On constraint-consistency, covariant operators, 
gauge-invariance, etc

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Abstract

We look at the covariant techniques and the ideas on constraints and gauge-invariance, which were recently employed in [1] to support earlier work by the same authors. That work was criticised in [2]. Using very simple and well known examples we show that, when adopted, the methods and views of [1] lead to basic-level mathematical problems, with analogous consequences for the physics. We provide a few simple rules that should help to avoid similar problems in the future.

1 Introduction

The subject of this note is related to the 1+3-covariant study of the interaction between gravitational waves and large-scale magnetic fields in cosmology. Our aim is not to discuss this interaction again. The literature is available to anyone who might be interested. Here, we will look at the mathematics (as recently explained in [1]) on which the physics of [3] was based upon.

The main issues have to do with the consistency of constraints, the use of 3-D commutation operators and gauge-invariance. The problems are technical but also elementary. The authors imposed inconsistent constraints, used incomplete commutation laws and presented as gauge-invariant results that are gauge-dependent by construction. We argue mainly by using simple and well known examples and provide a few basic rules that should help to avoid analogous problems in the future. Given that, the present brief communique may also have some pedagogical value.

2 Constraints

It was claimed (see § III.B in [3]) that, despite the presence of an inhomogeneous magnetic field, the electric component stays curl-free, provided it was so initially and the zero-order FRW model is spatially flat. The vanishing of curl$E_a$ does not depend on the scale, or the electrical conductivity, and applies to the second perturbative order. Even without looking at the mathematics, this claim sounds rather extreme.

A constraint is consistent if, once imposed, it holds at all times with no need for further restrictions. To check whether curl$E_a$ vanishes, we must look for sources in the propagation equation (i.e. the first time-derivative) of curl$E_a$. In [3] the authors considered instead the second time-derivative of this quantity, which on a spatially flat FRW background reads (see Eq. (29) in [3])

$$
(curlE_a)^\cdot = -\frac{7}{3} \Theta (curlE_a)^\cdot + D^2 curlE_a - \left[\frac{7}{9} \Theta^2 + \frac{1}{6} (\rho - 9p) + \frac{5}{3} \Lambda\right] curlE_a.
$$

Here, curl$E_a = \varepsilon_{abc} D^b E^c$, $\Theta$ is the background volume expansion, $\rho$ and $p$ are the matter density and pressure, $\Lambda$ is the cosmological constant, $D_a$ is the 3-D covariant derivative and $D^2 = D^a D_a$ is the
associated Laplacian. The absence of explicit source terms in Eq. (1) misled the authors into claiming that curl$E_a$ will remain zero, if it was so initially (see § III.B in [3] and also [1]). This conclusion is incorrect, as the following simple example shows.

Consider a perturbed FRW universe with dust. It is well known that the second time-derivative of the density perturbation (here represented by $\Delta$) reads

$$\ddot{\Delta} = -\frac{2}{3} \Theta \dot{\Delta} + \frac{1}{2} \rho \Delta. \quad (2)$$

Just like Eq. (1), the above contains no explicit source terms. Then, if we were to adopt [1, 3], $\Delta$ will stay zero if it was initially zero and there will be no density perturbations. We all know that this is not the case. To look for sources one must check the first time-derivative of $\Delta$, namely the expression

$$\dot{\Delta} = -Z. \quad (3)$$

This ensures that density perturbations are sourced by those in the expansion (represented here by $Z$). For the same reasons, when dealing with curl$E_a$, one must look for sources in the first time-derivative of this quantity. The latter gives (see Eq. (1) in [2])

$$\text{curl curl} E_a^{(2)} = -\Theta \text{curl} E_a^{(2)} + \mathcal{R}_{ab}^{(1)} B_b^{(1)} - D^2 B_a^{(2)} - \text{curl} J_a^{(2)}, \quad (4)$$

where $\mathcal{R}_{ab}$ is the 3-Ricci tensor, $J_a$ is the 3-current and the indices (1), (2) indicate the perturbative order of the variable. Thus, even if curl$E_a = 0$ initially, it will not remain so once the magnetic and the current perturbations kick in (unless further constraints – on the sources – are imposed). The zero electric-curl claim made in [3] is unsustainable.

Note that, in line with [3], the $B$-field vanishes to zero order has a homogeneous first-order component ($B_a^{(1)}$) and an inhomogeneous second-order part ($B_a^{(2)}$). Thus, $B_a = B_a^{(1)} + B_a^{(2)}$. Splitting the variables like that is largely avoided in covariant calculations, but we will adopt it here to agree with [1].

### 3 Commutators

The 3-D commutators have the form $2D_{[a}D_{b]} = D_aD_b - D_bD_a$ and monitor the commutation of 3-D covariant derivatives. How to use these operators became an issue, when [1] sought to remove the source terms from the right-hand side of Eq. (1). One can imagine situations where some of these sources are negligible relative to the rest, but not all three of them simultaneously. When the Laplacian is the only quantity left in the right-hand side of (1), in particular, we cannot ignore it (even on large – finite– scales) because it vanishes only asymptotically (i.e. at infinity). Here, we will focus on the curvature term because the argument used to remove this quantity suffers at the elementary level.

The curvature term in Eq. (1) appears when analysing the quantity curl curl$B_a$ by means of the covariant 3-D commutator $D_{[a}D_{b]}$. This operator applies to the full variable and the commutation takes place prior to any decomposition (into background, first-order perturbation, etc – the reasons will become clear below). It is argued in [1] (see Eq. (3) there), that we must instead split the variables first and commute their 3-D gradients afterwards. Splitting the magnetic field into $B_a = B_a^{(1)} + B_a^{(2)}$, the authors involve only the inhomogeneous $B_a^{(2)}$-part and arrive at the second-order formula

$$\text{curl curl} B_a = \text{curl curl} B_a^{(2)} = -D^2 B_a^{(2)} + \mathcal{R}_{ab}^{(0)} B_b^{(2)} = -D^2 B_a^{(2)}, \quad (5)$$

Initially, refers to the initial perturbed hypersurface and not to the FRW background, as [1] seems to suggest.
since \( R^{(0)}_{ab} = 0 \) (see Eq. (3) in \[1\]). The term \( R^{(1)}_{ab} B^k_b \) of \[1\] does not appear above, although it is also second order and it should have been included.

We will accept, for argument’s sake, the recipe leading to expression (5) and test it on a well-known case. Assume a perturbed FRW universe containing a single perfect fluid with nonzero pressure. Following \[1\], we decompose the density as \( \rho = \rho^{(0)} + \rho^{(1)} \), where \( \rho^{(0)} \) is the homogeneous (zero-order) part and \( \rho^{(1)} \) the inhomogeneous perturbation. Then, in line with \[1\] and Eq. (5), only \( \rho^{(1)} \) takes part in the operation and the linear commutator for the 3-gradients of \( \rho \) reads

\[
D_{[a}D_{b]}\rho = D_{[a}D_{b]}\rho^{(1)} = -\dot{\rho}^{(1)}\omega^{(0)}_{ab} = 0, \tag{6}
\]

since \( \omega^{(0)}_{ab} = 0 \). However, almost everyone using the covariant equations knows that the appropriate linear expression is

\[
D_{[a}D_{b]}\rho = D_{[a}D_{b]}\rho^{(1)} = -\dot{\rho}^{(0)}\omega^{(1)}_{ab} \neq 0, \tag{7}
\]

obtained easily when we commute first and split afterwards. The commutation recipe employed in \[1\] is incomplete. Note that if we were to use relation (6) instead of (7), the linear propagation equation of the vorticity would be given by

\[
\dot{\omega}_{ab} = -\frac{2}{3}\Theta\omega_{ab}, \tag{8}
\]

instead of the familiar expression

\[
\dot{\omega}_{ab} = -\frac{2}{3}\left(1 - \frac{3}{2}c_s^2\right)\Theta\omega_{ab}. \tag{9}
\]

The extra term in the parentheses results from the right-hand side of Eq. (7) – \( c_s^2 \) is the square of the adiabatic sound-speed. Similar problems will emerge in several other expressions, if we were to follow the commutation rule of \[1\], and relativistic cosmology would require drastic revision.

## 4 Gauge-invariance

An additional issue is that of gauge-invariance. In \[1, 3\] the authors consider the aforementioned \( B \)-field, which has nonzero first-order value and is therefore gauge-dependent at second order by well-known theorems. It is noted that the first-order field has

\[
\dot{B}^{(1)}_a + \frac{2}{3}\Theta B^{(1)}_a = 0, \tag{10}
\]

which makes the quantity

\[
\beta_a = \dot{B}_a + \frac{2}{3}\Theta B_a = \dot{B}^{(2)}_a + \frac{2}{3}\Theta B^{(2)}_a, \tag{11}
\]

a gauge-independent variable at second order. This is true, although there should be a \( 2\Theta^{(1)} B^{(1)}_a / 3 \) term somewhere in the above (to simplify things let’s say that \( \Theta^{(1)} = 0 \)). The real problem starts when the authors claim that they can integrate Eq. (11), with respect to \( B^{(2)}_a \), and obtain gauge-independent results (see § IV in \[1\]). This is an interesting idea, but it does not sound right. The gauge ambiguity resides in \( B^{(2)}_a \) itself, by construction, due to the nonzero first-order value of \( B_a \).

Let us assume, for argument’s sake again, that \[1\] are right and apply their principle to a familiar case. Consider a perturbed FRW universe containing dust with density \( \rho = \rho^{(0)} + \rho^{(1)} \). This quantity is the archetype of a gauge-dependent perturbation. To zero order,

\[
\dot{\rho}^{(0)} + \Theta\rho^{(0)} = 0, \tag{12}
\]

The reader might wonder why it should make a difference which operation (splitting or commuting) is done first. It should not in principle, but if we first split and then commute the chances of doing something wrong increase considerably.
which means that the quantity
\[ \dot{q} = \dot{\rho} + \Theta \rho = \dot{\rho}^{(1)} + \Theta \rho^{(1)} \] (13)
is a gauge-invariant first-order perturbation. Following [1], one can integrate the above and obtain gauge-independent results for \( \rho^{(1)} \). The same principle could be applied to every gauge-dependent perturbation. If this were true, there should be no need to introduce complicated variables to describe gauge-invariant perturbations and the gauge problem would have been a rather trivial one. Unfortunately, it is not that simple. Nevertheless, if the authors still believe in their case, they should communicate their findings without delay.

5 Numerical results

It is also worth mentioning the numerical evaluation of Eq. (14) (see expression (7) in [1]). The latter gives the strength of a gravitationally amplified \( B \)-mode that passes through reheating and enters the horizon sometime in the radiation era
\[ B = \tilde{B}_0 \left[ 1 + \left( \frac{a_{RH}}{a_0} \right)^2 \left( \frac{\sigma}{H} \right)_0 + \left( \frac{\lambda_{\tilde{B}}}{\lambda_H} \right)^2 \left( \frac{\sigma}{H} \right)_{RH} + \left( \frac{\lambda_{\tilde{B}}}{\lambda_H} \right)^2 \left( \frac{\sigma}{H} \right)_0 \left( \frac{\sigma}{H} \right)_{RH} \right] \left( \frac{a_0}{a} \right)^2. \] (14)
Here \( \sigma/H \) describes the shear anisotropy, \( \lambda_{\tilde{B}} \) and \( \lambda_H \) are the scale of the \( B \)-field and the horizon size, while the indices 0 and \( RH \) indicate evaluation at the end of inflation and reheating respectively (with \( \tilde{B} = B^{(1)} \)). The physics leading to the above result was explained in [2]. It was clarified there that the amplification is of the superadiabatic nature, takes place outside the horizon and defers from epoch to epoch. This view has also been adopted in [1], but the authors dropped the last term in the brackets claiming that it is of the third order. This is not so, as a bit of algebra can easily show. What is more important, is that the quantities in the brackets are numbers and what decides which stays or goes is their value.

As explained in [2] (see Eq. (10) there), when there is substantial magnetic amplification during both the reheating and the radiation epochs, the third term in (14) prevails. All these are of uncertain practical value, however, as long as the gauge-issue remains unresolved.

6 Conclusions

We close by summarising this communique in four basic rules: (i) To check the consistency of a constraint we look at the first (not the second) time-derivative of the constrained quantity; (ii) To avoid mistakes, we first commute the 3-D gradients and then split the involved variables (not the other way around); (iii) The gauge-ambiguity and the arbitrariness are in the variable that describes the perturbation; (iv) When evaluating sums like that of Eq. (11), one should always ensure that all the terms are properly accounted for.

References

[1] G. Betschart, C. Zunckel, P.K.S. Dunsby and M. Marklund, gr-qc/0702104.
[2] C.G. Tsagas, Phys. Rev. D, to appear (gr-qc/0503042).
[3] G. Betschart, C. Zunckel, P.K.S. Dunsby and M. Marklund, Phys. Rev. D 72, 123514 (2005).

3It helps to recall that the amplified \( B \)-field at the end of reheating, acts as the background source for the gravito-magnetic interaction in the radiation era (see [2]).