Hybrid phenomenology in a chiral approach

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Abstract We calculate masses and decays of the (lightest) hybrid nonet with exotic quantum numbers $J^{PC} = 1^{-+}$ and the nonet of their chiral partners with $J^{PC} = 1^{++}$ in the framework of the extended Linear Sigma Model (eLSM). As an input, we identify $\pi_{1}^{hyb} = \pi_{1}(1600)$ as a low-lying hybrid. We investigated interaction terms which fulfill chiral symmetry. For what concerns $\pi_{1}^{hyb}$, the most important decays are $\pi_{1}^{hyb} \rightarrow b_{1}\pi, \pi_{1}(1600) \rightarrow \rho\pi\eta, \pi_{1}^{hyb} \rightarrow \rho\pi$, and $\pi_{1}^{hyb} \rightarrow K K^{*}(892)$. The decays $\pi_{1}^{hyb} \rightarrow \eta\pi$ and $\pi_{1}^{hyb} \rightarrow \eta'\pi$ are expected to be small but nonzero: they follow from a chirally symmetric interaction term that breaks explicitly the axial anomaly. For all the other members of the two hybrid nonets (for which no experimental candidates exist yet), we report decay ratios that may guide ongoing and future experiments.

1 Introduction

The search for hybrids is an important part of experimental as well as theoretical hadronic physics, see, e.g., Refs.\textsuperscript{[1,2]} for reviews. Lattice QCD predicts a rich spectrum of hybrids below 5 GeV\textsuperscript{[3–8]}, but up to now no predominantly hybrid state could be unambiguously assigned to one of the mesons listed in the PDG\textsuperscript{[9]}. Yet, two states with “exotic”\textsuperscript{1} quantum numbers $J^{PC} = 1^{-+}$ are listed below 2 GeV: $\pi_{1}(1400)$ and $\pi_{1}(1600)$. Recent results by COMPASS confirmed the state $\pi_{1}(1600)$ and led to a revival of interest in this topic\textsuperscript{[10]}. At the Jefferson Lab (JLAB), the GlueX\textsuperscript{[11]} and CLAS12\textsuperscript{[12]} experiments are actively searching for more states. At the ongoing BESIII experiment\textsuperscript{[13–15]}, hybrids can be deter-

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\textsuperscript{1} Here and in the following we use the term “exotic” to indicate quantum numbers that are not possible for quark–antiquark states in the nonrelativistic quark model. The term “crypto-exotic” is reserved for mesons with non-exotic quantum numbers, but valence content is beyond the nonrelativistic quark model, such as hybrids, glueballs, and tetraquarks.
mined through decays of charmonia. In the future, one expects new insights by the Panda experiment at FAIR [16].

In the context of flavor multiplets, besides the hybrid meson $\pi_1$, one expects a full nonet of such states. Hence also the kaonic state $K_1$ and two isoscalar states $\eta_1$ should exist. In frameworks based on chiral symmetry, an additional nonet of chiral partners should also emerge: these are so-called pseudovector crypto-exotic hybrid states with quantum numbers $J^{PC} = 1^{+-}$. Based on the success of chiral models in the ordinary meson sector, it seems natural to study hybrids in such a framework, in particular since to our knowledge this has not yet been done before.

In this work, we use the so-called extended Linear Sigma Model (eLSM) [17–19] for this purpose. The eLSM masses and decays of a range of hadrons up to and above 2 GeV have been described in Refs. [17,20]. In particular, in Ref. [17] a fit to various experimental quantities has shown that a good description of PDG data is achieved. In addition, this fit allowed to fix univocally the parameters of the model and thus to make other predictions/postdictions. Besides conventional $\bar{q}q$-states, various non-conventional gluonic mesons were already studied in the eLSM. The scalar glueball appears naturally in the eLSM as a consequence of dilatation invariance as well as its anomalous breaking. The resulting dilaton/glueball field mixes with conventional light mesons and, as shown in Ref. [18], is predominantly contained in the resonance $f_0(1710)$. The eLSM has been also applied to the study of the pseudoscalar glueball(s) [21–23] and the vector glueball [24]. Moreover, in the low-energy domain the eLSM has been shown to be compatible with chiral perturbation theory [25] for what concerns low-energy pions (most notably, pion–pion scattering). On the other edge, the inclusion of charmed mesons was presented in Refs. [26,27].

The eLSM has been also successfully applied in the baryonic sector within the so-called mirror assignment [28–31], where pion–nucleon scattering and baryonic decays turn out to be in agreement with data. One additional advantage of the eLSM is that it can be easily employed at finite temperature [32,33] and density [31,34,35], allowing for the description of the chiral phase transition in the medium.

As discussed in Refs. [17,18], the general strategy regarding the Lagrangian construction in eLSM involves implementing symmetries of relevance for dynamics of low-energy QCD, in particular dilatation and chiral and dilatation ones and their breaking patterns [17,18]. Dilatation (or scale) invariance is a symmetry of the classical QCD Lagrangian that holds in the chiral limit (that is, when setting the bare masses of the quarks $u, d, s$ to zero). This symmetry is broken by quantum fluctuations and by the running coupling which decreases for increasing energy (asymptotic freedom), see, e.g., Ref. [36]: as a consequence, an energy scale $\Lambda_{QCD}$ emerges. At the hadronic level, these features are described by a dilaton/glueball field, which mimics the breaking of dilatation invariance through an appropriate logarithmic potential that involves an energy scale $\Lambda$ [37,38]. We assume that—in the chiral limit and neglecting the second anomaly of QCD, the chiral or axial anomaly (see below)—this is the only way dilatation invariance is broken. As a consequence, in this limit all other interaction terms are dilatation invariant: this requirement strongly constrains the possible terms that can be included in the eLSM Lagrangian. Next, it is also required that the eLSM embodies another key feature of QCD in the low-energy domain: chiral symmetry, based on the right- and left-handed groups $U_R(3) \times U_L(3)$, and its spontaneous breaking into $U_V(3)$. The pions and kaons appear as quasi-Goldstone bosons and condensate of scalar fields (scalar $\bar{q}q$ configuration) form in the vacuum. The masses of the chiral partners (such as scalar and pseudoscalar mesons, but also vector and axial-vector meson, as well as pseudovector and orbitally excited vector mesons) are not degenerate: the mass differences are proportional to the aforementioned chiral condensates. Finally, terms that are linked to the chiral (or axial
anomaly [39]: this is the second anomaly of QCD, responsible, for example, for the large mass of the \( \eta' \) meson) appear: they can break explicitly the \( U_A = R-L \) symmetry and/or include the Levi-Civita tensor in the interaction (Wess–Zumino-type terms [40]) shall also be added: they typically also break scale invariance and are important in some decay channels.

In this article, the eLSM setup is extended to hybrids by following the same strategy related to symmetries outlined above. We construct the chiral multiplet for the hybrid nonets with \( J^{PC} = 1^-+ \) and \( J^{PC} = 1^{+-} \) and determine the interaction terms which satisfy chiral symmetry. As a consequence, the spontaneous breaking of chiral symmetry is responsible for the mass differences between the low-lying \( 1^±+ \) exotic hybrids and the heavier \( 1^{+-} \) crypto-exotic hybrids, just as among standard quark–antiquark chiral partners. The possible decays of the hybrids in the two multiplets are described by four interaction terms. Two of these fulfill chiral dilatation invariance and therefore should be dominant. The third and the fourth terms break dilatation invariance and are linked to the chiral anomaly. In particular, the third term involves the Levi-Civita tensor and the fourth term breaks explicitly the axial anomaly \( U(1)_A \). We work out the resulting decays and identify promising channels for the experimental discovery of these states.

As mentioned above, two hybrid candidates \( \pi_1(1400) \) and \( \pi_1(1600) \) are listed in the PDG [9]. However, in the recent theoretical analysis of Ref. [41] it was suggested that these two states could correspond to a single resonant pole, with mass and width close to the original \( \pi_1(1600) \). Indeed, our chiral multiplet—just as other models and lattice studies—has space for only one such \( \pi_1 \)-state: we then adopt the interpretation of Ref. [41] and use the mass of the \( \pi_1(1600) \) as an input that fixes the masses of hybrids in our framework.

This paper is organized as follows: In Sect. 2 we present the standard quark–antiquark nonets in the eLSM and construct the new hybrid nonets and their transformation properties. In Sect. 3 we introduce the effective Lagrangian and discuss the interaction terms that lead to the hybrid decays. In Sect. 4 we present and discuss our results, and in Sect. 5 we outline our conclusions and outlook. Technical details of our calculations are relegated to several appendices.

2 Chiral multiplets

In this section, we briefly review the assignment of (pseudo)scalar, (axial-)vector and pseudovector fields, which are the basic ingredients of the eLSM. Then, we show how to build two nonets of hybrid states with quantum numbers \( J^{PC} = 1^-+ \) and \( J^{PC} = 1^{+-} \).

2.1 (Pseudo)scalar and (axial-)vector quark–antiquark multiplets

The nonets of (pseudo)scalar fields are introduced as

\[
P = \frac{1}{\sqrt{2}} \begin{pmatrix}
\eta_N + \pi^0 \\
\pi^+ \\
\pi^- \\
K^0 \\
K^-
\end{pmatrix},
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sigma_N + \sigma^0 \\
\sigma^+ \\
\sigma^- \\
K^+_S \\
K^-_S
\end{pmatrix}.
\]  

The matrix \( P \) contains the light pseudoscalar nonet \( \{\pi, K, \eta, \eta'\} \) with quantum numbers \( J^{PC} = 0^{++} [9] \), where \( \eta \) and \( \eta' \) arise via the mixing \( \eta = \eta_N \cos \theta_p + \eta_S \sin \theta_p, \eta' = -\eta_N \sin \theta_p + \eta_S \cos \theta_p \) with \( \theta_p \simeq -44.6^\circ [17] \). Using other values for the mixing angle such as \( \theta_p = -41.4^\circ [42] \) changes only slightly the results presented in this work. The matrix \( S \) contains the scalar fields \( \{a_0(1450), K^*_0(1430), \sigma_N, \sigma_S\} \) with \( J^{PC} = 0^{++} \). These
are identified with states above 1 GeV [17]: the nonstrange bare field \( \sigma_N \equiv |\bar{u}u + \bar{d}d| / \sqrt{2} \) corresponds predominantly to the resonance \( f_0(1370) \) and the bare field \( \sigma_S \equiv |\bar{s}s| \) predominantly to \( f_0(1500) \). As already indicated above, the state \( f_0(1710) \) is dominated by the scalar glueball. For details of the mixing see Ref. [18]. Evidence for a large gluonic component in \( f_0(1710) \) has also been found on the lattice [43] and in the holographic QCD study of Refs. [44–46].

In the eLSM, the nonet of the light scalar states \{\( a_0(980) \), \( K_0^0(700) \), \( f_0(500) \), \( f_0(980) \)\} turns out to be non-\( q\bar{q} \). One possibility is a nonet of light tetraquark states [47–57] and/or a nonet of dynamically generated states [58–61]). Moreover, these two configurations can mix with each other, making a clear distinction quite difficult. Nevertheless, there is an agreement toward the interpretation of the light scalar nonet as a nonet of states made up with four quarks.

The scalar and pseudoscalar matrices are combined into the matrix

\[
\Phi = S + i P ,
\]

which has a simple transformation under chiral transformations \( U_L(3) \times U_R(3) \): \( \Phi \to U_L \Phi U_R^\dagger \), where \( U_L \) and \( U_R \) are unitary \( U(3) \) matrices. Under parity \( P \) the matrix \( \Phi \) transforms as \( \Phi \to \Phi^\dagger \) and under charge conjugation \( C \) as \( \Phi \to \Phi' \). The matrix \( \Phi \) is used as a building block in the construction of the eLSM Lagrangian, see Appendix A and Tables 1 and 2. For a detailed report of the transformation properties, we refer to Ref. [62].

A comment on the currents is in order: the quantity \( \bar{q}i\gamma^5q \) is a pseudoscalar \( (J^{PC} = 0^{-+}) \), as a simple check on the transformations at the quark level shows. Then, in the framework of a nonrelativistic approach it corresponds to \( L = 0 \) and \( S = 0 \), for which \( P = (-1)^{L+1} = -1 \) and \( C = (-1)^{L+S} = +1 \). One may also (and in more detail) verify this correspondence by studying \( \bar{q}i\gamma^5q \) in the nonrelativistic limit, by taking the dominant components within the Dirac representation for the spinors. For what concerns the object \( \bar{q}q \), this is clearly a scalar object \( (J^{PC} = 0^{++}) \); in the nonrelativistic language, it is obtained by choosing \( L = 1 \) and \( S = 1 \) coupled to \( J = 0 \). Also in this case, even if not so obvious at a first sight, a nonrelativistic decomposition of \( \bar{q}q \) shows that \( L = 1 \) and the spin triplet configuration emerge.

We now turn to vector and axial-vector fields, described by:

\[
V^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\rho + \rho^0}{\sqrt{2}} & K^{\ast +} \\ \rho^- & \frac{\rho - \rho^0}{\sqrt{2}} K^{\ast 0} \end{pmatrix} \mu , \quad A^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} f_{1N} + a_0^0 & a_1^+ \\ a_1^- & f_{1N} - a_0^0 \end{pmatrix} \frac{K_{1A}^+}{\sqrt{2} K_{1A}^0} \frac{f_{1S}}{f_{1S}} \]

The elements of the matrix \( V^\mu \) are the vector states \{\( \rho(770) \), \( K^*(892) \), \( \omega(782) \), \( \phi(1020) \)\} with \( J^{PC} = 1^{-+} \), and the elements of the matrix \( A^\mu \) the axial-vector states \{\( a_1(1230) \), \( K_{1A} (1270) \), \( f_1(1285) \), \( f_1(1420) \)\} with \( J^{PC} = 1^{++} \). Here, \( K_{1A} \) is a mixture of the two physical states \( K_{1}(1270) \) and \( K_{1}(1400) \), see also Sect. 2.2. We neglect (the anyhow small) strange–nonstrange mixing; hence, \( \omega_N \equiv \omega(782) \) and \( f_{1N} \equiv f_1(1285) \) are regarded as purely nonstrange mesons of the type \( \sqrt{1/2}(\bar{u}u + \bar{d}d) \), while \( \omega_S \equiv \phi(1020) \) and \( f_{1S} \equiv f_1(1420) \) are regarded as purely \( \bar{s}s \) states.

Next, one defines the right-handed and left-handed combinations:

\[
R^\mu = V^\mu - A^\mu \quad \text{and} \quad L^\mu = V^\mu + A^\mu .
\]

Under chiral transformation they transform as \( R^\mu \to U_R R^\mu U_R^\dagger \) and \( L^\mu \to U_L L^\mu U_L^\dagger \). Details of the currents and transformations are shown in Tables 1 and 2.
The vector current $\bar{q}\gamma\mu q$ with $J^{PC} = 1^{--}$ corresponds to $L = 1$ and $S = 0$, while the axial-vector current $\bar{q}\gamma^5\gamma\mu q$ with $J^{PC} = 1^{++}$ to $L = 1$ and $S = 1$, coupled to $J = 1$. Again, these correspondences can be also verified by studying the nonrelativistic limits of the quark–antiquark currents $\bar{q}\gamma\mu q$ and $\bar{q}\gamma^5\gamma\mu q$.

The eLSM Lagrangian includes the multiplets $S$, $P$, $V$, and $A$ presented above. In addition, a dilaton/glueball field is also present in order to describe dilatation symmetry and its anomalous breaking. The details of the eLSM (together with its symmetries, most notably chiral and dilatation symmetries and their anomalous, explicit, and spontaneous breaking terms) are briefly summarized in Appendix A and extensively presented in Refs. [17,18] for $N_f = 3$. An extension to $N_f = 4$ can be found in Refs. [26,27] and a study of mesons at finite temperature can be found in Refs. [32,33].

### 2.2 Pseudovector and excited vector mesons

Since we are interested in hybrids, it is important to consider also the pseudovector mesons with quantum numbers $J^{PC} = 1^{+−}$ and the excited vector mesons with quantum numbers $J^{PC} = 1^{−±}$, since they are important decay products of hybrids. To this end we introduce the matrices (see Ref. [24] for technical details):

$$
B^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{b_1 + b_0}{\sqrt{2}} & \frac{b_1^+}{\sqrt{2}} & K_{1,B}^{++} \\
\frac{b_1 - b_0}{\sqrt{2}} & \frac{b_1^0}{\sqrt{2}} & K_{1,B}^{*0} \\
K_{1,B}^{1+} & K_{1,B}^{0+} & h_{1,S}
\end{pmatrix}
,$$

$$
V_E^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{\omega_{E,N} + \rho_E}{\sqrt{2}} & \frac{\rho_E^+}{\sqrt{2}} & K_{E}^{++} \\
\frac{\omega_{E,N} + \rho_E}{\sqrt{2}} & \frac{\rho_E^-}{\sqrt{2}} & K_{E}^{*0} \\
\rho_E & -\omega_{E,N} - \rho_E & \omega_{E,S}
\end{pmatrix}.
$$

(5)

Here, $B^\mu$ contains the pseudovector states $\{b_1(1230), K_{1,B}(h_{1170}), h_{1}(1380)\}$. In the quark model these states emerge from $L = 1$, $S = 0$ coupled to $J^{PC} = 1^{−±}$ (hence, pseudovector states as axial-vector states with negative $C$-parity). For simplicity, the strange–nonstrange isoscalar mixing is again neglected, thus $h_{1,N} \equiv h_{1}(1170)$ is a purely nonstrange state, while $h_{1,S} \equiv h_{1}(1380)$ is a purely strange–antistrange state. Note, these states are distinguished from the axial-vector states of Eq. (3) due to $C$-parity. However, $C$-parity does not apply for kaonic states and mixing arises. The kaonic fields $K_{1,A}$ from Eq. (3) and $K_{1,B}$ from Eq. (5) mix and generate the two physical resonances $K_1^{+(1270)}$ and $K_1^{+(1400)}$:

$$
\begin{pmatrix}
K_1^{+}(1270) \\
K_1^{+}(1400)
\end{pmatrix}^\mu = \begin{pmatrix}
\cos \varphi & -i \sin \varphi \\
-i \sin \varphi & \cos \varphi
\end{pmatrix}
\begin{pmatrix}
K_{1,A}^{+} \\
K_{1,B}^{+}
\end{pmatrix}^\mu.
$$

(6)

The mixing angle reads $\varphi = (56.3 ± 4.2)^\circ$ [63]. The same transformations hold for $K_0^{+}(1090)$ and $K_0^{+}(1400)$, while for the other kaonic states one has to take into account that $K_1^−(1270) = K_1^{+(1270)}$ and $\bar{K}_1^0(1270) = K_0^{+(1270)}$ (and so for $K_1^−(1400)$). Notice that the imaginary number $i$ is a consequence of the specific mixing term between the fields $K_{1,A}^+$ and $K_{1,B}^+$ fields, that must be invariant under charge-conjugation and parity, and the specific convention used in Ref. [63]. According to this setup, under $C$-transformation the field $K_1^{+}(1270)$ changes into $K_1^−(1270)$, while $K_1^{+}(1400)$ into $−K_1^−(1400)$. Since the kaonic fields are not eigenstates of the charge conjugation operator $C$, other choices are possible, which however do not change the physical results.

The chiral partners of the pseudovector mesons are excited vector mesons which arise from the combination $L = 2$, $S = 1$ coupled to $J^{PC} = 1^{−±}$. The corresponding fields listed are given by $\{\rho(1700)$, $K^*(1680)$, $\omega(1650)$, $\phi(1930)\}$. The experimental evidence of the first three states is compiled by the PDG, while the putative new state $\phi(1930)$ is expected to couple predominantly to $K$ and $K^*$ according to the study of Ref. [64]. The question mark
in \( \phi(1930) \) means that presently this state (and the corresponding mass of 1930 MeV) is only a theoretical prediction.

We then build the matrix

\[ \tilde{\Phi}^{\mu} = V_E^{\mu} - i B^{\mu}, \]

which under chiral transformations changes as \( \tilde{\Phi}^{\mu} \rightarrow U_L \tilde{\Phi}^{\mu} U_R^\dagger \) (it is a so-called heterochiral multiplet, just as the standard (pseudo)scalar \( \Phi \)), under parity as \( \tilde{\Phi}^{\mu} \rightarrow \tilde{\Phi}^{\dagger \mu} \), and under charge conjugations as \( \tilde{\Phi}^{\mu} \rightarrow - \tilde{\Phi}^{\dagger \mu} \), see Tables 1 and 2 for details. As shown in Ref. [65], further chiral multiplets can be built in an analogous way.

The currents for these fields involve derivatives. The pseudovector current \( \bar{q} \gamma^5 \partial^\mu q \) with \( J^{PC} = 1^{-+} \) corresponds to \( L = 1 \) and \( S = 0 \). Intuitively, it is obtained from \( \bar{q} \gamma^5 q \) (with \( L = 0, S = 0 \)) by adding a derivative, which increases the angular momentum of one unit; hence, \( L = 1 \). Similarly, the excited vector current \( \bar{q} \partial^\mu q \) with \( J^{PC} = 1^{--} \) corresponds to \( L = 2 \) and \( S = 1 \) coupled to \( J = 1 \). Again, it is obtained from the object \( \bar{q} q \) (which has \( L = S = 1 \)) upon adding the derivative, lifting \( L \) to 2. Thus, this is the nonet of orbitally excited vector mesons. Such statements can be checked, just as in the previous cases, by a nonrelativistic study of the quark–antiquark currents \( \bar{q} \gamma^5 \partial^\mu q \) and \( \bar{q} \partial^\mu q \).

### 2.3 Hybrid multiplets

In this subsection, we introduce hybrids. The currents of exotic hybrid states with quantum numbers \( J^{PC} = 1^{-+} \) are given by

\[ \Pi_{ij}^{hyb,\mu} = \frac{1}{\sqrt{2}} \tilde{q}_j G^{\mu \nu} \gamma_\nu q_i, \]

where \( G^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - g_{QCD} [A^\mu, A^\nu] \) is the gluonic field tensor. Thus, these currents can be understood as “vector currents with the addition of one gluon,” which is responsible for the switch of the \( C \)-parity. Note, the emerging quantum numbers are exotic (not allowed for a local quark–antiquark current). According to lattice QCD, these are the lightest hybrid states [3–6].

In other words, it is important to stress there is no way to build a local current with \( J^{PC} = 1^{-+} \) by combining a quark and an antiquark, such as \( \bar{q} \Gamma q \) where \( \Gamma \) is a combination of Dirac matrices and/or derivatives (see the previous subsection for specific examples). In the nonrelativistic language, one cannot construct a \( \bar{q} q \) object with \( J^{PC} = 1^{-+} \) [the conditions \( P = (-1)^{L+S} \) and \( C = (-1)^{L+S} \) together with \( J = |L - S|, \ldots, L + S \) cannot be fulfilled simultaneously]. Nevertheless, besides having exotic quantum numbers, we still have a nonet of states, just as for a regular quark–antiquark nonet. Thus, for what concerns the construction of effective models, the nonet \( \Pi_{ij}^{hyb,\mu} \) can be used as a building block of the model.

The chiral partners of \( \Pi_{ij}^{hyb,\mu} \) are the pseudovector states \( B_{ij}^{hyb,\mu} \), which have the quantum numbers \( J^{PC} = 1^{++} \) and are given by

\[ B_{ij}^{hyb,\mu} = \frac{1}{\sqrt{2}} \tilde{q}_j G^{\mu \nu} \gamma^5 \gamma_\nu q_i. \]

The quantum numbers are the same as for pseudovector mesons, even if the underlying currents are utterly different. We thus refer to these states as crypto-exotic mesons, since we do have a nonet of non-conventional hybrid states, but the quantum numbers are also allowed for normal quark–antiquark states (see above). A mixing of configurations with the same quantum numbers is in principle possible. Yet, in the present case the mass differences...
Table 1 Summary of the quark–antiquark and hybrid nonets and their properties

| Nonet     | $J^P_C$ | Current                                                                 | Assignment            | $P$         | $C$     |
|-----------|---------|-------------------------------------------------------------------------|-----------------------|-------------|---------|
| $P$       | 0$^-^+$ | $P_{ij} = \frac{1}{\sqrt{2}} \bar{q}_i j \gamma^5 q_j$                 | $\pi, K, \eta, \eta'$ | $-P(t, -x)$ | $P^I$   |
| $S$       | 0$^+^+$ | $S_{ij} = \frac{1}{\sqrt{2}} \bar{q}_j q_i$                            | $a_0(1450), K^*_0(1430), f_0(1370), f_0(1510)$ | $S(t, -x)$ | $S^I$   |
| $V^\mu$   | 1$^-^-$ | $V^\mu_{ij} = \frac{1}{\sqrt{2}} \bar{q}_j (\gamma^\mu q_i)$           | $\rho(770), K^*(892), \omega(785), \phi(1024)$ | $V_\mu(t, -x)$ | $-V^\mu,I$ |
| $A^\mu$   | 1$^+^+$ | $A^\mu_{ij} = \frac{1}{\sqrt{2}} \bar{q}_j \gamma^5 \gamma^\mu q_i$   | $a_1(1230), K_{1,A}, f_1(1285), f_1(1420)$ | $-A_\mu(t, -x)$ | $A^\mu,I$ |
| $B^\mu$   | 1$^+^+$ | $B^\mu_{ij} = \frac{1}{\sqrt{2}} \bar{q}_j \gamma^5 \gamma^5 \gamma^\mu q_i$ | $b_1(1230), K_{1,B}, h_1(1170), h_1(1380)$ | $-B_\mu(t, -x)$ | $-B^\mu,I$ |
| $V^\mu_E$ | 1$^-^-$ | $V^\mu_{E,ij} = \frac{1}{\sqrt{2}} \bar{q}_j \gamma^5 \gamma^\mu q_i$ | $\rho(1700), K^*(1680), \omega(1650), \phi(1930)$ | $V_\mu(t, -x)$ | $-V^\mu_E,I$ |
| $\Pi^{hyb,\mu}$ | 1$^-^+$ | $\Pi^{hyb,\mu}_{ij} = \frac{1}{\sqrt{2}} \bar{q}_j G^{\mu\nu} \gamma^\nu q_i$ | $\pi_1(1600), K_1(?)$, $\eta_1(?)$, $\eta_1(?)$ | $\Pi^{hyb}_\mu(t, -x)$ | $\Pi^{hyb,\mu,I}$ |
| $B^{hyb,\mu}$ | 1$^+^+$ | $B^{hyb,\mu}_{ij} = \frac{1}{\sqrt{2}} \bar{q}_j G^{\mu\nu} \gamma^\nu \gamma^5 q_i$ | $b_1(2000)$, $K_{1,B}(?)$, $h_1(?)$, $h_1(?)$ | $-B^{hyb}_\mu(t, -x)$ | $-B^{hyb,\mu,I}$ |
Table 2 Transformation properties of the chiral multiplets

| Chiral multiplet | Current | $U_R(3) \times U_L(3)$ | $P$ | $C$ |
|-----------------|---------|------------------------|-----|-----|
| $\Phi = S + i P$ | $\sqrt{2} \bar{q}_R, j q_L, i$ | $U_L \Phi U_R^\dagger$ | $\Phi^\dagger$ | $\Phi^\dagger$ |
| $R^\mu = V^\mu - A^\mu$ | $\sqrt{2} \bar{q}_R, j \gamma^\mu q_R, i$ | $U_R R^\mu U_R^\dagger$ | $L_{\mu}$ | $-(L^\mu)^\dagger$ |
| $L^\mu = V^\mu + A^\mu$ | $\sqrt{2} \bar{q}_L, j \gamma^\mu q_L, i$ | $U_L R^\mu U_L^\dagger$ | $R_{\mu}$ | $-(R^\mu)^\dagger$ |
| $\tilde{\phi}^\mu = V^\mu - i B^\mu$ | $\sqrt{2} \bar{q}_L, j \gamma^\mu q_L, i$ | $U_L \tilde{\phi}^\mu U_R^\dagger$ | $\tilde{\phi}_{\mu}^\dagger$ | $-\tilde{\phi}_{\mu}^\dagger$ |
| $R_{hyb, \mu} = \Pi_{hyb, \mu} - B_{hyb, \mu}$ | $\sqrt{2} \bar{q}_R, j G^{\mu \nu} q_R, i$ | $U_R R_{hyb, \mu} U_R^\dagger$ | $L_{\mu, hyb}$ | $(L_{\mu, hyb})^\dagger$ |
| $L_{hyb, \mu} = \Pi_{hyb, \mu} + B_{hyb, \mu}$ | $\sqrt{2} \bar{q}_L, j G^{\mu \nu} q_L, i$ | $U_L R_{hyb, \mu} U_L^\dagger$ | $R_{\mu, hyb}$ | $(R_{\mu, hyb})^\dagger$ |

between conventional $\bar{q} q$ pseudovector and hybrid mesons are large enough to neglect such a mixing.

In terms of matrices, we have

$$
\Pi_{hyb, \mu} = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc}
\frac{1}{2} \left( \frac{h_{1,N}^{hyb} + \pi_1^{hyb}}{\eta_{1,N}^{hyb} - \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{\eta_1^{hyb} + \pi_1^{hyb}}{\eta_1^{hyb} - \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{K_1^{hyb} + \pi_1^{hyb}}{\eta_1^{hyb} - \eta_1^{hyb}} \right) \\
\frac{1}{2} \left( \frac{h_{1,N}^{hyb} - \pi_1^{hyb}}{\eta_{1,N}^{hyb} + \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{\eta_1^{hyb} - \pi_1^{hyb}}{\eta_1^{hyb} + \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{K_1^{hyb} - \pi_1^{hyb}}{\eta_1^{hyb} + \eta_1^{hyb}} \right) \\
\frac{1}{2} \left( \frac{h_{1,N}^{hyb} + \pi_1^{hyb}}{\eta_{1,N}^{hyb} - \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{\eta_1^{hyb} + \pi_1^{hyb}}{\eta_1^{hyb} - \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{K_1^{hyb} + \pi_1^{hyb}}{\eta_1^{hyb} - \eta_1^{hyb}} \right)
\end{array} \right) \mu,
$$

$$
B_{hyb, \mu} = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc}
\frac{1}{2} \left( \frac{h_{1,N}^{hyb} + b_1^{hyb}}{\eta_{1,N}^{hyb} - b_1^{hyb}} \right) & \frac{1}{2} \left( \frac{b_1^{hyb} + \pi_1^{hyb}}{\eta_1^{hyb} - \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{K_1^{hyb} + \pi_1^{hyb}}{\eta_1^{hyb} - \eta_1^{hyb}} \right) \\
\frac{1}{2} \left( \frac{h_{1,N}^{hyb} - b_1^{hyb}}{\eta_{1,N}^{hyb} + b_1^{hyb}} \right) & \frac{1}{2} \left( \frac{b_1^{hyb} - \pi_1^{hyb}}{\eta_1^{hyb} + \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{K_1^{hyb} - \pi_1^{hyb}}{\eta_1^{hyb} + \eta_1^{hyb}} \right) \\
\frac{1}{2} \left( \frac{h_{1,N}^{hyb} + b_1^{hyb}}{\eta_{1,N}^{hyb} - b_1^{hyb}} \right) & \frac{1}{2} \left( \frac{b_1^{hyb} + \pi_1^{hyb}}{\eta_1^{hyb} - \eta_1^{hyb}} \right) & \frac{1}{2} \left( \frac{K_1^{hyb} + \pi_1^{hyb}}{\eta_1^{hyb} - \eta_1^{hyb}} \right)
\end{array} \right) \mu.
$$

For the hybrid states contained in $\Pi_{hyb, \mu}$, the field $\pi_1^{hyb}$ is assigned to $\pi_1(1600)$ [41], as already discussed in the introduction. For the other members of the nonet, no experimental candidates are yet known. In Sect. 4 we will present our estimate for their masses and decays.

For the chiral partners contained in $B_{hyb, \mu}$ again no candidate exists. In a lattice simulation no states below 2.4 GeV have been found [5], but this result has to be interpreted with caution due to the large pion masses (about 400 MeV) used in the simulation. We estimate the mass of the chiral partner of $\pi_1$, the so-called $b_1^{hyb}$ state, to have a mass in the 2–2.5 GeV range, once that the pion mass converges to the physical value. For definiteness, we shall assign it to an hypothetical state to the lower limit $b_1(2000?)$ state, but our results do not change much when increasing this mass up to 2.4 GeV. The masses of the other members of the pseudovector crypto-exotic nonet then follow as a consequence of this assumption. [In this work we restrict to hybrids containing only the light quarks $u, d$, and $s$. It should be also stressed that hybrid mesons can be also realized with heavy quarks, see the lattice studies in Refs. [7,8]: the corresponding currents have a similar form as in Eqs. (8) and (9).]

For completeness, in Tables 1 and 2 we summarize all relevant properties and transformations of the nonets introduced in this section.
3 The Lagrangian terms involving hybrid mesons

In this section we present the enlarged eLSM Lagrangian involving hybrids. We start form the general form

\[ \mathcal{L}_{\text{enlarged}}^{eLSM} = \mathcal{L}_{eLSM} + \mathcal{L}_{\text{hybrid}}^{eLSM} \]  

where \( \mathcal{L}_{eLSM} \) is the standard part, built under chiral and dilatation symmetries, as well as their spontaneous and explicit breaking features (see Appendix A). Next, the hybrid part is written as:

\[ \mathcal{L}_{\text{hybrid}}^{eLSM} = \mathcal{L}_{\text{hybrid--quadratic}}^{eLSM} + \mathcal{L}_{\text{hybrid--linear}}^{eLSM}. \]  

These terms will be discussed separately in this section. Before doing so, a general comment is in order: In this work, the masses and decays are calculated at tree-level. This is in agreement with the basic strategy of the eLSM, which amounts to include as much as possible mesonic (and baryonic) interpolating fields. As a matter of fact, this strategy has shown to be quite successful in describing some aspects of the low-energy QCD phenomenology [17,29]. Namely, we recall that in the mesonic sector it was possible to describe the meson phenomenology up to 1.7 GeV. In the extension of Ref. [20] also mesons above 2 GeV have been considered. We are therefore confident that a linear chiral model, which by construction includes chiral partners, can give useful results also for the yet unknown hybrid states (one nonet at about 1.7 GeV and one just above 2 GeV), that are the subject of the present work.

Of course, as a matter of principle, the necessity of unitarization and its influence on statements made in this paper represent a relevant question. To this end, we note that in Ref. [66] it was shown that, as long as the width-to-mass ratio \( \Gamma/M \) is sufficiently small (smaller than 0.2, the ratio in the case of the \( \rho \) meson), the effect of loops—which is large-\( N_c \) suppressed—is not expected to change the picture; later on, in Ref. [67] the next-to-leading triangle diagram for two-body decays has been shown to be negligible.

On the other hand, there are indeed some cases where the role of loops can be important. This is generally true for scalar mesons. For instance, in the eLSM the \( a_0(1450) \) is a predominantly \( \bar{q}q \) state, and the \( a_0(980) \) is not part of the model. When including loops it is possible to show that the \( a_0(980) \) emerges as an additional companion pole as a kind of four-quark state [68] (see also Ref. [69]). Similarly, the resonance \( K_0^*(700) \) is dynamically generated through loops and is the companion state of the predominantly \( \bar{q}q \) state \( K_0^*(1430) \) [70], see also Ref. [71].

In conclusion, while loops are relevant for (relatively) broad resonances, the tree-level results represent a clear and well-definite setup to get meaningful results such as decay ratios. The inclusion of loop effects should be performed in the future when a better experimental knowledge will be available.

3.1 Quadratic terms in the hybrid fields hybrid kinetic terms and masses

The quadratic term for the hybrid fields can be decomposed as

\[ \mathcal{L}_{eLSM}^{\text{hybrid--quadratic}} = \mathcal{L}_{eLSM}^{\text{hybrid--kin}} + \mathcal{L}_{eLSM}^{\text{hybrid--mass}}, \]  

where one has the usual vectorial kinetic term

\[ \mathcal{L}_{eLSM}^{\text{hybrid--kin}} = -\text{Tr} \left( L_{\mu\nu}^{hyb,2} + R_{\mu\nu}^{hyb,2} \right) = -\text{Tr} \left( V_{\mu\nu}^{hyb,2} + A_{\mu\nu}^{hyb,2} \right), \]  

with

\[ V_{\mu\nu}^{hyb} = \partial_\mu V_{\nu}^{hyb} - \partial_\nu V_{\mu}^{hyb} \quad \text{and} \quad A_{\mu\nu}^{hyb} = \partial_\mu A_{\nu}^{hyb} - \partial_\nu A_{\mu}^{hyb}. \]  


Moreover, we consider the term describing the masses of hybrids as

\[
L^\text{hybrid-mass}_{\text{eLSM}} = m^1_1 G^2 \phi^2 \mu \left( L^\text{hyb,2}_{\text{hyb,2}} + R^\text{hyb,2}_{\text{hyb,2}} \right) + \text{Tr} \left( \Delta^\text{hyb} \left( L^\text{hyb,2}_{\text{hyb,2}} + R^\text{hyb,2}_{\text{hyb,2}} \right) \right)
\]

\[
+ \frac{h^\text{hyb}_1}{2} \text{Tr} \left( \Phi^4 \right) \text{Tr} \left( L^\text{hyb,2}_{\text{hyb,2}} + R^\text{hyb,2}_{\text{hyb,2}} \right) + h^\text{hyb}_2 \text{Tr} \left[ \left| L^\text{hyb}_1 \Phi \right|^2 + \left| \Phi R^\text{hyb}_1 \right|^2 \right]
\]

\[
+ 2 h^\text{hyb}_3 \text{Tr} \left( L^\text{hyb}_1 \Phi R^\text{hyb}_1, \Phi \right),
\]

(16)

which fulfills both chiral and dilatation invariances. Note, the dilaton field \( G \) as well as its vacuum’s expectation value \( G_0 \) enter into these expressions, see Appendix A and Refs. [17–19].

The masses of hybrids can be calculated from the previous expressions by taking into account that the (pseudo)scalar field \( \Phi \) has a nonzero condensate or vacuum’s expectation value (v.e.v.): \( \Phi \equiv \Phi_0 = \text{diag} \{ \Phi_N / 2, \Phi_S / 2, \Phi_S / \sqrt{2} \} \). This condensate reflects the spontaneous breaking of chiral symmetry, which intuitively is a consequence of the Mexican-hat form for the (pseudo)scalar potential. The quantity \( \phi_N \) corresponds then to the v.e.v. of \( \sqrt{1/2(\bar{u}u + \bar{d}d)} \), while \( \phi_S \) to the v.e.v. of \( \bar{s}s \). Quite interestingly, the term proportional to \( h^\text{hyb}_3 \) turns out to be particularly important, since—together with the scalar condensates \( \phi_N \) and \( \phi_S \)—it generates a mass difference between the \( 1^{-} \) and \( 1^{+-} \) hybrid nonets upon shifting the masses of the latter upward (see below). Note, the second term in Eq. (16) models the direct contribution of the nonzero bare quark masses

\[
\Delta^\text{hyb} = \text{diag} \{ \delta^\text{hyb}_N, \delta^\text{hyb}_N, \delta^\text{hyb}_S \}
\]

(17)

and breaks flavor symmetry explicitly when \( \delta^\text{hyb}_S \neq \delta^\text{hyb}_N \).

The masses of the \( 1^{-} \) exotic hybrid mesons read:

\[
m^2_{\pi N^+_1} = m^1_1 + \frac{1}{2} \left( h^\text{hyb}_1 + h^\text{hyb}_2 + h^\text{hyb}_3 \right) \phi^2_N + \frac{h^\text{hyb}_1}{2} \phi^2_S + 2 \delta^\text{hyb}_N,
\]

(18)

\[
m^2_{K^0} = m^1_1 + \frac{1}{4} \left( 2 h^\text{hyb}_1 + h^\text{hyb}_2 \right) \phi^2_N + \frac{1}{\sqrt{2}} \phi_N \phi_S h^\text{hyb}_3
\]

\[
+ \frac{1}{2} \left( h^\text{hyb}_1 + h^\text{hyb}_2 \right) \phi^2_S + \delta^\text{hyb}_N + \delta^\text{hyb}_S,
\]

(19)

\[
m^2_{h^\text{hyb}_{1,N}} = m^2_{\pi^+_1},
\]

(20)

\[
m^2_{h^\text{hyb}_{1,S}} = m^1_1 + \frac{h^\text{hyb}_1}{2} \phi^2_N + \left( \frac{h^\text{hyb}_1}{2} + h^\text{hyb}_2 + h^\text{hyb}_3 \right) \phi^2_S + 2 \delta^\text{hyb}_S,
\]

(21)

while the squared masses of the crypto-exotic pseudovector hybrid states are:

\[
m^2_{K^0_{1,\pi}} = m^1_1 + \frac{1}{2} \left( h^\text{hyb}_1 + h^\text{hyb}_2 - h^\text{hyb}_3 \right) \phi^2_N + \frac{h^\text{hyb}_1}{2} \phi^2_S + 2 \delta^\text{hyb}_N,
\]

(22)

\[
m^2_{K^0_{1,B}} = m^1_1 + \frac{1}{4} \left( 2 h^\text{hyb}_1 + h^\text{hyb}_2 \right) \phi^2_N - \frac{1}{\sqrt{2}} \phi_N \phi_S h^\text{hyb}_3
\]

\[
+ \frac{1}{2} \left( h^\text{hyb}_1 + h^\text{hyb}_2 \right) \phi^2_S + \delta^\text{hyb}_N + \delta^\text{hyb}_S,
\]

(23)

\[
m^2_{h^\text{hyb}_{1,N}} = m^2_{\pi^+_1},
\]

(24)
\[ m_{h_{1S}^{\text{hyb}}}^2 = m_{1}^{\text{hyb}2} + \frac{h_{1}^{\text{hyb}}}{2} \phi_{N}^2 + \left( \frac{h_{1}^{\text{hyb}}}{2} + h_{2}^{\text{hyb}} - h_{3}^{\text{hyb}} \right) \phi_{S}^2 + 2\delta_{S}^{\text{hyb}}. \]  

(25)

Note, these equations are formally equal to the mass expressions for vector and axial-vector fields reported in Ref. [17] upon replacing \( h_{k} \rightarrow h_{k}^{\text{hyb}} \), \( \delta_{k} \rightarrow \delta_{k}^{\text{hyb}} \), and \( m_{1} \rightarrow m_{1}^{\text{hyb}} \), this is expected, since the terms are built following the same rules. There is however an important difference: there is no \( g_{1}^{\text{hyb}} \), since such a term is not possible for the hybrid multiplet, see Appendix B.

In particular, we get the (exact) relations:

\[ m_{b}^{\text{hyb}} \sim m_{1}^{\text{hyb}2} + 2h_{3}^{\text{hyb}} \phi_{N}^2, \]  

(26)

\[ m_{K_{1,B}}^{\text{hyb}} \sim m_{1}^{\text{hyb}2} - \sqrt{2} \phi_{N} \phi_{S} h_{3}^{\text{hyb}}, \]  

(27)

\[ m_{1S}^{\text{hyb}} \sim m_{1}^{\text{hyb}2} - h_{3}^{\text{hyb}} \phi_{S}^2 \]  

(28)

Hence, only the parameter \( h_{3}^{\text{hyb}} \) is responsible for the mass splitting of the hybrid chiral partners.

Altogether, six parameters appear in the expressions for the hybrid masses, but some simplifications are possible:

(a) The parameters \( h_{1}^{\text{hyb}} \) and \( m_{1}^{\text{hyb}} \) are not independent since they always appear in the combination \( m_{1}^{\text{hyb},2} + \frac{1}{2} h_{1}^{\text{hyb}} (\phi_{N}^2 + \phi_{S}^2) \). Hence, without loss of generality, we can set \( h_{1}^{\text{hyb}} = 0 \). (In addition, the parameter \( h_{1}^{\text{hyb}} \propto N^{-2} \) is large-\( N_{c} \) suppressed.)

(b) Only the difference \( \delta_{S}^{\text{hyb}} - \delta_{N}^{\text{hyb}} \) is physical. In fact, one can write

\[
\text{Tr} \left[ \Delta_{\text{hyb}}^{\mu} \left( L_{\mu}^{\text{hyb},2} + R_{\mu}^{\text{hyb},2} \right) \right] = \text{Tr} \left[ \left( \Delta_{\text{hyb}}^{\mu} - \delta_{N}^{\text{hyb}} \right) \left( L_{\mu}^{\text{hyb},2} + R_{\mu}^{\text{hyb},2} \right) \right] + \text{Tr} \left[ \delta_{N}^{\text{hyb}} \left( L_{\mu}^{\text{hyb},2} + R_{\mu}^{\text{hyb},2} \right) \right]
\]  

(29)

and the last term can be absorbed into the one proportional to \( m_{1}^{\text{hyb},2} \) (when \( G \) is set equal to the condensate \( G_{0} \)). Therefore, for what concerns masses, we set \( \delta_{N}^{\text{hyb}} = 0 \). Moreover, considering that

\[ \phi_{S}^2 - 2\phi_{N}^2 \simeq 0 \]  

(30)

(this equation is exact in the \( U(3)_{V} \) limit), one can neglect the corresponding combinations in the expressions for the masses. As a result, the parameter \( h_{3}^{\text{hyb}} \) no longer appears and we are left with three independent parameters

\[ m_{1}^{\text{hyb}}, h_{3}^{\text{hyb}}, \delta_{S}^{\text{hyb}}. \]  

(31)

We then obtain the following simple equations for the masses of the hybrid states:

\[ m_{K_{1}}^{\text{hyb}} \simeq m_{1}^{\text{hyb}2} + \delta_{S}^{\text{hyb}}, \]  

(32)

\[ m_{K_{1,N}}^{\text{hyb}} \simeq m_{1}^{\text{hyb}2}, \]  

(33)

\[ m_{1S}^{\text{hyb}} \simeq m_{1}^{\text{hyb}2} + 2\delta_{S}^{\text{hyb}}, \]  

(34)

\[ m_{1B}^{\text{hyb}} \simeq m_{1}^{\text{hyb}2} - 2h_{3}^{\text{hyb}} \phi_{N}^2, \]  

(35)
Since the $s$-quark contribution is solely related to the strange constituent quark, we shall use the numerical value obtained in the fit of Ref. [17]

$$\delta S^\text{hyb} \simeq \delta S = 0.151 \text{ GeV}^2,$$

which leaves us with two parameters that are fixed in the next section.

### 3.2 Linear terms in the hybrid fields: hybrid decays

The Lagrangian terms which generate decays of the hybrid states into pseudovector and excited vector states as well as into (axial-)vector and (pseudo)scalar mesons are given by:

$$\mathcal{L}_\text{LSM}^\text{hybrid-linear} = i \lambda_1^\text{hyb} G \text{Tr} \left[ L^\text{hyb}_\mu \left( \tilde{\Phi}^\dagger \phi^\dagger - \Phi \tilde{\Phi}^\dagger \phi \right) + R^\text{hyb}_\mu \left( \tilde{\Phi}^\dagger \phi^\dagger - \Phi \tilde{\Phi}^\dagger \phi \right) \right]$$

$$+ i \lambda_2^\text{hyb} \text{Tr} \left( \left[ L^\text{hyb}_\mu , L^\text{hyb}_\nu \right] \Phi \phi^\dagger + \left[ R^\text{hyb}_\mu , R^\text{hyb}_\nu \right] \Phi \phi^\dagger \right)$$

$$+ \alpha^\text{hyb} \text{Tr} \left( \tilde{L}^\text{hyb}_\mu \phi R^\text{hyb}_\mu \phi^\dagger - \tilde{R}^\text{hyb}_\mu \phi R^\text{hyb}_\mu \phi^\dagger \right)$$

$$+ \beta_A^\text{hyb} \left( \text{det} \Phi - \text{det} \Phi^\dagger \right) \text{Tr} \left( L^\text{hyb}_\mu \left( \partial^\mu \phi \phi^\dagger - \phi \partial^\mu \phi^\dagger \right) \right)$$

$$- \lambda_1^\text{hyb} G \left( \tilde{\Phi}^\dagger \phi^\dagger - \Phi \tilde{\Phi}^\dagger \phi \right) \right).$$

These terms are invariant under $SU(3)_R \times SU(3)_L$, $C$, and $P$ transformations. The first three terms are invariant under $U(3)_R \times U(3)_L$, while the last breaks $U_A(1)$: this is a typical term caused by the axial anomaly [65]. In addition, the first two terms are also dilatation invariant: the two coupling constants $\lambda_1^\text{hyb}$ and $\lambda_2^\text{hyb}$ are dimensionless. The third term, proportional to $\alpha^\text{hyb}$, involves the Levi-Civita tensor and carries the dimension Energy$^{-2}$, while the fourth $\beta_A^\text{hyb}$ has dimension Energy$^{-3}$. In Appendix C we report the proof of the invariance properties for each of these terms.

Let us now consider the first term closely. Upon condensation of the glueball field $G$, the effective coupling $\lambda_1^\text{hyb} G_0$ has dimension energy. In terms of the physical nonets, the first term reads

$$\mathcal{L}_\text{LSM,1}^\text{hybrid-linear} = i 2 \lambda_1^\text{hyb} G \left[ \text{Tr} \left[ \Gamma^\text{hyb}_\mu \left[ P, B^\mu \right] \right] + \text{Tr} \left[ \Pi^\text{hyb}_\mu \left[ V^\mu, S \right] \right] \right]$$

$$+ 2 \lambda_1^\text{hyb} G \left[ \text{Tr} \left[ B^\mu_\mu \left[ P, V^\mu \right] \right] + \text{Tr} \left[ B^\text{hyb}_\mu \left[ B^\mu, S \right] \right] \right].$$

It generates decays of the type $\Pi^\text{hyb} \rightarrow B P$, in particular:

$$\pi_1 \rightarrow b_1 (1230) \pi.$$

These decay channels of exotic hybrids are expected to be dominant. Correspondingly, also the decay $B^\text{hyb} \rightarrow V_E P$ takes place. Note that further decays of the form $\Pi^\text{hyb} \rightarrow V_E S$ and $B^\text{hyb} \rightarrow B^\mu S$ cannot take place because they are kinematically forbidden.

We now turn to the second term. When the matrix $\Phi$ condenses, $\Phi \tilde{\Phi}^\dagger = \Phi_0^2$, this term vanishes: there is no mixing between (axial-)vector mesons and vectorial hybrid states, in agreement with the fact that they have a different $C$-parity. A related important consideration is the lack of a term that generates a mixing of the hybrid states with (pseudo)scalar mesons, see Appendix B for details. As a consequence, no shift of the hybrid fields and no additional renormalization factor for (pseudo)scalar states needs to be performed. The necessary shifts
are those of the “standard eLSM” that were studied in Ref. [17] and are summarized in Appendix A. The second term can be cast into the form:

\[ \mathcal{L}_{\text{eLSM,2}}^{\text{hybrid-linear}} = 2i \lambda_2^\text{hyb} \text{Tr} \left[ \left( \left[ \Pi_2^\mu, V^\mu \right] + [B_2^\mu, A_2^\mu] \right) \left( S^2 + P^2 \right) \right] - 2 \lambda_2^\text{hyb} \text{Tr} \left[ \left( \left[ \Pi_1^\mu, A_2^\mu \right] + [B_2^\mu, V^\mu] \right) \left[ P, S \right] \right]. \] (43)

Thus, we get decays of the types \( \Pi_2^\text{hyb} \rightarrow VPP \) and \( \Pi_1^\text{hyb} \rightarrow A_2^\mu PS \). The decay channel into \( VPP \) is potentially relevant. For the nonet \( B_2^\text{hyb} \), decays into \( A_3^\mu PP \) are expected. As a next step, one has to perform the transformations described in Appendix A (shifts of \( S \) and \( A_2^\mu \) and redefinition of \( P \), and other decays emerge, such as the one into two pseudoscalar states. The decays \( \pi_1 \rightarrow \eta \pi \) and \( \pi_1 \rightarrow \eta' \pi \), however, do not follow from this term.

The third term in Eq.(39) breaks dilation invariance but leads to two interesting decay channels: \( \Pi_2^\text{hyb} \rightarrow VPV \) and \( \Pi_1^\text{hyb} \rightarrow AP \). In fact, the most relevant decay terms read:

\[ \mathcal{L}_{\text{eLSM,3}}^{\text{hybrid-linear}} = i \alpha_2^\text{hyb} \phi_1 \left\{ \text{Tr} \left[ \tilde{\Gamma}_2^{hyb} \left( P, V^{\mu \nu} \right) \right] - \text{Tr} \left( \tilde{B}_2^{hyb} \left( \left\{ P, A_3^{\mu \nu} \right\} \right) \right) \right\} + \ldots, \] (44)

where \( \phi_N \) is the condensate of \( \sigma_N \). Hence, this term is responsible for \( \pi_1 \rightarrow \rho \pi \). This is the channel in which \( \pi_1(1600) \) was recently observed at COMPASS [10]. Terms that make use of the Levi-Civita tensor (here into \( \tilde{L}_2^{hyb} = \frac{1}{4} \epsilon_{\mu \nu \rho \sigma} L_2^{hyb, \rho \sigma} \) and \( \tilde{R}_2^{hyb} = \frac{1}{4} \epsilon_{\mu \nu \rho \sigma} R_2^{hyb, \rho \sigma} \)) are linked to the axial anomaly and are typically not negligible, even if the corresponding coupling constant is not dimensionless.

As a last step, we consider the fourth term in Eq.(39). This term breaks explicitly the \( U_A(1) \) symmetry because of the involvement of the determinant. Considering that \( \det \Phi - \det \Phi^\dagger = i Z_\pi \sqrt{\frac{3}{2}} \phi_1^3 \eta_0 \) [30], one has:

\[ \mathcal{L}_{\text{eLSM,4}}^{\text{hybrid-linear}} = -Z_\pi \sqrt{\frac{3}{2}} \phi_1^3 \eta_0 \text{Tr}(\Pi_3^\mu \partial_\mu P) + \ldots, \] (45)

and hence, decays of the type \( \Pi_3^\text{hyb} \rightarrow P \eta_0 \) emerge. Since \( \eta_0 \) is a combination of \( \eta \) and \( \eta' \), the decays \( \pi_1 \rightarrow \eta \eta \) and \( \pi_1 \rightarrow \eta \eta' \) follow. Note, experimentally the decay \( \pi_1(1400) \rightarrow \eta \eta \) and the decay \( \pi_1(1600) \rightarrow \eta \eta' \) have been seen in experiments. If these resonances ultimately correspond to a unique hybrid state [41], it means that both decay channels have been measured. Similar decay terms appear for the other members of the nonet. It is interesting to notice that this fourth term does not lead to two-body decays for the nonet of chiral partners \( B_2^\text{hyb} \).

As a last remark, we recall that the structure \( \det \Phi - \det \Phi^\dagger \) mixes with the pseudoscalar glueball [21,22]. Hence, the following interaction term is possible:

\[ \mathcal{L}_{\text{eLSM,G}}^{\text{hybrid-linear}} = i \tilde{\rho}_2^\text{hyb} \hat{G} \text{Tr}(L_2^\mu \partial_\mu (\partial_\mu \Phi \cdot \Phi^\dagger - \Phi \cdot \partial_\mu \Phi^\dagger) - R_2^\mu \partial_\mu (\partial_\mu \Phi^\dagger \cdot \Phi - \Phi^\dagger \cdot \partial_\mu \Phi)) \]

\[ = -2 \tilde{\rho}_2^\text{hyb} \hat{G} \phi_N \text{Tr}(\Pi_3^\mu \partial_\mu P) + \ldots. \] (46)

An interesting consequence is the decay \( \hat{G} \rightarrow \pi_1^\text{hyb} \pi \). According to lattice QCD, the mass of the pseudoscalar glueball \( \hat{G} \) may be in the range around 2.6 GeV [72]; therefore, this decay is kinematically allowed. The detailed study of this term is left for the future, when the pseudoscalar glueball will be supported by concrete experimental candidates.
Table 3: A: Masses of the exotic $J^{PC} = 1^{-+}$ and $J^{PC} = 1^{++}$ hybrid mesons. B: Ratios for the decay of vector and pseudovector hybrid mesons into pseudoscalar, pseudovector, and excited vector mesons (term proportional to $\lambda_{1}^{hyb}$ in Eq. (39))

| A | Resonance | Mass [MeV] | B | Ratio | Value |
|---|---|---|---|---|---|
| $\pi_{1}^{hyb}$ | 1660 [input using $\pi_{1}(1600)$ [9]] | $\Gamma_{K_{1}}^{hyb} \rightarrow K_{1}(1170)$ \(\rightarrow \frac{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}$ | 0.050 |
| $\eta_{1,N}^{hyb}$ | 1660 | $\Gamma_{K_{1}}^{hyb} \rightarrow K_{1}(1170)$ \(\rightarrow \frac{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}$ | 0.065 |
| $\eta_{1,S}^{hyb}$ | 1751 | $\Gamma_{K_{1}}^{hyb} \rightarrow K_{1}(1170)$ \(\rightarrow \frac{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}$ | 0.19 |
| $K_{1}^{hyb}$ | 1707 | $\Gamma_{K_{1}}^{hyb} \rightarrow K_{1}(1170)$ \(\rightarrow \frac{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}$ | 0.16 |
| $\eta_{1,N,B}^{hyb}$ | 2000 [input set as an estimate] | $\Gamma_{K_{1}}^{hyb} \rightarrow K_{1}(1170)$ \(\rightarrow \frac{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}$ | |
| $\eta_{1,N,B}^{hyb}$ | 2000 | $\Gamma_{K_{1}}^{hyb} \rightarrow K_{1}(1170)$ \(\rightarrow \frac{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}$ | |
| $K_{1,B}^{hyb}$ | 2063 | $\Gamma_{K_{1}}^{hyb} \rightarrow K_{1}(1170)$ \(\rightarrow \frac{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}$ | |
| $\eta_{1,S,B}^{hyb}$ | 2126 | $\Gamma_{K_{1}}^{hyb} \rightarrow K_{1}(1170)$ \(\rightarrow \frac{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}{\Gamma_{\pi_{1}}^{hyb} \rightarrow \pi_{b_{1}}}$ | |

4 Results

4.1 Masses

We compute the masses of vector and pseudovector hybrid mesons by using Eqs. (18–25) in which $\pi_{1}$ is identified with $\pi_{1}(1600)$ (with mass $1660^{+15}_{-11}$ MeV) and the mass of $b_{1}^{hyb}$ is set to 2 GeV. Moreover, we use $\delta_{S}^{hyb} \simeq \delta_{S} = 0.151$ GeV$^{2}$. Then, we obtain $\eta_{3}^{hyb} = -45.7$. The results for the other hybrid states are reported in Table 3. We thus expect the other members of the nonet of $\eta_{1,N}^{hyb}$, $\eta_{1,S}^{hyb}$, and $K_{1}^{hyb}$ to be also well below 2 GeV. Eventually, they can be also very broad, just as $\pi_{1}(1600)$, rendering their experimental discovery quite challenging, but—just as $\pi_{1}(1600)$—not impossible.

In particular, one may observe that our model predicts the state $\eta_{1,N}^{hyb}$ to have the same mass of $\pi_{1}^{hyb} \equiv \pi_{1}(1600)$. The nonstrange-strange mixing is expected to be small, since the hybrid mesons under study are grouped into left- and right-handed current, and thus build a so-called homochiral multiplet according to the classification of Ref. [65]. We then do not expect a sizable shift of $\eta_{1,N}^{hyb}$ and $\eta_{1,S}^{hyb}$ in Table 3.

In the lattice work of Ref. [6] the isoscalar hybrids were investigated together with the isovector state $\pi_{1}^{hyb}$. There is a predominantly nonstrange hybrid meson which corresponds to $\eta_{1,N}$ state: its mass is about 100-150 MeV heavier than $\pi_{1}^{hyb}$ for $m_{\pi} = 391$ MeV. The value of this mass difference in the limit of the physical pion mass is not yet settled. Our prediction concerning the similar mass about $\eta_{1,N}^{hyb}$ and $\pi_{1}^{hyb}$ is upheld and should be verified in future lattice evaluations. Admittedly, if the large mass difference presently found in Ref. [6] shall be confirmed in the future, it will be hard to understand it in the framework of our model.

Moreover, there is a predominantly strange–antistrange hybrid meson about 300 MeV heavier than $\pi_{1}^{hyb}$, which corresponds to $\eta_{1,S}^{hyb}$. It is interesting to notice that the fields found...
on the lattice show a small but nonzero nonstrange-strange mixing, that is not included in our model. The small mixing is in agreement with the “homochiral” nature of the chiral multiplet mentioned above. In the future, it will be also interesting to include the effect of this mixing into the eLSM, which can generate a small mass difference between $\eta_{1, N}^{hyb}$ and $\pi_{1}^{hyb}$. In the lattice work of Ref. [5] a kaonic hybrid, corresponding to $K_{1}^{hyb}$, could be identified (the pion mass was about 400 MeV). In conclusions, there are robust lattice candidates for the whole nonet of hybrid states \{$\pi_{1}^{hyb}, K_{1}^{hyb}, \eta_{1, N}^{hyb}, \eta_{1, S}^{hyb}$\}. We regard this result as an important support of our model. New lattice results about hybrid mesons with a pion mass approaching the physical value would be very useful to confirm the overall picture and investigate eventual discrepancies.

4.2 Decays

The coupling constants $\{\lambda_{1}^{hyb}, \lambda_{2}^{hyb}, \alpha^{hyb}, \rho_{A}^{hyb}\}$ entering the Lagrangian of Eq. (39) are not known and cannot be determined as long as a clear experimental information about (at least some of) the decay rates of hybrids is missing. Nevertheless, we can build ratios of decays, since the values of the coupling constant cancel out. Various decay ratios are reported in the Tables 3–6 and are independent on the parameters of the model. In each table, a reference decay has been chosen for building the ratios. Any other desired ratio can be constructed by dividing entries in the tables.

In the present work, we restrict to large-$N_c$ dominant interaction terms and we neglect decay terms that break flavor symmetry. Hence, the ratio that we obtain is subject to these uncertainties and approximations. While in Ref. [17] an agreement at the 10% level was obtained, this is definitely too optimistic in the present study of hybrids, which involves heavier states and a poor experimental knowledge. Similar to Ref. [20] that dealt with heavier states, we rather expect—as an educated guess—an agreement of the order 20–30%, but this value should be seen only as a rough estimate, since a determination of the accuracy is not possible at the present stage. At the same time, it must be remarked that the aim of this work is not (and cannot be yet) a precise calculation of hybrid decays, but rather the determination of some useful decay ratios that can help toward the identification of possible hybrid candidates in the (hopefully near) future. The results can tell us which decay channels are favored and which turns out to be suppressed according to the underlying symmetries used in the model. In the future, when more precise data will be available, one may study in more detail to which extent the underlying symmetries (and their breaking patterns) are still applicable for these unconventional mesonic states.

Next, we move to the presentation of each decay channel step by step. In (the right part of) Table 3 we report the decays of the $1^{++}$ and $1^{+-}$ hybrid states into a pseudovector and a pseudoscalar. The by far dominant decay is $\pi_{1}^{hyb} \rightarrow b_{1} \pi$, that we use as our reference decay. This decay mode should indeed one of the dominant decays of the broad state $\pi_{1}(1600)$ that sizably contributes to the broad decay width of this state. This is in agreement with the model results of Ref. [73] and the lattice results of Ref. [74].

The second term of the Lagrangian (39) contains two- and three-body decays. The dominant decay channel is $b_{1}^{hyb} \rightarrow \pi \pi \eta$. For what concerns the state $\pi_{1}^{hyb}$, one expects a quite small decay $\pi_{1}^{hyb} \rightarrow K^{*}K\pi$, since the coupling is proportional to the combination $\phi_{N} - \sqrt{2}\phi_{S}$, that vanishes in the flavor limit $U_{Y}(3)$ (where $\phi_{N} = \sqrt{2}\phi_{S}$). The decay $\pi_{1}^{hyb} \rightarrow K K$ vanishes even if an interaction Lagrangian is present (see Appendix C.2 for details). This is in agreement with the fact that $\pi_{1}^{hyb,0} \rightarrow K K$ would violate $C$-parity, while
Table 4  A: Ratios for the two-body decay of vector and pseudovector hybrid mesons into axial-vector and pseudo(scalar) mesons (term proportional to $\lambda_2^{hyb}$ in Eq. (39)). B: Ratios for the three-body decay of vector and pseudovector hybrid mesons into axial(vector) and pseudo-scalar mesons (term proportional to $\lambda_2^{hyb}$ in Eq. (39))

| A | B |
|---|---|
| $\Gamma^{0_{hyb}}_{1, B} \to K^- \pi^+ / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to K^0 \pi^0 / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.0041 | 0.0046 |
| $\Gamma^{0_{hyb}}_{1, B} \to K^0 \eta / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to K^0 \rho^- / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.0027 | 0.1832 |
| $\Gamma^{0_{hyb}}_{1, B} \to K^0 \rho^0 / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to K^0 \rho^0 / \Gamma^{0_{hyb}}_{1, B}$ |
| 3.6 $\cdot$ 10$^{-7}$ | 0.0046 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \pi^- / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^- \pi^+ / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.24 | 0.024 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \rho^- / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.0083 | 0.022 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \rho^0 / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \rho^0 / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.0111 | 0.201 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \omega / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \omega / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.0082 | 0.0012 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \omega' / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \omega' / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.015 | 0.14 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \omega'' / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \omega'' / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.0090 | 0.0090 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta' / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta' / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.17 | 0.17 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta'' / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta'' / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.59 | 0.59 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta''' / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta''' / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.015 | 0.015 |
| $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta'''' / \Gamma^{+_{hyb}}_{1, B}$ | $\Gamma^{+_{hyb}}_{1, B} \to \pi^+ \eta'''' / \Gamma^{+_{hyb}}_{1, B}$ |
| 0.00031 | 0.00031 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^+ / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^+ / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.10 | 0.000034 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \eta' / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \eta' / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.029 | 0.029 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \eta'' / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \eta'' / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.0016 | 0.0016 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \eta''' / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \eta''' / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.00092 | 0.00092 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \eta'''' / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \eta'''' / \Gamma^{0_{hyb}}_{1, B}$ |
| 1.7 $\times$ 10$^{-6}$ | 1.7 $\times$ 10$^{-6}$ |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.17 | 0.17 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.016 | 0.016 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.00012 | 0.00012 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.0036 | 0.0036 |
| $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ | $\Gamma^{0_{hyb}}_{1, B} \to \pi^- \pi^- / \Gamma^{0_{hyb}}_{1, B}$ |
| 0.48 | 0.48 |
**Table 5** A: Ratios for the decay of vector and pseudovector hybrid mesons into pseudoscalar, pseudovector, and excited vector mesons (term proportional to $\lambda_2^{hyb}$ in Eq. (39)). B: Ratios for the decay of vector and pseudovector hybrid mesons into pseudoscalar, pseudovector, and excited vector mesons (term proportional to $\rho^{hyb}$ in Eq. (39))

| Ratio | Value     | Ratio   | Value     |
|-------|-----------|---------|-----------|
| $\Gamma_1^{hyb} \to f_{1N}K^0\pi^0/\Gamma_1^{hyb}$ | 0.0012  | $\Gamma_1^{hyb} \to f_{1N}K^0\pi^0/\Gamma_1^{hyb}$ | 0.61  |
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.0062  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 1.6   |
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.032  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.0011|
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.00053  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.0011|
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.029  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 3.8   |
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.60  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.60   |
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.59  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 1.7801  |
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.60  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 1.78  |
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.010  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.029 |
| $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 0.046  | $\Gamma_1^{hyb} \to f_{1S}K^0\pi^0/\Gamma_1^{hyb}$ | 1.57  |

**Table 6** Ratios for the decay of vector hybrid mesons into two pseudoscalar mesons (term proportional to $\beta_A^{hyb}$ in Eq. (39))

| Ratio | Value |
|-------|-------|
| $\Gamma_1^{hyb} \to K^0\eta/\Gamma_1^{hyb}$ | 12.7  |
| $\Gamma_1^{hyb} \to K^0\eta/\Gamma_1^{hyb}$ | 0.69  |
| $\Gamma_1^{hyb} \to K^0\eta/\Gamma_1^{hyb}$ | 5.3   |
| $\Gamma_1^{hyb} \to K^0\eta/\Gamma_1^{hyb}$ | 2.2   |
| $\Gamma_1^{hyb} \to K^0\eta/\Gamma_1^{hyb}$ | 1.57  |
\[ \pi_{1}^{hyb,+} \rightarrow K^+ K^0 \] and \[ \pi_{1}^{hyb,-} \rightarrow K^- K^0 \] would violate G-parity. The by far largest decay of \( \pi_{1}^{hyb} \) for this term is the channel \( \pi_{1}^{hyb} \rightarrow \pi \rho \eta \) (then, a \( \pi \pi \pi \eta \) final state).

The third term of the Lagrangian (39) describes decays into vector-pseudoscalar and axial-vector-pseudoscalar pairs. Two decays of \( \pi_{1}^{hyb} \) are expected to be sizable:

\[ \pi_{1}^{hyb} \rightarrow \rho \pi \text{ and } \pi_{1}^{hyb} \rightarrow K^* K \] (47)

Other interesting and potentially large decays are \( \eta_{1N} \rightarrow K^* K, \eta_{1N} \rightarrow K^* K, b_{1}^{hyb} \rightarrow a_{1} \pi, \) see Table 5 for the full list.

The fourth and the last term describes the decays of the \( 1^{-+} \) hybrid nonet states into two pseudoscalar states, one of which is either the \( \eta \) or the \( \eta' \). The term explicitly breaks the axial symmetry \( U_{A}(1) \) (although it preserves chiral symmetry, see Appendix A), thus the flavor blind state \( \eta_0 \) plays a crucial role. It is interesting to observe that this term does not lead to two-body decays of the \( 1^{+-} \) hybrid states (see Appendix C4 for more details). In particular, the decays

\[ \pi_{1}^{hyb} \rightarrow \eta \pi \text{ and } \pi_{1}^{hyb} \rightarrow \eta' \pi \] (48)

are a consequence of this decay channel, with the decay channel \( \pi_{1}^{hyb} \rightarrow \eta' \pi \) being favored (this is due to the fact that \( \eta' \) is closer to the flavor singlet, while the meson \( \eta \) is closer to the octet configuration): the ratio \( \Gamma_{\pi_{1}^{hyb} \rightarrow \eta' \pi}/\Gamma_{\pi_{1}^{hyb} \rightarrow \eta \pi} \) equals 12.7 (a large ratio was also predicted in Ref. [75]). At present, the decay modes \( \pi_{1}(1600) \rightarrow \eta' \pi \) and \( \pi_{1}(1400) \rightarrow \eta \pi \) have been observed. As already discussed, if \( \pi_{1}(1600) \) and \( \pi_{1}(1400) \) corresponds to the same state [41] (see also Ref. [76]), then both decay modes have been measured. The determination of the ratio in the future would constitute an important test of our approach. The summary of the results for this term are presented in Table 6. Notice also that the decay into the identical pair \( \eta \eta \) does not take place, since the amplitude for this process vanishes exactly when the direct and the crossed tree-level Feynman diagrams are taken into account. This is expected because the two identical \( \eta \) mesons in \( L = 1 \) configuration have positive parity, at odd with the initial state which has negative parity; for more details on this point, we refer to Appendix C.4.

As a last remark, we note that different decays listed in the tables can, by further decay of some products, lead to the same final state. For instance, in our calculations, the width \( \Gamma_{b_{1}^{0,hyb} \rightarrow \pi^{+} a_{0}^{-}} \) is evaluated under the assumption that \( a_{0} \equiv a_{0}(1450) \) is stable. However, in reality \( a_{0}(1450) \) is not stable and may decay further into \( \pi \eta \), thus the following decay chain takes place: \( b_{1}^{0,hyb} \rightarrow \pi^{+} a_{0}^{-} (1450) \rightarrow \pi^{+} \pi^{-} \eta \). On the other hand, we also have the direct decay channel \( b_{1}^{0,hyb} \rightarrow \pi^{+} \pi^{-} \eta \). More in general, the following reactions

\[ b_{1}^{0,hyb} \rightarrow \pi^{+} \pi^{-} \eta, \quad b_{1}^{0,hyb} \rightarrow \pi^{+} a_{0}^{-} \rightarrow \pi^{+} \pi^{-} \eta, \quad b_{1}^{0,hyb} \rightarrow \pi^{-} a_{0}^{-} \rightarrow \pi^{-} \pi^{+} \eta, \ldots \] (49)

end up in the same final state, where dots refer to other possible decay chains. In principle, one should first perform the sum of the related amplitudes and the square, obtaining:

\[ \Gamma_{\text{total}}^{b_{1}^{0,hyb} \rightarrow \pi^{+} \pi^{-} \eta} = \Gamma_{\text{direct}}^{b_{1}^{0,hyb} \rightarrow \pi^{+} \pi^{-} \eta} + \Gamma_{b_{1}^{0,hyb} \rightarrow \pi^{+} a_{0}^{-} \rightarrow \pi^{+} \pi^{-} \eta} + \Gamma_{b_{1}^{0,hyb} \rightarrow \pi^{-} a_{0}^{-} \rightarrow \pi^{-} \pi^{+} \eta} + \ldots + \text{“mixed terms”}, \] (50)

where mixed terms refer to interference effects between the amplitudes. Fortunately, such interference effects are typically small, since the overlap of different channels is suppressed, as the explicit calculation presented in Ref. [77] shows. More in details, one can study the distributions in the Dalitz plot \( \pi^{+} \pi^{-} \eta \) in order to distinguish the contributions. Summarizing,
in our work we do not consider these effects, since they are not expected to sizably change the result and they go beyond the accuracy of our tree-level approach. Yet, they should be included once concrete candidates and accurate experimental results will be available.

5 Conclusions and Outlook

In this work we have studied masses and decays of the lightest hybrid nonet with \( J^{PC} = 1^{-+} \) and of its chiral partner nonet with \( J^{PC} = 1^{+-} \). To this end, we have embedded the hybrid state into a chiral multiplet and coupled it to the chiral model called eLSM. Upon assigning the resonance \( \pi_1(1600) \) to the isovector member of the lightest hybrid nonet, we have made predictions for some masses of hybrid states and for branching ratios of \( \pi_1(1600) \) and the members of this multiplet as well as for their chiral partners. The main results are reported in Tables 3, 4, 5 and 6.

For what concerns the masses, there are three hybrid states with \( J^{PC} = 1^{-+} \), denoted as \( K_1 \) as well as \( \eta_1 \) and \( N_\pi \) and \( \eta_1 \), \( S \). Their discovery is then possible, provided that these states are not too wide. For what concerns decays, we have introduced four chirally invariant effective interaction terms describing the masses and the two- and three-body decays of hybrids.

The interaction Lagrangian describing the hybrid-meson decays into other mesons is presented in Eq. (39). The first and the second terms in the interaction Lagrangian fulfill the chiral and dilatation symmetries and for this reason are expected to deliver the dominant contributions to the decays of the hybrid states. In particular, the first term of our approach describes decays of the \( J^{PC} = 1^{-+} \) state into pseudovector (\( J^{PC} = 1^{+-} \)) and pseudoscalar states, such as \( \pi_1(1600) \to b_1(1230)\pi \to \omega \pi \pi \). Hence, the final state \( \omega \pi \pi \) represents a promising channel for the confirmation of this hybrid candidate. Analogous decays of the other exotic hybrids have been obtained as a prediction. In addition, the decays of crypto-exotic hybrids into the scalar and orbitally excited vector mesons could be evaluated. According to the second term, \( \pi_1(1600) \) decays into \( K K \pi \), \( \rho \pi \eta \), and \( K K \), but only the decay into \( \rho \pi \eta \) is expected to be sizable. The third term of the interaction Lagrangian breaks dilatation invariance and generates also three-body and two-body decays. The latter are important, since they contain the process \( \pi_1 \to \rho \pi \to \pi \pi \pi \), thanks to which the \( \pi_1(1600) \) was seen at COMPASS. Decays of other member of the multiplet and their chiral partners are presented as predictions. Finally, the decays \( \pi_1 \to \eta \pi \) and \( \pi_1 \to \eta' \pi \) emerge from the fourth Lagrangian term which breaks axial and dilatation symmetries (but still fulfills chiral symmetry). These decay modes, even if subleading, are seen in experiment due to the very clean nature of their decay products. It is quite remarkable that, within our setup, the only way to obtain such decays goes through the axial anomaly. A breaking of flavor symmetry in the first three decay terms does not lead to decays into \( \eta \pi \) and \( \eta' \pi \).

Summarizing, for the resonance \( \pi_1(1600) \) we expect the following decays:

\[
\begin{align*}
\pi_1(1600) & \to \pi b_1, \quad \pi_1(1600) \to \rho \pi \eta, \quad \pi_1(1600) \to \rho \pi \pi, \quad \pi_1(1600) \\
& \to K^*(892)K, \quad \pi_1(1600) \to \eta' \pi, \quad \pi_1(1600) \to \eta \pi.
\end{align*}
\]

(51)

It is hard at the present stage to determine which one of them is the largest. We expect the \( \pi b_1 \) mode to be quite large, in agreement with the lattice study of Ref. [74]. At the same time, the latter two decays are expected to be small, since they break explicitly the axial anomaly.

At present, the results of these papers are at the tree level. As a possible outlook, one can calculate the spectral function of \( \pi_1(1600) \). One may start with the dominant terms discussed in this work and calculate loops, following the techniques described in Ref. [68–70,78–80].
This can be quite important, as shown in the recent work of Ref. [41]. Another possibility is to study other hybrid nonets (such as, for instance, tensor hybrids) by repeating the steps presented in this work.

Summarizing, the confirmation of $\pi_1(1600)$ as a genuine hybrid state as well as the discovery of the other members of the nonet and its chiral partners would represent a step forward in our understanding of QCD, for which both theoretical and experimental efforts are worth to be spent.

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Appendix A: Details of the eLSM

The Lagrangian of the eLSM for (pseudo)scalar and (axial-)vector states, constructed upon requiring chiral symmetry ($U(3)_R \times U(3)_L$), dilatation invariance, as well as under charge conjugation $C$ and parity $P$ symmetries, reads:

$$L_{eLSM} = L_{dil} + \text{Tr}[(D_\mu \Phi) \dagger (D^\mu \Phi)] - m_0^2 \left( \frac{G}{G_0} \right)^2 \text{Tr}(\Phi \dagger \Phi)$$

$$- \left[ \lambda_1 \text{Tr}(\Phi \dagger \Phi) \right]^2 - \lambda_2 \text{Tr}(\Phi \dagger \Phi)^2$$

$$- \frac{1}{4} \text{Tr}[(L^{\mu \nu})^2 + (R^{\mu \nu})^2] + \text{Tr} \left( \left( \frac{m_1^2}{2} \left( \frac{G}{G_0} \right)^2 + \Delta \right) (L_\mu^2 + R_\mu^2) \right)$$

$$+ \text{Tr}[H(\Phi + \Phi \dagger)]$$

$$+ c_1 (\det \Phi - \det \Phi \dagger)^2 + i \frac{g_2}{2} \{ \text{Tr}(L_{\mu \nu}[L^\mu, L^\nu]) + \text{Tr}(R_{\mu \nu}[R^\mu, R^\nu]) \}$$

$$+ \frac{h_1}{2} \text{Tr}(\Phi \dagger \Phi) \text{Tr}(L_\mu^2 + R_\mu^2) + h_2 \text{Tr}[|L_\mu \Phi|^2 + |\Phi R_\mu|^2]$$

$$+ 2h_3 \text{Tr}(L_\mu \Phi R^\mu \Phi) + L_{eLSM}^\Phi \cdots ,$$

where $D^\mu \Phi = \partial^\mu \Phi - ig_1 (L^\mu \Phi - \Phi R^\mu)$ and the dilaton (i.e., the scalar glueball) Lagrangian is

$$L_{dil} = \frac{1}{2} (\partial_\mu G)^2 - \frac{1}{4} \frac{m_0^2}{G_0} \left( G^4 \ln \left| \frac{G}{\Lambda} \right| - G^4 \frac{4}{4} \right) ,$$

(A2)

see Refs. [17, 18] for details. We recall that the two diagonal matrices $H$ and $\Delta$ parameterize the explicit breaking of chiral symmetry due to nonzero quark masses. Moreover, the term proportional to $c_1$ describes the axial anomaly. Finally, $L_{eLSM}^\Phi$ contains the kinetic as well as interaction terms for the chiral multiplet $\Phi^\mu = V^\mu_E - i B^\mu$, whose detailed form was not yet unexplored (but is not relevant for us).

Thank to $L_{dil}$, one can describe the breaking of dilatation invariance. In the chiral limit ($\Delta = H = 0$) and neglecting terms linked to the axial anomaly ($c_1 = 0$ and in the hybrid sector $\alpha_{hyb}^A = \beta_{hyb}^A = 0$), the parameter $\Lambda$ is the only dimensionful parameter of Eq. (A1). Namely, all other quantities are described by dimensionless parameters, e.g., $h_k, h_k^{hyb}$ with $k = 1, 2, 3$. In this way only a finite number of terms is possible. The breaking of dilata-
tion invariance due to quark masses, parameterized by $H$ and $G$, is rather small, but terms describing the chiral anomaly may be non-negligible.

The field $G$ develops a nonzero vacuum’s expectation value $G_0$ (note, $G_0 = \Lambda$ in the limit in which the glueball decouples from (pseudo)scalar fields, i.e., $m_0 = 0$); hence, a shift is needed:

$$G \rightarrow G_0 + G.$$  \hspace{1cm} (A3)

Next, for $m_0^2 < 0$ (realized in nature), spontaneous breaking of chiral symmetry takes place. As a consequence, one has to perform the shift of the scalar-isoscalar quark–antiquark fields by their vacuum expectation values

$$\sigma_N \rightarrow \sigma_N + \phi_N \text{ and } \sigma_S \rightarrow \sigma_S + \phi_S.$$  \hspace{1cm} (A4)

In matrix form:

$$S \rightarrow \Phi_0 + S \text{ with } \Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_N \sqrt{2} & 0 & 0 \\ 0 & \phi_N \sqrt{2} & 0 \\ 0 & 0 & \phi_S \end{pmatrix}. \hspace{1cm} (A5)$$

Note, one can rewrite $\Phi_0$ as

$$\Phi_0 = \frac{\phi_N}{2} 1_3 + \left( \frac{\phi_S}{\sqrt{2}} - \frac{\phi_N}{2} \right) \text{diag}(0, 0, 1), \hspace{1cm} (A6)$$

where the first term is dominant and the second is a flavor breaking correction since $\phi_N \simeq \sqrt{2}\phi_S$. In addition, one has also to “shift” the axial-vector fields

$$a_1^\mu \rightarrow a_1^\mu + Z_\pi w_\pi \partial^\mu \pi, \quad K_{1, A}^{+ \mu} \rightarrow K_{1, A}^{+ \mu} + Z_K w_k \partial^\mu K, \quad \ldots$$

$$f_{1, N}^\mu \rightarrow f_{1, N}^\mu + Z_{\eta_N} w_{\eta_N} \partial^\mu \eta_N, \quad f_{1, S}^\mu \rightarrow f_{1, S}^\mu + Z_{\eta_S} w_{\eta_S} \partial^\mu \eta_S.$$  \hspace{1cm} (A7)

and to consider the wave-function renormalization of the pseudoscalar fields:

$$\pi \rightarrow Z_\pi \pi, \quad K^+ \rightarrow Z_K K^+, \quad \ldots$$  \hspace{1cm} (A8)

$$\eta_N \rightarrow Z_{\eta_N} \eta_N, \quad \eta_S \rightarrow Z_{\eta_S} \eta_S.$$  \hspace{1cm} (A9)

The constants entering into the previous expressions are:

$$Z_\pi = Z_{\eta_N} = \frac{m_{a_1}}{\sqrt{m_{a_1}^2 - g_1^2 \phi_N^2}}, \quad Z_K = \frac{2m_{K_{1, A}}}{\sqrt{4m_{K_{1, A}}^2 - g_1^2 (\phi_N + \sqrt{2}\phi_S)^2}},$$

$$Z_{\eta_S} = \frac{m_{f_{1S}}}{\sqrt{m_{f_{1S}}^2 - 2g_1^2 \phi_S^2}}, \hspace{1cm} (A10)$$

and:

$$w_\pi = w_{\eta_N} = \frac{g_1 \phi_N}{m_{a_1}^2}, \quad w_K = \frac{g_1 (\phi_N + \sqrt{2}\phi_S)}{2m_{K_{1, A}}^2}, \quad w_{\eta_S} = \frac{\sqrt{2}g_1 \phi_S}{m_{f_{1S}}^2}. \hspace{1cm} (A11)$$

The numerical values of the renormalization constants are $Z_\pi = 1.709, Z_K = 1.604, Z_{\eta_S} = 1.539$ \cite{17}, while those of the $w$-parameters are: $w_\pi = 0.683$ GeV$^{-1}, w_K = 0.611$ GeV$^{-1}$, $w_{\eta_S} = 0.554$ GeV$^{-1}$. Moreover, the condensates $\phi_N$ and $\phi_S$ read

$$\phi_N = Z_\pi f_\pi = 0.158 \text{ GeV}, \quad \phi_S = \frac{2Z_K f_K - \phi_N}{\sqrt{2}} = 0.138 \text{ GeV}, \hspace{1cm} (A12)$$
where the standard values \( f_\pi = 0.0922 \) GeV and \( f_K = 0.110 \) GeV have been used \([9]\). The previous expressions can be summarized by the matrix replacements

\[
P \rightarrow \mathcal{P} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{Z_\pi}{\sqrt{2}} (\eta_N + \pi^0) & Z_\pi \pi^+ & Z_K K^+ \\
Z_\pi \pi^- & \frac{Z_\pi}{\sqrt{2}} (\eta_N - \pi^0) & Z_K K^0 \\
Z_K K^- & Z_K K^0 & Z_{\eta_S} \eta_S
\end{pmatrix},
\]

(A13)

and

\[
A^\mu \rightarrow \mathcal{A}^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix}
f_{1N} + \frac{a_1^0}{\sqrt{2}} & a_1^+ & K_{1,1}^+ \\
f_{1N} - \frac{a_1^0}{\sqrt{2}} & K_{1,1}^0 & f_{1S} \\
K_{1,1}^- & K_{1,1}^0 & f_{1S}
\end{pmatrix}^\mu
\]

(A14)

Note, in the \( U_V(3) \) limit (in which all three bare quark masses are equals) some simplifications take place (useful for cross-check of the results): \( \Phi_N = \sqrt{2} \Phi_S \), \( Z_\pi = Z_K = Z_{\eta_S} \), and \( w = w_\pi = w_K = w_{\eta_S} \), out of which \( P \rightarrow P = Z \mathcal{P} \) and \( A^\mu \rightarrow \mathcal{A}^\mu = A^\mu + Z \partial^\mu P \).

The chiral anomaly is described by the term \( c_1 (\det \Phi - \det \Phi)^2 \) in Eq. (A1). This term is not invariant under \( U_3(R) \times U_3(R) \), according to which \( \Phi \rightarrow U_L \Phi U_R^\dagger \). However, we consider the \( U_3(1) \) transformation \( \Phi \rightarrow e^{i\alpha} \Phi \) (obtained for \( U_L = 1_3 e^{i\alpha/2} \) and \( U_R = U_R^\dagger \), the determinant implies that this symmetry is broken: in this way the \( U_3(1) \) anomaly is taken into account. As a consequence, this term generates a term proportional to \( \tilde{\eta}_2 \), which shifts up the mass of the flavor singlet configuration, in a way analogous to the one originally described in Ref. \([39]\). Thus, the masses of \( \eta \) and \( \eta' \) can be correctly described \([17]\).

For more details and other anomalous terms, see Ref. \([51, 65]\).

The eLSM has been enlarged to four flavors in Refs. \([26, 27]\). Interestingly, charmed meson masses and large-\( N_c \) dominant decays can be described relatively well (even if one is far from the natural domain of chiral symmetry). In the end, we also recall that the pseudoscalar glueball can be coupled to the eLSM via the chiral Lagrangian \( \mathcal{L}_{\tilde{G}} = i c \tilde{G} \Phi \tilde{G} \), which reflects the axial anomaly in the pseudoscalar-isoscalar sector, see details and results in Refs. \([21, 22]\). In a recent extension, the very same Lagrangian is used to study the decay of an hypothetical excited pseudoscalar glueball \([23]\).

### Appendix B: Absence of shift for vector hybrid states

There is no allowed term which mixes the hybrid nonets with (pseudo)scalar mesons. Namely, one may start from the general chirally invariant Lagrangian term involving hybrid fields as well as \( \Phi \) and \( \partial^\mu \Phi \)

\[
\mathcal{L}_{\text{test}} = \alpha \text{Tr}[(\partial^\mu \Phi) (\Phi^\dagger L^\mu_{hyb})] + \beta \text{Tr}[(\partial^\mu \Phi) (R^\mu_{hyb} \Phi^\dagger)]
\]

(B1)

Note, other terms can be always recasted in a combination of the previous one. For instance,

\[
\text{Tr}[(\partial^\mu \Phi^\dagger) L^\mu_{hyb} \Phi] = \text{Tr}\left[\Phi \left(\Phi^\dagger L^\mu_{hyb} \Phi\right)\right] - \text{Tr}\left[\Phi^\dagger \left(\partial^\mu L^\mu_{hyb}\Phi\right)\right] - \text{Tr}\left[\Phi^\dagger L^\mu_{hyb} \left(\partial^\mu \Phi\right)\right]
\]

\[
= -\text{Tr}[\Phi^\dagger L^\mu_{hyb} \left(\partial^\mu \Phi\right)] = -\text{Tr}[\left(\partial^\mu \Phi\right) \Phi^\dagger L^\mu_{hyb}]
\]

(B2)
where a full derivative has been neglected and \( \partial^\mu R^\text{hyb}_\mu = \partial^\mu L^\text{hyb}_\mu = 0 \) (since they are divergenceless vector fields).

Under parity transformation:

\[
\mathcal{L}_{\text{test}} \xrightarrow{P} \alpha \text{Tr} \left( \partial^\mu \Phi \right) \Phi R^\text{hyb}_\mu + \beta \text{Tr} \left( \partial^\mu \Phi \right) L^\text{hyb}_\mu \Phi
\]

\[
= -\alpha \text{Tr} \left[ \Phi^\dagger \left( \partial^\mu \Phi \right) R^\text{hyb}_\mu \right] - \beta \text{Tr} \left[ \Phi^\dagger L^\text{hyb}_\mu \left( \partial^\mu \Phi \right) \right]
\]

\[
= -\beta \text{Tr} \left[ \left( \partial^\mu \Phi \right) \Phi^\dagger L^\text{hyb}_\mu \right] - \alpha \text{Tr} \left[ \left( \partial^\mu \Phi \right) R^\text{hyb}_\mu \Phi^\dagger \right]
\]

(B3)

where again similar manipulations have been applied. Therefore, if we impose parity invariance, the condition \( \beta = -\alpha \) follows.

Next, we consider \( C \)-parity, according to which Eq. (B1) transforms into

\[
\mathcal{L}_{\text{test}} \xrightarrow{C} \alpha \text{Tr} \left[ \left( \partial^\mu \Phi^\dagger \right) \left( \Phi^\dagger \right)^\dagger \left( R^\text{hyb}_\mu \right)^\dagger \right] + \beta \text{Tr} \left[ \left( \partial^\mu \Phi^\dagger \right) \left( L^\text{hyb}_\mu \right)^\dagger \left( \Phi^\dagger \right)^\dagger \right]
\]

\[
= \alpha \text{Tr} \left[ \left( R^\text{hyb}_\mu \Phi^\dagger \left( \partial^\mu \Phi \right) \right)^\dagger \right] + \beta \text{Tr} \left[ \left( \Phi^\dagger \left( L^\text{hyb}_\mu \right) \left( \partial^\mu \Phi \right) \right)^\dagger \right]
\]

\[
= \beta \text{Tr} \left[ \left( \partial^\mu \Phi \right) \Phi^\dagger L^\text{hyb}_\mu \right] + \alpha \text{Tr} \left[ \left( \partial^\mu \Phi \right) R^\text{hyb}_\mu \Phi^\dagger \right],
\]

(B4)

out of which \( \beta = \alpha \) assures invariance under \( C \).

It is then clear that the only solution is

\[
\alpha = \beta = 0 \rightarrow \mathcal{L}_{\text{test}} = 0,
\]

(B5)

i.e., the simultaneous requirement of invariance under \( P \) and \( C \) cannot be fulfilled. In particular, it is the different transformation of hybrids under \( C \)-parity that forbids this interaction. The only interaction involving one hybrid field and two (pseudo)scalar ones does not contain derivatives and is the one of Eq. (39).

The implications are important: there is no mixing such as the \( a_1 \pi \) one discussed above. The fields entering Eq. (10) are already the physical ones.

### Appendix C: Decay rates for hybrid mesons

We present the explicit expressions for the two- and three-body decay rates for hybrid mesons.

1. First term of the Lagrangian of Eq. (39)

The first term of the effective Lagrangian (39)

\[
\mathcal{L}_{\text{eLSM,1}}^{\text{hybrid–linear}} = i \lambda_1^\text{hyb} G \text{Tr} \left[ L^\text{hyb}_\mu \left( \Phi^\mu \Phi^\dagger - \Phi \Phi^\dagger \right) + R^\text{hyb}_\mu \left( \Phi^\mu \Phi - \Phi^\dagger \Phi^\dagger \right) \right]
\]

describes the interaction of hybrid mesons with pseudovector and excited vector mesons and (pseudo)scalar mesons.

Let us first verify the invariance under \( P \) and \( C \). Under parity the first term transforms as:

\[
\text{Tr} \left( L^\text{hyb}_\mu \Phi^\mu \Phi^\dagger \right) \xrightarrow{P} \text{Tr} \left( R^{\text{hyb},\mu} \Phi^\dagger \Phi^\mu \right),
\]

(C1)

which equals the third term; similarly, the second converts into the fourth; hence, invariance under \( P \) is guaranteed.

Next, under \( C \) the first term transforms as:

\[
\text{Tr} \left( L^\text{hyb}_\mu \Phi^\mu \Phi^\dagger \right) \xrightarrow{C} -\text{Tr} \left( R^{\text{hyb},\mu} \Phi^\mu \Phi^\dagger \right) = -\text{Tr} \left( \Phi^\dagger \Phi^\mu R^\text{hyb}_\mu \right) = -\text{Tr} \left( R^\text{hyb}_\mu \Phi^\dagger \Phi^\mu \right),
\]

(C2)
and hence, the first term converts into the fourth and the second into the third. Invariance under is also fulfilled.

As a last check, we show that the Lagrangian is Hermitian. For the first term (including the $i$ in front), one has:

$$\left\{ i \text{Tr} \left[ L_{\mu}^{hyb} (\Phi^{\dagger} - \Phi \Phi^{\dagger}) \right] \right\}^\dagger = -i \text{Tr} \left[ (\Phi^{\dagger} - \Phi \Phi^{\dagger}) L_{\mu}^{hyb,\dagger} \right] = \text{Tr} \left[ L_{\mu}^{hyb,\dagger} (\Phi^{\dagger} - \Phi \Phi^{\dagger}) \right]$$

(C3)

$$= -i \text{Tr} \left[ L_{\mu}^{hyb} (\Phi^{\dagger} - \Phi \Phi^{\dagger}) \right] = i \text{Tr} \left[ L_{\mu}^{hyb} (\Phi \Phi^{\dagger} - \Phi \Phi^{\dagger}) \right].$$

(C4)

A similar expressions holds for the second term.

In terms of the physical nonets with defined $J^{PC}$, the Lagrangian can be rewritten as:

$$L^{\text{hybrid-linear}}_{eLSM,1} = i 2 \lambda_1^{hyb} G \left\{ \text{Tr} \left[ \Gamma_{\mu}^{hyb} \{ P, B^{\mu} \} \right] + \text{Tr} \left[ \Pi_{\mu}^{hyb} \{ V_E^{\mu}, S \} \right] \right\}$$

$$+ 2 \lambda_1^{hyb} G \left\{ \text{Tr} \left[ B_{\mu}^{hyb} \{ P, V_E^{\mu} \} \right] + \text{Tr} \left[ B_{\mu}^{hyb} \{ B^{\mu}, S \} \right] \right\}. \quad \text{(C5)}$$

This expression shows that following decays for the hybrid nonet $\Pi_{\mu}^{hyb}$ are possible

$$\Pi_{\mu}^{hyb} \rightarrow P B^{\mu} \text{ and } \Pi_{\mu}^{hyb} \rightarrow S V_E^{\mu}.$$ \quad \text{(C6)}$$

However, the second is not relevant for our purposes because the corresponding decay channels are kinematically forbidden.

For the hybrid nonet $B_{\mu}^{hyb}$ we get

$$B_{\mu}^{hyb} \rightarrow P V_E^{\mu} \text{ and } \Pi_{\mu}^{hyb} \rightarrow B^{\mu} S,$$ \quad \text{(C7)}$$

where, as above, the second term leads to kinematically forbidden decays.

Note, for completeness we check also the invariance $i 2 \lambda_1^{hyb} G \text{Tr} \left[ \Pi_{\mu}^{hyb} \{ P, B^{\mu} \} \right]$ under $P$ and $C$ and $\dagger$:

$$\text{Tr} \left[ \Pi_{\mu}^{hyb} \{ P, B^{\mu} \} \right] \rightarrow \text{Tr} \left[ \Pi_{\mu}^{hyb,\dagger} \{ -P, -B^{\mu} \} \right] = \text{Tr} \left[ \Pi_{\mu}^{hyb} \{ P, B^{\mu} \} \right]; \quad \text{(C8)}$$

$$\text{Tr} \left[ \Pi_{\mu}^{hyb} \{ P, B^{\mu} \} \right] \rightarrow \text{Tr} \left[ \Pi_{\mu}^{hyb,\dagger} \{ P^{t}, -B^{\mu,t} \} \right] = -\text{Tr} \left[ \Pi_{\mu}^{hyb,\dagger} \{ P^{t}, B^{\mu,t} \} \right]; \quad \text{(C9)}$$

$$\left\{ i \text{Tr} \left[ \Pi_{\mu}^{hyb} \{ P, B^{\mu} \} \right] \right\}^\dagger = -i \text{Tr} \left[ \{ P, B^{\mu} \}^\dagger \Pi_{\mu}^{hyb} \right]$$

$$= i \text{Tr} \left[ \Pi_{\mu}^{hyb} \{ P, B^{\mu} \} \right].$$ \quad \text{(C10)}$$

Similar check for the dominant decay term of $B_{\mu}^{hyb}$ are also reported:

$$\text{Tr} \left[ B_{\mu}^{hyb} \{ P, V_E^{\mu} \} \right] \rightarrow \text{Tr} \left[ -B_{\mu}^{hyb,\dagger} \{ -P, -V_E^{\mu} \} \right] = \text{Tr} \left[ B_{\mu}^{hyb} \{ P, V_E^{\mu} \} \right]; \quad \text{(C11)}$$

$$\text{Tr} \left[ B_{\mu}^{hyb} \{ P, V_E^{\mu} \} \right] \rightarrow \text{Tr} \left[ -B_{\mu}^{hyb,\dagger} \{ P^{t}, -V_E^{\mu,t} \} \right] = \text{Tr} \left[ B_{\mu}^{hyb} \{ P, V_E^{\mu} \} \right]; \quad \text{(C12)}$$

$$\left\{ \text{Tr} \left[ B_{\mu}^{hyb} \{ P, V_E^{\mu} \} \right] \right\}^\dagger = \text{Tr} \left[ \{ P, V_E^{\mu} \}^\dagger B_{\mu}^{hyb} \right].$$
After performing the field transformations in Eq. (A4), Eq. (A8) and Eq. (A9), it is calculate the corresponding terms describing the decays. For instance, for the case of the state $\pi_1^{hyb} \pi b_1$ interaction, the following explicit Lagrangian term:

$$\mathcal{L}_{eLSM-\lambda_1^{hyb}} = \lambda_1^{hyb} G_0 Z_\pi (\pi^{-0}_{b_1} - \pi^{0}_{b_1} - \pi^{+}_{b_1} + \cdots)$$

Then, the average modulus squared decay amplitude is given by

$$|M_{\pi_1^{hyb} \rightarrow b_1 \pi}|^2 = \frac{1}{3} \frac{G_0^2 \lambda_1^{hyb} Z_\pi^2}{2 + \frac{(m_{\pi}^2 + m_{b_1}^2 - m_{\pi}^2)^2}{4m_{A}^2 m_{B_1}^2}}$$

and hence, the decay width reads:

$$\Gamma_{\pi_1^{hyb} \rightarrow b_1 \pi} = \frac{k_1}{8\pi m_{\pi}^2} \left\{ \frac{1}{3} \frac{G_0^2 \lambda_1^{hyb} Z_\pi^2}{2 + \frac{(m_{\pi}^2 + m_{b_1}^2 - m_{\pi}^2)^2}{4m_{A}^2 m_{B_1}^2}} \right\}$$

Similar expressions hold for all other possible decay widths described by the first term. The results are listed in Table 3.

2. Second term of the Lagrangian of Eq. (39)

The second term of the effective Lagrangian (39)

$$\mathcal{L}_{eLSM, 2}^{hyb-linear} = i\lambda_2^{hyb} \text{Tr}([L_{\mu}^{hyb}, L_{\mu}^\dagger] \Phi \Phi^\dagger + [R_{\mu}^{hyb}, R_{\mu}^\dagger] \Phi^\dagger \Phi)$$

generates two- and three-body decays for hybrid mesons into (axial-)vector mesons and (pseudo)scalar mesons.

Let us first check the invariance under $P$ and $C$. Under $P$ the first term transforms into

$$\text{Tr} \left( [L_{\mu}^{hyb}, L_{\mu}^\dagger] \Phi \Phi^\dagger \right) \rightarrow \text{Tr} \left( [R_{\mu}^{hyb}, R_{\mu}^\dagger] \Phi^\dagger \Phi \right)$$

and therefore $P$ is conserved since the first term goes into the second. Under $C$, the first term transforms as:

$$\text{Tr} \left( [L_{\mu}^{hyb}, L_{\mu}^\dagger] \Phi \Phi^\dagger \right) \overset{C}{\rightarrow} \text{Tr} \left( [R_{\mu}^{hyb}, -R_{\mu}^\dagger] \Phi^\dagger \Phi \right) = \text{Tr} \left( -R_{\mu}^{hyb} R_{\mu}^\dagger + R_{\mu}^\dagger R_{\mu}^{hyb} \right) (\Phi \Phi^\dagger)^\dagger$$

$$\overset{C}{\rightarrow} \text{Tr} \left( [R_{\mu}^{hyb}, R_{\mu}^\dagger] (\Phi \Phi^\dagger)^\dagger \right) = \text{Tr} \left( (\Phi \Phi^\dagger) [R_{\mu}^{hyb}, R_{\mu}] \right) = \text{Tr} \left( [R_{\mu}^{hyb}, R_{\mu}] (\Phi \Phi^\dagger) \right).$$

Hence, $C$ is also conserved.

Last, we check that the matrix that the Lagrangian is Hermitian:

$$\left\{ i \text{Tr} \left( [L_{\mu}^{hyb}, L_{\mu}^\dagger] \Phi \Phi^\dagger \right) \right\}^\dagger = -i \text{Tr} \left( \Phi^\dagger [L_{\mu}^\dagger, L_{\mu}^{hyb}] \right) = i \text{Tr} \left( [L_{\mu}^{hyb}, L_{\mu}^\dagger] \Phi \Phi^\dagger \right).$$

In terms of the nonets with defined $J^{PC}$ we get:

$$\mathcal{L}_{eLSM, 2}^{hyb-linear} = 2i\lambda_2^{hyb} \text{Tr} \left( [\Pi_{\mu}^{hyb}, V_{\mu}] (P^2 + S^2) \right) - 2\lambda_2^{hyb} \text{Tr} \left( [\Pi_{\mu}^{hyb}, A_{\mu}] [P, S] \right)$$

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Out of the first decay above, 

\[ B_{hyb} \] 

(Basically, out of \( PP \) the first is relevant, the second is suppressed, but the third can be relevant due to the shift \( A^\mu \rightarrow Zw,\partial^\mu P \) and the condensation of \( S \), since \( \Pi_{\mu}^{hyb} \rightarrow PP, PSS, \Pi_{\mu}^{hyb} \rightarrow A^\mu PS \).

As an example of a two-body decay, let us consider the case 

\[ A^{\mu} \rightarrow Zw,\partial^\mu P \] 

three-body decays described by this interaction term. The same transformations can be checked for the other terms. Next, we turn to the two- and three-body decays described by this interaction term.

For the hybrid nonet \( B_{hyb}^{\mu} \) one has:

\[ B_{hyb}^{\mu} \rightarrow VP, B_{hyb}^{\mu} \rightarrow A^\mu PP, B_{hyb}^{\mu} \rightarrow A^\mu SS. \]

Out of the first decay above, \( B_{hyb}^{\mu} \rightarrow VP \) emerges upon condensation of one field \( S \) (but turns out to be suppressed), and out of the second, \( B_{hyb}^{\mu} \rightarrow PPP \) is realized when \( A^\mu \) is shifted, see Appendix 1.

Next, for both terms we verify the invariance under \( C, P \) and \( \dagger \):

\[
\begin{align*}
\text{Tr} & \left( [\Pi_{\mu}^{hyb}, V^\mu](P^2 + S^2) \right) \rightarrow \text{Tr} \left( [\Pi_{\mu}^{hyb}, V^\mu]PP \right) \times (P^2 + S^2) \right) \right.
\end{align*}
\]

\[
= \text{Tr} \left( [\Pi_{\mu}^{hyb}, V^\mu]PP \right) \times (P^2 + S^2) \right) \right.
\]

\[
\text{Tr} \left( [\Pi_{\mu}^{hyb}, V^\mu]PP \right) \times (P^2 + S^2) \right) \right.
\]

\[
\left\{ 2i\lambda_2^{hyb} \text{Tr} \left( [\Pi_{\mu}^{hyb}, V^\mu]PP \right) \times (P^2 + S^2) \right) \right.
\]

\[
= 2i\lambda_2^{hyb} \text{Tr} \left( [\Pi_{\mu}^{hyb}, V^\mu]PP \right) \times (P^2 + S^2) \right) \right.
\]

\[
= 2i\lambda_2^{hyb} \left\{ 2 \left( [\Pi_{\mu}^{hyb}, V^\mu]PP \right) \times (P^2 + S^2) \right) \right.
\]

The same transformations can be checked for the other terms. Next, we turn to the two- and three-body decays described by this interaction term.

\[ a. \text{ Two-body decay rates} \]

As an example of a two-body decay, let us consider the case \( b_{1\mu}^{hyb,0} \rightarrow a_0^\pi^+ \), which is described by the following part of the Lagrangian:

\[
\mathcal{L}_{b_{1\mu}a_0\pi} = \lambda_2^{hyb} Z_\pi w_\pi \phi_N \left[ b_{1\mu}^{0hyb} (a_0^- \partial^\mu \pi^+ - a_0^+ \partial^\mu \pi^-) + b_{1\mu}^{+hyb} (a_0^0 \partial^\mu \pi^- - a_0^- \partial^\mu \pi^0) \right.
\]

\[
+ b_{1\mu}^{-hyb} (a_0^0 \partial^\mu \pi^- - a_0^- \partial^\mu \pi^0) \right.\]

We compute the decay width as

\[
\Gamma_{b_{1\mu}^{0hyb} \rightarrow a_0^\pi^+} = \frac{|k_1|}{24\pi m_{b_{1\mu}^{0hyb}}^2 \lambda_2^{hyb} Z_\pi w_\pi^2 \phi_N^2} \left[ m_{a_0}^2 + \frac{(m_{b_{1\mu}^{0hyb}}^2 + m_\pi^2 - m_{a_0}^2)^2}{4m_{b_{1\mu}^{0hyb}}^2} \right].
\]

The other decay channels of this type are calculated in a similar way and the results are listed in (the left part of) Table 4.
Next, we show why the decays $\pi_1^{hyb} \to K \bar{K}$, $\eta_{1,N}^{hyb} \to K \bar{K}$, and $\eta_{1,S}^{hyb} \to K \bar{K}$ vanish. For $\pi_1^{0,hyb} \to K \bar{K}$, the corresponding Lagrangian is
\[
\mathcal{L}_{\pi_1^{0,hyb}, K \bar{K}} = -\frac{2}{3} \lambda_2^{hyb} Z_K w_K (\phi_N - \sqrt{2} \phi_S) \pi_{1}^{0,\text{hyb}, \mu} \left[ K^0 \partial_\mu K^0 + K^0 \partial_\mu \bar{K}^0 - K + \partial_\mu K^- - K^- \partial_\mu K^+ \right].
\] (C30)
This term follows from $-2\lambda_2^{hyb} Tr (\Pi_\mu^{hyb}, A^\mu [P, S])$ upon shifting $A^\mu$ according to Eq. (A14) and setting $S = \Phi_0$ as in Eq. (A5) (thus, flavor symmetry is slightly broken). One can verify that this term is parity and $C$-invariant thanks to the combination $K^0 \partial_\mu K^0 + K^0 \partial_\mu \bar{K}^0$. Moreover, isospin is also conserved, since the quantity $[K^0 \partial_\mu K^0 + K^0 \partial_\mu \bar{K}^0 - K + \partial_\mu K^- - K^- \partial_\mu K^+]$ has $I = 1$ (intuitively, it is proportional to $\propto \bar{u}u - \bar{d}d$), in agreement with the fact that the $\pi_1^{0,\text{hyb}, \mu}$ state has also isospin 1. Yet, this interaction term does not lead to any decay $\pi_1^{0,hyb}$; for instance, upon considering $\pi_1^{hyb} \to K^+ K^-$, one can easily verify this point by considering that the decay amplitude is proportional to
\[
\varepsilon_\mu^{(a)}(p) \cdot (k_1^\mu + k_2^\mu) = \varepsilon_\mu^{(a)}(p) \cdot p^\mu = 0,
\] (C31)
where $p = (m_{\pi_1}, 0)$ is the four-momentum of $\pi_1^{hyb}$ and the $k_1$ and $k_2$ are the four-momenta of the decaying particles. Note, the same result can be also obtained by rewriting the relevant Lagrangian term as:
\[
\pi_1^{0,\text{hyb}, \mu} \left[ K^+ \partial_\mu K^- + K^- \partial_\mu K^+ \right] = \pi_1^{0,\text{hyb}, \mu} \partial_\mu \left[ K^+ K^- \right] = \partial_\mu \left[ \pi_1^{0,\text{hyb}, \mu} K^+ K^- \right] - \left[ \partial_\mu \pi_1^{0,\text{hyb}, \mu} \right] K^+ K^- = \left[ \partial_\mu \pi_1^{0,\text{hyb}, \mu} \right] K^+ K^-,
\] (C32)
where the full derivative can be neglected and the term proportional to $\partial_\mu \pi_1^{0,\text{hyb}, \mu}$ implies that no decay for $\pi_1^{hyb}$ is allowed as a consequence of the $\varepsilon_\mu^{(a)}(p) \cdot p^\mu = 0$. Hence, $\Gamma_{\pi_1^{hyb,0} \to K \bar{K}} = 0$. Indeed, this result is in agreement with $C$-parity, since this decay would violate it: the initial state $\pi_1^{hyb,0}$ has by definition $C$-parity +1, while the final kaon–antikaon would necessarily have $C = -1$. This can be seen by considering the two-kaon state (schematically) as
\[
|K^+(k_1) K^-(\bar{k}_1)| \rangle_{L = 1},
\] (C33)
where $L = 1$ assures that parity is $P = (-1)^L = -1$. Under $C$-transformation this ket changes into
\[
|K^-(k_1) K^+(\bar{k}_1)| \rangle_{L = 1} = -|K^-(k_1) K^+(\bar{k}_1)| \rangle_{L = 1},
\] (C34)
where in the last passage we took into account that the switching the momenta is equivalent to a parity transformation. Hence, the $K^+ K^-$ final state has $C = -1$.

It is indeed interesting to notice that a Lagrangian term proportional to $[\partial_\mu \pi_1^{0,\text{hyb}, \mu}] K^+ K^-$ does not lead to any decay of $\pi_1^{0,\text{hyb}}$ and, in general, vanishes for any process in which $\pi_1^{0,\text{hyb}, \mu}$ is on-shell. Yet, the Lagrangian term cannot be set to zero. In principle, it has a non-vanishing contribution to processes in which $\pi_1^{0,\text{hyb}, \mu}$ appear as a virtual particle. It has, for instance, a very small but nonzero contribution to $K \bar{K}$ scattering.
Similarly, $\pi^{+\text{hyb}}_1 \rightarrow K^+ \bar{K}^0$ and $\pi^{-\text{hyb}}_1 \rightarrow K^- \bar{K}^0$ vanish since the interaction term have a form analogous to Eq. (C30). This is in agreement with $G$-parity, where $G = C e^{i \pi f_2}$, $f_2$ being the second component of the isospin operator $I$. The state $\pi^{+\text{hyb}}_1$ is not an eigenstate of $C$ parity, but is an eigenstate of $G$-parity with eigenvalue $-1$. Yet, the final state

$$|K^+(k_1)\bar{K}^0(-k_1)|L = 1 \rangle \quad (C35)$$

has $G$ parity $+1$, as one can verify by recalling that under $G$ the transformations $K^+ \rightarrow \bar{K}^0$ and $\bar{K}^0 \rightarrow -K^+$ hold. Quite interestingly, the decays $\pi^{+\text{hyb}}_1 \rightarrow K^+ \bar{K}^0$ and $\pi^{-\text{hyb}}_1 \rightarrow K^- \bar{K}^0$ become possible if isospin breaking is considered. Namely, the decay amplitudes would be proportional to $\phi_U - \phi_D$, where $\phi_U$ and $\phi_D$ correspond to the $\bar{u}u$ and $\bar{d}d$ condensates, respectively (in the present version of the model $\phi_U = \phi_D = \phi_N/\sqrt{2}$). The inclusion of isospin violation of the eLSM is an interesting subject on its own that can be investigated in the future.

Finally, we can summarize the result as

$$\Gamma_{\pi^{\text{hyb}}_1 \rightarrow \bar{K}K} = 0. \quad (C36)$$

Next, the decay $\eta^{\text{hyb}}_{1,N} \rightarrow \bar{K}K$ also vanishes, since the Lagrangian reads

$$L_{\eta^{\text{hyb}}_{1,N} \bar{K}K} = \frac{-1}{4} \lambda^{\text{hyb}}_2 Z_K \mu K (\phi_N - \sqrt{2} \phi_S) \eta^{\text{hyb}, \mu}_{1,N} \left[ \bar{K}^0 \partial_\mu K^0 + K^0 \partial_\mu \bar{K}^0 + K_+ \partial_\mu K^- + K^- \partial_\mu K_+ \right]$$

thus leading to the same discussion above. The same argument applies for $\eta^{\text{hyb}}_{1,S} \rightarrow \bar{K}K$. Both decays would violate $C$-parity.

### b. Tree-body decay rates

We present the decay amplitudes for the three-body decay rates, which are extracted from the Lagrangian (C16) and are used to compute for the three-body decay widths. We use the following notations:

$$k_1 \cdot k_2 = \frac{m_1^2 - m_2^2 - m_3^2}{2},$$

$$k \cdot k_1 = m_1^2 + \frac{m_1^2 - m_2^2 - m_3^2}{2} + \frac{m_2^2 - m_2^2 - m_3^2}{2},$$

$$k \cdot k_2 = k_1 \cdot k_2 + \frac{m_1^2 - m_2^2 - m_3^2}{2} + \frac{m_2^2 - m_2^2 - m_3^2}{2},$$

$$m_1^2 = M^2 + m_1^2 + m_2^2 + m_3^2 - m_1^2 - m_2^3.$$

The decay amplitude for $\pi^{\text{hyb}}_1 \rightarrow K^* K \pi$ channel

$$-i M_{\pi^{\text{hyb}}_1 \rightarrow K^* K \pi} = \frac{1}{16} \lambda^{\text{hyb}}_2 Z_K^2 Z_\pi^2 \left[ 2 + \frac{(k \cdot k_1)^2}{M_2^2 M_1^2} \right], \quad (C37)$$

where the quantities $k$, $k_1$, $k_2$, and $k_3$ refer to the fields $\pi^{\text{hyb}}_1$, $K^*$, $\bar{K}^0$, and $\pi^0$, respectively.

For instance, the decay width $b^{\text{hyb}}_1 \rightarrow \bar{K}^0 K^0 \pi^0$ reads

$$\Gamma_{b^{\text{hyb}}_1 \rightarrow \bar{K}^0 K^0 \pi^0} = \frac{F^2_{b^{\text{hyb}}_1 \bar{K} K \pi}}{96 (2\pi)^3 M^3} \int_{(m_3)^{\text{min}}}^{(m_3)^{\text{max}}}(M-m_3)^2 \int_{(m_2)^{\text{min}}}^{(m_2)^{\text{max}}}(M-m_2)^2 \int_{(m_1)^{\text{min}}}^{(m_1)^{\text{max}}}(M-m_1)^2,$$

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\[
\left[ \frac{|k \cdot k_2 - k \cdot k_1|^2}{M^2} - (m_1^2 + m_2^2 - 2k_1 \cdot k_2) \right] \, dm_{23}^2 \, dm_{12}^2 \, , \quad (C38)
\]

and
\[
F_{\text{hyb} \lambda K \pi}^{\text{hyb}} = \frac{1}{4} \lambda_2^\text{hyb} Z_\pi Z_K \, , \quad (C39)
\]

Analogous expressions hold for the other channels and the results can be found in the right part of Table 4 and the left part of Table 5.

3. Third term of the Lagrangian of Eq. (39)

The third term of the effective Lagrangian (39) generate two-body decays for hybrid mesons into (axial-)vector mesons and (pseudo)scalar mesons, which are written as
\[
\mathcal{L}_{\text{eLSM,}3}^{\text{hybrid-linear}} = \alpha^{\text{hyb}} \mathcal{R} (\tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}} \Phi R^{\mu \nu} \Phi^\dagger - \tilde{\mathcal{R}}^{\text{hyb}} \Phi^\dagger L^{\mu \nu} \Phi) \, .
\]

We first check the invariance under \( P \) and \( C \). Parity is conserved because the first term transforms into the second:
\[
\text{Tr}(\tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}} \Phi R^{\mu \nu} \Phi^\dagger) \overset{P}{\to} \text{Tr}(\tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}} \Phi R^{\mu \nu} \Phi^\dagger L^{\mu \nu} \Phi) \, . \quad (C40)
\]

Note, the extra minus is due to the fact that the hybrid field is dual.

Under \( C \) one has
\[
\text{Tr}(\tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}} \Phi R^{\mu \nu} \Phi^\dagger) \overset{C}{\to} \text{Tr}(\tilde{\mathcal{R}}^{\text{hyb}} \Phi^\dagger \Phi (R^{\mu \nu} \Phi^\dagger) \, , \quad (C41)
\]

and hence, \( C \)-invariance is preserved. As a last point, we check hermiticity:
\[
\text{Tr}(\tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}} \Phi R^{\mu \nu} \Phi^\dagger)^\dagger = \text{Tr}(\Phi R^{\mu \nu} \Phi^\dagger \tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}}) = \text{Tr}(\Phi R^{\mu \nu} \Phi^\dagger \tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}}) = \text{Tr}(\tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}} \Phi^\dagger R^{\mu \nu} \Phi) \, . \quad (C42)
\]

In terms of the nonets, we isolate the following relevant terms relevant for the two-body decays (obtained considering one condensation of the field \( \Phi \)):
\[
\mathcal{L}_{\text{eLSM,}3}^{\text{hybrid-linear}} = 2\alpha^{\text{hyb}} \mathcal{R} (\tilde{\mathcal{L}}_{\mu \nu}^{\text{hyb}} (-i \Phi_0 V^{\mu \nu} P + i PV^{\mu \nu} \Phi_0)
+ 2\alpha^{\text{hyb}} \mathcal{R} (\tilde{B}_{\mu \nu}^{\text{hyb}} (i \Phi_0 A^{\mu \nu} P - i PA^{\mu \nu} \Phi_0) + \cdots \quad (C43)
\]

Then, we obtain decays of the type \( \tilde{\Pi}^{\text{hyb}} \to PV \) and \( \tilde{B}_{\mu \nu}^{\text{hyb}} \to AP \). Using Eq. (A6), the flavor-invariant piece is given by
\[
\mathcal{L}_{\text{eLSM,}3}^{\text{hybrid-linear}} = 2i\alpha^{\text{hyb}} \tilde{\Phi} N \frac{2}{2} \left\{ \text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) - \text{Tr}(\tilde{B}_{\mu \nu}^{\text{hyb}} [P, A^{\mu \nu}]) \right\} + \cdots
\]

For completeness, we verify the correctness of the first term upon checking the invariance under \( C, P \), and \( \dagger \) :
\[
\text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) \overset{C}{\to} \text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) = \text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) \, ; \quad (C44)
\]
\[
\text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) \overset{P}{\to} \text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [-P, V^{\mu \nu}]) = \text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) \, ; \quad (C45)
\]
\[
\left\{ i\text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) \right\} ^\dagger = -i\text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) \right\} ^\dagger = i\text{Tr}(\tilde{\Pi}_{\mu \nu}^{\text{hyb}} [P, V^{\mu \nu}]) \, . \quad (C46)
\]
As an example, let us consider the following explicit term in the Lagrangian:

\[ \mathcal{L}^{\text{hybrid-linear}}_{eLSM-q^{hyb}} = 2i\alpha^{hyb}\phi N Z_{\pi} \tilde{\pi}^{+\mu\nu} (\pi^{-\rho,\mu\nu} + \pi^0 \rho^{-\rho,\mu\nu}) + \cdots, \]  

(C47)

for which the explicit decay width reads:

\[ \Gamma_{\pi^{\mu\nu} \to \rho\pi} = \frac{2k_1}{8\pi m^2_{\pi^{hyb}}} \left[ \left(2\alpha^{hyb}\phi N Z_{\pi}\right)^2 \frac{8}{3} \frac{m^2_{\pi^{hyb}} k_1^2}{\pi^2_{\pi^{hyb}}} \right] \]  

(C48)

Similar decay widths hold for the other channels. The corresponding results can be found in the right part of Table 5.

4. Fourth term of the Lagrangian of Eq. (39): the anomaly term

Finally, we consider the fourth (and the last) term of Eq. (39):

\[ \mathcal{L}^{\text{hybrid-linear}}_{eLSM,4} = \beta^{hyb}_A (\det \Phi - \det \Phi^\dagger) \text{Tr}(L^{hyb}_\mu (\partial^\mu \Phi \cdot \Phi^\dagger) - R^{hyb}_\mu (\partial^\mu \Phi^\dagger \cdot \Phi^\dagger)). \]  

(C49)

As for the other cases, we check the transformation properties. Under parity, the first term transforms as:

\[ (\det \Phi - \det \Phi^\dagger) \text{Tr}(L^{hyb}_\mu (\partial^\mu \Phi \cdot \Phi^\dagger)) \overset{P}{\longrightarrow} (\det \Phi^\dagger - \det \Phi) \text{Tr}(R^{hyb}_\mu (\partial^\mu \Phi^\dagger \cdot \Phi)) \]  

(C50)

therefore, the first term transforms into the third. Similarly, the second converts into the fourth, assuring that \( P \) is preserved.

Next, one has

\[ (\det \Phi - \det \Phi^\dagger) \text{Tr}(L^{hyb}_\mu (\partial^\mu \Phi \cdot \Phi^\dagger)) \overset{C}{\longrightarrow} (\det \Phi^\dagger - \det \Phi^\prime) \text{Tr}(R^{hyb}_{\mu\tau} (\partial^\mu \Phi^\dagger \cdot \Phi^\dagger)) \]  

which shows that the first term converts into the fourth. Similarly, the second goes into the third.

Finally, we show that the Lagrangian is Hermitian. For the first term:

\[ \left\{ (\det \Phi - \det \Phi^\dagger) \text{Tr}(L^{hyb}_\mu (\partial^\mu \Phi \cdot \Phi^\dagger)) \right\}^\dagger = (\det \Phi^\dagger - \det \Phi) \text{Tr}(\Phi \partial^\mu \Phi^\dagger L^{hyb}_\mu) \]  

(C51)

which shows that the first converts into the second.

In terms of the fields, we recall that [30]

\[ \det \Phi - \det \Phi^\dagger = i \frac{Z_{\pi}}{2} \sqrt{\frac{3}{2}} \phi_N^3 \eta_0 + \cdots, \]  

(C53)

where dots refer to flavor breaking corrections and to terms involving two or more fields. Then, upon condensation of one \( \Phi \) and using Eq. (A6):

\[ \mathcal{L}^{\text{hybrid-linear}}_{eLSM,4} = i \beta^{hyb}_A \frac{Z_{\pi}}{4} \sqrt{\frac{3}{2}} \phi_N^3 \eta_0 \text{Tr}(L^{hyb}_\mu (\partial^\mu \Phi - \partial^\mu \Phi^\dagger) - R^{hyb}_{\mu\tau} (\partial^\mu \Phi^\dagger - \partial^\mu \Phi)) + \cdots \]  

(C54)
\[ i\beta^\text{hyb}_A Z_\pi \sqrt{\frac{3}{2} \phi_N^3} \eta_0 \text{Tr} \left( \frac{L^\text{hyb}_{\mu} (2i \partial^\mu P) - R^\text{hyb}_{\mu} (-2i \partial^\mu P)}{4} \right) + \cdots \]  
\[ - \beta^\text{hyb}_A Z_\pi \sqrt{\frac{3}{2} \phi_N^3} \eta_0 \text{Tr} \left( \Pi^\text{hyb}_{\mu} \partial^\mu P \right) + \cdots \]  
(C55)

which described the decay \( \Pi^\text{hyb}_{\mu} \rightarrow P \eta_0 \) and hence \( \Pi^\text{hyb}_{\mu} \rightarrow P \eta \) and \( \Pi^\text{hyb}_{\mu} \rightarrow P \eta' \).

As an example, let us report the explicit form of decay width of the process \( \pi^1 \rightarrow \pi \eta \):

\[ \Gamma_{\pi^1 \rightarrow \pi \eta} = \frac{k_1}{8\pi m_{\pi^1}^2} \left( \beta^\text{hyb}_A Z_\pi \frac{\sqrt{3}}{2} \phi_N^3 \right)^2 \frac{1}{3} \left[ \frac{1}{3} \mp \frac{m_{\pi^1}^2 + m_\eta^2 - m_\eta^2}{m_{\pi^1}^2} \right]^2 . \]  
(C57)

The other decays listed in Table 6 are calculated following the same steps.

Note, the decays \( \eta_{1,N} \rightarrow \eta \eta \) and \( \eta_{1,S} \rightarrow \eta \eta \) do not take place because the final state consists of identical particles. The total amplitude is the sum of two terms—the direct and the crossed Feynman diagrams—which cancel each other. More precisely, the Lagrangian above contains a terms proportional to \( \eta_{\mu \nu}^1, \eta_{\nu \mu}^1, \eta_{\nu \nu}^1 \), \( (\partial_{\mu} \eta_{\nu \mu}^1, \eta_{\nu \nu}^1) \eta_{\nu \nu}^2 / 2 = (\partial_{\mu} \eta_{\nu \mu}^1, \eta_{\nu \nu}^1) \eta_{\nu \nu}^2 / 2 , \) which does not lead to any decay, since the first term on the r.h.s. is a full derivative and the second term contains \( (\partial_{\mu} \eta_{\nu \mu}^1, \eta_{\nu \nu}^1) \) which vanishes as a consequence of the Proca equation.

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