^4} -F_A \wedge *F_A \]
\[ = \frac{1}{8} \int_{S^4} \left( da \wedge *_h da + (1 - |\Phi|^2) *_h (1 - |\Phi|^2) + 2 \, \text{Re} \, d_a \Phi \wedge *_h \text{Re} \, d_a \Phi + 2 \, \text{Im} \, d_a \Phi \wedge *_h \text{Im} \, d_a \Phi \right) \wedge (dQ \wedge Q \, dQ). \]
Note that the left factor of the integrand is independent of the variable \( Q \). Since \( Q \) parametrizes the unit 2-sphere with volume form \( \frac{1}{2} (dQ \wedge Q \, dQ) \), we can integrate out a factor of \( \int_{S^2} dQ \wedge Q \, dQ = 8\pi \), leaving an integral over hyperbolic space. The action then becomes
\[ \|F_A\|^2 = \pi \left( \|F_a\|^2_h + 2 \|d_a \Phi\|^2_h + \|1 - |\Phi|^2\|^2_h \right). \]
\[ ^4 \text{After adjusting to the slightly different notation of [13], using } a' = -a \text{ and } \Phi' = \bar{\Phi}, \text{ the reader will find that equations (11.5a) and (11.5b) on p. 99 of [13] should be switched.} \]
which we recognize as the U(1) Yang-Mills-Higgs action on hyperbolic space, at least up to a constant. From this action, we see that a finite-energy SO(3)-invariant connection on $S^4$ corresponds to a pair $(ia, \Phi)$ over $\mathcal{H}^2$ satisfying the boundary conditions

$$d_a \Phi(x) \to 0, \quad |\Phi(x)| \to 1,$$

as $|x| \to \infty$.

Next we examine the relationship between the Chern classes of an SO(3)-invariant connection $A$ on $S^4$ and those of the corresponding connection $ia$ over $\mathcal{H}^2$. Computing $c_2(A)$, we first note that the negative definite form $\xi \mapsto \text{Tr}(\xi^2)$ on the Lie algebra $\mathfrak{su}(2)$ corresponds to $\xi \mapsto 2\xi^2$ on $\mathfrak{sp}(1)$. In this quaternionic notation we therefore have

$$c_2(A) = -\frac{1}{4\pi^2} \int_{S^4} F_A \wedge F_A$$

$$= -\frac{1}{16\pi^2} \int_{S^4} [(1 - |\Phi|^2) da - 2 \text{Re} d_a \Phi \wedge \text{Im} d_a \Phi] \wedge (dQ \wedge Q dQ)$$

$$= \frac{i}{2\pi} \int_{\mathcal{H}^2} F_a = c_1(a).$$

Here we again integrate out the $S^2$ factor $dQ \wedge Q dQ$, and on the last line we apply Stokes’ theorem with the integrand

$$d(i\bar{\Phi} d\Phi) = i d \bar{\Phi} \wedge d \Phi - \Phi \bar{\Phi} da - (\Phi d \bar{\Phi} + \bar{\Phi} d \Phi) \wedge a$$

$$= -|\Phi|^2 da - 2 \text{Re} d_a \Phi \wedge \text{Im} d_a \Phi,$$

assuming that $i\bar{\Phi} d\Phi$ vanishes at infinity.

### 2.2. Another Harmonic Function ansatz.

We now return to the harmonic function ansatz that we discussed in Section 1. If we begin with a SO(3)-invariant harmonic super-potential, then the resulting SD or ASD connection will also exhibit SO(3) symmetry, and from the previous section we know that such a connection is equivalent to a hyperbolic vortex or anti-vortex. In this section, we take advantage of this dimensional reduction to provide a similar harmonic function ansatz constructing solutions to the vortex equation over hyperbolic space.

Our first step is to examine the relationship between the Laplacian on hyperbolic space $\mathcal{H}^2$ and the SO(3)-symmetric Laplacian on $\mathbb{R}^4$.

**Lemma 3.** An SO(3)-invariant function $\rho$ on $\mathbb{R}^4$ is harmonic if and only if it can be written as $\rho = r^{-1}\phi$, where $\phi$ is a harmonic function on $\mathcal{H}^2$.

**Proof.** Using the quaternionic notation $x = t + rQ$ introduced in §2.1, if $\rho = \rho(r,t)$ is an SO(3)-invariant function on $\mathbb{R}^4$, then its Laplacian is

$$\Delta \rho = -\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}\right) \rho.$$

To cancel the unwanted linear term, we put $\rho = r^{-1}\phi$, where $\phi$ is a function on $\mathcal{H}^2$. The Laplacian $\Delta$ then becomes

$$\Delta r^{-1} \phi = -\frac{1}{r} \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2}\right) \phi = r^{-3} \Delta_h \phi,$$

where the Laplacian $\Delta_h$ on $\mathcal{H}^2$ is

$$\Delta_h = -r^2 \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2}\right)$$
These two Laplacians are therefore related by \( \Delta = r^{-3} \Delta_h r \) and thus \( \rho \) is harmonic on \( \mathbb{R}^4 \) if and only if \( \phi \) is harmonic on \( \mathcal{H}^2 \).

We recall from Theorem 1 that the connection given by equation (3),

\[
A = -\text{Im} \left( \frac{\partial}{\partial x} \log \rho \, dx \right),
\]

is self-dual if and only if \( \rho \) is harmonic. In our current notation, the quaternionic differential and partial derivative in this expression are

\[
dx = dt + Q \, dr + r \, dQ, \quad \frac{\partial}{\partial x} = \frac{1}{2} \left( \frac{\partial}{\partial t} - Q \frac{\partial}{\partial r} - \cdots \right),
\]

where we have left out the portions of the partial derivative in the \( Q \) directions as these vanish when applied to \( \text{SO}(3) \)-invariant functions. Taking \( \rho = r^{-1} \phi \), we see that \( \log \rho = \log \phi - \log r \).

Expanding equation (3) using these expressions, our \( \text{SO}(3) \)-invariant self-dual connection becomes

\[
A = \frac{1}{2} Q \left[ \left( \frac{\partial}{\partial r} \log \phi - \frac{1}{r} \right) dt - \frac{\partial}{\partial t} \log \phi \, dr \right]
- r \frac{\partial}{\partial t} \log \phi \, dQ + \left( r \frac{\partial}{\partial r} \log \phi - 1 \right) Q \, dQ.
\]

As we did in §2.1, we can extract from this connection the \( U(1) \) connection

\[
\text{(15)} \quad d_a = d + i \left[ \frac{\partial}{\partial t} \log \phi - \frac{1}{r} \right] dt - \frac{\partial}{\partial t} \log \phi \, dr
\]

with curvature

\[
F_a = -i \left[ \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} \right] \log \phi + \frac{r}{r^2} \right] dt \wedge dr = -i \left( 1 - \Delta_h \log \phi \right) r^{-2} dt \wedge dr,
\]

and the complex Higgs field

\[
\text{(16)} \quad \Phi = r \left( -\frac{\partial}{\partial t} \log \phi + i \frac{\partial}{\partial r} \log \phi \right)
\]

with norm

\[
|\Phi|^2 = r^2 \left[ \left( \frac{\partial}{\partial t} \log \phi \right)^2 + \left( \frac{\partial}{\partial r} \log \phi \right)^2 \right] = |\nabla \log \phi|^2_{\mathcal{H}^2}.
\]

Writing the pair \((a, \phi)\) using complex notation with \( z = t + ir \), we obtain the hyperbolic space analogue of Theorem 1.

**Theorem 4.** Given a positive real-valued super-potential \( \phi \) on the hyperbolic upper half-plane \( \mathcal{H}^2 \), the connection and Higgs field pair \((a, \Phi)\) defined by

\[
\text{(17)} \quad \bar{\partial}_a = \bar{\partial} + \bar{\partial} \log \phi + \frac{d\bar{z}}{z - \bar{z}} \quad \Phi = i (z - \bar{z}) \frac{\partial}{\partial z} \log \phi,
\]

In general, the conformal Laplacian is \( L_g = \Delta + kR \), where \( R \) is the scalar curvature and \( k \) is a constant depending on the dimension. Taking the metric \( g' = e^{2f} g \), it we have

\[
L_{g'} = e^{-(d+2)f/2} L_g e^{(d-2)f/2}.
\]

Here the Euclidean metric on \( \mathbb{R}^4 \) is \( g = dt^2 + dr^2 + r^2 dS^2 \), where \( dS^2 \) is the metric on \( S^2 \). Taking the conformally equivalent metric \( g' = r^{-2} (dt^2 + dr^2) + dS^2 \) on \( \mathcal{H}^2 \times S^2 \), Lemma 2 tells us that \( R' = 0 \) since \( r^{-1} \) is harmonic. It follows that \( \Delta' = r^3 \Delta r^{-1} \).
satisfies the vortex equations (4) and (7) and the pair \((a', \Phi')\) defined by

\[
\partial' a = \partial + \partial \log \phi - \frac{dz}{z - \bar{z}} \tag{18}
\]

\[
\Phi' = -i (z - \bar{z}) \frac{\partial}{\partial \bar{z}} \log \phi, \tag{19}
\]

satisfies the anti-vortex equations (11) and (12) if and only if the super-potential \(\phi\) is harmonic.

**Proof.** Recalling that \(*_{h} 1 = r^{-2} dt \wedge dr\) with our hyperbolic metric, we see that the second of the vortex equations \(iF_a = *_{h} (1 - |\Phi|^2)\) reduces to

\[
\frac{1}{\phi} \Delta_h \phi = \Delta_h \log \phi - |\nabla \log \phi|^2_h = 0.
\]

Using the complex form (17), it is easy to verify that the holo morphicity condition \(\bar{\partial} a \Phi = 0\) likewise reduces to \(\Delta_h \log \phi = 0\). Hence the pair \((a, \Phi)\) satisfies the vortex equations if and only if the super-potential \(\phi\) is harmonic. Similarly, the pair \((a', \phi')\) is derived by dimensional reduction from the ASD connection (2), and so it satisfies the anti-vortex equations if and only if \(\phi\) is harmonic. \(\square\)

Computing the Chern class \(c_1\) for a pair \((a, \Phi)\) satisfying the second of the vortex equations (10), we have

\[
c_1(a) = \frac{i}{2\pi} \int_{\mathcal{H}^2} F_a = \frac{1}{2\pi} \int_{\mathcal{H}^2} *_{h} (1 - |\Phi|^2) \cdot
\]

For the vortex over the upper half-plane \(\mathcal{H}^2\) constructed in (15) and (16) using a harmonic super-potential \(\phi\), this Chern class takes the form

\[
c_1(a) = \frac{1}{2\pi} \int_{\mathbb{R}^2_+} (1 - \Delta_h \log \phi) r^{-2} dt \wedge dr = \frac{1}{2\pi} \int_{\mathbb{R}^2_+} \left( \frac{1}{r^2} - 4 \left| \frac{\partial}{\partial z} \log \phi \right|^2 \right) dt \wedge dr.
\]

Likewise, if we use the above ansatz to construct the anti-vortex corresponding to a harmonic super-potential, then the Chern class switches sign.

As in §1.3, we note that the curvature \(F_a\) of a vortex is gauge invariant. Therefore, if two harmonic super-potentials \(\phi_1\) and \(\phi_2\) over hyperbolic space yield gauge equivalent vortices, then they must satisfy the equations

\[
\left| \frac{\partial}{\partial z} \log \phi_1 \right| = \left| \frac{\partial}{\partial z} \log \phi_2 \right|
\]

and \(\Delta_h \log \phi_1 = \Delta_h \log \phi_2\).

Note that it is significantly simpler to calculate \(c_1\) directly from the vortex construction on \(\mathcal{H}^2\) than it is by invoking dimensional reduction and computing the equivalent Chern class \(c_2\) for the corresponding SO(3)-invariant instanton over \(S^4\). Indeed, by comparing the above expression for \(c_1(a)\) with the expression (4) for \(c_2(A)\), we obtain a circuitous proof of the identity

\[
\int_{\mathbb{R}^2_+} (1 - \Delta_h \log \phi) r^{-2} dt \wedge dr = -\int_{\mathbb{R}^2_+} \left( \frac{1}{r^2} \Delta \Delta \log \phi \right) dt \wedge dr
\]

for a harmonic function \(\phi\) defined on the upper half-plane, where \(\Delta_h\) is the Laplacian on \(\mathcal{H}^2\) given by (14) and \(\Delta\) is the Laplacian on \(\mathbb{R}^4\) given by (13).

### 2.3. Conformal Transformations Revisited.

Instead of relying on dimensional reduction to derive the vortex ansatz of Theorem 4, we present here an interpretation of this construction that is entirely intrinsic to hyperbolic space. As we did in §1.3, we can treat the super-potential as a conformal transformation and then compute the Levi-Civita connection of the resulting metric. Since we are working on two-dimensional hyperbolic space, we can take advantage of complex notation to simplify our task.
Let $\bar{\partial}$ be the standard holomorphic structure on the complex upper half-plane. Choosing a holomorphic tangent frame (i.e., a single holomorphic section $e$), consider the Hermitian metric $g$ specified by $(e,e)_g = \rho^2$, where $\rho$ is a smooth nonzero real-valued function. With respect to our holomorphic frame $e$, the unique connection compatible with both the holomorphic structure $\bar{\partial}$ and the metric $g$ is specified by the $(1,0)$-form
\[ a = \rho^{-2}(\partial \rho^2) = 2 \partial \log \rho. \]

To express this in the form of a unitary connection (in this case given by a purely imaginary complex 1-form), we must switch to a tangent frame that is orthonormal with respect to the metric $g$. In terms of the unitary frame $e' = \rho^{-1}e$, the connection $a$ then becomes
\[ a' = \partial \log \rho - \bar{\partial} \log \rho = 2i \Im \partial \log \rho = -2i \Im \bar{\partial} \log \rho \]
and the new holomorphic structure is $\bar{\partial}' = \bar{\partial} - \partial \log \rho$, which we observe is compatible with the connection $a'$. In either frame, the curvature of this connection is given by
\[ F_a = 2 \bar{\partial} \partial \log \rho = -i \Delta \log \rho \, d\mu, \]
where the volume form $d\mu$ and the Laplacian $\Delta$ are both taken here with respect to the Euclidean metric on $\mathbb{R}^2$.

When working with $H^2$, the hyperbolic metric $h$ on the upper half-plane corresponds to the function $\rho = r^{-2}$. Taking a conformal transformation, we consider the metric $h'$ specified by a function of the form $\rho = \phi^2/r^2$ with $\phi$ harmonic. The resulting unitary connection $a$ then splits into the $(0,1)$ and $(1,0)$ components
\[ \bar{\partial}_a = \bar{\partial} - \bar{\partial} \log \phi - \frac{d\bar{z}}{2ir}, \quad \partial_a = \partial + \partial \log \phi - \frac{dz}{2ir}, \]
noting that $r = (z - \bar{z})/2i$. The curvature of this connection is then
\[ iF_a = (\Delta \log \phi - r^{-2}) \, d\mu = (\Delta_h \log \phi - 1) \, d\mu_h \]
where $d\mu_h = r^{-2}d\mu$ is the volume form and $\Delta_h = r^2 \Delta$ is the Laplacian for the hyperbolic metric $h$. It is then easy to show that the complex Higgs field $\Phi$ defined by
\[ \Phi = 2r \frac{\partial}{\partial \bar{z}} \log \phi = 2 \frac{r}{\phi} \frac{\partial}{\partial \bar{z}} \phi \]
satisfies the anti-vortex equations $\partial_\phi \Phi = 0$ and $*_h iF_a = |\Phi|^2 - 1$ if the super-potential $\phi$ is harmonic. We observe that this pair $(a, \Phi)$ agrees with the anti-vortex (18) and (19) constructed by dimensional reduction of an anti-self-dual connection over the 4-sphere. Similarly, we can construct the vortex given by (17) by reversing orientation, thereby exchanging the holomorphic and anti-holomorphic structures $\bar{\partial}$ and $\partial$.

2.4. The Symmetric ‘t Hooft Construction. In (12) as an illustration of the harmonic function ansatz, we constructed the ‘t Hooft instantons. These are the instantons formed by taking the superposition of multiple copies of the basic instanton with varying scales and distinct centers. For our super-potential, we used a sum of the Green’s functions of the Laplacian, centered at the given points and weighted according to the corresponding scales. If we impose SO(3) symmetry on this class of instantons, we see that all of the centers must lie on a single real line. In fact, as we will demonstrate in the following section, all SO(3)-invariant instantons can be constructed in this manner—as the superposition of basic instantons on a line. In this section, we examine the hyperbolic vortices associated to these symmetric ‘t Hooft instantons by dimensional reduction.

We begin with the basic instanton with unit scale centered at the origin, which we recall is given by the $\mathbb{R}^4$ super-potential $\rho = 1 + |x|^{-2}$. The corresponding super-potential for hyperbolic space
$H^2$ is then
\[ \phi = r \rho = r + \frac{r}{r^2 + t^2} = \text{Im} \left( z - \frac{1}{z} \right) = \frac{(z - \bar{z}) (1 + z \bar{z})}{2i z \bar{z}}. \]

Taking the complex partial derivatives of its logarithm, we obtain
\[
\frac{\partial}{\partial z} \log \phi = + \frac{1}{z - \bar{z}} + \frac{\bar{z}}{1 + z \bar{z}} - \frac{1}{z} \quad \frac{\partial}{\partial \bar{z}} \log \phi = - \frac{1}{z - \bar{z}} + \frac{z}{1 + z \bar{z}} - \frac{1}{\bar{z}}.
\]

Inserting these expressions into the formula (17) gives us the connection and Higgs field
\[
\bar{\partial}_a = \bar{\partial} - \frac{d \bar{z}}{\bar{z} (1 + z \bar{z})} \quad \Phi = i \frac{\bar{z} (1 + z^2)}{z (1 + z \bar{z})},
\]
satisfying the vortex equations (9) and (10). Note that as $z$ approaches the real axis, the Higgs field obeys the boundary condition $|\Phi| \to 1$.

One of the primary results concerning solutions to the vortex equations is that they are uniquely specified up to gauge equivalence by the zeros of the Higgs field (see [13, Chapter III]). In the example above, we see that $\Phi$ vanishes at the point $z = i$. If we alter the scale of our basic instanton and translate it along the real axis, the $H^2$ super-potential becomes
\[ \phi = \text{Im} \left( z - \frac{\lambda}{z - a} \right) \]
with $\lambda > 0$ and $a$ real. The corresponding vortex is then
\[
\bar{\partial}_a = \bar{\partial} - \frac{\lambda d \bar{z}}{(\bar{z} - a) (1 + |z - a|^2)} \quad \Phi = i \frac{(\bar{z} - a) (1 + (z - a)^2)}{(z - a) (1 + |z - a|^2)},
\]
and we see that $\Phi$ vanishes at the point $z = a + i\sqrt{\lambda}$. Most generally, given a set of $k$ complex points $\{z_i\}$ in the upper half-plane, the super-potential
\[ \phi = \text{Im} \left( z - \sum_{i=1}^{k} \frac{(\text{Im} z_i)^2}{z - \text{Re} z_i} \right) \]
generates the unique hyperbolic vortex with Higgs field vanishing at the points $\{z_i\}$. Hence the centers of the instantons correspond to the real parts of the complex zeros, while the scales correspond to their imaginary parts.

2.5. The Equivariant ADHM Construction. In this section we shall use an SO(3) equivariant version of the ADHM construction in order to provide an alternative construction for the symmetric 't Hooft instantons discussed in the previous section. In addition, since the ADHM construction actually generates all possible anti-self-dual connections on bundles over $S^4$, we will then be able to show that every symmetric instanton must be gauge equivalent to one constructed using the 't Hooft ansatz. By dimensional reduction, this gives us a complete classification of hyperbolic vortices, proving that such vortices are uniquely determined up to a gauge transformation by the zeros of their Higgs fields. This is to be contrasted with Euclidean vortices, in which case the classification theorem may be proved using approximation techniques (see [13, Chapter III]), but no explicit construction for the vortex solutions is known.

Here we use the quaternionic version of the construction as discussed in [11]. When dealing with quaternionic vector spaces and linear maps, we use the convention that scalar multiplication acts on the right. Recall from [2, 14] that under our Sp(1)-action, we may view $S^4$ as the quaternionic projective space $P(\mathbb{H}^2)$, where $\mathbb{H}$ is the fundamental representation with Sp(1) acting by left quaternion multiplication.
To construct an Sp(1)-invariant ASD connection on the bundle $E \to S^4$ with Chern class $c_2(E) = -k$, we introduce the $k + 1$ dimensional quaternionic Sp(1) representation $V$ given by
\begin{equation}
V = \text{Ker} \mathcal{D}_A^* : \Gamma(S^4, E \otimes S^- \otimes S^-) \to \Gamma(S^4, E \otimes S^+ \otimes S^-)
\end{equation}
and the $k$ dimensional real Sp(1) representation $W$ given by
\begin{equation}
W = (\text{Ker} \mathcal{D}_A^* : \Gamma(S^4, E \otimes S^-) \to \Gamma(S^4, E \otimes S^+))^*\mathbb{R},
\end{equation}
where $A$ is an arbitrary connection on $E$ (the spaces $V, W$ are independent of the connection), $S^\pm$ are the two quaternionic half-spin bundles, and $\mathcal{D}_A^*$ is the adjoint of the Dirac operator with coefficients in $E \otimes S^-$ and $E$ respectively. Using these spaces $V, W$ the ADHM data consists of the three maps:
- an arbitrary Sp(1) equivariant inclusion $W \otimes \mathbb{H}_1 \hookrightarrow V$
- an Sp(1) equivariant $\mathbb{H}$-linear map $B : W \otimes \mathbb{H}_1 \to W \otimes \mathbb{H}_1$ satisfying $B^* = \bar{B}$ (i.e., $B$ is represented by a symmetric matrix)
- an Sp(1) equivariant $\mathbb{H}$-linear map $\Lambda : W \otimes \mathbb{H}_1 \to V / W \otimes \mathbb{H}_1$.

If we fix the inclusion $W \otimes \mathbb{H}_1 \hookrightarrow V$, then we say that two sets of ADHM data $(B, \Lambda)$ and $(B', \Lambda')$ are equivalent if
\[ B' = UB^{-1}, \quad \Lambda' = v\Lambda U^{-1} \]
for suitable $U \in O(W)$ and $v \in \text{Sp}(V / W \otimes \mathbb{H}_1)$. From the ADHM data, we construct an Sp(1) equivariant family of $\mathbb{H}$-linear maps $v(x) : W \otimes \mathbb{H}_1 \to V$ parametrized by $x \in \mathbb{H}$, given by
\[ v(x) = \begin{pmatrix} \Lambda \\ B - xI \end{pmatrix} \]
relative to the decomposition $V = (V / W \otimes \mathbb{H}_1) \oplus (W \otimes \mathbb{H}_1)$.

**Theorem 5 (ADHM).** There is a one-to-one correspondence between equivalence classes of ADHM data $(B, \Lambda)$ satisfying the two conditions
- non-degeneracy: $v(x)$ is injective for all $x \in \mathbb{H}$
- ADHM condition: $\Lambda^* \Lambda + B^* B : W \otimes \mathbb{H}_1 \to W \otimes \mathbb{H}_1$ is real,

and gauge equivalence classes of Sp(1)-invariant ASD connections on $E$.

To construct the connection associated to a set of ADHM data, we first observe that the non-degeneracy condition implies that $f : x \mapsto \text{Coker } v(x) \subset V$ is a smooth map from $S^4$ to the quaternionic projective space $P(V)$ (we map the point at $\infty$ to the line $V / W \otimes \mathbb{H}_1$). We then define the corresponding vector bundle $E$ and connection $A$ to be the pullback of the canonical quaternionic line bundle over $P(V)$ with its standard connection (induced by orthogonal projection from the trivial flat connection on $V$). The fact that $A$ is ASD follows from the ADHM condition. For a complete proof of the non-equivariant version of this theorem, see [1, §3.3] or [2]. The proof of equivariant version then proceeds with minimal modification.

We now give an even more precise description of Sp(1)-invariant instantons, starting by examining the characters of the representations $V$ and $W$.

**Lemma 6.** The ADHM representations $V, W$ described in (21) and (22) corresponding to the bundle $E \to S^4$ with $c_2(E) = -k$ are given by
\[ V = \mathbb{H}_1^{k+1}, \quad W = \mathbb{R}_0^k, \]
where $\mathbb{H}_1$ is the fundamental representation of Sp(1) acting by left multiplication and $\mathbb{R}_0$ is the trivial real representation.
Given a set of $k$ distinct points $\{z_1, \ldots, z_k\}$ in the complex upper half-plane, there exists a finite action solution $(a, \Phi)$ to the hyperbolic vortex equations, unique up to gauge transformation, such that $\{z_1, \ldots, z_k\}$ is the zero set of the Higgs field $\Phi$. The Chern class of such a vortex is then $c_1(a) = k$, obtained by counting the zeros of the Higgs field.

2.6. Symmetric Gauge Transformations. Now that we understand the relationship between symmetric instantons and hyperbolic vortices, we examine how the notion of gauge equivalence behaves under this dimensional reduction. Starting with a symmetric $SU(2)$ gauge transformation on $S^4$, we compute the resulting $U(1)$ gauge transformation on hyperbolic space $H^2$. Then, continuing
to work in the simpler $\mathcal{H}^2$ picture, we examine the conditions under which two hyperbolic vortices given by the harmonic function ansatz of $2.2$ are gauge equivalent.

Using the quaternionic notation $x = t + rQ$, the most general Sp(1)-invariant gauge transformation on $S^4$ has the form

$$g = e^{Q\chi(t,r)} = \cos \chi(t, r) + Q \sin \chi(t, r),$$

where $\chi$ is a real-valued function on $\mathcal{H}^2$. Computing its differential, we have

$$dg^{-1} = Q \, d\chi \, + \, \sin \chi \, dQ \, e^{-Q\chi} = Q \, d\chi \, - \, \frac{1}{2} \, Q \, (e^{2Q\chi} - 1) \, dQ.$$ 

Applying this gauge transformation to the general SO(3)-invariant connection (2.1), we obtain

$$g(A) = \frac{1}{2} \, e^{Q\chi} \, [Qa + (\Phi_1 + Q\Phi_2) \, dQ] \, e^{-Q\chi} \, - \, dg^{-1}$$

$$= \frac{1}{2} \, [Q \, (a - 2 \, d\chi) \, + \, (e^{2Q\chi} \, [\Phi_1 + Q \, (\Phi_2 + 1)] \, - \, Q) \, dQ] \, ,$$

noting that $gQ = Qg$ while $g \, dQ = dQ \, g^{-1}$. The associated connection $ia$ and Higgs field $\Phi = \Phi_1 + i \, (\Phi_2 + 1)$ over $\mathcal{H}^2$ then transform according to

$$g(a) = a - 2 \, d\chi, \quad g(\Phi) = e^{2i\chi} \Phi.$$ 

Hence, the corresponding gauge transformation over $\mathcal{H}^2$ is simply $g = e^{2i\chi}$.

Suppose that we have two gauge equivalent hyperbolic vortices $(a_+, \Phi_+)$ and $(a_-, \Phi_-)$ constructed by the harmonic function ansatz of $2.2$ using equation (17) with the super-potentials $\phi_+$ and $\phi_-$ respectively. If these two vortices satisfy $a_- = g(a_+)$ and $\Phi_- = g(\Phi_+)$ with a gauge transformation of the form $g = e^{2i\chi}$ as discussed above, then we obtain the system of differential equations

\begin{align}
(23) \quad & \frac{\partial}{\partial z} \log \phi_- = \frac{\partial}{\partial z} \log \phi_+ - \frac{\partial}{\partial \bar{z}} 2i\chi \\
(24) \quad & \frac{\partial}{\partial z} \log \phi_- = e^{2i\chi} \frac{\partial}{\partial z} \log \phi_+. 
\end{align}

Note that equation (23) implies that $\chi$ is harmonic,

$$\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \chi = 0,$$

as it is the imaginary part of a holomorphic function. Taking the conjugate of (24) and inserting it into (23), we can eliminate either $\log \phi_+$ or $\log \phi_-$ from these equations to obtain

\begin{align*}
\frac{\partial}{\partial z} \log \phi_+ &= \frac{e^{2i\chi}}{e^{2i\chi} - 1} \frac{\partial}{\partial z} 2i\chi \\
\frac{\partial}{\partial z} \log \phi_- &= \frac{1}{e^{2i\chi} - 1} \frac{\partial}{\partial z} 2i\chi \\
\frac{\partial}{\partial z} \log \phi_- &= \frac{1}{e^{2i\chi} - 1} \frac{\partial}{\partial z} 2i\chi.
\end{align*}

These equations may also be written in either of the two simpler forms

$$\frac{\partial}{\partial z} \log \phi_\pm = \frac{e^{\pm i\chi}}{\sin \chi} \frac{\partial}{\partial \bar{z}} \chi \quad \frac{\partial}{\partial z} \log \phi_\pm = \frac{e^{\mp i\chi}}{\sin \chi} \frac{\partial}{\partial \bar{z}} \chi$$

or

\begin{align}
(25) \quad & \frac{\partial}{\partial z} \log \phi_\pm = \frac{\partial}{\partial z} \log (e^{\pm 2i\chi} - 1) \\
& \frac{\partial}{\partial z} \log \phi_\pm = \frac{\partial}{\partial z} \log (e^{\mp 2i\chi} - 1). 
\end{align}

Substituting these formulae for the partial derivatives into (17), we can express the two gauge equivalent hyperbolic vortices completely in terms of the gauge transformation without reference.
to their super-potentials. Furthermore, computing the Laplacian of the super-potentials $\phi_1, \phi_2$ in terms of the function $\chi$, we have

$$\frac{1}{\phi_{\pm}} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \phi_{\pm} = \frac{\partial}{\partial \bar{z}} \log \phi_{\pm} + \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \log \phi_{\pm} = \frac{e^{\pm i \chi} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \chi}{\sin \chi \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \chi}.$$

Hence, the requirements that $\phi_1, \phi_2$ be harmonic reduce simply to the condition that $\chi$ be harmonic, which we have already established as a corollary to equation (23). We have thus proved

**Theorem 9.** Let $\chi$ be a real-valued harmonic function on the hyperbolic upper half-plane $\mathcal{H}^2$. The two pairs $(a_+, \Phi_+)$ and $(a_-, \Phi_-)$ given by

$$\partial a_{\pm} = \partial + \bar{\partial} \log (e^{\pm 2i \chi} - 1) + \frac{d\bar{z}}{z - \bar{z}},$$

$$\Phi_{\pm} = i (z - \bar{z}) \frac{\partial}{\partial z} \log (e^{\mp 2i \chi} - 1),$$

then satisfy the hyperbolic vortex equations (9) and (10) and are related by the gauge transformation $g = e^{2i \chi}$. Conversely, any two gauge equivalent hyperbolic vortices constructed via the harmonic function ansatz of Theorem 4 can be expressed in this form.

### 2.7. The Unit Disc Model

Until now, we have always used the upper half-plane model for hyperbolic space $\mathcal{H}^2$. In some circumstances, it will be more convenient to use the unit disc model. While the upper half-plane arises naturally by the dimensional reduction technique discussed in the previous sections, the calculations in the following section become much simpler and exhibit significantly more symmetry if we can work on the unit disk. Here we make the transition between the two coordinate systems, showing how the formulae of the previous sections behave under the transformation.

Letting $z$ be the complex coordinate for the upper half-plane and $w$ the coordinate for the unit disk, these two models are related by the conformal transformation

$$w = \frac{i - z}{i + z}, \quad z = \frac{i - w}{1 + w}.$$  

This map takes the upper half-plane to the interior of the unit disc, mapping the real axis to the unit circle. In particular, the point $z = i$ maps to the origin, while the origin maps to $w = 1$ and the point at infinity maps to $w = -1$. The positive imaginary axis in $z$-coordinates maps to the interval $(-1, 1)$ on the real axis in $w$-coordinates.

The hyperbolic metric on the unit disk is given by

$$h = \frac{4}{(1 - |w|^2)^2} dw d\bar{w}.$$  

The vortex equations remain fixed under this change of coordinates; equation (9) is preserved because the transformation is holomorphic, while equation (10) is a relationship between coordinate-invariant scalar quantities. To compute the new connection and Higgs fields generated by the harmonic function ansatz, we first note that

$$z - \bar{z} = 2i \frac{1 - |w|^2}{|1 + w|^2},$$

and that the partial derivatives transform according to

$$\frac{\partial}{\partial w} = -\frac{2i}{(1 + w)^2} \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \bar{w}} = \frac{2i}{(1 + w)^2} \frac{\partial}{\partial \bar{z}}.$$  

Converting the formula (17) for the hyperbolic vortex associated to a harmonic super-potential $\phi$ to $w$-coordinates on the unit disc, Theorem 4 then becomes
Theorem 10. Given a positive real-valued super-potential \( \phi \) over the hyperbolic disc \( \mathcal{H}^2 \), the \( U(1) \) connection and Higgs field pair \((a, \Phi)\) defined by

\[
\bar{\partial}_a = \bar{\partial} + \bar{\partial} \log \phi + \frac{d\bar{w}}{1 - |w|^2} \frac{1}{1 + \bar{w}} \quad \Phi = -i \left(1 - |w|^2\right) \frac{1 + w}{1 + \bar{w}} \frac{\partial}{\partial w} \log \phi,
\]

satisfies the vortex equations (24) and (25) if and only if the super-potential \( \phi \) is harmonic.

The Chern class \( c_1(a) \) of this connection is given in these coordinates by

\[
c_1(a) = \frac{1}{2\pi} \int_{\mathcal{H}^2} *_h \left(1 - |\Phi|^2\right) = \frac{1}{2\pi} \int_{D^2} \left(\frac{4}{(1 - |w|^2)^2} - 4 \left|\frac{\partial}{\partial w} \log \phi\right|^2\right) d\mu,
\]

where \( d\mu \) is the volume element on the disc.

We now return to the simple hyperbolic vortex constructed in §2.4, which we obtained by a dimensional reduction of the basic \( c_2 = 1 \) instanton. In \( w \)-coordinates on the unit disc, the super-potential (20) becomes

\[
\phi = 2 \frac{1 - |w|^4}{1 - |w|^2}.
\]

Computing the partial derivatives of its logarithm, we obtain

\[
\frac{\partial}{\partial w} \log \phi = \frac{2w}{1 - |w|^4} \frac{1 - \bar{w}^2}{1 - w^2} \quad \frac{\partial}{\partial \bar{w}} \log \phi = \frac{2\bar{w}}{1 - |w|^4} \frac{1 - w^2}{1 - \bar{w}^2},
\]

which when inserted into formula (27) above give the vortex

\[
\bar{\partial}_a = \bar{\partial} + \frac{d\bar{w}}{1 + |w|^2} \frac{1}{1 - \bar{w}} + \frac{\partial}{\partial w} \log \phi \quad \Phi = -\frac{2iw}{1 + |w|^2} \frac{1 - \bar{w}}{1 - w}.
\]

Note that here the Higgs field \( \Phi \) vanishes only at the origin, where it has a simple zero, and so the Chern class of this vortex should be \( c_1(a) = 1 \).

Using (28) to explicitly calculate this Chern class, we obtain the integral

\[
c_1(a) = \frac{1}{2\pi} \int_{D^2} \left(\frac{4}{(1 - |w|^2)^2} - \frac{16 |w|^4}{(1 - |w|^4)^2}\right) d\mu.
\]

Taking polar coordinate \( r, \theta \) on the unit disc, this integral becomes

\[
c_1(a) = \int_0^1 \left(\frac{4r}{(1 - r^2)^2} - \frac{16 r^3}{(1 - r^4)^2}\right) dr = \left[\frac{2}{1 - r^2} - \frac{4}{1 - r^4}\right]_{r=0}^{r=1}.
\]

To evaluate this expression at \( r = 1 \), we substitute \( r = 1 - x \) and expand it about \( x = 0 \), giving us

\[
\frac{2}{1 - (1 - x)^2} - \frac{4}{1 - (1 - x)^4} = \frac{2}{2x - x^2 + O(x^3)} - \frac{4}{4x - 6x^2 + O(x^3)} = \left(\frac{1}{x} + \frac{1}{2} + O(x)\right) - \left(\frac{1}{x} + \frac{3}{2} + O(x)\right) = -1 + O(x).
\]

We therefore see that the Chern class of this vortex is indeed \( c_1(a) = 1 \) as we predicted by counting the zeros of the Higgs field.

The formulae from §2.6 giving the two super-potentials in terms of the gauge transformation remain unchanged, except for replacing all the \( z \)'s with \( w \)'s. In particular, if the vortices determined
by the super-potentials $\phi_+$ and $\phi_-$ are gauge equivalent by a transformation of the form $g = e^{2i\chi}$, then the differential equations \[ (23) \quad \frac{\partial}{\partial \bar{w}} \log \phi_- = \frac{\partial}{\partial \bar{w}} \log \phi_+ - \frac{\partial}{\partial w} 2i\chi \] and \[ (31) \quad \frac{\partial}{\partial w} \log \phi_- = e^{2i\chi} \frac{\partial}{\partial w} \log \phi_+ . \]

The unit disc version of Theorem 9 is then

**Theorem 11.** Let $\chi$ be a real-valued harmonic function on the hyperbolic unit disc $\mathbb{H}^2$. The two pairs $(a_+, \Phi_+)$ and $(a_-, \Phi_-)$ given by

\[ \bar{\partial}_{a_\pm} = \bar{\partial} + \bar{\partial} \log (e^{\pm 2i\chi} - 1) + \frac{d\bar{w}}{1 - |w|^2} \frac{1 + w}{1 + \bar{w}} \] \[ \Phi_\pm = -i (1 - |w|^2) \frac{1 + w}{1 + \bar{w}} \frac{\partial}{\partial w} \log (e^{\mp 2i\chi} - 1) \] then satisfy the hyperbolic vortex equations (9) and (10) and are related by the gauge transformation $g = e^{2i\chi}$. Conversely, any two gauge equivalent hyperbolic vortices constructed via the harmonic function ansatz of Theorem 10 can be expressed in this form.

### 3. Holonomy Singularity

#### 3.1. The Forgács, Horváth, Palla Instanton

In this section, we construct the singular instanton described by P. Forgács, Z. Horváth, and L. Palla in [10]. In order to obtain a connection on $S^4 \setminus S^2$ with a holonomy singularity, Forgács et al. patch together two non-singular connections on overlapping simply connected regions using a gauge transformation. (This process is not unlike the clutching construction, which creates “twisted” vector bundles given their local trivializations and transition functions.) These two non-singular solutions are generated by the harmonic function ansatz of Section 1 and since the super-potentials they use are SO(3)-invariant, the resulting connection can be analyzed in terms of the dimensional reduction to hyperbolic space $\mathcal{H}^2$ discussed in Section 2.

Using the quaternionic notation $x = t + rQ$ with $t$, $r$ real, $r > 0$ and $Q$ pure imaginary satisfying $Q^2 = -1$, we want to construct an SO(3)-invariant self-dual connection singular along the 2-sphere $t = 0, r = 1$. By dimensional reduction, this translates into a vortex over the hyperbolic upper half-plane with non-trivial holonomy around the point $z = i$. If instead we work using the unit disc model of hyperbolic space, our task takes the more symmetric form of finding a hyperbolic vortex on the punctured disc with a holonomy singularity at the origin.

We therefore set out to construct two gauge equivalent hyperbolic vortices on the punctured disc, using the harmonic function ansatz of \([2.2]\). For the two simply connected regions, we let $P_1$ be the disc with a cut along the positive real axis, and let $P_2$ be the disc with a cut along the negative real axis. The areas of overlap are then the upper and lower half-discs, excluding the real axis. Let $(a_1, \Phi_1)$ and $(a_2, \Phi_2)$ be the hyperbolic vortices corresponding to the super-potentials $\phi_1$ and $\phi_2$ on the regions $P_1$ and $P_2$ respectively.

In light of Theorem 9, we begin by examining the gauge transformation between these two vortices, rather than focusing on the super-potentials. Here our gauge transformation $g$ is specified by a real-valued harmonic function $\chi$, and we take

\[ g = \begin{cases} 
 e^{+2i\chi} & \text{for } \text{Im } w > 0 \\
 e^{-2i\chi} & \text{for } \text{Im } w < 0.
\end{cases} \]

In other words, on the lower half-disc we use the inverse of the gauge transformation that we use on the upper half-disc. In the notation of \([2.6]\) switching the sign of $2i\chi$ simply interchanges the resulting gauge equivalent super-potentials $\phi_+$ and $\phi_-$ determined by equation \([2.5]\). Note that
although the resulting $g$ is undefined along the real axis, this does not pose a problem for our construction. Indeed, we use $g$ directly only on regions excluding the real axis, and we will find that the two hyperbolic vortices $(a_1, \Phi_1)$ and $(a_2, \Phi_2)$ constructed from $g$ are continuous across the negative and positive axes respectively.

In [10], Forgács et al. use the gauge transformation specified by

$$2\chi = \left( \frac{\pi}{2} + 2 \arctan \frac{T_2}{1-T_1} + 2 \arctan \frac{T_1}{1-T_2} \right).$$

We will define the $T_i$ below, but before doing so we first study the behavior of this gauge transformation for general values of $T_i$. In particular, we would like to coerce $T_1$ and $T_2$ into being the real and imaginary parts of a holomorphic (or anti-holomorphic) function $f(w)$. Then where $|f(w)| = 1$, the arguments of the arctans resemble the half-angle formula for $\tan(\theta)$. In such circumstances we have

$$\arctan \frac{T_2}{1-T_1} = \text{Im} \log (1 - \bar{f}) \quad \arctan \frac{T_1}{1-T_2} = \text{Im} \log (1 + i f),$$

and we can write $\chi$ in terms of $f$ using

$$2\chi = \text{Im} \log \left( i \frac{(1 + i f)^2}{(1-f)^2} \right).$$

We then see that $\chi$ is indeed harmonic as it is the imaginary part of a holomorphic (or anti-holomorphic) function. Exponentiating, we obtain

$$e^{2i\chi} = i \frac{(1 - \bar{f})(1 + if)}{(1-f)(1-if)},$$

and the expressions $e^{2i\chi} - 1$ and $e^{-2i\chi} - 1$ take the form

$$e^{+2i\chi} - 1 = -\frac{(1 - i)(1 - f \bar{f})}{(1-f)(1-if)}, \quad e^{-2i\chi} - 1 = -\frac{(1 + i)(1 - f \bar{f})}{(1-f)(1+if)},$$

which we shall use in equation (25). Note that if $|f| = 1$ then $\bar{f} = f^{-1}$, and we observe that $e^{\pm 2i\chi} = 1$.

In equation (32) above, the $T_i$ are defined by

$$T_1 = \frac{1}{2S^6} \sqrt{(S + S_-)^2 - 4} \left\{ \frac{1}{4} [4 - (S - S_-)^2]^2 + S^2 S_-^2 - 3(z + \bar{z})^2 \right\},$$

$$T_2 = \frac{1}{2S^6} \sqrt{4 - (S - S_-)^2} \left\{ \frac{1}{4} [4 - (S + S_-)^2]^2 + S^2 S_-^2 - 3(z + \bar{z})^2 \right\},$$

with

$$S = |z + i|, \quad S_- = |z - i|,$$

using the complex coordinate $z$ on the upper half-plane. With formulae such as these, it is not surprising that the mathematical community was incredulous. Changing to the complex coordinate $w$ on the unit disc via the conformal transformation (20), we have

$$S = \frac{2}{|w + 1|}, \quad S_- = 2 \frac{|w|}{|w + 1|} = |w|S.$$

Calculating the various components of the $T_i$, we obtain

$$\sqrt{(S + S_-)^2 - 4} = 2 \frac{\sqrt{|w|} - \sqrt{|w|}}{|w + 1|} = |\sqrt{|w|} - \sqrt{|w|}|S,$$

$$\sqrt{4 - (S - S_-)^2} = 2 \frac{\sqrt{|w|} + \sqrt{|w|}}{|w + 1|} = |\sqrt{|w|} + \sqrt{|w|}|S.$$
and
\[ z + \bar{z} = -2i \frac{w - \bar{w}}{|w + 1|^2} = -\frac{i}{2} (w - \bar{w}) S^2. \]

Putting these pieces together, the \( T_i \) are given much more simply by
\[
T_1 = \frac{1}{2} \sqrt{w - \overline{w}} \left[ \frac{1}{4} \left( \sqrt{w} - \sqrt{\overline{w}} \right)^4 + w\bar{w} + \frac{3}{4} (w - \bar{w})^2 \right] \\
= \frac{1}{2} \left( w^{1/2} - \bar{w}^{1/2} \right) \left( w^2 + w^{3/2}w^{1/2} + w\bar{w} + w^{1/2}\bar{w}^{3/2} + \bar{w}^2 \right) \\
= \pm \frac{i}{2} \left( w^{5/2} - \bar{w}^{5/2} \right) = \left( \text{Im} \, w^{5/2} \right) \left( \text{sign} \, \text{Im} \, w^{1/2} \right),
\]
and
\[
T_2 = \frac{1}{2} \sqrt{w + \overline{w}} \left[ \frac{1}{4} \left( \sqrt{w} + \sqrt{\overline{w}} \right)^4 + w\bar{w} + \frac{3}{4} (w - \bar{w})^2 \right] \\
= \frac{1}{2} \left( w^{1/2} + \bar{w}^{1/2} \right) \left( w^2 - w^{3/2}\bar{w}^{1/2} + w\bar{w} - w^{1/2}\bar{w}^{3/2} + \bar{w}^2 \right) \\
= \pm \frac{1}{2} \left( w^{5/2} + \bar{w}^{5/2} \right) = \left( \text{Re} \, w^{5/2} \right) \left( \text{sign} \, \text{Re} \, w^{1/2} \right).
\]

Hence, we see that \( T_1 \) and \( T_2 \) are indeed the imaginary and real parts of the function \( f(w) \) defined by
\[
f(w) = \begin{cases} 
  w^{5/2} & 0 \leq \arg w \leq \pi \\
  -\bar{w}^{5/2} & \pi \leq \arg w \leq 2\pi
\end{cases}
\]
on the region \( P_1 \) or equivalently
\[
f(w) = \begin{cases} 
  \bar{w}^{5/2} & -\pi \leq \arg w \leq 0 \\
  w^{5/2} & 0 \leq \arg w \leq \pi
\end{cases}
\]
on the region \( P_2 \). This function \( f(w) \) is holomorphic on the upper half-disc and anti-holomorphic on the lower half-disc, and we note that \( f(w) \) is well defined and continuous over the whole unit disc.

We now have all that we need to calculate the gauge equivalent connections and Higgs fields \( a_i, \Phi_i \) satisfying \( a_2 = g(a_1) \) and \( \Phi_2 = g(\Phi_1) \). We first consider the super-potential \( \phi_1 \) defined on the region \( 0 < \arg w < 2\pi \). Using the notation of (2.6) on the upper half-disc we have \( \phi_1 = \phi_+ \) and \( f = w^{5/2} \), giving us
\[
\frac{\partial}{\partial \overline{w}} \log \phi_1 = \frac{\partial}{\partial \overline{w}} \log \frac{1 - |w|^5}{(1 - w^{5/2})(1 - i \, w^{5/2})} = \frac{5}{2} \frac{i \, w^{3/2}}{1 - |w|^5} \frac{1 + i \, w^{5/2}}{1 - i \, w^{5/2}}.
\]

On the lower half-disc, we obtain the same expression
\[
\frac{\partial}{\partial \overline{w}} \log \phi_1 = \frac{\partial}{\partial \overline{w}} \log \frac{1 - |w|^5}{(1 + w^{5/2})(1 - i \, w^{5/2})} = \frac{5}{2} \frac{i \, w^{3/2}}{1 - |w|^5} \frac{1 + i \, w^{5/2}}{1 - i \, w^{5/2}},
\]
although this time we take \( \phi_1 = \phi_- \) and \( f = -\bar{w}^{5/2} \). Similarly, considering the super-potential \( \phi_2 \) defined for \( -\pi < \arg w < \pi \), on the upper half-disc with \( \phi_2 = \phi_+ \) and \( f = w^{5/2} \) we have
\[
\frac{\partial}{\partial w} \log \phi_2 = \frac{\partial}{\partial w} \log \frac{1 - |w|^5}{(1 - w^{5/2})(1 + i \, w^{5/2})} = \frac{5}{2} \frac{\bar{w}^{3/2}}{1 - |w|^5} \frac{1 - w^{5/2}}{1 - \bar{w}^{5/2}},
\]
while on the lower half-disc, we again get the same expression
\[
\frac{\partial}{\partial w} \log \phi_2 = \frac{\partial}{\partial w} \log \frac{1 - |w|^5}{(1 - \bar{w}^{5/2})(1 - i \, w^{5/2})} = \frac{5}{2} \frac{\bar{w}^{3/2}}{1 - |w|^5} \frac{1 - w^{5/2}}{1 - \bar{w}^{5/2}}.
\]
transformation extend continuously across the negative and positive real axes respectively, even though the gauge transformation \( g \) between them does not.

For the purposes of our vortex construction, we need not know the super-potentials explicitly; rather, all we require are the complex partial derivatives of their logarithms which we computed above. Nevertheless, here we present the super-potentials as given in [10]. There the two super-potentials take center stage, defined on their own instead of being constructed from the gauge transformation as we have done. Forgács et al. define \( \phi_1, \phi_2 \) by

\[
\phi_1 = \frac{S^5 - S^5}{S^5 + S^5 - 2 S^5 T_1}, \quad \phi_2 = \frac{S^5 - S^5}{S^5 + S^5 - 2 S^5 T_2}.
\]

Simplifying these expressions and writing them using the unit disc model of hyperbolic space, we obtain

\[
\phi_1 = \frac{1 - |w|^5}{1 - 2 \text{Im} w^{5/2} + |w|^5} = \frac{1 - |w|^5}{|1 + i w^{5/2}|^2},
\]

\[
\phi_2 = \frac{1 - |w|^5}{1 - 2 \text{Re} w^{5/2} + |w|^5} = \frac{1 - |w|^5}{|1 - w^{5/2}|^2}.
\]

The reader may want to verify that the partial derivatives \( \frac{\partial}{\partial w} \log \phi_1 \) and \( \frac{\partial}{\partial w} \log \phi_2 \) agree with those calculated on the previous page.

We now compute the Chern class \( c_1 \) of the vortex patched together from the two super-potentials \( \phi_1 \) and \( \phi_2 \). Using equation (28), we have

\[
c_1(a) = \frac{1}{2\pi} \int_{D^2} \left( \frac{4}{(1 - |w|^2)^2} - \frac{25|w|^3}{(1 - |w|^5)^2} \right) d\mu.
\]

Taking polar coordinates \( r, \theta \) on the disc \( D^2 \), this integral becomes

\[
c_1(a) = \int_0^1 \left( \frac{4r}{(1 - r^2)^2} - \frac{25r^4}{(1 - r^5)^2} \right) dr = \left[ \frac{2}{1 - r^2} - \frac{5}{1 - r^5} \right]_{r=0}^{r=1},
\]

and evaluating this expression by the method used at the end of [27] shows that \( c_1(a) = 3/2 \). We can also arrive at this same result by naively counting the zeros (with multiplicity) of the Higgs field, as both \( \frac{\partial}{\partial w} \log \phi_1 \) and \( \frac{\partial}{\partial w} \log \phi_2 \) vanish to order 3/2 at the origin but are otherwise nonzero.

3.2. A Family of Singular Vortices. In this section we will generalize the construction of [10] to produce a family of hyperbolic vortices with varying Chern class \( c_1 \). This family includes both the standard \( c_1 = 1 \) vortex of [27] and the fractionally charged vortex of the previous section, as well as vortices with arbitrary real \( c_1 \). We will continue to work using the unit disc model of hyperbolic space.

Observing the resemblance between the two super-potentials [28] and [35], we consider a more general super-potential of the form

\[
\phi = \frac{1 - |w|^{2c}}{|1 - w^c|^2},
\]

where \( c \) is a nonzero real constant. For non-integral \( c \), this \( \phi \) is not well defined over the whole unit disc; rather, we must restrict it to a simply-connected cut disc. If we loop around the origin once

---

6Actually, [10] uses +2 \( S^5 T_1 \) and +2 \( S^5 T_2 \) in the denominators of \( \phi_1 \) and \( \phi_2 \) respectively. The resulting super-potentials are still harmonic, and they do indeed generate gauge equivalent hyperbolic vortices. However, this choice of sign is not consistent with the gauge transformation they use, given here by [28].
in the positive direction, crossing our cut in the unit disc, then this super-potential becomes

\[
\phi' = \frac{1 - |w|^{2c}}{|1 - \epsilon w^c|^2},
\]

introducing a factor of \( \epsilon = e^{2\pi i c} \) in the denominator. In both of these cases, we note that \( \phi \) and \( \phi' \) vanish on on the unit circle, except at the roots of \( w^a = 1 \) (or \( w^a = \bar{\epsilon} \)) where they have simple poles.

Taking the logarithmic derivatives of the super-potential \( \phi \), we obtain

\[
\frac{\partial}{\partial w} \log \phi = \frac{c w^{c-1} - \bar{w}^c}{1 - |w|^{2c}} \quad \text{and} \quad \frac{\partial}{\partial w} \log \phi = \frac{c \bar{w}^{c-1} - w^c}{1 - |\bar{w}|^{2c}},
\]

which we can use in equation (37) to construct a hyperbolic vortex \((a, \Phi)\). After looping around the origin, the logarithmic derivatives become

\[
\frac{\partial}{\partial w} \log \phi' = \frac{c \epsilon w^{c-1} - \bar{\epsilon} \bar{w}^c}{1 - |w|^{2c}} \quad \text{and} \quad \frac{\partial}{\partial w} \log \phi' = \frac{c \bar{\epsilon} \bar{w}^{c-1} - \epsilon w^c}{1 - |\bar{w}|^{2c}},
\]

and we let \((a', \Phi')\) be the corresponding hyperbolic vortex.

In order to construct a single hyperbolic vortex over the whole of the unit disc, we would like to find a gauge transformation \( g \) taking the vortex \((a, \Phi)\) to the vortex \((a', \Phi')\). Using equation (31) for gauge equivalent Higgs fields, we have

\[
\frac{\partial}{\partial w} \log \phi' = g \frac{\partial}{\partial w} \log \phi,
\]

and so our gauge transformation must be

\[
g = \epsilon \frac{1 - \bar{\epsilon} \bar{w}^c}{1 - \epsilon w^c}.
\]

We must also verify that this gauge transformation \( g \) satisfies equation (30) for gauge equivalent connections

\[
\frac{\partial}{\partial w} \log \phi' = \frac{\partial}{\partial w} \log \phi - \frac{\partial}{\partial w} \log g,
\]

which we leave to the reader. Hence when we loop around the origin, we obtain a vortex that is gauge equivalent to our original one, and so this vortex is well defined over the punctured disc.

Suppose that instead of defining \( \epsilon = e^{2\pi i c} \), we use an arbitrary constant \( \epsilon \) with \(|\epsilon| = 1\) in equations (37) and (38). In this case, the gauge transformation \( g \) still maps between the two vortices corresponding to the super-potentials \( \phi \) and \( \phi' \). Indeed, if we take \( c = 5/2 \) and \( \epsilon = -i \), then the resulting \( \phi, \phi' \), and \( g \) correspond to the super-potentials (35) and (35) and the gauge transformation (33) we used in (3.1).

Writing out the connection \( a \) and Higgs field \( \Phi \) of this vortex explicitly using formula (27), we obtain

\[
\tilde{\partial}_a = \tilde{\partial} + \left( \frac{c \bar{w}^{c-1} - w^c}{1 - |w|^{2c}} + \frac{1}{1 - |\bar{w}|^{2c}} \right) d\bar{w}
\]

\[
\Phi = -ic w^{c-1} \frac{1 - |w|^{2c}}{1 - |\bar{w}|^{2c}} + 1 + w \frac{1 - \bar{w}^c}{1 - |\bar{w}|^{2c}}.
\]

The Chern class \( c_1(a) \) of this vortex is given by equation (28), yielding the integral

\[
c_1(a) = \frac{1}{2\pi} \int_{D^2} \left( \frac{4}{(1 - |w|^2)^2} - \frac{4c^2 |w|^{2c-2}}{(1 - |w|^{2c})^2} \right) d\mu
\]

\[
= \int_0^1 \left( \frac{4r}{(1 - r^2)^2} - \frac{4c^2 r^{2c-1}}{(1 - r^{2c})^2} \right) dr = \left[ \frac{2}{1 - r^2} - \frac{2c}{1 - r^{2c}} \right]_{r=0}^{r=1},
\]
where we have used polar coordinates \( r, \theta \) on the unit disc. Evaluating this final expression by the method used at the end of \( \S 2.7 \) we have \( c_1(a) = c - 1 \).

For \( c = 2 \), this construction yields yields the standard \( c_1 = 1 \) hyperbolic vortex which we discussed in \( \S 2.7 \). If we take \( c = 5/2 \) then we obtain the \( c_1 = 3/2 \) vortex given by the super-potential \( \Phi \) on the cut disc \( P_2 \). Note that with our construction, it is no longer necessary to cover the disc with two overlapping regions as we did in \( \S 3.1 \). Rather, it is sufficient to take a single vortex on the cut disc and then study how it behaves across that cut. For integer values of \( c \), the vortex is continuous across the cut, while for other values the vortex changes by a gauge transformation. The flat \( c = 1 \) vortex in this family is

\[
\tilde{\phi}_a = \tilde{\phi} + \frac{2}{1 - \bar{w}^2} \quad \Phi = -i \frac{1 + w}{1 + \bar{w}} \frac{1 - \bar{w}}{1 - w},
\]

which we readily see has \( |\Phi| = 1 \) and \( F_a = 0 \), and we therefore have \( c_1 = 0 \) as expected. We observe that this vortex is equivalent to the standard flat vortex \( (a = 0, \Phi = 1) \) using the gauge transformation \( \Phi = 1 \) with \( \epsilon = -1 \).

To compute the holonomy around loops circling the origin, we introduce polar coordinates \( r, \theta \) on the unit disc. In these coordinates, the complex differentials \( dw \) and \( d\bar{w} \) are

\[
dw = \frac{w}{|w|} dr + iw \, d\theta, \quad d\bar{w} = \frac{\bar{w}}{|w|} dr - iw \, d\theta.
\]

In a small neighborhood of the origin, our connection is approximated by

\[
a \approx c \left( \bar{w}^{-1} dw - w^{-1} d\bar{w} \right) = -ic \left( w^c + \bar{w}^{-c} \right) d\theta + \cdots
\]

(only the \( d\theta \) term is needed for calculating the holonomy). The contribution to the holonomy around the circle \(|w| = r\) due to the connection \( a \) is then given by the loop integral

\[
\int_{|w|=r} a = \int_0^{2\pi} -icr^c \left( e^{-ic\theta} + e^{-ic\theta} \right) d\theta = r^c \left( -e^{2\pi ic} + e^{-2\pi ic} \right) = -2r^c \sin 2\pi ic.
\]

For \( c > 0 \), this expression vanishes as \( r \to 0 \), and thus the limit holonomy around the origin comes entirely from the gauge transformation \( g \) given by \( \Phi = 1 \). Near the origin, we have \( g \approx \epsilon = e^{2\pi ic} \).

Hence, the limit of the holonomy around small loops centered at the origin is \( e^{2\pi ic} \).

By the dimensional reduction technique of Section 2, our family of singular hyperbolic vortices corresponds to a similar family of \( \text{SO}(3) \)-invariant self-dual connections over \( S^1 \setminus S^2 \). Using the results and quaternionic notation of \( \S 2.6 \) we see that the holonomy around small loops linking the singular surface \( S^2 \) is \( e^{2\pi Qc} \). Viewing these connections as \( \text{SU}(2) \) connections on a bundle \( E \), we then obtain a splitting \( E = \mathcal{L} \oplus \mathcal{L}^* \) on a neighborhood of \( S^2 \) with respect to which the holonomy takes the standard form

\[
\begin{pmatrix}
    e^{2\pi ic\alpha} & 0 \\
    0 & e^{-2\pi ic\alpha}
\end{pmatrix}
\]

for a constant \( \alpha \) in the range \([0, 1/2)\). In the \( \text{SO}(3) \)-symmetric case, the complex line bundle \( \mathcal{L} \) on \( S^2 \) has Chern class \( c_1(L) = -1 \). For our family of singular solutions, the holonomy parameter is \( \alpha = (c - |c|)/2 \), where \( |c| \) is the greatest integer less than or equal to \( c \).

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