Independence of the metric in the fine $C^0$-topology of a function space

Francisco J. Gonzalez-Acuña

January 17, 2014

The following is a translation of the paper “Independencia de la métrica en la $C^0$ topología fina de un espacio de funciones”. Vol. 3 (1963) pp. 29-35. Anales del Instituto de Matemáticas (Universidad Nacional Autónoma México). I thank Francisco Marmolejo for writing this translation.

1 Introduction

Several topologies can be given to the space of continuous functions $F(X,Y)$ from a topological space $X$ into a metrizable space $Y$. One of them is defined as follows: let $d$ be a bounded distance consistent with the topology of $Y$. A distance $d^*$ can be defined on $F(X,Y)$ by:

$$d^*(f,g) = \sup_{x \in X} d[f(x), g(x)].$$

The topology of $F(X,Y)$ determined by $d^*$ is called the $d^*$-topology. This topology depends not only on the topologies of $X$ and $Y$ but also on the metric $d$ that is chosen for $Y$.

In §2 we define de fine $C^0$-topology on $F(X,Y)$. It is shown in [1,§5] that this topology does not depend on the chosen metric for $Y$ in the case where $X$ paracompact. We show here (Theorem 1) the same result without imposing any condition on $X$.

In §3 some properties of the fine $C^0$-topology are mentioned and it is compared with other topologies.

2 The fine $C^0$-topology

Definition 1. Let $X$ be a topological space, $(Y,d)$ metric, $f:X \to Y$ continuous, $\delta:X \to \mathbb{R}^+ = \{\text{positive reals}\}$ continuous. $g:X \to Y$ continuous is a $\delta$-approximation of $f$ if $d[f(x), g(x)] < \delta(x)$ for all $x \in X$. The $\delta$-neighborhood of $f$ is the set of all the $\delta$-approximations of $f$. If $F(X,Y)$ is the set of continuous functions from $X$ to $Y$ and $f \in F(X,Y)$, we define a neighborhood of $f$ as a subset of $F(X,Y)$ that contains some $\delta$-neighborhood of $f$. 

This defines a topology on $F(X,Y)$ given that the intersection of the $\delta$-neighborhood of $f$ and the $\eta$-neighborhood of $f$ is the $\min(\delta, \eta)$-neighborhood of $f$ and furthermore if $g$ is in the $\delta$-neighborhood of $f$ and $\eta(x) = \delta(x) - d[f(x), g(x)]$, then the $\eta$-neighborhood of $g$ is contained in the $\delta$-neighborhood of $f$. We will call this topology the fine $C^0$-topology of $F(X,Y)$, and we denote by $T_d(X,Y)$ the resulting topological space.

The next theorem establishes that the topology on $T_d(X,Y)$ depends only on the topologies of $X$ and $Y$ and not on the metric on $Y$.

**Theorem 1.** If $d_1$ and $d_2$ are two equivalent distances on $Y$, then $T_{d_1}(X,Y)$ and $T_{d_2}(X,Y)$ are the same topological space.

**Proof.** Let $W_1$ be a $\delta_1$-neighborhood of $f: X \to Y$ on $T_{d_1}(X,Y)$. We will show that there exists $W_2$, $\delta_2$-neighborhood of $f$ in $T_{d_2}(X,Y)$, contained in $W_1$. The proof that for every $\delta_2$ neighborhood of $W_2$ of $f$ there is a $\delta_1$-neighborhood contained in $W_2$ being analogous.

We define $\delta_2: X \to \mathbb{R}^+$ as

$$\delta_2 = G' \circ (f \times \delta_1),$$

where $G'$ is a continuous function from $Y \times \mathbb{R}^+$ to $\mathbb{R}^+$ that satisfies:

$$G'(y, t) \leq \sup \{r | B_2(y, r) \subset B_1(y, t)\} = G(y, t)$$

$$(B_i(y, r) = \{ y' \in Y | d_i(y, y') < r \} \ i = 1, 2).$$. Let’s show that there is such a $G'$.

The function $G$, that is positive since $d_1$ and $d_2$ are equivalent, satisfies:

$$G(y', t) \geq G(y, t - \varepsilon) - \varepsilon \quad \text{if} \quad d_i(y, y') < \varepsilon, \quad i = 1, 2.$$ 

Indeed, if $w \in B_2(y', G(y, t - \varepsilon) - \varepsilon)$ and $d_i(y, y') < \varepsilon$ $i = 1, 2$,

$$d_2(w, y') + \varepsilon < G(y, t - \varepsilon)$$
$$d_2(w, y) < G(y, t - \varepsilon)$$
$$B_2(y, s) \subset B_1(y, t - \varepsilon) \quad \text{with} \quad d_2(w, y) < s < G(y, t - \varepsilon)$$
$$d_1(w, y) < t - \varepsilon$$

$$w \in B_1(y', t)$$

and

$$B_2(y', G(y, t - \varepsilon) - \varepsilon) \subset B_1(y', t)$$

that is

$$G(y', t) \geq G(y, t - \varepsilon) - \varepsilon.$$ 

Since $G$ is not decreasing in the second variable, for $(y, t) \in Y \times \mathbb{R}^+$ there is $\varepsilon$ such that $G(y, t - \varepsilon) - \varepsilon > 0$ and thus every point $(y, t)$ of $Y \times \mathbb{R}^+$ has a neighborhood in which $G$ has a positive lower bound, namely the neighborhood $[B_1(y, \varepsilon) \cap B_2(y, \varepsilon)] \times (a, \infty)$ with $0 < a < t$ and $G(y, a - \varepsilon) - \varepsilon > 0$. 

2
Hence, since the domain of $G$ is paracompact, (every metrizable is paracompact), there is a locally finite open cover $\{V_\alpha\}$ of this domain such that $G$ has a positive lower bound $\varepsilon_\alpha$ on each $V_\alpha$. Let $\{\phi_\alpha\}$ be a partition of unity subordinated to $\{V_\alpha\}$. Define $G'(y, t) = \sum_\alpha \varepsilon_\alpha \phi_\alpha(y, t)$, a continuous function with values in $\mathbb{R}^+$. We have that

$$G'(y, t) = \sum_{(y, t) \in V_\alpha} \varepsilon_\alpha \phi_\alpha(y, t) \leq \max_{(y, t) \in V_\alpha} \{\varepsilon_\alpha\} \leq G(y, t).$$

If $W_2$ is the $\delta_2$-neighborhood of $f$ in $T_{d_2}(X, Y)$, $W_2 \subset W_1$ since, if $g \in W_2$,

$$d_2[g(x), f(x)] < \sup \{r | B_2(f(x), r) \subset B_1(f(x), \delta_1(x))\}$$

$B_2(f(x), s) \subset B_1(f(x), \delta_1(x))$ with

$$d_2[g(x), f(x)] < s < \sup \{r | B_2(f(x), r) \subset B_1(f(x), \delta_1(x))\}$$

$d_1[f(x), g(x)] < \delta_1(x)$

and

$g \in W_1$

This completes the proof.

We can thus omit the subindex $d$ in $T_d(X, Y)$.

3 Properties and comparison with other topologies

From now on $Y$ will always denote a metrizable space.

The space $T(X, Y)$ is always Tychonoff ($T_1$ and completely regular) since if $f \in T(X, Y)$ and $\delta$ is any positive continuous function, the function from $T(X, Y)$ to the reals defined by

$$g \mapsto \min \left[ \sup_{x \in X} \left\{ \frac{d[f(x), g(x)]}{\delta(x)} \right\}, 1 \right],$$

where $d$ is a distance in $Y$ consistent with the topology of $Y$, is continuous, has value 1 outside of the $\delta$-neighborhood of $f$, and has value 0 if and only if $g = f$.

A topology in $F(X, Y)$ is called admissible if the function $(f, x) \mapsto f(x)$ defined on $F(X, Y) \times X$ turns out to be continuous when $F(X, Y)$ is endowed with that topology. The fine $C^0$-topology and the $d^*$ topology are admissible. The compact-open topology is coarser than the $d^*$-topology, which in turn is coarser than the fine $C^0$-topology [2].

In case $X$ is compact these topologies are identical (see for example [3, Chap. 7, Theorem 11]). We will see that if $X$ is $T_1$, not countably compact and $Y$ has a subspace homeomorphic to the reals, then the three topologies are all different.
Theorem 2. If \( X \) is countably compact (every open countable covering has a finite subcovering), the fine \( C^0 \)-topology of \( F(X, Y) \) coincides with the \( d^* \)-topology. If \( X \) is \( T_1 \) and normal, \( Y \) has a subspace homeomorphic to \( \mathbb{R} \) and \( T(X, Y) \) satisfies the first countability axiom, then \( X \) is countably compact.

Proof. To show the first statement it suffices to show that every positive continuous function \( \delta \) on a countably compact space \( X \) has a positive lower bound.

If \( X \) is countably compact, \( \delta(X) \) is as well. A countably compact in \( \mathbb{R} \) is compact, so \( \delta(X) \) has a positive lower bound.

Assume now that \( X \) is \( T_1 \), normal and not countably compact, and that \( Y \) contains \( \mathbb{R} \) as a subspace. We will show that \( T(X, Y) \) does not satisfy the first countability axiom.

\( T(X, \mathbb{R}) \) is a subspace of \( T(X, Y) \) so it will suffice to show that \( T(X, \mathbb{R}) \) does not satisfy the first countability axiom.

Since \( X \) is not countably compact there is a sequence of distinct points \((x_i)\) without a cluster point in \( X \). \( \{x_i\} \) is closed in \( X \) and discrete since \( X \) is a \( T_1 \) space.

Let \( f \in T(X, \mathbb{R}) \) be the function identically 0, and let \( \{\delta_i\}_{i=1,2,\ldots} \) be any countable family of positive continuous functions on \( X \). By the normality of \( X \) we can define \( \delta : X \to \mathbb{R}^+ \) continuous such that \( \delta(x_i) = \frac{1}{2}\delta_i(x_i) \), \( i = 1, 2, \ldots \).

The \( \delta \)-neighborhood of \( f \) does not contain any \( \delta_i \)-neighborhood of \( f \). Indeed, since \( X \) is normal, for every \( i \) there is \( g \in T(X, \mathbb{R}) \) such that \( g(x_i) = \delta_i(x) \), \( g(X) = [0, \delta(x_i)] \) and \( g(X - V) = \{0\} \) where \( V \) is a neighborhood of \( x_i \) in which \( \delta_i \) is bigger than \( \frac{1}{2}\delta_i(x_i) \). \( g \) is then in the \( \delta_i \)-neighborhood of \( f \) but not in the \( \delta \)-neighborhood of \( f \) (taking the usual distance in \( \mathbb{R} \)).

Therefore the \( \delta_i \)-neighborhoods of \( f \) do not form a fundamental system of neighborhoods of \( f \). This completes the proof of the theorem.

Condition \( T_1 \) can not be omitted from the statement of the theorem since, if \( X \) is the set of natural numbers with the topology in which the open sets are the sets of the form \( \{1, 2, \ldots, n\} \), the empty set and the whole \( X \), then \( X \) is normal, not countably compact and \( T(X, Y) \) satisfies the first countability axiom.

The proposition is also not valid if normal is replaced by completely regular. The following is a counterexample: \( X = \Omega' \times \omega' - \{(\Omega, \omega)\} \) where \( \Omega' \) and \( \omega' \) are the set of ordinals not bigger that the first uncountable ordinal \( \Omega \) and the set of ordinals not bigger than the first infinite ordinal \( \omega \) respectively, both spaces with the order topology. \( X \) is \( T_1 \), completely regular, not countably compact and \( T(X, Y) \) satisfies the first countability axiom.

Corollary 1. If \( X \) is \( T_1 \), normal, not countably compact and \( Y \) has a subspace homeomorphic to \( \mathbb{R} \), then the compact-open topology, the \( d^* \)-topology and the fine \( C^0 \)-topology on \( F(X, Y) \) are all different. See [2,§3]
References

[1] Whitehead, J.H.C.: Manifolds with transverse fields in euclidean space, Ann. of Math. vol. 73 (1961), pp 154-212.

[2] Jackson, J.R.: Comparison of topologies on function spaces. Proc. Amer. Math. Soc. vol. 3 (1952), pp 156-158.

[3] Kelley, J. L.: General topology (New York, 1955).