Loop-erased random walk, uniform spanning forests and bi-Laplacian Gaussian field in the critical dimension

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Abstract

For $N = n(\log n)^{1/4}$, on the ball in $\mathbb{Z}^4$ with radius $N$ we construct a $\pm 1$ spin model coupled with the wired spanning forests. We show that as $n \to \infty$ the spin field has bi-Laplacian Gaussian field on $\mathbb{R}^4$ as its scaling limit. The precise fluctuation magnitude of the field is $\sqrt{3 \log N}$. In order to prove such results, we show that the escape probability of the loop-erased random walk (LERW) renormalized by $(\log n)^{1/3}$ converges almost surely and in $L^p$ for all $p > 0$. This improves the known mean field picture of 4D LERW. As an application, we provide a construction of the bi-infinite LERW, thus solving a question of Benjamini-Lyons-Peres-Schramm in the case of $\mathbb{Z}^4$.

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1 Introduction

This paper connects three natural objects that have four as the critical dimension: the bi-Laplacian Gaussian field, the uniform spanning trees (UST), and the loop-erased random walk (LERW). We will construct a sequence of random fields on the integer lattice $\mathbb{Z}^d$ ($d \geq 4$) using UST and show that they converge in distribution to the bi-Laplacian field (Theorem 1.1). The core of the analysis is based on fine estimates of the non-intersection probability of a LERW and simple random walks (SRW) on

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Our next main result, Theorem 3.1, establishing the asymptotics of these intersection probabilities, thus completes the mean field picture of LERW in dimension four. Moreover, using Theorem 3.1 we are able to sample the two sided LERW in \( \mathbb{Z}^4 \) (see Section 4), which can be viewed as a stationary ergodic process embedded in \( \mathbb{Z}^4 \).

### 1.1 A spin field from USF

We will now describe the construction of the random field on (a rescaling of) \( \mathbb{Z}^d \). For each positive integer \( n \), let \( N = N_n = n(\log n)^{1/4} \). Let \( A_N = \{ x \in \mathbb{Z}^d : |x| \leq N \} \). We will construct a \( \pm 1 \) valued random field on \( A_N \) as follows. Recall that a **wired spanning tree** on \( A_N \) is a tree on the graph \( A_N \cup \{ \partial A_N \} \) where we have viewed the boundary \( \partial A_N \) as “wired” to a single point. Such a tree produces a **spanning forest** on \( A_N \) by removing the edges connected to \( \partial A_N \). We define the uniform spanning forest (USF) on \( A_N \) to be the forest obtained by choosing the wired spanning tree of \( A_N \cup \{ \partial A_N \} \) from the uniform distribution. (Note this is not the same thing as choosing a spanning forest uniformly among all spanning forests of \( A_N \).) The field is then defined as follows. Let \( a_n \) be a sequence of positive numbers (we will be more precise later).

- Choose a USF on \( A_N \). This partitions \( A_N \) into (connected) components.
- For each component of the forest, flip a fair coin and assign each vertex in the component value \( 1 \) or \( -1 \) based on the outcome. This gives a field of spins \( \{ Y_{x,n} : x \in A_N \} \). If we wish we can extend this to a field on \( x \in \mathbb{Z}^d \) by setting \( Y_{x,n} = 0 \) for \( x \not\in A_N \).
- Let \( \phi_n(x) = a_n Y_{nx,n} \) which is a field defined on \( L_n := n^{-1} \mathbb{Z}^d \).

This random function is constructed in a manner similar to the Edward-Sokal coupling of the FK-Ising model [8]. The content of our main theorem which we now state is that for \( d \geq 4 \), we can choose \( a_n \) such that \( \phi_n \) converges to the bi-Laplacian Gaussian field on \( \mathbb{Z}^d \). If \( h \in C_0^\infty(\mathbb{R}^d) \), we write

\[
\langle h, \phi_n \rangle = n^{-d/2} \sum_{x \in L_n} h(x) \phi_n(x).
\]

**Theorem 1.1.**

- If \( d \geq 5 \), there exists \( a > 0 \) such that if \( a_n = a n^{(d-4)/2} \), then for every \( h_1, \ldots, h_m \in C_0^\infty(\mathbb{R}^d) \), the random variables \( \langle h_j, \phi_n \rangle \) converge in distribution to a centered joint Gaussian random variable with covariance

\[
\int \int h_j(z) h_k(w) |z - w|^{4-d} \, dz \, dw.
\]
If $d = 4$, if $a_n = \sqrt{3 \log n}$, then for every $h_1, \ldots, h_m \in C_0^\infty(\mathbb{R}^d)$ with
\[ \int h_j(z) \, dz = 0, \quad j = 1, \ldots, m, \]
the random variables $\langle h_j, \phi_n \rangle$ converge in distribution to a centered Gaussian random variable with variance
\[ - \int \int h_j(z) h_k(w) \log |z - w| \, dz \, dw. \]

Remark 1.2.

- Gaussian fields on $\mathbb{R}^d$ with correlations as above is called $d$-dimensional bi-Laplacian Gaussian field. [19]
- For $d = 4$, we could choose the cutoff $N = n(\log n)^\alpha$ for any $\alpha > 0$. We choose $\alpha = \frac{1}{4}$ for concreteness. For $d > 4$, we could do the same construction with no cutoff ($N = \infty$) and get the same result.
- We will prove the result for $m = 1, h_1 = h$. The general result follows by applying the $k = 1$ result to any linear combination of $h_1, \ldots, h_m$.
- We will do the proof only for $d = 4$; the $d > 4$ case is similar but easier (see [24] for the $d \geq 5$ case done in detail). In the $d = 4$ proof, we will have a slightly different definition of $a_n$ but we will show that $a_n \sim \sqrt{3 \log n}$.

Our main result uses strongly the fact that the scaling limit of the loop-erased random walk in four or more dimensions is Brownian motion, that is, has Gaussian limits [17, 16]. Although we do not use explicitly the results in those papers, they are used implicitly in our calculations where the probability that a random walk and a loop-erased walk avoid each other is comparable to that of two simple random walks and can be given by the expected number of intersections times a non-random quantity. This expected number of intersections is the discrete biharmonic function that gives the covariance structure of the bi-Laplacian random field.

Gaussian fluctuations have been observed and studied for numerous physical systems. Many statistical physics models with long correlations are known to converge to the Gaussian free field. Typical examples come from domino tilings [10], random matrix theory [5, 21] and random growth models [4]. Our model can be viewed as a statistical mechanics model whose critical dimension is four, in the sense that the correlation functions are scale invariant when $d = 4$. Such Gaussian limits were also expected, but only partially proved for Ising and $\Phi^4$ models on $\mathbb{Z}^d$ ($d \geq 4$), which is at or above their critical dimensions [7]. (This is sometimes referred to as quantum triviality.) Theorem 1.1 confirms the quantum triviality for our spin model.
The discrete bi-Laplacian Gaussian field (in physics literature, this is known as the membrane model) is studied in [22, 11, 12], whose continuous counterpart is the bi-Laplacian Gaussian field. Our model can be viewed as another natural discrete object that converges to the bi-Laplacian Gaussian field. In one dimensional case, Hammond and Sheffield constructed a reinforced random walk with long range memory [9], which can be associated with a spanning forest attached to \( \mathbb{Z} \). Our construction can also be viewed as a higher dimensional analogue of “forest random walks”.

1.2 Intersection probabilities for loop-erased walk

By Wilson’s algorithm of sampling USF (see Section 2.1), it will be clear later that the hardest part of the proof of Theorem 1.1 is the sharp asymptotics of the intersection probability of a SRW and a LERW for \( d = 4 \), which gives our second main result. As stated here, this is a combination of Theorem 3.1, Proposition 4.1, and Remark 4.2.

**Theorem 1.3.** Let \( S, S', S'' \) be four independent simple random walks on \( \mathbb{Z}^4 \) starting from the origin and \( \eta = LE[W[0, \infty)] \). As a sequence of random variables which are measurable with respect to \( \eta \),

\[
X_n = (\log n)^{\frac{1}{3}} \mathbb{P}\{S[1, n^2] \cap \eta = \emptyset | \eta\}
\]

converge to a nontrivial random variable \( X_\infty \) almost surely and in \( L^p \) for all \( p > 0 \). Moreover,

\[
\lim_{n \to \infty} (\log n) \mathbb{P}\{(S[1, n^2] \cup S'[1, n^2]) \cap \eta = \emptyset, S''[0, n^2] \cap \eta = \{0\}\} = \frac{1}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{24}.
\]

(1)

\( X_\infty \) can be thought of as the renormalized escape probability of 4D LERW. In [16], building on the work on slowly recurrent sets [15], the first author proved that \( \mathbb{E}[X_n^p] \asymp 1 \) for all \( p > 0 \). In this paper we perform a finer analysis based on slowly recurrent sets (Section 3.2) and special properties of 4D LERW to establish the limit.

The explicit constant in (1) is obtained by a first passage path decomposition of the intersection of a SRW and a LERW. Combined with the asymptotic result, this gives the exact evaluation (see Section 3.7.2). The constant \( \frac{1}{3} \) is universal and is the reciprocal of the number of SRW’s involved (three in this case). However \( \pi^2/8 \) comes from the bi-harmonic Green function of \( \mathbb{Z}^4 \) evaluated at \( (0, 0) \), thus is lattice-dependent.

To see the role of Theorem 1.3 in Theorem 1.1, we note that the two-point correlation of the field depends on the probability

\[
\mathbb{P}\{LE(S^x) \cap S^y = \emptyset\}.
\]

where \( S^x, S^y \) are two independent random walks started from \( x, y \in A_N \) and stopped at \( \partial A_N \). By the first passage decomposition mentioned above, this probability is asymptotically equal to the product of the Green function of \( A_N \) evaluated at \( x, y \) times the
non-intersection probability in (1) (Lemma 2.4). Then (1) together with the Green function estimate (Lemma 2.2) gives the convergence of the variance in Theorem 1.1. The Gaussian fluctuation follows from the moment method that is carried out in Section 2.3.

We finally remark that our proof strategy should apply to the escape probability of the range of 4D SRW. In this case, we believe that a similar but even slightly simpler (since one does not have to deal with the fact that loop-erasures of subpaths are not always the subpaths of loop-erasures) argument as in our Section 3 gives

\[
\lim_{n \to \infty} (\log n)^{\frac{1}{2}} \mathbb{P}\{S[1, n^2] \cap W[0, n^2] = \emptyset | W\} \text{ exists almost surely and in } L^p(p > 0).
\]

\[
\lim_{n \to \infty} (\log n)\mathbb{P}\{S[1, n^2] \cap W[0, n^2] = \emptyset, S'[0, n^2] \cap W[1, n^2] = \emptyset\} = \frac{1}{2} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{16}.
\]

This would also be an improvement of [15].

To be more speculative, we could also consider the 4D self-avoiding walk (SAW) which is defined as the limit of the measure on paths obtained from the uniform distribution on all SAW of length \(n\) starting at the origin. This limit is not known to exist. But assume that it exists and denote the path by \(Y[0, \infty)\). Then we expect that \(Y[0, \infty)\) is a slowly recurrent set that is “thinner” than \(W[0, \infty)\) but “thicker” than \(LE(W[0, \infty))\). We are far from proving anything here, but we conjecture that

\[
\mathbb{P}\{S[1, n^2] \cap Y[0, \infty) = \emptyset\} \sim c (\log n)^{-3/8},
\]

and if \(\tilde{Y}\) is another independent infinite SAW,

\[
\mathbb{P}\{\tilde{Y}[1, n^2] \cap Y[0, \infty) = \emptyset\} \sim c (\log n)^{-1/4}.
\]

1.3 Two sided LERW

In [3], Benjamini-Lyons-Peres-Schramm raised the following conjecture:

Conjecture 1.4 (Conjecture 15.12 in [3]). Let \(T_0\) be the component of the origin in the USF on \(\mathbb{Z}^d\), and let \(\xi\) be the unique (infinite) ray from 0 in \(T_0\). Then the sequence of “bushes” observed along \(\xi\) converges in distribution.

Since USF can be sampled from LERW via Wilson’s algorithm, this conjecture can be reduced to the following: as the base point moving along \(\xi\) (which is an infinite one sided LERW), the law of \(\xi\) seen from the based point converges in distribution. The limiting distribution, if it exists, should be given by the two sided LERW.

The existence of the two sided LERW when \(d \geq 5\) is proved in [13] based on the existence of global cut points of the simple random walk. As an application of results proved in Section 3 we show in Section 4 that for \(d = 4\), the two sided LERW measure exists, and can be obtained as the limit of a finite size LERW measure.

As shown in Section 4, \(X_\infty\) in Theorem 1.3 is the Radon-Nikodym derivative between the two sided LERW restricted to non-negative times and the usual LERW. The
existence of the two-sided LERW in $\mathbb{Z}^2$ can be deduced from the result in [1]. A big difference in $d = 2$ compared to $d \geq 4$ case is that the marginal distribution of one side of the path is not absolutely continuous with respect to the usual LERW. We do not believe that the existence of the two-sided LERW has been established in three dimensions. We expect that it exists and its marginal distribution will not be absolutely continuous with respect to the one-sided LERW.

2 Gaussian limits for the spin field

In this section, we start by reviewing some known facts of UST and random walk Green’s function, then proving Theorem 1.1 by applying main estimates of LERW. The proof of these estimates will be deferred to Section 3.

2.1 Uniform spanning trees

Here we review some facts about the uniform spanning forest (that is, wired spanning trees) on finite subsets of $\mathbb{Z}^d$ and loop-erased random walks (LERW) on $\mathbb{Z}^d$. Most of the facts extend to general graphs as well. For more details, see [18, Chapter 9].

Given a finite subset $A \subset \mathbb{Z}^d$, the uniform wired spanning tree in $A$ is a subgraph of the graph $A \cup \{\partial A\}$, choosing uniformly random among all spanning trees of $A \cup \{\partial A\}$. (A spanning tree $T$ is a subgraph such that any two vertices in $T$ are connected by a unique simple path in $T$). We define the uniform spanning forest (USF) on $A$ to be the uniform wired spanning tree restricted to the edges in $A$. One can also consider the uniform spanning forest on all of $\mathbb{Z}^d$ [20, 3], but we will not need this construction.

The uniform wired spanning tree, and hence the USF, on $A$ can be generated by Wilson’s algorithm [25] which we recall here:

- Order the elements of $A = \{x_1, \ldots, x_k\}$.

- Start a simple random walk at $x_1$ and stop it when in reaches $\partial A$ giving a nearest neighbor path $\omega$. Erase the loops chronologically to produce a self-avoiding path $\eta = LE(\omega)$. Add all the edges of $\eta$ to the tree which now gives a tree $T_1$ on a subset of $A \cup \{\partial A\}$ that includes $\partial A$.

- Choose the vertex of smallest index that has not been included and run a simple random walk until it reaches a vertex in $T_1$. Erase the loops and add the new edges to $T_1$ in order to produce a tree $T_2$.

- Continue until all vertices are included in the tree.

Wilson’s theorem states that the distribution of the tree is independent of the order in which the vertices were chosen and is uniform among all spanning trees. In particular we get the following.
If \( x, y \in A \), let \( S^x, S^y \) be two independent SRWs starting from \( x, y \) respectively. Then the probability that \( x, y \) are in the same component of the USF equals to
\[
P\{LE(\omega^x) \cap \omega^y = \emptyset \}.
\] (2)

Using this characterization, we can see the three regimes for the dimension \( d \). Let us first consider the probabilities that neighboring points are in the same component. Let \( q_N \) be the probability that a nearest neighbor of the origin is in a different component as the origin when \( A = A_N \). Then
\[
q_\infty := \lim_{N \to \infty} q_N > 0, \quad d \geq 5
\]

\[
q_N \approx (\log N)^{-1/3}, \quad d = 4.
\]

For \( d < 4 \), \( q_N \) decays like a power of \( N \). For far away points, we have

- If \( d > 4 \), and \( |x| = n \), the probability that 0 and \( x \) are in the same component is comparable to \( |x|^{1-d} \). This is true even if \( N = \infty \).

- If \( d = 4 \) and \( |x| = n \), the probability that 0 and \( x \) are in the same component is comparable to \( 1/\log n \). However, if we chose \( N = \infty \), the probability would equal to one.

The last fact can be used to show that the uniform spanning tree in all of \( \mathbb{Z}^4 \) is, in fact, a tree. For \( d < 4 \), the probability that 0 and \( x \) are in the same component is asymptotic to 1 and our construction is not interesting. This is why we restrict to \( d \geq 4 \).

### 2.2 Estimates for Green’s functions

Let \( G_N(x, y) \) denote the usual random walk Green’s function on \( A_N \), and let
\[
G^2_N(x, y) = \sum_{z \in \mathbb{Z}^d} G(x, z) G(z, y),
\]
\[
G^2_N(x, y) = \sum_{z \in A_N} G_N(x, z) G_N(z, y),
\]
\[
\tilde{G}^2_N(x, y) = \sum_{z \in A_N} G_N(x, z) G(z, y),
\]

Note that
\[
G^2_N(x, y) \leq \tilde{G}^2_N(x, y) \leq G^2(x, y).
\]

Also, \( G^2(x, y) < \infty \) if and only if \( d > 4 \).

The Green’s function \( G(x, y) \) is well understood for \( d > 2 \). We will use the following asymptotic estimate [18, Theorem 4.3.1]:

\[ \quad \]
\[ G(x) = C_d |x|^{2-d} + O(|x|^{-d}), \quad \text{where} \quad C_d = \frac{d \Gamma(d/2)}{(d-2) \pi^{d/2}}. \]  
(3)

(Here and throughout we use the convention that if we say that a function on \( \mathbb{Z}^d \) is \( O(|x|)^{-r} \) with \( r > 0 \), we still imply that it is finite at every point. In other words, for lattice functions, \( O(|x|^{-r}) \) really means \( O(1 \wedge |x|^{-r}) \). We do not make this assumption for functions on \( \mathbb{R}^d \) which could blow up at the origin.) It follows immediately, and will be more convenient for us, that

\[ G(x) = C_d I_d(x) \left[ 1 + O(|x|^{-2}) \right], \]
(4)

where we write

\[ I_d(x) = \int_V \frac{d^d \zeta}{|\zeta - x|^{d-2}}, \]

and \( V = V_d = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d : |x_j| \leq 1/2 \} \) denotes the \( d \)-dimensional cube of side length one centered at the origin. One aesthetic advantage is that \( I_d(0) \) is not equal to infinity.

**Proposition 2.1.** If \( d \geq 5 \), there exists \( c_d \) such that

\[ G^2(x, y) = c_d |x-y|^{4-d} + O \left( |x-y|^{2-d} \right). \]

Moreover, there exists \( c < \infty \) such that if \( |x|, |y| \leq N/2 \), then

\[ G^2(x, y) - G_N^2(x, y) \leq c N^{4-d}. \]

We note that it follows immediately that

\[ G^2(x, y) = c_d J_d(x-y) \left[ 1 + O \left( |x-y|^{-2} \right) \right], \]

where

\[ J_d(x) = \int_V \frac{d^d \zeta}{|\zeta - x|^{d-4}}. \]

The second assertion shows that it does not matter in the limit whether we use \( G^2(x, y) \) or \( G_N^2(x, y) \) as our covariance matrix.

**Proof.** We may assume \( y = 0 \). We use (4) to write

\[
G^2(0, x) = C_d^2 \sum_{z \in \mathbb{Z}^d} I_d(z) I_d(z-x) \left[ 1 + O(|z|^{-2}) + O(|z-x|^{-2}) \right]
\]

\[ = C_d^2 \int_V \int_V \frac{d\zeta_1 d\zeta_2}{|\zeta_1 - z|^{d-2} |\zeta_2 - (z-x)|^{d-2}} + O(|x|^{2-d}) \]

\[ = C_d^2 |x|^{4-d} \int \int \frac{d\zeta_1 d\zeta_2}{|\zeta_1|^{d-2} |\zeta_2 - u|^{d-2}} + O(|x|^{2-d}), \]

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where \( u \) denotes a unit vector in \( \mathbb{R}^d \). This gives the first expression with

\[
c_d = C_d^2 \int \int \frac{d\zeta_1 \, d\zeta_2}{|\zeta_1|^{d-2} |\zeta_1 - u|^{d-2}}.
\]

For the second assertion, Let \( S^x, S^y \) be independent simple random walks starting at \( x, y \) stopped at times \( T^x, T^y \), the first time that they reach \( \partial A_N \). Then,

\[
G^2(x, y) - G^2_N(x, y) \leq \mathbb{E} \left[ G^2(S^x(T^x), y) \right] + \mathbb{E} \left[ G^2(x, S^y(T^y)) \right] ,
\]

and we can apply the first result to the right-hand side. \( \square \)

**Lemma 2.2.** If \( d = 4 \), there exists \( c_0 \in \mathbb{R} \) such that if \( |x|, |y|, |x - y| \leq N/2 \), then

\[
G^2_N(x, y) = \frac{8}{\pi^2} \log \left[ \frac{N}{|x - y|} \right] + c_0 + O \left( \frac{|x| + |y| + 1}{N} + \frac{1}{|x - y|} \right).
\]

**Proof.** Using the martingale \( M_t = |S_t|^2 - t \), we see that

\[
\sum_{w \in A_N} G_N(x, w) = N^2 - |x|^2 + O(N). \tag{5}
\]

Using (4), we see that

\[
\sum_{w \in A_N} G_N(0, w) = O(N) + \frac{2}{\pi^2} \int_{|s| \leq N} \frac{d^4 s}{|s|^2} = 2N^2 + O(N).
\]

Let \( \delta = N^{-1}[1 + |x| + |y|] \) and note that \( |x - y| \leq \delta N \). Since

\[
\sum_{|w| < N(1 - \delta)} G_N(x - y, w) \, G_N(0, w) \leq \sum_{w \in A_N} G_N(x, w) \, G_N(y, w)
\]

\[
\leq \sum_{|w| \leq N(1 + \delta)} G_N(x - y, w) \, G_N(0, w),
\]

it suffices to prove the result when \( y = 0 \) in which case \( \delta = (1 + |x|)/N \). Recall that

\[
\min_{N \leq |z| \leq N + 1} G(x, z) \leq G(x, w) - G_N(x, w) \leq \max_{N \leq |z| \leq N + 1} G(x, z).
\]

In particular,

\[
G_N(0, w) = G(0, w) - \frac{2}{\pi^2 N^2} + O(N^{-3}),
\]

\[
G_N(x, w) = G(x, w) - \frac{2}{\pi^2 N^2} + O(\delta N^{-2}).
\]
Using (5), we see that
\[
\sum_{w} G_N(x, w) G_N(0, w) = \sum_{w \in A_N} \left[ G(x, w) - \frac{2}{\pi^2 N^2} + O(\delta N^{-2}) \right] G_N(0, w) \\
= O(\delta) - \frac{2}{\pi^2} + \sum_{w \in A_N} G(x, w) G_N(0, w).
\]

Similarly, we write,
\[
\sum_{w \in A_N} G(x, w) G_N(0, w) = \sum_{w \in A_N} G(x, w) \left[ G(0, w) - \frac{2}{\pi^2 N^2} + O(N^{-3}) \right],
\]
and use (5) to see that
\[
\sum_{w \in A_N} G(x, w) G_N(0, w) = -\frac{2}{\pi^2} + O(\delta^2) + \sum_{w \in A_N} G(x, w) G(0, w).
\]

Hence it suffices to show that there exists \(c'\) such that
\[
\sum_{w \in A_N} G(x, w) G(0, w) = \log(\frac{N}{|x|}) + c' + O(|x|^{-1}) + O(\delta).
\]

Define \(\epsilon(x, w)\) by
\[
\epsilon(x, w) = G(x, w) G(0, w) - \left( \frac{2}{\pi^2} \right)^2 \int_{S_w} \frac{d^4s}{|x - s|^2 |s|^2},
\]
where \(S_w\) is the unit cube centered at \(w\). The estimate for the Green’s function implies that
\[
|\epsilon(x, w)| \leq \begin{cases} 
  c |w|^{-3} |x|^{-2}, & |w| \leq |x|/2 \\
  c |w|^{-2} |x - w|^{-3}, & |x - w| \leq |x|/2 \\
  c |w|^{-5}, & \text{other } w,
\end{cases}
\]
and hence,
\[
\sum_{w \in A_N} G(x, w) G(0, w) = O(|x|^{-1}) + \sum_{w \in A_N} \left( \frac{2}{\pi^2} \right)^2 \int_{S_w} \frac{d^4s}{|x - s|^2 |s|^2} \\
= O(|x|^{-1}) + \left( \frac{2}{\pi^2} \right)^2 \int_{|s| \leq N} \frac{d^4s}{|x - s|^2 |s|^2}.
\]
(The second equality had an error term of \(O(N^{-1})\) but this is smaller than the \(O(|x|^{-1})\) term already there.) It is straightforward to check that there exists \(c'\) such that
\[
\int_{|s| \leq N} \frac{d^4s}{|x - s|^2 |s|^2} = 2\pi^2 \log \left( \frac{N}{|x|} \right) + c' + O\left( \frac{|x|}{N} \right).
\]
2.3 Proof of Theorem 1.1

Here we give the proof of the theorem leaving the proof of two lemmas, Lemma 2.3 and 2.4 for Section 3. The proofs for the $d = 4$ and $d > 4$ cases are similar with a few added difficulties for $d = 4$. We will only consider the $d = 4$ case here. We fix $h \in C_0^\infty$ with $\int h = 0$ and allow implicit constants to depend on $h$. We will write just $\langle h, \phi_n \rangle$ for $\langle h, \phi_{n,x} \rangle$.

Let $L_n = n^{-1} \mathbb{Z}^d \cap \{|x| \leq K\}$ and

$$\langle h, \phi_n \rangle = n^{-2} a_n \sum_{x \in L_n} h(x) Y_{nx} = n^{-2} a_n \sum_{nx \in A_n K} h(x) Y_{nx},$$

where $a_n$ is still to be specified. We will now give the value. Let $\omega$ be a LERW from 0 to $\partial A_N$ in $A_N$ and let $\eta = LE(\omega)$ be its loop-erasure. Let $\omega_1$ be another simple random walk starting at the origin stopped when it reaches $\partial A_N$ and let $\omega'_1$ be $\omega_1$ with the initial step removed. Let $u(\eta) = \mathbb{P}\{\omega_1 \cap \eta = \{0\} \mid \eta\}$, $\tilde{u}(\eta) = \mathbb{P}\{\omega'_1 \cap \eta = \emptyset \mid \eta\}$.

We define

$$b_n = \mathbb{E}\left[u(\eta) \tilde{u}(\eta)^2\right].$$

In [16] it was shown that $b_n \asymp 1/\log n$ if $d = 4$. One of the main goals of Section 3 is to give the following improvement, which is an immediate consequence of Theorem 3.1.

Lemma 2.3. If $d = 4$ and $b_n$ is defined as above, then

$$b_n \sim \frac{\pi^2}{24 \log n}.$$

We can write the right-hand side as

$$\frac{1}{(8/\pi^2)} \frac{1}{3 \log n},$$

where $8/\pi^2$ is the nonuniversal constant in Lemma 2.2 and $1/3$ is a universal constant for loop-erased walks. If $d \geq 5$, then $b_n \sim c n^{4-d}$. We do not prove this here.

For $d = 4$, we define

$$a_n = \sqrt{\pi^2/(8b_n)} \sim \sqrt{3 \log n}.$$

Let $q_N(x, y)$ be the probability that $x, y$ are in the same component of the USF on $A_N$, and note that

$$\mathbb{E}[Y_{x,n} Y_{y,n}] = q_N(x, y),$$

$$\mathbb{E}[\langle h, \phi_n \rangle^2] = n^{-4} \sum_{x \in L_n} h(x) h(y) a_n^2 q_N(nx, ny).$$

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An upper bound for \( q_N(x, y) \) can be given in terms of the probability that the paths of two independent random walks starting at \( x, y \), stopped when they leave \( A_N \), intersect. This gives
\[
q_N(x, y) \leq \frac{c \log[N/|x - y|]}{\log N}.
\]
In particular,
\[
q_N(x, y) \leq c \frac{(\log \log n)^2}{\log n}, \quad |x - y| \geq n \delta_n.
\]

To estimate \( \mathbb{E}[\langle h, \phi_n \rangle^2] \) we will use the following lemma, which will be shown as a consequence of Proposition 3.28 in Section 3.

**Lemma 2.4.** There exists a sequence \( r_n \) with \( r_n \leq O(\log \log n) \), a sequence \( \epsilon_n \downarrow 0 \), such that if \( x, y \in L_n \) with \( |x - y| \geq 1/\sqrt{n} \),
\[
| a_n^2 q_N(nx, ny) - r_n + \log |x - y| | \leq \epsilon_n.
\]

It follows from Lemma 2.4, (6), and the trivial inequality \( q_N \leq 1 \), that
\[
\mathbb{E}[\langle h, \phi_n \rangle^2] = o(1) + n^{-4} \sum_{x, y \in L_n} [r_n - \log |x - y|] h(x) h(y)
= o(1) - n^{-4} \sum_{x, y \in L_n} \log |x - y| h(x) h(y)
= o(1) - \int h(x) h(y) \log |x - y| dx dy,
\]
which shows that the second moment has the correct limit. The second equality uses \( \int h = 0 \) to conclude that
\[
\frac{r_n}{n^2} \sum_{x, y \in L_n} h(x) h(y) = o(1).
\]

We now consider the higher moments. It is immediate from the construction that the odd moments of \( \langle h, \phi_n \rangle \) are identically zero, so it suffices to consider the even moments \( \mathbb{E}[\langle h, \phi_n \rangle^{2k}] \). Let \( \delta_n = \exp\{-(\log \log n)^2\} \); this is a function that decays faster than any power of \( \log n \). We fix \( k \geq 1 \) and allow implicit constants to depend on \( k \) as well. Let \( L_s = L_{n,s}^k \) be the set of \( \bar{x} = (x_1, \ldots, x_{2k}) \in L_{n,s}^k \) such that \( |x_j| \leq K \) for all \( j \) and \( |x_i - x_j| \geq \delta_n \) for each \( i \neq j \). We write \( h(\bar{x}) = h(x_1) \ldots h(x_{2k}) \).

Note that \( \#L_s \asymp n^{8k} \) and \( \#(L_{2k}^k \setminus L_s) \asymp k^2 n^{8k} \delta_n \). In particular,
\[
n^{-8k} a_n^{2k} \sum_{\bar{x} \in L_{n,s}^k} h(x_1) h(x_2) \cdots h(x_{2k}) = o_{2k}(\sqrt{\delta_n}).
\]

Then we see that
\[
\mathbb{E}[\langle h, \phi_n \rangle^{2k}] = n^{-8k} a_n^{2k} \sum_{\bar{x} \in L_{2k}^k} h(\bar{x}) \mathbb{E}[Y_{nx_1} \cdots Y_{nx_{2k}}]
= O(\sqrt{\delta_n}) + n^{-8k} a_n^{2k} \sum_{\bar{x} \in L_s} h(\bar{x}) \mathbb{E}[Y_{nx_1} \cdots Y_{nx_{2k}}].
\]
**Lemma 2.5.** For each $k$, there exists $c < \infty$ such that the following holds. Suppose $x \in I_{2^k}^{n,s}$ and let $\omega^1, \ldots, \omega^{2k}$ be independent simple random walks started at $nx_1, \ldots, nx_{2k}$ stopped when they reach $\partial A_N$. Let $N$ denote the number of integers $j \in \{2, 3, \ldots, 2k\}$ such that

$$\omega^j \cap (\omega^1 \cup \cdots \cup \omega^{j-1}) \neq \emptyset.$$ 

Then,

$$\mathbb{P}\{N \geq k + 1\} \leq c \left[ \frac{(\log \log n)^3}{\log n} \right]^{k+1}.$$

Conditioned on Lemma 2.5 we now prove Theorem 1.1 by verifying Wick’s formula. We write $y_j = nx_j$ and write $Y_j$ for $Y_{y_j}$. To calculate $\mathbb{E}[Y_1 \cdots Y_{2k}]$ we first sample our USF which gives a random partition $\mathcal{P}$ of $\{y_1, \ldots, y_{2k}\}$. Note that $\mathbb{E}[Y_1 \cdots Y_{2k} | \mathcal{P}]$ equals 1 if it is an “even” partition in the sense that each set has an even number of elements. Otherwise, $\mathbb{E}[Y_1 \cdots Y_{2k} | \mathcal{P}] = 0$. Any even partition, other than a partition into $k$ sets of cardinality 2, will have $N \geq k + 1$. Hence

$$\mathbb{E}[Y_1 \cdots Y_{2k}] = O \left( \left[ \frac{(\log \log n)^3}{\log n} \right]^{k+1} \right) + \sum \mathbb{P}(\mathcal{P}_y),$$

where the sum is over the $(2k-1)!!$ perfect matchings of $\{1, 2, \ldots, 2k\}$ and $\mathbb{P}(\mathcal{P}_y)$ denotes the probability of getting this matching for the USF for the vertices $y_1, \ldots, y_{2k}$.

Let us consider one of these perfect matchings that for convenience we will assume is $y_1 \leftrightarrow y_2, y_3 \leftrightarrow y_4, \ldots, y_{2k-1} \leftrightarrow y_{2k}$. We claim that

$$\mathbb{P}(y_1 \leftrightarrow y_2, y_3 \leftrightarrow y_4, \ldots, y_{2k-1} \leftrightarrow y_{2k}) = O \left( \left[ \frac{(\log \log n)^3}{\log n} \right]^{k+1} \right) + \mathbb{P}(y_1 \leftrightarrow y_2) \mathbb{P}(y_3 \leftrightarrow y_4) \cdots \mathbb{P}(y_{2k-1} \leftrightarrow y_{2k}).$$

Indeed, this is just inclusion-exclusion using our estimate on $\mathbb{P}\{N \geq k + 1\}$.

If we write $\epsilon_n = \epsilon_{n,k} = (\log \log n)^{3(k+1)} / \log n$, we now see from symmetry that

$$\mathbb{E} \left[ \langle h, \phi_n \rangle^{2k} \right] = O(\epsilon_n) + n^{-8k}a_n (2k-1)!! \sum_{x \in L_s} \mathbb{P}\{nx_1 \leftrightarrow nx_2, \ldots, nx_{2k-1} \leftrightarrow nx_{2k}\} = O(\epsilon_n) + (2k-1)!! \left[ \mathbb{E} \left( \langle h, \phi_n \rangle^2 \right) \right]^k.$$

### 2.4 Proof of Lemma 2.5

Here we fix $k$ and let $y_1, \ldots, y_{2k}$ be points with $|y_j| \leq Kn$ and $|y_i - y_j| \geq n \delta_n$, where we recall $\log \delta_n = -(\log \log n)^2$. Let $\omega^1, \ldots, \omega^{2k}$ be independent simple random walks starting at $y_j$ stopped when they get to $\partial A_N$. We let $E_{i,j}$ denote the event that $\omega^i \cap \omega^j \neq \emptyset$, and let $R_{i,j} = \mathbb{P}(E_{i,j} | \omega^j)$.
Lemma 2.6. There exists $c < \infty$ such that for all $i, j$, and all $n$ sufficiently large,

$$
\mathbb{P}\left\{ R_{i,j} \geq c \frac{(\log \log n)^3}{\log n} \right\} \leq \frac{1}{(\log n)^{4k}}.
$$

Proof. We know that there exists $c < \infty$ such that if $|y - z| \geq n \delta_n^2$, then the probability that simple random walks starting at $y, z$ stopped when they reach $\partial A_N$ intersect is at most $c(\log \log n)^2/\log n$. Hence there exists $c_1$ such that

$$
\mathbb{P}\left\{ R_{i,j} \leq c_1 \frac{(\log \log n)^2}{\log n} \right\} \geq \frac{1}{2}. \quad (7)
$$

Start a random walk at $z$ and run it until one of three things happens:

- It reaches $\partial A_N$
- It gets within distance $n \delta_n^2$ of $y$
- The path is such that the probability that a simple random walk starting at $y$ intersects the path before reaching $\partial A_N$ is greater than $c_1(\log \log n)^2/\log n$.

If the third option occurs, then we restart the walk at the current site and do this operation again. Eventually one of the first two options will occur. Suppose it takes $r$ trials of this process until one of the first two events occur. Then either $R_{i,j} \leq rc_1(\log \log n)^2/\log n$ or the original path starting at $z$ gets within distance $\delta_n^2$ of $y$.

The latter event occurs with probability $O(\delta_n) = o((\log n)^{-4k})$. Also, using (7), we can see the probability that it took at least $r$ steps is bounded by $(1/2)^r$. By choosing $r = c_2 \log \log n$, we can make this probability less than $1/(\log n)^{4k}$. \qed

Proof of Lemma 2.5. Let $R$ be the maximum of $R_{i,j}$ over all $i \neq j$ in $\{1, \ldots, 2k\}$. Then, at least for $n$ sufficiently large,

$$
\mathbb{P}\left\{ R \geq c \frac{(\log \log n)^3}{\log n} \right\} \leq \frac{1}{(\log n)^{3k}}.
$$

Let

$$
E_{i,j} = \bigcup_{i=1}^{j-1} E_{i,j}.
$$

On the event $R < c(\log \log n)^3/\log n$, we have

$$
\mathbb{P}\left\{ E_{i,j} \mid \omega^1, \ldots, \omega^{j-1} \right\} \leq \frac{c(j-1)(\log \log n)^3}{\log n}.
$$

If $N$ denotes the number of $j$ for which $E_{i,j}$ occurs, we see that

$$
\mathbb{P}\{N \geq k + 1\} \leq c \left[ \frac{(\log \log n)^3}{\log n} \right]^{k+1}.
$$

\qed
3 Loop-erased walk in $\mathbb{Z}^4$

The purpose of this section is to improve results about the escape probability for loop-erased walk in four dimensions.

We start by setting up our notation. Let $S$ be a simple random walk in $\mathbb{Z}^4$ defined on the probability space $(\Omega, \mathcal{F}, P)$ starting at the origin. Let $G(x, y)$ be the Green’s function for simple random walk which we recall satisfies

$$G(x) = \frac{2}{\pi^2 |x|^2} + O(|x|^{-4}), \quad |x| \to \infty.$$  

Using this we see that there exists a constant $\lambda$ such that as $r \to \infty$,

$$\sum_{|x| \leq r} G(x)^2 = \frac{8}{\pi^2} \log r + \lambda + O(r^{-1}). \quad (8)$$

It will be easiest to work on geometric scales. For $m < n$ we let $C_n, A(m, n)$ be the discrete balls and annuli defined by

$$C_n = \{ z \in \mathbb{Z}^4 : |z| < e^n \}, \quad A(m, n) = C_n \setminus C_m = \{ z \in C_n : |z| \geq e^m \}.$$  

Let

$$\sigma_n = \min \{ j \geq 0 : S_j \not\in C_n \},$$

and let $\mathcal{F}_n$ denote the $\sigma$-algebra generated by $\{ S_j : j \leq \sigma_n \}$. If $V \subset \mathbb{Z}^4$, we write

$$H(x, V) = \mathbb{P}^x \{ S[0, \infty) \cap V \neq \emptyset \}, \quad H(V) = H(0, V), \quad \operatorname{Es}(V) = 1 - H(V),$$

$$\overline{H}(x, V) = \mathbb{P}^x \{ S[1, \infty) \cap V \neq \emptyset \}, \quad \overline{\operatorname{Es}}(V) = 1 - \overline{H}(0, V).$$

Note that $\overline{\operatorname{Es}}(V) = \operatorname{Es}(V)$ if $0 \not\in V$ and otherwise a standard last-exit decomposition shows that

$$\operatorname{Es}(V^0) = G_{\mathbb{Z}^4 \setminus V^0}(0, 0) \overline{\operatorname{Es}}(V),$$

where $V^0 = V \setminus \{0\}$ and $G_{\mathbb{Z}^4 \setminus V^0}$ is the Green function on the graph $\mathbb{Z}^4 \setminus V^0$.

We have to be a little careful about the definition of the loop-erasures of the random walk and loop-erasures of subpaths of the walk. We will use the following notations.

- $\hat{S}[0, \infty)$ denotes the (forward) loop-erasure of $S[0, \infty)$ and

$$\Gamma = \hat{S}[1, \infty) = \hat{S}[0, \infty) \setminus \{0\}.$$  

- $\omega_n$ denotes the finite random walk path $S[\sigma_n - 1, \sigma_n]$

- $\eta^n = LE(\omega_n)$ denotes the loop-erasure of $S[\sigma_n - 1, \sigma_n]$.

- $\Gamma_n = LE(S[0, \sigma_n]) \setminus \{0\}$, that is, $\Gamma_n$ is the loop-erasure of $S[0, \sigma_n]$ with the origin removed.
Note that $S[1, \infty)$ is the concatenation of the paths $\omega_1, \omega_2, \ldots$. However, it is not true that $\Gamma$ is the concatenation of $\eta^1, \eta^2, \ldots$, and that is one of the technical issues that must be addressed in the proof.

Let $Y_n, Z_n$ be the $F_n$-measurable random variables

$$Y_n = H(\eta^n), \quad Z_n = \Es[\Gamma_n], \quad G_n = G_{Z^4 \setminus \Gamma_n}(0, 0).$$

As noted above,

$$\Es(\Gamma_n \cup \{0\}) = G_n^{-1} Z_n.$$  

It is easy to see that $1 \leq G_n \leq 8$, and using transience, we can see that with probability one

$$\lim_{n \to \infty} G_n = G_\infty := G_{Z^4 \setminus \Gamma}(0, 0).$$

**Theorem 3.1.** For every $0 \leq r, s < \infty$, there exists $c_{r,s} < \infty$, such that

$$\Es[Z_r^n G_n^{-s}] \sim \frac{c_{r,s}}{n^{r/3}}.$$  

Moreover, $c_{3,2} = \pi^2/24$.

Our methods do not compute the constant $c_{r,s}$ except in the case case $r = 3, s = 2$.

The proof of this theorem requires several steps which we will outline now. For the remainder of this paper we fix $r > 0$ and allow constants to depend on $r$. We write

$$p_n = \Es[Z^n_r], \quad \hat{p}_n = \Es[Z_n^3 G_n^{-2}].$$

Here are the steps in order with a reference to the section where the argument appears. Let

$$\phi_n = \prod_{j=1}^{n} \exp \{-\Es[H(\eta^j)]\}.$$

- **Section 3.1.** Show that $\Es[H(\eta^n)] = O(n^{-1})$, and hence,

$$\phi_n = \phi_{n-1} \exp\{-\Es[H(\eta^n)]\} = \phi_{n-1} \left[1 + O\left(\frac{1}{n}\right)\right].$$

- **Section 3.2.** Show that

$$p_{n+1} = p_n \left[1 + O\left(\frac{\log^4 n}{n}\right)\right]. \quad (9)$$

- **Section 3.4.** Find a function $\tilde{\phi}_n$ such that there exists $\tilde{c}_r$ with

$$p_n^r = \tilde{c}_r \tilde{\phi}_n^r \left[1 + O\left(\frac{\log^2 n}{n}\right)\right].$$
• **Section 3.5.** Find $\hat{c}, u > 0$ such that
  \[
  \tilde{\varphi}_n = \hat{c} \phi_n^{1 + O(n^{-u})}.
  \]

  Combining the previous estimates we see that there exist $c', u > 0$ such that
  \[
  \tilde{p}_n = c' \phi^{1 + O(n^{-u})}.
  \]

• **Section 3.6.** Show that there exists $c', u > 0$ such that
  \[
  \mathbb{E} \left[ Z_n^r G^{-s}_n \right] = c' \phi^{1 + O(n^{-u})}.
  \]

• **Section 3.7.** Use a path decomposition to show that
  \[
  \mathbb{E} \left[ H(\eta^n) \right] = \frac{8}{\pi^2} \hat{p}_n \left[ 1 + O(n^{-u}) \right],
  \]
  and combine the last two estimates to conclude that
  \[
  \hat{p}_n \sim \frac{\pi^2}{24n}.
  \]

  The error estimates are probably not optimal, but they suffice for proving our theorem. We say that a sequence $\{\epsilon_n\}$ of positive numbers is *fast decaying* if it decays faster than every power of $n$, that is
  \[
  \lim_{n \to \infty} n^k \epsilon_n = 0,
  \]
  for every $k > 0$. We will write $\{\epsilon_n\}$ for fast decaying sequences. As is the convention for constants, the exact value of $\{\epsilon_n\}$ may change from line to line. We will use implicitly the fact that if $\{\epsilon_n\}$ is fast decaying then so is $\{\epsilon'_n\}$ where
  \[
  \epsilon'_n = \sum_{m \geq n} \epsilon_m.
  \]

### 3.1 Preliminaries

In this subsection we prove some necessary lemmas about simple random walk in $\mathbb{Z}^4$. Most results will be frequently used in the rest of the paper except that Lemmas 3.7 and 3.8 will only be used in Section 3.5, thus can be skipped at the first reading. We first recall the following facts about intersections of random walks in $\mathbb{Z}^4$ (see [14, 15]).

**Proposition 3.2.** There exist $0 < c_1 < c_2 < \infty$ such that the following is true. Suppose $S$ is a simple random walk starting at the origin in $\mathbb{Z}^4$ and $\alpha \geq 2$. Then
  \[
  \frac{c_1}{\sqrt{\log n}} \leq \mathbb{P} \{ S[0, n] \cap S[n + 1, \infty] = \emptyset \} \leq \mathbb{P} \{ S[0, n] \cap S[n + 1, 2n] = \emptyset \} \leq \frac{c_2}{\sqrt{\log n}},
  \]
  
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\[
c_1 \frac{\log \alpha}{\log n} \leq \mathbb{P} \{ S[0, n] \cap S[n(1 + \alpha^{-1}), \infty) \neq \emptyset \} \leq c_2 \frac{\log \alpha}{\log n}.
\]
Moreover, if \( S^1 \) is an independent simple random walk starting at \( z \in \mathbb{Z}^4 \),
\[
\mathbb{P} \{ S[0, n] \cap S^1[0, \infty) \neq \emptyset \} \leq c_2 \frac{\log a}{\log n},
\]
where
\[
a = \max \left\{ 2, \frac{\sqrt{n}}{|z|} \right\}.
\]
An important corollary of the proposition is that
\[
\sup_n n \mathbb{E} [H(\eta^n)] \leq \sup_n n \mathbb{E} [H(\omega_n)] < \infty,
\]
and hence
\[
\exp \{-\mathbb{E} [H(\eta^n)]\} = 1 - \mathbb{E} [H(\eta^n)] + O(n^{-2}),
\]
and if \( m < n \),
\[
\phi_n = \phi_m \left[ 1 + O(m^{-1}) \right] \prod_{j=m+1}^{n} \left[ 1 - \mathbb{E} [H(\eta^j)] \right].
\]

**Corollary 3.3.** There exists \( c < \infty \) such that if \( n > 0, \alpha \geq 2 \) and we let \( m = m_{n, \alpha} = (1 + \alpha^{-1}) n \),
\[
Y = Y_{n, \alpha} = \max_{j \geq m} H(S_j, S[0, n]),
\]
\[
Y' = Y'_{n, \alpha} = \max_{0 \leq j \leq n} H(S_j, S[m, 2n]),
\]
then for every \( 0 < u < 1 \),
\[
\mathbb{P} \left\{ Y \geq \frac{\log \alpha}{(\log n)^u} \right\} \leq \frac{c}{(\log n)^{1-u}}, \quad \mathbb{P} \left\{ Y' \geq \frac{\log \alpha}{(\log n)^u} \right\} \leq \frac{c}{(\log n)^{1-u}}.
\]

**Proof.** Let
\[
\tau = \min \left\{ j \geq m : H(S_j, S[0, n]) \geq \frac{\log \alpha}{(\log n)^u} \right\}.
\]
The strong Markov property implies that
\[
\mathbb{P} \{ S[0, n] \cap S[m, \infty) \neq \emptyset | \tau < \infty \} \geq \frac{\log \alpha}{(\log n)^u},
\]
and hence Proposition 3.2 implies that
\[
\mathbb{P} \{ \tau < \infty \} \leq \frac{c_2}{(\log n)^{1-u}}.
\]
This gives the first inequality and the second can be done similarly by looking at the reversed path.
Lemma 3.4. There exists $c < \infty$ such that for all $n$:

- if $\phi$ is a positive (discrete) harmonic function on $C_n$ and $x \in C_{n-1}$, 
  \[
  |\log[\phi(x)/\phi(0)]| \leq c |x| e^{-n}. \tag{12}
  \]

- if $m < n$, 
  \[
  P\{S[\sigma_n, \infty) \cap C_{n-m} \neq \emptyset | F_n\} \leq c e^{-2m}; \tag{13}
  \]

- If $m < n$ and $V \subset C_{n-m}$, then 
  \[
  P\{S[\sigma_n, \infty) \cap V \neq \emptyset | F_n\} \leq c e^{-2m} H(V). \tag{14}
  \]

Proof. The inequalities (12) and (13) are standard estimates, see, e.g., [18, Theorem 6.3.8, Proposition 6.4.2]. To get (14), we first use (13) to see that if $w \in \partial C_n$,

\[
P^w \{S[\sigma_n, \infty) \cap C_{n-m+1} \neq \emptyset\} = O(e^{-2m}).
\]

We then invoke the Harnack principle to see that for $z \in \partial C_{n-m+1}$,

\[
H(z, V) \leq c H(V). \tag*{\blacksquare}
\]

Lemma 3.5. Let $U_n$ be the event that there exists $k \geq \sigma_n$ with

\[
LE(S[0, k]) \cap C_{n-\log^2 n} \neq \emptyset
\]

Then $P(U_n)$ is fast decaying.

Proof. By the loop-erasing process, we can see that the event $U_n$ is contained in the event that either

\[
S[\sigma_n - \frac{1}{2} \log^2 n, \infty) \cap C_{n-\log^2 n} \neq \emptyset
\]

or

\[
S[\sigma_n, \infty) \cap C_{n-\frac{1}{2} \log^2 n} \neq \emptyset.
\]

The probability that either of these happens is fast decaying by (13). \tag*{\blacksquare}

Proposition 3.6. If $\Lambda(m, n) = S[0, \infty) \cap A(m, n)$, then the sequences

\[
P\left\{ H[\Lambda(n - 1, n)] \geq \frac{\log^2 n}{n} \right\}, \quad P\left\{ H(\omega_n) \geq \frac{\log^4 n}{n} \right\},
\]

are fast decaying.
Proof. Using Proposition 3.2 and Markov inequality we can see that there exists \( c < \infty \) such that for all \( |z| \geq e^{n-1} \),

\[
\mathbb{P}^z \left\{ H(S[0, \infty)) \geq \frac{c}{n} \right\} \leq \frac{1}{2}.
\]

Using this and the strong Markov property, we can by induction that for every positive integer \( k \),

\[
\mathbb{P} \left\{ H[\Lambda(n - 1, n)] \geq \frac{ck}{n} \right\} \leq 2^{-k}.
\]

Setting \( k = \lceil c^{-1} \log^2 n \rceil \), we see that the first sequence is fast decaying.

For the second, we use (13) to see that \( \mathbb{P} \{ \omega_n \not\subset A(n - \log^2 n, n) \} \) is fast decaying, and if \( \omega_n \subset A(n - \log^2 n, n) \),

\[
H(\omega_n) \leq \sum_{n-\log^2 n \leq j \leq n} H[\Lambda(j - 1, j)].
\]

Lemma 3.7. Let

\[
\sigma_n^- = \sigma_n - \lfloor n^{-1/4} e^{2n} \rfloor, \quad \sigma_n^+ = \sigma_n + \lfloor n^{-1/4} e^{2n} \rfloor,
\]

\[
S_n^- = S[0, \sigma_n^-], \quad S_n^+ = S[\sigma_n^+, \infty),
\]

\[
R = R_n = \max_{x \in S_n^-} H(x, S_n^+) + \max_{y \in S_n^+} H(y, S_n^-),
\]

Then, for all \( n \) sufficiently large,

\[
\mathbb{P}\{R_n \geq n^{-1/3}\} \leq n^{-1/3}.
\]

Our proof will actually give a stronger estimate, but (13) is all that we need and makes for a somewhat cleaner statement.

Proof. Using the fact that \( \mathbb{P}\{\sigma_n \leq (k + 1) e^{2n} \mid \sigma_n \geq k e^{2n}\} \) is bounded uniformly away from zero, we can see that there exists \( c_0 \) with

\[
\mathbb{P}\{\sigma_n \geq c_0 e^{2n} \log n\} \leq n^{-1}.
\]

Let \( N = N_n = \lceil c_0 e^{2n} \log n \rceil, k = k_n = \lfloor n^{-1/4} e^{2n}/4 \rfloor \) and let \( E_j = E_{j,n} \) be the event that either

\[
\max_{0 \leq i \leq jk} H(S_i, S[(j + 1)k, N]) \geq \log \frac{n}{n^{1/4}},
\]

or

\[
\max_{(j+1)k \leq i \leq N} H(S_i, S[0, jk]) \geq \log \frac{n}{n^{1/4}}.
\]

By Corollary 3.3 with \( u = \frac{1}{4} \), we see that \( \mathbb{P}(E_i) \leq O(n^{-3/4}) \) and hence

\[
\mathbb{P}\left[ \bigcup_{j,k \leq N} E_j \right] \leq O\left( \frac{\log n}{n^{1/2}} \right).
\]
The next lemma will use the capacity \( \text{cap}(V) \) of a subset \( V \subset \mathbb{Z}^4 \). We will not review the definition; the key fact that we will use is the following (see [18, Proposition 6.5.1]):

- If \( V \subset C_n \) and \( z \not\in C_n + 1 \), then
  \[
  H(z, V) \simeq \frac{\text{cap}(V)}{|z|^2}.
  \]  
  \hspace{1cm} (16)

**Lemma 3.8.** Let

\[
L[j, m] = \text{cap}(S[j, m]) \quad \text{and} \quad \bar{L}(n; k) = \max_{j \leq n} L[j, j + k].
\]

Then for every \( u < \infty \)

\[
\mathbb{P} \left\{ \bar{L}(n^u e^{2n}; n^{-1/4} e^{2n}) \geq 2 n^{-11/10} e^n \right\}
\]

is fast decaying.

**Proof.** Let \( k = \lceil n^{-1/4} e^{2n} \rceil \). Let \( U \) denote the event in (17). Note that

\[
U \subset \bigcup_{i=1}^{n^u+1} \left\{ L[ik, (i+1)k] \geq n^{-11/10} e^n \right\}.
\]

and since the events in the union are identically distributed,

\[
\mathbb{P}(U) \leq n^{u+1} \mathbb{P} \left\{ L[ik, (i+1)k] \geq n^{-11/10} e^n \right\}.
\]

Hence it suffices to show that

\[
\mathbb{P} \left\{ L[0, k] \geq n^{-11/10} e^n \right\} \text{ is fast decaying.}
\]

Using the estimate on intersection probability and Markov inequality, we can see that there exists \( c_0 \) such that

\[
\mathbb{P} \left\{ L[0, k] \geq c_0 n^{-9/8} e^n \right\} \leq \frac{1}{2},
\]

and hence by the Markov property and the subadditivity of capacity,

\[
\mathbb{P} \left\{ L[0, k] \geq c_0 m n^{-9/8} e^n \right\} \leq 2^{-m} \quad \forall m \in \mathbb{N}.
\]
3.2 Sets in $\mathcal{X}_n$

Simple random walk paths in $\mathbb{Z}^4$ are “slowly recurrent” sets in the terminology of [15]. In this section we will consider a subcollections $\mathcal{X}_n$ of the collection of slowly recurrent sets and give uniform bounds for escape probabilities for such sets. Throughout this section constants, explicit and implicit in $O(\cdot)$ and $o(\cdot)$ notations depend only on $n$.

Given a subset $V \subset \mathbb{Z}^4$ and $m \in \mathbb{N}$ we write

$$V_m = V \cap A(m - 1, m), \quad h_m = h_{m,V} = H(V_m).$$

Using the Harnack inequality, we can see that there exist $0 < c_1 < c_2 < \infty$ such that

$$c_1 h_m \leq H(z, V_m) \leq c_2 h_m, \quad \forall z \in C_{m-2} \cup A(m+1, m+2).$$

**Definition 3.9.** Let $\mathcal{X}_n$ denote the collection of subsets $V$ of $\mathbb{Z}^4$ such that for all integers $m \geq \sqrt{n}$,

$$H(V_m) \leq \frac{\log^2 m}{m} .$$

Note that $\mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots$. The following is an immediate corollary of Proposition 3.6.

**Proposition 3.10.** $\mathbb{P}\{S[0, \infty) \not\in \mathcal{X}_n\}$ is a fast decaying sequence.

Let $E_m$ denote the event

$$E_m = E_{m,V} = \{S[1, \sigma_m] \cap V = \emptyset\} .$$

We write $h_{m}(z)$ for the harmonic measure of $\partial C_m$ for random walk starting at the origin,

$$h_{m}(z) = \mathbb{P}\{S(\sigma_m) = z\}, \quad z \in \partial C_m .$$

If $V \subset \mathbb{Z}^4$ and $\mathbb{P}(E_m) > 0$, we write

$$h_{m}(z; V) = \mathbb{P}\{S(\sigma_m) = z \mid E_m\}, \quad z \in \partial C_m .$$

Note that by the strong Markov property

$$\mathbb{P}\{S[\sigma_m, \sigma_{m+1}] \cap V \neq \emptyset\} = \sum_{z \in \partial C_m} h_{m}(z) \mathbb{P}^{x}\{S[0, \sigma_{m+1}] \cap V \neq \emptyset\} ,$$

$$\mathbb{P}(E_{m+1} \mid E_m) = \mathbb{P}\{S[\sigma_m, \sigma_{m+1}] \cap V \neq \emptyset \mid E_m\} = \sum_{z \in \partial C_m} h_{m}(z; V) \mathbb{P}^{x}\{S[0, \sigma_{m+1}] \cap V \neq \emptyset\} .$$

**Proposition 3.11.** There exists $c < \infty$ such that if $V \in \mathcal{X}_n$, $m \geq n/10$, and $\mathbb{P}(E_{m+1} \mid E_m) \geq 1/2$, then

$$\mathbb{P}(E_{m+2} \mid E_{m+1}) \leq \frac{c \log^2 n}{n} .$$

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Proof. As above, we write
\[ P(E_{m+2}^c \mid E_{m+1}) = \sum_{z \in C_{m+1}} h_{m+1}(z) P^z \{ S[0, \sigma_{m+2}] \cap V \neq \emptyset \}. \]

Using \( P(E_{m+1} \mid E_m) \geq 1/2 \), we claim that \( h_{m+1}(z; V) \leq c h_{m+1}(z) \) for all \( z \in \partial C_{m+1} \). Indeed, we have
\[
h_{m+1}(z; V) = \frac{P\{ S(\sigma_{m+1}) = z, E_{m+1} \}}{P(E_{m+1})} \leq 2 \frac{P\{ S(\sigma_{m+1}) = z, E_{m+1} \}}{P(E_m)} \leq 2 P\{ S(\sigma_{m+1}) = z \mid E_m \},
\]
and the Harnack principle shows that
\[
P\{ S(\sigma_{m+1}) = z \mid E_m \} \leq \sup_{w \in \partial C_m} P^w \{ S(\sigma_{m+1}) = z \} \leq c P\{ S(\sigma_{m+1}) = z \}.
\]

Therefore,
\[
P(E_{m+2}^c \mid E_{m+1}) \leq c \sum_{z \in C_{m+1}} h_{m+1}(z) P^z \{ S[0, \sigma_{m+2}] \cap V \neq \emptyset \} = c P\{ S[\sigma_{m+1}, \sigma_{m+2}] \cap V \neq \emptyset \} \leq c \sum_{k=1}^{m+2} P\{ S[\sigma_{m+1}, \sigma_{m+2}] \cap V_k \neq \emptyset \}.
\]

Note that
\[
P\{ S[\sigma_{m+1}, \sigma_{m+2}] \cap (V_m \cup V_{m+1} \cup V_{m+2}) \neq \emptyset \} \leq H(V_m \cup V_{m+1} \cup V_{m+2}) \leq \frac{c \log^2 n}{n}.
\]

Using (13), we see that for \( \lambda \) large enough,
\[
P\{ S[\sigma_{m+1}, \sigma_{m+2}] \cap C_{m-\lambda \log m} \neq \emptyset \} \leq c n^{-2}.
\]

For \( m - \lambda \log m \leq k \leq m - 1 \), we estimate
\[
P\{ S[\sigma_{m+1}, \sigma_{m+2}] \cap V_k \neq \emptyset \} \leq P[E'] P\{ S[\sigma_{m+1}, \sigma_{m+2}] \cap V_k \neq \emptyset \mid E' \},
\]
where \( E' = \{ S[\sigma_{m+1}, \sigma_{m+2}] \cap C_{k+1} \neq \emptyset \} \). The first probability on the right-hand side is \( \exp\{ -O(m-k) \} \) and the second is \( O(\log^2 n/n) \). Summing over \( k \) we get the result. \( \square \)

Definition 3.12. Let \( \mathcal{X}_n \) denote the set of \( V \subset \mathcal{X}_n \) such that \( P(E_n) \geq 2^{-n/4} \).

The particular choice of \( 2^{-n/4} \) in this last definition is rather arbitrary but it is convenient to choose a particular fast decaying sequence. For typical sets in \( \mathcal{X}_n \) one expects that \( P(E_n) \) decays as a power in \( n \), so “most” sets in \( \mathcal{X}_n \) with \( P(E_n) > 0 \) will also be in \( \mathcal{X}_n \).
Proposition 3.13. There exists $c < \infty$ such that if $V \in \tilde{X}_n$, then

$$\mathbb{P}(E_{j+1}^c \mid E_j) \leq \frac{c \log^2 n}{n}, \quad \frac{3n}{4} \leq j \leq n.$$  

Proof. It suffices to consider $n$ sufficiently large. If $\mathbb{P}(E_{m+1} \mid E_m) < 1/2$ for all $n/4 \leq m \leq n/2$, then $\mathbb{P}(E_n) < (1/2)^{n/4}$ and $V \not\in \tilde{X}_n$. If $\mathbb{P}(E_{m+1} \mid E_m) \geq 1/2$ for some $n/4 \leq m \leq n/2$, then (for $n$ sufficiently large) we can use Proposition 3.11 and induction to conclude that $\mathbb{P}(E_{k+1} \mid E_k) \geq 1/2$ for $m \leq k \leq n$. The result then follows from Proposition 3.11.

It follows from this proposition that there exists $n_0$ (independent of $V$) such that for $n \geq n_0$, $\tilde{X}_n \subset \tilde{X}_{n+1}$. We fix the smallest such $n_0$ and set

$$\tilde{X} = \bigcup_{j=n_0}^{\infty} \tilde{X}_j.$$  

Combining the last two propositions we get

$$\mathbb{P}(E_{n+k}^c \mid E_n) \leq \frac{c k \log^2 n}{n}, \quad V \in \tilde{X}_n.$$  

The next proposition is the key to the analysis of slowly recurrent sets. It says that the distribution of the first visit to $\partial C_n$ given that one has avoided the set $V$ is very close to the unconditioned distribution. We would not expect this to be true for recurrent sets that are not slowly recurrent.

Proposition 3.14. There exists $c < \infty$ such that if $V \in \tilde{X}_n$ and $z \in \partial C_n$,

$$h_m(z; V) \leq h_m(z) \left[1 + \frac{c \log^3 n}{n}\right].$$  

Moreover,

$$\sum_{z \in \partial C_n} |h_m(z) - h_m(z; V)| \leq \frac{c \log^3 n}{n}.$$  

Proof. Let $k = \lfloor \log n \rfloor$. Applying Proposition 3.11 $k$ times, we can see that

$$\mathbb{P}(E_n^c \mid E_{n-k}) \leq \frac{c \log^3 n}{n}.$$  

Consider a random walk starting on $\partial C_{n-k}$ with the distribution $h_{n-k}(\cdot; V)$ and let $\nu$ denote the distribution of the first visit to $\partial C_n$. In other words, $\nu$ is the distribution of the first visit to $\partial C_n$ conditioned on the event $E_{n-k}$. Using Lemma 3.4, we see that for $z \in \partial C_n$,

$$\nu(z) = h_m(z) [1 + O(n^{-1})].$$  

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Using (21), we see that if \( z \in \partial C_n \),
\[
\begin{align*}
\text{hm}_n(z; V) &= \mathbb{P}\{S(\sigma_n) = z \mid E_n\} \leq \frac{\mathbb{P}(E_{n-k})}{\mathbb{P}(E_n)} \mathbb{P}\{S(\sigma_n) = z \mid E_{n-k}\} \\
&= \frac{\nu(z)}{1 - \mathbb{P}(E_n \mid E_{n-k})} \leq \text{hm}_n(z) \left[ 1 + O\left(\frac{\log^3 n}{n}\right)\right].
\end{align*}
\]
Since \( \text{hm}_n(\cdot) \) and \( \text{hm}_n(\cdot; V) \) are probability measures on \( \partial C_n \),
\[
\sum_{z \in \partial C_n} |\text{hm}_n(z) - \text{hm}_n(z; V)| = 2 \sum_{z \in \partial C_n} [\text{hm}_n(z; V) - \text{hm}_n(z)] +
\leq \frac{c \log^3 n}{n} \sum_{z \in \partial C_n} \text{hm}_n(z)
\leq \frac{c \log^3 n}{n}.
\]

**Proposition 3.15.** There exists \( c < \infty \) such that the following holds. If \( V \in \tilde{X}_{n^4} \) and
\[
V^*_n = V \cap \left\{ e^{(n-1)^4 + 4(n-1)} \leq |z| \leq e^{n^4 - 4n} \right\},
\]
then,
\[
\mathbb{P}(E_{(n+1)^4}) = \mathbb{P}(E_{n^4}) \left[ 1 - H(V^*_n) + O\left(\frac{\log^2 n}{n^3}\right)\right].
\]

**Proof.** Let \( \tau_n = \sigma_{n^4} \). Using (14), we can see that
\[
\mathbb{P}\{S[\tau_n, \tau_{n+1}] \cap V^*_n \neq \emptyset\} = H(V^*_n) + O(e^{-2n}).
\]

Using (20), we see that
\[
\mathbb{P}\{S[\tau_n, \tau_{n+1}] \cap V^*_n \neq \emptyset \mid E_{n^4}\}
\]
\[
= \mathbb{P}\{S[\tau_n, \tau_{n+1}] \cap V^*_n \neq \emptyset\} \left[ 1 + O\left(\frac{\log^3 n}{n^4}\right)\right].
\]
Hence, it suffices to prove that
\[
\mathbb{P}\{S[\tau_n, \tau_{n+1}] \cap (V \setminus V^*_n) \neq \emptyset \mid E_{n^4}\} = O\left(\frac{\log^2 n}{n^3}\right),
\]
and using (19) we see that we only need to show
\[
\mathbb{P}\{S[\tau_n, \tau_{n+1}] \cap (V \setminus V^*_n) \neq \emptyset\} = O\left(\frac{\log^2 n}{n^3}\right),
\]

Note that $V \setminus V^*_n$ is contained in the union of $C_{n^4-4n}$ and $O(n)$ sets of the form $V_m$ with $m \geq n^4 - 4n$. Using the definition of $X_{n^4}$, we see that

$$H \left( (V \setminus V^*_n) \cap \left\{ e^{n^4-4n} \leq |z| \leq e^{(n+1)^4} \right\} \right) = O \left( \frac{\log^2 n}{n^3} \right),$$

and Lemma 3.4 shows that

$$\mathbb{P} \{ S[\tau_n, \tau_{n+1}] \cap C_{n^4-4n} \neq \emptyset \} = O(e^{-2n}).$$

**Corollary 3.16.** If $V \in \tilde{X}_{n^4}$, $m \geq n$, and $m^4 \leq k \leq (m+1)^4$, then

$$\mathbb{P}(E_k) = \mathbb{P}(E_{n^4}) \exp \left\{ - \sum_{j=n+1}^{m} H(V^*_j) \right\} \left[ 1 + O \left( \frac{\log^4 n}{n} \right) \right].$$

**Proof.** Using the previous proposition and the estimate $H(V^*_j) = O(\log^2 j/j)$, we see that

$$\frac{\mathbb{P}(E_{m^4})}{\mathbb{P}(E_{n^4})} = \prod_{j=n+1}^{m} \left[ 1 - H(V^*_j) + O \left( \frac{\log^2 j}{j^3} \right) \right] = \prod_{j=n+1}^{m} \left[ e^{-H(V^*_j)} + O \left( \frac{\log^4 j}{j^2} \right) \right] = \left[ 1 + O \left( \frac{\log^4 n}{n} \right) \right] \exp \left\{ - \sum_{j=n+1}^{m} H(V^*_j) \right\}.$$

Also, [18] shows that for $m^4 \leq k \leq (m+1)^4$,

$$\mathbb{P}(E_k) = \mathbb{P}(E_{m^4}) \left[ 1 - O \left( \frac{\log^2 m}{m} \right) \right].$$

We give one application to the loop-erased walk. We do not give the optimal error term.

**Corollary 3.17.** For all $n$,

$$p_{n+1} = p_n \left[ 1 + O \left( \frac{\log^4 n}{n} \right) \right].$$

In particular, if $n^4 \leq m \leq (n+1)^4$,

$$p_m = p_{n^4} \left[ 1 + O \left( \frac{\log^4 n}{n} \right) \right].$$
Proof. Let \( k = \lceil \log^2 n \rceil \). Using Lemma 3.5, we can see that except for an event of fast decaying probability: \( \Gamma_n, \Gamma_{n+1} \in \mathcal{X}_{n-k}; \Gamma_n \cap C_{n-k} = \Gamma_{n+1} \cap C_{n-k}; \) and \( H(\Gamma_n \Delta \Gamma_{n+1}) = O(\log^2 n/n) \). Note that \( p_m \) decays like a power law in \( m \) (and hence is not fast decaying). Therefore Corollary 3.16 applies and

\[
p_n = p_{n-k} \left[ 1 + O \left( \frac{\log^4 n}{n} \right) \right].
\]

From this we see that a similar formula holds for \( p_{n+1} \).

### 3.3 Loop-free times

One of the technical nuisances in the analysis of the loop-erased walk is that if \( j < k \), it is not necessarily the case that

\[
LE(S[j, k]) = LE(S[0, \infty)) \cap S[j, k].
\]

However, this is the case for special times called \textit{loop-free} times. We say that \( j \) is a \textit{(global) loop-free time} if

\[
S[0, j] \cap S[j + 1, \infty) = \emptyset.
\]

Proposition 3.2 shows that the probability that \( j \) is loop-free is comparable to \((\log j)^{-1/2}\).

From the definition of chronological loop erasing we can see the following. If \( j < k \) and \( j, k \) are loop-free times, then for all \( m \leq j < k \leq n \),

\[
LE(S[m, n]) \cap S[j, k] = LE(S[0, \infty)) \cap S[j, k] = LE(S[j, k]).
\]

It will be important for us to give upper bounds on the probability that there is no loop-free time in a certain interval of time. If \( m \leq j < k \leq n \), let \( I(j, k; m, n) \) denote the event that for all \( j \leq i \leq k - 1 \),

\[
S[m, i] \cap S[i + 1, n] \neq \emptyset.
\]

Proposition 3.2 gives a lower bound on \( P[I(n, 2n; 0, 3n)] \),

\[
P[I(n, 2n; 0, 3n)] \geq P\{S[0, n] \cap S[2n, 3n] \neq \emptyset \} \asymp \frac{1}{\log n}.
\]

The next lemma shows that \( P[I(n, 2n; 0, 3n)] \asymp 1/\log n \) by giving the upper bound.

**Lemma 3.18.** There exists \( c < \infty \) such that

\[
P[I(n, 2n; 0, \infty)] \leq \frac{c}{\log n}.
\]
Let $E = E_n$ denote the complement of $I(n, 2n; 0, \infty)$. We need to show that $\mathbb{P}(E) \geq 1 - O(1/\log n)$.

Let $k_n = \lfloor n/(\log n)^{3/4} \rfloor$ and let $A_i = A_{i,n}$ be the event that

$$A_i = \{S[n + (2i - 1)k_n, n + 2ik_n] \cap S[n + 2ik_n + 1, n + (2i + 1)k_n] = \emptyset\}.$$ 

and consider the events $A_1, A_2, \ldots, A_r$ where $\ell = \lceil (\log n)^{3/4}/4 \rceil$. These are $\ell$ independent events each with probability greater than $c(\log n)^{-1/2}$. Hence the probability that none of them occurs is $\exp\{-O((\log n)^{1/4})\} = o((\log n)^{-3})$, that is,

$$\mathbb{P}(A_1 \cup \cdots \cup A_\ell) \geq 1 - o((\log n)^{-3}).$$

Let $B_i = B_{i,n}$ be the event

$$B_i = \{S[0, n + (2i - 1)k_n] \cap S[n + 2ik_n, \infty) = \emptyset\},$$

and note that

$$E \supset (A_1 \cup \cdots \cup A_\ell) \cap (B_1 \cap \cdots \cap B_\ell).$$

Since $\mathbb{P}(B_\ell^c) \leq c \log \log n \log n$, we see that

$$\mathbb{P}(B_1 \cap \cdots \cap B_\ell) \geq 1 - \frac{c\ell \log \log n}{\log n} \geq 1 - O\left(\frac{\log \log n}{(\log n)^{1/4}}\right),$$

and hence,

$$\mathbb{P}(E) \geq \mathbb{P}[(A_1 \cup \cdots \cup A_\ell) \cap (B_1 \cap \cdots \cap B_\ell)] \geq 1 - O\left(\frac{\log \log n}{(\log n)^{1/4}}\right). \quad (23)$$

This is a good estimate, but we need to improve on it.

Let $C_j, j = 1, \ldots, 5$, denote the independent events (depending on $n$)

$$I \left(\frac{1 + 3(j-1) + 1}{15}, n \left[1 + \frac{3(j-1) + 2}{15}\right]; n + \frac{(j-1)n}{5}, n + \frac{jn}{5}\right).$$

By (23) we see that $\mathbb{P}[C_j] \leq o\left(1/(\log n)^{1/5}\right)$, and hence

$$\mathbb{P}(C_1 \cap \cdots \cap C_5) \leq o\left(\frac{1}{\log n}\right).$$

Let $D = D_n$ denote the event that at least one of the following ten things happens:

$$S \left[0, n \left(1 + \frac{j-1}{5}\right)\right] \cap S \left[n \left(1 + \frac{3(j-1)+1}{15}\right), \infty\right) \neq \emptyset, \quad j = 1, \ldots, 5.$$

Each of these events has probability comparable to $1/\log n$ and hence $\mathbb{P}(D) \asymp 1/\log n$.

Also,

$$I(n, 2n; 0, \infty) \subset (C_1 \cap \cdots \cap C_n) \cup D.$$

Therefore, $\mathbb{P}[I(n, 2n; 0, \infty)] \leq c/\log n$. \qed
Corollary 3.19.

1. There exists $c < \infty$ such that if $0 \leq j \leq j + k \leq n$, then
   \[ \mathbb{P}[I(j, j + k; 0, n)] \leq \frac{c \log(n/k)}{\log n}. \]

2. There exists $c < \infty$ such that if $0 < \delta < 1$ and
   \[ I_{\delta,n} = \bigcup_{j=0}^{n-1} I(j, j + \delta n; 0, n), \]
   then
   \[ \mathbb{P}[I_{\delta,n}] \leq \frac{c \log(1/\delta)}{\delta \log n}. \quad (24) \]

3. Let $\tilde{I}(m, n)$ denote the event that there is no loop-free point $j$ with $\sigma_m \leq j \leq \sigma_n$. There exists an integer $\ell < \infty$ such that for all $n$,
   \[ \mathbb{P}\{\tilde{I}(n - \ell k, n + \ell k) \mid \mathcal{F}_n - 3\ell k\} \leq O(n^{-1}), \quad (25) \]
   where $k = k_n = \lfloor \log n \rfloor$.

Proof.

1. We will assume that $k \geq n^{1/2}$. Note that $I(j, j + k; 0, n)$ is contained in the union of the following three events:
   \[ I(j, j + k; j - k, j + 2k), \]
   \[ \{S[0, j - k] \cap S[j, n] \neq \emptyset\}, \]
   \[ \{S[0, j] \cap S[j + k, n] \neq \emptyset\}. \]
   Since $k \geq n^{1/2}$ the probability of the first event is $O(1/\log n)$ and the probabilities of the second two events are $O(\log(n/k)/\log n)$.

2. We can cover $I_{\delta,n}$ by the union of $O(1/\delta)$ events of the form $I(j, j + k; 0, n)$, each of which has probability $O(\log(1/\delta)/\log n)$.

3. Using standard estimates, we can find $\ell$ such that with probability $1 - O(n^{-1})$, we have
   \[ \sigma_{n-\ell k} \leq e^{2n} \leq 2e^{2n} \leq \sigma_{n+\ell k}, \quad S[\sigma_{n-\ell k}, \infty) \cap C_{n-2\ell k} = \emptyset. \]
   On this event $\tilde{I}(n - \ell k, n + \ell k) \subset I(e^{2n}, 2e^{2n}; 0, \infty)$. Hence let the event
   \[ U = \tilde{I}(n - \ell k, n + \ell k) \cup \{S[\sigma_{n-\ell k}, \infty) \cup C_{n-2\ell k} \neq \emptyset\}, \]
   then $\mathbb{P}(U) \leq O(n^{-1})$. Note that the conditional distribution of $U$ given $\mathcal{F}_{n-2\ell k}$ depends only on $S(\sigma_{n-2\ell k})$. Hence, using Lemma 3.4 we see that the conditional distribution of $U$ given $\mathcal{F}_{n-3\ell k}$ is bounded by a constant times the unconditional distribution, and
   \[ \mathbb{P}[U \mid \mathcal{F}_{n-3\ell k}] \leq O(n^{-1}). \]

Proof. \qed
3.4 Along a subsequence

It will be easier to prove the main result first along the subsequence \( \{n^4\} \). We let \( \tau_n = \sigma_{n^4}, q_n = p_{n^4}, G_n = F_{n^4}, Q_n = Z_{n^4} \). Let \( \tilde{\eta}_n \) denote the (forward) loop-erasure of 
\[ S[\sigma_{(n-1)^4} + (n-1)^4], \sigma_{n^4} \]
and
\[ \tilde{\Gamma}_n = \tilde{\eta}_n \cap A((n-1)^4 + 4(n-1), n^4 - 4n), \]
\[ \Gamma^*_n = \Gamma \cap A((n-1)^4 + 4(n-1), n^4 - 4n). \]

We state the main result that we will prove in this subsection.

**Proposition 3.20.** There exists \( c_0 \) such that as \( n \to \infty \),
\[ q_n = \left[ c_0 + O \left( \frac{1}{n} \right) \right] \exp \left\{ -r \sum_{j=1}^{n} \tilde{h}_j \right\}, \]
where
\[ \tilde{h}_j = \mathbb{E} \left[ H(\tilde{\Gamma}_j) \right]. \]

**Proof.** Proposition 3.15 implies that, except for an event of fast decaying probability,
\[ Q_{n+1} = Q_n \left[ 1 - H(\Gamma^*_n) + O \left( \frac{\log^2 n}{n^3} \right) \right], \]
and hence,
\[ \mathbb{E} \left[ Q_{n+1}^r \right] = \mathbb{E} \left[ Q_n^r \left[ 1 - r H(\Gamma^*_n) \right] \right] + \mathbb{E} \left[ Q_n^r \right] O \left( \frac{\log^2 n}{n^3} \right). \]

Moreover, we can use (25) to see that \( \mathbb{P} \{ \Gamma^*_n + 1 \neq \tilde{\Gamma}_n + 1 | G_n \} \leq O(n^{-4}) \) which implies that
\[ \mathbb{E} \left[ Q_n^r | H(\Gamma^*_n + 1) - H(\tilde{\Gamma}_n + 1) \right] \leq O(n^{-4}) \mathbb{E} \left[ Q_n^r \right]. \]

Hence,
\[ \mathbb{E} \left[ Q_{n+1}^r \right] = \mathbb{E} \left[ Q_n^r \left[ 1 - r H(\tilde{\Gamma}_n + 1) \right] \right] + \mathbb{E} \left[ Q_n^r \right] O \left( \frac{\log^2 n}{n^3} \right). \]

Using Lemma 3.4 we can see that
\[ \mathbb{E}[H(\tilde{\Gamma}_n + 1) | G_n] = \mathbb{E}[H(\tilde{\Gamma}_n + 1)] [1 + o(n^{-4})] = \tilde{\eta}_{n+1} [1 + o(n^{-4})] \]
Recall \( p_n = \mathbb{E}[Z_n^r] \) and \( q_n = p_{n^4} \) and combine all of above, we get
\[ q_{n+1} = q_n \left[ 1 - r \tilde{h}_{n+1} + O \left( \frac{\log^2 n}{n^3} \right) \right] = q_n \exp \left\{ -r \tilde{h}_{n+1} \right\} \left[ 1 + O \left( \frac{1}{n^2} \right) \right]. \]
The last inequality uses $\tilde{h}_{n+1} = O(n^{-1})$. In particular, if $m > n$,

$$q_m = q_n \left[ 1 + O\left(\frac{1}{n}\right)\right] \exp\left\{-r \sum_{j=n+1}^{m} \tilde{h}_j\right\}. \quad \square$$

**Corollary 3.21.** There exists $c_0 < \infty$ such that as $m \to \infty$,

$$p_m = \left[c_0 + O\left(\frac{\log m}{m^{1/4}}\right)\right] \exp\left\{-r \sum_{j=1}^{\lfloor m^{1/4}\rfloor} \tilde{h}_j\right\}.$$

**Proof.** This follows immediately from the previous proposition and Corollary 3.17. \quad \square

### 3.5 Comparing hitting probabilities

The goal of this section is to prove the following.

**Proposition 3.22.** There exists $c < \infty$, $u > 0$ such that

$$\left| \mathbb{E}\left[H(\tilde{\Gamma}_n) - \sum_{j=(n-1)^4+1}^{n^4} \mathbb{E}[H(\eta_j)]\right]\right| \leq \frac{c}{n^{1+u}}.$$

**Proof.** The strategy is to define for each $n$ a random subset $U = U(n) \subset \mathbb{Z}^4$, of the form

$$U = \bigcup_{j=(n-1)^4+1}^{n^4} U_j, \quad (26)$$

such that the following four conditions hold:

$$U \subset \tilde{\Gamma}_n, \quad (27)$$

$$U_j \subset \eta^j, \quad j = (n-1)^4 + 1, \ldots n^4, \quad (28)$$

$$\mathbb{E}\left[H(\tilde{\Gamma}_n \setminus U) + \sum_{j=(n-1)^4+1}^{n^4} \mathbb{E}\left[H(\eta_j \setminus U_j)\right]\right] \leq O(n^{-(1+u)}), \quad (29)$$

$$\max_{(n-1)^4 < j \leq n^4} \max_{x \in U_j} H(x, U \setminus U_j) \leq n^{-u}. \quad (30)$$

We will first show that finding such a set gives the result. Taking expectations and using (27)–(29), we get

$$\mathbb{E}\left[H(\tilde{\Gamma}_n)\right] = O(n^{-(1+u)}) + \mathbb{E}\left[H(U)\right],$$

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and
\[ \sum_{j=(n-1)^4+1}^{n^4} \mathbb{E}[H(U_j)] = O(n^{-(1+u)}) + \sum_{j=(n-1)^4+1}^{n^4} \mathbb{E}[H(\eta^j)]. \]

Also it follows from (30) and the strong Markov property that
\[ H(U) = \left[ 1 - O(n^{-u}) \right] \sum_{j=(n-1)^4+1}^{n^4} H(U_j), \]
and hence that
\[ \mathbb{E}[H(U)] = \left[ 1 - O(n^{-u}) \right] \sum_{j=(n-1)^4+1}^{n^4} \mathbb{E}[H(U_j)] \]
\[ = -O(n^{-(1+u)}) + \sum_{j=(n-1)^4+1}^{n^4} \mathbb{E}[H(U_j)]. \]

This shows that we need only find the sets \( U_j \).

Let \( \sigma_j^\pm = \sigma_j \pm \lfloor j^{-1/4} e^{2j} \rfloor \) as in Lemma 3.7 and \( \tilde{\omega}_j = S[\sigma_{j-1}^+, \sigma_j^-] \). We will set \( U_j \) equal to \( \eta^j \cap \tilde{\omega}_j \) unless one of the following six events occurs in which case we set \( U_j = \emptyset \). We assume \( (n-1)^4 < j \leq n^4 \).

1. If \( j \leq (n-1)^4 + 8n \) or \( j \geq n^4 - 8n \).
2. If \( H(\omega_j) \geq j^{-1} \log^2 j \).
3. If \( \omega_j \cap C_{j-8 \log n} \neq \emptyset \).
4. If \( H(\omega_j \setminus \tilde{\omega}_j) \geq j^{-1-u} \).
5. If it is not true that there exist loop-free points in both \( [\sigma_{j-1}, \sigma_j^+ \] and \( [\sigma_j^-, \sigma_j] \).
6. If \( \sup_{x \in \tilde{\omega}_j} H(x, S[0, \infty) \setminus \omega_j) \geq j^{-1/3} \).

We need to show that \( U \) defined as in (26) satisfies (27)–(30).

The definition of \( U_j \) immediately implies (28). Combining conditions 1 and 3, we see that (for \( n \) sufficiently large which we assume throughout) that \( U_j \subset A((n-1)^4 + 6n, n^4 - 6n) \). Moreover, if there exists loop-free points in \( [\sigma_{j-1}, \sigma_j^+] \) and \( [\sigma_j^-, \sigma_j] \), then \( \tilde{\eta}_n \cap \tilde{\omega}_j = \eta_j \cap \tilde{\omega}_j \). Therefore, (27) holds. Also, condition 6 immediately gives (30).

In order to establish (29) we first note that
\[ (\tilde{\Gamma}_n \cup \eta^{(n-1)^4+1} \cup \ldots \cup \eta^{n^4}) \setminus U \subset \bigcup_{j=(n-1)^4+1}^{n^4} V_j, \]
where
\[ V_j = \begin{cases} \omega_j & \text{if } U_j = \emptyset \\ \omega_j \setminus \tilde{\omega}_j & \text{if } U_j = \eta_i \cap \tilde{\omega}_j. \end{cases} \]

Hence, it suffices to find \( u \) such that for all \( j \),
\[ \sum_{(n-1)^4 < j \leq n^4} (\mathbb{E} [H(\omega_j); U_j = \emptyset] + \mathbb{E} [H(\omega_j \setminus \tilde{\omega}_j)]) \leq c n^{-1-u}. \]

To estimate \( \mathbb{E} [H(\omega_j \setminus \tilde{\omega}_j)] \), we use (16) and Lemma 3.8 to see that except for an event of fast decaying probability
\[ H(\omega_j \setminus \tilde{\omega}_j) \leq O\left(\frac{j}{\log j}\right), \tag{31} \]
and hence \( \mathbb{E}[H(\omega_j \setminus \tilde{\omega}_j)] \leq O(j^{-11/10}) \) and
\[ \sum_{(n-1)^4 < j \leq n^4} \mathbb{E} [H(\omega_j \setminus \tilde{\omega}_j)] \leq c \sum_{j = (n-1)^4}^{n^4} j^{-11/10} \leq c n^{-\frac{7}{5}}. \]

We write the event \( \{ U_j = \emptyset \} \) as a union of six disjoint events
\[ \{ U_j = \emptyset \} = E_1^j \cup \cdots \cup E_6^j, \]
where \( E_i^j \) is the event that the \( i \)th condition in the definition of \( U_j \) holds but none of the previous ones hold.

1. Since for each \( j \), \( \mathbb{E}[H(\omega_j)] \asymp j^{-1} \),
\[ \sum_{(n-1)^4 < j \leq n^4} \mathbb{E}[H(\omega_j); E_i^j] = O(n^{-3}). \]

2. By Proposition 3.6 \( \mathbb{P}\{H(\omega_j) \geq j^{-1} \log^2 j\} \) is fast decaying in \( j \). This takes care of the summation of \( \mathbb{E}[H(\omega_j); E_2^j] \).

On the event \( E_3^j \cup \cdots \cup E_6^j \), we have \( H(\omega_j) < j^{-1} \log^2 j \). Hence,
\[ \mathbb{E} \left[ H(\omega_j) 1_{E_3^j \cup \cdots \cup E_6^j} \right] \leq \frac{\log^2 j}{j} \mathbb{P}(E_3^j \cup \cdots \cup E_6^j). \]

In particular, it suffices to prove that \( \mathbb{P}(E_i^j) \leq j^{-u} \) for some \( u \) for \( i = 3, 4, 5, 6 \).

3. The standard estimate in Lemma 3.4 gives
\[ \mathbb{P}(E_3^j) \leq \mathbb{P}\{\omega_j \cap C_{\log j} \neq \emptyset\} \leq O(j^{-2}). \]

4. The bound on \( \mathbb{P}(E_4^j) \) is already done in (31).
5. Let $I = I_{δ,n}$ be as in (24) substituting in $n = e^{2j^{-1/8}}$ and $δ = j^{-7/16}$ so that $\delta n = e^{2j^{-3/8}}$. Using (24), we have $\mathbb{P}(I) = o(j^{-1/4})$. Note that the event that there is no loop-free point in $[\sigma_{j-1}, \sigma_{j-1}^+]$ or no loop-free point in $[\sigma_j^-, \sigma_j]$ is contained in the union of $I$ and the two events:

$$\{\sigma_{j+1} \geq e^{2j^{-1/16}}\},$$

$$\{S[e^{2j^{-1/16}} \infty) \cap S[0, \sigma_{j+1}] \neq \emptyset\}.$$  

The probability of the first event is fast decaying and the probability of the second is $o(1/j)$. Hence $\mathbb{P}(E_5^n) = o(j^{-1/4}).$

6. In Lemma 3.7 we showed that $\mathbb{P}(E_6^n) = O(j^{-1/3}).$

### 3.6 $s \neq 0$

The next proposition is easy but it is important for us.

**Proposition 3.23.** For every $r, s$ there exists $c'_{r,s}, u > 0$ such that

$$\mathbb{E} \left[ Z_r^n G_n^{-s} \right] = c'_{r,s} \phi_n^r \left[ 1 + O(n^{-u}) \right].$$

**Proof.** The case when $s = 0$ follows from Corollary 3.21 and Proposition 3.22.

For the general case, recall that $1 \leq G_n \leq 8$ and $G_n$ converge to $G_\infty$ almost surely. Moreover Lemma 3.5 implies that there exists a fast decaying sequence $\{\epsilon_n\}$ such that if $m \geq n$, $\mathbb{P}\{|G_n - G_m| \geq \epsilon_n\} \leq \epsilon_n$. Therefore

$$\mathbb{E} \left[ \phi_n^{-r} Z_n^r G_n^{-s} \right] - \mathbb{E} \left[ \phi_n^{-r} Z_n^r G_\infty^{-s} \right] \text{ is fast decaying}$$

and

$$|\mathbb{E} \left( \phi_n^{-r} Z_n^r G_\infty^{-s} \right) - \mathbb{E} \left( \phi_m^{-r} Z_m^r G_\infty^{-s} \right)| \leq c \left| \mathbb{E} \left( \phi_n^{-r} Z_n^r \right) - \mathbb{E} \left( \phi_m^{-r} Z_m^r \right) \right|.$$  

Take $m \to \infty$ we conclude the proof.  

### 3.7 Determining the constant

The goal of this section is to prove

$$\hat{p}_n \sim \frac{\pi^2}{24n}. \quad (32)$$

We will give the outline of the proof using the lemmas that we will prove afterwards. By Proposition 3.23,

$$\hat{p}_n = c'_{3,2} \left[ 1 + O(n^{-u}) \right] \exp \left\{-3 \sum_{j=1}^{n} h_j \right\}, \quad (33)$$

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where \( h_j = \mathbb{E}[H(\eta_j)] \), and \( c_{3,2}' \) is an unknown constant. We show in Section 3.7.2 that
\[
h_n = \frac{8}{\pi^2} \hat{p}_n + O(n^{-1-u}).
\]
It follows that the limit
\[
\lim_{n \to \infty} \left[ \log \hat{p}_n + \frac{24}{\pi^2} \sum_{j=1}^{n} \hat{p}_j \right],
\]
exists and is finite. Using this, Lemma 3.24 and \( \hat{p}_{n+1} = \hat{p}_n \left[ 1 + O(\log^4 n/n) \right] \) (see Corollary 3.17) we can conclude Theorem 3.1. In particular, this and (33) implies
\[
\phi_3^n = \exp \left\{ -3 \sum_{j=1}^{n} h_j \right\} \sim \frac{c}{n} \quad \text{for some constant } c > 0. \tag{34}
\]

3.7.1 A lemma about a sequence

**Lemma 3.24.** Suppose \( \beta > 0, p_1, p_2, \ldots \) is a sequence of positive numbers with \( p_{n+1}/p_n \to 1 \), and
\[
\lim_{n \to \infty} \left[ \log p_n + \beta \sum_{j=1}^{n} p_j \right]
\]
exists and is finite. Then
\[
\lim_{n \to \infty} n p_n = 1/\beta.
\]

**Proof.** It suffices to prove the result for \( \beta = 1 \), for otherwise we can consider \( \tilde{p}_n = \beta p_n \).

Let
\[
a_n = \log p_n + \sum_{j=1}^{n} p_j.
\]
The hypothesis imply that \( \{a_n\} \) is a Cauchy sequence.

We first claim that for every \( \delta > 0 \), there exists \( N_\delta > 0 \) such that if \( n \geq N_\delta \) and \( p_n = (1 + 2\epsilon)/n \) with some \( \epsilon \geq \delta \), then there does not exists \( r > n \) with
\[
p_k \geq \frac{1}{k}, \quad k = n, \ldots, r - 1,
p_r \geq \frac{1 + 3\epsilon}{r}.
\]
Indeed, suppose these inequalities hold for some \( n, r \). Then,
\[
\log(p_r/p_n) \geq \log \frac{1 + 3\epsilon}{1 + 2\epsilon} - \log(r/n),
\]
\[
\sum_{j=n+1}^{r} p_j \geq \log(r/n) - O(n^{-1}).
\]
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and hence for \( n \) sufficiently large,

\[
a_r - a_n \geq \frac{1}{2} \log \frac{1 + 3\epsilon}{1 + 2\epsilon} \geq \frac{1}{2} \log \frac{1 + 3\delta}{1 + 2\delta},
\]

Since \( a_n \) is a Cauchy sequence, this cannot be true for large \( n \).

We next claim that for every \( \delta > 0 \), there exists \( N_\delta > 0 \) such that if \( n \geq N_\delta \) and \( p_n = (1 + 2\epsilon)/n \) with some \( \epsilon \geq \delta \), then there exists \( r \) such that

\[
\frac{1 + \epsilon}{k} \leq p_k < \frac{1 + 3\epsilon}{k}, \quad k = n, \ldots, r - 1,
\]

\[
p_r < \frac{1 + \epsilon}{r}. \tag{35}
\]

To see this, we consider the first \( r \) such that \( p_r < \frac{1 + \epsilon}{r} \). By the previous claim, if such an \( r \) exists, then (35) holds for \( n \) large enough. If no such \( r \) exists, then by the argument above for all \( r > n \),

\[
a_r - a_n \geq \log \frac{1 + \epsilon}{1 + 2\epsilon} + \frac{\epsilon}{2} \log(r/n) - (1 + \epsilon) O(n^{-1}).
\]

Since the right-hand side goes to infinity as \( r \to \infty \), this contradicts the fact that \( a_n \) is a Cauchy sequence.

By iterating the last assertion, we can see that for every \( \delta > 0 \), there exists \( N_\delta > 0 \) such that if \( n \geq N_\delta \) and \( p_n = (1 + 2\epsilon)/n \) with some \( \epsilon \geq \delta \), then there exists \( r > n \) such that

\[
p_r < \frac{1 + 2\delta}{r}, \quad \text{and} \quad p_k \leq \frac{1 + 3\epsilon}{k}, \quad k = n, \ldots, r - 1.
\]

Let \( s \) be the first index greater than \( r \) (if it exists) such that either

\[
p_k \leq \frac{1}{k} \quad \text{or} \quad p_k \geq \frac{1 + 2\delta}{k}.
\]

Using \( p_{n+1}/p_n \to 1 \), we can see, perhaps by choosing a larger \( N_\delta \) if necessary, that

\[
\frac{1 - \delta}{k} \leq p_s \leq \frac{1 + 4\delta}{k}.
\]

If \( p_s \geq (1 + 2\delta)/k \), then we can iterate this argument with \( \epsilon \leq 2\delta \) to see that

\[
\limsup_{n \to \infty} n p_n \leq 1 + 6\delta.
\]

The \( \liminf \) can be done similarly. \( \square \)
3.7.2 Exact relation

We will prove the following. Let $S, W$ be simple random walks with corresponding stopping times $\sigma_n$ and let $G_n = G_{C_n}$. We will assume that $S_0 = w, W_0 = 0$. Let $\eta = LE(S[0, \sigma_n])$. Let

$$\hat{G}_n^2(w) = \sum_{z \in \mathbb{Z}^4} G(0, z) G_n(w, z) = \sum_{z \in C_n} G(0, z) G_n(w, z).$$

Note that we are stopping the random walk $S$ at time $\sigma_n$ but we are allowing the random walk $W$ to run for infinite time.

**Lemma 3.25.** If $w \in C_n$, then

$$\hat{G}_n^2(w) = \frac{8}{\pi^2} \left[ n - \log |w| \right] + O(e^{-n}) + O(|w|^{-2} \log |w|).$$

In particular,

$$\hat{G}_n^2(w) = \frac{8}{\pi^2} + O(e^{-n}), \quad \forall w \in \partial C_{n-1}. \quad (36)$$

**Proof.** Let $f(x) = \frac{8}{\pi^2} \log |x|$ and note that

$$\Delta f(x) = \frac{2}{\pi^2 |x|^2} + O(|x|^{-4}) = G(x) + O(|x|^{-4}).$$

where $\Delta$ denotes the discrete Laplacian. Also, we know that for any function $f$,

$$f(w) = E^w[f(S_{\sigma_n})] - \sum_{z \in C_n} G_n(w, z) \Delta f(z).$$

Since $e^n \leq |S_{\sigma_n}| \leq e^n + 1,$

$$E^w[f(S_{\sigma_n})] = \frac{8n}{\pi^2} + O(e^{-n}).$$

Therefore,

$$\sum_{z \in C_n} G_n(w, z) G(z) = \frac{8}{\pi^2} \left[ n - \log |w| \right] + O(e^{-n}) + \epsilon,$$

where

$$|\epsilon| \leq \sum_{z \in C_n} G_n(w, z) O(|z|^{-4}) \leq \sum_{z \in C_n} O(|w - z|^{-2}) O(|z|^{-4}).$$

We split the sum into three pieces.

$$\sum_{|z| \leq |w|/2} O(|w - z|^{-2}) O(|z|^{-4}) \leq c |w|^{-2} \sum_{|z| \leq |w|/2} O(|z|^{-4}) \leq c |w|^{-2} \log |w|.$$
\[ \sum_{|z-w| \leq |w|/2} O(|w-z|^{-2}) O(|z|^{-4}) \leq c |w|^{-4} \sum_{|x| \leq |w|/2} O(|x|^{-2}) \leq c |w|^{-2}, \]

If we let \( C'_n \) the the set of \( z \in C_n \) with \( |z| > |w|/2 \) and \( |z-w| > |w|/2 \), then
\[ \sum_{z \in C'_n} O(|w-z|^{-2}) O(|z|^{-4}) \leq \sum_{|z| > |w|/2} O(|z|^{-6}) \leq c |w|^{-2}. \]

\[ \square \]

**Proposition 3.26.** There exists \( \alpha < \infty \) such that if \( n^{-1} \leq e^{-n} |w| \leq 1 - n^{-1} \), then
\[ \left| \log \mathbb{P}\{W[0, \infty] \cap \eta \neq \emptyset\} - \log \hat{G}_n^2(w) \hat{p}_n \right| \leq c \frac{\log^\alpha n}{n}. \]

In particular,
\[ \mathbb{E}[H(\eta^n)] = \frac{8 \hat{p}_n}{\pi^2} \left[ 1 + O\left( \frac{\log^\alpha n}{n} \right) \right]. \]

The second assertion follows immediately from the first and (36). We can write the conclusion of the proposition as
\[ \mathbb{P}\{W[0, \infty] \cap \eta \neq \emptyset\} = \hat{G}_n^2(w) \hat{p}_n \left[ 1 + \theta_n \right], \]
where \( \theta_n \) throughout this subsection denotes an error term that decays at least as fast as \( \log^\alpha n/n \) for some \( \alpha \) (with the implicit uniformity of the estimate over all \( n^{-1} \leq e^{-n} |w| \leq 1 - n^{-1} \)). \( \theta_n \) may vary line by line.

We start by giving the sketch of the proof which is fairly straightforward. On the event \( \{W[0, \infty] \cap \eta \neq \emptyset\} \) there are typically many points in \( W[0, \infty] \cap \eta \). We focus on a particular one. This is analogous to the situation when one is studying the probability that a random walk visits a set. In the latter case, one usually focuses on the first or last visit. In our case with two paths, the notions of “first” and “last” are a little ambiguous so we have to take some care. We will consider the first point on \( \eta \) that is visited by \( W \) and then focus on the last visit by \( W \) to this first point on \( \eta \).

To be precise, we write
\[ \eta = [\eta_0, \ldots, \eta_m], \]
\[ i = \min\{t : \eta_t \in W[0, \infty)\}, \]
\[ \rho = \max\{t \geq 0 : S_t = \eta_i\}, \]
\[ \lambda = \max\{t : W_t = \eta_i\}. \]

Then the event \( \{\rho = j; \lambda = k; S_\rho = W_\lambda = z\} \) is the event that:
\[ I : \quad S_j = z, \quad W_k = z, \]
II: \( LE(S[0, j]) \cap (S[j + 1, \sigma_n] \cup W[0, k] \cup W[k + 1, \infty]) = \{z\} \),

III: \( z \not\in S[j + 1, \sigma_n] \cup W[k + 1, \infty] \).

Using the slowly recurrent nature of the random walk paths, we expect that as long as \( z \) is not too close to 0, \( w \), or \( \partial C_n \), then I is almost independent of (II and III) and 

\[ \mathbb{P}\{\text{II and III}\} \sim \hat{p}_n. \]

This then gives 

\[ \mathbb{P}\{\rho = j; \lambda = k; S_\rho = W_\lambda = z\} \sim \mathbb{P}\{S_j = W_k = z\} \hat{p}_n, \]

and summing over \( j, k, z \) gives 

\[ \mathbb{P}\{W[0, \infty] \cap \eta \neq \emptyset \} \sim \hat{p}_n \hat{G}^2_n(w). \]

The following proof makes this reasoning precise.

**Proof of Proposition 3.26.** Let \( V \) be the event 

\[ V = \{w \not\in S[1, \sigma_n], 0 \not\in W[1, \infty]\}. \]

A simple argument based of the last visit of the origin shows that there is a fast decaying sequence \( \{\epsilon_n\} \) such that 

\[ |\mathbb{P}(V) - G(0, 0)^{-2}| \leq \epsilon_n, \]

\[ |\mathbb{P}\{W[0, \infty] \cap \eta \neq \emptyset \mid V\} - \mathbb{P}\{W[0, \infty] \cap \eta \neq \emptyset\}| \leq \epsilon_n. \]

Hence it will suffice to show that 

\[ \mathbb{P}\{V \cap \{W[0, \infty] \cap \eta \neq \emptyset\}\} = \frac{\hat{G}^2_n(w)}{G(0, 0)^2} \hat{p}_n [1 + \theta_n]. \quad (37) \]

Let \( E(j, k, z), E_z \) be the events 

\[ E(j, k, z) = V \cap \{\rho = j; \lambda = k; S_\rho = W_\lambda = z\}, \quad E_z = \bigcup_{j, k=0}^{\infty} E(j, k, z). \]

Then, 

\[ \mathbb{P}\{V \cap \{W[0, \infty] \cap \eta \neq \emptyset\}\} = \sum_{z \in C_n} \mathbb{P}(E_z). \]

Let 

\[ C'_n = C'_{n, w} = \{z \in C_n : |z| \geq n^{-4}e^n, |z - w| \geq n^{-4}e^n, |z| \leq (1 - n^{-4})e^n\}. \]
We can use the easy estimate
\[ \mathbb{P}(E_z) \leq G_n(w, z) G(0, z), \]
to see that
\[ \sum_{z \in C_n \setminus C'_n} \mathbb{P}(E_n) \leq O(n^{-6}), \]
so it suffices to estimate \( \mathbb{P}(E_z) \) for \( z \in C'_n \).

We will translate so that \( z \) is the origin and will reverse the paths \( W[0, \lambda], S[0, \rho] \). Using the fact that reverse loop-erasing has the same distribution as loop-erasing, we see that \( \mathbb{P}[E(j, k, z)] \) can be given as the probability of the following event where \( \omega_1, \ldots, \omega_4 \) are independent simple random walks starting at the origin and \( l_i \) denotes the smallest index \( l_i \) such that \( |\omega_{l_i} - (-z)| \geq e^n \).

\[
(\omega_3[1, l_3] \cup \omega_4[1, \infty)) \cap LE(\omega_1[0, j]) = \emptyset,
\]
\[
\omega_2[0, k] \cap LE(\omega_1[0, j]) = \{0\},
\]
\[
j < l_1, \quad \omega_1(j) = x, \quad x \notin \omega_1[0, j - 1],
\]
\[
\omega_2(k) = y, \quad y \notin \omega_2[0, k - 1],
\]
where \( x = w - z, y = -z \). Note that
\[ n^{-1} e^n \leq |y|, |x - y|, |x| \leq e^n [1 - n^{-1}], \]

We now rewrite this. We fix \( x, y \) and let \( C^y_n = y + C_n \). Let \( W^1, W^2, W^3, W^4 \) be independent random walks starting at the origin and let \( T^i = T^i_n = \min\{j : W^i_j \notin C^y_n\} \),

\[
\tau^1 = \min\{j : W^1_j = x\}, \quad \tau^2 = \min\{k : W^2_k = y\}.
\]

Note that
\[ \mathbb{P}\{\tau^1 < T^1\} = \frac{G_n(0, x)}{G_n(x, x)} = \frac{G_n(0, x)}{G(0, 0)} + o(e^{-n}), \]
\[ \mathbb{P}\{\tau^2 < \infty\} = \frac{G(0, y)}{G(y, y)}. \]

Let \( p'_n(x, y) \) be the probability of the event
\[ \hat{\Gamma} \cap (W^2[1, \tau^2] \cap W^3[1, T^3]) = \emptyset, \quad \hat{\Gamma} \cap W^4[0, T^4] = \{0\}, \]
where \( W^1, W^2, W^3, W^4 \) are independent starting at the origin; \( W^3, W^4 \) are usual random walks; \( W^1 \) has the distribution of random walk conditioned that \( \tau^1 < T^2 \) stopped at \( \tau^1 \); and \( W^2 \) has the distribution of random walk conditioned that \( \tau^2 < \infty \) stopped at \( \tau^2 \). Moreover,
\[ \hat{\Gamma} = \hat{\Gamma}_n = LE(W^1[0, \tau^1]). \]
Then in order to prove (37), it suffices to prove that

\[ p'_n(x, y) = \hat{p}_n \left[ 1 + \theta_n \right]. \tag{38} \]

We write \( Q \) for the distribution of \( W_1, W_2, W_3, W_4 \) under the conditioning. Then consider two events \( E_1, E_2 \) as follows. Let \( \hat{W} = W^2[1, \tau^2] \cup W^3[1, T^3] \cup W^4[0, T^4] \) and let \( E_0, E_1, E_2 \) be the events

\[ E_0 = \{ 0 \not\in W^2[1, \tau^2] \cup W^3[1, T^3] \}, \]
\[ E_1 = E_0 \cap \{ \hat{W} \cap \hat{\Gamma} \cap C_{n-\log^3 n} = \{ 0 \} \}, \]
\[ E_2 = E_1 \cap \{ \hat{W} \cap \Theta_n = \emptyset \}, \]

where

\[ \Theta_n = W^1[0, \tau^1] \cap A(n - \log^3 n, 2n). \]

Since \( \hat{\Gamma} \cap C_{n-\log^3 n} \subset \hat{\Gamma} \subset \Theta_n \cup (\hat{\Gamma} \cap C_{n-\log^3 n}) \), we have

\[ Q(E_2) \leq p'_n(x, y) \leq Q(E_1). \]

Now to prove (38), it suffices to show

\[ Q(E_1) = \hat{p}_n \left[ 1 + \theta_n \right], \tag{39} \]
\[ Q(E_1 \setminus E_2) \leq n^{-1} \theta_n. \tag{40} \]

For any path \( \omega = [0, \omega^1, \ldots, \omega^k] \) with \( 0, \omega^1, \ldots, \omega^{k-1} \in C_{n}^y \setminus \{ x \} \),

\[ Q\{ [W^1_0, \ldots, W^1_k] = \omega \} = \mathbb{P}\{ [S_0, \ldots, S_k] = \omega \} \frac{\phi_x(\omega_k)}{\phi_x(0)}, \tag{41} \]

where

\[ \phi_x(z) = \phi_{x,n}(z) = \mathbb{P}^z\{ \tau^1 < T^1_n \}. \]

Similarly if \( \phi_y(z) = \mathbb{P}^z\{ \tau^2 < \infty \} \) and \( \omega = [0, \omega^1, \ldots, \omega^k] \) is a path with \( y \not\in \{ 0, \omega^1, \ldots, \omega^{k-1} \} \),

\[ Q\{ [W^2_0, \ldots, W^2_k] = \omega \} = \mathbb{P}\{ [S_0, \ldots, S_k] = \omega \} \frac{\phi_y(\omega_k)}{\phi_y(0)}. \]

Using (12), we can see that if \( \zeta \in \{ x, y \} \), and if \( |z| \leq e^n e^{\frac{-1}{2} n} \),

\[ \phi_{\zeta}(z) = \phi_{\zeta}(0) \left[ 1 + O(\epsilon_n) \right]. \]

where \( \epsilon_n \) is fast decaying.

Let \( \sigma_k \) be defined for \( W^1 \) as before,

\[ \sigma_k = \min\{ j : |W^1_j| \geq e^k \}. \]
Using this we see that $W^1, S$ can be coupled on the same probability space such that, except perhaps on an event of probability $O(\epsilon_n)$,

$$\hat{\Gamma} \cap C_{n-\log^3 n} = LE (S[0, \infty)) \cap C_{n-\log^3 n}.$$ 

This concludes (39) since we have

$$Q(E_1) = p_{n-\log^3 n} + O(\epsilon_n) = p_n [1 + \theta_n].$$

To prove (40), consider the following events whose union covers $E_1 \setminus E_2$:

$$F^2 = E_1 \cap \{W^2[1, \tau^2] \cup \Theta_n \neq \emptyset\}, \quad F^3 = E_1 \cap \{W^3[1, T^3] \cup \Theta_n \neq \emptyset\}, \quad F^4 = E_1 \cap \{W^4[1, T^4] \cup \Theta_n \neq \emptyset\}.$$ 

We are now going to prove $Q(F_i) \leq n^{-1} \theta_n$ for $i = 2, 3, 4$, thus proving (40).

Find a fast decaying sequence $\bar{\epsilon}_n$ as in Proposition 3.6 such that

$$\bar{\epsilon}_n = \frac{\epsilon_1}{100}$$

which is also fast decaying and let $\rho = \rho_n = \min\{j : |W^1_j - x| \leq e^n \delta_n\}$. Using the Markov property we see that

$$\mathbb{P} \left\{ H \left( S[0, \infty) \cap A(m-1, m) \right) \geq \frac{\log^2 m}{m} \right\} \leq \bar{\epsilon}_m, \quad \forall m \geq \sqrt{n}.$$ 

Let $\delta_n = \epsilon_n^{1/100}$ which is also fast decaying and let $\rho = \rho_n = \min\{j : |W^1_j - x| \leq e^n \delta_n\}$.

Using the Markov property we see that

$$\mathbb{P} \left\{ W^1[\rho, \tau^1] \not\subseteq \{|z - x| \leq e^n \sqrt{\delta_n}\} \right\} = O(\delta_n),$$

and since $|x| \geq e^{-n} n^{-1}$, we know that

$$H \left( \{|z - x| \leq e^n \sqrt{\delta_n}\} \right) \leq n^2 \delta_n.$$ 

Using (41), we see that

$$Q \left\{ H \left( W^1[0, \tau^1] \cap A(m-1, m) \right) \geq \frac{\log^2 m}{m} \right\} \leq \delta_m, \quad \forall m \geq \sqrt{n}.$$ 

Now a union bound gives

$$Q \left\{ H \left( W^1[0, \tau^1] \cap A(n-\log^4 n, n+1) \right) \geq \frac{\log^7 n}{n} \right\}$$

is fast decaying. Using Proposition 3.13 we see that there exists a fast decaying sequence $\epsilon_n$ such that, except for an event of $Q$ probability at most $\epsilon_n$, either

$$\text{Es}[\hat{\Gamma}] \leq \epsilon_n \quad \text{or} \quad \text{Es}[\hat{\Gamma}] = \text{Es}[\hat{\Gamma} \cap C_{n-\log^3 n} \cdot [1 + \theta_n]].$$

Combined with (42) this gives $Q(F^3), Q(F^4) \leq n^{-1} \theta_n$. But we still need a similar result to conclude $Q(F^2) \leq n^{-1} \theta_n$. More precisely, we need to show that except for an event of $Q$ probability at most $\epsilon_n$, either

$$\text{Es}[\hat{\Gamma}] \leq \epsilon_n \quad \text{or} \quad \text{Es}[\hat{\Gamma}] = \text{Es}[\hat{\Gamma} \cap C_{n-\log^3 n} \cdot [1 + \theta_n]],$$

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where
\[ \text{Es}^y[\hat{\Gamma}] = Q \left\{ W^2[1, \tau^2] \cap \hat{\Gamma} = \emptyset \mid \hat{\Gamma} \right\}. \]

Since the measure \( Q \) is almost the same as \( P \) for the paths near zero, we can use Proposition 3.13 to reduce this to showing that with \( Q \) probability at least \( 1 - \epsilon_n \),
\[ Q\{W^2[0, \tau^2] \cap (\hat{\Gamma} \setminus C_n - \log^3 n) \neq \emptyset \mid \hat{\Gamma} \} \leq \theta_n. \]

This can be done with a similar argument using the fact that the probability that random walks starting at \( y \) and \( x \) ever intersect is less than \( \theta_n \).

\[ \square \]

**Corollary 3.27.** If \( n^{-1} \leq e^{-n} |w| \leq 1 - n^{-1} \), then
\[ \mathbb{P}\{W[0, \infty) \cap \eta \neq \emptyset\} \sim \frac{\pi^2 \hat{G}_n^2(w)}{24 n}. \]

More precisely,
\[ \lim_{n \to \infty} \max_{n^{-1} \leq e^{-n} |w| \leq 1 - n^{-1}} \left| 24 n \mathbb{P}\{W[0, \infty) \cap \eta \neq \emptyset\} - \pi^2 \hat{G}_n^2(w) \right| = 0. \]

By a very similar proof one can show the following. Let
\[ G_n^2(z) = \sum_{z \in \mathbb{Z}^4} G_n(0, z) G_n(w, z) = \sum_{z \in C_n} G_n(0, z) G_n(w, z), \]
and
\[ \sigma_n^W = \min\{j : W_j \notin C_n\}. \]

We note that
\[ \hat{G}_n^2(w) - G_n^2(w) = \sum_{z \in C_n} [G(0, z) - G_n(0, z)] G_n(w, z) = O(1). \]

The following can be proved in the same way.

**Proposition 3.28.** There exists \( \alpha < \infty \) such that if \( n^{-1} \leq e^{-n} |w| \leq 1 - n^{-1} \), then
\[ |\log \mathbb{P}\{W[0, \sigma_n^W] \cap \eta \neq \emptyset\} - \log[G_n^2(w) \hat{p}_n]| \leq c \frac{\log^{\alpha} n}{n}. \]

We note that if \( |w| = n^{-r} e^n \), then
\[ G_n^2 \sim \hat{G}_n^2(w) \sim \frac{\log(1/r)}{n}. \]

This gives Lemma [2.4] in the case that \( x \) or \( y \) equals zero. The general case can be done similarly.
4 Two-sided loop-erased random walk

The two-sided loop-erased random walk gives the distribution of a loop-erased random walk viewed at an interior point where we see two path — the future and the (reversal of the) past. In this section we describe the two-sided LERW in $\mathbb{Z}^4$.

We first consider $d \geq 5$ where this was first described in [13]. Suppose $W, S$ are independent simple random walks starting at the origin in $\mathbb{Z}^d$. Let $\hat{W}^1[0, \infty)$ denote the loop-erasure of the first path. Since $d \geq 5$, the event $J = \{ S[1, \infty) \cap \hat{W}^1[0, \infty) = \emptyset \}$ has positive probability and hence we can condition with respect to this event. The distribution of $(\hat{W}^1, \hat{S})$ with respect to this event is the two-sided loop-erased random walk. Using the reversibility of the loop-erased walk, it is not difficult to see that this distribution is the limit as $n \to \infty$ of the measure of the loop-erased walk viewed at $\hat{S}(n)$, that is the distribution of $(\tilde{W}, \tilde{S})$, where

\[
\tilde{W}(j) = \hat{S}(n - j) - \hat{S}(n), \quad j = 0, \ldots, n,
\]

\[
\tilde{S}(j) = \hat{S}(n + j) - \hat{S}(n), \quad j = 0, 1, \ldots.
\]

The marginal distribution on $\tilde{W}^1[0, \infty)$ in the two-sided measure is absolutely continuous with respect to the original distribution with Radon-Nikodym derivative

\[
Z^* := \frac{Z}{\mathbb{E}(Z)}
\]

where $Z$ is the $W$-measurable random variable

\[
Z = \mathbb{P}\{ S[1, \infty) \cap \tilde{W}^1[0, \infty) = \emptyset \mid \tilde{W}^1[0, \infty) \}.
\]

Indeed, we have a uniform upper bound, $Z^* \leq 1/\mathbb{P}(J)$. In order to sample from the two-sided LERW, we could do the following:

- sample $\tilde{W}^1$ using the marginal distribution;
- given $\tilde{W}^1$, sample a simple random walk $S$ conditioned so that $S[1, \infty) \cap \tilde{W}^1[0, \infty) = \emptyset$;
- erasing loops from $S$ to get $\hat{S}$.

We will use our results to do a similar construction for $d = 4$. Here it is more difficult because $Z$ as defined in [13] is zero with probability one. We use the notation of Section 3.2 where we choose our slowly recurrent set $V$ to the (random) set $\hat{W}^1[0, \infty)$. We let $Z_n$ be the $W$-measurable random variable

\[
Z_n = \mathbb{P}\{ S[1, \sigma_n] \cap \tilde{W}^1[0, \infty) = \emptyset \mid \tilde{W}^1[0, \infty) \}.
\]
Proposition 4.1. If $d = 4$, then with probability one the limit

$$Z = \lim_{n \to \infty} n^{1/3} Z_n,$$

exists with probability one and in $L^p$ for all $p > 0$.

Remark 4.2. Note that $Z_n$ here is not exactly the same as the $Z_n$ in Section 3 since we are using the entire loop-erasure rather than the loop-erasure of $W[0, \sigma_n]$. If we use $\tilde{Z}_n$ for the random variable in Section 3, the estimates in that section (especially Lemma 3.5 and Proposition 3.6) show that, except for an event of fast decaying probability

$$\tilde{Z}_n = Z_n \left[ 1 + O \left( \frac{\log n}{n} \right) \right],$$

so we can use $Z_n$ or $\tilde{Z}_n$ in the definition of $Z$. Similar, except for an event of fast decaying probability,

$$(\log n)^2 \leq e^{-2n} \sigma_n \leq (\log n)^2,$$

and hence we could also replace $Z_n$ with

$$\mathbb{P}\{ S[1, e^{2n}] \cap \hat{W}^1[0, \infty) = \emptyset \mid \hat{W}^1[0, \infty) \}.$$

Before giving the proof let us describe how it can be applied to sample the two-sided LERW in $d = 4$. For any fixed $n$, we can do the following:

- sample $\hat{W}^1[0, n]$ using the marginal distribution, with Radon-Nikodym derivative $Z_n / \mathbb{E} Z_n$ (this density is always non-degenerate for fixed $n$);
- given $\hat{W}^1$, sample a simple random walk $S$ conditioned so that $S[1, \infty) \cap \hat{W}^1[0, n] = \emptyset$;
- erasing loops from $S$ to get $\hat{S}$.

By sending $n \to \infty$ and applying (44) we obtained the two-sided LERW.

Proof. Using Theorem 3.1 we can see that for each $p > 0$, there exists $c_p$ such that

$$\lim_{n \to \infty} \mathbb{E} [Z_n^p] = c_p.$$

In particular, for each $p$, $\{Z_n^p\}$ is uniformly integrable and hence it suffices to establish the convergence in (44) with probability one.

We write $V$ for the (random) set $\hat{W}^1[0, \infty)$. Using Proposition 3.6 we can see that $\mathbb{P}\{ V \notin \mathcal{X}_n \}$ is fast decaying and hence with probability one $V \in \mathcal{X}_n$ for all $n$ sufficiently large.

Suppose first that $Z_n = 0$ for some $n$ or that

$$\limsup_{n \to \infty} \frac{Z_{n+1}}{Z_n} < 1.$$
Then $Z_n$ decays exponentially in $n$ and $Z = 0$. Hence, we may assume that
\[
\limsup_{n \to \infty} \frac{Z_{n+1}}{Z_n} = 1.
\]
Using the argument as in Proposition 3.13 we can see that for all $n$ sufficiently large, $Z_n > 0$ and $Z_{n+1} \geq Z_n/2$ and hence that $V \in \tilde{X}_n$ for all $n$ sufficiently large. We will establish the limit for $V \in \tilde{X}_n$. In this case, Corollary 3.16 implies that there exists a (random) $Y_1 \in (0, \infty)$ such that
\[
Z_n \sim Y_1 \exp \left\{ - \sum_{j=1}^{\lfloor n^{1/4} \rfloor} H(V_j^*) \right\},
\]
where
\[
V_n^* = V \cap \left\{ e^{(n-1)^4 + 4(n-1)} \leq |z| \leq e^{n^4 - 4n} \right\}.
\]
Let $\tilde{V}_n^*$ be the intersection of $\left\{ e^{(n-1)^4 + 4(n-1)} \leq |z| \leq e^{n^4 - 4n} \right\}$ and the loop-erasure of $W$ from the first visit to $\partial C_{(n-1)^4 + (n-1)}$ stopped at the first visit to $\partial C_{n^4 - n}$. The probability that $\tilde{V}_n^* \neq V_n^*$ decays faster than $n^{-1+u}$ for some $u > 0$ and hence by the Borel-Cantelli lemma, with probability one, for all $n$ sufficiently large $\tilde{V}_n^* = V_n^*$. From this, we see that there exists $Y_2 \in (0, \infty)$ such that with probability one
\[
Z_n \sim Y_2 \exp \left\{ - \sum_{j=1}^{\lfloor n^{1/4} \rfloor} H(\tilde{V}_j^*) \right\}.
\]
Our next claim is that with probability one there exists $Y_3$ such that
\[
\exp \left\{ - \sum_{j=1}^{n} H(\tilde{V}_j^*) \right\} \sim Y_3 \exp \left\{ - \sum_{j=1}^{n} \mathbb{E}[H(\tilde{V}_j^*)] \right\}.
\]
Indeed (12) and the strong Markov property implies that
\[
\mathbb{E} \left[ H(\tilde{V}_n^*) \mid \mathcal{F}_{(n-1)^4} \right] = \mathbb{E} \left[ H(\tilde{V}_n^*) \right] + O(e^{-n}).
\]
Proposition 3.22 now gives that
\[
\exp \left\{ - \sum_{j=1}^{n} \mathbb{E}[H(\tilde{V}_j^*)] \right\} \sim c \exp \left\{ - \sum_{j=1}^{n} \mathbb{E}[H(\eta_j)] \right\},
\]
and (34) shows that
\[
\exp \left\{ - \sum_{j=1}^{n} \mathbb{E}[H(\eta_j)] \right\} \sim c \left[ \exp \left\{ -3 \sum_{j=1}^{n} \mathbb{E}[H(\eta_j)] \right\} \right]^{1/3} \sim c' n^{-1/3}.
\]
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