Integrating Geometry in General 2D Dilaton Gravity with Matter

W. Kummer\textsuperscript{1}, H. Liebl\textsuperscript{1}, and D.V. Vassilevich\textsuperscript{2}\textsuperscript{1}
\textsuperscript{1}Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstr. 8–10, A-1040 Wien
Austria
\textsuperscript{2}Institut für Theoretische Physik, Universität Leipzig
Augustusplatz 10, D-04109 Leipzig
Germany

Abstract

General 2d dilaton theories, containing spherically symmetric gravity and hence the Schwarzschild black hole as a special case, are quantized by an exact path integral of their geometric (Cartan-) variables. Matter, represented by minimally coupled massless scalar fields is treated in terms of a systematic perturbation theory. The crucial prerequisite for our approach is the use of a temporal gauge for the spin connection and for light cone components of the zweibeine which amounts to an Eddington Finkelstein gauge for the metric. We derive the generating functional in its most general form which allows a perturbation theory in the scalar fields. The relation of the zero order functional to the classical solution is established. As an example we derive the effective (gravitationally) induced 4-vertex for scalar fields.

\textsuperscript{*}e-mail: wkummer@tph.tuwien.ac.at
\textsuperscript{†}e-mail: liebl@tph16.tuwien.ac.at
\textsuperscript{‡}e-mail: Dmitri.Vassilevich@itp.uni-leipzig.de

On leave from Dept. of Theoretical Physics, St. Petersburg University, 198904 St. Petersburg, Russia
1 Introduction

Already for a long time the unsolved problem of quantizing gravity has been thought to allow a point of attack in the realm of two-dimensional theories of gravity. After all, the Schwarzschild solution for the black hole (BH) can be treated by reducing 4d gravity in terms of spherical coordinates\cite{1}. However, even in that reduced phase space, interactions with scalar matter — moreover simplified to minimal coupling — seemed to be not manageable as a quantum theory. If studies desired to reach beyond semiclassical approaches eventually they had to rely again on computations in a given curved background. This is the case also for several most recent papers dealing with this subject\cite{2}.

The two-dimensional dilaton BH\cite{3} (DBH) for the first time provided hope that some break-through could be achieved: The classical solvability of a theory with a global structure coinciding with the Penrose diagram for the Schwarzschild BH seemed to offer the possibility that by integrating out the geometric variables (metric) exactly, matter could be treated as a perturbation in a systematic manner. This hope was supported by the fact that the DBH model had been derived from string theory and thus powerful methods of conformal field theory were expected to be applicable in the quantization program\cite{4}. The use of the conformal gauge in all this work therefore seemed to be most convenient. Nevertheless, also within the 2d DBH the program of a perturbative treatment of matter in a quantum exact (nonperturbative) integration of the geometry could not be achieved. One loop matter, represented by a Polyakov term was plugged back into the classical equations of motion, again leading to a basically semiclassical — albeit improved — treatment of quantum effects around a given more dynamical background representing a model of the BH. Also right from the beginning of those studies the authors working in this field were well aware of a basic shortcoming of the DBH with respect to Hawking radiation: In that model Hawking temperature and Hawking flux are not determined by the black hole’s mass, but only depend on a cosmological constant. As noted later by M.O. Katanaev and two of the present authors\cite{5} another serious deviation of the Schwarzschild BH affecting possible applications to final black hole evaporation at times of order inverse Planck mass and thus also to the information paradox, stems from the null-completeness of the (non-null incomplete) singularity.

As a consequence of the flurry of interest generated by the DBH of\cite{3} and in continuation of even earlier attempts, also generalized dilaton theories\cite{6}...
\( L_{(2)} = \sqrt{-g} \left( -X R - V(X) + \frac{U(X)}{2} (\nabla X)^2 \right) \) \hspace{1cm} (1)

with or without (minimal) coupling to matter \([1]\)

\( L^{(m)} = \frac{1}{2} \sqrt{-g} g^{\mu \nu} \partial_\mu S \partial_\nu S \) \hspace{1cm} (2)

received considerable attention. In (1) and (2) \( g_{\mu \nu} \) is the 2d metric, \( R \) the Ricci scalar. \( U \) and \( V \) are general functions of the dilaton field \( X \). Spherically reduced gravity (SRG) is the special case \( U_{SRG} = -(2X)^{-1}, V_{SRG} = -2 \), the DBH follows with \( U_{DBH} = -(X)^{-1}, V_{DBH} = 2 \lambda^2 X \). Some aspects of classical and (in particular cases) quantum integrability of such models were considered recently in \([7]\).

Another development started with the inclusion of nonvanishing torsion in a twodimensional model of gravity \([8]\). In connection with that model the advantages of a temporal gauge \( (\omega_0 = 0) \) for the components of the spin connection \( \omega^a_{\mu b} = \omega_{\mu} e^a_{b} \) and for the light cone components of the zweibein \( e^a_{\mu} \) \( (e^a_{0} = e^a_{0} - 1 = 0) \) were first realized in connection with its classical solution \([9]\) and for the path integral in the quantum case \([10]\). The Dirac quantization could also be carried through \([11]\) for that model. Because this gauge leads to an Eddington Finkelstein form of the 2d metric, it will be referred to as EF-gauge below.

Closely related to the intriguing properties of that gauge is the fact that actually all matterless 2d-models of gravity with arbitrary powers of torsion and curvature can be summarized in an action \([12]\)

\[
L_{(2)} = \int_{M_2} \left[ X^+ D_\omega + X^- D_\omega + X d\omega + \epsilon \mathcal{V}(X^+ X^-, X) \right] \hspace{1cm} (3)
\]

\[
\mathcal{V} = X^+ X^- U(X) + V(X) \hspace{1cm} (4)
\]

where \( D_\omega = de^a + (\omega \wedge e)^a \) is the torsion two form, the scalar curvature \( R \) is related to the spin connection \( \omega \) by \( R \) = \( *d\omega \) and \( \epsilon \) denotes the volume two form \( \epsilon = \frac{1}{2} \epsilon_{ab} e^a \wedge e^b = d^2 x \det e^a = d^2 x \). Our conventions are determined by \( \eta = \text{diag}(1,-1) \) and \( \epsilon^{ab} \) by \( \epsilon^{01} = -\epsilon^{10} = 1 \). We also have to stress that even with Greek indices, \( \epsilon^{\mu \nu} \) is always understood to be the antisymmetric Levi-Civitá symbol and never the corresponding tensor. In \([13]\) we have shown
that (3) is quantum equivalent to the generalized dilaton theory (1) with $U$ and $V$ representing the same functions in both actions. This represented the generalization of the classical equivalence first established for the model of (8) with a dilaton theory in (14). It should be noted that a removal of the kinetic term in (1) by a conformal transformation was avoided in that step, because this would imply drastic changes of global properties already at the classic level (14). The essential differences between quantum theories related by conformal transformations has been pointed out repeatedly, e.g. in ref. (15). It is especially relevant for the DBH which otherwise could be reduced formally to a theory in a flat background. This is the origin of its classical solvability, however did not turn out to be helpful to obtain a quantum solution in the end.

In previous work the present authors (16) have shown that the strategy of an exact path integral for the geometric variables with matter as a perturbation can be pursued to obtain 2-loop corrections in the scalar field for the class of theories (3) with $U(X) = 0$. This exercise showed how our approach can be used to determine loop corrections for the Polyakov term. But the restriction to $U(X) = 0$ in that work eliminated dilaton theories (1) and especially also those “physical” ones where vanishing of the absolutely conserved quantity $C$ (8–10,12,14,17) implies (classically) a flat background. DBH and Schwarzschild BH (SRG) belong precisely to this latter class.

In the present paper we report the successful extension of our formalism to an arbitrary theory (3) with kinetic term for the dilaton field ($U \neq 0$). To the best of our knowledge therefore here for the first time a systematic quantization of gravity is available at least in $d = 2$, including even the S-wave part of $d = 4$ General Relativity, i.e. the quantum theory of the Schwarzschild BH. We stress that we achieve this goal without any additional assumption on the quantum behavior of the BH (13), remaining strictly within “orthodox” quantum field theory.

An important part of our paper also deals with aspects which are related to the conserved quantity $C$ referred to above. It is found to be intimately connected with 'homogeneous’ solutions of first order differential equations which we had not considered in (10) but which represent the classical background to zero order in the scalar loops. An arbitrariness in the generating functional leads us to introduce a contribution which generalizes the corresponding Lagrangian in the second reference of (10) for the special model (8). It turns out to be the only surviving term if the sources, introduced for the momenta, are set to zero.
In Section 2 we perform the path integral to the point when the matter fields $S$ are to be integrated. As expected from our previous work the 'geometric' part of the path integral can be done exactly — and is trivial in the sense that only classical effects remain, if — as a consequence of our specific regularization — global quantum fluctuations are suppressed. Quantum interactions are then induced by the matter fields only.

Contact to the classical conservation law and to the classical (EF) solution of the zweibeine and spin connection is made in Section 3. In the (Gaussian) integral of the scalars which is the subject of Section 4 we have to confront the problem of the (generalized) Polyakov term in the EF gauge. In that gauge a technical problem arises for the determinant defining the measure and for the (effective) d’Alembertian. This we solve in two ways. We either introduce a further auxiliary field, acting as a token contribution for that component of the metric which in the EF gauge cancels anyhow in the kinetic term of the $S$-fields. The other approach uses path integrals of ad hoc ghost fields. In the present paper we try to attain a sufficient level of rigor. This is why we pay attention to the asymptotics and the path integral measure. We also give two different forms of generating functional for the effective matter theory because different forms may appear to be more convenient for different applications.

In Section 5 we derive the four vertex of the scalar fields, induced by gravity.

Our results are summarized in Section 6, together with an outlook on the wide range of possible applications of our approach. Also several open (but in principal solvable) problems are listed.

In Appendix A the absence of anomalous contributions to the measure resulting from the geometric variables is demonstrated in the background field formalism. Appendix B discusses some aspects of the UV and IR regularization.

## 2 Dilaton Path Integral with Scalar Matter

Expressing (3) in terms of the components $e^a_\mu, \omega_\mu$ and rewriting the matter Lagrangian (2) as

\[
L^{(m)} = \frac{1}{2} \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} e^{a}_\mu e^{b}_\nu \partial_\alpha S \partial_\beta S
\]
the extended Hamiltonian in the sense of Batalin and Vilkovisky \cite{19} can be constructed following the line described in our previous work \cite{13} for the matterless case. In the EF gauge

\[ e_0^+ = \omega_0 = 0, \quad e_0^- = 1 \]  

(6)

after integrating out the two types of ghosts, the gauge-fixed “coordinates” \( q \) and their respective canonical momenta (primary constraints), the generating functional for Green functions reads

\[
W = \int (dS)(dP)(d^3q)(d^3p) \sqrt{\det \left( \frac{q_3}{q_2} \right)} \det F \exp i \int \left( \mathcal{L}^{\text{eff}} + \mathcal{L}_{(s)} \right) d^2 x ,
\]

(7)

where some explanation of the notation is needed. The first point is that to simplify (7) and our formulas below we use the shorthand notation for “coordinates”, “momenta” and sources

\[
q_i = (\omega_1, e_1^-, e_1^+) ,
\]

\[
\bar{q}_i = (\omega_0, e_0^-, e_0^+) ,
\]

\[
p_i = (X, X^+, X^-) ,
\]

\[
j_i = (j, j^+, j^-) ,
\]

\[
J_i = (J, J^+, J^-) .
\]

(8)

In the EF gauge (4) \( q_3 = e_1^+ = \det e = \sqrt{-\tilde{g}} \). Together with the only other nonvanishing component of the zweibein \( q_2 = e_1^- \) the product \( 2q_2q_3 \) represents the Killing norm of \( g_{\mu\nu} \) in that gauge. Furthermore, in (7) the introduction of \( [\det (q_3/q_2)]^{1/2} \) is a consequence of the required covariance of the measure for the final S-integration \cite{20}.

The second determinant

\[
\det F = \det[\partial_0 + p_2 U(p_1)]
\]

(9)

is a remnant of the preceding functional integrations \cite{16} in extended phase space, which include the ones with respect to \( \bar{q}_i \) in (8), fixed by the gauge-fermion \cite{19} according to (6). Another careful treatment of the Faddeev–
Popov determinant in the background field approach can be found in Appendix A, where the absence of gauge anomalies for the measure is demonstrated. Finally in (cf. (4) for the definition of $V$)

$$L_{\text{eff}}(1) = \dot{q}_i p_i + \dot{S} P + q_1 p_2 - q_3 V + \frac{1}{4q_2} (P - \partial_1 S)^2$$

the last three terms are the ones remaining from (minus) the extended Hamiltonian, and $P = \partial L^{(m)}/\partial \dot{S}$. The source term reads

$$L_{(s)} = j_i q_i + J_i p_i + SQ .$$

We also retain $\hbar \neq 1$ in order to keep track of loop orders in a simple manner.

After performing the Gaussian integration with respect to the momenta $P$ one arrives at

$$W = \int (dS)(d^3q)(d^3p) \sqrt{\det q_3 \det F} \exp i \int \left( \frac{L_{\text{eff}}^{(2)}}{\hbar} + L_{(s)} \right) d^2x$$

where as compared to (10) the new effective Lagrangian is

$$L_{\text{eff}}^{(2)} = -q_i \dot{p}_i + q_1 p_2 - q_3 V - q_2 (\partial_0 S)^2 + (\partial_0 S)(\partial_1 S) .$$

The cancellation of the determinant of $q_2$ in the measure should be noted. It is well known that the correct diffeomorphism invariant measure for a scalar field $S$ on a curved background $e^\mu_a$ is $d((-g)^{1/4}S) = d(\sqrt{e} S)$. Note, that $e = e^+_1 = q_3$ in the EF gauge. Therefore (12) indeed contains the correct measure. We want to be able to use the same order of simple integrations as in the matterless case (first $\int (d^3q)$, then $\int (d^3p)$). The factor $\sqrt{\det q_3}$ prohibits this for an immediate $q_3$ integration.

One way (approach a)) to solve this problem consists by introducing a new field $f$ and by representing the path integral (12) as

$$W = \int (df) \delta(f - \frac{1}{i} \frac{\delta}{\delta j_3}) \widetilde{W}$$

where in

$$\widetilde{W} = \int (d\tilde{S})(d^3q)(d^3p) \det F \exp i \int \left[ \left( \frac{L_{\text{eff}}^{(2)}}{\hbar} + L_{(s)} \right) d^2x \right]_{S = f^{-\frac{1}{2}} \tilde{S}}$$
the $\sqrt{\det f}$ has been absorbed in a new variable $\tilde{S}$ for the scalars. The representation (14) allows us to integrate the geometric variables as in the matterless case since the action remains linear in the $q_i$. An important feature of (15) is that $(d\tilde{S})$ is just the standard Gaussian measure independent of $q_i$.

Integrating out the $S$-field in section 4, we therefore will be able to use the definition of the Polyakov action

$$\int (d\tilde{S}) \exp \left( i \int d^2 x \frac{\tilde{S}}{g^{1/4}} \partial \mu \sqrt{g} g^{\mu \nu} \partial _\nu \tilde{S} \right) = \det^{-\frac{1}{2}} \left( \frac{1}{g^{1/4}} \partial \mu \sqrt{g} g^{\mu \nu} \partial _\nu \frac{1}{g^{1/4}} \right)$$

where $g_{\mu \nu}$ are arbitrary functions of sources and fields, which should only be independent of $S$.

Another way (approach b)) to achieve the same technical advantage starts from the path integral identity

$$\sqrt{\det q_3} = \int (d\phi)(dc)(d\bar{c}) \exp \left( i \int q_3 (\phi^2 + \bar{c}c) \right) ,$$

where $\phi$ is a scalar, $c$ and $\bar{c}$ represent anticommuting Grassmann fields. Then instead of (14)

$$W = \int (d\phi)(dc)(d\bar{c})\tilde{\tilde{W}}$$

where

$$\tilde{\tilde{W}} = \int (dS)(d^3 q)(d^3 p) \det F \exp i \int d^2 x \left( \frac{\tilde{L}_{\text{eff}}^{(2)}}{\hbar} + \mathcal{L}_{(s)} \right)$$

and

$$\tilde{L}_{\text{eff}}^{(2)} = \mathcal{L}_{(2)}^{\text{eff}} + \hbar \tilde{q_3} \tilde{l} , \quad \tilde{l} = \phi^2 + \bar{c}c .$$

The main advantage of the EF gauge is the fact that due to the linearity in $q_i$ in both approaches the $(d^3 q)$ integration in $\tilde{W}$ or $\tilde{\tilde{W}}$ can be done first –
as in the case without matter fields – leading to three $\delta$-functions:

$$\delta \left( -\nabla_0 \left( p_1 - \hat{B}_1 \right) \right) \quad (21)$$

$$\delta \left( -\nabla_0 \left( p_2 - \hat{B}_2 \right) \right) \quad (22)$$

$$\delta \left( -F \left( p_3 - \hat{B}_3 \right) \right). \quad (23)$$

where $F$ is the differential operator in \(\text{(12)}\). $\hat{B}_i$ will be given below. The symbol $\nabla_0 = \partial_0 - i(\mu - i\varepsilon) = \partial_0 - i\tilde{\mu}$ describes an IR regularized ‘derivative’, related to Green functions $\nabla_0^{-1}$, $\nabla_0^{-2}$ with proper asymptotics (cf. Appendix B). Below we will be careful to mark the difference with respect to $\nabla_0 = \partial_0 + i\tilde{\mu}$ which appears e.g. in partial integrations like $\int A(\nabla_0 B) = -\int (\nabla_0 A)B$ \cite{11}. Beside the IR and UV asymptotics for a proper definition of consistent Green functions also the allowed range of the variable $x^0$ is crucial. For example, a singularity at $x^0 = 0$ – as in the case of the Schwarzschild black hole for our choice of coordinates – require appropriate boundary conditions for the half line $x^0 \geq 0$. This point will be discussed in more detail in future work. Here we mainly concentrate on the general formalism.

Using these three $\delta$-functions the remaining integrations over $(d^3p)$ yield directly

$$p_i = \hat{B}_i. \quad (24)$$

This simply means that in the phase-space (path-) integral \(\text{(7)}\) only classical paths contribute to the $p$-s, fixing them by \(\text{(24)}\) in terms of the sources (and some homogeneous solutions, representing a classical background, as will be demonstrated below).

Solving \(\text{(21)}\) and \(\text{(22)}\) we observe that the terms $\hat{B}_1$ and $\hat{B}_2$ allow for homogeneous solutions ($\nabla_0 \bar{p}_1 = \nabla_0 \bar{p}_2 = 0$) and the $S$-dependent parts can be separated easily:

$$\hat{B}_1 = \bar{p}_1 + \nabla_0^{-1} \bar{p}_2 + \hbar (\nabla_0^{-1} j_1 + \nabla_0^{-2} j_2) - \nabla_0^{-2} (\partial_0 S)^2 \quad (25)$$

$$\hat{B}_2 = \bar{p}_2 + \hbar \nabla_0^{-1} j_2 - \nabla_0^{-1} (\partial_0 S)^2 \quad (26)$$
For the computation of $\hat{B}_3$ it is convenient to use an exponential form of the differential operator $\hat{F} = F(\hat{B}_1, \hat{B}_2)$ defined in (9)

\[
\hat{F} = e^{-\hat{T}} \nabla_0 e^{\hat{T}}
\]

\[
\hat{T} = \nabla_0^{-1} (\hat{U} \hat{B}_2)
\]

\[
\hat{U} = U(\hat{B}_1)
\]

Including the homogeneous solution from the operator $\hat{F}$ we deduce ($\nabla_0 \bar{p}_3 = 0$) for approach a)

\[
\hat{B}_3 = e^{-\hat{T}} \left[ \nabla_0^{-1} e^{\hat{T}} (h j_3 - V(\hat{B}_1)) + \bar{p}_3 \right] + \text{terms } O(S^2)
\]

\[
\tilde{\hat{B}}_3 = \hat{B}_3 + e^{-\hat{T}} \nabla_0^{-1} e^{\hat{T}} h \tilde{l}
\]

In approach b) only (30) must be replaced by the expression

As we shall demonstrate in Section 3 the homogeneous solutions $\bar{p}_i$ will provide the essential ingredient to recover the classical solutions of the theory. It should be emphasized that by performing the $p_3$ intepration the term $\det F$ in the path integral measure has been cancelled. Thus no Faddeev-Popov type determinant appears finally in our gauge – not too surprisingly in view of its 'axial' character (for more details on the measure we refer to Appendix A).

We now argue that (for approach a)) in

\[
\tilde{\mathcal{W}} = \int (dS) \exp i/h \int \mathcal{L}_{(3)}^{\text{eff}}
\]

\[
\mathcal{L}_{(3)}^{\text{eff}} = h J_i \hat{B}_i + h SQ + \partial_0 S (\partial_1 S) + \hat{\mathcal{L}}^{HK}
\]

we have to add another term

\[
\hat{\mathcal{L}}^{HK} = \tilde{g} e^{\hat{T}} (h j_3 - \hat{V})
\]
three terms proportional to $J_i$: For $J_1$ and $J_2$ the (nonlocal) $\nabla_0^{-1}$-factors in $[25, 26]$ (before adding the homogeneous solution) could also have been attached to $J_{1,2}$.

E.g. for the simplest expression of this type in $J_2 B_2$

$$\int_{x'} \int_x J_{2x} \nabla_{xx'}^{-1} j_{2x'} = - \int_{x'} \int_x \nabla_{xx'}^{-1} J_{2x'} j_{2x}$$

also the r.h.s. of this equation could be taken to represent the “correct” one. But then a homogeneous solution may be added: $\nabla^{-1} J_2 \to \nabla^{-1} J_2 - \tilde{g}_2$ with $\tilde{\nabla}_0 \tilde{g}_2 = 0$. Proceeding in a similar way for $J_1 B_1$ produces additional contributions ($\tilde{\nabla} \tilde{g}_1 = 0$) as

$$J_1 \hat{B}_1 + J_2 \hat{B}_2 \to J_1 \hat{B}_1 + J_2 \hat{B}_2 + \tilde{g}_1 (j_1 + \nabla_0^{-1} j_2) + \tilde{g}_2 j_2.$$  

These terms, however, must be irrelevant because they would only contribute to couplings of the sources of $e_1 -$ and $\omega_1$ to some 'external fields' $\tilde{g}_{1,2}$. The situation is different for a similar contribution from $J_3 \hat{B}_3$. There after an analogous reordering $\int J_3 \hat{B}_3 = - \int \left( \tilde{\nabla}_0^{-1} J_3 e^{-\hat{T}} \right) e^{\hat{T}} (h j_3 - \hat{V})$ a homogeneous solution ($-\tilde{g}_3/h$) is added which leads to the additional expression (35). There are several further reasons why $\hat{L}^{HK}$ must be present: 1) It is impossible that all the dynamics disappear together with the sources of the momenta; indeed the classical equations of motion for the $q_i$ (cf. (45) to (47) below) acquire their “quantum” counterpart from $L^{HK}$ alone! 2) For the special case of the Katanaev-Volovich model it was shown, keeping there $J_i = 0$ from the start and following the traditional sequence of integrations (first a Gaussian integral of $(d^3 p)$ and then another Gaussian integral for the $\int d^3 q$), that precisely an effective action of type (35) is produced (cf. the second ref. [10]). 3) In the matterless classical case on the mass shell by the simple method explained in section 3.1 $(h j_3 - \hat{V}) e^{\hat{T}}$ may be expressed as the derivative of the conserved quantity $C$. This is reminiscent of $C$ appearing at the boundary for a quasilocal energy definition [17].

For approach b) the same arguments are valid. Here the additional term $\tilde{l}$ which enters the total Lagrangian like an addition to $\tilde{j}_3$ modifies (35) to

$$\tilde{L}^{HK} = \tilde{g} e^{\hat{T}} (h j_3 + h \tilde{l} - \hat{V})$$

(38)
3 Classical and Quantum Equations of Motion, Conservation Law

As seen already in the previous section, after integrating the geometric variables the path integral contains many terms with inverse derivatives $\nabla_0^{-1}$. These inverse derivatives (regularized $x^0$-integrals) allow several integration constants, or homogeneous modes. In this section we relate the homogeneous modes to each other and to the conserved quantity $C$ by means of equations of motion and Slavnov–Taylor like identities.

Since a nontrivial quantum theory only emerges through the interaction with matter, we have to show how classical dynamics is reproduced in the path integral formalism if the matter terms are disregarded to zero loop order, the order considered in this section.

3.1 Classical Equations of Motion

Variation of $\delta \omega_1$, $\delta e_i^\pm$ (or $\delta q_i$) in (3), and fixing the gauge according to (6) yields the classical equations of motion for the $p_i$

$$\nabla_0 p_1 - p_2 = 0 \quad (39)$$
$$\nabla_0 p_2 = 0 \quad (40)$$
$$(\nabla_0 + p_2 U(p_1)) p_3 + V(p_1) = 0 \quad . \quad (41)$$

The analogous variation of $\delta \omega_0$, $\delta e_0^\pm$ in (3) results in

$$\nabla_1 p_1 + p_3 q_3 - p_2 q_2 = 0 \quad (42)$$
$$\nabla_1 p_2 + p_2 q_1 - (V(p_1) + U p_2 p_3) q_3 = 0 \quad (43)$$
$$(\nabla_1 + p_2 U) p_3 - q_3 p_3 + V q_2 = 0 \quad , \quad (44)$$

and finally variation of $\delta X$, $\delta X^\pm$ (or $\delta p_i$) provides a third triple of equations of motion:

$$\tilde{\nabla}_0 q_1 - p_2 p_3 q_3 \frac{\partial U}{\partial p_1} - q_3 \frac{\partial V}{\partial p_1} = 0 \quad (45)$$
$$\tilde{\nabla}_0 q_2 + q_1 - q_3 p_3 U = 0 \quad (46)$$
$$(\tilde{\nabla}_0 - U p_2) q_3 = 0 \quad . \quad (47)$$

As can be seen we distinguished the regularized $\nabla_\mu$ from $\tilde{\nabla}_\mu$ appearing after partial integration. The peculiar property of all covariant 2d models to
provide one set of equations involving the momenta alone [21] is clear from (39-41).

As expected, the solution of eqs. (39-41) coincides with the quantities $B_i^{(0)} = B_i(j = 0)$ as defined in (25,26) and (30) for $j_i = 0$ and $S = 0$:

\[
p_1 = B_1^{(0)} = \bar{p}_1 + \nabla_0^{-1}\bar{p}_2 \tag{48}
\]
\[
p_2 = B_2^{(0)} = \bar{p}_2 \tag{49}
\]
\[
p_3 = B_3^{(0)} = e^{-T^{(0)}} \left( \bar{p}_3 - \nabla_0^{-1}e^{T^{(0)}} V^{(0)} \right) \tag{50}
\]
\[
T^{(0)} = \nabla_0^{-1} \left( B_2^{(0)} U^{(0)} \right) \tag{51}
\]
\[
V^{(0)} = V(B_1^{(0)}) \tag{52}
\]
\[
U^{(0)} = U(B_1^{(0)}) . \tag{53}
\]

The zero component of the absolute conservation law [9,11–13,17,21–23] is obtained by linear combination of eqs. (39-41)

\[
\nabla_0(p_2p_3) + p_2^2 p_3 U + V(\nabla_0 p_1) = 0 \tag{54}
\]

and thus

\[
\nabla_0 C = \nabla_0 \left[ e^{Q(p_1)} p_2 p_3 + w(p_1) \right] = 0 \tag{55}
\]

follows with

\[
Q(x) = \int_{y_0}^{x} U(y)dy \tag{56}
\]
\[
w = \int_{x_0}^{x} V(y)e^{Q(y)}dy . \tag{57}
\]

In an analogous way one derives from the classical eqs. (42-44) $\nabla_1 C = 0$ so that

\[
C = C_0 = \text{const} . \tag{58}
\]

Therefore $p_2p_3$ and $p_1$ always depend on each other. The constant $C_0$ alone (for fixed lower limits of the integrals [56,57]) labels independent classical solutions, as seen e.g. from the classification from their global properties [8,23]. If matter is added the (classical) conservation law generalizes to
\( \nabla_\mu (C + C^{(m)}) = 0 \) \[17, 22\] which will not be directly relevant for our present work.

In terms of \((48 - 50)\) \(C_0\) of \((55)\) is evaluated easily. Assuming fixed lower limits of all integrations as in \((56, 57)\) we may rewrite \((51)\) with a homogeneous solution \(\bar{t} (\nabla_0 \bar{t} = 0)\) as

\[
T^{(0)} = \int_{B^{(0)}} U(y) dy = Q^{(0)} + \bar{t}
\]

(59)

where \(Q\) has been defined already in \((56)\). Employing the same trick as going from \((51)\) to \((59)\) for the last term in \((50)\) produces \(w(p_1)\) of \((57)\), up to a function \(\bar{w} (\nabla_0 \bar{w} = 0)\), i.e.

\[
\nabla^{-1}_0 \left( e^{T^{(0)} V^{(0)} p_2^{(0)}} \right) = w^{(0)}(p_1) + \bar{w} . \tag{60}
\]

This we could also have used to simplify \((50)\). Therefore the result for

\[
C_0 = e^{-t} \bar{p}_2 \bar{p}_3 - \bar{w} \tag{61}
\]

because of \((58)\), i.e. because of other e.o.m-s, must actually be a constant. The factor \(e^{-t}\) may be absorbed by a redefinition of \(\bar{p}_3\), because it does not contribute to the second term of \((54)\). Of course, \(\bar{t}\) and \(\bar{w}\) may also be dropped as long as the lower limits in \((57)\) and \((59)\) have not been fixed beforehand.

Instead of solving \((12-17)\) directly it is much more convenient to solve the e.o.m-s from \((3)\) and \((4)\) in the formalism of exterior derivatives. By a trivial generalization of the steps in \([12]\) and especially in \([17]\) the general solutions for \(q_i\) are obtained from

\[
e^+ = e^{Q(X)} X^+ df \tag{62}
\]

\[
e^- = \frac{dX}{X^+} + X^- e^{Q(X)} df \tag{63}
\]

\[
\omega = - \frac{dX^+}{X^+} + \nu e^{Q(X)} df \tag{64}
\]

\[
X^+ X^- e^{Q(X)} = C_0 - w(X) \tag{65}
\]

by introducing the specific gauge \((6)\) for the 0-components of the 1-forms. Identifying the \(p_i\) and \(q_i\) according to \((8)\) and introducing new arbitrary
functions \( \tilde{q}_i \) \((\nabla_0 \tilde{q}_i = 0)\) one obtains

\[
e^+_1 = \tilde{q}_3 e^{Q(0)}
\]

\[
e^-_1 = \tilde{q}_2 - \frac{\tilde{q}_3}{\tilde{p}_2} (w^{(0)} + \bar{w}) + \int_{x'} \tilde{q}_1 \nabla^{-1}_{0xx'}
\]

\[
\omega_1 = \tilde{q}_1 + \frac{\tilde{q}_3}{\tilde{p}_2} V^{(0)} e^{Q(0)} - \frac{\tilde{q}_3}{\tilde{p}_2} U^{(0)} (w^{(0)} + \bar{w}) + \tilde{p}_3 \tilde{q}_3 e^{-\bar{t}} U^{(0)}
\]

where \( Q^{(0)} \) etc. are defined as in (51) and (56). The new arbitrary functions \( \tilde{q}_i \) are restricted by

\[
\nabla_1 \tilde{p}_1 + \tilde{p}_3 \tilde{q}_3 e^{-\bar{t}} - \tilde{p}_2 \tilde{q}_2 = 0
\]

\[
\nabla_1 \tilde{p}_2 + \tilde{q}_1 \tilde{p}_2 = 0
\]

and (71). It can be easily verified that (70) follows by inserting the solutions for \( p_i \) and \( q_i \) into (44), similarly (69) is a consequence of (43), if (70) is used again. (61) corresponds to a linear combination of (42, 43, 44), as indicated already above.

The gauge \( 6 \) leaves undetermined residual transformations. One derives easily that local Lorentz boosts \( \gamma = \tilde{\gamma}(x) \) \((\nabla_0 \tilde{\gamma} = 0)\)

\[
e^+_1 = q_3 \rightarrow e^\gamma q_3
\]

\[
e^-_1 = q_2 \rightarrow e^{-\gamma} q_2
\]

\[
\omega_1 = q_1 \rightarrow q_1 - \nabla_1 \tilde{\gamma}
\]

and \( \tilde{\gamma} \)-dependent diffeomorphisms \((\nabla_0 \bar{x}^\mu = 0)\)

\[
x^0(x') = \bar{x}^0(x') + \nabla^{-1}_0 e^\gamma(x')
\]

\[
x^1(x') = \bar{x}^1(x')
\]

still may be used to simplify the solutions.

Applying the Lorentz boost (72) to \( \tilde{p}_2 \) this quantity can be fixed as \( \tilde{p}_2 = \pm 1 \). Eq. (74) enables us to make \( \tilde{p}_1 = 0 \). Removing the regularization \( (\tilde{\mu} \rightarrow 0) \) in the manner described in the Appendix B at the same time requires some care. For small coordinate values \( x^0 \) the combination \( \tilde{p}_1 + \nabla_0^{-1} \tilde{p}_2 \) in this residual gauge simply becomes \( x^0 \).

The component \( g_{11} \) of the metric coincides with the Killing norm in the EF gauge. It is expressed as \( g_{11} = 2 e^+_1 e^-_1 = 2 q_2 q_3 \) in terms of (66), (67). In
this special gauge from (70) \( \ddot{q}_1 = 0 \) and with (69) the usual EF-form of the line element in terms of coordinates \( x^0 \) and \( x^1 \)

\[
ds^2 = 2q_3 dx^1 \left( dx^0 + [c_0 - w(x^0)] dx^1 \right)
\]  

(76)
is obtained.

### 3.2 “Quantum” Equations of Motion

The triple of equations (39 - 41) can also be verified to be an immediate consequence of the generating functional

\[
W^{(0)}(j, J) = \int (d^3q)(d^3p) \det F e^i \int_x \dot{q}_i \dot{p}_i + q_1 p_2 - q_3 V(p_1, p_2, p_3) + j q_i + J_i p_i
\]

(77)

which follows in the matterless case from (10) and (11) where terms with \( S \) and \( P \) and the improved related measure are dropped. Integrating \((d^3q)\) and \((d^3p)\) leads to

\[
W^{(0)} = \exp \frac{i}{\hbar} \int \left[ \hbar J_i B_i + \mathcal{L}^{HK} \right] d^2x
\]

(78)

\[
\mathcal{L}^{HK} = \tilde{g} e^T (\hbar j_3 - V)
\]

(79)

The “quantum” e.o.m.s for the \( p_i \) are obtained by varying the \( q_i \) in \((d^3q_i)\). In this way e.g. for \( \delta q_1 \) a relation

\[
\delta W^{(0)} = 0 = i \int (d^3q)(d^3p)[(\mathcal{L}^{12}) + \text{terms with } j, J] \det F e^i \int_x \ldots
\]

(80)

follows, where \((\mathcal{L}^{12})\) means the left hand side of \((\mathcal{L}^{12})\). Replacing \( p_i \to \frac{1}{\hbar} \frac{\delta}{\delta \dot{q}_i} \) the square bracket may be pulled outside the integral which can be evaluated as before. At \( J_i = j_i = 0 \) we thus obtain (apart from a factor \( W^{(0)} \)) eq. (39) with \( p_i \) replaced by \( B_i^{(0)} \). In a similar manner from \( \delta q_2 \) and \( \delta q_3 \) eqs. (40) and (41) follow with the same replacements. Because \( B_i^{(0)} \) indeed are the solutions of the classical equations their “quantum” versions are fulfilled identically, in other words the “expectation values”

\[
\langle p_i \rangle = B_i^{(0)}
\]

(81)
coincide with classical solutions (for vanishing external sources). In order to obtain the quantum version of (55) we may proceed as in equations (39-41) of which (54) has been a consequence ($p_i \rightarrow \frac{1}{i} \delta J \rightarrow B_1^{(0)}$):

$$C = e^{Q(B_1^{(0)})} B_2^{(0)} B_3^{(0)} + w(B_1^{(0)})$$

(82)

with $\nabla_0 C = 0$. In the classical case $\nabla_1 C = 0$ is a consequence of relations (42-44). These relations are nothing else but the constraints appearing in the Hamiltonian [13]. They clearly do not follow directly from the gauge fixed action but are related to Slavnov-Taylor like identities. These are obtained by the available gauge transformations (local Lorentz transformations $\delta e_{\pm} = \pm \delta \gamma(x) e_{\pm}$ etc. and diffeomorphisms $\delta \zeta^\mu(x)$) in the variables of the path integral (77). A straightforward computation shows that $\delta \gamma(x)$ indeed produces the Lorentz constraint (42), $\delta \zeta^0$ yields (43), whereas for $\delta \zeta^1$ the identity $q_i \partial_1 p_i = 0$ follows which is easily verified from (42), (43) and (44).

Thus the expectation values of $q_i$

$$\langle q_i \rangle = \frac{1}{W(0)} \frac{\delta W(0)}{i \delta j_i} \bigg|_{j=j=0} = \delta \frac{1}{i \delta j_i} \int \mathcal{L}^{HK} d^2 x \bigg|_{j=0}$$

(83)

must be identical to the classical solutions (66)-(68). We re-emphasize the importance of $\mathcal{L}^{HK}$ especially for the present case.

Of course, $\nabla_1 C = 0$ then also is true “quantum mechanically”.

According to the quantum point of view the values of $C_0$ must be related to a superselection rule. The “full” generating functional ($C$ now runs through all constant values) reads

$$W^{\text{total}} = \int d\mathcal{C} W(0)(\mathcal{C}, J, j).$$

(84)

But each “expectation value” of an operator $O_{C_0} \delta(C_0 - \mathcal{C})$ related to each superselection sector with fixed $C_0$ leads to the use of just the $W(0)(C_0, J, j)$ as discussed above.

In view of the general nature of this argument, it should be possible to apply it to the case with matter ($S$-fields) as well. But at the classical level already the determination of the matter contribution $C^{(m)}$ to $\mathcal{C}$ is nontrivial, because that contribution (in contrast to $\mathcal{C} = \mathcal{C}^{\text{geom}}$ treated in this section) is nonlocal and in general cannot be obtained explicitly by solving the equations of motion [23]. On the other hand, the perturbative theory in terms of
scalar loops described below can be conjectured to take care of this order by order if matter fields are retained in eq. (77).

As a consequence of the preceding comparison between the classical and the “quantum” version for the matterless case we also see that the first order form of the action (3) – which has been crucial to obtain a simple solution for the geometric part of all 2d covariant theories – in the quantum case quite naturally suggests the introduction of sources $J_i$ for the momenta. Of course, for the subsequent computation of correlation functions of “physical” fields we always have $J_i = 0$. However, those sources remain a crucial tool for a simple formulation of the central conservation law (55).

We conclude this section with an apology to the reader that we found it necessary to deal in considerable detail with the equivalence between a “quantum” and a classical formulation at all. However, we feel that it is a somewhat unusual situation indeed to obtain a basically classical solution from an (exact) generating functional. Using this section as a basis we will also have to continue below at the quantum level when we proceed to take interactions with scalar matter into account, where the scalar fields induce loop corrections to the exact geometric part.

4 Integration of Scalars

4.1 Generating functional

For a perturbation theory in terms of the scalar field with exact geometric integrations it is sufficient to collect systematically the terms of $O(S^2)$ and perform a Gaussian integration in order to obtain the propagator of $S$. Terms $O(S^{2(n+1)})$, $n > 0$ yield the interaction Lagrangian $L_{\text{int}}^{(S)}$ (cf. (90) below).

We first treat approach a). For $\hat{B}_1$ and $\hat{B}_2$ the terms $O(S^2)$ can be read off from (25,26). For $\hat{B}_3$ they can be summarized in the nonlocal expression $H_{xy}$ as

$$\hat{B}_3 = B_3 + \int_y H_{xy} (\partial_0 S(y))^2$$

(85)

where $B_3$ is defined in (40). From the definition of $\hat{T} = T(\hat{B}_1, \hat{B}_2)$ one
computes

\[ \hat{T}_x = T(B_1, B_2)_x - \int_y G_{xy} (\partial_0 S(y))^2 \]  
\[ (86) \]

\[ G_{xy} = \int_z \nabla^{-1}_{0xy} (B_1 U' B_2 \nabla^{-2}_{0xy} + U_z \nabla^{-1}_{0xy}) \]  
\[ (87) \]

The prime in \( U \) denotes a differentiation with respect to the argument. With

\[ \hat{V}_x = V(\hat{B}_1)_x = V(B_1)_x - V'_x \int_y \nabla^{-2}_{0xy} (\partial_0 S(y))^2 \]  
\[ (88) \]

one arrives at

\[ H_{xy} = e^{-T_x} \int_z \nabla^{-1}_{0yz} e^{T_z} \left[ (G_{zy} - G_{xy}) (h j_3 - V)_z + V'_y \nabla^{-2}_{0xy} \right] - \bar{g} e^{-T_x} G_{xy} \]  
\[ (89) \]

Together with an analogous expansion of \( \hat{L}^{HK} \) for \( \mathcal{L}_{\text{eff}}^{(3)} \) in (33) this yields

\[ \frac{\mathcal{L}_{\text{eff}}^{(4)}}{\hbar} = J_i B_i + \frac{\bar{g} e^T}{\hbar} (h j_3 - V) + \]  
\[ \frac{1}{\hbar} \left( (\partial_0 S)(\partial_1 S) - E_1^- (\partial_0 S)^2 \right) + SQ + \frac{\mathcal{L}_{\text{int}}^{(S)}}{\hbar} \]  
\[ (90) \]

The abbreviation

\[ E_1^-(x) = - \int_y [h (J_1(y) \nabla^{-2}_{0y} + J_2(y) \nabla^{-1}_{0y} + J_3(y) H_{yx}) + \]  
\[ + \bar{g}[e^{T(x')}(h j_3 - V)]_y G_{yx} - V'_y \nabla^{-2}_{0yx}] \]  
\[ (91) \]

\[ (92) \]

indicates the role of this quantity, replacing \( e_1^- \) in the EF-gauge version of the term \( \sqrt{-\bar{g} S \square S} \). The \( S \)-integration to be performed now becomes

\[ \tilde{W} = \int (dS) \sqrt{\det f} \exp \int_x \frac{i}{\hbar} \left( -\frac{1}{2} \left[ S \sqrt{-\bar{g} \square S} \right]_{\text{EF}} + SQ + \mathcal{L}_{\text{int}}^{(S)} \right) \]  
\[ (93) \]

where it should be noted that according to (90) in the EF gauge the quadratic expression in \( S \) is independent of \( e_1^+ \), with only \( e_1^- = E_1^- \) determining the “background” for that integral. \( e_1^- = E_1^+ = f \) only enters through the
measure $\sqrt{\det f}$. Thus (93) corresponds to the standard Polyakov integral with the metric determined by $E_1^-$ from (91) and $E_1^+ = f$:

$$g_{\mu\nu} = f \begin{pmatrix} 0 & 1 \\ 1 & 2E_1^- \end{pmatrix}$$

(94)

Completing the square in $S$ the Gaussian integral (93) yields the Polyakov term $L_P$ and the propagator $\Delta_{xy} = (f(x)^{-1/2} \Box^{-1} f(y))^{-1/2}$ for $S$:

$$\tilde{W} = \exp \left[ i \int_x \left[ \frac{\hbar}{2} \left( \int_y Q_x \Delta_{xy}^{-1} Q_y \right)^{EF} + L_P^{EF} \bigg/ \hbar + J_i B_i + \frac{\bar{g} e^T}{\hbar} (h j_3 - V(B_1)) + \frac{L_{int}^{(S)}}{\hbar} \right] \right]$$

(95)

In (95) EF means that the Polyakov action

$$L_P = -\frac{\hbar}{96\pi} \int_x \int_y \sqrt{-g} R_x \Box_{xy}^{-1} R_y .$$

(96)

is to be understood to depend on the (source-dependent!) metric (94):

$$L_P(E_1^+, E_1^-) = -\frac{\hbar}{96\pi} \int_x \int_y (\partial_0^2 E_1^- - \Gamma \ln E_1^+) x \Gamma_{xy}^{-1} (\partial_0^2 E_1^- - \Gamma \ln E_1^+) y$$

$$\Gamma = \partial_t \partial_0 - \partial_0 E_1^- \partial_0$$

(97)

It obviously does not possess the simple form of the conformal gauge. The factorization of the contribution from the Polyakov term in (95) means that this one-loop contribution from the scalars appears disconnected from the propagator part of $O(Q^2)$. (95) is a function of external sources and $f$. But $f$ cancels in the propagator and thus resides in the Polyakov term only. We may rewrite (95) as

$$\tilde{W}(f, j, J, Q) = \exp \left( \frac{i}{\hbar} L_{int}^{(S)} \left( \frac{1}{i} \frac{\delta}{\delta Q} \right) \right) \tilde{W}_{prop} \tilde{W}_0(j, J) \exp \left[ \frac{i}{\hbar} \int L_P^{EF} \right] .$$

(98)

We recall that $L_{int}^{(S)}$ is obtained from (34) by keeping the terms of order $S^4$ and higher, and $S$ is replaced by the functional derivative $\frac{1}{i} \frac{\delta}{\delta Q}$. The other
factors in (98) are

\[ -i \ln \tilde{W}_0(j, J) = \int \left[ J_i B_i + \frac{\tilde{g}}{h} e^{T_i(B_1)} (h j_3 - V(B_1)) \right] \]

(99)

\[ -i \ln \tilde{W}_{\text{prop}}(j, J, Q) = \frac{\hbar}{2} \int_x \int_y Q_x \triangle^{-1}_{xy} Q_y \]

(100)

As a final step in approach a) \( \tilde{W} \) has to be integrated with a \( \delta \)-function as in (14),

\[ W = \int (df) \delta(f - \frac{\delta}{i \delta j_3}) \tilde{W} = \int (df) \delta(f - \frac{1}{i W \delta j_3}) . \]

(101)

Using (98) with (99) we may write

\[ \frac{1}{i W \delta j_3} \delta \tilde{W} = f^{(0)} + h Y(f, \text{sources}) \]

(102)

where

\[ f^{(0)} = \tilde{g} e^{T_i(B_1, B_2)} \]

(103)

represents the “background” value of \( q_3 = e_1^+ \) consisting of the classical background together with sources \( j_1, j_2 \) for \( e_1^- \) and \( \omega_1 \) in \( B_1, B_2 \). \( h Y \) is the remainder, being of higher order in \( \hbar \). The \( \delta \)-function in (101) is solved by some \( f = \hat{f} \) obeying

\[ f = f^{(0)} + h Y(f, \text{sources}) . \]

(104)

In general no exact solution of this equation is available. Thus an iterative solution must be sought. The most obvious one would consist in an expansion in \( \hbar \). To lowest order in \( \hbar \) we would have \( f = f^{(0)} + h f^{(1)} + \ldots \) with \( f^{(1)} = Y(f^{(0)}, \ldots) \). Possibly one may be able to do better by including some part of \( Y \) already to lowest order. This would correspond in spirit to the semiclassical approach which uses the Polyakov action as a contribution to the classical one (cf. e.g. the fourth and fifth reference in [3]). When such a (approximate) solution \( f = \hat{f} \) to the vanishing argument of the \( \delta \)-function in (101) has been
obtained it yields

\[
\ln W = -\ln \det \left[ \delta_{xx'} - \left( \frac{\delta}{\delta f(x')} \frac{1}{iW} \frac{\delta W}{\delta j_3(x)} \right)_{f=f} \right] \tag{105}
\]

\[
= -\int_x \ln \left[ 1 - \left( \frac{\delta}{\delta f(x')} \frac{1}{iW} \frac{\delta W}{\delta j_3(x)} \right)_{f=f} \right] \tag{106}
\]

This is true whether \( \hat{f} \) is known exactly or perturbatively. Of course, in a perturbative expansion in \( \hbar \) (or in some other “small” parameter) the determinant (or the log) can be expanded as well. E.g. to lowest nontrivial (first) order in \( \hbar \) (106) becomes

\[
\ln W \equiv \left( -\int_x \frac{1}{W^2} \frac{\delta W}{\delta f(x')} \frac{1}{i} \frac{\delta W}{\delta j_3(x)} + \int_x \frac{1}{W} \frac{\delta^2 W}{\delta j_3(x) \delta f(x)} \right)_{f=f^{(0)}} \tag{107}
\]

The first term in (107) can be interpreted as the coupling of the external field \( f^{(0)} \) to \( e_1^+ \). The second one is a “self-loop” contribution.

Clearly the necessity of an iterative (perturbative) solution for \( f \) in approach a) conceals our program to start from an exact solution of the geometric part: It is apriori unclear whether this expansion is related to the one in scalar loops. Therefore, we now turn to approach b). The generating functional (32) after integrating over \( q_i \) and \( p_i \) becomes

\[
\tilde{W} = \int (dS) \exp \frac{i}{\hbar} \int d^2 x \tilde{\mathcal{L}}^{\text{eff}}_{(3)} \tag{108}
\]

where

\[
\tilde{\mathcal{L}}^{\text{eff}}_{(3)} = \hbar (J_1 \hat{B}_1 + J_2 \hat{B}_2 + J_3 \tilde{\hat{B}}_3 + SQ) + (\partial_0 S)(\partial_1 S) + \tilde{\mathcal{L}}^{\text{HK}} \tag{109}
\]

According to (32) and (38) the ghost contribution \( \tilde{t} \) from \( \hat{B}_3 \) as well as from \( \tilde{\mathcal{L}}^{\text{HK}} \) still appear linearly. Therefore, the identity (19) may be simply used backwards:

\[
\tilde{W} = \int (dS) \sqrt{\det E^+_1} \exp \frac{i}{\hbar} \int d^3 x \tilde{\mathcal{L}}^{\text{eff}}_{(3)} \tag{110}
\]
$\mathcal{L}_{\text{eff}}^{(3)}$ is precisely the expression (34) again and in the measure

$$E_1^+ = h e^{-\hat{T} \nabla_0^{-1} \hat{T}} J_3 + \tilde{g} e^{\hat{T}}$$

(111)

even at $J_3 = 0$ shows a dependence on the scalar field $S$. Thus $S$ itself influences the measure in the path integral and backreaction is fully taken into account. Of course, (110) with (111) cannot be integrated exactly. But we are interested in a loop expansion for $S$ only. Therefore, from the determinant in (110) as well as from $\mathcal{L}_{\text{eff}}^{(3)}$ terms $O(S^2)$ will contribute together to a Polyakov type action and to a propagator for $S$. Again higher order terms $O(S^{2n})$, $n > 2$, may be collected in an interaction Lagrangian where $S$ is replaced by $\frac{1}{i} \delta / \delta Q$ as in (98).

The expansion for $\mathcal{L}_{\text{eff}}^{(3)}$ has been given already above. The additional contributions from $\sqrt{\det E_1^+}$ are determined by (for simplicity we restrict $J_3 = 0$) in (111)

$$\sqrt{\det E_1^+} = \sqrt{\det e^{T}} \exp \left[ - \frac{i}{2} \int_x \int_y G_{yx} (\partial_0 S)_x^2 + \ldots \right]$$

(112)

Thus the classical background enters – beside the sources $j_i$ (cf. eqs. (86) and (87)) – as expected, to lowest loop order in the proper definition of the $S$-measure. On the other hand, for the generalized Polyakov term (with $E_1^+ = e^{T}$), for the propagator of the scalar $S$ and for scalar vertices additional terms will emerge.

The difference between the $S^2$ terms in the two approaches a) and b) comes from the exponential (112). The contribution from (112) is of order $\hbar^0$, compared to the order $\hbar^{-1}$ contribution from $\mathcal{L}_{\text{eff}}^{(3)}$. In approach a) a term like (112) appears as a one-loop effect (tadpole). To lowest order in $\hbar$, which is considered below, the propagators for the scalar coincide in both approaches. Comparing the two approaches, b) clearly has the advantage of being formally exact, avoiding the iterative solution in the geometric variable $e_1^+ = \sqrt{-g}$ of (104) whose relation to the expansion in scalar loops is not obvious. We must admit though that there is always a danger involved in manipulating the measure for the $S$-integral. One verifies easily that the formula (97) for the Polyakov action from the $S$-integral is not consistent with arbitrarily extracting (part of) the (background) measure before the integral is done. Therefore case b) and the situation with a general background field $f$ (case a)) superficially seem to be different. Another important remark concerns a
necessary ultra violet regularization of the scalar propagator. In approach a) it sometimes may be convenient to use an $f$-dependent regularization. For example, this might be a large eigenvalue cut off for an $f$-dependent differential operator. Hence, though the first two terms in (98) formally do not depend on $f$ one cannot pull them out of the $f$-integral.

5 Effective scalar theory

After integrating out all geometric degrees of freedom we are left with an effective theory of the scalar field $S$ with non-local self-interaction. In this section we take a step back from our somewhat involved quantum expressions and show that again our lowest order contributions reproduce their classical counterparts. We also derive the effective $S^4$ vertices for spherically reduced gravity (SRG).

5.1 Effective propagator of scalars

The propagator $[\sqrt{-g\Box}]_{xx'}^{-1} = \Delta^{-1}_{xx'}$ in (100) only depends on $E^{-1}_1$. An exact evaluation (to this order in the matter fields!) still requires the solution of

$$2\partial_0 (\partial_1 - E^{-1}_1 \partial_0) \Delta^{-1}_{xx'} = \delta^2(x - x')$$

which may be reduced to finding the inverse of $\vartheta = \partial_1 - E^{-1}_1 \partial_0$ in

$$\Delta^{-1}_{xx'} = \frac{1}{2} \int_{x''} \vartheta^{-1}_{x''x} \nabla^{-1}_{0x''x'}.$$

As noted in (16) we may write

$$\vartheta^{-1}_{xx''} = P^{-1}_1 \nabla^{-1}_1 P$$

$$P = \mathcal{P} \exp \left( - \int_{x'} \nabla_{1xx'} E^{-1}_1 (x'^1, x'^0) \partial_0 \right)$$

where $P$ contains the path ordering $\mathcal{P}$ and is local in the overall variable $x^1$. Whenever $W$ is used to calculate correlation functions after (functional) differentiations with respect to the sources in $E^{-1}_1$, those sources are set to
zero. Therefore, the relevant \( E_1^- \) for such computations is \( (\tilde{g}' = \tilde{g} + O(\mu) \) in the second term)

\[
E_1^- \big|_{j=J=0} = E_1^{-(0)} = - \int_{y} \tilde{g} e^{T(0)y} (V^{(0)} + U^{(0)}V^{(0)}) y \nabla_{y_{0x_1}}^{-2}.
\]

(117)

In (117) the index \( (0) \) indicates a dependence on the “classical” solution \( B_1^{(0)}, B_2^{(0)} \) of section 3. The first term on the r.h.s. of (117) then indeed (after the integrations implied by \( \nabla^{-2}_{0x_1} \)) coincides with the classical solution of \( q_2 = e_1^- \).

Higher order scalar vertices can be obtained in the same way by straightforward calculations.

5.2 Effective scalar interaction in spherically reduced gravity

Certainly the most interesting case among 2D dilaton gravities is SRG, which corresponds to \( U_{SRG}(X) = -(2X)^{-1} \) and \( V_{SRG}(X) = -2 \). Here the previous formulae for the effective interaction vertices must be modified. Due to the singularity in \( U(X) \) at \( X = 0 \) the right hand side of (117) become divergent. One may try to introduce definitions of inverse derivatives valid on the half line \( \mathbb{R}^+ \) (see, e.g. [25]). This is not likely to work either, because higher vertices involve higher derivatives of \( U \), which are more and more singular at \( X = 0 \) and cannot be made square integrable with any reasonable weight function. However, as we demonstrate below, all integration constants in \( \nabla_{0x_1}^{-1} \) can be recovered if one takes scalar fields to be localized at certain points \( x_1^0, \ldots, x_n^0 \) in the “time” variable and compares the geometric fields in the “past” with known empty space classical solutions. Thus a kind of causality condition will be used.

We restrict ourselves to approach a). Since this approach does not mix different orders in \( \hbar \), it is easier to handle in the present context. We consider first the general case before specializing to SRG.

The effective vertex of order \( 2(n+1) \) has the form:

\[
\int dx_1 \ldots dx_{n+1} S^{2(n+1)}(x_1, \ldots, x_{n+1})(\partial_0 S)^2(x_1) \ldots (\partial_0 S)^2(x_{n+1})
\]

(118)

We consider the vertex for vanishing sources \( j_i \) only. Then in approach a) \( S^{2(n+1)} \) is given by the \( (n+1) \)th functional derivative of \( \mathcal{L}^{HK} \) with respect
to \( j_2 \) because \((\partial_0 S)^2\) enters in the combination \([h j_2 - (\partial_0 S)^2]\):

\[
S^{2(n+1)} = (-\hbar)^{-n-1} \frac{1}{(n+1)!} \frac{\delta^{n+1}}{\delta j_2^{n+1}} \mathcal{L}^{HK} |_{j=0}
\]  

(119)

For vanishing sources \( j_1 = j_3 = 0 \), \( T \) in \( \mathcal{L}^{HK} \) can be expressed in a similar way as (53)

\[
T = \int_{y_0}^{B_1} U(y) dy ,
\]

(120)

where \( B_1 = B_1(j_2) \) only and thus according to (25), (26) \( \nabla_0 B_1 = B_2 \). Here the arbitrariness in the choice of the inverse derivative \( \nabla_0^{-1} \) in (28) is taken into account by an, at first, arbitrary lower limit of integration. In the limit of a vanishing \( \mu \), the IR regulator, \( y_0 \) can depend on \( x^1 \) only, and, therefore, does not depend on sources. For simplicity, we put \( \tilde{g} = 1 \). Let us evaluate the first derivative in (119) explicitly:

\[
S^{2(n+1)} = (-\hbar)^{-n} \frac{1}{(n+1)!} \frac{\delta^n}{\delta j_2^n} E_1^- |_{j=0}
\]

\[
E_1^- = -\nabla_0^{-2} (e^T (V' + UV))
\]

(121)

As a next step, we embed the \( E_1^- \) defined in (121) into a classical system described by the equations of motion (39)-(41), (45)-(47) in the presence of an external source \( j_2 \). We do not need the constraint equations (42)-(44) here. In the presence of \( j_2 \) only the equation (10) is modified:

\[
\dot{p}_2 = \hbar j_2
\]

(122)

We simply assume that the equations (39)-(41) are satisfied and use them to define the functions \( p_1 \), \( p_2 \) and \( p_3 \). Note, that these equations were found to hold if they are interpreted as equations for expectation values of corresponding quantum fields in the presence of external sources, i.e. only \( j_2 \) here. The rest of the equations of motion (45)-(47) yields

\[
\partial_0^2 q_2 = -q_3 (V' + UV)
\]

\[
q_3 = e^T ,
\]

(123)

where integration constants in the definition of \( q_3 \) are encoded again in the lower limit \( y_0 \) of the integral (120). Hence, \( E_1^- (j_2) \) can be identified with
a classical solution for $q_2$. Fixing the ambiguity in the definition of $\nabla_0^2$ is equivalent to fixing integration constants in the classical field equation (123). As we shall demonstrate below, this is equivalent to choosing asymptotics of the classical gravitational background.

To obtain the $n$th functional derivative of $E_1^-$ it is sufficient to take $j_2$ localized at $n$ different points:

$$j_2(x) = -\sum_{k=1}^{n} c_k \delta(\bar{y}_k - x)$$  \hspace{1cm} (124)

then we can expand $E_1^-(j_2, x)$ in a power series of $c_k$. In the resulting sum the coefficient of the term with $(-1)^n \prod_{k=1}^{n} c_k$ will give the desired functional derivative. Indeed, consider the expansion of $E_1^-$

$$E_1^- = \sum_n \frac{1}{n!} \int dx_1 \ldots dx_n E^n(x_1, \ldots, x_n) j_2(x_1) \ldots j_2(x_n).$$  \hspace{1cm} (125)

For $j_2$ given by the equation (124), we have

$$E_1^-(j_2) = (-1)^n \prod_{k=1}^{n} c_k E^n(\bar{y}_1, \ldots, \bar{y}_n) + \text{other terms}. \hspace{1cm} (126)$$

We must now fix a particular solution $E_1^-(j_2, x)$. This is done by some “causality” condition. For $x^0$ in the asymptotic region, $x^0 > \bar{y}_k^0$, $k = 1, \ldots, n$, the function $E_1^-(j_2, x)$ must coincide with a fixed vacuum solution $E_1^-(0, x)$. Given a vacuum solution $E_1^-(0, x)$, the function $E_1^-(j_2, x)$ is uniquely defined and non-singular.

As demonstrated above and at the end of Appendix B, by using the residual gauge freedom and after removing the regularization one can choose the vacuum values $p_2(j = 0, x) = \bar{p}_2 = 1$, $p_1(j = 0, x) = x^0$. For $q_3$ we have from (123) for $U_{SRG}$ and $V_{SRG}$

$$q_3 = e_1^+ = \exp \int_{y_0}^{p_1} \frac{dy}{-2y} = \sqrt{\frac{y_0}{p_1}}$$  \hspace{1cm} (127)

where $y_0$ contributes to an irrelevant scale factor. We put $y_0 = 1$. In the absence of sources this amounts just to $e_1^+ = \sqrt{1/x^0}$. The equation for $q_2$ yields

$$q_2 = e_1^- (0, x) = 4\sqrt{x^0} - m_\infty + bx^0$$  \hspace{1cm} (128)
Nonzero values of $b$ correspond to Rindler coordinates referring to an uniformly accelerated frame. Therefore we set $b = 0$. The effective metric now reads
\begin{equation}
    ds^2 = \frac{2dx^0 dx^1}{\sqrt{x^0}} + 2 \left( 4 - \frac{m_\infty}{\sqrt{x^0}} \right) (dx^1)^2 . \tag{129}
\end{equation}
After a change of the coordinate $z = \sqrt{x^0}$ the metric (129) becomes
\begin{equation}
    ds^2 = 4dzdx^1 + 2 \left( 4 - \frac{m_\infty}{z} \right) (dx^1)^2 , \tag{130}
\end{equation}
which, up to trivial numerical factors, coincides with the Schwarzschild black hole in Eddington–Finkelstein coordinates with the 'radial' variable $z$ and $m_\infty$ proportional to the mass of the black hole.

With $j_2$ defined in (124) the classical equations of motion have unique solutions with given asymptotics for large $x^0$:
\begin{align*}
    p_2(x) &= 1 + \hbar \sum_k c_k \theta(\bar{y}_k^0 - x^0) \\
    p_1(x) &= x^0 + \hbar \sum_k c_k (\bar{y}_k^0 - x^0) \theta(\bar{y}_k^0 - x^0) \tag{131}
\end{align*}
The solution for $q_3$ in SRG is given again by (127). Proper asymptotic behavior is achieved by taking $y_0 = 1$.

Consider the quartic scalar interaction ($n = 1$) in (118) for SRG. For $x^0 > \bar{y}^0$ the function $E_1^-(j_2, x)$ must be a solution of (123) coinciding with (128). For $x^0 < \bar{y}^0$ we have:
\begin{equation}
    E_1^- = \frac{4}{(1 + \hbar c_1)^2} \sqrt{(1 + \hbar c_1)x^0 - \hbar c_1 \bar{y}^0 - m_1 - a_1 x^0} \tag{132}
\end{equation}
where the integration constants $m_1$ and $a_1$ are defined by the requirement that $E_1^-$ and its first derivative are continuous at $x^0 = \bar{y}^0$. Since we are interested in terms which are linear in $c_1$ only, we arrive at
\begin{equation}
    m_\infty - m_1 = 6\hbar c_1 \sqrt{\bar{y}^0}, \quad a_1 = -\frac{2\hbar c_1}{\sqrt{\bar{y}^0}} \tag{133}
\end{equation}
In this linear order in $c_1$ we also have:
\begin{equation}
    E_1^- = -\hbar c_1 \left( 6\sqrt{x^0} - 6\sqrt{\bar{y}^0} + 2 \left[ \frac{\bar{y}^0}{\sqrt{x^0}} - \frac{x^0}{\sqrt{\bar{y}^0}} \right] \right) \tag{134}
\end{equation}
This yields the vertex function:

$$S^4(x, \bar{y}) = \frac{1}{2} \left( 6\sqrt{x^0} - 6\sqrt{\bar{y}^0} + 2 \left[ \frac{\bar{y}^0}{\sqrt{x^0}} - \frac{x^0}{\sqrt{\bar{y}^0}} \right] \right) \theta(\bar{y}^0 - x^0) \quad (135)$$

The interaction (135) must be local in the $x^1$ coordinate. A corresponding $\delta$-function is not written explicitly in (135).

Note, that in the presence of matter fields the constraint equations are modified. Therefore, the equations (42) - (44) with zero right hand sides are not satisfied here.

6 Summary and Outlook

The central result of our present paper is that for all 2d covariant theories a quantum field theory can be formulated which treats the geometric part (Cartan variables or metric) exactly. Interactions with matter are included in a systematic manner by a loop expansion which automatically takes into account backreaction, order by order. The crucial ingredient of our successful approach is the use of a light cone gauge for Cartan variables which amounts to an Eddington-Finkelstein gauge for the metric.

In that gauge the remaining geometric variables appear linearly in the action of the path integral. We found that the proper covariant choice of the measure for the scalar fields for general dilaton theories can also be transformed into a similar linear contribution, either by introducing an auxiliary scalar field or by the introduction of ghost fields. Both types of fields are later integrated out again.

This linearity in the geometric variables produces delta-functions for the associated momenta so that the formal path integral in phase space - to zero loop order in the matter fields - reduces to the classical solution. Matter fields produce from their Gaussian integral a (generalized) Polyakov action, depending on external sources. Higher order vertices of the scalars can be computed systematically.

Our present work generalizes ref. [16] in an essential manner. There we did not include yet dilaton theories with kinetic term for the dilaton field. But only the latter theories contain spherically reduced gravity, i.e. the interaction of black holes with S-wave matter, as a special case.

The main purpose of our paper has been to set up the general structure of our approach which perhaps is the first one to develop at least for some
sector of gravity theories the general consequences of an “orthodox” quantum field theory. In this respect our basic philosophy shares certain similarities with the $S$-matrix approach by 't Hooft [26]. We are fully aware of the fact that the usual difficulties of principle for pure quantum gravity (especially nonrenormalizability) cannot be swept under the rug. But here we have a well-defined framework, fixed by quantum experience in Minkowski space. Any necessary modification from the unification of gravity and quantum theory must show up in its precise relation to orthodox quantum theory. So several of the many open question left by our work may already have deeper implications. We list just a few:

Performing the path integral for the geometric variables we integrated the field variable $q_3 = e^+_1$ from $-\infty$ to $+\infty$. But in the EF gauge used here this quantity determines the volume element. Therefore we really integrated over “negative” (and vanishing) volumes as well. Geometrically this seems to be very doubtful, but from the point of view of quantum field theory zweibeine are fields to be varied over their whole range.

In Minkowski quantum field theory path integrals refer to space-time extending over the whole $R^4$. Here we find that e.g. for the Schwarzschild black hole one coordinate is cut off at the singularity given by the classical background, where the latter is not inserted “by hand” but appears naturally in this formalism. This clearly precludes the application of $S$-matrix concepts to, say, the decay of a given black hole. On the other hand, the quantum formation and disappearance of some intermediate black hole like object in the scattering matrix element of two initial scalars into two final ones may be a process to be calculated in a Minkowski background. In order to obtain theoretical information concerning the eventual self-extinguishing of a black hole by something like single quantum Hawking radiation it may be necessary to develop a sort of “quantum optics” for multiquantal states of matter fields. Again our present work should be a suitable starting point.

It has been noticed some time ago [11] that a 2d theory of gravity which allows multiply connected, topologically nontrivial, classical solutions may carry quantum fluctuations as zero modes in some compact direction. While such configurations are unlikely in the phase space of SRG they may well play a role when an additional $U(1)$ gauge field is introduced as well. From technical experience with this case [17,22] our approach should basically work as well. But then these additional quantum fluctuations - not even induced by matter! - will contribute to the geometric part of the path integral.

The explicit technical machinery has been already complicated enough in
our present work for minimally coupled scalar matter. In fact it is quite easy to see that the basic steps work equally well for SRG with the dilaton field coupled to the scalars. Only in that case the scalar (nonminimal) couplings of full SRG are properly taken into account. This will be, among other things, the object of future work.

These open questions certainly do not form an exhaustive list. In any case we believe that the range of topics to be explored, starting from our present results is not negligible.

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Appendix A: Background field formalism

In this Appendix we rederive the main result of our previous paper [13] in the framework of a background field formalism, namely that dilaton gravity without matter does not have any loop effects. We show that a suitable choice of IR regularization all anomalous terms are cancelled. Hence the path integral measure which we are using for the geometric variables is indeed gauge invariant.

Consider the first order action (3) in component notation:

\[
\mathcal{L} = e^- (\partial_0 - \omega_0) X^+ + e^+_0 (\partial_1 + \omega_1) X^+ \\
+ e^+_1 (\partial_0 + \omega_0) X^- + e^-_0 (\partial_1 - \omega_1) X^- \\
- \omega_1 \partial_0 X + \omega_0 \partial_1 X \\
+ (e^+_0 e^-_1 - e^+_1 e^-_0) (V(X) + X^+ X^- U(X)).
\]

Before introducing background fields we determine the gauge symmetries of the action (136). The simplest way to derive them is to use the canonical formalism. The canonical coordinates are taken to be \( q_i \) and the corresponding canonical momenta \( p^i \) (cf. [8]). The Poisson brackets have the form

\[
\{ q_i(x^1), p^k(y^1) \} = \delta_i^k \delta, \quad \delta = \delta(x^1 - y^1). \]
For the remaining variables \( q_i = (\omega_0, e^{-0}, e^{+0}) \) the canonical momenta \( p_i \) vanish (primary constraints). The \( q \)'s generate the secondary constraints

\[
\begin{align*}
G_1 &= -p_2q_2 + p_3q_3 + \partial_1 p_1 \\
G_2 &= \partial_1 p_2 + q_1 p_2 - q_3(V(p_1) + p_2 p_3 U(p_1)) \\
G_3 &= \partial_1 p_3 - q_1 p_3 + q_2(V(p_1) + p_2 p_3 U(p_1)).
\end{align*}
\]

(138)

Their Poisson brackets are

\[
\begin{align*}
\{G_1, G_2\} &= -G_2\delta \\
\{G_1, G_3\} &= G_3\delta \\
\{G_2, G_3\} &= -[(V''(p_1) + (p_2)(p_3)U''(p_1))G_1 + \\
&\quad + (p_3)U(p_1)G_2 + (p_2)U(p_1)G_3]\delta.
\end{align*}
\]

(139)

The constraints (138) generate the gauge transformations of the action (136):

\[
\delta z(p, q) = \{z, G_i\}\xi^i, \\
\delta q^k = -\xi^k - \bar{q}^k C_{ji}^{k}\xi^j
\]

(140) (141)

where \( \xi^i \) is a parameter, \( z(p, q) \) is an arbitrary function of \( p_i \) and \( q^i \). The structure functions \( C_{ij}^{k} \) are defined through \( \{G_i, G_j\} = C_{ij}^{k}G_k\delta \) by (139). There are no ternary constraints.

In the background field formalism one should decompose all fields into quantum fluctuations and background values

\[
e \to e + E, \quad \omega \to \omega + \Omega, \quad X \to X + Y,
\]

(142)

where \( E, \Omega, Y \) denote background fields. Our gauge choice for the fluctuations is

\[
\bar{e}_0^\pm = \omega_0 = 0.
\]

(143)

Now the strategy is as follows. We subtract the classical action together with all terms linear in the fluctuations. Next we integrate over quantum fields. We anticipate the result that the effective action will be just classical one. Hence it does not generate any tadpole graphs. Otherwise, the following
result would be true at one loop only. From (142) and (143) the action (136) is replaced by

\[ \mathcal{L} \rightarrow e_1^-(\partial_0 - \Omega_0)X^+ + E_0^-\omega_1X^+ \\
+ e_1^+(\partial_0 + \Omega_0)X^- - E_0^+\omega_1X^- - \omega_1\partial_0X \]

(144)

\[ +(E_0^+ E_1^- - E_1^+ E_0^-)V_2 + (E_0^+ e_1^- - E_1^+ e_0^-)V_1 , \]

\[ V_1 = V(X + Y) + U(X + Y)(X^+ + Y^+)(X^- + Y^-) - V(Y) - U(Y)Y^+Y^- , \]

\[ V_2 = V_1 - (V'(Y) + U'(Y)Y^+Y^-)X - U(Y)(X^+Y^- + X^-Y^+). \]

Integration over \( \omega_1, e_1^- \) and \( e_1^+ \) gives delta functions:

\[ \delta_{\omega_1} = \delta(-\partial_0 X + E_0^- X^+ - E_0^+ X^-) \]

\[ \delta_{e_1^-} = \delta(-\partial_0 X^+ - \Omega_0 X^+ + E_0^+ V_1) \]

\[ \delta_{e_1^+} = \delta(-\partial_0 X^- + \Omega_0 X^- - E_0^- V_1) \]

(145)

For generic values of the background fields \( E, \Omega \) the only solution with proper (vanishing) asymptotics is

\[ X^\pm = X = 0. \]

(146)

Hence there are no quantum corrections except for two Jacobian factors. The first one, which appears due to integration over \( X \)'s in the delta functions becomes

\[ J = \det^{-1}(-\partial_0 \delta_i^k + M_i^k) , \]

(147)

\[ M = \begin{pmatrix} 0 & E_0^- & -E_0^+ \\
E_0^+(V'(Y) + U'(Y)Y^+Y^-) & -\Omega_0 + E_0^+ U(Y)Y^- & E_0^+ U(Y)Y^+ \\
-E_0^-(V'(Y) + U'(Y)Y^+Y^-) & -E_0^- U(Y)Y^- & \Omega_0 - E_0^- U(Y)Y^+ \end{pmatrix} . \]

(148)

The second Jacobian factor is the Faddeev–Popov determinant.

Taking (146) into account, one can easily find linearized gauge transformations

\[ \delta_{\omega_1} = -\dot{\xi}^1 - E_0^+(V'(Y) + Y^+Y^-U'(Y))\xi^2 + E_0^-(V'(Y) + Y^+Y^-U'(Y))\xi^3 , \]

\[ \delta_{e_1^-} = -\dot{\xi}^2 - E_0^-\xi^1 + \Omega_0\xi^2 - E_0^+Y^-U(Y)\xi^2 + E_0^-Y^-U(Y)\xi^3 , \]

\[ \delta_{e_1^+} = -\dot{\xi}^3 + E_0^+\xi^1 - E_0^+Y^+U(Y)\xi^2 + E_0^-Y^+U(Y)\xi^3 - \Omega_0\xi^3 . \]
This immediately gives us the Faddeev–Popov determinant

\[ J_{FP} = \det(\delta \bar{q}^i / \delta \xi^k) = \det(-\partial_0 \delta_i^k - M_i^k) \]  \hspace{1cm} (149)

Notice that the arguments of the determinants in (147) and (149) differ only by the sign in front of the matrix \( M_i^k \). Introducing ghost fields \( \bar{c}_i, c_k \) this determinant is generated by the ghost action

\[ \int (\bar{D}\bar{c})(Dc) \exp \left[ i \int_M d^2x (-\bar{c}^i(-\partial_0 \delta_i^k - M_i^k)c_k) \right] \]  \hspace{1cm} (150)

To evaluate the determinants (147) and (149) an IR regularization is needed. According to our prescription \( \partial_0 \) must be replaced by \( \nabla_0 \) in (147). Consider

\[ -\ln J = \ln \det(-\nabla_0 \delta_i^k + M_i^k) = \ln \det(-\nabla_0) + \sum_{n=1} \text{Tr} \left( \nabla_0^{-1} M \right)^n \frac{1}{n} \]  \hspace{1cm} (151)

The first term in (151) is independent of background fields and will be neglected in what follows. The remaining terms are

\[ \text{Tr}(\nabla_0^{-1} M)^n = \int dx_1 \ldots dx_n \text{tr} \left[ (\nabla_0^1)_{x_n x_1} M(x_1) \ldots (\nabla_0^1)_{x_{n-1} x_n} M(x_n) \right] \]  \hspace{1cm} (152)

Due to the presence of the step function \( x_n \leq x_1 \leq \ldots \leq x_n \) in \( \nabla_0^{-1} \) only coinciding points contribute to (152). Hence we get

\[ \text{tr} \int dx \ln(1 + \theta(0)M(x)) \]  \hspace{1cm} (153)

The term (153) is not gauge invariant and thus would lead to diffeomorphism and Lorentz anomalies. However we can use a freedom in choosing the IR regularization of the ghost action (150). The proper choice is to replace \( \partial_0 \) by \( \tilde{\nabla}_0 \). With this regularization we have

\[ \ln J_{FP} = -\ln J \]  \hspace{1cm} (154)

and find that all anomalous contributions to the effective action have cancelled.
Appendix B: Regularized Inverse Derivatives

In the preceding sections we frequently encountered inverse derivative operators. Here we shall define a proper infrared regularization scheme and list the corresponding calculation rules which where used in the main text. We restrict ourselves here to the case $x^0 \in \mathbb{R}$. Two regularized Green functions $\nabla^{-1}_{0x'}$ and $\tilde{\nabla}^{-1}_{0x'}$ are introduced to replace $\partial^{-1}_{0}$ as

$$\partial^{-1}_{0} \to \begin{cases} \lim_{\mu \to 0} (\partial_{0} - i\mu)^{-1} = \lim_{\mu \to 0} (\nabla^{-1}_{0}) \\ \lim_{\mu \to 0} (\partial_{0} + i\mu)^{-1} = \lim_{\mu \to 0} (\tilde{\nabla}^{-1}_{0}) \end{cases} \tag{155}$$

where $\mu = \mu_{0} - i\varepsilon$. $\mu_{0} \to +0$ represents the IR regularization, proper asymptotic behavior (cf. (156), (156) below) is provided by $\varepsilon \to +0$. Note that a partial integration transforms $\nabla^{-1}_{0}$ into $\tilde{\nabla}^{-1}_{0}$ and also that $\tilde{\nabla}^{-1}_{0}$ is not the complex conjugate of $\nabla^{-1}_{0}$. The inverse operators are defined as the Green functions $\nabla_{0}$ and $\tilde{\nabla}_{0}$ and are calculated straightforwardly:

$$\left(\nabla^{-1}_{0}\right)_{xy} = -\theta(y - x)e^{i\mu(x - y)}$$

$$\left(\tilde{\nabla}^{-1}_{0}\right)_{xy} = \theta(x - y)e^{-i\mu(x - y)} \tag{156}$$

$\theta$ denotes the step function. The inverse squared operators are defined as the Green functions of $(\nabla_{0})^{2}$ and $(\tilde{\nabla}_{0})^{2}$ and are given by

$$\left(\nabla^{-2}_{0}\right)_{x,y} = (y - x)\theta(y - x)e^{i\mu(x - y)}$$

$$\left(\tilde{\nabla}^{-2}_{0}\right)_{x,y} = (x - y)\theta(x - y)e^{-i\mu(x - y)} \tag{157}$$

Using (156) to (157) the following identities may be verified easily:

$$\nabla^{-1}_{0}\nabla^{-1}_{0} = \nabla^{-2}_{0} \quad \tilde{\nabla}^{-1}_{0}\tilde{\nabla}^{-1}_{0} = \tilde{\nabla}^{-2}_{0}$$

$$\nabla_{0}\nabla^{-2}_{0} = \nabla^{-1}_{0} \quad \tilde{\nabla}_{0}\tilde{\nabla}^{-2}_{0} = \tilde{\nabla}^{-1}_{0}$$

$$\nabla_{0}\tilde{\nabla}^{-2}_{0} = \tilde{\nabla}^{-1}_{0} - 2i\mu\tilde{\nabla}^{-2}_{0} \quad \tilde{\nabla}_{0}\nabla^{-2}_{0} = \nabla^{-1}_{0} + 2i\mu\nabla^{-2}_{0} \tag{158}$$

$$\nabla_{0}\tilde{\nabla}^{-1}_{0} = \delta(x - y) - 2i\mu\tilde{\nabla}^{-1}_{0} \quad \tilde{\nabla}_{0}\nabla^{-1}_{0} = \delta(x - y) + 2i\mu\nabla^{-1}_{0}$$

$$\nabla^{-2}_{0xy} = \nabla^{-2}_{0}(-x)(-y) = \tilde{\nabla}^{-2}_{0yx}$$

$$\nabla^{-1}_{0xy} = -\tilde{\nabla}^{-1}_{0yx} \tag{158}$$

Note that in the main text only these types of operations appeared and therefore the limit $\mu \to 0$ does not cause any obviously divergent expressions, because $\mu$ does not appear with negative powers. Therefore this regularization
scheme seems much superior to the one used in [10]. Nevertheless, some care is necessary at $\mu \to 0$ in special cases (see below).

This regularization introduces an IR-cutoff and thus eliminates possible global quantum fluctuations which may occur in certain backgrounds. For the classical solutions which enter our formulas to zero order no such regularization is necessary. Therefore in that case a formal replacement

$$\int \nabla_{0xy}^{-1} \bar{F}(y^1)dx^0dx^1 \to x^0 \bar{F}(y^1) + \bar{G}(y^1) := \bar{x}^0$$ (159)

is possible. Using the residual gauge transformations of the EF gauge the complete r.h.s of (159) may be even identified with a new coordinate $\bar{x}^0$. In quantum expressions we can expect (159) to be true only for a restricted range of $x^0$ (or $\bar{x}^0$). A naive application of (156) e.g. in the expression for $B_1^{(0)} (\nabla_0 \bar{p}_{1,2} = 0)$

$$B_1^{(0)} = \bar{p}_1 + \nabla_0^{-1} \bar{p}_2$$ (160)

for $\bar{p}_{1,2} = \hat{p}_{1,2}(x^1)e^{i\bar{\mu}x^0}$ indeed yields an ill-defined (divergent) result even before the regularization is removed. This can be circumvented by introducing a different $\bar{\mu}' \to \mu' - i\varepsilon'$ for homogenous solutions as

$$\bar{p}_{1,2} = \hat{p}_{1,2}(x^1)e^{i\bar{\mu}'x^0}.$$ (161)

Now (160) is well-defined. Taking the limits in the sequence $\varepsilon' \to 0$, $\varepsilon \to 0$, $\mu \to 0$ for small $x^0$ then leads to the desired expression of type (159) if a factor $\frac{1}{\mu'}$ is absorbed in $\hat{p}_1$.

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