FALTINGS’ LOCAL-GLOBAL PRINCIPLE FOR THE MINIMAXNESS OF LOCAL COHOMOLOGY MODULES

Mohammad Reza Doustimehr and Reza Naghipour
Department of Mathematics, University of Tabriz, Tabriz, Iran, and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
Tehran, Iran

The concept of Faltings’ local-global principle for the minimaxness of local cohomology modules over a commutative Noetherian ring $R$ is introduced, and it is shown that this principle holds at level 2. We also establish the same principle at all levels over an arbitrary commutative Noetherian ring of dimension not exceeding 3. These generalize the main results of Brodmann et al. in [6]. Moreover, it is shown that if $M$ is a finitely generated $R$-module, $\alpha$ an ideal of $R$ and $r$ a non-negative integer such that $\alpha^i H^i_\alpha(M)$ is skinny for all $i < r$ and for some positive integer $t$, then for any minimax submodule $N$ of $H^r_\alpha(M)$, the $R$-module $\text{Hom}_R(R/\alpha, H^r_\alpha(M)/N)$ is finitely generated. As a consequence, it follows that the associated primes of $H^r_\alpha(M)/N$ are finite. This generalizes the main results of Brodmann-Lashgari [5] and Quy [16].

Key Words: Cofinite module; Local cohomology; Local-global principle; Minimax module.

2010 Mathematics Subject Classification: 13D45; 14B15; 13E05.

1. INTRODUCTION

Throughout this paper, let $R$ denote a commutative Noetherian ring (with identity) and $\alpha$ an ideal of $R$. For an $R$-module $M$, the $i$th local cohomology module of $M$ with support in $V(\alpha)$ is defined as

$$H^i_\alpha(M) = \lim_{n \geq 1} \text{Ext}^i_R(R/\alpha^n, M).$$

Local cohomology was first defined and studied by Grothendieck. We refer the reader to [7] or [11] for more details about local cohomology. An important theorem in local cohomology is Faltings’ local-global principle for the finiteness dimension of local cohomology modules [10, Satz 1], which states that for a positive integer $r$, the $R_\alpha$-module $H^i_{\alpha R}(M_\alpha)$ is finitely generated for all $i \leq r$ and for all $\mathfrak{p} \in \text{Spec } R$ if and only if the $R$-module $H^i_\alpha(M)$ is finitely generated for all $i \leq r$.

Received August 28, 2013. Communicated by S. Goto.
Address correspondence to Prof. Reza Naghipour, Department of Mathematics, University of Tabriz, P.O. Box 51666-16471, Tabriz, Iran; E-mail: naghipour@ipm.ir and naghipour@tabrizu.ac.ir
Another formulation of Faltings’ local-global principle, particularly relevant for this paper, is in terms of the generalization of the finiteness dimension $f_\alpha(M)$ of $M$ relative to $\alpha$, where

$$f_\alpha(M) := \inf\{i \in \mathbb{N}_0 | H^i_\alpha(M) \text{ is not finitely generated}\}. \quad (\dagger)$$

With the usual convention that the infimum of the empty set of integers is interpreted as $\infty$. For any non-negative integer $n$, the $n$th finiteness dimension $f^n_\alpha(M)$ of $M$ relative to $\alpha$ is defined by

$$f^n_\alpha(M) := \inf\{f_{\alpha R_p}(M_p) | p \in \text{Supp}(M/\alpha M) \text{ and } \dim R/p \geq n\}.$$

Note that $f^n_\alpha(M)$ is either a positive integer or $\infty$ and that $f^0_\alpha(M) = f_\alpha(M)$. The $n$th finiteness dimension $f^n_\alpha(M)$ of $M$ relative to $\alpha$ has been introduced by Bahmanpour et al. in [4] and they showed that

$$f^n_\alpha(M) = \inf\{i \in \mathbb{N} : H^i_\alpha(M) \text{ is not minimax}\}.$$

We shall show that

$$f^1_\alpha(M) = \inf\{i \in \mathbb{N}_0 : \alpha^tH^i_\alpha(M) \text{ is not minimax for all } t \in \mathbb{N}\}.$$

This motivates to introduce the notion of the $\beta$-minimaxness dimension $\mu^\beta_\alpha(M)$ of $M$ relative to $\alpha$, by

$$\mu^\beta_\alpha(M) = \inf\{i \in \mathbb{N} : b^tH^i_\alpha(M) \text{ is not minimax for all } t \in \mathbb{N}\},$$

where $b$ is a second ideal of $R$. Note that $\mu^\beta_\alpha(M) = f^1_\alpha(M)$.

Recall that the $b$-finiteness dimension of $M$ relative to $\alpha$ is defined by

$$f^n_b(M) := \inf\{i \in \mathbb{N}_0 | b^t \not\subseteq \text{Rad}(0:R H^i_\alpha(M))\} = \inf\{i \in \mathbb{N}_0 | b^nH^i_\alpha(M) \neq 0 \text{ for all } n \in \mathbb{N}\}.$$

Brodmann et al. in [6] defined and studied the concept of the local-global principle for annihilation of local cohomology modules at level $r \in \mathbb{N}$ for the ideals $\alpha$ and $b$ of $R$. We say that the local-global principle for the annihilation of local cohomology modules holds at level $r$ if for every choice of ideals $\alpha$, $b$ of $R$ and every choice of finitely generated $R$-module $M$, it is the case that

$$f^n_{\alpha R_p}(M_p) > r \quad \text{for all } p \in \text{Spec } R \iff f^n_b(M) > r.$$

It is shown in [6] that the local-global principle for the annihilation of local cohomology modules holds at levels 1, 2, over an arbitrary commutative Noetherian ring $R$ and at all levels whenever $\dim R \leq 4$. 
We say that the local-global principle for the minimaxness of local cohomology modules holds at level \( r \in \mathbb{N} \) if for every choice of ideals \( \mathfrak{b}, \mathfrak{a} \) of \( R \) with \( \mathfrak{b} \subseteq \mathfrak{a} \) and every choice of finitely generated \( R \)-module \( M \), it is the case that

\[
\mu_{R_{\mathfrak{a} \mathfrak{b}}}(M_\mathfrak{p}) > r \quad \text{for all } \mathfrak{p} \in \text{Spec } R \iff \mu_{\mathfrak{a}}^\mathfrak{b}(M) > r.
\]

Our main result in Section 2 is to introduce the concept of Faltings' local-global principle for the minimaxness of local cohomology modules over a commutative Noetherian ring \( R \), and we show that this principle holds at level 2. We also establish the same principle at all levels over an arbitrary commutative Noetherian ring of dimension not exceeding 3. Our tools for proving the main result in Section 2 is the following theorem.

**Theorem 1.1.** Let \( R \) be a Noetherian ring, and let \( \mathfrak{b} \) be a second ideal of \( R \) such that \( \mathfrak{b} \subseteq \mathfrak{a} \). Let \( M \) be a finitely generated \( R \)-module, and let \( r \) be a positive integer such that the local cohomology modules \( H^0_{\mathfrak{a}}(M), \ldots, H^{r-1}_{\mathfrak{a}}(M) \) are \( \mathfrak{a} \)-cofinite. Then

\[
\mu_{R_{\mathfrak{a} \mathfrak{b}}}(M_\mathfrak{p}) > r \quad \text{for all } \mathfrak{p} \in \text{Spec } R \iff \mu_{\mathfrak{a}}^\mathfrak{b}(M) > r.
\]

Pursuing this point of view further we establish the following consequence of Theorem 1.1 which is an extension of the results of Brodmann et al. in [6, Corollary 2.3] and Raghavan in [17] for an arbitrary Noetherian ring.

**Corollary 1.2.** Let \( R \) be a Noetherian ring, \( M \) a finitely generated \( R \)-module and \( \mathfrak{a}, \mathfrak{b} \) two ideals of \( R \) such that \( \mathfrak{b} \subseteq \mathfrak{a} \) and \( \mathfrak{a}M \neq M \). Set \( r \in \{1, \text{grade}_M \mathfrak{a}, f_\mathfrak{a}(M), f_\mathfrak{a}^1(M), f_\mathfrak{a}^2(M)\} \). Then

\[
\mu_{R_{\mathfrak{a} \mathfrak{b}}}(M_\mathfrak{p}) > r \quad \text{for all } \mathfrak{p} \in \text{Spec } R \iff \mu_{\mathfrak{a}}^\mathfrak{b}(M) > r.
\]

In Section 3, we explore an interrelation between this principle and the Faltings’ local-global principle for the annihilation of local cohomology modules, and show that the local-global principle for the annihilation of local cohomology modules holds at level 2 over \( R \) and at all levels whenever \( \text{dim } R \leq 3 \). These generalize and reprove the main results of Brodmann et al. in [6].

An \( R \)-module \( L \) is said to be a FSF module if there is a Finitely generated submodule \( N \) of \( L \) such that \( \text{Support of quotient module } L/N \) is a Finite set. The class of FSF modules was introduced by Quy [16] and he has given some properties of these modules. Another main result in Section 3 is the following proposition.

**Proposition 1.3.** Let \( R \) be a Noetherian ring and \( M \) a finitely generated \( R \)-module. Let \( \mathfrak{a} \) be an ideal of \( R \) and \( r \) a positive integer such that the \( R \)-modules \( \mathfrak{a}^iH^0_{\mathfrak{a}}(M), \ldots, \mathfrak{a}^{r-1}H^{r-1}_{\mathfrak{a}}(M) \) are FSF for some \( i \in \mathbb{N}_p \). Then, for any minimax submodule \( N \) of \( H^0_{\mathfrak{a}}(M) \), the \( R \)-module \( \text{Hom}_R(R/\mathfrak{a}, H^0_{\mathfrak{a}}(M)/N) \) is finitely generated, and the \( R \)-modules \( H^0_{\mathfrak{a}}(M), \ldots, H^{r-1}_{\mathfrak{a}}(M) \) are \( \mathfrak{a} \)-cofinite.

We will call a module skinny or weakly Laskerian module, if each of its homomorphic images has only finitely many associated primes (cf. [9] and [18]). By
using Proposition 1.3, we obtain the following corollary which is a generalization of the main results of Brodmann and Lashgari [5] and Quy [16].

**Corollary 1.4.** Let \( R \) be a Noetherian ring and \( M \) a finitely generated \( R \)-module. Let \( \alpha \) be an ideal of \( R \) and \( r \) a positive integer such that the \( R \)-modules \( \alpha^iH^{(i)}_\alpha(M), \ldots, \alpha^rH^{(r)}_\alpha(M) \) are skinny. Then, for any minimax submodule \( N \) of \( H^s_\alpha(M) \), the set \( \text{Ass}_{R}H^s_\alpha(M)/N \) is finite.

Throughout this paper, \( R \) will always be a commutative Noetherian ring with nonzero identity and \( \alpha \) will be an ideal of \( R \). Recall that an \( R \)-module \( L \) is called \( \alpha \)-cofinite if \( \text{Supp}L \subseteq V(\alpha) \) and \( \text{Ext}^j_R(R/\alpha, L) \) is finitely generated for all \( j \geq 0 \). The concept of \( \alpha \)-cofinite modules were introduced by Hartshorne [12]. An \( R \)-module \( L \) is said to be minimax if there exists a finitely generated submodule \( N \) of \( L \), such that \( L/N \) is Artinian. The class of minimax modules was introduced by H. Zöschinger [19], and he has given ([19, 20]) many equivalent conditions for a module to be minimax. We shall use \( \text{Max}R \) to denote the set of all maximal ideals of \( R \). Also, for any ideal \( \mathfrak{a} \) of \( R \), we denote \( \{ p \in \text{Spec} R : p \supseteq \mathfrak{a} \} \) by \( V(\mathfrak{a}) \). Finally, for any ideal \( \mathfrak{b} \) of \( R \), the radical of \( \mathfrak{b} \), denoted by \( \text{Rad}(\mathfrak{b}) \), is defined to be the set \( \{ x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N} \} \). For any unexplained notation and terminology, we refer the reader to [7] and [13].

### 2. LOCAL-GLOBAL PRINCIPLE FOR MINIMAXNESS OF LOCAL COHOMOLOGY

In this section we introduce the concept of Faltings' local-global principle for the minimaxness of local cohomology modules over a commutative Noetherian ring \( R \), and we show that this principle holds at level 2. We also establish the same principle at all levels over an arbitrary commutative Noetherian ring of dimension not exceeding 3. We begin with the following lemmas which are needed in this section.

**Lemma 2.1.** Let \( R \) be a Noetherian ring, \( \alpha \) an ideal of \( R \), and \( M \) an arbitrary \( R \)-module. Then \( \alpha M \) is minimax if and only if \( M/(0:_{M} \alpha) \) is minimax.

**Proof.** This follows easily from the definition. \( \square \)

The next lemma, which is a generalization of [7, Lemma 9.1.2], states that the \( R \)-modules \( H^0_\alpha(M), \ldots, H^{s-1}_\alpha(M) \) are minimax if and only if there is an integer \( s \in \mathbb{N} \) such that \( \alpha^iH^0_\alpha(M), \ldots, \alpha^iH^{s-1}_\alpha(M) \) are minimax.

**Lemma 2.2.** Let \( R \) be a Noetherian ring, \( \alpha \) an ideal of \( R \), and \( M \) a finitely generated \( R \)-module. Let \( s \) be a positive integer. Then the following statements are equivalent:

(i) \( H^i_\alpha(M) \) is minimax for all \( i < s \);
(ii) There exists a positive integer \( t \) such that \( \alpha^tH^i_\alpha(M) \) is minimax for all \( i < s \).

**Proof.** The implication (i) \( \implies \) (ii) is obviously true. In order to show (ii) \( \implies \) (i), we proceed by induction on \( s \). If \( s = 1 \), there is nothing to show. Suppose that
s > 1 and the case s − 1 is settled. By inductive hypothesis the R-module $H_i^s(M)$ is minimax for all $i < s − 1$, and so it is enough for us to show that the R-module $H_i^{s−1}(M)$ is minimax. To this end, since by virtue of Lemma 2.1, the R-module $H_i^{s−1}(M)/(0 :_{H_i^{s−1}(M)}  \alpha')$ is minimax, there exists a finitely generated submodule $N$ of $H_i^{s−1}(M)$ such that the R-module

$$H_i^{s−1}(M)/N + (0 :_{H_i^{s−1}(M)}  \alpha'),$$

is Artinian. On the other hand, in view of [3, Theorem 2.3], the R-module $(0 :_{H_i^{s−1}(M)}  \alpha')$ is finitely generated, and hence $N + (0 :_{H_i^{s−1}(M)}  \alpha')$ is also finitely generated. Therefore, $H_i^{s−1}(M)$ is minimax, as required.

**Corollary 2.3.** Let $R$ be a Noetherian ring, $\alpha$ an ideal of $R$, and $M$ a finitely generated $R$-module. Let $f_i^v(M)$ denote the 1-th finiteness dimension of $M$ relative to $\alpha$. Then

$$f_i^v(M) = \inf \{i \in \mathbb{N}_0 : \alpha'H_i^v(M) \text{ is not minimax for all } t \in \mathbb{N}\}.$$

**Proof.** The result follows immediately from [4, Corollary 2.4] and Lemma 2.2. □

Now, we introduce the notion of the $v$-minimaxness dimension $\mu_i^v(M)$ of $M$ relative to $\alpha$, as a generalization of the $v$-finiteness dimension $f_i^v(M)$ of $M$ relative to $\alpha$.

**Definition 2.4.** Let $M$ be a finitely generated module over a Noetherian ring $R$, and let $v, \alpha$ be two ideals of $R$ such that $v \subseteq \alpha$. We define the $v$-minimaxness dimension $\mu_i^v(M)$ of $M$ relative to $\alpha$ by

$$\mu_i^v(M) := \inf \{i \in \mathbb{N}_0 : v'H_i^v(M) \text{ is not minimax for all } t \in \mathbb{N}\}.$$

Note that, since $\Gamma_i(M)$ is minimax, we can write

$$\mu_i^v(M) := \inf \{i \in \mathbb{N}_0 : v'H_i^v(M) \text{ is not minimax for all } t \in \mathbb{N}\},$$

that $\mu_i^v(M)$ is either a positive integer or $\infty$, and, by Corollary 2.3, $\mu_i^v(M) = f_i^v(M)$.

We can also introduce the Faltings’ local-global principle for the minimaxness of local cohomology modules which is a generalization of the Faltings’ local-global principle for the annihilation of local cohomology modules.

**Definition 2.5.** Let $R$ be a commutative Noetherian ring, and let $r$ be a positive integer. We say that the Faltings’ local-global principle for the minimaxness of local cohomology modules holds at level $r$ over the ring $R$ if, for every choice of ideals $\alpha, v$ of $R$ with $v \subseteq \alpha$ and for every choice of finitely generated $R$-module $M$, it is the case that

$$\mu_i^{v_h}(M_v) > r \text{ for all } v \in \text{Spec}R \iff \mu_i^v(M) > r.$$
The following theorem plays a key role in the proof of the main result of this section.

**Theorem 2.6.** Let $R$ be a Noetherian ring, and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of $R$ such that $\mathfrak{b} \subseteq \mathfrak{a}$. Let $M$ be a finitely generated $R$-module, and let $r$ be a positive integer such that the local cohomology modules $H^0_{\mathfrak{b}}(M), \ldots, H^{r-1}_{\mathfrak{b}}(M)$ are $\alpha$-cofinite. Then

$$\mu^{\mathfrak{b}}_{\alpha R} (M_v) > r \text{ for all } v \in \text{Spec}R \iff \mu^{\mathfrak{b}}_{\alpha} (M) > r.$$

**Proof.** Let $i$ be an arbitrary non-negative integer such that $i < r$. It is sufficient for us to show that there is a non-negative integer $t_0$ such that $t^i R^i (M)$ is minimax. To do this, in view of [3, Theorem 2.3], the set $\text{Ass}_R (b^i H^i_{\mathfrak{b}}(M))$ is finite, for all $t \in N_0$. Thus for all $t \in N_0$, the set $\text{Supp} b^i H^i_{\mathfrak{b}}(M)$ is a closed subset of $\text{Spec} R$ (in the Zariski topology), and so the descending chain

$$\cdots \supseteq \text{Supp}(b^i H^i_{\mathfrak{b}}(M)) \supseteq \text{Supp}(b^{i+1} H^{i+1}_{\mathfrak{b}}(M)) \supseteq \cdots$$

is eventually stationary. Therefore, there is a non-negative integer $t_0$ such that for each $t \geq t_0$,

$$\text{Supp}(b^i H^i_{\mathfrak{b}}(M)) = \text{Supp}(b^{t_0} H^{t_0}_{\mathfrak{b}}(M)).$$

Let $\mathfrak{m}$ be a maximal ideal of $R$. Since $\mu^{\mathfrak{b}}_{\alpha R} (M_v) > r$, for all $v \in \text{Spec}R$, it follows that there is an integer $u \geq t_0$ such that $(b R_{\mathfrak{m}}) u^i H^i_{\alpha R_{\mathfrak{m}}} (M_{\mathfrak{m}})$ is minimax. Now, let $v \in \text{Spec}R$ be such that $v \supseteq \mathfrak{m}$. Then it follows from

$$(b R_{\mathfrak{m}}) u^i H^i_{\alpha R_{\mathfrak{m}}} (M_{\mathfrak{m}})\big|_{R_{\mathfrak{m}}} \cong (b^v H^v_{\alpha} (M))_v$$

and the definition of minimax modules that $(b^v H^v_{\alpha} (M))_v$ is a finitely generated $R_v$-module (note that a module $L$ which is minimax has the property that the localization $L_v$ is a finitely generated $R_v$-module for each non-maximal prime ideal $v$). Now, as $(b^v H^v_{\alpha} (M))_v$ is $\alpha R_v$-torsion, there is an integer $v \geq 1$ such that $(b^{v+1} H^{v+1}_{\alpha} (M))_v = 0$, and so $v \notin \text{Supp} b^v H^v_{\alpha} (M)$. Therefore, $\text{Supp} b^v H^v_{\alpha} (M) \subseteq \text{Max} R$. Furthermore, in view of hypothesis and [3, Theorem 2.3], the $R$-module $\text{Hom}_R(R/\alpha, b^v H^v_{\alpha} (M))$ is finitely generated. Thus, as $\text{Hom}_R(R/\alpha, b^v H^v_{\alpha} (M)) \subseteq \text{Max} R$, it follows that $\text{Hom}_R(R/\alpha, b^v H^v_{\alpha} (M))$ is Artinian. As $b^v H^v_{\alpha} (M)$ is $\alpha$-torsion, it yields from Melkersson’s theorem [14, Theorem 1.3] that $b^v H^v_{\alpha} (M)$ is Artinian. Hence $b^v H^v_{\alpha} (M)$ is minimax, as required. □

**Corollary 2.7.** Let $R$ be a Noetherian ring, $M$ a finitely generated $R$-module, and $\alpha$ an ideal of $R$ with $\dim M/\alpha M \leq 1$. Let $\mathfrak{b}$ be a second ideal of $R$ such that $\mathfrak{b} \subseteq \alpha$. Then, for any integer $r$,

$$\mu^{\mathfrak{b}}_{\alpha R} (M_v) > r \quad \text{for all } v \in \text{Spec}R \iff \mu^{\mathfrak{b}}_{\alpha} (M) > r.$$

**Proof.** The assertion follows from [4, Corollary 3.5] and Theorem 2.6. □
Corollary 2.8. The local-global principle (for the minimaxness of local cohomology modules) holds over any (commutative Noetherian) ring \( R \) with \( \dim R \leq 2 \) at all levels \( r \in \mathbb{N} \).

**Proof.** The result follows easily from [8, Corollary 5.2] and Theorem 2.6. □

Our next corollary is a generalization of a result of Raghavan [17].

Corollary 2.9. The local-global principle (for the minimaxness of local cohomology modules) holds at level 1 (over any commutative Noetherian ring).

**Proof.** The assertion follows from Theorem 2.6. □

Our next corollary is a generalization of a result of Brodmann et al. [6].

Corollary 2.10. Let \( R \) be a Noetherian ring, and let \( \alpha, \beta \) be two ideals of \( R \) such that \( \beta \subseteq \alpha \). Let \( M \) be a finitely generated \( R \)-module such that \( \alpha M \neq M \). Then

\[
\mu_{\beta R_{\alpha}}(M_{\alpha}) > \text{grade}_{\alpha \alpha} \quad \text{for all } \varpi \in \text{Spec} R \iff \mu_{\alpha}^{\beta}(M) > \text{grade}_{\alpha \alpha}.
\]

**Proof.** The assertion follows from the definition of \( \text{grade}_{\alpha \alpha} \) and Theorem 2.6. □

The next result is a generalization of Corollary 2.10.

Corollary 2.11. Let \( R \) be a Noetherian ring, and let \( \alpha, \beta \) be two ideals of \( R \) such that \( \beta \subseteq \alpha \). Let \( M \) be a finitely generated \( R \)-module, and that \( r \in \{ f_{\beta}(M), f_{\alpha}^{1}(M), f_{\alpha}^{2}(M) \} \). Then

\[
\mu_{\alpha R_{\beta}}(M_{\beta}) > r \quad \text{for all } \varpi \in \text{Spec} R \iff \mu_{\beta}^{\alpha}(M) > r.
\]

**Proof.** The assertion follows from [4, Theorems 2.3 and 3.2] and Theorem 2.6. □

We are now ready to state and prove the main theorem of this section, which shows that Faltings’ local-global principle for the minimaxness of local cohomology modules is valid at level 2 over any commutative Noetherian \( R \). This generalizes the main result of Brodmann et al. in [6, Theorem 2.6].

**Theorem 2.12.** The local-global principle (for the minimaxness of local cohomology modules) holds over any (commutative Noetherian) ring \( R \) at level 2.

**Proof.** Let \( M \) be a finitely generated \( R \)-module such that \( \mu_{\alpha R_{\beta}}(M_{\beta}) > 2 \) for all \( \varpi \in \text{Spec} R \). We must show that \( \mu_{\beta}^{\alpha}(M) > 2 \). To do this, in view of Corollary 2.9, it is enough for us to show that there exists a non-negative integer \( t \) such that the \( R \)-module \( \beta^{t}H_{\alpha}^{2}(M) \) is minimax. To do this, let \( M' = M/\Gamma_{\beta}(M) \). We first show that the \( R \)-module \( \text{Hom}_{R}(R/\alpha, H_{\alpha}^{2}(M')) \) is minimax. The short exact sequence

\[
0 \longrightarrow \Gamma_{\beta}(M) \longrightarrow M \longrightarrow M' \longrightarrow 0,
\]

is exact at each level.
induces the exact sequence

\[
H^1_\alpha(M) \longrightarrow H^1_\alpha(M') \longrightarrow H^2_\alpha(\Gamma_\alpha(M)) \longrightarrow H^2_\alpha(M) \longrightarrow H^2_\alpha(M').
\]

(\dagger)

Next, since the set \(\text{Ass}_R H^1_\alpha(M)\) is finite, it follows from the proof of Theorem 2.6 that, for every \(\nu \in \text{Spec} R\) with \(\dim R/\nu > 0\), there exists a non-negative integer \(u\) such that \((\nu^u H^1_\alpha(M))_\nu = 0\). Furthermore, there exists a non-negative integer \(v\) such that \(\nu^v H^1_\alpha(\Gamma_\alpha(M)) = 0\) for all \(i \geq 0\). Thus it follows from the exact sequence obtained by the localization of the exact sequence \((\dagger)\) at the prime ideal \(\nu\) with \(\dim R/\nu > 0\) and \([7, \text{Lemma 9.1.1}]\) that \((\nu^v H^1_\alpha(M_\nu)) = 0\), for some integer \(k \in \mathbb{N}_0\). Moreover, by \([7, \text{Lemma 2.1.1}]\), there exists \(x \in b\) which is a non-zerodivisor on \(M'\). Then \(x^j H^1_{\alpha b}(M_\nu) = 0\), and if we consider the long exact sequence of local cohomology modules (with respect to \(\alpha R_\nu\)) induced by the short exact sequence

\[
0 \longrightarrow M'_\nu \xrightarrow{x^j} M'_\nu \longrightarrow M'_\nu/x^j M'_\nu \longrightarrow 0,
\]

we see that \(H^1_{\alpha b}(M'_\nu)\) is a homomorphic image of \(H^0_{\alpha b}(M'_\nu/x^j M'_\nu)\), and so it is a finitely generated \(R_\nu\)-module, for all \(\nu \in \text{Spec} R\) with \(\dim R/\nu > 0\). It therefore follows from \([4, \text{Proposition 2.2}]\) that \(H^1_\alpha(M')\) is minimax; and hence, by \([3, \text{Theorem 2.3}]\), the \(R\)-module \(\text{Hom}_R(R/\alpha, H^2_\alpha(M'))\) is finitely generated.

Since \(s_h R_\nu(M_\nu) > 2\), analogous to the proof of Theorem 2.6, for each \(\nu \in \text{Spec} R\) with \(\dim R/\nu > 0\), there is \(u_\nu \in \mathbb{N}_0\) such that \((\nu^u H^2_\alpha(M))_\nu = 0\). Thus it follows from the exact sequence obtained by the localization of the exact sequence \((\dagger)\) at the prime ideal \(\nu\) with \(\dim R/\nu > 0\) and \([7, \text{Lemma 9.1.1}]\) that \((\nu^v H^2_\alpha(M'))_\nu = 0\), for some integer \(v_\nu \in \mathbb{N}_0\). Since \(\text{Hom}_R(R/\alpha, H^2_\alpha(M'))\) is finitely generated, it follows from the proof of Theorem 2.6 that there is an integer \(k \in \mathbb{N}_0\) such that \(\nu^k H^2_\alpha(M')\) is Artinian, and so \(\text{Supp}(\nu^k H^2_\alpha(M'))\) is a finite subset of \(\text{Max} R\). Now, let \(s := v + k\). Then \(\text{Supp}(\nu^s H^2_\alpha(M')) \subseteq \text{Supp}(\nu^k H^2_\alpha(M')) \subseteq \text{Max} R\). Let \(\text{Supp}(\nu^s H^2_\alpha(M')) := \{m_1, \ldots, m_r\}\). Then by assumption, for each integer \(j\) with \(1 \leq j \leq r\), there is a non-negative integer \(s_j \geq s\) such that \((\nu^{s_j} H^2_\alpha(M))_{m_j}\) is a minimax \(R_{m_j}\)-module. If we set \(t = \max\{s_1, \ldots, s_r\}\), then \(\text{Supp}(\nu^t H^2_\alpha(M)) \subseteq \{m_1, \ldots, m_r\}\) and \((\nu^t H^2_\alpha(M))_{m_j}\) is minimax, for all \(j\) with \(1 \leq j \leq r\). Therefore, by \([2, \text{Theorem 3.3}]\) and \([1, \text{Proposition 2.2}]\), \(\nu^t H^2_\alpha(M)\) is a minimax \(R\)-module, as required.

\[\square\]

**Corollary 2.13.** The local-global principle (for the minimaxness of local cohomology modules) holds over any (commutative Noetherian) ring \(R\) with \(\dim R \leq 3\).

**Proof.** The assertion follows from Corollary 2.9, Theorem 2.12, and \([7, \text{Exercise 7.1.7}]\). \[\square\]

### 3. Annihilation and Associated Primes of Local Cohomology Modules

The main goal of this section is to explore an interrelation between the Faltings’ local-global principle for the minimaxness and annihilation of local cohomology modules, and to show the local-global principle for the annihilation of local cohomology modules holds at level 2 over \(R\) and at all levels whenever
dim \( R \leq 3 \). These reprove the main results of Brodmann et al. in [6]. Moreover, it will be shown in this section that the subjects of the previous section can be used to prove a finiteness result about local cohomology module. In fact, we will generalize the main results of Brodmann–Lashgari and Quy. The main result is Theorem 3.6.

The following result describes a relation between the local-global principle for the minimaxness and the annihilation (of local cohomology modules) over a commutative Noetherian ring \( R \).

**Proposition 3.1.** The local-global principle for the minimaxness (of local cohomology modules) implies the local-global principle for the annihilation (of local cohomology modules) over any (commutative Noetherian) ring \( R \).

**Proof.** Let \( r \) be a non-negative integer, and suppose that the local-global principle for the minimaxness of local cohomology modules holds at level \( r \). Let \( M \) be a finitely generated \( R \)-module such that \( f_{aR}(M) > r \) for all \( \mathfrak{p} \in \text{Spec} R \). We must show that \( f_{\mathfrak{p}}^*(M) > r \). To this end, as \( f_{aR}(M) > r \), it follows that \( \mu_{aR}(M) > r \). Therefore, by hypothesis \( \mu_{aR}(M) > r \), and hence there exists \( t \in \mathbb{N} \) such that \( \mathfrak{p}^t H_i^\mathfrak{p}(M) \) is minimax for all \( i \leq r \). Thus the set \( \text{Ass}_R(\mathfrak{p}^t H_i^\mathfrak{p}(M)) \) is finite. Let \( \text{Ass}_R(\mathfrak{p}^t H_i^\mathfrak{p}(M)) := \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_s \} \). By assumption, for each \( 1 \leq j \leq s \), there is an integer \( t_j \geq t \) such that \( (\mathfrak{p}^t H_i^\mathfrak{p}(M))_{\mathfrak{p}_j} = 0 \). Set \( k := \max\{t_1, \ldots, t_s\} \). Then \( \text{Ass}_R(\mathfrak{p}^t H_i^\mathfrak{p}(M)) = \emptyset \), and hence \( \mathfrak{p}^t H_i^\mathfrak{p}(M) = 0 \). Therefore, \( f_{\mathfrak{p}}^*(M) > r \), as required. \( \Box \)

As a consequence of previous proposition and Theorem 2.12, the following corollary shows that the local-global principle for the annihilation of local cohomology modules holds at level 2 over \( R \). This reproves the main result of Brodmann et al. in [6].

**Corollary 3.2.** The local-global principle (for the annihilation of local cohomology modules) holds over any (commutative Noetherian) ring \( R \) at level 2.

**Proof.** The assertion follows from Proposition 3.1 and Theorem 2.12. \( \Box \)

In [16], P. H. Quy introduced the class of FSF modules and has given some properties of these modules. An \( R \)-module \( L \) is said to be a FSF module if there is a finitely generated submodule \( N \) of \( L \) such that support of quotient module \( L/N \) is finite. It is shown in [2, Theorem 3.3] that an \( R \)-module \( L \) is FSF if and only if it is skinny.

**Proposition 3.3.** Let \( R \) be a Noetherian ring, \( M \) a finitely generated \( R \)-module, \( \alpha \) an ideal of \( R \), and \( r \) a positive integer such that the \( R \)-modules \( \alpha^t H_i^\alpha(M), \ldots, \alpha^{t-r+1} H_i^\alpha(M) \) are FSF for some \( t \in \mathbb{N} \). Then the \( R \)-module \( \text{Hom}_R(R/\alpha, H_i^\alpha(M)) \) is finitely generated and the \( R \)-modules \( H_i^\alpha(M), \ldots, H_i^{t-r+1} H_i^\alpha(M) \) are \( \alpha \)-cofinite. In particular, the set \( \text{Ass}_R H_i^\alpha(M) \) is finite.

**Proof.** Since for each \( i < r \), \( \alpha^t H_i^\alpha(M) \) is FSF, it follows that there is a finitely generated submodule \( N_i \) of \( \alpha^t H_i^\alpha(M) \) such that the set \( \text{Supp}(\alpha^t H_i^\alpha(M)/N_i) \) is finite,
and so \( \dim(a'H_i(M)/N) \leq 1 \), for each \( i < r \). Therefore, for each \( \mathfrak{p} \in \text{Spec}R \) with \( \dim R/\mathfrak{p} > 1 \), we have

\[
(aR_\mathfrak{p})^tH^t_{\alpha R_\mathfrak{p}}(M_\mathfrak{p}) \cong (a'H_i(M))_\mathfrak{p} \cong (N)_\mathfrak{p}.
\]

Hence the \( R_\mathfrak{p} \)-module \( (aR_\mathfrak{p})^tH^t_{\alpha R_\mathfrak{p}}(M_\mathfrak{p}) \) is finitely generated, for each \( i < r \). Now, as \( (aR_\mathfrak{p})^tH^t_{\alpha R_\mathfrak{p}}(M_\mathfrak{p}) \) is \( \alpha R_\mathfrak{p} \)-torsion, so there exists a non-negative integer \( s \) such that \( (aR_\mathfrak{p})^{t+s}H^t_{\alpha R_\mathfrak{p}}(M_\mathfrak{p}) = 0 \). Thus, by \([7, \text{Proposition 9.1.2}]\), \( H^t_{\alpha R_\mathfrak{p}}(M_\mathfrak{p}) \) is a finitely generated \( R_\mathfrak{p} \)-module, for every \( \mathfrak{p} \in \text{Spec}R \) with \( \dim R/\mathfrak{p} > 1 \) and for all \( i < r \).

It therefore follows from \([4, \text{Proposition 3.1}]\) that \( \text{Hom}_{R}(R/\alpha, H^0(M)) \) is finitely generated and the \( R \)-modules \( H^0(M), \ldots, H^{r-1}(M) \) are \( \alpha \)-cofinite.

**Corollary 3.4.** Let \( R \) be a Noetherian ring, \( M \) a finitely generated \( R \)-module, \( \alpha \) an ideal of \( R \), and \( r \) a positive integer such that the \( R \)-modules \( \alpha'H^0(M), \ldots, \alpha'H^{r-1}(M) \) are skinny. Then the \( R \)-modules \( H^0(M), \ldots, H^{r-1}(M) \) are \( \alpha \)-cofinite and for any ideal \( \mathfrak{b} \) of \( R \) with \( \mathfrak{b} \subseteq \alpha \),

\[
\mu^\mathfrak{b}_{\alpha R}(M_\mathfrak{p}) > r \quad \text{for all } \mathfrak{p} \in \text{Spec}R \iff \mu^\mathfrak{b}_{\alpha}(M) > r.
\]

**Proof.** Apply \([2, \text{Theorem 3.3}]\), Proposition 3.3, and Theorem 2.6. \( \square \)

**Corollary 3.5.** Let \( R \) be a Noetherian ring, \( M \) a finitely generated \( R \)-module, \( \alpha \) an ideal of \( R \), and \( r \) a positive integer such that the \( R \)-modules \( \alpha'H^0(M), \ldots, \alpha'H^{r-1}(M) \) have finite support, for some \( t \in \mathbb{N}_0 \). Then the \( R \)-modules \( H^0(M), \ldots, H^{r-1}(M) \) are \( \alpha \)-cofinite and for any ideal \( \mathfrak{b} \) of \( R \) with \( \mathfrak{b} \subseteq \alpha \),

\[
\mu^\mathfrak{b}_{\alpha R}(M_\mathfrak{p}) > r \quad \text{for all } \mathfrak{p} \in \text{Spec}R \iff \mu^\mathfrak{b}_{\alpha}(M) > r.
\]

**Proof.** Since the set \( \text{Supp}(\alpha'H^i(M)) \) is finite, for every \( i < r \), it follows that the \( R \)-module \( \alpha'H^i(M) \) is FSF for each \( i < r \). Now the assertion follows from Corollary 3.4. \( \square \)

The following theorem, which is one of our main results, generalizes the main results of Brodmann and Lashgari \([5]\) and Quy \([16]\).

**Theorem 3.6.** Let \( R \) be a Noetherian ring, \( M \) a finitely generated \( R \)-module, and \( \alpha \) an ideal of \( R \). Let \( r \) be a non-negative integer such that \( \alpha'H^i(M) \) is FSF for all \( i < r \) and for some \( t \in \mathbb{N}_0 \). Then, for any minimax submodule \( N \) of \( H^r(M) \), the \( R \)-module \( \text{Hom}_{R}(R/\alpha, H^r(M)/N) \) is finitely generated. In particular, the set \( \text{Ass}_{R}(H^r(M)/N) \) is finite.

**Proof.** In view of Proposition 3.3, \( \text{Hom}_{R}(R/\alpha, H^r(M)) \) is finitely generated. On the other hand, according to Melkersson \([15, \text{Proposition 4.3}]\), \( N \) is \( \alpha \)-cofinite. Now, the exact sequence

\[
0 \to N \to H^r(M) \to H^r_a(M)/N \to 0
\]
induces the exact sequence,
\[ \text{Hom}_R(R/\alpha, H_\alpha^r(M)) \to \text{Hom}_R(R/\alpha, H_\alpha^r(M)/N) \to \text{Ext}_R^1(R/\alpha, N). \]
Consequently, \( \text{Hom}_R(R/\alpha, H_\alpha^r(M)/N) \) is finitely generated, as required. □

In [3, Theorem 2.3], Bahmanpour and Naghipour showed that, if \( H_0^0(M), \ldots, H_{r-1}^0(M) \) are minimax, then \( H_0^0(M), \ldots, H_{r-1}^0(M) \) are \( \alpha \)-cofinite. The following corollary provides a slight generalization of [3, Theorem 2.3].

**Corollary 3.7.** Let \( R \) be a Noetherian ring, \( M \) a finitely generated \( R \)-module, \( \alpha \) an ideal of \( R \), and \( r \) a positive integer such that the \( R \)-modules \( \alpha^i H_\alpha^0(M), \ldots, \alpha^i H_\alpha^{r-1}(M) \) are skinny. Then the set \( \text{Ass}_R H_\alpha^r(M) \) is finite.

**Proof.** Apply [2, Theorem 3.3] and Theorem 3.6. □

**ACKNOWLEDGMENTS**

The authors would like to thank Professor Hossein Zakeri for his reading of the first draft and valuable discussions.

**FUNDING**

We also would like to thank from the Institute for Research in Fundamental Sciences (IPM) for its financial support.

**REFERENCES**

[1] Aghapour, M., Melkersson, L. (2010). Finiteness properties of minimax and coatomic local cohomology modules. Arch. Math. 94:519–528.

[2] Bahmanpour, K., On the category of weakly laskerian cofinite modules. Math. Scand. To appear.

[3] Bahmanpour, K., Naghipour, R. (2008). On the cofiniteness of local cohomology modules. Proc. Amer. Math. Soc. 136:2359–2363.

[4] Bahmanpour, K., Naghipour, R., Sedghi, M. (2013). Minimaxness and cofiniteness properties of local cohomology modules. Comm. Algebra 41:2799–2814.

[5] Brodmann, M. P., Lashgari, F. A. (2000). A finiteness result for associated primes of local cohomology modules. Proc. Amer. Math. Soc. 128:2851–2853.

[6] Brodmann, M. P., Rothaus, Ch., Sharp, R. Y. (2000). On annihilators and associated primes of local cohomology modules. J. Pure and Appl. Algebra 153:197–227.

[7] Brodmann, M. P., Sharp, R. Y. (1998). Local Cohomology: An Algebraic Introduction with Geometric Applications. Cambridge: Cambridge University Press.

[8] Cuong, N. T., Goto, S., Hoang, N. V., On the Cofiniteness of Generalized Local Cohomology Modules. Preprint.

[9] Divaani-Aazar, K., Mafi, A. (2005). Associated primes of local cohomology modules. Proc. Amer. Math. Soc. 133:655–660.

[10] Faltings, G. (1981). Der Endlichkeitssatz in der lokalen Kohomologie. Math. Ann. 255:45–56.

[11] Grothendieck, A. (1966). Local Cohomology. Notes by R. Hartshorne. Lecture Notes in Math., vol. 862. New York: Springer.
[12] Hartshorne, R. (1970). Affine duality and cofiniteness. *Invent. Math.* 9:145–164.
[13] Matsumura, H. (1986). *Commutative Ring Theory.* Cambridge, UK: Cambridge Univ. Press.
[14] Melkersson, L. (1990). On asymptotic stability for sets of prime ideals connected with the powers of an ideal. *Math. Proc. Cambridge Philos. Soc.* 107:267–271.
[15] Melkersson, L. (2005). Modules cofinite with respect to an ideal. *J. Algebra* 285:649–668.
[16] Quy, P. H. (2010). On the finiteness of associated primes of local cohomology modules. *Proc. Amer. Math. Soc.* 138:1965–1968.
[17] Raghavan, K. N. (1994). Local-global principle for annihilation of local cohomology. *Contemporary Math.* 159:329–331.
[18] Robbins, H. (2012). Associated primes of local cohomology and $S_2$-ification. *J. Pure and Appl. Algebra* 216:519–523.
[19] Zöschinger, H. (1986). Minimax modules. *J. Algebra.* 102:1–32.
[20] Zöschinger, H. (1988). Über die maximalbedingung für radikalvolle untermoduln. *Hokkaido Math. J.* 17:101–116.