On the Classicality of Broda’s SU(2) Invariants of 4-Manifolds

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Abstract: Recent work of Roberts [R] has shown that the surgical 4-manifold invariant of Broda [B1] and (up to an unspecified normalization factor) the state-sum 4-manifold invariant of Crane-Yetter [CY] are equivalent to the signature of the 4-manifold. Subsequently Broda [B2] defined another surgical invariant of 4-manifolds in which the 1- and 2- handles are treated differently. We use a refinement of Roberts’ techniques developed in [CKY] to identify the normalization factor to show that the “improved” surgical invariant of Broda [B2] also depends only on the signature and Euler character.

As a starting point, let us first observe that the construction of Crane-Yetter [CY] does not really depend on the use of labels chosen from the irreps of \( U_q(sl_2) \) at the principal \( r^\text{th} \) root of unity: the simple objects of any artinian semi-simple tortile category (cf. [S, Y]) in which all objects are self-dual and the fusion rules are multiplicity free will suffice. In particular, if we restrict to the integer spin (bosonic) irreps, we obtain a construction of a different invariant of 4-manifolds.

In what follows, we use Temperley-Lieb recoupling theory (cf. [KL,L,R]). In particular, arcs are labelled with elements of \( \{0, 1, \ldots, r-2\} \) (twice the spin), \( A = e^{2\pi i/4r}, q = A^2 \), \( \Delta(n) = (-1)^n \frac{2^{n+1} - q^n - q^{n-1}}{q-q^{-1}} \), \( \theta(a,b,c) \) denoted the evaluation of the theta-net with edge labelled \( a, b, \) and \( c \), and \( 15j - j \) denotes the evaluation of the Temperley-Lieb version of the Crane-Yetter quantum 15j-symbol (with indices suppressed).

We then adopt the following further notational conventions:

Arcs labelled \( \omega \) denote the linear combination of arcs labelled \( 0, 1, \ldots, r-2 \) in which the coefficient of \( i \) is \( \Delta(i) \). Arcs labelled \( \tilde{\omega} \) denote the linear combination of arcs labelled \( 0, 2, \ldots, 2\lfloor \frac{r-2}{2} \rfloor \) (even integers) in which the coefficient of \( i \) is \( \Delta(i) \). \( N \) denotes the sum of the squares of the \( \Delta(i)'s \), \( \tilde{N} \) denotes the sum of the squares of the \( \Delta(i)'s \) for \( i \) even. Let \( \kappa \) be as in [KL,R], the evaluation of an \( \omega \) labelled 1-framed unknot divided by the positive square root of \( N \), and let \( \tilde{\kappa} \) be the evaluation of an \( \tilde{\omega} \) labelled 1-framed unknot divided by \( \tilde{N} \).

If \( L \) is a framed link, then \( \tilde{\omega}(L) \) denotes the evaluation of the link with all components labelled \( \tilde{\omega} \). If \( L \) is a set of 4-manifold surgery instructions (cf. Kirby [K]), that is a link \( L \) with a

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1Supported by National Science Foundation grant #DMS-9106476
2Supported by National Science Foundation grant #DMS-9205277 and the Program for Mathematics and Molecular Biology of the University of California at Berkeley, Berkeley, CA
3This use of bosonic is a hideous abuse of language—everything in sight has braid statistics—the “bosons” of this paper are the result of q-deforming honest bosons.
distinguished 0-framed unlink $\hat{L}$, then $\mathcal{B}'(\mathcal{L})$ denotes the evaluation of the link $L$ with all components of $\hat{L}$ (one-handle attachments) colored $\omega$ and all other components of $L$ (two-handle attachments) colored $\tilde{\omega}$.

We then have

**Lemma 1** $\tilde{\omega}(L)$ is invariant under handle-sliding. $\mathcal{B}'\mathcal{L}$ is invariant under handle-sliding of 1- and 2-handles 1-handles and of 2-handles over 2-handles.

**proof:** This follows immediately from handle-sliding over components labelled $\omega$ and the analysis given in Remark 17 §12.6 of Kauffman/Lins [KL] once it is observed that pairs of bosons only couple to produce bosons. $\square$

And

**Lemma 2** (The bosonic encirclement lemma)

\[
\sum_{j \text{ even}} \sum_{j} = 0
\]

whenever $n$ is even and non-zero.

**proof:** This follows from the same proof as the encirclement lemma of Lickorish [L] (cf. also Kauffman/Lins [KL]) with the “auxiliary loop” labelled 2 instead of 1. $\square$

Let

\[
\text{CY}_B(W) = \tilde{N}^{-n_0-n_1} \sum \prod \Delta(\lambda(\sigma)) \prod \frac{\Delta(\lambda(\sigma))}{\theta(\lambda(\tau), \lambda(\tau_0), \lambda(\tau_2))\theta(\lambda(\tau), \lambda(\tau_1), \lambda(\tau_3))} \prod \text{4-simplexes}
\]

de the bosonic Crane-Yetter invariant.

Let $|L|$ (resp. $\nu(L)$, $\sigma(L)$) denote the number of components of a link $L$ (resp. the nullity of the linking matrix of $L$, the signature of the linking matrix of $L$).

We can then define a purely bosonic version of Broda’s original invariant by

\[
BR_B(W) = \frac{\tilde{\omega}(L)}{N^{|L|+\sigma(L)}}
\]

where $L$ is the underlying link of a surgery presentation of $W$; while a bosonic version of the Reshetikhin/Turaev [RT] 3-manifold invariant is given by

\[
I_B(M) = \tilde{\kappa}^{-\sigma(L)} \tilde{N}^{-1|L|+1} \tilde{\omega}(L)
\]

where $L$ is a framed link giving surgery instructions for $M$.

Applying the two lemmas above in an analysis otherwise identical to that of given by Roberts [R] of the original Broda invariant [B1] shows that
Proposition 3

\[ Br_B(W) = \tilde{\kappa}^{\sigma(W)} \]

Similarly it follows from the bosonic encirclement lemma that

\[ CY_B(W) = \tilde{N}^{n_0-n_1-n_3} \bar{\omega}(L) \]

where \( n_d \) is the number of \( d \)-simplexes in a triangulation, and \( L \) is the link derived from a triangulation by putting a 0-framed unknot in each tetrahedron, and a loop around each 2-simplex (running mostly through 4-simplexes but linking each tetrahedron’s unknot) after the manner of Roberts [R].

It then follows as in [CKY] that

Proposition 4

\[ CY_B(W) = \tilde{\kappa}^{\sigma(W)} N^{\chi(W)/2} \quad (*) \]

Now, Broda’s new invariant is defined by

\[ B(W) = \frac{B^!(L)}{N^{n(L)-\nu(L)}} \]

For convenience we first analyse a slightly different normalization (for which the proof of invariance is effectively identical to that for \( B(W) \): let

\[ B(W) = \frac{B^!(L)}{N^{n(L)-|L| |N|L|}} \]

Now, it follows from the original encirclement lemma of Lickorish [L] that

\[ CY_B(W) = \tilde{N}^{n_0-n_1-n_3} B^!(L) \quad (**) \]

where \( L \) is the surgery instructions given by associating the link \( L \) to the triangulation as above, and letting \( \hat{L} \) be the unlink of loops in the tetrahedra.

Observe that \( B \) is multiplicative under connected sum, and that \( B(S^1 \times S^3) = \tilde{N} \) (an easy calculation). As shown in Roberts [R], \( L \) is a surgery presentation for \( W \# (\#^{n_4-1} S^1 \times S^3) \).

From this and the fact that for \( L \), \(|L - \hat{L}| = n_2 \) and \(|\hat{L}| = n_3 \), we see that

\[ \frac{B^!(L)}{N^{n_2-n_3 N_n^3}} = B(W \# (\#^{n_4-1} S^1 \times S^3)) \]

\[ = B(W) \tilde{N}^{n_4-1}. \]

Thus

\[ B^!(L) = B(W) \tilde{N}^{n_2-n_3+n_4-1} N_n^3. \quad (***) \]

It then follows from (*) , (** ) and (***) that

\[ B(W) = \tilde{\kappa}^{\sigma(W)} N^{\chi(W)/2-1} \]

To return to Broda’s [B2] original normalization, note that

\[ B(W) = B(W)(\tilde{N} N^{-\frac{1}{2}})^{|L-L| - |L| - \nu(L)} \]

From which we obtain
**Theorem 5** If $W$ is a connected closed oriented smooth 4-manifold, then

$$B(W) = \kappa(W) \left( \frac{\bar{N}}{N} \right)^{\chi(W) - 1}$$

**proof:** It suffices to shown that if $W$ is given by the surgery instruction $\mathcal{L}$, then

$$|L - \dot{L}| - |\dot{L}| - \nu(L) = \chi(W) - 2.$$ But this follows immediately from the observation that $\nu(L)$ is the number of 3-handles attached in completing the construction of $W$. □

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