Floating flocks: 2D long-range uniaxial order in 3D active fluids

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Uniaxial active units cannot break rotation symmetry in bulk, momentum-conserved, incompressible fluids to form nematic or polar phases. This has led to the defining image of active suspensions composed of elongated particles as a spontaneously evolving, spatiotemporally chaotic state. In contrast, I show that two-dimensional, long-ranged ordered active nematic and polar phases can exist at the boundary of a momentum-conserved, incompressible fluid or at a fluid-fluid interface. The active flows that “anti-screen” orientational fluctuations in the bulk, destroying the ordered phase, screen angular fluctuations of interfacial uniaxial states leading to long-range order. This implies that while a transition to a uniaxial phase is impossible in the bulk, it is allowed at boundaries of momentum conserving fluids. Specifically, I show that active species diffusing in the bulk have boundary-associated long-range ordered nematic phases if they are immotile or polar states if they are motile. When the active species are constrained to live at the boundary, immotile units do not order, but motile particles still form a long-range-ordered polar phase. I exactly characterise the hydrodynamic properties of these phases. This is the first prediction of stable, uniaxial, active phases in Stokesian fluids and of a nematic phase displaying long-range order in two dimensions, and has functional consequences for active transport.

I. INTRODUCTION

Active matter theories describe systems whose constituents convert a sustained supply of energy, from either an ambient or on-board fuel source, into work [1–12]. The microscopic energy input leads to macroscopic forces and currents dramatically modifying the mechanics and statistical mechanics of active systems relative to their dead or passive counterparts. This has spectacular effects on the phases [13–19] and phase behaviours [12, 20–22] that active systems display.

One of the more notable consequences of the microscopic drive on the phase behaviour of active systems is the threshold-free instability of uniaxial phases – both nematic and polar – in incompressible, bulk Stokesian fluids [1, 21, 22] first discussed in a seminal paper by Simha and Ramaswamy [21]. This bulk Simha-Ramaswamy instability – driven by active flow and defect proliferation into a spatiotemporally chaotic state [23, 24] – has become the defining image of active suspensions. The impossibility of a bulk uniaxial state implies that active suspensions do not break rotation symmetry i.e., there is no true bulk isotropic-nematic or isotropic-polar transition. Instead, activity constantly stirs the bulk Stokesian fluid via a random stress with a correlation length vanishing as the inverse of the square root of activity. This has led to the popular belief that uniaxial phases do not exist in momentum-conserved active fluids.

In this article, I re-examine this belief and show that while bulk uniaxial ordering is indeed forbidden in active suspensions, two-dimensional nematic and polar order exist at interfaces or boundaries of momentum-conserved fluids. Interface-associated ordered phases have been been examined in detail in passive systems [25–32]. However, while in passive magnetic systems [28, 30, 31] and nematics [29, 33], such transitions may precede the bulk transition, the unusual active uniaxial, interfacial phase exists even though a bulk uniaxial phase is impossible for any parameter value. Furthermore, in passive interfacial ordering, the interface-associated phase is less stable than the bulk phase – in bulk three-dimensional systems with a boundary, the two-dimensional interface-associated phase displays only quasi-long range order [24] while the bulk, three-dimensional phase has long-range order since Goldstone fluctuations are more dominant in two dimensions than in three dimensions. In contrast, I show that both nematic and polar phases at interfaces or boundaries of bulk fluids display long-range order (LRO). This stabilisation of interfacial order in momentum-conserved systems is due to the active fluid flows; that is, while active flows destabilise bulk uniaxial order, they can anomalously stabilise interfacial order. In other words, while active flows “anti-screen” fluctuations of bulk orientational order, they screen fluctuations of interfacial uniaxial states in the same system.

In bulk fluids, the ratio of active stress and viscosity in- stabilities of bulk orientational order, they can anomalously stabilise interfacial order. This stabilisation of interfacial order in momentum-conserved fluids is due to the active fluid flows; that is, while active flows destabilise bulk uniaxial order, they can anomalously stabilise interfacial order. In other words, while active flows “anti-screen” fluctuations of bulk orientational order, they screen fluctuations of interfacial uniaxial states in the same system.

In bulk fluids, the ratio of active stress and viscosity introduces an inverse timescale [6]. However, this need not be proportional to the growth rate of an instability; it could be proportional to the relaxation rate of a passive mode. It turns out to be proportional to the growth rate of the instability of an oriented phase in bulk fluids only because of incompressibility. The crucial distinction that allows active flows to stabilise interfacial order while destabilising bulk order is that the two-dimensional flow-field is not incompressible, with the fluid escaping into the third dimension, while the three-dimensional flow is [26, 34, 36, 37].

Beyond the fundamental physical interest of realising a boundary-associated two-dimensional long-range ordered phase in momentum-conserved active fluids, when no corresponding bulk phase exists, such states may have important experimental and biological consequences. Some
of the more widely used experimental systems for studying pattern formation in active uniaxial systems are composed of motor-microtubule filaments either at a two-fluid interface \[11, 12\] or forming a self-assembled layer immersed in a bulk fluid \[13, 14\]. While it had been assumed that uniaxial phases are forbidden in these geometries \[15, 17\], this article demonstrates that this conclusion is contingent on experimental details. Swimmers, such as bacteria \[18, 51\], generally aggregate at interfaces \[39\] and can form effectively two-dimensional uniaxial phases. In cellular systems, uniaxial ordering is likely to be associated with interfaces or membranes – such as in the cellular cortex – and may be relevant for active transport \[52 – 55\]. Finally, there has recently been a great deal of interest in the possibility of forming protocells by microphase separation in active fluids. If such protocells contain elongated particles – for instance, if the phase separation leads to droplets that are rich in elongated active units in a background that is poor in them – then there can be an ordered boundary layer of orientable filaments in the droplet or a protocortex.

I now briefly summarise the key results of the paper.

### A. Summary of results

In this article, I examine active uniaxial orientational order – both nematic and polar – at the interface between two dissimilar fluids, at an air-fluid boundary and at an immersed surface, such as a membrane, in a fluid. I demonstrate that the Simha-Ramaswamy instability is tamed in all these bulk momentum-conserved geometries and both active nematic and polar phases display stable long-ranged order (LRO) when the active units can freely diffuse into the bulk fluid. The hydrodynamic properties of these LRO uniaxial phases belong to a new universality class which I exactly characterise. This constitutes the first prediction of stable uniaxial active phases in momentum-conserved systems and of a two-dimensional LRO active nematic phase. In contrast, when the active units are confined to interface, membrane or the boundary layer, homogeneous nematic order is generically destroyed. The form of the instability, for a compressible interfacial species is however distinct from that of an essentially incompressible one, with the growth rate of the fastest-growing mode becoming independent of activity at large active drive. Hence, the lengthscale of the structures formed beyond the instability in this regime are expected to be independent of activity. An LRO polar phase can still exist even when the motile units are confined to an interface with the self-propelled particles “outrunning” the instability that immotile nematic order suffers, for large enough propulsion speeds. The mechanism by which the polar phase escapes the instability characteristic of the nematic state is conceptually similar to the one for inertial polar suspensions in a bulk active fluid \[57\]. The concentration fluctuations in this motile phase violate the law of large numbers – i.e., display giant number fluctuations (GNF) – with the R.M.S. number fluctuations in a region containing on average \(\langle N \rangle\) swimmers scaling as \(\langle N \rangle^{3/4}\) instead of \(\sqrt{\langle N \rangle}\) as it would in all equilibrium systems. Together, these results, summarised in Table I, present a complete description of boundary-associated uniaxially ordered phases in active, momentum-conserved systems.

### B. Organisation of the paper

The rest of the paper is organised as follows. In Sec. II, I present the hydrodynamic equations for all the models that I describe in this article. The hydrodynamic description of a nematic (NNC model) and polar phase (PNC model) at an interface between two fluids or at the boundary of a bulk fluid, with the active units being free to diffuse in the fluid, are presented respectively in Sec. II.A and Sec. II.B. The description for nematic (NC model) and polar phases (PC model) when the active units are constrained to be at the interface or the boundary are presented in Sec. II.C and Sec. II.D respectively. In Sec. III, I demonstrate that when active particles can diffuse in the bulk, i.e., for NNC and PNC models described in Sec. II.A and Sec. II.B, interfacial nematic and polar phases display stable long-ranged order. Sec. IV demonstrates that for NC model, i.e., when active units are constrained to lie at the interface or a boundary of a bulk fluid, a nematic phase is generically destabilised i.e., there is no isotropic-nematic transition. However, I show that the character of the instability, and hence the structures expected to result beyond it, are qualitatively distinct from the Simha-Ramaswamy instability of active nematic in bulk fluids. The motility-driven escape form this instability, and the consequent polar long-range ordered state with giant number fluctuations formed by interface or boundary-associated self-propelled particles described by the PC model, is discussed in Sec. V. Finally, in Sec. VI I present some concluding remarks and discuss possible experimental realisations and consequences for current experiments.
II. MODEL DESCRIPTION

In this section, I describe the models that I consider in detail. I examine both active nematic and polar order at an interface between two fluids or at the surface of a semi-infinite fluid. The interface or surface is assumed to be clean i.e., not coated by surfactants, and is taken to be situated at $z = 0$. For both nematic and polar order, I consider two situations: i. the number of active units at the interface is not conserved i.e., the active units can dissolve in the bulk fluid and ii. the number of active units at the interface is conserved. In all of these cases, the interfacial ordering is driven by the interfacial fluid flow. For slow flows, the interfacial fluid velocity is obtained purely in terms of the interfacial force densities via a linear relation of the form $\mathbf{v} = \mathbf{M} \cdot \mathbf{f}^s$, where $\mathbf{v}$ is the interfacial velocity field, $\mathbf{M}$ is a mobility tensor, $\mathbf{f}^s$ is the interfacial force density which will be different for each of the models I will consider. The mobility tensor can be calculated using the Stokes equations for the bulk, three-dimensional fluid, as I show now.

The fluids above and below the interface support no uniaxial order even when active units dissolve in the bulk fluid, irrespective of the bulk density of the active units [21]. This is because of the instability of the bulk active nematic fluids which implies that the active units essentially lead to a conserving noise which is correlated over finite spatial and temporal scales [21, 22]. I will show in the Appendix A that such noises in the bulk fluids above and below the interface do not modify the hydrodynamic behaviour of the interfacial ordered state and therefore, neglect them here. With these considerations, the Stokes equations for the fluid velocities above and below the interface, $\mathbf{V}_t$ and $\mathbf{V}_b$, respectively, are

$$\eta_t \nabla^2 \mathbf{V}_t = \nabla \Pi_t$$

and

$$\eta_b \nabla^2 \mathbf{V}_b = \nabla \Pi_b$$

where $\eta_t$ and $\eta_b$ are the viscosities of the fluids above and below the interface respectively and the pressures $\Pi_t$ and $\Pi_b$ are determined by the incompressibility constraints $\nabla \cdot \mathbf{V}_t = 0$ and $\nabla \cdot \mathbf{V}_b = 0$. These equations have to be solved with boundary conditions $\mathbf{V}_t(0) = \mathbf{V}_b(0) = \mathbf{v}$ where $\mathbf{v}$ is the surface velocity, and $\lim_{z \to \infty} \mathbf{V}_t(z) \to 0$ and $\lim_{z \to -\infty} \mathbf{V}_b(z) \to 0$. I show in the Appendix B that the solution for the three-dimensional velocity fields, Fourier transformed in the $x - y$ plane, in terms of the interfacial velocity field, are

$$\mathbf{V}_{tq}(z) = e^{-i|q||z|} [\mathbf{v} - z(i\hat{z} + \mathbf{q}_\perp)(\mathbf{q}_\perp \cdot \mathbf{v})], \quad (3)$$

and

$$\mathbf{V}_{bq}(z) = e^{-i|q||z|} [\mathbf{v} - z(i\hat{z} - \mathbf{q}_\perp)(\mathbf{q}_\perp \cdot \mathbf{v})] \quad (4)$$

where $\mathbf{q}_\perp \equiv (q_x, q_y)$ is the in-plane wavevector and $\mathbf{q} \equiv \mathbf{q}_\perp/|\mathbf{q}_\perp|$. Eqs. (3) and (4) imply that while $\nabla \cdot \mathbf{V}_b = \mathbf{V}_t = \mathbf{V}_b = \mathbf{v}$, $\nabla \cdot \mathbf{V}_b \neq 0$, i.e., the interfacial flow is not itself incompressible with compressional or dilational in-plane flows balanced by $\partial_z (\mathbf{V}_t)_z$ and $\partial_z (\mathbf{V}_b)_z$ at $z = 0$. This will turn out to be crucial for the existence of interfacial active uniaxial phases.

Eqs. (3) and (4) also directly yield the difference of the tangential stresses of the top and the bottom fluid at $z = 0$ which has to be balanced by the surface forces. Separating out the interfacial viscous forces from those arising due to interfacial director distortions and concentration fluctuations of the ordered active species and writing the latter as $\mathbf{F} = i\mathbf{q}_\perp \cdot \sigma^s$, where $\sigma^s$ is the interfacial particle-phase stress, I get

$$\eta_t \partial_z (\mathbf{V}_{tq})_\perp|_{z=0} - \eta_b \partial_z (\mathbf{V}_{bq})_\perp|_{z=0} - \eta_s q^2_\perp \mathbf{v} - \eta_b \mathbf{q}_\perp \cdot (\mathbf{q}_\perp \cdot \mathbf{v}) = -i\mathbf{q}_\perp \cdot \sigma^s. \quad (5)$$

Defining $\gamma = (\eta_t + \eta_b)/2$ and the Saffmann-Delbrück lengths $\ell_s = \eta_s / 2\eta$ and $\ell_b = \eta_b / 2\eta$, I obtain an expression of the in-plane velocity fields in terms of the in-plane particle-phase stress or force in the form $\mathbf{v} = i\mathbf{M} \cdot (\mathbf{q}_\perp \cdot \sigma^s) = \mathbf{M} \cdot \mathbf{F}^s$ with a mobility

$$\mathbf{M} = \frac{1}{2\eta |q_\perp|^3} \left[2 + (\ell_b + 3\ell_s)q_\perp + \ell_s(\ell_b + \ell_s)q_\perp^2 \right] \left( q_x^2(1 + \ell_s |q_\perp|) + q_y^2(2 + (\ell_s + \ell_b)|q_\perp|) \right) - q_x q_y (1 + \ell_b |q_\perp|)$$

$$- q_x q_y (1 + \ell_b |q_\perp|) q_\perp^2 \left[2 + (\ell_s + \ell_b) |q_\perp| \right] \right). \quad (6)$$

For in-plane scales much larger than the Saffmann-Delbrück ones, i.e., when $\ell_s |q_\perp|, |\ell_b q_\perp| \ll 1$, that is relevant for the hydrodynamic theory of an interfacial ordered state, this reduces to

$$\mathbf{M} = \frac{1}{4\eta |q_\perp|^3} \left( q_x^2 + 2q_y^2 - q_x q_y \right) - \frac{1}{8\eta q_\perp^2} \left( \ell_b - 3\ell_s \right) q_x q_y + \mathcal{O}(\ell_b, \ell_s, \eta_b) \quad (7)$$

Thus, it can be performed when there is no fluid above the interface i.e., no equation for $\mathbf{V}_t$. This describes the behaviour of a phase associated with the boundary of a semi-infinite bulk fluid. In this case, the effective mobility is still given
by (7) with an effective viscosity \( \eta = \eta_f/2 \). Therefore, (7)
can be used to obtain the velocity field at the interface between
two fluids, at the boundary of a semi-infinite bulk fluid and at an immersed surface – such as a membrane –
in a bulk fluid simply by modifying the definition of the parameter \( \eta \): \( \eta = (\eta_b + \eta_f)/2 \) in the first case, \( \eta = \eta_f/2 \) in
the second and \( \eta = \eta_f \), where \( \eta_f \) is the viscosity of the
bulk fluid, in the third.

Having specified how the interfacial or boundary velocity field can be calculated from the knowledge of
interfacial or boundary-associated particle-phase stresses, I now need to construct these stresses and the equations
of motion of the hydrodynamic variables which are the concentration field \( c \) of the active particles and the apolar
order parameter \( Q \) or the the polar order parameter \( p \) for
the apolar and polar ordered phase respectively. These
will be different for each of the four models I consider –
apolar and polar ordered phase when active particles
diffuse in the bulk and when they are localised at
the interface – and I will now construct the equations of
motion for each of these cases separately.

A. NNC Model: Interfacial active nematic composed of nematogens diffusing in the bulk

In this section, I construct a model to describe a nematic phase at a two-fluid interface, at the boundary of
a bulk fluid or at an immersed surface in a fluid, with the active particles diffusing in the bulk. The nematogens
are constrained to lie parallel to the interface – the directions in the plane of the interface form a degenerate
set of easy axes [25] – and have no component along the normal. The in-plane nematic order, which I take to be
along \( \hat{x} \), is characterised by the two-dimensional apolar order parameter

\[
Q = \frac{S}{2} \left( \frac{\cos 2\theta \sin 2\theta}{\sin 2\theta - \cos 2\theta} \right)
\]

(8)

where \( \theta \) is the deviation of the local nematic order from \( \hat{x} \) and \( S \) is the magnitude of the nematic order whose
steady state value is \( S_0 = \langle S \rangle \).

Since the particles in the NNC model can diffuse in the bulk, the number of nematogens at the interface is not
conserved but the total number at the interface and in the bulk, \( N_{\text{tot}} \), is. This describes two distinct situations
i. when the active nematogens themselves can diffuse in the bulk fluid and ii. when monomers can associate to
form active nematogens at the interface as well as dissociate and diffuse in the bulk. These two situations are
depicted in Fig. 1 respectively at a boundary of a bulk fluid and at a two-fluid interface. The former is relevant
for systems in which elongated active swimmers, such as bacteria, may form an ordered phase at an interface or
the boundary of a bulk fluid, but can move away from the interface into the bulk fluid. The latter is relevant
when monomers associate or dissociate at a boundary or an immersed surface to form polymers which then be-
come active due to the action of motors. This is the case, for instance in the cellular cortex [41, 60]. Further, this is
also relevant for a widely used experimental geometry for studying pattern formation in active nematics [41–44]
which consists of a layer of microtubule filaments acted on by kinesin motors either between two fluids [41, 42]
or self-assembled in a fluid [43, 44] if additionally micro-
tubule filaments in the layer are allowed form and break
up with the monomers diffusing in the bulk fluid.

I take the concentration of active particles, or in the case of active polymers formed from monomers, the con-
centration of monomers in the polymeric state, at the in-
terface or the boundary to be \( c(r_\perp, t) \), where \( r_\perp \equiv (x, y) \). The concentration of active particles or monomers in the bulk
is \( \rho(r, t) \) with \( r \equiv (x, y, z) \). The total number of active
particles, or monomers composing active particles,
\[
\int d^3r_\perp [c + \int dz \rho] = N_{\text{tot}} \text{ in the system is constant.}
\]

For a surface-associated nematic phase at the boundary of a semi-infinite fluid, \( \rho \) diffuses in the bulk below the
interface. Even for an interface between two fluids, I assume that \( \rho \) only diffuses in the fluid below the interface, for
convenience. However, this assumption may be easily re-
laxed and the argument presented here works even when
the particles diffuse in the fluids both above and below the
interface. The bulk density \( \rho \) is conserved inside the fluid and gets converted to \( c \) only at the boundary. Therefore,
in the bulk, \( \rho \) obeys a conservation equation of the form

\[
\partial_t \rho + \nabla \cdot \mathbf{V}_b = D_\rho \nabla^2 \rho
\]

(9)

with a boundary condition at the surface of the bulk fluid

(a) Uniaxial phase of active particles

(b) Interfacial uniaxial phase of active particles that diffuse in the bulk fluid

FIG. 1. Two experimental situations described by the NNC
model: a. A nematic phase formed by polymeric elongated
active particles which associate and dissociate at the interface
between two fluids, and b. An apolar phase formed by elon-
gated active units at the boundary of a bulk fluid where the
active units are exchanged between the bulk and the interface
diffuse in the bulk fluid.
\[ z = 0, \ D_p \partial_z \rho|_{z=0} = j \] where \( j \) is the flux of \( \rho \) into and out of the bulk. The surface concentration field \( c \) has an equation of motion \( 61 \) \( 02 \)

\[ \partial_t c = -\nabla \cdot (c\mathbf{v}) - \nabla \cdot \mathbf{J}^a + D_c \nabla^2 c + j \quad (10) \]

where \( \mathbf{J}^a \) is the active current which, for a nematic system has the form, \( \mathbf{J}^a = gq_1(c)\nabla \cdot (gq_2(c)\mathbf{Q}) \) where \( gq_1(c) \) and \( gq_2(c) \) are arbitrary functions of the concentration and \( j \) is the flux of particles into and out of the interface. Eqs. \( 9 \) along with its boundary condition and \( 10 \) together imply that there is an exchange of particles between the bulk fluid and the surface. The current \( j \) is generally modelled as \( 61 \) \( 02 \)

\[ j = k_a \rho|_{z=0} \left( 1 - \frac{c}{c_{\text{max}}} \right) - k_d c \quad (11) \]

where \( k_a \) is the adsorption rate, \( k_d \) is the desorption rate, \( c_{\text{max}} \) is the maximum interfacial concentration and \( \rho|_{z=0} \) is the bulk density adjacent to the interface. The specific form of \( j \) is unimportant; it simply models the adsorption and desorption of the particles to and from the interface.

In the steady state, \( \rho = \rho_0 \) everywhere and \( j = 0 \) implying that \( c_0 = k_a \rho_0 c_{\text{max}} / (k_d c_{\text{max}} + k_a \rho_0) \). Eq. \( 10 \) along with \( 11 \) imply that deviations from \( \delta c \) of \( c \) from this steady-state value are not hydrodynamic but relax at a finite time as

\[ \partial_t \delta c = k_a \rho_0 \left( 1 - \frac{c}{c_{\text{max}}} \right) - k_d \delta c \quad (12) \]

This implies that \( \delta c \) adjusts in a finite time to preserve \( j = 0 \) and is slaved to the fluctuations of the bulk density adjacent to the interface via

\[ \delta c = \frac{c_{\text{max}} k_a}{k_a \rho_0 + k_d c_{\text{max}}} \rho|_{z=0}. \quad (13) \]

The bulk fluid in the NNC model, therefore, acts as a reservoir for the concentration of the particles at the interface which is not globally conserved, but locally fixed – a local excess or deficit of \( c \) decays primarily by the exchange with the bulk, i.e., by diffusion into or out of the bulk fluid, and not by diffusion within the boundary layer. Further, since \( j \) relaxes to 0 in a finite time, the bulk density field \( 9 \) effectively has a no-flux condition at the interface and, to the linear order, is not affected by fluctuations of either the velocity field or the order parameter field.

The bulk fluid equation has a contribution from the bulk density of the form \( \rho \nabla \delta F_p / \delta \rho \) where \( F_p \) is the bulk free energy of the particles. The linearised force due to density fluctuations is therefore \( -\rho_0 \nabla \delta F_p / \delta \rho \) and can be absorbed into a redefinition of the pressure \( \Pi_0 \). There are possible active forces involving \( \rho \) of the form \( \nabla \cdot (\nabla \rho \nabla \rho) \) \( 19 \) \( 63 \) \( 64 \) but for a density field that is uniform on average, these contribute only at nonlinear order and therefore, do not affect the linear theory that will be described here. This implies that bulk density fluctuations do not modify \( \mathbf{v}_p \) at linear order which retains its form in \( 4 \).

Since \( c \) is not a hydrodynamic variable and \( \rho \) does not couple to the other fields at the linear order, I need to only consider the dynamics of \( \mathbf{Q} \) which is coupled to the in-plane velocity field \( \mathbf{v} \), to describe the hydrodynamic behaviour of the interfacial nematic phase in the NNC model. Further, \( \mathbf{v} \) is completely determined in terms of the particle-phase stress \( \sigma^\tau \) which, in the hydrodynamic limit, can be written purely in terms of \( \mathbf{Q} \). The standard dynamical equation for \( \mathbf{Q} \) is

\[ \dot{\mathbf{Q}} = \mathbf{Q} \cdot \mathbf{Q} - \mathbf{Q} - \lambda \mathbf{A} - \lambda_1 (\mathbf{Q} \cdot \mathbf{A})^{ST} - \Gamma \mathbf{H} + \xi^Q, \quad (14) \]

where the overdot denotes the convected derivative \( \partial_t + \mathbf{v} \cdot \nabla \), \( \mathbf{Q} = (1/2)[\nabla \mathbf{v} - (\nabla \mathbf{v})^T] \) is the planar vorticity tensor at the interface, \( \mathbf{A} = (1/2)[\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \) is the planar strain rate tensor at the interface, with \( |\lambda| > 1 \) describing nematogens with tendency to align under an imposed shear flow and \( |\lambda| < 1 \) describing flow tumbling, the superscript \( ST \) denotes symmetrized, traceless part of a tensor, \( \mathbf{H} = [\delta F_Q / \delta \mathbf{Q}]^{ST} \) and \( \xi^Q \) is a non-conserving noise with the correlation

\[ \langle \xi^Q_{ij}(r, t) \xi^Q_{kl}(r', t') \rangle = 2 \Delta^Q Q_{ij} Q_{kl} \delta(t - t') \quad (15) \]

Finally, \( \Gamma \) controls the relaxation to the equilibrium steady state, given by the free energy \( F_Q \), in the absence of activity. The standard Landau-de Gennes free energy for a two-dimensional nematic, written here in a simplified one Frank constant approximation, is \( F_Q = \int d\Omega d\mathbf{Q} \), with \( f_Q = (\alpha/2)(\mathbf{Q} \cdot \mathbf{Q}) + (\beta/2)(\mathbf{Q} \cdot \mathbf{Q})^2 + (1/2)[\nabla \mathbf{Q}]^2 \), which supports an orientationally ordered phase for \( \alpha < 0 \). In the in-plane, particle phase stress is

\[ \sigma^\tau = -\zeta Q - \lambda H - 2(\mathbf{Q} \cdot \mathbf{H}) A - \lambda_1 (\mathbf{Q} \cdot \mathbf{H})^{ST} - \nabla \cdot \mathbf{Q} \cdot \partial F_Q / \partial \mathbf{Q} \quad (16) \]

Here, the term with the coefficient \( \zeta \) is the active stress \( 1 \) \( 21 \) while the remaining contributions are required by Onsager symmetry to ensure that the equilibrium distribution is recovered in the limit \( \zeta = 0 \). An active stress with \( \zeta > 0 \) denotes an extensile suspension while one with \( \zeta < 0 \) signifies contractility \( 65 \). Eqs. \( 14 \), \( 16 \) along with the definition of the mobility \( 7 \) are the dynamical equations for model NNC which describes the dynamics of an interfacial nematic layer when active nematogens can diffuse in the bulk fluid.

**B. PNC Model: Interfacial active polar fluid composed of motile particles that diffuse in the bulk**

The PNC model describing an interfacial polar phase composed of motile particles that can diffuse in the bulk fluid – as can be realised in a system of elongated active Brownian particles \( 66 \) at the boundary of a bulk fluid or at a two-fluid interface – shares similarities with model NNC. In particular, as in model NNC, the interfacial concentration field \( c \) is nonhydrodynamic in this
model for precisely the same reason. Therefore, to describe the motile phase in the PNC model, I need to specify the dynamics of a polar order parameter \( \mathbf{p} \) which couples to the in-plane velocity field. The polar order parameter is taken to be dispensed, on average, along \( \hat{x} \): \( \mathbf{p} = \rho \cos \theta, \sin \theta \) where \( \rho \) is the magnitude of the polar order and \( \theta \) denotes the local deviation of the polarisation from \( \hat{x} \). The in-plane velocity field is determined by a \( \mathbf{p} \)-dependent stress \( \sigma^p \) and the effective mobility \( \eta^p \). The dynamics of \( \mathbf{p} \) is described by

\[
\dot{\mathbf{p}} + v_0 \mathbf{p} \cdot \nabla \mathbf{p} + \Omega \cdot \mathbf{p} = -\lambda_p \mathbf{A} - \lambda_p \nabla_p^2 \mathbf{v} - \Gamma_p \mathbf{h} + \xi^p \tag{17}
\]

where \( v_0 \) denotes active self-advection due to the motility of the polar particles, \( F_p = \int d\mathbf{r}_p [(\alpha / 2) \mathbf{p}^2 + \beta / 4 \mathbf{p}^2 + (K / 2) (\nabla \mathbf{p})^2] + K_p \mathbf{p}^2 \nabla \mathbf{p} \cdot \mathbf{p} = \int d\mathbf{r}_p F_p \) is the standard free energy for polar liquid crystals, \( \mathbf{h} = \delta F_p / \delta \mathbf{p} \) and the noise \( \xi^p \) has the correlation \( \langle \xi^p_i (\mathbf{r}, t) \xi^p_j (\mathbf{r}', t') \rangle = 2 \Delta^p \delta_{ij} \delta (\mathbf{r} - \mathbf{r}', \delta (t - t')) \). There are additional, permitted active terms in the polarisation equation at the same order in gradients and fields of the form \( \mathbf{p} \nabla \cdot \mathbf{p} \) and \( \nabla \mathbf{p} \) [11] [12] [67]. However, these do not affect the hydrodynamic fluctuations of the polar phase [67] and therefore, I disregard them. The particle phase stress tensor for a polar state is \[68 \text{ and } 69\]

\[
\sigma^p = -c_0 \left( \frac{\mathbf{p} \mathbf{p} + \mathbf{p} \mathbf{p}^T}{2} \right) + c_p [\nabla \mathbf{p} + (\nabla \mathbf{p})^T] - \lambda [\mathbf{p} \mathbf{h}]^T
\]

\[
+ \lambda_p [\nabla \mathbf{p}]^T - 2 [\mathbf{p} \mathbf{h}^A] - \nabla \mathbf{p} \cdot \frac{\partial f_p}{\partial (\nabla \mathbf{p})} \tag{18}
\]

where the superscripts \( S \) and \( A \) describe symmetric and antisymmetric parts of a tensor respectively and the final term is Ericksen stress [70]. Eqs. \[17\] and \[18\] together with the definition of mobility \( \eta^p \) completely specify the dynamics of an interfacial polar state where the motile swimmers can move in and out of the interface.

C. NC Model: Interfacial active nematic composed of nematogenic species living at the interface

The models NNC and PNC, described in secs. [11A] and [11B] refer to systems in which the active species is free to move away from the interface into the bulk fluid. However, in some experimental and biological systems, active particles may be localised at the interface between two fluids, as in the experiments of [41–44] or at the surface of a bulk fluid. In this case, the concentration \( c \) of active particles at the interface is conserved unlike in IIA and IIB. The NC model describes an interfacial active nematic in which the concentration of the active particles at the interface is conserved; see Fig. 2. The effective free energy coupling \( c \) and \( \mathbf{Q} \) is \( \delta F_{Qc} = \delta F_{Qc} + \int d\mathbf{r}_c [g(c) + A_Q \mathbf{Q} \cdot \nabla \mathbf{Q} \cdot \mathbf{Q}] \) where \( g(c) \) is an arbitrary function of concentration. The dynamics of the \( \mathbf{Q} \) tensor is still described by \[14\] with \( \mathbf{H} = [\delta \delta F_{Qc} / \delta \mathbf{Q}]^T \). The concentration dynamics has the form

\[
\partial_t c = -\nabla \cdot (cv) + \Gamma_{c} \nabla^2 \delta F_{Qc} / \delta c + \nabla \cdot \mathbf{J}^c + \xi_c \tag{19}
\]

where \( \mathbf{J}^c = \zeta_{Qc} \nabla \cdot [g_{Q1}(c) \nabla \cdot \{g_{Q2}(c) \mathbf{Q}\}] \) for an apolar system, with \( g_{Q1}(c) \) and \( g_{Q2}(c) \) being arbitrary functions of the concentration field, and the conserving, spatiotemporally white noise \( \xi_c \) has the correlation \( \langle \xi_c (\mathbf{r}, t_1) \xi_c (\mathbf{r}', t') \rangle = -2 \Delta^c \delta (\mathbf{r} - \mathbf{r}', \delta (t - t')) \). The second term in \[19\] controls the relaxation to equilibrium in the absence of activity while the final term is the active curvature current [11] [20] [71]. Finally, the particle phase stress tensor is also modified from \[16\]: the active stress coefficient \( \zeta_c \) can depend on the concentration \( -\zeta_c (c) \mathbf{Q} \) and there is an additional isotropic pressure-like term \( -\Pi^c (c) \mathbf{I} \), where \( \mathbf{I} \) is the rank two identity tensor, which, in general, has both active and passive contributions. Further active nonlinear terms of the form \( \nabla_c \mathbf{c} \nabla_c \mathbf{c} \) [19] [63] [64] are also allowed in the stress but they do not affect the stability of a homogeneous ordered phase and I will not consider their effects in this work. The conserved concentration dynamics crucially modifies the phase behaviour of the oriented state in the NC model.

D. PC Model: Interfacial active polar fluid composed of polar species living at the interface

The PC model describes a floating flock of purely interfacial or boundary-associated motile units as depicted
in Fig. 3. The polarisation dynamics of the flockers is described by eq. [17] with a modified definition of $h = \delta F_p c / \delta p$ with $F_p = F_p + \int dr \left[ \gamma (c) + \gamma_p \cdot \nabla \cdot c \right]$, where $F_p$ has been defined in Sec. 11B. $g(c)$ is an arbitrary function of the concentration field and the final term is the spontaneous splay energy familiar in studies of polymer liquid crystals [22]. The dynamics of the concentration field is described by eq. [19] with, however, a distinct active current $J_p = \nu c \frac{\partial g}{\partial c}$ proportional to the polarisation itself. This models the active motility of the flockers. Finally, as in the NC model (sec. 11C), the particle phase stress tensor is modified from eq. [18] due to the concentration dependence of the active stresses $\zeta (c) (p p - \nu^2 I / 2) + \zeta_p (c) \left[ \nabla \cdot (p + (\nabla \cdot p) I) \right]$ [23] and the presence of a surface pressure-like isotropic stress $-\Pi(c) I$. This completes the description of the four distinct models and physical situations that I will examine in this paper. In the next sections, I use these models to examine the possibility of interfacial, orientationally ordered active phases.

III. LONG-RANGE ORDERED NEMATIC AND POLAR PHASE IN MODELS NNC AND PNC

In this section, I will demonstrate that long-range ordered active nematic and polar flockings exist at the interface between two similar or dissimilar fluids or at the boundary of a semi-infinite fluid medium when the active units can diffuse into the bulk. That is, both models NNC and PNC can support two-dimensional long-range order. This atypical purely two-dimensional ordered phase exists even though the transition to a uniaxial phase is not possible in the bulk fluid – that is, a surface transition to a long-range ordered uniaxial phase is possible even when there is no ordinary or extraordinary transition to a phase which is ordered both in the bulk and at the boundary [25, 33, 74, 77].

A. Nematic phase in the NNC model

To demonstrate the stability of nematic phase at the interface between two fluids or at the boundary of a bulk fluid, I expand eqs. [11], and [16] of the NNC model described in Sec. 11A with [7] to linear order in fluctuations about a perfectly ordered phase characterised by $S_0 = 1$. Writing $S = 1 + \delta S$, the equation of motion for $\delta S$ from [14], to lowest order in gradients is simply $\partial_\tau \delta S = -\Gamma_\delta (\delta S)$. This implies that the fluctuations of the order parameter magnitude relax within a finite time $\sim (\Gamma_\delta)^{-1}$ and therefore do not affect the long-time, large-scale behaviour of the ordered phase. The angle field, in contrast, is a Goldstone mode of corresponding to broken rotation symmetry and, therefore, hydrodynamic; for a state oriented along $\hat{x}$, the linearised angular dynamics from [14] is

$$\partial_\tau \theta = \Omega_{zy} \lambda (c) + \Gamma_\theta K \nabla^2 \theta + \xi,$$  

where $\Gamma_\theta = \Gamma / 4$ and $\langle \xi (r_{\perp}, t) \xi (r_{\perp}, t') \rangle = 2 \Delta \delta (r_{\perp} - r_{\perp}) \delta (t - t') \delta (z)$, with $\Delta = \Delta^2 / 4$. Similarly, expanding the interfacial particle phase force density $f^* = i q_{\perp} \cdot \sigma^s \perp \phi$ [16], to lowest order in wavenumbers, I obtain $f^* = -i c (q_{\perp} \hat{\theta} \hat{x} + q_{\perp} \hat{\theta} \hat{y})$. Combining this with the lowest order in wavenumber $q^{-1}$ part of the mobility [7] yields the in-plane velocity field and the relaxation rate of the angular fluctuations to leading order in wavenumbers:

$$\partial_\tau \theta = -\frac{\zeta}{\eta} q_{\perp}^2 (\lambda - 1) + q_{\perp}^2 (\lambda + 1) \frac{\theta}{|q_{\perp}|^2} + O(q_{\perp}^2) + \xi,$$

where in the second equality, $\theta$ is the angle between $q_{\perp}$ and the mean ordering direction $\hat{\theta}$. Eq. [21] implies that the relaxation rate for angular fluctuations is positive for all $\phi$ when $|\lambda| > 1$ i.e., flow-aligning $-\lambda > 0$. To see this, first consider $\xi > 0$, $\lambda > 1$. Then, for the relaxation rate to be positive, the term within the square brackets has to be $< 0$ for all $\phi$ i.e., all of its extreme values must be negative. The extreme values of this term, which can be expressed as $[\cos 2\phi (\lambda - 1) + \cos 2\phi (\lambda + 1)] = 0$ at $\phi = (2n + 1) \pi / 2, n \pi$ or at $\cos 2\phi = 1 / \lambda$ and all of them are negative for $\xi > 0$ and $\lambda > 1$: $[\cos 2\phi (\lambda - 1) + \cos 2\phi (\lambda + 1)]_{|\phi = \pi / 2} = -1 - \lambda < 0$, and $[\cos 2\phi (\lambda - 1) + \cos 2\phi (\lambda + 1)]_{\cos 2\phi = 1 / \lambda} = (1 / \lambda) - (\lambda / 2) - 1(1 / \lambda) = 1(2\lambda) - \lambda / 2 < 0$. A similar argument shows that the relaxation rate is positive, i.e., the planar ordered phase is stable, when $\xi < 0$ and $\lambda < -1$. This directly demonstrates that, contrary to popular belief, a two-dimensional planar nematic phase is realised in the fully momentum-conserved NNC model. This is not in contradiction with the Simha-Ramaswamy instability which only forbids bulk orientational order in Stokesian, momentum-conserved fluid and not interfacial order. As discussed in the introduction, the ratio of active stress and viscosity is an inverse timescale in a bulk fluid, which is only proportional to the growth rate of the instability of an oriented state because of incompressibility [77]. The unusual stability of the nematic state is a result of the two-dimensional interfacial fluid velocity not being incompressible [34, 35] unlike in [11, 15, 16] – a non-vanishing dilational or compressive flow in the plane is compensated by a non-vanishing $z$-gradient of the three-dimensional velocity field as can be seen from [3] and [11]. Indeed, an in-plane director distortion leads to a compressive or dilational flow $i q_{\perp} \cdot v = (\zeta (\theta / 2 \eta) / q_{\perp} [q_{\perp} / |q_{\perp}|] = (\zeta (\theta / 4 \eta) / |q_{\perp}| \sin 2 \theta$ which leads to the final term in the square brackets in [21] ensuring the stability of the nematic order.

In contrast, the nematic phase is unstable to fluctuations in flow-tumbling systems $|\lambda| < 1$ for either sign of activity – extensile systems ($\xi > 0$) are unstable for $\phi \approx 0$, i.e., bend, and contractile systems ($\xi < 0$) are unstable for splay, $\phi \approx \pi / 2$ – or even in flow-aligning
systems when $\zeta\lambda < 0$. A stability diagram, summarising the parameter values for which a stable nematic state exists in the NNC model is displayed in Fig. 4

When the nematic state is linearly destabilised in the NNC model, a patterned state is expected to appear \([11, 45, 46]\) whose wavenumber, which is expected to coincide with the fastest growing mode, is $q^*_1 = \text{Max}[\zeta (2 \cos 2\phi - \lambda (1 + \cos^2 2\phi))/(8\eta I \delta K_r(\phi))]$, where $\text{Max}$ denotes the maximum value of a function and $\Gamma_r K_r(\phi)$ denotes the coefficient of the $O(q^2)$ term in \([21]\), with $K_r(\phi)$ being the effective activity-renormalised Frank elasticity, which in passive systems would have been just $-\Gamma_r K$ but here will have active corrections as displayed in the Appendix C.

Concentrating on the stable nematic phase in the NNC model, eq. \([21]\) implies that the relaxation rate of angular fluctuations scales as $|q_\perp|$ along all directions of the wavevector space unlike the relaxation rate of passive nematics or active nematics on substrates both of which scale as $\sim q^2_\perp$ \([11, 20, 21]\). This is due to a combination of activity and momentum-conserved fluid dynamics-induced long-range interactions. While the interaction is not long-enough ranged (i.e., it decays too fast) to make the orientational Goldstone mode massive \([63, 29]\), which remains hydrodynamic $\omega \propto -i|q_\perp|$, it drastically reduces the director fluctuations. From Eq. \([21]\), the static structure factor of angular fluctuations

\[
\langle |\theta(q_\perp, t)|^2 \rangle = \frac{8\eta \Delta}{\zeta |q_\perp| \lambda [1 + \cos^2(2\phi)] - 2 \cos(2\phi)}. \tag{22}
\]

This diverges as $\sim 1/|q_\perp|$ along all directions of the wavevector space – more slowly than in two-dimensional rotation symmetry broken states with short-range interactions where the divergence is stronger ($\sim 1/q^2_\perp$) leading to the destruction of LRO \([80]\). The depression of the order parameter from its perfectly ordered value $S_\theta = 1$ due to fluctuations can be obtained using $\langle S \rangle / S_\theta = \langle \cos 2\theta \rangle = e^{-W}$ where $W = 2 \theta(r_\perp, t)^2 = 2 \int (d^2q_\perp/4\pi)^2 \langle |\theta(q_\perp, t)|^2 \rangle$ evaluates to

\[
W = 2 \int d|q_\perp| \int \frac{2\eta \Delta d\phi}{\pi^2 \zeta [\lambda (1 + \cos^2(2\phi)] - 2 \cos(2\phi)} \propto \Delta \Lambda, \tag{23}
\]

with $\Lambda$ being a wavenumber cut-off. Since $W$ is finite, $\langle S \rangle$ does not generically vanish due to fluctuations even in infinite systems, unlike in, for instance, X-Y model \([80]\), implying that at least for small enough $\Delta$, a long-range ordered nematic phase exists. More formally, the Debye-Waller factor $e^{-2W}$ is nonzero even for infinite systems implying long range order. This atypical two-dimensional long-range order is due to the reduction of fluctuations due to the fluid-induced long-ranged interactions. Surprisingly, while active fluid flows destroys orientational order in the bulk, it stabilises interfacial order.

This discussion of the existence of long-range interfacial nematic order in the NNC model considered only linear fluctuations. I now demonstrate that it remains qualitatively correct even taking nonlinearities into account. I do this using the standard renormalisation group logic. For this, I rescale lengths, time and the angle field as $x \rightarrow bx$, $y \rightarrow b^2y$, $t \rightarrow b^2t$ and $\theta \rightarrow b^2\theta$, where $b$ is the anisotropy exponent, $z$ is the dynamical exponent and $\chi$ is the roughness exponent. Within the linear theory, these exponents can be determined from the knowledge of the static \([22]\) and dynamic structure factors

\[
\langle |\theta(q_\perp, \omega)|^2 \rangle = \frac{128\eta b^2 \Delta}{(8\eta \omega)^2 + \zeta |q_\perp| (\lambda (1 + \cos^2(2\phi) - 2 \cos(2\phi))]^2. \tag{24}
\]

From \([22]\) scales the same way along all directions of the wavevector space, the anisotropy exponent $\mu = 1$. Finally, balancing $\omega$ against the damping term in \([24]\) yields $\omega \propto q_\perp$ and therefore, the dynamical exponent $z = 1$ within the linearised theory. The values of these linear exponents may also be obtained more formally by demanding that size of fluctuations of $\theta$ remain fixed upon rescaling i.e., by ensuring that the relaxation rate and the noise strength $\Delta$ remain unchanged. Since the relaxation rate must remain unchanged along all directions of wavevector space $\mu = 1$. Since the relaxation rate $\sim |q_\perp|$, $z = 1$ as well. Under rescaling, the noise strength scales as $\Delta \rightarrow \Delta b^{z-\mu-1-2x}$ implying that for it to remain unchanged, $\chi = -1/2$. Furthermore, since Eq. \([22]\) scales the same way along all directions of the wavevector space, the anisotropy exponent $\mu = 1$. Finally, balancing $\omega$ against the damping term in \([24]\) yields $\omega \propto q_\perp$ and therefore, the dynamical exponent $z = 1$ within the linearised theory. The values of these linear exponents may also be obtained more formally by demanding that size of fluctuations of $\theta$ remain fixed upon rescaling i.e., by ensuring that the relaxation rate and the noise strength $\Delta$ remain unchanged. Since the relaxation rate must remain unchanged along all directions of wavevector space $\mu = 1$. Since the relaxation rate $\sim |q_\perp|$, $z = 1$ as well. Under rescaling, the noise strength scales as $\Delta \rightarrow \Delta b^{z-\mu-1-2x}$ implying that for it to remain unchanged, $\chi = -1/2$. Therefore, $\mu = 1$. Unsurprisingly, both arguments yield the same values of the linear exponents. With these exponents in hand, I can now assess the importance of possible nonlinear terms in \([21]\).
by simply checking whether the coefficients of these grow or decay under rescaling using the linear exponents. The most relevant nonlinearities that can appear in \[21\] are from terms that have one power of the velocity field \(v\) and one power of \(\theta\), along with a gradient operator, such as the one due to advection \(v \cdot \nabla \theta\). This scales as \(q_{\perp}(\theta^2)q\) since \(v \sim \Phi(\phi)\theta\) where \(\Phi(\phi)\) is a vector function that depends on \(\phi\), but not on \(|q_{\perp}|\). The most relevant nonlinearity not involving the velocity field contains two powers of the gradient and is quadratic in \(\theta\) \[51, 53\] and is therefore subdominant to this. The coefficient of the most dominant nonlinearity \(q_{\perp}(\theta^2)q\) scales under rescaling as \(b^{\gamma-1+\kappa} = b^{\kappa}\). Since within the linear theory, \(\chi < -1/2 < 0\), this clearly decays under rescaling and, therefore, is irrelevant. This implies that this mode has no relevant nonlinearity and the linear theory is exact i.e., \[21\], \[23\] and \[24\] describe the exact long-distance, large time properties of a boundary-associated active nematic phase in a momentum-conserved system. In particular, the low-noise, long-range ordered nematic state predicted on the basis of the linear theory survives even upon taking nonlinearities into account.

### B. Defect interactions in the nematic phase of NNC model

The discussion of stability and long-range order of interfacial active nematics described by the NNC model considered only director fluctuations about the ordered state and ignored topological defects. Yet, in active nematics in bulk, incompressible fluids, dynamics of topological defects are thought to drive the eventual spatiotemporally chaotic state \[23, 24, 54\]. Indeed, \(+1/2\) defects in nematics are geometrically polar \[70, 80, 85\] and, therefore, are rendered motile by activity \[86–90\]. Since \(-1/2\) defects have a three-fold symmetry, \(+1/2\) defect-pairs in active nematics generically unbind at low noise strengths \[80\]. While a full examination of defect dynamics in the interfacial nematic phase, including their advection by the active flow leading to non-reciprocal defect interactions \[13, 91\], is beyond the scope of this paper, I now argue that the long-range interaction that stabilises the ordered phase to director fluctuations also makes the separation of \(+1/2\) defect much less likely than in other active nematic systems, both wet and dry.

The modification of the attractive interaction between \(+1/2\) defect pairs due to the long-ranged fluid dynamics-mediated active force can be understood via an analogy with passive two-dimensional \(X-Y\) model with dipolar interactions \[92, 94\]. This analogy is not perfect: the dipolar interaction, in real space, has the free energy density \(F_{\text{dip}} = \int d^d r \int \partial_i \theta(r) \partial_j \theta(r) \frac{\kappa}{|r-r'|}\). Therefore, the relaxation rate for the spin fluctuations in two-dimensional dipolar \(X-Y\) model vanishes at small wavenumbers as \(\sim q_{\perp}\) for most directions of the wavevector space. Unlike the interfacial active nematics, however, the dispersion relation is \(\omega \sim -i \kappa q_{\perp}^2/|q_{\perp}|\) and therefore, vanishes as \(q_{\perp}^2\) for fluctuations in the ordering direction \[15, 63, 94\]. This implies that the two models belong to distinct universality classes with the dynamical exponent \(z = 2\), the anisotropy exponent \(\mu = 3/2\) and the roughness exponent \(\chi = -1/4\) for the dipolar \(X-Y\) model. The higher value of \(\chi\) in the dipolar \(X-Y\) model compared to interfacial active nematics demonstrates that it has a lower degree of order, as expected, since the relaxation rate of fluctuations in the latter scale as \(|q_{\perp}|\) in all directions of the wavevector space. Nevertheless, the imperfect analogy between the two models proves fruitful in understanding defect interactions in interfacial nematics. The linear dynamics of the interfacial nematic \[21\] can be derived from an effective free energy

\[
F_{\text{eff}}[\theta] = \frac{1}{2} \int d|q_{\perp}| \int d\phi \left[ \frac{|C(\phi)|}{4\Gamma \eta q_{\perp}} + \kappa r_{\perp}^2 \right] q_{\perp}^2 |\theta|^2,
\]

where \(C(\phi) = \cos(2\phi)|1 - \lambda \cos(2\phi)| - (\lambda/2) \sin^2(2\phi)\). Importantly, the first term in the square bracket has the same spatial character as the dipolar energy in a two-dimensional \(X-Y\) model with dipolar interactions \[92, 94\], but as advertised, a different angular character, with \(C(\phi)\) being strictly positive in the stable phase being considered here (in dipolar magnets, the corresponding \(C(\phi)\) vanishes as \(\phi = 0\)). Ignoring, for the moment, the motility of \(+1/2\) defect, I use an argument presented in \[93\] for vortex interactions in passive dipolar \(X-Y\) model to obtain the defect interactions in the NNC model. The singular part of the angle field due to a defect at the origin with charge \(s\) is \(\theta(|r_{\perp}|, \phi') = 2\pi s \int_{|q_{\perp}|} e^{i|q_{\perp}||r_{\perp}| \cos(\phi-\phi')}/q_{\perp}^2\). From this, following \[93\] the defect interaction energy between two defects (which can be easily generalised to arbitrary number of defects), with charges \(s_1\) and \(s_2\) and positions \(0\) and \(|r_{\perp}|(\cos \phi', \sin \phi')\) is

\[
F_{\text{def}} = \pi^2 s_1 s_2 \int_{q_{\perp}} e^{-i|q_{\perp}||r_{\perp}| \cos(\phi-\phi')} \left[ \frac{K_r}{q_{\perp}^4} + \frac{|C(\phi)|}{4\Gamma \eta q_{\perp}} |q_{\perp}| \right].
\]

As is well-known, the infrared divergences cancel for a charge-neutral system finally yielding a defect interaction energy (for one of the defects at the origin) which, when written in real space has the form

\[
F_{\text{int}} = -s_1 s_2 \left[ \frac{\pi K_r}{2} \ln \left( \frac{|r_{\perp}|}{a} \right) + \frac{\zeta - \alpha}{\Gamma \eta} |r_{\perp}| \right]
\]

where \(a\) is the size of the defect core and \(\alpha > 0\) is a positive constant whose determination requires a detailed calculation far beyond the scope of this heuristic argument. The interaction energy scaling as \(|r_{\perp}|\) is stronger than the usual logarithmic potential that binds defects in passive systems with short-range interactions and is equivalent to the potential between vortices in dipolar \(X-Y\) models. In effect, here activity effectively leads to an additional elasticity which diverges as \(1/q_{\perp}^4\) as \(q_{\perp} \to 0\) or as \(r_{\perp} \to \infty\) leading to the extra contribution. In passive dipolar \(X-Y\) models an interaction
energy of the form (27) immediately implies that defects are bound in the low temperature ordered phase. In passive dipolar $X-Y$ models, this immediately implies that defects are bound in the low temperature ordered phase. The situation is more complicated here because of two aspects of defect dynamics in active systems which I disregarded here: i. Self-propulsion of $+1/2$ defects. Within a one-dimensional approximation, this leads to an effective repulsive potential between oppositely charged defects that scales as $|r_{\perp}|$ and should compete against the attractive potential $\propto \zeta$ here. This repulsive interaction also depends linearly on $\zeta$ with the full effective potential becoming $F_{\perp,1/2} = F_{\text{int}}|_{\xi_{1,2} = -1/4} - |v||r_{\perp}|$ where $|v| \propto \zeta$ is the motility of the $+1/2$ defect [85] [89]. Therefore, understanding whether $\pm 1/2$ defect pairs remain bound at low noise strengths in interfacial active nematics would require a detailed calculation of the coefficients of the defect motility and attractive interaction. However, [95] demonstrates that at intermediate noise strengths fluctuations isotropise the direction of motion of the $+1/2$ defects making them diffusive and weakening the effect of motility such that it simply renormalises the effective Frank elasticity. While that calculation ignores all effects of fluid flow, a similar argument should hold in this case as well implying that at least at intermediate noise strengths, the defects should remain strongly bound making the long-range ordered phase described here possible. ii. In addition to this, defects in an active system are also dragged by the velocity field generated by other defects [94] which leads to non-reciprocal interaction between defects [13] [111] [96]. A consideration of these complexities will be discussed elsewhere. Notwithstanding these caveats, the present heuristic discussion suggests that the long-range ordered nematic state discussed here should be observable in experiments and will not be inevitably destroyed due to the unbinding of defects.

C. Polar phase in the NNC model

The linear stability of a polar phase in the PNC model (Sec. IIB) can be examined by using arguments similar to the NNC model. Expanding (17) and (18) with the mobility [7] in angular fluctuations $p = p_0(\cos \theta, \sin \theta)$ to leading order in wavenumber, about a perfectly ordered state with $p_0 = 1$, I obtain the angular dynamics

$$\partial_\theta \theta = \frac{\zeta |q_{\perp}|}{4\eta} \left[ \cos(2\phi)[1 - \lambda \cos(2\phi)] - \frac{\lambda}{2} \sin^2(2\phi) \right] \theta - iv_P(|q_{\perp}| \cos \phi) \theta - \Gamma_k K^p q_{\perp}^2 \theta + \xi,$$  

(28)

where $K^p$ is the effective Frank elasticity renormalised by activity displayed in Appendix C and $\langle \xi(r_{\perp}, t)\xi(r'_{\perp}, t') \rangle = 2\Delta^p \delta(r_{\perp} - r'_{\perp})\delta(t - t')\delta(z)$. The only qualitative difference between this and (21) is the propagating part $\propto v_P$, which is due to the motility of the polar particles. However, this doesn’t affect the conditions for the stability of the polar phase which are equivalent to the ones discussed for the nematic phase in the NNC model: the interfacial polar phase is stable when $|\lambda| > 1$ and $\zeta \lambda > 0$. Further, the static structure factor of angular fluctuations has the same form as (22) implying that the polar order in long-ranged. The argument presented for irrelevance of all nonlinearities in the NNC model also remains valid in the PNC model implying that the exact exponents for the polar phase are $s = m = 1$ and $c = -1/2$. Thus, just as its nematic counterpart, a motile polar phase is stable and displays long-range order at the boundary of a bulk fluid even though bulk polar ordering is forbidden.

IV. GENERIC INSTABILITY OF A NEMATIC PHASE IN THE NC MODEL

In the last section, I demonstrated that NNC and PNC models, in which active particles can diffuse from the bulk to the interface, support long-ranged, interfacial nematic and polar order respectively. In this section, I will consider a homogeneous nematic phase in which the active nematogens are constrained to float on the surface or the interface. This situation is described by the NC model (sec. II C) in a state with perfect nematic order, $S_0 = 0$, and a mean concentration $c_0$. Expanding the osmotic pressure $\Pi_{\perp}(c) \approx A(c)c_0$ to linear order in fluctuations about $c_0$ and the active force $\nabla_\perp \cdot (\zeta(\xi) Q) \approx \zeta(c_0)(\partial_\theta \delta x + \partial_\phi \phi) + \zeta_0(c_0)(\partial_\delta \delta x - \partial_\phi \phi y)$ to linear order in $\theta$ and $\delta c$, where $\zeta_0 = (1/2)\partial_\phi \zeta(c)_{c=c_0}$, and solving for the velocity field I obtain the coupled equations of motion for the in-plane concentration from (19) and the angle field, to leading order in wavenumbers:

$$\partial_\theta \theta = -\frac{\zeta q_{\perp}^2 (\lambda - 1) + q_{\perp}^2 (\lambda + 1)}{4\eta |q_{\perp}|^3} \theta - iv_P(|q_{\perp}| \cos \phi) \theta - \Gamma_k K^p q_{\perp}^2 \theta + \xi,$$

(29)

$$-q_{\perp} q_{\parallel} \left[ A \lambda - (\lambda - 2) \zeta_0 \right] + q_{\perp}^2 \left[ (A \lambda + (\lambda + 2) \zeta_0) \right] \frac{\partial_\delta \delta c}{4\eta |q_{\perp}|^3}$$

and

$$\partial_\delta \delta c = -\zeta c_0 q_{\parallel} q_{\perp} \theta - \frac{c_0 q_{\parallel}^2 (A \lambda + \zeta_0) + c_0 q_{\perp}^2 (A \lambda - \zeta_0)}{4\eta |q_{\perp}|^4} \delta c,$$

(30)

where $K^p$ is the effective Frank elasticity renormalised by activity displayed in Appendix C and $\langle \xi(r_{\perp}, t)\xi(r'_{\perp}, t') \rangle = 2\Delta^p \delta(r_{\perp} - r'_{\perp})\delta(t - t')\delta(z)$. The only qualitative difference between this and (21) is the propagating part $\propto v_P$, which is due to the motility of the polar particles. However, this doesn’t affect the conditions for the stability of the polar phase which are equivalent to the ones discussed for the nematic phase in the NNC model: the interfacial polar phase is stable when $|\lambda| > 1$ and $\zeta \lambda > 0$. Further, the static structure factor of angular fluctuations has the same form as (22) implying that the polar order in long-ranged. The argument presented for irrelevance of all nonlinearities in the NNC model also remains valid in the PNC model implying that the exact exponents for the polar phase are $s = m = 1$ and $c = -1/2$. Thus, just as its nematic counterpart, a motile polar phase is stable and displays long-range order at the boundary of a bulk fluid even though bulk polar ordering is forbidden.
The eigenfrequencies of the coupled concentration and angular dynamics, from \(29\) and \(30\) are:

\[
\omega_\pm = -\frac{i|q_\perp|}{8\eta} \left[ (A_r + \zeta_1 \cos 2\phi)c_0 - \zeta \cos 2\phi(1 - \lambda \cos 2\phi) + \frac{\zeta \lambda}{2} \sin^2 2\phi \right]
\]

\[\pm \sqrt{\left((A_r + \zeta_1 \cos 2\phi)c_0 - \zeta \cos 2\phi(1 - \lambda \cos 2\phi) + \frac{\zeta \lambda}{2} \sin^2 2\phi\right)^2 + 4c_0 \zeta_1 (A_r \cos 2\phi)(1 - \lambda \cos 2\phi)} \right]. \quad (31)\]

For \(A_r \neq 0\), at least one of these eigenfrequencies has a positive imaginary part for some \(\phi\) (except in a special case discussed in Appendix [2]), implying that the homogeneous nematic phase is generically destabilised in the NC model, i.e., when the active units are constrained to live at the interface. In particular, for large \(A_r c_0 \gg \zeta_1\), which can be accessed by enhancing the concentration of active units, the film becomes essentially incompressible \([27]\) since the dynamics is extremely sensitive to departures of \(c\) from \(c_0\). In this case, \(\omega_+ = -(i\zeta_1 c_0/4\eta)A_r|q_\perp|\) signals an infinitely fast relaxation of the concentration fluctuations as \(A_r \to \infty\) and \(\omega_- = (i\zeta c_0/4\eta)\cos 2\phi(1 - \lambda \cos 2\phi)\) reduces to the eigenfrequency obtained for the dynamics of the angle field in an incompressible two-dimensional interfacial layer \([11, 45, 46, 98]\) which has the same angular character as the Simha-Ramaswamy instability \([21]\). The works of \([11, 45, 46, 98]\) model a popular active matter system consisting of a thin layer of motors and microtubules at an oil-water interface which was shown to be essentially incompressible in two dimensions \([29]\). This calculation demonstrates that the effective incompressibility is due to the high concentration of motors and microtubules (or motors, microtubules and surfactants) in that experiment.

At low \(A_r\), however, the character of the generic instability diverges from the Simha-Ramaswamy instability. First, for \(A_r = 0\), \(\omega_-\) vanishes to \(O(q_\perp)\) (i.e., \(\omega_- \sim q_\perp^2\) and is controlled by the diffusivity and is, therefore, stabilising) while \(\omega_+\) has the same value as the eigenfrequency in \([21]\) to \(O(q_\perp)\), and is stable along all wavevector directions for \(\zeta \lambda > 0\) and \(|\lambda| > 0\) since the equation for angular fluctuations is decoupled from that of the concentration fluctuations at \(O(q_\perp)\). This implies that nematic order at the interface is stable, even when the nematogens are constrained to be at the surface, in the limit of infinite compressibility in the NC model. However, at small but non-zero \(A_r \to 0\), the \(\omega_-\) is generically unstable and increases linearly with \(A_r\):

\[
\lim_{A_r \to 0} \omega_- = \frac{iA_r c_0|q_\perp|}{8\eta} \left[ -\frac{\lambda}{4} - \frac{\lambda - 4 \cos 2\phi + 3 \lambda \cos 4\phi}{-4 \cos 2\phi + \lambda(3 + \cos 4\phi)} \right], \quad (32)
\]

where, for simplicity, I have additionally taken \(\zeta_1 = 0\), which doesn’t qualitatively affect the discussion. Importantly, this eigenfrequency becomes independent of \(\zeta\), and therefore, of its sign, at small \(A_r\) (i.e., when \(\zeta \lambda \gg A_r c_0\)). Of course, \(\omega_-\) vanishes when \(\zeta = 0\). The R.H.S. of \(32\) vanishes generically for \(\phi = \pi/4\) implying that it is unstable even at small \(A_r\) either for \(\phi \gtrless \pi/4\) or for \(\phi \lesssim \pi/4\) depending only on the value of \(\lambda\). Therefore, \(\omega_-\) crosses over from being independent of \(\zeta\) when \(\zeta \lambda \gg A_r c_0\) to linearly depending on \(\zeta\) when \(\zeta \lambda \ll A_r c_0\). However, in both limits, it is unstable either just above or below \(\phi = \pi/4\). This crossover is best examined by expanding \(31\) (still in the simplifying \(\zeta_1 = 0\) case) near \(\phi = \pi/4\):

\[
\omega_-(\phi \approx \pi/4) \approx \frac{i|q_\perp|}{8\eta} \frac{A_r c_0 \zeta}{2A_r c_0 + \zeta \lambda} \left(\phi - \frac{\pi}{4}\right). \quad (33)
\]

FIG. 5. A log-log plot of the activity length scale \(\ell_a\) – the lengthscale associated with the fastest growing mode in active uniaxial fluids living at the interface between two fluids or at the boundary of a bulk medium – as a function of activity \(\zeta\). For an essentially incompressible system, i.e., \(A_r \to \infty\), \(\ell_a\) scales as \(1/\zeta\) but for any finite \(A_r\), \(\ell_a\) scales as \(1/\zeta\) only till \(\zeta \sim 2A_r c_0/\lambda\) saturating beyond that.
contrast, \( \omega_+ \) is not generically destabilising near \( \phi = \pi/4 \)

\[
\omega_+ (\phi \approx \pi/4) \approx -\frac{i|q_\perp|}{2} \left[ \frac{2A_r c_0 + \zeta \lambda}{4\eta} + \frac{\zeta^2 \lambda}{2A_r c_0 \eta + \zeta \eta \lambda} \left( \phi - \frac{\pi}{4} \right) \right]
\]  (34)

and does not vanish in the \( A_r = 0 \) limit, instead going to a constant value which is stabilising when \( \zeta \lambda > 0 \) as can be seen from Fig. 5. The observation that the character of the active instability can be controlled by changing the concentration of the active particles (which controls \( A_r \)), has interesting consequences for the experiments on motor-microtubule films \[41, 98\]. Often, these experiments measure the "activity length scale" \( \ell_a \) which is the lengthscale associated with the fastest growing mode. This discussion demonstrates that the dependence of \( \ell_a \sim K_r \eta (2A_r c_0 + \zeta \lambda)/A_r c_0 \zeta \) on activity is, in general, more complicated than in \[41, 98\]. In particular, for small enough \( A_r \), \( \ell_a \) saturates at a finite value as activity is increased, while for larger \( A_r \), it scales as \( 1/\zeta \) (see Fig. 5). This implies that the fastest growing mode in the motor-microtubule experiments \[41\] should become independent of activity as the density of the active particles is reduced at least for sufficiently small activity (such that the active modification of \( K_r \) remains unimportant). Since activity is generally controlled by the concentration of motors, this implies that at low microtubule concentration, \( \ell_a \) should become independent of the kinesin concentration.

V. STABLE POLAR PHASE PC MODEL: OUTRUNNING THE GENERIC INSTABILITY

The last section demonstrated that when active particles are associated with an interface in the NC model, a nematic state is generically destabilised. In this section, I will demonstrate that a polar species living at the surface can outrun this instability and a motile ordered state exists at fluid-fluid or fluid-air interfaces of momentum conserved systems in the PC model (sec. [IV]). As in the earlier sections, solving for the velocity field and expanding the equations of motion of the polarisation field and the concentration field in angular fluctuations \( \theta \) about a perfectly ordered state with \( p_0 = 1 \) and \( \delta c \) – the local deviation of the concentration field about its mean value \( c_0 \) – to leading order in wavevectors, using the definitions in Sec. [IV] leads to

\[
\partial_t \theta = -\frac{\zeta}{4\eta} \frac{q_\perp^2 (\lambda - 1) + q_\perp^4 (\lambda + 1)}{|q_\perp|^3} \theta - \frac{q_\perp q_\parallel [q_\perp^2 (A_r \lambda - (\lambda - 2) \zeta \lambda) + q_\perp^4 (A_r \lambda + (\lambda + 2) \zeta \lambda)]}{4\eta |q_\perp|^3} \delta c - i \Gamma_\parallel \gamma q_\parallel \delta c - iv_p q_\perp \theta + \xi,
\]

and

\[
\partial_t \delta c = \frac{\zeta c_0 q_\perp q_\parallel}{2\eta |q_\perp|} \theta - \frac{c_0 q_\perp^2 (A_r + \zeta \lambda) + c_0 q_\perp^2 (A_r - \zeta \lambda)}{4\eta |q_\perp|} \delta c - i \tilde{v}_c q_\perp \delta c - iv_p q_\parallel \theta
\]

where, as in Sec. [IV], \( \zeta_1 = (1/2) \partial_\perp \zeta(c_c) |_{c = c_0} \), \( \tilde{v}_c \) is defined as \( \tilde{v}_c = \partial_\perp [c \zeta(c_c)] |_{c = c_0} \). The noise in the concentration equation is ignored as for the NC model (see sec. [IV]) since it turns out to be irrelevant. I now examine the mode structure implied by (35) and (36) in some limits. First, when \( A_r c_0 \gg \zeta, \zeta_1, \tilde{v}_c, \eta, v_p \eta \), the active system is effectively incompressible and as in the NC model (see sec. [IV]), the imaginary part of one of the two eigenfrequencies diverge with \( A_r c_0 \), signifying an infinitely fast relaxation of the concentration fluctuations while the imaginary part of the other eigenvalue, to \( \mathcal{O}(q_\perp) \), has the angular character of the Sinha-Ramaswamy instability implying a generic destruction of the ordered state in the PC model:

\[
\omega_+ = \left[ \bar{v}_c + \frac{v_p \lambda}{c_0} \sin^2 \phi \right] \cos \phi - \frac{ic_0}{4\eta} A_r \right] |q_\perp|, \quad (37)
\]

\[
\omega_- = \left[ \sqrt{v_p - \frac{v_p \lambda \sin^2 \phi}{c_0}} \cos \phi + \frac{i \zeta}{4\eta} (\cos 2\phi - \lambda \cos^2 2\phi) \right] |q_\perp|.
\]

However, unlike in for the NC model (sec. [IV]), the ordered state is not unstable in the entire parameter range of the PC model. Specifically, for large \( v_p \) or large \( \bar{v}_c \) (i.e., for \( v_p, \bar{v}_c \gg A_r c_0/\eta \)), disturbances can outrun the exponential growth, stabilising the ordered phase. In both limits, the eigenfrequencies for all \( \phi \neq (2n+1)\pi/2 \), where \( n \) is an integer, the eigenfrequencies to leading order in \( v_p \) and \( \bar{v}_c \) are

\[
\omega_+ = \left[ \bar{v}_c \cos \phi - \frac{ic_0}{4\eta} (A_r + \zeta_1 \cos 2\phi) \right] |q_\perp|, \quad (39)
\]

and

\[
\omega_- = \left[ v_p \cos \phi + \frac{i \zeta}{4\eta} \left( \cos 2\phi - \lambda \cos^2 2\phi - \frac{\lambda}{2} \sin^2 2\phi \right) \right] |q_\perp| \quad (40)
\]
This implies that for ζ1 ≪ Ar both of these eigenfrequencies are stabilising when |λ| > 1 and ζλ > 0. Of course, for this expansion, the closer φ is to π/2, the large vp or \( \bar{v}_c \) must be for it to hold. Further, for the ordered state to be stable, the fluctuations along φ = (2n + 1)π/2 must also be stabilising:

\[
\omega_\pm(\phi = \pi/2) = -\frac{i|q_\perp|v_p}{8\eta} \left( (A_r - \zeta_1)c_0 + \zeta(1 + \lambda) \right)
\]

\[
\pm \sqrt{\{\zeta(1 + \lambda) - (A_r - \zeta_1)c_0\}^2 - 64\nu_c\gamma\eta^2}. \tag{41}
\]

These eigenfrequencies are not generically destabilising i.e., the imaginary part of either of these is not generically positive; in particular, they are stabilising when \( \gamma v_c > 0 \), \( \zeta\lambda > 0 \) and |λ| > 1. This is the advertised outrunning of the active instability by motility which allows for a stable interfacial polar state to exist in the PC model, i.e., even when the motile particles are localised at the interface. The mechanism for the survival of a polar phase in the PC model has conceptual similarities to the suppression of the bulk instability in extensile polar fluids due to inertia \[57\].

The stability boundary of the polar phase is best examined in a simplifying limit \( \bar{v}_c = 0, v_c = 0, \gamma = 0 \) and \( \zeta_1 = 0 \), in terms of the non-dimensional parameters \( \mathcal{R}_1 = A_r c_0/4v_p\eta \) and \( \mathcal{R}_2 = \zeta/4v_p\eta \). The eigenfrequencies, in terms of \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), for all values of \( v_p \), are in all directions of the wavevector space as in Sec. III implying long-range polar order in two dimensions. Restoring motility \( v_c \neq 0 \), the dynamic structure factor of concentration fluctuations implied by \[35\] and \[36\] in the stable polar state is

\[
\langle |\delta c(q_\perp, \omega)|^2 \rangle = \frac{2(\zeta^2 c_0^2 \sin^2 \phi + 16\eta^2 v_c^2 \sin^2 \phi)\Delta|q_\perp|^2}{16\eta^2\{(\omega - c_+)^2 + \kappa_+^2\}\{(\omega - c_-)^2 + \kappa_-^2\}} \tag{43}
\]

where \( -\text{Im}[\omega_\pm] = \kappa_\pm \) and \( \text{Re}[\omega_\pm] = c_\pm \) with both \( c_\pm \) and \( \kappa_\pm \) being \( O(|q_\perp|) \). Here, \( \Delta \) is the strength of the non-conserved noise in \[35\] which enters the concentration equation via the motility and is dominant in the small wave number regime. Integrating \( \int_0^\infty (d\omega/2\pi)\langle |\delta c(q_\perp, \omega)|^2 \rangle \) over all frequencies yield the static structure factor of the concentration fluctuations:

\[
\langle |\delta c(q_\perp, t)|^2 \rangle = \frac{(\zeta^2 c_0^2 \sin^2 \phi + 16\eta^2 v_c^2 p_0^2 \sin^2 \phi)\Delta|q_\perp|^2(\kappa_+ + \kappa_-)}{16\eta^2\kappa_+\kappa_-\{(c_+ - c_-)^2 + (\kappa_+ + \kappa_-)^2\}} \sim \frac{1}{|q_\perp|} \tag{44}
\]
The $1/|q_\perp|$ divergence of the static structure factor of concentration fluctuations implies that the R.M.S. number fluctuations $\sqrt{\langle N^2 \rangle}$ in a region with $\langle N \rangle$ particles on average scales as $\langle N \rangle^{3/4}$, instead of as $\langle N \rangle^{1/2}$ as it would in all equilibrium systems not at a critical point, and thus violates the law of large numbers. This implies giant number fluctuations, which had earlier been discussed both for polar and nematic active systems in contact with substrates \[1\] \[\text{[20, 67, 71] even in this fully-momentum conserved system. However, in the PC model, the divergence of the concentration static structure factor is milder ($\sim 1/|q_\perp|$ instead of $\sim 1/q_\perp^2$) than in systems on substrates and, as a result, the number fluctuations are less violent (though still larger than in equilibrium and, therefore, giant) due to the coupling of the two-dimensional film with the bulk fluid medium.

Finally, (35) and (36) also yields the exact exponents of the polar phase. The dynamical exponent $z$, the anisotropy exponent $\mu$ and the roughness exponent $\chi$ remain unchanged from Sec. \[\text{[14]}\] $z = \mu = 1$, $\chi = -1/2$. Defining $\chi_c$ as the exponent characterising the concentration fluctuations via $\delta c \to b^\kappa \delta c$, it is clear from (44) that $\chi_c = -1/2$ as well. This is expected since (35) and (36) imply that concentration and angular fluctuations scale the same way. As in Sec. \[\text{[11]}\] there is no relevant nonlinearity: the lowest order nonlinearities, arising either from the flow couplings or advection or motility in either equation must scale as $q_\perp (\theta^2)_{q_\perp}$, $q_\perp (\delta c^2)_{q_\perp}$ or $q_\perp (\delta c \theta)_{q_\perp}$, all of which are evidently irrelevant. Together, this implies that due to motility, polar active particles living at an interface between two fluids or the surface of a fluid, form stable flocks displaying giant number fluctuations in the PC model, despite momentum conservation.

VI. CONCLUSIONS

In this article, I have presented a comprehensive study of interfacial uniaxial ordering in a variety of geometries. I have shown that when active particles can diffuse into the bulk fluid (NNC and PNC models), LRO, two-dimensional nematic and polar order can be associated with an interface between two fluids or a boundary of a bulk fluid. When active units are constrained to be at the interface, nematic order is not possible (NC model) but two-dimensional LRO polar order can still exist (PC model), with giant number fluctuations, at a two-fluid interface or at a boundary layer of a bulk fluid. In appendix \[\text{[F]}\] I further demonstrate that small fluctuations of the interface cannot generically destroy this ordering. This is particularly relevant for order on membranes immersed in a bulk fluid.

This result is surprising because it was popularly believed that Simha-Ramaswamy instability forbids uniaxial ordering in all momentum-conserved active systems, leading to permanently evolving, spatio-temporally chaotic states. The examination of surface-associated ordered states in momentum-conserved systems demonstrates that it only forbids bulk uniaxial ordering. While in bulk fluids, the active stress leads to a wavenumber-independent growth rate of orientational fluctuations – a sort of “negative mass” as in gravitational Jeans instability – because of incompressibility, at two-dimensional interfaces in three-dimensional fluids, which are not incompressible, because fluid can escape into the third dimension, it stabilises order by suppressing orientational fluctuations. As a result, surface-associated active polar and nematic states display two-dimensional long-range order, even in momentum-conserved fluids.

I now discuss some possible experimental realisations of these results. Self-assembled motor-microtubule layers at an oil-water interface \[\text{[11, 42, 100]}\] or in a bulk fluid \[\text{[43, 44]}\] have been one of the mainstays of studying active pattern formation. Here, the number of capped microtubule filaments at the interface is essentially constant and nematic state should be generically unstable. Further, due to the surfactants present in these systems \[\text{[93, 101]}\] the interfacial layer is essentially incompressible in two dimensions. Therefore, the theory presented by \[\text{[45, 46]}\], which is equivalent to the $A_\tau \to \infty$ case considered in Sec. \[\text{[IV]}\] in this article, describes the instability of the interfacial nematic state in these experiments. However, the value of $A_\tau$ in these experiments can, in principle, be varied by not using surfactants and lowering concentrations of microtubule filaments. In that case, the activity-independent growth rate predicted for interfacial order with a small $A_\tau$, in Sec. \[\text{[IV]}\] may be observed in this system. More radically, a nematic phase may be realised in this setup by allowing the microtubule filaments to associate and dissociate at the interface and the monomers to diffuse in the fluid. A further complication is that most of these experiments \[\text{[11, 44, 98, 100]}\] are performed in a geometry where the film or the two-fluid interface is in a channel that is tightly confined in the $z$ direction. This cuts-off the long-range hydrodynamic interactions and may suggest that the stabilisation mechanism described in this article may not operate at in-plane scales larger than the height of the channel, implying that the uniaxial phase is destabilised at large enough in-plane scales. I demonstrate in appendix \[\text{[F]}\] that this not the case; a stable uniaxial phase can be realised at arbitrarily high values of activity even in a film floating in a confined channel or at a two-fluid interface in a channel. However, the suppression of long-range fluid interaction by the boundaries qualitatively modifies the decay rate of the angular fluctuations which now scales as $\sim q_\perp^2$, implying that orientational order observed in this geometry would only be quasi-long-ranged as opposed to long-ranged.

Beyond experiments on cellular extracts, the theory presented here may also be tested in bacterial fluids. Bacteria can aggregate at air-water interfaces \[\text{[43, 51]}\] and may be a natural candidate for discovering and examining the stable LRO, boundary-associated active nematic phase predicted in Sec. \[\text{[11]}\]. Interfacial active nematics can also be created by impregnating passive interfacial...
nematics with bacteria.

Active polar phases may be associated with membranes in cellular systems and can have important consequences for signalling and transport. In this context, polar ordering of short actomyosin filaments associated with the cell membrane immersed in a bulk, cytoskeletal, cortical fluid may be especially relevant since they have been argued to be involved in the active transport leading to nanoclustering of cell-surface molecules.

A route to spontaneous generation of surfaces is via phase separation which is common in active systems and this paper demonstrates that such surfaces can naturally host a flocking wetting layer. A more intriguing possibility for interface associated active uniaxial order may arise in protocells. In active systems, small droplets can form due to the segregation of molecules in a complex mixture by phase separation [103–105]. Further, these droplets can grow and separate due to chemical activity thus forming protocells. If some of the components in these protocells are elongated, they can, in principle, form a stable and aligned boundary layer at the edge of the protocell leading to the creation of an aligned protocortex.

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Appendix A: Noise stemming from the velocity coupling in the angle and concentration equations

In this appendix, I will calculate the noises that enter the angle field equation and the concentration equation due to the coupling with the velocity field. The Stokes equation has a conserving noise that I ignored in the main paper. I will now show that it is irrelevant. For simplicity, I will consider an immersed interface in a fluid with the same value of viscosity both above and below the interface. As shown in the main paper, the results for an interface between two dissimilar fluids or an air-fluid boundary have the same form except that the expression of the effective viscosity η in terms of the viscosities of the fluids above and below the interface (or only below the interface for the boundary of a bulk medium) are modified. I consider a spatiotemporally uncorrelated noise in the velocity equation. The equations of motion of the bulk three-dimensional fluid are

\[- \eta \nabla^2 \mathbf{v} = - \nabla \Pi + \mathbf{f} \delta(z) + \xi^v,\]

where \(\mathbf{f}^v\) is the surface force density, the noise \(\xi^v\) has the correlation \(\langle \xi^v(r,t)\xi^v(r',t') \rangle = -2\Delta v^3 \nabla^2 \delta(r-r')\delta(t-t')\) and I have suppressed the in-plane viscosities (i.e., effectively assumed that the Saffman-Delbrück lengths \(l_p\) are 0). The in-plane velocity fields at \(z = 0\), including the stochastic part, are obtained the integrating (A1) over all \(q_z\):

\[
v_x = \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} f_{q_z}^x q_y + f_s^x(q_y^2 + q_z^2) + \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \frac{1}{\eta q^4} [q_x (\xi^v_x q_y + \xi^v_y q_z) - \xi^v_y (q_y^2 + q_z^2)] = \frac{f_s^x (q_y^2 + 2q_z^2) - f_s^z q_z q_y + \xi^v_x}{4\eta |q_x|^3},
\]

\[
v_y = \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} f_{q_z}^y q_y + f_s^y(q_y^2 + q_z^2) + \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \frac{1}{\eta q^4} [q_y (\xi^v_x q_z + \xi^v_y q_y) - \xi^v_y (q_x^2 + q_z^2)] = \frac{f_s^y (q_x^2 + 2q_z^2) - f_s^z q_z q_y + \xi^v_y}{4\eta |q_y|^3},
\]

and

\[
V_z|_{(z=0)} = - \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} q_z^2 \mathbf{f}^x + \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \frac{1}{\eta q^4} [q_x (\xi^v_x q_x + \xi^v_y q_y) - \xi^v_y q_z^2] = \tilde{\xi}^v_x,
\]

where \(\mathbf{q} \equiv (q_x, q_z)\). Note that the non-stochastic part of the interfacial velocity field is exactly the one obtained in the main paper. The correlations of the noises in these equations are

\[
\langle \xi^v_x(q_x, t)\tilde{\xi}^v_x(q_x', t') \rangle = \frac{2\Delta v^3 (2q_x^2 + q_z^2)}{4\eta^2 |q_x|^3} \delta(q_x + q_x') \delta(t-t'),
\]

\[
\langle \xi^v_x(q_x, t)\tilde{\xi}^v_y(q_y', t') \rangle = \frac{2\Delta v^3 q_x q_y}{4\eta^2 |q_x|^3} \delta(q_x + q_y') \delta(t-t'),
\]

\[
\langle \xi^v_x(q_x, t)\tilde{\xi}^v_y(q_y', t') \rangle = \frac{2\Delta v^3 q_x q_y}{4\eta^2 |q_x|^3} \delta(q_x + q_y') \delta(t-t').
\]
with cross-correlations between $\bar{\xi}_x$ or $\bar{\xi}_y$ and $\bar{\xi}_z$ being 0.

This vanishes at small $q_\perp$, in all directions and, therefore, it is subdominant to the non-conserving noise in the angle field $\xi$ which is retained in the main paper. This also implies that interfacial passive nematic order cannot be facilitated by an active but isotropic interfacial force density, unlike in bulk fluids [106].

The noise $\bar{\xi}_c$ that enters the concentration equation due to the velocity coupling $\propto -c_0 \nabla_\perp \cdot v$ has the correlation

$$
\langle \bar{\xi}(q_\perp,t)\bar{\xi}(q'_\perp,t') \rangle = \frac{2\Delta''\bar{\xi}(q_\perp)\bar{\xi}(q'_\perp)|q_\perp + q'_\perp|\delta(t-t')}{4\bar{\xi}^2|q_\perp|^3\delta(\omega + \omega')}.
$$

Even this coloured noise doesn’t qualitatively modify the angular fluctuations. Writing the relaxation rate of angular fluctuations in [21] as $-\kappa(\phi)|q_\perp|$, the static structure factor of angular fluctuations becomes

$$
\langle \theta(q_\perp,t)^2 \rangle = \frac{\Delta''}{\kappa(\phi)|q_\perp|} + \frac{\Delta''\bar{\xi}(q_\perp)}{8\bar{\xi}^2|q_\perp|^4\kappa(\phi)[1 + \tau_\alpha|q_\perp|\kappa(\phi)]}.
$$

For $\tau_\alpha|q_\perp|\kappa(\phi) \ll 1$, which is always realised for small enough $|q_\perp|$ (since $\kappa(\phi) > 0$ for all $\phi$ for stable interfacial order), the second term does not diverge at small $|q_\perp|$ unlike the first one. Therefore, it can be neglected in comparison to the first implying that bulk active fluctuations are irrelevant for the hydrodynamic properties of the interfacial phase. Spatially correlated active noises can also be treated the same way and also turns out to be irrelevant. Similar considerations apply to the noise in the concentration equation $\bar{\xi}_c$.

**Appendix B: Detailed derivation of the three-dimensional velocity fields in terms of the interfacial velocity field**

In this appendix, I present the detailed steps required to obtain the mobility tensor $\bar{\epsilon}^{\perp}$ used in the main text. I consider a thin fluid film containing active units at the flat interface of two incompressible fluids with viscosity $\eta_t$ and $\eta_b$. The equations of motion of the two fluids and the boundary conditions are given by (1) and (2). The incompressibility constrain implies that $\nabla^2 \Pi_t = \nabla^2 \Pi_b = 0$ and $\nabla \times \bar{V}_t = \nabla \times \bar{V}_b = 0$. Fourier transforming in the $x$ and $y$ directions, transverse to the interface, this yields

$$
(q_4^2 + \partial^2_x - 2q_2^2)\bar{V}_{tq}(z) = 0,
$$

and

$$
(q_4^2 + \partial^2_x - 2q_2^2)\bar{V}_{bq}(z) = 0,
$$

where the subscript $q$ is explicitly retained. These equations have to be solved with boundary conditions

$$
\bar{V}_{tq}(0) = \bar{V}_{bq}(0) = \bar{v},
$$

and $\lim_{z \to \infty} \bar{V}_{tq}(z) = 0$ and $\lim_{z \to -\infty} \bar{V}_{bq}(z) = 0$. The general solutions of this form are

$$
\bar{V}_{tq}(z) = e^{\pm|q_\perp|z}(\bar{v} + z\bar{T}),
$$

and

$$
\bar{V}_{bq}(z) = e^{\pm|q_\perp|z}(\bar{v} + z\bar{B}),
$$

where $\bar{T}$ and $\bar{B}$ are $q_\perp$-dependent quantities which have to be determined. Taking the curl of (1) and (2) and using these solutions yield

$$
(q_4^2 + \partial^2_x - 2q_2^2)\bar{V}_{tq}(z) = 0
$$

$$
\implies (-|q_\perp|\hat{z} + iq_\perp) \times \bar{T} = 0
$$

$$
\implies q_\perp \times \bar{T} = 0
$$

(B5)

and similarly,

$$
(-|q_\perp|\hat{z} + iq_\perp) \times \bar{B} = 0
$$

$$
\implies q_\perp \times \bar{B} = 0
$$

(B6)
i.e., both \( \mathbf{T}_\perp \) and \( \mathbf{B}_\perp \) must be vectors parallel to \( \mathbf{q}_\perp \). Taking the divergence of the velocity fields yield
\[
(i\mathbf{q}_\perp \cdot \mathbf{v} + T_z) + (i\mathbf{q}_\perp \cdot \mathbf{T}_\perp - T_\perp|q_\perp|)z = 0 \tag{B7}
\]
and
\[
(i\mathbf{q}_\perp \cdot \mathbf{v} + B_z) + (i\mathbf{q}_\perp \cdot \mathbf{B}_\perp + B_\perp|q_\perp|)z = 0, \tag{B8}
\]
This implies that \( T_z = -i\mathbf{q}_\perp \cdot \mathbf{v}, B_z = -i\mathbf{q}_\perp \cdot \mathbf{v} \)
\[
\mathbf{q}_\perp \cdot \mathbf{T}_\perp = -\mathbf{q}_\perp \cdot \mathbf{v}|q_\perp| \implies \mathbf{T}_\perp = -\frac{\mathbf{q}_\perp \mathbf{q}_\perp}{|q_\perp|} \cdot \mathbf{v}, \tag{B9}
\]
where I have used the fact that \( \mathbf{T}_\perp \) is a vector parallel to \( \mathbf{q}_\perp \) in the last step, and \( \mathbf{B}_\perp = \mathbf{q}_\perp \frac{\mathbf{v}}{|q_\perp|} \). Therefore, the velocity field at the top and the bottom of the interface can be expressed in terms of the interfacial velocity field as
\[
\mathbf{V}_{tg}(z) = e^{-i|q_\perp|z}[\mathbf{v} - z(i\hat{z} + \hat{q}_\perp)](\mathbf{q}_\perp \cdot \mathbf{v}) \tag{B10}
\]
and
\[
\mathbf{V}_{bg}(z) = e^{i|q_\perp|z}[\mathbf{v} - z(i\hat{z} - \hat{q}_\perp)](\mathbf{q}_\perp \cdot \mathbf{v}) \tag{B11}
\]
which are (3) and (4) of the main text, respectively. This further implies that
\[
\partial_z \mathbf{V}_{tg}|_{z=0} = -\frac{q_\perp^2 I + q_\perp \mathbf{q}_\perp}{|q_\perp|} \cdot \mathbf{v} - i\mathbf{q}_\perp \cdot \mathbf{v} \hat{z} \tag{B12}
\]
and
\[
\partial_z \mathbf{V}_{bg}|_{z=0} = \frac{q_\perp^2 I + q_\perp \mathbf{q}_\perp}{|q_\perp|} \cdot \mathbf{v} - i\mathbf{q}_\perp \cdot \mathbf{v} \hat{z} \tag{B13}
\]
where \( I \) is the 2 \times 2 identity matrix and the \( \hat{z} \) component could have been directly obtained by the three-dimensional incompressibility constraint and the fact that \( \mathbf{V}_{tg}(z = 0) = \mathbf{V}_{bg}(z = 0) = \mathbf{v}. \) The mobility tensor used in the main text directly follows from this calculation.

Appendix C: Higher order corrections to the angular dynamics equations

In this appendix, I will calculate the higher order correction to the angular dynamics described in Sec. III for both the NNC and PNC models and present an explicit expression for \( K_r \). I will first calculate this for the NNC model (nematic) and, next, for the PNC model (polar).

1. Nematic

The correction to the Frank elastic coefficient in a nematic i.e., the \( O(q_\perp^4) \) correction to \(-\Gamma_0 K_{q_\perp}^z \) passive term in (21) arising from the fluid coupling is purely active. It arises from expanding the mobility (7) to \( O(q_\perp^0) \). Using (7) and the active stress, the \( O(q_\perp^0) \) term in (21) is \(-\Gamma_0 K_r(\phi)q_\perp^z \theta \)

\[
K_r = K + \frac{\zeta}{32\eta}\left[8\ell_s \cos 2\phi - \lambda(\ell_b + 5\ell_s - (\ell_b - 3\ell_s) \cos 4\phi)\right] \tag{C1}
\]
That is, a non-zero Saffmann-Delbrück length yields an active correction to the effective Frank elasticity. This activity-renormalised elasticity controls the wavelength of the fastest growing mode when the nematic state is linearly unstable.

The lowest order passive correction to the angular dynamics due to the coupling with the fluid arises only at \( O(q_\perp^2) \). The linearised passive surface force density due to director distortions is

\[
f^s_p = -\frac{1 + \lambda}{2} \frac{\partial_y \mathbf{v}_\perp^2 \theta \hat{x}}{\partial y} - \frac{1 - \lambda}{2} \frac{\partial_y \mathbf{v}_\perp^2 \theta \hat{y}}{\partial y}. \tag{C2}
\]
This yields \( O(q_\perp^2) \) term in the angular dynamics equation

\[
- \left[ q_\perp^2 (\lambda - 1)^2 + q_\perp^2 (\lambda + 1)^2 + 2q_\perp^2 g_\perp^2 \right] \frac{8\eta|q_\perp|^3}{K_{q_\perp}^z} \theta, \tag{C3}
\]
which is obviously always stabilising. Using (A9), I can now check that the static structure factor of angular fluctuations reduces to its equilibrium value, despite the coupling to the fluid, in the absence of activity. In a passive system, the angular fluctuations equation would be

\[
\partial_t \theta = \left[ q_\perp^2 (\lambda - 1)^2 + q_\perp^2 (\lambda + 1)^2 + 2q_\perp^2 g_\perp^2 \right] \frac{8\eta|q_\perp|^3}{K_{q_\perp}^z} \theta + \zeta + \xi \tag{C4}
\]

Taking \( \Delta = \Gamma_0 T \) and \( \Delta^r = \eta T \), where \( T \) is the temperature, as is required for passive systems,

\[
\langle \zeta(q_\perp, t)\xi(q_\perp', t) \rangle + \langle \xi(q_\perp, t)\xi(q_\perp', t) \rangle = 2T \delta(q_\perp + q_\perp')
\]
\[
\delta(t-t') \left[ q_\perp^2 (\lambda - 1)^2 + q_\perp^2 (\lambda + 1)^2 + 2q_\perp^2 g_\perp^2 \right] \frac{8\eta|q_\perp|^3}{K_{q_\perp}^z} + \Gamma_0 \right] \tag{C5}
\]

This clearly follows the fluctuation-dissipation relation since the noise correlator is \( 2T \) times the dissipative coefficient in front of \( \delta F/\delta \theta = K_{q_\perp}^z \theta \) in (C4) and the angular static structure factor is

\[
\langle \theta(q_\perp, t)^2 \rangle = \frac{T}{K_{q_\perp}^z} \tag{C6}
\]

This demonstrates that the long-range order of nematics is a consequence of activity – a passive nematic at the interface between two fluids only has quasi-long range order and its static correlator can be calculated from the equipartition theorem (as in all equilibrium systems).
2. Polar

The angular dynamics in a polar state has both relaxing and propagating parts both at \( O(q_{\perp}) \) [28] and at \( O(q_{\parallel}^2) \). The relaxational part at \( O(q_{\parallel}^2) \) is \(-\Gamma_{\theta} \kappa_{\parallel} \theta q_{\parallel}^2 \theta\) where \( \kappa_{\parallel} (\phi) \) is given by [31] with \( \Gamma_{\theta} \) replaced by \( \Gamma_{\parallel} \). The \( O(q_{\perp}^4) \) correction to the propagative part of the angular dynamics arises from the polar active stress with the coefficient \( \Gamma_{\parallel} \) [18] which leads to a surface force density 
\[
\gamma_{\parallel} [\partial_x \partial_y \partial_x + (\partial_x^2 + 2 \partial_y^2)] \theta |. 
\]
This, along with the \( O(q_{\perp}^{-1}) \) part of [7] leads to a correction to \( v_p \) in [21]:

\[
\omega_{\pm} = -\frac{1}{8 \eta} \left[ (c_0 \zeta_1 - \zeta) \cos 2 \phi + c_0 A_r - \frac{A_r \zeta}{2 \zeta_1} (1 + \cos^2 2 \phi) \right] \pm \sqrt{\left\{ (\zeta - c_0 \zeta_1) \cos 2 \phi - c_0 A_r + \frac{A_r \zeta}{2 \zeta_1} (1 + \cos^2 2 \phi) \right\}^2 + 4 \frac{c_0 \zeta}{\zeta_1} \{ \zeta_1 + A_r \cos 2 \phi \}^2} \]  
(D1)

Appendix D: Stable interfacial nematic composed of particles living at the interface

In Sec. IV it was mentioned that there is a special point in the parameter space at which the eigenfrequencies [31] are both stabilising. This special point is accessed when \( A_r > |\zeta_1|, \lambda = -A_r/\zeta_1 \) and \( |\zeta_1| < 0 \). In this case, the eigenfrequencies become

\[
\langle \theta(\mathbf{r}_\perp, t) \rangle^2 \approx S \int |q_{\perp}| d|q_{\perp}| \int \frac{1}{\delta \phi} (\delta \phi)^2 |q_{\perp}| + B^2 |q_{\perp}|^2 \]  
(D4)

where \( S \) and \( B \) are constants. I have extended the range of the angular integral to \( \pm \infty \) since it is dominated by \( \delta \phi \ll 1 \). The angular integral evaluates to

\[
\int_{-\infty}^{\infty} d\phi (\delta \phi)^2 |q_{\perp}| + B^2 |q_{\perp}|^2 = \frac{\pi}{B |q_{\perp}|^{3/2}} \]  
(D5)

and therefore, \( \langle \theta(\mathbf{r}_\perp, t) \rangle^2 \propto q_{\perp}^{-1/2} d|q_{\perp}| \) which clearly converges in the long wavelength limit, implying long-range order. However, unlike the nematic state in Sec. III this implies a roughness exponent \( \chi = -1/4 \) and not \( \chi = -1/2 \). Since the static structure factor is not isotropically \( 1/|q_{\perp}| \) in all directions, the anisotropy and dynamical exponents are also not 1.

The static structure factor of concentration fluctuations can be calculated similarly:

\[
\langle |\delta c(\mathbf{r}_\perp, t) |^2 \rangle = \frac{\zeta^2 c_0^2 \sin^2 2 \phi \Delta |q_{\perp}|^2}{16 \eta^2 (\kappa_{\parallel} + \kappa_{\perp})}. \]  
(D6)

This implies real space concentration fluctuations \( \langle |\delta c(\mathbf{r}_\perp, t) |^2 \rangle \propto q_{\perp}^{-1/2} d|q_{\perp}|. \) This implies that the R.M.S. number fluctuations in \( \sqrt{\langle \delta N \rangle^2} \) in a region containing an average \( N \) particles scales as \( N^{7/8} \). Since the angular fluctuations are softer than \( 1/|q_{\perp}| \), the concentration fluctuations are larger than \( N^{3/4} \) found for a polar flock composed of interface-associated swimmers in the PC model (see sec. V).

Therefore, the stable, long-range ordered active nematic formed by interface-associated nematogens and realised at this special point has distinct hydrodynamic...
properties. However, it is only realised for a perfect tuning of parameter values and is unlikely to be realised in any experimental system.

\[
n \cdot N = 0 = \frac{1}{\sqrt{1 + \delta n \cdot \delta n}} \frac{1}{\sqrt{1 + (\nabla_{\perp} h)^2}} \left[ \delta n_z - \delta n_y \partial_y h - (1 + \delta n_x) \partial_x h \right] = \delta n_z \approx \partial_x h
\]  

where the final approximate equality is obtained by retaining only the linear terms. The interface fluctuations are controlled by a free energy of the form \( F_{\text{int}} = (12) \int \delta x \left[ \frac{\zeta}{2} (\nabla_{\perp} h)^2 + \kappa (\nabla_{\perp}^2 h)^2 \right] \) where \( \zeta \) is the surface tension and \( \kappa \) is the bending modulus. Denoting \( \delta n_y = \theta \) for continuity of notation, the active force \( \sim \nabla_{\perp} \cdot (\delta n) = \partial_y \theta \hat{x} + \partial_x \theta \hat{y} + \partial_z^2 h \hat{z} \), to linear order, where I have used \( \delta n_z \approx \partial_x h \). Putting all these together, the linearised force balance equation, in the notation of Appendix A, is

\[
- \eta \nabla^2 V = - \nabla \Pi - \zeta (\partial_y \theta \hat{x} + \partial_x \theta \hat{y}) \delta(z)
\]

\[
+ \left( \zeta \partial_x^2 h + \frac{\delta F_{\text{int}}}{\delta h} \right) \dot{\delta}(z) + \xi_v, \quad \text{(E2)}
\]

while the dynamics of \( \theta \) is still described by \( \text{(20)} \) and equation of motion for the height fluctuations is simply

\[
\partial_t h = V_z |_{z=0}.
\]

Solving for the velocity field as in Appendix A, \( v_x = -(i \zeta q_x^2/2\eta q_{\perp}^3) \theta + \xi_v \) and \( v_y = -(i \zeta q_y^2/2\eta q_{\perp}^3) \theta + \xi_v \) which yields \( \text{(21)} \) (ignoring, as ear-

**Appendix E: Effect of interfacial fluctuations on the ordered state**

All the calculations presented in the article assumed a flat interface or surface and tacitly ignored fluctuations of the interface. In this appendix, I will demonstrate that small interfacial fluctuations do not affect the ordered state. For simplicity, I will consider an immersed interface with viscosity-matched fluids above and below the interface with viscosity \( \eta \) and consider only nematic order with nematogens free to diffuse out of the interface. This can be generalised easily to all the cases considered in the article without any modification of the qualitative results [107] beyond the definition of \( \eta \) (as discussed in the article, \( \eta = (\eta_1 + \eta_b)/2 \) for two dissimilar fluid with \( \eta_b \) being the viscosity of the fluid above the interface and \( \eta_1 \) being the viscosity of the fluid below the interface and \( \eta = \eta_b/2 \) for the boundary of a bulk fluid). The order is confined to the tangent plane and since the interfacial fluctuations are assumed to be small, I use the Monge gauge to parametrise the membrane displacement away from the \( z = 0 \) plane; that is, a point on the interface is parametrised by the three-dimensional position vector \( \mathbf{R} = (x, y, h(x, y)) \); see Fig. 7. With this parametrisation, the normal to the interface is \( \mathbf{N} = \frac{-\hat{x} + \hat{y} h}{\sqrt{1 + (h')^2}} \). The three component nematic director, with in plane order along \( \hat{x} \), is denoted by \( \mathbf{n}(\mathbf{r}_{\perp}, t) = (n_x, n_y, n_z) = \frac{-\hat{x} + \hat{y} h}{\sqrt{1 + (h')^2}} \). Since the director is confined to the tangent plane, \( n_z \approx 0 \) in comparison to \( \zeta \) and

\[
V_z(z = 0) = -\frac{1}{4\eta |q_{\perp}|} (\zeta q_x^2 + \kappa q_x^4 - \zeta q_y^2) h + \tilde{\xi}_v. \quad \text{(E3)}
\]

This implies that to linear order, the angular dynamics is unaffected by the height fluctuations, while the linearised equation for the height fluctuations is

\[
\partial_t h = -\frac{1}{4\eta |q_{\perp}|} (\zeta q_x^2 + \kappa q_x^4 - \zeta q_y^2) h + \tilde{\xi}_v, \quad \text{(E4)}
\]

i.e., activity yields an effective surface tension like term for fluctuations along the ordering direction. Therefore, a flat interface is destabilised for \( \zeta > \zeta_c \) i.e. when extensile activity is greater than the surface tension. In that case, the interface may have an undulated conformation as in [49]. This situation is likely to be realised for an ordered phase on a self-assembled membrane. Conversely, when \( \zeta < \zeta_c \), the flat interface results described in this article are realised. In this case, the static structure factor of

FIG. 7. Order on a fluctuating interface or immersed surface in a fluid. The small surface fluctuations are parametrised by the height field \( h \) which measures the local displacement of the surface from a flat, fiducial one.
height fluctuations is
\[
\langle |h(q_\perp, t)|^2 \rangle = \frac{\Delta^v}{\eta(q_\perp^2 + \kappa q_\perp^4 - \zeta q_\perp^2)}. \tag{E5}
\]

This implies that small fluctuations of an interface or a boundary cannot modify the conclusions regarding interfacial order described in this article. In particular, nematically ordered fluids above and below the interface or the film, with viscosity \( \eta \), which can be easily generalised to fluids with two different viscosities, without any qualitative modification of the results, by simply replacing the viscosity with \( (\eta_b + \eta)/2 \). I consider a film floating at \( z = 0 \) in channel extending from \( z = -H \) to \( z = -H \) i.e., of height \( 2H \) along the \( z \) direction. The constitutive equation for the three-dimensional velocity field is

\[
\eta \nabla^2 \mathbf{V} = \nabla \mathbf{h} + \zeta (\partial_y \theta \hat{x} + \partial_x \theta \hat{y}) \delta(z) \tag{F1}
\]

while the dynamics of \( \theta \) is still described by (20). The channel geometry is effectively taken into account by taking \( V = 0 \) for \(-2\pi/H < q_s < 2\pi/H \). Therefore, the integral over \( q_s \) to obtain \( v_x \) and \( v_y \) will not be from \( (-\infty, \infty) \) but from \( (-\infty, -2\pi/H) \cup (2\pi/H, \infty) \). This yields

\[
v_x = \int_{-\infty}^{\infty} \frac{dq_{x}}{2\pi} f_x(q_x q_y + f_x^2(q_y^2 + q_z^2)} - \int_{-2\pi/H}^{2\pi/H} \frac{dq_x}{2\pi} f_x(q_x q_y + f_x^2(q_y^2 + q_z^2)} = -\frac{f_x^2 q_x q_y + f_x^2 (q_y^2 + 2q_y^2)}{4\eta |q_s|^3} \\
+ \frac{H q_x f_s \cdot q_s}{\eta q_s^2 (4\pi^2 + H^2 q_s^2)} - \tan^{-1} \left( \frac{2\pi}{H |q_s|} \right) \frac{-f_x^2 q_x q_y + f_x^2 (q_y^2 + 2q_y^2)}{2\eta |q_s|^3} \tag{F2}
\]

\[
v_y = \int_{-\infty}^{\infty} \frac{dq_y}{2\pi} f_y(q_x q_y + f_y^2(q_x^2 + q_z^2)} - \int_{-2\pi/H}^{2\pi/H} \frac{dq_y}{2\pi} f_y(q_x q_y + f_y^2(q_x^2 + q_z^2)} = -\frac{f_y^2 q_x q_y + f_y^2 (q_x^2 + 2q_x^2)}{4\eta |q_s|^3} \\
+ \frac{H q_y f_s \cdot q_s}{\eta q_s^2 (4\pi^2 + H^2 q_s^2)} - \tan^{-1} \left( \frac{2\pi}{H |q_s|} \right) \frac{-f_y^2 q_x q_y + f_y^2 (q_x^2 + 2q_x^2)}{2\eta |q_s|^3} \tag{F3}
\]

where \( f_x^2 = -i\zeta q_y \theta \) and \( f_y^2 = -i\zeta q_x \theta \). Using these values for \( v \) in (20), I recover (21) for \( H \gg 1/|q_s| \), i.e., when the in-plane scales are much smaller than the confinement scale, as expected. Interestingly, for \( H \ll 1/|q_s| \) i.e., for in-plane scales much larger than the confinement height, the angular dynamics becomes

\[
\partial_t \theta = \left[ \frac{H q_s^2 \zeta (\cos 2\phi - \lambda)}{4\pi^2 \eta} - K q_s^2 \right] \theta, \tag{F4}
\]

where \( \phi \) is the angle between the ordering direction, \( \hat{x} \), and \( q_s \). The first term is stabilising when \( \lambda \zeta > 0 \) and \( |\lambda| > 1 \) which is exactly the condition for stability of the orientational order in an unbounded fluid. Therefore, orientational order in a film floating in a channel [43] or at a two-fluid interface in a channel can remain stable even at arbitrarily high active drive. However, due to the long-range fluid interactions being cut-off by the confinement at the scale \( H \), the relaxation rate of angular fluctuations now scales as \( \sim q_s^2 \) for \( |q_s| \ll 1/H \) instead of as \( |q_s| \) as it would in an unbounded system. This implies that the stable nematic phase in a film or at a two fluid interface in a channel is only quasi-long range ordered instead of being truly long ranged.

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should be compressible at large-enough scales due to dimensional medium, whether surfactant-coated or not, only when the surface is treated with surfactants. However, all two-dimensional films in a three-dimensional medium, whether surfactant-coated or not, should be compressible at large-enough scales due to fluctuations. In fact, if there were no constraint of incompressibility, the bulk velocity would be proportional to $\lambda |\dot{\theta}|$ where $\lambda$ is a bulk active stress, in a two-dimensional fluid which would lead to relaxation rate for angular fluctuations $\lambda |\dot{\theta}|$, where $\lambda$ is a bulk active stress, in a two-dimensional fluid which would lead to relaxation rate for angular fluctuations $\lambda |\dot{\theta}|$. This relaxation rate is positive and hence stabilising when $\lambda > 1$ and $|\dot{\theta}| > 1$. A similar effect stabilises interfacial flow because the interface becomes effectively incompressible only when the surface is treated with surfactants. However, all two-dimensional films in a three-dimensional medium, whether surfactant-coated or not, should be compressible at large-enough scales due to fluctuations. In fact, if there were no constraint of incompressibility, the bulk velocity would be proportional to $\lambda |\dot{\theta}|$, where $\lambda$ is a bulk active stress, in a two-dimensional fluid which would lead to relaxation rate for angular fluctuations $\lambda |\dot{\theta}|$. This relaxation rate is positive and hence stabilising when $\lambda > 1$ and $|\dot{\theta}| > 1$. A similar effect stabilises interfacial order.
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