Theory of tuning ‘tunnelling-probability’ through potential barrier by acoustically augmented phonons (AAP)

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Abstract
This paper attempts to theoretically establish the technical possibility of manually controlling the tunnelling probability of phonons through potential barrier by superimposing Ultra High Frequency (UHF) acoustic wave over the source of phonons transmission, which results into Acoustically Augmented Phonons (AAP). Tuneable high fidelity acoustic piezoelectric transducer emitters may be used to tune the tunnelling. If the probability of reflectivity of phonon (P) during tunnelling through the potential barrier is given by P(R)_p and that of Acoustically Augmented Phonons (AAP) is P(R)_AAP then it is analytically obtained that P(R)_AAP = ∝ P(R)_p. Where ∝ is dimensionless number and is defined as the augmentation factor which is dependent on structure based damping coefficient, wave number of superimposed UHF acoustic wave and phonon’s kinetic energy. Enhancement of tunnelling probability by mechanically superimposing UHF acoustic wave over the in situ phonons is independent of initial UHF acoustic wave’s amplitude. It is noted that the range of ‘tunnelling enhancement AAP’S wave numbers’ (or frequencies) gradually broadens with increase in the kinetic energy of phonons and also gradually tends to a limit. Thus, existence of almost zero reflectivity of tunnelling probability through the potential barrier by Acoustically Augmented Phonons (AAP) is theoretically obtained. A general observation is that to achieve very low reflectivity of phonon tunnelling probability ‘structural damping factor’ must be small enough than superimposed UHF of acoustic wave number.

1. Introduction

Phonons are a quantum mechanical version of a special type of vibrational motion, known as normal modes in classical mechanics, in which each part of a lattice oscillates with the same frequency. These normal modes are important because, according to a well-known result in classical mechanics, any arbitrary vibrational motion of a lattice can be considered as a superposition of normal modes with various frequencies; in this sense, the normal modes are the elementary vibrations of the lattice. Although normal modes are wave-like phenomena in classical mechanics, they acquire certain particle-like properties when the lattice is analyzed using quantum mechanics. They are then known as phonons.

Acoustic phonons have been extensively studied particularly in current century. Concerning the coupling to acoustic phonons, photoluminescence measurements in single II–VI CdTe/ZnTe QD’s (Besombes et al 2001) have shown that, by increasing the temperature, the line shape progressively deviates from the expected Lorentzian profile, with the appearance of acoustic phonon sidebands.

Concept of tuning the width of quantum well was used by Petukhov et al (2002) on spin-dependent resonant tunnelling, which dramatically enhanced tunnelling magnetoresistance. They considered double-barrier structures comprising a semiconductor quantum well between two insulating barriers and two ferromagnetic electrodes. By tuning the width of the quantum well, the lowest resonant level can be moved into the energy interval where the density of states for minority spins is zero. This leads to a great enhancement of the
magnetoresistance, which exhibits a strong maximum as a function of the quantum well width. They demonstrate that magnetoresistance exceeding 800% is achievable in GaMnAs/AlAs/GaAs/AlAs/GaMnAs double-barrier structures. Favero et al (2003) presented experimental and theoretical study of the existence of acoustic phonon sidebands in the emission line of single self-assembled InAs/GaAs quantum dots.

Alexander et al (2007) discussed the reduction of the size of electronic devices below the acoustic phonon mean free path creates a new situation for the phonons propagation and interaction. From one side, it may complicate heat removal from the down-scaled devices. From the other side, it opens up an opportunity for engineering phonon spectrum in nanostructured materials, and achieving enhanced operation of nanoscale devices. They review the development of the nanoscale phonon engineering concept and discuss possible device applications. The focus of the review is on tuning the phonon spectrum in the acoustically mismatched nano- and heterostructures in order to change the ability of semiconductors to conduct heat or electric current. New approaches for the electron–phonon scattering rates suppression and improvement of the carrier mobility as well as for formation of the phonon stop-bands are discussed. The phonon engineering concept can be potentially as powerful as the band gap engineering, which led to some ground-breaking developments in the electronics.

Tian et al (2010) studied Phonon wave-packet interference and phonon tunnelling based energy transport across nanostructured thin films. The molecular dynamics based phonon wave-packet technique is used to study phonon transport across mass-mismatched fcc thin films. Transport behaviour of normally incident longitudinal acoustic phonon wave packets with wave vectors ranging in magnitude from 2% to 50% of the first Brillouin zone boundary is examined as a function of thin film thickness when the phonon mean free path exceeds film thickness. The results indicate that for thin film to bulk solid mass ratios up to a factor of 6, the transmission of energy through the thin film can be well described by treating the thin film as a bulk solid.

In the present paper an attempt has been made to theoretically establish the feasibility of tuning (increasing or decreasing) of tunnelling-probability with the help of Acoustically Augmented Phonons (AAP). For completeness section 2 begins with basic solutions of Schrödinger equation. Readers interested in details may refer appendix A. Mathematical details of superposition of ultrahigh frequency acoustic wave over the in situ phonon, is presented in section 3. Resulting phonon state is termed as Acoustically Augmented Phonon (AAP). Mathematical expression for the important Phonon augmentation factor (termed as ∝-factor) has been derived. Discussion is presented in section 4 and lastly results are summarised and concluded in section 5. To help readers compare the order of a few parameters and analyse, one table has been included in appendix B. All units in this paper are in MKS system only.

2. Quantum mechanical tunnelling through potential barrier

We consider simple case of a potential barrier (figure 1) to study the penetration of wave function through $V_0$ barrier i.e. the ability of particles to ‘tunnel’ through the barrier of $V_0$, exceeding their kinetic energy $E$.

It is well known (Mathews and Venkatesan 2007) that if wave function $\psi(x, t) = U'(x) T(t)$ then Schrödinger equation gives

$$T(t) \propto e^{-i(\frac{\pi}{2})}.$$

Where $\psi(x, t)$ is wave function and $U'(x)$ and $T(t)$ are its components. $x$ and $t$ are distance and time respectively.

Let $U'(x) = U_k(x)$ for kinetic energy $E$. Where $U_k(x)$ is solution of time independent Schrödinger equation for specific value of $E$. $E$ must be real; for imaginary part exists then wave function would vanish for large time. It is set of all the admissible values of $E > 0$ from Eigen value Energy Spectrum. If $C_−/C_+$ is the amplitude of reflection at $x = a$ then,
\[
\frac{C_-}{C_+} = \frac{-[(k^2 + r^2) \sinh (r(b - a))]}{-[(k^2 + r^2) \sinh (r(b - a)) + 2irk \cosh (r(b - a))]} e^{2iak} = Z e^{2iak} 
\]  
(1)

Where
\[
Z = \frac{B}{C + iD} \text{(say)} 
\]  
(2)

(Refer appendix A for the solution).

On right hand side of equation (1) k and r are having dimension of \( \text{m}^{-1} \) and a and b are having dimension of \( \text{m} \). Z is dimensionless. Hence expression \( (C_-/C_+) \) is dimensionless.

B is the numerator of coefficient of \( e^{2iak} \) in equation (1) and C and D are the real and imaginary parts of its denominator. Equation (2) gives the amplitude of reflection. Reflection probability (\( P(R)_p \)) is given by \( \left| \frac{C_-}{C_+} \right|^2 \).

\[
\left| \frac{C_-}{C_+} \right|^2 = \frac{(k^2 + r^2)^2 \sinh^2 (r(b - a))}{(k^2 + r^2)^2 \sinh^2 (r(b - a)) + 4r^2k^2 \cosh^2 (r(b - a))} 
\]

\[
= \left\{ \frac{k^2 + r^2}{k^2 + r^2} + \frac{4r^2k^2}{(k^2 + r^2)^2} \coth^2 (r(b - a)) \right\}^{-1} 
\]

= Square of modulus of \( Z \) in equation (2) = \( P(R)_p \) (Say)  
(3)

3. Ultrasonic wave superposition

Now let at any instant if one dimensional ultrasonic source \( U(x) \) is triggered at \( x = 0 \) to move towards the potential barrier the acoustic wave will travel from left to right, then \( U(x) \) may be defined as

\[
U(x) = \begin{cases} 
A e^{-\beta x} e^{iny}, & x \geq 0 \\
0, & x < 0 
\end{cases} 
\]  
(4)

Where A is the initial wave amplitude (\( \text{m} \)) at \( x = 0 \). \( \beta \) is the damping coefficient per unit length. It is material property e.g. if initially the amplitude was \( A_1 \) and after a meter it became \( A_2 \) then \( \beta = (A_2/A_1) \text{ m}^{-1} \). \( \lambda \) is wavelength of acoustic wave and \( n \) is the wave number then \( n = 2\pi/\lambda \) of UHF source in \( \text{m}^{-1} \). Desired high fidelity acoustic wave frequency can be generated (Thomas 1982) by the piezoelectric transducer.

If \( u_1, u_2 \) and \( u_3 \) are the \( x \) dependent components of wave functions, while transiting the regions I, II and III, respectively. Then the final wave output after the superposition of UHF acoustic wave could be obtained by taking the convolution of \( u_1, u_2 \) and \( u_3 \) (refer equations (A1)–(A3) of appendix A) with equation (4) respectively. This would result into Acoustically Augmented Phonons (AAP). Boundary conditions remain unchanged. Hence region wise three expected AAP’s solutions are mentioned below:

**Region I**

\[
U(x)^*u_1 = \int_{-\infty}^{\infty} A e^{-\beta x} e^{iny} \{ C_+ e^{i(kx-y)} - C_- e^{-i(kx-y)} \} dy, \quad (Convolution) 
\]

For \( x < 0 \), \( U(x) = 0 \), we have

\[
= C_+ A e^{ikx} \int_0^{\infty} e^{-i(\beta - in + ik)y} dy + C_- A e^{-ikx} \int_0^{\infty} e^{-i(\beta - in - ik)y} dy 
\]

\[
= C_+ A e^{ikx} \frac{e^{-i(\beta - in + ik)y}}{-i(\beta - in + ik)} \bigg|_0^{\infty} + C_- A e^{-ikx} \frac{e^{-i(\beta - in - ik)y}}{-i(\beta - in - ik)} \bigg|_0^{\infty} 
\]

\[
= \frac{C_+ A e^{ikx}}{\beta - in + ik} + \frac{C_- A e^{-ikx}}{\beta - in - ik} = \frac{C_+ A e^{ikx}}{\beta - i(n-k)} + \frac{C_- A e^{-ikx}}{\beta - i(n+k)}, \quad [\because \beta > 0] 
\]  
(5)

**Region II**

\[
U(x)^*u_2 = \int_{-\infty}^{\infty} A e^{-\beta x} e^{iny} \{ A_+ e^{i(kx-y)} + A_- e^{-i(kx-y)} \} dy, \quad (Convolution) 
\]
For \( x < 0 \), \( U(x) = 0 \), we have

\[
U(x) = A_+ e^{\alpha x} \int_{0}^{\infty} e^{-(\beta + i\eta + r - r')} dy + A_- e^{-\alpha x} \int_{0}^{\infty} e^{-(\beta + i\eta + r') - r} dy
\]

\[
= A_+ e^{\alpha x} \left[ \int_{0}^{\infty} e^{-(\beta + i\eta + r)} dy \right]_{\eta = 0}^{\beta + i\eta} + A_- e^{-\alpha x} \left[ \int_{0}^{\infty} e^{-(\beta - i\eta - r)} dy \right]_{\eta = 0}^{\beta - i\eta}
\]

\[
= \frac{A_+ e^{\alpha x}}{\beta - in + r} + \frac{A_- e^{-\alpha x}}{\beta - in - r} = \frac{A_+ e^{\alpha x}}{(\beta + r) - in} + \frac{A_- e^{-\alpha x}}{(\beta - r) - in}, \quad [\because \beta > 0]
\]

**Region III**

\[
U(x)_{u_3} = \int_{-\infty}^{\infty} A e^{-\beta e^{im}D_r e^{ik(x-y)/\alpha}} dy, \quad \text{(Convolution)}
\]

For \( x < 0 \), \( U(x) = 0 \), we have

\[
U(x)_{u_3} = AD_+ e^{ikx} \int_{0}^{\infty} e^{-(\beta - i\eta - ik)} dy
\]

\[
= AD_+ e^{ikx} \left[ \int_{0}^{\infty} e^{-(\beta - i\eta - ik)} dy \right]_{\eta = 0}^{\beta - i\eta} = \frac{AD_+ e^{ikx}}{\beta - i(n - k)}, \quad [\because \beta > 0]
\]

If \( u_{1A}, u_{2A} \) and \( u_{3A} \) denote \( x \)-dependent wave function components in regions I, II and III (figure 1), respectively, after the superposition of UHF acoustic wave and \( C_+ \), \( C_- \) are constants of corresponding version of equation (A1) (appendix A) after the superposition of UHF acoustic wave then Acoustically Augmented Phonons (AAP) solutions for three regions can be summarized as under,

**Region I** \((0 < x < a)\)

\[
u_{1A} = \frac{AC_+ e^{ikx}}{\beta - i(n - k)} + \frac{AC_- e^{-ikx}}{\beta - i(n + k)}
\]

**Region II** \((a < x < b)\)

\[
u_{2A} = \frac{AA_+ e^{ikx}}{(\beta + r) - in} + \frac{AA_- e^{-ikx}}{(\beta - r) - in}
\]

**Region III** \((b < x)\)

\[
u_{3A} = \frac{AD_+ e^{ikx}}{\beta - i(n - k)}
\]

Amplitude of reflectivity for AAP may be given by

\[
\frac{C_+}{C_-} = \frac{\frac{AC_+}{\beta - i(n + k)}}{\frac{AC_-}{\beta - i(n - k)}} = \left( \frac{C_+}{C_-} \right) \left( \frac{\beta - i(n - k)}{\beta - i(n + k)} \right) = B \left( \frac{\beta - i(n - k)}{\beta - i(n + k)} \right) = \frac{B^2(\beta^2 + n^2 + k^2 - 2nk)}{(C + iD)(\beta^2 + n^2 + k^2 + 2nk)}
\]

It may be noted that left hand side term is dimensionless. Probability of AAP reflectivity \((P(R)_{AAP})\) is given by

\[
\left| \frac{C_+}{C_-} \right| = \left| \frac{\beta - i(n - k)}{\beta - i(n + k)} \right| = \frac{B^2(\beta^2 + n^2 + k^2 - 2nk)}{(C^2 + D^2)(\beta^2 + n^2 + k^2 + 2nk)}
\]

Equation (12) indicates that probability of reflectivity depends on augmentation probability factor \((\beta^2 + n^2 + k^2 - 2nk)/(\beta^2 + n^2 + k^2 + 2nk) = \infty\) due to AAP, \(\infty\) is dimensionless number and such that \([\beta^2/}(\beta^2 + 4 n^2)] < \infty < 1\) for all \(\beta, n, k > 0\). We will analyse \(\infty\)-factor in section 4, in detail.

Comparing (3) and (12) we see that

\[
P(R)_{AAP} = \infty. P(R)_{PA}, \text{ where } \alpha = \frac{\beta^2 + n^2 + k^2 - 2nk}{\beta^2 + n^2 + k^2 + 2nk}
\]

It is noted that \(\alpha\)-factor is independent of \(r\) and amplitude \((A)\) of UHF acoustic wave. It is function of \(\beta, n\) and \(k\) only. \(r\) relates to energy deficit, \((V_0 - E)\). Hence reflectivity is independent of height of the barrier \(V_0\). \(\beta\) depends on medium and \(n\) characterizes the UHF acoustic wave.
4. Discussion

$\alpha$-factor in equation (13) is result of superimposition of acoustic wave over the phonons. As UHF acoustic wave can be artificially generated the technique might be utilized for regulating the tunnelling process, hence might be of immense utility in design and development of equipments based on tunnelling principles.

It can be noted from figure 2 that as the value of $k$ increases, range of tuneable acoustic wave numbers (frequencies) also broadens which finally approaches a limit for specific value of $n$; where almost no reflectivity is observed; since $\alpha$ reaches its lowest value ($\approx [\beta^2/(\beta^2 + 4 n^2)$]) in the limit. This expression is structure specific and can be varied by tuning $n$.

Since $[\beta^2/(\beta^2 + 4 n^2)] \approx 0$ for the $\beta \ll n$ (figure 2), hence the limiting condition for achieving minimum reflectivity vis-a-vis maximum tunnelling is that $\beta$ must be small enough than the magnitude of $n$.

Figure 2 shows value of $\alpha$-factor as $n$ varies for discrete $k$ values and fixed $\beta = 10^4$. It may be noted that $n \ll k$ is necessary condition for decreasing trend though not sufficient. Also with the increase of value of $n$ along $x$-axis, $\alpha$ decreases and reaches its lowest value before again rising, for all values of $k$. The lowest value on the curve for any specific value of $k$ could be the indicator of the lowest probability of reflectivity in tunnelling process by the Acoustically Augmented Phonons (AAP), corresponding to specific $n$. With this condition it may be stated that probability of tunnelling increases as $n$ tends to $k$ from left. Secondly tunnelling probability by the Acoustically Augmented Phonons (AAP) may be made to tend to its highest value as $k$ is made to increase monotonically. As $n$ increases $\alpha$-factor initially shows decreasing trend but broadest range of decrease of $\alpha$-factor is observed when $k = 10^{5.5}$ and $n \ll k, \beta = 10^4$. The resulting curve shown in figure 2 may be simulated by six order polynomial;

$$\alpha = 3 \times 10^{-6} n^6 - 2 \times 10^{-5} n^5 - 0.000 n^4 + 0.005 n^3 - 0.024 n^2 + 0.033 n + 0.972,$$

with coefficient of determination $R^2 = 1$. Hence given any specific medium parameter $\beta$ equation (14) signifies scope of achieving minimum probability of reflectivity vis-a-vis optimum tunnelling scope by any artificially superimposed UHF acoustic wave. Equation (14) remains unchanged for specific medium parameter $\beta$. Hence it is material dependent.

To support tunnelling these observations could be integrated as most favourable combination of UHF acoustic wave number and phonon parameters as;

$$\beta < n \leq k.$$  \(\text{(15)}\)

Table in appendix B shows values of $\alpha$ as $\beta$ varies for discrete $k$ values and fixed $n = 10^4$. Corresponding graphs are shown in figure 3. Bold values in the table (appendix B) indicate significant rise in $\alpha$ values. For these $\alpha$ values corresponding $\beta/k$ values are shown in the adjacent columns.

It may be noted that till order of $\beta$ is one order less than $k$ or $O(\beta) < (1/10)O(k)$ there is no significant change in $\alpha$ values vis-a-vis AAP probability of reflectivity for any $k$ value. It however, rapidly rises as if $O(\beta) > (1/10)O(k)$. Hence one of the favourable conditions for decreased reflectivity, favouring tunnelling, could be...
Summing up favourable conditions from equations (15) and (16) it may be stated that most favourable condition for reduced reflectivity vis-a-vis enhanced tunnelling is

\[ \beta < \frac{\beta}{k} \]

Hence, the general condition for enhancing the tunnelling probability by mechanically superimposing UHF acoustic wave number \( n \) while the structural damping coefficient per unit length of any medium is \( \beta \), over the \( in \) \( situ \) phonons is \( \beta \ll n \leq k \). Such phenomena may be realized in the spatial confinement in low-dimensional structures such as thin film, nanowires etc.

Reverse of the enhancement favouring conditions would lead to decrease in tunnelling. Either of the case may be achieved by simply varying the superimposed UHF wave number (frequency) through 'frequency-regulator' attached with high fidelity acoustic-transducer (Thomas 1982).

5. Conclusions

It is possible to tune tunnelling by Acoustically Augmented Phonons (AAP)—generated by superposing the ultrahigh frequency acoustic waves over the \( in \) \( situ \) phonons. Probability of reflectivity \( (P(R)_{AAP}) \) from a potential barrier for AAP is independent of the height of the barrier \( V_0 \) and initial amplitude \( 'A' \) of UHF acoustic wave. If probability of reflectivity of phonon from the potential barrier is \( P(R)_p \) then its relation with \( P(R)_{AAP} \) is given by,

\[ P(R)_{AAP} = \infty \cdot P(R)_p \]

Where \( \infty = \left( \frac{\beta + n^2 + k^2 - 2nk}{\beta + n^2 + k^2 + 2nk} \right) \) is non-dimensional Augmentation Probability Factor.

The range of tunnelling-enhancement wave numbers (frequencies) of UHF waves increase as the kinetic energy of phonons increase and tends to a limit. The lowest point on the curve for any specific value of \( k \) could be the indicator of the lowest probability of reflectivity in tunnelling process by corresponding Acoustically Augmented Phonons (AAP), specific to superimposed UHF wave number ‘n’. Necessary but not sufficient condition for \( \infty \)-factor to reduce is \( n \leq k \). With this condition it may be stated that probability of reflectivity decreases vis-a-vis that of tunnelling increases, as \( n \) tends to \( k \) from left. Tunnelling probability by the Acoustically Augmented Phonons (AAP) may be made to tend to its highest limit, with almost no reflectivity, as \( k \) is made to increase monotonically. Till order of \( \beta \) is one order less than \( k \) or \( O(\beta) < \left( \frac{1}{10} \right) O(k) \) there is no significant change in \( \infty \) values vis-a-vis AAP probability of reflectivity for any \( k \) value. It however, rapidly rises when;

\[ O(\beta) > \left( \frac{1}{10} \right) O(k). \]

General condition for enhancing the tunnelling probability by mechanically superimposing UHF acoustic wave number \( n \) while the damping coefficient per unit length of any medium is \( \beta \), over the \( in \) \( situ \) phonons, may be
Given any specific structure (with medium parameter \( \beta = 10^4 \text{ m}^{-1} \)) six order polynomial

\[
\alpha = 3 \times 10^{-6}n^6 - 2 \times 10^{-5}n^5 - 0.0004n^4 + 0.005n^3 - 0.024n^2 + 0.033n + 0.972,
\]

signifies scope of achieving minimum probability of reflectivity vis-a-vis optimum tunnelling scope by any artificially superimposed UHF acoustic wave. Polynomial will vary with \( \beta \) i.e. with material structures.

The research might be of immense utility in design and development of equipments based on tunnelling principles.

### Appendix A

Mathews and Venkatesan (2007) have described regions I, II and III as shown in figure 1. Potential function (on \( y \)-axis of figure 1) may be defined by

\[
V(x) = 0 \text{ for } 0 < x < a \text{ (Region – I)} \\
V(x) = V_0 \text{ for } a < x < b \text{ (Region – II)} \\
V(x) = 0 \text{ for } b < x \text{ (Region – III)}
\]

If \( E \) and \( m \) are the kinetic energy and mass of the particle and \( h = \frac{h}{2\pi} \) (Plank’s Constant) then for \( E > 0 \), \( \frac{2mE}{h^2} \) is positive hence we can write \( \frac{2mE}{h^2} = k^2 \) and \( \frac{2m(V_0 - E)}{h^2} = r^2 \), \( V_0 > E > 0 \) hence \( k, r > 0 \)

Standard equation and region wise solution of Schrodinger equation for wave function motion from left to right in figure 1, for each region is as under

- **Region – I**
  \[
  \frac{d^2u}{dx^2} + k^2u = 0 \\
  u_1 = C_+e^{ikx} + C_-e^{-ikx}(A1)
  \]

- **Region – II**
  \[
  \frac{d^2u}{dx^2} - r^2u = 0 \\
  u_2 = A_1e^{ax} + A_2e^{-ax}(A2)
  \]

- **Region – III**
  \[
  \frac{d^2u}{dx^2} + k^2u = 0 \\
  u_3 = D_1e^{ax} + D_2e^{-ax}
  \]

Where \( A_1, A_2, C_+, C_-, D_1, D_2 \) are all constants and \( u_1, u_2 \) and \( u_3 \) denote \( x \)-dependent components of wave functions in regions I, II and III (figure 1), respectively without superposition of UHF acoustic wave. Boundary conditions at \( x = a \) and \( x = b \) are given by the continuity of \( u \) and its derivatives with respect to \( x \).

- **Region – I**
  \[
  u_1 = u_2 \text{ and } \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x}
  \]
  \[
  u_2 = u_3 \text{ and } \frac{\partial u_2}{\partial x} = \frac{\partial u_3}{\partial x}
  \]
  \[
  \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x}
  \]

- **Region – II**
  \[
  \frac{\partial u_2}{\partial x} = \frac{\partial u_3}{\partial x}
  \]
  \[
  \frac{\partial u_3}{\partial x} = \frac{\partial u_2}{\partial x}
  \]

Solving equations with help of boundary conditions we get,

\[
\frac{C_-}{C_+} = \frac{i(r^2 + k^2)(1 - e^{2i(r-b-a)})e^{2iak}}{i(r^2 - k^2 + 2irk) + i(r^2 - k^2 - 2irk)e^{2i(r-b-a)}}
\]

\[
= \frac{i(r^2 + k^2)(1 - e^{2i(r-b-a)})e^{2iak}}{i[-(r^2 - k^2 + 2irk) + (r^2 - k^2 - 2irk)e^{2i(r-b-a)}]}
\]

\[
= \frac{(r^2 + k^2)(1 - e^{2i(r-b-a)})e^{2iak}}{-(k^2 + r^2)(e^{2i(b-a)} - 1) - 2irk(e^{2i(r-b-a)} + 1)}
\]

\[
= \frac{[k^2 + r^2] \sinh (r(b-a)) + 2irk \cosh (r(b-a))] e^{2iak}}{k^2 + r^2 \sinh (r(b-a)) + 2irk \cosh (r(b-a))} = Z e^{2iak}
\]

Let \( Z = \frac{8}{C_+ + D_2} \) (say).
Appendix B

Data corresponding to figure 3 is in table below

Table of $\beta$ versus $\propto$ for different k values; $n = 10^4$. First column shows the variable $\beta$ values. Thereafter each set of columns correspond to $\propto$ and $\beta/k$ values for corresponding k.

| $K$ | $\beta$ | $\propto$ | $\beta/k$ | $\propto$ | $\beta/k$ | $\propto$ | $\beta/k$ | $\propto$ | $\beta/k$ | $\propto$ | $\beta/k$ | $\propto$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $K = 10^{3.5}$ | $K = 10^{3.75}$ | $K = 10^4$ | $K = 10^{4.25}$ | $K = 10^5$ | $K = 10^{5.5}$ |
| $10$ | $0.269$ | $0.074$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ |
| $17.782$ | $0.269$ | $0.074$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ |
| $31.622$ | $0.269$ | $0.074$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ |
| $56.234$ | $0.269$ | $0.074$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ |
| $100$ | $0.269$ | $0.074$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ | $0.003$ | $0.078$ |

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References

Alexander A B, Pokatilov E P and Nika D L 2007 Phonon engineering in hetero- and nanostructures *Journal of Nanoelectronics and Optoelectronics* 2 140–70

Besombes L, Kheng K, Marsal L and Mariette H 2001 Acoustic phonon broadening mechanism in single quantum dot emission *Phys. Rev. B* 63 155307

Faveri O, Cassabois G, Ferreira R, Danson D, Voisin C, Tignon J, Delalande C, Bastard G and Roussignol P H 2003 Acoustic phonon sidebands in the emission line of InAs/GaAs quantum dots *Phys. Rev. B* 68 233301

Mathews P M and Venkatesan K 2007 *A Textbook of Quantum Mechanics* (New Delhi: Tata McGraw-Hill) 37th Reprint

Petukhov A G, Chantis A N and Demchenko D O 2002 Resonant enhancement of tunneling magnetoresistance in double-barrier magnetic heterostructures *Phys. Rev. Lett.* 89 107205

Thomas M P Jr 1982 An improved piezoelectric acoustic emission transducer *The Journal of the Acoustical Society of America* 71 1163

Tian Z T, White B E Jr and Sun Y 2010 Phonon wave-packet interference and phonon tunneling based energy transport across nanostructured thin films *Appl. Phys. Lett.* 96 263113