Bayesian Estimation of Stress-Strength $P(T < X < Z)$ for Dagum Distribution

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Abstract. In this paper, the reliability formula is derived for the stress-strength model of the probability $P(T < X < Z)$ for a component's strength $X$ falling between two stresses $T$ and $Z$, based on Dagum Distribution with unknown parameter $\beta$ and known and common parameters $\lambda$ and $\delta$. Bayesian estimation is discussed to estimate the reliability under complete data by using Gamma prior based on two loss function (weighted and quadratic loss functions), and the comparison between these estimators based on a simulation study using mean square error criteria (MSE) for each of the small, medium and large samples. The most important conclusion is that this comparison confirms that the performance of the estimator according to the weighted loss function works better for the most experiments studied.

1. Introduction
Camilo Dagum introduced Dagum distribution in (1977) for modelling data for personal income as an alternative to the Pareto and log-normal models. This distribution was widely used in various fields such as, income and fortune data, meteorological data, reliability and survival analysis [1]. The Dagum distribution belongs to the Burr model with an additional scale parameter. The Dagum distribution is also called the inverse Burr, particularly in the actuarial literature, as it is the reciprocal transformation of the Burr XII [2]. One of the most important characteristic of Dagum distribution is that it can have a monotonically decreasing hazard function. This trait has led to the study of the model in several fields. In fact, in recent years Dagum distribution was studied from reliability point of view and used for the analysis of survival data [1]. Kleiber and Kotz in (2003) presented a comprehensive review of the Dagum model's origins and applications [3]. Domma et al. in (2011) used censored samples to estimate the parameters of the Dagum distribution [4].

For any random variable X that follows Dagum distribution the cumulative density function (cdf) is given by: [1]

\[ F(x) = \left(1 + \lambda x^{-\delta}\right)^{-\beta} ; x > 0; \beta, \lambda, \delta > 0 \]  

and the probability density function (pdf):

\[ f(x) = \beta \lambda \delta x^{-\delta - 1} \left(1 + \lambda x^{-\delta}\right)^{-\beta - 1} ; x > 0; \beta, \lambda, \delta > 0 \]

Where $\lambda$ is the scale parameter and the shape parameters are $\beta, \delta$.

The main aim of this paper is to obtain a mathematical formula for the reliability $R$ of the probability that component's strength is between two stresses based on Dagum distribution in section 2. Then
Bayesian estimation is used to estimate this reliability in section 3. A simulation study was conducted to compare the performance of the estimator according to the weighted loss function of the reliability in section 4, based on nine experiments of shape parameter values and at different sample sizes of (15) for small, (30) for medium and (90) for large sample sizes. The comparison is made by the mean square error criteria (MSE), and the conclusions are discussed in section 5.

2. Reliability formula
The stress-strength model in the reliability studies describes the life of a component which has a random strength $X$ and is subjected to random stress $Y$. where the reliability of the stress component, when the applied stress is greater than its strength $X$.

$$R = P(T < X < Z)$$

$$= \int_0^\infty P(T < x, Z > x/X = x) dF_x(x)$$

$$= \int_0^\infty H_T(x) (1 - G_Z(x)) f(x) dx$$

(3)

Suppose that $T$ and $Z$ be two independent random stress variables with cumulative density functions $H_T(t), G_Z(z)$ from Dagum distribution as $DD(\beta_1, \lambda, \delta)$ and $DD(\beta_2, \lambda, \delta)$; respectively. Let $X$ be a random strength variable from $DD(\beta, \lambda, \delta)$. Assuming that $X$ is independent from $T$ and $Z$, then the densities of $T$ and $Z$ given by:

$$H_T(t) = (1 + \beta t^{-\delta})^{-\beta_1} \quad t > 0; \beta_1, \lambda, \delta > 0$$

(4)

$$G_Z(z) = (1 + \beta z^{-\delta})^{-\beta_2} \quad z > 0; \beta_2, \lambda, \delta > 0$$

(5)

Now, from equation (1):

$$R = \int_0^\infty (1 + \lambda x^{-\delta})^{-\beta_1} (1 - (1 + \lambda x^{-\delta})^{-\beta_2}) \beta \lambda \delta x^{-\delta - 1} (1 + \lambda x^{-\delta})^{-\beta - 1} dx$$

$$= \int_0^\infty (1 + \lambda x^{-\delta})^{-\beta_1} \beta \lambda \delta x^{-\delta - 1} (1 + \lambda x^{-\delta})^{-\beta - 1} dx - \int_0^\infty (1 + \lambda x^{-\delta})^{-\beta_1} (1 + \lambda x^{-\delta})^{-\beta_2} \beta \lambda \delta x^{-\delta - 1} (1 + \lambda x^{-\delta})^{-\beta - 1} dx$$

$$= \beta \int_0^\infty \lambda \delta x^{-\delta - 1} (1 + \lambda x^{-\delta})^{-(\beta + \beta_1 + \beta_2)} dx - \beta \int_0^\infty \lambda \delta x^{-\delta - 1} (1 + \lambda x^{-\delta})^{-(\beta + \beta_1 + \beta_2)} dx$$

Since $f(x)$ is probability density function, then we can rewrite equation (2) as:

$$\int_0^\infty \lambda \delta x^{-\delta - 1} (1 + \lambda x^{-\delta})^{-\beta - 1} dx = \frac{1}{\beta}$$

So then,  

$$R = \frac{\beta}{\beta + \beta_1} - \frac{\beta}{\beta + \beta_1 + \beta_2} = \frac{\beta \beta_2}{(\beta + \beta_1)(\beta + \beta_1 + \beta_2)}$$

(6)

3. Bayesian Estimation
In this section, the Bayesian estimation is performed, under complete data using Gamma Prior based on two loss functions (weighted and quadratic loss functions).

3.1. The Likelihood function
Let \( x_1, x_2, \ldots, x_n \) be a strength random sample, and let \( t_1, t_2, \ldots, t_m \) and \( z_1, z_2, \ldots, z_m \) be a stress random samples from Dagum Distribution with unknown shape parameters \( \beta > 0, \beta_1 > 0, \beta_2 > 0 \) and common Known shape and scale parameters \( \lambda > 0, \delta > 0 \), respectively. Therefore the likelihood function of \((\beta, \beta_1, \beta_2)\), for observed sample is given by: [1]

\[
L(\mathbf{x} | \beta, \lambda, \delta) = (\beta \lambda \delta)^n \prod_{i=1}^{n} x_i^{\delta-1} (1 + \lambda x_i^{-\delta})^{-\beta-1}
\]

\[
= \beta^n \lambda^n \delta^n \prod_{i=1}^{n} x_i^{\delta-1} \prod_{i=1}^{n} \left(1 + \lambda x_i^{-\delta}\right)^{-1} \prod_{i=1}^{n} \left(1 + \lambda x_i^{-\delta}\right)^{-\beta}
\]

\[
= A \beta^n \prod_{i=1}^{n} \left(1 + \lambda x_i^{-\delta}\right)^{-\beta}
\]

\[
L(\mathbf{x} | \beta, \lambda, \delta) = A \beta^n e^{-\beta \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta})}
\]

Where \( A = \lambda^n \delta^n \prod_{i=1}^{n} x_i^{\delta-1} \prod_{i=1}^{n} \left(1 + \lambda x_i^{-\delta}\right)^{-1} \)

\[
L(\mathbf{t} | \beta_1, \lambda, \delta) = A_1 \beta_1^m e^{-\beta_1 \sum_{j=1}^{m} \ln(1 + \lambda t_j^{-\delta})}
\]

\[
L(\mathbf{z} | \beta_2, \lambda, \delta) = A_2 \beta_2^m e^{-\beta_2 \sum_{k=1}^{m} \ln(1 + \lambda z_k^{-\delta})}
\]

Thus, \( A_1 = \lambda^m \delta^m \prod_{j=1}^{m} t_j^{\delta-1} \prod_{j=1}^{m} \left(1 + \lambda t_j^{-\delta}\right)^{-1} \) and \( A_2 = \lambda^m \delta^m \prod_{k=1}^{m} z_k^{\delta-1} \prod_{k=1}^{m} \left(1 + \lambda z_k^{-\delta}\right)^{-1} \)

3.2. The posterior distribution

For any parameter \( \theta \) the posterior probability density function can be get from merged the probability function \( \prod(\theta) \) and the likelihood function for the observations \( L(\mathbf{x} | \theta) \) as bellow: [11]

\[
P(\theta | \mathbf{x}) = \frac{L(\mathbf{x} | \theta) \prod(\theta)}{\int L(\mathbf{x} | \theta) \prod(\theta) d\theta}
\]

3.3. Types of loss function that using in this paper

The type of loss function that we will consider is two types of loss function

3.3.1. Weighted loss function. The Weighted loss function can be defined as follows: [12]

\[
L(\hat{\theta}, \theta) = \left(\frac{\hat{\theta} - \theta}{\theta}\right)^2
\]

\[
\hat{\theta}_W = \frac{1}{E(\theta^{-1} | \mathbf{x})}
\]

3.3.2. Quadratic loss function. The quadratic loss function can be defined as follows: [13]

\[
L(\hat{\theta}, \theta) = \left(\frac{\hat{\theta} - \theta}{\theta}\right)^2
\]

\[
\hat{\theta}_Q = \frac{E(\theta^{-1} | \mathbf{x})}{E(\theta^{-2} | \mathbf{x})}
\]

3.4. Bayesian estimation under Gamma prior

The most widely used prior distribution of the parameters \((\beta, \beta_1, \beta_2)\) is the Gamma distribution with hyper-parameters \(a, b, b_1\) and \(b_2\) with pdf given by:

\[
g(\beta) = \frac{e^{-\beta}}{\Gamma(a)} \beta^{a-1} e^{-b\beta}, \beta > 0; \ a, b > 0
\]

\[
g(\beta_1) = \frac{e^{-\beta_1}}{\Gamma(a_1)} \beta_1^{a_1-1} e^{-b_1\beta_1}, \beta_1 > 0; \ a_1, b_1 > 0
\]

\[
g(\beta_2) = \frac{e^{-\beta_2}}{\Gamma(a_2)} \beta_2^{a_2-1} e^{-b_2\beta_2}, \beta_2 > 0; \ a_2, b_2 > 0
\]
Now to find the posterior distribution under the assumption of Gamma’s prior we substitute equation (7), (8), (9), (13), (14) and (15) in (10) as below:

\[ P(\beta, \beta_1, \beta_2 | x, t, z) = \frac{L(x, t, z | \beta, \beta_1, \beta_2) \Pi(\beta, \beta_1, \beta_2)}{\int_{\beta_1} \int_{\beta_2} L(x, t, z | \beta, \beta_1, \beta_2) \Pi(\beta, \beta_1, \beta_2) d\beta d\beta_1 d\beta_2} \]

Where

\[ L(x, t, z | \beta, \beta_1, \beta_2) = \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \frac{\Gamma(a+b)}{\Gamma(a)} \beta_1^{a-1} e^{-\beta_1 x_1} \frac{\Gamma(a+b)}{\Gamma(a)} \beta_2^{a-1} e^{-\beta_2 x_2} \]

\[ = A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]

\[ = \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]

Since we have \( \int_0^\infty x^a e^{-bx} dx = \frac{\Gamma(a)}{b^a} \), then:

\[ \int_0^\infty A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]

\[ = A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]

\[ = A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]

So, then:

\[ \int_0^\infty \int_0^\infty \int_0^\infty L(x, t, z | \beta, \beta_1, \beta_2) \Pi(\beta, \beta_1, \beta_2) d\beta d\beta_1 d\beta_2 = \int_0^\infty \int_0^\infty \int_0^\infty A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]

And the posterior function will be:

\[ P(\beta, \beta_1, \beta_2 | x, t, z) = \frac{A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2}}{A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2}} \]

\[ = \frac{A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2}}{A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2}} \]

\[ = \frac{A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2}}{A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2}} \]

\[ = A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]

\[ = A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]

\[ = A_1 A_2 \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-bx} \beta_1^{a-1} e^{-\beta_1 x_1} \beta_2^{a-1} e^{-\beta_2 x_2} \]
Where \( M = \left( \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta}) + b \right) \)
\[ M_1 = \left( \sum_{i=1}^{m} \ln(1 + \lambda t_i^{-\delta}) + b_1 \right) \]
\[ M_2 = \left( \sum_{k=1}^{m} \ln(1 + \lambda x_k^{-\delta}) + b_2 \right) \]

The Bayes estimators under two different loss functions are expressed as follows:
1. Weighted loss function: from equation (11), the Bayes estimator for the system reliability will be as follows:

\[
\hat{R}_W = \frac{1}{E(R^{-1}|x, t, z)}
\]

\[
E(R^{-1}|x, t, z) = \int_0^\infty \int_0^\infty \int_0^\infty R^{-1} P(\beta, \beta_1, \beta_2|x, t, z) d\beta d\beta_1 d\beta_2
\]

\[
E(R^{-1}|x, t, z) = \int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{\beta_2}{(\beta + \beta_1)(\beta + \beta_2 + \beta_3)} \right)^{-1} P(\beta, \beta_1, \beta_2|x, t, z) d\beta d\beta_1 d\beta_2
\]

\[
E(R^{-1}|x, t, z) = \frac{M^{n+a} M_1^{m+a} M_2^{m+a}}{\Gamma(n+a) \Gamma(m+a) \Gamma(m+a)}
\]

\[
\int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{\beta_2}{(\beta + \beta_1)(\beta + \beta_2 + \beta_3)} \right)^{-1} P(\beta, \beta_1, \beta_2|x, t, z) d\beta d\beta_1 d\beta_2
\]

\[
E(R^{-1}|x, t, z) = \frac{M^{n+a} M_1^{m+a} M_2^{m+a}}{\Gamma(n+a) \Gamma(m+a) \Gamma(m+a)}
\]

\[
\int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{\beta_2}{(\beta + \beta_1)(\beta + \beta_2 + \beta_3)} \right)^{-1} P(\beta, \beta_1, \beta_2|x, t, z) d\beta d\beta_1 d\beta_2
\]
\[ E(R^{-1}|\chi, t, z) = \frac{(n+a)}{M} \frac{M_2}{(m+a-1)} + 2 \frac{(m+a)}{M_1} \frac{M_2}{(m+a-1)} + 1 + \frac{M}{(n+a-1)} \frac{(m+a)(m+a)}{M_1^2} \frac{M_2}{(m+a-1)} \]

And \( \hat{R}_W \), can be written as:

\[ \hat{R}_W = \left( \frac{(n+a)}{M} \frac{M_2}{m+a+1} + 2 \frac{(m+a)}{M_1} \frac{M_2}{m+a+1} + 1 + \frac{M}{(n+a-1)} \frac{(m+a)(m+a)}{M_1^2} \frac{M_2}{m+a+1} \right)^{-1} \]

2. Quadratic loss function: from equation (12), the Bayes estimator for the system reliability will be as follows:

\[ \hat{R}_Q = \frac{E(R^{-1}|x)}{E(R^{-2}|x)} \]

\[ E(R^{-2}|\chi, t, z) = \int_0^\infty \int_0^\infty R^{-2} P(\beta, \beta_1, \beta_2|\chi, t, z) d\beta d\beta_1 d\beta_2 \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{\beta \beta_2}{(\beta + \beta_1)(\beta + \beta_1 + \beta_2)} \right)^2 P(\beta, \beta_1, \beta_2|\chi, t, z) d\beta d\beta_1 d\beta_2 \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty (\beta \beta_2 - 2 + 4 \beta \beta_2 - 2 + 2 \beta_1 \beta_2 - 2 + 6 \beta_1 \beta_2 - 1 + 1 + 2 \beta_1 \beta_2 + 4 \beta_1 \beta_2 - 1) \]

\[ \frac{M^{m+a} M_1^{m+a} M_2^{m+a}}{\Gamma(n+a) \Gamma(m+a) \Gamma(m+a)} \beta_1^{m+a-1} \beta_2^{m+a-1} e^{-\beta} e^{-\beta_1 M_1 e^{-\beta_2 M_2}} d\beta_1 d\beta_2 \]

\[ E(R^{-2}|\chi, t, z) = \frac{M^{m+a} M_1^{m+a} M_2^{m+a}}{\Gamma(n+a) \Gamma(m+a) \Gamma(m+a)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (\beta \beta_2 - 2 + 4 \beta \beta_2 - 2 + 2 \beta_1 \beta_2 - 2 + 6 \beta_1 \beta_2 - 1 + 1 + 2 \beta_1 \beta_2 + 4 \beta_1 \beta_2 - 1) \]

\[ \frac{M^{m+a} M_1^{m+a} M_2^{m+a}}{\Gamma(n+a) \Gamma(m+a) \Gamma(m+a)} \beta_1^{m+a-1} \beta_2^{m+a-1} e^{-\beta} e^{-\beta_1 M_1 e^{-\beta_2 M_2}} d\beta_1 d\beta_2 \]

\[ E(R^{-2}|\chi, t, z) = \frac{M^{m+a} M_1^{m+a} M_2^{m+a}}{\Gamma(n+a) \Gamma(m+a) \Gamma(m+a)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (\beta \beta_2 - 2 + 4 \beta \beta_2 - 2 + 2 \beta_1 \beta_2 - 2 + 6 \beta_1 \beta_2 - 1 + 1 + 2 \beta_1 \beta_2 + 4 \beta_1 \beta_2 - 1) \]

\[ \frac{M^{m+a} M_1^{m+a} M_2^{m+a}}{\Gamma(n+a) \Gamma(m+a) \Gamma(m+a)} \beta_1^{m+a-1} \beta_2^{m+a-1} e^{-\beta} e^{-\beta_1 M_1 e^{-\beta_2 M_2}} d\beta_1 d\beta_2 \]
For the nine different experiments, a simulation study is conducted by using MATLAB 2020 to calculate the mean by the equation: Mean \( \frac{1}{N} \sum_{i=1}^{N} \hat{R}_i \). Calculate the mean by the equation: Mean \( \frac{1}{N} \sum_{i=1}^{N} \hat{R}_i \). The results are recorded in the tables from 1 to 3. The comparison of these estimator’s performance based on the MSE values, where the value of MSE decreases with increasing sample sizes for (weighted and quadratic loss functions) for each of the experiments in the three tables, has been observed as:

- In tables (1), (2) and (3), for experiments (1), (2), (4), (5), (7) and (8), for all sample sizes the best value of MSE at the weighted loss function, followed by Quadratic loss function.

- In tables (1), (2) and (3), for experiments (3), (6) and (9), for sample sizes (15,15), (30,30), (90,90) and (30,15) the best value of MSE at Quadratic loss function, followed by weighted loss function.
loss function, but for sample sizes (15, 90) and (30, 90) the best value of MSE at the weighted loss function, followed by Quadratic loss function. Thus, the estimators according to the weighted loss function give better performance than those of Quadratic loss function through the small values of the MSE for most of the experiments.

Table 1: Simulation results when $\lambda = 2, \delta = 2$.

| n,m  | RW   | RQ   | Best |
|------|------|------|------|
| 15,15| Mean | 0.1612 | 0.1526 | RW |
|      | MSE  | 0.0015 | 0.0016 |
| 30,30| Mean | 0.1628 | 0.1581 |
|      | MSE  | 8.3310e-04 | 8.7600e-04 |
| 90,90| Mean | 0.1647 | 0.1630 |
|      | MSE  | 2.7330e-04 | 2.8090e-04 |
| 30,15| Mean | 0.1639 | 0.1553 |
|      | MSE  | 0.0017 | 0.0018 |
| 15,90| Mean | 0.1603 | 0.1584 |
|      | MSE  | 3.5400e-04 | 3.8210e-04 |
| 30,90| Mean | 0.1636 | 0.1618 |
|      | MSE  | 3.1660e-04 | 3.2620e-04 |

Table 2: Simulation results when $\lambda = 2, \delta = 2$.

| n,m  | RW   | RQ   | Best |
|------|------|------|------|
| 15,15| Mean | 0.3041 | 0.2947 | RW |
|      | MSE  | 0.0018 | 0.0018 |
| 30,30| Mean | 0.3146 | 0.3096 |
|      | MSE  | 0.0017 | 0.0018 |
| 90,90| Mean | 0.3183 | 0.3165 |
|      | MSE  | 6.5130e-04 | 6.6770e-04 |
| 30,15| Mean | 0.3099 | 0.3008 |
|      | MSE  | 0.0033 | 0.0036 |
| 15,90| Mean | 0.3062 | 0.3036 |
|      | MSE  | 0.0010 | 0.0011 |
| 30,90| Mean | 0.3142 | 0.3122 |
|      | MSE  | 7.1440e-04 | 7.5430e-04 |

Exp.2: $\beta = 0.9, \beta_1 = 0.3, \beta_2 = 0.9, R = 0.3214$

Exp.3: $\beta = 2.5, \beta_1 = 2.5, \beta_2 = 1.0, R = 0.0833$

Exp.4: $\beta = 1.2, \beta_1 = 0.6, \beta_2 = 0.6, R = 0.1667$
Gamma prior $a = 4, b = 2, b_1 = 1.2, b_2 = 0.8$

| n,m  | Mean RW      | MSE RW      | Mean RQ      | MSE RQ      | Best |
|------|--------------|-------------|--------------|-------------|------|
| 15,15| 0.1581       | 0.1495      |              |             |      |
|      | 0.0016       | 0.0018      |              |             | RW   |
| 30,30| 0.1623       | 0.1576      |              |             |      |
|      | 8.0900e-04   | 8.5630e-04 |              |             | RW   |
| 90,90| 0.1666       | 0.1649      |              |             |      |
|      | 2.9290e-04   | 2.9380e-04 |              |             | RW   |
| 30,15| 0.1618       | 0.1533      |              |             |      |
|      | 0.0018       | 0.0019      |              |             | RW   |
| 15,90| 0.1608       | 0.1588      |              |             |      |
|      | 3.6050e-04   | 3.8680e-04 |              |             | RW   |
| 30,90| 0.1641       | 0.1623      |              |             |      |
|      | 3.0120e-04   | 3.1120e-04 |              |             | RW   |

Exp.5: $\beta = 0.9, \beta_1 = 0.3, \beta_2 = 0.9, R = 0.3214$

Gamma prior $a = 4, b = 2, b_1 = 1.2, b_2 = 0.8$

| n,m  | Mean RW      | MSE RW      | Mean RQ      | MSE RQ      | Best |
|------|--------------|-------------|--------------|-------------|------|
| 15,15| 0.3068       | 0.2974      |              |             |      |
|      | 0.0037       | 0.0041      |              |             | RW   |
| 30,30| 0.3120       | 0.3070      |              |             |      |
|      | 0.0018       | 0.0020      |              |             | RW   |
| 90,90| 0.3192       | 0.3175      |              |             |      |
|      | 5.2110e-04   | 5.3370e-04 |              |             | RW   |
| 30,15| 0.3074       | 0.2982      |              |             |      |
|      | 0.0032       | 0.0036      |              |             | RW   |
| 15,90| 0.3060       | 0.3035      |              |             |      |
|      | 0.0011       | 0.0012      |              |             | RW   |
| 30,90| 0.3133       | 0.3113      |              |             |      |
|      | 6.9850e-04   | 7.4080e-04 |              |             | RW   |

Exp.6: $\beta = 2.5, \beta_1 = 2.5, \beta_2 = 1.0, R = 0.0833$

Gamma prior $a = 4, b = 2, b_1 = 1.2, b_2 = 0.8$

| n,m  | Mean RW      | MSE RW      | Mean RQ      | MSE RQ      | Best |
|------|--------------|-------------|--------------|-------------|------|
| 15,15| 0.0866       | 0.0792      |              |             |      |
|      | 6.8680e-04   | 6.4260e-04 |              |             | RQ   |
| 30,30| 0.0852       | 0.0812      |              |             |      |
|      | 3.4080e-04   | 3.2810e-04 |              |             | RQ   |
| 90,90| 0.0836       | 0.0822      |              |             |      |
|      | 1.1230e-04   | 1.1180e-04 |              |             | RQ   |
| 30,15| 0.0870       | 0.0798      |              |             |      |
|      | 6.5730e-04   | 6.0550e-04 |              |             | RQ   |
| 15,90| 0.0821       | 0.0806      |              |             |      |
|      | 1.1390e-04   | 1.1940e-04 |              |             | RW   |
| 30,90| 0.0823       | 0.0808      |              |             |      |
|      | 1.2440e-04   | 1.2790e-04 |              |             | RW   |

Table 3: Simulation results when $\lambda = 0.5, \delta = 5$

Exp.7: $\beta = 1.2, \beta_1 = 0.6, \beta_2 = 0.6, R = 0.1667$
### Gamma prior $a = 4, b = 2, b_1 = 1.2, b_2 = 0.8$

| n,m  | RW   | RQ   | Best |
|------|------|------|------|
| 15,15| Mean | 0.1611 | 0.1525 | RW |
|      | MSE  | 0.0016 | 0.0017 | RW |
| 30,30| Mean | 0.1640 | 0.1593 | |
|      | MSE  | 8.2410e-04 | 8.5500e-04 | RW |
| 90,90| Mean | 0.1663 | 0.1646 | |
|      | MSE  | 3.1140e-04 | 3.1610e-04 | RW |
| 30,15| Mean | 0.1633 | 0.1547 | |
|      | MSE  | 0.0016 | 0.0017 | RW |
| 15,90| Mean | 0.1605 | 0.1586 | |
|      | MSE  | 3.3990e-04 | 3.6750e-04 | RW |
| 30,90| Mean | 0.1631 | 0.1613 | |
|      | MSE  | 3.0870e-04 | 3.2210e-04 | RW |

Exp.8: $\beta = 0.9, \beta_1 = 0.3, \beta_2 = 0.9, R = 0.3214$

### Gamma prior $a = 4, b = 2, b_1 = 1.2, b_2 = 0.8$

| n,m  | RW   | RQ   | Best |
|------|------|------|------|
| 15,15| Mean | 0.3056 | 0.2962 | |
|      | MSE  | 0.0035 | 0.0040 | RW |
| 30,30| Mean | 0.3122 | 0.3072 | |
|      | MSE  | 0.0017 | 0.0018 | RW |
| 90,90| Mean | 0.3177 | 0.3159 | |
|      | MSE  | 5.8870e-04 | 6.0720e-04 | RW |
| 30,15| Mean | 0.3112 | 0.3021 | |
|      | MSE  | 0.0029 | 0.0033 | RW |
| 15,90| Mean | 0.3077 | 0.3052 | |
|      | MSE  | 0.0009 | 0.0010 | RW |
| 30,90| Mean | 0.3136 | 0.3116 | |
|      | MSE  | 7.7620e-04 | 8.1820e-04 | RW |

Exp.9: $\beta = 2.5, \beta_1 = 2.5, \beta_2 = 1.0, R = 0.0833$

### Gamma prior $a = 4, b = 2, b_1 = 1.2, b_2 = 0.8$

| n,m  | RW   | RQ   | Best |
|------|------|------|------|
| 15,15| Mean | 0.0875 | 0.0801 | |
|      | MSE  | 6.3100e-04 | 5.7820e-04 | RQ |
| 30,30| Mean | 0.0852 | 0.0812 | |
|      | MSE  | 3.7520e-04 | 3.6170e-04 | RQ |
| 90,90| Mean | 0.0838 | 0.0824 | |
|      | MSE  | 1.1890e-04 | 1.1800e-04 | RQ |
| 30,15| Mean | 0.0886 | 0.0814 | |
|      | MSE  | 6.4050e-04 | 5.6960e-04 | RQ |
| 15,90| Mean | 0.0824 | 0.0808 | |
|      | MSE  | 1.2140e-04 | 1.2610e-04 | RW |
| 30,90| Mean | 0.0822 | 0.0808 | |
|      | MSE  | 1.1990e-04 | 1.2360e-04 | RW |

### Conclusions

In this paper, Bayesian estimation was presented for estimating the reliability $P(T<X<Z)$ as each of $T$, $Z$ and $X$ follow Dagum distribution with different parameters under complete data using Gamma prior based on two loss function (weighted and quadratic loss functions). Simulation results that appeared confirm that the performance of estimator according to weighted loss function is much better than the estimator of quadratic loss function for the most experiments and for all sample sizes.

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