Learning with Safety Constraints: 
Sample Complexity of Reinforcement Learning for 
Constrained MDPs

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Abstract

Many physical systems have underlying safety considerations that require that the policy employed ensures the satisfaction of a set of constraints. The analytical formulation usually takes the form of a Constrained Markov Decision Process (CMDP), where the constraints are some function of the occupancy measure generated by the policy. We focus on the case where the CMDP is unknown, and RL algorithms obtain samples to discover the model and compute an optimal constrained policy. Our goal is to characterize the relationship between safety constraints and the number of samples needed to ensure a desired level of accuracy—both objective maximization and constraint satisfaction—in a PAC sense. We explore generative model based class of RL algorithms wherein samples are taken initially to estimate a model. Our main finding is that compared to the best known bounds of the unconstrained regime, the sample complexity of constrained RL algorithms are increased by a factor that is logarithmic in the number of constraints, which suggests that the approach may be easily utilized in real systems.

1 Introduction

Markov Decision Processes (MDPs) are used to model a variety of systems for which stationary control policies are appropriate. In many physical systems, restrictions may be placed on functions of the probability with which states may be visited. For example, in power systems, the frequency must be kept within tolerable limits, and allowing it to go outside these tolerances often might be unsafe. Similarly, in communication systems the number of transmissions that may be made in a time interval is limited by an average radiated power constraint due to interference and human safety considerations. The Constrained-MDP (CMDP) framework is used to model such circumstances [1]. Here, the idea is that any stationary policy will generate an occupancy measure over the states, and the constraint must be evaluated over this occupancy measure. It is also well known that randomization is needed in optimal policies for CMDP problems.

In this paper, our objective is to design simple algorithm to solve CMDP problems under an unknown model. Whereas the goal of a typical model-based RL approach would take as few samples as possible to quickly determine the optimal policy, minimizing the number of samples taken is even more important in the CMDP setting. This because constraints are violated during the learning process, and it might be critical to keep the number of such violations as low as possible due to safety considerations mentioned earlier, and yet ensure that the system objectives are maximized. Hence, determining how the joint metrics of objective maximization and safety violation evolve over time as the model becomes more and more accurate is crucial to understand the efficacy of a proposed RL algorithm for CMDPs.

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Main Contributions: Our goal is to analyze the sample complexity of solving CMDPS to a desired accuracy with a high probability in both objective and constraints in the context of finite horizon (episodic) problems. We focus on two figures of merit in a probably-approximately-correct (PAC) sense corresponding to objective maximization and constraint satisfaction. Our main contributions are as follows:

(i) We develop a model-based algorithm, namely, a generative approach that obtains samples initially creates a model. The estimated model might have no solution, and we utilize a confidence-ball around the estimate to ensure that a solution may be found with high probability (assuming that the real model has a solution).

(ii) The algorithm follows the general pattern of model construction or update, followed by a solution using linear programming (LP) of the CMDP generated in this manner, with the addendum that the LP is extended to account for the fact that a search is made over the entire ball of feasible models given the current samples.

(iii) We develop PAC-type sample complexity bound for the algorithm, accounting for both objective maximization and constraint satisfaction. The general intuition is that the model accuracy should be higher than in the unconstrained case. Our main finding agrees with this intuition, and further discovers that the increase in the sample complexity is by a logarithmic factor in the number of constraints.

The results indicate that the number of constraints is not a major concern in solving unknown CMDPs via RL, hence raising the possibility of practical applications.

Related Work: Much work in the space of CMDP has been driven by problems of control, and many of the algorithmic approaches and applications have taken a control-theoretic view [1–5]. The approach taken is to study the problem under a known model, and showing asymptotic convergence of the solution method proposed. Extending such approaches to the context on an unknown model has also mostly focused on asymptotic convergence [7–10] under Lagrangian methods to show zero eventual duality gap.

A parallel theme has been related to the constrained bandit case, wherein the the underlying problem, while not directly being an MDP, bears a strong relation to it. Work such as [11–13] consider such constraints, either in a knapsack sense, or on the type of controls that may be applied in a linear bandit context.

Closest to our work are parallel works on CMDPs where the reward is unknown [14] and on regret analysis of CMDPs [15]. In particular, [15] explores algorithms and themes similar to ours, but focuses on characterizing objective and constrained regret under different flavors of online algorithms. Our work can be seen as the complement to these, and provides a sample complexity view point. Our discovery of a general principle of logarithmic increase in sample complexity with the number of constraints also distinguishes our work.

2 Problem Formulation and Solution Overview

2.1 CMDP Formulation

We consider a general finite-horizon CMDP formulation. There are a set of states \( S \) and set of actions \( A \). Reward matrix is denoted by \( r(s, a) \) for any state-action pair \( (s, a) \). We also use cost matrix \( c(i, s, a) \) and constraint vector \( C \) to formulate \( N \) constraints where \( i \in \{1, \ldots, N\} \). The probability of reaching another state \( s' \) while being at state \( s \) taking action \( a \) is determined by transition kernel \( P(s'|s, a) \). As the CMDP is has finite horizon, the length of each episode; or horizon, is considered

\[ \text{We use } |.| \text{ to indicate the cardinality of any set.} \]
to be a fixed value $H$. So, CMDP is $M$ is defined by the tuple $M = \langle S, A, P, r, c, \bar{C}, H \rangle$. Next, in order to choose an action from the set $A$, we define policy $\pi$ which is a mapping from state space $S$ to action space $A$. These criteria are briefed in Assumption [1]

**Assumption 1.** We assume $S$ and $A$ are finite sets with cardinalities $|S|$ and $|A|$. Further, we assume that the immediate reward $r(s,a)$ is taken from the interval $[-1,1]$ and immediate cost lies in $[0,1]$. We also make an assumption that there are $N$ constraints which for each $i \in \{1, \ldots, N\}$, $\bar{C}_i \in [-\bar{C}_{\text{max}}, \bar{C}_{\text{max}}]$.

Further, we consider cumulative finite horizon criteria for both objective function and cost functions of constraints with identical horizon $H$. Therefore, the objective function under policy $\pi$ starting from initial state $s_0$ would be

$$V_0^\pi(s_0) = \mathbb{E}\left[ \sum_{h=0}^{H-1} r(s_h, a_h); a_h \sim \pi(s_h, h) \right], \quad (1)$$

where action $a_h$ is chosen according to policy $\pi$ and expectation $\mathbb{E}[.]$ is defined w.r.t transition kernel $P$. Similarly, the $i^{th}$ cost function under policy $\pi$ would be formulated as

$$C_{i,0}^\pi(s_0) = \mathbb{E}\left[ \sum_{h=0}^{H-1} c(i, s_h, a_h); a_t \sim \pi(s_h, h) \right]. \quad (2)$$

Now, the general finite-horizon CMDP problem is formulated as

$$\max_{\pi} V_0^\pi(s_0) \text{ s.t. } C_{i,0}^\pi(s_0) \leq \bar{C}_i, \quad \forall i \in \{1, \ldots, N\}. \quad (3)$$

**Assumption 2.** We assume that there exists some policy $\pi$ that satisfies the constraints in (3).

This assumption does not limit us since this article studies learning feasible CMDPs.

**Notation:** Along analyzing learning procedure of feasible CMDPs, we make use of some notations which we introduce them here.

The operator $P_\pi f(s) = \mathbb{E}[f(s_{h+1})|s_h = s] = \sum_{s' \in S} P_\pi(s'|s)f(s')$ takes any function $f : S \rightarrow \mathbb{R}$ and returns the expected value of $f$ with respect to ; w.r.t., the next time step. Please note that $P_\pi(s'|s) = \sum_{a \in A} P(s'|s,a)\pi(a|s)$. For convenience, we define the multi-step version $P_\pi^h f(s) = P_\pi P_\pi \ldots P_\pi f$ which is repeated $h$ times.

The local variance of the value function at time step $h$ under policy $\pi$ is

$$\sigma_{h}^2(s) = \mathbb{E}[(V_{h+1}^\pi(s_{h+1}) - P_\pi V_{h+1}^\pi(s))^2].$$

Similarly, we define $\sigma_{i,h}^2$ as local variance of $i^{th}$ cost function at time-step $h$ under policy $\pi$.

**CMDP Solution Overview:** Let $\pi^*$ be the optimal solution to CMDP problem (3). It is discussed in [1] that unlike MDP solution, $\pi^*$ might not always be a deterministic policy over episodes. However, the optimal solution is a stationary policy. Furthermore, $\pi^*$ also depends on initial state $s_0$; or on initial state distribution generally, in contrast to optimal solutions of MDP problems.

**Linear programming (LP)** is one technique to solve CMDP problem (3) [4] [11]. To convert CMDP problem (3) to a linear programming problem, we introduce occupation measures. The finite-horizon state-action occupation measure under policy $\pi$ is defined as

$$\mu(s,a,\pi, h) := \mathbb{P}(s_h = s, a_h = a), \quad (4)$$
where the probability is calculated w.r.t. underlying transition kernel under policy \( \pi; P_\pi \). It is shown that both objective function and cost functions could be restated as functions of occupation measures. Then, the problem would become to find the optimal occupation measures. This procedure may be accomplished by creating a Linear Program that is equivalent to the CMDP problem [1].

**Constrained RL:** The Constrained RL problem formulation is identical to CMDP optimization problem of [3], but without being aware of values of transition kernel \( P \). Thus, although the problem is quite similar to CMDP, but the solution approach is more intricate. Our goal is to provide model-based algorithms and determine the sample complexity results.

We provide a generative model-based algorithms to solve the Constrained RL problem. Here, we have an estimated model obtained via sampling state action pairs in an offline manner. Since there might be no solution to the CMDP corresponding to this estimated model, we cannot directly solve for an optimal policy using the LP approach. Rather, we need to expand the space of models around the estimated model to increase the probability that a feasible policy will exist somewhere within the ball, and then to search for a feasible policy. As the number of samples increases and the estimated model converges to the true model, the ball can be tightened. We will show in the analysis of the algorithms that we develop that this indeed will take place with high probability.

This general approach is consistent with the upper-confidence (UC) style of model construction, and its use for enhancing the feasibility set is presented in [15]. Now, given that there is now a whole ensemble of models that must be considered while determining the policy, a modified LP called the extended-LP formulation (ELP) is needed to obtain a candidate feasible solution (if it exists within the ball). ELP is another Linear Programming with different occupancy measures \( q(s, a, s', h) = P(s'|s, a) = \mu(s, a, h) \), where \( \mu(s, a, h) \) is defined as [4]. Then, it outputs an optimistic model which all variables related to this model is denoted by \( \sim \), like \( \tilde{P} \) or \( \tilde{V} \). Finally, optimistic transition kernel and optimistic policy is

\[
\tilde{P}(s'|s, a) = \frac{q(s, a, s', h)}{\sum yq(s, a, y, h)} \quad \tilde{\pi}^*(s, a, h) = \frac{\sum zq(s, a, s', h)}{\sum b, s' q(s, b, s', h)}
\]

The details of ELP are presented in Appendix 5.

Given this solution approach for an estimated model, we are now ready to design the algorithm for learning such models and computing the attendant optimal policies. The approach taken will be to characterize the number of samples needed to ensure both objective maximization and constraint satisfaction in a PAC sense.

### 3 Optimistic- Generative Model Based Learning

According to Optimistic-GMBL, we sample each state-action pair \( n \) number of times uniformly across all state-action pairs, count the number of times each transition occurs \( n(s', s, a) \) for each next state \( s' \), and construct an empirical model of transition kernel denoted by \( \hat{P}(s'|s, a) = n(s', s, a) / n \). Then, Optimistic-GMBL creates a class of CMDPs using the empirical model. This class is denoted by \( \mathcal{M} \) and contains CMDPs with identical reward, cost matrices, \( \bar{C} \) and horizon to true CMDP, but with transition kernels close to true model. Finally, Optimistic-GMBL maximizes the objective function among all possible transition kernels, while satisfying constraints (if feasible). This class of transition kernels is defined as

\[
\mathcal{M}_{\delta P} := \{ M' : r'(s, a) = r(s, a), c'(i, s, a) = c(i, s, a), H' = H, \}
\]

\[
|P'(|s, a) - \hat{P}(|s, a)| \leq \sqrt{\frac{2 \hat{P}(|s, a)(1 - \hat{P}(|s, a))}{n} \log \frac{1}{\delta P} + \frac{2}{3n} \log \frac{1}{\delta P}}.
\]
if \( n > 1 \) : \( |P'(s'|s,a)(1 - P'(s'|s,a)) - \hat{P}(s'|s,a)(1 - \hat{P}(s'|s,a))| \leq \frac{2 \log 1/\delta P}{n - 1} \forall (s,a,i), \) \hspace{1cm} (6)

where \( \delta P \) is defined in Algorithm 1. For any \( M' \in \mathcal{M} \), objective function \( V_0^\pi(s_0) \) and cost functions \( C_{i,0}^\pi(s_0) \) are computed w.r.t. the corresponding transition kernel \( P' \). Now, consider the optimistic planning problem below

\[
\max_{\pi, M' \in \mathcal{M}_{\delta P}} V_0^\pi(s_0) \quad \text{s.t.} \quad C_{i,0}^\pi(s_0) \leq \bar{C}_i \quad \forall i \in \{1, \ldots, N\}, \hspace{1cm} (7)
\]

We will show that true CMDP lies inside the \( \mathcal{M} \) with high probability. So, the problem (7) would be feasible with high probability, since the original CMDP problem is assumed to be feasible according to Assumption 2.

Finally, Optimistic-GMBL uses Extended-LP to solve the optimistic planning problem (7). This method gets \( \mathcal{M}_{\delta P} \) and outputs \( \bar{\pi} \) for the optimal solution. Algorithm 1 describes Optimistic-GMBL.

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**Algorithm 1 Optimistic-GMBL**

1. Input: accuracy \( \epsilon, \delta \).
2. Set \( \delta P = \frac{(3N+6)||A||H}{\delta} \).
3. Set \( n(s',s,a) = 0 \) \( \forall (s,a,s') \).
4. for each \( (s,a) \in S \times A \) do
5. Sample \( (s,a), n = 288||S||H^3 \frac{1}{\epsilon^2} \log \frac{(3N+6)||A||H}{\delta} \) and update \( n(s',s,a) \).
6. \( \hat{P}(s'|s,a) = \frac{n(s',s,a)}{n} \forall s' \).
7. Construct \( \mathcal{M}_{\delta P} \) according to (5).
8. Output \( \bar{\pi} = \text{ELP}({\mathcal{M}_{\delta P}}) \).

Now, we present the sample complexity result of Optimistic-GMBL.

**Theorem 1.** Consider any finite-horizon CMDP \( M = < S, A, P, r, c, \bar{C}, H > \) satisfying assumptions 1 and 2 and CMDP problem formulation of (3) with optimal solution \( \pi^* \). Then, for any \( \epsilon \in (0, 12 \sqrt{\frac{\delta H}{|S|}}) \) and \( \delta \in (0, 1) \), algorithm 1 creates a model CMDP \( \hat{M} = < S, A, \hat{P}, r, c, \bar{C}, H > \) and outputs policy \( \bar{\pi}^* \) such that

\[
P(V_0^{\bar{\pi}^*}(s_0) \geq V_0^{\pi^*}(s_0) - \epsilon) \geq 1 - \delta \quad \text{and} \quad P(C_{i,0}^{\bar{\pi}^*}(s_0) \leq \bar{C}_i + \epsilon) \geq 1 - \delta \quad \forall i \in \{1, 2, \ldots, N\},
\]

with at least total sampling budget of

\[
288||S||^2|A|H^3 \frac{1}{\epsilon^2} \log \frac{(3N+6)||A||H}{\delta}.
\]

The proof requires use of lemmas which are presented in Appendix B.

**Proof.** Let \( \delta' \in (0, 1) \). Considering the concentration inequality around \( \hat{P} \) and Lemma 2 we have

\[
V_0^{\pi^*}(s_0) - 3\sqrt{\frac{8||S||H^3 \log 1/\delta'}{n}} \leq \hat{V}_0^{\pi^*}(s_0) \leq V_0^{\pi^*}(s_0) + 3\sqrt{\frac{8||S||H^3 \log 1/\delta'}{n}}
\]

w.p. at least \( 1 - 3||S|||A||H\delta' \) and

\[
V_0^{\bar{\pi}^*}(s_0) - 3\sqrt{\frac{8||S||H^3 \log 1/\delta'}{n}} \leq \bar{V}_0^{\bar{\pi}^*}(s_0) \leq V_0^{\bar{\pi}^*}(s_0) + 3\sqrt{\frac{8||S||H^3 \log 1/\delta'}{n}}
\]
w.p. at least $1 - 3|S||A|H\delta'$ according to Lemma 5. On the other hand, we know that $\tilde{V}_0^\pi^*(s_0) \leq \tilde{V}_0^\pi^*(s)$. Thus, by combining these results we get

$$V^\pi_0(s_0) - 3\sqrt{\frac{8|S|H^3\log1/\delta'}{n}} \leq \tilde{V}_0^\pi^*(s) \leq V^\pi_0(s) \leq 3\sqrt{\frac{8|S|H^3\log1/\delta'}{n}}.$$

It yields that $V^\pi_0(s_0) \geq V^\pi_0(s_0) - 6\sqrt{\frac{8|S|H^3\log1/\delta'}{n}}$ w.p. at least $1 - 6|S||A|H\delta'$ by Lemma 1.

On the other hand, for any $i \in \{1, \ldots, N\}$ we have

$$C_{i,0}^\pi^*(s_0) \leq \tilde{C}_{i,0}^\pi^*(s_0) + 3\sqrt{\frac{8|S|H^3\log1/\delta'}{n}} \leq \bar{C}_i + 3\sqrt{\frac{8|S|H^3\log1/\delta'}{n}}$$

w.p. at least $1 - 3|S||A|H\delta'$ according to Lemma 5. By applying Lemma 1 we get that all statements for value and cost functions hold w.p. at least $1 - (3N + 6)|S||A|H\delta'$. Hence, putting $\epsilon = 6\sqrt{\frac{8|S|H^3\log1/\delta'}{n}}$ and $\delta = (3N + 6)|S||A|H\delta'$ concludes the proof. Please note that $\epsilon < 12\sqrt{\frac{2H}{\delta'}}$ would satisfy the assumption in Lemma 8.

**Corollary 1.** In case of $N = 0$, the problem would become regular unconstrained MDP. And, the sample complexity result with $N = 0$ would also hold for unconstrained case.

### 4 Conclusion

This paper introduced the notion of sample complexity in objective maximization and constraint satisfaction for understanding the performance of RL algorithms for safety-constrained applications. We developed an offline GMBL algorithm. The main finding of a logarithmic factor increase in sample complexity over the unconstrained regime suggests value of the approach to real systems.

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5 Appendix

In this section, we provide ELP formulation beside additional technical lemmas with their proofs if they are not proved in existing literature. In each lemma or proposition below, we assume that we have \( n \) samples from each state-action \((s, a)\) pair.

5.1 Extended-LP

To elaborate on ELP, we present generic LP formulation equivalent to CMDP problem (3). Let \( \mu \) be any generic occupation measure defined as (4). Then, the equivalent LP would be

\[
\max_{\mu} \sum_{s,a,h} \mu(s, a, h) r(s, a)
\]

s.t.

\[
\sum_{s,a,h} \mu(s, a, h) c(i, s, a) \leq C_i \quad \forall i \in \{1, \ldots, N\},
\]

\[
\sum_{a} \mu(s, a, h) = \sum_{s', a'} P(s'|s, a') \mu(s', a', h - 1) \quad \forall h \in \{1, \ldots, H - 1\},
\]

\[
\sum_{a} \mu(s_0, a, 0) = 1, \quad \sum_{a} \mu(s, a, 0) = 0 \quad \forall s \in S\setminus\{s_0\},
\]

\[
\mu(s, a, h) \geq 0 \quad \forall s \in S, a \in A, h \in \{0, 1, \ldots, H - 1\}.
\]

It is proved that the linear programming (8) is equivalent to CMDP problem of (3), and the optimal policy computed by linear programming is also the solution to CMDP problem in [1]. Now given the estimated model \( \hat{P} \), we get the ELP formulation if we define new occupancy measure \( q(s, a, s', h) = P(s'|s, a) \mu(s, a, h) \) as mentioned in section 2. Then, the ELP is

\[
\max_{q} \sum_{s,a,s',h} q(s, a, s', h) r(s, a)
\]

s.t.

\[
\sum_{s,a,s',h} q(s, a, s', h) c(i, s, a) \leq C_i \quad \forall i \in \{1, \ldots, N\},
\]

\[
\sum_{a,s'} q(s, a, s', h) = \sum_{s', a'} q(s', a', s, h - 1) \quad \forall h \in \{1, \ldots, H - 1\},
\]

\[
\sum_{a,s'} q(s_0, a, s', 0) = 1, \quad \sum_{a,s'} q(s, a, s', 0) = 0 \quad \forall s \in S\setminus\{s_0\},
\]

\[
q(s, a, s', h) - (\hat{P}(s'|s, a) + \beta(s, a, s')) \sum_{y} q(s, a, y, h) \leq 0,
\]

\[-q(s, a, s', h) + (\hat{P}(s'|s, a) - \beta(s, a, s')) \sum_{y} q(s, a, y, h) \leq 0,
\]

\[q(s, a, s', h) \geq 0 \quad \forall s, s' \in S, a \in A, h \in \{0, 1, \ldots, H - 1\},
\]

where \( \beta(s, a, s') \) is related to confidence set which depends on the algorithm. The details of ELP regarding the time and space complexity is briefed in [15], so we do not present them here.

5.2 Lemmas and Propositions

Lemma 1. Consider the events \( E_1, E_2, \ldots, E_K \) such that for each \( k \), \( \mathbb{P}(E_k) \geq (1-\delta_k) \). Then, \( \mathbb{P}(\cap_{k=1}^{K} E_k) \geq (1 - \sum_{k=1}^{K} \delta_k) \).
Proof. We have $P(\cup_{k=1}^{K}E_k) \leq \sum_{k=1}^{K} P(E_k) \leq \sum_{k=1}^{K} \delta_k$. Now, $P(\cap_{k=1}^{K}E_k) = P((\cup_{k=1}^{K}E_k)^c) = 1 - P(\cup_{k=1}^{K}E_k) \geq 1 - \sum_{k=1}^{K} \delta_k$. \qed

Lemma 2. Let $\delta_p \in (0, 1)$. Assume $p, \hat{p}, \tilde{p} \in [0, 1]$ satisfy $P(p \in P_{\delta_p}) \geq 1 - \delta_p$ and $\tilde{p} \in P_{\delta_p}$, where

$$P_{\delta_p} := \{ p' \in [0, 1] : |p' - \hat{p}| \leq \sqrt{\frac{2\tilde{p}(1 - \hat{p})}{n} \log \frac{1}{\delta_1}} + \frac{2}{3n} \log \frac{1}{\delta_p},$$

if $n > 1 : |p'(1 - p') - \tilde{p}(1 - \hat{p})| \leq \frac{2 \log 1/\delta_p}{n - 1} \}. \]

Then,

$$|p - \hat{p}| \leq \sqrt{\frac{8\tilde{p}(1 - \hat{p})}{n} \log \frac{1}{\delta_p}} + \frac{16 \log 1/\delta_p}{3(n - 1)} \] w.p. at least $1 - \delta_p$. \]

Proof.

$$|p - \hat{p}| \leq |p - \tilde{p}| + |\tilde{p} - \hat{p}| \leq \sqrt{\frac{2\tilde{p}(1 - \hat{p})}{n} \log \frac{1}{\delta_p}} + \frac{4}{3n} \log \frac{1}{\delta_p} \] \leq 2 \sqrt{\frac{2\tilde{p}(1 - \hat{p})}{n} \log \frac{1}{\delta_p}} + \frac{4}{3n} \log \frac{1}{\delta_p} \leq 2 \sqrt{\frac{2\tilde{p}(1 - \tilde{p})}{n} \log \frac{1}{\delta_p}} + \frac{4}{3n} \log \frac{1}{\delta_p} \leq \sqrt{\frac{8\tilde{p}(1 - \tilde{p})}{n} \log \frac{1}{\delta_p}} + \frac{16 \log 1/\delta_p}{3(n - 1)} \]

The first term in the first line is true w.p. at least $1 - \delta_p$, hence the proof is complete. \qed

Lemma 3. Suppose there are two CMDPs $M = \{ S, A, P, r, c, \bar{C}, H \}$ and $M' = \{ S, A, P', r, c, \bar{C}, H \}$ satisfying assumption. Then, under any policy $\pi$

$$V_0^\pi - V_0'^\pi = \sum_{h=0}^{H-2} P^h_{\pi}(P_{\pi} - P'_{\pi})V^\pi_{h+1}$$ and $V_0^\pi - V_0'^\pi = \sum_{h=0}^{H-2} P^h_{\pi}(P_{\pi} - P'_{\pi})V'^\pi_{h+1}$. \]

and for any $i \in \{1, \ldots, N\}$,

$$C_{i, 0}^\pi - C_{i, 0}'^\pi = \sum_{h=0}^{H-2} P^h_{\pi}(P_{\pi} - P'_{\pi})C_{i, h+1}^\pi$$ and $C_{i, 0}^\pi - C_{i, 0}'^\pi = \sum_{h=0}^{H-2} P^h_{\pi}(P_{\pi} - P'_{\pi})C_{i, h+1}'^\pi$. \]

Proof. We only prove the first statement of value function since the proof procedure for cost is identical. For a fixed $h$ and $s$

$$V^\pi_h(s) - V'^\pi_h(s) = r_{\pi}(s) + \sum_{s'} P_{\pi}(s'|s)V^\pi_{h+1}(s') - r_{\pi}(s) + \sum_{s'} P'_{\pi}(s'|s)V'^\pi_{h+1}(s') \] = \sum_{s'} P_{\pi}(s'|s)V^\pi_{h+1}(s') - \sum_{s'} P'_{\pi}(s'|s)V'^\pi_{h+1}(s') + \sum_{s'} P_{\pi}(s'|s)V^\pi_{h+1}(s') - \sum_{s'} P'_{\pi}(s'|s)V'^\pi_{h+1}(s') \] = \sum_{s'} (P_{\pi}(s'|s) - P'_{\pi}(s'|s))V^\pi_{h+1}(s') + \sum_{s'} P_{\pi}(s'|s)(V^\pi_{h+1}(s') - V'^\pi_{h+1}(s')).$$

Because $V^\pi_{H-1}(s) = V'^\pi_{H-1}(s) = r_{\pi}(s)$, if we expand the second term until $h = H - 1$, we get the result. \qed
Lemma 4. Let \( \delta_P \in (0, 1) \). Suppose there are two CMDPs \( M = < S, A, P, r, c, C, H > \) and \( M' = < S, A, P', r, c, C, H > \) satisfying assumption [1]. Further assume

\[
|P(s'|s, a) - P'(s'|s, a)| \leq c_1 + c_2 \sqrt{P'(s'|s, a) - (1 - P'(s'|s, a))}
\]

w.p. at least \( 1 - \delta_P \) for all \( s, s' \in S, a \in A \). Then, under any policy \( \pi \)

\[
|\sum_{s'} (P_\pi(s'|s) - P'_\pi(s'|s))V_{h+1}^{\pi}(s')| \leq |S|c_1 \|V_{h+1}^{\pi}\|_\infty + c_2 \sqrt{|S|\sigma_{h}^{\pi}(s)}
\]

for any \( (s, a) \in S \times A \) and \( h \in [0, H - 2] \) w.p. at least \( 1 - |S||A|\delta_P \), and

\[
|\sum_{s'} (P_\pi(s'|s) - P'_\pi(s'|s))C_{i,h+1}^{\pi}(s')| \leq |S|c_1 \|C_{i,h+1}^{\pi}\|_\infty + c_2 \sqrt{|S|\sigma_{i,h}^{\pi}(s)}
\]

for any \( (s, a) \in S \times A, i \in \{1, \ldots, N\} \) and \( h \in [0, H - 2] \) w.p. at least \( 1 - |S||A|\delta_P \).

Proof. We only prove the statement of value function since the proof procedure for cost is identical. Fix state \( s \) and define for this fixed state \( s \) the constant function \( \bar{V}^{\pi}(s') = \sum_{s''} P_\pi(s''|s)V_{h+1}^{\pi}(s') \) as the expected value function of the successor states of \( s \). Note that \( \bar{V}^{\pi}(s') \) is a constant function and so \( \bar{V}^{\pi}(s') = \sum_{s''} P_\pi^{\prime}(s''|s)\bar{V}^{\pi}(s'') = \sum_{s''} P_\pi^{\prime}(s''|s)\bar{V}^{\pi}(s'') \).

\[
|\sum_{s'} (P_\pi(s'|s) - P'_\pi(s'|s))V_{h+1}^{\pi}(s')| = |\sum_{s'} (P_\pi(s'|s) - P'_\pi(s'|s))V_{h+1}^{\pi}(s') + \bar{V}^{\pi}(s) - \bar{V}^{\pi}(s)|
\]

\[
\leq \sum_{s'} |P_\pi(s'|s) - P'_\pi(s'|s)||V_{h+1}^{\pi}(s') - \bar{V}^{\pi}(s')| + |\bar{V}^{\pi}(s) - \bar{V}^{\pi}(s)|
\]

\[
\leq \sum_{s'} (c_1 + c_2 \sqrt{P'_\pi(s'|s) - (1 - P'_\pi(s'|s))})|V_{h+1}^{\pi}(s') - \bar{V}^{\pi}(s')|
\]

\[
\leq |S|c_1 \|V_{h+1}^{\pi}\|_\infty + c_2 \sqrt{|S|} \sum_{s'} (P'_\pi(s'|s) - (1 - P'_\pi(s'|s)))|V_{h+1}^{\pi}(s') - \bar{V}^{\pi}(s')|^2
\]

\[
\leq |S|c_1 \|V_{h+1}^{\pi}\|_\infty + c_2 \sqrt{|S|} \sum_{s'} (P'_\pi(s'|s) - (1 - P'_\pi(s'|s)))|V_{h+1}^{\pi}(s') - \bar{V}^{\pi}(s')|^2
\]

Inequality (9) holds w.p. at least \( 1 - |S||A|\delta_P \), since we used the assumption and applied the triangle inequality and Lemma [1]. We then applied the assumed bound and bounded \( |V_{h+1}^{\pi}(s') - \bar{V}^{\pi}(s')| \) by \( \|V_{h+1}^{\pi}\|_\infty \) as all value functions are non-negative. In inequality (10), we applied the Cauchy-Schwarz inequality and subsequently used the fact that each term is non-negative and that \( (1 - P'_\pi(s'|s)) \leq 1 \). The final equality follows from the definition of \( \sigma_{h}^{\pi}(s) \). \( \square \)

Lemma 5. Let \( \delta_P \in (0, 1) \). Suppose there are two CMDPs \( M = < S, A, P, r, c, C, H > \) and \( M' = < S, A, P', r, c, C, H > \) satisfying assumption [1]. Further assume

\[
|P(s'|s, a) - P'(s'|s, a)| \leq \frac{\alpha}{\sqrt{n}}
\]
for all \( s, s' \in S, a \in A \) w.p. at least \( 1 - \delta_p \). Then, under any policy \( \pi \)

\[
\|V^\pi_0 - V'^\pi_0\|_\infty \leq |S|H^2a\frac{1}{\sqrt{n}}.
\]

w.p. at least \( 1 - |S||A|H\delta_p \), and for any \( i \in \{1, \ldots, N\} \)

\[
\|C^\pi_{i,0} - C'^\pi_{i,0}\|_\infty \leq |S|H^2a\frac{1}{\sqrt{n}}
\]

w.p. at least \( 1 - |S||A|H\delta_p \).

Proof. We prove the statement of value function since the proof procedure for cost is identical. Let

\[
\Delta_h = \max_s |V^\pi_h(s) - V'^\pi_h(s)|.
\]

Then

\[
\Delta_h = |V^\pi_h(s) - V'^\pi_h(s)| = |r_\pi(s) + \sum_{s'} P_\pi(s'|s)V^\pi_{h+1}(s') - (r_\pi(s) + \sum_{s'} P'_\pi(s'|s)V'^\pi_{h+1}(s'))|
\]

\[
= \sum_{s'} |(P_\pi(s'|s) - P'_\pi(s'|s))V^\pi_{h+1}(s') + \sum_{s'} P'_\pi(s'|s)V'^\pi_{h+1}(s')|
\]

\[
\leq \sum_{s'} |(P_\pi(s'|s) - P'_\pi(s'|s))|H + \Delta_{h+1}
\]

\[
\leq |S|Ha\frac{1}{\sqrt{n}} + \Delta_{h+1}.
\]

Thus,

\[
\Delta_h \leq |S|Ha\frac{1}{\sqrt{n}} + \Delta_{h+1}
\]

w.p. at least \( 1 - |S||A|\delta_p \) according to Lemma 4. If we expand this recursively, we get

\[
\Delta_0 \leq |S|H^2a\frac{1}{\sqrt{n}}.
\]

since \( \Delta_{H-1} = \max_s |r_\pi(s) - r_\pi(s)| = 0 \). By applying Lemma 3 we get the result holds w.p. at least \( 1 - |S||A|H\delta_p \). Hence the proof is complete. \( \square \)

Lemma 6. Let \( \delta_p \in (0,1) \). Suppose there are two CMDPs \( M =< S, A, P, r, c, C, H > \) and \( M' =< S, A, P', r, c, C, H > \) satisfying assumption[2] Further assume

\[
|P(s'|s, a) - P'(s'|s, a)| \leq \frac{a}{\sqrt{n}}
\]

w.p. at least \( 1 - \delta_p \) for all \( s, s' \in S, a \in A \). Then if \( n \geq |S|H^2 \), under any policy \( \pi \)

\[
\|\sigma^\pi_0 - \sigma'^\pi_0\|_\infty \leq \frac{2\sqrt{|S|H^2a}}{n^{1/4}},
\]

w.p. at least \( 1 - 2|S||A|H\delta_p \), and for any \( i \in \{1, \ldots, N\} \)

\[
\|\sigma^\pi_{i,0} - \sigma'^\pi_{i,0}\|_\infty \leq \frac{2\sqrt{2|S|H^2a}}{n^{1/4}}
\]

w.p. at least \( 1 - 2|S||A|H\delta_p \).
Proof. We prove the statement of value function since the proof procedure for cost is identical. Fix a state \( s \). Then,

\[
\sigma_0^2(s) = \sigma_0^2(s) - \mathbb{E}'[(V_1^\pi(s) - P_\pi^s V_1^\pi(s))^2] + \mathbb{E}'[(V_1^\pi(s) - P_\pi^s V_1^\pi(s))^2] \\
\leq \sum_{s'} (P_\pi(s'|s) - P_\pi^s(s'|s)) V_1^\pi(s') - \left[ \sum_{s'} P_\pi(s'|s) V_1^\pi(s') \right]^2 - \left[ \sum_{s'} P_\pi^s(s'|s) V_1^\pi(s') \right]^2 \\
+ \left[ \mathbb{E}'[(V_1^\pi(s) - V^\pi(s) - P_\pi^s(V_1^\pi(s)))^2] + \mathbb{E}'[(V_1^\pi(s) - P_\pi^s(V_1^\pi(s)))^2] \right],
\]

where we applied triangular inequality in the last line. And, please note that \( \mathbb{E}' \) means expectation w.r.t. transition kernel \( P'_{\pi} \). It is straightforward to show that \( \text{Var}_{s' \sim P'_{\pi}(s|s)}(V_1^\pi(s') - V^\pi(s')) \leq \| V_0^\pi - V_0^\pi \|_\infty^2 \) implying

\[
\sigma_0^2(s) \leq \sum_{s'} (P_\pi(s'|s) - P_\pi^s(s'|s)) V_1^\pi(s') - \left[ \sum_{s'} P_\pi(s'|s) V_1^\pi(s') \right]^2 - \left[ \sum_{s'} P_\pi^s(s'|s) V_1^\pi(s') \right]^2 \\
+ \left( \| V_0^\pi - V_0^\pi \|_\infty + \sigma_0^2(s) \right)^2.
\]

Now, if we use Lemma 5 we get

\[
\sigma_0^2(s) \leq \left| \sigma_0'(s) \right| + \frac{|S|H^2 a^2}{\sqrt{n}} + \frac{2|S|H^2 a^2}{\sqrt{n}} \leq \left| \sigma_0'(s) \right| + \frac{|S|H^2 a^2}{\sqrt{n}} + \frac{\sqrt{2} |S|H^2 a^2}{n^{1/4}}^2
\]

because for any \( x, y > 0 \) we have \( a^2 + b^2 \leq (x + y)^2 \). And, the assumption on \( n \), dominates the term with \( \frac{1}{\sqrt{n}} \) over \( \sqrt{n} \). Eventually, the result follows by taking square root from both sides and applying Lemma 1.



**Lemma 7.** \[16\] The variance of the value function defined as \( \Sigma_{\pi}^2(s) = \mathbb{E}[(\sum_{t=0}^{H-1} r(s_t) - V_0^\pi(s_t))^2] \) satisfies a Bellman equation \( \Sigma_{\pi}^2(s) = \sigma_1^2(s) + \sum_{s' \in S} P_\pi(s'|s) \Sigma_{\pi}^2(s') \) which gives \( \Sigma_{\pi}^2(s) = \sum_{h=0}^{H-1} (P^h_{\pi} \sigma_{\pi}^2) \). Since \( 0 \leq \Sigma_{\pi}^2(s) \leq H^2 \), it follows that \( 0 \leq \sum_{h=0}^{H-1} (P^h_{\pi} \sigma_{\pi}^2) \leq H^2 \) for all \( s \in S \).

**Corollary 2.** The result of Lemma 2 also holds for variance of cost functions.

**Lemma 8.** Let \( \delta_P \in (0, 1) \). Suppose there are two CMDPs \( M = < S, A, P, c, C, H > \) and \( M' = < S, A, P', c, C', H > \) satisfying assumption \[1\] Further assume

\[
|P(s'|s, a) - P'(s'|s, a)| \leq \frac{c_1}{n} + \frac{c_2}{\sqrt{n}} \sqrt{P'(s'|s, a) - (1 - P'(s'|s, a))}
\]

w.p. at least \( 1 - \delta_P \) for all \( s, s' \in S, a \in A \). Then if \( n \geq \frac{(c_1^2 + c_2^2)}{\sqrt{n}} \), under any policy \( \pi \)

\[
\| V_0^\pi - V_0^\pi \|_\infty \leq 3c_2 \sqrt{\frac{|S|H^3}{n}}
\]

w.p. at least \( 1 - 3|S||A|H^2 \delta_P \), and for any \( i \in \{1, \ldots, N\} \),

\[
\| C_i^{\pi} - C_i^{\pi} \|_\infty \leq 3c_2 \sqrt{\frac{|S|H^3}{n}}
\]

w.p. at least \( 1 - 3|S||A|H^2 \delta_P \).
Proof. We only prove the statement of value function since the proof procedure for cost is identical. Let fix state $s$:

$$|V_0^\pi(s) - V_0'^\pi(s)| = \left| \sum_{h=0}^{H-2} P^h_\pi (P_\pi - P'_\pi)V_{h+1}^\pi(s) \right|$$  \hspace{1cm} (11)$$

$$\leq \sum_{h=0}^{H-2} P^h_\pi |(P_\pi - P'_\pi)V_{h+1}^\pi(s)| \leq \sum_{h=0}^{H-2} P^h_\pi (|S|c_1 \sqrt{n} \|V_{h+1}^\pi\|_\infty + c_2 \sqrt{|S|}) (s)$$  \hspace{1cm} (12)$$

$$\leq |S|H^2 \frac{c_1}{n} + \frac{c_2}{\sqrt{n}} \sqrt{|S|} \sum_{h=0}^{H-1} (P^h_\pi \sigma_h^\pi)(s)$$  \hspace{1cm} (13)$$

$$\leq \frac{|S|H^2 c_1}{n} + c_2 \sqrt{|S|} \sum_{h=0}^{H-1} (P^h_\pi \sigma_h^\pi + 2\sqrt{2|S|H^2 c_2}) (s)$$  \hspace{1cm} (14)$$

$$\leq \frac{|S|H^2 c_1}{n} + c_2 \sqrt{|S|} \sum_{h=0}^{H-1} (P^h_\pi \sigma_h^\pi) (s) + c_2 H \sqrt{\frac{|S|2|S|H^2 c_2}{n^{1/4}}}$$  \hspace{1cm} (15)$$

$$\leq \frac{|S|H^2 c_1}{n} + c_2 \sqrt{|S|} \frac{H^2}{n} + 2\sqrt{2|S|H^2 c_2} \sqrt{\frac{|S|2|S|H^2 c_2}{n^{1/4}}}$$  \hspace{1cm} (16)$$

$$\leq 3c_2 \sqrt{\frac{|S|H^3}{n}}. $$  \hspace{1cm} (17)$$

In equation (11), we used Lemma 3. Then, we applied Lemma 3 to obtain inequality (12). Next, we bound $\|V_{h+1}^\pi\|_\infty$ by $H$ in inequality (13). To get inequality (14), we use Lemma 6 since we can bound $P(\cdot|s,a) - P'(\cdot|s,a)$ by $2c_2 \sqrt{n}$. And, we applied Cauchy-Scharwz inequality to get inequality (15). Finally, inequality (17) follows from the fact that $n \geq \frac{(|S|H)^2}{c_1 c_2}$. Since the result is true for every $s \in S$, hence the proof is complete. \qed