GLOBAL EXISTENCE FOR LAPLACE REACTION-DIFFUSION EQUATIONS

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Abstract. We study the initial-boundary value problem for a Laplace reaction-diffusion equation. After constructing local solutions by using the theory of abstract degenerate evolution equations of parabolic type, we show global existence under suitable assumptions on the reaction function. We also show that the problem generates a dynamical system in a suitably set universal space and that this dynamical system possesses a Lyapunov function.

1. Introduction. We study the initial-boundary value problem for a Laplace reaction-diffusion equation

\[
\begin{align*}
& m(x) \frac{\partial u}{\partial t} = a \Delta u + m(x) f(u) \quad \text{in } \Omega \times (0, \infty), \\
& u = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
& u(x, 0) = u_0(x) \quad \text{in } \Omega,
\end{align*}
\]

(1)
in a three-dimensional bounded domain \( \Omega \) of \( C^2 \) class. Here, \( m(x) \) is a given function in \( L^\infty(\Omega) \) such that

\[
0 \leq m(x) \leq 1 \quad \text{and} \quad m(x) \not\equiv 0.
\]

(2)
The function \( f(u) \) is a real valued \( C^3 \) function defined for \( -\infty < u < \infty \). It is assumed that

\[
-D_1 u^{p-1} + 1 \leq f(u) \leq D_2 u(1 + u)^{-1}, \quad 0 \leq u < \infty,
\]

(3)
\[
-D_3 u^{p-1} + 1 \leq f'(u) \leq D_4, \quad 0 \leq u < \infty,
\]

(4)
with some exponent \( p \geq 2 \) and some constants \( D_i > 0 (i = 1, 2, 3, 4) \). Note that (3) implies \( f(0) = 0 \). On the unknown function \( u = u(x, t) \) we impose the homogeneous Dirichlet conditions on the boundary \( \partial \Omega \). The initial function \( u_0(x) \geq 0 \) is a nonnegative function in \( \Omega \).

Such an elliptic-parabolic equation arises in the study of heat conduction in the composite mediums consisting of several materials that have their own heat conductivity. Let \( \Omega \subset \mathbb{R}^3 \) denote such a composite medium and let \( \Omega \) be divided into the direct sum of subdomains \( \Omega_i, 1 \leq i \leq n \), \( \Omega_i \) denoting a material with a

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constant heat conductivity $a_i > 0$. Then the equation describing heat conduction
in $\Omega$ is given by
\[
\frac{\partial u}{\partial t} = \nabla \cdot [a(x)\nabla u] + f(u) \quad \text{in } \Omega \times (0, \infty),
\] (5)
where $a(x)$ is a step function such that $a(x) \equiv a_i$ for $x \in \Omega_i$, $1 \leq i \leq n$, and where $f(u)$ denotes a nonlinear heat controller. For a test function $\varphi(x) \in C^\infty_0(\Omega)$, we have
\[
\langle \nabla \cdot [a(x)\nabla u], \varphi \rangle = -\langle a(x)\nabla u, \nabla \varphi \rangle = -\sum_{i=1}^{n} a_i \int_{\Omega_i} \nabla u \cdot \nabla \varphi \, dx = \sum_{i=1}^{n} a_i \int_{\Omega_i} \Delta u \varphi \, dx - \sum_{i=1}^{n} a_i \frac{\partial u}{\partial n_i} \varphi \, dx,
\]
where $n_i$ denotes the outer normal vector of $\partial \Omega_i$. We here assume on each interface $\Gamma_{ij} = \partial \Omega_i \cap \partial \Omega_j \neq \emptyset$ that
\[
a_i \frac{\partial u}{\partial n_i} + a_j \frac{\partial u}{\partial n_j} = 0 \quad \text{on } \Gamma_{ij} \times (0, \infty),
\] (6)
which means that the flux $a(x)\nabla u$ is continuous on the interface $\Gamma_{ij}$ and is called the continuity condition on the interface. For the details, see Carslaw-Jaeger [2] and Hahn-Özişik [8]. Under this assumption the heat equation then takes the form
\[
\frac{\partial u}{\partial t} = a \Delta u + f(u), \quad \text{in } \Omega \times (0, \infty).
\] (7)
Conversely, if a function $u$ defined in $\Omega \times (0, \infty)$ satisfies (5) and (7) at the same time, then $u$ is a solution to (5) satisfying the continuity condition (6). That is, the problem of solving (5) under (6) is reduced to that of solving (5) and (7) simultaneously.

We want to consider, furthermore, the case where some material may possess extremely larger conductivity than others, say, (for simplicity) $a_i = \infty$ for some $i$. In such a subdomain, the equation is no longer a heat equation but is a Laplace equation. Then, instead of (7), it is convenient to rewrite the equation into the form
\[
m(x) \frac{\partial u}{\partial t} = a \Delta u + m(x)f(u), \quad \text{in } \Omega \times (0, \infty),
\]
where $a = \min_{1 \leq i \leq n} a_i$ is a positive number and $m(x)$ is the function $a/a(x)$ for $x \in \Omega$. Clearly, $m(x)$ satisfies (2).

Under the continuity condition (6), the linear problems were mainly studied until now on the basis of Fourier analysis. Deconinck-Pelloni-Sheils [4] and de Monte [3] construct solutions for one-dimensional linear equations. Mikhailov-Özişik [14] and Salt [15] construct solutions for two and three-dimensional linear equations. Meanwhile, Sheils-Deconinck [16] constructs a mapping from the initial functions to the trace functions of the solutions on the interfaces. We hope that the techniques obtained in this paper together with those of handling (5) will open researches for the nonlinear problems, i.e., (5)-(6).

As for analytical or numerical researches on the general diffusion equations with discontinuous coefficients (not necessarily under the continuity condition on interfaces), we want to quote [1, 9, 10, 11, 12, 13] and references therein.

First, we construct a unique local solution for (1). Indeed, the local solution can be constructed under rather more general conditions than (3)-(4); for example, $C^2$
regularity of $f(u)$ is sufficient. We will regard the equation of (1) as a degenerate evolution equation of parabolic type (see (8) below) whose linear problems have been systematically studied by the monograph [7]. Use of the multivalued linear operators introduced in [6] enable us to rewrite the degenerate equation into a multivalued evolution equation (see (14)) but of nondegenerate form. The reduced multivalued evolution equation can then be solved locally by analogous techniques to the usual (single valued) evolution equations. Those are described in the last section of paper.

Second, we show global existence of solutions for suitable initial functions $u_0(x)$. Under (3)-(4), we establish a priori estimates for local solutions. The reduction of (8) into (14) enables us also to use the theory of infinite-dimensional dynamical systems developed by Temam [17] and others. It is shown that (14) generates a dynamical system whose universal space is suitably determined. It is also shown that every global solution is uniformly bounded and has a nonempty $\omega$-limit set and that there exists a Lyapunov function for this dynamical system, namely, if a global solution is not stationary, then the value of the Lyapunov function decreases strictly as $t$ increases.

Throughout the paper, $\Omega$ denotes a $C^2$ bounded domain in $\mathbb{R}^3$. For $s \geq 0$, $H^s(\Omega)$ is the complex Sobolev space with exponent $s$. As usual, $H^0(\Omega) = L^2(\Omega)$. For $s > 0$, $H^s_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ (space of infinitely differentiable functions in $\Omega$ with compact support) in the topology of $H^2(\Omega)$. We shall also use the Sobolev space $H^{-s}(\Omega) = [H^s_0(\Omega)]'$ with negative exponent $-s$. The diffusion coefficient $a > 0$ is a fixed constant.

2. Local solutions. We begin with constructing local solutions to (1) by employing the general theory of semilinear abstract degenerate evolution equations reviewed in Section 5.

2.1. Abstract formulation. Let us formulate (1) as the Cauchy problem for an abstract evolution equation of the form (46), i.e.,

\[
\begin{cases}
M\frac{du}{dt} + Lu = Mf(u), & 0 < t < \infty, \\
u(0) = u_0,
\end{cases}
\tag{8}
\]

in the underlying space $Y \equiv L_2(\Omega)$.

Here, $L$ is a realization of $-a\Delta$ in $L_2(\Omega)$ under the homogeneous Dirichlet conditions on $\partial\Omega$ with $\mathcal{D}(L) \equiv H^2(\Omega) \cap H^1_0(\Omega)$. By the estimates of elliptic operators, it is known that

\[
\|u\|_{H^2} \leq C(\|Lu\|_{L_2} + \|u\|_{L_2}), \quad u \in \mathcal{D}(L).
\tag{10}
\]

Of course, $L$ is a self-adjoint operator of $Y$. By the Poincaré inequality, there exists a positive constant $c$ such that

\[
(-a\Delta u, u) = a\|\nabla u\|_{L_2}^2 \geq ac\|u\|_{L_2}^2, \quad u \in H^2(\Omega) \cap H^1_0(\Omega).
\tag{11}
\]

Hence, $L$ is positive definite in $Y$ and satisfies (47)-(48). According to [18, Theorem 16.12], the domains of its fractional powers $L^\theta$ are characterized by

\[
\mathcal{D}(L^\theta) = \begin{cases}
H^{2\theta}(\Omega) & \text{if } 0 \leq \theta < \frac{1}{2}, \\
H^{2\theta}_0(\Omega) = \{u \in H^{2\theta}(\Omega); u|_{\partial\Omega} = 0\} & \text{if } \frac{1}{2} < \theta \leq 1.
\end{cases}
\tag{12}
\]

In particular, we have $\mathcal{D}(L^{\frac{1}{2}}) = H^1_0(\Omega) = H^1_D(\Omega)$. 
Meanwhile, the second Banach space $X$ is set by

$$X \equiv H^1_0(\Omega),$$

(13)

noting that (49) is verified with $\alpha = \frac{1}{2}$ (due to (12)). The operator $M$ is then a multiplicative operator by the function $m(x)$ from $H^1_0(\Omega)$ into $L_2(\Omega)$. As verified in [7, Example 3.4], $M$ and $L$ satisfy (50)-(51) with some angle $\omega < \frac{\pi}{2}$. Notice that these conditions may fail in $L_2(\Omega)$; so, the settings (9) and (13) are essential.

Finally, $f(u) \equiv f(\text{Re}(u(x)))$ denotes a nonlinear operator with $D(f) \equiv D(L^\beta) = H^{2\beta}_D(\Omega)$ (due to (12)), where $\beta$ is some fixed exponent such that $\frac{3}{4} < \beta < 1$. It is known that $H^{2\beta}(\Omega) \subset C(\Omega)$ and that $u \in H^{2\beta}_D(\Omega)$ if and only if both Re $u$ and Im $u$ belong to $H^{2\beta}_D(\Omega)$ (due to [18, Theorem 1.34]). Then $f$ is a mapping from $D(f)$ into $X$. Moreover, since

$$\nabla[f(u) - f(v)] = [f'(\text{Re} u) - f'(\text{Re} v)] \nabla \text{Re} u + f'(\text{Re} v) \nabla \text{Re} (u - v),$$

we observe that

$$\|\nabla[f(u) - f(v)]\|_{L_2} \leq \max_{|r| \leq \|u\|_{L_2} + \|v\|_{L_2}} |f''(r)| \|u - v\|_{C} \|\nabla u\|_{L_2}$$

$$+ \max_{|r| \leq \|u\|_{L_2}} |f'(r)| \|\nabla (u - v)\|_{L_2}, \quad u, v \in D(f).$$

From this it is readily verified that the Lipschitz condition (52) takes place. In this way, all the structural assumptions (47)∼(52) in Section 5 are fulfilled by the three operators $L, M$ and $f$.

The problem (8) is equivalently rewritten in the form

$$\begin{cases}
\frac{du}{dt} + Au \ni f(u), & 0 < t < \infty, \\
u(0) = u_0,
\end{cases}$$

(14)

in the space $X$. Here, $A \equiv M^{-1}L$ is a multivalued linear operator of $X$ given by

$$D(A) = \{u \in H^1_0(\Omega); \exists f \in H^1_0(\Omega) \text{ such that } m(x)f = -a\Delta u\}.$$ 

We fix the third exponent $\tilde{\beta}$ in such a way that $\tilde{\beta}$ satisfies

$$2\beta - 1 < \tilde{\beta} < 1.$$ 

Then, $\alpha, \beta$ and $\tilde{\beta}$ satisfy the relation (57). By virtue of (56) in Proposition 3 ($\theta = \beta$), we have $D(A^{\frac{1}{2}}) \subset D(L^\beta)$ and

$$\|u\|_{C} \leq C\|L^\beta u\|_{L_2} \leq C\|A^{\frac{1}{2}}u\|_{H^1_0}, \quad u \in D(A^{\frac{1}{2}}).$$

(15)

Theorem 5.1 in Section 5 then provides that for any $u_0 \in D([M^{-1}L]^{\tilde{\beta}})$ (\(\subset H^{2\tilde{\beta}}_D(\Omega)\)) there exists a unique local solution to (8) in the function space:

$$u \in C([0, T]; H^{2\tilde{\beta}}_D(\Omega)) \cap C((0, T]; H^{1\tilde{\beta}}_D(\Omega)) \cap C^1((0, T]; H^1_0(\Omega)),$$

$T > 0$ being determined by the norm $\|[M^{-1}L]^{\tilde{\beta}}u_0\|_{H^1_0}$ alone. In addition, (67) of Theorem 5.2 yields that

$$u \in C^\sigma([0, T]; H^{2\tilde{\beta}}_D(\Omega)) \cap C^{1+\tilde{\beta}}((0, T]; H^1_0(\Omega))$$

(16)

with some exponents such that $0 < \tilde{\beta} < \sigma$. 
2.2. Higher temporal regularities of solutions. In order to obtain the global existence, however, we need higher temporal regularities of solutions. These regularities are obtained by using the techniques established by \[\text{[5]}\]. In fact, the local solution \(u\) actually enjoys:

\[
u \in C^1((0, T]; H^2_0(\Omega)) \cap C^2((0, T]; H^1_0(\Omega)). \tag{17}\]

Fix time \(0 < t_0 < T\). Putting \(f(t) = f(u(t))\) for \(t_0 \leq t \leq T\), let us regard \(u\) as a solution to the linear equation

\[
M \frac{du}{dt} + Lu = Mf(t), \quad t_0 < t \leq T,
\]

in \(L_2(\Omega)\) on an interval \([t_0, T]\). Then, the derivative of \(u(t)\) is naturally expected to satisfy the linear equation

\[
M \frac{dv}{dt} + L v = Mf'(t), \quad t_0 < t \leq T, \tag{18}
\]

in \(L_2(\Omega)\) for an unknown function \(v = v(t)\), where \(f'(t) = f'(\text{Re}\, u(t))[\text{Re}\, u'(t)]\). Since \(\nabla\{f'(\text{Re}\, u(t))[\text{Re}\, u'(t)]\} = f''(\text{Re}\, u(t))[\text{Re}\, u'(t)]\nabla\text{Re}\, u(t) + f'(\text{Re}\, u(t))\nabla\text{Re}\, u'(t)\), \(16\) yields that \(f'(t)\) is an \(H^2(\Omega)\)-valued Hölder continuous function with exponent \(\theta\) on the interval \([t_0, T]\). Then, we can apply \[\text{[7, Theorems 3.10 and 3.11 (\(\alpha = \beta = 1\))]}\] to \(18\) with the initial value \(u'(t_0) \in \overline{D(A)}\). So, \(18\) has a unique solution \(v\) in the space:

\[
v \in C([t_0, T]; H^1_0(\Omega)) \cap C((t_0, T]; H^2_0(\Omega)) \cap C^1([t_0, T]; H^1_0(\Omega)). \tag{19}\]

Meanwhile, let \(h > 0\) be a small variable, and put \(u_h(t) = h^{-1}|u(t + h) - u(t)|\) and \(f_h(t) = h^{-1}[f(t + h) - f(t)]\). Clearly, \(u_h\) satisfies

\[
M \frac{du_h}{dt} + Lu_h = Mf_h(t), \quad t_0 < t \leq T - h.
\]

Furthermore, put \(w(t) = u_h(t) - v(t)\), and observe that \(w\) satisfies

\[
M \frac{dw}{dt} + Lw = M[f_h(t) - f'(t)], \quad t_0 < t \leq T - h.
\]

By \[\text{[7, Theorem 3.7 (\(\alpha = \beta = 1\))]}\], \(w(t)\) must be represented as

\[
w(t) = e^{-(t-t_0)A}[u_h(t_0) - u'(t_0)] + \int_{t_0}^{t} e^{-(t-s)A}[f_h(s) - f'(s)]ds, \quad t_0 \leq t \leq T - h.
\]

This shows that, as \(h \to 0\), \(w(t) \to 0\) in \(H^1_0(\Omega)\) for each \(t_0 \leq t < T\). Hence, \(u'(t) = v(t)\) for every \(t_0 \leq t < T\). By continuity, it is the same at \(t = T\), too. We have thus verified that \(u'\) lies in the space \(19\). Since \(t_0 > 0\) is arbitrary, we conclude the desired regularity \(17\) of \(u\).

2.3. Nonnegativity of solutions. This subsection is devoted to showing nonnegativity of \(u(t)\) in \(\Omega\) under the condition that \(u_0 \geq 0\) in \(\Omega\).

First, \(u(t)\) is a real function. Indeed, if \(u(t)\) is a solution to \(8\), then its complex conjugate \(\bar{u}(t)\) is also a solution lying in \(16\) and having the same initial value. Uniqueness of solution then implies that \(u(t) = u(t)\).

Next, introduce a cutoff function \(H(u)\), \(-\infty < u < \infty\), defined by \(H(u) = \frac{1}{2}u^2\) for \(-\infty < u < 0\) and by \(H(u) \equiv 0\) for \(0 \leq u < \infty\). Put

\[
\varphi(t) = \int_{\Omega} m(x) H(u(x, t))dx, \quad 0 \leq t \leq T.
\]
For $0 < t \leq T$, $\varphi(t)$ is differentiable with the derivative
\[
\varphi'(t) = \int_\Omega m(x)H'(u(t)) \frac{\partial u}{\partial t} \, dx = \int_\Omega H'(u(t))[a \Delta u(t) + m(x)f(u(t))] \, dx
\]
\[
= -a \int_\Omega \nabla H'(u(t)) \cdot \nabla u(t) \, dx + \int_\Omega m(x)H'(u(t))f(u(t)) \, dx.
\]
Since $H''(u) \geq 0$, we have
\[
-a \int_\Omega \nabla H'(u(t)) \cdot \nabla u(t) \, dx = -a \int_\Omega H''(u(t))|\nabla u(t)|^2 \, dx \leq 0.
\]
Meanwhile, since $|f(u)| \leq C|u|$ for $|u| \leq \max_{0 \leq t \leq T} \|u(t)\|_C$ due to $f(0) = 0$, and since $H'(u)u = 2H(u)$ for every $-\infty < u < \infty$, it follows that
\[
\varphi'(t) \leq C \int_\Omega m(x)|H'(u(t))| |u(t)| \, dx \leq C \varphi(t).
\]
So, $\varphi(t) \leq e^{Ct} \varphi(0)$ for $0 < t \leq T$. But, since $\varphi(0) = 0$, we have $\varphi(t) \equiv 0$, which means that for any $0 \leq t \leq T$, $m(x)H(u(x,t)) = 0$ almost everywhere in $\Omega$.

In view of this fact, multiply the equation of (8) by $H(u(t))$ and integrate the product in $\Omega$. Then,
\[
0 = a \int_\Omega [\Delta u(t)]H(u(t)) \, dx = -a \int_\Omega H'(u(t))|\nabla u(t)|^2 \, dx.
\]
Consider the set $\Omega_t^- = \{x \in \Omega; u(x,t) < 0\}$ which is an open subset of $\Omega$ due to (15). Since $H'(u(x,t)) \leq 0$ for $x \in \Omega_t^-$, $\nabla u(t)$ vanishes identically in $\Omega_t^-$, and hence $u(x,t)$ must be a negative constant in each connected component of $\Omega_t^-$. Thereby, if $\Omega_t^- \neq \emptyset$, then $\Omega_t^-$ must coincide with $\Omega$ because on the boundary of $\Omega_t^-$, $u(x,t)$ takes negative values. But this contradicts the homogeneous Dirichlet conditions on $\partial \Omega$. Thus,
\[
\text{if } u_0(x) \geq 0, \text{ then } u(t) \geq 0 \text{ for any } 0 < t \leq T. \quad (20)
\]

3. **Global solutions.** For constructing global solutions, the essential thing is to establish the a priori estimates for local solutions. By the smoothing effect of solutions observed by (16)-(17), there is no loss of generality to assume that $u_0 \in D(A) = D(M^{-1}L)$. Since $X$ is a Hilbert space, the compatibility condition
\[
[f(u_0) - Au_0] \cap \overline{D(A)} \neq \emptyset. \quad (21)
\]
is fulfilled automatically (see (40)). By Remark 2, $u$ is differentiable even at $t = 0$ and $u'(0)$ is determined by the relation (65).

We are then led to assume that $u$ is a local solution of (8) lying in
\[
0 \leq u \in C([0, T_u]; H^2_D(\Omega)) \cap C^1([0, T_u]; H^1_D(\Omega))
\]
\[
\cap C^1((0, T_u]; H^2_D(\Omega)) \cap C^2((0, T_u]; H^1_D(\Omega)). \quad (22)
\]

**Proposition 1.** There exists a constant $\tilde{C} > 0$ such that the estimate
\[
\|A^\beta u(t)\|_{H^1_D} \leq \tilde{C}(\|u'(0)\|^2_{L^p} + \|A^\beta u_0\|^2_{H^1_D} + 1)^p, \quad 0 \leq t \leq T_u, \quad (23)
\]
holds true for any local solution in (22), the constant $\tilde{C}$ being independent of $T_u$. 
Proof. The proof is carried out by several steps.

**Step 1.** Multiply the equation of (8) by \(2u(t)\) and integrate the product in \(\Omega\). Then, it follows by (3) and (20) that
\[
\frac{d}{dt} \int_{\Omega} m[u(t)]^2 dx + 2a \int_{\Omega} |\nabla u(t)|^2 dx \\
= 2 \int_{\Omega} m f(u(t)) u(t) dx \leq 2D_2 \|m\|_{L_{\infty}} \int_{\Omega} u(t) dx \\
\leq ac \int_{\Omega} u(t)^2 dx + \frac{D_2^2 \|m\|_{L_{\infty}}^2 |\Omega|}{ac}.
\]
Here, we use (11) to obtain that
\[
\frac{d}{dt} \int_{\Omega} m[u(t)]^2 dx + a \int_{\Omega} |\nabla u(t)|^2 dx \leq C_1, \tag{24}
\]
where \(C_1 = \frac{D_2^2 \|m\|_{L_{\infty}}^2 |\Omega|}{ac}\). In view of (11) again,
\[
\frac{d}{dt} \int_{\Omega} m[u(t)]^2 dx + \frac{ac}{\|m\|_{L_{\infty}}} \int_{\Omega} m[u(t)]^2 dx \leq C_1, \tag{25}
\]
which will be used later.

**Step 2.** Similarly, multiply the equation of (8) by \(2pu(t)^{2p-1}\) and integrate the product in \(\Omega\). Then, after some calculations as above,
\[
\frac{d}{dt} \int_{\Omega} m[u(t)]^{2p} dx + 2(2p-1)p^{-1}a \int_{\Omega} |\nabla u(t)|^{2p} dx = 2p \int_{\Omega} m f(u(t)) |u(t)|^{2p-1} dx \\
\leq 2pD_2 \|m\|_{L_{\infty}} \int_{\Omega} u(t)^{2p-1} dx \\
\leq 2pD_2 \|m\|_{L_{\infty}} \int_{\Omega} \{\varepsilon [u(t)]^{2p} + C_\varepsilon\} dx
\]
with any number \(\varepsilon > 0\). Taking \(\varepsilon\) as \(\varepsilon = \frac{(2p-1)ac}{2pD_2 \|m\|_{L_{\infty}}}\), we apply (11) to \(u(t)^p\). Then,
\[
\frac{d}{dt} \int_{\Omega} m[u(t)]^{2p} dx + (2p-1)p^{-1}ac \int_{\Omega} u(t)^{2p} dx \leq 2pD_2 \|m\|_{L_{\infty}} C_\varepsilon |\Omega|.
\]
Furthermore,
\[
\frac{d}{dt} \int_{\Omega} m[u(t)]^{2p} dx + \frac{(2p-1)p^{-1}ac}{\|m\|_{L_{\infty}}} \int_{\Omega} m[u(t)]^{2p} dx \leq 2pD_2 \|m\|_{L_{\infty}} C_\varepsilon |\Omega|.
\]
Solving this differential inequality, we conclude that
\[
\int_{\Omega} m[u(t)]^{2p} dx \leq C_2 [e^{-d_1 t} \|u_0\|_{L_{2p}}^{2p} + 1], \quad 0 \leq t \leq T_u, \tag{26}
\]
where \(d_1 = \frac{(2p-1)ac}{p \|m\|_{L_{\infty}}}\).

**Step 3.** Now, multiply the equation of (8) by \(\frac{\partial u}{\partial t}(t)\) and integrate the product in \(\Omega\). Then,
\[
a \frac{d}{dt} \int_{\Omega} |\nabla u(t)|^2 dx + 2 \int_{\Omega} m \left|\frac{\partial u}{\partial t}(t)\right|^2 dx = 2 \int_{\Omega} m f(u(t)) \frac{\partial u}{\partial t}(t) dx \\
\leq \int_{\Omega} m \left|\frac{\partial u}{\partial t}(t)\right|^2 dx + \int_{\Omega} m|f(u(t))|^2 dx.
\]
Therefore, by (3),
\[
\alpha \frac{d}{dt} \int_\Omega |\nabla u(t)|^2 dx + \int_\Omega m \left| \frac{\partial u}{\partial t}(t) \right|^2 \, dx \leq \int_\Omega m \{D_1 u(t)[u(t)]^{p-1} + D_2 \}^2 dx \\
\leq \int_\Omega 3m \{D_1^2 u(t)^{2p} + D_1^2 u(t)^2 + D_2^2 \} dx \\
\leq \int_\Omega 3m \{(D_1^2 + 1)[u(t)]^{2p} + C_3 \} dx.
\]
Since we already know (26), it follows that
\[
ad \frac{d}{dt} \int_\Omega |\nabla u(t)|^2 dx + \int_\Omega m \left| \frac{\partial u}{\partial t}(t) \right|^2 \, dx \leq C_4(\|u_0\|_{L^p}^{2p} + 1). \quad (27)
\]
We take a summation of this inequality, (24) and (25). Then, the following differential inequality for \(\psi_1(t) = \int_\Omega [2m|u(t)|^2 + a|\nabla u(t)|^2] \, dx\) is obtained, i.e.,
\[
\frac{d}{dt} \psi_1(t) + d_2 \psi_1(t) + \int_\Omega m \left| \frac{\partial u}{\partial t}(t) \right|^2 \, dx \leq C_5(\|u_0\|_{L^p}^{2p} + 1)
\]
with \(d_2 = \min\{\frac{\mu_0}{2m\|u\|_{L^\infty}}, 1\}\). Solving this inequality, we conclude that
\[
\int_\Omega \{2m|u(t)|^2 + a|\nabla u(t)|^2\} \, dx \leq C_0[e^{-d_2t}\|u_0\|_{H^1_0}^2 + \|u_0\|_{L^p}^{2p} + 1], \quad 0 \leq t \leq T_u.
\]
**Step 4.** In view of (22), \(u_t = \frac{\partial u}{\partial t}\) is seen to satisfy
\[
m(x) \frac{\partial u_t}{\partial t} - a \Delta u_t = m(x)f'(u(t))u_t.
\]
Multiply this equation by \(2u_t\) and integrate the product in \(\Omega\). Then, by (4),
\[
\frac{d}{dt} \int_\Omega m|u_t(t)|^2 dx + 2a \int_\Omega |\nabla u_t(t)|^2 dx = 2 \int_\Omega mf'(u(t))u_t(t)^2 dx \\
\leq 2D_4 \int_\Omega m|u_t(t)|^2 dx. \quad (28)
\]
After multiplying (27) with a constant \(2D_4 + 1\), we add the equation to (28). Then, a differential inequality for \(\psi_2(t) = (2D_4 + 1)\psi_1(t) + \int_\Omega m|u_t(t)|^2 dx\) is obtained, i.e.,
\[
\frac{d}{dt} \psi_2(t) + d_2 \psi_2(t) + 2a \int_\Omega |\nabla u_t(t)|^2 \, dx \leq C_7(\|u_0\|_{L^p}^{2p} + 1).
\]
Hence,
\[
\int_\Omega m|u_t(t)|^2 dx \leq C_8[e^{-d_2t}[\|u'(0)\|_{L^2} + \|u_0\|_{H^1_0}^2] + \|u_0\|_{L^p}^{2p} + 1], \quad 0 \leq t \leq T_u.
\]
As a direct consequence of this estimate and (26), it is observed that
\[
a^2||\Delta u(t)||_{L^2}^2 = \|mu_t(t) - mf(u(t))\|_{L^2}^2 \leq \|m||_{L_{\infty}} \|m^\frac{1}{2} u_t(t)\|_{L^2}^2 + \|m^\frac{1}{2} f(u(t))\|_{L^2}^2 \\
\leq C_9[\|u'(0)\|_{L^2}^2 + \|u_0\|_{H^1_0}^2 + \|u_0\|_{L^p}^{2p} + 1].
\]
Hence, since \(|\nabla u(t)||_{L^2}\) has already been estimated above, we obtain on account of (10) and (11) that
\[
\|u(t)||_{H^2}^2 \leq CC_9[\|u'(0)\|_{L^2}^2 + \|u_0||_{H^1_0}^2 + \|u_0||_{L^p}^{2p} + 1], \quad 0 \leq t \leq T_u. \quad (29)
\]
Step 5. As verified in the proof of Theorem 5.1, \( u(t) \) satisfies the integral equation
\[
    u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(u(s))ds
\]
on the interval \([0, T_u]\). Applying the operator \( A^{\tilde{\beta}} \) to this equation, we see that
\[
    A^{\tilde{\beta}}u(t) \equiv A^{\tilde{\beta}}e^{-tA}u_0 + \int_0^t A^{\tilde{\beta}}e^{-(t-s)A}f(u(s))ds.
\]
Then, by (43),
\[
    \|A^{\tilde{\beta}}e^{-tA}u_0\|_{H^1_0} \leq C_{10}e^{-\delta t}\|A^{\tilde{\beta}}u_0\|_{H^1_0}, \quad 0 \leq t \leq T_u. \tag{30}
\]
Meanwhile, by (43) and (45),
\[
    \|A^{\tilde{\beta}}e^{-tA}\|_{L(H^1_0)} = e^{-\frac{\delta}{2}A}A^{\frac{\beta}{2}}A^{-\frac{\beta}{2}}A\|_{L(H^1_0)} \leq C_{11}t^{-\beta}e^{-\frac{\delta}{2}t}, \quad 0 < t < \infty.
\]
Since it is easily observed by (3), (4) and (15) that
\[
    \|f(u)\|_{H^1_0}^2 = \|
        \nabla f(u)\|_{L^2}^2 + \|f(u)\|_{L^2}^2 \leq \|f'(u)\|_{L^2}^2 + \|f(u)\|_{L^2}^2
\]
\[
    \leq C_{12}([\|u\|_{L^\infty}^{p-1} + 1]\|u\|_{H^1_0}^2 + \|u\|_{L^\infty}^{p} \|u\|_{L^2}^2 + 1)^2
\]
\[
    \leq C_{13}(\|u\|_{H^2}^2 + 1)^2, \quad u \in H^1_0(\Omega),
\]
(29) yields that
\[
    \|f(u(t))\|_{H^1_0}^2 \leq C_{14}(\|u'(0)\|_{L^2} + \|u_0\|_{H^1_0}^2 + \|u_0\|_{L^2}^{2p} + 1)^{2p}, \quad 0 \leq t \leq T_u.
\]
Therefore,
\[
    \left\| \int_0^t A^{\tilde{\beta}}e^{-(t-s)A}f(u(s))ds \right\|_{H^1_0} \leq C_{15} \int_0^t (t-s)^{-\beta}e^{-\frac{\delta}{2}(t-s)}ds
\]
\[
    \times (\|u'(0)\|_{L^2}^2 + \|u_0\|_{H^1_0}^2 + \|u_0\|_{L^2}^{2p} + 1)^{p}, \quad 0 \leq t \leq T_u.
\]
Ultimately, using (15), we obtain that
\[
    \left\| \int_0^t A^{\tilde{\beta}}e^{-(t-s)A}f(u(s))ds \right\|_{H^1_0} \leq C_{16}(\|u'(0)\|_{L^2}^2 + \|A^{\tilde{\beta}}u_0\|_{H^1_0}^{2p} + 1)^{p}, \quad 0 \leq t \leq T_u.
\]
This jointed with (30) yields the desired estimate (23). \( \Box \)

This Proposition 1 readily provides the global existence of solutions.

Theorem 3.1. Under (2)~(4), let \( u_0 \in \mathcal{D}(A^{\tilde{\beta}}) (\subset H^2_{00}(\Omega)) \) and \( u_0 \geq 0 \). Then, (8) possesses a unique global solution \( u \) in the function space:
\[
    0 \leq u \in C^\sigma([0, \infty); H^{2\beta}_0(\Omega)) \cap C^1((0, \infty); H^2_0(\Omega)) \cap C^2((0, \infty); H^1_0(\Omega)).
\]
Moreover, the global solution satisfies the estimates
\[
    \|A^{\tilde{\beta}}u(t)\|_{H^1_0} \leq \psi(\|A^{\tilde{\beta}}u_0\|_{H^1_0}), \quad 0 \leq t < \infty, \tag{31}
\]
\[
    \|u'(t)\|_{H^1_0} + \|Au(t)\|_{H^1_0} \leq \psi(\|A^{\tilde{\beta}}u_0\|_{H^1_0})(t^{\tilde{\beta}-1} + 1), \quad 0 < t < \infty, \tag{32}
\]
\( \psi(\cdot) > 0 \) being some continuous increasing function.
Proof. First, let us apply Theorem 5.1 to obtain a local solution on an interval \([0, T_{u0}]\) in the space (16)-(17). In addition, as verified by (59) and (60), the local solution \(u\) satisfies (31) and (32) locally on the interval \([0, T_{u0}]\).

Second, we reset a new initial time by the \(T_{u0}\) and a new initial value \(u(T_{u0}).\) (Note that the condition (21) is satisfied.) Theorem 5.1 again ensures that the original local solution can be extended beyond the time \(T_{u0}\) as an \(X\)-valued strict solution, see Remark 2.

Third, we repeat such an extension procedure. Let \(u\) be any local solution on an interval \([0, T_u]\). Applying Theorem 5.1 with initial time \(T_u\) and with initial value \(u(T_u),\) we can extend this local solution to another one on an interval \([0, T_u + \tau]\), here \(\tau > 0\) is determined by \(\|A^\beta u(T_u)\|_{H^1_u}\). Since \(\|A^\beta u(t)\|_{H^1_u}\) is uniformly bounded owing to (23) in Proposition 1, the extension length \(\tau\) is independent of \(T_u;\) this means that the original local solution can be extended on the whole interval \([0, \infty)\).

Finally, let us verify the global estimates. As mentioned in Step 1, (31) is the case on the interval \([0, T_{u0}]\). Meanwhile, it is true that

\[
\|u'(T_{u0})\|_{L^2} \leq \|u'(T_{u0})\|_{H^1_u} \leq \psi(\|A^\beta u_0\|_{H^1_u}) (T_{u0})^{\beta - 1}.
\]

Hence, (23) actually means that (31) is the case on the half line \([T_{u0}, \infty)\).

To verify (32), let \(0 < s < \infty\) and apply (60) with initial time \(s\) and initial value \(u(s).\) Then, there is a time length \(\tau > 0\) and a constant \(C_u(s) > 0\) such that

\[
\|u'(t)\|_{H^1_u} + \|Au(t)\|_{H^1_u} \leq C_u(s)(t - s)^{\beta - 1}, \quad s < t \leq s + \tau.
\]

In particular, it is observed that

\[
\|u'(t)\|_{H^1_u} + \|Au(t)\|_{H^1_u} \leq C_u(s)2^{1 - \beta} \tau^{\beta - 1}, \quad s + \frac{\tau}{2} \leq t \leq s + \tau.
\]

Here, the length \(\tau\) is determined by the norm \(\|A^\beta u(s)\|_{H^1_u}\) alone. But in view of (31) verified now, that is independent of \(s\). It is the same for \(C_u(s).\) Hence, (32) is the case on the half line \([\frac{\tau}{2}, \infty)\). As mentioned in Step 1, we already know that (32) holds true in the neighborhood of the initial time.

\[\square\]

4. Dynamical system. Knowing that (8) has a unique global solution for every initial value \(u_0 \in D(A^\beta)\) satisfying \(u_0(x) \geq 0,\) where \(A = M^{-1}L,\) let us construct a dynamical system in the underlying space \(D(A^\beta).\)

Set a phase space by

\[K = \{u_0 \in D(A^\beta); u_0(x) \geq 0 \text{ a.e. in } \Omega\},\]

\(K\) being equipped with the norm \(\|A^\beta \cdot\|_{X}\).

For \(u_0 \in K,\) let \(u(t; u_0)\) be the global solution of (8), and set

\[S(t)u_0 = u(t; u_0), \quad 0 \leq t < \infty.\]

By Theorem 3.1, \(S(t)\) maps \(K\) into itself. By uniqueness of solution, \(S(t)\) satisfies the semigroup property \(S(t + s) = S(t)S(s)\) for \(0 \leq s, t < \infty.\) Meanwhile, \(S(t)\) is locally Lipschitz continuous on \(K.\) In fact, put

\[K_R = \{u_0 \in K; \|A^\beta u_0\|_{X} \leq R\}.\]

Then, by (31), all the trajectories \(S(t)u_0\) starting from \(K_R\) remain in a bounded subset of \(K_{\psi(R)}\). In addition, thanks to Theorem 5.3, there is \(\tau_R > 0\) such that

\[\|A^\beta[S(t)\tilde{u}_0 - S(t)\tilde{v}_0]\|_{H^1_u} \leq C_{\psi(R)}\|A^\beta[\tilde{u}_0 - \tilde{v}_0]\|_{H^1_u}, \quad 0 \leq t \leq \tau_R; \quad \tilde{u}_0, \tilde{v}_0 \in K_{\psi(R)}\].
For $0 < t < \infty$, let $t = n\tau_R + t_0$ with $0 \leq t_0 < \tau_R$ and some integer $n$. Then, we have
\[
\|A^\beta[S(t)u_0 - S(t)v_0]\|_{H_0^1} \leq \|A^\beta[S(\tau_R)^nS(t_0)u_0 - S(\tau_R)^nS(t_0)v_0]\|_{H_0^1} \leq C_n^{n+1}\|A^\beta[u_0 - v_0]\|_{H_0^1},
\]
where $u, v \in K_R$.

Meanwhile, as $X$ is a Hilbert space, Remark 1 shows that $S(t)u_0$ is continuous at $t = 0$ with respect to the graph norm of $A^\beta$. We have thus verified that $S(t)$ is a continuous nonlinear semigroup on $K$ and $(S(t), K, D(A^\beta))$ defines a dynamical system.

The dynamical system is shown to possesses a Lyapunov function. Indeed, for $u_0 \in K$ and $u(t; u_0) = S(t)u_0$, multiply the equation of (8) by $\frac{\partial u}{\partial t}$ and integrate the product in $\Omega$. Then,
\[
\int_\Omega m^2 |\frac{\partial u}{\partial t}|^2 dx + \frac{a}{2} \int_\Omega |\nabla u|^2 dx = \frac{d}{dt} \int_\Omega m F(u(t)) dx,
\]
or
\[
\frac{d}{dt} \int_\Omega \left[ \frac{a}{2} |\nabla u|^2 - m F(u(t)) \right] dx = - \int_\Omega m \left| \frac{\partial u}{\partial t} \right|^2 dx,
\]
where $F(u) = \int_0^u f(v) dv$ is a primitive function of $f(u)$. This then shows that the function
\[
\Psi(u) = \int_\Omega \left[ \frac{a}{2} |\nabla u|^2 - m F(u) \right] dx, \quad u \in D(A^\beta),
\]
plays the role of a Lyapunov function to $(S(t), K, D(A^\beta))$.

The following properties of $\Psi(\cdot)$ are verified.

**Proposition 2.** For any $u_0 \in K$, the value $\Psi(S(t)u_0)$, $0 \leq t < \infty$, is uniformly bounded from below; therefore, $\Psi(S(t)u_0)$ has a limit as $t \to \infty$. If $\frac{d}{dt}[\Psi(S(t)u_0)] = 0$ at some time $t = \bar{t}$, then $\bar{u} = S(\bar{t})u_0$ is a stationary solution of (8).

**Proof.** It follows from (3) that $F(u) = \int_0^u f(v) dv \leq D_2 u$, $0 \leq u < \infty$. Therefore, it is seen from (20) that
\[
\int_\Omega m F(u(t)) dx \leq D_2 \|m\|_{L_\infty} \int_\Omega u(t) dx \leq D_2 \|m\|_{L_\infty} (\varepsilon \|u(t)\|_{L_2}^2 + (4\varepsilon)^{-1}|\Omega|)
\]
with any number $\varepsilon > 0$. Hence, (11) yields that
\[
\Psi(u(t)) \geq \left( \frac{ac}{2} - D_2 \|m\|_{L_\infty} \varepsilon \right) \|u(t)\|_{L_2}^2 - D_2 \|m\|_{L_\infty} (4\varepsilon^{-1}|\Omega|), \quad 0 \leq t < \infty.
\]

Let us prove the second assertion. Assume that $\frac{d}{dt}[\Psi(S(t)u_0)] = 0$ at some time $t = \bar{t}$. From (33) it follows that $m(x)\frac{\partial u}{\partial t}(\bar{t}) = 0$ in $\Omega$; naturally, it is the same for $m(x)\frac{\partial u}{\partial t}(\bar{t})$. Hence,
\[
a\Delta u(\bar{t}) + m(x) f(u(\bar{t})) = 0 \quad \text{in } \Omega,
\]
which means that $u(\bar{t})$ is a stationary solution of (8). \qed

By the standard arguments, the following theorem is proved.

**Theorem 4.1.** For any $u_0 \in K$, the $\omega$-limit set $\omega(u_0)$ of the trajectory $S(t)u_0$ in $D(A^\beta)$ is a nonempty set and consists of stationary solutions of (8).
Proof. Let \( u_0 \in K \) and fix time \( t_0 > 0 \). From (32), \( \sup_{t_0 \leq t < \infty} \| AS(t)u_0 \|_X < \infty \). In the meantime, \( D(A) \) is compactly embedded in \( D(A^\theta) \) by the Lemma below. Hence, \( \omega(u_0) \) is non empty.

Let \( \bar{\pi} \in \omega(u_0) \). Then, by the first assertion of Proposition 2, we see that \( \Psi(\bar{\pi}) = \inf_{0 \leq t < \infty} \Psi(S(t)u_0) \). But, since \( \omega(u_0) \) is an invariant set of \( S(t) \), i.e., \( S(t)[\omega(u_0)] = \omega(u_0) \), \( \Psi(S(t)\bar{\pi}) = \inf_{0 \leq t < \infty} \Psi(S(t)u_0) \) for all \( t \geq 0 \). Then, the second assertion of Proposition 2 yields that \( \bar{\pi} \) must be in \( D(A) \) and must be a stationary solution.

**Lemma 4.2.** For any \( 0 \leq \theta < 1 \), the embedding from \( D(A) \) into \( D(A^\theta) \) is compact.

**Proof.** We notice that \( D(A) \subset D(L) = H^2_0(\Omega) \) is compactly embedded in \( H^1_0(\Omega) = X \). Then, the desired result is verified by the moment inequality (42).

5. Abstract degenerate evolution equations of parabolic type.

5.1. **Multivalued linear operators.** Let \( X \) be a complex Banach space with norm \( \| \cdot \| \) and let \( 2^X \) denote the family of all subsets of \( X \). An operator \( A : D(A) \to 2^X - \{ \emptyset \} \), where \( D(A) \) is a linear subspace of \( X \), is called a multivalued linear operator of \( X \) if \( A \) satisfies

\[
\begin{aligned}
&\begin{cases}
Au + Av \subset A(u + v), \\
\lambda Au \subset A(\lambda u),
\end{cases}
&u, v \in D(A), \\
&\lambda \in \mathbb{C}, \ u \in D(A),
\end{aligned}
\]

see [6]. When \( A \) is a multivalued linear operator, \( A0 \) is always a linear subspace of \( X \), and for \( u \in D(A) \) it is true that \( Au = f + A0 \) with any \( f \in Au \). Analogously to the single valued linear operators (i.e., \( A0 = \{0\} \)), a number \( \lambda \in \mathbb{C} \) is said to belong to the resolvent set \( \rho(A) \) of \( A \) if the inverse of \( \lambda - A \) is single valued and is a bounded linear operator of \( X \). On the contrary, if \( \lambda \notin \rho(A) \), then \( \lambda \) is said to belong to the spectrum \( \sigma(A) \) of \( A \). The \( \mathcal{L}(X) \) valued analytic function \( \lambda(A) \) defined in \( \rho(A) \) is called the resolvent of \( A \). Moreover, \( A \) is said to be a sectorial operator of \( X \) if there exists an open sectorial domain such that

\[
\sigma(A) \subset \Sigma = \{ \lambda \in \mathbb{C} : |\arg \lambda| < \omega \},
\]

where \( 0 < \omega < \pi \), and that \( (\lambda - A)^{-1} \) satisfies

\[
\| (\lambda - A)^{-1} \|_{\mathcal{L}(X)} \leq \frac{D}{|\lambda|}, \quad \lambda \notin \Sigma,
\]

with some constant \( D > 0 \).

If \( 0 \in \rho(A) \), that is \( A^{-1} \in \mathcal{L}(X) \), the graph

\[
\mathcal{G}(A) = \{(f, u) \in X \times X : f \in Au\}
\]

is a closed subspace of \( X \times X \). Furthermore, \( \{0\} \times A0 \) is a closed subspace of \( \mathcal{G}(A) \). Then, under \( 0 \in \rho(A) \), \( \mathcal{D}(A) \) becomes a Banach space with the graph norm

\[
\| u \|_{\mathcal{D}(A)} = \| Au \|_X \equiv \inf_{f \in Au} \| f \|, \quad u \in \mathcal{D}(A),
\]

see [7, Proposition 1.1].

If \( A \) is sectorial, \( (\lambda - A)^{-1} \) satisfies the optimal decay estimate on the half line \( (-\infty, 0] \); moreover, it is seen that

\[
\lim_{\lambda \to -\infty} \lambda(\lambda - A)^{-1} f = f, \quad f \in \overline{D(A)}.
\]

From this it is proved that \( A0 \cap \overline{D(A)} = \{0\} \). (In fact, if \( f \in A0 \), then \( f \in (\lambda - A)0 \) for every \( \lambda \leq 0 \) and hence \( (\lambda - A)^{-1} f = 0 \); therefore, if \( f \in \overline{D(A)} \) in addition, then
(37) implies \( f = 0 \). This property furthermore provides that for any \( u \in D(A) \) and any \( f \in X \), it holds that

\[
[f - Au] \cap \overline{D(A)} \quad \text{is a singleton if it is not empty.} 
\]  

When \( X \) is a reflexive Banach space, it is known that

\[
X = A0 + \overline{D(A)}, 
\]

see Remark to [7, Proposition 2.1]. Hence, if \( g \in Au \), then \( f - g = f' + f'' \) with \( f' \in A0 \) and \( f'' \in \overline{D(A)} \), i.e., \( f - g - f' = f'' \); on one hand, we have \( f - g - f' \in f - Au \); on the other hand, \( f'' \in \overline{D(A)} \). Thus, the condition that

\[
[f - Au] \cap \overline{D(A)} \quad \text{is a singleton} 
\]

holds automatically for any \( u \in D(A) \) and any \( f \in X \).

When \( A \) is sectorial, its fractional powers for negative exponents are defined by the integrals

\[
A^{-x} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-x}(\lambda - A)^{-1}d\lambda, \quad x > 0, 
\]

in \( \mathcal{L}(X) \), where \( \Gamma = \Gamma_+ \cup \Gamma_0 \cup \Gamma_- \) is an integral contour lying in \( \rho(A) \) such that \( \Gamma_+: \lambda = re^{i\omega} \) for \( \infty > r \geq \delta, \Gamma_0: \lambda = \delta e^{i\omega} \) for \( \omega \geq \vartheta \geq -\omega, \Gamma_-: \lambda = re^{-i\omega} \) for \( \delta \leq r < \infty, \delta > 0 \) being a sufficiently small radius. It is easy to verify that \( A^{-x} \) satisfy the exponential law \( A^{-(r+x')} = A^{-x}A^{-x'} \). The fractional powers for positive exponents are defined by \( A^x = [A^{-x}]^{-1}, x > 0 \); but of course \( A^x \) are multivalued linear operators of \( X \). They also satisfy the law \( A^{x+x'} = A^xA^{x'} \) in the sense of multivalued operators, see [7, Theorem 1.10]. As noticed by (36), each \( \mathcal{D}(A^x) \) is a Banach space with the norm \( \|A^x\| \). For \( 0 < x < y \), it is clear that \( \mathcal{D}(A^y) \subset \mathcal{D}(A^x) \) with continuous embedding. Moreover, the moment inequality

\[
\|A^xu\| \leq C_{x,y} \|A^yu\|^{x/y}\|u\|^{1-x/y}, \quad u \in \mathcal{D}(A^y), \tag{42}
\]

holds true. Indeed, if \( f \in A^yu, \) then \( u = A^{-y}f = A^{-x}A^{-y}f; \) thereby, \( A^{-y}f \in A^xu \). Therefore,

\[
\|A^xu\| \leq \|A^{-y}f\| \leq C_{x,y}\|A^{-y}f\|^{1-x/y}\|f\|^{x/y} = C_{x,y}\|u\|^{1-x/y}\|f\|^{x/y}. 
\]

But, since \( f \in A^yu \) is arbitrary, we observe (42) to be true.

Let now \( A \) be a sectorial operator with angle \( \omega < \pi \). Then the analytic semigroup \( e^{-tA} \) generated by \(-A\) is given by the integral

\[
e^{-tA} = \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda t}(\lambda - A)^{-1}d\lambda, \quad t > 0, 
\]

in \( \mathcal{L}(X) \), the integral contour \( \Gamma \) being as above, with the norm estimate

\[
\|e^{-tA}\|_{\mathcal{L}(X)} \leq Ce^{-\delta t}, \quad 0 \leq t < \infty. 
\]

If \( f \in A0 \), then \( e^{-tA}f = 0 \) for all \( t > 0 \); therefore, as \( t \to 0 \), \( e^{-tA}f \) does not converge to \( f \) in general. As a matter of fact, it is only verified like (37) that

\[
\lim_{t \to 0} e^{-tA}f = f, \quad f \in \overline{D(A)}, \tag{44}
\]

see the second Remark to [7, Theorem 3.5]. It is seen that \( A^x e^{-tA} \) is single valued for every \( t > 0 \), although \( A^x \) is multivalued. Moreover, \( A^x e^{-tA} \) satisfies the norm estimate

\[
\|A^x e^{-tA}\|_{\mathcal{L}(X)} \leq Ct^{-x}, \quad 0 \leq x < \infty, 0 < t < \infty, \tag{45}
\]

see [7, Proposition 3.2].
5.2. **Semilinear degenerate evolution equations.** Let $Y$ be a complex Banach space with norm $\|\cdot\|_Y$. Consider the Cauchy problem for an abstract degenerate equation

\[
\begin{cases}
M \frac{du}{dt} + Lu = Mf(u), & 0 < t < \infty, \\
u(0) = u_0,
\end{cases}
\]  
(46)

in $Y$. Here, $L$ is a sectorial operator of $Y$. That is, $L$ is a densely defined, closed linear operator whose spectrum is contained in an open sectorial domain

\[
\sigma(L) \subset \Sigma' = \{ \lambda \in \mathbb{C}; \ |\arg\lambda| < \omega' \}
\]  
(47)

with $0 < \omega' < \pi$ and whose resolvent satisfies the estimate

\[
\|(\lambda - L)^{-1}\|_{L(Y)} \leq \frac{D'}{|\lambda|}, \quad \lambda \notin \Sigma',
\]  
(48)

with some constant $D' > 0$.

Meanwhile, $M$ is a bounded linear operator from $X$ into $Y$, where $X$ is another Banach space with norm $\|\cdot\|_X$ such that

\[
D(L) \subset X \subset D(L^\alpha) \quad \text{(continuously)}
\]  
(49)

with some $0 \leq \alpha < 1$. It is assumed that $M$-spectrum of $L$ is contained in an open sectorial domain

\[
\sigma_M(L) \subset \Sigma = \{ \lambda \in \mathbb{C}; \ |\arg\lambda| < \omega \}
\]  
(50)

with some angle $0 < \omega < \frac{\pi}{2}$, and that the $M$-resolvent $(\lambda M - L)^{-1}$ of $L$ satisfies

\[
\|(\lambda M - L)^{-1}M\|_{L(X)} \leq \frac{D}{|\lambda|}, \quad \lambda \notin \Sigma,
\]  
(51)

with some constant $D > 0$.

Finally, $f$ is a nonlinear operator from $D(f) \supset D(L^\alpha)$ into $X$. We assume that there is an exponent $\beta$ such that $\alpha \leq \beta < 1$ for which it holds that $D(L^\beta) \subset D(f)$ together with the Lipschitz condition

\[
\|f(u) - f(v)\|_X \leq \varphi(\|L^\beta u\|_Y + \|L^\beta v\|_Y)\|L^\beta (u - v)\|_Y, \quad u, v \in D(L^\beta),
\]  
(52)

where $\varphi(\cdot)$ is some continuous increasing function. It clearly follows that

\[
\|f(u)\|_X \leq \|f(0)\|_X + \varphi(\|L^\beta u\|_Y)\|L^\beta u\|_Y, \quad u \in D(L^\beta).
\]  
(53)

The initial value $u_0$ is taken in $D(L^\beta)$. Under these structural assumptions (47)-(52), one can show local existence of strict solution for (46).

5.3. **Semilinear multivalued evolution equations.** It is often essentially convenient to treat the Cauchy problems of degenerate evolution equations like (46) as those of non-degenerate evolution equations introducing the multivalued linear operators. Rewrite (46) into the form

\[
\begin{cases}
\frac{du}{dt} + Au \ni f(u), & 0 < t < \infty, \\
u(0) = u_0,
\end{cases}
\]  
(54)

in the Banach space $X$. Here, $A = M^{-1}L$ is a multivalued linear operator of $X$ defined by

\[
\begin{align*}
D(A) &= \{ u \in D(L); \ \exists f \in X \text{ such that } Mf = Lu \}, \\
Au &= \{ f \in X; \ Mf = Lu \}.
\end{align*}
\]  
(55)
It is easy to see that \((\lambda - A)^{-1} = (\lambda M - L)^{-1}M\). Consequently, \((50)\) and \((51)\) imply that \((34)\) and \((35)\) hold for \(A\). Therefore, \(-A\) generates an analytic semigroup \(e^{-tA}\) on \(X\).

For \(0 < \theta \leq 1\), the fractional power \(A^\theta = [M^{-1}L]^\theta\) is defined. It is difficult to know \(\mathcal{D}(A^\theta)\) in any precise way. However, since \(\mathcal{D}(A^0) = X \subset \mathcal{D}(L^\alpha)\) due to \((49)\) and \(\mathcal{D}(A^1) = \mathcal{D}(A) \subset \mathcal{D}(L)\) due to \((55)\), we can compare the domains \(\mathcal{D}(A^\theta)\) and \(\mathcal{D}(L^\theta)\) as follows.

**Proposition 3.** For \(0 < \tilde{\theta} < 1\) and \(\alpha < \theta < (1 - \tilde{\theta})\alpha + \tilde{\theta}\), it is true that \(\mathcal{D}(A^\theta) \subset \mathcal{D}(L^\theta)\) with the estimate
\[
\|L^\theta u\|_Y \leq C_{\tilde{\theta},\theta} \|A^\tilde{\theta}u\|_X, \quad u \in \mathcal{D}(A^\tilde{\theta}),
\]
with some constant \(C_{\tilde{\theta},\theta} > 0\). Note that \(\|A^\tilde{\theta}u\|_X\) is defined by \((36)\).

**Proof.** Since
\[
L(\lambda - A)^{-1} = L(\lambda M - L)^{-1}M = (L - \lambda M + \lambda M)(\lambda M - L)^{-1}M = -M + \lambda M(\lambda M - L)^{-1}M = \lambda M(\lambda + (\lambda - A)^{-1}),
\]
it follows that
\[
\|L(\lambda - A)^{-1}\|_{\mathcal{L}(X,Y)} \leq (D + 1)\|M\|_{\mathcal{L}(X,Y)}, \quad \lambda \notin \Sigma.
\]
Meanwhile, by \((49)\) it follows that
\[
\|L^\alpha(\lambda - A)^{-1}\|_{\mathcal{L}(X,Y)} \leq \|L^\alpha\|_{\mathcal{L}(X,Y)}\|\lambda - A\|^{-1}_{\mathcal{L}(X)} \leq D\|L^\alpha\|\|\lambda\|^{-1}, \quad \lambda \notin \Sigma.
\]
The moment inequality then yields for \(0 < \theta < 1\) that
\[
\|L^\theta(\lambda - A)^{-1}\|_{\mathcal{L}(X,Y)} \leq C\|L^\alpha(\lambda - A)^{-1}\|^{(1-\theta)/(1-\alpha)} \times \|L(\lambda - A)^{-1}\|^{[\theta-\alpha]/(1-\alpha)} \leq C\|\lambda\|^{-(1-\theta)/(1-\alpha)}, \quad \lambda \notin \Sigma.
\]
By the definition \((41)\), we see that
\[
L^\theta A^{-\tilde{\theta}} = \frac{1}{2\pi i} \int \lambda^{-\tilde{\theta}}L^\theta(\lambda - A)^{-1}d\lambda.
\]
Obviously, the integral is convergent in \(\mathcal{L}(X,Y)\) if \((1 - \tilde{\theta})\alpha + \tilde{\theta} > \theta\). Meanwhile, \(L^\theta A^{-\tilde{\theta}} \in \mathcal{L}(X,Y)\) immediately implies the desired inequality \((56)\).

Let us fix an exponent \(0 < \tilde{\beta} < 1\) so that
\[
\beta < (1 - \tilde{\beta})\alpha + \tilde{\beta}.
\]
By the proposition, we have \(\mathcal{D}(A^\tilde{\beta}) \subset \mathcal{D}(L^\beta) \subset \mathcal{D}(f)\).

It is now ready to construct local solution for \((54)\). Since the equation in \((54)\) is a semilinear parabolic equation (although it is multivalued), we can use analogous techniques as in the proof of \([18, \text{Theorem 4.1(}\beta = \eta)\]).

**Theorem 5.1.** Under \((47)\)~\((52)\), for any \(u_0 \in \mathcal{D}(A^\tilde{\beta})\), \((54)\) possesses a unique local solution \(u\) in the function space:
\[
u \in \mathcal{C}([0,T_{u_0}];\mathcal{D}(L^\beta)) \cap \mathcal{C}([0,T_{u_0}];\mathcal{D}(A)) \cap \mathcal{C}^1((0,T_{u_0});X),
\]
\(T_{u_0} > 0\) being determined by the norm \(\|A^\tilde{\beta}u_0\|_X\) alone.
Moreover, the local solution \( u \) satisfies the estimates
\[
\|A^\beta u(t)\| \leq C_{u_0}, \quad 0 \leq t \leq T_{u_0},
\]
\[
\|u'(t)\|_X + \|Au(t)\|_X \leq C_{u_0} t^{\beta - 1}, \quad 0 < t \leq T_{u_0},
\]
where the radius \( C_{u_0} > 0 \) being determined by the norm \( \|A^\beta u_0\|_X \) alone.

**Proof.** For \( 0 < T < \infty \), we set a Banach space \( \mathcal{X}(T) \) by
\[
\mathcal{X}(T) = C([0, T]; D(L^\beta))
\]
equipped with the norm \( \|u\|_X = \max_{0 \leq t \leq T} \|L^\beta u(t)\|_Y \). In addition, we set a closed ball \( \mathcal{B}(T) \) of \( \mathcal{X}(T) \) by
\[
\mathcal{B}(T) = \{u \in \mathcal{X}(T); \|u\|_X \leq R\},
\]
where the radius \( R > 0 \) will be specified below.

For \( u \in \mathcal{B}(T) \), we define a mapping
\[
[\Phi u](t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(u(s))ds, \quad 0 \leq t \leq T.
\]

Let us verify that, if \( R \) is suitably chosen and if \( T \) is sufficiently small, then \( \Phi \) is a contraction of \( \mathcal{X}(T) \) which maps \( \mathcal{B}(T) \) into itself.

**Step 1.** The function \([\Phi u](t)\) is seen to be a Hölder continuous function with values in \( D(L^\beta) \). To show this, we need to introduce an auxiliary exponent \( \tilde{\beta}' \) such that \( 0 < \tilde{\beta}' < \tilde{\beta} \) but \( \beta < (1 - \tilde{\beta}')\alpha + \tilde{\beta}' \) (see (57)). Then, it follows by Proposition 3 that
\[
D(A^\beta) \subset D(A^{\tilde{\beta}'}) \subset D(L^\beta).
\]

Let \( g_0 \) be any element such that \( g_0 \in A^\beta u_0 \). Then, since \( u_0 = A^{-\beta} g_0 \), we have
\[
L^\beta[e^{-tA} - e^{-sA}]u_0 = L^\beta[A^{\beta'} e^{-(t-s)A} - 1]A^{\tilde{\beta}' - \beta} e^{-sA}g_0.
\]

Therefore, by [7, Theorem 3.5], we obtain that
\[
\|L^\beta[e^{-tA} - e^{-sA}]u_0\|_Y \leq C\|g_0\|_X (t-s)^\sigma, \quad 0 < s < t \leq T,
\]
with the exponent \( \sigma = \tilde{\beta} - \tilde{\beta}' \).

Meanwhile, we write
\[
\int_s^t e^{-(t-\tau)A}f(u(\tau))d\tau - \int_0^s e^{-(s-\tau)A}f(u(\tau))d\tau
\]
\[
= \int_s^t e^{-(t-\tau)A}f(u(\tau))d\tau + [e^{-(t-s)A} - 1] \int_0^s e^{-(s-\tau)A}f(u(\tau))d\tau.
\]

Then, in view of (45) and (53), the first term in the right hand side is estimated by
\[
\|L^\beta \int_s^t e^{-(t-\tau)A}f(u(\tau))d\tau\|_Y
\]
\[
= \|L^\beta A^{-\beta'} \int_s^t A^{\tilde{\beta}'} e^{-(t-\tau)A}f(u(\tau))d\tau\|_Y
\]
\[
\leq C \int_s^t (t-\tau)^{-\tilde{\beta}'} \|f(0)\|_X + \varphi(R)Rd\tau
\]
\[
\leq C[\|f(0)\|_X + \varphi(R)R(t-s)^{1-\tilde{\beta}'}, \quad 0 < s < t \leq T.
\]
Similarly, the second term is estimated by
\[
\left\|L^\beta [e^{-(t-s)A} - 1] \int_0^s e^{-(s-\tau)A} f(u(\tau)) d\tau \right\|_Y \\
= \left\|L^\beta A^{-\beta'} [e^{-(t-s)A} - 1] A^{\beta'} - \beta \int_0^s A^{\beta'} e^{-(s-\tau)A} f(u(\tau)) d\tau \right\|_Y \\
\leq CT^1 - \beta' \left\|f(0)\right\|_X + \varphi(R)R(t-s)^\sigma, \quad 0 \leq s < t \leq T.
\]
Hence, we have observed that
\[
[\Phi u] \in C^\sigma([0,T]; D(L^\beta)) \quad \text{(with } \sigma = \tilde{\beta} - \tilde{\beta}'). \quad (61)
\]
In particular, \(\Phi\) is a mapping from \(B(T)\) into \(X(T)\).

**Step 2.** Let us verify that \(\Phi\) can map \(B(T)\) into itself. Using (53) and arguing in a similar way as above, we easily verify that
\[
\left\|L^\beta [\Phi u](t)\right\|_Y \leq C' \|g_0\|_X + C^\prime T^{1-\beta'} \left\|f(0)\right\|_X + \varphi(R)R, \quad 0 \leq t \leq T,
\]
with some positive constants \(C'\) and \(C''\). Then, choose now \(R\) in such a way that
\[
R = C' \|g_0\|_X + 1.
\]
Furthermore, diminish \(T > 0\) in such a way that
\[
C'' T^{1-\beta'} \left\|f(0)\right\|_X + \varphi(R)R \leq 1.
\]
Then, \(\Phi\) maps \(B(T)\) into itself.

**Step 3.** In the meantime, \(\Phi\) can be a contraction of \(X(T)\). In fact, for \(u, v \in B(T)\),
\[
[\Phi u](t) - [\Phi v](t) = \int_0^t e^{-\tau A} [f(u(\tau)) - f(v(\tau))] d\tau.
\]
Therefore, after some computations,
\[
\left\|L^\beta \{[\Phi u](t) - [\Phi v](t)\}\right\|_Y \leq C \varphi(2R) \int_0^t (t-\tau)^{-\beta'} \left\|L^\beta [u(\tau) - v(\tau)]\right\|_Y d\tau \\
\leq C \varphi(2R) T^{1-\beta'} \max_{0 \leq \tau \leq T} \left\|L^\beta [u(\tau) - v(\tau)]\right\|_Y, \quad 0 \leq t \leq T.
\]
This shows that \(\Phi\) is a contraction provided we further diminish \(T > 0\).

**Step 4.** By the fixed point theorem for contraction mappings, we conclude that \(\Phi\) has a unique fixed point \(u = [\Phi u]\) in \(B(T)\). (61) jointed with (52) then implies that \(f(u) \in C^\sigma([0,T]; X)\). Thanks to [7, Theorem 3.7] on the linear multivalued equations, we obtain that \(u\) has the regularity \(u \in C^1((0,T]; X)\) together with \(u \in C((0,T]; D(A))\) and satisfies the multivalued equation of (54). Moreover, according to the second Remark to [7, Theorem 3.7], \(u'(t)\) is represented as
\[
u'(t) = -A e^{-tA} u_0 + \int_0^t A e^{-(t-s)A} [f(u(t)) - f(u(s))] ds + e^{-tA} f(u(t)). \quad (62)
\]
It is then verified that \(u\) is a strict solution to (54) belonging to (58).

Let us verify the estimates (59) and (60). Since
\[
A^\beta u(t) \geq e^{-tA} y_0 + \int_0^t A^\beta e^{-(t-s)A} f(u(s)) ds,
\]
we see that \(\|A^\beta u(t)\|_X \leq C(\|g_0\|_X + 1)\) by the definition of the graph norm (36). Thereby, we obtain (59). Meanwhile, it follows from (62) that
\[
\|u'(t)\|_X \leq \|A^{1-\beta} e^{-tA} g_0\|_X + \int_0^t \|A e^{-(t-s)A} \|_\mathcal{L}(X) \|f(u(t)) - f(u(s))\|_X \, ds + \|e^{-tA} f(u(t))\|_X.
\]
Then, due to (45) with \(x = 1 - \tilde{\beta}\) and 1,
\[
\|u'(t)\|_X \leq C \left[ t^{\beta-1} \|g_0\|_X + \int_0^t (t-s)^{-1} \|f(u(t)) - f(u(s))\|_X \, ds + 1 \right].
\]
As mentioned above, \(u = \Phi(u)\) and (61) yield that \(u(t)\) is Hölder continuous with values in \(\mathcal{D}(L^{\beta})\) at the exponent \(\sigma\); therefore, (52) implies that \(f(u(t))\) is Hölder continuous with values in \(X\) at the same exponent. Hence, we obtain that
\[
\|u'(t)\|_X \leq C(t^{\beta-1} \|g_0\|_X + 1), \quad 0 < t \leq T_{u_0},
\]
which is the first estimate of (60).

Noting that \(Au(t) \equiv -u'(t) + f(u(t))\), the second one of (60) is also observed.

We remember that all the constants appearing in the arguments were determined by the norm \(\|g_0\|_X\) alone. But, since \(g_0\) was any element of \(A^\beta u_0\), it is possible to assert that \(T\) and \(C_{u_0}\) are determined by \(\|A^\beta u_0\|_X\) alone.

**Step 5.** It remains to see uniqueness of the solution to (54) in the space (58). So, let \(v\) be any other solution lying in (58). Then, thanks to [7, Theorem 3.7] again, \(v(t)\) must be equal to \([\Phi v](t)\) for any \(0 \leq t \leq T_{u_0}\). Thereby,
\[
u(t) - v(t) = \int_0^t e^{-(t-s)A} [f(u(s)) - f(v(s))] \, ds,
\]
\[
\|L^{\beta}[u(t) - v(t)]\|_Y \leq C \int_0^t (t-s)^{-\beta} \|L^{\beta}[u(s) - v(s)]\|_Y \, ds, \quad 0 \leq t \leq T_{u_0}.
\]
For \(0 < S \leq T_{u_0}\), we see that
\[
\|L^{\beta}[u(t) - v(t)]\|_Y \leq C \int_0^t (t-s)^{-\beta} \, ds \max_{0 \leq s \leq S} \|L^{\beta}[u(s) - v(s)]\|_Y \leq CS^{1-\beta} \|u - v\|_{X(S)} , \quad 0 \leq t \leq S.
\]
This means that, if \(S > 0\) is sufficiently small, then \(u(t) = v(t)\) for all \(0 \leq t \leq S\). Consequently,
\[
u(t) - v(t) = \int_S^t e^{-(t-s)A} [f(u(s)) - f(v(s))] \, ds, \quad S \leq t \leq T_{u_0}.
\]
We can repeat this argument to conclude that \(u(t) = v(t)\) for all \(0 \leq t \leq T_{u_0}\). □

**Remark 1.** We remark that the solution \(u\) may not be continuous at \(t = 0\) with respect to the graph norm of \(\mathcal{D}(A^{\beta})\). Indeed, from
\[
A^{\beta} u(t) \equiv e^{-tA} g_0 + \int_0^t A^{\beta} e^{-(t-s)A} f(u(s)) \, ds,
\]
it is observed that \(u(t) \to u_0\) in \(\mathcal{D}(A^{\beta})\) if \(e^{-tA} g_0 \to g_0\) in \(X\) as \(t \to 0\). But, in view of (44), this is the case only when \(g_0 \in \overline{\mathcal{D}(A)}\), i.e.,
\[
[A^{\beta} u_0] \cap \overline{\mathcal{D}(A)} \neq \emptyset.
\]
(63)
It is however observed that, when \( X \) is a reflexive Banach space, this condition is automatically fulfilled for any \( u_0 \in D(A^\beta) \). Indeed, if \( g_0 \in A^\beta u_0 \), then \( g_0 = f' + f'' \) with \( f' \in A^{0} \) and \( f'' \in D(A) \) due to (39); since \( A^0 = A^{\beta}0 \) in general, on one hand, we have \( g_0 - f' \in A^\beta u_0 \); on the other hand, \( f'' \in D(A) \). Hence, (63) is the case. □

**Remark 2.** Similarly, the solution \( u \) may not be differentiable at \( t = 0 \) even if \( u_0 \) is taken in \( D(A) \). However, if \( u_0 \in D(A) \) satisfies the compatibility condition

\[
[f(u_0) - Au_0] \cap \overline{D(A)} \neq \emptyset,
\]

then \( u \) is differentiable at \( t = 0 \), too, and satisfies the equation of (54) at the initial time. In view of (44) this fact is verified by the second Remark to [7, Theorem 3.7]. As mentioned by (40), when \( X \) is a reflexive Banach space, this condition is automatically fulfilled for any \( u_0 \in D(A) \).

**Remark 3.** On the other hand, if (64) takes place, then \([f(u_0) - Au_0] \cap \overline{D(A)} \) consists of a single element \( u_1 \) due to (38). Since the solution satisfies the relation

\[
u'(0) + Au_0 \ni f(u_0) \quad \text{at} \quad t = 0,
\]

and since \( u'(0) \in \overline{D(A)} \), it must hold that \( u'(0) = u_1 \). In other words, \( u'(0) \) satisfying (65) is uniquely determined by \( u_0 \).

It is immediate to verify that the solution of (54) constructed above gives a unique solution to (46) lying in the function space:

\[u \in C([0,T];D(L^\beta)) \cap C((0,T];D(L)) \cap C^1((0,T];X).\]

(66)

In fact, if \( u \) is a solution of (54), then it follows from \( \frac{\partial u}{\partial x}(t) - f(u(t)) \in M^{-1}Lu(t) \) that \( M \left[ \frac{\partial u}{\partial x}(t) - f(u(t)) \right] = Lu(t) \). In addition, \( u \) naturally belongs to (66). Conversely, if \( u \) is a solution to (46) lying in (66), then \( u \) actually belongs to (58) and satisfies the multivalued equation of (54).

Next, we notice higher regularities of the local solution. These properties often play an important role (see [5]).

**Theorem 5.2.** Under (47)\textendash (52), let \( u_0 \in D(A^{\beta}) \) and let \( u \) be the local solution obtained in Theorem 5.1. Then, \( u \) actually belongs to

\[u \in C^\sigma((0,T_{u_0}];D(L^\beta)).\]

(67)

For any exponent \( 0 < \bar{\theta} < \sigma \) (\( = \bar{\beta} - \bar{\beta}' \)), its derivative enjoys the regularities:

\[u' \in C((0,T_{u_0}];D(A^{\bar{\beta}})) \cap C^\bar{\beta}((0,T_{u_0}];X).\]

(68)

**Proof.** Due to (61), \( u = [\Phi u] \in C^\sigma([0,T_{u_0}];D(L^\beta)) \) is already observed.

Put \( f(t) = f(u(t)) \). Then, \( u \in C^\sigma([0,T_{u_0}];D(L^\beta)) \) together with (52) provides that \( f \in C^\sigma([0,T_{u_0}];X) \). Meanwhile, as noticed above, \( u'(t) \) is known to be written as (62). So,

\[u'(t) = A^{-\bar{\beta}} \left[ -A^{1+\bar{\beta}} e^{-tA} u + \int_0^t A^{1+\bar{\beta}} e^{-(t-s)A} [f(t) - f(s)] ds + A^{\bar{\beta}} e^{-tA} f(t) \right].\]

This means that \( u'(t) \in D(A^{\bar{\beta}}) \) for any \( 0 < t \leq T_{u_0} \) and \( u' \) is continuous with respect to the graph norm of \( D(A^{\bar{\beta}}) \).

Let \( 0 < t_0 < T_{u_0} \). Consider \( u \) as a solution to the linear equation

\[
\frac{du}{dt} + Au \ni f(t), \quad t_0 < t \leq T_{u_0},
\]
on an interval \([t_0, T_0]\). Since \(u'(t_0) \in f(t_0) - Au(t_0)\) belongs to \(\mathcal{D}(A^\beta)\), \(u(t_0)\) satisfies the condition \([f(t_0) - Au(t_0)] \cap \mathcal{D}(A^\beta) \neq \emptyset\). Then, we can apply [7, Theorem 3.15] on the temporal maximal regularity of solutions of linear equations to obtain that \(u' \in C^\beta([t_0, T_0]; X)\). As \(t_0 > 0\) is arbitrary, we conclude \(u' \in C^\beta((0, T_0]; X)\). 

Finally, let us show continuous dependence of local solutions on the initial values. Set a subset of initial values

\[ K_r = \{ u_0 \in \mathcal{D}(A^\beta); \| A^\beta u_0 \|_X \leq r \} \]

for \(r > 0\). Theorems 5.1 and 5.2 provide existence of local solutions lying in (67) and (68) for every \(u_0 \in K(r)\) on a unified interval \([0, T_r]\).

**Theorem 5.3.** There exists a constant \(C_r > 0\) such that, for any pair of \(u_0, v_0 \in K_r\) and their local solutions \(u, v\), respectively, it holds that

\[
\|A^\beta[u(t) - v(t)]\|_X \leq C_r \| A^\beta(u_0 - v_0) \|_X, \quad 0 \leq t \leq T,
\]

provided that \(0 < T(\leq T_R)\) is suitably diminished.

**Proof.** Let \(g_0 \in A^\beta u_0\) and \(h_0 \in A^\beta v_0\). Then, \(u_0 = A^{-\beta} g_0\) and \(v_0 = A^{-\beta} h_0\). So,

\[
u(t) - v(t) = A^{-\beta} \left[ e^{-tA}(g_0 - h_0) + \int_0^t A^\beta e^{-(t-s)A}[f(u(s)) - f(v(s))] ds \right].
\]

The norm is then estimated by

\[
\|A^\beta[u(t) - v(t)]\|_X \leq C \|g_0 - h_0\|_X + C_r \int_0^t \|L^\beta[u(s) - v(s)]\|_Y ds
\]

\[
\leq C \|g_0 - h_0\|_X + C_r T^{1-\beta} \|u - v\|_{\mathcal{B}([0,T];\mathcal{D}(A^\beta))}, \quad 0 \leq t \leq T.
\]

Since \(g_0 - h_0 \in A^\beta(u_0 - v_0)\), the Lipschitz condition (69) holds true if \(T > 0\) is sufficiently small. 

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