Client-Server Sessions in Linear Logic

Zesen Qian\(^1\), G. A. Kavvos\(^2\), and Lars Birkedal\(^1\)

\(^1\) Aarhus University, Denmark
\(^2\) University of Bristol, UK

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Abstract

We introduce coexponentials, a new set of modalities for Classical Linear Logic. As duals to exponentials, the coexponentials codify a distributed form of the structural rules of weakening and contraction. This makes them a suitable logical device for encapsulating the pattern of a server receiving requests from an arbitrary number of clients on a single channel. Guided by this intuition we formulate a system of session types based on Classical Linear Logic with coexponentials, which is suited to modelling client-server interactions. We also present a session-typed functional programming language for server-client programming, which we translate to our system of coexponentials.

1 Introduction

The programme of session types Honda et al. (1998); Vasconcelos (2012) aims to formulate behavioural type systems that capture the notion of a session—a structured, concurrent interaction between communicating agents. Very little is usually assumed about these agents: their only shared resource is usually a set of channels through which they can send and receive messages. On the other hand, ever since its inception it has been clear that linear logic Girard (1987) has a deep and mystifying relationship with concurrency. Abramsky (1994) argued that process calculi and linear logic should be in a Curry-Howard correspondence Bellin and Scott (1994). Consequently, one should be able to use formulas of linear logic as types that specify concurrent interactions, thereby constructing a system of session types that is logically motivated. Session types and linear types have recently undergone a swift rapprochement beginning with the work of Caires and Pfenning Caires and Pfenning (2010); Caires et al. (2016).

Despite these advances, the \(\pi\)-calculi that have been developed as process calculi for Linear Logic suffer from dire expressive poverty. The typable processes are free of deadlock and nondeterminism, at the price of being unable to model even benign forms of race. One striking omission is that it is difficult to write down a well-typed process that represents two distinct clients being served by a server listening on a single channel. The goal of the present paper is to introduce a logical device, namely the strong coexponential modalities, that will allow us to give a linear type to this extremely common pattern of concurrent interaction.

1.1 The problem

Caires and Pfenning (2010) proposed a Curry-Howard correspondence in which Intuitionistic Linear Logic is used as a type system for \(\pi\)-calculus Milner et al. (1992). This correspondence allows one to interpret formulas of linear logic as session types, i.e., as specifications of disciplined communication over a named channel. A few years later Wadler (2014) extended this interpretation to Classical Linear Logic (CLL). Wadler’s system, which is called Classical Processes (CP), perfectly corresponds to Girard’s original one-sided sequent system for CLL (1987). Its typing judgments are of the form \(P \vdash \Gamma\), where \(P\) is a \(\pi\)-calculus process, and \(\Gamma\) is a list \(x_1 : A_1, \ldots, x_n : A_n\) of name-session type pairs, with \(A_i\) a formula of Classical Linear Logic. The operational semantics of CP led Wadler to the following interpretation of the connectives.

\[
\begin{align*}
\otimes & \quad \text{output} \\
\& & \quad \text{offer a choice} \\
! & \quad \text{server} \\
\& & \quad \text{offer a choice} \\
? & \quad \text{client}
\end{align*}
\]
We follow a convention by which the multiplicative connectives \( \otimes, \Box \) associate to the right. Thus a type like \( A \otimes B \Box C \) can be read as: output a (channel of type) \( A \), then input a (channel of type) \( B \), and proceed as \( C \). While the interpretation of the first four connectives is intuitive, something seems to have gone awry with the exponentials (Wadler, 2014, §3.4). We claim that the computational behaviour of exponentials in CP does not in fact accommodate what we would think of as client-server interaction.

To begin, we consider the following aspects to be the main characteristics of a client-server architecture (van Steen and Tanenbaum, 2017, §§2.3, 3.4):

(i) There is a server process, which repeatedly provides a service any number of clients.

(ii) There is a pool of client processes, each of which requests the said service.

(iii) There is a unique end point at which the clients may issue their requests to the server.

(iv) The underlying network is inherently unreliable: clients may be served out-of-order, i.e., in a nondeterministic manner.

While Wadler’s interpretation faithfully captures (i) and (iii), it does not immediately enable the representation of (ii). Because of its deterministic behaviour, CP is incapable of modelling (iv).

A CP term \( S \vdash x : !A \) can indeed ‘serve’ sessions of type \( A \) over the channel \( x \). However, the reading of a term \( C \vdash y : ?A \) as a process which behaves as a pool of clients along channel \( y \) is not so crisp. Recall the three rules of ?, namely weakening, dereliction, and contraction. In CP:

\[
\frac{Q \vdash \Gamma}{Q \vdash \Gamma, x : ?A} \quad \frac{Q \vdash \Gamma, y : A}{Q \vdash \Gamma, x : ?A} \quad Q \vdash \Gamma, x ; ?A, y : ?A \quad Q[y/x] \vdash \Gamma, x : ?A
\]

Wadler interprets these rules as client formation. Weakening stands for the empty case of a pool of no clients. Dereliction represents a single client following session \( A \). Finally, contraction enables one to aggregate two client pools together: two sessions that are both of type \(?A\) can be collapsed into one, now communicating along the shared channel \( x \).

We argue that, of those interpretations, only the one for dereliction is tenable. In the case of weakening, we see that at least one process is involved in the premise; hence, the ‘pool’ formed has at least one client in it, albeit one that does not communicate with the server. Likewise, contraction does not aggregate different clients, but different sessions owned by the same client. Beginning with a single process \( P \vdash x : A, y : A \) we can use dereliction twice followed by contraction to obtain \(?w[x], ?w[y], P \vdash w : ?A\). This process will ask for two channels that communicate with session \( A \). Nevertheless, the result is still a single process, and not a pool of clients. Dually, the type \(!A\) merely connotes a shared channel: a non-linearized, non-session channel which is used to spawn an arbitrary number of new sessions, each one of type \( A \) (Caiares and Pfennig, 2010, §3).

More alarmingly, there is no way to combine two distinct processes \( P \vdash z : A \) and \( Q \vdash w : A \) into a single process \( \text{pool}(x : z, P, w, Q) \vdash x : ?A \) communicating along a shared channel. As a remedy, Wadler introduces the \text{Mix} rule:

\[
\text{Mix} \quad \begin{array}{c}
P \vdash \Gamma \\
Q \vdash \Delta
\end{array} \quad \frac{P \mid Q \vdash \Gamma, \Delta}{\vdash \Gamma}
\]

\text{Mix} was used carefully considered for inclusion in Linear Logic, but was ultimately left out (Girard, 1987, §V.4). Informally, it allows two completely independent, non-intercommunicating processes to run ‘in parallel.’ We may then use contraction to merge them into a single client pool:

\[
\frac{P \vdash z : A}{?x[z], P \vdash x : ?A} \quad \frac{Q \vdash w : A}{?y[w], P \vdash y : ?A} \quad \frac{\vdash \Gamma}{\frac{\vdash \Gamma}{?d}} \quad \frac{\vdash \Gamma}{\frac{\vdash \Gamma}{?d}} \quad \text{Mix} \quad \frac{\vdash \Gamma}{\vdash \Gamma}
\]

The operational semantics of the \text{Mix} rule in CP are studied by Atkey et al. (2016). To formulate them correctly one needs also to add the rule

\[
\text{Mix0} \quad \begin{array}{c}
\vdash \\
\text{stop}
\end{array}
\]
Mix₀ has a flavour of inconsistency to it, but is advantageous on two levels. On the technical level, it let
us show that the operational semantics, which adds a reaction \( P \mid Q \rightarrow P' \mid Q \) whenever \( P \rightarrow P' \), is
well-behaved (terminating, deadlock-free, and deterministic). In terms of computational interpretation,
Mix₀ represents a stopped process. This solves the second problem we pointed out above, viz. the
formation of a vacuously empty client pool:

\[
\begin{align*}
\text{stop} \vdash & \cdot \\
\text{stop} \vdash & x : \?A
\end{align*}
\]

Nevertheless, Mix and Mix₀ are unbecoming rules. To begin, they are respectively equivalent to \( \bot \rightarrow 1 \) and \( 1 \rightarrow \bot \), thereby conflating the two units. Moreover, it is well-known (Bellin, 1997, §1.1)
Girard (1987); Abramsky et al. (1996); Wadler (2014); Atkey et al. (2016) that Mix is equivalent to

\[
A \otimes B \rightarrow A \otimes B
\]

where \( C \rightarrow D \equiv C^\perp \otimes D \).

Admitting this implication is unwise. At first glance, (⋆) merely weakens the separation between
these connectives, and hence damages the interpretation of \( \otimes \) as input, and \( \otimes \) as output. However, we
argue that deeper problems lurk just beneath the surface. Abramsky et al. (1996, §3.4.2) describe a
perspective on CLL which reads \( A \otimes B \) as connected concurrency (information necessarily flows between
A and B (Girard, 1987, §V.4)) and \( A \otimes B \) as disjoint concurrency (no information flow between A and B
whatevsoever). The implication (⋆) makes \( \otimes \) a special case of \( \otimes \). Hence, flow between the components
of \( A \otimes B \) is permitted, but not obligatory (Abramsky and Jagadeesan, 1994, §3.2). Thus, (⋆) allows us to
pretend that there is flow of information between two clients.¹

Nevertheless, generating the actual flow of information is seemingly impossible. Using Mix we can
put together two clients \( C_i \vdash c_i : A \), and get a single process \( C_0 \mid C_1 \vdash c_0 : A, c_1 : A \). As the comma
stands for \( \otimes \), we can only cut this with a server \( S \vdash s : A^\perp \otimes A^\perp \). But, by the interpretation of \( \otimes \) as
disjoint concurrency, we see that the two client sessions will be served by disjoint server components. In
other words, the server will not allow information to flow between clients, which does not conform to
our usual conception of a stateful server! To enable this kind of flow, a server must use \( \otimes \). As we cannot
cut a \( \otimes \) (in the server) with another \( \otimes \) (in the client pool), we are compelled to also accept the converse
implication \( A \otimes B \rightarrow A \otimes B \) in order to convert one of the two \( \otimes \)'s to \( \otimes \). This forces \( \otimes = \otimes \), which
inescapably leads to deadlock (Atkey et al., 2016, §4.2).

Requiring \( \otimes = \otimes \), a.k.a. compact closure Barr (1991); Abramsky et al. (1996), is often deemed
necessary for concurrency. In fact, Atkey et al. (2016) argue that this conflation of dual connectives
(\( 1 = \bot \), \( \otimes = \otimes \), and so on) is the source of all concurrency in Linear Logic. The objective of this
paper is to argue that there is another way: we aim to augment the Caires-Pfenning interpretation
of propositions-as-sessions with a certain degree of concurrency without adding Mix. We also wish to
introduce just enough nondeterminism to convincingly model client-server interactions in a style that
satisfies points (i)–(iv).

We shall achieve both of these goals with the introduction of coexponentials.

1.2 Roadmap

First, in §2 we discuss the expression of the usual exponential modalities of linear logic (!?!) as least and
greatest fixed points. This leads us to a different definition of !, which we call the strong exponential. By
taking a ‘multiplicative dual’ of these fixed point expressions, we reach two novel modalities, the strong
coexponentials, for which we write \( \emptyset \) and \( \boxempty \). We refine coexponentials back into a weak form that is similar
to the usual exponentials, and show that they coincide with weak coexponentials in the presence of Mix
and the Binary Cut rule.

Following that, in §3 we introduce a process calculus with strong coexponentials, which we call
CSLL. This new system is in the style of Kokke et al. (2019a), which replaces the one-sided sequents with
hypersequents. It is argued that coexponentials enable a new abstraction, viz. the collection of an
arbitrary number of clients following session A into a client pool, which communicates on a channel that

¹This is evident in the Abramsky-Jagadeesan game semantics for MLL+MIX: a play in \( A \otimes B \) projects to plays for \( A \)
and \( B \), but the Opponent can switch components at will. The fully complete model consists of history-free strategies, so
there can only be non-stateful Opponent-mediated flow of information between \( A \) and \( B \).
follows session ¥A. Conversely, the rules for ¥ express the formation of a server, which can be cut with a client pool to serve it requests.

In §4 we present an extended example which illustrates the computational behaviour of coexponentials, i.e. an implementation of the Compare-and-Set (CAS) synchronization primitive. Our system neatly encapsulates racy yet atomic behaviour implicit in such operations.

In §5 we explore the implications of coexponentials in a session-typed functional language. We extend Wadler’s GV with client-server interactions and translate them to coexponentials in CSLL. We take advantage of the higher-level notation to give several examples that would be tedious to program directly in CSLL.

We survey related work in §6, and make some concluding remarks in §7.

2 Exponentials, fixed points, and coexponentials

2.1 Exponentials as fixed points

The exponential (or ‘of course’) modality of linear logic ! is used to mark a replicable formula. While describing a combinatory presentation of linear logic, Girard and Lafont (1987, §3.2) noticed that !A can be expressed as a fixed point

!A ≡ 1 & A & (!A ⊗ !A)

The three additive conjuncts on the RHS correspond to the three rules of the dual connective ?!, namely weakening, dereliction, and contraction. As &, is a negative connective, the choice of conjunct rests on the ‘user’ of the formula,² who may pick one of the three conjuncts at will.

One may thus be led to believe that, were we to allow fixed points for all functions, we could obtain !A as the fixed point of a functor. Baelde (2012, §2.3) discusses this in the context of a system of higher-order CLL with least and greatest fixed points. Using the functors

\[
F_A(X) \overset{\text{def}}{=} 1 \& A \& (X \otimes X) \\
G_A(X) \overset{\text{def}}{=} \bot \oplus A \oplus (X \otimes X)
\]

one defines

\[
!A \overset{\text{def}}{=} \nu F_A \\
?A \overset{\text{def}}{=} \mu G_A
\]

where μ and ν stand for the least and greatest fixed point respectively. Just by expanding the fixed point rules, one then obtains certain derivable rules. While those for ? are the usual ones—weakening, dereliction, and contraction—the rule for ! is radically different:

\[
\text{StrongExp} \\
\vdash \Gamma, B  \\
\vdash B^\bot, 1  \\
\vdash B^\bot, A  \\
\vdash B^\bot, B \otimes B \\
\vdash \Gamma, !A
\]

As foreshadowed by the use of a greatest fixed point, this rule is coinductive. To prove !A from context Γ one must use it to construct a ‘seed’ value (or ‘invariant’) of type B. Moreover, this value must be discarable (\(\vdash B^\bot, 1\)), derelictable (\(\vdash B^\bot, A\)), and copiable (\(\vdash B^\bot, B \otimes B\)). This is eerily reminiscent of the free commutative comonoids used to build certain categorical models of Linear Logic (Melliès, 2009, §7.2). Because of the arbitrary choice of ‘seed’ type B, the system using this rule does not produce good behaviour under cut elimination: the normal forms do not satisfy the subformula property (Baelde, 2012, §3): not all detours are eliminated. We call the modality introduced by StrongExp the strong exponential.

Baelde shows that the standard ! rule can be derived from StrongExp. But while the strong exponential can simulate the standard exponential, it also enables a host of other computational behaviours under cut elimination. Put simply, the standard exponential ensures uniformity: each dereliction of !A into an A must be reduced to the very same proof of A every time. This makes sense in at least two ways. First, when we embed intuitionistic logic into linear logic through the Girard translation, we expect that in a proof of \((A \rightarrow B)^o \overset{\text{def}}{=} !A^o \rightarrow B^o\) each use of the antecedent !A produces the same proof of A. Second, we know that one way to construct the exponential in many ‘degenerate’ models of linear logic Barr (1991); Mellîès et al. (2018) is through the formula

\[
!A \overset{\text{def}}{=} \&_{n \in \mathbb{N}} A^{\otimes n} / \sim_n
\]

²Also known as external choice. In the language of game semantics, the opponent.
where $A^\otimes n \overset{\text{def}}{=} A \otimes \cdots \otimes A$, and $A^\otimes n / \sim_n$ stands for the equalizer of $A^\otimes n$ under its $n!$ symmetries. 

Decoding the categorical language, this means that we take one $\&$ component for each multiplicity $n$, and each component consists of exactly $n$ copies of the same proof of $A$. In contrast, the $!$ rules derived from their fixed point presentation merely create an infinite tree of occurrences of $A$, and not all of them need be proven in the same way.

### 2.2 Deriving Coexponentials

Both exponentials (qua fixed points) are given by a tree where each fork is marked with a connective ($\otimes$ for $!$, $\&$ for $?$). The leaves of the tree are either marked with $A$, or with the corresponding unit. Turning this process on its head leads to two dual modalities, which we call the coexponentials.

More concretely, we define two functors by dualising the connective that adorns forks. We must not forget to change the units accordingly: we swap $1$ (the unit for $\otimes$) with $\bot$ (the unit for $\&$). Let

$$H_A(X) \overset{\text{def}}{=} 1 \& A \& (X \& X)$$

$$K_A(X) \overset{\text{def}}{=} \bot \& \bot \& 1 \& \bot \& (X \& X)$$

The strong coexponentials are then defined by

$$\nu H_A \quad \mu K_A$$

We define $(\nu A) \overset{\text{def}}{=} \mu H_A$, and vice versa. We obtain the following derived rules.

The rules for $\nu$ are distributed forms of the structural rules, while the $\mu$ rule gives a strong coexponential, analogous to the strong version of $!$ described in the previous section. The corresponding ‘weak’ coexponential is given by replacing the above $\mu$ rule with

$$\vdash \bigotimes \nu \Gamma, A$$

$$\vdash \bigotimes \nu \Gamma, 1$$

$\nu \Gamma$ stands for the context obtained by applying $\nu$ to every formula in $\Gamma$, and $\bigotimes$ folds this context with a tensor. Unfortunately, the presence of this folding operation means that this rule is not well-behaved in proof-theoretic terms.

### 2.3 Exponentials vs. Coexponentials under Mix and Binary Cuts

In fact, we can show that, in the presence of additional rules, (weak) exponentials and (weak) coexponentials are interderivable up to provability. This result provides strong evidence that coexponentials are the right abstraction for our purposes, essentially by showing that Wadler (2014) and others implicitly use them in modelling server-client interactions within CLL.

The requisite rules are Mix, and one of the binary cut or multicut rules:

**BiCut**

$$\Gamma, A, B \vdash \Delta, A^\bot, B^\bot$$

$$\vdash \Gamma, \Delta$$

**MultiCut**

$$\Gamma, A_1, \ldots, A_n \vdash \Delta, A_1^\bot, \ldots, A_n^\bot$$

$$\vdash \Gamma, \Delta$$

**BiCut** cuts two formulas at once, and MultiCut an arbitrary number. These rules were first proposed in the context of Linear Logic by Abramsky (1993b) in the compact setting ($\otimes = \&$). They are logically equivalent, but only the second one satisfies cut elimination (Atkey et al., 2016, §4.2). We recall some folklore facts regarding the interderivability of certain formulas and Mix-like inference rules. Recall that $C \multimap D \overset{\text{def}}{=} C^\bot \bowtie D$. The following statements may be found across the relevant literature Girard (1987); Abramsky et al. (1996); Bellin (1997); Wadler (2014); Atkey et al. (2016).

**Lemma 1.** The following rules are logically interderivable.
(i) The axiom \(1 \rightarrow \bot\) and the Mix0 rule.

(ii) The axiom \(\bot \rightarrow 1\) and the Mix rule.

(iii) The axiom \(A \otimes B \rightarrow A \otimes B\) and the Mix rule.

(iv) The axiom \(A \not\otimes B \rightarrow A \otimes B\) and the BiCut rule.

(v) BiCut and MultiCut.

Moreover, Mix0 is derivable from the axiom rule \(\vdash A \not\otimes, A\) and BiCut.

Armed with this, we can prove that

**Theorem 1.** In CLL with Mix and BiCut, exponentials and coexponentials coincide up to provability. That is: if we replace ? and ! in the rules for the exponentials with \(\not\otimes\) and \(\not\otimes\) respectively, the resultant rule is provable using the coexponential rules, and vice versa.

This result extends to strong exponentials vs. strong coexponentials. The proof is even simpler: under Mix and BiCut we have \(\otimes = \not\otimes\), so \(F_A, H_A\) and \(G_A, K_A\) are pairwise logically equivalent.

### 3 Processes

In the rest of the paper we will argue that the logical observations we made in §2 have an interesting computational interpretation as server-client interaction. To this end we will introduce a process calculus for CLL equipped with a bespoke form of strong coexponentials. Our system shall introduce a certain amount of nondeterminism, yet it will remain Mix-free.

We first explain how the coexponentials capture the intuitive shape of client pool formation (§3.1). Following that, we briefly discuss three technical design decisions that pertain to the coexponentials used in our system (§§3.2–3.4). Finally, we introduce the system in §3.5, and its metatheory in §3.6.

#### 3.1 \(\not\otimes\) means client, \(\not\otimes\) means server

Recall the three rules for \(\not\otimes\), namely

\[
\begin{align*}
\vdash \not\otimes^{\underline{\text{w}}} & & \vdash \Gamma, A \vdash \not\otimes & & \vdash \Gamma, \not\otimes A \vdash \Delta, \not\otimes A \\
\vdash \not\otimes^{\underline{\text{d}}} & & \vdash \Gamma, \not\otimes A & & \vdash \Gamma, \Delta, \not\otimes A \\
\vdash \not\otimes^{\underline{\text{c}}} & & \vdash \Gamma, \not\otimes A & & \vdash \Gamma, \Delta, \not\otimes A
\end{align*}
\]

We can read \(\not\otimes A\) as the session type of a channel shared by a pool of clients.

- \(\not\otimes^{\underline{\text{w}}}\) allows the vacuous formation of a empty client pool.
- \(\not\otimes^{\underline{\text{d}}}\) allows the formation of a client pool consisting of exactly one client.
- \(\not\otimes^{\underline{\text{c}}}\) rule can be used to aggregate two client pools together.

The last point requires some elaboration. Each premise can be seen as a client pool with an external interface (\(\Gamma\) and \(\Delta\) respectively). \(\not\otimes^{\underline{\text{c}}}\) allows us to combine these into a single process. This new process still behaves as a client pool, but it also retains both external interfaces. In contrast, the \(\not\otimes^{\underline{\text{c}}}\) rule only allowed us to collapse two shared channel names belonging to a single process. Moreover, it did not allow us to mix two external interfaces—one had to use Mix for that.

Finally, the ‘weak’ \(\not\otimes\) rule, i.e.,

\[
\begin{align*}
\vdash \not\otimes^{\underline{\text{w}}} A \\
\vdash \not\otimes^{\underline{\text{d}}} \Gamma, A \\
\vdash \not\otimes^{\underline{\text{c}}} \Gamma, \not\otimes A
\end{align*}
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\begin{align*}
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\vdash \not\otimes \not\otimes^{\underline{\text{d}}} \Gamma, \not\otimes A \\
\vdash \not\otimes \not\otimes^{\underline{\text{c}}} \Gamma, \not\otimes A
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\end{align*}
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\vdash \not\otimes \not\otimes^{\underline{\text{c}}} \Gamma, \not\otimes A
\end{align*}
\]
3.2 Design Decision #1: Server State and the Strong Rules

The first change with respect to the above interpretation is the switch to the strong rule:

\[ \vdash \Gamma, B \vdash B^\perp, \perp \vdash B^\perp, A \vdash B^\perp, B \vdash B \vdash B \vdash \Gamma, B \vdash \Gamma, iA \]

This rule strongly evokes the structure of a ‘stateful’ server serving A’s, with external interface Γ. Within
the server there exists an internal server protocol B. This comes with four ingredients: a process that
provides a B, interacting along Γ (initialization); a way to silently consume B (finalization); a way
to ‘convert’ a B to an A (serving a client); and a way to fork one B into two connected B’s (forking two
subservers).

We decide to use this strong rule in order to avoid the uniformity property that was discussed in
§2.1: the weak coexponential rule gives trivial servers providing identical A’s to all clients. In contrast,
this rule will allow a server to provide a different A each time it is called upon to do so.

3.3 Design Decision #2: Replacing Trees with Lists

The strong coexponential rule arose by taking the greatest fixedpoint of

\[ H_A(X) \overset{\text{def}}{=} \bot \& A \& (X \& X) \]

As discussed in §§2.1 and 2.2, this rule represents a tree-like structure. Nothing stops us from replacing
it with a list-like structure.\(^3\) We use the functors

\[ H'_A(X) \overset{\text{def}}{=} \bot \& (A \& X) \]

\[ K'_A(X) \overset{\text{def}}{=} 1 \oplus (A \otimes X) \]

and acquire the strong server rule derived from \( H'_A \), viz.

\[ \vdash \Gamma, B \vdash B^\perp, \perp \vdash B^\perp, A \vdash B \vdash B \vdash \Gamma, iA \]

The main benefit is that the resulting system more closely reflects the pattern of server-client interac-
tion: clients form a queue rather than a tree, and servers no longer have to fork subprocesses. This rule
also requires fewer ingredients: an initialization of the internal protocol, a finalization, and a component
that spawns a session to serve one additional client.

To optimize this further, we make the \( \& \) implicit, and replace \( \perp \) with a general \( \Delta \) in the finalization:

\[ \text{SERVER} \]

\[ \vdash \Gamma, B \vdash B^\perp, \Delta \vdash B^\perp, A, B \]

\[ \vdash \Gamma, \Delta, iA \]

This second rule can be immediately derived from the first one:

\[ \vdash \Gamma, B \vdash B^\perp, \Delta \vdash B^\perp, B \vdash B^\perp, B \vdash B^\perp, B \vdash B^\perp, A \vdash B^\perp, B, B \vdash B^\perp, B, B \vdash B^\perp, B \vdash B^\perp, B \vdash B^\perp, B \vdash B^\perp, A \vdash B^\perp, A \vdash (B \otimes B^\perp) \vdash \Gamma, \Delta, iA \]

There is a surreptitious twist here: the ‘new’ internal server protocol is not B, but \( B \otimes B^\perp \). This leads
to internal back-and-forth communication in the server. Γ is consumed to produce a B. This is ‘passed’
to each process serving each client. Finally, it is reflected back to the initialization process, and ‘finalized’
to an \( \Delta \). The \( \perp \) rule is invertible, so instantiating \( \Delta \overset{\text{def}}{=} \perp \) in \text{SERVER} gives back the preceding rule.

Hence, these two rules are logically equivalent.

\(^3\)It is worth noting that Girard considered list-like exponentials (1987, §V.5(ii)), but rejected them as they were not
able to reproduce contraction. This is not a requirement for modelling client-server interaction.
3.4 Design Decision #3: Nondeterminism Through Permutation

Using list-shaped rules for $j$ forces us to revise the rules for $i_c$. To define a cut elimination procedure they must now match the dual functor $K'_A$, and hence become

$$
\Gamma \vdash j A \\
\frac{\Gamma, i A \vdash \Delta, A}{\Gamma, \Delta, i A}
$$

The cut elimination procedure for these rules leads to a confluent dynamics. This is unsatisfactory from the perspective of client-server interaction: a proper model requires some nondeterminism in the order in which clients are served. There are many ways to introduce this kind of behaviour. We choose the simplest one: we identify derivations up to permutation of client formation in pools. That is, we quotient them under the least congruence $\equiv$ generated from

$$
\frac{\Gamma, i A \vdash \Delta, A}{\Gamma, \Delta, i A} \\
\frac{\Delta, i A \vdash \Sigma, A}{\Gamma, \Delta, \Sigma, i A} \\
\frac{\Gamma, i A \vdash \Sigma, A}{\Gamma, \Delta, \Sigma, i A}
$$

This amounts to quotienting lists up to permutation. Thus, when a client pool interacts with a server, the cut elimination procedure may silently choose to serve any of the constituent clients.

Trees and nondeterminism The careful reader might notice that the original, tree-like ‘distributed contraction’ rule $i_c$ inherently supported a certain amount of nondeterminism: if we were to quotient derivations up to permutation of the premises of $i_c$, then the cut elimination procedure would have some choice of whether to serve the left or right subtree first. Switching to list-like functors forbids this move, and seemingly imposes a much stricter discipline.

Nevertheless, the tree structure is awkward and rigid. For example, consider a client pool whose tree structure is informally $[[c_0, c_1], [c_2, c_3]]$. As nondeterministic choices are only made at each node, the clients cannot be served in any order. For example, if $c_0$ is served first then $c_1$ must be served next, as it is in the same subtree. From a conventional client-server perspective this is arguably not a sufficient amount of nondeterminism. In contrast, our formulation allows full permutations of the client pool.

3.5 Introducing CSLL

Based on the above considerations, we introduce the system CSLL of Client-Server Linear Logic.

Following recent presentation of CLL-based systems of session types Kokke et al. (2019a), CSLL is structured around hyperenvironments. Intuitively, the logical system underlying CSLL is not one-sided sequent calculus like CP, but a hypersequent system Avron (1991). In this kind of presentation process constructors are more finely decoupled. For example, the original CP output/$\otimes$ constructor $x[y].(P | Q)$ has a combination of a parallel composition with an output prefix. Hypersequent systems allow us to separately type to these two constructs, and brings the language closer to $\pi$-calculus.

One-sided sequent systems for CLL—such as Girard’s original presentation (1987)—use sequents of the form $\vdash \Gamma$ where $\Gamma$ is an environment, i.e. an unordered list of formulas. We assign distinct names to each formula. The environment $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ stands for $A_1 \otimes \ldots \otimes A_n$. Hence, a comma stands for $\otimes$. Environments are identical up to permutation. We write $\cdot$ for the empty one.

A hyperenvironment adds another layer: it is an unordered list of environments. We separate environments by vertical lines. If each environment $\Gamma_i$ stands for the formula $A_i$, the hyperenvironment $\mathcal{G} = \Gamma_1 | \cdots | \Gamma_n$ stands for the formula $A_1 \otimes \cdots \otimes A_n$. Hence, $|$ stands for $\otimes$. Hyperenvironments are identical up to permutation, and we write $\emptyset$ for the empty one. We also stipulate that variable names be distinct within and across environments.

The syntax and the type system of CSLL are defined in Fig. 1. The types are the formulas of CLL. Processes have the following binding structure:

- $x$ and $y$ are bound in $P$ within $\nu xy. P$.
- $x$ and $y$ are bound in $P$ within $y(x). P$, $y[x]. P$.
- $x$ is bound in $P$ within $x[\text{in}]. P$, $x[\text{inr}]. P$.
- $x$ is bound in both $P$ and $Q$ within $x.\text{case}(P; Q)$.
\[ \begin{align*}
A, B, \ldots & \quad ::= \quad \text{1} \mid \perp \mid A \otimes B \mid A \oplus B \mid A \& B \mid \varrho A \mid \varpi A \\
\Gamma, \Delta, \ldots & \quad ::= \quad \cdot \mid \Gamma, x : A \\
\mathcal{G}, \mathcal{H}, \ldots & \quad ::= \quad \emptyset \mid \mathcal{G} \mid \Gamma \\
P, Q, \ldots & \quad ::= \quad \text{stop} \\
\quad & \quad | \quad x \leftrightarrow y \\
\quad & \quad | \quad \nu xy. P \\
\quad & \quad | \quad P \mid Q \\
\quad & \quad | \quad y. \text{case}\{ P ; Q \} \\
\quad & \quad | \quad y(x). P \mid y[x]. P \\
\quad & \quad | \quad y(). P \mid y[]. P \\
\quad & \quad | \quad \iota x[]. P \\
\quad & \quad | \quad \iota x[y]. P \\
\quad & \quad | \quad \nu \{z', w', y'. Q\}(z, w). P \\
\quad & \quad | \quad ?x[]. P \mid ?y[y]. P \mid ?x[y_0, y_1]. P \\
\quad & \quad | \quad !x\{\tilde{y}\}. P
\end{align*} \]

\( \text{M-True} \):
\[ P \vdash \mathcal{G} \]
\[ x[]. P \vdash \mathcal{G} \mid x : 1 \]
\( \text{WhyNotW} \):
\[ P \vdash \mathcal{G} \mid \Gamma \]
\[ ?x[]. P \vdash \mathcal{G} \mid \Gamma, x : ?A \]
\( \text{WhyNotD} \):
\[ P \vdash \mathcal{G} \mid \Gamma, y : A \]
\[ ?x[y]. P \vdash \mathcal{G} \mid \Gamma, x : ?A \]
\( \text{WhyNotC} \):
\[ P \vdash \mathcal{G} \mid \Gamma, y_0 : ?A, y_1 : ?A \]
\[ ?x[y_0, y_1]. P \vdash \mathcal{G} \mid \Gamma, x : ?A \]
\( \text{OfCourse} \):
\[ P \vdash \mathcal{G} \]
\[ \lambda x\{\tilde{y}\}. P \vdash \mathcal{G} \mid x : B \]
\( \text{Claro} \):
\[ P \vdash \mathcal{G} \mid \Gamma, i : B \mid \Delta, f : B^\perp \]
\[ Q \vdash z : B^\perp, z' : B, y' : A \]

\[ \nu y(z, z', y'. Q)(i, f). P \vdash \mathcal{G} \mid \Gamma, \Delta, y : ?A \]

\( \text{Cut} \):
\[ \frac{P \vdash \mathcal{G} \mid \Gamma, x : A \mid \Delta, y : A^\perp}{\nu xy. P \vdash \mathcal{G} \mid \Gamma, \Delta} \]

\( \text{Ax} \):
\[ \frac{\nu xy. P \vdash \mathcal{G} \mid \Gamma, \Delta}{x \leftrightarrow y \vdash x : A^\perp, y : A} \]

\( \text{HMix0} \):
\[ P \vdash \mathcal{G} \]
\[ \text{stop} \vdash \emptyset \]

\( \text{HMix2} \):
\[ P \mid Q \vdash \mathcal{G} \mid \mathcal{H} \]

\( \text{Par} \):
\[ P \vdash \mathcal{G} \mid \Gamma, x : A, y : B \]
\[ y(x). P \vdash \mathcal{G} \mid \Gamma, y : A \otimes B \]

\( \text{Tensor} \):
\[ P \vdash \mathcal{G} \mid \Gamma, x : A \mid \Delta, y : B \]
\[ y[x]. P \vdash \mathcal{G} \mid \Gamma, x : A \oplus B \]

\( \text{PlusL} \):
\[ P \vdash \mathcal{G} \mid \Gamma, x : A \]
\[ x[\mathcal{G}]. P \vdash \mathcal{G} \mid \Gamma, x : A \oplus B \]

\( \text{PlusR} \):
\[ Q \vdash \mathcal{G} \mid \Gamma, y : B \]
\[ y[\mathcal{G}]. Q \vdash \mathcal{G} \mid \Gamma, y : A \oplus B \]

\( \text{With} \):
\[ P \vdash \Gamma, x : A \]
\[ Q \vdash \Gamma, y : B \]
\[ x. \text{case}\{ P ; Q \} \vdash \Gamma, x : A \& B \]

\( \text{M-False} \):
\[ P \vdash \mathcal{G} \mid \Gamma \]
\[ \Gamma, x : \perp \vdash x(). P \vdash \mathcal{G} \mid \Gamma, x : \perp \]

Figure 1: The syntax and type system of CSLL.
• Within \( \pi_y \{ z, z', y', Q \} \langle i, f \rangle \). \( P, i \) and \( f \) are bound in \( P \), \( z, z' \), and \( y' \) are bound in \( Q \).

• \( x \) and \( y \) are bound in \( P \) within \( \forall x[y], P \).

As is usual, processes are identified up to \( \alpha \)-equivalence.

In all processes that involve a dot not in curly braces, we call the part that precedes it the \textit{prefix} of the process, and the part that succeeds it the \textit{continuation}. E.g. in the process \( y(x). P \), the prefix is \( y(x) \), and the continuation is \( P \). We write \( \eta_y \) for an arbitrary prefix communicating on channel \( y \), and \( \text{BN}(\eta_y) \) for the variables that it binds in its continuation. For example, we may write \( \eta_y \equiv y(x) \), and \( \text{BN}(\eta_y) = x \). We write \( \text{FN}(P) \) for the free names in a process \( P \).

A generic judgment of the type system has the shape \( P \vdash \mathcal{G} \) where \( P \) is a process, and \( \mathcal{G} \) is a hyperenvironment. Most rules are identical to HCP, and in the interest of brevity we only discuss those that are important. Similarly to HCP, the typing cannot be inferred from the terms alone. For example, in \textbf{M-False} the term \( x(). P \) does not specify the environment \( \Gamma \) on which it acts.

Hyperenvironment components are introduced by the nullary and binary \textit{hypermix} rules, \textsc{HMix0} and \textsc{HMix2}. These rules are ‘Mix’ rules only in name. \textsc{HMix2} forms the disjoint parallel composition of two processes: their environments are joined with \( | \), which stands for \( \otimes \).\(^4\) \textsc{HMix0} is the stopped process; its hyperenvironment is the empty one, which stands for the unit of \( \otimes \), namely \( 1 \).\(^5\)

The \textbf{Cut} and \textbf{Tensor} rules eliminate hyperenvironment components. The premises of \textbf{Cut} ensure that the two variables that are being connected—viz. \( x \) and \( y \)—are in different ‘parallel components’ of \( P \). Notice that the external environments of these two components, namely \( \Gamma \) and \( \Delta \), are then brought together in the conclusion. A similar pattern permeates the \textbf{Tensor} and \textbf{M-True} rules. It is instructive to follow the derivation of the original CP rules for \( \otimes \) and \( 1 \), which we will silently use:

\[
\begin{align*}
\frac{P \vdash \Gamma, y : A \quad Q \vdash \Delta, x : B}{P \mid Q \vdash \Gamma, y : A \mid \Delta, x : B} \\
\frac{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}{\text{stop} \vdash \emptyset} \\
\end{align*}
\]

The exponential rules \textbf{WhyNotW}, \textbf{WhyNotD}, \textbf{WhyNotC} and \textbf{OfCourse} are formulated in the style of Kokke et al. (2019a). In \textbf{OfCourse} we use vector notation \( \langle \cdot \rangle \) as a shorthand for lists of names and types. Note that—in contrast to all previous systems—we notate \( P \) as a parameter rather than as the continuation in the process \( !x[\langle y, P \rangle] \). This because \( P \) does not behave like a continuation. For example, it has its own distinct commuting conversion.

The coexponential rules \textbf{QueW}, \textbf{QueA} and \textbf{Claro} follow the patterns described in §§3.1–3.4. The rule \textbf{QueW} (\( W \) stands for ‘weaken’) constructs an empty client pool. The rule \textbf{QueA} (\( A \) stands for ‘absorb’) combines a client and a pool into a slightly larger pool. The interfaces of the client pool and the client are necessarily disjoint, as they are separated by \( | \) in the premise. All the processes in the resultant pool race to communicate with a server at the single endpoint \( x \).

Correspondingly, \textbf{Claro} constructs a process that offers a service at the single endpoint \( y \). Its continuation \( P \) functions as both the initialization and the finalization of the server, over channels \( i \) and \( f \) respectively. In terms of the \textbf{Server} rule of §3.3, it combines the premises \( \vdash \Gamma, B \) and \( \vdash B^+, \Delta \) into one process. However, these functionalities continue to be disjoint components of \( P \), as their interfaces are separated by \( | \) in the premise. The process \( Q \) is a ‘worker process’ which is spawned every time a client is to be served.

### 3.6 Operational Semantics and Metatheory

**Definition 1.** Canonical terms are defined by the following clauses.

- \( \pi_x.P \) is canonical whenever \( P \) is.
- \( P \mid Q \) is canonical if both \( P \) and \( Q \) are canonical.
- \( \text{stop} \) and \( x \leftrightarrow y \) are canonical.
- \( y.\text{case}\{ P; Q \} \) and \( !x[\langle y, P \rangle] \) are canonical.

\(^4\text{Mix} \) would join them with a comma, which would stand for a \( \otimes \).

\(^5\text{Mix0} \) would stand for the unit of \( \otimes \), namely \( 1 \).
\[ P \mid \text{stop} \equiv P \]  
\[ P \mid Q \equiv Q \mid P \]  
\[ P \mid (Q \mid R) \equiv (P \mid Q) \mid R \]  
\[ x \leftrightarrow y \equiv y \leftrightarrow x \]  
\[ \nu xy. (P \mid Q) \equiv \nu xy. P \mid \nu xy. Q \ (x, y \not\in \text{FN}(P)) \]  
\[ \nu xy. vzw. P \equiv vzw. \nu xy. P \]  
\[ \pi_x. (P \mid Q) \equiv \pi_x. P \mid \pi_x. Q \ (\text{BN}(\pi_x) \cap \text{FN}(P) = \emptyset) \]  
\[ \nu xy. \pi_z. P \equiv \pi_z. \nu xy. P \ (z \neq x, y) \]  
\[ \pi_x. \pi_y. P \equiv \pi_y. \pi_x. P \ (y \not\in \text{BN}(\pi_x) \text{ and } x \not\in \text{BN}(\pi_y)) \]

In particular, \( \nu xy. P \) is not canonical; it is a cut.

The above notion of canonicity is not definitive. For example, \( \pi_x. P \) could have been considered canonical regardless of the canonicity of \( P \) (similar to weak head normal form for \( \lambda \)-calculus). However, we choose to react \( P \) further to make the ‘final result’ of an interaction visible in later examples. In addition, we could require terms such as \( P \) and \( Q \) in \( \text{y.case}\{P;Q\} \) be canonical for the whole term to be canonical, but we choose not to so as to reduce the number of reaction rules.

We define the notion of \emph{structural equivalence} \( P \equiv Q \) to be the least congruence between processes induced by the clauses in Fig. 2. Furthermore, we define the \emph{reaction relation} \( P \rightarrow Q \) between processes to be the least relation induced by the clauses in Fig. 3.

The structural equivalence and the reaction semantics largely mirror the notions of the same name in the \( \pi \)-calculus Milner (1992, 1999). Those that differ are justified \emph{via} linear logic. \text{Res-Pre} and \text{Pre-Pre} can be seen as identifications arising from \emph{proof nets}, in which the corresponding proofs would be graphically identical. \text{OfCourse-Comm} and \text{With-Comm} are commuting conversions for \text{OfCourse} and \text{With} respectively. \text{Pre-Par} with \text{Res-Pre} combine into a kind of commuting conversion for prefixes. We take the former as reaction rules, and the latter as structural equivalences. The reason is that we wish to make equivalence preserve canonicity. For example, in \text{With-Comm} the LHS is not canonical, but the RHS is.

The overwhelming majority of these commuting conversions is used in previous works on the relationship between linear logic and \( \pi \)-calculus to obtain cut elimination (Wadler, 2014, §3.6) (Bellin and Scott, 1994, §3). Perhaps the only exception is \text{Pre-Pre}, which allows us to swap any two noninterfering prefixes. It can be justified computationally as an observational equivalence arising from the semantics of Atkey (2017, §5). Finally, Kokke et al. (2019a) view it as a session-theoretic version of \emph{delayed actions} Merro and Sangiorgi (2004).

The structural equivalence \text{Que-Que} allows us to commute the position of two clients in the pool, thereby imitating racing—as discussed in §3.4. Note that to fully exploit the nondeterminism induced by \text{Que-Que} the other structural equivalences are necessary. For example, the two clients in \( \iota[x_0]. y(y'). \iota[x_1]. P \) cannot be permuted without using \text{Pre-Pre} first.

\text{Pre} corresponds to eliminating non-top-level cuts in Linear Logic; it is not standard in either \( \pi \)-calculus or CP. Nevertheless, we choose to include it in order to strengthen our notion of canonical form, which in turn elucidates the examples in §4. In contrast, the reaction rules for the exponentials are standard; see Kokke et al. (2019a).

The rule \text{Claro-QueW} corresponds to serving an empty client pool. In this case we simply ‘short-circuit’ the initialization and finalization channels of \( P \). Likewise, the rule \text{Claro-QueA} is the reaction caused by a nonempty pool of clients. The pool offers a fresh channel \( x' \) on which the new client expects to be served. The server then spawns a worker process \( Q \), and the channel \( y' \) on which it will serve the new client is connected to \( x' \), as expected. The initialization channel \( i \) of the server continuation is connected to the \( z \) channel, on which the worker process expects to receive the ‘current state’ of the server. Once \( Q \) serves the client, it will send the ‘next state’ of the server on \( z' \). Thus, we re-instantiate

Figure 2: The structural equivalence of CSLL processes.
the server with $z'$ as the new initialization channel. Note that the ‘server state’ we discuss here does not conform to the usual intuition of an immutable value; it could be a session type itself, as demonstrated by the example in §5.5.

We have the following metatheoretic results.

**Lemma 2.** If $P \equiv Q$, then $P \vdash \mathcal{G}$ if and only if $Q \vdash \mathcal{G}$.

**Theorem 2** (Preservation). If $P \vdash \mathcal{G}$ and $P \rightarrow Q$, then $Q \vdash \mathcal{G}$.

**Theorem 3** (Progress). If $R \vdash \mathcal{G}$ then either $R$ is canonical, or there exists $R'$ such that $R \rightarrow R'$.

## 4 An example: Compare-and-Set

We now wish to demonstrate the client-server features of CSLL. To do so we produce an implementation of the quintessential example of a synchronization primitive, the *Compare-and-Set operation* (CAS) (Herlihy and Shavit, 2012, §5.8). Higher-level examples are given in §5.

A register that supports compare-and-set comes with an operation $\text{CAS}(e, d)$ which takes two values: the *expected* value $e$, and the *desirable* value $d$. The function compares the expected value $e$ with the register. If the two differ, the value of the register remains put, and $\text{CAS}(e, d)$ returns false. But if they are found equal, the register is updated with the desirable value $d$, and $\text{CAS}(e, d)$ returns true. When multiple clients are trying to perform CAS operations on the same register, these must be performed in a *wait-free* manner.

We follow previous work Girard (1987); Abramsky (1993a); Atkey et al. (2016); Kokke et al. (2019a) and define the type of Boolean sessions to be $\mathbf{2} \triangleleft \mathbf{1} \oplus \mathbf{1}$. We have the following derivable constants:

$$tt_z \triangleq z[\text{inl}], \text{stop} \vdash z : \mathbf{2}$$

$$ff_z \triangleq z[\text{inr}], \text{stop} \vdash z : \mathbf{2}$$

Moreover, we obtain the following derivable ‘elimination’ rule (we write derivable rules in blue):

$$\frac{P \vdash \Gamma}{\text{if}(z; P; Q) \triangleq z[\text{case}(z(.); P; z(.); Q)] \vdash z : \mathbf{2}_\bot, \Gamma}$$

Hence, we can eliminate a Boolean channel in any environment $\Gamma$. The induced reactions are:

$$\nu xy. (tt_x | \text{if}(y; P; Q)) \rightarrow^* \text{stop} | P \equiv P$$

$$\nu xy. (ff_x | \text{if}(y; P; Q)) \rightarrow^* \text{stop} | Q \equiv P$$

**Figure 3:** The operational semantics of CSLL processes.
We can now implement a register with a CAS operation. To begin, each client communicates with the register along a channel of type
\[ A \equiv \mathbb{2} \otimes \mathbb{2}^\perp \otimes \mathbb{1} \]
Thus, a client outputs three channels. On the first two it shall receive the expected and desirable values. On the third it will input a boolean, namely the success flag of the CAS operation. Following that, it will accept an end-of-session signal. Curiously, this last step is necessary for our implementation to type-check.

As a minimal example we will construct a pool of two racing clients, one performing CAS(tt, tt), and the other one CAS(tt, ff). Initially \( x_1 \) is ahead in the client pool.

\[
\begin{align*}
C_0 & \overset{\text{def}}{=} x_0[x_1], x_0[x_2], (tt_{x_1} | \tt_{x_2} | x_0 \leftrightarrow r_0) \vdash x_0: 2 \otimes 2^\perp \otimes 1, r_0: 2 \otimes \perp \\
C_1 & \overset{\text{def}}{=} x_1[x_1], x_1[x_2], (tt_{x_1} | \tt_{x_2} | x_1 \leftrightarrow r_1) \vdash x_1: 2 \otimes 2^\perp \otimes 1, r_1: 2 \otimes \perp
\end{align*}
\]

Note that each client forwards the result it receives to an individual channel \( r_i \). By the \textsc{Quea} rule these two channels are preserved in the final interface of the pool.

Next we define the CAS register process, for which we use the \( \iota \) connective. This requires two components: the initialization and finalization process \( P \), and the worker process \( Q \) that serves one client. To begin, we pick the internal server state to be \( B \overset{\text{def}}{=} 2 \). We initialize the register to false, and forward the final state of the register to \( u \).

\[
P \overset{\text{def}}{=} (\text{ff} | f \leftrightarrow u) \vdash i : 2 \mid f : 2^\perp, u : 2
\]

Finally, we define \( Q \). We begin by receiving the input and output channels from a client, and do a case analysis on the current state of the register:

\[
Q \overset{\text{def}}{=} ^y(y_e), ^y(y_d), \text{if}(z ; R_1 ; R_0) \vdash z : 2^\perp, y' : 2^\perp \otimes 2^\perp \otimes \perp, z' : 2
\]

We have carefully named the channels so that \( y_e : 2^\perp \) and \( y_d : 2^\perp \) carry the expected and desirable values. \( z' \) and \( w' \) carry the internal register, before and after the operation. The continuations \( R_0 \) and \( R_1 \) do a case analysis on the expected and desired value:

\[
\begin{align*}
R_1 & \overset{\text{def}}{=} \text{if}(z ; \text{if}(y_d ; S_{111} ; S_{110}); \text{if}(y_d ; S_{101}; S_{100})) \vdash y_e : 2^\perp, y_d : 2^\perp, y' : 2^\perp \otimes \perp, z' : 2 \\
R_0 & \overset{\text{def}}{=} \text{if}(y_e ; \text{if}(y_d ; S_{011} ; S_{010}); \text{if}(y_d ; S_{001}; S_{000})) \vdash y_e : 2^\perp, y_d : 2^\perp, y' : 2^\perp \otimes \perp, z' : 2
\end{align*}
\]

Two further case analyses lead to an exhaustive eight cases, each of which is handled by a separate process \( S_{ijkl} \). We only give \( S_{110} \) here, the rest being analogous:

\[
S_{110} \overset{\text{def}}{=} ^y(y_e), (\tt_{y_e} | y() . \tt_{z_1} \vdash y' : 2^\perp \otimes \perp, z' : 2
\]

In this case, the expected value (true) matches the register state (true), so the process outputs true to the race channel \( y_e \) (the CAS operation succeeds), and the register is set to the desired value (false). We must not forget to receive an end-of-session signal on \( y \), as required by the session type. We let \( \text{server} \overset{\text{def}}{=} y(y_e, z', y'.Q)(i, f) \). \( P \vdash y : (2^\perp \otimes 2^\perp \otimes 2^\perp \otimes \perp), u : 2 \), and cut:

\[
\begin{align*}
& \nu x. y. (\text{clients} | \text{server}) \\
& \equiv \nu x. y. (\iota[x_1], \iota[x_0], \iota[z_1], \iota[z_2]) . (C_0 | C_1) | \text{server} \\
& \to \nu x. y. x_0.y'. (C_0 | \iota[x_1], \iota[z_1], \iota[z_2]) . \nu x . y_1 . Q \{ z', f \}. \nu y . (P | Q) \quad (C_0 \text{ is accepted}) \\
& \to \nu x . y_1 . (\tt_{y_1} | r_1() . \nu x . y_1.y' \nu x . y_1 . (C_1 | \iota[x_1]) \cdot \text{stop} \cdot \nu y . (z', y', Q)(z', f). (\nu y' . z . (P' | Q)) \quad (C_1 \text{ is accepted}) \\
& \to \nu x . y_1 . (\tt_{y_1} | r_1() . \nu x . y_1.y' \nu x . y_1 . (C_1 | \iota[x_1]) \cdot \text{stop} \cdot \iota[y . (z', y', Q)(z', f). (\nu y' . z . (P' | Q))) \quad \text{(server starts to finalize)} \\
& \to \nu x . y_1 . (\tt_{y_1} | r_1() . \nu x . y_1.y_1 | r_1() . (\nu z . y' . f . P')) \quad \text{(C1 performs CAS)} \\
& \to \nu x . y_1 . (\tt_{y_1} | r_1() . \nu x . y_1.y_1 | r_1() . (\nu z . y' . f . P')) \quad \text{(server finalizes)} 
\end{align*}
\]

where \( P' = \text{tt}_{x_1} \mid f \leftrightarrow u \) and \( P'' = \text{ff}_{x_1} \mid f \leftrightarrow u \). This corresponds to the scenario where \( C_0 \) wins the firs race, and hence the CAS operation of both clients succeeds. There is another reaction sequence: if \( C_1 \) wins the first race, we end up with \( r_1(y_e); (\text{ff}_{y_e} | r_1() . r_0(y_e); (\tt_{y_e} | r_0() . \tt_u)) \).
The coexponentials play a central rôle here: ¡ is used to represent the fact that this register provides a server session at a unique end point, and ¿ is used to collect requests for a CAS operation to this single end point. We see that every feature of client-server interaction, as described in points (i)–(iv) of §1.1, is modelled.

The fact we are able to implement a synchronization primitive like CAS shows that the server-client rules also provide an additional safeguard, namely that server acceptance is atomic. While the actual CAS is not an atomic operation—as many things are happening in parallel—the causal flow of information ensures that the state implicitly remains atomic. To illustrate the type of atomicity we have, consider an alternative reaction sequence where the two clients are immediately accepted before any other reaction. Fig. 4 shows the process topology of the scenario where $C_0$ is accepted immediately before $C_1$. Each client is connected to the one of the two worker processes $Q$ with client protocol $A$, and the worker processes are connected to each other and $P$ with internal server protocol $B$. Which specific worker process a client connects to is determined by the client’s position in the queue, before the coexponential reaction Claro-QueA takes place. The clients’ positions in the layout also determine the final result of the reaction up to structural equivalence, even before the computation of the output takes place.

5 A session-typed language for server-client programming

As the example of the previous section shows, CSLL is a particularly low-level language. This is a feature of essentially all variants of linear logic as used for session typing, including Kokke et al.’s HCP (2019a, Example 2.1), and Wadler’s CP (Atkey, 2017, §2.1) (Atkey et al., 2016, §3.1). Consequently, the need arises for higher-level notation to help us write richer examples that illustrate the degree of channel sharing in CSLL. We follow the lead of Wadler (2014, §4) and introduce a higher-level, session-typed functional language, which we call CSGV.

CSGV is a linear $\lambda$-calculus augmented with session types and communication primitives. It is based on the influential work of Gay and Vasconcelos (2010). Over the past decade many variations of this language have been proposed; see e.g. Lindley and Morris (2015, 2016, 2017) and Fowler et al. (2019). CSGV extends the version given by Wadler with primitives for client-server interaction. Like the approach in loc. cit. we do not directly endow CSGV with a semantics. Instead, we formulate a type-preserving translation into CSLL, which indirectly provides an execution mechanism. Naturally, the client-server primitives translate to the coexponential rules of CSLL.

5.1 Source Language and the translation

Types The types of CSGV consist of standard functional types and session types. While the former are used to classify values, the latter are used to describe the behaviour of channels. Compared to Wadler (2014) we have added sum types, and session types for server-client shared channels.

\[
T, \ldots := T \rightarrow T \mid T + T \mid T \otimes T \mid \text{Unit} \mid T_S
\]
\[
T_S, \ldots := !T.T_S \quad \text{(output value of type } T, \text{ then behave as } T_S)
\]
\[
\text{?}T.T_S \quad \text{(input value of type } T, \text{ then behave as } T_S)
\]
\[
T_S \oplus T_S \quad \text{(select from options)}
\]
\[
T_S \& T_S \quad \text{(offer choice)}
\]
\[
\text{end} \mid \text{end} \quad \text{(end-of-session)}
\]
\[
\text{i}T_S \quad \text{(request } T_S \text{ session)}
\]
\[
\text{e}T_S \quad \text{(serve } T_S \text{ session)}
\]
Both the functional types and the session types of CSGV are translated to the linear types of CSLL. The functional part closely follows Wadler in using the ‘call-by-value’ embedding of intuitionistic logic into linear logic Benton and Wadler (1996); Maraist et al. (1995, 1999). The session types are translated as follows:

\[
\begin{align*}
[T.T_S] & \overset{\text{def}}{=} [T] \perp \otimes [T_S] \\
[T.S \& U_L] & \overset{\text{def}}{=} [T.S] \oplus [U_L] \\
[T.S \oplus U_L] & \overset{\text{def}}{=} [T_S] \& [U_L] \\
iT_S & \overset{\text{def}}{=} i[T_S] \\
jT_S & \overset{\text{def}}{=} j[T_S]
\end{align*}
\]

As noted by Wadler (2014, §4.1), the connectives translate to the dual of what one might expect. The reason is that channels are used in the opposite way. Consider the session type !T.S: sending a value in CSGV is translated as inputting a channel on which you can send it in CSLL. Similarly, providing a service |S is interpreting as inputting a channel on which the result will be served.

Duality

We define duality on session types in the standard way; it is obviously an involution.

\[
\begin{align*}
\overline{!T.S} & \overset{\text{def}}{=} ?T.S \\
\overline{T.S \& U_L} & \overset{\text{def}}{=} \overline{T.S} \oplus \overline{U_L} \\
\overline{T.S \oplus U_L} & \overset{\text{def}}{=} \overline{T.S} \& \overline{U_L} \\
\overline{iT_S} & \overset{\text{def}}{=} \overline{iT_S} \\
\overline{jT_S} & \overset{\text{def}}{=} \overline{jT_S}
\end{align*}
\]

The translation is a homomorphism of involutions:

**Lemma 3.** \([\overline{T.S}] = [\overline{T_S}]^+\).

Thus, connecting channels in CSGV will be translated to cuts in linear logic.

**Definition 2.** The set of unlimited types is defined inductively as follows.

- unit and \(T \rightarrow U\) are unlimited.
- \(T + U\) and \(T \otimes U\) are unlimited whenever \(T\) and \(U\) are.
- All other types are linear.

Values of unlimited types can be discarded and duplicated, because they are translated to CSLL types that admit weakening and contraction. Categorical considerations (Melliès, 2009, §6.5) lead us to consider \(T \otimes U\) unlimited whenever \(T\) and \(U\) are, which is finer-grained than loc. cit.

**Terms**

CSGV is a linear \(\lambda\)-calculus, extended with constructs for sending and receiving messages.

\[
L, M, N ::= x \mid \ast \mid \lambda x. N \mid MN \mid (M, N) \mid \text{let } (x, y) = M \text{ in } N \\
| \text{inl } M \mid \text{inr } M \mid \text{match } L \text{ with } x.\{M, N\} \\
| \text{send } M \mid \text{rec } M \mid \text{receive } M \\
| \text{select}_L \mid \text{select}_R \mid \text{case } L \text{ of } x.\{M, N\} \\
| \text{terminate } M \\
| \text{connect}(x. M; y. N) \\
| \text{end client pool} \\
| \text{link } x'. M \\
| \text{extract client interface} \\
| \text{serve } y\{L, z. M, f. N\} \\
\]

**Typing rules**

The environments of CSGV are given by \(\Gamma, \ldots ::= \cdot \mid \Gamma, x : T\). The translation of types is extended to environments pointwise.

Selected typing rules of CSGV are given in Figs. 5 and 6. Most rules follow Wadler (2014, §4.1) to the letter, and are therefore omitted. In the interest of economy we also give the translation to CSLL at the same time. The translation is defined by induction on the typing derivations of CSGV. As the purpose of a CSGV program is the computation of a value of a distinguished type, the translation must privilege a single name over which this value will be returned. Thus, given a choice of name \(z\) and a typing derivation \(\Gamma \vdash M : T\), we write \([\Gamma \vdash M : T]_z\) for its translation into CSLL. Somewhat abusively
\[
\begin{align*}
\text{RECV} & \quad \frac{\Gamma \vdash M : T \cdot T_S}{\Gamma \vdash \text{recv} M : T \otimes T_S} \\
& \quad \frac{}{\Gamma \vdash \text{recv} M : T \otimes T_S} \quad \text{def} \quad [M], \vdash \exists \ell [F]^{\perp}, z : [T] \otimes [T_S]
\end{align*}
\]

\[
\begin{align*}
\text{SEND} & \quad \frac{\Gamma \vdash \Delta : N : T \cdot T_S}{\Gamma, \Delta \vdash \text{send} M N : T_S} \\
& \quad \frac{}{\Gamma, \Delta \vdash \text{send} M N : T_S} \quad \text{def} \quad [M], \vdash \exists \ell x : [F]^{\perp}, x : [T_S] \otimes [N], \vdash \exists \ell x, z : [T]^{\perp} \otimes [T_S]
\end{align*}
\]

\[
\begin{align*}
\text{CONN} & \quad \frac{\Gamma, x : T_S \vdash M : \text{end}_1, \Delta, y : T_S \vdash N : T}{\Gamma, \Delta \vdash \text{connect}(x, \text{M}; y, \text{N}) : T} \\
& \quad \frac{}{\Gamma, \Delta \vdash \text{connect}(x, M; y, N) : T} \quad \text{def} \quad [M], \vdash \exists \ell x, y : [F]^{\perp}, x : [T_S]^{\perp}, y : [T_S], z : [T]
\end{align*}
\]

\[
\begin{align*}
\text{REQW} & \quad \frac{\Gamma \vdash \text{eof}_z : \text{end}_1}{x : \exists \ell T_S \vdash \text{eof}_x : \text{end}_1} \\
& \quad \frac{}{\Gamma \vdash \text{eof}_x : \text{end}_1} \quad \text{def} \quad \text{stop} \vdash \emptyset \quad \text{stop} \vdash x \exists \ell T_S^{\perp} \\
& \quad \frac{}{\text{stop} \vdash x \exists \ell T_S^{\perp}} \quad \text{z}() \quad \text{stop} \vdash x \exists \ell T_S^{\perp}, z : \perp
\end{align*}
\]

\[
\begin{align*}
\text{REQA} & \quad \frac{\Gamma, x : \exists \ell T_S \vdash M : \text{end}_1, \Gamma, x : \exists \ell T_S \vdash \text{fork}_x, x' : M : \exists \ell T_S}{\Gamma, x : \exists \ell T_S \vdash M : \text{end}_1} \\
& \quad \frac{}{\Gamma, x : \exists \ell T_S \vdash \text{fork}_x, x' : M : \exists \ell T_S} \quad \text{def} \quad [M], \vdash \exists \ell x, \perp : [F]^{\perp}, x : [T_S]^{\perp}, \perp : [T_S]
\end{align*}
\]

\[
\begin{align*}
\text{SERV} & \quad \frac{\Delta \vdash \Delta, \Sigma, y : T_S \vdash M : T, \Sigma, f : T \vdash N : U}{\Delta, \Sigma, y : T_S \vdash \text{serve}(L, z, M, f, N) : U} \\
& \quad \frac{}{\Delta, \Sigma, y : T_S \vdash \text{serve}(L, z, M, f, N) : U} \quad \text{def} \quad [M], \vdash \exists \ell z, y, [F]^{\perp}, z : [T_S]^{\perp}, y : [T_S]^{\perp}, z', [T]
\end{align*}
\]

\[
\begin{align*}
\text{OBJ} & \quad \frac{[L], \vdash \exists \ell \Delta, x' : [F]^{\perp}, x : [T]^{\perp}, \perp, \perp : [U]}{[L], \vdash \exists \ell \Delta, x' : [F]^{\perp}, x : [T]^{\perp}, \perp, \perp : [U]} \\
& \quad \frac{[L], \vdash \exists \ell \Delta, x' : [F]^{\perp}, x : [T]^{\perp}, \perp, \perp : [U]}{[L], \vdash \exists \ell \Delta, x' : [F]^{\perp}, x : [T]^{\perp}, \perp, \perp : [U]} \quad \text{def} \quad [M], \vdash \exists \ell z, y, [F]^{\perp}, z : [T_S]^{\perp}, y : [T_S]^{\perp}, z', [T]
\end{align*}
\]

Figure 5: CSGV Typing Rules and Translation to CSLL: linear session part

Figure 6: CSGV Typing Rules and Translation to CSLL: shared session part
we will sometimes also write \([M] \vdash \Gamma \vdash \Delta, z : [T] \) for the translated term. This slight abuse of notation also reveals the intended typing.

The novelty here is in the CSGV rules for client-server interaction, and their translation into CSLL. A name of shared client type \(\forall T S \) can be seen as a form of ‘capability’ for talking to the server. \(\texttt{ReqW} \) discards this capability, signalling the end of the client pool. \(\texttt{ReqA} \) uses it to spawn a fresh channel \(x'\) on which a client \(M \) will talk to a server, and returns the capability back to the caller. The client \(M \) itself has type end; it does not return valuable information, but uses values and channels found in \(\Gamma \).

Dually, \(\forall T S \) is the type of a server channel. \texttt{Serv} constructs a server from three components. \(L \) computes the initial state of the server. Given the current state in \(z\), and a client channel \(y\), \(M \) serves the client listening on \(y\), and then returns the next state of the server. \(N \) finalizes the server. Note that the so-called server ‘state’ here could well be a channel itself, enabling bidirectional interleaving communication—a design we will explore in §5.5.

The \texttt{Serv} typing rule is quite restrictive, in that it does not allow anything from the environments \(\Delta \) and \(\Sigma \) to be used in the term \(M \) which computes the next state of the server. Fortunately, the following derivative rule allows us to weave some non-linear values of types \(\bar{V} \) in the server.

\[
\begin{align*}
\Gamma \vdash L : T & \quad \bar{v} : \bar{V}, z : T, y : T S \vdash M : T & \quad \Sigma, f : T \vdash N : U \quad \bar{V} \text{ unlimited} \\
\bar{v} : \bar{V}, \Delta, \Sigma, y : \forall T S \vdash & \texttt{serve} y\{(\bar{v}, L), \bar{z}', \bar{f}'\} : \bar{V} \quad \text{serve} y\{(\bar{v}, M), \bar{f}'\} : \bar{V} \in N \quad : U \\
& \texttt{serve}' y\{L, z, M, f, N\} \\
\end{align*}
\]

We will make crucial use of this derivable rule in a couple of our examples. We also also adopt the common shorthands let \(x = M \) in \(N = (x, N)M \) and let \(x = M \) in \(N = (x, N)M \) for fresh \(z : \star \).

### 5.2 Functional Data Structure Server

Our primitives can be used to protect a shared functional data structure. Without loss of generality, we consider a server whose state is a purely functional queue \(T \) with operations

\[
\begin{align*}
\texttt{enq} : T \otimes A & \rightarrow T & \texttt{deq} : T & \rightarrow T \otimes (Unit + A) & \texttt{empty} : T
\end{align*}
\]

In particular, \texttt{deq} could return \texttt{Unit} if the queue is empty. The server will talk to a client via a channel of type \(T S \equiv (?A.end_{T}) \& (!\texttt{Unit} + A).\texttt{end} \) and \(!\texttt{Unit} + A) \). One client receives an \(A\) along \(r_{0}\), and enqueues it. The other one dequeues an element, and sends it along \(r_{1}\).

\[
\begin{align*}
L & \equiv \text{empty} \\
M_{\text{enq}} & \equiv \text{let } (v, y') = \texttt{recv } y' \text{ in} \\
& \quad \text{let } _{\text{deq}} = \texttt{terminate } y' \text{ in } \texttt{enq}(z, v) \\
M_{\text{deq}} & \equiv \text{let } (v, z') = \texttt{deq } z \text{ in} \\
& \quad \text{let } _{\text{deq}} = \texttt{terminate (send } v y' \text{) in } z' \\
M & \equiv \text{case } y' \{M_{\text{enq}}, M_{\text{deq}}\}
\end{align*}
\]

We then define server \( \equiv \texttt{serve } y\{L, z, M, f, f\} \), and see that

\[
\begin{align*}
\texttt{r}_0 : ?A\texttt{.end} & , \texttt{r}_1 : !(\texttt{Unit} + A) . \texttt{end} \vdash \texttt{connect}(x . \text{clients}; y . \text{server}) : T
\end{align*}
\]

### 5.3 Nondeterminism

Unsurprisingly, the races in our system suffice to implement nondeterministic choice. We define \(B \equiv \texttt{Unit} + \texttt{Unit} \). We implement \(\texttt{tt} \) and \(\texttt{ff} \) by the obvious injections, and the conditional by

\[
\Gamma, \Delta \vdash B : B \quad \Delta \vdash M : V \quad \Delta \vdash N : V \quad +E
\]

The clients \(C_{0}, C_{1} \) respectively send \(\texttt{ff} \) and \(\texttt{tt} \) over a channel. We also define a server with a pair of Booleans as internal state. The first component records whether the server has ever received a value. When a value is received it is stored in the second component, and any further values received are discarded.
$C_0 \equiv \text{send } ff \ x_0$

$C_1 \equiv \text{send } tt \ x_1$

$M \equiv \text{let } (z_0, z_1) = z \text{ in let } (v, y') = \text{recv } y \text{ in let } \bot = \text{terminate } y' \text{ in if } z_0 \text{ then } z \text{ else } (tt, v)$

We define a server $\equiv \text{serve } \{ (ff, ff), z. M, f. N \}$ beginning from $(ff, ff)$. We then have that

$$\vdash \text{flip } \equiv \text{connect}(x. \text{clients}; y. \text{server}) : \mathbb{B}$$

This program is translated to $[\text{flip}]_y \vdash y : \mathbb{2}$, with reactions $[\text{flip}]_y \rightarrow^* ff_y$ and $[\text{flip}]_y \rightarrow^* tt_y$. We can use this to implement a nondeterministic choice operator:

$$\frac{P \vdash \Gamma \quad Q \vdash \Gamma}{\text{choose}(P, Q) \equiv \nu x. y. ([\text{flip}]_y | if(x; P; Q)) \vdash \Gamma}$$

such that $\text{choose}(P, Q) \rightarrow^* P$ and $\text{choose}(P, Q) \rightarrow^* Q$.

### 5.4 Fork–join Parallelism

**Fork-join parallelism** (Conway, 1963) is a common model of parallelism in which child processes are *forked* to perform computation simultaneously. Once they have finished, they are *joined* by the parent process, which collects their work and produces the final result. We assume a ‘heavyweight’ function $h : A \rightarrow B$ that will run on forked processes, and a relatively less expensive function $g : B \rightarrow B \rightarrow B$ that will combine their answers. We also assume an initial value $g_0 : B$, and a list of ‘tasks’ $xs : [A]$ to process. $[A]$ is the type of lists of $A$, and is supported by the operations:

$$\text{nil} : [A] \quad \text{cons} : A \rightarrow [A] \rightarrow [A] \quad \text{fold}_C : C \rightarrow (C \rightarrow A \rightarrow C) \rightarrow [A] \rightarrow C$$

Let

$$\text{clients} \equiv \text{let } y = \text{fold}_C \ z_0 \rightarrow \text{fork}_x x_0. (\lambda x. \lambda v. \text{fork}_x x'. (\text{let } v' = h \ v \text{ in send } v' x')). x \rightarrow \text{eof}_y$$

$$M \equiv \text{let } (v, y') = \text{recv } y \text{ in let } \bot = \text{terminate } y' \text{ in } g \ v$$

The client protocol is $T_S \equiv !B. \text{end}_1$. To form the client pool, we begin with a shared client channel $c : \exists T_S$. We fold over the list $xs : [A]$, adding a forked process for each ‘task’ $v : A$ to the client pool. Each one of these forked processes will compute $h \ v : B$, and send it over its fresh channel $x' : T_S$. We have $c : \exists T_S \vdash \text{clients} : \text{end}_2$.

We let $\text{server} \equiv \text{serve } \{ g_0, z. M, f. f \}$. The server begins with internal state $g_0 : B$. It nondeterministically receives the result of a computation of $h$ from each client, and ‘merges’ it into its state using $g$. In the end, it returns the result. We have $z : B, y : T_S \vdash M : B$, and thus $y : T_S \vdash \text{server} : B$. We use $\text{serve}$ to pass unlimited parameters to the server internals.

Putting this system together, we get

$$\vdash \text{fork-join}(h, g_0, g, xs) \equiv \text{connect}(x. \text{clients}; y. \text{server}) : B$$

The fork-join paradigm is often used in industrial parallelization frameworks (Dagum and Menon, 1998; Reinders, 2007; Blumofe et al., 1995; Leijen et al., 2009). The background languages and type systems usually do not use any logical devices for concurrency. In particular, concurrent behaviour is not controlled by the type system, as it is here. Note that fork-join requires each spawned process to be independent of each other and only communicate with the parent process, which is precisely captured by the linearity restriction of our system.

Another parallel computation model is that of *async-finish*. It is more expressive than fork-join, as it allows spawned processes to spawn further processes. The whole tree is then joined at the root process, with no regard to the spawning thread of each child. Our system(s) does not support that: in the REQA rule, the spawned process $M$ is only given a channel $x' : T_S$, which cannot be used to spawn further processes in the same pool. However, it is well-known that is nested parallelism is still possible, but each
child has to spawn its own instance of a fork-join computation, which does not interfere with the root process.

An even more expressive model is that of futures (Halstead Jr, 1984). A future is a first-class value that represents a computation running in parallel to the current process. At any point it can be forced to obtain its result; if it has not finished an error may be returned, or the process forcing it may block. While fork-join or async-finish spawned processes are independent of each other, futures may be passed around freely (in any reasonably expressible language) and introduce rich interactions. This seems to be in violation of the linearity restriction of our system(s), and thus cannot be expressed. Nevertheless, the Conn rule can be seen as a very restricted form of future, where the spawned process can only communicate with the parent process. More discussions about the difference between these models is given by Acar (2016).

5.5 Keynes’ beauty contest

Until this point we have seen only relatively simple examples of server-client interaction. In all cases, the ‘internal server protocol’ we have used has consisted of an unlimited type, the values of which we can replicate and discard. This leads to the false impression that clients access the server one-by-one in a sequential manner, so that clients that connect later are unable to influence the information observed by the earlier ones. In this section we present an example that shows this to be untrue. In particular, if the ‘internal server protocol’ consists of a session type itself, then we witness bidirectional, interleaving behaviour. This distinguishes our systems from those based on manifest sharing Balzer and Pfenning (2017).

We present a server implementing the umpire in a Keynesian beauty contest (Keynes, 1936, §12). Keynes’ beauty contest works as follows. A newspaper runs a beauty contest in which readers have to pick the prettiest faces from a set of photographs. The competitors are not those pictured, but the readers themselves: if they pick the faces which are judged to be the prettiest by the majority, they will win a prize. Thus, the readers are incentivized to estimate the aesthetics of the majority.

We will implement a restricted version of this scenario, where a pool of clients votes for a Boolean value. The server then counts the votes, and awards a payoff of 0 or 1 (represented by \( \uparrow\) and \( \wedge\) respectively) to each client, indicating whether they voted for the winner. This is obviously impossible if the server handles requests sequentially. In fact, the server will be implemented by spawning a network of interconnected processes, each of which will handle one vote.

We first define the following derived rule. Informally, this rule expresses that a process that uses a channel of type \( T_S \) is also exposing a channel of dual type \( \overline{T_S} \).

\[
\Gamma, x : T_S \vdash M : \text{end} \quad y : \overline{T_S} \vdash y : \overline{T_S} \\
\Gamma \vdash \text{inv}_2(M) \overset{\text{def}}{=} \text{connect}(x.M; y, y) : \overline{T_S}
\]

The client session type is \( C_S \overset{\text{def}}{=} \exists b. \neg b. \text{end}, \) and the internal server protocol is \( T_S \overset{\text{def}}{=} ?(N \otimes N). \neg b. \text{end}, \) where \( N \) is the type of natural numbers. We assume a bunch of standard functions:

- \( \text{zero} : N \)
- \( \text{succ} : N \rightarrow N \)
- \( \leq : N \rightarrow N \rightarrow \mathbb{B} \)
- \( \text{eq} : \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B} \)

where eq checks Boolean values for equality. We let

\[
\begin{align*}
L & \overset{\text{def}}{=} \text{let } w' = \text{send}(\text{zero}, \text{zero}) w \text{ in } \\
& \text{send initial state} \\
& \text{let } (w, w'') = \text{recv } w' \text{ in } w'' \\
& \text{receive final value} \\
N & \overset{\text{def}}{=} \text{let } (s, f') = \text{recv } f \text{ in } \\
& \text{receive final count} \\
& \text{let } (n_0, n_1) = s \text{ in } \\
& \text{unpack state} \\
& \text{let } f'' = \text{send } (n_0 \leq n_1) f' \text{ in } \\
& \text{compute winner and notify the last worker process} \\
& \text{terminate } f'' \\
M & \overset{\text{def}}{=} \text{let } (s, z') = \text{recv } z \text{ in } \\
& \text{get state} \\
& \text{let } (n_1, n_f) = s \text{ in } \\
& \text{unpack state} \\
& \text{let } (b, y') = \text{recv } y \text{ in } \\
& \text{receive a vote} \\
& \text{let } s' = \text{if } b \text{ then } (\text{succ } n_f, n_f) \text{ else } (n_f, \text{succ } n_f) \text{ in } \\
& \text{increment the right counter} \\
& \text{let } w' = \text{send } s' w \text{ in } \\
& \text{pass new state to next worker process}
\end{align*}
\]
We define \( \text{serve} \overset{\text{def}}{=} \text{serve}\{\text{inv}_w(L), z. \text{inv}_w(M), f. N\} \). The components are typed as

\[
\begin{align*}
  w : T_S & \vdash L : \text{end}_1 \\
  w : T_S, z : T_S, y : C_S & \vdash M : \text{end}_1 \\
  y : i T_S & \vdash \text{server} : \text{unit}
\end{align*}
\]

The details of this protocol are subtle. The construct \( \text{inv}_w(\cdot) \) allows us to use programs which only have side-effects as internal server state, by inverting the polarity of one of the channels. The server is initialized by \( L \), which sets the state to be \((0, 0)\). It then listens on the same channel to receive the winner, which it promptly discards. The server finalization \( N \) receives the final tally of the votes, computes the winner, and sends it back the result, and closes the channel.

The component \( M \) is used to communicate with each competitor. It receives the state of the server, the competitor’s vote, and increments the appropriate tally. It then passes on this new state to the next worker process \( M' \), which will communicate with the next competitor. This sets up an entire network of worker processes \( M_i \), one to serve each competitor. When the competitors have all cast their votes, \( N \) computes the winner, and sends it back to the last worker process. This process then tells the competitor whether they won, closes the channel to the competitor, and passes on the result to the worker process serving the previous competitor, and so on. At the very end, the winner is passed to the initialization process \( L \).

We can then define a number of competitors \( x_i : !B.?B. \text{end}_1 \vdash C_i : \text{end}_1 \) who will cast their votes by sending a Boolean value and receive a payoff along \( x_i \). These can be combined into a client pool, much in the same way as in previous examples.

If we have two such competitors \( C_0 \) and \( C_1 \) merged in a pool, and we connect them to server, we will obtain a process topology of the form illustrated in the schematic diagram of Fig. 7. Compared with Fig. 4 this diagram is intuitive but loose on accuracy. Details such as \( \text{end}_1 \) and \( \text{end}_1 \) are left out. We have also spelled out the protocols internals. For example, the server internal protocol \( T_S \) is indicated by a forward arrow \( N \otimes N \) and a backward arrow \( \mathbb{B} \).

### 6 Related work

**Hypersequents and Session Types.** Hypersequents were introduced to process calculi and Classical Linear Logic by Montesi and Peressotti (2018). Another version of that system was studied in detail by Kokke et al. (2019a). A reaction semantics similar to the one used here was given in a later paper (Kokke et al., 2019b).

**Clients, Servers, and Races in Linear Logic.**

Typing client-server interaction has been a thorn in the side of session types and Linear Logic. All previous attempts rely on some version of the Mix rule. Both Wadler (2014, §3.4) and Caires and Pérez (2017, Ex. 2.4) use Mix to combine clients into client pools. Kokke et al. implicitly use Mix to type an otherwise untypable client pool in HCP (Kokke et al., 2019a, Ex. 3.7). Remarkably, none of these calculi demonstrate stateful server behaviour, as we predicted using a semantic argument in §1.1.

Atkey et al. (2016) explore the additional power bestowed upon CP by conflating dual connectives. The conflation of \( ? \) and \( ! \) leads to the notion of access point, a dynamic match-making communication

\[ \text{let } (b', w') = \text{recv } w \text{ in } \text{let } z = \text{terminate } (\text{send } (eq b b') y') \text{ in } \text{let } z' = \text{terminate } (\text{send } b' z') \text{ in } w'' \]

(receive winner from next worker process)
(tell competitor if they won, close channel)
(forward winner on, close channel)

Figure 7: Layout of the voting game after coexponentials reactions but before other reactions. Boxes represents processes whose names are at the center of the boxes. Arrows represents directed messages between processes with types of the data annotated. Labels on edges of boxes are the names of the channels to the processes.

---

\[ \text{let } (b', w') = \text{recv } w \text{ in } \text{let } z = \text{terminate } (\text{send } (eq b b') y') \text{ in } \text{let } z' = \text{terminate } (\text{send } b' z') \text{ in } w'' \]

(receive winner from next worker process)
(tell competitor if they won, close channel)
(forward winner on, close channel)

\[ \text{let } (b', w') = \text{recv } w \text{ in } \text{let } z = \text{terminate } (\text{send } (eq b b') y') \text{ in } \text{let } z' = \text{terminate } (\text{send } b' z') \text{ in } w'' \]

(receive winner from next worker process)
(tell competitor if they won, close channel)
(forward winner on, close channel)
service on a single end point. In fact, the rules look eerily close to the list-like formulation of our servers and generators. Access points prove too powerful: they introduce stateful nondeterminism, racy communication, and general recursion. This impairs the safety of CP by introducing deadlock and livelock. Our work shows that we can still safely obtain the former two features without introducing the third.

Adding nondeterminism to CLL in a controlled fashion is complex. Atkey et al. (2016) express a form of nondeterministic local choice in CP by conflating \( \otimes \) and \( \oplus \). The resultant form of nondeterministic choice cannot induce the racy behaviour normally exhibited in the \( \pi \)-calculus (Kokke et al., 2019c, §2). Caires and Pérez (2017) present a dual-context system based on CLL+Mix in which the same kind of nondeterministic local choice is expressed through a new set of modalities, \( \oplus \) and \( \otimes \). These bear a similarity to the coexponential modalities presented here, but they are used for nondeterminism instead. Their \( \otimes \) modality has a monadic flavour, and hence can be used to encapsulate nondeterminism ‘in the monad’ in the usual manner in which we isolate effects.

Kokke et al. (2019c) drew inspiration from Bounded Linear Logic (Girard et al., 1992) to formulate a system for nondeterministic client-server interaction. They use types of the form \( ?_n A \) (standing for \( n \) copies of \( A \) delimited by \( \otimes \)) and \( !_n A \) (standing for \( n \) copies of \( A \) delimited by \( \oplus \)). \( !_n A \) represents a pool of \( n \) disjoint clients with protocol \( A \), and \( ?_n A \) a server that can serve exactly \( n \) clients with protocol \( A \). While this is consistent with disjoint-vs.-connected concurrency, their system is limited to serving a specific number of clients in each session. Thus, it fails to satisfy criterion (i) in §1.1, and does not form a satisfactory model.

**Fixed Points in Linear Logic.** Inductive and coinductive types—presented proof-theoretically as least and greatest fixed points—were introduced in the context of higher-order Classical Linear Logic by Baelde (2012). Baelde formulates a weakly normalization cut elimination procedure, which albeit does not satisfy the subformula property. Ehrlhard and Jafarrahmani (2021) study categorical models for a slight extension of the propositional fragment of Baelde’s system, which allow them to infer certain facts about the behaviour of (co)inductive linear data types.

The structure of Baelde’s system has been used to extend CP with inductive and coinductive types by Lindley and Morris (2016). This is our starting point in §2.2, but with significant alterations along the way. First, our system is based on HCP. Second, the server rule has been reformulated to support hyperenvironments, and server finalization (§3.3). Third, our client pools allow permutation in order to enable nondeterminism (§3.4). Finally, our reaction semantics are tailored to the specific setting, and are consequently simpler. In a separate strand of work, Toninho et al. (2014) introduce coinduction in a system of session types based on Intuitionistic Linear Logic (ILL); see Lindley and Morris (2016, §§1, 7) for a comparison.

**Manifest sharing.** Closely related to our work is the notion of manifest sharing (Balzer and Pfenning, 2017). This work starts from a very different premise: a channel is either linear (as the usual channels in session types), or shared between processes. This leads to an ILL-based system, SILL\(_S\), with two modes and two modalities shifting between them (Reed, 2009). The switch to a shared channel is punctuated by the modalities. Thus, sharing manifests in the types. In some ways, SILL\(_S\) is a much stronger system, as it features equi-synchronizing recursive session types. The price to pay is the introduction of deadlock. Balzer et al. (2019) develop an additional layer of the type system that protects from it.

Our work attempts to solve the expressivity problem of LL-based session types beginning from Curry-Howard: we seek the minimal extension to Linear Logic that will enable us to write server and client processes. Unlike manifest sharing, we remain committed to CLL and its duality. The result is that our system has simpler rules, avoiding the notions of linear and shared channels (as (co)exponentials internalise them), as well as the lock-like primitives used to introduce modalities in Balzer and Pfenning (2017). Moreover, we have remained committed to the goal of retaining the good properties ensured by cut elimination in CLL (e.g. deadlock freedom). A drawback of this approach is that our system inherits the linearity constraint from linear logic, and is thus unable to express circular structures (such as Dijkstra’s dining philosophers).

Both systems provide atomicity, but in radically different ways. In SILL\(_S\) users access the service in a mutually exclusive manner. This is not compatible with the usual view of typical client-server interaction, where multiple clients need to access the server simultaneously in order to exchange information. A common workaround is to decompose an interleaved session into a ‘stateless’ protocol consisting of

---

\(\text{2This is an intentional clash with external and internal choice in Linear Logic.}\)
several mini-sessions. Every client is then required to send a ‘cookie’ to identify themselves across mini-
sessions. In our system accesses to the shared service are concurrent, but causally atomic (§4). As a
result, interleaved sessions can be expressed natively (§5.5).

**Differential Linear Logic.** The rules for \( \downarrow \) given in §2.2 are almost the same as the coweakening,
codereliction and cocontraction rules that are added to ! in Differential Linear Logic (DiLL) (Ehrhard,
2018). DiLL is equipped with nondeterministic reduction and formal sums, and is thus believed to
have something to do with concurrency. Ehrhard and Laurent (2010) have produced an embedding of the
finitary \( \pi \)-calculus into DiLL, though that encoding has been criticized (Mazza, 2018). A type of
client-server interactions—namely the encoding of ML-style reference cells into session types—has been
encoded by Castellan et al. (2020) in a system based on the rules of DiLL. This work relies on both the
costructural rules and Mix, so it is not clear which device primarily augments expressive power. Our
work shows that something akin to the costructural rules of DiLL arises from the wish to form client
pools. The exact relationship between coexponentials and DiLL remains to be determined.

**Multiparty Session Types.** There is a nontrivial connection between our work and Multiparty Ses-
sion Types (Honda et al., 2008, 2016; Coppo et al., 2016), which comprise a \( \pi \)-calculus and a behavioural
type system specifying interaction between multiple agents. The kinds of protocols expressed by mul-
tiparty session types are ‘fully’ choreographed, and involve a fixed number of participants. As such,
you cannot model interactions with an arbitrary number of clients; nor can they introduce a controlled
amount of nondeterminism. Some of these expressive limitations have been remedied in systems of Dy-
namic Multirole Session Types (Denielou and Yoshida, 2011), which come at the price of introducing
roles that parties can dynamically join or leave, and a notion of quantification over participants with
a role. Our system captures certain use-cases of roles using only tools from linear logic, with little
additional complexity.

Closer to our work is the approach of Carbone et al. (2017) to multiparty session types through
cohesion proofs. In op. cit. the authors develop Multiparty Classical Processes, a version of CP with
role annotations and the MCut rule. The latter is a version of the MultiCut rule annotated with a
cohesion judgment derived from Honda et al. (2008), which generalises duality and ensures that roles
match appropriately. MCP does not allow dynamic sessions with arbitrary numbers of participants,
and hence cannot model client-server interactions. MCP was later refined into the system of Globally-
governed Classical Processes (GCP) by Carbone et al. (2016). Unlike these calculi, our work does not
require any consideration of coherence or local vs. global types.

**7 Conclusions and Further Work**

We presented the system of Client-Server Linear Logic, which features a novel form of modality, the
coelexponentials. We then showed how CSLL can be used to model client-server interactions without
falling down the slippery slope of introducing Mix. We comment on some directions for future work.

**Termination.** It would be interesting to establish a termination result for CSLL. This would prove
that the resulting calculus does not generate livelock. We expect this proof to be somewhat involved,
which is why most work on Linear Logic and session types either fails to produce a proof, or defers to
Girard’s proof for CLL (Wadler, 2014; Aschieri and Genco (2019).

**Syntax.** The weak \( \downarrow \) rule listed in §2.2 is expressed by folding \( \otimes \) over the set of formulas. This obstructs
a particular commuting conversion in cut elimination. Similarly, presentation of the strong exponential
and its computational interpretation is omitted due to its unsatisfactory rules. We believe these issues
are due to the limitation of sequent calculus, and alternative techniques are necessary to solve them.

**References**

Samson Abramsky. 1993a. Computational interpretations of linear logic. *Theoretical Computer Science*
111, 1-2 (1993), 3–57. https://doi.org/10.1016/0304-3975(93)90181-R ISBN: 0304-3975.

Samson Abramsky. 1993b. Interaction categories. In *Theory and Formal Methods 1993*. Springer, 57–69.
Samson Abramsky. 1994. Proofs as processes. *Theoretical Computer Science* 135, 1 (1994), 5–9.

Samson Abramsky, Simon J Gay, and Rajagopal Nagarajan. 1996. Interaction categories and the foundations of typed concurrent programming. In *Deductive Program Design (Nato ASI Subseries F)*, Manfred Broy (Ed.). Springer-Verlag Berlin Heidelberg, 35–113. http://www.springer.com/us/book/9783540609476

Samson Abramsky and Radha Jagadeesan. 1994. Games and Full Completeness for Multiplicative Linear Logic. *The Journal of Symbolic Logic* 59, 2 (1994), 543. https://doi.org/10.2307/2275407

arXiv:1311.6057

Umut A Acar. 2016. *Parallel Computing: Theory and Practice.* http://www.cs.cmu.edu/afs/cs/academic/class/15210-f15/www/tapp.html

Federico Aschieri and Francesco A. Genco. 2019. Par Means Parallel: Multiplicative Linear Logic Proofs as Concurrent Functional Programs. *Proc. ACM Program. Lang.* 4, POPL, Article 18 (Dec. 2019), 28 pages. https://doi.org/10.1145/3371086

Robert Atkey. 2017. Observed communication semantics for classical processes. In *European Symposium on Programming*. Springer, 56–82.

Robert Atkey, Sam Lindley, and J. Garrett Morris. 2016. Conflation Confers Concurrency. In *A List of Successes That Can Change the World*, Sam Lindley, Conor McBride, Phil Trinder, and Don Sannella (Eds.). Lecture Notes in Computer Science, Vol. 9600. Springer International Publishing, 32–55.

Arnon Avron. 1991. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence* 4, 3-4 (1991), 225–248.

David Baelde. 2012. Least and greatest fixed points in linear logic. *ACM Transactions on Computational Logic* 13, 1 (2012), 1–44.

Stephanie Balzer and Frank Pfenning. 2017. Manifest sharing with session types. *Proceedings of the ACM on Programming Languages* 1, ICFP (2017), 1–29. https://doi.org/10.1145/3110281

Stephanie Balzer, Bernardo Toninho, and Frank Pfenning. 2019. Manifest Deadlock-Freedom for Shared Session Types. In *Programming Languages and Systems*, Luís Caires (Ed.), Vol. 11423. Springer International Publishing, Cham, 611–639. https://doi.org/10.1007/10.1007/978-3-030-17184-1_22

Michael Barr. 1991. *-*Autonomous categories and linear logic. *Mathematical Structures in Computer Science* 1, 2 (1991), 159–178. https://doi.org/10.1017/S0960129500001274

Gianluigi Bellin. 1997. Subnets of proof-nets in multiplicative linear logic with MIX. *Mathematical Structures in Computer Science* 7, 6 (1997), 663–669. https://doi.org/10.1017/S0960129597002326

G. Bellin and P. J. Scott. 1994. On the π-calculus and linear logic. *Theoretical Computer Science* 135, 1 (1994), 11–65. https://doi.org/10.1016/0304-3975(94)00104-9

N Benton and P Wadler. 1996. Linear logic, monads and the lambda calculus. In *Proceedings 11th Annual IEEE Symposium on Logic in Computer Science*. IEEE. https://doi.org/10.1109/LICS.1996.561458

Robert D Blumofe, Christopher F Joerg, Bradley C Kuszmaul, Charles E Leiserson, Keith H Randall, and Yuli Zhou. 1995. Cilk: An efficient multithreaded runtime system. *ACM SigPlan Notices* 30, 8 (1995), 207–216.

Luís Caires and Jorge A. Pérez. 2017. Linearity, Control Effects, and Behavioral Types. In *Programming Languages and Systems. ESOP 2017*, Hongseok Yang (Ed.). Springer Berlin Heidelberg, Berlin, Heidelberg, 229–259. https://doi.org/10.1007/978-3-662-54434-1_9

Luís Caires and Frank Pfenning. 2010. Session Types as Intuitionistic Linear Propositions. In *CONCUR 2010 - Concurrency Theory (Lecture Notes in Computer Science, Vol. 6269)*, Paul Gastin and François Laroussinie (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 222–236.
Luís Caires, Frank Pfenning, and Bernardo Toninho. 2016. Linear logic propositions as session types. *Mathematical Structures in Computer Science* 26, 3 (2016), 367–423. https://doi.org/10.1017/S0960129514000218

Marco Carbone, Sam Lindley, Fabrizio Montesi, Carsten Schürmann, and Philip Wadler. 2016. Coherence Generalises Duality: A Logical Explanation of Multiparty Session Types. In 27th International Conference on Concurrency Theory (CONCUR 2016) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 59), Josée Desharnais and Radha Jagadeesan (Eds.). Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 33:1–33:15. https://doi.org/10.4230/LIPIcs.CONCUR.2016.33

Marco Carbone, Fabrizio Montesi, Carsten Schürmann, and Nobuko Yoshida. 2017. Multiparty session types as coherence proofs. *Acta Informatica* 54 (2017), 243–269. https://doi.org/10.1007/s00236-016-0285-y

Simon Castellan, Nobuko Yoshida, and Léo Stefanesco. 2020. Game Semantics: Easy as Pi. arXiv:2011.05248 [cs] (Nov. 2020). arXiv:2011.05248 http://arxiv.org/abs/2011.05248

Melvin E Conway. 1963. A multiprocessor system design. In *Proceedings of the November 12-14, 1963, fall joint computer conference*. 139–146.

Mario Coppo, Mariangiola Dezani-Ciancaglini, Nobuko Yoshida, and Luca Padovani. 2016. Global progress for dynamically interleaved multiparty sessions. *Mathematical Structures in Computer Science* 26, 2 (2016), 238–302. https://doi.org/10.1017/S0960129514000188

Leonardo Dagum and Ramesh Menon. 1998. OpenMP: an industry standard API for shared-memory programming. *IEEE computational science and engineering* 5, 1 (1998), 46–55.

Pierre-Malo Denielou and Nobuko Yoshida. 2011. Dynamic Multirole Session Types. In *Proceedings of the 38th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages - POPL ’11*. ACM Press, 435. https://doi.org/10.1145/1926385.1926435

Thomas Ehrhard. 2018. An introduction to differential linear logic: proof-nets, models and antiderivatives. *Mathematical Structures in Computer Science* 28, 7 (2018), 995–1060. https://doi.org/10.1017/S0960129516000372

Thomas Ehrhard and Farzad Jafarrahmani. 2021. Categorical models of Linear Logic with fixed points of formulas. arXiv:2011.10209 [cs.LO]

Thomas Ehrhard and Olivier Laurent. 2010. Interpreting a finitary pi-calculus in differential interaction nets. *Information and Computation* 208, 6 (2010), 606–633. https://doi.org/10.1016/j.ic.2009.06.005

Simon Fowler, Sam Lindley, J. Garrett Morris, and Sára Decova. 2019. Exceptional asynchronous session types: session types without tiers. *Proceedings of the ACM on Programming Languages* 3, POPL (2019). https://doi.org/10.1145/3290341

Simon J Gay and Vasco T Vasconcelos. 2010. Linear type theory for asynchronous session types. *Journal of Functional Programming* 20, 1 (2010), 19.

Jean-Yves Girard. 1987. Linear logic. *Theoretical Computer Science* 50, 1 (1987), 1–101. https://doi.org/10.1016/0304-3975(87)90045-4

J. Y. Girard and Y. Lafont. 1987. Linear logic and lazy computation. In *TAPSOFT ’87 (Lecture Notes in Computer Science, Vol. 250)*, Hartmut Ehrig, Robert Kowalski, Giorgio Levi, and Ugo Montanari (Eds.). Springer-Verlag, Berlin/Heidelberg, 52–66. https://doi.org/10.1007/BFb0014972

Jean-Yves Girard, Andre Scedrov, and Philip J. Scott. 1992. Bounded linear logic: a modular approach to polynomial-time computability. *Theoretical Computer Science* 97, 1 (1992), 1–66. https://doi.org/10.1016/0304-3975(92)90386-T

Robert H Halstead Jr. 1984. Implementation of Multilisp: Lisp on a multiprocessor. In *Proceedings of the 1984 ACM Symposium on LISP and functional programming*. 9–17.
Maurice Herlihy and Nir Shavit. 2012. The Art of Multiprocessor Programming (revised first ed.). Morgan Kaufmann.

Jonas Kastberg Hinrichsen, Jesper Bengtson, and Robbert Krebbers. 2019. Actris: Session-type based reasoning in separation logic. Proceedings of the ACM on Programming Languages 4, POPL (2019), 1–30.

Kohei Honda, Vasco T Vasconcelos, and Makoto Kubo. 1998. Language primitives and type discipline for structured communication-based programming. In Programming Languages and Systems: Proceedings of the 7th European Symposium on Programming (ESOP’98) (Lecture Notes in Computer Science, Vol. 1381). Springer, Berlin, Heidelberg, 122–138. https://doi.org/10.1007/BFb0053567

KoheiHonda, Nobuko Yoshida, and Marco Carbone. 2008. Multiparty Asynchronous Session Types. In Proceedings of the 35th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages. ACM Press. https://doi.org/10.1145/1328438.1328472

KoheiHonda, NobukoYoshida, and MarcoCarbone. 2016. Multiparty Asynchronous Session Types. J. ACM 63, 1 (2016), 1–67. https://doi.org/10.1145/2827695

JohnMaynardKeynes. 1936. The General Theory of Employment, Interest and Money. Macmillan & Co. Ltd., London.

Wen Kokke, FabrizioMontesi, and MarcoPeressotti. 2019a. Better late than never: a fully-abstract semantics for classical processes. Proceedings of the ACM on Programming Languages 3, POPL (2019), 1–29.

Wen Kokke, Fabrizio Montesi, and Marco Peressotti. 2019b. Taking linear logic apart. (2019). https://arxiv.org/abs/1904.06848

Wen Kokke, J. GarrettMorris, and PhilipWadler. 2019c. Towards Races in Linear Logic. arXiv:1909.13376 [cs.LO]

DaanLeijen, WolframSchulte, and SebastianBurchhardt. 2009. The design of a task parallel library. Acm Sigplan Notices 44, 10 (2009), 227–242.

SamLindley and J. GarrettMorris. 2015. A semantics for propositions as sessions. In European Symposium on Programming Languages and Systems. Springer, 560–584.

SamLindley and J. Garrett Morris. 2016. Talking bananas: structural recursion for session types. In Proceedings of the 21st ACM SIGPLAN International Conference on Functional Programming - ICFP 2016. ACM Press, Nara, Japan, 434–447. https://doi.org/10.1145/2951913.2951921

SamLindley and J. Garrett Morris. 2017. Lightweight Functional Session Types. In Behavioural Types: from Theory to Tools, Simon Gay and Antonio Ravara (Eds.). River Publishers. https://doi.org/10.13052/rp-9788793519817

J. Maraist, M. Odersky, D.N. Turner, and P. Wadler. 1999. Call-by-name, call-by-value, call-by-need and the linear lambda calculus. Theoretical Computer Science 228, 1-2 (1999), 175–210. https://doi.org/10.1016/S0304-3975(98)00358-2

JohnMaraist, MartinOdersky, David N. Turner, and PhilipWadler. 1995. Call-by-name, Call-by-value, Call-by-need, and the Linear Lambda Calculus. Electronic Notes in Theoretical Computer Science 1 (1995), 370–392. https://doi.org/10.1016/S1571-0661(04)00022-2

DamianoMazza. 2018. The true concurrency of differential interaction nets. Mathematical Structures in Computer Science 28, 7 (Aug. 2018), 1097–1125. https://doi.org/10.1017/S0960129516000402

Paul-Andr´e Melli`es. 2009. Categorical Semantics of Linear Logic. In Panoramas et synth`eses 27: Interactive models of computation and program behaviour, Pierre-Louis Curien, Hugo Herbelin, Jean-Louis Krivine, and Paul-Andr´e Melli`es (Eds.). Soci´et ´e Math´ematique de France. http://www.pps.univ-paris-diderot.fr/~sim$mellies/papers/panorama.pdf

Paul-Andr´e Melli`es, NicolasTabareau, and ChristineTasson. 2018. An explicit formula for the free exponential modality of linear logic. Mathematical Structures in Computer Science 28, 7 (2018). https://doi.org/10.1017/S0960129516000426
A Coexponentials and Logical equivalences

We may derive the following logical equivalences about coexponentials, which are dual to similar laws for the exponentials.

\[ \begin{align*}
\vdash ! A & \equiv ! A \\
\vdash A & \equiv ! A \otimes ! A \\
\vdash ! ! A & \equiv ! A \\
\vdash ! A, ! A & \equiv ! A \\
\vdash ! A \otimes ! A & \equiv ! A \\
\vdash ! A, ! A & \equiv ! ! A \\
\vdash A \otimes A & \equiv ! ! A \\
\vdash A & \equiv ! A \\
\vdash A & \equiv ! A \\
\vdash ! A & \equiv ! A \\
\vdash ! A & \equiv ! A \\
\vdash ! A & \equiv ! A \\
\vdash A & \equiv ! A \\
\vdash A & \equiv ! A \\
\end{align*} \]

Theorem 4. In CLL with Mix and BiCut, exponentials and coexponentials coincide up to provability. That is: if we replace ? and ! in the rules for the exponentials with \(\tilde{\cdot}\) and \(\tilde{\cdot}\) respectively, the resultant rule is provable using the coexponential rules, and vice versa.

Proof. We first show that the exponential rules are derivable using coexponential rules under the substitution \(\tilde{\cdot}\)\(\tilde{\cdot}\). The weakening rule \(\vdash \Gamma, ?A \vdash w\) is mapped to the derivation \(\vdash \Gamma, \tilde{\cdot}A \vdash \tilde{\cdot}w\) Mix.

The dereliction rule \(\tilde{\cdot}d\) is just \(\tilde{\cdot}d\), and the contraction rule \(\vdash \Gamma, ?A ?c\) is mapped to

\[ \begin{align*}
\vdash \Gamma, \tilde{\cdot}A, \tilde{\cdot}A & \vdash A^+ \tilde{\cdot}A \\
\vdash \Gamma, \tilde{\cdot}A, \tilde{\cdot}A & \vdash A^+ \tilde{\cdot}A \\
\vdash \Gamma, \tilde{\cdot}A, \tilde{\cdot}A & \vdash \tilde{\cdot}A \\
\vdash \Gamma, \tilde{\cdot}A, \tilde{\cdot}A & \vdash \tilde{\cdot}A \\
\end{align*} \]

BiCut

This leaves promotion. The forklores (iii–v) can be generalised to a bi-implication

\[ A_1 \otimes \ldots \otimes A_n \to A_1 \otimes \cdots \otimes A_n \]

\[ A_1 \otimes \cdots \otimes A_n \to A_1 \otimes \ldots \otimes A_n \]
and hence sequents \( \vdash \otimes \Delta \upharpoonright, \otimes \Delta \) and \( \vdash \otimes \Delta \downharpoonleft, \Delta \) for any \( \Delta \). With these in hand, we can interpret the promotion rule \( \vdash ?\Gamma, A \downharpoonleft \) by the derivation

\[
\begin{array}{c}
\vdash \otimes ?\Gamma, \otimes i \upharpoonright \\
\vdash \otimes ?\Gamma, A \\
\vdash \otimes i \Delta, A \\
\vdash ?\Gamma, i \Delta, A \\
\vdash \otimes ?\Gamma, i \Delta, A \\
\end{array}
\]

\[
\begin{array}{c}
\text{CUT} \\
\text{CUT}
\end{array}
\]

In the opposite direction, we show that the coexponentials rules are derivable using exponentials rules under the substitution \( \vartriangleright \mapsto \). As the folklores ensure \( \text{Mix}_0 \) is derivable in this system, we can interpret the weakening rule \( \vdash \vartriangleright A \downharpoonright \) by \( \vdash ?A \downharpoonright \). The dereliction rule \( \vartriangleright d \) is simply \( ?d \), and the contraction rule \( \vdash \Gamma, \vartriangleright A \downharpoonright \) is interpreted by the derivation

\[
\begin{array}{c}
\vdash \vartriangleright A \\
\vdash \Delta, \vartriangleright A \\
\vdash \Gamma, \Delta, \vartriangleright A \\
\vdash \otimes \vartriangleright A \\
\end{array}
\]

\[
\begin{array}{c}
\text{Mix} \\
\text{\&c}
\end{array}
\]

Finally, the rule \( \vdash \otimes \vartriangleright A \downharpoonright \) is interpreted in a way similar to promotion, but with the cuts replacing \( \otimes \) with \( \otimes \) happening in the opposite order.

\[\square\]

### B Translation of CSGV to CSLL: Omitted rules

Of the functional fragment of CSGV the types are translated to CSLL:

\[
\begin{align*}
[T \rightarrow U] & \equiv [T] \upharpoonright \otimes [U] \\
[T \rightarrow U] & \equiv ![\otimes T \upharpoonleft \otimes [U]] \\
[T + U] & \equiv [T] \otimes [U] \\
[T \otimes U] & \equiv [T] \otimes [U] \\
\text{Unit} & \equiv 1
\end{align*}
\]

Omitted rules of CSGV with their translation into CSLL are shown in Figs. 8 and 9. Note that some translations use Lemma 11.

### C CSLL: Metatheoretic Proofs

**Lemma 4.** If \( P \equiv Q \), then \( P \vdash \mathcal{G} \) if and only if \( Q \vdash \mathcal{G} \).

**Proof.** By induction on \( P \equiv Q \). We prove one direction, the other one being entirely analogous. Moreover, the congruence cases are trivial. \( P \mid \text{stop} \equiv P \), commutativity, and associativity follow from the structure of hyperenvironments. Link-commutativity follows from the involutive property of \( (- \downharpoonright) \).

**Case (Res-Par).**

Then \( P = \nu xy. (R \mid S) \) and \( Q = R \mid \nu xy. S \) where \( x, y \notin \text{Fn}(R) \). We must then have that \( R \vdash \mathcal{H} \) where \( x, y \notin \mathcal{H} \) (using Lemma 8) and \( S \vdash I \mid \Gamma, x : A \mid \Delta, y : A \upharpoonright, \Delta \), where \( \mathcal{G} = \mathcal{H} \mid I \mid \Gamma, \Delta \). Hence, we can derive that \( Q = R \mid \nu xy. S \vdash \mathcal{G} \).

**Case (Res-Res).**

Then \( P = \nu xy. \nu zw. R \) and \( Q = \nu zw. \nu xy. R \) for some \( R \). We must invert \( P \vdash \mathcal{G} \). This generates many cases: for example, it could be that \( R \vdash \mathcal{G} \mid \Gamma, x : A \mid \Delta, y : A \upharpoonright \mid \Sigma, z : B \) where \( \mathcal{G} = \mathcal{G} \mid \Gamma, \Delta, \Sigma \), whence \( Q = \nu zw. \nu xy. R \vdash \mathcal{G} \). The other cases are similar.
Figure 8: Translation from CSGV to CSLL, functional fragments
\[
\begin{align*}
\text{\textsc{SelectL}} & \quad \Gamma \vdash M : T_S \oplus U_S \quad \Gamma \vdash \text{select}_L M : T_S \\
& \quad x \leftrightarrow y \vdash x : [T_S]^\perp, y : [T_S] \\
& \quad \forall x, y, z, w : [T_S] \oplus U_S, x \leftrightarrow y \vdash x : [T_S]^\perp, y : [T_S] \\
& \quad \forall x, y, z, w : [T_S] \oplus U_S, x \leftrightarrow y \vdash x : [T_S]^\perp, y : [T_S]
\end{align*}
\]

\[
\text{\textsc{Case}}(\text{Res-Pre}) \quad \Gamma \vdash L : T_S \& U_S \quad \Delta, x : T_S \vdash M : V \quad \Delta, x : U_S \vdash N : V
\]

\[
\left[\begin{array}{c}
\Gamma, \Delta \vdash \text{case L of x}.\{N, M\} : V \\
\end{array}\right]_{\text{def}} = \left[\begin{array}{c}
\left[\begin{array}{c}
[M]_x \vdash [\Delta]^\perp, \underline{x : [T_S]}^\perp, z : [V] \\
[N]_x \vdash [\Delta]^\perp, \underline{x : [U_S]}^\perp, z : [V]
\end{array}\right]_{\text{cut}}\end{array}\right]
\]

\[
\text{\textsc{Case}}(\text{Pre-Par}) \quad \Gamma \vdash \text{case L of x}.\{N, M\} : V
\]

\[
\left[\begin{array}{c}
\Gamma, \Delta \vdash \text{case L of x}.\{N, M\} : V \\
\end{array}\right]_{\text{def}} = \left[\begin{array}{c}
\left[\begin{array}{c}
[M]_x \vdash [\Delta]^\perp, \underline{x : [T_S]}^\perp, z : [V] \\
[N]_x \vdash [\Delta]^\perp, \underline{x : [U_S]}^\perp, z : [V]
\end{array}\right]_{\text{cut}}\end{array}\right]
\]

Figure 9: Translation from CSGV to CSLP, omitted rules of linear fragments

\begin{align*}
\text{\textsc{Case}}(\text{Res-Pre}). & \quad \text{We show the case for } \nu y. z[w]. P \equiv z[w]. \nu x. y. P, \text{ with that of other prefixes being similar. We have} \\
& \quad P \vdash \Sigma, y : C^\perp \mid \Gamma, z : A, x : C \mid \Delta, w : B \\
& \quad z[w], P \vdash \Sigma, y : C^\perp \mid \Gamma, z : B \otimes A, x : C, \Delta \\
& \quad \nu x. y. z[w]. P \vdash \Sigma, \Gamma, z : B \otimes A, \Delta \\
& \quad \text{and therefore} \\
& \quad P \vdash \Sigma, y : C^\perp \mid \Gamma, z : A, x : C \mid \Delta, w : B \\
& \quad \nu x. y. z[w]. P \vdash \Sigma, \Gamma, z : B \otimes A, \Delta \\
& \quad \text{the other case has } x, y, z, w \text{ in separate environments and are simpler.} \\
\text{\textsc{Case}}(\text{Pre-Par}). & \quad \text{We show the case for } x[y].(P \mid Q) \equiv P \mid x[y]. Q, \text{ with that of other prefixes being similar. We have} \\
& \quad P \vdash \Sigma, y : C^\perp \mid \Gamma, x : A \mid \Delta, y : B \\
& \quad P \mid Q \vdash \Sigma, y : C^\perp \mid \Gamma, x : A \mid \Delta, y : B \\
& \quad x[y].(P \mid Q) \vdash \Sigma, \Gamma, x : B \otimes A \\
& \quad \text{and therefore} \\
& \quad P \vdash \Sigma, y : C^\perp \mid \Gamma, x : A \mid \Delta, y : B \\
& \quad P \mid x[y]. Q \vdash \Sigma, \Gamma, x : B \otimes A \\
& \quad P \mid x[y]. Q \vdash \Sigma, \Gamma, x : B \otimes A \\
& \quad \text{Lemma 5. If } P \equiv Q, \text{ then } P \text{ is canonical if and only if } Q \text{ is canonical.} \\
& \quad \text{Proof. Straightforward by induction on } P \equiv Q.
\end{align*}

\begin{align*}
\text{Lemma 6 (Separation). If } T \vdash \Gamma_0 \mid \cdots \mid \Gamma_{n-1}, \text{ then there exist } T_i \vdash \Gamma_i \text{ for } 0 \leq i < n \text{ such that} \\
& \quad T \equiv T_0 \mid \cdots \mid T_{n-1}. \text{ Moreover, if } T \text{ is canonical, then every } T_i \text{ is canonical.} \\
& \quad \text{Proof. We prove the first claim by induction on } T \vdash \Gamma_0 \mid \cdots \mid \Gamma_{n-1}. \text{ We show only the following cases; all other cases are either trivial or similar.} \\
\text{Case}(\text{HMix2}). & \quad \text{Then } T = P \mid Q, \text{ and after appropriately reordering the hyperenvironment we have} \\
& \quad P \vdash \Gamma_0 \mid \cdots \mid \Gamma_{m-1} \text{ and } Q \vdash \Gamma_m \mid \cdots \mid \Gamma_{n-1} \text{ with } m \leq n. \text{ By the IH we have } T_i \vdash \Gamma_i \text{ for } 0 \leq i < n, \\
& \quad \text{with } P \equiv T_0 \mid \cdots \mid T_{m-1}, \text{ and } Q \equiv T_m \mid \cdots \mid T_{n-1}. \text{ We then have } P \mid Q \equiv T_0 \mid \cdots \mid T_{n-1} \equiv T, \text{ as } \equiv \text{ is a congruence.}
\end{align*}
CASE(Cut). Then $T = \nu xy. P$, and after appropriately reordering the hyperenvironment we have

$$P \vdash \Gamma_0 \mid \ldots \mid \Gamma_{n-2} \mid \Delta_0, x : A \mid \Delta_1, y : A^\bot$$

where $\Gamma_{n-1} = \Delta_0, \Delta_1$. By the IH we have $P_i \vdash \Gamma_i$ for $0 \leq i < n-1$, $P_{n-1} \vdash \Delta_0, x : A$, and $P_n \vdash \Delta_1, y : A^\bot$, with $P \equiv P_0 \mid \cdots \mid P_n$. The result follows, as $\nu xy. (P_{n-1} \mid P_n) \vdash \Gamma_{n-1}$, and by (Res-Par)

$$\nu xy. P \equiv \nu xy. (P_0 \mid \cdots \mid P_{n-1}) \mid P_n \equiv P_0 \mid \cdots \mid P_n$$

CASE(Tensor). Then $T = x[y]. P$, and after appropriately reordering the hyperenvironment we have

$$P \vdash \Gamma_0 \mid \ldots \mid \Gamma_{n-2} \mid \Delta_0, x : A \mid \Delta_1, y : B$$

where $\Gamma_{n-1} = \Delta_0, \Delta_1$. By the IH we have $P_i \vdash \Gamma_i$ for $0 \leq i < n-1$, $P_{n-1} \vdash \Delta_0, x : A$ and $P_n \vdash \Delta_1, y : B$ with $P \equiv P_0 \mid \cdots \mid P_n$. The result follows, as $x[y]. (P_{n-1} \mid P_n) \vdash \Gamma_{n-1}, x : B \otimes A$, and by ()

The second claim follows by Lemma 5, and the fact subterms of canonical terms are canonical.

Lemma 7 (Local Progress). If $P \vdash \Gamma, x : A$ and $Q \vdash \Delta, y : A^\bot$ and both $P$ and $Q$ are canonical, then there exists an $R$ such that $\nu xy. (P \mid Q) \longrightarrow R$.

Proof. By induction on $P$ and $Q$. Note the two are symmetric which we will exploit to omit some cases. The type judgment implies neither $P$ nor $Q$ can be stop. They cannot be of the form $\nu xy. S$ either, for they would not be canonical.

- If $P = P_0 \mid P_1$, then it must be that $P_1 = \text{stop}$ without loss of generality. We have that $P_0 \mid \text{stop} \equiv P_0$ by Par-Unit. Apply induction hypothesis on $P_0$ we get $\nu xy. (P_0 \mid Q) \longrightarrow R$. Use Eq we have $\nu xy. (P \mid Q) \longrightarrow R$. Similar when $Q = Q_0 \mid Q_1$.

- If $P = a \leftrightarrow b$, it must be that $x = b$, so we can reduce by Link. Similar for $Q$.

- The remaining scenarios are where $P = \pi_z. P'$, or $P = \pi_z. \text{case}(P_0 \mid P_1)$ or $P = \pi_z. \pi_z. P'$, and similarly $Q = \pi_w. Q'$, or $Q = \pi_w. \text{case}(Q_0 \mid Q_1)$ or $Q = \pi_w. \pi_w. Q'$, and thus $\pi_w. \pi_w. Q'$. If $z = x$ and $w = y$, then one of the reaction axioms applies; otherwise we can assume WLOG that $z \neq x$ (and of course $z \neq y$), and take cases of $P$.

- If $P = \pi_z. P'$, we have that $\nu xy. (\pi_z. P' \mid Q) \equiv \pi_z. \nu xy. (P' \mid Q)$ by Pre-Par and Res-Pre. By induction hypothesis we have $\nu xy. (P' \mid Q) \longrightarrow R$, we therefore have $\nu xy. (P \mid Q) \longrightarrow \pi_z. R$ by Pre and Eq.

- If $P = \pi_z. \text{case}(P_0 \mid P_1)$, the commuting conversion Case-Comm applies.

- If $P = \pi_z. \pi_z. P'$ where $x \in \pi_z$. We know $x : A = ?B$ for some $B$ and thus $y : A^\bot = !B^\bot$. We check if $w = y$. If so, we have $Q = \pi_w. \pi_w. Q'$, and thus OfCourse-Comm applies; otherwise, we know that $Q$ cannot be of the form $\pi_w. \pi_w. Q'$ and thus OfCourse requirement that $\pi_w. \pi_w. Q'$ (because $y : !B^\bot$ breaks OfCourse requirement that $\pi_w. \pi_w. Q'$), which means $Q = \pi_w. Q'$ or $Q = \pi_w. \text{case}(Q_0 \mid Q_1)$ and can be handled similarly as the previous two cases.

Theorem 5 (Progress). If $R \vdash G$ then either $R$ is canonical, or there exists $R'$ such that $R \longrightarrow R'$.

Proof. By induction on $R \vdash G$.

- If $R = \text{stop}$ or $x \leftrightarrow y$, then it is canonical.

- Suppose $R = P \mid Q$. If both $P$ and $Q$ are canonical, then so is $R$. Otherwise, if $P$ is not canonical, then by the IH we have $P \longrightarrow P'$ for some $R'$, and thus $P \mid Q \longrightarrow P' \mid Q$ by ParL. Similarly for $Q$.

- Suppose $R = \pi_y. P$. If $P$ is canonical then so is $R$. Otherwise $P$ is not canonical, and by the IH $P \longrightarrow P'$, and thus $\pi_y. P \longrightarrow \pi_y. P'$ for some $P'$ by Pre.

- Suppose $R = y. \text{case}(P \mid Q)$ or $R = \pi_y. \pi_w. P$, then it is canonical.
• Suppose $R = \nu xy. P$, with $P \vdash \mathcal{G} | \Gamma, x : A | \Delta, y : A^\bot$. If $P$ is not canonical then by the IH we have $P \rightarrow P'$ for some $P'$, and thus $\nu xy. P \rightarrow \nu xy. P'$ by Res. If $P$ is canonical, by Lemma 6 we have that $P \equiv P_0 | \cdots | P_n | P_{n+1}$ where $P_n \vdash \Gamma, x : A$ and $P_{n+1} \vdash \Delta, y : A^\bot$. Note that both $P_n$ and $P_{n+1}$ are canonical. Hence we have $R \equiv \nu xy. (P_0 | \cdots | P_{n+1})$. By Res-Par and Lemma 8 we obtain $R \equiv P_0 | \cdots | P_{n-1} | \nu xy. (P_n | P_{n+1})$. Local progress (Lemma 7) yields $\nu xy. (P_n | P_{n+1}) \rightarrow R'$, which gives $R \rightarrow P_0 | \cdots | P_{n-1} | R'$ by ParL.

\[\square\]

**Definition 3** (Free Names). The free names $\text{FN}(P)$ of a process $P$ is inductively defined as follows.

\[
\begin{align*}
\text{FN}(\text{stop}) &= 0 \\
\text{FN}(P | Q) &= \text{FN}(P) \cup \text{FN}(Q) \\
\text{FN}(\nu xy. P) &= \text{FN}(P) \setminus \{x, y\} \\
\text{FN}(x \rightarrow y) &= \{x, y\} \\
\text{FN}(y(x) \cdot P) &= \text{FN}(y[x]. P) = \text{FN}(P) \setminus \{x\} \\
\text{FN}(x \cdot P) &= \text{FN}(\text{inr}. P) = \text{FN}(x \cdot \text{case}(P; Q)) = \text{FN}(P) \\
\text{FN}(\text{inl}. P) &= \text{FN}(x \cdot P) \cup \{x\} \\
\text{FN}(x[x'] \cdot P) &= \text{FN}(P) \setminus \{x'\} \cup \{x\} \\
\text{FN}(y\{z', w', y', Q\}(z, w). P) &= \text{FN}(P) \cup \{y\}
\end{align*}
\]

The free names $\text{FN}(\Gamma)$ of an environment $\Gamma$ is defined to be the names in $\Gamma$. The free names $\text{FN}(\mathcal{G})$ of an hyperenvironment $\mathcal{G}$ is defined to be the union of the free names of each environment in $\mathcal{G}$. Note that we stipulated names in each environment must not overlap.

**Lemma 8.** If $P \vdash \mathcal{G}$, then $\text{FN}(P) = \text{FN}(\mathcal{G})$.

**Proof.** Straightforward by induction on $P \vdash \mathcal{G}$. \[\square\]

**Lemma 9.** If $P \vdash \mathcal{G} \mid \Gamma, y : A$ and $x \notin \mathcal{G}, \Gamma$, then $P[x/y] \vdash \mathcal{G} \mid \Gamma, x : A$.

**Proof.** Straightforward by induction on $P \vdash \mathcal{G} \mid \Gamma, y : A$. \[\square\]

**Theorem 6** (Preservation). If $P \vdash \mathcal{G}$ and $P \rightarrow Q$, then $Q \vdash \mathcal{G}$.

**Proof.** By induction on $P \rightarrow Q$. We show the nontrivial cases of top-level cuts, and the commuting conversions.

**Case** (Eq). Suppose $P \equiv P' \rightarrow Q' \equiv Q$. Then the result follows by the IH and two applications of Lemma 2.

**Case** (Case-Comm). The redex is $\nu xy. (z \cdot \text{case}(P_0; P_1) \mid Q)$ and typed.

\[
\begin{align*}
P_0 \vdash \Gamma, x : C, z : A & \quad P_1 \vdash \Gamma, x : C, z : B \\
\text{z.case}(P_0; P_1) \vdash \Gamma, x : C, z : A \land B & \quad Q \vdash \Delta, y : C^\bot
\end{align*}
\]

and therefore

\[
\begin{align*}
\nu xy. (\text{z.case}(P_0; P_1) \mid Q) \vdash \Gamma, \Delta, z : A & \quad \nu xy. (P_1 \mid Q) \vdash \Gamma, \Delta, z : B
\end{align*}
\]

**Case** (OfCourse-Comm). The redex is $\nu xy. (\lambda x \cdot \nu \bar{v}. P) \mid y \cdot \nu \bar{v}. Q)$ and typed.

\[
\begin{align*}
P \vdash \bar{w} :: \tilde{\mathcal{B}}, z : A, x : C & \quad Q \vdash \bar{v} :: \tilde{\mathcal{D}}, y : C^\bot \\
\lambda x \cdot \nu \bar{v}. P \vdash \bar{w} :: \tilde{\mathcal{B}}, z : A, x : C & \quad \nu \bar{v}. Q \vdash \bar{v} :: \tilde{\mathcal{D}}, y : C^\bot
\end{align*}
\]
and therefore
\[ P \vdash \vec{w} : ?\vec{B}, z : \text{A}, x : ?C \quad !y \{\vec{v}, Q\} \vdash \vec{w} : ?\vec{D}, y : !C' \]
\[ \nu xy. (P \mid ! y \{\vec{v}, Q\}) \vdash \vec{w} : ?\vec{B}, z : \text{A}, \vec{v} : ?\vec{D} \]
\[ !z \{\vec{v}, \vec{w}, \nu xy. (P \mid ! y \{\vec{v}, Q\})\} \vdash \vec{w} : ?\vec{B}, z : !\text{A}, \vec{v} : ?\vec{D} \]

**CASE (Link).** Then the redex is \( \nu xy. (z \leftrightarrow x \mid P) \) and the last steps of the typing derivation must have been
\[ z \leftrightarrow x \vdash z : \text{A}^\perp, x : \text{A} \quad P \vdash \mathcal{G} \mid \Gamma, y : \text{A}^\perp \]
\[ z \leftrightarrow x \mid P \vdash z : \text{A}^\perp, x : \text{A} \quad \nu xy. (z \leftrightarrow x \mid P) \vdash \mathcal{G} \mid \Gamma, y : \text{A}^\perp \]
and therefore \( P[z/y] \vdash \mathcal{G} \mid \Gamma, z : \text{A}^\perp \) by Lemma 9 because \( z \notin \text{FN}(P) \). (otherwise the redex would not be well-typed)

**CASE (One-Bot).** Then the redex is \( \nu xy. (x() \mid P \mid y[]. Q) \) and the last steps of the typing derivation must have been
\[ P \vdash \mathcal{G} \mid \Gamma \quad Q \vdash \mathcal{H} \]
\[ x(). P \vdash \mathcal{G} \mid \Gamma, x : \bot \]
\[ y[]. Q \vdash \mathcal{H} \mid y : 1 \]
\[ \nu xy. (x(). P \mid y[]. Q) \vdash \mathcal{G} \mid \mathcal{H} \mid \Gamma \]
Hence, we have
\[ P \vdash \mathcal{G} \mid \Gamma \quad Q \vdash \mathcal{H} \]
\[ P \mid Q \vdash \mathcal{G} \mid \mathcal{H} \mid \Gamma \]

**CASE (Tensor-Par).** Then the redex is \( \nu xy. x[z]. P \mid y(w). Q \) and the last steps of the typing derivation must have been
\[ P \vdash \mathcal{G} \mid \Gamma, z : A \mid \Delta, x : B \]
\[ Q \vdash \mathcal{H} \mid \Sigma, w : A', y : B'_\perp \]
\[ x[z]. P \vdash \mathcal{G} \mid \Gamma, \Delta, x : A \otimes B \]
\[ y(w). Q \vdash \mathcal{H} \mid \Sigma, y : A' \otimes B'_\perp \]
\[ \nu xy. (x[z]. P \mid y(w). Q) \vdash \mathcal{G} \mid \mathcal{H} \mid \Gamma, \Delta, \Sigma \]
so that \( \mathcal{G} = \Gamma, \Delta, \Sigma \). Therefore, we can infer that
\[ P \vdash \mathcal{G} \mid \Gamma, z : A \mid \Delta, x : B \quad Q \vdash \mathcal{H} \mid \Sigma, w : A', y : B'_\perp \]
\[ \nu xy. \nu zw. (P \mid Q) \vdash \mathcal{G} \mid \mathcal{H} \mid \Gamma, \Delta, \Sigma \]

**CASE (PlusL-With).** Then the redex is \( \nu xy. (x[]. P \mid y. \text{case}(Q_l; Q_r)) \), and the last steps of the typing derivation must have been
\[ P \vdash \mathcal{G} \mid \Gamma, x : A \]
\[ Q_l \vdash \Delta, y : A'_\perp \quad Q_r \vdash \Delta, y : B'_\perp \]
\[ \nu xy. (x[]. P \mid y. \text{case}(Q_l; Q_r)) \vdash \mathcal{G} \mid \Gamma, \Delta \]
Hence,
\[ P \vdash \mathcal{G} \mid \Gamma, x : A \quad Q_l \vdash \Delta, y : A'_\perp \]
\[ \nu xy. (P \mid Q_l) \vdash \mathcal{G} \mid \Gamma, \Delta \]

**CASE (Claro-QueW).** This is the case of an empty client pool. The redex must be
\[ \nu xy. (\varnothing[x]. S \mid ! y \{z, z', y', Q\}(i, f). P) \]
and the last steps in the typing derivation must have been
\[ \mathcal{D} \vdash \mathcal{G} \mid x : \varnothing A \]
\[ P \vdash \mathcal{H} \mid \Delta, i : B \mid \Sigma, f : B'_\perp \]
\[ Q \vdash z : B'_\perp, z' : B, y' : A'_\perp \]
\[ \nu xy. (\varnothing[x]. S \mid ! y \{z, z', y', Q\}(i, f). P) \vdash \mathcal{G} \mid \mathcal{H} \mid \Delta, \Sigma \]
\[ \nu xy. (\mathcal{D} \mid ! y \{z, z', y', Q\}(i, f). P) \vdash \mathcal{G} \mid \mathcal{H} \mid \Delta, \Sigma \]
\[ \text{Cut} \]
where $D \overset{\text{def}}{=} S \vdash \emptyset, S \vdash \emptyset | x : iA$. Hence,

$$
\begin{align*}
P & \vdash H | \Delta, i : B | \Sigma, f : B^\perp \\
S & \vdash \emptyset \\
\nu f \cdot P & \vdash H | \Delta, \Sigma
\end{align*}
$$

$S \vdash \emptyset \iff \nu f \cdot P \vdash H | \Delta, \Sigma$

**Case** (Claro-QueA). Then the redex is

$$
\nu xy, (\nu x[x']). S \vdash \emptyset | xy \{z, z', y', Q(i, f), P\}
$$

The last few steps in the typing derivation must have been

$$
\begin{align*}
D & \vdash \emptyset | \Gamma, \Gamma', x : iA \\
P & \vdash H | \Delta, i : B | \Sigma, f : B^\perp \\
Q & \vdash z : B^\perp, z' : B, y' : A^\perp \\
\nu xy \cdot (\nu x[x'] \cdot S \vdash \emptyset | \Gamma, \Gamma', x : iA) & \vdash \emptyset | \Gamma, \Gamma', \Delta, \Sigma
\end{align*}
$$

where

$$
D \overset{\text{def}}{=} S \vdash \emptyset | \Gamma, x : iA | \Gamma', x' : A
$$

Therefore,

$$
\begin{align*}
S & \vdash \emptyset | \Gamma, x : iA | \Gamma', x' : A \\
\nu xy \cdot \nu x[x'][S \vdash \emptyset | \Gamma', x' : A] & \vdash \emptyset | \Gamma, \Gamma', \Delta, \Sigma
\end{align*}
$$

where $D$ is

$$
\nu xz(P | Q) \vdash H | \Delta, z' : B, y' : A^\perp | \Sigma, f : B^\perp \\
\nu y(z, z', y', Q(i, f) \cdot P) \vdash H | \Delta, y' : A^\perp, \Sigma, y : iA^\perp
$$

**Case** (OfCource-WhyNotW).

$$
\begin{align*}
P & \vdash \emptyset | \Gamma \\
\nu xy \cdot (?x[x'] \cdot P \vdash \emptyset | \Gamma, x : ?A) & \vdash \emptyset | \Gamma, ?z : ?B, y : A^\perp \\
l y(\emptyset Q) & \vdash \emptyset | \Gamma, ?z : ?B, y : !A^\perp
\end{align*}
$$

and therefore

$$
P \vdash \emptyset | \Gamma \\
\nu xy \cdot (?x[x'] \cdot P \vdash \emptyset | \Gamma, x : ?A) & \vdash \emptyset | \Gamma, ?z : ?B
$$

**Case** (OfCource-WhyNotD).

$$
\begin{align*}
P & \vdash \emptyset | \Gamma, x' : A \\
\nu xy \cdot (?x[x'] \cdot P \vdash \emptyset | \Gamma, x : ?A) & \vdash \emptyset | \Gamma, ?z : ?B, y : A^\perp \\
l y(\emptyset Q) & \vdash \emptyset | \Gamma, ?z : ?B, y : !A^\perp
\end{align*}
$$

and therefore

$$
P \vdash \emptyset | \Gamma, x' : A \\
\nu xy \cdot (?x[x'] \cdot P \vdash \emptyset | \Gamma, x : ?A) & \vdash \emptyset | \Gamma, ?z : ?B
$$

**Case** (OfCource-WhyNotC).

$$
\begin{align*}
P & \vdash \emptyset | \Gamma, y_0 : ?A, y_1 : ?A \\
\nu xy \cdot (?x[y_0, y_1] \cdot P \vdash \emptyset | \Gamma, x : ?A) & \vdash \emptyset | \Gamma, ?z : ?B, y : A^\perp \\
l y(\emptyset Q) & \vdash \emptyset | \Gamma, ?z : ?B, y : !A^\perp
\end{align*}
$$

and therefore

$$
P \vdash \emptyset | \Gamma, y_0 : ?A, y_1 : ?A \\
\nu xy \cdot (?x[y_0, y_1] \cdot P \vdash \emptyset | \Gamma, x : ?A) & \vdash \emptyset | \Gamma, ?z : ?B
$$

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and therefore

\[
\mathcal{D} \vdash \Gamma, \vec{z}_0 : ?\vec{B}, \vec{z}_1 : ?\vec{B} \\
?\vec{z}[\vec{z}_0, \vec{z}_1], \mathcal{D} \vdash \Gamma, \vec{z} : ?\vec{B}
\]

where \(\mathcal{D}\) is

\[
\begin{align*}
&\mathcal{P} \vdash \Gamma, y_0 : ?A, y_1 : ?A & Q[z_0/\vec{z}] \vdash \vec{z}_0 : ?\vec{B}, y : A^\perp & Q[z_1/\vec{z}] \vdash \vec{z}_1 : ?\vec{B}, y : A^\perp \\
&\forall y_0 : \vec{z}_0, \forall y_1 : \vec{z}_1, (P \mid Q[z_0/\vec{z}] \mid Q[z_1/\vec{z}]) \vdash \Gamma, \vec{z}_0 : ?\vec{B}, y_1 : ?A
\end{align*}
\]

Lemma 10. \([\overline{T_S}] = [T_S]^\perp\).

\(\square\)

Proof. By simple induction. \(\square\)

Lemma 11. If \(T\) is unlimited, we have the following derivable rule in CSLL.

\[
\text{Positive} \\
\mathcal{P} \vdash \Gamma, x : ?[T]^\perp \\
\overline{\lambda x'}. \mathcal{P} \vdash \Gamma, x : [T]^\perp
\]

The above lemma means that all unlimited types enjoy contraction and weakening. Some session types, such as \textit{end}, also enjoy such properties: see e.g. (Gay and Vasconcelos, 2010, §5). However, in order to retain the good properties of termination and deadlock-freedom, we insist that all channels are used linearly, and carefully closed at the end.

Proof. It is given by the well-known fact that \(!A, \ 1\) and \(0\) are always positive, that \(\otimes, \oplus\) preserve positivity, and that our system (without server and client) is equivalent to linear logic in terms of expressivity (Kokke et al., 2019a, §2.3). More concretely, we derive the rule by induction on \(T\).

- If \(T\) is Unit we have

\[
\begin{array}{l}
y[y]. \text{stop} \vdash y : 1 \\
\ \upharpoonright\ \downarrow y[y]. \text{stop} \vdash y : 1 \\
\end{array}
\]

\[
x() \upharpoonright y[y]. \text{stop} \vdash y : 1, x : \perp
\]

Cut \(y\) of this with \(x'\) of \(P\) and we are done.

- If \(T\) is \(U \rightarrow V\), we have

\[
\begin{array}{l}
y \leftrightarrow x \vdash y : !(U \otimes [V]^\perp), x : ?([U]^\perp \otimes [V]) \\
\ \uparrow y[x, y \leftrightarrow x] \vdash y : !(U \otimes [V]^\perp), x : ?([U]^\perp \otimes [V])
\end{array}
\]

Cut \(y\) of this with \(x'\) of \(P\) and we are done.

- If \(T\) is \(U + V\), and both \(U\) and \(V\) are unlimited. First we apply Lemma 6 on \(P\) and acquire \(P_0 \vdash \mathcal{G}\) and \(P_1 \vdash \Gamma, x : ?([U]^\perp \& [V]^\perp)\), and we have \(\mathcal{D}_U\) defined as

\[
\begin{array}{l}
y \leftrightarrow x \vdash y : [U], x : [U]^\perp \\
y[y]. \text{inl} \vdash y : [U] \oplus [V], x : [U]^\perp \\
\ \uparrow x'[x], y[y]. \text{inl} \vdash x' : ?[U]^\perp \\
\ \uparrow !x'[x]. y[y[x', y[y]. \text{inl} \vdash y : !(U \oplus [V]), x : [U]^\perp
\end{array}
\]

and similarly for \(\mathcal{D}_V\). Finally we have

\[
\begin{array}{l}
\mathcal{D}_U \vdash y : !(U \oplus [V]), x : [U]^\perp \\
\mathcal{D}_V \vdash y : !(U \oplus [V]), x : [V]^\perp \\
\mathcal{D} \vdash y : !(U \oplus [V]), x : [U]^\perp \& [V]^\perp \\
\end{array}
\]

Cut \(y\) of this with \(x'\) of \(P_1\) then combine with \(P_0\) and we are done.
• If $T$ is $U \otimes V$, and both $U$ and $V$ are unlimited, we have

\[
\frac{v \leftrightarrow v' \vdash v : [V], v : [V]^\perp \quad u \leftrightarrow u' \vdash u : [U], u : [U]^\perp}{\vdash v'[u'], (v \leftrightarrow v' \mid u \leftrightarrow u') \vdash v' : [U] \otimes [V], u : [U]^\perp, v : [V]^\perp}
\]

\[
\frac{\vdash u'[u], ?u'[u'], ?v'[u'], (v \leftrightarrow v' \mid u \leftrightarrow u') \vdash v' : [U] \otimes [V], u : [U]^\perp, v : [V]^\perp}{\vdash v(u). ?u[u'], ?v'[u']. \lambda(v, u, v) \cdot (v \leftrightarrow v' \mid u \leftrightarrow u') \vdash v' : !([U] \otimes [V]), v : [U]^\perp \otimes [V]^\perp}
\]

Cut $v'$ of this with $x$ of $P$, rename $v$ to $x$ and we are done.

\qed

## D More Examples

### D.1 Compare-And-Set

We now recover the example of CAS server/client in CSGV. We define the server-client protocol to be $T_S \stackrel{\text{def}}{=} \lambda!B. !B. ?B. \text{end}_r$. The choice of end$_r$ vs. end$_l$ is purely driven by well-typedness.

\[
C_0 \stackrel{\text{def}}{=} \text{let } x_0 = \text{send } x_0 \text{ in } \text{let } x_r = \text{send } \texttt{tt} x_0 \text{ in } \text{let } \_ = \text{terminate (send } r \text{ in } x') \\
C_1 \stackrel{\text{def}}{=} \ldots \text{ clients} \stackrel{\text{def}}{=} \text{let } x = \text{fork}_x x_0. C_0 \text{ in } \text{let } x = \text{fork}_x x_1. C_1 \text{ in } \text{eof}_x
\]

typed as:

\[
x_0 : T_S, r_0 : !B. \text{end}_r \vdash C_0 : \text{end}_r \\
x_1 : T_S, r_1 : !B. \text{end}_r \vdash C_1 : \text{end}_r \\
x : ?T_S, r_0 : !B. \text{end}_r, r_1 : !B. \text{end}_r \vdash \text{clients} : \text{end}_r
\]

where $\overline{T_S} = ?B. ?B. !B. \text{end}_r$. Finally we have

\[
r_0 : !B. \text{end}_r, r_1 : !B. \text{end}_r \vdash \text{connect}(x, \text{clients}; y, \text{server}) : B
\]

### D.2 List Shuffling

We use the racing behaviour of clients to shuffle a list. We define server/client protocol to be $T_S \stackrel{\text{def}}{=} !A. \text{end}_r$, meaning each client sends a value of $A$ and ends the session. Each clients are defined the same way: they simply take the value $A$ from the environment and send it over the channel. Clients are forked by folding the list. The server simply receives values from clients and reforms the list.

\[
L \stackrel{\text{def}}{=} \text{nil} \\
M \stackrel{\text{def}}{=} \text{let } (v, y') = \text{recv } y \text{ in } \text{let } \_ = \text{terminate } y' \text{ in } \text{cons } v z' \\
N \stackrel{\text{def}}{=} z \\
S \stackrel{\text{def}}{=} \text{serve } y(L, z', M, z, N)
\]

\[
C \stackrel{\text{def}}{=} \text{send } v x' \\
\text{ clients} \stackrel{\text{def}}{=} \text{let } y = \text{fold}_1^{T_S} x \lambda x. \lambda v. (\text{fork}_x x', C) \text{ l in } \text{eof}_y \\
\text{ server} \stackrel{\text{def}}{=} \text{serve } y(L, z', M, z, N)
\]

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and we have the following typing

\[
v : A, x' : T_S \vdash C : \text{end}; \\
l : [A], x : \exists T_S \vdash \text{clients} : \text{end}; \\
z : [A] \vdash N : [A] \\
L : [A], y : \exists T_S \vdash \text{server} : [A]
\]

and finally we define shuffling as

\[
l : [A] \vdash \text{connect}(x. \text{clients}; y. \text{server}) : [A]
\]

### D.3 Merge Sort

Using fork-join, we can define parallel merge sort. We first have to assume general recursion (and therefore not expressible in the vanilla CSGV), and two functions on lists of \( A \). split splits a list of \( A \) into several (supposedly two) lists, and merge merges several sorted lists into one list. isend returns \( \text{tt} \) if the list is empty or singleton.

\[
\vdash \text{split} : [A] \rightarrow [[A]] \\
\vdash \text{merge} : [[A]] \rightarrow [A] \\
\vdash \text{isend} : [A] \rightarrow B
\]

And we define merge sort as follows. We first check if the list \( l \) is too short to sort; if not we split the list and sort each of the sub-lists. The sorted sub-lists are collected into \( l' \) which we will merge. Note that the racing behaviour of client/server means \( l' \) could be any ordering, which does not matter for merge sort. For scenarios where it does matter, each sub-result should be accompanied by its index to get re-ordered.

\[
\text{sort } l \overset{\text{def}}{=} \begin{cases} \text{isend } l & \text{then } l \\ \text{else} \\ \text{let } l' = \text{fork-join}(\text{sort}, \text{cons}, \text{nil}, (\text{split } l)) \text{ in} \\ \text{merge } l' \end{cases}
\]

which gives us

\[
\vdash \text{sort} : [A] \rightarrow [A]
\]

### D.4 Map-Reduce

The purpose of the map-reduce model is to transform input of type \([A]\) into output of type \([D]\) using two functions (using the functorial formulation given by Hinrichsen et al. (2019)). Take the example of counting the frequency of each word in an article which contains several paragraphs \([A]\). The map function \( f \) counts the frequency \( C \) of each word \( B \) in a paragraph. The reduce function \( g \) takes a word \( B \) and its frequency \( [C] \) in all paragraphs and simply returns the word with the sum frequency \( D \overset{\text{def}}{=} B \otimes C \).

In the end we hope to get \([D]\).

\[
f : A \rightarrow [B \otimes C] \\
g : B \otimes [C] \rightarrow D
\]

We first define parallelized flatMap with fork-join:

\[
\vdash \text{flatMap}_{A,B} \overset{\text{def}}{=} \lambda f. \lambda l. \text{fork-join}(f, \text{nil}, \text{concat}, l) : (A \rightarrow [B]) \rightarrow [A] \rightarrow [B]
\]

where concat is the standard function that concatenate two lists. Based on this we define map-reduce:

\[
f, g \vdash \text{map-reduce} \overset{\text{def}}{=} (\text{flatMap}_{B \otimes [C], D} g) \circ \text{group}_{B,C} \circ (\text{flatMap}_{A, B \otimes C} f) : [A] \rightarrow [D]
\]

where \( \vdash \text{group}_{B,C} : [B \otimes C] \rightarrow [B \otimes [C]] \) is the standard function that groups list of pairs by their keys.

There is a notable difference between our version of map-reduce and the version in Hinrichsen et al. (2019) (and other related literatures). Usually a fixed number of threads (that usually corresponds to the number of cpu cores/nodes) are spawned, who will then repeatedly retrieve tasks from and send result to the main thread. In our version, however, the number of threads is the number of tasks, and each thread will handle one task only. The former approach seems lower-level, allowing optimizing the number of threads according to the hardware reality. Our language is higher-level, and it is up to the implementation to coordinate threads with cores/nodes. Implementing it at a lower-level seems to be difficult because of the linearity constraints.
D.5 Interleaving clients

Another interleaving clients example (but simpler than the beauty contest example) is one where each client submits a boolean to the server, who calculates the XOR of all the submissions and sends the result back to all clients. The internal protocol of the server, as well as the server interface, will be \( T_S \defeq \mathbb{T} \mathbb{B}. ! \mathbb{B}. \mathbb{B}. \mathbb{E}. \). We define

\[
L \defeq \begin{align*}
& \text{let } w' = \text{send } ff \text{ in} \\
& \text{let } (w''') = \text{recv } w' \text{ in} \\
& w'''
\end{align*}
\]

(send initial value)

\[
M \defeq \begin{align*}
& \text{let } (s, z'') = \text{recv } z' \text{ in} \\
& \text{let } (b, y') = \text{recv } y \text{ in} \\
& \text{let } s' = \text{xor } (s, b) \text{ in} \\
& \text{let } w' = \text{send } s' \text{ w in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (b, y') = \text{recv } y \text{ in} \\
& \text{let } s' = \text{xor } (s, b) \text{ in} \\
& \text{let } w' = \text{send } s' \text{ w in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (b, y') = \text{recv } y \text{ in} \\
& \text{let } s' = \text{xor } (s, b) \text{ in} \\
& \text{let } w' = \text{send } s' \text{ w in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (b, y') = \text{recv } y \text{ in} \\
& \text{let } s' = \text{xor } (s, b) \text{ in} \\
\end{align*}
\]

(recv the boolean from client)

\[
N \defeq \begin{align*}
& \text{let } (f, z') = \text{recv } z \text{ in} \\
& \text{let } z'' = \text{send } f \text{ z' in} \\
& \text{terminate } z''
\end{align*}
\]

(send the final to previous worker process)

\[
\text{server} \defeq \text{serve } y \{ \text{inv}_w(L), z', \text{inv}_w(M), z, N \}
\]

We omit defining the clients as they will be very similar to the ones in previous examples.

D.6 Symbol Generator

Another simple scenario is where server acts like a generator of unique symbols (essentially natural numbers \( \mathbb{N} \)) and clients race to acquire those symbols. The server protocol is \(!N. \mathbb{E}. \), meaning the server simply sends a number to the client and ends the session; the server internal state is \( \mathbb{N} \).

\[
L \defeq \begin{align*}
& \text{let } w' = \text{send } ff \text{ in} \\
& \text{let } (w''') = \text{recv } w' \text{ in} \\
& w'''
\end{align*}
\]

(starts with zero)

\[
M \defeq \begin{align*}
& \text{let } s = \text{send } s \text{ w in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
& \text{let } (f, w'') = \text{recv } w' \text{ in} \\
\end{align*}
\]

(output the final counter)

\[
\text{server} \defeq \text{serve } y \{ \text{inv}_w(L), z', \text{inv}_w(M), z, N \}
\]

We omit defining the clients as they would be similar to previous ones; but we note that it is impossible for a process to act as multiple clients and aggregate two symbols. The reason is that informally speaking, in our system clients are not allowed to communicate with each other besides via the server as indicated by the functor. More concretely, supposed we are to define a process acting as multiple clients, it would be typed as \( \Gamma, x_0 : ?N. \mathbb{E}. x_1 : ?N. \mathbb{E}. \vdash K : T \); but there is no way in CSGV to combine \( x_0 \) and \( x_1 \).