An elementary transformation of vector bundles in $\mathbb{P}^n$

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Abstract

By considering the equivalence between the category of locally free sheaves and the category of algebraic vector bundles, we show how elementary transformations of vector bundles can be used to prove a case of the maximal rank hypothesis. We in turn show how this can be applied in the study of minimal free resolutions.

Key words: locally free sheaves, vector bundles, elementary transformations, maximal rank hypotheses, minimal free resolutions.

1 Introduction

In order to understand the geometry of the ideal $I_S$ of $s$ points in $\mathbb{P}^n$, one source of information is the Hilbert function $h_d(I_S)$, which gives the number of degree $d$ generators of $I_S$. The relation among these generators is captured in the free resolution of $I_S$. Conjectures have been made as to the prescribed form of the minimal free resolution for the ideal of points in $\mathbb{P}^n$. One such conjecture is the minimal resolution conjecture by Lorenzini [3] for the ideal of points in general position. Hirschowitz and Simpson [2] in their study of minimal free resolutions showed that in order to prove that an ideal of points in general position has a minimal free resolution of the expected form, it suffices to show that some evaluation map is of maximal rank.

In this paper, we make use of the equivalence between the category of locally free sheaves and the category of algebraic vector bundles to describe an elementary transformation in $\mathbb{P}^n$, and explain how this elementary transformation can be used to prove a case of maximal rank hypothesis.

The paper is organized as follows. In the next section, we give an overview of category theory with a view of building notation and developing the language used in the future sections. In section 3 we introduce locally free sheaves and relate them to vector bundles. We end the section by defining an elementary transformation of algebraic vector bundles. The definitions and properties we come across in these sections follow largely from [1] and [6]. Our main results are found in section 4 where we present an elementary transformation in $\mathbb{P}^n$, and prove that the diagram is indeed a diagram of elementary transformation. As a conclusion to this section, we show how our results can be used in studying minimal free resolutions.

2 Some category theory

Definition 2.1. A category $\mathcal{C}$ consists of a class of objects, denoted by $\text{Obj}(\mathcal{C})$, together with a set of morphisms between any pair $X, Y \in \text{Obj}(\mathcal{C})$. The set of morphisms from $X$ to $Y$ is denoted by $\text{Hom}(X, Y)$ and satisfy the following properties;

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a. given \( f_1 \in \text{Hom}(X,Y) \) and \( f_2 \in \text{Hom}(Y,Z) \), then the composition \( f_2 \circ f_1 \in \text{Hom}(X,Z) \).

b. for every \( X \in \text{Obj}(\mathcal{C}) \) there exist a morphism \( \text{id}_X \mapsto \text{Hom}(X,X) \) which is both right and left identity of the composition.

c. given \( f_1 \in \text{Hom}(X,Y), \ f_2 \in \text{Hom}(Y,Z) \) and \( f_3 \in \text{Hom}(Z,W) \), \( f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ f_3 \).  
For every ordered triple \((X,Y,Z)\in \text{Obj}(\mathcal{C})\).

Given a category \( \mathcal{C} \), we can define the **opposite category** \( \mathcal{C}^{op} \) by reversing the sense of maps in the category \( \mathcal{C} \). That is, the objects in the category \( \mathcal{C}^{op} \) is the same as the objects in the category \( \mathcal{C} \) and for any pair \( X,Y \), \( \text{Hom}_\mathcal{C}(X,Y) = \text{Hom}_{\mathcal{C}^{op}}(Y,X) \).

Given two categories \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) we can define a function \( F : \mathcal{C}_1 \to \mathcal{C}_2 \) between the two categories. Such a function is called a **functor**. Functors can be covariant or contravariant.

A **covariant functor** from a category \( \mathcal{C}_1 \) to a category \( \mathcal{C}_2 \) consists of a map \( F \) from \( \text{Obj}(\mathcal{C}_1) \) to \( \text{Obj}(\mathcal{C}_2) \) together with, for every pair \( (X,Y) \in \text{Obj}(\mathcal{C}_1) \), a function \( F_{X,Y} : \text{Hom}(X,Y) \to \text{Hom}(F(X),F(Y)) \), such that \( F \) commutes with composition and carries \( \text{id}_X \) to \( \text{id}_{F(X)} \).

A **contravariant functor** \( \mathcal{C}_1 \) to a category \( \mathcal{C}_2 \) is a functor \( F \) defined as \( F : \mathcal{C}_1^{op} \to \mathcal{C}_2 \). In other words it is a functor that reverses the sense of the morphisms.

Given two functors \( F_1 \) and \( F_2 \) from \( \mathcal{C}_1 \) to \( \mathcal{C}_2 \), a **natural transformation** of \( F_1 \) to \( F_2 \) consists of, for each \( X \in (\mathcal{C}_1) \), a morphism \( \phi_X : F_1(X) \to F_2(X) \) such that for every morphism \( f \in \text{Hom}(X,Y) \), the diagram

\[
\begin{array}{ccc}
F_1(X) & \xrightarrow{F_1(f)} & F_1(Y) \\
\downarrow{\phi_X} & & \downarrow{\phi_Y} \\
F_2(X) & \xrightarrow{F_2(f)} & F_2(Y)
\end{array}
\]

is commutative. A **natural isomorphism** of functors is a natural transformation for which \( \phi_X \) is an isomorphism. The data of functors \( F : \mathcal{C}_1 \to \mathcal{C}_2 \) and \( F' : \mathcal{C}_2 \to \mathcal{C}_1 \) for which \( F \circ F' \) is naturally an isomorphism to the identity function on \( \text{Id}_{\mathcal{C}_2} \) on \( \mathcal{C}_2 \) and \( F' \circ F \) is naturally isomorphic to \( \text{Id}_{\mathcal{C}_1} \) is called an **equivalence of categories**. We say that two categories are equivalent if there exist an equivalence between them.

A category \( \mathcal{C} \) is said to be **enriched** over a category \( \mathcal{D} \) if for every \( X,Y \in \text{Obj}(\mathcal{C}) \), \( \text{Hom}(X,Y) \in \text{Obj}(\mathcal{D}) \) and the composition \( \text{Hom}(Y,Z) \times \text{Hom}(X,Y) \to \text{Hom}(X,Z) \) is a morphism in \( \mathcal{D} \).

An abelian category is a category enriched with the category of abelian groups with some extra conditions.

**Definition 2.2.** An abelian category is a category \( \mathcal{C} \), such that: for each \( X,Y \in \text{Obj}(\mathcal{C}) \), \( \text{Hom}(X,Y) \) has a structure of an abelian group, and the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and finally, every morphism can be factored into an epimorphism followed by a monomorphism.

The category \( \text{Ab} \) of abelian groups and the category \( \text{R-Mod} \) of (left) \( R \)-modules over a ring \( R \) are examples of abelian categories.

Given a category \( \mathcal{C} \), one can construct a **complex**. That is, a sequence

\[
\cdots \to X_{i-1} \xrightarrow{\delta_i} X_i \xrightarrow{\delta_{i+1}} X_{i+1} \to \cdots
\]

For which \( \text{im}(\delta_i) \subseteq \ker(\delta_{i+1}) \). If \( \text{im}(\delta_i) = \ker(\delta_{i+1}) \) for all \( i \) then the complex is called an **exact sequence**. The following properties hold for exact sequences in an abelian category.
Theorem 2.3 (Five lemma). Consider the commutative diagram below with exact rows.

\[
\begin{array}{cccccc}
X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & Y_4
\end{array}
\]

a. If \( f_1 \) and \( f_3 \) are monomorphisms and \( f_0 \) is an epimorphism, then \( f_2 \) is a monomorphism.

b. If \( f_1 \) and \( f_3 \) are epimorphisms and \( f_4 \) is a monomorphism, then \( f_2 \) is an epimorphism.

Theorem 2.4 (Snake lemma). Consider the diagram below whose rows are short exact sequences.

\[
\begin{array}{cccccc}
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & 0
\end{array}
\]

Then there exists a canonical homomorphism \( \delta : \ker(f_3) \rightarrow \text{coker}(f_1) \) called the connecting homomorphism, such that

\[
0 \longrightarrow \ker(f_1) \longrightarrow \ker(f_2) \longrightarrow \ker(f_3) \xrightarrow{\delta} \text{coker}(f_1) \longrightarrow \text{coker}(f_2) \longrightarrow \text{coker}(f_3) \longrightarrow 0
\]

is exact, where all the maps other than \( \delta \) are the obvious ones induced by the diagram.

Let \( F : C_1 \rightarrow C_2 \) be a covariant functor between two abelian categories \( C_1 \) and \( C_2 \). The functor \( F \) is called additive if it commutes with addition of morphisms. \( \text{Hom} \) and the tensor product are examples of additive functors. An additive functor sends complexes to complexes, but does not generally send exact sequences to exact sequences.

Definition 2.5. Let \( F \) be a functor;

a. \( F \) is left exact if for a given exact sequence \( 0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0 \) the sequence \( 0 \rightarrow F(X_1) \rightarrow F(X_2) \rightarrow F(X_3) \) is exact.

b. \( F \) is right exact if for a given exact sequence \( X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0 \rightarrow 0 \) the sequence \( F(X_1) \rightarrow F(X_2) \rightarrow F(X_3) \rightarrow 0 \) is exact.

c. \( F \) is exact if it is both left exact and right exact.

The \( \text{Hom} \) functor is left exact while the tensor product functor is right exact. We now give the definition of cohomological functors.

Definition 2.6 (Cohomological functor). A cohomological functor (or \( \delta \)-functor) between abelian categories \( C_1 \) and \( C_2 \) is a sequence of functors

\[
T^i : C_1 \rightarrow C_2 \quad i = 0, 1, \ldots
\]

plus for each short exact sequence

\[
0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0
\]

in \( C_1 \) a morphism \( \delta^i : T^i(Z) \rightarrow T^{i+1}(X) \) functorial in the sequence, such that the sequence

\[
0 \longrightarrow T^0(X) \longrightarrow T^0(Y) \longrightarrow T^0(Z) \xrightarrow{\delta^0} T^1(X) \longrightarrow T^1(Y) \longrightarrow T^1(Z) \xrightarrow{\delta^1} T^2(X) \longrightarrow \cdots
\]

is exact.
A cohomological functor $T$ is said to be **universal** if given any other cohomological functor $U$ and a natural transformation $f^0 : T^0 \to U^0$, there is a unique sequence of natural transformations $f^i : T^i \to U^i$ starting with $f^0$ which commute with the $\delta_i$. Given $T^0$, any two extensions of it to a universal cohomological functor are naturally isomorphic.

Let $I$ be an object in an abelian category $C$. Then $I$ is **injective** if the functor $\text{Hom}(\_ , I) : C^{\text{op}} \to \text{Ab}$ is exact. Since the $\text{Hom}$ functor is left exact it suffices to show that if $0 \to X \to Y$ is a monomorphism, then for any morphism $X \to I$ we can find some morphism $Y \to I$ so that the diagram below commute.

\[
\begin{array}{ccc}
0 & \to & X \\
\downarrow & & \downarrow \\
& & Y \\
& \downarrow & \downarrow \\
& I & \to I \\
\end{array}
\]

If for every object $X$ in category $C$ there exists a monomorphism $X \to I$, where $I$ is an injective element, then we say that the category $C$ has enough injectives. For an abelian category with enough injectives, any universal cohomology functor can be computed using injective resolutions. Thus it is possible to define the right derived functor of $F$ as follows; for any object $X$, if $I^*$ is an injective resolution of $X$, set $R^iF(X) = H^i(F(I^*))$.

### 3 Sheaves

**Definition 3.1** (Presheaf of abelian groups). Let $X$ be a topological space. A presheaf $\mathcal{F}$ of abelian groups on $X$ is an assignment of each open set $U \subset X$ an abelian group $\mathcal{F}(U)$, and to every inclusion $V \subset U$ of open subsets of $X$ a morphism of abelian groups $r^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ called the restriction of $\mathcal{F}$ to $V$ subject to the following conditions.

- **a.** $\mathcal{F}(U) = 0 \iff U = \emptyset$, the empty set,
- **b.** $r^U_U$ is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$ and
- **c.** for any three open sets $U$, $V$, and $W$ such that $W \subset V \subset U$, $r^U_W = r^V_W \circ r^U_V$

In other words, a presheaf $\mathcal{F}$ of abelian groups is a contravariant functor $\mathcal{F} : \text{Top}(X) \to \text{Ab}$, where $\text{Top}(X)$ is a category whose objects are open sets and morphisms inclusion maps in which $\text{Hom}(U, V) = \emptyset$ if $V$ is not a subset of $U$ and $\text{Hom}(U, V)$ has only one element if $V$ is a subset of $U$. The category $\text{Ab}$ in the definition of presheaves of abelian groups can be replaced by any category $\mathcal{C}$ to obtain a presheaf with values in the fixed category $\mathcal{C}$. We refer to $\mathcal{F}(U)$ as the sections of the presheaf $\mathcal{F}$ over the open set $U$.

If in addition the presheaf in the definition above satisfies the following conditions:

- **i.** for any open set $U$, if $V_i$ is an open cover for $U$ and if $s \in \mathcal{F}(U)$ is an element such that $r^U_{V_i}(s) = 0$ for all $i$, then $s = 0$;
- **ii.** for an open set $U$, if $V_i$ is an open cover for $U$, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each $i$, with the property that for each $i, j$, $r^U_{V_i \cap V_j}(s_i) = r^U_{V_i \cap V_j}(s_j)$ then there is an element $s \in \mathcal{F}(U)$ such that $r^U_{V_i}(s) = s_i$ for each $i$.

Then $\mathcal{F}$ is called a **sheaf**.

**Definition 3.2.** Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on $X$. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U$ such that whenever $V \subset U$ is an inclusion, the diagram

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\
\downarrow r^U_V & & \downarrow r^U_V \\
\mathcal{F}(V) & \xrightarrow{\phi(U)} & \mathcal{G}(V)
\end{array}
\]
commute. The presheaf kernel of $\phi$, presheaf cokernel of $\phi$, and presheaf image of $\phi$ to be the presheaves given by $U \mapsto \ker(\phi(U))$, $U \mapsto \coker(\phi(U))$ and $U \mapsto \im(\phi(U))$ respectively.

**Remark 3.3.** If $\phi$ is a morphism of sheaves, then the presheaf kernel of $\phi$ is a sheaf. However, the presheaf cokernel and presheaf image of $\phi$ are not necessarily sheaves.

Let $X = \text{Spec} R$ the prime spectrum of $R = k[x_0, \ldots, x_n]$ endowed with the Zariski topology. In this topology, open sets are the distinguished open sets $D(f)$ for $f \in R$, where $D(F)$ is the set of all functions in $R$ that do not vanish outside the vanishing set of $f$. We can then define a ring $\mathcal{O}_X(U)$ for every open set $U \subset X$. We call $\mathcal{O}_X(U)$ the ring of regular function in the neighbourhood of $U$ and the assignment to every open set $U$ the ring $\mathcal{O}_X(U)$ the structure sheaf of $X$ denoted by $\mathcal{O}_X$. The pair $(X, \mathcal{O}_X)$ is a ringed space. Cohomology of sheaves can be defined by taking the derived functors of the global section functor. This is possible because for any ring $R$ every $R$-module is isomorphic to a submodule of some injective $R$-module.

Let $(X, \mathcal{O}_X)$ be a ringed space and consider the category $\textbf{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules. This category has enough injectives. Consequently then the category $\textbf{Ab}(X)$ of sheaves of abelian groups on $X$ has enough injectives.

**Definition 3.4.** Suppose $X$ is a topological space. Denote by $\Gamma(X, \_)$ the global section functor from $\textbf{Ab}(X)$ to $\textbf{Ab}$. We can then define the cohomology functor $H^i(X, \_)$ as the right derived functors of $\Gamma(X, \_)$.

For any sheaf $\mathcal{F}$, the groups $H^i(X, \mathcal{F})$ are referred to as the cohomology groups of $\mathcal{F}$.

For a given short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0,$$

there is a long exact sequence induced by the cohomology functor given by;

$$0 \longrightarrow H^0(X, \mathcal{F}_1) \longrightarrow H^0(X, \mathcal{F}_2) \longrightarrow H^0(X, \mathcal{F}_3) \longrightarrow H^1(X, \mathcal{F}_1) \longrightarrow H^1(X, \mathcal{F}_2) \longrightarrow H^1(X, \mathcal{F}_3) \longrightarrow \cdots$$

If $X$ is a noetherian topological space of dimension $n$, then $H^i(X, \mathcal{F}) = 0$ for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on $X$.

The dimension of the cohomology group $H^i(X, \mathcal{F})$ is denoted by $h^i(X, \mathcal{F})$. If $X = \mathbb{P}^n$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(d)$ is the sheaf of $p$-forms of degree $d$, then the dimension of the cohomology group $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ is given by the Botts formula.

**Theorem 3.5 (Botts formula [5]).**

$$h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} \binom{d+n-p}{d} \binom{d-1}{p} & \text{for } i = 0, \ 0 \leq p \leq n, \ d > p \\ 1 & \text{for } d = 0, \ 0 \leq p = i \leq n \\ \binom{-d-p}{-d} \binom{-d-1}{n-p} & \text{for } i = n, \ 0 \leq p \leq n, \ d < p - n \\ 0 & \text{otherwise} \end{cases}$$

In particular if $p = 0$, we have

$$h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} \binom{d+n}{d} & \text{for } i = 0, \ d \geq 0 \\ \binom{-d-1}{-d-1-n} & \text{for } d \leq -n - 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $X = \text{Spec} R$ and $\mathcal{O}_X$ be the structure sheaf of $X$. A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ on $X$ such that for each open set $U \subset X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module. An $\mathcal{O}_X$-module which is isomorphic to a direct sum of copies of $\mathcal{O}_X$ is called a free $\mathcal{O}_X(U)$-module. If the open sets $U$ such that $\mathcal{F}|_U$ is an $\mathcal{O}_X(U)$-module forms an open cover for the topological space $X$ then $\mathcal{F}$ is a locally free sheaf. The rank of $\mathcal{F}$ on is the number of copies of the structure sheaf needed whether finite or infinite. For a connected topological space $X$, the rank of a locally free sheaf is the same everywhere. A locally free sheaf of rank 1 is also called an invertible sheaf.
Definition 3.6. Let $X$ be a variety, a vector bundle over $X$ is a variety $E$ with a map $\pi : E \to X$ such that the following conditions hold:

a. For each $p \in X$, we have $\pi^{-1}(p)$ is isomorphic to $\mathbb{A}^n$ for some $n$.

b. There exists a cover $U_i$ of $X$ such that $\pi^{-1}(U_i)$ is isomorphic to $U_i \times \mathbb{A}^n$.

Given any locally free sheaves, one can define a vector bundle and conversely. By viewing locally free sheaves as vector bundles we can define elementary transformations.

Elementary transformation of Vector bundles [4]

Let $\mathcal{F}$ be a vector bundle on $X$. If we define a surjective map $\psi : \mathcal{F} \to \mathcal{F}'$, where $\mathcal{F}'$ is a vector bundle on a divisor $X'$ of $X$, then $\mathcal{E} = \ker(\psi)$ is a vector bundle on $X$. The procedure of obtaining $\mathcal{E}$ from $\mathcal{F}$ is called the elementary transformation of $\mathcal{F}$ along $\mathcal{F}'$ and is denoted by $\mathcal{E} = \text{elm}_{\mathcal{F}'}(\mathcal{F})$. We call $\mathcal{E}$ the elementary transform of $\mathcal{F}$ along $\mathcal{F}$. For the given $\psi : \mathcal{F} \to \mathcal{F}'$, we have the following exact, commutative diagram which is called the display of the elementary transformation:

\[
\begin{array}{cccccc}
0 & 0 & \downarrow & \downarrow & \\
\mathcal{F}(-X) & \mathcal{F}(-X) & \downarrow & \downarrow & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \mathcal{E} & \mathcal{F} & \mathcal{F}' & \to 0 & \\
\downarrow & \psi & \downarrow & \downarrow & \\
0 & \mathcal{F}'' & \mathcal{F}|_{X'} & \mathcal{F}' & \to 0 & \\
\downarrow & \downarrow & \\
0 & 0 & & & & \\
\end{array}
\]

where $\mathcal{F}''$ is the kernel of $\psi|_{X'}$ and $\mathcal{F}(-X) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-X)$. The leftmost vertical exact sequence gives us the inverse of the given transformation, that is, $\mathcal{F}(-X) = \text{elm}_{\mathcal{F}''}(\mathcal{E})$.

Remark 3.7. There is an equivalence of categories between the category of algebraic vector bundles and the category of locally free sheaves, given by associating to an algebraic vector bundle $F \mapsto X$, the sum $F$ of sections of $F$.

4 Main

In this section, we prove our main result, that is, we use the equivalence in category between the category of locally free sheaves and the category of algebraic vector bundles to give an elementary transformation in $\mathbb{P}^n$. Now that $\mathcal{O}_X$-modules form an abelian group, but locally free sheaves along with reasonably natural maps between them (those arising as maps of $\mathcal{O}_X$-modules) do not form an abelian category, we will enlarge our notion of nice $\mathcal{O}_X$-modules to quasi-coherent sheaves. In fact our locally free sheaves have finite rank, and so we will talk about coherent sheaves. We will conclude this section by describing how the elementary transformation in $\mathbb{P}^n$ can be used to prove the minimal resolution conjecture, especially when the points under consideration are in general position.
Theorem 4.1. There exist an elementary transformation of vector bundles on $\mathbb{P}^n$ comprising of the following exact sequences.

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & \ & \ & \ & \ & \ & \\
& & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\Omega^p_{\mathbb{P}^n}(p + 1) & \rightarrow & \Omega^p_{\mathbb{P}^n}(p + 1) & \rightarrow & \Omega^p_{\mathbb{P}^n}(p + 2) & \rightarrow & \Omega^p_{\mathbb{P}^n-1}(p + 2) & \rightarrow & 0 \\
& & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^{n+1}} \oplus (n + 1) & \rightarrow & \Omega^p_{\mathbb{P}^n}(p + 1) & \rightarrow & \Omega^p_{\mathbb{P}^n-1}(p + 2) & \rightarrow & 0 \\
& & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & \Omega^p_{\mathbb{P}^n-1}(p + 1) & \rightarrow & \Omega^p_{\mathbb{P}^n-1}(p + 2) & \rightarrow & \Omega^p_{\mathbb{P}^n-1}(p + 2) & \rightarrow & 0 \\
& & & & & & & & \\
\end{array}
\]

The proof of this theorem above follows from the following set of claims.

Claim 4.2. 

i) The kernel of the map $\Omega^p_{\mathbb{P}^n} \rightarrow \Omega^p_{\mathbb{P}^n-1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-p - 2)^{\oplus (n+1)}$.

ii) The sequence $0 \rightarrow \Omega^p_{\mathbb{P}^n}(p) \rightarrow \Omega^p_{\mathbb{P}^n-1}(p + 2) \rightarrow \Omega^p_{\mathbb{P}^n-1}(p + 2) \rightarrow 0$ is exact.

iii) The kernel of the map $\mathcal{O}_{\mathbb{P}^n}(n+1) \rightarrow \Omega^p_{\mathbb{P}^n-1}(2)$ is isomorphic to $\Omega^p_{\mathbb{P}^n-1}(2)$.

Proof. 

i) We prove the first claim by induction on $p$. For the base case, we set $p = 0$ and prove that the kernel of the map $\Omega^p_{\mathbb{P}^n} \rightarrow \Omega^p_{\mathbb{P}^n-1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-p - 2)^{\oplus (n+1)}$.

Consider the Euler sequence.

\[
0 \rightarrow \Omega^p_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}^n-1} \rightarrow 0
\]

We can construct a commutative diagram of exact sequences on $\mathbb{P}^n$:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Omega^p_{\mathbb{P}^n} & \rightarrow & \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} & \rightarrow & \mathcal{O}_{\mathbb{P}^n} & \rightarrow & 0 \\
& & & \downarrow e & \downarrow b & \downarrow c & \ & \ & \\
0 & \rightarrow & \Omega^p_{\mathbb{P}^n-1} & \rightarrow & \mathcal{O}_{\mathbb{P}^n-1}(-1)^{\oplus n} & \rightarrow & \mathcal{O}_{\mathbb{P}^n-1} & \rightarrow & 0 \\
& & & & & & & & \\
\end{array}
\]

where $\mathbb{P}^n-1$ is identified with the locus of $\mathbb{P}^n$ where the last coordinate vanish. The rows are the Euler sequences, $e$ is the restriction of forms, $c$ is the restriction of functions and $b$ is the restriction on the first $n$ summands of $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}$ and is 0 on the last summand. Considering the kernels of $e$, $b$ and $c$ we get a commutative diagram with exact rows and
columns,

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & & \\
0 & \to & K & \to & \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1) & \xrightarrow{\alpha} & \mathcal{O}_{\mathbb{P}^n}(-1) & \to & 0 \\
\downarrow & & & & & & & \\
0 & \to & \Omega_{\mathbb{P}^n} & \to & \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} & \xrightarrow{S_n} & \mathcal{O}_{\mathbb{P}^n} & \to & 0 \\
\downarrow & & & & & & & \\
0 & \to & \Omega_{\mathbb{P}^{n-1}} & \to & \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\oplus n} & \xrightarrow{} & \mathcal{O}_{\mathbb{P}^{n-1}} & \to & 0 \\
\downarrow & & & & & & & \\
0 & & & & & & & 0
\end{array}
\]  

and we need to prove that \( K \cong \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus n} \). It suffices to show that \( \alpha \) sends the last summand \( \mathcal{O}_{\mathbb{P}^n}(-1) \) of \( \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \) isomorphically to \( \mathcal{O}_{\mathbb{P}^n}(-1) \). As this summand is isomorphic to the codomain, it suffices to prove that the restriction of \( \alpha \) to the summand is \( \mathcal{O}_{\mathbb{P}^n}(-1) \) is generically injective. This is true because the restriction of \( S_n \) to the last summand is an isomorphism outside of \( \mathbb{P}^{n-1} \). (homomorphism of line bundles on the complement of \( \mathbb{P}^{n-1} \)). As a consequence the first row of (4.2) splits and \( K \cong \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus n} \).

Suppose now that the kernel of the map \( \Omega_{\mathbb{P}^n}/\mathcal{O}_{\mathbb{P}^{n-1}} \) induced by the restriction of forms is isomorphic to \( \mathcal{O}_{\mathbb{P}^{n}}(-p - 1)^{\oplus (n)} \). Consider the exact sequence below obtained from the Euler sequence of \( \mathbb{P}^n \).

\[
0 \to \Omega_{\mathbb{P}^n}^{+1} \to \wedge^{p+1}(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}) \to \mathcal{O}_{\mathbb{P}^n}^p \to 0
\]

\[
\mathcal{O}_{\mathbb{P}^n}(-p - 1)^{\oplus (n+1)}
\]

We can then construct the commutative diagram below with exact rows;

\[
\begin{array}{cccccccc}
0 & \to & \Omega_{\mathbb{P}^n}^{+1} & \to & \wedge^{p+1}(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}) & \to & \mathcal{O}_{\mathbb{P}^n}^p & \to & 0 \\
\downarrow{c} & & \downarrow{b} & & \downarrow{c} & & \downarrow & & \\
0 & \to & \Omega_{\mathbb{P}^{n-1}}^{+1} & \to & \wedge^{p+1}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n}) & \to & \mathcal{O}_{\mathbb{P}^{n-1}}^p & \to & 0
\end{array}
\]

where \( c \) and \( c \) are restrictions of forms and \( b \) is described as follows. Recall that \( \mathbb{P}^{n-1} \) is the locus where the last coordinate vanishes and decompose \( \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \) as the direct sum of the first \( n \) summands and the last summand, that is, \( \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} = \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \). The domain of \( b \) is \( \wedge^{p+1}\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \)

\[
\mathcal{O}_{\mathbb{P}^n}(-p - 1)^{\oplus (n) + 1} \oplus \mathcal{O}_{\mathbb{P}^n}(-p - 1)^{\oplus (n+1)}
\]

The codomain of \( b \) is \( \wedge^{p+1}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n}) = \mathcal{O}_{\mathbb{P}^n}(-p - 1)^{\oplus (n+1)} \). Thus \( b \) sends \( \wedge^{p}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n}) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \) to zero and its restriction to \( \wedge^{p+1}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n}) = \mathcal{O}_{\mathbb{P}^n}(-p - 1)^{\oplus (n+1)} \) is natural. In particular, the kernel of \( b \) is \( \wedge^{p+1}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n}) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \wedge^{p}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n}) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{O}_{\mathbb{P}^n}(-p - 2)^{\oplus (n+1)} \oplus \mathcal{O}_{\mathbb{P}^n}(-p - 1)^{\oplus (n+1)} \). Considering the kernel of \( a \), \( b \) and \( c \) we get a
It suffices to prove that the first row splits. To do this, we prove that the restriction of \( \alpha \) to \( \wedge^p \mathcal{O}_{\mathbb{P}^n-1}(-1)^{\oplus n} \otimes \mathcal{O}_{\mathbb{P}^n-1} \) is an isomorphism on \( \mathcal{O}_{\mathbb{P}^n}(-p-1)^{\oplus (p+1)} \). Since we know that \( \wedge^p \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n} \otimes \mathcal{O}_{\mathbb{P}^n-1} \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^n}(-p-1)^{\oplus (p+1)} \), it suffices to show that \( \alpha \) is of maximal rank on all open subsets. This is because on the complement of \( \mathbb{P}^{n-1} \) the morphism \( \gamma \) injects \( \wedge^p \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n} \otimes \mathcal{O}_{\mathbb{P}^n-1} \) in \( \Omega_{\mathbb{P}^n}^{p+1} \).

ii) To prove the exactness of the bottom row, consider the exact sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \Omega_{\mathbb{P}^n}^1|_{\mathbb{P}^n-1} \rightarrow \Omega_{\mathbb{P}^n-1}^1 \rightarrow 0.
\]

Taking the \((p+1)\)-th exterior product and tensoring by \( \mathcal{O}(p+2) \) we get the desired exact sequence.

iii) Finally, to show that the kernel of the map \( \mathcal{O}_{\mathbb{P}^n}^{\oplus (p+1)} \rightarrow \Omega_{\mathbb{P}^n-1}^{p+1}(2) \) is isomorphic to \( \Omega_{\mathbb{P}^n}^{p+1}(2) \), we first recall that given a commutative diagram whose lines are two exact sequences.

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]

there always exists a map \( A \rightarrow D \) still making the extended diagram commutative. Let \( a, b \) and \( c \) be the 3 vertical maps appearing in this extended diagram. By the Snake lemma, we have an exact sequence

\[
0 \rightarrow \text{ker}(a) \rightarrow \text{ker}(b) \rightarrow \text{ker}(c) \rightarrow \text{coker}(a) \rightarrow \text{coker}(b) \rightarrow \text{coker}(c) \rightarrow 0.
\]

Since \( c \) is an isomorphism, \( \text{ker}(a) = \text{ker}(b) \). As the second column of the diagram is obtained by tensoring \( \Omega_{\mathbb{P}^n}^{p+1}(p+2) \) with the exact sequence

\[
0 \rightarrow I \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n-1} \rightarrow 0
\]

defining the ideal of \( \mathbb{P}^{n-1} \) in \( \mathbb{P}^n \), we have \( \text{ker}(a) = \text{ker}(b) = \Omega_{\mathbb{P}^n}^{p+1}(p+1) \).
In conclusion, let’s see how the results above can be applied in the study of minimal free resolution.

Suppose \( X \) is a smooth projective variety and \( X' \) is a non-singular divisor of \( X \). Let \( F \) be a locally free sheaf on \( X \) and

\[
0 \longrightarrow F'' \longrightarrow F|_{X'} \longrightarrow F' \longrightarrow 0
\]

be an exact sequence of locally free sheaves on \( X' \). The kernel \( E \) of \( F \rightarrow F' \) is a locally free sheaf on \( X \) and we have another exact sequence of locally free sheaves on \( X' \)

\[
0 \longrightarrow F'(-X') \longrightarrow E|_{X'} \longrightarrow F'' \longrightarrow 0
\]

and as well exact sequences of coherent sheaves on \( X \)

\[
0 \longrightarrow E \longrightarrow F \longrightarrow F' \longrightarrow 0
\]

and

\[
0 \longrightarrow F(-X) \longrightarrow E \longrightarrow F'' \longrightarrow 0.
\]

**Theorem 4.3** (Differential method of Horace). Suppose we are given a surjective morphism of vector spaces, \( \lambda : H^0(X', F') \rightarrow L \) and suppose that there exist a point \( Z' \in X' \) such that \( H^0(X', F') \rightarrow L + F'|_{Z'} \) and suppose that \( H^1(X, E) = 0 \). Then there exists a quotient \( \mathcal{E}(Z') \rightarrow D(\lambda) \) with a kernel contained in \( F'(Z') \) of dimension \( \dim(D(\lambda)) = \text{rk}(\mathcal{F}) - \dim(\ker \lambda) \) having the following property. Let \( \mu : H^0(X, F) \rightarrow M \) be a morphism of vector spaces, then there exist \( Z \in X' \) such that if \( H^0(X, \mathcal{E}) \rightarrow M \oplus D(\lambda) \) is of maximal rank then \( H^0(X, F) \rightarrow M \oplus L \oplus F(Z) \) is also of maximal rank.

**Remark 4.4.** The idea of the theorem is illustrated in the diagram below.

\[
\begin{array}{cccc}
0 & \longrightarrow & H^0(X, E) & \longrightarrow & H^0(X, F) & \longrightarrow & H^0(X', F') & \longrightarrow & 0 \\
\alpha_1 & & \alpha_2 & & \alpha_3 & & \\
0 & \longrightarrow & M \oplus D(\lambda) & \longrightarrow & M \oplus L \oplus F(Z) & \longrightarrow & L \oplus D'(\lambda)|_Z & \longrightarrow & 0
\end{array}
\]

The key point is that if the map \( \alpha_3 \) is bijective, then \( \alpha_2 \) will be bijective provided that \( \alpha_1 \) is bijective.

The elementary transformation in theorem \([4.3]\) above can be used in proving inductively that the map

\[
H^0 \Big( \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d + p + 1) \Big) \longrightarrow \bigoplus_{i=1}^{s} \Omega^{p+1}_{\mathbb{P}^n}(d + p + 1)|_{P_i}
\]

is of maximal rank for a fixed \( p \), for all non-negative integers \( d \geq m \). To see this, set \( X = \mathbb{P}^n \), \( X' = \mathbb{P}^n-1 \), \( F = \Omega^{p+1}_{\mathbb{P}^n} \), \( F' = \Omega^{p}_{\mathbb{P}^n-1} \) and \( E = \mathcal{O}_{\mathbb{P}^n}(2) \otimes_{\mathbb{P}^n}^{\mathbb{P}^n+1} \) in the diagram under remark \([4.4]\). We can also construct a diagram similar to this using the sequence

\[
0 \longrightarrow H^0(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \otimes_{\mathbb{P}^n}^{\mathbb{P}^n+1}(d - 1)) \longrightarrow H^0(\mathbb{P}^n-1, \Omega^{p}_{\mathbb{P}^n-1}) \longrightarrow 0.
\]

Call the three vertical maps in this second diagram \( \alpha'_1 \), \( \alpha'_2 \) and \( \alpha'_3 \). By construction of the map \( \alpha'_1 \) coincides with the map \( \alpha'_2 \). In reference to remark \([4.4]\) we have that \( \alpha_2 \) is bijective provided \( \alpha_1 \) is, and \( \alpha_1 \) is bijective provided that \( \alpha'_1 \) is. Bijectivity of \( \alpha_2 \) is a statement on forms of degree \( d + 1 \) and bijectivity of \( \alpha'_1 \) is a statement of forms of degree \( d \). It therefore follows that if we can prove such bijectivity for \( d = m \), and also prove that the bijectivity of \( \alpha_1 \) implies bijectivity of \( \alpha_2 \) and bijectivity of \( \alpha'_1 \) implies bijectivity of \( \alpha_1 \), then it will follow by induction that the map \([4.4]\) is of maximal rank for all \( d \geq m \).
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