**u-Deformed WZW Model and Its Gauging**

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**Abstract.** We review the description of a particular deformation of the WZW model. The resulting theory exhibits a Poisson–Lie symmetry with a non-Abelian cosymmetry group and can be vectorially gauged.

**Key words:** gauged WZW model; Poisson–Lie symmetry

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1 Introduction

The theory of Poisson–Lie symmetric deformations of the standard WZW models [6] was developed in [2, 3, 4] and it is based on the concept of the twisted Heisenberg double [5]. This contribution is a review of a part of our work [4]. It is intended to the attention of those readers who are interested just in the direct description and gauging of one particular example of the Poisson–Lie WZW deformation and do not wish to go through the general theory of the twisted Heisenberg doubles exposed in [4].

2 u-deformed WZW model

Let $K$ be a connected simple compact Lie group and denote by $(\cdot, \cdot)_K$ the negative-definite Ad-invariant Killing form on its Lie algebra $K$. Let $LK$ be the group of smooth maps from a circle $S^1$ into $K$ (the group law is given by pointwise multiplication) and define a non-degenerate Ad-invariant bilinear form $(\cdot | \cdot)$ on $LK \cong \text{Lie}(LK)$ by the following formula

$$ (\alpha | \beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma (\alpha(\sigma), \beta(\sigma))_K. \tag{1} $$

Let $P_H : LK \to H$ be the orthogonal projector to the Cartan subalgebra $H$ of $K$ and let $U : H \to H$ be a skew-symmetric linear operator with respect to the inner product $(\cdot, \cdot)_K$. We denote by $u$ the composition $U \circ P_H$.

The $u$-deformed WZW model is a dynamical system whose phase space $P$ is the direct product $P = LK \times LK$, its symplectic form $\omega_u$ reads

$$ \omega_u = \frac{1}{2}(dJ_L \wedge |r_{LK}|) - \frac{1}{2}(dJ_R \wedge |r_{LK}|) + \frac{1}{2}(u(dJ_L) \wedge |dJ_L|) + \frac{1}{2}(u(dJ_R) \wedge |dJ_R|) \tag{2} $$

and its Hamiltonian $H$ is given by

$$ H = -\frac{1}{2K}(J_L|J_L) - \frac{1}{2K}(J_R|J_R). $$

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Here $k$ is a positive integer, $r_{L,K} = dgg^{-1}$ and $l_{L,K} = g^{-1}dg$ stand for the right and the left-invariant Maurer–Cartan forms on the group manifold $L,K$ and the $L,K$-valued functions $J_L, J_R$ on $P$ are defined as

$$J_L(\chi, g) = \chi, \quad J_R(\chi, g) = -\text{Ad}_g^{-1}\chi + kg^{-1}\partial_g g, \quad g \in L,K, \quad \chi \in L,K.$$ (3)

If $U = 0$, the $u$-deformed WZW model becomes just the standard WZW model in the formulation [1][2].

Consider the standard actions of the loop group $L,K$ on the phase space $P$:

$$h \triangleright_L (\chi, g) = (k\partial_\mu hh^{-1} + h\chi h^{-1}, hg), \quad h, g \in L,K, \quad \chi \in L,K,$$

$$h \triangleright_R (\chi, g) = (\chi, gh^{-1}), \quad h, g \in L,K, \quad \chi \in L,K.$$ (4)

It was established in [4] that these actions can infinitesimally be expressed via the Poisson bivector $\Pi_u$, corresponding to the symplectic form $\omega_u$:

$$\xi_L f = (\Pi(u)(df, J_L^r r_B)\xi), \quad \xi_R f = (\Pi(u)(df, J_R^r r_B)\xi).$$ (5)

The reader can recognize in [3] the defining relations of the $L,K$-Poisson–Lie symmetries with the cosymmetry group equal to $B$ (cf. [2 eq. (5.30)]).

It is insightful to detail the fundamental relations (4) in the standard Cartan basis $H_{\mu,n} = H_{\mu} e^{i\sigma n} \in L,K^C, E_{\alpha,n} = E_{\alpha} e^{i\sigma n} \in L,K^C, n \in \mathbb{Z}$. We have

$$H_{\mu,n} = \{f, J_{\mu,n}^L\}_u, \quad E_{\alpha,n} = \{f, J_{\alpha,n}^L\}_u + \langle \alpha, U(H^\mu) \rangle J_{\alpha,n}^L \{f, J_{\mu,0}^L\}_u, \quad (6)$$

$$H_{\mu,n} = \{f, J_{\mu,n}^R\}_u, \quad E_{\alpha,n} = \{f, J_{\alpha,n}^R\}_u + \langle \alpha, U(H^\mu) \rangle J_{\alpha,n}^R \{f, J_{\mu,0}^R\}_u, \quad (7)$$

where

$$J_{\mu,L,R} = (J_{\mu,L,R}|E^\alpha e^{i\sigma n}), \quad J_{\mu,L,R} = (J_{\mu,L,R}|H^\mu e^{i\sigma n}).$$

For completeness, note that $E^\alpha$ are the step generators of the complexified Lie algebra $K^C$ and $H^\mu$ are the orthonormalized generators of the Cartan subalgebra $H^C$:

$$[H^\mu, E^\alpha] = \langle \alpha, H^\mu \rangle E^\alpha, \quad [E^\alpha, E^{-\alpha}] = \alpha^\vee, \quad [E^\alpha, E^\beta] = \epsilon^{\alpha\beta} E^{\alpha+\beta},$$

$$(H^\mu, H^\nu)_K = \delta^{\mu\nu}, \quad (E^\alpha, E^{-\alpha})_{K^C} = \frac{2}{|\alpha|^2}, \quad (E^\alpha)^\dagger = E^{-\alpha}, \quad (H^\mu)^\dagger = H^\mu.$$ (5)

The coroot $\alpha^\vee$ is defined as

$$\alpha^\vee = \frac{2}{|\alpha|^2} \langle \alpha, H^\mu \rangle H^\mu.$$ (5)

We observe, that the actions $\triangleright_{L,R} $ are not Hamiltonian, unless $u = 0$. This suggests that the current algebra brackets cannot be the same as they are in the non-deformed WZW model. Indeed, $u$-corrections are present and we underline them for the better orientation of the reader:

$$\{J_{\mu,n}^L, J_{\nu,m}^L\}_u = k\delta^{\mu\nu} r_{m+n,0}, \quad \{J_{\mu,n}^L, J_{\alpha,n}^L\}_u = \langle \alpha, H^\mu \rangle J_{\alpha,n+m}^L,$$

$$\{J_{\alpha,n}^L, J_{\mu,n}^L\}_u = \langle \alpha, H^\mu \rangle J_{\mu,n+m}^L + i k n \delta_{m+n,0}.\]$$
\{J_L^\alpha, J_L^\beta\}_u = c^{\alpha\beta} J_L^{\alpha+\beta,m+n} - \langle \alpha, U(H\mu) \rangle \langle \beta, H\mu \rangle J_L^{\alpha,m} J_L^\beta,n, \quad (8)
\{J_R^{\mu,m}, J_R^{\nu,n}\}_u = -k^{\mu\nu} \delta_{m+n,0}, \quad \{J_R^{\mu,m}, J_R^{\alpha,n}\}_u = \langle \alpha, H\mu \rangle J_R^{\alpha,n+m}, \quad (9)
\{J_R^{\mu,m}, J_R^{\alpha,n}\}_u = \frac{2}{|\alpha|^2} (\langle \alpha, H\mu \rangle J_R^{\mu,m+n} - i k n \delta_{m+n,0}), \quad (10)
\{J_L, J_R\}_u = 0. \quad (11)

Note that the brackets of the left currents differ from those of the right currents just by the sign in front of the parameter \(k\).

The relations (8), (9) and (10) almost determine the Poisson bracket \(\{\cdot, \cdot\}_u\), corresponding to the symplectic form \(\omega_u\). The remaining relation, which completes the description of \(\{\cdot, \cdot\}_u\), is as follows:

\[ \{\phi, \psi\}_u = U(H_{\mu,0})_L \phi H_{L,0}^{\mu,0} \psi - H_{R,0}^{\mu,0} \phi U(H_{\mu,0})_R \psi. \]

Here \(\phi, \psi\) are functions on \(P\) which depend only on \(LK\) but not on \(L\).

### 3 Symplectic reduction

The symplectic reduction of a dynamical system \((P, \omega)\) consists in singling out a particular set of observables \(\phi_i \in \text{Fun}(P)\) called first class constraints. One just requires from \(\phi_i\) that on the common locus \(L\), where all \(\phi_i\) vanish, also all Poisson brackets \(\{\phi_i, \phi_j\}\) vanish. This requirement and the Frobenius theorem guarantee that the kernels of the restriction of the symplectic form \(\omega\) to \(L\) form an integrable distribution on \(L\). Under certain conditions, the set of integrated surfaces of this distribution is itself a manifold \(P_r\), which is called the reduced symplectic manifold. The reduced symplectic form \(\omega_r\) on \(P_r\) is uniquely fixed by a condition that the pull-back of \(\omega_r\) to \(L\) coincides with the restriction of the symplectic form \(\omega\) to \(L\).

In many interesting situations, the integrated surfaces of the integrable distribution can be naturally identified with orbits of a Lie group acting on \(L\). This is the reason why the symplectic reduction is sometimes called the gauging of that Lie group action. As an warm-up example, let us first perform the (vectorial) gauging of the standard WZW model corresponding to the choice \(u = 0\) in the formula (2).

Let \(\mathcal{Y}\) be a subset of the set of all positive roots of the Lie algebra \(\mathcal{K}^\mathbb{C}\) and suppose that the complex vector space \(S^\mathbb{C}\)

\[ S^\mathbb{C} = \text{Span}\{E^\gamma, E^{-\gamma}, [E^\gamma, E^{-\gamma}]\}, \quad \gamma \in \mathcal{Y} \]

is the Lie subalgebra of \(\mathcal{K}^\mathbb{C}\) (as an example take the block diagonal embedding of \(sl_3\) in \(sl_4\)). The complex Lie algebra \(S^\mathbb{C}\) has a natural compact real form \(S\) consisting of the anti-Hermitean elements of \(S^\mathbb{C}\). Consider the corresponding compact semi-simple group \(S\) and view it as the subgroup of \(K\).

For the first class constraints, we take

\[ \phi^\gamma, n = J_L^\gamma,n + J_R^\gamma,n, \quad \phi^{\nu,n} = J_L^{\nu,n} + J_R^{\nu,n}, \quad (12) \]

where \(\gamma \in \pm \mathcal{Y}\) and \(\nu\) is such that \(H^\nu\) is in the Cartan subalgebra \(\mathcal{H}_S\) of \(S\). For \(u = 0\), we obtain

\[ \{\phi^{\mu,m}, \phi^{\nu,n}\}_u = 0, \quad \{\phi^{\mu,m}, \phi^{\alpha,n}\}_u = \langle \alpha, H^\mu \rangle \phi^{\alpha,n+m}, \quad \{\phi^{\alpha,m}, \phi^{\beta,n}\}_u = c^{\alpha\beta} \phi^{\alpha+\beta,m+n}. \]

We immediately observe that the Poisson brackets of the first class constraints vanish on the common locus \(L = \{p \in P; \phi^\gamma, n(p) = 0, \phi^{\nu,n}(p) = 0\}\), therefore the symplectic reduction can be
performed. As the result of analysis, it turns out that the integrated surfaces of the integrable distribution are given as the orbits of the following action of the loop group $L S$ on $L$:

$$s \triangleright (\chi, g) = (k \partial s s^{-1} + s \chi s^{-1}, s g s^{-1}), \quad s \in L S, \quad (\chi, g) \in L.$$

If $u \neq 0$, the Poisson brackets of the constraints (12) do not vanish on the common locus $L$ and, therefore, they cannot serve as the base for a symplectic reduction. It is not difficult to find a way out from the trouble, however. For that, we take an inspiration from the case $u = 0$ where the sum of the left and right currents can be interpreted as the product in the Abelian cosymmetry group $L K$ (the group multiplication is the addition in the vector space $L K$). Thus, for $u \neq 0$, it looks plausible to use the product in the non-Abelian cosymmetry group $B$. This gives the following constraints:

$$\phi^{\gamma,n}_u \equiv (J_L \bullet J_R|E^{\gamma,n}) = J_{L}^{\gamma,n} + e^{-(\gamma, U(H^\nu))} J_{R}^{\gamma,n},$$

$$\phi^{\mu,n}_u \equiv (J_L \bullet J_R|H^{\mu,n}) = J_{L}^{\mu,n} + J_{R}^{\mu,n},$$

where, again, $\gamma \in \pm \Upsilon$ and $\nu$ is such that $H^\nu$ is in $H_S$. Suppose, moreover, that it holds for all $\gamma \in \Upsilon$:

$$(\gamma \circ U)(H^\Upsilon_\perp) = 0,$$

where the subscript $\perp$ stands for the orthogonal complement with respect to the restriction of the Killing–Cartan form $(\cdot, \cdot)_K$ to $H$. Then the Poisson brackets of the constraints $\phi_u$ vanish on the common locus $L_u = \{ p \in P; \phi^{\gamma,n}_u(p) = 0, \phi^{\mu,n}_u(p) = 0 \}$, as it is obvious from the following explicit formulas:

$$\{ \phi^{\mu,m}_u, \phi^{\nu,n}_u \}_u = 0, \quad \{ \phi^{\alpha,n}_u, \phi^{\alpha,n}_u \}_u = \langle \alpha, H^\alpha \rangle \phi^{\alpha,n+m}_u,$$

$$\{ \phi^{\mu,m}_u, \phi^{\alpha,n}_u \}_u = 2 \langle \alpha, H^\mu \rangle \phi^{\alpha,n+m}_u,$$

$$\{ \phi^{\beta,n}_u, \phi^{\gamma,m}_u \}_u = c^{\alpha\beta} c^{\beta\gamma} \phi^{\alpha+m\gamma+n}_u \phi^{\beta,n}_u.$$

The symplectic reduction now can be performed and the question arises whether we can identify the orbits of a $L_S$ action on $L_u$ which would coincide with the integrated surfaces of the integrable distribution. The answer is affirmative (4) and it reads:

$$s \triangleright_u (\chi, g) = (s \chi s^{-1} + k \partial s s^{-1}, s g s^{-1}), \quad s_L = e^{-u(s J_L s^{-1} + k \partial s s^{-1})} s e^{u(J_L)}.$$

We conclude that the reduced symplectic manifold $P_u$ can be identified with the coset space $L_u/L_S$.

4 Outlook

We believe that the $u$-deformed WZW model may become a useful laboratory for the study of possible generalizations of the standard axioms of the conformal field theory in two dimensions.

[1] Balog J., Fehér L., Palla L., Chiral extensions of the WZNW phase space, Poisson–Lie symmetries and groupoids, Nucl. Phys. B, 2000, V.568, 503–542, hep-th/9910046

[2] Klimčík C., Quasitriangular WZW model, Rev. Math. Phys., 2004, V.16, 679–808, hep-th/0103118

[3] Klimčík C., Poisson–Lie symmetry and $q$-WZW model, in Proceedings of the 4th International Symposium “Quantum Theory and Symmetries” (August 15–21, 2005, Varna), Sofia, Heron Press, 2006, V.1, 382–393, hep-th/0511003

[4] Klimčík C., On moment maps associated to a twisted Heisenberg double, Rev. Math. Phys., 2006, V.18, 781–821, math-ph/0602048

[5] Semenov-Tian-Shansky M., Poisson Lie groups, quantum duality principle and the twisted quantum double, Theor. Math. Phys., 1992, V.93, 1292–1307, hep-th/9304042

[6] Witten E., Non-Abelian bosonisation in two dimensions, Comm. Math. Phys., 1984, V.92, 455–472.