Gravity, Geometry and the Quantum

Abhay Ashtekar

Institute for Gravitational Physics and Geometry
Physics Department, Penn State, University Park, PA 16802-6300
Institute for Theoretical Physics, University of Utrecht,
Princetonplein 5, 3584 CC Utrecht, The Netherlands

After a brief introduction, basic ideas of the quantum Riemannian geometry underlying loop quantum gravity are summarized. To illustrate physical ramifications of quantum geometry, the framework is then applied to homogeneous isotropic cosmology. Quantum geometry effects are shown to replace the big bang by a big bounce. Thus, quantum physics does not stop at the big-bang singularity. Rather there is a pre-big-bang branch joined to the current post-big-bang branch by a ‘quantum bridge’. Furthermore, thanks to the background independence of loop quantum gravity, evolution is deterministic across the bridge.

I. INTRODUCTION

General relativity and quantum theory are among the greatest intellectual achievements of the 20th century. Each of them has profoundly altered the conceptual fabric that underlies our understanding of the physical world. Furthermore, each has been successful in describing the physical phenomena in its own domain to an astonishing degree of accuracy. And yet, they offer us strikingly different pictures of physical reality. Our past experience in physics tells us that these two pictures must be approximations, special cases that arise as appropriate limits of a single, universal theory. That theory must therefore represent a synthesis of general relativity and quantum mechanics. This would be the quantum theory of gravity that we invoke when faced with phenomena, such as the big bang and the final state of black holes, where the worlds of general relativity and quantum mechanics must unavoidably meet.

Remarkably, the necessity of a quantum theory of gravity was pointed out by Einstein already in 1916. In a paper in the Preussische Akademie Sitzungsberichte he wrote:

Nevertheless, due to the inneratomic movement of electrons, atoms would have to radiate not only electromagnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in Nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics but also the new theory of gravitation.

Ninety years later, our understanding of the physical world is vastly richer but a fully satisfactory unification of general relativity with quantum physics still eludes us. Indeed, the problem has now moved to the center-stage of fundamental physics. (For a brief historical account of the evolution of ideas see, e.g., [1].)
A key reason why the issue is still open is the lack of experimental data with direct bearing on quantum gravity. As a result, research is necessarily driven by theoretical insights on what the key issues are and what will ‘take care of itself’ once this core is understood. As a consequence, there are distinct starting points which seem natural. Such diversity is not unique to this problem. However, for other fundamental forces we have had clear-cut experiments to weed-out ideas which, in spite of their theoretical appeal, fail to be realized in Nature. We do not have this luxury in quantum gravity. But then, in absence of strong experimental constraints, one would expect a rich variety of internally consistent theories. Why is it then that we do not have a single one? The reason, I believe, lies the deep conceptual difference between the description of gravity in general relativity and that of non-gravitational forces in other fundamental theories. In those theories, space-time is given a priori, serving as an inert background, a stage on which the drama of evolution unfolds. General relativity, on the other hand, is not only a theory of gravity, it is also a theory of space-time structure. Indeed, in general relativity, gravity is encoded in the very geometry of space-time. Therefore, a quantum theory of gravity has to simultaneously bring together gravity, geometry and the quantum. This is a band new adventure and our past experience with other forces can not serve as a reliable guide.

Loop quantum gravity (LQG) is an approach that attempts to face this challenge squarely (for details, see, e.g., [2, 3, 4]). Recall that Riemannian geometry provides the appropriate mathematical language to formulate the physical, kinematical notions as well as the final dynamical equations of any classical theory of relativistic gravity. This role is now assumed by quantum Riemannian geometry. Thus, in LQG both matter and geometry are quantum mechanical ‘from birth’.

In the classical domain, general relativity stands out as the best available theory of gravity. Therefore, it is natural to ask: Does quantum general relativity, coupled to suitable matter (or supergravity, its supersymmetric generalization) exist as consistent theories non-perturbatively? In particle physics circles the answer is often assumed to be in the negative, not because there is concrete evidence which rules out this possibility, but because of the analogy to the theory of weak interactions. There, one first had a 4-point interaction model due to Fermi which works quite well at low energies but which fails to be renormalizable. Progress occurred not by looking for non-perturbative formulations of the Fermi model but by replacing the model by the Glashow-Salam-Weinberg renormalizable theory of electro-weak interactions, in which the 4-point interaction is replaced by $W^\pm$ and $Z$ propagators. It is often assumed that perturbative non-renormalizability of quantum general relativity points in a similar direction. However this argument overlooks a crucial and qualitatively new element of general relativity. Perturbative treatments pre-suppose that space-time is a smooth continuum at all scales of interest to physics under consideration. This assumption is safe for weak interactions. In the gravitational case, on the other hand, the scale of interest is the Planck length and there is no physical basis to pre-suppose that the continuum approximation should be valid down to that scale. The failure of the standard perturbative treatments may largely be due to this grossly incorrect assumption and a non-perturbative treatment which correctly incorporates the physical micro-structure of geometry may well be free of these inconsistencies.

Are there any situations, outside LQG, where such physical expectations are borne out by
detailed mathematics? The answer is in the affirmative. There exist quantum field theories (such as the Gross-Neveu model in three dimensions) in which the standard perturbation expansion is not renormalizable although the theory is exactly soluble! Failure of the standard perturbation expansion can occur because one insists on perturbing around the trivial, Gaussian point rather than the more physical, non-trivial fixed point of the renormalization group flow. Interestingly, thanks to the recent work by Lauscher, Reuter, Percacci, Perini and others, there is now growing evidence that situation may be similar with general relativity (see [5] and references therein). Impressive calculations have shown that pure Einstein theory may also admit a non-trivial fixed point. Furthermore, the requirement that the fixed point should continue to exist in presence of matter constrains the couplings in physically interesting ways [6].

Let me conclude this introduction with an important caveat. Suppose one manages to establish that non-perturbative quantum general relativity (or, supergravity) does exist as a mathematically consistent theory. Still, there is no a priori reason to assume that the result would be the ‘final’ theory of all known physics. In particular, as is the case with classical general relativity, while requirements of background independence and general covariance do restrict the form of interactions between gravity and matter fields and among matter fields themselves, the theory would not have a built-in principle which determines these interactions. Put differently, such a theory would not be a satisfactory candidate for unification of all known forces. However, just as general relativity has had powerful implications in spite of this limitation in the classical domain, quantum general relativity should have qualitatively new predictions, pushing further the existing frontiers of physics. In section III we will see an illustration of this possibility.

II. QUANTUM RIEMANNIAN GEOMETRY

In a recent short review [1] I have provided a semi-qualitative description of the quantum Riemannian geometry. To complement that discussion, in this section I will provide a concise but more mathematical summary.

The starting point of LQG is a Hamiltonian formulation of general relativity based on spin connections [7]. Since all other basic forces of nature are also described by theories of connections, this formulation naturally leads to an unification of all four fundamental forces at a kinematical level. Specifically, the phase space of general relativity is the same as that of a Yang-Mills theory. The difference lies in dynamics: whereas in the standard Yang-Mills theory the Minkowski metric features prominently in the definition of the Hamiltonian, there are no background fields whatsoever once gravity is switched on.

Let us focus on the gravitational sector of the theory. Then, the phase space \( \Gamma_{\text{grav}} \) consists of canonically conjugate pairs \( (A^i_a, \Pi^i_{ab}) \), where \( A^i_a \) is a connection on a 3-manifold \( M \) and \( \Pi^i_{ab} \) a 2-form, both of which take values in the Lie-algebra \( \text{su}(2) \). The connection \( A \) enables one to parallel transport chiral spinors (such as the left handed fermions of the standard electro-weak model) along curves in \( M \). Its curvature is directly related to the electric and
magnetic parts of the space-time \textit{Riemann tensor}. The dual $P^a_i$ of $P^i_{ab}$ plays a double role.\footnote{\footnotesize $P^a_i$ is a vector density, defined via $3 \int_M P^i_{ab} \omega^a_c = \int_M P^i_a \omega^a_c$ for any 1-form $\omega$ on $M$.}

Being the momentum canonically conjugate to $A$, it is analogous to the Yang-Mills electric field. In addition, $E^a_i := 8\pi G \gamma P^a_i$, has the interpretation of an orthonormal triad of frame field (with density weight 1) on $M$, where $\gamma$ is the ‘Barbero-Immirzi parameter’ representing a quantization ambiguity. Each triad $E^a_i$ determines a positive definite ‘spatial’ 3-metric $q_{ab}$, and hence the Riemannian geometry of $M$. This dual role of $P$ is a reflection of the fact that now SU(2) is the (double cover of the) group of rotations of the orthonormal spatial triads on $M$ itself rather than of rotations in an ‘internal’ space associated with $M$.

To pass to quantum theory, one first constructs an algebra of ‘elementary’ functions on $\Gamma_{\text{grav}}$ (analogous to the phase space functions $x$ and $p$ in the case of a particle) which are to have unambiguous operator analogs. The holonomies

$$h_e(A) := \mathcal{P} \exp (-\int_e A)$$  \hspace{1cm} (2.1)

associated with a (piecewise analytic) curve/edge $e$ on $M$ is a (SU(2)-valued) configuration function on $\Gamma_{\text{grav}}$. Similarly, given a (piecewise analytic) 2-surface $S$ on $M$, and a su(2)-valued (test) function $f$ on $M$,\n
$$P_{S,f} := \int_S \text{Tr} (f \mathcal{P})$$  \hspace{1cm} (2.2)

is a momentum-function on $\Gamma_{\text{grav}}$, where Tr is over the su(2) indices. The symplectic structure on $\Gamma_{\text{grav}}$ enables one to calculate the Poisson brackets $\{h_e, P_{S,f}\}$. The result is a linear combination of holonomies and can be written as a Lie derivative,

$$\{h_e, P_{S,f}\} = \mathcal{L}_{X_{S,f}} h_e,$$  \hspace{1cm} (2.3)

where $X_{S,f}$ is a derivation on the ring generated by holonomy functions, and can therefore be regarded as a vector field on the configuration space $A$ of connections. This is a familiar situation in classical mechanics of systems whose configuration space is a manifold. Functions $h_e$ and vector fields $X_{S,f}$ generate a Lie algebra. As in quantum mechanics on manifolds, the first step is to promote this algebra to a quantum algebra by demanding that the commutator be given by $i\hbar$ times the Lie bracket. The result is a $\star$-algebra $\mathfrak{a}$, analogous to the algebra generated by operators $\exp i\lambda \hat{x}$ and $\hat{p}$ in quantum mechanics. By exponentiating also the momentum operators $\hat{P}_{S,f}$ one obtains $\mathfrak{W}$, the analog of the quantum mechanical Weyl algebra generated by $\exp i\lambda \hat{x}$ and $\exp i\mu \hat{p}$.

The main task is to obtain the appropriate representation of these algebras. In that representation, \textit{quantum} Riemannian geometry can be probed through the momentum operators $\hat{P}_{S,f}$, which stem from classical orthonormal triads. As in quantum mechanics on manifolds or simple field theories in flat space, it is convenient to divide the task into two parts. In the first, one focuses on the algebra $\mathcal{C}$ generated by the configuration operators $\hat{h}_e$ and finds all its representations, and in the second one considers the momentum operators $\hat{P}_{S,f}$ to restrict the freedom.

$\mathcal{C}$ is called the holonomy algebra. It is naturally endowed with the structure of an Abelian $C^*$ algebra (with identity), whence one can apply the powerful machinery made available
by the Gel’fand theory. This theory tells us that $C$ determines a unique compact, Hausdorff space $\tilde{A}$ such that the $C^*$ algebra of all continuous functions on $A$ is naturally isomorphic to $C$. $\tilde{A}$ is called the Gel’fand spectrum of $C$. It has been shown to consist of ‘generalized connections’ $\tilde{A}$ defined as follows: $\tilde{A}$ assigns to any oriented edge $e$ in $M$ an element $\tilde{A}(e)$ of $SU(2)$ (a ‘holonomy’) such that $\tilde{A}(e^{-1}) = [\tilde{A}(e)]^{-1}$; and, if the end point of $e_1$ is the starting point of $e_2$, then $\tilde{A}(e_1 \circ e_2) = \tilde{A}(e_1) \cdot \tilde{A}(e_2)$. Clearly, every smooth connection $\tilde{A}$ is a generalized connection. In fact, the space $A$ of smooth connections has been shown to be dense in $\tilde{A}$ (with respect to the natural Gel’fand topology thereon). But $\tilde{A}$ has many more ‘distributional elements’. The Gel’fand theory guarantees that every representation of the $C^*$ algebra $C$ is a direct sum of representations of the following type: The underlying Hilbert space is $H = L^2(\tilde{A}, d\mu)$ for some measure $\mu$ on $\tilde{A}$ and (regarded as functions on $\tilde{A}$) elements of $C$ act by multiplication. Since there are many inequivalent measures on $\tilde{A}$, there is a multitude of representations of $C$. A key question is how many of them can be extended to representations of the full algebra $a$ (or $a^W$) without having to introduce any ‘background fields’ which would compromise diffeomorphism covariance. Quite surprisingly, the requirement that the representation be cyclic with respect to a state which is invariant under the action of the group of (piecewise analytic) diffeomorphisms on $M$ singles out a unique irreducible representation. This result was recently established for $a$ by Lewandowski, Okołów, Sahlmann and Thiemann [8], and for $a^W$ by Fleischhack [5]. It is the quantum geometry analog to the seminal results by Segal and others that characterized the Fock vacuum in Minkowskian field theories. However, while that result assumes not only Poincaré invariance but also specific (namely free) dynamics, it is striking that the present uniqueness theorems make no such restriction on dynamics. The requirement of diffeomorphism invariance is surprisingly strong and makes the ‘background independent’ quantum geometry framework surprisingly tight.

This unique representation was in fact introduced already in the mid-nineties [10, 11, 12] and has been extensively used in LQG since then. The underlying Hilbert space is given by $H = L^2(\tilde{A}, d\mu_o)$ where $\mu_o$ is a diffeomorphism invariant, faithful, regular Borel measure on $\tilde{A}$, constructed from the normalized Haar measure on $SU(2)$. Typical quantum states can be visualized as follows. Fix: (i) a graph $\alpha$ on $M$, and, (ii) a smooth function $\psi$ on $[SU(2)]^n$. Then, the function

$$\Psi_{\alpha}(\tilde{A}) := \psi(\tilde{A}(e_1), \ldots, \tilde{A}(e_n))$$

(2.4)

on $\tilde{A}$ is an element of $H$. Such states are said to be cylindrical with respect to the graph $\alpha$ and their space is denoted by $\text{Cyl}_{\alpha}$. These are ‘typical states’ in the sense that $\text{Cyl} := \cup_{\alpha} \text{Cyl}_{\alpha}$ is dense in $H$. Finally, as ensured by the Gel’fand theory, the holonomy (or configuration) operators $\hat{h}_e$ act just by multiplication. The momentum operators $\hat{P}_{S,f}$ act as Lie-derivatives:

$$\hat{P}_{S,f} \Psi = -i\hbar \mathcal{L}_{X_{S,f}} \Psi.$$ 

Given any graph $\alpha$ in $M$, and a labelling of each of its edges by a non-trivial irreducible representation of $SU(2)$ (i.e., by a non-zero half integer $j$), one can construct a finite dimensional Hilbert space $H_{\alpha,j}$ which can be thought of as the state space of a spin system ‘living on’ the graph $\alpha$. The full Hilbert space admits a simple decomposition: $H = \bigoplus_{\alpha,j} H_{\alpha,j}$. This is called the spin-network decomposition [13, 14]. The geometric operators discussed in Rovelli’s talk leave each $H_{\alpha,j}$ invariant. Therefore, the availability of this decomposition
greatly simplifies the task of analyzing their properties\cite{2,15,16}.

Key features of this representation which distinguish it from, say, the standard Fock representation of the quantum Maxwell field are the following. While the Fock representation of photons makes a crucial use of the background Minkowski metric, the above construction is manifestly ‘background independent’. Second, the connection itself is not represented as an operator (valued distribution). Holonomy operators, on the other hand, are well-defined. Finally, and most importantly, the Hilbert space $\mathcal{H}$ and the associated holonomy and (smeared) triad operators only provide a kinematical framework —the quantum analog of the phase space. Thus, while elements of the Fock space represent physical states of photons, elements of $\mathcal{H}$ are not the physical states of LQG. Rather, like the classical phase space, the kinematic setup provides a home for formulating quantum dynamics. In the Hamiltonian framework, the dynamical content of any background independent theory is contained in its constraints. In quantum theory, the Hilbert space $\mathcal{H}$ and the holonomy and (smeared) triad operators thereon provide the necessary tools to write down quantum constraint operators. The physical states are solutions to these quantum constraints. Thus, to complete the program, one has to: i) obtain the expressions of the quantum constraints; ii) solve the constraint equations; iii) construct the physical Hilbert space from the solutions (e.g. by the group averaging procedure); and iv) extract physics from these physical sectors (e.g., by analyzing the expectation values, fluctuations of and correlations between Dirac observables). While strategies have been developed —particularly through Thiemann’s ‘Master constraint program’\cite{17}— to complete these steps, important open issues remain in the full theory. However, as section III illustrates, the program has been completed in mini and midi superspace models, leading to surprising insights and answers to some long-standing questions.

A more detailed treatment of quantum geometry along the lines presented here can be found in, e.g.,\cite{2}.

III. APPLICATION: HOMOGENEOUS ISOTROPIC COSMOLOGY

As emphasized in Sec. I, a central feature of general relativity is its encoding of the gravitational field in the Riemannian geometry of space-time. This encoding is directly responsible for the most dramatic ramifications of the theory: the big-bang, black holes and gravitational waves. However, it also leads one to the conclusion that space-time itself must end and classical physics must come to a halt at the big-bang and black hole singularities. A central question is whether the situation improves when gravity is treated quantum mechanically. Analysis of models within LQG strongly suggests that the answer is in the affirmative. Because of space limitation, I will restrict myself to the big bang singularity and that too only in the simplest setting of homogeneous, isotropic cosmology.

Let us begin with a short list of long-standing questions that any satisfactory quantum gravity theory is expected to answer:

• How close to the Big Bang does a smooth space-time of general relativity make sense? In particular, can one show from first principles that this approximation is valid at the onset of inflation?
Is the Big-Bang singularity naturally resolved by quantum gravity? Or, is some external input such as a new principle or a boundary condition at the Big Bang essential?

Is the quantum evolution across the ‘singularity’ deterministic? Since one needs a fully non-perturbative framework to answer this question in the affirmative, in the Pre-Big-Bang \[^{18}\] and Ekpyrotic/Cyclic \[^{19, 20}\] scenarios, for example, so far the answer is in the negative.

If the singularity is resolved, what is on the ‘other side’? Is there just a ‘quantum foam’, far removed from any classical space-time, or, is there another large, classical universe?

For many years, these and related issues had been generally relegated to the ‘wish list’ of what one would like the future, satisfactory quantum gravity theory to eventually address. However, Since LQG is a background independent, non-perturbative approach, it is well-suited to address them. Indeed, starting with the seminal work of Bojowald some five years ago \[^{21}\], notable progress has been made in the context of symmetry reduced, minisuperspaces. In this section I will summarize the state of the art, emphasizing recent developments. For a comprehensive review of the older work see, e.g., \[^{22}\].

Consider the spatially homogeneous, isotropic, $k=0$ cosmologies with a massless scalar field. It is instructive to focus on this model because every of its classical solutions has a singularity. There are two possibilities: In one the universe starts out at the big bang and expands, and in the other it contracts into a big crunch. The question is if this unavoidable classical singularity is naturally tamed by quantum effects. This issue can be analyzed in the geometrodynamical framework used in older quantum cosmology. Unfortunately, the answer turns out to be in the negative. For example, if one begins with a semi-classical state representing an expanding classical universe at late times and evolves it back via the Wheeler-DeWitt equation, one finds that it just follows the classical trajectory into the big bang singularity \[^{25, 26}\].

In loop quantum cosmology (LQC), the situation is very different \[^{24, 25, 26}\]. This may seem surprising at first. For, the system has only a finite number of degrees of freedom and von Neumann’s theorem assures us that, under appropriate assumptions, the resulting quantum mechanics is unique. The only remaining freedom is factor-ordering and this is generally insufficient to lead to qualitatively different predictions. However, for reasons we will now explain, LQC does turn out to be qualitatively different from the Wheeler-DeWitt theory \[^{23}\].

Because of spatial homogeneity and isotropy, one can fix a fiducial (flat) triad $\varpi_i^a$ and its dual co-triad $\omega_i^a$. The SU(2) gravitational spin connection $A_i^a$ used in LQG has only one component $c$ which furthermore depends only on time; $A_i^a = c \, \omega_i^a$. Similarly, the triad $E_i^a$ (of density weight 1) has a single component $p$; $E_i^a = p \, (\det \omega) \, \varpi_i^a$. $p$ is related to the scale factor $a$ via $a^2 = |p|$. However, $p$ is not restricted to be positive; under $p \to -p$ the metric remains unchanged but the spatial triad flips the orientation. The pair $(c, p)$ is ‘canonically conjugate’ in the sense that the only non-zero Poisson bracket is given by:

$$\{c, p\} = \frac{8 \pi G \gamma}{3}.$$

\[(3.1)\]
where as before \( \gamma \) is the Barbero-Immirzi parameter.

Since a precise quantum mechanical framework was not available for full geometrodynamics, in the Wheeler-DeWitt quantum cosmology one focused just on the reduced model, without the benefit of guidance from the full theory. A major difference in LQC is that although the symmetry reduced theory has only a finite number of degrees of freedom, quantization is carried out by closely mimicking the procedure used in full LQG, outlined in section III. Key differences between LQC and the older Wheeler-DeWitt theory can be traced back to this fact.

Recall that in full LQG diffeomorphism invariance leads one to a specific kinematical framework in which there are operators \( \hat{\mathcal{H}} \) representing holonomies and \( \hat{P}_{S,f} \) representing (smeared) momenta but there is no operator(-valued distribution) representing the connection \( A \) itself [8, 9]. In the cosmological model now under consideration, it is sufficient to evaluate holonomies along segments \( \mu e^a_i \) of straight lines determined by the fiducial triad \( e^a_i \). These holonomies turn out almost periodic functions of \( c \), i.e. are of the form \( N(\mu)(c) := \exp i\mu(c/2) \). (The \( N(\mu) \) are the LQC analogs of the spin-network functions of LQG.) In the quantum theory, then, we are led to a representation in which operators \( \hat{N}(\mu) \) and \( \hat{p} \) are well-defined, but there is no operator corresponding to the connection component \( c \). This seems surprising because our experience with quantum mechanics suggests that one should be able to obtain the operator analog of \( c \) by differentiating \( \hat{N}(\mu) \) with respect to the parameter \( \mu \). However, in the representation of the basic quantum algebra that descends to LQC from full LQG, although the \( \hat{N}(\mu) \) provide a 1-parameter group of unitary transformations, the group fails to be weakly continuous in \( \mu \). Therefore one can not differentiate and obtain the operator analog of \( c \).

In quantum mechanics, this would be analogous to having well-defined (Weyl) operators corresponding to the classical functions \( \exp i\mu x \) but no operator \( \hat{x} \) corresponding to \( x \) itself. This violates one of the assumptions of the von-Neumann uniqueness theorem. New representations then become available which are inequivalent to the standard Schrödinger one. In quantum mechanics, these representations are not of direct physical interest because we need the operator \( \hat{x} \). In LQC, on the other hand, full LQG naturally leads us to a new representation, i.e., to new quantum mechanics. This theory is inequivalent to the Wheeler-DeWitt type theory already at a kinematical level. In particular, just as we are led to complete the space \( \mathcal{A} \) of smooth connections to the space \( \hat{\mathcal{A}} \) of generalized connections in LQG, in LQC we are led to consider the Bohr compactification \( \hat{\mathcal{R}}_{\text{Bohr}} \) of the ‘\( c \)-axis’. The gravitational Hilbert space is now \( L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}}) \), rather than the standard \( L^2(\mathbb{R}, d\mu) \) used in the Wheeler-DeWitt theory [23], where \( d\mu_{\text{Bohr}} \) is the LQC analog of the measure \( d\mu \) selected by the uniqueness results [8, 9] in full LQG. While in the semi-classical regime LQC is well approximated by the Wheeler-DeWitt theory, important differences manifest themselves at the Planck scale. These are the hallmarks of quantum geometry [2, 22].

The new representation also leads to a qualitative difference in the structure of the Hamiltonian constraint operator: the gravitational part of the constraint is a difference operator, rather than a differential operator as in the Wheeler-DeWitt theory. The derivation can be summarized briefly as follows. In the classical theory, the gravitational part of the constraint is given by \( \int d^3x e^{ik} e^{-1} E_i^a E_j^b F_{abk} \) where \( e = |\det E|^{1/2} \) and \( F_{ab} \) the
curvature of the connection $A_i^a$. The part $\epsilon^{ijk}e^{-1}E_i^aE_j^b$ of this operator involving triads can be quantized [21, 23] using a standard procedure introduced by Thiemann in the full theory [4]. However, since there is no operator corresponding to the connection itself, one has to express $F_{ab}^k$ as a limit of the holonomy around a loop divided by the area enclosed by the loop, as the area shrinks to zero. Now, quantum geometry tells us that the area operator has a minimum non-zero eigenvalue, $\Delta$, and in the quantum theory it is natural to shrink the loop only till it attains this minimum. There are two ways to implement this idea in detail (see [23, 25, 26]). In both cases, it is the existence of the ‘area gap’ $\Delta$ that leads one to a difference equation. So far, most of the LQC literature has used the first method [23, 25]. In the resulting theory, the classical big-bang is replaced with a quantum bounce with a number of desirable features. However, it also has one serious drawback: at the bounce, matter density can be low even for physically reasonable choices of quantum states (for details, see [25]); i.e. the theory predicts certain departures from classical general relativity even in the low curvature regime. The second and more recently discovered method [26] cures this problem while retaining the physically appealing features of the first and, furthermore, has a more direct motivation. Due to space limitation, I will confine myself only to the second method.

Let us represent states as functions $\Psi(v, \phi)$, where $\phi$ is the scalar field and the dimensionless real number $v$ represents geometry. Specifically, $|v|$ is the eigenvalue of the operator $\hat{V}$ representing volume (essentially the cube of the scale factor):

$$\hat{V}|v\rangle = K \left( \frac{8\pi\gamma}{6} \right)^{\frac{1}{2}} |v| |\ell_\text{Pl}|^3 |v\rangle$$

where $K = \frac{3\sqrt{3\sqrt{3}}}{2\sqrt{2}}$ (3.2)

Then, the LQC Hamiltonian constraint assumes the form:

$$\partial^2_\phi \Psi(v, \phi) = [B(v)]^{-1} \left( C^+(v) \Psi(v + 4, \phi) + C^0(v) \Psi(v, \phi) + C^-(v) \Psi(v - 4, \phi) \right)$$

$$= -\Theta \Psi(v, \phi)$$

(3.3)

where the coefficients $C^\pm(v)$, $C^0(V)$ and $B(v)$ are given by:

$$C^+(v) = \frac{3\pi K G}{8} |v + 2| \left| |v + 1| - |v + 3| \right|$$

$$C^-(v) = C^+(v - 4) \quad \text{and} \quad C^0(v) = -C^+(v) - C^-(v)$$

$$B(v) = \left( \frac{3}{2} \right)^3 K |v| \left| |v + 1|^{1/3} - |v - 1|^{1/3} \right|^3.$$ 

(3.4)

Now, in each classical solution, $\phi$ is a globally monotonic function of time and can therefore be taken as the dynamical variable representing an internal clock. In quantum theory there is no space-time metric, even on-shell. But since the quantum constraint (3.3) dictates how $\Psi(v, \phi)$ ‘evolves’ as $\phi$ changes, it is convenient to regard the argument $\phi$ in $\Psi(v, \phi)$ as emergent time and $v$ as the physical degree of freedom. A complete set of Dirac observables is then provided by the constant of motion $\hat{p}_\phi$ and operators $\hat{v}|_{\phi_0}$ determining the value of $v$ at the ‘instant’ $\phi = \phi_0$. 
Physical states are the (suitably regular) solutions to Eq (3.3). The map $\hat{\Pi}$ defined by $\hat{\Pi} \Psi(v, \phi) = \Psi(-v, \phi)$ corresponds just to the flip of orientation of the spatial triad (under which geometry remains unchanged); $\hat{\Pi}$ is thus a large gauge transformation on the space of solutions to Eq. (3.3). One is therefore led to divide physical states into sectors, each providing an irreducible, unitary representation of this gauge symmetry. Physical considerations [25, 26] imply that we should consider the symmetric sector, with eigenvalue +1 of $\hat{\Pi}$.

To endow this space with the structure of a Hilbert space, one can proceed along one of two paths. In the first, one defines the action of the Dirac observables on the space of suitably regular solutions to the constraints and selects the inner product by demanding that these operators be self-adjoint [27]. A more systematic procedure is the ‘group averaging method’ [28]. The technical implementation [25, 26] of both these procedures is greatly simplified by the fact that the difference operator $\Theta$ on the right side of (3.3) is independent of $\phi$ and can be shown to be self-adjoint and positive definite on the Hilbert space $L^2(\mathbb{R}^\text{Bohr}, B(v) d\mu_\text{Bohr})$.

The final result can be summarized as follows. Since $\Theta$ is a difference operator, the physical Hilbert space $H_\text{phy}$ has sectors $H_\epsilon$ which are superselected; $H_\text{phy} = \bigoplus_\epsilon H_\epsilon$ with $\epsilon \in (0, 2)$. The overall predictions are insensitive to the choice of a specific sector (for details, see [25, 26]). States $\Psi(v, \phi)$ in $H_\epsilon$ are symmetric under the orientation inversion $\hat{\Pi}$ and have support on points $v = |\epsilon| + 4n$ where $n$ is an integer. Wave functions $\Psi(v, \phi)$ in a generic sector solve (3.3) and are of positive frequency with respect to the ‘internal time’ $\phi$: they satisfy the ‘positive frequency’ square root
\[
-i \partial_\phi \Psi = \sqrt{\Theta} \Psi.
\]
of Eq (3.3). The physical inner product is given by:
\[
\langle \Psi_1 | \Psi_2 \rangle = \sum_{v \in \{ |\epsilon| + 4n \}} B(v) \bar{\Psi}_1(v, \phi_o) \Psi_2(v, \phi_o)
\]
and is ‘conserved’, i.e., is independent of the ‘instant’ $\phi_o$ chosen in its evaluation. On these states, the Dirac observables act in the expected fashion:
\[
\hat{p}_\phi \Psi = -i \hbar \partial_\phi \Psi
\]
\[
\hat{v}|_{\phi_o} \Psi(v, \phi) = e^{i \sqrt{\Theta}(\phi - \phi_o)} v \Psi(v, \phi_o)
\]

To construct semi-classical states and for numerical simulations, it is convenient to express physical states as linear combinations of the eigenstates of $\hat{p}_\phi$ and $\Theta$. To carry out this step, it is convenient to consider the Wheeler-DeWitt theory first. Let us begin with the observation that, for $v \gg 1$, there is a precise sense [26] in which the difference operator $\Theta$ approaches the Wheeler DeWitt differential operator $\Theta_0$, given by
\[
\Theta_0 \Psi(v, \phi) = 12\pi G \left( v \partial_v\left( v \partial_v \Psi(v, \phi) \right) \right)
\]
Thus, if one ignores the quantum geometry effects, Eq (3.3) reduces to the Wheeler-DeWitt equation
\[
\partial_\phi^2 \Psi = -\Theta \Psi.
\]
Note that the operator $\Theta$ is positive definite and self-adjoint on the Hilbert space $L^2_s(\mathbb{R}, \mathbb{B}(v)dv)$ where the subscript $s$ denotes the restriction to the symmetric eigenspace of $\Pi$ and $\mathbb{B}(v) := K^{-1}$ is the limiting form of $B(v)$ for large $v$. Its eigenfunctions $\xi_k$ with eigenvalue $\omega^2(\geq 0)$ are 2-fold degenerate on this Hilbert space. Therefore, they can be labeled by a real number $k$:

$$\xi_k(v) := \frac{1}{\sqrt{2\pi}} e^{ik\ln|v|}$$

(3.10)

where $k$ is related to $\omega$ via $\omega = \sqrt{12\pi G|k|}$. They form an orthonormal basis on $L^2_s(\mathbb{R}, \mathbb{B}(v)dv)$. A ‘general’ positive frequency solution to (3.9) can be written as

$$\Psi(v, \phi) = \int_{-\infty}^{\infty} dk \tilde{\Psi}(k) \xi_k(v) e^{i\omega\phi}$$

(3.11)

for suitably regular $\tilde{\Psi}(k)$.

The complete set of eigenfunctions $e_k(v)$ of the discrete operator $\Theta$ is also labelled by a real number $k$ and detailed numerical simulations show that $e_k(v)$ are well-approximated by $\xi_k(v)$ for $v \gg 1$. The eigenvalues $\omega^2(k)$ of $\Theta$ are again given by $\omega = \sqrt{12\pi G|k|}$. Finally, the $e_k(v)$ satisfy the standard orthonormality relations $\langle e_k | e_{k'} \rangle = \delta(k, k')$. A physical state $\Psi(v, \phi)$ can therefore be expanded as:

$$\Psi(v, \phi) = \int_{-\infty}^{\infty} dk \tilde{\Psi}(k) e^{(s)}_k(v) e^{i\omega(k)\phi}$$

(3.12)

where $\tilde{\Psi}(k)$ is any suitably regular function of $k$, and $e^{(s)}_k(v) = (1/\sqrt{2})(e_k(v)+e_k(-v))$. Thus, as in the Wheeler-DeWitt theory, each physical state is characterized by a free function $\tilde{\Psi}(k)$. The difference between the two theories lies in the functional forms of the eigenfunctions $e_k(v)$ of $\Theta$ and $\xi_k(v)$ of $\Theta$. While $e_k(v)$ is well approximated by $\xi_k(v)$ for large $v$, the differences are very significant for small $v$ and they lead to very different dynamics.

With the physical Hilbert space and a complete set of Dirac observables at hand, we can now construct states which are semi-classical at late times — e.g., now — and evolve them numerically ‘backward in time’. There are three natural constructions to implement this idea in detail, reflecting the freedom in the notion of semi-classical states. In all cases, the main results are the same [25, 26]. Here I will report on the results obtained using the strategy that brings out the contrast with the Wheeler-DeWitt theory most sharply.

As noted before, $p_\phi$ is a constant of motion. For the semi-classical analysis, we are led to choose a large value $p_\phi^* (\gg \sqrt{G}\hbar)$. In the closed model, for example, this condition is necessary to ensure that the universe can expand out to a macroscopic size. Fix a point $(v^*, \phi_o)$ on the corresponding classical trajectory which starts out at the big bang and then expands, choosing $v^* \gg 1$. We want to construct a state which is peaked at $(v^*, p_\phi^*)$ at a ‘late initial time’ $\phi = \phi_o$ and follow its ‘evolution’ backward. At ‘time’ $\phi = \phi_o$, consider then the function

$$\Psi(v, \phi_o) = \int_{-\infty}^{\infty} dk \tilde{\Psi}(k) \xi_k(v) e^{i\omega(\phi_o-\phi^*)}, \quad \text{where} \quad \tilde{\Psi}(k) = e^{-\frac{(k-k^*)^2}{2\sigma^2}}$$

(3.13)
FIG. 1: The figure on left shows the absolute value of the wave function $\Psi$ as a function of $\phi$ and $v$. Being a physical state, $\Psi$ is symmetric under $v \rightarrow -v$. The figure on the right shows the expectation values of Dirac observables $\hat{v}\phi$ and their dispersions. They exhibit a quantum bounce which joins the contracting and expanding classical trajectories marked by fainter lines. In this simulation, the parameters in the initial data are: $v^* = 5 \times 10^4$, $p^*_\phi = 5 \times 10^3\sqrt{G\hbar}$ and $\Delta p\phi/p\phi = 0.0025$.

where $k^* = -p^*_\phi/\sqrt{12\pi G\hbar^2}$ and $\phi^* = -\sqrt{1/12\pi G \ln(v^*)} + \phi_o$. In the Wheeler-DeWitt theory one can easily evaluate the integral in the approximation $|k^*| \gg 1$ and calculate mean values of the Dirac observables and their fluctuations. One finds that, as required, the state is sharply peaked at values $v^*, p^*_\phi$. The above construction is closely related to that of coherent states in non-relativistic quantum mechanics. The main difference is that the observables of interest are not $v$ and its conjugate momentum but rather $v$ and $p_\phi$ — the momentum conjugate to ‘time’, i.e., the analog of the Hamiltonian in non-relativistic quantum mechanics. Now, one can evolve this state backwards using the Wheeler-DeWitt equation (3.9). It follows immediately from the form (3.11) of the general solution to (3.9) and the fact that $p_\phi$ is large that this state would remain sharply peaked at the chosen classical trajectory and simply follow it into the big-bang singularity.

In LQC, we can use the restriction of (3.13) to points $v = |\epsilon| + 4n$ as the initial data and evolve it backwards numerically. Now the evolution is qualitatively different (see Fig.1). The state does remains sharply peaked at the classical trajectory till the matter density reaches a critical value:

$$\rho_{\text{crit}} = \frac{\sqrt{3}}{16\pi^2\gamma^3G^2\hbar},$$

which is about 0.82 times the Planck density. However, then it bounces. Rather than following the classical trajectory into the singularity as in the Wheeler-DeWitt theory, the state ‘turns around’. What is perhaps most surprising is that it again becomes semi-classical
and follows the ‘past’ portion of a classical trajectory, again with \( p_{\phi} = p_{\phi}^* \), which was headed towards the big crunch. Let us we summarize the forward evolution of the full quantum state. In the distant past, the state is peaked on a classical, contracting pre-big-bang branch which closely follows the evolution dictated by Friedmann equations. But when the matter density reaches the Planck regime, quantum geometry effects become significant. Interestingly, they make gravity repulsive, not only halting the collapse but turning it around; the quantum state is again peaked on the classical solution now representing the post-big-bang, expanding universe. Since this behavior is so surprising, a very large number of numerical simulations were performed to ensure that the results are robust and not an artifact of the special choices of initial data or of the numerical methods used to obtain the solution \([25, 26]\).

For states which are semi-classical at late times, the numerical evolution in exact LQC can be well-modelled by an effective, modified Friedman equation:

\[
\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho \left[ 1 - \frac{\rho}{\rho_{\text{crit}}} \right] \tag{3.15}
\]

where, as usual, \( a \) is the scale factor. In the limit \( \hbar \to 0 \), \( \rho_{\text{crit}} \) diverges and we recover the standard Friedmann equation. Thus the second term is a genuine quantum correction. Eq. (3.15) can also be obtained analytically from (3.3) by a systematic procedure \([29]\). But the approximations involved are valid only well outside the Planck domain. It is therefore surprising that the bounce predicted by the exact quantum equation (3.3) is well approximated by a naive extrapolation of (3.15) across the Planck domain. While there is some understanding of this seemingly ‘unreasonable success’ of the effective equation (3.15), further work is needed to fully understand this issue.

Finally let us return to the questions posed in the beginning of this section. In the model, LQC has been able to answer all of them. One can deduce from first principles that classical general relativity is an excellent approximation till very early times, including the onset of inflation in standard scenarios. Yet quantum geometry effects have a profound, global effect on evolution. In particular, the singularity is naturally resolved without any external input and there is a classical space-time also in the pre-big-bang branch. LQC provides a deterministic evolution which joins the two branches.

**IV. DISCUSSION**

Even though there are several open issues in the formulation of full quantum dynamics in LQG, detailed calculations in simple models have provided hints about the general structure. It appears that the most important non-perturbative effects arise from the replacement of the local curvature term \( F_{ab}^i \) by non-local holonomies. This non-locality is likely to be a central feature of the full LQG dynamics. In the cosmological model considered in section \([III]\) it is this replacement of curvature by holonomies that is responsible for the subtle but crucial differences between LQC and the Wheeler-DeWitt theory.\(^2\)

\(^2\) Because early presentations emphasized the difference between \( B(v) \) of LQC and \( \tilde{B}(v) = K v^{-1} \) of the Wheeler-DeWitt theory, there is a misconception in some circles that the difference in quantum dynamics
By now a number of mini-superspaces and a few midi-superspaces have been studied in varying degrees of detail. In all cases, the classical, space-like singularities are resolved by quantum geometry provided one treats the problem non-perturbatively. For example, in anisotropic mini-superspaces, there is a qualitative difference between perturbative and non-perturbative treatments: if anisotropies are treated as perturbations of a background isotropic model, the big-bang singularity is not resolved while if one treats the whole problem non-perturbatively, it is resolved.

A qualitative picture that emerges is that the non-perturbative quantum geometry corrections are ‘repulsive’. While they are negligible under normal conditions, they dominate when curvature approaches the Planck scale and halt the collapse that would classically have lead to a singularity. In this respect, there is a curious similarity with the situation in the stellar collapse where a new repulsive force comes into play when the core approaches a critical density, halting further collapse and leading to stable white dwarfs and neutron stars. This force, with its origin in the Fermi-Dirac statistics, is associated with the quantum nature of matter. However, if the total mass of the star is larger than, say, 5 solar masses, classical gravity overwhelms this force. The suggestion from LQC is that near Planck densities a new repulsive force is induced that is associated with the quantum nature of geometry may come into play which is strong enough to prevent the formation of singularities irrespective of how large the mass is. Since this force is negligible until one enters the Planck regime, predictions of classical relativity on the formation of trapped surfaces, dynamical and isolated horizons would still hold. But assumptions of the standard singularity theorems would be violated. There may be no singularities, no abrupt end to space-time where physics stops. Non-perturbative, background independent quantum physics could continue.

The major weakness of the current status of LQG is that so far one has been able to obtain detailed dynamical predictions only in symmetry reduced models. These results do provide valuable hints for the full theory but a large number of ambiguities still remain there. A fascinating question is whether the singularity resolution due to quantum geometry is a rather general feature which is largely insensitive to these ambiguities. When matter satisfies the appropriate energy conditions in general relativity, the Raychaudhuri equation captures the attractive nature of gravity in a particularly convenient fashion, providing a central ingredient to the singularity theorems. Is there a general equation in quantum geometry which implies that gravity effectively becomes repulsive near generic space-like singularities, thereby halting the classical collapse? If so, one could construct robust arguments, establishing general ‘singularity resolution theorems’ for broad classes of situations in quantum gravity, without having to analyze models, one at a time.

---

is primarily due to the non-trivial ‘inverse volume’ operator of LQC. This is not correct. In deed, in the model considered here, qualitative features of quantum dynamics, including the bounce, remain unaffected if one replaces by hand $B(v)$ with $\overline{B}(v)$ in the LQC evolution equation.
Acknowledgments: I would like to thank Martin Bojowald, Jerzy Lewandowski, and especially Tomasz Pawlowski and Parampreet Singh for collaboration and numerous discussions. This work was supported in part by the NSF grants PHY-0354932 and PHY-0456913, the Alexander von Humboldt Foundation, the Krammers Chair program of the University of Utrecht and the Eberly research funds of Penn State.

[1] A. Ashtekar, Gravity and the Quantum, New J. Phys. 7 (2005) 198; arXiv:gr-qc/0410054.
[2] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A status report, Class. Quant. Grav. 21, R53-R152 (2004), arXiv:gr-qc/0404018.
[3] C. Rovelli Quantum Gravity, (CUP, Cambridge, 2004).
[4] T. Thiemann, Introduction to Modern Canonical Quantum General Relativity (CUP, Cambridge, at press).
[5] O. Luscher and M. Reuter, Asymptotic Safety in Quantum Einstein Gravity: nonperturbative renormalizability and fractal spacetime structure, arXiv:hep-th/0511260.
[6] R. Percacci and D. Perini, Asymptotic Safety of Gravity Coupled to Matter, Phys. Rev. D68, 044018 (2003).
[7] A. Ashtekar, New variables for classical and quantum gravity Phys. Rev. Lett. 57 2244-2247 (1986)
New Hamiltonian formulation of general relativity Phys. Rev. D36, 1587-1602 (1987).
[8] J. Lewandowski, A. Okolow, H. Sahlmann and T. Thiemann, Uniqueness of diffeomorphism invariant states on holonomy flux algebras, arXiv:gr-qc/0504147.
[9] C. Fleischhack, Representations of the Weyl algebra in quantum geometry, arXiv:math-ph/0407006.
[10] A. Ashtekar and J. Lewandowski, Representation theory of analytic holonomy algebras, in Knots and Quantum Gravity, ed J. Baez, (Oxford U. Press, Oxford, 1994)
Projective techniques and functional integration, Jour. Math. Phys. 36, 2170-2191 (1995)
A. Ashtekar and J. Lewandowski, Differential geometry on the space of connections using projective techniques Jour. Geo. & Phys. 17, 191–230 (1995)
[11] J. C. Baez, Generalized measures in gauge theory, Lett. Math. Phys. 31, 213-223 (1994).
[12] D. Marolf and J. Mourão, On the support of the Ashtekar-Lewandowski measure, Commun. Math. Phys. 170, 583-606 (1995).
[13] C. Rovelli and L. Smolin, Spin networks and quantum gravity Phys. Rev. D52 5743–5759 (1995).
[14] J. C. Baez, Spin networks in non-perturbative quantum gravity, in The Interface of Knots and Physics ed Kauffman L (American Mathematical Society, Providence) pp. 167–203 (1996).
[15] C. Rovelli and L. Smolin, Discreteness of area and volume in quantum gravity Nucl. Phys. B442 593–622 (1995); Erratum: Nucl. Phys. B456 753 (1995).
[16] A. Ashtekar and J. Lewandowski, Quantum theory of geometry I: Area operators, Class. Quant. Grav. 14 A55–A81 (1997)
Quantum theory of geometry II: Volume Operators Adv. Theo. Math. Phys. 1 388–429 (1997).
[17] T. Thiemann, The phoenix project: Master constraint program for loop quantum gravity, arXiv:gr-qc/0305080; T. Thiemann and B. Dittrich, Testing the master constraint program for loop quantum gravity I. General Framework, Class. Quantum. Grav. 23 1025-1066 (2006).

[18] M. Gasperini and G. Veneziano, The pre-big bang scenario in string cosmology, Phys. Rept. 373, 1 (2003) arXiv:hep-th/0207130.

[19] J. Khoury, B. A. Ovrut, P. J. Steinhardt, N. Turok, The Ekpyrotic Universe: Colliding Branes and the Origin of the Hot Big Bang, Phys. Rev. D64 (2001) 123522 hep-th/0103239.

[20] J. Khoury, B. . Ovrut, N. Seiberg, P. J. Steinhardt, N. Turok, From Big Crunch to Big Bang, Phys.Rev. D65 (2002) 086007 hep-th/0108187.

[21] M. Bojowald, Absence of singularity in loop quantum cosmology, Phys. Rev. Lett. 86, 5227-5230 (2001), arXiv:gr-qc/0102069. Isotropic loop quantum cosmology, Class. Quantum. Grav. 19, 2717-2741 (2002), arXiv:gr-qc/0202077.

[22] M. Bojowald, Loop quantum cosmology, Liv. Rev. Rel. 8, 11 (2005), arXiv:gr-qc/0601085.

[23] A. Ashtekar, M. Bojowald, J. Lewandowski, Mathematical structure of loop quantum cosmology, Adv. Theo. Math. Phys. 7, 233-268 (2003), gr-qc/0304074.

[24] A. Ashtekar, T. Pawlowski and P. Singh, Quantum Nature of the Big Bang, Phys. Rev. Lett. 96, 141301 (2006) arXiv:gr-qc/0602086.

[25] A. Ashtekar, T. Pawlowski and P. Singh, Quantum Nature of the Big Bang: An analytical and numerical Investigation, Phys. Rev. D73, 124038 (2006) arXiv:gr-qc/0604013.

[26] A. Ashtekar, T. Pawlowski and P. Singh, Quantum Nature of the Big Bang: Improved dynamics. gr-qc/0607039.

[27] A. Ashtekar, Lectures on non-perturbative canonical gravity, Notes prepared in collaboration with R. S. Tate (World Scientific, Singapore, 1991), Chapter 10.

[28] D. Marolf, Refined algebraic quantization: Systems with a single constraint arXives:gr-qc/9508015

Quantum observables and recollapsing dynamics, Class. Quant. Grav. 12 (1995) 1199-1220.

[29] J. Willis, On the low energy ramifications and a mathematical extension of loop quantum gravity, Ph.D. Dissertation, The Pennsylvania State University (2004); A. Ashtekar, M. Bojowald and J. Willis, Corrections to Friedmann equations induced by quantum geometry, IGPG preprint (2004).

[30] M. Bojowald, H. H. Hernandez and H. A. Morales-Tecotl, Perturbative degrees of freedom in loop quantum gravity: Anisotropies, Class. Quant. Grav., 18 L117–L127 (2001) arXiv:gr-qc/0511058.

[31] A. Ashtekar and B. Krishnan Isolated and dynamical horizons and their applications, Living Rev. Rel. 10 1-78 (2004), gr-qc/0407042

[32] N. Dadhich, Singularity: Is Raychaudhuri equation required once again? to appear in: Raychaudhuri Memorial Volume.