FRIEDRICHS LEARNING: WEAK SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS VIA DEEP LEARNING

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Abstract. This paper proposes Friedrichs learning as a novel deep learning methodology that can learn the weak solutions of PDEs via a minimax formulation, which transforms the PDE problem into a minimax optimization problem to identify weak solutions. The name “Friedrichs learning” is to highlight the close relation between our learning strategy and Friedrichs theory on symmetric systems of PDEs. The weak solution and the test function in the weak formulation are parameterized as deep neural networks in a mesh-free manner, which are alternately updated to approach the optimal solution networks approximating the weak solution and the optimal test function, respectively. Extensive numerical results indicate that our mesh-free Friedrichs learning method can provide reasonably good solutions for a wide range of PDEs defined on regular and irregular domains, where conventional numerical methods such as finite difference methods and finite element methods may be tedious or difficult to be applied, especially for those with discontinuous solutions in high-dimensional problems.

Key words. Partial Differential Equation; Friedrichs’ System; Minimax Optimization; Weak Solution; Deep Neural Network; High Dimensional Complex Domain.

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1. Introduction. High-dimensional PDEs and PDEs defined on complex domains are important tools in physical, financial, and biological models, etc. [52, 19, 68, 25, 67]. Generally speaking, they do not have closed-form solutions making numerical solutions of such equations indispensable in real applications. First, developing numerical methods for high-dimensional PDEs has been a challenging task due to the curse of dimensionality in conventional discretization. Second, conventional numerical methods rely on mesh generation that requires profound expertise and programming skills without the use of commercial software. In particular, for problems defined in complicated domains, it is challenging and time-consuming to implement conventional methods. As an efficient parametrization tool for high-dimensional functions [8, 17, 58, 57, 64, 41, 43, 62, 63] with user-friendly software (e.g., TensorFlow and PyTorch), neural networks have been applied to solve PDEs via various approaches recently. The idea of using neural networks to solve PDEs dates back to the 1990s [51, 26, 15, 50] and was revisited and popularized recently [16, 32, 18, 48, 11, 53, 42, 41, 60, 54, 69, 7, 56, 47, 44].

Many network-based PDE solvers are concerned with the classical solutions that are differentiable and satisfy PDEs in common sense. Unlike classical solutions, weak solutions are functions for which the derivatives may not always exist but which are nonetheless deemed to satisfy the PDE in some precisely defined sense. These solutions are crucial because many PDEs in modeling
Friedrichs Learning for Weak Solutions of PDEs

Real-world phenomena do not have sufficiently smooth solutions. Motivated by the seminal work in [7], we propose Friedrichs learning as an alternative method that can learn the weak solutions of elliptic, parabolic, and hyperbolic PDEs in $L^2(\Omega)$ via a novel minimax formulation devised and analyzed in Section 2.3. Since the formulation is closely related to the work of Friedrichs theory on symmetric systems of PDEs (cf. [24]), we call our learning strategy the Friedrichs learning. The main idea is to transform the PDE problem into a minimax optimization problem to identify weak solutions. Note that no regularity for the solution is required in Friedrichs learning, which is the main advantage of the proposed method, making it applicable to a wide range of PDE problems, especially those with discontinuous solutions. In addition, Friedrichs learning is capable of solving PDEs with discontinuous solutions without a priori knowledge of the location of the discontinuity.

Although Friedrichs learning may not be able to provide highly accurate solutions, it could solve a coarse solution without a priori knowledge of the discontinuity. This rough estimation of the discontinuity could serve as a good initial guess of conventional computation approaches for highly accurate solutions following the Int-Deep framework in [40]. Finally, theoretical results are provided to justify the Friedrichs learning framework for various PDEs.

The main philosophy of Friedrichs learning is to reformulate a PDE problem into a minimax optimization, the solution of which is a test deep neural network (DNN) that maximizes the loss and a solution DNN that minimizes the loss. For a high-order PDE, we first reformulate it into a first-order PDE system by introducing auxiliary variables, the weak form of which naturally leads to a minimax optimization using integration by parts according to the theory of Friedrichs’ system [24]. The above-mentioned feature is the crucial difference from existing deep learning methods for weak solutions [18, 69]. Let us introduce the formulation of Friedrichs learning using first-order boundary value problems (BVPs) with homogeneous boundary conditions without loss of generality. The initial value problems (IVPs) can be treated as BVPs, where the time variable is considered to be one more spatial variable. The non-homogeneous boundary conditions can be easily transferred to homogeneous ones by subtracting the boundary functions from the solutions.

In the seminal results by Friedrichs in [24] and other investigations in [5, 23], an abstract framework of the boundary value problem of the first-order system was established, which is referred to as Friedrichs’ system in the literature. Let us introduce the concept of Friedrichs’ system using a concrete and simple example and illustrate the main idea and intuition of the Friedrichs learning proposed in this paper. A more detailed abstract framework of Friedrichs learning will be discussed later in Section 2. Let $r \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with Lipschitz boundary $\partial \Omega$. The notation $(\cdot)^\top$ denotes the transpose of a vector or a matrix throughout the paper. We assume: 1) $A_k \in [L^\infty(\Omega)]^{r \times r}$, $\sum_{k=1}^d \partial_k A_k \in [L^\infty(\Omega)]^{r \times r}$, $A_k = A_k^\top$ a.e. in $\Omega$ for $k = 1, \ldots, d$, and $C \in [L^\infty(\Omega)]^{r \times r}$; 2) the full coercivity holds true, i.e., $C + C^\top - \sum_{k=1}^d \partial_k A_k \geq 2\mu_0 I_r$ a.e. in $\Omega$ for some $\mu_0 > 0$ and the identity matrix $I_r \in \mathbb{R}^{r \times r}$. Then the first-order differential operator $T : \mathcal{D} \to L$ with $L = [L^2(\Omega)]^r$ and $\mathcal{D} = [C_0^\infty(\Omega)]^r$ defined by $T u := \sum_{k=1}^d A_k \partial_k u + C u$ is called the Friedrichs operator, while the first-order system of PDEs $T u = f$ is called the Friedrichs’ system, where $f$ is a given data function in $L$ and the space $C_0^\infty(\Omega)$ consists of all infinitely differentiable functions with compact support in $\Omega$. Throughout this paper, the bold font will be used for vectors and matrices in concrete examples. In our abstract framework, PDE solutions are considered as elements of a Hilbert space, so they will not be denoted as bold letters.

Friedrichs [24] also introduced an abstract framework for representing boundary conditions via matrix-valued boundary fields. First, let $A_n := \sum_{k=1}^d n_k A_k \in [L^\infty(\partial \Omega)]^{r \times r}$, where $n = (n_1, \ldots, n_d) \in \mathbb{R}^d$ is the unit outward normal direction on $\partial \Omega$, and let $M : \partial \Omega \to \mathbb{R}^{r \times r}$ be a matrix field on the boundary. Then a homogeneous Dirichlet boundary condition of Friedrichs’ system is prescribed by $(A_n - M)u = 0$ on $\partial \Omega$ by choosing an appropriate $M$ to ensure the well-posedness of
Friedrichs’ system. In real applications, $\mathbf{M}$ is given by physical knowledge. Let $V := \mathcal{N}(\mathbf{A}_n - \mathbf{M})$ and $V^* := \mathcal{N}(\mathbf{A}_n + \mathbf{M})$, where $\mathcal{N}$ is the null space of the argument. It has been proved that $\mathbf{u}$ solves the BVP

$$Tu = f \text{ in } \Omega \quad \text{and} \quad (\mathbf{A}_n - \mathbf{M})\mathbf{u} = \mathbf{0} \text{ on } \partial\Omega, \quad (1.1)$$

if and only if $\mathbf{u}$ solves the minimax problem

$$\min_{\mathbf{u} \in V} \max_{\mathbf{v} \in V^*} \mathcal{L}(\mathbf{u}, \mathbf{v}) := \frac{|(\mathbf{u}, \tilde{T}\mathbf{v})_L - (f, \mathbf{v})_L|}{\|\tilde{T}\mathbf{v}\|_L},$$

where $\tilde{T}: \mathcal{D} \to L$ is the formal adjoint of $T$. Hence, in our Friedrichs learning, DNNs are applied to parametrize $\mathbf{u}$ and $\mathbf{v}$ to solve the above minimax problem to obtain the solution of the BVP (1.1). Friedrichs learning also works for other kinds of boundary conditions.

This paper is organized as follows. In Section 2, we devise and analyze Friedrichs minimax formulation for weak solutions of PDEs. In Section 3, several concrete examples of PDEs and their minimax formulations are provided. In Section 4, network-based optimization is introduced to solve the minimax problem in Friedrichs formulation. In Section 5, a series of numerical examples are provided to demonstrate the effectiveness of the proposed Friedrichs learning. Finally, we conclude this paper in Section 6.

2. Friedrichs Minimax Formulation for Weak Solutions. In this section, we shall first recall some standard notations frequently used later on. Then, briefly review Friedrichs’ system in a Hilbert space setting [23, 12], followed by introducing and analyzing Friedrichs minimax formulation for weak solutions which is the foundation of Friedrichs learning.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with the Lipschitz boundary. Let $D_j = \frac{\partial}{\partial x_j}$ be the partial derivative operation with respect to $x_j$ in the weak sense. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with each $\alpha_i$ being a non-negative integer, denote $D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_d^{\alpha_d}$. For a non-negative integer $k$ and a real number with $1 \leq p \leq \infty$, define the Sobolev space $W^{k,p}(\Omega)$ as a vector space consisting of all functions $v \in L^p(\Omega)$ such that $D^\alpha v \in L^p(\Omega)$ for all multi-indices $\alpha$ with $|\alpha| = \sum_{j=1}^d \alpha_j \leq k$, which is equipped with the following norm:

$$\|v\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha v|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty; \quad \|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \text{esssup}_\Omega |D^\alpha u|,$$

where $\text{esssup}\Omega$ is the essential supremum for a function in $\Omega$. When $p = 2$, $W^{k,2}(\Omega)$ is simply written as $H^k(\Omega)$. In addition, let $H^0_0(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H^k(\Omega)$, while $H^{-k}(\Omega)$ denotes the dual space of $H^k_0(\Omega)$. We refer the reader to the monograph [1] for details about Sobolev spaces and their properties.

Let $L$ denote a real Hilbert space, which is equipped with the inner product $\langle \cdot, \cdot \rangle_L$ and the induced norm $\|\cdot\|_L$. For any two vectors in an Euclidean space, we use $\langle \cdot, \cdot \rangle$ to represent their natural inner product and denote by $\|\cdot\|_p$ the related $\ell_p$ norm for $1 \leq p \leq \infty$; most of these symbols will appear in Sections 4 and 5. For a vector space $W$ and its dual space $W'$, the notation $\langle \cdot, \cdot \rangle_{W \times W'}$ represents the duality pair between $W$ and $W'$. For any two Hilbert spaces $X$ and $Y$, denote by $\mathcal{L}(X,Y)$ the vector space consisting of all continuous linear operators from $X$ into $Y$.

2.1. An Abstract Framework of Friedrichs’ System. First of all, we recall some basic results on Friedrichs’ system developed in [12, 23] for later use in order to be self-contained. Let $L$ be a real Hilbert space, and dual space of $L$, denoted by $L'$, can be identified naturally with $L$ by the Riesz representation theorem. For a dense subspace $\mathcal{D}$ of $L$, we consider two linear operators...
$T : \mathcal{D} \to L$ and $\tilde{T} : \mathcal{D} \to L$ satisfying the following properties: for any $u, v \in \mathcal{D}$, there exists a positive constant $C$ such that

\begin{align}
(2.1) \quad & (Tu, v)_L = (u, \tilde{T}v)_L, \\
(2.2) \quad & \|(T + \tilde{T})u\|_L \leq C\|u\|_L.
\end{align}

It is worth noting that the two operators $T$ and $\tilde{T}$ are given simultaneously. Due to the property (2.1), we often call $\tilde{T}$ as the formal adjoint of $T$ and vice versa. Since the operators $T$ and $\tilde{T}$ play the same roles, we will focus on the forthcoming discussion for $T$, which can be applied to $\tilde{T}$ in a straightforward way. As shown in [6, Sect. 5.5], write $W_0$ as the completion of $\mathcal{D}$ with respect to the scalar product $(\cdot, \cdot)_L = (\cdot, \cdot)_L + (T, T)_L$. Then, we have by (2.1) that

\begin{equation}
\mathcal{D} \subset W_0 \subset L = L' \subset W_0' \subset \mathcal{D}'.
\end{equation}

In addition, in view of (2.2), we know $W_0$ is also the completion of $\mathcal{D}$ with respect to the scalar product $(\cdot, \cdot)_{\tilde{T}} = (\cdot, \cdot)_L + (T', T')_L$. Thus, $\tilde{T}$ can be extended from $\mathcal{D}$ to $W_0$, and its true adjoint $(\tilde{T})^* \in \mathcal{L}(L; W_0')$ can be viewed as the extension of $T$ to $L$. When there is no confusion caused, we still use the notation $T$ for this extension operator. This argument applies to $\tilde{T}$ as well.

We provide an example to make the above abstract treatment more accessible. Let $\Omega = (a, b)$. Choose $\mathcal{D} = C_0^\infty(\Omega)$ and $L = L^2(\Omega)$. Let $Tv = v'$ and $\tilde{T}v = -v'$ for all $v \in C_0^\infty(\Omega)$. In this case, we have

\begin{equation}
(v, w)_T = (v, w)_{\tilde{T}} = \int_a^b (vw + v'w')dx, \quad \forall \, v, w \in C_0^\infty(\Omega),
\end{equation}

so, by definition, the completion of $C_0^\infty(\Omega)$ with respect to the induced norm is exactly the Sobolev space $H_0^1(\Omega)$. Hence, according to Theorem 1.4.4.6 in [27, p. 31], if we understand the derivative operator in the sense of distributions, we know $(\tilde{T})^* \in \mathcal{L}(L^2(\Omega); H^{-1}(\Omega))$. In other words, the derivative operator $(\cdot)'$ can be viewed as a continuous linear operator from $L^2(\Omega)$ into $H^{-1}(\Omega)$.

Next, as given in [23, Lemma 2.1], define a graph space $W$ by

\begin{equation}
W = \{u \in L; Tu \in L\},
\end{equation}

which is a Hilbert space with respect to the graph norm $\|\cdot\|_T = (\cdot, \cdot)_T^{1/2}$. In addition, owing to (2.2), we have

\begin{equation}
W = \{u \in L; \tilde{T}u \in L\}.
\end{equation}

That means $W$ is also a graph space associated with $\tilde{T}$.

The abstract framework of Friedrichs’ system concerns the solvability of the problem

\begin{equation}
Tu = f \in L,
\end{equation}

and its solution falls in the graph space $W$. Obviously, the problem (2.4) may not be well-posed since its solution in $W$ may not be unique. We are interested in constructing a subspace $V \subseteq W$ such that $T : V \to L$ is an isomorphism. A standard way is carried out as follows. We first define a self-adjoint boundary operator $B \in \mathcal{L}(W, W')$ as follows (cf. [23]):

\begin{equation}
\langle Bu, v \rangle_{W' \times W} = (Tu, v)_L - (u, \tilde{T}v)_L, \quad \forall \, u, v \in W.
\end{equation}

This operator plays a key role in the forthcoming analysis. Moreover, the identity (2.5) can be reformulated in the form

\begin{equation}
(Tu, v)_L = (u, \tilde{T}v)_L + \langle Bu, v \rangle_{W' \times W},
\end{equation}

\begin{equation}
\langle Bu, v \rangle_{W' \times W} = (Tu, v)_L - (u, \tilde{T}v)_L, \quad \forall \, u, v \in W.
\end{equation}
which is usually regarded as an abstract integration by parts formula (cf. [23]).

Furthermore, we assume that there exists an operator \( M \in \mathcal{L}(W, W') \) such that

\[
\langle Mw, w \rangle_{W' \times W} \geq 0, \quad \forall w \in W,
\]

(2.6)

\[
W = \mathcal{N}(B - M) + \mathcal{N}(B + M),
\]

(2.7)

where \( \mathcal{N} \) is the null space of its argument. Meanwhile, let \( M^* \in \mathcal{L}(W, W') \) denote the adjoint operator of \( M \) given by \( \langle M^*u, v \rangle_{W' \times W} = \langle Mu, v \rangle_{W' \times W}, \forall u, v \in W \).

To find \( V \) such that the problem (2.4) is well-posed, we should make an additional assumption for \( L \) as follows; i.e.,

\[
((T + \tilde{T})v, v)_L \geq 2\mu_0\|v\|_L^2, \quad \forall v \in L,
\]

(2.8)

where \( \mu_0 \) is a positive constant. Then we choose

\[
V = \mathcal{N}(B - M), \quad V^* = \mathcal{N}(B + M^*).
\]

(2.9)

We have the following important theory for Friedrichs’ system [23, Lemma 3.2 and Theorem 3.1].

**Theorem 2.1.** Assume (2.2), (2.8), (2.6) and (2.7) hold true. Let \( V \) and \( V^* \) be given by (2.9).

The following statements hold true:

1. For any \( v \in W \), it holds

\[
\mu_0\|v\|_L \leq \|Tv\|_L, \quad \mu_0\|v\|_L \leq \|\tilde{T}v\|_L.
\]

(2.10)

2. For any \( f \in L \), problem (2.4) has a unique solution in \( V \). In other words, \( T \) is an isomorphism from \( V \) onto \( L \). Moreover, \( \tilde{T} \) is an isomorphism from \( V^* \) onto \( L \).

### 2.2. First Order PDEs of Friedrichs Type

As a typical application of the above framework, we restrict \( L \) to be the space of square integral (vector-valued) functions over an open and bounded domain \( \Omega \subset \mathbb{R}^d \) with Lipschitz boundary, \( D \) to be the space of test functions, and \( T \) to be a first-order differential operator with its formal adjoint \( \tilde{T} \). In particular, we take \( L = [L^2(\Omega)]^r, \) \( r \in \mathbb{N} \) and \( D = [C_0^\infty(\Omega)]^r. \) \( D \) is thus dense in \( L. \) Consider \( T : D \rightarrow L \) as follows

\[
Tu = \sum_{k=1}^d A_k \partial_k u + Cu = f, \quad \forall u \in D.
\]

(2.11)

The standard assumptions are imposed on \( A_k \) and \( C \) for Friedrichs’ system [20, 21, 24]:

\[
C \in [L^\infty(\Omega)]^{r \times r},
\]

(2.12)

\[
A_k \in [L^\infty(\Omega)]^{r \times r}, \quad k = 1, \ldots, d \quad \text{and} \quad \sum_{k=1}^d \partial_k A_k \in [L^\infty(\Omega)]^{r \times r}
\]

(2.13)

\[
A_k = A_k^T, \quad \text{a. e. in } \Omega, \quad k = 1, \ldots, d.
\]

(2.14)

The formal adjoint \( \tilde{T} : D \rightarrow L \) of \( T \) can be defined by

\[
\tilde{T}u = -\sum_{k=1}^d A_k \partial_k u + (C^T - \sum_{k=1}^d \partial_k A_k)u, \quad \forall u \in D.
\]

(2.15)
It is easy to see that $T$ and $\tilde{T}$ satisfy (2.1)-(2.2). All the results in this section hold true for Friedrichs’ system satisfying (2.12)-(2.14).

For an abstract Friedrichs’ system, one may find the explicit representation of $B$, but it is very difficult to derive the operator on $M$ which is governed by the conditions (2.6) and (2.7). Assume $\mathcal{B} = \sum_{k=1}^{d} n_k A_k$ is well-defined a.e. on $\partial \Omega$ where $n = (n_1, \ldots, n_d)^T$ is the unit outward normal vector of $\partial \Omega$. For simplicity of notations, we set $\mathcal{H}^s = [H^s]^r$ with $H^s$ being the usual Sobolev space of order $s$, and $C^1 = [C^1]^r$ with $C^1$ being the space of continuously differentiable functions, similarly notate $C_0^\infty = [C_0^\infty]^r$.

**Lemma 2.2**. [13, 47] For $u, v \in \mathcal{H}^1(\Omega) \subset W(\Omega)$, there holds

$$\langle Bu, v \rangle_{W'(\Omega) \times W(\Omega)} = \langle Bu, v \rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial \Omega) \times \mathcal{H}^{\frac{1}{2}}(\partial \Omega)},$$

where $W(\Omega) = \{ u \in L(\Omega); Tu \in L(\Omega) \}$ and $W'(\Omega)$ is the dual space of $W(\Omega)$. Specifically, $\langle Bu, v \rangle_{W'(\Omega) \times W(\Omega)} = \int_{\partial \Omega} v^T B u d s$, for any $u, v \in C_0^\infty(\mathbb{R}^d)$.

If $\Omega$ has segment property [4], $C^1(\Omega)$ is thus dense in $\mathcal{H}^1(\Omega)$ and further is dense in $W(\Omega)$. Therefore, the representation could be uniquely extended to the whole space $W(\Omega)$ in the sense that for any $u \in W(\Omega)$ and $v \in \mathcal{H}^1(\Omega)$,

$$\langle Bu, v \rangle_{W'(\Omega) \times W(\Omega)} = \langle Bu, v \rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial \Omega) \times \mathcal{H}^{\frac{1}{2}}(\partial \Omega)}.$$  

The coercivity condition on $T$ dictated by the positiveness condition on the coefficients $A_k$ and $C$ [20, 21, 22] is needed to show the well-posedness of PDEs of Friedrichs type. After some direct manipulation, the abstract coercivity condition (2.8) is equivalent to the following full coercivity for Friedrichs PDEs:

$$C + C^T - \sum_{k=1}^{d} \partial_k A_k \geq 2 \mu_0 I_r, \quad \text{a.e., in } \Omega,$$

where $\mu_0$ is a positive constant and $I_r$ is the $r \times r$ identity matrix. If a system does not satisfies the coercivity condition (2.17) we can introduce a feasible transformation so that the modified system satisfies this condition. In [12], the authors introduced the so-called partial coercivity condition to study the mathematical theory of the corresponding system. Readers are referred to [12] for more details.

### 2.3. Friedrichs Minimax Formulation

Throughout this subsection, we assume all the conditions given in Theorem 2.1 hold true. Recall that $V = \mathcal{N}(B - M)$ and $V^* = \mathcal{N}(B + M^*)$ with $M \in \mathcal{L}(W, W')$ satisfying conditions (2.6)-(2.7). For a given $f \in L$, find the solution $u \in V$ such that

$$Tu = f,$$

or equivalently,

$$\langle Tu, v \rangle_L = \langle f, v \rangle_L, \quad \forall v \in L.$$

In most cases, $T$ is a differential operator whose action on a function should be understood in the sense of distributions. $u$ is thus called the weak solution of the primal variational equation (2.19).

We restrict $v \in V^* \subset L$. From (2.5),

$$\langle Tu, v \rangle_L = (u, \tilde{T} v)_L + \langle Bu, v \rangle_{W' \times W}$$

$$= (u, \tilde{T} v)_L + \langle \frac{B - M}{2} u, v \rangle_{W' \times W} + \langle \frac{B + M}{2} u, v \rangle_{W' \times W}$$

$$= (u, \tilde{T} v)_L + \langle u, \frac{B + M^*}{2} v \rangle_{W' \times W} = (u, \tilde{T} v)_L,$$

where $\tilde{T}$ is a differential operator whose action on a function should be understood in the sense of distributions.
where we used $u \in V = \mathcal{N}(B - M)$ and $v \in V^* = \mathcal{N}(B + M^*)$. This, combined with (2.19), gives

\begin{equation}
(2.20) \quad (u, \tilde{T}v)_L = (f, v)_L, \quad \forall v \in V^*.
\end{equation}

For $u \in V$, (2.20) is equivalent to (2.19). For $u \in L$ satisfying (2.20), $u$ is called the weak solution of the dual variational equation (2.20).

For $u \in V$, $v \in V^*$, we define

\begin{equation}
(2.21) \quad \mathcal{L}(u, v) := \frac{|(u, \tilde{T}v)_L - (f, v)_L|}{\|T v\|_L}.
\end{equation}

According to the estimate (2.10), we have

\[ |(u, \tilde{T}v)_L - (f, v)_L| \leq \|u\|_L \|T v\|_L + \|f\|_L \|v\|_L \leq \left(\|u\|_L + \frac{1}{\mu_0} \|f\|_L\right) \|T v\|_L, \]

where $\mu_0$ is given in (2.8). Therefore, the functional $\mathcal{L}(u, v)$ is bounded with respect to $v \in V^*$ for a fixed $u \in L$.

Thus we can reformulate the problem (2.18) or equivalently the problem (2.19) as the following minimax problem formally:

\begin{equation}
(2.22) \quad \min_{u \in V} \max_{v \in V^*} \mathcal{L}(u, v) := \min_{u \in V} \max_{v \in V^*} \frac{|(u, \tilde{T}v)_L - (f, v)_L|}{\|T v\|_L},
\end{equation}

to identify the weak solution of the primal variational equation (2.19).

**Theorem 2.3.** Assume all the conditions given in Theorem 2.1 hold true. Then $u$ is the unique weak solution of the primal variational equation (2.19) if and only if $u$ is the unique solution that solves the minimax problem (2.22).

**Proof.** On the one hand, if $u \in V$ is a weak solution of (2.19), we have from (2.20) that $\mathcal{L}(u, v) = 0$ for all $v \in V^*$. Thus, $u$ is a solution to the minimax problem (2.22).

On the other hand, if $u$ is a solution of the minimax problem (2.22), then

\[ \max_{v \in V^*} \mathcal{L}(u, v) = \max_{v \in V^*} \frac{|(u, \tilde{T}v)_L - (f, v)_L|}{\|T v\|_L} = 0. \]

Thus, we have $\mathcal{L}(u, v) = 0$ for all $v \in V^*$. This implies

\[ (u, \tilde{T}v)_L - (f, v)_L = 0, \quad \forall v \in V^*. \]

Since $u$ is in $V$, the above equation gives

\[ (Tu - f, v)_L = 0, \quad \forall v \in V^*. \]

Observing that $D$ belongs to $V^*$ and is dense in $L$, the above equation implies that $u$ is a weak solution of the primal variational equation (2.19).

Finally, under the conditions given in Theorem 2.1 it is well known that the weak solution $u$ of the primal variational equation (2.19) exists and is unique. This completes the proof of this theorem.

\[ \square \]

Note that the above discussion and Theorem 2.3 are concerned with the weak solution of the primal variational equation (2.19) with a solution $u$ being in $V$. It is also of interest to discuss
In this case, it is easy to check that the conditions (2.6) and (2.7) hold true. By (2.11),

$$\tag{3.3} \mu u + \beta \cdot \nabla u = f,$$

we define the inflow and outflow boundary for the advection-reaction equation (3.1):

Thus, the full coercivity condition in (2.17) holds true. The graph space $W$

$$\tag{3.4} W = \{ w \in L^2(\Omega); \beta \cdot \nabla w \in L^2(\Omega) \}.$$

We define the inflow and outflow boundary for the advection-reaction equation (3.1):

$$\tag{3.3} \partial \Omega^- = \{ x \in \partial \Omega; \beta(x) \cdot n(x) < 0 \}, \quad \partial \Omega^+ = \{ x \in \partial \Omega; \beta(x) \cdot n(x) > 0 \}.$$

To enforce boundary conditions, we choose from the physical interpretation that

$$\tag{3.4} V = \{ v \in W; v|_{\partial \Omega^-} = 0 \}, \quad V^* = \{ v \in W; v|_{\partial \Omega^+} = 0 \}.$$

In this case, it is easy to check that the conditions (2.6) and (2.7) hold true. By (2.11),

$$\tilde{T}v = -\sum_{i=1}^d \left( \beta_i \frac{\partial v}{\partial x_i} + \frac{\partial}{\partial x_i} \beta_i v \right) + C^T v = -\beta \cdot \nabla v - (\nabla \cdot \beta) v + \mu v.$$
The minimax problem is thus given as follows

\[
\min_{u \in V} \max_{v \in V^*} \mathcal{L}(u, v) = \min_{u \in V} \max_{v \in V^*} \frac{|(u, -\langle \beta \cdot \nabla v + (\nabla \cdot \beta) \psi \rangle + \mu v, \Omega) - (f, v, \Omega)|}{\|\beta \cdot \nabla v + (\nabla \cdot \beta) \psi - \mu v\|_\Omega}.
\]

Note that if the coercivity condition (3.2) does not hold true, we can introduce a transformation \( u = e^{\lambda_0 t} \tilde{u} \), so that the advection-reaction equation (3.1) in \( \tilde{u} \) satisfies (3.2) for sufficiently large constant \( \lambda_0 > 0 \).

### 3.2. Scalar Elliptic PDEs

Consider the second-order PDE to find \( u \) satisfying

\[
- \Delta u + \mu u = f, \quad \text{in } \Omega,
\]

where \( \Omega \subset \mathbb{R}^d, \mu \in L^\infty(\Omega) \) is positive and uniformly bounded away from zero, \( f \in L^2(\Omega) \). This PDE can be rewritten into a first-order PDE system by introducing an auxiliary function \( v \); i.e.,

\[
v + \nabla u = 0, \quad \mu u + \nabla \cdot v = f.
\]

This first order system could be formulated into a Friedrichs’ system with \( r = d + 1 \). The Hilbert space \( L \) is chosen as \( L = [L^2(\Omega)]^r \). Let \( \tilde{u} = (v^T, u)^T \in L \). For \( k = 1, 2, \ldots, d, A_k = \begin{bmatrix} 0 & e^k \\ (e^k)^T & 0 \end{bmatrix} \), \( C = \begin{bmatrix} I_d & 0 \\ 0 & \mu \end{bmatrix} \), where \( e^k \) is the \( k \)-th canonical basis of \( \mathbb{R}^d \). Since \( \mu > 0 \) and has a lower bound away from zero, the full coercivity condition (2.17) is satisfied. The graph space is

\[
W = H(\text{div}; \Omega) \times H^1(\Omega).
\]

One possible choice of the Dirichlet boundary condition is as follows

\[
V = V^* = H(\text{div}; \Omega) \times H^1_0(\Omega) = \{ (v^T, u)^T \in W; \ u|_{\partial \Omega} = 0 \}.
\]

The choices of boundary conditions are not unique, obviously. By introducing auxiliary variables, the second-order linear PDE can be reformulated into a first-order PDE system. Finally, the weak solution of (3.5) can be found by solving the equivalent minimax problem in (2.22).

Denote the test function by \( \psi = (\psi_1^T, \psi_u)^T \) in the space \( V^* \). The minimax problem can be presented as

\[
\min_{\tilde{u} \in V} \max_{\psi \in V^*} \mathcal{L}(\tilde{u}, \psi) = \min_{u \in V} \max_{v \in V^*} \frac{|(-v, \psi_u - \nabla \psi_u) | + (u, \mu \psi_u - \nabla \cdot \psi_{\tilde{u}}) | - (f, \psi_u) |}{\|(\psi_u - \nabla \psi_u)^T, \mu \psi_u - \nabla \cdot \psi_{\tilde{u}}\|_\Omega}.
\]

To reduce the computational cost, we will reformulate the above formulation into a minimax problem in a primal form. To this end, letting \( \psi_v = \nabla \psi_u \), and noting that \( \tilde{u} = ((\nabla u)^T, u)^T \), we have by a direct manipulation that

\[
\mathcal{L}(\tilde{u}, \psi) = \frac{|(u, \mu \psi_u - \Delta \psi_u) | - (f, \psi_u) |}{\|\mu \psi_u - \Delta \psi_u\|_\Omega},
\]

which induces the following minimax problem

\[
(3.7) \quad \min_{u \in H^1_0(\Omega)} \max_{\psi_u \in H^1_0(\Omega)} \mathcal{L}(u, \psi_u) = \min_{u \in H^1_0(\Omega)} \max_{\psi_u \in H^1_0(\Omega)} \frac{|(u, \mu \psi_u - \Delta \psi_u) | - (f, \psi_u) |}{\|\mu \psi_u - \Delta \psi_u\|_\Omega}.
\]

In fact, we can derive the above minimax problem in a rigorous way. From (3.5), we have

\[
(-\Delta u + \mu u, \psi_u) = (f, \psi_u), \quad \forall \psi_u \in H^1_0(\Omega),
\]

which, from the usual integration by parts twice, gives

\[
(u, \mu \psi_u - \Delta \psi_u) = (f, \psi_u), \quad \forall \psi_u \in H^1_0(\Omega).
\]

This will naturally give the minimax problem (3.7).
3.3. Maxwell’s Equation in the Diffusion Regime. The Maxwell’s equations in $\mathbb{R}^3$ in the diffusive regime could be considered as

\begin{equation}
\mu \mathbf{H} + \nabla \times \mathbf{E} = \mathbf{f}, \quad \sigma \mathbf{E} - \nabla \times \mathbf{H} = \mathbf{g},
\end{equation}

with $\mu$ and $\sigma$ being two positive functions in $L^\infty(\Omega)$ and uniformly bounded away from zero. Three-dimensional functions $\mathbf{f}, \mathbf{g}$ lie in the space $[L^2(\Omega)]^3$ and the solution functions $(\mathbf{H}^\top, \mathbf{E}^\top)^\top$ are in the space $[L^2(\Omega)]^3 \times [L^2(\Omega)]^3$. In Equation (2.11), set $r = 6$ and let $\mathbf{A}_k \in \mathbb{R}^{6 \times 6}$ and $\mathbf{C}$ be $
abla \times \mathbf{H}$ is defined as follows:

\begin{equation}
\mathbf{C} = \begin{bmatrix}
\mu \cdot \mathbf{I}_3 & 0 \\
0 & \sigma \cdot \mathbf{I}_3
\end{bmatrix}, \quad \text{for } k = 1, 2, 3. \text{ Here, the entries of } \mathcal{R}^k_{ij} = \text{sign}(i - j) \text{ if } i = k+1(\text{mod } 3) \text{ and } \mathcal{R}^k_{ij} = 0 \text{ otherwise.}
\end{equation}

The graph space is defined as $W = \mathcal{H}(\text{curl}; \Omega) \times \mathcal{H}(\text{curl}; \Omega)$. One example of the boundary condition is $V = V^* = \mathcal{H}(\text{curl}; \Omega) \times \mathcal{H}_0(\text{curl}; \Omega)$. The function pair $u := (\mathbf{H}^\top, \mathbf{E}^\top)^\top \in W$ is in $V$ whenever $\mathbf{E} \times \mathbf{n}|_{\partial \Omega} = 0$. Let $\psi = (\psi^H, \psi^E)^\top$ be the test function in $V^*$. Then the minimax problem in (2.22) becomes

\begin{equation}
\min_{u \in V} \max_{\psi \in V^*} \frac{|(-\nabla \times \psi^E + \mu \psi^H)|}{\|(-\nabla \times \psi^E + \mu \psi^H)^\top, (\nabla \times \psi^H + \sigma \psi^E)^\top \|_\Omega}.
\end{equation}

4. Deep Learning-Based Solver. To complete the introduction of Friedrichs learning, we introduce a deep learning-based method to solve the minimax optimization in (2.22) or (2.23) for the weak solution of (2.18) or (2.20) in this section. For simplicity, we will focus on the minimax optimization (2.22) to identify the weak solution of (2.18).

4.1. Overview. In the deep learning-based method, one solution DNN, $\phi_s(x; \theta_s)$, is applied to parametrize the weak solution $u$ in (2.22) and another test DNN, $\phi_t(x; \theta_t)$, is used to parametrize the test function $\psi$ in (2.22). Here, $\theta_s$ and $\theta_t$ are the parameters to be identified such that

\begin{equation}
(\theta_s, \theta_t) = \arg \min_{\theta_s} \max_{\theta_t} L(\phi_s(x; \theta_s), \phi_t(x; \theta_t))
\end{equation}

under the constraints

\begin{equation}
\phi_s(x; \theta_s) \in V \quad \text{and} \quad \phi_t(x; \theta_t) \in V^*.
\end{equation}

For simplicity, we use $L(\theta_s, \theta_t)$ for short to represent $L(\phi_s(x; \theta_s), \phi_t(x; \theta_t))$ from now on.

4.2. Network Implementation and Approximation Theory. Now, we will introduce the network structures of the solution DNN and test DNN used in the previous section. In this paper, all DNNs are chosen as ResNet \[35\] defined as follows. Let $\phi(x; \theta)$ denote such a network with an input $x$ and parameter $\theta$, which is defined recursively using a nonlinear activation function $\sigma$ as follows:

\begin{equation}
h_0 = V x, \quad g_\ell = \sigma(W_\ell h_{\ell-1} + b_\ell), \quad h_\ell = \tilde{U}_\ell h_{\ell-2} + U_\ell g_\ell, \quad \ell = 1, 2, \ldots, L, \phi(x; \theta) = a^\top h_L,
\end{equation}

where $V \in \mathbb{R}^{m \times d}, \quad W_\ell \in \mathbb{R}^{m \times m}, \quad \tilde{U}_\ell \in \mathbb{R}^{m \times m}, \quad U_\ell \in \mathbb{R}^{m \times m}, \quad b_\ell \in \mathbb{R}^m \quad \text{for} \quad \ell = 1, \ldots, L, \quad a \in \mathbb{R}^m, \quad h_{-1} = 0$. Throughout this paper, $\tilde{U}_\ell$ is set as an identity matrix in the numerical implementation of ResNets for the purpose of simplicity. Furthermore, as used in \[18\], we set $\tilde{U}_\ell$ as the identity matrix when $\ell$ is even and set $\tilde{U}_\ell = 0$ when $\ell$ is odd, i.e., each ResNet block has two layers of activation functions. $\theta$ consists of all the weights and biases $\{W^\ell, b^\ell\}_{\ell=0}^L$. The number $m$ and $L$ are called the
width and the depth of the network, respectively. The activation function \( \sigma \) is problem-dependent. For example, if the DNN as a test function is required to be continuously differentiable, the Tanh activation function can be chosen to guarantee that our DNN is in \( C^\infty \); if it is desired that \( \phi(x; \theta) \) is in the \( H^1 \) space, the activation function ReLU(\( x \)) could be used, where \( \text{ReLU}(x) := \max\{0, x\} \).

ResNets contain fully connected neural networks (FNNs) as special examples when \( \bar{U}_\ell = 0 \) and \( U_\ell \) is the identity matrix for all \( \ell \). Here, we quote existing approximation theory to briefly justify the application of neural networks as a parametrization tool in this paper. Of particular interest here is the approximation theory for Sobolev spaces \( W^{n,p} \) for numerical PDEs. The following lemma is proved in [31] to describe the approximation power of neural networks quantitatively.

**Lemma 4.1 (Theorem 4.9 of [31]).** Let \( d \in \mathbb{N}, k \in \mathbb{N}_0, n \in \mathbb{N}_{\geq k+1}, \) and \( 1 \leq p \leq \infty \). There exist constants \( L, C, \) and \( \tilde{\epsilon} \) such that, for every \( \epsilon \in (0, \tilde{\epsilon}) \) and every \( f \in \{ f \in W^{n,p}((0,1)^d) : \| f \|_{W^{n,p}((0,1)^d)} \leq 1 \} \), there exist a FNN \( \phi \) with at most \( L \) layers and nonzero weights at most

\[
M = \begin{cases} 
C\epsilon^{-d/(n-k)}, & \text{max}\{0, x\} \text{ activation function}, \\
C\epsilon^{-d/(n-k-1)}, & \text{Tanh activation function}, 
\end{cases}
\]

such that

\[
\| \phi - f \|_{W^{k,p}((0,1)^d)} \leq \epsilon.
\]

The approximation theory in Lemma 4.1 justifies the application of Tanh, ReLU, and the power of ReLU as activation functions in FNNs to approximate target functions in Friedrichs learning. Since ResNets of depth \( L \) and width \( m \) contain FNNs of depth \( L \) and width \( m \) as special cases, Lemma 4.1 can also provide a lower bound of the approximation capacity of ResNets to justify the application of ResNets in our numerical examples. Lemma 4.1 is asymptotic in the sense that it requires sufficiently large network width and depth. For quantitative results in terms of a finite width and depth, the reader is referred to [37].

In theory, the target function space of neural network approximation in Friedrichs learning may be as large as the \( L^p \) space, which is not covered by Lemma 4.1. Recently, the approximation capacity of neural networks for \( L^p \) spaces has been characterized in [61].

**4.3. Unconstrained Minimax Problem.** When the domain becomes relatively complex, the penalty method may be employed to solve the constrained minimax optimization in (4.1). For this purpose, we shall introduce a distance to quantify how good the solution DNN is and test how DNN satisfies its constraints. Such a distance is specified according to the boundary conditions. Denote by \( \text{dist}(\phi(x; \theta), V) \) the distance between a DNN \( \phi(x; \theta) \) and a space \( V \). Therefore, the penalty terms of boundary conditions can be written as

\[
L_b(\theta_s, \theta_t) := \lambda_1 \text{dist}(\phi_s(x; \theta_s), V) + \lambda_2 \text{dist}(\phi_t(x; \theta_t), V^*),
\]

where \( \lambda_1 \) and \( \lambda_2 \) are two positive hyper-parameters. Finally, the constraint minimax problem (4.1) can be formulated into the following unconstrained minimax problem

\[
(\bar{\theta}_s, \bar{\theta}_t) = \arg \min_{\theta_s} \max_{\theta_t} \left( L(\theta_s, \theta_t) + L_b(\theta_s, \theta_t) \right),
\]

which can be solved to obtain the solution DNN \( \phi_s(x; \bar{\theta}_s) \) as the weak solution of the given PDE in (2.18) by Friedrichs Learning.
4.4. Special Networks for Different Boundary Conditions. As discussed in \cite{29, 28}, it is possible to build special networks to satisfy various boundary conditions automatically, which can simplify the unconstrained optimization \(4.5\) into

\[
(\hat{\theta}_s, \hat{\theta}_t) = \arg \min_{\theta_s} \max_{\theta_t} L(\theta_s, \theta_t). \tag{4.6}
\]

This optimization problem \(4.6\) is easier to solve compared to \(4.5\) since two hyperparameters \(\lambda_1\) and \(\lambda_2\) in \(4.4\) are dropped. Note that for a regular PDE domain, e.g., a hypercube or a ball, it is simple to construct such special networks satisfying various boundary conditions automatically.

Let us take the case of a homogeneous Dirichlet boundary condition as an example. For other cases, the readers are referred to \cite{29, 28}. A DNN satisfying the Dirichlet boundary condition \(\psi(x) = g(x)\) on \(\partial \Omega\) can be constructed by \(\phi(x; \theta) = h(x)\phi(x; \theta) + b(x)\), where \(\hat{\phi}\) is a generic network as in \(4.2\), and \(h(x)\) is a specifically chosen function such that \(h(x) = 0\) on \(\partial \Omega\), and \(b(x)\) is chosen such that \(b(x) = g\) on \(\partial \Omega\). For example, if \(\Omega\) is a \(d\)-dimensional unit ball, then \(\phi(x; \theta)\) can take the form \(\phi(x; \theta) = (|x|^2 - 1)\hat{\phi}(x; \theta) + b(x)\). For another example, if \(\Omega\) is the \(d\)-dimensional hyper-cube \([-1, 1]^d\), then \(\phi(x; \theta)\) can take the form \(\phi(x; \theta) = \prod_{i=1}^{d} (x_i^2 - 1)\hat{\phi}(x; \theta) + b(x)\).

4.5. Network Training. Once the solution DNN and test DNN have been set up, the rest is to train them to solve the minimax problem in \(4.5\). The stochastic gradient descent (SGD) method or its variants (e.g., RMSProp \cite{26} and Adam \cite{27}) is an efficient tool to solve this problem numerically. Although the convergence of SGD for the minimax problem is still an active research topic \cite{24, 14, 66}, empirical success shows that SGD can provide a good approximate solution. The training algorithm and main numerical setup are summarized in Algorithm \ref{alg:training}.

In Algorithm \ref{alg:training}, the outer iteration loop takes \(n\) iterations. Each inner iteration loop contains \(n_s\) steps of \(\theta_s\) updates and \(n_t\) steps of \(\theta_t\) updates. In each inner iteration for updating \(\theta_s\), we generate two new sets of random samples \(\{x^{1}_{i}\}_{i=1}^{N_1} \subset \Omega\) and \(\{x^{2}_{i}\}_{i=1}^{N_2} \subset \partial \Omega\) following uniform distributions. In most of the examples, the Latin Hyper-cube Sampling method is employed to generate random points in order to simulate the distributional characteristics even for the relatively small number of samples. We define the empirical loss of these training points for the Friedrichs’ system \(2.11\) as

\[
L_{t}(\theta_s, \theta_t) := \hat{L}(\theta_s, \theta_t) + \hat{L}_b(\theta_s, \theta_t), \tag{4.7}
\]

where \(\hat{L}(\theta_s, \theta_t) := \frac{\hat{L}_n(\theta_s, \theta_t)}{\hat{L}_d(\theta_s, \theta_t)}\) with

\[
\hat{L}_n(\theta_s, \theta_t) = \frac{A(\Omega)}{N_1} \sum_{i=1}^{N_1} \left( \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (-A_j \phi_t(x^1_i; \theta_t), \phi_s(x^1_i; \theta_s)) + \frac{A(\Omega)}{N_1} \sum_{i=1}^{N_1} (C^T \phi_t(x^1_i; \theta_t), \phi_s(x^1_i; \theta_s)) \right)
- \frac{A(\Omega)}{N_1} \sum_{i=1}^{N_1} (f(x^1_i), \phi_t(x^1_i; \theta_t)) + \frac{A(\partial \Omega)}{N_2} \sum_{i=1}^{N_2} \left( \sum_{j=1}^{d} A_j n_j \phi_s(x^2_i; \theta_s), \phi_t(x^2_i; \theta_t) \right),
\]

\[
\hat{L}_d(\theta_s, \theta_t) = \frac{A(\Omega)}{N_1} \sum_{i=1}^{N_1} \left( \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (-A_j \phi_t(x^1_i; \theta_t)) + C^T \phi_t(x^1_i; \theta_t) \right)^2,
\]

where \((\cdot, \cdot)\) denotes the inner product of two vectors, \(\| \cdot \|_2\) denotes the 2-norm of vectors, \(A(\cdot)\) is denoted as the area or volume of the integral region, \(\frac{\partial}{\partial x_j}\) denotes the partial derivative with respect to the \(j\)-th argument of a function in \(x\), and \(\{A_j\}_{j=1}^{d}\) has been introduced in Section \ref{sec:2.2}. As for
the boundary loss, let us take the Dirichlet boundary condition \( u(x) = g_d(x) \) as an example. In this case, the boundary loss can be formulated as

\[
\hat{L}_b(\theta_s, \theta_t) := \frac{A(\partial \Omega)}{N_s} \sum_{i=1}^{N_s} \| \phi_s(x_i^2, \theta_s) - g_d(x_i^2) \|^2.
\]

As mentioned in Section 4.4, if the solution DNN and test DNN are both built to satisfy their boundary conditions automatically, \( \hat{L}_b(\theta_s, \theta_t) \) is zero.

Next, we compute the gradient of \( L_t(\theta_s, \theta_t) \) with respect to \( \theta_s \), denoted by \( g_s \), which is known as the gradient descent direction. The gradient is evaluated via the autograd in PyTorch, which is essentially computed by processing a sequence of chain rules since the loss function is the composition of several simple functions with explicit formulas. For specific classes of PDEs, the computational cost of gradients can be reduced via recent development [10]. Besides, optimizers will use \( \eta \) together with some historical gradient information to output a real descent direction, say \( \tilde{g}_s \). Thus, \( \theta_s \) can update along the direction \( \tilde{g}_s \) as \( \theta_s \leftarrow \theta_s - \eta \tilde{g}_s \). In each outer iteration of Algorithm 1 we repeatedly sample new training points and update \( \theta_s \) for \( n_s \) steps.

In each inner iteration, \( \theta_t \) can be updated similarly to maximize the empirical loss \( L_t(\theta_s, \theta_t) \). In each inner iteration for updating \( \theta_t \), we generate random samples and evaluate the gradient of the empirical loss with respect to \( \theta_t \), denoted by \( g_t \). Similar to the update of \( \theta_s \), \( \theta_t \) can be updated via one step of ascent with a step size \( \eta_t \) as follows: \( \theta_t \leftarrow \theta_t + \eta \tilde{g}_t \). In each outer iteration, we repeatedly sample new training points and update \( \theta_t \) for \( n_t \) steps.

We would like to emphasize that minimax optimization problems are generally more challenging to solve than minimization problems arising in network-based PDE solvers in the strong form. Note that, when we fix the test DNN \( \phi_t(x; \theta_t) \), the loss function in (4.1) is a convex functional with respect to the solution DNN \( \phi_s(x; \theta_s) \), but not with respect to the parameters \( \theta_s \) on it. Hence, the difficulty of the minimization problem when the test DNN is fixed is the same as the network-based least squares method. An appropriate choice of step size is crucial to improve the solution. Moreover, in the extra step of updating test function DNN for a fixed solution DNN, the maximization problem over the test DNN is not convex neither in the parameter space nor in the DNN space, which makes the optimization even difficult.

To further facilitate the convergence of Friedrichs learning, a restarting strategy is employed to obtain the restarted Friedrichs learning in Example 5.1, which is in the same spirit as typical restarted iterative solvers in numerical linear algebra, e.g., the restarted GMRES [10], or the restart strategies in optimization [2, 33, 59, 39]. For simplicity and without loss of generality, the restarted Friedrichs learning is introduced for PDEs with Dirichlet boundary conditions. For other boundary conditions, the restarted Friedrichs learning can be designed similarly. We stress the fact that except for the example in 5.1, the Friedrichs learning algorithm performs well enough without a restarting strategy, so we do not implement the restarting method in the subsequent experiments.

5. Numerical Experiments. In this section, all hyperparameters are listed in Table 5.1. We set the solution DNN \( \phi_s(x, \theta_s) \) as a fully connected ResNet with ReLU activation functions, depth 7, and width \( m_s \), where \( m_s \) is problem dependent. The activation of \( \phi_s(x, \theta_s) \) is chosen as ReLU due to its capacity to approximate functions with low regularity and its good numerical performance. The test DNN \( \phi_t(x, \theta_t) \) has the same structure with depth 7 and width \( m_t \). To ensure the smoothness of \( \phi_t(x, \theta_t) \), we employ the Tanh activation function. The optimizers for updating \( \phi_s(x, \theta_s) \) and \( \phi_t(x, \theta_t) \) are chosen as Adam and RMSProp, respectively. All of our experiments share the same setting for network structures and optimizers. During the pre-training phase, we always set the learning rate to be larger than the following training phase. Thereafter, to ensure an effective and stable training process, the learning rate in the optimization is updated in an
Algorithm 1 Restarted Friedrichs Learning for Weak Solutions of PDEs.

Require: The desired PDE.
Ensure: Parameters $\theta_t$ and $\theta_s$ solving the minimax problem in (4.5).

Set iteration parameters $n$, $n_s$, and $n_t$. Set sample size parameters $N_1$ and $N_2$. Set step sizes $\eta_s^{(k)}$ and $\eta_t^{(k)}$ in the $k$-th iteration. Set the restart index set $\Theta_s$ and $\Theta_t$.

Initialize $\phi_s(x; \theta_s^{0,0})$ and $\phi_t(x; \theta_t^{0,0})$.

for $k = 1, \ldots, n$ do
    if $k \in \Theta_s$ then
        Keep a copy $b(x) = \phi_s(x, \theta_s^{k-1,0})$ and randomly re-initialized $\theta_s^{k-1,0}$.
        if the penalty method for boundary conditions is used then
            Set a new DNN $\phi_s(x, \theta_s^{k-1,0}) = \hat{\phi}_s(x, \theta_s^{k-1,0}) + b(x)$ with a generic DNN $\hat{\phi}_s(x, \theta_s^{k-1,0})$.
        else
            Set a new DNN $\phi_s(x, \theta_s^{k-1,0}) = h(x)\hat{\phi}_s(x, \theta_s^{k-1,0}) + b(x)$ with a generic DNN $\hat{\phi}_s(x, \theta_s^{k-1,0})$ and $h(x)$ in (5.4).
        end if
    end if
    for $j = 1, \ldots, n_s$ do
        Generate uniformly distributed sample points $\{x_i^{1}\}_{i=1}^{N_1} \subset \Omega$ and $\{x_i^{2}\}_{i=1}^{N_2} \subset \partial\Omega$.
        Compute the gradient of the loss function in (4.7) at the point $(\theta_s^{k-1,j-1}, \theta_t^{k-1,0})$ with respect to $\theta_s$ and denote it as $g(\theta_s^{k-1,j-1}, \theta_t^{k-1,0})$.
        Update $\theta_s^{k-1,j} \leftarrow \theta_s^{k-1,j-1} - \eta_s^{(k)} g(\theta_s^{k-1,j-1}, \theta_t^{k-1,0})$ with a step size $\eta_s^{(k)}$.
    end for
    if $k \in \Theta_t$, re-initialize $\theta_t^{k-1,0}$ randomly.
    for $j = 1, \ldots, n_t$ do
        Generate uniformly distributed sample points $\{x_i^{1}\}_{i=1}^{N_1} \subset \Omega$ and $\{x_i^{2}\}_{i=1}^{N_2} \subset \partial\Omega$.
        Compute the gradient of the loss function in (4.7) at $(\theta_s^{k,0}, \theta_t^{k-1,j-1})$ with respect to $\theta_t$ and denote it as $g(\theta_s^{k,0}, \theta_t^{k-1,j-1})$.
        Update $\theta_t^{k-j} \leftarrow \theta_t^{k-1,j-1} + \eta_t^{(k)} g(\theta_s^{k,0}, \theta_t^{k-1,j-1})$ with a step size $\eta_t^{(k)}$.
    end for
    if Stopping criteria is satisfied then
        Return $\theta_s = \theta_s^{k,0}$ and $\theta_t = \theta_t^{k,0}$.
    end if
end if
end for

The desired PDE.

Throughout this section, special networks satisfying boundary conditions automatically are used to avoid tuning the parameters $\lambda_1$ and $\lambda_2$ in (4.4); the inner iteration numbers are set as $n_s = 1$ and $n_t = 1$. The values of other parameters listed in Table 5.1 will be specified later.

To measure the solution accuracy, the following discrete relative $L^2$ error at uniformly dis-
| Notation | Meaning |
|----------|---------|
| $d$      | the dimension of the problem |
| $n_p$    | the number of pre-training iterations |
| $n$      | the number of outer iterations |
| $\eta^p_s$ | the pre-training learning rate for optimizing the solution network |
| $\eta^p_t$ | the pre-training learning rate for optimizing the test network |
| $\eta_s^{(0)}$ | the initial learning rate for optimizing the solution network |
| $\eta_t^{(0)}$ | the initial learning rate for optimizing the test network |
| $\nu_s$ | the decaying rate for $\eta_s$ |
| $\nu_t$ | the decaying rate for $\eta_t$ |
| $m_s$ | the width of each layer in the solution network |
| $m_t$ | the width of each layer in the test network |
| $n_s$ | the number of inner iterations for the solution network |
| $n_t$ | the number of inner iterations for the test network |
| $N$ | the number of training points inside the domain |
| $N_b$ | the number of training points on the domain boundary |
| $\Theta_s$ | the restart index set of the solution network |
| $\Theta_t$ | the restart index set of the test network |

Table 5.1
Parameters in the model and algorithm.

The distributed test points in the domain is applied; i.e.,

$$
e_{L^2}(\theta_s) := \left( \frac{\sum_i \| \phi_s(x_i; \theta_s) - u^*(x_i) \|^2_2}{\sum_i \| u^*(x_i) \|^2_2} \right)^{\frac{1}{2}},$$

where $u^*$ is the exact solution. In the case when the true solution is continuous, the following discrete relative $L^\infty$ error at uniformly distributed test points in the domain is also applied; i.e.,

$$
e_{L^\infty}(\theta_s) := \frac{\max_i(\| \phi_s(x_i; \theta_s) - u^*(x_i) \|_\infty)}{\max_i(\| u^*(x_i) \|_\infty)},$$

where $\| \cdot \|_\infty$ denotes the $L^\infty$-norm of a vector. In most examples, we choose at least 10,000 testing points for error evaluation. When the dimension is high or the value of the target function surges, we may choose 50,000 or even 100,000 testing points.

5.1. Advection-Reaction Equation with Plain Discontinuity. In the first example, we identify the weak solution in $L^2(\Omega)$ of the advection-reaction equation in (3.1) with discontinuous solutions. Following Example 2 in [38], we choose the velocity $\beta = (\frac{1}{10}, \frac{9}{10})^\top$ and $\mu = 1$ in the domain $\Omega = [-1, 1]^2$. We choose the right-hand-side function $f$ and the boundary function $g$ such that the exact solution is

$$u^*(x, y) = \begin{cases} 
\sin(\pi(x+1)^2/4)\sin(\pi(y - \frac{9}{10}x)/2) & \text{for } -1 \leq x \leq 1, \frac{9}{10}x < y \leq 1, \\
e^{-5(x^2 + (y - \frac{2}{10}x)^2)} & \text{for } -1 \leq x \leq 1, -1 \leq y < \frac{9}{10}x.
\end{cases}$$

The exact solution is visualized in Figure 5.1(b). The discontinuity of the initial value function will propagate along the characteristic line $y = \frac{9}{10}x$. Hence, the derivative of the exact solution does
not exist along that line. Classical network-based least square algorithms in the strong form will encounter a large residual error near the characteristic line and hence its accuracy may not be very attractive, which motivates our Friedrichs Learning in the weak form.

As discussed in [38], a priori knowledge of the characteristic line is crucial for conventional finite element methods with adaptive mesh to obtain high accuracy. In [38], the streamline diffusion method (SDFEM) can obtain a solution with $O(10^{-2})$ accuracy using $O(10^4)$ degrees of freedom when the mesh is aligned with the discontinuity, i.e., when the priori knowledge of the characteristic line is used in the mesh generation. The discontinuous Galerkin method (DGFEM) in [38] can obtain $O(10^{-8})$ accuracy under the same setting. When the mesh is not aligned with the discontinuity, e.g., when the characteristic line is not used in mesh generation, DGFEM converges as slow as SDFEM and the accuracy is not better than $O(10^{-2})$ with $O(10^4)$ degrees of freedom according to the discussion in [38].

As a deep learning algorithm, Friedrichs Learning is a mesh-free method and the weak solution can be identified without the priori knowledge of the characteristic line. By the discussion in Section 4.4, a special network $\phi_s(x, \theta_s)$ is constructed as follows to fulfill the boundary condition of the solution:

\begin{equation}
\phi_s(x, \theta_s) = \cos(-\frac{\pi}{4} + \frac{\pi}{4} x) \cos(-\frac{\pi}{4} + \frac{\pi}{4} y) \hat{\phi}_s(x, \theta_s) + b(x, y),
\end{equation}

where $b(x, y)$ is constructed directly from the boundary condition as

\begin{equation}
b(x, y) = \begin{cases} 
0, & \text{for } -1 \leq x \leq 1, -0.4 + x/2 < y \leq 1, \\
-e^{-5[(-1)^2 + (y+9/10)^2]} + e^{-5[(-1)^2 + (y+9/10)^2]} - e^{-5[(-1)^2 + (y+9/10)^2]} & \text{for } -1 \leq x \leq 1, -1 \leq y \leq -0.4 + x/2,
\end{cases}
\end{equation}

satisfying $b(x, y) = u(x, y)$ on the inflow boundary $\partial\Omega^-$. For test function, we fix its structure so that $\phi_t(x, \theta_t) = 0$ on $\partial\Omega^+$ defined in (3.3).

First of all, the restarting strategy as introduced at the end of section 4 for pre-training the base function is employed. The special network structure satisfying the Dirichlet boundary conditions for solution DNN $\phi_s$ is constructed as

\begin{equation}
\phi_s(x; \theta_s) = h(x) \hat{\phi}_s(x; \theta_s) + b(x),
\end{equation}

where $b(x)$ satisfies the boundary condition which also can be regarded as an initial guess; $h(x) = 0$ on the Dirichlet boundary. We observe that if $b(x)$ is closer to the true solution, it is easier to train a generic DNN $\hat{\phi}_s$ to obtain the solution DNN $\phi_s$ that approximates the true solution more accurately. Therefore, after a few rounds of outer iterations in the original Friedrichs learning, we obtain a rough solution DNN, which can be served as a better $b$ function in (5.4) to construct a new solution DNN. After that, we will continue training to obtain a more accurate solution.

Secondly, we choose $b(x, y)$ to be discontinuous along a random line rather than the true discontinuous line of the exact solution. This could be a reasonable reproduction of the real application scenarios. Indeed, our choice of $b(x, y)$ above actually makes the problem more challenging. The true solution is discontinuous along the characteristic line, the blue line in Figure 5.1(a) and $b(x, y)$ is discontinuous along the orange line in Figure 5.1(a). Hence, to make the solution DNN $\phi_s$ in (5.2) approximate the true solution well, one algorithm needs to find and correct these two lines automatically and the DNN $\hat{\phi}_s$ in (5.2) should be approximately discontinuous along these two lines. As shown by Figure 5.1(d), with Friedrichs learning the solution DNN $\phi_s$ has a configuration similar to the true solution in Figure 5.1(b), which means that it has successfully learned these two
The parameters for the Friedrichs learning solver of the experiment in Section 5.1.

| Parameters | \(n\) | \(m_s\) | \(m_t\) | \(N\) | \(N_b\) | \(\Theta_s\) |
|------------|--------|--------|--------|------|------|--------|
| Value      | 50,000 | pre-train 50, after 250 | 150    | 90,000 | 45,000 | \{1,000\} |

| Parameters | \(\eta^{(0)}_s\) | \(\eta^{(0)}_t\) | \(\nu_s\) | \(\nu_t\) | \(\Theta_t\) |
|------------|------------------|------------------|-------------|-------------|-------------|
| Value      | 3e-4             | 3e-3             | 9,000       | 9,000       | 327,700     | \(\emptyset\) |

**Table 5.2**

The parameters for the Friedrichs learning solver of the experiment in Section 5.1.

| Parameters | \(n\) | \(m_s\) | \(N\) | \(\eta^{(0)}_s\) | \(\nu_s\) |
|------------|--------|--------|------|------------------|-------------|
| Value      | 50,000 | 250    | 90,000 | 1e-3             | 10,000      |

**Table 5.3**

The parameters of the comparative experiment in Section 5.1.

This feature can be significant because no prior knowledge of the discontinuity of the exact solution is needed during the training, as long as the boundary condition is satisfied.

Thirdly, we can observe the mechanism of Friedrichs learning from Figure 5.1(e), where the test DNN \(\phi_t\) surges and has a larger magnitude near these two lines to emphasize the error of the solution DNN \(\phi_s\). It can make the update of the configuration of \(\phi_s\) more focused on these two lines than other places, which in turn facilitates the expected convergence of the solution DNN.

The whole training process can be divided into two phases due to restarting. In Phase I of pre-training, we train a ResNet of width 50 for 1,000 outer iterations to get a rough solution with an \(L^2\) relative error 2.76e-1. All other parameters are shown in Table 5.2. As shown in Figure 5.1(c), the rough solution has already captured basically the shape of the solution. In Phase II of training, we set this rough solution as a base function \(b(x)\) and again set up a ResNet of width 150. It is shown that 50,000 outer iterations are enough to make the \(L^2\) error of the solution DNN decrease to 2.27e-2, as shown in Figure 5.1(d) and Figure 5.1(f). Our method is comparable with the SDFEM in [38] considering the same order of degrees of freedom summarized in Table 5.2. However, SDFEM in [38] requires the priori knowledge of the characteristic line while our method does not. Therefore, from the perspective of practical computation, our method would be more convenient in real applications.

To compare Friedrichs Learning and the DNN-based least square (LS) algorithm [15, 50, 60], we conduct comparative experiments with very similar hyper-parameters shown in Table 5.3. After 50,000 iterations we obtain a solution with the relative error in \(L^2\) norm which is 3.29e-2 as shown in (5.1(f)). It is worth pointing out that the iteration shown is the outer iteration, and the computation of Friedrichs learning costs about twice as much as the LS approach for each iteration. Though Friedrichs learning is more accurate, the DNN-based least square algorithm and the Friedrichs learning have errors of the same order in this numerical test.

### 5.2. Advection-Reaction Equation with Curved Discontinuity

Consider a domain \(\Omega = \{(x, y)|x^2 + y^2 \leq 1, y \geq 0\}\). The velocity \(\beta = (\sin \theta, -\cos \theta)^T = (y/\sqrt{x^2 + y^2}, -x/\sqrt{x^2 + y^2})\) with \(\theta\) being the polar angle and \(\mu = 0\). The Dirichlet boundary condition on the inflow boundary is given as \(u(x,0) = 1\) for \(-1 \leq x \leq -\frac{1}{2}\), \(u(x,0) = 0\) for \(-\frac{1}{2} < x < 0\). The true solution is

\[
(5.5) \quad u^*(x, y) = \begin{cases} 
0, & x^2 + y^2 < 1/4 \\
1, & x^2 + y^2 \geq 1/4
\end{cases}
\]

Again, without the prior knowledge of the characteristic line, to create a network satisfying the
Friedrichs Learning for Weak Solutions of PDEs

(a) The characteristic line (blue) of the exact solution and the line (orange) along which \( b(x, y) \) in (5.2) is discontinuous.

(b) Exact solution.

(c) The solution DNN right before restarting.

(d) The point-wise error of approximate solution at epoch 50,000 by Friedrichs learning.

(e) The test DNN value at epoch 50,000.

(f) The relative \( L^2 \) error curve by DNN-based least square and Friedrichs learning.

**Fig. 5.1.** Numerical results of Equation (3.1) when the exact solution is chosen as (5.1).

boundary condition, we choose a solution DNN \( \phi_s \) as

\[
\phi_s(x, \theta_s) = \left( \frac{\pi}{2} - \arctan \left( \frac{-x}{y} \right) \right) \sin \left( \frac{\pi}{2} r \right) \hat{\phi}_s(x, \theta_s) + b(x, y),
\]

where

\[
b(x, y) = \begin{cases} 
0, & x \geq -1/2 \\
1, & x < -1/2 
\end{cases}, \text{ and } r = \sqrt{x^2 + y^2}.
\]

\( \phi_s \) will be applied as the solution network of Friedrichs learning. Similarly,

\[
\phi_t(x, \theta_t) = \left( -\frac{\pi}{2} - \arctan \left( \frac{-x}{y} \right) \right) \hat{\phi}_t(x, \theta_t).
\]

By applying Friedrichs learning with \( \phi_s \) and \( \phi_t \) as the solution and test DNN, respectively, we get an approximate solution with an \( L^2 \) relative error 2.48e-2 with the iteration error visualized in Figure 5.3(b). Figure 5.2(a) shows the point-wise error after 100,000 iterations by Friedrichs learning. Friedrichs learning can capture the discontinuous locations well with sharp characterization. The test function value is relatively large around the discontinuous place, resulting in a greater weight for samples around there, which can help to obtain a more accurate PDE solution. Our experiments are implemented on the graphic card Nvidia Tesla P100 with CUDA; in this example, for 10,000 iterations it will take about 50 minutes and cost twice as much as the Least Square methods.

As a comparison with traditional PDE solvers, note that the same PDE was solved by the adaptive least-squares finite element method (LSFEM) in [55] with the same order of degrees of
The parameters for the Friedrichs learning solver of the experiment in Section 5.2.

| Parameters | $n$  | $m_s$ | $m_t$ | $N$   | $N_b$ |
|------------|------|-------|-------|-------|-------|
| Value      | 100,000 | 150   | 150   | 45,000 | 5,000 |

| Parameters | $\eta_s^{(0)}$ | $\eta_t^{(0)}$ | $\nu_s$ | $\nu_t$ | parameter number |
|------------|-----------------|-----------------|---------|---------|------------------|
| Value      | 3e-4            | 3e-3            | 15,000  | 15,000  | 113,850          |

Table 5.4

The parameters for the Friedrichs learning solver of the experiment in Section 5.2.

freedom ($\approx 1.1 \times 10^5$) as in Friedrichs learning. The $L^2$ relative error of LSFEM is $4.59e-2$, which is larger than the one by Friedrichs learning. We would like to emphasize that LSFEM in [55] has applied extra computational resources to adaptively generate discretization mesh, without which the error would be poorer. Besides, the DGFEM\(^1\) with adaptive mesh is also applied to solve the same PDE with the same order of degrees of freedom (107,332) as in Friedrichs learning. The $L^2$ relative error of DGFEM is $2.05e-2$, which is very similar to the error by Friedrichs learning. Following the idea in [55] to visualize the solution, we project the approximate solutions by DGFEM and Friedrichs learning to the radius axis in Figure 5.3(b) and plot the scatters corresponding to the angle $\theta$ ranging from 0 to $\pi$, the points chosen is the same as DGFEM following the software built-in functions. This visualization makes it easier to compare the solutions near the discontinuous location. It is easy to see that the solution by DGFEM has a larger error than the one by Friedrichs learning near the discontinuous location.

DNN-based least square is also applied to solve the same problem as a comparison. Two options of DNN-based least square are tested: one with $\phi_s$ as the solution network so that there is no penalty terms to enforce the boundary condition in the loss function; another one with a standard neural network as the solution network and, hence, a penalty term in the loss function is added to enforce the boundary condition. The first option, i.e., DNN-based least square with the special network structure described in (5.6) to parametrize the PDE solution, fails to find a reasonable solution even though the optimization loss is almost zero as shown by Figure 5.2(b). One possible reason is due to the fact that the square loss in the strong form is 0 for $b(x,y)$, since DNN-based least square samples points randomly in the “interior” but not on the discontinuous line with probability almost 1. Therefore, even if the generic network $\phi_s(x,\theta_s)$ is not 0 at the beginning, no information of the discontinuity is captured by the strong form in DNN-based least square and, hence, the solution network will converge to 0, resulting in a fake solution satisfying the equation almost everywhere in the strong sense. However, this solution is mathematically wrong in the weak sense. For instance, the derivatives across the discontinuity contain Dirac’s delta functions.

The second option of DNN-based least square can provide a meaningful solution and serves as a good baseline for Friedrichs learning. Figure 5.2(a) shows the point-wise error after 100,000 iterations by DNN-based least square with a boundary penalty term and Friedrichs learning. Friedrichs learning can capture the location of discontinuous line with better accuracy than DNN-based least square. The error curve of DNN-based least square in the $L^2$ norm is shown in 5.2(b) (the red line) and the iteration error cannot be improved anymore at the early beginning. DNN-based least square with a boundary penalty term provides a solution with an $L^2$ error 9.35e-2 after 100,000 iterations and this error is almost 4 times as the error by Friedrichs learning.

5.3. Green’s Function. The next example is to identify the Green’s function of the Laplacian operator by solving

$$\Delta u(x) = \delta_0(x),$$

\(^1\)Available at [https://github.com/dealii/dealii](https://github.com/dealii/dealii)
where $\delta_0(x)$ is the Dirac’s delta function at the origin. In this example, we solve the above equation on a 3D unit ball $\Omega = \{ x \in \mathbb{R}^3 \mid \|x\|_2 \leq 1 \}$. The true solution is

$$u^*(x) = \frac{1}{8\pi \|x\|_2},$$

and the given Dirichlet boundary condition is $u(x) = \frac{1}{8\pi}$ on $\partial \Omega$. Although the exact solution is in $H^1$ and has strong singularity near the origin, Friedrichs learning can provide an approximate solution with a small error as shown in Figure 5.4(a) and 5.4(b). Figure 5.4(b) visualizes the point-wise relative error of the solution by Friedrichs learning. We can see that, except for those
locations that are very close to the origin, the relative errors are not greater than $1e^{-1}$. In Table 5.7, we summarize the relative $L^2$ errors of the solution by Friedrichs learning in the region of $\Omega \setminus B(0, \varepsilon)$ with $\varepsilon$ equal to 0.001, 0.01, 0.1, 0.2, respectively. Therefore, the solution is accurate when the location is not very close to the origin.

As a comparison, the DNN-based least square method cannot find a meaningful solution for the Green’s function. The right hand side function of (5.8) is a Dirac Delta function and, hence, cannot be captured by the discrete analog of the least square loss function via random sampling. Therefore, even if the DNN-based least square method can be applied to form an optimization problem, the minimizer of this problem will return a constant function as a solution, which has a large error.

![Cross section of the Green's function](image1)
![Projected point-wise relative error](image2)
![Relative $L^2$ and maximum error curve](image3)

(a) The cross section of the Green’s function at $x_3 = 0$. The Green’s function has strong singularity near the origin.
(b) The projected point-wise relative error by Friedrichs learning on the slice $x_3 = 0$.
(c) The relative $L^2$ and maximum error curve with respect to the iteration number.

**Fig. 5.4.** *Numerical results of Equation (3.1) when the exact solution is chosen as (5.9).*

### 5.4. High-Dimensional Advection-Reaction Equation.

We consider a 10D advection equation with discontinuity in the domain $[0, 1]^{10}$. In particular, we find $u = u(x)$ such that

\[
(5.10) \quad 2\left(1 + \exp \left(-\left(\sum_{i=3}^{10} x_i \right)^2 \right)\right) u_{x_1} + \exp(2x_1)u_{x_2} = 0,
\]
where \( u_{x_1} = \frac{\partial u}{\partial x_1} \) and \( u_{x_2} = \frac{\partial u}{\partial x_2} \). The exact solution is

\[
(5.11) \quad u^*(x) = g \left( \exp(2x_1) - 4 \left( 1 + \exp \left( - \sum_{i=3}^{10} x_i^2 \right) \right) x_2 \right),
\]

where

\[
g(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}
\]

The Dirichlet boundary condition is given on the inflow boundary \( \{ x | x_1 = 0 \text{ or } x_2 = 0 \} \).

Figure 5.5(a) and Figure 5.5(c) show that Friedrichs learning can identify the location of low regularization by test DNNs in this high-dimensional problem. After 50,000 outer iterations, we obtain an approximate solution with a relative \( L^2 \) error 4.034e-2. As a comparison, the DNN-based least square is also applied to solve the same problem and the relative \( L^2 \) error is 1.015e-1, which is much larger than the one by Friedrichs learning. In Figure 5.5(d), we observe that DNN-based least square is not stable in optimization due to the curved discontinuity, and stops ultimately at a solution with a large error.

5.5. Maxwell Equations. In the last example, we consider Maxwell equations (3.8) defined in the domain \( \Omega = [0, \pi]^3 \). Let \( H \) and \( E \) be the solutions of the Maxwell equations (3.8) with \( \mu = \sigma = 1 \). Let \( f, g \in \left[ L^2(\Omega) \right]^3 \) be \( f = (0, 0, 0)^T \) and \( g = (3 \sin y \sin z, 3 \sin z \sin x, 3 \sin x \sin y)^T \).

The boundary condition is set as \( E \times n = 0 \), which is an ideal conductor boundary condition. The exact solutions to these equations are \( H^* = (\sin x(\cos z - \cos y), \sin y(\cos x - \cos z), \sin x(\cos y - \cos x))^T \) and \( E^* = (\sin y \sin z, \sin z \sin x, \sin x \sin y)^T \). Considering test functions \( (\varphi_H^T, \varphi_E^T)^T \) in the space \( V^* = V \) mentioned in (3.4), we set up DNNs to satisfy the boundary conditions \( \varphi_E \cdot n = 0 \) and \( \varphi_H \times n = 0 \), where \( n \) is the unit outward normal direction to the boundary. Note that the domain is a cube, the normal vector is parallel to one of the unit vectors. The boundary condition above is indeed a Dirichlet boundary. For example, \( S_1 = \{ x = \pi \} \cap \partial \Omega \) on the right surface, implies that \( E_2|_{S_1} = E_3|_{S_1} = 0 \). It is worth pointing out that the Dirichlet boundary for \( (E_i^T, (\varphi_H)_i)^T \) closes the faces of the cube as shown in Figure 5.6(a). Here, we denote by \( E_i(i = 1, 2, 3) \) the \( i \)-th component of the vector \( E \) and the same applies to other notations.

| \( \varepsilon \) | mean |
|---|---|
| 0.2 | 3.47e-2 |
| 0.1 | 4.43e-2 |
| 0.01 | 8.16e-2 |
| 0.001 | 9.39e-2 |

Table 5.7

The relative \( L^2 \) errors by Friedrichs learning in the region of \( \Omega \| B(0, \varepsilon) \) for the Green’s function experiment in Section 5.3

| Parameters | \( n \) | \( m_s \) | \( m_t \) | \( N \) | \( N_b \) | \( \eta_s^p \) | Value |
|---|---|---|---|---|---|---|---|
| Value | 50,000 | 150 | 150 | 45,000 | 5,000 | 3e-4 |

| Parameters | \( \eta_s^p \) | \( \eta_s^{(0)} \) | \( \eta_t^{(0)} \) | \( \nu_s \) | \( \nu_t \) | parameter number |
|---|---|---|---|---|---|---|
| Value | 3e-3 | 5e-5 | 5e-4 | 20,000 | 20,000 | 115,050 |

Table 5.8

The parameters for the Friedrichs learning solver of the experiment in Section 5.4
(a) The projected point-wise error by Friedrichs learning on the slice $x_i = \frac{1}{2}, i = 3, 4, \ldots, 10$.

(b) The projected point-wise error by DNN-based least square on the slice $x_i = \frac{1}{2}, i = 3, 4, \ldots, 10$.

(c) The projected point-wise test function value on the slice $x_i = \frac{1}{2}, i = 3, 4, \ldots, 10$.

(d) The relative $L^2$ error curve with respect to the iteration number by DNN-based least square and Friedrichs learning.

**Fig. 5.5.** Numerical results of Equation 3.1 when the exact solution is chosen as 5.11.

| Parameters | $n$ | $N$ | $\eta_s^{(0)}$ | $\nu_s$ | $m_s$ |
|------------|-----|-----|--------------|--------|-----|
| Value      | 50,000 | 45,000 | 1e-3         | 20,000 | 150 |

**Table 5.9**
The parameters of the comparative experiment in Section 5.4.

To solve the Maxwell equations by Friedrichs learning, we initialize sub-networks of width $m_s$ for vector functions and each sub-network decides one output value of the vector function. The test networks are set up similarly. We list all the parameters used in this experiment in Table 5.10.

After 20,000 outer iterations, we obtain an $L^2$ relative error $1.766e-2$ and an $L^\infty$ relative error $3.467e-2$. Figure 5.6(c) and Figure 5.6(d) illustrate the absolute difference between $E_1$ and $(\phi_E)_1$ and the absolute difference between $H_1$ and $(\phi_H)_1$ after 20,000 outer iterations.

**6. Conclusion.** Friedrichs learning was proposed as a new deep learning methodology to learn the weak solutions of PDEs via Friedrichs seminal minimax formulation. Extensive numerical results imply that our mesh-free method provides reasonably accurate solutions for a wide range of PDEs defined on regular and irregular domains in various dimensions, where classical numerical methods may be difficult to be employed. In particular, Friedrichs learning infers the solution
(a) The boundary conditions of \((E_1, (\phi_\mathcal{H})_1)\).

(b) The relative error versus iterations.

(c) The absolute difference between \(E_1\) and \((\phi_E)_1\) after 20,000 outer iterations.

(d) The absolute difference between \(H_1\) and \((\phi_\mathcal{H})_1\) after 20,000 outer iterations.

**Fig. 5.6. Numerical results of Maxwell equations in (3.8).**

| Parameters | \(n\) | \(m_s\) | \(m_t\) | \(N\) |
|------------|-------|-------|-------|------|
| Value      | 20,000| 250   | 50    | 50,000|
| Parameters | \(\eta_w^{(0)}\) | \(\eta_0^{(0)}\) | \(\nu_s\) | \(\nu_t\) |
| Value      | 3e-6  | 3e-3  | 8,000 | 15,000|

**Table 5.10**  
The parameters for Friedrichs learning solver of the experiment in Section 5.5.

without the knowledge of the location of discontinuity when the solution is discontinuous. Our numerical experiments show that Friedrichs learning can solve PDEs with a discontinuous solution to \(O(10^{-2})\) accuracy, while the DNN-based least square method can typically only get \(O(10^{-1})\) accuracy. This demonstrates the advantage of the loss function in Friedrichs learning over the naive least square loss function. Compared with traditional FEM methods, Friedrichs learning performs as well as DGFEM with adaptive mesh when no prior knowledge about the discontinuous location is known. Friedrichs learning is better than LSFEM with adaptive mesh when no prior knowledge about the discontinuous location is known. In the future, it is interesting to develop adaptive Friedrichs learning to further reduce the error or the network size.
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