EMBEDDING OF GROUPS AND QUADRATIC EQUATIONS
OVER GROUPS

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Abstract. We prove that, for every integer \( n \geq 2 \), a finite or infinite countable group \( G \) can be embedded into a 2-generated group \( H \) in such a way that the solvability of quadratic equations of length at most \( n \) is preserved, i.e., every quadratic equation over \( G \) of length at most \( n \) has a solution in \( G \) if and only if this equation, considered as an equation over \( H \), has a solution in \( H \).

1. Introduction

It is a classical result of Higman, B. Neumann, and H. Neumann \([6]\) that every finite or infinite countable group can be embedded into a 2-generated group. In this note, we are concerned with such an embedding that would preserve the solvability of every quadratic equation of bounded length.

We start with definitions. Let \( G \) be a finite or infinite countable group and let

\[
G = \langle a_1, a_2, \ldots \mid R_1 = 1, R_2 = 1, \ldots \rangle
\]

be a presentation for \( G \) by means of generators \( a_1, a_2, \ldots \) and defining relations \( R_1 = 1, R_2 = 1, \ldots \), where \( R_1, R_2, \ldots \) are nonempty cyclically reduced words over the alphabet \( A^{\pm 1} := \{a_1^{\pm 1}, a_2^{\pm 1}, \ldots \} \). If \( U \) is a word over \( A^{\pm 1} \) and the image of \( U \) in \( G \) is trivial, we write \( U \cong 1 \) or say that \( U = 1 \) in \( G \).

Let \( \mathcal{X} \) be a finite or infinite countable set, called a set of variables, \( \mathcal{X}^{-1} := \{x^{-1} \mid x \in \mathcal{X}\} \), and \( \mathcal{X}^{\pm 1} := \mathcal{X} \cup \mathcal{X}^{-1} \). Let \( \mathcal{F}(\mathcal{X}) \) denote the free group with the free base \( \mathcal{X} \) and let \( G * \mathcal{F}(\mathcal{X}) \) denote the free product of \( G \) and \( \mathcal{F}(\mathcal{X}) \). Elements of \( G * \mathcal{F}(\mathcal{X}) \) can be regarded as words over the alphabet \( \mathcal{Y}^{\pm 1} \), where \( \mathcal{Y} := \mathcal{A} \cup \mathcal{X} \).

A word \( W = y_1 \ldots y_\ell \) over \( \mathcal{Y}^{\pm 1} \), where \( y_1, \ldots, y_\ell \in \mathcal{Y}^{\pm 1} \), is called reduced if \( \ell > 0 \), i.e., \( W \) is not empty, and \( W \) contains no subwords of the form \( yy^{-1} \) or \( y^{-1}y \), where \( y \in \mathcal{Y} \). A word \( W \) over \( \mathcal{Y}^{\pm 1} \) is cyclically reduced if \( W \) is reduced and every cyclic permutation of \( W \) is reduced. The length of a word \( W = y_1 \ldots y_\ell \) over \( \mathcal{Y}^{\pm 1} \) is \( \ell = |W| \) and the \( \mathcal{X} \)-length \( |W|_\mathcal{X} \) of \( W \) is the number of all occurrences of letters of \( \mathcal{X}^{\pm 1} \) in the word \( W \). For example, \( |a_1 x_1 x_2 a_2^{-1} x_1^{-1}|_\mathcal{X} = 3 \) if \( a_1, a_2 \in \mathcal{A} \) and \( x_1, x_2 \in \mathcal{X} \).

An equation over \( G \) is a formal expression \( W = 1 \), where \( W \) is a cyclically reduced word over \( \mathcal{Y}^{\pm 1} \) with \( |W|_\mathcal{X} > 0 \). The length of an equation \( W = 1 \) over \( G \) is the number \( |W|_\mathcal{X} \). The total length of an equation \( W = 1 \) over \( G \) is \( |W| \). An equation \( W = 1 \) over \( G \) is called quadratic if, for every letter \( x \in \mathcal{X} \), the sum of the number of occurrences of \( x \) in \( W \) and the number of occurrences of \( x^{-1} \) in \( W \) is either 2 or 0.

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We say that an equation \( W = 1 \) over \( G \) has a solution if there exists a homomorphism \( \psi_W : G \ast \mathcal{F}(X) \to G \) which is identical on \( G \) and which takes the word \( W \in G \ast \mathcal{F}(X) \) to the identity, i.e., \( \psi_W|_G = \text{id}_G \) and \( \psi_W(W) = 1 \) in \( G \). Let \( x_1, \ldots, x_k \) be all letters of \( X \) that occur in \( W \) or in \( W^{-1} \). A solution tuple to the equation \( W = 1 \), defined by a homomorphism \( \psi_W : G \ast \mathcal{F}(X) \to G \), is a tuple \( (U_1, \ldots, U_k) \), where \( U_1, \ldots, U_k \) are some words over \( \mathcal{A}^{\pm 1} \), such that \( \psi_W(x_j) = U_j \) in \( G \) for every \( j = 1, \ldots, k \). The length of a solution tuple \( (U_1, \ldots, U_k) \) to the equation \( W = 1 \) is the sum \( \sum_{j=1}^k |U_j| \).

If \( \mu : G \to H \) is a group monomorphism and \( W = 1 \) is an equation over \( G \), then we can use \( \mu \) and \( W = 1 \) to obtain an equation over \( H \) by replacing every letter \( a_i^\varepsilon \in \mathcal{A}^{\pm 1} \) by \( \mu(a_i^\varepsilon) \). This new equation over \( H \) is denoted by \( \mu(W) = 1 \).

**Theorem 1.1.** Let \( n \geq 2 \) be an integer and let \( G \) be a finite or infinite countable group. Then there exists an embedding \( \mu_n : G \to H \) of \( G \) into a 2-generated group \( H = \langle h_1, h_2 \rangle \), that preserves the solvability of every quadratic equation \( W = 1 \) over \( G \) of length \( |W|_X \leq n \), i.e., for every \( W = 1 \) over \( G \) of length at most \( n \), the equation \( W = 1 \) has a solution in \( G \) if and only if \( \mu_n(W) = 1 \) has a solution in \( H \).

We remark that the embedding \( \mu_n : G \to H \) of Theorem 1.1 has additional properties that are of interest even in the case when \( G \) is already a 2-generated group. For example, a solution tuple to a quadratic equation \( W = 1 \) over \( G \) such that \( |W|_X \leq n \) may be arbitrarily long relative to the original alphabet \( \mathcal{A} \) whereas the equation \( \mu_n(W) = 1 \) has a relatively short solution tuple in \( H \) with respect to the alphabet \( \{h_1, h_2\} \). This and other technical properties of the embedding \( \mu_n \), that could be useful for potential future applications, are recorded in the following.

**Theorem 1.2.** The embedding \( \mu_n : G \to H \) of Theorem 1.1 can be constructed in such a way that \( \mu_n \) has the following properties.

(a) Fix an enumeration \( W_1 = 1, W_2 = 1, \ldots \) of all quadratic equations over \( G \) such that, for every \( i \geq 1 \), \( |W_i|_X \leq n \) and \( W_i = 1 \) has a solution in \( G \). Then there is a constant \( C > 0 \) such that, for every \( i \geq 1 \), there exists a solution tuple to the equation \( \mu_n(W_i) = 1 \) over \( H \) whose length, in generators \( h_1, h_2 \) of \( H \), does not exceed \( Cn^4i \).

(b) Assume that the presentation \( \{1,1\} \) for \( G \) is recursively enumerable. Then defining relations of the 2-generated group \( H = \langle h_1, h_2 \rangle \) can be recursively enumerated.

(c) Assume that the presentation \( \{1,1\} \) for \( G \) is decidable and there is an algorithm that detects whether a quadratic equation over \( G \) of length at most \( n \) has a solution in \( G \). Then the 2-generated group \( H = \langle h_1, h_2 \rangle \) has a decidable set of defining relations and the embedding \( \mu_n : G \to H \) can be effectively constructed.

As an example of a quadratic equation, consider the equation \( xu_1x^\varepsilon u_2 = 1 \), where \( \varepsilon = \pm 1 \) and \( u_1, u_2 \) are some reduced (or possibly empty if \( \varepsilon = 1 \)) words over \( \mathcal{A}^{\pm 1} \). Note that if \( \varepsilon = -1 \) then this equation has a solution if and only if the elements of \( G \), represented by the words \( u_1, u_2^{-1} \), are conjugate in \( G \). If \( \varepsilon = 1 \), then this equation has a solution if and only if the element of \( G \), represented by the word \( u_1^{-1}u_2 \), is a square in \( G \), i.e., there is a word \( T \) over \( \mathcal{A}^{\pm 1} \) with \( u_1^{-1}u_2 \overset{G}{\equiv} T^2 \). According to Theorem 1.1 applied with \( n = 2 \), if \( G \) is a finite or infinite countable
group, then $G$ embeds into a 2-generated group $H$, \( \mu_2 : G \to H \), in which two elements of \( \mu_2(G) \) are conjugate if and only if they are conjugate in $G$ and every element of \( \mu_2(G) \) is a square in $H$ if and only if it is a square in $G$. This is reminiscent of an embedding result of Ol’shanskii and Sapir \cite{13} that states that a finitely generated group $G$ with the solvable conjugacy problem can be embedded into a finitely presented group $K$ with the solvable conjugacy problem, \( \sigma : G \to K \), in such a way that two elements of $\sigma(G)$ are conjugate in $K$ if and only if they are conjugate in $G$.

It would be of interest to find out whether Theorem \ref{thm:embedding} generalizes to arbitrary equations of bounded length and whether one could drop the upper bound on the length of quadratic equations in Theorem \ref{thm:embedding}. The first question seems to be technically relevant to the following interesting problem.

**Problem 1.3.** For given integer $n > 0$, does there exist a real number \( \lambda > 0 \) such that if a presentation (1.1) satisfies the small cancelation condition \( C'(\lambda) \), for every relation $R = 1$ of (1.1), $|R| > \lambda^{-1}$ and $R$ is not a proper power, then every equation $W = 1$ over $G$ of total length $|W| \leq n$ has a solution in $G$ if and only if the equation $W = 1$, considered as an equation over the free group $F(A)$, has a solution in $F(A)$?

We remark that for quadratic equations of total length $\leq n$ this problem would likely have a positive solution and a proof would be analogous to the proof of Theorem \ref{thm:embedding} with additional consideration of contiguity subdiagrams between boundary paths of faces of type $F3$ and boundary paths of a surface diagram $\Delta$. We also mention that the arguments of Frenkel and Klyachko \cite{5}, which are used to prove that a nontrivial commutator cannot be a proper power in a torsion-free group $G$ that satisfies the small cancelation condition $C'(\lambda)$ with $\lambda \ll 1$, might be useful for making some progress in nonquadratic case.

At the suggestion of the referee, we mention that connections between compact surfaces and solutions of quadratic equations in free groups, free products and in hyperbolic groups were first studied by Culler \cite{2} and Ol’shanskii \cite{11}. Earlier work on quadratic equations in free groups and in free products was done by Edmunds \cite{2, 3}, Comerford and Edmunds \cite{1}, see also articles cited in \cite{2, 3, 1}. The bound of Theorem 1.2(a) is reminiscent of bounds on the length of a minimal solution of quadratic equations in free groups obtained by Lysenok and Myasnikov \cite{10} and by Kharlampovich and Vdovina \cite{8}.

2. Group Presentations and Diagrams

Fix an even integer $n \geq 2$. Since we consider quadratic equations $W = 1$ of length $|W| \leq n$, we may assume that the cardinality of $X$ is $n$, $|X| = n$. Since $G$ is finite or countably infinite, we can choose an enumeration

\[ W_1 = 1, \ W_2 = 1, \ldots, \]

of all quadratic equations over $G$ such that, for every $i \geq 1$, $|W_i| \leq n$ and $W_i = 1$ has a solution in $G$. Let $\bigcup_{i=1}^{\infty} X_i$ be an infinite countable alphabet consisting of disjoint copies $X_i$, $i = 1, 2, \ldots$, of $X$. Let $W_i(X_i)$ denote the word over the alphabet $A^{\pm 1} \cup X_i^{\pm 1}$ obtained by rewriting $W_i$ so that every letter $b \in A^{\pm 1}$ of $W_i$ is unchanged and every letter $y$ of $W_i$, such that $y \in X_i^{\pm 1}$, is replaced with $\beta_i(y) \in X_i^{\pm 1}$, where
\[ \beta_i : \mathcal{X}^{\pm 1} \to \mathcal{X}_i^{\pm 1} \] is a bijection such that \( \beta_i(\mathcal{X}) = \mathcal{X}_i \) and \( \beta_i(x^{-1}) = \beta_i(x)^{-1} \) for every \( x \in \mathcal{X} \).

Consider the following group presentation
\[ \mathcal{G}_1 = \langle \bigcup_{i=1}^\infty \mathcal{X}_i \cup \mathcal{A} \mid R_1 = 1, R_2 = 1, \ldots, W_1(\mathcal{X}_1) = 1, W_2(\mathcal{X}_2) = 1, \ldots \rangle \] (2.2)
whose generating set is \( \bigcup_{i=1}^\infty \mathcal{X}_i \cup \mathcal{A} \) and whose defining relations are those of (1.1) and \( W_i(\mathcal{X}_i) = 1, i = 1, 2, \ldots. \)

**Lemma 2.1.** There is a natural embedding of the group \( G \) into the group \( \mathcal{G}_1 \) given by presentation (2.2), denoted \( \nu_1 : G \to \mathcal{G}_1. \) Furthermore, if \( W = 1 \) is an equation over \( G \) then \( W = 1 \) has a solution in \( G \) if and only if the equation \( \nu_1(W) = 1 \) has a solution in the group \( \mathcal{G}_1. \)

**Proof.** Denote \( \mathcal{X}_i = \{x_{i,1}, \ldots, x_{i,n}\} \) for \( i = 1, 2, \ldots. \) Since the equation \( W_i(\mathcal{X}_i) = 1 \) has a solution in \( G \), there exists a homomorphism \( \psi_i : G * \mathcal{F}(\mathcal{X}_i) \to G \) such that \( \psi_i \) is identical on \( G \) and \( \psi_i(W_i(\mathcal{X}_i)) = 1. \) Let \( U_{i,1}, \ldots, U_{i,n} \) be words over \( \mathcal{A} \) such that \( \psi_i(x_{i,j}) = U_{i,j} \) in \( G, j = 1, \ldots, n. \) Then the map \( \psi_\infty(x_{i,j}) := U_{i,j}, \) where \( 1 \le j \le n, i = 1, 2, \ldots, \) and \( \psi_\infty(a) := a \) for all \( a \in \mathcal{A} \) induces a homomorphism \( \psi_\infty : \mathcal{G}_1 \to G \) which is identical on \( G. \) Hence, the group \( G \) embeds in \( \mathcal{G}_1. \) The existence of this homomorphism \( \psi_\infty : \mathcal{G}_1 \to G \) also implies that, for an arbitrary equation \( W = 1 \) over \( G, \) the equation \( W = 1 \) has a solution in \( G \) if and only if the equation \( \nu_1(W) = 1 \) over \( \mathcal{G}_1 \) has a solution in \( \mathcal{G}_1. \)

Denote \( M := 24n. \) For every \( i \ge 1, \) consider a word \( V_i \) over the alphabet \( \{h_1, h_2\} \) defined by the formula
\[ V_i = V_i(h_1, h_2) := h_1h_2^{M+1}h_1h_2^{M+2} \cdots h_1h_2^{M(i+1)-1}h_1h_2^{M(i+1)}h_1. \] (2.3)

The literal (or letter-by-letter) equality of two words \( U, V \) is denoted \( U \equiv V. \) In the following lemma, we establish a small cancelation condition for the words \( V_i, i = 1, 2, \ldots. \)

**Lemma 2.2.** Let \( U \) be a subword of both words \( V_i \) and \( V_j, \) defined by (2.3), so \( V_i \equiv V_iU_iV_i2 \) and \( V_j \equiv V_jU_jV_j2. \) Then either \( |U| < \frac{4}{3} \min\{|V_i|, |V_j|\} \) or \( i = j \) and \( V_{i,1} \equiv V_{j,1}. \)

**Proof.** Suppose that \( U \) is a subword of the word \( V_i, \) where \( i = 1, 2, \ldots, \) and \( |U| \ge \frac{4}{3} |V_i|. \) Then
\[ |U| \ge \frac{4}{3} |V_i| > 4(Mi + 2) > 2M(i + 1) + 2. \] (2.4)
Since every maximal power of \( h_2 \) in \( V_i \) is no longer than \( M(i + 1), \) it follows from (2.4) that \( U \) contains a subword of the form \( h_1h_2^kh_1, \) where \( Mi + 1 \le k \le M(i + 1). \) Now our claim follows from the fact that each word \( V_1, V_2, \ldots \) contains a unique subword of the form \( h_1h_2^kh_1, \) where \( Mi + 1 \le k \le M(i + 1). \)

Let \( \bigcup_{i=1}^\infty \mathcal{X}_i = \{x_1, x_2, \ldots\} \) be an enumeration of elements of \( \bigcup_{i=1}^\infty \mathcal{X}_i \) compatible with the enumeration of sets \( \mathcal{X}_i, \) i.e., if \( x_j \in \mathcal{X}_k, \) \( x_j' \in \mathcal{X}_{k'} \) and \( k < k', \) then \( j < j'. \) Using this enumeration, new generators \( h_1, h_2 \) and the words \( V_i(h_1, h_2), \) we extend the presentation (2.2) as follows
\[ \mathcal{G}_2 = \langle \bigcup_{i=1}^\infty \mathcal{X}_i \cup \mathcal{A} \cup \{h_1, h_2\} \mid R_1 = 1, R_2 = 1, \ldots, W_1(\mathcal{X}_1) = 1, W_2(\mathcal{X}_2) = 1, \ldots, x_iV_i^{2i-1} = 1, a_iV_i^{2i+1} = 1, i = 1, 2, \ldots. \rangle \] (2.5)
To study this group presentation and quadratic equations over $\mathbb{Z}_2$, we will use diagrams over the presentation (2.5). We start with basic definitions.

Let $\Delta$ be a finite 2-complex and let $\Delta(i)$ denote the set of closures of $i$-cells of $\Delta$, $i = 0, 1, 2$. The elements of $\Delta(i)$ are called vertices, edges, faces of $\Delta$ if $i = 0, 1, 2$, resp. We also consider the set $\tilde{\Delta}(1)$ of oriented 1-cells of $\Delta$. If $e \in \tilde{\Delta}(1)$, then $e^{−1}$ denotes $e$ with opposite orientation. For every $e \in \tilde{\Delta}(1)$, let $e_−, e_+$ denote the initial, terminal, resp., vertices of $e$. In particular, $(e^{−1})_− = e_+$ and $(e^{−1})_+ = e_−$. Note that $e \neq e^{−1}$.

A path $p = e_1 \ldots e_\ell$ in $\Delta$ is a sequence of oriented edges $e_1, \ldots, e_\ell$ of $\Delta$ with $(e_i)_+ = (e_{i+1})_−$, $i = 1, \ldots, \ell − 1$. The length of a path $p = e_1 \ldots e_\ell$ is $|p| = \ell$. The initial vertex of $p$ is $p_− := (e_1)_−$ and the terminal vertex of $p$ is $p_+ := (e_\ell)_+$. A path $p$ is called closed if $p_− = p_+$. A path $p$ is called reduced if $|p| > 0$ and $p$ contains no subpath of the form $ee^{−1}$, where $e$ is an edge. A cyclic path is a closed path with no distinguished initial vertex. A path $p = e_1 \ldots e_\ell$ is called simple if the vertices $(e_1)_−, \ldots, (e_\ell)_−, (e_\ell)_+$ are all distinct. A closed path is simple if the vertices $(e_1)_−, \ldots, (e_\ell)_−$ are all distinct.

A diagram $\Delta$ over presentation (2.5) is a connected finite 2-complex which is equipped with a labeling function

$$\varphi : \tilde{\Delta}(1) \to \cup_{i=1}^\infty \chi_i^{\pm 1} \cup A^{\pm 1} \cup \{h_1^{\pm 1}, h_2^{\pm 1}, 1\}$$

such that, for every $e \in \tilde{\Delta}(1)$, one has $\varphi(e^{−1}) = \varphi(e)^{−1}$, where $1^{−1} := 1$, and, for every face $\Pi$ of $\Delta$, if $\partial \Pi = e_1 \ldots e_\ell$ is a boundary path of $\Pi$, where $e_1, \ldots, e_\ell \in \tilde{\Delta}(1)$, then the label $\varphi(\partial \Pi) := \varphi(e_1) \ldots \varphi(e_\ell)$ of $\partial \Pi$ has one of the following three forms.

(F1) $\varphi(\partial \Pi) = 1^\ell$.

(F2) $\ell = 4$ and $\varphi(\partial \Pi)$ is a cyclic permutation of a word $y_1y^{−1}1$, where $y \in \cup_{i=1}^\infty \chi_i \cup A \cup \{h_1, h_2\}$.

(F3) $\varphi(\partial \Pi)$ is a cyclic permutation of one of the words $R^{\pm 1}$, where $R = 1$ is a relation of the presentation (2.5).

A face $\Pi$ of $\Delta$ is said to have type $F1$, $F2$, $F3$ if $\varphi(\partial \Pi)$ has the form (F1), (F2), (F3), resp. The set of faces of type $Fj$ is denoted $\Delta_j(2)$, $j = 1, 2, 3$.

An edge $e \in \tilde{\Delta}(1)$ is called an $a$-edge, $x$-edge, $h$-edge, $1$-edge if $\varphi(e) \in A^{\pm 1}$, $\varphi(e) \in \cup_{i=1}^\infty \chi_i^{\pm 1}$, $\varphi(e) \in \{h_1^{\pm 1}, h_2^{\pm 1}\}$, $\varphi(e) = 1$, resp. An edge $e \in \tilde{\Delta}(1)$ is termed essential if $e$ is not a 1-edge.

We will say that $\Delta$ is a surface diagram of type $(k, k')$ over (2.5) if $\Delta$ is a diagram over (2.5) and $\Delta$, as a topological space, is homeomorphic to a compact (orientable or nonorientable) surface that has Euler characteristic $k$ and contains $k'$ punctures. This surface is called the underlying surface for $\Delta$. In particular, $\Delta$ is called a disk diagram if $\Delta$ is a surface diagram of type $(1, 1)$, hence, the underlying surface for $\Delta$ is a disk.

If $\Delta$ is a surface diagram and the underlying surface is orientable, then a fixed orientation of the underlying surface makes it possible to define positive (=counter-clockwise) and negative (=clockwise) orientation for boundaries of faces of $\Delta$ and for connected components of $\partial \Delta$. Regardless of whether the underlying surface is orientable or not, we always consider the boundary $\partial \Pi$ of a face $\Pi$ of $\Delta$ or a connected component $c$ of the boundary $\partial \Delta$ of $\Delta$ as a cyclic path which is called a boundary path of $\Pi$ or a boundary path of $\Delta$, resp. Note that $(\partial \Pi)^{−1}$ or $c^{−1}$ are also boundary paths of $\Pi$ or $\Delta$, resp., with the opposite orientation.
Suppose that $\Delta$ is a surface diagram over (2.5). Making refinements of $\Delta$ by using faces of type $F_1$, $F_2$ if necessary (informally, we “thicken” boundary paths of faces of type $F_3$ and $\partial \Delta$, this should be evident; more formal details can be found in [12]), we may assume that the following property holds for $\Delta$.

(A) Suppose that each of $c_1, c_2$ is either a boundary path of a face of type $F_3$ in $\Delta$ or a boundary path of $\Delta$. Then $c_1, c_2$ are closed simple paths and either $c_1$ is a cyclic permutation of one of $c_2, c_2^{-1}$ or $c_1, c_2$ have no common vertices.

Note that the property (A) implies that if an essential edge $e$ of $\Delta$ belongs to a boundary path of a face of type $F_3$ or $e$ belongs to a boundary path of $\Delta$, then $e$ also belongs to a boundary path of a face of type $F_2$.

From now on we always assume, unless stated otherwise, that a diagram is a surface diagram over (2.5) with the property (A).

Recall that the literal (or letter-by-letter) equality of the words $U, V$ is denoted $U \equiv V$.

Lemma 2.3. Let $W$ be a nonempty word over the alphabet

$$\bigcup_{i=1}^{\infty} X_i^{\pm 1} \cup A^{\pm 1} \cup \{h_1^{\pm 1}, h_2^{\pm 1}, 1\}$$

and let $G_2$ be the group defined by presentation (2.5). Then $W \equiv 1$ if and only if there is a surface diagram $\Delta$ of type $(1, 1)$, called a disk diagram, over presentation (2.5) such that $\varphi(\partial \Delta) \equiv W$.

Proof. The proof is straightforward, for details the reader is referred to [12], [7], see also [9]. As in [12], faces of type $F_1$, $F_2$ make it possible to “thicken” the diagram and turn its underlying topological space into a disk.

We finish this section with a lemma about graphs on surfaces.

Lemma 2.4. Let $S$ be a compact surface whose Euler characteristic is $\chi(S) = k$ and let $\Psi$ be a finite graph on $S$ that has the property (B) with parameter $m = 2$. If $V_\Psi$ and $E_\Psi$ denote the number of vertices and nonoriented edges of $\Psi$, resp., then $E_\Psi \leq 3(V_\Psi - k)$.

Proof. Note that the property (B) with parameter $m = 2$ can be stated less formally by saying that the partial cell decomposition of $S$, defined by the graph $\Psi$, contains no 1- and 2-gons whose interiors contain no vertices of $\Psi$. Preserving this condition, i.e., preserving the property (B) with parameter $m = 2$, we will draw as many new edges in $\Psi$ as possible and obtain a graph $\Psi'$ with $V_{\Psi'} = V_{\Psi}$, $E_{\Psi'} \geq E_{\Psi}$. Note that $\Psi'$ is connected and if $c$ is a connected component of $\partial \Delta$ then there is a closed simple path $e_{c,1} \ldots e_{c,k_c}$, where $e_{c,1}, \ldots, e_{c,k_c}$ are edges of $\Psi'$, such that $e_{c,1} \ldots e_{c,k_c}$ and $c$ bound an annulus $A_c$ whose interior contains no vertices of $\Psi'$. Hence, taking
As in Sect. 2, let $\Delta$ be a surface diagram over presentation (2.5) with property (A). Consider a relation $\sim_2$ on the set $\Delta_2(2)$ of faces of type $F_2$ so that $\Pi_1 \sim_2 \Pi_2$ if and only if there is an essential edge $e$ such that $e$ belongs to $(\partial \Pi_1)_{\pm 1} := (\partial \Pi_1 \cup \partial \Pi_1)^{-1}$ and $e$ belongs to $(\partial \Pi_2)_{\pm 1}$. It is easy to see that this relation is reflexive and symmetric on $\Delta_2(2)$. The transitive closure of this relation $\sim_2$ is an equivalence relation on $\Delta_2(2)$ which we denote by $\sim$. Let $[\Pi]_\sim$ denote the equivalence class of a face $\Pi$ of type $F_2$ relative to this equivalence relation. For every $\Pi \in \Delta_2(2)$, we consider a minimal subcomplex $B_\Pi = B([\Pi]_\sim)$ of $\Delta$ that contains all faces of $[\Pi]_\sim$. It follows from definitions that there exists a surface diagram $A_\Pi$ of type $(1,1)$ (meaning that $A_\Pi$ is a disk) or of type $(0,1)$ (meaning that $A_\Pi$ is an annulus) and a continuous cellular map $\mu_\Pi: A_\Pi \to B_\Pi$ such that $\mu_\Pi$ preserves dimension of cells, $\varphi$-labels of edges, and $\mu_\Pi(A_\Pi) = B_\Pi$. We also require that $A_\Pi$ consists of faces of type $F_2$ and their number $|A_\Pi(2)|$ equals the number $|B_\Pi(2)|$ of faces in $B_\Pi$. Note that $\mu_\Pi$ need not be injective and this is the reason we consider an "ideal" preimage $A_\Pi$ of the subcomplex $B_\Pi$.

If $A_\Pi$ is a disk, then $\partial A_\Pi = s_1f_1s_2f_2$, where $f_1, f_2$ are essential edges with $\varphi(f_1) = \varphi(f_2)^{-1} \neq 1$, and $s_1, s_2$ are simple paths consisting of 1-edges with $|s_1| = |s_2| = |A_\Pi(2)|$, see Fig. 1(a). In this case, we say that $B_\Pi$ is a band between the edges $e_1, e_2$ and that $\partial B_\Pi = u_1e_1u_2e_2$, where $e_i = \mu_\Pi(f_i)$, $u_i = \mu_\Pi(s_i)$, $i = 1, 2$, is a standard boundary path of the band $B_\Pi$. Clearly, $e_1, e_2$ are essential edges with $\varphi(e_1) = \varphi(f_1) = \varphi(e_2)^{-1} \neq 1$ and $|u_1| = |s_1| = |u_2|$ but $u_1, u_2$ need not be simple paths. If $\varphi(e_1)^{\pm 1} = y$, where $y \in \bigcup_{i=1}^\infty X_i \cup A \cup \{h_1, h_2\}$, then we may also specify that $B_\Pi$ is a $y$-band.

Since we neither fix a base vertex for $\partial B_\Pi$, nor fix an orientation for $B_\Pi$, it follows that if $\partial B_\Pi = u_1e_1u_2e_2$ is a standard boundary path for a band $B_\Pi$, then $ue_2u_1e_1$ and $u_2^{-1}e_1^{-1}u_1^{-1}e_2^{-1}$ are also standard boundary paths for $B_\Pi$. We also observe that a standard boundary path of a band $B$ need not be the topological boundary of $B$ but it can be turned into the topological boundary (of a deformed space) by an arbitrarily small deformation of $B$ which pushes $B$ into its interior.

On the other hand, if $A_\Pi$ is an annulus, then $\partial A_\Pi = s_1 \cup s_2$, where $s_1, s_2$ are cyclic simple paths consisting of 1-edges, $|s_1| = |s_2| = |A_\Pi(2)|$, see Fig. 1(b). In this case, we say that $B_\Pi$ is an annulus and that $\partial B_\Pi = u_1 \cup u_2$, where $u_i = \mu_\Pi(s_i)$, $i = 1, 2$, are boundary paths of the annulus $B_\Pi$.

3. Contiguity Subdiagrams

$A_e$ out of $S$ and adding back the cycle $e_{c,1} \ldots e_{c,k_e}$ for every connected component $c$ of $\partial \Delta$, we obtain a surface $S'$ such that $\chi(S') = \chi(S) = k$. In addition, it follows from definitions that $S' \setminus \Psi'$ is a collection of open disks. Indeed, if a connected component of $S' \setminus \Psi'$ were different from a disk, then one could draw an additional edge in $\Psi'$ without creating a 1- or 2-gon, contrary to the maximality of $\Psi'$. Hence, the graph $\Psi'$ defines a cell decomposition of $S'$ and

$$V_{\Psi'} - E_{\Psi'} + F_{\Psi'} = \chi(S') = k,$$

where $F_{\Psi'}$ is the number of faces of the cell decomposition of $S'$ defined by $\Psi'$. Since there are no 1- and 2-gons in this decomposition, every face has 3 edges in its boundary path which implies that $3F_{\Psi'} \leq 2E_{\Psi'}$ or $F_{\Psi'} \leq \frac{2}{3}E_{\Psi'}$. Hence, it follows from (2.5) that

$$V_{\Psi'} - \frac{1}{3}E_{\Psi'} \geq k \text{ or } E_{\Psi'} \leq 3(V_{\Psi'} - k).$$

Since $V_{\Psi'} = V_\Psi$, $E_{\Psi'} \geq E_\Psi$, our claim is proved. □
Note that if $B$ is a band and $\partial B = u_1 e_1 u_2 e_2$ is a standard boundary path of $B$, then each of the essential edges $e_1, e_2$ belongs either to a boundary path of $\Delta$ or to a boundary path of a face of type $F_3$. If, say, $e_i$ belongs to $c_i$, where $i = 1, 2$ and $c_i$ is a boundary path of $\Delta$ or is a boundary path of a face of type $F_3$, then we say that $B([\Pi]_*)$ is a band between $c_1$ and $c_2$.

Let $B$ be a band between edges $e_1$ and $e_2$. Let $o_1 \in e_1, o_2 \in e_2$ be interior points of edges $e_1, e_2$ and let $\ell(B)$ be a simple arc such that $\ell(B)$ is contained in $B$, the boundary points of $\ell(B)$ are $o_1, o_2$ and the intersection of $\ell(B)$ with every face $\Pi$ of $B$ consists of a single arc which is properly embedded in $\Pi$ and the boundary points of the arc are interior points of essential edges of $\partial \Pi$. Such an arc $\ell(B)$ is called a connecting line for $B$. It follows from definitions that if $B$ is a band between edges $e_1$ and $e_2$, then a connecting line $\ell(B)$ for $B$ connects interior points of $e_1, e_2$ through faces of $B$ of type $F_2$.

Let $s$ be either a subpath of $\partial \Pi$ (where $\Pi$ is a face of type $F_3$ in $\Delta$) or a subpath of $\partial \Delta$ such that $s$ consists of $h$-edges and $s$ is maximal with respect to this property. Such $s$ is called an $h$-section of $\Delta$.

Suppose that $s_1, s_2$ are $h$-sections of $\Delta$, not necessarily distinct, and $B_1, B_2$ are bands between $s_1, s_2$, perhaps $B_1 = B_2$, whose standard boundary paths are $\partial B_i = u_i e_{i1} u_{i2} e_{i2}, i = 1, 2$, where $e_{i1}, e_{i2}$ are essential edges of $\partial B_i$. Also, assume that $e_{i1}, e_{i2}$ are edges of $s_i$ so that $s_1 = s_{11} e_{11} s_{12} e_{12} s_{13}$ and $e_{21}, e_{22}$ are edges of $s_2$ so that $s_2 = s_{21} e_{21} s_{22} e_{22} s_{23}$, see Fig. 2.

Note that the path $p = u_{11} e_{11} s_{12} e_{21} u_{22} e_{22} s_{23} e_{12}$ is closed. Furthermore, assume that there exists a connected subcomplex $\Gamma'$ of $\Delta$ such that $\Gamma'$ contains $B_1, B_2, p$. $\Gamma'$ has no faces of type $F_3$ with $h$-edges, and the path $p$ is nullhomotopic in $\Gamma'$.
Denote $\Gamma \land s_1 := e_{11}s_{12}e_{21}$ and $\Gamma \land s_2 := e_{22}s_{22}e_{12}$ and call these paths contiguity arcs of $\Gamma$. If $B_1 = B_2$, then $\Gamma \land s_1 := e_{11}$ and $\Gamma \land s_2 := e_{12}$. Since $\Gamma$ contains no faces of type $F3$ with $h$-edges, $s_1, s_2$ are $h$-sections and $u_{11}, u_{12}$ consist of 1-edges, it follows that $\varphi(e_{11}s_{12}e_{21}) = \varphi(e_{22}s_{22}e_{12})^{-1}$ and, by definitions and property (A), there exists a simple path $t$, $|t| > 0$, that connects $(u_{11})_+ \in s_2$ with $(u_{11})_- \in s_1$ and consists of 1-edges. A factorization of $\partial \Gamma$ of the form

$$\partial \Gamma = u_{11}(e_{11}s_{12}e_{21})u_{22}(e_{22}s_{22}e_{12})$$

is called a standard boundary path of the contiguity subdiagram $\Gamma$.

A contiguity subdiagram $\Gamma$ between $h$-sections $s_1, s_2$ is called maximal if there is no contiguity subdiagram $\Gamma'$ between $s_1, s_2$ such that $\Gamma \land s_1$ is a subpath of $\Gamma' \land s_1$, for both $i = 1, 2$, and $|\Gamma \land s_1| + |\Gamma \land s_2| < |\Gamma' \land s_1| + |\Gamma' \land s_2|$. In the following lemma, we record simple facts about bands and contiguity subdiagrams.

**Lemma 3.1.** Suppose that $e$ is an edge of an $h$-section of a surface diagram $\Delta$ and $B$ is an $h$-band in $\Delta$. Then the following are true.

(a) There is an $h$-band one of whose essential edges is $e$.

(b) There is a unique maximal contiguity subdiagram $\Gamma$ that contains $B$.

(c) There is a unique maximal contiguity subdiagram one of whose contiguity arcs contains $e$.

**Proof.** (a) Suppose that $e$ belongs to a boundary path of $\Pi$, where $\Pi$ is a face of type $F3$ in $\Delta$. Then it follows from property (A) that if $o$ is an interior point of $e$ then a regular neighborhood $N$ of $o$ in $\Delta$ consists of two parts separated by the arc $N \cap e$, one of which is in $\Pi$ and the other of which is in a face $\Pi'$ of type $F2$. Then $B_{\Pi'}$ is a desired $h$-band. If $e$ is on $\partial \Delta$ then, again by property (A), there is a face $\Pi''$ of type $F2$ whose boundary path contains $e$. Then $B_{\Pi''}$ is a desired $h$-band.

(b) Let $B$ be a band between $h$-sections $s_1, s_2$. Then there exists a contiguity subdiagram $\Gamma$ between $s_1$ and $s_2$ that contains $B$. For example, $\Gamma = B$. If $\Gamma_1, \Gamma_2$ are two contiguity subdiagrams between $s_1$ and $s_2$ that contain $B$, then it is easy to check that there is also a contiguity subdiagram $\Gamma_0$ that contains both $\Gamma_1$ and $\Gamma_2$. This implies the uniqueness of a maximal contiguity subdiagram that contains $B$.

(c) This follows from parts (a)–(b). \qed

Let $\Delta$ be a surface diagram over presentation $\langle k, k' \rangle$ of type $(k, k')$. Consider the set $\mathcal{E}_h$ of all maximal contiguity subdiagrams between $h$-sections in $\Delta$. It follows from Lemma 3.1 that, for every edge $e$ of an $h$-section $s$ of $\Delta$, there is a unique maximal contiguity subdiagram $\Gamma \in \mathcal{E}_h$ whose contiguity arc contains $e$, i.e., $e$ belongs to $\Gamma \land s$.

For every $\Gamma \in \mathcal{E}_h$, we pick a connecting line $\ell(\Gamma)$, where $B = B(\Gamma)$ is a band that defines $\Gamma$. Denote $\ell(\Gamma) := \ell(B)$ and call $\ell(\Gamma)$ a connecting line of $\Gamma$. For every face $\Pi$ of type $F3$, whose boundary path $\partial \Pi$ contains $h$-edges, we pick a vertex $v_{\Pi}$ in the interior of $\Pi$. Then we connect each point in $(\bigcup_{\Gamma \in \mathcal{E}_h} \ell(\Gamma)) \cap \partial \Pi$ to $v_{\Pi}$ by drawing simple arcs in $\Pi$ such that the arcs’ pairwise intersections are $\{v_{\Pi}\}$ and each arc intersects $\partial \Pi$ only at its endpoint different from $v_{\Pi}$. The union of all such arcs and connecting lines $\ell(\Gamma)$, $\Gamma \in \mathcal{E}_h$, is a graph on $\Delta$, denoted $\Psi_h$, whose vertex set is the union of the set $\{v_{\Pi}\} \Pi \in \Delta_3(2), \partial \Pi$ has $h$-edges and the set of those boundary points of connecting lines $\ell(\Gamma)$, $\Gamma \in \mathcal{E}_h$, that belong to $\partial \Delta$. Note that
the set of nonoriented edges of $\Psi_h$ is in bijective correspondence with the set $\mathcal{C}_h$ of maximal contiguity subdiagrams and that each edge of $\Psi_h$ is obtained from $\ell(\Gamma)$, where $\Gamma \in \mathcal{C}_h$, by extending $\ell(\Gamma)$ into a face $\Pi$ of type F3 whenever a point of $\partial \ell(\Gamma)$ belongs to $\partial \Pi$.

Now we will define reduced diagrams over the presentation (2.5). We say that a pair of distinct faces $\Pi_1, \Pi_2$ of type F3 with $h$-edges in a surface diagram $\Delta$ over (2.5) forms a reducible pair if there is a simple path $t$ such that $t$ connects some vertices $t_- \in \partial \Pi_1, t_+ \in \partial \Pi_2$, $t$ consists of 1-edges, $|t| > 0$, and the label $\varphi(\partial \Gamma)$ of the boundary path $\partial \Gamma = t \partial \Pi_2 t^{-1} \partial \Pi_1$ of the subdiagram $\Gamma$, consisting of $t, \Pi_1, \Pi_2$, is equal to 1 in the free group whose free base is the alphabet $\bigcup_{i=1}^{\infty} \mathcal{X}_i \cup \mathcal{A} \cup \{h_1, h_2\}$, see Fig. 3.

$$\begin{array}{cc}
\Pi_1 & \Pi_2 \\
\xrightarrow{t} & \\
\end{array}$$

Fig. 3

It is easy to see that if $\Pi_1, \Pi_2$ form a reducible pair in $\Delta$, then one can perform a surgery on $\Delta$ that replaces the subdiagram $\Gamma$, whose boundary path is $\partial \Gamma = t \partial \Pi_2 t^{-1} \partial \Pi_1$, by a subdiagram that consists of faces of type F1–F2. If $\Delta'$ is obtained from $\Delta$ by this surgery, then $\varphi(\partial \Delta')$ is identical to $\varphi(\partial \Delta)$ (in fact, the surgery does not affect the boundary of $\Delta$) and $|\Delta'(2)| = |\Delta(2)| - 2$. Hence, by induction on the number $|\Delta(2)|$ of faces of type F3, every diagram $\Delta$ can be turned into a diagram $\Delta$ without reducible pairs and with no change in $\varphi(\partial \Delta)$. A diagram $\Delta$ will be called reduced if $\Delta$ contains no reducible pairs.

**Lemma 3.2.** Suppose that $\Delta$ is a reduced surface diagram of type $(k, k')$, there are no $h$-edges contained in $\partial \Delta$, $\Delta$ contains a face of type F3 whose boundary path has $h$-edges, and the graph $\Psi_h$ is defined as above. Then there exists a vertex in $\Psi_h$ whose degree is positive and is at most $\max\{12(1 - k), 12\}$.

**Proof.** Let $v_{\Pi}$ be a vertex of $\Psi_h$, let $f$ be an oriented edge of $\Psi_h$ such that $f_- = f_+ = v_{\Pi}$ and $f$ bounds a disk on $\Delta$. It follows from the definition of relations in (2.5) that if $e_1, e_2$ are $h$-edges of $\partial \Pi$, then either $\varphi(e_1), \varphi(e_2) \in \{h_1, h_2\}$ or $\varphi(e_1), \varphi(e_2) \in \{h_1^{-1}, h_2^{-1}\}$. On the other hand, let $\Gamma \in \mathcal{C}_h$ be the contiguity subdiagram that $f$ passes through and let $B$ denote the bond that contains the connecting line $\ell(\Gamma)$. If $e_3, e_4$ are $h$-edges of $\partial B$, then it follows from the fact that $f$ bounds a disk on $\Delta$ that $\varphi(e_3) = \varphi(e_2)^{-1}$, hence, the inclusions $e_3, e_4 \in \partial \Pi$ are impossible. Thus, there is no 1-gon in the partial cell decomposition of $\Delta$ defined by $\Psi_h$.

Now assume that the property (B) fails for $\Psi_h$ with parameter $m = 3$. This means that there are three distinct edges $f_1, f_2, f_3$ in $\Psi_h$ such that

$$(f_1)_- = (f_2)_- = (f_3)_- = v_{\Pi}, \quad (f_1)_+ = (f_2)_+ = (f_3)_+ = v_{\Pi'},$$

where $\Pi, \Pi'$ are some faces of type F3 with $h$-edges, such that both paths $f_1 f_2^{-1}, f_2 f_3^{-1}$ bound disks on $\Delta$ whose interiors contain no vertices of $\Psi_h$. Let $f_i$ be the extension of the connecting line $\ell(\Gamma_i)$, where $\Gamma_i \in \mathcal{C}_h$, $i = 1, 2, 3$, and $s, s'$ be $h$-sections of the faces $\Pi, \Pi'$, resp. Then it is not difficult to check that either $\Gamma_1, \Gamma_2$ or $\Gamma_2, \Gamma_3$ are contained in a contiguity subdiagram $\Gamma'$ between $s$ and $s'$, contrary to the maximality of contiguity subdiagrams $\Gamma_1, \Gamma_2, \Gamma_3$. This contradiction proves that the property (B) holds for $\Psi_h$ with $m = 3$. 
Consider those pairs \{f, f'\} of oriented edges of \( \Psi_h \) for which the property (B) with \( m = 2 \) fails. Note that the property (B) with \( m = 3 \) for \( \Psi_h \) implies that every oriented edge \( e \) of \( \Psi_h \) is contained in at most one such pair \{f, f'\}. For each such pair \{f, f'\}, we remove edges \((f')^\pm 1\) (or \(f^\pm 1\)) from \( \Psi_h \). Doing this results in a graph \( \hat{\Psi}_h \) which, as follows from definitions, has the property (B) with \( m = 2 \). Therefore, Lemma 2.4 applies to \( \hat{\Psi}_h \) and yields that \( E_{\hat{\Psi}_h} \leq 3(V_{\hat{\Psi}_h} - k) \), where \( V_{\hat{\Psi}_h} \) denote the number of vertices, nonoriented edges, resp., in \( \hat{\Psi}_h \). Note that \( V_{\hat{\Psi}_h} = V_{\Psi_h} \) and \( E_{\hat{\Psi}_h} \leq 2E_{\Psi_h} \). Hence, \( E_{\Psi_h} \leq 6(V_{\Psi_h} - k) \). If \( d \) is the minimal positive degree of a vertex in \( V_{\Psi_h} \), then it is easy to see from definitions that \( d > 0 \) and \( dV_{\Psi_h} \leq 2E_{\Psi_h} \). Thus \( dV_{\Psi_h} \leq 12(V_{\Psi_h} - k) \) and
\[
d \leq 12(1 - \frac{k}{V_{\Psi_h}}) \leq \max\{12(1 - k), 12\},
\]
as desired. \( \square \)

4. Proofs of Theorems

Proof of Theorem 1.1. First we observe that the group \( \mathcal{G}_2 \), given by presentation (2.4), can also be presented by generators and relations in the following form
\[
\langle h_1, h_2 \mid \hat{R}_1 = 1, \hat{R}_2 = 1, \ldots, \hat{W}_1 = 1, \hat{W}_2 = 1, \ldots \rangle,
\]
where, for every possible \( i = 1, 2, \ldots \), the defining words \( \hat{R}_i, \hat{W}_i \) result from rewriting of the words \( R_i, W_i(X_i) \), resp., of presentation (2.5) so that letters \( a_{j_1}^r, x_{j_2}^s \), where \( a_{j_1} \in A \), \( x_{j_2} \in X \), \( r \leq 1 \), \( s \in \mathbb{Z} \), are replaced with the words \( V_{12j_1}^{\varepsilon_1}, V_{12j_2}^{\varepsilon_2} \) over \( \{h_1^\pm 1, h_2^\pm 1\} \), see (2.3).

Now we will show that the group \( G \) given by the presentation (2.1) naturally embeds into the group \( \mathcal{G}_2 \) given by (2.5). Assume that \( U_0 \) is a cyclically reduced word over \( A^{\pm 1} \) and \( U'_0 = 1 \) in \( \mathcal{G}_2 \). By Lemma 2.3 there is a disk diagram \( \Delta_0 \) over (2.5) such that \( \varphi(\partial \Delta_0) \equiv U_0 \). Without loss of generality, we may assume that \( \Delta_0 \) is reduced. Note that a boundary path of \( \Delta_0 \) contains no \( h \)-edges. If \( \Delta_0 \) contains no face of type F3 whose boundary path has \( h \)-edges then, turning \( h \)-edges into \( 1 \)-edges by relabeling, we may assume that \( \Delta_0 \) contains no \( h \)-edges. Hence, we may suppose that \( \Delta_0 \) is a disk diagram over the presentation (2.2). Then it follows from Lemmas 2.1, 2.3 that \( U_0 = 1 \) in \( G \). Thus, if \( U_0 \) is not trivial in \( G \), then \( \Delta_0 \) must contain a face of type F3 with \( h \)-edges. Therefore, Lemma 3.2 applies to \( \Delta_0 \) and yields the existence of a vertex \( \eta_1 \), where \( \Pi \) is a face of type F3 with \( h \)-edges, whose degree \( d \) in the graph \( \Psi_0 \) is positive and is at most \( \max\{12(1 - k), 12\} = 12 \) as \( k = \chi(\Delta_0) = 1 \). It follows from the definition of the graph \( \Psi_h \) and Lemmas 2.4, 3.1 that there are \( d \leq 12 \) maximal contiguity subdiagrams \( \Gamma_1, \ldots, \Gamma_d \) between an \( h \)-section \( q \) of \( \Pi \) and some \( h \)-sections of \( \Delta_0 \) so that every edge of \( q \) is contained in exactly one of the contiguity arcs \( \Gamma_i \cap q \), \( i = 1, \ldots, d \). Therefore, there is an index \( i^* \) such that \( |\Gamma_{i^*} \cap q| \geq \frac{1}{12}|q| \). Since \( \partial \Delta_0 \) contains no \( h \)-edges, it follows that \( \Gamma_{i^*} \) is a contiguity subdiagram between \( q \) and \( q' \), where \( q' \) is an \( h \)-section of a face \( \Pi' \). Denote \( q_{1i} := \Gamma_i \cap q \) and \( q_{1ii} := \Gamma_i \cap q' \). Since \( \varphi(q_{1i}) \equiv \varphi(q_{1ii})^{-1} \) and \( |q_{1i}| \geq \frac{1}{12}|q| > \frac{1}{12}|q| \) as \( n \geq 2 \) and \( M = 24n \geq 48 \), it follows from Lemma 2.2 that \( \varphi(q) \equiv \varphi(q')^{-1} \). Hence, by the definition of relations in (2.5) and by the definition of a contiguity subdiagram, we have that \( \varphi(\partial \Pi) \equiv \varphi(\partial \Pi')^{-1} \) and the faces \( \Pi, \Pi' \) form a reducible pair. This contradiction to the fact that \( \Delta_0 \) is reduced proves that
Consider a quadratic equation \( W = 1 \) over \( G \) of length \( \ell \leq n \). We need to prove that the equation \( W = 1 \) has a solution in the group \( G \) given by (5.1) if and only if the equation \( \nu_2(W) = 1 \) has a solution in the group \( S_2 \) given by (5.5).

First assume that \( W = 1 \) has a solution in \( G \). By Lemma 5.1, the equation \( \nu_1(W) = 1 \) has a solution in the group \( S_1 \) given by (5.2). Since \( G \) naturally embeds in \( S_2 \), it follows from the definition of presentations (5.2), (5.3) that there is a homomorphism \( S_1 \to S_2 \) which is identical on \( G \). Hence, we may conclude that the equation \( \nu_2(W) = 1 \) has a solution in the group \( S_2 \), as desired.

Conversely, suppose that the equation \( \nu_2(W) = 1 \) has a solution in the group \( S_2 \). Our goal is to show that \( W = 1 \) has a solution in \( G \). Let

\[
W = t_1^{\varepsilon_1}U_1t_2^{\varepsilon_2}U_2\ldots t_\ell^{\varepsilon_\ell}U_\ell,
\]

where \( t_1, \ldots, t_\ell \in \cup_{i=1}^{\infty} X_i, \varepsilon_1, \ldots, \varepsilon_\ell \in \{\pm 1\} \), and \( U_1, \ldots, U_\ell \) are some reduced or empty words over \( A^{\pm 1} \). Since \( \nu_2(W) = 1 \) has a solution in \( S_2 \), there are nonempty words \( T_1, \ldots, T_\ell \) over the alphabet \( \cup_{i=1}^{\infty} X_i^{\pm 1} \cup A^{\pm 1} \cup \{h_1^{\pm 1}, h_2^{\pm 1}, 1\} \) such that

\[
T_1^{\varepsilon_1}U_1T_2^{\varepsilon_2}U_2\ldots T_\ell^{\varepsilon_\ell}U_\ell \equiv 1.
\]

Note that we would use the letter 1 for the trivial element of \( S_2 \). By Lemma 5.2, there is a disk diagram \( \Delta \) over presentation (5.5) such that

\[
\varphi(\partial\Delta) = T_1^{\varepsilon_1}U_1T_2^{\varepsilon_2}U_2\ldots T_\ell^{\varepsilon_\ell}U_\ell.
\]

Since \( W = 1 \) is a quadratic equation, there is a permutation

\[
\tau : \{1, \ldots, \ell\} \to \{1, \ldots, \ell\}
\]

such that \( \tau^2 = 1 \), \( \tau(i) \neq i \) and \( t_i = t_{\tau(i)} \) for every \( i \in \{1, \ldots, \ell\} \). Hence, we may assume that \( T_i \equiv T_{\tau(i)} \) for every \( i \in \{1, \ldots, \ell\} \). Denote

\[
\partial\Delta = r_1^{\varepsilon_1}u_1r_2^{\varepsilon_2}u_2\ldots r_\ell^{\varepsilon_\ell}u_\ell,
\]

where \( r_1, u_i \) are paths of \( \partial\Delta^{\pm 1} \) such that \( \varphi(r_1) \equiv T_1, \varphi(u_i) \equiv U_i \) for every \( i = 1, \ldots, \ell \). Now we construct a surface diagram \( \tilde{\Delta} \) from \( \Delta \) by attaching the path \( r_i \) to \( r_{\tau(i)} \) for every \( i = 1, \ldots, \ell \). Note that \( \chi(\Delta) = 1 - \frac{1}{2} \ell \) and \( \Delta \) has \( k' \) connected components in its boundary \( \partial\Delta \), \( 1 \leq k' \leq \ell \). Thus, \( \Delta \) is a surface diagram of type \((1 - \frac{1}{2}, k')\).

Let \( c_1, \ldots, c_{k'} \) be connected components of \( \partial\Delta \). Note that each \( c_j \) is a product of some paths in the set \( \{u_1^{\delta_1}, \ldots, u_\ell^{\delta_\ell}\} \), where \( \delta_1, \ldots, \delta_\ell \in \{\pm 1\} \), and each \( u_j^{\delta_j} \) occurs in one of \( c_1, \ldots, c_{k'} \) exactly once. If \( \Delta \) contains a reducible pair of faces, then we remove this pair by the surgery described above and obtain a surface diagram \( \Delta' \) with unchanged boundary paths and \( |\Delta_3(2)| = |\Delta_3(2)| - 2 \), where \( |\Delta_3(2)| \) is the number of faces of type F3 in \( \Delta \). It is not difficult to check that there exists a disk diagram \( \Delta' \) such that

\[
\partial\Delta' = (r_1')^{\varepsilon_1}u_1'(r_2')^{\varepsilon_2}u_2'\ldots (r_\ell')^{\varepsilon_\ell}u_\ell',
\]

where \( r_i', u_i' \) are paths of \( \partial(\Delta')^{\pm 1} \) such that \( \varphi(r_i') \equiv \varphi(r_{\tau(i)}) \equiv T_i' \), \( \varphi(u_i') \equiv \varphi(u_i) \) for every \( i = 1, \ldots, \ell \). Moreover, the diagram \( \Delta' \) can be obtained from \( \Delta' \) in the same manner as \( \tilde{\Delta} \) was obtained from \( \Delta \), in particular, \( |\Delta_3(2)| = |\Delta_3'(2)| \). Hence,
by induction on the number $|\Delta_3(2)|$ of faces of type F3 in $\Delta$, we may assume that the surface diagram $\Delta$ is reduced.

Suppose that $\Delta$ contains no faces of type F3 with $h$-edges. Then $\Delta$ also has this property, hence we can turn $h$-edges of $\Delta$ into 1-edges by relabeling and obtain thereby a disk diagram $\Delta$ from $\Delta$ with no $h$-edges. Such a diagram $\Delta$ could be regarded as a diagram over presentation (2.2). The existence of such $\Delta$ over (2.2) means that the equation $\nu_1(W) = 1$ has a solution in the group $\mathcal{G}_1$ given by (2.2). By Lemma 2.1, the equation $W = 1$ has a solution in $G$, as required.

Hence, we may assume that $\Delta$ contains faces of type F3 with $h$-edges. Clearly, $\Delta$ also has this property and we may consider the graph $\Psi_h = \Psi_h(\Delta)$ on $\Delta$ as defined before. Since $\partial\Delta$ contains no $h$-edges, Lemma 3.2 applies to the graph $\Psi_h$ on $\Delta$ and yields the existence of a vertex $\nu_1$, where $\Pi$ is a face of $\Delta$, whose positive degree is at most

$$\max\{12(1-k), 12\} = \max\{6\ell, 12\} = 6\ell \leq 6n$$

as $\ell \geq 2$. As above, it follows from the definition of the graph $\Psi_h$ and Lemmas 2.4, 3.1 that there are $d \leq 6n$ maximal contiguity subdiagrams $\Gamma_1, \ldots, \Gamma_d$ between an $h$-section $q$ of $\Pi$ and some $h$-sections of $\Delta$ so that every edge of $q$ is contained in exactly one of the contiguity arcs $\Gamma_i \cap q$, $i = 1, \ldots, d$. Therefore, there is an index $i^*$ such that $|\Gamma_i \cap q| \geq \frac{1}{4n} |q|$. Let $\Gamma_i^* = \Gamma_i \cap q$ be a contiguity subdiagram between $q$ and $q'$, where $q'$ is an $h$-section of a face $\Pi'$. Denote $q_{I_1} := \Gamma_i \cap q$ and $q_{I_1'} := \Gamma_i \cap q'$. Since $\phi(q_{I_1}) \equiv \phi(q_{I_1'})^{-1}$ and $|q_{I_1}| \geq \frac{1}{4n} |q| = \frac{1}{4} |q|$ as $M = 24n$, it follows from Lemma 2.2 that $\phi(q) \equiv \phi(q')^{-1}$. Hence, by the definition of relations in (2.5) and by the definition of a contiguity subdiagram, we have that $\phi(\partial\Pi) \equiv \phi(\partial\Pi')^{-1}$ and the faces $\Pi, \Pi'$ form a reducible pair in $\Delta$. This contradiction to the fact that $\Delta$ is reduced proves that it is impossible that $\Delta$ contains faces of type F3 with $h$-edges. Hence, the equation $W = 1$ has a solution in $G$, as desired.

Thus, the group $\mathcal{G}_2$ has all of the required properties of the group $H$ of the statement of Theorem 1.1 and the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.3 (a) Let $W_1 = 1, W_2 = 1, \ldots$ be the enumeration, fixed in (2.1), of all quadratic equations over $G$ such that, for every $i \geq 1$, $|W_i|_{\infty} \leq n$ and $W_i = 1$ has a solution in $G$. Recall that the enumeration $\bigcup_{i=1}^{\infty} X_i = \{x_1, x_2, \ldots\}$ has the property that if $x_j \in X_k, x_j' \in X_{k'}$ and $k < k'$ then $j < j'$. This property implies, for every $W_i = 1$, that if $x_{k_1}, \ldots, x_{k_t}$ are the letters of $\bigcup_{i=1}^{\infty} X_i$ that appear in $W_i(x_i)^{\pm 1}$, then $k_1, \ldots, k_t \leq ni$. Thus, in view of the relations $x_j^{-1}V_{2j} = 1$ of the presentation (2.3), it follows that $(V_{2k_1}, \ldots, V_{2k_t})$ is a solution tuple to the equation $\nu_2(W_i) = 1$ over $\mathcal{G}_2$. Since $|V_k| \leq (M(k + 1) + 1)M$ and $\ell \leq n$, we further obtain that

$$\sum_{j=1}^{t} |V_{2k_j}| \leq nM(2ni + 1) + 1 \leq 3n^2M^2i = Cn^4i,$$

where $C = 3 \cdot 24^2$ as $M = 24n$.

(b) Since the presentation (1.1) of $G$ is recursively enumerable, it follows that the set of all words $U$ over $A^{\pm 1}$ such that $U \subseteq G$ is also recursively enumerable. More generally, we can analogously obtain that all quadratic equations $W = 1$ over $G$ of length $\leq n$ that have solutions in $G$ can be recursively enumerated. The last
observation means that we can create a recursive enumeration (2.1). Now we can use constructions of (2.5), (4.1) and see that defining relations of the presentation (4.1) for $G_2$ can be recursively enumerated as well.

(c) The existence of an algorithm that detects whether a quadratic equation $W = 1$ over $G$ of length $\leq n$ has a solution enables us to effectively write down all quadratic equations $W = 1$ over $G$ of length $\leq n$ that have solutions in $G$. Hence, we can effectively create an enumeration (2.1) and, using constructions of (2.5), (4.1), write down all relations of the form $\hat{W}_1 = 1, \hat{W}_2 = 1, \ldots$ in the presentation (4.1). Since the presentation (4.1) of $G$ is decidable, we can also effectively write down all relations of the form $\hat{R}_1 = 1, \hat{R}_2 = 1, \ldots$ in the presentation (4.1). Hence, the presentation (4.1) is decidable. Since the map $a_i \to V_{2i+1}, i = 1, 2, \ldots$, extends to the embedding $\mu_n : G \to H$ and the set of defining relations of presentation (4.1) is recursive, we see that the embedding $\mu_n : G \to H$ can be effectively constructed. Theorem (1.2) is proven.

References

[1] L. P. Comerford and C. C. Edmunds, Quadratic equations over free groups and free products, J. Algebra 68(1981), 276–297.
[2] M. Culler, Using surfaces to solve equations in free groups, Topology 20(1981), 133–145.
[3] C. C. Edmunds, On the endomorphism problem for free groups, Comm. in Algebra 3(1957), 1–20.
[4] C. C. Edmunds, On the endomorphism problem for free groups II, Proc. London Math. Soc. 38(1979), 153–168.
[5] E. V. Frenkel and A. A. Klyachko, Commutators cannot be proper powers in metric small-cancellation torsion-free groups, http://arxiv.org/abs/1309.0571
[6] G. Higman, B. H. Neumann, and H. Neumann, Embedding theorems for groups, J. London Math. Soc. 24(1949), 247–254.
[7] S. V. Ivanov, The free Burnside groups of sufficiently large exponents, Internat. J. Algebra Comp. 4(1994), 1–308.
[8] O. Kharlampovich and A. Vdovina, Linear estimates for solutions of quadratic equations in free groups, Internat. J. Algebra Comp. 22(2012), 1250004-1–1250004-16.
[9] R. C. Lyndon and P. E. Schupp, Combinatorial group theory. Springer-Verlag, 1977.
[10] I. G. Lysenok and A. G. Myasnikov, A polynomial bound for solutions of quadratic equations in free groups, Trud. Mat. Inst. Steklova 274(2011), 148–190.
[11] A. Yu. Ol’shanskii, Diagrams of homomorphisms of surface groups, Sibirsk. Mat. Zh. 30(1989), 150–171.
[12] A. Yu. Ol’shanskii, Geometry of defining relations in groups, Nauka, Moscow, 1989; English translation: Math. and Its Applications, Soviet series, vol. 70, Kluwer Acad. Publ., 1991.
[13] A. Yu. Ol’shanskii and M. V. Sapir, The conjugacy problem and Higman embeddings, Memoirs Amer. Math. Soc. 170(2004), 133pp.

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