A MATHEMATICAL STUDY OF DIFFUSIVE LOGISTIC EQUATIONS WITH MIXED TYPE BOUNDARY CONDITIONS

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Dedicated to the memory of Professor Rosella Mininni (1963–2020)

Abstract. The purpose of this paper is to provide a careful and accessible exposition of static bifurcation theory for a class of mixed type boundary value problems for diffusive logistic equations with indefinite weights, which model population dynamics in environments with spatial heterogeneity. We discuss the changes that occur in the structure of the positive solutions as a parameter varies near the first eigenvalue of the linearized problem, and prove that the most favorable situations will occur if there is a relatively large favorable region (with good resources and without crowding effects) located some distance away from the boundary of the environment. A biological interpretation of main theorem is that an initial population will grow exponentially until limited by lack of available resources if the diffusion rate is below some critical value; this idea is generally credited to the English economist T. R. Malthus. On the other hand, if the diffusion rate is above this critical value, then the model obeys the logistic equation introduced by the Belgian mathematical biologist P. F. Verhulst. The approach in this paper is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in partial differential equations.

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1. **Introduction and main results.** Let $D$ be a bounded domain of Euclidean space $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial D$; its closure $\overline{D} = D \cup \partial D$ is an $N$-dimensional, compact smooth manifold with boundary. In this paper we study the following semilinear elliptic boundary value problem:

\[
\begin{aligned}
-\Delta u &= \lambda (m(x) - h(x) u) u \quad \text{in } D, \\
Bu &= a(x') \frac{\partial u}{\partial n} + b(x')u = 0 \quad \text{on } \partial D.
\end{aligned}
\]

Here:

1. $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_N^2}$ is the usual Laplacian.
2. $\lambda$ is a real parameter.
3. $m(x)$ is a real-valued, continuous function on $\overline{D}$.
4. $h(x)$ is a non-negative, $C^1$ function on $\overline{D}$.
5. $a(x')$ and $b(x')$ are non-negative, smooth functions on $\partial D$.
6. $n = (n_1, n_2, \ldots, n_N)$ is the unit outward normal to the boundary $\partial D$ (see Figure 1).

![Figure 1. The bounded domain $D$ and the unit outward normal $n$ to $\partial D$](image-url)
The main purpose of this paper is to study the existence and uniqueness of positive solutions of the semilinear problem (1), substantially improving the previous work [59], [60], [62] and [63]. We study the semilinear problem (1) under the following three conditions on the functions \( m(x), a(x') \) and \( b(x') \):

(M.1) The function \( m(x) \) takes a positive value in \( D \).

(H.1) \( a(x') \geq 0 \) on \( \partial D \) and \( b(x') \geq 0 \) on \( \partial D \).

(H.2) \( a(x') + b(x') > 0 \) on \( \partial D \), and \( b(x') \neq 0 \) on \( \partial D \).

---

| Term | Biological interpretation |
|------|---------------------------|
| \( D \) | Terrain |
| \( x \) | Location of the terrain |
| \( u(x) \) | Population density of a species inhabiting the terrain |
| \( \Delta \) | A member of the population moves about the terrain via the type of random walks occurring in Brownian motion |
| \( \frac{1}{\lambda} \) | Rate of diffusive dispersal |
| \( m(x) \) | Intrinsic growth rate |
| \( h(x) \) | Coefficient of intraspecific competition |

Table 1. A biological meaning of each term

We discuss our motivation and some of the modeling process leading to the semilinear problem (1) (see Tables 1 and 2).

The basic interpretation of the various terms in the semilinear logistic equation

\[
-\Delta u = \lambda (m(x) - h(x) u) u \quad \text{in } D
\]

is that \( u(x) \) represents the population density of a species inhabiting the region \( D \). The members of the population are assumed to move about \( D \) via the type of random walks occurring in Brownian motion that is modeled by the diffusive term \((1/\lambda)\Delta\); hence \( d = 1/\lambda \) represents the diffusion rate, so small values of \( \lambda \) the population spreads more rapidly than for larger values of \( \lambda \). The local rate of change in the population density is described by the density dependent term \( m(x) - h(x) u \).

In this term, \( m(x) \) describes the rate at which the population would grow or decline at the location \( x \) in the absence of crowding or limitations on the availability of resources. The sign of \( m(x) \) will be positive on favorable habitats for population growth and negative on unfavorable ones. Specifically, \( m(x) \) may be considered as a food source or any resource that will be good in some areas and bad in some others. The term \(-h(x)u\) describes the effects of crowding on the growth rate of the population at the location \( x \); these effects are assumed to be independent of those determining the growth rate at low densities. The size of \( h(x) \) describes the strength of the crowding effects.

Condition (M.1) implies that there exists a region endowed with a nice food source.

On the other hand, in terms of biology, the functions \( a(x') \) and \( b(x') \) measure the hostility of the exterior of the domain. For example, if \( a(x') \equiv 0 \) and \( b(x') \equiv 1 \) on \( \partial D \), then the (Dirichlet) boundary condition \( B \) represents that \( D \) is surrounded by a completely hostile exterior such that any member of the population which reaches the boundary dies immediately; in other words, the exterior of the domain is deadly to the population. If \( a(x') \equiv 1 \) and \( b(x') \equiv 0 \) on \( \partial D \), then the (Neumann) boundary
Boundary Condition | Biological interpretation
--- | ---
Dirichlet case $(a(x') \equiv 0, b(x') \equiv 1)$ | Completely hostile (deadly) exterior
Neumann case $(a(x') \equiv 1, b(x') \equiv 0)$ | Barrier
Robin or mixed-type case $(a(x') + b(x') > 0)$ | Hostile but not completely deadly exterior

**Table 2.** A biological meaning of boundary conditions

Condition $B$ represents that the boundary acts as a barrier, that is, individuals reaching the boundary simply return to the interior. If the exterior is hostile but not completely deadly, then the mixed type boundary condition

$$Bu = a(x') \frac{\partial u}{\partial n} + b(x')u = 0 \text{ on } \partial D$$

results (see Afrouzi–Brown [2]). Our boundary condition $B$ is a smooth linear combination of the Dirichlet and Neumann boundary conditions. We remark that $B$ is non-degenerate (or coercive) if and only if either $a(x') > 0$ on $\partial D$ (the regular Robin case) or $a(x') \equiv 0$ and $b(x') > 0$ on $\partial D$ (the Dirichlet case).

Condition (H.2) implies that the exterior of the domain is not totally reflective, that is, the boundary condition $B$ is not the pure Neumann condition. More precisely, condition (H.2) implies that the boundary portion

$$M := \{x' \in \partial D : a(x') = 0\}$$

is deadly to the population, while the exterior of $M$

$$\partial D \setminus M = \{x' \in \partial D : a(x') > 0\}$$

acts as a barrier to the population (see Figure 2).

(Figure 2. The boundary portion $M$ is deadly and its complement $\partial D \setminus M$ is a barrier)

It should be emphasized that the semilinear problem (1) is a degenerate elliptic boundary value problem from an analytical point of view (see [14, Chapitre V, condition (4.5)], [30, Chapter XX, Definition 20.1.1], [76, Chapter II, Condition 11.1]). In fact, this is due to the fact that the so-called Shapiro–Lopatinski complementary condition is violated at the points $x' \in \partial D$ where $a(x') = 0$ (see [30, p. 232, Section 20.1], [76, p. 148, Section 11], [66, Section 6.6]).
We provide a simple example of such functions \( a(x') \) and \( b(x') \) in the plane \( \mathbb{R}^2 \) \((N = 2)\):

**Example 1.1** (K. Umezu). Let

\[
D = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}
\]

be the unit disk with the boundary \( \partial D = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\} \). For a local coordinate system \( x_1 = \cos \theta, x_2 = \sin \theta \) with \( \theta \in [0, 2\pi] \) on the unit circle \( \partial D \), we define functions \( a(x_1, x_2) \) and \( b(x_1, x_2) \) as follows:

\[
a(x_1, x_2) := a(\cos \theta, \sin \theta)
\]

\[
= \begin{cases} 
  e^{\frac{\theta}{2} - \frac{1}{3}} \left(1 - e^{\frac{\theta}{2} + \frac{1}{3}}\right) & \text{for } \theta \in \left[0, \frac{\pi}{2}\right], \\
  1 & \text{for } \theta \in \left[\frac{\pi}{2}, \pi\right], \\
  e^{\frac{\theta}{2} + \frac{1}{3}} \left(1 - e^{-\frac{\theta}{2} - \frac{1}{3}}\right) & \text{for } \theta \in \left[\pi, \frac{3\pi}{2}\right], \\
  0 & \text{for } \theta \in \left[\frac{3\pi}{2}, 2\pi\right],
\end{cases}
\]

and let

\[
b(x_1, x_2) := 1 - a(x_1, x_2) = 1 - a(\cos \theta, \sin \theta).
\]

Here

\[
M = \{(\cos \theta, \sin \theta) : a(\cos \theta, \sin \theta) = 0\} = \left\{(\cos \theta, \sin \theta) : \theta \in \left[\frac{3\pi}{2}, 2\pi\right]\right\}.
\]

Some remarks are in order:

**Remark 1.1.**

1. Amann [4] studied the boundary condition \( B \) in the non-degenerate case where the boundary \( \partial D \) is the disjoint union of the two closed subsets \( M \) and \( \partial D \setminus M \), each of which is an \((N - 1)\) dimensional compact smooth manifold.

2. On the other hand, García-Melián–Rossi–Sabina de Lis [25] studied a discontinuous linear combination of the Dirichlet and Neumann boundary conditions for the negative Laplacian \( A = -\Delta \) (see [36, p. 41, Assumptions (H.1) and (H.3)]. More precisely, by letting \( \Gamma_0 = \text{Int} \{x \in \partial D : a(x') = 0\} \) and \( \Gamma_1 = \text{Int} \{x \in \partial D : b(x') = 0\} \) they considered the following case:

   (a) The open sets \( \Gamma_0 \) and \( \Gamma_1 \) are non-empty and disjoint.

   (b) The closures \( \overline{\Gamma_0} \) and \( \overline{\Gamma_1} \) are compact manifolds equipped with a common \((N - 2)\) dimensional, closed boundary \( \gamma = \overline{\Gamma_0} \cap \overline{\Gamma_1} \).

   (c) \( \partial D = \Gamma_0 \cup \gamma \cup \Gamma_1 \).

3. Ladyzhenskaya–Solonnikov–Ural’tseva [33] and Ladyzhenskaya–Ural’tseva [34] are the classics for linear and quasilinear partial differential equations.

The crucial point in our approach is how to generalize the variational approach ([36]) to the degenerate case. This paper is an outgrowth of our research on the subject during the past two decades (see [58] through [71]).

**1.1. The linearized mixed-type boundary value problem.** First, we study the following linearized mixed-type boundary value problem:

\[
\begin{align*}
Au := (-\Delta + c(x)) u &= g & \text{in } D, \\
Bu = a(x') \frac{\partial u}{\partial n} + b(x') u &= \varphi & \text{on } \partial D.
\end{align*}
\]
Here

(7) \( c(x) \) is a real-valued, continuous function on \( \overline{D} \).

For simplicity, we only consider the case where

\[
\begin{equation}
\tag{3}
\begin{aligned}
\text{if } k \text{ is a positive integer and } 1 < p < \infty, \text{ we define the Sobolev space }
\end{aligned}
\end{equation}
\]

the space \( B \)

\[
\begin{aligned}
\text{and the boundary space }
\end{aligned}
\]

Here \( \{\partial D\} \) is the space of the boundary values \( u\mid_{\partial D} \) of functions \( u \in W^{k,p}(D) \), and the boundary space \( B^{k-1/p,p}(\partial D) \) is the space of the boundary values \( u\mid_{\partial D} \) of functions \( u \in W^{k,p}(D) \).

In the space \( B^{k-1/p,p}(\partial D) \), we define a norm

\[
\|\varphi\|_{B^{k-1/p,p}(\partial D)} = \inf \left\{ \|u\|_{W^{k,p}(D)} : u \in W^{k,p}(D), u\mid_{\partial D} = \varphi \right\}.
\]

The space \( B^{k-1/p,p}(\partial D) \) is a Banach space with respect to the norm \( \|\cdot\|_{B^{k-1/p,p}(\partial D)} \); more precisely, it is a Besov space (cf. [1], [7], [73], [74]).

If conditions (H.1) and (H.2) are satisfied, then we can introduce a subspace of the Besov space \( B^{1-1/p,p}(\partial D) \) that is associated with the boundary condition

\[
Bu = a(x') \frac{\partial u}{\partial n} + b(x')u
\]

in the following way: We let

\[
B_{1-1/p,p}^{\star}(\partial D) := a(x')B_{1-1/p,p}(\partial D) + b(x')B_{2-1/p,p}(\partial D)
\]

\[
= \left\{ \varphi = a(x')\varphi_1 + b(x')\varphi_2 : \varphi_1 \in B_{1-1/p,p}(\partial D), \varphi_2 \in B_{2-1/p,p}(\partial D) \right\},
\]

and define a norm

\[
\|\varphi\|_{B_{1-1/p,p}^{\star}(\partial D)} = \inf \left\{ \|\varphi_1\|_{B_{1-1/p,p}(\partial D)} + \|\varphi_2\|_{B_{2-1/p,p}(\partial D)} : \varphi = a(x')\varphi_1 + b(x')\varphi_2 \right\}.
\]

It is easy to verify (see [66, Lemma 6.8]) that the space \( B_{1-1/p,p}^{\star}(\partial D) \) is a Banach space with respect to the norm \( \|\cdot\|_{B_{1-1/p,p}^{\star}(\partial D)} \).

We remark that the space \( B_{1-1/p,p}^{\star}(\partial D) \) is an “interpolation space” between the Besov spaces \( B_{2-1/p,p}(\partial D) \) and \( B_{1-1/p,p}(\partial D) \). In fact, we have the assertions

\[
B_{1-1/p,p}^{\star}(\partial D) = \begin{cases} 
B_{2-1/p,p}(\partial D) & \text{if } a(x') \equiv 0 \text{ on } \partial D \text{ (the Dirichlet case),} \\
B_{1-1/p,p}(\partial D) & \text{if } a(x') > 0 \text{ on } \partial D \text{ (the regular Robin case),}
\end{cases}
\]

and, for general \( a(x') \), the continuous injections

\[
B_{2-1/p,p}(\partial D) \subset B_{1-1/p,p}^{\star}(\partial D) \subset B_{1-1/p,p}(\partial D).
\]

Our first main result of this paper is stated as follows (cf. [56, Theorem 1.4], [63, Theorem 1.1], [66, Theorem 1.1]):
Theorem 1.1. Assume that conditions (H.1), (H.2) and (3) are satisfied. Then the mapping
\[ A := (A, B): W^{2,p}(D) \longrightarrow L^p(D) \oplus B^{1-1/p-0}(\partial D) \]
is an algebraic and topological isomorphism for all \( N < p < \infty \). In particular, for any \( f \in L^p(D) \) and any \( \varphi \in B^{1-1/p-0}(\partial D) \), there exists a unique solution \( u \in W^{2,p}(D) \) of the mixed-type boundary value problem (2).

1.1.2. Existence and uniqueness theorem for problem (2) in the framework of H"older spaces. Moreover, Theorem 1.1 remains valid in the framework of H"older spaces under the additional condition that
\[ c(x) \in C^\theta(\bar{D}) \text{ for } 0 < \theta < 1. \] (4)

More precisely, we can introduce a subspace of the H"older space \( C^{1+\theta}(\partial D) \) for \( 0 < \theta < 1 \) that is associated with the degenerate boundary condition
\[ Bu = a(x') \frac{\partial u}{\partial \nu} + b(x')u \]
in the following way: We let
\[ C^{1+\theta}(\partial D) := a(x') C^{1+\theta}(\partial D) + b(x') C^{2+\theta}(\partial D) \]
\[ = \{ \varphi = a(x') \varphi_1 + b(x') \varphi_2 : \varphi_1 \in C^{1+\theta}(\partial D), \varphi_2 \in C^{2+\theta}(\partial D) \}, \]
and define a norm
\[ |\varphi|_{C^{1+\theta}(\partial D)} = \inf \{ |\varphi_1|_{C^{1+\theta}(\partial D)} + |\varphi_2|_{C^{2+\theta}(\partial D)} : \varphi = a(x') \varphi_1 + b(x') \varphi_2 \}. \]

Then it is easy to verify (see [66, Lemma 6.8]) that the space \( C^{1+\theta}(\partial D) \) is a Banach space with respect to the norm \( |\cdot|_{C^{1+\theta}(\partial D)} \). We remark that the space \( C^{1+\theta}(\partial D) \) is an “interpolation space” between the spaces \( C^{2+\theta}(\partial D) \) and \( C^{1+\theta}(\partial D) \). More precisely, we have the assertions
\[ C^{1+\theta}_*(\partial D) = \begin{cases} C^{2+\theta}(\partial D) & \text{if } a(x') \equiv 0 \text{ on } \partial D \text{ (the Dirichlet case)}, \\ C^{1+\theta}(\partial D) & \text{if } a(x') > 0 \text{ on } \partial D \text{ (the regular Robin case)}, \end{cases} \]
and, for general \( a(x') \), the continuous injections
\[ C^{2+\theta}(\partial D) \subset C^{1+\theta}_*(\partial D) \subset C^{1+\theta}(\partial D). \]

The second main result of this paper is stated as follows (see [56, Theorem 1.1]):

Theorem 1.2. Assume that conditions (H.1), (H.2), (3) and (4) are satisfied. Then the mapping
\[ A := (A, B): C^{2+\theta}(\bar{D}) \longrightarrow C^\theta(\bar{D}) \oplus C^{1+\theta}_*(\partial D) \]
is an algebraic and topological isomorphism for all \( 0 < \theta < 1 \). In particular, for any \( f \in C^\theta(\bar{D}) \) and any \( \varphi \in C^{1+\theta}_*(\partial D) \) there exists a unique solution \( u \in C^{2+\theta}(D) \) of the mixed-type boundary value problem (2).
1.2. The linearized eigenvalue problem. Now, in order to study the semilinear problem (1), we consider the following linearized eigenvalue problem with the weight function 

\[ m(x) \in C(D): \]

\[
\begin{cases}
  -\Delta \varphi = \lambda m(x) \varphi & \text{in } D, \\
  B \varphi = 0 & \text{on } \partial D.
\end{cases}
\]

(5)

The next theorem asserts that the first eigenvalue of the mixed-type problem (5) is algebraically simple and its corresponding eigenfunction is positive, which is a generalization of a result due to Manes–Micheletti [38] (see [19, Theorem 1.13], [36, Theorem 9.1, part (c)]) to the degenerate case (cf. [55, Theorem 1], [62, Theorem 1.2], [63, Theorem 1.2]):

**Theorem 1.3.** Assume that conditions (M.1), (H.1) and (H.2) are satisfied. Then the first eigenvalue \( \lambda_1(m) \) of the mixed-type problem (5) is positive and algebraically simple, and its corresponding eigenfunction \( \psi_1(x) \in W^{2,p}(D), N < p < \infty \), may be chosen to be positive everywhere in \( D \):

\[
\begin{cases}
  -\Delta \psi_1 = \lambda_1(m) m(x) \psi_1 & \text{in } D, \\
  \psi_1 > 0 & \text{in } D, \\
  B \psi_1 = 0 & \text{on } \partial D.
\end{cases}
\]

(6)

Moreover, no other eigenvalues \( \lambda_j(m) \) for \( j \geq 2 \) have positive eigenfunctions.

Here it should be noticed that we have, by Sobolev's imbedding theorem (see [1, Theorem 4.12, Part II]),

\[ W^{2,p}(D) \subset C^1(D), \]

since \( 2 - N/p > 1 \) for \( N < p < \infty \).

A biological interpretation of Theorem 1.3 is that if there is a favorable region, then the models we consider predict persistence for a population, since the existence of the first positive eigenvalue is equivalent to the existence of a positive density function describing the distribution of the population of \( D \). It is worthwhile to point out here that the first eigenvalue \( \lambda_1(m) \) will tend to be smaller in situations where favorable and unfavorable habitats are closely intermingled (producing cancellation effects), and larger when the favorable region consists of a relatively small number of relatively large isolated components.

By the Rayleigh principle, we can prove that the first eigenvalue \( \lambda_1(m) \) is characterized by the variational formula (cf. [19], [36], [46])

\[
\lambda_1(m) = \inf \left\{ \frac{\int_D (-\Delta \phi, \phi)_{L^2(D)} - \int_D m(x) |\phi|^2 \, dx}{\int_D m(x) |\phi|^2 \, dx} : \phi \in H^2(D), \ B \phi = 0, \ \int_D m(x) |\phi|^2 \, dx > 0 \right\}. \quad (7)
\]

Here \( H^2(D) = W^{2,2}(D) \).

More precisely, by combining Theorem 1.3 with [64, Propositions 3.4 and 3.5] we can obtain a generalization of [36, Theorem 7.7 and Proposition 8.3] to the degenerate case:

**Theorem 1.4.** Let \( m(x) \in C(\overline{D}) \). Assume that conditions (M.1), (H.1) and (H.2) are satisfied. Then we have the following three assertions:

(i) The mixed-type problem (5) with weight has an infinite sequence of eigenvalues

\[ 0 < \lambda_1(m) < \lambda_2(m) \leq \lambda_3(m) \leq \ldots. \]

(ii) The first eigenvalue \( \lambda_1(m) \) is given by the variational formula (7).
(iii) The first eigenvalue $\lambda_1(m)$ is strictly decreasing with respect to the weight $m(x)$ in the following sense: If $\hat{m}(x) \leq m(x)$ in $D$, then the first eigenvalues $\lambda_1(m)$ and $\lambda_1(\hat{m})$ satisfy

$$\lambda_1(\hat{m}) \geq \lambda_1(m).$$

If, in addition, $\hat{m}(x) < m(x)$ on a subset of positive measure in $D$, we have the strict inequality

$$\lambda_1(\hat{m}) > \lambda_1(m).$$

1.3. The semilinear boundary value problem (1). A solution $u(x) \in C^2(\overline{D})$ of the semilinear boundary value problem (1) is said to be non-trivial if it does not identically equal zero on $\overline{D}$. We call a non-trivial solution $u$ of the semilinear problem (1) a positive solution if $u(x) \geq 0$ on $\overline{D}$.

In this paper we discuss the changes that occur in the structure of the positive solutions of the semilinear problem as the parameter $\lambda$ varies near the first eigenvalue $\lambda_1(m)$ under the condition that:

(M.2) The function $m(x)$ attains both positive and negative values in $D$.

We consider the case where $h(x) > 0$ near the boundary $\partial D$. More precisely, our structural condition on the $C^1$ function $h(x)$ is formulated as follows. We let

$$D_0(h):=\text{the interior of the set } \{x \in D : h(x) = 0\},$$

and assume that (see Figure 3)

(Z.1) The open set $D_0(h)$ consists of a finite number of connected components with smooth boundary, say $D_0^i(h), 1 \leq i \leq \ell$, which are bounded away from the boundary $\partial D$:

$$D_0(h) = \bigcup_{i=1}^{\ell} D_0^i(h).$$

In other words, we are interested in the changes that occur in the structure of the habitat inside $D$ rather than boundary effects. The complement

$$\overline{D} \setminus D_0(h) = \{x \in \overline{D} : h(x) > 0\}$$

is supposed to correspond to a location marked by the competition for existence among living things in nature.

\[\text{Figure 3. The structural condition (Z.1) on the function } h(x)\]
Remark 1.2. In the Dirichlet case, the structural condition (Z.1) can be weakened such that the function \( h(x) \) may vanish on the boundary \( \partial D \) (see [23, Theorem 3.5]).

We consider the Dirichlet eigenvalue problem in each connected component \( D_i(h) \) for \( 1 \leq i \leq \ell \),

\[
\begin{aligned}
-\Delta \varphi &= \lambda m(x) \varphi \quad \text{in } D_i(h), \\
\varphi &= 0 \quad \text{on } \partial D_i(h).
\end{aligned}
\]

(9)

In this paper we assume that

(Z.2) Each set \( \{ x \in D_i(h) : m(x) > 0 \} \) has positive measure for \( 1 \leq i \leq \ell \), and let

\[ \lambda_1 \big( D_i(h) \big) = \text{the first eigenvalue} \]

of problem (9).

By applying Theorem 1.3 with

\[ D := D_i(h) \quad \text{for } 1 \leq i \leq \ell, \]

(M.1) := (Z.2),

\[ a(x') \equiv 0, \quad b(x') \equiv 1, \]

we obtain that the first eigenvalue \( \lambda_1 \big( D_i(h) \big) \) is positive and algebraically simple:

\[ \lambda_1 \big( D_i(h) \big) > 0 \quad \text{for } 1 \leq i \leq \ell. \]

Moreover, by the Rayleigh principle we know that the first eigenvalue \( \lambda_1 \big( D_i(h) \big) \)

is given by the variational formula (cf. [19], [36], [46])

\[
\begin{aligned}
\lambda_1 \big( D_i(h) \big) &= \inf \left\{ \frac{\int_{D_i(h)} \left| \nabla \psi \right|^2 \, dx}{\int_{D_i(h)} m(x) |\psi|^2 \, dx} : \psi \in H_0^1(D_i(h)), \quad \int_{D_i(h)} m(x) |\psi|^2 \, dx > 0 \right\}.
\end{aligned}
\]

(10)

Here \( H_0^1(D_i(h)) = W^{1,2}_0(D_i(h)) \) is the closure of smooth functions with compact support in \( D_i(h) \) in the Sobolev space \( H^1(D_i(h)) = W^{1,2}(D_i(h)) \).

By virtue of assertion (10), we can associate with the open set \( D_0(h) \) a positive number \( \mu_1 \big( D_0(h) \big) \) as follows:

\[
\mu_1 \big( D_0(h) \big) = \min \left\{ \lambda_1 \big( D_0^1(h) \big), \lambda_1 \big( D_0^2(h) \big), \ldots, \lambda_1 \big( D_0^\ell(h) \big) \right\}.
\]

(11)

It should be noticed that the eigenvalue \( \mu_1 \big( D_0(h) \big) \) tends to be larger in situations where favorable and unfavorable habits are closely intermingled, and smaller when the favorable region consists of a relatively small number of relatively large isolated components. Indeed, the eigenvalue \( \mu_1 \big( D_0(h) \big) \) is strictly decreasing with respect to the domain \( D_0(h) \) (see [13], [36, Proposition 8.2]).

Now we can state our main result that is a generalization of Fraile et al. [23, Theorem 3.5] to the degenerate case (cf. [22], [49], [52, Theorem 3.2], [62, Theorem 1.3]):

**Theorem 1.5.** Let \( m(x) \in C^0(\bar{D}) \) for \( 0 < \theta < 1 \) and \( h(x) \in C^1(\bar{D}) \). If conditions (M.1), (Z.1), (Z.2), (H.1) and (H.2) are satisfied, then the semilinear problem (1) has a unique positive solution

\[ u(\lambda) \in C^{2+\theta}(\bar{D}) \]
for every $\lambda \in (\lambda_1(m), \mu_1(D_0(h)))$. For any $\lambda \geq \mu_1(D_0(h))$, there exists no positive solution of the semilinear problem (1). Moreover, we have the assertions (see Figure 4)

$$\lim_{\lambda \uparrow \mu_1(D_0(h))} \|u(\lambda)\|_{L^2(D)} = +\infty,$$

and

$$\lim_{\lambda \downarrow \lambda_1(m)} \|u(\lambda)\|_{C^{2+\eta}(\overline{D})} = 0.$$ (13)

Some remarks are in order:

**Remark 1.3.** (1) We recall that assertion (12) holds true for $N \geq 3$ (see [60, Lemma 4.1]). Indeed, the proof of [60, Lemma 4.1] is based on Moser’s technique [39] for $N \geq 3$.

(2) We can prove the strong assertion

$$\lim_{\lambda \uparrow \mu_1(D_0(h))} \|u(\lambda)\|_{C(\overline{D})} = +\infty,$$

by using the super-sub-solution method, just as in the proof of Fraile et al. [23, Theorems 3.5 and 4.6].

(3) It should be emphasized that an estimate of the growth rate of the total size

$$\|u(\lambda)\|_{L^1(D)} = \int_D u(\lambda) \, dx$$

of the positive steady states $u(\lambda)$ as $\lambda \uparrow \mu_1(D_0(h))$ is of crucial importance from the viewpoint of population dynamics as in estimate (17) of Theorem 1.6 (see Section 9).

Rephrased, Theorem 1.5 asserts that the models we consider predict persistence for a population if its diffusion rate $d = 1/\lambda$ is below the critical value $1/\lambda_1(m)$ depending on the coefficient $m(x)$ which describes the growth rate and if it is above the critical value $1/\mu_1(D_0(h))$ depending on the coefficient $h(x)$ which describes the strength of the crowding effects:

(i) The model predicts persistence for the population if $\lambda_1(m) < \lambda < \mu_1(D_0(h))$.

(ii) The model predicts extinction for the population if $0 < \lambda < \lambda_1(m)$.
The size of the first eigenvalue $\lambda_1(m)$ is of crucial importance; increasing the eigenvalue $\lambda_1(m)$ imposes a more stringent condition on the diffusion rate $d = 1/\lambda$ if the population is to persist, since $0 < d = 1/\lambda < 1/\lambda_1(m)$.

The dynamics of a population inhabiting a strongly heterogeneous environment are modelled by diffusive logistic equations of the form

\[
\begin{cases}
\frac{\partial w}{\partial t} = d \Delta w + (m(x) - h(x))w & \text{in } D \times (0, \infty), \\
Bw = a(x') \frac{\partial w}{\partial n} + b(x')w = 0 & \text{on } \partial D \times (0, \infty),
\end{cases}
\]

To study problem (14), we may view it as generating a dynamical system. The semilinear parabolic initial boundary value problem (14) admits a unique solution for sufficiently small times. However, comparison theorems based on the maximum principle guarantee the existence of global solutions in time, since the nonlinearity we are dealing with is sublinear. We can show that problem (14) admits a unique positive steady state which is a global attractor for non-negative solutions provided $d = 1/\lambda$ is sufficiently small, so that the population persists. Furthermore, we can show that the zero solution is a global attractor for non-negative solutions if $d = 1/\lambda$ is sufficiently large, so that the population tends to extinction.

Another biological interpretation of Theorem 1.5 is that, for a species with a given rate $d = 1/\lambda$ of diffusion, the most favorable situations occur if there is a relatively large favorable region (with good resources and without crowding effects) located some distance away from the boundary of $D$ if the population is to persist. Indeed, an initial population $u_0$ grows exponentially until limited by lack of available resources if the diffusion rate $d = 1/\lambda$ is below the critical value $1/\mu_1(D_0(h))$; this idea is generally credited to the English economist Thomas Robert Malthus (1776–1834). On the other hand, if the diffusion rate $d = 1/\lambda$ is above the critical value $1/\mu_1(D_0(h))$, then the model obeys the logistic equation introduced by the Belgian mathematical biologist Pierre François Verhulst (1804–1849) around 1840 (see [35]):

1. Malthusian theory holds true for $\lambda > \mu_1(D_0(h))$.
2. Verhulst theory holds true for $0 < \lambda < \mu_1(D_0(h))$.

The situation of Theorem 1.5 may be represented schematically by the following bifurcation diagram (Figure 5):

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{The bifurcation diagram of Theorem 1.5: Malthus versus Verhulst}
\end{figure}
1.4. **Heuristic approach to problem (1) via the Semenov approximation.**

For simplicity, we assume that

\[(Z.3) \ h(x) > 0 \text{ on } \overline{D},\]

and further that the weight function \(m(x)\) satisfies condition \((M.1)\). First, we rewrite the semilinear problem (1) in the form

\[
\begin{cases}
-\Delta u = \lambda (m(x) - h(x) u) u & \text{in } D, \\
u > 0 & \text{in } D, \\
Bu = 0 & \text{on } \partial D.
\end{cases}
\]  

(15)

Namely, we consider the semilinear problem (1) as the mixed type eigenvalue problem (15) with the weight \(m(x) - h(x) u\).

However, Theorem 1.3 asserts that the first eigenvalue \(\lambda_1(m)\) is the unique eigenvalue of the eigenvalue problem (1) corresponding to a positive eigenfunction \(\psi_1(x)\). Hence we assume that the solution \(u\) is of the form

\[u = C(\lambda) \psi_1 \text{ for } \lambda > \lambda_1(m),\]

where \(C(\lambda)\) is a non-zero constant. This trick is due to the Russian chemist Nikolay Nikolayevich Semenov (see [51], [71]).

Then we have the formulas

\[-\Delta u = -C(\lambda) \Delta \psi_1 = C(\lambda) \lambda_1(m) m(x) \psi_1 \text{ in } D\]

and

\[\lambda (m(x) - h(x) u) u = \lambda (m(x) - h(x) u) C(\lambda) \psi_1 \text{ in } D.\]

In view of formula (15), this implies that

\[\lambda m(x) - \lambda h(x) u = \lambda_1(m) m(x) \text{ in } D,\]

so that

\[u = u(\lambda) = \frac{m(x)}{h(x)} \left(1 - \frac{\lambda_1(m)}{\lambda}\right) \text{ in } D.\]

Therefore, we obtain that the bifurcation solution curve \((\lambda, u(\lambda))\) of the semilinear problem (1) is “formally” given by formula, called the Semenov approximation (see Figure 6),

\[u(\lambda) = \frac{m(x)}{h(x)} \left(1 - \frac{\lambda_1(m)}{\lambda}\right) \text{ for all } \lambda > \lambda_1(m). \quad (16)\]

In fact, the next theorem is a generalization of Cantrell–Cosner [11, Theorem 2.1] to the degenerate case:

**Theorem 1.6.** Let \(m(x) \in C^\theta(\overline{D})\) for \(0 < \theta < 1\) and \(h(x) \in C^1(\overline{D})\). If conditions \((M.1)\), \((Z.3)\), \((H.1)\) and \((H.2)\) are satisfied, then the semilinear problem (1) has a unique positive solution \(u(\lambda) \in C^{2+\theta}(\overline{D})\) for every \(\lambda > \lambda_1(m)\). Moreover, we can give an estimate of the growth rate of the total size \(\|u(\lambda)\|_{L^1(D)}\) of the positive steady states \(u(\lambda)\)

\[\int_D u(\lambda) \, dx \leq \left(1 - \frac{\lambda_1(m)}{\lambda}\right) |D|^{2/3} \left(\frac{\int_D (m^+(x))^3 \, dx}{\min_{x \in \overline{D}} h(x)}\right)^{1/3} \text{ for all } \lambda > \lambda_1(m). \quad (17)\]

Here \(|D|\) is the volume of the domain \(D\) and

\[m^+(x) = \max\{m(x), 0\} \text{ for } x \in D.\]
Rephrased, Theorem 1.6 asserts that the models we consider predict persistence for a population if its diffusion rate \( d = 1/\lambda \) is below the critical value \( 1/\lambda_1(m) \), and predict extinction for a population if the diffusion rate \( d = 1/\lambda \) is above the critical value \( 1/\lambda_1(m) \). The situation may be represented schematically by the bifurcation diagram below (Figure 7). The size of \( \lambda_1(m) \) is of crucial importance; increasing \( \lambda_1(m) \) imposes a more stringent condition on the diffusion rate \( d = 1/\lambda \) if the population is to persist, since \( 0 < d = 1/\lambda < 1/\lambda_1(m) \).

**Remark 1.4.** It should be emphasized that \( \lambda(h) = +\infty \) if the function \( h(x) \) satisfies condition (Z.3). In Subsection 9.2, by using the maximum principle as in Cantrell–Cosner [11] we will give a uniform bound on the positive steady states \( u(\lambda) \):

\[
\max_{\partial D} u(\lambda) \leq \ell := \frac{\max_{\partial D} m^+}{\min_{\partial D} h} = \frac{\max_{\partial D} m}{\min_{\partial D} h} \quad \text{for all } \lambda > \lambda_1(m).
\]

By inequality (18), we find that the quantity \( \ell \) is the *carrying capacity* of the environment (see Figure 7).

**Figure 7.** The bifurcation diagram of Remark 1.4 under condition (Z.3) (Verhulst theory)

1.5. **An outline of the paper.** The present paper is amply illustrated; 4 tables and 32 figures (3 flowcharts of the proof) are provided with appropriate captions in such a fashion that a broad spectrum of readers could understand our problem and main results. The rest of this paper is organized as follows.
We make use of the theory of positive operators in ordered Banach spaces to study positive solutions of the semilinear problem (1) (cf. [3], [31]). In Section 2 we summarize the basic definitions and results about ordered Banach spaces and the well-known Kreǐn–Rutman theorem for strongly positive, compact linear operators (Theorem 2.1) that enter naturally in connection with elliptic eigenvalue problems.

Section 3 is devoted to static bifurcation theory for the nonlinear equation

$$F(\lambda, u) = 0.$$  \hfill (20)

More precisely, $F(\lambda, u)$ is a nonlinear operator, depending on a real parameter $\lambda$, which operates on the unknown vector $u$. By making use of bifurcation theory from a simple eigenvalue of the linearized problem, essentially due to Crandall–Rabinowitz [16] and Rabinowitz [44], we discuss the changes that occur in the structure of solutions of the nonlinear equation $F(\lambda, u) = 0$ as $\lambda$ varies near the first eigenvalue of the linearized problem (Theorems 3.3 and 3.4), which will play an essential role in the study of the semilinear problem (1) in Sections 7 and 8.

In Section 4 we study the mixed-type boundary value problem (2), and we prove Theorems 1.1 and 1.2. Our proof is carried out as in Taira [56], [62] and [66]. The main technique used is the $L^p$ theory of pseudo-differential operators that may be considered as a modern version of classical potentials.

Section 5 is devoted to the proof of Theorem 1.3, just as in the proof of Brown–Lin [9, Theorem 3.5] by using Theorem 4.9.

In Section 6 we study the inequalities among the first eigenvalues $\mu_D(\lambda)$, $\mu_N(\lambda)$ and $\mu_1(\lambda)$ subject to Dirichlet, Neumann and mixed type boundary conditions, respectively (Theorem 6.3).

An overview of theorems for eigenvalue problems with indefinite weights may be visualized as in Table 3 below.

| Problems | Conditions | Theorems |
|----------|------------|----------|
| (6) (mixed type case) | (M.1) (H.1), (H.2) | Theorem 1.3 for $\lambda_1(m)$ |
| (55) (Dirichlet case) | (M.1) | Theorem 6.1 for $\gamma_1(m)$ |
| (58) (Neumann case) | (M.2) | Theorem 6.2 for $\nu_1(m)$ |
| (56), (59), (61) | (M.1), (M.2) (H.1), (H.2) | Theorem 6.3 for $\mu_D(\lambda)$, $\mu_N(\lambda)$, $\mu_1(\lambda)$ |

Table 3. An overview of theorems for eigenvalue problems with indefinite weights

Sections 7 and 8 are devoted to the proof of Theorem 1.5. Our approach to the semilinear problem (1) is a modification of that of Ouyang [41] adapted to the present context (see [54]), which goes back to Moser’s technique [39].

In Section 7, by using Theorems 1.3 and 1.4 we prove that if there exists a positive solution $u(\lambda) \in C^2(\overline{D})$ of the semilinear problem (1), then it follows that $\lambda > \lambda_1(m)$ (Lemma 7.1). The existence of positive solutions of the semilinear problem (1) near the point $(\lambda_1(m), 0)$ follows by applying local static bifurcation theory from a simple eigenvalue due to Crandall–Rabinowitz [16] (Theorem 3.3). Next, by making use of the implicit function theorem (Theorem 3.1) we prove that there exists a critical
value \( \bar{\lambda}(h) \in (\lambda_1(m), \mu_1(D_0(h))] \) such that the semilinear problem (1) has a positive solution \( u(\lambda) \) for all \( \lambda \in (\lambda_1(m), \bar{\lambda}(h)) \) (Lemmas 7.2, 7.3 and formula (88)).

The fundamental formula
\[
\bar{\lambda}(h) = \mu_1(D_0(h)) = \min \left\{ \lambda_1\left(D_0^1(h)\right), \lambda_1\left(D_0^2(h)\right), \ldots, \lambda_1\left(D_0^\ell(h)\right) \right\}
\]
follows from the uniqueness of a bifurcation point and the comparison principle (Theorem 8.3) in Section 8.

Section 9 is devoted to the proof of Theorem 1.6 and Remark 1.4. More precisely, we consider the semilinear problem (1) under the condition (Z.3), and prove an estimate of the growth rate of the total size \( \|u(\lambda)\|_{L^1(D)} \) of the positive steady states \( u(\lambda) \).

The last Section 10 is devoted to open problems in numerical analysis. In the near future, we would like to apply Theorem 1.5 to provide numerical solutions of diffusive logistic equations with mixed type boundary conditions, generalizing Fleming [22] and García-Melían et al [24].

The ecological conclusion of the present paper is that, for a species with a given rate of diffusion, the best environments are those where the favorable regions are relatively large and few in a number, and the worst are those where favorable and unfavorable regions are closely intermingled, producing cancellation effects. An overview of main theorems for diffusive logistic problems may be visualized as in Table 4 below.

| Problems            | Conditions  | Theorems                     |
|---------------------|-------------|------------------------------|
| (1) (mixed type case) | (M.1)       | Theorem 1.5 for \( u(\lambda) \) |
|                     | (Z.1), (Z.2) |                              |
|                     | (H.1), (H.2) |                              |
| (91) (Dirichlet case) | (M.1)       | Theorem 8.1 for \( v(\lambda) \) |
|                     | (Z.1), (Z.2) |                              |
| (93) (Neumann case)  | (M.2)       | Theorem 8.2 for \( w(\lambda) \) |
|                     | (Z.1), (Z.2) |                              |
| (1), (91), (93)     | (M.1), (M.2) | Theorem 8.3 for \( v(\lambda), w(\lambda), u(\lambda) \) |
|                     | (Z.1), (Z.2) |                              |
|                     | (H.1), (H.2) |                              |

Table 4. An overview of existence theorems for diffusive logistic problems

2. **Theory of ordered Banach spaces.** A general class of semilinear second-order elliptic boundary value problems satisfies the maximum principle. Roughly speaking, this additional information means that the operators associated with the boundary value problems are compatible with the natural ordering of the underlying function spaces. Consequently, we are led to the study of nonlinear equations in the framework of ordered Banach spaces. This section is devoted to a review of some important topics from nonlinear functional analysis that forms a necessary background for the proof of Theorem 1.3.

For detailed studies of nonlinear analysis, the reader is referred to Ambrosetti–Malchiodi [5], Brown [10], Chang [12], Chow and Hale [15], Deimling [20], Drábek–Milota [21], Nirenberg [40], Pao [42] and Runst–Sickel [47].
2.1. Ordered Banach spaces and the Kreĭn–Rutman theorem. In this subsection we recall some basic definitions and results concerning ordered Banach spaces (see [3], [21]).

Let $X$ be a non-empty set. An ordering $\leq$ in $X$ is a relation in $X$ which is reflexive, transitive and antisymmetric. A non-empty set together with an ordering is called an ordered set.

Let $V$ be a real vector space. An ordering $\leq$ in $V$ is said to be linear if the following two conditions are satisfied:

1. If $x, y \in V$ and $x \leq y$, then we have $x + z \leq y + z$ for all $z \in V$.
2. If $x, y \in V$ and $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A real vector space together with a linear ordering is called an ordered vector space.

If we let $Q = \{x \in V : x \geq 0\}$, then it is easy to verify that the set $Q$ has the following two conditions:

3. If $x, y \in Q$, then $\alpha x + \beta y \in Q$ for all $\alpha, \beta \geq 0$.
4. If $x \neq 0$, then at least one of $x$ and $-x$ does not belong to $Q$, that is, $Q \cap (-Q) = \{0\}$.

The set $Q$ is called the positive cone of the ordering $\leq$.

Let $E$ be a Banach space $E$ with a linear ordering $\leq$. The Banach space $E$ is called an ordered Banach space if the positive cone

$$P = \{x \in E : x \geq 0\}$$

is closed in $E$. We say that $P$ is generating if, for each $x \in E$ there exist vectors $u, v \in P$ such that $x = u - v$. It is to be expected that the topology and the ordering of an ordered Banach space are closely related if the norm is monotone: If $0 \leq u \leq v$, then $\|u\| \leq \|v\|$.

For $x, y \in E$, we write

$$x \geq y \text{ if } x - y \in P,$$

$$x > y \text{ if } x - y \in P \setminus \{0\}.$$

If the interior $\text{Int}(P)$ is non-empty, then we write

$$x \gg y \text{ if } x - y \in \text{Int}(P).$$

Here we give two simple but important examples of ordered Banach spaces:

**Example 2.1.** Let $E = \mathbb{R}^N$, and let

$$\mathbb{R}^{N,+} = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_i \geq 0 \text{ for all } 1 \leq i \leq N\}.$$

For any $u, v \in \mathbb{R}^N$, we write $u \leq v$ if $v - u \in \mathbb{R}^{N,+}$. Then it is easy to see that $(\mathbb{R}^N, \mathbb{R}^{N,+}, \leq)$ is an ordered Banach space and that the norm is monotone. Moreover, the positive cone $\mathbb{R}^{N,+}$ is generating. We remark that

$$\text{Int}(\mathbb{R}^{N,+}) = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_i > 0 \text{ for all } 1 \leq i \leq N\}.$$

**Example 2.2.** Let $E = C(\overline{D})$ be the set of real-valued, continuous functions on the closure $\overline{D}$, and let

$$P = \{u \in C(\overline{D}) : u(x) \geq 0 \text{ on } \overline{D}\}.$$
For any \( u, v \in C(D) \), we write \( u \leq v \) if \( v - u \in P \). Then it is easy to see that \((C(D), P)\) is an ordered Banach space and that the norm is monotone. Moreover, the positive cone \( P \) is generating. We remark that

\[
\text{Int}(P) = \{ u \in C(D) : u(x) > 0 \text{ on } D \}\,.
\]

A linear operator \( K : E \to E \) is said to be positive if \( K \) maps \( P \) into itself:

\[
x \geq 0 \implies Kx \geq 0.
\]

A linear operator \( K : E \to E \) is said to be strictly positive if \( K \) maps \( P \setminus \{0\} \) into itself:

\[
x > 0 \implies Kx > 0.
\]

A linear operator \( K : E \to E \) is said to be strongly positive if \( Kx \) belongs to \( \text{Int}(P) \) for every \( x \in P \setminus \{0\} \):

\[
x > 0 \implies Kx \gg 0.
\]

A linear operator \( K : E \to E \) is said to be compact if it maps bounded sets in \( K \) into relatively compact sets in \( E \) (see [50, Section 4.3]).

Then the well-known Kreǐn–Rutman theorem for strongly positive, compact linear operators reads as follows (see [32, Theorem 6.3], [31, Chapter 2], [12, Theorem 3.6.12], [21, Chapter 6, Proposition 6.3.34]):

**Theorem 2.1 (Kreǐn–Rutman).** Let \((E, P)\) be an ordered Banach space with non-empty \( \text{Int}(P) \) and \( K : E \to E \) a linear operator. If \( K \) is strongly positive and compact, then we have the following five assertions:

(i) The spectral radius

\[
spr(K) := \lim_{n \to \infty} \sqrt[n]{\|K^n\|}
\]

is positive and \( spr(K) \) is the unique eigenvalue of \( K \) having a positive eigenvector \( x \): \( Kx = spr(K)x \). In other words, there exists no other eigenvalue with a positive eigenvector.

(ii) The eigenvalue \( spr(K) \) is algebraically simple and \( x \in \text{Int}(P) \).

(iii) The eigenvalue \( spr(K) \) is greater than all the remaining eigenvalues \( \lambda \) of \( K \):

\[
spr(K) > |\lambda|.
\]

The eigenvalue \( spr(K) \) is called the principal eigenvalue of \( K \).

(iv) The spectral radius \( spr(K) \) is an algebraically simple eigenvalue of the dual operator \( K^* : E^* \to E^* \) having a strictly positive eigenvector \( x^* \in P^* \setminus \{0\} \). Namely, the vector \( x^* \) satisfies the conditions

\[
K^*x^* = spr(K)x^*
\]

and

\[
\langle y, x^* \rangle > 0 \quad \text{for all } y \in P \setminus \{0\}.
\]

(v) For every continuous linear operator \( L \) satisfying \( L \geq K \), we have the assertion

\[
spr(L) \geq spr(K).
\]

If the operator \( L - K \) is strongly positive, then we have the assertion

\[
spr(L) > spr(K).
\]

The Kreǐn–Rutman theorem is a generalization of the classical Perron–Frobenius theorem in linear algebra.
2.2. Application of the Kreĭn–Rutman theorem. As an application of the Kreĭn–Rutman theorem, we consider the following non-homogeneous operator equation: For a given $h > 0$ in $E$, find an element $u \in E$ such that
\[
\lambda u - Ku = h,
\] (19)
where $\lambda$ is a real parameter.

The next theorem will play an important role in the proof of Theorem 1.3 in Section 4 (see [3], [28]):

**Theorem 2.2.** Let $K : E \to E$ be a strongly positive, compact linear operator and let $\text{spr}(K)$ be its principal eigenvalue as in Theorem 2.1. Then we have the following three assertions:

(i) If $\lambda > \text{spr}(K)$, then the operator equation (19) has a unique positive solution $u$ and $u \gg 0$.

(ii) If $\lambda < \text{spr}(K)$, then the operator equation (19) has no positive solution.

(iii) If $\lambda = \text{spr}(K)$, then the operator equation (19) has no solution.

3. Elements of bifurcation theory. This section is devoted to static bifurcation theory for the nonlinear equation $F(\lambda, u) = 0$. By making use of bifurcation theory from a simple eigenvalue of the linearized problem, essentially due to Crandall–Rabinowitz [16], [17] and Rabinowitz [44], we discuss the changes that occur in the structure of the solutions of $F(\lambda, u) = 0$ as $\lambda$ varies near the first eigenvalue of the linearized problem (Theorems 3.3 and 3.4), which will play an essential role in the study of the semilinear problem (1) in Sections 7 and 8. For detailed studies of bifurcation theory, the reader is referred to Ambrosetti–Malchiodi [5], Ambrosetti–Prodi [6], Brown [10], Chang [12], Chow–Hale [15], Deimling [20], Drábek–Milota [21], Nirenberg [40] and Sattinger [48].

3.1. Local bifurcation theory. This subsection is devoted to local static bifurcation theory from a simple eigenvalue essentially due to Crandall–Rabinowitz [16].

3.1.1. Differentiability. Let $X$, $Y$ be Banach spaces. Let $U$ be a subset of $X$. A map $f : U \to Y$ is said to be completely continuous if it is continuous on $U$ and maps bounded sets in $U$ into relatively compact sets in $Y$ (see [21, Definition 5.2.2]).

Let $U$ be an open set in $X$ and let $f : U \to Y$ be a map. We say that the map $f$ is (Fréchet) differentiable at a point $x \in U$ if there exist a continuous linear operator $A : X \to Y$ and a map $\psi$ defined for all sufficiently small $h$ in $X$, with values in $Y$, such that
\[
\begin{align*}
  f(x+h) &= f(x) + Ah + ||h||\psi(h), \\
  \lim_{h \to 0} \psi(h) &= 0.
\end{align*}
\]
We remark that the continuous linear operator $A$ is uniquely determined by $f$ and $x$. The operator $A$ is called the (Fréchet) derivative of $f$ at $x$, and is denoted by $Df(x)$ or $f'(x)$. A map $f$ is said to be (Fréchet) differentiable on $U$ if it is (Fréchet) differentiable at every point of $U$. In this case, the derivative $Df$ is a map of $U$ into the Banach space $B(X,Y)$ of all continuous (bounded) linear operators:
\[
Df : U \to B(X,Y), \quad u \mapsto Df(u).
\]
If in addition $Df$ is continuous from $U$ into $B(X,Y)$, we say that $f$ is of class $C^1$. 
If the derivative $Df$ is differentiable at a point $x \in U$ (resp. in $U$), we say that $f$ is \textit{twice differentiable} at $x$ (resp. in $U$). The derivative of $Df$ at $x$ is called the \textit{second derivative} of $f$ at $x$, and is denoted by $D^2f(x)$. This is an element of the Banach space $B(X,B(X,Y))$ which can be naturally identified with the space $B_2(X,Y) = B(X;X,Y)$ of all continuous bilinear mappings of $X \times X$ into $Y$.

By induction on $k$, we define a $k$ times differentiable mapping $f$ of $U$ into $Y$ as a $(k-1)$ times differentiable mapping whose $(k-1)$-th derivative $D^{k-1}f$ is differentiable in $U$. The derivative $D^k f(x)$ at a point $x \in U$ can be identified with an element of the space $B_k(X,Y)$ of all continuous $k$-linear mappings of $X \times \cdots \times X$ into $Y$. A map $f : U \to Y$ is said to be of class $C^r$ $(r \geq 2)$ in $U$ if all the derivatives $D^k f$ exist and are continuous in $U$ for $1 \leq k \leq r$.

Here it is worthwhile pointing out that if $X = \mathbb{R}$, then the space $B(X,Y)$ can be identified with the space $Y$; so the space $B_k(\mathbb{R},Y)$ can be identified with the space $Y$ for general $k \geq 2$.

Now we assume that the Banach space $X$ is the product space of two Banach spaces $X_1$ and $X_2$: \[ X = X_1 \times X_2. \]

For each point $x = (x_1, x_2) \in U \subset X$, one can consider the partial mappings
\[
F_1 : u_1 \mapsto f(u_1, x_2), \\
F_2 : u_2 \mapsto f(x_1, u_2)
\]
of open subsets of $X_1$ and $X_2$ respectively into $Y$. We say that $f$ is \textit{differentiable with respect to the first (resp. second) variable} if the mapping $F_1(u_1)$ (resp. $F_2(u_2)$) is differentiable at $x_1$ (resp. at $x_2$). The derivative $DF_1(x_1)$ (resp. $DF_2(x_2)$) is an element of the Banach space $B(X_1,Y)$ (resp. $B(X_2,Y)$), and is called the \textit{partial (Fréchet) derivative} of $f$ at $(x_1, x_2)$ with respect to the first (resp. second) variable. We write
\[
D_{x_1} f(x_1, x_2) = DF_1(x_1), \\
D_{x_2} f(x_1, x_2) = DF_2(x_2).
\]

We can define inductively the partial (Fréchet) derivatives $D^k_{x_1} D^\ell_{x_2} f$ for general $k$ and $\ell$.

The process of linearization provides a key link between the linear and nonlinear theories of partial differential equations. Our basic tool is the implicit function theorem (cf. [6, Chapter 2, Section 2], [12, Theorem 1.2.1], [21, Chapter 4, Section 4.2], [40, Theorem 2.7.2]):

\begin{theorem}[the implicit function theorem] \textbf{Theorem 3.1} \textit{(the implicit function theorem).} Let $X$, $Y$, $Z$ be Banach spaces, and let $f$ be a $C^r$ map $(r \geq 1)$ of an open subset $U \times V$ of $X \times Y$ into $Z$. Assume that the derivative $D_0 f(x_0, y_0) : Y \to Z$ is an algebraic and topological isomorphism at a point $(x_0, y_0)$ of $U \times V$. Then there exist neighborhoods $U_0$ of $x_0$ and $W_0$ of $f(x_0, y_0)$ and a unique $C^r$ map $g : U_0 \times W_0 \to V$ such that \[ f(x, g(x, w)) = w \quad \text{for all } (x, w) \in U_0 \times W_0. \]
\end{theorem}

The inverse mapping theorem provides a criterion for a map to be a local $C^r$ diffeomorphism in terms of its derivative (see [40, Corollary 2.7.3]; [12, Theorem 1.2.3]):
Theorem 3.2 (the inverse mapping theorem). Let $X, Y$ be Banach spaces, and let $f$ be a $C^r$ map ($r \geq 1$) of an open subset $U$ of $X$ into $Y$. Assume that the derivative $$f'(x_0): X \to Y$$ is an algebraic and topological isomorphism at a point $x_0$ of $U$. Then the map $f$ is a $C^r$ diffeomorphism of some neighborhood of $x_0$ onto some neighborhood of $f(x_0)$.

3.1.2. Local bifurcation from a simple eigenvalue. Let $X, Y$ be Banach spaces. We consider a map $F(w)$ of a neighborhood of $(0,0)$ in the Banach space $\mathbb{R} \times X$ into the Banach space $Y$. Bifurcation theory offers information about solutions of the equation $F(w) = 0$ in $Y$.

Assume that there is a curve $\Gamma$ in the space $\mathbb{R} \times X$ given by $\Gamma = \{w(t) : t \in I\}$, where $I$ is an interval, such that $F(w) = 0$ for all $w \in \Gamma$. In many situations the curve $\Gamma$ is of the form $\{w = (\lambda, u) : \lambda \in \mathbb{R}, u \in X\}$. Namely, $F(\lambda, u)$ is a nonlinear operator, depending on a real parameter $\lambda$, which operates on the unknown vector $u$.

Now we study the nonlinear equation of the form

$$F(\lambda, u) = 0 \quad \text{for } \lambda \in \mathbb{R} \text{ and } u \in X.$$  \hspace{1cm} (20)

One of the first questions to be answered is whether or not the nonlinear equation $F(\lambda, u) = 0$ has any solution $u$ for a given value of $\lambda$. If it does, the question of how many solutions it has arises, and then how this number varies with $\lambda$. Of particular interest is the process of bifurcation whereby a given solution of $F(\lambda, u) = 0$ splits into two or more solutions as $\lambda$ passes through some critical value $\lambda^*$.

First, the points $\{(\lambda, 0)\}$ are considered as trivial solutions of the nonlinear equation $F(\lambda, u) = 0$. Thus we are concerned about solutions $\{(\lambda, u)\}$ of the nonlinear equation for which $u \neq 0$. More precisely, we define $\mathcal{S}$ to be the set of non-trivial solutions

$$\mathcal{S} := \{(\lambda, u) \in \mathbb{R} \times X : F(\lambda, u) = 0, \ u \neq 0\},$$

and let

$$\overline{\mathcal{S}} := \text{the closure of } \mathcal{S} \text{ in the Banach space } \mathbb{R} \times X.$$ 

Then a point $(\lambda^*, 0) \in \overline{\mathcal{S}}$ is called a bifurcation point for the nonlinear equation $F(\lambda, u) = 0$ with respect to the trivial solution curve $\{(\lambda, 0)\}$.

The next theorem, due to Crandall–Rabinowitz [16, Theorem 1.7], gives sufficient conditions in order that a point $(\lambda^*, 0)$ is a bifurcation point for the nonlinear equation (20) (see [6, p. 93, Theorem 4.1], [12, Theorem 1.3.3], [15, Chapter 6, Theorem 6.1], [21, Theorem 4.3.22], [40, Theorem 3.2.2]):

Theorem 3.3 (Crandall–Rabinowitz). Let $F(\lambda, u)$ be a $C^2$ map of a neighborhood of $(\lambda^*, 0)$ in a Banach space $\mathbb{R} \times X$ into a Banach space $Y$. Assume that the following four conditions are satisfied (see Figure 8):

(a) $F(\lambda, 0) = 0$ near $\lambda = \lambda^*$.

(b) The null space $V := N(F_u(\lambda^*, 0))$ of the partial Fréchet derivative $F_u(\lambda^*, 0)$ is one dimensional, spanned by a non-zero vector $u^* \in X$:

$$V = N(F_u(\lambda^*, 0)) = \text{span}[u^*].$$

(c) The range $R(F_u(\lambda^*, 0))$ of $F_u(\lambda^*, 0)$ has codimension one in the space $Y$. More precisely, there is a functional $\psi \in Y^*$ such that

$$R(F_u(\lambda^*, 0)) = \{y \in Y : \langle \psi, y \rangle = 0\}.$$

(d) $F_{u,\lambda}(\lambda^*, 0)u^* \notin R(F_u(\lambda^*, 0)).$
Then the point \((\lambda^*, 0)\) is a bifurcation point for the nonlinear equation \(F(\lambda, u) = 0\). In fact, the set of solutions of \(F(\lambda, u) = 0\) near \((\lambda^*, 0)\) consists of two continuous curves \(\Gamma_1\) and \(\Gamma_2\) intersecting only at the point \((\lambda^*, 0)\) (see Figure 9). Furthermore, the trivial solution curve \(\Gamma_1\) may be parametrized by \(\lambda\) as

\[
\Gamma_1 = \{(\lambda, 0) : |\lambda - \lambda^*| < \delta\},
\]

while the non-trivial solution curve \(\Gamma_2\) may be parametrized by a variable \(t\) as

\[
\Gamma_2 = \{(\lambda^* + \mu(t), tu^* + \gamma(\lambda^* + \mu(t), tu^*)) : |t| < \varepsilon, t \neq 0\}.
\]

Here the functions \(\mu(t)\) and \(\gamma(\lambda^* + \mu(t), tu^*)\) satisfy the conditions

\[
\mu(0) = 0, \quad \gamma(\lambda^*, 0) = 0.
\]

**Figure 8.** Conditions (b) and (d) in Theorem 3.3

**Figure 9.** The bifurcation curves \(\Gamma_1\) and \(\Gamma_2\) of the nonlinear equation (20) in Theorem 3.3

### 3.2. Global bifurcation from a simple eigenvalue.

In [31, Chapter 4] Krasnosel’skii gives a general sufficient condition for a point to be a bifurcation point in the framework of compact operators. It should be noticed that Rabinowitz ([44, Theorem 1.10]) proves a global version of Krasnosel’skii theorem (see [3, Theorem 18.3], [5, Chapter 4, Section 4.3], [10, Chapter 22], [12, Theorem 3.5.4], [20, Theorem 29.2], [21, Chapter 5, Section 5.8A], [40, Chapter 3, Section 3.4]).

The next theorem asserts the existence of global solution branches for positive mappings due to Dancer [18, Corollary to Theorem 2]:
Theorem 3.4 (Dancer). Let $E$ be an ordered Banach space with total positive cone $P$, that is, $P - P = E$. We consider a completely continuous map

$$A : \mathbb{R} \times E \rightarrow E.$$ 

Assume that

$$\begin{align*}
A(0, x) &= 0 \quad \text{for } x \in E, \\
A(\lambda, 0) &= 0 \quad \text{for } \lambda \in \mathbb{R},
\end{align*}$$

and further that there exist a linear operator $B : E \rightarrow E$ and a nonlinear map $f : \mathbb{R} \times E \rightarrow E$ such that

$$A(\lambda, x) = \lambda Bx + f(\lambda, x) \quad \lambda \in \mathbb{R} \text{ and } x \in E,$$

where

$$\|f(\lambda, x)\| = o(\|x\|) \quad \text{as } x \rightarrow 0 \text{ in } E, \text{ locally uniform in } \lambda \in \mathbb{R}.$$ 

We let

$$C_P(B) := \{ \lambda \in [0, \infty) : \text{there exists an } x \in P \text{ with } \|x\| = 1 \text{ such that } \lambda Bx = x \}.$$ 

Then we have the following two assertions (see Figure 10):

(i) If the spectral radius $\text{spr}(B)$ is positive, then the component of the set

$$\mathcal{D}_P(A) = \{ (\lambda, x) \in [0, \infty) \times P : x = A(\lambda, x), \ x \neq 0 \} \cup (C_P(B) \times \{0\})$$

containing the point $(1/\text{spr}(B), 0)$ is unbounded in $\mathbb{R} \times E$.

(ii) The point $(1/\text{spr}(B), 0)$ is a bifurcation point of the nonlinear equation

$$F(\lambda, x) := x - A(\lambda, x) = x - \lambda Bx - f(\lambda, x) = 0$$

(21)

to the trivial solution.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{The point $(1/\text{spr}(B), 0)$ is a bifurcation point of the nonlinear equation (21) to the trivial solution in Theorem 3.4}
\end{figure}

Some remarks are in order:
Remark 3.1. (1) If the operator $B : E \to E$ is compact, then it follows from an application of the Riesz–Schauder theory (see [77, Chapter X, Section 5, Theorem 2]) that the set $C_P(B)$ is an at most countable set with no finite limit point.

(2) If $P$ has non-empty interior $\text{Int} P$ and if $B$ is a strongly positive, then it follows from an application of Theorem 2.1 that $\text{spr}(B)$ is positive. Hence, Theorem 3.4 applies.

4. Elliptic boundary value problems and the maximum principle. This section is devoted to the study of the non-homogeneous boundary value problem

\begin{equation}
\begin{cases}
    Au := (-\Delta + c(x))u = g & \text{in } D, \\
    Bu = a(x') \frac{\partial u}{\partial n} + b(x')u = \varphi & \text{on } \partial D.
\end{cases}
\end{equation}

(2)

We prove two existence and uniqueness theorems for the mixed-type boundary value problem (2) in the framework of $L^p$ Sobolev spaces (Theorem 1.1) and in the framework of Hölder spaces (Theorem 1.2), respectively. The uniqueness result in Theorem 1.1 follows from a variant of the Bakel’man and Aleksandrov maximum principle in the framework of Sobolev spaces due to Bony [8]. Furthermore, we will prove the positivity of the resolvent $R_c$ of the homogeneous boundary value problem (27) (Lemma 4.7 and Proposition 4.1), which allows us to apply the Kreĭn–Rutman theory (Theorems 2.1 and 2.2) to the operator equation (31) in the ordered Banach space $C_e(D)$ (Theorem 4.8). Especially, we can characterize the eigenvalues and positive eigenfunctions of the resolvent $R_c$ (Theorem 4.9).

4.1. Proof of Theorem 1.1. This subsection is devoted to the proof of Theorem 1.1 under conditions (H.1), (H.2) and (3). In order to prove Theorem 1.1, it suffices to show that the mapping

$A = (A, B) : W^{2,p}(D) \to L^p(D) \oplus B^{1-1/p,p}_r(\partial D)$

is bijective. Indeed, since the inverse $A^{-1}$ is a closed operator, we find from Banach’s closed graph theorem (see [50, Theorem 3.10], [77, Chapter II, Section 6, Theorem 1]) that $A^{-1}$ is continuous.

The proof of Theorem 1.1 is divided into four steps.

Step 1. First, we consider the following linear elliptic boundary value problem

\begin{equation}
\begin{cases}
    A_0u := -\Delta u = g & \text{in } D, \\
    Bu = a(x') \frac{\partial u}{\partial n} + b(x')u = \varphi & \text{on } \partial D.
\end{cases}
\end{equation}

(22)

If we associate with the linear problem (22) a continuous linear operator

$A_0 := (A_0, B) : W^{2,p}(D) \to L^p(D) \oplus B^{1-1/p,p}_r(\partial D)$,

then we have the following theorem (see [65, Theorem 1.1]):

**Theorem 4.1.** If conditions (H.1) and (H.2) are satisfied, then the mapping $A_0$ is an algebraic and topological isomorphism for all $1 < p < \infty$.

In particular, we have, by Theorem 4.1,

$\text{ind } A_0 = \dim \mathcal{N}(A_0) - \text{codim } \mathcal{R}(A_0) = 0.$

(23)
Step 2. If $C$ is the multiplication operator by the function $c(x) \in C(\overline{D})$, then it follows from an application of the Rellich–Kondrachov theorem (see [1, Theorem 6.3, Part I], [26, Theorem 7.26]) that the mapping
\[ C : W^{2,p}(D) \rightarrow L^p(D) \]
is compact.

Therefore, we obtain from formula (23) that the mapping
\[ A = A_0 + (C, 0) : W^{2,p}(D) \rightarrow L^p(D) \oplus B^{1-1/p,p}(\partial D) \]
is a Fredholm operator with index zero:
\[ \text{ind } A = \text{ind } A_0 = 0. \tag{24} \]

Indeed, it suffices to note that the index is stable under compact perturbations (see [27, Theorem 2.6], [50, Theorem 5.10]). In particular, the Fredholm alternative holds true for the operator $A$.

Step 3. On the other hand, we will show that the uniqueness result in Theorem 1.1 follows from a variant of the Bakel’man–Aleksandrov maximum principle in the framework of Sobolev spaces due to Bony [8, Théorème 2] (see also [75, Lemmas 3.25 and 3.26 and Theorem 3.27], [68, Section 8], [69, Part IV]):

**Theorem 4.2** (the weak maximum principle). Assume that condition (3) is satisfied. If a function $v \in W^{2,p}(D)$, $N < p < \infty$, satisfies the condition
\[ Av(x) \leq 0 \quad \text{in } D, \]
then we have the inequality
\[ \max_D v \leq \max_{\partial D} v^+, \]
where
\[ v^+(x) = \max\{v(x), 0\}. \]

**Theorem 4.3** (the Hopf boundary point lemma). Assume that condition (3) is satisfied and that a function $v \in W^{2,p}(D)$, $N < p < \infty$, satisfies the condition
\[ Av(x) \leq 0 \quad \text{in } D. \]
If $v(x)$ attains a strict local non-negative maximum at a point $x'_0$ of $\partial D$, then we have the inequality
\[ \frac{\partial v}{\partial n}(x'_0) > 0. \]

**Theorem 4.4** (the strong maximum principle). Assume that condition (3) is satisfied and that a function $v \in W^{2,p}(D)$, $N < p < \infty$, satisfies the condition
\[ Av(x) \leq 0 \quad \text{in } D. \]
If $v(x)$ attains a non-negative maximum at a point $x_0$ of $D$, then it is a constant.

Here we recall that, for $N < p < \infty$,
\[ W^{2,p}(D) \subset C^1(\overline{D}). \]

By applying the maximum principle, we can obtain a uniqueness theorem for the mixed-type boundary value problem (2) in the framework of Sobolev spaces of $L^p$ style:
Theorem 4.5. Assume that conditions (H.1), (H.2) and (3) are satisfied. If a function \( u \in W^{2,p}(D), \ N < p < \infty \), satisfies the conditions

\[
\begin{align*}
    Au &= 0 \quad \text{in } D, \\
    Bu &= 0 \quad \text{on } \partial D,
\end{align*}
\]

then it follows that \( u(x) \equiv 0 \) in \( D \).

Proof. The proof is based on a reduction to absurdity. Assume, to the contrary, that \( u(x) \not\equiv 0 \) in \( D \). Without loss of generality, we may assume that there exists a point \( x_0 \in \overline{D} \) such that

\[ u(x_0) = \max_{x \in \overline{D}} u(x) > 0. \]

(a) If \( x_0 \in D \), then it follows from an application of Theorem 4.4 with \( v := -u \) that

\[ u(x) \equiv u(x_0) > 0 \quad \text{for } x \in D. \]

Hence we have, for any point \( x' \in \partial D \),

\[ 0 = Bu(x') = a(x') \frac{\partial u}{\partial n}(x') + b(x')u(x') = b(x')u(x_0), \]

and so

\[ u(x_0) = 0, \]

since \( b(x') \not\equiv 0 \) on \( \partial D \). This is a contradiction.

(b) If \( x_0 \in \partial D \), then we may assume that \( u(x) \) attains a strict positive maximum at a point \( x_0 \), that is,

\[
\begin{align*}
    u(x_0) &= \max_{x \in \partial D} u(x) > 0, \\
    u(x) &< u(x_0) \quad \text{for } x \in D.
\end{align*}
\]

Thus it follows from an application of Theorem 4.3 with \( v := u \) that

\[ \frac{\partial u}{\partial n}(x_0) > 0. \]

However, we have, by condition (H.2),

\[ 0 = Bu(x_0) = a(x_0) \frac{\partial u}{\partial n}(x_0) + b(x_0)u(x_0) > 0. \]

This is also a contradiction.

The proof of Theorem 4.5 is complete. \( \square \)

Step 4. Theorem 4.5 asserts that the mapping

\[ A : W^{2,p}(D) \longrightarrow L^p(D) \oplus B_1^{1-1/p,p}(\partial D) \]

is injective for \( N < p < \infty \). Hence, we find from formula (24) that \( A \) is also surjective for \( N < p < \infty \).

Summing up, we have proved that the mapping

\[ A = (A,B) : W^{2,p}(D) \longrightarrow L^p(D) \oplus B_1^{1-1/p,p}(\partial D) \]

is an algebraic and topological isomorphism for \( N < p < \infty \).

The proof of Theorem 1.1 is complete. \( \square \)
4.2. **Proof of Theorem 1.2.** In this subsection we prove Theorem 1.2 under conditions (H.1), (H.2), (3) and (4), which will play an important role in the proof of Theorem 1.5 in Sections 7 and 8.

In order to prove Theorem 1.2, it suffices to show that the mapping

\[ A = (A, B) : C^{2+\theta}(\overline{D}) \rightarrow C^\theta(\overline{D}) \oplus C^1_{\lambda}(\partial D) \]

is bijective. Indeed, since the inverse \( A^{-1} \) is a closed operator, we find from Banach’s closed graph theorem (see [50, Theorem 3.10], [77, Chapter II, Section 6, Theorem 1]) that \( A^{-1} \) is continuous.

The proof is divided into three steps.

**Step 1.** If we associate with the linear problem (22) a continuous linear operator

\[ A_0 = (-\Delta, B) : C^{2+\theta}(\overline{D}) \rightarrow C^\theta(\overline{D}) \oplus C^1_{\lambda}(\partial D), \]

then we have the following theorem (see [65, Theorem 1.1]):

**Theorem 4.6.** If conditions (H.1) and (H.2) are satisfied, then the mapping \( A_0 \) is an algebraic and topological isomorphism for \( 0 < \theta < 1 \).

In particular, we have, by Theorem 4.6,

\[ \text{ind} A_0 := \dim \mathcal{N}(A_0) - \text{codim} \mathcal{R}(A_0) = 0. \quad (25) \]

**Step 2.** If \( C \) is the multiplication operator by the function \( c(x) \in C^\theta(\overline{D}) \), then it follows from an application of the Ascoli–Arzelà theorem (see [26, Lemma 6.36]) that the mapping

\[ C : C^{2+\theta}(\overline{D}) \rightarrow C^\theta(\overline{D}) \]

is compact.

Therefore, we obtain from formula (25) that the mapping

\[ A = A_0 + (C, 0) : C^{2+\theta}(\overline{D}) \rightarrow C^\theta(\overline{D}) \oplus C^1_{\lambda}(\partial D) \]

is a Fredholm operator with index zero:

\[ \text{ind} A = \text{ind} A_0 = 0. \quad (26) \]

Indeed, it suffices to note that the index is stable under compact perturbations (see [27, Theorem 2.6], [50, Theorem 5.10]).

**Step 3.** By applying Theorem 4.5 to our situation, we find that the mapping

\[ A = (A, B) : C^{2+\theta}(\overline{D}) \rightarrow C^\theta(\overline{D}) \oplus C^1_{\lambda}(\partial D) \]

is injective for \( 0 < \theta < 1 \). Hence, we find from formula (26) that \( A \) is also surjective for \( 0 < \theta < 1 \).

Summing up, we have proved that the mapping \( A \) is an algebraic and topological isomorphism for \( 0 < \theta < 1 \).

The proof of Theorem 1.2 is complete.

4.3. **Positivity of the resolvent.** In order to apply the Kreîn–Rutman theorem (Theorem 2.1) in Section 5, we study the following *homogeneous* linear elliptic boundary value problem:

\[
\begin{aligned}
Au := (-\Delta + c(x)) u &= g & \text{in } D, \\
Bu := a(x') \frac{\partial u}{\partial n} + b(x') u &= 0 & \text{on } \partial D.
\end{aligned}
\] (27)
By using Theorem 1.1, we find that the linear problem (27) has a unique solution \( u \in W^{2,p}(D) \) for any \( g \in L^p(D) \). Therefore, we can introduce a continuous linear operator (resolvent)

\[
R_c := (-\Delta + c(x))^{-1} : L^p(D) \rightarrow W^{2,p}(D)
\]

by the formula \( u = R_c g \). Moreover, by the Ascoli–Arzelà theorem (see [26, Lemma 6.36]) it follows that the resolvent \( R_c \), considered as an operator

\[
R_c = (-\Delta + c(x))^{-1} : C(\overline{D}) \rightarrow C^1(\overline{D}),
\]

is compact if \( N < p < \infty \). Indeed, by Sobolev’s imbedding theorem (see [1, Theorem 4.12, Part II]) it suffices to note that the space \( W^{2,p}(D) \) is continuously imbedded into \( C^{2-N/p}(\overline{D}) \) with \( 2 - N/p > 1 \), for all \( N < p < \infty \). The situation can be visualized as in Figure 11.

**Figure 11.** The mapping properties of the resolvent \( R_c \) in the spaces \( C(\overline{D}), W^{2,p}(D) \) and \( C^1(\overline{D}) \).

Now we introduce an ordered Banach subspace of \( C(\overline{D}) \) that combines the good properties of the resolvent \( R_c \). To do this, we need the following lemma (see [72, Lemma 2.1], [62, Lemma 3.7]):

**Lemma 4.7.** Assume that conditions (H.1), (H.2) and (3) are satisfied. If \( v \in C(\overline{D}) \) and if \( v(x) \geq 0 \) but \( v(x) \not\equiv 0 \) on \( \partial D \), then the function

\[
u = R_c v = (-\Delta + c(x))^{-1} v
\]

satisfies the following three conditions:

(a) \( u(x') = 0 \) on \( M = \{x' \in \partial D : a(x') = 0\} \).

(b) \( u(x) > 0 \) on \( \overline{D} \setminus M \).

(c) For the outward normal derivative \( \partial u/\partial n \) of \( u \), we have the inequality

\[
\frac{\partial u}{\partial n}(x') < 0 \quad \text{on } M.
\]

In particular, the resolvent \( R_c : C(\overline{D}) \rightarrow C(\overline{D}) \) is positive.

4.4. **The ordered Banach space** \( C_e(\overline{D}) \). Now we introduce an ordered Banach subspace \( C_e(\overline{D}) \) of \( C(\overline{D}) \) which combines the good properties of the homogeneous boundary value problem (27) with the good properties of the natural ordering of \( C(\overline{D}) \).

If we let

\[
e(x) := (R_c 1)(x) = (-\Delta + c(x))^{-1} 1(x) \quad \text{for } x \in \overline{D}, \tag{28}
\]

then it follows from an application of Theorem 1.1 that the function \( e(x) \in W^{2,p}(D) \), \( N < p < \infty \), is the unique solution of the linear elliptic boundary value problem

\[
\begin{cases}
(-\Delta + c(x)) e = 1 & \text{in } D, \\
Be = 0 & \text{on } \partial D.
\end{cases}
\]
Moreover, it follows from an application of Lemma 4.7 with \( v := 1 \) that the function 
\[
e(x) = (R_c \cdot 1)(x)
\]

satisfies the conditions
\[
\begin{cases}
  e(x) > 0 & \text{on } \overline{D} \setminus M, \\
  e(x') = 0 & \text{on } M, \\
  \frac{\partial e}{\partial n}(x') < 0 & \text{on } M,
\end{cases}
\]

where 
\[
M = \{ x' \in \partial D : a(x') = 0 \}.
\]

By using the function \( e(x) \) defined by formula (28), we introduce a subspace 
\[ C_c(\overline{D}) \]
by the formula
\[
C_c(\overline{D}) := \{ u \in C(\overline{D}) : \text{there is a constant } \alpha > 0 \text{ such that } -\alpha e(x) \leq u(x) \leq \alpha e(x) \text{ in } D \}
\]
with the norm
\[
\|u\|_c = \inf \{ \alpha > 0 : -\alpha e(x) \leq u(x) \leq \alpha e(x) \text{ in } D \}.
\]

By rescaling, we may assume that
\[
\|e\|_{C(\overline{D})} = \max_{x \in \overline{D}} e(x) = 1.
\]

Then we have the inequality
\[
\|v\|_{C(\overline{D})} \leq \|v\|_{C_c(\overline{D})} \quad \text{for all } v \in C_c(\overline{D}).
\]

This implies that the injection
\[
C_c(\overline{D}) \hookrightarrow C(\overline{D})
\]
is continuous.

If we let
\[
P_c := \{ u \in C_c(\overline{D}) : u \geq 0 \text{ on } \overline{D} \},
\]
then it is easy to verify that the space \( C_c(\overline{D}) \) is an ordered Banach space having the positive cone \( P_c \) with non-empty interior \( \text{Int}(P_c) \). Indeed, every \( C^1 \) function \( u(x) \in C_c(\overline{D}) \) which satisfies the three conditions
\[
\begin{cases}
  u(x) > 0 & \text{on } \overline{D} \setminus M, \\
  u(x') = 0 & \text{on } M, \\
  \frac{\partial u}{\partial n}(x') < 0 & \text{on } M
\end{cases}
\]
belongs to the interior \( \text{Int}(P_c) \).

This setting has the advantages that it takes into consideration in an optimal way the \textit{a priori} information given by the maximum principle and that it is amenable to the methods of abstract functional analysis (see [3], [28]). In fact, we have the following proposition (see [72, Proposition 2.2], [62, Proposition 3.8]):

**Proposition 4.1.** If conditions (H.1), (H.2) and (3) are satisfied, then the resolvent
\[
R_c = (-\Delta + c(x))^{-1}
\]
maps \( C(\overline{D}) \) compactly into \( C_c(\overline{D}) \). Moreover, the resolvent \( R_c \), considered as an operator \( R_c : C_c(\overline{D}) \rightarrow C_c(\overline{D}) \), is strongly positive, that is, \( R_c v \in \text{Int}(P_c) \) for all \( v \in P_c \setminus \{0\} \).
Remark 4.1. More precisely, if we let
\[ \mathcal{C}_B^1(\mathcal{D}) = \{ u \in \mathcal{C}^1(\mathcal{D}) : Bu = 0 \text{ on } \partial \mathcal{D} \}, \]
then it follows from Figure 11 that the resolvent \( R_c \) maps \( \mathcal{C}(\mathcal{D}) \) compactly into \( \mathcal{C}_B^1(\mathcal{D}) \), and further that the inclusion mapping
\[ \iota : \mathcal{C}_B^1(\mathcal{D}) \rightarrow \mathcal{C}(\mathcal{D}) \]
is continuous. Hence we have the following assertion:
\[ R_c = (-\Delta + c(x))^{-1} : \mathcal{C}(\mathcal{D}) \rightarrow \mathcal{C}_B^1(\mathcal{D}) \quad \text{compactly} \quad \mathcal{C}_B^1(\mathcal{D}) \rightarrow \mathcal{C}(\mathcal{D}). \] (30)

Now we consider the following homogeneous boundary value problem (analogous to the operator equation (19)): For a given function \( h \in \mathcal{P}_e \), find a function \( u(x) \) such that
\[ \begin{align*}
\{ (-\Delta + c(x) - \lambda) u &= h \quad \text{in } \mathcal{D}, \\
Bu &= 0 \quad \text{on } \partial \mathcal{D} \}
\end{align*} \] (31)
\[ \iff \frac{1}{\lambda} u - R_c u = h \quad \text{in } \mathcal{C}(\mathcal{D}). \]

Then, by combining Theorems 2.1 and 2.2 with
\[ E := \mathcal{C}(\mathcal{D}), \quad P := \mathcal{P}_e, \]
\[ K := R_c = (-\Delta + c(x))^{-1}, \]
\[ \lambda := \frac{1}{\lambda} \]
and Proposition 4.1 we obtain the main result of this subsection (see [28, Theorem 16.6], [63, Theorem 2.2]):

**Theorem 4.8.** Assume that conditions (H.1), (H.2) and (3) are satisfied. Then the principal eigenvalue \( r(R_c) := \lim_{n \to \infty} \sqrt[n]{\| R_c^n \|} \) of \( R_c = (-\Delta + c(x))^{-1} \) is positive, and we have the following three assertions:
(i) If \( 0 < \lambda < 1/r(R_c) \), then the boundary value problem (31) has a unique positive solution \( u \) and \( u \in \text{Int}(P_e) \).
(ii) If \( \lambda > 1/r(R_c) \), then the boundary value problem (31) has no positive solution.
(iii) If \( \lambda = 1/r(R_c) \), then the boundary value problem (31) has no solution.

### 4.5. Spectral analysis of the resolvent via the Kreĭn–Rutman theorem.
Now we consider the resolvent \( R_c \) as an operator in the ordered Banach space \( \mathcal{C}_e(\mathcal{D}) \), and prove important results concerning its eigenfunctions and corresponding eigenvalues (Theorem 4.9).

First, Proposition 4.1 tells us that the resolvent
\[ R_c = (-\Delta + c(x))^{-1} : \mathcal{C}_e(\mathcal{D}) \rightarrow \mathcal{C}_e(\mathcal{D}) \]
is strongly positive and compact under conditions (H.1), (H.2) and (3) (see Figure 12).

Moreover, we find that all the eigenvalues of \( R_c \) are positive. Indeed, if \( \mu \) is a non-zero eigenvalue of \( R_c \), that is, if we have the formula
\[ R_c v = \mu v \quad \text{for some non-zero function } v \in W^{2,p}(\mathcal{D}) \text{ with } N < p < \infty, \]
then it follows that
\[
\begin{aligned}
&\left\{ \begin{array}{l}
Av = (-\Delta + c(x))v = \frac{1}{\mu} v \quad \text{in } D, \\
Bv = a(x')\frac{\partial v}{\partial n} + b(x')v = 0 \quad \text{on } \partial D.
\end{array} \right.
\end{aligned}
\]
Hence, by applying Green’s formula we obtain from conditions (H.1) and (H.2) that
\[
\frac{1}{\mu} \int_D |v(x)|^2 \, dx = (Av, v)_{L^2(D)} = - \int_D \Delta v(x) \cdot \overline{v(x)} \, dx + \int_D c(x)|v(x)|^2 \, dx
\]
\[
= \int_D |\nabla v|^2 \, dx + \int_D c(x)|v(x)|^2 \, dx - \int_{\partial D} \frac{\partial v}{\partial n} \cdot \overline{v} \, d\sigma
\]
\[
\geq \int_D |\nabla v|^2 \, dx + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} |v|^2 \, d\sigma
\]
\[
\geq 0.
\]
This implies that
\[
\mu > 0.
\]
Therefore, we have proved that \( R_c \) has a countable number of positive eigenvalues, \( \mu_j \), which may accumulate only at 0 (see [77, Chapter X, Section 5, Theorem 2]). Hence they may be arranged in a decreasing sequence
\[
\mu_1 \geq \mu_2 \geq \ldots \geq \mu_j \geq \ldots \rightarrow 0,
\]
where each eigenvalue is repeated according to its multiplicity.

The next theorem, a sharper version of the Krein–Rutman theorem (Theorem 2.1), characterizes the eigenvalues and positive eigenfunctions of the resolvent \( R_c \) (cf. [31]):

**Theorem 4.9.** If conditions (H.1), (H.2) and (3) are satisfied, then the resolvent \( R_c = (-\Delta + c(x))^{-1} \), considered as an operator
\[
R_c: C_c(D) \rightarrow C_c(D),
\]
has the following two spectral properties:

(i) The largest eigenvalue \( \mu_1 \) of \( R_c \) is algebraically simple and has a positive eigenfunction \( \psi_1(x) \):
\[
\left\{ \begin{array}{l}
R_c \psi_1 = \mu_1 \psi_1, \\
\psi_1 \in \text{Int} (P).
\end{array} \right.
\]

(ii) No other eigenvalues, \( \mu_j, j \geq 2 \), have positive eigenfunctions.
5. **Proof of Theorem 1.3.** This section is devoted to the proof of Theorem 1.3 under conditions (M.1), (H.1) and (H.2), which is inspired by Brown and Lin [9, Theorem 3.5].

5.1. **Eigenvalue problems with indefinite weight function.** This subsection is devoted to the study of the mixed-type problem (5) with an indefinite weight function $m(x) \in C(D)$:

$$
\begin{align*}
  -\Delta \varphi &= \lambda m(x) \varphi \quad \text{in } D, \\
  B\varphi &= 0 \quad \text{on } \partial D.
\end{align*}
$$

First, we introduce a densely defined, selfadjoint operator $\mathfrak{A}$ from the Hilbert space $L^2(D)$ into itself as follows.

(a) The domain of definition $D(\mathfrak{A})$ is the space

$$
D(\mathfrak{A}) = \left\{ v \in H^2(D) = W^{2,2}(D) : Bv := a(x') \frac{\partial v}{\partial n} + b(x')v = 0 \text{ on } \partial D \right\}.
$$

(b) $\mathfrak{A}v = -\Delta v$ for every $v \in D(\mathfrak{A})$.

Then the next theorem is a special case of Theorem 4.9 with $c(x) \equiv 0$ and assertion (32) (cf. [57, Theorem 0]):

**Theorem 5.1.** Assume that conditions (H.1) and (H.2) is satisfied. Then we have the following three spectral properties of the selfadjoint operator $\mathfrak{A}$ in the Hilbert space $L^2(D)$:

(i) The spectrum of $\mathfrak{A}$ contains only the discrete eigenvalues

$$
0 < \gamma_1 < \gamma_2 \leq \ldots,
$$

and their corresponding eigenfunctions $\varphi_j \in C^\infty(D)$ such that

$$
\begin{align*}
  -\Delta \varphi_j &= \gamma_j \varphi_j \quad \text{in } D, \\
  B\varphi_j &= 0 \quad \text{on } \partial D.
\end{align*}
$$

(ii) The first eigenvalue $\gamma_1$ is algebraically simple and its corresponding eigenfunction $\varphi_1$ may be chosen to be positive everywhere in $D$:

$$
\begin{align*}
  -\Delta \varphi_1 &= \gamma_1 \varphi_1 \quad \text{in } D, \\
  \varphi_1 > 0 \quad &\text{in } D, \\
  B\varphi_1 &= 0 \quad \text{on } \partial D.
\end{align*}
$$

(iii) Moreover, no other eigenvalues, $\gamma_j, j \geq 2$, have positive eigenfunctions.

If we introduce a linear operator

$$
T(\lambda) := \mathfrak{A} - \lambda m(x) I \quad \text{for } \lambda \geq 0,
$$

then it follows that $T(\lambda)$ is selfadjoint in $L^2(D)$ and further that the eigenvalues and eigenfunctions of $T(\lambda)$ correspond to those of the linear eigenvalue problem

$$
\begin{align*}
  (-\Delta - \lambda m(x)) v &= \mu(\lambda) v \quad \text{in } D, \\
  Bv &= 0 \quad \text{on } \partial D.
\end{align*}
$$

Furthermore, by adapting the proof of Theorem 4.9 to our situation (see Remark 5.1 below) we obtain the following generalization of Theorem 5.1 for $\lambda \geq 0$:

**Theorem 5.2.** Assume that conditions (M.1), (H.1) and (H.2) are satisfied. If $\lambda \geq 0$, then we have the following three spectral properties of $T(\lambda)$:
(i) The spectrum of $T(\lambda)$ contains only the discrete eigenvalues
\[ \mu_1(\lambda) < \mu_2(\lambda) \leq \ldots, \]
and their corresponding eigenfunctions $\phi_j \in W^{2,p}(D)$ for $N < p < \infty$ such that
\[
\begin{cases}
(\Delta - \lambda m(x)) \phi_j = \mu_j(\lambda) \phi_j & \text{in } D, \\
B\phi_j = 0 & \text{on } \partial D.
\end{cases}
\]

(ii) The first eigenvalue $\mu_1(\lambda)$ of the operator $T(\lambda)$ is algebraically simple and its corresponding eigenfunction $\phi_1 \in W^{2,p}(D)$ for $N < p < \infty$ may be chosen to be positive in $D$:
\[
\begin{cases}
(\Delta - \lambda m(x)) \phi_1 = \mu_1(\lambda) \phi_1 & \text{in } D, \\
\phi_1 > 0 & \text{in } D, \\
B\phi_1 = 0 & \text{on } \partial D.
\end{cases}
\]

(iii) Moreover, no other eigenvalues, $\mu_j(\lambda)$, $j \geq 2$, have positive eigenfunctions.

Proof. First, by rescaling we may assume that $|m(x)| < 1$ on $\overline{D}$. Then it is easy to see that the eigenvalue problem (35) is equivalent to the eigenvalue problem
\[
\begin{cases}
(\Delta + \lambda (1 - m(x))) v = (\mu(\lambda) + \lambda) v & \text{in } D, \\
Bv = 0 & \text{on } \partial D,
\end{cases}
\]
where
\[ \lambda (1 - m(x)) \geq 0 \quad \text{on } \overline{D}. \]

By applying Theorem 4.9 with
\[ c(x) := \lambda (1 - m(x)), \quad \mu := \mu(\lambda) + \lambda, \]
we can obtain the following two assertions:

(A) The spectrum of the eigenvalue problem (35') contains only the discrete eigenvalues
\[ 0 < \gamma_1(\lambda) < \gamma_2(\lambda) \leq \ldots \quad \text{for } \lambda \geq 0, \]
and their corresponding eigenfunctions $\phi_j \in W^{2,p}(D)$ for $N < p < \infty$ such that
\[
\begin{cases}
(\Delta + \lambda (1 - m(x))) \phi_j = \gamma_j(\lambda) \phi_j & \text{in } D, \\
B\phi_j = 0 & \text{on } \partial D.
\end{cases}
\]

(B) The first eigenvalue $\gamma_1(\lambda)$ is algebraically simple and its corresponding eigenfunction $\phi_1(x)$ may be chosen to be positive in $D$:
\[
\begin{cases}
(\Delta + \lambda (1 - m(x))) \phi_1 = \gamma_1(\lambda) \phi_1 & \text{in } D, \\
\phi_1 > 0 & \text{in } D, \\
B\phi_1 = 0 & \text{on } \partial D.
\end{cases}
\]

Therefore, the desired formulas (36) and (37) follow from two assertions (A) and (B), respectively, by taking
\[ \mu_j(\lambda) := \gamma_j(\lambda) - \lambda, \quad j = 1, 2, \ldots. \]
Indeed, we have, by formula (38),
\[
\begin{cases}
(-\Delta + \lambda (1 - m(x))) v = \gamma_j(\lambda) v & \text{in } D, \\
Bv = 0 & \text{on } \partial D
\end{cases}
\] 
\[
\iff \begin{cases}
(-\Delta + \lambda (1 - m(x))) v = (\mu_j(\lambda) + \lambda) v & \text{in } D, \\
Bv = 0 & \text{on } \partial D
\end{cases}
\]

The proof of Theorem 5.2 is complete.

Some remarks are in order:

**Remark 5.1.**

1. By formula (38) and assertion (A), it follows that 
   \( \mu_j(0) = \gamma_j(0) = \gamma_j > 0 \) for all \( j \geq 1 \).

2. It should be emphasized that the first eigenvalue \( \mu_1(\lambda) = \gamma_1(\lambda) - \lambda \) is non-negative for \( \lambda \geq \lambda_1(m) \) (see Figure 13).

5.2. **End of proof of Theorem 1.3.** The proof of Theorem 1.3 is divided into six steps.

**Step I.** If \( \lambda \geq 0 \), we let (see the definition (34))

\[
Q_\lambda(v) = (T(\lambda)v, v)_{L^2(D)} = (Av, v)_{L^2(D)} - \lambda \int_D m(x)|v|^2 \, dx
\]

\[
= - \int_D \Delta v \cdot \nabla v \, dx - \lambda \int_D m(x)|v|^2 \, dx \quad \text{for } v \in D(T(\lambda)) = D(\mathcal{A}).
\]

Then the next lemma characterizes the range of possible eigenvalues corresponding to non-negative eigenfunctions:

**Lemma 5.3.** If there exists a non-negative eigenfunction \( \psi(x) \) corresponding to the mixed-type problem (5), then we have, for all \( v \in D(\mathcal{A}) \),

\[
Q_\lambda(v) \geq 0.
\]  

**Proof.** The proof of lemma 5.3 is divided into two steps.

(1) First, we prove the following claim for the eigenvalue problem (36):

**Claim 5.1.** The eigenfunctions \( \phi_j(x) \) corresponding to the eigenvalues \( \mu_j(\lambda) \) for \( j \geq 2 \) are orthogonal to the eigenfunction \( \phi_1(x) \) corresponding to the eigenvalue \( \mu_1(\lambda) \):

\[
\int_D \phi_1(x) \cdot \overline{\phi_j(x)} \, dx = 0 \quad \text{for all } j \geq 2.
\]

**Proof.** By formula (36) and Green's formula, it follows that

\[
(\mu_1(\lambda) - \mu_j(\lambda)) \int_D \phi_1 \cdot \overline{\phi_j} \, dx = \int_D T(\lambda) \phi_1 \cdot \overline{\phi_j} \, dx - \int_D \phi_1 \cdot \overline{T(\lambda)\phi_j} \, dx
\]

\[
= \int_D (-\Delta - \lambda m(x)) \phi_1 \cdot \overline{\phi_j} \, dx - \int_D \phi_1 \cdot (-\Delta - \lambda m(x)) \overline{\phi_j} \, dx
\]

\[
= - \int_D \Delta \phi_1 \cdot \overline{\phi_j} \, dx + \int_D \phi_1 \cdot \overline{\Delta \phi_j} \, dx
\]

\[
= - \int_{\partial D} \frac{\partial \phi_1}{\partial n} \cdot \overline{\phi_j} \, d\sigma + \int_{\partial D} \phi_1 \cdot \frac{\partial \phi_j}{\partial n} \, d\sigma,
\]
where $d\sigma$ is the surface element of $\partial D$. However, the eigenfunctions $\phi_1(x)$ and $\phi_j(x)$ satisfy the boundary conditions
\[
\left(\frac{\partial \phi_1}{\partial n}, \frac{\partial \phi_j}{\partial n}\right) = (0, 0) \quad \text{on } \partial D.
\]
Thus it follows that
\[
\left|\frac{\partial \phi_1}{\partial n} \phi_1 - \frac{\partial \phi_j}{\partial n} \phi_j\right| = 0 \quad \text{on } \partial D,
\]
since $(a(x'), b(x')) \neq (0, 0)$ on $\partial D$.

Therefore, we obtain from formula (41) that
\[
(\mu_1(\lambda) - \mu_j(\lambda)) \int_D \phi_1 \cdot \phi_j \, dx = -\int_{\partial D} \frac{\partial \phi_1}{\partial n} \cdot \phi_j \, d\sigma + \int_{\partial D} \phi_1 \cdot \frac{\partial \phi_j}{\partial n} \, d\sigma = 0 \quad \text{for all } j \geq 2.
\]
This proves the desired orthogonal condition (40), since $\mu_1(\lambda) - \mu_j(\lambda) < 0$ for all $j \geq 2$.

The proof of Claim 5.1 is complete.

(2) If $\psi(x)$ is a non-negative eigenfunction corresponding to the mixed-type problem (5), that is, if the function $\psi(x)$ satisfies the conditions
\[
\begin{cases}
(-\Delta - \lambda m(x)) \psi = 0 \cdot \psi & \text{in } D, \\
\psi \geq 0 & \text{in } D, \\
B\psi = 0 & \text{on } \partial D,
\end{cases}
\]
then it follows that $\psi(x)$ is a positive eigenfunction corresponding to the eigenvalue $\mu_1(\lambda) = 0$ of the eigenvalue problem (35). However, we have, by the eigenvalue problem (37),
\[
\int_D \psi(x) \cdot \phi_1(x) \, dx > 0.
\]
In view of Claim 5.1, we find from assertions (42) that
\[
\mu_1(\lambda) = 0.
\]
On the other hand, by the spectrum theorem it follows that
\[
(T(\lambda)v, v)_{L^2(D)} \geq \mu_1(\lambda) (v, v)_{L^2(D)} \quad \text{for all } v \in D(T(\lambda)).
\]
Summing up, we obtain from assertions (43) and (44) that
\[
Q_\lambda(v) = (\mathcal{A}v, v)_{L^2(D)} - \lambda \int_D m(x)|v|^2 \, dx = (T(\lambda)v, v)_{L^2(D)} \geq 0 \quad \text{for all } v \in D(\mathcal{A}).
\]
This proves the desired assertion (39).

The proof of Lemma 5.3 is complete.

**Step II.** Now we consider the Rayleigh quotient
\[
K(v) = \frac{(\mathcal{A}v, v)_{L^2(D)}}{\int_D m(x)|v|^2 \, dx},
\]
and let
\[
\lambda_1(m) := \inf \left\{ K(v) : v \in D(\mathcal{A}), \int_D m(x)|v|^2 \, dx > 0 \right\}.
\]
Since we have, by condition (H.2),

\[ a(x') = 0 \implies v(x') = 0, \]

it follows from an application of Green’s formula that

\[
(\mathfrak{A}v, v)_{L^2(D)} = -\int_D \Delta v \cdot \bar{v} \, dx = \int_D |\nabla v|^2 \, dx - \int_{\partial D} \frac{\partial v}{\partial n} \cdot \bar{v} \, d\sigma
\]

\[
= \int_D |\nabla v|^2 \, dx + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} \cdot |v|^2 \, d\sigma
\]

\[ \geq 0, \]

so that, by the definition (45),

\[ \lambda_1(m) \geq 0. \]

More precisely, we have the following lemma:

**Lemma 5.4.** The quantity \( \lambda_1(m) \) can be estimated as follows:

\[
\lambda_1(m) \geq \frac{\gamma_1}{\|m^+\|_{L^\infty(D)}}. \tag{46}
\]

Here \( \gamma_1 > 0 \) is the first eigenvalue of the operator \( \mathfrak{A} \) (see Theorem 5.1), and

\[ m^+(x) = \max\{m(x), 0\} \quad \text{for} \quad x \in \partial D. \]

**Proof.** By the spectrum theorem, it follows that

\[
(\mathfrak{A}v, v)_{L^2(D)} \geq \gamma_1(v, v)_{L^2(D)} \quad \text{for all} \quad v \in D(\mathfrak{A}). \tag{47}
\]

If \( v \in D(\mathfrak{A}) \) satisfies the condition

\[
\int_D m(x)|v(x)|^2 \, dx > 0,
\]

then we have the inequality

\[
\int_D m(x)|v(x)|^2 \, dx \leq \int_D m^+(x)|v(x)|^2 \, dx \leq \|m^+\|_{L^\infty(D)} \int_D |v(x)|^2 \, dx
\]

\[ = \|m^+\|_{L^\infty(D)} (v, v)_{L^2(D)}. \]

Therefore, we find from inequality (47) that

\[
K(v) = \frac{(\mathfrak{A}v, v)_{L^2(D)}}{\int_D m(x)|v|^2 \, dx} \geq \frac{\gamma_1(v, v)_{L^2(D)}}{\int_D m(x)|v|^2 \, dx} \geq \frac{\gamma_1}{\|m^+\|_{L^\infty(D)}} \quad \text{for all} \quad v \in D(\mathfrak{A}).
\]

By definition (45), this proves the desired estimate (46).

The proof of Lemma 5.4 is complete. \( \square \)

**Step III.** We begin by considering the case where \( \lambda > \lambda_1(m) \):

**Lemma 5.5.** If \( \lambda > \lambda_1(m) \), then \( \lambda \) is not an eigenvalue of the mixed-type problem (5) possessing a non-negative eigenfunction.

**Proof.** If \( \lambda > \lambda_1(m) \), we can find a function \( v \in D(\mathfrak{A}) \) such that

\[
\int_D m(x)|v|^2 \, dx > 0,
\]

\[
(\mathfrak{A}v, v)_{L^2(D)} < \lambda \int_D m(x)|v|^2 \, dx.
\]
Hence we have the inequality
\[ Q_\lambda(v) = (\mathfrak{A}v, v)_{L^2(D)} - \lambda \int_D m(x) |v|^2 \, dx < 0. \]

Therefore, it follows from an application of Lemma 5.3 that the mixed-type problem (5) does not admit a non-negative eigenfunction for \( \lambda > \lambda_1(m) \).

The proof of Lemma 5.5 is complete.

**Step IV.** Next we consider the case where \( 0 < \lambda < \lambda_1(m) \):

**Lemma 5.6.** If \( 0 < \lambda < \lambda_1(m) \), then we have, for all \( v \in D(\mathfrak{A}) \),
\[ Q_\lambda(v) \geq \gamma_1 \left( 1 - \frac{\lambda}{\lambda_1(m)} \right) (v, v)_{L^2(D)}. \tag{48} \]

**Proof.** If \( 0 < \lambda < \lambda_1(m) \), we can write \( Q_\lambda(v) \) in the form
\[ Q_\lambda(v) = (\mathfrak{A}v, v)_{L^2(D)} - \lambda \int_D m(x) |v|^2 \, dx \]
\[ = \frac{\lambda}{\lambda_1(m)} (\mathfrak{A}v, v)_{L^2(D)} - \lambda \int_D m(x) |v|^2 \, dx + \left( 1 - \frac{\lambda}{\lambda_1(m)} \right) (\mathfrak{A}v, v)_{L^2(D)} \]
\[ = \frac{\lambda}{\lambda_1(m)} Q_{\lambda_1(m)}(v) + \left( 1 - \frac{\lambda}{\lambda_1(m)} \right) (\mathfrak{A}v, v)_{L^2(D)}. \]

However, it is clear that
\[ (\mathfrak{A}v, v)_{L^2(D)} \geq \lambda_1(m) \int_D m(x) v^2 \, dx \quad \text{for all} \quad v \in D(\mathfrak{A}), \]
so that
\[ Q_{\lambda_1(m)}(v) = (\mathfrak{A}v, v)_{L^2(D)} - \lambda_1(m) \int_D m(x) |v|^2 \, dx \geq 0 \quad \text{for all} \quad v \in D(\mathfrak{A}). \tag{50} \]

Therefore, we obtain from formula (49) and inequality (47) that
\[ Q_\lambda(v) \geq \left( 1 - \frac{\lambda}{\lambda_1(m)} \right) (\mathfrak{A}v, v)_{L^2(D)} \]
\[ \geq \gamma_1 \left( 1 - \frac{\lambda}{\lambda_1(m)} \right) (v, v)_{L^2(D)} \quad \text{for all} \quad v \in D(\mathfrak{A}). \]

This proves the desired inequality (48).

The proof of Lemma 5.6 is complete.

**Step V.** By combining Lemmas 5.5 and 5.6, we have the following proposition:

**Proposition 5.1.** If \( \lambda > 0 \) and \( \lambda \neq \lambda_1(m) \), then \( \lambda \) is not an eigenvalue of the mixed-type problem (5) possessing a non-negative eigenfunction.

**Proof.** By Lemma 5.5, it suffices to consider the case where \( 0 < \lambda < \lambda_1(m) \). The proof is based on a reduction to absurdity.

Assume, to the contrary, that if there exists a non-negative eigenfunction \( v \in W^{2, p}(D) \) for \( N < p < \infty \) corresponding to the mixed-type problem (5), then we have the formulas
\[
\begin{cases}
-\Delta v = \lambda m(x) v & \text{in } D, \\
v \geq 0 & \text{in } D, \\
Bv = 0 & \text{on } \partial D.
\end{cases}
\]
This implies that
\[ v \in D(\mathfrak{A}), \]
\[ Q_\lambda(v) = (\mathfrak{A}v, v)_{L^2(D)} - \lambda \int_D m(x) |v|^2 \, dx = 0. \]
However, it follows from an application of Lemma 5.6 that
\[ 0 = Q_\lambda(v) \geq \gamma_1 \left( 1 - \frac{\lambda}{\lambda_1(m)} \right) (v, v)_{L^2(D)} \text{ for } 0 < \lambda < \lambda_1(m), \]
so that
\[ v(x) \equiv 0 \text{ in } D. \]
This contradiction proves the desired assertion for \( 0 < \lambda < \lambda_1(m). \)

The proof of Proposition 5.1 is complete. \( \square \)

Step VI. Finally, the next theorem proves Theorem 1.3:

**Theorem 5.7.** Assume that condition (M.1), (H.1) and (H.2) are satisfied. Then we have the following four assertions:

(i) \( \lambda_1(m) \) is an eigenvalue of the mixed-type problem (5).
(ii) \( \lambda_1(m) \) is algebraically simple.
(iii) \( \lambda_1(m) \) admits a positive eigenfunction \( \psi_1(x) \) (see the linear eigenvalue problem (6)).
(iv) No other eigenvalues, \( \lambda_j(m), j \geq 2 \), have positive eigenfunctions.

**Proof.** The proof of Theorem 5.7 is divided into four steps.

(1) First, we consider the following eigenvalue problem:

[\begin{align*}
-\Delta w - \lambda_1(m) m(x) w &= \mu w \quad \text{in } D, \\
Bw &= 0 \quad \text{on } \partial D.
\end{align*}]  \tag{51}

Then it is easy to see that \( \lambda_1(m) \) is an eigenvalue of the mixed-type problem (5) with corresponding eigenfunction \( w(x) \) if and only if \( \mu = 0 \) is an eigenvalue of the linear problem (51) with corresponding eigenfunction \( w(x) \).

To prove assertion (i), we introduce a densely defined, selfadjoint operator
\[ S : L^2(D) \longrightarrow L^2(D) \]
by the formula
\[ S := T(\lambda_1(m)) = \mathfrak{A} - \lambda_1(m) m(x)I. \]

It suffices to show that the first eigenvalue \( \mu_1(\lambda_1(m)) \) of the operator \( S \) is equal to zero, that is, \( \mu_1(\lambda_1(m)) = 0 \) (see formula (38) and Figure 13).

By the Rayleigh principle, it follows that
\[ \mu_1(\lambda_1(m)) = \inf \left\{ \frac{(Sv, v)_{L^2(D)}}{\int_D |v|^2 \, dx} : v \in D(\mathfrak{A}), \ v \neq 0 \right\} \quad \tag{52} \]
\[ = \inf \left\{ \frac{(\mathfrak{A}v, v)_{L^2(D)} - \lambda_1(m) \int_D m(x) |v|^2 \, dx}{\int_D |v|^2 \, dx} : v \in D(\mathfrak{A}), \ v \neq 0 \right\} \]
\[ = \inf \left\{ \frac{Q_{\lambda_1(m)}(v)}{\int_D |v|^2 \, dx} : v \in D(\mathfrak{A}), \ v \neq 0 \right\}. \]

By virtue of inequality (50), it follows from formula (52) that
\[ \mu_1(\lambda_1(m)) \geq 0. \]
The next claim proves the desired assertion (i):

**Claim 5.2.** $\mu_1(\lambda_1(m)) = 0$.

*Proof.* By definition (45) of $\lambda_1(m)$, we can find a sequence $\{v_j\} \subset D(\mathcal{A})$ such that

$$\int_D m(x) |v_j|^2 \, dx = 1,$$

$$(\mathcal{A}v_j, v_j)_{L^2(D)} \to \lambda_1(m) \quad \text{as} \; j \to \infty.$$  

Then we have the assertion

$$Q_{\lambda_1(m)}(v_j) = (\mathcal{A}v_j, v_j)_{L^2(D)} - \lambda_1(m) \int_D m(x) |v_j|^2 \, dx \to 0 \quad \text{as} \; j \to \infty. \quad (53)$$

On the other hand, it follows that

$$1 = \int_D m(x) |v_j|^2 \, dx \leq \|m^+\|_{L^\infty(D)} \int_D |v_j|^2 \, dx,$$

so that

$$\int_D |v_j|^2 \, dx \geq \frac{1}{\|m^+\|_{L^\infty(D)}}. \quad (54)$$

Therefore, by combining assertions (53) and (54) we obtain that

$$\frac{Q_{\lambda_1(m)}(v_j)}{\int_D |v_j|^2 \, dx} \to 0 \quad \text{as} \; j \to \infty.$$  

By formula (52), this proves that $\mu_1(\lambda_1(m)) = 0$.

The proof of Claim 5.2 is complete. $\square$

(2) Secondly, we recall that $\lambda_1(m)$ is an eigenvalue of the mixed-type problem (5) with corresponding eigenfunction $w(x)$ if and only if zero is an eigenvalue of the operator $\mathcal{S}$ with corresponding eigenfunction $w(x)$. However, Claim 5.2 tells us that zero is the first eigenvalue of $\mathcal{S} = T(\lambda_1(m))$. Therefore, the desired assertions (ii) and (iii) follow from an application of Theorem 4.9 to our situation, just as in the proof of Theorem 5.2.

(3) Thirdly, Proposition 5.1 proves the desired assertion (iv).

(4) Finally, the variational formula (7) is an immediate consequence of the definition (45).

Now the proof of Theorem 5.7 (and hence that of Theorem 1.3) is complete. $\square$
6. The comparison theorem for the first eigenvalues. In this section we study the inequalities among the first eigenvalues $\mu_D(\lambda)$, $\mu_N(\lambda)$ and $\mu_1(\lambda)$ subject to Dirichlet, Neumann and mixed type boundary conditions, respectively, in the case where $h(x) \equiv 0$ on $\partial D$.

First, we consider the Dirichlet eigenvalue problem with an indefinite weight function $m(x) \in C^\theta(D)$ for $0 < \theta < 1$ and a positive parameter $\lambda$

$$
\begin{cases}
-\Delta \phi = \lambda m(x) \phi & \text{in } D, \\
\phi = 0 & \text{on } \partial D.
\end{cases}
$$

(55)

The next theorem asserts the existence of the first positive eigenvalue of the Dirichlet problem (55) with an indefinite weight function (see [38], [19], [61, Theorem 1.2]):

Theorem 6.1 (the Dirichlet case). If condition (M.1) is satisfied, then the first eigenvalue $\gamma_1(m)$ of the Dirichlet problem (55) is positive and algebraically simple, and its corresponding eigenfunction $\phi_1(x) \in C^{2+\theta}(D)$ may be chosen to be positive everywhere in $D$:

$$
\begin{cases}
-\Delta \phi_1 = \gamma_1(m) m(x) \phi_1 & \text{in } D, \\
\phi_1 > 0 & \text{in } D, \\
\phi_1 = 0 & \text{on } \partial D.
\end{cases}
$$

Moreover, no other eigenvalues, $\gamma_j(m)$, $j \geq 2$, have positive eigenfunctions.

If $v \in C^{2+\theta}(D)$ is a positive eigenfunction corresponding to the first eigenvalue $\mu_D(\lambda)$ of the Dirichlet problem

$$
\begin{cases}
(-\Delta - \lambda m(x)) v = \mu_D(\lambda) v & \text{in } D, \\
v > 0 & \text{in } D, \\
v = 0 & \text{on } \partial D,
\end{cases}
$$

(56)

then it is easy to see that $\lambda$ is the first eigenvalue $\gamma_1(m)$ of the Dirichlet problem (55) with corresponding positive eigenfunction if and only if $\mu_D(\lambda) = 0$ is an eigenvalue of problem (56) with corresponding positive eigenfunction:

$$
\lambda = \gamma_1(m) \text{ in the Dirichlet problem (55)} \iff \mu_D(\lambda) = 0 \text{ in the Dirichlet problem (56)}.
$$

(57)

Secondly, we consider the Neumann eigenvalue problem with an indefinite weight function $m(x) \in C^\theta(D)$ and a positive parameter $\lambda$

$$
\begin{cases}
-\Delta \phi = \lambda m(x) \phi & \text{in } D, \\
\frac{\partial \phi}{\partial n} = 0 & \text{on } \partial D.
\end{cases}
$$

(58)

The next theorem asserts the existence of the first eigenvalue of the Neumann problem (58) with an indefinite weight function (see [22, Section 3], [9, Theorem 3.13], [53, Theorems 2 and 3], [70, Theorem 1.2]):

Theorem 6.2 (the Neumann case). If condition (M.2) is satisfied, then the Neumann eigenvalue problem (58) admits a unique non-zero, eigenvalue $\mu_1(m)$ having a positive eigenfunction. More precisely, we have the following two assertions:
(i) If \( \int_D m(x) \, dx \neq 0 \), then the Neumann eigenvalue problem (58) admits a unique positive eigenvalue \( \nu_1(m) \) with a positive eigenfunction \( \theta_1(x) \in C^{2+\theta}(\overline{D}) \) in \( D \):

\[
\begin{align*}
-\Delta \theta_1 &= \nu_1(m) \, m(x) \theta_1 & \text{in } D, \\
\theta_1 &> 0 & \text{in } D, \\
\frac{\partial \theta_1}{\partial n} &= 0 & \text{on } \partial D.
\end{align*}
\]

The eigenvalue \( \nu_1(m) \) is algebraically simple. Moreover, the eigenvalue 0 is algebraically simple and has a positive eigenfunction \( \phi_1(x) \equiv 1 \) in \( D \).

Namely, we have the assertions

\[
\begin{align*}
\nu_1(m) &> 0 & \text{if } \int_D m(x) \, dx < 0, \\
\nu_1(m) &< 0 & \text{if } \int_D m(x) \, dx > 0.
\end{align*}
\]

(ii) If \( \int_D m(x) \, dx = 0 \), then the eigenvalue \( \nu_1(m) = 0 \) of the Neumann eigenvalue problem (58) is the only eigenvalue having the positive eigenfunction \( \phi_1(x) \equiv 1 \) in \( D \). We remark that the eigenvalue 0 is not algebraically simple (see [28, Example 28.6]).

If \( w(x) \in C^{2+\theta}(\overline{D}) \) is a positive eigenfunction corresponding to the first eigenvalue \( \mu_N(\lambda) \) of the Neumann problem

\[
\begin{align*}
(-\Delta - \lambda m(x)) w &= \mu_N(\lambda) w & \text{in } D, \\
w &> 0 & \text{in } D, \\
\frac{\partial w}{\partial n} &= 0 & \text{on } \partial D,
\end{align*}
\]

then it is easy to see that \( \lambda \) is the first eigenvalue \( \nu_1(m) \) of problem (58) with corresponding positive eigenfunction if and only if \( \mu_N(\lambda) = 0 \) is an eigenvalue of problem (59) with corresponding positive eigenfunction:

\[
\lambda = \nu_1(m) \quad \text{in the Neumann problem (58) (60)}
\]

\[
\iff \quad \mu_N(\lambda) = 0 \quad \text{in the Neumann problem (59)}.
\]

Thirdly, we consider the following mixed-type eigenvalue problem:

\[
\begin{align*}
(-\Delta - \lambda m(x)) u &= \mu_1(\lambda) u & \text{in } D, \\
u &> 0 & \text{in } D, \\
Bu + a(x') \frac{\partial u}{\partial n} + b(x') u &= 0 & \text{on } \partial D.
\end{align*}
\]

Then we find from Claim 5.2 (see Figure 13) that \( \lambda \) is the first eigenvalue \( \lambda_1(m) \) of the mixed-type problem (5) with corresponding positive eigenfunction if and only if \( \mu(\lambda) = \mu_1(\lambda) = 0 \) is an eigenvalue of the mixed-type problem (61) with corresponding positive eigenfunction:

\[
\lambda = \lambda_1(m) \quad \text{in the mixed-type problem (5) (62)}
\]

\[
\iff \quad \mu_1(\lambda) = 0 \quad \text{in the mixed-type problem (61)}.
\]

The main result of this subsection is the following inequalities among the first eigenvalues \( \mu_D(\lambda) \), \( \mu_N(\lambda) \) and \( \mu_1(\lambda) \) (see [28, Proposition 17.7]):
Theorem 6.3. Assume that conditions (H.1) and (H.2) are satisfied. Then we have the following inequalities for the first eigenvalues $\mu_D(\lambda)$, $\mu_N(\lambda)$ and $\mu_1(\lambda)$ of problems (56), (59) and (61), respectively (see Figures 14, 15 and 16):

(i) If condition (M.1) is satisfied, then it follows that
$$\mu_1(\lambda) < \mu_D(\lambda)$$ for all $\lambda \geq 0$. \hfill (63)

In particular, we have, by assertions (57) and (62),
$$0 < \lambda_1(m) < \gamma_1(m).$$

(ii) If condition (M.2) is satisfied, then it follows that
$$\mu_N(\lambda) < \mu_1(\lambda)$$ for all $\lambda \geq 0$. \hfill (64)

In particular, by assertions (60) and (62) it follows from Theorem 6.2 that we have the inequalities
\[
\begin{cases}
0 < \nu_1(m) < \lambda_1(m) & \text{if } \int_D m(x) \, dx < 0, \\
\nu_1(m) = 0 < \lambda_1(m) & \text{if } \int_D m(x) \, dx = 0, \\
\nu_1(m) < 0 < \lambda_1(m) & \text{if } \int_D m(x) \, dx > 0.
\end{cases}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{eigenvalues_graph.png}
\caption{The first eigenvalues $\mu_D(\lambda)$, $\mu_N(\lambda)$ and $\mu_1(\lambda)$ in the case $\int_D m(x) \, dx < 0$}
\end{figure}

**Proof.** The proof of Theorem 6.3 is divided into two steps.

**Step I.** First, we will prove inequality (63) under condition (M.1). If we take a constant $c_0 > 0$ so large that
\[
\begin{align*}
c_0 + \mu_D(\lambda) &> 0, \\
c_0 - \lambda m(x) &> 0 \quad \text{in } D,
\end{align*}
\hfill (65a)
\hfill (65b)
\]
then, by applying [65, Theorem 1.2] to our situation we can find a unique solution $u(x) \in C^{2+\theta}(\overline{D})$ of the mixed-type problem
\[
\begin{cases}
(-\Delta - \lambda m(x) + c_0) u = (\mu_D(\lambda) + c_0) v & \text{in } D, \\
Bu = 0 & \text{on } \partial D.
\end{cases}
\]
Here \( v \in C^{2+\theta}(\overline{D}) \) is a positive eigenfunction of the Dirichlet problem (56). By condition (65a), it follows from an application of the maximum principle (see Lemma 4.7) that

\[
\text{If } \int_D m(x) \, dx = 0 \quad \text{then} \quad u(x) > 0 \quad \text{in } D.
\]

Moreover, we have the following claim:

**Claim 6.1.** \( u(x) \geq v(x) \) in \( D \).

**Proof.** The proof is based on a reduction to absurdity. Assume, to the contrary, that

\[
\alpha = \min_{\overline{D}} (u - v) < 0.
\]

Since we have the assertions

\[
\begin{cases}
(-\Delta - \lambda m(x) + c_0) v = (\mu_D(\lambda) + c_0) v & \text{in } D, \\
v > 0 & \text{in } D, \\
v = 0 & \text{on } \partial D,
\end{cases}
\]

Figure 15. The first eigenvalues \( \mu_D(\lambda) \), \( \mu_N(\lambda) \) and \( \mu_1(\lambda) \) in the case \( \int_D m(x) \, dx = 0 \)

Figure 16. The first eigenvalues \( \mu_D(\lambda) \), \( \mu_N(\lambda) \) and \( \mu_1(\lambda) \) in the case \( \int_D m(x) \, dx > 0 \)
it follows that
\[
\begin{cases}
(\Delta + \lambda m(x) - c_0) (u - v) = 0 & \text{in } D, \\
u - v \geq 0 & \text{on } \partial D.
\end{cases}
\]
This implies that the function \( u(x) - v(x) \) may take its negative minimum \( \alpha \) at an interior point of \( D \). Thus, by applying the strong maximum principle (Theorem 4.4) we obtain that
\[ u(x) - v(x) \equiv \alpha \quad \text{in } D. \]
Hence we have, by condition (65b),
\[
0 = (\Delta + \lambda m(x) - c_0) (u - v) = (\lambda m(x) - c_0) \alpha > 0 \quad \text{in } D.
\]
This contradiction proves the desired inequality
\[ u(x) \geq v(x) \quad \text{in } D. \]
The proof of Claim 6.1 is complete. \( \square \)

By Claim 6.1, it follows that
\[
(-\Delta - \lambda m(x) + c_0) u = (\mu_D(\lambda) + c_0) v \leq (\mu_D(\lambda) + c_0) u \quad \text{in } D.
\]
Hence we have the assertions
\[
\begin{cases}
((-\Delta + (c_0 - \lambda m(x))) - (\mu_D(\lambda) + c_0)) (-u) \geq 0 & \text{in } D, \\
-u < 0 & \text{in } D, \\
B(-u) = 0 & \text{on } \partial D.
\end{cases}
\]
Therefore, the desired inequality (63) follows by applying Theorem 4.8 with
\[
u := -u, \\
c(x) := c_0 - \lambda m(x), \quad \lambda := \mu_D(\lambda) + c_0, \\
r(R_e) := \frac{1}{\mu_1(\lambda) + c_0}.
\]

**Step II.** Next we will prove inequality (64) under condition (M.2). Let \( u(x) \) be a positive eigenfunction corresponding to the first eigenvalue \( \mu_1(\lambda) \) of the mixed-type problem (61). If we take a constant \( d_0 > 0 \) so large that
\[
d_0 + \mu_1(\lambda) > 0, \quad (66a)
\]
\[
d_0 - \lambda m(x) > 0 \quad \text{in } D, \quad (66b)
\]
then we can find a unique solution \( w(x) \in C^{2+\theta}(\overline{D}) \) of the Neumann problem
\[
\begin{cases}
(-\Delta - \lambda m(x) + d_0) w = (\mu_1(\lambda) + d_0) u & \text{in } D, \\
\frac{\partial w}{\partial n} = 0 & \text{on } \partial D.
\end{cases}
\]
By the maximum principle, it follows that
\[ w(x) > 0 \quad \text{in } D. \]
Moreover, we have the following claim:

**Claim 6.2.** \( w(x) \geq u(x) \) in \( D \).
Proof. The proof is based on a reduction to absurdity. Assume, to the contrary, that
\[ \beta = \min_D (w - u) < 0. \]

We remark that
\[ (\Delta + \lambda m(x) - d) (w - u) = 0 \quad \text{in } D. \]

(a) If the function \( w(x) - u(x) \) takes its negative minimum \( \beta \) at an interior point \( x_0 \in D \), then, by applying the strong maximum principle (Theorem 4.4) we obtain that
\[ w(x) - u(x) \equiv \beta \quad \text{in } D. \]

Hence we have, by condition (66b),
\[ 0 = (\Delta + \lambda m(x) - d_0) (w - u) = (\lambda m(x) - d_0) \beta > 0 \quad \text{in } D. \]

This is a contradiction.

(b) If the function \( w(x) - u(x) \) takes its negative minimum \( \beta \) at a boundary point \( x_0' \in \partial D \), then, by applying the Hopf boundary point lemma (Theorem 4.3) we obtain that
\[ \frac{\partial w}{\partial n}(x_0') - \frac{\partial u}{\partial n}(x_0') = \frac{\partial (w - u)}{\partial n}(x_0') < 0. \]

This implies that
\[ \frac{\partial u}{\partial n}(x_0') > 0, \tag{67} \]

since we have the formula
\[ \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial D. \]

On the other hand, we have the assertion
\[ 0 = Bu(x_0') = a(x_0') \frac{\partial u}{\partial n} + b(x_0') u(x_0'). \]

However, it follows from condition (H.2) that
\[ a(x_0') > 0. \tag{68} \]

Indeed, it suffices to note that if \( a(x_0') = 0 \), then we have, by condition (H.2),
\[ u(x_0') = 0, \]

and so
\[ 0 \leq w(x_0') = w(x_0') - u(x_0') = \beta < 0. \]

This is also a contradiction.

Therefore, by combining assertions (67) and (68) we find that
\[ 0 < \frac{\partial u}{\partial n}(x_0') = - \frac{b(x_0')}{a(x_0')} u(x_0') \leq 0. \]

This contradiction proves the inequality
\[ w(x) \geq u(x) \quad \text{in } D. \]

The proof of Claim 6.2 is complete. \( \square \)
By Claim 6.2, it follows that
\[
(-\Delta - \lambda m(x) + d_0) w = (\mu_1(\lambda) + d_0) u \leq (\mu_1(\lambda) + d_0) w \quad \text{in } D.
\]
Hence we have the assertions
\[
\begin{cases}
((\Delta + (d_0 - \lambda m(x))) - (\mu_1(\lambda) + d_0)) (-w) \geq 0 & \text{in } D, \\
-w < 0 & \text{in } D, \\
\frac{\partial(-w)}{\partial n} = 0 & \text{on } \partial D.
\end{cases}
\]
Therefore, the desired inequality (64) follows by applying Theorem 4.8 with
\[
u := -w, \\
c(x) := d_0 - \lambda m(x), \quad \lambda := \mu_1(\lambda) + d_0, \\
r(R_n) := \frac{1}{\mu_N(\lambda) + d_0}.
\]
Now the proof of Theorem 6.3 is complete. \qed

7. **Proof of Theorem 1.5 –(i)–.** This section and the next section are devoted to the proof of Theorem 1.5. Our approach to the semilinear problem (1) is a modification of that of Ouyang [41] adapted to the present context. The idea of proof of part (i) of Theorem 1.5 can be visualized in the diagram below (see Figure 17).

![Figure 17. A flowchart of proof of Theorem 1.5, part (i)](image)

Part (i) of the proof is divided into three steps.

**Step I.** First, we begin with the following lower bound on the parameter \(\lambda\) for the existence of positive solutions of the semilinear problem (1):

**Lemma 7.1.** Assume that conditions (M.1), (H.1) and (H.2) are satisfied. If there exists a positive solution \(u(\lambda) \in C^2(\bar{D})\) of the semilinear problem (1) for \(\lambda > 0\), then we have the inequality
\[
\lambda > \lambda_1(m).
\]
Proof. Let \( u \in C^2(\overline{D}) \) be a positive solution of the semilinear problem (1):
\[
\begin{align*}
-\Delta u &= \lambda (m(x) - h(x)u) \quad \text{in } D, \\
u &\geq 0 \quad \text{in } D, \\
Bu &= 0 \quad \text{on } \partial D.
\end{align*}
\]
Then, by applying Theorem 1.3 with the weight \( m(x) := m(x) - h(x)u \) we obtain that
\[
\lambda = \lambda_1 (m(x) - h(x)u).
\] (70)
However, since \( h(x) \geq 0 \) and \( u \geq 0 \) in \( D \), it follows that
\[
m(x) \geq m(x) - h(x)u \quad \text{in } D.
\]
Therefore, by using inequality (8) for \( m(x) := m(x) \) and \( \tilde{m}(x) := m(x) - h(x)u \) we have the strict inequality
\[
\lambda_1 (m(x) - h(x)u) > \lambda_1 (m(x)) = \lambda_1 (m).
\] (71)
The desired lower bound (69) follows by combining assertions (70) and (71).

The proof of Lemma 7.1 is complete.

Step II. Secondly, we construct a positive solution \( u(\lambda) \) of the semilinear problem (1) for every \( \lambda > \lambda_1 (m) \).

By using the resolvent \( R_0 \) for the mixed-type boundary value problem (2) with \( c(x) \equiv 0 \), we transform the semilinear problem (1) into a nonlinear operator equation in the ordered Banach space \( C_c(\overline{D}) \) (see [3]). It follows from an application of Proposition 4.1 with \( c(x) \equiv 0 \) and Figures 11 and 12 that a function \( u(x) \in W^{2,p}(D) \) for \( N < p < \infty \) is a solution of the semilinear problem
\[
\begin{align*}
-\Delta u &= \lambda (m(x) - h(x)u) \quad \text{in } D, \\
Bu &= 0 \quad \text{on } \partial D
\end{align*}
\] (1)
if and only if it satisfies the nonlinear operator equation
\[
u = \lambda R_0 (m(x) u - h(x) u^2) = \lambda (-\Delta)^{-1} (m(x) u - h(x) u^2) \quad \text{in } C_c(\overline{D}).\] (72)
Moreover, just as in the proof of Hess–Kato [29, Theorem 2] we extend the function
\[
f(x, s) = m(x)s - h(x) s^2
\]
as an odd function in the variable \( s \) as follows:
\[
\tilde{f}(x, s) = \begin{cases} 
m(x)s - h(x) s^2 & \text{if } s > 0, \\
m(x)s + h(x) s^2 & \text{if } s \leq 0.
\end{cases}
\] (73)
Then we associate with the function \( \tilde{f}(x, s) \) the Nemytskii operator \( \tilde{F}(u) \) defined by the formula
\[
\tilde{F}(u)(x) = \tilde{f}(x, u(x)) \quad \text{for } x \in \overline{D},
\]
and consider instead of the nonlinear operator equation (72) the following equation:
\[
u = \lambda R_0 \left( \tilde{F}(u) \right) = \lambda (-\Delta)^{-1} \left( \tilde{F}(u) \right) \quad \text{in } C_c(\overline{D}).\] (74)
We remark that \( u(x) \) is a solution of the nonlinear operator equation (74) if and only if \( -u(x) \) is a solution. Hence we may identify positive solutions with negative solutions in this section.
Substep II-1. The proof of Theorem 1.5 is based on Theorem 3.3 due to Crandall–Rabinowitz [16, Theorem 1.7].

We shall apply Theorem 3.3 (see Figure 8) with
\[ X = Y := C_e(D), \quad P := P_e, \]
\[ F(\lambda, u) := u - \lambda R_0 \left( \bar{f}(x, u(x)) \right) = u - \lambda R_0 \left( \bar{F}(u) \right), \]
\[ F_u(\lambda_1(m), 0) := I - \lambda_1(m)R_0 \cdot (m(x)) \cdot = I - \lambda_1(m)R_0 M, \]
\[ F_{u,\lambda}(\lambda_1(m), 0) := - R_0 \cdot (m(x)) \cdot = - R_0 M, \]
\[ \lambda^* := \lambda_1(m), \quad u^* := \psi_1(x). \]

Substep II-2. The next lemma proves the existence of positive solutions of the semilinear problem (1) emanating from the point \((\lambda_1(m), 0)\):

**Lemma 7.2.** Assume that conditions (M.1), (H.1) and (H.2) are satisfied. Then there exists a positive bifurcation solution curve \((\lambda, u(\lambda))\) of the semilinear problem (1) starting at the point \((\lambda_1(m), 0)\).

**Proof.** The proof of Lemma 7.2 is divided into four steps visualized as in the diagram below (see Figure 18).

![Figure 18. A flowchart of proof of Lemma 7.2](image)

(1) The Crandall–Rabinowitz local bifurcation theorem (Theorem 3.3) may be employed to assert that the algebraic simplicity of the eigenvalue \(\lambda_1(m)\) guarantees the existence of the continuum of non-trivial solutions of the semilinear problem (1) emanating from the point \((\lambda_1(m), 0)\), which can be expressed as the union of two subcontinua intersecting at the point \((\lambda_1(m), 0)\) (see Figure 19 below).

To do this, it suffices to verify the following two assertions:

1a) \(\dim N(I - \lambda_1(m)R_0 M) = \text{codim} R(I - \lambda_1(m)R_0 M) = 1.\)

1b) \(R_0 M\psi_1 \notin R(I - \lambda_1(m)R_0 M)\).

**Proof of Assertion (1a).** First, since the operator
\[ R_0 M = (-\Delta)^{-1} (m(x)) : C_e(D) \to C(D) \to C_e(D) \]
is compact, by applying the Riesz–Schauder theory [77, Chapter X, Section 5, Theorem 3] we obtain that the index of the operator
\[
I - \lambda_1(m)R_0 M : C_c(\overline{D}) \longrightarrow C_c(\overline{D})
\]
is equal to zero, that is,
\[
\text{ind} \ (I - \lambda_1(m)R_0 M) = \dim \mathcal{N} \ (I - \lambda_1(m)R_0 M) - \text{codim} \ (I - \lambda_1(m)R_0 M) = 0.
\]
However, Theorem 1.3 tells us that the null space
\[
\mathcal{N} \ (I - \lambda_1(m)R_0 M) = \mathcal{N} \ (-\Delta - \lambda_1(m) M)
\]
is one dimensional, spanned by the positive eigenfunction \(\psi_1(x)\). In particular, we have the formula
\[
\dim \mathcal{N} \ (I - \lambda_1(m)R_0 M) = 1.
\]
Therefore, the desired assertion (1a) follows by combining assertions (75) and (76).

**Proof of Assertion (1b).** The proof is based on a reduction to absurdity. We assume, to the contrary, that
\[
R_0 M \psi_1 \in \mathcal{R} \ (I - \lambda_1(m)R_0 M),
\]
or equivalently,
\[
M \psi_1 = \frac{1}{\lambda_1(m)} \Delta \psi_1 \in \mathcal{R} \ (-\Delta - \lambda_1(m) M).
\]
Then we have, by Figure 8 with \(u^* := \psi_1\) and Green’s formula,
\[
0 = (M \psi_1, \psi_1)_{L^2(D)} = \frac{1}{\lambda_1(m)} (-\Delta \psi_1, \psi_1)_{L^2(D)}
\]
\[
= \frac{1}{\lambda_1(m)} \left\{ \int_D |\nabla \psi_1|^2 \, dx + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} \cdot \psi_1^2 \, d\sigma \right\}.
\]
This implies that
\[
\nabla \psi_1(x) \equiv 0 \quad \text{in} \ D
\]
and further that
\[
\psi_1(x') = 0 \quad \text{if} \ b(x') \neq 0.
\]
Therefore, we obtain from condition (H.2) that
\[
\psi_1(x) \equiv 0 \quad \text{in} \ D.
\]
This contradiction proves the desired assertion (1b).

Moreover, since the Nemytskii operator \(\tilde{F}(u)\) is an odd function of \(u\) (see formula (73)), we can prove that these subcontinua are locally the strictly positive and negative solutions of the semilinear problem (1) as in Figure 19. Here recall that we may identify positive solutions with negative solutions in the nonlinear operator equation (74).
The proof is based on a reduction to absurdity. We assume, to the contrary, that there exists a sequence $(\lambda_j, u_j)$, with $\lambda_j > 0$ and $u_j \in C_c(\overline{D})$, such that

$$u_j = \lambda_j R_0 \left( \tilde{F}(u_j) \right),$$

$$\lambda_j \to \lambda_1(m) \quad \text{as} \quad j \to \infty,$$

$$u_j \to 0 \quad \text{in} \quad C_c(\overline{D}) \quad \text{as} \quad j \to \infty,$$

$$u_j \in C_c(\overline{D}) \setminus \text{Int} \ (P_e).$$

If we let

$$v_j := \frac{u_j}{\|u_j\|_e},$$

then it follows that

$$v_j \in C_c(\overline{D}) \setminus \text{Int} \ (P_e),$$

$$\|v_j\|_e = 1,$$

and

$$v_j = \lambda_j \frac{R_0(\tilde{F}(u_j))}{\|u_j\|_e} = \lambda_j R_0 \left( \tilde{F}(v_j) \right) = \lambda_j \left( -\Delta \right)^{-1} (m(x)v_j - h(x)v_j |u_j|). \quad (77)$$

By the compactness of the resolvent $R_0 : C_c(\overline{D}) \to C_c(\overline{D})$, we may choose a subsequence, denoted again by $\{v_j\}$, which converges to some function $v$ in $C_c(\overline{D})$:

$$v_j \to v \quad \text{in} \quad C_c(\overline{D}) \quad \text{as} \quad j \to \infty.$$

Therefore, by passing to the limit in formula (77) we obtain that

$$v \in C_c(\overline{D}) \setminus \text{Int} \ (P_e),$$

$$\|v\|_e = 1,$$

and further from assertion (29) that

$$v = \lambda_1(m) R_0 (m(x)v) = \lambda_1(m) \left( -\Delta \right)^{-1} (m(x)v), \quad (78)$$

since $u_j \to 0$ in $C_c(\overline{D})$ as $j \to \infty$. 

**Figure 19.** The set of solutions of the semilinear problem (1) consists of a *pitchfork* near $\lambda = \lambda_1(m)$.
By virtue of Figure 11, it follows from formula (78) that the limit function \( v \in W^{2,p}(D) \) for \( N < p < \infty \) satisfies the conditions
\[
\begin{cases}
-\Delta v = \lambda_1(m)(m(x)v) & \text{in } D, \\
Bv = 0 & \text{on } \partial D.
\end{cases}
\]
However, we arrive at a contradiction (perhaps by changing sign in \( v \))
\[
v \in \text{Int}(P_c),
\]
since \( \lambda_1(m) \) is an algebraically simple eigenvalue of the mixed-type problem (5) having a positive eigenfunction \( \psi_1 \in \text{Int}(P_c) \) (see Theorem 1.3).

(3) We show that these subcontinua are globally the strictly positive and the strictly negative solutions of the semilinear problem (1). Namely, we show that the subcontinuum \( \mathcal{C} \) of positive solutions of the semilinear problem (1) stays in the set \( (0, \infty) \times \text{Int}(P_c) \). Our proof is based on the global bifurcation theorem (Theorem 3.4) due to Dancer [18].

Indeed, we assume, to the contrary, that there exists a point \( (\lambda_0, u_0) \in \mathbb{R} \times C_c(D) \)
such that
\[
\begin{align*}
\lambda_0 &> 0, \\
u_0 &\in P_c \setminus \{0\}, \\
u_0 &\in \partial(P_c), \\
u_0 &= \lambda_0 R_0 \left( \tilde{F}(u_0) \right) = \lambda_0 R_0 \left( F(u_0) \right) = \lambda_0 \left( -\Delta \right)^{-1} \left( F(u_0) \right).
\end{align*}
\]
If we let
\[
c_0 := \max_{0 \leq s \leq \|u_0\|_{C_c(D)}} |m(x) - h(x)s| + 1,
\]
then it follows that
\[
( -\Delta + \lambda_0 c_0 ) u_0 = \lambda_0 \left( F(u_0) + c_0 u_0 \right) = \lambda_0 u_0 \left( m(x) - h(x)u_0 + c_0 \right) \geq \lambda_0 u_0 \quad \text{in } D,
\]
and that
\[
Bu_0 = 0 \quad \text{on } \partial D.
\]
However, by applying Proposition 4.1 with \( c(x) := \lambda_0 c_0 \), we arrive at a contradiction
\[
u_0 = R_{\lambda_0 c_0} \left( \lambda_0 \left( F(u_0) + c_0 u_0 \right) \right) = \left( -\Delta + \lambda_0 c_0 \right)^{-1} \left( \lambda_0 u_0 \left( m(x) - h(x)u_0 + c_0 \right) \right) \in \text{Int}(P_c).
\]

(4) Finally, we apply the global bifurcation theorem (Theorem 3.4) with
\[
\begin{align*}
E &= C_c(D), \\
P &= P_c, \\
x &= u \in P_c, \\
Bx &= R_0 M u = R_0 \left( m(x)u \right) = \left( -\Delta \right)^{-1} \left( m(x)u \right), \\
spr(B) &= \text{spr} \left( R_0 M \right) = \left\{ \frac{1}{\lambda_1(m)} \right\}, \\
C_P(B) &= \left\{ \frac{1}{\text{spr}(B)} \right\} = \{ \lambda_1(m) \}, \\
f(\lambda, x) &= -\lambda R_0 \left( h(x)u^2 \right).
\end{align*}
\]
In fact, by Theorem 2.1 and Theorem 1.3 it is easy to verify that
\[
\begin{cases}
\lambda R_0 M \psi = \psi & \text{in } D, \\
\psi > 0 & \text{in } D
\end{cases}
\]

\[\iff\]
\[
\begin{cases}
-\Delta \psi = \lambda m(x) \psi & \text{in } D, \\
\psi > 0 & \text{in } D, \\
B \psi = 0 & \text{on } \partial D
\end{cases}
\]

\[\implies \text{spr} (R_0 M) = \frac{1}{\lambda} = \frac{1}{\lambda_1(m)}.\]

Therefore, by applying Theorem 3.4 to our situation we obtain that the subcontinuum \(C\) of positive solutions of the semilinear problem (1) emanating from \((\lambda_1(m), 0)\) is unbounded.

The proof of Lemma 7.2 is complete. \(\square\)

**Remark 7.1.** It should be emphasized that the subcontinuum \(C\) of positive solutions of the semilinear problem (1) may admit a secondary bifurcation, just as in Figure 10.

**Substep II-3.** Now we show that the subcontinuum \(C\) of positive solutions of the semilinear problem (1) has no secondary bifurcation point. More precisely, we prove the existence of a critical value \(\lambda(h)\in (\lambda_1(m), +\infty)\) such that the subcontinuum \(C\) may be parametrized as a \(C^1\) curve in the following form (see Figure 20):

\[C = \{ (\lambda, u(\lambda)) : \lambda_1(m) < \lambda < \lambda(h) \}. \]

![Figure 20](image)

**Figure 20.** The critical value \(\lambda(h)\) of the positive bifurcation solution curve \(C = \{(\lambda, u(\lambda))\}\)

To do so, we introduce a mapping

\[H(\lambda, v) : \mathbb{R}^+ \times C_e(D) \to C_e(D)\]

defined by the formula

\[H(\lambda, v) := \lambda (\lambda I - \Delta)^{-1} \left( \tilde{F}(v) + v \right) \quad \text{for } \lambda > 0 \text{ and } v \in C_e(D). \] (79)
Here it should be noticed that, by applying Theorems 1.1 and 1.2 with \( c(x) \equiv \lambda \) we have the following diagram for the resolvent \( R_\lambda = (\lambda I - \Delta)^{-1} \) just as in Figure 12:

\[
\begin{array}{c}
\text{Figure 21. The mapping properties of the resolvent } R_\lambda = (\lambda I - \Delta)^{-1} \text{ in the spaces } C(\overline{D}), C_e(\overline{D}) \text{ and } C^1_B(\overline{D})
\end{array}
\]

Moreover, it is easy to see that
\[
\begin{align*}
-\Delta u &= \lambda \tilde{F}(u) \quad \text{in } D, \\
Bu &= 0 \quad \text{on } \partial D
\end{align*}
\]
\[
\Leftrightarrow u = \lambda R_0 \left( \tilde{F}(u) \right) = \lambda (-\Delta)^{-1} \left( \tilde{F}(u) \right) \quad \text{in } C_e(\overline{D}) \quad (74)
\]
\[
\Leftrightarrow u = H(\lambda, u) = \lambda (\lambda I - \Delta)^{-1} \left( \tilde{F}(u) + u \right) \quad \text{in } C_e(\overline{D}). \quad (80)
\]

Then we have the following lemma:

**Lemma 7.3.** There exists a positive constant \( \overline{\lambda}(h) \), satisfying the condition
\[
\overline{\lambda}(h) \in (\lambda_1(m), +\infty], \quad (81)
\]
such that we have a positive solution \((\lambda, u(\lambda))\) of the nonlinear operator equation
\[
u = H(\lambda, u) \quad \text{for all } \lambda \in (\lambda_1(m), \overline{\lambda}(h)).
\]

**Proof.** Let \((\lambda^*, u^*)\) be a reference point for \( \lambda^* > \lambda_1(m) \) and \( u^* \in \text{Int } (P_e) \). By rescaling, we may assume that
\[
m(x) - 2h(x)s + 1 > 0 \quad \text{for } x \in \overline{D} \text{ and } 0 \leq s \leq \|u^*\|_{C(\overline{D})} + 1. \quad (82)
\]
Then, by applying Proposition 4.1 to our situation we obtain from condition (82) that the partial Fréchet derivative
\[
H_e(\lambda^*, u^*) = \lambda^* (\lambda^* I - \Delta)^{-1} (\tilde{F}'(u^*) + I) : C_e(\overline{D}) \rightarrow C_e(\overline{D}) \quad (83)
\]
is *strongly positive* and *compact*. Indeed, by formula (79) it suffices to note that
\[
\tilde{F}'(u^*)(x) = m(x) - 2h(x)|u^*(x)| \quad \text{for } x \in \overline{D}.
\]

The next claim guarantees the *bijectivity* of the Fréchet derivative \( H_e(\lambda^*, u^*) \) at the reference point \((\lambda^*, u^*)\):

**Claim 7.1.** If \( r^* = \text{spr } (H_e(\lambda^*, u^*)) \) is the principal eigenvalue of \( H_e(\lambda^*, u^*) \), then it follows that \( 0 < r^* < 1 \).

**Proof.** The proof is based on a reduction to absurdity. Assume, to the contrary, that
\[
r^* \geq 1.
\]
By the Kreĭn–Rutman theorem (Theorem 2.1), it follows that there exists a function \(w \in \text{Int} (P_e)\) such that
\[
H_e(\lambda^*, u^*)w = r^*w \quad \text{for } \lambda^* > \lambda_1(m) \text{ and } u^* \in \text{Int} (P_e).
\]
However, we can find a number \(t_0 > 0\) such that
\[
u^* - t_0 r^* w \in \partial P_e,
\]
since \(u^*, w \in \text{Int} (P_e)\). Then we have the assertion
\[
H(\lambda^*, u^* - t_0 r^* w) \in P_e.
\]
Indeed, it suffices to note that the function \(H(\lambda, \cdot)\) is increasing and \(H(\lambda, 0) = 0\).

On the other hand, it follows from formulas (79) and (83) that
\[
u^* - t_0 r^* w = H(\lambda^*, u^*) - t_0 H_e(\lambda^*, u^*)w
= \lambda^* (\lambda^* I - \Delta)^{-1} \left( (m(x)u^* - h(x)(u^*)^2 + u^* - t_0 (m(x) - 2h(x)u^* + 1) w \right)
= H(\lambda^*, u^* - t_0 w) + \lambda^* t_0^2 (\lambda^* I - \Delta)^{-1} (h(x)w^2)
\geq H(\lambda^*, u^* - t_0 r^* w) + \lambda t_0^2 (\lambda^* I - \Delta)^{-1} (h(x)w^2),
\]
since \(u^* - t_0 w \geq u^* - t_0 r^* w\) for \(r^* \geq 1\). Moreover, it follows that
\[
\lambda t_0^2 (\lambda^* I - \Delta)^{-1} (h(x)w^2) \in \text{Int} (P_e),
\]
since \(h(x)w^2 > 0\) in \(D\) and since the resolvent
\[
R_{\lambda^*} = (\lambda^* I - \Delta)^{-1} : C_\epsilon (\overset{-}{\mathcal{D}}) \to C_\epsilon (\overset{-}{\mathcal{D}})
\]
is strongly positive (see the proof of Proposition 4.1 with \(c(x) \equiv \lambda^*\)).

Therefore, by combining assertions (85), (86) and (87) we obtain that
\[
u^* - t_0 r^* w \in \text{Int} (P_e).
\]
This contradicts condition (84).

The proof of Claim 7.1 is complete.

By Claim 7.1, it follows that the partial Fréchet derivative \(I - H_e(\lambda^*, u^*)\) is invertible at the reference point \((\lambda^*, u^*)\) in \(C_\epsilon (\overset{-}{\mathcal{D}})\). Hence, by using the implicit function theorem (see Theorem 3.1) we can find a positive bifurcation solution curve \((\lambda, u(\lambda))\) of the nonlinear operator equation
\[
u = H(\lambda, u) \quad \text{in } C_\epsilon (\overset{-}{\mathcal{D}})
\]
beyond the reference point \((\lambda^*, u^*)\) (see Figure 22).

Summing up, we can define the critical value \(\overline{\lambda}(h)\) in condition (81) by the formula (see Figure 20)
\[
\overline{\lambda}(h) := \sup \left\{ \lambda : \text{the nonlinear operator equation } u = H(\lambda, u) \text{ has a positive solution } (\lambda, u(\lambda)) \text{ for } \lambda > \lambda_1(m) \right\}.
\]

The proof of Lemma 7.3 is complete.

\textbf{Step III.} Thirdly, we will prove the strong assertion (13) under the additional condition that
\[
m(x) \in C^\theta (\overset{-}{\mathcal{D}}) \quad \text{for } 0 < \theta < 1.
\]
To do this, we let
\[
C_{2+\theta}^B(\mathcal{D}) := \{ u \in C^{2+\theta}(\mathcal{D}) : Bu = 0 \text{ on } \partial\mathcal{D} \}.
\]

By condition (89), we can associate with the semilinear problem (1) a nonlinear mapping \( G(\lambda, u) \) of \( \mathbb{R} \times C_{2+\theta}^B(\mathcal{D}) \) into \( C^\theta(\mathcal{D}) \) as follows:
\[
G : \mathbb{R} \times C_{2+\theta}^B(\mathcal{D}) \rightarrow C^\theta(\mathcal{D})
\]
\[
(\lambda, u) \mapsto -\Delta u - \lambda m(x) u + \lambda h(x) u^2.
\]

It is clear that a function \( u \in C^{2+\theta}(\mathcal{D}) \) is a solution of the semilinear problem (1) if and only if \( G(\lambda, u) = 0 \).

By assertion (30), we have the mapping properties of the resolvent
\[
R_c = (-\Delta + c(x))^{-1}
\]
in the spaces \( C(\mathcal{D}), C_c(\mathcal{D}) \) and \( C^1_B(\mathcal{D}) \) as in Figure 12. Moreover, by virtue of Theorem 1.2 with \( c(x) \equiv 0 \) we have the following mapping properties of the negative Laplacian \(-\Delta\) and the resolvent
\[
R_0 = (-\Delta)^{-1}
\]
as in Figure 23.
By virtue of Figures 23 and 12, it is easy to verify that
\[ F(\lambda, u) = 0 \text{ in } \mathbb{R} \times C_c(\overline{D}) \]
\[ \iff u = \lambda R_0 \left( m(x) u - h(x) u^2 \right) \text{ in } \mathbb{R} \times C_c(\overline{D}) \]
\[ \iff u = \lambda R_0 \left( m(x) u - h(x) u^2 \right) \text{ in } \mathbb{R} \times C^1_B(\overline{D}) \]
\[ \iff u = \lambda R_0 \left( m(x) u - h(x) u^2 \right) \text{ in } \mathbb{R} \times C^{2+\theta}_B(\overline{D}) \]
\[ \iff -\Delta u = \lambda \left( m(x) u - h(x) u^2 \right) \text{ in } \mathbb{R} \times C^\theta(\overline{D}) \]
\[ \iff G(\lambda, u) = 0 \text{ in } \mathbb{R} \times C^\theta(\overline{D}). \]

Moreover, by Lemma 7.2 and Figures 12 and 23 we find that
\[ u(\lambda) \to 0 \text{ in } C_c(\overline{D}) \text{ as } \lambda \downarrow \lambda_1(m) \]
\[ \implies m(x) u(\lambda) - h(x) u(\lambda)^2 \to 0 \text{ in } C(\overline{D}) \text{ as } \lambda \downarrow \lambda_1(m) \]
\[ \implies u(\lambda) = \lambda R_0 \left( m(x) u(\lambda) - h(x) u(\lambda)^2 \right) \to 0 \text{ in } C^\theta_B(\overline{D}) \text{ as } \lambda \downarrow \lambda_1(m) \]
\[ \implies m(x) u(\lambda) - h(x) u(\lambda)^2 \to 0 \text{ in } C^\theta(\overline{D}) \text{ as } \lambda \downarrow \lambda_1(m). \]

This proves the desired assertion (13) under conditions (M.1), (H.1) and (H.2).

Now the proof of part (i) of Theorem 1.5 is complete.

8. Proof of Theorem 1.5 – (ii) –. It remains to characterize explicitly the critical value \( \lambda(h) \) given by formula (88) (formula (81) in Lemma 7.3) as follows:
\[ \lambda(h) = \mu_1(D_0(h)) = \min \left\{ \lambda_1 \left( D_1^0(h) \right), \lambda_1 \left( D_2^0(h) \right), \ldots, \lambda_1 \left( D_\ell^0(h) \right) \right\}. \tag{90} \]

The proof of formula (90) is divided into four steps.
**Step I.** First, we consider the logistic *Dirichlet* problem

\[
\begin{align*}
-\Delta v &= \lambda (m(x) - h(x)v) v & \text{in } D, \\
v &= 0 & \text{on } \partial D.
\end{align*}
\]  

(91)

Then we have the following generalization of Cantrell–Cosner [11, Theorems 2.1 and 2.3], Hess [28, Theorem 27.1] and Hess–Kato [29, Theorem 2] to the case where \( h(x) \) may vanish in \( D \) (see Remark 8.1):

**Theorem 8.1** (the Dirichlet case). Let \( m(x) \in C^\theta(D) \) for \( 0 < \theta < 1 \) and \( h(x) \in C^1(D) \). If conditions (M.1), (Z.1) and (Z.2) are satisfied, then the logistic Dirichlet problem (91) has a unique positive solution \( v(\lambda) \in C^{2+\theta}(D) \) for every \( \lambda \in (\gamma_1(m), \mu_1(D_0(h))) \). For any \( \lambda \geq \mu_1(D_0(h)) \), there exists no positive solution of the logistic Dirichlet problem (91). Moreover, we have the assertions

\[
\begin{align*}
\lim_{\lambda \uparrow \mu_1(D_0(h))} \|v(\lambda)\|_{L^2(D)} &= +\infty, \\
\lim_{\lambda \downarrow \gamma_1(m)} \|v(\lambda)\|_{C^{2+\theta}(D)} &= 0.
\end{align*}
\]  

(92a), (92b)

A biological interpretation of Theorem 8.1 is that a population will grow exponentially until limited by lack of available resources if the diffusion rate \( d = 1/\lambda \) is below the critical value \( 1/\mu_1(D_0(h)) \); this idea is generally credited to the English economist Thomas Robert Malthus (1776–1834). On the other hand, if the diffusion rate \( d = 1/\lambda \) is above the critical value \( 1/\mu_1(D_0(h)) \), then the model obeys the logistic equation introduced by the Belgian mathematical biologist Pierre François Verhulst (1804–1849) around 1840. The situation of Theorem 8.1 may be represented schematically as in the following bifurcation diagram (Figure 25):

**Figure 25.** The bifurcation diagram of Theorem 8.1 (the Dirichlet case)

**Remark 8.1.** López-Gómez and Sabina de Lis [37] analyze the pointwise growth to infinity of positive solutions of the logistic Dirichlet problem under the condition that \( m(x) \equiv 1 \) in \( D \) (see [37, Theorems 4.2 and 4.3]). Moreover, García-Melián et al [24] study the pointwise behavior and the uniqueness of positive solutions of nonlinear elliptic boundary value problems of general sublinear type, and give the exact limiting profile of the positive solutions (see [24, Theorem 3.1, Corollary 3.3 and Theorem 6.4]). Their numerical computations confirm and illuminate Figure 25.

**Step II.** Next we consider the logistic *Neumann* problem

\[
\begin{align*}
-\Delta w &= \lambda (m(x) - h(x)w) w & \text{in } D, \\
\frac{\partial w}{\partial n} &= 0 & \text{on } \partial D.
\end{align*}
\]  

(93)
Then we have the following generalization of Hess [28, Theorem 27.1] and Senn [52, Theorem 2.4] to the case where \( h(x) \) may vanish in \( D \):

**Theorem 8.2** (the Neumann case). Let \( m(x) \in C^\theta(D) \) for \( 0 < \theta < 1 \) and \( h(x) \in C^1(D) \). Assume that conditions (M.2), (Z.1) and (Z.2) are satisfied. Then we have the following two assertions:

(i) If \( \int_D m(x) \, dx < 0 \), the logistic Neumann problem (93) has a unique positive solution \( w(\lambda) \in C^{2+\theta}(D) \) for every \( \lambda \in (\nu_1(m), \mu_1(D_0(h))) \). For any \( \lambda \geq \mu_1(D_0(h)) \), there exists no positive solution of the logistic Neumann problem (93). Moreover, we have the assertions

\[
\lim_{\lambda \uparrow \mu_1(D_0(h))} \| w(\lambda) \|_{L^2(D)} = +\infty,
\]

\[
\lim_{\lambda \uparrow \nu_1(m)} \| w(\lambda) \|_{C^{2+\theta}(\overline{D})} = 0.
\]

In a neighborhood of the point \((0,0)\) the solution set of the semilinear problem (93) just consists of the two lines of trivial solutions (see Figure 26).

(ii) \( \int_D m(x) \, dx \geq 0 \), the logistic Neumann problem (93) has a unique positive solution \( w(\lambda) \in C^{2+\theta}(D) \) for every \( \lambda \in (0, \mu_1(D_0(h))) \). For any \( \lambda \geq \mu_1(D_0(h)) \), there exists no positive solution of the logistic Neumann problem (93). Moreover, we have the assertions

\[
\lim_{\lambda \downarrow 0} \| w(\lambda) \|_{L^2(D)} = +\infty,
\]

\[
\lim_{\lambda \downarrow 0} \| w(\lambda) - c \|_{C^{2+\theta}(\overline{D})} = 0,
\]

where

\[
c = \begin{cases} 0 & \text{if } \int_D m(x) \, dx = 0, \\ \frac{\int_D m(x) \, dx}{\int_D h(x) \, dx} & \text{if } \int_D m(x) \, dx > 0. \end{cases}
\]

More precisely, if \( \int_D m(x) \, dx > 0 \), there occurs a secondary bifurcation from the line \( \{0\} \times \mathbb{R} \) of trivial solutions at the point \((0,c)\) (see Figure 28). If \( \int_D m(x) \, dx = 0 \), there are two curves bifurcating at the point \((0,0)\); the line \( \{0\} \times \mathbb{R} \) of trivial solutions and the curve \( \{(\lambda, w(\lambda)) : \lambda > 0\} \) (see Figure 27).

**Remark 8.2.** Theorem 8.1 is proved by [60, Theorem 1.2] and Theorem 8.2 is proved by [60, Theorem 7.2], respectively. In the proof we make use of comparison theorems based on the maximum principle just as in Fraile et al. [23, Theorem 3.7], Pao [42, Chapter 5, Theorem 4.4] and Sattinger [48, Theorem 2.6.2]. Moreover, we make use of Reddinger [45, Satz] on the compactness of a bounded regular solution orbit for semilinear parabolic problems.

In the context of population dynamics, the behavior of solutions of the logistic Neumann problem (93) is similar to that of the logistic Dirichlet problem (91) with homogeneous Dirichlet condition if \( \int_D m(x) \, dx < 0 \). Hence, there is a positive eigenvalue with positive eigenfunction to act as a bifurcation point for positive steady states.

If \( \int_D m(x) \, dx \geq 0 \), then there will exist positive steady states for all values of \( d \). More precisely, if \( \int_D m(x) \, dx > 0 \), there occurs a secondary bifurcation from the line \( \{0\} \times \mathbb{R} \) of trivial solutions at the point \((0,c)\). If \( \int_D m(x) \, dx = 0 \), there are two curves bifurcating at the point \((0,0)\); the line \( \{0\} \times \mathbb{R} \) of trivial solutions and the curve \( \{(\lambda, w(\lambda)) : \lambda > 0\} \) (see [28, Example 28.6]).
A biological interpretation is that when the environment has an impassable boundary and is on the average unfavorable, then high diffusion rates have the same effect (that is, the ultimate extinction of the population) as they always have when the boundary is deadly; but if the boundary is impassable and the environment is on the average neutral or favorable, then the population can persist, no matter what its rate of diffusion.

The situation of Theorem 8.2 may be represented schematically as in the three bifurcation diagrams below (Figures 26, 27 and 28).

![Figure 26](image1)

**Figure 26.** The bifurcation diagram of Theorem 8.2 in the case $\int_D m(x)\,dx < 0$ and $\nu_1(m) > 0$ (the Neumann case)

![Figure 27](image2)

**Figure 27.** The bifurcation diagram of Theorem 8.2 in the case $\int_D m(x)\,dx = 0$ and $\nu_1(m) = 0$ (the Neumann case)

![Figure 28](image3)

**Figure 28.** The bifurcation diagram of Theorem 8.2 in the case $\int_D m(x)\,dx > 0$ and $\nu_1(m) < 0$ (the Neumann case)

**Step III.** The next *comparison principle* for the semilinear problems (1), (91) and (93) plays an essential role in the proof of formula (90):
Theorem 8.3. Assume that conditions (H.1), (H.2), (Z.1) and (Z.2) are satisfied. Then we have the following inequalities for the positive solutions \( u(\lambda) \), \( v(\lambda) \) and \( w(\lambda) \) of the semilinear problems (1), (91) and (93), respectively (see Figures 30, 31 and 32 below):

(i) If condition (M.1) is satisfied, then it follows that
\[
0 \leq v(\lambda) \leq u(\lambda) \quad \text{on } D.
\]  
(ii) If condition (M.2) is satisfied, then it follows that
\[
0 \leq u(\lambda) \leq w(\lambda) \quad \text{on } D.
\]

Proof. The proof of Theorem 8.3 is divided into two steps.

Step 1. First, we will prove inequality (95) under condition (M.2) in part (ii). The proof is based on a reduction to absurdity. Let
\[
\varphi(x) := u(\lambda)(x) - w(\lambda)(x),
\]
and assume, to the contrary, that the open set
\[
D^+ := \{ x \in D : \varphi(x) > 0 \} = \{ x \in D : u(\lambda)(x) > w(\lambda)(x) \}
\]
is non-empty (see Figure 29).

![Figure 29. The open subset \( D^+ \) with boundary \( \partial D^+ \)](image)

Then it follows that
\[
0 = -\Delta \varphi - \lambda \left( m(x)u(\lambda) - h(x)u(\lambda)^2 \right) + \lambda \left( m(x)w(\lambda) - h(x)w(\lambda)^2 \right)
= -\Delta \varphi - \lambda m(x)\varphi + \lambda h(x) \left( u(\lambda)^2 - w(\lambda)^2 \right)
= -\Delta \varphi - \lambda m(x)\varphi + \lambda h(x) \left( u(\lambda) + w(\lambda) \right) \varphi \quad \text{in } D.
\]

Hence we have the inequality
\[
\Delta \varphi + \lambda m(x)\varphi = \lambda h(x) \left( u(\lambda) + w(\lambda) \right) \varphi \geq 0 \quad \text{in } D^+.
\]  
Let \( x_0 \) be a point of the closure \( \overline{D^+} \) such that
\[
\varphi(x_0) = \max_{\overline{D^+}} \varphi(x) > 0.
\]

Without loss of generality, we may assume that
\[
\max_{x \in \overline{D}} m(x) < 1.
\]

If we let
\[
\Phi(x, t) := e^{-\lambda t} \varphi(x) \quad \text{for } x \in D \text{ and } t \geq 0,
\]
then it follows from inequality (96) that the function $\Phi(x,t)$ satisfies the inequality
\[
\frac{\partial \Phi}{\partial t} - \Delta \Phi + \lambda (1 - m(x)) \Phi = e^{-\lambda t}(-\Delta \varphi - \lambda m(x)\varphi) \leq 0 \quad \text{in } D^+ \times (0, T). \tag{99}
\]
Here we remark, by condition (98), that
\[
\lambda (1 - m(x)) \geq 0 \quad \text{in } D,
\]
and further from condition (97) that
\[
\max_{D^+ \times [0,T]} \Phi(x,t) = \max_D \varphi(x) = \varphi(x_0) > 0. \tag{100}
\]

(a) We consider the case where $x_0 \in D^+$: By applying the parabolic maximum principle (see [43, Chapter 3, Section 3, Theorems 5 and 7]) to our situation, we obtain from conditions (99) and (100) that
\[
\varphi(x) = \Phi(x,0) \equiv \Phi(x_0,0) = \varphi(x_0) > 0 \quad \text{for all } x \in D^+.
\]
This is a contradiction, since we have the formula
\[
\varphi(x) = 0 \quad \text{on } \partial D^+ \cap D.
\]

(b) Next we consider the case where $x_0' \in \partial D \cap \partial D^+$: Then it follows from an application of the Hopf boundary point lemma of parabolic type (see [43, Chapter 3, Section 3, Theorems 6 and 7]) that
\[
\frac{\partial \varphi}{\partial n}(x_0') > 0. \tag{101}
\]
However, we have the boundary condition
\[
0 = B \varphi(x_0') = a(x_0') \frac{\partial \varphi}{\partial n}(x_0') + b(x_0') \varphi(x_0').
\]
Thus, by combining conditions (97) and (101) we obtain that
\[
a(x_0') = b(x_0') = 0.
\]
This contradicts condition (H.2).

Therefore, we have proved the desired inequality (95), since the set $D^+$ is empty.

**Step 2.** Secondly, we will prove inequality (94) under condition (M.1) in part (i). The proof is based on a reduction to absurdity. Let
\[
\psi(x) := v(\lambda)(x) - u(\lambda)(x),
\]
and assume, to the contrary, that the open set
\[
E^+ := \{x \in D : \psi(x) > 0\} = \{x \in D : u(\lambda)(x) > w(\lambda)(x)\}
\]
is non-empty (see Figure 29). Then it follows that
\[
0 = -\Delta \psi - \lambda (m(x)v(\lambda) - h(x)v(\lambda)^2) + \lambda (m(x)u(\lambda) - h(x)u(\lambda)^2)
\]
\[
= -\Delta \psi - \lambda m(x)\psi + \lambda h(x) (v(\lambda)^2 - u(\lambda)^2)
\]
\[
= -\Delta \psi - \lambda m(x)\psi + \lambda h(x) (v(\lambda) + u(\lambda)) \psi \quad \text{in } D.
\]
Hence we have the assertion
\[
\Delta \psi + \lambda m(x)\psi = \lambda h(x)(v(\lambda) + u(\lambda)) \psi \geq 0 \quad \text{in } E^+. \tag{102}
\]
Let $x_0$ be a point of the closure $\overline{E^+}$ such that
\[
\psi(x_0) = \max_{\overline{E^+}} \psi(x) > 0. \tag{103}
\]
If we let

$$\Psi(x,t) := e^{-\lambda t}\psi(x) \quad \text{for } x \in D \text{ and } t \geq 0,$$

then it follows from inequality (102) that the function $\Psi(x,t)$ satisfies the inequality

$$\frac{\partial \Psi}{\partial t} - \Delta \Psi + \lambda (1 - m(x)) \Psi = e^{-\lambda t}(-\Delta \psi - \lambda m(x)\psi) \leq 0 \quad \text{in } D^+ \times (0,T). \quad (104)$$

Here we recall from condition (98) that

$$\lambda (1 - m(x)) \geq 0 \quad \text{in } D,$$

and further from condition (103) that

$$\max_{E^+ \times [0,T]} \Psi(x,t) = \max_{E^+} \psi(x) = \psi(x_0) > 0. \quad (105)$$

(a) We consider the case where $x_0 \in E^+$: By applying the parabolic maximum principle (see [43, Chapter 3, Section 3, Theorem 7]) to our situation, we obtain from conditions (104) and (105) that

$$\psi(x) = \Psi(x,0) = \Psi(x_0,0) = \psi(x_0) > 0 \quad \text{for all } x \in E^+.$$

This is a contradiction, since we have the assertion

$$\psi(x) = 0 \quad \text{on } \partial E^+ \cap D.$$

(b) Next we consider the case where $x'_0 \in \partial D \cap \partial E^+$: Then we have, by condition (103),

$$0 < \psi(x'_0) = v(\lambda)(x'_0) - u(\lambda)(x'_0) = -u(\lambda)(x'_0) \leq 0.$$

This is also a contradiction.

Therefore, we have proved the desired inequality (94), since that the set $E^+$ is empty. 

Now the proof of Theorem 8.3 is complete. \hfill \Box

**Step IV.** The desired formula (90) follows by combining Theorem 6.3, Lemma 7.3, Theorems 8.1, 8.2 and 8.3. Indeed, it suffices to note the three bifurcation diagrams below (Figures 30, 31 and 32).

Finally, the desired assertion (12) follows by combining inequality (94) and assertion (92a) under conditions (M.1), (Z.1), (Z.2), (H.1) and (H.2).

Now the proof of Theorem 1.5 is complete. \hfill \Box

**Figure 30.** The bifurcation diagrams of Theorem 8.3 in the case $\int_D m(x) \, dx < 0$ and $\nu_1(m) > 0$
9. Proof of Theorem 1.6 and Remark 1.4. In this section we consider the semilinear problem (1) under the condition (Z.3), and prove an estimate of the growth rate of the total size $\|u(\lambda)\|_{L^1(D)}$ of the positive steady states $u(\lambda)$ (Theorem 1.6 and Remark 1.4).

9.1. Proof of Theorem 1.6. In view of Lemma 7.2, it suffices to prove estimate (17) of the total size of the positive steady states $u(\lambda)$.

First, we have, by Green’s formula,

$$\lambda \left( \int_D m(x) u(\lambda)^2 \, dx - \int_D h(x) u(\lambda)^3 \, dx \right) = - \int_D \Delta u(\lambda) \cdot u(\lambda) \, dx$$

$$= \int_D |\nabla u(\lambda)|^2 \, dx - \int_{\partial D} \frac{\partial u(\lambda)}{\partial \mathbf{n}} u(\lambda) \, d\sigma$$

$$= \int_D |\nabla u(\lambda)|^2 \, dx + \int_\text{\{a(x')\neq0\}} \frac{b(x')}{a(x')} \cdot u(\lambda)^2 \, d\sigma,$$

and so

$$0 < \frac{1}{\lambda} \left( \int_D |\nabla u(\lambda)|^2 \, dx + \int_\text{\{a(x')\neq0\}} \frac{b(x')}{a(x')} \cdot u(\lambda)^2 \, d\sigma \right) + \int_D h(x) u(\lambda)^3 \, dx$$
By applying the variational formula (7), we obtain that
\[ (-\Delta u(\lambda), u(\lambda))_{L^2(D)} = \int_D |\nabla u(\lambda)|^2 \, dx + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} \cdot u(\lambda)^2 \, d\sigma \]  
(107)
\[
\geq \lambda_1(m) \int_D m(x) u(\lambda)^2 \, dx.
\]

Hence it follows from formula (106), inequality (107) and the variational formula (7) that
\[
\int_D h(x) u(\lambda)^3 \, dx = \int_D m(x) u(\lambda)^2 \, dx - \frac{1}{\lambda} \left( \int_D |\nabla u(\lambda)|^2 \, dx + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} \cdot u(\lambda)^2 \, d\sigma \right)
\]
\[
= \int_D m(x) u(\lambda)^2 \, dx - \frac{1}{\lambda} \left( -\Delta u(\lambda), u(\lambda) \right)_{L^2(D)}
\]
\[
\leq \int_D m(x) u(\lambda)^2 \, dx - \frac{\lambda_1(m)}{\lambda} \int_D m(x) u(\lambda)^2 \, dx
\]
\[
= \left( 1 - \frac{\lambda_1(m)}{\lambda} \right) \int_D m(x) u(\lambda)^2 \, dx \quad \text{for all } \lambda > \lambda_1(m).
\]

Secondly, we have, by Hölder’s inequality,
\[
\int_D m(x) u(\lambda)^2 \, dx \leq \int_D m^+(x) u(\lambda)^2 \, dx
\]
\[
\leq \left( \int_D (m^+(x))^3 \, dx \right)^{1/3} \left( \int_D u(\lambda)^3 \, dx \right)^{2/3}
\]
\[
= \|m^+\|_{L^3(D)} \left( \|u(\lambda)\|_{L^3(D)} \right)^2 \quad \text{for all } \lambda > \lambda_1(m).
\]

By combining inequalities (108) and (109), we obtain that
\[
\min_{x \in D} h(x) \cdot (\|u(\lambda)\|_{L^3(D)})^3
\leq \int_D h(x) u(\lambda) \, dx \leq \left( 1 - \frac{\lambda_1(m)}{\lambda} \right) \int_D m(x) u(\lambda)^2 \, dx
\]
\[
\leq \left( 1 - \frac{\lambda_1(m)}{\lambda} \right) \|m^+\|_{L^3(D)} \left( \|u(\lambda)\|_{L^3(D)} \right)^2 \quad \text{for all } \lambda > \lambda_1(m).
\]

This proves that
\[
\|u(\lambda)\|_{L^3(D)} \leq \left( 1 - \frac{\lambda_1(m)}{\lambda} \right) \|m^+\|_{L^3(D)} \frac{\|u(\lambda)\|_{L^3(D)}}{\min_{x \in D} h(x)} \quad \text{for all } \lambda > \lambda_1(m).
\]
(110)

On the other hand, by Hölder’s inequality it follows that
\[
\int_D u(\lambda) \, dx \leq \left( \int_D u(\lambda)^3 \, dx \right)^{1/3} \left( \int_D \, dx \right)^{2/3}
\]
\[
= |D|^{2/3} \|u(\lambda)\|_{L^3(D)} \quad \text{for all } \lambda > \lambda_1(m).
\]
(111)

Therefore, the desired estimate (17) follows by combining inequalities (10) and (11).
Now the proof of Theorem 1.6 is complete. \hfill \Box

9.2. **Proof of Remark 1.4.** Now we are in a position to prove the uniform estimate (18).

If we let
\[
 w(x) \equiv \ell := \frac{\max_{D} m^+}{\min_{D} h},
\]
then we have the inequalities
\[
 -\Delta w - \lambda m(x) w + \lambda h(x) w^2 = -\lambda m(x) \ell + \lambda h(x) \ell^2 \geq \lambda \ell(h(x) \ell - m(x)) \geq 0 \quad \text{in } D
\]
and
\[
 Bw = b(x') \ell \geq 0 \quad \text{on } \partial D.
\]
Namely, it follows that the constant function \( w(x) \) is a *supersolution* of the semilinear problem (1). Therefore, by using a comparison theorem based on the maximum principle we obtain the uniform estimate (18) as follows:

\[
\begin{aligned}
 -\Delta w - \lambda m(x) w + \lambda h(x) w^2 \geq 0 &= -\Delta u(\lambda) - \lambda m(x) u(\lambda) + \lambda h(x) u(\lambda)^2 \quad \text{in } D, \\
 Bw \geq 0 &= Bu(\lambda) \quad \text{on } \partial D \\
 \implies w \equiv \ell \geq u(\lambda) \quad \text{in } D.
\end{aligned}
\]

Indeed, it suffices to apply [67, Lemma 3.1] with
\[
 f(x, z, p) = f(x, z) := \lambda (1 - m(x)) z + \lambda h(x) z^2 \quad \text{for } x \in \overline{D} \text{ and } z \geq 0
\]
under the condition
\[
 |m(x)| < 1 \quad \text{on } \overline{D},
\]
just as in the proof of Theorem 5.2.

The proof of Remark 1.4 is complete. \hfill \Box

10. **Open problems in numerical analysis.** The ecological conclusion of the present paper is that, for a species with a given rate of diffusion, the best environments are those where the favorable regions are relatively large and few in a number, and the worst are those where favorable and unfavorable regions are closely intermingled, producing cancellation effects.

In the near future, we would like to apply Theorem 1.5 to provide numerical solutions of diffusive logistic equations with mixed type boundary conditions, generalizing Fleming [22] and García-Melián et al [24]. Moreover, we leave the numerical analysis of the critical value
\[
 \mu_1(D_0(h)) = \min \{ \lambda_1(D_1^1(h)), \lambda_1(D_2^1(h)), \ldots, \lambda_1(D_\ell^1(h)) \}
\]
in the general case for future study.

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