Numerical analysis of a wave equation for lossy media obeying a frequency power law

Katherine Baker *  Lehel Banjai †

December 9, 2020

Abstract

We study a wave equation with a nonlocal time fractional damping term that models the effects of acoustic attenuation characterized by a frequency dependence power law. First we prove existence of unique solutions to this equation with particular attention paid to the handling of the fractional derivative. Then we derive an explicit time stepping scheme based on the finite element method in space and a combination of convolution quadrature and second order central differences in time. We conduct a full error analysis of the mixed time discretization and in turn the fully space time discretized scheme. A number of numerical results are presented to support the error analysis for both smooth and nonsmooth solutions.

Keywords: fractional calculus; wave equation; convergence; existence and uniqueness.

1 Introduction

We are interested in the initial boundary problem given by a wave equation with the addition of a time fractional damping term on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d \) with boundary \( \partial \Omega \),

\[
\frac{1}{c^2} \partial_t^2 u - \Delta u + \frac{\alpha}{c} \partial_t^{\gamma+1} u = f, \quad \text{in } \Omega \times [0, T] \\
u(\cdot, 0) = u_0 \quad \text{in } \Omega \\
\partial_t u(\cdot, 0) = v_0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega \times [0, T],
\]

*Maxwell Institute for Mathematical Sciences, Department of Mathematics, Heriot-Watt University, Edinburgh, UK, EH14 4AS. (kb54@hw.ac.uk)
†Maxwell Institute for Mathematical Sciences, Department of Mathematics, Heriot-Watt University, Edinburgh, UK, EH14 4AS. (l.banjai@hw.ac.uk)
Here $\gamma \in (-1, 1) \setminus \{0\}$, $c(x) > 0$ is the wave speed, and
\[
a_\gamma = -\alpha_0 \frac{4}{\pi} \Gamma(-\gamma - 1) \Gamma(\gamma + 2) \cos((\gamma + 1)\pi/2),
\]  
(2)
for some constant $\alpha_0 > 0$. Note that $a_\gamma > 0$ for $\gamma \in (-1, 1)$ and that $a_\gamma \to \infty$ as $\gamma \to \pm 1$. In this problem $\partial_t^\gamma u$ denotes the Caputo fractional derivative; see Definition 2.2 and we only consider $\gamma \neq 0$ throughout. For the rest of the paper, to simplify notation we set $c \equiv 1$, however we do keep track of the constant $a_\gamma$.

The interest in this problem stems from the modelling of acoustic attenuation that occurs as a wave propagates through lossy media [28, Chapter 4]. In applications, this is applied to modelling high intensity focused ultrasound therapy (HIFU) where the lossy media is biological tissue. HIFU is a noninvasive and nonionising medical treatment that focuses multiple high intensity acoustic pressure waves on a region, ablating it away ([30]). It is known that acoustic attenuation obeys a frequency dependence characterized by the following power law,
\[
S(\mathbf{x} + \Delta \mathbf{x}) = S(\mathbf{x}) e^{-\alpha(\omega)|\mathbf{x}|}
\]  
(3)
where $S$ is the amplitude, $\Delta \mathbf{x}$ is the wave propagation distance, $\omega$ denotes frequency and $\alpha(\omega)$ the attenuation coefficient which defined by
\[
\alpha(\omega) = \alpha_0 |\omega|^\gamma
\]
for $\gamma$ the frequency power exponent and $\alpha_0$ a constant relating to the media ([7]). Values of $\gamma$ have been determined by many experiments and field measurements, their results conclude that for most media $\gamma \in (0, 2)$ ([10]). In this paper we restrict our attention to $\gamma \in (-1, 1)$. For the range $\gamma \in (1, 2)$ (and potentially also (0, 1)) we prefer changing the weak damping term $\partial_t^{\gamma+1}$ with the strong damping of the form $(-\Delta)^{\gamma-1}$ ([6, 12, 13, 32]). The techniques developed in this paper, can be used to analyze the strongly damped case for the full range $\gamma \in (0, 2)$; see ([1]). Another model that we can consider is the nonlinear fractional Westervelt equation ([12]), this contains the strong damping term mentioned above and describes the nonlinear behaviour that the high intensity focusing causes during HIFU. Similar models appear also in fractional order viscoelasticity, similar to both the strong damping case [14, 25] and the weak damping case investigated here [23].

[27] derived a wave equation with a convolution type operator to incorporate the effects of acoustic attenuation and showed that it adheres to the frequency dependence power law ([3]. [7] praise this model for its simplicity, due to it only containing two parameters, but criticise it for being difficult to implement initial conditions. To overcome this they refined the model to include the Caputo fractional derivative as the damping term. Furthermore within their paper they show that even with this modification the solutions still obey the power law.

Issues that arise within this model largely stem from the convolution based fractional derivative that requires us to store the full history when numerically solving with a time stepping scheme, making it expensive to compute, particularly in 3D. In the literature this is countered by replacing the fractional time
derivative with a fractional Laplacian ([8, 31]). However, using efficient quadrature methods ([2]) will also be sufficient in reducing memory requirements and allows us to remain using the simpler operator.

The paper is structured in the following way. Section 2 begins by outlining necessary definitions and lemmas required to prove existence and uniqueness of solutions to (1). In Section 3 we develop a numerical scheme to approximate solutions of (1) using finite element methods in space and a combination of second order central difference and convolution quadrature in time. In Section 4 we study the convergence analysis of the mixed time discretization used which allows us to determine the convergence order of the full scheme. Lastly, in Section 5 we show results from the implementation of our scheme to support the theory from the previous section for both smooth and non smooth solutions of (1).

2 A damped wave equation model

In this section we describe the model with the fractional in time weak damping and prove existence and uniqueness of the solution. In order to do this we will require some properties of fractional calculus, that we list first. For more detail on fractional calculus see textbooks by [9] and [22].

Before proceeding we introduce the following notation:

- $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product
- $\| \cdot \| = \| \cdot \|_{L^2(\Omega)}$ denotes the $L^2(\Omega)$ norm
- $\| \cdot \|_p$ denotes the $H^p(\Omega)$ norm
- $\mathcal{C}^k(0, T; X)$ denotes the space of functions $f : (0, T) \to X$, that are $k$ times continuously differentiable in time with an associated Hilbert space $X$.
- The space of $H^1(\Omega)$ functions with a zero trace on $\partial \Omega$ is denoted by $H^1_0(\Omega)$.

We first give the definition of the Riemann Liouville fractional integral.

**Definition 2.1.** For $\beta > 0$ the Riemann Liouville fractional integral of a function $f$ is defined by

$$I^\beta_t f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} f(\tau) d\tau, \quad t > 0,$$

where $\Gamma$ denotes the Gamma function.

If $f(t) = t^\mu$ for $\mu > -1$ we have that

$$I^\beta_t t^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \beta + 1)} t^{\beta + \mu}. \quad (4)$$

The Caputo fractional derivative is obtained by applying the Riemann Liouville fractional integral to a classical derivative of the function.
Definition 2.2. For $\gamma > -1$ and $n = \lceil \gamma \rceil$ the Caputo fractional derivative of a function $f$ is defined by

$$
\partial_t^\gamma f(t) := \begin{cases} 
\frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\tau)^{n-\gamma-1} f^{(n)}(\tau) \, d\tau = I_t^{n-\gamma} f^{(n)}(t) & \gamma \notin \mathbb{N} \\
\frac{d^n}{dt^n} f(t) & \gamma \in \mathbb{N},
\end{cases}
$$

for $t > 0$.

Again, an explicit formula can be given in case where $f(t) = t^\mu$ for $\mu > 0$ we have that

$$
\partial_t^\gamma t^\mu = \begin{cases} 
0 & \gamma > \mu \\
\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\gamma)} t^{\mu-\gamma} & \gamma \leq \mu.
\end{cases} \tag{5}
$$

Lemma 2.1. (a) For $\gamma > 0$ and any $f \in C^1[0,T]$

$$
\partial_t^\gamma f(t) = \partial_t^{\gamma-1} \partial_t f(t), \quad t \in [0,T].
$$

(b) For $\gamma > 0$ and any $f \in C^n[0,T]$ such that $f^{(n-1)}(0) = 0$, $n = \lceil \gamma \rceil$,

$$
\partial_t^\gamma f(t) = \partial_t \partial_t^{\gamma-1} f(t), \quad t \in [0,T].
$$

(c) The semi group property holds for fractional integrals: for any $f \in C[0,T]$ and $\alpha, \beta \geq 0$

$$
I_t^\alpha I_t^\beta f(t) = I_t^{\alpha+\beta} f(t), \quad t \in [0,T].
$$

From the definition of the fractional derivative we see that the model (1) is a Volterra integro differential equation. To proceed with the analysis we will need a result on the corresponding ordinary differential equation.

Theorem 2.2. For $f \in C[0,T]$, $\gamma \in (-1,1)$, $\lambda \in \mathbb{C}$, $a_\gamma > 0$, and $u_0$, $v_0 \in \mathbb{R}$ there exists a unique solution $u \in C^2[0,T]$ to the initial value problem,

$$
u'' + \lambda u + a_\gamma \partial_t^{\gamma+1} u = f \quad \text{in } [0,T]
$$

$$
u(0) = u_0
$$

$$
\nu'(0) = v_0. \tag{6}
$$

Furthermore, for $\gamma \in (-1,0)$

$$
u''(t) = f(t) - \lambda u_0 - \frac{a_\gamma}{\Gamma(1-\gamma)} t^{-\gamma} v_0 + O(t),
$$

and for $\gamma \in (0,1)$

$$
u''(t) = f(t) - \lambda u_0 - \frac{a_\gamma}{\Gamma(2-\gamma)} (f(0) - \lambda u_0) t^{1-\gamma} + o(t^{1-\gamma})
$$

as $t \to 0^+$. 

4
Proof. For $\gamma = 0$ we omit the proof since in this case the result follows from standard ODE theory.

Using the definition of the Caputo derivative in (6) gives

$$u'' + \lambda u + \frac{a_{\gamma}}{\Gamma(n - \gamma)} \int_0^t (t - \tau)^{n - \gamma - 1} u^{(n+1)}(\tau) d\tau = f, \quad (7)$$

where $n = \lceil \gamma \rceil$. Next, define $v = u''$, i.e.,

$$u(t) = u_0 + tv_0 + \int_0^t (t - \tau)v(\tau) d\tau. \quad (8)$$

We first consider the case $\gamma \in (0, 1)$, i.e., $n = 1$. By substituting $v$ into (7) we obtain a Volterra integral equation for $v$:

$$v(t) = f - \lambda u_0 - \lambda tv_0 - \int_0^t \lambda(t - \tau)v(\tau) d\tau - \frac{a_{\gamma}}{\Gamma(1 - \gamma)} \int_0^t (t - \tau)^{-\gamma} v(\tau) d\tau.$$

This Volterra equation for $v$ can be written in the form investigated in [5, Section 6.1.2], where the existence of a unique solution $v \in C[0, T]$ is proved; see Theorem [A.2] in the appendix. Hence $u$ defined by (8) is in $C^2[0, T]$ and solves the original equation. The asymptotic behaviour of $u''$ follows from Theorem [A.2] giving

$$u''(t) = f(t) - \lambda u_0 - \frac{a_{\gamma}}{\Gamma(2 - \gamma)} v(0)t^{1-\gamma} + o(t^{1-\gamma})$$

$$= f(t) - \lambda u_0 - \frac{a_{\gamma}}{\Gamma(2 - \gamma)} (f(0) - \lambda u_0)t^{1-\gamma} + o(t^{1-\gamma}).$$

If $\gamma \in (-1, 0)$, the equation is given by

$$u'' + \lambda u + \frac{a_{\gamma}}{\Gamma(-\gamma)} \int_0^t (t - \tau)^{-\gamma - 1} u'(\tau) d\tau = f. \quad (9)$$

Again we wish to let $v = u''$ but to do so directly we must write the integral term in the above equation as

$$\frac{1}{\Gamma(-\gamma)} \int_0^t (t - \tau)^{-\gamma - 1} u'(\tau) d\tau = \frac{1}{\Gamma(1 - \gamma)} t^{-\gamma} v_0 + \frac{1}{\Gamma(1 - \gamma)} \int_0^t (t - \tau)^{-\gamma} v(\tau) d\tau.$$

Now we substitute the above equation and (8) into the ODE (9) and we have,

$$v(t) = f - \lambda u_0 - \lambda tv_0 - \frac{a_{\gamma}}{\Gamma(1 - \gamma)} t^{-\gamma} v_0 - \int_0^t \lambda(t - \tau)v(\tau) d\tau$$

$$- \frac{a_{\gamma}}{\Gamma(1 - \gamma)} \int_0^t (t - \tau)^{-\gamma} v(\tau) d\tau.$$

This is now a Volterra integral equation with a continuous kernel. Hence, see [5, Theorem 2.1.5] and Theorem [A.2] in the appendix, a unique solution $v \in C[0, T]$ exists and consequently also the solution $u \in C^2[0, T]$ of the original equation. The asymptotic behaviour of $u''$ again follows from Theorem [A.2].
Next we show that the time-fractional term satisfies a positivity result that will in turn imply its damping properties.

**Lemma 2.3.** Let $f \in C([0, T]; L^2(\Omega))$, $g \in C^1([0, T]; L^2(\Omega))$ and let $\gamma \in (0, 1)$. Then the following hold:

(a) $$\int_0^T (\partial_t I_\gamma g(t), g(t)) \, dt \geq \frac{(T/2)^{1-\gamma}}{\Gamma(\gamma)} \int_0^T \|g(t)\|^2 \, dt.$$

(b) $$\int_0^T (I_\gamma f(t), f(t)) \, dt \geq \frac{(T/2)^{-\gamma}}{\Gamma(1-\gamma)} \int_0^T \|I_\gamma f(t)\|^2 \, dt.$$

**Proof.** The proof of an analogue of (a) for Caputo fractional derivatives is given in [4]. The result can also be deduced from Lemma 1.7.2 in [29] and Lemma 3.1 in [23], which for our case gives

$$\int_0^T (\partial_t I_\gamma g(t), g(t)) \, dt \geq \frac{1}{2\Gamma(\gamma)} \int_0^T ((T-t)^{\gamma-1} + t^{\gamma-1}) \|g(t)\|^2 \, dt.$$

Minimizing the kernel gives the required result

$$\int_0^T (\partial_t I_\gamma g(t), g(t)) \, dt \geq \frac{(T/2)^{\gamma-1}}{\Gamma(\gamma)} \int_0^T \|g(\tau)\|^2 \, d\tau.$$

This inequality but with a different constant is given in [21, Theorem A.1]. A numerical test indicates that the constant derived here is larger than the constant in [21, Theorem A.1] for all $\nu \in (0, 1)$.

The second part of the lemma follows by using the semigroup property, see Lemma 2.1(c),

$$\int_0^T (I_\gamma f(t), f(t)) \, dt = \int_0^T (I_\gamma f(t), \partial_t I_\gamma^{-\gamma} I_\gamma f(t)) \, dt$$

and then applying part (a).

Next we investigate the existence and uniqueness of the solution of the weak formulation of (11): Find $u(t) \in H_0^1(\Omega)$ such that

$$(\partial_t^2 u, v) + (\nabla u, \nabla v) + (a, \partial_t^{-1} u, v) = (f, v), \quad \text{for all } v \in H_0^1(\Omega). \quad (10)$$

**Theorem 2.4.** Given $u_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$ and $\gamma \in (-1, 0)$ a unique weak solution $u \in L^\infty(0, T; H_0^1(\Omega))$ of (11) exists and we have that $\partial_t^{\gamma+1} u \in L^\infty(0, T; L^2(\Omega))$, $\partial_t^{\gamma} u \in L^2(0, T; H^{-1}(\Omega))$ and $\partial_t u \in L^\infty(0, T; L^2(\Omega))$. 

6
Proof. We follow the standard proof for the wave equation as described, e.g., in [11, 17], with modifications required due to the fractional damping term.

Let \( \{ \omega_k \}_{k=1}^m \) be an orthogonal basis of \( H^1_0(\Omega) \) and an orthonormal basis of \( L^2(\Omega) \) with corresponding eigenvalues \( \{ \lambda_k \}_{k=1}^m \). We look for \( u_m \) of the form

\[
    u_m(t) = \sum_{k=1}^m d_k^m(t) \omega_k
\]

satisfying

\[
    (\partial_t^2 u_m, \omega_k) + (\nabla u_m, \nabla \omega_k) + \left( a_\gamma \partial_\gamma^{\gamma+1} u_m, \omega_k \right) = (f, \omega_k), \quad (11)
\]

and

\[
    d_k^m(0) = (u_0, \omega_k) \quad \text{and} \quad \frac{d}{dt} d_k^m(0) = (v_0, \omega_k), \quad (12)
\]

for \( k = 1, 2, \ldots, m \). As this problem is equivalent to

\[
    \partial_t^2 d_k^m(t) + \lambda_k d_k^m(t) + a_\gamma \partial_\gamma^{\gamma+1} d_k^m(t) = (f, \omega_k),
\]

Theorem 2.2 shows that a unique solution \( d_k^m \in C^2[0, T] \) exists for all \( k \leq m \).

Testing (11) with \( \partial_t u_m \), integrating in time, and using Lemma 2.3b we obtain

\[
    E(T; u_m) \leq E(0; u_m) + \int_0^T (f, \partial_t u_m) \, dt,
\]

where the energy is given by

\[
    E(t; v) = \frac{1}{2} \| \partial_t v \|^2 + \frac{1}{2} \| \nabla v \|^2.
\]

Applying Cauchy Schwarz inequality, the definition of the energy and the Gronwall inequality in the usual way we obtain that the energy is bounded independently of \( m \)

\[
    E(T; u_m) \leq C \left( \| \nabla u_0 \|^2 + \| v_0 \|^2 + \int_0^T \| f \|^2 \, dt \right), \quad (13)
\]

for a constant \( C = C(T) \).

In the standard way, see [11], we obtain a bound on \( \partial_t^2 u_m \)

\[
    \int_0^T \| \partial_t^2 u_m \|_{-1}^2 \, dt \leq C \left( \| \nabla u_0 \|^2 + \int_0^T \| f \|^2 + \| \partial_\gamma^{\gamma+1} u_m \|^2 \, dt \right). \quad (14)
\]

It remains to bound the term containing the fractional derivative.

Recalling that \( \gamma + 1 \in (0, 1) \), then using the definition of the Caputo derivative, see Definition 2.2, Young’s convolution inequality and (13) we deduce that

\[
    \int_0^T \| \partial_\gamma^{\gamma+1} u_m \|^2 \, dt \leq C \int_0^T \| \partial_t u_m \|^2 \, dt \leq C \left( \| \nabla u_0 \|^2 + \| v_0 \|^2 + \| f \|_{L^2(0, T; L^2(\Omega))}^2 \right), \quad (15)
\]
Returning to (14) we obtain the bound

\[ \int_0^T \| \partial_{tt}^2 u_m \|_{L^2}^2 \leq C \left( \| \nabla u_0 \|^2 + \| v_0 \|^2 + \| f \|^2_{L^2(0, T; L^2(\Omega))} \right). \]

Hence \( u_m \) has a subsequence that converges weakly to a \( u \) in the following spaces

\[ u \in L^\infty(0, T; H_0^1(\Omega)), \quad \partial_t u \in L^\infty(0, T; L^2(\Omega)), \]
\[ \partial_{tt}^2 u \in L^2(0, T; H^{-1}(\Omega)), \quad \partial_t^{\gamma+1} u \in L^\infty(0, T; L^2(\Omega)). \]

Further, in the usual way \( u \) is a weak solution of the original problem (11) and satisfies the initial condition (11).

Finally we will show that \( u \) is a unique weak solution. To do so we let \( f = 0 \), \( u_0 = 0 \) and \( v_0 = 0 \) and show that the weak solution is \( u \equiv 0 \). For a fixed \( s > 0 \), let

\[ v(t) = \begin{cases} 0 & 0 \leq t \leq s \\ \int_0^s u(\tau) \, d\tau & s < t \leq T \end{cases}. \]

Testing (11) with \( v \) we obtain

\[ 0 = (\partial_t^2 u, v) + (\nabla u, \nabla v) + (a_\gamma \partial_t^{\gamma+1} u, v). \]

Then integrating with respect to time,

\[ 0 = \int_0^s (\partial_t^2 u, v) + (\nabla u, \nabla v) + (a_\gamma \partial_t^{\gamma+1} u, v) \, dt = -\int_0^s (\partial_t u, \partial_t v) - (\nabla u, \nabla v) + (a_\gamma \partial_t^\gamma u, \partial_t v) \, dt \]

where we used Lemma 2.1b. Now by the definition of \( v \),

\[ \int_0^s \frac{1}{2} \partial_t \| u \|^2 \, dt - \int_0^s \frac{1}{2} \partial_t \| \nabla v \|^2 \, dt = -\int_0^s (a_\gamma \partial_t^\gamma u, u) \, dt \]

and thus

\[ \frac{1}{2} \| u(s) \|^2 + \frac{1}{2} \| \nabla v(0) \|^2 = -\int_0^s (a_\gamma \partial_t^\gamma u, u) \, dt \leq 0 \]

implying \( u \equiv 0 \) as \( a_\gamma > 0 \).

We now prove the corresponding result for \( \gamma \in (0, 1) \).

**Theorem 2.5.** Given \( u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \), \( v_0 \in H_0^1(\Omega) \), \( \partial_t f \in L^2(0, T; L^2(\Omega)) \) and \( \gamma \in (0, 1) \) a unique weak solution \( u \in L^\infty(0, T; H^1(\Omega)) \) of (11) exists and we have that \( \partial_t^{\gamma+1} u \in L^\infty(0, T; L^2(\Omega)), \partial_t^2 u \in L^2(0, T; H^{-1}(\Omega)) \) and \( \partial_t u \in L^\infty(0, T; L^2(\Omega)). \)

**Proof.** The only difficulty in extending the proof of Theorem 2.4 to the case of \( \gamma \in (0, 1) \) is the bound on \( \partial_t^{\gamma+1} \) in (15). To circumvent this problem we note that due to the additional smoothness assumption on \( f \), the solution \( u_m \in C^2[0, T] \) of (11) satisfies the time differentiated equation

\[ \partial_t^3 u_m - \Delta \partial_t u_m + a_\gamma \partial_t^{\gamma+1} u_m = \partial_t f. \]
Now testing with $\partial_t^2 u_m$ and using Lemma 2.3, we obtain the energy bound
\[ E(t; \partial_t u_m) \leq C \left( E(0; \partial_t u_m) + \int_0^t \| \partial_t f \|^2 \, dt \right). \]

As
\[ E(0; \partial_t u_m) = \frac{1}{2} \| \partial_t^2 u(0) \|^2 + \frac{1}{2} \| \nabla \partial_t u(0) \|^2 = \frac{1}{2} \| \Delta u_0 + f(0) \|^2 + \frac{1}{2} \| \nabla v_0 \|^2 \]
we have obtained a bound on $\partial_t^2 u$ and consequently using again Young's inequality
\[ \int_0^t \| \partial_t^{\gamma+1} u \|^2 \, dt \leq C \int_0^t \| \partial_t^2 u \|^2 \, dt \leq C \left( \| \Delta u_0 \|^2 + \| f(0) \|^2 + \| \nabla v_0 \|^2 + \int_0^t \| \partial_t f \|^2 \, dt \right). \]

Using this bound in the proof of Theorem 2.4 we obtain the result.

**Remark 1.** The fractional Zener wave equation investigated in [23] is of a similar form to the Szabo equation with $\gamma \in (0, 1)$; see equation (2.5) in [23]. With similar approach to ours, the authors prove uniqueness and existence of the Zener model. The solution in [23] is understood in a weaker sense, namely they require lower regularity of the data but do not give bounds on the second derivative in time of the solution.

**Remark 2.** As is usual in PDEs with time fractional derivatives, we expect a singularity at $t = 0$ even for smooth data. Namely, due to Theorem 2.2, we expect for smooth $f$ and $\gamma \in (0, 1)$
\[ u(x, t) = u_0(x) + tv_0(x) + t^2 w_0(x) + t^{3-\gamma} z_0(x) + o(t^{3-\gamma}). \]

The right hand side is then
\[ f(t) = \partial_t^2 u - \Delta u + a_\gamma \partial_t^{\gamma+1} u \]
\[ = 2w_0 + (3 - \gamma)(2 - \gamma)t^{1-\gamma} z_0 - \Delta u_0 + \frac{2a_\gamma}{\Gamma(2 - \gamma)} t^{1-\gamma} w_0 + o(t^{1-\gamma}) \]
If $f$ is to be smooth, we need to match the term $t^{1-\gamma}$ by setting $z_0 = -\frac{2a_\gamma}{\Gamma(3-\gamma)} w_0$.

On the other hand if $\gamma \in (-1, 0)$ we have
\[ u(x, t) = u_0(x) + tv_0(x) + t^2 w_0(x) + t^{2-\gamma} z_0(x) + o(t^{2-\gamma}). \]

The right hand side is then
\[ f(t) = 2w_0 + (2 - \gamma)(1 - \gamma)t^{-\gamma} z_0 - \Delta u_0 + \frac{a_\gamma}{\Gamma(1 - \gamma)} t^{-\gamma} v_0 + o(t^{-\gamma}) \]
If $f$ is to be smooth, we need to match the term $t^{-\gamma}$ by setting $z_0 = -\frac{a_\gamma}{\Gamma(3-\gamma)} v_0$.

Alternatively, for $u$ to be smooth we would need $f$ to have a singularity of the type $t^{1-\gamma}$ for $\gamma \in (0, 1)$ and $t^{-\gamma}$ for $\gamma \in (-1, 0)$ with further weaker singularities at $t = 0$.
3 Fully discrete system

To obtain the fully discrete system we will use a finite element method in space and a combination of leapfrog and BDF2 based convolution quadrature discretization in time. The motivation for using leapfrog is to obtain an explicit scheme, whereas the main motivation for using convolution quadrature are its excellent stability properties ([19, 18]) and the ability to evaluate it very efficiently ([2, 24]). An alternative discretization of the fractional time derivative is the L1 scheme ([26, 16]). This scheme also has the required stability property, i.e., it preserves the positivity property of the fractional derivative; see [26, Lemma 3.1].

3.1 Spatial semidiscretization

To discretize in space, we make use of a piecewise linear Galerkin finite element method. Namely let \( V^h \) be a family of finite dimensional subspaces of \( H^1_0(\Omega) \) parametrized by the meshwidth \( h > 0 \). We assume that these spaces satisfy the following approximation property

\[
\inf_{v^h \in V^h} \| v - v^h \|_1 \leq Ch \| v \|_2
\]

and

\[
\inf_{v^h \in V^h} \| v - v^h \| \leq Ch^2 \| v \|_2
\]

for some constant \( C > 0 \) independent of \( h \). Furthermore, we assume that an inverse inequality holds

\[
\sup_{v \in V^h} \| \nabla v \| \leq C_{\text{inv}} h^{-1} \| v \|, \quad (17)
\]

for some constant \( C_{\text{inv}} \).

The semidiscrete problem then reads: Find \( u^h(t) \in V^h \) such that

\[
(\partial_t^2 u^h, v) + (\nabla u^h, \nabla v) + (a_{\gamma} \partial_{\gamma+1}^1 t u^h, v) = (f, v), \quad \text{for all } v \in V^h.
\]

We denote projections of the initial data \( u_0 \) and \( v_0 \) onto \( V^h \) by \( u_0^h \) and \( v_0^h \) respectively; the projections will be specified later on.

3.2 Time discretization

In time we discretize using the explicit, leapfrog scheme

\[
\frac{1}{\kappa}(u^h_{n+1} - 2u^h_n + u^h_{n-1}, v) + (\nabla u^h_n, \nabla v) + (a_{\gamma} \partial_{\gamma+1}^1 n u^h_n, v) = (f(t_n), v),
\]

where \( t_n = n\kappa \) for some fixed time step \( \kappa > 0 \), \( u^h_n \in V^h \) is an approximation of \( u(t_n) \) and \( w_n \) is an approximation of \( \partial_{\gamma+1}^1 u(t_n) \) which we describe next.

First of all we define the central difference operator

\[
\tilde{\partial}_t u^h(t_n) := \begin{cases} 
  v^h_0 & n = 0 \\
  \frac{1}{\kappa}(u^h_{n+1} - u^h_{n-1}) & n \geq 1 
\end{cases}
\]
We extend the definition also to continuous functions \( g \in C[0, T] \) with a given time derivative at \( t = 0 \):

\[
\tilde{\partial}_t g(t) := \begin{cases} 
\partial_t g(0) & t = 0 \\
\frac{1}{2\kappa}(g(t + \kappa) - g(t - \kappa)) & t \geq \kappa \\
\frac{1}{\kappa}(\tilde{\partial}_t g(\kappa) - \partial_t g(0))t + \partial_t g(0) & t \in [0, \kappa] 
\end{cases}
\]

i.e., \( \tilde{\partial}_t g(t) \) is exact at \( t = 0 \), the central difference quotient for \( t \geq \kappa \) and is the linear interpolant of \( \tilde{\partial}_t g \) for \( t \in [0, \kappa] \).

It remains to discretize the fractional derivative \( \partial^{\gamma}_t \). The formula should be computable efficiently and should retain the positivity property of the fractional derivative as described in Lemma 2.3. All this is satisfied by convolution quadrature introduced by Lubich in [18], which we describe next.

Convolution quadrature (CQ) is based on an \( A(\theta) \)-stable linear multistep method. For a continuous function \( g \), the CQ formula for \( \partial^{\gamma}_t \) is given by

\[
\partial^{\gamma}_\kappa g(t_n) := \sum_{j=0}^{\infty} \omega_{n-j}(g(t_j) - \chi_{\gamma} g(0)) = \sum_{j=0}^{\infty} \omega_j(g(t_n - t_j) - \chi_{\gamma} g(0)), \quad (18)
\]

where

\[
\chi_{\gamma} = \begin{cases} 
0 & \gamma \in (-1, 0] \\
1 & \gamma \in (0, 1)
\end{cases}
\]

and \( \omega_j \) are convolution weights defined below. The term \( \chi_{\gamma} \omega_n g(0) \) above is added to the standard definition of convolution quadrature in order to correct for the fact that we are using CQ to compute Caputo rather than Riemann Liouville fractional derivatives.

Next, we define the convolution weights. As central differences are approximations of order 2, we restrict our discussion to second order, BDF2 based CQ. The corresponding convolution weights are then given by

\[
\left( \frac{\delta(\zeta)}{\kappa} \right)^\gamma = \sum_{j=0}^{\infty} \omega_j \zeta^j, \quad \delta(\zeta) = \frac{3}{2} - 2\zeta + \frac{1}{2} \zeta^2.
\]

We can also define the approximation at intermediate values of \( t \) by

\[
\partial^{\gamma}_\kappa g(t) := \sum_{j \geq 0, t_j \leq t} \omega_j(g(t - t_j) - \chi_{\gamma} g(0)) = \sum_{j=0}^{\infty} \omega_j(g(t - t_j) - \chi_{\gamma} g(0)), \quad (19)
\]

where we define \( g(t) \equiv 0 \) for \( t < 0 \). Further, it is clear that definition (18) extends to sequences by

\[
\partial^{\gamma}_\kappa g(t_n) := \sum_{j=0}^{n} \omega_{n-j}(g_j - \chi_{\gamma} g_0) = \sum_{j=0}^{n} \omega_j(g_{n-j} - \chi_{\gamma} g_0).
\]

From [20] Theorem 2.1] we have the useful estimate

\[
|\omega_n - \kappa t_n^{-\mu-1}| \leq Ct_n^{-\mu-1-p\kappa^{p+1}}, \quad p = 0, 1, 2, \quad (20)
\]
for $t \in (0, T]$.

As mentioned above, most of the results in the literature analyse CQ as an approximation to Riemann Liouville derivatives ([18]). However, as the Caputo and Riemann Liouville derivatives of order $\gamma$ are equivalent for $\gamma \in (-1, 0)$ and for $\gamma \in (0, 1)$ are equivalent if $g(0) = 0$, we can deduce from [20, Theorem 2.2] the following result.

**Lemma 3.1.** Let $\gamma \in (-1, 1)$, $\kappa \in (0, \bar{\kappa})$ be the time step for some sufficiently small $\bar{\kappa} > 0$, and $g(t) = t^\beta$ for $\beta \in \mathbb{R}$. Then for BDF2 based CQ it holds

$$|\partial^\gamma \kappa g(t) - \partial^\gamma t g(t)| \leq C \begin{cases} 0 & \beta = 0, \gamma \in (0, 1) \\ t^{-\gamma - 1} \kappa, & \beta = 0, \gamma \in (-1, 0) \\ t^{-\gamma + \beta - \kappa p}, & \beta \geq 1, \end{cases} \quad (21)$$

and $p = 1, 2$.

**Proof.** Note that for $\beta = 0$ and $\gamma \in (0, 1)$ we have $\partial^\gamma \kappa g \equiv \partial^\gamma t g \equiv 0$. The remaining cases follow directly from [20, Theorem 2.2].

Next we add correction terms that integrate lower order terms exactly and do not destroy the convergence for the higher order terms. For $\gamma \in (-1, 0)$ it is sufficient to correct for constant functions, whereas for $\gamma \in (0, 1)$ we will need to correct for linear terms also.

Correction terms were introduced by Lubich in [18] and are of the form

$$\partial^\gamma \kappa g(t) = \sum_{j \geq 0, t_j \leq t} \omega_j g(t - t_j) + w_0(t) g(0) + w_1(t) g(t_1). \quad (22)$$

Here, the correction terms $w_j$ are chosen so that

$$w_0(t) = \begin{cases} t^{-\gamma} \frac{\Gamma(1 - \gamma)}{\Gamma(2 - \gamma)} - \sum_{j \geq 0, t_j \leq t} \omega_j, & \text{for } \gamma \in (-1, 0) \\ - \sum_{j \geq 0, t_j \leq t} \omega_j - w_1(t), & \text{for } \gamma \in (0, 1) \end{cases}$$

and

$$w_1(t) = \begin{cases} 0 & \text{for } \gamma \in (-1, 0) \\ \kappa^{-1} t^{1-\gamma} \frac{\Gamma(1 - \gamma)}{\Gamma(2 - \gamma)} - \sum_{j \geq 0, t_j \leq t} (t - t_j) \omega_j, & \text{for } \gamma \in (0, 1). \end{cases}$$

Furthermore, denote $w_{nj} := w_j(t_n)$ and hence

$$\partial^\gamma \kappa g(t_n) = \sum_{j=0}^n \omega_{n-j} g(t_j) + w_{n0} g(0) + w_{n1} g(t_1).$$

Note that by definition $\partial^\gamma \kappa$ already contains the correction for constants if $\gamma \in (0, 1)$. In that case the above just adds the correction for linear functions.
Lemma 3.2. Under the conditions of Lemma [3.1] it holds that for \( g(t) = t^\beta \)

\[
|\partial_{\kappa}^\gamma g(t) - \partial_t^\gamma g(t)| \leq C \begin{cases} 
0 & \beta = 0, \\
0 & \beta = 1, \gamma \in (0,1) \\
t^{-\gamma+\beta-2\kappa^2} & \beta > 1, \gamma \in (0,1) \\
t^{-\gamma+\beta-2\kappa^2 + t^{-\gamma-1}\kappa^2+1} & \beta \geq 1, \gamma \in (-1,0) 
\end{cases} 
\]  

(23)

with \( C > 0 \) independent of \( \kappa \in (0, \kappa) \).

Proof. For \( \gamma < 0 \), the proof follows directly from Lemma [3.1] and definition of \( \partial_{\kappa}^\gamma \). For \( \gamma > 0 \), Lemma [3.1] and the definition of \( w_1(t) \) imply that \( w_1(t) \leq Ct^{-\gamma-1}\kappa \) and hence \( w_n g(t_1) = t^{-\gamma-1}O(\kappa^{\gamma+1}) \) for \( g(t) = t^\beta \) and \( \beta > 1 \). This in turn implies the final missing result.

Using the above shorthand for the CQ approximation, we can write the fully discretized systems as

\[
\frac{1}{h_n} (u_{n+1}^h - 2u_n^h + u_{n-1}^h, v) + (\nabla u_n^h, \nabla v) + (a_\gamma \partial_\kappa^\gamma \bar{\partial}_{\kappa} u_n^h(t_n), v) = (f(t_n), v) 
\]  

(24)

or when including the correction

\[
\frac{1}{h_n} (u_{n+1}^h - 2u_n^h + u_{n-1}^h, v) + (\nabla u_n^h, \nabla v) + (a_\gamma \partial_\kappa^\gamma \bar{\partial}_{\kappa} u_n^h(t_n), v) = (f(t_n), v), 
\]  

(25)

\( n = 1, \ldots, N - 1 \). The coupling of the two time discretizations is similar to the FEM-BEM coupling in [3]. As both of the above schemes are explicit, we will see during the course of the analysis that the following CFL condition is required

\[
\kappa \leq \frac{\sqrt{2}h}{C_{\text{inv}}}. 
\]  

(26)

It remains to describe the choice of initial data \( u_0^h, u_1^h \). To do this we require the Ritz projection, denoted by \( R_h : H_0^1(\Omega) \to V^h \) and defined by

\[
(\nabla R_h u, \nabla v) = (\nabla u, \nabla v) \quad \text{for all } v \in V^h 
\]

and the \( L^2 \) projection is denoted by \( P_h : L^2(\Omega) \to V^h \)

\[
(P_h u, v) = (u, v) \quad \text{for all } v \in V^h. 
\]

We have the approximation property, see, e.g., [13],

\[
\|R_h u - u\| \leq C h^s \|u\|_s \quad \text{for } s = 1, 2 \text{ and } \forall u \in H^s \cap H_0^1. 
\]  

(27)

We then define the initial data by

\[
u_0^h = R_h u_0 \quad \text{and} \quad u_1^h = R_h (u_0 + \kappa v_0) + \frac{1}{2} \kappa^2 P_h \partial_\kappa^2 u(0). 
\]  

(28)

Using the PDE and the fact that \( \partial_\kappa^2 u \) is continuous we see that \( P_h \partial_\kappa^2 u(0) \in V^h \) is the solution of

\[
(P_h \partial_\kappa^2 u(0), v) = (f(0), v) - (\nabla u_0, \nabla v) \quad \text{for all } v \in V^h. 
\]
Note also that by definition
\[ \bar{\partial}_t u_h(0) = R_h \partial_t u(0) = R_h v_0. \]

The reason for using the mixed approximation \( \partial_t^\gamma \bar{\partial}_t \) instead of a fully CQ approximation \( \partial_t^{\gamma+1} \) is to conserve the sign of the damping term.

**Lemma 3.3.** Given a sequence \( v_0, \ldots, v_N \in L^2(\Omega) \), we have

(a) For \( \gamma \in (-1, 0) \)
\[
\sum_{n=0}^{N} (\partial_{\kappa}^\gamma v(t_n), v_n) \geq 0
\]

and
\[
\sum_{n=0}^{N} (\partial_{\kappa}^\gamma v(t_n), v_n) \geq - \sum_{n=0}^{N} \omega_{n0}(v_0, v_n).
\]

(b) For \( \gamma \in (0, 1) \)
\[
\sum_{n=0}^{N} (\partial_{\kappa}^\gamma v(t_n), v_n) \geq - \sum_{n=0}^{N} \omega_n(v_0, v_n)
\]

and
\[
\sum_{n=0}^{N} (\partial_{\kappa}^\gamma v(t_n), v_n) \geq - \sum_{n=0}^{N} \omega_{n0}(v_0, v_n) - \sum_{n=0}^{N} \omega_{n1}(v_1, v_n).
\]

**Proof.** The proof follows directly from the frequency domain estimate
\[ \text{Re}(s^\gamma v, v) \geq 0 \]

for \( v \in L^2(\Omega) \) and the Herglotz theorem ([3, Theorem 2.3]). \( \square \)

4 Convergence analysis

We start with analyzing the error of the mixed approximation \( \partial_t^\gamma \bar{\partial}_t \) when applied to \( t^\beta \). We will first need a simple technical lemma.

**Lemma 4.1.** Let \( t_n = nk \) with \( \kappa > 0 \) and \( t_N = T \). Given \( \eta \neq -1 \) there exists a constant \( C > 0 \) independent of \( T \) and \( \kappa \) such that
\[
\kappa \sum_{n=1}^{N} t_n^\eta \leq C \max(T^{\eta+1}, \kappa^{\eta+1}).
\]

**Proof.** Consider first the case \( \eta \leq 0 \). Then \( t^\eta \) is a decreasing function and
\[
\kappa \sum_{n=1}^{N} t_n^\eta \leq \kappa^{\eta+1} + \int_{\kappa}^{T} t^\eta dt = \frac{1}{\eta + 1} (T^{\eta+1} + \kappa^{\eta+1}).
\]
If $\eta \geq 0$, then $t^\eta$ is an increasing function and we have instead

$$
\kappa \sum_{n=1}^N t_n^\eta \leq \int_\kappa^{T+\kappa} t^\eta dt = \frac{1}{\eta+1} \left( (T + \kappa)^{\eta+1} - \kappa^{\eta+1} \right).
$$

Hence in both cases the sum is bounded by $C \max(T^{\eta+1}, \kappa^{\eta+1})$ with the constant depending on $\eta$.

\[\square\]

**Lemma 4.2.** For $g(t) = t^\beta$ and $\beta = 0, 1$ or $\beta \in [2, 3]$ we have

$$
|\partial_\kappa^\gamma \tilde{\partial}_t g(t) - \partial_\kappa^{\gamma+1} g(t)| \leq C \left\{ \begin{array}{ll}
0 & \beta = 0, \\
0 & \beta = 1, \gamma \in (0, 1) \\
t^{-\gamma-1}\kappa & \beta = 1, \gamma \in (-1, 0) \\
t^{-\gamma+\beta-3}\kappa^2 & \beta \in [2, 3] \\
t^{-\gamma+\beta-3}\kappa^2 + t^{\beta-3}\kappa^{\min(2-\gamma, 2)} & \beta \in (2, 3),
\end{array} \right.
$$

and constant $C > 0$ independent of $\kappa \in (0, \kappa)$ for some small enough $\kappa > 0$.

**Proof.** Throughout the proof, $C > 0$ denotes a generic constant allowed to change from one step to another.

First note that the error is 0 for $\beta = 0$ since $\partial_t g \equiv \tilde{\partial}_t g \equiv 0$. For $\beta > 0$, split the error as

$$
|\partial_\kappa^\gamma \tilde{\partial}_t g(t) - \partial_\kappa^{\gamma+1} g(t)| \leq \underbrace{|\partial_\kappa^\gamma \tilde{\partial}_t g(t) - \partial_\kappa^\gamma \tilde{\partial}_t g(t)|}_{A_\beta} + \underbrace{|(\partial_\kappa^\gamma \tilde{\partial}_t g(t) - \partial_\kappa^\gamma \tilde{\partial}_t g(t))|}_{B_\beta}.
$$

As $\tilde{\partial}_t g(t) - \partial_t g(t) = 0$ for $t \geq 0$ if $\beta = 1, 2$, we have that $A_1 = A_2 = 0$ and for $\gamma > 0$ also $A_3 = 0$.

For $\beta \in [2, 3]$ we apply Newton’s generalized binomial theorem to $\tilde{\partial}_t g(t) = \frac{1}{\kappa^2}((t + \kappa)^\beta - (t - \kappa)^\beta)$ and see that for $t > \kappa$

$$
\tilde{\partial}_t g(t) - \partial_t g(t) = \sum_{k=1}^\infty \binom{\beta}{k} \kappa^{\beta-k} t^{k-1} \leq \kappa^2 t^{-3} \sum_{k=3}^\infty \binom{\beta}{k} t^{\beta-k+3} \kappa^{k-3} \leq C\kappa^2 t^{-3} \sum_{k=0}^\infty \binom{\beta}{k+3} t^{\beta-k} \kappa^k \leq C\kappa^2 t^{-3} \sum_{k=0}^\infty \binom{\beta}{k} t^{\beta-k} \kappa^k = C\kappa^2 t^{-3} (t + \kappa)^\beta \leq C\kappa^2 t^{\beta-3}.
$$
For $t \in [0, \kappa]$ we have instead

$$\partial_t g(t) - \partial_t g(t) = 2^{\beta-1} \kappa^{\beta-2} t - \beta t^{\beta-1} = O(\kappa^{\beta-1}).$$

As the convolution weights satisfy, see (20), $|\omega_j| \leq C \kappa^{-\gamma}$, $j \geq 1$, with $|\omega_0| \leq C \kappa^{-\gamma}$ we have for $t \in [t_n, t_{n+1})$, $n \geq 1$,

$$A_\beta \leq C \kappa^2 \left( |\omega_0| t^{\beta-3} + |\omega_n| \kappa^{\beta-3} + \sum_{j=1}^{n-1} |\omega_j| (t - t_j)^{\beta-3} \right)$$

$$\leq C \kappa^2 \left( \kappa^{-\gamma} t^{\beta-3} + t_n^{-\gamma-1} \kappa^{\beta-2} + \kappa \sum_{j=1}^{n-1} t_j^{-\gamma-1} (t - t_j)^{\beta-3} \right)$$

$$\leq C \kappa^2 \left( \kappa^{-\gamma} t^{\beta-3} + t_n^{-\gamma-1} \kappa^{\beta-2} + t_n^{\beta-3} \max(1, \kappa^{-\gamma}) \right) = C(t^{\beta-3} \kappa^{\min(2-\gamma, 2)}), \quad t \geq \kappa,$$

where we have used [19] Lemma 5.3] to bound the discrete convolution in the following way:

$$\sum_{j=1}^{n-1} t_j^{-\gamma-1} (t - t_j)^{\beta-3} \leq \kappa^{\beta-3-\gamma} \sum_{j=1}^{n-1} j^{-\gamma-1} (n - j)^{\beta-3}$$

$$\leq C \kappa^{\beta-3-\gamma} n^{\max(-\gamma-1, \beta-3, \beta-\gamma-3)}$$

$$\leq C \kappa^{-\gamma} \kappa^{\beta-3} n^{\max(-\gamma-\beta+2, 0, -\gamma)}$$

$$\leq C t_n^{\beta-3} \max(1, \kappa^{-\gamma}),$$

where $C$ as always is allowed to depend on $T$. For $t \in (0, \kappa)$ we obtain a similar bound as $A_\beta \leq C \kappa^{-\gamma+\beta-1} \leq C t^{\beta-3} \kappa^{-\gamma}.$

For $\beta = 0, B_0 = 0$ and for $\beta = 1$ or $\beta \geq 2$ we have from (21) with $\partial_t g = \beta t^{\beta-1}$ that

$$B_\beta \leq C \begin{cases} 0 & \beta = 1, \gamma \in (0, 1) \\ t^{-\gamma-1} \kappa & \beta = 1, \gamma \in (-1, 0) \\ t^{-\gamma+\beta-3} \kappa^2 & \beta \geq 2. \end{cases}$$

Combining all the cases gives the stated result. 

\textbf{Remark 3.} With a different extension of $\partial_t g(t)$ for $t \in (0, \kappa)$, it may be possible to improve the estimate for $\beta \in (2, 3)$. However, as we will see in Remark 4 and Lemma A.3, this estimate is sufficient to obtain an optimal global error.

We now prove the corresponding lemma for the corrected quadrature.

\textbf{Lemma 4.3.} For $g(t) = t^\beta$ and $\beta = 0, 1$ or $\beta \in [2, 3]$ we have

$$\left| \partial_{t_n}^\beta \partial_t g(t) - \partial_{t_n}^{\beta+1} g(t) \right| \leq C \begin{cases} 0 & \beta = 0, 1 \\ 0 & \beta = 2, \gamma \in (0, 1) \\ t^{-\gamma+\beta-3} \kappa^2 & \beta = 2, \gamma \in (0, 1) \\ t^{-\gamma+\beta-3} \kappa^2 & \beta \in [2, 3], \gamma \in (-1, 0) \\ t^{-\gamma+\beta-3} \kappa^2 & \beta = 3, \gamma \in (0, 1) \\ t^{-\gamma+\beta-3} \kappa^2 + t^{-\gamma-1} \kappa^3 & \beta > 2, \beta \neq 3, \gamma \in (-1, 0) \\ t^{-\gamma+\beta-3} \kappa^2 + t^{-\gamma-1} \kappa^3 + t^{\beta-3} \kappa^{\min(2-\gamma, 2)} & \beta \in (2, 3), \gamma \in (0, 1) \end{cases}$$
for \( t \in (0, T) \), \( C > 0 \) independent of time step \( \kappa \in (0, \bar{\kappa}) \) for some small enough \( \bar{\kappa} > 0 \).

**Proof.** The proof is along the same lines as the proof of Lemma 4.2 where using Lemma 3.2 we note that

\[
B_\beta \leq C \begin{cases} 
0 & \beta = 1, \\
t^{-\gamma+\beta-3\kappa^2} & \beta = 2, \gamma \in (0, 1) \\
t^{-\gamma+\beta-3\kappa^2} + t^{-\gamma-1} \kappa^\beta & \beta > 2, \gamma \in (0, 1).
\end{cases}
\]

and as before \( A_\beta = 0 \) for \( \beta = 1, 2, 3 \) and \( A_\beta = t^{\beta-3}O(\kappa^\min(2, 2-\gamma)) \) for \( \beta \in (2, 3) \).

We next investigate the error for general, smooth functions.

**Lemma 4.4.** There exists a constant \( C > 0 \) independent of \( \kappa \in (0, \bar{\kappa}) \) for some small enough \( \bar{\kappa} > 0 \), but depending on \( \gamma \) and \( T \) such that

(a) For \( \gamma \in (-1, 0) \) and \( t \in (0, T) \) if \( u \in C^3[0, T] \)

\[
\left| \partial_t^\gamma \partial_t u(t) - \partial_t^{\gamma+1} u(t) \right| \leq C \{ \partial_t u(0) t^{-\gamma-1} \kappa \\
+ \left( u^{(2)}(0) t^{-\gamma-1} + \max_{t \in [0, T]} |u^{(3)}(t)| \right) \kappa^2 \}.
\]

(b) For \( \gamma \in (0, 1) \) and \( t \in [0, T] \) if \( u \in C^4[0, T] \)

\[
\left| \partial_t^\gamma \partial_t u(t) - \partial_t^{\gamma+1} u(t) \right| \leq C \{ u^{(2)}(0) t^{-\gamma-1} \\
+ u^{(3)}(0) t^{-\gamma} + \max_{t \in [0, T]} |u^{(4)}(t)| \} \kappa^2.
\]

**Proof.** Let \( \hat{\beta} = \lceil \gamma \rceil + 3 \) and consider the Taylor expansion of \( u \) with the Peano kernel remainder term

\[
u(t) = \sum_{k=0}^{\hat{\beta}-1} \frac{u^{(k)}(0)}{k!} t^k + \frac{1}{(\hat{\beta} - 1)!} \int_0^T (t - \tau)^{\hat{\beta}-1} u^{(\hat{\beta})}(\tau) d\tau.
\]

where \( (t - \tau)_+ = \max(t - \tau, 0) \) and \( t \in [0, T] \). The problem then reduces to analysing the error for polynomials

\[
\left| \partial_t^\gamma \partial_t u(t) - \partial_t^{\gamma+1} u(t) \right| \leq \sum_{k=0}^{\hat{\beta}-1} \frac{u^{(k)}(0)}{k!} |r_{k, \gamma}(t)| \\
+ \frac{1}{(\hat{\beta} - 1)!} \max_{t \in [0, T]} |u^{(\hat{\beta})}(t)| \int_0^T |r_{\hat{\beta}-1, \gamma}(\tau)| d\tau,
\]

where \( r_{k, \gamma}(t) = \partial_t^k \partial_t g(t) - \partial_t^{k+1} g(t) \), with \( g(t) = t^k \). Hence applying the result of Lemma 4.2 finishes the proof.
The corresponding result with the corrected quadrature is stated next.

**Lemma 4.5.** For $u \in C^6[0, T]$, $\gamma \in (-1, 1)$ and $\bar{\rho} = [\gamma] + 3$

$$\left| \partial_{\bar{\rho}}^\gamma \tilde{\partial}_t u(t) - \partial_{\bar{\rho}}^{\gamma+1} u(t) \right| \leq C \left( u^{(\bar{\rho}-1)}(0) \bar{\rho}^{\gamma-4} + \max_{t \in [0, T]} \left| u^{(\bar{\rho})}(t) \right| \right) \kappa^2,$$

for $t \in [\kappa, T]$, $\kappa \in (0, \bar{\kappa})$, a constant $C > 0$ which may depend on $T$ and $\gamma$.

**Theorem 4.6.** Let $u$ be the solution of (10) and $u_n \in \mathcal{V}_h$, $n = 0, \ldots, N$, the solution of the fully discrete system (24). If $u$ is sufficiently smooth we have that

$$\left\| \frac{u_n - u_{n-1}}{\kappa} - u\left( t_n - \frac{1}{2} \kappa \right) \right\| + \left\| \frac{u_n + u_{n-1}}{2} - u\left( t_n - \frac{1}{2} \kappa \right) \right\| = O \left( u^{(k)}(0) \kappa^q + \kappa^2 + h^2 \right)$$

and

$$\left\| \frac{u_n + u_{n-1}}{2} - u\left( t_n - \frac{1}{2} \kappa \right) \right\|_1 = O \left( u^{(k)}(0) \kappa^q + \kappa^2 + h \right)$$

where,

$$k = 1 + \lceil \gamma \rceil \quad \text{and} \quad q = \left\{ \begin{array}{ll} 1 - \gamma & \gamma \in (-1, 0) \\ 2 - \gamma & \gamma \in (0, 1) \end{array} \right.$$

The constants implicit in the error estimate are independent of $\kappa \in (0, \bar{\kappa})$, for sufficiently small $\bar{\kappa}$.

**Proof.** Let $e_n^h = R_h u(t_n) - u_n^h$ be the error which satisfies,

$$\frac{1}{\kappa^2} \left( e_{n+1}^h - 2e_n^h + e_{n-1}^h, w \right) + \left( \nabla e_n^h, \nabla w \right) + \left( a_r \partial_{\kappa}^2 \tilde{\partial}_t e_n^h(t_n), w \right) = \left( \delta_n + a_r \varepsilon_n, w \right),$$

for all $w \in \mathcal{V}_h$, where

$$\delta_n = \frac{1}{\kappa^2} R_h (u(t_{n+1}) - 2u(t_n) + u(t_{n-1})) - \partial_t^2 u(t_n)$$

and

$$\varepsilon_n = R_h \partial_{\kappa}^2 \tilde{\partial}_t u(t_n) - \partial_t^{\gamma+1} u(t_n).$$

Mimicking the continuous case, we test with $w = \tilde{\partial}_t e_n^h$

$$E_{n+1}^e - E_n^e = \kappa \left( \delta_n + a_r \varepsilon_n, \tilde{\partial}_t e_n^h \right) - \kappa \left( a_r \partial_{\kappa}^2 \tilde{\partial}_t e_n^h(t_n), \tilde{\partial}_t e_n^h \right)$$

(31)

where

$$E_n^e = \frac{1}{2} \left\| \frac{e_n^h - e_{n-1}^h}{\kappa} \right\|^2 + \frac{1}{2} \left\langle \nabla e_n^h, \nabla e_{n-1}^h \right\rangle$$

(32)

$$= \frac{1}{2} \left\| \frac{e_n^h - e_{n-1}^h}{\kappa} \right\|^2 + \frac{1}{4} \left\langle \nabla e_n^h, \nabla e_{n-1}^h \right\rangle - \frac{1}{8} \left\| \nabla e_n^h - \nabla e_{n-1}^h \right\|^2 + \frac{1}{8} \left( \left\| \nabla e_n^h \right\|^2 + \left\| \nabla e_{n-1}^h \right\|^2 \right)$$

(33)
Under the CFL condition (26), we have that the energy is nonnegative and bounded below as

\[ E^e_n \geq \frac{1}{4} \left( e_n^h - e_{n-1}^h \right)^2 + \frac{1}{2} \left( \nabla e_n^h + \nabla e_{n-1}^h \right)^2. \]

As \( \bar{\partial}_t e^h(0) = 0 \), see (28), from Lemma 3.3 it follows that

\[ \kappa \sum_{n=1}^{N-1} (\bar{\partial}_n^\gamma \bar{\partial}_t e^h(t_n), \bar{\partial}_t e^h_n) \geq 0. \]

Hence, by taking the sum over \( n = 1, \ldots, J, J \leq N - 1 \), of (31) we obtain the estimate

\[ E^e_J \leq E^e_1 + \kappa \sum_{n=1}^J (\delta_n + a_\gamma \varepsilon_n, \bar{\partial}_t e^h_n). \]

Using the Cauchy Schwarz inequality we have,

\[ E^e_J \leq E^e_1 + \kappa \sum_{n=1}^J \| \delta_n + a_\gamma \varepsilon_n \| \| \bar{\partial}_t e^h_n \| \leq E^e_1 + \left( \kappa \sum_{n=1}^{N-1} \| \delta_n + a_\gamma \varepsilon_n \| \right)^2 + \frac{1}{2} \max_{m \leq N} E^e_m \]

due to \( \| \bar{\partial}_t e^h_n \|^2 \leq E^e_{n+1} + E^e_n \leq 2 \max_m E^e_m \). Hence

\[ \frac{1}{2} \max_{m \leq N} E^e_m \leq E^e_1 + \left( \kappa \sum_{n=1}^{N-1} \| \delta_n + a_\gamma \varepsilon_n \| \right)^2 \]

\[ \leq E^e_1 + \left( \kappa \sum_{n=1}^J \| \delta_n \| + a_\gamma \kappa \sum_{n=1}^{N-1} \| \varepsilon_n \| \right)^2. \]  

(34)

It remains to bound \( \delta_n, \varepsilon_n \) and the initial energy. For a sufficiently smooth \( u \) a Taylor expansion shows that

\[ \| \delta_n \| = O(h^2 + \kappa^2). \]  

(35)

To bound \( \varepsilon_n \) we split the error into two parts

\[ \| \varepsilon_n \| \leq \| R_h \partial_2^\gamma \bar{\partial}_t u(t_n) - R_h \partial_2^\gamma \partial_t u(t_n) \| + \| R_h \partial_2^\gamma \partial_t u(t_n) - \partial_2^\gamma \partial_t u(t_n) \|. \]

(\(A\))

(\(B\))

Using the Poincaré inequality and the definition of the Ritz projection we have

\[ A \leq C \| \partial_2^\gamma \bar{\partial}_t u(t_n) - \partial_2^\gamma \partial_t u(t_n) \|_1. \]

Hence by Lemma 4.4 we have for \( \gamma \in (-1,0) \)

\[ A \leq C \gamma^{-1} \left( \| \partial_t u(0) \|_1 \kappa + \| \partial_2^\gamma u(0) \|_1 \kappa^2 \right) + O(\kappa^2) \]

(36)
and for $\gamma \in (0, 1)$

$$A \leq C(\|\partial_t^2 u(0)\|_1 \kappa^2 t_n^{-\gamma} + \|\partial_t^3 u(0)\|_1 \kappa^2 t_n^{-\gamma}) + O(\kappa^2).$$

(37)

By (27) we have that $B \leq Ch^2 \|\partial_t^3 \partial_t u(t_n)\|_2$. Thus from Lemma 4.1 we have that if $\gamma \in (-1, 0)$

$$a_\gamma \kappa \sum_{n=1}^{N-1} \|\varepsilon_n\| \leq C(\|\partial_t u(0)\|_1 \kappa + \|\partial_t^2 u(0)\|_1 \kappa^2) + O(\kappa^2) + O(h^2)$$

and for $\gamma \in (0, 1)$

$$a_\gamma \kappa \sum_{n=1}^{N-1} \|\varepsilon_n\| \leq C(\|\partial_t^2 u(0)\|_1 \kappa^{2-\gamma} + \|\partial_t^3 u(0)\|_1 \kappa^{3-\gamma}) + O(\kappa^2) + O(h^2).$$

It remains to estimate the initial error

$$E_1^\gamma = \frac{1}{2} \left| \frac{e^h - e^h_0}{\kappa} \right|^2 + \frac{1}{2} (\nabla e^h_1, \nabla e^h_0).$$

Recalling the definition of the initial data (28) and that $e_n^h = R_h u(t_n) - u_n^h$ we have for some $\xi \in (0, \kappa)$

$$\left\| \frac{e^h_1 - e^h_0}{\kappa} \right\| = \left\| R_h \left( \frac{u_0}{\kappa} + v_0 + \frac{\kappa}{2} u''(0) + \frac{\kappa^2}{6} u'''(\xi) \right) - \left( \frac{u_0}{\kappa} + v_0 + \frac{\kappa}{2} P_h u''(0) - \frac{1}{\kappa} (R_h u_0 - u_0) \right) \right\|$$

$$= \left\| R_h \left( v_0 + \frac{\kappa}{2} u''(0) + \frac{\kappa^2}{6} u'''(\xi) \right) - \left( v_0 + \frac{\kappa}{2} P_h u''(0) \right) \right\|$$

$$\leq \|R_h v_0 - v_0\| + \frac{\kappa}{2} \|R_h u''(0) - P_h u''(0)\| + \frac{\kappa^2}{6} \|u'''(\xi)\|.$$

Properties of the $L^2$ projection tell us that $\|R_h u''(0) - P_h u''(0)\| \leq \|R_h u''(0) - u''(0)\|$. With this inequality and (27) we conclude that

$$\left\| \frac{e^h_1 - e^h_0}{\kappa} \right\| \leq C \left( h^2 \|v_0\|_2 + \kappa h \|u''(0)\|_1 + \kappa^2 \max_{t \in [0, \kappa]} \|u'''(t)\| \right).$$

As $R_h$ is the Ritz projection we have that $(\nabla e^h_1, \nabla e^h_0) = 0$ and therefore

$$E_1^\gamma = \mathcal{O}(h^4 + \kappa^4 + \kappa^2 h^2) = \mathcal{O}(h^4 + \kappa^4)$$

Combining the result together with the triangle inequality applied to $u(t_n) - u^h_n = u(t_n) - R_h u(t_n) + e^h_n$ and the approximation properties of the Ritz projection (27) gives the result.

We next turn to the corrected scheme.
Theorem 4.7. Under the assumptions of Theorem 4.7
\[
\left\| \frac{u_n - u_{n-1}}{\kappa} - u'(t_n - \frac{1}{2}\kappa) \right\| + \left\| \frac{u_n + u_{n-1}}{2} - u(t_n - \frac{1}{2}\kappa) \right\| = O(\kappa^2 + h^2)
\]
and
\[
\left\| \frac{u_n + u_{n-1}}{2} - u(t_n - \frac{1}{2}\kappa) \right\| = O(\kappa^2 + h).
\]

The constants implicit in the error estimate are independent of $\kappa \in (0, \bar{\kappa})$, for sufficiently small $\bar{\kappa}$.

Proof. The proof is a modification of its noncorrected counterpart. The variation lies in the use of Lemma 4.5 and more critically Lemma 3.3.

In particular using the same notation we have that
\[
\kappa \sum_{n=1}^{N-1} (\partial_{t_n}^\gamma \bar{\partial}_t e^h(t_n), \bar{\partial}_t e^h(t_n)) \geq -\kappa \sum_{n=1}^{N-1} \omega_{n1}(\bar{\partial}_t e^h(t_1), \bar{\partial}_t e^h(t_n)).
\]

For $\gamma \in (-1,0)$, $w_{n1} = 0$ so the proof can proceed in the same way. For $\gamma \in (0,1)$, let us first investigate the case $N = 2$, i.e.,
\[
\kappa \sum_{n=1}^{N-1} (\partial_{t_n}^\gamma \bar{\partial}_t e^h(t_n), \bar{\partial}_t e^h(t_n)) \geq -\kappa \omega_{11} \| \bar{\partial}_t e^h(t_1) \|^2
\]
\[
\geq -C\kappa^{1-\gamma} \max_{m=1,2} E_m^e \geq -\frac{1}{4} \max_{m=1,2} E_m^e
\]
for sufficiently small $\kappa > 0$. Hence, proceeding as before and using the same notation, but applying Lemma 4.5 we obtain that
\[
\kappa \sum_{n=1}^{N-1} \| \delta_n + a_\gamma \varepsilon_n \| = \kappa \| \delta_1 + a_\gamma \varepsilon_1 \| = O(\kappa^3) + O(\kappa h^2)
\]
and hence
\[
\| e^h(t_2) \| = O(\kappa^3) + O(\kappa h^2).
\]
For \( n > 1 \) we proceed as

\[
\kappa \sum_{n=1}^{N-1} \left( \partial_{x_n}^2 \bar{e}_h(t_n), \bar{e}_h(t_n) \right) \geq -\kappa \sum_{n=1}^{N-1} \omega_n \left( \bar{e}_h(t_1), \bar{e}_h(t_n) \right) \geq -C \kappa^2 \sum_{n=1}^{N-1} t_n^{-\gamma-1} \left( \bar{e}_h(t_1), \bar{e}_h(t_n) \right) \geq -C \max_n \| \bar{e}_h(t_n) \| \kappa^2 \| \bar{e}_h(t_1) \| \sum_{n=1}^{N-1} t_n^{-\gamma-1} \geq -C \max_n \| \bar{e}_h(t_n) \| \| \bar{e}_h(t_1) \| \sum_{n=1}^{N-1} t_n^{-\gamma} \geq -C \max_n \| \bar{e}_h(t_n) \| \| \bar{e}_h(t_1) \| \sum_{n=1}^{N-1} t_n^{-\gamma} \geq -C \max_n \| \bar{e}_h(t_n) \| \| \bar{e}_h(t_1) \| \sum_{n=1}^{N-1} t_n^{-\gamma} \geq -C \max \left( \frac{\kappa \sum_{n=1}^{N-1} \| \bar{e}_n \|}{C} \right) + O(\kappa^2 + h^4),
\]

where in the last step we used the result for \( n = 1 \).

Proceeding with the proof with this alteration we arrive at the result.

**Remark 4.** The above two theorems assume a sufficiently smooth solution. However, as we have seen in Remark 2 unless the right hand side is of a special form we expect that \( u \) has the singular behaviour at \( t \to 0 \) of the form \( t^{2-\gamma+\lceil \gamma \rceil} \).

To investigate the expected convergence rate, we need to investigate \( \| \varepsilon_n \| \) and \( \| \delta_n \| \), where we used the notation from the the proofs of Theorem 4.6 and 4.7.

We first consider \( \| \delta_n \| \) and again using Newton’s generalized binomial theorem, see Lemma 4.3, find that \( \| \delta_n \| \leq C t^{\gamma-\gamma+\lceil \gamma \rceil} \). Hence, by Lemma 4.1

\[
\kappa \sum_{n=1}^{N-1} \| \delta_n \| \leq C \kappa^3 \sum_{n=1}^{N-1} t_n^{-\gamma} \leq C \kappa^{1-\gamma+\lceil \gamma \rceil}.
\]

For \( \gamma \in (-1,0) \), Lemma 4.2 and Lemma 4.3 with Lemma 4.1 imply that

\[
\kappa \sum_{n=1}^{N} \| \varepsilon_n \| = O(\kappa) \text{ for the uncorrected scheme and } \kappa \sum_{n=1}^{N} \| \varepsilon_n \| = O(\kappa^2) \text{ for the corrected one.}
\]

For \( \gamma \in (0,1) \), the same arguments imply that \( \kappa \sum_{n=1}^{N} \| \varepsilon_n \| = O(\kappa^2-\gamma) \) for both the corrected and uncorrected scheme, which is in both cases equal to the error due to the approximation of the second derivative.
5 Numerical Results

5.1 Smooth solution

First we consider the problem of approximating solutions to (1) in 1D on the interval $\Omega = [0, 1]$ with $h = 6\kappa$ using the two schemes (24) and (25). We construct the right hand side $f$ so that the exact solution is given by

$$u(x, t) = (\sin(24t) + \cos(12t)) \sin(x).$$  \hspace{1cm} (38)

We measure the error in the following norm

$$\text{Error} = \max_n \left\| \frac{u_n - u_{n-1}}{\kappa} - u'(t_n - \frac{1}{2}\kappa) \right\| + \left\| \frac{u_n + u_{n-1}}{2} - u(t_n - \frac{1}{2}\kappa) \right\|$$  \hspace{1cm} (39)

to compare with the theoretical results in Theorem 4.6 and 4.7.

In Fig. 1 we see that the numerical experiments agree with the predicted convergence rates for various values of $\gamma$, except that for $\gamma = 0.25$, Fig. 1c, we seem to obtain a higher than expected convergence rate $O(h^{2.75})$ in contrast to the predicted rate $O(h^{1.75})$. However, by increasing the value of $\alpha_0$ in (2) the predicted convergence rate becomes visible; see Fig. 2.
Figure 1: Plot of mesh size $h$ against the maximum error, see (39), produced in 1D numerical experiments approximating the exact solution (38). The dashed lines represent the expected convergence rates determined by Theorems 4.6 and 4.7.
Figure 2: A repeat of the experiment in Fig. 1c with $\alpha_0$ increased from 1 to 20.

Next we perform an experiment in 2D on the domain $\Omega = [-1,1] \times [-1,1]$ with $h = 10\kappa$. The right hand side is chosen so that the exact solution is given by

$$u(x, t) = (\sin(24t) + \cos(12t)) \sin(\pi x) \sin(\pi y).$$  \hspace{1cm} (40)$$

In 2D experiments we use the $L^2$ error defined as In Fig. 3 we show the convergence of the $L^2$ error

$$\text{Error} = \max_n \| u_n - u(t_n) \|.$$ 

and achieve the expected convergence orders for $\gamma = 0.7$ with and without correction terms.
Figure 3: Plot of mesh size $h$ against the maximum $L^2$ error produced in 2D numerical experiments approximating the exact solution (40). The dashed lines represent the expected convergence rates determined by Theorems 4.6 and 4.7.

5.2 Nonsmooth case

In the next set of figures we study the case discussed in Remark 2 in 1D on the domain $\Omega = [0, 1]$ with $h = 6x$. We now choose the right hand side so that the exact solution is

$$u(x, t) = \left(1 + t + t^2 - \frac{a_1}{\Gamma(3 - \gamma + [\gamma])} t^{2+\gamma-\gamma}\right) \sin(\pi x).$$

The error norm is again as in (39).

The results shown in Fig. 4 generally agree with our claims from Remark 4, and in some cases we actually achieve a higher convergence rate than expected. More specifically, for $\gamma = 0.25$ we observe second order convergence with and without correction terms, and when $\gamma = 0.75$ we have a rate of $O(h^{1.35})$ with corrected CQ. By increasing $a_0$ we would see the expected convergence rates in these two cases, similarly to the adjustment we see in Fig. 2.
Figure 4: Plot of mesh size $h$ against the maximum error, see (39), produced in 1D numerical experiments approximating the exact solution (40). The dashed lines represent the expected convergence rates determined by Theorems 4.6 and 4.7.

5.3 Damping in 2D

We end the section on numerical experiments, by illustrating the damping effect for the fractional term. Fig. 5 shows the profile of our approximation of the solution of the PDE (1) with

$$u_0 = e^{-10(x^2+y^2)}, \quad v_0 = 0 \quad \text{and} \quad f = 0$$

at the point $(0,0)$ on the domain $\Omega = [-1,1] \times [-1,1]$. The first plot has no fractional derivative included, i.e., $a_\gamma = 0$, and the remaining have varying $\gamma$s. In this experiment, the damping effect seems to be strongest for $\gamma = 0.25$. 
Figure 5: The profile of a solution to (1) at one point on a 2D mesh with varying levels of damping introduced through changing the order $\gamma$ of the fractional derivative. For the case with no damping we remove the fractional derivative.

References

[1] K. Baker and L. Banjai. A numerical study of a strongly damped fractional wave equation. In preparation, 2020.

[2] L. Banjai and M. López-Fernández. Efficient high order algorithms for fractional integrals and fractional differential equations. Numer. Math., 141(2):289–317, 2019.

[3] L. Banjai, C. Lubich, and F.-J. Sayas. Stable numerical coupling of exterior and interior problems for the wave equation. Numer. Math., 129(4):611–646, 2015.

[4] L. Banjai and C. Makridakis. A posteriori analysis of subdiffusion problems. In preparation, 2020.

[5] H. Brunner. Collocation methods for Volterra integral and related functional differential equations, volume 15 of Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 2004.
[6] M. Caputo. Linear models of dissipation whose $Q$ is almost frequency
independent. II. *Fract. Calc. Appl. Anal.*, 11(1):4–14, 2008. Reprinted
from Geophys. J. R. Astr. Soc. 13 (1967), no. 5, 529–539.

[7] W. Chen and S. Holm. Modified Szabo’s wave equation models for lossy
media obeying frequency power law. *J. Acoust. Soc. Am.*, 114(5):2570–
2574, 2003.

[8] W. Chen and S. Holm. Fractional laplacian time-space models for linear
and nonlinear lossy media exhibiting arbitrary frequency power-law depen-
dency. *J. Acoust. Soc. Am.*, 115(4):1424–1430, 2004.

[9] K. Diethelm. *The analysis of fractional differential equations*, volume
2004 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010. An
application-oriented exposition using differential operators of Caputo type.

[10] F. Duck. *Physical properties of tissue: a comprehensive reference book*.
Academic Press, 1990.

[11] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies
in Mathematics*. American Mathematical Society, Providence, RI, second
dition, 2010.

[12] S. Holm, S. P. Näsholm, F. Prieur, and R. Sinkus. Deriving fractional
acoustic wave equations from mechanical and thermal constitutive equa-
tions. *Comput. Math. Appl.*, 66(5):621–629, 2013.

[13] J. F. Kelly and R. J. McGough. Approximate analytical time-domain
Green’s functions for the Caputo fractional wave equation. *J. Acoust. Soc.
Am.*, 140(2):1039–1047, 2016.

[14] S. Larsson and F. Saedpanah. The continuous Galerkin method for an
integro-differential equation modeling dynamic fractional order viscoelas-
ticity, *IMA J. Numer. Anal.* 30, 964–986, 2010.

[15] S. Larsson and V. Thomée. *Partial differential equations with numerical
methods*, volume 45 of *Texts in Applied Mathematics*. Springer-Verlag,
Berlin, 2009. Paperback reprint of the 2003 edition.

[16] Y. Lin and C. Xu. Finite difference/spectral approximations for the time-
fractional diffusion equation. *J. Comput. Phys.*, 225(2):1533–1552, 2007.

[17] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems
and applications. Vol. I*. Springer-Verlag, New York-Heidelberg, 1972.
Translated from the French by P. Kenneth, Die Grundlehren der math-
ematischen Wissenschaften, Band 181.

[18] C. Lubich. Discretized fractional calculus. *SIAM J. Math. Anal.*, 17(3):704–
719, 1986.
[19] C. Lubich. Convolution quadrature and discretized operational calculus. I. Numer. Math., 52(2):129–145, 1988.

[20] C. Lubich. Convolution quadrature revisited. BIT, 44(3):503–514, 2004.

[21] W. McLean. Fast summation by interval clustering for an evolution equation with memory. SIAM J. Sci. Comput., 34(6):A3039–A3056, 2012.

[22] K. B. Oldham and J. Spanier. The fractional calculus. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. Theory and applications of differentiation and integration to arbitrary order, With an annotated chronological bibliography by Bertram Ross, Mathematics in Science and Engineering, Vol. 111.

[23] Lj. Oparnica and E. Sülü. Well-posedness of the fractional Zener wave equation for heterogeneous viscoelastic materials. Fract. Calc. Appl. Anal., 23(1): 126–166, 2020.

[24] A. Schädle, M. López-Fernández, and C. Lubich. Fast and oblivious convolution quadrature. SIAM J. Sci. Comput., 28(2):421–438, 2006.

[25] F. Saedpanah. Well-posedness of an integro-differential equation with positive type kernels modeling fractional order viscoelasticity. European Journal of Mechanics - A/Solids, 44, 201–211, 2014.

[26] Z.-z. Sun and X. Wu. A fully discrete difference scheme for a diffusion-wave system. Appl. Numer. Math., 56(2):193–209, 2006.

[27] T. Szabo. Time domain wave equations for lossy media obeying a frequency power law. J. Acoust. Soc. Am., 96(1):491–500, 1994.

[28] T. Szabo. Diagnostic Ultrasound Imaging: Inside Out, Second Edition. Academic Press, 2014.

[29] K. Šiškova Inverse Source Problems in Evolutionary PDE’s. Doctoral Thesis, Ghent University, 2018.

[30] G. Ter Haar. Hifu tissue ablation: Concept and devices. Adv. Exp. Med. Biol., 880:3–10, 2016.

[31] B. Treeby and B. Cox. Modeling power law absorption and dispersion for acoustic propagation using the fractional laplacian. J. Acoust. Soc. Am., 127(5):2741–2748, 2010.

[32] M. Wismer. Finite element analysis of broadband acoustic pulses through inhomogenous media with power law attenuation. J. Acoust. Soc. Am., 120(6):3493–3502, 2006.
Consider the linear Volterra equation of the second kind
\[ v(t) = g(t) + \int_0^t (t - \tau)^{-\gamma} k(t - \tau)v(\tau) d\tau, \quad (42) \]
for \( \gamma \in (-1, 1) \) and \( k \in C[0,T] \). Before stating an existence result, we need a technical lemma the proof of which is elementary.

**Lemma A.1.** Let \( g, k \in C[0,T] \). Then for \( \beta > -1 \)
\[ \int_0^t (t - \tau)^{\beta} k(t - \tau)g(\tau)d\tau = g(0)k(0) \frac{t^{\beta+1}}{\beta+1} + o(t^{\beta+1}) \quad \text{as } t \to 0^+. \]

**Proof.** As \( k \) and \( g \) are continuous, there exists constants \( C_k > 0, C_g > 0 \) such that \( |k(t)| \leq C_k \) and \( |g(t)| \leq C_g \). Further for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
\[ \max\{|k(t) - k(0)|, |g(t) - g(0)|\} \leq \varepsilon \quad \text{for } t \in (0, \delta). \]
Hence \[ \left| \int_0^t (t - \tau)^{\beta} k(t - \tau)g(\tau)d\tau - g(0)k(0) \frac{t^{\beta+1}}{\beta+1} \right| \leq (C_k + C_g) \frac{1}{\beta+1} t^{\beta+1} \varepsilon, \]
for any \( t \in (0, \delta). \)

**Theorem A.2.** If \( g \in C[0,T] \), then (42) has a unique solution \( v \in C[0,T] \). Furthermore,
\[ v(t) = g(t) + g(0)k(0) \frac{t^{1-\gamma}}{1-\gamma} + o(t^{1-\gamma}) \quad \text{as } t \to 0^+. \]

**Proof.** For \( \gamma \in (-1, 0) \), the kernel \( (t - \tau)^{-\gamma} k(t - \tau) \) is continuous and hence the existence of the continuous solution follows from [5, Theorem 2.1.5]. Whereas, for \( \gamma \in (0, 1) \), [5, Theorem 6.1.2] gives the existence of the continuous solution.

The technical lemma together with the fact that \( v(0) = g(0) \) and (42) gives the form of solution for \( t \to 0^+ \).

The following technical lemma is needed to investigate the error for the nonsmooth solution.

**Lemma A.3.** Let \( g(t) = t^\beta \) for \( \beta > 2 \). Then there exists a constant \( C > 0 \) such that for \( \kappa > 0 \)
\[ \left| g''(t) - \frac{1}{\kappa^2} (g(t+\kappa) - 2g(t) + g(t-\kappa)) \right| \leq Ct^{\beta-4}\kappa^2, \]
for \( t \geq \kappa \).
Proof. Denote the error by

\[ e(t) = \frac{1}{\kappa^2} (g(t - \kappa) - 2g(t) + g(t + \kappa)) - \beta(\beta - 1)t^{\beta - 2}. \]

and first note that for \( t = \kappa \)

\[ e(t) = (2\beta - 2 - \beta(\beta - 1))\kappa^{\beta - 2} = (2\beta - 2 - \beta(\beta - 1))t^{\beta - 4}\kappa^2. \]

For \( t \geq 2\kappa \)

\[ e(t) = 2 \sum_{k=2}^{\infty} \binom{\beta}{k} \frac{t^{\beta - k}\kappa^{k - 2} - \beta(\beta - 1)t^{\beta - 2}}{k \text{ even}} \]

\[ = 2 \sum_{k=4}^{\infty} \binom{\beta}{k} \frac{t^{\beta - k}\kappa^{k - 2}}{k \text{ even}} \]

\[ = 2t^{-4} \kappa^2 \sum_{k=4}^{\infty} \binom{\beta}{k} t^{\beta - k}\kappa^{k - 4} \]

\[ \leq 2t^{-4} \kappa^2 \sum_{k=0}^{\infty} \binom{\beta}{k+4} t^{\beta - k}\kappa^{k} \]

\[ \leq Ct^{-4} \kappa^2 \sum_{k=0}^{\infty} \binom{\beta}{k} t^{\beta - k}\kappa^{k} \]

\[ = Ct^{-4} \kappa^2 (t + \kappa)^{\beta} \leq Ct^{\beta - 4}\kappa^2. \]

\[ \square \]

Acknowledgements

Katherine Baker was supported by The Maxwell Institute Graduate School in Analysis and its Applications, a Centre for Doctoral Training funded by the UK Engineering and Physical Sciences Research Council (grant EP/L016508/01), the Scottish Funding Council, Heriot-Watt University and the University of Edinburgh.

We also acknowledge discussions with David Sinden.