No free lunch for markets with multiple numé raires

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Abstract We consider a new framework, that of a global market with a finite number of submarkets, where there is a tradable numéraire for each submarket, but no tradable numéraire for the global market. Under a global no arbitrage condition, we show the existence of a common density from which equivalent (local) martingale measures are constructed for each submarket. We also introduce several superreplication prices, depending on the chosen type of hedging: on the global market, on a given submarket or on all submarkets separably. We prove duality results on these prices that allow to assess differences in characteristics between the submarkets, such as liquidity or credit quality. The results are applied in concrete situations, in particular in a Brownian setup.

Keywords Multiple numé raires · No free lunch · Martingale measure · Superreplication price · Illiquidity · Multicurve model · Credit spread

1 Introduction

The concept of no arbitrage (NA) is fundamental in the modern theory of mathematical finance. Let us consider a one period market with $d+1$ assets with price process $(S_t)_{t \in \{0,1\}}$. Let $\Phi \in \mathbb{R}^{d+1}$ be the number of shares purchased at time zero. It is an arbitrage if $\Phi.S_0 \leq 0$, $\Phi.S_1 \geq 0$ P-a.s. and $P(\Phi.S_1 > 0) > 0$, where $P$ is the probability measure prevailing on the state space. This condition has a good mathematical characterization in terms of martingale measures, which is called the fundamental theorem of asset pricing (FTAP). To state the FTAP, one has to consider a specific financial asset called numéraire, which allows to deflate the other assets. They are then denominated in units of the numéraire as units of account. It is important to note that the NA
condition does not depend on the numéraire, but the martingale measures do. To see this, let \( \hat{S} = (S^0, S) \) where \( S^0 \) is the numéraire and \( S \) is the price process of the \( d \) other assets. If \( S^0 \) is a tradable asset, and only in this case, NA is equivalent to the following property: any investment \( \Phi \) in the other \( d \) traded assets which yields with positive probability a better result than investing the same amount (divided by the initial value of the numéraire) in the numéraire must be exposed to some downside risk, i.e., setting \( \hat{S} = S/S^0 \) there is not \( \Phi \in \mathbb{R}^d \) such that \( \Phi.(\hat{S}_1 - \hat{S}_0) \geq 0 \) \( \mathbb{P} \)-a.s. and \( \mathbb{P}(\Phi.(\hat{S}_1 - \hat{S}_0) > 0) > 0 \). Then, the FTAP asserts that NA is equivalent to the existence of a probability measure \( Q \) equivalent to \( \mathbb{P} \) and such that \( \hat{S} \) is martingale under \( Q \).

In general market, under integrability conditions, the FTAP asserts that an appropriate notion of no arbitrage is equivalent to the existence of equivalent local martingale (or risk-neutral) probability measures for the price process of the assets deflated by the tradable numéraire. One usually takes the numéraire to be a savings account (or zero-coupon bond) expressed in the currency of the country. The FTAP was initially formalised in [10,11], while [4] established it in a general discrete-time setting, and [5] did so in continuous-time models. The literature on the subject is vast, and we refer to [6,13] for a general overview. The FTAP is essential for pricing issues, namely, for computing the superreplication price, which is, for a given claim, the minimum selling price needed to superreplicate it by trading in the market. This is the hedging price with no risk, and to the best of our knowledge, it was first introduced in [1] in the context of transaction costs. In complete markets, the superreplication cost is just the cash flow expectation computed under the unique risk-neutral measure. However, in incomplete markets, the superreplication cost is equal to the supremum of those expectations computed under the different risk-neutral probability measures. This is the so-called dual formulation of the superreplication price or superhedging theorem (see, for instance, [15] or [3]).

In this paper, we assume that there is a global market made of several submarkets and we suppose that one cannot trade between submarkets. So, there is not one asset that is traded in all the submarkets and that could be used as a numéraire in the FTAP. Still we consider the absence of arbitrage at the level of the global market and not only in each submarket. Here, we want to capture the situation where each submarket is arbitrage free but where, without further assumption, there may be arbitrage in the global market.

One may think of a big company with several Business Units (BU). Each BU acts as a separate part of the company and has some form of autonomy in its operations. Each BU is implemented around a single area of activity with its own objectives and resources. Nevertheless, the total wealth of the company is the sum of the wealth of each BU and an arbitrage opportunity may occur at the level of the company even if every BU is arbitrage free. One may also think of a financial institution with several trading desks specialized by financial products or market segments: stocks, currencies, commodities, bonds, etc. As argued by [12], cognitive restrictions require traders to restrict attention to a given subset of assets. Each desk is independent but the P&L of the financial institution is the sum of the P&L of each desks.
So, we suppose that the economic agent (the company or the financial institution) starts with an endowment in each submarket (BU or desks), trades separately in each submarket and that her total wealth is the sum of her wealth in each submarket. Note that we do not allow borrowing on a submarket against others. This is justified by the fact that a company does not allocate its wealth by betting on one BU against another or even that a bank has risk limits and cannot invest massively on a desk. Going back to the example of a one period model, we suppose that there are two submarkets: the submarket $\tau_1$ with $d_1 + 1$ assets with price process $(\hat{S}^{\tau_1}_{t \in \{0,1\}})$ and the submarket $\tau_2$ with $d_2 + 1$ assets with price process $(\hat{S}^{\tau_2}_{t \in \{0,1\}})$. Then, $(\hat{\Phi}^{\tau_1}, \hat{\Phi}^{\tau_2}) \in \mathbb{R}^{d_1 + d_2 + 2}$ is an arbitrage in the global market if $\hat{\Phi}^{\tau_1}\cdot \hat{S}^{\tau_1}_{0} \leq 0$ and $\hat{\Phi}^{\tau_2}\cdot \hat{S}^{\tau_2}_{0} \leq 0$ and $\hat{\Phi}^{\tau_1}\cdot \hat{S}^{\tau_1}_{1} + \hat{\Phi}^{\tau_2}\cdot \hat{S}^{\tau_2}_{1} \geq 0 \mathbb{P}$-a.s. and $\mathbb{P}(\hat{\Phi}^{\tau_1}\cdot \hat{S}^{\tau_1}_{1} + \hat{\Phi}^{\tau_2}\cdot \hat{S}^{\tau_2}_{1} > 0) > 0$.

Our purpose is to extend the FTAP and the Superhedging Theorem when no tradable numéraire exists for the global market but when there is a tradable numéraire per submarket. Under the assumption that there is no arbitrage in the global market, there exist risk-neutral measures (or state price deflators) that depend on the submarket but are constructed from a common random variable (see Theorem 1). If we apply the classical FTAP separately to each submarkets (which are arbitrage free), we obtain the existence of submarket-dependent risk-neutral measures, but we are not able to find some common factor between them. We find that in general, it is not possible to identify a common risk-neutral measure. Nevertheless, we prove in Section 6 that this is true when the spreads between the different submarkets are deterministic (and of course when there is a common numéraire). In this section, we also show by examples and by counter-examples that there is no relationship between the completeness of all submarkets and that of the global market.

The initial motivation for this modelisation was the multicurve setting of the post-2007 interest rate market. Since 2007, significant spreads are observed between the zero-coupon (ZC) curves associated to different frequencies (or tenors) and the market practice for interest rate swap valuation has evolved and consider several ZC curves which are tenor based. This means that each curve is build using instruments, like Forward Rate Agreement (FRA), with the same tenor. In Section 6.1, we show by a text-book no-arbitrage argument that if one assume the existence of ZC bonds traded in the whole market for all maturities, then there should not co-exist the FRA of different tenors. This justifies why one should not assume that there is a tradable numéraire available for the entire interest rate multicurve market.

Our model could also apply to other situations. One may incorporate the risks implicit in interbank transactions like for example liquidity risk or credit risk. For that we assume that the different submarkets correspond to different classes of investments with the same level of liquidity or of credit spread. The credit spread may be defined as the market unit that remunerates investors for the risk of default inherent in any debt instrument not considered to be risk free. Consider as a final example the case of multi-currency markets. Usually, one specific currency is chosen, and the other currencies are expressed
using this specific currency and then deflated using the associated numéraire. Our model is different since the numéraires associated with each currency are used jointly to express the value of the portfolio. Taking the example of the European Union, the German and French markets use their own numéraires, and there is a significant spread between their sovereign rates (OAT in France and Bund in Germany). A European financial institution has positions in both countries, each of which will be discounted using its own numéraire, and the institution will at the end of the day compute its wealth by summing its positions in both countries.

A natural question is how to price a contract $H$. The payoff of this contract may be associated to the activities of one or of several BUs or of financial products treated by several desks. We define different superreplication prices that depend on the way the superhedging is performed. If one uses only the submarket $\tau$, the price $\hat{\pi}^\tau(H)$ is the classical superreplication price in the market $\tau$. If one invests the initial wealth in the submarket $\tau$ and then uses all the other submarkets to hedge, the price is called $\pi^\tau(H)$. Now, if the initial investment and the hedging strategy use all submarkets, the price is called $\pi(H)$. We also introduce the minimum cost $\underline{\pi}(H)$ (resp. $\overline{\pi}(H)$) for which it is possible to find one specific submarket where $H$ can be superreplicated using only assets from this submarket (resp. to superreplicate $H$ in all submarkets).

This means that the same claim can have various superhedging prices depending on how it is hedged. So, considering submarkets without the possibility to neither trade between them nor borrow on one again the others allows to model assets having the same payoff but different initial prices. In the example of a one period model with two arbitrage-free submarkets and $d_1 = d_2 = 1$, assume that $S^\tau_0 = S^\tau_1$. Then, one may have $S^\tau_2 > S^\tau_0$. Clearly, considering that both assets can be traded together, there is an arbitrage opportunity. But if it is not possible to short in the market $\tau_2$ and be long in the market $\tau_1$, there is no arbitrage in the global market. Moreover, $S^\tau_2 - S^\tau_0$ is a measure of the difference of features between submarkets $\tau_1$ and $\tau_2$, for example a difference of liquidity.

In Theorem 2, we provide some inequalities and equalities between the different superreplication prices. We give lower and upper bounds for $\pi(H)$, but in the general case, we do not get an exact duality formulation. We show that $\overline{\pi}(H) \geq \underline{\pi}(H) \geq \min_{\tau \in T} \pi^\tau(H) \geq \pi(H)$. This proves that using all submarkets together to superreplicate $H$ yields the lowest price when requiring that superreplication holds true in all the submarkets separably yields the highest price. Taking the example of the credit risk markets, the price $\pi(H)$ corresponds to a hedge through markets with different credit risk levels as D assets while $\underline{\pi}(H)$ ensures a hedge in all markets and in particular with AAA assets. We propose in Proposition 2 and Section 6.2 particular types of markets, where the duality formulation for $\pi(H)$ is exact. This is the case when the spreads between the different submarkets are deterministic and we find that $\pi(H) = \pi^{\tau_{\text{max}}}(H)$, where $\tau_{\text{max}}$ is the submarket where the spread is maximum.
When there are two submarkets $\tau_1$ and $\tau_2$ with one risky asset per submarket, we are able to fully compute in Proposition 3 the prices of $S^\tau_1$ and $S^\tau_2$ and also of $S^\tau_1 - S^\tau_2$, the instrument which allows to be short of the asset of the submarket $\tau_2$ and to be long of the asset of the submarket $\tau_1$. This instrument is like a Basis swap in the interest rate market. We find that $\pi(S^\tau_2)$ is the infimum between $\pi^\tau_1(S^\tau_1)$ and $\pi^\tau_2(S^\tau_1)$. We also show that our measure of difference of features between submarkets $\tau_1$ and $\tau_2$ is indeed relevant as the initial investment(s) is (are) made in the submarket(s) where this measure equals zero.

Then, we propose several economic illustrations. First, we show the appropriateness of our model by detailing the case of the multi-curve market. Then, we prove that in the case of constant numéraire spreads, a common martingale measure exists for the whole market. We also discuss completeness issues. We finish with a Brownian illustration, where under the assumption of a time dependent Vasicek model for the numéraire spread, we provide a characterisation of the sets of martingale measures. When there are two sub-markets, each with only one risky asset, we fully calculate the different superreplication prices.

The paper is structured as follows: Section 2 presents the framework and notations of the paper. In Section 3, we present the FTAP, while in Section 5, we study various superreplication prices depending on the choice of the submarket(s) allowed in the superreplicating portfolio. Section 6 proposes some economic illustrations and we conclude in Section 7. Finally, Section A collects the proofs.

2 The model

Let $\mathcal{T}$ be the finite set of all submarkets for a given global market. For all $\tau \in \mathcal{T}$, the stochastic processes $S^\tau = (S^\tau_t)_{t \geq 0}$ and $S^{\tau,0} = (S^{\tau,0}_t)_{t \geq 0}$ are $\mathbb{R}^d$, (for some given $(d_\tau)_{\tau \in \mathcal{T}} \subset \mathbb{N}$) and $(0, \infty)$-valued, respectively, have càdlàg trajectories and are adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$. For all $\tau \in \mathcal{T}$, we denote by $\tilde{S}^\tau = S^\tau / S^{\tau,0}$ the $\mathbb{R}^d$-valued adapted price process of the risky assets of the submarket $\tau$ deflated (or discounted) by the numéraire $S^{\tau,0}$, which is tradable only in the submarket $\tau$. We note that $(S, S^0) = (S^\tau, S^{\tau,0})_{\tau \in \mathcal{T}}$ and $\tilde{S} = (\tilde{S}^\tau)_{\tau \in \mathcal{T}}$. We assume that the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions (right continuous, $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_{\infty} = \mathcal{F}$).

We denote by $L^1 = L^1(\Omega, \mathcal{F}, \mathbb{P})$ the set of integrable, $\mathbb{R}$-valued and $\mathcal{F}$-adapted random variables, by $L^+_1 = L^+_1(\Omega, \mathcal{F}, \mathbb{P})$ the set of non-negative elements of $L^1$ and by $L^+_{\geq 0} = L^+_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P})$ the set of $X \in L^1$ such that $\mathbb{P}(X > 0) = 1$. The same notations apply for $L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$: the set of essentially bounded, $\mathbb{R}$-valued and $\mathcal{F}$-adapted random variables. Moreover, $x \land y = \min(x, y)$.

In order to prove our main results, we need to ensure that the wealth process belongs to $L^\infty$. So, we first make a technical assumption.
Assumption 1 For all \( \tau \in \mathcal{T} \), the process \( \tilde{S}^\tau \) is locally bounded.

Note that if \( \tilde{S}^\tau \) is an adapted càdlàg process with uniformly bounded jumps, then it is locally bounded.

In the setup of the paper, the wealth process cannot be discounted as there is no common tradable numéraire. As we will see in (3) below, \( S_t^{\tau,0} \) appears in the expression of the wealth process at time \( t \), which needs to be bounded. So, we have to guarantee that \( S^{\tau,0} \) remains bounded by some real number \( M \).

We fix some real number \( T_0 > 0 \) which will be the time horizon of the market if all numéraires \( S^{\tau,0} \) are essentially bounded. In this case, we define \( M \) as the maximum of their norms. Otherwise, we will choose as time horizon for the market a finite stopping time \( T \) guaranteeing that all the \( S^{\tau,0} \) are bounded.

Let \( M \) be any fixed positive real number, \( T_M^\tau = \inf \{ t > 0 \mid S_t^{\tau,0} > M \} \) and \( T = T_0 \wedge \inf \{ T_M^\tau \mid \tau \in \mathcal{T} \} \). Then, \( T_M^\tau \) and \( T \) are stopping times. It is clear that \( \mathbb{P}(\{ T < \infty \}) = 1 \). We also remark that \( S_t^{\tau,0} \leq M \) on \( \{ t \leq T \} \) for all \( \tau \in \mathcal{T} \).

Indeed, fix some \( \tau \in \mathcal{T} \). If \( \omega \in \{ T_M^\tau = \infty \} \), then \( S_t^{\tau,0}(\omega) \leq M \) for all \( t > 0 \) and, in particular, if \( t \leq T(\omega) \). If now \( \omega \in \{ T_M^\tau < \infty \} \), then \( S_t^{\tau,0}(\omega) \leq M \) if \( t \leq T(\omega) \leq T_M^\tau(\omega) \). Then, for all \( \tau \in \mathcal{T} \), we assume that \( S_t^{\tau} = S_t^{\tau,0} \) and \( S_t^{\tau,0} = S_t^{\tau,0} \) on \( \{ t > T \} \).

In the sequel, in order to avoid to many fractions we also write \( \tilde{S}_t^{\tau,0} = S_t^{\tau,0} / S_0^{\tau,0} \). Indeed, as there are several numéraires, there is no reason to suppose that \( S_0^{\tau,0} = 1 \) for all \( \tau \in \mathcal{T} \).

We will use simple trading strategies to define the no free lunch condition as in [10], [16] and [11]. Using this kind of strategy, it is not possible to construct doubling strategies.

Definition 1 For all \( \tau \in \mathcal{T} \), a (simple) strategy in the submarket \( \tau \) is given by \( \mathbb{R}^d \times \mathbb{R} \)-valued processes \( (\Phi^\tau, \Phi^{\tau,0}) = (\Phi_j^\tau, \Phi_j^{\tau,0})_{j \geq 0} \) of the form

\[
\Phi^\tau = \sum_{j=1}^{n_\tau} \varphi_j^\tau \mathbb{1}_{[\delta_j^{-1}, \delta_j]} \quad \Phi^{\tau,0} = \sum_{j=1}^{n_\tau} \varphi_j^{\tau,0} \mathbb{1}_{(\delta_j^{-1}, \delta_j]},
\]

where \( n_\tau \geq 1 \) and \( 0 = \delta_0^\tau \leq \delta_1^\tau \leq \ldots \leq \delta_{n_\tau}^\tau < \infty \) are \( n_\tau + 1 \) finite stopping times and \( \varphi_j^\tau \) (resp. \( \varphi_j^{\tau,0} \)) are \( \mathbb{R}^d \) (resp. \( \mathbb{R} \))-valued, \( F_{\delta_{j-1}^{-1}} \)-measurable random variables for all \( j \in \{1, \ldots, n_\tau \} \). We denote the global strategy by \( (\Phi, \Phi^0) = (\Phi_j^\tau, \Phi_j^{\tau,0})_{\tau \in \mathcal{T}} \).

Definition 2 A strategy \( (\Phi, \Phi^0) \) is self-financing if for all \( \tau \in \mathcal{T} \) and all \( j \in \{1, \ldots, n_\tau \} \)

\[
\Phi_{\delta_j^{-1}}^{\tau,0} S_{\delta_j^{-1}}^{\tau,0} + \Phi_{\delta_j^{-1}}^{\tau,0} S_{\delta_j^{-1}}^{\tau} = \Phi_{\delta_j^{-1}}^{\tau} S_{\delta_j^{-1}}^{\tau} + \Phi_{\delta_j^{-1}}^{\tau,0} S_{\delta_j^{-1}}^{\tau,0},
\]

(1)

The economic agent trades in each submarket separately. This is why the self-financing condition is given submarket by submarket and we do not define a global self-financing condition (i.e., summing on \( \tau \) on the left- and right-hand sides of (1)). Nevertheless, the total wealth of the agent is the sum of her wealth in each submarket.
Let \((\Phi, \Phi^0)\) be a self-financing strategy. We denote by \(V^{\Phi, \Phi^0}\) the \(\mathbb{R}\)-valued adapted wealth process obtained by summing the wealth of each submarket, i.e., on \(\{t \leq \delta^r_n\}\)

\[
V^{\Phi, \Phi^0}_t = \sum_{\tau \in T} \left( \phi^r_\tau S^r_t + \phi^r_\tau 0 S^r_0 \right),
\]

(2)

Note that on \(\{t \geq T\}\), \(V^{\Phi, \Phi^0}_t = V^{\Phi, \Phi^0}_T\). Using the self-financing condition, it is easy to see that on \(\{t \leq \delta^r_n\}\)

\[
V^{\Phi, \Phi^0}_t = \sum_{\tau \in T} S^r_0 \left( \varphi^0_\tau S^r_0 + \varphi^r_\tau 0 + \sum_{j=1}^{n_\tau} \varphi^j_\tau \left( S^r_{\delta^r_{j-1} \land t} - S^r_{\delta^r_j \land t} \right) \right)
\]

\[
= \sum_{\tau \in T} x^\tau S^r_0 + \sum_{\tau \in T} S^r_0 \sum_{j=1}^{n_\tau} \varphi^j_\tau \left( S^r_{\delta^r_{j-1} \land t} - S^r_{\delta^r_j \land t} \right) = V^{x, \Phi}_t,
\]

(3)

where \(x^\tau = \varphi^0_\tau S^r_0 + \varphi^r_\tau 0 S^r_0\) is the initial wealth in the submarket \(\tau\) and \(x = (x^\tau)_{\tau \in T}\) the associated vector. Conversely, starting from some vector of initial wealth \(x = (x^\tau)_{\tau \in T}\) and some strategy in the risky assets \(\Phi\), it is possible, submarket by submarket using the self-financing condition, to construct some \(\Phi^r,0\) such that on \(\{t \leq \delta^r_n\}\)

\[
S^r_0 \left( \frac{x^\tau}{S^r_0} + \sum_{j=1}^{n_\tau} \varphi^j_\tau \left( S^r_{\delta^r_{j-1} \land t} - S^r_{\delta^r_j \land t} \right) \right) = \phi^r_\tau S^r_t + \phi^r_\tau 0 S^r_0
\]

(4)

and \(x^\tau = \varphi^0_\tau S^r_0 + \varphi^r_\tau 0 S^r_0\). This implies that \(V^{x, \Phi}_t = V^{\Phi, \Phi^0}_t\). We see in this argument that the assumption that all the \(S^{r,0}\) are tradable is crucial. Note that on \(\{T > \delta^r_n\}\), we add one step to the strategy ending at the stopping time \(\delta^r_n\), \(\phi^r_\tau = 0\) and using the self-financing condition, we have that

\[
\phi^r_\tau = \phi^r_{\delta^r_n} + \left( \phi^r_{\delta^r_n} - \phi^r_T \right) \frac{S^r_{\delta^r_n}}{S^r_{\delta^r_n}} = \phi^r_{\delta^r_n} \frac{S^r_{\delta^r_{j-1}}}{S^r_{\delta^r_n}} + \phi^r_T \frac{S^r_{\delta^r_{j-1}}}{S^r_{\delta^r_n}}
\]

\[
= \frac{x^\tau S^r_{\delta^r_{j-1}} + S^r_0 \sum_{j=1}^{n_\tau} \varphi^j_\tau \left( S^r_{\delta^r_{j}} - S^r_{\delta^r_{j-1}} \right)}{S^r_{\delta^r_n}}
\]

Thus, we obtain that

\[
V^{x, \Phi}_x = \sum_{\tau \in T} \left( \phi^r_\tau S^r_t + \phi^r_\tau 0 S^r_0 \right) = \sum_{\tau \in T} x^\tau S^r_t + \sum_{\tau \in T} S^r_0 \sum_{j=1}^{n_\tau} \varphi^j_\tau \left( S^r_{\delta^r_{j}} - S^r_{\delta^r_{j-1}} \right)
\]

The same formula holds true on \(\{T \leq \delta^r_n\}\) using (3), because \(S^r_{\delta^r_n \land T} = S^r_{\delta^r_n}\) on \(\{\delta^r_j < T\}\) and \(S^r_{\delta^r_j \land T} = S^r_{\delta^r_j}\) on \(\{\delta^r_j \geq T\}\).
**Definition 3** A strategy \((\Phi, \Phi^0)\) as in Definition 1 is called admissible if the processes \((\tilde{S}_\tau, \tilde{S}_0^\tau)_{\tau > 0}\) and the random variables \((\varphi_{\tau j}, \varphi_0^\tau)_{j \in \{1, \ldots, n\}}\) are uniformly bounded for all \(\tau \in T\) and if the initial wealth in each submarket is non-negative, i.e., \(x^\tau = \varphi_0^\tau S_0^\tau + \varphi_0 S_0^\tau \geq 0\) for all \(\tau \in T\).

The assumption that \(x^\tau\) is non-negative is made because it is not permitted to borrow on a submarket against the others.

We introduce now the set of contingent claims available at the stopping time \(T\) at cost zero (i.e., \(x^\tau = \varphi_0^\tau S_0^\tau + \varphi_0 S_0^\tau = 0\)) via an admissible self-financing simple strategy using only the submarket \(\tau \in T\)

\[
K^\tau = \left\{ S_T^\tau, \varphi_{\tau j} \left( \tilde{S}_j^\tau - \tilde{S}_{j-1}^\tau \right) \Bigg| n_\tau \geq 1, \ 0 = \delta_{0}\leq \ldots \leq \delta_{n_\tau} < \infty, \right.

\left. (\tilde{S}_0^\tau, \tilde{S}_0^\tau)_{\tau > 0}\text{uniformly bounded, } (\varphi_{\tau j})_{j \in \{1, \ldots, n_\tau\}} \mathcal{F}_{\tilde{S}_{j-1}^\tau}\text{-adapted and bounded} \right\}
\]

which belongs to \(L^\infty\) using Assumption 1 and the definition of \(T\).

We denote by \(K\) the set of contingent claims available at the stopping time \(T\) at total cost zero via an admissible self-financing simple strategy

\[
K = \left\{ V_T^{\Phi, \Phi^0} | \Phi, \Phi^0, \varphi_0^\tau S_0^\tau + \varphi_0 S_0^\tau = 0 \ \forall \tau \right\} = \left\{ \sum_{\tau \in T} W^\tau | W^\tau \in K^\tau \right\}
\]

which belongs to \(L^\infty\) using again Assumption 1 and the definition of \(T\). Note that it is equivalent to assume that \(\varphi_0^\tau S_0^\tau + \varphi_0^\tau S_0^\tau = 0\) for all \(\tau \in T\) or that \(\sum_{\tau \in T}(\varphi_0^\tau S_0^\tau + \varphi_0^\tau S_0^\tau) = 0\) because the admissibility condition requires \(\varphi_0^\tau S_0^\tau + \varphi_0^\tau S_0^\tau \geq 0\) for all \(\tau \in T\). We finally define

\[
C = K - L^\infty_+ = \{ V - Z | V \in K, Z \in L^\infty_+ \}.
\]

3 Fundamental theorem of asset pricing

The classical no arbitrage (NA) condition for simple strategies stipulates that \(K \cap L^\infty_+ = \{ 0 \}\), and this is equivalent to \(C \cap L^\infty_+ = \{ 0 \}\). It is well known that in continuous-time models, stipulating the (NA) condition is insufficient to obtain an equivalent local martingale measure (see, for example, Proposition 5.1.7 in [6]). Therefore, following [16], we use the no free lunch (NFL) condition.

**Definition 4** The process \((S, S^0) = (S^\tau, S_0^\tau)_{\tau \in T}\) satisfies the no free lunch (NFL) condition if

\[
\bar{C} \cap L^\infty_+ = \{ 0 \},
\]

where \(\bar{C}\) is the closure of \(C\) taken with respect to the weak-star topology of \(L^\infty\).
Definition 4 ensures the NFL condition for the global market. It is clear from the definition that this implies the NFL in each submarket $\tau$, i.e.,

$$\bar{C}_\tau \cap L^\infty_+ = \{0\} \text{ with } C_\tau = K_\tau - L^\infty_+.$$ 

The reverse implication does not hold true: there may be a free lunch in the global market while the different submarkets satisfy the NFL condition. Consider a one period market with finite $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and two submarkets $\tau_1$ and $\tau_2$ each with one risky asset such that $S_{\tau_1}^0 = S_{\tau_2}^0 = 0$, $S_{\tau_1}^1 = -I_{\omega_2} + I_{\omega_3}$ and $S_{\tau_2}^1 = I_{\omega_1} + I_{\omega_2} - I_{\omega_3}$, where $I_\omega$ is 1 if $\omega = \omega_i$ and zero else. The numéraires are such that $S_{\tau_1}^{1,0} = S_{\tau_2}^{1,0} = 1$ and $S_{\tau_1}^{1,1} = \frac{3}{2}$ and $S_{\tau_2}^{1,1} = 1$.

Then, both submarkets satisfy the NFL condition. However, the global market admits an arbitrage: if we buy one unit of asset $S_{\tau_1}$ and of asset $S_{\tau_2}$, the initial cost is $x = 0$ and

$$V_1 = S_{\tau_1}^{1,0} \left( \frac{S_{\tau_1}^{1,1}}{S_{\tau_1}^{1,1} - 0} \right) + S_{\tau_2}^{1,0} \left( \frac{S_{\tau_2}^{1,1}}{S_{\tau_2}^{1,1} - 0} \right) = I_{\omega_1}.$$ 

**Definition 5** We denote by $\mathcal{X}_\tau$ the set of $X^* \in L^1_{>0}$ such that for all stopping times $\beta_1 \leq \beta_2 \leq T$ and all $\mathbb{R}^{d_\tau}$-valued, $F_{\beta_1}$-measurable random variables $\varphi$,

$$\mathbb{E}\left( X^* S_{T}^{\beta_1, \beta_2} \varphi \left( \tilde{S}_{\beta_2} - \tilde{S}_{\beta_1} \right) \right) = 0. \quad (7)$$

We set $\mathcal{X}^* = \cap_{\tau \in \mathcal{T}} \mathcal{X}_\tau$.

**Theorem 1** Under Assumption 1, $(S, S^0)$ satisfies the no free lunch (NFL) condition if and only if $\mathcal{X}^* \neq \emptyset$.

**Proof** See Section A.1. $\square$

As NFL condition implies NFL in each submarket $\tau$ for all $\tau \in \mathcal{T}$, the classical Kreps-Yan theorem (see [16], [20]) shows that $\mathcal{X}^* \neq \emptyset$. Theorem 1 also proves that this holds true since $\mathcal{X}^* \subseteq \mathcal{X}_\tau$ for all $\tau \in \mathcal{T}$. However, starting from the fact that $\mathcal{X}^* \neq \emptyset$ for all $\tau \in \mathcal{T}$, Kreps-Yan theorem does not show that the global no free lunch holds true or that there exists some $X^*$ belonging to all $\mathcal{X}^*$.

One may wonder whether the fact that each submarket $\tau$ is complete implies that the global market is complete and vice versa. In Section 6.3, we will precise the notion of completeness for the global market and give examples and counter-examples for both implications.

### 4 Martingale measures

The condition $\mathcal{X}^* \neq \emptyset$ is equivalent to the existence for all $\tau \in \mathcal{T}$ of an equivalent local martingale $\mathbb{Q}^\tau$ (or risk-neutral) measure $\mathbb{Q}^\tau$ for the process $\tilde{S}^\tau$.

1 We obtain local martingale instead of true martingale because the $\tilde{S}^\tau$ are locally bounded.
(i.e., $\tilde{S}^\tau$ is a local martingale under $Q^\tau$ and $Q^\tau \sim P$). Let $X^* \in X^*$. Those martingale measures are defined by

$$\frac{dQ^\tau}{dP} |_{\mathcal{F}_T} = \frac{X^* S_T^{\tau,0}}{E(X^* S_T^{\tau,0})}.$$  \hspace{1cm} (8)

In the financial literature, a state price deflator is a process $(D_t)_{t \geq 0}$ such that $(D_t S_t^\tau)_{t \geq 0}$ is a local martingale under the initial probability measure $P$. Here, $(D_t^\tau)_{t \geq 0}$ is a state price deflator, where

$$D_t^\tau = \frac{E \left( X^* S_T^{\tau,0} | \mathcal{F}_t \right)}{S_T^{\tau,0}}.$$  

Therefore, Theorem 1 shows that the martingale measures or the state price deflators are dependent of the submarket but constructed from a common random variable $X^*$.  

We now study the problem called “change of numéraire”. We use “” to emphasize that the word “numéraire” in this paragraph has not the same meaning as in the rest of the paper. Indeed, a “numéraire” $Z$ does not need to be tradable as $S_T^{\tau,0}$ is assumed to be. It is sometimes convenient to change the “numéraire” because of modelling considerations. Indeed, when we change the “numéraire”, we change the risk-neutral measures, which may simplify the model, facilitating calculations of prices. Therefore, we now propose to characterize the NFL condition using new “numéraires”. Proposition 1 will be used to prove our second main result on pricing issues.

**Definition 6** Let $Z \in L^\infty_\infty$, $Q \in Q^{\tau,Z}$ if and only if $Q \sim P$, $\frac{dQ}{dP}/Z \in L^1$ and

$$\left( E_Q \left( \frac{S_T^{\tau,0}}{Z} | \mathcal{F}_t \right) \tilde{S}^\tau_t \right)_{t \geq 0}$$

is a local martingale under $Q$.

We set $Q^Z = \cap_{\tau \in T} Q^{\tau,Z}$.

- Let $\hat{\tau} \in T$. We denote by $\hat{Q}^\tau = Q^{\tau, S_T^{\tau,0}}$ the usual uni-market set of local martingale measures for $S^{\tau,0}$ and we denote by $Q^\tau = Q^{S_T^{\tau,0}}$. Then, $Q^\tau \subset \hat{Q}^\tau$.

- For $Z = \sum_{\tau \in T} \lambda_{\tau} S_T^{\tau,0}$ where $\lambda = (\lambda_{\tau})_{\tau \in T} \in \Lambda^T$, we denote $Q^{\lambda,Z} = Q^Z$ where

$$\Lambda^T = \left\{ (\lambda_{\tau})_{\tau \in T} \subset \mathbb{R}^+ | P \left( \sum_{\tau \in T} \lambda_{\tau} S_T^{\tau,0} > 0 \right) = 1 \right\}.$$  

- For $Z = \max_{\tau \in T} S_T^{\tau,0}$, we denote $Q^{\max} = Q^Z$.

- For $Z = \min_{\tau \in T} S_T^{\tau,0}$, we denote $Q^{\min} = Q^Z$.

**Proposition 1** Assume that Assumption 1 holds true.

1. If $(S, S^0)$ satisfies the NFL condition, then $Q^Z$ is not empty for all $Z \in L^\infty_\infty$. Conversely, if $Q^Z$ is not empty for some $Z \in L^\infty_\infty$, then $(S, S^0)$ satisfies the
NFL condition.

2. Fix $Z \in L_\infty$. Then, we get that

$$Q^Z = \left\{ Q \mid \exists X^* \in \mathcal{X}^*, \frac{dQ}{dP} = \frac{X^*Z}{\mathbb{E}(X^*Z)} \right\}$$  \hspace{1cm} (9)

Proof See Section A.2. \hspace{1cm} \Box

Remark 1 Let $\tilde{\tau} \in \mathcal{T}$. Then, as in Proposition 1 or using Kreps-Yan Theorem, $(S, S^0)$ satisfies the NFL condition in the submarket $\tilde{\tau}$ if and only if $\tilde{Q}^\tau \neq \emptyset$. Moreover, $\tilde{Q}^\tau = \left\{ Q \mid \exists X^\tau \in \mathcal{X}^\tau, \frac{dQ}{dP} = \frac{X^\tau S^\tau_T}{\mathbb{E}(X^\tau S^\tau_T)} \right\}$.

5 Superhedging theorem

5.1 Definitions

We now turn to pricing issues of some contingent claim $H$. In the classical case, the superreplication cost for $H$ is the minimum selling price needed to superreplicate it by trading in the (uni-market) market. In complete markets, the superreplication cost is just the cash flow expectation computed under the unique (local) martingale measure. However, in incomplete markets, this is no longer the case, as the risk-neutral measure is no longer unique.

Here, as we have several submarkets that coexist, several choices of superhedging are possible. The first definition of price assumes that all submarkets are used to cover $H$. Specifically, the initial wealth $x$ is divided among the different submarkets. Then, in each submarket $\tau$ starting from initial wealth $x^\tau$, one can find some hedging strategy (using only the instruments of submarket $\tau$) such that when liquidating all those portfolios, one is in a position to superreplicate $H$. Recalling (5) and (6), this can written as follows

$$\pi(H) = \inf \left\{ \sum_{\tau \in \mathcal{T}} x^\tau \mid x^\tau \geq 0, \forall \tau \in \mathcal{T}, \exists W \in \bar{C} \cap L_\infty, \sum_{\tau \in \mathcal{T}} x^\tau S^\tau_T + W \geq H \text{ a.s.} \right\} \hspace{1cm} (10)$$

with the convention that $\pi(H) = +\infty$ if the above set is empty. The same convention applies to the prices defined in (11) to (14).

We can think of the minimum costs of other types of hedging strategies. First, we consider the case in which it is possible to find one specific submarket where $H$ can be superreplicated using only assets from this submarket

$$\mathcal{Z}(H) = \inf \left\{ x \mid x \geq 0, \exists \tau \in \mathcal{T}, \exists W \in \bar{C} \cap L_\infty, x S^\tau_T + W \geq H \text{ a.s.} \right\} \hspace{1cm} (11)$$

Recall that $\bar{C} = K - L_\infty$.

We can also consider the minimal initial cost for which it is possible to superreplicate $H$ in all submarkets, i.e., that for each submarket, $H$ can be superreplicated using only assets from this submarket

$$\overline{\pi}(H) = \inf \left\{ x \mid x \geq 0, \forall \tau \in \mathcal{T}, \exists W \in \bar{C} \cap L_\infty, x S^\tau_T + W \geq H \text{ a.s.} \right\} \hspace{1cm} (12)$$
Finally, we introduce prices which are related to some specific submarket $\tau$, for any $\tau \in \mathcal{T}$. The superreplication price $\pi^\tau(H)$ is obtained investing the entire initial wealth in the submarket $\tau$ but trading in the other submarkets too

$$\pi^\tau(H) = \inf \left\{ x | x \geq 0, \exists W \in \bar{C} \cap L^\infty, xS^\tau_0 + W \geq H \ a.s. \right\}.$$  

(13)

It is clear that $\pi^\tau(H) \geq \pi(H)$. We introduce the classical superreplication price of $H$ in the submarket $\tau$:

$$\hat{\pi}^\tau(H) = \inf \left\{ x | x \geq 0, \exists W \in \bar{C}^\tau \cap L^\infty, xS^\tau_0 + W \geq H \ a.s. \right\}.$$  

(14)

The superreplication price $\hat{\pi}^\tau(H)$ in the submarket $\tau$ alone is higher than $\pi^\tau(H)$: even if the entire initial wealth $\pi^\tau(H)$ is invested in the submarket $\tau$, trading may also take place in the other submarkets. This is not the case for $\hat{\pi}^\tau(H)$, where all the trading activity is made in the submarket $\tau$.

It is clear that the same claim may have various prices. This is because they are associated with different choices of initial endowments and/or hedging strategies and that one cannot trade between submarkets. We prove in Theorem 2 below that $\pi(H) \geq \hat{\pi}(H) \geq \min_{\tau \in \mathcal{T}} \pi^\tau(H) \geq \pi(H)$. This shows that using all submarkets together to superreplicate $H$ yields the lowest price when requiring that superreplication holds true in all the submarkets separately.

We show that $\pi(H)$ corresponds to the maximum of all the superreplication prices $\hat{\pi}^\tau(H)$, while $\hat{\pi}(H)$ corresponds to the minimum of $\hat{\pi}^\tau(H)$. Note also that the following classical dual representation holds true

$$\hat{\pi}^\tau(H) = \sup_{Q \in \hat{\mathcal{Q}}^\tau} E_Q \left( \frac{H}{S^\tau_T} \right).$$  

(15)

This well-known result may be proven as below (recall that the NFL condition implies the NFL condition in every submarket $\tau$).

With those definitions, we can measure the difference of features between the submarket $\tau$ stand alone and the submarket $\tau$ as part of the global market: $\hat{\pi}^\tau(H) - \pi(H)$ measures the gain to invest and to trade in the whole market and not only in $\tau$, while $\hat{\pi}^\tau(H) - \pi^\tau(H)$ measures the gain to trade in the whole market and not only in $\tau$, while investing all the initial wealth in $\tau$. Below and in Section 6, we will compute those differences of features for several types of markets.

Remark 2 In the different definitions of the superreplication prices, we take the intersection of $C^\tau$ or $\bar{C}$ with $L^\infty$ so that the infima are minima.
5.2 Dual Characterization

We now state our second main result on the dual formulation of the superreplication prices, called the superhedging theorem.

**Theorem 2** Assume that Assumption 1 and that the NFL condition hold true. Assume that
\[
\sum_{\tau \in \mathcal{T}} \lambda_\tau S_T^{\tau,0} \in L^\infty
\]
for some \((\lambda_\tau)_{\tau \in \mathcal{T}} \in \Lambda^T\). Then, the infimum in (10) is attained, and there exists \((\check{x}_\tau)_{\tau \in \mathcal{T}}\) such that \(\check{x}_\tau \geq 0\) for all \(\tau \in \mathcal{T}\) and
\[
\pi(H) = \sum_{\tau \in \mathcal{T}} \check{x}_\tau.
\]
Moreover, the following holds true:
\[
\sup_{Q \in \mathcal{Q}^{lc,\Lambda}} \mathbb{E}_Q \left( \frac{H}{\sum_{\tau \in \mathcal{T}} \lambda_\tau S_T^{\tau,0}} - \sum_{\tau \in \mathcal{T}} \check{x}_\tau \frac{S_T^{\tau,0}}{\sum_{\tau \in \mathcal{T}} \lambda_\tau S_T^{\tau,0}} \right) = 0.
\]
(16)

Now, if \(\frac{H}{\min_{\tau \in \mathcal{T}} S_T^{\tau,0}} \in L^\infty\), we have that
\[
\sup_{Q \in \mathcal{Q}^{max}} \mathbb{E}_Q \left( H \frac{\max_{\tau \in \mathcal{T}} S_T^{\tau,0}}{\sum_{\tau \in \mathcal{T}} \lambda_\tau S_T^{\tau,0}} \right) \leq \pi(H) \leq \sup_{Q \in \mathcal{Q}^{min}} \mathbb{E}_Q \left( H \frac{\min_{\tau \in \mathcal{T}} S_T^{\tau,0}}{\sum_{\tau \in \mathcal{T}} \lambda_\tau S_T^{\tau,0}} \right).
\]
(17)

We also obtain that
\[
\pi(H) = \min_{\tau \in \mathcal{T}} \hat{\pi}_\tau(H) = \min_{\tau \in \mathcal{T}} \sup_{Q \in \hat{\mathcal{Q}}} \mathbb{E}_Q \left( \frac{H}{S_T^{\tau,0}} \right),
\]
(18)
\[
\pi(H) = \max_{\tau \in \mathcal{T}} \hat{\pi}_\tau(H) = \max_{\tau \in \mathcal{T}} \sup_{Q \in \hat{\mathcal{Q}}} \mathbb{E}_Q \left( \frac{H}{S_T^{\tau,0}} \right),
\]
(19)
\[
\pi^*(H) = \sup_{Q \in \hat{\mathcal{Q}}} \mathbb{E}_Q \left( \frac{H}{S_T^{\tau,0}} \right).\]
(20)

The infima in (11), (12), (13) and (14) are attained. Finally, one can compare the initial values of the different hedging strategies
\[
\pi(H) \leq \min_{\tau \in \mathcal{T}} \pi^*(H) \leq \pi(H) \leq \pi(H).
\]
(21)

**Proof** See Section A.3. \(\blacksquare\)

In the next proposition, we propose specific types of submarkets where the superreplication price \(\pi(H)\) has an exact dual representation. We will apply Proposition 2 to concrete economic cases in Section 6.2.

**Proposition 2** 1. Assume that there exist \((\lambda_\tau)_{\tau \in \mathcal{T}} \in \Lambda^T\) and \((c_\tau)_{\tau \in \mathcal{T}} \in (0, \infty)^T\) such that for all \(\tau \in \mathcal{T}\), for all \(Q \in \mathcal{Q}^{lc,\Lambda}\)
\[
\mathbb{E}_Q \left( \frac{S_T^{\tau,0}}{\sum_{\tau \in \mathcal{T}} \lambda_\tau S_T^{\tau,0}} \right) = c_\tau.
\]
Assume that \( \sum_{\tau \in T} H_{\tau} S_T^\tau \in L^\infty \) and let \( \tau_{\text{max}} \) be the \( \tau \in T \) such that \( c_\tau \) is maximum. Then,

\[
\pi(H) = \frac{1}{c_{\tau_{\text{max}}}} \sup_{Q \in \mathcal{Q}^{lc,\lambda}} E_Q \left( \frac{H}{\sum_{\tau \in T} \lambda_\tau S_T^\tau} \right).
\]

2. Assume now that there exist some \( c_{\tau_{\text{max}}} \) and some \( (c_\tau)_{\tau \in T} \in (0,1] \) such that \( \frac{H}{S_T^\tau} \in L^\infty \), \( E_Q \left( \frac{S_T^\tau}{\max S_T^\tau} \right) = c_\tau \) for all \( \tau \in T \) and for all \( Q \in Q_{\tau_{\text{max}}} \).

Then,

\[
\pi(H) = \pi_{\tau_{\text{max}}}(H).
\]

**Proof** Let \( Q \in \mathcal{Q}^{lc,\lambda} \). We compute in (16)

\[
E_Q \left( \frac{H}{\sum_{\tau \in T} \lambda_\tau S_T^\tau} \right) - \sum_{\tau \in T} \hat{x}_\tau E_Q \left( \frac{S_T^\tau}{\sum_{\tau \in T} \lambda_\tau S_T^\tau} \right) = E_Q \left( \frac{H}{\sum_{\tau \in T} \lambda_\tau S_T^\tau} \right) - \sum_{\tau \in T} c_\tau \hat{x}_\tau.
\]

So taking the supremum over all \( Q \in \mathcal{Q}^{lc,\lambda} \), we get that

\[
\sum_{\tau \in T} c_\tau \hat{x}_\tau = \sup_{Q \in \mathcal{Q}^{lc,\lambda}} E_Q \left( \frac{H}{\sum_{\tau \in T} \lambda_\tau S_T^\tau} \right).
\]

Thus,

\[
\pi(H) = \inf \left\{ \sum_{\tau \in T} x_\tau \mid \sum_{\tau \in T} c_\tau x_\tau = \sup_{Q \in \mathcal{Q}^{lc,\lambda}} E_Q \left( \frac{H}{\sum_{\tau \in T} \lambda_\tau S_T^\tau} \right) \right\}
\]

\[
= \frac{1}{c_{\tau_{\text{max}}}} \sup_{Q \in \mathcal{Q}^{lc,\lambda}} E_Q \left( \frac{H}{\sum_{\tau \in T} \lambda_\tau S_T^\tau} \right)
\]

The second part of the proposition is obtained choosing \( \lambda_{\tau_{\text{max}}} = 1 \) and \( \lambda_\tau = 0 \) for all \( \tau \neq \tau_{\text{max}} \). Then, \( \mathcal{Q}^{lc,\lambda} = \mathcal{Q}^\tau_{\tau_{\text{max}}} \), \( c_{\tau_{\text{max}}} = 1 \) and the result follows.

The proof above shows that all of the initial wealth is invested in the market \( \tau_{\text{max}} \). If we go back to the example of the credit risk market, \( \pi(H) \), which is the cheapest superhedging price, is obtained by investing all the initial allocation in the market where the credit spread is the largest, which is intuitive because it is the submarket where the risk is maximum.

We now focus on the situation of 2. in Proposition 2. In this case,

\[
\hat{x}_{\tau_{\text{max}}}(H) - \pi(H) = \sup_{Q \in \mathcal{Q}^\tau_{\tau_{\text{max}}}} E_Q \left( \frac{H}{S_T^{\tau_{\text{max}},0}} \right) - \sup_{Q \in \mathcal{Q}^\tau_{\tau_{\text{max}}}} E_Q \left( \frac{H}{S_T^{\tau_{\text{max}},0}} \right) \geq 0,
\]

since \( \mathcal{Q}^\tau_{\tau_{\text{max}}} \subset \mathcal{Q}^\tau_{\tau_{\text{max}}} \) and the difference of features (for example liquidity) is related to the size difference between both sets \( \mathcal{Q}^\tau_{\tau_{\text{max}}} \) and \( \mathcal{Q}^\tau_{\tau_{\text{max}}} \). We give a concrete illustration of this size difference just after Proposition 4.
5.3 Focus on the prices of the risky assets

In this section, we give with some elements on the prices of the risky assets. For that we assume that for all \( \tau \in \mathcal{T} \), the process \( \tilde{S}^\tau \) is a true martingale under any \( Q \in \tilde{Q}^\tau \). For example, we may assume that \( \tilde{S}^\tau \) is bounded or that \( \mathbb{E}_Q \sup_{0 \leq s \leq T} [\tilde{S}_T^\tau] < \infty \) for all \( t > 0 \) or even that the quadratic variation of \( \tilde{S}^\tau \) is integrable under \( Q \). Then, for all \( j \in \{1, \ldots, d_\tau\} \), using the dual formula given in (15), we get that

\[
\hat{\pi}^\tau(S_{T,j}^\tau) = \sup_{Q \in \tilde{Q}^\tau} \mathbb{E}_Q \left( \frac{S_{T,j}^\tau}{S_T^\tau} \right) = S_0^{\tau,0} \sup_{Q \in \tilde{Q}^\tau} \mathbb{E}_Q \left( \tilde{S}_{T,j}^\tau \right) = S_0^{\tau,j}. \tag{23}
\]

In the context of the situation of 2. in Proposition 2, (22) implies that

\[
\pi(S_{T,max,j}^\tau) = \sup_{Q \in \mathcal{Q}^{\tau,max}} \mathbb{E}_Q \left( \frac{S_{T,max,j}^\tau}{S_{T,max,0}^\tau} \right) = S_0^{\tau,max,j},
\]

as \( \tilde{S}_{T,max}^\tau \) is a \( Q \)-martingale for all \( Q \in \mathcal{Q}^{\tau,max} \subset \mathcal{Q}^\tau \). Thus, the illiquidity cost for \( S_{T,max}^\tau \) is zero.

The case of submarkets with only one risky asset is of one interest. Indeed, one may assume that each business unit only trades on one risky asset. We furthermore assume that there are only two submarkets \( \tau_1 \) and \( \tau_2 \). We also introduce the contingent claim \( S_T^{\tau_1} - S_T^{\tau_2} \) which allows to be short of the asset \( \tau_2 \) and long of the asset \( \tau_1 \). This instrument is like a Basis swap in the interest rate market. In this context, we are able to fully compute all the superreplication prices and also the illiquidity costs. We obtain that the first inequality in (21) is in fact an equality.

**Proposition 3** Assume that \( \mathcal{T} = \{\tau_1, \tau_2\} \) and that for \( i \in \{1, 2\}, d_{\tau_i} = 1 \) and that \( \tilde{S}^\tau \) is a true martingale under any \( Q \in \tilde{Q}^\tau \). Then,

\[
\begin{align*}
\pi^{\tau_1}(S_{T_1}^\tau) &= \hat{\pi}^{\tau_1}(S_{T_1}^\tau) = S_0^{\tau_1} \tag{24} \\
\pi^{\tau_2}(S_{T_2}^\tau) &= S_0^{\tau_1} \sup_{Q \in \mathcal{Q}^{\tau_2}} \mathbb{E}_Q \left( \frac{S_{T,0}^{\tau_1}}{S_{T,0}^{\tau_2}} \right) = \frac{S_0^{\tau_1}}{\inf_{Q \in \mathcal{Q}^{\tau_1}} \mathbb{E}_Q \left( \frac{S_{T,0}^{\tau_2}}{S_{T,0}^{\tau_1}} \right)} \tag{25}
\end{align*}
\]

\[\hat{\pi}^{\tau_1}(S_{T_1}^\tau) - \pi^{\tau_1}(S_{T_1}^\tau) = (\pi^{\tau_1}(S_{T_1}^\tau) - \pi^{\tau_2}(S_{T_2}^\tau))_+, \text{ where } x_+ = \max(x, 0). \]

If \( \pi(S_{T_1}^\tau) = \pi^{\tau_2}(S_{T_1}^\tau) \), then \( \pi(S_{T_1}^\tau) = \pi^{\tau_2}(S_{T_2}^\tau) \).

If \( \pi^{\tau_2}(S_{T_2}^\tau) < \pi^{\tau_2}(S_{T_2}^\tau) \), then \( \pi(S_{T_2}^\tau - S_{T_2}^\tau) = +\infty \). Else,

\[
\pi(S_{T_2}^\tau - S_{T_2}^\tau) = \pi(S_{T_2}^\tau) - \max \left( \sup_{Q \in \mathcal{Q}^{\tau_2}} \mathbb{E}_Q \left( \frac{S_{T,0}^{\tau_1}}{S_{T,0}^{\tau_2}} \right), 1 \right). \tag{26}
\]

We now distinguish between three cases.

**Case 1** : \( \pi(S_{T_1}^\tau) = \pi^{\tau_2}(S_{T_1}^\tau) \) and \( \pi(S_{T_2}^\tau) = \pi^{\tau_2}(S_{T_2}^\tau) \). This is equivalent to
\sup_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S_t^{1,0}}{S_{\tau|^2}} \right) < 1. \text{ Then, all the initial investments are made in the market } \tau_2 \text{ and } \pi(S_{\tau_1} - S_{\tau_2}) = \pi(S_{\tau_1}) - \pi(S_{\tau_2}).

Case 2: \pi(S_{\tau_1}) = \pi(S_{\tau_1}) \text{ and } \pi(S_{\tau_2}) = \pi(S_{\tau_2}). \text{ This is equivalent to } \inf_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S_t^{1,0}}{S_{\tau|^2}} \right) > 1. \text{ Then, all the initial investments are made in the market } \tau_1 \text{ and } \\
\pi(S_{\tau_1} - S_{\tau_2}) = \pi(S_{\tau_1}) - \pi(S_{\tau_2}) \frac{\inf_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S_t^{1,0}}{S_{\tau|^2}} \right)}{\sup_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S_t^{1,0}}{S_{\tau|^2}} \right)}.

Case 3: \pi(S_{\tau_1}) = \pi(S_{\tau_1}) \text{ and } \pi(S_{\tau_2}) = \pi(S_{\tau_2}). \text{ This is equivalent to } \sup_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S_t^{1,0}}{S_{\tau|^2}} \right) \geq 1 \text{ and } \inf_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S_t^{1,0}}{S_{\tau|^2}} \right) \leq 1. \text{ Then, the initial investments for the assets } S_{\tau_1} \text{ and } S_{\tau_2} \text{ (resp. } S_{\tau_1} \text{ and } S_{\tau_2}) \text{ are made in the market } \tau_1 \text{ (resp. } \tau_2) \text{ and } \\
\pi(S_{\tau_1} - S_{\tau_2}) = \pi(S_{\tau_1}) - \pi(S_{\tau_2}) \frac{1}{\sup_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S_t^{1,0}}{S_{\tau|^2}} \right)}.

Proof See Section A.4. □

We give some interpretations of Proposition 3 for a difference of liquidity but it may be a difference of credit risk. For that, we assume that \( \tilde{S}_{\tau_1} \) and \( \tilde{S}_{\tau_2} \) are bounded.

We have found that the price of \( S_{\tau_1} \), \( \pi(S_{\tau_1}) \), is the minimum between its initial value, \( \pi(S_{\tau_1}) = S_{\tau_1}^{\tau_1} \), which corresponds to the case where the trading takes place in \( \tau_1 \), i.e.,

\[ \pi(S_{\tau_1}) S_{\tau_1}^{\tau_1,0} + S_{\tau_1}^{\tau_1,0}(\tilde{S}_{\tau_1} - \tilde{S}_{\tau_0}) = S_{\tau_1} \]

and \( \pi(S_{\tau_1}) \), which corresponds to the case where the initial investment is made in \( \tau_2 \) and there exists \( W \in \mathcal{C} \cap L^\infty \) such that

\[ \pi(S_{\tau_2}) S_{\tau_2}^{\tau_2,0} + W = S_{\tau_2}. \]

When \( \pi(S_{\tau_1}) = \pi(S_{\tau_1}) \), the illiquidity cost \( \pi(S_{\tau_1}) - (\pi(S_{\tau_2}) = 0. \text{ So, the market } \tau_1 \text{ is liquid. When } \pi(S_{\tau_2}) = \pi(S_{\tau_2}), \text{ we get that } \pi(S_{\tau_2}) = \pi(S_{\tau_2}). \text{ So, the market } \tau_2 \text{ is liquid, while the market } \tau_1 \text{ is illiquid as } \pi(S_{\tau_1}) - (\pi(S_{\tau_2}) > 0. \text{ Note that in the first situation, the market } \tau_2 \text{ may be illiquid or liquid as we will see below.}

Now, we comment on the price of the Basis swap and how the investments are made. Case 1 corresponds to the situation where the asset \( S_{\tau_1} \) is liquid and the asset \( S_{\tau_2} \) is liquid. Proposition 3 shows that all the initial investments
are made in the liquid market \( \tau_2 \). Recalling the definition of the different superreplication prices, there exist \( W, W_1 \in \mathcal{C} \cap L^\infty \) such that
\[
\pi(S_T^{\tau_1}) \tilde{S}_T^{\tau_1,0} + W_1 = S_T^{\tau_1} \\
\pi(S_T^{\tau_2}) \tilde{S}_T^{\tau_2,0} + S_T^{\tau_2,0}(\tilde{S}_T^{\tau_2} - S_T^{\tau_2}) = S_T^2 \\
(\pi(S_T^{\tau_1}) - \pi(S_T^{\tau_2})) \tilde{S}_T^{\tau_2,0} + W = S_T^{\tau_1} - S_T^{\tau_2}.
\]
So, replicating the illiquid asset \( S_T^{\tau_2} \) by buying the assets \( S_T^{\tau_2} \) and \( S_T^{\tau_1} - S_T^{\tau_2} \) does not provide a cheaper price. As expected, there is no free lunch.

The case 2 corresponds to the situation where the asset \( S_T^{\tau_1} \) is liquid and the asset \( S_T^{\tau_2} \) is illiquid. Proposition 3 proves that all the initial investments are made in the liquid market \( \tau_1 \) and there exists \( W \in \mathcal{C} \cap L^\infty \) such that
\[
\pi(S_T^{\tau_1}) \tilde{S}_T^{\tau_1,0} + S_T^{\tau_1,0}(\tilde{S}_T^{\tau_1} - \tilde{S}_0) = S_T^{\tau_1} \\
\pi(S_T^{\tau_2}) \tilde{S}_T^{\tau_2,0} + W_2 = S_T^{\tau_2}
\]
\[
\left( \pi(S_T^{\tau_1}) - \frac{S_T^{\tau_2}}{\sup_{Q \in \mathcal{Q}^{\tau_2}} \mathbb{E}_Q \left( \frac{\tilde{S}_T^{\tau_2,0}}{S_T^{\tau_2}} \right)} \right) \tilde{S}_T^{\tau_2,0} + W = S_T^{\tau_1} - S_T^{\tau_2}.
\]
So, setting \( W = W_2 + W - S_T^{\tau_1,0}(\tilde{S}_T^{\tau_1} - \tilde{S}_0) \in \mathcal{C} \cap L^\infty \), we obtain that
\[
\left( \frac{\inf_{Q \in \mathcal{Q}^{\tau_1}} \mathbb{E}_Q \left( \frac{\tilde{S}_T^{\tau_1,0}}{S_T^{\tau_1}} \right)}{\sup_{Q \in \mathcal{Q}^{\tau_2}} \mathbb{E}_Q \left( \frac{\tilde{S}_T^{\tau_2,0}}{S_T^{\tau_2}} \right)} \right) \tilde{S}_T^{\tau_1,0} - \frac{S_T^{\tau_2}}{\sup_{Q \in \mathcal{Q}^{\tau_2}} \mathbb{E}_Q \left( \frac{\tilde{S}_T^{\tau_2,0}}{S_T^{\tau_2}} \right)} \geq \pi(0) = 0.
\]
This is not a free lunch since
\[
\frac{\inf_{Q \in \mathcal{Q}^{\tau_1}} \mathbb{E}_Q \left( \frac{\tilde{S}_T^{\tau_1,0}}{S_T^{\tau_1}} \right)}{\sup_{Q \in \mathcal{Q}^{\tau_2}} \mathbb{E}_Q \left( \frac{\tilde{S}_T^{\tau_2,0}}{S_T^{\tau_2}} \right)} \geq \pi(0) = 0.
\]

Finally, in case 3 both assets are liquid. In this case, the initial investments are made in both markets. There exists \( W \in \mathcal{C} \cap L^\infty \) such that
\[
\pi(S_T^{\tau_1}) \tilde{S}_T^{\tau_1,0} + S_T^{\tau_1,0}(\tilde{S}_T^{\tau_1} - \tilde{S}_0) = S_T^{\tau_1} \\
\pi(S_T^{\tau_2}) \tilde{S}_T^{\tau_2,0} + S_T^{\tau_2,0}(\tilde{S}_T^{\tau_2} - \tilde{S}_0) = S_T^{\tau_2}
\]
\[
\left( \pi(S_T^{\tau_1}) - \frac{S_T^{\tau_2}}{\sup_{Q \in \mathcal{Q}^{\tau_2}} \mathbb{E}_Q \left( \frac{\tilde{S}_T^{\tau_2,0}}{S_T^{\tau_2}} \right)} \right) \tilde{S}_T^{\tau_2,0} + W = S_T^{\tau_1} - S_T^{\tau_2}.
\]
So, setting \( \bar{W} = W + S_T^{\tau_2,0}(\tilde{S}_T^{\tau_2} - \tilde{S}_0) - S_T^{\tau_1,0}(\tilde{S}_T^{\tau_1} - \tilde{S}_0) \in \mathcal{C} \cap L^\infty \), we obtain that
\[
\left( -\frac{S_T^{\tau_2}}{\sup_{Q \in \mathcal{Q}^{\tau_2}} \mathbb{E}_Q \left( \frac{\tilde{S}_T^{\tau_2,0}}{S_T^{\tau_2}} \right)} \right) \tilde{S}_T^{\tau_2,0} + \pi(S_T^{\tau_2}) \tilde{S}_T^{\tau_2,0} + \bar{W} = 0.
Again, as \( \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q \left( \frac{S_{T_1}^r}{S_{T_2}^r} \right) \geq 1 \), this is not a free lunch because

\[
S_0^r - \frac{S_0^r}{\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q \left( \frac{S_{T_1}^r}{S_{T_2}^r} \right)} \geq \pi(0) = 0.
\]

An illustration of these results in the Brownian framework is given in Section 6.4.2 below.

6 Economic illustrations

We now propose some economic illustrations. First, we provide some further evidence for our choice of model (self-financing and no free lunch conditions, definition of the set \( K \)) coming from the multicurve financial market. Then, we give some explicit cases where there exists a common martingale measure. We also discuss completeness issues. Finally, we give a complete illustration of our result in a Brownian setting.

6.1 Multicurve financial market

In this section, we focus on the case of the multicurve financial market, which gives a good justification for our modeling choices. Before the financial crisis, interest rate swaps (denominated in the same currency) were priced using the same zero-coupon (ZC) curve used as a numéraire, regardless of the priced swaps’ payment structure frequency. In Europe, typical ZC curves were constructed using instruments with a 6-month payment frequency. In 2007, important distortions began to appear in the swap rates indexed by different frequencies (or tenors): 1 month, 3 month, 6 month and 1 year. Therefore, the market practice for interest rate swap valuation has evolved and considers several ZC curves that are tenor based while retaining the assumption of the existence of a common martingale measure. For example the 3-month ZC curve is built using only instruments with a tenor of 3 months such as the forward rate agreements (FRA) 3 months in 3 months or 3 months in 6 months or such as swaps rates against Euribor 3M for several maturities.

Usually, in the fixed income theory, in order to state the no-arbitrage condition and then obtain FTAP, one takes the ZC bonds as basic tradable underlying assets. In the presence of multiple tenors, the existence of such tradable ZC bonds is more than questionable. Firstly, if such bonds exist and are tradable, why do several ZC curves co-exist? There should be arbitrage opportunities between, say, the ZC bond in the 3-month tenor with maturity 6 months and the one in the 6-month tenor with the same maturity. In particular, if their prices are not equal at time 0, there is an arbitrage opportunity, because they both pay 1 euro at time 6 months. So, it appears clearly that in the multicurve market the basic tradable underlying assets cannot be the ZC bonds of
several tenors. So, one may use FRA of different tenors as basic risky assets, see [8], [9] and the references therein. We argue below that if one assumes the existence of ZC bonds traded in the global market for all maturities (for example, associating a tradable zero-coupon bond to the overnight indexed swap (OIS) rate as in [8]), then there should not co-exist FRA of different tenors by a textbook no-arbitrage argument as before. Indeed, let $T > 0$ denote a finite time horizon for the whole market and fix $I \leq M \leq T$. We recall that an FRA is an over-the-counter derivative that allows the holder to lock in at any date $0 \leq t \leq I$ the interest rate between the inception date $I$ and the maturity $M$ at a fixed value $R$. At the maturity $M$, a payment based on $R$ is exchanged for a cash flow indexed on the underlying floating rate (generally the spot Libor rate $L(I; I, M)$), so giving the payoff

$$(M - I)(L(I; I, M) - R).$$

The FRA rate $R(t; I, M)$ will be the rate for which the FRA is at pair. We assume that the notional amount is equal to one for simplicity. Assuming that the ZC bonds are denoted by $(B(s, t))_{0 \leq s \leq t \leq T}$ and the FRA are tradable, we can implement the following strategy: A time $t$ buy one bond maturing $I$, short $B(t, I)$ bonds maturing $M$ and enter an FRA with inception date $I$ and maturity $M$. At time $I$, place one euro on Libor. The net cash at $t$ is $-B(t, I) + \frac{B(t, I)}{B(t, M)}B(t, M) = 0$, at time $I$ it is $1 - 1 = 0$ and at time $M$, it is $-(M - I)(L(I; I, M) - R(t; I, M)) + (1 + (M - I)L(I; I, M)) - \frac{B(t, I)}{B(t, M)} = 1 + (M - I)R(t; I, M) - \frac{B(t, I)}{B(t, M)}$. So, by no-arbitrage the FRA rate is

$$R(t; I, M) = \frac{1}{M - I} \left( \frac{B(t, I)}{B(t, M)} - 1 \right).$$

The formula above shows that as the ZC bonds are not tenor dependent so are the FRA rates. So, we see that the existence of a tradable ZC bonds for all maturities is not consistent with a multicurve financial market, where FRA of different tenors are the basic assets.

Here, we have addressed this issue by assuming that there is a tradable numéraire for each tenor $\tau$ (the ZC of tenor $\tau$) which allows to deflate only the instruments of tenor $\tau$. The submarket of tenor $\tau$ consists only in the instruments having tenor $\tau$ and we model the case where there is no one numéraire available for trading to every submarkets. For example, the 3-month numéraire will be constructed using the OIS rate, but taking into account some roll-over risk, which will be of course different from, say, the 6-month tenor. The fact that both numéraires are traded in two different submarkets and that it is not allowed to borrow on one submarket again others avoid the arbitrage opportunity described above.

We have seen that in general, it is not possible to identify a common risk-neutral measure, as usually assumed in practice and in the literature (see [9] and [2] and the references therein). Nevertheless, we will see in Section 6.2 below that this is true when the spreads between the different submarkets are
deterministic (and in particular when there is a common numéraire) or when
the numéraires are constant. Note that the existence of a common martingale
measure in the post-crisis market is justified by some FTAP in [8] assuming
that the overnight indexed swap ZC bonds are freely tradable for all maturi-
ties. The authors also show the existence of martingale measures for a given
numéraire assuming that this numéraire is traded.

An alternative approach to ours would be to incorporate the risks implicit
in interbank transactions like for example liquidity risk. This can be done
introducing illiquidity cost in the self-financing condition (1). To the best of
our knowledge this has not been done in this context.

6.2 Examples of submarkets with a common martingale measure

We present two situations where for a specific choice of numéraires \( (S^\tau, 0)_{\tau \in T} \),
there exists a (local) martingale measure common to all submarkets. Using
Proposition 2, we provide an explicit formula for the price \( \pi(H) \). Assume
that \( (S, S^0) \) satisfies the no free lunch condition. Then, Theorem 1 shows that
\( X^* \neq \emptyset \). Let \( X^* \in X^* \).

Example 1 The first situation is the one where the numéraires \( S^\tau, 0 \) are deter-
mministic for any \( \tau \in T \). Then, (8) implies that
\[
\frac{dQ^\tau}{dP} \bigg|_F = \frac{X^*}{\mathbb{E}(X^*)}
\]
and the (local) martingale measures clearly do not depend on the submarket.

Let \( \tau_{\text{max}} \) be the \( \tau \) such that \( S^{\tau_{\text{max}}, 0}_T \) is maximum. Then, \( Q^{\tau_{\text{max}}} = Q^{\text{max}} \),
\[
ce_{\tau} = \frac{S^{\tau, 0}_T}{S^{\tau_{\text{max}}, 0}_T} \leq 1 \quad \text{and (22) implies that for } H \in L^\infty
\]
\[
\pi(H) = \pi^{\tau_{\text{max}}}(H) = \sup_{Q \in Q^{\tau_{\text{max}}}} \mathbb{E}_Q \left( \frac{H}{S^{\tau_{\text{max}}, 0}_T} \right) = \sup_{Q \in Q^{\text{max}}} \mathbb{E}_Q \left( \frac{H}{\max_{\tau \in T} S^{\tau, 0}_T} \right)
\]
and \( \pi(H) \) and the lower bound in (17) are equal.

Example 2 In this example, we consider the case where the spreads are deter-
mministic. First, we rewrite the numéraires \( S^{\tau, 0} \) for each \( \tau \in T \) as follows:
\[
S^{\tau, 0}_t = S^{\tau, 0}_0 \exp \left( \int_0^t (r_u + s^{\tau}_u) \, du \right),
\]
where \( r \) is some short-term submarket-free rate\(^2\) and \( s^{\tau} \) denotes the short-term
spread specific\(^3\) to the submarket \( \tau \). Let
\[
\frac{dP^*}{dP} \bigg|_F = \frac{X^* \exp \left( \int_0^T r_u \, du \right)}{\mathbb{E} \left( X^* \exp \left( \int_0^T r_u \, du \right) \right)}.
\]

\(^2\) For the multi-curve interest rates model, \( r \) may be the Overnight Indexed Swap (OIS)
rate, which is a reference basis rate (see, e.g., [9]).

\(^3\) In the example of credit risk markets, \( s^{\tau} \) is the default intensity of the class of risk \( \tau \).
As the spreads are assumed to be deterministic, (8) shows that \( Q^\tau = P^* \) for all \( \tau \in T \). This is in particular the case if we assume that there exists a common numéraire for all \( \tau \in T \) (i.e., \( s^\tau = 0 \)).

Let \( \tau_{\text{max}} \) be the \( \tau \) such that \( \exp(\int_0^T s_u^\tau \, du) \) is maximum. Then, \( \max_{\tau \in T} S_T^{\tau,0} = \bar{S}_T^{\tau_{\text{max}},0} \). So, \( Q^{\tau_{\text{max}}} = Q^{\text{max}}, c_\tau = \exp(\int_0^T (s^\tau_u - s^\tau_{\text{max}} u) \, du) \leq 1 \), and (22) implies that

\[
\pi(H) = \pi^{\tau_{\text{max}}}(H) = \sup_{Q \in Q^{\tau_{\text{max}}}} \mathbb{E}_Q \left( \frac{H}{S_T^{\tau_{\text{max}},0}} \right) = \sup_{Q \in Q^{\text{max}}} \mathbb{E}_Q \left( \frac{H}{\max_{\tau \in T} S_T^{\tau,0}} \right).
\]

where \( Q \in P^R \) if and only if \( Q \sim P, \frac{dQ}{dP} e^{-\int_0^T r_u \, du} \in L^1 \) and for all \( \tau \in T, (S_t^\tau \exp(-\int_0^T r_u \, du))_{t \geq 0} \) is a local martingale under \( Q \). It is clear that \( Q^{\tau_{\text{max}}} = P^R \) and again \( \pi(H) \) and the lower bound in (17) are equal.

6.3 Completeness

As mentioned after Theorem 1, we now discuss completeness. To do so, we provide simple examples that allow a better understanding of the link between the completeness of all submarkets \( \tau \) considered as separate markets and that of the global market. The completeness of the submarket \( \tau \) is well known. It amounts to saying that the set \( \hat{Q}^\tau \) of local martingale measures for the discounted asset \( \tau \) is a singleton. Now, we define the completeness of the global market as follows: each set \( Q^\tau \) is a singleton.

For simplicity, we consider the case of Example 1, where the numéraires \( S^{\tau,0} \) are deterministic for all \( \tau \in T \). In this case, the (local) martingale measures for the global market clearly do not depend on the submarket. Proposition 1 shows that for all \( \tau \in T \)

\[
Q^\tau = \left\{ P^* | \exists X^* \in X^*, \frac{dP^*}{dP} = \frac{X^*}{\mathbb{E}(X^*)} \right\}.
\]

Moreover, Remark 1 shows that for all \( \tau \in T \),

\[
\mathcal{Q}^\tau = \left\{ Q^\tau | \exists X^\tau \in X^\tau, \frac{dQ^\tau}{dP} = \frac{X^\tau}{\mathbb{E}(X^\tau)} \right\}.
\]

Now, Definition 6 implies that for deterministic numéraires

\[
Q^\tau = \cap_{\tau' \in T} Q^\tau, S_T^{\tau,0} = \cap_{\tau' \in T} \mathcal{Q}^\tau'.
\]

(27)

So, the only case where the completeness of all submarkets \( \tau \) implies the completeness of the global market is when the sets \( \mathcal{Q}^\tau \) reduce to the same singleton. Otherwise, even if every submarket \( \tau \) is complete, the global market
will not satisfy the NFL as $\mathcal{Q}^\tau$ is empty. We provide a concrete example of that in Example 3.

We present also in Example 4 a situation where the global market is complete but each submarket $\tau$ is incomplete. The intuition beyond is that there are more constraints in the definition of $\mathcal{Q}^\tau$ than in the one of $\hat{\mathcal{Q}}^\tau$.

**Example 3** We consider two Binomial models, i.e., for $i \in \{1, 2\}$, $S_1^\tau = S_0^\tau (1 + U^i)$, where $U^i$ can take only two values: $u_i$ with probability $p > 0$ in the “up” state and $d_i$ with probability $1 - p > 0$ in the “down” state. We suppose that $d_0 = 1/2$, $u_1 = 2$, $d_2 = 5/2$ and $u_2 = 4$. Moreover, $S_1^{\tau_i, 0} = S_0^{\tau_i, 0} (1 + r_i)$, where $r_1 = 1$ and $r_2 = 3$. Then, it is easy to see that the markets $\tau_1$ and $\tau_2$ are arbitrage free and complete and that the global market is also arbitrage free. Indeed, we have that

$$\hat{\mathcal{Q}}^{\tau_1} = \hat{\mathcal{Q}}^{\tau_2} = \mathcal{Q}^{\tau_1} = \mathcal{Q}^{\tau_2} = \{\mathbb{P}^*\},$$

where $\mathbb{P}^*$ is the probability that gives the weight $1/3$ to the “up” state and $2/3$ to the “down” state.

Making a choice of coefficients such that $d_i < r_i < u_i$ for $i \in \{1, 2\}$ but $\frac{u_i - d_i}{u_i - d_1} \neq \frac{u_2 - d_2}{u_1 - d_2}$ illustrates a case where the markets $\tau_1$ and $\tau_2$ are arbitrage free and complete and the global market is not arbitrage free (the set $\mathcal{X}^*$ is empty).

**Example 4** We consider two Trinomial models, i.e., for $i \in \{1, 2\}$, $S_1^\tau = S_0^\tau (1 + U^i)$, where $U^i$ can take only three values $\{u_i, 0, d_i\}$ with probability $p$ in the “up” state, $v$ in the “middle” state and $q$ in the “down” state. We assume that $-1 < d_i < 0 < u_i$, $p, v, q \in (0, 1)$ and $p + v + q = 1$. Moreover, $S_1^{\tau_i, 0} = S_0^{\tau_i, 0} (1 + r_i)$ and we assume that $d_i < r_i < u_i$. The set $\hat{\mathcal{Q}}^\tau$ is characterized by the following equations

$$p \frac{1 + u_i}{1 + r_i} + v \frac{1}{1 + r_i} + q \frac{1 + d_i}{1 + r_i} = 1$$

$$p + v + q = 1.$$

The preceding system admits the following set of solutions

$$v \in \left(0, 1 - \frac{r_i}{u_i}\right), \quad p = \frac{r_i - (1 - v)d_i}{u_i - d_i} \quad q = \frac{(1 - v)u_i - r_i}{u_i - d_i}$$

and, in general, the submarkets $\tau_1$ and $\tau_2$ are arbitrage free but incomplete.

Now, for the global market, recalling (27), we have to solve the system

$$p \frac{1 + u_1}{1 + r_1} + v \frac{1}{1 + r_1} + q \frac{1 + d_1}{1 + r_1} = 1$$

$$p \frac{1 + u_2}{1 + r_2} + v \frac{1}{1 + r_2} + q \frac{1 + d_2}{1 + r_2} = 1$$

$$p + v + q = 1.$$
This system admits the following unique solution
\[ p = \frac{r_2d_1 - r_1d_2}{d_1u_2 - d_2u_1}, \]
\[ v = \frac{(u_2 - d_2)(u_1 - r_1) - (u_1 - d_1)(u_2 - r_2)}{d_1u_2 - d_2u_1}, \]
\[ q = \frac{r_1u_2 - r_2u_1}{d_1u_2 - d_2u_1}. \]

We still have to check that \( 0 < v < 1 - \max(\frac{d_1}{u_2}, \frac{d_2}{u_1}) \). We propose below an example where this condition holds true. We choose \( d_1 = \frac{1}{4}, u_1 = 2 \) and \( r_1 = 1 \) together with \( d_2 = 3, u_2 = 15 \) and \( r_2 = \frac{29}{4} \). We find that \( v = \frac{1}{4} < 1 - \max(\frac{1}{4}, \frac{29}{15}) \), \( p = \frac{5}{12} \) and \( q = \frac{1}{3} \). So, all the conditions are satisfied: the submarkets \( \tau_1 \) and \( \tau_2 \) are arbitrage free but incomplete whereas the global market is complete.

6.4 Brownian model

6.4.1 General model

Let \( W \) be a \( d \)-dimensional Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\), adapted to the filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\), where we suppose that \( d \geq \sum_{\tau \in \mathcal{T}} d_\tau \). We assume in the rest of the section (and of the next one) that all processes have càdlàg trajectories and are adapted to this filtration. We also use the notation \( \langle \cdot, \cdot \rangle \) for the scalar product. We postulate that the risky assets \( S^\tau \) follow the following SDE.

**Assumption 2** Let \( \tau \in \mathcal{T} \), there exist an \( \mathbb{R}^{d_\tau} \)-valued process \((M^\tau_t)_{t \geq 0}\) such that \( \int_0^T \|M^\tau_t\| dt < \infty \) a.s., a \( d_\tau \times d_\tau \) matrix-valued process \((H^\tau_t)_{t \geq 0}\) such that \( \sum_{i=1}^{d_\tau} \sum_{j=1}^{d_\tau} \int_0^T |(H^\tau_t)_{i,j}|^2 dt < \infty \) a.s. and
\[
dS^\tau_t = \text{diag}(S^\tau_t) \left( M^\tau_t dt + \langle H^\tau_t, dW_t \rangle \right),
\]
where \( \text{diag}(S^\tau_t) \) is the \( d_\tau \times d_\tau \) diagonal matrix with the \( d_\tau \)-vector \( S^\tau_t \) on the diagonal.

We adopt a spread form for the numéraires \( S^{\tau,0} \) for each \( \tau \in \mathcal{T} \), i.e.,
\[
S^{\tau,0}_t = S^{\tau,0}_0 \exp \left( \int_0^t (r_u + s^\tau_u) du \right),
\]
where the process \( r \) is the reference risk-free rate and the process \( s^\tau \) denotes the short-term spread for the tenor \( \tau \). We suppose that \( \int_0^T |r_t| dt < \infty \) a.s. and \( \int_0^T |s^\tau_t| dt < \infty \) a.s.

We assume that the density process \( dQ^\tau/d\mathbb{P} \) related to the measure change has the following property.
Assumption 3 Let $\tau \in \mathcal{T}$. For every $Q^\tau \in \mathcal{Q}^\tau$, there exists an $\mathbb{R}^d$-valued process $(A^\tau_t)_{t \geq 0}$ such that $\int_0^T \|A^\tau_t\|^2 dt < \infty$ a.s. and

$$
\frac{dQ^\tau}{dP} \bigg|_{\mathcal{F}_t} = \exp \left( \int_0^T (A^\tau_t, dW_t) - \frac{1}{2} \int_0^T \|A^\tau_t\|^2 dt \right). \tag{30}
$$

Then, the process

$$
dW^\tau_t = dW_t - A^\tau_t dt \tag{31}
$$

is a $Q^\tau$-Brownian motion by Girsanov’s theorem.

Let $\tau \in \mathcal{T}$ and $Q^\tau \in \mathcal{Q}^\tau$ as in (30). Definition 6 implies that $\bar{S}^\tau$ is a $Q^\tau$ local martingale. Using Itô’s formula, we find that for all $\tau \in \mathcal{T}$,

$$
d\bar{S}^\tau_t = \text{diag}(\bar{S}^\tau_t) \left( (M^\tau_t - (r_t + s^\tau_t)1_{d_\tau}) dt + (H^\tau_t, dW_t) \right)
= \text{diag}(\bar{S}^\tau_t) \left( (M^\tau_t - (r_t + s^\tau_t)1_{d_\tau} + (H^\tau_t, A^\tau_t)) dt + (H^\tau_t, dW^\tau_t) \right),
$$

where $1_{d_\tau}$ is the $d_\tau$-vector with all coordinates equal to one. As $W^\tau$ is a $Q^\tau$-Brownian motion, $A^\tau_t$ must satisfy for all $\tau \in \mathcal{T}$ that

$$
(H^\tau_t, A^\tau_t) = (r_t + s^\tau_t)1_{d_\tau} - M^\tau_t \text{ a.s.} \tag{32}
$$

Then,

$$
d\bar{S}^\tau_t = \text{diag}(\bar{S}^\tau_t)(H^\tau_t, dW^\tau_t). \tag{33}
$$

We want to find the other relations that the $A^\tau_t$ should satisfy. Let $\tau^* \in \mathcal{T}$.

Recalling Definition 6, $Q^{\tau^*} \in \mathcal{Q}^{\tau^*}$ if and only if $Q^{\tau^*} \sim P$, $\frac{dQ^{\tau^*}}{dP} / \bar{S}^{\tau^*,0}_{\mathcal{T}} \in L^1$ and for all $\tau \in \mathcal{T}$

$$
\left( \mathbb{E}_{Q^{\tau^*}} \left( \frac{\bar{S}^{\tau^*,0}_{\mathcal{T}}}{\bar{S}^{\tau^*,0}_{\mathcal{T}}} \bigg| \mathcal{F}_t \right) \bar{S}^\tau_t \right)_{t \geq 0}
$$

is a local martingale under $Q^{\tau^*}$.

So, we need to compute $X_t = \mathbb{E}_{Q^{\tau^*}} \left( \frac{\bar{S}^{\tau^*,0}_{\mathcal{T}}}{\bar{S}^{\tau^*,0}_{\mathcal{T}}} \bigg| \mathcal{F}_t \right) = \mathbb{E}_{Q^{\tau^*}} \left( \exp \left( \int_0^T s_t du \right) \bigg| \mathcal{F}_t \right)$,

where $s = s^\tau - s^{\tau^*}$ denotes the difference between the spreads $s^\tau$ and $s^{\tau^*}$. For that, we assume that the spreads follow an affine Hull-White (time dependent Vasicek) dynamic under their respective measures $Q^\tau \in \mathcal{Q}^\tau$, where the coefficient $b_t$ does not depend from $\tau$.

Assumption 4 Let $\tau \in \mathcal{T}$, let $Q^\tau \in \mathcal{Q}^\tau$ satisfying Assumption 3 and let $W^\tau$ be the $Q^\tau$-Brownian motion defined in (31). The spread $s^\tau$ follows

$$
da^\tau_t = (a^\tau_t - b_t s^\tau_t) dt + (\Sigma^\tau_t, dW^\tau_t),
$$

where $t \mapsto a^\tau_t$, $b_t$ are nonrandom, real-valued, positive functions of time and $t \mapsto \Sigma^\tau_t$ is a nonrandom, $\mathbb{R}^d_+$-valued function of time.
Let $\tau, \tau^* \in \mathcal{T}$ and let $\mathbb{Q}_\tau^* \in \mathcal{Q}^\tau$ and $\mathbb{Q}^{\tau^*} \in \mathcal{Q}^{\tau^*}$, both satisfying Assumption 3. Let $\Lambda^\tau$ (resp. $\Lambda^{\tau^*}$) be the measure change associated to $\mathbb{Q}_\tau^*$ (resp. $\mathbb{Q}^{\tau^*}$) in Assumption 3. We set $\Lambda = \Lambda^\tau - \Lambda^{\tau^*}$. Using that $dW_t^\tau = dW_t^{\tau^*} - \Lambda dt$, which follows from (31), we obtain that

$$ds_t^\tau = (a_t^\tau - \langle \Sigma_t^\tau, \Lambda_t \rangle - b_t s_t^\tau) dt + \langle \Sigma_t^\tau, dW_t^{\tau^*} \rangle$$

and hence, setting $\Sigma_t = \Sigma_t^\tau - \Sigma_t^{\tau^*}$ and $a_t = a_t^\tau - a_t^{\tau^*}$, we get that

$$ds_t = (a_t - \langle \Sigma_t, \Lambda_t \rangle - b_t s_t) dt + \langle \Sigma_t, dW_t^{\tau^*} \rangle.$$  

At this stage, in order to obtain explicit results and apply the standard affine machinery, we have to assume that $T$ is deterministic and that the spread $s$ has an affine Vasicek-type dynamic with time-dependent coefficients.

Assumption 5 The stopping time $T$ is deterministic and for all $\tau \in \mathcal{T}$, the function $t \mapsto \langle \Sigma_t^\tau, \Lambda_t \rangle$ is a deterministic function of time.

Assuming that $T$ is deterministic is indeed an approximation that allows to obtain explicit results. The computation of the law of $T$ and the adaptation of the affine machinery would be very technical and would not add much to the article message. Then, using the standard affine machinery (see for example [7, Theorem 10.4] or formula (6.5.8)-(6.5.11) in [19]), we can calculate explicitly that

$$\mathbb{E}_{\mathbb{Q}_\tau^*} \left( \exp \left( \int_t^T s_u du \right) \bigg| \mathcal{F}_t \right) = \exp \left( m^\Lambda(t, T) + \tilde{b}(t, T) s_t \right),  \quad (34)$$

where $m^\Lambda(t, T)$ and $\tilde{b}(t, T)$ are deterministic functions obtained by solving the associated system of Riccati equations:

$$\frac{\partial \tilde{b}(t, T)}{\partial t} = b_t \tilde{b}(t, T) - 1, \quad \tilde{b}(t, T) = \int_t^T e^{\int_t^s b_u du} ds$$

$m^\Lambda(t, T) = \frac{1}{2} \int_t^T \| \Sigma_t \|^2 \tilde{b}^2(s, T) ds + \int_t^T a_s \tilde{b}(s, T) ds - \int_t^T \langle \Sigma_s, \Lambda_s \rangle \tilde{b}(s, T) ds.$

Note that $\tilde{b}(t, T) \geq 0$.

So, (34) implies that $X_t = \exp \left( \int_0^t s_u du \right) \exp(m^\Lambda(t, T) + \tilde{b}(t, T) s_t)$ and Itô’s formula yields that

$$dX_t = X_t \left( \left( s_t - \frac{1}{2} \| \Sigma_t \|^2 \tilde{b}^2(t, T) - (a_t - \langle \Sigma_t, \Lambda_t \rangle) \tilde{b}(t, T) + b_t \tilde{b}(t, T) s_t - s_t \right) dt 
+ \tilde{b}(t, T) ds_t + \frac{1}{2} \tilde{b}^2(t, T) d < s >_t \right)$$

$$= X_t \tilde{b}(t, T) \langle \Sigma_t, dW_t^{\tau^*} \rangle.$$

Moreover, recalling (33), for all $\tau \in \mathcal{T} \setminus \{\tau^*\}$

$$dS_t^\tau = \text{diag}(S_t^\tau) \left( -(H_t^\tau, \Lambda_t) dt + \langle H_t^\tau, dW_t^{\tau^*} \rangle \right).$$
So, using the integration by part formula, we get that
\[
d(X_t \tilde{S}_t^\tau) = X_t \text{diag}(\tilde{S}_t^\tau) \left( \left( -\langle H_t^\tau, \Lambda_t \rangle + \tilde{b}(t, T) \langle H_t^\tau, \Sigma_t \rangle \right) dt + \tilde{b}(t, T) \langle \Sigma_t, dW_t^\tau \rangle \right) dt + \langle H_t^\tau, dW_t^\tau \rangle.
\]
As \( X \tilde{S}^\tau \) is a \( Q^\tau \) local martingale, we get that for all \( \tau \in \mathcal{T} \setminus \{\tau^*\} \)
\[
\langle H_t^\tau, \Lambda_t \rangle = \tilde{b}(t, T) \langle H_t^\tau, \Sigma_t \rangle \text{ a.s.} \quad \quad (35)
\]
So, we have proved the following proposition.

**Proposition 4** Assume that Assumptions 2, 3, 4 and 5 hold. Then, \( Q^\tau \in \mathbb{Q}^\tau \) if and only if \( \frac{dQ^\tau}{dP}\bigm|_{\tilde{S}^\tau} \in L^1 \) and
\[
\langle H_t^\tau, \Lambda_t \rangle = (r_t + s_t^\tau)1_{d+} - M_t^\tau \text{ a.s.}
\]
\[
\langle H_t^\tau, \Lambda_t^\tau \rangle = \langle H_t^\tau, \Lambda_t^\tau \rangle - \tilde{b}(t, T) \langle H_t^\tau, \Sigma_t^\tau - \Sigma_t^\tau \rangle \text{ a.s. for all } \tau \in \mathcal{T} \setminus \{\tau^*\},
\]
where for all \( \tau \in \mathcal{T}, \Lambda^\tau \) is the measure change associated to \( Q^\tau \in \mathbb{Q}^\tau \) by Assumption 3.

Note that if \( \hat{Q}^\tau \) satisfies Assumption 3 (denoting by \( \hat{\Lambda}^\tau \) the measure change associated to \( \hat{Q}^\tau \)), then under Assumption 2, \( \hat{Q}^\tau \in \mathbb{Q}^\tau \) if and only if \( \frac{d\hat{Q}^\tau}{d\hat{P}}\bigm|_{\tilde{S}^\tau} \in L^1 \) and
\[
\langle H_t^\tau, \hat{\Lambda}_t^\tau \rangle = (r_t + s_t^\tau)1_{d+} - M_t^\tau \text{ a.s.}
\]
We see that \( \hat{Q}^\tau \) may be much larger than \( Q^\tau \).

Now, we give the link between the spread \( s \) and the risk premium process \( \Lambda \).

**Proposition 5** Assume that Assumptions 3, 4 and 5 hold. Let \( \tau, \tau^* \in \mathcal{T} \) and let \( Q^\tau \in \mathbb{Q}^\tau \) and \( Q^\tau^* \in \mathbb{Q}^\tau^* \). Let \( \Lambda^\tau \) (resp. \( \Lambda^\tau^* \)) be the measure change associated to \( Q^\tau \) (resp. \( Q^\tau^* \)) in Assumption 3. Then, the process \( \Lambda = \Lambda^\tau - \Lambda^\tau^* \) satisfies the following equalities for all \( t \geq 0 \).
\[
\langle \Lambda_t, dW_t^\tau \rangle = \tilde{b}(t, T) \langle \Sigma_t, dW_t^\tau \rangle
\]
\[
\| \Lambda_t \| = \tilde{b}(t, T) \| \Sigma_t \|. \quad (36)
\]
We see from Proposition 5, that \( t \mapsto \Lambda_t \) is a deterministic function of time. We also see that (35) and (36) provide \( \sum_{\tau \in \mathcal{T}} d_\tau + 1 \) equations in order to get the \( d \) coordinates of \( \Lambda \). Thus, if we assume that \( d = \sum_{\tau \in \mathcal{T}} d_\tau + 1 \), we may be able to characterize \( \Lambda \). This will be done in the next section.
Proof From Assumption 3, we have
\[
\frac{dQ^T}{dQ^{T^*}} |_{\mathcal{F}} = \exp \left( \int_0^T \langle A_t^r - A_t^r^*, dW_s \rangle - \frac{1}{2} \int_0^T (\|A_t^r\|^2 - \|A_t^r^*\|^2) ds \right) \\
= \exp \left( \int_0^T \langle A_s, dW_s^r \rangle - \frac{1}{2} \int_0^T \|A_s\|^2 ds \right).
\]

As \(W^r\) is a \(Q^{T^*}\) Brownian motion, we get that
\[
\mathbb{E}_{Q^{T^*}} \left( \frac{dQ^T}{dQ^{T^*}} |_{\mathcal{F}} \right) = \exp \left( \int_0^t \langle A_s, dW_s^r \rangle - \frac{1}{2} \int_0^t \|A_s\|^2 ds \right). 
\tag{37}
\]

Using (8), we obtain
\[
\frac{dQ^T}{dQ^{T^*}} |_{\mathcal{F}} = \frac{\mathbb{E}(X^* S_T^{r^*, 0})}{\mathbb{E}(X^* S_T^{r^*, 0})} S_T^{r^*, 0} = C \exp \left( \int_0^T s_u du \right), 
\tag{38}
\]

where \(C = \frac{\mathbb{E}(X^* S_T^{r^*, 0})}{\mathbb{E}(X^* S_T^{r^*, 0})}\) is a constant. Hence, combining (34), (37) and (38), we obtain that
\[
\int_0^t \langle A_s, dW_s^r \rangle - \frac{1}{2} \int_0^t \|A_s\|^2 ds = \ln C + \int_0^t s_u du + m^A(t, T) + \tilde{b}(t, T) s_t.
\]

Thus,
\[
\langle A_t, dW_t^r \rangle - \frac{1}{2} \|A_t\|^2 dt = s_t dt + dm^A(t, T) + s_t \tilde{b}(t, T) + 
\tag{39}
\tilde{b}(t, T) \left( (a_t - \langle \Sigma_t^r, A_t \rangle) - b_t s_t \right) dt + \langle \Sigma_t, dW_t^r \rangle.
\]

So, we see that \(\langle A_t, dW_t^r \rangle = \tilde{b}(t, T) \langle \Sigma_t, dW_t^r \rangle\) and that
\[
- \frac{1}{2} \|A_t\|^2 = s_t - \frac{1}{2} \|\Sigma_t\|^2 \tilde{b}^2(t, T) - (a_t - \langle \Sigma_t^r, A_t \rangle) \tilde{b}(t, T) + 
\tag{40}
\tilde{b}(t, T) (a_t - \langle \Sigma_t^r, A_t \rangle) b_t s_t + b_t \tilde{b}(t, T) s_t - s_t
\]

which concludes the proof. \(\square\)

6.4.2 Application

In order to illustrate Proposition 3, we assume now that there are only two submarkets with one risky asset in each. Let \(d = 3\) and \(W = (W_0, W_1, W_2)\) be a 3-dimensional Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\). Again, in order to obtain explicit results, we make an approximation and assume that \(T\) is deterministic.

We propose below an incomplete model where each submarket is driven by its own Brownian motion and by the common noise \(W_0\). We are in the situation mentioned after Proposition 5 and we will be able to calculate \(A\) as a solution of (40). This will allow us to calculate the different prices of Proposition 3.
Assumption 6 For $i \in \{1, 2\}$, we consider that $t \mapsto \mu_{i,t}, \eta_{i,t}, \rho_{i,t}, \kappa_{i,t}, \sigma_{i,t}$ are deterministic functions of time such that $0 < \rho_{i,t} < 1$, $|\kappa_{i,t}| \leq 1$, $\sigma_{i,t} > 0$, $\eta_{i,t} > 0$, $\int_0^T |\mu_{i,t}| dt < \infty$, $\int_0^T \eta_{i,t}^2 dt < \infty$ and $\int_0^T \sigma_{i,t}^2 dt < \infty$. Then, we set

$$M_t^{\tau_1} = \mu_{1,t}$$
$$M_t^{\tau_2} = \mu_{2,t}$$

$H_t^{\tau_1} = (\eta_{1,t} \sqrt{1 - \rho_1^2}, \eta_{1,t} \rho_{1,t}, 0)$ and $H_t^{\tau_2} = (\eta_{2,t} \sqrt{1 - \rho_2^2}, 0, \eta_{2,t} \rho_{2,t})$

$\Sigma_t^{\tau_1} = (\sigma_{1,t} \sqrt{1 - \kappa_1^2}, \sigma_{1,t} \kappa_{1,t}, 0)$ and $\Sigma_t^{\tau_2} = (\sigma_{2,t} \sqrt{1 - \kappa_2^2}, 0, \sigma_{2,t} \kappa_{2,t})$.

Under these conditions, for $i \in \{1, 2\}$, $\tilde{S}_t^{\tau_i}$ is a true martingale under any $Q \in \tilde{Q}^{\tau_i}$, see (33). We may take more general assumptions as some Novikov condition $\mathbb{E}_{Q_t} \left[ \int_0^T \sigma^{\tau_i}_t dW_t \right] < \infty$, but we wanted to keep it simple. Then, we set

$$\Sigma_t = \Sigma_t^{\tau_1} - \Sigma_t^{\tau_2} = (\Sigma_1,t, \Sigma_2,t, \Sigma_3,t)$$

$$= \left( \sigma_{1,t} \sqrt{1 - \kappa_1^2} - \sigma_{2,t} \sqrt{1 - \kappa_2^2}, \sigma_{1,t} \kappa_{1,t}, -\sigma_{2,t} \kappa_{2,t} \right).$$

Let $Q_1 \in \tilde{Q}^{\tau_1}$ and $Q_2 \in \tilde{Q}^{\tau_2}$. Let $\Lambda^{\tau_1}$ (resp. $\Lambda^{\tau_2}$) be the measure change associated to $Q_1$ (resp. $Q_2$) in Assumption 3. We set $\Lambda = \Lambda^{\tau_1} - \Lambda^{\tau_2}$ and $s = s^{\tau_1} - s^{\tau_2}$.

We first want to compute $\pi_2(S_t^{\tau_1})$ and recalling (25), (29) and (34)

$$\mathbb{E}_{Q_2} \left[ \frac{S_t^{\tau_1,0}}{S_t^{\tau_2,0}} \right] = \mathbb{E}_{Q_2} \left[ \exp \left( \int_0^T s_u du \right) \right] = \exp(m^{\Lambda}(0,T) + \tilde{b}(0,T)s_0) \cdot 39$$

So, we need to determine the possible values for $\Lambda$. Using Proposition 4 for $\tau = \tau_1$ and $\tau^* = \tau_2$ and for $\tau = \tau_2$ and $\tau^* = \tau_1$ and Proposition 5 for $\tau = \tau_1$ and $\tau^* = \tau_2$, we get that

$$\langle H_t^{\tau_1}, A_t \rangle = \tilde{b}(t,T) \langle H_t^{\tau_1}, \Sigma_t \rangle$$
$$\langle H_t^{\tau_2}, A_t \rangle = \tilde{b}(t,T) \langle H_t^{\tau_2}, \Sigma_t \rangle$$
$$\|A_t\|^2 = \tilde{b}^2(t,T) \|\Sigma_t\|^2,$$

where $A_t = A_t^{\tau_1} - A_t^{\tau_2} = (A_{1,t}, A_{2,t}, A_{3,t})$. We now solve (40). Setting $\sqrt{1 - \rho_{i,t}} - 1 = r_{i,t}$ for $i \in \{1, 2\}$, after some computations, we get that

$$A_t^{\tau_1} (r_{1,t}^2 + r_{2,t}^2 + 1) - 2\tilde{b}(t,T) A_t (\Sigma_{1,t} (r_{1,t}^2 + r_{2,t}^2) + \Sigma_{2,t} r_{1,t} + \Sigma_{3,t} r_{2,t})$$
$$+ \tilde{b}^2(t,T) (\Sigma_{1,t}^2 (r_{1,t}^2 + r_{2,t}^2) - 1) + 2 \Sigma_{1,t} (\Sigma_{2,t} r_{1,t} + \Sigma_{3,t} r_{2,t}) = 0.$$
The two solutions of the quadratic equation are
\[ A_1^t = \tilde{b}(t, T) \Sigma_t \]
\[ A_2^t = -\frac{\tilde{b}(t, T)}{r_{1,t}^2 + r_{2,t}^2 + 1} \left( \Sigma_{1,t} (r_{1,t}^2 + r_{2,t}^2 - 1) + 2 \Sigma_{2,t} r_{1,t} + 2 \Sigma_{3,t} r_{2,t} + 2 \Sigma_{3,t} r_{1,t} + \Sigma_{2,t} (-r_{1,t}^2 + r_{2,t}^2 + 1) - 2 \Sigma_{3,t} r_{1,t} r_{2,t} + 2 \Sigma_{1,t} r_{2,t} + 2 \Sigma_{2,t} r_{1,t} r_{2,t} + \Sigma_{3,t} (r_{1,t}^2 - r_{2,t}^2 + 1) \right). \]

So, solving (40) provides two possible values for \( \Lambda \). Recalling (39), we now compute the associated values of \( m^A(t, T) \). First, we find that
\[ (\Sigma_{t}^{\tau_1}, A_{1}^{t}) = \tilde{b}(t, T) \left( \sigma_{1,t}^2 - \sigma_{1,t} \sqrt{1 - \kappa_{1,t}^2} \sqrt{1 - \kappa_{2,t}^2} \right). \]
\[ (\Sigma_{t}^{\tau_2}, A_{2}^{t}) = \tilde{b}(t, T) \left( \sigma_{1,t}^2 - \sigma_{1,t} \sqrt{1 - \kappa_{1,t}^2} \sqrt{1 - \kappa_{2,t}^2} \right) - \frac{2 \tilde{b}(t, T)}{r_{1,t}^2 + r_{2,t}^2 - 1} (\hat{\sigma}_{1,t}^2 - \hat{\sigma}_{1,t} \hat{\sigma}_{2,t}). \]

where \( \hat{\sigma}_{i,t} = \sqrt{1 - \kappa_{i,t}^2} - \kappa_{i,t} \sqrt{\frac{1}{r_{i,t}^2} - 1} \). for \( i \in \{1, 2\} \). So, we get that
\[ m^{A_{1}}(t, T) = \frac{1}{2} \int_{t}^{T} \| \Sigma_{s} \|^{2} \tilde{b}^2(s, T) ds + \int_{t}^{T} (a_{1,s}^{\tau_{1}} - a_{1,s}^{\tau_{2}}) \tilde{b}(s, T) ds - \int_{t}^{T} (\Sigma_{s}^{\tau_{1}}, A_{1}^{s}) \tilde{b}(s, T) ds \]
\[ = - \int_{t}^{T} \frac{\sigma_{1,s}^2 - \sigma_{2,s}^2}{2} \tilde{b}^2(s, T) ds + \int_{t}^{T} (a_{1,s}^{\tau_{1}} - a_{1,s}^{\tau_{2}}) \tilde{b}(s, T) ds \]
\[ m^{A_{2}}(t, T) = \frac{1}{2} \int_{t}^{T} \| \Sigma_{s} \|^{2} \tilde{b}^2(s, T) ds + \int_{t}^{T} (a_{1,s}^{\tau_{1}} - a_{1,s}^{\tau_{2}}) \tilde{b}(s, T) ds - \int_{t}^{T} (\Sigma_{s}^{\tau_{1}}, A_{2}^{s}) \tilde{b}(s, T) ds \]
\[ = m^{A_{1}}(t, T) + 2 \int_{t}^{T} \frac{\tilde{b}(s, T)}{r_{1,s}^2 + r_{2,s}^2 - 1} (\hat{\sigma}_{1,s}^2 - \hat{\sigma}_{1,s} \hat{\sigma}_{2,s}) ds. \]

Using (39), we get that
\[ \sup_{Q^{*} \in Q^{*}} \mathbb{E}_{Q^{*}} \left( \frac{S_{T}^{\tau_{1},0}}{S_{T}^{\tau_{2},0}} \right) = \max \left( \exp(m^{A_{1}}(0, T) + \tilde{b}(0, T) s_{0}, \exp(m^{A_{2}}(0, T) + \tilde{b}(0, T) s_{0}) \right) \]
\[ = \exp \left( \tilde{b}(0, T) s_{0} + \max \left( m^{A_{1}}(0, T), m^{A_{2}}(0, T) \right) \right) = \exp Y^{M} \]
\[ \inf_{Q^{*} \in Q^{*}} \mathbb{E}_{Q^{*}} \left( \frac{S_{T}^{\tau_{1},0}}{S_{T}^{\tau_{2},0}} \right) = \exp \left( \tilde{b}(0, T) s_{0} + \min \left( m^{A_{1}}(0, T), m^{A_{2}}(0, T) \right) \right) = \exp Y^{m}, \]

where, with the notations \( x_{+} = \max(x, 0) \) and \( x_{-} = \max(-x, 0) \), we have set
\[ Y^{M} = \tilde{b}(0, T) s_{0} + m^{A_{1}}(0, T) + 2 \int_{0}^{T} \frac{\tilde{b}(s, T)}{r_{1,s}^2 + r_{2,s}^2 - 1} (\hat{\sigma}_{1,s}^2 - \hat{\sigma}_{1,s} \hat{\sigma}_{2,s}) ds \]
\[ Y^{m} = \tilde{b}(0, T) s_{0} + m^{A_{1}}(0, T) - 2 \int_{0}^{T} \frac{\tilde{b}(s, T)}{r_{1,s}^2 + r_{2,s}^2 - 1} (\hat{\sigma}_{1,s}^2 - \hat{\sigma}_{1,s} \hat{\sigma}_{2,s}) ds. \]
Using Proposition 3, we obtain that
\[
\begin{align*}
\pi \tau_2(S_T^\tau) &= S_0^\tau \exp Y^M \text{ and } \pi \tau_1(S_T^\tau) = S_0^\tau \exp (-Y^m) \\
\pi(S_T^\tau) &= \pi \tau_1(S_T^\tau) + \pi \tau_2(S_T^\tau) = S_0^\tau \exp (-Y^M) \\
\pi(S_T^\tau) &= S_0^\tau \exp (-Y^m).
\end{align*}
\]

If \( S_0^\tau \exp Y^M < S_0^\tau \), then \( \pi(S_T^\tau - S_T^\tau) = +\infty \). Else, recalling (26)
\[
\pi(S_T^\tau - S_T^\tau) = S_0^\tau \exp (-Y^M) - S_0^\tau \exp (-Y^M).
\]

We now illustrate the discussion after Proposition 3. Case 3 corresponds to \( Y^m < 0 \) and \( Y^M \geq 0 \). The cost of illiquidity is equal to zero in both markets, and
\[
\pi(S_T^\tau) \exp(-Y^M) + \pi(S_T^\tau - S_T^\tau) = \pi(S_T^\tau).
\]

Case 2 corresponds to \( Y^m > 0 \). The cost of illiquidity for \( S_T^\tau \) is \( S_0^\tau (1 - \exp(-Y^m)) > 0 \), while the cost of illiquidity for \( S_T^\tau \) is 0 and
\[
\pi(S_T^\tau) \exp \left\{-2 \left| \int_0^T \frac{b(s,T)}{\rho_{1,s} + \rho_{2,s}^2} - 1 (\hat{\sigma}_{1,s} - \sigma_{1,s}, \sigma_{2,s}) ds \right| \right\} + \pi(S_T^\tau - S_T^\tau) = \pi(S_T^\tau).
\]

Finally, Case 1 occurs when \( Y^M < 0 \). Then, \( \pi(S_T^\tau) = S_0^\tau \exp Y^M \) and \( \pi(S_T^\tau) = S_0^\tau \). The cost of illiquidity of \( S_T^\tau \) is equal to zero, while the cost of illiquidity of \( S_T^\tau \) is \( S_0^\tau (1 - \exp Y^M) > 0 \). Moreover, \( \pi(S_T^\tau - S_T^\tau) + \pi(S_T^\tau) = \pi(S_T^\tau) \). Thus, we don’t get a cheaper price for the illiquid asset being long of the liquid asset and of the Basis swap \( S_T^\tau - S_T^\tau \).

We now give conditions on the parameters that allow us to obtain the signs of \( Y^M \) and \( Y^m \). We have that \( Y^M \geq 0 \) if \( s_0^\tau \geq s_0^\tau \), \( \sigma_{1,s} \leq \sigma_{2,s} \), and \( a_{s}^\tau \geq a_{s}^\tau \) for all \( s \). This is an example where the spread of the market \( \tau_\tau \) dominates the one of the market \( \tau_\tau \). Indeed, \( s_0^\tau \) starts with a bigger initial value, has a higher drift and a smaller standard deviation. If we add the condition \( \hat{\sigma}_{1,s} - \sigma_{1,s}, \sigma_{2,s} \geq 0 \), for all \( s \), then we also have that \( Y^m \geq 0 \).

Symmetrically, we have that \( Y^m \leq 0 \) if \( s_0^\tau \leq s_0^\tau \), \( a_{s}^\tau \leq a_{s}^\tau \), and \( \sigma_{1,s} \leq \sigma_{2,s} \) for all \( s \). If we add the condition \( \hat{\sigma}_{1,s} - \sigma_{1,s}, \sigma_{2,s} < 0 \), for all \( s \), then \( Y^M \leq 0 \).

Let
\[
\hat{s}_s = \sqrt{1 - \frac{\sigma_{s}^2}{\sigma_{s}^2}} - \frac{1}{\sqrt{1 - \frac{\sigma_{s}^2}{\sigma_{s}^2}}} - 1
\]
and
\[
\sqrt{1 - \frac{\sigma_{s}^2}{\sigma_{s}^2}} - \frac{1}{\sqrt{1 - \frac{\sigma_{s}^2}{\sigma_{s}^2}}} - 1.
\]

Then, \( \hat{s}_s - \sigma_{1,s}, \sigma_{2,s} \leq 0 \) if and only if \( \frac{\sigma_{s,s}}{\sigma_{s,s}} \leq \hat{s}_s \).

Thus, to get that \( Y^M \leq 0 \), we can assume that \( s_0^\tau \leq s_0^\tau \), and for all \( s \), \( r_{s}^\tau \leq r_{s}^\tau \), and \( 1 \leq \frac{\sigma_{s,s}}{\sigma_{s,s}} \leq \hat{s}_s \).

Sufficient conditions for \( Y^M \geq 0 \) are \( s_0^\tau \geq s_0^\tau \), and for all \( s \), \( a_{s}^\tau \geq a_{s}^\tau \), and \( \hat{s}_s \leq \frac{\sigma_{s,s}}{\sigma_{s,s}} \leq 1 \).

We now consider the case where the coefficients are not time dependent i.e., \( b_t = b \), \( a_{i,t} = a_i \), \( \sigma_{i,t} = \sigma_i \), \( \rho_{i,t} = \rho_i \), \( \eta_{i,t} = \eta_i \) and \( \kappa_{i,t} = \kappa_i \) for all
\( t \) and \( i \in \{1, 2\} \). This allows to obtain closed formula for which we provide numerical examples where \( Y^M \geq 0 \) and \( Y^m \leq 0 \). If \( b > 0 \), we obtain that

\[
Y^M = \frac{s_0}{b} \left( 1 - e^{-b(T-t)} \right) - \frac{\sigma_1^2 - \sigma_2^2}{2} \left( \frac{T}{b^2} - \frac{2}{b^3} \left( 1 - e^{-bT} \right) + \frac{1}{2b^4} \left( 1 - e^{-2bT} \right) \right) + \\
\left( \frac{T}{b} - \frac{1}{b^2} \left( 1 - e^{-bT} \right) \right) \left( a^{\tau_1} - a^{\tau_2} + \frac{2 \left( \hat{\sigma}_1^2 - \hat{\sigma}_1 \hat{\sigma}_2 \right)}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2 - 1} \right).
\]

If \( b = 0 \), we get that \( \tilde{b}(t, T) = T - t \) and

\[
Y^M = s_0 T - \frac{\sigma_1^2 - \sigma_2^2 T^3}{2} + \frac{T^2}{2} \left( a^{\tau_1} - a^{\tau_2} + \frac{2 \left( \hat{\sigma}_1^2 - \hat{\sigma}_1 \hat{\sigma}_2 \right)}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2 - 1} \right).
\]

We obtain \( Y^m \) changing in the equations above \( \left[ \hat{\sigma}_1^2 - \hat{\sigma}_1 \hat{\sigma}_2 \right]_+ \) by \( \left[ \hat{\sigma}_1^2 - \hat{\sigma}_1 \hat{\sigma}_2 \right]_- \).

Now, we illustrate the three situations of liquidity/illiquidity with a numerical example. Assume that \( T = 2 \), \( s_0^T = 3.4 \), \( s_0^m = 3 \), \( a_1 = 0.4 \), \( a_2 = 0.8 \), \( b = 0.1 \), \( \sigma_1 = 0.3 \), \( \sigma_2 = 0.4 \), \( \rho_1 = 0.9 \), \( \rho_2 = 0.8 \), \( \kappa_1 = 0.5 \) and \( \kappa_2 = -0.7 \). We find that \( Y^M = 0.0564 \), \( Y^m = -0.0639 \) and

\[
\begin{align*}
\pi^{\tau_1}(S_T^T) &= S_0^T \exp Y^M = 3.5972 \quad \text{and} \quad \pi^{\tau_1}(S_T^m) = S_0^m \exp (-Y^m) = 3.1981 \\
\pi(S_T^T) &= 3.4 \land 3.5972 = 3.4 \quad \text{and} \quad \pi(S_T^m) = 3 \land 3.1981 = 3 \\
\pi(S_T^T - S_T^m) &= 0.5645.
\end{align*}
\]

Now, if we change \( \kappa_2 \) into \( \kappa_2 = 0.7 \), without changing the values of the other coefficients, we find that \( Y^M = 0.0999 \), \( Y^m = 0.0564 \) and

\[
\begin{align*}
\pi^{\tau_1}(S_T^T) &= S_0^T \exp Y^M = 3.7572 \quad \text{and} \quad \pi^{\tau_1}(S_T^m) = S_0^m \exp (-Y^m) = 2.8355 \\
\pi(S_T^T) &= 3.4 \land 3.7572 = 3.4 \quad \text{and} \quad \pi(S_T^m) = 3 \land 2.8355 = 2.8355 \\
\pi(S_T^T - S_T^m) &= 0.6852.
\end{align*}
\]

Finally, if furthermore we set \( \sigma_1 = 0.4 \) and \( \sigma_2 = 0.3 \), and keep the same values for the other coefficients, we find that we find that \( Y^M = -0.0044 \), \( Y^m = -0.1047 \) and

\[
\begin{align*}
\pi^{\tau_1}(S_T^T) &= S_0^T \exp Y^M = 3.3850 \quad \text{and} \quad \pi^{\tau_1}(S_T^m) = S_0^m \exp (-Y^m) = 3.3311 \\
\pi(S_T^T) &= 3.4 \land 3.3850 = 3.3850 \quad \text{and} \quad \pi(S_T^m) = 3 \land 3.3311 = 3 \\
\pi(S_T^T - S_T^m) &= 0.3850.
\end{align*}
\]

### 6.5 Extensions

The definition of NFL is tailored to easily obtain the existence of equivalent (local) martingale measures. However, the economic interpretation of this condition is not very natural. A free lunch is some non-negative and non-zero random variable for which one can find a convergent net in \( C \). If one wants
to return to the notion of sequence and take into account some budget constraints, one has to introduce the notion of no free lunch with vanishing risk (NFLVR) and consider general (and not simple) strategies; see [5]. The version of Theorem 1 with the NFLVR condition instead of the NFL condition is left for further research. Note that the definition of NFLVR (and in particular the admissibility of the trading strategies) has to be carefully generalized since there are several numéraires that are used together. Finally, recall that it is shown in [5] that for continuous and locally bounded processes with finite time horizons, it is sufficient to consider no free lunch with bounded risk for simple integrands (together with the NA condition) to obtain equivalent local martingale measures. One may also study the no unbounded profit with bounded risk (NUPBR) condition of [14], which is equivalent to the no arbitrage of the first kind (NA1) condition, but again those definitions have to be adapted carefully because there are several numéraires. Recall that the NFLVR condition is equivalent to the NA condition plus the NA1 condition.

7 Conclusion

We propose a model, where several submarkets coexist, each one with its own numéraire, but where no tradeable numéraire exists for the whole market. Nevertheless, we consider the No Free Lunch condition at the level of the global market. This modelling is meaningful for credit or liquidity risk and for the multicurve markets.

We prove that there is a common factor from which equivalent submarket-dependent local martingale measures are constructed. Next, we introduce several superhedging prices depending on the chosen type of initial wealth allocation as well as the selected submarket(s) for hedging. Those prices allow to measure the difference of features between submarkets. We provide dual relationships for the super-replication prices, as well as a comparison between them.

In the special case of two submarkets each with a single risky asset, we also calculate the price of a Basis swap. We find that our measure of the difference of features is indeed relevant: the initial investment is made in the submarket where this measure is equal to zero.

Then, we provide several economic illustrations. First, we show that in the case where the numéraire spreads are constant, a common local martingale measure exists for the whole market. Then, we show with several examples and counter-examples, that there is no implication between completeness of each submarket taken separately and completeness of the whole market. We also propose a Brownian illustration, where under the assumption of a time dependent Vasicek model for the numéraires spread, we provide a characterization of the sets $Q^*$ of (local) martingale measures. In the special case mentioned above, we fully compute the different superreplication prices and give conditions on the parameters to obtain one of the three possible config-
urations: market 1 is illiquid and market 2 is liquid, market 1 is liquid and market 2 is illiquid or finally both markets are liquid.

A Appendix

We now present the proofs of Theorems 1 and 2 and of Propositions 1 and 3.

A.1 Proof of Theorem 1

First, we prove that

$$N_{FL} \iff \exists X^* \in L^1_{\geq 0}, \mathbb{E}(X^*V) = 0, \forall V \in K. \quad (41)$$

On the top of that, we show that

$$X^* = \{ X^* \in L^1_{\geq 0} | \mathbb{E}(X^*V) = 0, \forall V \in K \}. \quad (42)$$

The proof of (41) is very similar to the proof of the Kreps-Yan theorem (see [16], [20]) given by [17] and [6] and is given for the convenience of the reader. We first prove $\Leftarrow$ in (41). For every $W \in C$, $\mathbb{E}(X^*W) \leq 0$. Let $W \in C \cap L^\infty_+$. As $W \rightarrow \mathbb{E}(X^*W)$ is a weak-star continuous functional, $\mathbb{E}(X^*W) \leq 0$ follows. Thus, $W = 0$ $\mathbb{P}$-a.s. since $\mathbb{P}(X^* > 0) = 1$.

We prove $\Rightarrow$ in (41) in two steps. First, we prove that for any fixed $Y \in L^\infty_+$, $Y \neq 0$, there exists some $X \in L^1$ such that $\mathbb{E}(XY) > 0$ and $\mathbb{E}(XW) \leq 0$ for all $W \in C \cap L^\infty$. As $\{Y\} \cap C \cap L^\infty = \emptyset$, we can use the Hahn-Banach theorem (see Theorem II.9.2 in [18]) and find some $X \in L^1$ and $a < b$ such that $\mathbb{E}(XY) \geq b$ and $\mathbb{E}(XW) \leq a$ for all $W \in C \cap L^\infty$. As $0 \in \bar{C}$ we have that $b > a \geq 0$ and thus $\mathbb{E}(XY) > 0$. As $C \cap L^\infty$ is a cone, $\mathbb{E}(XW) \leq 0$ for all $W \in C \cap L^\infty$. Finally, as $-L^\infty_+ \subset C \cap L^\infty$, $\mathbb{E}(XW) \geq 0$ for all $W \in L^\infty_+$, and thus $X \in L^1_+$. The second step is the so-called exhaustion argument. Let

$$D = \{ X \in L^1_+ | \mathbb{E}(XW) \leq 0, \forall W \in C \}.$$

Then, $D$ is non-empty ($0 \in D$), and there exists some $X^* \in D$ such that

$$\mathbb{P}(X^* > 0) = \sup_{X \in D} \mathbb{P}(X > 0). \quad (43)$$

Indeed, let $(X_n)_{n \geq 1}$ be a maximizing sequence for (43) and $X^* = \sum_{n \geq 1} \frac{1}{n + 1} \mathbb{1}_{\{X_n > 0\}} X_n$. Then, $X^* \in D$ and (43) holds true.

Note that, $\mathbb{P}(X^* > 0) = 1$. Else, $\mathbb{P}(X^* = 0) > 0$, and applying step 1 with $Y = \mathbb{1}_{\{X^* = 0\}}$ we obtain some $X_0 \in D$ such that $\mathbb{E}(YX_0) > 0$. This last inequality implies that $\mathbb{P}(\{X^* = 0\} \cap \{X_0 = 0\}) > 0$, and thus $\mathbb{P}(X^* + X_0 > 0) > \mathbb{P}(X_0 > 0)$, which contradicts (43).

From the first two steps, we have found some $X^* \in L^1_+$ such that $\mathbb{E}(X^*W) \leq 0$ for all $W \in C$ and as for all $V \in K$, as $\pm V \in C$, $\mathbb{E}(X^*V) = 0$, which concludes the proof of (41).

Now we show $\Rightarrow$ in (42). Let $X^* \in X^*$ and $V = \sum_{r \in T} \sum_{j=1}^{n_r} S^r_n \varphi_j (\bar{S}^r_{\delta_j} - \bar{S}^r_{\delta_{j-1}}) \in K$.

$$\mathbb{E}(X^*V) = \sum_{r \in T} \sum_{j=1}^{n_r} \mathbb{E} \left( X^* S^r_n \varphi_j (\bar{S}^r_{\delta_j} - \bar{S}^r_{\delta_{j-1}}) \right) = 0$$

using (7). For the other inclusion, fix some $\tau \in T$, some stopping times $0 \leq \beta_1 \leq \beta_2 \leq T$ and some $\mathbb{R}$-valued, $\mathcal{F}_{\beta_1}$-measurable random variable $\varphi$. Then, $\mp S^r_n \varphi (\bar{S}^r_{\beta_2} - \bar{S}^r_{\beta_1}) \in K$ and we obtain that (7) holds true. $\square$
A.2 Proof of Proposition 1

Lemma 1 Assume that Assumption 1 holds true and that $X^* \neq \emptyset$. If $X^* \in \mathcal{X}$, then for all $Z \in L_{\infty 0}^\infty$, $Q_Z \in Q^Z$ where $Q_Z$ is defined by

$$\frac{dQ_Z}{dP} = \frac{X^*Z}{E(X^*Z)}, \quad (44)$$

Proof Let $X^* \in \mathcal{X}$ and fix some $Z \in L_{\infty 0}^\infty$. Let $Q_Z$ be defined by (44). As $\mathbb{P}(X^* > 0) = 1$ and $\mathbb{P}(Z > 0) = 1$, we obtain that $Q_Z \sim \mathbb{P}$ and $(dQ_Z/dP) / Z \in L_1$, as $X^* \in L_1$. Fix some $\tau \in T$. Then, for all stopping times $\beta_1 \leq \beta_2 \leq T$ and all $\mathbb{R}^\tau$-valued, $F_{\beta_1}$-measurable random variables $\varphi^\tau$, we obtain from (7) that

$$0 = E(X^* S_{\beta_2}^\tau - S_{\beta_1}^\tau) = E(X^*Z)E_{Q_Z} \left( \frac{S_{\beta_2}^\tau - S_{\beta_1}^\tau}{Z} \varphi^\tau \right).$$

Thus, $E_{Q_Z} \left( \frac{S_{\beta_2}^\tau - S_{\beta_1}^\tau}{Z} \varphi^\tau \right) = 0$, which implies that

$$E_{Q_Z} \left( E_{Q_Z} \left( \frac{S_{\beta_2}^\tau - S_{\beta_1}^\tau}{Z} \varphi^\tau \big| F_{\beta_1} \right) \right) = E_{Q_Z} \left( \frac{S_{\beta_2}^\tau - S_{\beta_1}^\tau}{Z} \varphi^\tau \big| F_{\beta_1} \right), \quad (45)$$

and $Q_Z \in Q^Z$. $\square$

Proof of Proposition 1 First, we prove that if $Q^Z \neq \emptyset$ for some $Z \in L_{\infty 0}^\infty$ then

$$E_0 \left( \frac{W}{Z} \right) \leq 0, \forall Q \in Q^Z, \forall W \in C \cap L^\infty. \quad (46)$$

Let $V = \sum_{\tau \in T} S_{\beta_2}^\tau \left( \sum_{j=1}^{n_\tau} \varphi_j^\tau \left( S_{\beta_j}^\tau \right) \right) \in K$;

$$E_0 \left( \frac{V}{Z} \right) = \sum_{\tau \in T} \sum_{j=1}^{n_\tau} E_0 \left( \varphi_j^\tau S_{\beta_2}^\tau \big| F_{\beta_1} \right) - E_0 \left( \varphi_j^\tau S_{\beta_1}^\tau \big| F_{\beta_1} \right) = 0.$$

Thus, (46) follows as $W \mapsto E \left( \frac{dQ}{dP} \frac{W}{Z} \right)$ is a weak-star continuous functional (recall that $\frac{dQ}{dP} / Z \in L_1$).

1. Assume that the NFL condition holds true. Theorem 1 implies that there exists some $X^* \in \mathcal{X}$ and Lemma 1 implies that $Q^Z$ is not empty for all $Z \in L_{\infty 0}^\infty$.

Now, assume that there exists some $Z \in L_{\infty 0}^\infty$ such that $Q^Z \neq \emptyset$. Let $W \in C \cap L^\infty$ and let $Q \in Q^Z$. Then, (46) implies that $\frac{W}{Z} = 0$ Q-a.s. As $Q \sim P$ and $\mathbb{P}(Z > 0) = 1$, $W = 0$ P-a.s., and the NFL condition follows.

2. Assume first that $X^* = \emptyset$. Then, for all $Z \in L_{\infty 0}^\infty$, the set $\{ Q \mid \exists X^* \in \mathcal{X}, \frac{dQ}{dP} = \frac{X^*Z}{E(X^*Z)} \} = \emptyset$.

Moreover, Theorem 1 and the contraposition of the second part of 1. imply that $Q^Z = \emptyset$ and both sets in (9) are empty. Assume now that for some $Z \in L_{\infty 0}^\infty$, $Q^Z = \emptyset$. Then, the contraposition of the first part of 1. and Theorem 1 imply that $X^* = \emptyset$ and again both sets in (9) are empty. We now assume that both sets are non-empty.

We have proved the reverse inclusion in Lemma 1. Now, let $Q \in Q^Z$. Then, for all stopping times $\beta_1 \leq \beta_2 \leq T$, Definition 6 shows that (45) is satisfied for all $\tau \in T$ with $Q$ instead of $Q_Z$. This implies that for all $\mathbb{R}^\tau$-valued, $F_{\beta_1}$-measurable random variables $\varphi^\tau$

$$0 = E_0 \left( \frac{S_{\beta_2}^\tau - S_{\beta_1}^\tau}{Z} \varphi^\tau \right) = \frac{1}{S_{\beta_2}^\tau} E_0 \left( \frac{dQ}{dP} \frac{S_{\beta_2}^\tau}{Z} \varphi^\tau \left( S_{\beta_2}^\tau - S_{\beta_1}^\tau \right) \right).$$

Then, $X^* = \frac{dQ}{dP} \frac{1}{Z} \in X^*$ and $\frac{dQ}{dP} = X^*Z / E(X^*Z)$.
A.3 Proof of Theorem 2

Let \((\lambda_\tau)_{\tau \in \mathcal{T}} \in A^T\) be such that \[ H = \frac{\sum_{\tau \in \mathcal{T}} \lambda_\tau \pi^0_{\tau}}{\sum_{\tau \in \mathcal{T}} \lambda_\tau} \in L^\infty.\] Note that this implies that \(H \in L^\infty\).

The infima are attained in \(\pi(H), \pi^*(H),\) and \(\hat{\pi}^*(H).\) Let

\[ F = \left\{ x \in \mathbb{R} \mid \exists (x^\tau)_{\tau \in \mathcal{T}} \subset (0, \infty) \text{ s.t. } x = \sum_{\tau \in \mathcal{T}} x^\tau, \left( H - \sum_{\tau \in \mathcal{T}} x^\tau \hat{S}^0_{\tau} \right) \in \hat{C} \cap L^\infty \right\}. \]

It is clear that \(F\) is nonempty and that \(\pi(H) = \inf F.\) We show now that the set \(F\) is closed.

Let \((\hat{y}_n)_{n \geq 1} \subset F\) be such that \(\hat{y}_n\) goes to \(\hat{y}.\) For all \(n \geq 1,\) there exist \((\hat{y}_n^\tau)_{\tau \in \mathcal{T}}\) with \(\hat{y}_n^\tau \geq 0,\)

\[ y_n = \sum_{\tau \in \mathcal{T}} \hat{y}_n^\tau \] and

\[ Y_n = \left( H - \sum_{\tau \in \mathcal{T}} \hat{y}_n^\tau \hat{S}^0_{\tau} \right) \in \hat{C} \cap L^\infty. \]

Let \(\varepsilon > 0.\) For \(n\) large enough, \(0 \leq \hat{y}_n^\tau \leq \hat{y}_n \leq \hat{y} + \varepsilon\) for all \(\tau \in \mathcal{T},\) and we can extract a sub-sequence of that we still denote by \((\hat{y}_n^\tau)_{\tau \in \mathcal{T}}\) that converges to some \((\hat{y}^\tau)_{\tau \in \mathcal{T}}\) such that \(\hat{y}^\tau \geq 0\) for all \(\tau \in \mathcal{T}\) and \(\hat{y} = \sum_{\tau \in \mathcal{T}} \hat{y}^\tau.\) It is clear that \((Y_n)_{n \geq 1}\) converges a.s. and also weak-star to \(\hat{Y} = \left(H - \sum_{\tau \in \mathcal{T}} \hat{y}^\tau \hat{S}^0_{\tau}\right).\) Indeed, let \(A \in L^1,\)

\[ \left| \mathbb{E} \left( A \left( Y_n - \hat{Y} \right) \right) \right| \leq \sum_{\tau \in \mathcal{T}} \mathbb{E} \left( |A| \hat{S}^0_{\tau} |\hat{y}_n - \hat{y}^\tau| \right) \leq \varepsilon \sum_{\tau \in \mathcal{T}} \mathbb{E} \left( |A| \hat{S}^0_{\tau} \right). \]

Thus, \(\hat{Y} \in \hat{C} \cap L^\infty\) and \(\hat{y} \in F.\) As \(\pi(H) = \inf F, \pi(H)\) belongs to \(F.\) From now, we denote \(\pi(H) = \sum_{\tau \in \mathcal{T}} \hat{z}^\tau \in F.\)

Using the same kind of arguments, we obtain that the infima are attained in (13) and (14).

Proof of (47)

For all \(x = \sum_{\tau \in \mathcal{T}} x^\tau \in F\) and all \(Z \in L_{2,0}^\infty,\)

\[ \sup_{Q \in \mathcal{Q}^Z} \mathbb{E}_Q \left( \frac{H}{Z} - \sum_{\tau \in \mathcal{T}} x^\tau \frac{\hat{S}^0_{\tau}}{Z} \right) \leq 0. \] \hspace{1cm} (47)

Indeed, there exist \(W \in \hat{C} \cap L^\infty \sum_{\tau \in \mathcal{T}} x^\tau \hat{S}^0_{\tau} + W \geq H\) a.s. So, (46) implies that for all \(Q \in \mathcal{Q}^Z\) (which is non-empty, see Proposition 1),

\[ \sum_{\tau \in \mathcal{T}} x^\tau \mathbb{E}_Q \left( \frac{\hat{S}^0_{\tau}}{Z} \right) \geq \mathbb{E}_Q \left( \frac{H}{Z} \right). \]

Proof of (16) Set for all \(\tau \in \mathcal{T}, \hat{z}^\tau = \hat{z}^\tau - \varepsilon \frac{\lambda_\tau}{\sum_{\tau \in \mathcal{T}} \lambda_\tau}.\) Note that one may choose \(\varepsilon\) such that \(\hat{z}^\tau \geq 0.\) Then, \(\sum_{\tau \in \mathcal{T}} \hat{z}^\tau = \pi(H) - \varepsilon,\) and thus

\[ \left\{ H - \sum_{\tau \in \mathcal{T}} \hat{z}^\tau \hat{S}^0_{\tau} \right\} \cap \hat{C} \cap L^\infty = \emptyset. \]

As in the proof of Theorem 1 using the Hahn-Banach argument, we obtain the existence of \(X \in X^*\) such that

\[ \mathbb{E} \left( \hat{X} H \right) > \mathbb{E} \left( \hat{X} \sum_{\tau \in \mathcal{T}} \hat{z}^\tau \hat{S}^0_{\tau} \right). \] \hspace{1cm} (48)

Let \(d\hat{Q}/d\hat{P} = \left( \hat{X} \sum_{\tau \in \mathcal{T}} \lambda_\tau \hat{S}^0_{\tau} \right) / \mathbb{E} \left( \hat{X} \sum_{\tau \in \mathcal{T}} \lambda_\tau \hat{S}^0_{\tau} \right).\) Proposition 1 implies that \(\hat{Q} \in \mathcal{Q}_{\mathcal{E}, \lambda}\) and

\[ \mathbb{E}_\hat{Q} \left( \frac{H}{\sum_{\tau \in \mathcal{T}} \lambda_\tau \hat{S}^0_{\tau}} \right) > \mathbb{E}_\hat{Q} \left( \frac{\sum_{\tau \in \mathcal{T}} \hat{z}^\tau \hat{S}^0_{\tau}}{\sum_{\tau \in \mathcal{T}} \lambda_\tau \hat{S}^0_{\tau}} \right) = \mathbb{E}_\hat{Q} \left( \frac{\sum_{\tau \in \mathcal{T}} \hat{z}^\tau \hat{S}^0_{\tau}}{\sum_{\tau \in \mathcal{T}} \lambda_\tau \hat{S}^0_{\tau}} \right) - \frac{\varepsilon}{\sum_{\tau \in \mathcal{T}} \lambda_\tau}. \]
This implies that
\[
\sup_{\mathcal{Q} \in \mathcal{Q}_m} \mathbb{E}_Q \left( \frac{H}{\sum_{\tau \in T} \lambda^{	au} S^\tau_{\tau}} - \frac{\sum_{\tau \in T} X^\tau S^\tau_{\tau}}{\sum_{\tau \in T} \lambda^\tau S^\tau_{\tau}} \right) \geq \mathbb{E}_Q \left( \frac{H}{\sum_{\tau \in T} \lambda^\tau S^\tau_{\tau}} - \frac{\sum_{\tau \in T} X^\tau S^\tau_{\tau}}{\sum_{\tau \in T} \lambda^\tau S^\tau_{\tau}} \right) > 0.
\]

Therefore, letting $\varepsilon$ go to zero and using (47) for $Z = \sum_{\tau \in T} \lambda^\tau S^\tau_{\tau}$, we obtain that (16) holds true.

**Proof of (17)** We assume now that $\frac{H}{\min_{\tau \in T} S^\tau_{\tau}} \in L^\infty$, which implies that $\frac{H}{S^\tau_{\tau}} \in L^\infty$ for all $\tau \in T$ and also that $\frac{H}{\sum_{\tau \in T} \lambda^\tau S^\tau_{\tau}} \in L^\infty$ for all $(\lambda^\tau)_{\tau \in T} \in \Lambda^T$. Using (47) for $Z = \max_{\tau \in T} X^\tau$, we obtain that if $x = \sum_{\tau \in T} x^\tau \in F$,
\[
\sup_{\mathcal{Q} \in \mathcal{Q}_m} \mathbb{E}_Q \left( \frac{H}{\max_{\tau \in T} X^\tau} - \sum_{\tau \in T} x^\tau \right) \leq 0,
\]
and thus
\[
\sum_{\tau \in T} x^\tau \geq \sup_{\mathcal{Q} \in \mathcal{Q}_m} \mathbb{E}_Q \left( \frac{H}{\max_{\tau \in T} X^\tau} \right)
\]
and taking the infimum on all $x = \sum_{\tau \in T} x^\tau \in F$ on the left-hand side, we obtain that
\[
\pi(H) \geq \sup_{\mathcal{Q} \in \mathcal{Q}_m} \mathbb{E}_Q \left( \frac{H}{\max_{\tau \in T} X^\tau} \right).
\]

Now, we use (48) to obtain that
\[
\mathbb{E} \left( \hat{X} H \right) > \mathbb{E} \left( \hat{X} \sum_{\tau \in T} \hat{z}^\tau S^\tau_{\tau} \right) \geq \mathbb{E} \left( \hat{X} \min_{\tau \in T} S^\tau_{\tau} \sum_{\tau \in T} z^\tau \right).
\]

Let $d\mathcal{Q}_m/d\mathcal{P} = \left( \frac{\hat{X} \min_{\tau \in T} S^\tau_{\tau}}{\hat{X} \min_{\tau \in T} S^\tau_{\tau}} \right) / \mathbb{E} \left( \hat{X} \min_{\tau \in T} S^\tau_{\tau} \right)$. Then, from Proposition 1, $\mathcal{Q}_m \in \mathcal{Q}_m$ and
\[
\sup_{\mathcal{Q} \in \mathcal{Q}_m} \mathbb{E}_Q \left( \frac{H}{\min_{\tau \in T} S^\tau_{\tau}} \right) \geq \mathbb{E}_{\mathcal{Q}_m} \left( \frac{H}{\min_{\tau \in T} S^\tau_{\tau}} \right) \geq \sum_{\tau \in T} z^\tau.
\]

Thus, letting $\varepsilon$ go to 0, we obtain that
\[
\sup_{\mathcal{Q} \in \mathcal{Q}_m} \mathbb{E}_Q \left( \frac{H}{\min_{\tau \in T} S^\tau_{\tau}} \right) \geq \pi(H)
\]
and this achieves the proof of (17) (note that here we do not need the infimum in $\pi(H)$ to be attained; it is enough to choose $\hat{z}^\tau$ such that $\sum_{\tau \in T} \hat{z}^\tau < \pi(H)$ and to let $\sum_{\tau \in T} \hat{z}^\tau$ go to $\pi(H)$).

**Proof of (20)** Let $x \geq 0$ such that there exists $W \in \mathcal{C} \cap L^\infty$, satisfying $x \hat{S}^\tau_{\tau} + W \geq H$ a.s. Using (47), $\sup_{\mathcal{Q} \in \mathcal{Q}_m} \mathbb{E}_Q \left( \frac{H}{\hat{S}^\tau_{\tau}} \right) \leq x$ and the first inequality in (20) is proved. For the reverse one, let $\varepsilon > 0$. As $(H - (\pi(H) - \varepsilon) \hat{S}^\tau_{\tau}) \cap \mathcal{C} \cap L^\infty = \emptyset$, the proof of $\Rightarrow$ in (41)
shows that there exists some $\hat{\mathcal{X}} \in \mathcal{X}^*$ such that $\mathbb{E} \left( \hat{\mathcal{X}} \left( H - (\pi^*(H) - \varepsilon)\mathcal{S}_{T}^{\varepsilon,0} \right) \right) > 0$. Let $d\hat{Q}/d\mathbb{P} = \left( \hat{\mathcal{X}}\mathcal{S}_{T}^{\varepsilon,0} \right) / \mathbb{E} \left( \hat{\mathcal{X}}\mathcal{S}_{T}^{\varepsilon,0} \right)$. Proposition 1 implies that $\hat{Q} \in \mathcal{Q}^r$ and
\[
\sup_{\varepsilon \in \mathcal{Q}^r} \mathbb{E}_{\hat{Q}} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right) \geq \mathbb{E}_{\hat{Q}} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right) > \pi^*(H) - \varepsilon
\]
and the proof is complete letting $\varepsilon$ go to zero.

**Proof of (18)** Let $x \geq 0$ such that there exist $\tau \in \mathcal{T}, W \in \mathcal{C}^r \cap L^\infty$ satisfying $x\mathcal{S}_{T}^{\varepsilon,0} + W \geq H$ a.s. As the NFL condition implies the NFL in the submarket $\tau$, Remark 1 implies that $\hat{\mathcal{Q}}^r$ is nonempty and as in (46) that for all $Q \in \hat{\mathcal{Q}}^r$, $x \geq \mathbb{E}_{\hat{Q}} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right)$. As this is true for all $Q \in \hat{\mathcal{Q}}^r$, one obtains that
\[
x \geq \sup_{Q \in \hat{\mathcal{Q}}^r} \mathbb{E}_{Q} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right) \geq \min_{\tau \in \mathcal{T}} \sup_{Q \in \mathcal{Q}^r} \mathbb{E}_{Q} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right).
\]
Thus,
\[
\pi(H) \geq \min_{\tau \in \mathcal{T}} \sup_{Q \in \mathcal{Q}^r} \mathbb{E}_{Q} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right) \quad (49)
\]
Let $z < \pi(H)$; then, for all $\tau \in \mathcal{T}$, $\left\{ H - z\mathcal{S}_{T}^{\varepsilon,0} \right\} \cap \mathcal{C}^r \cap L^\infty = \emptyset$. Following the same arguments as above, one finds $\mathcal{X}^r \in \mathcal{X}^*$ such that
\[
z < \mathbb{E} \left( \mathcal{X}^r H \right) / \mathbb{E} \left( \mathcal{X}^r \mathcal{S}_{T}^{\varepsilon,0} \right) = \mathbb{E}_{\mathcal{Q}^r} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right) \leq \sup_{Q \in \hat{\mathcal{Q}}^r} \mathbb{E}_{Q} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right) \quad \text{where} \quad \frac{d\hat{Q}}{d\mathbb{P}} = \frac{\mathcal{X}^r \mathcal{S}_{T}^{\varepsilon,0}}{\mathbb{E} \left( \mathcal{X}^r \mathcal{S}_{T}^{\varepsilon,0} \right)} \in \hat{\mathcal{Q}}^r
\]
see Remark 1. Letting $z$ go to $\pi(H)$, one obtains that for all $\tau \in \mathcal{T}$,
\[
\pi(H) \leq \sup_{Q \in \hat{\mathcal{Q}}^r} \mathbb{E}_{Q} \left( \frac{H}{\mathcal{S}_{T}^{\varepsilon,0}} \right) \quad (50)
\]
As this is true for all $\tau \in \mathcal{T}$, the other inequality in (49) is proven, and (18) holds true (recall (15)).

**Proof of (19)** It is clear from (12) and (14) that $\pi(H) \geq \max_{\tau \in \mathcal{T}} \hat{\pi}^*(H)$. For the reverse inequality, fix $\varepsilon > 0$. Let $\hat{\tau} \in \mathcal{T}$. There exists $W \in \mathcal{C}^r \cap L^\infty$ such that
\[
\left( \max_{\tau \in \mathcal{T}} \hat{\pi}^*(H) + \varepsilon \right) \mathcal{S}_{T}^{\varepsilon,0} + W \geq \left( \hat{\pi}^*(H) + \varepsilon \right) \mathcal{S}_{T}^{\varepsilon,0} + W \geq H \text{ a.s.}
\]
Therefore, $\max_{\tau \in \mathcal{T}} \hat{\pi}^*(H) + \varepsilon \geq \pi(H)$, and the reverse inequality is proven by letting $\varepsilon$ go to zero. Now, (19) follows from (15).

**Proof of the fact that the infima in the definitions of the different superreplication prices are minima** We have already proved that the infima are attained for $\pi(H)$, $\pi^*(H)$, and $\hat{\pi}^*(H)$. As the set $\mathcal{T}$ is finite and (18) holds true, there exists some $\hat{\tau} \in \mathcal{T}$ such that $\pi(H) = \hat{\pi}(H)$. Now, as the infimum is attained in $\pi(H)$, there exists $W \in \mathcal{C}^r \cap L^\infty$ such that
\[
\pi(H) \mathcal{S}_{T}^{\varepsilon,0} + W \geq H \text{ a.s.}
\]
So, the infimum is attained in (11). For all $\tau \in \mathcal{T}$, the infimum is attained in $\hat{\pi}^*(H)$ and there exists $\mathcal{W}^* \in \mathcal{C}^r \cap L^\infty$ such that
\[
\left( \max_{\tau \in \mathcal{T}} \hat{\pi}^*(H) \right) \mathcal{S}_{T}^{\varepsilon,0} + \mathcal{W}^* \geq \hat{\pi}^*(H) \mathcal{S}_{T}^{\varepsilon,0} + \mathcal{W}^* \geq H \text{ a.s.}
\]
Recalling (19), the infimum is also attained in (12).

**Proof of (21)** Fix \( \tau \in T \). It is clear that \( \pi^\tau(H) \geq \pi(H) \) and as this true for all \( \tau \in T \), the first inequality in (21) holds true. The second one follows as \( \pi^\tau(H) \leq \hat{\pi}^\tau(H) \) for all \( \tau \in T \) and recalling (18). \( \blacksquare \)

### A.4 Proof of Proposition 3

In this section, recall that the local martingale are assumed to be true martingale.

**Lemma 2** Let \( Z \in \mathcal{L}^\infty \) and \( Q \in \mathcal{Q}^Z \). Then, for all \( \tau \in T \),

\[
E_Q \left( \frac{S_{\tau}}{Z} \right) = S_0^\tau E_Q \left( \frac{\tilde{S}_{\tau,0}^\tau}{Z} \right). \tag{51}
\]

Assume that \( \frac{H}{Z} \in \mathcal{L}^\infty \) and \( \frac{\tilde{Z}}{Z} \in \mathcal{L}^\infty \). Then,

\[
\sup_{Q \in \mathcal{Q}^Z} E_Q \left( \frac{H}{Z} \right) = \frac{1}{\inf_{Q \in \mathcal{Q}^H} E_Q \left( \frac{\tilde{Z}}{Z} \right)}. \tag{52}
\]

**Proof** As \( E_Q \left( \frac{S_{\tau}}{Z} \right) = S_0^\tau E_Q \left( \frac{\tilde{S}_{\tau,0}^\tau}{Z} \right) \), (51) follows from Definition 6 (with true martingale). Then, if \( \frac{H}{Z} \in \mathcal{L}^\infty \), (9) implies that

\[
\sup_{X^* \in \mathcal{X}^*} E_{(X^*H)} = \sup_{Q \in \mathcal{Q}^Z} E_Q \left( \frac{H}{Z} \right). \tag{53}
\]

Note that the same equality holds true by changing suprema by infima or by changing respectively \( \mathcal{Q}^Z \) by \( \mathcal{Q}^{\tau,X} \) and \( X^* \) by \( X^* \). Assume furthermore that \( \frac{\tilde{Z}}{Z} \in \mathcal{L}^\infty \), using (53), we get that

\[
\sup_{Q \in \mathcal{Q}^Z} E_Q \left( \frac{H}{Z} \right) = \sup_{X^* \in \mathcal{X}^*} E_{(X^*H)} = \frac{1}{\inf_{X^* \in \mathcal{X}^*} E_{(X^*Z)}} = \frac{1}{\inf_{Q \in \mathcal{Q}^H} E_Q \left( \frac{\tilde{Z}}{Z} \right)}.
\]

\( \blacksquare \)

**Proof of Proposition 3** Note that (24) holds true since recalling (20) and (23)

\[
\pi^{\tau_1}(S_{\tau_2}^T) = \sup_{Q \in \mathcal{Q}^{\tau_1}} E_Q \left( \frac{S_{\tau_1}^T}{\tilde{S}_{\tau_1,0}^T} \right) = S_0^{\tau_1} = \hat{\pi}^{\tau_1}(S_{\tau_2}^T).
\]

We choose in Theorem 2 \( \lambda_{\tau_2} = 1 \) and \( \lambda_{\tau_1} = 0 \). Then, \( \mathcal{Q}^{lc,\lambda} = \mathcal{Q}^{\tau_2} \).

**Computation of \( \pi(S_{\tau_1}^T) \)** Let \( Q \in \mathcal{Q}^{\tau_2} \). Using (51) we get that

\[
E_Q \left( \frac{S_{\tau_1}^T}{\tilde{S}_{\tau_1,0}^T} \right) = S_0^{\tau_1} E_Q \left( \frac{\tilde{S}_{\tau_1,0}^T}{\tilde{S}_{\tau_1,0}^T} \right). \tag{54}
\]

So, (25) follows from (52) and (54). Then, we compute in (16)

\[
E_Q \left( \frac{S_{\tau_1}^T}{\tilde{S}_{\tau_1,0}^T} \right) - \sum_{\tau \in T} \tilde{\pi}^T E_Q \left( \frac{S_{\tau}^T,0}{\tilde{S}_{\tau,0}^T} \right) = (S_0^{\tau_1} - \tilde{\pi}^{\tau_1}) E_Q \left( \frac{\tilde{S}_{\tau_1,0}^T}{\tilde{S}_{\tau_1,0}^T} \right) - \tilde{\pi}^{\tau_2} . \tag{55}
\]
Thus, we get that
\[ \hat{x}^* = \sup_{Q \in Q^2} \left( S^{T_1}_0 - \hat{x}^* \right) \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right). \]

If \( S^{T_1}_0 - \hat{x}^* < 0, \) \( \hat{x}^* < 0. \) Thus, assume that \( 0 \leq \hat{x}^* \leq S^{T_1}_0. \) Then,
\[ \hat{x}^* = (S^{T_1}_0 - \hat{x}^*) \sup_{Q \in Q^2} \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) \]
and we conclude using (24) and (25).

\[ \pi(S^{T_1}_T) = \sup_{Q \in Q^2} \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) \]
This is equivalent to \( \pi(S^{T_2}_T) = \pi^2(S^{T_2}_T). \)

Assume that \( \pi(S^{T_1}_T) = \pi^2(S^{T_2}_T). \) This is equivalent to \( \sup_{Q \in Q^2} \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) \leq 1. \) If \( \pi(S^{T_2}_T) < \pi^2(S^{T_2}_T), \) i.e.,
\[ 1 > \sup_{Q \in Q^1} \mathbb{E}_Q \left( \frac{S^{T_2}_0}{S^{T_1}_0} \right) = 1, \]
see (52) and we get a contraction. Thus, \( \pi(S^{T_1}_T) \geq \pi^2(S^{T_2}_T) \) and \( \pi(S^{T_2}_T) = \pi^2(S^{T_2}_T). \)

Then,
\[ \hat{x}^* = \pi^2(S^{T_2}_T) - \pi(S^{T_1}_T) = \pi^2(S^{T_2}_T) - \pi^2(S^{T_1}_T) \wedge \pi^2(S^{T_1}_T). \]

**Computation of \( \pi(S^{T_1}_T - S^{T_2}_T) \)**
Let \( Q \in Q^2, \) using (54), we compute in (16)
\[ \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) - \sum_{\tau \in T} \hat{x}^* \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) = (S^{T_1}_0 - \hat{x}^*) \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) - S^{T_1}_0 - \hat{x}^* S^{T_2}_0. \]

Thus, we get that
\[ \hat{x}^* = \sup_{Q \in Q^2} \left( S^{T_1}_0 - \hat{x}^* \right) \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) - S^{T_1}_0. \]

If \( S^{T_1}_0 - \hat{x}^* < 0, \) then \( \hat{x}^* < 0. \) Thus, assume that \( 0 \leq \hat{x}^* \leq S^{T_1}_0. \) Then,
\[ \hat{x}^* = (S^{T_1}_0 - \hat{x}^*) \sup_{Q \in Q^2} \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) - S^{T_1}_0. \]
As \( \hat{x}^* \geq 0, \) we need to impose that (recall (25))
\[ 0 \leq \hat{x}^* \leq S^{T_1}_0 - \frac{S^{T_2}_0}{\sup_{Q \in Q^2} \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right)} = \frac{S^{T_1}_0}{\pi^2(S^{T_2}_T)} (\pi^2(S^{T_2}_T) - S^{T_2}_0). \]

For that we impose that \( \pi^2(S^{T_2}_T) \geq S^{T_2}_0. \) Else, \( \pi(S^{T_1}_T - S^{T_2}_T) = +\infty. \) Then, we get that
\[ \pi(S^{T_1}_T - S^{T_2}_T) = \inf \left\{ \pi^2(S^{T_1}_T - S^{T_2}_T) - \frac{\hat{x}^*}{S^{T_1}_0} \left( S^{T_1}_0 - \pi^2(S^{T_1}_T) \right) \mid 0 \leq \hat{x}^* \leq \frac{S^{T_1}_0}{\pi^2(S^{T_2}_T)} (\pi^2(S^{T_2}_T) - S^{T_2}_0) \right\}, \]

**Case \( \sup_{Q \in Q^2} \mathbb{E}_Q \left( \frac{S^{T_1}_0 - \hat{x}^*}{S^{T_2}_0} \right) < 1. \)**
This is equivalent to \( \pi^2(S^{T_2}_T) < S^{T_1}_0. \) Then, \( \pi(S^{T_1}_T) = \pi^2(S^{T_1}_T) \) and also \( \pi(S^{T_2}_T) = S^{T_1}_0. \)
This is the case where the asset \( \tau_2 \) is liquid and the asset \( \tau_1 \) is illiquid. In this case, \( \hat{x}^1 = 0 \), all the initial investments are made in the liquid market \( \tau_2 \) and
\[
\pi(S^{\tau_1}_T - S^{\tau_2}_T) = \pi^{\tau_2}(S^{\tau_1}_T) - \pi(S^{\tau_2}_T) = \pi(S^{\tau_1}_T) - \pi(S^{\tau_2}_T).
\]
Note that in this case we need to have that \( S^{\tau_2}_0 \leq S^{\tau_1}_0 \).

Case \( \sup_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S^{\tau_1}_T}{S^{\tau_2}_T} \right) \geq 1. \)
This is equivalent to \( \pi^2(S^{\tau_1}_T) \geq S^{\tau_1}_0 \). Then, \( \pi(S^{\tau_1}_T) = S^{\tau_1}_0 \), the asset \( \tau_1 \) is liquid. Moreover, \( \hat{x}^1 = \frac{s^T_0}{\pi(S^{\tau_1}_T)} (\pi^{\tau_2}(S^{\tau_1}_T) - S^{\tau_2}_T) \) and \( \hat{x}^2 = \left( 1 - \frac{s^T_0}{S^0_0} \right) \pi^{\tau_2}(S^{\tau_1}_T) - S^0_0 = 0 \). Thus,
\[
\pi(S^{\tau_1}_T - S^{\tau_2}_T) = \hat{x}^1 = \pi(S^{\tau_1}_T) - \frac{S^0_T}{\sup_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S^{\tau_1}_T}{S^{\tau_2}_T} \right)}.
\]

Case \( \inf_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S^{\tau_1}_T}{S^{\tau_2}_T} \right) \leq 1. \) Then, \( \pi(S^{\tau_2}_T) = S^{\tau_2}_0 \), the asset \( \tau_2 \) is also liquid.
\[
\pi(S^{\tau_1}_T - S^{\tau_2}_T) = \pi(S^{\tau_1}_T) - \pi(S^{\tau_2}_T) \frac{1}{\sup_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S^{\tau_1}_T}{S^{\tau_2}_T} \right)}.
\]

Case \( \inf_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S^{\tau_1}_T}{S^{\tau_2}_T} \right) > 1. \) Then, \( \pi(S^{\tau_2}_T) = \pi^1(S^{\tau_2}_T) \), the asset \( \tau_2 \) is illiquid, the initial investments are made in the market \( \tau_1 \) and we get that
\[
\pi(S^{\tau_1}_T - S^{\tau_2}_T) = \inf_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S^{\tau_1}_T}{S^{\tau_2}_T} \right) \frac{\inf_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S^{\tau_1}_T}{S^{\tau_2}_T} \right)}{\sup_{Q \in \mathcal{Q}_2} \mathbb{E}_Q \left( \frac{S^{\tau_1}_T}{S^{\tau_2}_T} \right)}.
\]

\( \Box \)

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