ELLIPTIC EQUATIONS IN SOBOLEV SPACES WITH MORREY DRIFT AND THE ZEROTH-ORDER COEFFICIENTS

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Abstract. We consider elliptic equations with operators $L = a^{ij}D_{ij} + b^i D_i - c$ with $a$ being almost in VMO, $b$ in a Morrey class containing $L_d$, and $c \geq 0$ in a Morrey class containing $L_{d/2}$. We prove the solvability in Sobolev spaces of $Lu = f \in L^p$ in bounded $C^{1,1}$-domains, and of $\lambda u - Lu = f$ in the whole space for any $\lambda > 0$. Weak uniqueness of the martingale problem associated with such operators is also discussed.

1. Introduction

This paper is a natural continuation of [12] in which the main coefficients are almost in VMO, the drift and the zeroth-order coefficients of the equations are allowed to have the least possible powers of integrability and the equations still admit solutions in Sobolev spaces. In case the main part of the operator has smooth coefficients some results of [12] can be found in Theorem 10 of [18]. Also it is worth mentioning that in [2] estimates similar to those in [12] are obtained but on the right in these estimates the zeroth order norm of the unknown function is present.

Next relaxation of integrability condition is to express it in terms of Morrey spaces. Then one can be naturally interested in solutions also in Morrey, rather than Sobolev, spaces. An investigation to this effect is presented in [17] in the framework of linear and fully nonlinear equations, albeit with bounded zeroth-order coefficient. Here we confine ourselves to the case of linear equations and, following the scheme in [12], deal with Sobolev space solutions.

Let $\mathbb{R}^d$ be a $d$–dimensional Euclidean space of points $x = (x^1, ..., x^d)$ with $d \geq 3$. We are working with a uniformly elliptic operator

$$Lu(x) = a^{ij}(x)D_{ij}u(x) + b^i(x)D_iu(x) - c(x)u(x), \quad D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j,$$

with measurable coefficients acting on functions given on $\mathbb{R}^d$.

One of our goals is to prove the unique solvability in the classical Sobolev class $W^2_p(\Omega)$ of the equation $Lu = f$ given in a domain $\Omega \subset C^{1,1}$ with

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boundary data $u = g$ on $\partial \Omega$, where $g \in W^2_0(\Omega)$. We also deal with equation $(\lambda - L)u = f$ in $\Omega = \mathbb{R}^d$. The coefficients $b$ and $c$ are allowed to be quite singular. As an example consider the equation

$$\Delta u - \frac{\beta}{|x|^2} x^i D_i u - \frac{\gamma(x)}{|x|^2} u = f$$

(1.1)

in the unit ball $B_1$ in $\mathbb{R}^d$ with zero boundary condition. Our results show that if $|\beta|$ and $\sup |\gamma|$ are small enough, then this problem has a unique solution in $0 W^2_p(B_1)$ ($= W^2_p(B_1) \cap \{u|_{\partial B_1} = 0\}$, $B_R = \{|x| < R\}$) as long as $f \in L^p(B_1)$ and $p \in (1,d/2)$. Observe that the smallness assumption on $|\beta|$ is essential. Indeed, if $\gamma = 0$, $2d - 2\beta = 0$ and $f \equiv 0$, equation (1.1) with zero boundary condition on $\partial B_1$ has two solutions: one is $1 - |x|^2$ and the other equal to zero.

If $\beta = 0$ and $\sup |\gamma|$ is not small our Example 9.1 shows that one can lose uniqueness of even $C^{1,1}$-solutions. In Example 9.2 we show that even if $\gamma$ is small, one can find $p \in (1,2)$ close to 1 such that there will be no uniqueness in $W^2_p(B_1)$. In both cases the effect can be attributed to eigenvalue-like phenomenon and our guess is that the smallness condition on $|\gamma|$ can be replaced with the boundedness of $\gamma \geq 0$. However, the restriction $p < d/2$ is essential because $|x|^{-2} \notin L^p_{\text{loc}}$ if $p \geq d/2$. There is also a peculiar feature of the operator on the left in (1.1) if $\beta = 0$ and $\gamma \equiv 2d > 0$ which is that it does not satisfy Harnack’s inequality. Indeed, applied to $|x|^2$ it yields zero and this “harmonic” function is nonnegative and vanishes at the origin.

We show that (1.1) is solvable even if $\Delta$ is replaced with $a^{ij} D_{ij}$, if it is a uniformly elliptic operator with $a^{ij}$ almost in VMO. In short, we can deal with the solvability in $W^2_p$ for $L$ with the drift term with summability below $L_d$ and the zeroth-order coefficient with summability below $L_{d/2}$. This is a step forward in comparison with the setting, for instance, in [5], $(p = 2)$ or in [6] (general $p$, spaces with weights) where $b$ is at least in $L_d$, $c$ is at least in $L_{d/2}$, and $a^{ij}$ are not so general. Our setting is more general, however, the goals and results are somewhat different.

The probabilistic aspects related to $L$ (without $c$) are investigated in [7] with much more general $b$ than in the present paper (and with $a = (\delta^{ij})$).

In [20] in case of second-order equations the authors consider singular $c$ of certain class but $1/|x|^2$ does not belong to this class. In [19] we find a treatment in terms of weak solutions of equations like (1.1) with coefficients in Kato classes. However, our coefficients are way out of those classes. It seems like at the moment there are no results covering existence and uniqueness of solutions even for (1.1).

There is a huge general literature about elliptic equations with singular coefficients. All kinds of issues are investigated. But the closest to our results and methods the author could find in the literature are those in [12], the methods of which we use frequently, and also in [3], which contain
plenty of information with an extensive list of references and the history of the subject containing results that are beyond the scope of this article. For instance, in [3] the power of summability $p$ of $D^2 u$ can be any number in $(1, \infty)$. In our results we have a restricted range of $p$, but $b$ and $c$ are in Morrey classes containing $L_d$ and $L_{d/2}$, respectively.

The article is organized as follows. Section 2 contains main results. Section 3 is devoted to auxiliary results closely related to Chiarenza-Frasca paper [1]. In Section 4 we prove the first existence theorem, derive some interior estimates and deal with better regularity of solutions. In Section 5 we present two results proved by probabilistic means. These are used in Sections 6 and 7 to prove the solvability of $Lu - \lambda u = f$ with the smallest $\lambda$ generally possible in domains and in the whole space. Section 8 deals with weak uniqueness of solutions of stochastic equations.

In conclusion a few notation. If $\Omega$ is a bounded domain, by $W^2_p(\Omega)$ we denote the usual Sobolev space obtained by closing $C^2(\bar{\Omega})$ with respect to the norm $\|u\|_{W^2_p(\Omega)}$. The space $\dot{W}^2_p(\Omega)$ is obtained by closing $C^2(\bar{\Omega}) \cap \{u : u|_{\partial\Omega} = 0\}$ with respect to the same norm. In a natural way these definitions extend in case $\Omega = \mathbb{R}^d$, where we write $C^\infty_0, L_p, W^2_p$ in place of $C^\infty_0(\mathbb{R}^d)$, $L_p(\mathbb{R}^d), W^2_p(\mathbb{R}^d)$. By $Du$ we denote the gradient of a function $u$ and $D^2 u$ its Hessian. By $|\Gamma|$ we denote the volume of $\Gamma \subset \mathbb{R}^d$ and set

$$\int_{\Gamma} f \, dx := \frac{1}{|\Gamma|} \int_{\Gamma} f \, dx.$$ 

Set $B_r(x)$ to be the open ball in $\mathbb{R}^d$ of radius $r$ centered at $x$, $B_r = B_r(0)$, $\mathbb{B}$ the collection of open balls. For $B \in \mathbb{B}$ we define $r_B$ as the radius of $B$.

2. Main results

We have some parameters $\delta \in (0, 1]$ and $R_0, R_a, K \in (0, \infty)$, which are fixed, and $\theta_a, \theta_b, \theta_c \in (0, \infty)$, the values of which are specified later.

Introduce $a(x) = (a^{ij}(x))$, 

$$\text{osc} (a, B) = |B|^{-2} \int_{x,y \in B} |a(x) - a(y)| \, dx \, dy,$$

$$a^\#_\rho = \sup_{B \in \mathbb{B}, r_B \leq \rho} \text{osc} (a, B).$$

Assumption 2.1. (i) The matrices $a(x)$ are symmetric and satisfy

$$\delta^{-1} |\xi|^2 \geq a^{ij}(x) \xi^i \xi^j \geq \delta |\xi|^2$$

for all $\xi, x \in \mathbb{R}^d$.

(ii) For $\rho \leq R_a$ we have $a^\#_\rho \leq \theta_a$.

The assumptions on $b$ and $c$ depend on the power

$$p \in (1, d]$$
of summability of the second-order derivatives we want to expect. Recall that the Muckenhoupt space $A_1$ consists of functions such that $M|f| \leq N|f|$ for a constant $N$, where $M$ is the Hardy-Littlewood maximal operator. The smallest constant $N$ is denoted by $[f]_{A_1}$.

**Assumption 2.2** $(q_b, p, \theta_b)$. We have $q_b \in [p, d]$ and for each ball $B$ with $r_B \leq R_0$ it holds that

$$
\left( \int_B |b|^{q_b} \, dx \right)^{1/q_b} \leq \theta_b r_B^{-1}.
$$

(2.2)

In addition, if $p = q_b$, then $[|b|^{p}]_{A_1} \leq K$.

**Assumption 2.3.** We have $q_c \in [p, \infty)$, and either

$$q_c > \frac{d}{2}, \quad \|c\|_{L_{q_c}} \leq K,
$$

(2.3)

or $q_c \leq \frac{d}{2}$ ($d \geq 3$) and for each ball $B$ with $r_B \leq R_0$ it holds that

$$
\left( \int_B |c|^{q_c} \, dx \right)^{1/q_c} \leq \theta_c r_B^{-2}.
$$

(2.4)

In addition, if $p = q_c \leq \frac{d}{2}$, then $[|c|^{p}]_{A_1} \leq K$.

**Remark 2.4.** (i) If $|b| = 1/|x|$, condition (2.2) is satisfied for any $q_b < d$ (with an appropriate $\theta_b$) and, if $|c| = 1/|x|^2$, condition (2.4) is satisfied with any $q_c < d/2$. In addition, recall that for $u = 1/|x|^r$, $0 < r < d$, we have $u \in A_1$.

(ii) Conditions (2.2) and (2.4) are satisfied with any $\theta_b, \theta_c > 0$ if $|b|$ and $|c|$ are bounded by a constant, say $K$, on the account of choosing $R_0$ sufficiently small (depending on $\theta_b, \theta_c$, and $K$).

(iii) Conditions (2.2) and (2.4) are satisfied with any $\theta_b, \theta_c > 0$ if $|b| \in L_d$ and $c \in L_{d/2}$ on the account of choosing $R_0$ sufficiently small (depending on $\theta_b, \theta_c$). Indeed, for instance, in case of $b$ by Hölder’s inequality

$$
\left( \int_B |b|^{q_b} \, dx \right)^{1/q_b} \leq \left( \int_B |b|^{d} \, dx \right)^{1/d} r_B^{-1}.
$$

This shows that the results of the present article generalize the corresponding results in [12].

(iv) If (2.2) holds with an exponent $\hat{q}_b > q_b$ in place of $q_b$, then it holds as is due to Hölder’s inequality.

(v) We are going to say that Assumption 2.3 is satisfied with, say $\theta_c = 1$ (or any other value) if (2.3) holds.

Our first main result is about existence and uniqueness of solutions for equations $Lu - \lambda u = f$ for $\lambda$ large. Here we do not have extra restrictions on $q_b$.

**Theorem 2.5.** Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ of class $C^{1,1}$ or $\Omega = \mathbb{R}^d$. Suppose that Assumption 2.1 is satisfied with $\theta_a = \theta_a(d, \delta, p)$ from Lemma 4.1 and Assumptions 2.2 and 2.3 are satisfied with $\theta_b, \theta_c$ introduced
in Definition 4.2. Then there exist $\lambda_0 \geq 1, N_0$, depending only on $p, q_b, q_c, d, \delta, R_a, R_0, K,$ and $\Omega$, such that, for any $u \in W^2_p(\Omega)$ and $\lambda \geq \lambda_0$,

$$||D^2u||_{L_p(\Omega)} + \sqrt{\lambda}||Du||_{L_p(\Omega)} + \lambda||u||_{L_p(\Omega)} \leq N_0||Lu - \lambda u||_{L_p(\Omega)}.$$  \hfill (2.5)

Furthermore, for any $f \in L_p(\Omega)$ there exists a unique $u \in W^2_p(\Omega)$ such that $Lu - \lambda u = f$.

This theorem is proved in the beginning of Section 4.

Next, we need parameters $d_0 = d_0(d, \delta) \in (d/2, d)$ and $\hat{b}(d, \delta) > 0$ introduced in [15] in order to be able to reduce $\lambda$ down from $\lambda_0$. Set

$$P = (d, K, \delta, R_a),$$

take $\tilde{\theta}_b(p, q_b, P)$ introduced before Theorem 4.9, and define

$$\theta_b(p, q_b, P, \Omega) = \tilde{\theta}_b(p, q_b, P, \Omega) \wedge \hat{b}(d, \delta).$$

**Assumption 2.6.** We have a domain $\Omega$ which is either bounded and of class $C^{1,1}$ or $\Omega = \mathbb{R}^d$. For a number $r$ we have $q_b \geq r > d/2$, $q_b > d_0$, $r \geq p$, Assumption 2.1 is satisfied with $\theta_a = \theta_a(d, \delta, p)$ from Lemma 4.1, Assumption 2.2 is satisfied with $\theta_b = \tilde{\theta}_b(p, q_b, P, \Omega)$, and Assumption 2.3 is satisfied with $\theta_c$ introduced in Definition 4.2. Moreover if $r > p$, Assumptions 2.2 ($q_b, p(n), \theta_b(n)$), $n = 0, 1, \ldots, m$, are satisfied, where $p(n) (\in [p, r])$ are specified in the proof of Theorem 4.9, $\theta_b(n) = \tilde{\theta}_b(p(n), q_b, P, \Omega)$.

**Remark 2.7.** The role of $r$ may need an explanation. If $p = q_b > d/2$, there is only one possibility, $r = p$, and there is no need in the second part of Assumption 2.6. Observe that in this case we require $|b|^p \in A_1$. If $q_b > p > d/2$ one can take $r = p$ and again there is no need in the second part of Assumption 2.6. This part comes into real play only if $p \leq d/2$ when we can take $r$ as close to $d/2$ as we wish ($q_b > d_0 > d/2$).

**Theorem 2.8.** Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ of class $C^{1,1}$ and suppose that $c \geq 0$ and Assumption 2.6 is satisfied. Then there exists a constant $N$ depending only on $d, \delta, p, q_b, q_c, r, R_0, R_a, K,$ and $\Omega$, such that for any $\lambda \geq 0$ and $u \in W^2_p(\Omega)$

$$||u||_{W^2_p(\Omega)} \leq N\|\|u||_{L_p(\Omega)}.$$  \hfill (2.6)

Furthermore, for any $f \in L_p(\Omega)$ there exists a unique $u \in W^2_p(\Omega)$ such that $\lambda u - Lu = f$ in $\Omega$.

This theorem is proved in Section 6.

In the whole space we have the following.

**Theorem 2.9.** Suppose that $\Omega = \mathbb{R}^d$, $c \geq 0$, $\varepsilon_0 \in (0, R_0^{-2}]$, $\lambda \geq \varepsilon_0$, Assumption 2.6 is satisfied and it is also satisfied if $\Omega$ is any ball of radius $R = R(\varepsilon_0 R_0, d, \delta)$ introduced in (7.1). Then for any $f \in L_p$ there exists a
unique \( u \in W^2_p \) such that \( \lambda u - Lu = f \). Moreover, there exists a constant \( N \), depending only on \( \varepsilon_0, d, \delta, p, q_b, q_c, r, R_0, R_a, K \), such that

\[
\|u\|_{W^2_p} \leq N\|f\|_{L^p}.
\]  

This theorem is proved in Section 7.

### 3. AUXILIARY RESULTS

Introduce the Morrey space \( E_{r,\beta}, r \geq 1, \beta > 0 \) as the set of functions \( f \) with finite norm

\[
\|f\|_{E_{r,\beta}} = \sup_{B \in \mathbb{B}} r_B^\beta \|f\|_{L^r(B)}, \quad \text{where} \quad \|f\|_{L^r(B)} = \int_B |f|^r \, dx.
\]

Just in case, observe that, if \( r \beta > d \), the space \( E_{r,\beta} \) consists of only one function \( f = 0 \).

**Lemma 3.1.** Let \( 1 \leq p < r < \infty, \beta > 0 \), nonnegative \( \gamma \in E_{r,\beta} \). Then there exists \( \hat{\gamma} \geq \gamma \) such that \( \hat{\gamma} \in E_{r,\beta}, \hat{\gamma}^p \in A_1 \),

\[
\|\hat{\gamma}\|_{E_{r,\beta}} \leq N\|\gamma\|_{E_{r,\beta}}, \quad [\hat{\gamma}^p]_{A_1} \leq N,
\]

where the constants \( N \) depend only on \( d, p, r, \beta \).

Actually, for \( s = (p + r)/2 \) one can define \( \hat{\gamma} = (M(\gamma^s))^{1/s} \), where \( M \) is the maximal operator. Then the result follows from Lemma 1 of [1].

Next comes a result which for \( R_0 = \infty \) coincides with the result one obtains from the proof of the Theorem of [1].

**Lemma 3.2.** Assume \( \theta > 0, 1 < r \leq d \), and a nonnegative \( \gamma \) are such that for all \( \rho \leq R_0, |x| \leq R_0 \)

\[
\|\gamma\|_{L^r(B_\rho(x))} \leq \theta \rho^{-1}, \quad [\gamma^r]_{A_1} \leq K.
\]

Then for any \( u \in C^\infty_0(B_{R_0}) \) we have

\[
I := \|\gamma u\|_{L^r} \leq N \theta^r \|Du\|_{L^r},
\]  

where \( N = N(d, r, K) \).

**Proof.** Changing scales allows us to assume that \( R_0 = 1 \). Then set \( \hat{\gamma} = \gamma I_{B_1} \) and observe that for all \( x \in B_1 \) and \( \rho > 0 \)

\[
\|\hat{\gamma}\|_{L^r(B_\rho(x))} \leq N \theta \rho^{-1}.
\]

Indeed, if \( \rho \leq 1 \), this is obvious, and if \( \rho > 1 \),

\[
\|\hat{\gamma}\|_{L^r(B_\rho(x))} \leq (1/\rho)^{d/r} \|\gamma\|_{L^r(B_1)} \leq N \theta \rho^{-d/r} \leq N \theta \rho^{-1}.
\]

Then, we follow the arguments in [1] and for \( |x| \leq 1 \) define

\[
V(x) = \int_{\mathbb{R}^d} \frac{\hat{\gamma}^r(y)}{|x - y|^{d-2}} \, dy.
\]

Notice that

\[
V(x) = N \int_0^\infty \frac{1}{\rho^{d-2}} \left( \frac{\partial}{\partial \rho} \int_{B_\rho(x)} \hat{\gamma}^r(y) \, dy \right) \, d\rho.
\]
where
\[
\frac{1}{\rho^{d-2}} \int_{B_{\rho}(x)} \tilde{\gamma}^r(y) \, dy \leq N \min \left( \rho^2 [\gamma^r]_{A_1 \gamma^r(x)}, \rho^{2-d} \right),
\]
which shows that we can integrate by parts and get
\[
V(x) = N \int_{0}^{A} \rho \int_{B_{\rho}(x)} \tilde{\gamma}^r \, dy \, d\rho + N \int_{A}^{\infty} \rho^{1-d} \int_{B_{\rho}(x)} \tilde{\gamma}^r \, dy \, d\rho,
\]
where \( A > 0 \) is any number. For small \( \rho \) we use that \( \tilde{\gamma} \leq \gamma \) and \( \gamma^r \in A_1 \), and for large \( \rho \) we use that \( \tilde{\gamma} \in L_r \). Then we see that \( V \) is well defined.

Similarly,
\[
|DV(x)| \leq N \int_{0}^{\infty} \frac{1}{\rho^{d-1}} \left( \frac{\partial}{\partial \rho} \int_{B_{\rho}(x)} \tilde{\gamma}^r(y) \, dy \right) \, d\rho
\]
\[
= N \int_{0}^{A} \int_{B_{\rho}(x)} \tilde{\gamma}^r \, dy \, d\rho + N \int_{A}^{\infty} \int_{B_{\rho}(x)} \tilde{\gamma}^r \, dy \, d\rho
\]
\[
\leq N \gamma^r(x) \int_{0}^{A} d\rho + N \theta^r \int_{A}^{\infty} \rho^{-r} \, d\rho = N A \gamma^r(x) + N A^{1-r} \theta^r,
\]
where \( A > 0 \) is any number. For \( A^{-r} \theta^r = \gamma^r(x) \) we conclude that on \( B_1 \) we have \( |DV| \leq N \theta \gamma^r \). Next, we use the fact that \( \Delta V = -N \gamma^r \) in \( B_1 \) and integrating by parts and using Hölder’s inequality we get
\[
I = N \int_{\mathbb{R}^d} |u|^r \Delta V \, dx \leq N \int_{\mathbb{R}^d} |DV| |u|^{r-1} |Du| \, dx
\]
\[
\leq N \theta \int_{\mathbb{R}^d} |\gamma u|^{r-1} |Du| \, dx \leq N \theta I^{1-1/r} \|Du\|_{L_r}.
\]
This leads to (3.1) and proves the theorem.

**Theorem 3.3.** Under Assumption 2.2 suppose that \( p = q_0 \). Then there exists a constant \( N = N(d, p, K) \) such that for any \( u \in C_0^\infty \) we have
\[
\|bu\|_{L_p}^p \leq N \theta_0^p (\|Du\|_{L_p}^p + R_0^{-p} \|u\|_{L_p}^p).
\]

**Proof.** Take \( \zeta \in C_0^\infty(B_{R_0}), \zeta \geq 0 \), such that
\[
\int_{B_{R_0}} \zeta^p \, dx = 1, \quad \int_{B_{R_0}} |D\zeta|^p \, dx = N(d) R_0^{-p}.
\]
By shifting the origin we get from Lemma 3.2 that for any \( x \)
\[
\int_{\mathbb{R}^d} |b(y)u(y)\zeta(x+y)|^p \, dy
\]
\[
\leq N \theta_0^p \int_{\mathbb{R}^d} (\zeta^p(x+y)|Du(y)|^p + |D\zeta(x+y)|^p |u(y)|^p) \, dy.
\]
Integrating through this relation over \( \mathbb{R}^d \) leads to (3.3) and proves the theorem.
Lemma 3.4. Assume \( \theta > 0, \ 1 < r \leq d/2, \) and nonnegative \( \gamma \) are such that for all \( \rho \leq R_0, |x| \leq R_0 \)

\[
\|\gamma\|_{L^r(B_\rho(x))} \leq \theta \rho^{-2}, \quad [\gamma^\gamma]_{A_1} \leq K.
\]

Then for any \( u \in C_0^\infty(B_{R_0}) \) we have

\[
I := \|\gamma u\|_{L^r} \leq N\theta^r\|D^2u\|_{L^r},
\]

(3.4)

where \( N = N(d, r, K) \).

Proof. Again we may assume that \( R_0 = 1 \). For the function \( V \) from the proof of Lemma 3.2 we have

\[
|DV(x)| \leq N\gamma^r(x) \int_0^A d\rho + N\theta^r \int_A^\infty \rho^{-2r} \ d\rho = NA\gamma^r(x) + NA^{-2r}\theta^r,
\]

which for \( A^{-2r}\theta^r = \gamma^r \) yields \(|DV| \leq N\gamma^{r-1/2}\theta^{1/2} \). Hence

\[
I = N \int_{\mathbb{R}^d} |u|^r \Delta V \ dx \leq N \int_{\mathbb{R}^d} |DV| |u|^{r-1} |Du| \ dx
\]

\[
\leq N\theta^{1/2} \int_{\mathbb{R}^d} |\gamma u|^{r-1} |\gamma^{1/2} Du| \ dx \leq N\theta^{1/2} J^{1-1/r} J^{1/r}
\]

with

\[
J := \int_{\mathbb{R}^d} |\gamma^{1/2} Du|^r \ dx \leq N\theta^{r/2} \int_{\mathbb{R}^d} |D^2 u|^r \ dx,
\]

where the inequality follows from Lemma 3.2. This yields (3.4) and proves the lemma.

Quite similarly to Theorem 3.3 we get the following.

Theorem 3.5. Under Assumption 2.3 suppose that \( p = q_c \leq d/2 \). Then there exists a constant \( N = N(d, p, K) \) such that for any \( u \in C_0^\infty \) we have

\[
\|\gamma u\|_{L^p}^p \leq N\theta^p (\|D^2 u\|_{L^p}^p + R_0^{-p}\|Du\|_{L^p}^p + R_0^{-2p}\|u\|_{L^p}^p).
\]

(3.5)

By sending \( R_0 \to \infty \) in Lemma 3.4 we arrive at the following.

Lemma 3.6. Assume \( 1 < r < \infty, \gamma \in E_{r,2}, \) and \(|\gamma|^{r} \in A_1\). Then for any \( u \in C_0^\infty \)

\[
\|\gamma u\|_{L^r} \leq N \|\gamma\|_{E_{r,2}} \|D^2 u\|_{L^r},
\]

(3.6)

where \( N \) depends only on \( d, r \) and \(|\gamma|^{r} \). A_1.\)

Formally speaking in Lemma 3.6 we have to assume that \( r \leq d/2 \) since this is one of the assumptions of Lemma 3.4. However, if \( r > d/2 \) the space \( E_{r,2} \) is trivial consisting of only zero function. The same comment applies to the corollary below.

Corollary 3.7. Assume \( 1 < r < q < \infty \) and \( \gamma \in E_{q,2} \). Then for any \( u \in C_0^\infty \)

\[
\|\gamma u\|_{L^r} \leq N \|\gamma\|_{E_{q,2}} \|D^2 u\|_{L^r},
\]

(3.7)

where \( N \) depends only on \( d, r, q \).
Indeed, by Lemma 3.1 we can replace $\gamma$ with $\tilde{\gamma}$ and then by Lemma 3.6 get (3.7) with $\|\tilde{\gamma}\|_{E_{r,2}}$ in place of $\|\gamma\|_{E_{r,2}}$. After that it only remains to observe that $\|\tilde{\gamma}\|_{E_{r,2}} \leq \|\tilde{\gamma}\|_{E_{q,2}} \leq N\|\gamma\|_{E_{q,2}}$.

The following corollary of embedding theorems is well known.

**Lemma 3.8.** Let $R \in (0, \infty]$, $1 < s \leq r < \infty$, $u \in W^1_s(B_R)$

$$1 + \frac{d}{r} - \frac{d}{s} > 0.$$ 

Then with $N = N(d, s, r)$

$$\|u\|_{L_r(B_R)} \leq N\|Du\|_{L_s(B_R)} \|u\|_{L_s(B_R)}^{1 + d/r - d/s} + NR^{d/r - d/s}\|u\|_{L_s(B_R)}.$$  

(3.8)

The following theorem, in particular, generalizes and implies Corollary 3.7 when $R = R_0 = \infty$.

**Theorem 3.9.** Suppose that Assumptions 2.2 and 2.3 are satisfied and let $u \in C_0^\infty$, $R \in (0, \infty]$. Then

(i) For $1 < p \leq q_c \leq d$ there exists a constant $N_0 = N_0(p, q_c, d, K)$ such that

$$\|bu\|_{L_p(B_R)} \leq N_0\theta_b\|Du\|_{L_p(B_R)} + N_0\theta_b(R^{-1} + R_0^{-1})\|u\|_{L_p(B_R)},$$  

(3.9)

$$\|b|Du\|_{L_p(B_R)} \leq N_0\theta_b\|D^2u\|_{L_p(B_R)} + N_0\theta_b(R^{-1} + R_0^{-1})\|Du\|_{L_p(B_R)}.$$  

(3.10)

(ii) For $q_c > d/2$ there exists a constant $N = N(p, q_c, d)$ such that

$$\|cu\|_{L_p(B_R)} \leq N\theta_c\big((\|D^2u\|_{L_p(B_R)}^{d/(2q_c)})\|u\|_{L_p(B_R)}^{1 - d/(2q_c)} + R^{-d/q_c}\|u\|_{L_p(B_R)}\big).$$  

(3.11)

(iii) For $q_c \leq d/2$ there exists a constant $N_0 = N_0(p, q_c, d, K)$ such that

$$\|cu\|_{L_p(B_R)} \leq N_0\theta_c\big((\|D^2u\|_{L_p(B_R)} + (R_0^{-2} + R^{-2})\|Du\|_{L_p(B_R)}\big).$$  

(3.12)

Proof. Estimate (3.9) for $p < q_b$ and $R = \infty$ is proved as Lemma 3.5 in [14]. If $R = \infty$ and $p = q_b$ (and $[b]_{A_1} \leq K$) it is proved in Theorem 3.3. If $R < \infty$, take any extension operator $\Pi_R$ which extends smooth functions in $B_R$ to $C_0^\infty$ functions and is such that

$$\|\Pi u\|_{L_p} \leq N\|u\|_{L_p(B_R)},$$  

$$\|D\Pi u\|_{L_p} \leq N\|Du\|_{L_p(B_R)} + NR^{-1}\|u\|_{L_p(B_R)},$$  

$$\|D^2\Pi u\|_{L_p} \leq N\|D^2u\|_{L_p(B_R)} + NR^{-2}\|u\|_{L_p(B_R)}$$  

(the latter will be needed while dealing with (iii)), where $N = N(d, p)$. By the way, the fact that $N$ can be chosen independent of $R$ is easily proved by rescaling. Then after applying (3.9) with $R = \infty$ to $\Pi_R u$ we obtain (3.9) as is. Estimate (3.10) is an obvious corollary of (3.9).

To prove (ii), if $p < q_c$, we write

$$\|cu\|_{L_p(B_R)} \leq \|c\|_{L_{q_c}(B_R)}\|u\|_{L_{p,q_c/(q_c-p)}(B_R)}.$$  

Then we use embedding theorems (see (3.8)) and observe that

$$\|Du\|_{L_p(B_R)} \leq N\|D^2u\|_{L_p(B_R)}^{1/2}\|u\|_{L_{p}(B_R)}^{1/2} + NR^{-1}\|u\|_{L_p(B_R)},$$  

(3.13)
If $p = q_c$, then $p > d/2$, and the result follows again by embedding theorem $(\sup |u| \leq N\|u\|_{W^2_p})$.

As in the case of (i) while proving (iii) we reduce the general situation to the one where $R = \infty$. Then, if $p = q_c$, we get the result by Theorem 3.5. If $p < q_c$, to prove (iii) take $\zeta \in C^\infty_0(B_{R_0})$, $\zeta \geq 0$, such that

$$
\int_{B_{R_0}} \zeta^{2p} \, dx = 1, \quad \zeta + R_0|D\zeta| + R_0^2|D^2\zeta| \leq N(d)R_0^{-d/(2p)}.
$$

(3.14)

We claim that for any $B \in \mathbb{B}$ we have

$$
\left( \int_{B} |c\zeta|^q \, dx \right)^{1/q} \leq NR_0^{-d/(2p)} \theta c^{-2}.
$$

(3.15)

Indeed, if $r_B \leq R_0$ it suffices to use that $\zeta \leq NR_0^{-d/(2p)}$. In case $r_B > R_0$, it suffices to use that

$$
\int_{B} |c\zeta|^q \, dx = NR_0^{-d} \int_{B} |c\zeta|^q \, dx \leq NR_0^{-q_c/(2p)}r_B^{-d} \int_{B_{R_0}} |c|^q \, dx
$$

$$
= NR_0^{-q_c/(2p)}r_B^{-d} \int_{B_{R_0}} |c|^q \, dx \leq NR_0^{-q_c/(2p)}r_B^{-d}\theta c^{-2}R_0^{-2q_c}
$$

$$
= NR_0^{-q_c/(2p)}(R_0/r_B)^{-d-2q_c}\theta c^{-2} \leq NR_0^{-q_c/(2p)}\theta c^{-2}.
$$

Now, in light of (3.15) by Corollary 3.7

$$
\int_{\mathbb{R}^d} |c\zeta u|^p \, dx \leq NR_0^{-d/2}\theta c\int_{\mathbb{R}^d} |D^2 u|^p \, dx.
$$

We plug in here $\zeta(\cdot + y)$ and $\zeta(\cdot + y)u$ in place of $\zeta$ and $u$, respectively. Then we get

$$
\int_{\mathbb{R}^d} \zeta^p(x + y)|cu|^p \, dx \leq NR_0^{-d/2}\theta c\int_{\mathbb{R}^d} \left[ \zeta^p(x + y)|D^2 u| + |D\zeta(x + y)|^p |Du| + |D^2\zeta(x + y)|^p |u|^p \right] \, dx.
$$

After integrating through with respect to $y$ and using (3.13) and that by Hölder’s inequality and (3.14)

$$
\int_{\mathbb{R}^d} \zeta^p \, dy \leq NR_0^{-d/2}, \quad \int_{\mathbb{R}^d} |D\zeta|^p \, dy \leq NR_0^{-d/2-p}, \quad \int_{\mathbb{R}^d} |D^2\zeta|^p \, dy \leq NR_0^{-d/2-2p},
$$

we come to (3.12). The theorem is proved.

Remark 3.10. Theorem 3.9 will still hold if we replace balls $B_R$ with half-balls. To see this it suffices to just extend our functions across the flat part to the whole ball. Actually the boundary of “half balls” even need not to be flat, as long as it allows one to extend the functions $u$ across the border to $\bar{u}$ not much distorting the $L_p$ norms of $u$, $Du$, $D^2u$. Therefore we can consider $B_R(x) \cap \Omega$, where $\Omega$ is a bounded domain of class $C^{1,1}$ and $x \in \partial\Omega$. Of
course, in this situation $R$ should be sufficiently small, $R = R(d, p, R_0, \Omega)$. However, having it small enough, we can have
\[
\|\hat{u}\|_{L_p(B_R(x))} \leq N_0 \|u\|_{L_p(B_R(\Omega))},
\]
\[
\|D\hat{u}\|_{L_p(B_R(x))} \leq N_0 \|Du\|_{L_p(B_R(\Omega))} + N_1 \|u\|_{L_p(B_R(\Omega))},
\]
\[
\|D^2\hat{u}\|_{L_p(B_R(x))} \leq N_0 \|D^2u\|_{L_p(B_R(\Omega))} + N_1 \|u\|_{L_p(B_R(\Omega))},
\]
where $N_0 = N(d, p)$, $N_1 = N_1(d, p, \Omega)$. This and partitions of unity lead to the following result.

**Theorem 3.11.** Suppose that Assumptions 2.2 and 2.3 are satisfied, $\Omega$ is a bounded domain in $\mathbb{R}^d$ of class $C^{1,1}$ and let $u \in W^2_p(\Omega)$.

(i) For $1 < p \leq q_b \leq d$ there exist constants $N_0 = N_0(p, q_b, d, K)$ and $N_1 = N_1(p, q_b, d, K, R_0, \Omega)$ such that
\[
\|bu\|_{L_p(\Omega)} \leq N_0 \theta_b \|Du\|_{L_p(\Omega)} + N_1 \theta_b \|u\|_{L_p(\Omega)},
\]
\[
\|b\| |Du|\|_{L_p(\Omega)} \leq N_0 \theta_b \|D^2u\|_{L_p(\Omega)} + N_1 \theta_b \|Du\|_{L_p(\Omega)}.
\]

(ii) For $q_c > d/2$ there exists a constant $N = N(p, q_c, d, \Omega)$ such that
\[
\|cu\|_{L_p(\Omega)} \leq NK \left( \|D^2u\|_{L_p(\Omega)}^{d/(2q_c)} \|u\|_{L_p(\Omega)}^{1-d/(2q_c)} + \|u\|_{L_p(\Omega)} \right).
\]

(iii) For $q_c \leq d/2$ there exist constants $N_0 = N_0(p, q_c, d, K)$ and $N_1 = N_1(p, q_c, d, K, R_0, \Omega)$ such that
\[
\|cu\|_{L_p(\Omega)} \leq N_0 \theta_c \|D^2u\|_{L_p(\Omega)} + N_1 \theta_c \|u\|_{L_p(\Omega)}.
\]

Of course, a simple consequence of Theorem 3.11 is that $L$ is a bounded operator from $W^2_p(\Omega)$ into $L_p(\Omega)$. In particular, the problem of solving
\[
(\lambda - L)u = f \in L_p(\Omega) \text{ in } W^2_p(\Omega) \text{ with boundary condition } u - g \in W^0_p(\Omega),
\]
where $g \in W^2_p(\Omega)$, reduces to solving $(\lambda - L)w = h \in L_p(\Omega)$ in $W^2_p(\Omega)$ by using the substitution $w = u - g$, $h = f - (\lambda - L)g$.

**Remark 3.12.** If $\Omega$ is as in Theorem 3.11 and we have a sequence of $c_n$, $n = 1, 2, \ldots$, such that each $c_n$ satisfies Assumption 2.3 with the same $R_0, q_c, K, \theta_c$ and $c_n \rightarrow c$ in $L_p(\Omega)$, then for any $u \in W^2_p(\Omega)$ we have $c_n u \rightarrow cu$ in $L_p(\Omega)$.

Indeed, for any smooth $v$ we have $(c_n - c)u = (c_n - c)v + (c_n - c)(u - v)$, where the first term tends to zero in $L_p(\Omega)$ because $c_n \rightarrow c$ in $L_p(\Omega)$ and $v$ is bounded and the $L_p(\Omega)$-norm of the second term is dominated by a constant independent of $n$ times the $W^2_p(\Omega)$-norm of $u - v$, which can be made arbitrarily small on the account of choosing $v$ appropriately.

In case $\Omega = \mathbb{R}^d$ one can approximate $u \in W^2_p(\Omega)$ by functions with compact support. Therefore, in this case we need the convergence $c_n \rightarrow c$ in $L_p$ only in each ball. Similar observation is valid also for $bu$ or $b^i D_i u$. 
4. Proof of Theorem 2.5 and interior estimates

Set

\[ L_0 u = a^{ij} D_{ij} u. \]

Here is a particular case of Theorem 8 of [4] if \( \Omega \in C^{1,1} \) or is a slight restatement of part of Theorem 6.4.1 of [11] if \( \Omega = \mathbb{R}^d \).

Lemma 4.1. Let \( s \in (1, \infty) \) and let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) of class \( C^{1,1} \) or \( \Omega = \mathbb{R}^d \). There exists \( \theta_a = \theta_a(d, \delta, s) > 0 \) such that, if Assumption 2.1 is satisfied with this \( \theta_a \), then there exist \( \lambda_0 \geq 1, \hat{N} \), depending only on \( d, \delta, s, R_a, \) and \( \Omega \), such that, for any \( u \in \dot{W}^2_s(\Omega) \) and \( \lambda \geq \lambda_0 \),

\[
\| D^2 u \|_{L^s(\Omega)} + \sqrt{\lambda} \| Du \|_{L^s(\Omega)} + \lambda \| u \|_{L^s(\Omega)} \leq \hat{N} \| L_0 u - \lambda u \|_{L^s(\Omega)}. \tag{4.1}
\]

Furthermore, for any \( f \in L_s(\Omega) \) there exists a unique \( u \in \dot{W}^2_s(\Omega) \) such that \( L_0 u - \lambda u = f \).

Recall that

\[ P = (d, K, \delta, R_a). \]

Definition 4.2. Let \( \Omega \) be as in Lemma 4.1.

(i) For \( 1 < p \leq q_b \) introduce \( \theta_b = \theta_b(p, q_b, P, \Omega) > 0 \) so that

\[ \theta_b \hat{N} N_0 \leq 1/4, \]

where \( \hat{N} \) is taken from Lemma 4.1 with \( s = p \) and \( N_0 \) is from Theorem 3.11 (i) if \( \Omega \) is bounded and is from Theorem 3.9 (i) if \( \Omega = \mathbb{R}^d \).

(ii) If \( q_c > d/2 \), the value of \( \theta_c \) is irrelevant, just set \( \theta_c = 1 \). If \( q_c \leq d/2 \) introduce \( \theta_c = \theta_c(p, q_c, P, \Omega) > 0 \) so that

\[ \theta_c \hat{N} N_0 \leq 1/4, \]

where \( \hat{N} \) is taken from Lemma 4.1 with \( s = p \) and \( N_0 \) is from Theorem 3.11 (iii) if \( \Omega \) is bounded and is from Theorem 3.9 (iii) if \( \Omega = \mathbb{R}^d \).

Theorems 3.9, 3.11, Lemma 4.1, perturbation method, and the method of continuity immediately lead to the proof of Theorem 2.5 about existence and uniqueness of solutions for equations \( Lu - \lambda u = f \) for \( \lambda \) large.

We denote the solution from Theorem 2.5 by \( R_{\lambda+c} f \).

Remark 4.3. By taking \( \lambda = \lambda_0 \) in (2.5) we see that for the same kind of \( N \) as in (2.5) and any \( u \in \dot{W}^2_p(\Omega) \)

\[
\| u \|_{\dot{W}^2_p(\Omega)} \leq N \left( \| Lu \|_{L^p(\Omega)} + \| u \|_{L^p(\Omega)} \right). \tag{4.2}
\]

The next result, the proof of which is left to the reader, is a standard consequence of Theorem 2.5 combined with Remark 3.12.

Theorem 4.4. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) of class \( C^{1,1} \) or \( \Omega = \mathbb{R}^d \). Let \( a^n, b^n, c^n, n = 1, 2, \ldots, \) be a sequence of symmetric \( d \times d \)-matrix valued,
Lemma 4.6. Suppose $\mathbb{R}^d$-valued, and real-valued, respectively, measurable functions, satisfying Assumptions 2.1, 2.2, and 2.3 with the same $\delta, p, q_b, q_c, R_a, R_0, \theta_0, \theta_b, \theta_c, K$ as in Theorem 2.5. Let $f \in L_p(\Omega)$ and suppose that $a^n \to a$ on $\mathbb{R}^d$ (a.e.) and

$$\|b - b^n\|_{L_p(\Omega \cap B)} + \|c^n - c\|_{L_p(\Omega \cap B)} \to 0$$

as $n \to \infty$ for any ball $B$. Let $\lambda \geq \lambda_0$, where $\lambda_0$ is taken from Theorem 2.5, and introduce $u^n$ as unique $W^{2,p}_p(\Omega)$-solutions of $\lambda u^n - L^n u^n = f$, where the operator $L^n$ is constructed from $a^n, b^n, c^n$. Then

$$\lim_{n \to \infty} \|u^n - R_{\lambda+c} f\|_{W^{2,p}_p(\Omega)} = 0.$$

By using approximation by bounded functions and properties of solutions of equations with bounded coefficients we easily arrive at the following.

Corollary 4.5. If $c \geq 0$, then we can add one more statement in Theorem 2.5: For any $f \in L_p(\Omega)$ we have $|R_{\lambda+c} f| \leq R_{\lambda+c} f| \leq R_{\lambda} f|$ (a.e.).

If $\Omega$ is bounded, obviously, $R_{\lambda+c}$ is independent of $p$. The same holds if $\Omega = \mathbb{R}^d$. To show this, for a moment, denote by $R^{p}_{\lambda+c}$ what was before called $R_{\lambda+c}$.

Lemma 4.6. Suppose $\Omega = \mathbb{R}^d$ and assumptions of Theorem 2.5 are satisfied with $p_1, p_2$ in place of $p$ and

$$0 \leq \frac{d}{p_1} - \frac{d}{p_2} < 1$$

Take $\lambda_0$ as the greater of $\lambda_0$’s corresponding to $p_1$ and $p_2$. Then $R^{p_1}_{\lambda+c} = R^{p_2}_{\lambda+c}$ for $\lambda \geq \lambda_0$.

Proof. Since $L_{p_1} \cap L_{p_2}$ is dense in $L_{p_1}$ and $L_{p_2}$ it suffices to prove that $u := R^{p_1}_{\lambda+c} f = R^{p_2}_{\lambda+c} f =: v$ for $f \in L_{p_1} \cap L_{p_2}$. Take $\zeta \in C^\infty_0$ with support in $B_1$ such that $\zeta(0) = 1$ and set $\zeta_n(x) = \zeta(x/n)$

$$f_n = (\lambda_0 - L)(\zeta_n v),$$

so that

$$(\lambda_0 - L)(\zeta_n v - u) = (\zeta_n - 1)f - v(a^{ij} D_{ij} \zeta_n + b^i D_i \zeta_n) - 2a^{ij}D_i D_j \zeta_n =: g_n.$$ 

Clearly, to prove the lemma, it suffices to show that $g_n \to 0$ in $L_{p_1}$.

By the dominated convergence theorem $(\zeta_n - 1)f \to 0$ in $L_{p_1}$. By Theorem 3.9

$$\|b^n| D_{ij} \zeta_n\|_{L_{p_1}} \leq N n^{-1} \|b^n\|_{L_{p_1}(B_n)} \leq N n^{-1} (\|D v\|_{L_{p_1}(B_n)} + \|v\|_{L_{p_1}(B_n)}),$$

where the constants $N$ are independent of $n$. By Hölder’s inequality the last expression is dominated by

$$N n^{-1} n^{d/(p_2 - p_1)} (\|D v\|_{L_{p_2}(B_n)} + \|v\|_{L_{p_2}(B_n)}) \to 0$$

as $n \to \infty$. The remaining terms in $g_n$ tend to zero in $L_{p_1}$ owing to Hölder’s inequality. The lemma is proved.
To be able to move $\lambda$ to zero in (2.5), when $c \geq 0$, we need to do some preparations.

In case $s = p$, $\Omega = \mathbb{R}^d$, $\lambda = \lambda_0$, Lemma 4.1 yields

$$
\|u\|_{W^2_p} \leq N(d, \delta, p, R_a) (\|Lu\|_{L_p} + \|u\|_{L_p}).
$$

(4.3)

By using the method of proof of Theorem 9.4.1 of [11] we derive from (4.3) the following.

**Lemma 4.7.** Under Assumption 2.1 with $\theta_a = \theta_d(d, \delta, p)$ there exists a constant $N_0 = N_0(d, \delta, p, R_a)$ such that for any $R_1 < R_2$ and $u \in W^2_p(B_{R_2})$ we have

$$
\|u\|_{W^2_p(B_{R_1})} \leq N_0 (\|Lu\|_{L_p(B_{R_2})} + [1 + (R_2 - R_1)^{-2}] \|u\|_{L_p(B_{R_2})}).
$$

(4.4)

Next, we carry over Lemma 4.7 to the full operator $L$ basically mimicking the proof of Theorem 9.4.1 of [11] (originated in [8]).

**Theorem 4.8.** Under Assumption 2.1 with $\theta_a = \theta_d(d, \delta, p)$ there exist $\theta_b = \theta_b(p, q_b, P) > 0$ and $\theta_c = \theta_c(p, q_c, P) > 0$ such that if Assumptions 2.2 and 2.3 are satisfied with these $\theta_b, \theta_c$, there exists a constant $N_1$ depending only on $d, \delta, p, q_b, q_c, K, R_a, R_0$, such that for any $R_1 < R_2$ and $u \in W^2_p(B_{R_2})$ we have ($N_0$ is from Lemma 4.7)

$$
\|u\|_{W^2_p(B_{R_1})} \leq (8/7) N_0 \|Lu\|_{L_p(B_{R_2})} + N_1 [1 + (R_2 - R_1)^{-2}] \|u\|_{L_p(B_{R_2})}.
$$

(4.5)

Proof. We may assume that $R_1 \geq R_2 - R_1$. Set $\rho_0 = R_1$,

$$
\rho_m = R_1 + (R_2 - R_1) \sum_{j=1}^m 2^{-j}.
$$

By Lemma 4.7

$$
I_m := \|u\|_{W^2_p(B_{\rho_m})} \leq N_0 (\|Lu\|_{L_p(B_{\rho_{m+1}})} + [1 + 4^{m+1} (R_2 - R_1)^{-2}] \|u\|_{L_p(B_{\rho_{m+1}})}) + J,
$$

where

$$
J = N_0 (\|bDu\|_{L_p(B_{\rho_{m+1}})} + \|cu\|_{L_p(B_{\rho_{m+1}})}).
$$

We estimate the norms of $bDu$ and $cu$ by using Theorem 3.9. By $N$ below we denote generic constants depending only on $d, \delta, p, q_b, q_c, K, R_a, R_0$. Then we get that for some $\theta_b$ and $\theta_c$ chosen appropriately in our assumptions we have

$$
J \leq (1/8) \|u\|_{W^2_p(B_{\rho_{m+1}})} + N(1 + R_1^{-2}) \|u\|_{L_p(B_{R_2})},
$$

where $R_1^{-2} \leq (R_2 - R_1)^{-2}$. Hence,

$$
I_m \leq (1/8) I_{m+1} + N_0 \|Lu\|_{L_p(B_{R_2})} + N [1 + 4^m (R_2 - R_1)^{-2}] \|u\|_{L_p(B_{R_2})}.
$$

By multiplying both parts of this inequality by $8^{-m}$, summing up for $m = 0, 1, \ldots$, and cancelling (finite) like terms we come to (4.5) and the theorem is proved.
Our next result is about better summability of \( D^2u \) if the right-hand side is summable to a higher power. It will be used to reduce \( \lambda_0 \) in Theorem 2.5 to any number > 0 in case \( \Omega = \mathbb{R}^d \). Introduce

\[
\theta_b(p, q_b, P, \Omega) = \theta_b(p, q_b, P, \Omega) \land \theta_b(p, q_b, P),
\]

where \( \theta_b(p, q_b, P, \Omega) \) is from Definition 4.2 and \( \theta_b(p, q_b, P) \) is from Theorem 4.8.

**Theorem 4.9.** Suppose \( c \equiv 0 \), for a number \( r \) we have \( q_b \geq r > d/2 \), \( r \geq p \), Assumption 2.1 is satisfied with \( \theta_a = \theta_a(d, \delta, p) \) and Assumption 2.2 is satisfied with \( \theta_b(p, q_b, d, K, \delta, R_a) \) from Theorem 4.8. Moreover, if \( r > p \), also suppose that Assumptions 2.2 \((q_b, p(n), \theta_b(n))\), \( n = 0, 1, \ldots, m \), are satisfied, where \( p(n)(\in [p, r]) \) are specified in the proof and \( \theta_b(n) = \theta_b(p(n), q_b, P, \mathbb{R}^d) \).

Then for any \( R \in (0, \infty) \) and \( u \in W^2_{p, \text{loc}}(B_{2R}) \) such that \( Lu \in L_r(B_{2R}) \) we have \( u \in W^2_{r, \text{loc}}(B_{2R}) \) and

\[
\|u\|_{W^2_{r}(B_{2R})} \leq N(\|Lu\|_{L_r(B_{2R})} + \|u\|_{L_p(B_{2R})}),
\]

where \( N \) depends only on \( R, p, r, q_b, d, \delta, R_a, R_0, K \).

Proof. If \( r = p \) the result follows from Theorem 4.8. Therefore we assume that \( r > p \). Introduce

\[
\gamma = 1 + \frac{2r - d}{d} \cdot \frac{p}{r}.
\]

Observe that \( \gamma > 1 \) and introduce \( p(n) = \gamma^n, n = 0, \ldots, m - 1 \), where \( m - 1 \) is the largest \( n \) such that \( p(n) \leq r \). Then set \( p(m) = r \).

Take \( \lambda_0 \) so large (see Theorem 2.5) that \( \lambda_0 - L \) is invertible as an operator acting from \( W^2_{p(n)} \) onto \( L_{p(n)} \) for all \( n \).

Also take \( n \in [0, m - 1] \), \( u \in W^2_{p(n), \text{loc}}(B_{2R}) \) and assume that \( Lu \in L_{p(n+1)}(B_{2R}) \). Then take \( \zeta \in C^\infty_0(B_{2R}) \) such that \( \zeta = 1 \) on \( B_R \) and denote

\[
f = (L - \lambda_0)u, \quad g = (L - \lambda_0)(\zeta u) = \zeta f + 2a^{ij}D_iuD_j\zeta + u(a^{ij}D_i\zeta + b^iD_i\zeta).
\]

Observe that for \( n < m - 1 \)

\[
\frac{d}{p(n)} - \frac{d}{p(n + 1)} = \frac{d}{p\gamma^{n+1}(\gamma - 1) < \frac{d}{p}(\gamma - 1) = \frac{2r - d}{r}} = 2 - \frac{d}{r} = 1
\]

and \( p(n) < p(n + 1) \leq r \leq d \). If \( n = m - 1 \) and \( p(m - 1) = r \), then the left-hand side of (4.8) vanishes for \( n = m - 1 \), and if \( p(m - 1) < r \), then for \( m \leq p(m - 1) \gamma \) and

\[
\frac{d}{p(m - 1)} - \frac{d}{p(m)} = \frac{d}{p(m - 1)} - \frac{d}{r} \leq \frac{d}{p(m - 1)} - \frac{d}{p(m - 1)\gamma} \\
\leq \frac{\gamma d}{r}(1 - \frac{1}{\gamma}) = \frac{2rd - d}{r} \cdot \frac{p}{r} < \frac{2rd - d}{r} \leq 1.
\]

It follows that, if \( \eta u \in W^2_{p(n)} \) for any \( \eta \in C^\infty_0(B_{2R}) \), then \( \eta Du \in L_{p(n+1)} \)

for any \( \eta \in C^\infty_0(B_{2R}) \). Furthermore, by Theorem 3.11 for \( \eta \in C^\infty_0(B_{2R}) \)
such that $\eta = 1$ on the support of $\zeta$ we have
\[ \|uB\zeta\|_{L^p(\Omega^{n+1})} \leq N\|b\eta\|_{L^p(\Omega^{n+1})} \leq N\|\eta u\|_{W^1_{p(n+1)}} \leq N\|\eta u\|_{W^2_{p(n)}}. \]
We conclude that $g \in L^p(\Omega^{n+1})$ and
\[ \|g\|_{L^p(\Omega^{n+1})} \leq N\|\eta u\|_{W^2_{p(n)}} + N\|Lu\|_{L^p(B_{2R})}. \]
Next by the choice of $\lambda_0$ the equation
\[(L - \lambda_0)w = g\]
has a solution in $W^2_{p(n+1)}$ which in addition is unique in $W^2_{p(n)}$. By Lemma 4.6
\[ w = \zeta u \in W^2_{p(n+1)} \quad \forall \zeta \in C^\infty_0(B_{2R}). \quad (4.9) \]
Again by the choice of $\lambda_0$
\[ \|u\|_{W^2_{p(n+1)}(B_R)} \leq \|\zeta u\|_{W^2_{p(n+1)}} \leq N\|g\|_{L^p(n+1)} \leq N\|\eta u\|_{W^2_{p(n)}} + N\|Lu\|_{L^p(B_{2R})}. \]
By iterating this we see that there exists $\eta \in C^\infty_0(B_{2R})$ such that
\[ \|u\|_{W^2_{p(n)}(B_{2R})} \leq N\|\eta u\|_{W^2_{p(n)}} + N\|Lu\|_{L^p(B_{2R})} \]
and to finish proving (4.6) it only remains to apply Theorem 4.8. The fact that $u \in W^2_{r,\text{loc}}(B_{2R})$ is obtained by changing the origin and $R$ allowing us to explore what is going on in a neighborhood of any point in $B_{2R}$. The theorem is proved.

**Corollary 4.10.** Under the assumptions of Theorem 4.9 if $u \in W^2_{p,\text{loc}}(B_{2R})$ satisfies $Lu = 0$ in $B_{2R}$, then $u \in W^2_{r,\text{loc}}(B_{2R})$. In particular, $u \in C^2_{2-d/r}(B_{2R})$.

The following will be instrumental in reducing $\lambda_0$ to zero in Theorem 2.5 in case $\Omega$ is a bounded domain.

**Theorem 4.11.** Let $\Omega$ be a bounded domain of class $C^{1,1}$. Suppose $c \equiv 0$, for a number $r$ we have $q_0 \geq r > d/2$, $r \geq p$, Assumption 2.1 is satisfied with $\theta_a = \theta_0(d, \delta, p)$ and Assumption 2.2 is satisfied with $\theta_b(p, q_0, P, \Omega)$ from Definition 4.2. Moreover if $r > p$, also suppose that Assumptions 2.2 ($q_0, p(n), \theta_b(n)$), $n = 0, 1, ..., m$, are satisfied, where $p(n) (\in [p, r])$ are specified in the proof of Theorem 4.9 and $\theta_b(n) = \theta_0(p(n), q_0, P, \Omega)$ from Definition 4.2.

Then there exists an integer $m_0$, depending only on $p$ and $d$, and there exist $\lambda_0 \geq 1$ and a constant $N$, depending only on $p, r, q_0, d, \delta, R_0, R_0, K$, and $\Omega$, such that for any $f \in L^p(\Omega)$ we have
\[ \sup_{x \in \Omega} |R_{x_0}^{m_0} f(x)| \leq N\|f\|_{L^p(\Omega)}. \quad (4.10) \]
If $r = p$ estimate (4.10) with $m = 1$ follows from embedding theorems. In case $r > p$ the proof of this theorem is achieved by almost literally repeating that of Theorem 2.16 of [12].
5. TWO AUXILIARY RESULTS USING PROBABILITY THEORY

Here we assume that \( c \equiv 0 \) and \( q_b \in (d_0, d] \), where \( d_0 = d_0(d, \delta) \in (d/2, d) \) is defined in [15]. We suppose that Assumption 2.1 (i) and Assumption 2.2 are satisfied with \( q_b > p = d_0 \) and \( \theta_b = \hat{b}(d, \delta) \), where \( \hat{b}(d, \delta) > 0 \) is defined in [15].

**Lemma 5.1.** Let \( \Omega \) be a bounded domain of class \( C^{1,1} \), \( \lambda \geq \nu > 0 \) and let \( f \in L_{d_0}(\Omega) \) and \( u \in W^{2}_{d_0}(\Omega) \cap C(\overline{\Omega}) \) satisfy \( \lambda u - Lu \leq 1 + f \) in \( \Omega \) and \( u \leq 0 \) on \( \partial \Omega \). Then

\[
\lambda u \leq \mu + N \lambda \|f\|_{L_{d_0}(\Omega)},
\]

where \( N \) depends only on \( d, \delta, R_0 \) and the diameter of \( \Omega \) and \( \mu < 1 \) is a constant depending only on \( \nu, d, \delta \), and the diameter of \( \Omega \).

**Proof.** In light of Theorem 3.11 (and Remark 3.12 and \( q_b \neq d_0 \)) we may assume that \( u, a, \) and \( b \) are smooth. In that case we can use some basic facts from stochastic calculus which are found, for instance, in [10]. For the reader’s orientation we sketch some of them. A \( d \)-dimensional Wiener process \( w_t \) is the mathematical model of Brownian motion and is a continuous random process with independent increments and independent coordinates such that \( w_i^t - w_i^s \) has normal distribution with zero mean and variance \( |t - s| \) for any \( i = 1, \ldots, d \). Itô proved that one can define the stochastic integral

\[
\int_0^t f_t \, dw_t
\]

for random \( \mathbb{R}^d \)-valued \( f_t \) as the limit of usual integral sums provided that \( f_t \), say, is measurable bounded and, for each \( t, f_t \) and the process \( w_{s+t} - w_t, \) \( s > 0, \) are independent. After that, by using Perron’s method of successive approximations, he showed that under our above assumptions on \( a \) and \( b \), for any \( x \), the equation

\[
x_t = x + \int_0^t \sqrt{2a(x_s)} \, dw_s + \int_0^t b(x_s) \, ds \tag{5.1}
\]

has a unique solution such that for each \( t, x_t \) and the process \( w_{s+t} - w_t, \) \( s > 0, \) are independent. Finally, what we need is Itô’s formula, which implies (see [9]) that if \( \Omega \) is a bounded domain \( u \in W^{2}_d(\Omega) \cap C(\overline{\Omega}) \) and \( c_t \geq 0 \) is measurable bounded and, for each \( t, c_t \) and the process \( w_{s+t} - w_t, s > 0, \) are independent, then for any \( x \in \Omega \)

\[
u(x) = E e^{-\phi_t} u(x_t) + E \int_0^\tau e^{-\phi_t} (c_t u(x_t) - Lu(x_t)) \, dt,
\]

where \( x_t \) is the solution of (5.1), \( \tau \) is its first exit time from \( \Omega \) and

\[
\phi_t = \int_0^t c_s \, ds.
\]
In our case with \( c_t = \lambda \) it follows that
\[
\lambda u(x) \leq E \int_0^\tau \lambda e^{-\lambda t} dt + v = 1 - E e^{-\lambda \tau} + v \leq 1 - E e^{-\nu \tau} + v,
\]
with
\[
v = \lambda E \int_0^\tau |f(x_t)| dt \leq N \lambda \|f\|_{L_{d_0}(\Omega)},
\]
where the inequality holds due to Theorem 1.2 of [15], which is applicable because of our condition on \( \theta_b \). This theorem also implies that \( E\tau \leq \hat{N} \), where \( \hat{N} \) depends only on \( d, \delta, R_0 \), and the diameter of \( \Omega \). Since this holds for any starting point \( x \in \Omega \), by Khasminskii’s lemma for \( \hat{\nu} = (2 \hat{N})^{-1} \) we have \( E e^{-\hat{\nu} \tau} \leq 2 \). Hence,
\[
P(\tau > T) \leq 2 e^{-\hat{\nu} T},
\]
and \( 1 - E e^{-\nu \tau} \leq 1 - e^{-\nu T}(1 - 2 e^{-\hat{\nu} T}) =: \mu \), where \( \mu < 1 \) for an appropriate choice of \( T \). The lemma is proved.

**Lemma 5.2.** Let \( \lambda, \rho \geq 0 \) and let \( f \in L_{d_0}(B_R) \) and \( u \in W^{2,d_0}(B_R) \) satisfy
\[
\lambda u - Lu \leq f \text{ in } B_R.
\]
Then
\[
u(0) \leq e^{\tilde{\xi}/2} e^{-\rho \sqrt{\lambda \tilde{\xi}/2}} \max_{\partial B_R} u_+ + N \|f\|_{L_{d_0}(B_R)},
\]
where \( N \) depends only on \( d, \delta, R_0, \) and \( R, \tilde{\xi} = \xi(d, \delta) \in (0, 1) \), and
\[
\hat{\lambda} = \lambda \min(1, \lambda R_0^2).
\]

Proof. Again we may assume that \( u, a, \) and \( b \) are smooth and keep going the argument in the previous proof. In light of Theorem 1.1 of [15] the assumption of Theorem 2.3 of [15] is satisfied with \( R = R_0 \). Therefore, for \( \rho \leq R_0 \)
\[
P(\tau \rho \geq \rho^2) \geq \tilde{\xi},
\]
where \( \tau \rho \) is the first time \( x_t \) deviates from its arbitrary starting point by distance \( \rho \). This by Corollary 2.5 of [16] leads to the fact that for \( \mu \in [0, 1] \) and \( \rho \leq R_0 \) we have
\[
E e^{-\mu \rho^2 - 2 \tau \rho} \leq e^{-\mu \tilde{\xi}/2},
\]
which by Theorem 2.6 of [16] yields that for any \( \lambda, \rho > 0 \)
\[
E e^{-\lambda \tau \rho} \leq e^{\tilde{\xi}/2} e^{-\rho \sqrt{\lambda \tilde{\xi}/2}}.
\]

After that it only remains to recall that by Itô’s formula
\[
u(0) = E \left( e^{-\lambda \tau_R} u(x_{\tau_R}) + \int_0^{\tau_R} e^{-\lambda t}(\lambda - L) u(x_t) dt \right)
\]
\[
\leq E e^{-\lambda \tau_R} \max_{\partial B_R} u_+ + E \int_0^{\tau_R} |f(x_t)| dt,
\]
where \( x_t \) is the solution of (5.1) with \( x = 0 \) and \( \tau_R \) is the first time it reaches \( \partial B_R \). The lemma is proved.
**Remark 5.3.** In Corollary 2.8 of [16] estimate (5.3) is shown to imply that for any \( m > 0 \) and \( 0 \leq s \leq t \) we have
\[
E \sup_{r \in [s,t]} |x_r - x_s|^m \leq N(|t - s|^{m/2} + |t - s|^m),
\]
where \( N = N(m, R_0, \xi) \).

**6. Proof of Theorem 2.8**

We repeat the short proof of Theorem 4.2 of [12]. In light of the method of continuity it suffices to prove the first assertion. If \( \lambda \geq \lambda_0 \), with \( \lambda_0 \) taken from Theorem 2.5, the result is known from Theorem 2.5 even without the restriction \( q_b > d_0 \). Therefore we will only concentrate on \( 0 \leq \lambda < \lambda_0 \).

Define
\[
f = \lambda u - Lu
\]
so that
\[
\lambda_0 u - Lu = (\lambda_0 - \lambda)u + f, \quad u = (\lambda_0 - \lambda)R_{\lambda_0+c}u + R_{\lambda_0+c}f,
\]
and by induction on \( n \)
\[
u = [(\lambda_0 - \lambda)R_{\lambda_0+c}]^n u + \sum_{i=0}^{n-1}[(\lambda_0 - \lambda)R_{\lambda_0+c}]^i R_{\lambda_0+c}f,
\]
where \( n \) is any integer \( \geq 1 \). We thus have the beginning of the Neumann series.

Introduce the constants \( N_1 \) and \( M_n \) so that
\[
\|R_{\lambda_0}g\|_{L_p(\Omega)} \leq N_1\|g\|_{L_p(\Omega)} \quad \forall g \in L_p(\Omega), \quad M_n = \sum_{i=0}^{n-1} \lambda_0^i N_1^{i+1}.
\]

Finally, let \( |\Omega| \) be the volume of \( \Omega \) and take \( m_0 \) from Theorem 4.11. For \( n > m_0 \), in light of Corollary 4.5
\[
\|u\|_{L_p(\Omega)} \leq |\Omega|^{1/p} \lambda_0^m \sup_{x \in \Omega} R_{\lambda_0}^{m-m_0} R_{\lambda_0}^{m_0} |u|_1(x) + M_n \|f\|_{L_p(\Omega)}.
\]

By Lemma 5.1 the above supremum is dominated by
\[
\lambda_0^{m_0-n} \mu^{n-m_0} \sup_{x \in \Omega} R_{\lambda_0}^{m_0} |u|_1(x),
\]
where \( \mu < 1 \), which by Theorem 4.11 is less than
\[
N_2 \lambda_0^{m_0-n} \mu^{n-m_0} \|u\|_{L_p(\Omega)}.
\]

Hence,
\[
\|u\|_{L_p(\Omega)} \leq N_2 |\Omega|^{1/p} \lambda_0^{m_0} \mu^{n-m_0} \|u\|_{L_p(\Omega)} + M_n \|f\|_{L_p(\Omega)}.
\]

We fix \( n \) so that \( N_2 |\Omega|^{1/p} \lambda_0^{m_0} \mu^{n-m_0} \leq 1/2 \) and then arrive at
\[
\|u\|_{L_p(\Omega)} \leq 2M_n \|f\|_{L_p(\Omega)}.
\]

Now to get (2.6) it only remains to refer to Remark 4.3. The theorem is proved.
7. Proof of Theorem 2.9

In light of Theorem 3.11 we may assume that \( a \) is smooth and \( b, c \) are bounded (the mollification of \( c \) might (?) ruin belonging of \( |c|^p \) to \( A_1 \) which is required if \( q_c = p \)). Next, we need a lemma.

**Lemma 7.1.** Let \( u \in W^{2d}_{2d} \) and \( f \in L_{2d} \). Assume that \( f = 0 \) outside \( B_1 \), \( \lambda \geq \varepsilon_0 \), and \( \lambda u - Lu = f \) in \( \mathbb{R}^d \). Then there exists a constant \( N \), depending only on \( \varepsilon_0, d, \delta, p, q_b, q_c, \tau, R_0, R_a, K \), such that
\[
\|u/v\|_{L_p} \leq N\|f\|_{L_p},
\]
where \( v(x) = e^{-\varepsilon_0 R_0 |x|\xi/2} \) with \( \xi \) taken from Lemma 5.2.

**Proof.** We follow the proof of Lemma 11.6.1 of [11]. Take \( R' = R'(\varepsilon_0 R_0, d, \delta) \geq 4 \) so that
\[
e^{\xi/2}e^{-(R' - 2)\varepsilon_0 R_0 \xi/2} \leq 1/2. \tag{7.1}
\]
Relying on classical results, define \( h \in W^{2d}_{2d}(B_{R'}) \) as a unique solution of
\[
\lambda h - Lh = 0 \quad \text{in} \quad B_{R'} \quad \text{with} \quad w := h - u \in W^{2d}_{2d}(B_{R'}).
\]
Then
\[
\lambda w - Lw = f.
\]
Notice that \( \lambda u - Lu = 0 \) outside \( B_1 \) and by the maximum principle
\[
|u(x)| \leq \max_{|x|=2} |u| \quad \text{for} \quad |x| \geq 2.
\]
Taking this into account, taking \( x \) as the new origin, and using Lemma 5.2 and the fact that \( \lambda \geq \varepsilon_0 \), we obtain
\[
|u(x)| \leq e^{\xi/2}e^{-(|x|-2)\varepsilon_0 R_0 \xi/2} \max_{|x|=2} |u| \quad \text{for} \quad |x| \geq 2. \tag{7.2}
\]
Also observe that by the maximum principle
\[
|h| \leq \max_{|x|=R'} |u| \quad \text{in} \quad B_{R'}.
\]
Now we claim that to prove the lemma, it suffices to prove that
\[
|w(x)| \leq N\|f\|_{L_p(B_{R'})} \quad \text{for} \quad |x| = 2. \tag{7.3}
\]
Indeed, if (7.3) holds, then
\[
\max_{|x|=2} |u| \leq \max_{|x|=2} |h| + \max_{|x|=2} |w| \leq \max_{|x|=R'} |u| + N\|f\|_{L_p(\mathbb{R}^d)} \leq e^{\xi/2}e^{-(R' - 2)\varepsilon_0 R_0 \xi/2} \max_{|x|=2} |u| + N\|f\|_{L_p(\mathbb{R}^d)},
\]
which for our choice of \( R' \) yields
\[
\max_{|x|=2} |u| \leq N\|f\|_{L_p(\mathbb{R}^d)}.
\]
Coming back to (7.2) and using that \( e^{(1+2\varepsilon_0 R_0)\xi/2} \leq N \) we get that
\[
\|u/v\|_{L_p(B_2')} \leq N\|f\|_{L_p(\mathbb{R}^d)}.
\]
The remaining part of the norm is also bounded by $N\|f\|_{L_p(\mathbb{R}^d)}$ since $|u| \leq |h| + |w|$, 

$$\max_{B_{R'}} |h| \leq \max_{|x|=R} |u| \leq \max_{|x|=2} |u| \leq N\|f\|_{L_p(\mathbb{R}^d)},$$

and by Theorem 2.8 we have 

$$\|w\|_{L_p(B_{R'})} \leq N\|f\|_{L_p(B_{R'})}.$$ 

Thus, indeed we need only prove (7.3).

By the maximum principle $|w| \leq \psi$, where $\psi$ is a $W^2_p(B_{R'})$-solution of $(L + c)\psi = -|f|$. So it suffices to estimate $\psi$ on $|x| = 2$. Take a point $x_0$ with $|x_0| = 2$ and observe that by embedding theorems we have 

$$|\psi(x_0)| \leq N\|\psi\|_{W^2_p(B_{1/2}(x_0))}.$$ 

Next, we use the local regularity result from Theorem 4.9. Then we find 

$$\|\psi\|_{W^2_p(B_{1/2}(x_0))} \leq N\|(L + c)\psi\|_{L_p(B_{1/2}(x_0))} + N\|\psi\|_{L_p(B_{1}(x_0))}.$$ 

Here the first term on the right is zero since $f = 0$ outside of $B_1$ and the second term is less than $N\|f\|_{L_p(B_{R'})}$ by Theorem 2.8. The lemma is proved.

**Proof of Theorem 2.9.** We follow the derivation of Theorem 11.6.2 of [11] from Lemma 11.6.1 of [11].

As usual, it suffices to prove the a priori estimate (2.7). As we said we may assume that $a$ is smooth and $b, c$ are bounded. Then we also may assume that $u \in C_0^\infty$. In that case $f := \lambda u - Lu$ also has compact support and is bounded.

Then let $\zeta$ be a $C_0^\infty$ function with unit integral and support in $B_1$. Define $\zeta^z(x) = \zeta(x - z)$ and let, for any $z \in \mathbb{R}^d$, $w(z) \in W^2_p$ be a unique solution of 

$$\lambda w(z) - Lw(z) = \zeta^z f.$$ 

Such functions $w(z)$ exist owing to Theorem 11.6.2 of [11]. Since the coefficients of $L$ are regular and $f$ is bounded $w(z) \in W^2_s$ for any $s > 1$. In particular, it is bounded and continuous and its first derivatives are bounded and continuous. These bounds are in terms of $W^2_s$ norms of $\zeta^z f$ and therefore are uniform with respect to $z$. Furthermore, for any $s > 1$, there are constants $N$ such that 

$$\|w^(y) - w^(z)\|_{W^2_s} \leq N\|(\zeta^y - \zeta^z)f\|_{L_s} \leq N|y - z|$$

for all $y$ and $z$. It follows by embedding theorems that $w(z)$ and its first derivatives derivatives in $x$ are Lipschitz continuous functions of $z$. Also $\zeta^z f(x) = 0$ and hence $w(z)(x) = 0$ for all $x$ if $|z|$ is large enough, say $|z| \geq R$, because $f$ has compact support. Therefore, the definition 

$$w = \int_{\mathbb{R}^d} w(z) \, dz \quad (= \int_{B_R} w(z) \, dz)$$
makes sense as the Bochner integral in $W^2_s$ and defines $w$ as an element of $W^2_s$. Integrating through in (7.4), we find that
\[ \lambda w - Lw = f, \]
which by Theorem 11.6.2 of [11] yields $w = u$.

Hence, by Hölder’s inequality, for $v$ taken from Lemma 7.1 and $v^\ast(x) = v(x - z)$,
\[ |u(x)|^p \leq \int_{\mathbb{R}^d} |w(z)/(v^z(x))|^p \, dz \|v\|_{L^p} = N_1^p \int_{\mathbb{R}^d} |w(z)/(v^z(x))|^p \, dz, \tag{7.5} \]
where $s = p/(p - 1)$ and $N_1$ depends only on $\varepsilon_0 R_0, \delta$, $p$, and $d$. In addition, by Lemma 7.1 we have
\[ \int_{\mathbb{R}^d} |w(z)/(v^z(x))|^p \, dx \leq N_2^p \int_{\mathbb{R}^d} |\zeta^z f|^p \, dx, \]
where $N_2$ is the constant called $N$ in Lemma 7.1. This and (7.5) yield
\[ \|u\|_{L^p} \leq N_3 \|\lambda u - Lu\|_{L^p}, \quad N_3 = N_1 N_2 \|\zeta\|_{L^p}. \]
After that it only remains to use that similarly to Remark 4.3
\[ \|u\|_{W^2_s} \leq N(\|\lambda u - Lu\|_{L^p} + \|u\|_{L^p}). \]
The theorem is proved.

One more result, proved in the next section, is the following stability theorem, which is nontrivial even if $b^n = b = 0$, $c^n = c = 0$.

**Theorem 7.2.** In addition to the assumptions of Theorem 2.9 suppose that $q_c > d_0, b \in L_{d_0}, c \in L_{d_0}$. Let $a^n, b^n, c^n, n = 1, 2, \ldots$, be sequences of smooth bounded functions with values in the set of symmetric $d \times d$ matrices having all eigenvalues in $[\delta, \delta^{-1}]$, in $\mathbb{R}^d$, and in $[0, \infty)$, respectively, such that $a^n \to a$ on $\mathbb{R}^d$ (a.e.) and
\[ \|b - b^n\|_{L_{d_0}} + \|c^n - c\|_{L_{d_0}} \to 0 \]
as $n \to \infty$. Suppose that $b^n$ satisfy Assumption 2.2 with $p = d_0$ and $\theta_b = \hat{b}(d, \delta)$. Take $\lambda > 0, f \in L_{d_0}$, and introduce $u^n$ as unique $W^2_{d_0}$-solutions of $\lambda u^n - L^n u^n = f$, where the operators $L^n$ are constructed from $a^n, b^n, c^n$. Then at each point of $\mathbb{R}^d$ we have $u^n \to u$ as $n \to \infty$, where $u \in W^2_{d_0}$ is a unique solution of $\lambda u - Lu = f$.

8. Weak uniqueness of solutions of stochastic equations

We suppose that the assumptions of Theorem 2.9 are satisfied and $q_b, q_c > d_0 = p = r$.

If $a$ and $b$ are smooth, as we have mentioned in Section 5, the results of [15] and [16] are applicable. In particular, take $x \in \mathbb{R}^d$ and on a probability space with a $d$-dimensional Wiener process $w_t$ define a process $x_t$ as a (unique) solution of
\[ x_t = x + \int_0^t \sqrt{2a(x_s)} \, dw_s + \int_0^t b(x_s) \, ds. \tag{8.1} \]
By Theorem 1.2 of [15] for any $R > 0$ and Borel nonnegative $f$

$$E \int_0^{\tau_R(x)} f(x_t) \, dt \leq N \|f\|_{L^d(B_R(x))}, \quad (8.2)$$

where $\tau_R(x)$ is the first time $x_t$ exits from $B_R(x)$ and $N$ depends only on $d, \delta, R$, and $R_0$.

Estimate (8.2) holds with the same $N$, if we take a sequence of smooth $a^n, b^n$, satisfying Assumptions 2.1 and 2.2 with $p = d_0$ and $\theta_b = \hat{b}(d, \delta)$ and converging to the original $a$ (a.e.) and $b$ in $L_{d_0}(B_R)$ for any $R$, and introduce $x^n_t$ by solving (8.1) with $a^n, b^n$ in place of $a, b$. Then (5.4) will also hold with the same constant for $x^n_t$ in place of $x_t$.

After that by using the Skorokhod embedding method, as always in such situation, and repeating the proofs of Theorem 1.1, 1.3, and 1.4 of [13] (where $b \in L^d$) and using the results of the preceding sections we come to the following.

**Theorem 8.1.** There is a probability space carrying a $d$-dimensional Wiener process $w_t$ such that equation (8.1) has a solution for which estimates (8.2) and (5.4) hold. Furthermore, all solutions (on any possible probability space) for which estimate (8.2) holds have the same finite-dimensional distribution. Finally, for any such solution, if $r \geq d_0$ and $u \in W^{r,2}$, then (a.s.) for all $t \geq 0$

$$u(x_t) = u(0) + \int_0^t \left[ \frac{1}{2} a_{ij}(x_s) D_{ij} u(x_s) + b^i(x_s) D^i u(x_s) \right] \, ds$$

$$+ \int_0^t D^i u(x_s) \sqrt{a(x_s)^{ik} dw^k_s} \quad (8.3)$$

and the last term is a square integrable martingale.

On the basis of Theorem 8.1 and what is said before it we can just repeat the proof of Theorem 6.4 of [12] and prove our Theorem 7.2.

9. Two Examples

**Example 9.1.** Take $\gamma_+ = 2d$, $4 \gamma_- > (d - 2)^2$, and set

$$k = d/2 - 1, \quad s = (1/2) \sqrt{4 \gamma_--(d-2)^2}.$$

Then take any $\varepsilon \in (0, 1)$ such that

$$\tan(s \ln \varepsilon) = \frac{s}{k+2} = \frac{2s}{d+2},$$

and set $\gamma(x) = \gamma_+ I_{x \leq \varepsilon} - \gamma_- I_{x > \varepsilon}$,

$$u(x) = \frac{|x|^2}{\varepsilon^2} I_{|x| \leq \varepsilon} + \frac{\varepsilon^k \sin(s \ln |x|)}{|x|^k \sin(s \ln \varepsilon)} I_{|x| > \varepsilon}.$$
One easily checks that $u \in C^{1,1}(B_1)$, $u = 0$ on $\partial B_1$ and
\[ \Delta u - \frac{\gamma(x)}{|x|^2} u = 0 \]
in $B_1$.

**Example 9.2.** Take $\gamma(x) \equiv \gamma < 0$ such that $4\varepsilon^2 := 4\gamma + (d - 2)^2 > 0$, define $2\lambda_\pm = -(d - 2) \pm 2\varepsilon \ (< 0)$ and also set $u_\pm(x) = |x|^\lambda_\pm$. Then both functions equal 1 on $\partial B_1$ and satisfy (1.1) in $B_1$ with $b = 0$ and $f = 0$. Furthermore, both functions are in $W^2_p(B_1)$ for $p < p_0$, where $p_0 = 2d/(d + 2 + 2\varepsilon) > 1$. Again uniqueness fails.

Furthermore, for any $\lambda > 0$, the function $v = u_+ - u_-$ is zero on $\partial B_1$ and satisfy
\[ \Delta v - \frac{\gamma}{|x|^2} v - \lambda v = f, \]
in $B_1$, where $f = -\lambda v \in L_{p_0}(B_1)$. But $v \not\in W^2_p(B_1)$, so that estimate (2.5) fails as $p \uparrow p_0$. This shows the necessity of the smallness assumption on $\theta_c$ depending on $p$ in Theorem 2.5.

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