Energy Constrained Depth First Search

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Abstract

Depth first search is a natural algorithmic technique for constructing a closed route that visits all vertices of a graph. The length of such route equals, in an edge-weighted tree, twice the total weight of all edges of the tree. This paper considers a variant of such search strategies where the length of each route is bounded by a positive integer $B$ (e.g. due to limited energy resources of the searcher). The objective is to cover all the edges of a tree $T$ using the minimum number of routes, each starting and ending at the root and each being of length at most $B$. To this end, we analyze the following natural greedy tree traversal process that is based on decomposing a depth first search traversal into a sequence of limited length routes. Given any arbitrary depth first search traversal $R$ of the tree $T$, we cover $R$ with routes $R_1, \ldots, R_l$, each of length at most $B$ such that: $R_i$ starts at the root, reaches directly the farthest point of $R$ visited by $R_{i-1}$, then $R_i$ continues along the path $R$ as far as possible, and finally $R_i$ returns to the root. We call the above algorithm piecemeal-DFS and we prove that it achieves the asymptotically minimal number of routes $l$, regardless of the choice of $R$. Our analysis also shows that the total length of the traversal (and thus the traversal time) of piecemeal-DFS is asymptotically minimum for any piecemeal exploration, again regardless of the choice of the original depth first search traversal $R$. The fact that $R$ can be chosen arbitrarily makes our approach suitable for construction of distributed and online algorithms, i.e., when the input tree $T$ is not known in advance, $R_i$’s can be constructed locally without any knowledge of the yet unvisited part of $T$. Moreover, our results show that depth first search is efficient for energy constrained exploration of trees, even though it is known that the same does not hold for the exploration of arbitrary graphs.

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1 Introduction

Graph-theoretic problems in which one wants to cover the entire graph with one or more routes satisfying certain objective is a well established and long studied topic in many areas of computer science. Particular problems vary depending on the research area or potential applications, including the study of simple graph traversals like DFS or BFS, for algorithmic purposes, to complex transportation problems with many variations of traveling salesman

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problem (TSP), or pursuit-evasion games like the watchman problem, and finally distributed monitoring of networks using mobile agents.

For one possible application of our results, consider a mobile robot that needs to explore an initially unknown tree. We assume that the tree is edge-weighted and the weight of each edge denotes the length of that edge. Starting from a single vertex (the root) of the tree, the robot must traverse all edges of and return to its initial location. Upon visiting a vertex $v$ for the first time, the robot discovers the edges incident to $v$ and can choose one of them to continue the exploration. Provided that the robot can remember the visited vertices and edges, a simple depth first search (DFS) is an efficient algorithm for exploring the tree, achieving the optimal cost of twice the sum of the lengths of edges in the tree.

Consider a more interesting scenario, when the robot has a limited source of energy (e.g. a battery) which allows it to traverse a path of length at most $B$ (we say such a robot is energy constrained). Naturally, we assume that each vertex of the tree is at distance at most $B/2$ from the root, otherwise the tree cannot be fully explored. In this case, exploration is possible if the robot can recharge its battery whenever it returns back to the starting location. Thus, the exploration is a collection of routes of the robot, each of which starts and ends at the root, and has length at most $B$. We are interested in the minimum number of such routes needed (i.e. the number of times the robot has to recharge) to completely explore the tree.

This model of exploration may be of interest for several reasons. One obvious reason is related to the capabilities of the robot; it may have a restricted fuel tank capacity or perhaps a harsh or risky environment enforces a return to its home-base every so often. A robot that returns periodically to the root can inform about new discoveries — in this way the knowledge is accumulated gradually at the base-station while the algorithm progresses. This may, for example, reduce the risk of having no data in case of robot failure prior to the end of exploration. From a different point of view, this process may be seen as a piecemeal learning, that is, one in which it is possible to have a trade off between exploration and utilization; the two phases representing parts in which learning occurs (exploration) and part in which accumulated knowledge is used (utilization). Finally, having many restricted-length routes covering a tree instead of a single long route may be potentially applied in scenarios in which one wants to minimize exploration time by strategies using multiple robots. In fact, when the robots are incapable of refueling, we can use several robots to explore the tree, each robot traversing a path of length $B$. In that case, it is important to minimize the number of robots used as well as the total energy cost for exploration.

Related work: There exists extensive literature on graph traversal and exploration, we survey here only the most relevant results on graph exploration by mobile agents. Exploration of general graphs by a single agent has been studied [24] who gave an algorithm of $m + O(n)$ steps. For exploration by $k$ agents, [19] provides an exploration algorithm taking $O(D + n/\log k)$ steps in trees of $n$ nodes and height $D$. This algorithm turns out to be $O(k/\log k)$ competitive [21] (where competitiveness is the ratio of the number of steps of an algorithm over the optimal number of steps). Authors in [9] give a $O(n/k + D^{k-1})$ time algorithm for tree exploration while [17] gives an algorithm for sparse trees with competitive ratio $O(D^{1-1/p})$, where $p$ is defined as the tree density. For some lower bound on exploration time, see [15, 19, 21]. For other recent results on exploration time see e.g. [9, 13, 14, 22, 23]. Other than optimizing time, exploration using little memory for the agents has also been studied, see e.g. [1, 12].

None of the results mentioned above consider any energy limitation for the agents. The energy constrained exploration problem was first studied under the name of Piecemeal Graph...
Exploration. It has been shown that, for general graphs, there exists an algorithm for energy constrained agents that has cost, i.e., the total number of edge traversals, of $O(m + n^{1+o(1)})$, where $n$ and $m$ are the number of nodes and edges of the graph [5]. This has been further improved (by an algorithm that is a combination of DFS and BFS) to $O(m + n \log^2 n)$ in [6].

For algorithms designed for a special class of grid graphs with ‘rectangular obstacles’ see [8]. In [15] has been shown that for general unknown weighted graphs there exists an algorithm of cost asymptotic to the sum of edge weights assuming that the route length is $2(1 + \beta)r$, where $r$ is the furthest distance from the starting node to other nodes, and $0 < \beta < 1$. Note that, unlike our results, the above strategies require the length of each route to be strictly larger than the actual length of the path to the farthest vertex. In other words, these algorithms fail in the extreme cases when the height of the tree is equal to half of the energy budget, which seem to be the most challenging cases.

The same tree exploration model as we study in this work has been considered in [11, 16] for unweighted trees and multiple agents, with one difference: each agent traverses a path of length at most $B$ such that the path starts at the root but may end at any node of the tree (in other words, agents do not have to return to the homebase). It has been shown in [11] that if the tree is not known in advance, then there exists an exploration algorithm (that minimizes the number of agents used) with competitive ratio of $O(\log B)$ and this is the best possible. On the other hand, it was shown that by allowing the route lengths to be a constant factor more than $B$, it is possible to explore the tree using the minimum number of agents [16]. Distributed algorithms for energy constrained agents has been a subject of recent investigation, see e.g. [2, 3, 7]. Authors in [10] also consider a model in which agents are allowed to transfer part of their energy to another agent.

There are many studies on variants of the traveling salesman problem, including the $k$-TSP [4, 20] related to the task of finding a bounded length route in a graph. Such results are out of scope for this paper.

Our Results and Outline: In this work we analyze a very natural process of partitioning a depth first search traversal $R_{\text{DFS}}$ of a tree into a sequence $S = (R_1, \ldots, R_k)$ of routes where each route $R_i$ has length at most $B$, starts and ends at the root of the tree (see Section 3 for a formal definition). We prove that the number of routes $k$ is asymptotically optimal (Theorem 1), that is, it is within a constant factor of the number of routes in any exploration strategy composed of routes of length at most $B$ that cover the entire tree. This fact, being intuitively expected for trees (although it does not hold in general graphs [15]) turns out to be nontrivial. Our approach is to consider another parameter of an exploration strategy: the cost (see Section 4) defined as the sum of the lengths of all routes in an exploration strategy. In order to prove our main result, we argue, in Section 5 that the cost of $S$ is asymptotically optimal (see Theorem 2). Then, in Section 4 we argue that the fact that $S$ has small cost implies that the number of routes in $S$ is expectedly small.

We emphasize that the above claim holds independently of the choice of the initial depth first search traversal $R_{\text{DFS}}$. The implications of this fact are two-fold. First, it provides a theoretical insight into such a partitioning of a depth first search traversals into bounded-length segments. Second, for an exploration algorithm design it means that the routes $R_i$ may be constructed without knowing $R_{\text{DFS}}$ in advance, or more precisely, the routes may be build in an online fashion based only on the knowledge of the subtree explored to date.

## Exploration strategies

In this work we consider edge-weighted rooted trees $T = (V, E, \omega: E \rightarrow \mathbb{R}_+)$, with root $r$. We define a route $R$ as sequence of nodes, $R = (v_0, v_1, \ldots, v_k)$, where $v_i$ is a vertex of $T$ for
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We write \( |S| \) to refer to the number of routes in \( S \).

### 3 Problem statement and DFS exploration

The formulation of the combinatorial problem, to which we refer as energy constrained tree exploration, we study in this work is as follows.

Energy Constrained Tree Exploration problem (ECTE)

Given an edge-weighted tree \( T \) and a real number \( B \), what is the minimum integer \( k \) such that there exists a \( B \)-exploration strategy that consists of \( k \) routes?

Our goal is to analyze a particular type of solution to this problem, namely, an exploration strategy that behaves like a depth first search traversal but adopted to the fact that route lengths are bounded by \( B \). Let \( R_{\text{DFS}} = (v_0, v_1, \ldots, v_l) \) be a route in \( T \) that covers the tree \( T \) and performs a depth first search traversal of \( T \). (Note that \( R_{\text{DFS}} \) is a route and thus we consider a depth first search traversal to have node repetitions.) We refer by \( \text{PDFS}(T) = (R_1, \ldots, R_k) \) (Piecemeal Depth First Search) to the following \( B \)-exploration strategy constructed iteratively for \( i := 1, \ldots, k \) (see also Figure 1 for an example):

(i) let \( v_{j_0} = r \),

(ii) \( R_i \) continues DFS exploration from where \( R_{i-1} \) stopped making progress (from the node \( v_{j_{i-1}} \)) as long as for currently visited \( v_p \):

\[
   d(r, v_{j_{i-1}}) + \ell((v_{j_{i-1}}, v_{j_{i-1}+1}, \ldots, v_p)) + d(v_p, r) \leq B,
\]

(1)

(iii) furthest \( v_p \) (for \( p \leq l \) that satisfies condition from (ii)) is denoted as \( v_{j_i} \), the vertex where \( R_i \) stopped making progress,

(iv) let \( R_i = P_i^{-1} \circ (v_{j_{i-1}}, v_{j_{i-1}+1}, \ldots, v_{j_i}) \circ P_i^R \), where \( P_i^{-1} \) is the path from \( r \) to \( v_{j_{i-1}} \), and \( P_i^R \) is the path from \( v_{j_i} \) to \( r \).

Such a strategy \( \text{PDFS}(T) \) is called a DFS \( B \)-exploration. We will say that the part of \( R_i \) containing the subsequence \( (v_{j_{i-1}}, \ldots, v_{j_i}) \) makes progress on the route \( R_{\text{DFS}} \).

We remark that different depth first search traversals \( R_{\text{DFS}} \) may result in different values of \( k \) (different number of routes) in the resulting DFS \( B \)-exploration, although for a particular choice of \( R_{\text{DFS}} \) the corresponding \( \text{PDFS}(T) \) is unique. In the rest of the work we fix the route \( R_{\text{DFS}} \) arbitrarily and thus \( \text{PDFS}(T) \) refers to the unique DFS \( B \)-exploration strategy obtained from \( R_{\text{DFS}} \).
4 Our results

The following theorem provides the first main result of this work.

\begin{theorem}
Let $T$ be a tree and let the longest path from the root to a leaf in $T$ be at most $B/2$. It holds $|\text{PDFS}(T)| \leq 10 |\mathcal{R}|$, where $\mathcal{R}$ is a $B$-exploration strategy that consists of the minimum number of routes.
\end{theorem}

Theorem 2 indeed implies Theorem 1.

\begin{proof}
We start with the following observation which relates the smallest possible number of routes in a $B$-exploration strategy and the minimum possible cost.

\begin{observation}
Given $T$ and $B$, $|\mathcal{R}| \geq \lceil \xi(\text{COPT}(T))/B \rceil$, where $\mathcal{R}$ is a $B$-exploration strategy with minimum number of routes.
\end{observation}

Proof. Each route of $\mathcal{R}$ is of length at most $B$. Thus, $|\mathcal{R}| \geq \xi(\mathcal{R})/B$. By definition of COPT($T$), $\xi(\mathcal{R}) \geq \xi(\text{COPT}(T))$, and since $|\mathcal{R}|$ is an integer, the claim follows.
\end{proof}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A route $R_{\text{DFS}}$ and the corresponding PDFS($T$) = ($R_1$, $R_2$, $R_3$) with $B = 20$: (a) a depth first search traversal $R_{\text{DFS}} = (a, b, c, b, d, e, d, f, d, b, a, g, a)$; (b)-(d) routes $R_1, R_2, R_3$ with lengths 18, 16 and 20, respectively.}
\end{figure}
Recall that $R_{\text{DFS}} = (v_0, \ldots, v_i)$ is the depth first search traversal of $T$ used to obtain $\text{PDFS}(T)$, and the $i$-th route $R_i$ in $\text{PDFS}(T) = (R_1, \ldots, R_k)$ makes progress on the depth first search traversal by traversing the part of $R_{\text{DFS}}$ that starts at $v_{j_i-1}$ and ends at $v_{j_i}$. By definition of $\text{PDFS}(T)$, extending $R_i$ so that it makes progress with the walk $(v_{j_i}, \ldots, v_{j_i}, v_{j_i+1})$ would exceed its length to be more than $B$ for each $i < k$, i.e., $\ell(R_i) + 2\omega([v_{j_i}, v_{j_i+1}]) > B$. Consider a tree $T'$ obtained from $T$ by subdividing the edge $[v_{j_i}, v_{j_i+1}]$ into two edges $[v_{j_i}, x_i]$ and $[x_i, v_{j_i+1}]$ with weights $(B - \ell(R_i))/2$ and $\omega([v_{j_i}, v_{j_i+1}]) - (B - \ell(R_i))/2$, respectively, for each $i \in \{1, \ldots, k-1\}$. Note that the common nodes of $T$ and $T'$ (that is, the nodes of $T$) are visited by $\text{PDFS}(T')$ and $\text{PDFS}(T')$ in the same order, both $\text{PDFS}(T)$ and $\text{PDFS}(T')$ are $B$-exploration strategies and the length of each route in $\text{PDFS}(T')$, except for the last one, is of length exactly $B$. The latter in particular implies

$$|\text{PDFS}(T')| = \left\lceil \frac{\xi(\text{PDFS}(T'))}{B} \right\rceil. \quad (2)$$

Note that

$$\xi(\text{COPT}(T)) = \xi(\text{COPT}(T')) \quad (3)$$

because, informally speaking, a strategy that minimizes the cost never reaches a node of degree two in order to return to previously visited node — thus, in particular, a traversal of $\{v_{j_i}, x_i\}$ is immediately followed by a traversal of $\{x_i, v_{j_i+1}\}$ and vice versa.

From Equation (2), Theorem 2 applied to $T'$, Equation (3) and Observation 3 (used in this order) we conclude that

$$|\text{PDFS}(T')| = \left\lceil \frac{\xi(\text{PDFS}(T'))}{B} \right\rceil \leq 10 \cdot \left\lceil \frac{\xi(\text{COPT}(T'))}{B} \right\rceil = 10 \cdot \left\lceil \frac{\xi(\text{COPT}(T))}{B} \right\rceil \leq 10 \cdot |R|,$$

and hence $|\text{PDFS}(T)| = |\text{PDFS}(T')|$ completes the proof of Theorem 1.

\section{Bounding the cost of $\text{PDFS}(T)$}

\subsection{Additional notation}

When referring to subtrees, we consider them always in the context of their distance from the root. More specifically, we consider the potential of a node $v$, denoted by $\varphi(v)$, to be defined as $\varphi(v) = B/2 - d_l(v, v)$. If $u$ is the parent of $v$ in $T$, then we say that $u$ is the higher endpoint of $\{u, v\}$ and $v$ is the lower endpoint of the edge $\{u, v\}$; we also say that $\{u, v\}$ is a downward edge of $u$. We define the potential of a tree $T$, denoted $\varphi(T)$, to be the potential of its root. Then, $2\varphi(T)$ is an upper bound on the total length of any route inside $T$. We say that a route reaches a potential $x$ in some subtree if it reaches a node having potential $x$. Additionally we denote $\omega(T) = \sum_{e \in E(T)} \omega(e)$, that is, the weight of $T$ to be the total weight of its edges. We denote by $T[e]$ the subtree of $T$ rooted at $v$ that contains $v$ and all its descendants, and by $T[v]$ the tree composed of the edge $e$ and $T[v]$ where $v$ is the lower endpoint of $e$.

We say that a tree $T$ is heavy if $\omega(T) > \varphi(T)$, and otherwise we say that $T$ is light. We extend this terminology to vertices and edges: a vertex $v$ or an edge $e$ is heavy if $T[v]$ or $T[e]$ is heavy, respectively. Additionally, by $\text{heavy} \deg(v)$ we denote the number of outgoing downward edges of $v$ that are heavy (note that if an edge is heavy, then both its endpoints are heavy as well). Observe that if $v$ is any vertex of $T$ and $T[v]$ is heavy, then one route is not enough to cover the entire $T[v]$ in any $B$-exploration strategy.
5.2 Adversarial DFS-exploration

In our analysis we will refer to a slightly modified DFS $B$-exploration, called a $B'$-adversarial DFS $B$-exploration, $B' \leq B$, and denoted by $\text{ADFS}_{B'}(T)$ for a given tree $T$, which is obtained by replacing Equation (1) for $i = 1$ in condition [ii] in the definition of the DFS $B$-exploration by the following one (see also Figure 2):

$$\ell((v_{j_0}, v_{j_0}+1, \ldots, v_p)) + d(v_p, r) \leq B'.$$

(4)

In other words, the length of the first route is bounded by $B'$ (Equation (4)) and the lengths of the remaining routes are bounded by $B$ (Equation (1) for $i > 1$). The intuition that lies behind such a generalization in which the first route may have smaller length is motivated by the behavior of DFS $B$-exploration strategies in the subtrees of the original tree $T$. In such a subtree rooted at $r'$, the first route ‘entering’ it might have reached $r'$ after visiting some vertices that lie outside of the shortest path between $r'$ and the root of $T$. However, each of the following routes reaching $r'$ takes this shortest path. Thus, if we analyze such a DFS $B$-exploration strategy within the subtree rooted at $r'$, the first route has possibly restricted length with respect to the subsequent routes.

For a given tree $T$, we define an adversarial DFS exploration of $T$, denoted by $\text{ADFS}(T)$, and defined to be an exploration strategy $\text{ADFS}_{B'}(T)$ that maximizes the cost, across all possible values of $B'$:

$$\xi(\text{ADFS}(T)) = \max_{0 \leq B' \leq 2\varphi(T)} \xi(\text{ADFS}_{B'}(T)).$$

(5)

In the following analysis, it will be convenient for us to use arguments that rely on the fact that $B'$, for our purposes, takes only one of the finite values from $[0, 2\varphi(T)]$. This is due to the above comment, namely, for $T[r]$, where $r$ is the root of $T$, we have that $B' = B$ and for any other subtree $T[v]$, the values of $B'$ interesting for us depend on the prefix of the route that starts at the root of $T$ and reaches $v$. Thus, a simple inductive argument allows us to conclude that the value of $B'$ depends on all possible DFS $B$-exploration strategies of $T$ (the number of those is finite). Hence, we denote by $B$ the finite set that consists all values $B' = B - x$ such that there exists a route in $T$ of length $x$. Thus, we can restate (5):

$$\xi(\text{ADFS}(T)) = \max_{B' \in B} \xi(\text{ADFS}_{B'}(T)).$$
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Note that it follows from the definition that
\[ \xi(\text{ADFS}(T)) \geq \xi(\text{PDFS}(T)). \]

Intuitively, if \( v \) is any node of the tree \( T \), then \( \text{ADFS}(T[v]) \) is the worst case scenario of how a DFS \( B \)-exploration may perform in \( T[v] \) in terms of the cost; this worst case is understood as considering the worst possible ending point of the route that (in the entire tree \( T \)) precedes the considered strategy \( \text{ADFS}(T[v]) \).

**Lemma 4.** If \( v \) is any node of \( T \) and \( e_1, e_2, \ldots, e_k \) are all downward edges of \( v \), then
\[ \xi(\text{COPT}(T[v])) = \sum_{1 \leq i \leq k} \xi(\text{COPT}(T[e_i])), \quad (6) \]
\[ \xi(\text{ADFS}(T[v])) \leq \sum_{1 \leq i \leq k} \xi(\text{ADFS}(T[e_i])). \quad (7) \]

**Proof.** Informally, equality in (6) for an optimal solution follows from the fact that \( \text{COPT}(T[v]) \) has the freedom to pick the length of each route to be an arbitrary number in \( B \). Any strategies for \( T[e_1], T[e_2], \ldots, T[e_k] \) can be translated into strategy for \( T[v] \): the latter one is constructed by simply concatenating the former exploration strategies. Similarly, if one takes an exploration strategy \( \text{COPT}(T[v]) \), then one can assume without affecting its cost that each of its routes has only two occurrences of the root: it is the first and last vertex of the route. But then, such a strategy \( \text{COPT}(T[v]) \) can be partitioned into the corresponding strategies for the trees \( T[e_1], T[e_2], \ldots, T[e_k] \).

We now prove (7). Consider an exploration strategy \( \text{ADFS}(T[v]) \). Each route of this strategy is of length at most \( 2\varphi(T) \). Obtain an exploration strategy \( S \) by partitioning each route in \( \text{ADFS}(T[v]) \) in such a way that the concatenation of all routes in \( S \) equals the concatenation of all routes in \( \text{ADFS}(T[v]) \) and no route in \( S \) has \( v \) as an internal vertex. Note that \( \xi(S) = \xi(\text{ADFS}(T[v])) \). Now, \( S \) can be partitioned into \( S_1, \ldots, S_k \) such that \( S_i \) is an \( b_i \)-adversarial DFS exploration strategy of \( T[e_i] \) for some \( b_i \leq 2\varphi(T[v]) \), i.e., \( S_i = \text{ADFS}_{b_i}(T[e_i]) \), for each \( i \in \{1, \ldots, k\} \) and the concatenation of \( S_1, \ldots, S_k \) gives \( S \). Thus,
\[ \xi(\text{ADFS}(T[v])) = \xi(S) = \sum_{i=1}^{k} \xi(\text{ADFS}_{b_i}(T[e_i])). \]

To conclude the proof, observe that by the definition of adversarial exploration
\[ \xi(\text{ADFS}_{b_i}(T[e_i])) \leq \xi(\text{ADFS}(T[e_i])), \quad i \in \{1, \ldots, k\}. \]  

A proof of the following lemma is in the appendix.

**Lemma 5.** For any tree \( T \), if \( T \) is light, then \( \xi(\text{ADFS}(T)) < 2 \cdot \xi(\text{COPT}(T)) \).

**Proof.** If \( T \) is light, then observe that \( \text{ADFS}(T) \) either consists of one route, in which case \( \xi(\text{ADFS}(T)) = \xi(\text{COPT}(T)) \), or it contains at least two routes but then the second route, having length up to \( 2\varphi(T) \), will explore all remaining vertices of \( T \) since \( \varphi(T) \geq \omega(T) \) holds for a light tree. Thus, in the latter case \( \text{ADFS}(T) \) has exactly two routes, one of them being strictly shorter than \( 2\omega(T) \), which gives \( \xi(\text{ADFS}(T)) < 4\omega(T) \leq 2 \cdot \xi(\text{COPT}(T)). \)
We will now state a theorem that immediately implies Theorem 2. (Note that it follows directly from definition that \(\xi(\text{PDFS}(T)) \leq \xi(\text{ADFS}(T))\).) The technical adjustment in Theorem 6 by the summands of \(8\varphi(T)\) and \(16\varphi(T)\), depending on the value of \text{heavydeg}(r)\) of the root of \(T\), is a key property that allows us to conduct an inductive proof given in Section 5.4.

\begin{theorem}
If \(T\) is heavy and \(r\) is its root, then:
\begin{enumerate}
  \item if \text{heavydeg}(r) = 1, then \(\xi(\text{ADFS}(T)) < 10 \cdot \xi(\text{COPT}(T)) - 8 \cdot \varphi(T)\),
  \item if \text{heavydeg}(r) \neq 1, then \(\xi(\text{ADFS}(T)) < 10 \cdot \xi(\text{COPT}(T)) - 16 \cdot \varphi(T)\).
\end{enumerate}
\end{theorem}

Our inductive argument, however, will require a certain structure of the tree \(T\), which can be formulated in the following way.

\((*)\) If the root \(r\) of \(T\) has heavy degree equal to one, then consider the longest path in \(T\) that connects \(r\) to such a \(r'\) that each internal vertex of the path has heavy degree equal to 1. (Equivalently, each edge of the path is heavy.) We then require that each vertex of this path, except for \(r'\), has at most one light edge incident to it.

In Section 5.3 we argue that we can without loss of generality assume that \(T\) has this structure. We achieve this by iteratively rearranging the tree \(T\), if needed, to obtain another tree with the desired structure and our rearrangement ensures that if the new tree satisfies Theorem 6 then so does \(T\). In this way we can focus in Section 5.4 on proving Theorem 6 for \(T\) with the above-mentioned structure.

5.3 Tree rearrangement

In this section we sketch our argument that the tree \(T\) may be assumed to have the desired structure introduced above. A complete formal proof is provided in the appendix.

Our goal is to construct a tree \(T_\varepsilon\), \(\varepsilon \in \mathbb{R}^+\), based on \(T\) such that \(T_\varepsilon\) has the required structure. The tree \(T_\varepsilon\) is obtained by selecting a vertex \(u\) in \(T\) with \text{heavydeg}(u) = 1 and having \(d > 1\) light downward edges \(e_1, \ldots, e_d\) and performing the following: we change points of attachments of the light edges so that the \(e_i\)'s are attached on the heavy edge within the distance of \(\varepsilon\) from \(u\) and no two light edges have a common endpoint in \(T_\varepsilon\). We will also need that \(T_\varepsilon\) ‘resembles’ \(T\) in the following way: a search strategy is valid for \(T\) if and only if a ‘very similar’ strategy is valid for \(T_\varepsilon\). This is achieved by selecting the order of attaching the light edges appropriately (by order we mean their distance from \(u\)). To simplify the statements considerably, it will be convenient to encode strategies in an uniform way so that we can apply the same strategy for both tree, without going into the details of tedious but straightforward conversions between a strategy for \(T\) and the corresponding strategy for \(T_\varepsilon\). We then define a collection \(C\) of all possible strategies (including adversarial ones and those that are not feasible for either \(T\) or \(T_\varepsilon\) because they contain routes that are too long or do not visit all vertices). Thanks to the construction of \(T_\varepsilon\) and \(\varepsilon\) being sufficiently small, our key argument is that a strategy in \(C\) is valid for \(T\) if and only if it is valid for \(T_\varepsilon\). We then finally provide the main result of this section which allows us to work with trees having the desired structure from now on.

\begin{lemma}
There exists \(\varepsilon > 0\) such that if Theorem 6 holds for \(T_\varepsilon\) that satisfies Property \((*)\), then Theorem 6 holds for \(T\).
\end{lemma}
5.4 Proof of Theorem 6

We will proceed by induction on the number of heavy edges in a tree. This is a valid approach since the parent of a heavy node is also heavy.

For the base case consider $T$ with no heavy edges. In particular we have that heavydeg($r$) = 0. Denote downward edges at $r$ by $e_1, e_2, \ldots, e_k$. For each $i \in \{1, \ldots, k\}$, $T[e_i]$ is light and hence by Lemma 3, $\xi(ADFS(T[e_i])) < 2 \cdot \xi(COPT(T[e_i]))$. Thus in particular, by (6) and (7) and $\xi(COPT(T)) \geq 2\omega(T)$

$$\xi(ADFS(T)) < 2\xi(COPT(T)) \leq 10\xi(COPT(T)) - 16\omega(T) \leq 10\xi(COPT(T)) - 16\varphi(T).$$

For the induction step, we assume that claim holds for all heavy proper subtrees of $T$. In the next subsections we consider two cases: when heavydeg($r$) > 1 and heavydeg($r$) = 1.

Case of heavydeg($r$) > 1

Let $e_1, \ldots, e_k$ be heavy downward edges at $r$ and $e'_1, \ldots, e'_k$ be light downward edges at $r$. By induction hypothesis (and precisely Theorem 6(i)) we obtain

$$\xi(ADFS(T[e_i])) < 10 \cdot \xi(COPT(T[e_i])) - 8\varphi(T[e_i])$$

for each $i \in \{1, \ldots, k\}$. Then, by (7) of Lemma 4 by Lemma 5 and by $\varphi(T) = \varphi(T[e_i])$ (used in this order):

$$\xi(ADFS(T)) \leq \sum_{1 \leq i \leq k} ADFS(T[e_i]) + \sum_{1 \leq i \leq j} ADFS(T[e'_i])$$

$$< \sum_{1 \leq i \leq k} (10\xi(COPT(T[e_i])) - 8\varphi(T)) + \sum_{1 \leq i \leq j} 2\xi(COPT(T[e'_i]))$$

$$\leq 10 \left( \sum_{1 \leq i \leq k} \xi(COPT(T[e_i])) + \sum_{1 \leq i \leq j} \xi(COPT(T[e'_i])) \right) - 8\varphi(T) \cdot k$$

$$\leq 10\xi(COPT(T)) - 16\varphi(T).$$

The last inequality is due to (6) and $k = \text{heavydeg}(r) \geq 2$.

Case of heavydeg($r$) = 1

From now on, assume that heavydeg($r$) = 1. Let $r'$ be a closest descendant of $r$ in $T$ that is heavy and satisfies heavydeg($r'$) $\neq 1$. Note that such a vertex always exists. Let $P$ denote the path connecting $r$ to $r'$. Additionally, we denote by $e_1, e_2, \ldots, e_k$ all light edges incident to vertices in $V(P) \setminus \{r'\}$ in the non-decreasing order of their potentials. (We remark here that the subtree rooted at $r'$ will be covered by the base case of the induction.)

Denote $\varphi_i = \varphi(e_i)$, $i \in \{1, \ldots, k\}$. Due to Lemma 2 and more precisely by the fact that thanks to Lemma 4, we assume that in the tree $T$ all edges $e_1, \ldots, e_k$ have pairwise different potentials) we then have:

$$\varphi(r) > \varphi_k > \cdots > \varphi_2 > \varphi_1 > \varphi(r') = \varphi_0.$$

We take for brevity

$$w_i = \omega(T[e_i]) = \frac{1}{2} \xi(COPT(T[e_i])) \quad \text{for } i \in \{1, \ldots, k\}, \quad w_0 = \frac{1}{2} \xi(COPT(T[r'])).$$
See Figure 3(a) that illustrates the path $P$ and placements of edges $e_i$ and the corresponding potentials.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure3a}
\caption{(a)\ \ 
\vspace{0.5cm}
\label{fig:3a}
\end{subfigure}
\hspace{0.5cm}
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure3b}
\caption{(b)\ \ 
\vspace{0.5cm}
\label{fig:3b}
\end{subfigure}
\caption{(a) the path $P$ (although it may contain nodes of degree two, they are trivial from the point of view of the analysis, thus omitted in figure); (b) illustration of $x_i$'s. Note that the first route (and possibly arbitrarily many more routes) always reaches $r'$}
\end{figure}

Let $c$ be the number of routes in $\text{COPT}(T)$, and we denote by $x_i$ the lowest potential $i$-th route reached on the path $P$ (we ignore potentials it reached in subtrees — see Figure 3(b)), where the routes are without loss of generality ordered so that $x_1 \leq x_2 \leq \cdots \leq x_c$.

Consider $j \in \{1, \ldots, c\}$. Informally speaking, the first $j$ routes of $\text{COPT}(T)$ need to cover all subtrees $T[e_i]$ such that the potential of the higher endpoint of $e_i$ is strictly smaller than $x_j$; otherwise some vertices would not be visited by $\text{COPT}(T)$. The total weight of these subtrees is $\sum_{i : \varphi_i < x_j} w_i$. Observe, that the total length of all parts of an $i$-th route that do not belong to the path $P$ is at most $2x_i$. Thus, the above total weight of the above-mentioned subtrees satisfies

$$
\sum_{i : \varphi_i < x_j} w_i \leq (x_1 + \cdots + x_{j-1}) \quad \text{for each } j \in \{1, \ldots, c+1\},
$$

where we denote $x_{c+1} = +\infty$ for the sake of simplicity.

We are now interested in bounding the cost of $\text{ADFS}(T)$ on the path $P$ with respect to $\sum_{i=1}^c 2(\varphi(r) - x_i)$, that is, with respect to the cost of $\text{COPT}(T)$ on the path $P$. To do this, we start by comparing $x_1, \ldots, x_c$ with $y_1, \ldots, y_d$ chosen by an appropriate greedy procedure:

$$
y_j = \min \{ y : \sum_{i : \varphi_i \leq y} w_i > y_1 + \cdots + y_{j-1} \}, \quad d = \min \{ i : y_1 + \cdots + y_i \geq w_0 + w_1 + \cdots + w_m \}, \quad (9)
$$

in other words, assigning $y_j$ to be the first value where (8) is violated given only $y_1, \ldots, y_{j-1}$. (Notice that from the definition we have that always $y_j \in \{\varphi_0, \varphi_1, \ldots, \varphi_k\}$. Moreover, if $y_j > \varphi_0$, then $y_{j+1} > y_j$.) We obtain the following lemma which says that, across all sequences satisfying (8), $y_i$ takes maximal values:

\begin{lemma}
It holds that $d \leq c$ and $y_j \geq x_j$ for each $j \in \{1, \ldots, d\}$.
\end{lemma}
Proof. We proceed by induction on \( j \).

For the inductive base, we have \( x_1 = y_1 = \varphi(r') \). For the inductive step, assume the claim holds for all indices smaller than \( j \). Suppose for a contradiction that \( x_j > y_j \). Then,

\[
\sum_{i : \varphi_i \leq y_j} w_i \leq \sum_{i : \varphi_i < x_j} w_i
\]

and by \([\text{9}]\) and the inductive assumption applied for all indices smaller than \( j \),

\[
\sum_{i : \varphi_i < x_j} w_i \leq y_1 + \ldots + y_{j-1}.
\]

These two inequalities give a contradiction with \([\text{9}]\).

Observe that ADFS(\( T \)) first traverses (in that order) some subset of light subtrees \( T[e_i] \), whose indices we denote by \( H \subseteq \{1, \ldots, k\} \), in a decreasing order of their indices. The above routes, none of which contains \( r' \), will form the first part of ADFS(\( T \)). Then, all vertices of \( T[r'] \) are visited (to those routes of ADFS(\( T \)) we refer at the second part of ADFS(\( T \)) and following that, remaining light subtrees \( T[e_i] \) for \( i \in H' = \{1, \ldots, k\} \setminus H \), in an increasing order of their indices (third part of ADFS(\( T \))). Note that there may exist a route that has a non-empty intersection with a tree \( T[e_i] \), \( i \in H \cup H' \), and also contains \( r' \) — this route belongs by definition to the second part of ADFS(\( T \)).

Denote by \( z_1, \ldots, z_p \) the lowest potentials reached by subsequent routes in ADFS(\( T \)) on the path \( P \) (ignoring as before the potentials they reach in subtrees), only in the first part of ADFS(\( T \)) in the reversed order of entering \( T \):

\[
\varphi(r) \geq z_p \geq \cdots \geq z_1 > \varphi(r').
\] (10)

We note that each route in the first part of ADFS(\( T \)) visits subtrees with a continuous segment of indices from \( H \), that is \( T[e_i] \) for \( i \in H \cap [i_1, i_2] \) for some integers \( i_1, i_2 \). We also note that due to the weight of a light tree, its vertices belong to at most two different routes.

Lemma 9. If for some \( i, j \) there is \( z_i > y_j \), then \( z_{i+1} \geq y_{j+1} \).

Proof. If \( y_{j+1} = y_j \) then the claim follows immediately from the fact that, by \([\text{10}]\), \( z_{i+1} \geq z_i \geq y_j = y_{j+1} \). Similarly, if \( z_i \geq y_{j+1} \) then by \([\text{10}]\) we have \( z_{i+1} \geq z_i \geq y_{j+1} \). Thus, assume that \( y_{j+1} > z_i \) and \( y_{j+1} > y_j \).

Let \( a > b \) be indices such that \( y_j = \varphi_a \) and \( y_{j+1} = \varphi_b \). We have from the way \( y_j \) is selected in \([\text{9}]\),

\[
w_0 + \cdots + w_b > y_1 + \cdots + y_j \geq w_0 + \cdots + w_{b-1},
\]

\[
w_0 + \cdots + w_a > y_1 + \cdots + y_{j-1} \geq w_0 + \cdots + w_{a-1}.
\]

Thus,

\[
y_j > w_{a+1} + \cdots + w_{b-1}.
\]

This inequality, informally speaking, certifies that the total weight of all subtrees \( T[e_s] \) with \( s \in \{a + 1, \ldots, b - 1\} \) is smaller than \( y_j \). By assumption \( z_i > y_j \). By the definition of the sequence \( z_1, \ldots, z_p \), the length of the \( i \)-th route in ADFS(\( T \)) restricted to the subtrees \( T[e_s] \) is at least \( 2z_i \). (Note that we are not using the fact that this route may avoid some subtrees \( T[e_s] \) with \( s \in \{a + 1, \ldots, b - 1\} \) as we analyze the first part of ADFS(\( T \)) which ‘avoids’ each subtree \( T[e_s] \) with \( s \in \{a + 1, \ldots, b - 1\} \cap H' \). Thus, the \( i \)-th route of ADFS(\( T \)) visits the node of \( P \) at potential \( \varphi_b \) and hence \( z_{i+1} \geq \varphi_b = y_{j+1} \) as required in the lemma.
We are now ready to bound the total cost of ADFS(T) with relation to COPT(T). The cost of COPT(T) can be decomposed:

$$\xi(\text{COPT}(T)) = O_{\text{light}} + O_{\text{deep}} + O_{\text{path}} + O_{\text{flat}}, \quad (11)$$

where:

- $O_{\text{light}}$ — is the cost restricted to light subtrees $T[e_i]$ with $i \in \{1, \ldots, k\}$,
- $O_{\text{deep}}$ — is the cost restricted to the subtree $T[r']$,
- $O_{\text{path}}$ — is the cost restricted to the path $P$ and the routes that do not contain $r'$, and
- $O_{\text{flat}}$ — is the cost restricted to the path $P$ and routes that do contain $r'$.

Similarly, we express the cost of ADFS(T) as a sum:

$$\xi(\text{ADFS}(T)) = D_{\text{light}} + D_{\text{deep}} + D_{\text{desc}} + D_{\text{flat}} + D_{\text{asc}}, \quad (12)$$

where:

- $D_{\text{light}}$ — is the cost of ADFS(T) restricted to light subtrees $T[e_i]$ with $i \in \{1, \ldots, k\}$,
- $D_{\text{deep}}$ — is the cost restricted to the subtree $T[r']$,
- $D_{\text{desc}}$ — is the cost restricted to the path $P$ in the first part of ADFS(T),
- $D_{\text{flat}}$ — is the cost restricted to $P$ and the routes that contain $r'$ (i.e., the second part of ADFS(T)), and
- $D_{\text{asc}}$ — is the cost restricted to the path $P$ in the third part of ADFS(T).

By Lemma 5, $\xi(\text{ADFS}(T[e_i])) < 2 \xi(\text{COPT}(T[e_i]))$ for each $i \in \{1, \ldots, k\}$ and therefore

$$D_{\text{light}} < 2 \cdot O_{\text{light}}. \quad (13)$$

Denote by $s$ the smallest index such that $y_{s+1} > \varphi_0 = y_s$ and let $H = \{j_1, \ldots, j_q\}$, $j_1 < j_2 < \cdots < j_q$. Recall that $H$ is the set of indices $i$ such that $T[e_i]$ is covered in the first part of ADFS(T). Since $z_{j_1} > y_s$, by iteratively applying Lemma 9 we obtain that

$$z_{j_i} \geq y_{s+i-1} \text{ for each } i \in \{1, \ldots, q\}.$$

Therefore, by Lemma 8:

$$D_{\text{desc}} = 2 \sum_{i=1}^{q} (\varphi(T) - z_{j_i}) \leq 2 \sum_{i=1}^{q} (\varphi(T) - y_{s+i-1}) \leq 2(\varphi(T) - \varphi_0) + 2 \sum_{i=2}^{q} (\varphi(T) - x_{s+i-1}) \leq 2\omega(P) + O_{\text{path}}. \quad (14)$$

By an analogous analysis, the same bound holds for the third part of ADFS(T):

$$D_{\text{asc}} \leq 2\omega(P) + O_{\text{path}}. \quad (15)$$

By the inductive assumption we have

$$D_{\text{deep}} < 10 \cdot O_{\text{deep}} - 16\varphi(r'). \quad (16)$$

We also get the following bounds by analyzing how much each particular route can overlap with $T[r']$. The first one follows from an observation that each route having a non-empty intersection with $T[r']$ may have length restricted to $T[r']$ at most $2\varphi(r')$. Thus, there
exist at least \(O_{\text{deep}}/(2\varphi(r'))\) such routes in \(\text{ADFS}(T)\) intersecting \(T[r']\) and each such route contributes at least \(2\omega(P)\) to \(O_{\text{flat}}\). Hence

\[
O_{\text{flat}} \geq \frac{\omega(P)}{\varphi(r')} \cdot O_{\text{deep}}. \tag{17}
\]

As for an upper bound, there exist at most \(\lceil D_{\text{deep}}/(2\varphi(r')) \rceil + 1\) routes in \(\text{PDFS}(T)\) containing \(r'\) (note that the first such route may include no other vertices except for \(r'\) from \(T[r']\)). Thus,

\[
D_{\text{flat}} \leq 2\omega(P) \cdot \lceil D_{\text{deep}}/(2\varphi(r')) \rceil + 1 \leq 4\omega(P) + \omega(P) \cdot D_{\text{deep}}/\varphi(r'). \tag{18}
\]

Equations (18), (16) and (17), used in that order, give us

\[
D_{\text{deep}} + D_{\text{flat}} \leq 4\omega(P) + \left( \frac{\omega(P)}{\varphi(r')} + 1 \right) \left( 10 \cdot O_{\text{deep}} - 16\varphi(r') \right)
\leq 10O_{\text{deep}} + 10O_{\text{flat}} - 12\omega(P) - 16\varphi(r').
\]

Then, by (13), (14) and (15) we have

\[
D_{\text{light}} + D_{\text{asc}} + D_{\text{desc}} \leq 2O_{\text{light}} + 4\omega(P) + 2O_{\text{path}}.
\]

The last two inequalities, (11), (12) and \(\varphi(r) = \omega(P) + \varphi(r')\) finally give

\[
\xi(\text{ADFS}(T)) \leq 10O_{\text{deep}} + 10O_{\text{flat}} + 2O_{\text{light}} + 2O_{\text{path}} - 8\omega(P) - 16\varphi(r')
\leq 10\xi(\text{COPT}(T)) - 8\varphi(r),
\]

which completes inductive proof of Theorem 6.

### 6 Conclusions and open problems

Our strategy \(\text{PDFS}(T)\) achieves the asymptotically minimum number of routes and also minimizes the cost up to a small constant. In particular, we provided an upper bound of 10 for the competitiveness of any online piecemeal exploration strategy. A trivial lower bound for the same problem is 3/2 (Consider the tree with three branches of length \(B/2, B/2, B\) starting from the root: Any online algorithm may cover the tree with 3 routes, while the optimal is 2 routes). This leaves a gap between the lower and upper bounds and the interesting open question of whether the strategy \(\text{PDFS}(T)\) is the best possible algorithm. Another open problem is to analyze similar strategies in other classes of graphs instead of trees.

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Appendix

A. Tree rearrangement

We start with an informal description providing a high level intuition that gives an overview of this section. Our first step (Section A.1) is to construct a tree $T_\varepsilon$, $\varepsilon \in \mathbb{R}^+$, based on $T$ such that $T_\varepsilon$ satisfies property \( \text{(P1)} \). We will also need that $T_\varepsilon$ ‘resembles’ $T$ in the following way: a search strategy is valid for $T$ if and only if a ‘very similar’ strategy is valid for $T_\varepsilon$. To simplify the statements considerably, it will be convenient to encode strategies in an uniform way so that we can apply the same strategy for both trees, without going into the details of tedious but straightforward conversions between strategy for $T$ and strategy for $T_\varepsilon$. We thus define (Section A.2) a collection $\mathcal{C}$ of all possible strategies (including adversarial ones and those that are not feasible for either $T$ or $T_\varepsilon$ because they contain routes that are too long or do not visit all vertices). Then in Section A.3 we select the right value of $\varepsilon$. The value of $\varepsilon$ and the construction of $T_\varepsilon$ will ensure that a strategy in $\mathcal{C}$ is valid for $T$ if and only if it is valid for $T_\varepsilon$. We then finally provide the main result of this section (Lemma 7) states that, if Theorem 6 holds for $T_\varepsilon$, then it holds for $T$, thus allowing us to restrict only to trees satisfying property \( \text{(P1)} \).

A.1 The construction of $T_\varepsilon$

We now construct the tree $T_\varepsilon = (V(T) \cup X, E(T_\varepsilon), \omega')$ based on $T = (V(T), E(T), \omega)$ and the construction depends on a parameter $\varepsilon > 0$ that will be fixed later. We now impose only a condition on $\varepsilon$ that is needed for the construction itself to be valid:

$$\varepsilon < \min\{\omega(e) \mid e \in E(T)\}.$$  

Select an arbitrary vertex $u$ in $T$ with $\text{heavydeg}(u) = 1$ (denote by \{u, v\} the heavy downward edge at $u$) and $d > 1$ light downward edges $e_1, \ldots, e_d$ at $v$. Subdivide the edge $e$ (see Figure 4 for an illustration) by replacing it by a path $P = (u = v_0, v_1, \ldots, v_{d-1}, v_d = v)$ with the following edge lengths: the first $d - 1$ edges have length $\varepsilon/(d - 1)$, i.e., $\omega'(v_i, v_{i+1}) = \varepsilon/(d - 1)$ for each $i \in \{0, \ldots, d - 2\}$, and for the last edge we set $\omega'(v_{d-1}, v_v) = \omega(v, v) - \varepsilon$. (Note that this preserves the distance between $u$ and $v$.) Then, the weight of each edge $e_i$ decreases by $\varepsilon$, $\omega'(e_i) = \omega(e_i) - \varepsilon$ and the higher endpoint of $e_i$ in $T_\varepsilon$ becomes $v_{i-1}$, $i \in \{1, \ldots, d\}$. (Note that this ensures that the distance between $u$ and the lower endpoint of $e_i$ or the distance between two children of $u$ is not greater in $T_\varepsilon$ than in $T$.) This construction allows us to assume (by permuting the edges $e_1, \ldots, e_d$ appropriately) that the DFS traversal of $T_\varepsilon$ visits the edges $e$ and $e_1, \ldots, e_d$ in the same order both in $T$ and in $T_\varepsilon$, ensuring that Condition \( \text{(P2)} \) is satisfied. Since the vertex $v$ is selected arbitrarily, we repeat the above modification for each such vertex $v$ obtaining the final tree $T_\varepsilon$.

We will require the following conditions to be satisfied:

- \( \text{(P1)} \) there are at most two downward edges at each vertex in $T_\varepsilon$ with $\text{heavydeg}(v) = 1$,
- \( \text{(P2)} \) there exists a DFS traversal of $T_\varepsilon$ that visits the vertices in $V(T)$ in the same order as the DFS traversal that we have fixed for $T$ in this work,
- \( \text{(P3)} \) $S \in \mathcal{C}$ is feasible for $T$ if and only if $S$ is feasible for $T_\varepsilon$.

Property \( \text{(P3)} \) will be proved in Lemma 13 and we now note:

\[ \textbf{Observation 10.} \text{ For each } \varepsilon > 0, \text{ the tree } T_\varepsilon \text{ satisfies Conditions } \text{(P1)} \text{ and } \text{(P2)}. \]
Observation 11. The strategies ADFS(T) and ADFS(T_\varepsilon) visit the nodes in V(T) in the same order.

A.2 Finding the right value of \varepsilon

We define a potential route as a following pair: R' = (L,v), where L = (l_1,\ldots,l_p) is a sequence of leaves and v \in V(T_\varepsilon). Then, R' translates to a route R in T as a concatenation of the following paths (in this order): the path from r to l_1, the path from l_i to l_{i+1}, i = 1,\ldots,p-1, the path from l_p to the closest ancestor x of v that belongs to V(T) and finally the path from x to r. The length of R is

\[ \ell(R) = d(r,l_1) + d(l_p,x) + d(x,r) + \sum_{1 \leq i < p} d(l_i,l_{i+1}). \]

R' translates to a route in T_\varepsilon in the same way, except that take x = v, i.e., v is not replaced by the ancestor. Then, a potential strategy is a sequence consisting of at most j potential routes, j \in \{1,\ldots,p\}, where p is the number of leaves in T.

Note that a potential strategy may not translate to a valid B-exploration strategy for T or T_\varepsilon because some nodes may not be explored and some routes may be too long. A potential strategy is feasible for T (respectively T_\varepsilon) if it translates to a valid B-exploration strategy for T (respectively T_\varepsilon). We denote by \mathcal{C} a collection of all potential strategies. Clearly, the size of \mathcal{C} is finite.

We conclude with the following:

Observation 12. For any route R that may appear in ADFS(T), COPT(T), ADFS(T_\varepsilon) and COPT(T_\varepsilon) there exists a potential route that translates to R.

A.3 The analysis of T_\varepsilon

In this section we argue that the construction of T_\varepsilon ‘preserves’ the problem: the minimum costs of adversarial DFS explorations of both T and T_\varepsilon, as well as COPT(T) and COPT(T_\varepsilon) remain close to each other for \varepsilon small enough. Intuitively speaking, this follows from a ‘continuity argument’ formalized in the remaining part of this section.

For the tree T_\varepsilon we define an interval denoted by I = (0,y), y < \min\{\omega(e) \mid e \in E(T)\}, such that for each \varepsilon \in (0,y), Condition [P3] holds for T_\varepsilon. We now prove that this interval is well defined, that is, y > 0. (Note that for \varepsilon = 0, T_\varepsilon and T are the same.)
Lemma 13. It holds $I \neq \emptyset$.

Proof. First we argue that there exists $y > 0$ such that $T_y$ fulfills Condition [P3]. We select $y$ based on the tree $T$ and the collection $C$. Consider any $B' \in B$. The number of potential strategies in $C$ is finite, and hence the number of potential strategies in $C$ that do not translate to feasible ones for $T$ (denote subset of those by $C'$) is also finite. For each $S \in C'$, define its deficiency $x(S)$ as follows: the $x(S)$ is the maximum value such that either the length of the first route in $S$ is $B' + x(S)$ or the length of some other route in $S$ in $T$ is $B + x(S)$. Intuitively, $S$ does not translate to a feasible $B'$-adversarial $B$-exploration strategy for $T$ because one of its routes exceeds the allowed length by $x(S)$ and no route exceeds it by more than $x(S)$. Take

$$y := \frac{1}{2n(m+1)} \cdot \min \{ \min \{ \omega(e) \mid e \in E(T) \}, \min \{ x(S') \mid S' \in C' \} \},$$

where $m$ is the number edges in $T$. Since by definition, $x(S) > 0$ for each $S \in C'$, we obtain that $y > 0$. Also, no route in any $S \in C'$ traverses an edge more than $2m$ times and hence the length of any route of $S$ in $T_y$ decreases by at most $2gmn$ with respect to its length in $T$. This implies that some route $R$ of $S$ has length in $T_y$ at least

$$\tilde{B} + x(S) - 2gmn \geq \tilde{B} + 2gm(m+1) - 2gmn > \tilde{B},$$

where take $\tilde{B} = B'$ if $R$ is the first route in $S$ and $\tilde{B} = B$ otherwise. Therefore, $S$ remains unfeasible in $T_y$. Since exploration strategy that is feasible in $T$ remains feasible in $T_y$ (recall that the length of each route is smaller in $T_y$ than in $T$), we have that Condition [P3] holds for $T_y$.

Finally, observe that substituting $y$ by any value $\varepsilon$ smaller than $y$ in the left hand side of (19) keeps this equation true. Therefore, Condition [P3] is satisfied by $T_\varepsilon$ for each $\varepsilon \in (0, y)$, which completes the proof. ▶

Before we state the mainlemma of this section, we prove these technical bounds:

Lemma 14. For each $\varepsilon \in I$ it holds:

(i) $\xi(\text{ADFS}(T)) \leq \xi(\text{ADFS}(T_\varepsilon)) + 4\varepsilon n^2$,

(ii) $\varphi(T_\varepsilon) \leq \varphi(T)$, and

(iii) $\xi(\text{COPT}(T_\varepsilon)) \leq \xi(\text{COPT}(T)).$

Proof. By Observation [11] both $\text{ADFS}(T)$ and $\text{ADFS}(T_\varepsilon)$ visit the leaves of both trees in the same order. Consider any edge $\{u, v\}$ in $T$ such that $v$ is its lower endpoint. We have that there exists a corresponding edge $\{u', v\}$ in $T_\varepsilon$. Moreover, $\omega(\{u, v\}) \leq \omega'(\{u', v\}) + \varepsilon$. Since for each edge traversal in $\text{ADFS}(T)$, a traversal of the corresponding edge occurs in $\text{ADFS}(T_\varepsilon)$ due to Condition [P3] we obtain that if $t$ is the number of edge traversals in $\text{ADFS}(T)$, then

$$\xi(\text{ADFS}(T)) \leq \xi(\text{ADFS}(T_\varepsilon)) + t\varepsilon \leq \xi(\text{ADFS}(T_\varepsilon)) + 4n^2\varepsilon.$$

The latter inequality follows from bounding each route in $\text{ADFS}(T_\varepsilon)$ to have at most $2n$ edges, bounding the number of routes by $n$ and observing that an edge is traversed at most twice in each route.

Property [II] is a direct consequence of the construction of $T_\varepsilon$: any path in $T$ corresponds to a path in $T_\varepsilon$ that connects the same common nodes and contains all common edges of the original path.

By construction of $T_\varepsilon$, the distance between any common nodes $u \in V(T)$ and $v \in V(T)$ is not greater in $T$ than in $T_\varepsilon$, which immediately gives

▶
Lemma 7. There exists \( \varepsilon > 0 \) such that if Theorem 6 holds for \( T_{\varepsilon} \), then Theorem 6 holds for \( T \).

Proof. We will analyze condition (i) in Theorem 6 and the proof for (ii) is analogous as we note at the end of the proof.

Define a parameter \( \tau(n') \) to be the maximum number for which an inequality

\[
\xi(\text{ADFS}(T')) \leq 10 \cdot \xi(\text{COPT}(T')) - 8 \cdot \varphi(T') - \tau(n')
\]

holds for each tree \( T' \) on at most \( n' \) nodes that satisfies Condition (i). Since the number of such trees is finite and the number of potential strategies in \( \mathcal{C} \) is finite for each tree \( T' \), we obtain that \( \tau(n') > 0 \). By Lemma 13, there exists \( \varepsilon \in \mathcal{I} \) such that \( \varepsilon > 0 \) and

\[
\varepsilon < \frac{\tau(2n)}{4n^2},
\]

\( n = |V(T)| \), such that \( T_{\varepsilon} \) satisfies Condition (P3). Suppose that Theorem 6(i) holds for \( T_{\varepsilon} \). Then, by Lemma 14 (in particular, (i) of Lemma 14 is used to obtain the first inequality below and (ii) and (iii) are used to obtain the third inequality below) we get:

\[
\xi(\text{ADFS}(T)) < \xi(\text{ADFS}(T_{\varepsilon})) + 4\varepsilon n^2 \\
\leq 10 \cdot \xi(\text{COPT}(T_{\varepsilon})) - 8 \cdot \varphi(T_{\varepsilon}) + 4\varepsilon n^2 - \tau(2n) \\
\leq 10 \cdot \xi(\text{COPT}(T)) - 8 \cdot \varphi(T) + 4\varepsilon n^2 - \tau(2n) \\
< 10 \cdot \xi(\text{COPT}(T)) - 8 \cdot \varphi(T).
\]

We can conduct the same argument for Theorem 6(ii) with the same value of \( \varepsilon \). Hence we obtain that Theorem 6 holds for \( T \). ▶