K-THEORY OF FLAG BOTT MANIFOLDS

BIDHAN PAUL AND VIKRAMAN UMA

Abstract. The aim of this paper is to describe the topological $K$-ring, in terms of generators and relations of a flag Bott manifold. We apply our results to give a presentation for the topological $K$-ring and hence the Grothendieck ring of algebraic vector bundles over flag Bott-Samelson varieties.

1. Introduction

A Bott tower $B_\bullet = \{B_j \mid 1 \leq j \leq m\}$, where $B_j = \mathbb{P}(1_C \oplus L_j)$ for a line bundle $L_j$ over $B_{j-1}$ for every $1 \leq j$, is a sequence of $\mathbb{P}^{1}_C$-bundles. Here $B_0 = *$ is a point. At each stage $B_j$ is a $j$-dimensional nonsingular toric variety with an action of a dense torus $(\mathbb{C}^*)^j$. We call $B_j$ a $j$-stage Bott manifold for $1 \leq j \leq m$.

The topology and geometry of these varieties have been widely studied recently (see [6]). One main motivation to study these varieties comes from its relation to the Bott-Samelson variety which arise as desingularizations of Schubert varieties in the full flag variety (see [4], [9], [13]). Indeed it has been shown by Grossberg and Karshon (see [12]) that a Bott-Samelson variety admits a degeneration of complex structures with special fibre a Bott manifold (also see [22]). Thus the topology and geometry of a Bott manifold gets related to those of the Bott-Samelson variety which in turn is equipped with rich connections with representation theory of semisimple Lie groups.

The construction of a Bott tower can be generalized in several directions. One such natural direction is to replace $\mathbb{P}^{1}_C$ with complex projective spaces of arbitrary dimensions. In other words at each stage $B_j$ is the projectivization of direct sum of finitely many complex line bundles. In this way we can construct the so called generalized Bott manifold and the generalized Bott tower which also has the structure of a nonsingular toric variety and has been studied widely (see [5], [20]).

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The construction of a Bott tower has recently been generalized in another natural direction of a flag Bott manifold by Kaji, Kuroki, Lee, Song and Suh (see [16], [19], [18]) by replacing $\mathbb{P}^1_C$ (which can be identified with the variety of full flags in $\mathbb{C}^2$) at every stage with the variety $\text{Flag} (\mathbb{C}^n)$ of full flags in $\mathbb{C}^n$ for any $n \geq 2$. More precisely, at the $j$th stage we let $B_j$ to be the flagification of direct sum of $n_j + 1$ complex line bundles over $B_{j-1}$. Thus $B_j$ is a flag bundle over $B_{j-1}$ with fibre the full flag manifold $\text{Flag}(\mathbb{C}^{n_j+1})$ of dimension $(n_j)(n_j+1)/2$.

Moreover, in [10] the flag Bott-Samelson variety has been introduced and studied by Fujita, Lee and Suh. The flag Bott-Samelson varieties are special cases of the more general notion of the generalized Bott-Samelson variety which was introduced by Jantzen [15] and studied by Perrin [23] to obtain small resolutions of Schubert varieties. Earlier in [24], Bott-Samelson varieties have been used by Sankaran and Vanchinathan to obtain resolutions of Schubert varieties in a partial flag variety $G/P$ by generalizing Zelevinsky’s results who constructed small resolutions for all Schubert varieties in Grassmann varieties [27]. The special property of the flag Bott-Samelson variety studied in [10] is that they are iterated bundles with fibres full flag varieties. In particular, similar to the Bott-Samelson variety the flag Bott-Samelson variety admits a one parameter family of complex structures which degenerates to a flag Bott manifold (see [10, Section 4]). Thus the geometry and topology of flag Bott-Samelson variety gets related with that of flag Bott manifolds since the underlying differentiable structure is preserved under the deformation and can therefore be studied using this degeneration.

There is a natural effective action of a complex torus $\mathbb{D}$ and the corresponding compact torus $\mathbb{T} \subseteq \mathbb{D}$ on the flag Bott manifold (see [16], [19, Section 3.1]).

The $\mathbb{T}$-equivariant cohomology of flag Bott manifolds has been studied by Kaji, Kuroki, Lee and Suh in [16].

In [7] Civan and Ray gave presentations for any complex oriented cohomology ring (in particular the $K$-ring) of a Bott tower. They also determine the $KO$-ring for several families of Bott towers.

In [26] P. Sankaran and the second named author describe the structure of the topological $K$-ring of a Bott tower, the topological $K$-ring of a Bott Samelson variety as well as the Grothendieck ring of a Bott Samelson variety.
Our main aim in this paper is to study the topological $K$-theory of flag Bott manifolds. By applying the classical results in [1] and [17] for $K$-ring of a flag bundle iteratively, in Theorem 4.4 we give a presentation for the $K$-ring of a flag Bott manifold in terms of generators and relations.

In Corollary 4.7, we apply our results to describe the topological $K$-ring of a flag Bott Samelson variety. This is essentially using the degeneration of the complex structures of a flag Bott Samelson variety to a flag Bott manifold.

Also since the generating line bundles of the topological $K$-ring are in fact algebraic line bundles on the flag Bott-Samelson variety we show using [25, Lemma 4.2] that the $K$-ring of algebraic vector bundles is isomorphic to the topological $K$-ring. Thus Corollary 4.7 also gives a presentation for the Grothendieck ring of flag Bott-Samelson variety.

Moreover, in Corollary 4.8 we show that our presentation in particular generalizes the presentation of the $K$-ring of Bott manifolds and the $K$-ring and Grothendieck ring of Bott-Samelson varieties in [26, Theorem 5.3, Theorem 5.4].

The flag Bott manifold has been defined for other Lie types by Kaji, Kuroki, Lee and Suh in [16] and their equivariant cohomology ring has been computed. Thus in another direction our results are an extension of their results to the $K$-ring of flag Bott manifolds of the $A$ type.

1.1. Organisation of the sections. In section 2 we recall the definition and construction of flag Bott manifolds from [19] and [18]. In particular, in subsection 2.1, we recall the construction of tautological line bundles on these manifolds which generate its Picard group.

In section 3 we recall the definition of a flag Bott-Samelson variety and its associated flag Bott manifold from [10].

In section 4 we study the topological $K$-theory of flag Bott manifolds and flag Bott Samelson varieties. In Theorem 4.4 which is our first main result we give a presentation for the topological $K$-ring of a flag Bott manifold in terms of generators and relations.

In subsection 4.1 we give the presentation for the topological $K$-ring and the Grothendieck ring of a flag Bott-Samelson variety as a corollary of Theorem 4.4 (see Corollary 4.7).
2. Flag Bott manifolds

Let $E$ be an $n$ dimensional holomorphic vector bundle over a compact complex manifold $X$. We define the flag bundle $Flag(E)$ as the bundle on $X$ whose fiber over each $x \in X$ is the full flag manifold $Flag(E_x)$. In particular, $Flag(E) = \bigsqcup_{x \in X} Flag(E_x)$. We recall the definition of flag Bott manifolds from [18, 19]. We broadly follow their notations and conventions.

**Definition 2.1.** An $m$-stage flag Bott tower $B_\bullet = \{B_j| 0 \leq j \leq m\}$ is a sequence of manifolds

$B_m \xrightarrow{p_m} B_{m-1} \xrightarrow{p_{m-1}} \ldots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0 = \{\ast\}$

which is defined recursively as follows:

(1) $B_0$ is a point.

(2) $B_j := Flag\left(\bigoplus_{l=1}^{n_j+1} \eta_l^{(j)}\right)$, where $\eta_l^{(j)}$ is a holomorphic line bundle over $B_{j-1}$ for each $1 \leq l \leq n_j + 1$ and $1 \leq j \leq m$.

We call $B_j$ as $j$-stage flag Bott manifold of the flag Bott tower $B_\bullet$.

Two flag Bott towers $B_\bullet = \{B_j| 0 \leq j \leq m\}$ and $B'_\bullet = \{B'_j| 0 \leq j \leq m_1\}$ are said to be isomorphic if $m = m_1$ and

$\begin{array}{ccc}
B_j \xrightarrow{\varphi_j} B'_j \\
p_j \downarrow \quad \downarrow p'_j \\
B_{j-1} \xrightarrow{\varphi_{j-1}} B'_{j-1}
\end{array}$

there exist a collection of diffeomorphisms $\varphi_j : B_j \rightarrow B'_j$ for each $1 \leq j \leq m$ such that the above diagram commutes for each $1 \leq j \leq m$.

First we give some trivial examples of flag Bott manifolds. Later we shall give a non-trivial example.

**Example 2.2.**

(1) The flag manifold $Flag(n+1) := Flag(\mathbb{C}^{n+1})$ is a 1-stage flag Bott tower.

(2) The usual product of flag manifolds $Flag(n_1+1) \times Flag(n_2+1) \times \cdots \times Flag(n_l+1)$ is an $l$-stage flag Bott manifold.
We now define the \( j \)-stage flag Bott manifold for each \( 1 \leq j \leq m \) as an orbit space under certain right action

\[
B_j^{quo} := \prod_{l=1}^{j} GL(n_l + 1) / \prod_{l=1}^{j} B_{GL(n_l+1)}
\]

where \( B_{GL(n_l+1)} \) denotes the Borel subgroup consisting of upper triangular matrices of \( GL(n_l + 1) \) for \( l = 1, 2, \ldots, j \).

\( B_1^{quo} \) is the flag manifold \( \text{Flag}(n_1+1) = GL(n_1+1)/B_{GL(n_1+1)} \). In order to define the flag Bott manifolds of higher stages we need a sequence of matrices with integer entries

\[
(2.1) \quad \Psi := (P_l^{(j)})_{1 \leq l < j \leq m} \in \prod_{1 \leq l < j \leq m} M_{n_j+1,n_l+1}(\mathbb{Z}).
\]

Let \( D(n_i+1) \subset GL(n_i+1) \) denotes the collection of diagonal matrices in \( GL(n_i + 1) \) for each \( i = 1, 2, \ldots, m \). Each \( P_l^{(j)} \) for \( 1 \leq l < j \leq m \) encodes a \( B_{GL(n_l+1)} \) action on \( GL(n_j+1) \) as follows.

Let

\[
P_l^{(j)} = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{n_j+1}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1,n_l+1} \\
a_{21} & a_{22} & \ldots & a_{2,n_l+1} \\
\vdots & \ddots & \ddots & \vdots \\
a_{n_j+1,1} & a_{n_j+1,2} & \ldots & a_{n_j+1,n_l+1}
\end{bmatrix} \in M_{n_j+1,n_l+1}(\mathbb{Z})
\]

Since the character group \( X^*(D(n_l+1)) \) is isomorphic to \( \mathbb{Z}^{n_l+1} \), we define a group homomorphism \( \pi_l^{(j)} : D(n_l+1) \rightarrow D(n_j+1) \) using the matrix \( P_l^{(j)} \) i.e.

\[
(2.2) \quad \pi_l^{(j)} : h \mapsto \text{diag}(h^{a_1}, h^{a_2}, \ldots, h^{a_{n_j+1}}) \in D(n_j+1)
\]

where, \( h^{a_i} = h_1^{a_{i1}} \cdot h_2^{a_{i2}} \cdots h_{n_l+1}^{a_{in_l+1}} \) for \( h := \text{diag}(h_1, h_2, \ldots, h_{n_l+1}) \) \( \in D(n_l+1) \) and \( a_i = (a_{i1}, a_{i2}, \ldots, a_{inin_l+1}) \) is the \( i \)-th row vector of \( P_l^{(j)} \) for each \( 1 \leq i \leq n_j+1 \).

We now define the homomorphism \( \Psi_l^{(j)} := \Psi(P_l^{(j)}) : B_{GL(n_l+1)} \rightarrow D(n_j+1) \) by

\[
(2.3) \quad \Psi_l^{(j)}(b) = \pi_l^{(j)} \circ \Gamma_l
\]

i.e for each \( b \in B_{GL(n_l+1)} \),

\[
\Psi_l^{(j)}(b) = \text{diag}(\Gamma_l(b)^{a_1}, \Gamma_l(b)^{a_2}, \ldots, \Gamma_l(b)^{a_{n_j+1}}) \in D(n_j+1)
\]
where $\Gamma_l : B_{GL(n_l+1)} \to D(n_l+1)$ is the canonical projection for each $1 \leq l \leq m$.

Now we define a right $\prod_{l=1}^{j} B_{GL(n_l+1)}$-action

$$\Phi^j_P : \prod_{l=1}^{j} GL(n_l+1) \times \prod_{l=1}^{j} B_{GL(n_l+1)} \longrightarrow \prod_{l=1}^{j} GL(n_l+1)$$

by $\Phi^j_P((g_1, g_2, \ldots, g_j), (b_1, b_2, \ldots, b_j)) :=$

$$\left( g_1 b_1, (\Psi^{(2)}_1(b_1))^{-1} g_2 b_2, (\Psi^{(3)}_1(b_1))^{-1} (\Psi^{(2)}_2(b_2))^{-1} g_3 b_3, \ldots, (\Psi^{(j)}_1(b_1))^{-1} (\Psi^{(j)}_2(b_2))^{-1} \cdots (\Psi^{(j)}_{j-1}(b_{j-1}))^{-1} g_j b_j \right)$$

**Lemma 2.3.** ([19, Lemma 2.6]) $\Phi^j_P$ in (2.4) is a free and proper right action for $1 \leq j \leq m$.

Hence by Lemma 2.3, the orbit space

$$B_j^{quo}(\mathfrak{P}) := \prod_{l=1}^{j} GL(n_l+1)/\Phi^j_P$$

is a complex manifold as $\prod_{l=1}^{j} B_{GL(n_l+1)}$ acts freely and properly on the complex manifold $\prod_{l=1}^{j} GL(n_l+1)$ (see [14, Proposition 2.1.13]) and $B_j^{quo}(\mathfrak{P}) := \{B_j^{quo}(\mathfrak{P}) | 0 \leq j \leq m\}$ is a flag Bott tower of height $m$ (see [19, Proposition 2.7]).

We write an element of $B_j^{quo}(\mathfrak{P})$ as $[g_1, g_2, \ldots, g_j]$ which is the orbit of $(g_1, g_2, \ldots, g_j) \in \prod_{l=1}^{j} GL(n_l+1)$ under $\prod_{l=1}^{j} B_{GL(n_l+1)}$ action and $p_j : B_{j}^{quo} \to B_{j-1}^{quo}$ is defined by

$$[g_1, g_2, \ldots, g_{j-1}, g_j] \mapsto [g_1, g_2, \ldots, g_{j-1}].$$

Since the character group $X^*(\prod_{l=1}^{j} D(n_l+1)) \simeq \bigoplus_{l=1}^{j} \mathbb{Z}^{n_l+1}$, we define a holomorphic line bundle on $B_j^{quo}$ for each integer vector $(v_1, v_2, \ldots, v_j) \in$
\[ \bigoplus_{l=1}^{j} \mathbb{Z}^{n_l+1} \] as an orbit space:

\[ \eta(v_1, v_2, \ldots, v_j) := \left( \prod_{l=1}^{j} GL(n_l + 1) \times \mathbb{C} \right) / \prod_{l=1}^{j} B_{GL(n_l+1)} \]

where we have the following right action

\[ (g_1, g_2, \ldots, g_j, w)(b_1, b_2, \ldots, b_j) := \left( \Phi_{j}^{\mathbb{C}} \left( (g_1, g_2, \ldots, g_j, (b_1, b_2, \ldots, b_j)) \right), b_1^{-v_1} \cdots b_j^{-v_j} w \right). \]

Here

\[ b_l^{-v_l} := \Gamma_l(b_l)^{-v_l} \]

for \( 1 \leq l \leq j \) and

\[ h^v := h_1^{v_1} \cdots h_{n_l+1}^{v_{n_l+1}} \]

for \( h \in D(n_l + 1) \) and \( v = (v_1, \ldots, v_{n_l+1}) \in \mathbb{Z}^{n_l+1} \).

**Proposition 2.4.** \((\text{see} \ [19, \text{proof of Prop.2.7}])\) For a collection of matrices

\[ \Psi := (P^{(j)}_{l})_{1 \leq l < j \leq m} \in \prod_{1 \leq l < j \leq m} M_{n_j+1, n_l+1}(\mathbb{Z}) \]

the \( j \)-stage flag Bott manifold \( B^{\text{quo}}_j \) of the flag Bott tower \( B^{\text{quo}}_j(\Psi) \) is the induced flag bundle \( \text{Flag}(\eta^{(j)}) \) over \( B^{\text{quo}}_{j-1} \), where

\[ \eta^{(j)} := \bigoplus_{k=1}^{n_{j+1}} \eta(v_{k,1}^{(j)}, v_{k,2}^{(j)}, \ldots, v_{k,j-1}^{(j)}) \]

and \( v_{k,l}^{(j)} \) is the \( k \)-th row vector of the matrix \( P^{(j)}_{l} \) for each \( 1 \leq l \leq j-1 \).

**Remark 2.5.** Note that \( \eta^{(j)} \) is a direct sum of \( n_j + 1 \) complex line bundles \( \eta^{(j)}_l \) for \( 1 \leq l \leq n_j + 1 \) on \( B^{\text{quo}}_{j-1} \). Thus the structure group of \( \eta^{(j)} \) is the complex torus \( D(n_j + 1) \). In particular this implies that for \( v \) in the fiber \( (\eta^{(j)})_{[g_1, \ldots, g_{j-1}]} \) of \( \eta^{(j)} \) over the point \( [g_1, \ldots, g_{j-1}] \in B^{\text{quo}}_{j-1} \) we have the identification

\[ \Psi_{j}^{(j)}(b_1)^{-1} \cdots \Psi_{j-1}^{(j)}(b_{j-1})^{-1} \cdot v \sim v \]

since \( \Psi_{l}^{(j)}(b_l) \in D(n_j + 1) \) for every \( 1 \leq l \leq j-1 \).

**Proof of Proposition 2.4.** Consider the map \( \varphi_j : B^{\text{quo}}_j \to \text{Flag}(\eta^{(j)}) \) defined by

\[ [g_1, g_2, \ldots, g_{j-1}, g_j] \mapsto ([g_1, g_2, \ldots, g_{j-1}], V) \]
where $V = (V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n) \subsetneq (\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]}$) is the full flag of $\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]}$ such that $V_k$ is spanned by the first $k$ columns of $g_j \in GL(n_j + 1)$. Note that
\[
[\Phi^q_j((g_1, \ldots, g_j), (b_1, \ldots, b_j))] \mapsto ([\Phi^q_{j-1}((g_1, \ldots, g_{j-1}), (b_1, \ldots, b_{j-1}))], V') = ([g_1, g_2, \ldots, g_{j-1}], V')
\]
for $(b_1, b_2, \ldots, b_j) \in \prod_{l=1}^j B_{GL(n_j+1)}$. Here
\[
V' = (V'_1 \subsetneq V'_2 \subsetneq \cdots \subsetneq V'_{n_j}) \subsetneq (\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]})
\]
is the full flag of $\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]}$ such that $V'_k$ is spanned by the first $k$ columns of $(\Psi^{(j)}_1(b_1))^{-1} \cdots (\Psi^{(j)}_{j-1}(b_{j-1}))^{-1} g_j \cdot b_j \in GL(n_j + 1)$. Since $b_j \in B_{GL(n_j+1)}$, the vector space spanned by the first $k$ column vectors of $g_j \cdot b_j$ is $V_k$ for $1 \leq k \leq n_j + 1$. Now, the column vectors of $g_j$ span the fibre $(\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]})$. Thus by Remark 2.5 it follows that we can identify any column vector $v$ of $g_j$ with $\Psi^{(j)}_j(b_1)^{-1} \cdots \Psi^{(j)}_{j-1}(b_{j-1})^{-1} \cdot v$ in $(\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]})$. Thus the flags $V'$ and $V$ in $(\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]}$ can be identified. It follows that $([g_1, g_2, \ldots, g_{j-1}], V')$ and $([g_1, g_2, \ldots, g_{j-1}], V)$ can be identified as elements of $Flag(\eta^{(j)})$. Hence $\varphi_j$ is well defined.

Let $f_j : Flag(\eta^{(j)}) \to B^{\text{gen}}_j$ be the map defined by
\[
([g_1, g_2, \ldots, g_{j-1}], V) \mapsto [g_1, g_2, \ldots, g_{j-1}, g_j]
\]
where
\[
V = (V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n) \subsetneq (\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]})
\]
is the full flag of $\eta^{(j)}_{[g_1, g_2, \ldots, g_{j-1}]}$ and $g_j \in GL(n_j + 1)$ is the matrix such that the first $k$ columns span the vector space $V_k$ for $1 \leq k \leq n_j + 1$. Then $f_j$ is the inverse of $\varphi_j$. It suffices therefore to show that $f_j$ is well defined. This follows since the element $([\Phi^q_j((g_1, \ldots, g_{j-1}), (b_1, \ldots, b_{j-1}))], V)$ maps to $[\Phi^q_j((g_1, \ldots, g_j), (b_1, \ldots, b_j))] = [g_1, \ldots, g_j]$. This again follows because the span of the first $k$ column vectors of $g_j$ can be identified with the span of the first $k$ column vectors of $\Psi^{(j)}_1(b_1)^{-1} \cdots \Psi^{(j)}_{j-1}(b_{j-1})^{-1} \cdot g_j \cdot b_j$.

Hence the map $\varphi_j$ is a diffeomorphism. Therefore the proposition follows.
Example 2.6. For \( n_1 = 2, n_2 = 1, \) and \( n_3 = 1, \) let
\[
P^{(2)}_1 = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}, \quad P^{(3)}_1 = \begin{bmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}, \quad P^{(3)}_2 = \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix}
\]
be the collection of matrices which determines the right action \( \Phi_j^{\Psi} \) of \( \prod_{i=1}^j BGL(n_i+1) \) on \( \prod_{i=1}^j GL(n_i + 1) \) as in (2.4) for \( j = 1, 2, 3. \) By Proposition 2.4, \( B^{q\omega}_o(\Psi) \) is isomorphic to the 3 stage flag Bott tower
\[
B_3 \xrightarrow{p_3} B_2 \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0 = \{ \text{a point} \}
\]
where the corresponding flag Bott manifolds are:
\[
B_1 = \text{Flag}(3) \\
B_2 = \text{Flag}(\eta((a_1, a_2, a_3)) \oplus \eta((b_1, b_2, b_3))) \\
B_3 = \text{Flag}(\eta((c_1, c_2, c_3), (f_1, f_2)) \oplus \eta((d_1, d_2, d_3), (0, 0))).
\]
The line bundle \( \eta((c_1, c_2, c_3), (f_1, f_2)) \) over \( B_2 \) is
\[
(GL(3) \times GL(2) \times \mathbb{C})/(BGL(3) \times BGL(2))
\]
where the right action of \((BGL(3) \times BGL(2))\) is given by
\[
(g_1, g_2, w) \cdot (b_1, b_2) := \left( \Phi_2^{\Psi}((g_1, g_2) \cdot (b_1, b_2)), b_1^{-(c_1, c_2, c_3)}b_2^{-(f_1, f_2)}w \right)
\]
(see (2.5)).

Theorem 2.7. ([19, Theorem 2.10]) For any flag Bott tower \( B_{\bullet} \) of height \( m, \) there is a sequence of matrices
\[
\Psi := (P^{(j)}_i)_{1 \leq i \leq j \leq m} \in \prod_{1 \leq i < j \leq m} M_{n_j+1, n_i+1}(\mathbb{Z})
\]
such that \( B^{q\omega}_{\bullet}(\Psi) \) is isomorphic to \( B_{\bullet} \) as flag Bott towers.

Definition 2.8. A flag Bott tower \( B_{\bullet} \) is said to be determined by the collection of matrices \( \Psi := (P^{(j)}_i)_{1 \leq i \leq j \leq m} \in \prod_{1 \leq i < j \leq m} M_{n_j+1, n_i+1}(\mathbb{Z}) \) if \( B_{\bullet} \) is isomorphic to \( B^{q\omega}_{\bullet}(\Psi) \) as flag Bott towers.

2.1. Tautological line bundles over a flag Bott manifold. We recall from Definition 2.1, that any point of \( B_j = \text{Flag}(\eta^{(j)}) \), where we write \( \eta^{(j)} := \bigoplus_{i=1}^{n_{j+1}} \eta^{(j)}_i \), can be interpreted as
\[
(p, V) = \left( p, V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n_j} \subset V_{n_j+1} = \eta^{(j)}_p \right)
\]
for \( p \in B_{j-1} \). For each \( 1 \leq k \leq n_j + 1 \), we define a sub bundle \( W_{j,k} \subseteq p^{(j)}_j(\eta^{(j)}) \) over \( B_j \) which has fiber \( V_k \) of the flag \( V \) over a point.
\((p, V) \in B_j\). Hence we have the quotient line bundle \(W_{j,k}/W_{j,k-1}\) over \(B_j\) for each \(1 \leq k \leq n_j + 1\).

**Lemma 2.9.** (see [19, Lemma 2.12]) Let \(B_j^{\text{quo}} := B_j^{\text{quo}}(\mathcal{P})\) be a flag Bott tower for a sequence of matrices \(\mathcal{P} := (P_l)_{1 \leq l < j < m} \in \prod_{1 \leq l < j < m} M_{n_j+1, n_i+1}(\mathbb{Z})\) defined as in (2.4). Then the line bundle \(W_{j,k}/W_{j,k-1} \rightarrow B_j^{\text{quo}}\) is isomorphic to the line bundle \(\eta(0, 0, \ldots, 0, e_k) \rightarrow B_j^{\text{quo}}\) as defined in (2.5), where \(e_k := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{n_j+1}\) has 1 in the \(k\)-th place with all other entries zero.

**Proof.** By Proposition 2.4, any element \(g = [g_1, g_2, \ldots, g_j] \in B_j^{\text{quo}}\) can be considered as a full flag
\[
(p_j(g), V) := (p_j(g), V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n_j} \subset V_{n_j+1} = \eta_j^{(p_j(g))})
\]
where \(\eta_j^{(p_j(g))} := \bigoplus_{k=1}^{n_j+1} \eta_j(v_j^{(p_j(g))}, v_j^{(p_j(g))}, \ldots, v_j^{(p_j(g))})\) and \(v_j^{(p_j(g))}\) is the \(k\)-th row vector of the matrix \(P_l^{(p_j(g))}\) for each \(1 \leq l \leq j - 1\). The fiber of \(W_{j,k}\) at \(g \in B_j^{\text{quo}}\) is the vector space \(V_k \subset \eta_j^{(p_j(g))}\) which is spanned by first \(k\) columns of \(g_j \in GL(n_j + 1)\) say, \(u_1, u_2, \ldots, u_k \in \eta_j^{(p_j(g))}\). Hence the fiber of the canonical line bundle \(W_{j,k}/W_{j,k-1}\) at \(g\) is \(V_k/V_{k-1}\) which is spanned by the class \(\overline{u}_k \in V_k/V_{k-1}\) of \(u_k \in \eta_j^{(p_j(g))}\).

Let \(b = (b_1, \ldots, b_j)\) be an element of \(\prod_{i=1}^j B_{GL(n_i+1)}\). Then it can be seen that the class \(\overline{u}_k \in V_k/V_{k-1}\) of the \(k\)-th column vector \(u_k'\) of the last coordinate
\[
(\Psi_1^{(j)}(b_1))^{-1}(\Psi_2^{(j)}(b_2))^{-1} \cdots (\Psi_{j-1}^{(j)}(b_{j-1}))^{-1} g_j b_j
\]
of \(\Phi_j^{(p_j(g), b)}\) is equal to \(b_k^{e_k} \cdot \overline{u}_k\). This is because the \(k\)th column vector \(u_k'\) is in the span of \(u_1, u_2, \ldots, u_k\) where the coefficient of \(u_k\) is
\[
(\Psi_1^{(j)}(b_1))^{-1}(\Psi_2^{(j)}(b_2))^{-1} \cdots (\Psi_{j-1}^{(j)}(b_{j-1}))^{-1} \cdot b_j^{e_k}.
\]
Now, by Remark 2.5 we have the equivalence
\[
(\Psi_1^{(j)}(b_1))^{-1}(\Psi_2^{(j)}(b_2))^{-1} \cdots (\Psi_{j-1}^{(j)}(b_{j-1}))^{-1} \cdot b_k^{e_k} \cdot u_k \sim b_k^{e_k} \cdot u_k
\]
in \(\eta^{(j)}_{[g_1, \ldots, g_{j-1}]}\). The result now follows by the definition (see (2.5)) of the holomorphic line bundle \(\eta(0, 0, \ldots, 0, e_k)\) on \(B_j^{\text{quo}}\). \(\square\)
Corollary 2.10. The line bundle $p^*_j \circ \cdots \circ p^*_{l+1}(W_{l,k}/W_{l,k-1}) \to B^{quo}_j$ is isomorphic to the line bundle
\[
\eta(0,0,\ldots,0,e_k,0,0,\ldots,0) \to B^{quo}_j
\]
as defined in (2.5), where $e_k := (0,0,\ldots,1,0,\ldots,0) \in \mathbb{Z}^{n_1+1}$ has 1 in the $k$-th place with all other entries zero.

Proof. Let $\eta(v_1,v_2,\ldots,v_l)$ be any holomorphic line bundle over $B^{quo}_l$ for an integer vector $(v_1,v_2,\ldots,v_l) \in \bigoplus_{i=1}^l \mathbb{Z}^{n_1+1}$ as defined in (2.5).

Then $p^*_l \cdots p^*_1(\eta(v_1,v_2,\ldots,v_l))$ is a holomorphic line bundle over $B^{quo}_{l+1}$, whose fiber over a point $[g_1,\ldots,g_l,g_{l+1}] \in B^{quo}_{l+1}$ is $\eta(v_1,v_2,\ldots,v_l)[g_1,\ldots,g_l]$ and, is defined by $(g_1,g_2,\ldots,g_{l+1},w) \cdot (b_1,b_2,\ldots,b_{l+1})$
\[
\times (\Phi^p_{l+1}(g_1,g_2,\ldots,g_{l+1}) \cdot (b_1,b_2,\ldots,b_{l+1})), b_1^{-v_1} \cdots b_l^{-v_l} w)
\]

for $(g_1,g_2,\ldots,g_{l+1},w) \in \prod_{i=1}^{l+1} GL(n_i+1) \times \mathbb{C}$ and $(b_1,b_2,\ldots,b_{l+1}) \in \prod_{i=1}^{l+1} B_{GL(n_i+1)}$.

Therefore, $p^*_l \cdots p^*_1(\eta(v_1,v_2,\ldots,v_l))$ is isomorphic to $\eta(v_1,v_2,\ldots,v_l,0)$ as line bundles over $B^{quo}_{l+1}$. Hence, the result follows from Theorem 2.9 by repeating the above procedure for higher $j > l + 1$.

\[\square\]

Lemma 2.11. Let $\eta(u_1,u_2,\ldots,u_j)$ and $\eta(v_1,v_2,\ldots,v_j)$ be two line bundles over $B^{quo}_j$ for $(u_1,u_2,\ldots,u_j), (v_1,v_2,\ldots,v_j) \in \bigoplus_{i=1}^j \mathbb{Z}^{n_1+1}$.

Then $\eta(u_1+v_1,u_2+v_2,\ldots,u_j+v_j) \simeq \eta(u_1,u_2,\ldots,u_j) \otimes \eta(v_1,v_2,\ldots,v_j)$ are isomorphic as line bundles over $B^{quo}_j$.

Proof. Let $\eta(u_1,u_2,\ldots,u_j)$ be a line bundle over $B^{quo}_j$ as defined in (2.5), respectively $\eta(v_1,v_2,\ldots,v_j)$. In which, right $\prod_{l=1}^j B_{GL(n_i+1)}$- action over $(\prod_{l=1}^j GL(n_l+1) \times \mathbb{C})$ is given as follows:

\[
(g_1,\ldots,g_j,w_1) \cdot (b_1,\ldots,b_j) := (\Phi^p_j((g_1,\ldots,g_j),(b_1,\ldots,b_j)), b_1^{-u_1} \cdots b_j^{-u_j} w_1),
\]

respectively,

\[
(g_1,\ldots,g_j,w_2) \cdot (b_1,\ldots,b_j) := (\Phi^p_j((g_1,\ldots,g_j),(b_1,\ldots,b_j)), b_1^{-v_1} \cdots b_j^{-v_j} w_2)
\]

for $(g_1,g_2,\ldots,g_j) \in \prod_{l=1}^j GL(n_l+1), (b_1,b_2,\ldots,b_j) \in \prod_{l=1}^j B_{GL(n_l+1)}$ and
Let \( \eta(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j) \) be a flag Bott tower. Then the Picard group \( \text{Pic}(B_j) \) is generated by the set of line bundles

\[
\{ L_{j,k} \mid 1 \leq k \leq n_j + 1 \} \cup \bigcup_{l=1}^{j-1} \{ p^*_l \circ \cdots \circ p^*_{l+1}(L_{l,k}) \mid 1 \leq k \leq n_l + 1 \}
\]

for each \( 1 \leq j \leq m \).

**Proof.** Let \( X \) be a flag bundle over \( Y \). Recall from (Example 19.1.11, [11]), that the cycle map \( c_X : A_k(X) \to H_{2k}(X) \) is an isomorphism if and only if \( c_Y \) is an isomorphism. Moreover, the cycle map is an isomorphism for an arbitrary flag manifold. Since, \( B_j \) is an iterated bundle of flags over a point, the cycle map \( c_{B_j} : A_k(B_j) \to H_{2k}(B_j) \) is an isomorphism. On the other hand, since flag Bott manifolds are smooth projective varieties, we have the following isomorphism:

\[
\text{Pic}(B_j) \xrightarrow{\sim} A_{\dim B_j - 1}(B_j) \xrightarrow{\sim} H_{2(\dim B_j) - 2}(B_j) \xrightarrow{\sim} H^2(B_j)
\]
where, the first isomorphism comes from (Example 2.1.1, [11]) and the last isomorphism is due to the well known Poincare duality. Hence, $c_1 : Pic(B_j) \to H^2(B_j)$ is an isomorphism by (2.6).

Now, using the result (Remark 21.18, [3]) on the cohomology ring of the induced flag bundle and an induction on the stages of $B_\bullet$, we see that $H^2(B_j)$ is generated by the first Chern classes of line bundles

$$\{ L_{j,k} | 1 \leq k \leq n_j + 1 \} \cup \bigcup_{l=1}^{j-1} \{ p_j^* \circ \cdots \circ p_{l+1}^* (L_{l,k}) | 1 \leq k \leq n_l + 1 \}$$

for each $1 \leq j \leq m$. Therefore, any cohomology class of degree 2 can be written as the first Chern class of a tensor product of the above line bundles. Hence the result follows. □

**Remark 2.15.** *(Description of the flag Bott manifold $B_j$ using compact Lie groups)* We consider the orbit space

$$\prod_{l=1}^{j} U(n_l + 1)/ \prod_{l=1}^{j} T(n_l + 1)$$

for compact unitary groups $U(n_l + 1)$ along with compact maximal torus $T(n_l + 1) \simeq (S^1)^{n_l+1}$ for each $1 \leq l \leq j$. The right action is similar to (2.4):

$$((g_1, g_2, \ldots, g_j), (t_1, t_2, \ldots, t_j)) := (g_1 t_1, (\Psi_1^{(2)}(t_1))^{-1} g_2 t_2, (\Psi_1^{(3)}(t_1))^{-1} g_3 t_3, \ldots, (\Psi_j^{(j)}(t_1))^{-1} g_j t_j)$$

where

$$(g_1, g_2, \ldots, g_j) \in \prod_{l=1}^{j} U(n_l + 1)$$

and

$$(t_1, t_2, \ldots, t_j) \in \prod_{l=1}^{j} T(n_l + 1).$$

Then the above manifold is a compact manifold which is diffeomorphic to $B_n$ since $U(n + 1)/T(n + 1)$ is diffeomorphic to the flag manifold $\text{Flag}(n + 1) = GL(n + 1)/B_{GL(n+1)}$ for each $n$. □

### 3. Flag Bott-Samelson Varieties

In this section, we recall the definition of flag Bott-Samelson varieties introduced in ([10, section 2.1]). In [15], flag Bott-Samelson varieties are considered in the more general setting of iterated fibrations of Schubert varieties without explicitly naming them. We also
recall the one-parameter family of complex structures on the flag Bott-
Samelson variety and its relation with flag Bott tower from [10, section
4] in Theorem 3.7.

3.1. **Definition of flag Bott-Samelson varieties.** Let $G$ be a simply
connected, semisimple algebraic group of rank $n$ over $\mathbb{C}$. Let $B \subset G$ be
a Borel subgroup and $T \subset B$ be a maximal torus. Let $\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{g}_\alpha$ be
the Cartan decomposition of the Lie algebra $\mathfrak{g}$ into root spaces where
$\mathfrak{h} := \text{Lie}(T)$. Let $\Phi \subset \mathfrak{h}^*$ denote the roots of $G$ and $\Phi^+ \subset \Phi$ be a set
of positive roots (corresponding to $B$), and $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi^+$
denote the set of simple roots. Let $\{\alpha_i^\vee\} \subset \mathfrak{h}^*$ denote the coroots and
$\{\omega_1, \ldots, \omega_n\} \subset \mathfrak{h}$ denote the fundamental weights which are characterized by
the relation $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker symbol. Similarly
let $\{\omega_i^\vee\} \subset \mathfrak{h}$ denote the fundamental coweights dual to the
simple roots.

Let $W$ denote the Weyl group of $G$. Let $s_i$ denote the simple
reflection in $W$ corresponding to the simple root $\alpha_i$. For a subset
$I \subset [n] := \{1, 2, \ldots, n\}$, we define the subgroup $W_I := \langle s_i | i \in I \rangle$ of
$W$. In particular, $W_\emptyset = \{1\}$ and $W_{[n]} = W$. We define, the para-
bolic subgroup $P_I := \bigcup_{w \in W_I} BwB = \overline{Bw_1B} \subset G$ where $w_1$ denotes the
longest element of $W_I$.

**Definition 3.1 (Flag Bott-Samelson variety).** Let $I = (I_1, \ldots, I_r)$ be
a sequence of subsets of $[n]$ and let $P_I = P_{I_1} \times \cdots \times P_{I_r}$. We define a
right action $\Theta : P_I \times B^r \rightarrow P_I$ given by

$$\Theta((p_1, \ldots, p_r), (b_1, \ldots, b_r)) = (p_1b_1, b_1^{-1}p_2b_2, \ldots, b_1^{-1}p_rb_r)$$

for $(p_1, \ldots, p_r) \in P_I$ and $(b_1, \ldots, b_r) \in B^r := \overline{B \times \cdots \times B}$. The flag
Bott-Samelson variety $F_I$ is defined to be the orbit space

$$F_I := P_I/\Theta.$$

**Remark 3.2.** If we take $I = ([n])$. Then we have $P_I = G$. Therefore,
the flag Bott-Samelson variety $F_I$ is the flag variety $G/B$.

Moreover, the flag Bott-Samelson variety is a Bott-Samelson variety
if each $|I_k| = 1$. Recall that a Bott-Samelson variety has a family of
complex structure which induces a toric degeneration [12, 22].

**Remark 3.3.** For the subsequence $I' = (I_1, \ldots, I_{r-1})$ of $I$, there is a
fibration structure on the flag Bott-Samelson variety $F_{I'}$ :

$$P_{I_r}/B \hookrightarrow F_I \rightarrow F_{I'}$$
where the projection map \( \pi : F_\gamma \to F'_\gamma \) is defined as
\[
[p_1, \ldots, p_r] \mapsto [p_1, \ldots, p_{r-1}].
\]
One can also represent \( F_\gamma \) as \( P_{I_1} \times_B F_{I_2} \), where \( I'' = (I_2, \ldots, I_r) \).

3.2. Line bundles over the flag Bott-Samelson variety. Let \( J \) be a sequence of subsets of \([n]\). An integral weight
\[
\chi \in \mathbb{Z} \omega_1 \bigoplus \cdots \bigoplus \mathbb{Z} \omega_n = X^\ast(T)
\]
induces a homomorphism \( e^\chi : B \to \mathbb{C}^\ast \) by composing with the canonical map \( \Gamma : B \to T \).

For \( \chi_1, \ldots, \chi_r \in X^\ast(T) \) we can define the one dimensional complex representation \( \mathbb{C}_{\chi_1, \ldots, \chi_r} \) of \( B^r \) where \( B^r \) acts on \( \mathbb{C} \) as follows:
\[
(b_1, \ldots, b_r) \cdot v := e^{\chi_1(b_1)} \cdots e^{\chi_r(b_r)} \cdot v.
\]

Let \( L_{J, \chi_1, \ldots, \chi_r} = P_{I_1} \times_B \mathbb{C}_{\chi_1, \ldots, \chi_r} \) denote the associated line bundle on \( F_J = P_{I_1}/B^r \).

We let
\[
L_{J, \chi} := L_{J, 0, \ldots, 0, \chi}.
\]

3.3. Complex structures on flag Bott-Samelson variety \( F_J \). Let \( \lambda = \sum_{i=1}^n a_i \omega_i^\vee \) for \( a_i > 0 \) for all \( 1 \leq i \leq n \). Then \( \lambda \) defines a one parameter subgroup \( \mathbb{C}^\ast \to T \) so that \( \alpha_i \circ \lambda(t) = t^{a_i} \) for \( t \in \mathbb{C}^\ast \) and \( 1 \leq i \leq n \). Then \( \langle \lambda, \alpha_i \rangle = a_i > 0 \) for every \( 1 \leq i \leq n \). We can choose \( a_i = a > 0 \) for all \( i \). We define \( \Gamma_t : B \to B \) by \( \Gamma_t(b) = \lambda(t) \cdot b \cdot \lambda(t)^{-1} \) for \( t \in \mathbb{C}^\ast \). We know by [12, Proposition 3.5] that \( \Gamma = \lim_{t \to 0} \Gamma_t \) where \( \Gamma : B \to T \) is the canonical surjection. We let \( \Gamma_0 := \Gamma \).

Using \( \Gamma_t \), we define a right action \( \Theta_t : P_J \times B^r \to P_J \) as
\[
\Theta_t((p_1, \ldots, p_r), (b_1, \ldots, b_r)) = (p_1 b_1, \Gamma_t(b_1)^{-1} p_2 b_2, \ldots, \Gamma_t(b_{r-1})^{-1} p_r b_r)
\]
for \( (p_1, \ldots, p_r) \in P_J \) and \( (b_1, \ldots, b_r) \in B^r \). Note that, \( \Theta_1 \) is same as the right action \( \Theta \) in (3.7). We consider the family of orbit spaces
\[
F_J^t := P_J/\Theta_t
\]
under the right action \( \Theta_t \) for \( t \in \mathbb{C} \).

Let \( L'_{J, \chi_1, \ldots, \chi_r} = P_J \times_{B^r} \mathbb{C}_{\chi_1, \ldots, \chi_r} \) denote the line bundle associated to the 1-dimensional representation \( \mathbb{C}_{\chi_1, \ldots, \chi_r} \) of \( B^r \) on \( F_J^t = P_J/B^r \) where
the action of $B^r$ on $P_J$ is via $\Theta_t$. We let

$$L^t_{j,\lambda} := L^t_{j,0,\ldots,0,\lambda}.$$  

**Proposition 3.4.** (see [10, Proposition 4.6]) Let $I = (I_1, \ldots, I_r)$ be a sequence of subsets of $[n]$. Then $\{F_I^t\}_{t \in \mathbb{C}}$ are all diffeomorphic. Moreover, $F_1^t = F_J$.

Henceforth we shall consider $I = (I_1, \ldots, I_r)$ be a sequence of subsets of $[n]$ such that the Levi subgroup $L_{I_k}$ of the parabolic subgroup $P_{I_k}$ has Lie type $A_{n_k}$ for all $1 \leq k \leq r$. We shall take an enumeration $I_k = \{u_{k,1}, \ldots, u_{k,n_k}\}$ which satisfies

$$(3.10) \quad \langle \alpha_{u_{k,s}}, \alpha_{u_{k,t}}^\vee \rangle = \begin{cases} 2 & \text{if } s = t, \\ -1 & \text{if } s - t = \pm 1, \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 3.5.** (see [10, Prop. 4.8]) Let $F_J$ be a flag Bott-Samelson variety. Let $I'$ denote the subsequence $(I_1, \ldots, I_{r-1})$ of $I$. Then $F_J^0$ is diffeomorphic to the following flag bundle over $F_{I'}^0$

$$F_J^0 \simeq \text{Flag}(L_{0,\chi_1}^0 \oplus \cdots \oplus L_{0,\chi_{nr}}^0 \oplus 1_{\mathbb{C}})$$

where $\chi_j = \alpha_{u_{r,j}} + \cdots + \alpha_{u_{r,n_r}} \in \mathfrak{h}^*$ for $1 \leq j \leq n_r$, $L_{j,\lambda}^0 := L_{j,0,\ldots,0,\lambda}$ and $1_{\mathbb{C}}$ denotes the trivial complex line bundle.

Moreover, $L_{j,\chi_1,\ldots,\chi_r}$ is isomorphic to the line bundle on the flag Bott manifold $F_J^0$ corresponding to the vector $(a_1, \ldots, a_r) \in \prod_{k=1}^r \mathbb{Z}^{n_k+1}$ where $a_k = (a_k(1), \ldots, a_k(n_k + 1)) \in \mathbb{Z}^{n_k+1}$ is defined by

$$a_k(l) = \langle \chi_k + \cdots + \chi_r, \alpha_{u_{k,l}}^\vee + \cdots + \alpha_{u_{k,n_k}}^\vee \rangle$$

for $1 \leq l \leq n_k$ and $a_k(n_k + 1) = 0$. Indeed, $(a_1, \ldots, a_r) \in \prod_{k=1}^r \mathbb{Z}^{n_k+1}$ can be identified with the first Chern class of the line bundle $L_{j,\chi_1,\ldots,\chi_r}^0$.

Recall by Lemma 2.9 that on the flag Bott manifold $F_J^0$ the tautological line bundles $L_{r,k}^0$ for $1 \leq k \leq n_r + 1$ correspond to the vector bundle $\eta(0, \ldots, 0, e_k)$, where $e_k \in \mathbb{Z}^{n_r+1}$ is the vector with 1 at the $k$th position and 0 elsewhere. Thus $L_{r,k}^0$ are associated to the characters $\epsilon_1, \ldots, \epsilon_{n_r+1}$ of the maximal torus $T_{SL^{n_r+1}}$ of $SL^{n_r+1}$ corresponding to the $n_r + 1$ coordinate projections. Now, $T_{SL^{n_r+1}}$ is of rank $n_r$ and $\text{Lie}(T_{SL^{n_r+1}})$
is free \( \mathbb{Z} \)-module on the fundamental weights \( \omega_{u_{r,1}}, \ldots, \omega_{u_{r,r}} \). Further, \( \epsilon_1 = \omega_{u_{r,1}}, \epsilon_2 = \omega_{u_{r,2}} - \omega_{u_{r,1}}, \ldots, \epsilon_{n_r} = \omega_{u_{r,n_r}} - \omega_{u_{r,1}} \) and 
\( \epsilon_{n_r+1} = -\omega_{u_{r,r}} \).

Thus we have the isomorphisms 
\[
\mathcal{L}_{r,1}^0 \cong \mathcal{L}_{j,\omega_{u_{r,1}}}^0, \quad \mathcal{L}_{r,2}^0 \cong \mathcal{L}_{j,\omega_{u_{r,2}} - \omega_{u_{r,1}}}^0, \quad \ldots
\]
\[
\mathcal{L}_{r,n_r}^0 \cong \mathcal{L}_{j,\omega_{u_{r,n_r}} - \omega_{u_{r,1}}}^0 \quad \text{and} \quad \mathcal{L}_{r,n_r+1}^0 \cong \mathcal{L}_{j, -\omega_{u_{r,n_r}}}^0 \quad \text{on} \ F_j^0.
\]

**Remark 3.6.** Under the diffeomorphism of \( F_j^0 \) and \( F_j^1 \), the line bundle \( \mathcal{L}_{j,\chi_1,\ldots,\chi_r} \) corresponds to \( \mathcal{L}_{j,\chi_1,\ldots,\chi_r}^0 \). In particular, \( \mathcal{L}_{r,1}^0 \) corresponds to \( \mathcal{L}_{j,\omega_{u_{r,1}}}^0 \), \( \mathcal{L}_{r,k}^0 \) corresponds to \( \mathcal{L}_{j,\omega_{u_{r,k}} - \omega_{u_{r,k-1}}}^0 \) for \( 2 \leq k \leq n_r \) and \( \mathcal{L}_{r,n_r+1}^0 \) corresponds to \( \mathcal{L}_{j, -\omega_{u_{r,n_r}}}^0 \). Thus under the diffeomorphism given by the deformation of complex structures, the tautological line bundles \( \mathcal{L}_{r,k}^0 \) for every \( 1 \leq k \leq n_r \) correspond to the topological restrictions of certain algebraic line bundles on the flag Bott-Samelson variety \( F_j \). Similarly for every subsequence \( j' = (I_1, \ldots, I_j) \) for \( 1 \leq j \leq r \), the tautological line bundles \( \mathcal{L}_{j,k} \) for \( 1 \leq k \leq n_j + 1 \) on the \( j \)-stage flag Bott manifold \( F_j^0 \) correspond respectively to the restrictions of the algebraic line bundles \( \mathcal{L}_{j,\omega_{u_{j,1}}} \), \( \mathcal{L}_{j,\omega_{u_{j,k}} - \omega_{u_{j,k-1}}} \) for \( 2 \leq k \leq n_j \) and \( \mathcal{L}_{j, -\omega_{u_{j,n_j}}} \) on the flag Bott-Samelson variety \( F_j \).

**Theorem 3.7.** ([10, Theorem 4.10]) The manifold \( F_j \) is an \( r \)-stage flag Bott manifold which is determined by a sequence of matrices
\[
\mathfrak{M} := (Q_t^{(j)})_{1 \leq l < j \leq r} \quad \prod_{1 \leq l < j \leq r} M_{n_l+1,n_l+1}(\mathbb{Z})
\]
in the sense of Definition 2.8, where \( Q_t^{(j)}(p,q) \) is
\[
\langle \alpha_{u_{j,p}} + \ldots + \alpha_{u_{j,n_j},} \alpha_{u_{l,q}} + \ldots + \alpha_{u_{l,n_l}} \rangle
\]
if \( 1 \leq p \leq n_j \) and \( 1 \leq q \leq n_l \), and 0 otherwise.

**Example 3.8.** Let \( G = \text{SL}(4) \). Consider the sequence \( J = (\{1,2\}, \{1,2\}) \).
Hence \( u_{1,1} = 1, u_{1,2} = 2, u_{2,1} = 1, u_{2,2} = 2 \). Then the manifold \( F_j \) is a 2-stage flag Bott manifold which is determined by a matrix
\[
Q_1^{(2)} := \begin{bmatrix}
\langle \alpha_1 + \alpha_2, \alpha_1^\vee + \alpha_2^\vee \rangle & \langle \alpha_1 + \alpha_2, \alpha_2^\vee \rangle & 0 \\
\langle \alpha_2, \alpha_1^\vee + \alpha_2^\vee \rangle & \langle \alpha_2, \alpha_2^\vee \rangle & 0 \\
0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

**Corollary 3.9.** Suppose that the flag Bott-Samelson variety is a Bott-Samelson variety i.e. \( |I_k| = 1 \quad \forall \ k = 1,2,\ldots,r \), so that \( n_1 = n_2 = \ldots = n_r = 1 \). Hence from Theorem 3.7,
\[
Q_l^{(j)} = \begin{bmatrix}
\langle \alpha_{u_{l,1}}, \alpha_{u_{l,1}}^\vee \rangle & 0 \\
0 & 0
\end{bmatrix}
\text{for each} \ 1 \leq l < j \leq r
\]
(see [12, Section 3.7]).
Let $E$ be a complex vector bundle of rank $n$ over a compact manifold $X$. Then the exterior power $\Lambda^i(E)$ is also a complex vector bundle over $X$ with fiber $\Lambda^i(E)_x = \Lambda^i(E_x)$ for each $x \in X$. Moreover, if $E$ is a direct sum of $n$ complex line bundles $E \simeq L_1 \oplus L_2 \oplus \cdots \oplus L_n$ over $X$, then we have the isomorphisms $\Lambda^1(E) \simeq L_1 \oplus L_2 \oplus \cdots \oplus L_n$, $\Lambda^2(E) \simeq \bigoplus_{i<j} L_i \otimes L_j$ etc. In particular, for each $1 \leq k \leq n$

$$\Lambda^k(E) \simeq \bigoplus_{i_1 < i_2 < \cdots < i_k} L_{i_1} \otimes L_{i_2} \otimes \cdots \otimes L_{i_k}$$

and the class $[\Lambda^k(E)]$ in $K(X)$ can be interpreted as $e_k([L_1], [L_2], \ldots, [L_n])$ where $e_k$ denotes the $k$-th elementary symmetric polynomial.

We recall from the definition of a flag bundle $\pi : \text{Flag}(E) \to X$ associated to an $n$ dimensional complex vector bundle $E$ over a compact manifold $X$, that we have a sequence of canonical sub-bundles $0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = \pi^*(E)$ such that the successive quotients $E_i/E_{i-1}$ are well defined line bundles over $\text{Flag}(E)$ with $\bigoplus_{i=1}^n (E_i/E_{i-1}) \simeq \pi^*E$. We shall denote the class of $(E_i/E_{i-1})$ in $K^*(\text{Flag}(E))$ by $h_i$.

We recall the following classical theorem describing $K^*(\text{Flag}(E))$ as $K^*(X)$-algebra.

**Theorem 4.1.** [17, Chapter IV : Theorem 3.6] Let

$$\varphi : K^*(X)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to K^*(\text{Flag}(E))$$

be the $K^*(X)$- algebra homomorphism sending $x_i$ to $h_i$. Then $\varphi$ is surjective and its kernel is the ideal $I$ in the Laurent polynomial ring $K^*(X)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by the elements

$$e_i(x_1, \ldots, x_n) - [\Lambda^i(E)],$$

where $e_i$ is the $i$-th elementary symmetric polynomial in the $x_i$'s. Hence, $\varphi$ induces an isomorphism

$$K^*(X)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/I \simeq K^*(\text{Flag}(E))$$

**Remark 4.2.** In [17] the ring $K^*(\text{Flag}(E))$ is expressed as the quotient of the polynomial ring $K^*(X)[x_1, \ldots, x_n]$ by the ideal $I'$ generated by the elements (4.11). Now, by [2, Section 2.6], we note that $(1 - [L_i])$ are nilpotent elements in $K^*(\text{Flag}(E))$ (since $X$ is finite dimensional being a compact manifold, the total space $\text{Flag}(E)$ is also finite dimensional). It follows that in $K^*(\text{Flag}(E))$, $[L_i]^{-1}$ can itself be expressed as a polynomial in $(1 - [L_i])$. For this reason, we
can as well express $K^*(\text{Flag}(E))$ as a quotient of the Laurent polynomial ring $K^*(X)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by the ideal $I$ generated by the elements 
(4.11), where $K^*(X)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ can be identified with the localization $K^*(X)[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ of $K^*(X)[x_1, \ldots, x_n]$ and $I$ is the extension of the ideal $I$ in $K^*(X)[x_1, \ldots, x_n]$. This is with a view to apply the Theorem 4.1 iteratively for the flag Bott manifolds. For, when the base space $X$ itself is a flag manifold, $[\Lambda^r(E)]$ may involve classes of negative powers of the tautological line bundles. Thus considering $K^*(\text{Flag}(E))$ as a quotient of the Laurent polynomial ring at each stage will facilitate our arguments as well as simplify the presentation for the $K$-ring of flag Bott manifolds.

4.1. $K$-ring of flag Bott manifolds. Recall from Proposition 2.4 that for every $j > 1$, $B_j^{\text{quo}} = \text{Flag}(\eta^{(j)})$ is a flag bundle over $B_{j-1}^{\text{quo}}$ where

\begin{equation}
\eta^{(j)} := \bigoplus_{k=1}^{n_j+1} \eta(v_{k,1}, v_{k,2}, \ldots, v_{k,j-1})
\end{equation}

is a direct sum of $n_j + 1$ line bundles over $B_{j-1}^{\text{quo}}$ and $v_{k,j}$ is the $k$-th row vector of the matrix $P^{(j)}_l$ for each $1 \leq l \leq j - 1$ for the collection of matrices

$\Psi := (P^{(j)}_l)_{1 \leq l < j \leq m} \in \prod_{1 \leq l < j \leq m} M_{n_j+1,m+1}(\mathbb{Z})$.

**Theorem 4.3.** We have an $K^*(B_{j-1}^{\text{quo}})$- algebra isomorphism :

$\frac{K^*(B_j^{\text{quo}}) [y_{j,1}^{\pm 1}, y_{j,2}^{\pm 1}, \ldots, y_{j,n_j+1}^{\pm 1}]}{I_j} \simeq K^*(\text{Flag}(\eta^{(j)})) = K^*(B_j^{\text{quo}})$

given by $y_{j,k} + I_j \mapsto [L_{j,k}]$ where $I_j$ is the ideal in the Laurent polynomial ring $K^*(B_{j-1}^{\text{quo}}) [y_{j,1}^{\pm 1}, y_{j,2}^{\pm 1}, \ldots, y_{j,n_j+1}^{\pm 1}]$ generated by the elements

$e_r(y_{j,1}, y_{j,2}, \ldots, y_{j,n_j+1}) - [\Lambda^r(\eta^{(j)})]$; for each $1 \leq r \leq n_j + 1$.

**Proof.** The theorem follows from Theorem 4.1. \qed

Let $y_j$ denote the collection of variables \{\$\$y_{j,1}^{\pm 1}, y_{j,2}^{\pm 1}, \ldots, y_{j,n_j+1}^{\pm 1}\$\} for every $1 \leq j \leq m$.

Let $R := \mathbb{Z}[y_j | 1 \leq j \leq m]$ denote the Laurent polynomial ring in the variables $y_{j,1}^{\pm 1}, y_{j,2}^{\pm 1}, \ldots, y_{j,n_j+1}^{\pm 1}$ for $1 \leq j \leq m$.

Let $e_r(y_j) \in R$ denote the $r$th elementary symmetric function in $y_{j,1}, y_{j,2}, \ldots, y_{j,n_j+1}$. 


Let $P_j^{(j)} = (P_j^{(j)}(r, s)) \in M_{n_j+1,n_l+1}(\mathbb{Z})$ for $1 \leq r \leq n_j + 1$, $1 \leq s \leq n_l + 1$.

Let $I_1$ denote the ideal in $R$ generated by the polynomials

$$e_r(y_1) - \binom{n_1 + 1}{r}, \text{ for every } 1 \leq r \leq n_1 + 1$$

and let $I_j$ denote the ideal in $R$ generated by the polynomials

$$e_r(y_j) - e_r\left(\prod_{s=1}^{j-1} \prod_{i=1}^{n_{j-1} + 1} y_{k,s}^{P_j^{(j)}(k,i)}\right) | 1 \leq k \leq n_j + 1)$$

for every $2 \leq j \leq m$ and $1 \leq r \leq n_j + 1$.

Let $I := I_1 + \cdots + I_m$ which is the smallest ideal in $R$ containing $I_1, \ldots, I_m$.

**Theorem 4.4.** Suppose that $B_* = \{B_j | 0 \leq j \leq m\}$ is an $m$-stage flag Bott manifold of Lie type $A$, determined by a set of integer matrices

$\Psi := (P_j^{(j)})_{1 \leq i \leq j \leq m} \in \prod_{1 \leq i < j \leq m} M_{n_j+1,n_l+1}(\mathbb{Z})$. The map from $R$ to $K^*(B_m)$ which sends $y_{m,k}$ to $[\mathcal{L}_{m,k}]$ for $1 \leq k \leq n_m + 1$ and $y_{l,k}$ to

$$[p_{m}^* \circ \cdots \circ p_{l+1}^*(\mathcal{L}_{l,k})]$$

for $1 \leq k \leq n_l + 1$ and $1 \leq l \leq m - 1$, induces an isomorphism

$$R/I \simeq K^*(B_m).$$

**Proof.** We prove this by induction on $m$. The result is true for $m = 1$ since $B_1$ is a flag manifold associated to direct sum of line bundles $\mathcal{L}_{1,k}$ for $1 \leq k \leq n_1 + 1$. In this case we know have the classical presentation given by

$$\mathbb{Z}[y_1]/I_1$$

(see [1, Proposition 2.7.13], [17, Chapter IV : Theorem 3.6]). Now, we assume by induction that the result is true for $B_{m-1}$. Thus

$$K^*(B_{m-1}) \simeq \mathbb{Z}[y_1, \ldots, y_{m-1}]/I_1 + \cdots + I_{m-1}$$

under the isomorphism which sends $y_{m-1,k}$ to $[\mathcal{L}_{m-1,k}]$ for $1 \leq k \leq n_{m-1} + 1$ and $y_{l,k}$ to $p_{m-1}^* \circ \cdots \circ p_{l+1}^*(\mathcal{L}_{l,k})$ for $1 \leq k \leq n_l + 1$ and $1 \leq l \leq m - 2$.

Now by Theorem 4.3 and the induction hypothesis we get that $K^*(B_m)$ is isomorphic to

$$(\mathbb{Z}[y_1, \ldots, y_{m-1}]/I_1 + \cdots + I_{m-1})[y_m]$$

modulo the ideal generated by the elements

$$e_r(y_{m,1}, y_{m,2}, \ldots, y_{m,n_m + 1}) - \left[\Lambda^r(\eta^{(m)})\right]; \text{ for each } 1 \leq r \leq n_m + 1.$$
The theorem will follow if we show that
\[ e_r \left( \prod_{s=1}^{m-1} \left( \prod_{i=1}^{n_s+1} y_{s,i}^{p_{s}^{(m)}(k,i)} \right) \right) | 1 \leq k \leq n_m + 1. \]
maps to $[\Lambda^r(\eta^{(m)})]$ under the isomorphism (4.13). Thus it suffices to show that $\prod_{s=1}^{m-1} \left( \prod_{i=1}^{n_s+1} y_{s,i}^{p_{s}^{(m)}(k,i)} \right)$ maps to $\eta(\nu_{k,1}, \ldots, \nu_{k,m-1})$ for $1 \leq k \leq n_m + 1$. Since $y_{m-1,k}$ maps to $[\mathcal{L}_{m-1,k}]$ for $1 \leq k \leq n_m + 1$ and $y_{s,i}$ maps to $p_{s}^{*} \circ \cdots \circ p_{s+1}^{*}(\mathcal{L}_{s,i})$ for $1 \leq s \leq m - 2$ and $1 \leq i \leq n_s + 1$, it further suffices to show that
\[ \eta(\nu_{k,1}, \ldots, \nu_{k,m-1}) \]
is isomorphic to
\[ \bigotimes_{s=1}^{m-2} \bigotimes_{i=1}^{n_{s+1}} (p_{s}^{*} \circ \cdots \circ p_{s+1}^{*}(\mathcal{L}_{s,i}))_{P_{s}^{(m)}(k,i)}^{m-1} \bigotimes_{i=1}^{m-1} \mathcal{L}_{m-1,i}^{P_{s}^{(m)}(k,i)}. \]

We recall that $\mathcal{L}_{m-1,j}$ is isomorphic to the line bundle $\eta(0, \ldots, 0, e_j)$ where $e_j \in \mathbb{Z}^{n_m+1}$ has 1 at the $j$-th place and 0 everywhere else and $p_{m-1}^{*} \circ \cdots \circ p_{s+1}^{*}(\mathcal{L}_{s,i})$ is isomorphic to the line bundle $\eta(0, \ldots, e_i, \ldots, 0)$ where $e_i \in \mathbb{Z}^{n_s+1}$ has 1 at the $i$-th place and 0 everywhere else. Also $\nu_{k,s}^{(m)} \in \mathbb{Z}^{n_s+1}$ is the $k$th row vector of the matrix $P_{s}^{(m)}$. Thus $\nu_{k,s}^{(m)} = \sum_{i=1}^{n_{s+1}} P_{s}^{(m)}(k, i) \cdot e_i$. This implies that
\[ (\nu_{k,s}^{(m)})_{s=1}^{m-1} = (\sum_{i=1}^{n_{s+1}} P_{s}^{(m)}(k, i) \cdot e_i)_{s=1}^{m-1}. \]

Now, the claim follows by repeated application of Lemma 2.11. \hfill $\Box$

**Example 4.5.** We now determine the presentation of the $K$ ring for the Example 2.6 in terms of generators and relations.

$K^*(B_0) \cong \mathbb{Z}$

$K^*(B_1) \cong \mathbb{Z}[y_{1,1}^{\pm 1}, y_{1,2}^{\pm 1}, y_{1,3}^{\pm 1}] / I_1$.

$K^*(B_2) \cong \mathbb{Z}[y_{1,1}^{\pm 1}, y_{1,2}^{\pm 1}, y_{1,3}^{\pm 1}, y_{2,1}^{\pm 1}, y_{2,2}^{\pm 1}] / I_1 + I_2$

$K^*(B_3) \cong \mathbb{Z}[y_{1,1}^{\pm 1}, y_{1,2}^{\pm 1}, y_{1,3}^{\pm 1}, y_{2,1}^{\pm 1}, y_{2,2}^{\pm 1}, y_{3,1}^{\pm 1}, y_{3,2}^{\pm 1}] / I_1 + I_2 + I_3$

where, $I_1$ is generated by the elements

\[ e_r(y_{1,1}, y_{1,2}, y_{1,3}) - \binom{3}{r}, \text{ for each } 1 \leq r \leq 3. \]

$I_2$ is generated by the elements

\[ e_r(y_{2,1}, y_{2,2}) - e_r(y_{1,1}^{a_1} y_{1,2}^{a_2} y_{1,3}^{a_3}, y_{1,1}^{b_1} y_{1,2}^{b_2} y_{1,3}^{b_3}) \]

for each $1 \leq r \leq 2$. 
\[ \mathcal{I}_3 \] is generated by the elements
\[ e_r(y_{3,1}, y_{3,2}) - e_r(y_{1,1}^{c_1} y_{1,2}^{c_2} y_{1,3}^{c_3} y_{2,1}^{f_1} y_{2,2}^{f_2}, y_{1,1}^{d_1} y_{1,2}^{d_2} y_{1,3}^{d_3} y_{2,1}^{g_1} y_{2,2}^{g_2}) \]
for each \( 1 \leq r \leq 2 \).

4.2. \textbf{K-ring of flag Bott-Samelson varieties.} Suppose that \( \mathcal{I} = (I_1, \ldots, I_r) \) be a sequence of subsets of \([n]\) such that the Levi subgroup \( L_{I_k} \) of the parabolic subgroup \( P_{I_k} \) has Lie type \( A_{n_k} \) for all \( 1 \leq k \leq r \). Take an enumeration \( I_k = \{u_{k,1}, \ldots, u_{k,n_k}\} \) which satisfies the conditions (3.10).

Let \( \mathcal{K}^*(F_3) \) denote the algebraic \( K \)-ring of the flag Bott-Samelson variety \( F_3 \). Let \( \mathcal{K}' := \mathbb{Z}[y_j \mid 1 \leq j \leq r] \) where \( y_j \) denotes the collection of variables \( \{y_{j,1}^{+1}, y_{j,2}^{+1}, \ldots, y_{j,n_j+1}^{+1}\} \) for every \( 1 \leq j \leq r \). Let \( \mathcal{I}_i \) be the ideal in \( \mathcal{K}' \) generated by the polynomials \( e_t(y_1) - \binom{n_1 + 1}{t} \) for each \( 1 \leq t \leq n_1 + 1 \). And for each \( 2 \leq j \leq r \), let \( \mathcal{I}_j \) be the ideal in \( \mathcal{K}' \) generated by the polynomials
\[ e_t(y_j) - e_t\left( \prod_{s=1}^{j-1} \left( \prod_{i=1}^{n_{s+1}} y_{s,i}^{Q_{s,i}^{(j,k,i)}} \right) | 1 \leq k \leq n_j + 1 \right) \]
for each \( 1 \leq t \leq n_j + 1 \), where \( Q_{s,i}^{(j,k,i)}(p,q) = \langle \alpha_{u_{j,p}} + \ldots + \alpha_{u_{j,n_j}}, \alpha_{u_{l,q}} + \ldots + \alpha_{u_{l,n_l}} \rangle \) if \( 1 \leq p \leq n_j \) and \( 1 \leq q \leq n_t \), and 0 otherwise. Let \( \mathcal{I}' \) denote the ideal \( \mathcal{I}_1 + \cdots + \mathcal{I}_r \) in \( \mathcal{K}' \).

\textbf{Remark 4.6.} Since the flag manifold has algebraic cell decomposition given by the Schubert cells the iterated flag bundles have cellular structure with cells only in even dimension. Thus it follows by [2, Section 2.5] that \( K^*(B_m) = K^0(B_m) \) i.e. \( K^{-1}(B_m) = 0 \). This can alternately be seen by Leray Hirsch theorem for \( K \)-theory (see [17, Theorem 1.3, Chapter IV]) or by the degeneracy of the Atiyah-Hirzebruch spectral sequence [2, Section 2.4]. Furthermore, since \( F_3 \) and \( F_3^0 \) are diffeomorphic, \( K^*(F_3) \simeq K^*(F_3^0) \). Thus it follows that \( K^*(F_3) = K^0(F_3) \).

As an immediate corollary of Theorem 4.4, we have the following presentation of the topological \( K^0 \)-ring (and hence the Grothendieck ring \( \mathcal{K}^0 \)) of \( F_3 \) in terms of generators and relations.

\textbf{Corollary 4.7.} The map \( \mathcal{K}' \rightarrow \mathcal{K}^0(F_3) \) which sends \( y_{j,k} \) to
\[ [p_r \circ \cdots \circ p_{j+1}(\mathcal{L}_{j,\omega_{u_{j,k}-\omega_{u_{j,k-1}}}})] \]
for \( 2 \leq k \leq n_j \),
\( y_{j,1} \) to
\[ [p_r^* \circ \cdots \circ p_{j+1}^*(\mathcal{L}_{j,\omega_{u_{j,1}}})] \] and
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\[ \text{for } 1 \leq j \leq r - 1, \text{ and } y_{r,k} \text{ to } \]
\[ [L_{j,k}] \text{ and } \]
\[ [L_{j,l}] \]

induces an isomorphism

(4.14) \[ \mathcal{K}^0(F_j) \simeq K^0(F_j) \simeq R'/I'. \]

Proof. From Proposition 3.4, \( F^0_j \) and \( F^1_j = F_j \) are diffeomorphic. Hence,

(4.15) \[ K^0(F_j) \simeq K^0(F^0_j). \]

From Theorem 3.7, the manifold \( F^0_j \) is an \( r \)-stage flag Bott manifold which is determined by a sequence of matrices \( \mathfrak{M} := (Q_{lj}^{(j)})_{1 \leq l < j \leq r} \in \prod_{1 \leq l < j \leq r} M_{n_j + 1, n_l + 1}(\mathbb{Z}) \) in the sense of Definition 2.8. Hence, from Theorem 4.4, the map from \( \mathcal{R}' \) to \( K^0(F^0_j) \) which takes \( y_{j,k} \) to \([p_r^* \circ \cdots \circ p_{j+1}^*(L_{j,k})]\) for \( 1 \leq j \leq r - 1 \) and \( y_{r,k} \) to \([L_{r,k}]\) induces an isomorphism of \( \mathbb{Z} \)-algebras

(4.16) \[ \mathcal{R}'/I' \simeq K^0(F^0_j). \]

Moreover, by [25, Lemma 4.2] and Remark 3.6 it follows that the forgetful map

(4.17) \[ f : \mathcal{K}^0(F_j) \to K^0(F_j) \]

is an isomorphism of rings where \( f([L_{j,\omega u_{j,k} - \omega u_{j,k-1}}]) = [L_{j,k}] \) for \( 2 \leq k \leq n_j \), \( f([L_{j,\omega u_{j,k}}]) = [L_{j,1}] \) and \( f([L_{j,\omega u_{j,k}}]) = [L_{j,n_j+1}] \). Now, the presentation (4.14) for the Grothendieck ring of \( F_j \) follows by (4.15), (4.16) and (4.17).

Let \( \mathcal{R}'' := \mathbb{Z}[y_j^{\pm 1}, j \leq r] \) and \( I_j'' \) is the ideal in \( \mathcal{R}'' \) generated by the elements \( y_{11} + y_{12} - 2 \) and \( y_{11} y_{12} - 1 \). For each \( 2 \leq j \leq r \) let \( I_j'' \) be the ideal generated by the polynomials \( y_{j1} y_{j2} - \prod_{l=1}^{j-1} y_{l1}^{cl_l} - 1 \) and
\( y_{j_1} \cdot y_{j_2} - \prod_{l=1}^{j-1} y_{l_1}^{c_{j_1}} \), such that

\[ c_{j_l} := \langle \alpha_{u_{j_1}}, \alpha_{u_{j_1}}^\vee \rangle \] 

for each \( l < j \).

Let \( \mathcal{I}'' \) denote the ideal \( \mathcal{I}' + \cdots + \mathcal{I}'' \) in \( \mathcal{R}'' \).

**Corollary 4.8.** Suppose that the flag Bott-Samelson variety \( F_j \) is a Bott-Samelson variety (cf. Corollary 3.9). Then the map from \( \mathcal{R}'' \) to \( K^0(F_j) \) which sends \( y_{j_1} \) to \([p_r^* \circ \cdots \circ p_{j+1}^*(L_{\omega_j})]\) for \( 1 \leq j \leq r - 1 \) and \( y_{r_1} \) to \([L_{\omega_r}]\) induces the following isomorphism of \( \mathbb{Z} \)-algebras:

\[
K^0(F_j) \cong K^0(F_j) \cong \mathcal{R}'' / \mathcal{I}''.
\]

Using the relations in \( \mathcal{I}'' \) we can further simplify this to

\[
K^*(F_j) \cong \mathbb{Z}[y_{j_1}^{\pm 1} \mid 1 \leq j \leq r] / ((y_{j_1} - 1)(y_{j_1} - y_{i_1}^{c_{j_1}} \cdots y_{j-1,1}^{c_{j-1,1}}); 1 \leq j \leq r),
\]

which matches with the presentation given by Sankaran and Uma ([26, Theorem 5.4]).

**Proof.** The isomorphism (4.18) directly follows from Theorem 4.4 and Corollary 3.9.

Now, from the relations in \( \mathcal{I}''' \) we get \( y_{j_2} = y_{i_1}^{-1} \) and from the relations in \( \mathcal{I}'' \) we get \( y_{j_2} = y_{j_1}^{-1}(\prod_{l=1}^{j-1} y_{l_1}^{c_{j_1}}) \). Moreover, \( y_{j_1} + y_{j_2} = \prod_{l=1}^{j-1} y_{l_1}^{c_{j_1}} + 1 \) for \( 1 \leq j \leq r \) (for \( j = 1 \) the right hand side is 2 so this holds by \( \mathcal{I}''' \)). This implies that

\[ y_{j_1}(\prod_{l=1}^{j-1} y_{l_1}^{c_{j_1}} + 1 - y_{j_1}) - (\prod_{l=1}^{j-1} y_{l_1}^{c_{j_1}}) = 0. \]

This simplifies to

\[ (y_{j_1} - 1)(\prod_{l=1}^{j-1} y_{l_1}^{c_{j_1}}) - y_{j_1}(y_{j_1} - 1) = 0 \]

which further simplifies to

\[ (y_{j_1} - 1)(\prod_{l=1}^{j-1} y_{l_1}^{c_{j_1}} - y_{j_1}) = 0. \]

Thus we can reduce the presentation to the simpler form (4.19). \( \square \)
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Department of Mathematics, Indian Institute of Technology, Madras, Chennai 600036, India

Email address: bidhan@small.iitm.ac.in

Department of Mathematics, Indian Institute of Technology, Madras, Chennai 600036, India

Email address: vuma@iitm.ac.in