On a Subclass of Close-to-Convex Functions

Yao Liang Chung1 · Maisarah Haji Mohd1 · See Keong Lee1

Received: 12 December 2017 / Revised: 30 January 2018 / Accepted: 26 May 2018 / Published online: 4 June 2018
© Iranian Mathematical Society 2018

Abstract In this paper, we introduce a subclass of close-to-convex functions defined in the open unit disk. We obtain the inclusion relationships, coefficient estimates and the Fekete–Szegö inequality. The results presented here would provide extensions of those given in earlier works.

Keywords Close-to-convex · Starlike function · Subordination

Mathematics Subject Classification 30C45

1 Introduction

Let $A$ denote the class of all analytic functions $f$ in the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$ with normalization $f(0) = 0$ and $f'(0) = 1$. The subclass of $A$ consisting of univalent functions is denoted by $S$. Let $P$ be the class of functions with positive real part consisting of all analytic functions $p : D \to \mathbb{C}$ satisfying $p(0) = 1$ and $\text{Re}(p(z)) > 0$.

An analytic function $f$ is said to be starlike of order $\alpha$ in $D$ if and only if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$$
for some $\alpha \ (0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all functions in $S$ which are starlike of order $\alpha$ in $\mathbb{D}$. The class $S^* := S^*(0)$ is the well-known class of starlike functions. In 1952, Wilfred Kaplan [7] generalized the concept of starlike function to that of a close-to-convex function. An analytic function $f$ is said to be close-to-convex in $\mathbb{D}$ if there exists a starlike function $g$ such that for any $z \in \mathbb{D}$, the inequality

$$\Re \left( \frac{zf'(z)}{g(z)} \right) > 0$$

holds. Note that the function $g$ need not be normalized. We let $K$ to denote the set of close-to-convex functions in $\mathbb{D}$. Kaplan [7] also showed that every close-to-convex function is univalent in $\mathbb{D}$.

A function $f \in A$ is said to be starlike with respect to symmetrical points in $\mathbb{D}$ if it satisfies

$$\Re \left( -z^2f'(z)g(z)g(-z) \right) > 0, \quad z \in \mathbb{D}.$$ 

This class was introduced by Sakaguchi [12] and is denoted by $S^*_s$. It is interesting to see that $f \in S^*_s$ is also close-to-convex in $\mathbb{D}$ because the function $(f(z) - f(-z))/2$ is starlike in $\mathbb{D}$. The constant $1/2$ is for normalization purpose. Motivated by the class of starlike functions with respect to symmetric points, Gao and Zhou [4] discussed a class $K_s$ of close-to-convex functions.

**Definition 1.1** [4] A function $f \in A$ belongs to $K_s$ if there exists a function $g(z) \in S^*(1/2)$ such that

$$\Re \left( -\frac{z^2f'(z)}{g(z)g(-z)} \right) > 0, \quad z \in \mathbb{D}.$$ 

Note that if $g(z) \in S^*(1/2)$, then $-g(z)g(-z)/z \in S^*$ [3].

Here, we recall the concept of subordination between analytic functions. An analytic function $f$ is subordinate to an analytic function $g$, written as $f(z) \prec g(z)$, if there exists an analytic function $w$ defined in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if $g$ is univalent in $\mathbb{D}$, then we have the following equivalence $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Using the concept of subordination, Wang et al. [15] introduced a general class $K_s(\varphi)$.

**Definition 1.2** [15] For a normalized function $\varphi$ with positive real part, the class $K_s(\varphi)$ consists of functions $f \in A$ satisfying

$$-\frac{z^2f'(z)}{g(z)g(-z)} < \varphi(z)$$

for some function $g(z) \in S^*(1/2)$.

Recently, Goyal and Singh [6] introduced and studied the following subclass of analytic functions:
Definition 1.3 [6] For a normalized function $\varphi$ with positive real part, a function $f \in \mathcal{A}$ is said to be in the class $K_s(\lambda, \mu, \varphi)$ if it satisfies the following subordination condition:

$$
\frac{z^2 f'(z) + (\lambda - \mu + 2\lambda\mu)z^3 f''(z) + \lambda\mu z^4 f'''(z)}{-g(z)g(-z)} < \varphi(z),
$$

where $0 \leq \mu \leq \lambda \leq 1$ and $g \in S^*(1/2)$.

Motivated by the aforementioned works, we now introduce the following subclass of analytic functions:

Definition 1.4 Let $\varphi$ be an analytic normalized function with positive real part, $g \in S^*((k-1)/k)$, and $g_k(z) = \prod_{v=0}^{k-1} e^{-v} g(\varepsilon^v z)$ and $0 \leq \mu \leq \lambda \leq 1$. A function $f \in \mathcal{A}$ is said to be in the class $K_s((k))_s(\lambda, \mu, \varphi)$ if

$$
\frac{z^k f'(z) + (\lambda - \mu + 2\lambda\mu)z^{k+1} f''(z) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)} < \varphi(z), \quad z \in \mathbb{D},
$$

holds for some positive integer $k$.

Remark 1.1 (a) For $\varphi(z) = (1 + Az)/(1 - Bz)$, $\mu = 0$, and $k = 2$, we have the class $K_s(\lambda, A, B)$ [14].

(b) When $A = 1 - 2\gamma$, $B = -1$ and $\lambda = \mu = 0$, we obtain the class $K_s^{(k)}(\gamma)$ [13]. In addition, if $k = 2$, then we obtain the class $K_s(\gamma)$ [10].

(c) When $A = \beta$, $B = -\alpha\beta$ and $\lambda = \mu = 0$, then we obtain the class $K_s^{(k)}(\alpha, \beta)$ in [15]. In addition, if $k = 2$, we obtain the class $K_s^{(k)}(\alpha, \beta)$ [16].

Before proving the main results, we recall some existing results that would be useful in the work here.

Lemma 1.1 [16] If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*((k-1)/k)$, then

$$
G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} B_n z^n \in S^*.
$$

(1.1)

Lemma 1.2 [11] Let $f(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ be analytic in $\mathbb{D}$ and $g(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$ be analytic and convex in $\mathbb{D}$. If $f < g$, then

$$
|c_k| \leq |d_k| \quad \text{where} \quad k \in \mathbb{N} := \{1, 2, 3, \ldots\}.
$$

Lemma 1.3 [17] Let $\gamma \geq 0$ and $f \in \mathcal{K}$. Then

$$
F(z) = \frac{1 + \gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt \in \mathcal{K}.
$$
2 Main Results

We first prove that the functions in the class $K_{s}^{(k)}(\lambda, \mu, \varphi)$ are close-to-convex.

**Theorem 2.1** Let $0 \leq \mu \leq \lambda \leq 1$. Then,

$$K_{s}^{(k)}(\lambda, \mu, \varphi) \subset \mathcal{K}. \tag{2.1}$$

**Proof** Let

$$F(z) = (1 - \lambda + \mu) f(z) + (\lambda - \mu) z f'(z) + \lambda \mu z^2 f''(z) \tag{2.2}$$

and $G_k$ as defined in (1.1). Then by Definition 1.4, we have the subordination

$$\frac{zF'(z)}{G_k(z)} < \varphi(z).$$

Since $\Re(\varphi(z)) > 0$, we have

$$\Re \left( \frac{zF'(z)}{G_k(z)} \right) > 0.$$

Also, since $G_k(z) \in \mathcal{S}^*$ (by Lemma 1.1), by the definition of close-to-convex function, we deduce that

$$F(z) = (1 - \lambda + \mu) f(z) + (\lambda - \mu) z f'(z) + \lambda \mu z^2 f''(z) \in \mathcal{K}.$$  

To show $f \in \mathcal{K}$, we consider three cases:

*Case 1:* $\mu = \lambda = 0$. It is then obvious that $f = F \in \mathcal{K}$.

*Case 2:* $\mu = 0, \lambda \neq 0$. Then we have

$$F(z) = (1 - \lambda) f(z) + \lambda z f'(z).$$

Solving this, we get

$$f(z) = \frac{1}{\lambda} z^{1-1/\lambda} \int_0^z t^{(1/\lambda)-2} F(t) dt.$$

Taking $\gamma = (1/\lambda) - 1$ in Lemma 1.3, we conclude that $f(z) \in \mathcal{K}$.

*Case 3:* $\mu \neq 0, \lambda \neq 0$. Then we have

$$F(z) = (1 - \lambda + \mu) f(z) + (\lambda - \mu) z f'(z) + \lambda \mu z^2 f''(z).$$

Let $G(z) = \frac{1}{(1-\lambda+\mu)} F(z)$, which is obviously close-to-convex, since $F$ is. Then,

$$G(z) = f(z) + \alpha z f'(z) + \beta z^2 f''(z), \tag{2.3}$$
where $\alpha = \frac{\lambda - \mu}{1 - \lambda + \mu}$ and $\beta = \frac{\lambda \mu}{1 - \lambda + \mu}$. Consider $\delta > 0$ and $\nu > 0$ satisfying

$$\delta + \nu = \alpha - \beta \quad \text{and} \quad \delta \nu = \beta.$$  

Then, (2.3) can be written as

$$G(z) = f(z) + (\delta + \nu + \delta \nu)zf'(z) + \delta \nu^2 f''(z).$$

Let $p(z) = f(z) + \delta zf'(z)$. Then,

$$p(z) + \nu z p'(z) = f(z) + (\delta + \nu + \delta \nu)zf'(z) + \delta \nu^2 f''(z) = G(z).$$

Since $p(z) + \nu z p'(z) = \nu z^{1-1/\nu} \left( z^{1/\nu} p(z) \right)'$, it follows that

$$G(z) = \nu z^{1-1/\nu} \left[ \delta z^{1+(1/\nu)-1/\delta} \left( z^{1/\delta} f(z) \right) \right]' .$$

Hence,

$$\delta z^{1+1/\nu-1/\delta} \left( z^{1/\delta} f(z) \right)' = \frac{1}{\nu} \int_0^z \frac{w^{(1/\nu)-1} G(w) \, dw}{w^{1/\nu}} .$$

Multiplying by $(1 + \nu)$ on both sides and dividing by $z^{1/\nu}$, we get

$$(1 + \nu) \delta z^{1-1/\delta} \left( z^{1/\delta} f(z) \right)' = \frac{1 + 1/\nu}{z^{1/\nu}} \int_0^z \frac{w^{(1/\nu)-1} G(w) \, dw}{w^{1/\nu}} .$$

By Lemma 1.3 with $\gamma = 1/\nu > 0$, we have

$$H(z) = \frac{1 + 1/\nu}{z^{1/\nu}} \int_0^z w^{(1/\nu)-1} G(w) \, dw \in \mathcal{K}.$$  

Further,

$$(1 + \nu) z^{1/\delta} f(z) = \frac{1}{\delta} \int_0^z t^{(1/\delta)-1} H(t) \, dt .$$

Multiplying by $(1 + \delta)$ on both sides and dividing by $z^{1/\delta}$, now we get

$$(1 + \delta)(1 + \nu) f(z) = \frac{1 + 1/\delta}{z^{1/\delta}} \int_0^z t^{(1/\delta)-1} H(t) \, dt .$$

By Lemma 1.3 with $\gamma = 1/\delta > 0$, we get $f \in \mathcal{K}$. This completes the proof. \hfill \Box

Next, we give the coefficient estimates of functions belonging to the class $\mathcal{K}_{s_{k}}^{(k)}(\lambda, \mu, \varphi)$. 

\hfill \copyright Springer
Theorem 2.2 Let \( 0 \leq \mu \leq \lambda \leq 1 \) and \( \varphi \) be a normalized analytic convex function in \( \mathbb{D} \). If \( f \in K_{\lambda}^{(k)}(\mu, \varphi) \), then

\[
|a_n| \leq \frac{1}{1 + (n - 1)(\lambda - \mu + n\lambda\mu)} \left( 1 + \frac{\varphi'(0)(n - 1)}{2} \right) \quad (n \in \mathbb{N}).
\]

Proof From the definition of \( K_{\lambda}^{(k)}(\mu, \varphi) \), we know that there exists an analytic function \( p \) with positive real part such that

\[
p(z) = \frac{z^k f'(z) + (\lambda - \mu + 2\lambda\mu)z^{k+1} f''(z) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)} = \frac{zF'(z)}{G_k(z)},
\]

or

\[
z f'(z) + (\lambda - \mu + 2\lambda\mu) z^2 f''(z) + \lambda\mu z^3 f'''(z) = p(z) G_k(z). \tag{2.4}
\]

By expanding both sides and equating the coefficients in (2.4), we get

\[
n|a_n| \left[ 1 + (n - 1)(\lambda - \mu + n\lambda\mu) \right] = B_n + p_{n-1} + p_1 B_{n-1} + \cdots + p_{n-2} B_2. \tag{2.5}
\]

Since \( G_k(z) \) is starlike, we have

\[
|B_n| \leq n, \quad \text{for all } n. \tag{2.6}
\]

Also, by Lemma 1.2, we know that

\[
|p_n| = \left| \frac{p^{(n)}(0)}{n!} \right| \leq |\varphi'(0)| \quad \text{for all } n \in \mathbb{N}. \tag{2.7}
\]

Combining (2.5), (2.6) and (2.7), we obtain

\[
n|a_n| \left[ 1 + (n - 1)(\lambda - \mu + n\lambda\mu) \right] \leq n + |\varphi'(0)| \left[ 1 + \sum_{n=2}^{n-1} k \right]
\]

or

\[
|a_n| \left[ 1 + (n - 1)(\lambda - \mu + n\lambda\mu) \right] \leq 1 + \frac{|\varphi'(0)|(n - 1)}{2}.
\]

This completes the proof. \( \Box \)

Setting \( \mu = 0 \) in Theorem 2.2,
Corollary 2.1 If \( f \in K_s^{(k)}(\lambda, \varphi) \), then
\[
|a_n| \leq \frac{1}{1 + \lambda(n - 1)} \left(1 + \frac{|\varphi'(0)|(n - 1)}{2}\right) \quad (n \in \mathbb{N}).
\]

Further, by letting \( \lambda = 0 \) in Corollary 2.1, we have

Corollary 2.2 If \( f \in K_s^{(k)}(\varphi) \), then
\[
|a_n| \leq 1 + \frac{|\varphi'(0)|(n - 1)}{2} \quad (n \in \mathbb{N}).
\]

Next the Fekete–Szegö inequality for function \( f \in K_s^{(k)}(\lambda, \mu, \varphi) \) is obtained. To prove our result, we need the following lemmas:

Lemma 2.1 [8] If \( p(z) = 1 + c_1z + c_2z^2 + \cdots \) is a function with positive real part, then for any complex number \( \mu \),
\[
|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\},
\]
and the result is sharp for the functions given by \( p(z) = (1 + z^2)/(1 - z^2) \) and \( p(z) = (1 + z)/(1 - z) \).

Lemma 2.2 [9] Let \( G(z) = z + b_2z^2 + b_3z^3 + \cdots \) be in \( S^* \). Then
\[
|b_3 - \lambda b_2^2| \leq \max\{1, |3 - 4\lambda|\},
\]
which is sharp for the Koebe function \( k \) if \( |\lambda - 3/4| \geq 1/4 \), and for \( (k(z^2))^{1/2} = z/(1 - z^2) \) if \( |\lambda - 3/4| \leq 1/4 \).

Theorem 2.3 Let \( \varphi(z) = 1 + q_1z + q_2z^2 + \cdots \) be an analytic function with positive real part on \( \mathbb{D} \). For \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) belonging to the class \( K_s^{(k)}(\lambda, \mu, \varphi) \) and \( \delta \in \mathbb{C} \), the following estimate holds:
\[
|a_3 - \delta a_2^2| \leq \frac{\max\{1, |3 - 4\alpha|\} + q_1 \max\{1, |2\beta - 1|\}}{3(1 + 2\lambda - 2\mu + 6\lambda\mu)}
+ 2q_1 \left( \frac{1}{3(1 + 2\lambda - 2\mu + 6\lambda\mu)} - \frac{\mu}{2(1 + \lambda - \mu + 2\lambda\mu)^2} \right),
\]
where
\[
\alpha = \frac{3(1 + 2\lambda - 2\mu + 6\lambda\mu)}{4(1 + \lambda - \mu + 2\lambda\mu)} \delta
\]
and
\[
\beta = \frac{1}{2} \left( 1 - \frac{q_2}{q_1} + \frac{3\delta q_2^2 d_1^2 (1 + 2\lambda - 2\mu + 6\lambda\mu)}{4(1 + \lambda - \mu + 2\lambda\mu)^2} \right).
\]
Proof If \( f \in K_s^{(k)}(\lambda, \mu, \varphi) \), then there exists an analytic function \( w \) analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
\frac{z^k f'(z) + (\lambda - \mu + 2\lambda \mu)z^{k+1} f''(z) + \lambda \mu z^{k+2} f'''(z)}{g_k(z)} = \varphi(w(z)).
\]

Define the function \( h \) by

\[
h(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + d_1 z + d_2 z^2 + \cdots.
\]

Since \( \omega \) is a Schwarz function, we see that \( \text{Re}\{h(z)\} > 0 \) and \( h(0) = 1 \). Also, we have

\[
\varphi(w(z)) = \varphi\left(\frac{h(z) - 1}{h(z) + 1}\right) = 1 + \frac{1}{2} q_1 d_1 z + \frac{1}{2} q_1 \left(d_2 - \frac{d_1^2}{2}\right) z^2 + \frac{1}{4} q_2 d_1^2 z^2 + \cdots. \tag{2.8}
\]

The series expansion of

\[
\frac{z^k f'(z) + z^{k+1} f''(z)(\lambda - \mu + 2\lambda \mu) + \lambda \mu z^{k+2} f'''(z)}{g_k(z)}
\]

is given by:

\[
1 + (2a_2(1 + \lambda + 2\lambda \mu - \mu) - B_2) z
+ (3a_3(1 + 2\lambda + 6\lambda \mu - 2\mu) - 2a_2(1 + \lambda + 2\lambda \mu - \mu)B_2 + B_2^2 - B_3) z^2 + \cdots. \tag{2.9}
\]

By comparing (2.8) and (2.9), we have

\[
a_2 = \frac{2B_2 + q_1 d_1}{4(1 + \lambda - \mu + 2\lambda \mu)} \quad \text{and} \quad a_3 = \frac{2B_2 q_1 d_1 + 2q_1 \left(d_2 - \frac{d_1^2}{2}\right) + q_2 d_1^2 + 4B_3}{12(1 + 2\lambda - 2\mu + 6\lambda \mu)}.
\]

Therefore, we have

\[
a_3 - \delta a_2^2 = \frac{1}{3(1 + 2\lambda - 2\mu + 6\lambda \mu)} (B_3 - \alpha B_2^2) + \frac{q_1}{6(1 + 2\lambda - 2\mu + 6\lambda \mu)} (d_2 - \beta d_1^2) + \frac{B_2 q_1 d_1}{2} \left( \frac{1}{3(1 + 2\lambda - 2\mu + 6\lambda \mu)} - \frac{\delta}{2(1 + \lambda - \mu + 2\lambda \mu)} \right),
\]

where

\[
\alpha = \frac{3\delta(1 + 2\lambda - 2\mu + 6\lambda \mu)}{4(1 + \lambda - \mu + 2\lambda \mu)}.
\]
and
\[ \beta = \frac{1}{2} \left( 1 - \frac{q_2}{q_1} + \frac{3\delta q_1^2 d_1^2 (1 + 2\lambda - 2\mu + 6\lambda \mu)}{4(1 + \lambda - \mu + 2\lambda \mu)^2} \right). \]

Taking the modulus on both sides, we have
\[
|a_3 - \delta a_2^2| \leq \frac{|B_3 - \alpha B_2^2|}{3(1 + 2\lambda - 2\mu + 6\lambda \mu)} + \frac{q_1 |d_2 - \beta d_1^2|}{6(1 + 2\lambda - 2\mu + 6\lambda \mu)} + \frac{q_1 |B_2||d_1|}{2} \left( \frac{1}{3(1 + 2\lambda - 2\mu + 6\lambda \mu)} - \frac{\mu}{2(1 + \lambda - \mu + 2\lambda \mu)^2} \right).
\]

Using Lemmas 2.1 and 2.2, we obtain
\[
|a_3 - \delta a_2^2| \leq \frac{\max \{1, |3 - 4\alpha|\}}{3(1 + 2\lambda - 2\mu + 6\lambda \mu)} + 2 \frac{q_1 \max \{1, |2\beta - 1|\}}{6(1 + 2\lambda - 2\mu + 6\lambda \mu)} + \frac{q_1 |B_2||d_1|}{2} \left( \frac{1}{3(1 + 2\lambda - 2\mu + 6\lambda \mu)} - \frac{\mu}{2(1 + \lambda - \mu + 2\lambda \mu)^2} \right).
\]

Also, using $|B_2| \leq 2$ and $|d_1| \leq 2$, the result is proved. \(\square\)

Lastly, we prove a sufficient condition for functions to belong to the class $K_s^{(k)}(\lambda, \mu, A, B)$. By setting $\varphi(z) = (1 + Az)/(1 + Bz)$ in Definition 1.4, we get the class $K_s^{(k)}(\lambda, \mu, A, B)$ as defined below.

**Definition 2.1** A function $f \in \mathcal{A}$ is said to be in the class $K_s^{(k)}(\lambda, \mu, A, B)$ if it satisfies the following subordination condition:
\[
\frac{z^k f'(z) + (\lambda - \mu + 2\lambda \mu) z^{k+1} f''(z) + \lambda \mu z^{k+2} f'''(z)}{g_k(z)} < \frac{1 + Az}{1 + Bz}, \tag{2.10}
\]

where $0 \leq \mu \leq \lambda \leq 1$, $k \geq 1$ is a fixed positive integer, and $g_k(z) = \prod_{v=0}^{k-1} e^{-v} g(\varepsilon^v z)$ with $\varepsilon = e^{2\pi i/k}$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*((k - 1)/k)$.

The condition in (2.10) is equivalent to
\[
\left| \frac{z^k f'(z) + (\lambda - \mu + 2\lambda \mu) z^{k+1} f''(z) + \lambda \mu z^{k+2} f'''(z)}{g_k(z)} - 1 \right| < \frac{A + B[z^k f'(z) + (\lambda - \mu + 2\lambda \mu) z^{k+1} f''(z) + \lambda \mu z^{k+2} f'''(z)]}{g_k(z)} \tag{2.11}
\]
or
\[
\left| z F'(z) - \frac{g_k(z)}{z^{k-1}} \right| < \left| - \frac{Ag_k(z)}{z^{k-1}} - Bz F'(z) \right|,
\]
From the above calculation, we obtain \( M \) is analytic in \( \mathbb{D} \) where \( F \) is defined in (2.2) and (1.1), respectively.

**Theorem 2.4** Let \(-1 \leq B < A \leq 1\) and \(0 \leq \mu \leq \lambda \leq 1\). If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic in \( \mathbb{D} \) and satisfies the inequality

\[
(1 + |B|) \sum_{n=2}^{\infty} n[1 + (n - 1)(\lambda - \mu + n\lambda\mu)]|a_n| + (1 + |A|) \sum_{n=2}^{\infty} |B_n| \leq A - B,
\]

where \( B_n \)'s are as in (1.1), then \( f \in K^{(k)}(\lambda, \mu, A, B) \).

**Proof** Let \( M \) be denoted by

\[
M = \left| zF'(z) - \frac{g_k(z)}{z^{k-1}} \right| - \left| - \frac{Ag_k(z)}{z^{k-1}} - BzF'(z) \right|
\]

\[
= \left| zf'(z) + (\lambda - \mu + 2\lambda\mu)z^2f''(z) + \lambda\mu z^3f'''(z) - \left(z + \sum_{n=2}^{\infty} B_n z^n\right) \right|
\]

\[
- \left| A \left(z + \sum_{n=2}^{\infty} B_n z^n\right) - B[zf'(z) + (\lambda - \mu + 2\lambda\mu)z^2f''(z) + \lambda\mu z^3f'''(z)] \right|
\]

\[
= \left| \sum_{n=2}^{\infty} n a_n z^n [1 + (n - 1)(\lambda - \mu + n\lambda\mu)] - \sum_{n=2}^{\infty} B_n z^n \right|
\]

\[
- \left| (A - B)z + A \sum_{n=2}^{\infty} B_n z^n - B \sum_{n=2}^{\infty} n a_n z^n [1 + (n - 1)(\lambda - \mu + n\lambda\mu)] \right|
\]

Then, for \(|z| = r < 1\), we have

\[
M \leq \sum_{n=2}^{\infty} n[1 + (n - 1)(\lambda - \mu + n\lambda\mu)]|a_n|r^n + \sum_{n=2}^{\infty} |B_n|r^n
\]

\[
- \left[ (A - B)r - |A| \sum_{n=2}^{\infty} |B_n|r^n - |B| \sum_{n=2}^{\infty} n[1 + (n - 1)(\lambda - \mu + n\lambda\mu)]|a_n|r^n \right]
\]

\[
< \left[ - (A - B) + (1 + |B|) \sum_{n=2}^{\infty} n[1 + (n - 1)(\lambda - \mu + n\lambda\mu)]|a_n| + (1 + |A|) \sum_{n=2}^{\infty} |B_n| \right] r
\]

\[
\leq 0.
\]

From the above calculation, we obtain \( M < 0 \). Thus, we have

\[
\left| zf'(z) + (\lambda - \mu + 2\lambda\mu)z^2f''(z) + \lambda\mu z^3f'''(z) - \frac{g_k(z)}{z^{k-1}} \right|
\]

\[
< \left| A \frac{g_k(z)}{z^{k-1}} - B[zf'(z) + (\lambda - \mu + 2\lambda\mu)z^2f''(z) + \lambda\mu z^3f'''(z)] \right|, \quad (2.12)
\]
which is equivalent to (2.11). Therefore, \( f \in K_s^{(k)}(\lambda, \mu, A, B) \). \( \Box \)

Setting \( \mu = 0 \) and \( \lambda = 0 \) in Theorem 2.4, we get

**Corollary 2.3** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \( D \) and \( -1 \leq B < A \leq 1 \). If

\[
(1 + |B|) \sum_{n=2}^{\infty} n|a_n| + (1 + |A|) \sum_{n=2}^{\infty} |B_n| \leq A - B,
\]

then \( f \in K_s^{(k)}(A, B) \).

**Remark 2.1** By taking \( A = \beta, B = -\alpha \beta \) in Corollary 2.3, we get the result obtained in [15, Theorem 5]. In addition, by taking \( A = 1 - 2\gamma, B = -1 \), we get the result obtained in [13, Theorem 2].

**Acknowledgements** The research of the first and second authors is supported by the USM Short Term Grant 304/PMATHS/6313192.

**References**

1. Ali, R.M., Lee, S.K., Subramanian, K.G., Swaminathan, A.: A third-order differential equation and starlikeness of a double integral operator. Abstr. Appl. Anal. **901235**, 10 (2011)
2. Cho, N.E., Kwon, O.S., Ravichandran, V.: Coefficient, distortion and growth inequalities for certain close-to-convex functions. J. Inequal. Appl. **2011**, 7–100 (2011)
3. Das, R.N., Singh, P.: On subclasses of schlicht mapping. Indian J. Pure Appl. Math. **8**(8), 864–872 (1977)
4. Gao, C., Zhou, S.: On a class of analytic functions related to the starlike functions. Kyungpook Math. J. **45**(1), 123–130 (2005)
5. Goodman, A.W.: Univalent functions, vol. 1. Mariner, Tampa (1983)
6. Goyal, S.P., Singh, O.: Certain subclasses of close-to-convex functions. Vietnam J. Math. **42**(1), 53–62 (2014)
7. Kaplan, W.: Close-to-convex schlicht functions. Michigan Math. J. **1**, 169–185 (1952)
8. Keogh, F.R., Merkes, E.P.: A coefficient inequality for certain classes of analytic functions. Proc. Am. Math. Soc. **20**, 8–12 (1969)
9. Kowalczyk, J., Les-Bomba, E.: On a subclass of close-to-convex functions. Appl. Math. Lett. **23**, 1147–1151 (2010)
10. Rogosinski, W.: On the coefficients of subordinate functions. Proc. London Math. Soc. (2) **48**, 48–82 (1933)
11. Sakaguchi, K.: On a certain univalent mapping. J. Math. Soc. Japan **11**, 72–75 (1959)
12. Seker, B.: On certain new subclass of close-to-convex functions. Appl. Math. Comput. **218**, 1041–1045 (2011)
13. Wang, Z.G., Gao, C.Y., Yuan, S.M.: On certain subclass of close-to-convex functions. Mat. Vesnik **58**(3–4), 119–124 (2006)
14. Wang, Z.G., Gao, C.Y., Yuan, S.M.: On certain subclass of close-to-convex functions. Mat. Vesnik **58**(3–4), 119–124 (2006)
15. Wang, Z.G., Gao, C.Y., Yuan, S.M.: On certain subclass of close-to-convex functions. Acta Math. Acad. Paedagog. Nyházi. (N.S.) **22**(2), 171–177 (2006).
16. Wu, Z.R.: The integral operator of starlikeness and the family of Bazilevic functions. Acta Math. Sin. **27**, 394–409 (1984). (in Chinese) (electronic)