Universality in Sherrington-Kirkpatrick’s Spin Glass Model

Philippe CARMONA,* Yueyun HU†

5th November 2018

Abstract

We show that the limiting free energy in Sherrington-Kirkpatrick’s Spin Glass Model does not depend on the environment.

1 Introduction

The physical system is an $N$-spin configuration $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$. Each configuration $\sigma$ is given a Boltzmann weight $e^{\beta \sqrt{N} H_N(\sigma) + h \sum_i \sigma_i}$, where $\beta = \frac{1}{T} > 0$ is the inverse of the temperature, $h$ is the intensity of the magnetic interaction, $H_N(\sigma)$ is the random Hamiltonian

$$H_N(\sigma) = H_N(\sigma, \xi) = \sum_{1 \leq i, j \leq N} \xi_{ij} \sigma_i \sigma_j,$$

and $(\xi_{ij})_{1 \leq i, j \leq N}$ is an i.i.d family of random variables, admitting order three moments, which we normalize:

$$E[\xi] = 0, \quad E[\xi^2] = 1, \quad E[|\xi|^3] < +\infty.$$ (1)

The object of interest is the random Gibbs measure

$$\langle f(\sigma) \rangle = \frac{1}{Z_N} 2^{-N} \sum_{\sigma} f(\sigma) e^{\beta \sqrt{N} H_N(\sigma, \xi) + h \sum_i \sigma_i},$$

and in particular the partition function

$$Z_N = Z_N(\beta, \xi) = 2^{-N} \sum_{\sigma} e^{\beta \sqrt{N} H_N(\sigma, \xi) + h \sum_i \sigma_i}.$$ 

We shall denote by $g = (g_{ij})_{1 \leq i, j \leq N}$ an environment of i.i.d Gaussian standard random variables ($\mathcal{N}(0, 1)$).

*P.Carmona: Laboratoire Jean Leray, UMR 6629, Université de Nantes, 92208, F-44322, Nantes cedex 03, e-mail: philippe.carmona@math.univ-nantes.fr
†Y.Hu: Laboratoire de Probabilités et Modèles Aléatoires (CNRS UMR-7599), Université Paris VI, 4 Place Jussieu, F-75252 Paris cedex 05, e-mail: hu@ccr.jussieu.fr
Recently, F. Guerra and F.L. Toninelli [1,2] gave a rigorous proof, at the mathematical level, of the convergence of free energy to a deterministic limit, in a Gaussian environment,

\[ \frac{1}{N} \log Z_N(\beta, g) \to \alpha_\infty(\beta) \quad \text{a.s. and in average.} \]

Talagrand [4] then proved that one can replace the Gaussian environment by a Bernoulli environment \( \eta_{ij}, \mathbb{P}(\eta_j = \pm 1) = \frac{1}{2} \), and obtain the same limit: \( \alpha_\infty(\beta) \). We shall generalize this result.

**Theorem 1.** Assume the environment \( \xi \) satisfies (1). Then,

\[ \frac{1}{N} \log Z_N(\beta, \xi) \to \alpha_\infty(\beta) \quad \text{a.s. and in average.} \]

Furthermore, the averages \( \alpha_N(\beta, \xi) \) defined by \( \frac{1}{N} \mathbb{E}[\log Z_N(\beta, \xi)] \) satisfy

\[ |\alpha_N(\beta, \xi) - \alpha_N(\beta, g)| \leq 9 \mathbb{E}[|\xi|^3] \frac{\beta^3}{\sqrt{N}}. \]

Therefore the limiting free energy \( \alpha_\infty(\beta) \) does not depend on the environment, hence the **Universality** in the title of this paper: this independence from the particular disorder was already clear to Sherrington and Kirkpatrick [3] although they had no mathematical proof of this fact (Guerra andToninelli [2] provided a physical proof in the case the environment is symmetric with a finite fourth moment).

Notice eventually that \( \alpha_\infty(\beta) \) can be determined in a Gaussian framework where Talagrand [5] recently proved that it is the solution of G. Parisi’s variational formula.

The universality property can be mechanically extended to the ground states, that is the supremum of the families of random variables:

\[ S_N(\xi) = \sup_\sigma \sum_{1 \leq i,j \leq N} \sigma_i \sigma_j \xi_{ij} = \sqrt{N} \lim_{\beta \to +\infty} \frac{1}{\beta} \log Z_N(\beta, \xi). \]

F. Guerra and F.L. Toninelli [1,2] proved that \( N^{-3/2} S_N(g) \) converges as and in average to a deterministic limit \( e_\infty \). Here is the generalization:

**Theorem 2.** Assume the environment \( \xi \) satisfies (1). Then,

\[ N^{-3/2} S_N(\xi) \to e_\infty \quad \text{a.s. and in average.} \]

Furthermore, the averages satisfy, for a universal constant \( C > 0 \),

\[ N^{-3/2} |\mathbb{E}[S_N(\xi)] - \mathbb{E}[S_N(g)]| \leq C(1 + \mathbb{E}[|\xi|^3]) N^{-1/6}. \]

We end this introduction by observing that we do not need the random variables \( \xi_{ij} \) to share the same distribution. They only need to be independent, to satisfy (1) and such that \( \sup_{ij} \mathbb{E}[|\xi_{ij}|^3] < +\infty \).
2 Comparison of free energies

Let us begin with an Integration by parts Lemma.

**Lemma 3.** Let \( \xi \) be a real random variable such that \( \mathbb{E}[|\xi|^3] < +\infty \) and \( \mathbb{E}[\xi] = 0 \). Let \( F : \mathbb{R} \to \mathbb{R} \) be twice continuously differentiable with \( \|F''\|_{\infty} = \sup_{x \in \mathbb{R}} |F''(x)| < +\infty \). Then

\[
|\mathbb{E}[\xi F(\xi)] - \mathbb{E}[\xi^2] \mathbb{E}[F'(\xi)]| \leq \frac{3}{2} \|F''\|_{\infty} \mathbb{E}[|\xi|^3].
\]

**Proof.** Observe first, that by Taylor’s formula,

\[
|F(\xi) - F(0) - \xi F'(0)| \leq \frac{\xi^2}{2} \|F''\|_{\infty},
\]

\[
|F'(\xi) - F'(0)| \leq |\xi| \|F''\|_{\infty}.
\]

Therefore,

\[
|\mathbb{E}[\xi F(\xi)] - \mathbb{E}[\xi^2] \mathbb{E}[F'(\xi)]| = |\mathbb{E}[\xi F(\xi)] - \mathbb{E}[\xi^2] \mathbb{E}[F'(\xi)] - F(0)\mathbb{E}[\xi]|
\]

\[
= |\mathbb{E}[\xi(F(\xi) - F(0) - \xi F'(0))]| - \mathbb{E}[\xi^2] \mathbb{E}[F'(0) - F'(\xi)]|
\]

\[
\leq \|F''\|_{\infty} \left( \frac{1}{2} \mathbb{E}[|\xi|^3] + \mathbb{E}[|\xi|] \mathbb{E}[\xi^2] \right)
\]

\[
\leq \|F''\|_{\infty} \left( \frac{1}{2} \mathbb{E}[|\xi|^3] + \mathbb{E}[|\xi|^3] \frac{3}{2} \mathbb{E}[|\xi|^3] \frac{3}{2} \right)
\]

\[
\leq \frac{3}{2} \|F''\|_{\infty} \mathbb{E}[|\xi|^3].
\]

\( \square \)

In the general framework, \( X = (X_1, \ldots, X_d) \) is a random vector defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that for any \( i : |X_i| \leq 1 \). The environment is an i.i.d family of random variables \( (\xi_1, \ldots, \xi_d) \) defined on \( (\Omega^{(\xi)}, \mathcal{F}^{(\xi)}, \mathbb{P}^{(\xi)}) \), distributed as a fixed random variable \( \xi \) satisfying (I). The Gibbs measure, partition function and averaged free energy are thus

\[
\langle f(X) \rangle = \frac{1}{Z(\beta, \xi)} \mathbb{E} \left[ f(X) e^{\beta \sum_{i=1}^d X_i \xi_i} \right]
\]

\[
Z(\beta, \xi) = \mathbb{E} \left[ e^{\beta \sum_{i=1}^d X_i \xi_i} \right], \quad \alpha(\beta, \xi) = \mathbb{E}[\log Z(\beta, \xi)].
\]

Observe that to define \( \alpha(\beta, \xi) \) we do not need to assume exponential moments for the random variable \( \xi \), since \( |\log Z(\beta, \xi)| \leq |\beta| \sum_{i=1}^d |\xi_i| \). We now approximate the derivative of the averaged free energy:

**Lemma 4.**

\[
\frac{\partial \alpha(\beta, \xi)}{\partial \beta} = \beta \mathbb{E} \left[ \sum_{i=1}^d (\langle X_i^2 \rangle - \langle X_i \rangle^2) \right] + 9d \mathbb{E}[|\xi|^3] O(\beta^2),
\]

where \( |O(\beta^2)| \leq \beta^2 \).
Remark 5. In a Gaussian random environment, the integration by parts formula is an exact formula, therefore the remainder $9d \times \mathbb{E}[\xi^3]O(\beta^2)$ vanishes.

Proof. We have
\[
\frac{\partial \alpha(\beta, \xi)}{\partial \beta} = \mathbb{E} \left[ \frac{1}{Z(\beta, \xi)} \mathbb{E} \left[ \sum_{i=1}^{d} X_i \xi_i e^{\beta \sum_{i=1}^{d} X_i \xi_i} \right] \right] = \mathbb{E} \left[ \sum_{i=1}^{d} \xi_i F_i(\xi_i) \right],
\]
with $F_i(z) = \mathbb{E} \left[ X_i e^{\beta X_i + \psi_i(X)} \right]/\mathbb{E} \left[ e^{\beta X_i + \psi_i(X)} \right]$ and $\psi_i(X) = \beta \sum_{j \neq i} X_j \xi_j$ independent of $\xi_i$.

If we define $\langle H \rangle^{(z)} = \mathbb{E} \left[ H e^{\beta X_i + \psi_i(X)} \right]/\mathbb{E} \left[ e^{\beta X_i + \psi_i(X)} \right]$, then
\[
\frac{\partial}{\partial z} \langle H \rangle^{(z)} = \beta \left( \langle H X_i \rangle^{(z)} - \langle H \rangle^{(z)} \langle X_i \rangle^{(z)} \right).
\]

Hence,
\[
F_i(z) = \langle X_i \rangle^{(z)}, \quad F_i'(z) = \beta \left( \langle X_i^2 \rangle^{(z)} - \left( \langle X_i \rangle^{(z)} \right)^2 \right)
\]
\[
F_i''(z) = \beta^2 \left[ \langle X_i^2 \rangle^{(z)} - 3 \langle X_i^2 \rangle^{(z)} \langle X_i \rangle^{(z)} + 2 \left( \langle X_i \rangle^{(z)} \right)^3 \right].
\]

Since $|X_i| \leq 1$, we have $\|F_i''\|_\infty \leq 6 \beta^2$, $0 \leq F_i'(z) \leq \beta$ and
\[
F_i(\xi_i) = \langle X_i \rangle, \quad F_i'(\xi_i) = \beta (\langle X_i^2 \rangle - \langle X_i \rangle^2).
\]

We infer from Lemma 3 that since $\mathbb{E}[\xi^2] = 1$,
\[
\mathbb{E}[\langle X_i \rangle \xi_i] = \mathbb{E}[\xi_i F_i(\xi_i)] = \beta \mathbb{E}[\langle X_i^2 \rangle - \langle X_i \rangle^2] + 9 \mathbb{E} [\xi^3] O(\beta^2),
\]
with $|O(\beta^2)| \leq \beta^2$. Therefore,
\[
\frac{\partial \alpha(\beta, \xi)}{\partial \beta} = \beta \mathbb{E} \left[ \sum_{i=1}^{d} (\langle X_i^2 \rangle - \langle X_i \rangle^2) \right] + 9d \mathbb{E}[\xi^3] \times O(\beta^2).
\]

The next step is the comparison of the averaged free energies for the environments $\xi$ and $g$ (standard normal).

Proposition 6. For any $\beta \in \mathbb{R}$,
\[
|\alpha(\beta, \xi) - \alpha(\beta, g)| \leq 9d \mathbb{E}[\xi^3] |\beta|^3.
\]

Proof. The interpolation technique of F. Guerra relies on the introduction of a two parameter Hamiltonian:
\[
Z(t, x) = \mathbb{E} \left[ e^{\sqrt{T} \sum_{i=1}^{d} X_i g_i + \sqrt{\pi} \sum_{i=1}^{d} X_i \xi_i} \right]
\]
and averaged free energy $\alpha(t, x) = \mathbb{E}[\log Z(t, x)]$ where the environments $g$ and $\xi$ are assumed to be independent of each other, $g$ being standard normal. By Lemma 4,
\[
\frac{\partial}{\partial t} \alpha = \frac{1}{2} \mathbb{E} \left[ \sum_{i=1}^{d} (\langle X_i^2 \rangle - \langle X_i \rangle^2) \right]
\]
\[
\frac{\partial}{\partial x} \alpha = \frac{1}{2} \mathbb{E} \left[ \sum_{i=1}^{d} (\langle X_i^2 \rangle - \langle X_i \rangle^2) \right] + 9d \mathbb{E}[\xi^3] O(\sqrt{x}),
\]
with $|O(\sqrt{x})| \leq \sqrt{x}$. We follow the path $x(s) = t_0 - s$, $0 \leq s \leq t_0$. Then,
\[
\left| \frac{d}{ds} \alpha(s, t_0 - s) \right| \leq 9d \mathbb{E} [\xi^3] \sqrt{t_0},
\]
and thus, integrating on $[0, t_0]$
\[
|\alpha(0, t_0) - \alpha(t_0, 0)| \leq 9d \mathbb{E} [\xi^3] t_0^{3/2}.
\]
This is the desired result for $\beta > 0$ (take $\beta = \sqrt{t_0}$). For negative $\beta$, we consider the environment $-\xi$ instead.

We shall now estimate the fluctuations of free energy, the environment is still constructed with i.i.d random variables $(\xi_1, \ldots, \xi_d)$ satisfying (P).

**Lemma 7.** There exists some universal constant $c > 0$ such that
\[
\mathbb{E} [\log Z(\beta, \xi) - \alpha(\beta, \xi)]^3 \leq c \mathbb{E} [\xi^3] |\beta|^3 d^{3/2}.
\]
Consequently, we have
\[
\mathbb{E} \left[ \sup_{(X_i)} \sum_{i=1}^d X_i \xi_i - \mathbb{E} \left( \sup_{(X_i)} \sum_{i=1}^d X_i \xi_i \right)^3 \right] \leq c \mathbb{E} [\xi^3] d^{3/2}.
\]

**Proof.** We shall use a martingale decomposition. Let $\mathcal{F}_k = \sigma\{\xi_1, \ldots, \xi_k\}, k \geq 1$, be the natural filtration generated by $(\xi_k)$. Consider the sequence of martingale difference
\[
\Delta_j := \mathbb{E} [\log Z(\beta, \xi) \mid \mathcal{F}_j] - \mathbb{E} [\log Z(\beta, \xi) \mid \mathcal{F}_{j-1}] \quad 1 \leq j \leq d,
\]
with $\mathcal{F}_0$ the trivial $\sigma$-field. Then
\[
\log Z(\beta, \xi) - \alpha(\beta, \xi) = \sum_{j=1}^d \Delta_j.
\]
Burkholder’s martingale inequality says that for some universal constant $c' > 0$,
\[
\mathbb{E} \left| \sum_{j=1}^d \Delta_j \right|^3 \leq c' \mathbb{E} \left( \sum_{j=1}^d \Delta_j^2 \right)^{3/2}.
\]
To estimate $\Delta_j$, we define $Z^{(j)} := \mathbb{E} \left[ e^{\beta \sum_{i=1, i \neq j}^d X_i \xi_i} \right]$ and an auxiliary random probability measure $\mathbb{Q}^{(j)}$ by
\[
\mathbb{Q}^{(j)} (F(X_1, \ldots, X_d)) := \frac{1}{Z^{(j)}} \mathbb{E} \left[ F(X_1, \ldots, X_d) e^{\beta \sum_{i=1, i \neq j}^d X_i \xi_i} \right], \quad \forall F(.) \geq 0.
\]
Then
\[
Z(\beta, \xi) = Z^{(j)} \mathbb{Q}^{(j)} e^{\beta X_j \xi_j}.
\]
Since $Z^{(j)}$ is independent of $\xi_j$, $\log Z^{(j)}$ has the same conditional expectation with respect to $\mathcal{F}_j$ as to $\mathcal{F}_{j-1}$. It follows that
\[
\Delta_j = \mathbb{E} \left( \log \mathbb{Q}^{(j)} (e^{\beta X_j \xi_j}) \mid \mathcal{F}_j \right) - \mathbb{E} \left( \log \mathbb{Q}^{(j)} (e^{\beta X_j \xi_j}) \mid \mathcal{F}_{j-1} \right).
\]
Using the fact that $|X_j| \leq 1$, we get $|\log Q_j(e^{\beta X_j}, \xi_j)| \leq \beta |\xi_j|$. This implies that

$$|\Delta_j| \leq \beta (|\xi_j| + E|\xi_j|).$$

It follows that

$$E \left| \log Z(\beta, \xi) - \alpha(\beta, \xi) \right|^3 \leq c' E \left( \sum_{j=1}^d \Delta_j^2 \right)^{3/2} \leq c' \beta^3 \left( \sum_{j=1}^d (|\xi_j| + E|\xi_j|)^2 \right)^{3/2} \leq c' \beta^3 \sqrt{d} \sum_{j=1}^d E(|\xi_j| + E|\xi_j|)^3 \leq c E|\xi|^3 \beta^3 d^{3/2},$$

where we used the convexity of the function $x \rightarrow x^{3/2}$ in the third inequality. Finally, considering $\frac{1}{\beta} \log Z(\beta, \xi)$ and letting $\beta \to \infty$, we obtain the second estimate and end the proof. \[\square\]

### 3 Application to Sherrington-Kirkpatrick’s model of spin glass

Observe that

$$Z_N(\beta, \xi) = 2^{-N} \sum_{\sigma} e^{\beta N H_N(\sigma, \xi) + h \sum_i \sigma_i} = E \left[ e^{\beta N H_N(\tilde{\tau}, \xi) + h \sum_i \tau_i} \right],$$

where $(\tau_i)_{1 \leq i \leq N}$ are i.i.d with distribution $P(\tau_i = \mp 1) = \frac{1}{2}$. We get rid of the magnetic field by introducing tilted laws:

$$P(\tilde{\tau}_i = \pm 1) = \frac{1}{2} e^{\pm h} \cosh(h), \text{ so that } E[f(\tilde{\tau}_i)] = \frac{E[f(\tau_i) e^{h\tau_i}]}{E[e^{h\tau_i}]}.$$  

With these notations we have

$$Z_N(\beta, \xi) = \cosh(h)^N E \left[ e^{\beta N H_N(\tilde{\tau}, \xi)} \right].$$

**Convergence of free energy : Theorem 1**

Applying Proposition 6 to $X_{ij} = \tilde{\tau}_i \tilde{\tau}_j$, $\beta \to \frac{\beta}{\sqrt{N}}$ and $d = N^2$ yields

$$|\alpha_N(\beta, \xi) - \alpha_N(\beta, g)| = \frac{1}{N} \left| \alpha(\frac{\beta}{\sqrt{N}}, \xi) - \alpha(\frac{\beta}{\sqrt{N}}, g) \right| \leq \frac{1}{N} 9N^2 E[|\xi|^3] \left( \frac{|\beta|}{\sqrt{N}} \right)^3 = 9E[|\xi|^3] |\beta|^3 \frac{1}{\sqrt{N}}.$$ \[2\]

Furthermore, the fluctuations can be controlled by Lemma 7

$$E \left[ \left( \frac{1}{N} \log Z_N(\beta, \xi) - \alpha_N(\beta, \xi) \right)^3 \right] \leq c E|\xi|^3 |\beta|^3 N^{-3/2},$$

this gives the a.s. convergence by Borel-Cantelli’s Lemma.
Convergence of ground state: Theorem 2

We have, restricting the sum to a configuration yielding a maximum Hamiltonian to get the lower bound,

\[ e^{\frac{\beta}{\sqrt{N}}S_N(\xi)} \geq Z_N(\beta, \xi) = 2^{-N} \sum_{\sigma} e^{\frac{\beta}{\sqrt{N}}H_N(\sigma, \xi)} \geq 2^{-N} e^{\frac{\beta}{\sqrt{N}}S_N(\xi)}. \]

Therefore,

\[ \frac{1}{\sqrt{N}} \mathbb{E}[S_N(\xi)] \geq \frac{1}{\beta} N \alpha_N(\beta, \xi) \geq \frac{1}{\sqrt{N}} \mathbb{E}[S_N(\xi)] - \frac{N \log 2}{\beta}. \]

Combining with inequality (2) yields, by taking \( \beta = N^{1/6} \)

\[ \frac{1}{N^{3/2}} |\mathbb{E}[S_N(g)] - \mathbb{E}[S_N(\xi)]| \leq \frac{2 \log 2}{\beta} + \frac{1}{\beta} |\alpha_N(\beta, \xi) - \alpha_N(\beta, g)| \]
\[ \leq \frac{2 \log 2}{\beta} + C \mathbb{E}[|\xi|^3] \frac{\beta^2}{\sqrt{N}} \]
\[ \leq C' (1 + \mathbb{E}[|\xi|^3]) N^{-1/6}. \]

The almost sure convergence follows in the same way from the control of fluctuations and Borel-Cantelli’s Lemma.

4 Some Extensions and Generalizations

4.1 The \( p \)-spin model of spin glasses

The partition function is

\[ Z_N(\beta, \xi) = 2^{-N} \sum_{\sigma} e^{\frac{\beta}{\sqrt{Np-1}}H_N(\sigma, \xi) + h \sum_{i} \sigma_i} = \mathbb{E} \left[ e^{\frac{\beta}{\sqrt{Np-1}}H_N(\tau, \xi) + h \sum_{i} \tau_i} \right], \]

where \( (\tau_i)_{1 \leq i \leq N} \) are i.i.d with distribution \( \mathbb{P}(\tau_i = \mp 1) = \frac{1}{2} \) (we get rid of the magnetic field by introducing tilted laws so we assume, without loss in generality, that \( h = 0 \)).

The Hamiltonian is

\[ H_N(\sigma, \xi) = \sum_{1 \leq i_1, \ldots, i_p \leq N} \sigma_{i_1} \ldots \sigma_{i_p} \xi_{i_1} \ldots i_p \]

where \( \xi_{i_1} \ldots i_p \) is an iid family of random variables with common distribution satisfying (1). Applying Proposition 6 to \( X_{i_1 \ldots i_p} = \tilde{\tau}_{i_1} \ldots \tilde{\tau}_{i_p}, \beta \rightarrow \frac{\beta}{\sqrt{Np-1}} \) and \( d = N^2 \) yields

\[ |\alpha_N(\beta, \xi) - \alpha_N(\beta, g)| \leq 9 \mathbb{E}[|\xi|^3] \frac{|\beta|^3}{N^{1/2}}. \]

4.2 Integration by parts and comparison of free energies

The more information we get on the random media, the more precise our comparison results can be. In particular, the more gaussian the environment looks like, the closer the
free energy is to the gaussian free energy. For example, we shall assume here that the random variable $\xi$ satisfies
\begin{equation}
E[|\xi|^4] < +\infty, \quad E[\xi] = E[\xi^3] = 0, \quad E[\xi^2] = 1.
\end{equation}

A typical variable in this class is the Bernoulli $P[\eta = \pm 1] = \frac{1}{2}$.

We get the approximate integration by parts formula

**Lemma 8.** Assume that the real random variable $\xi$ satisfies (3) and that the function $F : \mathbb{R} \to \mathbb{R}$ is of class $C^3$ with bounded third derivative $\|F^{(3)}\|_\infty < +\infty$. Then,
\begin{equation}
|E[\xi F(\xi)] - E[\xi^2] E[F'(\xi)]| \leq \|F^{(3)}\|_\infty E[\xi^4].
\end{equation}

**Proof.** This is again Taylor’s formula:
\begin{align*}
F(\xi) &= F(0) + \xi F'(0) + \frac{1}{2} \xi^2 F''(0) + O(|\xi^3| \|F^{(3)}\|_\infty) \\
F'(\xi) &= F'(0) + \xi F''(0) + O(\xi^2 \|F^{(3)}\|_\infty).
\end{align*}

\(\Box\)

Repeating, mutatis mutandis, the proof of Proposition 6 we obtain

**Proposition 9.** There exists a constant $C > 0$ such that for any environment $\xi$ satisfying (3), and for a Gaussian environment $g$,
\begin{equation}
|\alpha(\beta, \xi) - \alpha(\beta, g)| \leq C E[\xi^4] d\beta^4.
\end{equation}

In the framework of Sherrington-Kirkpatrick model of spin glass, this yields
\begin{equation}
|\alpha_N(\beta, \xi) - \alpha_N(\beta, g)| \leq C E[\xi^4] \frac{\beta^4}{N}.
\end{equation}

The ground state comparison is now
\begin{equation}
N^{-3/2} |E[S_N(\xi)] - E[S_N(g)]| \leq C (1 + E[|\xi|^4]) N^{-1/4}.
\end{equation}

This is of the same order than Talagrand’s result (Corollary 1.2 of [4]) established for Bernoulli random variables.

**Acknowledgements** We gratefully acknowledge fruitful conversations with Francesco Guerra, held during his stay at Université de Nantes as “Professeur Invité” in 2003 and 2004.

**References**

[1] Francesco Guerra, *Broken replica symmetry bounds in the mean field spin glass model.*, Commun. Math. Phys. 233 (2003), no. 1, 1–12 (English).

[2] Francesco Guerra and Fabio Lucio Toninelli, *The thermodynamic limit in mean field spin glass models.*, Commun. Math. Phys. 230 (2002), no. 1, 71–79 (English).
[3] David Sherrington and Scott Kirkpatrick, *Infinite-ranged model of spin-glasses.*, Phys. Rev. B 17 (1978), 4384–4403.

[4] Michel Talagrand, *Gaussian averages, Bernoulli averages, and Gibbs’ measures.*, Random Struct. Algorithms 21 (2002), no. 3-4, 197–204 (English).

[5] ______, *The generalized Parisi formula.*, C. R., Math., Acad. Sci. Paris 337 (2003), no. 2, 111–114 (English).