Abstract

The purpose of a program analysis is to compute an abstract meaning for a program which approximates its dynamic behaviour. A compositional program analysis accomplishes this task with a divide-and-conquer strategy: the meaning of a program is computed by dividing it into sub-programs, computing their meaning, and then combining the results. Compositional program analyses are desirable because they can yield scalable (and easily parallelizable) program analyses.

This paper presents algebraic framework for designing, implementing, and proving the correctness of compositional program analyses. A program analysis in our framework defined by an algebraic structure equipped with sequencing, choice, and iteration operations. From the analysis design perspective, a particularly interesting consequence of this is that the meaning of a loop is computed by applying the iteration operator to the loop body. This style of compositional loop analysis can yield interesting ways of computing loop invariants that cannot be defined iteratively. We identify a class of algorithms, the so-called path-expression algorithms [35, 37], which can be used to efficiently implement analyses in our framework. Lastly, we develop a theory for proving the correctness of an analysis by establishing an approximation relationship between an algebra defining a concrete semantics and an algebra defining an analysis.

1. Introduction

Compositional program analyses compute an approximation of a program’s behaviours by breaking a program into sub-programs, analyzing them, and then combining the results. A classical example of compositionality is interprocedural analyses based on the summarization approach [50], in which a summary is computed for every procedure and then used to interpret calls to that procedure when the program is analyzed. Compositionality is interesting for two main reasons. First is computational efficiency: compositionality is crucial to building scalable program analyses [11], and can be easily parallelized. Second, compositionality opens the door to interesting ways of computing loop invariants, based on the assumption that a summary of the loop body is available [3, 4, 26, 31].

These compositional analyses are presented using ad-hoc, analysis-specific arguments, rather than in a generic framework. This paper presents an algebraic framework for designing, implementing, and proving the correctness of such compositional program analyses.

We begin by describing the iterative framework for program analysis [7, 20] to illustrate what we mean by a program analysis framework, and to clarify what this paper provides an alternative to. In the iterative framework, an analysis designed by providing an abstract semantic domain $D$, a (complete) lattice defining the space of possible program properties and a set of transfer functions which interpret the meaning of program action as a function $D \rightarrow D$. The correctness of an analysis (with respect to a given concrete semantics) is proved by establishing an approximation relation between the concrete domain and the abstract domain (e.g., a Galois connection) and showing that the abstract transfer functions approximate the concrete ones. The result of the analysis is computed by via a chaotic iteration algorithm (e.g., [6]), which repeatedly applies transfer functions until a fixed point is reached (possibly using a widening operator to ensure convergence).

Let us contrast the iterative framework with the algebraic framework described in this paper. In the algebraic framework, an analysis designed by providing an abstract semantic domain $K$, an algebraic structure (called a Pre-Kleene algebra) defining the space of possible program properties which is equipped with sequencing, choice, and iteration operations; and a semantic function which interprets the meaning of a program action as an element of $K$. The correctness of an analysis (with respect to a given concrete semantics) is proved by establishing an soundness relation between the concrete semantic algebra and the abstract algebra. The result of the analysis is computed via a path expression algorithm [35, 37], which re-interprets a regular expression (representing a set of program paths) using the operations of the semantic algebra.

A summary of the difference between the our algebraic framework and the iterative framework is presented below:

| Iterative Framework | Algebraic Framework |
|---------------------|---------------------|
| (Complete) lattice  | Pre-Kleene Algebra   |
| Abstract transformers| Semantic function    |
| Chaotic iteration algorithm | Path-expression algorithm |
| Galois connection   | Soundness relation   |

Some program analyses make essential use of compositionality [3, 4, 26, 31] and cannot easily be described in the iterative framework. The primary contribution of this paper is a framework in which these (and similar) analyses can be described. Of course, these analyses can be (and indeed, have been) presented without the aid of a framework, which raises the question: why do we need program analysis frameworks? The practical benefits provided by our framework include the following:

- A means to study compositional analyses in an abstract setting. This allows us to prove widely-applicable theoretical results (e.g., Sections 5 and 7.2), and to design re-usable program analysis techniques (e.g., consider the wide variety of abstract domain constructions [9] that have been developed in the iterative framework).
- A clearly defined interface between the definition of a program analysis and the path-expression algorithms used to compute the result of an analysis. This allows for generic implement-

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1 Note that there are also iterative analyses which cannot be described in our framework - we are arguing an alternative rather than a replacement to the iterative framework.
A conceptual foundation upon which to build compositional analyses. For example, rather than ask an analysis designer to “find a way to abstract loops,” our framework poses the much more concrete, approachable problem of “design an iteration operator.”

In summary, the contributions of this paper are:

- We develop a generic, algebraic framework for compositional program analyses. This framework provides a unifying view for compositional program analyses, and in particular compositional loop analysis [37, 38]. Our work also unifies several uses of algebra in program analysis [5, 14, 23, 33].
- We develop a theory of abstract interpretation [7] that is specialized for algebraic analyses (Section 3). This allows an analysis designer to prove the correctness of an analysis by exhibiting a congruence relation between a concrete semantic algebra (defining a concrete semantics) and an abstract semantic algebra (defining an analysis).
- We develop a variant of our algebraic framework that is suitable for interprocedural program analyses (Sections 6 and 7). This framework includes a novel solution for handling local variables.
- We prove a pair of precision theorems (one intra-procedural (Section 4.1) and one interprocedural (Section 4.1)) which identify conditions under which the path expression algorithms compute an ideal (merge-over-path) solution to a program analysis.
- We develop a new program analysis, linear recurrence analysis, designed in our algebraic framework (Section 2). We exhibit experimental results which demonstrate that compositional loop analyses (well-suited to our algebraic framework) have the potential to generate precise abstractions of loops. The area of compositional loop analysis is relatively under-explored (as compared to iterative loop analysis), and our work provides a framework for exploring this line of research.

### 2. Overview and Motivating Example

The conceptual basis of our framework is Tarjan’s path expression algorithm [15, 28]. This algorithm takes two parameters: a program, and an interpretation (defining a program analysis). We may think of this algorithm as a two-step process. (1) For each vertex \( v \), compute a path expression for \( v \), which is a regular expression that represents the set of its control flow paths of the program which end at \( v \). (2) For each vertex \( v \), interpret the path expression for \( v \) (using the given interpretation) to compute an approximation of the executions of the program which end at \( v \).

Considering the program pictured in Figure 2, one possible path expression for \( v_0 \) (the control location immediately before the assertion) is as follows:

\[
\langle v_{\text{entry}}, v_1 \rangle \cdot \langle v_1, v_2 \rangle \cdot \langle v_2, v_3 \rangle \cdot \langle v_3, v_4 \rangle \cdot \langle v_4, v_5 \rangle \cdot \langle v_5, v_6 \rangle \cdot \langle v_6, v_4 \rangle^* \cdot \langle v_2, v_7 \rangle \cdot \langle v_7, v_2 \rangle^* \cdot \langle v_2, v_8 \rangle
\]

An interpretation consists of a semantic algebra and a semantic function. The semantic algebra consists of a universe which defines the space of possible program meanings, and sequencing, choice, and iteration operators, which define how to compose program meanings. The semantic function is a mapping from control flow edges to elements of the universe which defines the meaning of each control flow edge. The abstract meaning of a program is obtained by recursion on the path expression, by re-interpreting the regular operators with their counterparts in the semantic algebra.

### A Compositional Program Analysis for Loops

We sketch a compositional program analysis which can be defined and proved correct in our framework. This program analysis, which we call linear recurrence analysis, interprets the meaning of a program to be an arithmetic formula which represents its input/output behaviour. The analysis makes use of the power of Satisfiability Modulo Theories (SMT) solvers to synthesize interesting loop invariants.

This example serves two purposes. First, it motivates the development of our framework. The strategy this analysis uses to compute an invariant for a loop relies crucially on having access to a summary of the loop body (i.e., it relies on the compositional assumption of our framework); this makes it difficult to define this analysis in the typical iterative program analysis framework. Second, this analysis will be used as a running example to explain the formalism in the remainder of the paper. The analysis has not yet appeared elsewhere in the literature, but it bears conceptual similarity to “loop leaping” analyses [5, 28], and the compositional induction variable analysis in [5].

#### Universe

The universe of this semantic algebra, \( L \), is the set of arithmetic formulae with free variables in \( \text{Var} \cup \text{Var}' \) (quotiented by logical equivalence) where \( \text{Var} \) is the set of program variables (inputs), and \( \text{Var}' \) is a set of “primed” copies of program variables (outputs). Informally, each such formula represents a transition relation between states, and we say that a formula \( \phi_P \) approximates a program fragment \( P \) if all the input/output pairs of \( P \) belong to the transition relation of \( \phi_P \).

#### Semantic Function

The semantic function of linear recurrence analysis is defined in the obvious way (by considering the effect of the assignment label of each edge). For example, (again, considering Figure 2),

\[
\llbracket \phi_{\text{entry}} \cdot v_1 \rrbracket = q' \land q' = x \land t' = t \land x' = x \land y' = y
\]

\[
\llbracket v_1 \cdot v_2 \rrbracket = q' = 0 \land q' = r \land t' = t \land x' = x \land y' = y
\]

\[
\llbracket v_2 \cdot v_3 \rrbracket = r > y \land q' = q \land r' = r \land t' = t \land x' = x
\]

\[
\land y' = y
\]

#### Operators

Formally, the sequencing and choice operators are defined as follows:

\[
\phi \circ \psi = \exists x'. \phi[x'/x'] \land \psi[x'/x] \quad \text{Sequencing}
\]

\[
\phi \oplus \psi = \phi \lor \psi \quad \text{Choice}
\]

(Where \( \phi[x'/x] \) denotes \( \phi \) with each primed variable \( x' \) replaced by its double-primed counterpart \( x'' \), and \( \psi[x'/x] \) similarly replaces unprimed variables with double-primed variables).

The semantic function, sequencing, and choice operators are enough to analyze loop-free code. Interestingly, since these operations are precise, loop-free code is analyzed without loss of information. We can illustrate how loop free code is analyzed by considering the following example:

\[
\langle v_{\text{entry}}, v_1 \rangle \cdot \langle v_1, v_2 \rangle \cdot \langle v_2, v_3 \rangle \cdot \langle v_3, v_4 \rangle \cdot \langle v_4, v_5 \rangle \cdot \langle v_5, v_6 \rangle \cdot \langle v_6, v_4 \rangle^* \cdot \langle v_2, v_7 \rangle \cdot \langle v_7, v_2 \rangle^* \cdot \langle v_2, v_8 \rangle
\]
Considering how the meaning of the body of the inner-most loop is computed:
\[
\left(\langle v_4, v_5 \rangle \cdot \langle v_5, v_6 \rangle \right)_{xy}^L = \left(\langle v_4, v_5 \rangle \right)_{xy}^L \otimes_{\mathbb{Z}} \left(\langle v_5, v_6 \rangle \right)_{xy}^L
\]
\[
= t > 0 \land q' = q \land r' = r + 1 \land t' = t \land x' = x \land y' = y
\]
\[
\left(\langle v_4, v_5 \rangle \cdot \langle v_5, v_6 \rangle \cdot \langle v_6, v_4 \rangle \right)_{xy}^L = \left(\langle v_4, v_5 \rangle \cdot \langle v_5, v_6 \rangle \right)_{xy}^L \otimes_{\mathbb{Z}} \left(\langle v_6, v_4 \rangle \right)_{xy}^L
\]
\[
= t > 0 \land q' = q \land r' = r - 1 \land t' = t - 1 \land x' = x \land y' = y
\]
In the following, we will refer to the preceding formula as \( \varphi_{inner} \).

The final step in defining our analysis is to provide a definition of the iteration operator \( (\otimes_{\mathbb{Z}}) \). The idea behind the iteration operator of linear recurrence analysis is to use an SMT solver to extract linear recurrence relations, and then solve the recurrence relations to compute closed forms for (some of) the program variables.

Let \( \varphi \) be a formula (representing the body of a loop). The stratified linear induction variables of \( \varphi \) are defined as follows:

- If \( \models x' = x + n \) for some constant \( n \in \mathbb{Z} \), then \( x \) is a linear induction variable of stratum 0.
- If \( \models x' = x + f(y) \), where \( f \) is an affine function whose variables range over linear induction variables of strictly lower strata than \( n \), then \( x \) is a linear induction variable of stratum \( n \).

For example, in the formula \( \varphi_{inner} \), there are five induction variables (all at stratum 0) satisfying the following recurrences:

| Recurrence | Closed form |
|------------|-------------|
| \( q' = q + 0 \) | \( q_k = q \) |
| \( r' = r + 1 \) | \( r_k = r + k \) |
| \( t' = t - 1 \) | \( t_k = r - k \) |
| \( x' = x + 0 \) | \( x_k = x \) |
| \( y' = y + 0 \) | \( y_k = y \) |

Stratified induction variables can be detected automatically using an SMT solver. We use \( I(\varphi) \) to denote the set of all (stratified) induction variables in \( \varphi \). For each \( x \in I(\varphi) \), we may solve the recurrence associated with \( x \) to get a closed form, which is a function \( cL_\varphi \) with variables ranging over the program variables and a distinguished variable \( k \) indicating the iteration of the loop. Then \( \varphi_{\otimes_{\mathbb{Z}}} \) is defined as follows:

\[
\varphi_{\otimes_{\mathbb{Z}}} = \exists k. k \geq 0 \land \bigwedge_{x \in \ell(\varphi)} x' = cL_\varphi
\]

In the particular case of \( \varphi_{inner} \), we have:

\[
\varphi_{inner}^{\otimes_{\mathbb{Z}}} = \exists k. k \geq 0 \land q' = q \land r' = r - k \land t' = t - k \\
\land x' = x \land y' = y
\]

Describing how induction variables are detected and how recurrence relations are solved is beyond the scope of this paper (which is about the program analysis framework used for presenting this analysis, rather than the analysis itself).

**Computing the Meaning of the Program.** Using the summary for the inner loop computed previously, we may proceed to compute the abstract meaning of the body of the outer loop as:

\[
\varphi_{outer} = q' = q + 1 \land r' = r - y \land x' = x \land y' = y
\]

We apply the iteration operator to compute the meaning of the outer loop. We detect four induction variables, three of which \((x, y, q)\) are at stratum 0, and one of which \((r)\) is at stratum 1:

| Recurrence | Closed form |
|------------|-------------|
| \( q' = q + 1 \) | \( q_k = q + k \) |
| \( r' = r - y \) | \( r_k = r - yk \) |
| \( x' = x + 0 \) | \( x_k = x \) |
| \( y' = y + 0 \) | \( y_k = y \) |

We compute the abstract meaning of the outer loop as:

\[
\varphi_{outer}^{\otimes_{\mathbb{Z}}} = \exists k. k \geq 0 \land q' = q + k \land t' = r - ky \land x' = x \land y' = y
\]

Note the role of compositionality in computing the summary for the outer loop. Detection of stratified induction variables depends only on having access to a formula representing a loop body. The fact that the outer loop body contains another loop is completely transparent to the analysis.

Finally, an input/output formula for the entire program is computed:

\[
\varphi F = \left[\langle v_{entry}, v_1 \rangle \cdot \langle v_1, v_2 \rangle \right]_{xy}^L \otimes_{\mathbb{Z}} \varphi_{outer}^{\otimes_{\mathbb{Z}}} \otimes_{\mathbb{Z}} \left[\langle v_2, v_{exit} \rangle \cdot (v_{v_2}) \cdot \langle v_{v_2}, v_{exit} \rangle \right]_{xy}^L
\]

\[
= q' \geq 0 \land r' = x - q'y \land k \leq y \land x' = x \land y' = y
\]

This formula is strong enough to imply that the assertion at exit (the standard post-condition for the quotient/remainder computation) correct. This is particularly interesting because it requires
proving a non-linear loop invariant, which is out of scope for many state-of-the-art program analyzers (e.g., [23, 15]).

3. Preliminaries

In this section, we will introduce our program model and review the concept of path expressions [37, 38], which is the basis of our program analysis framework.

**Program model.** A program consists of a finite set of procedures \( \{ P_i \}_{i \leq n} \). A procedure is a pair \( P_i = (G_i, LV_i) \), where \( G_i \) is a flow graph and \( LV_i \) is a set of local variables. A flow graph is a finite directed graph \( G_i = (V_i, E_i, v^{entry}, v^{exit}) \) equipped with a distinguished entry vertex \( v^{entry} \in V_i \) and exit vertex \( v^{exit} \in V_i \). We assume that every vertex in \( V_i \) is reachable from \( v^{entry} \). \( v^{exit} \) has no incoming edges, \( v^{entry} \) has no outgoing edges, and that the vertices and local variables of each procedure are pairwise disjoint.

To simplify our discussion, we associate with each edge \( e \in E = \bigcup_{i \leq n} E_i \) an action \( \text{act}(e) \in \text{Act} \) that gives a syntactic representation of the meaning of that edge, where Act is defined as:

\[
\text{Act} ::= x \mid \text{assume}(t) \mid \text{return} \mid \text{call} \ i \ where \ 1 \leq i \leq N
\]

\( s, t \in \text{Exp} ::= x \mid n \ where \ n \in \mathbb{Z} \mid s \cdot t \ where \ \bullet \in \{+, -, \ast, /\} \)

\( x \in \text{Var} \supseteq \bigcup_{i \leq N} LV_i \)

We will sometimes use \( [\text{I}] \) in place of \( \text{assume}(t) \) as in Figure 2. Note that our model has no parameters or return values in procedure calls, but these can be modeled using global variables.

Let \( G = (V, E, v^{entry}, v^{exit}) \) be a flow graph. For an edge \( e \in E \), we use \( \text{src}(e) \) to denote the source of \( e \) and \( \text{tgt}(e) \) to denote its target. We define a path \( \pi = e_1 \cdots e_n \in E^* \) to be a finite sequence of edges such that for each \( i < n \), \( \text{tgt}(e_i) = \text{src}(e_{i+1}) \). We lift the src/tgt notation from edges to paths by defining \( \text{src}(\pi) = \text{src}(e_1) \) and \( \text{tgt}(\pi) = \text{tgt}(e_n) \). For any \( u, v \in V \), we define \( \text{Paths}_u[v, u, v] \) to be the set of paths \( \pi \) of \( G \) such that \( \text{src}(\pi) = u \) and \( \text{tgt}(\pi) = v \).

**Path expressions.** We will assume familiarity with regular expressions. We use the following syntax for the regular expressions RegExp over some alphabet \( \Sigma \): \( p, q \) denote regular expressions, \( e \) denotes the empty word, \( \emptyset \) denotes the empty language, \( p + q \) denotes choice, \( p \cdot q \) denotes sequencing, and \( p^\ast \) denotes iteration. A path expression for a flow graph \( G = (V, E, v^{entry}, v^{exit}) \) is a regular expression \( p \in \text{RegExp} \) over the alphabet of edges of \( G \), such that each word in the language generated by \( p \) corresponds to a path in \( G \). For any pair of vertices \( u, v \), there is a (not necessarily unique) path expression that generates \( \text{Paths}_u[v, u, v] \), the set of paths from \( u \) to \( v \). A particular case of interest is the path expressions \( \text{Paths}_u[v, u, v] \) for all control flow paths from the entry vertex of \( G \) to \( v \). Such a path expression can be seen as a succinct representation of the set of all computations of \( G \) ending at \( v \).

The (single-source) path expression problem for a flow graph \( G = (V, E, v^{entry}, v^{exit}) \) is to compute for each vertex \( v \in V \) a path expression \( P^v_{v^{entry}, v} \in \text{RegExp} \) representing the set of paths \( \text{Paths}_{v^{entry}, v} \). An efficient algorithm for solving this problem was given in [37]; a more recent path expression algorithm appears in [5]. Understanding the specifics of these algorithms is not essential to our development, but the idea behind both is to divide \( G \) into subgraphs, use Gaussian elimination to solve the path expression problem within each subgraph, and then combine the solutions. Kleene’s classical algorithm for converting a finite automaton to a regular expression [21] provides another way of solving path-expression problems that, while less efficient than [35, 37], should provide adequate intuition for readers not familiar with these algorithms.

4. Interpretations

In our framework, meaning is assigned to programs by an interpretation. We use interpretations to define both program analyses and the concrete semantics which justify their correctness. Interpretations are composed of two parts: (i) a semantic domain of interpretation, which defines a set of possible program meanings and which is equipped with operations for composing these meanings, and (ii) a semantic function that associates a meaning to each atomic program action. In this section, we define interpretations formally.

We will limit our discussion in this section to the intraprocedural variant of interpretations. We will assume that a program consists of a single procedure and that the flow graph associated with that procedure has no call or return actions. We will continue to make this assumption until Section 6 where we introduce our interprocedural framework.

Our semantic domains are algebraic structures endowed with the familiar regular operators of sequencing, choice, and iteration. These algebraic structures are used in place of the (complete) lattices typically used in the iterative theory of abstract interpretation. We are interested in two particular variations: pre-Kleene algebras (PKAs) and quantales [30]. Pre-Kleene algebras use weaker assumptions than the ones for Kleene algebras [24], while quantales make stronger ones. We make use of both variants for three reasons. (I) PKAs are desirable because we wish to make our framework as broadly applicable as possible, so that it can be used to design and prove the correctness of a large class of program analyses [3]. (II) Quantales are desirable because we can prove a precision theorem (Theorem 4) comparable to the classical coincidence theorem of dataflow analysis [19]. (III) Due to a result we will present in Section 5 it is particularly easy to prove the correctness of a program analysis over a PKA with respect to a concrete semantics over a quantale (as is the case, for example, in our motivating example of linear recurrence analysis).

We begin by formally defining pre-Kleene algebras:

**Definition 4.1 (PKA).** A pre-Kleene algebra (PKA) is a 6-tuple \( K = (\mathcal{K}, \circ, \oplus, \emptyset, \otimes, 1) \), where \( (\mathcal{K}, \circ, 0) \) forms join-semilattice with least element 0 (i.e., \( \circ \) is a binary operator that is associative, commutative, and idempotent, and which has 0 as its unit), \( (\mathcal{K}, \oplus, 1) \) forms a monoid (i.e., \( \oplus \) is an associative binary operator with unit 1), \( \emptyset \) is a unary operator, and such that the following hold:

\[
\begin{align*}
(a \circ b) \oplus (a \circ c) & \leq a \circ (b \oplus c) & \text{or} & \text{left pre-distributive} \\
(a \circ c) \oplus (b \circ c) & \leq (a \oplus b) \circ c & \text{or} & \text{right pre-distributive} \\
1 \oplus (a \circ a^\emptyset) & \leq a^\emptyset \\
1 \oplus (a^\emptyset \circ a) & \leq a^\emptyset
\end{align*}
\]

(11)

where \( \leq \) is the order on the semilattice \( (\mathcal{K}, \circ, 0) \). The priority of the operators is the standard one (\( \circ \) binds tightest, followed by \( \oplus \)).

As is standard in abstract interpretation, the order on the domain should be conceived as an approximation order: \( a \leq b \) iff \( a \) is approximated by \( b \) (i.e., if \( a \) represents a more precise property than \( b \)). Note that the pre-distributivity axioms for \( \oplus \) are equivalent to \( \circ \) being monotone in both arguments.

\footnote{In fact, the PKA assumptions are weaker than those of any similar algebraic structure that have been proposed for program analysis that we are aware of: Kleene algebras used in [3], the bounded idempotent semirings used in weighted pushdown systems [33], the flow algebras of [14], and the left-handed Kleene algebras of [23].}
Example 4.2. We return to our motivating example, linear recurrence analysis. We define the semantic algebra of this analysis as:

\[ \mathcal{L} = \langle L, \oplus, \wedge, \cdot, 0, 1 \rangle \]

The set of transition formulae \( L \) and the \( \oplus, \wedge, \cdot \) operators are defined as in Section 2. The constants \( 0 \) and \( 1 \) are defined as

\[ 0 = \false \quad 1 = \bigvee_{x \in \mathcal{V}} x = \star \]

It is easy to check that \( \mathcal{L} \) forms a PKA. We will show only that (I) holds as an example. Let \( a \in L \). Recalling the definition of \( \cdot \) from Section 2, we have

\[ a \cdot x = \exists k \geq 0 \land \bigvee_{x \in I(\phi)} x' = cl_k \]

where for each induction variable \( x \in I(\phi) \), \( cl_k \) is a closed form for an affine recurrence \( x' = x + f(y) \) such that \( \phi = x' = x + f(y) \).

It is easy to see that \( 1 \cdot x \leq \phi^* \) (since \( \leq \) is logical implication in this algebra and for all \( x \), \( cl_0[0/k] = x \)). It remains to show that \( \phi \cdot x \leq \phi^* \), noting that \( \phi \cdot x \) is monotone and (by the definition of \( I \) and \( cl_k \)) \( \phi \leq \bigvee_{x \in I(\phi)} x' = cl_x \). We have

\[ \phi \cdot x \leq \bigvee_{x \in I(\phi)} x' = x + f(y) \]

Quantales are a strengthening of the axioms of Kleene algebras that are particularly useful for representing concrete semantics. Intuitively, we may think of quantales as Kleene algebras where the choice and iteration operators are “precise.”

Definition 4.3 (Quantale). A quantale \( K = \langle K, \oplus, \cdot, 0, 1 \rangle \) is a PKA such that \( (K, \cdot, 0) \) forms a complete lattice, and for any \( a \in K \), \( a^\# = \bigoplus_{n \in \mathbb{N}} a^n \), and such that \( \cdot \) distributes over arbitrary sums: that is, for any index set \( I \), we have

\[ a \cdot \left( \bigoplus_{i \in I} b_i \right) = \bigoplus_{i \in I} (a \cdot b_i) \]

\[ \left( \bigoplus_{i \in I} a_i \right) \cdot b = \bigoplus_{i \in I} (a_i \cdot b) \]

Example 4.4. For any set \( A \), the set of binary relations on \( A \) forms a quantale. A case of particular interest is binary relations over the set \( \mathcal{E} = \mathcal{V} \to \mathcal{Z} \) of program environments. This quantale is the semantic algebra of the concrete semantics which justifies the correctness of linear recurrence analysis (i.e., it fills the typical role of state collecting semantics in the iterative abstract interpretation framework). Formally, this quantale is defined as:

\[ R = \langle R_{\mathcal{E} \times \mathcal{E}}, \oplus, \cdot, 0, 1 \rangle \]

where

\[ R \cdot S = \{ (\rho, \rho') : \exists \rho^\prime \cdot (\rho, \rho^\prime) \in R \land (\rho^\prime, \rho'') \in S \} \]

0 \cdot \mathcal{E} = \emptyset \quad R \oplus S = R \cup S \quad 1 \cdot \mathcal{E} = \{ (\rho, \rho) : \rho \in \mathcal{E} \}

We note that \( \mathcal{L} \), the semantic algebra of linear recurrence analysis defined in Example 4.2, is not an example of a quantale. In particular, it does not form a complete lattice and its iteration operator does not satisfy the condition that for all \( a, a^\# = \bigoplus_{n \in \mathbb{N}} a^n \).

It is also interesting to note that \( \mathcal{L} \) does not meet the assumptions of any of the Kleene-algebra-like structures that have been previously proposed for use in program analysis [3, 4, 5, 6].

Finally, we may formally define our notion of interpretation, which will be useful for defining both concrete semantics and program analyses in our framework.

Definition 4.5 (Interpretation). An interpretation is a pair \( I = \langle D, \llbracket \cdot \rrbracket \rangle \), where \( D \) is a set equipped with the regular operations (e.g., a pre-Kleene algebra or a quantale) and \( \llbracket \cdot \rrbracket : E \to D \). We will call \( D \) the domain of the interpretation and \( \llbracket \cdot \rrbracket \) the semantic function.

Interpretations assign meanings to programs by “evaluating” path expressions within the interpretation. For any interpretation \( I = \langle D, \llbracket \cdot \rrbracket \rangle \) and any path expression \( a \) in \( D \), we use \( \llbracket a \rrbracket \) to denote the interpretation of \( a \) within the domain \( D \), which can be defined recursively as:

\[ \llbracket e \rrbracket = [e] \quad \llbracket e + f \rrbracket = \llbracket e \rrbracket + \llbracket f \rrbracket \quad \llbracket 0 \rrbracket = 0 \quad \llbracket e \cdot f \rrbracket = \llbracket e \rrbracket \cdot \llbracket f \rrbracket \quad \llbracket 1 \rrbracket = 1 \]

A first example of an interpretation is the one defining linear recurrence analysis, obtained by pairing the PKA defined in Example 4.2 with the semantic function defined in Section 2. We finish this section with a second example, the relational interpretation, which we will use in the next section to justify the correctness of linear recurrence analysis in Section 5.

Example 4.6. In the relational interpretation, the meaning of a program is its input/output relation. That is, the meaning of a program is the set of all \( (\rho, \rho') \in \mathcal{E} \times \mathcal{E} \) such that there is an execution of the program that, starting from environment \( \rho \), terminates in environment \( \rho' \). Formally, the relational interpretation \( R \) is defined as:

\[ R = \langle R, \llbracket \cdot \rrbracket \rangle \]

where \( R \) is as in Example 4.4, the semantic function is defined by

\[ \llbracket a \rrbracket = \begin{cases} \{ (\rho, \rho') : \rho \in \mathcal{E} \} & \text{if } \text{act}(a) = x := t \\ \{ (\rho, \rho') : \rho \in \mathcal{E} \ Mon, \rho \not= t \neq 0 \} & \text{if } \text{act}(a) = [t] \end{cases} \]

(using \( \text{act}(a) \) to denote the integer to which the expression \( t \) evaluates in environment \( \rho \)).

4.1 Correctness and Precision

We now justify the correctness of using path-expression algorithms for program analysis. Given a control flow graph and a vertex \( v \) of the graph, a path expression \( P[v\text{entry}, v] \) is a regular expression representing the set of paths from \( v\text{entry} \) to \( v \). A reasonable assumption is that the interpretation of \( P[v\text{entry}, v] \) approximates the interpretation of each of these paths. The following theorem states that this is the case, if the semantic algebra of the interpretation is a PKA.

Theorem 4.7 (Correctness). Let \( I = \langle D, \llbracket \cdot \rrbracket \rangle \) be an interpretation where \( D \) is a PKA, \( G = \langle V, E, v\text{entry}, v\text{exit} \rangle \) be a flow graph,
and $v \in V$ be a vertex of $G$. Then for any path $e_1 \cdots e_n \in \text{Paths}[v^{\text{entry}}, v]$, we have

$$[e_1] \cdots [e_n] \leq \mathcal{I}[P[v^{\text{entry}}, v]]$$

**Proof.** We prove that for any word $w = e_1 \cdots e_n$ that is recognized by $P[v^{\text{entry}}, v]$ and any path expression $p$ that recognizes $w$, we have

$$[w] \leq \mathcal{I}[p]$$

(where $[w]$ denotes $[e_1] \cdots [e_n]$) by induction on $p$. The main result then follows since the language recognized by $P[v^{\text{entry}}, v]$ is exactly $\text{Paths}[v^{\text{entry}}, v]$.

Case $p = \emptyset$: contradiction, since $w$ is recognized by $p$.

Case $p = e$: then $n = 0$, and $\mathcal{I}[p] = 1 = [w]$.

Case $p = e$: for some edge $e$: then $n = 0, e_1 = e$, and $\mathcal{I}[p] = [e] = [w]$.

Case $p = q + r$: wlog, we may assume that $w$ is accepted by $q$. By the induction hypothesis, $[w] \leq \mathcal{I}[q]$, whence

$$[w] \leq \mathcal{I}[q] \sqcup \mathcal{I}[r] = \mathcal{I}[q + r]$$

Case $p = q$: then there exists $w_1, w_2$ such that $w = w_1 w_2$, $w_1$ is recognized by $q$, and $w_2$ is recognized by $r$. By the induction hypothesis, $[w_1] \leq \mathcal{I}[q]$ and $[w_2] \leq \mathcal{I}[r]$. By monotonicity of $\sqcup$,

$$[w] = [w_1] \sqcup [w_2] \leq \mathcal{I}[q] \sqcup \mathcal{I}[r]$$

Case $p = q^*$: Since $w$ is recognized by $q^*$, there exists some $n$ such that $w = w_1 \cdots w_n$ and each $w_i$ is recognized by $q$. By the induction hypothesis, $[w_i] \leq \mathcal{I}[q]$ for each $i$. It follows that

$$[w] = [w_1] \cdots [w_n] \leq \mathcal{I}[p]^n$$

Finally, we have $\mathcal{I}[p]^n \leq \mathcal{I}[p]^\# = \mathcal{I}[p^\#]$ by Lemma 5.3.

The preceding theorem is analogous to the classical result which was used to justify the use of chaotic iteration algorithms for data flow analysis: the fixed-point solution to an analysis approximates the merge-over-paths solution [20]. Our framework also accommodates a precision theorem, which is analogous to the coincidence theorem from dataflow analysis [19]. This result states that any path-expression algorithm computes exactly the least upper bound of the interpretations of the paths in the case that the semantic algebra of the interpretation is a quantale.

**Theorem 4.8** (Precision). Let $\mathcal{I} = (D, \Pi)$ be an interpretation where $D$ is a quantale, $G = \{V, E, v^{\text{entry}}, v^{\text{exit}}\}$ be a flow graph, and $v \in V$ be a vertex of $G$. Then

$$\bigoplus_{\pi \in \text{Paths}[v^{\text{entry}}, v]} [\pi] = \mathcal{I}[P[v^{\text{entry}}, v]]$$

where for a path $\pi = e_1 \cdots e_n$, we define $[\pi] = [e_1] \cdots [e_n]$.

Our precision theorem implies that for any path expressions $p$ and $p'$ that recognize the same set of paths, and any interpretation $\mathcal{I}$ over a quantale, we have $\mathcal{I}[p] = \mathcal{I}[p']$. This is important from the algorithmic perspective since it implies that the result of an analysis over a quantale is the same regardless of how path expressions are computed (e.g., we may use any of the algorithms appearing in [20] or [21]). While this “algorithm irrelevance” property would be undesirable in general, it would be un-intuitive if the concrete semantics of a program depended on such a detail.

5. Abstraction

We now develop a theory of abstract interpretation for (intraprocedural) compositional program analyses in our algebraic framework. There are a vast number of ways to approach the problem of establishing an approximation relationship between one interpretation and another [10]. This section presents an adaptation of one such method, soundness relations, to our framework.

Given two PKAs $\mathcal{C}$ and $\mathcal{A}$, a soundness relation is a relation $\sqsubseteq \subseteq \mathcal{C} \times \mathcal{A}$ that defines which concrete properties (elements of $\mathcal{C}$) are approximated by which abstract properties (elements of $\mathcal{A}$). That is, $c \sqsubseteq a$ indicates that $a \in \mathcal{A}$ is a sound approximation of $c \in \mathcal{C}$. We require that soundness relationships are congruent with respect to the regular operations; for example, if $a \in \mathcal{A}$ approximates $c \in \mathcal{C}$, then $c \circ a$ should approximate $c \circ c$. Formally, we define soundness relations as follows:

**Definition 5.1** (Soundness relation). Given two PKAs $\mathcal{C}$ and $\mathcal{A}$, $\sqsubseteq \subseteq \mathcal{C} \times \mathcal{A}$ is a soundness relation if $0_{\mathcal{C}} \sqsubseteq 0_{\mathcal{A}}$, $1_{\mathcal{C}} \sqsubseteq 1_{\mathcal{A}}$, and for all $c \sqsubseteq a$ and $c' \sqsubseteq a'$ we have the following:

$$c \circ c' \sqsubseteq a \circ a'$$

Algebraically speaking, $\sqsubseteq \subseteq \mathcal{C} \times \mathcal{A}$ is a soundness relation iff it is a subalgebra of the direct product $\mathcal{P}K\mathcal{A} \times \mathcal{A}$.

The main result on soundness relations is the following abstraction theorem, which states that if we can exhibit a soundness relation $\sqsubseteq$ between a concrete interpretation $\mathcal{C}$ and an abstract interpretation $\mathcal{A}$, then the interpretation of any path expression in $\mathcal{C}$ is approximated by its interpretation in $\mathcal{A}$ according to $\sqsubseteq$. From a program analysis perspective, we can think of the conclusion of this theorem as the program analysis defined by $\mathcal{A}$ is correct with respect to the concrete semantics defined by $\mathcal{C}$.

**Theorem 5.2** (Abstraction). Let $\mathcal{C}$ and $\mathcal{A}$ be interpretations over domains $\mathcal{C}$ and $\mathcal{A}$, $\sqsubseteq \subseteq \mathcal{C} \times \mathcal{A}$ be a soundness relation, and let $p \in \text{RegExp}_E$ be a path expression. If for all $e \in E$, $[e]_{\mathcal{C}} \sqsubseteq [e]_{\mathcal{A}}$, then $[p]_{\mathcal{C}} \sqsubseteq [p]_{\mathcal{A}}$. Moreover, this result holds even if $\mathcal{C}$ and $\mathcal{A}$ do not satisfy any of the PKA axioms.

**Proof.** Straightforward, by induction on $p$. 

A common special case of the soundness relation framework is when the concrete domain is a quantale and the abstract domain is a PKA (e.g., this is the case when proving the correctness of linear recurrence analysis w.r.t. the relational interpretation). Under this assumption, the following proposition establishes a set of conditions that imply that a given relation is a soundness relation, but which may be easier to prove than using Definition 5.1 directly. In particular, this proposition relaxes the analysis designer from having to prove congruence of the iteration operator.

**Proposition 5.3.** Let $\mathcal{C}$ be a quantale, $\mathcal{A}$ be a PKA, and let $\sqsubseteq \subseteq \mathcal{C} \times \mathcal{A}$ be a relation satisfying the following:

1. $1_{\mathcal{C}} \sqsubseteq 1_{\mathcal{A}}$
2. If $c \sqsubseteq a$ and $c' \sqsubseteq a'$, then $c \circ c' \sqsubseteq a \circ a'$
3. If $c \sqsubseteq a$ and $a \leq a'$, then $c \leq a'$.
4. For all $S \subseteq \mathcal{C}$, if for all $c \in S$, $c \sqsubseteq a$, then $\bigoplus_c S \sqsubseteq a$.

Then $\sqsubseteq$ is a soundness relation.

**Proof.** Congruence of $0$ and $\oplus$ are straightforward, so we just prove congruence of iteration. First, we prove a lemma:

**Lemma 5.4** (Star induction). Let $D$ be a PKA and let $p \in D$. Then for any $n \in \mathbb{N}$, $p^n \leq p^\ast$.

**Proof.** By induction on $n$. For the base case $n = 0$, we have $p^0 = 1 \leq p^\ast$. 

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by (11).

For the induction step, assume that \( p^n \leq p^* \) and calculate
\[
p^{n+1} = p^n \odot p \leq p^* \odot p \quad \text{induction hypothesis, monotonicity}
\]
\[
\leq p^* \quad \text{(12)}
\]
\[
\square
\]

Now we return to proving congruence of iteration. Suppose \( \models \) satisfies the conditions of Proposition 5.3 and that \( e \models a^x \) for every unprimed variable \( a \) by the third condition \( e^a \models a^x \) for all \( a \). It is easy to prove that \( e^a \models a^x \) for every unprimed variable \( a \), using the first two conditions for \( \models \). It follows from the Lemma 5.3 that \( a^n \models a^x \) for all \( n \), so we have by the third condition \( e^a \models a^x \) for all \( a \). Finally, the fourth condition implies \( e^c \models c^x \) (since \( c^x \) is a quantale, \( c^x \) is the least upper bound of the set \( \{e^n : n \in \mathbb{N}\} \)).

We conclude our discussion of soundness relations with an example of how they can be used in practice.

**Example 5.5** The correctness of linear recurrence analysis is proved with respect to the relational semantics (Examples 4.4 and 4.6), where the soundness relation \( \models \subseteq R \times \mathcal{L} \) is given by
\[
R \models \varphi \iff \forall (p,p') \in R, [p,p'] \models \varphi
\]
where \( [p,p'] \) denotes the structure such that \( p(x) = \rho(x) \) for every unprimed variable \( x \), and \( p(x') = \rho(x) \) for every primed variable \( x' \). We may show that \( e \models \) is a soundness relation by Proposition 5.3 Conditions 1 and 2 (congruence of 1 and \( \odot \)) are trivial. Conditions 3 and 4 follow easily from the definition of \( \models \).

Finally, it is easy to see that for any edge \( e \), we have \( [e]^\mathcal{R} \models [e]^\mathcal{R} \), so by Theorem 5.2 for any path expression \( p \in \text{RegExp} \), we have \( [p]^\mathcal{R} \models \mathcal{L}[p] \).

\[
\square
\]

### 6. Interprocedural interpretations

A major problem in program analysis is designing analyses that handle procedures both efficiently and precisely. One of the primary benefits of compositional program analyses is that they can be easily adapted to the interprocedural setting by employing procedure summarization. The idea behind this approach is to compute a summary for each procedure \( i \) which overapproximates its execution and then uses these summaries to interpret call actions (14).

One of the challenges involved in interprocedural analysis is the need to deal with local variables. The difficulty lies in the fact that the values of local variables before a procedure call are restored after the procedure call. Our approach uses an algebraic quantification operator, inspired by algebraic logic (14,17), to intuitively move variables out of scope. This is a binary operation \( \exists x.a \) that takes a variable \( x \) and a property \( a \), and yields a property where the variable \( x \) is “out of scope”, which is subject to the interpretation.

Interprocedural analysis presents another challenge that is specific to our algebraic approach: the regular operations (choice, sequencing, and iteration) are not sufficient for describing interprocedural behaviours, since the language of program paths with appropriately matched calls and returns (i.e., interprocedural paths) is not regular in the presence of recursion. Quantales already have sufficient structure to be appropriate semantic domains for interprocedural analyses, but PKAs need to be augmented with a widening operator.

We will now formally define interprocedural semantic domains and interpretations. We will explain the motivation behind these definitions and how they solve the problems of local variables and recursion in the rest of this section.

**Definition 6.1 (Quantified Pre-Kleene Algebra).** A quantified pre-Kleene algebra (QPKA) over a set of variables \( \text{Var} \) is a tuple
\[
\mathcal{K} = \langle K, \oplus, \ominus, \otimes, \exists, \forall, 0, 1 \rangle,
\]
where \( (K, \oplus, \ominus, \otimes, \exists, \forall, 0, 1) \) is a PKA, and for the quantification operator \( \exists : \text{Var} \times K \to K \) and the widening operator \( \forall : K \times K \to K \) the following hold:
\[
(\exists x.a) \oplus (\exists x.b) \leq \exists x.(a \ominus b) \quad \text{(Q1)}
\]
\[
\exists x.((\exists x.a) \ominus b) = (\exists x.a) \ominus (\exists x.b) \quad \text{(Q2)}
\]
\[
\exists x.\exists y.a = \exists y.\exists x.a \quad \text{(Q3)}
\]
\[
a \ominus b \leq a \ominus b \quad \text{(W1)}
\]
and for any sequence \( \langle a_i \rangle_{i \in \mathbb{N}} \) in \( K \), the sequence \( \langle w_i \rangle_{i \in \mathbb{N}} \) defined by
\[
w_0 = a_0
\]
\[
w_i = w_{i-1} \forall a_n \text{ eventually stabilizes (there exists some } k \in \mathbb{N} \text{ such that for all } j > k, w_k = w_j \).
\]

We discuss the intuition behind the quantification operator in Section 6.1. The conditions on quantification come from the treatment of existential quantifiers in algebraic logic (14,17) (hence our use of “quantification” for our scoping operator). The conditions for the widening operator are standard (14). We adapt quantales to the interprocedural setting in a similar fashion, except that we omit the widening operator and require that scoping distribute over arbitrary sums.

**Definition 6.2 (Quantified Quantale).** A quantified quantale over a set of variables \( \text{Var} \) is a tuple \( \mathcal{K} = \langle K, \oplus, \ominus, \otimes, 0, 1 \rangle \) such that \( (K, \oplus, \ominus, 0, 1) \) is a quantale and \( \exists : \text{Var} \times K \to K \) is an operator such that (Q2)-(Q4) hold and for any \( x \in \text{Var} \) and index set \( I \),
\[
(\exists x.\bigoplus_{i \in I} a_i) = \bigoplus_{i \in I} (\exists x.a_i) \quad \text{(Q1')}
\]

Since the order of quantification does not matter (due to (Q4)), for any finite set of variables \( X = \{x_1, \ldots, x_n\} \) we will write \( \exists X.a \) to denote \( \exists x_1 \exists x_2 \cdots \exists x_n.a \).

Recall that we use \( \mathcal{E} \) to denote the set of control flow edges of all procedures, \( E_i \) to denote the edges of procedure \( i \), and \( LV_i \) to denote the local variables of procedure \( i \). Interpretations can be adapted to interprocedural semantic domains as follows:

**Definition 6.3 (Interprocedural interpretation).** An interprocedural interpretation is a pair \( \mathcal{I} = (\mathcal{D}, \mathcal{I}) \), where \( \mathcal{D} \) is an interprocedural semantic domain (a quantified PKA or quantale), \( \mathcal{I} : \mathcal{E} \to \mathcal{D} \), and such that for any \( e \in E \), and variable \( x \in LV_i \) (with \( i \neq j \)), \( \exists x.[e] = [e] \) (i.e., the actions belonging to procedure \( i \) do not read from or write to the local variables of procedure \( j \)).

### 6.1 Local variables

We now define the quantification operator for the relational interpretation to provide intuition on how our framework deals with local variables. Any relation \( R \subseteq \text{Env} \times \text{Env} \) partitions the set of variables into a footprint and a frame. The variables in the footprint are the ones that are relevant to the relation (in the sense that the variable was read from or written to along some execution represented by the relation); the frame is the set of variables that are irrelevant. Variable quantification intuitively moves a variable from the footprint to the frame. Formally, we define this operator as:
\[
\exists^F x.R = \{(\rho[x \leftarrow n], \rho'[x \leftarrow n]) : (\rho, \rho') \in R, n \in \mathbb{Z}\}
\]
For example, consider the recursive procedure `foo` shown to the right. The variable `g` is global and `x` is local to the procedure `foo`. The input/output relation of this function is:

\[ R = \{(\rho, \rho') : \rho(x) = \rho(g) = \rho'(g) > 10\} \]

We wish to verify the correctness of this input/output relation by checking that if \( R \) is used as the interpretation of the recursive call, the interpretation of the path expression from the entry to the exit of `foo` is again \( R \). This check fails because the local variable `x` is conflated with the copy of `x` from the recursive call (i.e., we incorrectly determine that the value of `x` changes across the recursive call). The correct interpretation of the call is obtained if `x` is moved out of scope by using

\[ \exists^{\mathcal{D}} x. R = \{(\rho, \rho) : \rho(x) > 10\} \]

as the interpretation of the call edge.

A more subtle issue with local variables is illustrated in procedure `bar` shown to the right. Here, `flag` is a global variable and `x` is local. Suppose that we wish to compute an input/output relation representing the executions of `bar` which end at the assertion. The set of interprocedural paths ending at the assertion can be described by the following regular expression:

\[
(\text{assume}(x != 0); x := 0; \text{call bar})^*; \text{assume}(x == 0)
\]

The relational interpretation of this expression is:

\[
\{(\rho, \rho) : \rho(x) \geq 0\}
\]

This relation (incorrectly) indicates that the value of `flag` in the pre-state will be the same as the value of `flag` in the post-state (so if `bar` is called from state where `flag` is false, the assertion will not fail). The problem is that the `assume` instruction refers to a different instance of the local variable `x` than the one in `x := 0` in the regular expression above. By applying the quantification operator to move `x` out of scope in the loop, we arrive at the correct interpretation:

\[
\{(\rho, \rho) : \rho(x) \geq 0\} \cup \{(\rho, \rho[\text{flag} <- 0]) : \rho(x) \geq 0\}
\]

A significant benefit of our treatment of local variables using quantification operators is that we can handle both these potential issues with a single operator, which reduces the burden of designing and implementing an interprocedural analysis.

### 6.2 Recursion

As mentioned before, the behaviours of recursive programs may be non-regular languages, and beyond the expressive power of choice, sequencing, and iteration operators. Here we provide two solutions to this problem, which emerge from methods for defining a solution to a system of recursive equations in the classical theory of abstract interpretation. For (quantified) quantales, the underlying order forms a complete lattice, so we can use Tarski’s fixedpoint theorem to guarantee the existence of a least fixpoint solution to the equation system. For quantified PKAs, we use a widening operator to accelerate convergence of the Kleene iteration sequence associated with the equation system.

First, we factor the interpretation of call edges out of the interpretation of path expressions. Let \( \mathcal{F} = (\mathcal{D}, []\) \) be an (interprocedural) interpretation. A summary assignment (for a program consisting of \( N \) procedures) is mapping \( S : [1, N] \to \mathcal{D} \) that assigns each procedure a property in \( \mathcal{D} \) (i.e., the procedure summary). Given any path expression \( p \), we use \( \mathcal{F}(S)[p] \) to denote the interpretation of \( p \) within \( \mathcal{F} \), using \( S \) to interpret calls (this function is defined as \( \mathcal{F}[p] \) was in Section 4 except edges which correspond to call instructions are interpreted using \( S \) as the semantic function).

A summary assignment \( S \) is inductive if for all procedures \( i \in [1, N] \), we have (recalling that \( LV_i \) denotes the set of local variables of procedure \( i \))

\[
S(i) \geq \exists LV_i.\mathcal{F}(S)[p[i_{\text{entry}}, v_{i_{\text{exit}}}]]
\]

Conceptually, inductive summary assignments are the “viable candidates” for approximating the behaviour of each procedure. An equivalent way of describing inductive summary assignments is that they are the post-fixed points of the function

\[
F : ([1, N] \to \mathcal{D}) \to ([1, N] \to \mathcal{D}),
\]

defined as:

\[
F(S) = \lambda i.\exists LV_i.\mathcal{F}(S)[p[i_{\text{entry}}, v_{i_{\text{exit}}}]]
\]

In order to define the meaning of a program with procedures, we must be able to select a particular post-fixed point of \( F \). We will use \( \overline{S} \) to denote this uniquely-selected post-fixed point (regardless of the method used to select it). Our two approaches for selecting \( S \) are as follows:

**Quantified Quantales:** If the semantic domain \( \mathcal{D} \) of an interpretation is a quantale, then the underlying order \( (\mathcal{D}, \leq) \) is a complete lattice. It follows that the function space \( [1, N] \to \mathcal{D} \) is also a complete lattice (under the point-wise ordering). It is easy to show that (under the assumption that \( \mathcal{D} \) is a QPKA quantale), the function \( F \) defined above is monotone. Since \( F \) is a monotone function on a complete lattice, Tarski’s fixed point theorem guarantees that \( F \) has a least fixed point \( [39] \). We define \( \overline{S} \) to be this least fixed point.

**QPKAs:** If the semantic domain \( \mathcal{D} \) of an interpretation is a QPKA, then we have access to a widening operator \( \vee \). Consider the sequence \( (S_i)_{i \in [N]} \) defined by:

\[
S_1(i) = 0 \quad S_i(i) = S_{i-1}(i) \vee (\exists LV_i.\mathcal{F}(S_{i-1})[p[i_{\text{entry}}, v_{i_{\text{exit}}}]])
\]

Using the properties of the widening operator, we can prove that there exists some \( k \in \mathbb{N} \) such that \( S_k = S_{k+1} \), and that \( S_k \) is an inductive summary assignment. We define \( \overline{S} = S_k \). In practice, an efficient chaotic iteration strategy \( [6] \) may be employed to compute \( S_k \).

**Example 6.4** To extend linear recurrence analysis to handle procedures, we must define a quantification operator \( \exists^D \) and a widening operator \( \vee^D \). The quantification operator mirrors the one from the relational interpretation:

\[
\exists^D x. \varphi = (\exists x \exists^D x'. \varphi) \land x = x'
\]

As with the sequencing and choice operators, this quantification operator is precise (i.e., it loses no information from the perspective of the relational interpretation).

There are several possible choices for the widening operator \( \varphi_1 \cup^D \varphi_2 \) on \( ^D \). One natural choice is to leverage existing widening operators. For example, we may compute the best abstraction of \( \varphi_1 \) and \( \varphi_2 \) as convex polyhedra, and then apply a widening operator from that domain \( [12] \).

### 7. Interprocedural analysis

We now adapt the path expression algorithm to the interprocedural framework. Let \( \mathcal{F} = (\mathcal{D}, []) \) be an interpretation such that \( \mathcal{D} \) is equipped with a widening operator and let \( \mathcal{P} = \{P_i\}_{1 \leq i \leq N} \)
be a program, where \( P_i \) denotes the main procedure of \( P \). The \textit{interprocedural path expression problem} is to compute for each procedure \( P_i \) and each vertex \( v \) of \( P_i \) an approximation of the interprocedural paths from \( v_i^{\text{entry}} \) to \( v \). We use \( \mathcal{I}(v) \) to denote this approximation.

An \textit{interprocedural path} \( \pi = e_1 \cdots e_n \) is a sequence of edges such that for all \( i < n \), either \( \text{tgt}(e_i) = \text{src}(e_{i+1}) \), or \( \text{tgt}(e_i) \) has an outgoing edge which is labeled call \( k \) and \( \text{src}(e_{i+1}) = v_k^{\text{entry}} \) (for some \( k \)). This definition of interprocedural paths differs from the standard one \cite{Agarwal1997}: rather than requiring that every return edge be matched with a call edge, we allow each edge labeled call \( k \) to stand for the entire execution of procedure \( P_k \). The benefit of this approach is that the set of interprocedural paths from one vertex to another is a regular language (whereas for \cite{Agarwal1997} it is context-free, but not necessarily regular). Note that, since call edges are interpreted as complete executions of a procedure, transferring control from one procedure to another is not marked by a call edge.

Any interprocedural path from \( v_i^{\text{entry}} \) to a vertex \( v \) belonging to procedure \( P_i \) can be decomposed into an interprocedural path from \( v_i^{\text{entry}} \) to \( v_k^{\text{entry}} \) and an intraprocedural path from \( v_k^{\text{entry}} \) to \( v \). So we may compute \( \mathcal{I}(v) \) as

\[
\mathcal{I}(v) = \mathcal{I}(v_k^{\text{entry}}) \ominus \mathcal{I}(S)[P_{v_k^{\text{entry}}, v}]
\]

As a result of this decomposition, we need only to solve the interprocedural path expression problem for procedure entry vertices. This problem can be solved by applying Tarjan’s path-expression algorithm to the \textit{call graph} of \( P \).

The \textit{call graph} \( CG_P = (\mathcal{P}, CE) \) for a program \( P \) is a directed graph whose vertices are the procedures \( \{P_i\}_{1 \leq i \leq N} \), and such that there is an edge (called a \textit{call edge}) \( P_i \rightarrow P_j \in CE \) iff there exists some edge \( e \in E_i \) such that \( \text{act}(e) = \text{call } j \). To apply Tarjan’s algorithm to \( CG_P \), we must define an interpretation for call edges. The interpretation of a call edge \( P_i \rightarrow P_j \in CE \) should be an approximation of all paths that begin at the entry of \( P_i \) and end in a call of \( P_j \). The local variables of \( P_i \) go out of scope when the program enters \( P_i \), so variables in \( LV_i \) should be removed from the footprint of \( P_i \rightarrow P_j \). Formally,

\[
\mathcal{I}(S)[P_i \rightarrow P_j] = \exists LV_i. \bigoplus \{ \mathcal{I}(S)[P_{v_i^{\text{entry}}, \text{src}(e)}] : e \in E_i, \text{act}(e) = \text{call } j \}.
\]

We formalize the interprocedural program analysis algorithm we have developed in this section in Algorithm 1.

### Algorithm 1 Interprocedural path-expressions

**Require**: An interpretation \( \mathcal{I} \) and a program \( P = \{P_i\}_{1 \leq i \leq N} \)

**Ensure**: An array PathTo s.t. \( \forall k \in [1, N] \) and each vertex \( v \) of \( k \), \( \text{PathTo}[k][v] = \mathcal{I}(v) \).

**PathTo**

\[
S \leftarrow \lambda i.0
\]

**repeat**

\[
S' \leftarrow S \\
S \leftarrow \lambda i.S(i) \forall \forall L.V_i.\mathcal{I}(S)[P_{v_i^{\text{entry}}, \text{src}(v_i^{\text{exit}})}]
\]

**until** \( S = S' \)

**for all** \( i \in [1, N] \) \n
**do**

\[
\text{PathTo}[i][v_i^{\text{entry}}] \leftarrow \mathcal{I}(S)[P_{v_i^{\text{entry}}, P_i}]
\]

**end for**

**for all** \( i \in [1, N] \) \n
**do**

\[
\text{PathTo}[i][v] \leftarrow \text{PathTo}[i][v_i^{\text{entry}}] \ominus \mathcal{I}(S)[P_{v_i^{\text{entry}}, v}]
\]

**end for**

**return** PathTo

**Example: Interprocedural Linear Recurrence Analysis**

We now present an example of how our interprocedural analysis algorithm works using linear recurrence analysis on a simple program. Consider the program pictured in Figure 3 which includes two procedures: \textit{main} (the entry point of the program) and \textit{foo}. The action of this program is to set \( g \) to 20 and then decrement it 10 times. The decrement loop is implemented with a recursive procedure, \textit{foo}, which uses its parameter to keep track of the loop count (note that, since in our model procedures do not have parameters, we model parameter passing via assignment to the global variable \( p0 \)). We will illustrate how linear recurrence analysis can be used to prove that the assertion in \textit{foo} (that \( g \) is positive) will always succeed.

We begin by computing an inductive summary assignment \( S \). For the sake of this example, let us assume that we use an extremely imprecise widening operator, defined as

\[
\varphi \triangledown \varphi' = \begin{cases} 
\varphi & \text{if } \varphi \iff \varphi' \\
\text{true} & \text{otherwise}
\end{cases}
\]

First, we compute (using a path-expression algorithm \cite{Agarwal1997}), a path expression for each procedure, which represents the set of paths from the entry of that procedure to its exit. One possible result is:

\[
P_{v_{\text{foo}}, v_{\text{exit}}} = (v_{\text{foo}}, v_3) \left( (v_3, v_4) \langle v_4, v_5 \rangle \langle v_5, v_6 \rangle \langle v_6, v_{\text{foo}} \rangle \\
+ \langle v_3, v_7 \rangle \langle v_7, v_{\text{foo}} \rangle \right)
\]

We compute a summary assignment for each procedure by repeatedly interpreting these path expressions in \( \mathcal{I} \), using the summary for \textit{foo} to interpret calls to \textit{foo}, until the summary assign-
ment converges. This computation proceeds as follows:

\[ S_0(\text{foo}) = 0^\mathcal{L} = \text{false} \]
\[ S_0(\text{main}) = 0^\mathcal{L} = \text{false} \]
\[ S_1(\text{foo}) = 0^\mathcal{L} \lor (\exists x. S_1(x)[P_{\text{foo}}^{\text{entry}}, v_{\text{foo}}^{\text{exit}}]) \]
\[ = 0^\mathcal{L} \lor (\exists x. 10 \land x' = p0 \land p0' = p0 \land g' = g) \]
\[ = 0^\mathcal{L} \lor (p0 \geq 10 \land p0' = p0 \land g' = g \land x' = x) \]
\[ = \text{true} \]
\[ S_1(\text{main}) = 0^\mathcal{L} \lor (\exists x. S_1(x)[P_{\text{main}}, v_{\text{main}}]) \]
\[ = 0^\mathcal{L} \lor (\exists x. v_{\text{main}}) \]
\[ S_2(\text{main}) = 0^\mathcal{L} \lor (\exists x. S_2(x)[P_{\text{main}}, v_{\text{main}}]) \]

At \( S_2 \) the fixedpoint iteration converges, and we define our inductive summary assignment as \( S = S_2 \). Next, we compute summaries to the entry point of each procedure, using the path expression algorithm on the call graph shown to the right. The interpretation of the edges is given by the following:

\[ \mathcal{L}(S)[\text{main} \rightarrow \text{foo}] = g' = 20 \land p0' = 0 \land x' = x \]
\[ \mathcal{L}(S)[\text{foo} \rightarrow \text{foo}] = p0 < 0 \land p0' = p0 + 1 \land g' = g - 1 \land x' = x \]

which yields the following path-to-entry summaries:

\[ \mathcal{I}(v_{\text{main}}^{\text{entry}}) = 1^\mathcal{L} \]
\[ \mathcal{I}(v_{\text{foo}}^{\text{entry}}) = \mathcal{I}(S)[(\text{main} \rightarrow \text{foo}) \cdot (\text{foo} \rightarrow \text{foo})^*] \]
\[ = g' = 20 - p0' \land p0' \leq 10 \land x' = x \]

Finally, we may compute a summary which overapproximates the executions to \( v_7 \):

\[ \mathcal{I}(v_7) = \mathcal{I}(v_{\text{foo}}^{\text{entry}}) \circ \mathcal{L}(S)[P_{\text{foo}}^{\text{entry}}, v_{\text{foo}}^{\text{entry}}] \]
\[ = g' = 20 - p0' \land p0' = 10 \land x' = p0' \]

from which we may prove that the assertion in \( \text{foo} \) may never fail.

This example illustrates an interesting feature of our framework: interprocedural loops (which result from recursive procedures such as \( \text{foo} \)) are analyzed in exactly the same way as intraprocedural loops (by simply applying an iteration operator). This is a significant advantage of our generic algebraic framework over defining a compositional analysis by structural induction on the program syntax (as in [3, 4, 31]), which requires separate treatment for interprocedural loops.

### 7.1 Correctness and precision

It is easy to prove that Algorithm I computes an array PathTo such that for each procedure \( P_k \) and each vertex \( v \) of \( P_k \), PathTo[\( k \)][v] = \( \mathcal{I}(v) \). As in Section 3.1, we justify the correctness of our interprocedural path-expression algorithm with respect to interprocedural paths. We use IVP(\( v \)) to denote the set of interprocedurally valid paths (i.e., with matched calls and returns, as in [22]) to the vertex \( v \), and for any \( \pi \in \text{IVP}(v) \), we use \( \mathcal{I}[\pi] \) to denote the interpretation of \( \pi \) in \( \mathcal{I} \). The formal definition of \( \mathcal{I}[\pi] \), as well as all proofs presented in this section, can be found in the extended version of this paper [I].

First, we have that Algorithm I soundly approximates interprocedurally valid paths if the semantic algebra is a QPKA:

#### Theorem 7.1 (Correctness)

Let \( \mathcal{I} \) be an interpretation over a QPKA, \( P_k \) be a procedure, \( v \) be a vertex of \( P_k \), and let \( \pi \in \text{IVP}(v) \) be an interprocedural path to \( v \). Then \( \mathcal{I}[\pi] \subseteq \mathcal{I}(v) \).

Second, we have that Algorithm I computes exactly the sum over interprocedurally valid paths in the case that the semantic algebra is a quantified quantale.

#### Theorem 7.2 (Precision)

Let \( \mathcal{I} \) be an interpretation over a (quantified) quantale, \( P_k \) be a procedure, \( v \) be a vertex of \( P_k \). Then \( \mathcal{I}(v) = \bigoplus_{\pi \in \text{IVP}(v)} \mathcal{I}[\pi] \).

Remark that the precision theorem above also addresses the subtle issues involving local variables, as in [22, 28] (e.g., it shows that we do not lose correlations between local and global variables due to procedure calls, as [28] points out that [22] does).

### 7.2 Abstraction

In this section, we adapt the abstract interpretation framework of Section 5 to the interprocedural setting. The conditions of Proposition 5.3 are much easier to adapt to the interprocedural setting than the more general definition of a soundness relation, so we present this simplified version here. Under the assumptions of Proposition 5.3, all that is required to extend soundness relations to the interprocedural setting is an additional congruence condition for the quantification operator (which ensures that “abstract” quantification approximates “concrete” quantification).

#### Definition 7.3 (Interprocedural soundness relation)

Let \( \mathcal{C} \) be a quantified quantale and let \( A \) be a QPKA. A relation \( \vdash \subseteq \mathcal{C} \times A \) is a soundness relation if \( \vdash \) satisfies the conditions of Proposition 5.3 and for all \( c \vdash a \), and all \( x \in \mathcal{C} \), we have \( 3_x c \vdash \exists_x e.a \).

The abstraction theorem carries over to the interprocedural setting, and is stated below. The proof of this theorem is similar to the intraprocedural variant, except that we require some of the fixpoint machinery of classical abstract interpretation [I] due to the inductive summary assignment computation.

#### Theorem 7.4 (Abstraction)

Let \( \mathcal{C} \), \( \mathcal{I} \) be interpretations, where the domain of \( \mathcal{I} \) is a quantified quantale and the domain of \( \mathcal{I} \) is a QPKA. Let \( \vdash \subseteq \mathcal{C} \times A \) be a soundness relation, and let \( v \) be a vertex of some procedure. If for all \( x \in E \), \( e_x \vdash e_x^{\mathcal{I}} \), then \( \mathcal{C}(v) \vdash \mathcal{I}(v) \).

### 8. The algebraic framework in practice

In this section, we report on the algorithmic side of our framework, using the linear recurrence analysis presented in Section 2 as a case study. We present experimental results which demonstrate two key points:

- Algorithm I the algorithmic basis of our interprocedural framework, can effectively (and efficiently) solve real program analysis problems in practice.
- Compositional loop analyses (like linear recurrence analysis) can effectively compute accurate abstractions of loops. This indicates that compositional loop analyses, a relatively underexplored family of analyses, are an interesting direction for future work by the program analysis community. Our framework is particularly well suited for helping to explore this research program.

**Implementation** One of the stated goals of our framework is to provide a clearly defined interface between the definition of a program analysis and the algorithm used to compute the result of the analysis. The practical value of this interface is that allows for generic implementations of path expression algorithms to be developed independently of the program analysis. This is particularly
interesting because the choice of path expression algorithm affects both the speed and precision of an analysis.

Our interprocedural path expression algorithm (Algorithm 1) is implemented in OCaml as a functor parameterized by a module representing a QPKA. This allows any analysis which can be designed in our framework to be implemented as a plugin. Linear recurrence analysis is implemented as one such plugin. For our implementation of linear recurrence analysis, we use Z3 [13] to resolve SMT queries that result from applying the iteration operator and checking assertion violations.

**Linear Recurrence Analysis**
Let us refer to our implementation of the linear recurrence analysis as LRA. We compare the effectiveness of LRA in proving properties of program with loops with two state-of-the-art invariant generation tools: UFO [2] and INVGEN [15]. Table 1 presents the results of this comparisons. We ran the three tools on a suite consisting of 73 benchmarks from the INVGEN test suite and 17 benchmarks from the software verification competition (in particular, the safe, integer-only benchmarks from the Loops category). These benchmarks are small, but difficult. The total results show that LRA manages to prove more instances correct compared to UFO and INVGEN.

| Result | LRA | UFO | INVGEN |
|--------|-----|-----|--------|
| Safe   | 84  | 61  | 75     |
| Unsafe | 1   | 1   | 0      |
| False positive | 5 | 18 | 9 |
| False negative | 0 | 0 | 1 |
| Timeout | 1 | 5 | 0 |
| Crash  | 0   | 6   | 3      |

**Table 1.** Experimental Results comparing LRA analysis with UFO and INVGEN on a set of benchmarks from INVGEN and SV-Comp. INVGEN does not currently handle procedures, so benchmarks involving procedures are omitted from the INVGEN column.

9. **Conclusion and Related work**

In this paper, we presented an algebraic framework for compositional program analysis. Our framework can be used as the basis for analyses that use non-iterative methods of computing loop invariants and for efficient interprocedural analyses, even in the presence of recursion and local variables. We close with a comparison of our framework with previous approaches to compositional analysis, uses of algebra in program analysis, and interprocedural analysis.

9.1 **Compositional program analyses**

Perhaps the most prominent use of compositionality is in interprocedural analyses based procedure summarization. Summary-based analyses are not necessarily implementable in our framework since there are iterative methods for computing summaries that are beyond the scope of our framework. The literature on this subject is too vast to list here, but we note one particular example which can be implemented and proved correct in our framework: the affine equalities analysis of [29].

Compositional loop analyses use a summary of the body of the loop in order to compute loop invariants. The most common technique for presenting these analyses is by structural induction on program syntax [8, 12, 31]. An alternative technique is given in [22], which is based on graph rewriting. This technique is more generally applicable than induction on program syntax (e.g., it may be applied to programs with gotos), but is considerably more complicated: an analysis designer must develop a procedure which abstracts arbitrary loop-free programs (rather than a handful of simple syntactic constructors). Our framework allows all of these analyses to be implemented in a way that is both simple and generally applicable.

Elimination-style dataflow analyses are another source for compositional program analyses [14]. Elimination-style dataflow analyses were designed to speed up iterative dataflow analyses by computing loop summaries and thereby avoid repeatedly propagating dataflow values through loops. This style of analysis, the context in which Tarjan’s path expression algorithm was originally developed [37]. The analyses considered in this line of work typically limited to the class of dataflow analyses useful in compiler optimization (e.g., gen/kill analyses). These analyses can be implemented and proved correct in our framework, but we are primarily interested in a more general class of analysis (e.g., analyses for numerical invariant generation).

9.2 **Algebra in program analysis**

A number of papers have used algebraic structures as the basis of program analysis [3, 14, 23, 33]. A summary of the relative strength of the assumptions on these structures is presented in Figure 4. The notion of Pre-Kleene algebras introduced in this paper generalize all these structures, and thus provides a unifying foundation.

Weighted pushdown-systems (WPDSs) are a generic tool for implementing interprocedural program analyses [33]. Weighted pushdown systems extend pushdown systems with a weight on each rule that is drawn from a bounded idempotent semiring (a Kleene algebra without an iteration operator that must satisfy the ascending chain condition). Tarjan’s path-expression algorithm has also found a use in the WPDS method in improving the pre∗/post∗ algorithms that drive WPDS-based analyses [23].

There are two advantages of our framework over WPDSs. First, we compute loop summaries using an iteration operator, whereas WPDSs use fixed-point iteration. The advantage of our approach using an iteration operator was also noted in [27]. The second advantage is conceptual simplicity: our framework is based on familiar regular expression operations and procedure summaries, whereas WPDSs require more sophisticated automata theory.

There are two features of WPDSs that are not currently handled by our approach: backwards analysis and stack-qualified queries. Given the similarity between the WPDS framework and the interprocedural path-expression algorithm, we believe that our methodology could be adapted to handle these features as well.

Bouajjani et al present a generic methodology for proving that two program locations are not coreachable in a concurrent program with procedures [4]. Their method is based on developing a Kleene algebra [2] in which elements represent regular sets of paths; this allows the coreachability test to be reduces to emptiness-of-intersection of two regular languages. Their method allows for significant flexibility in designing the abstract domain, but ultimately their work does not intend to be a completely generic framework, but rather to solve a specific problem in concurrent program analysis.

Kot & Kozen develop an algorithm for implementing second-order (i.e., trace-based) program analyses based on left-handed Kleene algebra [2] (Kleene-algebras where sequencing is left distributive and right pre-distributive). Their work is an implementation of a path-expression algorithm, which provides an alternative to the one in [33, 37] (and which, interestingly, uses a matrix construction on left-handed Kleene algebras rather than explicitly using graphs). Their primary concern is with the implementation of

3 In fact, they are interested in two stronger forms: finite and commutative Kleene algebras. Finite KAs are quantales, but commutative KAs are incomparable.
analyses, rather than semantics-based justification of their correctness. Filipiuk et al present a program analysis framework most similar in spirit to ours, in which program analyses are defined by flow algebras: pre-distributive, bounded, idempotent semirings (i.e., Pre-Kleene algebras without an iteration operator) [14]. Of the works cited above, [14] is the only one which addresses the problem of proving the correctness of a program analysis with respect to a concrete semantics. Since flow algebras omit an iteration operator, they cannot express the non-iterative loop abstractions that PKAs can.

**Algebra in program semantics.** There is a line of work in program semantics which aims to enrich Kleene algebras with greater structure to make them useful as a basis for reasoning about programs: for example, [18][25]. In contrast, we use a weaker structure than Kleene algebras (at least in terms of their axiomatization), but we typically assume that the operations of interest are computationally effective so that we may compute the meaning of a program. Work on concurrent Kleene algebra (with an additional parallel composition operator) [18] suggests an interesting direction for future research, e.g. adapting our framework to handle concurrency.

### 9.3 Interprocedural analysis

The are a great number of approaches to interprocedural analysis; a nice categorization of these techniques can be found in [11]. Two particular techniques of interest is the summary-based approach of Sharir & Pnueli [36] and the method of Cousot & Cousot for analyzing (recursive) procedures [8]. Both of these techniques assume that the domain for procedure summaries is a function space over the structure of the universe. Some first-order domain. Our approach makes no such assumptions on the structure of the universe.

**Local variables.** Knoop & Steffan extend the coincidence theorem of [18][23] to handle local variables by using a merge function to combine information about local variables from the calling context with information about global variables from the returning context [22]. Lal et al. use a similar approach to handle local variables in WPDS-based analyses [28]. Cousot & Cousot use a different approach to handling local variables by explicitly managing the footprint frame of procedure summaries [8]; this is similar to the approach we use. In contrast to [8], our treatment of quantification is algebraic, whereas in [8] moving a variable from the footprint to the frame is a particular concrete operation.

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[1] Removed for blind reviewing. The results referenced in the paper are available in the appendix supplied in the supplementary material.

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A. Proof of Theorem 7.4

Proof. We begin with two general lemmas concerning interprocedural interpretations.

Lemma A.1. Let \( \mathcal{I} = (\mathcal{I},\llbracket \ldots \rrbracket) \) and \( \mathcal{J} = (\mathcal{J},\llbracket \ldots \rrbracket) \) be QPKA interpretations, let \( S_\mathcal{I} \) and \( S_\mathcal{J} \) be summary assignments, and let \( \sqsubseteq \subseteq \mathcal{I} \times \mathcal{J} \) be any relation that is coherent with respect to the QPKA operators and such that for all \( e \in E \),
\[
\mathcal{I}(S_\mathcal{I}[e]) \models \mathcal{J}(S_\mathcal{J}[e])
\]

Then for any path expression \( p \),
\[
\mathcal{I}(S_\mathcal{I}[p]) \models \mathcal{J}(S_\mathcal{J}[p])
\]

Moreover, this holds even if \( \mathcal{I} \) and \( \mathcal{J} \) do not satisfy any of the QPKA axioms.

Proof. By induction on \( p \).

Lemma A.2. Let \( \mathcal{I} = (\mathcal{I},\llbracket \ldots \rrbracket) \) and \( \mathcal{J} = (\mathcal{J},\llbracket \ldots \rrbracket) \) be QPKA interpretations, and let \( \sqsubseteq \subseteq \mathcal{I} \times \mathcal{J} \) be any relation that is coherent with respect to the QPKA operators and such that for all \( e \in E \),
\[
\mathcal{I}(\mathcal{S}_\mathcal{I}[e]) \models \mathcal{J}(\mathcal{S}_\mathcal{J}[e])
\]

where \( \mathcal{S}_\mathcal{I} \) and \( \mathcal{S}_\mathcal{J} \) are inductive summary assignments (as defined in Section 6). Then for any vertex \( v \),
\[
\mathcal{I}(v) \models \mathcal{J}(v)
\]

Moreover, this holds even if \( \mathcal{I} \) and \( \mathcal{J} \) do not satisfy any of the QPKA axioms.

Proof. This is essentially a corollary of Lemma A.1. Recall that
\[
\mathcal{I}(v) = \mathcal{I}(\mathcal{S}_\mathcal{I}[\mathcal{P}[P_1, P_k]] \odot \mathcal{J}(\mathcal{S}_\mathcal{J}[\mathcal{P}[v_{entry}, v]])
\]
\[
\mathcal{J}(v) = \mathcal{J}(\mathcal{S}_\mathcal{J}[\mathcal{P}[P_1, P_k]] \odot \mathcal{J}(\mathcal{S}_\mathcal{J}[\mathcal{P}[v_{entry}, v]])
\]

So we need only to prove that
\[
\mathcal{I}(\mathcal{S}_\mathcal{I}[\mathcal{P}[P_1, P_k]] \odot \mathcal{J}(\mathcal{S}_\mathcal{J}[\mathcal{P}[v_{entry}, v]])
\]

and
\[
\mathcal{J}(\mathcal{S}_\mathcal{J}[\mathcal{P}[P_1, P_k]] \odot \mathcal{J}(\mathcal{S}_\mathcal{J}[\mathcal{P}[v_{entry}, v]])
\]

The latter follows directly from Lemma A.1. It remains to show the former.

Recall that the interpretation of a call graph edge \( P_i \to P_j \) is defined in terms of the interpretations of the path expression \( \mathcal{P}[v_{entry}, e] \) describing paths to a vertex \( e \) that calls \( j \). It follows from Lemma A.1 that we must have
\[
\mathcal{I}(\mathcal{S}_\mathcal{I}[P_i \to P_j]) \models \mathcal{J}(\mathcal{S}_\mathcal{J}[P_i \to P_j])
\]

for all edges \( P_i \to P_j \) in the call graph. We may then prove by induction on \( \mathcal{P}[P_i, P_k] \) that
\[
\mathcal{I}(\mathcal{S}_\mathcal{I}[\mathcal{P}[P_i, P_k]]) \models \mathcal{J}(\mathcal{S}_\mathcal{J}[\mathcal{P}[P_i, P_k]])
\]

We can prove that for all \( i, \mathcal{S}_\mathcal{C}(i) \models \mathcal{S}_\mathcal{A}(i) \) in the standard way, by induction on the transfinite iteration sequence defining \( \mathcal{S}_\mathcal{C} \) (using Lemma A.1 for the induction step). We then have the main result by applying Lemma A.2.

B. Coincidence

We now consider the question of when the algorithm presented in Section 7 computes the ideal (merge-over-paths) solution to an analysis problem. This is analogous to the development of coincidence theorems \([19, 22, 28, 36]\) in the field of dataflow analysis, which state conditions under which the iterative algorithm for solving dataflow analyses computes the ideal solutions (i.e., when the join-over-paths solution to a dataflow analysis problem coincides with the minimum fixpoint solution \([19]\)). In this section, we formulate and prove a coincidence theorem for our interprocedural framework.

The first step towards formulating our coincidence theorem is to define the join-over-paths solution to an analysis. This requires us to define what a path is, and how to interpret it. Our treatment of interprocedural paths in Section 7 is not adequate for this purpose, because it relies on interpreting call edges using procedure summaries (which represent a set of paths, rather than a single path). In this section, we will take interprocedural paths to be words over an alphabet of control flow graph edges such that each return is properly matched with a call. For a formal definition of such a path, the reader may consult \([26]\). We use \( \mathcal{P}(v) \) to denote the set of all interprocedural paths to \( v \).

Now we must define the interpretation of an interprocedural path. Let \( \mathcal{I} = (\mathcal{D}, \llbracket \ldots \rrbracket) \) be an interpretation. We define the interpretation of a path within \( \mathcal{I} \) as a function that operates on stacks of abstract values (as in \([22, 28]\)). We use the notation \( \{V_1, a_1\}, \ldots, \{V_m, a_m\} \) to denote a stack of \( m \) activation records, where an activation record is a tuple \( (V_i, a_i) \) consisting of a set of variables \( V_i \) that are local to that activation record, and an abstract value \( a_i \). The first entry in this list corresponds to the top of the stack. We define our semantic function on edges so that each call edge corresponds to a push on this stack, each return edge corresponds to a pop, and each intraprocedural edge modifies the top of the stack (i.e., the “current” activation record). Formally, we have

\[
\mathcal{I}(e) = \begin{cases} ([V_1, a_1], \ldots, [V_m, a_m]) & \text{if } \text{act}(e) = \text{call } i \\ ([V_2, a_2 \odot (\exists V_1, a_1)], \ldots, [V_n, a_n]) & \text{if } \text{act}(e) = \text{return} \\ ([V_1, a_1 \odot [e]], \ldots, [V_n, a_n]) & \text{otherwise} \end{cases}
\]

In dataflow analysis literature, coincidence theorems typically relate the meet-over-paths and the maximum fixpoint solutions. We follow in the tradition of abstract interpretation, which uses the dual order.
Lemma B.2. For any interprocedural path \( \pi \), let \( stack(\pi) \) be the stack of activation records. Then \( stack(\pi) \) is the union of the activation records from all calls and returns in \( \pi \).

\[
stack(\epsilon) = [LV_1, 1] \\
stack(\pi a) = [a] \stack(\pi)
\]

Next, we define a function \( flatten \) that lowers a stack of activation records into a single abstract value, and existentially quantifies variables that are not in the scope of the current activation record.

\[
flatten(P_1, \ldots, P_n) = (\exists x_1, a_1) \cdot \cdots \cdot (\exists x_n, a_n) \cdot P_1 \cap \cdots \cap P_n
\]

Finally, we take the interpretation of an interprocedural path within \( \mathcal{F} \) to be \( flatten(stack(\pi)) \).

Recalling the definition of \( \mathcal{F}(v) \) from Section 7, we may state our coincidence theorem:

**Theorem B.1** (Coincidence). Let \( \mathcal{F} = (Q, [\cdot]) \) be an interpretation, where \( Q \) is a quantale. Then for any \( v \),

\[
\mathcal{F}(v) = \bigoplus_{\pi \in IPaths(v)} flatten(stack(\pi))
\]

**Proof.** The proof proceeds in two steps. First, we define a quantified word interpretation \( \mathcal{W} = (W, [\cdot]_W) \), for which we can show a coincidence result. Then, we show that any interpretation \( \mathcal{F} = (Q, [\cdot]) \) over a quantale “factors” through the quantified word interpretation, in the sense that there is a quantale homomorphism \( h \) such that

\[ \mathcal{F}(v) = h(\mathcal{W}(v)) \]

We first define the set of quantified words \( W \) to be the empty word or any word recognized by the following context free grammar:

\[
W ::= e \mid \exists x.W \mid \bigcap W \mid \bigcup W
\]

We then define a (weak) quantale

\[
W = \langle W, \cup, \cap, \cdot, \exists, \emptyset, \{\cdot\} \rangle
\]

where

\[
P \cap W Q = \{ab : a \in P, b \in Q\} \\
P \cdot W = \bigcup_{n \in \mathbb{N}} P^n \\
\exists x.P = \{\exists x.a : a \in P\}
\]

Note that \( W \) does not satisfy axioms (Q2)-(Q4). It is for this reason that we call \( W \) a weak quantale. However, the axioms (Q2)-(Q4) are irrelevant for this proof.

We define the quantified word interpretation as \( \mathcal{W} = (W, [\cdot]_W) \), where \([e]_W \) is defined as the word consisting of a single letter \( e \).

Next, we define a function \( h : W \to Q \) by

\[ h(P) = \bigoplus_{Q \in W} \{h(a) : a \in P\} \]

where

\[ \tilde{h}(a) = [e]_Q \]

\[ \tilde{h}(ab) = \tilde{h}(a) \cdot \tilde{h}(b) \]

\[ \tilde{h}(\exists x.a) = \exists x.\tilde{h}(a) \]

We will use the subscripts \( Q \) and \( W \) on \( stack \) and \( flatten \) to distinguish between which interpretation it is being applied in.

**Lemma B.2.** For any interprocedural path \( \pi \), \( h(flatten_Q(stack_Q(\pi))) = flatten_W(stack_W(\pi)) \)

**Proof.** By induction on \( \pi \).

**Lemma B.3.** For any \( v \), \( \mathcal{F}(v) = h(\mathcal{W}(v)) \)

**Proof.** Define a relation \( \sigma \subseteq W \times Q \) as the graph of the function \( h \). That is,

\[ \sigma = \{(P, h(P)) : P \in W\} \]

It is easy to show that \( \sigma \) is a congruence w.r.t. the QPKA operations.

Next, we must show that for any \( i \), \( S_W(i) \models S_Q(i) \).

We may prove this by induction on the transfinite iteration sequence defining \( S_W \) and \( S_Q \) (using Lemma A.1 for the induction step).

Finally, we may apply Lemma A.2 to get

\[ \mathcal{W}(v) \models \mathcal{F}(v) \]

whence \( \mathcal{F}(v) = h(\mathcal{W}(v)) \).

It is easy to see that \( \mathcal{W}(v) \) is exactly the set of interprocedural paths to \( v \), where calls and returns have been replaced by existential quantification of the appropriate variables. This can be expressed formally as

\[ w \in \mathcal{W}(v) \iff \exists \pi \in IPaths(v). \{w\} = flatten_W(stack_W(\pi)) \]

Or equivalently, as

\[ \mathcal{W}(v) = \bigcup_{\pi \in IPaths(v)} flatten_W(stack_W(\pi)) \]

We may then conclude

\[ \mathcal{W}(v) = \bigcup_{\pi \in IPaths(v)} flatten_W(stack_W(\pi)) \]

\[ h(\mathcal{W}(v)) = h \left( \bigcup_{\pi \in IPaths(v)} flatten_W(stack_W(\pi)) \right) \]

\[ \mathcal{F}(v) = h \left( \bigcup_{\pi \in IPaths(v)} flatten_W(stack_W(\pi)) \right) \]

**Lemma B.3**

\[ \mathcal{F}(v) = \bigoplus_{\pi \in IPaths(v)} h(flatten_W(stack_W(\pi))) \]

Def’n of \( h \)

\[ \mathcal{F}(v) = \bigoplus_{\pi \in IPaths(v)} flatten_Q(stack_Q(\pi)) \]

**Lemma B.2**

\[ \mathcal{F}(v) = h(\mathcal{W}(v)) \]

**C. Experimental results**

We present a detailed comparison of LRA, UFO, and INVGEN on our benchmark suite. This includes result only for those benchmarks where one of the answers among the three tools differed from the other two.
| Benchmark Name          | LRA   | UFO   | INVGEN |
|------------------------|-------|-------|--------|
| apache-escape-absolute | unsafe| unsafe| safe   |
| apache-get-tag         | safe  | unsafe| safe   |
| gulwani_cegar2         | safe  | safe  | unsafe |
| gulwani_fig1a          | safe  | safe  | unsafe |
| half                   | safe  | unsafe| unsafe |
| heapsort               | safe  | unsafe| safe   |
| heapsort2              | safe  | unsafe| safe   |
| heapsort3              | safe  | unsafe| safe   |
| id_trans               | unsafe| unsafe| safe (false negative) |
| large_const            | safe  | timeout| safe   |
| MADWiFi-encode_ieok    | safe  | unsafe| safe   |
| nested-if8             | safe  | unsafe| safe   |
| nested-d6              | safe  | unsafe| safe   |
| nested7                | safe  | crash | unsafe |
| nested8                | safe  | timeout| safe   |
| nested9                | safe  | unsafe| unsafe |
| rajamani_1             | safe  | timeout| safe   |
| sendmail-close-angle   | safe  | unsafe| safe   |
| seq-len                | safe  | safe  | unsafe |
| seq2                   | safe  | crash | safe   |
| split                  | unsafe| timeout| safe   |
| svd1                   | safe  | crash | safe   |
| svd2                   | safe  | crash | safe   |
| svd3                   | safe  | crash | safe   |
| svd4                   | unsafe| crash | safe   |
| up-nd                  | safe  | unsafe| safe   |
| up2                    | safe  | unsafe| safe   |
| up3                    | safe  | unsafe| safe   |
| up4                    | safe  | unsafe| safe   |
| up5                    | safe  | unsafe| safe   |
| for_infinite_loop_1    | safe  | safe  | crash  |
| for_infinite_loop_2    | safe  | safe  | crash  |
| sum01_safe             | safe  | unsafe| unsafe |
| sum02_safe             | unsafe| unsafe| unsafe |
| terminator02_safe      | safe  | unsafe| crash  |
| token_ring01_safe      | unsafe| safe  | –      |
| toy_safe               | timeout| timeout| –      |

Wrong Results (total) 6 29 14

**Figure 5.** Experimental Results of comparing LRA analysis with UFO and InvGen tools on the set of InvGen benchmarks, and the set of SVComp benchmarks. “–” indicates that the benchmark has function calls, which not currently supported by InvGen. The last row indicates how many wrong results each tool had (incorrect result, timeout, or crash) over the entire set of benchmarks.