Katětov functors

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Abstract

We develop a theory of Katětov functors which provide a uniform way of constructing Fraïssé limits. Among applications, we present short proofs and improvements of several recent results on the structure of the group of automorphisms and the semigroup of endomorphisms of Fraïssé limits.

Keywords: Katětov functor, amalgamation, Fraïssé limit.

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Contents

1 Introduction

2 The setup

3 Katětov functors
   3.1 Examples
   3.2 Sufficient conditions for the existence of Katětov functors

4 Katětov construction

5 Semigroup Bergman property

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1 Introduction

In this paper we present a uniform treatment of several apparently unrelated phenomena. One of the straightforward applications is universality of the groups of automorphisms of Fraïssé limits.

Papers [6] and [2] discuss some of the issues addressed in this paper, without realizing that what we deal with are actually functorial constructions.

Our principal motivation comes from Katětov’s construction of the Urysohn space [10], which we briefly present here in the case of the rational Urysohn space. Let $X$ be a metric space with rational distances. A Katětov function over $X$ is every function $\alpha : X \to \mathbb{Q}$ such that

$$|\alpha(x) - \alpha(y)| \leq d(x, y) \leq \alpha(x) + \alpha(y)$$

for all $x, y \in X$. Let $K(X)$ be the set of all Katětov functions over $X$. The sup metric turns $K(X)$ into a metric space. There is a natural isometric embedding $X \hookrightarrow K(X)$ which takes $a \in X$ to $d(a, \cdot) \in K(X)$. Hence we get a chain of embeddings

$$X \hookrightarrow K(X) \hookrightarrow K^2(X) \hookrightarrow K^3(X) \hookrightarrow \cdots$$

whose colimit is easily seen to be the rational Urysohn space.

It was first observed in [1] that the construction $K$ is actually functorial with respect to embeddings. Our principal observation is that more is true: if $\mathcal{A}$ is the category of all finite metric spaces with rational distances and nonexpansive maps, and $\mathcal{C}$ is the category of all countable metric spaces with rational distances and nonexpansive maps, then $K$ can be turned into a functor from $\mathcal{A}$ to $\mathcal{C}$ if we let $K$ act on morphisms as follows: for a nonexpansive map $f : X \to Y$ let $K(f) : K(X) \to K(Y)$ take $\alpha \in K(X)$ to $\tilde{\alpha} \in K(Y)$ where

$$\tilde{\alpha}(y) = \inf\{\alpha(x) + d_Y(f(x), y) : x \in X\}.$$ 

It is a matter of straightforward calculation to check that $\tilde{\alpha}$ is indeed a Katětov function over $Y$, that $K(id_X) = id_{K(X)}$ and that $K(g \circ f) = K(g) \circ K(f)$.

2 The setup

Let $\Delta = R \cup F \cup C$ be a first-order language where $R$ is a set of relational symbols, $F$ a set of functional symbols and $C$ a set of constant symbols. We
say that $\Delta$ is a purely relational language if $\mathcal{F} = \mathcal{C} = \emptyset$. For a $\Delta$-structure $A$ and $X \subseteq A$, by $(X)_A$ we denote the substructure of $A$ generated by $X$. We say that $A$ is finitely generated if $A = (X)_A$ for some finite $X \subseteq A$.

Let $\mathcal{C}$ be a category of $\Delta$-structures. A chain in $\mathcal{C}$ is a chain of objects and embeddings of the form $C_1 \hookrightarrow C_2 \hookrightarrow C_3 \hookrightarrow \cdots$. Note that although there may be other kinds of morphisms in $\mathcal{C}$, a chain always consists of objects and embeddings. For a $C \in \text{Ob}(\mathcal{C})$ let $\text{Aut}(C)$ denote the permutation group consisting of all automorphisms of $C$, and let $\text{End}_C(C)$ denote the transformation monoid consisting of all $C$-morphisms $C \rightarrow C$. Moreover, let $\text{age}(C)$ denote the class of all finitely generated objects that embed into $C$. We say that $\mathcal{A}$ has the joint-embedding property (briefly: (JEP)) if every two structures in $\mathcal{A}$ embed into a common structure in $\mathcal{A}$.

**Standing assumption.** Throughout the paper we assume the following. Let $\Delta$ be a first-order language, let $\mathcal{C}$ be a category of countably generated $\Delta$-structures and some appropriately chosen class of morphisms that includes all embeddings (and hence all isomorphisms). Let $\mathcal{A}$ be the full subcategory of $\mathcal{C}$ spanned by all finitely generated structures in $\mathcal{C}$. In particular, $\mathcal{A}$ is hereditary in the sense that given $A \in \text{Ob}(\mathcal{A})$, every finitely generated substructure$^1$ of $A$ is an object of $\mathcal{A}$. We also assume that the following holds:

- $\mathcal{C}$ has colimits of chains: for every chain $C_1 \hookrightarrow C_2 \hookrightarrow \cdots$ in $\mathcal{C}$ there is an $L \in \text{Ob}(\mathcal{C})$ which is a colimit of this diagram in $\mathcal{C}$;
- every $C \in \text{Ob}(\mathcal{C})$ is a colimit of some chain $A_1 \hookrightarrow A_2 \hookrightarrow \cdots$ in $\mathcal{A}$;
- $\mathcal{A}$ has only countably many isomorphism types; and
- $\mathcal{A}$ has the joint embedding property (JEP).

We say that $C \in \text{Ob}(\mathcal{C})$ is a one-point extension of $B \in \text{Ob}(\mathcal{C})$ if there is an embedding $j : B \hookrightarrow C$ and an $x \in C \setminus j(B)$ such that $C = (j(B) \cup \{x\})_C$. In that case we write $j : B \hookrightarrow C$ or simply $B \hookrightarrow C$.

The following lemmas are immediate consequences of the fact that $\mathcal{C}$ is a category of $\Delta$-structures and the fact that $\mathcal{A}$ is spanned by finitely generated objects in $\mathcal{C}$.

---

$^1$Recall that substructures of finitely generated structures may not be finitely generated. For example, the free group with 2 generators contains the free group with infinitely many generators.
Lemma 2.1 (Reachability)  (a) For all \( A, B \in \text{Ob}(A) \) and an embedding \( A \hookrightarrow B \) which is not an isomorphism, there exist an \( n \in \mathbb{N} \) and \( A_1, \ldots, A_n \in \text{Ob}(A) \) such that \( A \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n = B \).

(b) For all \( C, D \in \text{Ob}(C) \) and an embedding \( f : C \hookrightarrow D \) which is not an isomorphism, there exist \( C_1, C_2 \ldots \in \text{Ob}(C) \) such that

\[
\begin{array}{ccc}
C & \longrightarrow & C_1 \\
\downarrow & & \downarrow \\
D & \longleftarrow & C_2 \\
\end{array}
\]

is a colimit diagram in \( C \).

Lemma 2.2 Let \( C, D \in \text{Ob}(C) \) be structures such that \( f : C \hookrightarrow D \) and let \( A_1 \hookrightarrow A_2 \hookrightarrow \cdots \) be a chain in \( A \) whose colimit is \( C \). Then there exists a chain \( B_1 \hookrightarrow B_2 \hookrightarrow \cdots \) in \( A \) whose colimit is \( D \) and

\[
\begin{array}{ccc}
A_1 & \longrightarrow & A_2 \\
\downarrow & & \downarrow \\
B_1 & \longleftarrow & B_2 \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\downarrow & & \downarrow \\
C & \hookrightarrow & D \end{array}
\]

Proof. Without loss of generality we can assume that \( C \leq D \), and that \( A_1 \leq A_2 \leq \ldots \leq C \), so that \( C = \bigcup_{i \in \mathbb{N}} A_i \). Since \( D \) is a one-point extension of \( C \), there exists an \( x \in D \setminus C \) such that \( D = \langle C \cup \{x\} \rangle_D \). Put \( B_i = \langle A_i \cup \{x\} \rangle_D \).

The next lemma is rather obvious.

Lemma 2.3 (Factoring through the colimit) Let \( C_1 \hookrightarrow C_2 \hookrightarrow \cdots \) be a chain in \( C \) and let \( L \) be the colimit of the chain with the canonical embeddings \( \iota_k : C_k \hookrightarrow L \). Then for every \( A \in \text{Ob}(A) \) and every morphism \( f : A \rightarrow L \) there is an \( n \in \mathbb{N} \) and a morphism \( g : A \rightarrow C_n \) such that
\( f \circ g = \iota_n \). Moreover, if \( f \) is an embedding, then so is \( g \).

\begin{equation}
\begin{array}{c}
C_n \\
\downarrow \iota_n \\
A \xrightarrow{f} L
\end{array}
\end{equation}

**Lemma 2.4** For every \( C \in \text{Ob}(\mathcal{C}) \) we have that \( \text{age}(C) \subseteq \text{Ob}(\mathcal{A}) \).

**Proof.** Take any \( C \in \text{Ob}(\mathcal{C}) \), and let \( A_1 \hookrightarrow A_2 \hookrightarrow \cdots \) be a chain in \( \mathcal{A} \) whose colimit is \( C \). Take any \( B \in \text{age}(C) \). Then \( B \hookrightarrow C \), so by Lemma 2.3 there is an \( n \in \mathbb{N} \) and an embedding \( g : B \hookrightarrow A_n \) such that

\begin{equation}
\begin{array}{c}
A_n \\
\downarrow \\
B \xleftarrow{\eta^0} C
\end{array}
\end{equation}

Therefore, \( B \hookrightarrow A_n \in \text{Ob}(\mathcal{A}) \), so the assumption that \( \mathcal{A} \) is hereditary yields \( B \in \text{Ob}(\mathcal{A}) \). \( \square \)

### 3 Katětov functors

**Definition 3.1** A functor \( K^0 : \mathcal{A} \to \mathcal{C} \) is a Katětov functor if:

- \( K^0 \) preserves embeddings, that is, if \( f : A \to B \) is an embedding in \( \mathcal{A} \), then \( K^0(f) : K^0(A) \to K^0(B) \) is an embedding in \( \mathcal{C} \); and

- there is a natural transformation \( \eta^0 : \text{ID} \to K^0 \) such that for every one-point extension \( A \hookrightarrow B \) where \( A, B \in \text{Ob}(\mathcal{A}) \), there is an embedding \( g : B \hookrightarrow K^0(A) \) satisfying

\begin{equation}
\begin{array}{c}
A \xrightarrow{\eta^0_A} K^0(A) \\
\downarrow \\
B \xrightarrow{g}
\end{array}
\end{equation}

**Theorem 3.2** If there exists a Katětov functor \( K^0 : \mathcal{A} \to \mathcal{C} \) then there is a functor \( K : \mathcal{C} \to \mathcal{C} \) such that:
- $K$ is an extension of $K^0$ (that is, $K$ and $K^0$ coincide on $A$);
- there is a natural transformation $\eta : \text{ID} \to K$ which is an extension of $\eta^0$ (that is, $\eta_A = \eta^0_A$ whenever $A \in \text{Ob}(A)$);
- $K$ preserves embeddings;
- $K$ is continuous (that is, if $L$ is the colimit of the chain $C_1 \hookrightarrow C_2 \hookrightarrow \cdots$, then $K(L)$ is the colimit of the chain $K(C_1) \hookrightarrow K(C_2) \hookrightarrow \cdots$).

**Proof.**

Clearly, $K$ coincides with $K^0$ on $A$. Let us show how to extend $K$ to the whole of $C$. We first show how to define $K$ on objects. For each object $C$ of $C$ which is not an object of $A$ fix a chain $A^C_1 \hookrightarrow A^C_2 \hookrightarrow \cdots$ in $A$ whose colimit is $C$ with canonical embeddings $\iota_{n}^C$:

$$
\begin{array}{c}
A_1^C \hookrightarrow A_2^C \hookrightarrow \cdots \\
\downarrow \iota_1^C \uparrow \iota_2^C \\
C
\end{array}
$$

Then, let $K(C)$ to be the colimit of the chain $K^0(A^C_1) \hookrightarrow K^0(A^C_2) \hookrightarrow \cdots$ with canonical embeddings $\iota_{n}^{K(C)}$:

$$
\begin{array}{c}
K^0(A^C_1) \hookrightarrow K^0(A^C_2) \hookrightarrow \cdots \\
\downarrow \iota_1^{K(C)} \uparrow \iota_2^{K(C)} \\
K(C)
\end{array}
$$

Next, let us show how to define $K$ on morphisms. Let $f : B \to C$ be a morphism from $B \in \text{Ob}(A)$ to an object $C \in \text{Ob}(C)$ which is not an object of $A$. By Lemma 2.3 there is an $n$ and a morphism $g : B \to A^C_n$ such that $f \circ g = \iota_n^C$:

$$
\begin{array}{c}
A^C_n \\
\downarrow g \\
B \xrightarrow{f} C
\end{array}
$$

Now, choose the least such $n$ and let $K(f) = \iota_n^{K(C)} \circ K^0(g)$.
The rest of the proof is rather standard.  

We also say that \( K \) is a Katětov functor and from now on denote both \( K \) and \( K^0 \) by \( K \), and both \( \eta \) and \( \eta^0 \) by \( \eta \).

**Example 3.3** Consider \( \mathcal{A} \) the category of finite graphs, \( \mathcal{C} \) the category of all countable graphs. A typical Katětov functor assigns to a finite graph \( G \) the graph \( K(G) = G \cup \mathcal{P}(G) \), where each \( A \in \mathcal{P}(G) \) is connected precisely to its elements. Now let \( G \) be an infinite graph. Then \( K(G) = G \cup \text{Fin}(G) \), where \( \text{Fin}(X) \) is the family of all finite subsets of \( X \). Let \( H = G \cup \{v\} \), where \( v \) is connected to all the vertices of \( G \). Then there is no embedding of \( H \) extending \( \eta_G : G \to K(G) \).

**3.1 Examples**

**Example 3.4** A Katětov functor on the category of all finite metric spaces with rational distances and nonexpansive maps was described in Section 1. This is the original Katětov functor.

In the examples below let \( \mathcal{P}_2(X) = \{Y \subseteq X : |Y| = 2\} \), and let \( \mathcal{P}_{\text{fin}}(X) \) denote the set of all finite subsets of \( X \).

**Example 3.5** A Katětov functor on the category of all graphs and graph homomorphisms. Let \( \langle V, E \rangle \) be a graph, where \( E \subseteq \mathcal{P}_2(V) \). Put \( K(\langle V, E \rangle) = \langle V^*, E^* \rangle \) where

\[
V^* = V \cup \mathcal{P}_{\text{fin}}(V), \\
E^* = E \cup \{\{v, A\} : A \in \mathcal{P}_{\text{fin}}(V), v \in A\}.
\]

For a graph homomorphism \( f : \langle V_1, E_1 \rangle \to \langle V_2, E_2 \rangle \) let \( f^* = K(f) \) be a mapping from \( V_1^* \to V_2^* \) defined by \( f^*(v) = f(v) \) for \( v \in V_1 \) and \( f^*(A) = f(A) \) for \( A \in \mathcal{P}_{\text{fin}}(V_1) \). Then it is easy to show that \( f^* \) is a graph homomorphism from \( \langle V_1^*, E_1^* \rangle \) to \( \langle V_2^*, E_2^* \rangle \). Moreover, if \( f \) is an embedding, then so is \( f^* \).

**Example 3.6** A Katětov functor on the category of all \( K_n \)-free graphs and graph embeddings. Fix an integer \( n \geq 3 \). Let \( \langle V, E \rangle \) be a \( K_n \)-free graph, where \( E \) is the set of some 2-element subsets of \( V \). Put \( K(\langle V, E \rangle) = \langle V^*, E^* \rangle \) where

\[
V^* = V \cup V', \\
V' = \{A \in \mathcal{P}_{\text{fin}}(V) : \langle A, E \cap \mathcal{P}_2(A) \rangle \text{ is } K_{n-1}\text{-free}\}, \\
E^* = E \cup \{\{v, A\} : A \in V', v \in A\}.
\]
For a graph embedding \( f: \langle V_1, E_1 \rangle \to \langle V_2, E_2 \rangle \) let \( f^* = K(f) \) be a mapping from \( V_1^* \) to \( V_2^* \) defined by \( f^*(v) = f(v) \) for \( v \in V_1 \) and \( f^*(A) = f(A) \) for \( A \in V_1^* \). Then it is easy to show that \( f^* \) is a graph embedding from \( \langle V_1^*, E_1^* \rangle \) to \( \langle V_2^*, E_2^* \rangle \).

**Example 3.7** A Katětov functor on the category of all digraphs and digraph homomorphisms. Let \( (V, E) \) be a digraph, where \( E \subseteq V^2 \) is an irreflexive relation satisfying \((x, y) \in E \Rightarrow (y, x) \notin E\). Put \( K((V, E)) = \langle V^*, E^* \rangle \) where

\[
V^* = V \cup V',
\]

\[
E^* = E \cup \{\langle v, (A, B) \rangle : v \in V, (A, B) \in V', v \in A \}
\]

\[
\cup \{(\langle A, B \rangle, v) : v \in V, (A, B) \in V', v \in B\}.
\]

For a digraph homomorphism \( f: \langle V_1, E_1 \rangle \to \langle V_2, E_2 \rangle \) let \( f^* = K(f) \) be a mapping from \( V_1^* \) to \( V_2^* \) defined by \( f^*(v) = f(v) \) for \( v \in V_1 \) and \( f^*(\langle A, B \rangle) = \langle f(A), f(B) \rangle \) for \( (A, B) \in V_1^* \). Then it is easy to show that \( f^* \) is a digraph homomorphism from \( \langle V_1^*, E_1^* \rangle \) to \( \langle V_2^*, E_2^* \rangle \). Moreover, if \( f \) is an embedding, then so is \( f^* \).

**Example 3.8** A Katětov functor on the category of all linear orders and monotonous maps. Let \( 2 \) be the two element linear order \( 0 < 1 \) and let \( \text{hom}(\langle A_1, \leq_1 \rangle, \langle A_2, \leq_2 \rangle) \) denote the set of all monotonous maps \( \langle A_1, \leq_1 \rangle \to \langle A_2, \leq_2 \rangle \). For a linear order \( \langle A, \leq \rangle \) put \( K((\langle A, \leq \rangle)) = \langle A^*, \leq^* \rangle \) where

\[
A^* = A \cup A',
\]

\[
A' = \text{hom}(\langle A, \leq \rangle, 2),
\]

\[
\leq^* = \leq \cup \{\langle a, \varphi \rangle : a \in A, \varphi \in A', \varphi(a) = 0\}
\]

\[
\cup \{\langle \varphi, a \rangle : a \in A, \varphi \in A', \varphi(a) = 1\}
\]

\[
\cup \{\langle \varphi, \psi \rangle : \varphi, \psi \in A' \text{ and } \exists x \in A(\varphi(x) = 1 \land \psi(x) = 0)\}
\]

\[
\cup \{\langle \varphi, \varphi \rangle : \varphi \in A'\}.
\]

Then it is easy to see that \( \leq^* \) is a linear order on \( A^* \). For a monotonous map \( f: \langle A_1, \leq_1 \rangle \to \langle A_2, \leq_2 \rangle \) let \( f^* = K(f) \) be a mapping from \( A_1^* \) to \( A_2^* \) defined by \( f^*(a) = f(a) \) for \( a \in A_1 \) and for \( \varphi \in A_1^* \) we put \( f^*(\varphi) = \psi \) where \( \psi^{-1}(1) = f(\varphi^{-1}(1)) \). Then it is easy to show that \( f^* \) is a monotonous map from \( \langle A_1^*, \leq_1^* \rangle \) to \( \langle A_2^*, \leq_2^* \rangle \). Moreover, if \( f \) is an embedding, then so is \( f^* \).
Example 3.9 A Katětov functor on the category of all partially ordered sets and monotonous maps. Let 3 be the three element linear order 0 < \frac{1}{2} < 1. For a partially ordered set \( \langle A, \leq \rangle \) put \( K(\langle A, \leq \rangle) = \langle A^*, \leq^* \rangle \) where

\[
A^* = A \cup A',
\]

\[
A' = \{ \varphi \in \text{hom}(\langle A, \leq \rangle, 3) : \varphi^{-1}(0) \text{ and } \varphi^{-1}(1) \text{ are finite, and } \varphi^{-1}(0) \leq \varphi^{-1}(1) \},
\]

\[
\leq^* = \leq \cup \{ (a, \varphi) : a \in A, \varphi \in A', \varphi(a) = 0 \}
\]

\[
\cup \{ (\varphi, a) : a \in A, \varphi \in A', \varphi(a) = 1 \}
\]

\[
\cup \{ (\varphi, \psi) : \varphi, \psi \in A' \text{ and } \exists x \in A(\varphi(x) = 1 \land \psi(x) = 0) \}
\]

\[
\cup \{ (\varphi, \varphi) : \varphi \in A' \}.
\]

Then it is easy to see that \( \leq^* \) is a partial order on \( A^* \). For a monotonous map \( f : \langle A_1, \leq_1 \rangle \to \langle A_2, \leq_2 \rangle \) let \( f^* = K(f) \) be a mapping from \( A_1^* \) to \( A_2^* \) defined by \( f^*(a) = f(a) \) for \( a \in A_1 \) and for \( \varphi \in A' \) we put \( f^*(\varphi) = \psi \) where \( \psi^{-1}(0) = f(\varphi^{-1}(0)) \) and \( \psi^{-1}(1) = f(\varphi^{-1}(1)) \). Then it is easy to show that \( f^* \) is a monotonous map from \( \langle A_1^*, \leq_1^* \rangle \) to \( \langle A_2^*, \leq_2^* \rangle \). Moreover, if \( f \) is an embedding, then so is \( f^* \).

Example 3.10 A Katětov functor on the category of all tournaments and embeddings. For a finite set \( A \) and a positive integer \( n \) let \( A^{\leq n} \) be the set of all sequences \( \langle a_1, \ldots, a_k \rangle \) of elements of \( A \) where \( k \in \{0,1,\ldots,n\} \). In case of \( k = 0 \) we actually have the empty sequence \( \langle \rangle \), as we will be careful to distinguish the 1-element sequence \( \langle a \rangle \) from \( a \in A \). For a sequence \( s \in A^{\leq n} \) let \( |s| \) denote the length of \( s \). For a tournament \( T = \langle V, E \rangle \) let \( n = |V| \) and let \( T^{\leq n} \) be the tournament whose set of vertices is \( V^{\leq n} \) and whose set of edges is defined lexicographically as follows:

- if \( s \) and \( t \) are sequences such that \( |s| < |t| \), put \( s \to t \) in \( T^{\leq n} \);

- if \( s = \langle s_1, \ldots, s_k \rangle \) and \( t = \langle t_1, \ldots, t_k \rangle \) are distinct sequences of the same length, find the smallest \( i \) such that \( s_i \neq t_i \) and then put \( s \to t \) in \( T^{\leq n} \) if and only if \( s_i \to t_i \) in \( T \).

For a tournament \( T = \langle V, E \rangle \) put \( K(T) = \langle V^*, E^* \rangle \) where

\[
V^* = V \cup V^{\leq n},
\]

\[
E^* = E \cup E(T^{\leq n}) \cup \{ (v, s) : v \in V, s \in V^{\leq n}, v \text{ appears as an entry in } s \}
\]

\[
\cup \{ (s, v) : v \in V, s \in V^{\leq n}, v \text{ does not appear as an entry in } s \}.
\]
Then it is easy to see that \( \langle V^*, E^* \rangle \) is a tournament. For an embedding \( f : \langle V_1, E_1 \rangle \to \langle V_2, E_2 \rangle \) let \( f^* = K(f) \) be a mapping from \( V_1^* \) to \( V_2^* \) defined by \( f^*(v) = f(v) \) for \( v \in V_1 \) and for \( \langle s_1, \ldots, s_k \rangle \in V_1^{\leq n} \) we put \( f^*(\langle s_1, \ldots, s_k \rangle) = \langle f(s_1), \ldots, f(s_k) \rangle \). Then it is easy to show that \( f^* \) is an embedding from \( \langle V_1^*, E_1^* \rangle \) to \( \langle V_2^*, E_2^* \rangle \).

**Example 3.11** A Katětov functor on the category of all Boolean algebras.

For a finite set \( A \) let \( B(A) \) denote the finite Boolean algebra whose set of atoms is \( A \). For a finite Boolean algebra \( B(A) \) put \( K(B(A)) = B(\{0, 1\} \times A) \) and let \( \eta_{B(A)} : B(A) \hookrightarrow B(\{0, 1\} \times A) \) be the unique homomorphism which takes \( a \in A \) to \( \langle 0, a \rangle \vee \langle 1, a \rangle \in B(\{0, 1\} \times A) \). Clearly, \( \eta_{B(A)} \) is an embedding. Let us define \( K \) on homomorphisms between finite Boolean algebras as follows. Let \( f : B(A) \to B(A') \) be a homomorphism and assume that for \( a \in A \) we have \( f(a) = \bigvee S(a) \) for some \( S(a) \subseteq A' \), with the convention that \( \bigvee \emptyset = 0 \). Then for \( i \in \{0, 1\} \) let \( K(f)(\langle i, a \rangle) = \bigvee (\{i\} \times S(a)) \). This turns \( K \) into a functor from the category of finite Boolean algebras into itself which preserves embeddings and such that \( \eta : \text{ID} \to K \) is a natural transformation.

In order to see that \( K \) is indeed a Katětov functor, let us first note that \( j : B(A) \hookrightarrow B(A') \) if and only if \( |A'| = |A| + 1 \) and \( j \) maps \( |A| - 1 \) atoms of \( B(A) \) to atoms of \( B(A') \) while the remaining atom of \( B(A) \) is mapped onto the join of the remaining two atoms of \( B(A') \). Now let \( j : B(\{a_1, \ldots, a_n\}) \hookrightarrow B(\{b_0, b_1, \ldots, b_n\}) \) be a one-point extension such that \( j(a_1) = b_0 \vee b_1 \) and \( j(a_i) = b_i \) for \( i \geq 2 \). Then the embedding \( g : B(\{b_0, b_1, \ldots, b_n\}) \hookrightarrow K(B(\{a_1, \ldots, a_n\})) \) which makes the diagram (1) commute can be defined as the unique Boolean algebra homomorphism such that \( g(b_0) = \langle 0, a_1 \rangle \), \( g(b_1) = \langle 1, a_1 \rangle \) and \( g(b_i) = \langle 0, a_i \rangle \vee \langle 1, a_i \rangle \) for \( i \geq 2 \).

For \( n \in \mathbb{N} \) define \( \eta^n : \text{ID} \to K^n \) as \( \eta^n_C = \eta_{K^{n-1}(C)} \circ \cdots \circ \eta_K \circ \eta_C : C \to K^n(C) \).

**Lemma 3.12** Let \( K : A \to C \) be a Katětov functor. Then for every embedding \( g : A \hookrightarrow B \), where \( A, B \in \text{Ob}(A) \), there is an \( n \in \mathbb{N} \) and an embedding \( h : B \hookrightarrow K^n(A) \) satisfying \( h \circ g = \eta^n_A \).
Proof. If \( g \) is an isomorphism, take \( n = 1 \) and \( h = \eta_A \circ g^{-1} \). Assume, therefore, that \( g \) is not an isomorphism. Then by Lemma 2.1 (a) there exist an \( n \in \mathbb{N} \) and \( A_1, \ldots, A_n \in \text{Ob}(\mathcal{A}) \) such that

\[
A \leftrightarrow A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_n = B.
\]

It is easy to see that the diagram in Fig. 1 commutes: the triangles commute by the definition of a Katětov functor, while the parallelograms commute because \( \eta \) is a natural transformation. So, take \( h = K^{n-1}(f_1) \circ K^{n-2}(f_2) \circ \cdots \circ K(f_{n-1}) \circ f_n \).

\[\square\]

3.2 Sufficient conditions for the existence of Katětov functors

Let us now present some sufficient conditions for the existence of Katětov functors.

Let \( \Delta \) be a purely relational language, let \( A \) be a \( \Delta \)-structure, and let \( \{B_i : i \in I\} \) be a family of \( \Delta \)-structures such that \( A \) is a substructure of each of the \( B_i \)'s. Assume additionally that \( B_i \cap B_j = A \) whenever \( i \neq j \). The free amalgam of the \( B_i \)'s over \( A \) is the \( \Delta \)-structure \( C \) with universe \( \bigcup \{B_i : i \in I\} \) such that each \( B_i \) is a substructure of \( C \) and for every \( R \in \Delta \) we have that \( R^C = \bigcup \{R^{B_i} : i \in I\} \) (in other words, no tuple which meets \( B_i \setminus A \) and \( B_j \setminus A \) for some \( i \neq j \) satisfies any relation symbol in \( \Delta \)). The following result is implicit in [2] (see Definition 3.7 in [2] and the comment that follows).

**Theorem 3.13 (implicit in [2])** If \( A \) has free amalgamations then a Katětov functor \( K : \mathcal{A} \to \mathcal{C} \) exists.

The following theorem in a strengthening of the main result of [6]. We say that \( \mathcal{A} \) has one-point extension pushouts in \( \mathcal{C} \) if for every morphism \( f : A_0 \to A_1 \) in \( \mathcal{A} \) and a one-point extension \( g : A_0 \hookrightarrow A_2 \) in \( \mathcal{A} \) there exists a \( B \in \text{Ob}(\mathcal{A}) \), a one-point extension \( p : A_1 \hookrightarrow B \) and a morphism \( q : A_2 \to B \) such that \( p \circ f = q \circ g \) and this commuting square is a pushout square in \( \mathcal{C} \).

\[
\begin{array}{ccc}
A_0 & \hookrightarrow & A_2 \\
\downarrow f & & \downarrow q \\
A_1 & \hookrightarrow & B \\
\end{array}
\]
Figure 1: The proof of Lemma 3.12
Theorem 3.14 If $A$ has has one-point extension pushouts in $C$ then the Katětov functor $K : A \to C$ exists.

Proof. Let us first show that every countable source $(A \hookrightarrow B_n)_{n \in \mathbb{N}}$ has a pushout in $C$, where $A, B_1, B_2, \ldots \in \text{Ob}(A)$. Let $e_n : A \hookrightarrow B_n$ be the embeddings in this source. Let $P_2 \in \text{Ob}(A)$ together with the embeddings $f_2 : B_1 \hookrightarrow P_2$ and $g_2 : B_2 \hookrightarrow P_2$ be the pushout of $e_1$ and $e_2$. Next, let $P_3 \in \text{Ob}(A)$ together with the embeddings $f_3 : P_2 \hookrightarrow P_3$ and $g_3 : B_3 \hookrightarrow P_3$ be the pushout of $f_2 \circ e_1$ and $e_3$. Then, let $P_4 \in \text{Ob}(A)$ together with the embeddings $f_4 : P_3 \hookrightarrow P_4$ and $g_4 : B_4 \hookrightarrow P_4$ be the pushout of $f_3 \circ f_2 \circ e_1$ and $e_4$, and so on:

Let $P \in \text{Ob}(C)$ be the colimit of the chain $B_1 \hookrightarrow P_2 \hookrightarrow P_3 \hookrightarrow P_4 \hookrightarrow \cdots$. It is easy to show that $P$ is the pushout of the source $(A \hookrightarrow B_n)_{n \in \mathbb{N}}$.

Let us now construct the Katětov functor as the pushout of all the one-point extensions of an object in $A$. More precisely, for every $A \in \text{Ob}(A)$ let us fix embeddings $e_n : A \hookrightarrow B_n$, where $B_1, B_2, \ldots$ is the list of all the one-point extensions of $A$, where every isomorphism type is taken exactly once to keep the list countable. Define $K(A)$ to be the pushout of the source $(e_n : A \hookrightarrow B_n)_{n \in \mathbb{N}}$. This is how $K$ acts on objects.

Let us show how $K$ acts on morphisms. Take any morphism $h : A \to A'$ in $A$. Let $(e_i : A \hookrightarrow B_i)_{i \in I}$ be the source consisting of all the one-point extensions of $A$ (with every isomorphism type is taken exactly once), and let $(e'_j : A' \hookrightarrow B'_j)_{j \in J}$ be the source consisting of all the one-point extensions of $A'$ (with every isomorphism type is taken exactly once). By the assumption, for every $i \in I$ there exists an $m(i) \in J$ and a morphism $h_i : B_i \to B'_{m(i)}$ such that the following is a pushout square in $C$:

$$
\begin{array}{ccc}
A & \xrightarrow{e_i} & B_i \\
\downarrow{h} & & \downarrow{h_i} \\
A' & \xrightarrow{e'_i} & B'_{m(i)}
\end{array}
$$
Now, $K(A')$ is a pushout of the source $(e'_j : A' \hookrightarrow B'_j)_{j \in J}$, so let us denote the canonical embeddings $B'_j \hookrightarrow K(A')$ by $t'_j$, $j \in J$. Therefore, $(t'_m(i) \circ h_i : B_i \hookrightarrow K(A'))_{i \in I}$ is a compatible cone over the source $(e_i : A \hookrightarrow B_i)_{i \in I}$, so there is a unique mediating morphism $\tilde{h} : K(A) \to K(A')$. Then we put $K(h) = \tilde{h}$. □

4 Katětov construction

**Definition 4.1** Let $K : C \to C$ be a Katětov functor. A Katětov construction is a chain of the form:

$$
\xymatrix{ C \ar[r]^\eta_C & K(C) \ar[r]^{\eta_{K(C)}} & K^2(C) \ar[r]^{\eta_{K^2(C)}} & K^3(C) \ar[r] & \cdots }
$$

where $C \in \text{Ob}(C)$. We denote the colimit of this chain by $K^\omega(C)$. An object $L \in \text{Ob}(C)$ can be obtained by the Katětov construction starting from $C$ if $L = K^\omega(C)$. We say that $L$ can be obtained by the Katětov construction if $L = K^\omega(C)$ for some $C \in \text{Ob}(C)$.

Note that $K^\omega$ is actually a functor $C \to C$: for a morphism $f : A \to B$ let $K^\omega(f)$ be the unique morphism $K^\omega(A) \to K^\omega(B)$ from the colimit of the Katětov construction starting from $A$ to the competitive compatible cone with the tip at $K^\omega(B)$ and morphisms $(\hookrightarrow \circ K^n(f))_{n \in \mathbb{N}}$:

![Diagram](https://via.placeholder.com/150)

It is easy to show that $K^\omega$ preserves embeddings. Moreover, the canonical embeddings $\eta^\omega_A : A \hookrightarrow K^\omega(A)$ constitute a natural transformation $\eta^\omega : \text{ID} \to K^\omega$. Thus, we have:
Theorem 4.2 $K^{\omega} : C \rightarrow C$ is a Katětov functor.

Recall that a countable structure $L$ is ultrahomogeneous if every isomorphism between two finitely generated substructures of $L$ extends to an automorphism of $L$. More precisely, $L$ is ultrahomogeneous if for all $A, B \in \text{age}(L)$, embeddings $j_A : A \hookrightarrow L$ and $j_B : B \hookrightarrow L$, and for every isomorphism $f : A \rightarrow B$ there is an automorphism $f^* \circ j_A$ such that $j_B \circ f = f^* \circ j_A$.

\[
\begin{array}{ccc}
A & \xrightarrow{j_A} & L \\
\downarrow f & & \downarrow f^* \\
B & \xleftarrow{j_B} & L \\
\end{array}
\]

Analogously, we say that a countable structure $L$ is $C$-morphism-homogeneous, if every $C$-morphism between two finitely generated substructures of $L$ extends to a $C$-endomorphism of $L$. More precisely, $L$ is $C$-morphism-homogeneous if for all $A, B \in \text{age}(L)$, embeddings $j_A : A \hookrightarrow L$ and $j_B : B \hookrightarrow L$, and for every $C$-morphism $f : A \rightarrow B$ there is a $C$-endomorphism $f^* \circ j_A$ such that $j_B \circ f = f^* \circ j_A$. In particular, if $C$ is the category of all countable $\Delta$-structures with all homomorphisms between them, instead of saying that $L$ is $C$-morphism-homogeneous, we say that $L$ is homomorphism-homogeneous.

Theorem 4.3 If there exists a Katětov functor $K : A \rightarrow C$, then $A$ is an amalgamation class, it has a Fraïssé limit $L$ in $C$, and $L$ can be obtained by the Katětov construction starting from an arbitrary $C \in \text{Ob}(C)$. Moreover, $L$ is $C$-morphism-homogeneous.

Proof. Take any $C \in \text{Ob}(C)$, let

\[
C \xleftarrow{\eta_0} K(C) \xrightarrow{\eta_1} K^2(C) \xrightarrow{\eta_2} K^3(C) \xrightarrow{\eta_3} \cdots
\]

be the Katětov construction starting from $C$, and let $L \in \text{Ob}(C)$ be the colimit of this chain. Let $i_n : K^n(C) \hookrightarrow L$ be the canonical embeddings of the colimit diagram.

Let us first show that $\text{age}(L) = \text{Ob}(A)$. Lemma 2.4 yields $\text{age}(L) \subseteq \text{Ob}(A)$, so let us show that $\text{Ob}(A) \subseteq \text{age}(L)$. Take any $B \in \text{Ob}(A)$ and let $A_1 \hookrightarrow A_2 \hookrightarrow \cdots$ be a chain whose colimit is $C$. Since $A$ has (JEP) there is a $D \in \text{Ob}(A)$ such that $A_1 \hookrightarrow D \hookrightarrow B$. Lemma 3.12 then ensures that there is an $n \in \mathbb{N}$ such that $D \hookrightarrow K^n(A_1)$. On the other hand, $A_1 \hookrightarrow C$ implies $K^n(A_1) \hookrightarrow K^n(C)$. Therefore, $B \hookrightarrow D \hookrightarrow K^n(A_1) \hookrightarrow K^n(C) \hookrightarrow L$, so $B \in \text{age}(L)$. This completes the proof that $\text{age}(L) = \text{Ob}(A)$.
Next, let us show that \( L \) realizes all one-point extensions, that is, let us show that for all \( A, B \in \text{Ob}(A) \) such that \( A \hookrightarrow B \) and every embedding \( f : A \to L \) there is an embedding \( g : B \to L \) such that:

\[
\begin{array}{c}
A \xleftarrow{f} L \\
\downarrow \\
B \\
\end{array}
\]

Take any \( A, B \in \text{Ob}(A) \) such that \( A \hookrightarrow B \) and let \( f : A \to L \) be an arbitrary embedding. By Lemma 2.3 there is an \( n \in \mathbb{N} \) and an embedding \( h : A \hookrightarrow K^n(C) \) such that \( f \circ h = \iota_n \). Note that the following diagram commutes:

\[
\begin{array}{c}
A \xleftarrow{h} K^n(C) \xrightarrow{\iota_n} L \\
\downarrow \eta_A \downarrow \eta_{K^n(C)} \downarrow \iota_{n+1} \\
B \xleftarrow{j} K(A) \xrightarrow{\eta_{K}} K(\iota) K^{n+1}(C) \\
\end{array}
\]

(the triangle on the left commutes due to the definition of the Katětov functor, the parallelogram in the middle commutes because \( \eta \) is a natural transformation, while the triangle on the right commutes as a part of the colimit diagram for the chain (2)). Let \( g = \iota_{n+1} \circ K(h) \circ j \). Having in mind that \( f = \iota_n \circ h \), from the last commuting diagram we immediately get that the diagram (3) commutes for this particular choice of \( g \).

Therefore, \( L \) realizes all one-point extensions, so \( L \) is an ultrahomogeneous countable structure whose age is \( \text{Ob}(A) \). Consequently, \( L \) is the Fraïssé limit of \( \text{Ob}(A) \), whence we easily conclude that \( A \) is an amalgamation class. Moreover, the Fraïssé limit of \( A \) can be obtained by the Katětov construction starting from an arbitrary \( C \in \text{Ob}(C) \).

Finally, let us show that \( L \) is \( C \)-morphism-homogeneous. Take any \( A, B \in \text{age}(L) \), fix the embeddings \( j_A : A \to L \) and \( j_B : B \to L \), and let \( f : A \to B \) be an arbitrary morphism. Then

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\eta_A \downarrow \eta_B \\
K(\omega)(A) \xrightarrow{K(\omega)(f)} K(\omega)(B) \\
\end{array}
\]

Having in mind that \( K(\omega)(A) \) and \( K(\omega)(B) \) are colimits of Katětov constructions starting from \( A \) and \( B \), respectively, we conclude that both \( K(\omega)(A) \)
and $K^\omega(B)$ are isomorphic to $L$. Since $L$ is ultrahomogeneous, there exist isomorphisms $s : K^\omega(A) \to L$ and $t : K^\omega(B) \to L$ such that

$$
\begin{array}{ccc}
K^\omega(A) & \xrightarrow{s} & L \\
\downarrow_{\eta^\omega_A} & & \downarrow_{j_A} \\
A & \xleftarrow{\eta^\omega_A} & B
\end{array}
\quad
\begin{array}{ccc}
L & \xleftarrow{t} & K^\omega(B) \\
\downarrow_{\eta^\omega_B} & & \downarrow_{j_B} \\
B & \xrightarrow{\eta^\omega_B} & K^\omega(A)
\end{array}
$$

Putting diagrams (4) and (5) together we obtain

$$
\begin{array}{ccc}
K^\omega(A) & \xrightarrow{s} & L & \xleftarrow{\eta^\omega_A} \\
\downarrow_{\eta^\omega_A} & & \downarrow_{j_A} & \downarrow_{\eta^\omega_B} \\
A & \xleftarrow{f} & B \\
\downarrow_{\eta^\omega_B} & & \downarrow_{j_B} \\
L & \xleftarrow{t} & K^\omega(B)
\end{array}
\quad
\begin{array}{ccc}
K^\omega(f) & \xrightarrow{t \circ K^\omega(f) \circ s^{-1}} & L
\end{array}
$$

whence follows that $f^* = t \circ K^\omega(f) \circ s^{-1}$ is a $\mathcal{C}$-endomorphism of $L$ which extends $f$. So, $L$ is $\mathcal{C}$-morphism-homogeneous. \qed

Consequently, if the Katětov functor is defined on a category of countable $\Delta$-structures and all homomorphisms between $\Delta$-structures, the Fraïssé limit of $\mathcal{A}$ is both ultrahomogeneous and homomorphism-homogeneous.

**Example 4.4** Let $n \geq 3$ be an integer, let $\mathcal{C}_n$ be the category of all countable $K_n$-free graphs together with all graph homomorphisms, and let $\mathcal{A}_n$ be the full subcategory of $\mathcal{C}_n$ spanned by all finite $K_n$-free graphs. Then there does not exist a Katětov functor $K : \mathcal{A}_n \to \mathcal{C}_n$, for if there were one, the Henson graph $H_n$ – the Fraïssé limit of $\mathcal{A}_n$ – would be homomorphism-homogeneous, and we know this is not the case.

(Proof. Since $H_n$ is universal for all finite $K_n$-free graphs, it embeds both $K_{n-1}$ and the star $S_n$, which is the graph where one vertex is adjacent to $n - 1$ independent vertices. Let $f$ be a partial homomorphism of $H_n$ which maps the $n - 1$ independent vertices of the star $S_n$ onto the vertices of $K_{n-1}$. If $H_n$ were homomorphism-homogeneous, $f$ would extend to an endomorphism $f^*$ of $H_n$, so $f^*$ applied to the center of the star $S_n$ would produce a vertex adjacent to each of the vertices of $K_{n-1}$ inducing thus a $K_n$ in $H_n$, which is not possible.)

Note however that there exists a Katětov functor from the category $\mathcal{A}'_n$ of all finite $K_n$-free graphs together with all graph embeddings to the category.
\(C'_n\) of all countable \(K_n\)-free graphs together with all graph embeddings (see Example 3.6).

The following theorem shows that the existence of a Katětov functor for varieties of algebras understood as categories whose objects are the algebras of the variety and morphisms are embeddings is equivalent to the amalgamation property for the category of finitely generated algebras of the variety.

**Theorem 4.5** Let \(\Delta\) be an algebraic language and let \(\mathcal{V}\) be a variety of \(\Delta\)-algebras understood as a category whose objects are \(\Delta\)-algebras and morphisms are embeddings. Let \(\mathcal{A}\) be the full subcategory of \(\mathcal{V}\) spanned by all finitely generated algebras in \(\mathcal{V}\) and let \(\mathcal{C}\) be the full subcategory of \(\mathcal{V}\) spanned by all countably generated algebras in \(\mathcal{V}\). Assume additionally that there are only countably many isomorphism types in \(\mathcal{A}\). Then there exists a Katětov functor \(K: \mathcal{A} \to \mathcal{C}\) if and only if \(\mathcal{A}\) is the amalgamation class.

**Proof.**  \((\Rightarrow)\) Immediately from Theorem 4.3.  
\((\Leftarrow)\) Recall that a partial algebra consists of a set \(A\) and some partial operations on \(A\), where a partial operation is any partial mapping \(A^n \to A\) for some \(n\) (see [7] for further reference on partial algebras). Clearly, the class of all partial algebras of a fixed type is a free amalgamation class because we can simply identify the elements of the common subalgebra and leave everything else undefined.

According to Theorem 3.14 it suffices to show that \(\mathcal{A}\) has one-point extension pushouts in \(\mathcal{C}\). Take any \(A_0, A_1, A_2 \in \text{Ob}(\mathcal{A})\) such that \(A_0\) embeds into \(A_1\) and \(A_2\) is a one-point extension of \(A_0\). Without loss of generality we can assume that \(A_0 \leq A_1\) and \(A_0 \leq A_2\). Let \(G \subseteq A_0\) be a finite set which generates \(A_0\), choose \(x \in A_2 \setminus A_0\) so that \(G \cup \{x\}\) generates \(A_2\) and let \(H\) be a finite set disjoint from \(G\) such that \(G \cup H\) generates \(A_1\). Let \(S = A_1 \oplus_{A_0} A_2\) be the partial algebra which arises as the free amalgam of \(A_1\) and \(A_2\) over \(A_0\) in the class of all partial \(\Delta\)-algebras. Since \(\mathcal{A}\) has the amalgamation property, there is a \(C \in \text{Ob}(\mathcal{A})\) such that

\[
\begin{array}{c}
A_0 \rightarrowrightarrow A_2 \\
\downarrow \downarrow \downarrow \\
A_1 \rightarrowrightarrow C
\end{array}
\]

whence follows that \(C\) embeds the partial algebra \(S\) in the sense of [7, §28]. It is a well-known fact (see again [7, §28]) that if \(P\) is a partial algebra which
embeds into some total algebra from $\mathcal{V}$ then the free algebra $F_\mathcal{V}(P)$ exists in $\mathcal{V}$. Therefore, $F_\mathcal{V}(S)$ exists and belongs to $\mathcal{V}$. It is easy to see that $F_\mathcal{V}(S)$ is generated by $\{x\} \cup G \cup H$, so $F_\mathcal{V}(S)$ is a one-point extension of $A_1$. It clearly embeds $A_2$, so we have that

\[
\begin{array}{c}
A_0 \hookrightarrow A_2 \\
\downarrow \quad \quad \quad \quad \downarrow \\
A_1 \hookrightarrow F_\mathcal{V}(S)
\end{array}
\]

The universal mapping property, which is the defining property of free algebras, ensures that the above commuting square is actually a pushout square in $\mathcal{C}$. This completes the proof that $\mathcal{A}$ has one-point extension pushouts in $\mathcal{C}$. □

**Corollary 4.6** A Katětov functor exists for the category of all finite semilattices, the category of all finite lattices and for the category of all finite distributive lattices.

**Proof.** The proof follows immediately from the fact that all the three classes of algebras are well-known examples of amalgamation classes. □

The existence of a Katětov functor enables us to quickly conclude that the automorphism group of the corresponding Fraïssé limit is universal, as is the monoid of $\mathcal{C}$-endomorphisms. As an immediate consequence of Theorem 4.3 we have:

**Corollary 4.7** Let $K : \mathcal{A} \to \mathcal{C}$ be a Katětov functor and let $L$ be the Fraïssé limit of $\mathcal{A}$ (which exists by Theorem 4.3). Then for every $C \in \text{Ob}(\mathcal{C})$:

- $\text{Aut}(C) \hookrightarrow \text{Aut}(L)$;
- $\text{End}_\mathcal{C}(C) \hookrightarrow \text{End}_\mathcal{C}(L)$.

**Proof.** Since $K^\omega$ is a functor, we immediately get that $\text{Aut}(C) \hookrightarrow \text{Aut}(K^\omega(C))$ via $f \mapsto K^\omega(f)$ and that $\text{End}_\mathcal{C}(C) \hookrightarrow \text{End}_\mathcal{C}(K^\omega(C))$ via $f \mapsto K^\omega(f)$. But, $K^\omega(C) \cong L$ due to Theorem 4.3. □

**Corollary 4.8** For the following Fraïssé limits $L$ we have that $\text{Aut}(L)$ embeds all permutation groups on a countable set:
• $\mathbb{Q}$,
• the random graph (proved originally in [8]),
• Henson graphs (proved originally in [8]),
• the random digraph,
• the rational Urysohn space (follows also from [14]),
• the random poset,
• the countable atomless Boolean algebra,
• the random semilattice,
• the random lattice,
• the random distributive lattice.

For the following Fraïssé limits $L$ we have that $\text{End}(L)$ embeds all transformation monoids on a countable set:

• $\mathbb{Q}$,
• the random graph (proved originally in [3]),
• the random digraph,
• the rational Urysohn space,
• the random poset (proved originally in [5]),
• the countable atomless Boolean algebra.

Proof. Having in mind Corollary 4.7, in each case it suffices to show that the corresponding category $\mathcal{C}$ contains a countable structure whose automorphism group embeds $\text{Sym}(\mathbb{N})$ and whose endomorphism monoid embeds $\mathbb{N}^\mathbb{N}$ considered as a transformation monoid. For example, in case of the rational Urysohn space it suffices to consider the metric space $(\mathbb{N},d)$ where $d(m,n) = 1$ for all $m,n \in \mathbb{N}$, while in the case of the random Boolean algebra it suffices to consider the free Boolean algebra on $\aleph_0$ generators. □

The following example has a twofold purpose: it describes a Katětov functor in a case where the variety of algebras is not locally finite, and at the same time motivates the proof of the Theorem 4.10.
Theorem 4.3 tells us that the presence of a Katětov functor implies the amalgamation property for \( \mathcal{A} \). Unfortunately, in the general setting this is not the case. The following theorem gives a necessary and sufficient condition for a Katětov functor to exist. It depends on a condition that resembles the Herwig-Lascar-Solecki property (see [9, 13]).

**Definition 4.9** A partial morphism of \( C \in \text{Ob}(\mathcal{C}) \) is a triple \( \langle A, f, B \rangle \) where \( A, B \subseteq C \) are finitely generated and \( f : A \to B \) is a \( \mathcal{C} \)-morphism. We say that \( C \in \text{Ob}(\mathcal{C}) \) has the morphism extension property in \( \mathcal{C} \) if for any choice \( f_1, f_2, \ldots \) of partial morphisms of \( C \) there exist \( D \in \text{Ob}(\mathcal{C}) \) and \( m_1, m_2, \ldots \in \text{End}_\mathcal{C}(D) \) such that \( C \) is a substructure of \( D \), \( m_i \) is an extension of \( f_i \) for all \( i \), and the following coherence conditions are satisfied for all \( i, j \) and \( k \):

- if \( f_i = \langle A, \text{id}_A, A \rangle \) then \( m_i = \text{id}_D \),
- if \( f_i \) is an embedding, then so is \( m_i \), and
- if \( f_i \circ f_j = f_k \) then \( m_i \circ m_j = m_k \).

We say that \( \mathcal{C} \) has the morphism extension property if every \( C \in \text{Ob}(\mathcal{C}) \) has the morphism extension property in \( \mathcal{C} \).

**Theorem 4.10** The following are equivalent:

1. there exists a Katětov functor \( K : \mathcal{A} \to \mathcal{C} \);
2. \( \mathcal{A} \) has (AP) and \( \mathcal{C} \) has the morphism extension property;
3. \( \mathcal{A} \) has (AP) and the Fraïssé limit of \( \mathcal{A} \) has the morphism extension property in \( \mathcal{C} \).

**Proof.** (1) \( \Rightarrow \) (2): From Theorem 4.3 we know that \( \mathcal{A} \) is an amalgamation class, it has a Fraïssé limit \( L \) in \( \mathcal{C} \), and \( L \) can be obtained by the Katětov construction starting from an arbitrary \( C \in \text{Ob}(\mathcal{C}) \). Now, take any \( C \in \text{Ob}(\mathcal{C}) \) and let us show that \( C \) has the morphism extension property in \( \mathcal{C} \). Since \( L \) is universal for \( \text{Ob}(\mathcal{C}) \), without loss of generality we can assume that \( C \leq L \). For every finitely generated \( A \leq C \) fix an isomorphism \( j_A : K^\omega(A) \to L \) such that

\[
\begin{array}{ccc}
A & \xleftarrow{\eta_A} & K^\omega(A) \\
\downarrow & & \downarrow \\
C & \xleftarrow{\leq} & L
\end{array}
\]

21
(such an isomorphism exists because \( L \) is ultrahomogeneous). Now, for any family \( \langle A_i, f_i, B_i \rangle, \ i \in I \), of partial morphisms of \( C \) it is easy to see that \( L \) together with its endomorphisms \( m_i = j_{B_i} \circ K^\omega(j_i) \circ j_{A_i}^{-1}, \ i \in I \), is an extension of \( C \) and its partial morphisms \( f_i, \ i \in I \):

\[
\begin{array}{ccc}
A_i & \xleftarrow{j_{A_i}} & K^\omega(A_i) \\
f_i \downarrow & & \downarrow j_{A_i} \\
B_i & \xleftarrow{j_{B_i}} & K^\omega(B_i)
\end{array}
\]

The coherence requirements are satisfied since \( K^\omega \) is a functor which preserves embeddings.

(2) \( \Rightarrow \) (3): Trivial.

(3) \( \Rightarrow \) (1): Let \( L \) be the Fraïssé limit of \( \mathcal{A} \). For every \( A \in \text{Ob}(\mathcal{A}) \) fix an embedding \( j_A : A \hookrightarrow L \). Then every \( \mathcal{A} \)-morphism \( f : A \to B \) induces a partial morphism \( p(f) : j_A(A) \to j_B(B) \) of \( L \) by \( p(f) = j_B \circ f \circ j_A^{-1} \). Since \( L \) is a countable structure, there are only countably many partial morphisms \( p(f) \), say, \( p_1, p_2, \ldots \). By the assumption of (2) there exist \( D \in \text{Ob}(\mathcal{C}) \) and \( m_1, m_2, \ldots \in \text{End}_\mathcal{C}(D) \) such that \( L \) is a substructure of \( D \), \( m_i \) is an extension of \( p_i \) for all \( i \), and the coherence conditions are satisfied. Let \( e : L \leq D \) be the inclusion of \( L \) into \( D \).

Define a functor \( K : \mathcal{A} \to \mathcal{C} \) on objects by \( K(A) = D \) and on morphisms by \( K(f) = m_i \), where \( p(f) = p_i \). Let us show that \( K \) is indeed a functor. First, note that \( K(\text{id}_A) = \text{id}_D = \text{id}_{K(A)} \): let \( p(\text{id}_A) = p_i \); since \( p_i = p(\text{id}_A) = \text{id}_{j_A(A)} \) coherence requirements force that \( m_i = \text{id}_D \). Then, let us show that \( K(g \circ f) = K(g) \circ K(f) \), where \( f : A \to B \) and \( g : B \to C \). Let \( k \) and \( l \) be positive integers such that \( p(f) = p_k = j_B \circ f \circ j_A^{-1} \) and \( p(g) = p_l = j_C \circ g \circ j_B^{-1} \). Let \( s \) be an integer such that \( p_s = j_C \circ g \circ f \circ j_A^{-1} \). Then \( p_l \circ p_k = p_s \), so the coherence requirements imply that \( m_k \circ m_l = m_s \). Finally, \( K(g \circ f) = m_s = m_l \circ m_k = K(g) \circ K(f) \). The coherence requirements also ensure that \( K \) preserves embeddings.

Let us now show that the set of arrows \( \eta_A = e \circ j_A \) constitutes a natural transformation \( \eta : \text{ID} \to K \). Take any \( \mathcal{A} \)-morphism \( f : A \to B \). Then \( p(f) = p_i = j_B \circ f \circ j_A^{-1} \) is a partial morphism of \( L \) whose extension is \( m_i \). This is why the following diagram commutes (where the dashed arrow...
indicates a partial morphism):

\[
\begin{array}{c}
A \xrightarrow{j_A} L \xleftarrow{e} D \\
\downarrow f \downarrow \pi \downarrow m_i=K(f) \\
B \xleftarrow{j_B} L \xrightarrow{e} D \\
\end{array}
\]

Finally, let us show that \( K(A) \) embeds all one-point extensions of \( A \). Let \( A \hookrightarrow B \). Then there is an \( h : B \hookrightarrow L \) such that

\[
\begin{array}{c}
A \xrightarrow{j_A} L \\
\downarrow h \\
B \\
\end{array}
\]

since \( L \) is the Fraïssé limit of \( A \). Therefore,

\[
\begin{array}{c}
A \xrightarrow{j_A} L \xleftarrow{e} D = K(A) \\
\downarrow h \\
B \\
\end{array}
\]

which concludes the proof. \( \square \)

Note that the Henson graph \( H_n, n \geq 3 \), does not have the morphism extension property with respect to all graph homomorphisms (for otherwise there would be a Katetov functor defined on the category of all finite \( K_n \)-free graphs and all graph homomorphisms, and we know that such a functor cannot exist).

**Hypothesis.** Every Fraïssé limit has the morphism extension property with respect to embeddings.

## 5 Semigroup Bergman property

Following [11], we say that a semigroup \( S \) is *semigroup Cayley bounded with respect to a generating set \( U \) if \( S = U \cup U^2 \cup \ldots \cup U^n \) for some \( n \in \mathbb{N} \). We say
that a semigroup $S$ has the semigroup Bergman property if it is semigroup Cayley bounded with respect to every generating set.

A semigroup $S$ has Sierpiński rank $n$ if $n$ is the least positive integer such that for any countable $T \subseteq S$ there exist $s_1, \ldots, s_n \in S$ such that $T \subseteq \langle s_1, \ldots, s_n \rangle$. If no such $n$ exists, the Sierpiński rank of $S$ is said to be infinite.

A semigroup $S$ is strongly distorted if there exists a sequence of natural numbers $l_1, l_2, l_3, \ldots$ and an $N \in \mathbb{N}$ such that for any sequence $a_1, a_2, a_3, \ldots \in S$ there exist $s_1, \ldots, s_N \in S$ and a sequence of words $w_1, w_2, w_3, \ldots$ over the alphabet $\{x_1, x_2, \ldots, x_N\}$ such that $|w_n| \leq l_n$ and $a_n = w_n(s_1, \ldots, s_N)$ for all $n$.

**Lemma 5.1 ([11])** If $S$ is a strongly distorted semigroup which is not finitely generated, then $S$ has the Bergman property.

It was shown in [12] that $\text{End}(R)$, the endomorphism monoid of the random graph, is strongly distorted and hence has the semigroup Bergman property since it is not finitely generated. The idea from [12] was later in [4] directly generalized to classes of structures with coproducts. Here, we present a general treatment in the context of classes for which a Katětov functor exists, and where the (JEP) can be carried out constructively in the sense of the following definition.

**Definition 5.2** A category $\mathcal{C}$ has natural (JEP) if there exists a covariant functor $F : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ such that

- for all $C, D \in \text{Ob}(\mathcal{C})$ there exist embeddings $\lambda_C : C \hookrightarrow F(C, D)$ and $\rho_D : D \hookrightarrow F(C, D)$, and
- for every pair of morphisms $f : C \rightarrow C'$ and $g : D \rightarrow D'$ the diagram below commutes:

$$
\begin{array}{ccc}
C & \overset{\lambda_C}{\hookrightarrow} & F(C, D) & \overset{\rho_D}{\rightarrow} & D \\
\downarrow f & & \downarrow F(f, g) & & \downarrow g \\
C' & \overset{\lambda_{C'}}{\hookrightarrow} & F(C', D') & \overset{\rho_{D'}}{\leftarrow} & D \\
\end{array}
$$

We also say that $F$ is a natural (JEP) functor for $\mathcal{C}$.

A category $\mathcal{C}$ has retractive natural (JEP) if $\mathcal{C}$ has natural (JEP) and the functor $F$ has the following additional property: for every $C \in \text{Ob}(\mathcal{C})$ there exist morphisms $\rho^*_C, \lambda^*_C : F(C, C) \rightarrow C$ such that $\rho^*_C \circ \rho_C = \text{id}_C = \lambda^*_C \circ \lambda_C$.  

24
**Remark 5.3** Note that since $F$ is a covariant functor, the following also holds:

- $F(id_C, id_D) = id_{F(C,D)}$ for all $C, D \in \text{Ob}(C)$,
- for all $f_1 : B_1 \to C_1$, $f_2 : B_2 \to C_2$, $g_1 : C_1 \to D_1$, $g_2 : C_2 \to D_2$ we have $F(g_1 \circ f_1, g_2 \circ f_2) = F(g_1, g_2) \circ F(f_1, f_2)$, and

$$
\begin{array}{c}
A \xrightarrow{f_1} B \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
C \xrightarrow{f_3} D \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
Q \xrightarrow{g_3} S
\end{array}
$$

implies

$$
\begin{array}{c}
F(A) \xrightarrow{F(f_1, g_3)} F(C, Q) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
F(B) \xrightarrow{F(f_3, g_4)} F(D, S)
\end{array}
$$

**Example 5.4** Any category with coproducts (such as the category of graphs, posets, digraphs) has retractive natural (JEP): just take $F(C,D)$ to be the coproduct of $C$ and $D$.

**Example 5.5** The category of all countable metric spaces with distances in $[0,1]_Q = \mathbb{Q} \cap [0,1]$ and nonexpansive maps has retractive natural (JEP): take $F(C,D)$ to be the disjoint union of $C$ and $D$ where the distance between any point in $C$ and any point in $D$ is 1.

On the other hand, it is easy to show that the category of all countable metric spaces with distances in $\mathbb{Q}$ and nonexpansive maps does not have natural (JEP). Suppose, to the contrary, that there exists a functor $F$ which realizes the natural (JEP) in this category, let $U$ be the rational Urysohn space and let $W = F(U,U)$. Let $a_0, b_0 \in U$ be arbitrary but fixed, and let $\delta = d_W(\lambda_U(a_0), \rho_U(b_0))$. Take any $a, b \in U$, let $c_a : U \to U : x \mapsto a$ and $c_b : U \to U : x \mapsto b$ be the constant maps and put $\Phi = F(c_a, c_b)$. Then $d_W(\lambda_U(a), \rho_U(b)) = d_W(\lambda_U(c_b(a)), \rho_U(c_b(b))) = d_W(\Phi(\lambda_U(a)), \Phi(\rho_U(b))) \leq d_W(\lambda_U(a_0), \rho_U(b_0)) = \delta$, because $\Phi$ is nonexpansive. Now, for $a_1, a_2 \in U$ we have $d_U(a_1, a_2) = d_W(\lambda_U(a_1), \lambda_U(a_2)) \leq d_W(\lambda_U(a_1), \rho_U(b)) + d_W(\lambda_U(a_2), \rho_U(b)) \leq 2\delta$. Hence, $\text{diam}(U) \leq 2\delta$. Contradiction.

**Example 5.6** Let $\Delta$ be the language consisting of function symbols and constant symbols only so that $\Delta$-structures are actually $\Delta$-algebras, and assume that $\Delta$ contains a constant symbol 1. Then the category of $\Delta$-algebras has retractive natural (JEP): take $F(C,D)$ to be $C \times D$ where $\lambda_C : c \mapsto \langle c, 1^D \rangle$, $\rho_D : d \mapsto \langle 1^C, d \rangle$, $\lambda_C^* = \pi_1$ and $\rho_D^* = \pi_2$.  

25
Our aim in this section is to prove the following theorem:

**Theorem 5.7** Assume that there exists a Katětov functor $K : \mathcal{A} \to \mathcal{C}$ and assume that $\mathcal{C}$ has retractive natural (JEP). Let $L$ be the Fraïssé limit of $\mathcal{A}$ (which exists by Theorem 4.3). Then $\text{End}_\mathcal{C}(L)$ is strongly distorted and the Sierpiński rank of $\text{End}_\mathcal{C}(\mathcal{A})$ is at most 5. Consequently, if $\text{End}_\mathcal{C}(L)$ is not finitely generated then it has the Bergman property.

The proof of the theorem requires some technical prerequisites. Let us denote the functor which realizes (JEP) in $\mathcal{C}$ by $(\cdot, \cdot)$ so that $(\cdot, \cdot)$ denotes its action on objects, and $(\cdot, \cdot)$ its action on morphisms. For objects $C_1, C_2, C_3, \ldots, C_n$ and morphisms $f, g, f_1, f_2, f_3, \ldots, f_n$ of $\mathcal{C}$ let

$$[C_1, C_2, C_3, \ldots, C_n] = (((C_1, C_2), C_3), \ldots), C_n),$$
$$[f_1, f_2, f_3, \ldots, f_n] = (((f_1, f_2), f_3), \ldots), f_n),$$
$$[f, g]_n = [f, g, \ldots, g],$$
with $[f, g]_0 = f$.

Moreover, let

$$L_1 = L,$$
$$L_n = (L_{n-1}, L) = [L, L, \ldots, L],$$
for $n \geq 2$.

Let $C$ denote the colimit of the following chain in $\mathcal{C}$ with the canonical embeddings denoted by $\iota_n$:

$$L_1 \xrightarrow{\lambda_{L_1}} L_2 \xrightarrow{\lambda_{L_2}} L_3 \xrightarrow{\lambda_{L_3}} \cdots$$
$$\xleftarrow{\iota_1} \quad \xleftarrow{\iota_2} \quad \xleftarrow{\iota_3} \quad C$$

Let $L$ be the Fraïssé limit of $\mathcal{A}$, which exists by Theorem 4.3. We know that $K^\omega(C) \cong L$, so let us fix an isomorphism

$$\alpha : K^\omega(C) \xrightarrow{\cong} L.$$
The following diagram commutes because $(\cdot, \cdot)$ is a natural (JEP) functor:

\[
\begin{array}{c}
\begin{array}{c}
L_1 \xleftarrow{\lambda_{L_1}} L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} L_4 \xleftarrow{\lambda_{L_4}} \ldots \\
\downarrow{\text{id}_L} \quad \downarrow{[\lambda_{L_1}^{\ast}, \text{id}_L]} \quad \downarrow{[\lambda_{L_2}^{\ast}, \text{id}_L]} \quad \downarrow{[\lambda_{L_3}^{\ast}, \text{id}_L]} \\
L_1 \xleftarrow{\lambda_{L_1}} L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} \ldots
\end{array}
\end{array}
\]

so the following diagram also commutes:

\[
\begin{array}{c}
\begin{array}{c}
L_1 \xleftarrow{\lambda_{L_1}} L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} L_4 \xleftarrow{\lambda_{L_4}} \ldots \\
\downarrow{\text{id}_L} \quad \downarrow{[\lambda_{L_1}^{\ast}, \text{id}_L]} \quad \downarrow{[\lambda_{L_2}^{\ast}, \text{id}_L]} \quad \downarrow{[\lambda_{L_3}^{\ast}, \text{id}_L]} \\
L_1 \xleftarrow{\lambda_{L_1}} L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} \ldots
\end{array}
\end{array}
\]

Therefore, there is a compatible cone with the tip at $L$ and the morphisms $\text{id}_L$, $\lambda_{L_1}^{\ast}$, $\lambda_{L_1}^{\ast} \circ [\lambda_{L_2}^{\ast}, \text{id}_L]$, $\lambda_{L_1}^{\ast} \circ \lambda_{L_2}^{\ast} \circ [\lambda_{L_3}^{\ast}, \text{id}_L] \ldots$ over the chain $L_1 \hookrightarrow L_2 \hookrightarrow L_3 \hookrightarrow \ldots$. Since $C$ is a colimit of the chain, there is a unique $\beta : C \to L$ such that

\[
\begin{array}{c}
\begin{array}{c}
C \xleftarrow{\beta} L \\
\downarrow{\iota_1} \\
L_1 \xleftarrow{\lambda_{L_1}} L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} \ldots
\end{array}
\end{array}
\]

In particular,

\[
\beta \circ \iota_1 = \text{id}_L. \tag{6}
\]

As the next step in the construction, note that the following diagram commutes (again due to the fact that $(\cdot, \cdot)$ is a natural (JEP) functor):

\[
\begin{array}{c}
\begin{array}{c}
L_1 \xleftarrow{\lambda_{L_1}} L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} L_4 \xleftarrow{\lambda_{L_4}} \ldots \\
\downarrow{\rho_L} \quad \downarrow{[\rho_L, \text{id}_L]} \quad \downarrow{[\rho_L, \text{id}_L]} \quad \downarrow{[\rho_L, \text{id}_L]} \\
L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} L_4 \xleftarrow{\lambda_{L_4}} \ldots
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
L_1 \xleftarrow{\lambda_{L_1}} L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} L_4 \xleftarrow{\lambda_{L_4}} \ldots \\
\downarrow{\rho_L} \quad \downarrow{[\rho_L, \text{id}_L]} \quad \downarrow{[\rho_L, \text{id}_L]} \quad \downarrow{[\rho_L, \text{id}_L]} \\
L_1 \xleftarrow{\lambda_{L_1}} L_2 \xleftarrow{\lambda_{L_2}} L_3 \xleftarrow{\lambda_{L_3}} L_4 \xleftarrow{\lambda_{L_4}} \ldots
\end{array}
\end{array}
\]

27
Therefore, there is a compatible cone with the tip at \( C \) and the morphisms 
\[ \iota_2 \circ \rho_L, \iota_3 \circ [\rho_L, \text{id}_L], \iota_4 \circ [\rho_L, \text{id}_L] \ldots \]
over the chain \( L_1 \hookrightarrow L_2 \hookrightarrow L_3 \hookrightarrow \ldots \).
Since \( C \) is a colimit of the chain, there is a unique \( \sigma : C \to C \) such that

\[ \sigma \circ \iota_n = \iota_{n+1} \circ [\rho_L, \text{id}_L]_{n-1}, \text{ for all } n \geq 1. \]

An easy induction on \( n \) then suffices to show that

\[ \sigma^n \circ \iota_1 = \iota_{n+1} \circ [\rho_L, \text{id}_L]_{n-1} \circ \ldots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L, \text{ for all } n \geq 1. \] (7)

Also, there is a compatible cone with the tip at \( C \) and the morphisms 
\[ \iota_1 \circ \rho^*_L, \iota_2 \circ [\rho^*_L, \text{id}_L], \iota_3 \circ [\rho^*_L, \text{id}_L] \ldots \]
over the chain \( L_2 \hookrightarrow L_3 \hookrightarrow L_4 \hookrightarrow \ldots \), so there is a unique \( \tau : C \to C \) such that

\[ \tau \circ \iota_{n+1} = \iota_n \circ [\rho^*_L, \text{id}_L]_{n-1}, \text{ for all } n \geq 1. \]

Another easy induction on \( n \) suffices to show that

\[ \tau^n \circ \iota_{n+1} = \iota_1 \circ \rho^*_L \circ [\rho^*_L, \text{id}_L]_1 \circ \ldots \circ [\rho^*_L, \text{id}_L]_{n-1}, \text{ for all } n \geq 1. \] (8)

Let \( \overline{f} = (f_1, f_2, \ldots) \) be a sequence of \( C \)-endomorphisms of \( L \). As the final step, we shall now construct an endomorphism \( \varphi(\overline{f}) : C \to C \) which encodes the sequence \( \overline{f} \). Using once more the fact that \((\cdot, \cdot)\) is a natural (JEP) functor, we immediately get that the following diagram commutes:

\[
\begin{array}{cccccccc}
L_1 & \xrightarrow{\lambda_{L_1}} & L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} & L_4 & \xrightarrow{\lambda_{L_4}} & \ldots \\
\downarrow{f_1} & & \downarrow{[f_1, f_2]} & & \downarrow{[f_1, f_2, f_3]} & & \downarrow{[f_1, f_2, f_3, f_4]} & & \\
L_1 & \xrightarrow{\lambda_{L_1}} & L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} & L_4 & \xrightarrow{\lambda_{L_4}} & \ldots \\
\end{array}
\]
so there is a unique $\varphi(\overline{f}) : C \to C$ such that

$$
\xymatrix{
C \ar@{.>}[rr]^{\varphi(\overline{f})} \ar[dr]_{\lambda_{L_1}} & & C \\
L_1 \ar[ur]_{\lambda_{L_1}} & L_2 \ar[ur]_{\lambda_{L_2}} & L_3 \ar[ur]_{\lambda_{L_3}} & \cdots
}
$$

or, explicitly,

$$\varphi(\overline{f}) \circ \iota_n = \iota_n \circ [f_1, f_2, \ldots, f_n], \text{ for all } n \geq 1.$$  

**Lemma 5.8**  
(a) $\varphi(\overline{f}) \circ \iota_1 = \iota_1 \circ f_1$;  
(b) $\varphi(\overline{f}) \circ \iota_2 \circ \rho_L = \iota_1 \circ \rho_L \circ f_2$;  
(c) $\varphi(\overline{f}) \circ \iota_n \circ [\rho_L, \text{id}_L]_{n-2} \circ \ldots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L = \iota_n \circ [\rho_L, \text{id}_L]_{n-2} \circ \ldots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L \circ f_n$, for all $n \geq 3$.

**Proof.** (a) This is immediate from the construction of $\varphi(\overline{f})$.  
(b) It suffices to note that the diagram below commutes. The square on the left commutes because $(\cdot, \cdot)$ is natural, while the square on the right commutes by the construction of $\varphi(\overline{f})$.

$$
\xymatrix{
L_1 \ar[r]^{\rho_L} & L_2 \ar[r]^{\iota_2} & C \\
L_1 \ar[r]_{\rho_L} & L_2 \ar[r]_{\iota_2} & C
}
$$

(c) This follows by induction on $n$. Just to illustrate the main ideas (which are straightforward, anyhow) we show the case $n = 4$. The following diagram commutes:

$$
\xymatrix{
L_1 \ar[r]^{\rho_L} & L_2 \ar[r]^{[\rho_L, \text{id}_L]_1} & L_3 \ar[r]^{[\rho_L, \text{id}_L]_2} & L_4 \ar[r]^{\iota_4} & C \\
L_1 \ar[r]_{\rho_L} & L_2 \ar[r]_{[\rho_L, \text{id}_L]_1} & L_3 \ar[r]_{[\rho_L, \text{id}_L]_2} & L_4 \ar[r]_{\iota_4} & C
}
$$

The leftmost square commutes because $(\cdot, \cdot)$ is natural, while the rightmost square commutes by the construction of $\varphi(\overline{f})$. To see that the second square
in this row commutes, just apply the functor \((\cdot, \cdot)\) to the following two commutative squares (see Remark 5.3):

\[
\begin{array}{ccc}
L_1 & \xrightarrow{\rho_L} & L_2 \\
\downarrow f_3 & & \downarrow [f_2, f_3] \\
L_1 & \xrightarrow{\rho_L} & L_2
\end{array}
\quad
\begin{array}{ccc}
L & \xrightarrow{id_L} & L \\
\downarrow f_4 & & \downarrow f_4 \\
L & \xrightarrow{id_L} & L
\end{array}
\]

The same argument suffices to show that the third square in the row also commutes. \(\square\)

**Lemma 5.9**

(a) \(\beta \circ \varphi(f) \circ \iota_1 = f_1\);

(b) \(\beta \circ \tau^n \circ \varphi(f) \circ \sigma^n \circ \iota_1 = f_{n+1}\).

**Proof.** In order to make it easier to follow the calculations we underline the expression that is to be reduced in the following step.

(a) \(\beta \circ \varphi(f) \circ \iota_1 = \beta \circ \iota_1 \circ f_1 = f_1\), by Lemma 5.8 and (6).

(b) \(\beta \circ \tau^n \circ \varphi(f) \circ (\sigma^n \circ \iota_1) = \)

\[
\begin{aligned}
&[\text{by (7)}] = \beta \circ \tau^n \circ \varphi(f) \circ \iota_{n+1} \circ [\rho_L, \text{id}_L]_{n-1} \circ \ldots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L \\
&[\text{Lemma 5.8}] = \beta \circ \tau^n \circ \iota_{n+1} \circ [\rho_L, \text{id}_L]_{n-1} \circ \ldots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L \circ f_{n+1} \\
&[\text{by (8)}] = \beta \circ \iota_1 \circ \rho^*_L \circ [\rho^*_L, \text{id}_L]_1 \circ \ldots \circ [\rho^*_L, \text{id}_L]_{n-1} \circ [\rho^*_L, \text{id}_L]_1 \circ \rho_L \circ f_{n+1} \\
&[\text{by (6)}] = \rho^*_L \circ [\rho^*_L, \text{id}_L]_1 \circ \ldots \circ [\rho^*_L, \text{id}_L]_{n-1} \circ [\rho^*_L, \text{id}_L]_1 \circ \rho_L \circ f_{n+1} \\
&= \ldots = f_{n+1},
\end{aligned}
\]

since \([\rho^*_L, \text{id}_L]_j \circ [\rho_L, \text{id}_L]_j = \text{id}_L\), for all \(j\). \(\square\)

We are now ready to prove Theorem 5.7.

**Proof. (of Theorem 5.7)** We are going to show that \(\text{End}_C(K^\omega(C))\), which is isomorphic to \(\text{End}_C(L)\) because \(L \cong K^\omega(C)\), is strongly distorted and that the Sierpiński rank of \(\text{End}_C(K^\omega(C))\) is at most 5. Take any countable sequence \(f_1, f_2, \ldots \in \text{End}_C(K^\omega(C))\), and let us construct \(\tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}, \tilde{\tau}, \tilde{\varphi} \in \text{End}_C(K^\omega(C))\) as follows, with the notation introduced above.
Let \( \tilde{\alpha} = \eta^\omega C \circ \iota_1 \circ \alpha : K^\omega(C) \to K^\omega(C) \). By Lemma 5.2 (b) (for \( h = \alpha^{-1} \circ \beta \)) there exists a \( \tilde{\beta} : K^\omega(C) \to K^\omega(C) \) such that \( \tilde{\beta} \circ \eta^\omega C = \alpha^{-1} \circ \beta \). Next, let \( \tilde{\sigma} = K^\omega(\sigma) \) and \( \tilde{\tau} = K^\omega(\tau) \). Finally, let \( f_n^\alpha = \alpha \circ f_n \circ \alpha^{-1} \) and let \( \tilde{\varphi} = K^\omega(\varphi(\overline{g})) \) where \( \overline{g} = (f_1^\alpha, f_2^\alpha, \ldots) \). Then

\[
\tilde{\beta} \circ \tilde{\varphi} \circ \tilde{\alpha} = \tilde{\beta} \circ K^\omega(\varphi(\overline{g})) \circ \eta^\omega C \circ \iota_1 \circ \alpha
\]

[\( \eta^\omega \) is natural]

\[
\tilde{\beta} \circ \tilde{\varphi} \circ \tilde{\alpha} = \tilde{\beta} \circ \eta^\omega C \circ \varphi(\overline{g}) \circ \iota_1 \circ \alpha
\]

[definition of \( \tilde{\beta} \)]

\[
\tilde{\beta} \circ \tilde{\varphi} \circ \tilde{\alpha} = \alpha^{-1} \circ f_1^\alpha \circ \alpha = f_1,
\]

and

\[
\tilde{\beta} \circ \tilde{\tau} \circ \tilde{\varphi} \circ \tilde{\sigma} \circ \tilde{\alpha} = \tilde{\beta} \circ K^\omega(\tau \circ \varphi \circ \sigma) \circ \eta^\omega C \circ \iota_1 \circ \alpha
\]

[\( \eta^\omega \) is natural]

\[
\tilde{\beta} \circ \tilde{\tau} \circ \tilde{\varphi} \circ \tilde{\sigma} \circ \tilde{\alpha} = \tilde{\beta} \circ \eta^\omega C \circ \tau \circ \varphi \circ \sigma \circ \iota_1 \circ \alpha
\]

[definition of \( \tilde{\beta} \)]

\[
\tilde{\beta} \circ \tilde{\tau} \circ \tilde{\varphi} \circ \tilde{\sigma} \circ \tilde{\alpha} = \alpha^{-1} \circ \beta \circ \tau \circ \varphi \circ \sigma \circ \iota_1 \circ \alpha
\]

[Lemma 5.9]

\[
\tilde{\beta} \circ \tilde{\tau} \circ \tilde{\varphi} \circ \tilde{\sigma} \circ \tilde{\alpha} = \alpha^{-1} \circ f_{n+1}^\alpha \circ \alpha = f_{n+1}.
\]

This shows that every \( f_n \) belongs to the semigroup generated by \( \tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}, \tilde{\tau} \) and \( \tilde{\varphi} \), and we uniformly have that the length of the word representing \( f_n \) is \( 2n + 1 \). Therefore, \( \text{End}_C(K^\omega(C)) \) is strongly distorted and the Sierpiński rank of \( \text{End}_C(K^\omega(C)) \) is at most 5. Lemma 5.1 now yields that \( \text{End}_C(F) \) has the Bergman property if it is not finitely generated. 

**Corollary 5.10** For the following Fraïssé limits \( L \) we have that \( \text{End}_C(L) \) has the Bergman property:

- random graph,
- random digraph,
- rational Urysohn sphere (the Fraïssé limit of the category of all finite metric spaces with rational distances bounded by 1),
- random poset,
- random Boolean algebra (the Fraïssé limit of the category of all finite Boolean algebras)
References

[1] I. Ben Yaacov: The linear isometry group of the Gurarij space is universal. Proc. Amer. Math. Soc. 142 (2014), no. 7, 2459–2467.

[2] D. Bilge, J. Melleray: Elements of finite order in automorphism groups of homogeneous structures. Contrib. Discrete Math. 8 (2013), no. 2, 88–119.

[3] A. Bonato, D. Delić, I. Dolinka: All countable monoids embed into the monoid of the infinite random graph. Discrete Math. 310 (2010), 373–375.

[4] I. Dolinka: The Bergman property for endomorphism monoids of some Fraïssé limits. Forum Math. 26 (2014), no. 2, 357–376.

[5] I. Dolinka: The endomorphism monoid of the random poset contains all countable semigroups. Alg. Univers. 56 (2007), 469–474.

[6] I. Dolinka, D. Mašulović: A universality result for endomorphism monoids of some ultrahomogeneous structures. Proceedings of the Edinburgh Mathematical Society 55 (2012), 635–656.

[7] G. Grätzer: Universal algebra (2nd ed). Springer-Verlag, New York, 2008.

[8] C. W. Henson: A family of countable homogeneous graphs. Pac. J. Math. 38 (1971), 69–83.

[9] B. Herwig, D. Lascar: Extending partial automorphisms and the profinite topology on free groups. Trans. Amer. Math. Soc. 352 (2000), 1985–2021.

[10] M. Katétov: On universal metric spaces. General topology and its relations to modern analysis and algebra. VI (Prague, 1986), Res. Exp. Math., vol. 16, Heldermann, Berlin, 1988, 323–330.

[11] V. Maltcev, J. D. Mitchell, N. Ruškuc: The Bergman property for semigroups. J. London Math. Soc. 80 (2009), 212–232.

[12] Y. Péresse: Generating uncountable transformation semigroups. Ph.D. thesis, University of St Andrews, 2009.
[13] S. Solecki: Notes on a strengthening of the Herwig-Lascar extension theorem. 2009. Unpublished note, available at http://www.math.uiuc.edu/~ssolecki/papers/HervLascfin.pdf

[14] V. V. Uspenskij: On the group of isometries of the Urysohn universal metric space. Commentat. Math. Univ. Carolinae 31 (1990), 181–182.