Effective Inspiral Spin Distribution of Primordial Black Hole Binaries

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Abstract

We investigate the probability distribution of the effective inspiral spin, the mass ratio, and the chirp mass of primordial black hole (PBH) binaries, incorporating the effect of the critical phenomena of gravitational collapse. As a leading order estimation, each binary is assumed to be formed from two PBHs that are randomly chosen according to the probability distribution of single PBHs, and the accretion effects on the mass and spin of each PBH are not considered for simplicity. We find that, although the critical phenomena can lead to large spins on the low-mass tail, the effective inspiral spin of the binary is statistically very small, $\sqrt{\chi_{\text{eff}}^2} \sim 10^{-3}$. We also see that there is almost no anticorrelation between the effective inspiral spin and the mass ratio, which can be inferred from observations.

Unified Astronomy Thesaurus concepts: Black holes (162); Cosmology (343)

1. Introduction

The success in the direct observation of gravitational waves from compact binary coalescence has provided us with much information about astrophysics and cosmology. It reveals the abundant existence of massive black holes (see Gravitational-Wave Transient Catalog 3 (GWTC-3; Abbott et al. 2021a) for the latest data up to the end of LIGO–Virgo’s third observing run (O3)), shows us the detailed dynamics of a kilonova (Metzger 2020) and the properties of high-density nuclear matter (Abbott et al. 2018), constrains theories of modified gravity (Abbott et al. 2019, 2021b), will be possibly used to measure the Hubble constant of the universe (Abbott et al. 2021), etc. However, the origin of the source compact objects (black holes in particular) is not fully clear yet, represented by relatively large masses of black holes compared to the known astrophysical models of the black hole formation (e.g., see the implications of the $150 M_{\odot}$ binary black hole merger; Abbott et al. 2020a) and the compact binary merger in the low-mass gap (Abbott et al. 2020b). Candidate events of subsolar masses, which cannot be stellar black holes, are also reported (Phukon et al. 2021). In addition to the total mass of the binary, the effective inspiral spin $\chi_{\text{eff}} = (a_1 \cos \theta_1 + q a_2 \cos \theta_2)/(1 + q)$ and the mass ratio $q = M_2/M_1$ are other important characteristics of binaries to identify their origins, where $M_i, a_i$, and $\theta_i$ with $i = 1, 2$ are individual masses, individual dimensionless Kerr (spin) parameters, and the angles of individual spins with respect to the orbital angular momentum, respectively. The positive (negative) $\chi_{\text{eff}}$ indicates the alignment (antialignment) of their spins. Callister et al. (2021) and Abbott et al. (2021c) found broad distributions of $\chi_{\text{eff}}$ and $q$ in the observational data (GWTC-3), allowing the negative $\chi_{\text{eff}}$ in the posterior, particularly for less hierarchical ones $q \sim 1$ (note that there are only two candidate events, GW191109_010717 and GW200225_060421, with significant support; Abbott et al. 2021c). Such a spin antialignment is counterintuitive because progenitors’ spins are expected to be nearly aligned with their orbital angular momentum if they are isolated. They also remarkably reported a tendency of anticorrelation between the mean value of $\chi_{\text{eff}}$ and the mass ratio $q$, that is, the average $\chi_{\text{eff}}$ has a slightly larger positive value for smaller $q$, a hierarchical mass configuration. They noted that this tendency is also in an opposite sense to the standard astrophysical models (see Callister et al. 2021 and references therein).

In addition to the astrophysical black hole (ABH), the so-called PBH has been also extensively discussed as a candidate for merger black holes (see Bird et al. 2016; Clesse & García-Bellido 2017, and Sasaki et al. 2016 for the first proposals). PBHs are hypothetical black holes formed in the early universe without introducing massive stars contrary to the ordinary ABHs (Zel’dovich & Novikov 1967; Hawking 1971; Carr & Hawking 1974). While many formation mechanisms have been proposed, one main scenario is the collapse of an overdense region of the universe. If primordial density perturbations $\delta$ generated by cosmic inflation are large enough and exceed a threshold value $\delta_{\text{th}}$, they can gravitationally collapse directly into black holes soon after their horizon reentry (Carr 1975; Nadezhdin et al. 1978; Harada et al. 2013). Since the PBH masses are roughly given by the Hubble masses at their formation times, they can be distributed in the very broad range, $10^{-5}–10^{50}$ g, including both massive ones and subsolar ones (PBHs with masses smaller than $10^{15}$ g are considered to have evaporated away by the present epoch due to the Hawking radiation; Carr 2005).

If we focus on PBHs formed in the radiation-dominated era, the spins of PBHs have been thought to be small typically because they originate from the almost spherically symmetric contraction of the Hubble patch. Recently, the spin distribution of PBHs has been extensively studied (Chiba & Yokoyama 2017; Harada et al. 2017; He & Suyama 2019; Mirbabayi et al. 2020; Chongchitnan & Silk 2021; Eroshenko 2021; Flores & Kusenko 2021). In
particular, De Luca et al. (2020) and subsequently Harada et al. (2021) carefully investigated it based on the so-called peak theory (Bardeen et al. 1986) of cosmological perturbation. Harada et al. (2021) found that the rms of the initial value of the nondimensional Kerr parameter is given by a form proportional to $(M/M_{\text{H}})^{-1/3}$ with a typically small numerical factor of $O(10^{-3})$, where $M$ and $M_{\text{H}}$ are the PBH mass and the Hubble mass at the formation, respectively. This result implies that, for the ordinary formation of PBHs such that $M \sim M_{\text{H}}$, the spin parameter is very small as $\sim 10^{-3}$ in fact. However, according to numerical simulations, the so-called critical phenomena have been reported for the PBH formation on the other hand (Evans & Coleman 1994; Koike et al. 1995; Niemeyer & Jedamzik 1998, 1999; Yokoyama 1998; Green & Liddle 1999; Musco et al. 2005, 2009; Musco & Miller 2013; Escrivá 2020). That is, the resultant PBH mass is not necessarily given by the Hubble mass but in a scaling relation $M \propto M_{\text{H}}(\delta - \delta_{0})^{\nu}$ with the universal power $\nu \approx 0.36$. Therefore, the mass can be arbitrarily small as $M \ll M_{\text{H}}$ for $\delta \sim \delta_{0}$ and rapidly spinning PBHs could be formed in that case. Furthermore, PBHs basically have no correlation with each other because they are separated farther than the Hubble scale at their formation time, which makes the spin antialignment more natural for the PBH binaries.

In this paper, based on the above-mentioned observational and theoretical backgrounds, we investigate the probability distribution of the effective inspiral spin $\chi_{\text{eff}}$, mass ratio $q$, and chirp mass $M_{\text{ch}}$ of PBH binaries, taking account of the critical phenomena. In particular, as the mass-spin anticorrelation $(\langle a_{+}^{2}\rangle) \propto (M/M_{\text{H}})^{-1/3}$ is reported, it is interesting to see the correlation between $\chi_{\text{eff}}$ and $q$ as found in observations. While several scenarios have been proposed for the binary formation of PBHs (see, e.g., Nakamura et al. 1997 and Bird et al. 2016), we simply assume that each binary is formed from two randomly chosen PBHs, considering it as a leading order approximation in any case. Therefore, the probability distribution of the single PBH (Harada et al. 2021) can be straightforwardly extended to the binary system.

This paper is organized as follows. In Section 2, we derive the probability distribution of single PBHs incorporating the effect of the critical phenomena of gravitational collapse. For simplicity, we assume an almost monochromatic power spectrum of the density fluctuation that will collapse into a PBH. In Section 3, we derive and numerically estimate the probability distribution of PBH binaries formed from two randomly chosen PBHs. The conclusion is given in Section 4. We use units in which $c = 1$.

2. Distribution of Single PBHs

Let us discuss the statistics of single PBHs in this section. We focus on the PBH formation via the collapse of overdensities in the radiation-dominated universe. Due to the charge-neutrality of the universe, PBHs are basically assumed neutral electromagnetically, and hence the no-hair theorem tells us that PBHs are characterized only by their masses $M$ and spin vectors $a' = a(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^{\top}$, where we employ the dimensionless definition $a' = S'/GM^{2}$ with the angular momentum $S'$ and its norm $a$ is often called Kerr parameter. The PBH statistics are accordingly dictated by the probability distribution of their characteristics,

$$P(a, M, \theta, \phi) \, da \, dM \, d\theta \, d\phi,$$

and the statistical-isotropy assumption restricts its form as

$$P(a, M, \theta, \phi) \, da \, dM \, d\theta \, d\phi = \frac{1}{4\pi} P(a, M) \, da \, dM \, d\mu \, d\phi,$$

with $\mu = \cos \theta$, which consistently gives the distribution of the Kerr parameter and mass as

$$P(a, M) = \int P(a, M, \theta, \phi) \, d\theta \, d\phi.$$

We below derive this distribution $P(a, M)$.

2.1. Spin Distribution

PBHs are supposed to be formed by the collapse of rare highly overdense regions. According to the peak theory (Bardeen et al. 1986), if the density contrast $\delta$ follows the Gaussian distribution and is characterized by an almost monochromatic power spectrum $P_{0}(k) \approx \sigma_{0}^{2} \delta_{0}^{2} \delta(k - k_{0})$ with some scale $k_{0}$ as we assume throughout this paper, the spatial profile of such “high peaks” of the Gaussian random field is known to be typically spherically symmetric and given by (Yoo et al. 2018)

$$\delta_{pk}(r) \simeq \nu_{0} \frac{\sin k_{0}r}{k_{0}r},$$

with a (normalized) random Gaussian parameter $\nu$ following the distribution $P_{0}(\nu) = \frac{1}{\sqrt{2\pi}} e^{-\nu^{2}/2}$. The peak extremum is put at the origin $r = 0$ without loss of generality. The PBH characteristics (mass and spin as well as whether they are really formed or not) are parametrized by this $\nu$ parameter. In this subsection, we first review the spin distribution determined by $\nu$, following Harada et al. (2021; see also Heavens & Peacock 1988 and De Luca et al. 2019). Note that, although peaks of a Gaussian random field do not necessarily obey a Gaussian distribution, we assume a Gaussian distribution of $\nu$ as an approximation. The validity is discussed in Appendix B, and the appropriate normalization factor for the PBH case is given in Section 2.2.

Though the typical peak profile is almost spherically symmetric, a slight deviation from an exactly monochromatic spectrum can cause a tidal torque introducing a spin to a PBH. In the peak theoretical approach, Harada et al. (2021) revealed that the normalized spin parameter $h$, which is defined in Equation (A17) of Appendix A, is related to $a$ and $\nu$ as

$$h = \frac{a}{C(M, \nu)},$$

$$C(M, \nu) = 3.25 \times 10^{-2} \sqrt{1 - \gamma^{2}} \sigma_{0} \left( \frac{M}{M_{\text{H}}} \right)^{-1/3} \left( \frac{\nu}{10} \right)^{-1},$$

and follows the universal distribution$^{7}$

$$P_{h}(h) \, dh = 563h^{2} \times \exp[-12h + 2.5h^{1.5} + 8 - 3.2(1500 + h^{16})^{1/8}] \, dh,$$

$^{7}$ In the recent work (De Luca et al. 2019), another fitting distribution function is given as

$$P_{h}(h) \, dh = \exp[-2.37 - 4.12\ln h - 1.53(\ln h)^{2} - 0.13(\ln h)^{3}] \, dh.$$

However, since it is singular for $h \to 0$, we here adopt the original and regular fitting expression (6) given by Heavens & Peacock (1988).
which is a fitting formula found by Heavens & Peacock (1988; note that it is normalized so that \( \int_0^{\nu_{th}} P_{\nu}(h) \, dh = 1 \)). Here \( M \) is the total mass of the collapsing fraction, \( M_H \) is the horizon mass at the horizon reentry of the overdense region, and \( \gamma := \sigma_{\nu}/(\sigma_0 \sigma_2) \) with \( \sigma_{\nu}^2 := \int d k k^{3/2}P_{\nu}(k) \) characterizes the width of the power spectrum of the density contrast (\( \gamma = 1 \) for an exactly monochromatic spectrum). Throughout this paper, we assume \( \gamma = 0.85 \). Given \( M \) and \( \nu \), the PBH spin distribution can be deduced basically from this formula. See Appendix A for the brief introduction of \( h \) and \( C(M, \nu) \).

One should note that a PBH is not necessarily formed for a given \( \nu \) and thus the PBH formation condition should be imposed to obtain the spin distribution of PBHs. For an almost monochromatic spectrum (and thus for an almost uniform typical peak profile (4)), it is justified to judge the PBH formation just by whether \( \nu \) exceeds some threshold value \( \nu_{th} \) (see, e.g., Germani & Musco 2019). We basically neglect the spin dependence of \( \nu_{th} \) but just take account of the fact that \( a > 1 \) is not allowed for a BH.\(^8\) That is, we adopt the following simplified spin dependence in this paper:

\[
\nu_{th} = \begin{cases} 
\text{const.} & \text{for } 0 \leq a \leq 1, \\
\infty & \text{otherwise.}
\end{cases}
\] (7)

Therefore, the distribution of the PBH Kerr parameter \( a \) given \( M \) and \( \nu \) is simply obtained by the change of the variable \( h \rightarrow a = Ch \) for \( 0 \leq h \leq 1/C \) as

\[
P(a|M, \nu) \, da = \frac{P_{\nu}(a/C(M, \nu))}{C(M, \nu)N_{\nu}(M, \nu)} \, da,
\] (8)

with the normalization factor

\[
N_{\nu}(M, \nu) := \int_0^{1/C(M, \nu)} P_{\nu}(h) \, dh,
\] (9)

to ensure \( \int_0^1 P(a|M, \nu) \, da = 1 \). The simplified spin dependence (7) does not make a problem practically because the typical PBH spin is quite small as \( a \ll 1 \) as we will see below.

2.2. Critical Behavior

The PBH mass \( M \) also depends on \( \nu \). It is roughly equivalent to the horizon mass \( M_H \) at the horizon reentry of the overdense region, but in detail, it is often assumed to follow the scaling relation

\[
M(\nu) = KM_H(\nu \sigma_0 - \nu_{th} \sigma_0)\nu,
\] (10)

through which the PBH mass can be understood as a function of \( \nu \). Here \( \kappa \approx 0.36 \) is a universal power and \( K \) is a weakly profile-dependent coefficient (Evans & Coleman 1994; Escrivà 2020). Since \( K \) is of order unity in any case, we here set \( K = 1 \) for simplicity. Once \( M \) is related to \( \nu \), the joint probability \( P(a, M) \) for a PBH can be calculated as

\[
P(a, M) \, da \, dM = P(a|M(\nu)), \nu P_{\nu}(\nu) \, da \, d\nu, \] (11)

where

\[
P_{\nu}(\nu) = \frac{2}{\pi} \frac{e^{-\nu^2/2}}{\text{erfc}(\nu_{th}/\sqrt{2})},
\] (12)

is the Gaussian distribution of \( \nu \) for a PBH, i.e., it is defined only for \( \nu > \nu_{th} \) and normalized as \( \int_{\nu_{th}}^{\infty} P_{\nu}(\nu) \, d\nu = 1 \) with \( \text{erfc} \) denoting the complementary error function. For the derivation of Equation (12), see Appendix B.

The plot of \( P(\log_{10} a, \log_{10} M) = P(a, M)(\ln 10)^2 aM \) is shown in Figure 1 for \( \nu_{th} = 10 \) and \( \sigma = 0.192 \), which correspond to a 0.1% fraction of dark matter with \( M_H \sim M_5 \), as will be discussed in Section 2.3. The PBH spin and mass are mostly distributed in the range of \( 10^{-4} \leq a \leq 10^{-3} \) and \( 0.1 \leq M/M_H \leq 0.4 \). The expected value of \( a \) for each \( M \) defined by \( \langle a(M) \rangle = \int_0^1 aP(a, M) \, da \int_0^1 P(a, M) \, da \) is also plotted. The power law, \( \langle a \rangle \propto M^{-1/3} \), can be seen in the range, \( 10^{-8} \leq M/M_H \leq 0.3 \), as expected from the normalization \( C(M, \nu) \) (see Harada et al. 2021). This anticorrelation between \( a \) and \( M \) is because, as we can see from Equations (A14), (A19), and (A21), the magnitude of the total angular momentum of the collapsing fraction scales as \( S_{\text{ref}} \propto (M/M_0)^{5/3} \) and the corresponding Kerr parameter scales as \( a \propto A_{\text{ref}} = S_{\text{ref}}/(GM^2) \propto (M/M_0)^{-1/3} \). Also note that this power law is violated for the much smaller mass, \( M/M_H \lesssim 10^{-8} \), corresponding to the limit, \( \langle a \rangle \rightarrow 1 \), because of our assumption that a peak of the density fluctuation with \( a > 1 \) will not collapse into a PBH. The violation for \( M/M_H \gtrsim 0.3 \) appears because the factor \( \nu(M)^{-1} \) in \( C(M, \nu(M)) \) is not constant in this range, while it is almost constant, \( \nu(M)^{-1} \approx \nu_{th}^{-1} \), for \( M/M_H \lesssim 0.3 \).

2.3. PBH Abundance

In order to concretely specify the parameters, let us also review the current PBH abundance. The normalization of \( \nu \)'s distribution for a PBH (12) implies that a PBH can be formed with the probability \( \int_0^1 P_{\nu}(\nu) \, d\nu = \text{erfc}(\nu_{th}/\sqrt{2})/2 \) at each Hubble patch. Uniformly approximating the PBH mass by the horizon mass for simplicity, the ratio \( \beta \) of the PBH energy density to that of the background radiation at their formation

\[ figure \]
time is hence given by that probability:

\[
\beta = \frac{1}{2} \text{erfc} \left( \frac{\nu_0}{\sqrt{2}} \right).
\]  

(13)

After their formation, PBHs behave as nonrelativistic matters and their energy density decays as \( \propto a^{-3} \) where \( a \) is the scale factor of the universe. Accordingly, one can calculate the ratio of the current PBH energy density \( \rho_{\text{PBH},0} \) to that of the total cold dark matters \( \rho_{\text{DM},0} \) and it reads (see, e.g., Tada & Yokoyama 2019)

\[
f_{\text{PBH}} = \frac{\rho_{\text{PBH},0}}{\rho_{\text{DM},0}} \sim \left( \frac{\beta}{1.8 \times 10^{-9}} \right) \left( \frac{\Omega_{\text{DM}h^2}}{0.12} \right)^{-1} \left( \frac{g_*}{10.75} \right)^{-1/4} \left( \frac{M}{M_\odot} \right)^{-1/2},
\]

(14)

where \( \Omega_{\text{DM}h^2} \approx 0.12 \) is the current density parameter of the total cold dark matters (Aghanim et al. 2020), \( g_* \approx 10.75 \) is the effective degrees of freedom for the energy density of the radiation fluid at the formation time of solar mass PBHs, and \( M_\odot \approx 2 \times 10^{30} \) g is the solar mass. Here we approximated the effective degrees of freedom for entropy density by those for energy density, \( g_* \approx g_{*1} \), throughout this paper and assumed that PBHs were formed at around the time when \( k_0 \) reentered the horizon. It has been implied that \( f_{\text{PBH}} \sim 0.1\% \) accounts for the merger rate of BH binaries inferred from LIGO-Virgo’s observations (see, e.g., Ali-Haimoud et al. 2017 and Vaskonen & Veermäe 2020). From the formula (14), one sees that this abundance corresponds to \( \nu_0 \sim 10 \) for \( M \sim M_\odot \). Below, we will employ this value of \( \nu_0 \).

One can also infer the perturbation amplitude \( \sigma_0 \) from the value of \( \nu_0 \). The PBH formation is often judged by using the so-called compactness function \( C(r) \) which is defined by Equation (4.28) in Shibata & Sasaki (1999) or Equation (6.33) in Harada et al. (2015) for the constant-mean-curvature slicing. If the maximum \( C_m := \max (C(r)) r \) exceeds the threshold \( C_{m, \text{th}} \approx 2/5 \), which has been suggested by fully nonlinear numerical simulations (Shibata & Sasaki 1999; Harada et al. 2015; Germani & Musco 2019; Musco 2019), for some overdense region, that region is supposed to form a PBH. Assuming the peak profile (4), the maximum \( C_m \) corresponds to the central value \( \delta_{\text{pk}}(r = 0) \) by \( C_m \approx (5/24)\delta_{\text{pk}}(0) \) (see Harada et al. 2021 for details). In order for \( \nu_0 \) to correspond to \( C_m \), the perturbation amplitude \( \sigma_0 \) should be given by \( \sigma_0 \approx (24/5)(C_{m, \text{th}}/\nu_0) \approx 0.192 \).

3. Distribution of PBH Binaries

As only one PBH forms in one Hubble patch, basically PBHs have no correlation with each other before their formations. They randomly form in space and some of them make binaries by gravitationally catching each other through several proposed scenarios such as free falls of two near PBHs in the early universe (Nakamura et al. 1997) or gravitationally captures in galactic halos in the late universe (see, e.g., Bird et al. 2016). Two PBHs forming a binary can be assumed to be chosen randomly.

A black hole binary system seen by its merger gravitational waves (GWs) is characterized by the chirp mass \( M \), the mass ratio \( q \), and the effective inspiral spin \( \chi_{\text{eff}} \) defined by

\[
M = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}} \in (0, \infty),
\]

\[
q = \frac{M_2}{M_1} \in (0, 1],
\]

\[
\chi_{\text{eff}} = \frac{a_1 \mu_1 + q a_2 \mu_2}{1 + q} \in [-1, 1],
\]

(15)

respectively. Here, the quantities with subscripts 1 and 2 are those of the primary and secondary PBHs, respectively. The polar angles of spins, \( \theta_1 \) and \( \theta_2 \), are taken so that the axis coincides with the orbital angular momentum, \( L \). As mentioned, the primary (PBH1) and secondary (PBH2) PBHs are assumed to be chosen randomly according to their probability distribution, Equations (1) and (11). Moreover, for simplicity, we assume that the mass and spin angular momenta of PBHs are constant during the formation process of two isolated PBHs to a binary. Thus we can straightforwardly derive the probability distribution of PBH binaries, \( P(M, q, \chi_{\text{eff}}) dM \, dq \, d\chi_{\text{eff}} \) from this single-PBH distribution.

Thanks to the independence of PBH1 and PBH2, one first obtains the joint probability distribution of their intrinsic parameters \( w = (a_1, a_2, M_1, M_2, \mu_1, \mu_2, \phi_1, \phi_2) \) as a direct product of each probability,

\[
P(w) \, dw = \frac{2}{(4\pi)^2} \prod_{i=1}^{2} P(a_i, M_i) \, da_i \, dM_i \, d\mu_i \, d\phi_i,
\]

(16)

where we have normalized the PDF, \( P(w) \), so that its integration over \( a_i \in [0, 1] \), \( 0 < M_2 < M_1 < \infty \), \( \mu_i \in [-1, 1] \), and \( \phi_i \in [0, 2\pi) \) becomes unity. Note that the isotropy assumption (2) has been also used. According to the argument on the critical behavior in Section 2.2, the distribution of the variables \( a_i \) and \( M_i \) is read as that of \( a_i \) and \( \nu_i \) by using Equation (11). Thus, we also have

\[
P(w) \, dw' = \frac{1}{8\pi} \prod_{i=1}^{2} P(a_i | M_i(\nu_i), \nu_i) P_{\nu_i}(\nu_i) \, da_i \, d\nu_i \, d\mu_i \, d\phi_i,
\]

(17)

for the variables, \( w' = (a_1, a_2, \nu_1, \nu_2, \mu_1, \mu_2, \phi_1, \phi_2) \). Here, \( \nu_i \) is the peak value of each density fluctuation that forms each PBH of the binary. Noting the critical behavior (10), the Jacobian reads

\[
J_{w w'} = \left| \frac{dw}{dw'} \right| = M_1^{2} \kappa^{2} \sigma_0^{2} \left( \nu_1 \sigma_0 - \nu_2 \sigma_0 \right)^{\kappa-1} \left( \nu_2 \sigma_0 - \nu_3 \sigma_0 \right)^{-1},
\]

\[
= M_1 M_2 \kappa^{2} \sigma_0^{2} \left( \frac{M_1}{M_\odot} \right)^{-1/\kappa} \left( \frac{M_2}{M_\odot} \right)^{-1/\kappa},
\]

(18)

and the probability in \( w \) is given by

\[
P(w) = J_{w w'}^{-1} P(w') = \frac{1}{8\pi^{2} J_{w w'}} \prod_{i=1}^{2} P(a_i | M_i, \nu(M_i)) P_{\nu}(\nu(M_i))
\]

(19)
where
\[ \nu(M) = \frac{1}{\sigma_0} \left( \frac{M}{M_{\odot}} \right)^{1/5} + \nu_{\text{th}}. \] 
(20)

It can be further translated to the parameter set z = (M, q, \chi_{\text{eff}}, a_1, a_2, \mu_1, \phi_1, \phi_2). Recalling the definitions of the effective inspiral spin \chi_{\text{eff}}, the mass ratio q, and the chirp mass \mathcal{M} (15), the Jacobian from w to z can be then computed as
\[
J_{zw} = \frac{dz}{dw} = \frac{a_2 M_2}{a_2 q^{11/5}} \left( \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{3/5}} \right)^{1/5}
\]
(21)

where we used the inverse relation
\[
M_1(M, q) = q^{-3/5}(1 + q)^{1/5} M, \\
M_2(M, q) = q^{2/5}(1 + q)^{1/5} M.
\]
(22)

The probability in z is hence found as
\[
P(z) \propto J_{zw}^{-1} P(w) = \frac{1 + q}{a_2 q^{2/5} \sigma_0 \mathcal{M}} \left( \frac{(1 + q)^{2/5} M^2}{q^{1/5} M_{\odot}^2} \right)^{1/5}
\times \frac{1}{8\pi} \sum_{i=1}^{2} P(a_i M(M, q), \nu(M(M, q)) \nu(M(M, q))) \times P_e(\nu(M(M, q))).
\]
(23)

The probability only of \mathcal{M}, q, and \chi_{\text{eff}} is obtained as the integration over the rest variables a_1, a_2, \mu_1, \phi_1, and \phi_2: 
\[
P(M, q, \chi_{\text{eff}}) = \int P(z) da_1 da_2 d\mu_1 d\phi_1 d\phi_2.
\]
(24)

Note that the range of \mu_1, originally in (−1, 1), is now restricted by the other variables, \chi_{\text{eff}}, q, a_1, and a_2, as
\[
\mu_1 = \frac{(1 + q) \chi_{\text{eff}} - qa_2 \mu_2}{a_1} \\
\in \left[ \frac{(1 + q) \chi_{\text{eff}} - qa_2}{a_1}, \frac{(1 + q) \chi_{\text{eff}} + qa_2}{a_1} \right).
\]
(25)

because of the range of \mu_2 ∈ (−1, 1). As a result, we have
\[
\mu_1 \in \left[ \max \left[ -1, \frac{(1 + q) \chi_{\text{eff}} - qa_2}{a_1} \right], \min \left[ 1, \frac{(1 + q) \chi_{\text{eff}} + qa_2}{a_1} \right] \right).
\]
(26)

Finally we obtain the expression
\[
P(M, q, \chi_{\text{eff}}) = \frac{1 + q}{2q^{2/5} \sigma_0 \mathcal{M}} \left( \frac{(1 + q)^2 M^2}{q^{1/5} M_{\odot}^2} \right)^{1/5}
\times \int_0^1 da_1 \int_0^1 da_2 \Theta(T(a_1, a_2, \chi_{\text{eff}}, q)) T(a_1, a_2, \chi_{\text{eff}}, q)
\times \frac{1}{a_1 a_2} \prod_{i=1}^{2} P(a_i M(M, q), \nu(M(M, q)))
\times P_e(\nu(M(M, q))),
\]
(27)

where \Theta(x) is the Heaviside step function and
\[
T(a_1, a_2, \chi_{\text{eff}}, q) = \min[a_i, qa_2 + (1 + q) \chi_{\text{eff}}]
+ \min[a_i, qa_2 - (1 + q) \chi_{\text{eff}}].
\]
(28)

By integrating it over one of the three variables, one can further obtain the two-variable probabilities P(\chi_{\text{eff}}, q), P(M, \chi_{\text{eff}}), and P(M, q). The numerical results are shown in Figure 2. We take the parameters as \nu_{\text{th}} = 10 and \sigma_0 = 0.192 which correspond to f_{\text{PBH}} ≈ 0.1% for \mathcal{M}_\odot \sim \mathcal{M}_\odot PBHs as discussed in Section 2.3.

One can see that the effective spin is distributed in a very narrow region, |\chi_{\text{eff}}| ≤ 10^{-3}. The rms is given as \sqrt{\langle \chi_{\text{eff}}^2 \rangle} = 8.41 \times 10^{-4}. This would be because, although the effect of the critical phenomena allows each PBH to spin rapidly so that a ≈ 1 if the mass is very small, the probability of such a small mass is very low as can be seen from Figure 1. The mass ratio is broadly distributed as 0.1 ≤ q ≤ 1, and the chirp mass has a width 0.1 ≤ \mathcal{M}/M_{\odot} ≤ 0.3 due to the critical behavior (10) even though we assume an almost monochromatic power spectrum (i.e., a single value for M_{\odot}). In addition, in the plot of P(\chi_{\text{eff}}, M), we can see an anticorrelated behavior between \sqrt{\langle \chi_{\text{eff}}^2 \rangle} and \mathcal{M}. That is, for smaller \mathcal{M}, the rms of the effective...
spin, \( \sqrt{\langle \chi_{\text{eff}}^2 \rangle} (M) \), becomes larger. Actually, by numerical calculation, one can confirm that \( \sqrt{\langle \chi_{\text{eff}}^2 \rangle} (M) \) is monotonically decreasing with \( M \) in the range of \( 3 \times 10^{-4} \lesssim \langle \chi_{\text{eff}}^2 \rangle (M) \lesssim 3 \times 10^{-3} \). This is a result expected from the anticorrelation between \( \langle a \rangle \) and \( M \) for the single PBH distribution. On the other hand, we find that there is almost no correlation between \( \sqrt{\langle \chi_{\text{eff}}^2 \rangle} (q) \) and \( q \). In particular, \( \sqrt{\langle \chi_{\text{eff}}^2 \rangle} (q) \) cannot be large even for smaller \( q \). This is because, even if the secondary PBH has a very small mass and therefore has a Kerr parameter of order unity, \( a_2 \sim 1 \), its contribution to \( \chi_{\text{eff}} \) is suppressed by the very small mass ratio, \( q \), according to the definition (15).

In this paper, as discussed in Section 2.3, we have adopted the threshold value of the compaction function, \( C_{\text{m.th}} \approx 2/5 \), which leads to \( \sigma_0 \approx 0.192 \) under the assumption \( \nu_{\text{th}} = 10 \). However, this threshold value would include some uncertainty because the value of \( C_{\text{m.th}} \) slightly depends on the initial profile of the perturbation unlike the averaged one (Escrivà et al. 2020) and the nonzero angular momentum of the collapsing fraction of the universe should make the threshold value higher due to the centrifugal force against the gravitational contraction. We see how this uncertainty affects the resulting distribution in Appendix C by taking different values of \( \sigma_0 \) with the fixed value of \( \nu_{\text{th}} = 10 \), which we have determined by using Equation (14). We can see that the modification of \( \sigma_0 \) somehow, but not greatly, changes the widths of the distribution.

### 4. Conclusions

In this paper, we formulated the probability distribution of the characteristics (the effective inspiral spin \( \chi_{\text{eff}} \), the mass ratio \( q \), and the chirp mass \( M \) in particular) of PBH binaries, taking account of the critical phenomena of gravitational collapse. First, we have derived the distribution of the spin \( a \) and the mass \( M \) of single PBHs. It is basically featured by the scaling relation \( a \propto (M/M_1)^{-1/3} \) and PBHs with \( M \lesssim 10^{-8} M_1 \) can have spins of order unity, while the spin is rather suppressed for \( M \gtrsim M_1 \). Under the assumption that two PBHs of a binary are randomly chosen as a leading order approximation, this single PBH distribution is straightforwardly followed by the binary PBH distribution. The resultant probability in \( \chi_{\text{eff}}, q, \) and \( M \) is shown in Figure 2.

The first observation is the symmetry under \( \chi_{\text{eff}} \to -\chi_{\text{eff}} \), which is a direct consequence of our random-choice assumption. Because of the isotropic distribution of the spin of each PBH, the probability for realizing a binary configuration with the orbital angular momentum \( L \) should be the same as that with \(-L\). This symmetry appears in \( \chi_{\text{eff}} \) Equation (15), in terms of the polar angle of each spin, \( \theta \). That is, the reflection \( L \to -L \) corresponds to \( \theta \to \pi - \theta \) leading to the symmetry for \( \chi_{\text{eff}} \to -\chi_{\text{eff}} \). Actually, Equation (27) depends on \( \chi_{\text{eff}} \) only through the even function of \( \chi_{\text{eff}}, T(a_1, a_2, \chi_{\text{eff}}, q) \). This symmetry implies that at least a certain fraction of black hole binaries have negative values of \( \chi_{\text{eff}} \). The negative values of \( \chi_{\text{eff}} \) for black hole binaries with \( q \sim 1 \) have been indicated by the analyses in Callister et al. (2021), Abbott et al. (2021c), although there are only two candidate events (GW191109_010717 and GW200225_060421) with significant support (Abbott et al. 2021c). The PBH binary scenario would have the potential to explain those negative values.

However, the amplitude \( |\chi_{\text{eff}}| \) is found to be very small as \( |\chi_{\text{eff}}| \lesssim 8.41 \times 10^{-4} \), compared to the observed ones \( |\chi_{\text{eff}}| \sim 0.1 \). Furthermore, contrary to the anticorrelation between \( |\chi_{\text{eff}}| \) and \( M \) as expected from the anticorrelation between \( a \) and \( M \) in the single PBH distribution, we found almost no correlation between \( |\chi_{\text{eff}}| \) and \( q \). This would be because, even though the spin of the secondary PBH \( a_2 \) can be large enough if PBH2 is very light and the mass ratio \( q \) is very small, the contribution of \( a_2 \) to \( \chi_{\text{eff}} \) is suppressed by the factor \( q \) as can be seen in its definition (15). Therefore, it would be difficult to realize the observed anticorrelation between \( \chi_{\text{eff}} \) and \( q \) in our scenario.

For the free parameters of our system, we have taken the values as \( \sigma_0 = 0.192, \nu_{\text{th}} = 10, \) and \( \gamma = 0.85 \). We have fixed the values of \( \sigma_0 \) and \( \nu_{\text{th}} \), from the requirements of \( f_{\text{PBH}} \sim 0.1 \% \) and \( \nu_0 \approx 0.195 \) by that for the monochromatic spectrum, our procedure cannot be directly applied to the case \( \gamma \ll 1 \). Nevertheless, a different value of \( \gamma \) is allowed as long as \( \gamma \sim 1 \). As for the \( \sigma_0 \) dependence, in Appendix C, we have investigated the distribution for other values of \( \sigma_0 \) and have found that the width becomes somewhat broader for larger \( \sigma_0 \). We have also checked that changing the values of \( \nu_{\text{th}} \) and \( \gamma \) by about 10\% does not greatly affect the result. For example, if we take \( \nu_{\text{th}} = 8 \) with \( \sigma_0 = 0.195 \) and \( \gamma = 0.85 \), the shape of the distribution, Figure 2, does not change, and the width, i.e., the rms of \( \chi_{\text{eff}} \), increases only by a factor of the order of unity.

According to Equations (13) and (14), such a change of \( \nu_{\text{th}} \) arises if we change the values, \( f_{\text{PBH}} \sim 0.1 \% \) and \( M_1 \sim M_\odot \), by several orders of magnitude. Similarly, for \( \gamma \sim 0.75 \) with \( \sigma_0 = 0.195 \) and \( \nu_{\text{th}} = 10 \), the shape remains and the width increases only by a factor of the order of unity.

Although our formalism relies on the assumption of the almost monochromatic power spectrum, we can speculate about the case of a broad spectrum based on our formalism. First, as mentioned above, for a smaller value of \( \gamma \sim 1 \), the rms of \( \chi_{\text{eff}} \) becomes larger. Therefore one may expect that a broader power spectrum gives a broader distribution of \( \chi_{\text{eff}} \). Second, if the spectrum is broader, the mass distribution of single PBHs becomes broader. This results in a broader distribution of \( q \) and \( M \) for PBH binaries. However, if we take a completely broad spectrum, the typical profile of density contrast, Equation (4), and the spin distribution of single PBHs, Equations (5) and (6), would substantially change, and thus, it is necessary to derive them again. These effects on the distribution of PBH binaries are quite nontrivial, and here we leave these issues as future works.

As a further consideration, one may include the clustering effect on the PBH spatial distribution to alter the random-choice assumption. For example, it is known that PBHs are clustered when the source perturbations are non-Gaussian (see, e.g., Sasaki et al. 2018). Primordial non-Gaussianities may also change the peak statistics and then the spin distribution. Spin evolution through the accretion process is also interesting. Deluca et al. (2020) show that the evolution can be significant for massive PBHs \( \gtrsim 10 M_\odot \). The change of pressure of the background fluid is another possibility to enhance the PBH spins. The pressure \( p \) can be reduced from the radiational one \( p/3 \), where \( p \) is the energy density, during, e.g., the quantum chromodynamics (QCD) phase transition or the possible...
matter-dominated era in the early universe. The reduction of pressure allows a nonspherical collapse and the resultant PBH can have a large spin (Harada et al. 2017). Large initial spins, which induce nonvanishing $\chi_{\text{eff}}$, can enhance each merger GW (see, e.g., Hemberger et al. 2013 and Kuhnel 2020). Furthermore, the QCD phase corresponds to $\sim M_\odot$ PBHs and thus it would have a remarkable relation to individual merger GW events. We leave all these possibilities for future works.

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### Appendix A

#### Dimensionless Spin Parameter $h$

We have quoted the result for the spin distribution of a single PBH obtained in Harada et al. (2021). In this section, we briefly introduce the relevant quantities for the derivation, especially, the dimensionless spin parameter $h$. The parameter $h$ was first introduced by Heavens & Peacock (1988) and applied to the derivation of PBH spin distribution by De Luca et al. (2019).

Let us consider the $3+1$ decomposition of the spacetime,
\[ds^2 = -\alpha^2(\eta, x)\, dt^2 + \alpha^2(\eta, x)\left(dx^i + \beta^i(\eta, x)\, d\eta\right)^2 + (dx^i + \beta^i(\eta, x)\, d\eta)^2,\]
(A1)

with a background flat FLRW metric,
\[ds^2 = \alpha^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2).\]
(A2)

$a(\eta)$, $\alpha(\eta, x)$, and $\beta^i(\eta, x)$ denote the global scale factor, the lapse function, and the shift vector, respectively. We assume the matter field to be a single perfect fluid,
\[T_{ab} = \rho \alpha a^i a^b + p(\alpha a^i a^b + u_a u_b).\]
(A3)

On the background spacetime, there are rotational Killing vectors $\phi^i = \epsilon_{ijk}(x - x_{pk})\delta^i(\partial/\partial x^j)y^j$ ($i = 1, 2, 3$) tangent to a spacelike hypersurface $\eta = \text{const}$. For a region $\Sigma$ on the spacelike hypersurface, the conserved angular momentum $S_i(\Sigma)$ of the matter contained in $\Sigma$ can be defined as
\[S_i(\Sigma) := \frac{1}{16\pi G} \int_{\Sigma} \epsilon_{abcd} \nabla^c (\phi^d)_{,i}^j + \frac{1}{8\pi G} \int_{\Sigma} R^{ab} n_a d(\phi^c)_{,b} d\Sigma - \int_{\Sigma} T^{ab} n_a (\phi^c)_{,b} d\Sigma,\]
(A4)

where the Einstein equation is used in the last equality. For primordial black hole formation, we suppose $\Sigma$ to be a region that will collapse into a black hole, and the black hole mass and angular momentum are estimated as those of matter in $\Sigma$. Here we assume that the region $\Sigma$ is given by
\[\Sigma = \{x|\delta(x) > f \delta_{pk}\},\]
(A5)

with some positive constant $f$ less than unity.

Around the peak, the density contrast, which we assume to be a Gaussian random field, is expanded as
\[\delta \approx \delta_{pk} + \frac{1}{2} \zeta_{ij}(x - x_{pk})^j(x - x_{pk})^i,\]
(A6)

where
\[\zeta_{ij} := \frac{\partial^2 \delta}{\partial x^i \partial x^j} \bigg|_{x = x_{pk}}.\]
(A7)

Taking $x, y$, and $z$ axes as the principal directions of $\zeta_{ij}$, we have
\[\delta \approx \delta_{pk} - \frac{1}{2} \sigma_i (x - x_{pk})^i,\]
(A8)

where $\sigma_i$ is defined below Equation (6) and $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues of $-\zeta_{ij}/\sigma_2$. As a result, $\Sigma$ is given as an ellipsoid with the three axes,
\[a^2_i = \frac{2 - \sigma_i}{\sigma_2} \lambda_i.\]
(A9)

Expanding the fluid 3-velocity $v^i := u^i/u^0$ as
\[v^i - v_{pk}^i \approx v^j(x - x_{pk})^j,\]
(A10)

we obtain
\[S_i(\Sigma) \approx (1 + w)\alpha a^4 \epsilon_{ijk} v^k \int_{\Sigma} (x - x_{pk})^i(x - x_{pk})^j d^3x = (1 + w)\alpha a^4 \epsilon_{ijk} v^k J^j,\]
(A11)

where $w = p/\rho$ and
\[v^k := \frac{\partial v}{\partial x^k} \bigg|_{x = x_{pk}},\]
\[J^j := \int_{\Sigma} (x - x_{pk})^i (x - x_{pk})^j d^3x = \frac{4\pi}{15} a_1 a_2 a_3 \text{diag}(a_1^2, a_2^2, a_3^2).\]
(A12)

For PBH formation, we focus on a growing mode of the perturbation. The time dependence of the perturbation is investigated in Harada et al. (2021). According to it, the average of the spin magnitude is decomposed as
\[\sqrt{\langle S_i S^i \rangle} \approx S_{\text{ref}} \sqrt{\langle S_\delta S_\delta \rangle},\]
(A13)

where
\[S_{\text{ref}}(\eta) = (1 + w)\alpha a^4 \epsilon_{ijk} g(\eta)(1 - f)^{5/2} R_\ast^5,
\]
\[s_\ast = \frac{16\sqrt{2} \pi}{135 \sqrt{3}} \left(\frac{\nu}{\nu_f}\right)^{5/2} \frac{1}{\sqrt{\lambda}} \left(-\alpha_1 \hat{v}_{23}, \alpha_2 \hat{v}_{13}, -\alpha_3 \hat{v}_{12}\right),\]
\[\alpha_1 = \frac{1}{\lambda_3} - \frac{1}{\lambda_2}, \quad \alpha_2 = \frac{1}{\lambda_3} - \frac{1}{\lambda_1}, \quad \alpha_3 = \frac{1}{\lambda_2} - \frac{1}{\lambda_1},\]
\[\Lambda := \lambda_1 \lambda_2 \lambda_3, \quad R_\ast := \sqrt{3} \frac{\sigma_1}{\sigma_2}.\]
(A14)

The function $g(\eta)$ defined by
\[\langle (v^k v^i(\eta))^2 \rangle = g^2(\eta) \langle (v^k v^i)^2 \rangle,\]
(A15)
represents the time evolution of the velocity field for every \( k, l \), where the time-independent variable \( \nu_{i, j} \) is defined by

\[
\nu_{i, j} := -\frac{1}{\sigma_0} \int \frac{d^3k}{(2\pi)^3} \frac{k_i k_j}{k^2} e^{ik \cdot \mathbf{x}}. 
\]

For large \( \nu \) limit, the dimensionless spin parameter \( h \) is defined by

\[
h := \sqrt{\nu \eta/\sqrt{1 - \nu^2}}. 
\]

\( h \) is useful to investigate the probability distribution of the spin. Heavens & Peacock (1988) numerically derived the probability distribution as

\[
P_h(h) dh = 563h^2 \times \exp[-12h + 2.5h^{1.5} + 8 - 3.2(1500 + h^{16})/h] dh. 
\]

Recently, De Luca et al. (2019) gave another fitting formula that agrees with the above one very well. In this paper we adopt the former one because of the regular behavior in the limit \( h \to 0 \).

The total angular momentum \( S_i(\Sigma) \) will become that of the resulting PBH after the formation. The region \( \Sigma \), which will collapse into a PBH, would be specified when the contraction of matter is decoupled from the expansion of the universe. This is called turnaround. We denote the time of turnaround as \( \eta_{ta} \).

Defining the dimensionless reference spin value at turnaround as

\[
A_{\text{ref}}(\eta_{ta}) = \frac{S_{\text{ref}}(\eta_{ta})}{GM_{\text{a}}^2}, 
\]

where \( M_{\text{a}} \) is the mass inside \( \Sigma \) at the turnaround, we can estimate the initial dimensionless Kerr parameter \( a \) of the resulting PBH. For the radiation domination, Harada et al. (2021) found, in Equation (22) of it, the simple expression of \( A_{\text{ref}}(\eta_{ta}) \) as

\[
A_{\text{ref}}(\eta_{ta}) \approx \frac{1}{24\sqrt{3}\pi} x_{\text{a}}^2 (1 - f)^{-1/2} T_C(n_0, \eta_{ta}) I_{\text{ef}}. 
\]

where \( x_{\text{a}} = k_{0, \text{ta}} \), \( T_C(n_0, \eta) \) is the transfer function of the mode of the velocity field with wavelength \( k_0 \) in the conformal Newtonian gauge, and \( \sigma_{\text{H}} \) denotes \( \sigma_0 \) when the initial time of the evolution of the cosmological perturbation is set to the horizon entry. For radiation domination, the factors were numerically estimated as \( x_{\text{a}} \approx 2.14 \) and \( T_C(n_0, \eta_{ta}) \approx 0.622 \) in Section 3.3 of Harada et al. (2021). Using the relation between \( M_{\text{a}} \) and \( f \) derived in Harada et al. (2021),

\[
M_{\text{a}} \approx \frac{\sqrt{6}}{x_{\text{a}}} (1 - f)^{3/2} M_{\text{H}}, 
\]

and identifying \( M_{\text{a}} \) with \( M \) and \( \sigma_{\text{H}} \) with \( \sigma_0 \), we have

\[
A_{\text{ref}}(\eta_{ta}) \approx 2.28 \times 10^{-2} \sigma_0 \left( \frac{M}{M_{\text{H}}} \right)^{-1/3}. 
\]

Then, we obtain the Kerr parameter of a PBH by putting

\[
a = \sqrt{S_0^2/GM^2} = A_{\text{ref}} s_e = Ch 
\]

in terms of \( h \) with a coefficient \( C \), where

\[
C = \frac{2^{9/2}\pi}{5\gamma^3} \sqrt{1 - \gamma^2} A_{\text{ref}}(\eta_{ta}) \]

\[
= 3.25 \times 10^{-2} \sqrt{1 - \gamma^2} \sigma_0 \left( \frac{M}{M_{\text{H}}} \right)^{-1/3} \left( \nu \right)^{-1}. 
\]

We have set \( \gamma \approx 1 \) in the last equality.

**Appendix B**

**Correction Due to Peak Finding Condition**

We regard the density contrast \( \delta \) as a Gaussian random field. This implies that its scaling \( \delta/\sigma_0 \) is also a Gaussian random field. However, if we focus on its peaks, the probability distribution of the spin values, \( \nu = \delta_{pk}/\sigma_0 \), is not given by a Gaussian function because it is corrected by the peak finding condition.

According to Bardeen et al. (1986), the number density of the peaks with the value in \( (\nu, \nu + d\nu) \) is given by

\[
N_{pk}(\nu) d\nu = \frac{1}{(2\pi)^2} \left( \frac{\sigma_2}{\sqrt{3}\sigma_1} \right)^3 e^{-\nu^2/2} \times \left[ \int_0^\infty dx f(x) \exp[-(x - \gamma\nu)^2/(2(1 - \gamma^2))] \right] d\nu, 
\]

where \( x := -\nabla^2 \delta/\sigma_2 \) is the width of the peak and \( f(x) \) is a function behaving \( f(x) \to x^3 - 3x \) for large \( x \). Note that \( x \) is also a statistical variable that is, in general, independent of \( \nu \).

For the perfect correlation of \( \nu \) and \( x \), \( \gamma \to 1 \), it reduces to

\[
N_{pk}(\nu) d\nu = \frac{1}{(2\pi)^2} \left( \frac{\sigma_2}{\sqrt{3}\sigma_1} \right)^3 e^{-\nu^2/2} f(\nu) d\nu. 
\]

In a finite volume \( V \), the number of peaks that will collapse into PBHs, i.e., peaks of \( \nu > \nu_{th} \), is given by

\[
N_{\text{PBH}} = V \int_{\nu_{th}}^\infty N_{pk}(\nu) d\nu. 
\]

The number of peaks in the range \( (\nu, \nu + d\nu) \) in \( V \) is given by

\[
n_{\text{PBH}}(\nu) d\nu = V N_{pk}(\nu) d\nu. 
\]

Then, in the volume \( V \), the probability distribution for one to find a peak in the range \( (\nu, \nu + d\nu) \) from all the peaks greater than the threshold is given by

\[
P_p(\nu) d\nu = \frac{n_{\text{PBH}}(\nu) d\nu}{N_{\text{PBH}}} = \frac{e^{-\nu^2/2} f(\nu) d\nu}{\int_{\nu_{th}}^\infty e^{-\nu^2/2} f(\nu) d\nu}. 
\]

Therefore, the peak finding condition is given by a Gaussian function with the correction factor \( f(\nu) \). However, for peaks that will collapse into PBHs, the values of \( \nu \) are always large such that \( \nu > \nu_{th} \approx 10 \). Thus, the Gaussian factor \( e^{-\nu^2/2} \) rapidly decays for larger \( \nu \) in the range \( (\nu_{th}, \infty) \) and contributes to the probability distribution of PBH binaries, Equation (27), only if \( \nu > \nu_{th} \). Then, we can regard the factor \( f(\nu) \approx f(\nu_{th}) \) as a constant, which contributes to the overall factor. As a
conclusion, we approximate the probability distribution as

$$P_v(v) \, dv = \frac{e^{-v^2/2} f(v) \, dv}{\int_{v_{th}}^\infty e^{-v^2/2} f(v) \, dv} \simeq \frac{e^{-v^2/2} \, dv}{\int_{v_{th}}^\infty e^{-v^2/2} \, dv}, \quad (B6)$$

as in Equation (12).

As discussed at the end of Section 3, the current PBH binary model has an uncertainty because, for example, we have applied numerical results obtained in works that assume spherical symmetry. We estimate the effect of this uncertainty in Appendix C and find that it somehow, but not greatly, changes the widths of the distribution. Thus, the above approximation, Equation (B6), would not matter compared with this uncertainty.

Appendix C
Modification of $\sigma_0$

As a modification of the peak threshold due to the uncertainty of the value of $C_{m,th}$, we here show the numerical results of $P(M, q, \chi_{eff})$ with different values of $\sigma_0$. We take the values as $\sigma_0 = 0.128 \,(=(2/3) \times 0.192), 0.192 \,(\text{the value taken in the main part}),$ and $0.288 \,(=(3/2) \times 0.192)$. The results in Figures 3–5 show that the distribution is somehow broadened (narrowed) for larger (smaller) $\sigma_0$ with fixed $\nu_{th} = 10$. In particular, the widths in $\chi_{eff}$ of $P(\chi_{eff}, q)$ are about 0.001, 0.0015, and 0.002 for $\sigma_0 = 0.288, 0.192,$ and $0.128$, respectively. Thus, we conclude that the uncertainty in the value of $C_{m,th}$ does not greatly affect the distribution.

Figure 3. Contour plots of $\log_{10} P(\chi_{eff}, q)$ for $\sigma_0 = 0.128, 0.192$, and 0.288.

Figure 4. Contour plots of $\log_{10} P(\chi_{eff}, M)$ for $\sigma_0 = 0.128, 0.192$, and 0.288.

Figure 5. Contour plots of $\log_{10} P(M, q)$ for $\sigma_0 = 0.128, 0.192$, and 0.288.
