Critical Behaviors of 3D Black Holes with a Scalar Hair

A. Belhaj\textsuperscript{1,2}, M. Chabab\textsuperscript{2}, H. EL Moumni\textsuperscript{2}, M. B. Sedra\textsuperscript{3}

\textsuperscript{1}Département de Physique, Faculté Polydisciplinaire, Université Sultan Moulay Slimane, Béni Mellal, Morocco
\textsuperscript{2}High Energy Physics and Astrophysics Laboratory, FSSM, Cadi Ayyad University, Marrakesh, Morocco
\textsuperscript{3}Département de Physique, LHESIR, Faculté des Sciences, Université Ibn Tofail, Kénitra, Morocco.

May 7, 2014

Abstract

The principal focus of the present work concerns the critical behaviors of a class of three dimensional black holes with a scalar field hair. Since the cosmological constant is viewed as a thermodynamic pressure and its conjugate quantity as a volume, we examine such properties in terms of two parameters $B$ and $a$. The latters are related to the scalar field and the angular momentum respectively. In particular, we give the equation of state predicting a critical universal number depending on the $(B, a)$ moduli space. In the vanishing limit of the $B$ parameter, we recover the usual perfect gas behavior appearing in the case of the non rotating BTZ black hole. We point out that in a generic region of the $(B, a)$ moduli space, the model behaves like a Van der Waals system.
Thermodynamic behaviors of black holes in various dimensions have received a particular interest through recent important works [1-15]. The equation of state for certain black holes has been established using an analysis that involves some characteristics similar to the Van der Waals $P-V$ diagram [1, 12].

The $P-V$ criticality of four dimensional RN-AdS black holes with spherical configurations have been extensively studied. In particular, it has been shown a remarkable interplay between the behaviors of the RN-AdS black hole systems and the Van der Waals fluids. The $P-V$ criticality, the Gibbs free energy, the first order phase transition and the behavior near the critical points can be associated with the statistical liquid-gas systems. On the basis of [1], we have studied the critical behaviors of charged RN-AdS black holes in arbitrary dimensions of the spacetime[6]. In fact, we have presented a comparative study in terms of the dimension and the displacement of the critical points. We have realized that these parameters can be explored to control the transition between the small and the large black holes. More precisely, we have revealed that such behaviors vary in terms of the dimension of the spacetime in which the black hole belongs. A particular emphasis has been put on the three dimensional case corresponding to the BTZ black hole whose critical behaviors are associated with the ideal gas ones. More recently, a novel exact rotating black hole solution in $(2 + 1)$-dimensional gravity with a non-minimally coupled scalar field has been proposed in [16, 17] using an appropriate metric ansatz.

The present work concerns the critical behaviors of a class of three dimensional black holes with a scalar field hair. Since the cosmological constant is viewed as a thermodynamic pressure and its conjugate quantity as a volume, we examine such properties in terms of two parameters $B$ and $a$. These parameters correspond to the scalar field and the angular momentum respectively. Among our results, we derive the equation of state which predicts a critical universal number depending on the $(B, a)$ moduli space. If the $B$ parameter is set equal to zero, we recover the usual perfect gas behavior appearing in the case of the non rotating BTZ black hole. We point out that in a generic region of the $(B, a)$ moduli space, the model behaves like a Van der Waals system.

To start, we consider the following action

$$I = \frac{1}{2} \int d^3 x \sqrt{-g} \left[ R - g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi - \xi R \phi^2 - 2V(\phi) - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right]$$

(1)

where $F = dA$ is the field strength of the $A$ gauge field and where $V(\phi)$ is a scalar potential. In this action, $\xi$ describes the coupling between the gravity and the scalar field. Following [16, 17], the static and circularly symmetric solution can be obtained from the following metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\psi^2,$$

(2)

where the space-time variables are $-\infty < t < \infty, r \geq 0$ and $-\pi \leq \psi \leq \pi$. In this context, the metric reads as

$$f(r) = \left(3\beta - \frac{Q^2}{4}\right) + \left(2\beta - \frac{Q^2}{9}\right) \frac{B}{r} - \left(\frac{1}{2} + \frac{B}{3r}\right) \ln(r) + \frac{r^2}{\ell^2}$$

(3)

where $\Lambda$ has been chosen to be $-\frac{1}{\ell^2}$ due to the fact that in $3D$ the black hole horizon can exist only for a negative cosmological constant. The parameter $B$ corresponds to the scalar
field through \( \phi(r) = \pm \sqrt{\frac{8}{r+B}} \). It is worth noting that the parameter \( \beta \) is related to the black hole mass \( M \) and its charge \( Q \) as follows

\[
\beta = \frac{1}{3} \left( \frac{Q^2}{4} - M \right) \tag{4}
\]

Before discussing the corresponding thermodynamical quantities, we should define the Euclidean section \((t \to i\tau)\) of the solution and identify the period \( \beta \) of the imaginary time with the inverse of temperature \([18]\). In fact, performing the formula for the period, \( \beta = \frac{4\pi}{f'(r_+)} \), we get the Hawking temperature. It is given by

\[
T_H = \frac{9r_+ (B + r_+) (4r_+^2 - Q^2 \ell^2) - 4B^2Q^2\ell^2}{24\pi r_+^2 \ell^2 (2B + 3r_+)} \tag{5}
\]

A similar computation gives the following entropy function

\[
S = 4\pi r_+. \tag{6}
\]

To derive the equation of state, we identify the cosmological constant with the pressure by the relation

\[
\ell^2 = \frac{1}{8\pi P}. \tag{7}
\]

The calculation shows that the pressure function reads as

\[
P = \frac{B^2Q^2}{72\pi r_+^3 (B + r_+)} + \frac{Q^2}{32\pi r_+ (B + r_+)} + \frac{BO^2}{32\pi r_+^2 (B + r_+)} + \frac{T}{4 (B + r_+)} + \frac{BT}{6r_+ (B + r_+)}. \tag{8}
\]

This equation is plotted on figure 1.

![Figure 1: The P − V diagram of chargef (2 + 1) black hole with the scalar field hair, where the charge is equal to 1.](image)

It is observed from figure 1 that the corresponding black hole does not have any critical behavior. It looks like an ideal gas. This situation appears also in the case of BTZ black
hole showing a good agreement with the result obtained in [6]. We expect that this feature should be true for all no rotating 3D-black holes. More general behaviors can be obtained by implementing new physical parameters.

In what follow, we consider the 3D-dimensional gravity with a non-minimally coupled scalar field. In the absence of the Maxwell gauge field, the model can be described by the action

\[
I_R = \frac{1}{2} \int d^3x \sqrt{-g} \left[ R - g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi - \xi R \phi^2 - 2V(\phi) \right]
\]  

(9)

Fixing the coupling constant \(\xi = 1\) and taking the gravitational constant like \(\kappa = 8\pi G = 1\), the rotating black solution can be written as

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 \left( d\psi^2 + \omega(r)dt \right)^2
\]  

(10)

where the functions \(f\) and \(\omega\) are given respectively by

\[
f(r) = 3\beta + \frac{2B\beta}{r} + \frac{(3r + 2B)^2a^2}{r^4} + \frac{r^2}{\ell^2}
\]  

(11)

\[
\omega = \frac{(3r + 2B)a}{r^3}
\]  

(12)

with \(\beta = -\frac{M}{3}\). The new parameter \(a\) is a rotating parameter related to the angular momentum. In this case, the Hawking temperature reads as

\[
T_{Ha} = -\frac{3(B + r_+)(4a^2B_1 + 12a^2Br_+ + 9a^2r_+^2 - 8\pi Pr_+^5)}{2\pi r_+^5(2B + 3r_+)}.
\]  

(13)

Based on these quantities, the equation of state is

\[
P = \frac{(2B + 3r_+)(6a^2B_1 + 15a^2Br_+ + 9a^2r_+^2 + 2\pi r_+^5T)}{24\pi r_+^6(B + r_+)}.
\]  

(14)

For the small values of the \(B\) parameter, the equation of state (14) becomes

\[
P = \frac{9a^2}{8\pi r_+^4} + \frac{T}{4r_+} + B \left( \frac{3a^2}{2\pi r_+^5} - \frac{T}{12r_+^3} \right) + \mathcal{O}(B^2).
\]  

(15)

It is important to note that in the vanishing limit of the \(B\) parameter, the equation of state reduces to an equation describing the ideal gas. From (15), we get for the vanishing limit of parameters,

\[
P = \frac{T}{4r_+}.
\]  

(16)

To transform the equation of state from the geometric form (15) to physical one, we use the following redefinition

\[
Press = \frac{\hbar c}{\ell_p^2} P, \quad Temp = \frac{\hbar c}{k} T.
\]  

(17)

where the Planck length reads as \(\ell_p^2 = \frac{\hbar cG}{\kappa T}\). Multiplying the equation (15) with \(\frac{\hbar c}{\ell_p^2}\), we find

\[
Press = \frac{\hbar c}{\ell_p^2} P = \frac{\hbar c}{\ell_p^2} \left[ \frac{T}{4r_+} + \cdots \right] = k Temp \frac{T}{4\ell_p^2 r_+}.
\]  

(18)
If we compare this equation with the Van der Waals one \((P + \frac{a}{v^2})(v - b) = kT\), the specific volume can be then identified as
\[
v = 4\ell^2 r_+.
\] (19)

It follows also that the equation of state can be rewritten as
\[
P = \frac{T}{v} + \frac{288a^2}{\pi v^4} + B \left( \frac{1536a^2}{3\pi v^5} - \frac{4T}{3v^2} \right).
\] (20)

In fact, the discussion on the critical behavior depends on the \((B, a)\) moduli space. At the origin of this moduli space, the system describes a perfect gas system. For generic regions of the moduli space, the corresponding behaviors are shown in figure 2.

![Figure 2](image)

Figure 2: The 2 + 1 diagram of rotating (2 + 1) black holes with an scalar field hair, where \(T_c\) is the critical temperature and the charge is equal to 1.

It has been observed that for \(T > T_c\), the behavior looks like an extended Van der Waals gas. The system exhibits an inflection point. The corresponding critical point ensures a solution of the following conditions
\[
\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0.
\] (21)

The calculation gives the coordinates of the critical point given by
\[
T_c = \frac{9 \left( 5189 + 418\sqrt{154} \right) a^2}{50\pi B^3}, \quad v_c = \frac{4B}{9} \left( \sqrt{154} - 8 \right),
\]
\[
P_c = \frac{59049a^2 \left( 12B^4 + \left( \sqrt{154} - 8 \right) B^3 - 6 \left( 12 + \sqrt{154} \right) B + 8\sqrt{154} + 116 \right)}{8 \left( \sqrt{154} - 8 \right)^5 \pi B^3}.
\] (22)

From these equations we can see that the existence of the critical point is controlled by the two parameters \(a\) and \(B\). The discussion should be made in terms of these two parameters. In
this way, the \((B, a)\) moduli space together with the displacement of the critical point could be used to control the transition between the small and the large black holes. The corresponding general study could be reported elsewhere.

The critical behavior produces the following universal number

\[
\chi = \frac{p_{cv}}{T_c} = \frac{12B^4 + \left(\sqrt{154} - 8\right)B^3 - 6\left(12 + \sqrt{154}\right)B + 116 + 8\sqrt{154}}{116 + 8\sqrt{154}}. \tag{23}
\]

It is interesting to give some comments concerning this expression. First, it depends only on the parameter \(B\). More precisely, it depends on the value of the scalar field. Second, for the vanishing limit of the \(B\) parameter, it can be reduced to the usual equation \(\chi = 1\) describing an ideal gas.

To conclude, we have investigated the thermodynamical behaviors of a class of 3D-black holes with a scalar hair, in the presence of the cosmological constant considered as a thermodynamic pressure and its conjugate quantity as a volume. We have shown that the corresponding equation of state predicts a critical universal number depending on the \((B, a)\) moduli space. In the absence of the \(B\) parameter, the usual perfect gas behavior appearing in the case of non rotating BTZ black hole has been obtained. We have remarkably pointed out a good agreement of our analysis with the results given in [16].

It is worth noting that this study comes up with many open questions. In fact, the calculation has been made only for some thermodynamical quantities. To give a complete picture, others quantities should be implemented, including free energy, partition function an so on. Moreover, throughout this work, the scalar potential form has not been specified. This is an important issue that deserves a special investigation. We hope to replay to these questions in a forthcoming work.

References

[1] D. Kubiznak and R. B. Mann, \textit{P-V criticality of charged AdS black holes}, JHEP \textbf{1207} (2012) 033.

[2] S. Gunasekaran, D. Kubiznak, R. B. Mann, \textit{Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization}, arXiv:hep-th/1208.6251v2.

[3] B. P. Dolan, D. Kastor, D. Kubiznak, R. B. Mann, J. Traschen, \textit{Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter Black Holes}, arXiv:1301.5926.

[4] S. Hawking and D. N. Page, \textit{Thermodynamics of Black Holes in Anti-de Sitter Space}, Commun.Math.Phys. \textbf{83} (1987) 577.

[5] A. Belhaj, K. Bilal, A. El Boukili, M. Nach, M.B. Sedra, \textit{Solutions and thermodynamics of non commutative Liouville black hole}, Int.J.Geom.Meth.Mod.Phys.(2013).

[6] A. Belhaj, M. Chabab, H. El Moumni, M. B. Sedra, \textit{On Thermodynamics of AdS Black Holes in Arbitrary Dimensions}, CPL. Vol.\textbf{29}, No.10(2012)100401.

[7] A. Chamblin, R. Emparan, C. Johnson and R. Myers, \textit{Charged AdS black holes and catastrophic holography}, Phys.Rev. \textbf{D60} (1999) 064018.
[8] A. Chamblin, R. Emparan, C. Johnson and R. Myers, Holography, thermodynamics, and fluctuations of charged AdS black holes, Phys.Rev. D60 (1999) 104026.

[9] M. Cvetic, G. W. Gibbons, D. Kubiznak and C. N. Pope, Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume, Phys. Rev. D 84, 024037 (2011), arXiv:1012.2888 [hep-th].

[10] B. P. Dolan, D. Kastor, D. Kubiznak, R. B. Mann and J. Traschen, Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter Black Holes, arXiv:1301.5926 [hep-th].

[11] B. P. Dolan, Pressure and volume in the first law of black hole thermodynamics, Class. Quant. Grav. 28, 235017 (2011), [arXiv:1106.6260 [gr-qc].

[12] J. Liang and B. Liu, Thermodynamics of noncommutative geometry inspired BTZ black hole based on Lorentzian smeared mass distribution, EPL. 100 (2012) 30001.

[13] H. L. Li, S. Z. Yang, Hawking radiation from the charged BTZ black hole with backreaction, EPL.79(2007)20001.

[14] S. Chakraborty, N. Mazumder, R. Biswas, The generalized second law of thermodynamics and the nature of the entropy function, EPL. 91 (2010) 40007.

[15] C. Song-Bai, L. Xiao-Fang, L. Chang-Qing, P-V Criticality of an AdS Black Hole in f(R) Gravity, CPL. Vol.30, No.6(2013)060401.

[16] L. Zhao, W. Xu, B. Zhu, Novel rotating hairy black hole in (2+1)-dimensions, arXiv:1305.6001.

[17] W. Xu, L. Zhao, Charged black hole with a scalar hair in (2+1) dimensions, arXiv:1305.5446.

[18] D. Grumiller, W. Kummer, D.V. Vassilevich, Dilaton Gravity in Two Dimensions, Phys.Rept.369(2002)327, arXiv:hep-th/0204253.