Propagator norm and sharp decay estimates for Fokker-Planck equations with linear drift

A. Arnold, C. Schmeiser, and B. Signorello
Most recent ASC Reports

04/2020  G. Gantner, A. Haberl, D. Praetorius, and S. Schimanko  
Rate optimality of adaptive finite element methods with respect to the overall computational costs

03/2020  M. Faustmann, J.M. Melenk, M. Parvizi  
On the stability of Scott-Zhang type operators and application to multilevel preconditioning in fractional diffusion

02/2020  M. Buliček, A. Jüngeł, M. Pokorný, and N. Zamponi  
Existence analysis of a stationary compressible fluid model for heat-conducting and chemically reacting mixtures

01/2020  M. Braukhoff and A. Jüngeł  
Entropy-dissipating finite-difference schemes for nonlinear fourth-order parabolic equations

31/2019  W. Auzinger, M. Fallahpour, O. Koch, E.B. Weinmüller  
Implementation of a pathfollowing strategy with an automatic step-length control: New MATLAB package bvpsuite2.0

30/2019  W. Auzinger, O. Koch, E.B. Weinmüller, S. Wurm  
Modular version bvpsuite1.2 of the collocation MATLAB package bvpsuite1.1

29/2019  S. Kurz, D. Pauly, D. Praetorius, S. Repin, and D. Sebastian  
Functional a posteriori error estimates for boundary element methods

28/2019  P.-E. Druet and A. Jüngeł  
Analysis of cross-diffusion systems for fluid mixtures driven by a pressure gradient

27/2019  G. Dhariwal, F. Huber, A. Jüngeł, C. Kuehn, and A. Neamtu  
Global martingale solutions for quasilinear SPDEs via the boundedness-by-entropy method

26/2019  G. Di Fratta, M. Innerberger, and D. Praetorius  
Weak-strong uniqueness for the Landau-Lifshitz-Gilbert equation in micromagnetics
PROPAGATOR NORM AND SHARP DECAY ESTIMATES FOR
FOKKER-PLANCK EQUATIONS WITH LINEAR DRIFT

ANTON ARNOLD, CHRISTIAN SCHMEISER, AND BEATRICE SIGNORELLO

Abstract. We are concerned with the short- and large-time behavior of the $L^2$-propagator norm of Fokker-Planck equations with linear drift, i.e. $\partial_t f = \text{div}_x(D\nabla_x f + Cf)$. With a coordinate transformation these equations can be normalized such that the diffusion and drift matrices are linked as $D = C_S$, the symmetric part of $C$. The main result of this paper is the connection between normalized Fokker-Planck equations and their drift-ODE $\dot{x} = -Cx$: Their $L^2$-propagator norms actually coincide. This implies that optimal decay estimates on the drift-ODE (w.r.t. both the maximum exponential decay rate and the minimum multiplicative constant) carry over to sharp exponential decay estimates of the Fokker-Planck solution towards the steady state. A second application of the theorem regards the short time behaviour of the solution: The short time regularization (in some weighted Sobolev space) is determined by its hypocoercivity index, which has recently been introduced for Fokker-Planck equations and ODEs (see [5, 1, 2]). In the proof we realize that the evolution in each invariant spectral subspace can be represented as an explicitly given, tensored version of the corresponding drift-ODE. In fact, the Fokker-Planck equation can even be considered as the second quantization of $\dot{x} = -Cx$.

Keywords. Fokker-Planck equation, large-time behavior, sharp exponential decay, semigroup norm, regularization rate, second quantization

1. Introduction

We are going to study the large-time and short-time behavior of the solution of Fokker-Planck (FP) equations with linear drift and possibly degenerate diffusion for $g = g(t, y)$:

\begin{align}
\partial_t g &= -\tilde{L}g := \text{div}_y(D\nabla_y g + \tilde{C}yg), \quad y \in \mathbb{R}^d, \quad t \in (0, \infty), \\
g(t = 0) &= g_0 \in L^1_+ (\mathbb{R}^d), \\
\int_{\mathbb{R}^d} g_0(y) dy &= 1.
\end{align}

We assume that

- $\tilde{D} \in \mathbb{R}^{d \times d}$ is non-trivial, positive semi-definite, symmetric, and constant in $y$,
- $\tilde{C} \in \mathbb{R}^{d \times d}$ is positive stable, (typically non-symmetric,) and constant in $y$.

\textbf{Date:} March 1, 2020.
The goal of this study is to investigate the qualitative and quantitative large time behavior of the solution of (1.1). Several authors (see, e.g., [3, 6, 24, 1]) have addressed the following questions: Under which conditions is there a non trivial steady state \( g_\infty \)? In the affirmative case, does the solution \( g(t) \) converge to the steady state for \( t \to \infty \) in a suitable norm? Is the convergence exponential?

In particular, the large-time behavior of FP equations has been treated in [30] via spectral methods. Instead, entropy methods are used in [6]. From these previous studies it is well known that (under some assumptions that will be defined in the next section) the solution \( g(t) \) converges to the steady state \( g_\infty \) with an exponential decay rate, up to a multiplicative constant greater than one. In the degenerate case, where the diffusion matrix \( D \) is non-invertible, this property of the solution is known as hypocoercivity, as introduced in [31].

Optimal exponential decay estimates for the convergence of the solution to the steady state in both the degenerate and the non-degenerate cases has been shown in [3]. Special care is required when the eigenvalues of \( \tilde{C} \) with smallest real part are defective. This situation is covered in [4] and [22]. In both cases, the sharpness of the estimate refers only to the exponential decay rate of the convergence of the solution. The issue of finding the best multiplicative constant in the decay estimate for FP equations (1.1) is still open. This is one of the topics of this paper. Even for linear ODEs there are only partial results on this best constant, as for example in [21] and [3]. In particular, [3] gives the explicit best multiplicative constant in the two-dimensional case for \( \dot{x} = -Cx \), where \( C \) is a positive stable matrix. A very complete solution has been derived in [14] for a special case, the kinetic FP equation with quadratic confining potential. There the propagator norm is computed explicitly. The result can be written as an exponential decay estimate with time dependent multiplicative constant, whose maximal value is the result we are looking for. A related result based on Phi-entropies can be found in [12], where improved time dependent decay rates are derived.

The main result of this paper is equality of the propagator norms of the PDE on the orthogonal complement of the space of equilibria and of its associated drift ODE. The underlying norms are the \( L^2 \)-norm weighted by the inverse of the equilibrium distribution for the PDE, and the Euclidian norm for the ODE. This has two main consequences: First, the sharp (exponential) decay of the PDE is reduced to the same, but much easier question on the ODE level. The second consequence is that the hypocoercivity index (see [5, 1, 2]) of the drift matrix determines the short-time behavior (in the sense of a Taylor series expansion) both of the drift ODE and the FP equation. As a further consequence for solutions of the FP equation we determine the short-time regularization from the weighted \( L^2 \)-space to a weighted \( H^1 \)-space. This result can be seen as an illustration of the fact that for the FP equation hypocoercivity is equivalent to
hypoellipticity. Finally, it is shown that the FP equation can be considered as the second quantization of the drift ODE. This follows from the proof of the main theorem, where the FP evolution is decomposed on invariant subspaces, in each of which the evolution is governed by a tensorized version of the drift ODE.

The paper is organized as follows: In Section 2 we transform the FP operator $\tilde{L}$ to an equivalent version $L$ such that $D = CS$, the symmetric part of the drift matrix. The conditions for the existence of a unique positive steady state and for hypocoercivity are also set up. The main theorem is formulated in Section 3 together with the main consequences. The proof of the main theorem requires a long preparation that is split into Sections 4 and 5. In Section 4 we derive a spectral decomposition for the FP operator into finite-dimensional invariant subspaces. This allows to see an explicit link with the drift ODE $\dot{x} = -Cx$. In order to make this link more evident, we work with the space of symmetric tensors, presented in Section 5. In Section 6 we give the proof of the main theorem as a corollary of the fact that the propagator norm on each subspace is an integer power of the propagator norm of the ODE evolution. Finally, in Section 7 the FP operator is rewritten in the second quantization formalism.

2. Preliminary results

2.1. Equilibria – normalized form. The following theorem (from $[5]$, Theorem 3.1 or $[20]$, p. 41) states under which conditions on the matrices $\tilde{D}$ and $\tilde{C}$ there exists a unique steady state $g_\infty$ for (1.1) and it provides its explicit form. We denote the spectral gap of $\tilde{C}$ by $\mu(\tilde{C}) := \min\{\Re(\lambda) : \lambda \text{ is an eigenvalue of } \tilde{C}\}$.

**Definition 2.1.** We say that Condition $\tilde{A}$ holds for the Equation (1.1), iff

1. the matrix $\tilde{D}$ is symmetric, positive semi-definite,
2. there is no non-trivial $\tilde{C}^T$-invariant subspace of $\ker \tilde{D}$,
3. the matrix $\tilde{C}$ is positive stable, i.e. $\mu(\tilde{C}) > 0$.

Note that condition (2) is known as Kawashima’s degeneracy condition $[17]$ in the theory for systems of hyperbolic conservation laws. It also appears in $[16]$ as a condition for hypoellipticity of FP equations (see $[31]$, Section 3.3] for the connection to hypocoercivity).

**Theorem 2.2** (Steady state). There exist a unique ($L^1$-normalized) steady state $g_\infty \in L^1(\mathbb{R}^d)$ of (1.1), iff Condition $\tilde{A}$ holds. It is given by the (non-isotropic) Gaussian

$$g_\infty(y) = c_K \exp\left(-\frac{y^TK^{-1}y}{2}\right),$$

where the covariance matrix $K \in \mathbb{R}^{d \times d}$ is the unique, symmetric, and positive definite solution of the continuous Lyapunov equation

$$2\tilde{D} = \tilde{C}K + K\tilde{C}^T,$$
and $c_K = (2\pi)^{-d/2} (\det K)^{-1/2}$ is the normalization constant.

The natural setting for the evolution equation (1.1) is the weighted $L^2$-space $\tilde{\mathcal{H}} := L^2(\mathbb{R}^d, g^{-1})$ with the inner product

$$\langle g_1, g_2 \rangle_{\tilde{\mathcal{H}}} := \int_{\mathbb{R}^d} g_1(y) g_2(y) \frac{dy}{g_{\infty}(y)}.$$

Under Condition $\tilde{A}$ the FP equation (1.1) can be rewritten (see Theorem 3.5, [5]) as

$$\partial_t g = \text{div}_y \left( g_{\infty}(\tilde{D} + \tilde{R}) \nabla_y \left( \frac{g}{g_{\infty}} \right) \right), \quad y \in \mathbb{R}^d, \quad t \in (0, \infty),$$

where $\tilde{R} \in \mathbb{R}^{d \times d}$ is the anti-symmetric matrix $\tilde{R} = \frac{1}{2} (\tilde{C} K - K \tilde{C}^T)$.

The change of coordinates $x := K^{-1/2} y$, $f(x) := (\det K)^{-1/2} g(K^{1/2} x)$ transforms (1.1) into

$$\partial_t f = -L f := \text{div}_x (D \nabla_x f + C x f) = \text{div}_x \left( f_{\infty} C \nabla_x \left( \frac{f}{f_{\infty}} \right) \right),$$

where $D := K^{-1/2} \tilde{D} K^{-1/2}$, $C := K^{-1/2} \tilde{C} K^{1/2}$, and the steady state is the normalized Gaussian

$$f_{\infty}(x) = (2\pi)^{-d/2} e^{-|x|^2/2}.$$

This is due to the property

$$D = C_S := \frac{1}{2} \left( C + C^T \right),$$

which is a simple consequence of (2.2). We shall call a FP equation normalized, if the diffusion and drift matrices satisfy (2.6).

From now on we shall study the normalized equation (2.4) on the normalized version $\mathcal{H} := L^2(\mathbb{R}, f^{-1})$ of the Hilbert space $\tilde{\mathcal{H}}$. It is easily checked that

$$\|g(t)\|_{\tilde{\mathcal{H}}} = \|f(t)\|_{\mathcal{H}}, \quad \forall t \geq 0,$$

holds for the solutions of $g$ and $f$ of (1.1) and, respectively, (2.4), implying that the propagator norms are the same.

For later reference we now rewrite Condition $\tilde{A}$ in terms of the matrix $C$.

**Proposition 2.3.** The Equation (1.1) satisfies Condition $\tilde{A}$ iff its normalized version (2.4) satisfies Condition A, given by

1. the matrix $C_S$ is positive semi-definite,
2. there is no non-trivial $C^T$-invariant subspace of $\ker C_S$.

Condition A implies that the matrix $C$ is positive stable, i.e. $\mu(C) > 0$.

**Proof.** Equivalence of (1) with (1) of Definition 2.1 follows from $C_S = K^{-1/2} \tilde{D} K^{-1/2}$. For the second item, let us assume that (2) does not hold. Then, there exist $v \in \ker C_S$, $v \neq 0 \in \mathbb{R}^d$ such that

$$0 = C_S C^T v = (K^{-1/2} \tilde{D} K^{-1/2})(K^{1/2} \tilde{C}^T K^{-1/2}) v = K^{-1/2} \tilde{D} \tilde{C}^T (K^{-1/2} v).$$
This implies $\tilde{D}\tilde{C}^T(K^{-1/2}\nu) = 0$, since $K^{-1/2} > 0$. But this is a contradiction to (2) in Condition $\tilde{A}$ since it holds that $\nu \in \ker C_\delta$ iff $K^{-1/2}\nu \in \ker \tilde{D}$. With a similar argument the reverse implication can be proven.

For the proof that Condition $\tilde{A}$ implies positive stability of $C$ we refer to Proposition 1 and Lemma 2.4 in [1]. □

2.2. Convergence to the equilibrium: hypocoercivity. In [5], a hypocoercive entropy method was developed to prove the exponential convergence to $f_\infty$, for the solution to (2.4) with any initial datum $f_0 \in \mathcal{H}$. It employed a family of relative entropies w.r.t. the steady state, i.e. $e^{\psi}(f(t)|f_\infty) := \int_{\mathbb{R}^d} \psi(f(t)f_\infty) f_\infty^{-1} \, dx$, where the convex functions $\psi$ are admissible entropy generators (as in [6] and [9]).

**Definition 2.4.** Let $\{\lambda_m|1 \leq m \leq m_0\}$ be the set of eigenvalues of $C$ with $\Re(\lambda_m) = \mu(C) = \min\{|\Re(\lambda)|: \lambda \text{ is an eigenvalue of } C\}$.

1. We call the matrix $C$ non-defective if all $\lambda_m$, $1 \leq m \leq m_0$ are non-defective, i.e., their algebraic and geometric multiplicities coincide.
2. We call a FP equation (1.1) (non-)defective if its drift-matrix $\tilde{C}$ is (non-)defective, or equivalently, if the matrix $C$ in the normalized version (2.4) is (non-)defective.

For non-defective FP equations, the decay result from [5] provides the sharp exponential decay rate $\mu > 0$, but a sub-optimal multiplicative constant $c > 1$:

**Theorem 2.5** (Exponential decay of the relative entropy). Let $\psi$ generate an admissible entropy and let $f$ be the solution of (2.4) with normalized initial state $f_0 \in \mathcal{H}$ such that $e^{\psi}(f_0|f_\infty) < \infty$. Let $C$ satisfy Condition $A$. Then, if the FP equation is non-defective, there exists a constant $c \geq 1$ such that

$$
(2.8) \quad e^{\psi}(f(t)|f_\infty) \leq c^2 e^{-2\mu t} e^{\psi}(f_0|f_\infty), \quad t \geq 0.
$$

Choosing the admissible quadratic function $\psi(\sigma) = (\sigma - 1)^2$ yields the exponential decay of the $\mathcal{H}$-norm. For this particular choice of $\psi$, Theorem 2.5 holds also for $f_0 \in L^1(\mathbb{R}^d) \cap \mathcal{H}$, i.e. the positivity of the initial datum $f_0$ is not necessary.

**Corollary 2.6** (Hypocoercivity). Under the assumptions of Theorem 2.5 the following estimate holds with the same $\mu > 0$, $c > 1$:

$$
(2.9) \quad \|f(t) - f_\infty\|_{\mathcal{H}} \leq ce^{-\mu t}\|f_0 - f_\infty\|_{\mathcal{H}}, \quad t \geq 0.
$$

The hypocoercivity approach in [5] provides the optimal (i.e. maximal) value for $\mu$ and a computable value for $c$, which is however not sharp, i.e.

$$
(2.10) \quad c_{\min} := \min\left\{c \geq 1 : (2.9) \text{ holds for all } f_0 \in \mathcal{H} \text{ with } \int f_0 \, dx = 1\right\}.
$$
The central goal of this paper is the determination of $c_{\text{min}}$. Actually, we shall go much beyond this: The main result of this paper is to show that the $\mathcal{H}$-propagator norm of (stable) FP equations is equal to the (Euclidean) propagator norm of its corresponding drift ODE $\dot{x}(t) = -Cx(t)$. Hence, all decay properties of the FP equation (1.1) can be obtained from a simple linear ODE and sharp exponential decay estimates of an ODE carry over to the corresponding FP equation.

2.3. The best multiplicative constant for ODE. In [3] we analyzed the best decay constants for the (of course easier) finite dimensional problem

$$\dot{x}(t) = -Cx(t), \quad t > 0, \quad x(0) = x_0 \in \mathbb{C}^n,$$

where $C \in \mathbb{C}^{n \times n}$ is a positive stable and non-defective matrix. In this case we constructed a problem adapted norm as a Lyapunov functional. This allowed to derive a hypocoercive estimate for the Euclidean norm $\| \cdot \|_2$ of the solution:

$$\| x(t) \|_2 \leq ce^{-\mu t} \| x_0 \|_2, \quad t \geq 0.$$  \hspace{1em} (2.12)

Here $\mu > 0$ is the spectral gap of the matrix $C$ (and the sharp decay rate of the ODE (2.11)), and $c \geq 1$ is some constant.

In [3] we investigated, in the two dimensional case, the sharpness of the constant $c$. By analogy with (2.10), we define the best multiplicative constant for the hypocoercivity estimate of the ODE as

$$c_1 := c_1(C) := \min \{ c \geq 1 : \text{(2.12) holds for all } x_0 \in \mathbb{C}^n \}.$$  \hspace{1em} (2.11)

The explicit expression for the best constant $c_1$ depends on the spectrum of $C$. In particular, denoting by $\lambda_1, \lambda_2$ the two eigenvalues of $C$, we distinguish three cases:

1. $\Re(\lambda_1) = \Re(\lambda_2) = \mu$;
2. $\mu = \Re(\lambda_1) < \Re(\lambda_2)$, $\Im(\lambda_1) = \Im(\lambda_2)$;
3. $\mu = \Re(\lambda_1) < \Re(\lambda_2)$, $\Im(\lambda_1) \neq \Im(\lambda_2)$.

In [3] we treated all the cases for matrices in $\mathbb{C}^{2 \times 2}$. The corresponding explicit form of $c_1$ in the cases (1) and (2) is described in the next theorem. For the case (3) we have, instead, an implicit form, see Corollary 4.3 in [3].

**Theorem 2.7.** Let $C \in \mathbb{C}^{2 \times 2}$ be positive stable and non-defective with eigenvalues $\lambda_1, \lambda_2$. Denoting by $\alpha \in [0, 1]$ the cosine of the angle between the two eigenvectors of $C^T$, the best constant for (2.12) in the cases (1) and (2) is

$$c_1 = \sqrt{\frac{1 + \alpha}{1 - \alpha}} \quad \text{and, respectively,} \quad c_1 = \frac{1}{\sqrt{1 - \alpha^2}}.$$  \hspace{1em} (2.12)

For dimension $n \geq 3$, explicit expressions for the best constant $c_1$ seem to be unknown in general.
2.3.1. The defective case. So far we have discussed non-defective matrices \( C \in \mathbb{R}^d \). The remaining case has to be treated apart since we cannot obtain both the optimality of the multiplicative constant and the sharpness of the exponential decay at the same time if \( C \) is defective. Nevertheless, hypocoercive estimates hold (see Chapter 1.8 in [25] and Theorem 2.8 in [8]) with either reduced exponential decay rates or with the best decay rate \( \mu \), but augmented with a time-polynomial coefficient, as the following theorem claims (see Theorem 2.8 in [8] and Lemma 4.3 in [5]).

**Theorem 2.8.** Let \( C \in \mathbb{C}^d \) be a positive stable (possibly defective) matrix with spectral gap \( \mu > 0 \). Let \( M \) be the maximal size of a Jordan block associated to \( \mu \). Let \( x(t) \) be the solution of the ODE \( \frac{d}{dt} x(t) = -C x(t) \) with initial datum \( x_0 \in \mathbb{C}^d \). Then, for each \( \epsilon > 0 \) there exist a constant \( c_{\epsilon} \geq 1 \) such that

\[
\| x(t) \|_2 \leq c_{\epsilon} \epsilon^{-\mu t} \| x_0 \|_2, \quad \forall \; t \geq 0, \; x_0 \in \mathbb{C}^d.
\]

Moreover, there exists a polynomial \( p(t) \) of degree \( M - 1 \) such that

\[
\| x(t) \|_2 \leq p(t) e^{-\mu t} \| x_0 \|_2, \quad \forall \; t \geq 0, \; x_0 \in \mathbb{C}^d.
\]

As we did for the non-defective case, we define the best constant \( c_{1,\epsilon} \) for the estimate (2.13) with rate \( \mu - \epsilon \) as

\[
c_{1,\epsilon} := \min \left\{ c_{\epsilon} \geq 1 : \text{(2.13) holds for all } x_0 \in \mathbb{C}^d \right\}.
\]

We do not attempt to define an "optimal polynomial" \( p(t) \) in (2.14). In the next section it is shown that these ODE-results carry over to the corresponding FP equation (2.4).

3. Main results and applications

With the above review of ODE results we can state in this section one of the main results of this paper: The best decay constants in (2.9) for the FP equation (2.4) (and therefore also for (1.1)) coincide with the best constants for the ODE (2.11). This result is a corollary of the main theorem of this paper. As we have anticipated in Section 2 it claims that the propagator norm of the FP equation coincides with the propagator norm of its corresponding ODE (w.r.t. the Euclidean norm).

First we define the projection operator \( \Pi_0 \) that maps a function in \( \mathcal{H} \) into the subspace generated by the steady state \( f_\infty \).

**Definition 3.1.** Let \( f \in \mathcal{H} = L^2(\mathbb{R}, f_\infty^{-1}) \) and \( f_\infty \) the normalized Gaussian (2.5). We define the operator \( \Pi_0 : \mathcal{H} \rightarrow \mathcal{H} \) as

\[
\Pi_0 f := \langle f, f_\infty \rangle \mathcal{H} f_\infty,
\]

i.e., \( \Pi_0 \) projects \( f \) onto \( V_0 := \text{span}_{\mathbb{R}}(f_\infty) = \mathcal{N}(L) \).

**Remark 3.2.** Let \( f \in \mathcal{H} \). Then, the coefficient \( \langle f, f_\infty \rangle \mathcal{H} \) is equal to \( \int_{\mathbb{R}^d} f(x) \, dx \), by definition. Moreover, it is obvious from (2.4) that the "total
mass $\int_{\mathbb{R}^d} f(t, x) \, dx$ remains constant in time under the flow of the equation. Hence, $(\Pi_0 f)(t)$ is independent of $t$, if $f(t)$ solves (2.4). This implies $e^{-Lt} (1 - \Pi_0) = e^{-Lt} - \Pi_0$.

We introduce the standard definitions of operator norms.

**Definition 3.3.** Let $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathbb{R}^d \to \mathbb{R}^d$ be linear operators. Then

$$
\|A\|_{\mathcal{B}(\mathcal{H})} := \sup_{0 \neq f \in \mathcal{H}} \frac{\|Af\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}}, \quad \|B\|_{\mathcal{B}(\mathbb{R}^d)} := \sup_{0 \neq x \in \mathbb{R}^d} \frac{\|Bx\|_2}{\|x\|_2}.
$$

If $f(t)$ is the solution of the FP equation (2.4) with $f(0) = f_0 \in \mathcal{H}$, then

$$
\|e^{-Lt} (1 - \Pi_0)\|_{\mathcal{B}(\mathcal{H})} = \|e^{-Lt}\|_{\mathcal{B}(V_0^\perp)} = \sup_{0 \neq f_0 \in \mathcal{H}} \frac{\|f(t) - \Pi_0 f_0\|_{\mathcal{H}}}{\|f_0\|_{\mathcal{H}}}.
$$

If $x(t) \in \mathbb{R}^d$ is the solution of the ODE $\frac{d}{dt} x = -Cx$ with initial datum $x(0) := x_0$, then

$$
\|e^{-Ct}\|_{\mathcal{B}(\mathbb{R}^d)} = \sup_{0 \neq x_0 \in \mathbb{R}^d} \frac{\|x(t)\|_2}{\|x_0\|_2}.
$$

With these notations we can state the main result of this paper.

**Theorem 3.4.** Let Condition A hold for the FPE (2.4). Then the propagator norms of the FPE (2.4) and its corresponding ODE $\frac{d}{dt} x = -Cx$ are equal, i.e.,

$$
(3.1) \quad \|e^{-Lt}\|_{\mathcal{B}(V_0^\perp)} = \|e^{-Ct}\|_{\mathcal{B}(\mathbb{R}^d)}, \quad \forall t \geq 0.
$$

The proof of Theorem 3.4 will be prepared in the following two sections and finally completed in Section 6.

Theorem 3.4 can be seen as a generalization of a result in [14], where the propagator norm for the kinetic FP equation

$$
\partial_t g = -\tilde{L}_a g := -v \partial_x g + \partial_x (\partial_x g + (ax + v) g)
$$

(3.2)

$$
\text{div}_{(x,v)} \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \nabla_{(x,v)} g + \begin{pmatrix} 0 & -1 \\ a & 1 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} g \right),
$$

with $(x, v) \in \mathbb{R}^2$ and the parameter $a > 0$, has been computed explicitly.

**Theorem 3.5.** [14, Theorem 1.2] For any $a > 0$ and $t \geq 0$, it holds:

$$
(3.3) \quad \|e^{-\tilde{L}_a t}\|_{\mathcal{B}(V_0^\perp)} = c_a(t) \exp \left( -\frac{1 - \sqrt{(1-4a)x}}{2} t \right),
$$

where the non-negative factor $c_a(t)$ is given for $0 < a < 1/4$ by

$$
(3.4) \quad c_a(t) := \sqrt{e^{-2\theta t} + \frac{1 - \theta^2}{2\theta^2} (1 - e^{-\theta t})^2 + \frac{1 - e^{-2\theta t}}{2} \left( 1 + \frac{1}{\theta} \sqrt{1 + (\theta^{-2} - 1) \left( \frac{e^{\theta t} - 1}{e^{\theta t} + 1} \right)^2} \right)},
$$

with $\theta := \sqrt{4a - 1}$. 

with $\theta = \sqrt{1 - 4a}$, for $a > 1/4$ by

\begin{equation}
(3.5) \quad c_a(t) := \sqrt{1 + \frac{|e^{\theta t} - 1|}{2|\theta|^2} \left( |e^{\theta t} - 1| + \sqrt{|e^{\theta t} - 1|^2 + 4|\theta|^2} \right)},
\end{equation}

with $\theta := \sqrt{4a - 1}i$, and for $a = 1/4$ by

\begin{equation}
(3.6) \quad c_a(t) := \sqrt{1 + \frac{t^2}{2} + t \sqrt{1 + \left( \frac{t}{2} \right)^2}}.
\end{equation}

Note that there is a small typo in the formula for $c_a(t)$, $a < 1/4$ in [14] that corresponds to (3.4).

After normalization the drift matrix of (3.2) is given by

\begin{equation}
(3.7) \quad C_a := \begin{pmatrix} 0 & -\sqrt{a} \\ \frac{1}{\sqrt{a}} & 1 \end{pmatrix}.
\end{equation}

Its eigenvalues are $\lambda_{1,2} := \frac{1}{2} (1 \pm \theta)$, with $\theta$ as in Theorem 3.5 and the corresponding eigenvectors are $v_{1,2} = (\sqrt{a}, -\lambda_{1,2})^T$. This shows that the spectral gap is given by $\mu = \frac{1}{2} (1 - \sqrt{(1 - 4a)_+})$. It is easy to check that $C_a$ satisfies Condition A for each $a > 0$. We observe that the value $a = 1/4$ is critical in the sense that $C_{1/4}$ is defective.

With the approach of this work we can employ the results of Section 2.3 for obtaining the best possible constant $c_1$ in

\[
\|e^{-\tilde{L}a t}\|_{\mathcal{B}(V_0)} = \|e^{-C_a t}\|_{\mathcal{B}(\mathbb{R}^d)} \leq c_1 e^{-\mu t}.
\]

For $a \neq 1/4$ we apply Theorem 2.7 and note that for $0 < a < 1/4$ we are in case (2). We compute $\alpha = 2\sqrt{a}$, giving the optimal constant

\[
c_1 = (1 - 4a)^{-1/2}.
\]

which can also be obtained from (3.4) in the limit $t \to \infty$. For $a > 1/4$ we are in case (1) and obtain $\alpha = (2\sqrt{a})^{-1}$ and

\[
c_1 = \frac{2\sqrt{a} + 1}{\sqrt{4a - 1}}.
\]

The same is obtained as the maximal value of $c_a(t)$ in (3.5), taken whenever $|e^{\theta t} - 1| = 2$. Finally, for $a = 1/4$ the results of Theorems 2.8 and 3.5 agree with $c_a(t) \approx t$ as $t \to \infty$.

The plot in Figure 1 shows the right-hand side of (3.3) as a function of time for 3 values of $a$ ($a = 1/5$, $a = 1/4$, $a = 2$). Note the non-smooth behavior in the case $a = 2$. 

Figure 1. The propagator norm for equation (3.2) for 3 values of the parameter $a$. Solid curve (green) for $a = 2$, dashed curve (red) for $a = 1/4$, dotted curve (blue) for $a = 1/5$. The dash-dotted curve (green), gives the best exponential bound of the form $c_1 e^{-t/2}$ for the case $a = 2$.

3.1. Applications of Theorem 3.4

3.1.1. Long time behavior. One consequence of Theorem 3.4 is that all the estimates about the decay of the solutions of the ODE carry over to the corresponding FPE problem. In particular, it follows that the hypocoercive ODE estimates (2.12) and (2.13) hold also for solutions of the corresponding FP equation. Moreover, the best constants in the estimates are the same both for the FP case and for its corresponding drift ODE.

Theorem 3.6. Let $C \in \mathbb{R}^{d \times d}$ be non-defective and satisfy Condition A. Let $c_1$ be the best constant in the estimate (2.12) for the ODE (2.11). Then it is also the optimal constant $c_{\min}$ in the following hypocoercive estimate (3.8)

$$\|f(t) - f_\infty\|_\mathcal{H} \leq c_1 e^{-\mu t}\|f_0 - f_\infty\|_\mathcal{H}, \quad \forall t \geq 0, \forall f_0 \in \mathcal{H}, \int_{\mathbb{R}^d} f_0(x) \, dx = 1$$

for the Fokker-Planck equation (2.4).

Theorem 3.7. Let $C \in \mathbb{R}^{d \times d}$ be defective and satisfy Condition A. Let $M$ be the maximal size of a Jordan block associated to $\mu$. Let $\epsilon > 0$ be fixed and $c_{1,\epsilon}$ be the best constant in the estimate (2.13) for the ODE (2.11). Then the
following hypocoercive estimates holds
(3.9)
\[ \| f(t) - f_\infty \|_{\mathcal{H}} \leq c_{1,\epsilon} e^{-\epsilon t} \| f_0 - f_\infty \|_{\mathcal{H}}, \quad \forall t \geq 0, \forall f_0 \in \mathcal{H}, \int_{\mathbb{R}^d} f_0(x) \, dx = 1 \]
for the Fokker-Planck equation (2.4), and it is optimal with \( c_{1,\epsilon} \). Moreover,
(3.10)
\[ \| f(t) - f_\infty \|_{\mathcal{H}} \leq p(t) e^{-\mu t} \| f_0 - f_\infty \|_{\mathcal{H}}, \quad \forall t \geq 0, \forall f_0 \in \mathcal{H}, \int_{\mathbb{R}^d} f_0(x) \, dx = 1, \]
where \( p(t) \) is the polynomial of degree \( M - 1 \) appearing in (2.14).

We conclude that the quest to obtain the best decay for (1.1) is reduced to the knowledge of the best decay constants for the corresponding drift ODE.

3.1.2. Short time behavior. The second application of Theorem 3.4 concerns the short time behavior of the propagator norm of the FP operator. It is linked to the concept of hypocoercivity index, which describes the "structural complexity" of the matrix \( C \) and, more precisely, the intertwining of its symmetric and anti-symmetric parts. For the FP equation, the hypocoercivity index reflects its degeneracy structure. As we are going to illustrate in this section, this index represents the polynomial degree in the short time behavior of the propagator norm, both in the FP equation and in the ODE case. Moreover it describes the rate of regularization of the FP-solution from \( \mathcal{H} \) to a weighted Sobolev space \( \mathcal{H}^1 \).

In the literature the definition of hypocoercivity index is given both for FP equations and ODEs (see [5] and [2], respectively). We will see that these two concepts coincide when we consider the drift ODE associated to the FP equation. We first give the definition for the normalized FP equation and then it will be illustrated that the index is invariant for the general \( (D \neq C_S) \) equation (1.1).

**Definition 3.8.** We define \( m_{HC} \), the hypocoercivity index for the normalized FP equation (2.4) as the minimum \( m \in \mathbb{N}_0 \) such that
(3.11)
\[ T_m := \sum_{j=0}^{m} C_{AS}^j C_S (C_{AS}^T)^j > 0. \]

Here \( C_{AS} := \frac{1}{2} (C - C^T) \) denotes the anti-symmetric part of \( C \).

**Remark 3.9.** Lemma 2.3 in [5] states that the condition \( m_{HC} < \infty \) is equivalent to the FP-equation being hypoelliptic. This index can be seen as a measure of "how much" the drift matrix has to mix the directions of the kernel of the diffusion matrix with its orthogonal space in order to guarantee convergence to the steady state. For example, \( m_{HC} = 0 \) means, by definition, that the diffusion matrix \( D = C_S \) is positive definite, and hence coercive. In general, \( m_{HC} \) is finite when we are assuming Condition A (see Lemma 2.3, [5]).
For completeness, we give the definition of hypocoercivity index also for the non-normalized case. For simplicity we will denote it as well with $m_{HC}$. This is actually allowed since the next proposition will prove that these two definitions are unchanged under normalization.

**Definition 3.10.** We define $m_{HC}$ the hypocoercivity index for the FP equation (1.1) as the minimum $m \in \mathbb{N}_0$ such that

$$\bar{T}_m := \sum_{j=0}^{m} \tilde{C}^j \tilde{D}(\tilde{C}^T)^j > 0.$$  

**Proposition 3.11.** Let us consider the FP equation (1.1) and its normalized version (2.4). Let Condition $\tilde{A}$ (or, equivalently, Condition $A$) be satisfied. Then, the hypocoercivity indices of the two equations coincide, i.e., for any $m \in \mathbb{N}_0$

$$T_m > 0 \quad \text{if and only if} \quad \bar{T}_m > 0,$$

**Proof.** The proof is organized in two steps.

First we claim that it is equivalent to consider the full matrix $C$ instead of its anti-symmetric part in Definition 3.8. More precisely, for any $m \in \mathbb{N}_0$ (3.14)

$$\sum_{j=0}^{m} C_{AS} C_S (C_{AS}^T)^j > 0 \quad \text{if and only if} \quad \sum_{j=0}^{m} C_j C_S (C^T)^j > 0.$$  

This result has been proven in Lemma 3.4, [2].

The second step consists in proving that $T_m > 0$ iff

$$T_m := \sum_{j=0}^{m} C_j D(C^T)^j > 0,$$

where $C = K^{-1/2} \tilde{C} K^{1/2}$ and $D = K^{-1/2} \tilde{D} K^{-1/2} = C_S$ are the matrices appearing in the normalized equation and $K$ from (2.2). By substituting we get

$$T_m = \sum_{j=0}^{m} (K^{-1/2} \tilde{C} K^{1/2})^j K^{-1/2} \tilde{D} K^{-1/2} (K^{1/2} \tilde{C}^T K^{-1/2})^j$$

$$= K^{-1/2} \sum_{j=0}^{m} \tilde{C}^j \tilde{D}(\tilde{C}^T)^j K^{-1/2}$$

$$= K^{-1/2} \bar{T}_m K^{-1/2}.$$  

Then, it is immediate to conclude that the positivity of the two matrices is equivalent since $K > 0$.

Combining this last equivalence with (3.14) yields (3.13). □

**Remark 3.12.** We shall now compare the hypocoercivity index $m_{HC}$ of the normalized FP equation (2.4) to the commutator condition (3.5) in [31]. To this end we rewrite (2.4) for $h(x, t) := f(x, t)/f_\infty(x)$. In Hörmander form it reads

$$\partial_t h = -(A^* A + B) h,$$
where the adjoint is taken w.r.t. $L^2(f_\infty)$. Here, the vector valued operator $A$ and the scalar operator $B$ are given by

$$A = \sqrt{D} \cdot \nabla, \quad B = x^T \cdot C_{AS} \cdot \nabla.$$  

Following §3.3 in [31] we define the iterated commutators

$$C_0 := A, \quad C_k := [C_{k-1}, B].$$

They are vector valued operators mapping from $L^2(f_\infty)$ to $(L^2(f_\infty))^d$. Hence, the nabla operator in $B$ can be either the gradient or the Jacobian, depending on the dimensionality of the argument of $B$. One easily verifies that

$$C_k = pD \cdot C_k \cdot D \cdot \nabla, \quad k \in \mathbb{N}_0.$$  

We recall condition (3.5) from [31]: “There exists $N_c \in \mathbb{N}_0$ such that

$$\sum_{k=0}^{N_c} C_k^* C_k \text{ is coercive on } \ker(A^* A + B)^\perp.$$  

Note that $\ker(A^* A + B)$ consists of the constant functions, and its orthogonal is \{ $h \in L^2(f_\infty) : \int_{\mathbb{R}^d} h f_\infty dx = 0$ \}. The coercivity in (3.15) reads

$$\int_{\mathbb{R}^d} \nabla^T h \cdot T_{N_c} \cdot \nabla h f_\infty dx \geq \kappa \int_{\mathbb{R}^d} h^2 f_\infty dx$$  

for some $\kappa > 0$ and all $h \in \ker(A^* A + B)^\perp$, where $T_{N_c} := \sum_{k=0}^{N_c} (C_{AS}^T)^k DC_{AS}^k$. Clearly, the weighted Poincaré inequality (3.16) holds iff $T_{N_c} > 0$, see §3.2 in [6], e.g. Hence, the minimum $N_c$ for condition (3.15) to hold equals the hypocoercivity index $m_{HC}$ from Definition 3.8 above.

Next we shall link the hypocoercivity index of the FP equation with the hypocoercivity index $m_{HC}$ of its associated ODE $\dot{x}(t) = -C x(t)$, which is defined in the same way. At the ODE level, this index describes the short time decay of the propagator norm $\| e^{-C t} \|_{\mathcal{B}(\mathbb{R}^d)}$ as it is shown in the following theorem (see Theorem 3.2, [2]).

**Theorem 3.13.** Let $C$ satisfy Condition A. Then its (finite) hypocoercivity index is $m_{HC} \in \mathbb{N}_0$ if and only if

$$\| e^{-C t} \|_{\mathcal{B}(\mathbb{R}^d)} = 1 - ct^\alpha + O(t^{\alpha+1}), \quad \text{as } t \to 0^+,$$

for some $c > 0$, where $\alpha := 2m_{HC} + 1$.

**Remark 3.14.** We observe that, in the coercive case (i.e., $m_{HC} = 0$), the propagator norm satisfies an estimate of the form

$$\| e^{-C t} \|_{\mathcal{B}(\mathbb{R}^d)} \leq e^{-\lambda t}, \quad t \geq 0,$$

for some $\lambda > 0$.

In that case ($\alpha = 1$) Theorem 3.13 states that the propagator norm $\| e^{-C t} \|_{\mathcal{B}(\mathbb{R}^d)}$ behaves as $g(t) := 1 - ct$ for short times. With $c = \lambda$, this is the (initial part of the) Taylor expansion of the exponential function in (3.18).
Next we shall use this result to derive information about the short time behavior of the Fokker-Planck propagator norm $\|e^{-Lt}\|_{B(V_0^\perp)}$. By Theorem 3.4 the propagator norms of the FPE and the corresponding ODE coincide.

**Theorem 3.15.** Let $L$ be the Fokker-Planck operator defined in (2.4). Let $C$ satisfy Condition A. Then the finite hypocoercive index of (2.4) is $m_{HC} \in \mathbb{N}_0$ if and only if

$$\|e^{-Lt}\|_{B(V_0^\perp)} = 1 - ct^\alpha + O(t^{\alpha+1}), \quad t \to 0^+,$$

where $\alpha = 2m_{HC} + 1$, for some $c > 0$.

**Proof.** This result is an immediate corollary of Theorem 3.4 and Theorem 3.13 by recalling that the FP equation and its associated ODE have the same hypocoercivity index. □

**Remark 3.16.** As for the ODE case, the equality (3.19) shows that the index $m_{HC}$ describes how fast the propagator norm decays for short times. This is consistent with the fact that the coercive case ($m_{HC} = 0$) corresponds to the fastest behavior, i.e., with an exponential decay ($\alpha = 1$). In general, the bigger the index, the slower is the decay of the norm for short times.

**Example 3.17.** In Theorem 1.2 of [14] the authors derive the exact formula for the propagator norm of the FP equation associated to the matrix (3.7), see Theorem 3.5. From that they also conclude the short time behavior of this norm, depending on the parameter $a$. In the case $a > 0$, equality (2) in [14] implies

$$\|e^{-L_a t}\|_{B(V_0^\perp)} = 1 - \frac{a}{6} t^3 + o(t^3).$$

We note that this result is consistent with the equality (3.19). Indeed, it is easy to verify that for $a > 0$ the matrix $C_a$ has hypocoercivity index $m_{HC} = 1$. Hence the exponent in the polynomial short time behavior turns out to be $\alpha = 3$, as above. □

In the literature, the hypocoercivity index has also a second implication on the qualitative behavior of FPEs, namely the rate of regularization from some weighted $L^2$-space into a weighted $H^1$-space (like in non-degenerate parabolic equations). The following proposition was proven in [31] (see §7.3, §A.21 for the kinetic FP equation with $m_{HC} = 1$. The extension from Theorem A.12 is given without proof and includes a small typo.) and in [5, Theorem 4.8].

**Proposition 3.18.** Let $f(t)$ be the solution of (2.4). Let $C$ satisfy Condition A and $m_{HC}$ be its associated hypocoercivity index. Then, there exist $\tilde{c}, \delta > 0$, such that

$$\|f_\infty \nabla \left( \frac{f(t)}{f_\infty} \right) \|_{H^1} \leq \tilde{c} t^{-a/2} \|f_0\|_{H^1}, \quad 0 < t \leq \delta,$$

with $\alpha := 2m_{HC} + 1$ for all $f_0 \in \mathcal{H}$. 


So far we have seen that the hypocoercivity index of a FP equation determines both the short time decay and its regularization rate. An obvious question is now to understand the relation of these two qualitative properties. The following proposition shows that they are essentially equivalent for the family (2.4) of FP equations:

**Proposition 3.19.** Let $C$ satisfy Condition A, and let $f(t)$ be the solution of (2.4). We denote its propagator norm by \( \| e^{-Lt} \|_{\mathcal{H}(V_\perp^1)} =: \tilde{\eta}(t) \), $t \geq 0$.

(a) Assume that $\tilde{\eta}(t) = 1 - c t^\alpha + o(t^\alpha)$ as $t \to 0^+$ for some $c > 0$ and $\alpha > 0$. Then the regularization estimate (3.20) follows with the same $\alpha$, and for all $f_0 \in \mathcal{H}$. Moreover, this $\alpha$ in (3.20) is optimal (i.e. minimal).

(b) Let there exist some $\tilde{c}, \delta > 0$ and $\alpha > 0$ (not necessarily integer) such that (3.20) holds $\forall f_0 \in \mathcal{H}$. Then $\tilde{\eta}(t) \leq 1 - c_2 t^\alpha$ on $0 \leq t \leq \delta_2$, with some $\delta_2 > 0$ and some $c_2 > 0$. Moreover, if $\alpha$ is minimal in the assumed regularization estimate (3.20), then it is also minimal in the concluded decay estimate $\tilde{\eta}(t) \leq 1 - c_2 t^\alpha$.

The proof of Proposition 3.19 can be found in the Appendix, since it requires results that will be presented in the next sections.

**Remark 3.20.** Inequality (3.20) does not characterize the sharp regularization rate of the FP equation, it rather gives an upper bound to that rate. Hence, the conclusion $\tilde{\eta}(t) \leq 1 - c_2 t^\alpha$ is also just an upper bound for the short time behavior, rather than the dominant part of the Taylor expansion of $\tilde{\eta}(t)$.

**Remark 3.21.** Proposition 3.18 provides an isotropic regularization rate. We note that this result can be improved for degenerate, hypocoercive FP equations, which give rise to anisotropic smoothing. There the regularization is faster in the diffusive directions of $(\ker C S)^\perp$ than in the non-diffusive directions of $\ker C$. “Faster” corresponds here to a smaller exponent in (3.20).

An example of different speeds of regularization is given in [28 Section 11] for the solution $f(t, x, v)$ of a kinetic FP equation in $\mathbb{T}^d \times \mathbb{R}^d$ without confinement potential. In that case the short-time regularization estimate for the $v$-derivatives is the same as for the heat equation, since the operator is elliptic in $v$. But the regularization in $x$ has an exponent $3$ times as large; this corresponds, respectively, to the two cases $m_{HC} = 0, 1$ in (3.20). A more general result about anisotropic regularity estimates can be found in [31 Section A.21.2]. In an alternative description one can fix a uniform regularization rate in time, by considering different regularization orders (i.e. higher order derivatives) in different spatial directions in the setting of anisotropic Sobolev spaces. A definition of these functional spaces and an example of this behaviour is provided in [23], regarding the solution of a degenerate Ornstein-Uhlenbeck equation.
4. Solution of the FP Equation by Spectral Decomposition

In order to link the evolution in (2.4) to the corresponding drift ODE \( \dot{x} = -Cx \) we shall project the solution \( f(t) \in \mathcal{H} \) of (1.1) to finite dimensional subspaces \( \{V^{(m)}\}_{m \in \mathbb{N}_0} \subset \mathcal{H} \) with \( LV^{(m)} \subset V^{(m)} \). Then we shall show that, surprisingly, the evolution in each subspace can be based on the single ODE \( \dot{x} = -Cx \).

4.1. Spectral decomposition of the Fokker Planck operator. First we define the finite dimensional, \( L \)-invariant subspaces \( V^{(m)} \subset \mathcal{H} \). Let the dimension \( d \geq 1 \) be fixed. From §1 we recall that the (normalized) steady state of (2.4) is given by \( g_0(x) := f_\infty = \prod_{i=1}^d g(x_i), \) \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), where \( g(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \) is the one-dimensional (normalized) Gaussian. The construction and results about the spectral decomposition of \( L \) that we are going to summarize can be found in [5, Section 5].

Definition 4.1. Let \( \alpha = (\alpha_i) \in \mathbb{N}_0^d \) be a multi-index. Its order is denoted by \( |\alpha| = \sum_{i=1}^d \alpha_i \). For a fixed \( \alpha \in \mathbb{N}_0^d \) we define

\[
(4.1) \quad g_\alpha(x) := (-1)^{|\alpha|} \nabla_x^\alpha g_0(x),
\]

or, equivalently,

\[
(4.2) \quad g_\alpha(x) := \prod_{i=1}^d H_{\alpha_i}(x_i) g(x_i), \quad \forall x = (x_i) \in \mathbb{R}^d,
\]

where, for any \( n \in \mathbb{N}_0 \), \( H_n \) is the probabilists' Hermite polynomial of order \( n \) defined as

\[
H_n(y) := (-1)^n e^{y^2/2} \frac{d^n}{dy^n} e^{-y^2/2}, \quad \forall y \in \mathbb{R}.
\]

Lemma 4.2. Let \( \alpha = (\alpha_i) \in \mathbb{N}_0^d \). Then,

\[
(4.3) \quad \|g_\alpha\|_{\mathcal{H}} = \sqrt{\alpha!} = \sqrt{\alpha_1! \cdots \alpha_d!}.
\]

Proof. We compute

\[
\|g_\alpha\|_{\mathcal{H}}^2 := \int_{\mathbb{R}^d} \prod_{i=1}^d H_{\alpha_i}(x_i)^2 g(x_i)^2 g(x_i)^{-1} \, dx = \prod_{i=1}^d \int_{\mathbb{R}} H_{\alpha_i}(x_i)^2 g(x_i) \, dx_i = \prod_{i=1}^d \alpha_i!,
\]

where we have used the following weighted \( L^2 \)-norm of \( H_n \):

\[
(4.4) \quad \int_{\mathbb{R}} H_n(y)^2 g(y) \, dy = n!.
\]

Definition 4.3. We define the index sets \( S^{(m)} := \{\alpha \in \mathbb{N}_0^d : |\alpha| = m\}, m \in \mathbb{N}_0 \). For any \( m \in \mathbb{N}_0 \), the subspace \( V^{(m)} \) of \( \mathcal{H} \) is defined as

\[
(4.5) \quad V^{(m)} := \text{span}_{\mathbb{R}} \{g_\alpha : \alpha \in S^{(m)}\}.
\]
Remark 4.4. \( V^{(m)} \) has dimension
\[
\Gamma_m := |S^{(m)}| = \left( \frac{d + m - 1}{m} \right) < \infty.
\]

Let us consider some examples. If \( d = 2 \) we have
1. \( V^{(0)} = \{ \beta_1 g_0(x), \beta_1 \in \mathbb{R} \} \);
2. \( V^{(1)} = \text{span} \{ g_{(1,0)}, g_{(0,1)} \} = \text{span} \{ x_1 e^{-|x|^2/2}, x_2 e^{-|x|^2/2} \} \)
   \( = \{ (\beta_1 x_1 + \beta_2 x_2) g_0(x), \beta_1, \beta_2 \in \mathbb{R} \} \);
3. \( V^{(2)} = \text{span} \{ g_{(2,0)}, g_{(1,1)}, g_{(0,2)} \} \)
   \( = \{ [(\beta_1 x_1^2 - 1) + \beta_2 x_1 x_2 + \beta_3 (x_2^2 - 1)] g_0(x), \beta_i \in \mathbb{R}, i = 1,2,3 \} \);
4. \( V^{(3)} = \text{span} \{ g_{(3,0)}, g_{(2,1)}, g_{(1,2)}, g_{(0,3)} \} \)
   \( = \{ [(\beta_1 (-x_1^3 + 3 x_1) + \beta_2 (-x_1^2 x_2 + x_2) + \beta_3 (-x_2^2 x_1 + x_1) + \beta_4 (-x_3^2 + 3 x_2)] g_0(x), \beta_1, ..., \beta_4 \in \mathbb{R} \} \).

It is well known that \( \{ g_\alpha \}_{\alpha \in \mathbb{N}_0^d} \) forms an orthogonal basis of \( \mathcal{H} = L^2(\mathbb{R}^d, g_0^{-1}) \).

Hence, also the subspaces \( V^{(m)} \) are mutually orthogonal. This yields an orthogonal decomposition of the Hilbert space
\[
\mathcal{H} = \bigoplus_{m \in \mathbb{N}_0} V^{(m)}.
\]

Remark 4.5. In [18, §5] an alternative block diagonal decomposition of the FP solution operator (when considered in the flat \( L^2(\mathbb{R}^d) \)) into finite-dimensional subspaces is derived by using Wick quantization.

We also define the normalized version of the basis elements of the subspaces \( V^{(m)} \):

Definition 4.6 (Normalized basis). For each fixed \( \alpha \in \mathbb{N}_0^d \), we denote with \( \tilde{g}_\alpha \) the normalized function
\[
\tilde{g}_\alpha := \frac{g_\alpha}{\| g_\alpha \|_{\mathcal{H}}}. 
\]

The reason why we need both \( g_\alpha \) and \( \tilde{g}_\alpha \) is that we can obtain a "nicer" evolution of \( f(t) \) projected into \( V^{(m)} \) in terms of the matrix \( C \) with the first ones. Instead, the functions \( \tilde{g}_\alpha \) can be used for the equivalence of norms by Plancherel’s equality in the Hilbert space \( \mathcal{H} \).

Due to the orthogonal decomposition (4.6), we can write
\[
f(t,x) = \sum_{\alpha \in \mathbb{N}_0^d} \frac{\langle f(t), g_\alpha \rangle_{\mathcal{H}}}{\| g_\alpha \|_{\mathcal{H}}^2} g_\alpha(x) =: \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha(t) g_\alpha(x),
\]
or in terms of the normalized basis,
\[
f(t,x) = \sum_{\alpha \in \mathbb{N}_0^d} \langle f(t), \tilde{g}_\alpha \rangle_{\mathcal{H}} \tilde{g}_\alpha(x) =: \sum_{\alpha \in \mathbb{N}_0^d} \tilde{d}_\alpha(t) \tilde{g}_\alpha(x).
\]

The Fourier coefficients corresponding to a subspace \( V^{(m)} \) are grouped into vectors:
\[
d^{(m)} := (d_\alpha)_{\alpha \in S^{(m)}}, \quad \tilde{d}^{(m)} := (\tilde{d}_\alpha)_{\alpha \in S^{(m)}} \in \mathbb{R}^{\Gamma_m}.
\]
Plancherel’s Theorem then yields
\[ \| f \|_{\mathcal{H}}^2 = \sum_{m \geq 0} \left\| \hat{d}^{(m)} \right\|_2^2 = \sum_{m \geq 0} \sum_{\alpha \in S^{(m)}} |\hat{d}_\alpha|^2 = \sum_{m \geq 0} \sum_{\alpha \in S^{(m)}} |d_\alpha|^2 \| g_\alpha \|_\mathcal{H}^2, \]
where we have used the relation \( \hat{d}_\alpha = \| g_\alpha \|_\mathcal{H} d_\alpha \).

Moreover, we denote by \( (\Pi_m f) \in V^{(m)} \) the orthogonal projection of \( f \) into \( V^{(m)} \). It is given by
\[ (\Pi_m f) = \sum_{\alpha \in S^{(m)}} d_\alpha g_\alpha = \sum_{\alpha \in S^{(m)}} \tilde{d}_\alpha \tilde{g}_\alpha. \]

It follows that
\[ \| \Pi_m f \|_{\mathcal{H}} = \| \tilde{d}^{(m)} \|_2. \]

In the next proposition we shall see that the subspaces \( V^{(m)} \) are invariant under the action of the operator \( L \), by giving the explicit action of \( L \) on each basis element \( g_\alpha \). For this purpose we introduce a notation for shifted multi-indices.

**Definition 4.7.** Given \( \alpha = (\alpha_i) \in \mathbb{N}_0^d \) and \( l \in \langle d \rangle := \{1, \ldots, d\} \), we define the components of the multi-indices \( \alpha^{(l-)} \), \( \alpha^{(l+)} \in \mathbb{N}_0^d \) as
\[ \alpha_j^{(l\pm)} := \alpha_j \quad \text{for} \ j \neq l, \quad \alpha_l^{(l\pm)} := (\alpha_l \pm 1)_+. \]

So, for instance, if \( g_\alpha \in V^{(m)} \) and \( \alpha_l > 0 \), then \( g_{\alpha^{(l-)}} \in V^{(m-1)} \) and \( g_{(\alpha^{(l-)}\alpha^{(l+)})} \in V^{(m)} \). Note that cutting off negative values guarantees that \( \alpha^{(l-)} \) is always an admissible multi-index. This part of the definition will, however, not influence the following.

The next proposition specifies the action of the operator \( L \) on \( V^{(m)} \). It is taken from [5, Proposition 5.1 and its proof]:

**Proposition 4.8.** For every \( m \in \mathbb{N}_0 \), the subspace \( V^{(m)} \) is invariant under \( L \), its adjoint \( L^* \) and, hence, the solution operator \( e^{Lt} \), \( t \geq 0 \). Moreover, for each \( g_\alpha \),
\[ L g_\alpha = -\sum_{j,l=1}^d \alpha_l C_{jl} g_{(\alpha^{(j-)}\alpha^{(j+)})}, \]
where \( C_{jl} \) are the matrix elements of \( C \).

### 4.2. Evolution of the Fourier coefficients.

In this section we shall derive the evolution of \( \Pi_m f \) in terms of the Fourier coefficients \( d^{(m)} \).

**Proposition 4.9.** Let \( f \) satisfy the FP equation [2.4]. Then the coefficients in the expansion [4.7] satisfy
\[ \dot{d}_\alpha = -\sum_{j,l=1}^d \eta_{\alpha_j \geq 1} (\alpha^{(j-)})^{(l+)} C_{jl} d_{(\alpha^{(j-)}\alpha^{(j+)})}, \quad \alpha \in \mathbb{N}_0^d. \]
Proof. We substitute (4.7) into (2.4) and use (4.11): 
\[ \sum_{\alpha \in \mathbb{N}^d_0} d_\alpha g_\alpha = - \sum_{j,l=1}^d \sum_{\alpha : \alpha_l \geq 1} d_\alpha \alpha_l C_{jl} g_{\alpha(l-)^j(l+)} . \]
In the sum over \( \alpha \) on the right hand side we substitute 
\[ (\alpha^l)^{(j+)} = \beta \iff \alpha = (\beta^{(j-)})^{(l+)} , \]
leading to 
\[ \sum_{\alpha \in \mathbb{N}^d_0} d_\alpha g_\alpha = - \sum_{j,l=1}^d \sum_{\beta : \beta_j \geq 1} d_\beta (\beta^{(j-)}_l)^{(l+)} C_{jl} g_\beta , \]
completing the proof.

As the simplest example we shall first consider the evolution in \( V^{(1)} \). We use the notation \( S^{(1)} = \{ \alpha^{(1)}, \ldots, \alpha^{(d)} \} \) with \( \alpha^{(k)}_j = \delta_{jk} \), \( j, k = 1, \ldots, d \). In the right hand side of (4.12) with \( \alpha = \alpha^{(k)} \) obviously only the terms with \( j = k \) are nonzero, \( (\alpha^{(k)}(l-)^j)^{(l+)} = \alpha(l) \) and, thus, \( (\alpha^{(k-)^j})^{(l+)} = 1 \). This implies 
\[ \dot{d}^{(k)} = - \sum_{l=1}^d C_{kl} d^{(l)} \]
and therefore
\[ d^{(1)} = - C d^{(1)} \quad \text{for} \quad d^{(1)} = (d^{(1)}(1), \ldots, d^{(1)}(d)) . \]
We define \( h(t) := \| e^{-Ct} \|_{\mathbb{R}^d} \). Then (4.13) implies 
\[ h(t) = \sup_{0 \neq \tilde{d}^{(1)}(0) \in \mathbb{R}^d} \frac{\| \tilde{d}^{(1)}(t) \|_2}{\| \tilde{d}^{(1)}(0) \|_2} , \quad t \geq 0 . \]
To analyze the evolution in \( V^{(m)} \), \( m \geq 2 \), it turns out that the representation of \( d^{(m)} \) as a vector is not convenient. In the next section we shall rather represent it as a tensor. Not as a tensor of order \( d \), as the number of components of \( \alpha \) would indicate, but as a symmetric tensor of order \( m \) over \( \mathbb{R}^d \). This way it will be easier to characterize its evolution – in fact as a tensored version of (4.13).

5. SUBSPACE EVOLUTION IN TERMS OF TENSORS

5.1. Order-\( m \) tensors. In this subsection we briefly review some notations and basic results on tensors that will be needed. Most of their elementary proofs are deferred to the appendix. For more details we refer the reader to [10] and [19].

Let \( m \in \mathbb{N} \) be fixed.
Definition 5.1. For \( n_1, \ldots, n_m \in \mathbb{N} \), a function \( h : \langle n_1 \rangle \times \cdots \times \langle n_m \rangle \rightarrow \mathbb{R} \) is a (real valued) hypermatrix, also called order-\( m \) tensor or \( m \)-tensor, where \( \langle n_k \rangle := \{1, \ldots, n_k\}, \forall 1 \leq k \leq m \). We denote the set of values of \( h \) by an \( m \)-dimensional table of values, calling it \( A = (A_{i_1 \ldots i_m})^{i_1 \ldots \leq n_1}_{i_m = 1 \cdots i_m = 1} \), or just \( A = (A_{i_1 \ldots i_m}) \). The set of order-\( m \) hypermatrices (with domain \( \langle n_1 \rangle \times \cdots \times \langle n_m \rangle \)) is denoted by \( T^{n_1 \times \cdots \times n_m} \).

We will consider only the case in which \( n_1 = \cdots = n_m = d \), i.e., \( A = (A_{i_1 \ldots i_m})^d_{i_1 \ldots i_m = 1} \). In this case, we will denote \( T_d(m) := T^{d \times \cdots \times d} \) for simplicity. Also, since in our case the dimension \( d \) is fixed, we will denote it by \( T_d(m) \). Then \( A \in T_d(m) \) is a function from \( \langle d \rangle^m \) to \( \mathbb{R} \), denoted by \( A = (A_I)_{I \in \langle d \rangle^m} \).

It will be useful to define some operations on \( T_d(m) \):

Definition 5.2. It is natural to define the operations of entrywise addition and scalar multiplication that make \( T_d(m) \) a vector space in the following way: for any \( A, B \in T_d(m) \) and \( \gamma \in \mathbb{R} \)

\[
(A + B)_{i_1 \ldots i_m} := A_{i_1 \ldots i_m} + B_{i_1 \ldots i_m}, \quad (\gamma A)_{i_1 \ldots i_m} := \gamma A_{i_1 \ldots i_m}.
\]

Moreover, given \( m \) matrices \( B_1 = (b_{ij}^{(1)}) \), \( B_2 = (b_{ij}^{(2)}) \in \mathbb{R}^{d \times d} = T_2 \) and \( A \in T_d(m) \), we define the multilinear matrix multiplication by \( A' := (B_1, \ldots, B_m) \odot A \in T_d(m) \) where

\[
A'_{i_1 \ldots i_m} := \sum_{j_1, \ldots, j_m = 1}^d b_{i_1 j_1}^{(1)} \cdots b_{i_m j_m}^{(m)} A_{j_1 \ldots j_m}.
\]

For \( A \in T_d(m) \) and \( k \leq m \) matrices \( B_1, \ldots, B_k \in T_2 \), we also define the product \( A' := (B_1, \ldots, B_k) \odot A \in T_d(k) \) in the following way:

\[
A'_{i_1 \ldots i_m} := \sum_{j_1, \ldots, j_k = 1}^d b_{i_1 j_1}^{(1)} \cdots b_{i_k j_k}^{(k)} A_{j_1 \ldots j_k i_{k+1} \ldots i_m},
\]

i.e., the multiplication acts on the first \( k \)-indices of \( A \). For simplicity, when \( B_1 = \ldots = B_k := B \), we will denote \( (B_1, \ldots, B_k) \odot A \) by \( B \odot^k A \). For example, if \( d = 4 \) and given \( B = (b_{ij}) \in \mathbb{R}^{4 \times 4} \), \( A \in T_3 \),

\[
(B \odot A)_{i_1 i_2 i_3} = \sum_{j=1}^4 b_{i_1 j} A_{j i_2 i_3},
\]

and

\[
B \odot^3 A := (B, B, B) \odot A.
\]

Finally, we equip \( T_d(m) \) with an inner product:

Definition 5.3. Let \( A = (A_{i_1 \ldots i_m}), B = (B_{i_1 \ldots i_m}) \in T_d(m) \), we call \( \langle A, B \rangle_\mathcal{F} \in \mathbb{R} \) the Frobenius inner product between the \( m \)-tensors \( A \) and \( B \), defined by

\[
\langle A, B \rangle_\mathcal{F} := \sum_{i_1, \ldots, i_m = 1}^d A_{i_1 \ldots i_m} B_{i_1 \ldots i_m}.
\]
This induces a norm in $T^{(m)}$, called Frobenius norm in the natural way:

$$\|A\|_{\mathcal{F}} := \sqrt{\langle A, A \rangle_{\mathcal{F}}} = \left(\sum_{i_1, \ldots, i_m=1}^{d} (A_{i_1 \ldots i_m})^2\right)^{1/2} \geq 0.$$ 

**Definition 5.4.** The tensor $D = (D_I)_{I \in \langle d \rangle^m} \in T^{(m)}$ is called symmetric, if $\forall I \in \langle d \rangle^m$ it is true that $D_I = D_{\sigma(I)}$ for every permutation $\sigma$ acting on $\langle d \rangle^m$. $F^{(m)} \subset T^{(m)}$ (and occasionally $F_d^{(m)}$) denotes the set of symmetric $m$-tensors. Given $A \in T^{(m)}$, we define the symmetric part of $A$ as the symmetric tensor defined by

$$\text{Sym}A := \frac{1}{m!} \sum_{\sigma \in \mathcal{P}} \sigma(A) \in T^{(m)},$$

where $\mathcal{P}$ is the set of permutations acting on $\langle d \rangle^m$ and $\sigma(A)$ is the tensor with components $\sigma(A)_I := A_{\sigma(I)}$, $\forall I \in \langle d \rangle^m$.

**Remark 5.5.** For a symmetric tensor $D \in F^{(m)}$, clearly we do not need to define $D_I$ for each $I = (i_1, \ldots, i_d) \in \langle d \rangle^m$ since the value of $D_I$ depends only on the number of occurrences of each value in the index $I$. Therefore, we define the function $\varphi : \langle d \rangle^m \to S^{(m)}$ with

$$\varphi_k(I) := \sum_{j=1}^{m} \chi_k(i_j), \quad \forall k = 1, \ldots, d \quad \text{and for each} \quad I = (i_1, \ldots, i_m) \in \langle d \rangle^m.$$

Here, $\chi_k(i_j)$ is equal to one if $i_j = k$ and zero otherwise. Hence, the component $\varphi_k$ counts the occurrences of $k$ in the multi-index $I$. Then, $\forall I \in \langle d \rangle^m$ we define the multi-index $\varphi(I) \in S^{(m)}$ as $\varphi(I) = (\varphi_1(I), \ldots, \varphi_d(I))$. We observe that $\varphi(I)$ is in $S^{(m)}$, since $\sum_{k=1}^{d} \varphi_k(I) = m$, for any $I \in \langle d \rangle^m$.

For the computation of the Frobenius norm of a symmetric tensor it will be useful to introduce the following index classes:

**Remark 5.6.** For a fixed $I \in \langle d \rangle^m$ we define the class of $I$ under the action of $\varphi$ as

$$[I]_{\varphi} := \{J \in \langle d \rangle^m : \varphi(I) = \varphi(J)\},$$

and the set of classes

$$\langle d \rangle^m / \varphi := \{[I]_{\varphi} : I \in \langle d \rangle^m\}.$$

It is easy to show that there is a bijection between the quotient set $\langle d \rangle^m / \varphi$ and $S^{(m)}$ through the identification $[I]_{\varphi} \subset \langle d \rangle^m$ and $\alpha = \varphi(I)$, for each $\alpha \in S^{(m)}$. We observe that:

- If $\varphi(I) = \alpha = (\alpha_1, \ldots, \alpha_d)$, then $[I]_{\varphi}$ has exactly $\gamma_\alpha = \frac{m!}{\alpha_1! \cdots \alpha_d!}$ elements.
- If $D = (D_I)_{I \in \langle d \rangle^m}$ is symmetric, then $D_I = D_J$ if $I$ and $J$ are in the same class.

We will use these two properties in the proof of Proposition 5.18, for example to compute the Frobenius norm of a symmetric tensor.
Definition 5.7. Let $D = (D_I)$ be a symmetric $m$-tensor and $I \in \langle d \rangle^m$. Then, for any $\alpha = (\alpha_1, ..., \alpha_d) \in S^m$ we define
$$D_\alpha := D_I, \quad \text{if } \alpha = (\varphi_1(I), ..., \varphi_d(I)).$$
We observe that this notion is well-defined since $D$ is symmetric and the property $\varphi(I) = \varphi(\sigma(I))$ holds.

The previous definition shows that there is a one-to-one correspondence between the indices of a symmetric $m$-tensor and the elements of $S^m$. This implies that the dimension of $F^m$ is equal to the cardinality of $S^m$, i.e. $\Gamma_m$. Hence, for defining $D \in F^m$ we just need to define $D_\alpha$ for every $\alpha \in S^m$.

Next we define the order-$m$ outer product and discuss the rank-1 decomposition of tensors, using a result from algebraic geometry.

Definition 5.8. Let $v_i := (v_{1i}, ..., v_{di}), \ i = 1, ..., m$ be $m$ vectors in $\mathbb{R}^d$. We define $v_1 \otimes \cdots \otimes v_m \in T^m$ as the $m$-tensor with components
$$(v_1 \otimes \cdots \otimes v_m)_I := v_{1i_1} \cdots v_{mi_m}, \quad \forall I = (i_1, ..., i_m) \in \langle d \rangle^m.$$
We call this operation between $m$ vectors, the $m$-outer product.

In the special case of all the vectors $v_i = v \in \mathbb{R}^d, \ i = 1, ..., m$ equal, we denote
$$v^\otimes m := v \otimes \cdots \otimes v,$$
and we observe that the tensor $v^\otimes m$ is symmetric by definition.

Proposition 5.9 ([10], Lemma 4.2). Let $D \in F^m$. Then, there exist an integer $s \in [1, \Gamma_m], \text{ numbers } \lambda_1, ..., \lambda_s \in \mathbb{R}, \text{ and vectors } v_1, ..., v_s \in \mathbb{R}^d$ such that
$$(5.2) \quad D = \sum_{k=1}^s \lambda_k v_k^\otimes m.$$ 
The minimum $s$ such that (5.2) holds is called the symmetric rank of $D$.

Remark 5.10. In [10] the result is stated for complex tensors. In that case it is possible to choose all the coefficients $\lambda_i$ in (5.2) equal to one, due to the fact that $\mathbb{C}$ is a closed field. We remark that the same decomposition carries over to the real case, i.e. with real coefficients $\lambda_i$ and real vectors $v_i$, by using the same proof [11].

It is easy to see that this rank-1 decomposition persists under a (constant) multilinear matrix multiplication:

Lemma 5.11. Let $D \in F^m$ with decomposition (5.2), and let $B \in \mathbb{R}^{d \times d}$. Then it holds
$$(5.3) \quad B \otimes^m D = \sum_{k=1}^s \lambda_k (B v_k)^\otimes m.$$ 
For rank-1 tensors, their inner product simplifies as follows:
Lemma 5.12. Given \( v_k = (v_i^{(k)}) \in \mathbb{R}^d, k = 1, \ldots, 2m \), then
\[
\langle v_1 \otimes \cdots \otimes v_m, v_{m+1} \otimes \cdots \otimes v_{2m} \rangle_F = \prod_{i=1}^m \langle v_i, v_{i+m} \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathbb{R}^d \).

A special case of this lemma is given by

Corollary 5.13. Given \( v_1, v_2 \in \mathbb{R}^d \), then
\[
\langle v_1^m, v_2^m \rangle_F = (v_1, v_2)^m.
\]

Next we shall derive some results on matrix-tensor products \( B \otimes_k A \):

Lemma 5.14. Let \( B = B^T \in \mathbb{R}^{d \times d} \) be such that \( B \geq 0 \). Then, for any \( A \in T^{(m)} \)
\[
\langle A, B \otimes A \rangle_F \geq 0.
\]

For \( B \in \mathbb{R}^{d \times d} \), \( \| B \| \) we will denote in the sequel the spectral norm of \( B \).

Lemma 5.15. For any \( A \in T^{(m)} \), \( B \in \mathbb{R}^{d \times d} \) and \( 1 \leq k \leq m \),
\[
\| B \otimes_k A \|_F \leq \| B \| k \| A \|_F.
\]

5.2. Time evolution of the tensors \( D^{(m)}(t) \) in \( V^{(m)} \). Proposition 4.9 gives the time evolution of each vector \( d^{(m)} \). But for \( m \geq 2 \) it does not reveal its inherent structure. Therefore we shall now regroup the elements of \( d^{(m)} \) as an order-\( m \) tensor and analyze its evolution.

Definition 5.16. Let \( m \geq 1 \), \( t \geq 0 \), and \( d^{(m)}(t) = (d_\alpha(t))_{\alpha \in S^{(m)}} \in \mathbb{R}^{\Gamma^m} \) be the solution of the ODE \( \frac{d}{dt} d^{(m)} = -C^{(m)} d^{(m)} \). Then we define the symmetric \( m \)-tensor \( D^{(m)}(t) = (D_\alpha^{(m)}(t))_{\alpha \in S^{(m)}} \) as
\[
D_\alpha^{(m)}(t) := \frac{d_\alpha(t)}{\gamma_\alpha},
\]
where \( \gamma_\alpha := \frac{m!}{\alpha_1! \cdots \alpha_d!} \), for \( \alpha = (\alpha_1, \ldots, \alpha_d) \).

For \( m = 1 \) we of course have \( D^{(1)} = d^{(1)} \). We illustrate this definition for the case \( m = d = 2 \) with \( \Gamma_2 = 3 \):
\[
d^{(2)} = \begin{pmatrix} d_{(2,0)} \\ d_{(1,1)} \\ d_{(0,2)} \end{pmatrix}, \quad D^{(2)} = \begin{pmatrix} d_{(2,0)} \\ d_{(1,1)} \\ \frac{d_{(1,1)}}{2} \\ d_{(0,2)} \end{pmatrix} \in F^{(2)}_2 \subset T^{(2)}_2 = \mathbb{R}^{2 \times 2}.
\]

Elementwise, the evolution of \( D_\alpha^{(m)} \) easily carries over from Proposition 4.9.

Proposition 5.17. For any \( \alpha \in S^{(m)} \), the element \( D^{(m)}_\alpha(t) \) evolves according to
\[
D^{(m)}_\alpha(t) = -\sum_{j,l=1}^d \alpha_j C_{jl} D^{(m)}_{(\alpha^{(j-l)})^{(l+j)}}.
\]
Proof. From (4.12) we obtain by substituting the definition (5.8) on both sides:

\[ D_a^{(m)} = -\frac{1}{\gamma_{a_j}} \sum_{i=1}^{d} \delta_{a_i \geq 1} \gamma_{(a^{(j-1)}(l^+))} C_{ij} D_{(a^{(j-1)}(l^+))}^{(m)} \]

The claim (5.9) then follows from the relation

\[ \gamma_a \alpha_j = \gamma_{(a^{(j-1)}(l^+))} \forall \alpha \in \mathbb{N}_0 \text{ with } \alpha_j \geq 1, \]

which can be obtained as follows: It is trivial for \( l = j \), and for \( l \neq j \) it follows from the definition of \( \gamma_a \) and from the observation that \( (a^{(j-1)}(l^+)) = \alpha_l + 1 \) and \( (a^{(j-1)}(l^+)) = \alpha_j - 1 \).

The advantage of this new structure consists in two facts:

- The Frobenius norm \( \|D^{(m)}(t)\|_{\mathcal{F}} \) is proportional (uniformly in \( t \)) to the Euclidean norm \( \|\tilde{d}^{(m)}(t)\|_2 \) for which we want to prove a decay estimate like (4.14).
- The rank-1 decomposition of \( D^{(m)}(t) \) is compatible with the Fokker-Planck flow in \( V^{(m)} \). I.e., for each symmetric tensor \( D^{(m)}(0) \) (considered as an initial condition in \( V^{(m)} \)), we can decompose \( D^{(m)}(t) \) as a sum of order-\( m \) outer products of vectors that are solutions of the ODE \( \frac{d}{dt} \tilde{v}(t) = -C \tilde{v}(t) \).

Concerning the first property we have

**Proposition 5.18.** Given \( m \geq 1 \), then

\[ \|D^{(m)}(t)\|_{\mathcal{F}} = \frac{1}{\sqrt{m!}} \|\tilde{d}^{(m)}(t)\|_2, \quad \forall t \geq 0. \]  

Proof. We compute, using Remark 5.6

\[ \|D^{(m)}(t)\|^2_{\mathcal{F}} = \sum_{I \in \langle d \rangle^m} D_I^{(m)}(t)^2 = \sum_{a \in S^{(m)}} D_a^{(m)}(t)^2 \gamma_a, \]

where we used the identification \( D_a^{(m)}(t) := D_{(I)}^{(m)}(t) \) if \( a = \varphi(I) \) as well as \( \|I\varphi\| = \gamma_a \).

Then, using the definition of \( D^{(m)}(t), \tilde{d}_a(t) = \|g_a\|_{\mathcal{F}} d_a(t), \) and Lemma 4.2 we have

\[ \|D^{(m)}(t)\|^2_{\mathcal{F}} = \sum_{a \in S^{(m)}} \frac{d_a(t)^2}{\gamma_a} = \sum_{a \in S^{(m)}} \frac{\tilde{d}_a(t)^2}{\gamma_a \|g_a\|^2_{\mathcal{F}}} = \frac{1}{m!} \sum_{a \in S^{(m)}} \tilde{d}_a(t)^2 = \frac{1}{m!} \|\tilde{d}^{(m)}(t)\|^2_2, \]

concluding the proof.

Concerning the second property we find that the rank-1 decomposition of \( D^{(m)}(t) \) commutes with the time evolution by the Fokker-Planck equation:
Theorem 5.19. Let \( m \geq 1 \) be fixed and let \( D^{(m)} \in \mathcal{P}^{(m)} \), having the rank-1 decomposition \( D^{(m)} = \sum_{k=1}^{s} \lambda_k v_k^{ \otimes m} \) with symmetric rank \( s \), constants \( \lambda_1, \ldots, \lambda_s \in \mathbb{R} \) and \( s \) vectors \( v_k := (v_j^{(k)})_{j=1}^{d} \in \mathbb{R}^d \). Then, \( D^{(m)}(t), t > 0 \), the solution to (5.9) with initial condition \( D^{(m)}(0) = D^{(m)} \) has the decomposition

\[
D^{(m)}(t) = \sum_{k=1}^{s} \lambda_k [v_k(t)]^{ \otimes m},
\]

where all vectors \( v_k(t) \in \mathbb{R}^d, k = 1, \ldots, s \) satisfy the ODE \( \frac{d}{dt} v_k(t) = -C v_k(t) \) with initial condition \( v_k(0) = v_k \). Moreover, \( D^{(m)}(t), t > 0 \) has the constant-in-t symmetric rank \( s \).

Proof. We shall compute the evolution of the symmetric \( m \)-tensor \( A(t) := \sum_{k=1}^{s} \lambda_k [v_k(t)]^{ \otimes m} \), using that \( \frac{d}{dt} v_k(t) = -C v_k(t) \). To this end we compute first the derivative \( \frac{d}{dt}(w(t)^{ \otimes m})_a \) if the vector \( w(t) = (w_1(t), \ldots, w_d(t))^T \in \mathbb{R}^d \) satisfies the ODE with \( C \):

Given \( \alpha = (\alpha_1, \ldots, \alpha_d) \in S^{(m)} \), we have

\[
\frac{d}{dt}(w(t)^{ \otimes m})_a = \frac{d}{dt} \prod_{j=1}^{d} w_j(t)^{\alpha_j} = \sum_{j=1}^{d} \alpha_j \left( w_1(t)^{\alpha_1} \ldots w_j(t)^{\alpha_j} \ldots w_d(t)^{\alpha_d} \right) \frac{d}{dt} w_j(t)
\]

\[
= - \sum_{j=1}^{d} \alpha_j \left( w_1(t)^{\alpha_1} \ldots w_j(t)^{\alpha_j} \ldots w_d(t)^{\alpha_d} \right) \sum_{l=1}^{d} C_{jl} w_l(t)
\]

\[
= - \sum_{j,l=1}^{d} \alpha_j C_{jl} \left( w_1(t)^{\alpha_1} \ldots w_j(t)^{\alpha_j} \ldots w_l(t)^{\alpha_l} \ldots w_d(t)^{\alpha_d} \right)
\]

\[
= - \sum_{j,l=1}^{d} \alpha_j C_{jl} \left( w(t)^{ \otimes m} \right)_{(\alpha_j \ldots \alpha_1)(\alpha_j \ldots \alpha_1)},
\]

and hence, by linearity

\[
\frac{d}{dt} (A(t))_a = - \sum_{j,l=1}^{d} \alpha_j C_{jl} (A(t))_{(\alpha_j \ldots \alpha_1)(\alpha_j \ldots \alpha_1)}.
\]

This ODE equals the evolution equation (5.9) for \( D^{(m)} \), and hence \( A(t) = D^{(m)}(t) \) follows.

Next we consider the symmetric rank of \( D^{(m)}(t), t > 0 \). If it would be smaller than \( s \), a reversed evolution to \( t = 0 \) would lead to a contradiction to the symmetric rank of \( D^{(m)} \).

This theorem allows to reduce the evolution of the tensors \( D^{(m)}(t) \) to the ODE for the vectors \( v_k(t) \). This will be a key ingredient for proving sharp decay estimates of \( D^{(m)} \) in the next section. Moreover it provides a compact formula for the evolution of \( D^{(m)}(t) \).
Corollary 5.20. Let \( m \geq 1 \) be fixed. Then, \( D^{(m)}(t), t > 0 \), the solution to (5.9) follows the evolution

(5.15) \[
\frac{d}{dt} D^{(m)}(t) = -m \text{Sym}(C \odot D^{(m)}(t)), \quad t > 0.
\]

Proof. We shall use the decomposition (5.13) for \( D^{(m)}(t) \). First, we compute the evolution of \([v(t)]^\otimes m\), if \( \frac{d}{dt} v(t) = -C v(t) \):

\[
\frac{d}{dt} ([v(t)]^\otimes m) = -\sum_{k=0}^{m-1} [v(t)]^\otimes k \otimes ((C v(t)) \otimes [v(t)]^\otimes (m-k-1))
\]

\[
= -m \text{Sym}\{ (C v(t)) \otimes [v(t)]^\otimes (m-1) \}.
\]

In the last equality we have used, with \( w := C v(t) \), the general formula

\[
\text{Sym}(w \otimes v^\otimes (m-1)) = \frac{1}{m} \sum_{k=0}^{m-1} (v^\otimes k \otimes w \otimes v^\otimes (m-k-1)), \quad \forall v, w \in \mathbb{R}^d
\]

that can be proven with a straightforward computation. By using the linearity of \( \text{Sym} \) in \( T^{(m)} \), we obtain

\[
\frac{d}{dt} D^{(m)}(t) = \frac{d}{dt} \sum_{k=1}^{n} \lambda_k [v_k(t)]^\otimes m = -m \left( \sum_{k=1}^{n} \lambda_k \text{Sym}\{ (C v_k(t)) \otimes [v_k(t)]^\otimes (m-1) \} \right)
\]

\[
= -m \text{Sym}\left( \sum_{k=1}^{n} \lambda_k (C v_k(t)) \otimes [v_k(t)]^\otimes (m-1) \right) = -m \text{Sym}(C \odot D^{(m)}(t)).
\]

\[ \square \]

6. Decay of the Subspace Evolution in \( V^{(m)} \)

First we shall rewrite our main decay result, Theorem 3.4, in terms of tensors for all subspaces \( V^{(m)} \). We recall \( h(t) := \| e^{-C t} \|_{\mathcal{B}(\mathbb{R}^d)} \), which satisfies

(6.1) \[
h(t) \leq 1, \quad t \geq 0.
\]

This follows from

\[
\frac{d}{dt} \| e^{-C t} x_0 \|_2^2 = -2 \langle C x_0, x_0 \rangle \leq 0, \quad x_0 \in \mathbb{R}^d.
\]

We have shown in (4.14) that the inequality (6.7), see below, holds with \( m = 1 \), since \( D^{(1)}(t) = d^{(1)}(t) \) satisfies the evolution \( d^{(1)} = -C d^{(1)} \). Next we extend the estimate (6.7) to general \( m \geq 1 \). To this end we will show in the next theorem that the propagator norm in each \( V^{(m)} \) is the \( m \)-th power of the propagator norm of the ODE \( \dot{x} = -C x \). This will be used to derive the decay estimates for \( \| e^{-L t} \|_{\mathcal{B}(\mathcal{F} \cap V_{0}^{(m)})} \).

Theorem 6.1. For each \( m \geq 1 \), \( D^{(m)}(0) \in F^{(m)} \), and \( D^{(m)}(t) \) defined as in (5.8), the following estimate holds:

(6.2) \[
\| D^{(m)}(t) \|_{\mathcal{F}} \leq h(t)^m \| D^{(m)}(0) \|_{\mathcal{F}}, \quad t \geq 0.
\]
Moreover,

\[ \sup_{0 \neq D^{(m)}(0) \in F^{(m)}} \frac{\|D^{(m)}(t)\|_{\mathcal{F}}}{\|D^{(m)}(0)\|_{\mathcal{F}}} = h(t)^m. \tag{6.3} \]

**Proof.** Given the initial condition \( D^{(m)}(0) \in F^{(m)} \), Theorem 5.19 provides its rank-1 decomposition as

\[ D^{(m)}(t) = \sum_{k=1}^{\lambda} \lambda_k [v_k(t)]^{\otimes m} = \sum_{k=1}^{\lambda} \lambda_k [e^{-Ct}v_k]^{\otimes m} = e^{-Ct} \circ^{m} D^{(m)}(0), \quad \forall t \geq 0, \tag{6.4} \]

with \( v_k(t) = e^{-Ct}v_k \), for \( k = 1, \ldots, s \), where we have used Lemma 5.11 in the last equality. Using (5.7) then yields:

\[ \|D^{(m)}(t)\|_{\mathcal{F}} = \|e^{-Ct} \circ^{m} D^{(m)}(0)\|_{\mathcal{F}} \leq \|e^{-Ct}\| \|D^{(m)}(0)\|_{\mathcal{F}}, \tag{6.5} \]

proving (6.2).

In order to prove the equality (6.3) we choose initial data of the form \( D^{(m)}(0) := v^{\otimes m}, v \in \mathbb{R}^d \). In this case the Frobenius norm factorizes, i.e. \( \|D^{(m)}(0)\|_{\mathcal{F}} = \|v\|^2 \) and

\[ \|D^{(m)}(t)\|_{\mathcal{F}} = \|(e^{-Ct}v)^{\otimes m}\|_{\mathcal{F}} = \|e^{-Ct}v\|^2. \]

We conclude by observing that

\[ \sup_{0 \neq v \in \mathbb{R}^d} \frac{\|e^{-Ct}v\|^2}{\|v\|^2} = h(t)^m. \tag{6.6} \]

\[ \square \]

The key step in the above proof is to write the evolution of the tensor \( D^{(m)}(t) \) as in (6.4), which allows for the simple estimate (6.5). In contrast, using the rank-1 decomposition in \( \|D^{(m)}(t)\|_{\mathcal{F}}^2 \) would not be helpful, since the vectors \( v_k(t) \) are in general not orthogonal.

We conclude this chapter with the proof of our main result, Theorem 3.4, by using Theorem 6.1.

**Proof of Theorem 3.4.** The first step consists in proving the inequality

\[ \|e^{-Lt}\|_{\mathcal{F}(\mathcal{X} \cap V_0^\perp)} \leq h(t), \forall t \geq 0. \tag{6.6} \]

We can derive the estimate (6.6) from the same ones that hold for the tensors \( D^{(m)}(t) \) at each level \( m \). More precisely, (6.6) holds if

\[ \|D^{(m)}(t)\|_{\mathcal{F}} \leq h(t) \|D^{(m)}(0)\|_{\mathcal{F}}, \quad t \geq 0, \quad D^{(m)}(0) \in F^{(m)}, \quad m \geq 1, \tag{6.7} \]

where \( D^{(m)}(t) \) is defined as in (5.8). Indeed,

\[ \|f(t) - f_0\|^2_{\mathcal{H}} = \sum_{m \geq 1} \|\Pi_m f(t)\|^2_{\mathcal{H}} = \sum_{m \geq 1} \|\tilde{d}^{(m)}(t)\|^2_2 = \sum_{m \geq 1} m! \|D^{(m)}(t)\|^2_{\mathcal{F}}, \quad t \geq 0, \tag{6.8} \]

where we have used the orthonormal decomposition of \( f(t) \), formulas (4.9), (5.12), and that the coefficient \( d_0(t) \equiv 1 \), (with the index \( 0 \in \mathbb{N}_0^0 \)), is
constant in time since \( L g_0 = 0 \) and the normalization \( \int_{\mathbb{R}^d} f_0 \, dx = 1 \). Let us assume (6.7). Then,

\[
\| f(t) - f_\infty \|_{L^2}^2 = \sum_{m \geq 1} m! \| D^{(m)}(t) \|_{F}^2 \leq h(t)^2 \sum_{m \geq 1} m! \| D^{(m)}(0) \|_{F}^2
\]

proving (6.6).

Next, the proof of (6.7) is a direct consequence of Theorem 6.1 and \( h(t) \leq 1 \), yielding

\[
\| D^{(m)}(t) \|_{F} \leq (h(t))^m \| D^{(m)}(0) \|_{F} \leq h(t) \| D^{(m)}(0) \|_{F}.
\]

Now that (6.6) has been proved, we need to show that it is actually an equality, in order to conclude the proof of (3.1). For this purpose, we observe that for \( m = 1 \), \( D^{(1)} \in \mathbb{R}^d \) evolves according to the ODE \( \dot{x} = -Cx \) (see (4.13)). Then, it is sufficient to choose an initial datum \( f_0 \in V^{(1)} \) to achieve the equality, concluding the proof. \( \square \)

7. Second Quantization

In this last section we are going to write the FP operator \( L \) in (2.4) in terms of the second quantization formalism. This “language” was introduced in quantum mechanics in order to simplify the description and the analysis of quantum many-body systems. The assumption of this construction is the indistinguishability of particles in quantum mechanics. Indeed, according to the statistics of particles, the exchange of two of them does not affect the status of the configuration, possibly up to a sign. Since we are dealing with symmetric tensors, we are going to consider the case in which the sign does not change, i.e. the wave function is identical after this exchange. This is the case of particles that are called bosons.

The functional spaces of second quantization are the so-called Fock spaces, that we are going to define in this section. When a single Hilbert space \( H \) describes a single particle, then it is convenient to build an infinite sum of symmetric tensorization of \( H \) in order to represent a system of (up to) infinitely many indistinguishable particles, i.e. the Fock space over \( H \).

In the first part of this section the definitions of the Boson Fock space and second quantization operators are given. These constructions will be needed in order to write the FP operator \( L \) as the second quantization of its corresponding drift matrix \( C \). This will be the main result of the second part of this section as an application of well known results in the literature.

7.1. The Boson Fock space. In the next definition we will use the notion of \( m \)-fold tensor product over a Hilbert space \( H \). This is a generalization of the space of order-\( m \) hypermatrices \( T^{(m)} \) defined in §5, where the Hilbert space was the finite dimensional space \( \mathbb{R}^d \). In the quantum mechanics literature, the role of the Hilbert space is often played by \( L^2(\mathbb{R}^3; \mathbb{C}) \),
in order to describe the wave function of a quantum particle. For a more complete explanation of tensor products of Hilbert spaces and Fock spaces we refer to §II.4 in [26].

In the literature, Fock spaces are mostly considered for Hilbert spaces over the field \( \mathbb{C} \). But since the FP equations (1.1) and (2.4) are posed on \( \mathbb{R}^d \) (and not over \( \mathbb{C}^d \)), we shall use here only real valued Fock spaces. Moreover, these FP equations are considered here only for real valued initial data, and hence real valued solutions.

**Definition 7.1.** Let \( H \) be a Hilbert space and denote by \( H^{(m)} := H \otimes H \otimes \cdots \otimes H \) \( (m \text{ times}) \), for any \( m \in \mathbb{N} \). Set \( H^{(0)} := \mathbb{C} \) (or \( \mathbb{R} \)) and define the Fock space over \( H \) as the completed direct sum

\[
\mathcal{F}(H) = \bigoplus_{m=0}^{\infty} H^{(m)}.
\]

Then, an element \( \psi \in \mathcal{F}(H) \) can be represented as a sequence \( \psi = \{ \psi^{(m)} \}_{m=0}^{\infty} \), where \( \psi^{(0)} \in \mathbb{C} \) (or \( \mathbb{R} \)), \( \psi^{(m)} \in H^{(m)} \), \forall m \in \mathbb{N} \), so that

\[
\| \psi \|_{\mathcal{F}(H)} := \sqrt{\sum_{m=0}^{\infty} \| \psi^{(m)} \|_{H^{(m)}}^2} < \infty.
\]

Here \( \| \cdot \|_{H^{(m)}} \) denotes the norm induced by the inner product in \( H^{(m)} \) (see Proposition 1, §II.4 in [26]).

As we anticipated, we will rather work with a subspace of \( \mathcal{F}(H) \), the so-called Boson Fock space that we are going to define. First we need to define the \( m \)-fold symmetric tensor product of \( H \) as follows:

Let \( P_m \) be the permutation group on \( m \) elements and let \( \{ \phi_k \}; \ k = 1, \ldots, \dim H \), be a basis for \( H \). For each \( \sigma \in P_m \), we define its corresponding operator (we will still denote it with \( \sigma \)) acting on basis elements of \( H^{(m)} \) by

\[
(7.3) \quad \sigma(\phi_{k_1} \otimes \phi_{k_2} \otimes \cdots \otimes \phi_{k_m}) := \phi_{\sigma(k_1)} \otimes \phi_{\sigma(k_2)} \otimes \cdots \otimes \phi_{\sigma(k_m)}.
\]

Then \( \sigma \) extends by linearity to a bounded operator on \( H^{(m)} \). With the previous definition (7.3) we can define the operator \( S_m := \frac{1}{m!} \sum_{\sigma \in P_m} \sigma \) that acts on \( H^{(m)} \). Its range \( S_m H^{(m)} \) is called the \( m \)-fold symmetric tensor product of \( H \). Let us see examples of \( S_m H^{(m)} \).

**Example 7.2.** Let us consider first the case \( H = L^2(\mathbb{R}) \) and \( H^{(m)} = L^2(\mathbb{R}) \otimes \cdots \otimes L^2(\mathbb{R}) \). Since \( H^{(m)} \) is isomorphic to \( L^2(\mathbb{R}^m) \), it follows that an element \( \psi^{(m)} \in S_m H^{(m)} \) is a function \( \psi^{(m)}(x_1, \ldots, x_m) \) in \( L^2(\mathbb{R}^m) \) left invariant under any permutation of the variables. It is used in quantum mechanics to describe the quantum states of \( m \) particles that are not distinguishable.

For our purposes, we will deal with \( H = \mathbb{R}^d \). In this case it is easy to check that \( S_m H^{(m)} \) corresponds to the space of symmetric \( m \)-tensors \( F^{(m)} \) that we defined in §5, equipped with the Frobenius norm. \( \square \)
Definition 7.3. The subspace of $\mathcal{F}(H)$,

\begin{equation}
\mathcal{F}_s(H) := \bigoplus_{m=0}^{\infty} S_m H^{(m)}
\end{equation}

is called the symmetric Fock space over $H$ or the Boson Fock space over $H$.

7.2. The second quantization operator. In order to write the Fokker-Planck solution operator in terms of the second quantization formalism, we need to define the second quantization operators (see §I.4 in [29] and §X.7 in [27]) acting on the Boson Fock space.

Let $H$ be a Hilbert space and $\mathcal{F}_s(H)$ be the Boson Fock space over $H$.

Let $A$ be a contraction on $H$, i.e., a linear transform of norm smaller than or equal to 1. Then there is a unique contraction (Corollary I.15, [29]) $\Gamma(A)$ on $\mathcal{F}_s(H)$ so that

\begin{equation}
\Gamma(A) \upharpoonright S_m H^{(m)} = A \otimes \cdots \otimes A \quad (m \text{ times}),
\end{equation}

where the operator $A \otimes \cdots \otimes A$ is defined on each basis element $\psi^{(m)} = \psi_{i_1} \otimes \cdots \otimes \psi_{i_m}$ of $S_m H^{(m)}$ as

\[(A \otimes \cdots \otimes A)(\psi^{(m)}) := (A\psi_{i_1}) \otimes \cdots \otimes (A\psi_{i_m}),\]

and equal to the identity when restricted to $H^{(0)}$. In order to prove the above existence of $\Gamma(A)$, the estimate $\|\Gamma(A) \upharpoonright S_m H^{(m)}\| \leq \|A\|^m$ is first showed in [29]. This allows to extend the operator $\Gamma(A)$ to the Boson Fock space by continuity, and by remaining a contraction. In the case $A = e^{-Ct}$ and $H = \mathbb{R}^d$, the operator $\Gamma(A)$ will be useful to show the link between the Fokker-Planck solution operator $e^{-Lt}$ and the second quantization operators, defined in the following way:

Definition 7.4. Let $H$ be a Hilbert space. Let $A$ be an operator on $H$ (with domain $G(A)$). The operator $d\Gamma(A)$ is defined as follows: Let $G_m(A) \subseteq S_m H^{(m)}$ be $G(A) \otimes \cdots \otimes G(A)$ and $G(d\Gamma(A)) := \bigoplus_{m=0}^{\infty} G_m(A)$ (incomplete direct sum):

\begin{equation}
d\Gamma(A) \upharpoonright S_m H^{(m)} := A \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes A, \quad m \in \mathbb{N},
\end{equation}

and $d\Gamma(A) \upharpoonright H^{(0)} := 0$. The operator $d\Gamma(A)$ is called the second quantization of $A$.

In [29] the following property of the second quantization operator can be found (see I.41):

Let $A$ generate a $C_0$-contraction semigroup on $H$. Then the closure of $d\Gamma(A)$ generates a $C_0$-contraction semigroup on $\mathcal{F}_s(H)$ and

\begin{equation}
e^{-d\Gamma(A)t} = \Gamma(e^{-At}) \quad \forall t \geq 0.
\end{equation}
7.3. **Application to the operator** $e^{-Lt}$. In the last part of this section we will show that the Fokker-Planck operator $L$ is the second quantization of $C$. First, we shall identify the Hilbert space $L^2(\mathbb{R}^d, f_{\infty}^{-1})$ with a suitable Fock space.

The spectral decomposition and the tensor structure that we introduced in §5 suggest to consider the Boson Fock space over the finite dimensional Hilbert space $\mathbb{R}^d$, whose elements have components in the space of symmetric tensors $F^{(m)}$. Indeed, we can define an isomorphism $\Psi$ between $L^2(\mathbb{R}^d, f_{\infty}^{-1})$ and $\mathcal{F}_s(\mathbb{R}^d)$ as follows:

Let $f \in L^2(\mathbb{R}^d, f_{\infty}^{-1})$. As we saw in §4, $f$ admits the decomposition $f(x) = \sum_{m \in \mathbb{N}_0} \sum_{s \in S^m} d_s \gamma^s(x)$, for some coefficients $d_s \in \mathbb{R}$. For each $m \geq 1$, we define the symmetric tensor $\tilde{D}^{(m)} \in F^{(m)}$ with components $\tilde{D}^{(m)} := d_s \gamma^s \in \mathbb{R}$ (see (5.8)), $\forall s \in S^m$. For $m = 0$ we choose $\tilde{D}^{(0)} := \langle f, f_{\infty} \rangle |_{L^2(f_{\infty}^{-1})}$.

Hence, by observing that $F^{(m)} = S_m H^{(m)}$, $H := \mathbb{R}^d$, we define the isometry

\[ \Psi : f \in L^2(\mathbb{R}^d, f_{\infty}^{-1}) \rightarrow \psi := \tilde{D}^{(m)} |_{m=0} \in \mathcal{F}_s(\mathbb{R}^d). \]

It remains to check that $\|\psi\|_{\mathcal{F}_s(\mathbb{R}^d)} < \infty$. This follows from the Plancherel’s equality together with (5.12). It leads to

\[ \|f\|_{L^2(f_{\infty}^{-1})}^2 = \sum_{m=0}^{\infty} \|\tilde{D}^{(m)}\|_{\mathcal{F}_s}^2 = \|\psi\|_{\mathcal{F}_s(\mathbb{R}^d)}^2. \]

Hence, up to an isomorphism, we can consider the FP operator $L$ also as acting on the Fock space $\mathcal{F}_s(\mathbb{R}^d)$. We conclude the section with the next proposition that allows to write $L$ in the second quantization formalism.

**Proposition 7.5.** Let $L$ be the Fokker-Planck operator defined in (2.4) and let $C \in \mathbb{R}^{d \times d}$ be its corresponding drift matrix. Then, $L$, now considered as acting on $\mathcal{F}_s(\mathbb{R}^d)$, is the second quantization of $C$, considered as an operator from the Hilbert space $\mathbb{R}^d$ to itself, i.e., $L = d\Gamma(C)$.

**Proof.** Due to the relation (7.7), it is sufficient to prove that the FP solution operator $e^{-Lt}$ (considered on $\mathcal{F}_s(\mathbb{R}^d)$) satisfies the equality

\[ e^{-Lt} = \Gamma(e^{-Ct}), \quad \forall t \geq 0, \]

or, equivalently, on each $S_m H^{(m)}$, $m \geq 1$,

\[ e^{-Lt} (\psi^{(m)}) = (e^{-Ct} \psi_{i_1}) \otimes \cdots \otimes (e^{-Ct} \psi_{i_m}), \]

for any $\psi^{(m)} = \otimes_{k=1}^m \psi_{i_k}$, basis element of $F^{(m)}$.

Given an initial condition $f_0 \in L^2(\mathbb{R}^d, f_{\infty}^{-1})$ and its corresponding solution $f(t) = e^{-Lt} f_0$ of (2.4), the isometry $\Psi$ maps then to

$\Psi f_0 = \psi_0 = (\tilde{D}^{(m)}(0))_{m=0}^{\infty} \in \mathcal{F}_s(H)$ and $\Psi f(t) = \psi(t) = (\tilde{D}^{(m)}(t))_{m=0}^{\infty} \in \mathcal{F}_s(H)$, respectively. Then, the factored evolution formula (6.4) for $D^{(m)}(t) = \sqrt{m!} \tilde{D}^{(m)}(t)$ proves the equality (7.10), for each $m \geq 1$. Since the generator of a $C_0$-semigroup is unique, we obtain $L = d\Gamma(C)$. \qed
While $C$ is a bounded operator with domain $G(C) = \mathbb{R}^d$, its second quantization $d\Gamma(C)$ is unbounded with dense domain $G(d\Gamma(C)) \subseteq \mathcal{F}_s(H)$, just like $L$ is unbounded on $L^2(\mathbb{R}^d, f_\infty^{-1})$.

Finally, our main result, Theorem 3.4 reads in the language of second quantization

\begin{equation}
(7.11) \quad \|e^{-d\Gamma(C)t} \upharpoonright \bigoplus_{m \in \mathbb{N}} S_m H^{(m)} \|_{\mathcal{B}(\mathcal{F}_s(H))} = \|e^{-Ct}\|_{\mathbb{L}^{d\times d}}, \quad t \geq 0.
\end{equation}

Note that the restriction to $\bigoplus_{m \in \mathbb{N}} S_m H^{(m)}$ corresponds to the restriction to $V_0^\perp$ in (3.1), the orthogonal of the steady state $f_\infty$.

**Remark 7.6.** Many aspects of the above analysis seem to rely importantly on the explicit spectral decomposition of the FP operator in §4.1, i.e. knowing the FP eigenfunctions (as Hermite functions). We remark that this situation in fact carries over to FP equations with linear coefficients plus a nonlocal perturbation of the form $\theta_f := \theta * f$ with the function $\theta(x)$ having zero mean, see Lemma 3.8 and Theorem 4.6 in [7]. For such nonlocally perturbed FP equations, surprisingly, one still knows all the eigenfunctions as well as its (multi-dimensional) creation and annihilation operators.

**APPENDIX A. DEFERRRED PROOFS**

**Proof of Lemma 5.11** We compute the components of the l.h.s. of (5.3). Using (5.2) with $v_k = (v_i^{(k)}) \in \mathbb{R}^d$, we have for any $(i_1, \ldots, i_m) \in \langle d \rangle^m$:

\[
(B \otimes^m D)_{i_1 \ldots i_m} = \sum_{j_1, \ldots, j_m=1}^d B_{i_1 j_1} \cdots B_{i_m j_m} D_{j_1 \ldots j_m} = \sum_{j_1, \ldots, j_m=1}^d B_{i_1 j_1} \cdots B_{i_m j_m} \sum_{k=1}^s \lambda_k v_j^{(k)} \cdots v_{j_m}^{(k)} = \sum_{k=1}^s \lambda_k (Bv_k)_{i_1} \cdots (Bv_k)_{i_m} = \left(\sum_{k=1}^s \lambda_k (Bv_k) \otimes^m\right)_{i_1 \cdots i_m},
\]

concluding the proof.

**Proof of Lemma 5.12** By definition,

\[
\langle v_1 \otimes \cdots \otimes v_m, v_{m+1} \otimes \cdots \otimes v_{2m} \rangle_{\mathcal{F}} = \sum_{i_1, \ldots, i_m=1}^d \langle v_1 \otimes \cdots \otimes v_m \rangle_{i_1 \ldots i_m} \langle v_{m+1} \otimes \cdots \otimes v_{2m} \rangle_{i_1 \ldots i_m}
\]

\[
= \sum_{i_1, \ldots, i_m=1}^d v_{i_1}^{(1)} \cdots v_{i_m}^{(m)} v_{i_1}^{(m+1)} \cdots v_{i_m}^{(2m)} = \left(\sum_{i_1=1}^d v_{i_1}^{(1)} v_{i_1}^{(m+1)} \right) \cdots \left(\sum_{i_m=1}^d v_{i_m}^{(m)} v_{i_m}^{(2m)} \right) = \langle v_1, v_{m+1} \rangle \cdots \langle v_m, v_{2m} \rangle.
\]
Proof of Lemma 5.14. We have

\[\langle A, B \otimes A \rangle_{d} = \sum_{i_{1}, \ldots, i_{m}=1}^{d} A_{i_{1}, \ldots, i_{m}} (B \otimes A)_{i_{1}, \ldots, i_{m}} = \sum_{j_{1}, \ldots, j_{m}=1}^{d} A_{i_{1}, \ldots, i_{m}} B_{j_{1}, j_{2}, \ldots, j_{m}} A_{j_{1}, j_{2}, \ldots, j_{m}}\]

\[= \sum_{i_{2}, \ldots, i_{m}=1}^{d} \langle x^{(i_{2}, \ldots, i_{m})}, B x^{(i_{2}, \ldots, i_{m})} \rangle,\]

where, for \(i_{2}, \ldots, i_{m}\) fixed, \(x_{i_{1}}^{(i_{2}, \ldots, i_{m})} := A_{i_{1}, i_{2}, \ldots, i_{m}}\) are vectors in \(\mathbb{R}^{d}\). The claim then follows from \(B \geq 0\). 

Proof of Lemma 5.13. First consider the Case \(k = 1\). We have

\[(A.1) \quad \| B \otimes A \|_{d}^{2} = \sum_{i_{1}, \ldots, i_{m}=1}^{d} (\sum_{j_{1}=1}^{d} B_{i_{1}, j_{1}} A_{j_{1}, i_{2}, \ldots, i_{m}})^{2} = \sum_{i_{2}, \ldots, i_{m}=1}^{d} \| B \|_{x^{(i_{2}, \ldots, i_{m})}}^{2} = \sum_{i_{1}, \ldots, i_{m}=1}^{d} \| B \|_{x^{(i_{1})}} \| A \|_{x^{(i_{1}, i_{2}, \ldots, i_{m})}}\]

\[(A.2) \quad \leq \sum_{i_{2}, \ldots, i_{m}=1}^{d} \| B \|^{2} \| x^{(i_{2}, \ldots, i_{m})} \|^{2} = \| B \|^{2} \sum_{i_{1}, \ldots, i_{m}=1}^{d} \| x^{(i_{1})} \|^{2} \| A \|^{2}\]

where, for \(i_{2}, \ldots, i_{m}\) fixed, \(x_{j_{1}}^{(i_{2}, \ldots, i_{m})} := A_{j_{1}, i_{2}, \ldots, i_{m}}\) are vectors in \(\mathbb{R}^{d}\). Note that the estimate \((A.1)\) would hold as well if the matrix-tensor product does not operate on the first index (as in \(B \otimes A\), but on the \(j\)-th index, with some \(1 \leq j \leq m\). Then \((5.7)\) follows by iterated applications of \((A.1)\). 

Proof of Proposition 3.19. (a) We recall that Theorem 3.4 and (6.1) imply

\[\hat{h}(t) = \| e^{-L t} \|_{(\mathcal{H}_{d} \cap V_{0}^{1})} = \| e^{-C t} \|_{2} = h(t) \leq 1, \quad t \geq 0.\]

Then, Theorem 6.1 implies \((6.2), \forall m \geq 1\). From (4.9) we recall

\[(A.4) \quad \left\| \frac{f(t)}{f_{\infty}} \right\|_{L^{2}(f_{\infty})}^{2} = \sum_{m \in \mathbb{N}_{0}} \left\| d^{(m)}(t) \right\|_{2}^{2} = \sum_{\beta \in \mathbb{N}_{0}^{d}} | d_{\beta}(t) |^{2},\]

and \(\frac{f(t)}{f_{\infty}} = \sum_{\beta \in \mathbb{N}_{0}^{d}} \bar{d}_{\beta}(t) \hat{g}_{\beta}\), where \(\hat{g}_{\beta} := \frac{\tilde{g}_{\beta}}{f_{\infty}}\) is an orthonormal basis of \(L^{2}(f_{\infty})\).

Using (4.2) and the formula \(H_{n}(x) = n H_{n-1}(x)\) for Hermite polynomials we compute, for any \(\beta \in \mathbb{N}_{0}^{d}\),

\[\partial_{x_{j}} \hat{g}_{\beta} = \frac{\beta_{j} H_{\beta_{j}-1}(x_{j})}{\sqrt{\beta_{j}!}} \prod_{i \neq j} H_{\beta_{i}}(x_{i}), \quad \text{and} \quad \| \partial_{x_{j}} \hat{g}_{\beta} \|_{L^{2}(f_{\infty})} = \sqrt{\beta_{j}},\]

where we used \(\| H_{n} \|_{L^{2}(f_{\infty})} = \sqrt{n!}\). This yields, with \((6.2)\) and \((5.12)\),

\[(A.5) \quad \left\| \nabla \left( \frac{f(t)}{f_{\infty}} \right) \right\|_{L^{2}(f_{\infty})}^{2} = \sum_{\beta \in \mathbb{N}_{0}^{d}} | d_{\beta}(t) |^{2} | \beta | = \sum_{m \in \mathbb{N}_{0}} m \| d^{(m)}(t) \|^{2} \leq \sum_{m \in \mathbb{N}_{0}} m (\tilde{h}(t))^{2m} \| d^{(m)}(0) \|^{2}, \quad t > 0.\]
From the hypothesis on $\hat{h}$, we deduce $\hat{h}(t) \leq 1 - c_1 t^\alpha$ on $0 \leq t \leq \delta$ for some $0 < c_1 \leq c$ and some $\delta > 0$. Then (A.5) can be estimated further by
\[
\sum_{m \in \mathbb{N}_0} m (1 - c_1 t^\alpha)^{2m} \| \tilde{d}^{(m)}(0) \|^2 \leq \frac{1}{e c_1} t^{-\alpha} \sum_{m \in \mathbb{N}_0} \| \tilde{d}^{(m)}(0) \|^2, \quad 0 \leq c_1 t^\alpha \leq 1.
\]
where we used the elementary inequality $m (1 - c_1 t^\alpha)^{2m} \leq \frac{1}{e c_1} t^{-\alpha}$, $m \in \mathbb{N}_0$. The main assertion of part (a) then follows from (A.4).

Finally we turn to the optimality of $\alpha$: If (3.20) would hold for all $f_0 \in \mathcal{H}$ with some $\alpha_1 \in (0, \alpha)$, then part (b) of this proposition would imply $\hat{h}(t) \leq 1 - c_2 t^\alpha$. But this would contradict the assumption $\hat{h}(t) = 1 - c t^\alpha + o(t^\alpha)$. Hence, $\alpha/2$ is indeed the minimal regularization exponent in (3.20).

(b) For $f_0 \in V^{(m)}$, $m \in \mathbb{N}$ we compute, by using (A.5) and (3.20),
\[
(A.6) \quad \left\| \nabla \left( \frac{f(t)}{f_\infty} \right) \right\|^2_{L^2(f_\infty)} = m \| \tilde{d}^{(m)}(t) \|^2 \leq \bar{c}^2 t^{-\alpha} \| \tilde{d}^{(m)}(0) \|^2, \quad 0 < t \leq \delta.
\]
Then, by taking in (A.6) the supremum w.r.t. the set $\{0 \neq \tilde{d}^{(m)}(0) \in \mathbb{R}^m\}$ and using (6.3), (5.12) we obtain the family of estimates
\[
(A.7) \quad \tilde{h}(\nu)^{2m} = \sup_{0 \neq \tilde{d}^{(m)} \in F^{(m)}} \| D^{(m)}(t) \|^2_\nu = \sup_{0 \neq \tilde{d}^{(m)} \in \mathbb{R}^m} \| \tilde{d}^{(m)}(t) \|^2 \leq \frac{\bar{c}^2}{m} t^{-\alpha},
\]
with $m \in \mathbb{N}$, $0 < t \leq \delta$.

Next we will show that this family of estimates for $\tilde{h}(\nu)$ implies $\hat{h}(t) \leq 1 - c_2 t^\alpha$ for $0 \leq t \leq \delta_2$, with some $c_2 > 0$, $\delta_2 > 0$ (see Figure 2 for the case $\alpha = 1$). For each $m \in \mathbb{N}$ and $t \in I_\delta := (0, \delta)$, we rewrite (A.7) as
\[
(A.8) \quad \tilde{h}(t) \leq \left( \frac{\bar{c}}{\sqrt{m}} t^{-\frac{\alpha}{2}} \right)^{\frac{1}{m}} = e^{-\frac{1}{2} \log(\bar{c} t^m)} =: g(m; t),
\]
with $\bar{c} := \hat{c}^{-2}$. For $t \in I_\delta$ fixed, we now consider the function $g(\mu; t)$ with continuous argument $\mu > 0$. $g(\cdot; t)$ has its unique minimum at $\mu_0(t) := \frac{e}{\bar{c}} t^{-\alpha}$ and it is strictly decreasing on $(0, \mu_0(t))$.

To estimate the minimum of $g$ for the discrete argument $m \in \mathbb{N}$, we consider: For $0 \leq t \leq t_1 := (\frac{e}{\bar{c}})^{1/\alpha}$ we have
\[
\frac{2}{\bar{c}} t^{-\alpha} \leq \left[ \frac{2}{\bar{c}} t^{-\alpha} \right] < \frac{2}{\bar{c}} t^{-\alpha} + 1 \leq \frac{e}{\bar{c}} t^{-\alpha} = \mu_0(t),
\]
with $[\cdot]$ denoting the ceiling function. We choose the index $m(t) := \left\lfloor \frac{2}{\bar{c}} t^{-\alpha} \right\rfloor \in \mathbb{N}$ and use the monotonicity of $g(\cdot; t)$ on $(0, \mu_0(t))$ to estimate:
\[
\tilde{h}(t) \leq \min_{m \in \mathbb{N}} g(m; t) \leq g(m(t); t) \leq g\left( \frac{2}{\bar{c}} t^{-\alpha}; t \right) = e^{-2c_2 t^\alpha},
\]
with $c_2 := \frac{\log(2) \tilde{c}}{8} > 0$.

With the elementary estimate $e^{-2c_2 t^\alpha} \leq 1 - c_2 t^\alpha$ on some $[0, \delta_2]$, we obtain
\[
\tilde{h}(t) \leq e^{-2c_2 t^\alpha} \leq 1 - c_2 t^\alpha, \quad t \in [0, \delta_2],
\]
with $\delta_2 := \min\{t_1, t_2^{1/\alpha}\}$.

Finally we turn to the minimality of $\alpha$: If $\tilde{h}$ would even satisfy the decay estimate $\tilde{h}(t) \leq 1 - \tilde{c}_2 t^{\alpha_1}$ with some $\alpha_1 \in (0, \alpha)$ and $\tilde{c}_2 > 0$, then (the proof of) part (a) of this proposition would imply the regularization estimate (3.20) with the exponent $\alpha_1/2$. But this would contradict the assumption on $\alpha$ being minimal in that estimate. □

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The family of decay estimates $h(t) \leq g(m; t)$, $m \in \mathbb{N}$ with $\alpha = 1$, $\tilde{c} = 4$ (solid, blue curves) implies $h(t) \leq e^{-2c_2 t}$, (dashed, green curve), and hence $h(t) \leq 1 - c_2 t$ (dotted, red line).}
\end{figure}

**Acknowledgement**

The authors were partially supported by the FWF (Austrian Science Fund) funded SFB #F65 and the FWF-doctoral school W 1245. The first author acknowledges fruitful discussions with Miguel Rodrigues that led to Proposition 3.19(b).

**References**

[1] F. Achleitner, A. Arnold, E. Carlen, *On multi-dimensional hypocoercive BGK models*, Kinetic and related models 11, no. 4 (2018), 953-1009.

[2] F. Achleitner, A. Arnold, E. Carlen, *The Hypocoercivity Index for the short and large time behavior of ODEs*, preprint (2020).

[3] F. Achleitner, A. Arnold, B. Signorello, *On optimal decay estimates for ODEs and PDEs with modal decomposition*, Stochastic Dynamics out of Equilibrium, Springer Proceedings in Mathematics and Statistics 282 (2019), 241-264, G. Giacomini et al. (eds.), Springer.
[4] A. Arnold, A. Einav, T. Wöhrer, On the rates of decay to equilibrium in degenerate and defective Fokker-Planck equations, J. Differential Equations 264, no. 11 (2018), 6843-6872.

[5] A. Arnold, J. Erb, Sharp entropy decay for hypocoercive and non-symmetric Fokker-Planck equations with linear drift. Preprint. https://arxiv.org/abs/1409.5425.

[6] A. Arnold, P. A. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. Comm. PDE 26, no. 1-2 (2001), 43-100.

[7] A. Arnold, D. Stürzer, Spectral analysis and long-time behaviour of a Fokker-Planck equation with a non-local perturbation, Rend. Lincei Mat. Appl. 25 (2014), 53-89. Erratum: Rend. Lincei Mat. Appl. 27 (2016), 147-149.

[8] A. Arnold, S. Jin, T. Wöhrer, Sharp Decay Estimates in Local Sensitivity Analysis for Evolution Equations with Uncertainties: from ODEs to Linear Kinetic Equations, Journal of Differential Equations 268, no. 3 (2019), 1156-1204.

[9] D. Bakry, M. Émery, Diffusions hypercontractives, Séminaire de probabilités (Strasbourg) 19 (1985), 177–206.

[10] P. Comon, G. Golub, L.-H. Lim, B. Mourrain, Symmetric tensors and symmetric tensor rank, SIAM J. Matrix Anal. Appl. 30, no. 3 (2008), 1254-1279.

[11] P. Comon, B. Mourrain, private communication, 26.9.2019.

[12] J. Dolbeault, X. Li, Phi-entropies for Fokker-Planck and kinetic Fokker-Planck equations, Math. Mod. and Meth. in Appl. Sci. 28 (2018), 2637-2666.

[13] S. Friedland, M. Stawiska, Best Approximation on Semi-algebraic Sets and k-Border Rank Approximation of Symmetric Tensors, preprint, arXiv:1311.1561 [math.AG], (2013).

[14] S. Gadat, L. Miclo, Spectral decompositions and L2-operator norms of toy hypocoercive semi-groups, Kinetic and Related Models 2 (2013), 317-372.

[15] M. Herda, L.M. Rodrigues, Large-time behavior of solutions to Vlasov-Poisson-Fokker-Planck equations: from evanescent collisions to diffusive limit, J. Stat. Phys. 170 (2018), 895-931.

[16] L. Hörmander, Hypoelliptic second order differential equations, Acta Math. 119 (1967), 147-171.

[17] S. Kawashima, Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications, Proc. Roy. Soc. Edinburgh Sect. A 106 (1987), 169-194.

[18] T. Lelièvre, F. Nier, G.A. Pavliotis, Optimal Non-reversible Linear Drift for the Convergence to Equilibrium of a Diffusion, Springer Science+Business Media New York (2013).

[19] L.-H. Lim, Tensors and hypermatrices, Chapter 15, 30 pp., in L. Hogben (Ed.), Handbook of Linear Algebra, 2nd Ed., CRC Press, Boca Raton, FL (2013).

[20] G. Metafune, D. Pallara, E. Priola, Spectrum of Ornstein-Uhlenbeck operators in $L^p$ spaces with respect to invariant measures, J. Funct. Anal. 196, no. 1 (2002), 40-60.

[21] L. Miclo, P. Monmarché, Étude spectrale minuitieuse de processus moins indécis que les autres. (French) [Detailed spectral study of processes that are less indecisive than others] Séminaire de Probabilités XIV, Lecture Notes in Math. 2078, Springer, Cham (2013), 459-481.

[22] P. Monmarché, Generalized $\Gamma$ calculus and application to interacting particles on a graph, Preprint. https://arxiv.org/abs/1510.05936.

[23] M. Ottobre, G.A. Pavliotis, K. Pravda-Starov, Some remarks on degenerate hypoelliptic Ornstein-Uhlenbeck operators, J. Math. Anal. Appl. 429 (2015), 676-712.

[24] L. Pareschi, G. Russo, G. Toscani, Fast spectral methods for the Fokker-Planck-Landau collision operator, J. Comput. Phys. 165, no.1 (2000), 216-236.
[25] L. Perko, *Differential Equations and Dynamical Systems*, Texts in Applied Mathematics 7, Springer Verlag (1991).
[26] M. Reed, B. Simon, *Methods of modern mathematical physics, Vol. 1*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London (1980).
[27] M. Reed, B. Simon, *Methods of modern mathematical physics, Vol. 2*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London (1975).
[28] C. Schmeiser, *Entropy methods*, https://homepage.univie.ac.at/christian.schmeiser/Entropy-course.pdf.
[29] B. Simon, *The $P(\Phi)^2$ Euclidean (Quantum) Field Theory*, Princeton University Press, Princeton (1974).
[30] B. Shizgal, *Spectral methods in chemistry and physics. Applications to kinetic theory and quantum mechanics. Scientific Computation*, Springer, Dordrecht (2015), xviii+415 pp.
[31] C. Villani, *Hypocoercivity*, Memoirs of the American Mathematical Society 202 (2009).

Institute for Analysis and Scientific Computing, TU Vienna, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria
Email address: anton.arnold@tuwien.ac.at

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
Email address: Christian.Schmeiser@univie.ac.at

Institute for Analysis and Scientific Computing, TU Vienna, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria
Email address: bsignore@tuwien.ac.at