Abstract. In this work the $H^1$-stability of an L2 method on general nonuniform meshes is established for the subdiffusion equation. Under some mild constraints on the time step ratio $\rho_k$, for example $0.4573328 \leq \rho_k \leq 3.5615528$ for all $k \geq 2$, a crucial bilinear form associated with the L2 fractional-derivative operator is proved to be positive semidefinite. As a consequence, the $H^1$-stability of L2 schemes can be derived for the subdiffusion equation. In the special case of graded mesh, such positive semidefiniteness holds when the grading parameter $1 < r \leq 3.2016538$ and therefore the $H^1$-stability of L2 schemes holds. Based on the above analysis, a graded mesh with varying grading parameter is proposed which ensures the $H^1$-stability of L2 scheme. To the best of our knowledge, this is the first work on the $H^1$-stability of L2 method on general nonuniform meshes for subdiffusion equation.

Key words. L2-type method, subdiffusion equation, $H^1$-stability, graded mesh, positive semidefiniteness

AMS subject classifications. 35R11, 65M12

1. Introduction. The time-fractional diffusion equation was derived from continuous time random walks [18, 4], where a fractional derivative in time is introduced to model the memory effect in diffusing materials.

In the past decade, many numerical methods have been proposed to solve the time-fractional diffusion equation. Some of these methods are on the uniform time meshes. For example, the L1 scheme of $(2 - \alpha)$-order has been well-developed by Langlands and Henry [11], Sun-Wu [23], and Lin-Xu [16], etc. Alikhanov [1] proposed the L2-1-s scheme that has second order accuracy in time for the time-fractional diffusion equation with variable coefficients. An L2 method of $(3 - \alpha)$-order on uniform meshes is studied in [3] by Gao-Sun-Zhang. In [17], a slightly different L2 fractional-derivative operator is analyzed by Lv-Xu for uniform meshes, where the optimal convergence $(3 - \alpha)$-order in time is obtained under strong regularity assumptions on the exact solution.

Recently, those methods on nonuniform time meshes for time-fractional diffusion equation have attracted more and more attention, in particular, on the graded meshes. In fact, the exact solution to the time-fractional diffusion equation could have low regularity in general near the initial time, which would deteriorate the convergence rate of the numerical solutions. This motivates researchers to consider nonuniform time meshes to obtain the desired sharp convergence rate under low regularity assumptions on the exact solution. For example, Stynes-Riordan-Gracia [22] prove the sharp error analysis of L1 scheme on graded meshes. Kopteva provides a different analysis framework of the L1 scheme on graded meshes in two and three spatial dimensions in [8]. Chen-Stynes [2] prove the second-order convergence of L2-1-s scheme on fitted meshes combining the graded meshes and quasiuniform meshes. Kopteva-Meng [10] provide sharp pointwise-in-time error bounds for quasi-graded temporal meshes with
arbitrary degree of grading for L1 and L2-1 schemes. Later Kopteva generalize this sharp pointwise error analysis to an L2-type scheme on quasi-graded meshes [9]. In the case of general nonuniform meshes, Liao-Li-Zhang establish the sharp error analysis for the L1 scheme of linear reaction-subdiffusion equations in [12, 13] and then Liao-Mclean-Zhang [14] consider the L2-1 scheme.

In addition to the L1, L2-1 and L2 methods on nonuniform meshes, we shall mention that the convolution quadrature methods with corrections can also overcome the convergence rate problem for time-fractional diffusion equation, see for example [6, 7] and the references therein.

In this work, we consider the $H^1$-stability of an L2 method (the same as [9]) on general nonuniform meshes for subdiffusion equation. For the L2 fractional-derivative operator denoted by $L_{\alpha}^k$, we prove that the following bilinear form

\begin{equation}
B_n(v, w) = \sum_{k=1}^{n} \langle L_{\alpha}^k v, \delta_k w \rangle, \quad \delta_k w := w^k - w^{k-1}, \quad n \geq 1,
\end{equation}

is positive semidefinite under mild restrictions (3.9) and (3.10) on the time step ratios $\rho_k := \tau_k / \tau_{k-1}$ with $\tau_k$ the kth time step ($k \geq 2$), see Theorem 3.2 for details. Note that the positive semidefiniteness of $B_n$ on general nonuniform meshes is unknown as stated in [5, Table 1]. In particular, if $0.4573328 \leq \rho_k \leq 3.5615528$, the mild restrictions are satisfied. As a consequence, the $H^1$-stability of the implicit L2 scheme for the subdiffusion equation with homogeneous Dirichlet boundary condition can be derived for all time.

\textbf{Theorem 1.1.} Assume that $f(t, x) \in L^\infty([0, \infty); L^2(\Omega)) \cap BV([0, \infty); L^2(\Omega))$ and $u^0 \in H^1_0(\Omega)$. If the nonuniform mesh $\{\tau_k\}_{k \geq 1}$ satisfies (3.9) and (3.10), then the numerical solution $u^n$ of the implicit L2 scheme

$$
L_{\alpha}^k u = \Delta u^k + f(t_k, x) \quad \text{in} \quad \Omega, \quad u^k = 0 \quad \text{on} \quad \partial \Omega,
$$

satisfies the following $H^1$-stability

$$
\|\nabla u^n\|_{L^2(\Omega)} \leq \|\nabla u^0\|_{L^2(\Omega)} + 2C_f C_\Omega,
$$

where $C_f$ depends on the source term $f$, $C_\Omega$ is the Sobolev embedding constant depending on $\Omega$ and the dimension of space.

Moreover, in the special case of graded meshes, we show that if the grading parameter $1 < r \leq 3.2016538$, then $B_n$ is positive semidefinite and similar $H^1$-stability of L2 scheme can be established.

If the exact solution of subdiffusion equation only has low regularity, our stability result says that the L2 scheme would be at least $H^1$-stable, as soon as the time meshes satisfy constraints (3.9)–(3.10). One following issue is how to design suitable time meshes not only ensuring the $H^1$-stability of numerical solutions, but also having the sharp error bound. In [22, 8], the authors state that the large value of $r$ in the graded mesh increases the temporal mesh width near the final time $t = T$ which can lead to large errors. This inspire us to propose a new type of graded mesh, called $r$-variable graded mesh, with varying grading parameter that is large near $t = 0$ and small near $t = T$. Some first numerical tests show that this $r$-variable graded mesh could perform well. However the rigorous proof of its convergence rate needs to be further studied.
This work is organized as follows. In Section 2, the derivation, explicit expression and reformulation of L2 fractional-derivative operator are provided. In Section 3, we prove the positive semidefiniteness of the bilinear form \( B_h \) under some mild restrictions on the time step ratios. In Section 4, we establish the \( H^1 \)-stability of the L2 scheme for the subdiffusion equation, based on the positive semidefiniteness result. In Section 5, the special case of graded meshes is discussed. In Section 6, we do some first numerical tests on a new nonuniform mesh, called \( r \)-variable graded mesh, where the L2 scheme on this mesh is \( H^1 \)-stable.

2. Discrete fractional-derivative operator. In this part we show the derivation, explicit expression and reformulation of L2 operator on general nonuniform mesh.

We consider the L2 approximation of the fractional-derivative operator defined by

\[
\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} \, ds.
\]

Take a nonuniform time mesh \( 0 = t_0 < t_1 < \ldots < t_{k-1} < t_k < \ldots \) with \( k \geq 1 \). When \( k = 1 \), we use the standard linear Lagrangian polynomial interpolating \( \{u^0, u^1\} \):

\[
H_1^k(t) := \frac{t-t_1}{t_0-t_1} u^0 + \frac{t-t_0}{t_1-t_0} u^1.
\]

When \( k \geq 2 \), for \( 1 \leq j \leq k-1 \), we use the standard quadratic Lagrangian polynomial interpolating \( \{u^j, u^{j+1}\} \):

\[
H_2^j(t) := \frac{(t-t_j)(t-t_{j+1})}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} u^{j-1} + \frac{(t-t_{j-1})(t-t_{j+1})}{(t_{j-1}-t_j)(t_j-t_{j+1})} u^j + \frac{(t-t_{j-1})(t-t_j)}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)} u^{j+1},
\]

while for \( j = k \), we use the quadratic Lagrangian polynomial \( H_2^{k-1}(t) \) defined in (2.1).

Let \( \tau_j = t_j - t_{j-1} \). At \( t = t_k \), the fractional derivative \( \partial_t^\alpha u(t) \) is approximated by the discrete fractional-derivative operator

\[
L_k^\alpha u = \frac{u^1 - u^0}{\Gamma(2-\alpha) \tau_1^\alpha},
\]

\[
L_k^\alpha u = \frac{1}{\Gamma(1-\alpha)} \left( \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \frac{\partial_s H_2^j(s)}{(t_k-s)\alpha} \, ds + \int_{t_{k-1}}^{t_k} \frac{\partial_s H_2^{k-1}(s)}{(t_k-s)\alpha} \, ds \right)
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \left( \sum_{j=1}^{k-1} \left( a_j^k u^{j-1} + b_j^k u^j + c_j^k u^{j+1} + a_k^k u^{j-2} + b_k^k u^{j-1} + c_k^k u^k \right) \right),
\]

where for \( 1 \leq j \leq k-1 \),

\[
a_j^k = \int_{t_{j-1}}^{t_j} \frac{2s-t_j-t_{j+1}}{\tau_j(\tau_j+\tau_{j+1})(t_k-s)^\alpha} \, ds = \int_0^1 \frac{-2\tau_j(1-\theta)-\tau_{j+1}}{(\tau_j+\tau_{j+1})(t_k-(t_{j+1}+\theta\tau_j))^\alpha} \, d\theta,
\]

\[
b_j^k = -\int_{t_{j-1}}^{t_j} \frac{2s-t_j-t_{j+1}}{\tau_j(\tau_j+\tau_{j+1})(t_k-s)^\alpha} \, ds = \int_0^1 \frac{2\tau_j\theta-\tau_j-\tau_{j+1}}{\tau_{j+1}(t_k-(t_{j+1}+\theta\tau_j))^\alpha} \, d\theta,
\]

\[
c_j^k = \int_{t_{j-1}}^{t_j} \frac{2s-t_j-t_{j+1}}{\tau_{j+1}(\tau_j+\tau_{j+1})(t_k-s)^\alpha} \, ds = \int_0^1 \frac{\tau_j^2(2\theta-1)}{\tau_{j+1}(\tau_j+\tau_{j+1})(t_k-(t_{j+1}+\theta\tau_j))^\alpha} \, d\theta,
\]
and

\[ a_k = \int_{t_{k-1}}^{t_k} \frac{2s - t_{k-1} - t_k}{\tau_{k-1} (\tau_k + \tau)} \frac{1}{(t_k - s)^\alpha} \, ds = \int_0^1 \frac{\tau_k^2 (2\theta - 1)}{\tau_{k-1} (\tau_k + \tau) (t_k - (t_{k-1} + \theta \tau_k))^\alpha} \, d\theta, \]

\[ b_k = - \int_{t_{k-1}}^{t_k} \frac{2s - t_{k-2} - t_k}{\tau_{k-1} \tau_k} \frac{1}{(t_k - s)^\alpha} \, ds = - \int_0^1 \frac{\tau_k (2\theta - 1) + \tau_{k-1}}{\tau_{k-1} (t_k - (t_{k-1} + \theta \tau_k))^\alpha} \, d\theta, \]

\[ c_k = \int_{t_{k-1}}^{t_k} \frac{2s - t_{k-2} - t_{k-1}}{\tau_k (\tau_{k-1} + \tau_k)} \frac{1}{(t_k - s)^\alpha} \, ds = \int_0^1 \frac{2\tau_k \theta + \tau_{k-1}}{(\tau_{k-1} + \tau_k) (t_k - (t_{k-1} + \theta \tau_k))^\alpha} \, d\theta. \]

It can be verified that \( a_j < 0, b_j > 0, c_j > 0 \) for \( 1 \leq j \leq k - 1 \), and \( a_k > 0, b_k < 0, c_k > 0 \). Furthermore, \( a_j + b_j + c_j = 0 \) always holds for \( 1 \leq j \leq k \).

Specifically speaking, we can figure out the explicit expressions of \( a_j \) and \( c_j \) as follows (note that \( b_j = -a_j - c_j \)): for \( 1 \leq j \leq k - 1 \),

\[
\begin{align*}
\alpha
a_j &= \frac{\tau_{j+1}}{(1 - \alpha) \tau_j (\tau_j + \tau_{j+1})} (t_k - t_j)^{1-\alpha} - \frac{2\tau_j + \tau_{j+1}}{(1 - \alpha) \tau_j (\tau_j + \tau_{j+1})} (t_k - t_{j-1})^{1-\alpha} \\
&\quad + \frac{2}{(2 - \alpha)(1 - \alpha) \tau_j (\tau_j + \tau_{j+1})} [(t_k - t_{j-1})^{2-\alpha} - (t_k - t_j)^{2-\alpha}], \\
\alpha
(2.4)
\end{align*}
\]

while for \( j = k \),

\[
\begin{align*}
\alpha
a_k &= \frac{\alpha \tau_k^2}{(2 - \alpha)(1 - \alpha) \tau_{k-1} (\tau_k + \tau_k)^\alpha}, \\
\alpha
(2.5)
c_k &= \frac{1}{(1 - \alpha) \tau_k^\alpha} + \frac{\alpha \tau_k}{(2 - \alpha)(1 - \alpha) (\tau_{k-1} + \tau_k) \tau_k^\alpha}.
\end{align*}
\]

We reformulate the discrete fractional derivative \( L_k^u \) in (2.2) as

\[
L_k^u = \int \frac{1}{\Gamma(2 - \alpha) \tau_1^{\alpha}} \delta_1 u, \\
L_k^u = \frac{1}{\Gamma(1 - \alpha)} \left( (\alpha \tau_{k-1} \delta_{k-1} u - \alpha \tau_k \delta_k u) - \alpha \tau_k \delta_k u + \sum_{j=2}^{k-1} d_j^{k-1} \delta_j u \right), \quad k \geq 2,
\]

where \( \delta_j u = u^j - u^{j-1} \) and \( d_j^k := c_j^{k-1} - a_j^k \). To establish the \( H^1 \)-stability of the \( L_2 \)-type method for fractional-order parabolic problem, we shall prove the positive semidefiniteness of \( B_n \), defined in (1.1).

3. Positive semidefiniteness of bilinear form \( B_n \).

**Lemma 3.1 (Properties of \( a_j, c_j \) and \( d_j \)).** Given a nonuniform mesh \( \{\tau_j\}_{j \geq 1} \), the following properties of the \( L_2 \) coefficients in (2.3) hold:

- **(P1)** \( a_j^k < 0, \ 1 \leq j \leq k - 1, \ k \geq 2; \)
- **(P2)** \( a_j^{k+1} - a_j^k \geq 0, \ 1 \leq j \leq k - 1, \ k \geq 2; \)
- **(P3)** \( a_j^{k+1} - a_{j+1}^k < 0, \ 1 \leq j \leq k - 2, \ k \geq 3; \)
- **(P4)** \( a_j^{k+1} - a_j^k < a_{j+1}^{k+1} - a_{j+1}^{k+1}, \ 1 \leq j \leq k - 2, \ k \geq 3; \)

Furthermore, if the nonuniform mesh (3.4) and (3.5) we have (P2) and (P4) hold.

where we use the form (3.2) for $a_j$ and $a_k$. Therefore (P3) holds. Moreover, for any fixed $s$, $(t_k - t_{j-1})^{-\alpha}$, $(t_k - t_{j-1} - s\tau_j)^{-\alpha-1}$, $(t_k - t_j + s\tau_j)^{-\alpha}$ and $(t_k - t_j - s\tau_j)^{-\alpha}$ all decrease w.r.t. $k$. As a consequence, (3.3) and (3.4) result in $a_{j+1}^k - a_j^k > 0$, $(a_{j+1}^k - a_j^k) - (a_{j+1}^{k+1} - a_j^{k+1}) > 0$, i.e., the properties (P2) and (P4) hold.

We now turn to prove the properties of $e_j^k$ and $d_j^k = c_j^k - a_j^k$. For $e_j^k$ in (2.3), we have

$$e_j^k = \frac{\tau_j^2}{\tau_j + \tau_{j+1}} \int_0^1 (t_k - (t_{j-1} + s\tau_j))^{-\alpha} d(s^2 - s)$$

and

$$d_j^k = \frac{\alpha \tau_j}{\tau_j + \tau_{j+1}} \int_0^1 (t_k - (t_{j-1} + s\tau_j))(1 - s)(t_k - t_j + s\tau_j)^{-\alpha-1} ds$$

and

$$a_j^k = \frac{1}{\tau_j + \tau_{j+1}} \int_0^1 \frac{-2\tau_j (1 - s) - \tau_{j+1}}{(t_k - (t_{j-1} + s\tau_j))^{\alpha}} ds = \int_0^1 \frac{-2\tau_j s - \tau_{j+1}}{(t_k - (t_{j-1} + s\tau_j))^{\alpha}} ds$$

It is easy to see $a_j^k < 0$, i.e., (P1) holds.

Combining (3.2) and (3.3), we have

$$a_{j+1}^k - a_j^k = \frac{\alpha \tau_j}{\tau_j + \tau_{j+1}} \int_0^1 (\tau_j + \tau_{j+1} + s\tau_j)(1 - s)(t_k - t_j + s\tau_j)^{-\alpha-1} ds$$

$$- \frac{\alpha \tau_{j+1}}{\tau_j + \tau_{j+1}} \int_0^1 (\tau_{j+1} + \tau_{j+2} - s\tau_j)(1 - s)(t_k - t_j - s\tau_j)^{-\alpha-1} ds < 0,$$

where we use the form (3.2) for $a_j^k$ and the form (3.3) for $a_j^{k+1}$. Therefore (P3) holds. Moreover, for any fixed $s$, $(t_k - t_{j-1})^{-\alpha}$, $(t_k - t_{j-1} - s\tau_j)^{-\alpha-1}$, $(t_k - t_j + s\tau_j)^{-\alpha}$ and $(t_k - t_j - s\tau_j)^{-\alpha-1}$ all decrease w.r.t. $k$. As a consequence, (3.3) and (3.4) result in $a_{j+1}^k - a_j^k > 0$, $(a_{j+1}^k - a_j^k) - (a_{j+1}^{k+1} - a_j^{k+1}) > 0$, i.e., the properties (P2) and (P4) hold.

We now turn to prove the properties of $e_j^k$ and $d_j^k = c_j^k - a_j^k$. For $e_j^k$ in (2.3), we have

$$e_j^k = \frac{\tau_j^2}{\tau_j + \tau_{j+1}} \int_0^1 (t_k - (t_{j-1} + s\tau_j))^{-\alpha} d(s^2 - s)$$

and

$$d_j^k = \frac{\alpha \tau_j}{\tau_j + \tau_{j+1}} \int_0^1 s(1 - s)(t_k - t_j + s\tau_j)^{-\alpha-1} ds > 0.$$
This is the property (P5). Since $a_j^k < 0$, we have $d_j^k = c_{j-1}^k - a_j^k > 0$ for $j \geq 2$ and the property (P7) holds. For any fixed $s$, $(t_k - t_j + st_j)^{-\alpha - 1}$ decreases w.r.t. $k$, implying that $c_{j+1}^k - c_j^k < 0$, i.e., the property (P6). Combining this with property (P2), the property (P8) holds.

We now prove the property (P9). Combining (3.4) and (3.5) gives

$$
(3.6) \\
d_{j+1}^k - d_j^k = \frac{\alpha \tau_j^3}{\tau_j+1(\tau_j + \tau_{j+1})} \int_0^1 s(1-s)(t_k - t_j + st_j)^{-\alpha - 1} ds \\
- \frac{\alpha \tau_{j-1}^3}{\tau_j(\tau_j-1 + \tau_j)} \int_0^1 s(1-s)(t_k - t_{j-1} + st_{j-1})^{-\alpha - 1} ds \\
+ \frac{\alpha \tau_j^3}{\tau_j + \tau_{j+1}} \int_0^1 (\tau_j + \tau_{j+1} + st_j)(1-s)(t_k - t_j + st_j)^{-\alpha - 1} ds \\
+ \frac{\alpha \tau_{j+1}^3}{\tau_{j+1} + \tau_{j+2}} \int_0^1 (\tau_{j+1} + \tau_{j+2} - st_{j+1})(1-s)(t_k - t_j - st_{j+1})^{-\alpha - 1} ds.
$$

Note that for any fixed $j$, $(t_k - t_j + st_j)^{-\alpha - 1} \geq 0$ decreases w.r.t. $s$, and $\int_0^1 (1 - 3s)(1-s) = 0$, which imply

$$
\text{(3.7)} \\
\int_0^1 (\tau_j + \tau_{j+1} + st_j)(1-s)(t_k - t_j + st_j)^{-\alpha - 1} ds \\
\geq \int_0^1 (4\tau_j + 3\tau_{j+1})s(1-s)(t_k - t_j + st_j)^{-\alpha - 1} ds.
$$

Using (3.7) and the fact $(t_k - t_j + st_j)^{-\alpha - 1} > (t_k - t_{j-1} + st_{j-1})^{-\alpha - 1}$, we can derive from (3.6) that

$$
(3.8) \\
d_{j+1}^k - d_j^k > \alpha \left( \frac{\tau_j^3}{\tau_j+1(\tau_j + \tau_{j+1})} - \frac{\tau_{j-1}^3}{\tau_j(\tau_j-1 + \tau_j)} + \frac{(4\tau_j + 3\tau_{j+1})\tau_j}{\tau_j + \tau_{j+1}} \right) \int_0^1 s(1-s)(t_k - t_j + st_j)^{-\alpha - 1} ds.
$$

The property (P9) holds if the following condition is satisfied

$$
\frac{\tau_j^3}{\tau_j+1(\tau_j + \tau_{j+1})} - \frac{\tau_{j-1}^3}{\tau_j(\tau_j-1 + \tau_j)} + \frac{(4\tau_j + 3\tau_{j+1})\tau_j}{\tau_j + \tau_{j+1}} \geq 0 \\
\iff \frac{1}{\rho_{j+1}(1 + \rho_{j+1})} - \frac{1}{\rho_{j}^2(1 + \rho_{j})} + \frac{4 + 3\rho_{j+1}}{1 + \rho_{j+1}} \geq 0 \\
\iff \frac{1}{\rho_{j+1}} \geq \frac{1}{\rho_{j}^2(1 + \rho_{j})} - 3.
$$

We now prove the last property (P10). The convexity of the function $t^{-\alpha - 1}$ gives

$$
(t_k - t_j + st_j)^{-\alpha - 1} - (t_{k+1} - t_j + st_j)^{-\alpha - 1} \\
> (t_k - t_{j-1} + st_{j-1})^{-\alpha - 1} - (t_{k+1} - t_{j-1} + st_{j-1})^{-\alpha - 1},
$$

and for fixed $j$, it is easy to see that $(t_k - t_j + st_j)^{-\alpha - 1} - (t_{k+1} - t_j + st_j)^{-\alpha - 1} > 0$ decreases w.r.t. $s$. Then we can get the following result similar to (3.8):

$$
(d_{j+1}^k - d_j^k) - (a_{j+1}^k - a_j^k) > \alpha \left( \frac{\tau_j^3}{\tau_j+1(\tau_j + \tau_{j+1})} - \frac{\tau_{j-1}^3}{\tau_j(\tau_j-1 + \tau_j)} + \frac{(4\tau_j + 3\tau_{j+1})\tau_j}{\tau_j + \tau_{j+1}} \right) \int_0^1 s(1-s)(t_k - t_j + st_j)^{-\alpha - 1} ds.
$$
Similar to the proof of (P9), $(d_{k+1}^j - d_j^k) - (d_{j+1}^{k+1} - d_j^{k+1}) > 0$, as soon as the condition (3.1) is satisfied. Therefore, (P10) is proved.

**Theorem 3.2.** Consider a nonuniform mesh $\{\tau_k\}_{k \geq 1}$ satisfying that

\begin{equation}
\rho_* < \rho_2, \quad \rho_* < \rho_3 < \rho^*, \quad 2 + \frac{2}{1 + \rho_3} + \frac{4\rho_2}{1 + \rho_2} - \frac{\rho_2^3}{(1 + \rho_2)^2} \geq 0,
\end{equation}

and for $k \geq 3$,

\begin{equation}
\begin{cases}
\rho_* < \rho_{k+1} \leq \frac{\rho_k^2(1 + \rho_k)}{1 - 3\rho_k^2(1 + \rho_k)}, & \rho_* < \rho_{k+1} < \xi_1, \\
\rho_* < \rho_{k+1} < \rho^*, & \xi_1 \leq \rho_{k+1} \leq \xi_2, \\
\rho_* < \rho_{k+1} \leq -\rho_{k+1}^2 + 4\rho_{k+1} + 2 \rho_k^2 - 3\rho_k - 1, & \xi_2 < \rho_{k+1} < \rho^*,
\end{cases}
\end{equation}

where $\rho_* \approx 0.356341$, $\rho^* \approx 4.155358$, $\xi_1 \approx 0.459770$, $\xi_2 \approx 3.532016$. The graphical illustration of these constraints are provided in Figure 1. Then for any function $u$ defined on $[0, \infty) \times \Omega$ and $n \geq 2$,

$$
B_n(u, u) = \sum_{k=1}^{n} \langle L_k u, \delta_k u \rangle \geq \sum_{k=1}^{n} \frac{g_k(\alpha)}{2\Gamma(3 - \alpha)} \|\delta_k u\|_{L^2(\Omega)}^2 \geq 0,
$$

where

\begin{align*}
g_k(\alpha) &= \begin{cases}
\frac{(2 - \alpha)(1 - \alpha)}{\alpha} \hat{g}(\alpha), & k = 1, 2, \\
\frac{\alpha}{\tau_k^2} \left( \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_{k+1} - 1 + \alpha}{(1 + \rho_{k+1})^\alpha} - \frac{2\rho_{k+1}^{2-\alpha}}{1 + \rho_{k+1}} - \frac{\rho_k(\rho_k - 2)}{1 + \rho_k} \right), & 3 \leq k \leq n - 1, \\
\frac{\alpha}{\tau_n^2} \left( \frac{(1 + \alpha)(2 - \alpha)}{\alpha} - \frac{\rho_n(\rho_n - 2)}{1 + \rho_n} \right), & k = n \neq 2,
\end{cases}
\end{align*}

are positive for all $\alpha \in (0, 1)$ and $\hat{g}(\alpha)$ is defined in (3.29).

**Proof.** According to (2.6), we can rewrite $B_n(u, u)$ in the following matrix form

$$
B_n(u, u) = \sum_{k=1}^{n} \langle L_k u, \delta_k u \rangle = \frac{1}{\Gamma(1 - \alpha)} \int_{\Omega} \psi \mathbf{M} \psi^T dx,
$$

\begin{figure}

**Fig. 1.** Feasible regions from (3.9) for $(\rho_2, \rho_3)$ (left) and (3.10) for $(\rho_k, \rho_{k+1})$ with $k \geq 3$ (right).


where $\psi = [\delta_1u, \delta_2u, \ldots, \delta_nu]$ and
\[(3.12)\]
\[M = \begin{pmatrix}
(1 - \alpha)^{-1}\tau_1^{-\alpha} & -a_1^2 & c_1^2 + c_2^2 \\
-a_1^3 & d_2^3 - a_3^3 & c_2^3 + c_3^3 \\
-a_1^4 & d_4^3 - a_4^4 & c_3^3 + c_4^3 \\
\vdots & \vdots & \ddots \\
-a_1^n & d_n^3 - a_n^3 & c_{n-1}^3 + c_n^3
\end{pmatrix}.
\]

We split $M$ as $M = A + B$, where
\[A = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 \\
-a_1^2 & \beta_2 & \beta_3 \\
-a_1^3 & d_2^3 & \beta_3 \\
\vdots & \vdots & \ddots \\
-a_1^n & d_n^3 & \beta_n
\end{pmatrix},
\]
and
\[B = \begin{pmatrix}
(1 - \alpha)^{-1}\tau_1^{-\alpha} - \beta_1 & c_1^2 + c_2^2 - \beta_2 \\
-c_1^3 & c_2^3 - \beta_3 \\
\vdots & \ddots \\
-a_1^n & c_{n-1}^n + c_n^3 - \beta_n
\end{pmatrix},
\]
with
\[
2\beta_1 = -a_1^2, \quad 2\beta_2 - d_2^3 = a_3^3 - a_1^3,
\]
\[
2\beta_k - d_k^{k+1} = d_{k-1}^k - d_{k-1}^{k+1}, \quad 3 \leq k \leq n - 1,
\]
\[
2\beta_n = d_n^{n-1}, \quad n \geq 3.
\]

Consider the following symmetric matrix $S = A + A^T + \varepsilon e_n^T e_n$, with small constant $\varepsilon > 0$ and $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^{1 \times n}$. According to Lemma 3.1, if the condition (3.1) holds, $S$ satisfies the following three properties:

1. $\forall 1 \leq j < i \leq n, \ |S|_{i,j} \geq |S|_{i,j}^{1}$;
2. $\forall 1 < j \leq i \leq n, \ |S|_{i,j-1} < |S|_{i,j}^{1}$;
3. $\forall 1 < j < i \leq n, \ |S|_{i,j-1} - |S|_{i,j} - |S|_{i,j}^{1} \leq |S|_{i,j} - |S|_{i,j}^{1}$.

From [19, Lemma 2.1], $S$ is positive definite. Let $\varepsilon \to 0$. We can claim that $A + A^T$ is positive semidefinite.

We now consider the following splitting of $B + B^T$:
\[B + B^T = \begin{pmatrix}
C & 0 \\
0 & 0
\end{pmatrix}_{n \times n} + \begin{pmatrix}
0 & 0 \\
0 & D
\end{pmatrix}_{n \times n},
\]
where
\[C = \begin{pmatrix}
2(1 - \alpha)^{-1}\tau_1^{-\alpha} - 2\beta_1 & -a_1^2 \\
-a_1^3 & 2c_1^2 + 2c_1^2 - 2\beta_2 - a_1^3
\end{pmatrix}_{2 \times 2},
\]
\[D = \begin{pmatrix}
\beta_2 & \beta_3 \\
-d_2^3 & \beta_3 \\
\vdots & \ddots \\
-d_n^3 & \beta_n
\end{pmatrix},
\]
\[D = \begin{pmatrix}
(1 - \alpha)^{-1}\tau_1^{-\alpha} - \beta_1 & c_1^2 + c_2^2 - \beta_2 \\
-c_1^3 & c_2^3 - \beta_3 \\
\vdots & \ddots \\
-a_1^n & c_{n-1}^n + c_n^3 - \beta_n
\end{pmatrix},
\]
\[
2\beta_1 = -a_1^2, \quad 2\beta_2 - d_2^3 = a_3^3 - a_1^3,
\]
\[
2\beta_k - d_k^{k+1} = d_{k-1}^k - d_{k-1}^{k+1}, \quad 3 \leq k \leq n - 1,
\]
\[
2\beta_n = d_n^{n-1}, \quad n \geq 3.
\]
The positive semidefiniteness of $\mathbf{B} + \mathbf{B}^T$ can be ensured if $\mathbf{C}$ and $\mathbf{D}$ are both positive semidefinite.

We first discuss about the positive semidefiniteness of $\mathbf{C}$ of size $2 \times 2$. Note that from (2.4), we have the following explicit expression of $a_1^2$:

$$a_1^2 = \frac{\tau_2^{2-\alpha}}{(1-\alpha)\tau_1(\tau_1 + \tau_2)} - \frac{(2\tau_1 + \tau_2)(\tau_1 + \tau_2)^{1-\alpha}}{(1-\alpha)\tau_1(\tau_1 + \tau_2)} + \frac{2[(\tau_1 + \tau_2)^{2-\alpha} - \tau_2^{2-\alpha}]}{(2-\alpha)(1-\alpha)\tau_1(\tau_1 + \tau_2)}$$

and from (3.2), we have another formula of $a_1^2$:

$$a_1^2 = -\tau_2^{2-\alpha} + \frac{\alpha\tau_1}{\tau_1 + \tau_2} \int_0^1 (\tau_1 + \tau_2 + s\tau_1)(1-s)(\tau_2 + s\tau_1)^{-\alpha-1} ds.$$ (3.15)

According to the definition $2\beta_1 = -a_1^2$ in (3.13), if $\tau_1 \geq \tau_2$, then by (3.14),

$$(\text{C})_{11} > 2(1-\alpha)^{-1}\tau_1^{1-\alpha} - (1-\alpha)^{-1}(\tau_1 + \tau_2)^{-\alpha} > (1-\alpha)^{-1}\tau_1^{1-\alpha} > 0,$$

while if $\tau_1 \leq \tau_2$, then by (3.15)

$$(\text{C})_{11} = 2(1-\alpha)^{-1}\tau_1^{1-\alpha} + a_1^2 > \frac{1+\alpha}{(1-\alpha)\tau_1^\alpha} > 0.$$ (3.16)

From $2\beta_2 = d_2^3 + a_1^3 - a_1^2$ in (3.13), $d_2^3 = c_1^3 + a_2^3$ and the properties (P5)–(P6) on $c_j^k$, we have

$$\|\text{C}\|_{22} = c_1^2 + 2c_2^2 + (a_1^2 + a_2^2 - a_1^2) - a_3^2 + (c_1^2 - c_1^3) > 2c_2^2 + (a_1^2 + a_2^3 - a_1^2) - a_3^3.$$ (3.17)

Note that (3.2) and (3.4) give

$$a_1^2 + a_2^3 - a_1^3 = -\tau_2^{2-\alpha} + \frac{\alpha\tau_1}{\tau_1 + \tau_2} \int_0^1 (\tau_1 + \tau_2 + s\tau_1)(1-s)(t_2 - t_1 + s\tau_1)^{-\alpha-1} ds$$

$$- \frac{\alpha\tau_1}{\tau_1 + \tau_2} \int_0^1 (\tau_1 + \tau_2 + s\tau_1)(1-s)(t_3 - t_1 + s\tau_1)^{-\alpha-1} ds$$

$$- \frac{\alpha\tau_2}{\tau_2 + \tau_3} \int_0^1 (\tau_2 + \tau_3 - s\tau_2)(1-s)(t_3 - t_1 - s\tau_2)^{-\alpha-1} ds$$

$$> -\tau_2^{2-\alpha} - \frac{\alpha\tau_2}{\tau_2 + \tau_3} \int_0^1 (1-s)(\tau_2 + \tau_3 - s\tau_2)^{-\alpha} ds$$

$$= -\tau_2^{2-\alpha} - \frac{\alpha}{(2-\alpha)(1-\alpha)\tau_2^\alpha} \left( -\rho_3 + 1 + \alpha \frac{\rho_3^{2-\alpha}}{1 + \rho_3} + \frac{2\rho_2^{2-\alpha}}{1 + \rho_3} \right).$$ (3.18)

Substituting (2.5) and (3.19) into (3.18) yields

$$\|\text{C}\|_{22} > \frac{\alpha}{(2-\alpha)(1-\alpha)\tau_2^\alpha} \left( (1+\alpha)(2-\alpha) + \rho_3 - 1 + \alpha \frac{2\rho_2^{2-\alpha}}{1 + \rho_3} + \frac{2\rho_2}{1 + \rho_2} \right).$$ (3.20)

$$\begin{pmatrix} a_1^3 & -a_1^3 \\ -a_1^3 & 2c_2^3 + 2c_3^3 - 2\beta_3 \\ \ddots & \ddots & \ddots \\ -a_n^3 & 2c_{n-1}^3 + 2c_n^3 - 2\beta_n \end{pmatrix} \in (n-1) \times (n-1)$$
In the case of \( \alpha \geq 0 \), \( \mathbf{C} \) is positive definite as soon as \(|\mathbf{C}|_{11} > 0\), \(|\mathbf{C}|_{12} > 0\), \(|\mathbf{C}|_{21} > 0\) and \(|\mathbf{C}|_{22} > 0\).

When \( \tau_1 \geq \tau_2 \), i.e. \( \rho_2 \leq 1 \), from (2.5), (3.16) and (3.20), we have

\[
|\mathbf{C}|_{11}|\mathbf{C}|_{22} - |\mathbf{C}|_{12}|\mathbf{C}|_{21} > \frac{\alpha}{(1 - \alpha)^2(2 - \alpha)(\tau_1 \tau_2)\alpha} h_1(\alpha),
\]

where

\[
h_1(\alpha) = \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_3 - 1 + \alpha}{(1 + \rho_3)^\alpha} - \frac{2\rho_3^2 - \alpha}{1 + \rho_3} + \frac{2\rho_2}{1 + \rho_2} - \frac{\alpha}{(2 - \alpha)(1 + \rho_2)} \left( \frac{\rho_2^2}{1 + \rho_2} \right)^2.
\]

Now we show that \( h_1(\alpha) \) decreases w.r.t. \( \alpha \) and \( h_1(1) \geq 0 \) under some constraints on \( \rho_2 \) and \( \rho_3 \). It is trivial to check that \(-\frac{\alpha}{(2 - \alpha)(1 + \rho_2)} (\frac{\rho_2^2}{1 + \rho_2})^2 \) decreases w.r.t. \( \alpha \) when \( \rho_2 \leq 1 \). Let

\[
q(\alpha) = \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_3 - 1 + \alpha}{(1 + \rho_3)^\alpha} - \frac{2\rho_3^2 - \alpha}{1 + \rho_3} + \frac{2\rho_2}{1 + \rho_2}.
\]

A direct calculation gives

\[
q'(\alpha) = -2/\alpha^2 - 1 + (1 + \rho_3)^{-\alpha} - \frac{(\rho_3 - 1 + \alpha) \ln(1 + \rho_3)}{(1 + \rho_3)^\alpha} + \frac{2\rho_3^2 - \alpha \ln(\rho_3)}{1 + \rho_3}.
\]

To show \( q'(\alpha) \leq 0 \), we consider the following several cases. In the case of \( 0 < \rho_3 \leq 1 \), we have

\[
q'(\alpha) \leq -2/\alpha^2 - 1 + (1 + \rho_3)^{-\alpha} (1 - (\rho_3 - 1 + \alpha) \ln(1 + \rho_3)) \leq -3 + (1 + \ln 2) \leq 0.
\]

In the case of \( 1 < \rho_3 \leq 4.5 \), we have

\[
q'(\alpha) \leq -2/\alpha^2 - 1 + 2^{-\alpha} + 2 \times \frac{4.5^{2-\alpha} \ln(4.5)}{1 + 4.5} \leq 0.
\]

So \( q(\alpha) \) decreases w.r.t. \( \alpha \) for \( 0 < \rho_3 \leq 4.5 \). As a consequence, \( h_1(\alpha) \) decreases w.r.t. \( \alpha \) for \( 0 < \rho_3 \leq 4.5 \). Since

\[
h_1(1) = 2 - \frac{\rho_3}{1 + \rho_3} + \frac{2\rho_2}{1 + \rho_2} - \frac{\rho_3^3}{(1 + \rho_2)^2} = 1 + \frac{1}{1 + \rho_3} + \frac{2\rho_2 + 2\rho_2^2 - \rho_3^3}{(1 + \rho_2)^2} > 0,
\]

we know that \( \mathbf{C} \) is positive definite for \( \alpha \in (0, 1) \) when \( \rho_2 \leq 1 \) and \( \rho_3 \leq 4.5 \).

When \( \tau_1 \leq \tau_2 \), i.e. \( \rho_2 \geq 1 \), from (2.5), (3.17) and (3.20), we have

\[
|\mathbf{C}|_{11}|\mathbf{C}|_{22} - |\mathbf{C}|_{12}|\mathbf{C}|_{21} > \frac{\alpha^2}{(1 - \alpha)^2(2 - \alpha)^2 h_2(\alpha)}.
\]

where

\[
h_2(\alpha) = \frac{(1 + \alpha)(2 - \alpha)}{\alpha} q(\alpha) - \left( \frac{\rho_2^3}{1 + \rho_2} \right)^2
\]

with \( q(\alpha) \) defined in (3.21). We want to impose some constraints on \( \rho_2 \) and \( \rho_3 \) s.t. \( h_2(\alpha) \geq 0 \) for \( \alpha \in (0, 1) \). First we have to impose \( h_2(1) \geq 0 \), i.e.,

\[
(3.23) \quad \rho_2^{-1} h_2(1) = 2 + \frac{2}{1 + \rho_3} + \frac{4\rho_2}{1 + \rho_2} - \frac{\rho_3^3}{(1 + \rho_2)^2} \geq 0,
\]
which is equivalent to
\[
\rho_3^3 - \left( 6 + \frac{2}{1 + \rho_3} \right) \rho_2^2 - \left( 8 + \frac{4}{1 + \rho_3} \right) \rho_2 - \left( 2 + \frac{2}{1 + \rho_3} \right) \leq 0.
\]
Solving this cubic inequality yields
\[
0 < \rho_2 \leq \psi(\rho_3),
\]
where \(\psi(\rho_3)\) is the unique positive root of the left-hand side of (3.24). Next, we show that under the constraint (3.23), \(\rho'_2(\alpha) \leq 0\) holds, so that \(\rho_2(\alpha) \geq 0\) for \(\alpha \in (0, 1)\). Note that (3.23) indicates
\[
\rho_3^3 \leq 4 \rho_2 \frac{1}{1 + \rho_2} \leq 2 + \frac{2}{1 + \rho_3} < 4 \quad \Rightarrow \quad \rho_2 < 9.331852.
\]
A direct computation gives \(\rho'_2(\alpha) = \rho_2^2 q(\alpha)p(\alpha)\), where \(q(\alpha)\) is defined in (3.21) and
\[
p(\alpha) = -2/\alpha^2 - 1 + \ln \rho_2 \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{(1 + \alpha)(2 - \alpha)q(\alpha)}{a q(\alpha)}.
\]
Recall that \(q'(\alpha) \leq 0\) for \(\rho_2 > 0\), \(0 < \rho_3 \leq 4.5\), implying that \(q(\alpha) \geq q(1) > 0\). We now prove that \(p(\alpha) \leq 0\) for any \(\alpha \in (0, 1)\), \(0.3 < \rho_3 \leq 4.5\) and \(0 < \rho_2 \leq \psi(\rho_3)\). The following three cases are discussed.

**Case 1:** \(0 < \alpha \leq 0.43\). We have the following estimate
\[
p(\alpha) \leq -2/\alpha^2 - 1 + \ln 9.4 \frac{(1 + \alpha)(2 - \alpha)}{\alpha} \leq 0, \quad 0 < \rho_2 \leq 9.4, \quad 0 < \rho_3 \leq 4.5,
\]
where we use the inequality (3.26).

**Case 2:** \(0.43 < \alpha \leq 0.7\). In this case, for any \(1 \leq \rho_2 < 9.4\),
\[
\begin{align*}
\frac{d}{d\alpha} \left( -2/\alpha^2 - 1 + \ln \rho_2 \frac{(1 + \alpha)(2 - \alpha)}{\alpha} \right) & = 4/\alpha^3 + \ln \rho_2 (-2/\alpha^2 - 1) \\
& > 4/\alpha^3 + \ln 9.4 (-2/\alpha^2 - 1) \geq 4/\alpha^3 - 4.5/\alpha^2 - 2.25 \geq (4/0.7 - 4.5)/0.7^2 - 2.25 \geq 0.
\end{align*}
\]
For any interval \((b, a) \subset (0.43, 0.7)\) and \(\alpha \in (b, a)\), we have the following upper bound
\[
p(\alpha) \leq \left( -2/\alpha^2 - 1 + \ln \rho_2 \frac{(1 + \alpha)(2 - \alpha)}{a} \right) + \frac{(1 + \alpha)(2 - \alpha)}{a q(b)} \left( -2/\alpha^2 - 1 \right)
\]
\[
+ (1 + \rho_3)^{-b} - \rho_3 \ln \left( 1 + \rho_3 \right) \frac{a}{(1 + \rho_3)^a} + \frac{(1 - b) \ln(1 + \rho_3)}{(1 + \rho_3)^b} + \frac{2 \rho_3^{\alpha - b} - 1}{1 + \rho_3} \ln(\rho_3)
\]
\[
= : \varphi_1(\rho_2, \rho_3) \leq \varphi_1(\psi(\rho_3), \rho_3).
\]
Here, \(\psi(\rho_3)\) is defined in (3.25) and we use the fact that \(\varphi_1(\rho_2, \rho_3)\) increases w.r.t. \(\rho_2\). We separate \((0.43, 0.7)\) into \((0.43, 0.6)\) and \((0.6, 0.7)\), and plot the upper bounds according to (3.27) on these two small intervals respectively (see the left-hand side of Figure 2). Both upper bounds are smaller than 0. So \(p(\alpha) \leq 0\) for \(0.43 < \alpha \leq 0.7\), \(0.3 < \rho_3 \leq 4.5\), \(0 < \rho_2 \leq \psi(\rho_3)\).

**Case 3:** \(0.7 < \alpha < 1\). For any interval \((b, a) \subset (0.7, 1)\) and \(\alpha \in (b, a)\),
\[
p(\alpha) \leq \left( -2/\alpha^2 - 1 + \ln \rho_2 \frac{(1 + \alpha)(2 - \alpha)}{b} \right) + \frac{(1 + \alpha)(2 - \alpha)}{a q(b)} \left( -2/\alpha^2 - 1 \right)
\]
\[
+ (1 + \rho_3)^{-b} - \rho_3 \ln \left( 1 + \rho_3 \right) \frac{a}{(1 + \rho_3)^a} + \frac{(1 - b) \ln(1 + \rho_3)}{(1 + \rho_3)^b} + \frac{2 \rho_3^{\alpha - b} - 1}{1 + \rho_3} \ln(\rho_3)
\]
\[
= : \varphi_2(\rho_2, \rho_3) \leq \varphi_2(\psi(\rho_3), \rho_3).
\]
Here, \( \psi(\rho_3) \) is defined in (3.25) and we use the fact that \( \varphi_2(\rho_2, \rho_3) \) increases w.r.t. \( \rho_2 \). We separate \((0.7, 1]\) into small intervals and plot the upper bounds according to (3.28) on all these small intervals respectively (see the right-hand side of Figure 2). All these upper bounds are smaller than 0. So \( p(\alpha) \leq 0 \) for \( 0.7 < \alpha \leq 1 \), \( 0.3 < \rho_3 \leq 4.5 \), \( 0 < \rho_2 \leq \psi(\rho_3) \). Combining Case 1–3, we derive that \( C \) is positive definite for \( \alpha \in (0, 1) \) when \( \rho_2 \geq 1 \), \( 0.3 \leq \rho_3 \leq 4.5 \) and (3.23) is satisfied.

Combining all above discussions for \( 0 < \rho_2 \leq 1 \) and \( \rho_2 \geq 1 \), we claim that if \( 0.3 \leq \rho_3 \leq 4.5 \) and (3.23) is satisfied, then \( C \) is positive definite. Moreover, the eigenvalues of \( C \) are

\[
\lambda_{1,2} = \frac{|C|_{11} + |C|_{22} \pm \sqrt{(|C|_{11} + |C|_{22})^2 - 4(|C|_{11}|C|_{22} - |C|_{12}|C|_{21})}}{2} \geq \frac{|C|_{11} + |C|_{22} - \sqrt{(|C|_{11} - |C|_{22})^2 + 4|C|_{12}|C|_{21}}}{2} =: \hat{g}(\alpha).
\]

We have studied the positive definiteness of \( C \) and now turn to analyze the positive semidefiniteness of \( D \). We aim to show that \( D \) is diagonally dominant under some constraints on \( \rho_k \), so that the positive semidefiniteness can be guaranteed.

For \( 3 \leq k \leq n - 1 \), we show \( 2c_{k-1}^k + 2c_k - 2\beta_k - a_k^k - a_{k+1}^k \geq 0 \) under some constraints on \( \rho_k \) and \( \rho_{k+1} \). From the definition \( 2\beta_k = d_k^{k+1} + d_{k-1}^k - a_{k-1}^k \) in (3.13) and \( d_k^k = c_{j-1}^k - a_j^k \), we have

\[
2c_k^{k-1} + 2c_k^k - 2\beta_k - a_k^k - a_{k+1}^k = 2c_{k-1}^k + 2c_k - d_k^{k+1} - d_{k-1}^k + d_{k-1}^k - a_k^k - a_{k+1}^k
\]

\[
= c_{k-1}^k + 2c_k^k + (c_{k-1}^k - c_{k-2}^k) - (c_{k-2}^k - c_{k-3}^k) + (a_{k-1}^k - a_{k-2}^k) + (a_{k-2}^k - a_{k-3}^k).
\]

From (3.5), (3.4) and (3.2), we have

\[
(c_k^k - c_{k-1}^k) - (c_{k-2}^k - c_{k-2}^k) + (a_{k-1}^k - a_{k-2}^k) + a_{k-1}^k
\]

\[
= \frac{\alpha(\tau_k^3)}{\tau_k(\tau_{k-1} + \tau_k)} \int_0^1 s(1-s) \left[ (t_k - t_{k-1} + s\tau_k)_{\alpha-1} - (t_{k+1} - t_{k-1} + s\tau_k)_{\alpha-1} \right] ds
\]

\[
= \frac{\alpha\tau_k^3}{\tau_k(\tau_{k-1} + \tau_k)} \int_0^1 s(1-s) \left[ (t_k - t_{k-2} + s\tau_{k-2})_{\alpha-1} - (t_{k+1} - t_{k-2} + s\tau_{k-2})_{\alpha-1} \right] ds
\]
\[
\begin{align*}
&\quad + \frac{\alpha \tau_{k-1}}{\tau_{k-1} + \tau_k} \int_0^1 (\tau_{k-1} + \tau_k + st_{k-1})(1-s) \left[\left(t_k - t_{k-1} + st_{k-1}\right)^{-\alpha-1}ight. \\
&\quad \quad - \left. (t_{k+1} - t_{k-1} + st_{k-1})^{-\alpha-1}\right] ds - \tau_{k-\alpha} \int_0^1 (1-s)(\tau_k + \tau_{k+1} - s\tau_k)^{-\alpha} ds \\
&\quad > - \tau_{k-\alpha} \int_0^1 (1-s)(\tau_k + \tau_{k+1} - s\tau_k)^{-\alpha} ds \\
&\quad = - \tau_{k-\alpha} \int_0^1 \left( \frac{\rho_{k+1} - 1 + \alpha}{(1 + \rho_{k+1})^\alpha} - \frac{2\rho_{k+1}^2 - \alpha}{1 + \rho_{k+1}} \right) \left( \frac{\rho_{k+1} - 1 + \alpha}{(1 + \rho_{k+1})^\alpha} + \frac{2\rho_{k+1}^2 - \alpha}{1 + \rho_{k+1}} \right) \right) \\
\text{as soon as (3.1) is satisfied for } j = k - 1. \text{ Here, the inequality in (3.31) is obtained similar to the proof of the property (P10) in Lemma 3.1. Combining this with (2.5) and (3.30) yields} \\
2c_{k-1}^n + 2\beta_{k-1} - 2\alpha_{k-1} - \alpha_{k+1}^n \geq c_{k-1}^n + \frac{\alpha h_3(\alpha)}{(2-\alpha)(1-\alpha)\tau_k^\alpha}, \\
\text{where} \\
h_3(\alpha) = \frac{(1+\alpha)(2-\alpha)}{\alpha} + \frac{\rho_{k+1} - 1 + \alpha}{(1 + \rho_{k+1})^\alpha} - \frac{2\rho_{k+1}^2 - \alpha}{1 + \rho_{k+1}} - \frac{\rho_k(\rho_k - 2)}{1 + \rho_k}.
\end{align*}
\]

A direct calculation gives
\[
h'_3(\alpha) = -2/\alpha^2 - 1 + (1 + \rho_{k+1})^{-\alpha} - \frac{(\rho_{k+1} - 1 + \alpha)\ln(1 + \rho_{k+1})}{(1 + \rho_{k+1})^\alpha} + \frac{2\rho_{k+1}^2 - \alpha}{1 + \rho_{k+1}} \ln(\rho_{k+1}),
\]
which is similar to \(q'(\alpha)\) in (3.22) (just replacing \(\rho_3\) by \(\rho_{k+1}\)). Therefore, we have 

\[
h'_3(\alpha) \leq 0 \quad \text{and then } h_3(\alpha) \geq h_3(1) \quad \text{when } 0 < \rho_{k+1} < 4.5 \text{. To ensure } \geq 0, \text{ it is sufficient to impose for } 3 \leq k \leq n - 1
\]

(3.32) 

\[
\frac{1}{\rho_k} \geq \frac{1}{\rho_{k-1}(1 + \rho_{k-1})} - 3, \quad 0 < \rho_{k+1} < 4.5, \quad h_3(1) = \frac{2 + \rho_{k+1}}{1 + \rho_{k+1}} - \frac{\rho_k(\rho_k - 2)}{1 + \rho_k} \geq 0.
\]

Now we show \(2c_{n-1}^n + 2\alpha_n - 2\beta_n - \alpha_n^2 \geq 0\) under some constraints on \(\rho_n\). From (3.13), (2.5), (3.2) and (3.5), we can get

(3.33) 

\[
2c_{n-1}^n + 2\alpha_n - 2\beta_n - \alpha_n^2 = c_{n-1}^n + 2c_{n-1} - a_{n-1}^2 + c_{n-1}^n - c_{n-2}^n + a_{n-1}^n
\]

\[
eq \left( c_{n-1}^n + \frac{2}{(1 - \alpha)\alpha} \right) + \frac{2\alpha\tau_n}{\alpha\tau_n + s(1-s)(t_n - t_{n-1} + s\tau_{n-1})^{-\alpha-1}} ds \\
+ \frac{\alpha\tau_{n-2}}{\tau_n(\tau_{n-1} + \tau_n)} \int_0^1 s(1-s)(t_n - t_{n-1} + s\tau_{n-1})^{-\alpha-1} ds \\
- \frac{\alpha\tau_{n-2}}{\tau_{n-1}(\tau_{n-2} + \tau_{n-1})} \int_0^1 s(1-s)(t_n - t_{n-2} + s\tau_{n-2})^{-\alpha-1} ds \\
- \tau_n + \frac{\alpha\tau_{n-1}}{\tau_{n-1} + \tau_n} \int_0^1 (\tau_{n-1} + \tau_n + s\tau_{n-1})(1-s)(t_n - t_{n-1} + s\tau_{n-1})^{-\alpha-1} ds \\
\geq c_{n-1}^n + \frac{\alpha}{(2 - \alpha)(1 - \alpha)\alpha} \left( 1 + \alpha \right) - \frac{\rho_n(\rho_n - 2)}{1 + \rho_n}.
\]
If (3.1) holds for $j = n - 1$. The proof of the last inequality in (3.33) is similar to the proof of property (P9) in Lemma 3.1. To ensure $2c_n^a + 2c_n^a - 2\beta_n - a_n^a \geq 0$, it is sufficient to impose

$$
\frac{1}{\rho_n} \geq \frac{1}{\rho_n(1 + \rho_n)} - 3, \quad (1 + \alpha)(2 - \alpha)/\alpha - \frac{\rho_n(\rho_n - 2)}{1 + \rho_n} \geq 0, \quad \forall \alpha \in (0, 1),
$$

that is,

(3.34)  \quad \frac{1}{\rho_n} \geq \frac{1}{\rho_n(1 + \rho_n)} - 3, \quad \rho_n \leq 2 + \sqrt{6}.

Combining the above discussions on $D$, we conclude that if (3.32) and (3.34) hold, then $D$ is diagonally dominant and positive semidefinite, satisfying

$$
D \geq (2 - \alpha)^{-1}(1 - \alpha)^{-1}\text{diag}(0, g_3(\alpha), \ldots, g_k(\alpha), \ldots, g_n(\alpha)),
$$

where $g_k(\alpha)$ is given in (3.11).

We now combine all the conditions for the positive semidefiniteness of $A + A^T$, $C$ and $D$, so that

$$
M + M^T = (A + A^T) + (B + B^T) \geq B + B^T \geq (2 - \alpha)^{-1}(1 - \alpha)^{-1}\text{diag}(g_1(\alpha), g_2(\alpha), \ldots, g_k(\alpha), \ldots, g_n(\alpha))
$$

is positive definite, where $g_k(\alpha)$ is given in (3.11). This gives

$$
B_n(u, u) = \frac{1}{\Gamma(1 - \alpha)} \int_\Omega \psi M\psi^T dx \geq \sum_{k=1}^n \frac{g_k(\alpha)}{2\Gamma(3 - \alpha)} \|\delta_k u\|_{L^2(\Omega)}^2 \geq 0.
$$

In fact, we have proved the following results:

- Positive semidefiniteness of $A + A^T$: (3.1) holds.
- Positive definiteness of $C$: $0.3 \leq \rho_3 \leq 4.5$ and (3.23) holds.
- Positive semidefiniteness of $D$: (3.32) holds for $3 \leq k \leq n - 1$ and (3.34) holds for $k = n$.

In the following content, we just simplify the above constraints for the positive semidefiniteness of $M + M^T$.

The condition (3.1) actually says that $(\rho_j, \rho_{j+1})$ lies on the right-hand side of the blue solid curve in Figure 3. Let $\rho_* \approx 0.356341$ be the root of $\rho(1 + \rho) = 1 - 3\rho^2(1 + \rho)$. It can be found that if $\rho_j \leq \rho_*$ for some $j$, then $\rho_* \geq \rho_j \geq \rho_{j+1} \geq \rho_{j+2} \geq \ldots$ and $\tau_j$ will shrink to 0 quickly as $j$ increases. This doesn’t make sense in practice. We shall impose $\rho_j > \rho_*, \forall j \geq 2$. As a consequence, we have the following constraints: for $j \geq 2$,

$$
\begin{cases}
\rho_* < \rho_{j+1} \leq \rho^2(1 + \rho_j)/(1 - 3\rho^2(1 + \rho_j)), & \rho_* < \rho_j < \eta_1, \\
\rho_* < \rho_{j+1}, & \eta_1 \leq \rho_j,
\end{cases}
$$

where $\eta_1 \approx 0.475329$ be the unique positive root of $1 - 3\rho^2(1 + \rho) = 0$.

Since $\rho_{k+1} > \rho_*$, we can obtain from the last inequality of (3.32): $\forall 3 \leq k \leq n - 1$

$$
\frac{2 + \rho_* - \rho_k(\rho_k - 2)}{1 + \rho_*} > 0 \quad \Rightarrow \quad \rho_k < \rho_* := \frac{1}{2} \left[ \frac{4 + 3\rho_*}{1 + \rho_*} + \sqrt{\left( \frac{4 + 3\rho_*}{1 + \rho_*} \right)^2 + \frac{4 + 2 + \rho_*}{1 + \rho_*}} \right].
$$
Then the last inequality of (3.32) gives for $3 \leq k \leq n-1$,

\[
\begin{cases}
\rho_* < \rho_{k+1} < \rho^*, \\
\rho_* < \rho_{k+1} \leq -\frac{\rho^2}{\rho_k^2 - 3\rho_k - 1}, \\
\rho_* < \eta_2 < \rho_k < \rho^*,
\end{cases}
\]  

(3.38)

where $\eta_2 = \frac{3 + \sqrt{17}}{2}$ is the positive root of $\rho^2 - 3\rho - 1 = 0$. Note that $\rho^* \approx 4.155358 \leq 2 + \sqrt{6}$, (3.38) implies $\rho_n \leq 2 + \sqrt{6}$, the constraint in (3.34).

Combining (3.23), (3.37) and (3.38), it is sufficient to impose (3.9) and (3.10) to ensure the positive semidefiniteness of $B_n$. In (3.10), $\xi_1 \approx 0.459770$ is the positive root of $\rho_2 = \rho^*$ and $\xi_2 \approx 3.532016$ is the positive root of $\frac{-\rho^2 + 4\rho + 2}{\rho^2 - 3\rho - 1} = \rho^*$.

**Corollary 3.3.** Let

\[
\rho_L \approx 0.457332766746115, \quad \rho_R = \frac{3 + \sqrt{17}}{2} \approx 3.561552812808830.
\]

If $\rho_k \in [\rho_L, \rho_R]$ for all $k \geq 2$, then for any function $u$ defined on $[0, \infty) \times \Omega$ and $n \geq 2$,

\[
\mathcal{B}_n(u, u) = \sum_{k=1}^{n} \langle L_k^\alpha u, \delta_k u \rangle \geq \sum_{k=1}^{n} \frac{g_k(\alpha)}{2\Gamma(3-\alpha)} \|\delta_k u\|^2 \geq 0,
\]

(3.40)

with $g_k(\alpha)$ given in (3.11).

**Proof.** For $k \geq 3$, we want to find the largest square subregion contained by the region shown in Figure 1. In Figure 4 we draw this square $[\rho_L, \rho_R]^2$. We first set $\rho_{k+1} = \rho_k$ in the inequality (3.32). Precisely speaking, we derive the quantities of $\rho_R$ and $\rho_L$ as follows: $\rho_R$ is the positive root of $1 + \frac{1}{1+\rho} - \frac{\rho^2}{1+\rho} = 0$ and $\rho_L$ is the positive root of $\frac{\rho^2(1+\rho)}{1-3\rho^2(1+\rho)} = \rho_R$. Clearly, for any $\rho_k \in [\rho_L, \rho_R], k \geq 3$, the condition (3.10) in Theorem 3.2 is satisfied. Moreover, it is easy to check that if $(\rho_2, \rho_3) \in [\rho_L, \rho_R]^2$, the condition (3.9) in Theorem 3.2 also holds. The Corollary 3.3 is proved. \qed
4. $H^1$-stability of L2-type method for subdiffusion equation. We consider the following subdiffusion equation:

\begin{equation}
\begin{aligned}
\partial^\alpha_t u(t, x) &= \Delta u(t, x) + f(t, x), \quad (t, x) \in (0, \infty) \times \Omega, \\
\quad u(t, x) &= 0, \quad (t, x) \in (0, \infty) \times \partial \Omega, \\
\quad u(0, x) &= u^0(x), \quad x \in \Omega,
\end{aligned}
\end{equation}

where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$. Given an arbitrary nonuniform mesh $\{\tau_k\}_{k \geq 1}$, the L2 scheme of this subdiffusion equation is written as

\begin{equation}
\begin{aligned}
L^\alpha_k u &= \Delta u^k + f^k, \quad \text{in } \Omega, \\
\quad u^k &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

where $f^k = f(t_k, \cdot)$.

**Theorem 4.1.** Assume that $f(t, x) \in L^\infty([0, \infty); L^2(\Omega)) \cap BV([0, \infty); L^2(\Omega))$ is a bounded variation function in time and $u^0 \in H^1_0(\Omega)$. If the nonuniform mesh $\{\tau_k\}_{k \geq 1}$ satisfies (3.9) and (3.10) (or simply $\rho_k \in [\rho_L, \rho_R]$ given in Corollary 3.3), then the numerical solution $u^n$ of the L2 scheme (4.2) satisfies the following $H^1$-stability

\begin{equation}
\|\nabla u^n\|_{L^2(\Omega)} \leq \|\nabla u^0\|_{L^2(\Omega)} + 2C_{\Omega}C_f,
\end{equation}

where $C_f$ depends on the source term $f$, $C_{\Omega}$ is the Sobolev embedding constant depending on $\Omega$ and the dimension $d$.

**Proof.** When $n = 1$, we have

\begin{equation}
\frac{\delta_1 u}{\Gamma(2 - \alpha)\tau_1^\alpha} = \Delta u^1 + f^1.
\end{equation}

Multiplying (4.4) with $\delta_1 u$ and integrating over $\Omega$ yield

\begin{equation}
\frac{\|\delta_1 u\|_{L^2(\Omega)}^2}{\Gamma(2 - \alpha)\tau_1^\alpha} = -\frac{1}{2}\|\nabla u^1\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla u^0\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\nabla \delta_1 u\|_{L^2(\Omega)}^2 + \langle f^1, \delta_1 u \rangle.
\end{equation}

**Fig. 4.** Region of $[\rho_L, \rho_R]^2$ given in Corollary 3.3 for all $k \geq 3$, which is a subregion of the region in Figure 1.
Applying Cauchy–Schwarz inequality, then we can derive
\[
\|\nabla u^1\|_{L^2(\Omega)}^2 \leq \|\nabla u^0\|_{L^2(\Omega)}^2 + 4\|f\|_{L^\infty((0,\infty); L^2(\Omega))} \max_{0 \leq k \leq 1} \|u^k\|_{L^2(\Omega)}
\]
\[
\leq \|\nabla u^0\|_{L^2(\Omega)}^2 + 4\|f\|_{L^\infty((0,\infty); L^2(\Omega))} C_\Omega \max_{0 \leq k \leq 1} \|\nabla u^k\|_{L^2(\Omega)},
\]
where \(C_\Omega\) is the Sobolev embedding constant depending on \(\Omega\) and the dimension.

We now consider the case \(n \geq 2\). Multiplying (4.2) with \(\delta_k u\), integrating over \(\Omega\), and summing up the derived equations over \(n\) yield
\[
\sum_{k=1}^n \langle L_k^n u, \delta_k u \rangle = \sum_{k=1}^n \langle \Delta u^k, \delta_k u \rangle + \sum_{k=1}^n \langle f^k, \delta_k u \rangle
\]
\[
= -\frac{1}{2} \|\nabla u^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u^0\|_{L^2(\Omega)}^2 - \frac{1}{2} \sum_{k=1}^n \|\nabla \delta_k u\|_{L^2(\Omega)}^2
\]
\[
+ \langle f^n, u^n \rangle - \langle f^1, u^0 \rangle - \sum_{k=2}^n \langle \delta_k f, u^{k-1} \rangle.
\]
Applying the Cauchy–Schwarz inequality gives
\[
\langle f^n, u^n \rangle - \langle f^1, u^0 \rangle + \sum_{k=2}^n \langle \delta_k f, u^{k-1} \rangle \leq (2\|f\|_{L^\infty((0,\infty); L^2(\Omega))} + \|f\|_{BV((0,\infty); L^2(\Omega))}) \max_{0 \leq k \leq n} \|u^k\|_{L^2(\Omega)}
\]
\[
\leq C_f C_\Omega \max_{0 \leq k \leq n} \|\nabla u^k\|_{L^2(\Omega)},
\]
where \(C_f = 2\|f\|_{L^\infty((0,\infty); L^2(\Omega))} + \|f\|_{BV((0,\infty); L^2(\Omega))}\). From Theorem 3.2, we then have for \(n \geq 2\),
\[
\|\nabla u^n\|_{L^2(\Omega)}^2 \leq \|\nabla u^0\|_{L^2(\Omega)}^2 + 2C_f C_\Omega \max_{0 \leq k \leq n} \|\nabla u^k\|_{L^2(\Omega)}.
\]
Note that (4.6) implies that (4.9) also holds for \(n = 1\). For any \(N \geq 1\), we take \(\max_{0 \leq k \leq n} \|\nabla u^k\|_{L^2(\Omega)}\) on both sides of (4.9), to obtain
\[
\max_{0 \leq n \leq N} \|\nabla u^n\|_{L^2(\Omega)}^2 \leq \|\nabla u^0\|_{L^2(\Omega)}^2 + 2C_f C_\Omega \max_{0 \leq n \leq N} \|\nabla u^n\|_{L^2(\Omega)},
\]
which indicates
\[
\max_{0 \leq n \leq N} \|\nabla u^n\|_{L^2(\Omega)} \leq C_f C_\Omega + \sqrt{(C_f C_\Omega)^2 + \|\nabla u^0\|_{L^2(\Omega)}^2}
\]
\[
\leq \|\nabla u^0\|_{L^2(\Omega)} + 2C_f C_\Omega.
\]

**Remark 4.2.** In [20, 21], it is proved that the L1 scheme on an arbitrary nonuniform mesh and the L2 scheme on uniform meshes are energy stable for time-fractional gradient flows, where the source term \(f\) depends on \(u\).

**Remark 4.3.** In [15], Liao-Zhang consider the BDF2 scheme with nonuniform meshes for the diffusion equation (\(\alpha = 1\)) and prove that the scheme is stable if \(\rho_k \leq (3 + \sqrt{17})/2\). Their energy stability result is similar to Theorem 4.1, but for integer-order diffusion equation.
5. Application to the case of graded mesh. Consider the subdiffusion equation in finite time:

\[ \partial_\alpha t u(t, x) = \Delta u(t, x) + f(t, x), \quad (t, x) \in (0, T] \times \Omega, \]
\[ u(t, x) = 0, \quad (t, x) \in (0, T] \times \partial \Omega, \]
\[ u(0, x) = u^0(x), \quad x \in \Omega. \]

The graded mesh with grading parameter \( r > 1 \) is given by

\[ t_j = \left( \frac{j}{K} \right)^r T, \quad \tau_j = t_j - t_{j-1} = \left[ \left( \frac{j}{K} \right)^r - \left( \frac{j-1}{K} \right)^r \right] T. \]

The L2 scheme of the subdiffusion equation is still written as

\[ L^\alpha_k u = \Delta u^k + f^k \quad \text{with} \quad f^k = f(t_k, \cdot). \]

Recall that the constraint (3.9) for \( k = 2 \) in Theorem 3.2 is

\[ 2 + \frac{2}{1 + \rho_3} + \frac{4\rho_2}{1 + \rho_2} - \frac{\rho_2^3}{(1 + \rho_2)^2} \geq 0, \]

which gives a restriction on \( r \) for the graded mesh (5.2),

\[ 1 < r \leq 3.1253645. \]

Moreover, it is easy to check that if (5.5) is satisfied, \( \rho_3 \in [\rho_L, \rho_R] \). Since \( \rho_k \) decreases w.r.t. \( k \geq 2 \), all constraints in Theorem 3.2 are satisfied when (5.5) holds. Therefore, the \( H^1 \)-stability can be established if \( 1 < r \leq 3.1253645 \) according to Theorem 4.1.

However, we can provide an even better result on the constraint of \( r \) by improving the splitting of \( B + B^T \) in the proof of Theorem 3.2.

Theorem 5.1. Assume that \( f(t, x) \in L^\infty([0, T]; L^2(\Omega)) \cap BV([0, T]; L^2(\Omega)) \) is a bounded variation function in time and \( u^0 \in H^1_0(\Omega) \). If the graded mesh defined by (5.2) satisfies \( 1 < r \leq 3.2016538 \), then the numerical solution \( u^n \) of the L2 scheme (5.3) satisfies the following \( H^1 \)-stability:

\[ \| \nabla u^n \|_{L^2(\Omega)} \leq \| \nabla u^0 \|_{L^2(\Omega)} + 2C_f C_\Omega, \]

where \( C_f \) depends on the source term \( f \), \( C_\Omega \) is the Sobolev embedding constant depending on \( \Omega \) and the dimension \( d \).

Proof. We only need to prove the positive semidefiniteness of \( B_n(u, u) \) for the graded mesh. As in the proof of Theorem 3.2, \( A + A^T \) is positive semidefinite for the graded mesh due to \( \rho_k > 1 \). Now we consider the following splitting

\[ B + B^T = \left( \begin{array}{cc} C_0 & 0 \\ 0 & 0 \end{array} \right)_{n \times n} + \left( \begin{array}{cc} 0 & 0 \\ 0 & D_0 \end{array} \right)_{n \times n}, \]

where

\[ C_0 = \left( \begin{array}{cc} C_1 & 0 \\ 0 & 0 \end{array} \right)_{5 \times 5} + \left( \begin{array}{cc} 0 & 0 \\ 0 & C_2 \end{array} \right)_{5 \times 5} + \left( \begin{array}{cc} 0 & 0 \\ 0 & C_3 \end{array} \right)_{5 \times 5} + \left( \begin{array}{cc} 0 & 0 \\ 0 & C_4 \end{array} \right)_{5 \times 5}. \]
with

\[
C_1 = \begin{pmatrix} 
2(1 - \alpha)^{-1} & -\alpha_1 \\
-\alpha_1 & 2c_1^2 + 2c_2^2 - 2\beta_2 - 0.7013a_3^3 \\
\end{pmatrix}_{2 \times 2},
\]

\[
C_2 = \begin{pmatrix} 
0.7013a_3^3 & -\alpha_3^3 \\
-\alpha_3^3 & 2c_2^2 + 2c_3^2 - 2\beta_3 - 0.45473a_3^4 \\
\end{pmatrix}_{2 \times 2},
\]

\[
C_3 = \begin{pmatrix} 
0.45473a_4^4 & -\alpha_4^4 \\
-\alpha_4^4 & 2c_3^2 + 2c_4^2 - 2\beta_4 - 0.4131a_5^5 \\
\end{pmatrix}_{2 \times 2},
\]

\[
C_4 = \begin{pmatrix} 
0.4131a_5^5 & -\alpha_5^5 \\
-\alpha_5^5 & 2c_4^2 + 2c_5^2 - 2\beta_5 - a_6^6 \\
\end{pmatrix}_{2 \times 2},
\]

and

\[
D_0 = \begin{pmatrix} 
a_6^6 & -a_6^6 & \ldots & -a_6^6 \\
-a_6^6 & 2c_5^2 + 2c_6^2 - 2\beta_6 & -a_7^7 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
-a_7^7 & \ldots & \ldots & 2c_{n-1}^n + 2c_n^n - 2\beta_n \\
\end{pmatrix}_{(n-4) \times (n-4)}.
\]

Here, we consider the case of \( n \geq 5 \) by default, while in the case of \( n \leq 4 \), the proof is even simpler.

We now study the range of \( r \) to ensure the positive semidefiniteness of \( C_1, C_2, C_3, \) and \( C_4 \). Note that from (3.17), we have

\[
[C_1]_{11} = [C]_{11} > \frac{1 + \alpha}{(1 - \alpha)\tau_1^2} > 0.
\]

Similar as (3.20), we have the following inequalities

\[
[C_1]_{22} > \frac{\alpha}{(2 - \alpha)(1 - \alpha)\tau_2^2} \left( \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_2 - 1 + \alpha}{1 + \rho_2} + \frac{1.7013\rho_3^{2 - \alpha}}{1 + \rho_3} + \frac{2\rho_2}{1 + \rho_2} \right),
\]

\[
[C_2]_{22} > \frac{\alpha}{(2 - \alpha)(1 - \alpha)\tau_2^2} \left( \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_4 - 1 + \alpha}{1 + \rho_4} + \frac{1.45473\rho_4^{2 - \alpha}}{1 + \rho_4} + \frac{2\rho_4}{1 + \rho_4} \right),
\]

\[
[C_3]_{22} > \frac{\alpha}{(2 - \alpha)(1 - \alpha)\tau_4^2} \left( \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_5 - 1 + \alpha}{1 + \rho_5} + \frac{1.4131\rho_5^{2 - \alpha}}{1 + \rho_5} + \frac{2\rho_5}{1 + \rho_5} \right),
\]

\[
[C_4]_{22} > \frac{\alpha}{(2 - \alpha)(1 - \alpha)\tau_5^2} \left( \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_6 - 1 + \alpha}{1 + \rho_6} + \frac{2\rho_6^{2 - \alpha}}{1 + \rho_6} + \frac{2\rho_5}{1 + \rho_5} \right).
\]

From (2.5), (5.11) and (5.12), we have

\[
[C_1]_{11}[C_1]_{22} - [C_1]_{12}[C_1]_{21} > \frac{\alpha^2}{(1 - \alpha)^2(2 - \alpha)^22^2\kappa_1(\alpha)},
\]

\[
[C_2]_{11}[C_2]_{22} - [C_2]_{12}[C_2]_{21} > \frac{0.7013\alpha_3^3}{(1 - \alpha)(2 - \alpha)\tau_3^3}\kappa_2(\alpha),
\]

\[
[C_3]_{11}[C_3]_{22} - [C_3]_{12}[C_3]_{21} > \frac{0.45473\alpha_4^4}{(1 - \alpha)(2 - \alpha)\tau_4^4}\kappa_3(\alpha),
\]

\[
[C_4]_{11}[C_4]_{22} - [C_4]_{12}[C_4]_{21} > \frac{0.4131\alpha_5^5}{(1 - \alpha)(2 - \alpha)\tau_5^5}\kappa_4(\alpha),
\]
where

\[ \kappa_1(\alpha) = \frac{(1 + \alpha)(2 - \alpha)\rho_2^2}{\alpha} \left( \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_3 - 1 + \alpha}{(1 + \rho_3)^\alpha} - \frac{1.7013\rho_3^{-\alpha}}{1 + \rho_3} + \frac{2\rho_2}{1 + \rho_2} \right) \]

\[ - \left( \frac{\rho_2^2}{1 + \rho_2} \right)^2, \]

\[ \kappa_2(\alpha) = \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_4 - 1 + \alpha}{(1 + \rho_4)^\alpha} - \frac{1.45473\rho_4^{-\alpha}}{1 + \rho_4} + \frac{2\rho_3}{1 + \rho_3} - \frac{\rho_3^2}{0.7013(1 + \rho_3)}, \]

\[ \kappa_3(\alpha) = \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_5 - 1 + \alpha}{(1 + \rho_5)^\alpha} - \frac{1.4131\rho_5^{-\alpha}}{1 + \rho_5} + \frac{2\rho_4}{1 + \rho_4} - \frac{\rho_3^2}{0.45473(1 + \rho_4)}, \]

\[ \kappa_4(\alpha) = \frac{(1 + \alpha)(2 - \alpha)}{\alpha} + \frac{\rho_6 - 1 + \alpha}{(1 + \rho_6)^\alpha} - \frac{2\rho_6^{-\alpha}}{1 + \rho_6} + \frac{2\rho_5}{1 + \rho_5} - \frac{\rho_3^2}{0.4131(1 + \rho_5)}. \]

Here for the graded mesh with grading parameter \( r > 1, \)

\[ \rho_k = \frac{k^r - (k - 1)^r}{(k - 1)^r - (k - 2)^r}, \quad k \geq 2, \]

depends only on \( k \) and \( r. \) In Figure 5, we illustrate \( \kappa'_1(\alpha), \kappa'_2(\alpha), \kappa'_3(\alpha), \kappa'_4(\alpha) \) w.r.t. \( \alpha \in (0, 1) \) and \( r \in [1, 3.25] \) for the graded mesh. It can be observed that \( \kappa'_i(\alpha) \leq 0, \quad i = 1, 2, 3, 4 \) for \( \alpha \in (0, 1) \) and \( r \in [1, 3.25]. \) A more rigorous proof can be provided but is omitted here due to the length of this work. Thus for any fixed \( r \in [1, 3.25], \kappa_i \) decreases w.r.t. \( \alpha \in (0, 1). \)

![Figure 5](image)

**Fig. 5.** \( \kappa'_1(\alpha), \kappa'_2(\alpha), \kappa'_3(\alpha), \kappa'_4(\alpha) \) w.r.t. \( (\alpha, r) \) where \( \alpha \in (0, 1) \) and \( r \in [1, 3.25]. \)

To ensure \( \kappa_1(\alpha) \geq 0, \) we need to impose \( \kappa_1(1) \geq 0, \) i.e.,

\[ (5.13) \quad \rho_2^{-1} \kappa_1(1) = 4 - \frac{1.4026\rho_3}{1 + \rho_3} + \frac{4\rho_2}{1 + \rho_2} - \frac{\rho_3^2}{(1 + \rho_2)^2} \geq 0, \]

which results in

\[ (5.14) \quad 1 < r \leq 3.201653814682024. \]
Further, when (5.14) holds, it is easy to verify that \( \kappa_i(1) \geq 0 \), \( i = 2, 3, 4 \) implying that \( \kappa_i(\alpha) \geq 0 \) for all \( \alpha \in (0, 1) \). Since \( [C_i]_{11} > 0 \), \( i = 1, 2, 3, 4 \), the \( 2 \times 2 \) matrices \( C_i \) are positive semidefinite when (5.14) holds.

Similar to the proof of Theorem 3.2, the positive semidefiniteness of \( D_0 \) can be guaranteed because \( \rho_k \in (1, \rho_0] \subset [\rho_L, \rho_R] \) for any \( k \geq 6 \), when (5.14) holds. The proof is completed.

To better understand Theorem 5.1, we do a numerical test on the matrix \( M \) defined in (3.12) for the graded mesh. We take \( T = 1, n = K = 7, \alpha = 0.99999 \) and \( r = 3.20185 \) (slightly larger than 3.2016538 in Theorem 5.1). As a consequence, the symmetric matrix \( M + M^T \) has a negative eigenvalue, i.e., \( M + M^T \) is not positive semidefinite. This indicates that the constraint \( r \leq 3.2016538 \) in Theorem 5.1 is almost optimal to ensure the positive semidefiniteness of \( B_n(u, u) \).

6. Numerical tests. We propose a new type of graded mesh with varying grading parameter \( r_j \), called \( r \)-variable graded mesh. According to our theoretical analysis, the L2 scheme on this \( r \)-variable graded mesh is \( H^1 \)-stable. We do some first tests on the convergence rate of \( H^1 \)-error this \( r \)-variable graded mesh and compare it with the standard graded (where \( r \) is fixed).

Consider the following subdiffusion equation with zero Dirichlet boundary condition

\[
\partial^\alpha_t u(t, x) = \varepsilon^2 \Delta u(t, x) + f(t, x), \quad (t, x) \in (0, T] \times \Omega,
\]

where \( \varepsilon = 0.1, T = 1, \Omega = [0, 2\pi]^2, \) and \( f(t, x) = (\Gamma(1 + \alpha) + \varepsilon^2 t^\alpha) \sin(x) \sin(y) \). The exact solution of this subdiffusion equation is \( u(t, x) = t^\alpha \sin(x) \sin(y) \).

We use the finite (central) difference method for space discretization with grid spacing size \( h = 2\pi/1000 \). We compare two different nonuniform time meshes: the \( r \)-variable graded mesh (5.2) with fixed \( r = 2.8 \) and the following graded mesh with varying grading parameter \( r_j \):

\[
r_j = 3.1 - \frac{0.6(j - 6)}{K - 6}, \quad j \geq 6, \quad t_j = \left( \frac{j}{K} \right)^{r_j} T, \quad \tau_j = \left[ \left( \frac{j}{K} \right)^{r_j} - \left( \frac{j - 1}{K} \right)^{r_j - 1} \right],
\]

where \( K \geq 6 \) and \( 1_{j \geq 6} \) denotes the Heaviside function. Note that \( r_j = 3.1 \) for \( j = 1, \ldots, 6 \). A graphical illustration of \( \tau_j \) for \( K = 100 \) is given in Figure 6. According to Theorem 4.1, the L2 scheme on this \( r \)-variable graded mesh is \( H^1 \)-stable. In [22, Remark 5.6] and [8, Remark 2.5], the authors state that the large value of \( r \) in the graded mesh increases the temporal mesh width near \( t = T \) which can lead to large errors. The \( r \)-variable graded mesh avoid this problem by changing the grading parameter from large to small. However the rigorous proof of its convergence rate still needs to be studied.

We show the \( H^1 \)-errors at \( T = 1 \) with different \( K \) for the L2 scheme of the subdiffusion equation (6.1) in Table 1 for \( \alpha = 0.3, 0.5, 0.7 \). These \( H^1 \)-errors are also plotted in Figure 7. It can be observed that in this example the \( r \)-variable graded mesh performs better than the graded mesh.

Acknowledgements. C. Quan is supported by NSFC Grant 11901281, the Stable Support Plan Program of Shenzhen Natural Science Fund (Program Contract No. 20200925160747003), and Shenzhen Science and Technology Program (Grant No. RCYX20210609104358076).

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Fig. 6. Time steps for graded mesh with $r = 2.8$ and $r$-variable graded mesh (6.2), where $K = 100$.

Table 1

$H^1$-errors at $t = T$ for the graded mesh with $r = 2.8$ and the $r$-variable graded mesh with $r_j$ defined in (6.2) where $\alpha = 0.3$ (top), 0.5 (middle), 0.7 (bottom) respectively.

|        | K = 50 | K = 60 | K = 70 | K = 80 | K = 90 | K = 100 |
|--------|--------|--------|--------|--------|--------|---------|
| graded |        |        |        |        |        |         |
|        | 2.0219e-4 | 1.2304e-4 | 8.08539e-5 | 5.6248e-5 | 4.0812e-5 | 3.0776e-5 |
|        | 2.7243 | 2.7239 | 2.7174 | 2.7235 | 2.6789 |         |
| r-variable | 1.6057e-4 | 9.4700e-5 | 6.0753e-5 | 4.1624e-5 | 2.9302e-5 | 2.2258e-05 |
|        | 2.8964 | 2.8796 | 2.8318 | 2.9804 | 2.6997 |         |

|        | K = 50 | K = 60 | K = 70 | K = 80 | K = 90 | K = 100 |
|--------|--------|--------|--------|--------|--------|---------|
| graded |        |        |        |        |        |         |
|        | 7.3684e-4 | 4.6472e-4 | 3.1476e-4 | 2.2463e-4 | 1.6683e-4 | 1.2788e-4 |
|        | 2.5281 | 2.5275 | 2.5264 | 2.5254 | 2.5235 |         |
| r-variable | 6.7483e-4 | 4.1850e-4 | 2.7970e-4 | 1.9745e-4 | 1.4530e-4 | 11053 |
|        | 2.6205 | 2.6140 | 2.6078 | 2.6038 | 2.5955 |         |

|        | K = 50 | K = 60 | K = 70 | K = 80 | K = 90 | K = 100 |
|--------|--------|--------|--------|--------|--------|---------|
| graded |        |        |        |        |        |         |
|        | 1.4152e-3 | 9.2587e-4 | 6.4677e-4 | 4.7404e-4 | 3.6944e-4 | 2.8212e-4 |
|        | 2.3273 | 2.3272 | 2.3267 | 2.3261 | 2.3253 |         |
| r-variable | 1.4002e-3 | 9.0549e-4 | 6.2692e-4 | 4.5628e-4 | 3.4496e-4 | 2.6873e-4 |
|        | 2.3913 | 2.3849 | 2.3793 | 2.3744 | 2.3701 |         |

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Fig. 7. $H^1$-errors at $t = T$ of numerical solutions of $L2$ schemes on graded mesh ($r = 2.8$) and $r$-variable graded mesh (6.2) with $\alpha = 0.3, 0.5, 0.7$ (from left to right).

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