DOUBLE KODAIRA FIBRATIONS

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Abstract. The existence of a Kodaira fibration, i.e., of a fibration of a compact complex surface $S$ onto a complex curve $B$ which is a differentiable but not a holomorphic bundle, forces the geographical slope $\nu(S) = c_2^1(S)/c_2(S)$ to lie in the interval $(2, 3)$. But up to now all the known examples had slope $\nu(S) \leq 2 + 1/3$. In this paper we consider a special class of surfaces admitting two such Kodaira fibrations, and we can construct many new examples, showing in particular that there are such fibrations attaining the slope $\nu(S) = 2 + 2/3$. We are able to explicitly describe the moduli space of such class of surfaces, and we show the existence of Kodaira fibrations which yield rigid surfaces. We observe an interesting connection between the problem of the slope of Kodaira fibrations and a ‘packing’ problem for automorphisms of algebraic curves of genus $\geq 2$.

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1. Introduction.

It is well known that the Euler characteristic $e$ is multiplicative for fibre bundles and in 1957 Chern, Hirzebruch and Serre ([CHS57]) showed that the same holds true for the signature $\sigma$ if the fundamental group of the base acts trivially on the cohomology of the fibre.

In 1967 Kodaira [Kod67] constructed examples of fibrations of a complex algebraic surface over a curve where multiplicativity of the signature does not hold true, and in his honour such fibrations are nowadays called Kodaira Fibrations.

Definition 1.1 — A Kodaira fibration is a fibration $\psi : S \to B$ of a compact complex surface over a compact complex curve, which is a differentiable but not a holomorphic fibre bundle. We denote by $b$ the genus of the base curve $B$, and by $g$ the genus of the fibre $F$.

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It is well known (see section 2) that, if the fibre genus $g$ is $\leq 2$, and there are no singular fibres, then one has a holomorphic bundle. Likewise the genus $b$ of the base curve of a Kodaira fibration has to be $\geq 2$.

Atiyah and Hirzebruch ([At69], [Hirz69]) presented variants of Kodaira’s construction analysing the relation of the monodromy action to the non multiplicativity of the signature.

Other constructions of Kodaira fibrations have been later given by Gonzalez-Diez and Harvey and others (see [GD-H91], [Zaal95], [B-D] and references therein) in order to obtain fibrations over curves of small genus with fixed signature and fixed fibre genus.

A precise quantitative measure of the non multiplicativity of the signature is given by the geographic slope, i.e., the ratio $\nu := c_1^2(S)/c_2(S) = K_S^2/e(S)$ between the Chern numbers of the surface: for Kodaira fibred surfaces it lies in the interval $(2, 3)$, in view of the well known Arakelov inequality and of the improvement by Kefeng Liu ([Liu96]) of the Bogomolov-Miyaoka-Yau inequality $K_S^2/e(S) \leq 3$.

The basic problem we approach in this paper is: which are the slopes of Kodaira fibrations?

This problem was posed by Claude Le Brun who raised the question whether the slopes can be bounded away from 3: is it true for instance that for a Kodaira fibration the slope is at most $2 + 1/3 = 2,33\ldots$ (see [BHPV], page 221) and if one considers Kodaira fibrations obtained from a general complete intersection curve in the moduli space $M_g$ of curves of genus $g \geq 3$, one obtains a smaller slope (around 2,18).

Our main result in this direction is the following

**Theorem A** — There are Kodaira fibrations with slope equal to $2 + 2/3 = 2,66\ldots$.

Our method of construction is a variant of the one used by Kodaira, and is briefly described as follows: we consider branched coverings $S \rightarrow B_1 \times B_2$ branched on a smooth divisor $D \subset B_1 \times B_2$ such that the respective projections $D \rightarrow B_i$ are étale (unramified) for $i = 1, 2$. We denote these by double étale Kodaira Fibrations. The advantage of these is that we are able to completely describe their moduli spaces.

The starting point is the topological characterization of double Kodaira fibred surfaces (these are the surfaces admitting two different Kodaira fibrations), derived from [Kot99].

**Proposition 2.5** — Let $S$ be a complex surface. The datum of a double Kodaira fibration on $S$ is equivalent to the following data:

(i) Two exact sequences

$$
\begin{align*}
1 & \rightarrow \Pi_g \rightarrow \pi_1(S) \xrightarrow{\bar{\psi}_i} \Pi_{b_i} \rightarrow 1 \quad i = 1, 2\
\end{align*}
$$

where $\Pi_g$ denotes the fundamental group of a compact curve of genus $g$ and where $b_i \geq 2, g_i \geq 3$, such that

(ii) the composition homomorphism

$$
\Pi_g \xrightarrow{\bar{\psi}_1} \pi_1(S) \xrightarrow{\bar{\psi}_2} \Pi_{b_2}
$$

is neither zero nor injective, and

(iii) the Euler characteristic of $S$ satisfies

$$
e(S) = 4(b_1 - 1)(g_1 - 1) = 4(b_2 - 1)(g_2 - 1).
$$

The above characterization plays an important role in the explicit description of the moduli spaces of Kodaira fibrations.
Theorem 6.4 — Double étale Kodaira Fibrations form a closed and open subset in the moduli spaces of surfaces of general type.

Thus the moduli space of double étale Kodaira fibred surfaces $S$ is a union of connected components of the moduli spaces of surfaces of general type: we conjecture these connected components to be irreducible, and we prove this conjecture in the special case of standard double étale Kodaira fibred surfaces, where we have lots of concrete examples.

Let us explain how double étale Kodaira fibrations can be constructed starting from curves with automorphisms, and are indeed related to sets of étale morphisms between two fixed curves.

The simple reason for this is that each component of $D$ is an étale covering of each $B_i$, and thus we can take an étale cover $\tilde{B}_1 \to B_1$ dominating each of them; then the pullback $S' \to \tilde{B}_1$ of $S \to B_1$ has the property that $D'$ is composed of disjoint graphs of étale maps $\phi_i : \tilde{B}_1 \to B_2$.

The philosophy, as the reader may guess, is then: the larger the cardinality of $S = \{\phi_i\}$ compared to the genus of $B_2$, the bigger the slope, and conversely, once we find such a set $S$ we get (by the so-called tautological construction, described in section 4) plenty of corresponding double (étale) Kodaira fibrations. If by a further pullback we can achieve $B_1 = B_2 = B$ and $S \subset \text{Aut}(B)$ our question concerning the slope of double Kodaira fibrations is related to the following question.

Question B — Let $B$ be a compact complex curve of genus $b \geq 2$, and let $S \subset \text{Aut}(B)$ be a subset such that all the graphs $\Gamma_s, s \in S$ are disjoint in $B \times B$: which is the best upper bound for $|S|/(b-1)$?

We achieve $|S|/(b-1) = 3$, and in this way we obtain the slope $8/3$. Conversely, it is interesting to observe that the cited upper bound for the slope implies that $|S|/(b-1) < 8$.

It would be desirable to find examples with $|S|/(b-1) > 3$, for instance examples with $|S|/(b-1) = 4$ would yield a slope equal to 2.75. Even more interesting would be to find sharper upper bounds for the slope of Kodaira fibrations.

The consideration of double étale Kodaira fibrations related to curves with many automorphisms enables us also to prove the following interesting

Corollary 6.6 — There are double Kodaira fibred surfaces $S$ which are rigid.

The moduli space of some special Kodaira fibrations were described by Kas [Kas68] and Jost/Yau [J-Y83]; here, we prove the following general

Theorem 6.5 — The subset of the moduli space corresponding to standard double étale Kodaira fibred surfaces $S$ (those admitting a pullback branched in a union of graphs of automorphisms), is a union of connected components which are irreducible, and indeed isomorphic to the moduli space of pairs $(B,G)$, where $B$ is a curve of genus $b$ at least two and $G$ is a group of biholomorphisms of $B$ of a given topological type.

2. General set-up.

Definition 2.1 — A Kodaira fibration is a smooth fibration $\psi_1 : S \to B_1$ of a surface over a curve, which is not a holomorphic fibre bundle. $S$ is called a double Kodaira fibred surface if it admits a double Kodaira fibration, i.e., a surjective holomorphic map $\psi : S \to B_1 \times B_2$ yielding two Kodaira fibrations. Let $D \subset B_1 \times B_2$ be the branch divisor of $\psi$. If both projections $pr_{B_j}|_D : D \to B_j$ are étale we call $\psi : S \to B_1 \times B_2$ a double étale Kodaira Fibration.
Remark 2.2 — Observe that if $S$ admits a double étale Kodaira fibration, then
$S$ admits two Kodaira fibrations. Conversely, if $S$ admits two Kodaira fibrations
$\psi_i : S \rightarrow B_i, i = 1, 2$ then we consider the product morphism
$\psi_1 \times \psi_2 : S \rightarrow B_1 \times B_2$, and its branch locus $D$. A calculation in local coordinates shows that at a point $P$
of the ramification divisor $R$, there are local coordinates $(x, y)$ such that $\psi_1 \times \psi_2$
is locally given by $(x, x + f(x, y))$. For instance, if $f(x, y) = yx^2 - 1/4y^4$, $R$
is singular at $P$ $(y^4 = x^2)$ and $D$ is singular at the image point $(x^4 = z^3$ in suitable coordinates).

Note that a surface $S$ could admit more than two different Kodaira fibrations
in such a way that some pair of these (but not all) yield a double étale Kodaira fibration.

Remark 2.3 — A. Kas remarked in \cite{Kas68} that, if $\phi : S \rightarrow B$ is a Kodaira fibration, then
the genus of the base is at least two and the genus $g$ of the fibre is
at least three.

His argument runs as follows: since $\mathbb{P}^1$ is the only curve of genus zero we assume that
g $\geq 1$. The fibration induces a period mapping from the universal cover $\tilde{B}$ of $B$ to the Siegel upper halfspace of genus $g$, a bounded subset of $\mathbb{C}^n$ ($n = \frac{1}{2}g(g + 1)$).

If $\tilde{B}$ is $\mathbb{P}^1$ or $\mathbb{C}$, i.e., $B$ is rational or elliptic, such a map has to be constant by
compactness, resp. by Liouville’s theorem. By Torelli’s theorem all the fibres are
isomorphic, which contradicts our assumption. This settles the question for the base.

Since the $j$-invariant must be constant for a smooth elliptic fibration, the genus
of the fibre is at least two.

Consider now a smooth fibration $\phi : S \rightarrow B$ with fibres of genus two. Then
every fibre is a hyperelliptic curve and we get an induced hyperelliptic involution $\tau$ on $S$. The quotient $S/\tau$ is a $\mathbb{P}^1$-bundle over $B$ and the branch divisor $D$ of the
map $S \rightarrow S/\tau$ intersects every fibre in exactly six points. Hence the restriction of the projection $S/\tau \rightarrow B$ to $D$ is étale. Let $\mu : \pi_1(B) \rightarrow S_0$ be the corresponding
monodromy homomorphism and let $f : B' \rightarrow B$ the étale cover associated to the
kernel of $\mu$. We consider now the pullback of $S/\tau$ by $f$. By construction the monodromy of $f^*D \rightarrow B'$ is trivial hence every component of $f^*D$ comes from a
section $B' \rightarrow f^*(S/\tau)$. This implies that $f^*(S/\tau)$ is isomorphic to the product
$B' \times \mathbb{P}^1$ where $f^*D$ is given by six constant sections. Since every fibre of $\psi$
is a double cover of $\mathbb{P}^1$ branched over the same six points, all the fibres are isomorphic
and $\psi$ is a holomorphic bundle.

Hence the fibres of a Kodaira fibration have genus $g$ at least three.

In the case of a double Kodaira fibration the genus of the fibre is easily seen to be
at least four by Hurwitz’ formula, since a fibre of $\psi_1$ is a branched covering of a
curve of genus at least two, namely the base curve of the other fibration.

In particular, $S$ cannot contain rational or elliptic curves, since no such curve is
contained in a fibre or admits a non-constant map to the base curve. Hence $S$
is minimal and we see, using the superadditivity of Kodaira dimension, that $S$ is an
algebraic surface of general type.

Remark 2.4 — Let $S$ be a surface admitting two different smooth fibrations $\psi_i : S \rightarrow B_i$
where $b_i := \text{genus } (B_i) \geq 2$ and also the fibre genus satisfies $g_i \geq 2$. If
e.g. $\psi_1$ is a holomorphic fibred bundle map, then $S$ has an étale covering which
is isomorphic to a product of curves, $S \rightarrow B_1 \times B_2$ is étale, and also $\psi_2$ is a
holomorphic fibre bundle.

Proof. Let $F$ be a fibre of $\psi_1$. Since the genus of $F$ is at least two, the automorphism
group of $F$ is finite. Hence we can pull back $S$ by an étale map $f : \tilde{B}_i \rightarrow B_i$
to obtain a trivial bundle resulting in the diagram

\[
\begin{array}{ccc}
\tilde{B}_1 \times F & \overset{\phi}{\longrightarrow} & S \\
\downarrow & & \downarrow \psi \\
B_1 & \overset{\psi_1}{\longrightarrow} & B_1 \times B_2
\end{array}
\]

where \( \psi \) is induced by \( \psi_1 \) and \( \psi_2 \). By [Cat00], Rigidity-Lemma 3.8 there exists a map \( g : F \rightarrow B_2 \) such that \( \psi \circ \phi = f \times g \). Take \( x \in F \) and set \( g(x) := y \), \( S_y := \psi_2^{-1}(y) \). In the diagram

\[
\begin{array}{ccc}
\tilde{B}_1 \times \{x\} & \overset{\phi}{\longrightarrow} & S_y \\
\downarrow & & \downarrow \psi|_{S_y} \\
B_1 \times \{y\} & \overset{\psi|_{S_y}}{\longrightarrow} & B_1 \times \{y\}
\end{array}
\]

\( \phi \) and \( f \) are étale and consequently also \( \psi|_{S_y} \) is étale. Varying \( x \) we see that there can be no ramification points and \( \psi \) and \( g \) are étale. Now any fibre of \( \psi_2 \) is an étale covering of \( B_1 \) of fixed degree, corresponding to a fixed subgroup of \( \pi_1(B_1) \). Thus the fibres are all isomorphic and we have a holomorphic bundle.

\[ \square \]

We can now give a topological characterization of double Kodaira fibrations. We denote by \( \Pi_g \) the fundamental group of a complex curve of genus \( g \).

**Proposition 2.5** — Let \( S \) be a complex surface. The datum of a double Kodaira fibration on \( S \) is equivalent to the following data:

(i) Two exact sequences

\[ 1 \rightarrow \Pi_{g_i} \rightarrow \pi_1(S) \overset{\psi_i}{\longrightarrow} \Pi_{b_i} \rightarrow 1 \quad i = 1, 2 \]

with \( b_i \geq 2, g_i \geq 3 \), and such that

(ii) the composition map

\[ \Pi_{g_1} \rightarrow \pi_1(S) \overset{\psi_2}{\longrightarrow} \Pi_{b_2} \]

is neither zero nor injective, and

(iii) the Euler characteristic of \( S \) satisfies

\[ e(S) = 4(b_1 - 1)(g_1 - 1) = 4(b_2 - 1)(g_2 - 1). \]

**Proof.** Note that a holomorphic map \( f : C' \rightarrow C \) between algebraic curves is étale if and only if the induced map \( f_* \) on the fundamental groups is injective. In fact, in this case there is a covering space \( g : D \rightarrow C \) corresponding to the subgroup \( f_*(\pi_1(C')) \) in \( \pi_1(C) \) and by the lifting theorem we have a map \( f : C' \rightarrow D \) which induces an isomorphism of the fundamental groups. Hence \( f \) is of degree one and \( f = g \circ \tilde{f} \) is also étale. We will apply the previous observation to the map in (ii).

Therefore the only if part of our statement follows from remark 2.4.

Let’s consider the other direction. Using theorem 6.3 of [Cat00] conditions (i) and (iii) guarantee the existence of two curves \( B_i \) of genus \( b_i \) and of holomorphic submersions \( \psi_i : S \rightarrow B_i \) with \( \psi_i \circ \phi = \psi \) whose fibres have respective genera \( g_1, g_2 \).

Condition (ii) implies that the two fibrations are different and it remains to see that neither of the \( \psi_i \)'s can be a holomorphic bundle. But if it were so, by remark 2.4, then \( S \rightarrow B_1 \times B_2 \) would be étale and then the map in (ii) would be injective.

\[ \square \]
Remark 2.6 — Double Kodaira fibrations which are not double étale were constructed in [GD-H91] and [Zaal95], essentially with the same method. The map \( F : B \times B \to \text{Jac}(B) \), \((x, y) \mapsto x - y\) contracts the diagonal \( \Delta_B \subset B \times B \) and maps \( B \times B \) to \( Y := B - B \subset \text{Jac}(B) \). One takes \( \Gamma \subset Y \) to be a general very ample divisor, and \( D \subset \Gamma \times B \) as \( D := \bigcup_{x \in \Gamma} F^{-1}(x) \). The projection of \( D \) to \( \Gamma \) is étale of degree 2, while the projection of \( D \) to \( B \) is of degree equal to \( b := \text{genus}(B) \) but is not étale. The pair \( D \subset \Gamma \times B \) yields, as we shall explain in a forthcoming section, a 'logarithmic Kodaira fibration', and from it one can construct, via the tautological construction, an actual Kodaira fibration.

We shall be primarily interested in the case of double étale Kodaira fibrations. Given a holomorphic map \( \phi \) between two curves let us denote by \( \Gamma_\phi \) its graph.

Definition 2.7 — A double étale Kodaira Fibration \( S \to B_1 \times B_2 \) is said to be simple if there exist étale maps \( \phi_1, \ldots, \phi_m \) from \( B_1 \) to \( B_2 \) such that \( D = \bigcup_{k=1, \ldots, m} \Gamma_{\phi_k} \); i.e., if each component of \( D \) is the graph of one of the \( \phi_k \)'s.

We say that \( S \) is very simple if \( B_1 = B_2 \) and all the \( \phi_k \)'s are automorphisms.

Lemma 2.8 — Every double étale Kodaira Fibration admits an étale pullback which is simple.

Proof. Let \( S \to B_1 \times B_2 \) be a double étale Kodaira Fibration. The branch divisor \( D \) is smooth and we can consider the monodromy map \( \mu : \pi_1(B_1, b_1) \to S_m \) of the étale map \( p_1 : D \to B_1 \). Let \( f : B \to B_1 \) the (finite) covering associated to the kernel of \( \mu_1 \). By construction the monodromy of the pullback \( f^*D \to B \) is trivial, hence every component maps to \( B \) with degree 1 and the corresponding pullback \( f^*S \) is a simple Kodaira fibration. 

\[ \blacksquare \]

Remark 2.9 — Kollár remarked that it is not always possible to find a very simple étale cover of \( S \). But up to now we do not have a concrete example of this situation.

This motivates the following

Definition 2.10 — A double étale Kodaira fibration is called standard if there exist étale Galois covers \( B \to B_i, i = 1, 2 \), such that the étale pullback

\( S' := S \times_{(B_1 \times B_2)} (B \times B) \),

induced by \( B \times B \to B_1 \times B_2 \), is very simple.

3. Invariants of double étale Kodaira Fibrations

In this section we want to calculate some invariants of a double étale Kodaira fibration. First we need to fix some notation.

Let \( S \) be a double étale Kodaira fibration as in Definition 2.1. Let \( d \) be the degree of \( \psi : S \to B_1 \times B_2 \), let \( D \subset B_1 \times B_2 \) be the branch locus of \( \psi \) and let \( D_1, \ldots, D_m \) be the connected components of \( D \).

By assumption, the composition map

\( D_i \hookrightarrow B_1 \times B_2 \hookrightarrow B_j \)

is étale and we denote by \( d_{ij} \) its degree. Then the degree of \( \text{pr}_j|_{D} : D \to B_j \) is \( d_j = \sum_{i=1}^m d_{ij} \) and we get two formulas for the Euler characteristic of \( D_i \),

\[ e(D_i) = d_{i1}e(B_1) = d_{i2}e(B_2). \]
The canonical divisor of $B_1 \times B_2$ is $K_{B_1 \times B_2} = -e(B_1)B_2 - e(B_2)B_1$ and we calculate

\[
K_{B_1 \times B_2} \cdot D_i = -e(B_1)B_2 \cdot D_i - e(B_2)B_1 \cdot D_i \\
= -e(B_1)d_{i1} - e(B_2)d_{i2} = -2e(D_i)
\]

so that by adjunction

\[
D_i^2 = \text{deg}(K_{D_i}) - K_{B_1 \times B_2} \cdot D_i \\
= -e(D_i) + 2e(D_i) = e(D_i).
\]

We write

\[
\psi^{-1}(D_i) = \bigcup_{j=1}^{t_i} R_{ij}
\]

as a union of disjoint divisors and denote by $n_{ij}$ the degree of $\psi|_{R_{ij}} : R_{ij} \to D_i$ and by $r_{ij}$ the ramification order of $\psi$ along $R_{ij}$. Then

\[
K_S = \psi^* K_{B_1 \times B_2} + \sum_{i,j} (r_{ij} - 1)R_{ij} \text{ and } d = \sum_{j=1}^{t_i} n_{ij}r_{ij}.
\]

To summarize the situation we label the arrows in the following diagram by the degrees of the corresponding maps:

\[
\begin{array}{ccc}
R_{ij} & \xrightarrow{n_{ij}} & D_i & \xrightarrow{d_{i1}} & B_1 \\
\downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{d} & B_1 \times B_2 & & \\
\end{array}
\]

We can now calculate some invariants.

**Proposition 3.1** — In the above situation we have the following formulas

(i) Setting $\beta_i := \sum_{j=1}^{t_i} n_{ij}(r_{ij} - 1)$

\[
e_2(S) = d e_2(B_1 \times B_2) - \sum_{i=1}^{m} \beta_i e(D_i)
\]

\[
c_2^2(S) = 2e_2(S) - \sum_{i=1}^{m} \sum_{j=1}^{t_i} n_{ij}(r_{ij} - 1)(r_{ij} + 1) r_{ij} e(D_i)
\]

thus the signature is

\[
\sigma(S) = \frac{1}{3}(c_2^2(S) - 2e_2(S)) = -\frac{1}{3} \sum_{i=1}^{m} \sum_{j=1}^{t_i} n_{ij}(r_{ij} - 1)(r_{ij} + 1) r_{ij} e(D_i)
\]

(ii) (a) If $\psi : S \to B_1 \times B_2$ is a Galois covering then $r_{ij} = r_1$ and

\[
\frac{c_2^2(S)}{c_2(S)} = 2 + \frac{-\sum_{i=1}^{m} \frac{r_{ij}^2 - 1}{r_{ij}} e(D_i)}{e(B_1)e(B_2) - \sum_{i=1}^{m} \frac{1}{r_i} e(D_i)}.
\]

(b) If in addition $D$ is composed of graphs of étale maps from $B_1$ to $B_2$, i.e., $S$ is simple, we have

\[
\frac{c_2^2(S)}{c_2(S)} = 2 + \frac{1 - \frac{1}{m} \sum_{i=1}^{m} \frac{1}{r_i}}{2g - 2 + \frac{1}{m} \sum_{i=1}^{m} \frac{1}{r_i}}
\]

where $g$ is the genus of $B_2$. 

Proof. The first formula can be obtained by calculating the genus of a fibre $F$ of $S \to B_1$ using the Riemann-Hurwitz formula and using $c_2(S) = e(S) = e(B_1)e(F)$.

For the second one a rather tedious calculation of intersection numbers is needed so that we prefer to cite [Iz03] which gives us

$$c_2^2(S) = d \cdot e^2(B_1 \times B_2) - \sum_{i=1}^{m} \left( 2b_i e(D_i) + \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij} - 1)(r_{ij} + 1)}{r_{ij}} D_i^2 \right)$$

$$= 2d e(B_1)e(B_2) - \sum_{i=1}^{m} 2b_i e(D_i) - \sum_{i=1}^{m} \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij} - 1)(r_{ij} + 1)}{r_{ij}} e(D_i)$$

$$= 2c_2(S) - \sum_{i=1}^{m} \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij} - 1)(r_{ij} + 1)}{r_{ij}} e(D_i).$$

The formula for the signature is now obvious.

Let’s look at (ii). If the covering $\psi$ is Galois the stabilizers of $R_{ik}$ and $R_{id}$ are conjugate in the covering group and consequently $n_{ij} = n_i$ and $r_{ij} = r_i$ do not depend on $j$. Hence for every $i$

$$d = t_i n_i r_i \Leftrightarrow \frac{t_i n_i}{d} = \frac{1}{r_i} \text{ and } \beta_i = t_i n_i (r_i - 1) = d - t_i n_i$$

Plugging this into the above formulas we get (a):

$$\frac{c_2^2(S)}{c_2(S)} - 2 = -\sum_{i=1}^{m} \frac{t_i n_i (r_i - 1)(r_i + 1)}{r_i} e(D_i)$$

$$= -\sum_{i=1}^{m} \frac{t_i n_i (r_i - 1)(r_i + 1)}{r_i} e(D_i)$$

$$= d (e(B_1 \times B_2) - \frac{1}{d} \sum_{i=1}^{m} (d - t_i n_i) e(D_i))$$

$$= \frac{d (e(B_1)e(B_2) - \sum_{i=1}^{m} \frac{t_i n_i}{r_i} e(D_i))}{e(B_1)e(B_2) - \sum_{i=1}^{m} \frac{t_i n_i}{r_i} e(D_i)}$$

For (b) we further assume the components of $D$ to have all the same genus as $B_2$, i.e. $e(D_i) = e(B_2)$ for all $i$. Then

$$\frac{c_2^2(S)}{c_2(S)} - 2 = -\sum_{i=1}^{m} \frac{r_i^2 - 1}{r_i} e(B_2)$$

$$= \frac{-e(B_1) + m - \sum_{i=1}^{m} \frac{1}{r_i}}{e(B_1) e(B_2) - \sum_{i=1}^{m} \frac{1}{r_i}}$$

$$= \frac{2g - 2}{m} + 1 - \frac{1}{m} \sum_{i=1}^{m} \frac{1}{r_i}$$

The above formulas will allow us to give some upper bounds for the slope.

4. Tautological construction

**Definition 4.1** — A pair $(S, D)$ consisting of

(i) a smooth fibration $\psi : S \to B$ with fibre $F$ and

(ii) a divisor $D \subset S$ such that

(a) the projection $D \to B$ is étale and

\[ \text{Note that we have a slightly different notation.} \]
(b) the fibration of pointed curves \((F,F \setminus D)\) is not isotrivial, is called a log-Kodaira Fibration.

Our typical example of the above situation will be a divisor

\[D \subset B_1 \times B_2 := S\]

such that the first projection \(D \to B_1\) is étale and the second projection \(D \to B_2\)
is nowhere constant.

We shall now see that in order to construct Kodaira Fibrations it suffices to construct log-Kodaira Fibrations.

**Proposition 4.2** — Let \((S,D)\) be a log-Kodaira Fibration.

Then for all surjections \(\rho: \pi_1(F \setminus D) \to G\) there exist an étale covering \(f: B' \to B\) such that \(\rho\) is induced by a surjection \(\pi_1(S' \setminus D') = \pi_1(f^*(S \setminus D)) \to G\).

In geometric terms: for every Galois cover of the fibre \(\hat{F} \to F\), ramified exactly over \(F \cap D\), there is an étale cover \(f: B' \to B\) of the base and a Galois covering \(\hat{S}' \to S' := f^*S\), ramified over \(f^*D\), which extends \(\hat{F} \to F\), yielding the following diagram.

\[
\begin{array}{c}
\hat{F} \\
\downarrow \\
F \\
\downarrow \\
S \\
\downarrow \\
S'
\end{array}
\]

**Proof.** Set for convenience \(\hat{F} := F \setminus D\) and \(\hat{S} := S \setminus D\) and assume from now on that \(D \neq \emptyset\). The fundamental group \(\hat{F} := \pi_1(\hat{F})\) of the punctured fibre is then a free group on \(2g(F) + D \cdot F - 1\) generators. Setting \(\hat{\Gamma} := \pi_1(\hat{S})\) and \(\Pi := \pi_1(B)\), we have the exact sequence

\[1 \to \hat{F} \to \hat{\Gamma} \to \Pi \to 1\]

associated to the fibre bundle \(\hat{S} \to B\).

An étale base change \(f: B' \to B\) corresponds to a finite index subgroup \(\Pi' \hookrightarrow \Pi\) and yields a sequence \(1 \to \hat{F} \to \hat{\Gamma}' \to \Pi' \to 1\), associated to \(f^*\hat{S} \to B'\).

Given instead a finite index subgroup \(\hat{F} \hookrightarrow \hat{F}'\), corresponding to a branched cover \(\hat{F} \to F\) ramified over \(F \cap D\), we seek for an exact sequence

\[1 \to \hat{F} \to \hat{\Gamma} \to \hat{\Pi} \to 1,\]

where \(\hat{\Gamma}\) and \(\hat{\Pi}\) are finite index subgroups in \(\Gamma\) and \(\Pi\) respectively. It is necessary and sufficient that \(\hat{\Gamma}\) be contained in the normalizer of \(\hat{F}\) in \(\Gamma\) and \(F \cap \hat{\Gamma} = \hat{F}\).

The proof follows then from the following

**Lemma 4.3** — If the covering of the fibre is Galois, associated to \(1 \to \hat{F} \to \hat{F}' \to G \to 1\), then there is always a sequence as in [1].

**Proof.** Since \(\hat{F}\) is a normal subgroup, \(\gamma \in \Gamma\) operates on \(\hat{F}\) by conjugation and hence on \(\text{Hom}(\hat{F}, G)\) by \(\phi \mapsto \gamma(\phi) = \phi \circ \text{Int}_{\gamma^{-1}}\). Let \(\Gamma_\rho\) be the stabilizer of \(\rho\) under this action. For \(\gamma \in \Gamma_\rho\) holds \(\rho(\gamma x \gamma^{-1}) = \rho(x)\) and in particular \(\gamma\) normalizes \(\hat{F}\), the kernel of \(\rho\). Let \(\Gamma'\) be the subgroup of \(\Gamma\) generated by \(\hat{F}\) and \(\Gamma_\rho\). We have the sequence

\[1 \to \hat{F} \to \Gamma' \to \Pi' \to 1\]

Note that since \(\hat{F}\) is normal in \(\Gamma\) we can write every element \(\gamma' \in \Gamma'\) as a product \(\gamma' = fg\) where \(f \in \hat{F}\) and \(g \in \Gamma_\rho\).

Consider a tubular neighborhood \(N_0\) around a component \(D_0\) of the pullback of \(D\) associated to \(\Pi' \to \Pi\) and let \(\gamma_0\) be a small loop around \(D_0\) contained in \(N_0 \cap F\). We consider \(\gamma_0\) also as an element of \(\hat{F}\) and regard \(N_0\) as a small neighbourhood of the zero section in the normal bundle \(N_{D_0/S}\). The fundamental group \(\Pi'\) of \(D_0\)
is a group with generators $\alpha_i$, $\beta_i$ and the single relation $\prod_i[\alpha_i, \beta_i] = 1$ and we can write $\Gamma'' = \pi_1(N_0 \setminus D_0)$ as a central extension

$$1 \to \mathbb{Z} \to \Gamma'' \to 1$$

where $\prod_i[\alpha_i, \beta_i] = \gamma_0^k$ and $k = -D_0^2$. This is proven in [Cat06] with the following argument: pick a point $P \in D_0$ and write $D_0 = (D_0 \setminus P) \cup \Delta_P$ where $\Delta_P$ is a small disk around $P$. The $S^1$ bundle (homotopically equivalent to) $N_0 \setminus D_0$ restricted to these two open subsets is trivial and the $C^\infty$-Cocycle of $N_0$ with regard to this trivialisation can be given as $z^k$, where $z$ is a local coordinate in $P$ and $k = -c_1(N_{D_0/S}) = -D_0^2$. The fundamental group of $N_0$ is then calculated using the Seifert-van Kampen theorem.

By a further base change we may assume that $\gamma_0^k$ is in $\mathbb{F}$, e.g. by taking a pullback which makes $k$ divisible by the order of $G$ (the exponent of $G$ indeed suffices). The resulting diagram is

$$
\begin{array}{cccccc}
1 & \to & \mathbb{Z} & \to & \Gamma'' & \to & \Pi'' & \to & 1 \\
1 & \to & \mathbb{F} & \to & \Gamma'' & \to & \Pi'' & \to & 1
\end{array}
$$

Defining $\Gamma'''$ as the inverse image of $\Pi''$ in $\Gamma$ we have

$$
\begin{array}{cccccc}
1 & \to & \mathbb{Z} & \to & \Gamma'' & \to & \Pi'' & \to & 1 \\
1 & \to & \mathbb{F} & \to & \Gamma''' & \to & \Pi'' & \to & 1
\end{array}
$$

and may finally extend $\rho$ to $\Gamma'''$ as follows: After choosing arbitrary images $\rho''(\alpha_i) = \rho''(\beta_i)$ in $G$ and setting $\rho''(\gamma_0) = \rho(\gamma_0)$, $\rho$ extends to $\rho'' : \Gamma'' \to G$ since every such assignment is compatible with the relations in the group and the action of $\rho$ on $\gamma_0$.

We are now in the situation

$$
\begin{array}{ccc}
& \Gamma'' & \\
\mathbb{Z} & \searrow & G \\
& \Gamma''' & \nearrow \rho
\end{array}
$$

and writing each $\gamma \in \Gamma'''$ as a product $\gamma = fg$ with $f \in \mathbb{F}$ and $g \in \Gamma''$ we set

$$
\rho(\gamma) = \rho(f) \cdot \rho''(g).
$$

To see that this is well defined let $f_1g_1 = f_2g_2$. Then $f_2^{-1}f_1 = g_2g_1^{-1}$ is an element of $\mathbb{F}$ and of $\Gamma''$, i.e. a multiple of $\gamma_0$. Hence applying $\rho$ and $\rho''$ respectively, we get $\rho(f_2)^{-1}\rho(f_1) = \rho''(g_2)\rho''(g_1)^{-1}$ since the two homomorphisms act in the same way on $\gamma_0$. This implies $\rho(f_1g_1) = \rho(f_2g_2)$.

It remains to check that this defines a homomorphism. We consider two elements $f_1g_1$, $f_2g_2$ as above. Since $\Gamma'''$ is contained in $\Gamma''$ we can actually assume that the $g_i$’s can be written as a combination of the $\alpha_i$’s, $\beta_i$’s and are contained in $\Gamma''$, hence
provided we have chosen the images of the $\alpha_i$’s, $\beta_i$’s in the centralizer of $G$, which we can do. The desired group $\tilde{\Gamma}$ is the kernel of $\rho : \Gamma'' \to G$.

If $S$ is a double étale Kodaira fibration or a product of curves, one can easily see that also $S'$ is a double étale Kodaira fibration, provided that the restriction of the second projection to $D$ is étale. Moreover the following holds:

**Lemma 4.4** — Assume that we have a curve $B$ of genus at least two and a subset $S = \{\phi_1, \ldots, \phi_m\} \subset \text{Aut } B$ such that the graphs of these automorphisms are disjoint subsets of $B \times B$. If we construct a Kodaira fibration applying the tautological construction to this log-Kodaira fibration, then the resulting surface is in fact a standard Kodaira fibration.

**Proof.** Without loss of generality we may assume that $\phi_1 = \text{id}_B$, i.e., we identify the vertical and the horizontal part of the product $B \times B$ via the automorphism $\phi_1$. We fix a base point $x_0$ in $B$. It suffices to prove the following: for any étale Galois covering $B' \to B$ there exists another étale covering map $f : B'' \to B'$ such that the pullback of $D := \Gamma_{\phi_1} \cup \cdots \cup \Gamma_{\phi_m}$ under the map $f \times f : B'' \times B'' \to B \times B$ is composed of graphs of automorphisms of $B''$.

The fundamental group $\pi_1(B, x_0)$ can be considered as a subgroup of a Fuchsian group which acts on the upper half plane. Let $\Gamma$ be the maximal Fuchsian group which contains $\pi_1(B, x_0)$ as a normal subgroup. Then we have a sequence

$$1 \to \pi_1(B, x_0) \to \Gamma \to \text{Aut}(B) \to 1.$$  

The Galois covering $B' \to B$ corresponds to an inclusion $\pi_1(B', y_0) \subset \pi_1(B, x_0)$ where $y_0$ maps to $x_0$. Consider the Galois covering $B'' \to B'$ associated to the subgroup $\pi(B'', z_0) := \bigcap_{\gamma \in \Gamma} \gamma \pi_1(B', y_0) \gamma^{-1}$ which is the largest normal subgroup of $\Gamma$ contained in $\pi_1(B', y_0)$. It is in fact a finite index subgroup of $\pi_1(B', y_0)$ since $\pi_1(B', y_0)$ is of finite index in $\Gamma$. We have exact sequences

$$1 \to \pi_1(B, x_0) \to \Gamma \to \text{Aut}(B) \to 1$$

$$1 \to \pi_1(B', y_0) \to \Gamma \to G \to 1$$

where $G$ is a group of automorphisms of $B''$. 

$$\text{Gal}(B'' \to B)$$

$$1 \to \pi_1(B, x_0) \to \Gamma \to \text{Aut}(B) \to 1$$

$$1 \to \pi_1(B', y_0) \to \Gamma \to G \to 1$$

$$\text{Gal}(B'' \to B)$$

$$1 \to \pi_1(B, x_0) \to \Gamma \to \text{Aut}(B) \to 1$$

$$1 \to \pi_1(B', y_0) \to \Gamma \to G \to 1$$

$$\text{Gal}(B'' \to B)$$
Let $d$ be the degree of the covering $f : B'' \to B$. Then the degree of the map \( f \times f : B'' \times B'' \to B \times B \) is \( d^2 \) and it suffices to exhibit for any given automorphism \( \phi \) of \( B \) a set of \( d \) automorphisms of \( B'' \) such that their graphs map \( d \) to 1 to \( \Gamma_{\phi} \) under the map \( f \times f \). In order to do so pick \( \psi \in G \) such that \( \alpha(\psi) = \phi \) which means \( f \circ \psi = \phi \circ f \). Then for any \( \sigma \in \text{Gal}(B'' \to B) \) we have

\[
(f \times f)(\Gamma_{\sigma \psi}) = (f \times f)(\{(x, y) \in B'' \times B'' \mid y = \sigma \circ \psi(x)\})
\]

\[
= \{(f(x), f(\sigma \circ \psi(x))) \mid x \in B''\}
\]

\[
= \{(f(x), \phi(f(x))) \mid x \in B''\}
\]

and this map has in fact the same degree as \( f \).

\[\square\]

The reason why the monodromy problems mentioned in [2.9] do not occur in this case is that the horizontal and the vertical curve in the product are in fact identified via \( \alpha_1 \) and therefore, once we fix a basepoint on the curve during the tautological contraction, there is no ambiguity in the choice of the basepoint on the other curve.

5. Slope of double étale Kodaira Fibrations

Kefeng Liu proved in [Lin96] that the slope \( \nu \) of a Kodaira fibration \( S \) satisfies

\[
\nu := \frac{c_1^2(S)}{c_2(S)} < 3
\]

and LeBrun asked whether the better bound \( c_1^2(S) < 2.91 c_2(S) \) would hold.

We will now address the question about what can be said for double étale Kodaira fibrations. Our purpose here is twofold: to find effective estimates from below for the maximal slope via the construction of explicit examples and then to see whether one can prove also an upper bound for the slope of double Kodaira fibrations, using their explicit description.

To separate the numerical considerations from the geometrical problems we pose the following

**Definition 5.1** — Let \( B_1, B_2 \) be curves of genus at least two. An admissible configuration for \( B_1 \times B_2 \) is a tuple \( A = \{D, d, \{(t_i, \{r_{ij}, n_{ij}\})\} \} \) consisting of

- a smooth curve \( D = D_1 \cup \cdots \cup D_m \subset B_1 \times B_2 \) such that each component \( D_i \) maps étale to each of the factors,
- a positive integer \( d \), and positive integers \( t_i \), for all \( i = 1, \ldots, m \),
- for all \( i = 1, \ldots, m \), a \( t_i \)-tuple \( \{(r_{ij}, n_{ij})\}_{j=1}^{t_i} \), of pairs of positive integers with \( r_{ij} \geq 2 \), and such that

\[
d = \sum_{j=1}^{t_i} n_{ij} r_{ij}.
\]

We call the configuration Galois if \( r_{ij} \) does not depend on \( j \), and we then write \( A = \{D, d, \{(t_i, r_{ij}, n_{ij})\}\} \). If moreover \( D \) is made of graphs of étale maps \( \phi_k : B_1 \to B_2 \) (automorphisms if \( B_1 \cong B_2 \) we call \( A \) simple (resp.: very simple). Setting

\[
\beta_i := \sum_{j=1}^{t_i} n_{ij} (r_{ij} - 1)
\]

we define the abstract slope of \( A \) by

\[
\alpha(A) = 2 + \frac{-\sum_{i=1}^{m} \sum_{j=1}^{t_i} n_{ij} (r_{ij} - 1) (r_{ij} + 1) c(D_i)}{d \ e(B_1 \times B_2) - \sum_{i=1}^{m} \beta_i \ c(D_i)}.
\]

We have seen in section 3 that a double étale Kodaira fibration \( S \) gives rise to an admissible configuration \( A(S) \). If \( A \) is any admissible configuration and \( S \) a double étale Kodaira fibration with \( A(S) = A \) we say that \( S \) realizes \( A \).
5.1. Packings of graphs of automorphisms. In this section we let $B$ be a curve of genus $g$ and $G = \text{Aut}(B)$ its automorphism group. We want to study subsets of $G$ such that the corresponding graphs do not intersect. We can translate this into a group-theoretical condition:

**Lemma 5.2** — Let $P_1, \ldots, P_n$ be the points in $B$ which have a non trivial stabilizer $\Sigma_{P_i} \subset G$. Let $\pi_i : G \to G/\Sigma_{P_i}$ be the map that sends $\phi \in G$ to the left coset $\phi \Sigma_{P_i}$.

(i) Two automorphisms $\phi \neq \phi' \in G$ have intersecting graphs if and only if $\pi_i(\phi) = \pi_i(\phi')$ for some $i \in \{1, \ldots, n\}$.

(ii) A subset $S \subset G$ of cardinality $m$ has nonintersecting graphs if and only if for each $i \in \{1, \ldots, n\}$ the image of $S$ under the map

$$\pi_i : G \to G/\Sigma_{P_i}, \quad g \mapsto g \Sigma_{P_i}$$

has cardinality $m$. In particular:

$$m \leq \text{Minimum}\{|G/\Sigma_{P_i}|\}_{i=1,\ldots,n}$$

Note that, if $Q_1, \ldots, Q_r$ are the branch points of the quotient map $B \to B/G$ and $P'_i$ ($\forall i = 1, \ldots, r$) is an arbitrary point in the inverse image of $Q_i$, then the non trivial stabilizers of points are exactly all the subgroups conjugated to the stabilizers $\Sigma_{P'_i}$ ($i = 1, \ldots, r$).

**Proof.** Let $\phi, \phi'$ be two automorphisms of $B$. Their graphs intersect in some point $(P, Q) \in B \times B$ iff $\phi(P) = \phi'(P) = Q$. But this means $\phi^{-1} \circ \phi'(P) = P$, i.e., $\phi^{-1} \circ \phi' \in \Sigma_P$ or equivalently $\phi \Sigma_P = \phi' \Sigma_P$.

\[ \square \]

It is now a natural question to ask for the maximal possible $m$ that one can realize, given a curve $B$, or given a fixed genus $b$ (of $B$).

For the formulation of a partial result we introduce the following notation: we say that $B$ is of type $(\nu_1, \ldots, \nu_k)$ if $B/G$ has genus zero and the map $B \to B/G$ is a ramified covering, branched over $k$ points with respective multiplicities $\nu_i$.

We always order the branch points so that $\nu_1 \leq \cdots \leq \nu_k$.

**Proposition 5.3** —

(i) If the genus $g$ of $B$ is at least two, the maximal cardinality $m$ of a subset with nonintersecting graphs is smaller or equal to $3(g - 1)$ unless the type of $B$ occurs in the following table:

| type    | upper bound for $m$ | $|G|$   |
|---------|---------------------|--------|
| $(2,2,2,3)$ | $4(g - 1)$ | $12(g - 1)$ |
| $(2,2,7)$   | $12(g - 1)$ | $84(g - 1)$ |
| $(2,2,8)$   | $6(g - 1)$  | $48(g - 1)$ |
| $(2,3,9)$   | $4(g - 1)$  | $36(g - 1)$ |
| $(2,4,5)$   | $8(g - 1)$  | $40(g - 1)$ |
| $(2,4,6)$   | $4(g - 1)$  | $24(g - 1)$ |
| $(2,5,5)$   | $4(g - 1)$  | $20(g - 1)$ |
| $(3,3,4)$   | $6(g - 1)$  | $24(g - 1)$ |

(ii) If the genus of $B$ is small we get the following list:
If the genus of the curve is one, we can clearly produce such an arbitrarily large subset by choosing appropriate translations.

Proof. Part (i) is a case by case analysis using the previous Lemma. Let $B$ be a curve of genus $g \geq 2$, let $G$ be its automorphism group and let $h$ be the genus of $B/G$. Let $P_1, \ldots, P_k \in B/G$ be the branch points and $\nu_1 \leq \cdots \leq \nu_k$ be the corresponding ramification indices. Then we have the Hurwitz formula

$$2g - 2 = |G| \left( 2h - 2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{\nu_i} \right) \right)$$

and by the lemma a maximal subset as above has at most cardinality

$$\mu := \frac{|G|}{\nu_k} = \frac{2g - 2}{\nu_k \left( 2h - 2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{\nu_i} \right) \right)},$$

where we set $\nu_1 = 1$ if there is no ramification. Note that the denominator can never be zero since this would imply $g = 1$. We distinguish the following cases:

$h \geq 2$: Clearly

$$\mu \leq \frac{2g - 2}{\nu_k \left( 2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{\nu_i} \right) \right)} \leq g - 1.$$

$h = 1$: We have

$$\mu \leq \frac{2g - 2}{\nu_k \left( 1 - \frac{1}{\nu_k} \right)} \leq 2(g - 1)$$

$h = 0$: Also in this case we necessarily have ramification and

$$\mu \leq \frac{2g - 2}{\nu_k \left( -2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{\nu_i} \right) \right)}.$$

hence we have to check in which cases holds

$$0 < \lambda := \nu_k \left( -2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{\nu_i} \right) \right) \leq \frac{2}{3}.$$

Since $k \geq 5$ implies $\lambda \geq 1$ we have $k$ at most 4, and $\nu_k > 2$. If $k = 4$ then $\lambda \geq (1/2)\nu_k - 1$ thus $\nu_k = 3$ and one sees immediately that $(2, 2, 2, 3)$ is the only possibility. If $k = 3$ one can check that $1 - \sum_{i=1}^{3} 1/\nu_i \geq 1 - 1/2 - 1/3 - 1/7 = 1/42$, (which corresponds to $|G| = 84(g - 1)$), hence there are only finitely many cases for $\nu_k$ which are easy to consider and which yield exactly the remaining cases in the above table.

For part (ii) note that a finite group $G$ can occur as an automorphism group of a curve of type $(\nu_1, \ldots, \nu_k)$ iff there are distinct elements $g_1, \ldots, g_k$ in $G$ such that $g_1, \ldots, g_{k-1}$ generate $G$, $\prod_{i=1}^{k-1} g_i = 1$ and the order of $g_j$ is $\nu_j$. (cf. section 5.2 for a construction.) For all possible combinations of groups and generators up to the given genus, maximal subsets satisfying the conditions of the above Lemma were calculated using the program GAP (cf. [GAP04]).
Remark 5.4 — The bounds in the second table are sharp, that is, there exist examples that realize the given upper bound. The smallest group realizing $3(g-1)$ is $\text{SL}(2, \mathbb{Z}/3\mathbb{Z})$ acting on a curve of genus 2 of type $(3,3,4)$.

We will see in Remark 5.9 that the slope inequality obtained by Liu implies in fact the better bound $m < 8(g-1)$.

Question 5.5 — It is clear that we can realize the bound $m = 3(g-1)$ for arbitrary large genera $g$ by taking Galois étale coverings of the examples we have obtained.

Can one prove that $3(g-1)$ is an upper bound for all curves?

5.2. Bounds for the slope. Since the slope of a Kodaira fibration does not change under étale pullback, by lemma 2.8 it suffices to treat the slope for a simple configuration. We do this here for the Galois case.

Proposition 5.6 — Let $\mathcal{A} = (D_1 \cup \cdots \cup D_m, d, \{t_i, r_i, n_i\})$ be a simple, Galois configuration and let $g$ be the genus of the target curve $B_2$. If $m \leq 3(g-1)$ then $a(\mathcal{A}) \leq 2 + 2/3$ with equality if and only if $m = 3(g-1)$ and all the ramification indices $r_i$ are equal to three.

Note that we do not know any example of a possible (very) simple configuration with $m > 3(g-1)$.

We believe that the same result as above should hold also in the non Galois case.

Proof. First of all let’s assume that $m = 3(g-1)$ and let us calculate $a(\mathcal{A}) - 8/3$ in this case.

$$a(\mathcal{A}) - 8/3 = \frac{1}{3} - \frac{1}{m} \sum_{i=1}^{m} \frac{1}{r_i} - \frac{2}{3}$$

$$= \frac{3}{5} - \frac{1}{m} \sum_{i=1}^{m} \frac{1}{r_i} - \frac{4}{5} + \frac{1}{m} \sum_{i=1}^{m} \frac{1}{r_i}$$

and if we denote by $m_k$ the number of components $D_i$ of $D$ which have ramification index $r_i = k$

$$= \frac{1}{5} + \frac{1}{m} \sum_k \left( \frac{2m_k k - 3m_k}{k} \right) = \frac{1}{5} + \frac{1}{m} \sum_k \frac{m_k (2k-3)}{k}$$

The expression $\frac{m_k (2k-3)}{k}$ has a global maximum in $k = 3$ and hence for the numerator

$$- \frac{1}{3} + \frac{1}{m} \sum_k m_k \frac{2k-3}{k^2} \leq - \frac{1}{3} + \frac{1}{m} \sum_k m_k \frac{3}{9} = 0$$

with equality if and only if $m_3 = m$ and all other $m_k$’s are zero. Consequently $a(\mathcal{A}) - 8/3 \leq 0$ with equality if and only if all ramification indices are three. It remains to show that the abstract slope can only decrease if $m < 3(g-1)$ which follows by induction from the next lemma.

Lemma 5.7 — Let $\mathcal{A} = (D_1 \cup \cdots \cup D_{m+1}, d, \{t_i, r_i, n_i\})$ be a simple, Galois configuration with $m \leq 4(g(B_2) - 1)$ and let $\mathcal{A}' = (D_1 \cup \cdots \cup D_m, d, \{t_i, r_i, n_i\})$ be the configuration obtained by omitting the last component. Then $a(\mathcal{A}') < a(\mathcal{A})$.

Proof. Using again the formulas from proposition 3.1 we calculate

$$a(\mathcal{A}') - 2 < a(\mathcal{A}) - 2$$
Example 5.8 — We want now to construct an example of a double Kodaira fibration which actually realizes the slope $8/3$ thereby proving Theorem A. First of all we construct the curve mentioned in Remark 5.4.

Let $P_1, P_2, P_3$ be distinct points in $\mathbb{P}^1$ and let $\gamma_1, \gamma_2, \gamma_3$ be simple geometrical loops around these points. The fundamental group $\pi_1(\mathbb{P}^1 \setminus \{P_1, P_2, P_3\})$ is generated by the $\gamma_i$’s with the relation $\gamma_1 \gamma_2 \gamma_3 = 1$. Consider in $\text{SL}(2, \mathbb{Z}/3\mathbb{Z})$ the elements

$$g_1 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

and define $\rho : \pi_1(\mathbb{P}^1 \setminus \{P_1, P_2, P_3\}) \to \text{SL}(2, \mathbb{Z}/3\mathbb{Z})$ by $\gamma_i \mapsto g_i$. This map is well defined and surjective, because $g_1$ and $g_2$ generate $\text{SL}(2, \mathbb{Z}/3\mathbb{Z})$ and $g_1 g_2 g_3 = 1$. We define $B$ to be the ramified Galois cover of $\mathbb{P}^1$ associated to the kernel of $\rho$. By construction $\text{SL}(2, \mathbb{Z}/3\mathbb{Z})$ acts on $B$ as the Galois group of the covering and by the Riemann-Hurwitz formula

$$g(B) = \frac{|\text{SL}(2, \mathbb{Z}/3\mathbb{Z})|}{2} \left( \sum_{i=1}^{3} \left( 1 - \frac{1}{\text{ord}(g_i)} \right) - 2 \right) + 1 = \frac{24}{2} \left( 1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{4} \right) + 2 = 2$$

The subset

$$S = \left\{ \phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi_2 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \phi_3 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \right\} \subset \text{SL}(2, \mathbb{Z}/3\mathbb{Z})$$

satisfies the conditions of Lemma 5.2 since $\phi_2, \phi_3$ and $\phi_3 \circ \phi_2^{-1}$ have no fixed points being of order six and hence gives us $3 = 3(g(B) - 1)$ graphs of automorphisms in $B \times B$ which do not intersect. We denote the corresponding divisor by $D$.

In order to use the tautological construction we have to construct a ramified covering of a curve of genus two minus three points (which we denote for the sake of simplicity by $(B \setminus D)$) and Proposition 5.6 tells us that the ramification indices should all be equal to three.

Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be generators for $\pi_1(B)$ and let $\gamma_1, \gamma_2, \gamma_3$ simple geometrical loops around the three points such that

$$\pi_1(B \setminus D) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2, \gamma_3 \rangle / \langle [\alpha_1, \beta_1] = \gamma_1 \gamma_2 \gamma_3 \rangle$$

is a free group and we can define a map

$$\rho : \pi_1(B \setminus D) \to \mathbb{Z}/3\mathbb{Z}$$

$$\gamma_i \mapsto 1$$

$$\alpha_i, \beta_i \mapsto 0$$
which induces the desired ramified covering $F \to B$.

At this point we use the tautological construction, but we observe that in this case only the first étale covering $B' \to B$ is needed.

Indeed, the divisor $D = D_1 + D_2 + D_3$ has degree 3 on each fibre of the first projection $p : B \times B \to B$ and the homomorphism $\rho$ determines a simple cyclic covering of the fixed fibre $B^0 := \{x_0\} \times B$, ramified on the divisor $D \cap B^0$.

Therefore there is a divisor $M$ on $B \cong B^0$ such that the simple cyclic covering is obtained by taking the cubic root of $D$ in the line bundle corresponding to $M$, and in particular the following linear equivalence holds:

$$3M \equiv D|_{B^0}.$$ 

This linear equivalence determines $M$ up to 3-torsion, and the monodromy of $M$ is the same as the monodromy of $\rho$.

Therefore, if we take as before the étale covering $B' \to B$ associated to the stabilizer of $\rho$, and denote by $B'$ the pull back of $D$ on $B' \times B$, then on $B' \times B$ the divisor $D' = 3p_2^*(M)$ is trivial on each fibre of the first projection $p_1$; whence there is a divisor $L'$ on $B'$ such that $D = 3p_2^*(M) = p_1^*(L')$.

By intersecting with the fibres of the second projection we find that $\deg(L') = 0$, hence there is a divisor $M'$ on $B'$ such that $L' \equiv 3M'$, and we conclude that on $B' \times B$ we have the linear equivalence

$$D' \equiv 3(p_2^*(M) + p_1^*(L'))$$

and we can take the corresponding simple cyclic covering branched on $D'$ inside the line bundle corresponding to the divisor $p_2^*(M) + p_1^*(L')$.

We obtain in this way a double étale Kodaira fibration which is in fact a Standard Kodaira Fibration by Lemma 4.4. In particular we have a Kodaira fibration with base curve $B'$ and with fibre of genus $g = 7$ (since $2g = 2 = 3 \cdot 2 + 3 \cdot 2$).

Since the associated ramified covering is branched exactly over $D'$ with ramification index three at each component, the formula for the slope of a simple configuration calculated in Proposition 5.11 yields

$$c_2^2(S) \frac{c_2(S)}{c_1^2(S)} = 2 + \frac{1 - \frac{1}{3} \sum_{i=1}^{3} \frac{1}{3^i}}{\frac{1}{3} \sum_{i=1}^{3} \frac{1}{3^i}} = \frac{8}{3}.$$ 

**Remark 5.9** — We can also use this construction to give a partial answer to the question raised in 5.5. Knowing that the slope of a Kodaira fibration is strictly smaller than 3 it follows that $m < 8(g - 1)$. In fact, via a suitable base change we obtain a divisor $D' \subset B' \times B$ such that

(i) if $m$ is odd, then there is a component $D_1$ mapping to $B'$ with degree one,

(ii) setting $D'' := D'$ if $m$ is even, and $D'' := D' - D_1$ if $m$ is odd, then

(iii) we can take a double cover branched over $D''$.

The Kodaira fibration constructed in this way turns out, under the assumption $m \geq 8(g - 1)$, and in view of the above formulas, to have a slope $\geq 3$: this is a contradiction.

It follows in particular as a consequence: if $B$ is a curve of genus 2 and we have 8 étale maps from a fixed curve $C$ of arbitrary genus to $B$, then two of them have a coincidence point.

6. The moduli space

This section is devoted to the description of the moduli space of double étale Kodaira Fibrations. We start with some lemmas.
Lemma 6.1 — Let $B_1, B_2$ be curves of genus $g_i \geq 2$ resp. and let $C \subset B_1 \times B_2$ be an irreducible curve. Then

- $C$ is smooth and the restricted projections $p_i : C \to B_i$ are étale if and only if
- the negative of the selfintersection of $C$ attains its maximum possible value, i.e., iff
  \[-C^2 = 2m_i(b_i - 1) \quad (i = 1, 2)\]

where $m_1 = C \cdot \{*\} \times B_2$ and $m_2 = C \cdot B_1 \times \{*\}$.

Proof. "⇒" Let $p = p(C)$ be the arithmetic genus of $C$. Then

$$2p - 2 = K_{B_1 \times B_2} \cdot C + C^2 = 2(b_1 - 1)m_1 + 2(b_2 - 1)m_2 - 2(b_1 - 1)m_1\]

$$= 2m_i(b_i - 1) \quad (i \neq j)$$

Let $\tilde{C} \to C$ be the normalisation and let $g = g(\tilde{C})$ be the geometric genus of $C$. We have $2p - 2 \geq 2g - 2$ by the normalization sequence and on the other hand $2g - 2 \geq 2m_i(b_i - 1) = 2p - 2$ by the Hurwitz formula. Hence $g = p$, $C$ is smooth and equality holds in the last inequality, i.e., there is no ramification and the maps $p_i$ are étale.

Remark 6.2 — In general we see that $K_{B_1 \times B_2} \cdot C + C^2 = 2m_i(b_i - 1) + 2\delta + \rho_i$, where $\delta$ is the 'number of double points' and $\rho_i$ is the total ramification index of $C \to B_i$. So

$$-C^2 = 2m_j(b_j - 1) - 2\delta - \rho_i \quad (i \neq j)$$

Lemma 6.3 — Assume that we have a family of effective divisors $(D_t)_{t \in \mathcal{T}}$, $D_t \subset (B_1 \times B_2, t)$, such that the special fibre $D := D_0 = nC$ with $C$ as in Lemma 6.1. If $D'$ is another fibre ($D' = D_t$ for some $t$), then $D'$ is of the same type $D' = nC'$ (the integer $n$ being the same as before).

Proof. Write $D' = \sum_j r_jC_j$ as a sum of irreducible components, so that $C_i \cdot C_j \geq 0$ for $i \neq j$. Write also $m_1^i = C_i \cdot \{*\} \times B_{2,t}$ and $m_2^i = C_i \cdot B_{1,t} \times \{*\}$. We calculate

$$-D'^2 = \sum_j r_j^2(-C_j^2) - 2\sum_{i \neq j} r_jC_i \cdot C_j \leq \sum_j r_j^2(-C_j^2) \leq \sum_j r_j^2 2m_i^j(b_i - 1)$$

and also

$$-D'^2 = n^2(-C^2) = n^2 2m_i(b_i - 1).$$

Hence the following conditions hold:

$$\sum_j r_j^2 m_i^j \geq n^2 m_i, \quad nm_i = \sum_j r_j m_i^j.$$

Since $D$ is the special fibre every component $C_i$ tends to a positive multiple of $C$, we have $m_i^j \geq m_i$ and putting together the two inequalities yields

$$\sum_j r_j^2 m_i^j \geq n^2 m_i^j = \left(\sum_j r_j m_i^j\right)^2 \geq \sum_j r_j^2 m_i^j.$$
therefore in fact equality holds, there is only one summand and $D' = r_0 C_0$. To conclude the proof we look again at the conditions

$$n^2 m_i \leq r_0^2 m_i^0, \quad n m_i = r_0 m_i^0, \quad m_i \leq m_i^0.$$  

Combining the two inequalities with the equality in the middle we get $n \leq r_0 \leq n$ and we are done by observing that also $C' = C_0$ fulfills the conditions of Lemma 6.3.

\[\square\]

**Theorem 6.4** — Being a double étale Kodaira Fibration is a closed and open condition in the moduli space.

**Proof.** Due to the previous Lemma it remains to show the closedness. Assume then that we have a 1-parameter family of surfaces with general fiber $S_t$ a double étale Kodaira fibration. By the topological characterization (Proposition 2.5) also the special fibre $S_0$ is a double Kodaira fibration. Moreover, by Lemma 2.8 we may assume that $S_t$ is a branched covering of $B_{1,t} \times B_{2,t}$ branched over $D_t = \sum_i k_i D_{i,t}$, where the $D_{i,t}$'s are disjoint graphs of étale maps $\phi_i : B_{1,t} \to B_{2,t}$.

Now, $S_0 \to B_{1,0} \times B_{2,0}$ is branched over $D_0 := \sum_i k_i \nu_i D_{i,0}$ where $D_{i,t}$ tends to $\nu_i D_{i,0}$. Since however $D_{i,t}, (B_{1,t} \times \{\ast\}) = 1$ we have $\nu_i D_{i,0}, (B_{1,0} \times \{\ast\}) = 1$ which implies $\nu_i = 1$. Hence $D_{i,0}$ is the graph of a map $\phi_i' : B_{2,0} \to B_{2,0}$ and another application of Lemma 6.1 shows that also $\phi_i'$ is étale and $S_0$ is a double étale Kodaira Fibration.

\[\square\]

We can now describe the moduli space of Standard Kodaira Fibrations in detail. Let $S$ be a Standard Kodaira Fibration: then there exists a minimal common Galois cover $B'$ of $B_1, B_2$ yielding an étale pullback $S'$ which is very simple. We call $B'$ the simplifying covering curve. We have diagrams

\[
\begin{array}{ccc}
S' & \xrightarrow{\psi_2} & B' \\
\pi & & f_2 \\
\psi_1 & S & \psi_2 \\
\downarrow & & \downarrow \\
B' & \xrightarrow{f_1} & B_2 \\
B' & \xrightarrow{f_1} & B_1 \\
\end{array}
\quad
\begin{array}{ccc}
B' \times B' & \xrightarrow{B'} & B_1 \times B_2 \\
D' = \bigcup_{\varphi \in S} \Gamma_{\varphi} & & \\
\end{array}
\]

where $D$ is the ramification divisor of $\psi = \psi_1 \times \psi_2$ and $D' = \pi^* D$ is made of the graphs of a set of automorphisms $S \subset \text{Aut}(B')$. If we denote the Galois group of $f_i$ by $G_i$ ($i = 1, 2$) the following holds:

**Theorem 6.5** — Let $S$ be a standard Kodaira fibred surface and let $\mathcal{G}$ be the irreducible (and connected) component of the moduli space containing $[S]$. $\mathcal{G}$ is then isomorphic to the moduli space of the pair $(B', G)$, where $B'$ is the simplifying covering curve defined above and $G$ is the subgroup of $\text{Aut}(B')$ generated by $G_1, G_2$ and $S$.

**Proof.** Let us first consider the case where $S = S'$, i.e., where $S$ itself is very simple. By proposition 2.5 every deformation in the large of $S'$ is a branched cover at a product surface $B_1 \times B_2$. Moreover, clearly $B_1 = B_2$ if (*) there is a component of the branch locus mapping to both curves $B_1, B_2$ with degree 1. So let $S_t, t \in T$, be a family with connected parameter space $T$, having $S'$ as a fibre. It is clear that the set of points of $T$ where (*) holds is open. It is also closed because in the proof
of theorem 6.4 we have seen that the type of the branch divisor remains the same under specialization.

We have seen that \( \mathcal{N} \) parametrizes surfaces which are very simple and indeed a branched covering of a product \( B \times B \) branched on the union of graphs of automorphisms.

The automorphisms defining the components of the branch divisors in different fibers are clearly pairwise isotopic to each other and therefore we obtain a family of curves with automorphisms.

For each curve let \( G \) be the finite group generated by these automorphisms. This group has a faithful representation on the fundamental group of the curve, and therefore the group \( G \) remains actually constant.

\( G \) is a finite group and we have a faithful action on Teichmüller space \( \Sigma_b \). We use now Lemma 4.12 of [Cat00] (page 29) to the effect that the fixed locus of this action is a connected submanifold (diffeomorphic to an Euclidean space), hence the moduli space of such pairs \( (B,G) \) is irreducible.

Viceversa any element in this moduli space gives rise to a complex structure on the differentiable manifold underlying \( S' \).

Consider now the general case. It is clear that any deformation of \( S \) induces a deformation of \( B' \rightarrow B_1 \), hence any deformation of \( S \) yields a deformation of the pair \( (B',G) \).

Conversely, any deformation of the pair \( (B',G) \) yields a deformation of the pair \( D' \subset B' \times B' \) such that the group \( G_1 \times G_2 \) leaves \( D' \) and the monodromy of the unramified covering of \( (B' \times B') - D' \) invariant.

\[ \Box \]

**Corollary 6.6** — There exist double étale Kodaira fibred surfaces which are rigid.

**Proof.** Take the fibration constructed in Example 5.8: the automorphisms corresponding to the ramification divisor generate the whole triangle group of type \((3,3,4)\) and it is well known that pairs \( (B,G) \) yielding a triangle curve are rigid. Similarly for the other examples in proposition 5.3 which yield \( m = 3(g - 1) \).

\[ \Box \]

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