ON A FUNCTIONAL EQUATION FOR AN ELLIPTIC
DILOGARITHM

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Abstract. It is known due to S. Bloch that elliptic dilogarithm is subject of big
bunch of so-called Steinberg functional equation parameterized by rational functions
on an elliptic curve. We show that all of these equations follows from the case of
functions of degree three and antisymmetry relation.

1. The Introduction

1.1. Summary. We recall that Bloch-Wigner dilogarithm [Blo00] (Corollary 6.1.2) is
defined by the following formula:

\[ D(z) = \Im \text{Li}_2(z) + \arg(1 - z) \ln |z|. \]

This formula defines a real analytic function on \( \mathbb{C} \setminus \{0, 1\} \) that satisfies the relation
\( D(z) = -D(z^{-1}) \).

Let \( q \in \mathbb{C}^\times \) and \( E = \mathbb{C}^\times / q\mathbb{Z} \) be an elliptic curve over \( \mathbb{C} \). The elliptic version of this
function was defined by Spencer Bloch [Blo00] (Lemma 8.1.1) as the following sum

\[ D_q(z) = \sum_{k=-\infty}^{\infty} D(zq^k). \]

This function is closely connected to the classical non-holomorphic Eisenstein series.
From the relation \( D(z) = -D(z^{-1}) \) it easy follows the following antisymmetry relation:

(1.1) \[ D_q(z) + D_q(z^{-1}) = 0. \]

Now we want to formulate so-called Steinberg relations. Let \( f \) be a rational function
on \( E \) of degree \( n \). Let

\[ (f) = \sum_{k=1}^{n} (a_k[\alpha_k] - c_k[\gamma_k]), (1 - f) = \sum_{k=1}^{n} (b_k[\beta_k] - c_k[\gamma_k]) \]

be the divisors of the functions \( f \) and \( 1 - f \) correspondingly. As it follows from
[Blo00] (Theorem 8.1.2) we have the following relation for the function \( D_q(z) \)

(1.2) \[ \sum_{k,l=1}^{n} \left( a_k b_l D_q \left( \frac{\alpha_k}{\beta_l} \right) + b_k c_l D_q \left( \frac{\beta_k}{\gamma_l} \right) + c_k a_l D_q \left( \frac{\gamma_k}{\alpha_l} \right) \right) = 0. \]

Let us realize the elliptic curve \( E \) as the cubic plane curve given by the Weierstrass
equation \( y^2 = x^3 + ax + b \) for some \( a, b \in \mathbb{C} \). Let us assume that there are three
different lines \( l, m, n \subset \mathbb{P}^2(\mathbb{C}) \) intersecting at one point do not lying on \( E \). Let \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3 \) be the points of intersection of the lines \( l, m, n \) with \( E \).
It is not difficult to show using the relation (1.2) (see [GL08] (Lemma 3.29)), that there
is the following relation for the function $D_q(z)$:

$$(1.3) \quad \sum_{i,j=1}^{3} \left( D_q \left( \frac{\alpha_i}{\beta_j} \right) + D_q \left( \frac{\beta_i}{\gamma_j} \right) + D_q \left( \frac{\gamma_i}{\alpha_j} \right) \right)$$

Our main result is the following statement

**Theorem 1.1.** For any rational function $f$ on $E$ the relation $(1.2)$ can be represented as a linear combination of relations of the form $(1.1)$ and $(1.3)$.

It is useful to introduce some algebraic framework. Let us denote by $\mathbb{Q}[E]$ the group algebra of the elliptic curve $E$. It is a vector space freely generated by the points of $E$ with the following multiplication rule $[z] \cdot [w] = [zw]$, where $zw$ is the addition of the points $z$ and $w$ on the elliptic curve written multiplicatively. Now we can extend the map $D_q$ by linearity to the whole space $\mathbb{Q}[E]$ given by the following formula

$$D_q \left( \sum_{k=1}^{n} a_k \alpha_k \right) = \sum_{k=1}^{n} a_k D_q (\alpha_k) .$$

We will denote this map by the same letter $D_q$. Let us denote by $A$ the subspace of $\mathbb{Q}[E]$ generated by elements of the form $[a] + [a^{-1}]$. The relation $(1.1)$ is equivalent to the statement that the map $D_q$ is zero on the subspace $A$.

Let $K$ be the field of rational function on $E$. Let us introduce the map $\beta: \otimes^2 K^\times \otimes \mathbb{Q} \to \mathbb{Q}[E]$ defined by the formula $f \otimes g \mapsto \sum_{i,j} a_i b_j [x_i - y_j]$, where $\sum_i a_i [x_i], \sum_j b_j [y_j]$ are divisors of the functions $f$ and $g$. Now the relations $(1.2)$ is equivalent to the formula $D_q (\beta(f \otimes (1 - f))) = 0$. So in order to prove Theorem 1.1 we need to describe the subgroup of $\otimes^2 K^\times$ generated by elements of the form $f \otimes (1 - f)$. Since the map $\beta$ transform $S^2 K^\times$ to the space $A$ instead of the group $\otimes^2 K^\times$ we can pass to the quotient group $\Lambda^2 K^\times$.

Starting from here we replace the field of definition of $E$ from complex numbers $\mathbb{C}$ to any algebraically closed filed $k$ of zero characteristic.

**Theorem 1.1** follows from the following purely algebraic statement

**Proposition 1.2.** Let us denote by $St$ the subgroup of $\Lambda^2 K$ generated by the elements of the form $f \wedge (1 - f)$. Then $St$ is generated by elements of the form $g \wedge (1 - g)$ where $\deg g \leq 3$.

There are so-called Abel five term relations between elements of the form $f \wedge 1 - f$. This motivate the following definition

**Definition 1.3** (the pre-Bloch group). Let $\mathbb{Z}[K \setminus \{0, 1\}]$ be an abelian group freely generated by the set $K \setminus \{0, 1\}$ Let us denote by $B_2(F)$ the pre-Bloch group of the field $F$ defined as the quotient group of the space $\mathbb{Z}[F \setminus \{0, 1\}]$. by the subgroup $R_2(K)$ generated by the following elements

$$(1.4) \quad [x] - [y] + [y/x] + [(1 - x)/(1 - y)] - \left[ \frac{1 - x^{-1}}{1 - y^{-1}} \right] ,$$

where $x, y \in K \setminus \{0, 1\}$.

It is well-known that the map $\delta: B_2(K) \to \Lambda^2 K^\times$ given by the formula $\delta([x]) = x \wedge (1 - x)$ is well defined. **Proposition 1.2** follows from the following

**Theorem 1.4.** The group $B_2(K)$ is generated by elements of the form $[f]$ where $\deg f \leq 3$. 
1.2. Plan of the proof. Let us denote $F_nB_2(K)$ the subgroup of $B_2(K)$ generated by elements of the form $[f]$ where $\deg f \leq n$. In the second section we give a definition of a general function and prove the following

**Proposition 1.5** (Proposition 2.2). Let $n \leq 3$. The group $F_{n+1}B_2(K)$ is generated by the subspace $F_nB_2(K)$ and the elements of the form $[f]$, where $f$ is a general rational function of degree $n+1$.

The main result of the second section is the following

**Proposition 1.6** (Proposition 3.1). Let $n \leq 3$. Let $f$ be a general function of degree $n+1$. Then the element $[f]$ lies in $F_nB_2(K)$.

The derivation of Theorem 1.4 from Proposition 1.2. Since the group $B_2(K)$ is generated by elements of the form $[g]$ where $g$ is a general function of degree $n+1$. According to Proposition 3.1 such elements lie in the space $F_nB_2(K)$. So $F_nB_2(K) = F_{n+1}B_2(K)$. This mean that for any $n \geq 3$ we have $F_nB_2(K) = F_3B_2(K)$. Since $B_2(K) = \bigcup_{n=3}^{\infty} F_nB_2(K)$ the theorem follows.

**The derivation of Proposition 1.6 from Theorem 1.4**. Since the group $B_2(K)$ is generated by elements of the form $[f]$ where $\deg f \leq 3$ and the map $\delta$ is surjective, the group $S\delta$ is generated by elements of the form $f \wedge (1-f)$ with $\deg f \leq 3$.

Now we can give the proof of Theorem 1.4.

**The derivation of Theorem 1.4 from Proposition 1.2**. The map $\beta$ gives the canonical map $\Lambda^2K^\times \to \mathbb{Q}[E]/A$. It is easy to see that this map is zero on elements of the form $f \wedge (1-f)$ where $\deg f \leq 2$. Since the map $\beta$ is invariant under translations it follows that its image is generated by elements of the form $\beta(f \wedge 1-f)$ where $\deg f = 3$ and the sum of zeros of $f$ is equal to zero. In this case the element $D_4(\beta(f \wedge 1-f))$ is given by the formula similar to the formula (1.3).

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2. The Reduction to the Generic Case

We need the following

**Definition 2.1.** A rational function $f$ on elliptic curve $E$ is called general if there are points $\alpha_1, \alpha_2 \in f^{-1}(0), \beta_1, \beta_2 \in f^{-1}(1), \gamma_1, \gamma_2 \in f^{-1}(\infty)$ on $E$, such that the following conditions hold:

1. If the points $\alpha_1, \alpha_2$ are equal then the multiplicity of the poles of the function $f$ at the point $\alpha_1$ is at least 2. In the same way if the points $\gamma_1, \gamma_2$ are equal then the multiplicity of the pole of the function $f$ at the point $\gamma_1$ is at least 2,
2. The points $\beta_1$ and $\beta_2$ are different,
3. The point $\alpha_1 + \alpha_2 - \beta_1 - \beta_2$ is non zero,
4. The points $\alpha_1 + \alpha_2 - \gamma_1 - \beta_1, \alpha_1 + \alpha_2 - \gamma_1 - \beta_2$ are non-zero.

We have the following

**Proposition 2.2.** Let $n \leq 3$. The group $F_{n+1}B_2(K)$ is generated by the subspace $F_nB_2(K)$ and the elements of the form $[f]$, where $f$ is a general rational function of degree $n+1$. 

In order to prove this proposition we need the following

**Lemma 2.3.** Let \( n \geq 3 \). Let \( f \) be a function of degree \( n + 1 \) satisfying all but the third condition of Definition \([2.1]\). Then \([f] \in \mathcal{F}_nB_2(K)\).

**Proof.** The function \( f \) has zeros at \( \alpha_1, \alpha_2, \) a pole \( \gamma_1 \) and two points \( \beta_1, \beta_2 \) in which the function \( f \) takes value 1. Since the function \( f \) does not satisfy the third condition of Definition \([2.1]\), there is a function \( h \) of degree 2 on \( E \) with the same proprieties.

Let us substitute \( x = h, y = f \) into the formula \([1.4]\), we get

\[
[h] - [f] + [f/h] + [(1 - h)/(1 - f)] - \frac{1 - h^{-1}}{1 - f^{-1}}.
\]

All but the second term has the degree at most \( n \). So \([f] \in \mathcal{F}_nB_2(K)\). \(\square\)

**The proof of Proposition \([2.2]\).** Let \( a \in k \backslash \{0, 1\} \). If we substitute \( x = a, y = f \) into the formula \([1.4]\) we get the following relation in the group \( B_2(K) \)

\[
[a] - [f] + [f/a] + [(1 - a)/(1 - f)] - \frac{1 - a^{-1}}{1 - f^{-1}}.
\]

Let us prove that for all but finite numbers of values of \( a \) the element \([f/a]\) either is general or lies in the space \( \mathcal{F}_nB_2(K) \).

Let \( \alpha_1, \alpha_2, \gamma_1, \gamma_2 \) be two zeros and poles of the function \( f \) satisfying the first condition of Definition \([2.1]\). Let \( A_1 \subset \mathbb{P}^1 \) be the set of \( c \) critical points of the function \( f \), and \( A_2 = \{f(\alpha_1 + \alpha_2 - \gamma_1)\} \). Let \( a \not\in (A_1 \cup A_2) \). The set \( f^{-1}(a) \) has precisely \( n + 1 \) elements. Let \( \beta_1, \beta_2 \in f^{-1}(a) \) be two different elements. Since \( a \not\in A_2 \) the fourth condition of Definition \([2.1]\) holds. If the third condition is also holds then \( f/a \) is general. If it is does not hold then according to Lemma \([2.3]\) the element \([f/a]\) lies in \( \mathcal{F}_nB_2(K) \).

The cases of the fourth and the five terms in the formula \([2.1]\) are similar. Thus all but the second term of the formula \([2.1]\) lie in \( \mathcal{F}_nB_2(K) \) or have the form \([g]\) where \( g \) is general function of degree at most \( n + 1 \). The proposition is proved. \(\square\)

### 3. The Decreasing of Degree in Case of Generic Function

The goal of this section is to prove the following

**Proposition 3.1.** Let \( n \leq 3 \). Let \( f \) be a general function of degree \( n + 1 \). Then the element \([f]\) lies in \( \mathcal{F}_nB_2(K) \).

We will denote a divisor of a function \( f \) by \((f)\). Let us denote by \( \varphi \) the function of degree 2 on \( E \), satisfying \( \varphi(z) = z^{-2} + o(z) \) at \( z \to 0 \). It is easy to see that for arbitrary points \( \alpha, \beta \in \mathbb{E}, \alpha \neq \beta \), the function \( \varphi(z - \alpha) - \varphi(z - \beta) \) has the following divisor:

\[
(\varphi(z - \alpha) - \varphi(z - \beta)) = \left( \sum_{x \in \alpha + \beta} [x] \right) - 2[\alpha] - 2[\beta].
\]

We recall that the cross ratio of the four points of \( \mathbb{P}^1 \) is defined by the following formula

\[
[a, b, c, d] = \frac{a - c}{b - c} : \frac{a - d}{b - d} = \frac{(a - c)(b - d)}{(b - c)(a - d)}.
\]

Let \( \alpha, \beta, \gamma, \delta \) be four different points of \( E \). We define the function \( h_{\alpha, \beta, \gamma, \delta} \) by the following formula:
Therefore the function $h_{\alpha,\beta,\gamma,\delta}(z) = [\wp(z - \alpha), \wp(z - \beta), \wp(z - \gamma), \wp(z - \delta)]$.

It follows from the formula (3.1) that the function $h_{\alpha,\beta,\gamma,\delta}$ has the following divisor:

$$
(3.2) \quad (h_{\alpha,\beta,\gamma,\delta}) = \sum_{2x \in \{\alpha + \gamma, \beta + \delta\}} [x] - \sum_{2x \in \{\alpha + \beta, \gamma + \delta\}} [x].
$$

In particular the degree of function $h$ equals to 8. The group $E(2) := \{z \in E | 2z = 0\}$ acts in $E$ by translations. It follows from the formula (3.2) that the divisors of the functions $h_{\alpha,\beta,\gamma,\delta}$ and $1 - h_{\alpha,\beta,\gamma,\delta} = h_{\alpha,\gamma,\beta,\delta}$ are invariant under the group $E[2]$ and therefore the function $h_{\alpha,\beta,\gamma,\delta}$ is also invariant under the group $E[2]$. We need the following

**Lemma 3.2.** For any $m \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ there is $\mu \in E$, such that the following condition holds:

1. $h_{\alpha,\beta,\gamma,\delta}(\mu) = m$
2. $\mu \not\in \{\alpha, \beta\}$
3. $2 \mu \not\in \{\alpha + \gamma, \alpha + \delta, \alpha + \beta, \beta + \delta\}$

**Proof.** Since the function $h_{\alpha,\beta,\gamma,\delta}$ is invariant under the group $E[2]$, the set $h_{\alpha,\beta,\gamma,\delta}^{-1}(m)$ is also invariant under the group $E[2]$. The map $h_{\alpha,\beta,\gamma,\delta}$ is proper and hence surjective. It follows that the cardinality of the set $h_{\alpha,\beta,\gamma,\delta}^{-1}(\lambda)$ is at least 4. In particular there is $\mu \in h_{\alpha,\beta,\gamma,\delta}^{-1}(m)$ satisfying the first two condition of the lemma. The third condition follows from the formula (3.2) and the fact that $\lambda \not\in \{0, 1\}$. \qed

**Proposition 3.3.** Let $\alpha, \beta, \gamma, \delta$ be four different points on $E$ and $a, b, c, d$ be four different points on $\mathbb{P}^1$. Then there is a function $f$ of degree 2 on $E$ such that $f(\alpha) = a, f(\beta) = b, f(\gamma) = c, f(\delta) = d$.

**Proof.** Let $m$ be the cross-relation of the four points $a, b, c, d$ and $\mu$ be the point given by Lemma 3.2. Let us define the function $f$ by the following formula

$$
f(z) = \frac{\wp(z - \mu) - \wp(\alpha - \mu)}{\wp(z - \mu) - \wp(\beta - \mu)}.
$$

The divisor of this function is equal to $(f) = [\alpha] + [2\mu - \alpha] - [\beta] - [2\mu - \beta]$. It follows from the conditions of the previous lemma that the function $f$ satisfies the following two conditions:

1. The degree of $f$ equals to 2,
2. $f(\gamma), f(\delta) \not\in \{0, \infty\}$.

From the first condition of the previous lemma it follows that $g(\gamma)/g(\delta) = m$. Let $g$ be an element of the group $PSL_2(k)$ transforming the points $0, \infty, \lambda, 1$ to the points $a, b, c, d$. It is easy to see that the function $f(f(z)/f(d))$ satisfies the statement of the proposition. \qed

We have the following

**Corollary 3.4.** Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ be points on $E$, such that the following conditions hold:

1. The sets $\{\alpha_i\}_{i=1,2}, \{\beta_i\}_{i=1,2}, \{\gamma_i\}_{i=1,2}$ do not intersect.
2. $\beta_1 \neq \beta_2$
3. The points $\alpha_1 + \alpha_2 - \gamma_1 - \beta_1, \alpha_1 + \alpha_2 - \gamma_1 - \beta_2$ are non zero.
4. The point $\alpha_1 + \alpha_2 - \beta_1 - \beta_2$ is non zero.
Then one of the following statements hold.

1. There is a rational function $g$ on $E$ of degree 3 such that $g(\beta_1) = g(\beta_2) = 1$ and the function $g$ has the following divisor $(g) = [\alpha_1] + [\alpha_2] + [\alpha_3] - [\gamma_1] - [\gamma_2] - [\gamma_3]$, where $\alpha', \gamma'$ are some points on $E$.

2. There is a rational function $g$ on $E$ of degree 2 such that $g(\beta_1) = g(\beta_2) = 1$ and the function $g$ has the following divisor $(g) = [\tilde{\alpha}] + [\alpha'] - [\gamma_2] - [\gamma']$, where $\tilde{\alpha} \in \{\alpha_1, \alpha_2\}$ and $\alpha', \gamma'$ are some points on $E$.

Proof. From the first condition it follows that there is a function $\tilde{f}_1$ of degree 2 with the following divisor $[\alpha_1] + [\alpha_2] - [\gamma_1] - [\alpha_1 + \alpha_2 - \gamma_1]$. From the first and the third conditions it follows that $\tilde{f}_1(\beta_1) \not\in \{0, \infty\}$. Let us denote by $f_1$ the fraction $\tilde{f}_1/\tilde{f}_1(\beta_1)$. From the conditions of the corollary it follows that $g_1(\beta_2) \neq 0, 1, \infty$. According to Proposition 3.3 there is a function $g$ of degree 2 on $E$ taking values 0, $\infty$, 1, $f_1(\beta_2)^{-1}$ at the points $\alpha_1 + \alpha_2 - \gamma_1, \gamma_2, \beta_1, \beta_2$. It is easy to see that this function satisfies one of the statements of the corollary.

The proof of Proposition 3.1. Let $f$ be a general function of degree $n + 1$. Since the function $f$ is general the corresponding points from Definition 2.1 are satisfying the conditions of Corollary 3.4. Let $g$ be a function satisfying one of the statements of Corollary 3.4. Let us substitute $x = g, y = f$ into the relation (1.4)

$$[g] - [f] + [f/g] + [(1 - g)/(1 - f)] - [g(1 - x)/(g(1 - f))].$$

It is easy to see that all but the second term has degree $\leq n$.

References

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