Higher order mechanics on graded bundles

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Abstract
In this paper we develop a geometric approach to higher order mechanics on graded bundles in both, the Lagrangian and Hamiltonian formalism, via the recently discovered weighted algebroids. We present the corresponding Tulczyjew triple for this higher order situation and derive in this framework the phase equations from an arbitrary (also singular) Lagrangian or Hamiltonian, as well as the Euler–Lagrange equations. As important examples, we geometrically derive the classical higher order Euler–Lagrange equations and analogous reduced equations for invariant higher order Lagrangians on Lie groupoids.

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1. Introduction

It is certainly true that higher derivative theories have received less mathematical attention than first order theories and for some sound reasons. Recall the famous Ostrogradski theorem;

If a higher order time derivative Lagrangian is non-degenerate, then there is at least one linear instability in the Hamiltonian of this system.

In this context non-degeneracy means that the highest derivative term can be expressed in terms of canonical variables. The Ostrogradski instability leads to the fact that the associated Hamiltonian is not bounded from below. This by itself is not a problem as classically one can only measure energy differences. The difficulty lies in including quantum effects of
continuum theories with interactions. In such quantum field theories an ‘empty’ state would spontaneously decay into a collection of positive and negative energy particles, in accordance with energy conservation. Moreover, there are typically states with non-positive norm known as ghosts that require exorcising from the physical theory. Other problems include the presence of extra degrees of freedom and the well-posedness question of an initial-value formulation of the equations of motion. Such issues are of course not a really problem if we consider the theory to be an effective theory; at some scale ‘new physics’ enters the picture and this not dependent on higher order derivatives. Or in other words the effective theory is a kind of truncation of a complete theory that is first order. Indeed, the main source of higher order Lagrangians in particle physics and cosmology is via effective theories. In fact for effective theories one can use the equations of motion to reduce the order and reduce the theory to first order. For mechanical systems, higher order theories similarly arise as phenomenological models via various assumptions made in the modelling. In the context of classical mechanics, higher derivative theories will exhibit pathological derivations from Newton’s laws, but as we do not have external interactions here with continuous degrees of freedom they are not fundamentally sick as higher derivative quantum field theories. In short, higher derivative theories are important in physics even if they cannot be considered as fundamental theories. For a very nice review of the Ostrogradski instability the reader can consult [50].

One important technical distinction here is that higher derivative theories with degenerate Lagrangians are healthy in the sense that they do not suffer with the Ostrogradski instability. Degeneracy is an interesting feature of physical models as all theories that possess continuous local symmetries are degenerate, independently of the order. A closely related subject is the reduction of Lagrangian systems invariant under the action of some group(oid).

The challenge of describing mechanical systems on Lie groupoids and their reduction to Lie algebroids was first posted by Weinstein [49]. Many authors took up this challenge, for example see [4, 9, 10, 16, 18, 36, 40] where various approaches have been developed. The notion of the Tulczyjew triple for Lie algebroids, as we shall understand it, was first given in [16]. However, we wish to point out [36], where a different manifestation of the Tulczyjew triple for Lie algebroids is presented.

Extending the geometric tools of the Lagrangian formalism on tangent bundles to Lie algebroids was motivated by the fact that reductions usually push one out of the environment of tangent bundles. In a similar way, reductions of higher order tangent bundles, which is where higher order mechanical Lagrangians ‘live’, will push one into the environment of ‘higher Lie algebroids’.

Thus, general geometric methods that can handle both non-degenerate and degenerate higher order Lagrangians are very desirable in mathematical physics. In this paper we show how one can use weighted skew/Lie algebroids as described in [2] to achieve this goal in the context of the Lagrangian formulation of classical mechanics. Furthermore, we complete the Tulczyjew triple and present the complimentary Hamiltonian formalism. The mechanical systems we will be dealing with will be rather general, we will make no assumption about the degeneracy for example, and the generalised higher order velocities will take their values in graded bundles. Graded bundles are a natural higher order generalisation of the notion of a vector bundle, and were first defined and studied by Grabowski & Rotkiewicz in [27]. The graded bundles have already been used for a geometric formulation of some classical field theories in [20]. A canonical example of a graded bundle is the kth order tangent bundle and so our constructions can handle standard higher order Lagrangians. A geometric understanding of Lagrangian mechanics on higher tangent bundles close to our way of thinking was developed by de Leon & Lacomba [35] and independently by Crampin [11] (see also [8]).
For a geometric treatment of higher order systems of autonomous ordinary differential equations in terms of higher order tangent bundles the reader should consult [12]. A Tulczyjew triple approach to some higher order field theories can be found in [21].

A little more exotically, we show that higher order Lagrangian mechanics on Lie algebroids can naturally be accommodated within our framework. The study of such systems is motivated by the study of higher order systems on Lie groupoids with Lagrangians invariant under the groupoid multiplication. However, Lagrangian systems on Lie algebroids need not arise as the reduction of Lagrangian systems on Lie groupoids, and indeed not all Lie algebroids integrate to a Lie groupoid. Interestingly, higher derivative mechanics on Lie groupoids has received little attention in the literature. In fact we are only aware of three works in this direction [6, 32, 41], the latter has appeared after we had written the first version of this paper. The Lie group case has similarly not received a lot of attention; see for example [7, 14]. For structure-preserving numerical integration schemes for a class of higher order mechanical systems on Lie groups, which we do not discuss in this paper, see [3].

We do not consider the development of higher order mechanics on Lie groupoids and algebroids as a purely academic exercise. Recall that many interesting mechanical systems can be understood as the Euler–Poincaré equations on a Lie algebra. Similarly, a Lagrangian on a principal $G$-bundle that is invariant under the action of $G$ leads to Lagrangian system on the associated Atiyah bundle, which is canonically a Lie algebroid. In this paper we will provide a rather general setup that allows higher order versions of the Euler–Poincaré and Lagrange–Poincaré equations to appear geometrically. We certainly envisage applications of this geometric setup in the fields of engineering and the physical sciences. For example, there has been a renewed interest in higher order mechanics due to recent applications of optimal control theory in the biomedical sciences. In particular, the problem of trajectory planning and interpolation by variational curves requiring not only the simultaneous optimisation of the velocity and the acceleration, but also the higher order derivatives has found application in the emerging field of computational anatomy; see [14].

Our approach to higher order mechanics using weighted algebroids makes use of first order mechanics on Lie algebroids subject to affine (vakonomic) constraints [18]. The higher order flavour arises as underlying a weighted algebroid is a graded bundle. This mimics the approach to higher order mechanics on $T^k M$ by studying first order mechanics on $T(T^{k-1}M)$ and then using the natural embedding as a constraint.

As our approach is geometric we will understand the phase dynamics as implicit dynamics; this is quite standard for vakonomic constraints. That is the phase dynamics is a subset of the tangent bundle of the phase space of the system in question. Quite often people are satisfied with just the Euler–Lagrange equations rather than the full phase dynamics. However, the Euler–Lagrange equations, being only a ‘shadow’ of phase dynamics (Lagrange equations), do not carry the important information on how momenta are associated with velocities. Also, their solutions can come by gluing different phase solutions, therefore, apart from ‘good’ cases, they are physically unsatisfactory.

We remark that various higher order versions of Lie algebroids besides weighted algebroids have appeared in the literature. First we must mention the (prototype) higher Lie algebroids as defined by Jóźwikowski & Rotkiewicz [32], which are a direct generalisation of the kappa-relation $\kappa:TE \longrightarrow TE$ for Lie algebroids. Although this approach is motivated by geometric mechanics, it seems not to be quite suitable for the approach pursued in this paper.

Secondly, there is the more established notion of a higher or nonlinear Lie algebroid as defined by Voronov [47, 48] in terms of a weight one homological vector field on a non-negatively graded supermanifold. As it stands, it is not clear that this notion is related to the reduction of higher tangent bundles on Lie groupoids nor how it can be applied in geometric
mechanics. As we can view weighted algebroids as a special class of Voronov’s nonlinear algebroids (cf [2]) the link with higher order tangent bundles is clear. Thus in part, this paper establishes a link between geometric mechanics and nonlinear algebroids via weighted algebroids.

**Arrangement of paper:** In section 2 we briefly recall the necessary parts of theory of graded bundles, including the notion of the linearisation functor as needed in this work. We also present the bare minimum of the theory of weighted algebroids as tailored to the needs of the current paper. For a more complete account the reader is urged to consult [2]. In essence this first section if a review of the graded geometry needed in this paper is a review. In section 3 we recall the aspects affine phase spaces and vakonomic constraints necessary for the later sections of this paper. The new elements of this paper are to be found in section 4, where we present the Lagrangian and Hamiltonian formalism on graded bundles via weighed algebroids, as well as the corresponding Tulczyjew triple. We then look at an application of our formalism to higher order mechanics on a Lie algebroid in section 5. In particular we explicitly construct the higher order Euler–Lagrange equations for a higher order Lagrangian system on a Lie algebroid. As particular examples, we geometrically derive the second order Euler–Poincaré and Lagrange–Poincaré equations.

## 2. Graded bundles and weighted algebroids

In this section we recall parts of the theory of graded bundles, the linearisation functor and weighted algebroids as needed in later sections of this paper. The interested reader should consult the original literature [2, 26, 27] for details. A survey on related geometric brackets can be found also in [23].

### 2.1. Graded bundles

The general theory of graded supermanifolds in our understanding was initiated by Voronov in [47]. We will restrict our attention to just manifolds in this paper and will not deal with supermanifolds at all. The reason for this is rooted in our applications rather than any fundamental geometric reasons.

An important class of such manifolds are those that carry non-negative grading. We will furthermore require that this grading is associated with a smooth action \( h : \mathbb{R} \times F \to F \) of the monoid \((\mathbb{R}, \cdot)\) of multiplicative reals on a manifold \(F\); a homogeneity structure in the terminology of [27]. This action reduced to \(\mathbb{R}_{>0}\) is the one-parameter group of diffeomorphism integrating the weight vector field, thus the weight vector field is in this case \(h\)-complete [24] and only non-negative integer weights are allowed. Thus the algebra \(\mathcal{A}(F) \subset C^\infty(F)\) spanned by homogeneous functions is \(\mathcal{A}(F) = \bigoplus_{\mathbb{N}} \mathcal{A}(F)\).

Importantly, we have that for \( t \neq 0 \) the action \( h_t \) is a diffeomorphism of \( F \) and when \( t = 0 \) it is a smooth surjection \( \tau = h_0 \) onto \( F_0 = M \), with the fibres being diffeomorphic to \( \mathbb{R}^N \) (cf [27]). Thus, the objects obtained are particular kinds of polynomial bundles \( \tau : F \to M \), i.e. fibrations which locally look like \( U \times \mathbb{R}^N \) and the change of coordinates (for a certain choice of an atlas) are polynomial in \( \mathbb{R}^N \). For this reason graded manifolds with non-negative weights and \(h\)-complete weight vector fields are also known as graded bundles [27]. If the weight is constrained to be either zero or one, then the weight vector field is precisely a vector bundle structure on \( F \) and will be generally referred to as an Euler vector field. The principle canonical example of a graded bundle is the higher tangent bundle \( T^kM \); i.e. the \( k \)th jets (at zero) of curves \( \gamma : \mathbb{R} \to M \).
One can always pick an affine atlas of $F$ consisting of charts for which we have homogeneous local coordinates $(x^A, y^a_w)$, where $w(x^A) = 0$ and $w(y^a_w) = w$ with $1 \leq w \leq k$, for some $k \in \mathbb{N}$ known as the degree of the graded bundle. It will be convenient to group all the coordinates with non-zero weight together. The index $a$ should be considered as a ‘generalised index’ running over all the possible weights. The label $w$ in this respect largely redundant, but it will come in very useful when checking the validity of various expressions. The local changes of coordinates respect the weight and hence are polynomial for non-zero weight coordinates. Moreover, morphisms of graded bundles must respect the weight of the local coordinates.

A graded bundle of degree $k$ admits a sequence of polynomial fibrations, where a point of $F_l$ is a class of the points of $F$ described in an affine coordinate system by the coordinates of weight $\leq l$, with the obvious tower of surjections

$$F = F_k \xrightarrow{\iota^1_1} F_{k-1} \xrightarrow{\iota^1} \cdots \xrightarrow{\iota^1} F_2 \xrightarrow{\iota^1} F_1 \xrightarrow{\iota^1} F_0 = M,$$

where the coordinates on $M$ have zero weight. Note that $F_l \to M$ is a linear fibration and the other fibrations $F_l \to F_{l-1}$ are affine fibrations in the sense that the changes of local coordinates for the fibres are linear plus and additional additive terms of appropriate weight. The model fibres here are $\mathbb{R}^n$.

There is also a ‘dual’ sequence of submanifolds and their inclusions

$$M : = F_0 = F^{(k)} \subseteq F^{(k-1)} \subseteq \cdots \subseteq F^{(0)} = F_{k},$$

where we define, locally but correctly,

$$F^{[i]} := \{ p \in F_k | y^a_w = 0 \text{ if } w \leq i \}.$$  

It will also be useful to consider the submanifold $\bar{F}_l : = F^{[l-1]}$ which is in fact linearly fibred over $M$, with the linear coordinates carrying weight $l$. The module of sections of $\bar{F}_l$ is identified with the $C^\infty(M)$-module $\mathcal{A}(F)$.

The notion of a double vector bundle [44] (or a higher n-tuple vector bundle) is conceptually clear in the graded language in terms of mutually commuting weight vector fields; see [26, 27]. This leads to the higher analogues known as n-tuple graded bundles, which are manifolds for which the structure sheaf carries an $\mathbb{N}^n$-grading such that all the weight vector fields are h-complete and pairwise commuting. The local triviality of n-tuple graded bundles means that we can always equip an n-tuple graded bundle with an atlas such that the charts consist of coordinates that are simultaneously homogeneous with respect to the weights associated with each weight vector field (cf [27]). The changes of local coordinates must respect the weights. Similarly, morphisms between n-tuple graded bundles respect the weights of the local coordinates.

To set some useful notation, if $\Delta_M$ is a weight vector field on $M$, then we denote by $\mathcal{M}[\Delta_M \leq l]$ the base manifold of the locally trivial fibration defined by taking the weight $>l$ coordinates with respect to this complimentary weight to be the fibre coordinates. We have a natural projection that we will denote as

$$p^M_{\mathcal{M}[\Delta_M \leq l]} : \mathcal{M} \to \mathcal{M}[\Delta_M \leq l].$$

**Definition 2.1.1.** A double graded bundle $(D_k, \Delta^1, \Delta^2)$ such that $\Delta^1$ is of degree $k - 1$ and $\Delta^2$ is of degree 1, i.e. $\Delta^2$ is an Euler vector field, will be referred to as a graded-linear bundle of degree $k$, or for short a $\mathcal{GL}$-bundle.
It follows that $D_k$ is of total weight $k$ with respect to $\Delta := \Delta^1 + \Delta^2$ and a vector bundle structure

$$p^{D_k}_{B_{k-1}} : D_k \to B_{k-1}.$$ 

with respect to the projection onto the submanifold $B_{k-1} := D_k[\Delta^2 \leq 0]$ which inherits a graded bundle structure of degree $k - 1$.

The tangent bundle $TF_k$ of a graded bundle naturally has the structure of a double graded bundle. The first weight vector field is simply the tangent lift [29] of the weight vector field in $F_k$ and the second being the natural Euler vector field on the tangent bundle.

Consider the vertical bundle with respect to the projection $\tau : F_k \to M$. The weight vector fields on the vertical bundle $VF_k$ are simply the appropriate restrictions of those on the tangent bundle. Via passing to the total weight we can view $VF_k$ as a graded bundle of degree $k + 1$. However, it will be useful to shift the first component of bi-weight to allow us to consider the vertical bundle as a graded bundle of degree $k$.

Similarly to the case of the tangent bundle, the cotangent bundle $T^*F_k$ of a graded bundle also comes naturally with the structure of a double graded bundle. However, simply using the naturally induced weight would mean that the ‘momenta’ will have a negative component of their bi-degree and so the associated weight vector field cannot be complete; we would not remain in the category of graded bundles. Instead one needs to consider a phase lift of the weight vector field $\Delta_F$ (cf [24]). The $k$-phase lift of the weight vector field $\Delta_F$ essentially produces a shift in the induced bi-weight to ensure that everything is non-negative: it amounts to the following choice of homogeneous coordinates

$$\pi^{A,B}_{(0,0)}(x,y,z), \pi^{B,B}_{(n,0)}(y,0), \pi^{B}_{(1,1)}, \pi^{k-w+1}_{(k-w,1)}.$$ 

The linearisation functor takes a graded bundle and produces a double grade bundle for which the two side arrows are vector bundles. The basic idea is to mimic the canonical embedding $T^2M \subset T(T^{k+1}M)$ via ‘holonomic vectors’.

**Definition 2.1.2.** The linearisation of a graded bundle $F_k$ is the $GL$-bundle defined as

$$D(F_k) := VF_k\left[\Delta_{VF} \leq k - 1\right],$$

so that we have the natural projection $p^{VF}_{D(F_k)} : VF_k \to D(F_k)$.

Let us briefly describe the local structure of the linearisation. Consider $F_k$ equipped with local coordinates $(x^A, y^a, z^b)$, where the weights are assigned as $w(x) = 0$, $w(y) = w(1 \leq w < k)$ and $w(z) = k$. It will be convenient to single out the highest weight coordinates as well as the zero weight coordinates. In any natural homogeneous system of coordinates on the vertical bundle $VF_k$ one projects out the highest weight coordinates on $F_k$ to obtain $D(F_k)$. One can easily check in local coordinates that doing so is well-defined. Thus on $D(F_k)$ we have local homogeneous coordinates

$$\left(\tilde{x}^A, \frac{y^a}{\pi^{B}_{(1,1)}}, \frac{y^b}{\pi^{B}_{(k-1,1)}}, \frac{z^b}{\pi^{B}_{(k-1,1)}}\right).$$

(2.3)

Note that with this assignment of the weights the linearisation of a graded bundle of degree $k$ is itself a graded bundle of degree $k$ when passing from the bi-weight to the total weight. It is important to note that the linearisation has the structure of a vector bundle $D(F_k) \to F_{k-1}$.
hence the nomenclature ‘linearisation’. The vector bundle structure is clear from the construction by examining the bi-weight. In [2] it was shown that there is a canonical (total) weight preserving embedding

$$i : \tilde{F}_k \hookrightarrow D(F_k),$$

given by the image of the weight vector field $\Delta_F \in \text{Vect}(F)$ considered as a geometric section of $VF_k$. In natural local coordinates the nontrivial part of the embedding is given by

$$i^w(\tilde{x}_w^a) = w \tilde{x}_w^a, i^w(\tilde{z}_k^i) = k \tilde{z}_k^i.$$

Elements of $i(F_k)$ we refer to as holonomic vectors in $D(F_k)$.

**Definition 2.1.3.** The Mironian of a graded bundle is the double graded bundle defined as

$$\text{Mi}(F_k) := D^w(F_k) \left[ \Delta^1_{D^w(F)} + k \Delta^2_{D^w(F)} \leq k \right]$$

It is not hard to see that the local coordinates on the Mironian inherited from the coordinates on $D^w(F_k)$ are

$$\left( x^A, y^a_i, \pi_{\dot{i}} \right), \quad (0,0)^{(0,0)}, (0,1)^{(0,1)}$$

and so the Mironian of $F_k$ has the structure of a vector bundle over $F_{k-1}$. Moreover, we have the identification $\text{Mi}(F_k) \simeq F_{k-1} \times_M F_k^*$. 

**Example 2.1.4.** If $F_k = T^kM$, then $\text{Mi}(T^kM) = T^{k-1}M \times_M T^kM$.

**Remark 2.1.5.** To our knowledge the above example first appeared in the works of Miron, see [42]. His motivation, much like ours, was to develop a good notion of higher order Hamiltonian mechanics and this requires some notion of a dual of $T^kM$. Our more general notion of the Mironian of a graded bundle will similarly play a fundamental role in our formulation of higher order Hamiltonian mechanics.

Amongst all the possible fibrations of $T^kF_k$ we will use the following double graded structure

$$T^*F_k \xrightarrow{\pi_{T^k}} \text{Mi}(F_k) \xrightarrow{\pi_{T^k}} T^*F_k^* \xrightarrow{\pi_{T^k-1}} F_{k-1}$$

(2.4)

2.2. Weighted algebroids for graded bundles

We are now in a position to describe the main geometric object needed to define mechanics on a graded bundle. A weighted Lie algebroid should be understood as a Lie algebroid [16, 18, 31] carrying a compatible weight. For details the reader is urged to consult [2].
Definition 2.2.1. A weighted algebroid for $F_k$ is a morphisms of triple graded bundles

$$\epsilon: T^*D(F_k) \rightarrow TD^*(F_k)$$

covering the identity on the double graded bundle $D^*(F_k)$. The anchor map is the map $\rho: D(F_k) \rightarrow TF_{k-1}$ underlying the map $\epsilon$, see the diagram below. The anchor map induces a graded morphism $\hat{\rho} := \rho \circ \iota: F_k \rightarrow TF_{k-1}$ called the anchor map on $F_k$ or the $k$th anchor map. A graded bundle $F_k$ equipped with a graded morphism $\hat{\rho}: F_k \rightarrow TF_{k-1}$ we will call an anchored graded bundle. A weighted algebroid is this specified by the pair $(F_k, \epsilon)$. In this case we will also say that $F_k$ carries the structure of a weighted algebroid.

![Diagram](image)

The morphism (2.5) is known to be associated with a 2-contravariant tensor field $\Lambda_\epsilon$ on $D^*(F_k)$. If $\Lambda_\epsilon$ associated with $\epsilon$ is a bi-vector field on $D^*(F_k)$, we speak about a weighted skew algebroid for $F_k$. If the bi-vector field is a Poisson structure then we have a weighted Lie algebroid for $F_k$. We will from this point on focus on weighted skew/Lie algebroid structures as our examples will be based on these structures and they seem general enough to cover a wide range of potentially interesting situations.

We will adopt the following system of weight symbols and corresponding systems of coordinate indices on $F_k$ in order to condense our notation:

| weight | index |
|--------|-------|
| $0 \leq u \leq k-1$ | $\mu, \nu$ |
| $1 \leq U \leq k$ | $I, J$ |
| $0 \leq W \leq k$ | $\alpha, \beta$ |
| $1 \leq w \leq k-1$ | $a, b$ |
| $0$ | $A$ |
| $k$ | $i$ |

By convention the weight symbol will refer to the total weight of the given coordinate. Thus, we employ $X^u_\mu = (x^A, y^a_\mu)$ as coordinates on $F_{k-1}$, and $Y^U_I = (z^\alpha_I, z^a_I)$ as the linear coordinates on $D(F_k)$. Finally we shall employ adapted homogeneous coordinates

$$\left( X^u_\mu, Y^U_I, \sigma^{k+1-u}_u, \Pi^U_{j+1-I} \right)$$

on $T^*D(F_k)$. Similarly, on $TD^*(F_k)$ we employ adapted homogeneous local coordinates

$$\left( X^u_\mu, \Pi^U_I, \delta X^V_{u+1}, \delta \Pi^U_{i+1} \right)$$

In these coordinates, the map $\epsilon$, being the identity on $(X^u_\mu, \Pi^U_I)$, can be written in the compact form
\[ \delta X^\mu_j \circ e = \rho [u]^j_\mu (X) Y^l_{U-u}, \]
\[ \delta \Pi_{j}^{U+1} \circ e = \rho [u]_j^k (X) P^U_{\nu} + C [u]^{k}_{ij} (X) Y^l_{U-k} \Pi_{k}^{U+1-U-u}, \]  \hspace{1cm} (2.7)

where \( \rho [u] \) and \( C [u] \) are homogeneous parts of the structure functions of degree \( u = 0,...,k - 1 \).

**Example 2.2.2.** Consider the \( k \)th order tangent bundle of a manifold \( T^k M \). As proved in [2], \( T^k M \) carries the structure of a weighted Lie algebroid. Let us examine this structure explicitly. First, as we have already seen, \( D(T^k M) \cong T(T^{k-1} M) \) and thus \( D^*(T^k M) \cong T^*(T^{k-1} M) \). Then the weighted Lie algebroid for \( T^k M \) is a morphism between the triple graded bundles

\[ T^*(T^{k-1} M) \longrightarrow T(T^k M). \]

Let us now employ homogeneous \( \{ X^\mu_\nu, Y^\nu_U \} \) on \( D(T^k M) \). The weighted Lie algebroid structure is then specified by

\[ \delta X^\mu_j \circ e = \delta^\mu_j Y^\nu_U, \quad \text{and} \quad \delta \Pi_{j}^{U+1} \circ e = \delta^j Y^\nu_{U+1}, \]

in the notation introduced above.

**Example 2.2.3.** Let \( G \rightrightarrows M \) be an arbitrary Lie groupoid with source map \( s : G \to M \) and target map \( t : G \to M \). There is also the inclusion map \( \iota : M \to G \) and a partial multiplication \( (g, h) \mapsto g \cdot h \) which is defined on \( G^{(2)} = \{(g, h) \in G \times G : s(g) = t(h)\} \). Moreover, the manifold \( G \) is foliated by \( s \)-fibres \( G_s = \{ g \in G : s(g) = x \} \), where \( x \in M \). By as definition the source (and target) map is a submersion, the \( s \)-fibres are themselves smooth manifolds. Geometric objects associated with this foliation will be given the superscript \( \bar{s} \). For example, the distribution tangent to the leaves of the foliation will be denoted by \( T G_s \). Let us now consider the \( k \)th order version of this, that is the subbundle \( T^k G_s \subset T^k G \) consisting of all higher order velocities tangent to some \( s \)-leaf \( G_s \). That is we identify \( T^k G_s \) with the union of \( T^k G_{x} \) over all \( s \)-leaves \( G_s \). The relevant graded bundle (over \( \iota(M) \cong M \)) here is (cf [32])

\[ F^k_s = \mathcal{A}^k(G) := T^k G_s \bigg|_{\iota(M)}, \]

which of course inherits its graded bundle structure as a substructure of \( T^k G \) with respect to the projection \( \iota_s : \mathcal{A}^k(G) \to M \). The claim is that \( \mathcal{A}^k(G) \) canonically carries the structure of a weighted Lie algebroid, this was first proved in [2]. To see this weighted Lie algebroid we first need the linearisation. In [2] it was shown that the linearisation of \( \mathcal{A}^k(G) \) is given by

\[ D \big( \mathcal{A}^k(G) \big) = \left\{ (Y, Z) \in \mathcal{A}(G) \times_M T \mathcal{A}^{k-1}(G) \bigg| \ \tilde{\rho}(Y) = \iota \tau (Z) \right\}, \]

viewed as a vector bundle over \( \mathcal{A}^{k-1}(G) \) with respect to the obvious projection of part \( Z \) onto \( \mathcal{A}^{k-1}(G) \), where \( \tilde{\rho} : \mathcal{A}(G) \to TM \) is the standard anchor of the Lie algebroid and \( \tau : \mathcal{A}^{k-1}(G) \to M \) is the obvious projection. Note that this object is a canonical example of a Lie algebroid prolongation [4, 40, 43]. The notion of a prolongation of a Lie groupoid and its relation with Lie algebroid prolongations can be found in [45].

We shall pick local coordinates \( X^\mu_u = (x^A, y^\nu_u) \) on \( \mathcal{A}^{k-1}(G) \) and the corresponding linear coordinates \( Y^\mu_U = (y^\nu_U, y^\nu_{U+1}) \) on the linearisation \( D(\mathcal{A}^k(G)) \). Then, with the above choice of coordinates, the natural adapted coordinates on \( T^k D(\mathcal{A}^k(G)) \) are \( (X^\mu_u, Y^\nu_U, P^U_{\nu} \theta^{k-1-u}, \Pi^U_{U+1} \theta^{U+1-u}) \) and similarly let us employ local coordinates \( (X^\mu_u, \Pi^U_{U+1}, \delta X^\mu_U, \delta \Pi^U_{U+1}) \) on \( T D^*(\mathcal{A}^k(G)) \).
The weighted Lie algebroid structure is given by
\[
\delta X^i \circ c = \rho \left[ 0 \right]_j^i \alpha x i j Y^k, \\
\delta \Pi_{ij} \circ c = \rho \left[ 0 \right]_{ij}^{wc} \alpha x i j P^{l+1} + \delta^K_{ij} C \left[ 0 \right]^K_{ij} Y^l \Pi^{ij},
\]
where \( \rho \left[ 0 \right]_j^i = (\rho^a_i(x), \delta^b_c) \) and \( C \left[ 0 \right]_{ij}^K = C_{ab}^c(x) \). Here \( (\rho^a_i(x), C_{ab}^c(x)) \) are the structure functions of the Lie algebroid \( \mathcal{A}(G) \).

**Example 2.2.4.** A particular case of the above construction is the case of a Lie group \( G = G \). Then, we can identify \( G \) with \( g \times g \), where \( g \) is the Lie algebra of \( G \), and \( D(\mathcal{A}(G)) \) with \( g \times T_{g^{-1}} \) viewed as a vector bundle over \( g \). Here, \( g[i] \) is the space \( g \) with \( i \)th shift in the grading, so that linear functions on \( g \) have weight \( i \).

**Example 2.2.5.** Another particular case is the case of a pair groupoid \( G = M \times M \). It is easy to see that \( \mathcal{A}(M \times M) = T^1M \).

**Remark 2.2.6.** The weighted Lie algebroid \( D(\mathcal{A}(G)) \), without reference to any graded structure, first appeared in the works of Martínez [40] and Cariñena & Martínez [4], which they referred to as the prolongation of a Lie algebroid. A general prolongation has also been considered by Popescu and Popescu [43]. The motivation for these works was to understand geometric mechanics on Lie algebroids. Here we see that the prolongation of a Lie algebroid naturally appears in the context of weighted algebroids as do the ‘higher order’ prolongations.

Actually, as was shown in [32], the graded bundle \( \mathcal{A}(G) \) is completely determined by the Lie algebroid \( \mathcal{A} = \mathcal{A}(G) \), or better to say, by the anchored bundle structure of \( \tau : A \to M \). Moreover, in [32] the following iterative procedure to construct \( \mathcal{A}^k \) for an anchored bundle was presented:

\[
A^2 = \{ Z \in TA \mid \rho \circ \tau(Z) = T\tau(Z) \},
\]

\[
A^{k+1} = TA^k \cap T^kA \quad \text{for} \quad k \geq 2.
\]

In the latter condition we clearly understand \( TA^k \) (inductively) and \( T^kA \) as subsets of \( TT^{k-1}A \).

It is easy to see (cf [18 section 2]) that \( A^2 \) is the subbundle of \( TA \) of first jets (tangent prolongations) of admissible curves in the anchored bundle \( A \), i.e. curves satisfying

\[
\rho \circ \gamma = t(\tau \circ \gamma),
\]

where \( t \) denotes the tangent prolongation. In [18], \( A^2 \) was denoted \( T^1holA \) and its elements were called holonomic vectors. The set of \( A \)-holonomic vectors \( T^1holA \) can be equivalently characterized as the subset in \( TA \) which is mapped \( \text{via} \ T\rho : TA \to TM \) to classical holonomic vectors \( T^2M \), that justifies this name. In other words,

\[
A^2 = (T\rho)^{-1}(T^2M).
\]

The inductive definition gives easily the following.
Theorem 2.2.7. If $A$ is an anchored bundle, then $A^k$, $k \geq 1$, is the bundle of $(k-1)$-jets of admissible curves in $A$. Alternatively,

$$A^k = \left(T^{k-1} \rho\right)^{-1}(T^kM),$$

(2.12)

where we view $T^{k-1} \rho$ as the map

$$T^{k-1} \rho : T^{k-1}A \to T^{k-1}T^1M \cong TT^{k-1}M.$$

(2.13)

Proof. Indeed, the inductive characterization of $A^{k+1}$ (2.11) tells us that the elements of $A^{k+1}$ are jets of those $k$th tangent prolongations of curves in $A$ which project onto $(k-1)$th jets of admissible curves, thus are jets of admissible curves.

The inductive proof of (2.12) follows easily from the fact, for $k \geq 2$, that the map (2.13) is a part of

$$TT^{k-2} \rho : TT^{k-2}A \to T\left(TT^{k-2}M\right)$$

for which $TA^{k-1}$ can be characterized as the inverse image of $TT^{k-1}M$. Hence,

$$T^{k}A \cap TA^{k-1} = T^{k}A \cap \left(\left(TT^{k-2} \rho\right)^{-1}(TT^{k-1}M)\right) = \left(T^{k-1} \rho\right)^{-1}(T^{k}M).$$

Note we have made no reference to any Lie groupoid structure in this construction. Having defined $A^k$, we can use (2.8) to obtain $D(A^k)$ (just forgetting $G$).

The situation where we have some additional structure on $A$ is of course more interesting. If we suppose that $A$ is in fact a Lie algebroid, not necessarily integrable, then the $\mathcal{GL}$-bundle $D(A^k)$ is a weighted Lie algebroid of degree $k$, where the brackets are defined using projectable sections, see for example [9]. The space of projectable sections is closed under this Lie bracket; this follows from

$$\bar{\rho}([X, Y]) = \left[\rho(X), \bar{\rho}(Y)\right],$$

(2.14)

and the fact that $\bar{\rho}$ is a projection. Because we can chose a local basis consisting of projectable sections this Lie bracket extends to all sections.

Note that we do not have explicit reference to the Jacobi identity in defining this Lie bracket, the only essential piece for consistency is the compatibility of the anchor with the brackets (2.14), that is, we only require the underlying structure to be that of an almost Lie algebroid in the terminology of [25]. The condition (2.14) has appeared as a necessary condition for the possibility of developing an appropriate variational calculus already in [18] (see also [25, 32]).

The above observations lead to the following theorem.

Theorem 2.2.8. For any almost Lie algebroid $(A, [, ], \rho)$, the $\mathcal{GL}$-bundle $D(A)$ comes equipped with the structure of a weighted skew algebroid whose construction is outlined above.

For further examples and in particular slightly more general weighted algebroids, that is where we can relax the underlying double graded structure not to necessarily be associated with the linearisation of a graded bundle, see [2].
3. Affine phase spaces and constraints

We will later make use of the affine structure of the fibrations $F_k \to F_{k-1}$ and geometric mechanics on Lie algebroids with (affine) vakonomic constraints. In order for this paper to be relatively self-contained we review what we will need in the next section of this paper.

3.1. Affine phase spaces

Let $A$ be an affine space modelled on a vector space $V$. This means that the commutative group $V$ acts freely and transitively on $A$ by addition

$$A \times V \ni (a, v) \mapsto a + v.$$ 

In other words, the naturally defined differences $a_1 - a_2$ of points of $A$ belong to $V$. On affine spaces there are defined affine combinations of points, $ta_1 + (1 - t)a_2$, for all $a_1, a_2 \in A$ and $t \in \mathbb{R}$. Note that convex combinations are those affine combinations $ta_1 + (1 - t)a_2$ for which $0 \leq t \leq 1$. For the affine space $A$, we consider its affine dual, i.e. the space $A^* = \text{Aff}(A, \mathbb{R})$ of all affine maps from $A$ to $\mathbb{R}$.

All this can be extended to affine bundles $\tau : A \to N$ modelled on a vector bundle $\tau \to V$ over $N$. Any vector bundle is an affine bundle and fixing a section $a_0$ of $A$ induces an isomorphism of affine bundles $A$ and $V(A)$,

$$V(A) \ni v \mapsto a_0(x) + v \in A.$$ 

Using coordinates $(x^a)$ in the open set $\mathcal{O} \subset N$, a local section $a_0 : \mathcal{O} \to A$, and local base of sections $e_i : \mathcal{O} \to V(A)$, we can construct an adapted coordinate system $(x^a, y^i)$ in $\mathcal{O}$. An element $a \in A$ can be written as $a = a_0(x) + ye_i(x)$.

Definition 3.1.1. ([15, 17]) An AV-bundle (a bundle of affine values) is an affine bundle $\zeta : Z \to \mathcal{M}$ modelled on a trivial one-dimensional vector bundle $\mathcal{M} \times \mathbb{R}$.

Sections of the AV-bundle $\zeta$ are regarded as affine analogs of functions on a manifold $\mathcal{M}$. The bundle $\tau^* : A^* \to N$, where $A^* = \text{Aff}(A, \mathbb{R})$ is the set of all affine maps on fibres of $\tau$, is called the affine dual bundle. Instead of $A^*(\mathbb{R})$ we will write simply $A^i$.

Every affine map $\phi : A_1 \to A_2$ has a well-defined linear part, $\nu(\phi) : \nu(A_1) \to \nu(A_2)$, therefore there is a projection

$$\zeta : A^i \to \nu(A)^*.$$ 

The above bundle, denoted with $\nu(A)^*$, is a canonical example of an AV-bundle which is modelled on

$$\nu(A)^* \times \mathbb{R}.$$ 

Using the dual base sections $e^i : \mathcal{O} \to \nu(A)^*$, we construct an adapted coordinate system $(x^a, p_i, r)$ on $(\tau^*)^{-1}(\mathcal{O})$. An affine map $\phi$ on $A_2$ can be written as $\phi(a) = p_i e^i (a - a_0(q)) - r$, i.e. $r = -\varphi(a_0)$. The map $\zeta$ in coordinates reads $(x^a, p_i, r) \mapsto (x^a, p_i)$.

Remark 3.1.2. The choice of coordinate $r$ with the ‘minus’ sign may seem unnatural, but is well motivated (cf [15]). Assume for simplicity that the coordinates $x$ are not present, so we deal with just an affine space $A$. 

\[ \text{J. Phys. A: Math. Theor. 48 (2015) 205203} \]
Having coordinates \((p, r)\) on \(A^\dagger\) we want to identify (locally) sections of \(\zeta : A^\dagger = \text{Aff}(A, \mathbb{R}) \to \mathbb{R}^n(A)\) with functions on \(\mathbb{R}^n(A)\). For instance, any \(a \in A\) defines canonically a section \(\sigma_a\) of \(\zeta\) being the zero-level set of the function \(f_a : \text{Aff}(A, \mathbb{R}) \to \mathbb{R}, f_a(\phi) = \phi(a)\).

In our coordinates, \(f_a(p, r) = (p, a - a_0) - r\), so that the image of \(\sigma_a\) is \(\{(p, (p, a - a_0))\}\), so correctly the graph of \(a - a_0 \in \mathbb{R}(A)\) as the linear function on \(\mathbb{R}^n(A)\). Generally, in the introduced canonical coordinates we will interpret functions \(F : \mathbb{R}^n(A) \to \mathbb{R}\) as sections \((x, p) \mapsto (x, p, -F(x, p))\) of the AV-bundle \(\zeta\).

As we have already mentioned, in many constructions functions on a manifold can be replaced by sections of an AV-bundle over that manifold. We can obtain also an affine analog of the differential of a function and an affine version of the cotangent bundle as follows. Given an AV-bundle \(\zeta : Z \to \mathcal{M}\) and \(\sigma_1, \sigma_2 \in \text{Sec}(Z)\), \(\sigma_1 - \sigma_2\) may be seen as a map

\[\sigma_1 - \sigma_2 : \mathcal{M} \to \mathbb{R},\]

so the differential

\[d(\sigma_1 - \sigma_2)(m) \in T^*_m\mathcal{M}\]

is well defined.

**Definition 3.1.3.** The phase bundle \(PZ\) of an AV-bundle \(Z\) is the affine bundle of cosets \(d\sigma(m) = [(m, \sigma)]\) (‘affine differentials’) of the equivalence relation

\[(m_1, \sigma_1) \sim (m_2, \sigma_2) \iff m_1 = m_2, \quad d(\sigma_1 - \sigma_2)(m_1) = 0.\]

The projection \(P\zeta : PZ \to \mathcal{M}\) makes \(PZ\) into an affine bundle modelled on \(T^*\mathcal{M}\). Indeed, fixing a section \(\sigma_0 : \mathcal{M} \to Z\), we get a diffeomorphism

\[\psi : PZ \to T^*\mathcal{M}, \quad d\sigma(m) \mapsto d(\sigma - \sigma_0)(m).\]

Moreover, as the canonical symplectic form on \(T^*\mathcal{M}\) is linear and invariant with respect to translations by closed 1-forms, its pull-back does not depend on the choice of \(\sigma_0\), so \(PZ\) is canonically a symplectic manifold.

Like in the case of the cotangent bundle \(T^*\mathcal{M}\), where the image \(df(M)\) of the differential of a function \(f\) on \(\mathcal{M}\) yields a lagrangian submanifold, also the image \(d\sigma(M)\) of a section of \(\sigma\mathcal{M} \to Z\) of \(Z\) is a lagrangian submanifold in \(PZ\).

**Example 3.1.4.** For an affine bundle \(\tau : A \to N\), take \(Z = \text{AV}(A^\dagger)\). Then, \(P(\text{AV}(A^\dagger))\) is an affine bundle over \(\mathbb{R}^n(A)\) which is canonically symplectic. We will denote \(P(\text{AV}(A^\dagger))\), with some abuse of notation, with \(PA^\dagger\). It is easy to see that \(PA^\dagger\) is also canonically a vector bundle over \(A\). Both bundle structures make the affine phase bundle into a double affine bundle [17, 28]. Actually, one affine structure is linear, so we can speak about a vector-affine bundle. The situation is similar to that with the cotangent bundle \(T^*\mathcal{A}\) which, besides the vector fibration over \(A\) is canonically an affine bundle over \(\mathbb{R}^n(A)\). This is an affine analog.
of the well-known vector fibration of \( T^*E \) over \( E^* \), for a vector bundle \( E \). The local identification of \( A^\dagger \) with \( v^*(A) \) by means of the local section \( a_0 \) yields the local identification of \( PA^\dagger \) with \( T^*v^*(A) \). We can therefore use for \( PA^\dagger \) coordinates \( (x^a, p_j, \eta_a, y^i) \) pulled back from \( T^*v^*(A) \). The canonical symplectic form \( \omega_{A^\dagger} \) in coordinates reads \( \omega_{A^\dagger} = d\eta_a \wedge dx^a + dy^i \wedge dp_j \).

**Theorem 3.1.5.** There is a canonical isomorphism \( \mathcal{R}_A \) of vector-affine bundles

\[
\begin{align*}
\pi_A &\quad \pi\tau \\
{\mathcal{R}_A} & \quad {\mathcal{P}A^\dagger} \\
T^*A & \quad T^*\tau \\
{v^*(A)} & \quad {v^*(A)} \\
A \quad & \quad A \\
\tau \quad & \quad \tau \\
M \quad & \quad M
\end{align*}
\]

covering the identities over \( A \) and \( v^*(A) \), which is simultaneously an antisymplectomorphism.

More specifically, if \( A \) is the affine bundle \( \tau^k : F_k \to F_{k-1} \), then \( v^*(A) = F_{k-1} \times_M F_k^* \) is the Mironian \( M(F_k) \) of \( F_k \). In coordinates \( (x^\mu_u, X_k^i, \xi^k_u, \theta^k_i) \) in \( T^*F_k \) and \( (X^\mu_u, \theta^k_i, I^{-k+1-u}, X^i_j) \) in \( PF_k^* \) the isomorphism \( \mathcal{R}_F \) reads

\[
\left( X^\mu_u, \theta^k_i, I^{-k+1-u}, X^i_j \right) \circ \mathcal{R}_F = \left( X^\mu_u, X^i_j, -I^{k-u+1}_u, \theta^k_i \right).
\]

**Remark 3.1.6.** The above isomorphism is a particular case of the identification \( \mathcal{P}(A) \cong \mathcal{P}(A^\dagger) \) valid for any special affine bundle \( A \) [17]. Note also that the above diagram reduces to (2.4) if as the affine bundle \( \tau : A \to N \) we take \( \tau^k : F_k \to F_{k-1} \).

### 3.2. Mechanics on algebroids

For the standard concept and theory of Lie algebroids we refer to the survey article [37] (see also [22, 38]). It is well known that Lie algebroid structures on a vector bundle \( E \) correspond to linear Poisson tensors on \( E^* \). A 2-contravariant tensor \( \Pi \) on \( E^* \) is called linear if the corresponding mapping \( \Pi : T^*E^* \to TE^* \) induced by contraction, \( \Pi(\nu) = i_\nu \Pi \), is a morphism of double vector bundles. One can equivalently say that the corresponding bracket of functions is closed on (fiber-wise) linear functions. Further more recall that the cotangent bundles \( T^*E \) and \( T^*E^* \) are examples of double vector bundles which are canonically isomorphic with the isomorphism
being simultaneously an anti-symplectomorphism (cf \([31, 33, 34, 39]\)).

The commutative diagram
\[
\begin{array}{ccc}
\mathcal{T}^{*}E^{*} & \overset{\Pi_{\epsilon}}{\longrightarrow} & \mathcal{T}E^{*} \\
\mathcal{T}^{*}E & \overset{\epsilon}{\longrightarrow} & \mathcal{T}^{*}E
\end{array}
\]

(3.5)

describes a one-to-one correspondence between linear 2-contravariant tensors \(\Pi_{\epsilon}\) on \(E^{*}\) and morphisms \(\epsilon\) (covering the identity on \(E^{*}\)) of the following double vector bundles (cf \([16, 18, 30, 31]\)):

The morphisms (3.6) of double vector bundles covering the identity on \(E^{*}\) has been called an algebroid in \([31]\), while a Lie algebroid has turned out to be an algebroids for which the tensor \(\Pi_{\epsilon}\) is a Poisson tensor. If the latter is only skew-symmetric, we deal with a skew algebroid.

Combining (3.6) with (3.5) we get the algebroid version of the Tulczyjew triple in the form of the diagram:

The left-hand side is Hamiltonian with Hamiltonians being functions \(H : E^{*} \rightarrow \mathbb{R}\), the right-hand side is Lagrangian with Lagrangians being functions \(L : E \rightarrow \mathbb{R}\), and the phase dynamics \(\mathcal{D}\) lives in the middle and is understood as an implicit dynamics (Lagrange equation) on the phase space \(E^{*}\), i.e. a subset of \(\mathcal{T}E^{*}\). Solutions of the Lagrange equations are
'phase trajectories' of the system: \( \beta : \mathbb{R} \to E^* \) which satisfy \( t(\beta (t)) \in D \). Here \( t \) is the canonical tangent prolongation of the curve \( \beta \).

The dynamics \( D = D_{\beta} \) generated by the Lagrangian \( L \) is simply \( D = \Lambda_L^\mathcal{L}(E) \), where \( \Lambda_L^\mathcal{L} : E \longrightarrow T^*E \), \( \Lambda_L^\mathcal{L} = \circ \circ dL \) is the so called Tulczyjew differential.

Similarly, the Hamiltonian dynamics is \( \Pi = \circ \circ \mathcal{L}^H(\tilde{d} (\circ )) \), \( \Pi = \circ \circ \mathcal{L}^H(\tilde{d} (\circ )) \) is the so called Tulczyjew differential. \( \mathcal{L}^H(\tilde{d} (\circ )) \) is the Legendre map actually does not depend on the algebroid structure but only on the Lagrangian.

For details of mechanics on algebroids the reader should consult \[16\]. Also note that above formalisms can be generalized to include constraints (cf. \[19\]) and that a rigorous optimal control theory on Lie algebroids can be developed as well \[25\].

**Remark 3.2.1.** In this understanding of first order mechanics following Tulczyjew, there is no need for ad-hoc constructions nor Lie algebroid prolongations, Poincaré–Cartan forms, Cartan sections, and symplectic algebroids. However, as we shall see, for higher order mechanics on a Lie algebroid there is a fundamental role for Lie algebroid prolongations via the linearisation functor, at least in the formalism we shall develop here. However, our use of Lie algebroid prolongations significantly differs from the rôle they play in some approaches to the first order mechanics on Lie algebroids.

### 3.3. Vakonomic constraints

There is an extensive literature concerning vakonomic constraints in the Lagrangian and Hamiltonian formalism, see for example \[1\]. We will skip presenting older concepts and use the elegant geometric approach, based on some ideas of Tulczyjew (see e.g. \[46\]), as described in \[18\]. It is ideologically much simpler than many others and works very well also for mechanics on algebroids \[16\]. We will devote a page to present main points of this approach.

Let us recall first that with any submanifold \( S \) in \( E \) and any function \( L : S \to \mathbb{R} \) one can associate canonically a lagrangian submanifold \( S_L \) in \( T^*E \) defined by

\[
S_L = \{ \alpha_e \in T^*_eE : e \in S \text{ and } \langle \alpha_e, v_e \rangle = dL(v_e) \text{ for every } v_e \in T_eS \}.
\]

If \( S = E \), then \( S_L = dL(E) \), i.e. \( S_L \) reduces to the image of \( dL \). The vakonomically constrained phase dynamics is just \( D = \circ \circ \mathcal{L}^H(\tilde{d} (\circ )) \). With \( \mathcal{P}L : S \longrightarrow T^*E \) denote the relation

\[
\mathcal{P}L = \{ (e, \alpha) \in S \times T^*E : \alpha \in S_L \text{ and } \pi_L(\alpha) = e \}.
\]

Note that we use the notation \( \longrightarrow \) to reinforce the fact that we are dealing with relations rather than genuine maps. We will adopt this notation for relations throughout the remainder of this paper.

We say that a curve \( \gamma : \mathbb{R} \to S \) satisfies the vakonomic Euler–Lagrange equations associated with the Lagrangian \( L : S \to \mathbb{R} \) if and only if \( \gamma \) is \( \circ \circ \mathcal{P}L \)-related to an admissible curve, i.e. to a curve which is the tangent prolongation of a curve in \( E^* \) (cf \[18\]). In particular, the vakonomic Euler–Lagrange equations depend only on the the Lagrangian as a function on the constraint \( S \) that differs vakonomic constraints from nonholonomic of mechanical type
ones. As admissible curves in $\mathbb{T}E^*$ are exactly those whose tangent prolongations lie in the set $\mathbb{P} \subset \mathbb{T}E^*$ of holonomic vectors, the vakonomic Euler–Lagrange dynamics on $E$ can be described by means of the set $\lambda_L^{-1}(\mathbb{T}^2E^*)$, where $\lambda_L^c = e \circ P L$ is the Tulczyjew differential relation.

**Remark 3.3.1.** There is an alternative point of view [18] in which the vakonomic Euler–Lagrange equations are not equations on curves in $S$ but rather on curves in $S_L$. Their projections to $S$ give curves satisfying vakonomic Euler–Lagrange equation in the previous sense. The advantage of this approach is that the equations are ‘less implicit’ as the ‘Lagrange multipliers’ do not appear.

It is easy to see that the lagrangian submanifold $S_L$ can be obtained also as the image of $dL(S) \subset \mathbb{T}^*S$ under the symplectic relation $\iota_S : \mathbb{T}^*S \rightarrow \mathbb{T}^*E$ associated with the embedding $\iota_S : S \hookrightarrow E$. Denoting the composition of this relation with $\epsilon$ as $\epsilon_S$, we get the phase dynamics $D$ in the form $D = \epsilon_S(dL(S))$ that gives the Lagrangian part of our formalism completely analogous to the unconstrained one. We will sometimes drop the subscript $S$ if $S$ is fixed and the meaning of $\iota = \iota_S : S \rightarrow E$ etc is clear.

In the case when $S=A$ is an affine subbundle of $E$ (assume for simplicity that $A$ is supported on the whole $M$), the relation $\epsilon_A$ covers the map $\nu(\iota)^* : E^* \rightarrow \nu^*(A)$ which is dual to the map $\nu(\iota) : \nu(A) \rightarrow E$ associated with the affine embedding $\iota : A \rightarrow E$. With respect to the second fibration, it covers the restriction $\rho_A : A \rightarrow \mathbb{T}M$ of the anchor map $\rho : E \rightarrow \mathbb{T}M$. We can consider the Hamiltonian part just using theorem 3.1.5. In this way we get the reduced Tulczyjew triple for an affine vakonomic constraint:

![Diagram of the reduced Tulczyjew triple](image)

Here, the Hamiltonian is a section of the AV-bundle $\zeta : A^1 \rightarrow \nu(A)^*$. It is easy to see that in the case when the Lagrangian is hyperregular, i.e. the vertical derivative $d' L$ viewed as the Legendre map $\lambda_L : A \rightarrow \nu^*(A)$ is a diffeomorphism, the Lagrangian submanifold $dL(A) \subset \mathbb{T}^*A \simeq \mathcal{P}(A^1)$ has the Hamiltonian description, $dL(A) = \mathbb{H}(\nu^*(A))$, with the Hamiltonian section $H : \nu^*(A) \rightarrow A^1$ such that $H(v^*_x)$ is the unique affine function on $A_x$ with the linear part $v^*_x$ and satisfying

$$H(v^*_x)(\lambda_L^{-1}(v^*_x)) = L(\lambda_L^{-1}(v^*_x)).$$  \hspace{1cm} (3.9)

If the affine bundle $A$ is linear, $A^1 = \mathbb{A}^* \times \mathbb{R}$ is trivial, so the Hamiltonian (3.9) understood as a genuine function on $A^1$ looks like the completely classic one (cf remark 3.1.2):
3.4. Tulczyjew triples for higher order mechanics

Consider kth order mechanics, where in the ‘unreduced approach’ the Lagrangian \( L \) is understood as a function on the submanifold \( T^k Q \subset TT^{k-1} Q \). The unreduced Tulczyjew triple in this case is (cf [21])

\[
H(v^*_i) = \{ v^*_i, \lambda^{-1}_L(v^*_i) \} = L(\lambda^{-1}_L(v^*_i)).
\]

Starting from local coordinates \( q = (q^a) \) in \( Q \) and \( \bar{v} \) in \( T^{k-1} Q \), where \( v = (q, \dot{q}, \ddot{q},...,\dot{q}^{(k-1)}, \dot{q}^{(k)}) \), we get natural coordinates

\[
\begin{align*}
(v, \delta v) & \quad \text{in } TT^{k-1} Q, \\
(v, p) & \quad \text{in } T^*T^{k-1} Q, \\
(v, \delta v, \pi, \delta \pi) & \quad \text{in } T^*TT^{k-1} Q, \\
(v, p, \dot{v}, \dot{p}) & \quad \text{in } T^*TT^{k-1} Q, \\
(v, p, q, \psi) & \quad \text{in } T^*TT^{k-1} Q,
\end{align*}
\]

in which \( \iota^{TT^{k-1} Q} \) is just the identification of coordinates \( \delta v = \dot{v}, \pi = p, \delta \pi = p, \) and \( \Pi_{TT^{k-1} Q} \) corresponds to the identification \( q = v, \psi = -\dot{p} \). It will be convenient to write \( \dot{q}^i \) as \( v_i \), so that the full coordinate system on \( TT^{k-1} Q \) can be written as \( (v^a) \).

The submanifold \( S = T^i Q \subset TT^{k-1} Q \) is given by the condition \( \delta v_i = v_i, i = 1,...,k-1 \), with the embedding \( \hat{\rho}_k : T^i Q \rightarrow TT^{k-1} Q \), which in this case coincides with the anchor map,

\[
\hat{\rho}_k \left( q, \dot{q},...,\dot{q}^{(k)} \right) = \left( q, \dot{q},...,\dot{q}^{(k-1)}, \dot{q}^{(k)} \right) = (v, \delta v).
\]

We view \( S \) as an affine vakonomic constraint, so the reduced triple is (cf [21])
Hence, according to the general scheme, the Lagrangian function \( L = L(q^{(i)}, q^{(k)}) \) generates the following (Lagrangian) submanifold in \( T^*T^{k-1}Q \):

\[
\left( T^kQ \right)_L = \left\{ (v, \delta v, \pi, \delta \pi) : \delta v_{i-1} = v_i, \pi_i + \delta \pi_{i-1} = \frac{\partial L}{\partial q^i}, i < k, \pi_0 = \frac{\partial L}{\partial q^0}, \delta \pi_{k-1} = \frac{\partial L}{\partial \dot{q}^k} \right\}.
\]

Here, the conditions \( \delta v_{i-1} = v_i, i = 1, \ldots, k - 1 \), mean that \((v, \delta v)\) is in the image of \( \tilde{\rho}_k \), and partial derivatives of the Lagrangian are taken in the point \( \tilde{\rho}_k^{-1}(v, \delta v) \). The phase dynamics \( D = \epsilon((T^{k-1}Q)_L) \) is then

\[
D = \frac{\partial L}{\partial \dot{q}^i} p_{k-1} = \frac{\partial L}{\partial \dot{q}^i}.
\]

We understand \( D \) as an implicit first order differential equation for curves in \( T^*T^{k-1}Q \). A curve \( \gamma(t) = (q(t), \dot{q}(t), \ldots, \dot{q}^{(k)}(t)) \) is \( \tilde{\Lambda}_k \) related with a curve \( \beta(t) = (v(t), p(t), \dot{v}(t), \dot{p}(t)) \) in \( D \) if and only if

\[
\gamma(t) = \left( q(t), \dot{q}(t), \ldots, \dot{q}^{(k)}(t) \right), \quad \dot{\gamma}(t) = \left( \dot{q}(t), \ddot{q}(t), \ldots, \dot{q}^{(k)}(t) \right),
\]

and

\[
\dot{p}_i(t) + p_{i-1}(t) = \frac{\partial L}{\partial q^i}(\gamma(t)) \text{ for } i = 1, \ldots, k - 1, \quad \dot{p}_0(t) = \frac{\partial L}{\partial q^0}(\gamma(t)), \quad p_{k-1}(t) = \frac{\partial L}{\partial \dot{q}^k}(\gamma(t)).
\]

Assuming additionally that \( \beta \) is admissible, we get equations

\[
\frac{d}{dt}(q^i) = q^{(i+1)}, \quad i = 0, \ldots, k - 1;
\]

\[
p_{k-1} = \frac{\partial L}{\partial q^{(k)}}, \quad p_{i-1} = \frac{\partial L}{\partial q^i}(q^{(i)}, \ldots, q^{(k)}) - \frac{d}{dt}(p_i) \text{ for } i < k;
\]

\[
\frac{d}{dt}(p_0) = \frac{\partial L}{\partial \dot{q}^i}(q^{(i)}, \ldots, q^{(k)}),
\]

(3.10)
that can be rewritten as the Euler–Lagrange equations in the traditional form:

\[
\begin{align*}
\frac{d}{dt} q_i &= \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \ldots, k, \\
0 &= \frac{\partial L}{\partial q_i}(q, \ldots, \dot{q}_i) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i}(q, \ldots, \dot{q}_i) \right) + \cdots \\
&\quad + (-1)^k \frac{d^k}{dt^k} \left( \frac{\partial L}{\partial \dot{q}_i}(q, \ldots, \dot{q}_i) \right).
\end{align*}
\]

(3.11)

These equations can be viewed as a system of differential equations of order \(k\) on \(T^k Q\) or, which is the standard point of view, as ordinary differential equation of order \(2k\) on \(Q\).

4. Mechanics on graded bundles

The basic idea is to follow the construction of the Tulczyjew triple for the classical higher order mechanics in which we replace \(T^k M\) with an arbitrary graded bundle \(F_k\), and the canonical embedding \(T^k M \hookrightarrow T^k T^k M\) with the canonical embedding \(F_k \hookrightarrow D(F_k)\). Here the linearisation \(D(F_k)\) is equipped with a weighted algebroid structure \(\epsilon : T^k D(F_k) \rightarrow TD^k(F_k)\). For simplicity and with examples in mind we will work with skew/Lie weighted algebroids but general weighted algebroids can be used as well.

We therefore consider mechanics on the algebroid \(D(F_k)\) with the affine constraint \(F_k \subset D(F_k)\). In consequence, we understand the phase space of a mechanical system on \(F_k\) to be \(D^\psi(F_k)\) and the (implicit) phase equations as subsets of \(TD^\psi(F_k)\).

The corresponding reduced Tulczyjew triple mimics that of (3.8), where the affine bundle \(A\) is \(\tau^k : F_k \rightarrow F_{k-1}\), and the vector bundle \(E \rightarrow M\) is replaced with \(D(F_k) \rightarrow F_{k-1}\). Note that in this case \(\psi^\#(A)\) is the Mironian \(Mi(F_k) \simeq F_{k-1} \times_M F^*_k\). Denoting the affine dual of \(\tau^k : F_k \rightarrow F_{k-1}\) simply with \(F^*_k\), we get the triple as follows.

\[
\begin{align*}
\tau^k : F_k & \rightarrow F_{k-1}, \\
\psi^\#(A) : Mi(F_k) & \rightarrow F^*_k.
\end{align*}
\]

Here, we write \(\hat{\epsilon}\) instead of \(\epsilon_{F_k}\) and \(\hat{\rho}\) instead of \(\rho_{F_k}\). The relation \(\Pi_k\) can also be written as \(\hat{\epsilon} \circ R_k\), where \(R_k\) is the canonical isomorphism identifying the vector-affine bundles \(T^k F_k\) and \(P(F^*_k)\). Now, the generation of the phase dynamics out of a Lagrangian function or a Hamiltonian section and the construction of the Euler–Lagrange equations is subject of the general scheme for affine vakonomic constraints in algebroids, as described in section 3.3.
However, the fact that the algebroid is a weighted algebroid of the graded bundle $F_k$ puts additional flavour to the scheme, making the whole picture similar to the classical case $F_k = T^1M$.

### 4.1. The Lagrangian formalism on graded bundles

Let us employ natural homogeneous coordinates $\left(\bar{X}_W^\mu; \bar{F}_{k+1-W}^a\right)$ on $T^*F_k$, so that $\bar{X}_W^\mu$ serve as local coordinates on $F_{k-1}$. The reason for the ‘bar’ in the notation will become clear as we will be dealing with relations. In these local coordinates the section $dL : F_k \to T^*F_k$ is given by:

$$\bar{p}_{a}^{k+1-W} \circ dL = \frac{\partial L}{\partial \bar{X}_W^\mu}.$$ 

The canonical inclusion $\iota : F_k \hookrightarrow D(F_k)$ induces a graded symplectic relation $\mathcal{P}_L \hookrightarrow \mathcal{T}D(F_k)$. This, in turn, produces the graded relation $\mathcal{P}_L = r \circ dL : F_k \hookrightarrow \mathcal{T}D(F_k)$. By employing homogeneous local coordinates $(X_u^\mu, Y_U^I, P_p^{k+1-u}, \Pi_j^{k+1-l})$, this relation is given by:

$$X_u^\mu = \bar{X}_W^\mu, \quad Y_U^I = U\bar{X}_U^I,$n

$$P_p^{k+1} = \frac{\partial L}{\partial \bar{X}_p^{k+1-W}}(\bar{X}) - (k - W)P_p^{k+1},$$

where we employ the convention that coordinates with degrees outside the range are zero: $P_p^{1} = 0$ and $\Pi_j^{k+1} = 0$.

The Lagrangian produces the phase dynamics $D_L$ understood as the image of $F_k$ under the relation $L_{\epsilon} = e \circ \mathcal{P}_L$:

$$D_L = \Lambda_{\epsilon}^L(F_k) \subset TD^\times(F_k).$$

The relation $L_{\epsilon}$ we will refer to as the weighted Tulczyjew differential of $L$. Note that the phase dynamics explicitly depends on the weighted algebroid structure carried by $F_k$. In the preceding we will assume that this weighted algebroid structure is skew/Lie to slightly simplify the local expressions, though this specialisation is not fundamentally required in the formalism. In particular, note we do not need the Jacobi identity for the associated bracket structure on sections and so weighted skew algebroids will more than suffice for this formulation of higher order geometric mechanics.

By using natural homogeneous local coordinates $(X_u^\mu, \Pi_j^U, \delta X_{u+1}^\nu, \delta \Pi_j^{U+1})$ in $TD^\times(F_k)$, the relation $L_{\epsilon}^L : F_k \longrightarrow TD^\times(F_k)$ be described by:

$$X_u^\mu = \bar{X}_W^\mu,$n

$$\delta X_U^I = (U - u)\rho [u]_{\bar{X}_U^I}(\bar{X})X_U^I,$n

$$k\Pi_1^I = \frac{\partial L}{\partial \bar{X}_U^I}(\bar{X}),$$n

$$\delta \Pi^{U+1}_j = \rho [u]_{\bar{X}_U^I}(\bar{X})\left(\frac{\partial L}{\partial \bar{X}_U^{k-W+u}}(\bar{X}) - (k - U + u)\Pi^{U+1}_{j-U+u}\right) + U'C [u]_{\bar{X}_U^I}(\bar{X})\bar{X}_U^I\Pi^{U+1-U'-U}. $$

We understand the phase dynamics to be the first order implicit differential equation $D_L = L_{\epsilon}^L(F_k) \subset TD^\times(F_k)$ on the phase space $D^\times(F_k)$. A curve $\beta(t) = (X_u^\mu(t), \Pi_j^{U}(t)) \in D^\times(F_k)$ is a
solution of this equation if and only if its tangent prolongation
\[
\left( X^U(t), \Pi^U(t), \dot{X}^U(t), \Pi^{*U}(t) \right) \in T D^a(F_k)
\] (4.6)

lies in \( D_k \). Here, dots have the meaning of genuine time-derivatives. Therefore, a curve \( \gamma(t) = (\bar{X}^U(t)) \) in \( F_k \) satisfies the Euler–Lagrange equation if and only if it is \( \Lambda_f \)-related with an admissible curve (4.6), i.e.
\[
X^U_t = (U - u)\rho\{u\}^U(\bar{X})\bar{X}^U_{U-U^U},
\] (7.7)
\[
k\Pi^U_t = \frac{\partial L}{\partial \bar{X}^U_k}(\bar{X})
\] (7.8)
\[
\Pi^{U+1} = \rho\{u\}^U(\bar{X})\left( \frac{\partial L}{\partial \bar{X}^{U+1}_k}(\bar{X}) - (k - U + u)\Pi^{U+1-U^U} \right)
+ U'\rho\{u\}^U(\bar{X})\bar{X}^U_{U}^{U+1-U^U}. \] (7.9)
The first equation means that the curve \( \gamma : \mathbb{R} \rightarrow F_k \) is admissible, i.e.
\[
\hat{\rho} \circ \gamma = \tau^{-1}_{k-1},
\]
where \( \hat{\rho} := \rho \circ \iota : F_k \rightarrow TF_{k-1} \) is the anchor map, \( \gamma^{-1}_{k-1} = \tau^k \circ \gamma \) is the curve on \( F_{k-1} \) underlying \( \gamma \), and \( \gamma^{U}_{k-1} \) is the tangent prolongation of the said underlying curve.

The rest of equations defines an implicit differential equation for curves on \( F_k \), that is standard for vakonomic equations, on additional parameters \( \Pi^W \). These parameters are fixed if we understand Euler–Lagrange equations as equations on the lagrangian submanifold \( \subset T^*D(F_k) \). The latter equations are, in ‘good’ cases, of order \( k \). Indeed, if the matrix \( \rho \{u\}^U(\bar{X}) \) is invertible, we can express each \( \Pi^{U+1} \), \( U = 1, ..., k - 1 \), as a function \( \Pi^{U+1}_k = F^U_k(\Pi^U, \Pi^{U+1}, ..., \Pi^1, \bar{X}) \) of \( \Pi^U \) and of variables \( \Pi_U \) of lower weight and \( \bar{X} \). As
\[
k\Pi^U_t = \frac{\partial L}{\partial \bar{X}^U_k}(\bar{X}),
\]
we get inductively that \( \Pi^{U+1}_k \) is a function of
\[
\frac{d^U}{dU} \left( \frac{\partial L}{\partial \bar{X}^U_k}(\bar{X}) \right)
\]
and derivatives of \( \bar{X} \) of order \( < U, U = 1, ..., k - 1 \). Since, according to (4.9), \( \Pi^U_k \) is a function of variables \( \Pi \) and \( \bar{X} \), we get an equation on derivatives of \( \bar{X} \) of order \( \leq k \). In this situation a curve on \( F_k \) has at most one ‘prolongation’ to a corresponding curve in the lagrangian submanifold \( (F_k)_k \subset T^*D(F_k) \). The concepts of Euler–Lagrange equations understood as equations for curves on \( F_k \) or \( (F_k)_k \) coincide. This is the case of the standard higher order Lagrangian mechanics on \( T^kM \subset TT^kM \).

Observe additionally that the admissibility equation (7.7) puts additional relations on variables \( \bar{X} \) which are differential equations of order \( k \) in ‘good’ cases. Indeed, we get inductively the \( k \)th derivative of \( \bar{X} \) as a function of lower order derivatives of \( X \). Geometrically it means that the series of anchors \( F_k \rightarrow TF_{k-1} \) gives rise to a map \( \hat{\rho}^k : F_k \rightarrow T^kF_k \) and the \( k \)th prolongation of the curve \( \gamma_0 = \tau^k_0 \circ \gamma \) being the projection of the admissible curve \( \gamma \) on \( F_k \) equals \( \hat{\rho}^k \circ \gamma \).
In the canonical case $T^k M \subset T^{k+1} M$, admissible curves in $T^k M$ are just $k$th order prolongations of curves on $M$, so we get equations of order $2k$ on $M$.

**Example 4.1.1.** Let $g$ be a real finite-dimensional Lie algebra with the structure constants $c^{ij}_k$ relative to a chosen basis $e_i$, and put $F_2 = g_2 = g[1] \times g[2]$ (cf example 2.2.4). Note that this graded bundle is actually a graded space and its linearisation, carrying a canonical Lie algebroid structure, is an example of a weighted Lie algebra in the terminology of [2], as there are no coordinates of weight 0. The basis induces coordinates $(x^i, z^i, t)$ on $F_2$ and coordinates $(x^i, y^j, z^l)$ on $D(g_2) = g[1] \times T(g[1]) = g[1] \times g[1] \times g[2]$ for which the embedding $\iota : g_2 \hookrightarrow D(g_2)$ takes the form $\iota(x, z) = (x, y, z)$ and the vector bundle projection is $\pi(x, y, z) = x$. The Lie algebroid structure on $D(g_2)$ is the product of the Lie algebra structure on $g$ and the tangent bundle $T g$; the map $\epsilon : T^* D(g_2) \to TD^*(g_2)$ takes the form

$$\alpha \beta \gamma \beta \alpha \beta = \partial \partial = \partial \partial$$

that leads to the Euler–Lagrange equations on $g_2$:

$$\dot{x} = z,$$

$$\dot{y} = \mathrm{ad}^\gamma \beta,$$

$$\alpha = \frac{d}{dt}\left( \frac{\partial L}{\partial z} (x, z) \right).$$

Hence,

$$\beta = \partial \partial \left( x, z \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial z} (x, z) \right)$$

that leads to the Euler–Lagrange equations on $g_2$:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial x} (x, z) - \frac{d}{dt} \left( \frac{\partial L}{\partial z} (x, z) \right) \right) = \mathrm{ad}^\gamma \beta \left( \frac{\partial L}{\partial x} (x, z) - \frac{d}{dt} \left( \frac{\partial L}{\partial z} (x, z) \right) \right).$$

These equations are second order and induce the Euler–Lagrange equations on $g$ which are of order 3:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial x} (x, \dot{x}) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} (x, \dot{x}) \right) \right) = \mathrm{ad}^\gamma \beta \left( \frac{\partial L}{\partial x} (x, \dot{x}) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} (x, \dot{x}) \right) \right).$$

For instance, consider the ‘free’ Lagrangian $L(x, z) = 1/2 \sum I I (\dot{z}^i)^2$ induces on $g_2$ the equations (we do not use the summation convention loosing control on indices)

$$\dot{x} = z,$$

$$I I^k \dot{z}^i = \sum c^k_{ij} I_k x^i z^k,$$
which are equivalent to the equations

\[ I_j \ddot{x}_j = \sum_{i,k} c_{jk}^i I_k \dot{x}_k \]

on \( \mathfrak{g} \). The latter can be viewed as ‘higher Euler equations’ (for a rigid body if \( \mathfrak{g} = so(3, \mathbb{R}) \)).

### 4.2. The Hamiltonian formalism on graded bundles

For the system associated with a Hamiltonian section \( H : M_i \to F_k \) we obtain the phase dynamics \( \tilde{\mathcal{H}} \) understood as the image of \( H \) under the relation \( \Lambda_\mathcal{H} = \tilde{\Pi} \circ PH \). The map \( \tilde{\Pi} : T^* D^*(F_k) \to T^* D^*(F_k) \) encodes the structure of weighted algebroid on \( F_k \) while \( PH \) is the composition of \( \mathcal{H} : Mi(F_k) \to PF_k^\dagger \) with symplectic relation \( s : PF_k^\dagger \to T^* D^*(F_k) \). The relation \( s \) is the Hamiltonian counterpart of the relation \( r \) that we used in Lagrangian mechanics and which is just the phase lift of the inclusion \( F_k \hookrightarrow D(F_k) \). The relation \( s \) can be obtained as the composition of \( r \) with two isomorphisms \( R_k : T^* F_k \to PF_k^\dagger \) and \( R_{D(F_k)} : T^* D(F_k) \to T^* D^*(F_k) \), more precisely

\[ s = R_{D(F_k)} \circ r \circ R_k^{-1}. \]

The relation \( s \) can be however obtained independently of \( r \). First notice that the inclusion \( F_k \hookrightarrow D(F_k) \) is affine, i.e. the image of a fibre of \( r^i : F_k \to F_{k-1} \) is an affine subspace of the appropriate fibre of the vector bundle \( D(F_k) \to F_{k-1} \). It means that the vector hull \( \tilde{F}_k \) is a vector subbundle of \( D(F_k) \). The vector dual of \( \tilde{F}_k \) is the projection \( D^*(F_k) \to F_k^\dagger \).

The phase lift of that projection is one of the components of the relation \( s \). To get \( s \) we have to compose it with the symplectic reduction. Of course, one should check if the phase lift of projection and reduction are composable. This can be done in coordinates.

For the coordinate expression of \( s \) we will use coordinates:

\[
\begin{pmatrix}
X_u^i, \Theta^1, \Gamma^{i+1}_A, \Gamma_A^{k+1-w}, X_k^i
\end{pmatrix}
\]

on \( PF_k^\dagger \),

\[
\begin{pmatrix}
X_u^i, \Pi^{k+1-w}, \Pi_A^{i+1}, p_A^{k+1-w}, Y_u^a, Y_k^a
\end{pmatrix}
\]

on \( T^* D^*(F_k) \).

The relation \( s \) is described by the conditions

\[
\Theta^1 = k\Pi^1, \quad p_A^{k+1} = \Gamma_A^{k+1}, \quad Y_u^a = wX_u^a, \quad p_k^{k+1-w} = \Gamma_A^{k+1-w} + w\Pi_A^{k+1-w},
\]

so \( PH : Mi(F_k) \to T^* D^*(F_k) \) reads

\[
\Pi_A^{i+1} = \frac{\partial H}{\partial X_0^i}, \quad p_A^{k+1-w} = \frac{\partial H}{\partial X_u^a} + w\Pi_A^{k+1-w}, \quad Y_u^a = wX_u^a, \quad Y_k^a = k\frac{\partial H}{\partial \Theta^1}.
\]

Finally, the coordinate expression for \( \Lambda_\mathcal{H} \) is

\[
\delta X_u^i \circ \lambda_c = \rho[U - k\nu(X)k\frac{\partial H}{\partial \Theta^1} + \rho[U - w\nu(X)X_u^a] \quad (4.12)
\]

\[
\delta \Pi_A^{i+1} \circ \lambda_c = -\rho[U - k\nu(X)k\frac{\partial H}{\partial X_0^i} - \rho[U - w\nu(X)X_u^a] \left(\frac{\partial H}{\partial X_u^a} + w\Pi_A^{k+1-w}\right) \quad (4.13)
\]

\[ + C[u]_{\nu(X)}k\frac{\partial H}{\partial \Theta^1} \Pi^{k+1-w} + C[u]_{\nu(X)}wX_u^a \Pi^{k+1-w}.\]

Usually we are given a system with Lagrangian function \( L \) defined. The question whether there exists the corresponding Hamiltonian section such that \( D_H = D_L \) arises naturally. The
answer is very much the same as in the classical case for first order mechanics. It exists if
Lagrangian is hyperregular i.e. the Legendre map \( \lambda_L : F_k \to \text{Mi}(F_k) \), \( \lambda_L = T^{k+1} \circ dL \), is a
diffeomorphism. In case it is not, there exists a generating family of sections parameterized by
elements of \( F_k \). Using the correspondence between sections of \( F_k^* \to \text{Mi}(F_k) \) and functions on
\( F_k \) we can write the generating family of functions as
\[
h : F_k^1 \times F_k, \quad F_k \ni (\varphi, f) \mapsto \varphi(f) - L(f) \in \mathbb{R}.
\] (4.14)

**Example 4.2.1.** Let us consider the Hamiltonian side of the Lagrangian mechanics presented
in example 4.1.1, i.e. for \( F_2 = g_2 = \{ (x, z) \} \subset g \times Tg = \{ (x, y, z) \} \) and the ‘free’
Lagrangian \( L(x, z) = \frac{1}{2} \sum_i l_i (z)^2 \) on \( g_2 \). Since in this case the bundle \( \tau_2 : F_2 = g_2 \to F_1 = g[1] \)
is canonically linear with fibers \( g[2] \), the AV-bundle \( \zeta : g_2^1 \to \nu^v(g_2) = g[1] \times g^v[1] \) is
trivial, \( g_2^1 = g[1] \times g^v[1] \times \mathbb{R} \), and Hamiltonians can be understood as genuine functions on
\( g[1] \times g^v[1] \). Note that, due to our convention, a function \( H : g[1] \times g^v[1] \to \mathbb{R} \)
corresponds to the section \( (x, \theta, -H(x, \theta)) \) of \( g_2^1 \). Our Lagrangian is hyperregular,
\( \lambda_L : g_2 \to \nu^v(g_2) \), \( \lambda_L(x, z) = (x, l z) \),
where \( (lz)_i = l z^i \) (no summation convention here) is a diffeomorphism, so, according to
(3.9),
\[
H(x, \theta) = \left\{ \theta, -\frac{1}{T} \frac{\partial}{\partial \theta} \right\} - L(\lambda_L^{-1}(x, \theta)) = \sum_i \frac{\theta_i^2}{l_i} - \frac{1}{2} \sum_i l_i \left( \frac{\theta_i^2}{l_i} \right)^2 = \frac{1}{2} \sum_i \frac{\theta_i^2}{l_i}.
\]

5. Reductions

As remarked on in the introduction, higher order mechanics on Lie groups and Lie groupoids
has received very little attention in the literature. The need to understand higher order
mechanics on a Lie algebroid naturally appears in the context of reductions of higher order
theories on Lie groupoids with Lagrangians that are invariant with respect to the groupoid
multiplication. However, studying higher order Lagrangian mechanics on Lie algebroids
should be considered an interesting problem irrespective of any reduction. Indeed, via theorem 2.2.8
the results presented in the section will generalise quite directly to non-integrable
Lie algebroids and almost Lie algebroids. It appears that we cannot directly generalise these
constructions to skew algebroids, that is we cannot lose the compatibility of the anchor with
the brackets. This compatibility of the anchor and the brackets also features as an essential
ingredient in the variational approach developed by Jóźwikowski and Rotkiewicz [32]. We
will present in some detail the constructions for higher order Lagrangian mechanics on Lie
algebroids as we expect this to be a particularly rich source of concrete examples and
applications of our formalism. Moreover, this situation leads to a ‘good’ example and so we
can derive Euler–Lagrange equations explicitly.

5.1. Higher order Lagrangian mechanics on a Lie algebroid

Let us consider a Lie groupoid \( G \) and a Lagrangian systems on \( \mathcal{A}(G) = T^k G \mid _{Id} \). We will refer
to such systems as a \( k \)-th order Lagrangian system on the Lie algebroid \( A(G) \) as the structure is
completely defined by the underlying genuine Lie algebroid structure on $A(G)$, see example 2.2.3. The relevant diagram here is

$$
\begin{array}{ccc}
TD^*(A^k(G)) & \xleftarrow{\varepsilon} & T^*D(A^k(G)) \\
\downarrow & & \downarrow \\
TA^{k-1}(G) & \xleftarrow{\rho} & D(A^k(G)) \\
\downarrow & & \downarrow \\
T^*\Lambda^{k-1}(G) & \xleftarrow{\kappa} & \Lambda^k(G)
\end{array}
$$

Here,

$$
D\left(\Lambda^k(G)\right) \simeq \left\{(Y, Z) \in A(G) \times_M T\Lambda^{k-1}(G) \mid \tilde{\rho}(Y) = T\tilde{\tau}(Z)\right\},
$$

where $\tilde{\rho} : A(G) \to TM$ is the standard anchor of the Lie algebroid and $\tilde{\tau} : \Lambda^{k-1}(G) \to M$ is the obvious projection.

Following example 2.2.3, let us employ local coordinates $\Pi_{\mu\nu}^{-\cdots}$ on $T^*D(A(G))$ and similarly let us employ local coordinates $\left(X^\mu, \Pi^U_{i^{-1-1}}, \delta X^\nu, \delta \Pi^U_{j^{+1}}\right)$ on $TD^*(A(G))$. Using 4.5, the relation $\Lambda^k_f : \Lambda^k(G) \longrightarrow TD^*(A(G))$ is given by

$$
\begin{align*}
X^\mu &= \tilde{X}^\mu, \\
k\Pi^I &= \frac{\partial L}{\partial \tilde{X}^I}, \\
\delta X^\mu &= U\rho\left[0\right]_d(\tilde{x})\tilde{X}^\nu, \\
\delta \Pi^U_{j^{+1}} &= \rho\left[0\right]_d(\tilde{x})\left(\frac{\partial L}{\partial \tilde{x}^\nu} - (k - U)\Pi^U_{j^{+1}}\right) + \delta \Pi^U_{k^{+1}} C\left[0\right]_d(\tilde{x})\tilde{X}^I \Pi^U_{K}. 
\end{align*}
$$

Recall that the only non-zero components of $\rho\left[0\right]_d$ are $\rho_a^b$ and $\delta \Pi^U_{j}$, while $C\left[0\right]_d$ has only $C^a_{bc}$ non-vanishing and that $(\rho_a^b, C_{abc})$ are the structure functions of the Lie algebroid $A(G)$.

A curve $\gamma(t) = (X^\mu_a(t))$ in $\Lambda(G)$ satisfies the Euler–Lagrange equation if and only if it is $\Lambda^k_f$-related with an admissible curve (4.6). That is,

$$
\begin{align*}
\frac{d}{dt}X^\mu_{i^{-1-1}} &= U\rho\left[0\right]_d(\tilde{x})\tilde{X}^\nu, \\
k\Pi^I &= \frac{\partial L}{\partial \tilde{X}^I}(\tilde{X}), \\
\frac{d}{dt}\Pi^U_{j^{+1}} &= \rho\left[0\right]_d(\tilde{x})\left(\frac{\partial L}{\partial \tilde{x}^\nu} - (k - U)\Pi^U_{j^{+1}}\right) + \delta \Pi^U_{k^{+1}} C\left[0\right]_d(\tilde{x})\tilde{X}^I \Pi^U_{K}. 
\end{align*}
$$

Note that we have a ‘good’ case here, meaning that the components of the anchor $\rho_a^b$ are invertible, in this case trivially. For explicitness, let us revert back to coordinates $\left(x^A, y^a, z^b\right)$ on $\Lambda^k(G)$. Recursively we can write the momenta $\pi_a$ in terms of the coordinates on $\Lambda^k(G)$ as follows;
\[ \pi^1_a = \frac{1}{k} \frac{\partial L}{\partial \dot{q}_k^a}, \]

\[ (k - 1)\pi^2_b = \frac{\partial L}{\partial \dot{q}_k^b} - \frac{1}{k} \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}_k^b} \right), \]

\[ (k - 2)\pi^3_c = \frac{\partial L}{\partial \dot{q}_k^c} - \frac{1}{(k - 1)} \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}_k^c} \right) + \frac{1}{k(k - 1)} \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dddot{q}_k^c} \right) \]

\[ \vdots \]

\[ \pi^k_d = \frac{\partial L}{\partial \dot{q}_k^d} - \frac{1}{2!} \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}_k^d} \right) + \frac{1}{3!} \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dddot{q}_k^d} \right) - \cdots \]

\[ + (-1)^k \frac{1}{(k - 1)!} \frac{d^{k-2}}{dt^{k-2}} \left( \frac{\partial L}{\partial \dddot{q}_k^{d-1}} \right) - (-1)^k \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} \left( \frac{\partial L}{\partial \dddot{q}_k^d} \right). \]

which we recognise as the Jacobi–Ostrogradski momenta.

The remaining equation \( \frac{d}{dt} \pi^k_d = \rho^k_a (\tilde{x})(\dot{\tilde{x}}) + \tilde{g}_k^b C_{ba}(\tilde{x}) \pi^k_c \) can then be written as

\[ \rho^k_a (\tilde{x})(\dot{\tilde{x}}) - \left( \tilde{g}_{ik}^j \frac{d}{dt} - \tilde{g}_k^b C_{ba}(\tilde{x}) \right) \left( \frac{\partial L}{\partial \dot{q}_k^d} - \frac{1}{2!} \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}_k^d} \right) + \frac{1}{3!} \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dddot{q}_k^d} \right) - \cdots - (-1)^k \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} \left( \frac{\partial L}{\partial \dddot{q}_k^d} \right) \right) = 0, \]

which we define to be the kth order Euler–Lagrange equations on \( \mathcal{A}(\mathcal{G}) \), taking into account our choice of homogeneous coordinates.

The above Euler–Lagrange equations are in complete agreement with Jóźwikowski and Rotkiewicz [32], Colombo and de Diego [6] for the second order case, as well as Martínez [41] for the general higher order case. We clearly recover the standard higher Euler–Lagrange equations on \( T^k M \) as a particular example. Note that the geometric structure on \( \mathcal{A}(\mathcal{G}) \) is completely encoded in the Lie algebroid \( \mathcal{A}(\mathcal{G}) \) and so the nomenclature we have chosen is appropriate. If we restrict ourselves to Lagrangians that do not depend on higher order (generalised) velocities then we recover the standard Euler–Lagrange equations on a a Lie algebroid, as first derived by Weinstein [49] and generalized to (skew) algebroids in [16, 18].

### 5.2. Second order Hamel and Lagrange–Poincaré equations

In this section we briefly examine second order Lagrangian mechanics on the Atiyah algebroid and an Atiyah algebroid in the presence of a non-trivial connection. These examples give rise to generalisations of the Hamel and Lagrange–Poincaré equations.

**Example 5.2.1.** An important situation in physics is when a Lagrangian defined on a principal \( G \)-bundle is invariant under the action of \( G \). In this case if \( P \rightarrow M \) is the principal bundle in question, then the Lagrangian is a function on \( T^2 P \). The reduction of this system is a Lagrangian system on \( \mathcal{A} := T^2 P/G \), which is to be considered as the weighted (or higher order) version of the Atiyah algebroid. Let us be a little more specific and consider \( k = 2 \). Via a local trivialisation we can identify \( \mathcal{A} \approx T^2 M \times g[1] \times g[2] \) locally and thus employ homogeneous coordinates \( \{ \tilde{x}^\alpha, \tilde{v}_i^a, \tilde{y}_i^a, \tilde{w}_i^a, \tilde{z}_i^a \} \). Then, in hopefully clear notation, the phase dynamics of a second order Lagrangian on an Atiyah algebroid is specified by
\[ \delta x_1^A = \delta v_1^A, \quad \delta v_2^A = 2 \delta \xi_2^A, \quad \delta y_2^A = 2 \zeta_2^A, \]
\[ \delta \pi_{\alpha}^3 = \frac{\partial L}{\partial \delta x_{\alpha}^A}, \quad \delta \pi_{\alpha}^4 = \frac{\partial L}{\partial \delta \xi_{\alpha}^A} - \pi_{\alpha}^2, \]
\[ \delta \pi_{\alpha}^2 = \frac{\partial L}{\partial \delta v_{\alpha}^A} - \pi_{\alpha}^2, \quad \pi_{\alpha}^1 = \frac{1}{2} \frac{\partial L}{\partial \delta \xi_{\alpha}^A}, \quad \pi_{\alpha}^4 = \frac{1}{2} \frac{\partial L}{\partial \delta \xi_{\alpha}^A}. \]

Via inspect we see that the phase dynamics is essentially separated into a part to do with the base \( M \) and a part to do with the Lie group \( G \). Thus, using the general result on mechanics on a Lie algebroid and higher order tangent bundles as presented above, we arrive at the second order Hamel equations [32] (also see [5] for the first order case);

\[ \frac{\partial L}{\partial v^A} - \frac{d}{dt} \left( \frac{\partial L}{\partial v_t^A} \right) + \frac{1}{2!} \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \xi_t^A} \right) = 0, \]
\[ \left( \delta_{\alpha}^i \frac{d}{dt} - \delta_{\alpha}^b C_{ab}^{\gamma} \right) \left( \frac{\partial L}{\partial \gamma_t^c} - \frac{1}{2!} \frac{d}{dt} \left( \frac{\partial L}{\partial \xi_t^c} \right) \right) = 0. \]

(5.1)

For the case of \( \mathcal{L} = T^2G/G \) the above equations reduce to just the second equation which is the second order Euler–Poincaré equation. The \( k \)th order case follows directly.

**Proposition 5.2.2.** Let \( P \to M \) be a principal \( G \)-bundle, such that the Lie algebra \( \mathfrak{g} \) of \( G \) is abelian. Then given a higher order Lagrangian on the Atyiah algebroid \( \mathfrak{A} := TP/G \) the momentum

\[ \pi_k^A = \frac{\partial L}{\partial v^d_t^A} - \frac{1}{2!} \frac{d}{dt} \left( \frac{\partial L}{\partial v_t^d} \right) + \cdots + (-1)^k \frac{1}{(k-1)!} \frac{d^{k-1}}{dt^{k-1}} \left( \frac{\partial L}{\partial \xi_{k-1}^d} \right) - (-1)^k \frac{1}{k!} \frac{d^k}{dt^k} \left( \frac{\partial L}{\partial \xi_k^d} \right), \]

is a constant of motion.

**Proof.** Follows directly from the \( k \)th order Hamel equations upon setting \( C_{ab}^c = 0 \).

**Example 5.2.3.** For instance, let \( L \) be the Lagrangian governing the motion of the tip of a javelin (cf [8]) defined on \( T^3 \mathbb{R} \) by

\[ L(x, y, z) = \frac{1}{2} \left( \sum_{i=1}^3 (y')^2 - (z')^2 \right). \]

We can understand \( \mathcal{G} = \mathbb{R}^3 \) here as a commutative Lie group, and since \( L \) is \( G \)-invariant, we get immediately the reduction to the graded bundle \( \mathbb{R}^3[1] \times \mathbb{R}^3[2] \). The Euler–Lagrange equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} - \frac{1}{2} \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) \right) = 0 \]

give in this case

\[ \frac{dy^i}{dt} = \frac{1}{2} \frac{d^2 z^i}{dt^2}. \]
The Lagrangian is regular and we get, similarly as in example 4.2.1, \( \lambda_L(y, z) = (y, -z) \), and the Hamiltonian as a function on \( \mathbb{R}^3[1] \times \mathbb{R}^3[1] \) with coordinates \( (y, \theta) \):

\[
H(y, \theta) = \langle \theta, -\theta \rangle - L(y, -\theta) = -\frac{1}{2} \left( \sum_{i=1}^{3} (y_i')^2 + (\theta_i')^2 \right).
\]

Note that we do not have the minus sign if we view the Hamiltonian as a section of the corresponding AV-bundle.

We have derived the Hamel equations using a geometric reduction and not a reduction of the variational problem. In the context of the variational problem the Hamel equations arise as a special case of the Lagrange–Poincaré equations in the presence of a trivial connection on \( P \to P/G \) i.e. a trivial background Yang–Mills field. To take a non-trivial background into account we have to deform the weighted Atiyah algebroid using the Yang–Mills field following [16]. Details for the second order case are presented in the proceeding example.

**Example 5.2.4.** Following example 5.2.1, but now in the presence of a non-trivial background Yang–Mills field, the weighted Atiyah algebroid of degree two becomes deformed by the connection and the associated curvature. This deformation only effects the \( \delta \pi \) coordinates and leads to a modified phase dynamics given by

\[
\delta \pi^A_1 = \dot{\psi}^A_1, \quad \delta \pi^2_A = \frac{\partial L}{\partial \dot{\psi}^A_1} - \pi^2_A, \quad \delta \pi^A_2 = 2\dot{\psi}^A_2, \quad \delta \pi^A_3 = \frac{\partial L}{\partial \dot{\psi}^A_2} + \left( \dot{\psi}^b \mathcal{F}_{bA}^a + \dot{\psi}^b_c \mathcal{A}_c^a_A \right) \left( \pi^2_A + \pi^1_c \mathcal{C}_{ca}^c_A \right), \quad \delta \pi^2_A = 2\dot{\psi}^A_2,
\]

\[
\delta \pi^2_a = \frac{\partial L}{\partial \dot{\psi}^2_A} + \dot{\psi}_1^t \mathcal{C}_{bA}^t \mathcal{A}_b^t_A \pi^1_c - \pi^2_a, \quad \delta \pi^2_A = \frac{\partial L}{\partial \dot{\psi}^2_A} + \dot{\psi}_1^t \mathcal{C}_{bA}^t \mathcal{A}_b^t_A \pi^1_c - \pi^2_a,
\]

In the above \( A^A_h \) are the components of the connection and

\[
\mathcal{F}_{ab}^a = \frac{\partial A^a_h}{\partial x^A} - \frac{\partial A^a_h}{\partial x^B} + A^h_B A^a_h C^d_{ab}
\]

are the components of the associated curvature. After a direct and straightforward calculation, the associated Euler–Lagrange equations are

\[
\frac{\partial L}{\partial \dot{x}^A} + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}^A_1} \right) + \left( \dot{\psi}^b \mathcal{F}_{bA}^a + \dot{\psi}^b_c \mathcal{A}_c^a_A \right) \left( \delta \pi^2_A + \delta \pi^1_c \mathcal{C}_{ca}^c_A \right) \frac{\partial L}{\partial \dot{\psi}^A_2} - \delta \pi^2_a \frac{d}{dt} \frac{\partial L}{\partial \pi^2_a} + \frac{1}{2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}^2_A} + \dot{\psi}^t \mathcal{C}_{bA}^t \mathcal{A}_b^t_A \right) \frac{\partial L}{\partial \dot{\psi}^2_A} = 0,
\]

where we have defined the covariant derivative

\[
\frac{D}{Dt} \psi_a = \frac{d}{dt} \psi_a - \dot{\psi}_1^A C^b_{ab} A^h_A \psi_c
\]

for the appropriate objects.
The above pair of equations are, up to conventions, the second order Lagrange–Poincaré equations as defined in [13] using variational methods. It is clear that if we insist that the Lagrangian is independent of the weight two coordinates then we recover the classical Lagrange–Poincaré equations. If the connection is trivial then we are back to the second order Hamel equations of the previous example. If the Lie algebra $\mathfrak{g}$ is abelian then the second order Lagrange–Poincaré equations nicely simplify to

$$\frac{\partial L}{\partial \dot{x}^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}^i} \right) + \frac{1}{2} \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dot{z}^i} \right) = \eta_i \mathfrak{h}_A^B \mathfrak{h}_A^C \left( \frac{\partial L}{\partial \dot{x}^A} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^A} \right) \right).$$

The above equations describe a generalisation of the equations describing the (non-relativistic) Lorentz force. Further notice that we have, in accordance to earlier observations that $A^2 = \frac{\partial L}{\partial \dot{y}^i} - \frac{1}{2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}^i} \right)$ is a constant of motion.

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