We give analytic expression for the three-point function of three large classical non-BPS operators $\mathcal{N} = 4$ Super-Yang-Mills theory at weak coupling. We restrict ourselves to operators belonging to an $su(2)$ sector of the theory. In order to carry out the calculation we derive, by unveiling a hidden factorization property, the thermodynamical limit of Slavnov’s determinant.

I. INTRODUCTION

In the last ten years, starting with the pioneer paper by Minahan and Zarembo [1], a vast integrable structure has been unveiled in the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory [2]. There are hopes that together with the spectrum of states, the integrability can be used to compute the correlation functions of the theory. Of special interest are the correlation functions of one-trace operators in the classical limit when the length of the traces is very large. Such operators are dual to extended classical strings in the AdS$_5 \times$S$^5$ background, and knowing their correlation functions can shed light about the interactions at strong coupling.

Recently, Escobedo, Gromov, Sever and Vieira [3, 4] developed Bethe-Ansatz techniques for computing the tree-level structure coefficient $C_{123}$ and found an expression for the latter in terms of scalar products of Bethe states for the XXX$_{1/2}$ spin chain. In [4] an elegant analytic formula was derived for the classical limit of the structure coefficient of when one of the operators is protected (BPS). In this note we generalize the result of [4] to the case of three non-BPS classical operators. Our starting point will be the representation of the structure constant in terms of Slavnov-like determinants [5], proposed recently by Foda [6].

II. 3-POINT FUNCTIONS OF TRACE OPERATORS IN $\mathcal{N} = 4$ SYM

In a $su(2)$ sector of the SYM theory, the operators are made of two complex scalars $Z$ and $X$. We consider the correlation function of three single-trace operators of the type $\mathcal{O}_1 \sim \text{Tr}[Z_{L_1} X_{N_1} + \ldots]$, $\mathcal{O}_2 \sim \text{Tr}[Z_{L_2} X_{N_2} + \ldots]$, $\mathcal{O}_3 \sim \text{Tr}[Z_{L_3} X_{N_3} + \ldots]$, where the omitted terms are weighted products of the same constituents taken in different order. The weights are chosen so that the operator $\mathcal{O}_a$ is an eigenstate of the dilatation operator with dimensions $\Delta_a$. At tree level, the structure coefficient is a sum over all possible ways to perform the Wick contractions between the scalars and their conjugates. A non-zero result is obtained only if $N_1 = N_2 + N_3$ and the number of contractions $L_{ij}$ between operators $\mathcal{O}_i$ and $\mathcal{O}_j$ are $L_{12} = L_1 - N_3$, $L_{13} = N_3$, $L_{23} = L_3 - N_3$.

This problem is solved using the Algebraic Bethe Ansatz [3]. In the Bethe-Ansatz approach, the operator $\mathcal{O}_i$ is represented by a $N_i$-magnon Bethe eigenstate with energy $\Delta_i$ of the XXX$_{1/2}$ spin chain of length $L_i$ ($i = 1, 2, 3$). To simplify the presentation we consider only highest-weight states, but our method is valid in general. Such a state is completely characterized by the rapidities of the magnons $\mathbf{u} = \{u_a\}_{a=1}^N$ and will be denoted by $|\mathbf{n}\rangle$.

It is advantageous first to deform the problem by introducing impurities $\theta^{(n)} = \{\theta^{(n)}_j\}_{j=1}^{L_n}$ at the sites of the $n$-th spin chain ($n = 1, 2, 3$), and take the homogeneous limit $\theta^{(n)} \to 0$ at the very end. We denote the impurities associated with the contractions between the operators $\mathcal{O}_a$ and $\mathcal{O}_b$ by $\theta^{(mn)}$ so that $\theta^{(1)} = \theta^{(12)} \cup \theta^{(13)}$, etc. Then the tree level structure coefficient is given, up to a normalization and a phase factor, by [3]

$$C_{123}^0 = \frac{\langle \mathbf{u}|\mathbf{v}|z\rangle_{L_1} \langle \mathbf{w} \rangle_{L_2}}{\langle \mathbf{u}|\mathbf{v}|z\rangle_{L_1} \langle \mathbf{w} \rangle_{L_2}^2},$$

where $z = \theta^{(13)} + i/2$ and the r.h.s. should be evaluated in the homogeneous limit $z \to \pm i/2$. Here the symbol $\langle \mathbf{u}|\mathbf{v}\rangle_{L}$ stands for the scalar product of two Bethe states with rapidities $\mathbf{u} = \{u_a\}_{a=1}^N$ and $\mathbf{v} = \{v_a\}_{a=1}^N$ in a spin chain of length $L$. In the limit when all rapidities go to infinity, $C_{123}^0 \to C_{123}^{\text{BPS}}$.

We are interested in the classical limit $L_i \to \infty$, with $\alpha_i = N_i/L_i$ finite. As shown in [6], the regularization provided by the impurities allows to express the structure constant in terms of a ratio of determinants. In order to obtain the classical limit of $C_{123}^0$, we will first obtain the classical, or thermodynamical, limit of Slavnov’s determinant. In our approach it is essential to evaluate the classical limit before the homogeneous limit $z \to i/2$.

III. SLAVNOV’S DETERMINANT

1. Slavnov’s formula for the scalar product.

Assume that the length-$L$ N-magnon state with rapidities $\mathbf{u} = \{u_a\}_{a=1}^N$ a Bethe eigenstate. Then the rapidities $\mathbf{u}$ satisfy the Bethe equations, which depend on a set of impurities $\theta = \{\theta_j\}_{j=1}^{L_n}$, $n = 1, 2, 3$. The Bethe equations are equivalent to the conditions

$$e^{2ip_a(z)} = -1 \quad \text{for} \quad z \in \mathbf{u},$$

where the quasi-momentum $p_a$ is defined as

$$e^{2ip_a(z)} \equiv \kappa \left( \frac{Q_a(z + i\frac{1}{2})}{Q_a(z - i\frac{1}{2})} \right).$$

with $Q_a(z)$ the quantum number of the operator $\mathcal{O}_a$.

...
Here $Q_u$ and $Q_\theta$ are Baxter’s polynomials

$$Q_u(z) = \prod_{a=1}^{N} (z - u_a), \quad Q_\theta(z) = \prod_{j=1}^{L}(z - \theta_j). \tag{4}$$

We also introduced a twist $\kappa$, which does not spoil the integrability and allows to handle better the singularities. With this assumption, the scalar product $(u|v)_L$ with an arbitrary Bethe state with rapidities $v = \{v_a\}_{a=1}^N$ is evaluated, in certain normalization, by [5]

$$\langle u|v \rangle_L = \mathcal{J}_{u,v} \equiv \frac{\det_{ab} \Omega_k(u_a, v_b)}{\det_{ab} u_a - v_b + i} , \tag{5}$$

$$\Omega(u, v) = \frac{i}{u-v} \left( \frac{1}{u-v+i} - \frac{e^{2i\mu(u)}}{u-v-i} \right) . \tag{6}$$

An important particular case is the Gaudin-Izergin determinant, which gives the partition function of the 6-vertex model with domain-wall boundary conditions [7,8], and which we denote by $\mathcal{Z}_{u,z}$. Gaudin-Izergin determinant is equal to $\mathcal{J}_{u,v}$ with $N = L$, with the second set of rapidities frozen to $v = \theta + i/2 \equiv z$. Since $Q_\theta(v) = 0$ if $v-i/2 \in \theta$, the condition $v = z$ is equivalent to retaining only the first term in the definition (6). For any two sets $u$ and $v$, not necessarily satisfying Bethe equations, we define

$$\mathcal{Z}_{u,z} = \frac{\det_{ab} \Omega_k(u_a, v_b)}{\det_{ab} u_a - v_b + i} , \tag{7}$$

2. Factorization property of Slavnov’s determinant.

We will use an operator representation of Slavnov’s determinant (5), which we call factorization formula, because in the limit $N \to \infty$ it factorizes into a product of two computable functionals.

- **Factorization formula:** If $u \cap v = 0$, Slavnov’s determinant (5) is given by the expectation value

$$\mathcal{J}_{u,v} = (-1)^N \frac{\langle v | \mathcal{A}^+ \mathcal{U} | u \rangle \mathcal{J}_u | \mathcal{V}|u \rangle}{\langle v | u \rangle} , \tag{8}$$

where the functionals $\mathcal{A}^\pm [f]$ are defined by

$$\mathcal{A}^\pm [f] \equiv \frac{\det_{ab} (u_a^b - f(u_a)(u_a \pm i)^{b-1})}{\det_{ab} (u_a^{b-1})} , \tag{9}$$

and the functional arguments $\mathcal{U}, \mathcal{V}$ satisfy the algebra

$$\mathcal{U}(z) \mathcal{V}(w) = \mathcal{V}(w) \mathcal{U}(z) \left( 1 - \frac{1}{(z-w)^2+1} \right) . \tag{10}$$

We are interested in the classical limit $N \to \infty$, where the points of the set $u$ condense into a set of contours cuts $\Gamma_u = \bigcup_k \Gamma_u^k$ with linear density $\rho(u)$. We do not renormalize the $u$’s, so that $\rho \sim 1, u_a \sim N$. The distribution is characterized by the resolvent

$$G_u(z) = \sum_{j=1}^{N} \frac{1}{z - u_j} \simeq \int_{\Gamma_u} du \rho(u) \frac{1}{z - u} . \tag{15}$$

It is easy to see that the linear term in $f$ in (13) can be written as a contour integral,

$$\mathcal{A}^\pm [f] = 1 \pm \oint_{C_u} \frac{dz}{2\pi i} f(z) Q_u(z) + O(f^2) \tag{16}$$

$$\simeq 1 \pm \oint_{C_u} \frac{dz}{2\pi i} e^{iq(z)} + O(f^2) , \tag{16}$$

where the integration contour $C_u$ encircles $\Gamma_u$ anticlockwise and the function $q(z)$ is defined as

$$q(z) = -i \log [f(z)] \pm G_u(z) . \tag{17}$$

The Gaudin-Izergin determinant is evaluated by eq. (8) with $U = 0$. Then $\mathcal{Y}(u)$ can be treated as a c-number function $V(u)$ and eq. (8) becomes

$$\mathcal{Z}_{u,z} = (-1)^N A_u^{-}[V] , \quad V(u) = \frac{Q_u(u+i)}{Q_u(u)} . \tag{12}$$
By the functional relations (14), similar representation holds for $f$ large. For the complete solution we try an ansatz of the form

$$\mathcal{A}_u^\pm[f] = \exp\left[\oint_{C_u} \frac{dz}{2\pi} F^\pm(e^{i q z}(z))\right],$$

(18)

where the functions $F^\pm$ can be expanded as

$$F^\pm(\omega) = F_1^\pm \omega + F_2^\pm \omega^2 + F_3^\pm \omega^3 + \ldots,$$

(19)

with $F_1^\pm = \pm 1$. The coefficients $F_n$ can be determined by comparing with the exactly solvable case $f(z) = \kappa$, or $q^\pm(z) = -i \log \kappa \pm G_u(z)$, where [4]

$$\mathcal{A}_u^\pm[\kappa] = (1 - \kappa)^N.$$

(20)

To compare with (18), we perform the contour integration using the asymptotics $e^{i q z}(z) \simeq (1 \pm \kappa N z)$ at $z \to \infty$, and find $F_1^\pm = \pm 1/N^2$. Therefore

$$F^\pm(z) = \pm \frac{1}{N^2} \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \pm \text{Li}_2(z).$$

(21)

The functional equation for the dilogarithm,

$$\text{Li}_2(1/z) = -\text{Li}_2(z) - \pi^2/6 - \frac{1}{2} \log^2(-z),$$

(22)

is the scaling limit of (14).

4. Classical limit of the Slavnov and Gaudin-Izergin determinants and of the Gaudin norm

We will use the factorization formula (8) to find for the classical limit of the Slavnov determinant (5). In this limit we can consider $U$ and $V$ as $c$-number functions, since the they commute up to $O(N^{-2})$. Then we can use the functional relation (14) to write (8) in the form

$$\mathcal{A}_{u,v} = \mathcal{A}_v^+ [\kappa e^{i G_{u-v} - i G_\theta}] \mathcal{A}_u^-[e^{i G_v}].$$

(23)

Introduce, as in (15), the resolvents $G_u, G_v$ and $G_z$, associated respectively with the sets of points $u, v$ and $z$. The classical limit of Slavnov’s scalar product is obtained by substituting (18) in the factorization formula (23):

$$\log \mathcal{J}_{u,v} = \oint_{C_u} \frac{dz}{2\pi} \text{Li}_2(e^{i q z}) - \oint_{C_u} \frac{dz}{2\pi} \text{Li}_2(e^{i G_{u+G_\theta}}),$$

(24)

$$q \equiv G_u + G_v - G_\theta + \log \kappa.$$ 

(25)

The integration contours $C_u$ and $C_v$ encircle $\Gamma_u$ and $\Gamma_v$ anticlockwise.

The r.h.s. of (24) can be reformulated entirely in terms of the function $q(z)$ defined in (25). The Bethe equations (2) imply a boundary condition for the resolvent $G_u$,

$$2G_u(z) - G_\theta(z) + \log \kappa = 2\pi n_k \quad \text{for} \quad z \in \Gamma_u,$$

(26)

where $G_u$ is the half-sum of the values of the resolvent on both sides of $\Gamma_u$ and $n_k$ is the mode number associated with the $k$-th connected component $\Gamma^k_u \subset \Gamma_u$. Hence, if $q^{(1)}$ is the value of the function $q(z)$ on the physical sheet defined by (25), then the value of $q(z)$ on the second sheet is given by $q^{(2)} = -G_u + G_v$ and (24) can be written as

$$\log \mathcal{J}_{u,v} = \oint_{C_u} \frac{dz}{2\pi} \text{Li}_2(e^{i q(z)}),$$

(27)

(The minus sign is compensated by the change of the orientation of contour $C_u$ after it is moved to the first sheet.) The integral along $C_u$ is however ambiguous, because the integrand has two logarithmic cuts which start at two branch points on the first sheet and end at $z = \infty$ on the second sheet, after crossing the cut of the resolvent $G_u$ on $\Gamma_u$. The ambiguity is resolved by deforming the contour $C_u$ to a contour $C_u^\infty$ which encircles also the point $z = \infty$ on the second sheet [10]. In the case of a one-cut solution, the contour $C_u^\infty$ is depicted in Fig. 1, left.

With this prescription, eq. (24) reproduces the numerical data (for $\kappa = -1$ and $N$ up to 60) with precision $10^{-12}$. Another test of (27) is to send all the roots of $G_u$ to infinity. In this limit the integration goes only along the contour $\Gamma_u$ and the function $q$ in the integrand is given by $q = G_v - \frac{1}{2} G_\theta$. Then eq. (27) reproduces correctly the expression obtained in [4] for the scalar product of Bethe state and a vacuum descendental.

We will also need the classical limit of the Gaudin-Izergin determinant, for which (12) gives

$$\log \mathcal{Z}_{u,v} = -\oint_{C_u} \frac{dz}{2\pi} \text{Li}_2(e^{i G_{u-v} - i G_\theta}).$$

(28)

Finally, an expression for the square of the Gaudin norm can be formally obtained from (27) by taking $G_u = G_v = G$. When $\Gamma \to \Gamma_u$, the integration contour in (27) can be closed around $\Gamma_u = \Gamma_v$ as in Fig. 1, right, and $q$ in the integrand is replaced by $2p_u$, where

$$p_u = G_u - \frac{1}{2} G_\theta + \frac{1}{2} \log k$$

(29)
is the quasi momentum. Thus we find for the square of the Gaudin norm
\[ \log \mathcal{J}_{u,u} = \oint_{C_u} \frac{dz}{2\pi i} \text{Li}_2 \left( e^{2ip_u(z)} \right). \]  
(30)

One can check, using the fact that \( p(z) = \pm i\pi \rho(z) \) on the two edges of the cut, that the contour integral (30) can be transformed into (twice) the linear integral in eq. (2.15) of [4].

V. CLASSICAL LIMIT OF THE STRUCTURE CONSTANT

Now we can proceed with the computation of the classical limit of the structure constant (1), which we express in terms of the functional constants considered above,
\[ C_{123}^0 = \frac{\mathcal{J}_{u,v,z} \mathcal{J}_{z,w}^{1/2}}{\mathcal{J}_{u,v}^{1/2} \mathcal{J}_{v,w}^{1/2}}. \]  
(31)

In applying (27), (28) and (30) the only non-obvious point is the evaluation of \( \mathcal{J}_{u,v,z} \) with \( z = \theta^{(13)} + \frac{i}{2} \). This is the ‘restricted Slavnov product’ studied in [6, 11, 12], in which part of the magnon rapidities are frozen to the values of the impurities on a segment of the spin chain. In the original formulation (5), the restricted Slavnov product is given by a ratio of vanishing quantities, which necessitates to apply repeatedly l’Hôpital’s rule. In contrast, the factorized representation (8) is free of such complications. It is given by the r.h.s. of (27), with (for \( \kappa = 1 \))
\[ q = G_u + G_v \theta^{(12)} \theta^{(13)} \]
\[ = G_u + G_v - G_{\theta^{(12)}}. \]  
(32)

Expressing the resolvents \( G_u, G_v, G_w \) in terms of the three quasi-momenta \( p_u = G_u - \frac{1}{2} C_\theta^{(1)} \), \( p_v = G_v - \frac{1}{2} C_\theta^{(2)} \) and \( p_w = G_w - \frac{1}{2} C_\theta^{(3)} \), and taking the homogeneous limit \( \theta^{(n)} \to 0 \), replacing \( G_{\theta^{(n)}} \to L_{n/2z} \) (\( n = 1, 2, 3 \)), we finally obtain, up to a complex constant,
\[ \log C_{123}^0 \simeq - \sum_{n=u,v,w} \frac{1}{2} \oint_{C_n} \frac{dz}{2\pi i} \text{Li}_2 \left[ e^{2ip_n(z)} \right] \]
\[ + \oint_{C_u} \frac{dz}{2\pi i} \text{Li}_2 \left[ e^{ip_u(z) + ip_v(z) + iL_1/2z} \right] \]
\[ + \oint_{C_w} \frac{dz}{2\pi i} \text{Li}_2 \left[ e^{ip_w(z) + (L_2 - L_1)/2z} \right]. \]  
(33)

As it was pointed out by Gromov and Vieira in [13], the tree level solution for \( C_{123}^0 \) in presence of impurities \( \theta^{(1)}, \theta^{(2)}, \theta^{(3)} \) can be used to obtain the one-loop corrections. In this sense we have obtained also the correlator of three non-BPS classical fields at one-loop.

The method outlined in this note allows to handle the impurities in the classical limit and attack the problem in its full generality. The expression (33) can be used [14] to show that, at least in the classical limit, the two-loop result is obtained by changing the quasi-momenta \( p_u, p_v \) and \( p_w \) according to the three-loop Bethe ansatz equations [15]. It is natural to expect that the full structure coefficient in the \( SU(2) \) sector in SYM will be obtained from (33) by using the exact expression for the quasi-momenta upon inclusion of the dressing phase [16]. At least this possibility is worth of being explored and we hope to be able to report on this in a future publication.

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