Collective Bias Models in Two-Tier Voting Systems and the Democracy Deficit

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Abstract

We analyse optimal voting weights in two-tier voting systems. In our model, the overall population (or union) is split in groups (or member states) of different sizes. The individuals comprising the overall population constitute the first tier, and the council is the second tier. Each group has a representative in the council that casts votes on their behalf. By ‘optimal weights’, we mean voting weights in the council which minimise the democracy deficit, i.e. the expected deviation of the council vote from a (hypothetical) popular vote.

We assume that the voters within each group interact via what we call a local collective bias or common belief (through tradition, common values, strong religious beliefs, etc.). We allow in addition an interaction across group borders via a global bias. Thus, the voting behaviour of each voter depends on the behaviour of all other voters. This correlation may be stronger between voters in the same group, but is in general not zero for voters in different groups.

We call the respective voting measure a Collective Bias Model (CBM). The ‘simple CBM’ introduced in [12] and in particular the Impartial Culture and the Impartial Anonymous Culture are special cases of our general model.

We compute the optimal weights in the large population limit. Those optimal weights are unique as long as there is no ‘complete’ correlation between the groups. In this case, we obtain optimal weights which are the sum of a common constant equal for all groups and a summand which is proportional to the population of each group. If the correlation between voters in different groups is extremely strong, then the optimal weights are not unique. In fact, in this case, the weights are essentially arbitrary. We also analyse the conditions under which the optimal weights are negative, thus making it impossible to reach the theoretical minimum of the democracy deficit. This is a new aspect of the model owed to the correlation between votes belonging to different groups.

Keywords. Two-tier voting systems, probabilistic voting, collective bias models, democracy deficit, optimal weights, limit theorem.

2020 Mathematics Subject Classification. 91B12, 91B14, 60F05.

1 Introduction

We study voting in two-tier voting systems. Suppose the population of a state or union of states is subdivided into \( M \) groups (member states for example). Each group sends a representative to a council which makes decisions for the union. The representatives cast their vote (‘aye’ or ‘nay’) according to the majority (or to what they believe is the majority) in their respective group. Since the groups may differ in size, it is natural to assign different voting weights to the representatives, reflecting the size of the respective group. When a parliament such as a the House of Representatives in the U.S. is elected, usually the country is subdivided into a number of districts of roughly equal population, each of which votes on a representative for a single seat. This procedure is feasible within a country but may not be possible in other situations. Even in the U.S., no effort has been made to divide the states and reassemble them into districts of roughly equal size so that each of them could have the same number of senators

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without giving rise to questions whether that is the ‘right’ way to determine the number of senators. It is even less likely that sovereign countries – such as the members of the United Nations or the European Union – would be willing to submit to being divided into districts of equal size. Thus, it is not possible in practice to circumvent the question of how to assign voting weights to groups of different sizes.

To determine these weights is the problem of ‘optimal’ weights. How should the weights be assigned? One objective studied in the literature is to minimise the democracy deficit, i.e. the deviation of the council vote from a hypothetical referendum across the entire population. The democracy deficit was first studied for binary voting (the same setting which is considered in the present article) by Felsenthal and Machover \[8\]. Later on, it was also analysed in other settings by several authors (see e.g. \[7, 12, 28, 15, 25, 27\]). Other notions of optimal weights are based on welfare considerations or the criterion of equalising the influence of all voters belonging to the overall population. In the latter vein, we find the seminal article by Penrose \[25\], where the square root law was first established as the assignment rule for voting weights that equalises the probability of each voter’s being decisive in a two-tier voting system under the assumption of stochastically independent voting. Other contributions to the study of optimal voting weights under welfare and influence frameworks can be found in \[23, 1, 18, 19\].

Suppose the overall population is of size \(N\), whereas the group size is \(N_\lambda\), where the subindex \(\lambda\) stands for the group \(\lambda \in \{1, \ldots, M\}\). Let the two voting alternatives be encoded as \(\pm 1\), \(+1\) for ‘aye’ and \(-1\) for ‘nay’. The vote of voter \(i \in \{1, \ldots, N_\lambda\}\) in group \(\lambda\) will be denoted by \(X_{\lambda i}\).

**Definition 1.** For each group \(\lambda\), we define the voting margin \(S_\lambda := \sum_{i=1}^{N_\lambda} X_{\lambda i}\). The overall voting margin is \(S := \sum_{\lambda=1}^{M} S_\lambda\).

Each group casts a vote in the council:

**Definition 2.** The council vote of group \(\lambda\) is given by

\[
\chi_\lambda := \begin{cases} 
1, & \text{if } S_\lambda > 0, \\
-1, & \text{otherwise}.
\end{cases}
\]

The (representative of) group \(\lambda\) votes ‘aye’ if there is a majority in group \(\lambda\) on the issue in question. Each group \(\lambda\) is assigned a weight \(w_\lambda\). The weighted sum \(\sum_{\lambda=1}^{M} w_\lambda \chi_\lambda\) is the council vote. The council vote is in favour of a proposal if \(\sum_{\lambda=1}^{M} w_\lambda \chi_\lambda > 0\). Weights \(w_1, \ldots, w_M\) together with a relative quota \(q \in (0, 1)\) constitute a weighted voting system for the council, in which a coalition \(C \subset \{1, 2, \ldots, M\}\) is winning if

\[
\sum_{i \in C} w_i > q \sum_{i=1}^{M} w_i.
\]

We will exclusively consider the majority rule with \(q = 1/2\) in this article.

It is reasonable to choose the voting weights \(w_\lambda\) in the council in such a way, that the raw democracy deficit

\[
\left| S - \sum_{\lambda=1}^{M} w_\lambda \chi_\lambda \right|
\]

is as small as possible. For a given set of weights, each configuration of all \(N\) votes induces a certain raw democracy deficit. It is immediately clear that in general there is no choice of weights which makes this variable small uniformly over all possible distributions of Yes-No-votes across the overall population. All we can hope for is to make it small ‘on average’, more precisely we try to minimise the expected quadratic deviation of \(\sum_{\lambda=1}^{M} w_\lambda \chi_\lambda\) from \(S\).

To follow this approach, we have to clarify what we mean by ‘expected’ deviation, i.e. there has to be some notion of randomness underlying the voting procedure.

While the votes cast are assumed to be deterministic and rational, obeying the voters’ preferences which we do not model explicitly, the proposal put before them is assumed to be unpredictable, i.e. random.
Since each yes/no question can be posed in two opposite ways, one to which a given voter would respond ‘aye’ and one to which they would respond ‘nay’, it is reasonable to assume that each voter votes ‘aye’ with the same probability they vote ‘nay’.

This leads us to the following definition:

**Definition 3.** A voting measure is a probability measure \( P \) on the space of voting configurations \( \{-1, 1\}^N = \prod_{\lambda=1}^{M} \{-1, 1\}^{N_{\lambda}} \) with the symmetry property

\[
P(X_{11} = x_{11}, \ldots, X_{MN_M} = x_{MN_M}) = P(X_{11} = -x_{11}, \ldots, X_{MN_M} = -x_{MN_M})
\]

for all voting configurations \((x_{11}, \ldots, x_{MN_M}) \in \{-1, 1\}^N\).

By \( E \) we will denote the expectation with respect to \( P \).

The simplest and widely used voting measure is the \( N \)-fold product of the measures

\[
P_0(1) = P_0(-1) = \frac{1}{2}
\]

which models independence between all the individual votes \( X_{\lambda i} \). In this special case, known as the Impartial Culture (see e.g. [9], [10], or [20]), we have

\[
P(X_{11} = x_{11}, \ldots, X_{MN_M} = x_{MN_M}) = \prod_{\lambda=1}^{M} \prod_{i=1}^{N_{\lambda}} P_0(X_{\lambda i} = x_{\lambda i}) = \frac{1}{2^N}.
\]

This article treats the class of voting measures called the collective bias model (or common belief model, CBM) which extends the Impartial Culture considerably by allowing correlations both between voters in the same group as well as correlations across group borders. We introduce and discuss the CBM in Section 3.

Once a voting measure is given, the quantities \( X_{\lambda i}, S_\lambda, \chi_\lambda \), and the raw democracy deficit are random variables defined on the same probability space \( \{-1, 1\}^N \).

Now we can define the concept of democracy deficit which is a measure of how well the council votes follow the public opinion:

**Definition 4.** The democracy deficit given a voting measure \( P \) and a set of weights \( w_1, \ldots, w_M \) is defined by

\[
\Delta_1 = \Delta_1(w_1, \ldots, w_M) := E \left[ \left( S - \sum_{\lambda=1}^{M} w_{\lambda} \chi_{\lambda} \right)^2 \right].
\]

We call \((w_1, \ldots, w_M)\) optimal weights if they minimise the democracy deficit, i.e.

\[
\Delta_1(w_1, \ldots, w_M) = \min_{(v_1, \ldots, v_M) \in \mathbb{R}^M} \Delta_1(v_1, \ldots, v_M).
\]

Note that the democracy deficit depends on the voting measure. It is also worth pointing out that the democracy deficit is a differentiable function of the council weights. This facilitates the analysis required to find the optimal weights.

Instead of minimising the democracy deficit, we could ask the question of how to minimise the probability that the binary council decision differs from the decision made by a referendum. This would be a less strict criterion in the sense that for a favourable public opinion of 51%, a 51% percent vote in the council and a 100% vote would both be considered equally good. However, one could argue that a 100% vote in the council would not be a good representation of public opinion. In fact, the 49% minority might feel they are not represented in the council at all, giving rise to populist anti-elite sentiment among the minority. We argue that adjusting the voting outcomes in the council in such a way that they follow the popular opinion as closely as possible is a worthwhile goal.
If we multiply each weight by the same positive constant and keep the relative quota $q$ fixed, we obtain an equivalent voting system. If the weights $w_\lambda$ minimise the democracy deficit $\Delta_1$, then the (equivalent) weights $\sigma w_\lambda$ minimise the ‘renormalised’ democracy deficit $\Delta_\sigma$ defined by

$$\Delta_\sigma = \Delta_\sigma(v_1, \ldots, v_M) := \mathbb{E} \left[ \left( \frac{S}{\sigma} - \sum_{\lambda=1}^{M} v_{\lambda} \chi_\lambda \right)^2 \right].$$

It is, therefore, irrelevant whether we minimise $\Delta_1$ or $\Delta_\sigma$ as long as $\sigma > 0$. In this article, we will compute optimal weights as $N$ tends to infinity. As a rule, in this limit the minimising weights for $\Delta_1$ will also tend to infinity, it is therefore useful to minimise $\Delta_\sigma$ with an $N$-dependent $\sigma$ to keep the weights bounded. A particularly convenient choice is to normalize the weights $w_\lambda$ in such a way that $\sum_\lambda w_\lambda = 1$.

The rest of this paper is organised as follows: as a first step, in Section 2 we recall the CBMs with independent groups studied in the past and give an example of a CBM with correlated voting across group boundaries. Then, we formally define the CBM and give several more examples in Section 3. In Section 4 we discuss the problem of determining the optimal weights in order to minimise the democracy deficit. Section 5 contains the results concerning the large population behaviour of CBMs. In Sections 6 and 7 we calculate the optimal weights in the large population limit. Then, Section 8 discusses the optimal weights for some specific models introduced earlier, such as additive and multiplicative models. Sections 6 to 8 contain the main results of this article concerning the optimal weights for a large set of CBMs. The second part of our analysis of optimal weights concerns their non-negativity. Under independence of the groups, the optimal weights are always strictly positive. Thus, this is a new aspect owed entirely to the relaxation of the independence assumption and not previously analysed in the literature. Section 9 deals with the problem of negative optimal weights and conditions that rule them out. Section 10 presents an extension of the CBM with negative optimal weights and conditions that rule them out. Section 11 concludes the paper.

## 2 A Warm-Up

Before defining the CBM in its full generality, we first recall the CBMs treated in the past, where the biases in different groups are independent of each other, and hence the voters belonging to different groups act independently. Then, we give an example of a CBM with correlated groups. It is our hope that the informal description of these special cases before defining and analysing the CBM in its full generality will make the model and the article more accessible to a wider range of readers.

In [12], one of us introduced the CBM with groups still being independent. To distinguish it from the generalisation we are going to introduce below, we refer to the CBM with independent groups as the simple CBM for the rest of this paper. Let $T_1, \ldots, T_M$ be a collection of independent and identically distributed random variables with support in $[-1, 1]$. We will refer to a realisation of each of these variables as $t_\lambda$. Conditionally on $t_\lambda$, the voters in group $\lambda$ vote independently of each other and the probability of a ‘yes’ vote is $\frac{1 + t_\lambda}{2}$. Thus, a positive bias makes a +1 vote more likely from each of the voters. Since within each group all voters are subject to the same bias, there is, in fact, positive correlation between votes. There is, however, no correlation between votes in different groups. The group bias reflects some real-world influence on the voters’ decisions, such as cultural norms, institutions such as organised religions, or more recently social media and influencers, specific to each group. Suppose each of these entities has some stance on each issue that can be put to vote and these opinions aggregate to some public bias. This bias can be quantified: a value of $-1$ or close to it reflects a strong rejection; a value around 0 means neutrality or indifference; a positive value close to $+1$ reflects strong support of an issue. The bias affects the voting outcomes in such a way that, provided the population is large enough, a strong negative bias will result in a large negative vote $S_\lambda$. Similarly, a strong positive bias induces a large positive vote, and the absence of a substantial bias causes a small absolute voting margin $|S_\lambda|$, i.e. a voting outcome close to a tie, because the individual votes are nearly independent.

In order to obtain the distribution of votes in the overall population, we have to average out the votes given all possible values of the group bias variables. Due to the independence of the group bias variables
With respect to the distribution of the group bias variable $T_1, \ldots, T_M$, we can factor the probabilities and obtain for each voting configuration $(x_{11}, \ldots, x_{MN_M}) \in \{-1, 1\}^N$
\[
P(X_{11} = x_{11}, \ldots, X_{MN_M} = x_{MN_M}) = \prod_{\lambda=1}^M P(X_{\lambda 1} = x_{\lambda 1}, \ldots, X_{\lambda N_\lambda} = x_{\lambda N_\lambda})
\]
because the groups are independent. In accordance with the verbal description given in the last paragraph, the probabilities for each group’s voting configuration can be expressed as follows:
\[
P(X_{\lambda 1} = x_{\lambda 1}, \ldots, X_{\lambda N_\lambda} = x_{\lambda N_\lambda}) = E\left( \prod_{i=1}^{N_\lambda} P_{T_\lambda}(X_{\lambda i} = x_{\lambda i}) \right),
\]
where $P_{T_\lambda}$ is the probability measure on $\{-1, 1\}$ with $P_{T_\lambda}(1) = \frac{1+T_\lambda}{2}$ and the expectation $E$ is taken with respect to the distribution of the group bias variable $T_\lambda$. To recapitulate, we obtain the probability of a voting configuration by taking the (random) probabilities $P_{T_\lambda}(X_{\lambda i} = x_{\lambda i})$ for each individual vote and multiplying them together for all voters belonging to group $\lambda$. This multiplicative form of the probabilities is due to the conditional independence of the individual votes in the group for a given realisation of the group bias variable $T_\lambda = t_\lambda$. Next, we take the expectation of this product of probabilities over all possible realisations of $T_\lambda$. This gives us the probability of the voting configuration $(x_{\lambda 1}, \ldots, x_{\lambda N_\lambda})$ in group $\lambda$. Finally, multiplying the probabilities of each group’s voting configuration yields the probability of the overall voting configuration $(x_{11}, \ldots, x_{MN_M})$ under the simple CBM.

For the simple CBM, we can calculate the optimal weights in the council (see [12] for this result). These turn out to be proportional to the expected absolute value of the group voting margins, $w_\lambda \propto E(|S_\lambda|)$. It is known that as each group’s population diverges to infinity, we have $E(|S_\lambda|) / N_\lambda \rightarrow E|T_\lambda|$. The latter expectation is a characteristic of the underlying distribution of the bias variables $T_\lambda$. This implies (under the mild assumption $E|T_\lambda| > 0$) that in the large population limit the optimal weights in the council are proportional to each group’s population, i.e. $w_\lambda = CN_\lambda$, with the same positive multiplicative constant $C$ for each group.

Thus, the simple CBM yields the recommendation of assigning each group a number of votes in the council which is proportional to its population. This stands in contrast to Penrose’s square root law, which prescribes weights proportional to the square root of the population instead of the population itself. Evidently, proportionality favours larger groups at the expense of smaller ones and vice-versa for the square root law.

Next, we consider a model with bias variables $T_1, \ldots, T_M$ which are correlated, thus inducing correlated voting across groups boundaries. Let $Z$ be a uniformly distributed random variable on the interval $[-1/2, 1/2]$. We will call $Z$ the global bias variable. Let $Y_1, \ldots, Y_M$ be i.i.d. copies of $Z$. We define the group bias variables $T_1, \ldots, T_M$ by setting $T_\lambda := Y_\lambda + Z$ for each group $\lambda$. This is a special case of what we will call an ‘additive model’ in later sections.

In an additive model, the global bias is modified by a group-specific bias which may reinforce or counteract the global bias. By assuming all these variables are identically distributed, we assign equal influence to the global bias and group-specific attitudes. Of course, it is also possible to assume a stronger global or local bias, a topic we will explore in Section 8.3.

Even though $Z, Y_1, \ldots, Y_M$ are independent, due to the addition of the same variable $Z$ in the definition of each $T_\lambda$, the group bias variables $T_\lambda$ are not independent. Given a realisation $t_\lambda = y_\lambda + z$, the individual votes in group $\lambda$ each turn out positive with probability $\frac{1+T_\lambda}{2}$. Contrary to the simple CBM, we cannot factor the probabilities of the overall voting configuration into the probabilities of the group voting configurations. Instead, the probabilities can be expressed as
\[
P(X_{11} = x_{11}, \ldots, X_{MN_M} = x_{MN_M}) = E\left( \prod_{\lambda=1}^M \prod_{i=1}^{N_\lambda} P_{T_\lambda}(X_{\lambda i} = x_{\lambda i}) \right),
\]
where the expectation $E$ is taken with respect to the distributions of $Z, Y_1, \ldots, Y_M$. As the terms $P_{T_\lambda}(X_{\lambda i} = x_{\lambda i})$ each depend on two different random variables, $Y_\lambda$ and $Z$, there is no way to factor the expectation above.
In order to minimise the democracy deficit, we have to solve the linear equation system given by \( \mathbf{S} \). We omit the calculations and refer the reader to Section 2.1, where we will analyse a more general additive CBM with uniformly distributed bias variables. By Theorem 28, the optimal weight for each group is asymptotically given by

\[
 w_\lambda = D + C \frac{N_\lambda}{N},
\]

where we have simplified and normalised the weights. The positive constants \( C \) and \( D \) are common for all groups. Contrary to the simple CBM with independent groups, in this correlated example, we have a summand which is proportional to the size of the group, but we also have a constant summand \( D \) which is the same for each group and hence independent of the group’s size. This is qualitatively the same formula as the one employed for the composition of the U.S. Electoral College, where \( D = 2 \), the number of senators for each state, and \( C = 435 \) is the number of representatives, of which each state receives a number roughly proportional to its population. The new feature of the general CBM compared to the simple CBM concerning the problem of optimal council weights is the presence of the constant term \( D \). This functional form for the optimal weights applies not just to the special case discussed in this section but in general to CBMs with correlated groups. See Theorem 21 for the general result. As far as the authors of this article know, this is the first theoretical justification of the formula that determines the number of electors for each state in the U.S. Electoral College. We want to emphasise that it is the square root law which is more favourable to small groups than proportionality. However, the square root law is even better for small groups in most cases, the exception being when the difference in size between small and large groups is minuscule.

3 The Collective Bias Model

We recall from the last section that in the simple CBM, the votes \( X_{\lambda_1} \) within a group \( \lambda \) are correlated via a random variable \( T_\lambda \) with values in \([-1, 1]\), the local ‘collective bias’. The random variables \( T_\lambda \) model the influence of a cultural tradition in the respective group or the leverage of a strong political party (or religious group, etc.) within the group \( \lambda \). It is this central influence, which affects all voters within a given group equally, which induces positive correlation between votes within each group. Aside from this central influence, the voters make up their own minds. This is in contrast to models with interactions between voters such as those inspired by spin models from statistical mechanics, e.g. mean-field models (see [16] for a discussion of a mean-field model and the determination of the optimal weights). In the simple CBM, there is no correlation between votes in different groups, only correlation within groups.

In what follows, we will define the simple CBM in the same measure-theoretic language we will also employ for the general CBM. Given the bias \( T_\lambda = t_\lambda \), the per capita voting margin \( S_\lambda/N_\lambda \) inside group \( \lambda \) fluctuates around \( t_\lambda \). More precisely, suppose the bias variable \( T_\lambda \) is distributed according to the probability measure \( \rho \) on \([-1, 1]\). Then the simple CBM for the group \( \lambda \) is given by

\[
 P(X_{\lambda_1} = x_1, X_{\lambda_2} = x_2, \ldots, X_{\lambda N_\lambda} = x_{N_\lambda}) = \int P_t(x_1, \ldots, x_{N_\lambda}) \rho(dt),
\]

where

\[
 P_t(x) = \begin{cases} 
 \frac{1}{2}(1 + t), & \text{for } x = 1, \\
 \frac{1}{2}(1 - t), & \text{for } x = -1,
\end{cases}
\]

and

\[
 P_t(x_1, \ldots, x_{N_\lambda}) = P_t(x_1) P_t(x_2) \cdots P_t(x_{N_\lambda}).
\]

By \( E_t \) we denote the expectation with respect to \( P_t \). The definition of \( P_t \) implies that \( E_t(X) = t \). We call \( \rho \) the local bias measure of group \( \lambda \).
We remark that, due to de Finetti’s Theorem\(^1\), the simple CBM is the most general voting measure that is ‘anonymous’ in the sense that reordering the voters leaves the measure unchanged (see \([17]\) or \([14]\)). The ‘Impartial Anonymous Culture’, which underlies the Shapley-Shubik power index \([20]\) (see also \([10]\) or \([21]\)), is a particular case of \([2]\) where \(\rho\) is the uniform distribution on \([-1, 1]\). The Impartial Culture is another special case for which \(\rho = \delta_0\), the Dirac measure\(^2\) at \(t = 0\).

In the simple CBM, the voting results in different groups are independent, so the corresponding voting measure on \(\prod_{\lambda=1}^{M} (-1, 1)^{N_{\lambda}}\) is given by the product of the probabilities \([2]\).

\[
\mathbb{P}\left(\mathbf{X}_1 = x_1, \ldots, \mathbf{X}_M = x_M\right) = \prod_{\lambda=1}^{M} P_{\lambda}\left(x_\lambda\right) \rho(\mathrm{d}t_1) \cdots \rho(\mathrm{d}t_M),
\]

where \(\mathbf{X}_\lambda = (X_{\lambda,1}, \ldots, X_{\lambda, N_\lambda})\) and similarly for \(\mathbf{x}_\lambda\).

In this paper, we study the generalised collective bias model (CBMs with dependence across group boundaries) were first analysed in \([27]\)). In this model, there is an additional global bias variable \(Z\) with values in \([-1, 1]\) and with distribution \(\mu\). The global bias \(Z\) influences each of the groups in a similar way. This is implemented in the model by allowing the local bias measure \(\rho\) to depend on the value \(Z = z\). More precisely, the (generalised) collective bias model is given by:

**Definition 5.** Suppose \(\mu\) is a probability measure on \([-1, 1]\) and for every \(z \in \{-1, 1\}\) there is a probability measure \(\rho^z\) on \([-1, 1]\). Then we define the probability measure \(\mathbb{P}_{\mu, \rho}\) on \(\{-1, 1\}^N = \prod_{\lambda=1}^{M} \{-1, 1\}^{N_{\lambda}}\) by

\[
\mathbb{P}\left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M\right) = \left(\int P_{\lambda}\left(x_\lambda\right) \rho^z(\mathrm{d}t_1) \cdots \rho^z(\mathrm{d}t_M)\right) \mu(\mathrm{d}z),
\]

where \(\mathbf{x}_\lambda \in \{-1, 1\}^{N_{\lambda}}\).

We call \(\mathbb{P}_{\mu, \rho}\) the collective bias measure with global bias measure \(\mu\) and local bias measure \(\rho = \rho^z\) or the CBM(\(\mu, \rho\)) for short. If \(\mu\) and \(\rho\) are clear from the context, we simply write \(\mathbb{P}\) instead of \(\mathbb{P}_{\mu, \rho}\).

**Remark 6.** Technically speaking, \(\rho^z\) is a stochastic kernel (see e.g. \([17]\)), i.e.:

1. For every \(z \in [-1, 1]\), the quantity \(\rho^z\) is a probability measure on \([-1, 1]\).
2. For every Borel set \(A \subset [-1, 1]\), the function \(z \mapsto \rho^zA\) is measurable.

We could allow the kernels \(\rho^z\) to depend on the group \(\lambda\), and in Section \([10]\) we will come back to this generalisation, but for the moment we take the same local bias measure for all groups.

To ensure that \(\mathbb{P}_{\mu, \rho}\) is a voting measure, i.e. to satisfy \([1]\), we assume the following sufficient condition in what follows:

**Assumptions 7.** 1. \(\mu\) is symmetric, i.e. \(\mu A = \mu(-A)\),
2. for all \(z \in [-1, 1]\), the distributions \(\rho^z\) satisfy \(\rho^z A = \rho^{-z} (A)\) for all measurable sets \(A \subset [-1, 1]\).

The general framework of a CBM is given by a set of bias random variables that represent some cultural or political influence that acts on all voters. There is a global bias variable \(Z\) with distribution \(\mu\) which induces correlation between voters of different groups. Furthermore, there is a local bias variable \(T_\lambda\) for each group. Its conditional distribution given the realisation \(Z = z\) is \(\rho^z\). The group bias variable \(T_\lambda\) induces correlation between the voters belonging to that group. The result is correlated voting across group boundaries, as a rule with stronger correlation within each group to account for shared culture and preferences.

Conditionally on the realisations of \(Z = z\) according to \(\mu\) and \(T_\lambda = t_\lambda\) according to \(\rho^z\), all voters in group \(\lambda\) cast their vote independently, with a probability of voting ‘aye’ equal to \(\frac{1 + t_\lambda}{2}\). Hence, a value \(t_\lambda = 1\)

\(^1\)De Finetti’s Theorem states that an infinite sequence of exchangeable random variables can be represented as a mixture of i.i.d. random variables. The mixture is specified by a probability measure referred to as a de Finetti measure. De Finetti’s Theorem has been considerably generalised. See e.g. \([3]\).

\(^2\)The Dirac measure (or point mass) at \(x \in \mathbb{R}^M\), \(\delta_x\), is a probability measure which assigns any set \(A \subset \mathbb{R}^M\) the probability 1 if \(x \in A\) and 0 otherwise.
implies that all voters belonging to group $\lambda$ vote ‘aye’ almost surely. Similarly, $t_\lambda = -1$ implies all vote ‘nay’ almost surely. $t_\lambda = 0$ means there is no bias, and all voters in the group vote independently with probability $\frac{1}{2}$ for ‘aye’ (and the same probability for ‘nay’).

**Examples 8.** We discuss various examples (or classes of examples) of CBMs.

1. If the measures $\rho^z$ are independent of $z$, then the (generalised) CBM reduces to the simple CBM. The Impartial Anonymous Culture is a particular case of this class of examples.

2. If $\rho^z = \delta_0$, then all random variables $X_{\lambda i}$ are independent reflecting Impartial Culture.

3. If $\rho^z = \delta_z$, then we have a simple CBM for the union, i.e. for all $X_{\lambda i}$.

4. In the class of additive models, the ‘total bias’ prevailing within each group $T_\lambda$ is the sum of the global bias variable $Z$ and a local or group bias modifier variable $Y_\lambda$, i.e. $T_\lambda = Z + Y_\lambda$. Assume the bias modifiers $Y_\lambda$ are independent and identically distributed according to a fixed symmetric probability measure $\rho$. Then, for each realisation $Z = z$, the local measure $\rho^z$ is given by

$$\rho^z[a, b] = \rho[a - z, b - z].$$

So for this model class we have

$$\mathbb{P}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_M) = \int \left( \int P_{z+y_1}(\mathcal{Z}_1) \rho(dy_1) \cdots \int P_{z+y_M}(\mathcal{Z}_M) \rho(dy_M) \right) \mu(dz). \tag{4}$$

To ensure that $t_\lambda = z + y_\lambda \in [-1, 1]$ we assume that $\text{supp} \mu + \text{supp} \rho \subset [-1, 1]$, where $\text{supp} \mu$ stands for the support of the measure $\mu$. This kind of additive CBM was first introduced and analysed in Section 4.2 of [27]. Additive models are discussed in more detail in Section 5.1.

5. For a particular example of the additive model which we are going to discuss in some detail, we choose $\mu$ and $\rho$ as the uniform distribution on $[-g, g]$ and on $[-\ell, \ell]$, respectively, with $0 < g, \ell$ and $g + \ell \leq 1$.

In this case, the additive CBM-measure is given by

$$\frac{1}{2g} \int_{-g}^{+g} \left( \frac{1}{2\ell} \int_{-\ell}^{+\ell} P_{z}(x_1) \, dt_1 \cdots \frac{1}{2\ell} \int_{-\ell}^{+\ell} P_{z}(x_M) \, dt_M \right) \, dz.$$  

This example may be considered a ‘hierarchical’ version of Impartial Anonymous Culture.

6. In the class of multiplicative models, the total bias is the product of the global bias variable $Z$ and the group bias modifier variable $Y_\lambda$, i.e. $T_\lambda = ZY_\lambda$. We assume the $Y_\lambda$ are independent and identically distributed according to a fixed probability measure $\rho$. Then the local measure is $\rho^0 = \delta_0$ if $Z = 0$, and for $Z = z \neq 0,$

$$\rho^z[a, b] = \rho \left[ \frac{a}{z} \wedge \frac{b}{z}, \frac{a}{z} \vee \frac{b}{z} \right].$$

Above, we used the notation $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ for all real numbers $a$ and $b$.

This kind of multiplicative CBM was first introduced and analysed in Section 4.1 of [27]. We discuss the multiplicative model in Section 5.2.

7. In the CBM($\mu, \rho$), the measure $\rho$ must have support in $[-1, 1]$. Above, we assumed without loss of generality the same for the measure $\mu$. In the following example, it is more convenient to have more freedom in the choice of $\mu$.

Suppose that $\rho^z$ is the beta distribution $B(z, z, -1, 1)$, i.e. $\rho^z$ has the density

$$f_z(x) := \frac{\Gamma(2z)}{\Gamma(z)^2} \frac{1}{2^{2z-1}} (1 + x)^{z-1} (1 - x)^{z-1}$$
for \( x \in [-1, 1] \), where \( \Gamma \) is the Gamma function. For \( \mu \), we can take any probability distribution on \((0, \infty)\). Note that the symmetry condition (11) is satisfied.

For large \( z \), the measures \( \rho^z \) are more and more concentrated around 0. For \( z = 1 \), the measure \( \rho^z \) is the uniform distribution, and for small \( z > 0 \), \( \rho^z \) is more and more concentrated near the extreme positions +1 and −1. The measures \( \rho^z \) are intimately connected with Polya urn models which are discussed, for example, in [2] and [20].

In a sense, the parameter \( z \) reflects the ‘polarisation’ inside the society.

8. We end the presentation of examples with a rather pathological class, in fact one we are going to exclude below. Suppose that for \( \mu \)-almost all \( z \) either \( \rho^z = \delta_1 \) or \( \rho^z = \delta_{-1} \). Then the popular vote is always unanimous. So, in a sense, there is little randomness in this example.

4 Democracy Deficit and Optimal Weights

We want to choose the weights so that the democracy deficit is minimal. By taking partial derivatives of \( \Delta_\sigma \) with respect to each \( w_\lambda \), we obtain a system of linear equations that characterizes the optimal weights. Indeed, for \( \lambda = 1, \ldots, M \),

\[
\sum_{\nu=1}^{M} E (\chi_\lambda \chi_\nu) w_\nu = \frac{1}{\sigma} E (\chi_\lambda S) .
\]

(5)

Defining the matrix \( A \), the weight vector \( w \) and the vector \( b \) on the right hand side of (5) by

\[
A := (A_{\lambda \nu})_{\lambda,\nu=1,\ldots,M} := E (\chi_\lambda \chi_\nu),
\]

(6)

\[
w := (w_\lambda)_{\lambda=1,\ldots,M} := \frac{1}{\sigma} E (\chi_\lambda S),
\]

(7)

we may write (5) in matrix form as

\[
A w = b .
\]

(8)

Since the quantity \( b \) depends on \( \sigma \) (by a factor of \( \frac{1}{\sigma} \)), the optimal weights do as well.

A solution \( w \) of (5) is indeed a minimum if the matrix \( A \), the Hessian of \( \Delta \), is (strictly) positive definite.

In this case, the matrix \( A \) is invertible and consequently there is a unique tuple of optimal weights, namely the unique solution of (8).

If the groups vote independently of each other, the matrix \( A \) is diagonal. This happens for CBM(\( \mu, \rho \))-measures for which \( \rho \) is independent of \( z \). These cases are treated in [12].

It turns out that in the general case the matrix \( A \) is indeed invertible under rather mild conditions.

Definition 9. We say that a voting measure \( P \) on \( \prod_{\lambda=1}^{M} \{1, -1\}^{N_\lambda} \) is sufficiently random if

\[ P(\chi_1 = c_1, \ldots, \chi_M = c_M) > 0 \quad \text{for all } c_1, \ldots, c_M \in \{-1, 1\}. \]

(9)

Note that (11) is not very restrictive. For example, if the support \( \text{supp} P \) of the measure \( P \) is the whole space \( \{-1, 1\}^N \), then \( P \) satisfies (11). Moreover, for CBMs, we have:

Proposition 10. Suppose that \( P \) is a CBM(\( \mu, \rho \))-measure. Then \( P \) is sufficiently random if and only if

\[ \mu \left\{ z \mid \rho^z \{-1\} = 1 \text{ or } \rho^z \{1\} = 1 \right\} < 1. \]

(10)

Remark 11. If \( \mu \{ z \mid \rho^z \{-1\} = 1 \text{ or } \rho^z \{1\} = 1 \} = 1 \), then the voting result in each group is unanimous, so weights proportional to \( N_\lambda \) are optimal weights (not necessarily unique).
Proposition 12. Let \( P \) be a voting measure and let \( A \) be defined by (6).

1. The matrix \( A \) is positive semi-definite.
2. \( A \) is positive definite if \( P \) is sufficiently random.

Proof. For vectors \( x, y \in \mathbb{R}^M \), we will write \( (x, y) \) for the Euclidean inner product. For any vector \( x = (x_1, \ldots, x_M) \), we have

\[
(x, Ax) = \mathbb{E} \left( \left( \sum_{\lambda=1}^{M} x_\lambda \chi_\lambda \right)^2 \right) \geq 0.
\]

(11)

So \( A \) is positive semi-definite.

Suppose now that \( (x, Ax) = 0 \). Then

\[
\mathbb{E} \left( \left( \sum_{\lambda=1}^{M} x_\lambda \chi_\lambda \right)^2 \right) = 0.
\]

This implies that

\[
\sum_{\lambda=1}^{M} x_\lambda \chi_\lambda = 0 \quad \text{almost surely.} \tag{12}
\]

For a sufficiently random model, this is only possible if \( x = 0 \).

\[\square\]

Theorem 13. If the voting measure \( P \) is sufficiently random, the optimal weights minimising the democracy deficit \( \Delta_\sigma \) are unique and given by

\[
w = A^{-1} b.
\]

(13)

Definition 14. If \( w \) satisfies (13), we set

\[
\overline{w}_\nu := \frac{w_\nu}{\sum_{\lambda=1}^{M} w_\lambda},
\]

and call \( \overline{w}_\nu \) the normalised optimal weights.

While the weights \( w \) depend on \( \sigma \) through \( b = b_\sigma \), the normalised weights \( \overline{w} \) are independent of \( \sigma \). The \( \overline{w}_\nu \) sum up to 1.

For the rest of this paper, we shall always assume that our models are sufficiently random.

Given Theorem 13, one is tempted to believe that the problem of optimal weights is solved. Unfortunately, this is not the case, because it is practically impossible to compute the ingredients like \( \mathbb{E}(\chi_\lambda \chi_\nu) \) and \( \mathbb{E}(S\chi_\lambda) \) for finite (but fairly large) \( N \). A way out is to compute these quantities approximately for \( N \to \infty \), and this is what we are doing throughout the rest of this paper.

5 Asymptotics for the Collective Bias Model

For given \( \mu \) and \( \rho \) and for \( N = (N_1, \ldots, N_M) \), we denote by \( \mathbb{P}_N \) the CBM(\( \mu, \rho \))-measure on \( \prod_{\lambda=1}^{M} \{ -1, 1 \}^{N_\lambda} \).

In the following, we try to compute optimal weights for large \( N = \sum_\lambda N_\lambda \). More precisely, we consider (8) for \( N \to \infty \). This limit is always taken in the sense that

\[
\lim_{N \to \infty} \frac{N_\lambda}{N} = \alpha_\lambda > 0
\]

(14)
for each \( \lambda \), and we assume that each group’s population diverges to infinity as \( N \) goes to infinity. Observe that \( \sum_\lambda \alpha_\lambda = 1 \). The constants \( \alpha_\lambda \) represent the population of each group as a fraction of the overall population (at least asymptotically). Whenever the \( N_\lambda \) are clear from the context we write \( P_N, E_N \) instead of \( P_{N_\lambda}, E_{N_\lambda} \), etc. We also set

\[
(A_N)_\lambda := E_N(\chi_\lambda \chi_\nu), \quad (b_N)_\lambda := E_N\left(\frac{S}{N}\chi_\lambda\right),
\]
and

\[
s_N := E_N\left(\left(\frac{S}{N}\right)^2\right).
\]
(15)

Then

\[
\Delta_N(w) = s_N - 2(w, b_N) + (w, A_N w).
\]
(16)

In the above formulas, we set \( \sigma := N \). From now on, we assume that \( \mathbb{P} \) is sufficiently random, i.e. that (10) holds. Moreover, to avoid discussing different cases we also assume that \( \rho \) is not trivial in the sense that

\[
\mu\{z \mid \rho^z = \delta_0\} < 1.
\]
(17)

If (17) is violated, all voters act independently of each other. This is the ‘Impartial Culture’ and Penrose’s square root law holds (see e.g. [7] or [12]).

The following result is the key observation which allows us to evaluate important quantities asymptotically. This theorem explains the large population behaviour of a CBM.

**Theorem 15.** Suppose that the functions \( f_\lambda : [-1, 1] \to \mathbb{R}, \lambda = 1, \ldots, M \), are continuous on \( [-1,0) \cup (0,1] \), and assume that the limits \( f_\lambda(0+) = \lim_{t \to 0^+} f_\lambda(\alpha_\lambda t) \) and \( f_\lambda(0-) = \lim_{t \to 0^-} f_\lambda(\alpha_\lambda t) \) exist. Set

\[
I_z(f_\lambda) := \int_{[-1,0) \cup (0,1]} f_\lambda(\alpha_\lambda t) \rho^z(\text{d}t) + \frac{1}{2}(f_\lambda(0+) + f_\lambda(0-)) \rho^z\{0\}.
\]
Then

\[
E\left(f_1\left(\frac{1}{N}S_1\right) \cdots f_M\left(\frac{1}{N}S_M\right)\right) \to \int I_z(f_1) \cdots I_z(f_M) \mu(\text{d}z).
\]
(18)

We could handle functions \( f_\lambda \) with discontinuities (and left and right limits) in other points than 0 as well, but we need the result only in the above form. The proof below, however, works for the more general case as well. Theorem 15 says that the normalised voting margins \( S_\lambda/N \) follow a distribution given by \( \rho^z \) and \( \mu \) in the large population limit. We can take transformations \( f \) of these voting margins and their behaviour will be described by the distributions \( \rho^z \) and \( \mu \). Chief among these transformations will be the council vote \( \chi_\lambda = \chi_\lambda(S_\lambda) \) cast by each group which presents a point of discontinuity at 0.

**Proof.** By the strong law of large numbers, we get

\[
P_t\left(\lim_{n \to \infty} \frac{1}{N_\lambda} S_\lambda = t\right) = 1.
\]

So, if \( f \) is continuous on \([-1, 1]\), it follows that

\[
\int E_t\left(f\left(\frac{1}{N}S_\lambda\right)\right) \rho^z(\text{d}t) \to \int f(\alpha_\lambda t) \rho^z(\text{d}t)
\]
for all \( z \). From this, (18) follows for continuous \( f_\lambda \). To prove (18) in the general case, we observe that for \( N \to \infty \)

\[
P_0\left(\frac{1}{N_\lambda} S_\lambda > 0\right) \to \frac{1}{2} \quad \text{and} \quad P_0\left(\frac{1}{N_\lambda} S_\lambda < 0\right) \to \frac{1}{2}.
\]
**Definition 16.** We introduce the following notation for further use:

\[ m_1(\rho) = \int t \rho^z(dt), \quad m_2(\rho) = \int t^2 \rho^z(dt), \]

\[ m_1(\rho) = \int |t| \rho^z(dt), \quad d(\rho) = \rho^z(0, 1] - \rho^z[-1, 0). \]

Note that the above quantities depend on \( z \). These quantities are important characteristics of the measures \( \mu \) and \( \rho \). They measure the strength of the group bias for different values \( z \) of the global bias. E.g. a positive \( m_1(\rho) \) close to 1 implies that the group bias measure \( \rho^z \) induces, on average, a strong bias in favour of the issue being considered. Whereas \( m_1(\rho) \) can be interpreted as a measure of intra-group cohesion, the product of \( m_1 \) and \( d(\rho) \) is a measure of inter-group cohesion. These two measures will allow us to compare how strong the intra-group cohesion is versus the inter-group cohesion. These quantities will be used in the calculation of the optimal weights.

For any function \( \varphi \) on \([-1, 1]\), we introduce the shorthand notation

\[ \langle \varphi \rangle = \int \varphi(z) \mu(dz). \]

**Theorem 17.** Assume (10), (14) and (17). Then

\[ (A_N)_{\lambda \nu} \to a := \langle d(\rho)^2 \rangle, \lambda \neq \nu, \]

\[ (b_N)_{\lambda} \to b_\lambda := (m_1(\rho) - m_1(\rho)d(\rho)) \alpha_\lambda + \langle m_1(\rho)d(\rho) \rangle, \]

\[ s_N \to s := \sum_{\nu=1}^{M} \alpha_\nu^2 (\langle m_2(\rho) \rangle - (m_1(\rho))^2) + \langle m_1(\rho) \rangle. \]

Theorem 17 follows immediately from Theorem 15.

Informally speaking, Theorem 17 says that the minimisation problem (16) ‘converges’ to the minimisation problem

\[ \min \Delta_\infty(v_1, \ldots, v_M) = \sum_{\lambda=1}^{M} \lambda v_\lambda^2 + 2 \langle v, b \rangle + (v, Av). \]  

(19)

In the following, we try to explore the validity of this informal idea. The following theorem implies that for positive definite limiting coefficient matrices \( A \) the optimal weights of the finite population problem converge to the optimal weights of the asymptotic problem.

**Theorem 18.** The matrices \( A_N \) converge (in operator norm) to the matrix

\[ A_{\lambda \nu} = \begin{cases} 1, & \text{if } \lambda = \nu, \\ a, & \text{otherwise,} \end{cases} \]

(20)

with \( a = \langle d(\rho)^2 \rangle \).

Moreover, \( A \) is positive semi-definite. \( A \) is positive definite if \( a < 1 \). In this case,

\[ A_N^{-1} \to A^{-1} \]

(21)

and

\[ (A^{-1})_{\lambda \nu} = \frac{1}{D} \begin{cases} 1 + (M - 2)a, & \text{if } \lambda = \nu, \\ -a, & \text{otherwise,} \end{cases} \]

(22)

where \( D = (1 - a)(1 + (M - 1)a) \).

**Proof.** We note that \( 0 \leq a \leq 1 \). Since, for any \( x \in \mathbb{R}^M \),

\[ (x, Ax) = (1 - a) \sum_{\lambda=1}^{M} x_\lambda^2 + a \left( \sum_{\lambda=1}^{M} x_\lambda \right)^2, \]

(12)
we see that $A$ is positive semi-definite in general and positive definite if $a < 1$.

Let $I$ stand for the $M \times M$ identity matrix. To prove \(21\) we compute,

$$
\|A_N^{-1} - A^{-1}\| = \left\| A^{-1} \left( (I + (A_N - A)A^{-1})^{-1} - I \right) \right\|
$$

$$
\leq \|A^{-1}\| \sum_{k=1}^{\infty} \|A_N - A\|^k \|A^{-1}\|^k
$$

$$
= \frac{\|A^{-1}\|^2 \|A_N - A\|}{1 - \|A^{-1}\|\|A_N - A\|}.
$$

(23)

Since $\|A_N - A\|$ tends to 0, \(23\) goes to 0 as well. The claim \(22\) follows by direct calculation. \(\square\)

**Definition 19.** We say that the collective bias model CBM($\mu, \rho$) is tightly correlated if $a = \langle d(\rho)^2 \rangle = 1$.

As we will see, tight correlation implies that all groups end up voting unanimously in the council. For now, we characterise tight correlation in terms of the probabilities assigned by $\rho$ for different values $z$ of the global bias. The key idea is that $\rho^z$ assigns probability 1 to either $(0, 1)$ or $(0, 1)$ for $(\mu$-almost) all $z$, and thus all group biases will be of the same sign, inducing the aforementioned unanimous council vote.

**Proposition 20.** The collective bias model CBM($\mu, \rho$) is tightly correlated if and only if for $\mu$-almost all $z$ either $\rho^z(0, 1] = 1$ or $\rho^z[-1, 0) = 1$ holds.

*Proof.* Since $0 \leq d(\rho)^2 \leq 1$ for all $z$, we have $\xi := 1 - d(\rho)^2 \geq 0$ and $\int \xi \, d\mu = 0$ implies $\xi = 0 \, \mu$-almost surely. It follows that $|d(\rho)| = 1$ for $\mu$-almost all $z$, so $\rho^z(0, 1] = 1$ or $\rho^z[-1, 0) = 1$. \(\square\)

### 6 Optimal Weights

In this section, we investigate the asymptotics of the optimal weights of CBMs for large $N$. As above, we assume \(10, 14, \text{ and } 17\) for the rest of this paper.

The tightly correlated case needs a different treatment, so we first assume that the model CBM($\mu, \rho$) is not tightly correlated, i.e. that $a = \langle d(\rho)^2 \rangle < 1$, in this section. Section \(4\) discusses the tightly correlated case.

By Theorem \(13\) for fixed $N$, there are unique optimal weights $w_N$.

**Theorem 21.** If the model CBM($\mu, \rho$) is not tightly correlated, then the optimal weights $w^{(N)}$, i.e. the minima of $\Delta_N$, converge for $N \to \infty$ to the minima of $\Delta_\infty$ (defined in \(19\)), and these weights $w_\lambda$ are given by

$$
w_\lambda = C_1 \alpha_\lambda + C_2,
$$

(24)

with coefficients depending on $\mu, \rho$, and $M$ but not on the $\alpha_\lambda$.

More precisely,

$$
C_1 = \frac{1}{1 - a} \left( \langle \overline{m}_1(\rho) \rangle - \langle m_1(\rho) d(\rho) \rangle \right)
$$

(25)

and

$$
C_2 = \frac{1}{1 - a} \langle m_1(\rho) d(\rho) \rangle - a \langle \overline{m}_1(\rho) \rangle.
$$

(26)

Moreover,

$$
\sum_\lambda w_\lambda = \frac{\langle \overline{m}_1(\rho) \rangle + (M - 1) \langle m_1(\rho) d(\rho) \rangle}{1 + (M - 1) a}.
$$

(27)

Theorem \(21\) follows from Theorems \(17\) and \(18\) by a straightforward computation.
Corollary 22. Under the assumptions of Theorem 21, the normalised weights $\tilde{w}^{(N)}$ converge to

$$\tilde{w}_\lambda = C_1 \alpha_\lambda + C_2$$

(28)

Remark 23. 1. By Theorem 21, the optimal weights are always the sum of a term proportional to the size of the population and a term independent of the population. The weights of the states in the Electoral College of the U.S. constitution are precisely chosen in this fashion.

2. In the limit $a \to 0$, meaning that the groups are almost independent, the constant term in (24) tends to 0, so that $\tilde{w}_\lambda \to \alpha_\lambda$ which is the result for the simple CBM (see [12]).

3. The sum of the weights (27) is strictly positive and finite, even in the limit $a \to 1$. This indicates that the choice $\sigma = N$ is reasonable. In fact,

$$\lim_{a \to 1} \sum_\lambda w_\lambda = \langle m_1(\rho) \rangle.$$  

(29)

Corollary 24. Under the assumptions of Theorem 21, the minimal democracy deficit $\Delta_N$ is asymptotically of the form

$$\Delta_\infty = D_1 \sum_{\lambda=1}^M \alpha_\lambda^2 + D_2.$$  

Remark 25. The constants $D_1$ and $D_2$ depend on $\mu, \rho, M$ and can be computed from (22), (25), and (26).

7 Optimal Weights for Tight Correlations

Now we turn to the case of tightly correlated models, i.e. $a = 1$.

Then, in the limit $N \to \infty$, setting $\sigma := N$, equation (5) which describes the critical points of $\Delta_N$ tends to $\tilde{A}w = b$ with

$$\tilde{A}_{\lambda\nu} = 1 \quad \text{for all } \lambda, \nu.$$  

The matrix $\tilde{A}$ is degenerate. It has an $(M - 1)$-fold degenerate eigenvalue at 0 and a simple eigenvalue at $M$.

The democracy deficit $\Delta_N$ tends to

$$\Delta_\infty = \sum_{\lambda=1}^M \alpha_\lambda^2 \left( \langle m_2(\rho) \rangle - \langle m_1(\rho)^2 \rangle \right) + \langle m_1(\rho)^2 \rangle - 2 \langle m_1(\rho) \rangle \sum_{\lambda=1}^M w_\lambda + \left( \sum_{\lambda=1}^M w_\lambda \right)^2.$$  

(30)

[30] is an equation in $\sum_\lambda w_\lambda$. The extrema of $\Delta_\infty$ are all weights $w_\lambda$ such that

$$\sum_{\lambda=1}^M w_\lambda = \langle m_1(\rho) \rangle.$$  

This condition is in agreement with (29).

Theorem 26. Suppose $a = 1$. If

$$\sum_{\lambda=1}^M w_\lambda = \sum_{\lambda=1}^M v_\lambda,$$  

14
then
\[ \Delta_N(w) - \Delta_N(v) \to 0 \quad \text{as } N \to \infty. \]

In particular, any tuple of weights with \( \sum \lambda w_\lambda = \langle m_1(\rho) \rangle \) is close to the minimal democracy deficit in the sense that
\[ \Delta_N(w) \to \min_v \Delta_\infty(v). \]

Theorem 26 implies that for large systems with tight correlation ‘it doesn’t matter’ how the weights are distributed among the groups. This assertion is confirmed by the following observation:

**Theorem 27.** If the model CBM(\( \mu, \rho \)) is tightly correlated, then
\[ \mathbb{P}(S_\lambda > 0 \quad \text{or} \quad S_\lambda < 0 \quad \text{for all } \lambda) \to 1 \quad \text{as } N \to \infty. \]

Thus, in large tightly correlated systems, council votes are almost always unanimous! Consequently, for \( N \to \infty \), any \( w \) with \( \sum \lambda w_\lambda > 0 \) induces the same voting result in the council. This might be surprising at first. As the overall population goes to infinity, the probability of a unanimous council vote goes to 1. Hence, the limit of the optimality condition \( 5 \) is a linear equation system with an infinity of solutions. More precisely, any set of weights \( w_1, \ldots, w_M \) that sum to a fixed positive value given by the limit of \( 7 \) solves \( 5 \). As such, the assignation of the voting weights only serves the purpose of appropriately scaling the magnitude of the (unanimous) council vote to bring it in line with \( S/\sigma \). However, the constraint on the sum is not binding, as we know that any transformation of a weighted voting system that multiplies all weights by a positive constant while leaving the relative quota untouched is equivalent to the original voting system. Thus, a set of weights which sum to 1 is but a representative of an equivalence class of voting systems. The selection of the optimal weights when a unanimous council vote occurs with high probability is a trivial problem.

**Proof.** Set
\[ Z_+ = \{ z \in [-1, 1] \mid \rho^z(0, 1) = 1 \} \quad \text{and} \quad Z_- = \{ z \in [-1, 1] \mid \rho^z[-1, 0) = 1 \}. \]

Since the measure \( \mathbb{P} \) is tightly correlated, we have due to Proposition 20 that
\[ Z_+ \cup Z_- = [-1, 1] \quad \text{up to a set of } \mu \text{-measure 0}. \]

In particular, \( \rho^z \neq \delta_0 \) for \( \mu \)-almost all \( z \), so \( \mathbb{P}(S_\lambda = 0) \to 0 \) for any \( \lambda \).

Thus, it suffices to prove that for any given \( \nu \neq \lambda \)
\[ \mathbb{P}(S_\nu > 0, S_\lambda < 0) \to 0. \]

For \( t \in (0, 1] \), we have
\[ P_t(S_\lambda < 0) \to 0; \]

thus, for \( z \in Z_+ \),
\[ \int P_t(S_\lambda < 0) \rho^z(dt) \to 0 \]

and similarly, for \( z \in Z_- \),
\[ \int P_t(S_\nu > 0) \rho^z(dt) \to 0. \]

Hence
\[ \mathbb{P}(S_\nu > 0, S_\lambda < 0) \leq \int_{Z_+} \int P_t(S_\lambda < 0) \rho^z(dt) \mu(dz) + \int_{Z_-} \int P_t(S_\nu > 0) \rho^z(dt) \mu(dz) \to 0. \]
8 Specific Models

In this section, we analyse some models from Example 8. In these examples, we can compute relevant quantities explicitly.

8.1 Additive Models

We start with some additive models as in Example 8.4 with specific bias measures \( \mu \) and \( \rho \). We recall that for additive models the voting measure \( \mathbb{P}(x_1, x_2, \ldots, x_M) \) is given by

\[
\int \left( \int P_{z+y_1}(x_1) \rho(dy_1) \cdots \int P_{z+y_M}(x_M) \rho(dy_M) \right) \mu(dz).
\] (31)

\( \mathbb{P} \) is indeed a voting measure if both \( \mu \) and \( \rho \) are symmetric, i.e. \( \mu[a, b] = \mu[-b, -a] \) and similarly for \( \rho \). \( \mathbb{P} \) is sufficiently random except for the (pathological) case \( \mu = \frac{1}{2}(\delta_1 + \delta_{-1}) \) and \( \rho = \delta_0 \). \( \mathbb{P} \) is tightly correlated if (and only if) for \( \mu \)-almost all \( z \) either \( \rho(-z, 1] = 1 \) or \( \rho[-1, -z) = 1 \).

8.1.1 Uniform Distribution with Weak Global Bias

In our first example, we take \( \mu \) and \( \rho \) to be the uniform probability distribution on \([-g, g]\) (for ‘global’ bias) and \([-\ell, \ell]\) (‘local’ bias), respectively. We assume first that \( g \leq \ell \), indicating that the (average) global bias is not bigger than the (average) local bias. So the voting measure \( \mathbb{P}(x_1, \ldots, x_M) \) is given by

\[
\frac{1}{2g} \int_{-g}^{+g} \left( \frac{1}{2\ell} \int_{z-\ell}^{z+\ell} P_{t_1}(x_1) \, dt_1 \cdots \frac{1}{2\ell} \int_{z-\ell}^{z+\ell} P_{t_M}(x_M) \, dt_M \right) \, dz.
\] (32)

For this specific example, we can explicitly compute the relevant quantities from Definition 16 and Theorem 21. By a straightforward but tedious computation, we obtain:

\[
a = \frac{1}{3} \frac{g^2}{\ell^2} \leq \frac{1}{3}, \quad \langle m_1(\rho) \rangle = \frac{1}{3} \frac{g^2}{\ell},
\]

\[
\langle \overline{m}_1(\rho) \rangle = \frac{1}{6\ell} (3\ell^2 + g^2), \quad \langle \overline{m}_1(\rho) \rangle - \langle m_1(\rho) \rangle = \frac{1}{6\ell} (3\ell^2 - g^2),
\]

\[
\langle m_1(\rho) \rangle - a \langle \overline{m}_1(\rho) \rangle = \frac{g^2}{18\ell^3} (3\ell^2 - g^2).
\] (33)

This gives

**Theorem 28.** For the additive CBM in (32) with \( g \leq \ell \), the optimal weights are

\[
w_\lambda = \frac{1}{2} \frac{\ell}{\alpha_\lambda} + \frac{1}{2} \frac{g^2 \ell}{3\ell^2 + (M - 1)g^2}.
\] (34)

**Remark 29.**

1. If there is no global bias (meaning \( g \searrow 0 \)), we obtain the result for independent groups, i.e. the weights are proportional to \( \alpha_\lambda \).

2. The quantity \( \langle m_1 \rangle - a \langle \overline{m}_1 \rangle \) is non-negative. This is not always the case as we will see in Section 9.

8.1.2 Uniform Distribution with Strong Global Bias

Now, we turn to the case \( \ell \geq g \). In this case, we compute:

\[
a = 1 - \frac{2}{3} \frac{\ell}{g} < 1, \quad \langle m_1(\rho) \rangle = \frac{1}{6g} (3g^2 - \ell^2),
\]

\[
\langle \overline{m}_1(\rho) \rangle = \frac{1}{6g} (3g^2 + \ell^2), \quad \langle \overline{m}_1(\rho) \rangle - \langle m_1(\rho) \rangle = \frac{1}{3g} \ell^2,
\]

\[
\langle m_1(\rho) \rangle - a \langle \overline{m}_1(\rho) \rangle = \frac{\ell}{9g^2} (3g^2 - 3g\ell + \ell^2).
\] (35)
Theorem 30. For the additive CBM in (32), with \( \ell \leq g \), the optimal weights are

\[
w_\lambda = \frac{1}{2} \ell \alpha_\lambda + \frac{1}{2} \frac{3g^2 - 3g\ell + \ell^2}{3Mg - 2(M-1)\ell}. \tag{36}
\]

Remark 31. 1. For the case \( g = \ell \), formulae (33) and (30) agree.

2. In the limit \( \ell \to 0 \) we find \( a \to 1 \), i.e. we approach the tightly correlated case. In this case, the weights become constant, independent of the sizes of the groups. This limit case corresponds to Impartial Anonymous Culture for the union.

8.1.3 Global Bias Concentrated in Two Points

We now analyse the multiplicative models in Example 8.6. If \( a = 1 \), respectively, in more detail. The probability of each configuration \( \xi_\lambda \in \{-1,1\}^{N\times}, \lambda = 1, \ldots, M \), is

\[
\frac{1}{2g} \int_{-g}^{+g} \left( \frac{1}{\ell_2 - \ell_1} \int_{\ell_1}^{\ell_2} P_{zyi}((\xi_\lambda)^{xy}) \, dy_1 \cdots \frac{1}{\ell_2 - \ell_1} \int_{\ell_1}^{\ell_2} P_{zym}((\xi_\lambda)^{ym}) \, dy_M \right) \, dz. \tag{37}
\]

So the global bias measure is uniform on the interval \([-g, g]\), the local bias modifier \( \rho \) is uniform on the interval \([\ell_1, \ell_2]\). If \( \ell_1 \geq 0 \) (or \( \ell_2 \leq 0 \), which gives the same model class), the model is tightly correlated.

Assuming \( \ell_1 < 0 < \ell_2 \), we obtain

\[
a = \frac{\ell_2^2 + \ell_1^2}{\ell_2 - \ell_1} < 1, \tag{38}
\]

\[
\langle m_1(\rho) \rangle d(\rho) = \frac{g}{4} \frac{(\ell_2 + \ell_1)^2}{(\ell_2 - \ell_1)}, \tag{39}
\]

\[
\langle m_1(\rho) \rangle - \langle m_1(\rho) \rangle = -\frac{g}{2} \ell_2 (\ell_2 + \ell_1) \geq 0.
\]

So, for the optimal weights according to Theorem 31, we obtain

\[
w_\lambda = \frac{g}{8} (\ell_2 - \ell_1) \alpha_\lambda + \frac{g}{8} (\ell_2 - \ell_1) \frac{(\ell_2 + \ell_1)^2}{(\ell_2 - \ell_1)^2 + (M-1)(\ell_2 + \ell_1)^2}, \tag{38}
\]

or, equivalently,

\[
\bar{w}_\lambda = \alpha_\lambda + \frac{(\ell_2 + \ell_1)^2}{(\ell_2 - \ell_1)^2 + (M-1)(\ell_2 + \ell_1)^2}. \tag{39}
\]
For $\ell_1 \not\geq 0$ approaching the tightly correlated case, we get

$$\tilde{w}_\lambda = \alpha_\lambda + \frac{1}{M}.$$ 

Moreover, we observe that the formulae (38) and (39) make sense even in the tightly correlated case, i.e. for $\ell_1 \geq 0$.

Next we turn to the case where $\rho(0,1] = 1$ while maintaining the condition (17). Then there are only two possibilities: either the model is tightly correlated and the optimal weights are indeterminate. This is the case if and only if $\mu\{0\} = 0$. The complementary case is $0 < \mu\{0\} < 1$. We can interpret this as the existence of some fraction of the issues which are not subject to any global bias. The multiplicative structure of the local bias means all voters make up their own minds on these issues. We can determine the optimal weights in this case without placing any additional assumptions on the bias measures $\mu$ and $\rho$.

The key observation is that for $\mu$-almost all $z$ the equality $m_1(\rho)d(\rho) = \overline{m}_1(\rho)$ holds. The model is not tightly correlated, nor are the voters belonging to different groups independent. So we have $0 < a < 1$ and

$$w_\lambda = \frac{\langle \overline{m}_1(\rho) \rangle}{1 + (M-1)a} \quad \text{or, equivalently,} \quad \overline{w}_\lambda = \frac{1}{M}.$$ 

In conclusion, for this model, the optimal weights have to be chosen equal for all groups $\lambda$, no matter their size $\alpha_\lambda$. As for the intuition behind this result, let us recall the formula for optimal weights in non-tightly correlated models given in Theorem 21: the optimal weights are given by $C_1\alpha_\lambda + C_2$, with

$$C_1 = \frac{1}{1-a} \left( \langle \overline{m}_1(\rho) \rangle - \langle m_1(\rho) d(\rho) \rangle \right)$$

and

$$C_2 = \frac{1}{1-a} \left( \langle m_1(\rho) d(\rho) \rangle - a\langle \overline{m}_1(\rho) \rangle \right).$$

As mentioned after Definition 10, $\overline{m}_1(\rho)$ can be interpreted as a measure of intra-group cohesion, and $m_1(\rho)d(\rho)$ as a measure of inter-group cohesion. The equality of these two in the present example is intuitively due to the fact that $Z = 0$ induces zero cohesion both within each group as well as across group boundaries, and for $Z \neq 0$ we have a very strong correlation of all voters due to the assumption $\rho(0,1] = 1$ which implies that the sign of each group bias will always be the same as the sign of the global bias. Hence, we have $\langle m_1(\rho) d(\rho) \rangle = \langle \overline{m}_1(\rho) \rangle$, and $C_1 = 0$ implies there is no proportional component to the optimal weights. The constant component $C_2$, on the other hand, does not disappear, because the fraction of issues for which there is independent voting (i.e. those for which $Z = 0$), induces a non-tight correlation and $a < 1$. This example sheds some light on where the summands in the optimal weight formula in Theorem 21 come from: as mentioned previously, the constant component $C_2$ is induced by the correlation between votes belonging to different groups. Now we see that the proportional component is a manifestation of the stronger cohesion within each group when compared to inter-group cohesion.

9 Non-Negativity of the Weights

In applications on public voting procedures, negative weights would be rather absurd: the consent of such a voter could decrease the majority margin or even change an ‘aye’ to a ‘nay’. It seems likely that no group would accept being assigned a negative voting weight. Even if they did, this would not bring about a minimisation of the democracy deficit, since a group with negative weight would face incentives to misrepresent their true preferences. On the other hand, in an estimation problem, i.e. for estimating the magnitude of the voting margin, negative weights may make sense. It has been pointed out by an anonymous referee that negative weights may also make sense in an automated preference aggregation setting, when sincere voting can be assumed.

In Theorem 21 we identified the optimal weights $w_\lambda$ as

$$w_\lambda = C_1\alpha_\lambda + C_2.$$ (40)
The constant $C_1$ is always non-negative. Moreover, in all explicit examples in Section 8 the constant $C_2$ turned out to be non-negative as well.

In general, the constant $C_2$ is non-negative if and only if

$$a(\langle m_1(\rho) \rangle) \leq \langle m_1(\rho) \rangle d(\rho). \quad (41)$$

As it turns out, condition (41) can be violated under certain assumptions on the measures $\mu$ and $\rho$.

Consequently, for small $\alpha$, equation (40) prescribes negative weights.

### 9.1 An Example with Negative Optimal Weights

To see that (41) can be violated, we consider an additive model with $\mu = \frac{1}{3}(\delta_g + \delta_{-g})$ and $\rho = \frac{1}{3}(\delta_{-\ell_2} + \delta_{-\ell_1} + \delta_{\ell_1} + \delta_{\ell_2})$ and choose $0 < \ell_1 < g < \ell_2$ with $g + \ell_2 \leq 1$.

Then, for the additive model with $\mu$ and $\rho$ we compute:

$$a = \langle d(\rho)^2 \rangle = \frac{1}{4}, \quad \langle m_1 d(\rho) \rangle = \frac{1}{2}g, \quad \langle m_1(\rho) \rangle = \frac{1}{2}g + \frac{1}{2}\ell_2.$$

Consequently, the constant term $C_2$ in the optimal weight (40) is negative if $\ell_2 > 3g$. In this case, the optimal weight is negative for small $\alpha$.

An analogous result holds for uniform distributions both for $\mu$ (around $\pm g$) and for $\rho$ around $\pm \ell_1$ and $\pm \ell_2$, as long as these six intervals are small enough.

In the remainder of this section, we will focus on the additive model and the problem of negative weights. For simplicity’s sake, we will assume for the rest of Section 9 that the support of both $\mu$ and $\rho$ belongs to $[-1/2, 1/2]$.

### 9.2 Non-Negativity of $w$ in Additive Collective Bias Models with $\mu = \rho$

In this section, we consider the case where the central bias and the group modifiers of an additive CBM follow the same distribution. Of course, all bias variables and modifiers $Z$ and the $Y_\lambda$ are still assumed to be independent. So the random variables $Z, Y_1, \ldots, Y_M$ are all i.i.d. As noted in Section 2, the assumption of identically distributed $Z$ and $Y_\lambda$ reflects that global bias and local bias each have the same influence on the voters, with neither of the two dominating. We will use the notation

$$r := \langle m_1(\rho) d(\rho) \rangle, \quad m := \langle m_1(\rho) \rangle.$$

Recall that according to Theorem 21 the optimal weights are proportional to

$$w_\lambda = r - am + (1 + (M - 1)a)(m - r) \alpha_\lambda.$$

We prove that for this setup the optimal weights can never be negative.

**Theorem 32.** If $\mu = \rho$ in an additive CBM, the constant term in the optimal weights $r - am$ is non-negative and $r - am = 0$ holds if and only if $\mu = \delta_0$. Furthermore, $0 \leq a \leq 1/3$, where $a = 0$ holds if and only if $\mu = \delta_0$, and $a = 1/3$ if and only if $\mu$ has no atoms, i.e., for all $x \in \mathbb{R}$, $\mu \{x\} = 0$.

This theorem says – among other things – that the constant term in the optimal weights $r - am$ is 0 if and only if $\mu = \rho = \delta_0$. But the latter equality implies that all voters are independent, a case which we discarded earlier. (Note that if $\mu = \rho = \delta_0$, the optimal weights are not proportional to the group sizes. Instead, the square root law holds and the optimal weights are proportional to $\sqrt{\alpha_\lambda}$.) Hence, by Theorem 32 for all $\mu = \rho \neq \delta_0$, the optimal weights are the sum of a positive constant $r - am > 0$ and a term proportional to the group size $\alpha_\lambda$.

Under the assumption $\mu = \rho$, we consider $a = \mathbb{E}(X_1X_2)$ as a function of the measure $\mu$. So $a : \mathcal{M}_{\leq 1}([-1/2, 1/2]) \to \mathbb{R}_+$, where $\mathcal{M}_{\leq 1}([-1/2, 1/2])$ is the set of all sub-probability measures on $[-1/2, 1/2]$.

We will also write $\mathcal{M}_1([-1/2, 1/2])$ for the set of all probability measures. Similarly, $r$ is a function $r : \mathcal{M}_{\leq 1}([-1/2, 1/2]) \to \mathbb{R}_+$. To show the theorem, we consider the cases of discrete and continuous measures separately first and then show the general case.
Proposition 33. If $\mu$ is discrete, then we have $0 \leq a(\mu) < 1/3$. The supremum over all discrete measures of $a(\mu)$ is $1/3$. Within the class of discrete measures with at most $n$ points belonging to $\text{supp} \, \mu$, we have

$$a(\mu) \leq \begin{cases} \frac{(n-2)(n+2)}{3n^2}, & n \text{ even}, \\ \frac{(n-1)(n+1)}{3n^2}, & n \text{ odd}. \end{cases}$$

For measures $\mu$ with no atoms, we have

Proposition 34. If $\mu \in \mathcal{M}_{\leq 1}([-1/2, 1/2])$ has no atoms, then $a(\mu) \leq 1/3$. If $\mu \in \mathcal{M}_1([-1/2, 1/2])$, then $a(\mu) = 1/3$.

For the remainder of this article, we express $m$ as the sum of two terms:

$$m = E \left| T_1 \right| = E \left| Z + Y \right| = E \left( \left( \text{sgn}(Z + Y) \right) \cdot (Z + Y) \right) = E \left( Z \text{sgn}(Z + Y) \right) + E \left( Y \text{sgn}(Z + Y) \right).$$

The first of these summands equals $r$. The second one, we will call $s$ from now on. If $\mu = \rho$, then of course $r = s$, and the term $r - am$ equals $r(1 - 2a)$. For the proof of these results, we need the following auxiliary lemma:

Lemma 35. We can express the magnitudes $a(\mu)$ and $r(\mu)$ as

$$a(\mu) = 2 \int_{[0,1/2]} (\mu(-z, z))^2 \, \mu(dz), \quad r(\mu) = 2 \int_{[0,1/2]} z \mu(-z, z) \, \mu(dz).$$

Corollary 36. The terms $a$ and $r$ equal 0 if and only if $\mu = \delta_0$.

This follows easily from the representation of $a$ and $r$ given in Lemma 35.

The statements in this section are proved in the appendix.

9.3 Non-Negativity of $w$ in Additive Collective Bias Models with $\mu \neq \rho$

In this section, we will not assume the two measures $\mu$ and $\rho$ are equal. The random variables $Z, Y_1, \ldots, Y_M$ are all independent and $Y_1, \ldots, Y_M$ are i.i.d. copies of a random variable $Y$ that follows a distribution according to $\rho$. As we already know, $r - am < 0$ is possible in this case. We will give conditions under which this does not happen. Analogously to Lemma 35, we have these representations of $a, r,$ and $s$:

Lemma 37. We can express the magnitudes $a, r,$ and $s$ as

$$a = 2 \int_{[0,1/2]} (\rho(-z, z))^2 \, \mu(dz), \quad r = 2 \int_{[0,1/2]} z \rho(-z, z) \, \mu(dz), \quad s = 2 \int_{[0,1/2]} y \rho(-y, y) \, \rho(dy).$$

First we note that if the group modifiers override the central bias almost surely, then the groups are independent (but the voters within each group are still positively correlated!). In this case, the optimal weights are proportional to the group sizes.

Proposition 38. If $|Y|$ almost surely dominates $|Z|$, then we have $a = r = 0$ and $r - am = 0$.

This easily follows from Lemma 37.

Remark 39. If, instead, $|Z|$ almost surely dominates $|Y|$, then the CBM is tightly correlated. We note, however, that in that case any set of weights is optimal, among them weights proportional to the group sizes.

Now we turn to first order stochastic dominance which is a weaker form of the general concept of stochastic dominance.

Definition 40. We say that a random variable $X_1$ first order stochastically dominates a random variable $X_2$ if, for all $x \in \mathbb{R}$, $P(X_1 \leq x) \leq P(X_2 \leq x)$ holds. We will write $X_1 \succ X_2$ for this relation and FOSD for first order stochastic dominance.
This is weaker than almost sure dominance as it is possible to have $X_1$ first order stochastically dominate $X_2$ without $X_1 > X_2$ holding almost surely. We have the following sufficient conditions for the non-negativity of the optimal weights:

**Proposition 41.** If $|Z| > |Y|$ and $a \leq 1/2$, then $r - am \geq 0$. If $|Y| > |Z|$ and $s \leq 2r$, then $r - am \geq 0$.

The next idea is to assume that the measures $\mu$ and $\rho$ assign similar probabilities to each event.

**Proposition 42.** Suppose there are constants $c, C > 0$ such that, for all measurable sets $A$,

$$cpA \leq \mu A \leq C \rho A$$

holds. Then each of the following two conditions is individually sufficient for $r - am \geq 0$:

1. $c \geq \frac{2}{3-C}, C < 3,$
2. $C \leq c (3c^2 - 1)$.

If we assume additionally that $c = 1/C$, then a sufficient condition for $r - am \geq 0$ is given by

$$a \leq \frac{1}{1+C^2}.$$  

Earlier we saw that if both $\mu$ and $\rho$ are uniform distributions (we will write $U$ for a uniform distribution) on symmetric intervals around the origin, $r - am \geq 0$ holds. We can generalise this result as follows:

**Proposition 43.** Let $\rho = U[-1/2, 1/2]$ and $\mu \in M_1([-1/2, 1/2])$. Then $r - am \geq 0$ is satisfied and $r = am$ if and only if $\mu = 1/2 (\delta_{-1/2} + \delta_{1/2})$.

**Remark 44.** Since $\mu = 1/2 (\delta_{-1/2} + \delta_{1/2})$ implies that $|Z|$ almost surely dominates $|Y|$, we can disregard this case. Thus, this proposition implies that for $\rho = U[-1/2, 1/2]$ the optimal weights are given by a constant and a proportional part. We also note here that $\rho$ being uniform on the entire interval $[-1/2, 1/2]$ is important. For every $0 < \gamma < 1/2, \rho = U[-\gamma, \gamma]$, there is a $\mu$ such that $r - am$ is negative.

The last result in this section concerns a case where $\rho$ is some symmetric measure and $\mu$ is a contracted version of $\rho$ onto some shorter interval $[-c/2, c/2]$ for some $0 < c < 1$. Hence, the global bias tends to be weaker than the local bias modifier. For the rest of this section, assume the following conditions hold

**Assumptions 45.**  
1. $\rho$ has no atoms.
2. There is a function $g : [0, \infty) \to \mathbb{R}$ with the property that $\rho (0, xy) = g(x) \rho (0, y)$ holds for all $x \geq 0$ and all $y \in [0, 1/2]$ such that $xy \leq 1/2$.
3. $\mu (cA) = \rho A$ for some fixed $0 < c < 1$ and all measurable $A$.

The second point is a homogeneity condition. The last point is the aforementioned contraction property. Let $F_{\rho}$ be the distribution function of the sub-probability measure $\rho | [0, 1/2]$, i.e. $\rho$ constrained to the subspace $[0, 1/2]$. Note that due to property 1 above, $\rho [0, 1/2] = 1/2$, and hence $F_{\rho} (0) = 0$ and $F_{\rho} (1/2) = 1/2$.

The three properties in Assumptions 45 already determine that the measures $\rho$ and $\mu$ belong to a two-parameter family indexed by $(t, c) \in (0, \infty) \times (0, 1)$.

**Lemma 46.** If the second condition in Assumptions 45 is satisfied, then

1. For all $y \in [0, 1]$, $g(y) = 2F_{\rho}(y/2)$.
2. $\rho$ has no atoms, unless $\rho = \delta_0$.
3. $g$ is multiplicative: for all $x, y \geq 0$, $g(xy) = g(x)g(y)$.
4. $F_{\rho}$ has the form $F_{\rho}(y) = 2^{t-1}y^t$ for some fixed $t \geq 0$.  

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Remark 47. If \( t = 0 \) in the last point of the lemma, then \( \rho = \mu = \delta_0 \) and all voters are independent. We avoid this case by specifying the first condition in Assumptions \([45]\).

Now we state the theorem concerning the sign of the term \( r - am \).

**Theorem 48.** Let the conditions stated in Assumptions \([45]\) hold for \( \rho \) and \( \mu \). Then, by the last lemma, \( F_\rho(y) = 2^{-1} y^t \). If \( 0 < t < 1 \), then there is a unique \( c_0 \in (0,1) \) such that, for all \( c \in (0, c_0) \), \( r - am \) is negative, and, for all \( c \in [c_0,1) \), \( r - am \geq 0 \) with equality if and only if \( c = c_0 \). If \( t \geq 1 \), then \( r > am \).

The critical point \( c_0 \) for the regime \( t \in (0, 1) \) satisfies \( \lim_{t \to 1} c_0 = 0 \).

**Remark 49.** Note that \( t = 1 \) is the case of the uniform distributions \( \rho = \mathcal{U}[-\gamma, \gamma] \), \( \mu = \mathcal{U}[-\beta, \beta] \) with \( \gamma = 1/2 \) and \( 0 < \beta = c/2 < 1/2 \).

The proofs of these statements can be found in the appendix.

## 10 Extensions

An obvious extension to the general CBM framework is to allow different conditional distributions \( \rho_1^\ast \) for the different groups to account for more strongly or more weakly correlated groups. More precisely,

\[
\mathbb{P}(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_M) = \int \left( \int P_{t_1}(\vec{x}_1) \rho_1^{\ast}(dt_1) \cdots \int P_{t_M}(\vec{x}_M) \rho_1^{\ast}(dt_M) \right) \mu(dz) \tag{42}
\]

A large part of the analysis in Sections \([4] \) and \([5] \) can be done for this more general case as well. In fact, with the definitions \([9], [7], \) and \([15] \), the optimal weights \( w \) for this model again satisfy

\[
A_N w = b_N.
\]

In the limit \( N \to \infty \), using the same technique as in Section \([5] \) we obtain

\[
A w = b,
\]

with

\[
A_{\nu \nu} = \left\{ \begin{array}{ll}
1, & \text{if } \lambda = \nu, \\
\langle d(\rho_\lambda) d(\rho_\nu) \rangle, & \text{if } \lambda \neq \nu,
\end{array} \right.
\]

\[
b_\nu = \left( \langle \rho_\nu \rangle \right) - \langle m_1(\rho_\nu) d(\rho_\nu) \rangle \alpha_\nu + \sum_{\lambda=1}^{M} \langle m_1(\rho_\lambda) d(\rho_\nu) \rangle \alpha_\lambda.
\]

As in Proposition \([12] \) it is easy to see, that the matrix \( A \) is positive semi-definite. Moreover, we show

**Theorem 50.** The matrix \( A \) is positive definite, and hence invertible, if \( \langle d(\rho_\lambda)^2 \rangle < 1 \) for all but possibly one \( \lambda \).

**Proof.** For \( x \in \mathbb{R}^M \), we compute

\[
(x, Ax) = \sum_{\nu=1}^{M} (1 - \langle d(\rho_\nu)^2 \rangle) x_\nu^2 + \sum_{\nu, \lambda=1}^{M} \langle d(\rho_\nu) d(\rho_\lambda) \rangle x_\nu x_\lambda
\]

\[
= \sum_{\nu=1}^{M} (1 - \langle d(\rho_\nu)^2 \rangle) x_\nu^2 + \left( \sum_{\nu=1}^{M} x_\nu d(\rho_\nu) \right)^2 \tag{43}
\]

Both terms in \([43] \) are non-negative. If \( \langle d(\rho_\nu)^2 \rangle < 1 \) for all \( \nu \), the first sum in \([43] \) is strictly positive for \( x \neq 0 \), hence \( A \) is positive definite in this case. If all but one \( \nu^* \) have this property, the conclusion follows from considering that the first summand is 0 if and only if \( x_{\nu^*} \) is the only coordinate of \( x \) different than 0. But in that case, the second summand is positive.

\[ \square \]
We have two partial converses to Theorem 50.

**Proposition 51.** If \(| \langle d(\rho_\nu) d(\rho_\lambda) \rangle \rangle = 1\) for some \(\nu \neq \lambda\), then the matrix \(A\) is not invertible.

**Proof.** If \(| \langle d(\rho_\nu) d(\rho_\lambda) \rangle \rangle = 1\), then an application of the Cauchy-Schwarz inequality shows that \(\langle d(\rho_\nu)^2 \rangle = \langle d(\rho_\lambda)^2 \rangle = 1\). This implies that the matrix

\[
\begin{pmatrix}
\langle d(\rho_\nu)^2 \rangle & \langle d(\rho_\nu) d(\rho_\lambda) \rangle \\
\langle d(\rho_\nu) d(\rho_\lambda) \rangle & \langle d(\rho_\lambda)^2 \rangle
\end{pmatrix}
\]

is not invertible, hence \(A\) is not invertible. \(\square\)

**Proposition 52.** Let for \(\mu\)-almost all \(z\) \(\langle d(\rho_\lambda) \rangle = \langle d(\rho_\nu) \rangle, \lambda, \nu = 1, \ldots, M\). Then the matrix \(A\) being positive definite implies \(\langle d(\rho_\lambda)^2 \rangle < 1\) for all but possibly one \(\lambda\).

**Remark 53.** Additive CBMs satisfy the condition of all \(\langle d(\rho_\lambda) \rangle\) having the same sign for \(\mu\)-almost all \(z\) owing to the symmetry of each measure \(\rho_\lambda\). Multiplicative CBMs have this property if we assume a certain asymmetry for each \(\rho_\lambda\): If, for all \(\lambda, \rho_\lambda (0, 1] > 1/2\), or, for all \(\lambda, \rho_\lambda (0, 1] < 1/2\), then the condition holds. This can be interpreted as all group bias modifiers tending to reinforce the global bias, or, to the contrary, all tending to go against the global bias. Note, however, that this is a far weaker tendency than required by tight correlation.

**Proof.** Assume there are two distinct indices \(\lambda_1\) and \(\lambda_2\) such that \(\langle d(\rho_{\lambda_1})^2 \rangle, \langle d(\rho_{\lambda_2})^2 \rangle = 1\). We show that \(A\) is not positive definite. Define an \(x \in \mathbb{R}^M\) by setting \(x_{\lambda_1} = 1, x_{\lambda_2} = -1, \) and all other entries equal to 0. We calculate

\[
(x, Ax) = (1 - \langle d(\rho_{\lambda_1})^2 \rangle) + (1 - \langle d(\rho_{\lambda_2})^2 \rangle) + \langle (d(\rho_{\lambda_1}) - d(\rho_{\lambda_2})^2 \rangle
\]

\[
= \langle d(\rho_{\lambda_1})^2 \rangle - 2 \langle d(\rho_{\lambda_1}) d(\rho_{\lambda_2}) \rangle + \langle d(\rho_{\lambda_2})^2 \rangle
\]

By assumption, \(d(\rho_{\lambda_1})^2\) and \(d(\rho_{\lambda_2})^2\) are \(1\)-almost surely. Hence, \(d(\rho_{\lambda_1})\) and \(d(\rho_{\lambda_2})\) are 1 in absolute value and they have the same sign \(\mu\)-almost surely. Thus, the term \(\langle d(\rho_{\lambda_1}) d(\rho_{\lambda_2}) \rangle\) equals 1. \(\square\)

We will now consider a scenario in which there are two clusters of groups - think of them as parts of the overall population that tend to vote together. While voters within clusters tend to hold the same opinion, we will assume that there is antagonism between the two clusters.

Let groups \(1, \ldots, M_1\) belong to cluster \(C_1\), and \(M_1 + 1, \ldots, M\) to cluster \(C_2\). We set \(M_2 := M - M_1\). The fraction of the overall population belonging to groups in each cluster will be called \(\eta_i := \sum_{\lambda \in C_i} \alpha_\lambda, i = 1, 2\). The conditional distributions \(\rho_\lambda^i\) are identical within each cluster: In \(C_i\), all groups follow \(\rho_\lambda^i\). To obtain antagonistic behaviour, we will assume that

\[
\rho_\lambda^1 = \rho_\lambda^2^{-z}
\]

holds for all \(z\).

Now we have to distinguish the quantities

\[
r_{ij} := \langle m_1 (\rho_i) d(\rho_j) \rangle, \quad m_i := \langle m_1 (\rho_i) \rangle, \quad i, j = 1, 2.
\]

However, due to the antisymmetry condition \([44]\), we have the following equalities:

**Lemma 54.** Under the assumptions presented above, we have

\[
r := r_{ii} = -r_{ij} \quad \text{and} \quad m := m_i
\]

for all \(i, j = 1, 2, i \neq j\).

**Proof.** We omit the short calculation that yields the result. \(\square\)
The covariance matrix $A$ has block form

$$A = \begin{pmatrix} A_1 & B \\ B^T & A_2 \end{pmatrix},$$

where the $A_i$ are the covariance matrices of the groups belonging to cluster $i$. They have the form we know from (20), i.e. diagonal entries equal 1 and off-diagonal entries $0 < a < 1$. Of course, $A_i \in \mathbb{R}^{M_i \times M_i}$.

The matrix $B$ holds the covariances between groups of different clusters. Due to (44), all entries of $B$ are equal to $-a$.

We invert $A$ and obtain

$$\left(A^{-1}\right)_{\lambda \nu} = \frac{1}{D} \begin{cases} 1 + (M - 2) a, & \lambda = \nu, \\ -a, & \lambda, \nu \in C_i, \quad i = 1, 2, \quad \lambda \neq \nu, \\ a, & \lambda \in C_i, \quad \nu \in C_j, \quad i, j = 1, 2, \quad i \neq j, \end{cases}$$

where $D = (1 - a) (1 + (M - 1) a)$. Note that the entries within clusters are identical to those given in (22) for the model with identical conditional distributions $\rho^*_x$.

Using Lemma 54 we calculate the entries $\lambda \in C_i, i = 1, 2$, of $b$:

$$b_\lambda = (m - r) \alpha_\lambda + r (\eta_i - \eta_j).$$

In the formula above, the index $j$ is the cluster $\lambda$ does not belong to. Now a lengthy but straightforward calculation yields the optimal weights for each group $\nu$:

**Theorem 55.** Let $\lambda$ be a group in cluster $i = 1, 2$ and let $j$ be the other cluster. Then the optimal weight of group $\lambda$ is given by

$$w_\lambda = (A^{-1}b)_\lambda = D_1 \alpha_\lambda + D_2,$$

where the coefficients are

$$D_1 = \frac{m - r}{1 - a} \quad \text{and} \quad D_2 = \frac{(r - am) (\eta_i - \eta_j)}{(1 - a) (1 + (M - 1) a)}.$$

$D_1$ is equal to the coefficient $C_1$ given in (25) for the model with identical conditional distributions. We note that if both clusters have exactly half the overall population, then $D_2$ vanishes, and the optimal weights are proportional to the population of each group. If the two clusters represent different proportions of the overall population, then $w_\lambda$ is the sum of a proportional term $D_1 \alpha_\lambda$ and a constant $D_2$. If $\lambda$ belongs to the larger of the two clusters, then $D_2$ has the same sign as the coefficient $C_2$ in (26) in the identical conditional distribution model, and $D_2$ is a rescaled version of $C_2$ by the factor $(\eta_i - \eta_j)$. If $\lambda$ belongs to the smaller of the two clusters, then $D_2$ has the opposite sign compared to $C_2$ in (26) and it is once again rescaled.

11 Conclusion

We have defined and analysed a multi-group version of the CBM which allows for correlated voting across group boundaries. This CBM was then applied to the problem of calculating the optimal weights in a two-tier voting system. By the term ‘optimal weights’, we mean those council weights which minimise the democracy deficit, i.e. the expected quadratic deviation of the council vote from a hypothetical referendum over all possible issues which can be voted on. The main findings in this paper are:

- We determined the asymptotic behaviour of the CBM in Theorem 15. The theorem states that the global bias measure $\mu$ and the group bias measure $\rho$ describe the limiting distribution of the normalised voting margins.
• We distinguished the tightly correlated case from its complement. We characterised tight correlation in Proposition 20 in terms of the bias measures. Tight correlation means intuitively that there is perfect positive correlation between the different group votes in the council. This leads to non-unique optimal weights, as the assignation of weights does not matter if all groups vote alike anyway. We gave a sufficient condition for non-tight correlation between groups called ‘sufficient randomness’. This criterion states that all possible council votes occur with positive probability.

• In the non-tightly correlated case, we showed that the optimal council weights are uniquely determined in Theorem 13. The optimal weights are given by the sum of a constant term equal for all groups and a summand which is proportional to each group’s population as stated in Theorem 21.

• We analysed the optimal weights’ properties and showed that there are cases in which these weights are negative for the smallest groups. This is due to the fact that while the coefficient of the group’s size is positive, the constant term can have any sign, depending on the bias measures. We gave sufficient conditions for the non-negativity of the optimal weights as well as examples in which the weights are negative in Section 9.

Appendix

Proof of Proposition 33

We prove the claim for \( n = 2k + 1 \). The case of even \( n \) can be shown analogously. We prove by induction on \( k \) that

\[
a(\mu) \leq \frac{2k(2k + 2)}{3(2k + 1)^2},
\]

with equality if \( \mu \) is chosen to be the uniform distribution on the \( 2k + 1 \) points conforming the support of \( \mu \).

Base case: Let \( k = 1 \). Then the support of \( \mu \) consists of three points: 0 and two points \(-x_1, x_1\) such that \( 0 < x_1 \leq 1/2 \). The measure \( \mu \) is given by \( \beta_0 \delta_0 + \beta_1 (\delta_{-x_1} + \delta_{x_1}) \) and the constants satisfy \( \beta_0 + 2\beta_1 = 1 \). Set \( \beta := \beta_1 \). To show the upper bound (45), we solve the maximisation problem \( \max_\beta a(\mu) \). The first order condition is

\[
(1 - \beta)^2 - 2\beta(1 - \beta) = 0,
\]

which has two solutions: \( \beta = 1 \) and \( \beta = 1/3 \). The second order condition shows that \( \beta = 1 \) minimises \( a(\mu) \) and \( \beta = 1/3 \) maximises it. So, for \( k = 1 \), the uniform distribution maximises \( a(\mu) \) and, for the uniform distribution \( \mu_3 \) on \( \{-x_1, 0, x_1\} \),

\[
a(\mu_3) = \frac{8}{27} = \frac{2k(2k + 2)}{3(2k + 1)^2},
\]

and the upper bound (45) holds with equality.

Induction step: Assume that for some \( k \in \mathbb{N} \) and all sets \( \{-x_k, \ldots, 0, \ldots, x_k\}, 0 < x_1 < \cdots < x_k \leq 1/2 \), the uniform distribution \( \mu_{2k+1} \) maximises \( a(\mu) \) and \( a(\mu_{2k+1}) = \frac{2k(2k + 2)}{3(2k + 1)^2} \). We add another point \( 1/2 \geq x_{k+1} > x_k \) (if \( x_k = 1/2 \), then relabel the last two points) with probability \( 1 \geq \eta \geq 0 \) and solve the maximisation problem

\[
\max_{\mu, \eta} a((1 - 2\eta) \mu + \eta (\delta_{-x_{k+1}} + \delta_{x_{k+1}})),
\]
where \( \mu \) is any symmetric probability measure on \( \{ -x_k, \ldots, 0, \ldots, x_k \} \). Set \( \nu := (1 - 2\eta) \mu + \eta (\delta_{-x_k+1} + \delta_{x_k+1}) \) and we calculate

\[
a (\nu) / 2 = \int_{[0,1/2]} (\nu (-z, z))^2 \nu (dz)
\]

\[
= \int_{[0,x_k]} ((1 - 2\eta) \mu (-z, z))^2 (1 - 2\eta) \mu (dz) + \int_{[0,x_k]} (1 - 2\eta) \mu (-z, z))^2 \eta \delta_{x_k+1} (dz)
\]

\[
+ \int_{[x_k,1/2]} ((1 - 2\eta) + \eta \delta_{x_k+1} (-z, z))^2 (1 - 2\eta) \mu (dz) + \int_{[x_k,1/2]} (1 - 2\eta) + \eta \delta_{x_k+1} (-z, z))^2 \eta \delta_{x_k+1} (dz).
\]

The second summand is 0 because \( \delta_{x_k+1} (0, x_k) = 0 \). The third summand is 0 due to \( \mu (x_k, 1/2] = 0 \). We continue

\[
a (\nu) / 2 = \int_{[0,x_k]} ((1 - 2\eta) \mu (-z, z))^2 (1 - 2\eta) \mu (dz) + \int_{[x_k,1/2]} (1 - 2\eta) + \eta \delta_{x_k+1} (-z, z))^2 \eta \delta_{x_k+1} (dz)
\]

\[
= (1 - 2\eta)^3 \int_{[0,1/2]} (\mu (-z, z))^2 \mu (dz) + \eta (1 - 2\eta) + \eta \delta_{x_k+1} (-x_{k+1}, x_{k+1}))^2
\]

\[
= (1 - 2\eta)^3 a (\mu) / 2 + \eta (1 - \eta)^2.
\]

As we see, \( \mu \) and \( \eta \) can be chosen independently of each other to maximise \( a (\nu) \). By assumption, the maximising \( \mu \) is the uniform distribution on \( \{ -x_k, \ldots, 0, \ldots, x_k \} \mu_{2k+1} \). Hence,

\[
\max_{\nu} a (\nu) = \max_{\eta} (1 - 2\eta)^3 a (\mu_{2k+1}) + 2\eta (1 - \eta)^2.
\]

Since \( a (\mu_{2k+1}) \) is independent of the choice of \( \eta \), the first order condition is

\[
3 (1 - 2\eta)^2 a (\mu_{2k+1}) = (1 - \eta) (1 - 3\eta).
\]

The solutions of this quadratic equation are

\[
\eta = \frac{1 - 2 - 3a (\mu_{2k+1})}{3 (1 - 2a (\mu_{2k+1}))} \pm \frac{1}{3} \sqrt{\left( \frac{2 - 3a (\mu_{2k+1})}{1 - 2a (\mu_{2k+1})} \right)^2 - \frac{3 - 2 - 3a (\mu_{2k+1})}{2 (1 - 2a (\mu_{2k+1}))}}.
\]

By substituting \( a (\mu_{2k+1}) = \frac{2k(2k+2)}{3(2k+1)^2} \), we see that the root with the negative sign gives a negative \( \eta \). The positive root is \( \eta = \frac{1}{2k+1} \). This implies that the maximising measure \( \nu \) on \( \{ -x_{k+1}, \ldots, 0, \ldots, x_{k+1} \} \) is the uniform distribution \( \mu_{2(k+1)+1} \). This concludes the proof by induction that for finitely many points in the support, the uniform distribution maximises \( a \) and this maximum is given by the upper bound in [14].

Next we show that for discrete measures with finite support the upper bound \( 1/3 \) holds as well. If \( \sup \mu = \infty \), then \( \mu \) is of the form \( \beta_0 \delta_0 + \sum_{i=1}^\infty \beta_i (\delta_{-x_i} + \delta_{x_i}) \), where \( x_i > 0 \) for all \( i \in \mathbb{N} \). Set for each \( n \in \mathbb{N} \), \( \nu_n := \beta_0 \delta_0 + \sum_{i=1}^n \beta_i (\delta_{-x_i} + \delta_{x_i}) \). To obtain a contradiction, suppose that \( a (\mu) > 1/3 \) and set \( \tau := a (\mu) - 1/3 > 0 \). The sequence \( a (\nu_n) \) is monotonically increasing:

\[
a (\nu_n) = 2 \int_{[0,1/2]} (\nu_n (-z, z))^2 \nu_n (dz) \leq 2 \int_{[0,1/2] \cap \{ x_1, \ldots, x_n \}} (\nu_{n+1} (-z, z))^2 \nu_n (dz)
\]

\[
+ 2 (\nu_{n+1} (-x_{n+1}, x_{n+1}))^2 \beta_{n+1} = a (\nu_{n+1}).
\]

For any \( \varepsilon > 0 \), there is some \( m \in \mathbb{N} \) such that for all measurable sets \( A \subset [-1/2, 1/2] \) the inequality
Due to (46) and (47), we have
\[ a (\nu_m) = 2 \int_{(0,1/2]} (\nu_m (-z,z))^2 \nu_m (dz) > 2 \int_{(0,1/2]} (\mu (-z,z) - \varepsilon)^2 (\mu - \varepsilon) (dz) \]
and
\[ a (\mu) = a (\mu) - 2\varepsilon \int_{(0,1/2]} (\mu (-z,z))^2 dz - 4\varepsilon \int_{(0,1/2]} \mu (-z,z) \mu (dz) \]
\[ + 4\varepsilon^2 \int_{(0,1/2]} \mu (-z,z) dz + 2\varepsilon^2 \int_{(0,1/2]} \mu (dz) - 2\varepsilon^3 \int_{(0,1/2]} dz. \]

By letting \( \varepsilon \) go to 0, we see that \( a (\nu_n) \nearrow a (\mu) \) and there is an \( n \in \mathbb{N} \) such that \( a (\nu_n) > a (\mu) - \tau/2 > 1/3 \). This is a contradiction because the cardinality \( \text{supp } \nu_n \) equals \( 2n + 1 \) and therefore \( a (\nu_n) \leq 1/3 \).

Next we note that \( \sup \ a (\mu) \) over all discrete probability measures \( \mu \) is \( 1/3 \). This is easy to see because of the following facts:

**Lemma 56.** The sequences \( \left( \frac{(n-2)(n+2)}{3n^2} \right)_{n \text{ even}} \) and \( \left( \frac{(n-1)(n+1)}{3n^2} \right)_{n \text{ odd}} \) are monotonically increasing and their limit is equal to \( 1/3 \).

As we have proved, for uniform distributions \( \mu_n \), \( a (\mu_n) \) is equal to one of these expressions depending on the parity of \( n \). From this lemma, it follows that by choosing a discrete uniform distribution on either an even- or odd-numbered support we can get arbitrarily close to \( 1/3 \). This concludes the proof of Proposition 33.

Next we prove the result for continuous measures.

**Proof of Proposition 34**

Let \( \mu \in \mathcal{M}_1 ([0,1]) \) have no atoms. We show \( a (\mu) = 1/3 \) by approximating \( a (\mu) / 2 \) by a sum and then prove that the sum in question is a Riemann sum of the function \( x \mapsto 4x^2 \) on the interval \( (0,1/2) \).

Let \( \varepsilon > 0 \) be given. Then there is a partition \( \mathcal{P}_n = (I_1, \ldots, I_n) \) of \( (0,1/2) \) with the property that for all \( i = 1, \ldots, n \), \( \mu I_i < \varepsilon \). We assume the intervals are ordered from left to right. It is possible to choose at most \( n \leq \lceil 1/\varepsilon \rceil \) intervals for the partition \( \mathcal{P}_n \). Then we define the upper and lower sum

\[ U (\mathcal{P}_n) := \sum_{i=1}^{n} \left( \sum_{j=1}^{i} 2\mu I_j \right)^2 \mu I_i, \quad L (\mathcal{P}_n) := \sum_{i=1}^{n} \left( \sum_{j=1}^{i-1} 2\mu I_j \right)^2 \mu I_i. \]

For each summand \( i = 1, \ldots, n \), we have

\[ \left| \left( \sum_{j=1}^{i} 2\mu I_j \right)^2 \mu I_i - \left( \sum_{j=1}^{i-1} 2\mu I_j \right)^2 \mu I_i \right| \leq 2 \left| \sum_{j=1}^{i} 2\mu I_j - \sum_{j=1}^{i-1} 2\mu I_j \right| \mu I_i = 2 \cdot 2 \cdot (\mu I_i)^2 < 4\varepsilon^2. \]  

(46)

In the inequality above, we used that for all \( x, y \in [0,1] \) \( |x^2 - y^2| < 2|x - y| \) holds. Also for all \( i = 1, \ldots, n \) and all \( z \in I_i \)

\[ \sum_{j=1}^{i-1} 2\mu I_j \leq \mu (-z,z) \leq \sum_{j=1}^{i} 2\mu I_j, \]

and, therefore,

\[ L (\mathcal{P}_n) \leq \int_{(0,1/2]} (\mu (-z,z))^2 \mu (dz) \leq U (\mathcal{P}_n). \]  

(47)

Due to (46) and (47), we have

\[ 0 \leq U (\mathcal{P}_n) - L (\mathcal{P}_n) \leq n \cdot 4\varepsilon^2 \leq \lceil 1/\varepsilon \rceil \cdot 4\varepsilon^2 \leq 4\varepsilon (1 + \varepsilon). \]

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This shows that the upper and lower sum approximate \( a ( \mu ) / 2 \) well as we let the number of intervals in \( \mathcal{P}_n \) go to infinity.

The next step is to show \( U ( \mathcal{P}_n ) \) is an upper Riemann sum of the function \( x \mapsto 4x^2 \). For the partition \( \mathcal{P}_n \), there is a corresponding partition \( \mathcal{Q}_n = (J_1, \ldots, J_n) \) in which the intervals are once again assumed to be ordered from left to right and for each \( i = 1, \ldots, n \) the interval lengths \( |J_i| \) equal \( \mu J_i \). We define

\[
R ( \mathcal{Q}_n ) := \sum_{i=1}^{n} \sup_{x \in J_i} 4x^2 \cdot |J_i| .
\]

This is an upper Riemann sum of \( x \mapsto 4x^2 \). On the other hand, we have

\[
R ( \mathcal{Q}_n ) = \sum_{i=1}^{n} \left( 2 \sup_{x \in J_i} x \right)^2 |J_i| = \sum_{i=1}^{n} (2 \sup J_i)^2 |J_i| = \sum_{i=1}^{n} \left( \sum_{j=1}^{i} |J_j| \right)^2 |J_i|
\]

\[
= \sum_{i=1}^{n} \left( \sum_{j=1}^{i} 2\mu J_j \right)^2 \mu J_i = U ( \mathcal{P}_n ) \sim a ( \mu ) / 2 .
\]

Since \( R ( \mathcal{Q}_n ) \) is an upper Riemann sum of \( x \mapsto 4x^2 \), \( R ( \mathcal{Q}_n ) \sim \int_0^{1/2} 4x^2 \, dx = 1/6 \) holds as we let the number of intervals in the partition go to infinity and we are done.

Finally, we show the case of a general probability measure.

**Proof of Theorem 32**

Let \( \mu \in \mathcal{M}_1 ( [ -1/2, 1/2 ] ) \). We can express \( \mu \) as the sum of a discrete sub-probability measure \( \delta \) and a sub-probability measure \( \gamma \) that has no atoms. Both \( \delta \) and \( \gamma \) must satisfy the symmetry condition. Therefore, \( \delta \) must have the form \( \beta_0 \delta_0 + \sum_{i=1}^{\infty} \beta_i (\delta_{-x}, \delta_x) \). Similarly to the proof of Proposition 33, we truncate the sum to \( \delta_n = \beta_0 \delta_0 + \sum_{i=1}^{n} \beta_i (\delta_{-x}, \delta_x) \) choosing \( n \) large enough for a condition \( \delta A - \delta_n A < \varepsilon \) to hold for all measurable sets \( A \) and proceed with \( \delta_n \) instead of \( \delta \). Set \( \nu := \delta_n + \gamma \). Our strategy is to show that if we remove one pair of the points \( -x_i, x_i \) from supp \( \delta_n \) and add the probability mass \( 2\delta \{ x_i \} \) to \( \gamma \) as a uniform distribution on two small intervals around \( -x_i, x_i \), we obtain a new measure \( \nu^{(0)} \) and we increase \( a ( \nu ) < a ( \nu^{(0)} ) \). So by removing the \( 2n + 1 \) points in supp \( \delta_n \) in pairs (except for the origin where we remove a single point), we obtain a monotonically increasing finite sequence \( a ( \nu^{(i)} ) = 0, \ldots, n \).

After \( n + 1 \) steps, we have a sub-probability measure \( \nu^{(n)} \) with no atoms and the bound \( a ( \nu^{(n)} ) \leq 1/3 \) thus applies.

Let \( x = x_i \) for some \( i \in \{ 1, \ldots, n \} \) and set \( \alpha := \delta \{ x \} > 0 \). Let \( \varepsilon > 0 \) be given. Then we choose \( \eta > 0 \) with the properties

1. \( 2 \left| \int_{[0,x-\eta]} (\nu ( -z, z ) - \nu ( -\infty, \infty ) (\nu ( -z, z ) - \nu ( -\infty, \infty )) \, dz \right| < \varepsilon , \)
2. \( [x-\eta, x] \cap \text{supp} \delta_n = \emptyset , \)
3. \( \gamma (x-\eta, x) < \varepsilon , \)
4. \( \nu ( -x, x ) - \nu ( - (x-\eta), x-\eta ) < \varepsilon . \)

Next define the sub-probability measure

\[
\pi_\varepsilon := \delta_n - \alpha (\delta_{-x} + \delta_x) + \gamma - \gamma |(x-\eta, x) + \alpha (\mathcal{U} (x-\eta, x) + \mathcal{U} (-x, -x+\eta)) .
\]

Here \( \mathcal{U} \) stands for a uniform distribution. We remove the points \( -x, x \) from \( \delta_n \) as well as the continuous measure \( \gamma \) on the interval \( (x-\eta, x) \) and add in the probability mass \( 2\alpha \) on small intervals close to \( -x \) and \( x \), respectively. Note that by property 2 above, \( \gamma |(x-\eta, x) = \nu |(x-\eta, x) \). Also, by 3, \( \nu [ -1/2, 1/2 ] - \varepsilon < \pi_\varepsilon [-1/2, 1/2] . \)

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We divide \(a(\nu)\) into four summands
\[
a(\nu) = 2 \left[ \int_{(0,x-\eta]} (\nu(-z,z))^2 \nu(\text{d}z) + \int_{(x-\eta,x]} (\nu(-z,z))^2 \nu(\text{d}z) + \right. \\
+ (\nu(-x,x) + \alpha)^2 \alpha + \int_{(x,1/2]} (\nu(-z,z))^2 \nu(\text{d}z) \right].
\]

We define the terms
\[
A_{\nu} := \int_{(0,x-\eta]} (\nu(-z,z))^2 \nu(\text{d}z), \quad B_{\nu} := \int_{(x-\eta,x]} (\nu(-z,z))^2 \nu(\text{d}z), \quad C_{\nu} := \int_{(x,1/2]} (\nu(-z,z))^2 \nu(\text{d}z),
\]
and, analogously, we define \(A_{\pi_x}, B_{\pi_x}, C_{\pi_x}\) on the same intervals in each case. We note that
\[
B_{\nu} = \int_{(x-\eta,x)} (\nu(-z,z))^2 \nu(\text{d}z) + (\nu(-x,x) + \alpha)^2 \alpha.
\]
Let \(T := \int_{(x-\eta,x)} (\nu(-z,z))^2 \nu(\text{d}z)\). Due to property 1 above, we have
\[
|T| = \left| B_{\nu} - (\nu(-x,x) + \alpha)^2 \alpha \right| < \varepsilon.
\]

Then we calculate these terms:

\(A\): \(A_{\nu} = A_{\pi_x}\),

\(B\):
\[
B_{\pi_x} = \int_{(x-\eta,x]} (\pi_x(-z,z))^2 \pi_x(\text{d}z) = \int_{(x-\eta,x]} \left( \nu(-(x-\eta),x-\eta] + \frac{\alpha}{\eta} \cdot 2(z-(x-\eta)) \right)^2 \frac{\alpha}{\eta} \text{d}z,
\]
where we used property 2. We set \(y := \nu(-(x-\eta),x-\eta]\). By a change of variables \(u := y + \frac{2\alpha}{\eta} (z-(x-\eta))\), we obtain
\[
B_{\pi_x} = \int_{y}^{y+2\alpha} u^2 \cdot \frac{1}{2} \text{d}z = \frac{1}{6} \left( (y + 2\alpha)^3 - y^3 \right).
\]
The inequality \(\nu(-x,x) - \nu(-(x-\eta),x-\eta) > -\varepsilon\) holds. We calculate bounds for \(B_{\pi_x}\) in terms of \(B_{\nu}\):
\[
B_{\pi_x} - B_{\nu} = \frac{1}{6} \left( 60y^2 + 12\alpha^2y + 8\alpha^3 \right) - T - \left( (\nu(-x,x))^2 + 2\alpha \nu(-x,x) + \alpha^2 \right)^2 \alpha
\]
\[
= \alpha \left( y^2 - (\nu(-x,x))^2 \right) + 2\alpha^2 \left( y - \nu(-x,x) \right) + \frac{1}{3} \alpha^3 - T,
\]
so we obtain the bounds
\[
-\alpha \cdot 2\varepsilon - 2\alpha^2 \varepsilon - \varepsilon < B_{\pi_x} - \left( B_{\nu} + \frac{1}{3} \alpha^3 \right) < \alpha \cdot 2\varepsilon + 2\alpha^2 \varepsilon + \varepsilon,
\]
and hence
\[
-5\varepsilon < B_{\pi_x} - \left( B_{\nu} + \frac{1}{3} \alpha^3 \right) < 5\varepsilon.
\]

\(C\): Since \(\nu([-1/2,-x) \cup (x,1/2])\) is equal to \(\pi_x([-1/2,-x) \cup (x,1/2])\), we have
\[
0 \geq C_{\pi_x} - C_{\nu} > -2\varepsilon \int_{(x,1/2]} \nu(\text{d}z) \geq -2\varepsilon.
\]
In the second step above, we used that, for all \(x, y \in [0,1]\), \(|x^2 - y^2| < 2|x-y|\) is satisfied.
Putting together the three parts, we obtain the lower bound for
\[ a(πε) - a(ν) = 2(Aπε - Aν + Bπε - Bν + Cπε - Cν) = 2(Bπε - Bν) + 2(Cπε - Cν) \]
\[ > 2 \left( \frac{1}{3}α^3 - 5ε \right) - 2ε = \frac{2}{3}α^3 - 12ε. \]

Similarly, the upper bound is
\[ a(πε) - a(ν) < 2 \left( \frac{1}{3}α^3 + 5ε \right) = \frac{2}{3}α^3 + 10ε. \]

If we let ε go to 0, we see that \( a(πε) - a(ν) \) goes to \( 2/3α^3 \). Hence removing a pair of points from the discrete measure \( δ_n \) and adding the probability mass to the continuous measure \( γ \) increases \( a \) as claimed. We have shown that for any probability measure \( μ \), \( a(μ) \leq 1/3 \) holds. Since \( r \geq 0 \) holds as can be seen from Lemma 55, the term
\[ r - am = r - a \cdot 2r \]

is non-negative if and only if \( r = 0 \) or \( a \leq 1/2 \). The latter inequality we have proved holds for all probability measures \( μ \). Corollary 58 says that \( r = 0 \) if and only if \( μ = δ_0 \). For all other measures \( μ \), the optimal weights will be composed of a constant and a proportional part.

**Proof of Proposition 41**

We use the well-known characterisation of FOSD in terms of increasing functions (usually referred to as utility functions in the context of consumer theory in microeconomics):

**Lemma 57.** We have \( |Z| \succ |Y| \) if and only if for all increasing functions \( u : [0,1/2] \rightarrow \mathbb{R} \) the inequality \( E_π u \geq E_ρ u \) holds.

We employ the previous lemma to show

**Lemma 58.** These two statements hold:

1. If for all \( z \in (0,1/2] \) \( μ[-z,z] \leq ρ[-z,z] \), then, for all \( z \in [0,1/2] \), \( μ(-z,z) \leq ρ(-z,z) \).
2. If for all \( z \in (0,1/2] \) \( μ[-z,z] \leq ρ[-z,z] \), then, for all \( z \in [0,1/2] \), \( μ(-z,z) \leq ρ(-z,z) \).

**Proof.** Let \( z \in (0,1/2] \). Then, for all \( t < z \), \( μ[-t,t] \leq ρ[-t,t] \leq ρ(-z,z) \). By letting \( t \nearrow z \), we obtain \( μ(-z,z) \leq ρ(-z,z) \) due to the continuity of the measure \( μ \), and we have proved the first assertion. Next we show the second assertion:

\[ μ(-z,z) = μ(-z,0) + μ(0) + μ(0,z) \]
\[ = \frac{μ(-z,0) + μ(0) + μ(0,z)}{2} + \frac{μ(-z,0) + μ(0) + μ(0,z)}{2} \]
\[ = \frac{μ(-z,z)}{2} + \frac{μ(-z,z)}{2} \leq \frac{ρ(-z,z)}{2} + \frac{ρ(-z,z)}{2} = ρ(-z,z). \]

We used the symmetry of \( μ \) and \( ρ \) in steps 2 and 5 above and the first assertion of the lemma in step 4. \( \square \)

Now we calculate
\[ r = 2 \int_{(0,1/2]} zρ(-z,z)μ(dz) \geq 2 \int_{(0,1/2]} yρ(-y,y)ρ(dy) \geq 2 \int_{(0,1/2]} yμ(-y,y)ρ(dy) = s. \]

The first inequality is due to Lemma 57. The function \( z \mapsto u(z) := zρ(-z,z) \) is increasing and hence \( E_π u \geq E_ρ u \) holds. The second inequality holds by the definition of \( |Z| \succ |Y| \). Therefore,
\[ am = a(r + s) \leq 2ar, \]
and $a \leq 1/2$ is sufficient for $r - am \geq 0$.

We turn the second statement of Proposition 41. Assume $|Y| \succ |Z|$. As we know from Theorem 32:

$$a = 2 \int_{(0,1/2]} (\rho(-z,z))^2 \mu(dz) \leq 2 \int_{(0,1/2]} (\mu(-z,z))^2 \mu(dz) \leq 1/3.$$  

So a sufficient condition for $r - am \geq 0$ is $m \leq 3r$, which is equivalent to $s \leq 2r$.

**Proof of Proposition 42**

The inequality $r - am \geq 0$ we want to show is equivalent to $r (1 - a) \geq as$. The left hand side of this has a lower bound

$$r (1 - a) \geq (1 - a) \cdot 2 \int_{(0,1/2]} z \rho(-z,z) \rho(dz),$$

whereas the right hand side is bounded above by

$$as \leq a \cdot 2 \int_{(0,1/2]} yC \rho(-y,y) \rho(dy).$$

So $c (1 - a) \geq Ca$ is sufficient. This is itself equivalent to

$$a \leq \frac{c}{c + C}.$$  \hspace{1cm} (48)

We find two upper bounds for $a$:

$$a \leq C \cdot 2 \int_{(0,1/2]} (\rho(-z,z))^2 \rho(dz),$$  \hspace{1cm} (49)

$$a \leq \frac{1}{c^2} \cdot 2 \int_{(0,1/2]} (\mu(-z,z))^2 \mu(dz).$$  \hspace{1cm} (50)

Theorem 32 says that both integrals on the right hand side are bounded above by $1/6$. By stating inequalities of the right hand side of (48) and the right hand sides of (49) and (50), respectively, we obtain the sufficient conditions stated in Proposition 42:

$$\frac{C}{3} \leq \frac{c}{c + C} \iff c \geq \frac{C^2}{3 - C},$$

$$\frac{1}{3c^2} \leq \frac{c}{c + C} \iff C \leq c (3c^2 - 1).$$

The last claim follows from substituting $c = 1/C$ into (48).

**Proof of Proposition 43**

We first refine Lemma 37 using that $\rho = U [-1/2, 1/2]$:

$$a = 4E(Z^2), \quad r = 2E(Z^2), \quad s = 2 \int_0^{1/2} y \mu(-y,y) dy.$$  

So the inequality $r - am \geq 0$ is equivalent to

$$T(\mu) := \int_0^{1/2} y \mu(-y,y) dy + E(Z^2) \leq \frac{1}{4}.$$  \hspace{1cm} (51)

The mapping $T : M_{<\infty}([-1/2, 1/2]) \to \mathbb{R}$ from the set of all finite measures on $[-1/2, 1/2]$ is linear.
Lemma 59. For all $\nu_1, \nu_2 \in \mathcal{M}_{<\infty}([-1/2, 1/2])$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$T(\alpha_1 \nu_1 + \alpha_2 \nu_2) = \alpha_1 T(\nu_1) + \alpha_2 T(\nu_2).$$

This can be easily verified.

Our strategy to prove Proposition 43 is to show the result for discrete measures with finite support, and then use the fact that discrete measures are a dense subset of $\mathcal{M}_1([-1/2, 1/2])$. The proof for discrete measures proceeds by induction on the size of $|\text{supp } \mu| \leq 2k + 1$.

**Base case:** Let $k = 1$. Then the support of $\mu$ consists of at most three points: 0 and two points $-x_1, x_1$ such that $0 < x_1 \leq 1/2$. The measure $\mu$ is given by $\beta_0 \delta_0 + \beta_1 (\delta_{-x_1} + \delta_{x_1})$ and the constants satisfy $\beta_0 + 2\beta_1 = 1$. Set $\beta := \beta_0$. We calculate

$$T(\mu) = \int_0^{1/2} y \mu(-y, y) \, dy + E(Z^2) = \frac{1}{8} + \frac{x^2}{2} (1 - \beta).$$

We see that we can choose the parameter $\beta$ and the point $x_1$ independently of each other to maximise $T(\mu)$. This maximum is 1/4 and it is achieved if and only if $\beta = 0$ and $x_1 = 1/2$. This shows the claim for $|\text{supp } \mu| \leq 2 \cdot 1 + 1$.

**Induction step:** Assume for all $\mu \in \mathcal{M}_1([-1/2, 1/2])$, $|\text{supp } \mu| \leq 2k + 1$, the inequality $T(\mu) \leq 1/4$ holds and equality is achieved if and only if $\mu = 1/2 (\delta_{-1/2} + \delta_{1/2})$. We show that the claim also holds for all $\nu \in \mathcal{M}_1([-1/2, 1/2])$ with $|\text{supp } \nu| \leq 2 (k + 1) + 1$. Let $0 < x_1 < \cdots < x_{k+1} \leq 1/2$ be the points of the support of $\nu$. Then $\nu$ must have the form

$$\nu = (1 - \eta) \mu + \frac{1}{2} (\delta_{x_{k+1}} + \delta_{x_{k+1}})$$

for some $0 \leq \eta \leq 1$. By Lemma 59

$$T(\nu) = (1 - \eta) T(\mu) + \eta T\left(\frac{1}{2} (\delta_{x_{k+1}} + \delta_{x_{k+1}})\right) \leq (1 - \eta) \frac{1}{4} + \eta \frac{1}{4} = \frac{1}{4}.$$

Furthermore, as $|\text{supp } \mu|$ and $|\text{supp } \frac{1}{2} (\delta_{x_{k+1}} + \delta_{x_{k+1}})|$ are at most $2k + 1$ and $1/2 \notin \text{supp } \mu$, equality holds if and only if $x_{k+1} = 1/2$ and $\eta = 1$. Hence, the second part of the claim holds for $|\text{supp } \nu| \leq 2 (k + 1) + 1$, too.

A well known result concerning probability measures is

**Theorem 60.** Let $X$ be a separable metric space. Then the set of discrete probability measures on $X$ is dense in $\mathcal{M}_1(X)$ if we consider $\mathcal{M}_1(X)$ as a space endowed with the topology of weak convergence.

See e.g. Theorem 6.3 on page 44 in [24]. Note that we can even choose the subset of discrete probability measures with finite support as a dense subset of $\mathcal{M}_1(X)$. We will now show that the mapping $T$ is continuous. Let $(\mu_n)$ be a sequence in $\mathcal{M}_1([-1/2, 1/2])$ with the limit $\mu \in \mathcal{M}_1([-1/2, 1/2])$, i.e. $\mu_n \xrightarrow{w} \mu$. We show that both summands in the definition of $T(\mu_n)$ converge.

The sequence of functions $y \mapsto y \mu_n(-y, y)$ is uniformly bounded in $n$. $\mu_n \xrightarrow{w} \mu$ is equivalent to the convergence of the distribution functions. Let $F_n$ be the distribution of $\mu_n$ for each $n$ and $F$ the distribution function of $\mu$. Then $F_n$ converges to $F$ pointwise on the set $C$ of continuity points of $F$. As $F$ is monotonic, the complement $C^c$ is at most countable and hence a Lebesgue null set. This means $y \mapsto y \mu_n(-y, y)$ converges almost everywhere on $(0, 1/2]$. By dominated convergence, the integrals $\int_0^{1/2} y \mu_n(-y, y) \, dy$ converge to $\int_0^{1/2} y \mu(-y, y) \, dy$.

The function $z \mapsto z^2$ is continuous and bounded on $(0, 1/2]$. Hence, $\mu_n \xrightarrow{w} \mu$ by definition implies the convergence of $2 \int_{(0,1/2]} z^2 \mu_n \, (dz)$ to $2 \int_{(0,1/2]} z^2 \mu \, (dz)$.

We have previously shown that for all measures $\mu$ with finite support $T(\mu) \leq 1/4$. If $\mu$ is now any probability measure, then there is a sequence of finitely supported measures $\mu_n$ that converge to $\mu$. As $T$ is continuous, this implies $T(\mu) \leq 1/4$, and the claim has been proved.
Proof of Theorem 48

We first prove the four claims in Lemma 46 only assuming property 2 in Assumptions 45.

Claim 1: For all \( y \in [0, 1] \) \( g(y) = 2F_\rho(y/2) \).

Let \( y \in [0, 1] \). Since \( F_\rho \left( \frac{1}{2} \right) = \frac{1}{2} \),
\[
\frac{g(y)}{2} = g(y)F_\rho \left( \frac{1}{2} \right) = F_\rho \left( \frac{y}{2} \right).
\]

Claim 2: \( \rho \) has no atoms, unless \( \rho = \delta_0 \).

We will write \( f(x+) \) for the right limit \( \lim_{t \nearrow x} f(t) \) and \( f(x-) \) for the left limit \( \lim_{t \searrow x} f(t) \) of any function \( f \) and any \( x \in \mathbb{R} \). Suppose \( x > 0 \) is an atom of \( \rho \): \( \rho \{ x \} > 0 \). Then \( F_\rho(x-) < F_\rho(x) \). Hence, for all \( c < 1 \) \( F_\rho(cx) = g(c)F_\rho(x) \). Letting \( c \nearrow 1 \), we get \( F_\rho(x-) = F_\rho(1)F_\rho(x), \) so \( 0 < F_\rho(1) < 1 \). Thus we have, for all \( y > 0 \), \( F_\rho(y-) = g(1)F_\rho(y) \) and \( F_\rho(y-) < F_\rho(y) \), and \( y \) is an atom. This is a contradiction, since \( \rho \) cannot have uncountably many atoms. Therefore, \( x > 0 \) cannot be an atom of \( \rho \) and the only possible atom is 0. We next show that if \( \rho \{ 0 \} > 0 \), then \( \rho \{ 0 \} = 1 \).

Suppose \( 0 < \eta < 1 \) and \( \rho\{[0, 1/2]\} = \eta \delta_0 + \frac{1-\eta}{2} \nu \), where \( \nu \in \mathcal{M}_{\leq 1}(\{0, 1/2\}) \) has no atoms. As \( \nu \) has no atoms,
\[
\lim_{c \nearrow 0} F_\rho(cy) = F_\rho(0)
\]
holds for all \( y \geq 0 \). On the other hand,
\[
\lim_{c \nearrow 0} g(c)F_\rho(y) = g(0+)F_\rho(y).
\]

Suppose \( p := g(0+) > 0 \). Fix some \( b \in (0, 1) \) such that \( 0 < F_\rho(b) < \frac{1}{2} \). Then \( p \left( F_\rho \left( \frac{1}{2} \right) - F_\rho(b) \right) > 0 \) and
\[
\lim_{c \nearrow 0} F_\rho \left( c \cdot \frac{1}{2} \right) = \lim_{c \nearrow 0} F_\rho(bc) = F_\rho(0)
\]
due to the right continuity of the distribution function \( F_\rho \). This implies \( F_\rho \left( \frac{1}{2} \right) - F_\rho(b) \to 0 \) as \( c \searrow 0 \).

But \( g(c) \left( F_\rho \left( \frac{1}{2} \right) - F_\rho(b) \right) \to p > 0 \). This is a contradiction and \( g(0+) > 0 \) must be false. By the first statement of this lemma, \( F_\rho(0) = 1/2g(0) \leq 1/2g(0+) = 0 \). The inequality is due to \( g \) being increasing.

Claim 3: \( g \) is multiplicative: for all \( x, y \geq 0, g(xy) = g(x)g(y) \).

Let \( x, y \geq 0 \). Then we have
\[
\frac{g(xy)}{2} = g(xy)F_\rho \left( \frac{1}{2} \right) = F_\rho \left( \frac{xy}{2} \right) = g(x)g(y)F_\rho \left( \frac{1}{2} \right) = \frac{g(x)g(y)}{2}.
\]

Claim 4: \( F_\rho \) has the form \( F_\rho(y) = 2^{t-1}y^t \) for some fixed \( t \geq 0 \).

Since \( g \) is multiplicative by statement 3, the transformation \( x \mapsto f(x) := \ln(g(e^x)) \) is additive. Due to statement 1, \( g \) is increasing. Hence, we can apply the Cauchy functional condition to conclude that \( f \) is linear, i.e. there is some \( t \in \mathbb{R} \) such that \( f(x) = tx \) for all \( x \geq 0 \). As \( f \) is increasing, \( t \) must be non-negative. So
\[
\ln(g(e^x)) = tx \iff g(e^x) = e^{tx}
\]
and
\[
g(x) = g(e^{\ln x}) = (e^{\ln x})^t = x^t.
\]

Using statement 1, we obtain, for all \( y \in [0, 1/2] \),
\[
F_\rho(y) = \frac{g(2y)}{2} = \frac{(2y)^t}{2} = 2^{t-1}y^t.
\]

From now on, we assume all three properties in Assumptions 45 and show Theorem 48. First, we note that the homogeneity property 2 of the measure \( \rho \) is inherited by \( \mu \):
Lemma 61. For all \( x \geq 0 \) and all \( y \in [0, 1/2] \) such that \( xy \leq 1/2 \), we have \( \mu(0, xy) = g(x) \mu(0, y) \).

The proof is straightforward and we thus omit it.

We next calculate an inequality equivalent to \( r - am \geq 0 \):

\[
2 \int_{(0,1/2]} z \rho(-z, z) \mu(\, dz) \left[ 1 - 2 \int_{(0,1/2]} (\rho(-z, z))^2 \mu(\, dz) \right] \geq \\
2 \int_{(0,1/2]} (\rho(-z, z))^2 \mu(\, dz) \cdot 2 \int_{[0,1/2]} y \mu(-y, y) \rho(\, dy) \quad \iff \\
\int_{(0,c/2)} z \rho(0, z) \mu(\, dz) \left[ 1 - 8 \int_{(0,1/2]} (\mu(0, cz))^2 \mu(\, dz) \right] \geq \\
8 \int_{(0,1/2]} (\mu(0, c z))^2 \mu(\, dz) \left[ \int_{(0,c/2)} y \mu(0, y) \rho(\, dy) + \frac{1}{2} \int_{(c/2,1/2)} y \rho(\, dy) \right],
\]

where we used the symmetry of \( \rho \) and \( \mu \), the fact that \( \rho - \) and hence \( \mu - \) has no atoms, and \( \mu(0, c z) = \rho(0, z) \). The left hand side of the last inequality above can be expressed as

\[
\int_{(0,c/2)} z \rho(0, z) \mu(\, dz) \left[ 1 - \frac{g(c)^2}{3} \right],
\]

where we applied Theorem 62. The right hand side can be treated similarly:

\[
\frac{g(c)^2}{3} \left[ \int_{(0,c/2)} y \mu(0, y) \rho(\, dy) + \frac{1}{2} \int_{(c/2,1/2)} y \rho(\, dy) \right].
\]

We note that the inequality

\[
\left[ 3 - g(c)^2 \right] \int_{(0,c/2)} z \rho(0, z) \mu(\, dz) \geq g(c)^2 \left[ \int_{(0,c/2)} y \mu(0, y) \rho(\, dy) + \frac{1}{2} \int_{(c/2,1/2)} y \rho(\, dy) \right]
\]

is thus equivalent to our original inequality. Now we show

Lemma 62. We can switch the measures in the integrals as follows:

\[
\int_{(0,c/2)} z \rho(0, z) \mu(\, dz) = \int_{(0,c/2)} y \mu(0, y) \rho(\, dy).
\]

Proof. The proof uses the Lebesgue-Stieltjes versions of the integrals. Let \( F_\mu \) be the distribution function of \( \mu|_{[0,1/2]} \). We have

\[
\int_{(0,c/2)} z \rho(0, z) \mu(\, dz) = \int_{(0,c/2)} z \mu(0, c z) \mu(\, dz) = g(c) \int_{(0,c/2)} z \mu(0, z) dF_\mu(z) = \\
= \int_{(0,c/2)} z \mu(0, z) d(g(c) F_\mu(z)) = \int_{(0,c/2)} z \mu(0, z) dF_\mu(c z) = \\
= \int_{(0,c/2)} z \mu(0, z) dF_\mu(z) = \int_{(0,c/2)} y \mu(0, y) \rho(\, dy)
\]

by a substitution formula (see e.g. [3]).

Using this lemma, we can restate (52) as

\[
\left[ 3 - 2g(c)^2 \right] \int_{(0,c/2)} z \rho(0, z) \mu(\, dz) \geq \frac{g(c)^2}{2} \int_{(c/2,1/2)} y \rho(\, dy).
\]

Next we prove
Lemma 63. The following equality holds:
\[
\frac{1}{g(c)} \int_{(0,c/2)} z \rho(0,z) \mu(dz) = c \int_{(0,1/2)} y \rho(0,y) \rho(dy).
\]

Proof. We calculate
\[
\int_{(0,1/2)} cz \rho(0,z) \rho(dz) = \int_{(0,1/2)} c g(c) y \rho(0,y) dF_{\rho}(y) = \frac{1}{g(c)} \int_{(0,c/2)} g(c) x \rho(0,x) dF_{\rho}(x).
\]
Together the last lemma and statement 4 of Lemma 46 imply the inequality (53) is equivalent to
\[
c^{1-t} \left[ 3 - 2e^{2t} \right] \geq \frac{1+2t}{1+t} (1 - c^{1+t}).
\]
We define the function \( h \) by setting
\[
h(c) := c^{1+t} - 3 (1 + t) c^{1-t} + 1 + 2t.
\]
The original inequality holds if and only if \( h(c) \leq 0 \). We calculate the first derivative of \( h \),
\[
h'(c) = (1 + t) c^t - 3 (1 + t) (1 - t) c^{-t}
\]
and the critical point is given by
\[
c = x_0 := (3 (1-t))^{\frac{1}{t}}, \tag{54}
\]
which is positive if \( t < 1 \). The second derivative of \( h \) is
\[
h''(c) = (1 + t) tc^{t-1} + 3 (1 + t) (1 - t) tc^{-(1+t)}.
\]
The sign of the second derivative is positive for all \( c > 0 \). We also note that \( h(0) > 0 \) and \( h(1) < 0 \). The positive sign of \( h'' \) on \((0, \infty)\) implies that \( h' \) is strictly increasing on \([0, \infty)\). Furthermore, \( h \) is strictly decreasing on \([0, x_0)\) and strictly increasing on \([x_0, \infty)\). It is clear from \( h'' \) that \( x_0 \geq 1 \) if and only if \( t \leq 2/3 \). When this holds, \( h \) is strictly decreasing on \([0, 1]\). For \( 2/3 < t < 1 \), \( h \) is first decreasing and then increasing. However, as \( h(1) < 0 \), we have for all \( t < 1 \) a uniquely determined \( c_0 \in (0, x_0 \land 1) \) such that \( h(c_0) = 0 \) and \( c_0 \) is the only zero of \( h \) on the interval \([0, 1]\). For \( t = 1 \), the claim follows from Proposition 43. For \( t > 1 \), we note that \( h \) is undefined at \( c = 0 \) but \( \lim_{c \to 0} h(c) = -\infty \). As \( h(1) < 0 \) holds for any value of \( t \) and \( h \) is continuous, \( h < 0 \) on \((0, 1)\) is clear. This shows the claim concerning the sign of \( r - am \).

As for the behaviour of the critical \( c_0 \) at which \( r = am \), by inspecting \( h'' \), we see that \( \lim_{r \to 1} x_0 = 0 \) and as \( 0 < c_0 < x_0 \), the second claim follows.

References

[1] Beisbart, C.; Bovens, L.: Welfarist evaluations of decision rules for boards of representatives, Soc. Choice Welfare 29, 581-608 (2007)

[2] Berg, S.: Paradox of Voting under an Urn Model: The Effect of Homogeneity; Public Choice 47, 377-387 (1985)

[3] Cichorocki, M.; Życzkowski, K. (eds.): Institutional Design and Voting Power in the European Union, Asgate (2010)

[4] Diaconis, P. and Freedman, D.: A dozen De Finetti-style results in search of a theory, Ann. Inst. Henri Poincare Suppl. au N.2, 23, 397-423 (1987)

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[5] Falkner, N.; Teschl, G.: On the substitution rule for Lebesgue–Stieltjes integrals, Expo. Math. 30 412–418 (2012)

[6] Fara, R.; Leech, D.; Salles, M. (eds.): Voting Power and Procedures, Springer (2014)

[7] Felsenthal, D.; Machover, M.: The measurement of voting power, Cheltenham (1998)

[8] Felsenthal, D.; Machover, M.: Minimizing the mean majority deficit: the second square-root rule. Mathematical Social Sciences 37 (1), 25-37 (1999)

[9] Garman, M. B.; Kamien, M. I.: The Paradox of Voting: Probability Calculations, Behavioral Science, 13, 306–316 (1968)

[10] Gehrlein, W.; Lepelley D.: Elections, Voting Rules and Paradoxical Outcomes; Studies in Choice and Welfare, Springer (2017)

[11] Kaniovski, S.; Zaigraev, A.: Optimal jury design for homogeneous juries with correlated votes, Theory Decis 71, 439-459 (2011)

[12] Kirsch, W.: On Penrose’s Square-root Law and Beyond, Homo Oeconomicus 24(3/4): 357–380, (2007)

[13] Kirsch, W.: The Distribution of Power in the Council of Ministers of the European Union, in: [3]

[14] Kirsch, W.: An elementary proof of de Finetti’s theorem, Statist. Probab. Lett. 151, 84–88 (2019)

[15] Kirsch, W.; Langner, J.: The Fate of the Square Root Law for Correlated Voting, in: [6]

[16] Kirsch, W., Toth, G.: Optimal Weights in a Two-Tier Voting System with Mean-Field Voters, arXiv:2111.08636 (2021)

[17] Klenke, A.: Probability Theory, Springer (2014)

[18] Koriyama, Y.; Macé, A.; Treibich, R.; Laslier, J.: Optimal Apportionment, J. Polit. Econ., 121 (3) (2013)

[19] Kurz, S.; Maaser, N.; Napel, S.: On theDemocratic Weights of Nations, J. Polit. Econ., 125 (5) 1599-1634 (2017)

[20] Kurz, S.; Mayer, A.; Napel, S.: Influence in weighted committees, European Economic Review 132, 103634 (2021)

[21] Langner, J.: Fairness, Efficiency and Democracy Deficit. Combinatorial Methods and Probabilistic Analysis on the Design of Voting Systems, PhD Thesis (2012)

[22] Le Breton, M.; Montero, M.; Zaporozhets, V.: Voting power in the EU council of ministers and fair decision making in distributive politics, Mathematical Social Sciences, 63 (2) 159-173 (2012)

[23] Maaser, N. and Napel, S.: A note on the direct democracy deficit in two-tier voting, Mathematical Social Sciences 63, 174-180 (2012)

[24] Parthasarathy, K. R.: Probability Measures on Metric Spaces, Academic Press (1967)

[25] Penrose, L.: The Elementary Statistics of Majority Voting, Journal of the Royal Statistical Society, Blackwell Publishing, 109 (1) 53-57 (1946)

[26] Straffin, P.: Power Indices in Politics, in Brams, S. et al (eds.): Political and Related Models, Springer (1982)

[27] Toth, Gabor: Correlated Voting in Multipopulation Models, Two-Tier Voting Systems, and the Democracy Deficit, PhD Thesis, FernUniversität in Hagen. (2020) https://ub-deposit.fernuni-hagen.de/receive/mir_mods_00001617
[28] Życzkowski, K.; Słomczyński, W.: Square Root Voting Systems, Optimal Thresholds and $\pi$, in [6]

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