Local existence of solution to free boundary value problem for compressible Navier-Stokes equations

Jian Liu

Department of Mathematics, Capital Normal University
Beijing 100048, P.R. China. E-mail: liujian.maths@gmail.com

Abstract This paper is concerned with the free boundary value problem for multi-dimensional Navier-Stokes equations with density-dependent viscosity where the flow density vanishes continuously across the free boundary. A local (in time) existence of weak solution is established, in particular, the density is positive and the solution is regular away from the free boundary.

Key words Navier-Stokes equations; free boundary value problem; local existence

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1 Introduction

The compressible Navier-Stokes equations (CNS) with density-dependent viscosity coefficients are taken into granted recently. The prototype is the model of viscous Saint-Venat system used in geophysical flow [13] to simulate the motion of the surface in shallow water, of which the mathematical derivation is also made recently based on the motion of three dimensional incompressible viscous fluids on shallow region with free surface condition on the top and Navier type boundary condition at bottom of finite depth [6, 10].

In the present paper, we consider the general isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficients in $\mathbb{R}^N$, $N = 2, 3$, can be written for $t > 0$ as

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{U}) &= 0, \\
(\rho \mathbf{U})_t + \text{div}(\rho \mathbf{U} \otimes \mathbf{U}) - \text{div}(\mu(\rho)\mathbb{D}((\mathbf{U}))) - \nabla(\lambda(\rho)\text{div} \mathbf{U}) + \nabla P(\rho) &= 0,
\end{align*}
\]

where $\rho(\mathbf{x}, t)$, $\mathbf{U}(\mathbf{x}, t)$ and $P(\rho) = \rho^\gamma (\gamma > 1)$ stand for the fluid density, velocity and pressure, respectively, $\mathbb{D}(\mathbf{U}) = \frac{1}{2}(\nabla \mathbf{U} + \nabla \mathbf{U}^T)$ is the stress tensor, and $\mu(\rho)$ and $\lambda(\rho)$ are the Lamé viscosity coefficients satisfying $\mu(\rho) \geq 0$ and $\mu(\rho) + N\lambda(\rho) \geq 0$ for $\rho \geq 0$. Note here that the case $\gamma = 2$ and $\theta = 1$ in $[\square]$ corresponds to the viscous Saint-Venat system.
One of mathematical difficulties to investigate the existence and dynamics of solutions to \((1.1)\) is that the viscosity coefficients are density-dependent which leads to strong degeneracy in the appearance of vacuum \([5]\). Thus, it is natural and interesting to investigate the influence of vacuum state on the existence and dynamics of global solutions to \((1.1)\). One of the prototype problems is the time-evolution of the compressible viscous flow of finite mass expanding into infinite vacuum. This corresponds to free boundary value problem (FBVP) for the compressible Navier-Stokes equations \((1.1)\) for general initial data and variant boundary conditions imposed on the free surface. The study is fundamental issue of fluid mechanics and has attracted lots of research interests \([11, 17]\). These free boundary problems have been studied with rather abundant results concerned with the existence and dynamics of global solution for CNS \((1.1)\) in 1D, refer to \([4, 9, 14, 15, 19, 20]\) and references therein. As for related phenomena of vacuum vanishing and dynamics of free boundary, the reader can refer to \([8, 9]\).

The free boundary value problem for \((1.1)\) with stress free boundary condition has been investigated in \([7]\), where global existence of spherically symmetric weak solution is shown, in particular, the dynamics behaviors and the Lagrangian properties are also established therein. Chen-Zhang \([3]\) proved the local solutions of \((1.1)\) with spherically symmetric initial data between a solid core and a free boundary connected to a surrounding vacuum state. Under certain assumptions that are imposed on the spherically symmetric initial data, which between a solid core and a free boundary, Chen-Fang-Zhang established the global existence, uniqueness and continuous dependence on initial data of a weak solution in \([2]\). Wei-Zhang-Fang obtained the global existence and uniqueness of the spherically symmetric weak solution in \([18]\) with the symmetric center excluded.

In the present paper, we consider the free boundary value problem for multi-dimensional CNS \((1.1)\) in the case that where the fluid density connects with vacuum continuously. We show that a spherically symmetric weak solution, with the symmetric center included, exists locally in time, in particular the density is positive away from the free boundary but vanishes across the initial interface separating fluids and vacuum, and the free surface moves as particle pathes in radial direction. To this end, we need to employ the basic energy and the modified Bresch-Desjardins (BD) \([1]\) entropy to establish the expected boundary regularities of spherically symmetric solutions in Lagrangian coordinates so as to control the finite speed motion of free boundary within finite time. Then, in terms of the original equations instead of the spherically symmetric form, we are able to apply the higher order energy estimates to establish the necessary interior regularities of solutions away from the free boundary but with the symmetry center included. Then, the combination of both boundary estimates and interior estimates and the above leads to the desired local existence and uniqueness results of solutions.

The rest of this paper is as follows. In Section 2, we state the main results of this paper. In Sections 3-5 we establish boundary regularity and interior regularity, with which we can
prove the existence and uniqueness in Section 6.

## 2 Main results

For simplicity, the viscosity terms are assumed to satisfy $\mu(\rho) = \rho^\theta$, $\lambda(\rho) = \rho \mu'(\rho) - \mu(\rho) = (\theta - 1)\rho^\theta$ and $\mathbb{D}(U) = \frac{1}{2}(\nabla U + \nabla U^T)$ in (1.1). The pressure is assumed to be $P(\rho) = \rho^\gamma$. In this situation, (1.1) become

\[
\begin{cases}
\rho_t + \text{div}(\rho U) = 0, \\
(\rho U)_t + \text{div}(\rho U \otimes U) - \text{div}(\rho^\theta \mathbb{D}(U)) - (\theta - 1)\nabla(\rho^\theta \text{div} U) + \nabla \rho^\gamma = 0.
\end{cases}
\] (2.1)

Consider a spherically symmetric solution $(\rho, U)$ to (2.1) in $\mathbb{R}^N$ so that

\[
\begin{align*}
\rho(x, t) &= \rho(r, t), \quad \rho U(x, t) = \rho u(r, t) \frac{x}{r}, \quad r = |x|, \quad x \in \mathbb{R}^N,
\end{align*}
\] (2.2)

and (2.1) are changed to

\[
\begin{cases}
(r^{N-1}\rho)_t + (r^{N-1}\rho u)_r = 0, \\
(r^{N-1}\rho u)_t + (r^{N-1}\rho u^2)_r + r^{N-1}(\rho^\gamma)_r \\
- r^{N-1}(\theta \rho^\theta (u_r + \frac{N-1}{r} u))_r + (N - 1)r^{N-2}(\rho^\theta)_r u = 0,
\end{cases}
\] (2.3)

for $(r, t) \in \Omega_T$ with

\[
\Omega_T = \{(r, t) | 0 \leq r \leq a(t), \; 0 \leq t \leq T\}.
\] (2.4)

The initial data is taken as

\[
(\rho_0, U_0)(x) = (\rho_0(r), u_0(r) \frac{x}{r}), \quad r \in [0, a_0].
\] (2.5)

At the center of symmetry we impose the Dirichlet boundary condition

\[
u(0, t) = 0,
\] (2.6)

and the free surface $\partial \Omega_t$ moves in radial direction along the “particle path” $r = a(t)$ with the stress-free boundary condition

\[
\rho(a(t), t) = 0, \quad t > 0,
\] (2.7)

where $a(t)$ is the free boundary defined by

\[
a'(t) = u(a(t), t), \quad t > 0, \quad a(0) = a_0.
\] (2.8)

First, we define a weak solution to the FBVP (2.1)-(2.8) as follows.
Definition 2.1. \((\rho, U, a)\) with \(\rho \geq 0\) a.e. is said to be a weak solution to the free surface problem \((2.1) - (2.8)\) on \(\Omega_t \times [0, T]\), provided that it holds
\[
0 \leq \rho \in L^\infty(0, T; L^1(\Omega_t) \cap L^\infty(\Omega_t)), \quad \sqrt{\rho} U \in L^\infty(0, T; L^2(\Omega_t)), \\
\rho^\alpha \nabla U \in L^2(0, T; L^2(\Omega_t)), \quad a(t) \in C^0([0, T]),
\]
and the equations are satisfied in the sense of distributions. Namely, it holds for any \(t_2 > t_1 \geq 0\) and \(\phi \in C^1(\Omega_t \times [0, T])\) that
\[
\int_{\Omega_t} \rho^\alpha |\dot{\psi}|^2_{\Omega_t} \, dx = \int_{t_1}^{t_2} \int_{\Omega_t} (\rho \dot{\phi} + \rho U \cdot \nabla \phi) \, dx \, dt \tag{2.9}
\]
and for \(\psi = (\psi^1, \psi^2, \ldots, \psi^N) \in C^1(\bar{\Omega}_t \times [0, T])\) satisfying \(\psi(x, t) = 0\) on \(\partial \Omega_t\) and \(\psi(x, T) = 0\) that
\[
\int_{\Omega_t} m_0 \cdot \psi(x, 0) \, dx + \int_0^T \int_{\Omega_t} \rho^\gamma \nabla \psi \, dx \, dt - (\theta - 1) \int_0^T \int_{\Omega_t} \rho \nabla U \nabla \psi \, dx \, dt \\
- \int_0^T \int_{\Omega_t} \rho^\beta \nabla U : \nabla \psi \, dx \, dt + \int_0^T \int_{\Omega_t} [\rho U \cdot \partial_t \psi + \sqrt{\rho} U \otimes \sqrt{\rho} U : \nabla \psi] \, dx \, dt = 0 \tag{2.10}
\]
where \(m_0 = m_0 \frac{\dot{x}}{r}\). The free boundary condition \((2.7)\) is satisfied in the sense of continuity.

Notations: Throughout this paper, \(C\) and \(c\) denote generic positive constants, \(C_{f,g} > 0\) denotes a generic constant which may depend on the sub-index \(f\) and \(g\), and \(C_T > 0\) a generic constant dependent of \(T > 0\).

Before stating the main result, we need assume the initial data \((2.3)\) satisfies for \(0 < r_0 < r_2 < r_1 < r_1^+ < a_0\) that
\[
\begin{aligned}
&\int_0^{a_0} r^{N-1} \rho_0(r) \, dr = 1, \\
&\rho_*(a_0 - r)^\sigma \leq \rho_0(r) \leq \rho^*(a_0 - r)^\sigma, \quad r \in [0, a_0], \\
&\left(\rho_0^{\frac{1}{2}}(\rho_0 u_0 r)^{\frac{1}{2}}, \rho_0^{\frac{1}{2}} \rho_0 u_0 \right) \in L^2([r_2, a_0]), \quad u_0 \in H^1([r_2, a_0]), \\
&\rho_0 u_0^m \in L^1([r_2, a_0]), \\
&\left(\rho_0, U_0 \right) \in H^3([0, r_1^+]), \quad \left(\sqrt{\rho_0} U_0, \rho_0^{\gamma/2} \right) \in L^2([0, a_0]),
\end{aligned} \tag{2.11}
\]
where \(\rho_*\) and \(\rho^*\) are positive constants.

Meanwhile we list some assumptions on the constants \((\gamma, \theta, \beta, m)\) with \(\beta = \frac{\sigma}{1+\sigma}\).

(A1) Let \(\gamma, \theta\) satisfy
\[
\frac{N-1}{N} < \theta < \gamma, \quad \gamma > 1. \tag{2.12}
\]

(A2) Let \(\beta\) satisfying
\[
\frac{1}{2\gamma} < \beta < \min\left\{\frac{1}{2\theta}, \frac{1}{1+\theta}\right\}, \quad \beta(\theta - 1) < \frac{1}{3}. \tag{2.13}
\]
Theorem 2.1. Let $N = 2, 3, \gamma > 1$. Assume that (2.11) and $A_1 \sim A_3$ hold. Then, there exist a time $T_* > 0$ and $\rho_\pm > 0$ dependent of initial data, so that the FBVP (2.1)-(2.8) has a unique spherically symmetric weak solution for $t \in [0, T_*]$

$$(\rho, \rho U, a)(x, t) = (\rho(r, t), \rho w(r, t) \frac{x}{r}, a(t)), \quad r = |x|, $$

in the sense of Definition 2.1 for any $T \in (0, T_*)$ satisfying that

$$
\int_0^{a(t)} r^{N-1} \rho(r, t) dr = \int_0^{a_0} r^{N-1} \rho_0(r) dr, \quad (2.15)
$$

$$
c_0 \leq a(t) \leq 2a_0, \quad t \in [0, T_*], \quad \|a\|_{H^2([0, T_*])} \leq C, \quad (2.16)
$$

$$
(\rho, U) \in C^0(\Omega_t \times [0, T_*]), \quad \|U\|_{W^{1,\infty}(\Omega_t \times [0, T_*])} \leq C, \quad (2.17)
$$

$$
\rho_-(a(t) - r)^\sigma \leq \rho(r, t) \leq \rho_+(a(t) - r)^\sigma, \quad (r, t) \in [0, a(t)] \times [0, T_*], \quad (2.18)
$$

$$
\sup_{t \in [0, T_1]} \int_{\Omega_t} (\rho^\gamma + |\sqrt{\rho} U|^2)(x, t) dx + \int_0^T \int_{\Omega_t} \rho^\beta |\nabla U|^2 dx dt \leq C, \quad (2.19)
$$

$$
\sup_{t \in [0, T_1]} \| (\rho, U)(t) \|_{H^2(\Omega_t^{a(t)})} + \int_0^{T_*} (\| \rho(t) \|^2_{H^3(\Omega_t^{a(t)})} + \| \nabla U(t) \|^2_{H^3(\Omega_t^{a(t)})}) dt \leq C, \quad (2.20)
$$

$$
\sup_{t \in [0, T_1]} \int_{r_{x_2}(t)}^{a(t)} r^{N-1}(u^{2k} + u_t^2) dr + \int_0^{T_*} \int_{r_{x_2}(t)}^{a(t)} \rho^\beta r^{N-1}(u^{2k-2} u_r^2 + u_r^2 + r^{-2} u_t^2) dr dt \leq C, \quad (2.21)
$$

where $\Omega_t = \{0 \leq |x| \leq a(t)\}, \quad \Omega_t^{a(t)} = \{0 \leq |x| \leq r_{x_1}(t)\}, \quad r_{x_1}(t) \text{ is the particle path with } r_{x_i}(0) = r_i \ (i = 1, 2) \text{ and } 1 \leq k \leq 2m \text{ is an integer, and } C > 0 \text{ is a constant.}$

Remark 2.1. Theorem 2.1 yields the local existence of spherically symmetric weak solutions for two/three dimensional compressible Navier-Stokes equation with fluid density connecting with vacuum continuously. In particular, it applies to the viscous Saint-Venant model for shallow water (which is (2.11) with $N = 2, \mu(\rho) = \rho, \lambda(\rho) = 0, \text{ and } \gamma = 2$).

3 Basic energy estimates

The proof of Theorem 2.1 consists of the construction of approximate solutions, the basic a-priori estimates, and compactness arguments. We establish the a-priori estimates for any solution $(\rho, u, a)$ to FBVP (3.4)-(3.6) in this section.
Let us introduce the Lagrangian coordinates transform
\[ x(r, t) = \int_0^r \rho y^{N-1} dy = 1 - \int_r^{a(t)} \rho y^{N-1} dy, \quad \tau = t, \] (3.1)
which translates the domain \([0, T] \times [0, a(t)]\) into \([0, T] \times [0, 1]\) and satisfies
\[ \frac{\partial x}{\partial r} = \rho r^{N-1}, \quad \frac{\partial x}{\partial t} = -\rho u r^{N-1}, \quad \frac{\partial \tau}{\partial r} = 0, \quad \frac{\partial \tau}{\partial t} = 1, \] (3.2)
and
\[ r^N (x, \tau) = N \int_0^x \frac{1}{\rho} (y, \tau) dy = a(t)^N - N \int_x^1 \frac{1}{\rho} (y, \tau) dy, \quad \frac{\partial r}{\partial \tau} = u. \] (3.3)
In terms of (3.1)–(3.3), the free boundary value problem (2.1)–(2.8) is changed to
\[ \begin{cases} 
\rho \tau + \rho^2 (r^{N-1} u)_x = 0, \\
r^{1-N} u_{\tau} + (\rho^\gamma - \theta \rho^{\theta+1} (r^{N-1} u)_x) + \frac{N-1}{r} (\rho^\theta)_x u = 0,
\end{cases} \] (3.4)
for \((x, \tau) \in [0, 1] \times [0, T]\), with the initial data and boundary conditions given by
\[ (\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in [0, 1], \] (3.5)
\[ u(0, \tau) = 0, \quad \rho(1, \tau) = 0, \quad \tau \in [0, T], \] (3.6)
where \(r = r(x, \tau)\) is defined by
\[ \frac{d}{d\tau} r(x, \tau) = u(x, \tau), \quad x \in [0, 1], \quad \tau \in [0, T], \] (3.7)
and the fixed boundary \(x = 1\) corresponds to the free boundary \(a(\tau) = r(1, \tau)\) in Eulerian form determined by
\[ \frac{d}{d\tau} a(\tau) = u(1, \tau), \quad \tau \in [0, T], \quad a(0) = a_0. \] (3.8)
Note that in Lagrange coordinates the condition (2.11) is equivalent to
\[ \begin{cases} 
\rho_*(1 - x)^\beta \leq \rho_0(x) \leq \rho^*(1 - x)^\beta, \quad x \in [0, 1], \\
(\rho_0^{1+\theta} r^{N-1} u_{0x}) \in L^2([x_2, 1]), \quad \rho_0^{1/2} r^{N-1} u_0 \in H^1([x_2, 1]), \\
u_{0x}^m \in L^1([x_2, 1]), \\
(\rho_0, u_0) \in H^3([0, x_1^+]), \quad (u_0, \rho_0^{(\gamma-1)/2}) \in L^2([0, 1]),
\end{cases} \] (3.9)
where \(0 < x_2 = \int_0^{x_2} r^{N-1} \rho_0(r) dr < x_1^+ = \int_0^{x_1^+} r^{N-1} \rho_0(r) dr.\)

First, making use of similar arguments as [7] with modifications, we can establish the following Lemmas 3.1–3.3 which we omit the details.

**Lemma 3.1.** Let \(\gamma > 1, \ T > 0, \) and \((\rho, u, a)\) with \(\rho > 0\) be the solution to the FBVP (3.4)–(3.6) for \(\tau \in [0, T]\). Then, it holds
\[ \int_0^1 \left( \frac{1}{2} u^2 + \frac{1}{\gamma-1} \rho^{\gamma-1} \right) dx + [1 - N(1 - \theta)](N - 1) \int_0^\tau \int_0^1 \rho^{\theta-1} u^2 dx ds
+ [1 - N(1 - \theta)] \int_0^\tau \int_0^1 \rho^{1+\theta} (r^{N-1} u_x)^2 dx ds = E_0, \quad \tau \in [0, T], \] (3.10)
Under the same assumptions as Lemma 3.1, it holds that there is a time \( T > 0 \) with \( (\tau, \rho, u, a) \) satisfying (3.16) and (3.15) that
\[
\int_0^1 \left( \frac{1}{2} u^2 + \frac{1}{\gamma - 1} \rho^{-1} \right) dx + (\theta - 1 + \frac{1}{\gamma}) \int_0^\tau \int_0^1 \rho^{\theta + 1} [(r^{N-1} u_x)_x]^2 \, dx \, ds
+ \frac{N-1}{\gamma} \int_0^\tau \int_0^1 \rho^{\theta + 1} [r^{N-1} u_x - \frac{u}{\rho r}]^2 \, dx \, ds = E_0, \quad \tau \in [0, T].
\] (3.11)

**Lemma 3.3.** Under the same assumptions as Lemma 3.1, it holds that
\[
E_0^{-\frac{1}{N(N-1)}} \leq r(x, \tau) \leq \alpha(\tau), \quad (x, \tau) \in [0, 1] \times [0, T],
\] (3.12)
\[
E_0^{-\frac{1}{N(N-1)}} (x_2 - x_1)^{\frac{1}{N-1}} \leq r^N(x_2, \tau) - r^N(x_1, \tau), \quad 0 \leq x_1 < x_2 \leq 1, \quad \tau \in [0, T].
\] (3.13)

In particular, it holds for \( x = 1 \) that
\[
E_0^{-\frac{1}{N(N-1)}} \leq \alpha(\tau) \equiv r(1, \tau), \quad \tau \in [0, T].
\] (3.14)

Then, we have

**Lemma 3.4.** Let \( T > 0 \) and \( \gamma > 1 \). Let \((\rho, u, a)\) be the solution to FBVP (3.1)–(3.8) for \((x, \tau) \in [0, 1] \times [0, T]\). Assume further that it holds for some \( x_0 \in (0, 1) \)
\[
\frac{1}{2} \rho_-(1 - x)^\beta \leq \rho(x, \tau) \leq 2 \rho_+(1 - x)^\beta, \quad (x, \tau) \in [x_0, 1] \times [0, T],
\] (3.15)
\[
|\rho r^{N-1} u_x(x, \tau)| \leq 2 M_0, \quad (x, \tau) \in [x_0, 1] \times [0, T],
\] (3.16)

where \( \beta \in (0, 1), \rho_+ = 2 \rho^*, \rho_- = \frac{1}{2} \rho_*, \) and the constant \( M_0 > 0 \) is given by (5.5). Then, there is a time \( T_1 \in (0, T] \cap (0, 1) \) so that \((\rho, u, a)\) satisfies
\[
c_0 x^{-\frac{\gamma}{N(N-1)}} \leq r(x, \tau) \leq a(\tau) \leq 2 a_0, \quad (x, \tau) \in [0, 1] \times [0, T_1],
\] (3.17)
\[
\frac{1}{2} \rho_0(x) \leq \rho(x, \tau) \leq 2 \rho_0(x), \quad (x, \tau) \in [x_0, 1] \times [0, T_1],
\] (3.18)
\[
\int_0^x ((\rho^\gamma) r^{N-1})^2 \, dy + \int_0^\tau \int_0^x ((\rho^\gamma) r^{N-1})^2 \, dy \, ds \leq C E_x, \quad (x, \tau) \in (x_0, 1) \times [0, T],
\] (3.19)

with \( E_x := \int_0^x (u^2 + ((\rho^\gamma) r^{N-1})^2) \, dy + \int_0^1 \rho^\gamma \, dy \) and \( C_{x_0} > 0 \) a constant.

**Proof.** First of all, it follows directly from (3.16) and (3.15) that
\[
\|u\|_{L^\infty([x_0, 1] \times [0, T])} \leq (1 - x_0)^{-1} \int_{x_0}^1 |u| \, dx + \int_{x_0}^1 |u_x| \, dx
\leq C_{x_0} (E_0^{1/2} + M_0 \rho_-^{-1}) =: M_1,
\] (3.20)
(3.21)

which yields (3.17) with the help of (3.12) and
\[
r(x, \tau) \leq a(\tau) = a_0 + \int_0^\tau u(1, s) \, ds \leq a_0 + TM_1 \leq 2a_0,
\] (3.22)
for $\tau \in [0, T_{1,a}]$ with
\[
T_{1,a} =: a_0 M_1^{-1}. \tag{3.23}
\]
It follows from (3.14) that
\[
\rho(x, \tau) = \rho_0(x) \exp \left\{-\int_0^\tau \left( \rho r^{N-1} u_x + (N-1)u r \right)(x, s) ds \right\}, \tag{3.24}
\]
which together with (3.16), (3.17) and (3.21) yields (3.18) for $\tau \in [0, T_{1,b}]$ with $T_{1,b}$ determined by
\[
T_{1,b} =: \min \left\{ T_{1,a}, \frac{1}{2} M_1 E_0 \left( 2 a_0 r^{N-1} x_0^{N-1} \right) \ln 2 \right\}. \tag{3.25}
\]
Differentiating (3.14) with respect to $x \in [x_0, 1)$, substituting the resulted equation into (3.12) and using the fact $\frac{\partial r}{\partial \tau} = u$, we have
\[
(u + r^{N-1} (\rho^\theta)_x) + (\rho^\gamma) r r^{N-1} = 0. \tag{3.26}
\]
Multiplying (3.26) by $\phi(u + r^{N-1} (\rho^\theta)_x)$, where $\phi \in C^\infty([0, 1])$, $0 \leq \phi \leq 1$, $\phi(y) = 1$ for $y \in [0, x]$ and $\phi(y) = 0$ for $y > (1 + \eta)x$ with $\eta > 0$ small enough, and integrating the resulted equation over $[0, 1] \times [0, \tau]$ by parts, we obtain
\[
\int_0^1 \phi(u + r^{N-1} (\rho^\theta)_x) dx + \int_0^\tau \int_0^1 \phi((\rho^\theta)_x r^{N-1}) dx ds
\leq C \int_0^1 \phi(u + r^{N-1} (\rho^\theta)_x) dx + \int_0^\tau \int_0^1 \phi u r^{N-1} \rho^\gamma dx ds + \int_0^1 \phi(x) r^{N-1} (x, 0) dx
\leq C x_0 E_x + (2 a_0)^{N-1} M_1 r (\rho^\theta)^{2y}, \tag{3.27}
\]
where we have used (3.17) and (3.18). Choose
\[
T_{1,c} =: \min \left\{ T_{1,a}, T_{1,b}, \frac{1}{2} M_1 r (\rho^\theta)^{2y} E_x \right\}, \quad T_1 =: \min \left\{ T_{1,a}, T_{1,b}, T_{1,c} \right\},
\]
then the combination of (3.27) and (3.10) yield (3.19) for $\tau \in [0, T_1]$. \hfill \Box

## 4 Boundary regularities

This section is devoted to the boundary regularities of solutions to FBVP (3.4)—(3.8). To this end, we first establish the regularities of solution $(\rho, u, a)$ away from the symmetry center and the free boundary.

**Lemma 4.1.** Under the assumptions of Lemma 3.4, there is a time $T_2 \in (0, T_1]$ so that the solution $(\rho, u, a)$ satisfies for $x_1 \in (x_0, 1)$ and $\tau \in [0, T_2]$ that
\[
\| (\rho_x, u_x)(\tau) \|_{L^2([x_0, x_1])}^2 + \int_0^\tau \| (u_{xx}, u_s)(s) \|_{L^2([x_0, x_1])}^2 ds \leq C_4 \delta_4^2, \tag{4.1}
\]
\[
\| (\rho_{xx}, u_{xx}, u_x)(\tau) \|_{L^2([x_0, x_1])}^2 + \int_0^\tau \| (u_{xxx}, u_{xs})(s) \|_{L^2([x_0, x_1])}^2 ds \leq C_5 \delta_5^2, \tag{4.2}
\]
where $\delta_4, \delta_5$ are positive constants.
\[(\rho, u_0) \in H^3(I) \text{ with } [x_0, x_1] \subset (x_0^-, x_1^+), \text{ where } x_0^- \in (0, x_0) \text{ and } I =: [x_0^-, x_1^+], \]

where \( C_i > 0, (i = 4, 5, 6) \) are constants dependent of \( x_0 \) and \( x_1 \), but independent of \( M_0 \).

\[\delta_4 = \|(\rho_0, u_0)\|_{H^3(I)}, \ \delta_5 = \|(\rho_0, u_0)\|_{H^2(I)}, \ \text{and } \delta_6 = \|(\rho_0, u_0)\|_{H^3(I)}. \]

In addition, it holds

\[|\rho^{N-1} u(x, \tau)| \leq M_{0,a}, \ |u(x, \tau)| \leq M_{1,a}, \ (x, \tau) \in [x_0, x_1] \times [0, T_2], \]

with \( M_{0,a} = (2a_0)^N \rho^* (C_4^{\delta_4^2} + C_5^{\delta_5^2})^{1/2}, \ M_{1,a} = C_{20} (E_0^{1/2} + M_{0,a} \rho^*). \)

**Proof.** It is easy to verify that (4.1) follows from (4.1), (4.2), (3.18) and (3.20). What left is to show (4.1)-(4.3). Rewrite (3.4)2 as

\[r^{1-N} u_r + (\rho^\gamma - \theta \rho^{1+\gamma} r^{N-1} u_x) x + (1 - \theta)(N - 1)(\rho^\gamma) \frac{u}{r} - \theta(N - 1)\rho^\gamma \frac{u}{r} = 0, \]

Take inner product between (4.5) and \( \phi \rho^{1-N} u_r, \) where \( \phi = \psi^2(x) \) and \( \psi \in C^\infty([0, 1]) \) satisfies

\[0 \leq \psi \leq 1, \psi = 1 \text{ for } x \in [(1 - 2\eta)x_0, (1 + 2\eta)x_1], \ \text{and } \psi = 0 \text{ for } x \in [0, (1 - 3\eta)x_0] \cup [(1 + 3\eta)x_1, 1] \text{ with a fixed constant } \eta \in (0, 1) \text{ small enough so that } [(1 - 3\eta)x_0, (1 + 3\eta)x_1] \subset (x_0^-, x_1^+). \]

By lemma 3.4 and a direct computations, it follows

\[\int_0^1 \int_0^1 \left( \frac{d}{d\tau} \int_0^1 \left( 1 - \frac{1}{2} \rho \phi \rho^{2+\theta-N} r^{N-1} u_x^2 - \rho \phi^{1+\gamma} r^{N-1} u_x \right) dx + \frac{1}{2} \int_0^1 \phi \rho^{1-N} r^{1-N} u_x^2 dx \right) \frac{d\tau}{d\tau} \]

\[\leq C_{x_0, x_1} (M_0 + M_1 + (\rho^*)^{\gamma-\theta}) \int_0^1 \rho \phi^{2+\theta-N} r^{N-1} u_x^2 dx + C_{x_0, x_1} M_0 M_1 (\rho_* + \rho^*)^{1+\gamma-N} \]

\[+ C_{x_0, x_1} M_0 M_1 (\rho_* + \rho^*)^{1+2\theta-N} + C_{x_0, x_1} M_0^2 (\rho_* + \rho^*)^{2\theta-N} + C_{x_0, x_1} M_1^2 (\rho_* + \rho^*)^{2\theta-1-N} \]

\[+ C_{x_0, x_1} \rho^* E_{x_1}^y (M_0^2 + M_1^2 + (\rho_* + \rho^*)^{2\gamma+1-N}) \]

where \( C_{x_0, x_1} > 0 \) is a generic constant dependent of \( x_0, x_1 \), but independent of \( M_0, M_1 \).

Integrating (4.6) over \([0, \tau]\) to get

\[\int_0^1 \int_0^1 \phi \rho^{2+\theta-N} r^{N-1} u_x^2 dx + \int_0^1 \int_0^1 \phi \rho^{1-N} r^{1-N} u_x^2 dx ds \]

\[\leq C_{x_0, x_1} (M_0 + M_1 + (\rho^*)^{\gamma-\theta}) \int_0^1 \rho \phi^{2+\theta-N} r^{N-1} u_x^2 ds + \int_0^1 \int_0^1 \phi \rho^{1-N} r^{1-N} u_x^2 dx ds \]

\[\leq C_{x_0, x_1} M_0^2 + M_1^2 + a_0^{N-1} + C_{x_0, x_1} \rho^* E_{x_1}^y (M_0^2 + M_1^2) \]

\[+ C_{x_0, x_1} \rho^* E_{x_1}^y, \]

which together with Grönwall’s inequality leads to

\[\int_0^1 \int_0^1 \phi \rho^{2+\theta-N} r^{N-1} u_x^2 dx + \int_0^1 \int_0^1 \phi \rho^{1-N} r^{1-N} u_x^2 dx ds \leq C_{x_0, x_1, 1} \delta_4^2. \]

Due to the fact that \( \phi(x) = 1 \) for \( x \in [(1 - 2\eta)x_0, (1 + 2\eta)x_1] \), it follows

\[\int_{(1-2\eta)x_0}^{(1+2\eta)x_1} u_x^2 dx + \int_0^1 \int_{(1-2\eta)x_0}^{(1+2\eta)x_1} u_x^2 dx ds \leq C_{x_0, x_1, 1} \delta_4^2. \]
for some constant $C_{x_0,x_1,2} > 0$ and $\tau \in [0, T_{2,a}]$ with $T_{2,a}$ chosen as

$$T_{2,a} = \min\{T_1, K_1^{-1}\delta_4^2, K_2^{-1}\ln 2\},$$

$$K_1 = C_{x_0,x_1}(M_0^2 + M_1^2 + \sigma_0^{-N-1}) + C_{x_0,x_1}\rho_1\rho_1^{-N}E_{x_1}(M_0^2 + M_1^2),$$

$$K_2 = C_{x_0,x_1}(M_0 + M_1 + (\rho^*)^{\gamma-\theta}).$$

In addition, it follows from (4.9), (3.3) and Lemma 3.4 that

$$\int_0^\tau \int_{(1-2\eta)x_0}^{(1+2\eta)x_1} (u_{xx} + \rho_{xx}^2) dx ds \leq C_{x_0,x_1}\delta_4^2.$$  (4.10)

The combination of (4.9) - (4.10) and (3.19) gives rise to

$$\int_{(1-2\eta)x_0}^{(1+2\eta)x_1} (u_x^2 + \rho_x^2) dx + \int_0^\tau \int_{(1-2\eta)x_0}^{(1+2\eta)x_1} (u_{xx}^2 + \rho_{xx}^2 + u_s^2) dx ds \leq C_{x_0,x_1}\delta_4^2,$$  (4.11)

for $\tau \in [0, T_{2,a}]$, which implies (4.11) for any $T_2 \leq T_{2,a}$.

The higher order regularities of the solution can be obtained by applying the similar arguments as the proof of (4.11). Indeed, differentiating (4.3) with respect to $\tau$ gives

$$r^{1-N}u_{\tau r} - (1 - N)r^{-N}uu_{\tau r} + (\rho^\gamma - \theta \rho^{\theta+1}r^N u_x)_{xr}$$

$$+ (1 - \theta)(N - 1)((\rho^\theta u)_r)_x - \theta(N - 1)(\rho^\theta u)_x = 0,$$  (4.12)

taking inner product between (4.12) and $\phi u_{\tau r}$ over $[0, 1]$, where $\phi = \psi^2(x)$ and $\psi \in C^\infty([0, 1])$ satisfies $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ for $x \in [(1 - \eta)x_0, (1 + \eta)x_1]$, and $\psi(x) = 0$ for $x \in [0, (1 - 2\eta)x_0] \cup [(1 + 2\eta)x_1, 1]$, we can obtain

$$\frac{1}{2} \frac{d}{d\tau} \int_0^1 \phi r^{1-N}u_{\tau r}^2 dx + \frac{\theta}{2} \int_0^1 \phi r^{\theta+1}r^{N-1}u_{xxr}^2 dx$$

$$= \int_0^1 \phi_x (\rho^\gamma - \theta \rho^{\theta+1}r^{N-1}u_x)_{xr}u_{\tau r} dx + \frac{N - 1}{2} \int_0^1 \phi r^{-N}uu_{\tau r}^2 dx + \theta(N - 1) \int_0^1 \phi (\rho^\theta u)_r u_{\tau r} dx$$

$$+ \frac{1}{2} \int_0^1 \phi ((\rho^\gamma)_r - \theta(\rho^{\theta+1}r^{N-1})_r u_x)_x u_{\tau r} dx - \frac{\theta}{2} \int_0^1 \phi \rho^{\theta+1}r^{N-1}u_{xxr}^2 dx$$

$$+ (1 - \theta)(1 - N) \int_0^1 \phi ((\rho^\theta u)_r)^2 u_{\tau r} dx$$

$$\leq C_{x_0,x_1}(1 + M_1 + (\rho_0 + \rho_1)^\theta - 1) \int_0^1 \phi r^{1-N}u_{\tau r}^2 dx + C_{x_0,x_1}(1 + (\rho^*)^\theta+1)$$

$$+ (\rho_0 + \rho^*)^{\theta-1} \int_{(1-2\eta)x_0}^{(1+2\eta)x_1} u_{xxr}^2 dx + C_{x_0,x_1}(\rho_0 + \rho^*)^{\theta-1} \int_{(1-2\eta)x_0}^{(1+2\eta)x_1} \rho_{xxr}^2 dx$$

$$+ C_{x_0,x_1,3}(1 + (\rho^*)^{2(\gamma+1)} + (\rho^*)^4)(1 + M_0^4 + M_1^4),$$  (4.13)

which together with Lemma 3.4 and (4.11) yields

$$\int_0^1 \phi r^{1-N}u_{\tau r}^2 dx + \int_0^\tau \int_{(1-2\eta)x_0}^{(1+2\eta)x_1} \phi r^{\theta+1}r^{N-1}u_{xxr}^2 dx ds \leq C_{x_0,x_1}\delta_5^2, \quad \tau \in [0, T_{2,d}],$$  (4.14)
with \( T_{2,b} \) chosen as
\[
T_{2,b} =: \min\{ T_{2,a}, \ K_3^{-1}\delta_4^2, \ (C_{x_0}(1 + M_1 + (\rho_* + \rho^*)^{\gamma-1}))^{-1}\ln 2 \},
\]
\[
K_3 =: C_{x_0,x_1,3}(1 + (\rho^*)^{2(\gamma+1)} + (\rho^*)^{4宾}(1 + M_0^4 + M_1^4).
\]
This and (3.4), (3.17)–(3.18) imply that
\[
\int_{(1-\eta)x_0}^{(1+\eta)x_0} (u_x^2 + u_{xx}^2) dx + \int_0^\tau \int_{(1-\eta)x_0}^{(1+\eta)x_0} u_{xx}^2 dx ds \leq C_{x_0,x_1}\delta_5^2, \quad \tau \in [0,T_{2,b}]. \tag{4.15}
\]
Meanwhile, taking inner product between (3.20) and \( \phi(x)(u + r^{N-1}(\rho^\theta)_x) \) over \([0,1] \times [0,\tau]\), and using Lemma 3.1, we can obtain
\[
\int_0^1 \phi(u + r^{N-1}(\rho^\theta)_x)^2 dx + \frac{\gamma}{\theta} \int_0^\tau \int_0^1 \phi \rho^{\gamma-\theta}(u + r^{N-1}(\rho^\theta)_x)^2 dx ds
\]
\[
= \int_0^1 \phi(u + r^{N-1}(\rho^\theta)_x)^2 dx + \frac{2\gamma}{\theta} \int_0^\tau \int_0^1 \phi \rho^{\gamma-\theta}(u + r^{N-1}(\rho^\theta)_x) u_x dx ds
\]
\[- \frac{2\gamma}{\theta} \int_0^\tau \int_0^1 \theta r^{N-1}(u + r^{N-1}(\rho^\theta)_x)(\rho^{\gamma-\theta})_x(\rho^\theta)_x dx ds
\]
\[
\leq C_{x_0,x_1}((\rho^\gamma)^{\gamma-\theta} + (\rho^\gamma)^{\gamma-1} + (\rho_*)^{\gamma-3} + (\rho_*)^{2(\gamma-1)}
\]
\[
+ (\rho_*)^{\gamma-1} \int_0^\tau \max[(\rho^\gamma)_x r^{N-1}]^2 ds \delta_4^2. \tag{4.16}
\]
Integrating (3.20) over \([0,\tau]\) to get
\[
r^{N-1}(\rho^\theta)_x(x,\tau) = r^{N-1}(x,0)(\rho^\theta)_x - \int_0^\tau (\rho^\gamma)_x r^{N-1}(x,s) ds - u(x,\tau) + u(x,0), \tag{4.17}
\]
one deduces from (3.18), (3.21) and (4.16) that for any \((x,\tau) \in [(1-2\eta)x_0,(1+2\eta)x_1] \times [0,T_{2,b}],
\]
\[
\int_0^\tau [(\rho^\gamma)_x r^{N-1}]^2(x,s) ds
\]
\[
\leq C\tau (\rho^\gamma)^{2(\gamma-\theta)}(1 + M_1^2) + C(\rho^\gamma)^{2(\gamma-\theta)} \int_0^\tau \int_0^\delta [(\rho^\gamma)_x r^{N-1}]^2(x,s) ds ds, \tag{4.18}
\]
which implies for \((x,\tau) \in [(1-2\eta)x_0,(1+2\eta)x_1] \times [0,T_{2,c}],
\]
\[
\int_0^\tau [(\rho^\gamma)_x r^{N-1}]^2(x,s) ds \leq 2 \ln 2, \quad \tau \in [0,T_{2,c}], \tag{4.19}
\]
with \( T_{2,c} \) chosen as
\[
T_{2,c} =: \min\{ T_{2,b}, \ (C(\rho^\gamma)^{2(\gamma-\theta)}(1 + M_1^2))^{-1}\ln 2, \ (C(\rho^\gamma)^{2(\gamma-\theta)}))^{-1}\ln 2 \}.
\]
Using Lemma 3.1, (3.19) and (4.1) that
\[
\int_{(1-\eta)x_0}^{(1+\eta)x_0} \rho^2_{xx} dx + \int_0^\tau \int_{(1-\eta)x_0}^{(1+\eta)x_0} \rho^2_{xx} dx ds \leq C_{x_0,x_1}\delta_5^2, \quad \tau \in [0,T_{2,c}]. \tag{4.20}
\]
The combination of (4.15), (4.20), (4.1) and (3.42) gives rise to
\[
\int_{(1-\eta)x_0}^{(1+\eta)x_1} (u_x^2 + u_{xx}^2 + \rho_{xx}^2)\,dx + \int_0^\tau \int_{(1-\eta)x_0}^{(1+\eta)x_1} (u_{xxx}^2 + \rho_{xx}^2 + u_{xx}^2)\,dx\,ds \leq C_{x_0,x_1}\delta_6^2, \tag{4.21}
\]
for \(\tau \in [0, T_{2,c}]\), which implies (4.2) for any \(T_2 \leq T_{2,c}\).

Differentiating \(r^{N-1}(4.5)\) with respect to \(x\) to get
\[
u_{xxx} + (r^{N-1}(\rho - \theta)\rho^{\theta+1}r^{N-1}u_x)_x = (N-1)(1-\theta)(r^{N-2}(\rho)_x)_{xxx} - \theta(N-1)(r^{N-1}\rho^{\theta/\tau})_{xxx} = 0, \tag{4.22}
\]
and taking inner product between (4.22) and \(\phi(x)u_{xxx}\) over \([0, 1] \times [0, \tau]\), where \(\phi = \psi^2(x)\) and \(\psi \in C_0^\infty([0, 1])\) satisfies \(0 \leq \psi(x) \leq 1\) for \(x \in [x_0, x_1]\), and \(\psi(x) = 0\) for \(x \in [0, (1-\eta)x_0] \cup [(1+\eta)x_1, 1]\). We can obtain
\[
\frac{1}{2} \frac{d}{d\tau} \int_0^1 \phi u_{xxx}^2 \,dx - \int_0^1 \phi (r^{N-1}(\rho - \theta)\rho^{\theta+1}r^{N-1}u_x)_x u_{xxx} \,dx \\
= - \int_0^1 \phi_x (r^{N-1}(\rho - \theta)\rho^{\theta+1}r^{N-1}u_x)_x u_{xxx} \,dx - \int_0^1 \phi_x (r^{N-1}(\rho - \theta)\rho^{\theta+1}r^{N-1}u_x)_x u_{xxx} \,dx \\
+ \theta(N-1) \int_0^1 \phi (r^{N-1}\rho^{\theta/\tau})_{xxx} u_{xxx} \,dx + (N-1)(1-\theta) \int_0^1 \phi u_{xxx} (r^{N-2}(\rho)_x)_{xx} \,dx \\
+ (N-1)(1-\theta) \int_0^1 \phi u_{xxx} (r^{N-2}(\rho)_x)_{xx} \,dx. \tag{4.23}
\]

Lemma 3.4, (3.4), (4.11) and (4.21) lead to that
\[
\frac{d}{d\tau} \int_0^1 \phi (u_{xxx}^2 + \gamma \rho^{\gamma-3} \rho_{xxx}^2) \,dx + \int_0^1 \phi (\rho^{\theta+1}r^{2(N-1)}u_{xxx}^2) \,dx \\
\leq C_{x_0,x_1} (1 + \delta_4^2 + \delta_5^2 + \|u_{xx}(\tau)\|_{L^\infty([[(1-\eta)x_0,(1+\eta)x_1])}) \int_0^1 \phi (\rho^{\gamma-3} \rho_{xxx}^2 + u_{xxx}^2) \,dx \\
+ C_{x_0,x_1} (1 + \delta_4^2 + \delta_5^2)^2 + C_{x_0,x_1} \|u_{xx}(\tau)\|_{L^\infty([[(1-\eta)x_0,(1+\eta)x_1])}) \delta_6^2,
\]
we apply the Gronwall’s inequality, and the fact \(\psi(x) = 1\) for \(x \in [x_0, x_1]\), we can obtain
\[
\int_{x_0}^{x_1} (u_{xxx}^2 + \rho_{xxx}^2) \,dx + \int_0^\tau \int_{x_0}^{x_1} u_{xxx}^2 \,dx\,ds \leq C_{x_0,x_1} \delta_6^2, \quad \tau \in [0, T_{2,d}], \tag{4.24}
\]
with \(T_{2,d}\) determined by
\[
T_{2,d} = \min\{T_{2,c}, \ (C_{x_0,x_1} (1 + \delta_4^2 + \delta_5^2)^2)^{-1}\delta_6^2, \ (C_{x_0,x_1} (1 + \delta_4^2 + \delta_5^2)^2)^{-1}\ln 2\}.\]

By (4.24) and (3.42), we have
\[
\int_{x_0}^{x_1} u_{x\tau}^2 \,dx + \int_0^\tau \int_{x_0}^{x_1} u_{xx\tau}^2 \,dx\,ds \leq C_{x_0,x_1} \delta_6^2, \quad \tau \in [0, T_{2,d}]. \tag{4.25}
\]
The combination of (4.24) and (4.25) yields (4.3) with \(T_2 = T_{2,d}\).
Lemma 4.2. Under the same assumptions as Lemma 3.4, there is a time $T_3 \in (0, T_2]$ so that it holds for $(x_2, \tau) \in (x_0, x_1) \times [0, T_3]$

$$\int_{x_2}^{x_1} u^{2k}(x, \tau)dx + \int_0^{\tau} \int_{x_2}^{x_1} \rho^{\theta+1} u^{2k-2} r^{2N-2} u_x^2 dx ds \leq C_7, \quad k = 1, 2, \cdots, 2m,$$

(4.26)

$$\int_{x_2}^{x_1} (\rho^{\theta+1} u_x^2 + \rho^{\theta-1} u^2)dx + \int_0^{\tau} \int_{x_2}^{x_1} u_x^2 dx ds \leq C_8 + \int_0^{\tau} \int_{x_2}^{x_1} \rho^{\theta+3} u_x^4 dx ds,$$

(4.27)

$$\int_{x_2}^{x_1} \rho^{\theta+3} u_x^4 dx \leq C_9 + C(\int_{x_2}^{x_1} (\rho^{\theta+1} u_x^2 + \rho^{\theta-1} u^2 + u_s^2) dx)^2,$$

(4.28)

$$\int_{x_2}^{x_1} u_s^2 dx + \int_0^{\tau} \int_{x_2}^{x_1} (\rho^{\theta+1} r^{2N-2} u_x^2 + \rho^{\theta-1} r^{2} u_s^2) dx ds \leq C_{10},$$

(4.29)

where $C_i = C_i(\|\rho_0\|_{H^1([x_0, 1])}, \|u_0\|_{H^2([x_0, 1])}) > 0$ are constants, $i = 7, 8, 9, 10$.

Proof. We apply the arguments used in [3] to show (4.26). First we consider the case of $k = 1$. From Lemma 3.1, we obtain (4.26) with $k = 1$ easily. Assume (4.26) holds for $k = l - 1$,

$$\int_{x_2}^{x_1} u^{2(l-1)} dx + \int_0^{\tau} \int_{x_2}^{x_1} \rho^{\theta+1} u^{2l-2} r^{2N-2} u_x^2 dx ds \leq C.$$  

(4.30)

Now we need to prove (4.26) holds for $k = l$. Multiplying (3.4) by $\phi u^{2l-1}$ and integrating over $x$ from 0 to 1, where $\phi = \psi^2(x)$ and $\psi \in C^\infty([0, 1])$ satisfies $0 \leq \psi \leq 1$, $\psi = 1$ for $x \in [x_2, 1]$, and $\psi = 0$ for $x \in [0, x_0]$. We can obtain

$$\frac{d}{d\tau} \int_0^1 \frac{1}{2l} \phi u^{2l} dx$$

$$= -\theta \int_0^1 \phi \rho^{\theta+1} (r^{N-1} u^{2l-1})_x (r^{N-1} u)_x dx + \int_0^1 \phi \rho^{\gamma} (r^{N-1} u^{2l-1})_x dx$$

$$+ (N - 1) \int_0^1 \phi \rho^{\theta} (r^{N-2} u_x^2)_x dx - \theta \int_0^1 \rho^{\theta+1} r^{N-1} u_x^{2l-1} \phi_x (r^{N-1} u)_x dx$$

$$+ \int_0^1 \rho^{\theta} r^{N-1} u_x^{2l-1} \phi_x dx + (N - 1) \int_0^1 \rho^{\theta} r^{N-2} u_x^{2l} \phi_x dx$$

$$:= H_1 + H_2 + H_3 + H_4 + H_5 + H_6.$$  

(4.31)

Set

$$B_1^2 = \rho^{\theta+1} u_x^{2l-2} r^{2N-2} u_x^2 \geq 0, \quad B_2^2 = \rho^{\theta-1} u_x^{2l} \geq 0,$$

thus

$$H_1 + H_3 = -\theta (2l - 1) \int_0^1 \phi B_1^2 dx - 2(N - 1)(\theta - 1) l \int_0^1 \phi B_1 B_2 dx$$

$$+ ((N - 1)(N - 2) - \theta (N - 1)^2) \int_0^1 \phi B_2^2 dx.$$  

Inserting this to (4.31) and using Young’s inequality, we get

$$\frac{d}{d\tau} \int_0^1 \frac{1}{2l} \phi u^{2l} dx + \theta (2l - 1) \int_0^1 \phi B_1^2 dx$$
\[
\leq \varepsilon \int_0^1 \phi B_1^2 dx + C\varepsilon \int_0^1 \phi B_2^2 dx + \int_0^1 \rho^\gamma (r^{N-1}u^{2l-1})_x dx + |H_4| + |H_5| + |H_6|.
\]

(4.32)

By a direct computation, it follows

\[
\frac{d}{dT} \int_0^1 \phi u^2 dx + \int_0^1 \rho^\theta + 1 u^{2l-2} \tau^{2N-2} u_x^2 dx
\]

\[
\leq C_{x_0} (1 + M_{1,a}^2 + M_{1,a}^2 - M_{0,a} + M_{1,a}^2 - 2) + C_{x_0} (1 + M_{1}^2) \int_0^1 \phi u^2 dx,
\]

using Gronwall’s inequality and the fact that \( \phi = 1 \) for \( x \in [x_2, 1] \), we get

\[
\int_{x_2}^1 u^2 dx + \int_{x_2}^\tau \int_{x_2}^1 \rho^\theta + 1 u^{2l-2} \tau^{2N-2} u_x^2 dx ds \leq C, \quad \tau \in [0, T_{3,a}],
\]

(4.33)

with \( T_{3,a} \) determined by

\[
T_{3,a} = \min \{(C_{x_0} (1 + M_{1,a}^2 + M_{1,a}^2 - M_{0,a} + M_{1,a}^2 - 2))^{-1}, (C_{x_0} (1 + M_{1}^2))^{-1} \ln 2\},
\]

we get (4.26) immediately.

To show (4.27) we multiplying \( 3x \) by \( \phi r^{N-1} u_\tau \), where \( \phi = \psi^2(x) \) and \( \psi \in C^\infty([0, 1]) \) satisfies \( 0 \leq \psi \leq 1 \), \( \psi = 1 \) for \( x \in [x_2, 1] \), and \( \psi = 0 \) for \( x \in [0, x_0] \), integrate it over \([0, 1] \times [0, \tau] \) to have

\[
\int_0^\tau \int_0^1 \phi u_x^2 dx ds
\]

\[
= \int_0^\tau \int_0^1 \phi \rho^\gamma (u_x r^{N-1})_x dx ds - \int_0^\tau \int_0^1 \phi \rho^\theta + 1 (r^{N-1} u)_x (u_x r^{N-1})_x dx ds
\]

\[
+ (N - 1) \int_0^\tau \int_0^1 \phi \rho^\theta (r^{N-2} uu_x)_x dx ds + \int_0^\tau \int_0^1 \phi x \rho^\gamma u_x r^{N-1} dx ds
\]

\[
- \int_0^\tau \int_0^1 \phi x \rho^\theta + 1 (r^{N-1} u)_x u_x r^{N-1} dx ds + (N - 1) \int_0^\tau \int_0^1 \phi x \rho^\theta r^{N-2} uu_x dx ds
\]

\[
\leq C \delta^2 + C \int_0^\tau \int_0^1 \phi \rho^\theta + 3 u^4 x ds - C \int_0^1 \phi (\rho^\theta + 1 u^2 + \rho^\theta - 1 u^2) dx,
\]

which implies

\[
\int_0^\tau \int_{x_2}^1 u_x^2 dx ds + \int_{x_2}^1 (\rho^\theta + 1 u_x^2 + \rho^\theta - 1 u^2) dx
\]

\[
\leq C + C \int_0^\tau \int_{x_2}^1 (\rho^\theta + 3 u^4 x ds
\]

\[
\leq C + C \int_0^\tau \int_{x_2}^1 (\rho^\theta + 3 u^4 x ds,
\]

(4.34)

last we get (4.27), where we use the Lemma 3.1, Lemma 4.1, (3.19) and \( m > \frac{3}{4(1 + \beta - \beta)} \).
Now we begin to prove (4.28). Multiplying (3.4) with \( \phi \), where \( \phi = \psi^2(x) \) and \( \psi \in C^\infty([0,1]) \) satisfies \( 0 \leq \psi \leq 1 \), \( \psi = 1 \) for \( x \in [x_2,1] \), and \( \psi = 0 \) for \( x \in [0,x_0] \), then integrating over \([x_1,x]\), we have

\[
\int_x^1 \phi u_x^2 dx = \frac{1}{\theta} \phi x^{1-N} \rho^{\gamma-\theta-1} - (N-1) \phi \rho^{-1} u r^{-N} + \frac{1}{\theta} (N-1) \phi \rho^{-1} u r^{-N} + \frac{1}{\theta} (N-1) \phi \rho^{-1} u r^{-N} + \frac{1}{\theta} \rho^{-1-\theta} r^{1-N} \int_x^1 \phi (\frac{u}{r})^2 \rho^\theta dy - \frac{1}{\theta} \rho^{-1-\theta} r^{1-N} \int_x^1 \phi r^{1-N} u r dy \\
+ \frac{1}{\theta} \rho^{-1-\theta} r^{1-N} \int_x^1 \phi_y (\rho^\gamma - \theta \rho^{\theta+1} (r^{N-1} u)_y) dy \\
+ \frac{1}{\theta} (N-1) \phi \rho^{-1-\theta} r^{1-N} \int_x^1 \rho^\theta \phi_y \frac{u}{r} dy,
\]

integrating it over \([0,1]\), we have

\[
\int_0^1 \rho^{\theta+3} \phi^4 u_x^4 dx \\
\leq C \int_0^1 \phi^4 \rho^{4N-3\theta-1} dx + C \int_0^1 \phi^4 \rho^{4\theta-1} u^4 dx + C \int_0^1 \rho^{-3\theta-1} (\int_x^1 \phi (\frac{u}{r})^2 \rho^\theta dy)^4 dx \\
+ C \int_0^1 \rho^{-3\theta-1} (\int_x^1 \rho^{1-N} \phi u_x dy)^4 dx + C \int_0^1 \rho^{-3\theta-1} (\int_x^1 \rho^\theta \phi_y \frac{u}{r} dy)^4 dx \\
+ C \int_0^1 \rho^{-3\theta-1} (\int_x^1 \phi_y (\rho^\gamma - \theta \rho^{\theta+1} (r^{N-1} u)_y) dy)^4 dx \\
\leq C + C (\int_0^1 \phi^2 \rho^{\theta+1} u_x^2 dx)^2 + C (\int_0^1 \phi^2 \rho^{\theta-1} u^2 dx)^2 + C (\int_0^1 \phi^2 u_x dy)^2 \tag{4.35}
\]

which implies

\[
\int_0^1 \phi^4 \rho^{\theta+3} u_x^4 dx \leq C + C (\int_0^1 \phi^2 (\rho^{\theta-1} u^2 + u_x^2 + \rho^{\theta+1} u_x^2) dx)^2,
\]

and

\[
\int_{x_2}^1 \rho^{\theta+3} u_x^4 dx \leq C + C (\int_{x_2}^1 (\rho^{\theta-1} u^2 + u_x^2 + \rho^{\theta+1} u_x^2) dx)^2.
\]

Finally we prove (4.29). Differentiating equation (3.4) with respect to \( \tau \),

\[
u_{\tau \tau} = (r^{N-1}(\theta \rho^{\theta+1} (r^{N-1} u)_{\tau} - \rho^\gamma)_{\tau} - (N-1)(r^{N-2} u (\rho^\theta)_{\tau})_{\tau} = I + J, \tag{4.36}
\]

where

\[
I = r^{N-1}(\theta \rho^{\theta+1} (r^{N-1} u_{\tau}))_{\tau} - (N-1) r^{N-2} u_{\tau} (\rho^\theta)_{\tau};
\]

\[
J = (r^{N-1}(\theta \rho^{\theta+1} (r^{N-1} u_{\tau} - \rho^\gamma)_{\tau} - r^{N-1}(\theta \rho^{\theta+1} (r^{N-1} u_{\tau})))_{\tau} - (N-1)(N-2) r^{N-3} u^2 (\rho^\theta)_{\tau} - (N-1) r^{N-2} u (\rho^\theta)_{\tau}.
\]
Multiplying (4.36) by $\phi u_x$ and integrating over $x$ from 0 to 1, where $\phi = \psi^2(x)$ and $\psi \in C^\infty([0, 1])$ satisfies $0 \leq \psi \leq 1$, $\psi = 1$ for $x \in [x_2, 1]$, and $\psi = 0$ for $x \in [0, x_0]$, we have
\[
\frac{d}{dx} \int_0^1 \phi \frac{u_x^2}{2} dx + \left( \frac{1}{N} + \theta - 1 \right) \int_0^1 \phi \rho^{\theta + 1} (r^{N-1} u_x)_x^2 dx + \frac{N-1}{N} \int_0^1 \phi \rho^{\theta + 1} (r^{N-1} u_x - \rho^{-1} r^{-1} u_x)_x^2 dx \leq - \int_0^1 \phi \rho^{\theta + 1} r^{N-1} u_x (r^{N-1} u_x)_x dx + (N - 1) \int_0^1 \rho^{\theta} \phi r^{N-2} u_x^2 dx + \int_0^1 \phi J u_x dx
\]

\[
\leq \varepsilon \int_0^1 \phi \rho^{\theta + 1} r^{2(N-1)} u_x^2 dx + \varepsilon \int_0^1 \phi \rho^{\theta - 1} u_x^2 r^{-2} dx + C \int_0^1 \phi \rho^{\theta + 3} u_x^4 dx + C \int_0^1 \phi \rho^{3(1-\theta)} dx + C,
\]

after integrating over $[0, \tau]$ and choosing proper $\varepsilon$, which implies
\[
\int_0^1 u_x^2 dx + \int_0^\tau \int_0^1 (\rho^{\theta + 1} r^{2N-2} u_x^2 + \rho^{\theta - 1} r^{-2} u_x^2) dx ds \leq C \int_0^\tau \int_0^1 \rho^{\theta + 3} u_x^4 dx ds + \int_0^1 u_x^2(x, 0) ds + C
\]

\[
\leq C \int_0^\tau \left( \int_0^1 (\rho^{\theta - 1} u_x^2 + u_x^2 + \rho^{\theta + 1} u_x^2) dx \right)^2 ds + C,
\]

Using (4.27) and above we have
\[
\int_0^1 (\rho^{\theta + 1} u_x^2 + \rho^{\theta - 1} u_x^2 + u_x^2) dx + \int_0^\tau \int_0^1 (\rho^{\theta + 1} r^{2N-2} u_x^2 + \rho^{\theta - 1} r^{-2} u_x^2) dx ds \leq C_{11} + C_{12} \int_0^\tau \left( \int_0^1 (\rho^{\theta + 1} u_x^2 + \rho^{\theta - 1} u_x^2 + u_x^2) dx \right)^2 ds,
\]

by Gronwall’s inequality, we get
\[
\int_0^1 (\rho^{\theta + 1} u_x^2 + \rho^{\theta - 1} u_x^2 + u_x^2) dx + \int_0^\tau \int_0^1 (\rho^{\theta + 1} r^{2N-2} u_x^2 + \rho^{\theta - 1} r^{-2} u_x^2) dx ds \leq 2C_{11}, \quad 0 \leq \tau \leq T_3,
\]

where
\[
T_3 =: \min\{T_2, T_{3,a}, \frac{1}{2C_{11}C_{12}}\},
\]

thus we get (4.29).

\[\square\]

**Lemma 4.3.** Under the same assumptions as Lemma 3.4, the solution $(\rho, u, a)$ satisfies for the $x_2 \in (x_0, x_1)$ and $\tau \in [0, T_3]$ that
\[
\int_{x_2}^1 |u_x|^{\lambda_0} dx \leq C,
\]

where $\lambda_0$ is a constant satisfying:
\[
1 < \lambda_0 < \min\left\{ \frac{4m}{4m\beta + 1}, \frac{1}{\beta(\theta + 1)} \right\}.
\]
In addition, it holds
\[
| \rho r^{N-1} u_{x}(x, \tau) | \leq M_{0,b}, \quad | u(x, \tau) | \leq M_{1,b}, \quad (x, \tau) \in [x_2, 1] \times [0, T_3],
\]
where \( M_{0,b} \) and \( M_{1,b} \) are given by (4.40) and (4.44) respectively, and
\[
\rho \in C^0([x_2, 1] \times [0, T_3]), \quad u \in C^0([x_2, 1] \times [0, T_3]).
\]

**Proof.** From (3.41) and boundary conditions, we have
\[
u_x = \frac{1}{\theta} r^{1-N} \rho^\gamma - (N-1) \rho^{-1} w r^{-N} + \frac{1}{\theta} (N-1) \rho^{-1} w r^{-N}
\]
\[+ \frac{1}{\theta} (N-1) \rho^{-1} \theta r^{-1-N} \int_{x_2}^{1} \left( \frac{u}{r} \right) y \rho^\theta dy - \frac{1}{\theta} \rho^{-1} \theta r^{-1-N} \int_{x_2}^{1} r^{1-N} u_r dy,
\]
from (4.41), we obtain
\[
\int_{x_2}^{1} |u_x|^\lambda_0 dx \leq C \int_{x_2}^{1} (1 - x)^\lambda_0 f(x) dx + C(\int_{x_2}^{1} \rho^{-1} |x| \rho^\theta dx) \int_{x_2}^{1} u^{4m} dx \leq C_{13},
\]
so we obtain (4.37), where we use the fact \( \lambda_0 < \min\{\frac{1}{\beta(\theta+1)}, \frac{4m}{4m+1}\} \). Then we have
\[
\|u\|_{L^\infty([x_2, 1] \times [0, T_3])} \leq \frac{1}{1-x_2} \int_{x_2}^{1} |u| dx + \int_{x_2}^{1} |u_x| dx
\]
\[\leq C_{x_2} E_{0}^{\frac{1}{\lambda_0}} + (\int_{x_2}^{1} |u_x|^\lambda_0 dx)^{\lambda_0} (\int_{x_2}^{1} \lambda_0^{-1}) dx
\]
\[\leq C_{x_2} E_{0}^{\frac{1}{\lambda_0}} + C_{x_2} C_{13}^{\frac{1}{\lambda_0}} =: M_{1,b},
\]
with
\[
M_{1,b} =: \max\{C_{x_2} E_{0}^{\frac{1}{\lambda_0}} + C_{x_2} C_{13}^{\frac{1}{\lambda_0}}, \ M_{1,a}\}.
\]

From (4.41) for \( x \in [x_2, 1] \) we have
\[
| \rho r^{N-1} u_{x} | = \frac{1}{\theta} \rho^\gamma - (N-1) w r^{-1} + \frac{1}{\theta} (N-1) w r^{-1}
\]
\[+ \frac{1}{\theta} (N-1) \rho^{-1} \theta r^{-1-N} \int_{x_2}^{1} \left( \frac{u}{r} \right) y \rho^\theta dy - \frac{1}{\theta} \rho^{-1} \theta r^{-1-N} \int_{x_2}^{1} r^{1-N} u_r dy
\]
\[\leq C + C(\rho^\gamma + C(1-x)^{\frac{1}{\beta}-\theta} + C(1-x)^{\frac{1}{\beta(\theta+1)}-\theta} \]
\[\leq M_{0,b},
\]
(4.45)
with
\[ M_{0,b} =: \max\{C + C(\rho^*)^{\gamma-\theta}, M_{0,a}\}, \]  
which and (4.43) lead to (4.39).

We turn to prove (4.40). It is easy to verify
\[ \rho^\gamma \in L^\infty(0, T_3, H^1([x_2, 1])), \quad (\rho^\gamma)_x \in L^\infty(0, T_3, L^2([x_2, 1])). \]  
This implies \( \rho^\gamma \in C^0([x_2, 1] \times [0, T_3]) \) and the continuity of density \( \rho \in C^0([x_2, 1] \times [0, T_3]) \).

Indeed, it follows from (3.4),
\[ (\rho^\gamma)_x = -\gamma \rho^{\gamma+1}(r^{N-1}u)_x = -\gamma \rho^{\gamma+1}r^{N-1}u_x - \gamma(N - 1)\rho^\gamma ur^{-1} \in L^\infty(0, T_3, L^2([x_2, 1])). \]  

On the other hand, one derives from (4.17)
\[ \int_{x_2}^1 (r^{N-1}(\rho^\gamma)_x)^2 dx \leq C \int_{x_2}^1 (\rho^{\gamma-\theta} \rho_0x)^2 dx + C(\rho^*)^{2(\gamma-\theta)} \int_{x_2}^1 (r^{N-1}(\rho^\gamma)_x)^2 dx \]
\[ + C(\rho^*)^{2(\gamma-\theta)} \int_{x_2}^1 (u^2 + u_0^2) dx \]
\[ \leq C(\rho^*)^{2(\gamma-\theta)} + C(\rho^*)^{2(\gamma-\theta)} E_0 \]
\[ + C(\rho^*)^{2(\gamma-\theta)} \int_{x_2}^1 (r^{N-1}(\rho^\gamma)_x)^2 dx, \]
and then
\[ \int_{x_2}^1 (r^{N-1}(\rho^\gamma)_x)^2 dx \leq C(\rho^*)^{2(\gamma-\theta)}(E_0 + 1)(1 + C(\rho^*)^{2(\gamma-\theta)} e^{C(\rho^*)^{2(\gamma-\theta)}}), \]
which implies
\[ \int_{x_2}^1 (\rho^\gamma)_x^2 dx \leq C(\rho^*)^{2(\gamma-\theta)}(E_0 + 1)(1 + C(\rho^*)^{2(\gamma-\theta)} e^{C(\rho^*)^{2(\gamma-\theta)}}, \]
thus we obtain
\[ \rho^\gamma \in L^\infty(0, T_3, H^1([x_2, 1])), \]
this and (4.48) gives the half of (4.40). We can also show
\[ u \in L^\infty(0, T_3, W^{1,p}([x_2, 1])), \quad u_{\tau} \in L^\infty(0, T_3, L^2([x_2, 1])), \]
for any \( p \in (1, \beta^{-1}) \), so that
\[ \sup_{\tau \in [0, T_3]} \|u_x\|_{L^p([x_2, 1])} \leq \sup_{\tau \in [0, T_3]} \|\rho^{N-1}u\|_{L^1([x_2, 1])} \cdot \|\rho^{N-1}u_x\|_{L^\infty([x_2, 1] \times [0, T_3])} \leq C, \]
and
\[ \|u_{\tau}\|_{L^2([x_2, 1])}^2 = \int_{x_2}^1 u_\tau^2 dx \leq C, \]
this and (4.54) implies the continuity of velocity \( u \) on \([x_2, 1] \times [0, T_3] \).
5 Interior regularities

It is convenient to make use of (2.1) directly to investigate the interior regularities of solutions. Indeed, we have

Lemma 5.1. Under the assumptions of Theorem 2.7, there is a time $T_4 \in (0, T_3]$ so that the solution $(\rho, U)(x, t) = (\rho(r), u(r, t))$ to the FBVP (2.1) and (2.5) satisfies

$$
\|(\rho, U)(t)\|_{H^1(\Omega^m)}^2 + \int_0^t \|(\rho, U)(s)\|_{H^1(\Omega^m)}^2 ds \leq M, \quad t \in [0, T_4],
$$

(5.1)

where $\Omega^m = \{0 \leq |x| \leq r_{x_2}(t)\}$, $r_{x_2}(t)$ is the particle path with $r_{x_2}(0) = r_2 \in (r_0, r_1)$, and $M > 0$ is a constant given by (5.14). In particular, it holds

$$
\|\nabla U(t)\|_{L^\infty(\Omega^m)} \leq C_s M^{1/2}, \quad \|\nabla \rho(t)\|_{L^\infty((\Omega^m)^0)} \leq M_2, \quad t \in [0, T_4],
$$

(5.2)

$$
\frac{1}{2} \rho_0(r) \leq \rho(r, t) \leq 2 \rho_0(r), \quad (r, t) \in [0, r_{x_2}(t)] \times [0, T_4],
$$

(5.3)

with $C_s > 0$ the Sobolev constant for $\|f\|_{L^\infty} \leq C_s \|f\|_{H^2}$.

Proof. We first choose $T_{4,4} \leq T_3$ to be small and assume that it holds

$$
\|\nabla U(t)\|_{L^\infty(\Omega^m)} \leq 2M_0, \quad \|\nabla \rho(t)\|_{L^\infty(\Omega^m)} \leq 2M_2, \quad t \in (0, T_{4,4}],
$$

(5.4)

with

$$
M_0 = C_0 M^{1/2},
$$

(5.5)

and $M_2 = C_0 M^{1/2}$. It follows from (3.4) and (3.17)

$$
\|u(t)\|_{L^\infty([0, r_{x_2}(t)])} = \|U(t)\|_{L^\infty([0, r_{x_2}(t)])} \leq 2a_0 \|\nabla U(t)\|_{L^\infty([0, r_{x_2}(t)])} = 4a_0 M_0,
$$

(5.6)

which together with (4.4), (4.39), (3.18) and (3.24) (or (2.1)1) yields

$$
|u(r, t)| \leq M_1, \quad (r, t) \in [0, a(t)] \times [0, T_{4,4}],
$$

(5.7)

$$
\frac{1}{2} \rho_0(r) \leq \rho(r, t) \leq 2 \rho_0(r), \quad (r, t) \in [0, a(t)] \times [0, T_{4,4}],
$$

(5.8)

with $M_1 =: \max\{M_{1,4}, 4a_0 M_0\}$.

Corresponding to (3.10), we have the basic energy estimates

$$
\int_0^{a(t)} (\rho |U|^2 + \gamma^2) d\xi + \int_0^t \int_0^{a(t)} \left( |\rho \nabla U|^2 + |\rho^2 \nabla \log \rho|^2 \right) d\xi ds \leq E_0.
$$

(5.9)

Take derivative $\partial^\alpha$ with $1 \leq |\alpha| \leq 3$ to (2.1) to get

$$
(\partial^\alpha \log \rho)_{x_1} + U \cdot \nabla \partial^\alpha \log \rho + \nabla \cdot \partial^\alpha U(x(t), t) = g_\alpha,
$$

(5.10)

$$
\partial^\alpha U_{x_1} + U \cdot \nabla \partial^\alpha U + \gamma \rho^{-1} \nabla \partial^\alpha \log \rho = h_\alpha + f_\alpha + k_\alpha,
$$

(5.11)
where
\[ g_{\alpha} = -\partial^\alpha (U \cdot \nabla \log \rho) + U \cdot \nabla \partial^\alpha \log \rho, \quad h_{\alpha} = \partial^\alpha (\rho^{-1} \text{div} (\rho^\gamma \nabla U)) + (\theta - 1) \rho^{-1} \nabla (\rho^\gamma \text{div} U), \]
\[ f_{\alpha} = -(\partial^\alpha (U \cdot \nabla U) - U \cdot \nabla \partial^\alpha U), \quad k_{\alpha} = -\gamma (\partial^\alpha \rho^{-1} \nabla \log \rho - \rho^{-1} \nabla \partial^\alpha \log \rho). \]

Take inner product of (5.11) and \( \phi(x(r,t)) \partial^\alpha U \) over \([0, a(t)] \times [0, t] \), where \( \phi = \psi^2(x) \) and \( \psi \in C_0^\infty([0, 1]) \) satisfies \( 0 \leq \psi(y) \leq 1, \psi(y) = 1 \) for \( y \in [0, x_2] \), and \( \psi(y) = 0 \) for \( y \in [(1 - \eta)x_1, 1] \) with \( \eta > 0 \) small enough so that \((1 - \eta)x_1 > x_2\), and use the facts that \( \phi_t = \phi_x \rho N^{-1} u, \phi_r = \phi_x \rho N^{-1} \), make use of (5.10) and the relation
\[ \int \phi \rho^\gamma - 1 \partial^\alpha U \cdot \nabla \partial^\alpha \log \rho = - \frac{\delta}{a} \int \frac{\partial \phi \rho^\gamma - 1 \partial^\alpha \log \rho(2)}{2} - \frac{\gamma}{2} \int [(\phi \rho^\gamma - 1) + \nabla \cdot (\phi \rho^\gamma - 1 U)] \partial^\alpha \log \rho(2) \]

we have
\[ \int_0^{a(t)} \phi \left( \frac{1}{2} \partial^\alpha U \right)^2 + \frac{\gamma}{2} \rho^\gamma - 1 \partial^\alpha \log \rho(2) r^{-N-1} dr 
\]
\[ = \int_0^{a(0)} \phi \left( \frac{1}{2} \partial^\alpha U \right)^2 + \frac{\gamma}{2} \rho^\gamma - 1 \partial^\alpha \log \rho(2) r^{-N-1}(r, 0) dr 
\]
\[ - \frac{1}{2} \int_0^t \int_0^{a(s)} \phi \left( \partial^\alpha U \right)^2 r^{-N-1} dr ds - \int_0^t \int_0^{a(s)} \phi \partial^\alpha U \cdot (U \cdot \nabla \partial^\alpha U) r^{-N-1} dr ds 
\]
\[ + \frac{\gamma}{2} \int_0^t \int_0^{a(s)} \left( [\phi \rho^\gamma - 1] + \nabla \cdot (\phi \rho^\gamma - 1 U) \right) \partial^\alpha \log \rho(2) r^{-N-1} dr ds 
\]
\[ + \gamma \int_0^t \int_0^{a(s)} \partial^\alpha \log \rho \partial^\alpha U \cdot \nabla (\rho \rho^\gamma - 1) r^{-N-1} dr ds 
\]
\[ + \int_0^t \int_0^{a(s)} \phi \partial^\alpha U (\alpha + f\alpha + k\alpha) r^{-N-1} dr ds 
\]
\[ + \gamma \int_0^t \int_0^{a(s)} \phi r^N - 1 g\alpha r^{-1} \partial^\alpha \log \rho dr ds, \]

after a direct computation that
\[ \int_0^{a(t)} \phi \left( \partial^\alpha U \right)^2 + \rho^\gamma - 1 \partial^\alpha \log \rho(2) r^{-N-1} dr 
\]
\[ + \int_0^t \int_0^{a(s)} \phi \rho^\gamma - 1 \left( \nabla \partial^\alpha U \right)^2 + \left( \nabla \cdot \partial^\alpha U \right)^2 r^{-N-1} dr ds 
\]
\[ \leq C \left( \| U_0 \|_H^2 \right) + C_{x_0 \cdot x_1} \left( 1 + M_0 + M_1 + M_2 + (\rho^\gamma - 1) \right) \left( \delta^2_4 + \delta^2_5 \right) t 
\]
\[ + C_{x_0 \cdot x_1} \left( M_0^2 + M_1^2 + M_2^2 + \delta^2_4 + \delta^2_5 \right) \int_0^t \int_0^{a(s)} \phi \left( \partial^\alpha U \right)^2 + \rho^\gamma - 1 \partial^\alpha \log \rho(2) r^{-N-1} dr ds 
\]
\[ + C \left( 1 + M_0 + M_2 + \delta_4 + \delta_5 \right) \int_0^t \left( \sum_{|\alpha|=1}^3 \int_0^{a(s)} \phi \left( \partial^\alpha U \right)^2 + \rho^\gamma - 1 \partial^\alpha \log \rho(2) r^{-N-1} dr ds \right) ds, \]

which implies
\[ Y(t) + \sum_{|\alpha|=1}^3 \int_0^t \int_0^{a(s)} \phi \rho^\gamma - 1 \left( \nabla \partial^\alpha U \right)^2 + \left( \nabla \cdot \partial^\alpha U \right)^2 r^{-N-1} dr ds \]
\[ \leq K_6 + K_7 t + K_8 \int_0^t Y^2(s) ds, \quad (5.12) \]

where
\[ K_6 = C\| (U_0, \rho_0) \|_{H^3([0,r_1])}^2, \]
\[ K_8 = C_{x_0,x_1} (M_0^2 + M_1^2 + M_2^2 + \delta_4^2 + \delta_5^2) + C(1 + M_0 + M_2 + \delta_4 + \delta_5), \]
\[ K_7 = C_{x_0,x_1} (1 + M_0 + M_1 + M_2 + (\rho^s)^{-1} (\delta_4^2 + \delta_5^2) + C_{x_0,x_1} (M_0^2 + M_1^2 + M_2^2 + \delta_4^2 + \delta_5^2), \]
\[ T_{4,b} =: \min \{ K_7^{-1} C\| (U_0, \rho_0) \|_{H^3([0,r_1])}^2, T_{4,a} \}, \]
and
\[ Y(t) = \int_0^{a(t)} \phi(\| \partial U, \partial^2 U, \partial^3 U \|)^2 + |(\partial \log \rho, \partial^2 \log \rho, \partial^3 \log \rho|^2)(r,t) r^{N-1} dr. \]

We apply Gronwall’s inequality to have
\[ \sum_{|\alpha|=1} \int_0^{a(t)} \phi(\| \partial^\alpha U \|)^2 + |\partial^\alpha \log \rho|^2 r^{N-1} dr \]
\[ + \sum_{|\alpha|=1} \int_0^t \int_0^{a(s)} \phi \rho^{\theta-1} (|\nabla \partial^\alpha U|^2 + |\nabla \cdot \partial^\alpha U|^2) r^{N-1} dr ds \]
\[ \leq 4C\| (U_0, \rho_0) \|_{H^3([0,r_1])}^2, \quad t \in [0, T_4], \quad (5.13) \]
for \( T_4 =: \min \{ T_{4,b}, T_{4,c} \} \) with \( T_{4,c} = (4K_6K_8)^{-1}. \) The (5.13) together with (5.8), Lemma 4.4 and
\[ \| \rho \|_{H^3(\Omega^a)} \leq C_{x_0,x_1} (\| \log \rho \|_{H^3(\Omega^a)}^3, \]
leads to (5.1) with \( M \) given by
\[ M = C_{x_0,x_1} (4C\| (U_0, \rho_0) \|_{H^3(\Omega^a)}^2)^3. \quad (5.14) \]
The estimates (5.4) and (5.2) follow respectively from (5.1) and the Sobolev embedding theorem, and (5.3) follows from (3.24) and (5.1).

6 Proof of the main results

Proposition 6.1 (Existence and Uniqueness). Under the assumptions of Theorem 2.1 there exists a time \( T_* > 0 \) dependent of initial data, so that the FBVP (2.1) and (2.5) admits a unique solution
\[ (\rho, \rho U, a)(x, t) = (\rho(r, t), \rho u(r, t) \xi, a(t)), \quad r = |x|, \quad (r, t) \in [0, a(t)] \times [0, T_*], \]
which satisfies
\[ \| \nabla U \|_{L^\infty([0,r_{x_2}(t)] \times [0,T_*])} \leq M_0, \quad \| \rho r^{-1} u_x \|_{L^\infty([0,1] \times [0,T_*])} \leq M_0, \quad (6.1) \]
\[
\frac{1}{2} \rho_0(x) \leq \rho(x, \tau) \leq 2 \rho_0(x), \quad (x, \tau) \in [0, 1] \times [0, T_*],
\]
for constant \(M_0 > 0\) dependent of initial data, \(\rho_+ = 2\rho^*\) and \(\rho_- = \frac{1}{2}\rho^*\). In addition, \((\rho, \rho U, a)\) satisfies the inner regularities in Euler coordinates

\[
\int_0^t (\rho |U|^2 + \rho^2) dx + \int_0^t \int_0^t \rho^2 \nabla U^2 dx ds \leq C, \quad t \in [0, T_*],
\]

(6.3)

\[
\| (\rho, U) (t) \|_{H^3(\Omega^m)}^2 + \sum_{|\alpha| = 1}^3 \int_0^t \int_{\Omega^m} (|\partial^\alpha \rho|^2 + |\partial^\alpha+1 U|^2) dx ds \leq C_{in}, \quad t \in [0, T_*],
\]

(6.4)

where \(\Omega^m = \{0 \leq |x| \leq r_1(x, t)\}\), \(r_1(x, t)\) is defined by \(r_1'(x, t) = u(r_1(x, t), t)\) with \(r_1(0) = r_1 \in (r_2, a_0)\) and \(x_1 = 1 - \int_{r_1}^a \rho_0 r^{N-1} dr \in (x_2, 1)\), and \(C > 0\) and \(C_{in} > 0\) are constants, and the boundary regularities in Lagrange coordinates

\[
\int_{x_2}^1 u^2 dx + \int_0^\tau \int_{x_2}^1 \rho^{1+u^2-2} u^{2N-2} u^2 dx ds \leq C_b, \quad t \in [0, T_*],
\]

(6.5)

\[
\int_{x_2}^1 u^2 dx + \int_0^\tau \int_{x_2}^1 (\rho^{1+u^2-2} u^2 + \rho^{2-1} r^{2u^2}) dx ds \leq C_b, \quad t \in [0, T_*],
\]

(6.6)

with the integer \(1 \leq k \leq 2m\), and \(C_b > 0\) is a constant.

**Proof of the Theorem 2.1** With the estimates we have obtained in Sections 3-5, we can apply the method of difference scheme and compactness arguments as in [3, 8] and references therein, to prove the existence of weak solutions to the FBVP (2.1) and (2.5), we omit here. Next, we apply the idea in [3] to prove the uniqueness. Let \((\rho_1, u_1, r_1)\) and \((\rho_2, u_2, r_2)\) are two solutions to the FBVP (3.4)–(3.8), and denote

\[
(\rho, \omega, R)(x, \tau) = (\rho_1 - \rho_2, u_1 - u_2, \frac{r_1}{r_2} - 1)(x, \tau).
\]

Based on Proposition 6.1, we can derive the following estimates

\[
0 < c \rho_\varepsilon (\rho^*)^{-1} \leq \frac{\rho_1(x, \tau)}{\rho_2(x, \tau)} + \frac{\rho_2(x, \tau)}{\rho_1(x, \tau)} \leq C \rho_\varepsilon (\rho^*)^{-1}, \quad (x, \tau) \in [0, 1] \times [0, T_*],
\]

(6.7)

\[
\frac{|u_1|}{r_1} + \frac{|u_2|}{r_2} + |\rho_1^{1+\theta} r_1^{N-1} u_{1x}| + |\rho_2^{1+\theta} r_2^{N-1} u_{2x}| \leq C, \quad x \in [0, 1],
\]

(6.8)

\[
0 < C_{x_0}^{-1} (2a_0)^{-1} \leq \frac{r_1(x, \tau)}{r_2(x, \tau)} + \frac{r_2(x, \tau)}{r_1(x, \tau)} \leq 2C_{x_0} a_0, \quad (x, \tau) \in [x_0, 1] \times [0, T_*],
\]

(6.9)

\[
(\rho_\varepsilon (\rho^*)^{-1})^{1/N} \leq \frac{r_1(x, \tau)}{r_2(x, \tau)} + \frac{r_2(x, \tau)}{r_1(x, \tau)} \leq (\rho_\varepsilon (\rho^*)^{-1})^{1/N}, \quad (x, \tau) \in [x_0, 0] \times [0, T_*],
\]

(6.10)

with which, we can show the uniqueness of the solutions. Indeed, From (3.4) and using Young’s inequality, we have

\[
\frac{d}{dt} \int_0^1 \rho_1^{\theta-1} R^2 dx = \int_0^1 (2\rho_1^{\theta-1} RR + (\theta - 1)\rho_1^{\theta-2} \rho_1 R^2) dx
\]
\[(1 - \theta) \int_0^1 \rho_1^\theta (r_1^{-1} u_1)_x dx + 2 \int_0^1 \rho_1^{\theta - 1} R \left( \frac{u_1}{r_2} - \frac{r_1 u_2}{r_2^2} \right) dx \leq \varepsilon \int_0^1 \rho_1^{\theta - 1} \rho_1^\varepsilon dx + C_\varepsilon \int_0^1 \rho_1^{\theta - 1} R^2 dx, \quad (6.11)\]

and

\[
\frac{d}{d\tau} \int_0^1 \rho_1^{\theta - 3} \rho^2 dx = 2 \int_0^1 \rho_1^{\theta - 3} \rho (\rho_1 - \rho_2) dx + (\theta - 3) \int_0^1 \rho_1^{\theta - 4} \rho_1 \rho^2 dx
\]

\[
= (3 - \theta) \int_0^1 \rho_1^{\theta - 2} (r_1^{-1} u_1)_x \rho^2 dx + 2 \int_0^1 \rho_1^{\theta - 3} \rho (\rho_1^2 (r_1^{-1} u_1)_x - \rho_2^2 (r_2^{-1} u_2)_x) dx
\]

\[
\leq \varepsilon \int_0^1 \rho_1^{\theta - 1} \rho_1^\varepsilon dx + \varepsilon \int_0^1 \rho_1^{\theta + 1} r_1^{2(N - 1)} \rho_1^\varepsilon + C_\varepsilon \int_0^1 \rho_1^{\theta - 3} \rho^2 + C_\varepsilon \int_0^1 \rho_1^{\theta - 1} R^2 dx, \quad (6.12)\]

where \(\varepsilon > 0\) is chosen later and \(C_\varepsilon > 0\) a constant.

From the equation (3.4) and boundary condition, we get

\[
\frac{d}{d\tau} \int_0^1 \frac{1}{2} \omega^2 dx = \int_0^1 \left\{ -\theta \rho_1^{\theta + 1} (r_1^{-1} u_1)_x (r_1^{-1} \omega)_x - \theta \rho_2^{\theta + 1} (r_2^{-1} u_2)_x (r_2^{-1} \omega)_x \right\} dx
\]

\[
+ \int_0^1 \left\{ \rho_1^{\theta} (r_1^{-1} \omega)_x - \rho_2^{\theta} (r_2^{-1} \omega)_x \right\} dx
\]

\[
+ (N - 1) \int_0^1 \left\{ \rho_1^{\theta} (r_1^{-2} u_1 \omega)_x - \rho_2^{\theta} (r_2^{-2} u_2 \omega)_x \right\} dx.
\]

Using the similar argument as that in Lemma 3.2 and Cauchy-Schwartz inequality, we have

\[
\frac{d}{d\tau} \int_0^1 \frac{1}{2} \omega^2 dx \leq \varepsilon \int_0^1 \rho_1^{\theta + 1} r_1^{2(N - 1)} \rho_1^\varepsilon dx + \varepsilon \int_0^1 \rho_1^{\theta - 1} \rho_1^\varepsilon dx
\]

\[
+ C_\varepsilon \int_0^1 \rho_1^{\theta - 3} \rho^2 dx + C_\varepsilon \int_0^1 \rho_1^{\theta - 1} R^2 dx
\]

\[
- \frac{1}{2} \int_0^1 \{ (\theta - 1 + \frac{1}{N}) \rho_1^{\theta + 1} [(r_1^{-1} \omega)_x]^2
\]

\[
+ \frac{N - 1}{N} \rho_1^{\theta + 1} [r_1^{-1} \omega_x - \frac{\omega}{r_1 \rho_1}]^2 \} dx, \quad (6.13)\]

where \(\varepsilon > 0\) small enough and \(C_\varepsilon > 0\) is a constant.

Apply the Gronwall’s inequality to the summation of (6.11)–(6.13), we can finally obtain

\[
\int_0^1 (\omega^2 + \rho_1^{\theta - 1} R^2 + \rho_1^{\theta - 3} \rho^2) (x, \tau) dx \leq C \int_0^1 (\omega^2 + \rho_1^{\theta - 1} R^2 + \rho_1^{\theta - 3} \rho^2) (x, 0) dx = 0, \quad (x, \tau) \in [0, 1] \times [0, T_\varepsilon],
\]

which implies \((\rho_1, u_1, r_1) = (\rho_2, u_2, r_2)\).

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