Introduction to quantum matter

Frank Wilczek

Center for Theoretical Physics and Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail: wilczek@mit.edu

Received 16 August 2011
Accepted for publication 19 September 2011
Published 31 January 2012
Online at stacks.iop.org/PhysScr/T146/014001

Abstract
This paper records my introductory remarks at Nobel Symposium 148 on Quantum Matter and Graphene at Saltsjöbaden, Sweden, in June 2010. After some broad comments on the quantum theory of matter as a frontier of physics, and some slightly more particular comments on re-quantization, I report on the universal geometry that arises in a refined discussion of quantum-mechanical level crossing.

PACS numbers: 03.65.Vf, 32.80.Xx, 33.80.Be

The quantum theory of matter is still young. It has been over 100 years since its birth in the work of Max Planck; very close to 100 years since Bohr’s atomic model, which made it plausible that quantum theory held the key to understanding the deep structure of matter; and about 85 years since the foundational ideas of quantum mechanics were proposed by Heisenberg, Schrödinger, and other greats. While those ideas have been augmented with many new ones, and fruitfully applied in many directions, their essence remains unchanged. Yet such is their depth, power and strangeness that they continue to conjure up surprises and to tantalize us with their potential.

This Nobel Symposium is dedicated to the quantum theory of matter in general, with special emphasis on a spectacular success story of recent years: the discovery and exploration of graphene.

This introductory talk has three parts. I will start with a few general reflections on the quantum theory of matter, emphasizing some recently emerging themes. Then I will briefly introduce a remarkable class of developments that are close to my heart: the embodiment, in materials, of concepts—and even equations—that were studied earlier for their possible application to fundamental physics, or simply for their intrinsic beauty. Finally, to exhibit one distilled essence of that theme, I will go into some detail about a basic problem in elementary quantum mechanics, the problem of level crossing, and show how vortices, monopoles and even SU(2) instantons emerge in that context.

1. Quantum phenomenology and quantum engineering

The first task of quantum theory, as of all scientific theories, is to describe the natural world. The most basic ideas of quantum theory, including wave functions, entanglement (implicit already in having wave functions in configuration space), superposition, commutation relations, the Schrödinger equation, quantized angular momentum and spin, and Bose–Einstein and Fermi–Dirac statistics, emerged from confrontations with concrete phenomena of radiation and atomic spectroscopy. Fundamental insights and techniques for deploying those concepts, notably including the application of symmetry arguments and group theory, emerged from engagement with more intricate facts of spectroscopy (including molecular and nuclear spectroscopy), chemistry and the theory of solids.

Phenomenology continues to pose new challenges for the quantum theory of matter. Many specific issues remain outstanding, including the problem of strange metals and high-temperature superconductivity, and puzzling marvels from the biological world, notably photosynthesis. There is also a broader and more general challenge. There is every reason to think that for any practical question of chemistry and materials science we can write down appropriate equations, whose solution should provide the answer. In that precise and powerful—if ultimately limited—sense, the ‘theory of everything’ is in our grasp. The challenge, of course, is to solve the equations in useful detail, in as many contexts as possible.

After the natural world, a second frontier beckons. We need not be content with the world as we find it. We can...
build on our insights to imagine, and construct, materials and devices that Nature does not provide.

Semiconductor and microelectronic technology, laser technology and molecular design would be inconceivable without modern quantum theory. It is fair to call them examples of quantum engineering. Modern quantum theory underlies the rapid rise of graphene, and also fullerences and nanotubes. The main properties of that material, including its strength, remarkable band structure (and that of its bilayer variant), signature quantum Hall effect and others, are direct and fairly straightforward consequences of basic quantum theory; indeed they were largely anticipated in advance of graphene’s experimental discovery.

Quantum engineering, clearly, is already a diverse and highly successful enterprise. But it is still in its infancy. The Hilbert space of many-body systems is extremely large, and at present we can only reach small corners of it, with limited navigational control. A great task for the future will be to embody the rich possibilities of entanglement within systems that we can control and that are not cripplingly fragile. Quantum simulators, or ultimately general-purpose quantum computers, might enable us to enter a virtuous cycle, whereby better solutions to the equations of quantum theory lead to better quantum machines lead to better solutions of the equations . . . . At that point, quantum theory will have entered its adolescence.

2. Artificial atoms and ethers

Well before the emergence of modern ideas about atomic structure, in the 19th century, Thomson and Tait [1], impressed with the stability of hydrodynamic vortices, proposed that atoms are vortices, or knots of vortices, in ether. They produced models of such vortex atoms experimentally, in the form of smoke rings. Tait, especially, elaborated on the idea that different elemental atoms correspond to topologically distinct knotted vortices; his detailed tabulation of knots, in pursuit of that idea, inaugurated a still-flourishing field of mathematics.

As a theory of natural atoms, those ideas were, we now know, wide of the mark. But it is easy to understand their appeal: vortices, even in the form of smoke rings, can have an impressive degree of stability; they can be knotted into topologically distinct forms, which are also quasi-stable; and their interactions are complex and intricate, yet reproducible. These would be attractive features to embody in artificial ‘atoms’, meant to be building blocks for quantum engineering.

2.1. Designer atoms and re-quantization

Nor is that possibility entirely fanciful. Vortices in type II superconductors dominate their response to magnetic fields. They are quite tangible objects that experimenters work with, manipulate and even image. Several other quasi-macroscopic objects, including notably small magnetic domains, or the walls that separate them, share some of the same characteristics. The ‘artificial ethers’ in which these objects live are also subjects for creative engineering. They can, for example, be effectively two-dimensional (2D), or even 1D, if we work below the energy gap for motion in the transverse direction.

The study of artificial atoms inside artificial ethers has already brought us many interesting surprises. Particularly interesting, I think, is the phenomenon of re-quantization. That is, the quantum numbers of elementary excitations inside artificial ethers can be different from the quantum numbers of any elementary quanta we find in the vacuum. In particular, we can find fractional electric charge, electron number or angular momentum. The root of these phenomena is mixing between ordinary and topological quantum numbers2.

In quantum electrodynamics, we learn that there is a universal factor relating bare charges, that occur in the formulation of the theory as coefficients parameterizing interactions of quantum fields, to the observed (renormalized) charges of particles. Technically, this appears as a consequence of Ward’s identity. A heuristic explanation is that the dielectric response of long-wavelength virtual particles, which governs the renormalization of charge, can only be governed by structure that extends to spatial infinity, viz the value of the conserved, integrated charge.

If, however, there is additional structure—e.g. two distinct conserved quantities—then vacuum polarization can mix things up. Consider, to be concrete, artificial atoms in artificial ethers that have conserved topological, as well as ordinary, electric charges. Then the observed charge spectrum, in terms of renormalized charges, will be influenced by both. Since the renormalizing effects due to several atoms are additive, the observed charge will take the form

\[ \frac{Q}{e} = q + \frac{1}{\pi} \arctan \beta \ \tau = q + \frac{\theta}{2\pi} \cdot \tau. \] (1)

(The equivalent forms on the right-hand side are natural in different contexts [2, 3].) Here \( q \), representing the fundamental quantum numbers created by elementary fields, is an integer. But in the presence of topological invariants, typically parameterized by discrete quantities \( \tau \), the observed charges \( \frac{Q}{e} \) will not be integers. The charge spectrum, though modified, remains highly structured. We might say charge has been re-quantized, as opposed to de-quantized.

Here the notations \( \arctan \beta, \theta \) suggest angles; since an integer can be absorbed into a shift in \( q \), the physics of (1) is periodic in \( \arctan \beta \rightarrow \arctan \beta + 2\pi \) or \( \theta \rightarrow \theta + 2\pi \). In different physical examples, several cases have arisen:

- If the topological quantum number \( \tau \) is itself an additive integer, generally \( \theta \) is simply a number, characteristic of the material and disturbances in question.
- If the topological quantum number is an integer modulo \( m \), then we must have

\[ \theta = \frac{2\pi k}{m} \] (2)

with charges

\[ \frac{Q}{e} = q + \frac{k}{m} \tau, \] (3)

for some integer \( k \); this allows \( m \) of our atoms to annihilate yielding states with conventional charges. This situation arises in the fractional quantum Hall effect.

2 This viewpoint, elaborated below, was inspired by conversations with Sidney Coleman.
• In particular, if our atoms can annihilate in pairs, so \( m = 2 \), then the only nontrivial possibility is \( k = 1 \), implying half-integer charges for \( \tau = 1 \equiv -1 \). This situation is typical of self-conjugate (e.g. \( C \) invariant) atoms. Symmetry demands that the minimal charges \( \pm \frac{1}{2} \) must be realized by degenerate states; therefore, we must have zero-energy modes connecting them. Historically, this was the first case of re-quantization to be analyzed, by Jackiw and Rebbi [4].

2.2. Designer world-lines

Still more interesting and open-ended possibilities arise when we extend consideration of artificial ethers in two dimensions, to take the dimension of time into account. For then we are dealing with 3D space-time, and in those three dimensions we can imagine the world-lines of artificial atoms tangled into knots. That thought brings us, of course, into the world of anyons (as the preceding discussion alluded to the world of topological solitons).

3. Geometry of level crossing

Frequently in quantum mechanics we are led to consider families of Hamiltonians \( H(\lambda^j) \) that depend on a parameter. For example, in the context of the Born–Oppenheimer approximation the \( \lambda^j \) might be nuclear positions; or in band theory they might be labels for the band numbers, and quasi-momenta. For each value of \( \lambda^j \) the energy eigenspaces are simply linear manifolds, but when we consider how families of such spaces fit together nontrivial geometry comes into play.

In these and other applications, level crossings play an important role. As we will see, some canonical geometric and topological structures characterize quantum-mechanical level crossings. These structures are at the root of several phenomena of current interest, including the quantum Hall effect in general, the anomalous quantum Hall effect in graphene, quantized transport in general and, specifically, quantized transport in topological insulators.

3.1. The original case

The simplest case of level crossing was analyzed in the earliest days of quantum theory by von Neumann and Wigner (in 1927) [5]. It was only many years later, in 1963, that Herzberg and Longuet-Higgins [6] pointed out an additional subtlety, which is the real beginning of our story.

In this simplest case we focus on two nearby but otherwise isolated energy levels, and suppose that the relevant Hamiltonians are real (and, of course, Hermitian). Then the eigenstates can also be taken to be real. Shedding some trivial complications, by adding \( \lambda^j \)-dependent constants to the Hamiltonians \( H(\lambda^j) \), we can suppose that the crossing occurs at zero energy and that the average energy is zero. Then near the crossing, to linear order (and of course neglecting any other levels), the possible Hamiltonians and their associated energies can be parameterized as

\[
H(x, y) = \begin{pmatrix} x & y \\ y & -x \end{pmatrix} = \gamma \sigma_1 + \lambda \sigma_3
\]

\[
= \sqrt{x^2 + y^2} e^{-i \phi} \sigma_3 e^{i \frac{\pi}{4}} \sigma_1, \quad \phi \equiv \tan^{-1} \frac{y}{x},
\]

with energy eigenvalues

\[
E_{\pm} = \pm \sqrt{x^2 + y^2}.
\]

Here \( x \) and \( y \) are real numbers. From this, two simple but important consequences follow:

(i) If only one parameter is in play, generic crossing is avoided. Crossing is a robust phenomenon only within families of Hamiltonians that support at least two parameters, since it requires \( x = y = 0 \). The jargon for this is that crossing is a codimension-two phenomenon. If we have an \( n \)-dimensional parameter space, the expectation is that crossing will occur on an \( n - 2 \)-dimensional submanifold.

(ii) Near the crossing, the energy surface is not analytic, but has a conical singularity.

The third basic property was observed by Herzberg and Longuet-Higgins. Now we consider not only the energies, but also how the wave functions depend on \( x \) and \( y \). From equation (6) we have, continuing the upper (positive-energy) eigenfunction continuously from \( \phi = 0 \),

\[
\psi_+(x, y) = e^{-i \frac{\pi}{4} \sigma_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} \cos \frac{\phi}{2} \\ \sin \frac{\phi}{2} \end{pmatrix}.
\]

Therefore:

(iii) As \( \phi \) evolves from 0 to \( 2\pi \)—as we circle around the crossing—the sign of \( \psi_+(x, y) \) reverses.

The sign change is a discrete, topological feature of the quantum geometry. Thus the crossing leaves a robust signature, which extends far away in parameter space. This structure is known in other contexts, and has a name: we have found a \( Z_2 \) vortex.

This whole set-up relied on the restriction to real wave functions (and real Hamiltonians). This restriction is appropriate for \( T \)-invariant systems where spin-dependent interactions are negligible, and no internal symmetries are in play. If we allow ourselves to use complex wave functions, it seems that the topology disappears. We could, specifically, redefine

\[
\tilde{\psi}_+(x, y) \equiv e^{i \frac{\pi}{4}} \begin{pmatrix} \cos \frac{\phi}{2} \\ \sin \frac{\phi}{2} \end{pmatrix}
\]

and have eigenfunctions that both depend smoothly on \( \phi \) and remain invariant as \( \phi \to \phi + 2\pi \). But by going a little deeper we will recover and vastly generalize the nontrivial topology suggested by our earlier, ‘more natural’ choice.
3.2. Geometric matrix

3.2.1. A general formulation. The geometric matrix or Berry phase has been discussed in the literature, including the classic [7]. Here I will give a self-contained and slick but rather abstract derivation; if the whole subject is unfamiliar, you might want to consult the literature.

The proper dynamical setting for level-crossing problems is adiabatic evolution. We consider Hamiltonians with discrete levels, some of which may be multiply degenerate. (That will be all-important later.) As the parameters vary, the values of the energies associated with levels will change, as will their position in Hilbert space, but the ordering of levels will not change, except of course at crossings, where the adiabatic approximation breaks down. Thus we consider

$$H = S \Delta S^{-1}$$

(12)

with

$$\Delta = \begin{pmatrix} E_1 & 0 & 0 & \ldots \\ 0 & E_2 & 0 & \ldots \\ 0 & 0 & E_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(13)

a block-diagonal matrix, and $S$ a unitary transformation. This is a perfectly general form for a Hermitian Hamiltonian, of course. It is understood that $S$ and $\Delta$ are functions of time.

Now defining

$$\psi(t) = \begin{pmatrix} e^{-i/h \int_0^s E_1(s) ds} & 0 & 0 & \ldots \\ 0 & e^{-i/h \int_0^s E_2(s) ds} & 0 & \ldots \\ 0 & 0 & e^{-i/h \int_0^s E_3(s) ds} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tilde{\psi}(t)$$

(14)

or more briefly

$$\psi(t) = S(t) e^{-i/h \int_0^s \Delta(s) ds} \tilde{\psi}(t)$$

(15)

and applying the Schrödinger equation to the column vector $\tilde{\psi}$, we arrive at

$$\frac{d}{dt} \tilde{\psi} = -i \hbar \frac{\partial}{\partial s} \tilde{\psi}$$

(16)

In the product of matrices on the right-hand side, the $ij$ component will contain a factor

$$e^{i \hbar / \lambda(E_i - E_j)}$$

from the $\Delta$ integrals. If $S^{-1}\dot{S}$ is slowly varying and $E_i - E_j$ is not too close to zero, the contribution of this oscillatory term, which picks out frequency $\frac{E_i - E_j}{\lambda}$, to the evolution will be highly suppressed. The adiabatic theorem in its crude form—‘no quantum jumps’—instructs us to drop such terms. That principle does not give us a unique solution, however, even for infinitely slow changes of parameters.

Defining $\Pi^{(\kappa)}$ to be the projection operators that are restricted to the $\kappa$th level of degenerate eigenvalues, the adiabatic approximation in its crude form gives us the approximate equation

$$\frac{d}{dt} \tilde{\psi}^{(\kappa)} = -i \frac{\partial}{\partial s} \Pi^{(\kappa)} \tilde{\psi}^{(\kappa)}.$$  

(17)

It is useful at this point to introduce the gauge potentials

$$A_j^{(\kappa)} = -i \frac{\partial}{\partial \lambda_j} \Pi^{(\kappa)},$$

(18)

where the $\lambda^j$ parameterize the space of transformations $S$. In terms of the $A_j^{(\kappa)}$ we can solve (17) in terms of a path-ordered integral,

$$\tilde{\psi}^{(\kappa)}(t) = P \left[ \exp \int_{S(0)}^{S(t)} d\lambda / A_j^{(\kappa)} \right] \tilde{\psi}^{(\kappa)}(0),$$

(19)

and therefore

$$\psi^{(\kappa)}(t) = S(t) e^{-i/h \int_0^s \Delta(s) ds} P \left[ \exp \int_{S(0)}^{S(t)} d\lambda / A_j^{(\kappa)} \right]$$

$$\times S^{-1}(0) \psi^{(\kappa)}(0).$$

(20)

The characteristic factor

$$C(\text{path}) = P \left[ \exp \int_{S(0)}^{S(t)} d\lambda / A_j^{(\kappa)} \right]$$

(21)

connects the eigenspaces for energy $E_\kappa$, as they vary with $\lambda$, in a manner reminiscent of Wilson lines in gauge theory or parallel transport in Riemannian geometry. In the present context, this connection is known as the ‘Berry phase’, ‘geometric phase’ or ‘geometric matrix’ (which I will use). $C$ has several remarkable properties:

- It depends only on the geometry of the embeddings, and not on how fast they are run through.
- It remains nontrivial no matter how slow the adiabatic evolution is.
- It is of order $h^0$. The energy-frequency factors go as $h^{-1}$ (inside the exponential). Higher corrections, with positive powers of $\hbar$, are nonadiabatic, in the sense that they can become arbitrarily small as the evolution is taken arbitrarily slow.

Note that in equation (20) the geometric matrix operates inside the matrix $S$. This shows that it is more properly considered as a correction to the basis geometry than as a correction to the dynamics.

3.2.2. Back to the vortex. Returning to our example, if we use the wave functions from (11) then we find that the geometric matrix factor, integrated from $\phi = 0$ to $2\pi$, restores the minus sign that the original, totally real analysis suggested. The calculation is simple:

$$A_\phi = - \left( \cos \phi \frac{\partial}{\partial \phi} + \sin \phi \frac{\partial}{\partial \phi} \right) e^{-i/2} - \frac{\partial}{\partial \phi} e^{i/2} \cos \phi \frac{\partial}{\partial \phi} \sin \phi = -i/2,$$

$$e^{i/2} \phi A_\phi = e^{-i/2}.$$

(22)

The field strength associated with this gauge potential vanishes; nevertheless it is globally nontrivial, like the
time reversal symmetry $T$ is implemented by an antunitary transformation, and one may have $T^2 = \pm 1$. Depending on the complexity type of a representation, and the sign of $T^2$, it may or may not be necessary to double the representation in order to implement $T$ (generalized Kramers theorem). Hamiltonians invariant under the group and under $T$ can connect two multiplets of the same type only in restricted ways, which can be described by a finite number of parameters. We find that in different cases the geometric matrix field describes a vortex, monopole, or instanton, as indicated.

The general situation is indicated in table 1 and its caption. Here I will just spell out the second line a bit, for the special case of identity representation.

If we want to implement $T^2 = -1$, we cannot use a 1D space, since $|\psi\rangle$ and $T|\psi\rangle$ are orthogonal:

$$
(\langle\psi| (T|\psi\rangle) = -((\langle\psi| T^{12} |\psi\rangle)
= -((\langle\psi| T^4 |\psi\rangle) (T|\psi\rangle) = -\langle\psi| (T|\psi\rangle),
$$

(23)

where in the last equation we have used the antunitarity of $T$, viz $(\langle\beta|T |\alpha\rangle) = \langle\beta|\alpha\rangle$. For a minimal implementation we can take $T = K i \tau_2$ within a 2D space, where $K$ denotes complex conjugation. (A physical motivation for $T^2 = -1$ is a sort of converse to this construction: if we want $T$ to reverse all three spin operators $\sigma_i$, and yet to include complex conjugation, we must have $T = K i \tau_2$ acting on the spin variable, so that if the rest of $T$ is ‘normal’ we will have $T^2 = -1$.)

We have two complex-dimensional multiplets, and we want to consider crossing, so we have two doublets. Altogether then we have a 4D Hilbert space, with $T$ realized as

$$
T = K i \tau_2 \otimes 1.
$$

(24)

Now we can consider the space of possible Hamiltonians, i.e. Hermitian operators, invariant under $T$. Besides the identity, we easily identify five others,

$$
\Gamma_1 = \tau_1 \otimes \tau_2,
\Gamma_2 = \tau_2 \otimes \tau_1,
\Gamma_3 = \tau_3 \otimes \tau_2,
\Gamma_4 = 1 \otimes \tau_1,
\Gamma_5 = 1 \otimes \tau_3.
$$

(25)

3.3. The general case

The crossing problem is only nontrivial for states of the same symmetry. But given that restriction, we still have many possibilities beyond the original case considered by von Neumann and Wigner. Indeed we can consider different symmetry groups and representations, and crossing problems for multiplets in each class. (Here I am summarizing some results from a forthcoming paper with Kam Tuen Law [8]. Earlier, related work includes [9, 10].)

At first glance one might expect that the geometry could get very complicated, especially for multiplets of large dimensions, and that there are many cases to consider. It is pleasant that closer analysis reveals that the complexity is relatively modest, and that there is a remarkable degree of universality in the geometry governing different multiplet situations. (For sophisticates: the mathematical fact that keeps things under control is Schur’s lemma, and the key to proving the statements that follow is character theory.) This is not the place to enter into technicalities or proof; I will just indicate the general result and exemplify one interesting case.
and one can prove that these form a basis for all invariant Hamiltonians.

As the notation is meant to suggest, there is a pretty surprise here: these Hamiltonians obey the 5D Clifford algebra

\[ \{ \Gamma_j, \Gamma_k \} = 2\delta_{jk}. \]  

Thus we see that the space of relevant Hamiltonians near a crossing is the same as in equation (5),

\[ H(x^j) = x^j \cdot \Gamma_j, \]  

but now on a 5D rather than a 2D space.

In parallel with our earlier discussion, we can read off two basic conclusions: the crossing is highly avoided—codimension five!—and the energy function has a conical singularity:

\[ E(x^j) = \pm \sqrt{x \cdot x}. \]  

The most interesting issue, intellectually, is the geometric matrix. Since the eigenfunctions (as opposed to the eigenvalues) do not depend on the magnitude of \( \vec{x} \), and we excise \( \vec{x} = 0 \), the relevant parameter space is the sphere \( S^4 \). The positive and negative eigenspaces are each 2D, so we will get a gauge potential in the group U(2). Because the underlying Clifford algebra is SO(5) invariant, so will be the gauge structure we derive from it. Hence we can anticipate that a very special gauge structure will emerge.

Detailed calculation confirms that anticipation. Parallel transport, in the sense of Riemannian geometry, gives us an (SO(5) invariant) SO(4) gauge structure on \( S^4 \). Using the Lie algebra isomorphism \( \text{SO}(4) \to \text{SU}(2) \times \text{SU}(2) \), and projecting on either factor, we get (SO(5) invariant) SU(2) gauge potentials. The geometric matrices in the positive and negative eigenspaces are governed by precisely these two SU(2) potentials. The resulting gauge structures are the (sphere versions of) instantons and anti-instantons, famous in quantum field theory and topology. Because these gauge structures are topologically nontrivial, they leave robust signatures even far from the crossing.

3.4. Comments

(i) It might seem extravagant to consider structures that are so hard to get at, i.e. with such high codimension. Where are all those parameters supposed to come from? Well, they could be the positions of quasiparticles (as occurs, especially, in anyon physics), quasimomenta in a Brillouin zone, external electric or magnetic fields, or even 'conceptual parameters' that allow us to interpolate, theoretically, between materials with fundamentally different Hamiltonians (this is an important idea in the theory of topological insulators).

(ii) One can often relate the occurrence of level crossings, as a function of pumping parameters, to transport. In that context, topological invariants associated with the level crossing can govern the discrete parameters that appear in quantized transport.

(iii) A common source of level merger—a more general version of crossing—is the occurrence of enhanced symmetry. For example, in the context of band theory one often finds enhanced symmetry at special points in the Brillouin zone. If the symmetry of the Hamiltonian, as a function of parameters \( \lambda \), is enhanced from \( H \) to \( G \) at \( \lambda_0 \), then irreducible representations of \( G \) will be formed by merging \( H \)-multiplets that are nondegenerate away from \( \lambda_0 \). Interesting geometric matrices can arise in this way—and in graphene, specifically, they do.

References

[1] Epple M 1998 Arch. Hist. Exact Sci. 52 297
[2] Goldstone J and Wilczek F 1981 Phys. Rev. Lett. 47 986
[3] Witten E 1979 Phys. Lett. B 86 283
[4] Jackiw R and Rebbi C 1976 Phys. Rev. D 13 3398
[5] von Neumann J and Wigner E 1929 Phys. Z. 30 467
[6] Herzberg G and Longuet-Higgins H C 1963 Disc. Faraday Soc. 35 77
[7] Shapere A and Wilczek F 1989 Geometric Phases in Physics (Singapore: World Scientific)
[8] Law K T and Wilczek F 2012 in preparation
[9] Avron J, Sadun L, Segert J and Simon B 1989 Commun. Math. Phys. 124 595
[10] Johnson M and Aitchison I 1997 J. Phys. A: Math. Gen. 30 2085