A similarity structure on a connected manifold $M$ is a Riemannian metric on its universal cover $\tilde{M}$ such that the fundamental group of $M$ acts on $\tilde{M}$ by similarities. If the manifold $M$ is compact, we show that the universal cover admits a de Rham decomposition with at most two factors, one of which is Euclidean. Very recently, after Belgun and Moroianu conjectured that the number of factors was at most one, Matveev and Nikolayevsky found an example with two factors. When the non-flat factor has dimension 2, we give a complete classification of the examples with two factors. In greater dimensions, we make the first steps towards such a classification by showing that $M$ is a fibration (with singularities) by flat Riemannian manifolds; up to a finite covering of $M$, we may assume that these manifolds are flat tori. We also prove a version of the de Rham decomposition theorem for the universal covers of manifolds with locally metric connections. During the proof, we define a notion of transverse (not necessarily flat) similarity structure on foliations, and show that foliations endowed with such a structure are either transversally flat or transversally Riemannian. None of these results assumes analyticity.

1 Introduction

1.1 Similarity structures

A similarity $\phi : M_1 \to M_2$ of ratio $\lambda \in \mathbb{R}_{>0}$ between two Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ is a diffeomorphism such that $\phi^* g_2 = \lambda^2 g_1$. The similarity group $\text{Sim}(M)$ of a manifold $M$ is the group of all similarities from $M$ to itself.

A similarity structure on a connected manifold $M$ is a Riemannian metric $g$ on its universal cover $\tilde{M}$ such that $\pi_1(M)$ acts on $\tilde{M}$ as a subgroup of $\text{Sim}(\tilde{M})$; thus, the Riemannian metric is only defined locally “up to a constant” on the manifold $M$. Notice that the Levi-Civita connection $\nabla$ of $g$ does project to a connection $\nabla$ on $M$. Here are three fundamental examples (the first one is a simple particular case of the other two):

Example 1.1. Consider $N = (\mathbb{R}^n \setminus \{0\}, g)$, where $g$ is the restriction of the standard Euclidean metric of $\mathbb{R}^n$, and the subgroup $G$ of $\text{Diffeo}(N)$ generated by the similarity $\varphi : x \mapsto \lambda x$, with some $\lambda \in (0, 1)$. The metric $g$ induces a metric on the universal cover of $N$ (which is $N$ itself if $n \geq 3$). Thus, $M = N/G$ is naturally endowed with a similarity structure.
Example 1.2. Let \((M, g)\) be a connected Riemannian manifold, and \((\tilde{M}, \tilde{g})\) its universal cover. Any closed 1-form \(\omega\) on \(M\) lifts to an exact 1-form \(\tilde{\omega}\) on \(\tilde{M}\). Consider a primitive \(f\) of \(\tilde{\omega}\) and let \(\tilde{h} = e^f \tilde{g}\). Then the fundamental group \(\pi_1(M)\) acts on \((\tilde{M}, \tilde{h})\) by similarities, and thus \(\tilde{h}\) induces a similarity structure on \(M\). The group \(\pi_1(M)\) acts by isometries if and only if \(\omega\) is exact.

Example 1.3. Let \((N, g)\) be any compact connected Riemannian manifold. The Riemannian cone over \(N\) is the manifold \(C = N \times \mathbb{R}_{>0}\) endowed with the Riemannian metric \(t^2 g + dt^2\). Consider the subgroup \(G\) of \(\text{Diffeo}(C)\) generated by the similarity \(\varphi: (x, t) \mapsto (x, \lambda t)\) (with \(\lambda \in (0, 1)\)). Then \(M = C/G\) is a compact manifold with a similarity structure.

In 1979, Gallot studied the holonomy group of Riemannian cones [Gal79]. The holonomy group of a manifold \(M\) endowed with a connection \(\nabla\) at a point \(x \in M\), written \(\text{Hol}_x(\nabla)\), is the subgroup of \(GL(T_x M)\) obtained by the parallel transport along all loops based at \(x\). The manifold is said to have irreducible holonomy if there is no subspace of \(T_x M\) invariant by the holonomy group at \(x\): when \(M\) is connected, this property does not depend on the choice of \(x\). The holonomy group of a Riemannian manifold is the holonomy group of its Levi-Civita connection.

Theorem 1.4 (Gallot). If \((M, g)\) is the Riemannian cone over a compact connected Riemannian manifold \(N\), then either \(M\) is flat, or it has irreducible holonomy.

In 2014, Belgun and Moroianu [BM16] asked whether this result generalizes to all similarity structures on compact manifolds. In other words, assuming that a Riemannian manifold \(\tilde{M}\) has a compact quotient \(M\) such that \(\pi_1(M)\) acts by similarities, but not only by isometries, is it true that \(\tilde{M}\) is either irreducible or flat? In 2015, Matveev and Nikolayevsky [MN15a] answered negatively to this question by a counterexample. In this paper, we prove the following result:

Theorem 1.5. Consider a compact manifold \(M\) with a similarity structure, and its universal cover \(\tilde{M}\) equipped with the corresponding Riemannian structure \(g\). Assume that \(M\) is not globally Riemannian, i.e. \(\pi_1(M)\) is not a subgroup of \(\text{Isom}(\tilde{M})\). Then we are in exactly one of the following situations:

1. \(\tilde{M}\) is flat.
2. \(\tilde{M}\) has irreducible holonomy and \(\dim(\tilde{M}) \geq 2\).
3. \(\tilde{M} = \mathbb{R}^q \times N\), where \(q \geq 1\), \(\mathbb{R}^q\) is the Euclidean space, and \(N\) is a non-flat, non-complete manifold with irreducible holonomy.

In 2015, Matveev and Nikolayevsky [MN15b] proved Theorem 1.5 under the assumption that the manifold \((\tilde{M}, g)\) is analytic, and asked whether the theorem holds without this assumption. Here, we answer positively to this question, by a totally new proof.

Theorem 1.5 implies Gallot’s theorem. After admitting that Theorem 1.5 holds, let us prove Theorem 1.4 in a new way. Consider the universal cover \(\hat{C}\) of the cone \(C\) over \(N\), and its Cauchy completion \(\hat{\tilde{C}}\). Since \(\hat{C}\) is the cone over \(\tilde{N}\) (the universal cover of \(N\)), the difference \(\hat{C} \setminus \tilde{C}\) is a single point. Since \(C\) has a compact quotient with a similarity structure, Theorem 1.5 applies to \(C\). To obtain Theorem 1.4, assume that \(\hat{C} = \mathbb{R}^q \times M_1\) with \(q \geq 1\), and notice the following contradiction: \(\hat{C} = \mathbb{R}^q \times M_1\),
and $\tilde{M}_1 \setminus M_1 \neq \emptyset$, so the set $\tilde{C} \setminus \tilde{C} = \mathbb{R}^q \times (\tilde{M}_1 \setminus M_1)$ has infinite cardinal. Thus, Theorem 1.4 is proved.

**Theorem 1.5 implies Belgun and Moroianu’s theorem.** In the setting of Theorem 1.5, Belgun and Moroianu proved that, under an additional assumption on the lifetime of geodesics, only the first two cases are possible ([BM16], Theorem 1.4). We are now going to check that in the third case, this additional assumption cannot be satisfied. Hence, we will see that the proof of Theorem 1.5 contains a new proof of their theorem.

Consider a manifold $M$ with a similarity structure, which satisfies the assumptions of Theorem 1.5, and its universal cover $(\tilde{M}, g)$. If $X$ is a vector in the unit tangent bundle $S\tilde{M}$, denote by $L(X)$ the lifetime of the half-geodesic tangent to $X$, that is, the supremum of the times for which this half-geodesic is defined. Then, let:

$$\mu : \tilde{M} \rightarrow [0, +\infty], \quad \mu(x) = \sup \left\{ \{0\} \cup \{L(X) \mid X \in S_x\tilde{M}, \ L(X) < +\infty\} \right\},$$

where $S_x\tilde{M}$ is the set of unit length vectors tangent to $\tilde{M}$ at $x$.

Belgun and Moroianu proved that, if $\mu$ is locally bounded on $\tilde{M}$, then either Case 1 or Case 2 of Theorem 1.5 applies.

To obtain Belgun and Moroianu’s theorem from Theorem 1.5, we assume that the third case holds (i.e. $\tilde{M} = \mathbb{R}^q \times N$) and look for a contradiction. Let $X \in S\mathbb{R}^q$ and $Y \in SN$ such that the lifetime $L(Y)$ of the half-geodesic tangent to $Y$ in the manifold $N$ is finite. Then for $t \in (0, \pi/2)$, the lifetime of the half-geodesic tangent to $(\cos(t) \cdot X, \sin(t) \cdot Y)$ in the manifold $\tilde{M}$ is $L(Y)/\sin(t)$, which tends to $+\infty$ as $t \to 0$. Thus $\mu$ is not locally bounded and the proof is complete.

**Example 1.6.** It is not obvious how to construct examples which fall into the third category in Theorem 1.5. Let us give the recipe to construct an example:

1. Choose $q \geq 1$ and consider the torus $\mathbb{T}^{q+1} = \mathbb{R}^{q+1}/\mathbb{Z}^{q+1}$.
2. Consider a linear diffeomorphism of the torus $A \in GL_{q+1}(\mathbb{Z})$, such that there exists a number $\lambda \in (0, 1)$, and a decomposition $\mathbb{R}^{q+1} = E^s \oplus E^u$ invariant by $A$, and a positive definite symmetric bilinear form $b$ on $E^s$ satisfying the following:
   - (a) the stable subspace $E^s$ has dimension $q$, and $A|_{E^s}$ is a similarity of ratio $\lambda$, i.e. one may write $A|_{E^s} = \lambda \cdot O$, where $O \in O(E^s, b)$ is a linear mapping which preserves the form $b$;
   - (b) the unstable subspace $E^u$ is one-dimensional.
   (In particular, the diffeomorphism $A$ is Anosov.)
3. Construct the mapping torus $M$ of the diffeomorphism $A$ in the following way: take the quotient of $\mathbb{T}^{q+1} \times (0, +\infty)$ by the mapping $\Phi : (x, z) \mapsto (Ax, \lambda z)$.
4. Consider a basis $(e_1, \ldots, e_q)$ of $E^s$ which is orthonormal for $b$, and $e_{q+1} \in E^u$; the basis $(e_1, \ldots, e_q, e_{q+1})$ of $\mathbb{R}^{q+1}$ provides local coordinates $(x_1, \ldots, x_{q+1})$ in a neighborhood of each point of $\mathbb{T}^{q+1}$. Define a Riemannian metric $g$ on $\mathbb{T}^{q+1} \times (0, +\infty)$ by

$$g = dx_1^2 + \ldots + dx_q^2 + \varphi(z)dx_{q+1}^2 + dz^2,$$

where $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ is a smooth function such that for all $z \in (0, +\infty)$, $\varphi(\lambda z) = \lambda^{2q+2}\varphi(z)$. 

3
Notice that $\Phi^* g = \lambda^2 g$: thus the metric $g$ induces a similarity structure on $M$. The universal cover $\tilde{M}$ of $M$ is isometric to $\mathbb{R}^q \times N$, where $\mathbb{R}^q$ is the Euclidean space $(E^q, b)$, and $N = E^u \times (0, +\infty)$. Furthermore, the curvature of $g$ is given by $-\varphi''(z)/2$, where $\varphi''$ is the second derivative of $\varphi$, which is not everywhere zero (otherwise, $\varphi$ would be an affine mapping, which contradicts the assumption $\varphi(\lambda z) = \lambda^{2q+2} \varphi(z)$), so manifold $N$ is not flat. Thus, $M$ corresponds to the third case of Theorem 1.5.

Remark. It turns out that in Example 1.6, the only possible values for $q$ are 1 and 2. Indeed, for $q \geq 3$, Madani, Moroianu and Pilca [MMP] proved that it is impossible to construct a linear diffeomorphism $A$ satisfying Conditions (2a) and (2b).

We will say that two manifolds $M_1$ and $M_2$ with similarity structures are isomorphic if there is a diffeomorphism between $M_1$ and $M_2$ which lifts to a similarity between the universal covers $\tilde{M}_1$ and $\tilde{M}_2$. Matveev and Nikolayevsky’s example [MN15a] is isomorphic to Example 1.6 with the choice $q = 1$ and $\varphi(z) = z^4$. In this paper, we prove the following:

**Theorem 1.7.** Consider a manifold $M$ which corresponds to the third case of Theorem 1.5, and assume that $\dim(N) = 2$. Then $M$ is isomorphic to a manifold constructed in Example 1.6 for some choice of $q$, $A$ and $\varphi$. In particular, $M$ is the mapping torus of an Anosov diffeomorphism of the torus.

Theorem 1.7 gives a complete classification of the manifolds which correspond to the third case of Theorem 1.5, under the assumption that $\dim(N) = 2$. On the other hand, the manifolds corresponding to the first case (i.e. flat manifolds) were classified by Fried [Fri80].

**Fibration by flat tori.** In greater dimensions, the problem of classifying manifolds corresponding to the third case is still open, but we prove the following:

**Theorem 1.8.** In the third case of Theorem 1.5, consider the foliation $\tilde{F}$ induced by the submersion $\tilde{M} \to N$, and $F$ the foliation induced on $M$ by $\tilde{F}$. Then $F$ is a Riemannian foliation on $M$, and the closures of the leaves form a singular Riemannian foliation $\mathcal{F}$ on $M$, such that each leaf of $\mathcal{F}$ is a smooth manifold of dimension $d$ (which may depend on the leaf) with $q < d < q + n$, where $n = \dim(N)$. Moreover, on each leaf of $\mathcal{F}$, there is a flat Riemannian metric which is compatible with the similarity structure of $M$.

A Riemannian foliation is a foliation which has a Riemannian structure on its transversal which is compatible with the foliation: the reader may refer to [Mol88] for the general theory. The situation described in Theorem 1.8 induces a fibration with singularities, where the fibers are the leaves of $\mathcal{F}$ (in particular, there is a dense open set of $M$ which is a nonsingular fibration). To show that $\mathcal{F}$ is a nonsingular fibration as in Theorem 1.7, one would need to show that $\mathcal{F}$ is nonsingular (i.e. all the closures of the leaves of $\mathcal{F}$ have the same dimension), and that the leaf space of $\mathcal{F}$ is a smooth manifold. These questions are still open.

The closures of the leaves of a Riemannian foliation are always submanifolds (see [Mol88]), so the main difficulty in Theorem 1.8 is to prove that the closures of the leaves are flat. Moreover, we show that they have a structure of Riemannian manifold. This implies in particular that the closures of each leaf is finitely covered by a torus (by Bieberbach’s theorem: see Theorem 4.5). In fact, we show a more precise result:
Theorem 1.9. In the setting of Theorem 1.8, there exists a finite covering \( M' \to M \) with the following property: considering the foliation \( F' \) induced on \( M' \) by \( F \), the closures of the leaves of \( F' \) are flat tori.

Thus, \( M \) has a finite covering which is a fibration by tori with singularities. We do not know whether it is always possible to choose \( M' = M \) in Theorem 1.9. However, in the special case where \( q = 1 \), the answer is positive, as a consequence of the following:

Theorem 1.10 (Carrière, 1984, [Car84]). On a compact manifold, if \( F \) is a foliation of dimension 1 with a transverse Riemannian structure, then the closures of the leaves are tori.

1.2 De Rham decomposition

A de Rham decomposition of a connected Riemannian manifold \((M,g)\) is a family \((M_0,g_0),(M_1,g_1),\ldots,(M_k,g_k)\) of Riemannian manifolds \((k \geq 0)\), where \(M_0\) is flat, while \(M_1,\ldots,M_k\) are non-flat manifolds with irreducible holonomy, such that:

\[
(M,g) = (M_0,g_0) \times (M_1,g_1) \times \ldots \times (M_k,g_k).
\]

The de Rham decomposition theorem [dR52] states the following (see also [KN63]):

Theorem 1.11 (de Rham, 1952).
1. If a connected Riemannian manifold \((M,g)\) admits a de Rham decomposition, then it is unique up to the order of the factors.
2. (Local version.) Any point of a Riemannian manifold has a neighborhood which admits a de Rham decomposition.
3. (Global version.) Every complete, simply connected, connected Riemannian manifold admits a de Rham decomposition.

Notice that the universal cover of a manifold \(M\) with a similarity structure (which is not globally Riemannian) is never complete: otherwise, the similarities which are not isometries would have a fixed point (by the Banach fixed point theorem), but \(\pi_1(M)\) needs to act freely on \(\tilde{M}\). However, Theorem 1.5 states that \(\tilde{M}\) does admit a de Rham decomposition. More precisely, Theorem 1.5 may be rephrased as follows:

Theorem 1.12. If \((\tilde{M},g)\) is the universal cover of a compact, connected manifold \(M\) with a similarity structure, then \((\tilde{M},g)\) admits a de Rham decomposition. Furthermore, the number of factors in the decomposition is at most 2: if it is exactly two, then one of the factors is the Euclidean space.

In this paper, we also prove a new version of de Rham’s decomposition theorem in a more general framework.

Definition 1.13. A locally metric connection on a manifold \(M\) is a torsion-free connection \(\nabla\) which lifts to a connection \(\tilde{\nabla}\) on the universal cover such that \(\tilde{\nabla}\) is the Levi-Civita connection of a Riemannian metric \(g\). Equivalently, a locally metric connection is a torsion-free connection whose restricted holonomy group \(\text{Hol}^0(\nabla)\) (i.e. the subgroup of \(\text{Hol}(\nabla)\) given by parallel transport along loops which are homotopic to a constant) is a relatively compact subgroup of \(GL_n(\mathbb{R})\).
Example 1.14. If \( M \) is a manifold endowed with a similarity structure, the Levi-Civita connection \( \tilde{\nabla} \) of the Riemannian metric \( g \) on \( \tilde{M} \) induces a locally metric connection \( \nabla \) on \( M \). The locally metric connections obtained in this way are exactly those which preserve a conformal structure (see [BM16] for more details).

Unlike similarity structures, locally metric connections behave well with respect to the product structure: if \((M_1, \nabla_1)\) and \((M_2, \nabla_2)\) are two manifolds with locally metric connections, then the product connection \((\nabla_1, \nabla_2)\) is again a locally metric connection on \(M_1 \times M_2\), but this connection is not given by a similarity structure.

In this paper, we will prove the following generalization of Theorem 1.11:

**Theorem 1.15.** Consider a compact connected manifold \((M, \nabla)\), where \(\nabla\) is a locally metric connection, and a Riemannian metric \(g\) on its universal cover \((\tilde{M}, \tilde{\nabla})\) such that \(\tilde{\nabla}\) is the Levi-Civita connection of \(g\). Then \((\tilde{M}, g)\) admits a de Rham decomposition.

Again, it is important to notice that the metric \(g\) on the universal cover is almost never complete, thus Theorem 1.15 is not a consequence of Theorem 1.11. However, we will show that the following theorem applies:

**Theorem 1.16** (Ponge-Reckziegel, 1993). Let \(M\) be a simply connected Riemannian manifold, whose Levi-Civita connection \(\nabla\) is reducible: thus, the tangent bundle \(TM\) admits two complementary orthogonal distributions \(E'\) and \(E''\) invariant by parallel transport, which determine foliations \(F'\) and \(F''\). Assume that the leaves of \(F'\) are all complete. Then, \(M\) is globally isometric to a product of Riemannian manifolds \(M' \times M''\), and the foliations \(F'\) and \(F''\) are determined by the product structure.

Theorem 1.16 is a particular case of the main result of [PR93]. In fact, the classical proof of the de Rham theorem given in [KN63] also adapts directly to this case with very few changes.

### 1.3 Transverse similarity structures

The main tool in the proofs of Theorems 1.15 and 1.5 is the study of transverse similarity structures on foliations. Such foliations may be seen as a particular case of (transversally) conformal foliations, or a generalization of (transversally) Riemannian foliations.

If \((M, g)\) is a Riemannian manifold, its similarity pseudogroup \(\text{Sim}^{\text{loc}}(M)\) consists of all \(\phi: U \rightarrow V\) such that \(\phi^*g = \lambda^2 g\), where \(U\) and \(V\) are open subsets of \(M\), and \(\lambda \in \mathbb{R}^>\) is locally constant on \(U\). For any \(x \in M\), the number \(\lambda(x)\) is called the ratio of \(\phi\) at \(x\) (if \(M\) is connected, there is no need to specify the point \(x\)).

A foliation of a compact manifold \((M, \mathcal{F})\) is a covering \((U_i)_{1 \leq i \leq r}\) with the following structure:

1. For each \(i \in \{1, \ldots, r\}\), \(U_i\) is diffeomorphic to \(V_i \times T_i\), where \(V_i\) is an open ball of \(\mathbb{R}^p\) and \(T_i\) an open ball of \(\mathbb{R}^q\). This gives us natural projections \(f_i : U_i \rightarrow T_i\). The disjoint union \(T = \bigcup T_i\) is called the global transversal.
2. There exist transition maps which are diffeomorphisms \((\gamma_{ij})_{i,j} : f_i(U_i \cap U_j) \rightarrow f_j(U_i \cap U_j)\) such that \(f_j = \gamma_{ij} \circ f_i\) on \(U_i \cap U_j\).

The pseudogroup \(\Gamma\) spanned by the \((\gamma_{ij})\) is called the holonomy pseudogroup of the foliation.
Remark. In this paper, we use the notions of “holonomy group”, from the theory of Riemannian manifolds, and “holonomy pseudogroup”, from the theory of foliations: these two notions must not be confused.

A transverse similarity structure on the foliation $\mathcal{F}$ is a metric $g$ on the transversal $T$ such that the transition maps $\gamma_{ij}$ are local similarities (i.e. belong to $\text{Sim}_\text{loc}(T)$). The foliation is said to be transversally Riemannian (or simply Riemannian) if it is possible to choose $g$ such that the $\gamma_{ij}$ are isometries.

Our main result on transverse similarity structures is the following:

**Theorem 1.17.** Let $(M, \mathcal{F})$ be a compact foliated manifold with a transverse similarity structure. Then one of the following two facts occurs:

1. The transverse similarity structure on the foliation $\mathcal{F}$ is flat (i.e. the metric $g$ on the transversal $T$ is flat);
2. The foliation $\mathcal{F}$ is transversally Riemannian (i.e. there exists a metric $h$ on the transversal $T$ such that the transition maps are isometries).

We prove Theorem 1.17 in Section 2. Notice that we do not assume that the transverse similarity structure on the foliation is induced by a locally metric connection on $M$.

Foliations with a flat transverse similarity structure (i.e. those which correspond to the first case of Theorem 1.17) were completely classified by Ghys [Ghy91] when $M$ has dimension 3 and $\mathcal{F}$ has dimension 1. See also [Nis92] and [Asu97] on this subject.

**About the foliated Ferrand-Obata conjecture.** For transversally conformal foliations, there is an analogue of Theorem 1.17 (see [Tar04a]):

**Theorem 1.18** (Tarquini, 2004). Any transversally analytic conformal foliation of codimension $\geq 3$, on a compact connected manifold, is either transversally Möbius or Riemannian.

It is also believed that Theorem 1.18 should be valid without the analyticity assumption: this is the foliated Ferrand-Obata conjecture. Our Theorem 1.17 implies the following:

**Corollary 1.19.** The foliated Ferrand-Obata conjecture is true in the special case where the transverse conformal structure on the foliation is induced by a transverse similarity structure.

### 1.4 Structure of the paper

We start by proving Theorem 1.17 in Section 2. We use Theorem 1.17 to prove Theorems 1.5 and 1.15 in Section 3. Then, we show Theorem 1.8 in Section 4, and use Theorem 1.8 to prove Theorem 1.7 in Section 5.

### 2 Foliations with transverse similarity structures

In this section, we prove Theorem 1.17.

A foliation is said to be equicontinuous if there exists a Riemannian metric on the transversal such that its holonomy pseudogroup $\Gamma$ is equicontinuous. If the foliation
has a transverse similarity structure, equicontinuity is equivalent to the existence of a constant $m > 1$ such that the ratio of any $\gamma \in \Gamma$ at any $x \in M$ lies in the interval $[1/m, m]$.

The following proposition, which is proved in [Tar04b], is crucial in the proof of Theorem 1.17:

**Proposition 2.1.** Any equicontinuous foliation with a transverse similarity structure is Riemannian.

Now, our first step in the proof is based on a trick which was described in [FT02].

**Proposition 2.2.** Let $(M, g)$ be a Riemannian manifold whose Riemann tensor $R$ does not vanish anywhere (i.e. there is no $x \in M$ such that $R_x = 0$). Then $\text{Sim}_{\text{loc}}(M)$ preserves a smooth Riemannian metric.

**Proof.** If $R$ denotes the Riemann tensor, define $\|R\|_g(x)$ as the supremum of the values of $\|R_u(u, v)w\|_g$ when $u, v, w$ are vectors of $T_xM$ which have unit length for $g$. Then the metric $\|R\|_g$ is invariant by $\text{Sim}_{\text{loc}}(M)$. \(\square\)

Thus, if $(M, \mathcal{F})$ is a foliated manifold with a transverse similarity structure, either $\mathcal{F}$ is Riemannian, or the Riemann tensor of $(T, g)$ vanishes somewhere. Our aim is to show that, in the last case, the Riemann tensor vanishes in fact everywhere.

Until the end of this section, we consider a compact, connected foliated manifold $(M, \mathcal{F})$ with a transverse similarity structure (see Section 1.3). We consider a covering of $M$ by open sets $U_i$ which are diffeomorphic to $V_i \times T_i$, the projections $f_i$, the transversal $T$, the transition maps $\gamma_{ij}$, the holonomy pseudogroup $\Gamma$, and the metric $g$ on the transversal. This metric $g$ induces a distance $d_i$ on each $T_i$.

Intuitively, the holonomy pseudogroup may be defined as the set of local diffeomorphisms of the transversal obtained by “sliding along the leaves”.

**Definition 2.3.** A piecewise $C^1$ path $c : [a, b] \to M$ is vertical if for all $t_0 \in [a, b]$, for all $i \in \{1, \ldots, r\}$ such that $c(t_0) \in U_i$, the mapping $t \mapsto f_i(c(t))$ is constant in a neighborhood of $t_0$.

The leaf which contains $x \in M$ is defined as the set of all possible $c(t)$, where $c$ is a piecewise $C^1$ vertical path such that $c(0) = x$.

**Definition 2.4.** Consider a piecewise $C^1$ vertical path $c : [a, b] \to M$, and $i, j \in \{1, \ldots, r\}$ such that $c(a) \in U_i$, $c(b) \in U_j$. Define $x = f_i(c(a))$.

Choose a sequence of times $a = t_1 < \ldots < t_{p+1} = b$ and a sequence of indices $i_1, \ldots, i_p$, such that for all $l \in \{1, \ldots, p\}$ and all $t \in [t_l, t_{l+1}]$,

$$c(t) \in U_{i_l}.$$  

The holonomy germ $\gamma$ from $T_i$ to $T_j$ at $x$ obtained by sliding along $c$ is defined as the germ at $x$ of the diffeomorphism

$$\gamma_{i_pj} \circ \gamma_{i_{p-1}i_p} \circ \cdots \circ \gamma_{i_2i_1} \circ \gamma_{i_1}.$$  

The following two propositions are basic properties of holonomy pseudogroups: see for example Chapter 1 of [Mol88] for details.

**Proposition 2.5.** The holonomy germ is well-defined: it depends only on the path $c$ and the choice of $i$ and $j$. In particular, it does not depend on the choice of a sequence of times $t_1, \ldots, t_{p+1}$ or the choice of a sequence of indices $i_1, \ldots, i_p$.
Lemma 2.6. Consider an element of the holonomy pseudogroup

$$\gamma = \gamma_{i_{p-1}p} \circ \cdots \circ \gamma_{i_1i_2}$$

and an element \(x \in T_{i_1}\). Then there exists a piecewise \(C^1\) path \(c\) in \(M\) such that the germ of \(\gamma\) at \(x\) is the holonomy germ from \(T_{i_1}\) to \(T_{i_p}\) at \(x\) obtained by sliding along \(c\).

**Proof.** We construct \(c : [1, p + 1] \to M\) such that for each \(l \in \{1, \ldots, p\}\), and all \(t \in [l, l + 1]\):

$$f_l(c(t)) = \gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1i_2}(x).$$

This is possible because \(V_i\) is path-connected for each \(i\).

Lemma 2.7. There exists \(\epsilon_0 > 0\) such that for all \(x \in M\), there exists \(i \in \{1, \ldots, r\}\) which satisfies \(x \in U_i\) and \(d_i(f_i(x), \partial T_i) > \epsilon_0\) (see Section 1.3 for the notations).

**Proof.** Assume the contrary: there exists a sequence \((x_n)_{n \in \mathbb{N}}\) in \(M\) such that for all \(i \in \{1, \ldots, r\}\) with \(x_n \in U_i\), we have \(d_i(f_i(x_n), \partial T_i) \leq 1/n\). Since \(M\) is closed, we may assume that \(x_n\) converges to some \(x_\infty \in M\). Then \(x_\infty\) is in some \(U_{i_0}\), and for any large enough \(n\), \(x_n \in U_{i_0}\). Hence, \(d_{i_0}(f_{i_0}(x_n), \partial T_{i_0}) \to 0\), which contradicts the fact that \(x_\infty \in U_{i_0}\).

In the following, we fix this \(\epsilon_0\).

Definition 2.8. Let \(x \in M\), \(p \in \mathbb{N}\) and \(i_1, \ldots, i_p \in \{i, \ldots, r\}\). We will write

$$\gamma = \gamma_{i_{p-1}i_p} \circ \cdots \circ \gamma_{i_1i_2}$$

if:

1. \(\gamma = \gamma_{i_{p-1}i_p} \circ \cdots \circ \gamma_{i_1i_2}\);
2. For all \(l \in \{1, \ldots, p - 1\}\), \(d_i(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1i_2}(f_i(x)), \partial U_i) > \epsilon_0\);
3. For all \(l \in \{1, \ldots, p - 1\}\), the domain of \(\gamma_{i_{l+1}i_{l+2}}\) contains the ball \(B_g(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1i_2}(f_i(x)), \epsilon_0)\).

Here, \(B_g(y, r)\) denotes the ball of center \(y\) and radius \(r\) for the metric \(g\).

Lemma 2.9. Consider \(x \in M\) and an element \(\gamma = \gamma_{i_{p-1}i_p} \circ \cdots \circ \gamma_{i_1i_2}\) of the holonomy pseudogroup.

For each \(l \in \{1, \ldots, p\}\), write \(r_l\) the ratio of \(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1i_2}\) at \(f_i(x)\) (in particular \(r_1 = 1\)).

Then the domain of \(\gamma\) contains

$$B_g\left(f_i(x), \frac{\epsilon_0}{\max_{1 \leq i \leq p-1} r_i}\right).$$

**Proof.** We prove the lemma by induction on \(p\). For \(p = 1\) this results from Definition 2.8.
Consider an element \( \gamma = \gamma_{i_{p-1}} \otimes \cdots \otimes \gamma_{i_1} \) of the holonomy pseudogroup. Assume (induction hypothesis) that the domain of \( \tilde{\gamma} = \gamma_{i_{p-2}} \otimes \cdots \otimes \gamma_{i_1} \) contains the ball \( B_g \left( f_{i_1}(x), \frac{\epsilon_0}{\max_{1 \leq l \leq p-1} r_l} \right) \). Then
\[
\tilde{\gamma} \left( B_g \left( f_{i_1}(x), \frac{\epsilon_0}{\max_{1 \leq l \leq p-1} r_l} \right) \right) = B_g \left( \tilde{\gamma}(f_{i_1}(x)), \frac{r_{p-1} \epsilon_0}{\max_{1 \leq l \leq p-1} r_l} \right).
\]
Moreover, the domain of \( \gamma_{i_{p-1}} \) contains \( B_g(\tilde{\gamma}(f_{i_1}(x)), \epsilon_0) \) (by Definition 2.8), which itself contains \( B_g \left( f_{i_1}(x), \frac{\epsilon_0}{\max_{1 \leq l \leq p-1} r_l} \right) \).

Thus the domain of \( \gamma = \gamma_{i_{p-1}} \otimes \tilde{\gamma} \) contains the ball \( B_g \left( f_{i_1}(x), \frac{\epsilon_0}{\max_{1 \leq l \leq p-1} r_l} \right) \).

\[\text{Lemma 2.10.}\]
Let \( \gamma \in \Gamma, \ x \in M \) and \( i \in \{1, \ldots, r\} \), such that \( \gamma \) is defined on a neighborhood of \( f_i(x) \) in \( T_i \) and takes its values in \( T_j \). Then there exists \( \tilde{\gamma} = \gamma_{i_{p-1}} \otimes \cdots \otimes \gamma_{i_1} \) defined on a neighborhood of \( f_{i_1}(x) \), which has the same germ as \( \gamma_{j_{p-1}} \circ \gamma \circ \gamma_{i_1} \) at \( f_{i_1}(x) \).

**Proof.** It results from Lemma 2.6 that the germ of \( \gamma \) at \( f_i(x) \) is the holonomy germ from \( T_i \) to \( T_j \) at \( x \) obtained by sliding along a curve \( c : [a, b] \rightarrow M \), such that \( f_i(c(a)) = f_i(x) \), and \( f_j(c(b)) = \gamma(f_i(x)) \). For each \( l \in \{1, \ldots, r\} \), we define \( E_l \) as the set of all open subsets \( I \) of \([a, b]\) which are intervals such that for all \( t \in I \), \( c(t) \in I_l \) and \( d_t(1) > \epsilon_0 \). Lemma 2.7 implies that \( \bigcup_{1 \leq l \leq r} E_l \) is an open cover of \([a, b]\): it has a finite subcover \( \{(a_1, b_1), (a_2, b_2), \ldots, (a_{p-1}, b_{p-1}), (a_p, b_p)\} \), where \( a_1 = a \) and \( b_p = b \). We may assume that for all \( k \in \{1, \ldots, p-1\} \), \( a_k \leq a_{k+1} \leq b_k \leq b_{k+1} \).

For all \( k \in \{1, \ldots, p\} \), choose \( i_k \in \{1, \ldots, r\} \) such that \((a_k, b_k) \in E_{i_k}\), and define \( \tilde{\gamma} = \gamma_{i_{p-1}} \otimes \cdots \otimes \gamma_{i_1} \). Then, by Lemma 2.5, \( \tilde{\gamma} \) has the same germ as \( \gamma_{j_{p-1}} \circ \gamma \circ \gamma_{i_1} \) at \( f_{i_1}(x) \).

\[\text{Lemma 2.11.}\]
Let \( E \) be the set of all \( x \in M \) for which there exists \( m > 1 \) such that for all \( i \in \{1, \ldots, r\} \) with \( x \in U_i \), every \( \gamma \in \Gamma \) defined on \( f_i(x) \) has ratio \( \geq 1/m \) at \( f_i(x) \).

1. In the definition of \( E \), it is possible to choose \( m \) independently of \( x \).
2. If \( E \) is non-empty, then \( E = M \) and \( \Gamma \) is equicontinuous.

**Proof.** We start with the proof of the first statement. Assume that there is no uniform bound: then, there exist sequences \((x^n), (i^n), (j^n)\) and \((\gamma^n)\) such that \( x^n \in E \), \( \gamma^n \) is defined on a neighborhood of \( f_{i^n}(x^n) \) in \( T_{i^n} \), takes its values in \( T_{j^n} \), and the ratio of \( \gamma^n \) is \( \leq 1/n \) at \( f_{i^n}(x^n) \) (the exponents do not indicate exponentiation).

Let \( r_{\max} \) be the maximum ratio of \( \gamma_{ij} \) for \( i, j \in \{1, \ldots, r\} \).

For each \( n \), Lemma 2.10 gives us a \( \tilde{\gamma}^n = \gamma_{i_{p-1}^n} \otimes \cdots \otimes \gamma_{i_1^n} \), which has the same germ as \( \gamma_{j_{p-1}^n} \circ \gamma^n \circ \gamma_{i_1^n} \) at \( f_{i_1^n}(x^n) \). Notice that \( \tilde{\gamma}^n \) has ratio \( \leq r_{\max}^n/n \) at \( f_{i_1^n}(x^n) \) (because \( \gamma^n \) coincides with \( \gamma_{i_{p-1}^n} \circ \gamma^n \circ \gamma_{i_1^n} \) near \( f_{i_1^n}(x^n) \), and the ratio of \( \gamma^n \) is \( \leq 1/n \) at \( f_{i_1^n}(x^n) \)).

Choose \( q^n \in \{1, \ldots, p^n\} \) which minimizes the ratio of \( \gamma_{i_{p-1}^n} \circ \gamma_{i_1^n} \cdots \gamma_{i_1^n} \) at \( \gamma_{i_{p-1}^n} \circ \cdots \circ \gamma_{i_1^n}(f_{i_1^n}(x^n)) \) (in particular this ratio is \( \leq r_{\max}^n/n \), and write \( \tilde{\gamma}^n = \gamma_{i_{p-1}^n} \circ \gamma^n \circ \gamma_{i_1^n} \).
\[\gamma_{\iota_{n-1}} \circ \cdots \circ \gamma_{\iota_{1}} \circ y^n.\] Choose \(y^n\) such that \(f_{i_0}^{\iota_n}(y^n) = \gamma_{\iota_{n-1}} \circ \cdots \circ \gamma_{\iota_{1}}(f_{i_0}(x^n)).\) Notice that \(y^n \in E.\)

By Lemma 2.9, \(\tilde{\rho}^n\) is well-defined on \(B_{\delta}(f_{i_0}^{\iota_n}(y^n), \epsilon_0)\) and has ratio \(\leq \frac{r_{\max}^2}{n}\) at \(f_{i_0}^{\iota_n}(y^n).\)

Since \(M\) is compact, we may assume up to extraction that \((y^n)\) converges to a limit \(y \in M\) (and \(y \in U_i\) for some \(i\)): there exists \(n_0 > 0\) such that for all \(n \geq n_0,\) \(y^n \in U_i\) and \(d_i(f_i(y^n), f_i(y)) < \epsilon_0/(3m_{\max}).\) Thus, \(\tilde{\rho}^n\) is well-defined on \(f_{i_0}^{\iota_n}(y^n)\) for all \(n \geq n_0,\) which contradicts the fact that \(y^{n_0} \in E\) and ends the proof of the first statement.

To prove the second statement, first notice that for all \(x \in E,\) and all \(i \in \{1, \ldots, r\}\) such that \(x \in U_i,\) every \(\gamma \in \Gamma\) defined on \(f_i(x)\) (taking values in \(T_j\)) has ratio \(\leq m\) at \(f_i(x)\): otherwise, \(\gamma^{-1}\) would have ratio \(< 1/m\) at \(f_i(x),\) which contradicts the fact that \(\gamma(f_i(x)) \in f_i(E \cap U_j).\)

Since \(M\) is connected, it suffices to show that \(E\) is open and closed in \(M.\) Thus, \(\Gamma\) will be equicontinuous on \(M.\)

Let us show that \(E\) is open. Let \(x_0 \in E\) and \(i_1\) such that \(d_i(f_i(x_0), \partial T_i) > \epsilon_0).\)

Consider \(V\) a neighborhood of \(x_0\) such that \(V \subseteq U_{i_1}\) and \(f_{i_1}(V) \subseteq B_{\delta}(f_{i_1}(x_0), \epsilon_0/(2m)).\)

Let us show that \(V \subseteq E:\) let \(y_0 \in V,\) \(i \in \{1, \ldots, r\},\) and \(\gamma \in \Gamma\) defined on \(f_i(y_0),\)

taking its values in \(T_j.\)

With Lemma 2.10, there exists a \(\gamma = \gamma_{i_{p-1}} \circ \cdots \circ \gamma_{i_1}\), which has the same germ as \(\gamma_{i_{p-1}} \circ \cdots \circ \gamma_{i_1} \circ f_i(y_0).\)

Let us prove, by induction on \(l,\) that for all \(l \in \{1, \ldots, p\},\) the ratio of \(\gamma_{i_{p-1}} \circ \cdots \circ \gamma_{i_1} \circ f_i(y_0)\) is between \(1/m\) and \(m.\) For \(l = 1,\) there is nothing to prove. For \(l = 2,\) \(\gamma_{i_2}\) is well-defined on \(B_\delta(f_i(y_0), \epsilon_0)\), which contains \(f_i(x_0).\) Since \(x_0 \in E,\)

this implies that the ratio of \(\gamma_{i_2} \circ f_i(y_0)\) is between \(1/m\) and \(m.\) We now assume that the result is true for all \(l \leq l_0,\) where \(l_0 \in \{1, \ldots, p - 1\}.\) Then by Lemma 2.9, \(\gamma_{i_0} \circ f_i(y_0)\) is well-defined on \(B_\delta(f_i(y_0), \epsilon_0/m)\), which contains \(f_i(x_0).\) The fact that \(x_0 \in E\) now implies that the ratio of \(\gamma_{i_0} \circ f_i(y_0)\) is between \(1/m\) and \(m,\) which concludes the induction.

Therefore, the ratio of \(\gamma\) is between \(1/m\) and \(m\) at \(f_i(x_0).\) The ratio of \(\gamma\) is at least \(1/(r_{\max}^2 m)\) at \(f_i(y_0),\) so \(y_0 \in E,\) and \(E\) is open.

Now, we show that \(M \setminus E\) is open in \(M.\) Let \(x_0 \in M \setminus E,\) \(i \in \{1, \ldots, r\},\) and \(\gamma \in \Gamma\) defined on \(f_i(x_0)\) with ratio \(< 1/m.\) Then \(\gamma\) is defined on a connected open set \(W \subseteq T_i\) containing \(f_i(x_0),\) and \(f_i^{-1}(W)\) is an open set of \(M,\) containing \(x_0\) and contained in \(M \setminus E,\) so \(M \setminus E\) is open. \(\square\)

**End of the proof of Theorem 1.17.** Assume that \((T, g)\) is not flat, and let \(T'\) be the set of all \(y \in T\) at which the Riemann tensor of \(g\) is nonzero. Notice that \(T'\) is stable under the holonomy pseudogroup \(\Gamma.\) Now, Proposition 2.2 gives us a Riemannian metric \(g'\) on \(T'\) which is invariant by \(\text{Sim}_{\text{oc}}(T')\), and thus by the holonomy pseudogroup \(\Gamma.\) Hence, the set \(E\) defined in Lemma 2.11 is non-empty. By Lemma 2.11, \(\Gamma\) is equicontinuous. Finally, in view of Proposition 2.1, \(\mathcal{F}\) is a Riemannian foliation, and Theorem 1.17 is proved.
3 Decomposition theorems for locally metric connections

In this section, we prove Theorems 1.15 and 1.5.

We start with the following:

**Lemma 3.1.** Consider a connected Riemannian manifold \((M,g)\) with its Levi-Civita connection \(\nabla\). If \(\nabla\) has irreducible holonomy, then:

1. the only metrics whose Levi-Civita connection is \(\nabla\) are the metrics \(h_\lambda = \lambda^2 g\), \(\lambda > 0\);
2. \(\text{Aff}(M,g) = \text{Sim}(M,g)\).

**Proof.** 1. Let \(h\) be a metric whose Levi-Civita connection is \(\nabla\) and let \(x \in M\). Define the linear mapping \(F_x : T_x M \to T_x M\) in the following way: for all \(u \in T_x M\), \(F_x(u)\) is the unique vector such that \(g_x(u, \cdot) = h_x(F_x(u), \cdot)\). Since \(\text{Hol}_x(\nabla)\) preserves \(g_x\) and \(h_x\), the eigenspaces of \(F_x\) are invariant under \(\text{Hol}_x(\nabla)\). Since \(\nabla\) is irreducible, the only possible eigenspaces for \(F_x\) are \(\{0\}\) and \(T_x M\); but \(F_x\) is self-adjoint (for both metrics \(g\) and \(h\)), so \(F_x\) is a homothety. This shows that there exists \(\lambda_x > 0\) such that \(h_x = \lambda_x^2 g_x\).

Now, we prove that \(\lambda_x\) does not depend on \(x\): for \(x, y \in M\), choose any nonzero vector \(u \in T_x M\) and any path \(c : [0, 1] \to M\) joining \(x\) to \(y\): if \(v\) is obtained by the parallel transport of \(u\) along \(c\), we have \(h_y(v) = h_x(u) = \lambda_x^2 g_x(u) = \lambda_y^2 g_y(v)\), and thus \(\lambda_x = \lambda_y\).

2. For all \(\phi \in \text{Aff}(M,g)\), the metric \(\phi^*g\) is preserved by the Levi-Civita connection \(\nabla\) of \(g\), so \(\phi^*g\) is proportional to \(g\) and thus \(\phi\) is a similarity. \(\square\)

We are now ready to prove Theorem 1.15. Consider a compact manifold \((M,\nabla)\), where \(\nabla\) is locally metric, and its universal cover \((\hat{M}, \hat{\nabla})\), on which there is a metric \(\hat{g}\) preserved by \(\hat{\nabla}\). Fix \(x \in M\) and choose a preimage \(\hat{x} \in \hat{M}\).

Let \(E^0_x\) be the maximal linear subspace of the tangent space \(T_{\hat{x}} \hat{M}\) on which \(\text{Hol}_{\hat{x}}(\hat{\nabla})\) acts trivially, and define \(E^{>0}_x\) as the orthogonal complement of \(E^0_x\). The local theorem of de Rham (see for example [KN63]) states that there is a unique decomposition of \(E^{>0}_x\) (up to the order of the factors) into mutually orthogonal, invariant irreducible subspaces:

\[
E^{>0}_x = E^1_x \oplus \ldots \oplus E^k_x.
\]

This induces a decomposition \(T_x M = E^0_x \oplus E^{>0}_x\). Furthermore, since \(\pi_1(M)\) acts on \(\hat{M}\) by preserving the connection \(\hat{\nabla}\), this decomposition does not depend on the choice of the preimage \(\hat{x}\) of \(x\), up to the order of the factors. Thus, the holonomy group \(\text{Hol}_x(\nabla)\) acts on \(E^{>0}_x\) by permuting the factors: by considering a finite cover of \(M\), one may assume that \(\text{Hol}_x(\nabla)\) preserves the decomposition of \(T_x M\). Then, one may consider \(E'\) the distribution on \(\hat{M}\) obtained by parallel transport of \(E^1_x\), and \(E''\) obtained by parallel transport of \(E^0_x \oplus \ldots \oplus E^{k-1}_x\). These distributions induce transverse foliations \(\mathcal{F}'\) and \(\mathcal{F}''\) on \(M\). By pullback, one obtains distributions \(E'\) and \(E''\) on \(M\), and foliations \(\mathcal{F}'\) and \(\mathcal{F}''\) on \(M\).

The local version of De Rham’s theorem gives us a covering \((U_i)_{1 \leq i \leq r}\) of \(M\) compatible with the foliations \(\mathcal{F}'\) and \(\mathcal{F}''\), such that each \(U_i\) is diffeomorphic to \(V_i \times \mathbb{R}^k\).
For the proof of Theorem 1.16, $\hat{M}$ is globally the product of two Riemannian manifolds $M'$ and $M''$ whose tangent distributions are $\hat{E}'$ and $\hat{E}''$. The existence of the de Rham decomposition follows by induction on the dimension of $\hat{M}$. Thus, Theorem 1.15 is proved.

For the proof of Theorem 1.5, we will need the following propositions:

**Proposition 3.3.** Consider a connected Riemannian manifold $(M, g)$, and a similarity $\phi \in \text{Sim}(M)$. Assume that $\phi$ has a fixed point $x \in M$, and that its ratio is $r_{\phi} < 1$. Then the manifold $M$ is isometric to the Euclidean $\mathbb{R}^n$ for some $q \geq 0$.

**Proof.** First, let us prove that $M$ is flat. Choose any $y \in M$ and four vectors $a, b, c, d$ in $T_y \hat{M}$ of unit length for $g$. The point is that $\phi$ preserves $R$, i.e.

$$R(\phi_\ast a, \phi_\ast b)\phi_\ast c = \phi_\ast R(a, b)c.$$

Thus:

$$\langle R(a, b)c \mid d \rangle = r_{\phi}^{-2n} \langle \phi_\ast R(a, b)c \mid \phi_\ast d \rangle$$

$$= r_{\phi}^{-2n} \langle \phi_\ast R(\phi_\ast a, \phi_\ast b)\phi_\ast c \mid \phi_\ast d \rangle$$

$$\leq r_{\phi}^{-2n} r_{\phi}^{-4n} \|R\|_g \langle \phi_\ast (y) \rangle.$$
Since \( \phi^n(y) \) tends to the fixed point \( x \), the quantity \( \| R \|_g(\phi^n(y)) \) is bounded. Thus, \( \langle R(a,b)c \mid d \rangle = 0 \), and therefore, \( M \) is flat.

Now, since \( M \) is flat, the exponential map \( \exp_x : B(0, \epsilon) \to B_g(x, \epsilon) \) is an isometry for some \( \epsilon > 0 \) (where \( B(0, \epsilon) \) is the ball in \( T_xM \) of center \( 0 \) and radius \( \epsilon \) for the Euclidean metric \( g_x \), while \( B_g(x, \epsilon) \) is the ball in \( M \) of center \( x \) and radius \( \epsilon \) for the distance induced by \( g \)).

Thus, for all \( n \geq 0, \phi^{-n} \circ \exp_x \circ D_x \phi^n \) is an isometry from \( B(0, r_0^{-n} \epsilon) \) to \( B_g(x, r_0^{-n} \epsilon) \). Since \( \phi^n \) preserves the Levi-Civita connection of \( g \), we have

\[
\exp_x = \phi^{-n} \circ \exp_x \circ D_x \phi^n.
\]

Hence, \( \exp_x \) is an isometry from \( B(0, r_0^{-n} \epsilon) \) to \( B_g(x, r_0^{-n} \epsilon) \) for all \( n \geq 0 \). Since the balls \( B_g(x, r_0^{-n} \epsilon) \) cover \( M \), \( \exp_x \) is an isometry from \( \mathbb{R}^n \) to \( M \).

**Proposition 3.4.** Consider a complete connected Riemannian manifold \((M, g)\). If \( \text{Sim}(M) \) does not act properly on \( M \), then \( M \) is (globally) isometric to \( \mathbb{R}^q \) for some \( q \geq 0 \).

**Proof.** Since \( M \) is complete and connected, the isometry group \( \text{Isom}(M) \) acts properly on \( M \). In the same way, if \( \text{Sim}(M) \) does not act properly on \( M \), there exist a compact set \( K \subseteq M \) and a sequence \( (S_n) \) of similarities such that \( K \cap S_n(K) \neq \emptyset \) and the ratio of \( S_n \) (written \( r_n \)) tends to \( +\infty \) or \( 0 \) when \( n \to +\infty \). Considering \( S_n^{-1} \) instead of \( S_n \) if necessary, we may assume that \( r_n \to 0 \).

Let \( K' = \{ x \in M \mid d(x, K) \leq \epsilon \} \) for some small \( \epsilon > 0 \), where \( d \) is the distance induced by \( g \) in \( M \). Then \( S_n(K') = \{ x \in M \mid d(x, S_n(K)) \leq r_n \epsilon \} \); in particular, for some large enough \( n_0 > 0 \), \( S_{n_0}(K') \subseteq K' \). Thus, \( S_{n_0} \) has a fixed point and \( M \) is isometric to \( \mathbb{R}^q \) by Proposition 3.3.

**Proposition 3.5.** Consider the product of two connected Riemannian manifolds, denoted by \((M, h) = (M_1, h_1) \times (M_2, h_2)\), and a subgroup \( G \) of \( \text{Sim}(M) \) which preserves the product structure (i.e. which is a subgroup of \( \text{Sim}(M_1) \times \text{Sim}(M_2) \)), and acts on \( M \) in a cocompact way. Also assume that \( \text{Sim}(M) \) contains elements which are not isometries. Then, either \( M_1 = \mathbb{R}^q \) or \( M_2 = \mathbb{R}^q \), for some \( q \geq 0 \).

**Proof.** Assume that the conclusion is false. In view of Proposition 3.4, \( \text{Sim}(M_1) \) and \( \text{Sim}(M_2) \) act properly on \( M_1 \) and \( M_2 \) respectively.

Since \( G \) acts cocompactly on \( M \), there is a compact set \( K \subseteq M \) such that \( \text{Sim}(M) \cdot K = M \). We may assume that \( K = K_1 \times K_2 \), where \( \text{Sim}(M_1) \cdot K_1 = M_1 \) and \( \text{Sim}(M_2) \cdot K_2 = M_2 \).

Choose \( x_1 \in K_1 \). Since \( \text{Sim}(M_1) \) acts properly on \( M_1 \), there is a constant \( R > 1 \) such that for all \( \gamma \in \text{Sim}(M_1) \) satisfying \( \gamma(x_1) \in K_1 \), the ratio of \( \gamma \) is between \( R \) and \( 1/R \). Likewise, choose \( x_2 \in K_2 \). There is a constant, still called \( R \), such that the ratio of any \( \gamma \in \text{Sim}(M_2) \) satisfying \( \gamma(x_2) \in K_2 \) is between \( R \) and \( 1/R \).

We assumed that \( \text{Sim}(M) \) contains elements which are not isometries, so there exists \( \gamma_0 \in \text{Sim}(M_1) \) whose ratio is greater than \( R^3 \). And since \( G \cdot K = M \), there exists \( \gamma = (\gamma_1, \gamma_2) \in G \) such that \( \gamma(\gamma_0(x_1), x_2) \in K \). Then, \( \gamma_1 \circ \gamma_0(x_1) \in K_1 \), so the ratio of \( \gamma_1 \circ \gamma_0 \) is smaller than \( R \); hence, the ratio of \( \gamma_1 \) is smaller than \( 1/R^2 \). Meanwhile, \( \gamma_2(x_2) \in K_2 \), so the ratio of \( \gamma_2 \) is greater than \( 1/R \). But since \( (\gamma_1, \gamma_2) \in \text{Sim}(M) \), \( \gamma_1 \) and \( \gamma_2 \) should have the same ratio, which is impossible.
In the setting of Theorem 1.5, since the similarity structure induces a locally metric connection on \( M \), Theorem 1.15 implies that \( \tilde{M} \) admits a de Rham decomposition. Assuming that \( \tilde{M} \) is the product of two manifolds \( M_1 \) and \( M_2 \), there is a finite index subgroup of \( \pi_1(M) \) which preserves the product structure of \( M \); it acts cocompactly on \( M \) and contains elements which are not isometries. Thus, we may apply Proposition 3.5, which completes the proof of Theorem 1.5.

4 Closures of the leaves

This section is devoted to the proof of Theorem 1.8.

Consider \( \pi_1(M) \subseteq \text{Sim}(\mathbb{R}^q) \times \text{Sim}(N) \) and define \( P \) as the image of \( \pi_1(M) \) by the projection onto the second factor, i.e.
\[
P = \{ b \in \text{Sim}(N) \mid \exists a \in \text{Sim}(\mathbb{R}^q), (a, b) \in \pi_1(M) \}.
\]
Denote by \( \overline{P} \) the closure of \( P \) in \( \text{Sim}(N) \), and by \( P^0 \) the identity component of \( \overline{P} \).

In Example 1.6, \( P^0 \) is the group \( \mathbb{R} \) acting by translation on the first factor of \( N = \mathbb{E}^n \times (0, +\infty) \). In general, we will prove the following lemma:

**Lemma 4.1.** The group \( P^0 \) is abelian.

4.1 Generalities on lattices in Lie groups

Here, we state classical general facts about lattices in Lie groups: in this paper, a lattice is a discrete, cocompact subgroup of a Lie group.

**Proposition 4.2.** Let \( G \) be a Lie group, \( \Gamma \) a lattice in \( G \), and \( N \) a normal subgroup in \( G \). Then the following two properties are equivalent:

1. The group \( \Gamma \cap N \) is a lattice in \( N \).
2. The image of \( \Gamma \) by the projection \( G \to G/N \) is a lattice in \( G/N \).

**Proof.** We have the following chain of equivalences: \( \Gamma \cap N \) is a lattice in \( N \) \( \iff \) \( N/\Gamma \cap N \) is compact \( \iff \) the image of \( N \) in \( G/\Gamma \) is compact \( \iff \) the image of \( N \) in \( G/\Gamma \) is closed in \( G/\Gamma \) \( \iff \) the subgroup of \( G \) generated by \( N \cup \Gamma \) in \( G \) is closed \( \iff \) the image of \( \Gamma \) by the projection \( G \to G/N \) is closed in \( G/N \) \( \iff \) the image of \( \Gamma \) by the projection \( G \to G/N \) is a Lie subgroup of \( G/N \) \( \iff \) the image of \( \Gamma \) by the projection \( G \to G/N \) is discrete \( \iff \) the image of \( \Gamma \) by the projection \( G \to G/N \) is a lattice in \( G/N \). \( \square \)

Propositions 4.3, 4.4 and 4.7 give sufficient conditions for the two equivalent assertions of Proposition 4.2 to hold true.

The following proposition is classical: since the proof is short, we recall it here.

**Proposition 4.3.** Let \( G \) be a Lie group and \( \Gamma \) a lattice in \( G \). Then \( \Gamma \cap G^0 \) is a lattice in \( G^0 \) (where \( G^0 \) is the identity component of \( G \)).

**Proof.** The group \( \Gamma \cap G^0 \) is discrete since it is a subgroup of the discrete group \( \Gamma \). There remains to show that \( \Gamma \cap G^0 \) is cocompact in \( G^0 \).

The group \( G^0 \) acts on \( G \) by left translation. The orbits are the connected components of \( G \), so they are open. Thus, the orbits of the action of \( G^0 \) on \( G/\Gamma \) by left
translation are also open. Therefore, each orbit is closed (since its complement is a union of open orbits). In particular, the orbit \( G^0/(\Gamma \cap G^0) \) is closed in \( G/\Gamma \) which is compact, so \( G^0/(\Gamma \cap G^0) \) is compact.

The following proposition is also classical, see for example \([Rag72]\) page 40:

**Proposition 4.4.** If \( G \) is nilpotent and \( \Gamma \subseteq G \) is a lattice, \( \Gamma \cap Z(G) \) is a lattice in \( Z(G) \), where \( Z(G) \) is the center of \( G \).

We now recall Bieberbach’s theorem (see \([Bie11]\)):

**Theorem 4.5 (Bieberbach, 1911).** Consider a Euclidean space \( \mathbb{R}^q \), and a lattice \( G \) in \( \text{Isom}(\mathbb{R}^q) = O(\mathbb{R}^q) \rtimes \mathbb{R}^q \). Then \( G \cap \mathbb{R}^q \) is a lattice in \( \mathbb{R}^q \).

In order to state a more general version of Theorem 4.5, we define the **radical** and the **nilradical** of a Lie group.

**Definition 4.6.** The **radical** of a Lie group \( G \), written \( \text{Rad}(G) \), is the unique maximal normal connected closed solvable subgroup of \( G \). The **nilradical** of a Lie group \( G \), written \( \text{Nil}(G) \), is the unique maximal normal connected closed nilpotent subgroup of \( G \).

**Proposition 4.7.** Let \( G \) be a connected Lie group of the form \( G = S \rtimes R \), where \( R = \text{Rad}(G) \) and \( S \) is a semisimple Levi subgroup (such a decomposition always exists if \( G \) is simply connected). Assume that the kernel of the action of \( S \) on \( R \) does not contain compact factors. If \( \Gamma \) is a lattice in \( G \), then \( \Gamma \cap \text{Nil}(G) \) is a lattice in \( \text{Nil}(G) \).

Proposition 4.7 generalizes Theorem 4.5. It was initially stated in \([Rag72]\) as a corollary of a theorem by Auslander \([Aus61]\), but Raghunathan’s proof turned out to be flawed. A correct proof is available in the appendix of \([Sta89]\).

### 4.2 Proof of Lemma 4.1

**Lemma 4.8.** The group \( \overline{P} \) is a Lie group which acts properly on \( N \).

**Proof.** With Theorem 1.17, there exists a transverse Riemannian structure on \( \mathcal{F} \): it induces naturally a structure of complete Riemannian manifold on \( N \), such that \( P \) acts by isometries. Thus, \( \overline{P} \) is a closed subgroup of the group of isometries of \( N \) for this metric, so it is a Lie group which acts properly on \( N \).

**Lemma 4.9.** Every element of \( \overline{P} \) is the product of an element of \( P \) by an element of \( \overline{P}^0 \), i.e. \( \overline{P} = P \cdot \overline{P}^0 \).

**Proof.** Since \( P \subseteq \overline{P} \) and \( \overline{P}^0 \subseteq \overline{P} \), we have \( P \cdot \overline{P}^0 \subseteq \overline{P} \).

Conversely, consider some \( p \in \overline{P} \), and \((p_n)\) a sequence of elements of \( P \) converging to \( p \) in \( \overline{P} \). Then \((p_n)^{-1}p\) converges to the identity in \( \overline{P} \). Since \( \overline{P} \) is a Lie group, it is locally connected and thus \((p_n)^{-1}p \in \overline{P}^0\) for a large enough \( n \). Hence for this \( n \), \( p = p_n(p_n)^{-1}p \in P \cdot \overline{P}^0 \).

**Lemma 4.10.** Denote by \( f \) the covering \( \mathbb{R}^q \times N \to M \), and let \((a,x) \in \mathbb{R}^q \times N \). Then the leaf of \( \mathcal{F} \) containing \( f(a,x) \) is \( f(\mathbb{R}^q \times P^n) \), and the closure of this leaf (in \( M \)) is \( f(\mathbb{R}^q \times \overline{P}^0 \cdot x) \).
The set \( \gamma \) for any \( \gamma \in \pi_1(M) \) and \( y \in M \), \( f(\gamma y) = f(y) \). Thus, the leaf of \( \mathcal{F} \) containing \( f(a, x) \) is \( f(\mathbb{R}^q \times P x) \).

The closure of this leaf is \( \overline{f(\mathbb{R}^q \times P x)} \). Let us show that \( \overline{f(\mathbb{R}^q \times P x)} = f(\mathbb{R}^q \times \mathcal{P} x) \). If \( f(a, px) \in f(\mathbb{R}^q \times \mathcal{P} x) \), then there exists a sequence \((p_n)\) of elements of \( P \) such that \( p_n \to p \), and thus \( f(a, p_n x) \) is a sequence of elements of \( f(\mathbb{R}^q \times P x) \) converging to \( f(a, px) \). Conversely, for \( y \in \overline{f(\mathbb{R}^q \times P x)} \), there exists \( y_n \in f(\mathbb{R}^q \times P x) \) such that \( y_n \to y \). One may find \((a_n)\) a sequence in \( \mathbb{R}^q \) and \( p_n \) a sequence in \( P \) such that \( f(a_n, p_n x) = y_n \), and \((a_n, p_n x)\) converges in \( \mathbb{R}^q \times N \) to some point \((a, b)\). Furthermore, since \( \mathcal{P} \) acts properly on \( N \) (Lemma 4.8), one may assume (up to extraction) that \((p_n)\) converges to some \( p \in \mathcal{P} \). Hence, \( y = f(a, px) \).

Finally, \( f(\mathbb{R}^q \times \mathcal{P}^0 x) = f(\mathbb{R}^q \times P \cdot \mathcal{P}^0 x) = f(\mathbb{R}^q \times \mathcal{P} x) \) by Lemma 4.9.

Now let us notice the following:

**Lemma 4.11.** The group \( \pi_1(M) \) is a lattice in the Lie group \( \text{Sim}(\tilde{M}) \cap (\text{Sim}(\mathbb{R}^q) \times \mathcal{P}) \).

**Proof.** We will write \( S = \text{Sim}(\tilde{M}) \cap (\text{Sim}(\mathbb{R}^q) \times \mathcal{P}) \). Notice that \( S \) is a closed subgroup of the Lie group \( \text{Sim}(\mathbb{R}^q) \times \mathcal{P} \), so it is a Lie group.

The group \( \pi_1(M) \) is discrete, so there remains to show that it is cocompact.

Since \( M \) is compact, there is a compact set \( K_1 \subseteq \tilde{M} \) such that \( \pi_1(M) \cdot K_1 = \tilde{M} \). Define

\[
K_2 = \{ \phi \in S \mid \phi(K_1) \cap K_1 \neq \emptyset \}.
\]

The set \( K_2 \) is compact because the action of \( \text{Sim}(\tilde{M}) \) is proper (by Proposition 3.4). Then for all \( \psi \in S \) there exists \( \gamma \in \pi_1(M) \) such that \( \gamma(\psi(K_1)) \cap K_1 \neq \emptyset \); hence \( \gamma \circ \psi \in K_2 \). This proves that \( \pi_1(M) \cdot K_2 = S \), and thus \( \pi_1(M) \) is cocompact in \( S \).

We denote by \( \text{Isom}^+(\mathbb{R}^q) \) (resp. \( \text{Sim}^+(\mathbb{R}^q) \)) the group of orientation-preserving isometries (resp. similarities) of \( \mathbb{R}^q \).

Writing \( P_I = \mathcal{P}^0 \cap \text{Isom}(N) \), we have an exact sequence:

\[
0 \to P_I \to \mathcal{P}^0 \to \mathbb{R}_{>0}
\]

where \( r : \mathcal{P}^0 \to \mathbb{R}_{>0} \) gives the ratio of a similarity. Since \( \mathcal{P}^0 \) is connected, the mapping \( r \) is either surjective or constantly equal to 1. Furthermore, if \( r \) is surjective, it has a section because \( \mathcal{P}^0 \) is a Lie group. Thus, writing \( H \subseteq \mathbb{R}_{>0} \) the image of \( r \), we may write, up to isomorphism, \( \mathcal{P}^0 = H \times P_I \) (in particular, \( P_I \) is connected). In addition, \( \text{Sim}^+(\mathbb{R}^q) = \mathbb{R}_{>0} \rtimes \text{Isom}^+(\mathbb{R}^q) \). Considering the group \( T = \text{Sim}(\tilde{M}) \cap (\text{Sim}^+(\mathbb{R}^q) \times \mathcal{P}) \), we may write:

\[
T = H \ltimes (\text{SO}(q) \times \mathbb{R}^q) \times P_I.
\]

Denoting by \( \tilde{P}_I \) the universal cover of \( P_I \), and by \( \tilde{T} \) the universal cover of \( T \), we obtain:

\[
\tilde{T} = H \ltimes ((\tilde{\text{SO}}(q) \times \mathbb{R}^q) \times \tilde{P}_I),
\]

where \( \tilde{\text{SO}}(q) \) is the universal cover of \( \text{SO}(q) \).
The group $\pi_1(M)$ acts as a subgroup of $\text{Sim}(\hat{M}) \cap (\text{Sim}(\mathbb{R}^q) \times \mathbb{R})$. This subgroup is a lattice (see Lemma 4.11), so by Proposition 4.3, $\pi_1(M) \cap T$ is also a lattice in $T$.

Consider $\Gamma$ the subgroup of $\hat{T}$ defined as the pullback of $\pi_1(M) \cap T$ by the covering $\hat{T} \to T$: it is a lattice in $\hat{T}$. The image of $\pi_1(M) \cap T$ by the projection onto the second factor $\text{Sim}(\mathbb{R}^q) \times \mathbb{R} \to \mathbb{R}$ is $P \cap \mathbb{R}$ (by definition of $P$). Since $P \cap \mathbb{R}$ is dense in $\mathbb{R}$, this implies that the image of $\Gamma$ by the projection onto $H \times \hat{P}_1$ is dense in $H \times \hat{P}_1$.

Let $K$ be the maximal normal compact connected subgroup of $\hat{P}_1$ and write $\hat{P}_1 = K \times L$. Then $H \ltimes K$ is isomorphic to the direct product $H \times K$, so

$$\hat{T} = K \times \left(H \ltimes (\text{Sim}(q) \ltimes \mathbb{R}^q) \times L\right).$$

Lemma 4.12. The group $L$ is nilpotent and the group $H$ is trivial. In particular, $\mathbb{R}$ acts on $N$ by isometries.

Proof. Since $K$ is compact, the image $\Gamma_1$ of $\Gamma$ by the projection onto $H \ltimes (\text{Sim}(q) \ltimes \mathbb{R}^q) \times L$ is a lattice.

The nilradical of $\text{Isom}^+(\mathbb{R}^q)$ is $\text{Sim}(q) \ltimes \mathbb{R}^q$. Thus, the nilradical of the Lie group $H \ltimes (\text{Sim}(q) \ltimes \mathbb{R}^q) \times L$ contains $\mathbb{R}^q$. Since $H$ acts by homotheties, the action of a nontrivial element of $H$ on $\mathbb{R}^q$ is not unipotent, and thus the image of the nilradical by the projection onto $H$ is trivial. Hence, the nilradical of the Lie group $H \ltimes (\text{Sim}(q) \ltimes \mathbb{R}^q) \times L$ is $\mathbb{R}^q \times \text{Nil}(L)$, where $\text{Nil}(L)$ is the nilradical of $L$.

Thus, the image of $\Gamma_1$ by the projection onto $H \ltimes (\text{Sim}(q) \ltimes L/\text{Nil}(L))$ is a lattice by Proposition 4.7. Furthermore, $\Gamma_1$ contains the fundamental group of $\text{SO}(q)$, so the image of $\Gamma_1$ by the projection onto $H \ltimes (\text{Sim}(q) \ltimes L/\text{Nil}(L))$ is a lattice.

Since $\text{SO}(q)$ is compact, the image of $\Gamma_1$ by the projection onto $H \ltimes (L/\text{Nil}(L))$ is a lattice. But this image is dense. Therefore, $H$ and $L/\text{Nil}(L)$ are discrete. Since they are connected, $H = \{1\}$ and $L = \text{Nil}(L)$.

Lemma 4.13. The group $L$ is abelian.

Proof. Consider $\Gamma_2$ the intersection of $\Gamma_1$ with $\mathbb{R} \times L$. We use again the fact that the image of $\Gamma_1$ by the projection onto $H \ltimes (\text{Sim}(q) \ltimes L/\text{Nil}(L))$ is a lattice: it now means that the image of $\Gamma_1$ by the projection onto $\text{SO}(q)$ is a lattice, and therefore finite. Since the image of $\Gamma_1$ by the projection onto the second factor $L$ is dense, and $\Gamma_2$ has finite index in $\Gamma_1$, this implies that the image of $\Gamma_2$ by the projection onto $L$ is dense. In addition, still with Proposition 4.7, $\Gamma_2$ is a lattice in $\mathbb{R} \times L$.

By Proposition 4.4, the image of $\Gamma_2$ by the projection onto $L/\text{Z}(L)$ is a lattice in $L/\text{Z}(L)$, where $\text{Z}(L)$ is the center of $L$. This image is also dense, so $L/\text{Z}(L)$ is discrete. Since $L$ is connected, $L = \text{Z}(L)$ and $L$ is abelian.

Lemma 4.14. The group $K$ is abelian.

Proof. Consider the group $\Gamma_3 = \Gamma \cap (K \times \mathbb{R} \times L)$, and $\Gamma_4$ the image of $\Gamma_3$ by the projection onto $K$. Then for all $k, k' \in K$ and $\gamma, \gamma' \in \Gamma_2$, we may write

$$[(k, \gamma), (k', \gamma')] = ([k, k'], [\gamma, \gamma']) = ([k, k'], 1).$$

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Thus \([\Gamma_4, \Gamma_4] \subseteq \Gamma_3 \cap K = \Gamma \cap K\).

Since we already know that the image of \(\Gamma\) by the projection onto \(SO(q)\) is finite, we deduce that \(\Gamma_3\) is a subgroup of \(\Gamma\) of finite index, and thus \(\Gamma_4\) is dense in \(K\), which implies that \([K, K] \subseteq \Gamma \cap K\). But \(K\) is connected so \([K, K]\) is trivial: \(K\) is abelian (in fact, \(K\) is a compact abelian simply connected Lie group, so it is trivial).

Lemmas 4.12, 4.13 and 4.14 together imply that \(\hat{P}_f\) is abelian, and thus Lemma 4.1 is proved.

### 4.3 End of the proof of Theorem 1.8

Denote by \(\bar{F}\) the closure of a leaf in \(M\). By Lemma 4.10, \(\bar{F} = f(\mathbb{R}^q \times \mathbb{T}^d)\). We already know that \(\bar{F}\) is a submanifold of \(\hat{M}\) (see [Mol88]). Lemma 4.1 implies that \(\mathbb{R}^q \times \mathbb{T}^d\) is isometric to the product of a Euclidean space by a flat torus, so \(\bar{F}\) is flat. Moreover,

**Proposition 4.15.** The closures of the leaves are Riemannian manifolds, i.e. the Riemannian metric on the universal cover \(\hat{M}\) induces a Riemannian metric on \(\bar{F}\).

**Proof.** \(\mathbb{R}^q \times \mathbb{T}^d\) is complete, so every similarity of \(\mathbb{R}^q \times \mathbb{T}^d\) of ratio \(\neq 1\) has a fixed point (by the Banach fixed point theorem). Thus, the elements of \(\pi_1(M)\) with ratio \(\neq 1\) act freely on \(N/\mathbb{T}^d\), which proves the proposition. \(\square\)

To study the dimension of \(\bar{F}\), we will need the following lemma:

**Lemma 4.16.** The group \(\pi_1(M)\) acts freely on \(N\).

**Proof.** Consider some \(u \in \pi_1(M)\) with \(u \neq Id\), and write \(u = (u', u'')\), where \(u' \in \text{Sim}(\mathbb{R}^q)\) and \(u'' \in \text{Sim}(\mathbb{T}^d)\). Assume that \(u''\) has a fixed point \(a \in N\). Then \(u'\) has no fixed point (because \(\pi_1(M)\) acts freely on \(\hat{M}\)). Therefore, \(u'\) is an isometry of \(\mathbb{R}^q\), so one may write \(u'(x) = R_{u}x + t_{u}\) for \(x \in \mathbb{R}^q\), where \(R_{u} \in O(\mathbb{R}^q)\) and \(t_{u} \in \mathbb{R}^q\), with \(t_{u} \neq 0\).

Now, consider \(v \in \pi_1(M)\) with ratio \(\lambda \in (0, 1)\), and write \(v = (v', v'')\), where \(v' \in \text{Sim}(\mathbb{R}^q)\) and \(v'' \in \text{Sim}(\mathbb{T}^d)\). We have \(v'(x) = \lambda R_{v}x + t_{v}\) for \(x \in \mathbb{R}^q\), where \(R_{v} \in O(\mathbb{R}^q)\) and \(t_{v} \in \mathbb{R}^q\). Since \(v'\) has a fixed point in \(\mathbb{R}^q\), we may apply a translation in \(\mathbb{R}^q\) in order to assume that \(t_{v} = 0\).

Now for all \(k \in \mathbb{N}\) and \(x \in \mathbb{R}^q\), we have

\[
(v')^k(u')(v')^{-k}(x) = R_{u}^k R_{v}^{-k}x + \lambda^k R_{v}^k t_{u}.
\]

Furthermore, \((v'')^k u''(v'')^{-k}\) has a fixed point because \(u'\) has a fixed point. Thus, \(\{v^k u v^{-k} \mid k \in \mathbb{N}\}\) is an infinite, relatively compact subset of \(\pi_1(M)\), which contradicts the fact that \(\pi_1(M)\) is discrete. \(\square\)

We write \(d = \dim(\bar{F})\). Then \(\mathcal{F}\) induces a foliation of dimension \(q\) on \(\bar{F}\), so \(q \leq d \leq q + n\).

If \(\bar{F}\) has dimension \(q\), then \(\mathcal{F}\) has codimension 0 in \(F\), so \(\bar{F} = F\), and thus \(F\) is compact. But by Lemma 4.16, \(F\) is homeomorphic to \(\mathbb{R}^q\), so this is impossible.

If \(\bar{F}\) has dimension \(q + n\), then \(\bar{F}\) is open and closed in \(M\). Since \(M\) is connected, we have \(\bar{F} = M\), which contradicts the fact that \(M\) is not flat.

Therefore, \(q < d < q + n\) and Theorem 1.8 is proved.
4.4 End of the proof of Theorem 1.9

Define \( \Gamma_0 = \pi_1(M) \cap \left( \text{Sim}(\mathbb{R}^q) \times \overline{P} \right) \). For all \( x \in N \), consider the following subgroup of \( \pi_1(M) \):

\[
S_x = \left\{ p \in \pi_1(M) \mid p \cdot x \in \overline{P}^0 x \right\}.
\]

**Lemma 4.17.** The group \( \Gamma_0 \) is contained in \( \mathbb{R}^q \times \overline{P}^0 \). Moreover, it is a lattice in \( \mathbb{R}^q \times \overline{P}^0 \).

**Proof.** Since \( \overline{P}^0 \) is abelian, for all \( u, v \in \Gamma_0 \), the commutator \( uvu^{-1}v^{-1} \) acts trivially on \( N \). By Lemma 4.16, \( uvu^{-1}v^{-1} = \text{Id} \). Therefore, \( \Gamma_0 \) is abelian.

The closure of a leaf \( f(\mathbb{R}^q \times \overline{P}^0 x) \) (where \( x \in N \)) is isomorphic to \( (\mathbb{R}^q \times \overline{P}^0 x) / S_x \). In particular, since the closures of the leaves are closed Riemannian manifolds (by Proposition 4.15), \( S_x \) acts cocompactly by isometries on \( \mathbb{R}^q \times \overline{P}^0 x \). Also recall that \( \mathbb{R}^q \times \overline{P}^0 x \) is the product of a Euclidean space by a flat torus (Lemma 4.1). By applying Theorem 4.5 to the universal cover of \( \mathbb{R}^q \times \overline{P}^0 x \), we deduce that \( S_x \cap (\mathbb{R}^q \times \overline{P}^0) \) (which is a subgroup of \( \Gamma_0 \)) also acts cocompactly. In particular, the image of \( \Gamma_0 \) by the projection onto \( \text{Sim}(\mathbb{R}^q) \) contains translations: since it is abelian, it contains only translations. Thus, \( \Gamma_0 \) is a lattice in \( \mathbb{R}^q \times \overline{P}^0 \).

Now, consider the representation \( \rho : \pi_1(M) \to \text{Aut}(\Gamma_0) \) given by the action of \( \pi_1(M) \) onto \( \Gamma_0 \) by conjugation.

**Lemma 4.18.** There is a subgroup \( J \subseteq \pi_1(M) \) of finite index such that \( \rho(J) \) has no torsion.

**Proof.** The group \( \Gamma_0 \) is a lattice in \( \mathbb{R}^q \times \overline{P}^0 \), which implies that \( \Gamma_0 \) is a finitely generated abelian group. Therefore, \( \text{Aut}(\Gamma_0) \) is a subgroup of \( GL_m(\mathbb{Z}) \) for some \( m \in \mathbb{Z} \). By Selberg’s lemma (see for example [Alp87]), there is a subgroup of \( \rho(\pi_1(M)) \) of finite index which is torsion-free: the preimage \( J \) of this subgroup by \( \rho \) is the desired group.

In the following, we fix such a subgroup \( J \subseteq \pi_1(M) \).

**Lemma 4.19.** For all \( x \in N \), the group \( S_x \cap J \) is a subgroup of \( \Gamma_0 \).

**Proof.** Choose \( a \in S_x \cap J \). There exists an element \( t \in \mathbb{R}^q \times \overline{P}^0 \) such that the action of \( ta \) on \( \mathbb{R}^q \times N \) has a fixed point. Consider the subgroup \( H \) of \( \mathbb{R}^q \times \overline{P} \) generated by \( ta \): by Proposition 3.4, \( H \) is relatively compact in \( \mathbb{R}^q \times \overline{P} \). Then the image of \( H \) by the projection \( \mathbb{R}^q \times \overline{P} \to \overline{P}^0 \) is discrete and relatively compact, so it is finite. Thus, there exists \( n \geq 1 \) such that \( (ta)^n \in \mathbb{R}^q \times \overline{P}^0 \), which implies that \( \rho(a^n) \) is trivial. Since \( \rho(J) \) has no torsion, \( \rho(a) \) is trivial.

We have shown that all the elements of \( S_x \cap J \) act trivially on \( \overline{P}^0 \) by conjugation, which implies that \( S_x \cap J \subseteq \text{Sim}(\mathbb{R}^q) \times \overline{P}^0 \).}

Since \( J \) has finite index in \( \pi_1(M) \), there is a finite covering \( \tilde{M}/J \to M \): we will write \( M' = \tilde{M}/J \) and show that the closures of the leaves of the foliation \( \mathcal{F}' \) (induced by \( \mathcal{F} \) on \( M' \)) are tori. Denote by \( \overline{P}' \) the closure of a leaf of \( \mathcal{F}' \). By Lemma 4.10, \( \overline{P}' = f'(\mathbb{R}^q \times \overline{P}^0 x) \) for some \( x \in N \), where \( f' \) is the projection \( \tilde{M} \to M' \) and \( P' \)
is the image of $J$ by the projection onto $\text{Sim}(N)$. Since $P'$ has finite index in $P$, $\mathcal{T}_0$ has finite index in $\mathcal{T}$ and thus $\mathcal{T}_0 = \mathcal{T}$, so $\mathcal{T} = f'(\mathbb{R}^q \times \mathcal{T}_0)$: in other words, $\mathcal{T}' = (\mathbb{R}^q \times \mathcal{T}_0)/(S_x \cap J)$. Since $S_x \cap J$ is a subgroup of $\mathbb{R}^q \times \mathcal{T}_0$ (by Lemma 4.19 and Lemma 4.17), the group $(\mathbb{R}^q \times \mathcal{T}_0)/(S_x \cap J)$, which is the product of a linear space by a torus, acts transitively on $\mathcal{T}_0$ by isometries. Since $\mathcal{T}_0$ is compact, it is isometric to a flat torus, which ends the proof of Theorem 1.9.

5 Classification in dimension 2

In this section, we assume that $\dim N = 2$ and prove Theorem 1.7.

Lemma 5.1. The group $\mathcal{T}_0$ acts freely on $N$.

Proof. We assume that there exists $p \in \mathcal{T}_0$ which has a fixed point in $a \in N$ and look for a contradiction. Consider a one-parameter subgroup $G \subset \mathcal{T}_0$ which contains $a$. Then the flow induced by $G$ on $N$ has a closed orbit in $\mathbb{R}^2$, so it has a fixed point $x_0 \in N$ (here we use the fact that $\dim(N) = 2$). Thus, the closure of $G$ in $\mathcal{T}$ fixes $x_0$. Since $\mathcal{T}_0$ acts properly, $G$ is compact, so $G$ is a torus: it contains a closed Lie subgroup $H$ which is isomorphic to $\mathbb{R}/\mathbb{Z}$, and which fixes $x_0$.

Choose a point $x_1$ which is not fixed by $H$. Then $Hx_1$ defines a closed curve in $N$. Thus, $H$ has a fixed point $x_0' \in N$ such that the curve $Hx_0'$ in $N \setminus \{x_0\}$ is not homotopic to a constant. Since $N$ is connected, the homotopy class of the curve $Hx_0'$ in $N \setminus \{x_0\}$ does not depend on the choice of $x \in N \setminus \{x_0\}$. Therefore, $H$ has only one fixed point $x_0' = x_0$.

From now on, denote by $K(x)$ the curvature of $N$ at $x \in N$, and write

$$L_t = \{x \in N \mid K(x) = t\}$$

for all $t \in \mathbb{R}$. The curvature $K$ is not constant on $N \setminus \{x_0\}$ because $N$ is a non-flat manifold with similarities: thus there is some $x_2 \in N \setminus \{x_0\}$ such that $K(x_1) \neq K(x_2)$. Assume that the curve $Hx_2$ is outside $Hx_1$ (this is always possible up to a permutation of $x_1$ and $x_2$). Then the connected component $C$ of $N \setminus L_{K(x_2)}$ containing $x_1$ is bounded.

Choose a similarity $h \in P$. Since $P$ acts properly, $h^n C \cap C = \emptyset$ for some $n \in \mathbb{N}$. But $H(h^n x_1)$ (the orbit of $h^n x_1$ under the flow $H$) is contained in $L_{K(h^n x_1)}$, which does not intersect $L_{K(h^n x_2)}$, so it is contained in $h^n C$. Then $H(h^n x_1)$ is homotopic to a constant in $N \setminus \{x_0\}$: this contradiction ends the proof of the lemma.

Lemma 5.2. The Lie group $\mathcal{T}_0$ is isomorphic to $\mathbb{R}$.

Proof. For all $x \in N$, Theorem 1.8 implies that $\mathbb{R}^q \times \mathcal{T}_0 x$ has dimension $q + 1$, so $\mathcal{T}_0$ has dimension 1. Since $\mathcal{T}_0$ acts freely and transitively on $\mathcal{T}_0 x$, we deduce that $\mathcal{T}_0$ is diffeomorphic to $\mathcal{T}_0 x$, so it has dimension 1. Furthermore, $\mathbb{R}/\mathbb{Z}$ cannot act freely on the plane (if a flow has a closed orbit, then it has a fixed point), so $\mathcal{T}_0$ is isomorphic to $\mathbb{R}$.

In the following, we fix an identification of $\mathcal{T}_0$ with $\mathbb{R}$.
Lemma 5.3. There exists an open interval \((a, b)\) of \(\mathbb{R}\) and a diffeomorphism \(\Phi : \mathbb{R} \times (a, b) \to N\) which provides a system of coordinates \((y, z)\) on \(N\) (with \(y \in \mathbb{R}\), \(z \in (a, b)\)) such that:

- Writing \(y \in \mathbb{R}\) and \(z \in (a, b)\) the coordinates in \(N\) given by \(\Phi\), the action of \(p \in \mathbb{R}^0\) on \(N\) is \(p(y, z) = (y + p, z)\).
- The metric on \(N\) is given by \(\varphi(z)dy^2 + dz^2\), where \(\varphi : (a, b) \to \mathbb{R}_{>0}\) is a smooth function.

Proof. The group \(\mathbb{R}^0\) acts freely and properly on \(N\). Thus, \(N/\mathbb{R}^0\) has a natural structure of smooth manifold of dimension 1. Furthermore, \(N/\mathbb{R}^0\) is simply connected because \(N\) is, so \(N/\mathbb{R}^0\) is diffeomorphic to \(\mathbb{R}\). Thus, \(N \to N/\mathbb{R}^0\) is a fiber bundle over a contractible space, so it is trivial. Up to isometry, we may write \(N = (\mathbb{R}^2, g_N)\), where \(g_N\) is a Riemannian metric on \(\mathbb{R}^2\), and \(\mathbb{R}^0\) acts by translation on the first coordinate.

Denote by \(F_N\) the foliation induced by the submersion \(s : N \to N/\mathbb{R}^0\). Then \(F_N\) is the foliation induced on \(N\) by the closures of the leaves of \(F\): it is the standard foliation of \(\mathbb{R}^2\) by horizontal lines. Consider a vector field \(X\) on \(N\) (with the above identification, \(X : \mathbb{R}^2 \to \mathbb{R}^2\)) orthogonal to foliation \(F_N\) for the metric \(g_N\), such that all vectors have length 1 for the metric \(g_N\). Denoting by \(X_1\) and \(X_2\) the two coordinates of \(X\) in \(\mathbb{R}^2\), we may assume that \(X_2\) is everywhere positive. Let \(\gamma : (a, b) \to \mathbb{R}^2\) be a maximal integral curve of \(X\), where \((a, b)\) is an open interval of \(\mathbb{R}\), and denote by \(\gamma_1\) and \(\gamma_2\) the two coordinates of \(\gamma\) in \(\mathbb{R}^2\). Notice that \(\gamma_2\) is increasing, so \(\lim_{t \to b} \gamma_2(t)\) exists. If it is finite, \(X(\gamma(t))\) (which depends only on \(\gamma_2(t)\)) has a limit when \(t \to b\), so \(b = +\infty\); but the limit of \(X_2(\gamma(t))\) is positive, which contradicts the fact that \(\lim_{t \to b} \gamma_2(t)\) is finite. Thus, \(\lim_{t \to b} \gamma_2(t) = +\infty\) and for the same reason, \(\lim_{t \to a} \gamma_2(t) = -\infty\).

The mapping

\[\Phi : \mathbb{R}^0 \times (a, b) \to N, \quad (p, t) \mapsto p \cdot \gamma(t)\]

is bijective, and by the inverse function theorem, it is a diffeomorphism. Since \(\mathbb{R}^0\) is the Lie group \(\mathbb{R}\), this is the diffeomorphism announced in the statement of Lemma 5.3. With these coordinates, the metric on \(N\) is

\[g = \alpha_y(y, z)dy^2 + \alpha_z(y, z)dz^2 + \alpha_{yz}dydz.\]

For all \(p \in \mathbb{R}^0\), the curve \(p \circ \gamma\) has unit speed and is orthogonal to \(F_N\), so \(\alpha_z\) is everywhere 1 and \(\alpha_{yz}\) is everywhere 0. Also, the action of \(\mathbb{R}^0\) implies that \(\alpha_y\) depends only on \(z\). Thus, the metric has the desired form. \(\square\)

Lemma 5.4. Consider the mapping \(r : \mathbb{R} \to \mathbb{R}_{>0}\), which gives the ratio of a similitude. Then the image of \(\mathbb{R}\) by \(r\) is discrete.

Proof. If the image of \(\mathbb{R}\) is not discrete, then \(r\) is surjective, and thus \(r\) has a section (because \(\mathbb{R}\) is a Lie group): but then, the image of \(\mathbb{R}^0\) by \(r\) is \(\mathbb{R}\), which contradicts the fact that \(\mathbb{R}^0\) contains only isometries. \(\square\)

Now, each point of \(\tilde{M} = \mathbb{R}^q \times N\) has three coordinates \((x, y, z)\), where \(x \in \mathbb{R}^q\). We will denote by \(g_{\mathbb{R}^q}\) the standard Euclidean metric in \(\mathbb{R}^q\).
Lemma 5.5. Choose any $p \in \pi_1(M)$ with ratio $\lambda \neq 1$. Then:

- In Lemma 5.3, one may require that:
  - $(a, b) = (0, +\infty)$.
  - For all $z \in (0, +\infty)$, $\varphi(\lambda z) = \lambda^{2q+2}\varphi(z)$.
- The mapping $p$ has the form $p(x, y, z) = (p_0(x), p_1(y), p_2(z))$, where $p_0$ is a similarity of $\mathbb{R}^q$ of ratio $\lambda$, $p_1(y) = \lambda^{-q}y$ or $p_1(y) = -\lambda^{-q}y$, and $p_2(z) = \lambda z$.

Proof. Since $\mathcal{T}^0$ is normal in $\mathcal{T}$, the group $\mathcal{T}$ preserves the foliation $\mathcal{F}_N$, so the action of $p$ on $N$ preserves the foliation $\mathcal{F}_N$. Thus, $p$ also preserves $(\mathcal{F}_N)_{\mathcal{T}}$, and we may write $p(x, y, z) = (p_0(x), p_1(y), p_2(z))$. Since $p$ has ratio $\lambda$, we have

$$p^*(g_{\mathbb{R}^q} + \varphi(z)dy^2 + dz^2) = \lambda^2(g_{\mathbb{R}^q} + \varphi(z)dy^2 + dz^2),$$

but also

$$p^*(g_{\mathbb{R}^q} + \varphi(z)dy^2 + dz^2) = p_0^*g_{\mathbb{R}^q} + (p_1'(y))^2(\varphi(p_2(z))dy^2 + (p_2'(z))^2dz^2).$$

Thus, for all $(x, y, z) \in \mathbb{R}^q \times \mathcal{T}^0 \times (a, b)$, we have $(p_1'(y))^2 = \lambda^2\varphi(z)/\varphi(p_2(z))$. In particular, $p_1'(y)$ does not depend on $y$: we will write $p_1'(y) = \mu$. Hence $p_1$ is a similarity of $\mathbb{R}$ of ratio $\mu$; furthermore, $p_0$ is a similarity of $\mathbb{R}^q$ of ratio $\lambda$, and $p_2$ is a similarity of $(a, b)$ of ratio $\lambda$. Since $\Gamma_0$ is normal in $\pi_1(M)$, the mapping $p$ induces a diffeomorphism on $\mathbb{R}^q \times \mathcal{T}^0 / \Gamma_0$, which is compact by Lemma 4.17. Thus the mapping $(x, y) \mapsto (p_1(x), p_2(y))$ is volume-preserving, which implies that $|\lambda^q\mu| = 1$.

The similarity $p_1$ has ratio $\mu \neq 1$, and thus it has a fixed point in $\mathbb{R}$: up to a translation we may assume that this fixed point is 0, thus $p_1(y) = \pm \lambda y$. Since $P$ acts freely on $N$, this implies that $p_2$ has no fixed point, hence $(a, b) \neq \mathbb{R}$. Furthermore, $(a, b)$ cannot have finite length. Thus, $(a, b)$ is a half-line: it is isometric to $(0, +\infty)$. From now on, we will assume $(a, b) = (0, +\infty)$. Hence $p_2(z) = \lambda z$.

Considering again the equality $(p_1'(y))^2 = \lambda^2\varphi(z)/\varphi(p_2(z))$, we deduce that for all $z \in (0, +\infty)$, $\varphi(\lambda z) = (\lambda/\mu)^2\varphi(z)$.

Lemma 5.6. Choose $p \in \pi_1(M)$ which has ratio $\lambda < 1$, where $\lambda$ is maximal (this is made possible Lemma 5.4). Then $\pi_1(M)$ is generated by $\Gamma_0$ and $p$.

Proof. Choose another element $\hat{p} \in \pi_1(M)$, $\hat{p} = (\hat{p}_0, \hat{p}_1, \hat{p}_2)$, with ratio $\hat{\lambda}$. Then there exists $k \in \mathbb{Z}$ such that $\hat{\lambda} = \lambda^k$, and thus for all $z \in (0, +\infty)$, $p_2^{-k}\hat{p}_2(z) = z$. Since $\pi_1(M)$ acts freely on $N$ (by Lemma 4.16), $p_2^{-k}\hat{p}_1$ has no fixed point, so it is a translation, which means that $p^{-k}\hat{p} \in \Gamma_0$.

Finally, apply a linear map in order to assume that $\Gamma_0$ is the lattice $\mathbb{Z}^{q+1}$ in $\mathbb{R}^{q+1}$: then Lemmas 5.5 and 5.6 imply Theorem 1.7.

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References

[Alp87] Roger C. Alperin. An elementary account of Selberg’s lemma. *Enseign. Math. (2)*, 33(3-4):269–273, 1987.

[Asu97] Taro Asuke. Classification of Riemannian flows with transverse similarity structures. *Ann. Fac. Sci. Toulouse Math. (6)*, 6(2):203–227, 1997.

[Aus61] Louis Auslander. Bieberbach’s theorem on space groups and discrete uniform subgroups of Lie groups. II. *Amer. J. Math.*, 83:276–280, 1961.

[Bie11] Ludwig Bieberbach. Über die Bewegungsgruppen der Euklidischen Räume. *Math. Ann.*, 70(3):297–336, 1911.

[BM16] Florin Belgun and Andrei Moroianu. On the irreducibility of locally metric connections. *J. Reine Angew. Math.*, 714:123–150, 2016.

[Car84] Yves Carrière. Flots riemanniens. *Astérisque*, (116):31–52, 1984. Transversal structure of foliations (Toulouse, 1982).

[dR52] Georges de Rham. Sur la reductibilité d’un espace de Riemann. *Comment. Math. Helv.*, 26:328–344, 1952.

[Fri80] David Fried. Closed similarity manifolds. *Comment. Math. Helv.*, 55(4):576–582, 1980.

[FT02] Charles Frances and Cédric Tarquini. Autour du théorème de Ferrand-Obata. *Ann. Global Anal. Geom.*, 21(1):51–62, 2002.

[Gal79] S. Gallot. Équations différentielles caractéristiques de la sphère. *Ann. Sci. École Norm. Sup. (4)*, 12(2):235–267, 1979.

[Ghy91] Étienne Ghys. Flots transversalement affines et tissus feuilletés. *Mém. Soc. Math. France (N.S.)*, (46):123–150, 1991. Analyse globale et physique mathématique (Lyon, 1989).

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol I*. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[MMP] Farid Madani, Andrei Moroianu, and Mihaela Pilca. On Weyl-reducible locally conformally Kähler structures. *arXiv: 1705.10397*.

[MN15a] Vladimir S. Matveev and Yuri Nikolayevsky. A counterexample to Belgun–Moroianu conjecture. *C. R. Math. Acad. Sci. Paris*, 353(5):455–457, 2015.

[MN15b] Vladimir S Matveev and Yuri Nikolayevsky. Locally conformally berwald manifolds and compact quotients of reducible manifolds by homotheties. *arXiv: 1506.08935*, 2015.

[Mol88] Pierre Molino. *Riemannian foliations*, volume 73 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1988.

[Nis92] Toshiyuki Nishimori. A note on the classification of nonsingular flows with transverse similarity structures. *Hokkaido Math. J.*, 21(3):381–393, 1992.

[PR93] Ralf Ponge and Helmut Reckziegel. Twisted products in pseudo-Riemannian geometry. *Geom. Dedicata*, 48(1):15–25, 1993.

[Rag72] M. S. Raghunathan. *Discrete subgroups of Lie groups*. Springer-Verlag, New York-Heidelberg, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.
[Sta89] A. N. Starkov. Ergodic decomposition of flows on homogeneous spaces of finite volume. *Mat. Sb.*, 180(12):1614–1633, 1727, 1989.

[Tar04a] Cédric Tarquini. Feuilletages conformes. *Ann. Inst. Fourier (Grenoble)*, 54(2):453–480, 2004.

[Tar04b] Cédric Tarquini. Feuilletages de type fini compact. *C. R. Math. Acad. Sci. Paris*, 339(3):209–214, 2004.