TOPOLOGIZABLE AND POWER BOUNDED WEIGHTED COMPOSITION OPERATORS ON SPACES OF DISTRIBUTIONS

THOMAS KALMES

Abstract. We study topologizability and power boundedness of weighted composition operators on (certain subspaces of) $\mathcal{D}'(X)$ for an open subset $X$ of $\mathbb{R}^d$. For the former property we derive a characterization in terms of the symbol and the weight of the weighted composition operator, while for the latter property necessary and sufficient conditions on the weight and the symbol are presented. Moreover, for an unweighted composition operator a characterization of power boundedness in terms of the symbol is derived for the special case of a bijective symbol.

Keywords: Weighted composition operator; Topologizable operator; Power bounded operator

MSC 2020: 47B33, (47C05, 46F05)

1. Introduction

Recently, topologizability and power boundedness (see Definition 2.3 below) of (weighted) composition operators on various spaces of functions have been studied by several authors, see e.g. [1], [3], [4] [13], [14], [15]. In [9], a general approach within the framework of function spaces defined by local properties which are subspaces of continuous functions on a locally compact, $\sigma$-compact, non-compact Hausdorff space has been provided. By this general framework, many function spaces which appear in mathematical analysis are covered, and topologizability and power boundedness of weighted composition operators on such spaces are characterized in terms of the symbol and the weight of the operator. However, this general setting does not contain the space of distributions over an open subset of $\mathbb{R}^d$.

The objective of the present note is to characterize topologizability of weighted composition operators on spaces of distributions defined by local properties. Moreover, we investigate power boundedness in this setting as well, and characterize this property for unweighted composition operators on $\mathcal{D}'(X)$, $X \subseteq \mathbb{R}^d$ open, in terms of the symbol for the special case of a bijective symbol.

While the interest for power boundedness of an operator stems from its close relationship to (uniform) mean ergodicity, topologizable operators were introduced by Želazko in [17] (see also [2]). For a Hausdorff locally convex space $E$, in order that the algebra $L(E)$ of all continuous endomorphisms of $E$ (with composition as multiplication) is topologizable, i.e. $L(E)$ admits a locally convex topology for which multiplication is jointly continuous, $E$ is necessarily subnormed. The latter means that there is a norm on $E$ such that the corresponding topology is finer than the locally convex topology initially given on $E$, see [16] and references therein. In case of a sequentially complete $E$ it has been shown in [16] that this necessary condition on $E$ is also sufficient for the topologizability of $L(E)$. Motivated by this, in [17] it

Chemnitz University of Technology, Faculty of Mathematics, 09107 Chemnitz, Germany
E-mail address: thomas.kalmes@math.tu-chemnitz.de.
was investigated when for a given continuous linear operator $T$ on a locally convex Hausdorff space $E$ there is a unital subalgebra $A$ of $L(E)$ which contains $T$ and which admits a locally convex topology making $A$ into a topological algebra such that additionally the map

$$A \times E \to E, (S, x) \mapsto Sx$$

is continuous. By [17, Theorem 5] for a given $T \in L(E)$ there is such a subalgebra $A$ of $L(E)$ precisely when $T$ is topologizable.

Throughout, we use standard notation and terminology from functional analysis. For anything related to functional analysis which is not explained in the text we refer the reader to [3] and [6]. Moreover, we use common notation from the theory of distributions and linear partial differential operators. For this we refer the reader to [11].

By an open, relatively compact exhaustion $(X_n)_{n \in \mathbb{N}}$ of a topological space $X$ we understand a sequence of open subsets of $X$ such that $\overline{X}_n \subseteq X_{n+1}$ with compact closure $\overline{X}_n$ for all $n \in \mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} X_n = X$.

2. Weighted composition operators on spaces of distributions defined by local properties

As in [9] we are interested in topologizability of weighted composition operators - the precise definition of topologizability will be recalled below. However, contrary to [9] where weighted composition operators were considered on spaces of functions, in the present paper we consider these operators on spaces of distributions defined by local properties. As a general framework we choose the notion of sheaves. In what follows we always assume that the space of compactly supported smooth functions $\mathcal{D}(X)$ on an open set $X \subseteq \mathbb{R}^d$ is equipped with its standard locally convex topology (see e.g. [12, Chapter 6] or [7, Chapter 2.12]).

**Definition 2.1.** Let $\mathcal{D}$ be a sheaf of distributions on $\mathbb{R}^d$ defined by local properties, i.e.

- For every open subset $X \subseteq \mathbb{R}^d$ we have a subspace $\mathcal{D}(X)$ of $\mathcal{D}'(X)$ equipped with the relative topology inherited by the strong (dual) topology on $\mathcal{D}'(X)$ with respect to the dual pair $(\mathcal{D}(X), \mathcal{D}'(X))$ such that whenever $Y \subseteq \mathbb{R}^d$ is another open set with $Y \subseteq X$ the restriction mapping

$$r^Y_X : \mathcal{D}(X) \to \mathcal{D}(Y), u \mapsto u|_Y$$

is well-defined. Here we use the common abbreviation $u|_Y := u|_{\mathcal{D}(Y)}$ for $u \in \mathcal{D}'(X)$.

- (Localization) For an open set $X \subseteq \mathbb{R}^d$, for every open cover $(X_i)_{i \in I}$ of $X$, and for each $u, v \in \mathcal{D}(X)$ with $u|_{X_i} = v|_{X_i}$ ($i \in I$) we have $u = v$. (Note that this property always holds since $\mathcal{D}$ is a sheaf!)

- (Gluing) For an open set $X \subseteq \mathbb{R}^d$, for every open cover $(X_i)_{i \in I}$ of $X$, and for all $(u_i)_{i} \in \prod_{i \in I} \mathcal{D}(X_i)$ with $u_i|_{X_i \cap X_\kappa} = u_k|_{X_i \cap X_\kappa}$ ($i, \kappa \in I$) there is $u \in \mathcal{D}(X)$ with $u_i|_{X_i} = u_i$ ($i \in I$).

It follows from the above properties that for every open subset $X \subseteq \mathbb{R}^d$ and each open, relatively compact exhaustion $(X_n)_{n \in \mathbb{N}_0}$ of $X$ the space $\mathcal{D}(X)$ and the projective limit $\operatorname{proj}_{n \rightarrow \infty}(\mathcal{D}(X_n), \mathcal{D}(X_{n+1}))$ are algebraically isomorphic via the mapping

$$\mathcal{D}(X) \to \operatorname{proj}_{n \rightarrow \infty}(\mathcal{D}(X_n), \mathcal{D}(X_{n+1})), u \mapsto (r^a_X(u))_{a \in \mathbb{N}_0} = (u|_{X_a})_{a \in \mathbb{N}_0}.$$  

For $B \subseteq \mathcal{D}(X)$ bounded it follows (see e.g. [12, Theorem 6.5] or [7, Example 2.12.6]) that there is $n \in \mathbb{N}$ for which $B \subseteq \mathcal{D}(X_n)$ and that $B$ is bounded in $\mathcal{D}(X_n)$. 

From this we conclude that the above algebraic isomorphism between \( \mathcal{G}(X) \) and \( \text{proj}_{c_0}(\mathcal{G}(X_n), r^*_{X_n}) \) is a topological isomorphism.

For obvious reasons, \( \mathcal{G}(X) \), \( X \subseteq \mathbb{R}^d \) open, is called a space of distributions defined by local properties.

**Example 2.2.** Obviously, we can choose \( \mathcal{G} = \mathcal{D}' \). Moreover, for any polynomial \( P \in \mathbb{C}[X_1, \ldots, X_d] \) we can consider \( \mathcal{G} = \mathcal{D}'_P \), i.e. for every open \( X \subseteq \mathbb{R}^d \)

\[
\mathcal{D}'_P(X) = \{ u \in \mathcal{D}'(X); P(\partial)u = 0 \}.
\]

Whenever \( P \) is not hypoelliptic, \( \mathcal{D}'_P(X) \) and \( C^\infty_P(X) \) do not coincide, where \( C^\infty_P(X) = \{ f \in C^\infty(X); P(\partial)f = 0 \} \).

**Definition 2.3.** For a locally convex space \( E \) we denote by \( cs(E) \) the set of continuous seminorms on \( E \). Let \( T \) be a continuous linear operator on \( E \).

i) \( T \) is called topologizable if for every \( p \in cs(E) \) there is \( q \in cs(E) \) such that for all \( m \in \mathbb{N} \) there is \( \gamma_m > 0 \) such that

\[
p(T^m(x)) \leq \gamma_m q(x) \quad \text{for all } x \in E.
\]

ii) \( T \) is called power bounded if for every \( p \in cs(E) \) there is \( q \in cs(E) \) such that for all \( m \in \mathbb{N} \)

\[
p(T^m(x)) \leq q(x) \quad \text{for all } x \in E,
\]

i.e. if the set of iterates \( \{ T^m; m \in \mathbb{N} \} \) of \( T \) is equicontinuous.

Clearly, every power bounded operator is topologizable. Moreover, \( T \) is topologizable whenever there is a sequence \( \{ \alpha_m \}_{m \in \mathbb{N}} \) of strictly positive numbers such that the set \( \{ \alpha_m T^m; m \in \mathbb{N} \} \) is equicontinuous and then the sequences \( \{ \gamma_m \}_{m \in \mathbb{N}} \) in the definition of topologizability can be chosen independently of the involved seminorms \( p \) and \( q \).

**Definition 2.4.** Let \( X \subseteq \mathbb{R}^d \) be open, \( w \in C^\infty(X) \), and let \( \psi : X \to X \) be smooth such that \( \det J\psi(x) \neq 0 \) for every \( x \in X \), where \( J\psi(x) \) denotes the Jacobian. The **weighted composition operator** \( C_{w,\psi} \) on \( \mathcal{D}'(X) \) is defined as the unique continuous operator on \( \mathcal{D}'(X) \) which extends \( C_{w,\psi}(f) = w(f \circ \psi) \), see e.g. [5, Theorem 6.1.2]. The function \( \psi \) is called the **symbol** and \( w \) the **weight** of \( C_{w,\psi} \). In case that \( w = 1 \) we write \( C_\psi \) instead of \( C_{1,\psi} \) and \( C_\psi \) is simply called composition operator.

If \( C_{w,\psi}(\mathcal{G}(X)) \subseteq \mathcal{G}(X) \) it follows that \( C_{w,\psi} \) is a continuous operator on \( \mathcal{G}(X) \). We are interested to characterize when \( C_{w,\psi} \) is topologizable etc. on \( \mathcal{G}(X) \). For injective \( \psi \) one verifies

\[
(C^m_{w,\psi}(u), \varphi) = \langle u, \left( \varphi \frac{\prod_{j=0}^{m-1} w(\psi^j(\cdot))}{|\det J\psi^m(\cdot)|} \right) \circ (\psi^m)^{-1} \rangle
\]

for all \( u \in \mathcal{D}'(X), \varphi \in \mathcal{D}(X), m \in \mathbb{N} \).

**Definition 2.5.** Let \( X \) be a topological space and \( \psi : X \to X \) be a continuous mapping. \( \psi \) is said to have **stable orbits** if for every compact \( K \subseteq X \) there is another compact \( L \subseteq X \) such that \( \psi^m(K) \subseteq L \) for every \( m \in \mathbb{N} \).

Our first result gives a sufficient condition on the symbol \( \psi \) for the weighted composition operator \( C_{w,\psi} \) to be topologizable. Clearly, for every topologizable operator
Lemma 2.7. Let \( B_N \) be open, \( \varphi : X \to X \) be smooth and injective such that \( \det J\varphi(x) \neq 0 \) for all \( x \in X \). Assume that \( \varphi \) has stable orbits. Then \( C_{\varphi,\psi} \) is topologizable on \( \mathcal{D}'(X) \).

Proof. Let \( K \subseteq X \) be compact and choose \( L \subseteq X \) compact such that \( \psi^m(K) \subseteq L \) for all \( m \in \mathbb{N} \). For each \( m \in \mathbb{N} \)

\[
M_m : \mathcal{D}(K) \to \mathcal{D}(K), \varphi \mapsto \varphi \prod_{j=0}^{m-1} \frac{w(\psi^j(\cdot))}{\det J\psi(\psi^j(\cdot))}
\]

is continuous as is

\[
\Psi_m : \mathcal{D}(K) \to \mathcal{D}(\psi^m(K)), \varphi \mapsto \varphi \circ (\psi^m)^{-1}.
\]

Therefore, for every absolutely convex and bounded \( B \subseteq \mathcal{D}(K) \) we obtain together with the correctly defined continuous inclusions

\[
\forall m \in \mathbb{N} : \mathcal{D}(\psi^m(K)) \hookrightarrow \mathcal{D}(L)
\]

that

\[
\forall m \in \mathbb{N} : B_m := (\Psi_m \circ M_m)(B) \subseteq \mathcal{D}(L)
\]

is metrizable if it follows form Mackey’s countability condition (see e.g. [7] Proposition 2.6.3 or [11] Lemma 26.6 a]) that there are \( \tilde{B} \subseteq \mathcal{D}(L) \) bounded, absolutely convex and closed, \( (\alpha_m)_{m \in \mathbb{N}} \) in \((0, \infty)\) such that

\[
\forall m \in \mathbb{N} : B_m \subseteq \alpha_m \tilde{B}.
\]

Thus, we obtain for \( \varphi \in \tilde{B} \), \( u \in \mathcal{D}'(X) \)

\[
|\langle C_{\varphi,\psi}^m(u), \varphi \rangle| = |\langle u, (\Psi_m \circ M_m)\varphi \rangle| = \alpha_m |\langle u, \frac{1}{\alpha_m}(\Psi_m \circ M_m)\varphi \rangle| \leq \alpha_m \sup \{|\langle u, \phi \rangle| : \phi \in \tilde{B}\}
\]

which implies

\[
\forall m \in \mathbb{N} : \mathcal{D}'(X) : \sup \{|\langle C_{\varphi,\psi}^m(u), \varphi \rangle| : \varphi \in B \} \leq \alpha_m \sup \{|\langle u, \phi \rangle| : \phi \in \tilde{B}\}.
\]

Because every absolutely convex and bounded \( B \subseteq \mathcal{D}(X) \) is contained in \( \mathcal{D}(K) \) for a suitable compact \( K \subseteq X \) and is bounded in \( \mathcal{D}(K) \) (see e.g. [12] Theorem 6.5 or [7] Example 2.12.6) the proof is finished. □

The next result shows that under suitable additional hypothesis on \( \mathcal{G}(X) \) as well as on \( w \) and \( \psi \), topologizability of \( C_{\varphi,\psi} \) implies that \( \psi \) has stable orbits.

Lemma 2.7. Let \( \mathcal{G} \) be a sheaf of distributions defined by local properties, \( X \subseteq \mathbb{R}^d \) be open, \( w \in C^\infty(X), \psi : X \to X \) smooth and injective such that \( \det J\psi(x) \neq 0 \) for all \( x \in X \). Assume that \( C_{\varphi,\psi}(\mathcal{G}(X)) \subseteq \mathcal{G}(X) \) and that additionally the following conditions hold.

a) There is an open, relatively compact exhaustion \( (X_n)_{n \in \mathbb{N}} \) of \( X \) such that for each \( n \in \mathbb{N} \), every \( x \in X \setminus X_n \), and every \( \varepsilon > 0 \) for which \( B(x, \varepsilon) \subseteq X \setminus X_n \), the restriction

\[
r_{X_n \cup B(x, \varepsilon)} : \mathcal{G}(X) \to \mathcal{G}(X_n \cup B(x, \varepsilon))
\]

has dense range, where \( B(x, \varepsilon) \) denotes the open euclidean ball around \( x \) with radius \( \varepsilon \).
b) There is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is $\chi \in \mathcal{D}(B(0, \varepsilon))$ such that for all $x \in X$ with $\overline{B(x, \varepsilon)} \subseteq X$ there is $h \in \mathcal{D}(B(x, \varepsilon))$ satisfying

$$\langle h, \tau_x \chi \rangle \neq 0,$$

where $\tau_x \chi(y) := \chi(y - x)$.

c) For every $l \in \mathbb{N}_0$ the set $\{x \in X; w(\psi^l(x)) \neq 0\}$ is dense in $X$.

If $C_{w, \psi}$ is topologizable on $\mathcal{G}(X)$, then $\psi$ has stable orbits.

**Remark 2.8.** Before we present the technical proof of the above lemma we take a closer look at its additional assumptions a) - c).

Under the hypothesis on $\psi$ in the above lemma it follows that for every $x \in X$ there is an open neighborhood $U_x \subseteq X$ such that $\psi|_{U_x}$ is (injective and) open. Hence, if for $w \in C^\infty(X)$ the set $\{x \in X; w(x) \neq 0\}$ is dense in $X$, it follows from [3] Proposition 3.9] that the above hypothesis c) is fulfilled.

The above hypothesis b) is satisfied whenever $\mathcal{G}$ is invariant under translations (i.e. whenever for $u \in \mathcal{G}(X)$ it holds $\tau_x u \in \mathcal{G}(-x + X)$), where

$$\forall \varphi \in \mathcal{G}(-x + X) : (\tau_x u, \varphi) = (u, \tau_{-x} \varphi))$$

and $\mathcal{G}$ satisfies

$$\exists \varepsilon > 0 \forall \varepsilon \in (0, \varepsilon_0) \exists \chi \in \mathcal{D}(B(0, \varepsilon)), h \in \mathcal{D}(B(0, \varepsilon)) : \langle h, \chi \rangle \neq 0.$$
technical preparations have to be made which will be finished once we have proved (3) below.

For $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we define
\[
\delta_{m,n} := \text{dist}(\psi^m(X_n), \mathbb{R}^d \setminus \psi^m(X_{n+1}))
\]
and
\[
\delta_n := \delta_{0,n},
\]
so that $\delta_{m,n} > 0$. It follows from hypothesis c) that the set
\[
\bigcap_{i=0}^{m-1} \{ x \in X; w(\psi^i(x)) \neq 0 \}
\]
is dense in $X$ for every $m \in \mathbb{N}$. For $n, m, l, \delta \in \mathbb{N}$ with $l > \frac{2}{\delta n}$ it follows $B(x, \frac{\delta}{l}) \subseteq X_{n+1}$ whenever $x \in X_n$ and we define
\[
Y_{l,m,n} := \{ x \in X_n; \forall y \in B(x, \frac{2}{l}) : \prod_{j=0}^{m-1} w(\psi^j(y)) \neq 0 \}
\]
and
\[
X_{l,m,n} := \bigcup_{x \in Y_{l,m,n}} B(x, \frac{1}{l})
\]
so that $X_{l,m,n} \subseteq X_{n+1}$ then. We then have
\[
\bigcup_{x \in Y_{l,m,n}} B(x, \frac{1}{l}) \subseteq X_{l,m,n}.
\]
Indeed, if $y \in \bigcup_{x \in Y_{l,m,n}} B(x, \frac{1}{l})$ there are sequences $(x_k)_{k \in \mathbb{N}}$ in $Y_{l,m,n}$ and $(z_k)_{k \in \mathbb{N}}$ in $B(0, \frac{1}{l})$ such that $(x_k + z_k)_{k \in \mathbb{N}}$ converges to $y$. Since $Y_{l,m,n} \subseteq X_n$ we can assume without loss of generality that $(x_k)_{k \in \mathbb{N}}$ converges in $X_n$ and $(z_k)_{k \in \mathbb{N}}$ converges in $B(0, \frac{1}{l})$; we denote the limits by $x_0$ and $z_0$, respectively. For $v \in B(x_0, \frac{2}{l})$ and $k$ sufficiently large we have
\[
|x_k - x_0| < \frac{2}{l} - |x_0 - v|
\]
so that
\[
|x_k - v| \leq |x_k - x_0| + |x_0 - v| < \frac{2}{l},
\]
i.e. $v \in B(x_k, \frac{2}{l})$ hence $\prod_{j=0}^{m-1} w(\psi^j(v)) \neq 0$ because $x_k \in Y_{l,m,n}$. As $v \in B(x_0, \frac{2}{l})$ was chosen arbitrarily, it follows $x_0 \in Y_{l,m,n}$ so that
\[
y = x_0 + z_0 \in \bigcup_{x \in Y_{l,m,n}} B(x, \frac{2}{l}) = X_{l,m,n}
\]
showing (1).

Since the bijection
\[
(\psi^m(\chi_{n+1}))^{-1} : \psi^m(X_{n+1}) \to X_{n+1}
\]
is uniformly continuous, for $l > \frac{2}{\delta n}$ there is $\beta_{l,m} > 0$ such that
\[
(2) \quad \forall \psi^m(x), \psi^m(y) \in \psi^m(X_{n+1}), |\psi^m(x) - \psi^m(y)| \leq \beta_{l,m} : |x - y| < \frac{1}{l}.
\]
For every $l, m, n \in \mathbb{N}$ with $l > \frac{2}{\delta n}$, we choose - with $\varepsilon_0$ from hypothesis b) -
\[
\varepsilon_{l,m,n} \in (0, \min\{ \frac{1}{l}, \frac{\delta_{m,n}}{2}, \beta_{l,m}, \varepsilon_0 \})
\]
and $\chi_{l,m,n} := X_{l,m,n} \in \mathcal{D}(B(0, \varepsilon_{l,m,n}))$ according to hypothesis b).
Indeed, if \( y \in \mathbb{R}^d \) is such that \( \chi_{l,m,n}(\psi^m(y) - \psi^m(x_0)) \neq 0 \) it follows that \( \psi^m(y) \in B(\psi^m(x_0), \varepsilon_{l,m,n}) \subseteq \psi^m(X_n) + B(0, \varepsilon_{l,m,n}) \subseteq \psi^m(X_{n+1}) \),

where we have used \( \varepsilon_{l,m,n} < \delta_{l,m,n}/2 \) and the definition of \( \delta_{l,m,n} \) in the last inclusion. Because \( \psi^m \) is injective, we conclude \( y \in X_{n+1} \). Because, moreover

\[
|\psi^m(y) - \psi^m(x_0)| < \varepsilon_{l,m,n} \leq \beta_{l,m,n}
\]

we also have \( |y - x_0| < 1/l \) by (2). Hence

\[
y \in B(x_0, \frac{1}{l}) \subseteq \bigcup_{x \in Y_{l,m,n}} B(x, \frac{1}{l})
\]

so that the support of \( y \mapsto \chi_{l,m,n}(\psi^m(y) - \psi^m(x_0)) \) is contained in the closure of \( \cup_{x \in Y_{l,m,n}} B(x, 1/l) \) which proves (3).

We now fix \( n \in \mathbb{N} \). Recall that our objective is to prove the existence of \( k \in \mathbb{N} \) satisfying \( \psi^m(X_n) \subseteq X_k \) for all \( m \in \mathbb{N} \). Since for \( m, l \in \mathbb{N} \) with \( l > \frac{1}{2^m} \) we have \( X_{l,m,n} \subseteq X_{n+1} \), it follows from (1) and the relative compactness of \( X_{n+1} \) that the closure of \( \cup_{x \in Y_{l,m,n}} B(x, 1/l) \) is a compact subset of \( X_{l,m,n} \). Moreover, from the definition of \( Y_{l,m,n} \) it follows \( \psi^m(Y_{l,m,n}) \subseteq \psi^m(X_n) \) so that compactness of \( \psi^m(X_n) \) implies that

\[
\{ \chi_{l,m,n}(\psi^m(\cdot) - \psi^m(x_0)); x_0 \in Y_{l,m,n} \}
\]

is a bounded subset of \( \mathscr{D}(X_{l,m,n}) \). From the definition of \( X_{l,m,n} \) it follows that

\[
\mathscr{D}(X_{l,m,n}) \rightarrow \mathscr{D}(X_{l,m,n}), \varphi \mapsto \frac{|\det J\psi^m(\cdot)|}{\prod_{j=0}^{m-1} \varphi(\psi^j(\cdot))} \varphi
\]

is correctly defined and continuous so that

\[
\{ \chi_{l,m,n}(\psi^m(\cdot) - \psi^m(x_0)); x_0 \in Y_{l,m,n} \}
\]

is a bounded subset of \( \mathscr{D}(X_{l,m,n}) \), too. From the continuity of the inclusion \( \mathscr{D}(X_{l,m,n}) \hookrightarrow \mathscr{D}(X_{n+1}) \) \( (X_{l,m,n} \subseteq X_{n+1}) \), we derive that for all \( l, m \in \mathbb{N}, l > \frac{2^m}{\alpha_{l,m,n}} \),

\[
B_{l,m,n} := \{ \chi_{l,m,n}(\psi^m(\cdot) - \psi^m(x_0)); x_0 \in Y_{l,m,n} \}
\]

is a bounded subset of \( \mathscr{D}(X_{n+1}) \).

Because \( \mathscr{D}(X_{n+1}) \) is metrizable, it follows from Mackey’s countability condition (see e.g. [7] Proposition 2.6.3] or [11] Lemma 26.6 a)]) that there are a closed, absolutely convex, and bounded \( B \subseteq \mathscr{D}(X_{n+1}) \) and strictly positive \( \alpha_{l,m,n}(l, m \in \mathbb{N}, l > \frac{2^m}{\alpha_{l,m,n}}) \) such that

\[
(4) \quad B_{l,m,n} \subseteq \alpha_{l,m,n}B.
\]

Let \( B^\circ \) denote the polar of \( B \) with respect to the dual pair \( (\mathscr{D}(X), \mathscr{D}'(X)) \). Now, as \( \mathscr{D}(X) \) and \( \text{proj}_{\mathcal{F}}(\mathscr{D}(X), r^X_{X_k}) \) are topologically isomorphic, from the topologizability of \( C_{w,\psi} \) it follows that for the zero neighborhood \( B^\circ \cap \mathcal{F}(X) \) in \( \mathcal{F}(X) \) there is \( k \in \mathbb{N} \) and a zero neighborhood \( U_k \) in \( \mathcal{F}(X_k) \) such that for all \( m \in \mathbb{N} \) there are \( \gamma_m \) with

\[
(5) \quad C^m_{w,\psi}(r^X_{X_k})^{-1}(U_k) \subseteq \gamma_m(B^\circ \cap \mathcal{F}(X)) \subseteq \gamma_m B^\circ.
\]
We shall show that \( \psi^m(X_n) \subseteq X_k \) for all \( m \in \mathbb{N} \). Taking polars with respect to the dual pair \((\mathscr{D}(X), \mathscr{D}'(X))\), \(8\) together with the Bipolar Theorem (cf. \(11\) Theorem 22.13)) implies

\[
\forall m \in \mathbb{N} : (C^l_{w,\psi})^m(B) \subseteq \gamma_m \left( (r^X_k)^{-1}(U_k) \right)^\circ,
\]

where \( C^l_{w,\psi} \) denotes the transpose of \( C_{w,\psi} \) on \( \mathscr{D}(X) \). By \(4\) we deduce

\[
(6) \quad \forall l, m \in \mathbb{N}, l > 2/\delta_n : (C^l_{w,\psi})^m(B_{1,m,n}) \subseteq \alpha_{1,m,n} \gamma_m \left( (r^X_k)^{-1}(U_k) \right)^\circ.
\]

In order to show \( \psi^m(X_n) \subseteq X_k, m \in \mathbb{N} \), we argue by contradiction. We assume the existence of \( m_0 \in \mathbb{N} \) and

\[
x_0 \in \{ x \in X_n : \prod_{j=0}^{m_0-1} w(\psi^j(x)) \neq 0 \}
\]

such that \( \psi^{m_0}(x_0) \notin X_k \). Then there is \( l \in \mathbb{N}, l > 2/\delta_n \), with \( x_0 \in Y_{1,m_0,n} \) and because \( Y_{1+1,m_0,n} \subseteq Y_{1,m_0,n} \) we can have \( l_0 \) so large that \( x_0 \in Y_{l_0,m_0,n} \) and \( B(\psi^{m_0}(x_0), 1/l_0) \subseteq X(X_k) \) and such that according to hypothesis \(a)\) \( (r^X_k)^{-1}(U_k) \) has dense range for \( U := B(\psi^{m_0}(x_0), 1/l_0) \).

Choose \( h \in \mathscr{D}(B(\psi^{m_0}(x_0), 1/l_0)) \) for

\[
\varphi := \chi_{l_0,m_0,n}(\cdot - \psi^{m_0}(x_0)) = \chi_{l_0,m_0,n}(\cdot - \psi^{m_0}(x_0))
\]

according to hypothesis \(b)\) without loss of generality we assume that

\[
\delta_\varphi(h) := \langle h, \varphi \rangle = 1.
\]

By the properties of a sheaf, there is \( v \in \mathscr{D}(X_k \cup U) \) such that \( r^X_{X_k \cup U}(v) = 0 \) and \( r^Y_{X_k \cup U}(v) = 3\alpha_{l_0,m_0,n} \gamma_{m_0} h \). Since \( \varphi \in \mathscr{D}(U) \) we have \( \delta_\varphi(v) = 3\alpha_{l_0,m_0,n} \gamma_{m_0} \).

Because \( r^Y_{X_k \cup U} \) has dense range by hypothesis \(a)\) there is \( u \in \mathscr{D}(X) \) such that

\[
r^X_{X_k \cup U}(u) - v \in (r^X_{X_k \cup U})^{-1}(U_k) \cap \delta_\varphi^{-1}(B(0, \alpha_{l_0,m_0,n} \gamma_{m_0})),
\]

where \( U_k \) is the zero neighborhood in \( \mathscr{D}(X_k) \) from \(5\), so that

\[
(7) \quad \delta_\varphi(r^X_{X_k \cup U}(u)) \in B(3\alpha_{l_0,m_0,n} \gamma_{m_0}, \alpha_{l_0,m_0,n} \gamma_{m_0})
\]

as well as

\[
r^X_{X_k \cup U}(u) = r^X_{X_k \cup U}(r^X_{X_k \cup U}(u) - v) + r^X_{X_k \cup U}(v) = r^X_{X_k \cup U}(r^X_{X_k \cup U}(u) - v) + 0 \in U_k,
\]

that is

\[
u \in (r^X_{X_k \cup U})^{-1}(U_k).
\]

Because by definition of \( B_{l_0,m_0,n} \) and \( x_0 \in Y_{l_0,m_0,n} \) we have

\[
\frac{|\det J\psi^{m_0}(\cdot)|}{\prod_{j=0}^{m_0-1} w(\psi^j(\cdot))} \chi_{l_0,m_0,n}(\psi^{m_0}(\cdot) - \psi^{m_0}(x_0)) \in B_{l_0,m_0,n},
\]

it follows herefrom, \(8\), and \(9\)

\[
\alpha_{l_0,m_0,n} \gamma_{m_0} \geq \left| (C^l_{w,\psi})^{m_0} \left( \frac{|\det J\psi^{m_0}(\cdot)|}{\prod_{j=0}^{m_0-1} w(\psi^j(\cdot))} \chi_{l_0,m_0,n}(\psi^{m_0}(\cdot) - \psi^{m_0}(x_0)) \right), u \right|
\]

\[
= \left| \left( \frac{|\det J\psi^{m_0}(\cdot)|}{\prod_{j=0}^{m_0-1} w(\psi^j(\cdot))} \chi_{l_0,m_0,n}(\psi^{m_0}(\cdot) - \psi^{m_0}(x_0)) \right)^{-1}, u \right| = \left| \delta_\varphi(r^X_{X_k \cup U}(u)) \right| > 2\alpha_{l_0,m_0,n} \gamma_{m_0}
\]

which gives a contradiction.
Therefore,
\[ \forall m \in \mathbb{N} : \| \psi^m \| \left( \{ x \in X_n ; \prod_{l=0}^{m-1} w(\psi^l(x)) \neq 0 \} \right) \subseteq X_k. \]

Because \( \psi^m \) is continuous and because
\[ \{ x \in X_n ; \prod_{l=0}^{m-1} w(\psi^l(x)) \neq 0 \} = \bigcap_{l=0}^{m-1} \{ x \in X_n ; w(\psi^l(x)) \neq 0 \} \]

is dense in \( X_n \) we conclude
\[ \forall m \in \mathbb{N} : \psi^m(X_n) \subseteq X_k. \]

Because \( n \) was arbitrarily chosen and \( (X_n)_{n \in \mathbb{N}} \) is an open, relatively compact exhaustion of \( X \) it finally follows that \( \psi \) has stable orbits. \(\Box\)

Combining Proposition 2.6 and Lemma 2.7 we obtain a characterization of topologizability for weighted composition operators.

**Theorem 2.9.** Let \( X \subseteq \mathbb{R}^d \) be open, \( w \in C^\infty(X) \), \( \psi : X \to X \) smooth and injective such that \( \det J\psi(x) \neq 0 \) for all \( x \in X \). Assume that \( \mathcal{G} \) is a sheaf of distributions defined by local properties such that \( C_{w,\psi}(\mathcal{G}(X)) \subseteq \mathcal{G}(X) \) and that additionally the following conditions hold.

a) There is an open, relatively compact exhaustion \( (X_n)_{n \in \mathbb{N}} \) of \( X \) such that for each \( n \in \mathbb{N} \) and every \( x \in X \setminus X_n \) and every \( \varepsilon > 0 \) for which \( B(x, \varepsilon) \subseteq X \setminus X_n \) the restriction \( r_{X_n}^X \tau_B(x, \varepsilon) \) has dense range.

b) There is \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) there is \( \chi_{\varepsilon} \in \mathcal{G}(B(0, \varepsilon)) \) such that for all \( x \in X \) with \( B(x, \varepsilon) \subseteq X \) there is \( h \in \mathcal{G}(B(x, \varepsilon)) \) satisfying
\[ \langle h, \tau_x \chi_{\varepsilon} \rangle \neq 0, \]

where \( \tau_x \chi_{\varepsilon}(y) := \chi_{\varepsilon}(y - x) \).

c) For every \( l \in \mathbb{N}_0 \) the set \( \{ x \in X ; w(\psi^l(x)) \neq 0 \} \) is dense in \( X \).

Then for the weighted composition operator \( C_{w,\psi} \) on \( \mathcal{G}(X) \) the following are equivalent.

i) \( C_{w,\psi} \) is topologizable.

ii) \( \psi \) has stable orbits.

By Remark 2.8 the conditions a) and b) in the above theorem are satisfied for \( \mathcal{G} = \mathcal{G}' \) and \( \mathcal{G} = \mathcal{G}'_P \) for certain \( P \) while condition c) is fulfilled whenever \( \{ x \in X ; w(x) \neq 0 \} \) is dense in \( X \). In particular, we have the following.

**Corollary 2.10.** Let \( X \subseteq \mathbb{R}^d \) be open, \( w \in C^\infty(X) \), \( \psi : X \to X \) smooth and injective such that \( \det J\psi(x) \neq 0 \) for all \( x \in X \). Moreover, assume that \( \{ x \in X ; w(x) \neq 0 \} \) is dense in \( X \). Then, the following are equivalent.

i) The weighted composition operator \( C_{w,\psi} \) is topologizable on \( \mathcal{G}'(X) \).

ii) \( \psi \) has stable orbits.

Now, we turn our attention to power boundedness. For a smooth and injective \( \psi : X \to X \) with \( \det J\psi(x) \neq 0 \) for all \( x \in X \) it follows that \( \psi^m(X) \) is an open subset of \( \mathbb{R}^d \) and \( \psi^m : X \to \psi^m(X) \) is a diffeomorphism for every \( m \in \mathbb{N} \). In particular, \( (\psi^m)^{-1} : \psi^m(X) \to \mathbb{R}^d \) is a smooth function whose components we denote by \( (\psi^m)^{-1} \subseteq 1 \leq c \leq d \). For \( Y \subseteq \mathbb{R}^d \) open, \( K \subseteq Y \) compact, \( n \in \mathbb{N}_0 \), and \( f \in C^\infty(Y) \) we define \( \| f \|_{n,K} := \sup_{|\alpha| \leq n, x \in K} |\partial^\alpha f(x)| \). Thus, \( \| \cdot \|_{n,K} \) is a seminorm on \( C^\infty(Y) \) and the standard topology on \( C^\infty(Y) \) is the one generated by the set of seminorms \( \{ \| \cdot \|_{n,K} ; n \in \mathbb{N}_0, K \subseteq Y \text{ compact} \} \).
Theorem 2.11. Let $X \subseteq \mathbb{R}^d$ be open, $w \in C^\infty(X)$ be such that $\{x \in X; w(x) \neq 0\}$ is dense in $X$, $\psi : X \to X$ smooth and injective such that $\det J\psi(x) \neq 0$ for all $x \in X$. Then, among the following, i) implies ii) and ii) implies iii).

i) $\psi$ has stable orbits and for every compact set $K \subseteq X$ it holds

$$
\sup_{m \in \mathbb{N}} \left\| \left( \prod_{j=0}^{m-1} \frac{w(\psi^j(\cdot))}{\det J\psi(\psi^j(\cdot))} \right) \circ (\psi^m)^{-1} \right\|_{n,\psi^m(K)} < \infty,
$$

and

$$
\forall 1 \leq c \leq d : \sup_{m \in \mathbb{N}} \left\| (\psi^m)^{-1} \right\|_{n,\psi^m(K)} < \infty.
$$

ii) $C_{w,\psi}$ is power bounded on $\mathcal{D}'(X)$.

iii) $\psi$ has stable orbits and for every compact set $K \subseteq X$ holds.

Proof. Assume that i) is valid. For compact subset of $X$ for which $\psi^m(K) \subseteq L(K)$ holds for all $m \in \mathbb{N}$. Let $B \subseteq \mathcal{D}'(X)$ be bounded and let $K \subseteq X$ be compact such that $B \subseteq \mathcal{D}(K)$ is bounded. For fixed $u \in \mathcal{D}'(X)$ there are $r \in \mathbb{N}_0$ and $M_1 > 0$ such that

$$
\forall \phi \in \mathcal{D}'(L(K)) : \|u, \phi\| \leq M_1 \|\phi\|_{r,L(K)}.
$$

Moreover, as $B \subseteq \mathcal{D}(K)$ is bounded, there is $M_2 > 0$ such that

$$
\forall \phi \in B : \|\phi\|_{r,K} \leq M_2.
$$

Because i) holds, there is $M_3 > 0$ such that

$$
\sup_{m \in \mathbb{N}} \left\| \left( \prod_{j=0}^{m-1} \frac{w(\psi^j(\cdot))}{\det J\psi(\psi^j(\cdot))} \right) \circ (\psi^m)^{-1} \right\|_{n,\psi^m(K)} < M_3.
$$

Since for smooth function $f$ and $g$

$$
\|fg\|_{r,\psi^m(K)} \leq 2^r \|f\|_{r,\psi^m(K)} \|g\|_{r,\psi^m(K)}
$$

it follows for $\varphi \in B \subseteq \mathcal{D}(B)$ with $[9, Proposition 3.10 ii)]$ applied to $(\psi^m)^{-1}$ in place of $\psi$ and $m = 1$ in the context of the cited proposition

$$
\|C^m_{w,\psi}(u), \varphi\| = \|u, \left( \prod_{j=0}^{m-1} \frac{w(\psi^j(\cdot))}{\det J\psi(\psi^j(\cdot))} \right) \circ (\psi^m)^{-1} \varphi\| \\
\leq M_1 \left\| \left( \prod_{j=0}^{m-1} \frac{w(\psi^j(\cdot))}{\det J\psi(\psi^j(\cdot))} \right) \circ (\psi^m)^{-1} \right\|_{r,L(K)} \|\varphi\|_{r,\psi^m(K)} \\
= M_1 \left\| \left( \prod_{j=0}^{m-1} \frac{w(\psi^j(\cdot))}{\det J\psi(\psi^j(\cdot))} \right) \circ (\psi^m)^{-1} \right\|_{r,\psi^m(K)} \\
= M_1 \left\| \left( \prod_{j=0}^{m-1} \frac{w(\psi^j(\cdot))}{\det J\psi(\psi^j(\cdot))} \right) \circ (\psi^m)^{-1} \varphi \circ (\psi^m)^{-1} \right\|_{r,\psi^m(K)} \\
\leq 2^r M_1 \left\| \left( \prod_{j=0}^{m-1} \frac{w(\psi^j(\cdot))}{\det J\psi(\psi^j(\cdot))} \right) \circ (\psi^m)^{-1} \right\|_{r,\psi^m(K)} \|\varphi\|_{r,\psi^m(K)} \\
\leq 2^r M_1 M_3 M_4 \|\varphi\|_{r,K} (1 + \max_{1 \leq c \leq d} \| (\psi^m)^{-1}_{c} \|_{r,\psi^m(K)}) \\
\leq 2^r M_1 M_3 M_4 M_2 (1 + \max_{1 \leq c \leq d} \| (\psi^m)^{-1}_{c} \|_{n,\psi^m(K)})
$$

for a suitable constant $M_4$ which is independent of $\varphi$. Because of i) we thus obtain

$$
\forall u \in \mathcal{D}'(X) : \sup_{m \in \mathbb{N}} \sup \{|C^m_{w,\psi}(u), \varphi|; \varphi \in B\} < \infty.
$$
As the strong dual of the complete Schwartz space $\mathcal{D}(X)$ we have that $\mathcal{D}'(X)$ is ultrabornological (see e.g. [11 Proposition 24.23]), hence barrelled, so that (ii) implies ii).

Assume that ii) holds. Then $C_{w,ψ}$ is topologizable and from Remark 2.8 and Theorem 23.15 it follows that $ψ$ has stable orbits. As before, for $K ⊆ X$ compact we denote by $L(K)$ a compact subset of $X$ for which $ψ^m(K) ⊆ L(K)$ holds for every $m ∈ \mathbb{N}$.

We choose $φ ∈ \mathcal{D}(X)$ with $φ = 1$ in a neighborhood of $K$. Then

$$
∀ m ∈ \mathbb{N} : \{ φ \prod_{j=0}^{m-1} \frac{w(ψ^j(\cdot))}{|det Jψ(ψ^j(\cdot))|} \circ (ψ^m)^{-1} \} ⊆ \mathcal{D}(ψ^m(supp φ)) ⊆ \mathcal{D}(L(supp φ))
$$

and because

$$
∀ u ∈ \mathcal{D}'(X), m ∈ \mathbb{N} : \{ C_{w,ψ}^m(u), φ \} = \langle u, \{ φ \prod_{j=0}^{m-1} \frac{w(ψ^j(\cdot))}{|det Jψ(ψ^j(\cdot))|} \circ (ψ^m)^{-1} \} \rangle
$$

it follows from the power boundedness of $C_{w,ψ}$ that

$$
\{ \{ φ \prod_{j=0}^{m-1} \frac{w(ψ^j(\cdot))}{|det Jψ(ψ^j(\cdot))|} \circ (ψ^m)^{-1} \}; m ∈ \mathbb{N} \}
$$

is weakly bounded in $\mathcal{D}(X)$ and therefore, by Mackey’s Theorem (see e.g. [11 Theorem 23.15]), bounded in $\mathcal{D}(X)$. By the choice of $φ$ we have

$$
\{ φ \prod_{j=0}^{m-1} \frac{w(ψ^j(\cdot))}{|det Jψ(ψ^j(\cdot))|} \circ (ψ^m)^{-1}\}_{ψ^m(K)} = \{ \prod_{j=0}^{m-1} \frac{w(ψ^j(\cdot))}{|det Jψ(ψ^j(\cdot))|} \circ (ψ^m)^{-1}\}_{ψ^m(K)}
$$

for all $m ∈ \mathbb{N}$ so that (iii) follows. Thus, ii) implies iii). □

**Remark 2.12.** If $ψ : X → X$ is a diffeomorphism it is straightforward to calculate that the transpose of $C_{w,ψ}^m$ on $\mathcal{D}'(X)$ is given by the restriction of $C_{w,ψ^{-1}}^m$ to $\mathcal{D}(X)$, where

$$
w_ψ : X → \mathbb{C}, w_ψ(x) = \frac{w}{|det Jψ|} ψ^{-1}.
$$

Since $\mathcal{D}'(X)$ is the strong dual of the complete Schwartz space $\mathcal{D}(X)$ it follows that $\mathcal{D}'(X)$ is ultrabornological (see e.g. [11 Proposition 24.23]), hence barrelled. Thus, $C_{w,ψ}$ is power bounded if and only if $\{ C_{w,ψ}^m(u); m ∈ N_0 \}$ is bounded in $\mathcal{D}'(X)$ for every $u ∈ \mathcal{D}'(X)$. Because $\mathcal{D}(X)$ is reflexive, by Mackey’s Theorem (11 Theorem 23.15) it follows that the latter is equivalent to the boundedness of $\{ \{ C_{w,ψ}^m(u), φ \}; m ∈ N_0 \}$ for all $u ∈ \mathcal{D}'(X), φ ∈ \mathcal{D}(X)$. Applying Mackey’s Theorem once more, this in turn is equivalent to $\{ C_{w,ψ^{-1}}^m(φ); m ∈ N_0 \}$ being bounded in $\mathcal{D}(X)$. From the barrelledness of $\mathcal{D}(X)$ it finally follows that this is equivalent to $C_{w,ψ^{-1}}^m$ being power bounded on $\mathcal{D}(X)$.

As usual, for bijective $ψ$ we write $ψ^{-m}$ instead of $(ψ^{-1})^m, m ∈ \mathbb{N}$:

**Corollary 2.13.** Let $X ⊆ \mathbb{R}^d$ be open and $ψ : X → X$ be smooth and bijective. Then the following are equivalent:

i) The composition operator $C_ψ$ is power bounded on $\mathcal{D}'(X)$.

ii) $ψ$ has stable orbits and for every compact set $K ⊆ X$ it holds

$$
∀ 1 ≤ c ≤ d : \sup_{m∈\mathbb{N}} \| (ψ^{-m})_c \|_{n, ψ^m(K)} < ∞.
$$

**Proof.** Assuming that $C_ψ$ is power bounded on $\mathcal{D}'(X)$ it follows from Theorem 2.11 that $ψ$ has stable orbits. Thus, given $K ⊆ X$ compact we can choose $L ⊆ X$ compact such that $ψ^m(K) ⊆ L$ for every $m ∈ \mathbb{N}$. Additionally, we choose $φ ∈ \mathcal{D}(X)$
with $\phi = 1$ in a neighborhood of $L$. For $1 \leq c \leq d$ we define $\varphi_c(x) = x_c\phi(x)$ so that $\varphi_c \in \mathcal{D}(X)$. Since $C_{\varphi}$ is power bounded on $\mathcal{D}'(X)$ it follows from Remark 2.13 observing that $w_\varphi = 1$, that $\{C_{\varphi_{m-1}}(\varphi_c); m \in \mathbb{N}\}$ is a bounded subset of $\mathcal{D}(X)$. In particular, taking into account that $C_{\varphi_{m-1}}(\varphi_c) = \varphi_c \circ \varphi^{m-1} = (\varphi^{m})_c$ in a neighborhood of $K$, we obtain

$$\forall n \in \mathbb{N} : \infty > \sup_{m \in \mathbb{N}} \|C_{\varphi}^{m-1}(\varphi_c)\|_{n,L} \geq \sup_{m \in \mathbb{N}} \|C_{\varphi}^{m-1}(\varphi_c)\|_{n,\varphi^{m}(K)} = \sup_{m \in \mathbb{N}} \|\varphi^{m-1}c\|_{n,\varphi^{m}(K)},$$

so that ii) follows.

If on the other hand ii) holds, it follows from

$$\det J\varphi^{m-1} = \det \left( \prod_{j=1}^{m} (J\varphi)^{-1}(\varphi^{-1}(\cdot)) \right) = \left( \prod_{j=0}^{m-1} \frac{1}{\det J\varphi^{j}(\cdot)} \right) \circ (\varphi^{m})^{-1}$$

and the fact that for fixed $m \in \mathbb{N}$ and for every multi-index $\alpha \in \mathbb{N}^d_0$ the function $\partial^\alpha \det J\varphi^{m-1}$ is a polynomials in $\partial^\alpha(\varphi^{m})_c$, $1 \leq c \leq d$, $1 \leq |\beta| \leq |\alpha| + 1$, with integer coefficients independent from $m$ that for arbitrary $K \subseteq X$ compact

$$\sup_{m \in \mathbb{N}} \left( \prod_{j=0}^{m-1} \frac{w(\varphi^{j}(\cdot))}{\det J\varphi^{j}(\cdot)} \right) \circ (\varphi^{m})^{-1} ||_{n,\varphi^{m}(K)} < \infty.$$ 

Thus, ii) implies i) from Theorem 2.11 so that $C_{\varphi}$ is power bounded on $\mathcal{D}'(X)$. \qed

References

[1] María J. Beltrán-Meneu, M. Carmen Gómez-Collado, Enrique Jordá, and David Jornet. Mean ergodicity of weighted composition operators on spaces of holomorphic functions. J. Math. Anal. Appl., 444(2):1640–1651, 2016.
[2] José Bonet. A problem on the structure of Fréchet spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 104(2):427–434, 2010.
[3] José Bonet and Paweł Domański. Power bounded composition operators on spaces of analytic functions. Collect. Math., 62(1):69–83, 2011.
[4] M. C. Gómez-Collado, E. Jordá, and D. Jornet. Power bounded composition operators on spaces of meromorphic functions. Topology Appl., 203:141–146, 2016.
[5] Lars Hörmander. The analysis of linear partial differential operators. I. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].
[6] Lars Hörmander. The analysis of linear partial differential operators. II. Classics in Mathematics. Springer-Verlag, Berlin, 2005. Differential operators with constant coefficients, Reprint of the 1983 original.
[7] John Horváth. Topological vector spaces and distributions. Vol. I. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.
[8] T. Kalmas. An approximation theorem of Runge type for kernels of certain non-elliptic partial differential operators. arXiv:preprint, 2018.
[9] T. Kalmas. Power bounded weighted composition operators on function spaces defined by local properties. J. Math. Anal. Appl., 471(1-2):211–238, 2019.
[10] T. Kalmas. Surjectivity of differential operators and linear topological invariants for spaces of zero solutions. Rev. Mat. Complut., 32(1):37–55, 2019.
[11] Reinhold Meise and Dietmar Vogt. Introduction to functional analysis, volume 2 of Oxford Graduate Texts in Mathematics. The Clarendon Press Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.
[12] Walter Rudin. Functional analysis. McGraw-Hill Book Co., New York, 1973. McGraw-Hill Series in Higher Mathematics.
[13] Elke Wolf. Power bounded composition operators. Comput. Methods Funct. Theory, 12(1):105–117, 2012.
[14] Elke Wolf. Power bounded weighted composition operators. New York J. Math., 18:201–212, 2012.
[15] Elke Wolf. Power bounded composition operators in several variables. Rom. J. Math. Comput. Sci., 5(1):1–12, 2015.
[16] W. Żelazko. When is $L(X)$ topologizable as a topological algebra? *Studia Math.*, 150(3):295–303, 2002.

[17] Wieslaw Żelazko. Operator algebras on locally convex spaces. In *Topological algebras and applications*, volume 427 of *Contemp. Math.*, pages 431–442. Amer. Math. Soc., Providence, RI, 2007.