ASYMPTOTIC BOUNDARY KZB OPERATORS AND QUANTUM CALOGERO-MOSER SPIN CHAINS

N. RESHETIKHIN & J.V. STOKMAN

Abstract. Asymptotic boundary KZB equations describe the consistency conditions of degenerations of correlation functions for boundary Wess-Zumino-Witten-Novikov conformal field theory on a cylinder. In the first part of the paper we define asymptotic boundary KZB operators for connected real semisimple Lie groups $G$ with finite center. We prove their main properties algebraically using coordinate versions of Harish-Chandra’s radial component map. We show that their commutativity is governed by a system of equations involving coupled versions of classical dynamical Yang-Baxter equations and reflection equations.

We use the coordinate radial components maps to introduce a new class of quantum superintegrable systems, called quantum Calogero-Moser spin chains. A quantum Calogero-Moser spin chain is a mixture of a quantum spin Calogero-Moser system associated to the restricted root system of $G$ and an one-dimensional spin chain with two-sided reflecting boundaries. The asymptotic boundary KZB operators provide explicit expressions for its first order quantum Hamiltonians. We also explicitly describe the Schrödinger operator.

1. Introduction

This paper is part two of a sequel aimed at connecting harmonic analysis on affine symmetric spaces to boundary Wess-Zumino-Witten-Novikov (WZWN) conformal field theory (see [20] for part one).

1.1. Let $G$ be a noncompact real connected semisimple Lie group of real rank $r$ with finite center, $K = G^\Theta$ the compact subgroup of fixed points of a Cartan involution $\Theta$ of $G$, and $\mathfrak{h}_R$ a corresponding maximally noncompact Cartan subalgebra of the Lie algebra $\mathfrak{g}_R$ of $G$. We write $\mathfrak{g}$ and $\mathfrak{h}$ for the complexifications of $\mathfrak{g}_R$ and $\mathfrak{h}_R$. Let $\mathcal{M}_{1,n}$ be the moduli space of elliptic curves with $n$ marked points.

In [10] Felder and Wieczerkowski introduced the twisted WZWN conformal block bundle over $\mathcal{M}_{1,n} \times \mathfrak{h}$ with flat connection. The explicit realisation of the flat connection leads to explicit commuting first-order differential operators, called the Knizhnik-Zamolodchikov-Bernard (KZB) operators [1, 10]. In the limit when the period and marked points go to infinity the KZB flat connection over $\mathcal{M}_{1,n} \times \mathfrak{h}$ becomes a flat connection over $\mathfrak{h}$. We call the corresponding differential operators the asymptotic KZB operators. They were studied in [6].

In a similar way WZWN conformal field theory on a disc (or on a half plane, or on a strip) with $n$ marked points and Cardy type conformal boundary conditions [2, 24] leads to commuting asymptotic boundary KZB operators on $\mathfrak{h}$, see [20]. In this case $G$ is real split,
and $K = G^\Theta$ is the symmetry group at the boundary. One of the main goals of this paper is to introduce the asymptotic boundary KZB operators for all noncompact real connected semisimple Lie groups $G$ with finite center.

The resulting asymptotic boundary KZB operators (see Section 6) are first order commuting differential operators on $\mathfrak{a}$, where $\mathfrak{a}$ is the complexification of the $(-1)$-eigenspace of $\mathfrak{h}_\mathbb{R}$ with respect to the Cartan involution $\theta_\mathfrak{g}$ of $\mathfrak{g}_\mathbb{R}$ associated to $\Theta$. We construct these asymptotic boundary KZB operators and derive their commutativity using generalised Harish-Chandra $[11,3]$ radial component maps with respect to the Cartan decomposition $G = KAK$ of $G$, where $A := \exp(\mathfrak{a}_\mathbb{R})$ (see Section 4 for the description of the radial component maps). Furthermore, we show that the asymptotic boundary KZB operators are particular first order quantum Hamiltonians for a quantum superintegrable system whose algebra of Hamiltonians is isomorphic to a quotient of $\mathbb{Z}(U(\mathfrak{g})) \otimes \mathbb{C}^{n+1}$ (here $\mathbb{Z}(U(\mathfrak{g}))$ denotes the center of the universal enveloping algebra $U(\mathfrak{g})$). These quantum superintegrable systems are parametrized by a choice of $n$ representations of $G$ and two representations of $K$. We call them quantum Calogero-Moser spin chains because they can be regarded both as a system of interacting ”spin” particles and as a version of a spin chain of Gaudin type with reflecting boundary conditions.

One of the surprises is the fact that the commutativity of the asymptotic boundary KZB operators is equivalent to a hierarchy of coupled classical dynamical Yang-Baxter and reflection type equations for the local factors of the operators. These local factors are folded and contracted versions of a natural generalisation (5.10) of Felder’s $[9]$ trigonometric $r$-matrix.

For $G$ real split the coupled integrability equations decouple, reducing them to the mixed classical dynamical Yang-Baxter equations and associated classical dynamical reflection equations from $[20]$. This decoupling is caused by the fact that the centraliser $M = Z_K(A)$ of $A$ in $K$ is finite and discrete for real split $G$. In this case the mixed classical dynamical Yang-Baxter equations and the associated classical dynamical reflection equation are a direct consequence of the well known fact that Felder’s trigonometric $r$-matrix satisfies the classical dynamical Yang-Baxter equation (this will be explained in $[21]$).

1.2. We now describe the results in a bit more detail from the viewpoint of harmonic analysis. Harmonic analysis on symmetric spaces provides a natural representation theoretical context for a class of quantum superintegrable one-dimensional particle systems called quantum spin Calogero-Moser systems $[16,12,13,20,17]$. The main observation is as follows. For two finite dimensional $K$-representations $(\sigma_\ell, V_\ell), (\sigma_r, V_r)$ the space $C^\infty_{\sigma_\ell, \sigma_r}(G)$ of spherical functions consists of the smooth functions $f : G \to \text{Hom}(V_r, V_\ell)$ satisfying the equivariance property

$$f(k_\ell g k_\ell^{-1}) = \sigma_\ell(k_\ell)f(g)\sigma_r(k_r^{-1})$$

(see, e.g., $[11,3,23]$). A spherical function $f \in C^\infty_{\sigma_\ell, \sigma_r}(G)$ is uniquely determined by its radial component $f|_A$. The radial components of the biinvariant differential operators acting on $C^\infty_{\sigma_\ell, \sigma_r}(G)$ become vector-valued differential operators on $A$. They form an algebra of commuting differential operators acting on the space $C^\infty_{\sigma_\ell, \sigma_r}(G)|_A$. This algebra is the
algebra of quantum Hamiltonians of the associated quantum spin Calogero-Moser system. The Schrödinger operator of this system corresponds to the quadratic Casimir element $\Omega \in Z(U(\mathfrak{g}))$, regarded as a biinvariant differential operator on $C^\infty_{\sigma_{e_t,\sigma_{r}}} (G)$. The quantum integrals are the radial components of the action of $U(\mathfrak{g})^{K,\text{opp}} \otimes Z(U(\mathfrak{g})) U(\mathfrak{g})^{K}$ on $C^\infty_{\sigma_{e_t,\sigma_{r}}} (G)$ by left and right $G$-invariant differential operators, where $U(\mathfrak{g})^{K}$ is the algebra of Ad($K$)-invariant elements in $U(\mathfrak{g})$ and $U(\mathfrak{g})^{K,\text{opp}}$ is the associated opposite algebra.

The connection to asymptotic boundary WZWN conformal field theory arises when the left $K$-representation $\sigma_{e_t}$ is of the form $(\sigma_{e_t,n}, V_\ell \otimes U)$ with $U = U_1 \otimes \cdots \otimes U_n$ the tensor product of $n$ finite dimensional $G$-representations $(\tau_i, U_i)$ and $\sigma_{e_t,n}$ the diagonal $K$-action on $V_\ell \otimes U$. The analytic Weyl group $W := N_K(A)/M$ naturally acts on $A$, as well as on the space $(V_\ell \otimes U \otimes V_r^*)^M$ of $M$-invariant vectors in $V_\ell \otimes U \otimes V_r^*$. Restriction to $A$ now defines a linear isomorphism

$$ \left| A : C^\infty_{\sigma_{e_t,n},\sigma_{r}} (G) \xrightarrow{\sim} C^\infty (A; (V_\ell \otimes U \otimes V_r^*)^M)^W \right. $$

The space $C^\infty (A; (V_\ell \otimes U \otimes V_r^*)^M)^W$ is the smooth analogue of the space of asymptotic twisted conformal blocks for boundary WZWN conformal field theory on a cylinder [2, 24, 10, 6, 20], asymptotic in the sense that the insertion points have been sent to infinity within an appropriate asymptotic sector. Typical examples of spherical functions in $C^\infty_{\sigma_{e_t,n},\sigma_{r}} (G)$ are

$$ g \mapsto (\phi_\ell \otimes \text{id}_U) (\Psi_1 \otimes \text{id}_{U_2 \otimes \cdots \otimes U_n}) \cdots (\Psi_{n-1} \otimes \text{id}_{U_n}) \Psi_n \pi_n(g) \phi_r, $$

where $(\pi_i, \mathcal{H}_i)$ $(0 \leq i \leq n)$ are smooth $G$-representations, $\Psi_i \in \text{Hom}_{\mathcal{H}_i}(\mathcal{H}_i, \mathcal{H}_{i-1} \otimes U_i)$ $(1 \leq i \leq n)$ are $G$-intertwiners (asymptotic vertex operators) and $\phi_\ell \in \text{Hom}_K(\mathbb{H}_0, V_\ell)$, $\phi_r \in \text{Hom}_K(V_r, \mathcal{H}_n)$ are $K$-intertwiners (asymptotic boundary states). Functions of the form (1.2) appear naturally as limits of $n$-point functions in twisted conformal blocks for boundary WZWN conformal field theory on a cylinder, cf. [6, 10]. The twisting corresponds to the insertion of $\pi_n(g)$ in (1.2).

1.3. Replacing the $\Psi_i$ in (1.2) by secondary asymptotic field operators $(\pi_{i-1}(g_{i-1}) \otimes \text{id}_{U_i}) \Psi_i$ is providing smooth $V_\ell \otimes U \otimes V_r^*$-valued functions on $G^{\times (n+1)}$, equivariant with respect to the $K \times G^{\times n} \times K$-action on $G^{\times (n+1)}$. The corresponding vector space $C^\infty_{\sigma_{e_t,\mathcal{L},\sigma_{r}}}(G^{\times (n+1)})$ of $K \times G^{\times n} \times K$-equivariant functions is naturally isomorphic to $C^\infty_{\sigma_{e_t,n},\sigma_{r}} (G)$,

$$ C^\infty_{\sigma_{e_t,\mathcal{L},\sigma_{r}}}(G^{\times (n+1)}) \xrightarrow{\sim} C^\infty_{\sigma_{e_t,n},\sigma_{r}} (G), \quad f \mapsto f^\mathcal{L} $$

with $f^\mathcal{L}(g) := f(1, \ldots, 1, g)$. The upshot of this observation is that $C^\infty_{\sigma_{e_t,\mathcal{L},\sigma_{r}}}(G^{\times (n+1)})$ admits a natural $U(\mathfrak{g})^{K,\text{opp}} \otimes Z(U(\mathfrak{g}))^{\otimes (n-1)} \otimes U(\mathfrak{g})^{K}$-action in terms of coordinate-wise invariant differential operators. For the commutative subalgebra $Z(U(\mathfrak{g}))^{\otimes (n+1)}$ the action is given in terms of coordinate-wise biinvariant differential operators. Pushing this action through the two isomorphisms (1.3) and (1.4) provides algebras $A^\sigma_{\mathcal{L},\mathcal{L},\sigma_{r}} \subseteq B^\mathcal{L}_{\mathcal{L},\mathcal{L},\sigma_{r}}$ of $\mathcal{L}$-invariant differential operators on $A$ with coefficients in $\text{End}((V_\ell \otimes U \otimes V_r^*)^M)$, with $A^\sigma_{\mathcal{L},\mathcal{L},\sigma_{r}}$ commutative. They serve, after an appropriate gauge, as the algebras of quantum Hamiltonians and quantum integrals for a quantum superintegrable system, which we call the quantum
Calogero-Moser spin chain (see [18] for a detailed discussion of the underlying classical superintegrable system).

Gauged radial components of the action of the Casimir element $\Omega \in Z(U(g))$ as bi-invariant differential operator on the $j$th coordinate of $C^\infty_{\sigma_\ell,\tau}(G^{x(n+1)}) (0 \leq j \leq n)$ are

providing commuting second-order quantum Hamiltonians $\tilde{D}_{\Omega,j;M}^{\sigma_\ell,\tau}$ (0 ≤ $j$ ≤ $n$). We will derive explicit expressions for $\tilde{D}_{\Omega,j;M}^{\sigma_\ell,\tau}$ by computing $\tilde{D}_{\Omega,n;M}^{\sigma_\ell,\tau}$, which serves as the Schrödinger operator of the quantum Calogero-Moser spin chain, and the differences

$$\tilde{D}_{\Omega,i;M}^{\sigma_\ell,\tau} = \tilde{D}_{\Omega,i-1;M}^{\sigma_\ell,\tau} - \tilde{D}_{\Omega,i;M}^{\sigma_\ell,\tau} \quad (1 \leq i \leq n).$$

The vector-valued potential $V^{\alpha_\ell,\tau}$ (see (5.16)) of the Schrödinger operator $\tilde{D}_{\Omega,n;M}^{\sigma_\ell,\tau}$, which we will compute explicitly using standard techniques [11, 3, 12] from harmonic analysis on symmetric spaces, has the typical pairwise sinh$^{-2}$-interaction terms associated to the roots of the restricted root system of $G$. The first-order commuting differential operators $\tilde{D}_{\Omega,i;M}^{\sigma_\ell,\tau}$ are the asymptotic boundary KZB operators. They are of the form

$$\tilde{D}_{\Omega,i;M}^{\sigma_\ell,\tau} = \sum_{j=1}^{r} \tau_j(x_j) \partial_{x_j} - \kappa_i - \sum_{k=1}^{i-1} r^+_k - \sum_{k=i+1}^{N} r^-_k$$

with \(\{x_i\}_{i=1}^{r}\) an orthonormal basis of $a_\mathbb{R}$. The local factors $\kappa_i$, $r^+_k$, $r^-_k$ are $\text{End}((V_\ell \otimes U \otimes V^*_r)^M)$-valued functions on $a_\mathbb{R}$, acting nontrivially on the tensor components $V_\ell \otimes U_i \otimes V^*_r$, $U_k \otimes U_i$ and $U_i \otimes U_k$ within $(V_\ell \otimes U \otimes V^*_r)^M$, respectively. We will show that the operators $r^\pm_{ik}(a)$ are given by the action on $U_i \otimes U_k$ of $\theta$-(anti)symmetrised versions of the non-split analogue of Felder’s [9] classical dynamical $r$-matrix $r(a)$, and the core component $\kappa_i^{\text{core}}(a)$ of $\kappa_i(a)$ is the action on $U_i$ of a $\theta$-twisted contraction of $r(a)$ (see Proposition 6.10 and (6.8)).

As mentioned earlier, the operators $\tilde{\kappa}_i$, $r^\pm_{ki}$ and $r^-_{ik}$ are solutions of a hierarchy of integrability equations (see Theorem 6.12). It contains a classical dynamical reflection type equation for $\tilde{\kappa}$ relative to $(r^+, r^-)$ (see (6.15)) and three coupled classical dynamical Yang-Baxter-reflection type equations for the triple $(r^+, r^-, \tilde{\kappa})$ (see (6.16)).

In [20] formal $n$-point spherical functions were introduced when $G$ is real split. They provide common eigenfunctions for all the quantum Hamiltonians after a suitable gauge. In [20] this approach was used to arrive at the explicit form of the asymptotic boundary KZB operators, and to prove their commutativity. The theory of formal $n$-point spherical functions for non-split $G$ is yet to be developed, but we do shortly discuss global $n$-point spherical functions.

The results and techniques in this paper are closely related to the Etingof-Schiffmann-Varchenko theory on generalised trace functions, see, e.g., [6, 8, 7, 5] (our current trigonometric, non-affine level of the theory was discussed for generalised trace functions in [6]). From the harmonic analysis point of view the Etingof-Schiffmann-Varchenko theory is related to the symmetric pair $(G \times G, \text{diag}(G))$, with $\text{diag}(G)$ the diagonal embedding of $G$ into $G \times G$. The extension of the results in the current paper to the level of (quantum)
affine symmetric pairs is currently under investigation.

The content of the paper is as follows.

In Section 2 we introduce the space of global $n$-point spherical functions and study the coordinate-wise action of (bi)invariant differential operators on $C^\infty_{\sigma_\ell, \sigma_r}(G^{\times(n+1)}) \cong C^\infty_{\sigma_\ell, \sigma_r}(G)$. This part of the paper, which does not yet require radial component maps, will be developed for the more general context involving an arbitrary real connected Lie group $G$ and two closed Lie subgroups $K_\ell$ and $K_r$.

Section 3 is a short section in which we recall some basic facts about the structure theory of real semisimple Lie groups. This section is mainly meant to fix notations.

In Section 4 we introduce the coordinate radial component maps and prove their main algebraic properties.

In Section 5 we use the coordinate radial component maps to define the algebras of quantum Hamiltonians and quantum integrals for the quantum Calogero-Moser spin chain. We furthermore derive the explicit expression for the Schrödinger operator of the quantum Calogero-Moser spin chain.

In Section 6 we derive the explicit expressions for the asymptotic boundary KZB operators and we show that the local factors of the asymptotic boundary KZB operators satisfy coupled classical dynamical Yang-Baxter-reflection equations.

Finally, in Section 7 we explicitly compute the main structural ingredients in case $G = SU(p,r)$ with $1 \leq r \leq p$ (in this case the underlying restricted root system is of type BC$_r$).

**Conventions:** We write $\otimes_A$ for the tensor product over a complex associative algebra $A$. For $A = \mathbb{C}$ we simply denote it by $\otimes$. A similar convention is used for hom-spaces; $\text{Hom}_A(E, F)$ denotes the space of $A$-linear maps, and $\text{Hom}(E, F)$ the space of complex linear maps. Representations of a Lie group $G$ are complex, strongly continuous Hilbert space $G$-representations.

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2. $n$-Point Spherical Functions

Let $G$ be a real connected Lie group, and $K_\ell, K_r \subseteq G$ two closed Lie subgroups. Write for $n \in \mathbb{Z}_{\geq 1}$,

$$G_n := K_\ell \times G^{\times n} \times K_r.$$  

Elements in $G^{\times n}$ will be denoted by $h = (h_1, \ldots, h_n)$, and elements in $G^{\times(n+1)}$ by $g = (g_0, \ldots, g_n)$. Consider the left $G_n$-action on $G^{\times(n+1)}$ defined by

$$(k_\ell, h, k_r) \cdot g := (k_\ell g_0 h_1^{-1}, h_1 g_1 h_2^{-1}, \ldots, h_n g_n k_r^{-1}).$$
We will write omitted components in vectors with a cap. For instance, $(f)$ that $K$ stands for the dimensional smooth representations $(\sigma, V)$ of $K$ and $K_r$ respectively, and write $(\sigma_r^\ast, V_r^\ast)$ for the $K_r$-representation dual to $(\sigma, V)$.  

**Definition 2.1.** Let $C^\infty_{\sigma, \sigma^\ast}(G^{\times(n+1)})$ be the space of smooth functions $f : G^{\times(n+1)} \to V_\ell \otimes \bar{U} \otimes V_r^\ast$ satisfying  

$$f((k_\ell, k_r) \cdot g) = (\sigma(\ell) \otimes \tau_r^\ast(k_r)) f(g) \quad \forall (k_\ell, k_r) \in G, \quad \forall g \in G^{\times(n+1)}.$$  

**Remark 2.2.** For $n = 0$ the space $C^\infty_{\sigma, \sigma^\ast}(G)$ consists of the smooth functions $f : G \to V_\ell \otimes V_r^\ast$ satisfying  

$$f(k_\ell g k_r^{-1}) = (\sigma(\ell) \otimes \sigma_r^\ast(k_r)) f(g) \quad \forall k_\ell \in K_\ell, \quad \forall k_r \in K_r, \quad \forall g \in G.$$  

When $K_\ell = K_r$, the space $C^\infty_{\sigma, \sigma^\ast}(G)$ is called the space of $\sigma \otimes \sigma^\ast$-spherical functions on $G$ with respect to the Lie subgroup $K_\ell$ of $G$.  

**Remark 2.3.** We will use standard tensor-leg notations. For example, for $v_\ell \in V_\ell$, $u \in \bar{U}$, $\phi_r \in V_r^\ast$ and $g_1, g_2 \in G$ the tensor  

$$(id_{V_\ell} \otimes \tau_1(g_1) \otimes \tau_2(g_2) \otimes id_{U_\ell} \otimes \cdots \otimes id_{U_n} \otimes id_{V_r^\ast})(v_\ell \otimes u \otimes \phi_r)$$  

in $V_\ell \otimes \bar{U} \otimes V_r^\ast$ will be denoted by $(\tau_1(g_1) \otimes \tau_2(g_2))(v_\ell \otimes u \otimes \phi_r)$, or $\tau_1(g_1) \tau_2(g_2)(v_\ell \otimes u \otimes \phi_r)$. We will write omitted components in vectors with a cap. For instance, $(g_0, . . . , g_{j-1}, g_j, . . . , g_n)$ stands for the $n$-vector $(g_0, . . . , g_{j-1}, g_j, . . . , g_n)$ in $G^{\times n}$.  

We write $(\sigma_{\ell,n}, V_\ell \otimes \bar{U})$ for the finite dimensional $K_\ell$-representation defined by  

$$\sigma_{\ell,n}(k_\ell) := \sigma(\ell)(k_\ell) \otimes \tau_1(k_\ell) \otimes \cdots \otimes \tau_n(k_\ell) \quad \forall k_\ell \in K_\ell.$$  

For a smooth function $f : G^{\times(n+1)} \to V_\ell \otimes \bar{U} \otimes V_r^\ast$ we define $f^\flat : G \to V_\ell \otimes \bar{U} \otimes V_r^\ast$ by  

$$f^\flat(g) := f(1, . . . , 1, g) \quad (g \in G).$$  

**Lemma 2.4.** The linear map $f \mapsto f^\flat$ restricts to a linear isomorphism  

$$C^\infty_{\sigma, \sigma^\ast}(G^{\times(n+1)}) \xrightarrow{\sim} C^\infty_{\sigma_{\ell,n}, \sigma_r^\ast}(G)$$  

The preimage of $f^\flat \in C^\infty_{\sigma_{\ell,n}, \sigma_r^\ast}(G)$ is the function $f \in C^\infty_{\sigma, \sigma^\ast}(G^{\times(n+1)})$ defined by  

$$f(g) := (\tau_1(g_0^{-1}) \otimes \tau_2(g_1^{-1} g_0^{-1}) \otimes \cdots \otimes \tau_n(g_{n-1}^{-1} \cdots g_0^{-1})) f^\flat(g_0 g_1 \cdots g_n)$$  

for $g \in G^{\times(n+1)}$.  

**Proof.** Note that $f^\flat \in C^\infty_{\sigma_{\ell,n}, \sigma_r^\ast}(G)$ for $f \in C^\infty_{\sigma, \sigma^\ast}(G^{\times(n+1)})$ since  

$$(k_\ell, k_\ell, . . . , k_\ell) \cdot (1, . . . , 1, g) = (1, . . . , 1, k_\ell g k_\ell^{-1})$$  

in $G^{\times(n+1)}$ for $k_\ell \in K_\ell$, $k_r \in K_r$ and $g \in G$. Conversely, for $f^\flat \in C^\infty_{\sigma_{\ell,n}, \sigma_r^\ast}(G)$, it follows from the formula  

$$(1, g_0, g_0 g_1, . . . , g_0 g_1 \cdots g_{n-1}, 1) \cdot g = (1, . . . , 1, g_0 g_1 \cdots g_n) \quad (g \in G^{\times(n+1)})$$  

that $f$, defined by $(2.3)$, lies in $C^\infty_{\sigma_{\ell,n}, \sigma_r^\ast}(G^{\times(n+1)})$. The lemma now follows easily. \hfill $\square$
For a complex Lie algebra $L$ we write $U(L)$ for its universal enveloping algebra. It is a Hopf algebra, with counit $\epsilon : U(L) \rightarrow \mathbb{C}$ the unital algebra homomorphism satisfying $\epsilon(y) = 0 \ (y \in L)$, comultiplication $\Delta : U(L) \rightarrow U(L) \otimes U(L)$ the unital algebra homomorphism satisfying $\Delta(y) = y \otimes 1 + 1 \otimes y \ (y \in L)$, and with antipode $S : U(L) \rightarrow U(L)$ the unital anti-algebra isomorphism satisfying $S(y) = -y \ (y \in L)$. For $k \in \mathbb{Z}_{>0}$ let $\Delta^{(k)} : U(L) \rightarrow U(L)^{\otimes (k+1)}$ be the $k$th iterated comultiplication, defined inductively by $\Delta^{(1)} = \Delta$ and $\Delta^{(k)} = (\Delta \otimes \text{id}_{U(L)})\Delta^{(k-1)}$ for $k > 1$. We will use Sweedler’s notation for the expression of $\Delta^{(k)}(u) \ (u \in U(L))$ as sum of pure tensors,

$$\Delta^{(k)}(u) = \sum_{(u)} u_{(1)} \otimes \cdots \otimes u_{(k+1)}.$$ 

For a finite dimensional $U(L)$-module $(\sigma, V)$ we write $(\sigma^*, V^*)$ for the dual $U(L)$-module, defined by $(\sigma^*(u)f)(v) = -f(\sigma(u)v)$ for $u \in U(L)$, $v \in V$ and $f \in V^*$. We write $V^L$ for the space of $L$-invariant elements in $V$.

Write $g_{\mathbb{R}}$, $\mathfrak{t}_{\mathbb{R}}$, and $\mathfrak{t}_{\mathbb{R}}$ for the Lie algebras of $G$, $K_\ell$ and $K_\tau$, respectively. Their complexifications are denoted by $g$, $\mathfrak{t}_\ell$ and $\mathfrak{t}_\tau$. Differentiating the representations $(\sigma_\ell, V_\ell)$, $(\sigma_\tau, V_\tau)$ and $(\tau_i, U_i)$ turns $V_i$ into a left $U(\mathfrak{t}_\ell)$-module, $V_i$ into a left $U(\mathfrak{t}_\tau)$-module and the $U_i$ into left $U(g)$-modules. The corresponding representation maps will again be denoted by $\sigma_\ell, \sigma_\tau$ and $\tau_i$.

Let $V$ be a finite dimensional complex vector space. We denote by $C^\infty(G^{\times (n+1)}; V)$ the space of smooth $V$-valued functions on $G^{\times (n+1)}$. We write $C^\infty(G^{\times (n+1)})$ when $V = \mathbb{C}$.

We have a left $U(g)^{\otimes (n+1)}$-action on $C^\infty(G^{\times (n+1)}; V)$, defined by

$$u_0 \otimes \cdots \otimes u_n \mapsto u_0[0]u_1[1] \cdots u_n[n] \quad (u_j \in U(g))$$

with $u \mapsto u[j] \ (u \in U(g))$ the action of $U(g)$ on the $j^{th}$ coordinate of $f \in C^\infty(G^{\times (n+1)}; V)$ by left $G$-invariant differential operators. Concretely, the action $u \mapsto u[j]$ of $U(g)$ on $C^\infty(G^{\times (n+1)}; V)$ is determined by

$$(y[j]f)(g) := \frac{d}{dt}\bigg|_{t=0} f(g_0, \ldots, g_{j-1}, g_j \exp(ty), g_{j+1}, \ldots, g_n) \quad (y \in g_{\mathbb{R}}),$$

where $\exp : g_{\mathbb{R}} \rightarrow G$ is the exponential map.

Remark 2.5. We write $u \mapsto u[j]$ for the coordinate-wise action of $U(g)$ on $C^\infty(G^{\times (n+1)}; V)$ by right $G$-invariant differential operators, so that

$$(y[j]f)(g) := \frac{d}{dt}\bigg|_{t=0} f(g_0, \ldots, g_{j-1}, \exp(-ty)g_j, g_{j+1}, \ldots, g_n) \quad (y \in g_{\mathbb{R}}).$$

Note that

$$(u[j]f)(g) = (\text{Ad}_{g^{-1}}(S(u))[j]f)(g)$$

for $u \in U(g)$. In particular, $u[j]f = S(u)[j]f$ for $u \in Z(U(g))$, with $Z(U(g))$ the center of $U(g)$.
Let \( C_g \) be the conjugation action of \( g \in G \) on \( G \), defined by \( C_g(h) := ghg^{-1} \) for \( g, h \in G \). Differentiating \( C_g \) defines the adjoint action \( \text{Ad}_g \in \text{Aut}(g_{\mathbb{R}}) \) of \( g \in G \) on \( g_{\mathbb{R}} \). It extends to an action of \( G \) on \( U(g) \) by unital algebra automorphisms, which will also be denoted by \( g \mapsto \text{Ad}_g \).

**Proposition 2.6.** For \( f \in C_{\sigma_1, \ldots, \sigma_r}^\infty(G^{\times(n+1)}) \) and \( 0 \leq j < n \) we have

\[
(u[j]f)(g) = \sum_{(u)} A_{j,(u)}(g)(\text{Ad}_{g_{n-1}\cdots g_{j+1}}(u_{(n-j+1)})[n]f)(g_0, \ldots, \hat{g}_j, \ldots, g_n, C_{g_{n-1}\cdots g_{j+1}}^{-1}(g_j))
\]

for \( g \in G^{\times(n+1)} \) and \( u \in U(g) \), with \( A_{j,(u)}(g) \in \text{End}(V_{\ell} \otimes U \otimes V_{r}^*) \) given by

\[
A_{j,(u)}(g) := \tau_{j+1}(S(u(1)))\tau_{j+2}(S(u(2)))\cdots \tau_n(S(u_{(n-j)}))
\]

\[
\cdot \tau_n(g_{n-1}\cdots g_{j+1}) \tau_n(g_{j+1}^{-1}g_{j+2}^{-1}) \cdots \tau_n(g_{n-1}\cdots g_{j}^{-1}) \tau_n(S(u_{(n-j)}))\tau_n(g_{j}^{-1}g_{j+1} \cdots g_n).
\]

**Proof.** For the duration of the proof we use the shorthand notation

\[
F^{(j)}(g) := F(g_0, \ldots, \hat{g}_j, \ldots, g_n, C_{g_{n-1}\cdots g_{j+1}}^{-1}(g_j))
\]

for \( F \in C^\infty(U^{\times(n+1)}; V_{\ell} \otimes U \otimes V_{r}^*) \) and \( 0 \leq j < n \). One then easily checks that for \( y \in g_{\mathbb{R}} \),

\[
(2.4) \quad (y[j]F^{(j)})(g) = (\text{Ad}_{g_{n-1}\cdots g_{j+1}}(y)[n]F)^{(j)}(g).
\]

Consider now the standard filtration \( U(g) = \bigcup_{k \in \mathbb{Z}_{\geq 0}} U_k(g) \) of \( U(g) \). We prove the proposition for \( u \in U_k(g) \) by induction to \( k \). For \( k = 0 \) the result follows from the formula

\[
f(g) = (\tau_j(g_j^{-1}g_{j+1}) \cdots \tau_n(g_{n-1}^{-1}g_{j+1}^{-1}g_{j+2}^{-1}) \cdots \tau_n(g_{n-1}^{-1}g_{j}^{-1}g_{j+1} \cdots g_n)) f^{(j)}(g),
\]

which holds true since \( f \in C_{\sigma_1, \ldots, \sigma_r}^\infty(G^{\times(n+1)}) \). For the induction step, suppose the proposition is correct for \( u \in U_k(g) \), so

\[
(u[j]f)(g) = \sum_{(u)} A_{j,(u)}(g)(\text{Ad}_{g_{n-1}\cdots g_{j+1}}(u_{(n-j+1)})[n]f)^{(j)}(g).
\]

Then for \( y \in g_{\mathbb{R}} \) we have by (2.4),

\[
((yu)[j]f)(g) = \sum_{(u)} (y[j]A_{j,(u)})(g)(\text{Ad}_{g_{n-1}\cdots g_{j+1}}(u_{(n-j+1)})[n]f)^{(j)}(g)
\]

\[
+ \sum_{(u)} A_{j,(u)}(g)(\text{Ad}_{g_{n-1}\cdots g_{j+1}}(yu_{(n-j+1)})[n]f)^{(j)}(g).
\]

By the explicit expression for \( A_{j,(u)}(g) \) and the product rule we have

\[
(y[j]A_{j,(u)})(g) = \sum_{s=j+1}^n \left( \cdots \tau_s(g_{n-1}^{-1} \cdots g_{j+1}^{-1}) \tau_s(S(uy_{(s-j)})) \tau_s(g_{j}^{-1}g_{j+1} \cdots g_s) \otimes \cdots \right)
\]

where we specified only the tensor component that differs from the corresponding tensor component of \( A_{j,(u)}(g) \). Substituting into (2.5) proves the proposition for \( yu \in U_{k+1}(g) \). This completes the proof. \( \square \)
Lemma 2.9. The result now follows from Remark 2.8 and the observation that
\[
\tau_n(g)f(g_0, \ldots, g_{n-2}, g, g_n) = f(g_0, \ldots, g_{n-2}, 1, g g_n)
\]
since \( f \in C^\infty_{\sigma_t, \Sigma_\sigma_r}(G^{x(n+1)}) \).

Remark 2.8. A similar analysis can be done for the coordinate-wise action of \( U(\mathfrak{g}) \) by right \( G \)-invariant differential operators. It leads to the following analogue of Corollary 2.7. Suppose that \( f \in C^\infty_{\sigma_t, \Sigma_\sigma_r}(G^{x(n+1)}) \) and \( 0 \leq j < n \), then
\[
(u(j)f)(g_0, \ldots, g_{n-2}, g, g_n) = u_0[0]u_1[1] \cdots u_{n-1}[n-1]v_r[n]f
\]
for \( g \in G \) and \( u \in U(\mathfrak{g}) \).

For a closed Lie subgroup \( K \subseteq G \) write \( U(\mathfrak{g})^K \) for the subalgebra of \( \text{Ad}(K) \)-invariant elements in \( U(\mathfrak{g}) \), and \( U(\mathfrak{g})^{K, \text{opp}} \) for the opposite algebra. Note that \( S : U(\mathfrak{g})^K \to U(\mathfrak{g})^{K, \text{opp}} \) is an isomorphism of algebras.

Lemma 2.9. The space \( C^\infty_{\sigma_t, \Sigma_\sigma_r}(G^{x(n+1)}) \) is a \( U(\mathfrak{g})^{K, \text{opp}} \otimes Z(U(\mathfrak{g}))^{\otimes(n-1)} \otimes U(\mathfrak{g})^{K r} \)-module, with the action defined by
\[
(u \otimes (u_1 \otimes \cdots \otimes u_{n-1}) \otimes v_r) \ast f := S(v_r)(0)u_1[1] \cdots u_{n-1}[n-1]v_r[n]f
\]
for \( v_r \in U(\mathfrak{g})^{K r}, u_i \in Z(U(\mathfrak{g})) \) \((1 \leq i < n)\), \( v_r \in U(\mathfrak{g})^{K r} \) and \( f \in C^\infty_{\sigma_t, \Sigma_\sigma_r}(G^{x(n+1)}) \).

Furthermore, for \( u_0 \otimes \cdots \otimes u_n \in Z(U(\mathfrak{g}))^{\otimes(n+1)} \) we have the action
\[
(u_0 \otimes \cdots \otimes u_n) \ast f = u_0[0]u_1[1] \cdots u_n[n]f \quad (f \in C^\infty_{\sigma_t, \Sigma_\sigma_r}(G^{x(n+1)}))
\]
by coordinate-wise biinvariant differential operators.

Proof. The straightforward proof is left to the reader. \( \Box \)

Remark 2.10. For \( n = 0 \) the appropriate analogue of Lemma 2.9 is the statement that \( U(\mathfrak{g})^{K, \text{opp}} \otimes Z(U(\mathfrak{g})) U(\mathfrak{g})^{K r} \) acts on \( C^\infty_{\sigma_t}(G) \) by
\[
(v \otimes Z(U(\mathfrak{g})) v_r) \ast f := S(v_r)(1)v_r[1]f \quad (v \otimes Z(U(\mathfrak{g})) v_r, v_r \in U(\mathfrak{g})^{K r})
\]
for \( f \in C^\infty_{\sigma_t}(G) \), where \( u(1) f \) and \( u[1] f \) now denote the action of \( u \in U(\mathfrak{g}) \) on \( f \in C^\infty(G) \) as right and left \( G \)-invariant differential operator, respectively. This is a well defined action since \( S(u)(1) f = u[1] f \) for \( u \in Z(U(\mathfrak{g})) \), cf. Remark 2.5 (it is for this reason that we have taken the opposite algebra \( U(\mathfrak{g})^{K, \text{opp}} \) in Lemma 2.9).
We can now extend the definition of $n$-point spherical functions from [20] to the current context.

**Definition 2.11.** We call $f \in C_{\sigma_r,\Sigma_r}^\infty(G^{\times(n+1)})$ a $n$-point spherical function if there exist algebra homomorphisms $\chi_j : Z(U(\mathfrak{g})) \to \mathbb{C}$ ($0 \leq j \leq n$) such that

$$u[j]f = \chi_j(u)f \quad \forall u \in Z(U(\mathfrak{g})), \forall j \in \{0, \ldots, n\}.$$  

We write

$$C_{\sigma_r,\Sigma_r}^\infty(G^{\times(n+1)}; \chi) \subseteq C_{\sigma_r,\Sigma_r}^\infty(G^{\times(n+1)})$$

for the $U(\mathfrak{g})^{K_\ell} \otimes Z(U(\mathfrak{g}))^{\otimes(n-1)} \otimes U(\mathfrak{g})^{K_r}$-module of $n$-point spherical functions satisfying (2.7) with respect to the $(n+1)$-tuple $\chi := (\chi_0, \ldots, \chi_n)$ of characters of $Z(U(\mathfrak{g}))$.

Examples of $n$-point spherical functions $f \in C_{\sigma_r,\Sigma_r}^\infty(G^{\times(n+1)}; \chi)$ can be constructed using $G$-intertwiners $\Psi_i : \mathcal{H}_i \to \mathcal{H}_{i-1} \otimes U_i$ ($1 \leq i \leq n$), with $(\pi_j, \mathcal{H}_j)$ ($0 \leq j \leq n$) quasisimple smooth $G$-representations with central character $\chi_j$, together with a $K_\ell$-intertwiner $\phi_\ell : \mathcal{H}_0 \to V_\ell$ and a $K_r$-intertwiner $\phi_r : V_\ell \to H_\ell$, via the formula

$$f(\mathbf{g}) := (\phi_\ell \pi_0(g_0) \otimes \text{id}_U)(\Psi_1 \pi_1(g_1) \otimes \text{id}_{U_2 \otimes \cdots \otimes U_n}) \cdots (\Psi_{n-1} \pi_{n-1}(g_{n-1}) \otimes \text{id}_{U_n}) \Psi_\ell \pi_\ell(g_n) \phi_r.$$

See [20] for important explicit classes of such examples.

For special triples $(G, K_\ell, K_r)$, the action of the commuting differential operators $x[j]$ ($0 \leq j \leq N, x \in Z(U(\mathfrak{g}))$) on $C_{\sigma_r,\Sigma_r}^\infty(G^{\times(n+1)})$ gives rise to commuting vector-valued differential operators on a suitable abelian Lie subgroup $A \subseteq G$ by pulling the invariant differential operators through the restriction map $f \mapsto f|^A$ ($f \in C_{\sigma_r,\Sigma_r}^\infty(G^{\times(n+1)})$). Typically, one needs a natural parametrisation of the double $(K_\ell, K_r)$-cosets in $G$ in terms of orbits in $A$ with respect to a discrete group action on $A$. In the remainder of this paper we give a detailed analysis when $K := K_\ell = K_r$ is a maximal compact Lie subgroup of a real connected semisimple Lie group $G$ with finite center.

### 3. Structure theory of real semisimple Lie groups

We introduce standard facts and notations regarding the structure theory of real semisimple Lie groups. For more information see, e.g., [15] Chpt. VI], [23] Chpt. 9] and [3]. As a concrete example we will work out the structure theory in detail for $G = SU(r, p)$ in Section 7.

We fix from now on a real connected semisimple Lie group $G$ with finite center and Lie algebra $\mathfrak{g}_\mathbb{R}$. Fix a Cartan involution $\theta_\mathbb{R} \in \text{Aut}(\mathfrak{g}_\mathbb{R})$ and write

$$\mathfrak{g}_\mathbb{R} = \mathfrak{t}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$$

for the resulting Cartan decomposition, with $\mathfrak{t}_\mathbb{R} = \mathfrak{g}_\mathbb{R}^{\theta_\mathbb{R}}$ the fix-point Lie subalgebra with respect to $\theta_\mathbb{R}$ and $\mathfrak{p}_\mathbb{R}$ the $(-1)$-eigenspace of $\theta_\mathbb{R}$. Write $\mathfrak{m}_\mathbb{R} := Z\mathfrak{k}_\mathbb{R}(\mathfrak{a}_\mathbb{R})$ for the centraliser of $\mathfrak{a}_\mathbb{R}$ in $\mathfrak{t}_\mathbb{R}$. We fix the scalar product

$$(x, y) := -K_{\mathfrak{g}_\mathbb{R}}(x, \theta_\mathbb{R}(y)), \quad x, y \in \mathfrak{g}_\mathbb{R}$$

on $\mathfrak{g}_\mathbb{R}$, with $K_{\mathfrak{g}_\mathbb{R}}$ the Killing form of $\mathfrak{g}_\mathbb{R}$. 
Choose a maximal abelian subspace $a_R$ of $p_R$, which we view as Euclidean space with scalar product inherited from $(\cdot, \cdot)$. We endow the linear dual $a_R^*$ with the scalar product such that the linear isomorphism $a_R \xrightarrow{\sim} a_R^*$, $h \mapsto (h, \cdot)$, becomes an isomorphism of Euclidean spaces. We denote the scalar product on $a_R^*$ again by $(\cdot, \cdot)$.

For $\lambda \in a_R^*$ set

$$g^\lambda_R := \{ x \in g_R \mid [h, x] = \lambda(h)x \quad \forall h \in a_R \},$$

$$\Sigma := \{ \lambda \in a_R^* \setminus \{0\} \mid g^\lambda_R \neq \{0\} \}.$$ 

Then $\Sigma \subset a_R^*$ is a (possibly non-reduced) root system, called the restricted root system. Furthermore,

$$(3.1) \quad g_R = g^0_R \oplus \bigoplus_{\lambda \in \Sigma} g^\lambda_R$$

and $g^0_R = Z_{g_R}(a_R) = m_R \oplus a_R$. The decomposition (3.1) is orthogonal with respect to $(\cdot, \cdot)$, and $\theta(g^\lambda_R) = g^{-\lambda}_R$.

Fix a maximal abelian subspace $t_R$ of $m_R$. Then

$$\mathfrak{h}_R := t_R \oplus a_R$$

is a maximally noncompact $\theta_{\mathfrak{h}_R}$-stable Cartan subalgebra of $g_R$. Denote $g, g^\lambda, \mathfrak{k}, m, h, a, t, ...$ for the complexifications of the Lie algebras $g_R, g^\lambda_R, t_R, m_R, h_R, a_R, t_R, ...$. The root space decomposition of $g$ with respect to $h$ is denoted by

$$g = h \oplus \bigoplus_{\alpha \in R} g_\alpha$$

with

$$g_\beta := \{ x \in g \mid [h, x] = \beta(h)x \quad \forall h \in h \} \quad (\beta \in h^*)$$

and root system $R := \{ \alpha \in h^* \setminus \{0\} \mid g_\alpha \neq \{0\} \}$. We write $\theta \in \text{Aut}(g)$ for the complex linear extension of $\theta_{\mathfrak{h}_R}$ to an involution of $g$. Then $g = \mathfrak{k} \oplus p$ is the decomposition of $g$ in $(+1)$- and $(-1)$-eigenspaces of $\theta$. The involution $\theta$ restricts to an automorphism of $h$. The transpose of $\theta | h$ is a linear involution of $h^*$, which we will also denote by $\theta$. It restricts to an involution of $R$ satisfying $\theta |_{a_R} = -|_{a_R}$ for all $\alpha \in R$. We then have $\theta(g_\alpha) = g_{\theta \alpha}$.

For $\lambda \in \Sigma \cup \{0\}$ we write

$$R_\lambda := \{ \alpha \in R \mid \alpha |_{a_R} = \lambda \}.$$ 

Roots in $R_0$ are called imaginary roots (they are the roots in $R$ that take on purely imaginary values on $h_R$). Note that $\theta$ fixes $R_0$ point-wise and maps $R_\lambda$ to $R_{-\lambda}$ for $\lambda \in \Sigma$. Furthermore,

$$m = t \oplus \bigoplus_{\alpha \in R_0} g_\alpha, \quad g^\lambda = \bigoplus_{\alpha \in R_\lambda} g_\alpha \quad (\lambda \in \Sigma).$$

In particular, $R$ is the disjoint union of the $R_\lambda$ ($\lambda \in \Sigma \cup \{0\}$), $R_0$ is nonempty for $\lambda \in \Sigma$, and the restriction map

$$(3.2) \quad R \setminus R_0 \to \Sigma, \quad \alpha \mapsto \alpha |_{a_R}$$
is surjective. The strictly positive numbers
\[ \text{mtp}(\lambda) := \# R_\lambda, \quad \lambda \in \Sigma \]
are called the restricted root multiplicities. The Lie subalgebra \( m \) of \( g \) is reductive. The root system of the semisimple part \([m, m]\) of \( m \) with respect to the Cartan subalgebra \( t_{ss} := t \cap [m, m] \) is \( \{ \alpha_{|_{R_0}} \}_{\alpha \in R_0} \).

Choose a decomposition \( R = R^+ \cup R^- \) of the root system \( R \) into positive and negative roots such that

1. \( R_0 = R_0^+ \cup R_0^- \) with \( R_0^\pm := (R^\pm \cap R_0) \) is a decomposition of \( R_0 \) in positive and negative roots,
2. \( \Sigma = \Sigma^+ \cup \Sigma^- \) with \( \Sigma^\pm := \{ \alpha_{|_{R_0}} \mid \alpha \in R^\pm \setminus R_0^\pm \} \) is a decomposition of \( \Sigma \) in positive and negative roots
(see, e.g., [4, 11.1.16]).

Remark 3.1. Restricted root systems appear more generally for so-called normal \( \sigma \)-systems of roots, see, e.g., [22, §1.3]. A normal \( \sigma \)-system of roots consists of a root system \( R \) and an involutive isometry \( \sigma \) of the ambient Euclidean space, satisfying the additional properties that \( \sigma \) stabilises \( R \) and \( \sigma \alpha - \alpha \notin R \) for all \( \alpha \in R \).

In our current setup, \( R \) is a normal \((-\theta)\)-system of roots. Indeed, suppose that \( \alpha \in R \) satisfies \( \alpha + \theta \alpha \in R \). Then \( [e_\alpha, \theta(e_\alpha)] \in \mathfrak{p} \setminus \{0\} \). On the other hand, \( \alpha + \theta \alpha \in R_0 \) since the root \( \alpha + \theta \alpha \) vanishes on \( \mathfrak{a}_R \), and hence \( [e_\alpha, \theta(e_\alpha)] \in \mathfrak{k} \). This is a contradiction.

Consider the nilpotent Lie subalgebra
\[ n_R := \bigoplus_{\lambda \in \Sigma^+} g^\lambda_R \]
of \( g_R \). Its complexification within \( g \) is the nilpotent Lie subalgebra \( n = \bigoplus_{\alpha \in R^+ \setminus R_0^+} g_\alpha \). We then have the direct sum decompositions
\[ g_R = \mathfrak{k}_R \oplus \mathfrak{a}_R \oplus n_R, \quad g = \mathfrak{k} \oplus \mathfrak{a} \oplus n \]
as vector spaces (infinitesimal Iwasawa decomposition).

Let \( K, A, N \subset G \) be the connected Lie subgroups with Lie algebra \( \mathfrak{k}_R, \mathfrak{a}_R \) and \( n_R \), respectively. Then \( K \subset G \) is maximal compact, the multiplication map
\[ K \times A \times N \to G, \quad (k, a, n) \mapsto kan \]
is a diffeomorphism onto (the Iwasawa decomposition), and the exponential map \( \exp \) of \( G \) restricts to an isomorphism \( \exp : \mathfrak{a}_R \to A \) of Lie groups. We write \( \log : A \to \mathfrak{a}_R \) for its inverse.

Write \( M := Z_K(A) \) for the centraliser of \( A \) in \( K \). It is a closed, but not necessarily connected, Lie subgroup of \( K \) with Lie algebra \( \mathfrak{m}_R \). Note that \( M = Z_K(\mathfrak{a}) \) (the centraliser of \( \mathfrak{a} \) in \( K \) with respect to the adjoint action). Note that \( \theta_R \) and \( \theta \) are \( \text{Ad}(K) \)-linear, and \( g^\lambda \) is \( \text{Ad}(M) \)-stable (\( \lambda \in \Sigma \cup \{0\} \)). In particular, \( n_R \) and \( n \) are \( \text{Ad}(M) \)-stable.
The analytic Weyl group of $G$ is $W := N_K(A)/M$, with $N_K(A)$ the normaliser of $A$ in $K$. It acts naturally on $A$, as well as on the set $\Sigma$ of restricted roots. Denote by $W \backslash A$ the set of $W$-orbits in $A$, and write $K \backslash G/K$ for the double $(K, K)$-coset space of $G$. The map

$$W \backslash A \sim K \backslash G/K, \quad Wa \mapsto KaK$$

is a bijection (see, e.g., [15, VII.3]). The decomposition $G = KAK$ is sometimes called the Cartan decomposition of $G$ with respect to $K$.

For $\nu \in \mathfrak{h}^*$ write $h_\nu \in \mathfrak{h}$ for the unique element such that $K_\mathfrak{g}(h, h_\nu) = \nu(h)$ for all $h \in \mathfrak{h}$, with $K_\mathfrak{g}$ the Killing form of $\mathfrak{g}$. Choose root vectors $e_\alpha \in \mathfrak{g}_\alpha$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ ($\alpha \in R$). For $\alpha \in R$ we have

$$\theta(e_\alpha) = c_\alpha e_{\theta_\alpha}$$

for a unique nonzero scalar $c_\alpha \in \mathbb{C}^*$. Note that $c_\alpha = 1$ for $\alpha \in R_0$. Define for $\alpha \in R \setminus R_0$,

$$y_\alpha := e_\alpha + \theta(e_\alpha) = e_\alpha + c_\alpha e_{\theta_\alpha} \in \mathfrak{t}.$$

**Lemma 3.2.** For $\alpha \in R \setminus R_0$ we have $c_\alpha = c_\alpha^{-1} = c_{-\alpha}$. Furthermore, $y_\alpha = c_\alpha y_{\theta_\alpha}$.

**Proof.** We have $e_\alpha = \theta^2(e_\alpha) = c_\alpha e_{\theta_\alpha} e_\alpha$, hence $c_\alpha = c_\alpha^{-1}$. Furthermore,

$$\theta(h_\alpha) = \theta([e_\alpha, e_{-\alpha}]) = [\theta(e_\alpha), \theta(e_{-\alpha})] = c_\alpha c_{-\alpha} [e_{\theta_\alpha}, e_{-\theta_\alpha}] = c_\alpha c_{-\alpha} h_{\theta_\alpha}.$$

But $\theta(h_\alpha) = h_{\theta_\alpha}$ since $K_\mathfrak{g}(\theta(x), \theta(y)) = K_\mathfrak{g}(x, y)$ for $x, y \in \mathfrak{g}$, hence $c_{-\alpha} = c_{\alpha}^{-1}$. Finally,

$$y_\alpha = e_\alpha + \theta(e_\alpha) = \theta^2(e_\alpha) + c_\alpha e_{\theta_\alpha} = c_\alpha (e_{\theta_\alpha} + \theta(e_{\theta_\alpha})) = c_\alpha y_{\theta_\alpha}.$$

Consider the direct sum decomposition

$$\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{q}, \quad \mathfrak{q} := \bigoplus_{\alpha \in R^+ \setminus R^+_0} \mathbb{C} y_\alpha$$

as vector spaces. Note that $\mathfrak{q}$ is the complex linear span of the orthogonal complement $\mathfrak{q}_R$ of $\mathfrak{m}_R$ in $\mathfrak{k}_R$ with respect to $(\cdot, \cdot)$. Another description of $\mathfrak{q}$ is as follows. Let $\text{pr}_\mathfrak{k} : \mathfrak{g} \to \mathfrak{k}$ and $\text{pr}_\mathfrak{p} : \mathfrak{g} \to \mathfrak{p}$ be the projections along the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then $\text{pr}_\mathfrak{k} = \frac{1}{2}(\text{Id}_\mathfrak{g} + \theta)$ and $\text{pr}_\mathfrak{p} = \frac{1}{2}(\text{Id}_\mathfrak{g} - \theta)$, both projections are $\text{Ad}(K)$-equivariant, and $\mathfrak{q}$ is the image of $\mathfrak{u}$ under the projection $\text{pr}_\mathfrak{k}$. In particular, $\mathfrak{q}$ is $\text{Ad}(M)$-stable.

For $\lambda \in \mathfrak{a}^*$ the map $\xi_\lambda : A \to \mathbb{C}^*$, $a \mapsto a^\lambda := e^{\lambda(\log a)}$, defines a complex-valued multiplicative character of $A$. Note that $\xi_\lambda \xi_\nu = \xi_{\lambda + \nu}$ ($\lambda, \nu \in \mathfrak{a}^*$) and $\xi_0 \equiv 1$. Furthermore, for $\lambda \in \Sigma \cup \{0\}$ we have

$$\text{Ad}_a(x) = a^\lambda x a^{-\lambda} \quad (x \in \mathfrak{g}^\lambda, a \in A).$$

Write $A_{\text{reg}} := \{a \in A \mid a^\lambda \neq 1 \ \forall \lambda \in \Sigma\}$. For $a \in A_{\text{reg}}$ and $\alpha \in R_\lambda$ ($\lambda \in \Sigma$) a direct computation shows that

$$e_\alpha = \frac{a^\lambda \text{Ad}_a^{-1}(y_\alpha) - a^{2\lambda} y_\alpha}{1 - a^{2\lambda}},$$

$$\theta(e_\alpha) = \frac{a^\lambda \text{Ad}_a^{-1}(y_\alpha) - y_\alpha}{a^{2\lambda} - 1}.$$
This leads to the direct sum decomposition
\[(3.4) \quad g = a \oplus \text{Ad}_{a^{-1}}(q) \oplus \mathfrak{k} \quad (a \in A_{\text{reg}})\]
as vector spaces. It is this infinitesimal analogue of the Cartan decomposition that plays a crucial role in computing radial components of invariant differential operators, see \[11, 3\] and Section \[4\].

4. Generalised radial component maps

4.1. Differential operators. The conjugation action of \(N_K(A)\) on \(A\) descends to an action of the analytic Weyl group \(W = N_K(A)/M\). Its contragredient action on \(C^\infty(A)\) is explicitly given by \((wf)(a) := f(g^{-1}ag)\) for \(w = gM \in W, f \in C^\infty(A)\) and \(a \in A\). Note that the \(W\)-action on \(A\) preserves \(A_{\text{reg}}\), and hence \(C^\infty(A_{\text{reg}})\) admits the same \(W\)-module algebra structure.

Note furthermore that \(N_K(A)\) acts canonically on \(a^*\), and the action descends to an action of \(W\). Then \(w\xi_{\lambda} := \xi_{w\lambda}\) for \(w \in W\) and \(\lambda \in a^*\). We will write \(R\) be the unital subalgebra of \(C^\infty(A_{\text{reg}})\) generated by \((1 \pm \xi_\lambda)^{-1}\) for \(\lambda \in \Sigma\). It is a \(W\)-module subalgebra of \(C^\infty(A_{\text{reg}})\).

The action of \(N_K(A)\) and \(W\) on \(C^\infty(A)\) has the following twisted vector-valued analogue. Let \( (\sigma, V) \) be a finite dimensional \(N_K(A)\)-representation. The space \(C^\infty(A; V)\) of smooth \(V\)-valued functions on \(A\) is a left \(N_K(A)\)-module with action
\[(4.1) \quad (g \cdot f)(a) := \sigma(g)f(g^{-1}ag) \quad (g \in N_K(A))\]
for \(f \in C^\infty(A; V)\) and \(a \in A\). Let \(V^M \subseteq V\) be the subspace of \(M\)-invariant vectors in \(V\). It is a \(N_K(A)\)-subrepresentation of \(V\), and hence \(C^\infty(A; V^M)\) is a \(N_K(A)\)-submodule of \(C^\infty(A; V)\). The action of \(N_K(A)\) on \(C^\infty(A; V^M)\) descends to an action of \(W\). We write \(C^\infty(A; V^M)_W\) for the associated subspace of \(W\)-invariant functions.

The results from the previous paragraph also hold true for smooth \(V\)-valued functions on \(A_{\text{reg}}\). Then \(R \otimes V\) is a \(N_K(A)\)-submodule of \(C^\infty(A_{\text{reg}}; V)\), and \(R \otimes V^M\) is a \(W\)-submodule of \(C^\infty(A_{\text{reg}}; V^M)\).

We write \(h \mapsto \partial(h) (h \in U(a))\) for the action of \(U(a)\) on \(C^\infty(A)\) by constant coefficient differential operators. Concretely, for \(h \in a_{\mathbb{R}}\) the differential operator \(\partial(h)\) is the directional derivative \(\partial_h\) in direction \(h\),
\[(\partial_h f)(a) := \frac{d}{dt}\bigg|_{t=0} f(a \exp(th)) \quad (f \in C^\infty(A), \ a \in A).\]

We fix an orthonormal basis \(\{x_1, \ldots, x_r\}\) of \(a_{\mathbb{R}}\) with respect to \((\cdot, \cdot)\) once and for all, and we write
\[
\partial^m := \partial(x_1^{m_1} \cdots x_r^{m_r}) = \partial_{x_1}^{m_1} \cdots \partial_{x_r}^{m_r}
\]
for \(m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r\).

We now take \(V = \text{Hom}(E, F)\) with \(E\) and \(F\) two finite dimensional \(N_K(A)\)-representations, viewed as \(N_K(A)\)-representation with respect to the conjugation action. Then \(V\) is canonically isomorphic to the tensor product \(N_K(A)\)-representation \(F \otimes E^*\).
Definition 4.1. We write $\mathbb{D}(E, F)$ for the $N_K(A)$-module of differential operators $D = \sum_m q_m \partial^m$ on $A_{\text{reg}}$ with coefficients $q_m \in \mathcal{R} \otimes \text{Hom}(E, F)$.

Concretely, the $N_K(A)$-module structure on $\mathbb{D}(E, F)$ is given by

\begin{equation}
\sum_i (g \cdot q_i)\partial(\text{Ad}_g(h_i)) = g \cdot \left( \sum_i q_i\partial(h_i) \right)
\end{equation}

for $g \in N_K(A), q_i \in \mathcal{R} \otimes \text{Hom}(E, F)$ and $h_i \in U(a)$. The $N_K(A)$-action on differential operators $D \in \mathbb{D}(E, F)$, viewed as linear operators $D : C^\infty(A_{\text{reg}}; E) \rightarrow C^\infty(A_{\text{reg}}; F)$, is compatible with the $N_K(A)$-actions on $C^\infty(A_{\text{reg}}; E)$ and $C^\infty(A_{\text{reg}}; F)$,

\begin{equation}
\text{Ad}_g(f) = (\text{Ad}_g)(f).
\end{equation}

Note that the $M$-action on $\mathbb{D}(E, F)$ is the trivial extension to $\mathbb{D}(E, F)$ of the conjugation $M$-action on $\text{Hom}(E, F)$.

Definition 4.2. We write $\mathbb{D}_M(E, F)$ for the subspace of $M$-invariant differential operators in $\mathbb{D}(E, F)$. Concretely, it is the $N_K(A)$-module of differential operators $D = \sum_m q_m \partial^m$ with $q_m \in \mathcal{R} \otimes \text{Hom}_M(E, F)$.

The $N_K(A)$-action on $\mathbb{D}_M(E, F)$ descends to a $W$-action. We write $\mathbb{D}_M(E, F)^W$ for the subspace of $W$-invariant differential operators in $\mathbb{D}_M(E, F)$ (equivalently, it is the subspace of $N_K(A)$-invariant differential operators in $\mathbb{D}(E, F)$).

If $E = F$ then we write $\mathbb{D}(E), \mathbb{D}_M(E)$ and $\mathbb{D}_M(E)^W$ for the algebras $\mathbb{D}(E, E), \mathbb{D}_M(E, E)$ and $\mathbb{D}_M(E, E)^W$, respectively. Furthermore, for $D = \sum_m q_m \partial^m \in \mathbb{D}(E, F)$ we call

\begin{equation}
D_M := \sum_m q_m(\cdot)|_{E^M} \partial^m \in \mathbb{D}(E^M, F)
\end{equation}

the $M$-restriction of $D$. It satisfies

\begin{equation}
Df = D_Mf \quad (D \in \mathbb{D}(E, F), f \in C^\infty(A_{\text{reg}}; E^M)).
\end{equation}

Note that the assignment $\mathbb{D}(E, F) \rightarrow \mathbb{D}(E^M, F), D \mapsto D_M$ restricts to a linear map

$\mathbb{D}_M(E, F) \rightarrow \mathbb{D}(E^M, F^M),$

which is an algebra map $\mathbb{D}_M(E) \rightarrow \mathbb{D}(E^M)$ when $F = E$.

Typically $E$ and $F$ are going to be $N_K(A)$-subrepresentations of the tensor product representation $V_\ell \otimes V_r^*$, with $(\sigma_\ell, V_\ell)$ and $(\sigma_r, V_r)$ finite dimensional $K$-representations.

4.2. Radial component maps. This section recalls some well known facts about Harish-Chandra’s radial component maps. We follow closely the reference [3]. By [3, Thm. 2.4] there exist for $u \in U(g)$ unique elements

\begin{equation}
\Pi(u) = \sum_i f_i \otimes h_i \otimes (u_i \otimes_{U(m)} v_i) \in \mathcal{R} \otimes U(a) \otimes (U(t) \otimes_{U(m)} U(t)),
\end{equation}

\begin{equation}
\tilde{\Pi}(u) = \sum_i \tilde{f}_i \otimes \tilde{h}_i \otimes (\tilde{u}_i \otimes_{U(m)} \tilde{v}_i) \in \mathcal{R} \otimes U(a) \otimes (U(t) \otimes_{U(m)} U(t))
\end{equation}

for $u \in U(g)$.
such that
\[\sum_i f_i(a) \text{Ad}_a(u_i) v_i = u = \sum_i \tilde{f}_i(a) \tilde{u}_i \text{Ad}_a(\tilde{v}_i) \quad \forall a \in A_{\text{reg}}.\]

The two decompositions (4.3) of \(u \in U(\mathfrak{g})\) are called infinitesimal Cartan decompositions of \(u\) relative to \(a \in A_{\text{reg}}\). The maps \(\Pi, \tilde{\Pi} : U(\mathfrak{g}) \rightarrow \mathcal{R} \otimes U(\mathfrak{a}) \otimes (U(\mathfrak{t}) \otimes U(\mathfrak{m}) U(\mathfrak{t}))\) are called radial component maps. They provide the algebraic description of the radial components of the action of \(u \in U(\mathfrak{g})\) as left and right \(G\)-invariant differential operators on \(C^\infty(G)\) (see [3], Example 4.4, and Theorem 4.6).

**Remark 4.3.** Note that the coefficient algebra \(\mathcal{R}\) is different from the one used in [3]. The present choice is more convenient since \(\mathcal{R} \subseteq C^\infty(A_{\text{reg}})\) is \(W\)-invariant. The results in [3, §2–3] are valid for the coefficient ring \(\mathcal{R}\), with the same proofs.

**Example 4.4.** For \(\alpha \in R_\lambda\) with \(\lambda \in \Sigma\),
\[\Pi(e_\alpha) = \frac{\xi_\lambda}{1 - \xi_2\lambda} \otimes 1 \otimes (y_\alpha \otimes_U 1) - \frac{\xi_2\lambda}{1 - \xi_2\lambda} \otimes 1 \otimes (1 \otimes_U y_\alpha),\]
\[\tilde{\Pi}(e_\alpha) = \frac{1}{1 - \xi_2\lambda} \otimes 1 \otimes (y_\alpha \otimes_U 1) - \frac{\xi_\lambda}{1 - \xi_2\lambda} \otimes 1 \otimes (1 \otimes_U y_\alpha)\]
in view of the first line of (3.3).

Note that
\[\Pi(u) = \tilde{\Pi}(u) \quad \forall u \in U(\mathfrak{g})^A,\]
with \(U(\mathfrak{g})^A \subseteq U(\mathfrak{g})\) the subalgebra of \(\text{Ad}(A)\)-invariant elements in \(U(\mathfrak{g})\).

Fix two finite dimensional \(K\)-representations \((\sigma_\ell, V_\ell)\) and \((\sigma_r, V_r)\). Write \((V_\ell \otimes V^*_r)^m\) for the subspace of \(m\)-invariant elements in the tensor product \(\mathfrak{g}\)-module \(V_\ell \otimes V^*_r\), i.e.,
\[(V_\ell \otimes V^*_r)^m := \{ T \in V_\ell \otimes V^*_r \mid (\sigma_\ell(y) \otimes \text{id}_{V^*_r})T + (\text{id}_{V_\ell} \otimes \sigma^*_r(y))T = 0 \quad \forall y \in \mathfrak{m}\}.

Under the natural vector space identification \(V_\ell \otimes V^*_r \simeq \text{Hom}(V_r, V_\ell)\), the invariant subspace \((V_\ell \otimes V^*_r)^m\) corresponds to the subspace \(\text{Hom}_m(V_r, V_\ell)\) of \(m\)-intertwiners \(V_r \rightarrow V_\ell\). Note that \((V_\ell \otimes V^*_r)^m\) is a \(N_K(A)\)-subrepresentation of \(V_\ell \otimes V^*_r\) containing \((V_\ell \otimes V^*_r)^{\mathcal{R}}\). It may be a strict inclusion since \(M\) is not necessarily connected.

Consider now the linear map
\[\zeta_{\sigma_\ell, \sigma_r} : U(\mathfrak{g}) \otimes_U U(\mathfrak{m}) U(\mathfrak{g}) \rightarrow \text{Hom}((V_\ell \otimes V^*_r)^m, V_\ell \otimes V^*_r)\]
defined by
\[\zeta_{\sigma_\ell, \sigma_r}(u \otimes_U v)T := (\sigma_\ell(u) \otimes \sigma^*_r(S(v)))T \quad (u, v \in U(\mathfrak{g}), T \in (V_\ell \otimes V^*_r)^m).\]
The map is clearly well defined.

**Definition 4.5.** Let
\[\mathcal{R} \otimes U(\mathfrak{a}) \otimes (U(\mathfrak{g}) \otimes_U U(\mathfrak{m}) U(\mathfrak{g})) \rightarrow \mathbb{D}((V_\ell \otimes V^*_r)^m, V_\ell \otimes V^*_r), \quad z \mapsto L^z_{\sigma_\ell, \sigma_r}\]
be the linear map such that

$$L^{{\sigma}_{\prime},\sigma}_{z} := \zeta_{{\sigma}_{\prime},\sigma}(u \otimes U(m) v) f \partial(h)$$

for a pure tensor $z = f \otimes h \otimes (u \otimes U(m) v) \in \mathcal{R} \otimes U(a) \otimes (U(\mathfrak{t}) \otimes U(m) U(\mathfrak{t})).$

Write $f|_A \in C^\infty(A; V_\ell \otimes V_\ell^*)$ for the restriction of $f \in C^\infty(G; V_\ell \otimes V_\ell^*)$ to $A.$ Note that $|_A$ restricts to an injective linear map

$$|_A : C^\infty_{\sigma_{\prime},\sigma}(G) \hookrightarrow C^\infty_{\sigma_{\prime},\sigma}(G; (V_\ell \otimes V_\ell^*)^M).$$

Recall from Section 2 the actions $u[1]$ and $u\langle 1 \rangle$ of $u \in U(\mathfrak{g})$ on $C^\infty(G; V)$ as left and right $G$-invariant differential operators. Recall furthermore that

$$L^{{\sigma}_{\prime},\sigma}_{z;M} \in \mathcal{D}((V_\ell \otimes V_\ell^*)^M, V_\ell \otimes V_\ell^*)$$

denotes the restriction of $L^{{\sigma}_{\prime},\sigma}_{z;M}$ to $\mathcal{D}((V_\ell \otimes V_\ell^*)^M, V_\ell \otimes V_\ell^*).$ We now have the following theorem, which is essentially [3, Thm. 3.1].

**Theorem 4.6.** Let $u \in U(\mathfrak{g}), f \in C^\infty_{\sigma_{\prime},\sigma}(G)$ and $a \in A_{\text{reg}}.$ Then

$$\begin{align*}
(u[1]f)(a) &= (L^{{\sigma}_{\prime},\sigma}_{\Pi(u);M} f|_A)(a), \\
(u\langle 1 \rangle f)(a) &= (L^{{\sigma}_{\prime},\sigma}_{\Pi(S(u));M} f|_A)(a).
\end{align*}$$

**Proof.** By Remark 2.5 the second equality of (4.8) is equivalent to

$$\begin{align*}
(\text{Ad}_{a^{-1}} (u)[1] f)(a) &= (L^{{\sigma}_{\prime},\sigma}_{\Pi(u);M} f|_A)(a).
\end{align*}$$

By the definition of $\Pi(u)$ and $\Pi(u)$ as well as the definition of $L^{{\sigma}_{\prime},\sigma}_{z;M}$ (see Definition 4.5) it then suffices to prove that

$$\left((\text{Ad}_{a^{-1}} (u) h v) f\right)(a) = \zeta_{{\sigma}_{\prime},\sigma}(u \otimes U(m) v) \left(\partial(h) f\right)(a)$$

for $u, v \in U(\mathfrak{t}), h \in U(a)$ and $a \in A_{\text{reg}}.$ This is shown in the proof of [3, Thm. 3.1]. \qed

Note that

$$L^{{\sigma}_{\prime},\sigma}_{\Pi(u)} = L^{{\sigma}_{\prime},\sigma}_{\Pi(u)} (u \in U(\mathfrak{g})^A)$$

in view of (4.7).

The following proposition shows that the differential operators $L^{{\sigma}_{\prime},\sigma}_{\Pi(u);M}$ and $L^{{\sigma}_{\prime},\sigma}_{\Pi(u);M}$ are determined by the property (4.8) for all $f \in C^\infty_{\sigma_{\prime},\sigma}(G).$

**Proposition 4.7.** Suppose that $D \in \mathcal{D}((V_\ell \otimes V_\ell^*)^M, V_\ell \otimes V_\ell^*)$ satisfies $D(f|_A) = 0$ for all $f \in C^\infty_{\sigma_{\prime},\sigma}(G).$ Then $D = 0.$

**Proof.** This is shown as part of the proof of [3, Thm. 3.3]. \qed

The radial components satisfy the following equivariance property.

**Lemma 4.8.** For $u \in U(\mathfrak{g})$ and $g \in N_K(A)$ we have

$$L^{{\sigma}_{\prime},\sigma}_{\Pi(\text{Ad}_g(u))} = g \cdot L^{{\sigma}_{\prime},\sigma}_{\Pi(u)}, \quad L^{{\sigma}_{\prime},\sigma}_{\Pi(\text{Ad}_g(u))} = g \cdot L^{{\sigma}_{\prime},\sigma}_{\Pi(u)}$$

in $\mathcal{D}((V_\ell \otimes V_\ell^*)^M, V_\ell \otimes V_\ell^*).$
Proof. We only prove the first equality, the second is proved in a similar manner. Fix $g \in N_K(A)$ and write $w := gM \in W$. Using the notation (4.4) we have for $a \in A_{reg}$,
\begin{equation}
\text{Ad}_g(u) = \sum_i (w \cdot f_i)(wa)\text{Ad}(wa)^{-1}(\text{Ad}_g(u_i))\text{Ad}_g(h_i)\text{Ad}_g(v_i).
\end{equation}
Since $wf_i \in R$, $\text{Ad}_g(h_i) \in U(a)$ and $\text{Ad}_g(u_i), \text{Ad}_g(v_i) \in U(\mathfrak{t})$, formula (4.12) is an infinitesimal Cartan decomposition of $\text{Ad}_g(u)$ relative to $wa \in A_{reg}$ for each $a \in A_{reg}$. Hence
$$ \Pi(\text{Ad}_g(u)) = \sum_i w \cdot f_i \otimes \text{Ad}_g(h_i) \otimes (\text{Ad}_g(u_i) \otimes U(m) \text{Ad}_g(v_i)). $$
The lemma then follows from the definition of $L_{\tau,\sigma}^{\beta,\delta}$ (Definition 4.5) and the fact that
$$ \zeta_{\tau,\sigma}(\text{Ad}_g(u_i) \otimes U(m) \text{Ad}_g(v_i)) = (\sigma_{\ell}(g) \otimes \sigma_{\ell}^*(g)) \circ \zeta_{\tau,\sigma}(u_i \otimes v_i) \circ (\sigma_{\ell}(g^{-1}) \otimes \sigma_{\ell}^*(g^{-1})) $$
in $\text{Hom}((V_\ell \otimes V_\ell^*)^m, V_\ell \otimes U \otimes V_\ell^*)$.

4.3. Coordinate radial component maps. Let $n \in \mathbb{Z}_{\geq 0}$. Fix finite dimensional $G$-representations $(\tau_i, U_i)$ $(1 \leq i \leq n)$ and two finite dimensional $K$-representations $(\sigma_\ell, V_\ell)$, $(\sigma_r, V_r)$. Recall that $(\sigma_{\ell,n}, V_\ell \otimes U)$ is the $K$-representation with $K$ acting diagonally on $V_\ell \otimes U$.

Definition 4.9. For $u \in U(\mathfrak{g})$ and $0 \leq j \leq n$ define differential operators
$$ D_{u,j}^{\sigma_{\ell},\sigma_r} \in \mathcal{D}((V_\ell \otimes U \otimes V_r^*)^m, V_\ell \otimes U \otimes V_r^*) $$
by the formulas
$$ D_{u,j}^{\sigma_{\ell},\sigma_r} := \sum_{(u)} \tau_{j+1}(S(u(1))) \cdots \tau_{n}(S(u(n-j))) L_{\Pi(u_{n-j+1})}^{\sigma_{\ell,n},\sigma_r} \quad (0 \leq j < n), $$
$$ D_{u,n}^{\sigma_{\ell},\sigma_r} := I_{\Pi(u)}^{\sigma_{\ell,n},\sigma_r}. $$

The following lemma follows directly from Lemma 4.8.

Lemma 4.10. For $u \in U(\mathfrak{g})$, $0 \leq j \leq n$ and $g \in N_K(A)$ we have
$$ D_{\text{Ad}_g(u),j}^{\sigma_{\ell},\sigma_r} = g \cdot D_{u,j}^{\sigma_{\ell},\sigma_r} $$
in $\mathcal{D}((V_\ell \otimes U \otimes V_r^*)^m, V_\ell \otimes U \otimes V_r^*)$.

Let $U(\mathfrak{g})^m$ be the centraliser of $\mathfrak{m}$ in $U(\mathfrak{g})$. It contains $U(\mathfrak{g})^M$ as a subalgebra. The following is a direct consequence of Lemma 4.10.

Corollary 4.11. The differential operator $D_{u,j}^{\sigma_{\ell},\sigma_r} \in \mathcal{D}((V_\ell \otimes U \otimes V_r^*)^m, V_\ell \otimes U \otimes V_r^*)$ lies in
\begin{align*}
\mathcal{D}((V_\ell \otimes U \otimes V_r^*)^m) & \quad \text{if} \quad u \in U(\mathfrak{g})^m, \\
\mathcal{D}_M((V_\ell \otimes U \otimes V_r^*)^m) & \quad \text{if} \quad u \in U(\mathfrak{g})^M, \\
\mathcal{D}_M((V_\ell \otimes U \otimes V_r^*)^m)^W & \quad \text{if} \quad u \in U(\mathfrak{g})^{N_K(A)}.
\end{align*}
In particular, $D^σ_r;_u;j;_M ∈ \mathbb{D}((V_ℓ ⊗ U ⊗ V_τ^*)^M; V_ℓ ⊗ U ⊗ V_τ^*)$ lies in
\[ \mathbb{D}((V_ℓ ⊗ U ⊗ V_τ^*)^M) \quad \text{if} \quad u ∈ U(\mathfrak{g})^M; \]
\[ \mathbb{D}((V_ℓ ⊗ U ⊗ V_τ^*)^MW) \quad \text{if} \quad u ∈ U(\mathfrak{g})^{N_K(A)}. \]

Next we show that the differential operator $D^σ_r;_u;j;_M$ arises as the radial component of the action $u[j]$ on $C^∞_{σ,ℓ,σ_r}(G^{x(n+1)})$. The relevant restriction map is defined as follows.

**Definition 4.12.** Define the linear map
\[ \text{Res}^b : C^∞(G^{x(n+1)}; V_ℓ ⊗ U ⊗ V_τ^*) \to C^∞(A; V_ℓ ⊗ U ⊗ V_τ^*) \]
by $\text{Res}^b(f) := f^b|_A$. Concretely, for $f ∈ C^∞(G^{x(n+1)}; V_ℓ ⊗ U ⊗ V_τ^*)$ and $a ∈ A$,
\[ \text{Res}^b(f)(a) := f(1, \ldots, 1, a). \]

By Lemma 2.4 the restriction of $\text{Res}^b$ to $C^∞_{σ,ℓ,σ_r}(G^{x(n+1)})$ is an injective linear map
\[ \text{Res}^b : C^∞_{σ,ℓ,σ_r}(G^{x(n+1)}) \hookrightarrow C^∞(A; (V_ℓ ⊗ U ⊗ V_τ^*)^MW). \]

Theorem 4.6 now generalises as follows.

**Theorem 4.13.** For $u ∈ U(\mathfrak{g})$, $0 ≤ j ≤ n$, $f ∈ C^∞_{σ,ℓ,σ_r}(G^{x(n+1)})$ and $a ∈ A_{\text{reg}}$ we have
\[ \text{Res}^b(u[j]f)(a) = (D^σ_r;u;j;_M \text{Res}^b(f))(a). \]

**Proof.** Let $u ∈ U(\mathfrak{g})$, $f ∈ C^∞_{σ,ℓ,σ_r}(G^{x(n+1)})$ and $a ∈ A_{\text{reg}}$. Then $f^a ∈ C^∞_{σ,ℓ,n,σ_r}(G)$ by Lemma 2.4 and we have
\[ \text{Res}^b(u[n]f)(a) = (u[n]f)^b(a) = (u[1]f^a)(a) = (L^σ_{Π(a);_M}f^a|_A)(a) = (D^σ_r;u;n;_M \text{Res}^b(f))(a), \]
where the third equality follows from the first line of (4.8). This proves (4.13) for $j = n$.

Suppose that $0 ≤ j < n$. By Corollary 2.7 we have
\[ (u[j]f)^b(a) = \sum (u) τ_{j+1}(S(u(1))) \cdots τ_n(S(u(n-j)))(S(u(n-j+1))(n)f)^b(a). \]

Formula (4.13) then follows from the fact that
\[ (S(u(n-j+1))(n)f)^b(a) = (S(u(n-j+1))(1)f^a)(a) = (L^σ_{Π(a);_M} \text{Res}^b(f))(a), \]
where the last equality is due to the second equation of (4.8).

**Remark 4.14.** Replacing the role of the left $G$-invariant differential operators by right $G$-invariant differential operators gives
\[ \text{Res}^b(u(j)f)(a) = (\tilde{D}^σ_r;u;j;_M \text{Res}^b(f))(a) \]
for $u ∈ U(\mathfrak{g})$, $0 ≤ j ≤ n$, $f ∈ C^∞_{σ,ℓ,σ_r}(G^{x(n+1)})$ and $a ∈ A_{\text{reg}}$, with the differential operators $\tilde{D}^σ_r;u;j;_M ∈ \mathbb{D}((V_ℓ ⊗ U ⊗ V_τ^*)^m; V_ℓ ⊗ U ⊗ V_τ^*)$ given by
\[ \tilde{D}^σ_r;u;j;_M := \sum (u) τ_{j+1}(S(u(n-j+1))) \cdots τ_n(S(u(2)))L^σ_{Π(a)_M}. \]
Lemma 4.10 and Corollary 4.11 also apply for \( \hat{D}_{u,j}^{\sigma_t \bar{\sigma}_r} \).

Define the linear map

\[
(U(\mathfrak{g})^M)^{\otimes (n+1)} \to \mathbb{D}_M((V_\ell \otimes \underline{U} \otimes V_r^*)^M), \quad X \mapsto D_X^{\sigma_t \bar{\sigma}_r} \tag{4.15}
\]

by

\[
D_X^{\sigma_t \bar{\sigma}_r} := \hat{D}_{u_0,0}^{\sigma_t \bar{\sigma}_r} \circ \left( D_{u_1,0}^{\sigma_t \bar{\sigma}_r} \circ \cdots \circ D_{u_{n-1},0}^{\sigma_t \bar{\sigma}_r} \right) \circ D_{u_n,0}^{\sigma_t \bar{\sigma}_r}, \quad (X = u_0 \otimes \cdots \otimes u_n).
\]

Note that its restriction \( D_X^{\sigma_t \bar{\sigma}_r} \) to \( \mathbb{D}((V_\ell \otimes \underline{U} \otimes V_r^*)^M) \) is the composition of the restricted coordinate-wise radial components.

**Theorem 4.15.** The assignment \( X \mapsto D_X^{\sigma_t \bar{\sigma}_r} \) restricts to an algebra homomorphism

\[
U(\mathfrak{g})^{K,\text{opp}} \otimes Z(U(\mathfrak{g}))^{\otimes (n-1)} \otimes U(\mathfrak{g})^K \to \mathbb{D}((V_\ell \otimes \underline{U} \otimes V_r^*)^M)^W.
\]

**Proof.** Fix an element \( X \in U(\mathfrak{g})^{K,\text{opp}} \otimes Z(U(\mathfrak{g}))^{\otimes (n-1)} \otimes U(\mathfrak{g})^K \) and \( f \in C_{\sigma_t \bar{\sigma}_r}(G^{(n+1)}) \). We have \( D_X^{\sigma_t \bar{\sigma}_r} \in \mathbb{D}((V_\ell \otimes \underline{U} \otimes V_r^*)^M)^W \) by Corollary 4.11 and Remark 4.14. Furthermore, \( X \ast f \in C_{\sigma_t \bar{\sigma}_r}(G^{(n+1)}) \) by Lemma 2.9. By Theorem 4.13, Remark 4.14 and (2.6) we then have

\[
\text{Res}^b(X \ast f) = D_X^{\sigma_t \bar{\sigma}_r}(\text{Res}^b f).
\]

Choosing a second element \( Y \in U(\mathfrak{g})^{K,\text{opp}} \otimes Z(U(\mathfrak{g}))^{\otimes (n-1)} \otimes U(\mathfrak{g})^K \) we get

\[
D_X^{\sigma_t \bar{\sigma}_r}(\text{Res}^b f) = \text{Res}^b(X \ast (Y \ast f)) = D_{X \ast Y}^{\sigma_t \bar{\sigma}_r}(D_Y^{\sigma_t \bar{\sigma}_r}(\text{Res}^b f)). \tag{4.16}
\]

Lemma 2.4 and (4.16) now imply that

\[
D_X^{\sigma_t \bar{\sigma}_r}(F | A) = (D_X^{\sigma_t \bar{\sigma}_r} \circ D_Y^{\sigma_t \bar{\sigma}_r})(F | A) \quad \forall F \in C_{\sigma_t \bar{\sigma}_r}(G).
\]

applying Proposition 4.7 with respect to the two \( K \)-representations \((\sigma_{t,n}, V_\ell \otimes \underline{U})\) and \((\sigma_r, V_r)\) we conclude that

\[
D_X^{\sigma_t \bar{\sigma}_r} = D_X^{\sigma_t \bar{\sigma}_r} \circ D_Y^{\sigma_t \bar{\sigma}_r}
\]

in \( \mathbb{D}((V_\ell \otimes \underline{U} \otimes V_r^*)^M)^W \). \( \square \)

**Remark 4.16.** For \( n = 0 \), Theorem 4.15 should be read as follows: the assignment

\[
u \otimes Z(U(\mathfrak{g})) v \mapsto L_{\Pi(u),M}^{\sigma_t \sigma_r} \circ L_{\Pi(v),M}^{\sigma_t \sigma_r} \quad (u, v \in U(\mathfrak{g})^K)
\]

defines an algebra homomorphism \( U(\mathfrak{g})^{K,\text{opp}} \otimes Z(U(\mathfrak{g})) \to \mathbb{D}((V_\ell \otimes V_r^*)^M)^W \) (cf. 3. Thm. 3.3)). The balancing condition of the tensor product over the center is justified by (4.10), see also Remark 2.10.

By Theorem 4.15 we have the inclusion of algebras

\[
A^{\sigma_t \bar{\sigma}_r} \subseteq B^{\sigma_t \bar{\sigma}_r} \subseteq \mathbb{D}((V_\ell \otimes \underline{U} \otimes V_r^*)^M)^W \tag{4.17}
\]

with

\[
B^{\sigma_t \bar{\sigma}_r} := \{ D_X^{\sigma_t \bar{\sigma}_r} \mid X \in U(\mathfrak{g})^{K,\text{opp}} \otimes Z(U(\mathfrak{g}))^{\otimes (n-1)} \otimes U(\mathfrak{g})^K \}
\]

and \( A^{\sigma_t \bar{\sigma}_r} \) the commutative subalgebra

\[
A^{\sigma_t \bar{\sigma}_r} := \{ D_X^{\sigma_t \bar{\sigma}_r} \mid X \in Z(U(\mathfrak{g}))^{\otimes (n+1)} \}. \]
Furthermore, we have

\[ D_{X;M}^{\Delta_l \Delta_r} = D_{u_0;0;M}^{\Delta_l \Delta_r} \circ \cdots \circ D_{u_{n-1};n-1;M}^{\Delta_l \Delta_r} \circ D_{u_n;n;M}^{\Delta_l \Delta_r} \quad (X = u_0 \otimes \cdots \otimes u_n \in Z(U(\mathfrak{g}))^{(n+1)}) \]

due to the following lemma.

**Lemma 4.17.** For \( u \in Z(U(\mathfrak{g})) \) and \( 0 \leq j \leq n \) we have

\[ \hat{D}_{u,j;M}^{\Delta_l \Delta_r} = D_{u,j;M}^{\Delta_l \Delta_r} \]

in \( \mathbb{D}((V_\ell \otimes U \otimes V_r^*)^M)^W \).

**Proof.** Let \( u \in Z(U(\mathfrak{g})) \) and \( f \in C^\infty_{\Delta_l \Delta_r}(G^{(n+1)}) \). Then we have for \( a \in A_{reg} \),

\[ (\hat{D}_{u,j;M}^{\Delta_l \Delta_r} \text{Res}^\flat f)(a) = \text{Res}^\flat(S(u)(j)f)(a) = \text{Res}^\flat(u[j]f)(a) = (D_{u,j;M}^{\Delta_l \Delta_r} \text{Res}^\flat f)(a) \]

with the first equality by Remark 4.14, the second by Remark 2.5 and the third by Theorem 4.13. Now the conclusion follows by Lemma 2.4 and Proposition 4.7, cf. the proof of Theorem 4.15. \( \square \)

Suitable gauged versions of the subalgebras \( B^{\Delta_l \Delta_r} \) and \( A^{\Delta_l \Delta_r} \) will serve as the algebras of quantum integrals and quantum Hamiltonians of a quantum superintegrable system, see Subsection 5.

The following corollary of Theorem 4.15 and Lemma 4.17 is now immediate.

**Corollary 4.18.** For \( u, v \in Z(U(\mathfrak{g})) \) and \( 0 \leq i, j \leq n \) we have

\[ [D_{u,i;M}^{\Delta_l \Delta_r}, D_{v,j;M}^{\Delta_l \Delta_r}] = 0 \]

in \( \mathbb{D}((V_\ell \otimes U \otimes V_r^*)^M)^W \).

**Remark 4.19.** For finite dimensional \( \ell \)-modules \( (\sigma_l, V_\ell) \) and \( (\sigma_r, V_r) \) we expect that

\[ [D_{u,i;M}^{\Delta_l \Delta_r}, D_{v,j;M}^{\Delta_l \Delta_r}] = 0 \]

in \( \mathbb{D}_M((V_\ell \otimes U \otimes V_r^*)^m) \) for \( u, v \in Z(U(\mathfrak{g})) \) and \( 0 \leq i, j \leq n \). This follows from the theory of formal \( n \)-point spherical functions when \( G \) is real split, see [20].

## 5. The Quantum Calogero-Moser Spin Chain

Fix throughout this section finite dimensional \( K \)-representations \( (\sigma_l, V_\ell) \), \( (\sigma_r, V_r) \) and finite dimensional \( G \)-representations \( (\tau_i, U_i) \) (\( 1 \leq i \leq n \)).

### 5.1. A dynamical factorisation of the Casimir element

Denote the multiplication map of \( U(\mathfrak{g}) \) by \( \mu \in \text{Hom}(U(\mathfrak{g}) \otimes U(\mathfrak{g}), U(\mathfrak{g})) \).

**Definition 5.1.** The Casimir element \( \Omega \in Z(U(\mathfrak{g})) \) is

\[ \Omega := \mu(\varpi) \]

with \( \varpi \in S^2(\mathfrak{g})^G \) the \( G \)-invariant symmetric tensor associated to \( K_\mathfrak{g} \).
Recall that $\varpi = \sum b_s \otimes b'_s$ and $\Omega = \sum b_s b'_s$, where $\{b'_s\}_s$ is the basis of $\mathfrak{g}$ dual to $\{b_s\}_s$ with respect to $K_\mathfrak{g}$.

In the computations we will use a basis of $\mathfrak{g}$ compatible with the root space decomposition of $\mathfrak{g}$. It contains a basis $\{z_j\}_{j=1}^{\dim(h)}$ of $\mathfrak{h}$ such that

1. $K_\mathfrak{g}(z_j, z_{j'}) = \delta_{j,j'}$,  
2. $x_j := z_j$ $(1 \leq j \leq r)$ forms a basis of $\mathfrak{a}_R$,  
3. $iz_j$ $(r + 1 \leq j \leq \dim(h))$ forms a basis of $\mathfrak{t}_R$.

Recall that $e_\alpha \in \mathfrak{g}_\alpha$ are root vectors such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. Then $\{z_j\}_{j=1}^{\dim(h)} \cup \{e_{-\alpha}\}_{\alpha \in R}$ is the basis of $\mathfrak{g}$ dual to $\{z_j\}_{j=1}^{\dim(h)} \cup \{e_\alpha\}_{\alpha \in R}$ with respect to $K_\mathfrak{g}$ (cf. [14, Prop. 8.3 c]), hence

$$z_j \in \mathfrak{g}_\alpha$$

Note that $\varpi$ can be recovered from $\Omega$ by

$$\varpi = \frac{1}{2}(\Delta(\Omega) - \Omega \otimes 1 - 1 \otimes \Omega).$$

Write $\Omega_m := \mu(\varpi_m) \in Z(U(\mathfrak{m}))$, with $\varpi_m \in S^2(\mathfrak{m})^M$ the symmetric tensor associated to the nondegenerate bilinear form $K_\mathfrak{g}(\cdot, \cdot)|_{\mathfrak{m} \times \mathfrak{m}}$.

$$\varpi_m := \sum_{j=1}^{\dim(h)} z_j \otimes z_j + \sum_{\alpha \in \mathfrak{r}_0} e_{-\alpha} \otimes e_{\alpha}. \quad (5.3)$$

Finally, $\varpi' := \varpi - \varpi_m \in S^2(\mathfrak{g})^M$ and $\Omega' := \mu(\varpi') \in U(\mathfrak{g})^M$ admit the explicit expressions

$$\varpi' = \sum_{j=1}^{r} x_j \otimes x_j + \sum_{\lambda \in \Sigma} \varpi_\lambda, \quad \Omega' = \sum_{j=1}^{r} x_j^2 + \sum_{\lambda \in \Sigma} \Omega_\lambda \quad (5.4)$$

with, for a restricted root $\lambda \in \Sigma$,

$$\varpi_\lambda := \sum_{\alpha \in \mathfrak{r}_A} e_{-\alpha} \otimes e_{\alpha} \in \mathfrak{g}^{-\lambda} \otimes \mathfrak{g}^{\lambda}, \quad \Omega_\lambda := \mu(\varpi_\lambda) = \sum_{\alpha \in \mathfrak{r}_A} e_{-\alpha} e_{\alpha} \in U(\mathfrak{g}). \quad (5.5)$$

**Lemma 5.2.** We have $\varpi_\lambda \in (\mathfrak{g}^{-\lambda} \otimes \mathfrak{g}^{\lambda})^M$ and $\Omega_\lambda \in U(\mathfrak{g})^M$.

**Proof.** The restricted root spaces $\mathfrak{g}^{\lambda}$ and $\mathfrak{g}^{-\lambda}$ are $\text{Ad}(M)$-stable since $M = Z_K(\mathfrak{a})$. Let $g \in M$. The basis $\{\text{Ad}_g(e_{-\alpha})\}_{\alpha \in \mathfrak{r}_A}$ of $\mathfrak{g}^{-\lambda}$ is dual to the basis $\{\text{Ad}_g(e_\alpha)\}_{\alpha \in \mathfrak{r}_A}$ of $\mathfrak{g}^{\lambda}$ with respect to the perfect $\text{Ad}(M)$-invariant pairing $K_\mathfrak{g}(\cdot, \cdot)|_{\mathfrak{g}^{-\lambda} \times \mathfrak{g}^{\lambda}} : \mathfrak{g}^{-\lambda} \times \mathfrak{g}^{\lambda} \to \mathbb{C}$. Hence the associated 2-tensor $\sum_{\alpha \in \mathfrak{r}_A}\text{Ad}_g(e_{-\alpha}) \otimes \text{Ad}_g(e_\alpha)$ does not depend on $g \in M$. \hfill $\square$

The adjoint action of $N_K(A)$ on $(\mathfrak{g} \otimes \mathfrak{g})^M$ and $U(\mathfrak{g})^M$ factorises to a $W$-action. We write $w \cdot x$ for the action of $w \in W$ on $x \in (\mathfrak{g} \otimes \mathfrak{g})^M$ or $x \in U(\mathfrak{g})^M$. 
Lemma 5.3. We have

1. \( w \cdot \varpi_{\lambda} = \varpi_{w\lambda} \) and \( w \cdot \Omega_\lambda = \Omega_{w\lambda} \) for \( w \in W \) and \( \lambda \in \Sigma \).

2. \( \varpi_m \in S^2(m)^{\text{Ad}(A)} \) and \( \Omega_m \in U(m)^{\text{Ad}(A)} \).

3. \( \sum_{i=1}^{r} x_i \otimes x_i \in S^2(a)^{\text{Ad}(A)} \) and \( \sum_{i=1}^{r} x_i^2 \in U(a)^{\text{Ad}(A)} \).

Proof. (1) For \( w = gM \in W \) (\( g \in N_K(A) \)) the \( \text{Ad}_g \)-invariance of \( K_{\theta} \) implies that the basis \( \{ \text{Ad}_g(e_a) \}_{a \in R_\lambda} \) of \( g^{-\lambda} \) is dual to the basis \( \{ \text{Ad}_g(e_a) \}_{a \in R_\lambda} \) of \( g^{w\lambda} \) with respect to the perfect pairing \( K_{\theta}(\cdot, \cdot)|_{g^{w\lambda} \times g^{w\lambda}} \). The result now follows immediately, cf. the proof of Lemma 5.2.

(2) This follows from the \( \text{Ad}(N_K(A)) \)-invariance of \( m \) and \( K_{\theta}(\cdot, \cdot)|_{m \times m} \).

(3) The proof is similar to the proof of (2), now considering the \( \text{Ad}(N_K(A)) \)-invariant nondegenerate symmetric bilinear form \( K_{\theta}(\cdot, \cdot)|_{g^{\lambda} \times g^{\lambda}} \). We conclude that

\[
(\theta \otimes \text{id})(\varpi_{\lambda}) = - \sum_{i=1}^{\text{mtp}(\lambda)} b_{\lambda,i} \otimes b_{\lambda,i} \in S^2(g^\lambda)
\]

and

\[
(\theta \otimes \text{id})(\varpi_{\lambda}) = (\text{id} \otimes \theta)(\varpi_{-\lambda}).
\]

For \( \nu \in a^* \) we denote by \( t_\nu \) the unique element in \( a \) such that \( K_{\theta}(t_\nu, h) = \nu(h) \) for all \( h \in a \). Note that \( t_\lambda \in a_{\mathbb{R}} \) for \( \lambda \in \Sigma \).

Lemma 5.5. Let \( \lambda \in \Sigma \).

1. If \( \alpha \in R_0 \) then \( h_\alpha \in t \).

2. If \( \alpha \in R_\lambda \) then \( \text{pr}_p(h_\alpha) = t_\lambda \).

3. \( \sum_{\alpha \in R_\lambda} h_\alpha \in g^M \).

Proof. (1) This follows from the observation that \( t = \{ h \in h \mid K_{\theta}(h, a) = 0 \} \).

(2) Let \( \alpha \in R_\lambda \). Then \( \text{pr}_p(h_\alpha) \in a \) satisfies for all \( h \in a \),

\[
K_{\theta}(\text{pr}_p(h_\alpha), h) = K_{\theta}(h_\alpha, h) = \alpha(h) = \lambda(h),
\]

hence \( \text{pr}_p(h_\alpha) = t_\lambda \).

(3) Note that the \( G \)-equivariant linear map \( g \otimes g \to g \), \( x \otimes y \mapsto [x, y] \) maps \( \varpi_{\lambda} \) to \( -\sum_{\alpha \in R_\lambda} h_\alpha \). The result now follows from Lemma 5.2. \( \square \)
We set for \( \lambda \in \Sigma \),
\[
z_\lambda := \sum_{\alpha \in R_\lambda} \text{pr}_\xi (h_\alpha) \in \mathfrak{t}.
\]

**Corollary 5.6.** We have \( z_\lambda \in Z(\mathfrak{m}) \), \( z_{-\lambda} = -z_\lambda \) and
\[
\sum_{\alpha \in R_\lambda} h_\alpha = z_\lambda + \text{mtp}(\lambda)t_\lambda.
\]
Furthermore, \( w \cdot z_\lambda = z_{w\lambda} \) and \( w \cdot t_\lambda = t_{w\lambda} \) for \( w \in W \).

**Proof.** By Lemma 5.3 and Corollary 5.6 it is clear that the right-hand side of (5.8) lies in \( \mathfrak{m} \).

By Proposition 5.7.

Formula (5.7) is due to Lemma 5.5(2). It remains to prove the last identities.

Since \( \text{pr}_\xi \) and \( \text{pr}_p \) are \( \text{Ad}(K) \)-linear, it suffices to show that \( w \cdot (\sum_{\alpha \in R_\lambda} h_\alpha) = \sum_{\alpha \in R_{w\lambda}} h_\alpha \).

This follows from Lemma 5.3(1) and the proof of Lemma 5.5(3). \( \square \)

Consider the tensor product \( W \)-module algebra \( R \otimes U(\mathfrak{g})^M \). Its subalgebra \( (R \otimes U(\mathfrak{g})^M)_W \) of \( W \)-invariant elements contains \( U(\mathfrak{g})^{N_K(A)} \) as subalgebra. The Casimir \( \Omega \) decomposes as \( \Omega = \Omega_m + \Omega' \) with \( \Omega_m, \Omega' \in U(\mathfrak{g})^{N_K(A)} \). The second term \( \Omega' \in U(\mathfrak{g})^{N_K(A)} \) admits the following dynamical factorisation within \( (R \otimes U(\mathfrak{g})^M)_W \).

**Proposition 5.7.** We have
\[
\Omega' = \frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) z_\lambda + \sum_{j=1}^r x_j^2 + \frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) \text{mtp}(\lambda)t_\lambda + 2 \sum_{\lambda \in \Sigma} \frac{\Omega_\lambda}{1 - \xi_{2\lambda}}
\]
in \( (R \otimes U(\mathfrak{g})^M)_W \).

**Proof.** By Lemma 5.3 and Corollary 5.6 it is clear that the right-hand side of (5.8) lies in \( (R \otimes U(\mathfrak{g})^M)_W \). By (5.7) we have
\[
(5.9) \quad \Omega_{-\lambda} = \Omega_\lambda + z_\lambda + \text{mtp}(\lambda)t_\lambda
\]
for \( \lambda \in \Sigma \). Hence
\[
\sum_{\lambda \in \Sigma} \frac{\Omega_\lambda}{1 - \xi_{2\lambda}} = \sum_{\lambda \in \Sigma^+} \frac{z_\lambda + \text{mtp}(\lambda)t_\lambda}{1 - \xi_{-2\lambda}} + \sum_{\lambda \in \Sigma^+} \left( \frac{1}{1 - \xi_{2\lambda}} + \frac{1}{1 - \xi_{-2\lambda}} \right) \Omega_\lambda
\]
\[
= \sum_{\lambda \in \Sigma^+} \frac{z_\lambda + \text{mtp}(\lambda)t_\lambda}{1 - \xi_{-2\lambda}} + \sum_{\lambda \in \Sigma^+} \Omega_\lambda
\]
\[
= \sum_{\lambda \in \Sigma^+} \left( \frac{1}{1 - \xi_{-2\lambda}} - \frac{1}{2} \right) (z_\lambda + \text{mtp}(\lambda)t_\lambda) + \frac{1}{2} \sum_{\lambda \in \Sigma} \Omega_\lambda
\]
\[
= - \frac{1}{4} \sum_{\lambda \in \Sigma} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) (z_\lambda + \text{mtp}(\lambda)t_\lambda) + \frac{1}{2} \sum_{\lambda \in \Sigma} \Omega_\lambda,
\]
where we have used (5.9) in the first and third equality. Substituting this identity in the right-hand side of the dynamical expression (5.8) reduces it to the expression (5.4) of \( \Omega' \). \( \square \)
Remark 5.8. There is no dynamical expression of $\varpi'$ in $(\mathcal{R} \otimes (g \otimes g)^M)^W$ that leads to (5.8) by applying the multiplication map $\mu$ of $U(g)$. We can however write

$$\Omega' = \left( \frac{1}{2} \sum_{\lambda \in \sum} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) \varpi_{\lambda} - \Omega_m \right) + \left( \frac{1}{2} \sum_{\lambda \in \sum} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) \text{mtp}(\lambda) t_{\lambda} - 2\mu(r_{21}) \right)$$

involving the dynamical element

(5.10) $$r = -\frac{1}{2} \varpi_m - \frac{1}{2} \sum_{j=1}^{r} x_j \otimes x_j - \sum_{\lambda \in \sum} \frac{\varpi_{\lambda}}{1 - \xi_{-2\lambda}} \in (\mathcal{R} \otimes (g \otimes g)^M)^W$$

(here $r_{21}$ is obtained from $r$ by interchanging its tensor components). The 2-tensor $r$ will play an important role in the explicit description of the asymptotic boundary KZB operators, see Theorem 6.3 and Proposition 6.10. Note that $r$ satisfies the quasi-unitarity condition

$$r + r_{21} = -\varpi \in S^2(g)^G.$$

In the real split case $r$ is Felder’s [9] trigonometric solution of the classical dynamical Yang-Baxter equation. We will show in Subsection 5.3 that for arbitrary $G$, folded versions of $r$ satisfy coupled classical dynamical Yang-Baxter-reflection equations.

5.2. The radial component of the action of the Casimir element. We first define an explicit second order differential operator $H_{\sigma,\tau}^{\ell,\sigma,\tau} \in \mathbb{D}_M((V_\ell \otimes \bar{U} \otimes V_\tau)^m)^W$ and then show that the differential operator $D_{\sigma,\tau}^{\ell,\sigma,\tau}$ is equal to $H_{\sigma,\tau}^{\ell,\sigma,\tau}$. The technique to compute $D_{\sigma,\tau}^{\ell,\sigma,\tau}$ goes back to Harish-Chandra, see, e.g., [23, §9.1.2].

Write for $\lambda \in \sum$,

$$v_\lambda := 4(\text{pr}_\ell \otimes \text{pr}_\bar{U})(\varpi_\lambda) = \sum_{\alpha \in R_\lambda} y_{-\alpha} \otimes y_\alpha \in q \otimes q$$

and set $\Upsilon_\lambda := \mu(v_\lambda) = \sum_{\alpha \in R_\lambda} y_{-\alpha} y_\alpha \in U(\mathfrak{t})$.

Lemma 5.9. Let $\lambda \in \sum$.

1. $v_\lambda = v_{-\lambda}$ and $\Upsilon_\lambda = \Upsilon_{-\lambda}$.
2. $v_\lambda \in (q \otimes q)^M$ and $\Upsilon_\lambda \in U(\mathfrak{t})^M$.
3. $w \cdot v_\lambda = v_{w\lambda}$ and $w \cdot \Upsilon_\lambda = \Upsilon_{w\lambda}$ for $w \in W$.

Proof. (1) Observe that $(\text{pr}_\ell \otimes \text{pr}_\bar{U})(\varpi_\lambda) = (\text{pr}_\ell \otimes \text{pr}_\bar{U})((\theta \otimes \text{id})(\varpi_\lambda))$ is a symmetric tensor by Remark 5.4. Hence $v_\lambda = v_{-\lambda}$, and consequently $\Upsilon_\lambda = \Upsilon_{-\lambda}$.

(2) The $\text{Ad}(M)$-invariance of $v_\lambda$ follows from Lemma 5.2 and the fact that $\text{pr}_\ell$ is $\text{Ad}(K)$-equivariant. The $\text{Ad}(M)$-invariance of $\Upsilon_\lambda \in U(\mathfrak{t})$ now follows immediately.

(3) This follows from Lemma 5.3(1). \hfill \Box

Note that $S(\Upsilon_\lambda) = \Upsilon_\lambda$ in view of Lemma 5.9(1).
**Definition 5.10.** Define \( H^\sigma_{\ell, \Xi_r} \in \mathbb{D}_M((V_\ell \otimes U \otimes V^*_r)^W) \) by

\[
H^\sigma_{\ell, \Xi_r} := \sum_{j=1}^{r} \partial^2_{x_j} + \frac{1}{2} \sum_{\lambda \in \Sigma} \text{mtp}(\lambda) \left( \frac{\xi_\lambda + \xi_\lambda^-}{\xi_\lambda - \xi_\lambda^-} \right) \partial_{\lambda} + \sigma_r^*(\Omega_m) + \frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{\xi_\lambda + \xi_\lambda^-}{\xi_\lambda - \xi_\lambda^-} \right) \sigma_r^*(z_\lambda) - \sum_{\lambda \in \Sigma} \sigma_{\ell, n}(Y_\lambda) + \sigma_r^*(Y_\lambda) + (\xi_\lambda + \xi_\lambda^-)(\sigma_{\ell, n} \otimes \sigma_r^*)(v_\lambda).
\]

The linear operators \( \sigma_r^*(\Omega_m), \sigma_r^*(z_\lambda), \sigma_{\ell, n}(Y_\lambda), \sigma_r^*(Y_\lambda) \) and \( (\sigma_{\ell, n} \otimes \sigma_r^*)(v_\lambda) \) occurring in the definition of \( H^\sigma_{\ell, \Xi_r} \) are viewed as linear operators on \((V_\ell \otimes U \otimes V^*_r)^m\). Note that this is well defined and that \( H^\sigma_{\ell, \Xi_r} \) lies in \( \mathbb{D}_M((V_\ell \otimes U \otimes V^*_r)^W) \) in view of Lemma 5.9, Lemma 5.3 and Corollary 5.6.

We now have the following result, cf. [23, Prop. 9.1.2.11].

**Theorem 5.11.** We have

\[
D^\sigma_{\ell, \Xi_r} = H^\sigma_{\ell, \Xi_r}
\]

in \( \mathbb{D}_M((V_\ell \otimes U \otimes V^*_r)^m)^W \).

**Proof.** By Corollary 4.11 the differential operator \( D^\sigma_{\ell, \Xi_r} \) lies in \( \mathbb{D}_M((V_\ell \otimes U \otimes V^*_r)^m)^W \). Furthermore,

\[
D^\sigma_{\ell, \Xi_r} = L^\sigma_{\ell, n, \Xi_r} = L^\sigma_{\Pi(\Omega), \Xi_r} + \sigma_r^*(\Omega_m).
\]

Hence it suffices to show that \( L^\sigma_{\Pi(\Omega), \Xi_r} \) is the restriction to \( C^\infty(A_{\text{reg}}; V_\ell \otimes U \otimes V^*_r)^m \) of the differential operator

\[
\sum_{j=1}^{r} \partial^2_{x_j} + \frac{1}{2} \sum_{\lambda \in \Sigma} \text{mtp}(\lambda) \left( \frac{\xi_\lambda + \xi_\lambda^-}{\xi_\lambda - \xi_\lambda^-} \right) \partial_{\lambda} + \frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{\xi_\lambda + \xi_\lambda^-}{\xi_\lambda - \xi_\lambda^-} \right) \sigma_r^*(z_\lambda) - \sum_{\lambda \in \Sigma} \sigma_{\ell, n}(Y_\lambda) + \sigma_r^*(Y_\lambda) + (\xi_\lambda + \xi_\lambda^-)(\sigma_{\ell, n} \otimes \sigma_r^*)(v_\lambda).
\]

For this it suffices to prove that

\[
\Pi(\Omega') = \sum_{j=1}^{r} 1 \otimes x^2_j \otimes (1 \otimes U(m) 1) + \frac{1}{2} \sum_{\lambda \in \Sigma} \text{mtp}(\lambda) \left( \frac{\xi_\lambda + \xi_\lambda^-}{\xi_\lambda - \xi_\lambda^-} \right) \otimes t_\lambda \otimes (1 \otimes U(m) 1)
\]

\[
- \frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{\xi_\lambda + \xi_\lambda^-}{\xi_\lambda - \xi_\lambda^-} \right) \otimes 1 \otimes (z_\lambda \otimes U(m) 1) + \sum_{\lambda \in \Sigma} \left( \frac{\xi_\lambda + \xi_\lambda^-}{\xi_\lambda - \xi_\lambda^-} \right) \otimes 1 \otimes v_\lambda
\]

\[
- \frac{1}{2} \sum_{\lambda \in \Sigma} \frac{1}{(\xi_\lambda - \xi_\lambda^-)^2} \otimes 1 \otimes (\gamma_\lambda \otimes U(m) 1 + 1 \otimes U(m) \gamma_\lambda),
\]

where, in the fourth term, \( v_\lambda \in q \otimes q \) is viewed as element in \( U(\mathfrak{g}) \otimes U(m) U(\mathfrak{g}) \) via the canonical injection \( q \otimes q \hookrightarrow U(\mathfrak{g}) \otimes U(m) U(\mathfrak{g}) \), \( x \otimes y \mapsto x \otimes U(m) y \).
This in turn will follow from the identity
\[
\Omega' = \sum_{j=1}^{r} x_j^2 + \frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{1}{a^\lambda - a^{-\lambda}} \right) (\text{mtp}(\lambda) t_{\lambda} - z_{\lambda})
\]
\[
+ \sum_{\lambda \in \Sigma} \left( \frac{a^\lambda + a^{-\lambda}}{(a^\lambda - a^{-\lambda})^2} \right) \mu((\text{Ad}_{a^{-1}} \otimes \text{Id})(v_\lambda)) = \text{Ad}_{a^{-1}}(\Upsilon_\lambda) - \Upsilon_\lambda
\]
in $U(\mathfrak{g})$ for all $a \in \mathcal{A}_{\text{reg}}$.

Fix $a \in \mathcal{A}_{\text{reg}}$ and $\lambda \in \Sigma$. By applying (5.3) to both $e_\alpha$ and $e_\alpha$ and using Lemma 5.9(1) we get
\[
\Omega_\lambda = \sum_{\alpha \in R_\lambda} e_{-\alpha} e_\alpha = \left( \frac{a^\lambda + a^{-\lambda}}{(a^\lambda - a^{-\lambda})^2} \right) (\text{mtp}(\lambda) t_{\lambda} - z_{\lambda})
\]
In view of (5.4) it thus suffices to show that
\[
(5.12) \quad \sum_{\lambda \in \Sigma} \frac{a_{-\lambda}}{(a^\lambda - a^{-\lambda})^2} \sum_{\alpha \in R_\lambda} [y_\alpha, (\text{Ad}_{a^{-1}}(y_{-\alpha})] = \frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{a^\lambda + a^{-\lambda}}{a^\lambda - a^{-\lambda}} \right) (\text{mtp}(\lambda) t_{\lambda} - z_{\lambda}).
\]
Note that for $\alpha \in R_\lambda$,
\[
(5.13) \quad [y_\alpha, (\text{Ad}_{a^{-1}}(y_{-\alpha})] = a^\lambda h_\alpha + a^{-\lambda} \theta(h_\alpha) + a^\lambda [\theta(e_\alpha), e_{-\alpha}] + a^{-\lambda} [e_\alpha, \theta(e_{-\alpha})]
\]
\[
= (a^\lambda + a^{-\lambda}) \text{pr}_{1}(h_\alpha) + (a^\lambda - a^{-\lambda}) t_{\lambda} + a^\lambda [\theta(e_\alpha), e_{-\alpha}] + a^{-\lambda} [e_\alpha, \theta(e_{-\alpha})].
\]
When substituting into the left-hand side of (5.12), the first term of (5.13) contributes
\[
\sum_{\lambda \in \Sigma} \frac{a_{-\lambda}}{(a^\lambda - a^{-\lambda})^2} z_{\lambda} = \sum_{\lambda \in \Sigma^+} \frac{a_{-\lambda}}{(a^\lambda - a^{-\lambda})^2} \left( \frac{a^\lambda + a^{-\lambda}}{a^\lambda - a^{-\lambda}} \right) z_{\lambda}
\]
\[
= \sum_{\lambda \in \Sigma^+} \left( \frac{a_{-\lambda}}{a^\lambda - a^{-\lambda}} \right) z_{\lambda} = -\frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{a_{-\lambda}}{a^\lambda - a^{-\lambda}} \right) z_{\lambda}
\]
since $z_{-\lambda} = -z_{\lambda}$ (see Corollary 5.6), and the second term of (5.13) contributes
\[
\sum_{\lambda \in \Sigma} \text{mtp}(\lambda) \frac{a_{-\lambda}}{(a^\lambda - a^{-\lambda})^2} t_{\lambda} = \sum_{\lambda \in \Sigma^+} \text{mtp}(\lambda) \left( \frac{1}{a^2\lambda - 1} - \frac{1}{a^{-2\lambda} - 1} \right) t_{\lambda}
\]
\[
= \sum_{\lambda \in \Sigma^+} \text{mtp}(\lambda) \left( \frac{a^\lambda + a^{-\lambda}}{a^\lambda - a^{-\lambda}} \right) t_{\lambda} = \frac{1}{2} \sum_{\lambda \in \Sigma} \text{mtp}(\lambda) \left( \frac{a^\lambda + a^{-\lambda}}{a^\lambda - a^{-\lambda}} \right) t_{\lambda}
\]
since $t_{-\lambda} = -t_{\lambda}$ and $\text{mtp}(-\lambda) = \text{mtp}(\lambda)$. It thus suffices to show that
\[
(5.14) \quad \sum_{\lambda \in \Sigma} \frac{a_{-\lambda}}{(a^\lambda - a^{-\lambda})^2} \sum_{\alpha \in R_\lambda} (a^\lambda [\theta(e_\alpha), e_{-\alpha}] + a^{-\lambda} [e_\alpha, \theta(e_{-\alpha})]) = 0.
\]
This follows from the anti-symmetry of the Lie bracket and the fact that
\[ \sum_{\alpha \in R_\lambda} e_\alpha \otimes \theta(e_{-\alpha}) \in S^2(\mathfrak{g}^\lambda) \]
for all \( \lambda \in \Sigma \), which was observed in Remark 5.4.

\[ \square \]

Remark 5.12. In the real split case one can choose the root vectors \( e_\alpha \) in such a way that \( \theta(e_\alpha) = -e_{-\alpha} \) for \( \alpha \in R \). In that case one has \( \nu_\alpha = -y_\alpha \otimes y_\alpha \) and \( \Upsilon_\alpha = -y_\alpha^2 \) for \( \alpha \in R = \Sigma \), and Theorem 5.11 reduces to [20, Cor. 3.5].

5.3. The Schrödinger operator. The following explicit gauge will turn \( H^{\sigma_r \mathcal{L} \sigma_r} \) into a Schrödinger operator.

**Definition 5.13.** We write \( D \mapsto \tilde{D} \) for the \( W \)-equivariant \( \mathcal{R} \otimes \text{End}_M((V_\ell \otimes \overline{U} \otimes V_r^*)^m) \)-linear algebra automorphism of \( \mathbb{D}_M((V_\ell \otimes \overline{U} \otimes V_r^*)^m) \) satisfying

\[ \tilde{\partial}_h = \frac{1}{4} \sum_{\lambda \in \Sigma} \lambda(h) \text{mtp}(\lambda) \left( \frac{1 + \xi_\lambda}{1 - \xi_\lambda} \right) + \partial_h \quad \forall h \in \mathfrak{a}_\mathbb{R}. \]

**Remark 5.14.** Consider the positive Weyl chamber

\[ A^+ := \{ a \in A \mid a^\lambda > 1 \quad \forall \lambda \in \Sigma^+ \} \]

and define an analytic function \( \delta \) on \( A^+ \) by

\[ \delta(a) := a^\rho \prod_{\lambda \in \Sigma^+} (1 - a^{-2\lambda})^{\frac{\text{mtp}(\lambda)}{2}}, \quad \rho := \frac{1}{2} \sum_{\lambda \in \Sigma^+} \text{mtp}(\lambda) \lambda. \]

A straightforward computation then shows that

\[ (\tilde{D}(f))(a) = \delta(a)(Df)(a) \]

for \( D \in \mathbb{D}_M((V_\ell \otimes \overline{U} \otimes V_r^*)^m) \), \( f \in C^\infty(A^+; (V_\ell \otimes \overline{U} \otimes V_r^*)^m) \) and \( a \in A^+ \).

Define the potential \( V^{\sigma_r \mathcal{L} \sigma_r} \) by

\[ V^{\sigma_r \mathcal{L} \sigma_r} := -\sum_{\lambda \in \Sigma} \frac{k(\lambda)}{(\lambda_\lambda - \xi_{-\lambda})^2} + \frac{1}{2} \sum_{\lambda \in \Sigma} \left( \frac{\xi_{\lambda} + \xi_{-\lambda}}{\xi_{\lambda} - \xi_{-\lambda}} \right) \sigma_r^*(v_{\lambda}) \]

\[ -\sum_{\lambda \in \Sigma} \sigma_{\ell n}(\Upsilon_\lambda) + \sigma_r^*(\Upsilon_\lambda) + (\xi_{\lambda} + \xi_{-\lambda})(\sigma_{\ell n} \otimes \sigma_r^*)(v_{\lambda}) \]

with \( k(\lambda) := \text{mtp}(\lambda)(\text{mtp}(2\lambda) + \frac{\text{mtp}(\lambda)}{2} - 1)(t_\lambda, t_\lambda) \).

**Proposition 5.15.** The gauged differential operator \( \tilde{H}^{\sigma_r \mathcal{L} \sigma_r} \in \mathbb{D}_M((V_\ell \otimes \overline{U} \otimes V_r^*)^m) \) is explicitly given by

\[ \tilde{H}^{\sigma_r \mathcal{L} \sigma_r} = \sum_{j=1}^r \partial^2_{x_j} + V^{\sigma_r \mathcal{L} \sigma_r} + \sigma^*_r(\Omega_m) - (t_\rho, t_\rho). \]
Proof. The proof of the first statement is similar to the scalar case \(n = 0, \sigma_t = \sigma_r\) the trivial representation), cf. the proof of [13, Thm. 2.1.1].

Write \(A_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) and \(B_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) for the images of the algebras \(A_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) and \(B_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) (see (4.17)) under the isomorphism \(D \mapsto \tilde{D}\) of \(\mathbb{D}((V_\ell \otimes U \otimes V_r^*)^M)^W\). We have an inclusion of algebras 
\[
\tilde{A}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}} \subseteq \tilde{B}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}} \subseteq \mathbb{D}((V_\ell \otimes U \otimes V_r^*)^M)^W,
\]
with \(\tilde{A}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) commutative and containing \(\tilde{H}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}} = \tilde{D}_{\Omega, n; M}^{\sigma_t}\).

**Definition 5.16.** The quantum Calogero-Moser spin chain is the quantum superintegrable system of algebra of quantum Hamiltonians \(\tilde{A}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\), algebra of quantum integrals \(\tilde{B}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) and Schrödinger operator \(\tilde{H}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}} \in \tilde{A}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\).

The quantum Calogero-Moser spin chain are related to the Schrödinger operator \(\tilde{H}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) with \(\tilde{A}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) for the \(\nu\) of the images of the algebra of quantum Hamiltonians \(\tilde{A}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}}\) and Schrödinger operator \(\tilde{H}_{\sigma_t}^{\mathbb{Z}, \mathbb{Z}} = \tilde{D}_{\Omega, n; M}^{\sigma_t}\).

The quantum Calogero-Moser spin chain is a mixture of a quantum spin Calogero-Moser system of rank \(r\) and an one-dimensional spin chain with \(n\) internal sites and two reflecting boundaries. For the bosonic theory, the quantum state space is a suitable completion of \(C^\infty(A_{\text{reg}}; (V_\ell \otimes U \otimes V_r^*)^M)^W\).

**Corollary 5.17.** Let \(f \in C^\infty_{\sigma_t, \mathbb{Z}, \mathbb{Z}}(G^{\times(n+1)}; \chi)\) be a \(n\)-point spherical function (see Definition 2.11). Write 
\[
\tilde{f}^\Theta \in C^\infty(A_{\text{reg}}; (V_\ell \otimes U \otimes V_r^*)^M)^W
\]
for the \(W\)-invariant \((V_\ell \otimes U \otimes V_r^*)^M\)-valued function such that 
\[
\tilde{f}^\Theta(a) := \delta(a) f^\Theta(a) \quad \forall a \in A^+.
\]
For all \(X \in Z(U(\mathfrak{g}))^{\otimes(n+1)}\) we have 
\[
\tilde{D}_{X; M}^{\sigma_t, \mathbb{Z}, \mathbb{Z}}(\tilde{f}^\Theta) = \chi(X) \tilde{f}^\Theta,
\]
with \(\chi\) viewed as the character \(\chi_0 \otimes \cdots \otimes \chi_n\) of \(Z(U(\mathfrak{g}))^{\otimes(n+1)}\). In particular, 
\[
\tilde{H}_{M}^{\sigma_t, \mathbb{Z}, \mathbb{Z}}(\tilde{f}^\Theta) = \chi_n(\Omega) \tilde{f}^\Theta.
\]

See [20] for explicit examples of classes of \(n\)-point spherical functions when \(G\) is real split.

6. The asymptotic boundary KZB operators

Fix throughout this section finite dimensional \(K\)-representations \((\sigma_t, V_\ell), (\sigma_r, V_r)\) and finite dimensional \(G\)-representations \((\tau_i, U_i)\) \((1 \leq i \leq n)\).

The quadratic Hamiltonians \(\tilde{D}_{\sigma_t, \mathbb{Z}, \mathbb{Z}}^{\Omega, j; M} \in \tilde{A}_{\sigma_t, \mathbb{Z}, \mathbb{Z}}\) \((0 \leq j \leq n)\) of the quantum Calogero-Moser spin chain are related to the Schrödinger operator \(\tilde{H}_{\sigma_t, \mathbb{Z}, \mathbb{Z}}^{\mathbb{Z}, \mathbb{Z}}\) by 
\[
\tilde{D}_{\sigma_t, \mathbb{Z}, \mathbb{Z}}^{\Omega, j; M} = \tilde{H}_{\sigma_t, \mathbb{Z}, \mathbb{Z}}^{\mathbb{Z}, \mathbb{Z}} - 2 \sum_{i=j+1}^{n} \tilde{D}_{\sigma_t, \mathbb{Z}, \mathbb{Z}}^{\Omega, i; M} \in \tilde{A}_{\sigma_t, \mathbb{Z}, \mathbb{Z}}\quad (0 \leq j \leq n),
\]
with \( \tilde{D}^{\sigma_{i}, \xi_{i}} \in \tilde{A}^{\sigma_{i}, \xi_{i}} \) the quantum Hamiltonian obtained by gauging and \( M \)-restricting the differential operator

\[
D_{i}^{\sigma_{i}, \xi_{i}} := \frac{1}{2} \left( D_{\Omega_{i}, i}^{\sigma_{i}, \xi_{i}} - D_{\Omega_{i}, i-1}^{\sigma_{i}, \xi_{i}} \right) \in \mathbb{D}_{M}((V_{t} \otimes U \otimes V_{r}^{*})^{m})^{W} \quad (1 \leq i \leq n).
\]

**Definition 6.1.** The quantum Hamiltonians \( \tilde{D}^{\sigma_{i}, \xi_{i}} \in \tilde{A}^{\sigma_{i}, \xi_{i}} \) (1 ≤ i ≤ n) are called the asymptotic boundary KZB operators.

We now first derive explicit expressions for the asymptotic boundary KZB operators.

### 6.1. The derivation of the explicit expression

The asymptotic boundary KZB operators turn out to be first order differential operators, whose local terms we define now first. Write

\[
r^{+} := -\varpi_{m} - \sum_{\lambda \in \Sigma} \frac{2(pr_{\tau} \otimes \text{id})(\varpi_{\lambda})}{1 - \xi_{2\lambda}} \in (\mathcal{R} \otimes (\mathfrak{t} \otimes \mathfrak{g})^{M})^{W},
\]

\[
r^{-} := \sum_{j=1}^{r} x_{j} \otimes x_{j} + \sum_{\lambda \in \Sigma} \frac{2(pr_{p} \otimes \text{id})(\varpi_{\lambda})}{1 - \xi_{2\lambda}} \in (\mathcal{R} \otimes (\mathfrak{p} \otimes \mathfrak{g})^{M})^{W}
\]

(the \( M \)-invariance follows from Lemma 5.2, and the \( W \)-invariance from Lemma 5.3). Define \( \kappa_{\text{core}} \in (\mathcal{R} \otimes U(\mathfrak{g})^{M})^{W} \) by

\[
\kappa_{\text{core}} := \frac{1}{2} \Omega + \mu(r_{21}^{+}) = -\frac{1}{2} \Omega_{m} + \frac{1}{2} \Omega' - \sum_{\lambda \in \Sigma} \sum_{\alpha \in R_{\lambda}} \frac{e_{\alpha}y_{\alpha}}{1 - \xi_{2\lambda}}
\]

and \( \kappa \in (\mathcal{R} \otimes (U(\mathfrak{t}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{t}))^{M})^{W} \) by

\[
\kappa := 1 \otimes \kappa_{\text{core}} \otimes 1 - \left( \varpi_{m} + \sum_{\lambda \in \Sigma} \frac{2(pr_{\tau} \otimes \text{id})(\varpi_{\lambda})}{1 - \xi_{2\lambda}} \right) \otimes 1 \otimes \left( \sum_{\lambda \in \Sigma} \frac{2(id \otimes pr_{\tau})(\varpi_{\lambda})}{\xi_{-\lambda} - \xi_{\lambda}} \right).
\]

For 1 ≤ i < j ≤ n and 1 ≤ s ≤ n write

\[
r^{+}_{ij} := (\tau_{i} \otimes \tau_{j})(r^{+}), \quad \kappa_{s} := (\sigma_{\ell} \otimes \tau_{s} \otimes \sigma_{s}^{*})(\kappa),
\]

which we view as elements in \( (\mathcal{R} \otimes \text{End}_{M}((V_{t} \otimes U \otimes V_{r}^{*})^{m}))^{W} \).

**Definition 6.2.** Define first order differential operators \( D_{i}^{\sigma_{i}, \xi_{i}} \in \mathbb{D}_{M}((V_{t} \otimes U \otimes V_{r}^{*})^{m})^{W} \) (1 ≤ i ≤ n) by

\[
D_{i}^{\sigma_{i}, \xi_{i}} = \sum_{j=1}^{r} \tau_{i}(x_{j}) \partial_{x_{j}} - \kappa_{i} - \sum_{k=1}^{i-1} r_{ki}^{+} - \sum_{k=i+1}^{n} r_{ik}^{-}.
\]

The following theorem is the main result of this section.

**Theorem 6.3.** For 1 ≤ i ≤ n we have the identity

\[
\frac{1}{2} \left( D_{\Omega_{i}, i}^{\sigma_{i}, \xi_{i}} - D_{\Omega_{i}, i-1}^{\sigma_{i}, \xi_{i}} \right) = D_{i}^{\sigma_{i}, \xi_{i}}.
\]

in \( \mathbb{D}_{M}((V_{t} \otimes U \otimes V_{r}^{*})^{m})^{W} \).
Before discussing the proof of Theorem \[6.3\] we note some immediate consequences first.

**Corollary 6.4.** (1) The \( M \)-restricted gauged differential operators \( \tilde{D}_{i;M}^{\sigma_i \xi \sigma_r} \in \mathcal{A}_{\sigma_i \xi \sigma_r} \) are first order quantum Hamiltonians for the quantum Calogero-Moser spin chain. In particular,

\[
\tag{6.3}
[D_{i;M}^{\sigma_i \xi \sigma_r}, D_{j;M}^{\sigma_j \xi \sigma_r}] = 0, \quad [D_{i;M}^{\sigma_i \xi \sigma_r}, \tilde{H}_M^{\sigma_i \xi \sigma_r}] = 0
\]

for \( 1 \leq i, j \leq n \).

(2) The quadratic Hamiltonian \( \tilde{D}_{\Omega;j,M}^{\sigma_i \xi \sigma_r} \in \mathcal{A}_{\sigma_i \xi \sigma_r} \) for \( j \in \{0, \ldots, n\} \) is explicitly given by

\[
\tilde{D}_{\Omega;j,M}^{\sigma_i \xi \sigma_r} = \tilde{H}_M^{\sigma_i \xi \sigma_r} - 2 \sum_{i=j+1}^{n} \tilde{D}_{j;M}^{\sigma_i \xi \sigma_r}.
\]

(3) We have

\[
\tilde{D}_{i;M}^{\sigma_i \xi \sigma_r}(\vec{f}) = \frac{1}{2}(\chi_i(\Omega) - \chi_{i-1}(\Omega))\vec{f}
\]

for \( f \in C^\infty_{\sigma_i \xi \sigma_r}(G^{(n+1)}; \chi) \) and \( 1 \leq i \leq n \).

(4) We have in \( \mathbb{D}_M((V_\ell \otimes U \otimes V_r^*)^m)^W \),

\[
\tilde{D}_i^{\sigma_i \xi \sigma_r} = \sum_{j=1}^{r} \tau_i(x_j)\partial_{x_j} - \hat{\kappa}_i - \sum_{k=1}^{i-1} r^+_k - \sum_{k=i+1}^{n} r^-_k
\]

with

\[
\hat{\kappa}_i = \kappa_i - \frac{1}{4} \sum_{\lambda \in \Sigma} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) \text{mtp}(\lambda) \tau_i(t_\lambda).
\]

**Remark 6.5.** It follows from Corollary \[6.4\] that \( \tilde{D}_i^{\sigma_i \xi \sigma_r} \in \mathbb{D}_M((V_\ell \otimes U \otimes V_r^*)^m)^W \) is obtained from \( \mathcal{D}_i^{\sigma_i \xi \sigma_r} \) by replacing the core \( \kappa_{\text{core}} \) of \( \kappa \) by

\[
\tilde{\kappa}_{\text{core}} := \kappa_{\text{core}} - \frac{1}{4} \sum_{\lambda \in \Sigma} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) \text{mtp}(\lambda) t_\lambda.
\]

**Remark 6.6.** If the conjecture in Remark \[4.19\] is valid, then the differential operators \( \tilde{H}_r^{\sigma_i \xi \sigma_r}, \tilde{D}_r^{\sigma_i \xi \sigma_r} \) (\( 1 \leq i \leq n \)) already pairwise commute in \( \mathbb{D}_M((V_\ell \otimes U \otimes V_r^*)^m)^W \). This is indeed the case when \( G \) is real split, see \[20\] \S 6.

We now prove Theorem \[6.3\] in a couple of steps.

**Lemma 6.7.** For \( k \in \mathbb{Z}_{\geq 0} \),

\[
\Delta^{(k)}(\Omega) = \sum_{j=1}^{k+1} \Omega_j + 2 \sum_{1 \leq j < j' \leq k+1} \sum_{s=1}^{\dim(h)} (z_s)_j (z_s)_{j'} + 2 \sum_{1 \leq j < j' \leq k+1} \sum_{\alpha \in R} (e_{-\alpha})_j (e_{\alpha})_{j'}
\]

in \( U(\mathfrak{g})^{(k+1)} \), with the subindices indicating in which tensor component of \( U(\mathfrak{g})^{(k+1)} \) the elements are placed.

**Proof.** This follows by induction to \( k \) using the expression \[4.2\] for \( \Omega \). \( \square \)
Corollary 6.8. For $0 \leq i \leq n$ we have
\[
D_{\Omega,i}^{\sigma_i, \sigma_r} = L_{\Omega, \Omega}^{\sigma_i, \sigma_r} - 2 \sum_{j=i+1}^{n} \sum_{s=1}^{\text{dim}(\mathfrak{h})} \tau_j(z_s)L_{\Omega(z_s)}^{\sigma_i, \sigma_r} - 2 \sum_{j=i+1}^{n} \sum_{\alpha \in R} \tau_j(e_{-\alpha})L_{\Omega(e_{\alpha})}^{\sigma_i, \sigma_r} + 2 \sum_{j=i+1}^{n} \tau_j(\Omega) + 2 \sum_{i+1 \leq j < j' \leq n} (\tau_j \otimes \tau_{j'})(\omega),
\]
viewed as differential operators in $D((V_{\ell} \otimes U \otimes V_{r}^*)^m, V_{\ell} \otimes U \otimes V_{r}^*)$.

Proof. For $0 \leq i < n$ this follows directly from Definition 4.9, Lemma 6.7, the identity $S(\Omega) = \Omega$ and (5.1). For $i = n$ it follows from Definition 4.9 and (4.10).

Write for $\lambda \in \Sigma$,
\[
\omega^\lambda := 2(\text{id} \otimes pr_t)(\omega_\lambda) = \sum_{\alpha \in R_\lambda} e_{-\alpha} \otimes y_\alpha \in (\mathfrak{g} \otimes \mathfrak{t})^M.
\]
Furthermore, for $X = \sum_{t} a_t \otimes b_t \in \mathfrak{g} \otimes \mathfrak{t}$ we write $(\tau_i \otimes \sigma_{\ell,n})(X) := \sum_{t} \tau_i(a_t)\sigma_{\ell,n}(b_t)$, viewed as endomorphism of $V_{\ell} \otimes U \otimes V_{r}^*$ (cf. Remark 2.3).

Lemma 6.9. We have for $1 \leq i \leq n$,
\[
\frac{1}{2}(D_{\Omega,i}^{\sigma_i, \sigma_r} - D_{\Omega,i-1}^{\sigma_i, \sigma_r}) = \sum_{j=1}^{r} \tau_i(x_j)\partial_{x_j} - \frac{1}{2} \tau_i(\Omega) - \sum_{k=i+1}^{n} (\tau_i \otimes \tau_k)(\omega) + (\tau_i \otimes \sigma_{\ell,n})(\omega_m) + \sum_{\lambda \in \Sigma} \frac{\left( (\tau_i \otimes \sigma_{\ell,n})(\omega^\lambda) + \xi_\lambda (\tau_i \otimes \sigma_r^*)(\omega^\lambda) \right)}{1 - \xi_2\lambda},
\]
viewed as differential operators in $D((V_{\ell} \otimes U \otimes V_{r}^*)^m, V_{\ell} \otimes U \otimes V_{r}^*)$.

Proof. All identities below are in $D((V_{\ell} \otimes U \otimes V_{r}^*)^m, V_{\ell} \otimes U \otimes V_{r}^*)$, i.e., they hold true when acting on functions $f \in C^\infty(A_{reg} \setminus (V_{\ell} \otimes U \otimes V_{r}^*)^m)$. By Corollary 6.8,
\[
\frac{1}{2}(D_{\Omega,i}^{\sigma_i, \sigma_r} - D_{\Omega,i-1}^{\sigma_i, \sigma_r}) = \sum_{s=1}^{\text{dim}(\mathfrak{h})} \tau_i(z_s)L_{\Omega(z_s)}^{\sigma_i, \sigma_r} + \sum_{\alpha \in R} \tau_i(e_{-\alpha})L_{\Omega(e_{\alpha})}^{\sigma_i, \sigma_r} - \frac{1}{2} \tau_i(\Omega) - \sum_{k=i+1}^{n} (\tau_i \otimes \tau_k)(\omega).
\]
Now use that $L_{\Omega(z_s)}^{\sigma_i, \sigma_r} = \partial_{x_j}$ for $1 \leq j \leq r$, $L_{\Omega(z_s)}^{\sigma_{\ell,n}, \sigma_r} = \sigma_{\ell,n}(z_s)$ for $r + 1 \leq s \leq \text{dim}(\mathfrak{h})$, $L_{\Omega(e_{\alpha})}^{\sigma_{\ell,n}, \sigma_r} = \sigma_{\ell,n}(e_{\alpha})$ for $\alpha \in R_0$, and
\[
L_{\Omega(e_{\alpha})}^{\sigma_{\ell,n}, \sigma_r} = \frac{\sigma_{\ell,n}(y_\alpha) + \xi_\lambda \sigma_r^*(y_\alpha)}{1 - \xi_2\lambda}
\]
for $\alpha \in R_\lambda$ with $\lambda \in \Sigma$ (the last formula is a consequence of (4.6)). Substitute these equations in (6.7) and group the terms which act through $\tau_i$, through $\tau_i$ and $\tau_k$, through $\tau_i$ and $\sigma_{\ell,n}$, and through $\tau_i$ and $\sigma_r^*$. This directly leads to the desired result. $\square$
Proof of Theorem 6.3: We write the degree zero terms of the differential operator on the right-hand side of (6.6) in terms of local factors. By the fact that \( \overline{w}_m \in S^2(m) \), \( \Omega_m = \mu(\overline{w}_m) \) and the definition of \( \sigma_{\ell;n} \), we have

\[
(\tau_i \otimes \sigma_{\ell;n})(\overline{w}_m) = (\sigma_{\ell} \otimes \tau_i)(\overline{w}_m) + \sum_{k=1}^{i-1} (\tau_k \otimes \tau_i)(\overline{w}_m) + \tau_i(\Omega_m) + \sum_{k=i+1}^{n} (\tau_i \otimes \tau_k)(\overline{w}_m)
\]

in End\((V_\ell \otimes U \otimes V_r^*)\). Similarly, we have for \( \alpha \in R \setminus R_0 \),

\[
\tau_i(\sigma_{\ell;n}(y_\alpha)) = \sigma_{\ell}(y_\alpha)\tau_i(\sigma_{\ell;n}(\overline{w}_m)) + \sum_{k=1}^{i-1} \tau_k(y_\alpha)\tau_i(\sigma_{\ell;n}(\overline{w}_m)) + \tau_i(\sigma_{\ell;n}(\overline{w}_m)) + \sum_{k=i+1}^{n} \tau_i(\sigma_{\ell;n}(\overline{w}_m))
\]

Substituting these identities into (6.6) and comparing the resulting expression with the explicit formula for \( D_i^{\sigma_{\ell+2},\sigma_r} \) from Definition 6.2 formula (6.2) will follow from the identities

\[
\begin{align*}
 r^+_{ki} &= - (\tau_k \otimes \tau_i)(\overline{w}_m) - \sum_{\lambda \in \Sigma} \frac{1}{1 - \xi_{2\lambda}} \sum_{\alpha \in R_\lambda} \tau_k(y_\alpha)\tau_i(e_\alpha), & 1 \leq k < i, \\
r^-_{ik} &= (\tau_i \otimes \tau_k)(\overline{w}') - \sum_{\lambda \in \Sigma} \frac{1}{1 - \xi_{2\lambda}} \sum_{\alpha \in R_\lambda} \tau_i(e_\alpha)\tau_k(y_\alpha), & i < k \leq n, \\
\tau_i(\kappa_{\text{core}}) &= -\frac{1}{2} \tau_i(\Omega_m) + \frac{1}{2} \tau_i(\Omega') - \sum_{\lambda \in \Sigma} \frac{1}{1 - \xi_{2\lambda}} \sum_{\alpha \in R_\lambda} \tau_i(e_\alpha y_\alpha)
\end{align*}
\]

as endomorphisms of \( V_\ell \otimes U \otimes V_r^* \). Only the second equality requires proof; it follows from the fact that

\[
\overline{w}' - \sum_{\lambda \in \Sigma} \frac{1}{1 - \xi_{2\lambda}} \sum_{\alpha \in R_\lambda} e_\alpha \otimes y_\alpha = \sum_{j=1}^{r} x_j \otimes x_j + \sum_{\lambda \in \Sigma} \sum_{\alpha \in R_\lambda} e_\alpha \otimes \left( \frac{(1 - \xi_{2\lambda})e_\alpha - y_\alpha}{1 - \xi_{2\lambda}} \right)
\]

\[
= \sum_{j=1}^{r} x_j \otimes x_j - \sum_{\lambda \in \Sigma} \sum_{\alpha \in R_\lambda} e_\alpha \otimes \left( \frac{\xi_{2\lambda}e_\alpha + c_\alpha e_\theta_\alpha}{1 - \xi_{2\lambda}} \right)
\]

\[
= \sum_{j=1}^{r} x_j \otimes x_j + \sum_{\lambda \in \Sigma} \frac{1}{1 - \xi_{2\lambda}} \sum_{\alpha \in R_\lambda} (e_\alpha - c_\alpha^{-1} e_{-\theta_\alpha}) \otimes e_\alpha
\]

\[
= r^-, \quad \text{where we have used Lemma 6.2 for the third and fourth equality. It is clear that both sides of (6.2) lies in } \mathbb{D}_M((V_\ell \otimes U \otimes V_r^*)^m)^W. \text{ This completes the proof of the theorem.} \]

The local factors \( r^\pm \) and \( \kappa_{\text{core}} \) occurring in \( D_i^{\sigma_{\ell+2},\sigma_r} \in \mathbb{D}_M((V_\ell \otimes U \otimes V_r^*)^m)^W \) \((1 \leq i \leq n)\) decompose as follows.
Proposition 6.10. We have

\[ r^\pm = \pm r + (\id \otimes \theta) r_{21}, \]

\[ \kappa_{\text{core}} = \mu((1 \otimes \theta) r_{21}) + \frac{1}{4} \sum_{\lambda \in \Sigma} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) z_\lambda + \frac{1}{4} \sum_{\lambda \in \Sigma} \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \text{mtp}(\lambda) t_\lambda \]

with \( r \in (\mathcal{R} \otimes (\mathfrak{g} \otimes \mathfrak{g})^M)_W \) given by (5.10). Furthermore, \((\id \otimes \theta) r_{21} = (\theta \otimes \id) r\).

Proof. The alternative expressions for \( r^\pm \) follow by a straightforward computation (the derivation is similar to the last computation in the proof of Theorem 6.3). Proposition 5.7 implies that

\[ \kappa_{\text{core}} := \frac{1}{2} \sum_{\lambda \in \Sigma} \left( 1 + \frac{1}{2} \xi_{2\lambda} \right) z_\lambda - \frac{1}{2} \Omega_m \]

(6.8)

\[ + \frac{1}{2} \sum_{j=1}^{r} x_j^2 + \frac{1}{4} \sum_{\lambda \in \Sigma} \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \text{mtp}(\lambda) t_\lambda - \sum_{\lambda \in \Sigma} \frac{\mu((\id \otimes \theta) (\varpi_\lambda))}{1 - \xi_{2\lambda}}, \]

from which the alternative expression for \( \kappa_{\text{core}} \) easily follows. The symmetry \((\id \otimes \theta) r_{21} = (\theta \otimes \id) r\) follows from the fact that \((\theta \otimes \id) (\varpi_\lambda) \in S^2(\mathfrak{g})\) for \( \lambda \in \Sigma \), cf. Remark 5.4. \( \Box \)

Combined with (6.5) we conclude that

\[ \hat{\kappa}_{\text{core}} = \mu((1 \otimes \theta) r_{21}) + \frac{1}{4} \sum_{\lambda \in \Sigma} \left( \frac{1 + \xi_{2\lambda}}{1 - \xi_{2\lambda}} \right) z_\lambda. \]

(6.9)

Remark 6.11. For real split \( G \) an explanation of the folded structure of \( r^\pm \) and \( \kappa_{\text{core}} \), as described by Proposition 6.10, is given in [20, §6] using quantum field theoretic arguments and the theory of \( n \)-point spherical functions.

6.2. Coupled classical dynamical Yang-Baxter-reflection equations. As a consequence of the commutativity of the \( M \)-restricted asymptotic boundary KZB operators (cf. Corollary 6.4) we obtain coupled classical dynamical Yang-Baxter-reflection equations for its local factors \( r^\pm \) and \( \hat{\kappa} \). To formulate these equations, we define the classical dynamical reflection term \( \mathcal{A} \in \mathcal{R} \otimes (U(\mathfrak{t}) \otimes U(\mathfrak{g})^{\otimes 2} \otimes U(\mathfrak{t}))^M \) by

\[ \mathcal{A} := [\hat{\kappa}_1 + r_{12}, \hat{\kappa}_2 + r_{12}^+] - \sum_{j=1}^{r} ((x_j)_{11} \partial_{x_j} (\hat{\kappa}_2 + r_{12}^+) - (x_j)_{22} \partial_{x_j} (\hat{\kappa}_1 + r_{12}^-)), \]

(6.10)

with the sublabels numbering the two \( U(\mathfrak{g}) \) tensor legs within \( U(\mathfrak{t}) \otimes U(\mathfrak{g})^{\otimes 2} \otimes U(\mathfrak{t}) \), and we define mixed classical dynamical Yang-Baxter terms \( \text{MYB}(s) \in \mathcal{R} \otimes (U(\mathfrak{g})^{\otimes 3})^M \) (\( 1 \leq s \leq 3 \))
by
\[
\text{MYB}(1) := [r_{12}^+, r_{13}^+] + [r_{12}^+, r_{23}^+] - [r_{13}^+, r_{23}^-] - \sum_{j=1}^{r} ((x_j)_2 \partial_{x_j} (r_{13}^+) - (x_j)_3 \partial_{x_j} (r_{12}^+)),
\]
(6.11) \[
\text{MYB}(2) := [r_{12}^-, r_{13}^-] + [r_{12}^-, r_{23}^-] + [r_{13}^-, r_{23}^-] - \sum_{j=1}^{r} ((x_j)_1 \partial_{x_j} (r_{23}^-) - (x_j)_3 \partial_{x_j} (r_{12}^-)),
\]
(6.12) \[
\text{MYB}(3) := -[r_{12}^+, r_{13}^-] + [r_{12}^-, r_{23}^+] + [r_{13}^-, r_{23}^-] - \sum_{j=1}^{r} ((x_j)_1 \partial_{x_j} (r_{23}^+) - (x_j)_3 \partial_{x_j} (r_{13}^-)).
\]

In the following we will write
\[
\mathcal{A}_{ij} := (\sigma_i \otimes \tau_i \otimes \sigma^*_j)(\mathcal{A}), \quad 1 \leq i < j \leq n,
\]
\[
\text{MYB}(s)_{ijk} := (\tau_i \otimes \tau_j \otimes \tau_k)(\text{MYB}(s)), \quad 1 \leq i < j < k \leq n,
\]
which we view as element in \( \mathcal{R} \otimes \text{End}_M((V_i \otimes U \otimes V^*_r)^m) \) (the number \( n \) of \( G \)-representations in the tensor product representation \( U \) will always be clear from the context).

**Theorem 6.12.** For \( n \geq 2 \) and \( 1 \leq i < j \leq n \),

\[
(6.12) \quad (\mathcal{A}_{ij} + \sum_{k=1}^{i-1} \text{MYB}(1)_{kij} + \sum_{k=i+1}^{j-1} \text{MYB}(2)_{ikj} + \sum_{k=j+1}^{n} \text{MYB}(3)_{ijk})|_{(V_i \otimes U \otimes V^*_r)^M} = 0.
\]

The same holds true when \( \hat{\kappa} \) in \( \mathcal{A} \) is replaced by \( \kappa \).

**Proof.** By a lengthy computation one shows for \( 1 \leq i < j \leq n \),

\[
[\tilde{D}_i^{\tau_r} \mathcal{L}^{\sigma_r}, \tilde{D}_j^{\sigma_r} \mathcal{L}^{\sigma_r}] = \sum_{s=1}^{r} \left( [\tau_j(x_s), r_{ij}^-] - [\tau_i(x_s), r_{ij}^+] \right) \partial_{x_s}
\]
(6.13)
\[
+ \mathcal{A}_{ij} + \sum_{k=1}^{i-1} \text{MYB}(1)_{kij} + \sum_{k=i+1}^{j-1} \text{MYB}(2)_{ikj} + \sum_{k=j+1}^{n} \text{MYB}(3)_{ijk}
\]

in \( \mathcal{D}_{M}((V_i \otimes U \otimes V^*_r)^m) \). Using the explicit expression for \( r^\pm \) one verifies that

\[
[1 \otimes h, r^-] = [h \otimes 1, r^+] \quad \forall h \in \mathfrak{a},
\]

hence the first order term in the right-hand side of (6.13) vanishes and we are left with

\[
(6.14) \quad [\tilde{D}_i^{\tau_r} \mathcal{L}^{\sigma_r}, \tilde{D}_j^{\sigma_r} \mathcal{L}^{\sigma_r}] = \mathcal{A}_{ij} + \sum_{k=1}^{i-1} \text{MYB}(1)_{kij} + \sum_{k=i+1}^{j-1} \text{MYB}(2)_{ikj} + \sum_{k=j+1}^{n} \text{MYB}(3)_{ijk}
\]

in \( \mathcal{D}_{M}((V_i \otimes U \otimes V^*_r)^m) \). The result now follows since the left-hand side vanishes when acting on \( C^\infty(\mathcal{A}_{\text{reg}}; (V_i \otimes U \otimes V^*_r)^M) \) by Corollary 6.4.

Finally, replacing the role of \( \tilde{D}_i^{\tau_r} \mathcal{L}^{\sigma_r} \) by \( D_i^{\tau_r} \mathcal{L}^{\sigma_r} \) gives the result when \( \hat{\kappa} \) in \( \mathcal{A} \) is replaced by \( \kappa \). \( \square \)

For \( n = 2 \) we obtain the following special case of Theorem 6.12.
Corollary 6.13. The classical dynamical reflection term $\mathcal{A}$ (see (6.10)) acts as zero on $(V_\ell \otimes U_1 \otimes U_2 \otimes V^*_r)^M$,

\begin{equation}
\mathcal{A}|_{(V_\ell \otimes U_1 \otimes U_2 \otimes V^*_r)^M} = 0.
\end{equation}

The equation (6.15) are called the classical dynamical reflection equation for $\hat{\kappa}$ relative to the pair $(r^+, r^-)$.

For $n = 3$ we have

Corollary 6.14. The classical dynamical reflection term $\mathcal{A}$ (see (6.10)) and the mixed classical dynamical Yang-Baxter terms $\text{MYB}(s) \in \mathcal{R} \otimes (U(\mathfrak{g})^\otimes 3)^M$ (see (6.11)) satisfy for $1 \leq i < j \leq 3$,

\begin{equation}
(\mathcal{A}_{ij} + \text{MYB}(k))|_{(V_\ell \otimes U_1 \otimes U_2 \otimes U_3 \otimes V^*_r)^M} = 0
\end{equation}

where $\{k\} = \{1, 2, 3\} \setminus \{i, j\}$ and $\text{MYB}(k)$ acts in the natural way on the tensor components $U_1 \otimes U_2 \otimes U_3$ within $(V_\ell \otimes U_1 \otimes U_2 \otimes U_3 \otimes V^*_r)^M$.

The three equations (6.16) are called the coupled classical dynamical Yang-Baxter-reflection equations for the triple $(r^+, r^-, \hat{\kappa})$.

The classical dynamical reflection type equation (6.15) does not guarantee that $\mathcal{A}_{ij}$ in (6.16) vanishes when acting on $(V_\ell \otimes U_1 \otimes U_2 \otimes U_3 \otimes V^*_r)^M$. Hence in general the reflection term in the coupled classical dynamical Yang-Baxter-reflection equations (6.16) cannot be decoupled from the Yang-Baxter type terms.

Definition 6.15. We call the equations (6.12) for $n > 3$ the higher order coupled classical dynamical Yang-Baxter-reflection equations for the triple $(r^+, r^-, \hat{\kappa})$.

Remark 6.16. We expect that more generally,

\begin{equation}
\mathcal{A}_{ij} + \sum_{k=1}^{i-1} \text{MYB}(1)_{kij} + \sum_{k=i+1}^{j-1} \text{MYB}(2)_{ikj} + \sum_{k=j+1}^{n} \text{MYB}(3)_{ijk}
\end{equation}

vanishes when acting on $(V_\ell \otimes U \otimes V^*_r)^m$ for all $n \geq 2$. This is the case when $G$ is real split, see [20, Thm. 6.26] and Remark 4.19. In the real split case we have $m = 0$ and hence we conclude that in this case (6.17) vanishes identically on $V_\ell \otimes U \otimes V^*_r$ for all $n \geq 2$. These equations decouple, and are equivalent to the equations

\begin{equation}
\begin{aligned}
\mathcal{A}_{12} = 0 & \quad \text{in } \mathcal{R} \otimes \text{End}(V_\ell \otimes U_1 \otimes U_2 \otimes V^*_r), \\
\text{MYB}(k) = 0 & \quad \text{in } \mathcal{R} \otimes \text{End}(V_\ell \otimes U_1 \otimes U_2 \otimes U_3 \otimes V^*_r) \quad \text{for } k = 1, 2, 3,
\end{aligned}
\end{equation}

which are the classical dynamical reflection equation for $\hat{\kappa}$ relative to $(r^+, r^-)$ and the mixed classical dynamical Yang-Baxter equation for $(r^+, r^-)$ from [20, Thm. 6.31].

7. Example: $G = SU(p, r)$.

Write $e^{(m)}_{ij} \in \text{GL}(m; \mathbb{C})$ for the matrix unit with a one at entry $(i, j)$ and zeros everywhere else. Write $I_m \in \text{GL}(m; \mathbb{C})$ for the unit matrix, and $I_m^- := \sum_{i=1}^{m} e^{(m)}_{i,m+1-i} \in \text{GL}(m; \mathbb{C})$ for
the antidiagonal \( m \times m \)-matrix with ones on the antidiagonal. Fix \( 1 \leq r \leq p \) and write
\[
C := \begin{pmatrix}
\frac{1}{\sqrt{2}}I_r & 0 & -\frac{1}{\sqrt{2}}I_r \\
0 & I_{p-r} & 0 \\
-\frac{1}{\sqrt{2}}I_r & 0 & \frac{1}{\sqrt{2}}I_r
\end{pmatrix} \in \text{GL}(p + r; \mathbb{C})
\]
with 0 the zero matrix of the appropriate size, and consider \( G := \{ C^{-1}gC \mid g \in \text{SU}(p, r) \} \).
Then
\[
g_\mathbb{R} = \{ X \in \mathfrak{sl}(p + r; \mathbb{C}) \mid X^*J_{p,r} + J_{p,r}X = 0 \}
\]
with \( X^* = \overline{X}^T \) the adjoint of \( X \) and
\[
J_{p,r} := 
\begin{pmatrix}
0 & 0 & -I_r^{-} \\
0 & I_{p-r} & 0 \\
-I_r^{-} & 0 & 0
\end{pmatrix}.
\]
The root system \( R \subset \mathfrak{h}^* \) of \( \mathfrak{g} = \mathfrak{sl}(p + r; \mathbb{C}) \) is \( R = \{ \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq p + r \} \) with \( \epsilon_i(h) := \lambda_i \) for \( h = \sum_{j=1}^{p+r} \lambda_j e_{jj}^{(p+r)} \in \mathfrak{h} \). Take \( R^+ = \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq p + r \} \) as the set of positive roots. We will write use the shorthand notation \( \alpha_{ij} := \epsilon_i - \epsilon_j \) for the roots. Write \( \alpha_i := \alpha_{i,i+1} \) (\( 1 \leq i \leq p + q - 1 \)) for the associated basis elements of \( R \), then the involution \( '\theta \in \text{Aut}(R) \) is determined by
\[
\theta(\alpha_i) = \begin{cases} 
-\alpha_{i'} & (1 \leq i \leq r), \\
\alpha_i & (r + 1 \leq i < p), \\
-\alpha_{i'} & (p \leq i < p + r - 1),
\end{cases}
\]
which in turn determines the Satake diagram of \( G \).
Write \( \mathfrak{a}_R^* = \bigoplus_{j=1}^r \mathbb{R}f_j \) with \( f_j \in \mathfrak{a}_R^* \) defined by \( f_j(e_{ij}^{(p+r)} - e_{j'i'}^{(p+r)}) := \delta_{ij} \). The restricted root system \( \Sigma \) is a root system of type \( BC_r \) with the positive restricted roots given by
\[
\Sigma^+ = \{ f_i \pm f_j \}_{1 \leq i < j \leq r} \cup \{ f_j, 2f_j \}_{1 \leq j \leq r}.
\]
In fact, for \( 1 \leq i < j \leq r \) and \( 1 \leq k \leq r \) we have
\[
R_0 = \{ \alpha_{\ell m} \}_{r+1 \leq \ell \neq m \leq p}, \\
R_{f_i-f_j} = \{ \alpha_{ij}, \alpha_{j'i'} \}, \\
R_{f_i+f_j} = \{ \alpha_{ij'}, \alpha_{ij} \}, \\
R_{f_k} = \{ \alpha_{kk'} \}_{r+1 \leq \ell \leq p}, \\
R_{2f_k} = \{ \alpha_{kk'} \},
\]
(for \( r = 1 \) there are no restricted roots of the form \( f_i \pm f_j \) \( (i \neq j) \), and hence the middle line drops out). Write \( \xi_i := \xi_{f_i} \) for the multiplicative character of \( A \) associated to \( f_i \in \mathfrak{a}_R^* \).
The dependence on \( A \) of the local factors \( r^\pm, \kappa \) of the asymptotic boundary KZB operators
\[ \tilde{D}_{i:M}^{\sigma_i \xi^r} (1 \leq i \leq n), \text{ as well as of the potential } V^{\sigma_i \xi^r} \text{ of the Schrödinger operator } \tilde{H}^{\sigma_i \xi^r}_{M}, \]

are then described in terms of the multiplicative characters \( \xi_i \xi^r_j (1 \leq i < j \leq r) \) and \( \xi_k (1 \leq k \leq r) \), which are the characters associated to the indivisible positive restricted roots of \( \Sigma \). The equivariance of the operators is with respect to the hyperoctahedral group \( W \simeq S_r \rtimes (\pm 1)^r \).

Recall that \( \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{q} \) with \( \mathfrak{q} = \bigoplus_{\alpha \in \mathfrak{R}^+ \setminus \mathfrak{R}_0^+} \mathbb{C} y_\alpha \). To further describe the local factors of \( \tilde{D}_{i:M}^{\sigma_i \xi^r} \) and the potential \( V^{\sigma_i \xi^r} \) of \( \tilde{H}^{\sigma_i \xi^r}_{M} \) one needs the explicit description of \( \mathfrak{m} \) and of the standard basis elements \( y_\alpha (\alpha \in \mathfrak{R}^+ \setminus \mathfrak{R}_0^+) \) of \( \mathfrak{q} \). We have

\[ \mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{r+1 \leq i \neq j \leq p} \mathbb{C} e_{ij}^{(p+r)} \]

with \( \mathfrak{t} \) the \((+1)\)-eigenspace of \( \theta|_{\mathfrak{h}} \), which is explicitly described by

\[ \mathfrak{t} = \{ \sum_{i=1}^{r} \lambda_i (e_{ii}^{(p+r)} + e_{ii'}^{(p+r)}) + \sum_{j=r+1}^{p} \mu_j e_{jj}^{(p+r)} : 2 \sum_{i=1}^{r} \lambda_i + \sum_{j=r+1}^{p} \mu_j = 0 \} . \]

The elements \( y_\alpha (\alpha \in \mathfrak{R}^+ \setminus \mathfrak{R}_0^+) \) are explicitly given by

\[
\begin{align*}
    y_{\alpha_{ij}} &= e_{ij}^{(p+r)} + e_{i'i}^{(p+r)}, \\
    y_{\alpha_{jj'}} &= e_{jj}^{(p+r)} + e_{jj'}^{(p+r)}, \\
    y_{\alpha_{kk'}} &= e_{kk}^{(p+r)} + e_{kk'}^{(p+r)}, \\
    y_{\alpha_{kl}} &= e_{kl}^{(p+r)} + e_{kl'}^{(p+r)}, \\
    y_{\alpha_{kk'}} &= e_{kk}^{(p+r)} + e_{kk'}^{(p+r)},
\end{align*}
\]

for \( 1 \leq i < j \leq r, 1 \leq k \leq r \) and \( r+1 \leq \ell \leq p \) (the 1st-4th line describes \( y_\alpha \) for the roots \( \alpha \in \mathfrak{R}_1^i - \mathfrak{R}_j^i, \alpha \in \mathfrak{R}_1^i + \mathfrak{R}_j^i, \alpha \in \mathfrak{R}_k^i \) and \( \alpha \in \mathfrak{R}_k^i \) respectively).

Finally, since \( \mathfrak{s}(\mathfrak{gl}(p) \times \mathfrak{gl}(r)) = \text{Ad}(C) \mathfrak{g} \) the basis elements of \( \mathfrak{g} \) can be expressed in terms of the standard basis elements of \( \mathfrak{s}(\mathfrak{gl}(p) \times \mathfrak{gl}(r)) \), and vice versa. For instance, the center of \( \mathfrak{t} \) is spanned by

\[ \text{Ad}(C^{-1}) \begin{pmatrix} \frac{1}{p} I_p & 0 \\ 0 & -\frac{1}{r} I_r \end{pmatrix} = \begin{pmatrix} \left( \frac{1}{2p} - \frac{1}{2r} \right) I_r & 0 & -\left( \frac{1}{2p} + \frac{1}{2r} \right) I_r^- \\ 0 & \frac{1}{2r} I_{p-r} & 0 \\ -\left( \frac{1}{2p} + \frac{1}{2r} \right) I_r^- & 0 & \left( \frac{1}{2p} - \frac{1}{2r} \right) I_r \end{pmatrix}, \]

which can be expressed as

\[ \left( \frac{1}{2p} - \frac{1}{2r} \right) \sum_{i=1}^{r} (e_{ii}^{(p+r)} + e_{ii'}^{(p+r)}) + \frac{1}{p} \sum_{j=r+1}^{p} e_{jj}^{(p+r)} \right) - \left( \frac{1}{2p} + \frac{1}{2r} \right) \sum_{i=1}^{r} y_{\alpha_{ii'}} , \]

where the first term lies in \( \mathfrak{t} \) and the second term lies in \( \mathfrak{q} \).

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N.R.: Department of Mathematics, University of California, Berkeley, CA 94720, USA & ITMO University, Kronverskii Ave. 49, Saint Petersburg, 197101, Russia & KdV Institute for Mathematics, University of Amsterdam, Science Park 105-107, 1098 XG Amsterdam, The Netherlands

Email address: reshetik@math.berkeley.edu

J.S.: KdV Institute for Mathematics, University of Amsterdam, Science Park 105-107, 1098 XG Amsterdam, The Netherlands

Email address: j.v.stokman@uva.nl