Classification of materials for conducting spheroids based on the first order polarization tensor

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Abstract. Polarization tensor is an old terminology in mathematics and physics with many recent industrial applications including medical imaging, nondestructive testing and metal detection. In these applications, it is theoretically formulated based on the mathematical modelling either in electrics, electromagnetics or both. Generally, polarization tensor represents the perturbation in the electric or electromagnetic fields due to the presence of conducting objects and hence, it also describes the objects. Understanding the properties of the polarization tensor is necessary and important in order to apply it. Therefore, in this study, when the conducting object is a spheroid, we show that the polarization tensor is positive-definite if and only if the conductivity of the object is greater than one. In contrast, we also prove that the polarization tensor is negative-definite if and only if the conductivity of the object is between zero and one. These features categorize the conductivity of the spheroid based on its polarization tensor and can then help to classify the material of the spheroid.

1. Introduction
Polarization tensor (PT) has many applications in electric and electromagnetic systems. Previously, it has been used to improve electrical imaging [1] (for biomedical or industrial purposes) as well as investigating electrosensing fish [2, 3, 4, 5, 6, 7] and also metal detection for security screening [8, 9, 10] or landmine clearance [11, 12, 13]. It arises from the studies about the perturbation of electric or electromagnetic field in a free space (such as 2D or 3D) due to the conductivity contrast between the space itself and some small conducting objects.

In applied mathematics and analysis, previous studies by [1, 14, 15, 16, 17, 18, 19] have shown that the perturbation due to the presence of the conducting objects can be represented by asymptotic formulas and PT is actually the dominant term in the formula. The asymptotic formula and the corresponding PT are generally not the same for different systems. However, the PT for each system can be determined using explicit formula if the geometry and material properties of the presented conducting objects are known. Therefore, the PT caused by a conducting object is also referred as the PT for that object. Some examples regarding computation of the PT based on the explicit formula and according to the geometry as well as the material of a conducting object can be found in [17, 20, 21, 22, 23].
On the other hand, the PT is sometimes referred as the polarizibility tensor in engineering literatures and can be obtained from field measurements either in laboratory or fieldwork [8, 9, 10, 12, 13]. Since the measurements can be made as long as there is a conducting object perturbing the electric or electromagnetic field, any other information about the object is not needed to find the PT (see Makkonen et al. [10]). Thus, by making measurements with metal detectors, the PT is commonly used by electrical engineers to identify and characterize metallic targets. However, some measurements produce errors. So, if the geometry and material of the target are given, the PT for that target can also be computed by using the explicit formula derived from the asymptotic method and then compared with the PT obtained from field measurement. This will increase the chance of correctly identifying the target. Previous studies comparing the PT obtained by using explicit formula and field measurements were done by [5, 18, 24].

In the studies on electrosensing fish, the PT for a specific object is computed based on the explicit formula. Thus, it is very helpful to understand the properties of the PT before computing it to ensure accuracy. Besides, given a PT for some objects, Ahmad Khairuddin and Lionheart [6] and Khairuddin and Lionheart [25] have suggested that it is sometimes necessary to find an ellipsoid that has the similar PT with that objects. Similarly, in metal detection, it is necessary to use the PT obtained from metal detectors to identify and describe detected metallic targets [8, 9, 10, 11, 12, 13, 24]. In these two situations, it is important to understand what is described by the PT before applying it.

In this study, we will investigate one property of the PT which represents the material of a conducting object, in particular a conducting spheroid that perturbs an electric field in the three dimensional space. Basically, our finding suggests that the PT can be classified based on the conductivity of the spheroid and reversely, the conductivity can also be categorized based on the PT. In order to present these results, we first review the mathematical formulation for the related PT in the next section.

2. The First Order PT for a Spheroid

Let \( u \) be the electrical voltage with the presence of an isolated object \( B \) in \( \mathbb{R}^3 \) and define the conductivity \( \sigma(x) \) such that for any point \( x \in \mathbb{R}^3 \),

\[
\sigma(x) = \begin{cases} 
  k & \text{for } x \in B \\
  1 & \text{for } x \in \mathbb{R}^3 
\end{cases}
\]  

(1)

where \( k \) is a constant depending on the material of \( B \). In this case, \( u \) satisfies

\[
\nabla \cdot (\sigma \nabla u) = 0.
\]

(2)

The equation (1) suggests that there exists a conductivity contrast between \( \mathbb{R}^3 \) (conductivity equal to 1) and \( B \) (conductivity equal to \( k \)).

If a harmonic function in \( \mathbb{R}^3 \) denoted by \( H \) is the voltage without the object \( B \), then from [1],

\[
(u - H)(x) = -\nabla \Gamma(x) \cdot M \nabla H(0) + O(1/|x|^2) \text{ as } |x| \to \infty
\]

(3)

where the origin \( O \in B, \Gamma(x) = -(4\pi|x|)^{-1} \) and \( M \) is the PT called as the first order PT. This asymptotic formula represents the perturbation in the voltage due to a small conducting object in \( \mathbb{R}^3 \), where, the dominant term of the expansion is determined by \( M \). Here, \( M \) for \( B \) with conductivity \( k \) denoted by \( M(k, B) \) does not depend on the position of \( B \). Moreover, \( M \) is a real \( 3 \times 3 \) matrix by construction and [1] has shown that \( M \) is symmetric. For a general object \( B, M \) can be determined by solving system of integral equations [1].

Suppose that \( B \) is a spheroid, aligned with the coordinate axes in the Cartesian coordinate system and is given by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1 \) where \( a \) and \( b \) are the length of semi principal axes. By modifying
the explicit formula of $M$ for ellipsoid given in [1], Mohamad Yunus and Ahmad Khairuddin [26] have shown that $M$ for a spheroid with conductivity $k$ is alternatively given by

$$
M(k, B) = (k - 1)|B| \begin{bmatrix}
\frac{1}{(1 - d_1) + kd_1} & 0 & 0 \\
0 & \frac{2}{(1 + d_1) + k(1 - d_1)} & 0 \\
0 & 0 & \frac{2}{(1 + d_1) + k(1 - d_1)}
\end{bmatrix}
$$

(4)

where $|B|$ is the volume of $B$ and $d_1$ is a constant also known as depolarization factors for spheroid given in Milton [27]. Depolarization factors was previously used by [27] to study composites and also classically appeared in the studies of electromagnetism by [28, 29].

Here, if $B$ is prolate spheroid such that $a \geq b$, $d_1$ is given in [27] as

$$
d_1 = \frac{1 - \epsilon^2}{\epsilon^2} \left( \frac{1}{2\epsilon} \ln \left( \frac{1 + \epsilon}{1 - \epsilon} \right) - 1 \right)
$$

(5)

where $\epsilon = \sqrt{1 - \left( \frac{b}{a} \right)^2}$. On the other hand, if $a \leq b$ such that $B$ is an oblate spheroid, $d_1$ is given in [27] as

$$
d_1 = \frac{1}{\varphi^2} \left( 1 - \sqrt{1 - \frac{\varphi^2}{\varphi} \sin^{-1} \varphi} \right)
$$

(6)

where $\varphi = \sqrt{1 - \left( \frac{a}{b} \right)^2}$. Therefore, (4) together with (5) or (6) can be used to compute the first order PT for prolate or oblate spheroid.

Moreover, it is easy to see in (5) and (6) that $0 < \epsilon < 1$ and $0 < \varphi < 1$. Based on this, the following properties of $d_1$ in both (5) and (6) were previously proposed by Ahmad Khairuddin [24]. These propositions will be used to describe our main results in the next section.

**Proposition 1** If $B$ is a prolate spheroid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$ in the Cartesian coordinate system such that $a \geq b$, then $0 < d_1 < \frac{1}{3}$ when $0 < \epsilon < 1$.

**Proposition 2** If $B$ is an oblate spheroid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$ in the Cartesian coordinate system such that $a \leq b$, then $\frac{1}{3} < d_1 < 1$ when $0 < \varphi < 1$.

We also recall the following property of the first order PT given by Ammari and Kang [1].

**Proposition 3** The first order PT is positive definite matrix if $k > 1$ and it is negative definite matrix if $0 < k < 1$.

3. Main Results

Our main results in this study are actually the extensions of Proposition 3. Besides showing that the first order PT for spheroid is positive definite matrix if $k > 1$, we also show that if the first order PT for a spheroid is positive definite matrix then the spheroid must have conductivity, $k > 1$. Similarly, while showing that the first order PT for spheroid is negative definite matrix if $0 < k < 1$, we also show that if the first order PT for a spheroid is negative definite matrix then the spheroid must have conductivity, $0 < k < 1$.

Before proceeding further, to complete the proofs of our results, we recall the following definitions given in [24].
Definition 1 A square real matrix $M_{n \times n}$ is a positive definite matrix if and only if $x^T M x > 0$ for every nonzero $x \in \mathbb{R}^n$.

Definition 2 A square real matrix $M_{n \times n}$ is a negative definite matrix if and only if $x^T M x < 0$ for every nonzero $x \in \mathbb{R}^n$.

Our first result is presented in the following theorem.

Theorem 1

Let $B$ be a prolate spheroid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in the Cartesian coordinate system such that $a \geq b$. Suppose that the conductivity of $B$ is $k > 0$, $k \neq 1$.

i. $k > 1$ if and only if the first order PT of $B$ is a positive definite matrix.

ii. $k < 1$ if and only if the first order PT of $B$ is a negative definite matrix.

Proof:

Part (i)

Assume that $k > 1$ and we want to show that the first order PT for prolate spheroid is a positive definite matrix. This is actually the specific case of Proposition 3. The proof is given [1].

Next, assume that the first order PT of a prolate spheroid $B$ is a positive definite matrix such that $B$ is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a \geq b$. We want to show that $k > 1$. Since the first order PT denoted by $M$ is a $3 \times 3$ diagonal matrix as given in (4), for any $x_1, x_2, x_3 \in \mathbb{R}$, we then have

$$x^T M x = [x_1 \ x_2 \ x_3] M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$$x^T M x = M \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix} .$$

Let $\frac{|B|}{1-d_1+kd_1} = \alpha$ and $\frac{2|B|}{(1+d_1)+k(1-d_1)} = \beta$. Therefore,

$$x^T M x = (k-1) \begin{bmatrix} \alpha & \beta \\ \beta & \beta \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix},$$

$$x^T M x = (k-1)(\alpha x_1^2 + \beta x_2^2 + \beta x_3^2).$$

Since the first order PT is a positive definite matrix, we have $(k-1)(\alpha x_1^2 + \beta x_2^2 + \beta x_3^2) > 0$ by Definition 1. Here, $\alpha > 0$ since $|B| > 0$ and $1-d_1+kd_1 > 0$ for $k > 0$ and $0 < d_1 < \frac{1}{3}$ based on Proposition 1. Similarly, $\beta > 0$. Thus, $(\alpha x_1^2 + \beta x_2^2 + \beta x_3^2) > 0$ due to $x_1^2, x_2^2, x_3^2 > 0$. This means we must also have $(k-1) > 0$ which concludes that $k > 1$.

Part (ii)

Assume that $k < 1$ and we want to show that the first order PT for prolate spheroid is a negative definite matrix. This is actually the specific case of Proposition 3. The proof is given [1].

Next, assume that the first order PT of a prolate spheroid $B$ is a positive definite matrix such that $B$ is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a \geq b$. We want to show that $k < 1$. Similarly,

$$x^T M x = (k-1)(\alpha x_1^2 + \beta x_2^2 + \beta x_3^2)$$

where $\frac{|B|}{1-d_1+kd_1} = \alpha$ and $\frac{2|B|}{(1+d_1)+k(1-d_1)} = \beta$. 

Since the first order PT denoted by $M$ is a negative definite matrix, we have $(k-1)(\alpha x_1^2 + \beta x_2^2 + \beta x_3^2) < 0$ by Definition 2. Hence, $(\alpha x_1^2 + \beta x_2^2 + \beta x_3^2) > 0$ since $|B|$, $k > 0$, $0 < d_1 < \frac{1}{3}$ by Proposition 1 and $x_1^2, x_2^2, x_3^2 > 0$. Thus, we must have $(k-1) < 0$ which concludes that $k < 1$.

We also have the following theorem.

**Theorem 2**

Let $B$ be an oblate spheroid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$ in the Cartesian coordinate system such that $a \leq b$. Suppose that the conductivity of $B$ is $k > 0$, $k \neq 1$.

i. $k > 1$ if and only if the first order PT of $B$ is a positive definite matrix.

ii. $k < 1$ if and only if the first order PT of $B$ is a negative definite matrix.

**Proof**:
The proof of this theorem can be done by repeating the previous steps except now, we have to use Proposition 2 instead of Proposition 1 together with Definition 1 (for part (i)) and Definition 2 (for part (ii)).

4. Discussion and Conclusion

In this study, we have shown that the conductivity of a spheroid (prolate or oblate) can be classified based on its PT. If the spheroid has conductivity greater than 1 then its first order PT must be a positive definite matrix and if the spheroid has conductivity less than 1 then its first order PT must be a negative definite matrix. Moreover, we also show that if the first order PT of a spheroid is a positive definite matrix then the spheroid must have conductivity greater than 1 whereas if the first order PT of a spheroid is a negative definite matrix then the spheroid must have conductivity less than 1. These are achieved by using some previous properties related to the PT especially the depolarization factors. In the future, the theoretical results obtained during this study could be very useful to determine the material of a spheroid based on its conductivity in the related applications.

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