Abstract—This paper thoroughly explored the fundamental interactions between cell association, cell load and throughput in a green (energy-efficient) small cell network in which all base stations form a homogeneous Poisson point process (PPP) of intensity $\lambda_B$ and all users form another independent PPP of intensity $\lambda_U$. Cell voidness, usually disregarded due to rarity in cellular network modeling, is first theoretically analyzed under generalized (channel-aware) cell association (GCA). We showed that the void cell probability cannot be neglected any more since it is bounded above by $\exp(-\lambda_U/\lambda_B)$ that is typically not small in a small cell network. The accurate expression of the void cell probability for GCA was characterized and it was used to derive the average cell and user throughputs. We learned that cell association and cell load $\lambda_U/\lambda_B$ significantly affect these two throughputs. According to the average cell and user throughputs, the green cell and user throughputs are defined respectively to reflect whether the energy of a base station is efficiently used to transmit information or not. In order to achieve satisfactory throughput with certain level of greenness, cell load should be properly determined. We presented the theoretical solutions of the optimal cell loads that maximize the green cell and user throughputs, respectively, and verified their correctness by simulation.

I. INTRODUCTION

In recent years, we have witnessed a trend that mobile devices, such as smart phones and tablets, have been proliferating and relentlessly penetrating our daily life. Such powerful handsets have created a new dimension of transmitting information over newly developed wireless technologies. Accordingly, this situation has urged customers to expect more and more throughput to suffice their versatile wireless demands. To boost network throughput, one of the most effective approaches is to increase the spatial frequency reuse by deploying more cells in a given area so that there are fewer users sharing time and/or frequency resources in a cell [1]. Furthermore, as cell intensity increases, users are able to connect to a closer base station and thus desired signals suffer less path loss. Therefore, a small cell network consisting of highly dense picocells and/or femtocells has gained much attention since small cell base stations (BSs) can be deployed indefinitely and the throughput they achieve seems not to have a hard limit.

Migrating the topology of traditional macro cells to the one of small cells is the trend for the next generation of cellular systems since a small cell BS has much less power consumption than a traditional macro cell BS whose power consumption typically accounts for 80% of the total power consumption of a cellular system [2]. Cellular networks with a small cell topology absolutely help the entire information and communication technology (ICT) industry lessen CO$_2$ emission, which is estimated to account for about 6% of the global emission in 2020 [3]. Small cell BSs also have an advantage of low operational cost and hardware complexity. For example, they can quickly switch their different power control modes for saving power [4]. Due to dense deployment, however, small cell networks seriously suffer the void cell issue that is hardly noticed in the traditional macro BS network. The void cell problem originates user-centric cell association [5] and it could give rise to no users in a cell. To explain this void cell phenomenon, consider there are $m \times n$ users uniformly distributed in a unit-area network which is tessellated by $n$ equal-sized cells of BSs and thus the user intensity is $m \times n$ and the BS intensity is $n$. The probability of having no users in a cell is $(1 - \frac{1}{n})^{m/n}$ which is close to $e^{-m}$ as $n$ is very large. This tells us that the void cell probability is not negligible at any more in a small cell network because such a network usually has a small $m$, i.e., the ratio of the user intensity to the BS intensity is not large.

A. Motivation and Prior Work

Without modeling the impact of void cells, the analyses of network performance metrics, e.g., coverage probability and average throughput, are apparently underestimated since the void BSs do not contribute any interference. Despite the important phenomenon of void cells, there are very few works that study and model the effect of void cells. Almost all the prior works on cell association in cellular networks overlook this problem [5]–[10]. Although references [11]–[13] did consider the void cell impact on their models, the void cell probability they used is only valid for nearest BS association.

Since cell voidness essentially cannot be completely avoided and small cell BSs can easily switch their power control mode, the void BSs can be put in a “dormant” mode for saving power. Such energy-saving strategy entails another question, that is, how much denseness of BS deployment a small cell network needs to optimally exploit the performance of the energy-saving strategy. A very much dense small cell network is absolutely not green (energy-efficient) because its per-unit power throughput is significantly weakened due to too much interference. Previous works on designing the BS intensity

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with greenness in a heterogeneous or small cell network are still fairly minimal. The work in [14] numerically showed that there exists an optimal ratio of picocell to macro BS intensity that maximizes energy-efficient throughput without giving a rigorous theoretical explanation. In [15], a closed-form upper bound on the optimal BS intensity that satisfies some constraints on the user outage rate is found without considering the energy consumption of a BS.

The two aforementioned works did not study the optimal BS intensity from the viewpoint of an achievable green throughput. Although the average achievable throughput of a link was investigated in [5] and [16], it fails to characterize how energy is efficiently used to transported information on the per cell or per user basis. References [7] and [17] also presented some approximate average per-user achievable throughputs in a cell that are only valid for (biased) nearest BS association. In [18], an energy-efficient throughput metric was proposed based on the concept of outage capacity. None of these prior works on the average per-cell or per-user throughputs investigate the optimal cell load problem as well as consider the void cell issue in their models.

B. Contributions

To characterize the void cell probability in a small cell network with spatial randomness, in this paper we consider a single tier small cell network in which all BSs form a homogeneous Poisson point process (PPP) of intensity $\lambda_B$ and all users form another independent PPP of intensity $\lambda_U$. The fundamental interaction between user-centric cell association and void cell probability is first delved. All users associate with their serving BS by the proposed generalized cell association (GCA) scheme that is able to cover several cell association schemes, such as nearest cell association, maximum received power association, etc. We theoretically showed that the achievable lower bound on the void probability of a cell is $\exp \left( -\frac{\lambda_U}{\lambda_B} \right)$, where $\lambda_U/\lambda_B$ is termed “cell load”. An accurate closed-form expression for the void cell probability of GCA was derived under a more practical channel model that incorporates the effects of path loss, Rayleigh fading and log-normal shadowing. The derived void cell probability provides us some insights on how the void cell probability is affected by channel impairments and how its theoretical lower bound can be achieved by GCA. To the best of our knowledge, the fundamental relationship between cell association and void cell probability is first studied in this work.

Under the assumption that all the associated BSs form a thinning homogeneous PPP, the average cell and user throughputs were proposed to capture the impact of cell voidness. The average cell throughput is a per-cell average throughput metric, whereas the average user throughput is a per-user average throughput metric in a cell. The near closed-form expressions for these two throughputs parameterized by cell load $\lambda_U/\lambda_B$ were derived and they are the neatest results than any other similar throughput results in the literature. Most importantly, they directly reflect that the GCA schemes favoring large channel power can significantly benefit them. Also, we showed that there exists a unique optimal cell load that maximizes the average user throughput and it is accurately equal to the fixed point of a special function pertaining to the void cell probability.

A simple green power control, which switches a BS between active and dormant modes depending on whether the BS is void or not, is used to help the network save energy. The green cell and user throughputs were proposed and they aim to characterize how green the per-cell and per-user throughputs can be in the network. We showed that both the green cell and user throughputs can be maximized by a sole optimal cell load that can be accurately acquired by calculating the fixed point of the derived special function. Once the optimal cell load is found, the optimal BS intensity corresponding to any given user intensity is also readily obtained and deploying BSs based on this intensity can attain the maximum energy efficiency of information transportation. From these derived results, we can conclude that green power control schemes that keep large power consumption difference between active and dormant modes and cell association schemes that favor high channel power both significantly improve the optimal cell load and green throughputs.

II. System Model and Preliminaries

A. Network and Cell Association Models

We consider an infinitely large and planar small cell network in which all users form a homogeneous Poisson point process (PPP) $\Phi_U$ of intensity $\lambda_U$ whereas one-tier (small-sized) base stations that provide service to all users independently form another marked homogeneous PPP $\Phi_B$ of intensity $\lambda_B$ given by

$$\Phi_B \triangleq \{ (B_i, H_i, C_i, V_i) : B_i \in \mathbb{R}^2, H_i \in \mathbb{R}_{++}, C_i \subset \mathbb{R}^2, V_i \in \{0,1]\), \forall i \in \mathbb{N}_+ \}, \tag{1}$$

where $B_i$ denotes base station $i$ and its location, $H_i$ is used to characterize the downlink fading and shadowing channel power gain from $B_i$ to its service user, the cell region of $B_i$ is represented by $C_i$, $V_i$ is a void cell index that indicates whether or not the cell of $B_i$ has a user, i.e., whether $C_i \cap \Phi_U = \emptyset$ is true or not $V_i$ is equal to one if $C_i \cap \Phi_U \neq \emptyset$, otherwise zero. To characterize a much general statistic behavior of the channel power in the downlink, we assume that all channel power gains $H_i$’s are i.i.d. random variables and their probability intensity function (pdf) that characterizes the composite effect of Rayleigh fading and log-normal shadowing is given by [19]

$$f_H(h) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_0^\infty \frac{1}{x^2} \exp \left( -\frac{h}{x} - \left( \ln x - \mu_x \right)^2 \right) dx, \tag{2}$$

where $\mu_x$ and $\sigma^2$ are the mean and variance of log-normal shadowing, respectively. That is, the channel power gain $H$ is essentially a (composite) exponential-log-normal random variable.

Without loss of generality, our following analysis will be based on a typical user $U_0$ located at the origin. Each user associates with a base station in $\Phi_B$ by using a generalized (channel-aware) cell association (GCA) scheme, i.e., user $U_0$
This property is elaborated in the following theorem.

**Theorem 1 (Random Conservation Property of a PPP).** Suppose \( \Phi' \) is an independently marked PPP of intensity measure \( \lambda' \) on \( \mathbb{R}^d' \), which can be expressed as follows

\[
\Phi' \triangleq \{(X'_i, T_i) : X'_i \in \mathbb{R}^{d'}, T_i \in \mathbb{R}^{d \times d'}, \forall i \in \mathbb{N}_+ \},
\]

where \( X'_i \) denotes node \( i \) and its location and \( T_i : \mathbb{R}^{d'} \to \mathbb{R}^{d} \) is the non-singular mapping matrix (operator) of node \( X'_i \). For all \( i \neq j \), \( T_i \) and \( T_j \) are (element-wise) i.i.d. Let \( \hat{\Phi} \) be the mapped point process on \( \mathbb{R}^{d} \) generated by using the random mapping matrix of each node in \( \Phi' \), i.e., it is defined as

\[
\hat{\Phi} \triangleq \{(\hat{X}_i \triangleq T_i(X'_i)) : X'_i \in \Phi', T_i \in \mathbb{R}^{d \times d'}, \forall i \in \mathbb{N}_+ \}.
\]

Then \( \hat{\Phi} \) is also a PPP and for any bounded Borel set \( \mathcal{A} \subset \mathbb{R}^{d} \) its intensity measure \( \hat{\lambda} \) is given by

\[
\hat{\lambda}(\mathcal{A}) = \lambda'(A') \mathbb{E} \left[ \frac{1}{\sqrt{\text{det}(T^T T)}} \right].
\]

**Proof:** See Appendix A. \( \blacksquare \)

**Remark 1.** Theorem 1 is a generalization of the conservation property in [20] [21]. In a special case of \( d = d' = 2 \), all points in \( \Phi \) are mapped from their corresponding points in a homogeneous PPP \( \Phi' \) by scaling them with i.i.d. diagonal random matrices \( T_i = \text{diag}(T_1, T_2) \). In this case, \( \hat{\lambda} \) is equal to \( \lambda' \mathbb{E}[T^{-2}] \).

The random conservation property can significantly reduce the complexity of analyzing the statistics of some performance metrics induced by a PPP, especially a homogeneous PPP with i.i.d. marks. Hereupon the GCA scheme in (3) can be rewritten as

\[
B^*_0 = \arg \sup_{B_i \in \Phi_0} (W_i H_i \lVert B_i \rVert^{-\alpha}),
\]

where \( W_i \)'s are the i.i.d. random association weightings for BSs, \( \alpha > 2 \) is the path loss exponent and \( \lVert B_i \rVert \) denotes the Euclidean distance between \( B_i \) and the origin. Here GCA is “channel-aware” because its association weightings pertain to the channel power gain so that it can cover several BS association schemes. For example, if the channel power gain \( H_i \) is available \( W_i = 1/H_i \), GCA reduces to the popular nearest BS association. GCA becomes maximum received power association provided that all the weightings are the same constant, which is suitable for non-stationary users that cannot acquire the mean received power, e.g., they are moving very fast. GCA is essentially a “user-centric” scheme, that is, it is able to ensure every user to connect to a certain base station. No users are blocked out of the network. Nonetheless, user-centric cell association cannot, as we will show later, guarantee every cell is associated with at least one user, i.e., the probability of a void cell (a cell without users) is always bounded above zero and non-negligible, especially in a small cell network with large BS intensity. A simulation example of the void cell phenomenon in a small cell network with Voronoi tessellation is illustrated in Fig. 1.

**B. Random Conservation Property of a Homogeneous PPP**

In this subsection, we introduce the random conservation property of a homogeneous PPP, which specifies how the intensity of a homogeneous PPP is changed after all points of the PPP are transformed by i.i.d. random mapping matrices. This property is elaborated in the following theorem.

\[
\hat{\Phi} \triangleq \{X'_i : X'_i \in \mathbb{R}^{d'}, \forall i \in \mathbb{N}_+ \},
\]

where \( X'_i \) is the path loss exponent and \( X'_i \) is the non-singular mapping matrix (operator) of node \( X'_i \). For all \( i \neq j \), \( T_i \) and \( T_j \) are (element-wise) i.i.d. Let \( \hat{\Phi} \) be the mapped point process on \( \mathbb{R}^{d} \) generated by using the random mapping matrix of each node in \( \Phi' \), i.e., it is defined as

\[
\hat{\Phi} \triangleq \{X'_i \triangleq T_i(X'_i) : X'_i \in \Phi', T_i \in \mathbb{R}^{d \times d'}, \forall i \in \mathbb{N}_+ \}.
\]

Then \( \hat{\Phi} \) is also a PPP and for any bounded Borel set \( \mathcal{A} \subset \mathbb{R}^{d} \) its intensity measure \( \hat{\lambda} \) is given by

\[
\hat{\lambda}(\mathcal{A}) = \lambda'(A') \mathbb{E} \left[ \frac{1}{\sqrt{\text{det}(T^T T)}} \right].
\]
C. Power Consumption Model of a Small Cell BS

Although the cell voidness issue mentioned in Section II-A seems inevitable, its impact on network energy consumption can be alleviated by applying green power control at BSs. The green power control scheme for a small cell BS has two modes – the active mode is for non-void BSs, whereas the dormant mode is for void BSs for saving power. For base station $B_i$, the power consumption of such green power control can be characterized by the following expression:

$$\Psi_i = V_i(P_{on} + \delta P_i) + (1 - V_i)P_{off}, \tag{9}$$

where $\Psi_i$ is the power consumed by BS $B_i$, $P_{on}$ is the power consumed by the hardware of a BS in the active mode, $P_i$ denotes the transmit power of a base station, $\delta > 0$ is a scaling constant for the transmit power which usually depends on the power amplifier used by a BS, and $P_{off}$ denotes the power consumption of a dormant BS. Also, we assume $P_{on} > P_{off}$ since it is usually the case in practice. The average power consumption in (9) can be respectively written as

$$E[\Psi] = (1 - p_0)(P_{on} + \delta P_i) + p_0 P_{off} \tag{10}$$

in which $p_0 \triangleq P[V_i = 0]$ represents the void probability of a cell and its analytical result will be elaborated in Section III. Note that $p_0$ is certainly affected by cell association schemes adopted by users. In Section III, we will investigate how to characterize $p_0$ for GCA in (3) from a fundamental connectivity perspective. In addition, the transmit power $P_i$ of all BSs is assumed to be able to provide the minimum received power $P_{min}$ to their associated users by compensating the mean path loss, i.e., $P_i = P_{min} \inf \{E[|B_i|^\alpha]\}$. According to $|B_i|^\alpha$ in (6), we have

$$E[|B_i|^\alpha] = E \left[ \frac{1}{n!} \Gamma(1 + \frac{\alpha}{2}) \Gamma(\frac{\alpha}{2}) \frac{\pi^{\frac{\alpha}{2}} \lambda_\beta \Gamma(WH)^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2} - \frac{\alpha}{2}) \pi^{\frac{\alpha}{2}} r^\alpha dr}{\pi^{\frac{\alpha}{2}} \lambda_\beta \Gamma(WH)^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2} - \frac{\alpha}{2}) \pi^{\frac{\alpha}{2}} r^\alpha dr} \right] \geq E \left[ \frac{\Gamma(1 + \frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2} + \frac{\alpha}{2})} \right] \geq \frac{\Gamma(1 + \frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2} + \frac{\alpha}{2})}, \tag{11}$$

where $\Gamma(x) = \int_0^\infty t^{-1} e^{-t} dt$ is the Gamma function and $\zeta \triangleq (WH)^{\frac{\alpha}{2}} E\left[ (WH)^{\frac{\alpha}{2}} \right] \geq 1$. Thus, $P_i$ is given by

$$P_i = \frac{P_{min} \Gamma(1 + \frac{\alpha}{2})}{(\pi \lambda_\beta \zeta)^{\frac{\alpha}{2}}}. \tag{12}$$

By plugging (11) into (10), the explicit result of $E[\Psi]$ is acquired, which will be used to define the green cell and user throughputs in Section IV.

III. Void Cell Probability, Cell Load and Average Throughputs

A. The Void Probability of a Cell for GCA

As pointed out in the previous section, the GCA scheme or any “user-centric” association scheme cannot guarantee that there are at least one user in each cell, i.e., a void cell could exist in the network. This phenomenon can be intuitively interpreted by using a Poisson-Dirichlet (Voronoi) tessellation for a PPP of BSs. Suppose the Voronoi tessellation is used to determine the cell of each BS in $\Phi_B$ and all users adopt the nearest BS association scheme to connect with their serving BS. This nearest associating process can be viewed as the process of dropping all users in $\Phi_B$ on the large plane consisting of the Voronoi-tessellated cells formed by $\Phi_B$. Under this circumstance, the probability mass function (pmf) of the number of users in a cell of $\Phi_B$ can be expressed as

$$p_n \triangleq P[\nu(C) = n] = E \left[ \frac{(\lambda_\nu(C))^n}{n!} e^{-\lambda_\nu(C)} \right], \tag{12}$$

where $C$ denotes a Voronoi cell of a BS in $\Phi_B$, $\nu(C)$ represents the number of users in cell $C$, and $\nu(C)$ is the Lebesgue measure of $C$.

Unfortunately, the theoretical result of $p_n$ in (12) is unknown since the pdf of a Voronoi cell area is still an open problem [20]. However, it can be accurately approximated by using a Gamma distribution with some particular parameters [20], [23]. Reference [23] suggests the following Gamma distribution for $f_\nu(C)$:

$$f_\nu(C)(x) = \frac{\hat{\rho} \lambda_\nu x^{\hat{\rho} - 1} e^{-\hat{\rho} \lambda_\nu x}}{\Gamma(\hat{\rho})}, \tag{13}$$

and $\hat{\rho} = \frac{\alpha}{2}$ can achieve an accurate pdf of a Voronoi cell area. Substituting (13) into (12) yields the following result:

$$p_n = \frac{\lambda_\nu}{n!} \frac{\hat{\rho} \lambda_\nu}{\Gamma(\hat{\rho})} \int_0^\infty x^{n+\hat{\rho} - 1} e^{-\hat{\rho} \lambda_\nu x} dx = \frac{1}{n!} \frac{\Gamma(n + \hat{\rho})}{\Gamma(\hat{\rho})} \left( \frac{\lambda_\nu}{\hat{\rho} \lambda_\nu + \lambda_\nu} \right)^n \left( 1 + \frac{\lambda_\nu}{\hat{\rho} \lambda_\nu} \right)^{-\hat{\rho}}. \tag{14}$$

We call the term $\lambda_\nu/\hat{\rho} \lambda_\nu$ in (14) the cell load of the network since it represents the mean number of users in a Voronoi-tessellated cell. Hence, the pmf of the number of users in a cell for nearest BS association has an accurate closed-form expression. The void probability of a cell, $p_0$, can be found by $p_0$ with the case of $n = 0$, which is the last term of $p_n$ in (14), i.e.,

$$p_0 \triangleq P[V_i = 0] = \left( 1 + \frac{\lambda_\nu}{\hat{\rho} \lambda_\nu} \right)^{-\hat{\rho}} \tag{15}$$

and this indicates that the intensity of the void BSs is $\lambda_\nu p_0$ that is not negligible especially when cell load $\lambda_\nu/\lambda_\nu$ is small. Most importantly, the results in (13) and (14) are no longer accurate for all non-nearest BS association schemes since users do not necessarily associate with their nearest BS, such as the GCA scheme mentioned in Section II. However, an accurate void probability of a BS for the GCA scheme can be derived as shown in the following.

By Jensen’s inequality, the lower bound on $p_0$ in (15) is given by

$$p_0 = E \left[ e^{-\lambda_\nu(C)} \right] \geq \exp \left( -\frac{\lambda_\nu}{\lambda_\nu} \right). \tag{16}$$

This lower bound on the void probability reveals three crucial implications: (i) the void probability of a cell is always
bounded above zero such that there could be a certain number of void BSs in the small cell network; (ii) nearest BS association cannot achieve this lower bound since its void probability in (15) is always larger than the lower bound; (iii) from an energy-saving perspective, the lower bound represents the minimum percentage of all BSs in the network that can be explored to save energy. Later, we will theoretically show that, this lower bound can be achieved by GCA. To derive an accurate void probability of a cell for the GCA scheme, we approach this problem from a fundamental connectivity point of view and derive the bounds on $p_0$ as shown in the following theorem.

**Theorem 2.** If all users in $\Phi_U$ adopt the GCA scheme defined in (3) to associate with a BS in $\Phi_B$ defined in (1), the bounds on the void probability of a cell are given by

$$
\left(1 + \frac{\lambda_U}{\lambda_B}\right)^{-\zeta} \geq p_0 \geq \exp\left(-\frac{\lambda_U}{\lambda_B}\right) \quad (17)
$$

and $\zeta = \mathbb{E}[(W\mathcal{H})^{\frac{2}{a}}] \mathbb{E}[(W\mathcal{H})^{-\frac{2}{a}}] \geq 1$ as defined in Section II-C.

**Proof:** See Appendix B.

According to the proof of Theorem 2 the lower bound on $p_0$ is derived while considering the completely independence exists between the non-associated events of a BS whereas the upper bound is obtained by approaching the opposite case, i.e., all non-associated events of a BS are caused by considering all users in a cell located the same farthest distance from the BS and thus they are highly correlated. Accordingly, it is reasonably to conjecture that the upper bound is tightly close to the lower bound provided that the cross-correlations between all non-associated events are significantly weakened. On the other hand, mathematically we know $\lim_{\zeta \to \infty} \left(1 + \frac{\lambda_U}{\lambda_B}\right)^{-\zeta} = \exp\left(-\frac{\lambda_U}{\lambda_B}\right)$ and thus the bounds in (17) are fairly tight as $\zeta$ becomes large. This intuitively reveals that large $\zeta$ will “decorrelate” all non-associated events, which is an important observation since it lets us realize that the lower bound on $p_0$ is easily achieved by enlarging the $\frac{2}{a}$-fractional moment of $W/H_1$. For example, the lower bound on $p_0$ can be achieved by the maximum received power association scheme provided that channels have large shadowing power. In addition, the bounds in (17) can be tight for the case of a large cell load. This indicates that the void probability of a BS is reduced when more users join the network under a given BS intensity. In other words, when the network has a large user population the efficacy of reducing $p_0$ by using large $\zeta$ is apparently undermined such that the performance of GCA is similar to that of nearest BS association in this case.

Although the bounds on $p_0$ are characterized, an accurate result of $p_0$ is still needed since it will help us understand how many BSs per unit area are void and they should be switched to the dormant mode. The following proposition renders an accurate heuristic result of $p_0$ for GCA.

**Proposition 1.** The void probability of a BS can be accurately approximated by

$$
p_0 = \left(1 + \frac{\lambda_U}{\rho \lambda_B}\right)^{-\rho} \quad (18)
$$

where $\rho = \frac{7}{2}\mathbb{E}[(W\hat{H})^{\frac{2}{a}}] \mathbb{E}[(W\hat{H})^{-\frac{2}{a}}] = \frac{7}{2}\zeta$ if the GCA scheme is used in the network.

**Proof:** Since $(1 + x)^{-1} \geq (1 + x/\alpha)^{-\alpha}$ for $\alpha > 1$, we know

$$
\left(1 + \frac{\lambda_U}{\lambda_B \zeta}\right)^{-\zeta} > \left(1 + \frac{\lambda_U}{\lambda_B \rho}\right)^{-\rho}
$$

since $\rho > \zeta \geq 1$. Also, for nearest BS association $p_0$ with $\rho = \frac{7}{2}$ in (15) is an accurate void probability of a BS. Therefore, we can conclude that $p_0$ with $\rho = \frac{7}{2}\zeta$ is accurately the void probability of a BS for the GCA scheme since such $\rho$ reduces to $\frac{7}{2}$ as GCA reduces to nearest BS association (i.e., $W_i = 1/H_1$ for all $i$).

Fig. 2 shows the simulation result of the void probabilities for nearest BS association and maximum received power association. For nearest BS association, we have $\rho = \frac{7}{2}$ since $W_i = 1/H_1$, while $\rho$ for maximum received power association is given by

$$
\rho = \frac{7}{2}\mathbb{E}[(\hat{H})^{\frac{2}{a}}] \mathbb{E}[(\hat{H})^{-\frac{2}{a}}] = \frac{7\pi}{\alpha \sin(2\pi/\alpha)} \exp\left(-\frac{4\sigma^2_s}{\alpha^2}\right), \quad (19)
$$

which only depends on the shadowing power (variance) and path loss exponent. First, we see that the void probability of nearest BS association is no longer accurate if maximum received power association is used. For example, when $\frac{\lambda_B}{\lambda_U} \approx 2$, the void probabilities for the nearest BS and maximum received power association schemes with 8-dB shadowing are 0.2 and 0.14, respectively. The lower bound on the void probability is around 0.135. The void cell probability given in (18) indeed accurately coincides with the simulated result and is much closer to the lower bound. Next, as expected, large shadowing power (i.e., $\rho \gg 1$) indeed makes the void cell probability approach to its lower bound. Thus, GCA is able to achieve this lower bound.
B. Average Cell Throughput

Since void cells could exist in the network and they do not contribute network interference, the throughputs obtained in the previous works overlooking the void cell problem are certainly underestimated if interference is treated as noise. To effectively quantify how the void probability impacts the downlink throughput of a BS, we define the following average cell throughput assuming the channel input is Gaussian and treating interference as noise:

\[ \mathcal{T}_C \triangleq \sup \left\{ \mathbb{E} \left[ \log_2 \left( 1 + \frac{H_0 \|B_0^\alpha\|^\alpha}{I_0} \right) / \nu(C_0^\alpha) \right] \right\}, \]

where \( I_0 \triangleq \sum_{B_i \in \Phi_B \setminus B_0} V_i \|B_i\|^\alpha \) denotes the interference at user \( U_0 \) and \( \nu(C_0^\alpha) \) is the Lebesgue measure of cell \( C_0^\alpha \) associated by user \( U_0 \). Average cell throughput \( \mathcal{T}_C \) characterizes the trade-off relationship between average downlink channel rate and cell size due to the void probability – large void probability benefits average channel rate owing to interference reduction, yet it increases the average cell size of the associated BSs that cover the whole network such that more path loss is induced. The explicit result of \( \mathcal{T}_C \) is given in the following proposition.

**Proposition 2.** If every user associates with its cell by GCA and all the associated BSs are modeled as a homogeneous PPP and let \( \mathcal{L}_Z(s) \triangleq \mathbb{E}[e^{-sZ}] \) be the Laplace functional of random variable \( Z \), \( \mathcal{L}_Z(B_0^\alpha \setminus I_0) \) is given by

\[ \mathcal{L}_Z(B_0^\alpha \setminus I_0)(s) = \mathbb{E} \left[ \frac{G^2}{G^2 + (1 - p_0) \ell(s, G^2)} \right], \]

where \( p_0 \) is given by \( \lambda_B \left( \frac{1}{p_0} \right) \). For a sufficiently large positive integer \( n \), the average cell throughput in (20) can be accurately found by

\[ \mathcal{T}_C \approx \frac{\lambda_B - p \lambda_B}{(\ln 2)(1 - 1/p)(1 + (1 - p_0) \ell(s, G^2))}, \]

and \( \sup\{\mathcal{L}_Z(B_0^\alpha \setminus I_0)(s)\} \) is at most \( \frac{1}{1 + (1 - p_0)\ell(s, G^2)} \). For a sufficiently large positive integer \( n \), the average cell throughput in (20) can be accurately found by

\[ \mathcal{T}_C \approx \frac{\lambda_B - p \lambda_B}{(\ln 2)(1 - 1/p)(1 + (1 - p_0) \ell(s, G^2))}, \]

where \( \lambda_B \) is the average service rate of the BSs.

**Remark 3.** If the coverage probability is defined as the probability that the SIR of the user is no less than a threshold \( s \in \mathbb{R}^+ \), it follows that \( \mathcal{L}_Z(B_0^\alpha \setminus I_0)(s) \) is the user coverage probability in a cell. The result in (21) is much neat and more general if compared with the coverage probability in prior works since it indicates how void cell probability, fading and shadowing affect user’s coverage.

The average cell throughput in (22) is the nearest closed-form result than any other similar average throughput results in literature (such as [5] [9]). Its physical meaning is the per-cell spectrum efficiency of quantifying how the average spectrum efficiency of associated BSs changes along their average cell size \( 1/\lambda_B(1 - p_0) \) under the GCA scheme. Especially, it is the first cell-wise throughput result including the impact of the void cell probability. It possesses several important meanings that are addressed in the following, respectively.

- **BS intensity dominates \( \mathcal{T}_C \) in heavy cell load:** For the case of heavy cell load (i.e., \( \lambda_U/\lambda_B \gg 1 \)), we have \( p_0 \approx 0 \). Intuitively, decreasing \( p_0 \) increases more interference, however, it also induces less path loss in desired signals as well as the average cell size of the associated BSs.

- **User intensity dominates \( \mathcal{T}_C \) in light cell load:** For the case of light cell load (i.e., \( \lambda_U/\lambda_B \ll 1 \)), we have \( p_0 \approx 1 \) and \( \mathcal{T}_C \) is approximated by

\[ \mathcal{T}_C \approx \lambda_B \left[ (\ln 2)(1 - 1/p)(1 + (1 - p_0) \ell(s, G^2)) \right] \times \sum_{s=1}^\infty \frac{\omega_i ds}{s + e^{-(\sqrt{2s}x_i + \mu_i)}}, \]

which indicates \( \mathcal{T}_C \) becomes a linearly increasing function of \( \lambda_B \) such that the asymptotic scaling law \( \mathcal{T}_C \in \Theta(\lambda_B) \) holds. Thus, under the heavy cell load situation deploying more BSs can directly increase \( \mathcal{T}_C \).

**Proof:** See Appendix C.

- **Remark 2.** In order to make the derivations of \( \mathcal{T}_C \) tractable, we assume the associated BSs still form a homogeneous PPP in this paper even though in theory they are no longer a PPP due to the correlation between them (see the proof of Theorem 2). Nonetheless, our previous work in [25] showed that the associated BS can be still accurately approximated by a homogeneous PPP as long as the cell load is not pretty small.

The average cell throughput in (22) is the nearest closed-form result than any other similar average throughput results in literature (such as [5] [9]). Its physical meaning is the per-cell spectrum efficiency of quantifying how the average spectrum efficiency of associated BSs changes along their average cell size \( 1/\lambda_B(1 - p_0) \) under the GCA scheme. Especially, it is the first cell-wise throughput result including the impact of the void cell probability. It possesses several important meanings that are addressed in the following, respectively.

- **BS intensity dominates \( \mathcal{T}_C \) in heavy cell load:** For the case of heavy cell load (i.e., \( \lambda_U/\lambda_B \gg 1 \)), we have \( p_0 \approx 0 \). Intuitively, decreasing \( p_0 \) increases more interference, however, it also induces less path loss in desired signals as well as the average cell size of the associated BSs. In this context, \( \mathcal{T}_C \) is approximated by

\[ \mathcal{T}_C \approx \lambda_B \left[ (\ln 2)(1 - 1/p)(1 + (1 - p_0) \ell(s, G^2)) \right] \times \sum_{s=1}^\infty \frac{\omega_i ds}{s + e^{-(\sqrt{2s}x_i + \mu_i)}}, \]

where \( \omega_i \) is the monotonically increasing and concave function of \( \lambda_B \) such that the asymptotic scaling law \( \mathcal{T}_C \in \Theta(\lambda_B) \) holds. Thus, under the heavy cell load situation deploying more BSs can directly increase \( \mathcal{T}_C \).

- **User intensity dominates \( \mathcal{T}_C \) in light cell load:** For the case of light cell load (i.e., \( \lambda_U/\lambda_B \ll 1 \)), we have \( p_0 \approx 1 \) and \( \mathcal{T}_C \) is approximated by

\[ \mathcal{T}_C \approx \lambda_B \left[ (\ln 2)(1 - 1/p)(1 + (1 - p_0) \ell(s, G^2)) \right] \times \sum_{s=1}^\infty \frac{\lambda_B ds}{s + e^{-(\sqrt{2s}x_i + \mu_i)}}, \]

For a given \( \lambda_U \), we can observe that \( \mathcal{T}_C \) is a monotonically increasing and concave function of \( \lambda_B \) as \( \lambda_B \rightarrow \infty \). For a given \( \lambda_B \) and \( \lambda_U \rightarrow 0 \), we also have \( \mathcal{T}_C \in \Theta(\lambda_U) \). Accordingly, making more users join the network can efficiently boost \( \mathcal{T}_C \) in the light cell load situation. This scaling law corrects the flawed scaling result of the average throughput in the previous work without considering void probability.
Rayleigh fading and log-normal shadowing. The simulation parameters are
association (NBA) and maximum received power association (MRPA) under
Fig. 3. Simulation results of average cell throughput for nearest BS
\[ \lambda \]
All simulation curves show that
fectly coincide with their corresponding simulation results in
n terms of the average cell throughput since it has a larger
Most importantly, they indicate that maximum received power
eventually it will approaches to a constant regarding that
network with cell voidness, it fails to delineate how much
2 presents the per-cell spectrum efficiency of the small cell
we have already pointed out in above.
\[ \frac{\lambda}{4} \]
constant if the BS intensity is fixed. This is the desired situation we wish to
constant if the BS intensity is fixed. This is the desired situation we wish to
A numerical illustration of
\[ \mathcal{T}_C \]
Cell association significantly impacts \[ \mathcal{T}_C \]: Cell assocaition schemes have a considerable impact on \[ \mathcal{T}_C \] since \( p_0 \) and \( \ell(s, \zeta) \) in (21) both are affected by cell association weights. Specifically, a cell association scheme with fairly large \( \zeta \) certainly benefits \[ \mathcal{T}_C \] since \( \ell(s, \zeta) / \zeta \) decreases considerably and \( (1 - p_0) \) almost remains a constant in this case. General speaking, nearest cell association (\( W_i H_i = 1 \) and \( \zeta = 1 \)) is unable to achieve higher \[ \mathcal{T}_C \] than any other cell association schemes (with random \( W_i H_i \) and \( \zeta \gg 1 \)). In maximum received power association, for example, cell association results are highly impacted by \( \zeta \) that exponentially increases with shadowing power as shown in (19) such that \[ \mathcal{T}_C \] becomes much larger in a higher shadowing power environment. In this context, cell association schemes that favor BSs with stronger channel power surely benefits \[ \mathcal{T}_C \].
A numerical illustration of \[ \mathcal{T}_C \] for nearest BS association and maximum received power association is shown in Fig. 3. As can be seen, the analytical results in (23) with \( n = 6 \) perfectly coincide with their corresponding simulation results in different cell association schemes and shadowing conditions. All simulation curves show that \[ \mathcal{T}_C \] is a monotonic decreasing and convex function of the cell load for a given \( \lambda_U \) and eventually it will approaches to a constant regarding that \( \lambda_U \).
Most importantly, they indicate that maximum received power association evidently outperforms nearest BS association in terms of the average cell throughput since it has a larger \( \zeta \), as we have already pointed out in above.

C. Average User Throughput

Although the average cell throughput found in Proposition 2 presents the per-cell spectrum efficiency of the small cell network with cell voidness, it fails to delineate how much
Note that \[ \mathcal{T}_C \] will increase along the cell load and finally converge to a constant if the BS intensity is fixed. This is the desired situation we wish to have since \[ \mathcal{T}_C \] and cell load both increase at the same time.
of the downlink throughput resource in a cell can be fairly allocated to users either in the frequency or in the time domain.
To characterize the average throughput fairly shared by users, we define the following average user throughput
\[ \mathcal{T}_U \triangleq \sup \left\{ \mathbb{E} \left[ \log_2 \left( 1 + \frac{H_0 ||B_0||^{-\alpha}}{I_0} \right) \right] \middle| \mathbb{E} \left[ \Phi_U(C_0^s) \right] \right\}, \]
where \( \Phi_U(C_0^s) \) is the number of serviced users in a non-void cell. Note that the distribution of \( \Phi_U(C_0^s) \) is not equal to that of \( \Phi_U(A) \) for any Borel set \( A \) with \( \nu(A) = \nu(C_0^s) \) due to the cell voidness issue. The following proposition gives the explicit result of \( \mathcal{T}_U \).

Proposition 3. The average user throughput in (26) can be explicitly found as
\[ \mathcal{T}_U = \frac{\nu \lambda_1(1 - p_0)^2}{\nu \lambda_1(1 - p_0 - p_0^\rho)} \begin{cases} \sum_{i=1}^{n} \int_0^\infty \omega_i (1 + (1 - p_0)) \ell(s, \zeta) / \zeta \, ds, & \text{with shadowing} \\ \int_0^\infty (s + 1) [1 + (1 - p_0)] \ell(s, \zeta) / \zeta \, ds, & \text{without shadowing} \end{cases} \]
where \( \omega_i, x_i, \) and \( \ell(s, \zeta) \) are the same as given in Proposition 2.

Proof: According to the pdf of the area of associated BS \( C_0^s \) in (39), the pmf pf \( \Phi_U(C_0^s) \) is
\[ P \left[ \Phi_U(C_0^s) = n \right] = \mathbb{E} \left[ (\nu \lambda_1(1 - p_0))^n e^{-\lambda_1(1 - p_0)^n} \right| n \geq 1 \]
\[ = \frac{\nu \lambda_1(1 - p_0)^n}{n!} \int_0^\infty x^{n-1} e^{-\lambda_1(1 - p_0)^n} e^{-(\lambda_1 - \rho (1 - p_0) \lambda_1)x} \, dx \]
\[ = \frac{p_0 \Gamma(n + \rho)}{(1 - p_0) \Gamma(n) \lambda_1(1 - p_0) \lambda_1^n} \left( \frac{\lambda_1}{\lambda_1(1 - p_0) + \rho} \right)^n, \quad n \geq 1. \]
where \( p_0 \) is given in (18). As a result, the mean number of users in cell \( C_0^s \) is
\[ \mathbb{E} \left[ \Phi_U(C_0^s) \right] = \frac{p_0 \Gamma(n + \rho)}{(1 - p_0) \Gamma(n) \lambda_1(1 - p_0) \lambda_1^n} \left( \frac{\lambda_1}{\lambda_1(1 - p_0) + \rho} \right)^n = \frac{\lambda_1}{\lambda_1(1 - p_0)^2}. \]
The average user throughput can be calculated as follows
\[ \mathcal{T}_U = \frac{\lambda_1(1 - p_0)^2 \mathcal{T}_C}{\nu \lambda_1(1 - p_0) \lambda_1(1 - p_0) + \rho} \frac{\lambda_1}{\lambda_1(1 - p_0)^2}. \]
Substituting (22) into (28) results in (27).

The average user throughput in (27) is a linear scaling of the average cell throughput normalized by the user intensity such that it is virtually the function of cell load \( \lambda_1 / \lambda_B \). For the asymptotic behaviors of \( \mathcal{T}_U \), we have \( \mathcal{T}_U \in \Theta \left( \frac{\lambda_1}{\lambda_1(1 - p_0) \lambda_1(1 - p_0) + \rho} \right) \) and \( \mathcal{T}_U \to 0 \) in heavy cell load since there are too many users sharing the throughput resource in a cell. On the other hand, so light is the cell load that \( \mathcal{T}_U \) scales like \( \Theta(\lambda_1 / \lambda_B) \) and \( \mathcal{T}_U \to 0 \). Specifically, in moderate cell load there exists a sole optimal cell load that maximizes the average user throughput. Knowing this optimal cell load is important since it indicates when cell load and average user throughput increase at the same time.
and in this case more user throughput can be exploited by deploying fewer BSs for a given user intensity. The following proposition gives the accurate solution of the optimal cell load for $T_U$.

**Proposition 4.** The optimal cell load that maximizes the average user throughput in (27) is the fixed point of the following real-valued function parameterized by $\beta > 1$

$$L(v) = \left[ 1 - \left( 1 + \frac{v}{\rho} \right)^{-\rho} \right]^{1/\beta}, \quad v \geq 0. \quad (29)$$

Namely, if the optimal cell load is $v^*$, we have $L(v^*) = v^*$.

**Proof:** Since the cell load is only contained in the term $(1 - p_0)^2 \lambda_B / \lambda_U [1 + (1 - p_0) \ell(s, \zeta) / \zeta]$, we find its optimal value by merely calculating the derivatives of this term with respect to the cell load. Let $v$ denote the cell load, $V(v)$ represent this term parameterized by $v$ and $p_0 = 1 - p_0$. Then we have

$$V(v) = \frac{\tilde{p}_0^2}{v^2 + \tilde{p}_0 \ell(s, \zeta) / \zeta}.$$ 

Note that $\ell(s, \zeta) / \zeta$ is not function of $v$. Hence, the first-order derivative of $V(v)$ w.r.t. $v$ is

$$\frac{dV}{dv} = \frac{\tilde{p}_0^2 \tilde{p}_0^2 v^2 + \tilde{p}_0 \ell(s, \zeta) / \zeta - \tilde{p}_0^2 [1 + \tilde{p}_0 \ell(s, \zeta) / \zeta]}{v^2 [1 + \tilde{p}_0 \ell(s, \zeta) / \zeta]^2},$$

where

$$\tilde{p}_0 = -\frac{dp_0}{dv} = \left( 1 + \frac{v}{\rho} \right)^{-\rho - 1} = p_0^1 + \frac{1}{\rho} \geq 0.$$

Letting $\frac{dV}{dv} = 0$ yields the following differential equation of $\tilde{p}_0$ and its solution

$$\frac{d\tilde{p}_0}{dv} = \frac{\tilde{p}_0 [1 + \tilde{p}_0 \ell(s, \zeta) / \zeta]}{v^2 + \tilde{p}_0 \ell(s, \zeta) / \zeta} \Rightarrow \tilde{p}_0^2 = \frac{\tilde{p}_0^2}{1 + \tilde{p}_0 \ell(s, \zeta) / \zeta} = c(\tilde{p}_0) v,$$

where $c$ is an unknown function of $\tilde{p}_0$ that can be determined as follows. First, we can assign $c(v, \tilde{p}_0) = \tilde{p}_0^{2-1/\beta} / (1 + \tilde{p}_0 \ell(s, \zeta) / \zeta)$ for all $\beta \in \mathbb{R}^+$. Then, since $\tilde{p}_0$ should only depend on $v$, not on $\ell(s, \zeta) / \zeta$. Thereby, we acquire $\tilde{p}_0^{1/\beta} = v$. Since $\tilde{p}_0 = -\tilde{p}_0^{1+\frac{1}{\beta}}$, we know $\tilde{p}_0 \in [0, 1]$ and $\tilde{p}_0 < 0$. To make $\tilde{p}_0^{1/\beta} (v)$ have a fixed point at $v^*$, the first-order and second-order derivative constraints on $\tilde{p}_0^{1/\beta} (v)$, $\frac{1}{\beta} \tilde{p}_0^{1/\beta - 1} (0) \tilde{p}_0'(0) > 0$ and $\frac{1}{\beta} \left( 1 - \frac{1}{\beta} - 1 \right) \tilde{p}_0^{1/\beta - 2} (p_0^2) + \frac{1}{\beta} \tilde{p}_0^{1/\beta - 1} (p_0) < 0$ for all $v > 0$, must hold. Since they hold only when $\beta > 1$, $L(v)$ in (25) is obtained. Finally, we can show

$$\lim_{v \rightarrow 0} \frac{dV}{dv} > 0 \quad \text{and} \quad \lim_{v \rightarrow \infty} \frac{dV}{dv} < 0$$

and the solution of $\tilde{p}_0^{1/\beta} (v^*) - v^* = 0$ for $v^* \geq 0$ is unique since $v^* \geq 0$ and $\tilde{p}_0$ is a positive and one-to-one mapping function of $v$. This follows that $L(v^*) = (1 - p_0)^{1/\beta} = \tilde{p}_0^{1/\beta} = v^*$ and the fixed point of $L(v^*)$ is the optimal cell load that maximizes $T_U$.

**Remark 4.** For any GCA schemes with $\rho \gg 1$, (29) reduces to $L(v) \approx (1 - \exp(-\rho v))$, whose fixed point is $v = 0$. Hence, the optimal cell load asymptotically reduces to zero as $\rho$ goes to infinity.

Cell association also has a pivotal influence on the average user throughput same as the case in the average cell throughput, and this situation can be visually perceived from the numerical results of nearest BS and maximum received power associations in Fig. 4. Indeed, large shadowing power apparently increases $T_U$ as well. There exists only one optimal cell load that maximizes $T_U$ as shown in Proposition 4. For example, in the case of maximum received power association with $\alpha = 3.76$ and shadowing variance $\sigma_s^2 = 4$ dB, Proposition 4 delivers an optimal cell load $\approx 1$ for $\rho = 10.3$ and $\beta = 2$, which accurately coincides with the optimal cell load observed in the corresponding curve case of Fig. 4. In addition, we also observe that the optimal cell load happens later as shadowing power gets larger, which further implies that shadowing is instructive to the efficiency of deploying cells since higher average user throughput is attained by less deployment cost.

**IV. OPTIMAL GREEN CELL LOAD AND THROUGHPUT**

The average cell and user throughputs found in the previous section have a common drawback, that is, the energy cost is not integrated into them while considering to achieve higher throughput by deploying more cells. Deploying more cells may not necessarily lead to more total power consumption since the transmit power in (11) (that indicates $P_t \propto \lambda_B^{-2}$) decreases due to less path loss even though other hardware power consumptions increase in this case. In order to delve the fundamental relationship between throughput and power consumption, we define the following green cell throughput $\mathcal{G}_C$ (bits/Hz/joule) and green user throughput $\mathcal{G}_U$ (bits/Hz/joule/user):

$$\mathcal{G}_C = \frac{T_C}{\lambda_B E(\Psi)} \quad \text{and} \quad \mathcal{G}_U = \frac{T_U}{E(\Psi)}$$

in which $E(\Psi)$ is the average power consumption of a BS and it is already given in (10). The physical meanings of green cell and user throughputs are how much throughput can be transported by costing unit power consumption. Note that the
definitions of $G_C$ and $G_U$ are slightly different since $G_C$ aims to quantify an energy efficiency on the throughput of a BS whereas $G_U$ desires to characterize an energy efficiency on the throughput of a user. In this section, we would like to investigate the fundamental relationship between cell load and these two green throughputs in order to gain some insights on how to deploy BSs with an appropriate intensity for achieving high throughput efficiency. The green cell and user throughputs are studied in Sections II-V-A and II-V-B respectively.

A. Green Cell Throughput and Its Optimality

According to (30) and (22), the green cell throughput can be specifically expressed as

$$G_C = \sum_{i=1}^{\infty} \frac{\Gamma(1+\frac{\rho}{\beta})}{\Gamma(1+\frac{1}{\beta})} \frac{ds}{\ln(2)(1-1/\rho)(P_{on}+\delta P_t) + p_0 P_{off}/(1-p_0)},$$

where $P_t = \frac{P_{on}\Gamma(1+\frac{1}{\beta})}{(\pi^2 \lambda G_C)}$ as already given in (11). In heavy cell load, $p_0 \approx 0$ and $G_C$ approaches to a constant approximated by

$$G_C \approx \frac{\lambda U}{\beta} \frac{\sum_{i=1}^{\infty} \frac{\Gamma(1+\frac{\rho}{\beta})}{\Gamma(1+\frac{1}{\beta})} \frac{ds}{\ln(2)(1-1/\rho)P_{off}}}{\lambda U},$$

which can be viewed as the “least” green cell throughput merely affected by GCA and BS intensity since the increase in the power consumption of the active mode has a direct impact on $G_C$. In this case, deploying more BSs and using appropriate cell association schemes with a larger $\zeta$ can make cell throughput greener. In light cell load, $p_0 \approx 1$ and $G_C$ has the following approximation

$$G_C \approx \left(\frac{\lambda U}{\beta} \frac{\sum_{i=1}^{\infty} \frac{\Gamma(1+\frac{\rho}{\beta})}{\Gamma(1+\frac{1}{\beta})} \frac{ds}{\ln(2)(1-1/\rho)P_{off}}}{\lambda U}\right) \Theta(\lambda U/\beta P_{off}).$$

The asymptotic characteristics of $G_C$ are different from those of $G_U$. In heavy cell load, we have the scaling law of the “least” green user throughput, i.e., $\Theta(\lambda U/\beta U P_{on})$ as $p_0 \rightarrow 0$, which is dominated by the power consumption of the active mode and cell load. From the green user throughput point of view, heavy cell load should be avoided since any additional power consumption in the active mode completely gives rise to user throughput reduction. In light cell load, we have $G_U = \Theta(\lambda U/\beta U P_{off})$ as $p_0 \rightarrow 1$, which is the same as the asymptotic law of the greenest cell throughput and also suffers the low throughput problem while achieving high greenees. Light cell load should also be doded from the perspective of user’s quality of service.

Proposition 5. For a given user intensity $\lambda U$, there exists a $\beta > 1$ such that the unique optimal cell load that maximizes $G_C$ in (31) can be accurately found by the fixed point of the following real-valued function

$$\mathbb{L}_C(v) = \pi \zeta \lambda U \left(\frac{(P_{on} - P_{off})}{\delta P_{min} \Gamma(1+\frac{\rho}{\beta})}\right) \left\{ \left[ 1 - \left( 1 + \frac{v}{\rho} \right)^{-\beta} \right]^{-\frac{1}{\beta}} - 1 \right\}^{\frac{1}{\beta}} = 0 \quad \text{such that the optimal BS intensity is } \lambda_B = \lambda_U/v^*.$$ 

Proof: See Appendix B.

B. Green User Throughput and Its Optimality

Substituting (27) into (30) results in the explicit expression of the green user throughput given by

$$G_U = \frac{\lambda U (1-p_0) \sum_{i=1}^{\infty} \frac{\Gamma(1+\frac{\rho}{\beta})}{\Gamma(1+\frac{1}{\beta})} \frac{ds}{\ln(2)(1-1/\rho)P_{off}}}{\lambda U [(P_{on} + \delta P_t) + p_0 P_{off}/(1-p_0)]}.$$  

(33)

Namely, the optimal cell load is the unique solution of $\mathbb{L}_U(v^*) - v^* = 0$ such that the corresponding optimal BS intensity is $\lambda B = \lambda U/v^*$.

Proposition 6. For a given user intensity $\lambda U$, there exists a $\beta > 1$ such that the optimal cell load that maximizes $G_U$ in (31) can be accurately found by the fixed point of the following real-valued function

$$\mathbb{L}_U(v) = \pi \zeta \lambda U \left(\frac{(P_{on} - P_{off})}{\delta P_{min} \Gamma(1+\frac{\rho}{\beta})}\right) \left\{ \left[ 1 - \left( 1 + \frac{v}{\rho} \right)^{-\beta} \right]^{-\frac{1}{\beta}} - 1 \right\}^{\frac{1}{\beta}} = 0.$$  

(34)
which means $v^*$ is the unique maximizer of cell load for $G_U$ if and only if $L_U(v^*) - v^* = 0$ and the optimal BS intensity is $\lambda_B^* = \lambda_U/v^*$.

**Proof:** The proof is omitted here since it is similar to the proof of Proposition 5.

Obviously, the function in (34) for the optimal cell load of the green user throughput is similar to the function in (32) for the optimal cell load of the green cell throughput. Its fixed point also becomes larger if cell association schemes have a larger $\zeta$. Under the same conditions, since green cell throughput is essentially smaller than green cell throughput and the increasing rates of $G_C$ and $G_U$ w.r.t. the cell load are pretty much the same in the light cell load, the optimal cell load for the green user throughput tends to appear earlier than that for the optimal cell throughput. This phenomenon can be observed from Proposition 5 as well as the simulation results in the following subsection.

### C. Simulation Results

This subsection presents some simulation results to verify whether or not the previous results and discussions regarding the green cell and user throughputs are correct. We adopt the urban micro path loss model for 2GHz carrier frequency ($140.7 + 37.6 \log_{10} d$, $d$ in km) given in [22] incorporated with average penetration loss (20 dB for 80% users that are indoor and 0 dB for the rest that are outdoor), antenna gain 5 dBi [27]. The power consumption model of the small cell BS is based on the model of picocells in [22]. We summarize the common network parameters for all simulation figures as follows: $P_{on} = 6.8W$, $P_{off} = 4.3W$, $P_{min} = -106$ dBm, $\delta = 4.0$, $\mu_s = 0$ dB and $\alpha = 3.76$. For simplicity, only two GCA schemes, i.e., nearest BS association and maximum received power association, are used to do simulation and compare the simulation results.

Figs. 5 and 6 show the simulation results of the green cell and user throughputs, respectively. Both figures reveal
that maximum received power association outperforms nearest BS association, shadowing power helps improve the green cell and user throughputs, and only one optimal cell load exists for every different simulation cases, as expected. Also, stronger shadowing power lets the optimal cell load happen later. To further explain this point, the simulation results of the optimal BS intensities for green cell and user throughputs versus different user intensities are provided in Figs. [7] and [8] respectively. From these two figures, we see that the slopes of the green cell and user throughputs are initially both positive and reduce largely as the shadowing power increases (see the previous discussion for $G_C$ in light cell load), which is because large shadowing power still has a strong influence on the coverage probability in (21) but has very less impact on the void cell probability. Propositions 5 and 6 also indicate this phenomenon and thus stronger shadowing power indeed leads to a smaller optimal BS intensity and larger green cell and user throughputs for the same user intensity, which is desirable. Finally, we observe that optimal BS intensities found by (32) with $\beta \in (6,8)$ and (34) with $\beta \in (1,4)$ perfectly coincide with the numerical results in Figs. [7] and [8] The correctness of Propositions 5 and 6 is validated.

V. CONCLUSION

In this paper, we investigated the optimal cell load, average and green throughputs from the cell and user perspectives in a small cell network with cell voidness. The impacts from the void cell phenomenon are two-fold. Firstly, it indicates that the previous works on through rate or power are no longer accurate especially in a small cell network with dense BS deployment since no interferences are generated by the void cells. Secondly, it inspires the idea of green power control that makes the void BSs switch to the dormant mode for saving power. An accurate expression of the void cell probability is successfully derived for GCA. The average cell and user throughputs that reflect the impact of void cells are theoretically characterized and they can be effectively boosted if appropriate cell load and GCA schemes with strong channel power (such as fading and shadowing) are adopted. The proposed green power control scheme not only exploited power saving on the void BSs but also gave birth to the idea of defining the green cell and user throughputs on the basis of the average cell and user throughputs. The green cell and user throughputs can be maximized by their unique optimal cell load that can be significantly improved by green power control and GCA.

APPENDIX

A. Proof of Theorem 7

The following proof essentially generalizes the proof of Lemma 1 in [21]. First, we define the void probability of $\Phi'(\mathcal{A})$ for any bounded BS set $\mathcal{A} \subset \mathbb{R}^d$ as the probability that $\Phi'(\mathcal{A})$ is equal to zero where $\Phi'(\mathcal{A})$ denotes the number of nodes in $\Phi'$ enclosed in $\mathcal{A}$. That is,

$$
\mathbb{P}[\Phi'(\mathcal{A}) = 0] = \exp \left(-\int_{\mathcal{A}} \Lambda'(\text{d}X') \right)
$$

because $\Phi'$ is a PPP. Let $\mathcal{A}_i = \mathcal{T}_i(\mathcal{A}')$ where $\mathcal{A}_i \subset \mathbb{R}^d$ is a bounded Borel set and the probability that node $\hat{X}_i$ in $\hat{\Phi}$ is enclosed in $\mathcal{A}_i$ is

$$
\mathbb{P} \left[ \hat{X}_i \in \mathcal{A}_i \right] = \mathbb{P} \left[ \mathcal{T}_i(X'_i) \in \mathcal{A}_i \right] = \mathbb{P} \left[ \mathcal{T}_i^T \mathcal{T}_i(X'_i) \in \mathcal{T}_i^T(\mathcal{A}_i) \right] = \mathbb{P} \left[ X'_i \in (\mathcal{T}_i^T \mathcal{T}_i)^{-1}(\mathcal{A}') \right],
$$

where $(\mathcal{T}_i^T \mathcal{T}_i)^{-1}(\mathcal{A}') \triangleq \int_{\mathcal{A}} (\mathcal{T}_i^T \mathcal{T}_i)^{-1}(\text{d}X')$. By definition, $\tilde{\Lambda}(\mathcal{A})$ is the average of $\Phi(\mathcal{A})$ and it can be calculated by applying the Campbell theorem as shown in the following:

$$
\tilde{\Lambda}(\mathcal{A}) = \int_{\mathcal{A}} \Lambda(\text{d}\hat{X}) = \mathbb{E} \left[ \sum_{\hat{X}_i \in \hat{\Phi}} 1 \left( \hat{X}_i \in \hat{\Phi} \cap \mathcal{A}_i \right) \right]
$$

$$
= \mathbb{E} \left[ \sum_{X'_i \in \Phi'} 1 \left( X'_i \in \Phi' \cap (\mathcal{T}_i^T \mathcal{T}_i)^{-1}(\mathcal{A}') \right) \right]
$$

$$
= \mathbb{E} \left[ \int_{(\mathcal{T}_i^T \mathcal{T}_i)^{-1}(\mathcal{A}')} \mathbb{P}[X' \in \Phi'] \nu_{\text{d}}(\text{d}X') \right]
$$

$$
= \mathbb{E} \left[ \frac{1}{\sqrt{\det(\mathcal{T}_i^T \mathcal{T}_i)}} \int_{\mathbb{R}^d} \mathbb{P}[X' \in \Phi' \cap \mathcal{A}] \nu_{\text{d}}(\text{d}X') \right]
$$

where $(\ast)$ follows from the Jacobian determinant of two volumes. Thus, $\tilde{\Lambda}(\mathcal{A}) = \mathbb{E} \left[ \frac{1}{\sqrt{\det(\mathcal{T}_i^T \mathcal{T}_i)}} \Lambda'(\mathcal{A}') \right]$ and it follows that

$$
\mathbb{P}[\Phi'(\mathcal{A}') = 0] = \exp \left(-\mathbb{E} \left[ \frac{1}{\sqrt{\det(\mathcal{T}_i^T \mathcal{T}_i)}} \right]^{-1} \int_{\mathcal{A}} \tilde{\Lambda}(\text{d}\hat{X}) \right)
$$

$$
= \mathbb{P}[\Phi(\mathcal{A}) = 0]
$$

since $\nu_{\text{d}}(\mathcal{A}) = \nu_{\text{d}}(\mathcal{A}')$. Since the void probability of a point process completely characterizes the statistics of the process, $\Phi$ is also a PPP.

If $\Phi'$ is homogeneous, then its intensity measure is $\Lambda'(\mathcal{A}') = \lambda' \nu_{\text{d}}(\mathcal{A}')$ and thus the result in above can be further simplified to $\tilde{\Lambda}(\mathcal{A}) = \lambda' \mathbb{E} \left[ \frac{1}{\sqrt{\det((\mathcal{T}_i^T \mathcal{T}_i)^{-1})}} \right] \nu_{\text{d}}(\mathcal{A})$ and it does not depend on the location of any node in $\hat{\Phi}$. As a result, $\Phi$ is a homogeneous PPP with the intensity given in (7).

B. Proof of Theorem 2

Suppose each user in cell $C_l \subset \mathbb{R}^2$ of BS $B_l$ has its own cell association region which is also a bounded Borel set on $\mathbb{R}^2$. For example, the cell association region of user $U_j$ is denoted by $A_j$ and all cell association regions contain BS $B_l$. Since all users in $C_l$ adopt the RCA scheme, the probability that BS $B_l$ is not associated by user $U_j \in C_l$ can be expressed as

$$
\mathbb{P}[\mathcal{E}_j] = \lim_{\nu(A_j) \to \infty} \mathbb{P} \left[ W_l H_l \|U_j - B_l\|^{-\alpha} < \sup_{B_l \in \Phi \cap A_j \setminus B_l} \{ W_l H_l \|U_j - B_l\|^{-\alpha} \} \right],
$$

where $\mathcal{E}_j$ denotes the event that user $U_j$ is not associated with BS $B_l$ as its association region goes to infinity. According
to Theorem 1, $\Phi_B$ can be statistically mapped to another homogeneous PPP $\hat{\Phi}_B$ of intensity $\lambda_B = \lambda_B \mathbb{E}[|WH|^{1/2}]$ and $\Phi_B(A_j)$ is a Poisson random variable with parameter $\lambda_B \nu(A_j)$. Thus, it follows that

$$
P[\mathcal{E}_j] = \lim_{\nu(A_j)\to\infty} P \left[ \|\hat{U}_j - \hat{B}_i\|^{-\alpha} \right]
$$

$$
< \sup_{\hat{B}_i \in \Phi_B \cap A_j \setminus B_i} \left\{ \|\hat{U}_j - \hat{B}_i\|^{-\alpha} \right\}
$$

$$
= P \left[ \hat{D}_j > \hat{D}_j^* \right],
$$

where $\hat{U}_j = (W_i H_i)^{-\frac{1}{2}} U_j$, $\hat{D}_j$ is the distance from user $\hat{U}_j$ to BS $\hat{B}_i$, $\hat{D}_j^*$ is the distance from $U_j$ to its nearest BS in $\hat{\Phi}_B$. The void probability $p_0$ can be found by calculating the following

$$
p_0 = \mathbb{E} \left\{ \prod_{j=0}^{\infty} P[\mathcal{E}_j] \right\}
$$

Due to the complicate correlation effects between all events $\mathcal{E}_j$’s, it is analytically intractable to solve the exact result of $p_0$. However, the bounds on $p_0$ can be characterized in closed-form.

First, the lower bound on $p_0$ can be found as follows. For a given $\Phi_U(C_i)$ we know the following inequality

$$
P \left[ \bigcap_{j=0}^{l-1} \mathcal{E}_j \bigg| \Phi_U(C_i) \right] \geq \prod_{j=0}^{l-1} P[\mathcal{E}_j],
$$

which is obtained by considering $P[\mathcal{E}_j | \mathcal{E}_j] \geq P[\mathcal{E}_j]$ for $A_i \cap A_j \neq \emptyset$. Since the number of users in $C_i$ is a Poisson random variable with parameter $\lambda_U \nu(C_i)$, i.e., $P[\Phi_U(C_i) = n] = \frac{(\lambda_U \nu(C_i))^n}{n!} e^{-\lambda_U \nu(C_i)}$, the lower bound on $p_0$ can be given by

$$
p_0 \geq \mathbb{E} \left\{ \prod_{j=0}^{\infty} P[\mathcal{E}_j] \right\} = \sum_{n=0}^{\infty} \left( \prod_{j=0}^{\infty} P[\mathcal{E}_j] \right) \frac{(\lambda_U \nu(C_i))^n}{n!} e^{-\lambda_U \nu(C_i)} n!.
$$

(35)

According to the Slivnyak theorem [20], the statistic property evaluated at any point in a homogeneous PPP is the same such that $P[\mathcal{E}_j]$ is the same as $P[\mathcal{E}_0]$ evaluated at user $U_0$ (the origin) for all $j$. Accordingly, (35) can be simplified by

$$
p_0 \geq \exp \left( -\lambda_U \nu(C_i) \left( 1 - \mathbb{E}[P[\mathcal{E}_0]] \right) \right)
$$

(36)

and $P[\mathcal{E}_0] = P \left[ \hat{D}_0 > \hat{D}_0^* \right]$ in which $\hat{D}_0^*$ is the distance from the origin to the nearest BS in $\hat{\Phi}_B$. The pdf of $\hat{D}_0^*$ is $f_{\hat{D}_0^*}(r) = 2\pi \zeta \lambda_B x \exp(-\pi \zeta \lambda_B x^2)$ [28], whereby $P[\mathcal{E}_0]$ is simplified as

$$
P[\mathcal{E}_0] = 1 - \mathbb{E} \left[ \exp \left( -\pi \zeta \lambda_B (\hat{D}_0^* \right)^2) \right].
$$

(37)

Since all users in $C_i$ are uniformly distributed, we can know $f_{\hat{D}_0}(r) = \frac{2\pi \zeta \lambda_B}{\nu(C_i)}$, and thus we can calculate $P[\mathcal{E}_0]$ in closed-form as

$$
P[\mathcal{E}_0] = 1 - \int_{0}^{\frac{\lambda_B \nu(C_i)^{1/2}}{2\pi \zeta}} \exp \left( -\pi \zeta \lambda_B r^2 \right) \frac{2\pi \zeta \lambda_B}{\nu(C_i)} dr
$$

$$
= 1 - \exp \left( -\zeta \lambda_B \nu(C_i) \right) \frac{2\pi \zeta \lambda_B}{\nu(C_i)}.
$$

(38)

Apparently, $P[\mathcal{E}_0] \geq 1 - \frac{1}{\zeta \lambda_B \nu(C_i)}$. Then substituting it into (36) gives us the lower bound in (17).

Next, the upper bound on $p_0$ can be obtained as follows. The probability that user $U_j$ does not associate with BS $B_i$ is upper-bounded by

$$
P[\mathcal{E}_j] \leq \lim_{\nu(A_j)\to\infty} P \left[ W_i H_i \inf_{U_j \in C_i} \|U_j - B_i\|^{-\alpha} \right]
$$

$$
< \sup_{B_i \in \Phi_B \cap A_j \setminus B_i} W_i H_i \|U_j - B_i\|^{-\alpha},
$$

which is acquired by considering the farthest user of BS $B_i$; in cell $C_i$. According to Slivnyak’s theorem and Theorem 1, the upper bound on $p_0$ for a given $\Phi_U(C_i)$ can be written as

$$
P \left[ \bigcap_{j=0}^{l-1} \mathcal{E}_j \bigg| \Phi_U(C_i) \right] \leq \mathbb{P} \left[ \hat{R}_l \geq \hat{D}_0^* \right] \Phi_U(C_i),
$$

where $\hat{R}_l$ denotes the distance from BS $\hat{B}_i$ to its farthest user in $C_i = (WH)^{-\frac{1}{2}}$. Let $\Phi_U(C_i) \equiv \{ \hat{U}_j \in C_i : (W_i H_i)^{-\frac{1}{2}} U_j \in C_i \}$ and for given $\Phi_U(C_i)$ we have

$$
P \left[ \hat{R}_l \geq \hat{D}_0^* \right] = \mathbb{P} \left[ (\hat{R}_l)^2 \geq (\hat{D}_0^*)^2 \right]
$$

$$
= \mathbb{P} \left[ (W_i H_i)^{-\frac{1}{2}} \sum_{j=1}^{l-1} \hat{R}_{l,j} \geq (\hat{D}_0^*)^2 \right]
$$

$$
\leq \mathbb{P} \left[ \prod_{j=1}^{l-1} \mathbb{P} \left[ (W_i H_i)^{-\frac{1}{2}} \hat{R}_{l,j} \geq (\hat{D}_0^*)^2 \right] \right]
$$

$$
= (1 - \mathbb{E} \left[ \exp \left( -\pi \zeta \lambda_B \left[ (W_i H_i)^{-\frac{1}{2}} \hat{R}_{l,j} \right]^2 \right) \right] \Phi_U(C_i) / \zeta \lambda_B)
$$

(39)

$$
\leq 1 - \frac{\lambda_U}{\lambda_U + \mathbb{E}[\zeta \lambda_B \nu(C_i)]}.
$$

where (a) follows from that fact that $\{ (\hat{R}_l)^2, l = 1, 2, \cdots \}$ is an one-dimensional Poisson process of intensity $\lambda_U \mathbb{E}[|W H|^{1/2}]$ and thus $(\hat{R}_l)^2$ is equal to the sum of $\Phi_U(C_i)$ i.i.d. $(WH)^{-\frac{1}{2}} \hat{R}_0^2$ [29], (b) is due to first applying Jensen’s inequality on the term $(WH)^{-\frac{1}{2}}$ and then calculating the expectation regarding $\hat{R}_0^2$, and (c) is obtained from $(1 - x)^n \leq (1 - x)^{-1/n}$ for $x \in [0, 1]$ and $n > 0$ and $\mathbb{E}[\Phi_U(C_i)] = \Phi_U(C_i) / \zeta \lambda_B$. Therefore, we have

$$
P \left[ \bigcap_{j=0}^{l-1} \mathcal{E}_j \bigg| \Phi_U(C_i) \right] \leq 1 + \frac{\lambda_U}{\zeta \lambda_B}
$$

(40)

for any given $\Phi_U(C_i)$ and the upper bound in (17) follows. This completes the proof.

C. Proof of Proposition 3

According to [8], we can have the following CDF of $\|B_0^*\|

$$
P[\|B_0^*\| \leq x] = \mathbb{P} \left[ \|\hat{B}_0\| \leq x (W_0 H_0)^{-\frac{1}{2}} \right]
$$

$$
= 1 - \mathbb{E} \left[ \exp \left( -\pi \lambda_B \zeta G^2 x^2 \right) \right],
$$

(41)
where \( \hat{B}_0 \) is the nearest BS in \( \hat{\Phi}_B \) to the origin. Hence, for a given \( W_0, H_0, \| B_0^* \| \) can be viewed as the distance from the nearest BS in \( \Phi_B \) to the origin that is scaled by \( 1/G \). That means \( I_0 \| B_0^* \| = (G^{-a} I_0)(G^a \| B_0^* \|) = G^{-a} I_0 \| B_0 \| \) if \( B_0 \) is the nearest BS in \( \Phi_B \) to the origin. For a given \( \| B_0 \| = u \) and \( G \), the Laplace functional of \( \hat{G}_0 \) can be found by

\[
\mathcal{L}_{u^a I_0^G-s}(s|u) = \mathbb{E}\left[ e^{-s^a u^a/G^a I_0} \right]
\]

\[
= \exp\left[ -s^a \mathbb{E}\left[ \sum_{B_i \in \hat{\Phi}_B \setminus \{B_0\}} V_i u_i^2 / (\| B_i \|- u^2) \right] \right]
\]

\[
= \exp\left[ -s^a \mathbb{E}\left[ \int_0^\infty \left( 1 - e^{-s^a V H (1 + \frac{2}{\alpha})} \right) 2\pi s^a \right] \right]
\]

where \((a)\) is due to the fact that \( \{\| B_i \|^2, i = 1, 2, \ldots\} \) is an one-dimensional Poisson process of intensity \( \pi \lambda_B \), \((b)\) follows from the probability generation function of a homogeneous PPP \[20\] and \((c)\) is due to assuming the associated BSs are a thinning PPP of intensity \( \lambda_B \). By assuming \( Y \) is an exponential random variable with unit mean and variance, we have the following identity:

\[
\int_0^\infty \left( 1 - e^{-s^a V H (1 + \frac{2}{\alpha})} \right) dx = \int_0^\infty \mathbb{P}[Y \leq s^a V H (1 + \frac{2}{\alpha})] dx
\]

\[
= \int_0^\infty \mathbb{P}\left\{ x \leq \left( \frac{h}{G^a Y} \right) \right\} dx = \mathbb{E}_Y \left\{ \left( \frac{h}{G^a Y} \right) \right\}
\]

\[
= (sH)^{1/2} I_0^G - 2 \left( 1 - \frac{2}{\alpha} \right)
\]

which leads to the following result

\[
\mathcal{L}_{u^a I_0^G-s}(s|u) = \mathbb{E}\left[ e^{-s^a \mathbb{E}\left[ \sum_{B_i \in \hat{\Phi}_B \setminus \{B_0\}} V_i u_i^2 / (\| B_i \|- u^2) \right]} \right].
\]

Then \( \mathcal{L}_{\| B_0^* \|^a I_0}(s) \) in \[22\] can be obtained by calculating the following integral

\[
\mathcal{L}_{\| B_0^* \|^a I_0}(s) = \mathbb{E}\left[ \int_0^\infty \mathcal{L}_{u^a I_0^G-s}(s|u) e^{-s^a \mathbb{E}\left[ \sum_{B_i \in \hat{\Phi}_B \setminus \{B_0\}} V_i u_i^2 / (\| B_i \|- u^2) \right]} \right]
\]

\[
= \mathbb{E}\left[ \frac{G^2}{\| B_0 \|^2 + (1 - p_0) \| B_0 \|^2} \right].
\]

Since \( \mathcal{L}_{\| B_0^* \|^a I_0}(s) \) is concave for \( G^2 \), we can use Jensen’s inequality to obtain \( \sup \{\mathcal{L}_{\| B_0^* \|^a I_0}(s)\} = \frac{1}{1+(1-p_0)G^2 (\| B_0 \|^2)} \).

Now we find \( T_C \). According to \[13\], Proposition \[1\] and the assumption that all associated BSs are a homogeneous PPP, the pdf of the area of associated cell \( C_0^* \) can be written as

\[
f_{\nu}(C_0^*)(x) = \frac{\rho_{B_0}(1-p_0)^x}{\Gamma(p)} e^{-\rho_{B_0}(1-p_0)x}
\]

and the mean of \( 1/\nu(C_0^*) \) can be found by

\[
\mathbb{E}[1/\nu(C_0^*)] = \int_0^\infty 1/\nu(C_0^*)(x) dx = \frac{\rho_{B_0}(1-p_0)}{\rho - 1}.
\]

since \( \rho \geq \frac{2}{\alpha} \). Using this result, the average cell throughput in \[22\] can be expressed as

\[
T_C = \mathbb{E}\left[ \log_2 \left( 1 + \frac{H_0}{\| B_0^* \|^a I_0} \right) \right] = \frac{\lambda_B(1-p_0)}{(\ln 2)(\rho - 1)} \int_0^\infty \mathbb{P}\left[ \| B_0^* \|^a I_0 \geq (2^x - 1) \right] dx
\]

Moreover, we know

\[
\mathbb{P}[H > h] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{-\frac{h^2 - \frac{1}{2}h^2}{e^2}} dx = \mathbb{E}\left[ e^{-h^2 - z^2} \right] = \mathcal{L}_{e^2}(h),
\]

where \( Z \) is a normal random variable with mean \( \mu_s \) and variance \( \sigma_s^2 \). Then \( T_0 \) can be expressed as

\[
T_C = \frac{\rho \lambda_B(1-p_0)}{(\ln 2)(\rho - 1)} \int_0^\infty \mathbb{E}\left[ \| B_0^* \|^a I_0 \right] \mathcal{L}_{e^2}(h) ds.
\]

Using the above result and the definition of \( T_C \), the average cell throughput can be further written as

\[
T_C = \frac{\lambda_B(1-p_0)}{(\ln 2)(\rho - 1)} \int_0^\infty \frac{\mathcal{L}_{I_0}(s)(\| B_0 \|^a I_0)}{\mathcal{L}_e(s)} ds.
\]

Also, for any \( s > 0 \) we have

\[
\mathbb{E}\left[ \frac{1}{s + e^{-z}} \right] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \left( s + e^{-\sqrt{2\sigma_s}(s + \mu_s)} \right)^{-1} e^{-z^2} dx
\]

\[
= \sum_{i=1}^n \omega_i \left( s + e^{-\sqrt{2\sigma_s}(s + \mu_s)} \right)^{-1}
\]

where \((d)\) is obtained by using a Gauss-Hermite quadrature integration with a sufficiently large integer \( n \). Substituting \[43\] into \[42\] results in \[22\]. For the case of no shadowing, letting \( Z \) in \[22\] be zero gives \[23\]. This completes the proof.

\[D. \text{ Proof of Proposition 5}\]

The terms in \([G_s] \) that pertain to the cell load can be lumped into the following sole function

\[
\sum_q (\frac{\rho_{\hat{B}_0}}{\rho_{\hat{B}_0}})(s, \zeta) / (\rho_{\hat{B}_0} + \frac{\kappa}{\zeta}) = (1 - p_{\hat{B}_0})P_{off}
\]

where \( v \) represents cell load, \( p_{\hat{B}_0} = 1 - p_{\hat{B}_0}(v) \) and \( \kappa = \delta P_{min} \Gamma(1 + \frac{2}{\alpha})/(\pi \lambda_u (C)) \) is a constant for a given \( \lambda_u \). Then taking the derivative of \( \sum_q \) w.r.t. \( v \) and making it equal to zero yield the following differential equation:

\[
\frac{\partial}{\partial v} \left( \frac{\rho_{\hat{B}_0}}{\rho_{\hat{B}_0}} \right)(s, \zeta) - (s, \zeta) / (P_{off} - P_{off}) = \frac{\kappa(s, \zeta)}{\zeta} \frac{\rho_{\hat{B}_0}}{\zeta} - \frac{\kappa}{\zeta} \frac{v}{\zeta}
\]

\[
= \frac{\kappa}{\zeta} \left( 1 + \frac{\ell(s, \zeta)}{\rho_{\hat{B}_0}} \right)
\]

\[
\frac{d}{dv} [v^2 (1 + \rho_{\hat{B}_0}(s, \zeta))] = \frac{\kappa}{\zeta} [v^2 (1 + \rho_{\hat{B}_0}(s, \zeta))].
\]
Integrating both sides of the equation above and applying the appropriate boundary conditions of \( \tilde{\rho}_0 \) under the constraint that \( \tilde{\rho}_0 \) is not a function of \( \ell(s, \alpha) \) lead to

\[
\tilde{\rho}_0^{-1} (P_\text{on} - P_\text{off}) = \kappa \psi(1) + (P_\text{on} - P_\text{off}) \Rightarrow \psi = \left( P_\text{on} - P_\text{off} \right) \left( \tilde{\rho}_0^{-1} - 1 \right) \overline{L}_c(v).
\]

Since \( \tilde{\rho}_0 = 1 - (1 + v/\rho)^{-\beta} \), \( \overline{L}_c(v) \) in (32) has a fixed point at \( v^* \).

Now we need to show the uniqueness of the solution of the differential equation in (44), i.e., the fixed point of \( \overline{L}_c(v) \) is unique. First, we re-write (44) as

\[
\frac{d\tilde{\rho}_0}{\rho} = \frac{\kappa \beta}{P_\text{off}} \left( (P_\text{on} - P_\text{off}) \tilde{\rho}_0 - (P_\text{on} - P_\text{off}) \right) \frac{1}{\psi}. \]

Assume both \( \psi_1(v) \) and \( \psi_2(v) \) are two arbitrary solutions of (44) and let \( \tilde{\psi}(v) = \psi_1 - \psi_2 \). Since \( 0 \leq \tilde{\rho}_0 \leq 1 \), the following inequality always holds. By the Gronwall–Bellman lemma \([30]\), we know \( |\tilde{\psi}| \leq 0 \) and thus \( \tilde{\psi} = 0 \). Consequently, the solution is unique since any two arbitrary solutions \( \psi_1 \) and \( \psi_2 \) are equal.

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3Please refer to the proof of Proposition 4 for the details in how to determine the exact solution form of this kind of differential equation.