Multipartite Entanglement in Stabilizer Tensor Networks

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Tensor network models reproduce important structural features of holography, including the Ryu-Takayanagi formula for the entanglement entropy and quantum error correction in the entanglement wedge. Yet only little is known about their multipartite entanglement structure, which has been of considerable recent interest. In this work, we study tensor networks formed from random stabilizer states and show that here the tripartite entanglement question has a sharp answer: The average number of GHZ triples that can be extracted from a stabilizer tensor network is small, implying that the entanglement is predominantly bipartite. Previously, such a result was only known for single tensors. As a consequence, we obtain a new operational interpretation of the monogamy of the Ryu-Takayanagi mutual information and an entropic diagnostic for higher-partite entanglement. To establish our results, we derive a ferromagnetic spin model for the average tripartite entanglement of stabilizer tensors and develop novel techniques for evaluating higher moments of random stabilizer states.

Introduction.—In recent years, research in quantum gravity and quantum information theory has been inspired by a fruitful mutual exchange of ideas. Tensor networks in particular provide a common framework, rooted in the similarity between the structure of the tensor network and the bulk geometry in the holographic duality [1–3]. A paradigmatic example is the Ryu-Takayanagi formula, $S(A) \approx |\gamma_A|/4G_N$, which asserts that the entanglement entropy of a boundary region $A$ in a holographic state is in leading order proportional to the area of a corresponding minimal surface $\gamma_A$ in the bulk geometry [4, 5]. Likewise, in any tensor network the entanglement entropy of a boundary subsystem can be upper-bounded in terms of the size of a minimal cut through the network [6] (Fig. 1). This bound can be saturated not only through the choice of suitable tensors [7, 8] but is in fact a generic phenomenon in random tensor networks with large bond dimension [9, 10], the mechanism of which can be understood in terms of multipartite entanglement distillation. These tensor network models not only reproduce the Ryu-Takayanagi formula for the entanglement entropy, but they also implement several other significant features of holographic duality [7–9]. In many ways, these properties follow from the bipartite entanglement structure and can be therefore reduced to entropic considerations.

In this paper, we initiate a study of multipartite entanglement in random tensor network models. Our motivation is twofold: First, recent research in quantum gravity has raised profound questions regarding the multipartite entanglement in holographic states [11–13], in particular with regards to tripartite entanglement of Greenberger-Horne-Zeilinger (GHZ) type [14, 15]. Answers to these questions in the context of tensor network models will likely lead to new diagnostics applicable in holography. Second, we seek to understand the general mechanisms by which quantum information is encoded in tensor networks; an improved understanding of the entanglement structure may inform the design of tensor networks that adequately represent the physics. While it is possible to obtain partial information from the entanglement entropy of subsystems [11, 12, 16–18], many basic questions regarding the multipartite entanglement cannot be answered from entropic data. A striking example is that a pair of GHZ states cannot be entropically distinguished from three Bell pairs, even though their entanglement properties are vastly different [19].

We focus our considerations on stabilizer tensor networks, i.e., tensor networks that are obtained by contracting stabilizer states. Stabilizer states are a well-studied family of quantum states that can be highly entangled (even maximally so) but still have sufficient algebraic structure to admit an efficient classical description, which makes them a versatile tool in quantum information theory [20]. Of particular import in the present context is that their tripartite entanglement structure is well-understood – any tripartite stabilizer state is locally equivalent to a collection of bipartite Bell pairs and tripartite GHZ states [21, 22] (cf. [23–29]). Thus the tripartite entanglement in a stabilizer state can be precisely quantified.

Summary of Results.—Our main result is that the average amount of tripartite entanglement in random stabilizer networks is small. More precisely, for any tripartition the expected number of GHZ triples remains bounded as we take the limit of large bond dimensions (Theorem 1). This has a number of surprising consequences on the correlation and entanglement structure: (a) The number of Bell pairs that can be extracted between two subsystems $A$ and $B$ is roughly half the mutual information $I(A : B)$ (which in turn can be read off the geometry of the network using the Ryu-Takayanagi formula); (b) in particular, the mutual information measures quantum entanglement, proving a conjecture in [11] for stabilizer tensor networks; (c) the monogamy of the mutual information, $I(A : B) + I(A : C) \leq I(A : BC)$, established in [11] for holographic entropies, thus acquires an operational interpreta-

Figure 1. Stabilizer tensor networks. A tensor network state is obtained by placing random stabilizer states at the bulk vertices (blue) and contracting according to the edges of the graph. In the limit of large bond dimensions, the average entanglement entropy of a boundary region $A$ is proportional to the length of a minimal cut $\gamma_A$ through the network (dashed line) [9], $S(A) \approx S_{RT}(A)$, reproducing the Ryu-Takayanagi formula in holography.
tion as originating from the monogamy of quantum entanglement; (d) the tripartite information $I_{3} := I(A : B) + I(A : C) − I(A : BC)$ (i.e., the difference in the above inequality) provides a diagnostic for fourpartite entanglement; in fact, after extracting all Bell pairs we obtain a residual fourpartite entangled state with the entropies of a perfect tensor of size $−I_{3}/2$ [7], strengthening the picture provided by the holographic entropy cone [17] (Fig. 3).

Our results significantly extend previous work in the quantum information community on the multipartite entanglement of stabilizer states. Indeed, the situation considered in [27] can be understood in our setup as the special case of a network consisting of a single tensor.

We establish our results based on a number of technical contributions: First, we show that the average tripartite entanglement can be mapped onto a classical ferromagnetic spin model, the GHZ spin model. For large bond dimensions, this model is in its low-temperature (ordered) phase and hence the tripartite entanglement is determined by its minimal energy configurations (Fig. 2). Second, we develop new techniques for evaluating higher moments of random stabilizer states, including an explicit formula for the third moment of non-qubit stabilizer states. Our results generalize readily to arbitrary moments [30] and we expect that they will be of similar use as the recent breakthroughs [31–33]. Throughout this article, we measure entropies of p-level systems in units of $\log_{p}$ bits.

Random stabilizer networks.—We now describe the random stabilizer network model. Consider a connected graph with vertices $V$ and edges $E$ (parallel edges allowed). Let $V_{0}$ denote a subset of the vertices, which we will refer to as the boundary vertices; all other vertices are called bulk vertices and denoted by $V_{b}$. Given a choice of bond dimensions for all edges, we define a pure quantum state by placing tensors $|V_{x}\rangle$ at the bulk vertices and contracting according to the edges:

$$|\Psi\rangle = \left(\bigotimes_{x \in V_{b}} |V_{x}\rangle \right) \left(\bigotimes_{e \in E} |e\rangle\right)$$  (1)

Here, $|e\rangle \propto \sum_{i} |i\rangle$ denotes a normalized maximally entangled state corresponding to an edge $e$. The state $|\Psi\rangle$ is a tensor network state defined on the Hilbert space corresponding to the boundary vertices $V_{0}$, and in general unnormalized. We write $\rho = \Psi/\text{tr} \Psi$ for the normalized density matrix, where $\Psi = |\Psi\rangle\langle\Psi|$. See Fig. 1 for an illustration.

To build a stabilizer tensor network state, we choose bond dimensions of the form $D = p^{N}$, where $p$ is a fixed prime and $N$ some positive integer that we will later choose to be large (for simplicity of exposition, we choose all bond dimensions to be the same). Thus the Hilbert space associated with a single vertex is of dimension $D_{x} = p^{N \text{deg}(x)}$, where $\text{deg}(x)$ denotes the degree of the vertex (i.e., the number of incident edges), and the Hilbert spaces associated with the bulk vertices has dimension $D_{b} = p^{N_{b}}$, where $N_{b} = N \sum_{x \in V_{b}} \text{deg}(x)$. We now select each vertex tensor $V_{x}$ in (1) independently and uniformly at random from the set of stabilizer states. Thus $\Psi$ is obtained by partially projecting one stabilizer state onto another (viz., the random vertex tensors onto the maximally entangled pairs), which implies that either $\Psi$ is zero or again a stabilizer state. In the latter case, which occurs with high probability for large $N$, we say that $\Psi$ is a random stabilizer tensor network state. In any tensor network state, the entanglement entropy $S(A) = −\text{tr} \rho_{A} \log_{p} \rho_{A}$ of a boundary subsystem $A \subseteq V_{0}$ can always be upper bounded by $S_{RT}(A) := N \min \{\gamma_{A}\}$, where we minimize over all cuts $\gamma_{A}$ that separate the subsystem $A$ from its complement $\bar{A}$ in $V_{0}$ (Fig. 1). Formally, such a cut is defined by a subset of vertices $V_{A}$ that contains precisely those boundary vertices that are in $A$ such that the set of edges that leaves $V_{A}$ is $\gamma_{A}$.

The fundamental property of random tensor networks is that in the limit of large $N$ (or large $p$), this upper bound becomes saturated [9]. Thus these models reproduce the Ryu-Takayanagi formula in holography. More precisely, the average entanglement entropy of a boundary subsystem, conditioned on the tensor network state being nonzero, is given by

$$\langle S(A) \rangle_{\rho \neq 0} \approx S_{RT}(A).$$  (2)

Here and in the following, we write $\approx$ for equality up to order $O(1)$, independent of $N$. The central fact used to derive this is that random stabilizer states form a projective 2-design [34, 35], i.e., that their first and second moments agree with the Haar measure. For the reader’s convenience, and since the derivation in [9] focused on the case of large $p$, we give a succinct derivation in Appendix B. This result can be strengthened to show that in fact $S(A) \sim S_{RT}(A)$ with high probability [9].

Tripartite entanglement.—Any pure tripartite stabilizer state $\rho_{ABC}$ is locally equivalent to a tensor product of bipartite maximally entangled states, $|\Psi^{+}\rangle_{AB} \propto \sum_{i} |i\rangle_{A} \otimes |i\rangle_{B}$ etc., and tripartite GHZ states $|\text{GHZ}\rangle_{ABC} \propto \sum_{i} |i\rangle_{A} \otimes |i\rangle_{B} \otimes |i\rangle_{C}$ [21, 22]. That is, there exist a local unitary $U = U_{A} \otimes U_{B} \otimes U_{C}$ such that $U \rho_{ABC} U^\dagger$ is equal to

$$|\Phi^{+}_{AB} \rangle \otimes |\Phi^{+}_{AC} \rangle \otimes |\Phi^{+}_{BC} \rangle \otimes |\text{GHZ}_{ABC} \rangle$$  (3)

We suppress local states on $A$, $B$, and $C$ which do not impact the entanglement. The integers $a, b, c, g \geq 0$ are uniquely determined; thus they meaningfully characterize the bipartite and tripartite entanglement between subsystems $A, B$ and $C$. Our main result then is the following:

**Theorem 1** (Tripartite entanglement). Let $A, B, C$ denote a tripartition of the boundary (Fig. 2. (a)). Then the expected number of GHZ states in a random stabilizer network is of order $O(1)$ in the limit of large $N$. 

![Figure 2](image341x649 to 432x740)

**Figure 2. Tripartite entanglement and the GHZ spin model.** (a) Tripartition of the boundary. (b) Illustration of the spin model (with boundary conditions and minimal energy configuration) used to evaluate the GHZ content of a random stabilizer tensor network state.
A network state whose boundary is partitioned into four and more subsystems. We first consider the extraction of entanglement present between any two subsystems (blue lines). The residual state has approximately the entropies of a perfect tensor of the form (3),

\[ I(\rho) \approx I(A_i : A_j) / 2 \]

This decomposition is in one-to-one correspondence with the extreme rays of the holographic entropy cone [17].

In random stabilizer networks, however, Theorem 1 shows that the average number of maximally entangled pairs that can be extracted between any two parties (blue lines) is bounded. Thus the average number of maximally entangled pairs that can be extracted between any two parties (blue lines) is

\[ \langle g \rangle_{\not\equiv 0} \leq \#_k \log_p(p + 1) + \log_p(\#_A \#_B \#_C) + 4\delta, \]

with \#_k the number of minimal cuts for A, etc., \#_A the maximal number of components of any subgraph obtained by removing minimal cuts for A, B, and C [36], and \( \delta = (2p + 2)^{\sqrt{p}} / p^N \).

In most cases of interest, the minimal cuts are unique and there remains a single connected component after their removal, so that \( \langle g \rangle_{\not\equiv 0} \leq \log_p(p + 1) + 4\delta \) [37]. We note that Markov’s inequality implies that the number of GHZ triples in fact remains bounded with high probability. Theorem 1 vastly generalizes a bound in [27], which can be obtained as the special case where \( d = 2 \) and the graph has a single vertex.

In general, the mutual information is sensitive to both classical and quantum correlations. For a general stabilizer state of the form (3), \( I(A : B) = 2c + g \), where \( c \) is the number of maximally entangled pairs and \( g \) the number of GHZ triples (whose reduced state on AB is a classically correlated state). This decomposition is in one-to-one correspondence with the extreme rays of the holographic entropy cone [17].

We can therefore repeat the process and extract maximally entangled pairs between any pair of subsystems \( A_i \) and \( A_j \), until we obtain a residual state \( \tilde{\rho}_{A_1 \ldots A_k} \), whose bipartite mutual informations \( I(A_i : A_j) \) are all of order \( O(1) \).

We now specialize the preceding discussion to a fourpartite system (\( k = 4 \)). Here, the vanishing of the pairwise mutual informations implies that the entropies of the residual state will have the following simple form: \( S(A_i) \approx \frac{1}{2} S(A_i, A_j) \approx m \) for all \( i \neq j \), where \( m \geq 0 \) is some integer [17]. Ignoring the one-correction, stabilizer states with such entropies are four-partite perfect tensors. These are tensors that are unitaries from any pair of subsystems to the complement, a crucial property used in the explicit construction of holographic codes [7, 8].

Significantly, it is possible to determine \( m \) from the entropies of the original state, or, more specifically, from its tripartite information \( I_3 := I(A_1 : A_2) + I(A_1 : A_3) - I(A_1 : A_2 A_3) \), which is invariant under the extraction of the maximally entangled pairs (it also does not depend on the choice of \( A_1, A_2, A_3 \)). In short, we have established the following result:

**Theorem 2 (Fourpartite entanglement).** Let \( A_1, \ldots, A_4 \) denote a partition of the boundary into four subsystems. Then the random stabilizer network state is locally equivalent to

\[ \bigotimes_{i \neq j} (\Phi_{A_i A_j}^+)^{t_{ij}} \otimes \tilde{\rho}_{A_1 A_2 A_3 A_4}. \]  

(4)

For large \( N \), on average \( t_{ij} \approx \frac{1}{2} I(A_i : A_j) \) and the residual state \( \tilde{\rho} \) has approximately the entropies of a perfect tensor of size \( -I_3 / 2 \) (that is, \( S(A_i) \approx S(A_i, A_j) / 2 \approx -I_3 / 2 \)).

Our result provides a new interpretation of the tripartite information \( I_3 \) for random stabilizer networks – namely, as a measure of the entropy of the residual, genuinely fourpartite entangled state \( \tilde{\rho} \). Since entropies are always nonnegative, it follows that \( I_3 \lesssim 0 \); equivalently, the mutual information is monogamous, \( I(A : B) + I(A : C) \lesssim I(A : B C) \), as was proved for holographic entropies in [11] (cf. [38]). This can also be seen by observing that, in our setting, one half the mutual information is an entanglement measure; it is up to \( O(1) \) corrections equal to, e.g., the squashed entanglement \( E_{sq} \); therefore the monogamy of the mutual information also follows as a direct consequence of the monogamy of the latter.

Lastly, it is interesting to compare Theorem 2 with the classification of fourpartite holographic entropies in [17]. We find that there is a one-to-one correspondence between the building blocks of fourpartite entanglement in (4) and the extreme rays of the fourpartite holographic entropy cone defined in [17]. That is, the entropies of a four-partite holographic state can always be reproduced by states of the form (4) (up to rescaling). Theorem 2 elevates this result from the level of entropies to the level of quantum states for random stabilizer networks. It is an interesting question to see if this correspondence can be extended to higher number of parties, where the phase space of holographic entropies becomes significantly more complicated.

**Method: The GHZ spin model.**—We now sketch the proof of Theorem 1. Previous works such as [27] have calculated the GHZ content of multibit stabilizer states by using the algebraic formula from [21] in terms of dimensions of co-local...
stabilizer subgroups. Here, we proceed differently. The idea is to use that the partial transpose \( \rho_{AB}^T \) of the reduced state, which is sensitive to bipartite entanglement. A short calculation using (3) shows that \( \text{tr}(\rho_{AB}^T)^3 = p^{-2(a+b+c+g)} \). Thus the number of GHZ states contained in a tripartite stabilizer state can be computed as

\[
g = S(A) + S(B) + S(C) + \log_p(\text{tr}(\rho_{AB}^T)^3). \tag{5}
\]

In a random stabilizer network, we can upper-bound \( S(A) \leq S_{RT}(A) \) etc., and we know from the preceding section that this bound is not too lose. The main challenge is to upper-bound the expectation value \( \langle \text{tr}(\Psi_{AB}^T)^3 \rangle \), which is a third moment in the unnormalized random tensor network state (1).

We start with the multiqubit case \( (p = 2) \). Only in this case, we can use the recent result that multiqubit stabilizers are projective 3-designs [32, 33]. Thus we have that for each vertex tensor

\[
\langle |V_x\rangle |V_x|^{\otimes 3} = \frac{1}{D_x(D_x+1)(D_x+2)} \sum_{\pi \in S_3} R_x(\pi), \tag{6}
\]

where we sum over all permutations \( \pi \in S_3 \) and write \( R_x(\pi) \) for the corresponding permutation operator acting on three copies of the vertex Hilbert space. Using the analogous notation, we find that \( \text{tr}(\Psi_{AB}^T)^3 = \text{tr} \Psi^{\otimes 3} R_A(\zeta)R_B(\zeta^{-1}) \), where \( \zeta \) is the cyclic permutation that sends \( 1 \mapsto 2 \mapsto 3 \). A careful calculation then reveals that

\[
\langle \text{tr}(\Psi_{AB}^T)^3 \rangle \leq 2^{-3N_b} \sum_{\{\pi_x\}} 2^{-N} \sum_{(\pi_x,\pi_y)} d(\pi_x,\pi_y) \tag{7}
\]

where the sum is over all choices of permutations \( \pi_x \in S_3 \), subject to the boundary conditions \( \pi_x = \zeta \) for \( x \in A \), \( \pi_x = \zeta^{-1} \) in \( B \), and \( \pi_x = 1 \) in \( C \); the sum in the exponent is over all edges, and we define \( d(\pi_x,\pi_y) \) as the minimal number of transpositions required to go from one permutation to the other. We can interpret the right-hand side of (7) as the partition sum of a ferromagnetic spin model with permutation degrees of freedom at each vertex at inverse temperature \( N \) (Fig. 2, (b)).

For large \( N \), we are in the low-temperature (ordered) phase and the partition function is dominated by the minimal energy configuration:

\[
\sum_{\{\pi_x\}} 2^{-N} \sum_{(\pi_x,\pi_y)} d(\pi_x,\pi_y) \leq 2^{-NE_0} (\# + \delta),
\]

where \( E_0 \) denotes the minimal energy, \( \# \) the number of minimal energy configurations and \( \delta = 6\nu_6/2^N \). Now consider an arbitrary configuration \( \{s_x\} \), minimal or not. If we denote by \( V_A \) the \( \zeta \)-domain then the boundary conditions ensure that \( V_A \) is a cut separating \( A \) from \( BC \). While this cut is not necessarily minimal, we always have that \( N|\partial V_A| \geq S_{RT}(A) \), where \( |\partial V_A| \) denotes the number of edges that leaves \( V_A \). Likewise, the \( \zeta^{-1} \)-domain \( V_B \) is a cut for \( B \) and the identity domain \( V_C \) is a cut for \( C \), so that \( N|\partial V_B| \geq S_{RT}(B) \) and \( N|\partial V_C| \geq S_{RT}(C) \). For each edge leaving \( V_A \), the energy cost is at least 1, and it is 2 if the edge enters one of the domains \( V_B \) or \( V_C \) (since \( 1, \zeta, \zeta^{-1} \) are even permutations). Thus

the energy cost of an arbitrary configuration \( \{s_x\} \) can be lower bounded by \( NE_0(\# + \delta) \). The number of GHZ states contained in a tripartite stabilizer state can be computed as

\[
g = S(A) + S(B) + S(C) + \log_p(\text{tr}(\rho_{AB}^T)^3).
\]

degeneracy \( \# \leq 3^N \#_A \#_B \#_C \), since there are three possible transpositions to choose from for each component (Fig. 2, (b)). If we combine these estimates with (5) and some basic results for the trace, we obtain Theorem 1 for qubits.

For \( p \neq 2 \), the ensemble of stabilizer states no longer forms a projective 3-design. To generalize our preceding argument, we have derived a new, explicit formula for the third moment of a random stabilizer state \( |V⟩ \) in \( (C^p)^{\otimes n} \) when \( n \geq 2 \):

\[
\langle |V⟩|V⟩^{\otimes 3} = \frac{1}{p^n(p^n + 1)} \sum_{T \in \Sigma_3(p)} R(T) \tag{8}
\]

This formula can be understood as generalization of Eq. (6). The sum is over \( 2p + 1 \) many subspaces \( T \) of \( C^p \otimes C^p \) described explicitly in Appendix C. Just like the permutations, they can be classified as “even” or “odd”. Moreover, just like in the case of qubits, the operators \( R(T) \) act as a tensor product with respect to the \( n \) copies of the single-particle replica Hilbert space \( (C^p)^{\otimes n} \). These are the central properties used above for qubits, and they allow us to similarly obtain a classical ferromagnetic spin model, now with \( \Sigma_3(p) \) degrees of freedom. Theorem 1 follows as before by an analysis of the low-temperature behavior of this model. We refer to Appendix D for the technical details of this argument.

We emphasize that our formula can be used to evaluate arbitrary third moments, whereas previous works had only computed the frame potential [31–33]. We expect that (8) and its generalization to higher moments [30] will find many further applications in quantum information theory.

**Outlook.**—We have initiated a comprehensive study of multipartite entanglement in tensor network models of holography. Our results suggest several avenues for further investigation: First, it would be of mathematically interest to extend our analysis to stabilizer states of arbitrary local dimension and to establish sharp deviation bounds as in [27]. Second, tensor networks can also be used to define bulk-boundary mappings, or “holographic codes” [7, 8, 17]. In this case, the entanglement entropies of code states obtain a bulk correction, in agreement with the expectations of AdS/CFT [39], and it is natural to ask in which way the multipartite entanglement of typical code states is determined by the bulk [40]. Lastly, much less is known about the entanglement structure of non-stabilizer quantum states. Diagnostics such as moments of the partial transpose considered in this paper may provide a path towards generalizing our results to non-stabilizer states and lead to a more refined understanding of multipartite entanglement, both in tensor network models and in the AdS/CFT correspondence.

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[36] In the language of [7], #_a is the number of multipartite residual regions.
[37] Note that S(A) + S(B) + S(C) = 2(a + b + c) + 3g. It follows that if the sum of local entropies is odd then, necessarily, g > 0. This is all that can be said about the tripartite entanglement from the knowledge of the entropies alone, and it justifies that the upper bound in Theorem 1 is never smaller than log_p(p+1) ≥ 1, even when the minimal cuts are unique.
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[40] In light of our approach, it is natural to conjecture that S(Φ_{V_A}) + S(Φ_{V_B}) + S(Φ_{V_C}) + \log_p tr[\Phi^{⊗3} R_{V_A} (\zeta) R_{V_B} (\zeta^{-1}) R_{V_C} (\tau)]

It is the inserted bulk state, will play a significant role.
Appendix A: Quantization of the trace

In [9, App. F], it was shown that if \( |\phi_A \rangle \in (\mathbb{C}^p)^{\otimes a} \) and \( |\psi_{AB} \rangle \in (\mathbb{C}^p)^{\otimes (a+b)} \) are stabilizer states, with corresponding stabilizer groups \( G \) and \( H \), then the projection \( |\Psi \rangle_B = \langle \phi_A |\psi_{AB} \rangle \), if nonzero, is given by

\[
\Psi_B = \frac{|K|}{|H|} \sum_{g_B \in L} g_B,
\]

where \( K \) some subgroup of \( G \times H \) and \( L \) a commutative subgroup of the corresponding Weyl-Heisenberg group, implying that \( \Psi_B \) is again a stabilizer state. The order of both \( K \) and \( H \) is a power of \( p \), so that \( \text{tr} \, \Psi_B = |K|/|H| \) is necessarily quantized in powers of \( p \). Moreover, \( L \) was defined in [9] as the homomorphic image of \( K \), so that \( |K| \geq |L| \), and hence \( \text{tr} \, \Psi_B \geq |L|/|H| = p^b/p^{a+b} \), since \( |L| = p^b \) and \( |H| = p^{a+b} \). Thus we find that \( \text{tr} \, \Psi_B = p^k/p^b \), where \( k = 0, \ldots, a \).

Applied to the tensor network state \( |\Psi \rangle \) defined in (1), where the vertex tensors \( |V_b \rangle \) are stabilizer states, we note that \( |\Psi \rangle \) is obtained by projecting the collection of Bell pairs onto the tensor product \( \bigotimes_x |V_x \rangle \), which is a stabilizer state in \( (\mathbb{C}^p)^{\otimes N_b} \). Thus we obtain that \( |\Psi \rangle \) is either zero or again a stabilizer state, with trace \( \text{tr} \, \Psi = p^k/p^{N_b} \), where \( k = 0, \ldots, N_b \).

Appendix B: Proof of the Ryu-Takayanagi formula

We give a succinct derivation of the lower bound on the average entanglement entropy. The central fact that we will use is that random stabilizer states form a projective 2-design [34, 35]. Thus their first and second moments agree with the Haar measure; in other words, their entanglement spectra are flat, and thus

\[
\rho^A_0 \text{ is necessarily quantized in powers of } p \text{ or more. Thus,}
\]

\[
\langle \text{tr} \, \Psi^2 \rangle_A \leq p^{-2N_b} p^{-S_{\text{RT}}(A)} (\#_A + \varepsilon),
\]

where \( \#_A \) is the number of minimal cuts and \( \varepsilon := 2^{V}/N \). This calculation has two important consequences:

First, for \( A = \emptyset \) we have that \( \Psi_A = \text{tr} \, \Psi \), so the above can be used to bound the fluctuations of the trace of the unnormalized tensor network state (1). Here, \( \#_A = 1 \) as long as each connected component of the graph contains at least one boundary vertex (so in particular if the graph is connected), so that \( \langle \text{tr} \, \Psi^2 \rangle \leq p^{-2N_b}(1 + \varepsilon) \). From Appendix A we know that if \( \Psi \neq 0 \) then \( \text{tr} \, \Psi = p^k/p^{N_b} \) for some integer \( k = 0, 1, \ldots, N_b \). Let us write \( q_0 \) for the probability that \( \text{tr} \, \Psi = p^k/p^{N_b} \); we are interested in bounding \( q_0 \). Then we obtain the following two relations from the first and second moment of \( \text{tr} \, \Psi \) computed above:

\[
\sum_{k=0}^{n_V} q_k p^k = 1, \quad \sum_{k=0}^{n_V} q_k p^{2k} \leq 1 + \varepsilon.
\]

It follows that \( 1 + \varepsilon \geq q_0 + p \sum_{k=1}^{n_V} q_k p^{2k-1} \geq q_0 + p \sum_{k=1}^{n_V} q_k p = q_0 + p(q_0(1 - q_0) = 1 - p)q_0 + p \) and hence that \( q_0 \geq 1 - \frac{\varepsilon}{p - 1} \geq 1 - \varepsilon \). In other words,

\[
\text{Pr}(\Psi \neq 0) \geq \text{Pr}(\text{tr} \, \Psi = p^{-N_b}) = q_0 \geq 1 - \varepsilon.
\]

Thus we do not only find that \( \Psi \neq 0 \), but in fact that the trace is equal to its expected and minimal value with high probability as \( N \) or \( p \) becomes large.

Second, recall that the entanglement entropy can always be lower-bounded by the Rényi-2 entropy \( S_2(A) = -\log_p \text{tr} \, \rho^2_A \). For stabilizer states we in fact have equality, as their entanglement spectra are flat, and thus

\[
\langle S(A) \rangle_{\neq 0} = 2 \langle \log_p \text{tr} \, \Psi \rangle_{\neq 0} - \langle \log_p \text{tr} \, \Psi^2 \rangle_{\neq 0},
\]

where \( \langle \rangle_{\neq 0} \) denotes the expectation value for states where \( \Psi \neq 0 \).
where we write $\langle f \rangle_{\neq 0}$ for the expectation value of an observable $f$ conditioned tensor network state being nonzero ($\Psi \neq 0$). Using the fact that $\text{tr } \Psi \geq p^{-N_b}$ if $\Psi \neq 0$, Jensen’s inequality for the (concave) logarithm, and $\langle \text{tr } \Psi_A^2 \rangle = \langle \text{tr } \Psi_A^2 \rangle_{\neq 0} \text{Pr}(\Psi \neq 0)$, we can bound this as

$$\langle S(A) \rangle_{\neq 0} \geq -2N_b - \log_p(\text{tr } \Psi_A^2) + \log_p(1 - \varepsilon) \geq S_{RT}(A) - \log_p(\#_A + \varepsilon) + \log_p(1 - \varepsilon).$$

where we have plugged in the upper bound (B1) to obtain the second inequality. Since $\varepsilon$ is arbitrarily small for large enough $N$ or $p$, we obtain that

$$\langle S(A) \rangle_{\neq 0} \geq S_{RT}(A) - \log_p \#_A - 4\varepsilon,$$

where $\#_A$ is the number of minimal cuts. Thus the expected entanglement entropy of a boundary subsystem in a random stabilizer network is indeed close to saturating the Ryu-Takayanagi formula.

### Appendix C: Third moment of stabilizer states

In this section, we give a detailed proof of our formula (8) for the third moment of a random stabilizer state in $(\mathbb{C}^p)^{\otimes n}$ with local prime dimension $p$ and $n \geq 2$.

For $p = 2$, we can use the result that multiqubit stabilizer states form a projective 3-design [31–33]. Indeed, if we define $\Sigma_3(2)$ to be the permutation group $S_3$ and $R(\pi)$ to be the corresponding permutation operator on three copies of $(\mathbb{C}^p)^{\otimes n}$, then formula (8) is nothing but the familiar formula for the third moment of a projective 3-design. For odd primes $p \neq 2$, however, it is known that the stabilizer states do not form a 3-design. We will now develop new methods for this case.

Let $T$ denote a subspace of $\mathbb{F}_p^3 \oplus \mathbb{F}_p^3$. We define a corresponding operator $r(T) = \sum_{(x,y) \in T} |x\rangle\langle y|$ on $(\mathbb{C}^p)^{\otimes 3}$, where $|x\rangle = |x_1, x_2, x_3\rangle \in (\mathbb{C}^p)^{\otimes 3}$ denotes the computational basis vector associated with some $x \in \mathbb{F}_p^3$, and we consider the $n$-fold tensor power $R(T) := r(T)^{\otimes n}$, which is an operator on $((\mathbb{C}^p)^{\otimes 3})^{\otimes n} \cong (\mathbb{C}^p)^{\otimes 3n}$. We note that $R(T)$ is represented by a real matrix in the computational basis.

**Definition 3.** Let $T$ be a subspace of $\mathbb{F}_p^3 \oplus \mathbb{F}_p^3$. We say that $T$ is Lagrangian if $\bar{x} \cdot \bar{x}' = \bar{y} \cdot \bar{y}'$ for any two elements $(\bar{x}, \bar{y}), (\bar{x}', \bar{y}') \in T$ and $T$ is three-dimensional (the maximal possible dimension). We say that $T$ is stochastic if it contains the element $1_6 = (1, \ldots, 1)$. The set of Lagrangian and stochastic subspaces of $\mathbb{F}_p^3 \oplus \mathbb{F}_p^3$ will be denoted by $\Sigma_3(p)$.

For any permutation $\pi \in S_3$, the subspace $T_\pi = \{ (\pi \bar{y}, \bar{y}) : \bar{y} \in \mathbb{F}_p^3 \}$ is Lagrangian and stochastic; the corresponding action $R(T_\pi) = \sum_{\bar{y}} |\pi \bar{y}\rangle\langle \bar{y}|$ agrees with the usual permutation action of $S_3$ on $((\mathbb{C}^p)^{\otimes 3})^{\otimes n}$. Accordingly, we may identify $S_3$ with a subset of $\Sigma_3(p)$. For example, the subspace corresponding to the identity permutation is the diagonal subspace

$$\Delta := T_1 = \{ (\bar{y}, \bar{y}) : \bar{y} \in \mathbb{F}_p^3 \}.$$

We will give an explicit description of $\Sigma_3(p)$ in Eqs. (C5) and (C6) below.

The set of stabilizer states $\text{Stab}(n, p)$ on $(\mathbb{C}^p)^{\otimes n}$ is a single orbit of the Clifford group $\text{Cliff}(n, p)$. As a consequence, the third moment $\langle |V|V|^{\otimes 3} \rangle$ is an operator that commutes with $U^{\otimes 3}$ for any Clifford unitary $U \in \text{Cliff}(n, p)$, i.e., an element of the commutant of $\text{Cliff}(n, p)^{\otimes 3}$. For qubits, this commutant is generated by the permutation action $R(\pi) = R(T_\pi)$ for $\pi \in S_3$ (indeed, this implies that multiqubit stabilizer states form a 3-design). We will now show that an analogous statement holds true for $p \neq 2$ if we consider the operators $R(T)$ for $T \in \Sigma_3(p)$ instead of the permutation action; this will in turn be used to prove (8).

**Theorem 4.** Let $p \neq 2$ be a prime and $n \geq 2$. Then the operators $R(T)$ for $T \in \Sigma_3(p)$ are $2p + 2$ linearly independent operators that span the commutant of $\text{Cliff}(n, p)^{\otimes 3}$.

**Theorem 4** will be established by combining a number of intermediate results of independent interest. The most difficult step is to show that the operators $R(T)$ are indeed in the commutant of $\text{Cliff}(n, p)^{\otimes 3}$. We start with by defining the natural symmetry group of $\Sigma_3(p)$:

**Definition 5.** Let $O$ be a matrix acting on $\mathbb{F}_p^3 \oplus \mathbb{F}_p^3$. We say that $O$ is orthogonal if $\beta(Ov, Ow) = \beta(v, w)$ for all $v, w \in \mathbb{F}_p^3 \oplus \mathbb{F}_p^3$, where $\beta$ is the quadratic form defined by $\beta((\bar{x}, \bar{y}), (\bar{x}', \bar{y}')) = \bar{x} \cdot \bar{x}' - \bar{y} \cdot \bar{y}'$. We say that $O$ is stochastic if $O1_6 = 1_6$. The set of orthogonal and stochastic matrices on $\mathbb{F}_p^3 \oplus \mathbb{F}_p^3$ forms a group that will be denoted by $\mathcal{O}_{3,3}(p)$.

Note that the Lagrangian subspaces are defined with respect to the same form $\beta$. Using Witt’s theorem, it is thus not hard to see that $\mathcal{O}_{3,3}(p)$ acts transitively on $\Sigma_3(p)$. 
Above, we associated operators \( r(T) \) and \( R(T) \) to any subspace \( T \in \Sigma_3(p) \). We can similarly associate superoperators to any \( O \in O_{3,3}(p) \), acting on operators on \((\mathbb{C}^p)^{\otimes 3}\) and \((\mathbb{C}^p)^{\otimes 3n}\), respectively. They are defined by \( \rho(O) [\langle \vec{x} | \langle \vec{y} ] = |\vec{x}' \rangle \langle \vec{y}' | \), where \( \left( \begin{array}{c} \vec{x}' \\ \vec{y}' \end{array} \right) = O \left( \begin{array}{c} \vec{x} \\ \vec{y} \end{array} \right) \), and \( \mathcal{R}(O) = \rho(O)^{\otimes n} \). We record the following equivariance property for all \( O \in O_{3,3}(p) \) and \( T \in \Sigma_3(p) \):

\[
\mathcal{R}(O)[R(T)] = R(OT). \tag{C1}
\]

Importantly, the superoperators commute with conjugation by the third tensor power of Clifford unitaries:

**Lemma 6.** Let \( O \in O_{3,3}(p) \) and \( U \in \text{Cliff}(n, p) \). Then:

\[
\mathcal{R}(O) [U^{\otimes 3}(-)U^\dagger]^{\otimes 3} = U^{\otimes 3} \mathcal{R}(O) [-] U^\dagger. \tag{C2}
\]

**Proof.** Using state-channel duality, the statement of the lemma is equivalent to the following:

\[
[\Omega^{\otimes n}, U^{\otimes 3} \otimes U^{\otimes 3}] = 0,
\]

where \( \Omega = \sum_{v \in F_{p}^{2n}} |Ov \rangle \langle v| \) is an operator on \((\mathbb{C}^p)^{\otimes 3} \otimes (\mathbb{C}^p)^{\otimes 3}\).

It will be convenient to use the discrete phase space formalism [41]. Recall that the discrete phase space corresponding to the Hilbert space \((\mathbb{C}^p)^{\otimes N}\) is by definition \( \mathbb{F}_p^{2N} \cong \mathbb{F}_p^2 \otimes \mathbb{F}_p^N \), and that for each point \( z \in \mathbb{F}_p^N \) we have a phase space point operator \( \mathcal{A}(z) \); these operators form a basis of the space of operators on \((\mathbb{C}^p)^{\otimes N}\). We will first prove the weaker statement that \( \Omega^{\otimes n} \) and \( U^{\otimes 3} \otimes U^{\otimes 3} \) commute up to a phase. These are operators on \((\mathbb{C}^p)^{\otimes 3n} \otimes (\mathbb{C}^p)^{\otimes 3n} \) (i.e., \( N = 6n \)), with corresponding discrete phase space

\[
\mathbb{F}_p^{2N} \cong \mathbb{F}_p^2 \otimes \mathbb{F}_p^N \cong \mathbb{F}_p^2 \otimes \mathbb{F}_p^n \otimes \mathbb{F}_p^3 \otimes \mathbb{F}_p^2.
\]

We will denote elements in this phase space by \( x_{ABCD} \), where \( A, B, C, \) and \( D \) refer to the four tensor factors. The first two together make up the phase space for \( n \) qudits, the third corresponds to the three replicas, and the last to the two copies of \((\mathbb{C}^p)^{\otimes 3n}\).

To show that \( \Omega^{\otimes n} \) and \( U^{\otimes 3} \otimes U^{\otimes 3} \) commute up to a phase we will show that they have the same conjugation action on phase space point operators. We first consider the Clifford unitary \( U \in \text{Cliff}(n, p) \). Like any Clifford unitary, \( U \) can be parametrized by a symplectic matrix \( \Gamma \in \text{Sp}(2n, p) \) and a vector \( b \in \mathbb{F}_p^2 \otimes \mathbb{F}_p^n \) such that \( U \mathcal{A}(z) U^\dagger = \mathcal{A}(\Gamma z + b) \) (note that (only) here refers \( \mathcal{A}(z) \) to a phase space point operator on \((\mathbb{C}^p)^{\otimes n}\)); we will say that \( U \) acts on phase space point operators by \( z \mapsto \Gamma z + b \). It is easy to verify that \( U^{\otimes 3} \otimes U^{\otimes 3} \) is similarly a Clifford unitary, now in \( \text{Cliff}(N, p) \), whose action on phase space point operators is given by

\[
x_{ABCD} = (y_{ABC}, z_{ABCD}) \mapsto \mathcal{A}(\mathcal{A}(y_{ABC}) \mathcal{A}(z_{ABCD})) = \mathcal{A}(y_{ABC} \mathcal{A}(z_{ABCD})),
\]

where \( Z_A = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \). Using the controlled \( Z \) gate \( Z_{AD} = \text{diag}(1, 1, 1, -1) \) we can write this more succinctly as

\[
x_{ABCD} \mapsto Z_{AD} \mathcal{A}(y_{ABC} \mathcal{A}(z_{ABCD})). \tag{C3}
\]

On the other hand, the operator \( \Omega^{\otimes n} \) acts on phase space point operators by

\[
x_{ABCD} \mapsto Z_{AD} O_{CD} Z_{AD} x_{ABCD}, \tag{C4}
\]

where we regard \( O \in O_{3,3}(p) \) as an operator \( O_{CD} \) on \( \mathbb{F}_p^2 \otimes \mathbb{F}_p^2 \cong \mathbb{F}_p^2 \otimes \mathbb{F}_p^2 \). This is revealed by a short calculation using the explicit form of the phase space point operators, \( \langle v_{BCD} | \mathcal{A}(x_{ABCD}) | w_{BCD} \rangle = \exp(2\pi i p_{BCD} (v_{BCD} - w_{BCD}) \delta_{BCD} - (v_{BCD} + w_{BCD})/2) \), where \( x_{ABCD} = (p_{BCD}, q_{BCD}) \), and \( O_{CD} = Z_{D} O_{CD} Z_{D} \), which follows from the orthogonality of \( O \). Using that \( O \) is stochastic, it is easily verified that Eqs. (C3) and (C4) commute. This means that

\[
\Omega^{\otimes n} \left( U^{\otimes 3} \otimes U^{\otimes 3} \right) \mathcal{A}(x_{ABCD}) \left( U^{\dagger} \otimes U^{\dagger} \right) \Omega^{\otimes n} = \left( U^{\otimes 3} \otimes U^{\otimes 3} \right) \Omega^{\otimes n} \mathcal{A}(x_{ABCD}) \Omega^{\otimes n} \left( U^{\otimes 3} \otimes U^{\otimes 3} \right)
\]

for all \( x_{ABCD} \). Since the phase space point operators form a basis, this means that \( \Omega^{\otimes n} \) and \( U^{\otimes 3} \otimes U^{\otimes 3} \) commute up to a phase:

\[
\Omega^{\otimes n} \left( U^{\otimes 3} \otimes U^{\otimes 3} \right) \cong \left( U^{\otimes 3} \otimes U^{\otimes 3} \right) \Omega^{\otimes n}
\]

It remains to compare an arbitrary nonzero matrix element to establish Eq. (C2). For this, let \( \langle \vec{x} | U | \vec{y} \rangle \neq 0 \) be an arbitrary nonzero matrix element of the unitary \( U \), with \( \vec{x}, \vec{y} \in \mathbb{F}^n_p \). Since \( O \) is stochastic, \( \Omega^{\otimes n} | \vec{x} \rangle^{\otimes 6} - |(I \otimes \mathcal{O}) (\vec{x} \otimes 1_b) \rangle = |\vec{x} \otimes 1_b \rangle = |\vec{x} \rangle^{\otimes 6} \), and similarly \( |\vec{y} \rangle^{\otimes 6} \Omega_{\otimes n} = (\vec{y} \rangle^{\otimes 6} \). Therefore,

\[
|\vec{y} \rangle^{\otimes 6} \Omega_{\otimes n} \left( U^{\otimes 3} \otimes U^{\otimes 3} \right) |\vec{x} \rangle^{\otimes 6} = (\vec{y} \rangle^{\otimes 6} \left( U^{\otimes 3} \otimes U^{\otimes 3} \right) \Omega_{\otimes n} |\vec{x} \rangle^{\otimes 6} = |(\vec{y} | U | \vec{x} \rangle|^{\otimes 6} \neq 0,
\]

which concludes the proof of Eq. (C2) and the lemma. \(\Box\)
Since $O_{3,3}(p)$ acts transitively on $\Sigma_3(p)$, for any $T \in \Sigma_3(p)$ there exists some $O \in O_{3,3}(p)$ such that $O \Delta = T$. Since $R(\Delta)$ is the identity operator, it follows from Eq. (C1) that $R(O)[I] = R(T)$. Thus, Lemma 6 shows that $U^{\otimes 3} R(T) U^\dagger \otimes 3 = R(T)$ for all Clifford unitaries $U$. This shows that $R(T)$ is in the commutant of the third tensor power of Clifford unitaries:

**Corollary 7.** For all $T \in \Sigma_3(p)$ and $U \in \text{Cliff}(n,p)$, we have that $[R(T), U^{\otimes 3}] = 0$.

We now show that the operators $R(T)$ are linearly independent:

**Lemma 8.** If $n \geq 2$ then operators $R(T) = r(T)^{\otimes n}$, where $T \in \Sigma_3(p)$, are linearly independent.

**Proof.** Define $|T\rangle = \sum v_{eT} \in (C^p)^{\otimes n}$, so that $\langle v| T \rangle = \delta_{eT}$. It suffices to show that the vectors $|T\rangle^{\otimes n}$ are linearly independent as soon as $n \geq 2$. Each $T \in \Sigma_3(p)$ is three-dimensional and contains the vector $1_6 = (1, \ldots, 1)$. Extend $1_6$ by vectors $v_1, v_2$ to a basis of $T$. Then, if $T'$ is another element of $\Sigma_3(p)$,

$$\langle v_1 | \langle v_2 | (0)^{\otimes n-2} |T\rangle^{\otimes n} = \langle v_1 | T' \rangle \langle v_2 | T' \rangle = \delta_{T,T'}.$$

The first equality holds because any subspace contains the zero vector; the second equality holds because any subspace $T' \in \Sigma_3(p)$ that contains $v_1$ and $v_2$ must be equal to $T$. It is now immediate that the $|T\rangle^{\otimes n}$ are linearly independent.

We now list $2p + 2$ subspaces in $\Sigma_3(p)$. Each subspace is given in terms of three basis vectors (the rows of the below matrices). It is easy to verify that they define distinct Lagrangian stochastic subspaces in $\Sigma_3(p)$. Similar to the permutations, we divide them into two subsets, called *even* and *odd* (the reason for this will become clear momentarily):

- $T_{*,\text{even}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$,
- $T_{m,\text{even}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & m & 1 & 1 & m & -1 \\ m & 1 & -1 & m & -1 & 1 \end{bmatrix} \quad (m \in \mathbb{F}_p),$
- $T_{*,\text{odd}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$,
- $T_{m,\text{odd}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & m & 0 & 1 & m & 0 \\ -m & 1 & -1 & -m & -1 & 1 \end{bmatrix} \quad (m \in \mathbb{F}_p),$

We can now establish Theorem 4:

**Proof of Theorem 4.** The dimension of the commutant of $\text{Cliff}(n,p)^{\otimes 3}$ is known as the *third frame potential* of the Clifford group, denoted $\Phi_3$. It can be evaluated by counting the orbits of the diagonal action of the symplectic group on two copies of the phase space. The result is that $\Phi_3 = 2p + 2$ for $n \geq 2$ [32, eq. (9)]. But Corollary 7 and Lemma 8 show that the subspaces in Eq. (C5) give rise to $2p + 2$ linearly independent elements in the commutant. This concludes the proof.

The preceding proof shows that, for $p \neq 2$,

$$\Sigma_3(p) = \{ T_{m,\text{even}} : m \in \mathbb{F}_p \cup \{ \star \} \} \cup \{ T_{m,\text{odd}} : m \in \mathbb{F}_p \cup \{ \star \} \}.$$  \hspace{1cm} (C6)

For the subspaces in $\Sigma_3(p)$ that correspond to permutations, our notion of even and odd coincides with their usual definition for permutations. Indeed, $T_{*,\text{even}}$ is the subspace corresponding to the identity permutation, $T_{1,\text{even}}$ correspond to the two three-cycles, $T_{*,\text{odd}}$ corresponds to one of the transpositions, and $T_{0,\text{odd}}$ to the other two transpositions in $S_3$. Moreover, the following can be established by a direct calculation, generalizing the analogue property for permutations discussed in the main text:

$$\frac{1}{p^{3N}} \text{tr} R(T_x) R(T_y) = p^{-N(3 - \dim(T_x \cap T_y))} = p^{-N d(T_x, T_y)},$$  \hspace{1cm} (C7)

where

$$d(T_x, T_y) = \begin{cases} 0 & \text{if } T_x = T_y, \\ 1 & \text{if } T_x \neq T_y, \text{ with one subspace even and the other odd}, \\ 2 & \text{if } T_x \neq T_y, \text{ with both subspaces even or both odd}. \end{cases}$$  \hspace{1cm} (C8)

We note that $d(T_x, T_y)$ defines a metric on $\Sigma_3(p)$. Moreover,

$$\sum_{T \in \Sigma_3(p)} \text{tr} R(T) = p^{3n} + p^{p^n} + (p + 1)p^{2n} = p^n(p^n + 1)(p^n + p).$$  \hspace{1cm} (C9)

At last we establish our formula for the third moment of a random stabilizer state:
Proof of formula (8) for the third moment. Since the set of stabilizer states is an orbit of the Clifford group, we can evaluate the third moment of a random stabilizer state by instead averaging over the Clifford group:

\[ M_3 := \langle \langle V|V \rangle^{\otimes 3} \rangle = \langle U^{\otimes 3} |0\rangle^{\otimes 3n} (U^\dagger)^{\otimes 3n} \rangle \]

Here, \(|V\rangle\rangle\langle V|\) denotes a stabilizer state and \(U\) a Clifford unitary, each chosen uniformly at random. It is apparent from the right-hand side that \(M_3\) is in the commutant of \(\text{Cliff}(n, p)^{\otimes 3}\). By Theorem 4, we can therefore write \(M_3 = \sum_{T \in \Sigma_3(p)} \gamma_{T} R(T)\) for certain coefficients \(\gamma_T \in \mathbb{C}\). Now observe that, for all \(O \in O_{3,3}(p)\),

\[ \mathcal{R}(O)[M_3] = \langle \langle \mathcal{R}(O) [U^{\otimes 3} |0\rangle^{\otimes 3n} (U^\dagger)^{\otimes 3n} \rangle, \langle U^{\otimes 3} |0\rangle^{\otimes 3n} (U^\dagger)^{\otimes 3n} \rangle \rangle = \langle (U^{\otimes 3} |0\rangle^{\otimes 3n} (U^\dagger)^{\otimes 3n} \rangle = M_3. \]

Here we have used Lemma 6 and the fact that \(\mathcal{R}(O)\) is a Clifford unitary, chosen uniformly at random. It is apparent from the third moment. Now recall that the vertex Hilbert space is a tensor product \(\otimes \), so all coefficients \(\gamma_T\) must be equal. That is, \(M_3 \propto \sum_{T \in \Sigma_3(p)} R(T)\), and we obtain the normalization constant in (8) by comparing \(\text{tr} M_3 = 1\) with (C9).

Appendix D: Detailed derivation of the GHZ bound

In this section we give a detailed derivation of Theorem 1 which bounds the average number of GHZ states that can be extracted from a random stabilizer network. As in the main text, let \(\zeta\) denote the cyclic permutation \(1 \mapsto 2 \mapsto 3\), so that

\[ \text{tr}(\Psi_{AB}^{T_X})^3 = \text{tr} \Psi_{AB}^{\otimes 3} R_A(\zeta) R_B(\zeta^{-1}). \]

Here, \(R_X(T) = r(T)^{\otimes X}\) denotes the action of an element \(T \in \Sigma_3(p)\) on the three-fold copy of the Hilbert space corresponding to a subsystem \(X\); we recall that \(\Sigma_3(p)\) contains the permutation group \(S_3\). Explicitly, \(1 = T_{s,\text{even}}, \zeta = T_{1,\text{even}}\) and \(\zeta^{-1} = T_{-1,\text{even}}\), as is apparent from (C5). Using our formula (8) for the third moment of a random stabilizer state, we obtain that

\[ \langle \langle \text{tr}(\Psi_{AB}^{T_X})^3 \rangle \rangle = \langle \langle \Psi^{\otimes 3} \rangle \rangle R_A(\zeta) R_B(\zeta^{-1}) \sum_{x \in V} \text{tr} \left( \prod_{e} |e\rangle^{\otimes 3} \right) \left( \prod_{x \in V} R_x(T_x) \right) R_A(\zeta) R_B(\zeta^{-1}). \]

Multiplying out the right-hand product, we find that the above is in turn equal to

\[ \left( \prod_{x \in V} R_x(T_x) \right) \sum_{T_x \in \Sigma_3(p)} \text{tr} \left( \prod_{e} |e\rangle^{\otimes 3} \right) \left( \prod_{x \in V} R_x(T_x) \right) \]

where we sum over all assignments \(T_x \in \Sigma_3(p)\), subject to the boundary conditions that \(T_x = \zeta\) for \(x \in A\), \(T_x = \zeta^{-1}\) for \(x \in B\), and \(T_x = 1\) for \(x \in C\). Now recall that the vertex Hilbert space is a tensor product \(\bigotimes_e (\mathbb{C}^p)^{\otimes N}\), where \(e\) runs over the edges incident to \(x\), and that the representation \(R_x(T_x)\) factors correspondingly. Writing \(R_x(T_x) = \bigotimes_e R_{x,e}(T_x)\), we can evaluate the trace edge by edge:

\[ \left( \prod_{x \in V} R_x(T_x) \right) \sum_{T_x \in \Sigma_3(p)} \text{tr} |e\rangle^{\otimes 3} R_{x,e}(T_x) R_{y,e}(T_y) \]

Any maximally entangled state \(|\Phi^+\rangle_{AB}\) satisfies the identity \((X \otimes I) |\Phi^+\rangle_{AB} = (I \otimes X^t) |\Phi^+\rangle_{AB}\), where \(X^t\) denotes the transpose (in the computational basis, i.e., the basis that the maximally entangled state was defined in). Since \(|e\rangle^{\otimes 3}\) is a maximally entangled state on two copies of \((\mathbb{C}^p)^{\otimes 3N}\), we obtain that

\[ \text{tr} |e\rangle^{\otimes 3} R_{x,e}(T_x) R_{y,e}(T_y) = \frac{1}{p^{3N}} \text{tr} R(T_x) R(T_y)^t = \frac{1}{p^{3N}} \text{tr} R(T_x) R(T_y)^t \]
where we write \( R(T) \) for the representation of \( \Sigma_3(p) \) on the three-fold tensor power of \((\mathbb{C}^p)^{\otimes N}\); the second inequality holds as \( R(T) \) is represented by real matrices in the computational basis. According to Eq. (C7), the right-hand side is given by \( p^{-N|d(T_x,T_y)} \) and thus we obtain the following fundamental bound:

\[
\left\langle \text{tr}(\Psi_{AB}^{T_{AB}})^3 \right\rangle \leq p^{-3N_b} \sum_{\{T_x\}} p^{-N|\sum_{(x,y)} d(T_x,T_y)} \quad \text{(D1)}
\]

where the sum is over all choices of \( T_x \in \Sigma_3(p) \) such that \( T_x = \zeta \) in \( A \), \( T_x = \zeta^{-1} \) in \( B \), and \( T_x = 1 \) in \( C \). We note that (D1) reduces to (7) in the case of qubits \((p = 2)\).

To analyze (D1), we define the energy of a configuration by \( E[\{T_x\}] := \sum_{(x,y)} d(T_x,T_y) \) (cf. the main text for a justification of this terminology). We first consider an arbitrary configuration \( \{T_x\} \). If we denote by \( V_A = \{x : T_x = \zeta\} \) the domain where \( T_x \) is assigned the value \( \zeta \) then the boundary conditions imply that \( V_A \cap V_0 = A \); that is, \( V_A \) is a cut separating \( A \) and \( A = BC \). Likewise, the \( \zeta^{-1} \)-domain \( V_B \) is a cut for \( B \) and the identity domain \( V_C \) a cut for \( C \). These cuts are not necessarily minimal, and so we have that \( |\partial V_A| \geq S_{RT}(A)/N \) etc. Lastly, we write \( V' = V_B \setminus (V_A \cup V_B \cup V_C) \) for the remaining vertices. We now decompose the set of edges into (i) the set of edges \( E_1 \) that connect any of the domains \( V_A, V_B \) or \( V_C \) with \( V' \), (ii) the set of edges \( E_2 \) that go between any two of the domains \( V_A, V_B \), and \( V_C \), and (iii) the remaining edges \( E' \) (i.e., those within \( V' \)). We can then lower-bound the energy of the configuration as follows:

\[
E[\{T_x\}] = \sum_{\{xy\} \in E_1} d(T_x,T_y) + \sum_{\{xy\} \in E_2} d(T_x,T_y) + \sum_{\{xy\} \in E'} d(T_x,T_y) \geq |E_1| + 2|E_2|
\]

Indeed, the edges \( \{xy\} \in E_1 \) are by definition such that \( T_x \neq T_y \), hence \( d(T_x,T_y) \geq 1 \); for the edges in \( E_2 \) we in addition know that \( T_x \) and \( T_y \) are even, so that \( d(T_x,T_y) \geq 2 \) according to (C8). Furthermore, it is clear that

\[
|E_1| + 2|E_2| = |\partial V_A| + |\partial V_B| + |\partial V_C|
\]

since the right-hand side double-counts precisely those edges in \( E_2 \). Together, we find that

\[
E[\{T_x\}] \geq E_0 := (S_{RT}(A) + S_{RT}(B) + S_{RT}(C))/N.
\]

Equality holds if and only if the domains \( V_A, V_B \) and \( V_C \) are disjoint minimal cuts for \( A, B \) and \( C \), respectively, and if each connected components of \( V' \) is assigned an arbitrary odd element of \( \Sigma_3(p) \). It follows from Lemma 9 below that it is always possible to find disjoint minimal cuts for disjoint boundary regions; hence \( E_0 \) is achievable. Moreover, if we denote the number of minimal cuts for a boundary region \( A \) by \( \#_A \) and the maximal number of connected components of any subgraph \( V' \) obtained by removing minimal cuts by \( \#_b \), then we find that there are at most \( \# = (p + 1)^{\#_b} \#_A \#_B \#_C \) many configurations of energy \( E_0 \), for there are \( p + 1 \) odd elements in \( \Sigma_3(p) \). All other configurations have higher energy and hence are penalized by a factor of at least \( 1/p^N \) in (D1).

Thus we obtain the upper bound:

\[
\left\langle \text{tr}(\Psi_{AB}^{T_{AB}})^3 \right\rangle \leq p^{-3N_b} p^{-N|\partial V_A|} (\# + \delta) = p^{-3N_b - (S_{RT}(A) + S_{RT}(B) + S_{RT}(C))} (\# + \delta)
\]

where \( \delta = (2p + 2)^{V_6}/p^N \), since there are no more than \( |\Sigma_3(p)|^{V_6} = (2p + 2)^{V_6} \) non-minimal configurations, and hence

\[
\log_p \left\langle \text{tr}(\Psi_{AB}^{T_{AB}})^3 \right\rangle \leq -3N_b - (S_{RT}(A) + S_{RT}(B) + S_{RT}(C)) + \log_p \# + 2\delta.
\]

(D2)

At last we can bound the average number of GHZ states that can be extracted from a random stabilizer network state. Using (5) and \( \rho = \Psi/\text{tr} \Psi \), we obtain that

\[
\langle g \rangle \neq 0 \leq S_{RT}(A) + S_{RT}(B) + S_{RT}(C) + \log_p \left\langle \text{tr}(\Psi_{AB}^{T_{AB}})^3 \right\rangle \neq 0 - 3 \log_p \text{tr} \Psi \neq 0
\]

\[
\leq S_{RT}(A) + S_{RT}(B) + S_{RT}(C) + \log_p \left\langle \text{tr}(\Psi_{AB}^{T_{AB}})^3 \right\rangle + 2\delta + 3N_b
\]

\[
\leq \log_p \# + 4\delta
\]

where the first inequality uses \( S(X) \leq S_{RT}(X) \) and concavity of the logarithm, the second that \( \text{Pr}(\Psi \neq 0) \geq 1 - \delta \) ((B2) in Appendix B), \( \text{tr} \Psi \geq 1/p^{N_6} \) if \( \Psi \neq 0 \) (Appendix A) and that \( \delta \) is sufficiently small, and the last is obtained by plugging in (D2). This is the statement of Theorem 1.

**Lemma 9.** Let \( A \) and \( B \) be denote disjoint subsets of \( V_B, V_A \) and \( V_B \) minimal cuts for \( A \) and \( B \), respectively, and \( V_0 := V_A \cap V_B \). Then either \( V_A \setminus V_0 \) is a minimal cut for \( A \) or \( V_B \setminus V_0 \) is a minimal cut for \( B \).
Proof. Since $V_0 \cap V_0 = \emptyset$, it is clear that $V_A \setminus V_0$ is again a cut for $A$ and $V_B \setminus V_0$ again a cut for $B$. We now use that the cut function $c(W) := |\partial W|$ is symmetric and submodular, a fact that is well-known in graph theory. It follows that

$$|\partial V_A| + |\partial V_B| \geq |\partial (V_A \setminus V_0)| + |\partial (V_B \setminus V_0)|,$$

and hence that either $|\partial V_A| \geq |\partial (V_A \setminus V_0)|$ or $|\partial V_B| \geq |\partial (V_B \setminus V_0)|$. This implies the claim. $\square$