Non-perturbative Aspect of Zero Dimensional Superstring

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ABSTRACT

We discuss the non-perturbative aspect of zero dimensional superstring. The perturbative expansions of correlation functions diverge as \( \sum_l (3l)! \kappa^{2l} \), where \( \kappa \) is a string coupling constant. This implies there are non-perturbative contributions of order \( e^{C\kappa^{-\frac{2}{3}}} \). (Here \( C \) is a constant.) This situation contrasts with those of critical or non-critical bosonic strings, where the perturbative expansions diverge as \( \sum_l l! \kappa^{2l} \) and non-perturbative behaviors go as \( e^{C\kappa^{-1}} \). It is explained how such non-perturbative effects of order \( e^{C\kappa^{-\frac{2}{3}}} \) appear in zero dimensional superstring theory. Due to these non-perturbative effects, the supersymmetry in target space breaks down spontaneously.

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One of most important problems in superstring theories is how their supersymmetries break down. It seems that the spontaneous breakdown of supersymmetry never happens at classical level or perturbative level. In case of bosonic string, Gross and Periwal have proved that the perturbation theory diverges and is not Borel summable. It points out that the perturbative vacuum is unstable and the true vacuum is picked out by non-perturbative dynamics. This suggests that non-perturbative analysis will be necessary in order to solve the problem of the spontaneous breakdown of the supersymmetry in superstring theories. Few years ago, a great progress was made in the non-perturbative formulation of string theories in less than one dimension. These models can be regarded as an important solvable ‘toy’ model, which may give a clue to solve the dynamics of ‘realistic’ string models. In this development, string models which have a supersymmetry in one dimension was proposed by Marinari and Parisi. After that, the author constructed superstring models in less than one dimension and zero dimension. In this paper, we analyze zero dimensional superstring, which is the simplest superstring model. The possibility of the spontaneous breakdown of the supersymmetry has already been suggested in Ref.7. We now discuss the spontaneous breakdown of the supersymmetry by investigating the free energy. The perturbative expansions of correlation functions in zero dimensional superstring theory diverge as $\sum_l (3l)! \kappa^{2l}$. Here $\kappa$ is a string coupling constant. This implies that there will appear the non-perturbative contributions of order $e^{C\kappa^{-\frac{3}{4}}}$. Here $C$ is a constant. This forms a strange contrast to the cases of critical or non-critical bosonic string theories. The perturbative expansion of the free energy $F$ in critical bosonic string is given by $F \sim \sum_l l! \kappa^{2l}$ and we expect the non-perturbative effects of order $e^{C\kappa^{-1}}$, which have been also observed in case of non-critical bosonic strings. Similar non-perturbative effects were also found in Marinari-Parisi’s one dimensional superstring theory. In this paper, it will be explained how the non-perturbative effects of order $e^{C\kappa^{-\frac{3}{4}}}$ appear in zero dimensional superstring theory and how the supersymmetry breaks down spontaneously.

The partition function of zero dimensional superstring theory is given by the
path integrals of an $N \times N$ hermitian matrix $A_{ij}$ ($i, j = 1, \cdots, N$) and fermionic (anti-commuting) $N \times N$ hermitian matrices $\overline{\Psi}_{ij}$, $\Psi_{ij}$ ($i, j = 1, \cdots, N$).

$$Z = (\lambda N)^{-N^2} \int dA d\overline{\Psi} d\Psi \exp \lambda S(A, \overline{\Psi}, \Psi).$$  \hspace{1cm} (1)

The action $S(A, \overline{\Psi}, \Psi)$ has the following form:

$$S(A, \overline{\Psi}, \Psi) = N\left\{-\frac{1}{4} \text{tr}\left(\frac{\partial W(A)}{\partial A}\right)^2 - \frac{1}{2} \sum_{i,j,k,l=1}^{N} \overline{\Psi}_{ij} \Psi_{kl} \frac{\partial^2 W(A)}{\partial A_{ij} \partial A_{kl}}\right\}.$$  \hspace{1cm} (2)

Here $W(A)$ is a superpotential,

$$W(A) = \sum_{l=1}^{L} g_l \text{tr} A^l.$$  \hspace{1cm} (3)

The factor $(\lambda N)^{-N^2}$ in Eq.(1) appears due to the integration of the auxiliary field. The system is invariant under the following supersymmetry transformation in zero dimension:

$$\delta A = \epsilon \Psi + \epsilon \overline{\Psi},$$

$$\delta \Psi = \frac{1}{2} \epsilon \frac{\partial W}{\partial A},$$

$$\delta \overline{\Psi} = -\frac{1}{2} \epsilon \frac{\partial W}{\partial A}.$$  \hspace{1cm} (4)

By using the Nicolai mapping $[9]$,

$$\Gamma = \frac{1}{2} \frac{\partial W(A)}{\partial A},$$  \hspace{1cm} (5)

it has been shown that the invariance under the transformation (4) guarantees that the free energy $F = \ln Z$ of the matrix model, \textit{i.e.}, the vacuum amplitude of the corresponding string theory, vanishes in any order of the perturbation, $F = 0$. In case of $L = \text{odd}$, however, the supersymmetry breaks down spontaneously, and the non-perturbative partition function vanishes, $Z = 0$, and the free energy goes to infinity, $F \to \infty$. 

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In Ref.7, we have found that there exists a critical point by analyzing the correlation functions $\langle \text{tr} A^m \rangle$. The critical point appears in the large $N$ limit when the Nicolai mapping in Eq.(5) is degenerate:

$$\Gamma = -\frac{1}{n}g^{n-1}((g - A)^n + \frac{g}{n}.$$  \hspace{1cm} (6)

Here $g$ is a coupling constant. If we define $x$ by

$$x = \frac{2^2 n^2}{g^2 \lambda},$$  \hspace{1cm} (7)

the correlation functions $\langle \text{tr} A^m \rangle$ has the following form when the Nicolai mapping is degenerate (6),

$$\frac{1}{g^m N} < \text{tr}(g - A)^m > \sim \sum_l c_l N^{-2l} (1 - x)^{-3l + \frac{3}{2} + \frac{m}{n}}.$$  \hspace{1cm} (8)

Therefore, if we fix

$$\kappa^{-1} = N(1 - x)^{3/2}$$  \hspace{1cm} (9)

then by letting $x \to 1$ as $N \to \infty$, we obtain finite correlation functions up to multiplicative renormalization constants to all orders in the $\frac{1}{N}$ (i.e. genus) expansion and $\kappa$ can be regarded as a renormalized string coupling constant. An interesting point is the behavior of the coefficients $c_l$ in Eq.(8). By using the formulae in Ref.7, we can easily find, when $l$ is large,

$$c_l \sim (3l)!.$$  \hspace{1cm} (10)

The perturbative expansions of the correlation functions diverge as $\sum_l (3l)! \kappa^2$. Therefore we expect that there will appear non-perturbative contributions of order $e^{C \kappa^{-\frac{3}{2}}}$. In the following, we consider how such contributions appear in case $L$ in Eq.(3) is odd ($n$ in Eq.(6) is even) i.e., in case the supersymmetry breaks down spontaneously.
When \( L = \) odd, the partition function \( Z \) vanishes, therefore the expectation value of any operator diverges or vanishes in general and the non-perturbative theory is ill-defined. In order to obtain a well-defined theory, we modify the Nicolai mapping in Eq.(6) as follows:

\[
\frac{\partial W(A)}{\partial A} = \Gamma = -\frac{(e^{mg} - e^{mA})^n}{nm(e^{mg} - 1)^{n-1}} + \frac{e^{mg} - 1}{nm}.
\]

This modification does not change the behavior when \( A \sim g^* \) and the modified Nicolai mapping (11) reduces to the previous one in the limit of \( m \to 0 \). When \( A \) goes to \(+\infty\), \( \Gamma \) goes to \(-\infty\) but when \( A \) goes to \(-\infty\), \( \Gamma \) remains to be finite:

\[
A \to -\infty \implies \Gamma \to -\gamma_0
\]

\[
\gamma_0 = \frac{e^{mg}}{nm(e^{mg} - 1)} - \frac{e^{mg} - 1}{nm}
\]

Therefore the partition function of the modified theory is finite when \( m \) is finite.

The modified theory can be regarded as a regularized theory of the original theory (6). The modification (11) does not change the critical properties of the original theory. If we define

\[
\tilde{A} \equiv \frac{e^{mA} - 1}{m}, \quad \tilde{g} \equiv \frac{e^{mg} - 1}{m},
\]

the Nicolai mapping (11) can be rewritten by

\[
\Gamma = -\frac{1}{n\tilde{g}^{n-1}}(\tilde{g} - \tilde{A})^n + \frac{\tilde{g}}{n}.
\]

Then by using the argument given in Ref.7, we can find that the correlation function

* To be exact, when an eigenvalue \( a \) of the matrix \( A \) approaches to \( g \), the behavior of the Nicolai mapping does not change in the leading order w.r.t. \( a - g \) under the modification. The behavior is only relevant to the critical behavior of the correlation functions.
\frac{1}{\hat{g}^{mN}} < \text{tr}(\hat{g} - \hat{A})^m > \text{ shows the same critical behavior as the previous one in Eq.(9):}

\frac{1}{\hat{g}^{mN}} < \text{tr}(\hat{g} - \hat{A})^m > \sim \sum_l \zeta_l N^{-2l}(1 - x)^{-3l + \frac{3}{2} + \frac{m}{n}}.

(15)

Here we have redefined $x$ by

$$x \equiv \frac{2^2 n^2 m^2}{(e^{mg} - 1)^2 \lambda}.$$ 

(16)

Therefore the modified theory belongs to the same universality class as the original one.

Since the modification (11) is given in terms of the Nicolai mapping, i.e., the derivative of superpotential $W(A)$ with respect to $A$, the perturbative supersymmetry corresponding to Eq.(4) remains in the modified theory. The supersymmetry, however, breaks down spontaneously since the finite modified partition function depends on the coupling constants. In the following, we investigate if the breakdown of the supersymmetry remains after the double scaling limit and how the non-perturbative contribution of order $e^{C\kappa - \frac{\lambda}{\mu}}$ appears.

The modified partition function is given by

$$Z = (2\lambda N)^{\frac{N^2}{2}} \int d\Gamma \exp\{-\lambda N \text{tr}\Gamma\Gamma\}
= (2\lambda N)^{\frac{N^2}{2}} \prod_{\gamma_0}^\infty \prod_{n=1}^N d\gamma_n \prod_{m > l} (\gamma_m - \gamma_l)^2 \exp\left(-\frac{1}{2} \sum_{k=1}^N \gamma_k^2\right).

(17)

Here we have diagonalized the matrix $\Gamma$ by the unitary matrix $U$,

$$\Gamma = \frac{1}{\sqrt{2\lambda N}} U^{-1} \gamma U, \quad \gamma = \text{diag}(\gamma_1, \gamma_2, \cdots, \gamma_N).$$

(18)

By changing the variable: $\gamma_i = \frac{1}{2\lambda N \gamma_0} y_i + \gamma_0$, the Equation (17) can be rewritten
by
\[
Z = N^{-\frac{N^2}{2}} (\lambda \gamma_0^2) - N^2 e^{-N^2 \lambda \gamma_0^2} \\
\times \int_0^\infty \prod_{n=1}^N dy_n \prod_{m>t} (y_m - y_l)^2 \exp\left\{-\sum_{k=1}^N (y_k + \frac{y_k^2}{4N\lambda \gamma_0^2})\right\}
\]
\[
= N^{-\frac{N^2}{2}} (\lambda \gamma_0^2) - N^2 e^{-N^2 \lambda \gamma_0^2} \int [d\Phi] \exp\{-\text{tr}\Phi - \frac{\text{tr}\Phi^2}{4N\lambda \gamma_0^2}\}
\]
\[
= N^{-\frac{N^2}{2}} (\lambda \gamma_0^2) - N^2 e^{-N^2 \lambda \gamma_0^2} Z_0 < \exp\{-\frac{\text{tr}\Phi^2}{4N\lambda \gamma_0^2}\} > 0
\] (19)

Here $\Phi$ is a hermitian matrix whose eigenvalues are positive semi-definite. We define $Z_0$ and $< \cdots >_0$ by

\[
Z_0 \equiv \int [d\Phi] \exp\{-\text{tr}\Phi\}, \quad < O >_0 \equiv Z_0^{-1} \int [d\Phi] O \exp\{-\text{tr}\Phi\}.
\] (20)

Since the orthogonal polynomials whose measure is given by $\int_0^\infty dx e^{-x} \cdots$ are Laguerre’s polynomials,

\[
L_n(x) \equiv \sum_{m=0}^n (-1)^m \left(\begin{array}{c} n \\ n - m \end{array}\right) \frac{x^m}{m!},
\] (21)

we can calculate $Z_0$ straightforwardly,

\[
Z_0 = N! \prod_{n=1}^{N-1} (n!)^2.
\] (22)

Furthermore we know the general properties of expectation values

\[
< e^O > \leq e^{< O >} \quad \text{(convex inequality)}
\]

\[
< e^O > = e^{< O > + \frac{1}{2} < O^2 > + \frac{1}{3} < O^3 > + \cdots}
\]

\[
< O^2 >_c \equiv < O^2 > - < O >^2
\]

\[
< O^3 >_c \equiv < O^3 > - 3 < O > < O^2 >_c - < O >^3
\]

\[
\cdots
\] (23)
Therefore if we define
\[
z \equiv \lambda \gamma_0^2 ,
\]
we find
\[
< e^{-\frac{1}{4N^2} \text{tr} \Phi^2} >_0 = e^{-\frac{1}{4N^2} \text{tr} \Phi^2 >_0 + \frac{1}{2} \langle \text{tr} \Phi^2 \text{tr} \Phi^2 \rangle_c + O(z^{-3})} .
\]

Explicit calculation gives
\[
< \text{tr} \Phi^2 >_0 = 2N^3 \quad (\text{no higher order terms w.r.t. } \frac{1}{N}) .
\]

We can also estimate \(< (\text{tr} \Phi^2)^n >_c \) by using the factorization properties,
\[
< (\text{tr} \Phi^2)^n >_c \sim O(\frac{N^{3n-2(n-1)}}{N^n}) \sim O(N^{n+2}) .
\]

Then the partition function is given by
\[
Z = N^{-\frac{n^2}{2}} N! \prod_{n=1}^{N-1} (n!)^2 e^{-N^2 f(z) - g(z)} + O(N^-2)
\]
\[
f(z) = z + \frac{1}{2} \ln z + \frac{1}{2z} + O(z^{-2})
\]
\[
g(z) = O(z^{-2}) .
\]

Due to convex inequality, the function \(f(z)\) is bounded below by
\[
f(z) \geq z + \frac{1}{2} \ln z + \frac{1}{2z} \quad (29)
\]

In the following, we consider how the non-perturbative effect of order \(e^{C\kappa - \frac{\Phi}{4}}\) appears.

We begin with counting how many parameters (coupling constants) this theory has. We have three parameters \((\lambda, g, m)\) at first but one of these parameters are redundant since we can redefine (or rescale) the matrix field \(A\) by \(A \rightarrow e^t A\).
(t is a parameter of rescaling. Two parameters which are invariant under this redefinition are given by \((x, z)\) in Eqs.(16) and (24). Furthermore we know that the universality class does not change if we vary the parameter \(m\). Therefore there will be one more redundant parameter which we denote by \(\Lambda\). Since \(x\) is apparently a parameter specifying the theory, \(z\) is a function of \(x\) and \(\Lambda\) in general:

\[ z = z(x, \Lambda) . \]  

(30)

Since \(x = 1 - (N\kappa)^{-\frac{2}{3}}\) (Eq.(9)), we can expand \(N^2 f(z)\) in Eq.(28) when \(N\) is large:

\[ N^2 f(z(x, \Lambda)) = N^2 f(z(1, \Lambda)) + N^{\frac{4}{3}} \partial_x f(z(1, \lambda)) \partial_x z(1, \Lambda) \kappa^{-\frac{2}{3}} + \cdots \]  

(31)

The first term \(N^2 f(z(1, \Lambda))\) is essentially c-number since this term only depends on the redundant parameter \(\Lambda\). Therefore this term can be absorbed into the renormalization of the matrix field \(A\) and does not contribute to the expectation value of any operator. The second term \(N^{\frac{4}{3}} \partial_x f(z(1, \lambda)) \partial_x z(1, \Lambda) \kappa^{-\frac{2}{3}}\), however, has a physical meaning. We can make this term finite by adjusting the redundant parameter \(\Lambda\). Therefore we can find that a non-perturbative contribution of order \(e^{\kappa^{-\frac{2}{3}}}\) appears in the partition function,

\[ Z \sim e^{C \kappa^{-\frac{2}{3}}} . \]  

(32)

There is an ambiguity or freedom how to choose the redundant parameter \(\Lambda\). For example, we can choose \(\Lambda\) by

\[ z = c_1 - c_2 \frac{x}{\Lambda} . \]  

(33)

here \(c_1\) and \(c_2\) are arbitrary constants. If we adjust \(\Lambda \sim N^{\frac{2}{3}}\), we obtain a finite partition function and a finite free energy. Since the free energy does not vanish, the supersymmetry in zero dimensional target space breaks down spontaneously.
In summary, we have investigated the non-perturbative aspect of zero dimensional superstring. For this purpose, we have proposed a kind of regularization which is given in Eq.(11). This regularization has following properties.

1) This regularization makes the partition function to be finite when $N$ is finite.

2) The regularized model belongs to the same universality class as the original model.

3) This regularization keeps the supersymmetry perturbatively.

3) This regularization breaks the supersymmetry non-perturbatively.

By using this regularization, it has been explained how the non-perturbative contributions of order $e^{C\kappa^{-\frac{1}{2}}}$ appear. This contribution is consistent with the perturbative expansions of the correlation functions $\sim \sum_l (3l)!\kappa^{2l}$. Due to these contributions, the free energy does not vanish and the supersymmetry in zero dimensional target space breaks down spontaneously. These results do not depend on the details of the regularization. Any regularization which has the properties 1)–2) apparently gives the same results.

The zero dimensional superstring theory is closely related to the bosonic two dimensional gravity coupled with $c = -2$ conformal matter ($-2$ dimensional string theory).\textsuperscript{12–16} The $-2$ dimension string theory can be obtained from zero dimensional string theory by Parisi and Sourlas’s dimensional reduction mechanism.\textsuperscript{17} Parisi and Sourlas’s mechanism connects $D$ dimensional theory to $D - 2$ dimensional one. The correlation functions of $D$ dimensional theory, except vacuum amplitude, are identical with those of $D - 2$ dimensional one if the support of $D$ dimensional correlation functions is restricted to $D - 2$ hypersurface. Of course, this does not mean that these two theories are equivalent. We expect, however, the non-pertubative effects observed in this paper will also appear in the bosonic strings in $-2$ dimensions.

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