ON PRACTICAL STABILITY OF DIFFERENTIAL INCLUSIONS
USING LYAPUNOV FUNCTIONS

VOLODYMYR PICHKUR
Taras Shevchenko National University of Kyiv
Department of Computer Science and Cybernetics
Volodymyrska Str. 60, 01033, Kyiv, Ukraine

Abstract. In this paper we consider the problem of practical stability for differential inclusions. We prove the necessary and sufficient conditions using Lyapunov functions. Then we solve the practical stability problem of linear differential inclusion with ellipsoidal right-hand part and ellipsoidal initial data set. In the last section we apply the main result of this paper to the problem of practical stabilization.

1. Introduction. In many problems of practical importance it is necessary to investigate dynamic behavior of a system under state constrains on a finite time interval. Such problems are studied by methods of the practical stability theory. The main research technique in the theory of practical stability is the direct Lyapunov’s method and its generalizations. Note that practical stability of solution does not imply stability in Lyapunov’s sense, and conversely, unstable (in Lyapunov sense) solution can have good behavior on a finite interval. It means that Lyapunov functions being used in the practical stability problems do not obligatory satisfy the conditions of the second Lyapunov’s method theorems.

Different problems concerning stability analysis for differential inclusions are studied in [1, 3, 5, 10, 13, 11, 18, 20, 21]. In these works Lyapunov function is used to investigate global, strong and weak stability. Converse Lyapunov theorems are proved in [2, 5]. The stabilization problem on the basis of weak stability is considered in [20].

The concept of practical stability was introduced in [8, 17]. In works [3, 15, 10] the second Lyapunov method and methods of stability theory have been developed with respect to the problems of practical stability. Sufficient conditions of practical stability have been obtained for different types of practical stability, in some cases proved the necessary conditions. In [6, 7, 8, 12], the concept that studies the properties of maximum sets of initial conditions has been proposed and the effective numerical methods for different practical problems have been developed (for instance, for the charge beams optimization problem). This approach was effective in studying the practical stability problems of differential inclusions solutions. In [6, 12] the topological properties of optimal set of initial conditions (compactness, boundary and interior properties) have been studied both for strong and weak practical stability. In the case of linear differential inclusions and convex phase
constraints different techniques for describing such sets have been proposed (for example, support function, Minkowski function were used). In [19] similar results were obtained for discrete inclusions. But concerning practical stability of differential inclusions constructive generalization of the Lyapunov function method is still an open problem.

In this paper we study the problem of practical stability for differential inclusions. We prove the necessary and sufficient conditions using Lyapunov functions. Further we solve the practical stability problem of linear differential inclusion with ellipsoidal righthand part and ellipsoidal initial data set. In the last section we apply the main result of this paper to the problem of practical stabilization.

2. Necessary and sufficient conditions of practical stability. In this paper we introduce the following basic notations: $\mathbb{R}^n$ is an $n$-dimensional Euclidean space; $\langle x, y \rangle$ is the usual inner product of $x, y \in \mathbb{R}^n$, $\|x\| = \sqrt{\langle x, x \rangle}$; $K_r(a)$ is the ball in $\mathbb{R}^n$, $K_r(a) = \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$, $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$, and $E(a, Q)$ is the ellipsoid in $\mathbb{R}^n$,

$$E(a, Q) = \{x \in \mathbb{R}^n : \langle Q^{-1}(x - a), x - a \rangle \leq 1\},$$

where $Q$ is symmetric positive definite $n \times n$-matrix, $a \in \mathbb{R}^n$, $r > 0$; $M^*$ is the transpose of $n \times m$-matrix $M$; $intA$, $\partial A$ are respectively the set of inner points and the boundary, $A \subset \mathbb{R}^n$; $comp(\mathbb{R}^n)$ ($conv(\mathbb{R}^n)$) is the set of nonempty (convex) compact subsets of $\mathbb{R}^n$; $\alpha(A, B)$ is the Hausdorff distance for $A, B \subset \mathbb{R}^n$, $\|A\| = \alpha(A, 0)$, $c(A, \psi) = \sup_{x \in A} \langle x, \psi \rangle$, $\psi \in \mathbb{R}^n$.

We consider differential inclusion

$$\frac{dx}{dt} \in F(x,t),$$

where $x \in \mathbb{R}^n$ is an $n$-dimensional vector of phase coordinate, $(x,t) \in D, D \subset \mathbb{R}^{n+1}$ is a bounded domain. A set-valued mapping $F : D \to conv(\mathbb{R}^n)$ is measurable with respect to variable $t$ and satisfies the Lipschitz condition

$$\alpha(F(x,t), F(y,t)) \leq L(t) \|x - y\|.$$ (2)

Here $L(t)$ is a positive integrable function, $(x,t) \in D$, $(y,t) \in D$, $F(0,t) = 0$, $(0,t) \in D$. The map $F$ is integrably bounded. It means that there exists an integrable positive function $\lambda(\cdot)$ so that

$$F(x,t) \subseteq \lambda(t)K_1(0), (x,t) \in D.$$ (3)

A multifunction $\Phi : [t_0, T] \to comp(\mathbb{R}^n)$ prescribes state constraints, graph of the mapping $\Phi$ belongs to $D$, $0 \in int\Phi(t)$, $t \in [t_0, T]$. Let $x(t, z, s)$ be a solution of (1) corresponding to the Cauchy condition $x(s) = z$, $G_0 \subseteq \mathbb{R}^n$.

Definition 2.1. We say, that the zero solution of differential inclusion (1) is $\{G_0, \Phi(t), t_0, T\}$ - stable, if arbitrary solution $x(t, x_0, t_0)$ of (1) belongs to $\Phi(t)$ for any point $x_0 \in G_0$, $t \in [t_0, T]$.

Theorem 2.2. The zero solution of differential inclusion (1) is $\{G_0, \Phi(t), t_0, T\}$ - stable, if and only if there exists a continuous function $V : D \to \mathbb{R}^1$ such that the following conditions take place:

1) $G_0 \subseteq \{x \in \mathbb{R}^n : V(x, t_0) \leq 1\}$; (4)

2) $\{x \in \mathbb{R}^n : V(x, t) \leq 1\} \subseteq \Phi(t)$, $t \in [t_0, T]$; (5)
3). $V(x(t), t)$ is nonincreasing function, where $x(t)$ is a solution of the differential inclusion \[4\].

**Proof.** Necessity. We assume that the zero solution of differential inclusion \[1\] is \{$G_0, \Phi(t), t_0, T\}$ - stable. Consider the maximum set of initial conditions $G_*$ such that $x(t, x_0, t_0) \in \Phi(t), t \in [t_0, T]$, for all $x_0 \in G_*$. It means that $G_0 \subseteq G_*$. The set $G_*$ is compact \[6, 12\]. Therefore we may define a continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}^1$ having the following properties \[6\]:

1. $g(x) < 1$, if $x \in \text{int}G_*$;
2. $g(x) = 1$, if $x \in \partial G_*$;
3. $g(x) > 1, x \notin G_*$.

For instance, the function

$$g(x) = \begin{cases} 1 - \rho(x), & \text{if } x \in G_*; \\ 1 + \rho(x), & \text{if } x \notin G_* \end{cases}$$

satisfies the requirements. Here

$$\rho(x) = \min_{y \in \partial G_*} \|x - y\|$$

is the distance between $x \in \mathbb{R}^n$ and $\partial G_*$. We denote by $\Omega(y, t)$ the set of all solutions of the differential inclusion \[1\] defined on $[t_0, T]$ such that $x(s) = y$. The set $\Omega(y, t)$ is a compact subset of space $C([t_0, T]; \mathbb{R}^n)$ \[14]. Let

$$X(t_0, y, t) = \{x(t_0) : x(\cdot) \in \Omega(y, t)\}.$$

The set $X(t_0, y, t) \in \text{comp}(\mathbb{R}^n)$ and generates a continuous set-valued mapping with respect to variables $y, t \ [14]$. Consider a function

$$V(y, t) = \min_{x(\cdot) \in \Omega(y, t)} g(x(t_0)).$$

It is clear that $V(y, t) = \min_{z \in X(t_0, y, t)} g(z)$. The function $V(y, t)$ is continuous \[1\].

Let us show that the conditions 1 - 3 of the theorem hold. In fact, if $y \in G_0, t = t_0$, then

$$V(y, t) = g(y) \leq 1, \ y = x(t_0).$$

Therefore \[4\] is true.

To prove \[5\] we assume that $V(y, s) \leq 1$, where $s \in [t_0, T]$ is arbitrary. It means that there exists a solution $x_*(\cdot) \in \Omega(y, s)$ of differential inclusion \[1\] such that $x_*(s) = y$ and

$$V(y, t) = \min_{x(\cdot) \in \Omega(y, t)} g(x(t_0)) = g(x_*(t_0)) \leq 1.$$

Therefore $x_*(t_0) \in G_*$. From definition 2.1 it follows that $x_*(s) = y \in \Phi(s)$. Hence

$$\{x \in \mathbb{R}^n : V(x, s) \leq 1\} \subseteq \Phi(s)$$

so that \[5\] is true.

Finally, let us prove that $V(x(t), t)$ is nonincreasing function, where $x(t)$ is an arbitrary solution of the differential inclusion \[1\]. It is clear that

$$X(t_0, x(t_1), t_1) \subseteq X(t_0, x(t_2), t_2), \ t_0 \leq t_1 < t_2 \leq T.$$ 

Therefore

$$\min_{z \in X(t_0, x(t_2), t_2)} g(z) \leq \min_{z \in X(t_0, x(t_1), t_1)} g(z).$$
Remark 1. Let function $g$ in (4) take place. Taking any point $x$ a solution $x(t) = x(t, x_0, t_0)$. Since the function $V(x(t), t)$ is nonincreasing we see that

$$V(x(t), t) \leq V(x_0, t_0) \leq 1, \quad t \in [t_0, T].$$

Due to (5)

$$x(t) \in \{x \in \mathbb{R}^n : V(x, t) \leq 1\} \subseteq \Phi(t), \quad t \in [t_0, T].$$

Thus definition 2.1 holds and the zero solution of (1) is $\{G_0, \Phi(t), t_0, T\}$ - stable.

Following traditions the function $V(x, t)$ in theorem 2.2 is called the Lyapunov function.

**Remark 1.** Let $G_0 \in \text{comp}(\mathbb{R}^n)$. Define a continuous function $g_0 : \mathbb{R}^n \rightarrow \mathbb{R}^1$ such that

1. $g_0(x) < 1$, if $x \in \text{int}G_0$;
2. $g_0(x) = 1$, if $x \in \partial G_0$;
3. $g_0(x) > 1$, $x \notin G_0$.

To prove the necessity of theorem 2.2 we may use the function

$$V(y, t) = \min_{x(t) \in \Omega(y, t)} g_0(x(t)).$$

For instance, if $G_0 = K_r(0)$, then

$$V(y, t) = \min_{x(t) \in \Omega(y, t)} \|x(t_0)\| + r - 1.$$

**Remark 2.** If we claim that $F$ is upper semicontinuous set-valued function with respect to $x$ instead of the Lipschitz condition (2), then the sufficiency of theorem 2.2 holds

**Corollary 1.** Suppose that there exists a continuously differentiable function $V : D \rightarrow \mathbb{R}^1$ such that conditions (7), (8) hold and upper derivative due to differential inclusion (4)

$$\frac{dV(x, t)}{dt} = \frac{\partial V(x, t)}{\partial t} + \max_{v \in F(x, t)} \langle \text{grad}_x V(x, t), v \rangle \leq 0, \quad (x, t) \in D. \quad (6)$$

Then the zero solution of differential inclusion (1) is $\{G_0, \Phi(t), t_0, T\}$ - stable.

**Corollary 2.** Let there exists a continuous function $V : D \rightarrow \mathbb{R}^1$ such that conditions (4), (5) take place and for any solution $x(t)$ of the differential inclusion (7) the function $V(x(t), t)$ does not increase on

$$\{(x, t) \in D : V(x, t) \geq 1\}.$$

Then the trivial solution of differential inclusion (1) is $\{G_0, \Phi(t), t_0, T\}$ - stable.

**Proof.** We take any $x_0 \in G_0$ and a solution $x(t)$ of (1), $x(t_0) = x_0$. From (4) we have $V(x(t_0), t_0) \leq 1$. If $V(x(t), t) \leq 1, \quad t \in [t_0, T]$, then taking into account (5) we obtain $x(t) \in \Phi(t), \quad t \in [t_0, T]$. Now assume that we can find $s \in (t_0, T)$ so that $V(x(s), t) > 1$. Since $V(x(t), t)$ is continuous there exists $\tau \in (t_0, s)$ such that $V(x(\tau), \tau) = 1$. But

$$(x(t), t) \in \{(x, t) \in D : V(x, t) \geq 1\} \cup [\tau, s].$$
Therefore
\[ 1 = V(x(\tau), \tau) \geq V(x(s), s) > 1. \]

This contradiction implies \( V(x(t), t) \leq 1, t \in [t_0, T]. \) Thus from (5) we have \( x(t) \in \Phi(t), t \in [t_0, T]. \) It means that definition 2.1 holds and the trivial solution of (1) is \( \{G_0, \Phi(t), t_0, T\} - \) stable. \( \square \)

**Corollary 3.** Suppose that there exists a continuously differentiable function \( V : D \to \mathbb{R}^1 \) such that conditions (4), (5) hold and
\[
\left( \frac{dV}{dt} \right) = \frac{\partial V(x, t)}{\partial t} + \max_{v \in F(x, t)} \langle \text{grad}_x V(x, t), v \rangle \leq 0, (x, t) \in D, V(x, t) \geq 1. \quad (7)
\]

Then the zero solution of differential inclusion (7) is \( \{G_0, \Phi(t), t_0, T\} - \) stable.

3. **Practical stability of linear differential inclusion.** In this section we consider linear differential inclusion
\[
\frac{dx}{dt} \in A(t)x + E(0, H(t)), \quad (8)
\]
where \( x \in \mathbb{R}^n \) is an \( n \)-dimensional vector of phase coordinate, \( A(t) \) is a continuous \( n \times n \)-matrix, \( H(t) \) is a continuous symmetric positive definite \( n \times n \)-matrix. We denote by \( \Theta(t, s) \) a fundamental matrix of the linear system \( \frac{dx}{dt} = A(t)x \), normalized at the point \( s \) so that \( \Theta(s, s) = I \), where \( I \) is the identity \( n \times n \)-matrix. A continuous multifunction \( \Phi : [t_0, T] \to \text{conv}(\mathbb{R}^n) \) prescribe phase constraints,
\[
\int_{t_0}^{t} \Theta(t, s)E(0, H(s))ds \subset \text{int}\Phi(t), \quad t \in [t_0, T].
\]

This inclusion implies \( 0 \in \text{int}G \). Here \( G \) is the maximum set of initial conditions under the phase constraints \( \Phi(t), t \in [t_0, T]. \) It means that for arbitrary \( x_0 \in G \) any suitable solution of differential inclusion (8) belongs to \( \Phi(t) \) for all \( t \in [t_0, T]. \) Under the conditions of this section \( G \) is the convex compact set (9) [12].

Now we prove the following result.

**Theorem 3.1.** We suppose that \( G_0 = E(0, Q_0), \)
\[
\min_{t \in [t_0, T]} \min_{\psi \in S} \left( c(\Phi(t), \psi) - \sqrt{\langle Q(t)\psi, \psi \rangle} \right) \geq 0, \quad (9)
\]
an \( n \times n \)-matrix \( Q(t) \) is a positive definite solution of the matrix differential equation
\[
\frac{dQ(t)}{dt} = A(t)Q(t) + Q(t)A^*(t) + qQ(t) + q^{-1}H(t), \quad t \in [t_0, T], \quad (10)
\]
\[
Q(t_0) = Q_0, \quad (11)
\]
\( q > 0, Q_0 \) is a symmetric positive definite \( n \times n \)-matrix. Then the trivial solution of differential inclusion (8) is \( \{G_0, \Phi(t), t_0, T\} - \) stable.

**Proof.** We consider the function
\[
V(x, t) = \langle R(t)x, x \rangle,
\]
where \( R(t) = Q^{-1}(t). \) We have
\[
G_0 = E(0, Q_0) = \{ x \in \mathbb{R}^n : V(x, t_0) \leq 1 \}
\]
so that condition 4 holds.
Since $0 \in \text{int}\Phi(t)$ the support function $c(\Phi(t), \psi) > 0, \psi \in S$. From [10] it follows that
\[
\sqrt{(Q(t)\psi, \psi)} \leq c(\Phi(t), \psi),
\]
for all $\psi \in S, t \in [t_0, T]$. But the function $\sqrt{(Q(t)\psi, \psi)}$ is the support function of the set
\[
\{x \in \mathbb{R}^n : V(x, t) \leq 1\} = \{x \in \mathbb{R}^n : (R(t)x, x) \leq 1\}.
\]
Using support function properties we get
\[
\{x \in \mathbb{R}^n : V(x, t) \leq 1\} \subseteq \Phi(t), \ t \in [t_0, T].
\]
Thus condition (4) is true.

As far as $R(t) = Q^{-1}(t)$ after differentiating $Q(t)R(t) = I$ we obtain
\[
\frac{d}{dt}(Q(t)R(t)) = \frac{dQ(t)}{dt}R(t) + Q(t)\frac{dR(t)}{dt} = 0.
\]
Therefore from (10)
\[
\frac{dR(t)}{dt} = -R(t)\frac{dQ(t)}{dt}R(t) = -R(t)A(t)Q(t) + Q(t)A^*(t) + qQ(t) + q^{-1}H(t)R(t) =
\]
\[
-\frac{dQ(t)}{dt}A(t) - A^*(t)R(t) - qR(t) - q^{-1}R(t)H(t)R(t).
\]
Thus the matrix $R(t)$ satisfies the matrix differential equation
\[
\frac{dR(t)}{dt} + R(t)A(t) + A^*(t)R(t) + qR(t) + q^{-1}R(t)H(t)R(t) = 0, \ t \in [t_0, T],
\]
(12)
\[
Q(t_0) = Q_0^{-1}.
\]
From [7], [8]
\[
\left(\frac{dV}{dt}\right) = \left\langle \frac{dR(t)}{dt}x, x\right\rangle + \left\langle R(t)A(t)x, x\right\rangle + \max_{v \in E(0, H(t))} \left\langle (R(t) + R^*(t))x, v\right\rangle.
\]
Using (12), (14), we get
\[
\left(\frac{dV}{dt}\right) = -q \left\langle R(t)x, x\right\rangle - q^{-1} \left\langle R(t)H(t)R(t)x, x\right\rangle + 2\sqrt{(R(t)H(t)R(t)x, x)}.
\]
The Cauchy inequality implies
\[
2\sqrt{(R(t)x, x)(R(t)H(t)R(t)x, x)} \leq q \left\langle R(t)x, x\right\rangle + q^{-1} \left\langle R(t)H(t)R(t)x, x\right\rangle.
\]
Taking into account (15), we obtain
\[
\left(\frac{dV}{dt}\right) \leq 2\sqrt{(R(t)H(t)R(t)x, x)} \left(1 - \sqrt{(R(t)x, x)}\right).
\]
Hence
\[
\left(\frac{dV}{dt}\right) \leq 0
\]
if $V(x, t) = \langle R(t)x, x \rangle \geq 1$. From corollary 3 of theorem 2.2 it follows that the trivial solution of differential inclusion [8] is $\{G_0, \Phi(t), t_0, T\}$ - stable.
For instance, if $\Phi(t) = K_{r(t)}(0)$, $r(t) > 0$ is a continuous function on $[t_0, T]$, then

$$\max_{t \in [t_0, T]} \frac{\sqrt{\lambda_{\text{max}}(Q(t))}}{r(t)} \leq 1,$$

is completely equivalent to condition (9). Here $\lambda_{\text{max}}(Q(t))$ is the maximum eigenvalue of $Q(t)$.

Consider a particular case. Let the righthand part of differential inclusion (8) be time-independent. It means that $A(t) = A$, $H(t) = H$, $t \in [t_0, T]$, where $A$ is $n \times n$ matrix, $H$ is a symmetric positive definite $n \times n$-matrix. In this case the following statement is true.

**Theorem 3.2.** Suppose that

$$\min_{t \in [t_0, T]} \min_{\psi \in S} \left( c(\Phi(t), \psi) - \sqrt{\langle Q\psi, \psi \rangle} \right) \geq 0, \quad (16)$$

an $n \times n$-matrix $Q$ is positive definite and satisfies the matrix equation

$$AQ + QA^* + qQ + q^{-1}H = 0, \quad q > 0, \quad (17)$$

and $Q - Q^{-1}_0$ is a nonnegative definite matrix. Here $Q_0$ is a symmetric positive definite $n \times n$-matrix, $G_0 = E(0, Q_0)$.

Then the trivial solution of differential inclusion (8) is $\{G_0, \Phi(t), t_0, T\}$-stable.

**Proof.** The proof of theorem 3.2 is similar to the proof of the previous theorem. Let us consider the Lyapunov function

$$V(x, t) = \langle Rx, x \rangle,$$

where $R = Q^{-1}$. From (17) it follows that $R$ satisfies the matrix equation

$$RA + A^*R + qR + q^{-1}RHR = 0. \quad (18)$$

Equality (16) is equivalent to (4). From (18), (8) we get

$$\left( \frac{dV}{dt} \right) = \langle (RA + A^*R)x, x \rangle + 2 \max_{v \in E(0, H)} \langle Rx, v \rangle =$$

$$-q \langle Rx, x \rangle - q^{-1} \langle RHRx, x \rangle + 2\sqrt{\langle RHRx, x \rangle} \leq$$

$$2\sqrt{\langle RHRx, x \rangle} \left( 1 - \sqrt{\langle Rx, x \rangle} \right) \leq 0$$

when $V(x) = \langle Rx, x \rangle \geq 1$.

Since $Q - Q^{-1}_0$ is a nonnegative definite matrix we obtain $\langle Q^{-1}_0 \psi, \psi \rangle \leq \langle R\psi, \psi \rangle$, $\psi \in S$. Therefore

$$c(E(0, Q_0), \psi) \leq c(E(0, Q), \psi), \quad \psi \in S.$$

Using support function properties we get $G_0 = E(0, Q_0) \subseteq E(0, Q)$. But $E(0, Q) = \{x \in \mathbb{R}^n : V(x) \leq 1\}$ and condition (4) holds.

From corollary 3 of theorem 2.2 it follows that the trivial solution of differential inclusion (8) is $\{G_0, \Phi(t), t_0, T\}$-stable. \qed

For instance, let $\Phi(t) = K_r(0)$, $r > 0$, $t \in [t_0, T]$. Then (16) transforms to

$$\lambda_{\text{max}}(Q) \leq r^2.$$
4. On practical stabilization of differential inclusions. Given differential inclusion
\[
\frac{dx}{dt} \in F(x, t) + G(t)u(x, t),
\]
where, as it was above, \((x, t) \in D, D \subset \mathbb{R}^{n+1}\) is a bounded domain, a set-valued mapping \(F : D \rightarrow \text{conv} (\mathbb{R}^n)\) is measurable with respect to variable \(t\), upper semicontinuous with respect to \(x\), \(F(0, t) = 0, (0, t) \in D\) and satisfies \([3]\). Further, \(G(t)\) is integrable \(n \times m\)-matrix, \(u(x, t)\) is an \(m\)-dimensional control function, \(u(0, t) = 0\). We assume that \(u(x, t)\) is integrably bounded on \(D\), continuous with respect to variable \(x\) being measurable with respect to \(t\).

A multifunction \(\Phi : [t_0, T] \rightarrow \text{comp} (\mathbb{R}^n)\) prescribes phase constraints, the graph of the mapping \(\Phi\) belongs to \(D, 0 \in \text{int} \Phi(t), t \in [t_0, T], G_0 \subset \Phi(t_0)\). Suppose, that there exists a continuously differentiable function \(V : D \rightarrow \mathbb{R}^1\) such that
\[
\Phi(t) = \{x \in \mathbb{R}^n : V(x, t) \leq 1\}, \quad t \in [t_0, T],
\]
and \(V(0, t) = 0, \ grad_x V(x, t) = 0\) on \([t_0, T]\).

The problem of \(\{G_0, \Phi(t), t_0, T\}\) - stabilization for differential inclusion \([19]\) consists of finding the admissible control function \(u(x, t)\) such that the zero solution to \([19]\) is \(\{G_0, \Phi(t), t_0, T\}\) - stable.

**Theorem 4.1.** Suppose that \(W(x, t)\) is a continuous nonnegative function on \(D\) and
\[
|W(x, t) + \frac{\partial W(x, t)}{\partial t} + c(F(x, t), \ grad_x V(x, t))| \leq C \parallel G^*(t) \grad_x V(x, t) \parallel^2,
\]
\(C > 0\). Then control function
\[
u(x, t) = \begin{cases} (k(x, t)I + P)G^*(t) \grad_x V(x, t), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}
\]
solves the problem of \(\{G_0, \Phi(t), t_0, T\}\) - stabilization for differential inclusion \([19]\).
Here \(P = -P^*\) is an arbitrary \(m \times m\) - matrix,
\[
k(x, t) = -\frac{W(x, t) + \frac{\partial V(x, t)}{\partial t} + c(F(x, t), \ grad_x V(x, t))}{\parallel G^*(t) \grad_x V(x, t) \parallel^2}.
\]

**Proof.** Due to \([20]\), \(G_0 \subset \Phi(t_0)\) conditions \([4], [5]\) of theorem \([2.2]\) are fulfilled. We require that
\[
\left( \frac{dV}{dt} \right)^{[5]} = -W(x, t)
\]
so that \([5]\) is true. Hence
\[
\frac{\partial V(x, t)}{\partial t} + \max_{v \in F(x, t)} \langle \grad_x V(x, t), v + G(t)u(x, t) \rangle = -W(x, t)
\]
and
\[
\langle G^*(t) \grad_x V(x, t), u(x, t) \rangle = -W(x, t) - \frac{\partial V(x, t)}{\partial t} - c(F(x, t), \ grad_x V(x, t)).
\]

Plugging \([22]\) in \([24]\) we get \([23]\). From \([21]\) it follows that \(u(x, t)\) is continuous function with respect to \(x\) on \(D\). Thus all conditions of corollary \([1]\) hold. 
\(\square\)
As an example, consider differential inclusion
\[
\frac{dx}{dt} \in \Omega(t)x + G(t)u(x,t).
\] (25)

Here \( \Omega : [t_0, T] \to \text{comp}(\mathbb{R}^{n \times n}) \) is a measurable integrably bounded multifunction. In other words, there exists an integrable positive function \( \lambda(\cdot) \) so that \( \Omega(t) \subseteq \lambda(t)B, t \in [t_0, T], \) where \( B \) is the unit ball in \( \mathbb{R}^{n \times n}, \mathbb{R}^{n \times n} \) is space of \( n \times n \)-matrices with real components.

The solution of \( \{G_0, \Phi(t), t_0, T\} \) - stabilization problem for differential inclusion (25) is given by theorem 4.1, where
\[
|W(x, t) + \frac{\partial V(x, t)}{\partial t} + c(\Omega(t), \text{grad}_x V(x, t)x^*)| \leq C \|G^*(t)\text{grad}_x V(x, t)\|^2,
\]
\[
k(x, t) = -\frac{W(x, t) + \frac{\partial V(x, t)}{\partial t} + c(\Omega(t), \text{grad}_x V(x, t)x^*)}{\|G^*(t)\text{grad}_x V(x, t)\|^2}.
\]

Here \( c(\Omega(t), \Psi) = \max_{A \in \Omega(t)} tr(A^*\Psi) \) is a support function, \( \Psi \in \mathbb{R}^{n \times n}. \)

For instance, if \( G(t) = I \),
\[
\Omega(t) = \{A \in \mathbb{R}^{n \times n} : tr(A^*A) \leq r^2\}, \quad V(x, t) = \frac{1}{2} \langle Mx, x \rangle, \quad W(x, t) = \langle Nx, x \rangle,
\]
\( M, N \) are symmetric positive definite \( n \times n \)-matrices, \( r > 0 \), then
\[
k(x, t) = k(x) = -\frac{\langle Nx, x \rangle + r\|x\|\|Mx\|}{\|Mx\|^2}.
\]

In this case the control function (22) depends only on \( x \)
\[
u(x) = \begin{cases} 
2(k(x)I + P)G^*Mx, & \text{if } x \neq 0, \\
0, & \text{if } x = 0
\end{cases}
\]
and (21) is obviously true.

**Acknowledgments.** The author would like to thank prof. F. G. Garashchenko for fruitful discussions on the subject of the paper.

**REFERENCES**

[1] D. Angeli, B. Ingalls, E. D. Sontag and Y. Wang, Uniform global asymptotic stability of differential inclusions. *Journal of Dynamical and Control Systems*, 10 (2004), 391–412.

[2] E. Arzarello and A. Bacciotti, On stability and boundedness for lipschitzian differential inclusions: The converse of Lyapunov’s theorems. *Set-Valued Analysis*, 5 (1997), 377–390.

[3] J. P. Aubin and A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory* Berlin-Heidelberg-New York-Tokyo, Springer-Verlag, 1984.

[4] J. P. Aubin and H. Frankowska, *Set-valued Analysis* Boston, Birkhäuser, 2009.

[5] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory* Berlin - Heidelberg - New York, Springer, 2005.

[6] O. M. Bashnyakov, F. G. Garashchenko and V. V. Pichkur, *Practical Stability, Estimations and Optimization*, Kyiv : Taras Shevchenko National University of Kyiv, 2008.

[7] A. N. Bashnyakov, V. V. Pichkur and I. V. Hitko, On Maximal Initial Data Set in Problems of Practical Stability of Discrete System. *J. Automat. Inf. Scien.*, 43 (2011), 1–8.

[8] B. N. Bublik, F. G. Garashchenko and N. F. Kirichenko, Structural - Parametric Optimization and Stability of Bunch Dynamics, Kyiv: Naukova dumka, 1985.

[9] N. G. Chetaev, On certain questions related to the problem of the stability of unsteady motion. *J. Appl. Math. Mech.*, 24 (1960), 6–19.

[10] K. Deimling, *Multivalued Differential Equations* Berlin-New York: Walter de Gruyter, 1992.

[11] R. Gama and G. Smirnov, Stability and optimality of solutions to differential inclusions via averaging method. *Set-Valued and Variational Analysis*, 22 (2014), 349–374.
[12] F. G. Garashchenko and V. V. Pichkur, Properties of optimal sets of practical stability of differential inclusions. Part I. Part II. (Russian) Problemy Upravlen. Inform., (2006), 163–170.
[13] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Dordrecht-Boston-London: Kluwer Academic, 1988.
[14] A. F. Filippov, Differential Equations with Discontinuous Righthand Sides and Differential Inclusions, in *Nonlinear Analysis and Nonlinear Differential Equations* (eds. V. A. Trenogin and A. F. Filippov), Moscow: FIZMATLIT, (2003), 265–288.
[15] N. F. Kirichenko, *Introduction to the Stability Theory*, Kyiv: Vyshcha Shkola, 1978.
[16] V. Lakshmikantham, S. Leela and A. A. Martynyuk, *Practical Stability of Nonlinear Systems* Singapore: World Scientific, 1990.
[17] J. LaSalle and S. Lefshetz, *Stability by Lyapunov Direct Method and Application*, New York: Academic Press, 1961.
[18] A. Michel, K. Wang and B. Hu, *Qualitative Theory of Dynamical Systems. The Role of Stability-Preserving Mappings*, Marcel Dekker, Inc., New York, 1995.
[19] V. V. Pichkur and M. S. Sasonkina, Maximum set of initial conditions for the problem of weak practical stability of a discrete inclusion *J. Math. Sci.*, 194 (2013), 414–425.
[20] G. Smirnov, *Introduction to the Theory of Differential Inclusions*, American Mathematical Society, 2002.
[21] V. Veliov, Stability-like properties of differential inclusions *Set-Valued Analysis*, 5 (1997), 73–88.

Received January 2016; revised February 2016.

E-mail address: vpichkur@gmail.com