DISCRETE SERIES CHARACTERS AND THE LEFSCHETZ FORMULA FOR HECKE OPERATORS

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This paper consists of three independent but related parts. In the first part (§§1–6) we give a combinatorial formula for the constants appearing in the “numerator” of characters of stable discrete series representations of real groups (see §3) as well as an analogous formula for individual discrete series representations (see §6). Moreover we give an explicit formula (Theorem 5.1) for certain stable virtual characters on real groups; by Theorem 5.2 these include the stable discrete series characters, and thus we recover the results of §3 in a more natural way.

In the second part (§7) we use the character formula given in Theorem 5.1 to rewrite the Lefschetz formula of [GM] (for the local contribution at a single fixed point component to the trace of a Hecke operator on weighted cohomology) in the same spirit as that of Arthur’s Lefschetz formula [A]: in terms of stable virtual characters on real groups (see Theorem 7.14.B). We then sum the contributions of the various fixed point components and show that, in the case of middle weighted cohomology, the resulting global Lefschetz fixed point formula agrees with Arthur’s Lefschetz formula. This gives a topological proof of Arthur’s formula.

The third part of the paper (Appendices A and B) is purely combinatorial. In Appendix A we develop the combinatorics of convex polyhedral cones on which our results on characters of real groups are based. The methods of Appendix A are also used in Appendix B to prove a generalization of a combinatorial lemma of Langlands.

The formula for stable discrete series constants given in Theorem 3.1 is redundant, since it follows easily from Theorems 5.1 and 5.2. Nevertheless the proof of Theorem 3.1 is instructive and should probably not be skipped by readers interested in the case of individual discrete series constants. Theorem 3.2 is not redundant and in fact provides the link between our results on stable discrete series constants and individual discrete series constants (we return to this point later in the introduction). Because of the redundancy built into the paper, the reader who is mainly interested in the Lefschetz formula only needs to read §§5,7 and a little bit of Appendix A.

Let $G$ be a connected reductive group over $\mathbb{Q}$ and let $A_G$ denote the maximal $\mathbb{Q}$-split torus in the center of $G$. Let $K_G$ be a maximal compact subgroup of $G(\mathbb{R})$ and let $X_G$.

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denote the homogeneous space
\[ G(\mathbb{R})/(K_G \cdot A_G(\mathbb{R})^0). \]

Let \( K \) be a suitably small compact open subgroup of \( G(\mathbb{A}_f) \). We denote by \( S_K \) the space
\[ G(\mathbb{Q})\backslash[(G(\mathbb{A}_f)/K) \times X_G]. \]

Let \( E \) be an irreducible representation of the algebraic group \( G \) on a finite dimensional complex vector space. Then \( E \) gives rise to a local system \( E_K \) on \( S_K \).

Let \( P_0 = M_0N_0 \) be a minimal parabolic subgroup of \( G \), with Levi component \( M_0 \) and unipotent radical \( N_0 \). As usual by a standard parabolic subgroup of \( G \) we mean one that contains \( P_0 \). For any standard parabolic subgroup \( P \) we write \( P = MN \) where \( M \) is the unique Levi component of \( P \) containing \( M_0 \) and \( N \) is the unipotent radical of \( P \). The reductive Borel-Serre compactification \( \overline{S}_K \) of \( S_K \) is a stratified space whose strata are indexed by the standard parabolic subgroups of \( G \). The stratum indexed by \( G \) is the space \( S_K \). The stratum \( S^P_K \) indexed by standard \( P = MN \) is a finite union of spaces of the same type as \( S_K \), but for the group \( M \) rather than \( G \).

Let \( j : S_K \hookrightarrow \overline{S}_K \) denote the inclusion. Consider the object \( \mathbf{R}j_*E_K \) in the derived category of \( \overline{S}_K \). The restriction to the stratum \( S^P_K \) of the \( i \)-th cohomology sheaf of \( \mathbf{R}j_*E_K \) is the local system on \( S^P_K \) associated to the representation of \( M \) on
\[ H^i(\text{Lie}(N),E). \]

For any standard \( P = MN \) we write \( \mathfrak{A}_M \) for the real vector space
\[ X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} \]
and \( \mathfrak{A}^*_M \) for its dual. Let \( \nu \in \mathfrak{A}^*_M \) and suppose that the restriction of \( \nu \) to \( \mathfrak{A}_G \) coincides with the element of \( X^*(A_G) \) by which \( A_G \) acts on \( E \). For any standard parabolic subgroup \( P = MN \) we write \( \nu_P \in \mathfrak{A}^*_M \) for the restriction of \( \nu \) to the subspace \( \mathfrak{A}_M \) of \( \mathfrak{A}_M \). Then \( \nu \) determines a weight profile and hence a weighted cohomology complex (see [GHM]) \( E_K \) on \( S_K \) (an object in the derived category of \( \overline{S}_K \)). The restriction to \( S^P_K \) of the \( i \)-th cohomology sheaf of \( E_K \) is the local system on \( S^P_K \) associated to the representation of \( M \) on
\[ H^i(\text{Lie}(N),E)_{\nu_P}, \]
the subspace of \( H^i(\text{Lie}(N),E) \) on which \( A_M \) acts by weights \( \geq \nu_P \) (a weight \( \mu \in X^*(A_M) \subset \mathfrak{A}^*_M \) is \( \geq \nu_P \) if \( \mu - \nu_P \) takes non-negative values on the chamber in \( \mathfrak{A}_M \) determined by \( P \)).
The main result of [GM] is an explicit version of the Lefschetz formula for the alternating sum of the traces of the self-maps on

\[ H^i(S_K, E_K) \]

induced by a Hecke correspondence. One of our goals in this paper (see Theorem 7.14.B) is to rewrite the Lefschetz formula in [GM] in terms involving a stable virtual character \( \Theta_\nu \) on the group \( G(\mathbb{R}) \). If \( \nu \) is so positive that \( E_K \) coincides with the extension by zero of \( E_K \), then \( \Theta_\nu \) is just the character of the contragredient \( E^* \) of \( E \) (see 7.17). There is a similar (but more complicated) statement in case \( \nu \) is sufficiently negative (see 7.18).

In general \( \Theta_\nu \) is given by

\[
\Theta_\nu = \sum_P (-1)^{\dim(A_M/A_G)} i^G_M (\delta_P^{-1/2} \otimes (E_P^\nu)^*)
\]

where \( E_P^\nu \) is the following virtual finite dimensional representation of \( M \)

\[
\sum_i (-1)^i H^i(\text{Lie}(N), E)_{\geq \nu_P},
\]

and \( \delta_P \) is the usual modulus character

\[
\delta_P(x) = |\det(x; \text{Lie}(N))|
\]

on \( M(\mathbb{R}) \); the sum is taken over all standard parabolic subgroups \( P = MN \) and \( i^G_M(\cdot) \) denotes normalized parabolic induction from \( M(\mathbb{R}) \) to \( G(\mathbb{R}) \).

Since there are simple formulas for characters that are parabolically induced from finite dimensional representations of Levi subgroups, it is possible to determine the character \( \Theta_\nu \) explicitly (Theorem 5.1). In fact Theorem 5.1 is just what is needed to rewrite the Lefschetz formula of [GM] in terms of \( \Theta_\nu \) (the numbers \( L_\gamma \) that go into the definition of \( L^*_M(\gamma) \) occur as factors in the local Lefschetz numbers). If \( G(\mathbb{R}) \) has a discrete series and if \( \nu \) is the "upper middle" weight profile \( \nu_m \) of §5, then (see Theorem 5.2) \( \Theta_\nu \) agrees on all relevant maximal tori with

\[
(-1)^{\eta(G)} \sum_{\pi \in \Pi} \Theta_\pi
\]

where \( \Pi \) is the L-packet of discrete series representations of \( G(\mathbb{R}) \) having the same infinitesimal and central characters as \( E^* \) (and in fact \( \Theta_\nu \) is equal to this virtual character if \( P_0 \) remains minimal over \( \mathbb{R} \)), and our formula essentially coincides with Arthur’s formula for \( L^2 \)-Lefschetz numbers of Hecke operators [A] (see the remarks at the end of 7.19 for a detailed comparison with Arthur’s formula). This provides evidence for the agreement of middle weighted cohomology and \( L^2 \)-cohomology in this degree of generality (in the Hermitian symmetric case this agreement is known: middle weighted cohomology agrees with intersection cohomology of the Baily-Borel compactification [GHM] and this in turn agrees with \( L^2 \)-cohomology [L],[SS]).
This concludes our discussion of the global results in this paper. However it remains to summarize the results on real groups. The two theorems just mentioned (Theorems 5.1 and 5.2) together give a simple formula for the stable discrete series character

$$\sum_{\pi \in \Pi} \Theta_\pi$$

on any maximal torus in $G$ over $\mathbb{R}$. In particular we obtain a simple formula (Theorem 3.1) for the constants $d(w)$ ($w \in W$) appearing in stable discrete series character formulas (actually we prove Theorem 3.1 by a different method). Here $W$ is the Weyl group of a root system $R$ such that $-1 \in W$, for which we have fixed a system of positive roots. The formula for $d(w)$ is expressed as a sum over $W$, each term in the sum being 1, $-1$ or 0, and bears no obvious relation to the formula of Herb [He] for $d(w)$ in terms of two-systems. The terms in the sum depend on the finite dimensional representation $E$, although their sum does not, so that we in fact get finitely many different formulas, one for each cone in a certain decomposition of the positive Weyl chamber. In Theorem 3.2 we prove an unexpected symmetry for the function $d(w)$:

$$d(w^{-1}) = (-1)^{q(R)} \epsilon(w) d(w),$$

where $\epsilon$ is the usual sign function on the Weyl group. This symmetry, together with the formula for $d(w^{-1})$ as a sum over $W$, gives a second formula for $d(w)$ as a sum over $W$. Our fixed system of positive roots determines a certain subgroup $W_c$ of $W$ (the “compact” Weyl group). If in the second formula for $d(w)$ we replace the sum over $W$ by a sum over a coset of $W_c$ in $W$, the resulting expression turns out to be a formula (see §6) for the constants appearing in the characters of individual discrete series representations. Of course these constants were already known, implicitly by work of Harish-Chandra and explicitly by work of Hirai (or by combining formulas for the stable constants—Herb’s or ours—with Shalstad’s theory of endoscopy); what is perhaps interesting is the simplicity of our formula (again we in fact get finitely many different formulas, all giving the same result).

More should be said about Theorem 5.2, which expresses stable discrete series characters as linear combinations of characters induced from finite dimensional representations of Levi subgroups. J. Adams informs us that a result of this kind was known to G. Zuckerman when he wrote his 1974 Princeton thesis (certain examples are treated in the thesis, but a precise general statement is not given there). We do not know if our result is the one Zuckerman had in mind, though it seems unlikely that there could be two essentially different formulas. What we do know is that an inversion procedure due to Langlands (a simple special case of his combinatorial lemma, often applied in Arthur’s work on the trace formula) allows one to obtain from Theorems 5.2 and 5.3 an expression for the character of a finite dimensional representation as a linear combination of standard characters, and it is not hard to see that this inverted formula coincides with the one due to Zuckerman [Z] (see also [V]). Langlands’s inversion procedure works in both directions, so that one could also invert Zuckerman’s theorem to obtain our Theorem
5.2. However, this would result in a more complicated proof (ours uses only elementary
combinatorics and Harish-Chandra’s characterization of discrete series characters).

We wish to bring to the reader’s attention some related results of J. Franke [F],
G. Harder [H2], A. Nair [N] and M. Stern [St]. We would like to thank G. Harder for
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In this paper we use the following notation. For a finite set \( S \) we write \(|S|\) for the
cardinality of \( S \). For a subset \( A \) of a set \( S \) we write \( \chi_A \) for the characteristic function
of \( A \). For a subgroup \( H \) of a group \( G \) we write \( N_G(H) \) (respectively, \( \text{Cent}_G(H) \)) for
the normalizer (respectively, centralizer) of \( H \) in \( G \) (sometimes we allow \( H \) to be a subgroup
of a bigger group of which \( G \) is also a subgroup). For a free abelian group \( X \) of finite
rank we write \( X_\mathbb{R} \) for the real vector space \( X \otimes \mathbb{Z} \). Given an endomorphism \( A \) of a
finite dimensional vector space \( V \), we write \( \det(A;V) \) (respectively, \( \text{tr}(A;V) \)) for the
determinant (respectively, trace) of \( A \) whenever the name of \( A \) leaves doubt about which
vector space \( V \) we have in mind.

1. The function \( \psi_R(C_0, x, \lambda) \)

In this section we consider a root system \((X, X^*, R, R^\vee)\). Here \( X \) is a real vector space,
\( X^* \) its dual vector space, \( R \subset X^* \) a root system in \( X^* \) that spans \( X^* \), and \( R^\vee \subset X \)
the coroot system in \( X \). We write \( W = W(R) \) for the Weyl group of the root system
\( R \). For any Weyl chamber \( C \) in \( X \) we write \( C^\vee \) for the corresponding Weyl chamber in
\( X^* \). In this section we will define integers \( \psi_R(C_0, x, \lambda) \); in §3 we will see that in case
\(-1 \in W \) these integers are (essentially) the ones appearing in the formulas for stable
discrete series characters on real groups.

Let \( C_0 \) be a Weyl chamber in \( X \). We write \( \overline{C}_0 \) for the closure of \( C_0 \). Let \( \omega \in \overline{C}_0 \) be
a non-zero element in a 1-dimensional face of \( \overline{C}_0 \) (thus \( \omega \) is, up to a positive scalar, a
fundamental coweight for \( C_0 \)). Put

\[
R_\omega := \{ \alpha \in R \mid \alpha(\omega) = 0 \}.
\]

For any chamber \( C \) in \( X \) relative to \( R \) let \( \tilde{C} \) denote the unique chamber in \( X \) relative
to \( R_\omega \) that contains \( C \). For two chambers \( C_1, C_2 \) in \( X \) relative to \( R \) we write \( l_R(C_1, C_2) \)
or just \( l(C_1, C_2) \) for the number of root hyperplanes in \( X \) separating \( C_1 \) and \( C_2 \); thus
\( l(C_1, C_2) \) is the length with respect to \( C_1 \) of the unique element \( w \in W \) such that
\( wC_1 = C_2 \). We write \( R^+ \) for the set of roots in \( R \) that are positive on \( C_0 \). Finally we
write \( R_\omega^+ \) for \( R_\omega \cap R^+ \), the set of roots in \( R_\omega \) that are positive on \( \tilde{C} \).

**Lemma 1.1.** (a) The map \( C \mapsto \tilde{C} \) yields a bijection from the set of chambers in \( X \)
relative to \( R \) whose closures contain \( \omega \) to the set of chambers in \( X \) relative to \( R_\omega \). If
$C_1, C_2$ are chambers in $X$ relative to $R$ that contain $\omega$, then
\[ l(C_1, C_2) = l_{R_\omega}(\tilde{C}_1, \tilde{C}_2). \]

(b) Consider the difference $|R^+| - |R^+_\omega|$. If $\mathbb{R} \omega$ contains a coroot, this difference is odd. If $\mathbb{R} \omega$ does not contain a coroot and if $-1_{X/\mathbb{R} \omega} \in W(R_\omega)$, then this difference is even.

(c) There exists a unique chamber $C'_0$ in $X$ such that $-\omega \in \overline{C'_0}$ and $\tilde{C}_0 = \tilde{C}'_0$. Moreover
\[ \{ \alpha \in R^+ \mid \ker(\alpha) \text{ separates } C_0, C'_0 \} = R^+ \setminus R^+_{\omega}. \]

In particular
\[ l(C_0, C'_0) = |R^+| - |R^+_{\omega}|. \]

(d) Suppose that there exists a positive scalar $c$ such that $c\omega$ is a coroot $\alpha^\vee \in R^\vee$. Note that the corresponding root $\alpha$ belongs to $R^+$. Suppose that $C''$ is a chamber in $X$ satisfying the following three conditions:

1. $\alpha$ takes non-negative values on $C''$,
2. $\ker(\alpha)$ is a wall of $C''$,
3. $\tilde{C}'' = \tilde{C}_0$.

Then
\[ l(C_0, C'') = (|R^+| - |R^+_{\omega}| - 1)/2 \]
(note that by (b) the quantity on the right-hand side is an integer).

The assertion (a) is standard. Now we prove (b). First suppose that $\mathbb{R} \omega$ contains a coroot $\alpha^\vee$ and let $w \in W$ denote the reflection in $\alpha^\vee$. We consider the action of $w$ on the set $R/\pm$ obtained from $R$ by taking the quotient by the action of the group $\{\pm 1\}$. Since $w^2 = 1$, we see that $|R^+|$ has the same parity as the number of fixed points of $w$ on $R/\pm$. Let $\beta \in R$. Then $w\beta = \pm \beta$ if and only if $\beta \in R_\omega$ or $\beta^\vee \in \mathbb{R} \omega$ (of course these alternatives are mutually exclusive). Therefore the number of fixed points of $w$ on $R/\pm$ is $|R^+_{\omega}| + 1$, which shows that $|R^+| - |R^+_{\omega}|$ is odd, as desired.

Now suppose that $-1_{X/\mathbb{R} \omega} \in W(R_\omega)$. In other words we are supposing that there exists an element $w \in W$ of order 2 whose $+1$ eigenspace is $\mathbb{R} \omega$ and whose $-1$ eigenspace is the span of $R^\vee_\omega$. Again we consider the action of $w$ on $R/\pm$. Let $\beta \in R$. Then $w\beta = \pm \beta$ if and only if $\beta \in R_\omega$ or $\beta^\vee \in \mathbb{R} \omega$. Suppose further that $\mathbb{R} \omega$ contains no coroot. Then the number of fixed points of $w$ on $R/\pm$ is $|R^+_{\omega}|$, which shows that $|R^+| - |R^+_{\omega}|$ is even, as desired.

Now we consider (c). The existence and uniqueness of $C'_0$ follow from the first statement in (a), applied to both $\omega$ and $-\omega$. Next we prove the second statement in (c). Let $\alpha \in R^+$. Suppose first that $\alpha$ (strictly speaking, $\ker(\alpha)$) separates $C_0, C'_0$. Since $\tilde{C}_0 = \tilde{C}'_0$ it follows that $\alpha \notin R^+_{\omega}$. Conversely, suppose that $\alpha \notin R^+_{\omega}$. Then $\alpha(\omega) \neq 0$, so that $\alpha$ strictly separates $\omega, -\omega$. Since $\omega \in \overline{C}_0$ and $-\omega \in \overline{C}_0$ it follows that $\alpha$ separates $C_0, C'_0$.

Finally we consider (d). We are interested in roots $\beta \in R$ that separate $C_0, C''$. Using (1) and (3), we see that any such $\beta$ belongs to $R_0 := R \setminus (R_\omega \cup \{\pm \alpha\})$. To prove (d)
we must show that exactly half of the elements of $R_0$ separate $C_0, C''$. Let $s \in W$ be the reflection in the root $\alpha$. Then $s$ preserves both $R_\omega$ and $\{\pm \alpha\}$ and hence preserves $R_0$ as well. Let $\beta \in R_0$. We will be done if we can show that $\beta$ separates $C_0, C''$ if and only if $s\beta$ does not separate $C_0, C''$. Let $C'_0$ be as in (c) and note that $C'_0 = sC_0$. Of course $C_0, C''$ are separated by $\beta$ if and only if $C'_0 = sC_0, sC''$ are separated by $s\beta$. By (2) $sC''$ and $C''$ are separated only by $\pm \alpha$; therefore $sC'', C''$ are not separated by $s\beta$. Moreover $s\beta$ does separate $C_0, C'_0$ (use (c)). Therefore $C_0, C''$ are not separated by $s\beta$, as we wished to show. The proof of the lemma is now complete.

Now we prepare to define the main object of study in this section. For any two chambers $C_1$ and $C_2$ in $X$ we write $\varepsilon(C_1, C_2)$ for $(-1)^{l(C_1, C_2)}$. For any chamber $C$ in $X$ we define a function $\psi_C$ on $X \times X^*$ as follows. Let $\alpha_1, \ldots, \alpha_n$ (n = dim($X$)) be the simple roots in $R$ relative to $C$, and let $\omega_1, \ldots, \omega_n \in X$ be the basis for $X$ dual to the basis $\alpha_1, \ldots, \alpha_n$ for $X^*$. Let $x \in X$, $\lambda \in X^*$ and write

$$x = a_1 \omega_1 + \cdots + a_n \omega_n$$
$$\lambda = b_1 \alpha_1 + \cdots + b_n \alpha_n.$$  

Let $I = \{1, \ldots, n\}$ and define two subsets $I_x, I_\lambda$ of $I$ by

$I_x = \{i \in I \mid a_i \geq 0\}$
$I_\lambda = \{i \in I \mid b_i \geq 0\}$.

Then define $\psi_C(x, \lambda)$ by

$$\psi_C(x, \lambda) = \begin{cases} (-1)^{|I_\lambda|} & \text{if } I_\lambda = I \setminus I_x, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this function $\psi_C$ coincides with the function denoted by $\psi_C$ in Appendix A (see Lemma A.1).

Now let $C_0$ be a chamber in $X$. Define a function $\psi(C_0, \cdot, \cdot)$ on $X \times X^*$ by

$$\psi(C_0, x, \lambda) = \sum_C \varepsilon(C_0, C) \psi_C(x, \lambda),$$

where $C$ runs through the set of chambers in $X$. When it is necessary to stress the root system $R$ we will write $\psi_R(C_0, x, \lambda)$ instead. We have the following obvious property:

\begin{equation}
\psi(wC_0, x, \lambda) = \varepsilon(w)\psi(C_0, x, \lambda) = \psi(C_0, wx, w\lambda)
\end{equation}

for any $w \in W$, where $\varepsilon(w)$ denotes the sign of $w$.

As usual we say that an element $x \in X$ is regular if it lies on no root hyperplane. We say that an element $\lambda \in X^*$ is $R$-regular if it lies on no hyperplane of the form
\{ \lambda \in X^* \mid \lambda(\omega) = 0 \}; where \omega is a non-zero element of a 1-dimensional face of some closed Weyl chamber in \( X \). Of course this notion of regularity in \( X^* \) is in general different from the usual one, and we refer to the connected components in the set of \( \mathbf{R} \)-regular elements in \( X^* \) as \( \textit{R-chambers} \) in \( X^* \) to avoid confusion with the usual Weyl chambers in \( X^* \).

Suppose that \( \omega \) is a non-zero element in a 1-dimensional face of \( \mathcal{C}_0 \). We adopt the notation of Lemma 1.1 and the discussion preceding it (e.g., \( \mathbf{R}_\omega, R^+, R^+_\omega, \mathcal{C}_0, \mathcal{C}'_0 \)). Let \( Z \) denote the hyperplane \( \{ \lambda \in X^* \mid \lambda(\omega) = 0 \} \) in \( X^* \). We have the root system \( (X/\mathbf{R}\omega, Z, \mathbf{R}_\omega, R^+_\omega) \). Note that \( \mathcal{C}_0 = C_0 + \mathbf{R}\omega \) has the same image as \( C_0 \) in \( X/\mathbf{R}\omega \); we denote this chamber in \( X/\mathbf{R}\omega \) by \( \mathcal{C}_0 \).

**Lemma 1.2.** Let \( x \) be a regular element in \( X \). The function \( \psi(C_0, x, \cdot) \) on \( X^* \) is constant on \( \textit{R-chambers} \). Suppose that \( \lambda, \lambda' \) are \( \textit{R-regular} \) elements of \( X^* \) lying in adjacent \( \textit{R-chambers} \) separated only by the hyperplane \( Z \), and suppose further that \( \lambda(\omega) > 0 \), \( \lambda'(\omega) < 0 \). Then

\[
\psi(C_0, x, \lambda) - \psi(C_0, x, \lambda') = \begin{cases} -2\psi_{\mathbf{R}\omega}(C'_0, \tilde{x}, \check{\lambda}) & \text{if } \mathbf{R}\omega \text{ contains a coroot,} \\ 0 & \text{otherwise.} \end{cases}
\]

Here \( \tilde{x} \) denotes the image of \( x \) in \( X/\mathbf{R}\omega \) and \( \check{\lambda} \) denotes the unique point of \( Z \) lying on the line segment joining \( \lambda \) and \( \lambda' \). Moreover \( \tilde{x} \) is regular relative to \( \mathbf{R}_\omega \) and \( \check{\lambda} \) is \( \mathbf{R}_\omega \)-regular.

It is clear that \( \psi(C_0, x, \cdot) \) is constant on \( \textit{R-chambers} \). The statement regarding the regularity of \( \tilde{x} \) and \( \check{\lambda} \) is easy and will be left to the reader. By Corollary A.3

\[
\psi(C_0, x, \lambda) - \psi(C_0, x, \lambda')
\]

is equal to the sum over all chambers \( C \) such that \( \mathcal{C} \) contains \( \omega \) or \(-\omega\) of terms

\[
\pm \epsilon(C_0, C) \psi_{\mathcal{C}}(\tilde{x}, \check{\lambda}),
\]

where the sign is \(-\) if \( \mathcal{C} \) contains \( \omega \) and \( + \) if \( \mathcal{C} \) contains \(-\omega\). We have abused notation slightly by writing \( \mathcal{C} \) when we mean its image in \( X/\mathbf{R}\omega \); since \( \mathcal{C} \) contains \( \omega \) or \(-\omega\) this image coincides with the image of \( C \) in \( X/\mathbf{R}\omega \). By Lemma 1.1(c), for each chamber \( C \) such that \( \mathcal{C} \) contains \( \omega \), there exists a unique chamber \( C' \) such that \( \mathcal{C}' \) contains \(-\omega \) and \( \mathcal{C} = \mathcal{C}' \). Combining the terms for \( C, C' \), we get

\[
- \epsilon(C_0, C) (1 - \epsilon(C, C')) \psi_{\mathcal{C}}(\tilde{x}, \check{\lambda}).
\]

From Lemma 1.1(c) we see that \( \epsilon(C, C') = -1 \) if \( |R^+| - |R^+_\omega| \) is odd and is 1 otherwise. In the latter case each of the combined terms is 0 and so is their sum. In the former case the sum of the combined terms is

\[
-2 \sum \epsilon(C_0, C) \psi_{\mathcal{C}}(\tilde{x}, \check{\lambda}),
\]

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where \( C \) ranges through the set of chambers in \( X \) containing \( \omega \). It follows from Lemma 1.1(a) that this expression coincides with \(-2\psi_{R,\omega}(C_0^\omega, \bar{x}, \bar{\lambda})\).

Thus we have shown that

\[
\psi(C_0, x, \lambda) - \psi(C_0, x, \lambda') = \begin{cases} 
-2\psi_{R,\omega}(C_0^\omega, \bar{x}, \bar{\lambda}) & \text{if } |R^+| - |R_+^\omega| \text{ is odd,} \\
0 & \text{otherwise.}
\end{cases}
\]

(1.2)

From the equality (1.2) and Lemma 1.1(b) we see that Lemma 1.2 holds whenever \(-1_{X/R,\omega} \in W(R_\omega)\). Using only equality (1.2), we will prove Corollary 1.3 below. But then the general case of Lemma 1.2 will follow, since \(\psi_{R,\omega}(C_0^\omega, \bar{x}, \bar{\lambda}) = 0 \) if \(-1_{X/R,\omega} \notin W(R_\omega)\) (by Corollary 1.3).

**Corollary 1.3.** Suppose that \(-1_X \notin W\). Then \(\psi(C_0, x, \lambda) = 0\) for all regular \( x \in X \) and all \( R \)-regular \( \lambda \in X^* \).

We prove this by induction on \( \dim(X) \). If \( \dim(X) = 0 \), the statement is trivially true. Now assume that \( \dim(X) > 0 \). Fix a regular element \( x \in X \). There exists \( R \)-regular \( \lambda_0 \in X^* \) such that \( \lambda_0(x) > 0 \). By Proposition A.5 \(\psi(C_0, x, \lambda_0) = 0\). Therefore, to prove the corollary it would be enough to show that

\[
\psi(C_0, x, \lambda) - \psi(C_0, x, \lambda')
\]

vanishes whenever \( \lambda, \lambda' \) lie in adjacent \( R \)-chambers separated by the hyperplane

\[
Z = \{\lambda \in X^* \mid \lambda(\omega) = 0\}
\]

determined by a non-zero element \( \omega \) of some 1-dimensional face of the closure of some chamber in \( \bar{X} \); by (1.1) it is harmless to assume that \( \omega \in C_0 \). By equality (1.2), Lemma 1.1(b) and our induction hypothesis, this difference does vanish unless \(-1_{X/R,\omega} \in W(R_\omega)\) and \( R_\omega \) contains some coroot \( \alpha^\vee \). But the product of \(-1_{X/R,\omega} \) and reflection in the coroot \( \alpha^\vee \) is equal to \(-1_X \), and we are assuming that \(-1_X \notin W\). We conclude that the difference always vanishes, as desired.

We need to introduce more notation. Let \( P \) (respectively, \( Q \)) denote the lattice of coweights in \( X \) (respectively, the lattice in \( X \) generated by the coroots). For any chamber \( C \) in \( X \) we denote by \( \delta_C \in P \) the half-sum of the coroots that are positive for \( C \). Put

\[
\hat{A}_{\text{sc}} = \text{Hom}(P, \mathbb{C}^\times) \\
\hat{A}_{\text{ad}} = \text{Hom}(Q, \mathbb{C}^\times).
\]

The inclusion \( Q \subset P \) induces a surjection

\[
\hat{A}_{\text{sc}} \twoheadrightarrow \hat{A}_{\text{ad}}
\]

of complex tori, whose kernel we denote by \( Z^\vee \), so that we get an exact sequence

\[
1 \rightarrow Z^\vee \rightarrow \hat{A}_{\text{sc}} \rightarrow \hat{A}_{\text{ad}} \rightarrow 1.
\]
There are natural \( \mathbb{C}^\times \)-valued pairings \( \langle \cdot, \cdot \rangle \) between \( P \) and \( \hat{A}_{\text{sc}} \) and between \( Q \) and \( \hat{A}_{\text{ad}} \). Let \( s \in \hat{A}_{\text{sc}} \) and suppose that \( s^2 \in Z^\vee \). Define a root system \( R_s \) by

\[
\begin{align*}
R_s^\vee &= \{ \alpha^\vee \in R^\vee \mid \langle \alpha^\vee, s \rangle = 1 \} \\
R_s &= \{ \alpha \in R \mid \alpha^\vee \in R_s^\vee \}.
\end{align*}
\]

When we defined \( \psi_R(C_0, \cdot, \cdot) \) we insisted that \( R \) generate \( X^* \). Of course this was just a matter of convenience. In the general case the intersection of all the root hyperplanes in \( X \) is a linear subspace \( X_0 \) in \( X \). Defining \( \psi(C_0, \cdot, \cdot) \) as before, we find from (A.2) that \( \psi(C_0, x, \lambda) \) is 0 unless \( \lambda \) vanishes on \( X_0 \), in which case

\[
\psi(C_0, x, \lambda) = (-1)^{\dim(X_0)} \psi(\tilde{C}_0, \tilde{x}, \lambda)
\]

where \( \tilde{C}_0 \) (respectively, \( \tilde{x} \)) denotes the image of \( C_0 \) (respectively, \( x \)) in \( X/X_0 \); note that on the right-hand side \( \lambda \) is regarded as an element of \( (X/X_0)^* \). These remarks allow us to consider the function \( \psi_{R_s}(\tilde{C}_0, x, \lambda) \) obtained from \( (X, X^*, R_s, R_s^\vee) \), where \( \tilde{C}_0 \) denotes the unique chamber for \( R_s \) in \( X \) containing \( C_0 \).

**Lemma 1.4.** For all regular \( x \in X \) and all \( \lambda \in X^* \) there is an equality

\[
\sum_C \epsilon(C_0, C) \langle \delta_C - \delta_{C_0}, s \rangle \psi_C(x, \lambda) = \psi_{R_s}(\tilde{C}_0, x, \lambda),
\]

in which the sum runs over all chambers \( C \) in \( X \). In particular, if \( s^2 \neq 1 \), then the left-hand side of this equality vanishes for regular \( x \) and \( R \)-regular \( \lambda \).

Since \( s^2 \in Z^\vee \), the image of \( s^2 \) in \( \hat{A}_{\text{ad}} \) is 1, and therefore \( \langle \alpha^\vee, s \rangle = \pm 1 \) for every coroot \( \alpha^\vee \). By the definition of \( R_s \) we have \( \langle \alpha^\vee, s \rangle = 1 \) if \( \alpha \in R_s \) and \( \langle \alpha^\vee, s \rangle = -1 \) if \( \alpha \notin R_s \). Since \( \delta_C - \delta_{C_0} \) is the sum of the coroots in \( R^\vee \) that are positive on \( C \) and negative on \( C_0 \), we see that

\[
\epsilon(C_0, C) \langle \delta_C - \delta_{C_0}, s \rangle = \epsilon_{R_s}(\tilde{C}_0, \tilde{C})
\]

where \( \tilde{C} \) denotes the unique chamber of \( (X, R_s) \) containing \( C \). Therefore the left-hand side of the equality we are trying to prove is equal to the sum over chambers \( D \) for \( (X, R_s) \) of \( \epsilon(\tilde{C}_0, D) \) times

\[
\sum_C \psi_C(x, \lambda)
\]

where \( C \) runs through the chambers for \( (X, R) \) contained in \( D \). By Proposition A.4 the difference between

\[
\sum_C \psi_C(x, \lambda)
\]

and

\[
\psi_D(x, \lambda)
\]
is a sum of terms of the form \( \pm \psi_F(x, \lambda) \) where \( F \) is a proper face of the closure of some chamber \( C \) of \((X, R)\) contained in \( D \). Here \( \psi_F \) is as in Appendix A. But \( \psi_F(x, \lambda) \) vanishes unless \( x \in \text{span}(F) \) (see (A.2)). Therefore for regular \( x \) the left-hand side of the equality we are trying to prove is

\[
\sum_D \epsilon(C_0, D) \psi_D(x, \lambda)
\]

which, by definition, is \( \psi_{R_a}(\tilde{C}_0, x, \lambda) \).

It remains to prove the second statement of the lemma. If the rank of the root system \( R_s \) is smaller than that of \( R \), then \( \psi_{R_s}(\tilde{C}_0, x, \lambda) = 0 \) unless \( \lambda \) belongs to the proper linear subspace \( \text{span}(R_s) \) of \( X^* \). But the left-hand side of the equality of the lemma is constant on \( R \)-chambers in \( X^* \); therefore it vanishes for regular \( x \) and \( R \)-regular \( \lambda \). If \( s^2 \neq 1 \) and \( R_s \) has the same rank as \( R \), then \( -1_X \notin W(R_s) \) (since \( -1_X \) sends \( s \) to \( s^{-1} \) while all elements of \( W(R_s) \) fix \( s \)), and therefore by Corollary 1.3 \( \psi_{R_s}(\tilde{C}_0, x, \lambda) \) vanishes for all regular \( x \in X \) (regular for \( R_s \)) and all \( R_s \)-regular \( \lambda \in X^* \). Any \( x \in X \) that is regular for \( R \) is regular for \( R_s \), and again using that the left-hand side of the equality of the lemma is constant on \( R \)-chambers in \( X^* \), we see that it vanishes for regular \( x \) and \( R \)-regular \( \lambda \). This completes the proof of the lemma.

There is another result of this kind. With \( P, Q \) as before now put

\[
A_{sc} = Q \otimes \mathbb{C}^\times
\]

\[
A_{ad} = P \otimes \mathbb{C}^\times.
\]

The inclusion \( Q \subset \text{span}(R_s) \) induces a surjection

\[
A_{sc} \twoheadrightarrow A_{ad}
\]

of complex tori, whose kernel we denote by \( Z \), so that we get an exact sequence

\[
1 \to Z \to A_{sc} \to A_{ad} \to 1.
\]

There are natural \( \mathbb{C}^\times \)-valued pairings \( \langle \cdot, \cdot \rangle \) between \( Q^* \) and \( A_{sc} \) and between \( P^* \) and \( A_{ad} \) (\( P^*, Q^* \) are the free abelian groups dual to \( P, Q \) respectively). Note that \( Q^* \) is the lattice of weights in \( X^* \) and that \( P^* \) is the lattice in \( X^* \) generated by the roots. For any chamber \( C \) in \( X \) we write \( \rho_C \in Q^* \) for the half-sum of the roots that are positive for \( C \).

Let \( a \in A_{sc} \) and suppose that \( a^2 \in Z \). Define a root system \( R_a \) by

\[
R_a = \{ \alpha \in R \mid \langle \alpha, a \rangle = 1 \}.
\]

Let \( \tilde{C}_0 \) denote the unique chamber for \( R_a \) in \( X \) containing \( C_0 \).

**Lemma 1.5.** For all regular \( x \in X \) and all \( \lambda \in X^* \), there is an equality

\[
\sum_C \epsilon(C_0, C) \langle \rho_C - \rho_{C_0}, a \rangle \psi_C(x, \lambda) = \psi_{R_a}(\tilde{C}_0, x, \lambda),
\]

in which the sum runs over all chambers \( C \) in \( X \). In particular, if \( a^2 \neq 1 \), then the left-hand side of this equality vanishes for regular \( x \) and \( R \)-regular \( \lambda \).

The proof is essentially the same as that of Lemma 1.4.
2. The function \( \psi_R(C_0, x, \lambda) \) in case \(-1 \in W\)

We continue with \( X, X^*, R, R^\vee, W \) as in §1. We still assume (for convenience) that \( R \) generates \( X^* \), and we now add the assumption that \(-1_X \in W\). Let \( \alpha \) be a root and define a root system \( R_\alpha \) by

\[
R_\alpha^\vee = \{ \beta^\vee \in R^\vee \mid \langle \alpha, \beta^\vee \rangle = 0 \}
R_\alpha = \{ \beta \in R \mid \beta^\vee \in R_\alpha^\vee \}.
\]

Let \( Y \) denote the hyperplane \( \{ x \in X \mid \alpha(x) = 0 \} \); then \( R_\alpha^\vee \subset Y \). Let \( s_\alpha \) be the reflection in the root \( \alpha \). Since \(-1_X \) belongs to \( W \), so does \(-s_\alpha \). But \(-s_\alpha \) fixes \( \alpha \), hence belongs to \( W(R_\alpha) \). Since \(-s_\alpha \) acts by \(-1 \) on \( Y \), we conclude that \(-1_Y \in W(R_\alpha) \). Therefore \((Y, Y^*, R_\alpha, R_\alpha^\vee)\) satisfies the same conditions as \((X, X^*, R, R^\vee)\): \( R_\alpha \) generates \( Y^* \) and \(-1_Y \in W(R_\alpha) \). Note that \( \alpha^\vee \) lies in the kernel of every root for \( R_\alpha \). Therefore \( \alpha^\vee \) is a non-zero element in some 1-dimensional face of some chamber in \( X \), and \( \alpha^\vee \) can serve as the element \( \omega \) considered in §1. Note that \( R_\alpha = R_\omega \) with \( R_\omega \) as in §1.

There are two notions of chamber in \( Y \). Of course we have the usual Weyl chambers \( D \) in \( Y \) coming from the root system \( R_\alpha \); these are determined by the hyperplanes \( \beta = 0 \) \((\beta \in R_\alpha) \). There is a larger set of hyperplanes in \( Y \), namely those of the form \( \beta = 0 \) \((\beta \in R \setminus \{ \pm \alpha \}) \), and we will refer to the connected components \( E \) of the complement of this larger set of hyperplanes as chambers in \( Y \) relative to \( R \).

Fix a chamber \( C_0 \) in \( X \) having \( Y \) as a wall. As in §1 we write \( \tilde{C} \) for the unique chamber for \( X \) relative to \( R_\alpha = R_\omega \) that contains \( C \). It is easy to see that the map \( C \mapsto \tilde{C} \cap Y \) is a bijection from the set of chambers \( C \) in \( X \) having \( Y \) as a wall and lying on the same side of \( Y \) as \( C_0 \) to the set of closed chambers in \( Y \) relative to \( R \); note that the closure of \( \tilde{C} \cap Y \) is equal to \( \overline{C} \cap Y \). Using the chamber \( C_0 \), we obtain a function \( \psi(C_0, \cdot, \cdot) \) on \( X \times X^* \) as in §1.

**Lemma 2.1.** Fix \( \lambda \in X^* \). The function \( \psi(C_0, \cdot, \cdot, \lambda) \) on \( X \) is constant on the chambers in \( X \). Suppose that \( x, x' \) are regular elements lying in adjacent chambers separated only by the hyperplane \( Y \), and assume that \( x, C_0 \) lie on the same side of \( Y \) (so that \( x', C_0 \) lie on opposite sides of \( Y \)). Then

\[
\psi(C_0, x, \lambda) - \psi(C_0, x', \lambda) = 2\psi_{R_\alpha}(D_0, y, \lambda_Y)
\]

where \( \lambda_Y \in Y^* \) denotes the restriction of \( \lambda \) to \( Y \), \( y \in Y \) is the unique point of \( Y \) lying on the line segment joining \( x \) and \( x' \), and \( D_0 \) is the chamber \( \tilde{C}_0 \cap Y \) for \((Y, R_\alpha)\). Moreover \( y \) is regular in \( Y \), and if \( \lambda \) is \( R \)-regular then \( \lambda_Y \) is \( R_\alpha \)-regular in \( Y^* \).

It is clear that \( \psi(C_0, \cdot, \cdot, \lambda) \) is constant on chambers in \( X \). By Lemma A.2

\[
\psi(C_0, x, \lambda) - \psi(C_0, x', \lambda)
\]

is equal to

\[
2 \sum_C \varepsilon(C, C_0)\psi_{\tilde{C} \cap Y}(y, \lambda_Y),
\]
where \( C \) runs over the chambers in \( X \) having \( Y \) as a wall and lying on the same side of \( Y \) as \( C_0 \) (and \( x \)). The factor 2 arises since we have combined the contributions of \( C \) and the unique chamber adjacent to \( C \) across the wall \( Y \). We denote by \( C' \) the unique chamber for \( (X, R) \) that is contained in \( C \) and whose closure contains \( \omega \). Replacing \( \alpha \) by \( -\alpha \) if necessary, we may assume without loss of generality that \( \alpha \) is non-negative on \( C_0 \) and \( x \), and hence on any \( C \) appearing in the sum above. Applying Lemma 1.1(d) to both \( C_0 \) and any such \( C \) (both satisfy conditions (1) and (2)), we see that

\[
\epsilon(C_0, C) = \epsilon(C_0', C'),
\]

and then from Lemma 1.1(a) we see further that

\[
\epsilon(C_0, C) = \epsilon(\tilde{C}_0, \tilde{C}) = \epsilon(D_0, D)
\]

where \( D_0 = \tilde{C}_0 \cap Y \) and \( D = \tilde{C} \cap Y \).

We have now shown that the left-hand side of the equality we are trying to prove is equal to

\[
2 \sum_D \epsilon(D_0, D) \sum_E \psi_E(y, \lambda_Y)
\]

where \( D \) runs over the chambers of \( (Y, R_\alpha) \) and \( E \) runs over the chambers in \( Y \) relative to \( R \) such that \( E \subset D \) (the function \( \psi_E \) is the one attached in Appendix A to the closed convex polyhedral cone \( \overline{E} \) in \( Y \)). Each such \( D \) is the disjoint union of the corresponding \( E \)'s together with the relative interiors of some closed convex polyhedral cones \( F \) of lower dimension, each of which is contained in some root hyperplane other than \( Y \). It is clear that \( y \) lies on no root hyperplane of \( R \) other than \( Y \); therefore \( \psi_F(y, \lambda_Y) \) vanishes for all such \( F \) (see (A.2)). By Proposition A.4 the inner sum is equal to \( \psi_D(y, \lambda_Y) \), and therefore the whole expression is equal to \( 2\psi_{R_\alpha}(D_0, y, \lambda_Y) \). This proves the lemma, except for the last statement, which we leave to the reader.

Again let \( C_0 \) be a chamber in \( X \) and let \( R^+ \) be the set of roots in \( R \) that are positive for \( C_0 \). Let \( \epsilon : W \to \{\pm 1\} \) be the sign homomorphism. The longest element of \( W \) is \(-1_X\). On the one hand

\[
\epsilon(-1_X) = (-1)^{|R^+|}.
\]

On the other hand

\[
\epsilon(-1_X) = \det(-1_X) = (-1)^{\dim(X)}.
\]

Therefore \( |R^+| \) and \( \dim(X) \) have the same parity, and we can define an integer \( q(R) \) by

\[
q(R) := (|R^+| + \dim(X))/2.
\]

To understand the significance of the integer \( q(R) \) one should note that it is half the dimension of the symmetric space of the split semisimple real group \( G \) with root system \( R \) (a number that is traditionally denoted \( q(G) \)).
Let $C_0^\vee$ be the Weyl chamber in $X^*$ corresponding to $C_0$. From $C_0^\vee$ and the coroot system $R^\vee$ we get a function $\psi_{R^\vee}(C_0^\vee, \cdot, \cdot)$ on $X^* \times X$ (the roles of $X$ and $X^*$ are now reversed). We have the notions of regularity (for $x \in X$) and $R$-regularity (for $\lambda \in X^*$) from before. Applying these definitions to $R^\vee$ rather than $R$, we have the notions of regularity for $\lambda \in X^*$ and $R^\vee$-regularity for $x \in X$; note that the set of regular elements in $X^*$ is the union of the Weyl chambers in $X^*$. Since $-1_X \in W$ every coroot in $X$ is a non-zero element in a 1-dimensional face of some chamber in $X$ (we saw this during the discussion at the beginning of this section). Therefore, if $\lambda \in X^*$ is $R$-regular, it is automatically regular, and, similarly, if $x \in X$ is $R^\vee$-regular, it is automatically regular.

**Lemma 2.2.** For any $R^\vee$-regular $x \in X$ and any $R$-regular $\lambda \in X^*$ there is an equality

$$\psi_R(C_0, x, \lambda) = (-1)^{q(R)} \psi_{R^\vee}(C_0^\vee, \lambda, x).$$

We prove this by induction on $\dim(X)$. It is certainly true when $\dim(X) = 0$ (the empty root system). Now assume that $\dim(X) > 0$. Fix an $R^\vee$-regular element $x \in X$. There exists $R$-regular $\lambda_0 \in X^*$ such that $\lambda_0(x) > 0$, and the equality in the lemma holds for $x, \lambda_0$ since both sides of the equality vanish by Proposition A.5. Therefore it is enough to show that

$$(2.1) \quad \psi_R(C_0, x, \lambda) - \psi_R(C_0, x, \lambda') = (-1)^{q(R)}(\psi_{R^\vee}(C_0^\vee, \lambda, x) - \psi_{R^\vee}(C_0^\vee, \lambda', x))$$

whenever $\lambda, \lambda'$ are $R$-regular elements of $X^*$ lying in adjacent $R$-chambers. Let $Z$ denote the unique hyperplane separating these two adjacent $R$-chambers. Thus $Z$ is of the form

$$Z = \{\lambda \in X^* | \lambda(\omega) = 0\}$$

for some non-zero $\omega$ lying in a 1-dimensional face of the closure of some Weyl chamber in $X$; we may assume without loss of generality that this Weyl chamber coincides with $C_0$ (by property (1.1) changing $C_0$ changes both sides of (2.1) by the same sign). By switching $\lambda, \lambda'$ if necessary we may also assume that $\lambda(\omega) > 0$ and $\lambda'(\omega) < 0$.

First consider the case in which $R\omega$ does not contain a coroot. Then the left-hand side of (2.1) vanishes by Lemma 1.2, while the right-hand side vanishes because $\psi_{R^\vee}(C_0^\vee, \cdot, x)$ is constant on Weyl chambers in $X^*$, not just on $R$-chambers. We are left with the case in which $R\omega$ contains a coroot $\alpha^\vee$; replacing $\omega$ by a positive scalar multiple we may as well assume that $\omega = \alpha^\vee$. Since $\alpha^\vee \in C_0$, the root $\alpha$ is positive for $C_0$. By Lemma 1.2 the left-hand side of (2.1) is equal to

$$-2\psi_{R_\omega}(C_0^\omega, \bar{x}, \bar{\lambda})$$

(with notation as in that lemma).

Of course we are going to use Lemma 2.1 to evaluate the right-hand side of (2.1). However the hyperplane

$$Z = \{\lambda \in X^* | \lambda(\omega) = 0\} = \{\lambda \in X^* | \lambda(\alpha^\vee) = 0\}$$

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need not be a wall of $C_0^\vee$, so that we need to introduce another chamber $(C^\vee)^\prime$ in $X^*$, better suited to our purposes. We take $(C^\vee)^\prime$ to be any chamber in $X^*$ satisfying the following three conditions

1. $\alpha^\vee$ takes non-negative values on $(C^\vee)^\prime$,
2. $Z$ is a wall of $(C^\vee)^\prime$,
3. $((C^\vee)^\prime)^\sim = (C_0^\vee)^\sim$.

The notation $(\cdot)^\sim$ used in (3) has the following meaning: for any chamber $C^\vee$ in $X^*$ we write $(C^\vee)^\sim$ for the unique chamber in $X^*$ relative to $R^\vee = (R^\vee)^\alpha^\vee$ that contains $C^\vee$.

It is easy to see that $(C^\vee)^\prime$ exists: pick any chamber $E^\vee$ in $Z$ relative to $R^\vee$ contained in $(C^\vee)^\sim \cap Z$ and take for $(C^\vee)^\prime$ the unique chamber in $X^*$ satisfying (1) and (2) and having the property that the closure of $E^\vee$ is equal to the intersection of $Z$ with the closure of $(C^\vee)^\prime$.

By (1.1) and Lemma 2.1 the right-hand side of (2.1) is equal to

$$2(-1)^{q(R)}\epsilon(C_0^\vee, (C^\vee)^\prime)\psi_{R^\vee}(D_0^\vee, \tilde{\lambda}, \tilde{x}),$$

with $\tilde{\lambda}, \tilde{x}$ as before and $D_0^\vee$ the chamber $((C^\vee)^\prime)^\sim \cap Z$ in $Z$ for the root system $R^\vee$. It follows from Lemma 1.1(d) that

$$\epsilon(C_0^\vee, (C^\vee)^\prime) = (-1)^{(|R^+| - |R_+^\omega| - 1)/2};$$

since

$$q(R) - q(R_\omega) = (|R^+| - |R_+^\omega| + 1)/2,$$

we conclude that

$$(-1)^{q(R)}\epsilon(C_0^\vee, (C^\vee)^\prime) = -(-1)^{q(R_\omega)}.$$

By our induction hypothesis

$$(-1)^{q(R_\omega)}\psi_{R^\omega}(D_0^\vee, \tilde{\lambda}, \tilde{x}) = \psi_{R_\omega}(C_0^\omega, \tilde{x}, \tilde{\lambda}).$$

Of course we used that $D_0^\vee = (C_0^\omega)^\vee$ and that $\tilde{x}, \tilde{\lambda}$ are suitably regular. Therefore the right-hand side of (2.1) equals

$$-2\psi_{R_\omega}(C_0^\omega, \tilde{x}, \tilde{\lambda}),$$

which coincides with the expression we found for the left-hand side. This concludes the proof of the lemma.

### 3. Stable discrete series constants $\bar{c}_R$

In the theory of stable discrete series characters on real groups, which we will review briefly in §4, there appear integer-valued functions (see [K], [He], [He2])

$$\bar{c}_R : X_{\text{reg}} \times X_{\text{reg}}^* \to \mathbb{Z}$$
for every root system \((X, X^*, R, R^\vee)\) satisfying the two conditions of §2 \((R \text{ generates } X^* \text{ and } -1_X \in W, \text{ where } W = W(R) \text{ denotes the Weyl group of } R)\). Here \(X_{\text{reg}}\) and \(X_{\text{reg}}^*\) denote the sets of regular elements in \(X\) and \(X^*\) respectively (regular in the usual sense, so that \(X_{\text{reg}}, X_{\text{reg}}^*\) can also be described as the unions of the Weyl chambers in \(X, X^*\) respectively). The functions \(\tilde{c}_R\) satisfy the following five properties:

1. \(\tilde{c}_R(0,0) = 1\) if \(R\) is empty,
2. \(\tilde{c}_R(x, \lambda)\) depends only on the chamber in \(X\) in which \(x\) lies and the chamber in \(X^*\) in which \(\lambda\) lies,
3. \(\tilde{c}_R(x, \lambda) = 0\) unless \(\lambda(x) \leq 0\),
4. if \(x, x' \in X\) lie in adjacent chambers, separated only by the root hyperplane \(Y\), then
   \[\tilde{c}_R(x, \lambda) + \tilde{c}_R(x', \lambda) = 2\tilde{c}_{R_Y}(y, \lambda_Y),\]
   where \(R_Y \subset Y^*\) is the root system whose set of coroots is \(R^\vee \cap Y\), \(\lambda_Y\) is the restriction of \(\lambda\) to \(Y\), and \(y\) is the unique point of \(Y\) lying on the line segment joining \(x\) and \(x'\),
5. \(\tilde{c}_R(wx, w\lambda) = \tilde{c}_R(x, \lambda)\) for all \(w \in W(R)\).

It is well-known that the collection of functions \(\tilde{c}_R\) is characterized uniquely by properties (1), (3), (4) (this follows easily from an induction on \(\dim(X)\) as in the proofs of Lemma 2.2 and Corollary 1.3). Of course these properties are reminiscent of ones enjoyed by the functions \(\psi_R(C_0, x, \lambda)\) studied in §2. For \(x \in X_{\text{reg}}\) denote by \(C_x\) the unique chamber in \(X\) containing \(x\). We now define an integer-valued function \(m_R\) on \(X_{\text{reg}} \times X_{\text{reg}}^*\) by

\[m_R(x, \lambda) = \psi_R(C_x, x, \lambda).\]

**Theorem 3.1.** The functions \(\tilde{c}_R\) and \(m_R\) are equal.

We need only show that \(m_R\) satisfies properties (1), (3), (4) above. Property (1) is trivial. Property (3) follows from Proposition A.5. Property (4) follows from (1.1) and Lemma 2.1.

There is a more efficient way to encode the information in the function \(\tilde{c}_R\). Fix a Weyl chamber \(C_0\) in \(X\) and let \(C_0^\vee\) be the corresponding Weyl chamber in \(X^*\). Then define an integer-valued function \(d\) on \(W = W(R)\) by putting

\[d(w) := \tilde{c}_R(x_0, w\lambda_0) \quad (w \in W)\]

where \(x_0\) (respectively, \(\lambda_0\)) is any point in \(C_0\) (respectively, \(C_0^\vee\)). Of course \(d\) depends (in a simple way) on the choice of \(C_0\). Applying this construction to the root system \(R^\vee\) (and the chamber \(C_0^\vee\)) we get a function \(d^\vee\) on \(W = W(R^\vee) = W(R)\).

**Theorem 3.2.** For all \(w \in W\) there are equalities

1. \(d^\vee(w) = d(w)\)
2. \(d(w^{-1}) = (-1)^{q(R)}\epsilon(w)d(w)\)
where $e(w)$ denotes the sign of $w$.

Pick a $W$-equivariant isomorphism $j : X \to X^*$. Let $x_0 \in C_0$; then $\lambda_0 := j(x_0) \in C_0^\vee$. Since $\psi_R$ depends only on the root hyperplanes and not on the roots themselves, it is clear that
\begin{align}
\psi_R(C_0^\vee, \lambda_0, wx_0) = \psi_R(C_0, x_0, w\lambda_0),
\end{align}
and by Theorem 3.1 this just says that
\begin{align}
d^\vee(w) = d(w).
\end{align}
The equality (2) follows from Theorem 3.1 and Lemma 2.2 (use (1.1) and (3.1) as well).

4. Background material on stable characters

In this section we review some of the theory of characters of irreducible representations of real groups. Let $G$ be a connected reductive group over $\mathbb{R}$. Let $E$ be an irreducible finite dimensional complex representation of the algebraic group $G$. We are interested in irreducible representations $\pi$ of $G(\mathbb{R})$ (irreducible Harish-Chandra modules) having the same infinitesimal character as $E$. Harish-Chandra associated to any such $\pi$ its character $\Theta_\pi$, a real-analytic function on $G_{\text{reg}}(\mathbb{R})$, the set of regular semisimple elements in $G(\mathbb{R})$.

Let $T$ be a maximal torus in $G$. Let $B(T)$ denote the set of Borel subgroups of $G$ over $\mathbb{C}$ containing $T$. Let $R$ be the set of roots of $T$ in $G$. For $B \in B(T)$ denote by $\lambda_B \in X^*(T)$ the highest weight of $E$ relative to $B$, denote by $\rho_B \in X^*(T)_{\mathbb{R}}$ half the sum of the roots in $R$ that are positive for $B$ and denote by $\Delta_B$ the Weyl denominator
\begin{align}
\Delta_B = \prod_{\alpha > 0} (1 - \alpha^{-1})
\end{align}
for $T$ relative to $B$ (the index set is the subset of $R$ consisting of roots that are positive for $B$).

The character of $E$ on $T_{\text{reg}}(\mathbb{R}) := T(\mathbb{R}) \cap G_{\text{reg}}(\mathbb{R})$ is given by
\begin{align}
\text{tr}(\gamma; E) = \sum_{B \in B(T)} \lambda_B(\gamma) \cdot \Delta_B(\gamma)^{-1} \quad (\gamma \in T_{\text{reg}}(\mathbb{R})).
\end{align}
The character $\Theta_\pi$ of $\pi$ on $T_{\text{reg}}(\mathbb{R})$ is given by a similar expression
\begin{align}
\Theta_\pi(\gamma) = \sum_{B \in B(T)} n(\gamma, B) \lambda_B(\gamma) \Delta_B(\gamma)^{-1}
\end{align}
for certain integers $n(\gamma, B)$ depending on $(\gamma, B)$. Of course the invariance of $\Theta_\pi$ under conjugation by $G(\mathbb{R})$ implies that
\begin{align}
n(\gamma, B) = n(w\gamma w^{-1}, wBw^{-1}) \text{ for all } w \in \Omega(T(\mathbb{R}), G(\mathbb{R})),
\end{align}
where $\Omega(T(\mathbb{R}), G(\mathbb{R}))$ denotes the real Weyl group
$$N_{G(\mathbb{R})}(T)/T(\mathbb{R}).$$

For $\gamma \in T_{\text{reg}}(\mathbb{R})$ define subsets $R_\gamma$ and $R_\gamma^+$ of $R$ by

$$R_\gamma := \{ \alpha \in R \mid \alpha \text{ is real and } \alpha(\gamma) > 0 \},$$
$$R_\gamma^+ := \{ \alpha \in R \mid \alpha \text{ is real and } \alpha(\gamma) > 1 \}.$$

Note that $R_\gamma$ is a root system and that $R_\gamma^+$ is a positive system in $R_\gamma$. Moreover $R_\gamma$ depends only on the connected component $\Gamma$ of $T(\mathbb{R})$ in which $\gamma$ lies; thus we sometimes write $R_\Gamma$ instead of $R_\gamma$. Harish-Chandra [HC, Lemma 25] showed that

(4.3) $$n(\gamma_1, B) = n(\gamma_2, B) \text{ if } \Gamma_1 = \Gamma_2 \text{ and } R_{\gamma_1}^+ = R_{\gamma_2}^+,$$

where $\Gamma_i$ denotes the connected component of $T(\mathbb{R})$ in which $\gamma_i$ lies.

Of course any finite $\mathbb{Z}$-linear combination $\Theta$ of characters $\Theta_\pi$ as above can also be expressed in the form (4.1) for integers $n(\gamma, B)$ satisfying (4.2) and (4.3) (we refer to $\Theta$ as a virtual character on $G(\mathbb{R})$). We are particularly interested in virtual characters $\Theta$ on $G(\mathbb{R})$ that are stable in the sense that

$$\Theta(\gamma) = \Theta(\gamma')$$

whenever $\gamma, \gamma' \in G_{\text{reg}}(\mathbb{R})$ are stably conjugate. A virtual character $\Theta$ is stable if and only if the integers $n(\gamma, B)$ satisfy the following strengthening of (4.2) (for all $T$):

(4.4) $$n(\gamma, B) = n(w\gamma w^{-1}, wBw^{-1}) \text{ for all } w \in W(\mathbb{R}),$$

where $W$ is the Weyl group of $T_C$ in $G_C$ and $W(\mathbb{R})$ is the subgroup of $W$ consisting of all elements that are fixed by complex conjugation (of course $W(\mathbb{R})$ contains $\Omega(T(\mathbb{R}), G(\mathbb{R}))$).

Let $A$ be the maximal split subtorus of $T$ and let $M$ be the centralizer of $A$ in $G$, a Levi subgroup of $G$. As usual for $\gamma \in M(\mathbb{R})$ we define a real number $D_G^M(\gamma)$ by

$$D_G^M(\gamma) = \det(1 - \text{Ad}(\gamma); \text{Lie}(G)/\text{Lie}(M)).$$

We will need the following result of Arthur [A] and Shelstad.

**Lemma 4.1.** For any stable virtual character $\Theta$ on $G(\mathbb{R})$ the function

$$\gamma \mapsto |D_G^M(\gamma)|^{1/2}\Theta(\gamma)$$

on $T_{\text{reg}}(\mathbb{R})$ extends continuously to $T(\mathbb{R})$.

Let $\Gamma$ be a connected component of $T(\mathbb{R})$ and let $\Gamma_{\text{reg}}$ denote its intersection with $T_{\text{reg}}(\mathbb{R})$. To prove the lemma we must show that

$$|D_G^M(\gamma)|^{1/2}\Theta(\gamma)$$
extends continuously from $\Gamma_{\text{reg}}$ to $\Gamma$. Pick an element $a \in \Gamma$ such that $a^2 = 1$ (it is easy to see that such an element exists). The root system $R_{\Gamma}$ defined above is equal to the set of real roots $\alpha \in R$ such that $\alpha(a) = 1$; thus $a$ lies in the center of the connected reductive subgroup of $G$ containing $T$ with root system $R_{\Gamma}$, and we conclude that $a$ is fixed by the Weyl group $W(R_{\Gamma})$ of $R_{\Gamma}$. Thus $\Gamma$ is fixed by the subgroup $W(R_{\Gamma})$ of $\Omega(T(\mathbb{R}), G(\mathbb{R}))$, and since both $|D_M^G(\gamma)|^{1/2}$, $\Theta(\gamma)$ are invariant under $\Omega(T(\mathbb{R}), G(\mathbb{R}))$ (which normalizes $M$), it follows that the function

$$\frac{|D_M^G(\gamma)|^{1/2}}{\Theta(\gamma)}$$

on $\Gamma_{\text{reg}}$ is invariant under $W(R_{\Gamma})$. Let $T_c$ denote the maximal anisotropic subtorus of $T$. Then

$$\Gamma = a \cdot T_c(\mathbb{R}) \cdot \exp(\mathfrak{a}),$$

where

$$\mathfrak{a} = X_*(A)_{\mathbb{R}} = \text{Lie}(A(\mathbb{R})).$$

The Weyl group $W(R_{\Gamma})$ fixes $T_c(\mathbb{R})$ as well as $a$. Fix a positive system $R_{\Gamma}^+$ in $R_{\Gamma}$ and let $\mathcal{C}$ be the corresponding closed chamber in $\mathfrak{a}$. Then $\overline{\mathcal{C}}$ is a closed fundamental domain for the action of $W(R_{\Gamma})$ on $\mathfrak{a}$, and therefore a $W(R_{\Gamma})$-invariant function on $\Gamma$ is continuous if and only if its restriction to

$$\Gamma^+ := a \cdot T_c(\mathbb{R}) \cdot \exp(\overline{\mathcal{C}})$$

is continuous. Therefore it is enough to show that

$$\frac{|D_M^G(\gamma)|^{1/2}}{\Theta(\gamma)}$$

extends continuously to $\Gamma^+$. For any regular element $\gamma \in \Gamma^+$ we have $R_{\gamma}^+ = R_{\Gamma}^+$, and thus there are integers $m(B)$ ($B \in \mathcal{B}(T)$) such that for all regular $\gamma \in \Gamma^+$

$$\Theta(\gamma) = \sum_{B \in \mathcal{B}(T)} m(B) \lambda_B(\gamma) \Delta_B(\gamma)^{-1}.$$

The Weyl group $W_M$ of $T_C$ in $M_C$ is a subgroup of $W(\mathbb{R})$, and this subgroup fixes $A$ pointwise and hence preserves $\Gamma$ and $R_{\Gamma}^+$. Therefore it follows from (4.4) that

$$m(B) = m(wBw^{-1}) \quad \text{for all } w \in W_M. \quad (4.5)$$

Choose a parabolic subgroup $P = MN$ having $M$ as Levi component and having the property that every element of $R_{\Gamma}$ that appears in $\text{Lie}(N)$ is non-negative on $\overline{\mathcal{C}}$ (here $N$ denotes the unipotent radical of $P$). Put

$$\Delta_P = \prod_{\alpha}(1 - \alpha^{-1})$$
where \( \alpha \) runs through the roots of \( T \) in \( \text{Lie}(N) \). We claim that \( \Delta_P(\gamma) \) is non-negative for all \( \gamma \in \Gamma^+ \). Indeed, complex conjugation preserves the set of roots of \( T \) in \( \text{Lie}(N) \). If the complex conjugate \( \bar{\alpha} \) is different from \( \alpha \), then the contribution of \( \alpha, \bar{\alpha} \) to \( \Delta_P \) is \((1 - \alpha(\gamma)^{-1}) \) times its complex conjugate; this contribution is certainly non-negative. If \( \alpha \) is real, then \( \alpha(a) = \pm 1 \). If \( \alpha(a) = -1 \), then \( \alpha(\gamma)^{-1} \) is negative and therefore \( 1 - \alpha(\gamma)^{-1} \) is positive. If \( \alpha(a) = 1 \), then \( \alpha \in R_T \) and by our choice of \( P \) we have \( \alpha(\gamma) \geq 1 \), so that \( 1 - \alpha(\gamma)^{-1} \geq 0 \). It follows from the claim that

\[
|D_M^G(\gamma)|^{1/2} = \Delta_P(\gamma) \cdot \delta_P^{1/2}(\gamma),
\]

where \( \delta_P \) denotes the modulus character

\[
\delta_P(x) := |\det(x; \text{Lie}(N))|
\]
on \( M(\mathbb{R}) \). Therefore it is enough to show that

\[
\sum_{B \in B(T)} m(B) \cdot \Delta_P(\gamma) \cdot \lambda_B(\gamma) \cdot \Delta_B(\gamma)^{-1}
\]
extends continuously to \( \Gamma^+ \). But it follows immediately from (4.5) that this last expression is a linear combination of characters of irreducible finite dimensional representations of \( M \), and of course such a linear combination extends continuously to \( \Gamma^+ \) (and even to all of \( T(\mathbb{R}) \)). This completes the proof of the lemma.

For any stable virtual character \( \Theta \) on \( G(\mathbb{R}) \) we denote by \( \Phi_M^G(\gamma, \Theta) \) the (unique) continuous extension of

\[
|D_M^G(\gamma)|^{1/2} \Theta(\gamma)
\]
to \( T(\mathbb{R}) \) whose existence is asserted in the lemma we just proved. Sometimes it is convenient to extend \( \Phi_M^G(\gamma, \Theta) \) to a function on the set of all elliptic elements in \( M(\mathbb{R}) \) (in other words, the set of \( M(\mathbb{R}) \)-conjugates of elements in \( T(\mathbb{R}) \)) by taking the unique extension that is invariant under conjugation by \( M(\mathbb{R}) \).

The functions \( \Phi_M^G(\gamma, \Theta) \) behave simply under induction. Let \( Q = LU \) be a parabolic subgroup of \( G \) with Levi subgroup \( L \) and unipotent radical \( U \). Let \( \Theta_L \) be a stable virtual character on \( L(\mathbb{R}) \) and let \( \Theta = i^L_L(\Theta_L) \) be the virtual character on \( G(\mathbb{R}) \) obtained from \( \Theta_L \) by the usual normalized parabolic induction. Let \( T \subset M \subset G \) be as above. Then \( \Theta \) is stable and for all \( \gamma \in T(\mathbb{R}) \)

\[
(4.6) \quad \Phi_M^G(\gamma, \Theta) = \sum_{gL(\mathbb{R})} \Phi_{gL(\mathbb{R})}^{-1}(\gamma, \Theta_{gLg^{-1}}),
\]

where the sum runs over the set of cosets \( gL(\mathbb{R}) \) of \( L(\mathbb{R}) \) in \( G(\mathbb{R}) \) such that \( gLg^{-1} \supset M \), and where \( \Theta_{gLg^{-1}} \) denotes the virtual character on \( gL(\mathbb{R})g^{-1} \) obtained from \( \Theta_L \) on \( L(\mathbb{R}) \) via the isomorphism

\[
\text{Int}(g) : L \rightarrow gLg^{-1}
\]
(\text{Int}(g)(x) := gxg^{-1}). For regular $\gamma$ in $T(\mathbb{R})$ the formula (4.6) is just the usual formula for the character of a parabolically induced representation, and by continuity the formula remains valid on all of $T(\mathbb{R})$.

We finish this section by discussing stable discrete series characters. We now assume that there exists an elliptic maximal torus $T_e$ in $G$ (elliptic means that $T_e/Z$ is anisotropic, where $Z$ denotes the center of $G$). As usual we let $q(G)$ denote half the dimension of the symmetric space associated to the adjoint group of $G$ (our hypothesis on $G$ guarantees that this dimension is even). Let $\Pi$ be the L-packet consisting of all (isomorphism classes of) discrete series representations of $G(\mathbb{R})$ having the same infinitesimal and central characters as the finite dimensional representation $E$. Put

$$\Theta^E = (-1)^{q(G)} \sum_{\pi \in \Pi} \Theta_\pi,$$

where $\Theta_\pi$ denotes the character of $\pi$; then the virtual character $\Theta^E$ is stable [HC, Lemma 61], [S]. Any discrete series representation $\pi$ of $G(\mathbb{R})$ is obtained by induction from a discrete series representation of the normal subgroup

$$Z(\mathbb{R}) \text{im}[G_{sc}(\mathbb{R}) \to G(\mathbb{R})],$$

where $G_{sc}$ denotes the simply connected cover of the derived group $G_{\text{der}}$ of $G$. Therefore $\Theta^E$ is supported on this normal subgroup (of finite index).

Let $A \subset T \subset M \subset G$ be as above. Let $\gamma \in T_{\text{reg}}(\mathbb{R})$ and define $R_\gamma$ as above. The character value $\Theta^E(\gamma)$ is given by (4.1) for certain integers $n(\gamma, B)$. We will now review how these integers are related to the stable discrete series constants discussed in §3. Let $T_c$ denote the maximal anisotropic subtorus in $T$; note that $T = AT_c$. Let $L$ denote the centralizer of $T_e$ in $G$; then $L_C$ is a Levi subgroup of $G_C$ and $L$ contains $T$. Note that the roots of $T$ in $L$ are precisely the real roots of $T$.

Let $T(\mathbb{R})_1$ denote the maximal compact subgroup of $T(\mathbb{R})$. Then $T_c(\mathbb{R})$ is the identity component of $T(\mathbb{R})_1$, and there is a direct product decomposition

$$T(\mathbb{R}) = A(\mathbb{R})^0 \times T(\mathbb{R})_1.$$

We decompose our regular element $\gamma \in T(\mathbb{R})$ according to the decomposition (4.7):

$$\gamma = \exp(x) \cdot \gamma_1$$

for uniquely determined elements $x$ in $X_*(A)_{\mathbb{R}} = \text{Lie}(A)$ and $\gamma_1$ in $T(\mathbb{R})_1$. Let $J$ denote the identity component of the centralizer of $\gamma_1$ in $L$. The root system of $T$ in $J$ is precisely $R_\gamma$.

Of course we may as well assume that $\gamma$ belongs to

$$Z(\mathbb{R}) \text{im}[G_{sc}(\mathbb{R}) \to G(\mathbb{R})];$$

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otherwise $n(\gamma, B) = 0$ for all $B \in \mathcal{B}(T)$. In this case we claim that $-1$ belongs to the Weyl group of $R_\gamma$. In proving the claim we may as well assume that $\gamma$ lies in the image of $G_{sc}(\mathbb{R})$, and therefore we may as well assume that $G_{sc} = G$. Replacing $T_e$ by a conjugate, we may assume that $T_e$ is contained in $T_e$; then $T_e$ is contained in $L$. Therefore the connected center of $L$ is equal to $T_e$ (it contains $T_e$ and is contained in both $T$ and $T_e$). Moreover the maximal compact subgroups of $L(\mathbb{R})$ are connected since the derived group $L_{der}$ of $L$ is simply connected and $L/L_{der}$ is anisotropic. It follows that by conjugating $T$ in $L$ we may assume that $\gamma_1$ belongs to $T_e$ and hence that $T_e$ is contained in $J$. Therefore the connected center of $J$ is also equal to $T_e$. The maximal torus $T$ in $J$ is split modulo the connected center $T_e$ of $J$, and therefore its split component $A$ is a split maximal torus in $J_{der}$. But $J$ contains an anisotropic maximal torus, namely $T_e$, and therefore $-1 \in W(R_\gamma)$.

Thus the root system $R_\gamma$ in $X^*(A/\mathfrak{g})_\mathbb{R}$ is of the type considered in §3, and from this root system we obtain an integer-valued function $c$ on

$$(X_*(A/\mathfrak{g})_\mathbb{R})_{reg} \times (X^*(A/\mathfrak{g})_\mathbb{R})_{reg}$$

(see §3). The integer $n(\gamma, B)$ is given by

$$n(\gamma, B) = c(x, p(\lambda_B + \rho_B - \lambda_0))$$

where

$$p : X^*(T)_\mathbb{R} \to X^*(A)_\mathbb{R}$$

is the natural restriction map and $\lambda_0 \in X^*(T)_\mathbb{R}$ is obtained from the character $\lambda_0 \in X^*(\mathfrak{g})$ by which $\mathfrak{g}$ acts on $E$ by viewing $X^*(\mathfrak{g})_\mathbb{R}$ as a direct summand of $X^*(T)_\mathbb{R}$ in the usual way.

5. The stable virtual characters $\Theta_\nu$

Let $F$ be a subfield of $\mathbb{R}$ (the two examples we have in mind are $\mathbb{Q}$ and $\mathbb{R}$). Let $G$ be a connected reductive group over $F$. By a parabolic subgroup of $G$ we mean a parabolic subgroup of $G$ defined over $F$, and by a Levi subgroup of $G$ we mean a Levi component defined over $F$ of some parabolic subgroup of $G$.

Let $M$ be a Levi subgroup of $G$. We write $\mathcal{F}^G(M)$ for the set of parabolic subgroups of $G$ containing $M$, and we write $\mathcal{P}^G(M)$ for the subset of $\mathcal{F}^G(M)$ consisting of those parabolic subgroups for which $M$ is a Levi component; we often abbreviate $\mathcal{F}^G(M), \mathcal{P}^G(M)$ to $\mathcal{F}(M), \mathcal{P}(M)$. Let $A_M$ denote the maximal $F$-split torus in the center of $M$, and write $\mathfrak{a}_M$ for the real vector space

$$\mathfrak{a}_M := X_*(A_M)_\mathbb{R},$$

where the subscript $\mathbb{R}$ indicates that we have tensored $X_*(A_M)$ over $\mathbb{Z}$ with $\mathbb{R}$. By a root of $A_M$ we mean a non-zero weight of $A_M$ in $\text{Lie}(G)$. Any $P \in \mathcal{P}(M)$ determines a chamber $C_P$ in $\mathfrak{a}_M$, consisting of the points $x \in \mathfrak{a}_M$ such that

$$\langle x, \alpha \rangle > 0$$

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for every root of $A_M$ in $\text{Lie}(N)$, where $N$ denotes the unipotent radical of $P$. The map $P \mapsto C_P$ is a bijection from $\mathcal{P}(M)$ to the set of chambers in $\mathfrak{A}_M$ (a chamber in $\mathfrak{A}_M$ is a connected component of the complement in $\mathfrak{A}_M$ of the union of the root hyperplanes in $\mathfrak{A}_M$). Let $Q \in \mathcal{F}(M)$ and let $L$ denote the unique Levi component of $Q$ containing $M$. Then $\mathfrak{A}_L$ is a subspace of $\mathfrak{A}_M$, so that $C_Q$ can be regarded as a cone in $\mathfrak{A}_M$. Moreover $\mathfrak{A}_M$ is equal to the disjoint union

$$\mathfrak{A}_M = \bigsqcup_{Q \in \mathcal{F}(M)} C_Q.$$ 

We fix a minimal parabolic subgroup $P_0$ of $G$ and fix a Levi component $M_0$ of $P_0$ over $F$. We say that a parabolic subgroup $P$ of $G$ is standard if it contains $P_0$, and we say that $P$ is semistandard if it contains $M_0$. Thus $\mathcal{F}(M_0)$ is the set of semistandard parabolic subgroups. Given a semistandard parabolic subgroup $P$ of $G$, we write $N_P$ for the unipotent radical of $P$ and $M_P$ for the unique Levi component of $P$ containing $M_0$. We will often write $A_P, A_P$ rather than $A_{M_P}, A_{M_P}$. In fact we will often abbreviate $M_P, N_P$ to $M, N$, so that $P = MN$.

When we use $Q$ to denote a semistandard parabolic subgroup, we will often write $L, U$ instead of $M_Q, N_Q$, so that $Q = LU$.

Let $E$ be an irreducible representation of the algebraic group $G$ on a finite dimensional complex vector space, and let $\nu$ be an element in $(\mathfrak{A}_{P_0})^*$, the real vector space dual to $\mathfrak{A}_{P_0}$, such that the restriction of $\nu$ to $\mathfrak{A}_G$ coincides with the character by which $A_G$ acts on $E$ (this character is an element of $X^*(A_G)$, a lattice in $\mathfrak{A}_G^*$).

We are going to use $E, \nu$ to define a virtual representation of the real group $G(\mathbb{R})$, the character of which we will denote by $\Theta_\nu$. The first step is to define an element $\nu_P \in \mathfrak{A}_P^*$ for each semistandard parabolic subgroup $P = MN$. There is a unique standard parabolic subgroup $P' = M'N'$ conjugate to $P$ under $G(F)$. There exists $g \in G(F)$, unique up to right multiplication by $M(F)$, such that $gPg^{-1} = P'$ and $gMg^{-1} = M'$, and the inner automorphism $x \mapsto gxg^{-1}$ of $G$ induces isomorphisms

$$A_P \simeq A_{P'},$$

$$\mathfrak{A}_P \simeq \mathfrak{A}_{P'}$$

independent of the choice of $g$. Let $\nu' \in \mathfrak{A}_{P'}^*$ be the restriction of the linear form $\nu$ to the subspace $\mathfrak{A}_{P'}$ of $\mathfrak{A}_{P_0}$, and then use the isomorphism $\mathfrak{A}_P \simeq \mathfrak{A}_{P'}$ to transport $\nu'$ over to an element $\nu_P \in \mathfrak{A}_P^*$. This completes the definition of $\nu_P$. It is easy to see that if $P, Q \in \mathcal{F}^G(M_0)$ and $P \subset Q$, then $\nu_Q$ is the image of $\nu_P$ under the natural restriction map

$$\mathfrak{A}_P^* \to \mathfrak{A}_Q^*.$$
The next step is to use $E, \nu$ to define a virtual finite dimensional complex representation $E'_\nu$ of $M$ for any semistandard parabolic subgroup $P = MN$. We begin by considering the Lie algebra cohomology groups

$$H^i(\text{Lie}(N), E);$$

these are finite dimensional complex representations of $M$ (we use the usual left action of $M$). Of course the action of the split torus $A_P$ on $H^i(\text{Lie}(N), E)$ decomposes this space as a direct sum of weight subspaces

$$H^i(\text{Lie}(N), E)_{\mu},$$

where $\mu$ runs through $X^*(A_P)$ (a lattice in $\mathfrak{a}^*_P$). We write $C^*_P$ for the closed convex cone in $\mathfrak{a}^*_P$ dual to $C_P$; thus $C^*_P$ consists of all $\mu \in \mathfrak{a}^*_P$ such that $\langle x, \mu \rangle \geq 0$ for all $x \in C_P$. We write

$$H^i(\text{Lie}(N), E)_{\geq \nu_P}$$

for the subspace of $H^i(\text{Lie}(N), E)$ obtained by taking the direct sum of all the weight spaces $H^i(\text{Lie}(N), E)_{\mu}$ for $\mu \in X^*(A_P)$ such that $\mu - \nu_P \in C^*_P$; of course

$$H^i(\text{Lie}(N), E)_{\geq \nu_P}$$

is stable under the action of $M$. We write $E'_\nu$ for the virtual $M$-module

$$\sum_i (-1)^i H^i(\text{Lie}(N), E)_{\geq \nu_P}.$$

We now use a theorem of Kostant [Ko] to express $E'_\nu$ in terms of irreducible representations of $M$. Let $T$ be a maximal torus of $M$ over $\mathbb{C}$ and let $B_M$ be a Borel subgroup of $M$ over $\mathbb{C}$ containing $T$. For any $B_M$-dominant weight $\mu \in X^*(T)$ we let $V^M_\mu$ be an irreducible finite dimensional complex representation of $M$ with highest weight $\mu$. The set of Borel subgroups $B$ of $G$ over $\mathbb{C}$ containing $T$ and contained in $P$ is in natural bijection with the set of Borel subgroups of $M$ over $\mathbb{C}$ containing $T$. Thus our choice of $B_M$ determines a Borel subgroup $B$ of $G$ over $\mathbb{C}$ containing $T$ and contained in $P$, characterized by the equality

$$B_M = B \cap M.$$ 

Let $R$ (respectively, $R_M$) be the set of roots of $T$ in $G$ (respectively, $M$). The Borel subgroups $B, B_M$ determine positive systems $R^+, R^+_M$ in $R, R_M$, and of course

$$R^+_M = R_M \cap R^+. $$

Let $\lambda_B \in X^*(T)$ denote the highest weight (with respect to $B$) of the irreducible representation $E$ of $G$, and let $\rho_B \in X^*(T)_R$ denote half the sum of the roots in $R^+$. Let $W$ (respectively, $W_M$) denote the Weyl group of $T_\mathbb{C}$ in $G_\mathbb{C}$ (respectively, $M_\mathbb{C}$). Let
$W'$ denote the set of Kostant representatives for the cosets $W_M \setminus W$; thus $W'$ consists of the elements $w \in W$ such that
\[ w^{-1}(R^+_M) \subset R^+ \]
(obviously $W'$ depends on the choice of $B_M$). Kostant’s theorem on Lie($N$)-cohomology states that as an $M$-module $H^i(Lie(N), E)$ is isomorphic to
\[ \bigoplus_w V^M_{w(\lambda_B + \rho_B) - \rho_B} \]
where $w$ runs through the set of Kostant representatives of length $i$ (we use the length function on $W$ determined by $B$). Note that the weight $w(\lambda_B + \rho_B) - \rho_B$ is indeed $B_M$-dominant for any Kostant representative $w$.

Let
\[ \epsilon : W \to \{ \pm 1 \} \]
be the usual sign function on $W$ ($\epsilon(w)$ is $(-1)^l(w)$, where $l(w)$ denotes the length of $w$). We see from Kostant’s theorem that the virtual representation $E'_P$ is given by
\[ \sum_{w \in W'} \epsilon(w) \cdot \xi_{C_P^*}(p_M(w(\lambda_B + \rho_B) - \rho_B) - \nu_P) \cdot V^M_{w(\lambda_B + \rho_B) - \rho_B} \]
where $\xi_{C_P^*}$ denotes the characteristic function of the subset $C_P^*$ of $\mathfrak{a}_M^*$ and $p_M$ denotes the restriction map
\[ X^*(T)_\mathbb{R} = (X_*(T)_\mathbb{R})^* \to \mathfrak{a}_M^* \]
induced by the inclusion of $A_M$ in $T$.

Now we are ready to define the virtual character $\Theta_{\nu}$ on the real group $G(\mathbb{R})$. For any semistandard parabolic subgroup $P = MN$ we write $\delta_P$ for the modulus quasicharacter on $M(\mathbb{R})$ given by
\[ \delta_P(x) = |\det(x; Lie(N))| \]
for $x \in M(\mathbb{R})$. We write $(E'_P)^*$ for the contragredient of the virtual representation $E'_P$. Then
\[ \delta_P^{-1/2} \otimes (E'_P)^* \]
is a virtual representation of the real group $M(\mathbb{R})$, which we may induce from $P(\mathbb{R})$ to $G(\mathbb{R})$ to obtain a virtual representation
\[ i_P^G(\delta_P^{-1/2} \otimes (E'_P)^*) \]
of $G(\mathbb{R})$. We are using the usual normalized parabolic induction, which builds in a factor of $\delta_P^{1/2}$; if we used unnormalized induction we would simply be inducing $(E'_P)^*$ from $P(\mathbb{R})$ to $G(\mathbb{R})$. We write $\Theta_P^\nu$ for the (Harish-Chandra) character of
\[ i_P^G(\delta_P^{-1/2} \otimes (E'_P)^*) \]
and define a virtual character $\Theta_\nu$ on $G(\mathbb{R})$ by putting

$$\Theta_\nu := \sum_P (-1)^{\dim(A_P/A_G)} \Theta_\nu^P,$$

where $P$ runs over the set of standard parabolic subgroups of $G$. Note that $\Theta_\nu$ is stable. Indeed, the character of $E_\nu^\nu$ is obviously stable on $M(\mathbb{R})$, and stability is preserved by parabolic induction.

Now we fix a Levi subgroup $M$ of $G$ containing $M_0$, and we assume that $M_\mathbb{R}$ contains a maximal torus $T$ over $\mathbb{R}$ such that $T/A_M$ is anisotropic over $\mathbb{R}$. It follows that $A_M$ coincides with the maximal $\mathbb{R}$-split torus in the center of $M$, and this in turn implies that any parabolic subgroup of $G$ over $\mathbb{R}$ containing $M$ is automatically defined over $F$. Note that $A_M$ is the maximal $\mathbb{R}$-split torus in $T$ and that $T$ is elliptic in $M_\mathbb{R}$. The discussion following Lemma 4.1 applies to the stable character $\Theta_\nu$, and thus we obtain a continuous function $\Theta_\nu^G_{\mathbb{M}}(\gamma, \Theta_\nu)$ on $T(\mathbb{R})$. Sometimes we abbreviate $\Theta_\nu^G_{\mathbb{M}}(\gamma, \Theta_\nu)$ to $\Theta_\nu^{\mathbb{M}}(\gamma, \Theta_\nu)$.

We are now going to use $E;\nu$ to define another function $L_\nu^\nu(M)$ on $T(\mathbb{R})$ (with $M$ and $T$ as above); we will see in §7 that this function arises naturally in the Lefschetz trace formula for Hecke operators. Once the definition is complete our goal will be to show that $\Theta_\nu^G_{\mathbb{M}}(\gamma, \Theta_\nu)$ is in fact equal to $L_\nu^\nu(M)$. Let $\gamma \in T(\mathbb{R})$. There is a direct product decomposition

$$T(\mathbb{R}) = A_M(\mathbb{R})^0 \times T(\mathbb{R})_1,$$

where $T(\mathbb{R})_1$ denotes the maximal compact subgroup of $T(\mathbb{R})$. Therefore we can write $\gamma$ as

$$\gamma = \exp(x) \cdot \gamma_1$$

for unique elements $x \in A_M$ and $\gamma_1 \in T(\mathbb{R})_1$. The complex number $L_\nu^\nu(M)$ that we are in the process of defining has the form

$$L_\nu^\nu(M) := \sum_Q (-1)^{\dim(A_L/A_G)} \cdot |D^L_M(\gamma)|^{1/2} \cdot \delta^{-1/2}_Q(\gamma) \cdot L^\nu_Q(\gamma)$$

where the sum runs over $Q = LU$ in $\mathcal{F}(M)$ such that $x$ is contained in the subspace $\mathfrak{A}_L$ of $\mathfrak{A}_M$ and where $L^\nu_Q(\gamma)$ is a complex number we have yet to define. The factor $D^L_M(\gamma)$ was defined in §4, just before Lemma 4.1.

In order to define $L^\nu_Q(\gamma)$ we choose a Borel subgroup $B$ of $G$ over $\mathbb{C}$ containing $T$ and contained in $Q$, and we put $B_L := B \cap L$, a Borel subgroup of $L$ over $\mathbb{C}$ containing $T$; it turns out that $L^\nu_Q(\gamma)$ is independent of this choice. We now use the same notational system as we used when discussing Kostant’s theorem (though we are now using $Q, L$ instead of $P, M$). In particular we have the set $W'$ of Kostant representatives for the cosets $W_L \setminus W$, the irreducible representations

$$V^L_{\nu(\lambda_B + \rho_B) - \rho_B}$$

of $L$, and the restriction map

$$p_L : X^*(T)_{\mathbb{R}} \rightarrow \mathfrak{A}^*_L.$$
The open convex polyhedral cone $C_Q$ in $\mathfrak{A}_L$ determines a function

$$\varphi_{C_Q}(\cdot, \cdot)$$
on

on $\mathfrak{A}_L \times \mathfrak{A}_L^*$, as in the last part of Appendix A, and we will denote this function simply by $\varphi_Q(\cdot, \cdot)$. We define $L_Q^\nu(\gamma)$ by

$$L_Q^\nu(\gamma) := (-1)^{\dim(\mathfrak{A}_L)} \sum_{w \in W'} \epsilon(w) \cdot \text{tr}(\gamma^{-1}; V^L_{\rho_B} - \rho_B) \cdot \varphi_Q(-x, p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q).$$

**Theorem 5.1.** The two functions $\Phi_M(\gamma, \Theta_{\nu})$ and $L_M^\nu(\gamma)$ on $T(\mathbb{R})$ are equal.

By definition $\Phi_M(\gamma, \Theta_{\nu})$ is given by

$$\sum_Q (-1)^{\dim(\mathfrak{A}_L/\mathfrak{A}_G)} \cdot \Phi_M(\gamma, \Theta_{Q}^\nu)$$

where $Q = LU$ runs over the set of standard parabolic subgroups of $G$. Applying equation (4.6) to the induced character $\Theta_{Q}^\nu$ of $G(\mathbb{R})$, we see that $\Phi_M(\gamma, \Theta_{\nu})$ is equal to

$$\sum_Q (-1)^{\dim(\mathfrak{A}_L/\mathfrak{A}_G)} \sum_{Q'} \Phi_M^L(\gamma, \delta_Q^{-1/2} \otimes (E_Q^\nu)^*)$$

where the index set for the first sum is the same as before and the index set for the second sum is the set of parabolic subgroups $Q'$ over $\mathbb{R}$ containing $M$ such that $Q'$ is conjugate under $G(\mathbb{R})$ to $Q$. Since, as we remarked earlier, every parabolic subgroup of $G$ over $\mathbb{R}$ containing $M$ is automatically defined over $F$, we see that $\Phi_M(\gamma, \Theta_{\nu})$ is equal to

$$\sum_{Q \in \mathcal{F}(M)} (-1)^{\dim(\mathfrak{A}_L/\mathfrak{A}_G)} \Phi_M^L(\gamma, \delta_Q^{-1/2} \otimes (E_Q^\nu)^*)$$

(as usual $Q = LU$). Recalling the expression for $E_Q^\nu$ that we found using Kostant’s theorem, we see that $\Phi_M(\gamma, \Theta_{\nu})$ is equal to

$$\sum_{Q \in \mathcal{F}(M)} (-1)^{\dim(\mathfrak{A}_L/\mathfrak{A}_G)} \cdot |D_M^L(\gamma)|^{1/2} \cdot \delta_Q^{-1/2}(\gamma) \cdot \sum_{w \in W'} \epsilon(w) \cdot \text{tr}(\gamma^{-1}; V^L_{\rho_B} - \rho_B) \cdot \xi_{C_Q}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q).$$

The notation here is the same as that used during our discussion of Kostant’s theorem. In particular, given $Q \in \mathcal{F}(M)$ we must choose a Borel subgroup $B$ of $G$ over $\mathbb{C}$ containing $T$ and contained in $Q$ in order to define $W'$, the set of Kostant representatives.
Let $Q = LU$ be a parabolic subgroup in $\mathcal{F}(M)$. As usual $\mathfrak{A}_M$ is a disjoint union of convex cones $C_{Q'}$, one for each $Q' \in \mathcal{F}(M)$. But $M$ is also a Levi subgroup of $L$, and therefore $\mathfrak{A}_M$ is also a disjoint union of convex cones $C_{Q''}$, one for each $Q'' \in \mathcal{F}_L(M)$. For any $Q' \in \mathcal{F}(M)$ such that $Q' \subset Q$ we put $Q'' := Q' \cap L$, an element of $\mathcal{F}_L(M)$. The map $Q' \mapsto Q''$ sets up a bijection

$$\{Q' \in \mathcal{F}(M) \mid Q' \subset Q\} \cong \mathcal{F}_L(M),$$

and the convex cones $C_{Q'}, C_{Q''}$ in $\mathfrak{A}_M$ are related by the equality

$$C_{Q''} = C_{Q'} + \mathfrak{A}_L.$$

Recall that we have written $\gamma$ as

$$\gamma = \exp(x) \cdot \gamma_1$$

for uniquely determined $x \in \mathfrak{A}_M$ and $\gamma_1 \in T(\mathbb{R})_1$. For each parabolic subgroup $Q \in \mathcal{F}(M)$ we denote by $Q' = L'U'$ the unique element of $\mathcal{F}(M)$ such that

1. $Q' \subset Q$, and
2. $-x \in C_{Q'} + \mathfrak{A}_L$.

It follows from the second condition that $x$ belongs to $\mathfrak{A}_{L'}$.

Now let $Q_1 = L_1U_1$ be an element of $\mathcal{F}(M)$ such that $x \in \mathfrak{A}_{L_1}$. Pick a Borel subgroup $B$ in $G$ over $\mathbb{C}$ containing $T$ and contained in $Q_1$. We are interested in the terms in (5.4) indexed by parabolic subgroups $Q \in \mathcal{F}(M)$ such that $Q' = Q_1$. We have inclusions

$$T \subset B \subset Q_1 \subset Q,$$

so that we can (and do) use $B$ to define the set $W'$ of Kostant representatives for the cosets $W_L \backslash W_G$. As before we write $Q''$ for the element $Q_1 \cap L$ of $\mathcal{F}_L(M)$. Define a function $\Delta_{Q''}^L$ on $T(\mathbb{R})$ (a partial Weyl denominator for the group $L$) by

$$\Delta_{Q''}^L(\gamma) = \prod_{\alpha} (1 - \alpha(\gamma)^{-1})$$

where $\alpha$ runs through the set of roots of $T$ in Lie($N''$) ($N''$ denotes the unipotent radical of $Q''$). We claim that $\Delta_{Q''}^L(\gamma^{-1})$ is a non-negative real number (for $\gamma, Q, Q_1, Q''$ as in the discussion preceding the definition of $\Delta_{Q''}^L$). Since $Q''$ is defined over $\mathbb{R}$, the set of roots $\alpha$ of $T$ in Lie($N''$) is stable under complex conjugation. Complex conjugate pairs $\bar{\alpha}, \alpha$ with $\bar{\alpha} \neq \alpha$ make a non-negative contribution to $\Delta_{Q''}^L$, since $1 - \bar{\alpha}(\gamma)$ is complex conjugate to $1 - \alpha(\gamma)$. Let $\alpha$ be a root of $T$ in Lie($N''$) such that $\bar{\alpha} = \alpha$. It is enough to show that $1 - \alpha(\gamma)$ is non-negative. Since $Q' = Q_1$, the element $-x$ belongs to $C_{Q''}$, which implies that

$$\langle x, \alpha \rangle \leq 0.$$
Since $\alpha = \tilde{\alpha}$, the value of $\alpha$ on any element of $T(\mathbb{R})_1$ is $\pm 1$. Therefore $\alpha(\gamma_1) = \pm 1$ and

$$\alpha(\gamma) = \exp(\langle x, \alpha \rangle) \cdot \alpha(\gamma_1) \leq 1,$$

as desired.

It follows from the claim that

$$|D_{L_1}^L(\gamma)|^{1/2} \cdot \delta_q^{-1/2}(\gamma) = \delta_{q_1}^{-1/2}(\gamma) \cdot \Delta_{q''}(\gamma^{-1}).$$

Moreover, applying Kostant’s theorem to $L$ and its parabolic subgroup $Q''$, it is easy to see that for $w \in W''$

$$\Delta_{q''}(\gamma^{-1}) \cdot \text{tr}(\gamma^{-1}; V_{w(\lambda_B+\rho_B)-\rho_B}^L)$$

is equal to

$$\sum_{u \in W'_L} \epsilon(u) \cdot \text{tr}(\gamma^{-1}; V_{uw(\lambda_B+\rho_B)-\rho_B}^{L_1})$$

where $W'_L$ is the set of Kostant representatives for the cosets $W_{L_1} \setminus W_L$ (relative to the Borel subgroup $B \cap L$ of $L$). It is also easy to see that

$$W'_L W'$$

is the set $W''$ of Kostant representatives for the cosets $W_{L_1} \setminus W$ (relative to the Borel subgroup $B$ of $G$) and that for $w \in W'$, $u \in W'_L$

$$p_L(uw(\lambda_B + \rho_B)) = p_L(w(\lambda_B + \rho_B)).$$

Therefore the contribution of such a $Q$ to (5.4) is

$$(-1)^{\dim(A_L/A_G)} \cdot |D_M^L(\gamma)|^{1/2} \cdot \delta_{q_1}^{-1/2}(\gamma) \cdot \sum_{w \in W''} \epsilon(w) \cdot \text{tr}(\gamma^{-1}; V_{w(\lambda_B+\rho_B)-\rho_B}^{L_1})$$

$$\cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q).$$

Since all factors in this expression except for the first and last depend on $Q$ only through $Q_1$, we see that $\Phi_M(\gamma, \Theta_\nu)$ is equal to

$$\sum_{Q_1} (-1)^{\dim(A_{L_1}/A_G)} \cdot |D_M^L(\gamma)|^{1/2} \cdot \delta_{q_1}^{-1/2}(\gamma) \cdot \sum_{w \in W''} \epsilon(w) \cdot \text{tr}(\gamma^{-1}; V_{w(\lambda_B+\rho_B)-\rho_B}^{L_1})$$

$$\cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q),$$

(5.5)

where the index set for the first sum is the set of $Q_1 = L_1 U_1 \in \mathcal{F}(M)$ such that $x \in \mathbb{A}_{L_1}$ and the index set for the second sum is the set of $Q \in \mathcal{F}(M)$ such that $Q' = Q_1$. 

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Comparing (5.5) with the definition of $L_M^\nu(\gamma)$, we see that in order to prove that $\Phi_M(\gamma, \Theta_\nu)$ is equal to $L_M^\nu(\gamma)$, it is enough to prove the equality

$$\sum_Q (-1)^{\dim(A_{L_1}/A_L)} \cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q)$$

(5.6)

$$= (-1)^{\dim(A_{L_1})} \cdot \varphi_{Q_1}(-x, p_{L_1}(w(\lambda_B + \rho_B) - \rho_B) - \nu_{Q_1}).$$

The sum in (5.6) is taken over the set of $Q \in \mathcal{F}(M)$ such that $Q' = Q_1$, or, in other words, the set of $Q \in \mathcal{F}(M)$ such that $Q \supseteq Q_1$ and $-x \in C_{Q_1} + \mathfrak{A}_L$. Therefore the left-hand side of (5.6) is equal to

$$\sum_Q (-1)^{\dim(A_{L_1}/A_L)} \cdot \xi_{C_{Q_1} + \mathfrak{A}_L}(-x) \cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q),$$

where the sum is now taken over all $Q \in \mathcal{F}(M)$ such that $Q \supseteq Q_1$, and this is indeed equal to the right-hand side of (5.6), as one easily sees from Lemma A.6 and the fact that the restriction of $p_{L_1}(w(\lambda_B + \rho_B) - \rho_B) - \nu_{Q_1}$ to the subspace $\mathfrak{A}_L$ of $\mathfrak{A}_{L_1}$ is equal to $p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q$.

This concludes the proof of Theorem 5.1.

We will now make a particular choice for $\nu$, and we will show (Theorem 5.2) that with this choice the virtual character $\Theta_\nu$ becomes especially simple. Denote by $N_0$ the unipotent radical of our chosen minimal parabolic subgroup $P_0$. Let $\rho_0 \in \mathfrak{a}_{P_0}^*$ denote half the sum of the roots (counted with multiplicity) of $A_{P_0}$ in Lie$(N_0)$. As usual we regard $\mathfrak{a}_G^*$ as a direct summand of $\mathfrak{a}_{P_0}^*$. Define $\nu_m \in \mathfrak{a}_{P_0}^*$ by

$$\nu_m := -\rho_0 + \lambda_0,$$

where $\lambda_0 \in X^*(A_G) \subset \mathfrak{a}_G^*$ is the character by which $A_G$ acts in the representation $E$. From $\nu_m$ we obtain the virtual character $\Theta_{\nu_m}$ on $G(\mathbb{R})$.

Suppose first that there exists an elliptic maximal torus $T_e$ in $G$ over $\mathbb{R}$, so that $G(\mathbb{R})$ has a discrete series. As in §4 we put

$$\Theta^E = (-1)^{g(G)} \sum_{\pi \in \Pi} \Theta_{\pi},$$

where $\Pi$ is the L-packet of discrete series representations of $G(\mathbb{R})$ having the same infinitesimal and central characters as the finite dimensional representation $E$. Note that the contragredient of $\Theta^E$ is equal to $\Theta^{E^*}$, where $E^*$ denotes the contragredient of $E$.
Theorem 5.2. The virtual characters $\Theta_{\nu_m}$ and $\Theta^{E^*}$ agree on $T_{\text{reg}}(\mathbb{R})$ for any maximal torus $T$ in $G$ over $\mathbb{R}$ whose $\mathbb{R}$-split component is both defined and split over $\mathbb{F}$.

It is enough to prove the theorem when $\mathbb{F} = \mathbb{R}$, in which case we must show that $\Theta_{\nu_m}$ is equal to $\Theta^{E^*}$. We appeal to the characterization of $\Theta^{E^*}$ provided by Theorem 3 of [HC]. Clearly $\Theta_{\nu_m}$ and $\Theta^{E^*}$ are invariant distributions with the same infinitesimal and central characters. Moreover it is obvious from the definition of $\Theta_{\nu_m}$ that $\Theta_{\nu_m}$ agrees with $\Theta^{E^*}$ on $\mathcal{T}(\mathbb{R}) \setminus \mathcal{G}_{\text{reg}}(\mathbb{R})$ and with the virtual character $\Theta^{E^*}$ coincides with that of the finite dimensional representation $E^*$. Thus the only non-trivial point is to check the validity of the second condition in Harish-Chandra’s theorem:

$$\sup_{\gamma \in \mathcal{G}_{\text{reg}}(\mathbb{R})} |D(\gamma)|^{1/2} |\Theta_{\nu_m}(\gamma)| \omega(\gamma) < \infty,$$

where $\omega$ is the unique homomorphism from $G(\mathbb{R})$ to the group of positive real numbers whose restriction to $A_G(\mathbb{R})$ is equal to the absolute value of the quasi-character by which $A_G(\mathbb{R})$ acts in $E$, and where

$$D(\gamma) = \det(1 - \text{Ad}(\gamma); \text{Lie}(G)/\text{Lie}(T)),$$

where $T$ denotes the unique maximal torus of $G$ containing $\gamma$.

It is enough to check that for every maximal torus $T$ of $G$

$$\Phi_M(\gamma, \Theta_{\nu_m}) \cdot \omega(\gamma)$$

is bounded on $T_{\text{reg}}(\mathbb{R})$. Here $M$ is (as usual) the centralizer of the split component of $T$; we used that roots of $T$ in $M$ are imaginary and hence that

$$|\det(1 - \text{Ad}(\gamma); \text{Lie}(M)/\text{Lie}(T))|$$

is bounded on $T(\mathbb{R})$. But the boundedness of (5.7) follows directly from Theorem 5.1. Indeed, it is enough to check that for all $P = MN \in \mathcal{P}(M)$

$$\text{tr}(\gamma^{-1}; V^M_w(\lambda_B + \rho_B - \rho_B) \cdot \omega(\gamma) \cdot \delta_P^{-1/2}(\gamma))$$

is bounded on $T(\mathbb{R})$ whenever $w \in W'$ is such that

$$\varphi_P(-x, p_M(w(\lambda_B + \rho_B) - \lambda_0)) \neq 0.$$

By Proposition A.5 the condition (5.9) implies that

$$\langle x, p_M(w(\lambda_B + \rho_B) - \lambda_0) \rangle \geq 0.$$

Since

$$T(\mathbb{R}) = A_M(\mathbb{R})^0 \times T(\mathbb{R})_1$$
and \( T(\mathbb{R})_1 \) is compact, only the character by which \( A_M(\mathbb{R})^0 \) acts in \( V^M_{\ell_B+\rho_B-\rho_B} \) is relevant to the boundedness of (5.8). In fact the function (5.8) of \( \gamma \) transforms under \( A_M(\mathbb{R})^0 \) by the element

\[
p_M(-\omega(\lambda_B + \rho_B) + \rho_B + \lambda_0 - \rho_N) = p_M(-\omega(\lambda_B + \rho_B) + \lambda_0) \in \mathfrak{a}_M^*
\]

where \( \rho_N \) is half the sum of the roots of \( T \) in \( \text{Lie}(N) \), and thus (5.10) does imply that (5.8) is bounded on \( T(\mathbb{R}) \). This completes the proof.

Now we drop the assumption that \( G \) has an elliptic maximal torus over \( \mathbb{R} \). First, for arbitrary \( G \) and suitably regular \( \gamma \) we will rewrite \( \Phi_M(\gamma, \Theta_{\nu_m}) \) in terms of the functions \( \psi_R \) of §1. Second, for \( G \) having no elliptic maximal torus we will use this expression for \( \Phi_M(\gamma, \Theta_{\nu_m}) \) to show that it vanishes under a certain regularity hypothesis on the highest weight of \( E \).

Let \( T \subset M \subset G \) be as usual. Let \( \gamma \in T(\mathbb{R}) \) and write

\[
\gamma = \exp(x) \cdot \gamma_1
\]

with \( x \in \mathfrak{a}_M \) and \( \gamma_1 \in T(\mathbb{R})_1 \). Assume that \( x \) is regular in \( \mathfrak{a}_M \), in the sense that no root of \( A_M \) vanishes on \( x \). Then by (A.2) (or rather its analog for the function \( \varphi_Q \)) \( L'_M(\gamma) \) is given by

\[
L'_M(\gamma) = (-1)^{\dim(A_M)} \sum_{P \in \mathcal{P}(M)} \delta^{1/2}(\gamma) \cdot \sum_{w \in W'} \epsilon(w) \cdot \text{tr}(\gamma^{-1}; V^M_{\omega(\lambda_B + \rho_B) - \rho_B}) \cdot \varphi(-x, \rho_M(\omega(\lambda_B + \rho_B) - \rho_B) - \nu_P).
\]

Fix a Borel subgroup \( B_M \) of \( M \) over \( \mathbb{C} \) containing \( T \). For each \( P \in \mathcal{P}(M) \) we let \( B(P) \) denote the unique Borel subgroup of \( G \) over \( \mathbb{C} \) such that

\[
T \subset B(P) \subset P
\]

and \( B(P) \cap M = B_M \). Write \( P_x = MN_x \) for the unique element of \( \mathcal{P}(M) \) whose chamber in \( \mathfrak{a}_M \) contains \( -x \), and write \( B(x) \) for \( B(P_x) \). Let \( \mathcal{B}(T) \) denote the set of Borel subgroups of \( G \) over \( \mathbb{C} \) containing \( T \), and let \( \mathcal{B}(T)' \) denote the subset of \( \mathcal{B}(T) \) consisting of Borel subgroups \( B \) such that \( B \cap M = B_M \). Then

\[
L'_M(\gamma) = (-1)^{\dim(A_M)} \sum_{B \in \mathcal{B}(T)' \cap \mathcal{P}(M)} \sum_{P=MN \in \mathcal{P}(M)} \epsilon(B, B(P)) \cdot \delta^{1/2}(\gamma) \cdot \text{tr}(\gamma^{-1}; V^M_{\omega(\lambda_B + \rho_B) - \rho_B(P)}) \cdot \varphi(-x, \rho_M(\omega(\lambda_B + \rho_B - \rho_N) - \nu_P))
\]

where \( \epsilon(B, B(P)) = \epsilon(w) \) for the unique element \( w \in W \) such that \( wB(P)w^{-1} = B \) and where \( \rho_N \) is half the sum of the roots of \( T \) in \( \text{Lie}(N) \). Of course

\[
\rho_B(P) = \rho_M + \rho_N.
\]
where $\rho_M$ denotes half the sum of the roots of $T$ in $M$ that are positive for $B_M$. Thus

$$\rho_{B(P)} - \rho_{B(x)} = \rho_N - \rho_{N_x} \in X^*(T)$$

is trivial on the intersection of $T$ with the derived group of $M$, and therefore defines a homomorphism

$$M \to \mathbb{G}_m.$$ 

It follows that

$$\text{tr}(\gamma^{-1}; V_{\lambda_B + \rho_B - \rho_{B(P)}}^M) = \text{tr}(\gamma^{-1}; V_{\lambda_B + \rho_B - \rho_{B(x)}}^M) \cdot \langle \gamma^{-1}, \rho_{B(x)} - \rho_{B(P)} \rangle$$

and this shows that

$$L_M^L(\gamma) = (-1)^{\dim(A_G)} \sum_{B \in B(T)'} \text{tr}(\gamma^{-1}; V_{\lambda_B + \rho_B - \rho_{B(x)}}^M) \cdot \delta_{P_x}^{-1/2}(\gamma) \cdot \epsilon(B, B(x)) \cdot a_B$$

where

$$a_B = \sum_{P=MN \in \mathcal{P}(M)} \langle \gamma^{-1}, \rho_{B(x)} - \rho_{B(P)} \rangle \cdot \delta_{P_x}^{1/2}(\gamma) \cdot \delta_{P}^{-1/2}(\gamma) \cdot \epsilon(B(P), B(x))$$

and

$$\cdot \varphi_P(-x, p_M(\lambda_B + \rho_B - \rho_N) - \nu_P).$$

Let $P = MN$ be the element of $\mathcal{P}(M)$ opposite $P$. Note that

$$\langle \gamma^{-1}, \rho_{B(x)} - \rho_{B(P)} \rangle \delta_{P_x}^{1/2}(\gamma) \delta_{P}^{-1/2}(\gamma)$$

is equal to the sign of the real number

$$\langle \gamma^{-1}, \rho_{B(x)} - \rho_{B(P)} \rangle = \langle \gamma^{-1}, \rho_{N_x} - \rho_N \rangle.$$ 

But $\rho_{N_x} - \rho_N$ is the sum of all roots $\alpha$ of $T$ in $N_x \cap \overline{N}$, and since this set of roots is preserved by complex conjugation, the sign in question is

$$\prod_{\alpha} \text{sgn} \alpha(\gamma^{-1})$$

where $\alpha$ runs through the real roots of $T$ in $N_x \cap \overline{N}$. For such a real root

$$\text{sgn} \alpha(\gamma^{-1}) = \text{sgn} \alpha(\gamma_1).$$

Of course the sign $\epsilon(B(P), B(x))$ is $-1$ raised to the number of roots of $T$ in $N_x \cap \overline{N}$, and again since this set of roots is stable under complex conjugation, this sign is $-1$ raised to the number of real roots of $T$ in $N_x \cap \overline{N}$. Let $R_\gamma$ be the set of real roots $\alpha$ of $T$ in $G$ such that $\alpha(\gamma_1) = 1$ (or, equivalently, such that $\alpha(\gamma) > 0$). Of course $R_\gamma$ is a root system in $(\mathfrak{A}_M/\mathfrak{A}_G)^*$ (though it need not span that space). Let $\mathcal{C}$ be the set of
Weyl chambers in $\mathfrak{A}_M$ for the root system $R_\gamma$. For $C_1, C_2 \in \mathcal{C}$ let $\epsilon(C_1, C_2)$ be $-1$ raised to the number of root hyperplanes (for roots in $R_\gamma$) separating $C_1$ and $C_2$. Let $C_0$ be the unique element in $\mathcal{C}$ that contains $-x$. For $P \in \mathcal{P}(M)$ let $C_P$ denote the unique element of $\mathcal{C}$ that contains the chamber in $\mathfrak{A}_M$ determined by $P$. Then it follows from the discussion above that

$$a_B = \sum_{P=M \in \mathcal{P}(M)} \epsilon(C_0, C_P) \varphi_P(-x, p_M(\lambda_B + \rho_B - \rho_N) - \nu_P).$$

Now suppose that $\nu = \nu_m$. Then

$$p_M(\lambda_B + \rho_B - \rho_N) - \nu_P = p_M(\lambda_B + \rho_B - \lambda_0)$$

is independent of $P$. Since $x$ is regular, it follows from (A.13) and Lemma A.4 that $a_B$ is equal to

$$(5.11) \quad \sum_{C \in \mathcal{C}} \epsilon(C_0, C) \psi_{\mathfrak{g}}(-x, p_M(\lambda_B + \rho_B - \lambda_0)).$$

This is nothing but

$$(-1)^{\dim(A_G)} \psi_{R_\gamma}(C_0, -x, p_M(\lambda_B + \rho_B - \lambda_0),$$

(strictly speaking we only defined the functions $\psi_R$ for root systems spanning the vector space in which they lie, but of course the definition extends immediately to the general case). Note that by (A.2) each term of (5.11) vanishes unless $p_M(\lambda_B + \rho_B - \lambda_0)$ belongs to the span of $R_\gamma$.

Now suppose that the highest weight of $E$ satisfies the following property: for every proper Levi subgroup $M$ of $G$, for every maximal torus $T$ of $M$ over $\mathbb{C}$ and for every $B \in \mathcal{B}(T)$ the element

$$\lambda_B + \rho_B - \lambda_0$$

of $X^*(T)$ is non-trivial on $A_M$.

**Theorem 5.3.** Assume that $G$ contains no elliptic maximal torus over $\mathbb{R}$. Then, under the hypothesis above on the highest weight of $E$, the complex number $\Phi_M(\gamma, \Theta_{\nu_m})$ is 0, and consequently, if $F$ is $\mathbb{R}$, the virtual character $\Theta_{\nu_m}$ is 0.

By Theorem 5.1 we must show that $L_M^{\nu}$ is 0. By continuity it is enough to show this for $\gamma \in T(\mathbb{R})$ such that $x$ is regular in $\mathfrak{A}_M$. Then, by the discussion above, it is enough to show that (5.11) vanishes. As in §4 let $T_c$ denote the maximal anisotropic subtorus in $T$ and let $J$ denote the centralizer in $G$ of $T_c$ and $\gamma_1$. Then $R_\gamma$ is the root system $R_J$ of $T$ in $J$. Clearly $T_c$ is central in $J$ and $J/T_c$ is a split group with split maximal torus $T/T_c$. 

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Since by hypothesis \( G/A_G \) has no anisotropic maximal torus over \( \mathbb{R} \), the same is true of \( J/A_G \) and \( J/T_eA_G \). Since \( J/T_eA_G \) is a split group, either \( A_J \) is strictly bigger than \( A_G \) or \( A_J = A_G \) (in which case \( R_J \) spans \( (\mathfrak{A}_M/\mathfrak{A}_G)^* \) and \(-1\mathfrak{A}_M/\mathfrak{A}_G \) does not belong to the Weyl group of \( R_J \). In the first case \( A_J \) is of the form \( \mathfrak{A}_L \) for some proper Levi subgroup \( L \) of \( G \) containing \( M \), and therefore our hypothesis on the highest weight of \( E \) implies that every term of (5.11) vanishes. In the second case Corollary 1.3 (applied to the root system \( R_\circ \) in \( \mathfrak{A}_M/\mathfrak{A}_G \)) implies that (5.11) is 0, since our hypothesis on the highest weight of \( E \) ensures that \( p_M(\lambda_B + \rho_B - \lambda_0) \) is \( R_\gamma \)-regular (any intersection of root hyperplanes in \( \mathfrak{A}_M/\mathfrak{A}_G \) for roots in \( R_\gamma \) is of the form \( \mathfrak{A}_L/\mathfrak{A}_G \) for some Levi subgroup \( L \) of \( G \) containing \( M \), and \( R_\gamma \)-regularity is equivalent to non-vanishing on every such non-zero intersection).

6. Discrete series constants

In the beginning of §4 (see (4.1)–(4.3)) we reviewed the form taken by the character of an irreducible representation of a real reductive group. In this section we are concerned with the case of discrete series representations. We then refer to the integers \( n(\gamma, B) \) appearing in (4.1) as *discrete series constants* (however we will no longer use the notation \( n(\gamma, B) \)). In this section we give a simple formula for the discrete series constants. Because of a descent property satisfied by the constants (see [K,13.4]) it is enough to give the formula in the following special case.

Let \( G \) be a split semisimple simply connected group over \( \mathbb{R} \), and assume that \( G \) contains an anisotropic maximal torus \( T_e \). Let \( A \) be a split maximal torus in \( G \). We choose an isomorphism \( A' \simeq T_e \) over \( \mathbb{C} \) that is induced by an inner automorphism of \( G \) over \( \mathbb{C} \) and use it to identify the character groups of \( A \) and \( T_e \). We put

\[
X^* := X^*(T_e)_\mathbb{R} \simeq X^*(A)_\mathbb{R}
\]

and

\[
X := X_*(T_e)_\mathbb{R} \simeq X_*(A)_\mathbb{R}.
\]

The roots and coroots of \( T_e \) in \( G \) give us a root system \( (X, X^*, R, R_\circ) \). Of course the set \( R \) spans \( X^* \), and \(-1\) belongs to the Weyl group \( W = W(R) \).

Let \( \tau \) be a regular element in \( X^*(T_e) \). Associated to \( \tau \) is a discrete series representation \( \pi(\tau) \) of \( G(\mathbb{R}) \) having infinitesimal character \( \tau \) and having the same central character as the finite dimensional representation having infinitesimal character \( \tau \). We are interested in the constants needed to express the values of the character of \( \pi(\tau) \) at regular elements in the identity component \( A(\mathbb{R})^0 \) of \( A(\mathbb{R}) \); this is the special case alluded to above.

We need a little preparation before we can state our formula for these constants. There is a unique maximal compact subgroup \( K \) of \( G(\mathbb{R}) \) containing \( T_e(\mathbb{R}) \), and the roots of \( T_e \) in \( K \) form a subset \( R_c \) of \( R \) (such roots are said to be *compact*). We write \( W_c \) for the Weyl group of \( R_c \) and identify it with a subgroup of \( W \). It is not hard to see that the normalizer \( \tilde{W}_c \) of \( W_c \) in \( W \) is given by

\[
\tilde{W}_c = \{ w \in W \mid w(R_c) = R_c \}.
\]
Let $C$ be a chamber in $X$. The chamber $C$ determines a subset $R_C$ of $R$ in the following way. Let $\delta_C \in X$ denote the half-sum of the coroots that are positive for $C$, and put

$$R_C = \{ \alpha \in R \mid \alpha(\delta_C) \in 2\mathbb{Z} \};$$

note that no simple root (for $C$) belongs to $R_C$. We denote by $W_C$ the Weyl group of $R_C$. We identify $W_C$ with a subgroup of $W$, and let $\widetilde{W}_C$ denote its normalizer in $W$. As before we write $C^\vee$ for the Weyl chamber in $X^*$ corresponding to $C$.

It is known (see [AV,6.24(f)]) that there exists a chamber $C$ such that $R_C$ equals $R_e$. For such a chamber $C$ we have $W_C = W_e$ and $\widetilde{W}_C = \widetilde{W}_e$. The $\widetilde{W}_e$-orbit of $C$ is uniquely determined by the condition that $R_C$ equal $R_e$.

As before let $\tau$ be a regular element of $X^*(T_e)$ (actually, in the definition we are about to make we could just as well let $\tau$ be any regular element in $X^*(T_e)_R = X^*$). Let $C$ be a chamber in $X$ (for the time being we do not assume that $R_C = R_e$). Let $x$ be an $R^\vee$-regular element of $X$, and let $\lambda$ be an element of $X^*$ lying in the $W$-orbit $W \cdot \tau$ of $\tau$ (see §1 for the definition of $R^\vee$-regularity). We define an integer $b_R(\tau, C; x, \lambda)$ by

$$b_R(\tau, C; x, \lambda) = (-1)^{q(R)} \sum_{w \in W(\tau, C, \lambda)} \epsilon(x, wC)\psi_{wC^\vee}(\lambda, x).$$

In case $R_C = R_e$ these constants (for $\lambda \in W \cdot \tau$) are the ones needed to express the value of the character of $\pi(\tau)$ at the point $a \in A(\mathbb{R})^0$ obtained from $x \in X$ via the exponential map (we have identified $X_*(A)$ with $X_*(T_e)$ and thus we may view $X$ as the Lie algebra of $A(\mathbb{R})$). However for technical reasons it is best to define $b_R(\tau, C; x, \lambda)$ for any chamber $C$.

The expression (6.1) requires some explanation. The integer

$$q(R) = \frac{||R^+| + \dim(X)||}{2}$$

was used already in §2, and its interpretation in terms of $G$ was also given there. The index set for the sum is the coset

$$W(\tau, C, \lambda) := \{ w \in W \mid w^{-1}\lambda \in W_C \cdot \tau \}$$

of $W_C$ in $W$ (of course $W_C \cdot \tau$ denotes the orbit of $\tau$ under $W_C$). As in §1, for any two chambers $C_1, C_2$ in $X$ we write $\epsilon(C_1, C_2)$ for the sign of the Weyl group element $w$ such that $wC_1 = C_2$. The sign $\epsilon(x, wC)$ appearing in (6.1) is by definition $\epsilon(C_x, wC)$, where $C_x$ denotes the unique chamber in $X$ containing the (regular) element $x \in X$. Finally $\psi_{wC^\vee}(\lambda, x)$ is the function of $(\lambda, x) \in X^* \times X$ defined in §1; it is obtained from the coroot system $R^\vee$ and the Weyl chamber $wC^\vee$ in $X^*$.

Of course the integer $b_R(\tau, C; x, \lambda)$ depends only on the $R^\vee$-chamber of $X$ in which $x$ lies. But in fact we claim that $b_R(\tau, C; x, \lambda)$ depends only on the Weyl chamber of $X$.
in which \(x\) lies (\(x\) is still assumed to be \(R^\vee\)-regular). Indeed it follows from Lemma 1.5 (applied to \(R^\vee\)) that for all \(s \in \hat{A}_{sc}\) such that \(s^2 \in Z^\vee\)

\[
(6.2) \quad \sum_D \epsilon(D_0^\vee, D^\vee) \langle \delta_D - \delta_{D_0}, s \rangle \psi_{D^\vee}(\lambda, x)
\]

depends only on the Weyl chamber in which \(x\) lies. Here we are using the notation \(\delta_D, \hat{A}_{sc}, s, Z^\vee\) of Lemma 1.4 (since we are applying Lemma 1.5 to \(R^\vee\) rather than \(R\), we need the notation of Lemma 1.4), and we have applied Lemma 1.2 to the root system \(R_s^\vee\) of Lemma 1.4. Summing (6.2) over all \(s \in \hat{A}_{sc}\) such that \(s^2 \in Z^\vee\) we find that

\[
(6.3) \quad \sum_D \epsilon(D_0^\vee, D^\vee) \psi_{D^\vee}(\lambda, x)
\]

depends only on the Weyl chamber in which \(x\) lies, where the sum is now taken over all chambers \(D\) such that the element \(\delta_D - \delta_{D_0} \in Q\) lies in \(2Q\) (here \(Q \subset X\) denotes the lattice generated by \(R^\vee\)). This set of chambers can also be described as the set of chambers \(wD_0\) where \(w\) ranges through the stabilizer in \(W\) of the element \(\delta_{D_0} \in (\frac{1}{2}Q)/2Q\). We can think of \((\frac{1}{2}Q)/2Q\) as consisting of 4-torsion elements in a maximal torus \(A_{sc}\) in the semisimple simply connected complex group \(G_{sc}\) with root system \(R\), and by a theorem of Steinberg this stabilizer is the Weyl group of the root system of the centralizer of \(\delta_{D_0} \in A_{sc}\) in \(G_{sc}\), namely

\[
\{ \alpha \in R \mid \langle \delta_{D_0}, \alpha \rangle \in 2\mathbb{Z} \} = R_{D_0}.
\]

Therefore the sum in (6.3) is over \(wD_0\) (\(w \in W_{D_0}\)) and our claim has been proved (take \(D_0 = w_0C\) for any \(w_0 \in W(\tau, C, \lambda)\)).

In order to prove that the integers \(b_R(\tau, C; x, \lambda)\) are the ones appearing in the character formula for \(\pi(\tau)\) on \(A(\mathbb{R})^0\), we must show that they satisfy various properties. The first is that

\[
(6.4) \quad b_R(\tau, C; x, \lambda) = b_R(\tau, wC; x, \lambda) \quad \text{for all} \quad w \in \tilde{W}_C
\]

(in other words \(b_R(\tau, C; x, \lambda)\) only depends on the subset \(R_C\) determined by \(C\)). It is trivial that (6.4) holds for all \(w \in W_C\), but to prove it for \(w \in \tilde{W}_C\) we need to use Lemma 1.5 (applied to \(R^\vee\)), just as in the previous proof. Indeed, by Lemma 1.5 the expression (6.2) vanishes unless \(s^2 = 1\). Therefore summing (6.2) over \(\{ s \in \hat{A}_{sc} \mid s^2 = 1 \}\) yields the same result as summing over \(\{ s \in \hat{A}_{sc} \mid s^2 \in Z^\vee \}\). It follows that (drop the subscript 0 from \(D\))

\[
\sum_{w \in W_D} \epsilon(w) \psi_{wD^\vee}(\lambda, x)
\]

is equal to

\[
|Z^\vee|^{-1} \sum_{w \in W_D} \epsilon(w) \psi_{wD^\vee}(\lambda, x),
\]

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and it is clear that this expression is multiplied by $\epsilon(w)$ if $D$ is replaced by $wD$ for $w \in \widetilde{W}_D$. This proves (6.4) (take $D = w_0 C$ for any $w_0 \in W(\tau, C, \lambda)$).

The next two properties of $b_R(\tau, C; x, \lambda)$ are obvious.

(6.5) $b_R(w\tau, wC; x, \lambda) = b_R(\tau, C; x, \lambda)$ for all $w \in W$.

(6.6) $b_R(\tau, C; wx, w\lambda) = b_R(\tau, C; x, \lambda)$ for all $w \in W$.

Moreover it follows from Proposition A.5 that

$$b_R(\tau, C; x, \lambda) = 0 \text{ unless } \lambda(x) \leq 0,$$

and since $b_R(\tau, C; x, \lambda)$ depends only on the chamber $C_x$ containing $x$ we find that

(6.7) $b_R(\tau, C; x, \lambda) = 0 \text{ unless } \lambda \leq 0 \text{ on } C_x$.

There is one more elementary property of the constants:

(6.8) $b_R(\tau, C; x, \lambda) = 1$ if $R$ is empty.

The last property we need requires a bit more work. Suppose that $\alpha \in R$ and put $Y = \ker(\alpha) \subset X$. Define $R_\alpha, R^\vee_\alpha$ as in the discussion at the beginning of §2. Recall that $R_\alpha$ generates $Y^*$ and that $-1_Y \in W(R_\alpha)$. Let $s = s_\alpha \in W$ be the reflection in the root $\alpha$. Assume further that $C$ is a chamber in $X$ such that $\alpha$ belongs to the closure of $C^\vee$. Let $x, x'$ be $R^\vee$-regular elements in $X$ that lie in adjacent chambers separated by the wall $Y$. We are going to derive a formula for

$$b_R(\tau, C; x, \lambda) + b_R(\tau, C; x', \lambda)$$

in terms of the constants $b^R_\alpha$ associated to the root system $R_\alpha$.

To get a clean formula we need to use the constants for the root system $R_\alpha$ to define constants $b^R_{R_\alpha}(\tau, C; y, \lambda)$ for $R^\vee_\alpha$-regular $y \in Y$ and $\tau \in X^*$, $\lambda \in W \cdot \tau$, $C$ as before (subject to the requirement that $\alpha$ belongs to the closure of $C^\vee$). Write $W_\alpha$ for the Weyl group of $R_\alpha$. Then we define

$$b^R_{R_\alpha}(\tau, C; y, \lambda) = 0 \text{ unless } \lambda \in W_\alpha W_C \cdot \tau.$$

If $\lambda$ does belong to $W_\alpha W_C \cdot \tau$, choose $\tau' \in W_C \cdot \tau$ such that $\lambda \in W_\alpha \cdot \tau'$ and put

$$b^R_{R_\alpha}(\tau, C; y, \lambda) = b_{R_\alpha}(\hat{\tau}', C_Y; y, \hat{\lambda}).$$

Here $\hat{\tau}', \hat{\lambda} \in Y^*$ denote the restrictions of $\tau', \lambda$ to $Y$ and $C_Y$ is the chamber in $Y$ determined by $C$ (thus $C_Y = Y \cap \hat{C}$, where $\hat{C}$ is the unique chamber in $X$ relative to $R_\alpha$ that contains $C$). Since $\tau'$ is well-determined up to an element of $W_\alpha \cap W_C = W_{C_Y},$
we see from (6.4) that $b_{R\alpha}(\tau', C_Y; y, \tilde{\lambda})$ is independent of the choice of $\tau'$ (the equality $W_{C} \cap W_{C} = W_{C_Y}$ is a consequence of our assumption that $\alpha$ belongs to the closure of $C^\vee$). It is easy to see that for any $\lambda \in W \cdot \tau$ we have the formula

$$b_{R,\alpha}(\tau, C; y, \lambda) = (-1)^{q(R\alpha)} \sum_{w \in W_{\alpha} \cap W(\tau, C, \lambda)} \epsilon(y, wC_Y)\psi_{wC_Y}(\tilde{\lambda}, y).$$

Now we are ready to formulate the last property of our constants: for $\alpha, s, C, x, x'$ as above

$$b_{R}(\tau, C; x, \lambda) + b_{R}(\tau, C; x', \lambda) = b_{R}(\tau, C; y, \lambda) + b_{R}(\tau, C; s_\lambda),$$

where $y$ is the unique point of $Y$ lying on the line segment joining $x$ and $x'$. Note that by (6.6) the left-hand side of (6.10) can also be written as

$$b_{R}(\tau, C; x, \lambda) + b_{R}(\tau, C; s_\lambda).$$

Let us now prove (6.10). Assume without loss of generality that $\alpha(x) > 0$ and $\alpha(x') < 0$. Then by Corollary A.3 the left-hand side of (6.10) is equal to

$$(-1)^{q(R)} \sum_{w} \epsilon(x, wC) \cdot \psi_{(wC)^\vee}(\tilde{\lambda}, y) \cdot \eta(w)$$

where the sum is taken over the set of all $w \in W(\tau, C, \lambda)$ such that the closure of $wC^\vee$ contains either $\alpha$ or $-\alpha$, and where $\eta(w) = \pm 1$ is defined by

$$\eta(w) = \begin{cases} 
-1 & \text{if } wC^\vee \text{ contains } \alpha, \\
1 & \text{if } wC^\vee \text{ contains } -\alpha.
\end{cases}$$

Note that $wC^\vee$ contains $\alpha$ if and only if $w \in W_{\alpha}$, and $wC^\vee$ contains $-\alpha$ if and only if $sw \in W_{\alpha}$ (since $s\alpha = -\alpha$). Looking back at the proof of Lemma 2.2, we see that if $w \in W_{\alpha}$, then

$$(-1)^{q(R)} \epsilon(x, wC) = (-1)^{q(R\alpha)} \epsilon(y, wC_Y).$$

Therefore the left-hand side of (6.10) is equal to the difference of

$$(-1)^{q(R\alpha)} \sum_{w \in W_{\alpha} \cap W(\tau, C, \lambda)} \epsilon(y, wC_Y)\psi_{wC_Y}(\tilde{\lambda}, y)$$

and

$$(-1)^{q(R\alpha)} \sum_{w \in sW_{\alpha} \cap W(\tau, C, \lambda)} \epsilon(y, wC_Y)\psi_{(wC)^\vee}(\tilde{\lambda}, y).$$

Replacing $w$ by $sw$ in the second sum (and using that $(swC)_Y = (wC)_Y$ and $s\tilde{\lambda} = \tilde{\lambda}$), we see that the left-hand side of (6.10) is equal to the right-hand side of (6.10), as we wished to show.
The constants $b_R$ are determined uniquely by properties (6.4)-(6.8) and (6.10) (and the property that $b_R(\tau, C; x, \lambda)$ depends only on the chamber in which $x$ lies). To see this fix $\tau, C, \lambda$ and regard $b_R(\tau, C; x, \lambda)$ as a function of $x$. If $R$ is empty, $b_R$ is given by (6.8). If it is non-empty, the value of $b_R(\tau, C; x, \lambda)$ is given by (6.7) for $x$ in at least one chamber in $X$. Therefore it is enough to know

$$b_R(\tau, C; x, \lambda) + b_R(\tau, C; x', \lambda)$$

whenever $x, x'$ lie in adjacent chambers. But by (6.5) (which we use to put $C$ in good position relative to the wall separating $x, x'$) and (6.10) the sum above can be written in terms of the constants for a root system of lower rank, which we may assume have already been determined.

It is known (see [K,13.4]) that the discrete series constants satisfy these same properties; therefore they are equal to the constants $b(\tau, C; x, \lambda)$. Before making this statement more precise, we need to change the indexing of our constants in order to facilitate comparison with [K]. We now fix a chamber $C$ such that $R_C = R_c$ and define constants $c(w, \lambda, \Delta^+)$ as follows. For a regular element $\lambda \in X^*$, $w \in W$, and a system $\Delta^+$ of positive roots for $R$, we put

$$c(w, \lambda, \Delta^+) := b(\lambda, C; x, w\lambda),$$

where $x \in X$ is any $R^\vee$-regular element in the (positive) Weyl chamber in $X$ determined by $\Delta^+$. It follows from (6.4) that the right side in this definition is independent of the choice of chamber $C$ such that $R_C = R_c$. The constants $c(w, \lambda, \Delta^+)$ are those in [K] (see properties (13.32)-(13.34) in [K]). In other words our $\tau$ corresponds to Knapp’s $\lambda$, and our $\lambda$ corresponds to Knapp’s $w\lambda$.

7. Lefschetz formula on reductive Borel-Serre compactifications

(7.1) The group $G$. Let $G$ be a connected reductive group over $\mathbb{Q}$ and let $A_G$ denote the maximal $\mathbb{Q}$-split torus in the center of $G$. Choose a maximal $\mathbb{Q}$-split torus $A_0$ in $G$ and let $M_0$ denote its centralizer, a Levi subgroup of $G$. Fix a parabolic subgroup $P_0$ of $G$ over $\mathbb{Q}$ having $M_0$ as Levi component; then $P_0$ is a minimal parabolic subgroup of $G$ over $\mathbb{Q}$. See §5 for notation and terminology concerning parabolic and Levi subgroups. In particular for any standard parabolic subgroup $P$ we write $M$ for the unique Levi component of $P$ containing $M_0$ and $N$ for the unipotent radical of $P$; thus $P = MN$.

(7.2) The locally symmetric spaces $S_K$. Let $K$ be a suitably small compact open subgroup of $G(\mathbb{A}_f)$. Choose a maximal compact subgroup $K_G$ of $G(\mathbb{R})$ in good position relative to $M_0$, in the sense that the Cartan involution on $G$ associated to $K_G$ preserves $M_0$. For each standard parabolic subgroup $P = MN$ we denote by $K_M$ the intersection of $K_G$ with $M(\mathbb{R})$, a maximal compact subgroup in $M(\mathbb{R})$. We denote by $A_G(\mathbb{R})^0$ the identity component of the topological group $A_G(\mathbb{R})$, and we denote by $X_G$ the homogeneous space

$$G(\mathbb{R})/(K_G \cdot A_G(\mathbb{R})^0)$$

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for \( G(\mathbb{R}) \). We then denote by \( S_K \) the space
\[
G(\mathbb{Q}) \setminus [(G(\mathbb{A}_f) / K) \times X_G].
\]

(7.3) **The local system** \( E_K \) **on** \( S_K \). Let \( E \) be an irreducible representation of the algebraic group \( G \) on a finite dimensional complex vector space. Then \( E \) gives rise to a local system \( E_K \) on \( S_K \). By definition \( E_K \) is the sheaf of flat sections of the flat vector bundle
\[
G(\mathbb{Q}) \setminus [(G(\mathbb{A}_f) / K) \times X_G \times E]
\]
over \( S_K \).

(7.4) **The Hecke correspondence** \((c_1, c_2)\) **on** \( S_K \). Now fix an element \( g \) in \( G(\mathbb{A}_f) \), and let \( K' \) be any compact open subgroup of \( G(\mathbb{A}_f) \) that is contained in \( K \cap g^{-1} K g \). We use \( g, K' \) to form a Hecke correspondence on \( S_K \), as follows. The inclusion \( K' \subset K \) induces a surjection
\[
c_1 : S_{K'} \to S_K.
\]
The inclusion \( K' \subset g^{-1} K g \) induces a surjection
\[
S_{K'} \to S_{g^{-1} K g}
\]
which we compose with the canonical isomorphism (use the element \( g \))
\[
S_{g^{-1} K g} \simeq S_K
\]
to get a second surjection
\[
c_2 : S_{K'} \to S_K.
\]
There are canonical isomorphisms
(7.4.1)
\[
c_1^* E_K \simeq E_{K'} \simeq c_2^* E_K.
\]

(7.5) **The reductive Borel-Serre compactification** \( \overline{S}_K \) **of** \( S_K \). For any standard parabolic subgroup \( P = MN \) we denote by \( S_K^P \) the space
\[
S_K^P := M(\mathbb{Q}) \setminus [(N(\mathbb{A}_f) \setminus G(\mathbb{A}_f) / K) \times X_M],
\]
where \( X_M \) denotes the analog for \( M \) of \( X_G \), namely \( M(\mathbb{R}) / (K_M \cdot A_M(\mathbb{R})^0) \). Now we can make precise what it means for \( K \) to be suitably small: we require that for each standard \( P \) the group \( M(\mathbb{Q}) \) act freely on
\[
(N(\mathbb{A}_f) \setminus G(\mathbb{A}_f) / K) \times X_M.
\]
The reductive Borel-Serre compactification (see [GHM]) \( \overline{S}_K \) of \( S_K \) is a stratified space whose statra are indexed by standard parabolic subgroups \( P \) of \( G \), the stratum indexed by \( P \) being the manifold \( S_K^P \) described above.
(7.6) The weighted cohomology complex $\overline{E}_K$ on $\mathcal{S}_K$. Let $p$ be a weight profile (see §1.1 of [GHM]). Associated to the representation $E$ and the weight profile $p$ is a constructible complex of sheaves $W_p C^*(E)$ of complex vector spaces on $\mathcal{S}_K$ (see §1.3 of [GHM]). In this paper we will denote this complex of sheaves by $\overline{E}_K$; as the notation suggests, the restriction of $\overline{E}_K$ to $S_K$ may be identified with $E_K$.

(7.7) The Hecke correspondence $(\bar{c}_1, \bar{c}_2)$ on $\mathcal{S}_K$. The maps
\[ c_1, c_2 : S_{K'} \to S_K \]
have unique continuous extensions
\[ \bar{c}_1, \bar{c}_2 : \mathcal{S}_{K'} \to \overline{S}_K. \]
These maps carry $S^P_{K'}$ onto $S^P_K$; in fact, representing points of $S^P_{K'}$ by pairs $(x, x_\infty)$ where $x \in G(\mathbb{A}_f)$ and $x_\infty \in X_M$, we have that the image of the pair $(x, x_\infty)$ under $\bar{c}_1$ (respectively, $\bar{c}_2$) is the point of $S^P_K$ represented by $(x, x_\infty)$ (respectively, $(xg^{-1}, x_\infty)$).

It follows from the definition of weighted cohomology complexes that there are canonical isomorphisms
\[ \bar{c}_1^! \overline{E}_K \simeq \overline{E}_{K'} \simeq \bar{c}_2^* \overline{E}_K. \]

The Verdier dual of the weighted cohomology complex $\overline{E}_K$ is (a shift of) the weighted cohomology complex obtained from the contragredient of the representation $E$ and the weight profile $\bar{p}$ dual to $p$ (see §1.3 of [GHM]). Thus, applying Verdier duality to (7.7.1), we find that there are canonical isomorphisms
\[ \bar{c}_1^! \overline{E}_K \simeq \overline{E}_{K'} \simeq \bar{c}_2^* \overline{E}_K. \]

It follows that there is a canonical isomorphism
\[ \bar{c}_2^* \overline{E}_K \to \bar{c}_1^! \overline{E}_K, \]
obtained as the composition of the isomorphism (7.7.1) from $\bar{c}_2^* \overline{E}_K$ to $\overline{E}_{K'}$, and the isomorphism (7.7.2) from $\overline{E}_{K'}$ to $\bar{c}_1^! \overline{E}_K$. Thus there is a canonical extension, namely the morphism (7.7.3), of the Hecke correspondence $(\bar{c}_1, \bar{c}_2)$ to the weighted cohomology complex $\overline{E}_K$.

(7.8) The goal. The canonical morphism (7.7.3) induces self-maps on hypercohomology groups
\[ H^i(\mathcal{S}_K, \overline{E}_K) \to H^i(\mathcal{S}_K, \overline{E}_K). \]
These maps are obtained as the composition of the canonical pullback map
\[ H^i(\mathcal{S}_K, \overline{E}_K) \to H^i(\mathcal{S}_{K'}, \bar{c}_2^* \overline{E}_K), \]
the map
\[ H^i(S_{K'}, c_1^* E_K) \rightarrow H^i(S_{K'}, c_1^* E_K) \]
induced by (7.7.3), and the canonical proper pushforward map
\[ H^i(S_{K'}, c_1^* E_K) \rightarrow H^i(S_K, E_K). \]

The Lefschetz fixed point formula is a formula for the alternating sum of the traces of the self-maps (7.8.1). An explicit version of the Lefschetz formula (for the case at hand) is given in the theorem on page 474 of [GM]; our goal here is to rewrite that formula in terms of stable virtual characters on the group \( G(\mathbb{R}) \), using the results in §5 of this paper.

**7.9 Fixed points.** First we need to determine the fixed points of the correspondence. Of course a fixed point is an element \( x \) of \( S_{K'} \) such that \( c_1(x) = c_2(x) \). Let us fix a standard parabolic subgroup \( P = M N \) and determine the fixed points of the correspondence that lie in the subset \( S_{K'}^P \) of \( S_{K'} \). The group \( P(\mathbb{A}_f) \) acts on \( G(\mathbb{A}_f)/K' \) with finitely many orbits. Choose a set of representatives \( x_0 \in G(\mathbb{A}_f) \) for these orbits and put
\[ K'_P(x_0) = P(\mathbb{A}_f) \cap x_0 K' x_0^{-1} \]
\[ K'_M(x_0) = \text{image of } K'_P(x_0) \text{ in } M(\mathbb{A}_f). \]

Then \( S_{K'}^P \) is the disjoint union of the subsets
\[ S_{K'}^P(x_0) := M(\mathbb{Q}) \backslash [(M(\mathbb{A}_f)/K'_M(x_0)) \times X_M], \]
the disjoint union being indexed by the set of representatives \( x_0 \) chosen above.

A pair \((y, y_\infty) \in M(\mathbb{A}_f) \times X_M\) represents a fixed point in \( S_{K'}^P(x_0) \) of our Hecke correspondence if and only if there exists \( \gamma \in M(\mathbb{Q}) \) such that

1. \( \gamma y_\infty = y_\infty \), and
2. there exists \( n \in N(\mathbb{A}_f) \) such that \( y^{-1} n \gamma y \in x_0 K g x_0^{-1} \).

The conjugacy class of \( \gamma \) in \( M(\mathbb{Q}) \) depends only on the fixed point we started with. Now fix an element \( \gamma \in M(\mathbb{Q}) \) and denote by \( \text{Fix}(P, x_0, \gamma) \) the subset of \( S_{K'}^P(x_0) \) consisting of all fixed points of our correspondence for which the associated conjugacy class in \( M(\mathbb{Q}) \) is equal to that of \( \gamma \). The discussion above shows that \( \text{Fix}(P, x_0, \gamma) \) is equal to
\[ M(\mathbb{Q}) \backslash (Y_\infty \times Y_\infty) \]
where \( M_\gamma \) denotes the centralizer of \( \gamma \) in \( M \), \( Y_\infty \) denotes the subset of \( M(\mathbb{A}_f)/K'_M(x_0) \) consisting of elements in that set represented by elements \( y \in M(\mathbb{A}_f) \) such that \( y^{-1} \gamma y \) belongs to the image in \( M(\mathbb{A}_f) \) of \( P(\mathbb{A}_f) \cap x_0 K g x_0^{-1} \), and \( Y_\infty \) denotes the set of fixed points of \( \gamma \) in \( X_M \).

The group \( M_\gamma(\mathbb{Q}) \) acts freely on \( Y_\infty \times Y_\infty \). We write \( I \) for the identity component of \( M_\gamma \). The group \( I(\mathbb{A}_f) \) acts on \( Y_\infty \) with finitely many orbits. The space \( Y_\infty \) is empty.
unless \( \gamma \) is conjugate in \( M(\mathbb{R}) \) to an element of \( K_M \cdot A_M(\mathbb{R})^0 \), in which case \( I(\mathbb{R}) \) acts transitively on \( Y_\infty \) and in fact

\[
Y_\infty = I(\mathbb{R})/(K_I \cdot A_M(\mathbb{R})^0)
\]

for some maximal compact subgroup \( K_I \) of \( I(\mathbb{R}) \). In line with our usual notational conventions we write \( X_I \) for the homogeneous space

\[
I(\mathbb{R})/(K_I \cdot A_I(\mathbb{R})^0);
\]

note that \( Y_\infty \) maps onto \( X_I \) and is in fact a principal fiber bundle over that space for the vector group

\[
A_I(\mathbb{R})^0/A_M(\mathbb{R})^0.
\]

\textbf{(7.10) Euler characteristic of \( S_K \).} We need to recall Harder’s formula (see [H]) for the Euler characteristic of the space \( S_K \) (we should also note that this Euler characteristic coincides with the Euler characteristic with compact support of \( S_K \)). Harder’s formula involves several ingredients, which we now explain. Let us choose a Haar measure \( dg_f \) on \( G(\mathbb{A}_f) \). Then the Euler characteristic of \( S_K \) has the form

\[
\chi(G) \cdot \text{vol}(K)^{-1}.
\]

Of course \( \text{vol}(K) \) denotes the measure of \( K \) with respect to \( dg_f \). The quantity \( \chi(G) \) depends on \( G \) and the Haar measure \( dg_f \), but not on \( K \). Moreover \( \chi(G) \) is 0 unless the group \( G \) has a maximal torus \( T \) over \( \mathbb{R} \) such that \( T/A_G \) is anisotropic over \( \mathbb{R} \). Assume now that this condition is satisfied. Let \( \mathcal{D}(G) \) denote the finite set

\[
\mathcal{D}(G) := \ker[H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)]
\]

(as usual we write \( H^1(\mathbb{R}, G) \) as an abbreviation for \( H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G(\mathbb{C})) \)). Since \( G/A_G \) has an anisotropic maximal torus over \( \mathbb{R} \), there is an inner form \( \overline{G} \) of \( G \) over \( \mathbb{R} \) such that \( \overline{G}/A_G \) is anisotropic over \( \mathbb{R} \). We pick a Haar measure \( dg_\infty \) on \( G(\mathbb{R}) \) and transport it to the inner form \( \overline{G}(\mathbb{R}) \) in the usual way, by identifying the space of invariant top degree differential forms on \( G \) with the analogous space for \( \overline{G} \) (this identification is defined over \( \mathbb{R} \) since \( \overline{G} \) is an \emph{inner} form of \( G \)). We define \( q(G) \) to be half the real dimension of the symmetric space associated to the real points of the adjoint group of \( G \). Then \( \chi(G) \) is equal to

\[
(-1)^{q(G)} \text{vol}(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})) \text{vol}(A_G(\mathbb{R})^0 \backslash \overline{G}(\mathbb{R})^{-1} | \mathcal{D}(G)|.
\]

Of course we use \( dg_f \) and \( dg_\infty \) to get a Haar measure on \( G(\mathbb{A}) \); note that \( \chi(G) \) is independent of the choice of Haar measures on \( G(\mathbb{R}) \) and \( A_G(\mathbb{R})^0 \).

\textbf{(7.11) Euler characteristic with compact support of \( \text{Fix}(P, x_0, \gamma) \).} Assume that \( Y_\infty \) is non-empty. The Euler characteristic with compact support of the space \( \text{Fix}(P, x_0, \gamma) \) (which is one of the ingredients in the Lefschetz formula) is equal to

\[
|M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1}
\]

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times the Euler characteristic with compact support of
\[ I(\mathbb{Q}) \setminus (Y^\infty \times Y_\infty), \]
and this latter Euler characteristic with compact support is equal to \((-1)^{\dim(A_I/A_M)}\) times that of
\[ I(\mathbb{Q}) \setminus (Y^\infty \times X_I), \]
since the natural surjection
\[ I(\mathbb{Q}) \setminus (Y^\infty \times Y_\infty) \to I(\mathbb{Q}) \setminus (Y^\infty \times X_I) \]
is a principal fiber bundle under the vector group
\[ A_I(\mathbb{R})^0/A_M(\mathbb{R})^0. \]
It follows from Harder’s theorem (see (7.10)) that the Euler characteristic with compact support of \(\text{Fix}(P;x_0,\gamma)\) is equal to
\[ (-1)^{\dim(A_I/A_M)} |M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot \sum_y \text{vol}(I(\mathbb{A}_f) \cap yK'_M(x_0)y^{-1})^{-1}, \]
where the index set for the sum is the subset of
\[ I(\mathbb{A}_f) \setminus M(\mathbb{A}_f)/K'_M(x_0) \]
consisting of elements that can be represented by an element \(y \in M(\mathbb{A}_f)\) such that \(y^{-1}\gamma y\) belongs to the image in \(M(\mathbb{A}_f)\) of \(P(\mathbb{A}_f) \cap x_0Kgx_0^{-1}\). Of course we have chosen a Haar measure \(dI_f\) on \(I(\mathbb{A}_f)\). Let us fix a Haar measure \(dm\) on \(M(\mathbb{A}_f)\) as well. Define a locally constant compactly supported function \(f_{P,x_0}\) on \(M(\mathbb{A}_f)\) as follows: \(f_{P,x_0}\) is \(\text{vol}(K'_M(x_0))^{-1}\) times the characteristic function of the image in \(M(\mathbb{A}_f)\) of \(P(\mathbb{A}_f) \cap x_0Kgx_0^{-1}\). For any locally constant compactly supported function \(f\) on \(M(\mathbb{A}_f)\) write \(O_\gamma(f)\) for the orbital integral
\[ \int_{I(\mathbb{A}_f) \setminus M(\mathbb{A}_f)} f(m^{-1}\gamma m) \, dm/dI_f. \]
Then
\[ O_\gamma(f_{P,x_0}) = \sum_y \text{vol}(I(\mathbb{A}_f) \cap yK'_M(x_0)y^{-1})^{-1} \]
with the same index set as above. Therefore the Euler characteristic with compact support of \(\text{Fix}(P,x_0,\gamma)\) is equal to
\[ (-1)^{\dim(A_I/A_M)} |M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot O_\gamma(f_{P,x_0}). \]
(7.12) Lefschetz formula (qualitative version). We now need to recall the general form taken by the Lefschetz formula in [GM]. The formula is a sum of contributions, one for each connected component \( C \) of the fixed point set of the Hecke correspondence. We further decompose each such connected component into locally closed pieces

\[ C_P := C \cap S_{K'}^P. \]

There are two natural ways to break up the contribution of \( C \) to the Lefschetz formula as a sum of contributions from the pieces \( C_P \). In [GM] one of these two ways was chosen; it leads to the version of the Lefschetz formula given in that paper. However it is the other version that we are using here.

This alternative version differs in two respects from the one chosen in [GM]. The first is that it involves the Euler characteristic with compact support of \( \text{Fix}(P, x_0, \gamma) \) (rather than its Euler characteristic). The second is that neutral directions are treated as being contracting (rather than expanding); this change affects the definition of the set \( I(\gamma) \) appearing in (7.14), as we explain in more detail when we make the definition.

The subset \( \text{Fix}(P, x_0, \gamma) \) of the fixed point set is a disjoint union of certain sets of the form \( C_P \), and from [GM] we see that the total contribution of \( \text{Fix}(P, x_0, \gamma) \) to the Lefschetz formula is given by the product of three factors:

1. the Euler characteristic with compact support of \( \text{Fix}(P, x_0, \gamma) \),
2. the ramification index

\[ r(x_0) := [N(\mathbb{A}_f) \cap x_0 K x_0^{-1} : N(\mathbb{A}_f) \cap x_0 K' x_0^{-1}] \]

of the map \( \tilde{c}_1 \) at any point in \( S_{K'}^P(x_0) \),
3. a factor \( L_P(\gamma) \) that depends only on the \( G(\mathbb{R}) \)-conjugacy class of the pair \( (P, \gamma) \) (and, of course, the representation \( E \) and the weight profile \( p \) as well).

We will review the precise form of the factor \( L_P(\gamma) \) later. All that matters for the moment is the property stated in (3). The discussion above shows that the Lefschetz formula (for the alternating sum of the traces of the self-maps (7.8.1)) is given by the following sum

\[ \sum_P \sum_{\gamma} (-1)^{\dim(A_I/A_M)} \cdot |M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot L_P(\gamma) \cdot O_\gamma(f_P), \]

where \( f_P \) is the locally constant compactly supported function on \( M(\mathbb{A}_f) \) defined by

\[ f_P := \sum_{x_0} r(x_0) f_P, x_0. \]

In the sum defining \( f_P \), the index \( x_0 \) runs over a set of representatives for the orbits of \( P(\mathbb{A}_f) \) on \( G(\mathbb{A}_f)/K' \), as before. In the first sum in (7.12.1) \( P \) runs through the standard parabolic subgroups of \( G \), and in the second sum \( \gamma \) runs through the set of \( M(\mathbb{Q}) \)-conjugacy classes of elements \( \gamma \in M(\mathbb{Q}) \) such that the fixed point set of \( \gamma \) in \( X_M \) is non-empty.
(7.13) Some familiar harmonic analysis. Let $P = MN$ be a standard parabolic subgroup of $G$. In (7.11) we fixed a Haar measure $dm$ on $M(\mathbb{A}_f)$. Now we fix a Haar measure $dg$ on $G(\mathbb{A}_f)$ as well. Pick a compact open subgroup $K_0$ of $G(\mathbb{A}_f)$ such that

$$G(\mathbb{A}_f) = P(\mathbb{A}_f)K_0.$$ 

Choose Haar measures $dn$ on $N(\mathbb{A}_f)$ and $dk$ on $K_0$ so that the usual integration formula holds:

$$\int_{G(\mathbb{A}_f)} f(g) \, dg = \int_{M(\mathbb{A}_f)} \int_{N(\mathbb{A}_f)} \int_{K_0} f(mnk) \, dk \, dn \, dm$$

for any $f$ in $C^\infty_c(G(\mathbb{A}_f))$, the space of all locally constant compactly supported functions on $G(\mathbb{A}_f)$. Let $\delta_{P(\mathbb{A}_f)}$ denote the modulus function on $P(\mathbb{A}_f)$; thus, for $x \in P(\mathbb{A}_f)$ we have

$$\delta_{P(\mathbb{A}_f)}(x) := |\det(\text{Ad}(x); \text{Lie}(N) \otimes \mathbb{A}_f)|_{\mathbb{A}_f},$$

where $| \cdot |_{\mathbb{A}_f}$ is the normalized absolute value on $\mathbb{A}_f$. Given $f \in C^\infty_c(G(\mathbb{A}_f))$ we define a function $f_M \in C^\infty_c(M(\mathbb{A}_f))$ in the usual way, by putting

$$f_M(m) := \delta_{P(\mathbb{A}_f)}^{-1/2}(m) \int_{N(\mathbb{A}_f)} \int_{K_0} f(k^{-1}nmk) \, dk \, dn.$$

The function $f_M$ depends on $P$ and even on $K_0$, but its orbital integrals do not. It is worth noting that its orbital integrals are one of the ingredients in Arthur’s trace formula [A].

We are interested in a particular function $f^\infty \in C^\infty_c(G(\mathbb{A}_f))$, namely the Hecke operator associated to the Hecke correspondence in (7.4). Explicitly, $f^\infty$ is by definition $\text{vol}_d(K')^{-1}$ times the characteristic function of the coset $Kg$ (with $K,g,K'$ as in (7.4)). Applying the discussion above to $f^\infty$, we get $f^\infty_M \in C^\infty_c(M(\mathbb{A}_f))$ (defined by (7.13.2), with $f$ replaced by $f^\infty$). In (7.12.2) we defined a function $f_P \in C^\infty_c(M(\mathbb{A}_f))$.

Lemma 7.13.A. The functions $f^\infty_M$ and $\delta_{P(\mathbb{A}_f)}^{-1/2} \cdot f_P$ have the same orbital integrals.

It is equivalent to prove that $\delta_{P(\mathbb{A}_f)}^{1/2} \cdot f_M^\infty$ and $f_P$ have the same orbital integrals. Note that although $f_P$ depends on the choice of representatives $x_0$ for the double cosets

$$P(\mathbb{A}_f) \backslash G(\mathbb{A}_f) / K',$$

its orbital integrals do not. Suppose that we replace $K'$ by a compact open subgroup $K''$ of $G(\mathbb{A}_f)$ contained in $K'$. Then $f^\infty$ is multiplied by the index $[K' : K'']$, as is $f^\infty_M$. An easy calculation shows that $f_P$ is also multiplied by $[K' : K'']$, as long as we take representatives for

$$P(\mathbb{A}_f) \backslash G(\mathbb{A}_f) / K''$$

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of the form $x_0k'$, where $x_0$ is one of our previous representatives and $k' \in K'$. Therefore, by shrinking $K'$ if necessary, it is enough to prove the lemma in the case that $K'$ is contained in $K_0$. Then we have

$$\delta_{\mathcal{H}^f}(m) f_{M}^\infty(m) = \sum_{x \in \mathcal{H}^f / K'} \text{vol}_{dk}(K') \int_{N(\mathcal{H}^f)} f^\infty(x^{-1}nmx) \, dn.$$  

Thus $\delta_{\mathcal{H}^f} \cdot f_{M}^\infty$ has the same orbital integrals as the function of $m \in M(\mathcal{H}^f)$ given by

$$\sum_{x \in (\mathcal{H}^f \cap K_0) \setminus K_0 / K'} a(x) \int_{N(\mathcal{H}^f)} f^\infty(x^{-1}nmx) \, dn,$$

where

$$a(x) = \text{vol}_{dk}(K') \cdot [P(\mathcal{H}^f) \cap K_0 : P(\mathcal{H}^f) \cap xK'x^{-1}] = \text{vol}_{dg}(K') \cdot \text{vol}_{dm,n}(P(\mathcal{H}^f) \cap xK'x^{-1})^{-1}.$$  

Here we used that

$$\text{vol}_{dm,n}(P(\mathcal{H}^f) \cap K_0) = \text{vol}_{dg}(K_0) \cdot \text{vol}_{dk}(K_0)^{-1} = \text{vol}_{dg}(K') \cdot \text{vol}_{dk}(K')^{-1},$$

which follows from (7.13.1), applied to the characteristic function of $K_0$. From the equality

$$G(\mathcal{H}^f) = P(\mathcal{H}^f)K_0$$

it follows that

$$(P(\mathcal{H}^f) \cap K_0) \setminus K_0 / K' \simeq P(\mathcal{H}^f) \setminus G(\mathcal{H}^f) / K'.$$

Thus the elements $x$ used here can serve as the elements $x_0$ used to define $f_P$. Moreover it is easy to see that

$$\int_{N(\mathcal{H}^f)} f^\infty(x^{-1}nmx) \, dn$$

is 0 unless $m$ lies in the image in $M(\mathcal{H}^f)$ of

$$P(\mathcal{H}^f) \cap xKgx^{-1},$$

in which case it equals

$$\text{vol}_{dn}(N(\mathcal{H}^f) \cap xKx^{-1}) \cdot \text{vol}_{dg}(K')^{-1};$$

this shows that (with $x_0 = x$)

$$a(x) \int_{N(\mathcal{H}^f)} f^\infty(x^{-1}nmx) \, dn = r(x_0) f_{P,x_0}(m).$$

The proof of the lemma is now complete.
(7.14) Manipulation of the Lefschetz formula. From Lemma 7.13.A we see that the orbital integral $O_\gamma(f_P)$ appearing in (7.12.1) can be rewritten as

$$O_\gamma(f_P) = \delta_{P(A)}^{1/2}(\gamma) \cdot O_\gamma(f_M^\infty),$$

with $f_M^\infty$ as in (7.13). Moreover, since $\gamma \in M(\mathbb{Q})$ the product formula shows that

$$\delta_{P(A)}(\gamma) = \delta_{P(R)}^{-1}(\gamma),$$

where for $x \in P(\mathbb{R})$ we put

$$\delta_{P(R)}(x) = |\det(\text{Ad}(x); \text{Lie}(N) \otimes \mathbb{R})|.$$

Therefore the Lefschetz formula (7.12.1) can be rewritten as

$$(7.14.1) \sum \sum |M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot O_\gamma(f_M^\infty) \cdot (-1)^{\dim(A_I/A_M)} \cdot \delta_{P(R)}^{-1/2}(\gamma) \cdot L_P(\gamma),$$

with the two index sets for the sum the same as in (7.12.1).

The factors $|M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1}$ and $\chi(I)$ depend only on $M_\gamma$. The factor $O_\gamma(f_M^\infty)$ depends only on the $G(\mathbb{Q})$-conjugacy class of the pair $(M, \gamma)$. Therefore we could rewrite (7.14.1) by grouping together terms according to the $G(\mathbb{Q})$-conjugacy class of $(M, \gamma)$. However, there are still more terms that can be grouped together, and it is this phenomenon that we must study next.

The elements $\gamma$ appearing in (7.14.1) are semisimple, since they are required to fix some point in $X_M$. We now fix a semisimple element $\gamma \in G(\mathbb{Q})$ and consider the set $\mathcal{M}_\gamma$ of Levi subgroups of $G$ such that $\gamma \in M(\mathbb{Q})$. Recall that $M$ coincides with the centralizer $\text{Cent}_G(A_M)$ of $A_M$ in $G$ (where $A_M$ denotes the maximal $\mathbb{Q}$-split torus in the center of $M$, in line with the notational conventions established in (7.1)). Thus a Levi subgroup $M$ of $G$ contains $\gamma$ if and only if $A_M$ is contained in $G^0_\gamma$, the identity component of the centralizer of $\gamma$ in $G$. We are now going to define a map $M \mapsto M^*$ from $\mathcal{M}_\gamma$ to itself. Let $M \in \mathcal{M}_\gamma$. Put $I := M^0_\gamma$ and note that $A_I$ contains $A_M$. Then put

$$M^* := \text{Cent}_G(A_I) = \text{Cent}_M(A_I).$$

Clearly $M^*$ is a member of $\mathcal{M}_\gamma$ and $M^*$ is contained in $M$.

**Lemma 7.14.A.** The following statements hold.

1. $\gamma \in I \subset M^*$.
2. $I = (M^*)^0_\gamma$.
3. $A_{M^*} = A_I$.
4. $M^{**} = M^*$.
5. Suppose that $M^* = M$ and that $M_I$ is a Levi subgroup of $G$ containing $M$. Then $M^{*}_I = M$ if and only if $(M_I)^0_\gamma = M^0_\gamma$.

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The first statement is obvious. The second statement follows from the first and the fact that $M^*$ is contained in $M$. As for the third statement, the inclusion $A_{M^*} \supset A_I$ is trivial and the opposite inclusion follows from the obvious inclusion $A_{M^*} \subset A_{(M^*)_0}$, since $(M^*)_0 = I$ by the second statement. The fourth statement follows immediately from the second. Finally we verify the fifth statement. If $M^*_1 = M$, apply the second statement to $M_1$ to see that $(M^*_1)_0 = M_0$. Conversely, suppose that $(M^*_1)_0 = M_0$. Then, applying the third statement to both $M_1$ and $M$, we see that $A_{M^*_1} = A_M$, which in turn implies that $M^*_1 = M$.

The first and second statements of the lemma together allow us to apply the usual descent theory for orbital integrals. Put (as usual)

$$D^M_{M^*} (\gamma) := \det(1 - \text{Ad}(\gamma); \text{Lie}(M)/\text{Lie}(M^*)) \in \mathbb{Q}.$$ 

Since $M^*_0 = (M^*)_0$, it follows that

$$D^M_{M^*} (\gamma) \neq 0,$$

and from descent theory we have the equality

$$(7.14.2) \quad O_{\gamma}(f^\infty_M) = |D^M_{M^*} (\gamma)|^{1/2} \cdot O_{\gamma}(f^\infty_M).$$

Furthermore the product formula implies that

$$|D^M_{M^*} (\gamma)|^{1/2} = |D^M_{M^*} (\gamma)|^{-1/2}.$$

The Lefschetz formula (7.14.1) can now be rewritten as

$$(7.14.3) \quad \sum_{(P,M,\gamma)} |M_\gamma(Q)/I(Q)|^{-1} \cdot \chi(I) \cdot O_{\gamma}(f^\infty_M) \cdot (-1)^{\dim(A_I/A_M)} \cdot |D^M_{M^*} (\gamma)|^{1/2} \cdot \delta^{-1/2}_{P(\mathbb{R})}(\gamma) \cdot L_P(\gamma),$$

where the sum runs through a set of representatives for the $G(\mathbb{Q})$-conjugacy classes of triples $(P,M,\gamma)$ consisting of a parabolic subgroup $P$ of $G$, a Levi factor $M$ of $P$ and an element $\gamma \in M(\mathbb{Q})$, satisfying the condition that the fixed point set $X^\gamma_M$ of $\gamma$ on $X_M$ be nonempty. As usual we write $I$ for $M^*_0$. We may impose the additional condition that the real group $(I/A_I)_{\mathbb{R}}$ contain some anisotropic maximal $\mathbb{R}$-torus, since otherwise $\chi(I) = 0$ (see (7.10)).

For any triple $(P,M,\gamma)$ satisfying these conditions the real group $(M^*/A_{M^*})_{\mathbb{R}}$ contains some anisotropic maximal $\mathbb{R}$-torus, and moreover $\gamma$ is elliptic in $M^*(\mathbb{R})$ (in other words, is contained in some maximal $\mathbb{R}$-torus in $M^*$ that is anisotropic modulo $A_{M^*}$). Indeed, by assumption there exists a maximal torus $T$ in $I$ over $\mathbb{R}$ such that $T/A_I$ is anisotropic over $\mathbb{R}$. But $T$ is also a maximal torus in $M^*$ (use the second statement in Lemma
Therefore the Lefschetz formula (7.14.3) can be rewritten as

\[
\sum_{(M,\gamma)} \chi(I) \cdot O_{\gamma}(f_M^{\infty}) \cdot \sum_{(P_1,M_1)} |(M_1)_\gamma(Q)/I(Q)|^{-1} \cdot |D_{M_1}^M(\gamma)|^{1/2} \\
\cdot (-1)^{\dim(A_M/A_{M_1})} \cdot \delta_{P_1(R)}^{-1/2}(\gamma) \cdot L_{P_1}(\gamma).
\] (7.14.4)

In (7.14.4) \(I\) again denotes \(M_\gamma^0\), and the index sets for the sums are as follows. The first sum runs over a set of representatives for the \(G(Q)\)-conjugacy classes of pairs \((M,\gamma)\), where \(M\) is a Levi subgroup of \(G\) and \(\gamma \in M(Q)\), satisfying the conditions that \((M/A_M)_R\) contain some anisotropic maximal \(\mathbb{R}\)-torus and that \(\gamma\) be elliptic in \(M(R)\). Write \(N_G(M)\) for the normalizer of \(M\) in \(G\). Then the second sum in (7.14.4) runs over a set of representatives for the \(N_G(M)(Q) \cap G_\gamma(Q)\)-conjugacy classes of pairs \((P_1,M_1)\) consisting of a parabolic subgroup \(P_1\) in \(G\) and a Levi factor \(M_1\) of \(P_1\), satisfying the three conditions

1. \(M_1 \supset M\) (which implies that \(\gamma \in M_1(Q)\)),
2. \(M_1^\gamma = M\),
3. \(X_{M_1}^\gamma\) is non-empty.

Consider two pairs \((M,\gamma)\) and \((P_1,M_1)\) appearing in (7.14.4), and let \(I = M_\gamma^0\), as before. Our assumptions on \((M,\gamma)\) imply that \((I/A_I)_R\) contains some anisotropic maximal \(\mathbb{R}\)-torus and that \(A_I = A_M\) (note in passing that this implies that \(M^* = M\)). Let \(n(P_1,M_1)\) denote the number of elements in the conjugacy class of \((P_1,M_1)\) under \(N_G(M)(Q) \cap G_\gamma(Q)\). We are now going to check that

\[
|(M_1)_\gamma(Q)/I(Q)| \cdot n(P_1,M_1) = |(N_G(M)(Q) \cap G_\gamma(Q))/I(Q)|.
\] (7.14.5)

Indeed, it follows from the equality \(A_M = A_I\) that

\[
N_G(M)(Q) = N_G(A_M)(Q) = N_G(A_I)(Q).
\]

The stabilizer of \((P_1,M_1)\) in \(G(Q)\) is \(M_1(Q)\), and hence its stabilizer in \(N_G(M)(Q) \cap G_\gamma(Q)\) is \(N_{M_1}(M)(Q) \cap (M_1)_\gamma(Q)\). Since any element of \((M_1)_\gamma(Q)\) normalizes the split component \(A_I\) of \((M_1)_\gamma = I\), we see that

\[
N_{M_1}(M)(Q) \supset (M_1)_\gamma(Q),
\]

and therefore the stabilizer of \((P_1,M_1)\) in \(N_G(M)(Q) \cap G_\gamma(Q)\) is simply \((M_1)_\gamma(Q)\), which proves (7.14.5). Therefore the Lefschetz formula (7.14.4) can be rewritten as

\[
\sum_{(M,\gamma)} |(N_G(M)(Q) \cap G_\gamma(Q))/I(Q)|^{-1} \cdot \chi(I) \cdot O_{\gamma}(f_M^{\infty}) \cdot (-1)^{\dim(A_M/A_G} \cdot L_M(\gamma).
\] (7.14.6)
where the index set for the sum is the same as that for the first sum in (7.14.4), and where

\[ L_M(\gamma) := \sum_{(P_1, M_1)} (-1)^{\dim A_{M_1}/A_G} |D_{M_1}^M(\gamma)|^{1/2} \cdot \delta_{P_1(\mathbb{R})}^{-1/2}(\gamma) \cdot L_{P_1}(\gamma), \]

with the index set equal to the set of all pairs \((P_1, M_1)\) satisfying (1), (2), (3) (in other words, we no longer divide by the action of \(N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q})\) on the set of pairs).

Note that condition (2) is equivalent to the condition that the connected centralizers of \(\gamma\) in \(M\) and \(M_1\) coincide (recall that \(M\) and apply Lemma 7.14.A(5)), and this in turn is equivalent to the condition that \(D_{M_1}^M(\gamma)\) be non-zero. Therefore condition (2) can be dropped without changing (7.14.7).

Our assumptions on \((M, \gamma)\) imply that there exists a maximal torus \(T\) of \(G\) over \(\mathbb{R}\) containing \(\gamma\) such that \(T/A_M\) is anisotropic over \(\mathbb{R}\). The element \(\gamma\) can be written as

\[ \gamma = \exp(x) \cdot \gamma_1 \]

for unique elements \(x \in \mathfrak{A}_M\) and \(\gamma_1 \in T(\mathbb{R})_1\), where \(T(\mathbb{R})_1\) denotes the maximal compact subgroup of \(T(\mathbb{R})\) (see §5 for the definition of \(\mathfrak{A}_M\)). It is easy to see that \(M_1\) satisfies condition (3) if and only if \(x\) belongs to the subspace \(\mathfrak{A}_{M_1}\) of \(\mathfrak{A}_M\). We conclude that

\[ L_M(\gamma) = \sum_Q (-1)^{\dim(A_L/A_G)} \cdot |D_L^M(\gamma)|^{1/2} \cdot \delta_{Q(\mathbb{R})}^{-1/2}(\gamma) \cdot L_Q(\gamma) \]

where the sum ranges over all parabolic subgroups \(Q = LU\) containing \(M\) such that \(x\) lies in \(\mathfrak{A}_L\) (here, as usual, \(L\) denotes the unique Levi component of \(Q\) containing \(M\) and \(U\) denotes the unipotent radical of \(Q\)).

At this point we need to be more precise about the complex numbers \(L_Q(\gamma)\), and in order to do so we must first discuss weight profiles. As in §5 let \(\nu\) be an element in \((\mathfrak{A}_{P_0})^*\) whose restriction to \(\mathfrak{A}_G\) coincides with the character by which \(A_G\) acts on \(E\). Then \(\nu\) determines a weight profile (see §1.1 of [GHM]) in the following way. As in §5 \(\nu\) determines elements

\[ \nu_P \in \mathfrak{A}_P^* \]

for every \(P \in \mathcal{F}(M_0)\) and in particular for every standard parabolic subgroup \(P\). The weight profile we associate to \(\nu\) is the one such that the restriction to the stratum \(S^P_K\) (for standard \(P = MN\)) of the \(i\)-th cohomology sheaf of the weighted cohomology complex \(\mathbf{E}_K\) is the local system associated to the finite dimensional representation

\[ H^i(\text{Lie}(N), E)_{\geq \nu_P} \]

of \(M\) defined in §5. See Proposition 17.2 of [GHM] in order to see how to describe this weight profile in the language of that paper; note that in [GHM] the subspace

\[ H^i(\text{Lie}(N), E)_{\geq \nu_P} \]

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is denoted by

\[ H^i(n_P, E)_+ . \]

Now we are in a position to give a precise formula for \( L_Q(\gamma) \). Here, as above, \( Q = LU \) is a parabolic subgroup containing \( M \) such that \( x \) lies in the subspace \( A_L \) of \( A_M \). This formula is essentially the one given in [GM, §11], although that paper only discusses two particular weight profiles and uses the other natural way of breaking up the Lefschetz formula as a sum over strata of fixed point components. Let \( \alpha_1, \ldots, \alpha_n \in A^*_L \) be the simple roots of \( A_L \) in \( \text{Lie}(U) \). Let \( I = \{1, \ldots, n\} \) (we temporarily reuse the notation \( I \) in this way for the sake of compatibility with [GM]). Put

\[ I(\gamma) := \{ i \in I \mid \langle \alpha_i, x \rangle < 0 \} \]

(the set of "expanding directions"). In [GM] the set \( I(\gamma) \) is defined instead by the condition

\[ \langle \alpha_i, x \rangle \leq 0; \]

this, together with the use of Euler characteristics rather than Euler characteristics with compact support, is the other natural way of breaking up the Lefschetz formula.

Choose a Borel subgroup \( B \) of \( G \) over \( \mathbb{C} \) containing \( T \) and contained in \( Q \), and let \( W' \) be the corresponding Kostant representatives for the cosets \( W_L \backslash W \). Let \( \lambda_B, \rho_B \in X^*(T)_{\mathbb{R}} \) be as usual (see §5). Let \( t_1, \ldots, t_n \in A_L/A_G \) be the basis of \( A_L/A_G \) dual to the basis \( \alpha_1, \ldots, \alpha_n \) of \( (A_L/A_G)^* \). For \( w \in W' \) put

\[ I(w) = \{ i \in I \mid \langle p_L(w(-\lambda_B - \rho_B + \rho_B) + \nu_Q, t_i) \rangle > 0 \}. \]

In [GM] the set \( I(w) \) was defined using \( \leq \) rather than \( > \), but this was a misprint and should have been \( \geq \). If we take \( \nu = \nu_m \), our set \( I(w) \) still differs \( (\geq \) rather than \( \geq \) from the (corrected) one in [GM]: there are two middle weighted cohomology complexes, upper and lower, and the formula in [GM] arises from one of them while the formula here arises from the other.

The complex number \( L_Q(\gamma) \) is defined by

\[ (-1)^{|I(\gamma)|} \sum_w \epsilon(w) \cdot \text{tr}(\gamma^{-1}; V_{w(\lambda_B + \rho_B) - \rho_B}^L) \]

where the index set for the sum is the set of \( w \in W' \) such that \( I(w) = I(\gamma) \). Perhaps the following remarks will help the reader understand the number \( L_Q(\gamma) \). The local contribution to the Lefschetz fixed point formula from a connected component of the fixed point set is given (see [GM2]) by the (weighted) cohomology (of a regular neighborhood of the fixed point component) with supports which are compact in the expanding directions and which are closed in the contracting directions. The piece of weighted cohomology which is indexed by \( w \in W' \) (via Kostant’s theorem) contributes to stalk cohomology with compact supports in the directions \( I(w) \) and with degree shift by \( |I(w)| \). The Hecke
correspondence is expanding away from the stratum $S_K^Q$ in the directions $I(\gamma)$. Thus the contributions to the Lefschetz formula occur when $I(w) = I(\gamma)$.

As in §5 let $C_Q$ be the (open) chamber in $\mathfrak{A}_L$ corresponding to $Q \in \mathcal{P}(L)$; of course the image of the cone $C_Q$ in $\mathfrak{A}_L/\mathfrak{A}_G$ is generated by $t_1, \ldots, t_n$. Again as in §5 let $\varphi_Q$ denote the function

$$\varphi_{C_Q}(\cdot, \cdot)$$

on $\mathfrak{A}_L \times \mathfrak{A}_L^*$ determined by the open cone $C_Q$ (see the last part of Appendix A). By Lemma A.7 and the analog of (A.2) mentioned just before Lemma A.7 the value of $\varphi_Q(-x, p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q)$ is given by

$$(-1)^{\dim(A_G)} \cdot (-1)^{\dim(A_L/A_G) - |I(\gamma)|}$$

if $I(w) = I(\gamma)$ and is 0 otherwise. Therefore

$$L_Q(\gamma) = \sum_{w \in W'} \epsilon(w) \cdot (-1)^{\dim(A_L)} \cdot \varphi_Q(-x, p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q)$$

\[ \cdot \operatorname{tr}(\gamma^{-1}; V_{w(\lambda_B + \rho_B) - \rho_B}). \]

This expression for $L_Q(\gamma)$ coincides exactly with the one used to define $L'_Q(\gamma)$ in §5. Moreover, comparing the definition of $L'_M(\gamma)$ given in §5 with (7.14.8), we now see that the numbers $L'_M(\gamma)$ from §5 and $L_M(\gamma)$ from this section are equal. Applying Theorem 5.1 yields the following result (we now let $I$ once again denote $M^0$).

**Theorem 7.14.B.** The Lefschetz formula for the alternating sum of the traces of the self-maps (7.8.1) is given by

$$\sum_{(M, \gamma)} |(N_G(M)(Q) \cap G_\gamma(Q))/I(Q)|^{-1} \cdot \chi(I) \cdot O_\gamma(f_M^Q) \cdot (-1)^{\dim(A_M/A_G)} \cdot \Phi_M(\gamma; \Theta_\nu)$$

with $\Theta_\nu$ and $\Phi_M(\gamma; \Theta_\nu)$ as in §5, the index set for the sum being the same as that for the first sum in (7.14.4).

**Q-equivalence.** There are three special cases in which the statement of Theorem 7.14.B can be simplified, but before we can do this we need to make a couple of definitions. Let $\Theta, \Theta'$ be stable virtual characters on $G(\mathbb{R})$. We say that $\Theta$ and $\Theta'$ are Q-equivalent if they agree on $T_{\text{reg}}(\mathbb{R})$ for every maximal $\mathbb{R}$-torus $T$ in $G$ whose $\mathbb{R}$-split component is both defined and split over $\mathbb{Q}$. Note that $\Theta$ and $\Theta'$ are Q-equivalent if and only if the functions $\Phi_M(\cdot, \Theta)$ and $\Phi_M(\cdot, \Theta')$ coincide for every Levi subgroup $M$ of $G$ over $\mathbb{Q}$ such that $(M/A_M)_{\mathbb{R}}$ contains an anisotropic maximal $\mathbb{R}$-torus. Clearly the expression for the Lefschetz formula given in Theorem 7.14.B remains valid when $\Theta_\nu$ is replaced by any stable virtual character $\Theta'$ that is $\mathbb{Q}$-equivalent to $\Theta_\nu$. 

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(7.16) **The orientation character** $\chi_G$. We will also need the following sign character

$$\chi_G : G(\mathbb{R}) \to \{ \pm 1 \}.$$ 

Recall from 7.2 the real manifold

$$X_G = G(\mathbb{R})/(K_G \cdot A_G(\mathbb{R})^0),$$

on which $G(\mathbb{R})$ acts by left translations. Of course $X_G$ is diffeomorphic to a Euclidean space and hence is orientable. For $g \in G(\mathbb{R})$ we define $\chi_G(g)$ to be $-1$ if $g$ reverses the orientation of $X_G$ and $+1$ if $g$ preserves the orientation of $X_G$.

Let $P = MN$ be a parabolic subgroup of $G$ whose Levi component $M$ contains $M_0$. Suppose that $M$ contains a maximal $\mathbb{R}$-torus $T$ such that $T/A_M$ is anisotropic over $\mathbb{R}$. We claim that

$$\chi_G(\gamma) = \text{sgn} (\det (\gamma; \text{Lie}(N)))$$

for all $\gamma \in T(\mathbb{R})$.

Indeed, since $P(\mathbb{R})$ acts transitively on $X_G$, we see that as an $M(\mathbb{R})$-space $X_G$ is given by

$$X_G = \text{Lie}(N) \times X_M \times (A_M(\mathbb{R})^0/A_G(\mathbb{R})^0).$$

It follows that

$$\chi_G(m) = \chi_M(m) \cdot \text{sgn} (\det (m; \text{Lie}(N)))$$

for all $m \in M(\mathbb{R})$. Since $T(\mathbb{R})/A_M(\mathbb{R})$ is connected, and since $A_M(\mathbb{R})$ acts trivially on $X_M$, we see that $\chi_M$ is trivial on $T(\mathbb{R})$, so that (7.16.1) follows from (7.16.3).

(7.17) **Very positive** $\nu$. Now suppose that $\nu$ is sufficiently positive. Then the weighted cohomology complex $E_K$ is equal to $Rj_*E_K$, where $j$ denotes the inclusion of $S_K$ in $S_K$, and the global weighted cohomology is the cohomology with compact support of $S_K$ with coefficients in $E_K$. Moreover it is obvious from the definition of $\Theta_\nu$ that $\Theta_\nu$ is equal to (the character of) the representation $E^*$ contragredient to $E$ (for sufficiently positive $\nu$ the virtual modules $E^*_P$ used to define $\Theta_\nu$ are trivial except when $P = G$). Therefore in this case the expression for the Lefschetz formula given by Theorem 7.14.B involves the character of the representation $E^*$, as was also observed by J. Franke [F] and G. Harder.

(7.18) **Very negative** $\nu$. Now suppose that $\nu$ is sufficiently negative. Then the weighted cohomology complex $E_K$ is equal to the full direct image $Rj_*E_K$, and the global weighted cohomology is the ordinary cohomology of $S_K$ with coefficients in $E_K$. Moreover we claim that $\Theta_\nu$ is $\mathbb{Q}$-equivalent to (the character of) the virtual representation $(-1)^{\dim(X_G)}\chi_G \otimes E^*$ of $G(\mathbb{R})$, where $X_G$ is the space defined in 7.2 and $\chi_G$ is the orientation character defined in 7.16.
Let \( T \) be a maximal \( \mathbb{R} \)-torus in \( G \) whose \( \mathbb{R} \)-split component \( A \) is defined and split over \( \mathbb{Q} \). Let \( M \) be the centralizer of \( A \) in \( G \), a Levi subgroup of \( G \) over \( \mathbb{Q} \). Replacing \( T \) by a conjugate under \( G(\mathbb{Q}) \), we may assume that \( M \) contains \( M_0 \). We must show that for all \( \gamma \in T(\mathbb{R}) \)

\[
\Phi_M(\gamma, \Theta_\nu) = (-1)^{\dim(X_G)} \cdot \chi_G(\gamma) \cdot \Phi_M(\gamma, E^*).
\]

As usual we write

\[
\gamma = \exp(x) \cdot \gamma_1
\]

for unique elements \( x \in \mathfrak{A}_M \) and \( \gamma_1 \in T(\mathbb{R})_1 \). By continuity it is enough to prove (7.18.1) when \( \gamma \) is regular in \( T(\mathbb{R}) \) and \( x \) is regular in \( \mathfrak{A}_M \).

We use Theorem 5.1 to evaluate \( \Phi_M(\gamma, \Theta_\nu) \). Since we are assuming that \( \nu \) is sufficiently negative, the factor

\[
\varphi_Q(-x, p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q)
\]

entering into the definition of \( L_Q(\gamma) \) is 0 unless \( L = M \) and \( x \) lies in the chamber \( C_Q \) in \( \mathfrak{A}_M \) determined by \( Q \), in which case the factor is \((-1)^{\dim(A_M)}\) (use Lemma A.7 and the analog for \( \varphi_Q \) of (A.2) to evaluate \( \varphi_Q(-x, \mu) \) for very positive \( \mu \)). Therefore Theorem 5.1 tells us that \( \Phi_M(\gamma, \Theta_\nu) \) is equal to

\[
(-1)^{\dim(A_M/A_G)} \cdot \delta_Q^{-1/2}(\gamma)
\]

times

\[
\sum_{w \in W'} \epsilon(w) \cdot \text{tr}(\gamma^{-1}; V^M_{w(\lambda_B + \rho_B) - \rho_B})
\]

where \( Q = MN \) is the unique parabolic subgroup with Levi component \( M \) such that \( x \) lies in the chamber \( C_Q \). The number (7.18.2) is equal to

\[
\text{tr}(\gamma^{-1}; E) \cdot \text{tr}(\gamma; \wedge^* (\text{Lie}(N))),
\]

where \( \wedge^* (\text{Lie}(N)) \) denotes the virtual \( M \)-module

\[
\sum_{i \geq 0} (-1)^i \wedge^i (\text{Lie}(N))
\]

Therefore, in order to prove the equality (7.18.1) it is enough to prove that

\[
(-1)^{\dim(A_M/A_G)} \cdot \delta_Q^{-1/2}(\gamma) \cdot \prod_{\alpha}(1 - \alpha(\gamma)) = (-1)^{\dim(X_G)} \cdot \chi_G(\gamma) \cdot |D^G_M(\gamma)|^{1/2},
\]
where the product is taken over roots $\alpha$ of $T$ in $\text{Lie}(N)$. The two sides of (7.18.3) have the same absolute value, and therefore it is enough to prove that the number

\begin{equation}
(7.18.4) \quad (-1)^{\dim(X_G)} \cdot (-1)^{\dim(A_M/A_G)} \cdot \chi_G(\gamma) \cdot \prod_\alpha (1 - \alpha(\gamma))
\end{equation}

is positive. Using (7.16.1) we see that the number (7.18.4) has the same sign as

\begin{equation}
(-1)^{\dim(X_G)} \cdot (-1)^{\dim(A_M/A_G)} \cdot \prod_\alpha (\alpha^{-1}(\gamma) - 1).
\end{equation}

Complex conjugation preserves the set of roots $\alpha$ of $T$ in $\text{Lie}(N)$. If $\alpha$ is not real, then the product of $\alpha^{-1}(\gamma) - 1$ and $\bar{\alpha}^{-1}(\gamma) - 1$ is positive. If $\alpha$ is real, then $\alpha^{-1}(\gamma) - 1$ is negative (use that $x$ lies in the chamber $C_\mathbb{Q}$). Therefore the sign of the number (7.18.4) is equal to

\begin{equation}
(-1)^{\dim(X_G)} \cdot (-1)^{\dim(A_M/A_G)} \cdot (1 - \dim(N)).
\end{equation}

It follows from (7.16.2) that

$$\dim(X_G) = \dim(A_M/A_G) + \dim(N) + \dim(X_M).$$

Moreover $\dim(X_M)$ is even since $(M/A_M)_{\mathbb{R}}$ contains an anisotropic maximal $\mathbb{R}$-torus. Therefore the sign of the number (7.18.4) is indeed $+1$, as we wished to show.

(7.19) **Middle $\nu$.** Now suppose that $\nu = \nu_m$. Then $\overline{E}_K$ is the upper middle weighted cohomology complex. Assume further that $(G/A_G)_{\mathbb{R}}$ has an anisotropic maximal $\mathbb{R}$-torus. Then by Theorem 5.2 the virtual character $\Theta_\nu$ is $\mathbb{Q}$-equivalent to $\Theta^{E^\ast}$. Thus the expression for the Lefschetz formula given in Theorem 7.14.B essentially agrees with Arthur’s formula [A,Theorem 6.1] for the alternating sum of the traces of a Hecke operator on the $L^2$-cohomology groups of $S_K$ with coefficients in $E_K$. Actually Theorem 7.14.B appears at first to disagree with Arthur’s formula, which contains the factor $\Phi_M(\gamma, \Theta^E)$ rather than $\Phi_M(\gamma, \Theta^{E^\ast})$, but by replacing $\gamma$ by $\gamma^{-1}$ in one of the two formulas we come closer to agreement, since Arthur uses the usual left action of the Hecke algebra on the $L^2$-cohomology of $S_K$, while in this paper we have (implicitly) used the right action of the Hecke algebra on the upper middle weighted cohomology of $S_K$. In other words it is the right action of $f^\infty$ on $H^i(\overline{S}_K, \overline{E}_K)$ that coincides with the self-map defined by our Hecke correspondence $(\bar{c}_1, \bar{c}_2)$; of course the right action of $f^\infty$ coincides with the left action of the Hecke operator $f^\infty_r$ defined by

$$f^\infty_r(x) = f^\infty(x^{-1}) \quad (x \in G(\mathbb{A}_f)).$$

We should also note that the index set for the sum in our formula is somewhat different from the index set for the double sum in Arthur’s formula, and that our formula has the factor

$$|\{(N_G(M) \cap G_\gamma(Q))/I(Q)\}^{-1}$$
while Arthur’s has the factor

$$|W_0^M||W_0^G|^{-1}|\mu^M(\gamma)|^{-1};$$

however it is easy to see that these are just two different ways of writing the same thing.

However, our formula disagrees with Arthur’s in that we sum only over Levi subgroups $M$ such that $M/A_M$ contains an anisotropic maximal torus over $\mathbb{R}$. In [A] all Levi subgroups $M$ are allowed, although the term indexed by $M$ vanishes (due to the factor $\Phi_M$) unless $M/A_M$ contains an elliptic maximal torus over $\mathbb{R}$. Thus Arthur’s formula may have more non-zero terms than ours (the split component of the center of $M$ over $\mathbb{R}$ may be strictly bigger than $A_M$); however these extra terms in Arthur’s formula should not actually be there (the error occurs in equation (4.1) of [A], which is only valid if the split components of the center of $M$ over $\mathbb{Q}$ and $\mathbb{R}$ are the same).

A. Functions associated to convex polyhedral cones

Let $X$ be a finite dimensional real vector space and let $C$ be a closed convex polyhedral cone in $X$. Recall that this means that there exists a finite subset $S$ of $X$ such that $C$ is equal to the set of non-negative linear combinations of elements of $S$; equivalently, it means that there exists a finite subset $T$ of the dual vector space $X^*$ such that $C$ is equal to the intersection of the sets

$$\{x \in X \mid \lambda(x) \geq 0\}$$

where $\lambda$ ranges through $T$. We denote by $C^*$ the closed convex polyhedral cone in $X^*$ dual to $C$. Recall that $C^*$ is defined by

$$C^* := \{\lambda \in X^* \mid \lambda(x) \geq 0 \text{ for all } x \in C\}$$

and that the map

$$F \mapsto F^\perp := C^* \cap \{\lambda \in X^* \mid \lambda(x) = 0 \text{ for all } x \in F\}$$

sets up a bijection between the set $\mathcal{F}$ of (closed) faces of $C$ to the set $\mathcal{F}^*$ of (closed) faces of $C^*$. Moreover $C^{**} = C$ and $F^{\perp \perp} = F$. The map $F \mapsto F^\perp$ is order-reversing (we order faces by inclusion) and

$$\dim(F) + \dim(F^\perp) = \dim(X),$$

where $\dim(F)$ denotes the dimension of the linear span of $F$.

Define an integer-valued function $\psi_C$ on $X \times X^*$ by

$$\psi_C = \sum_{F \in \mathcal{F}} (-1)^{\dim(F)} \xi_{(F^\perp)^* \times F^*},$$

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where we have denoted by \( \xi_{(F^\perp)^* \times F^*} \) the characteristic function of the subset \( (F^\perp)^* \times F^* \) of \( X \times X^* \) (since \( F^\perp \) and \( F \) are themselves closed convex polyhedral cones, it makes sense to consider their dual cones \( (F^\perp)^* \) and \( F^* \)). Note that \( (F^\perp)^* \) is equal to \( C + \text{span}(F) \), where \( \text{span}(F) \) denotes the linear span of \( F \). Clearly

\[
(A.1) \quad \psi_C^*(\lambda, x) = (-1)^{\dim(X)} \psi_C(x, \lambda).
\]

Let \( X_1 \) denote the linear span of \( C \) and let \( X_2 \) denote the largest linear subspace of \( X \) contained in \( C \). In \( X^* \) we have the perpendicular subspaces

\[
X^*_i = \{ \lambda \in X^* | \lambda(x) = 0 \text{ for all } x \in X_i \} \quad (i = 1, 2).
\]

The cone \( C \) gives rise to a cone \( \tilde{C} \) in \( X_1/X_2 \) and it is easy to see that \( \psi_C(x, \lambda) \) is 0 unless \( x \in X_1 \) and \( \lambda \in X^*_2 \), in which case

\[
(A.2) \quad \psi_C(x, \lambda) = (-1)^{\dim(X_2)} \psi_{\tilde{C}}(\tilde{x}, \tilde{\lambda})
\]

where \( \tilde{x}, \tilde{\lambda} \) denote the images of \( x, \lambda \) under the natural surjections

\[
X_1 \to X_1/X_2 \\
X^*_2 \to X^*_2/X^*_1 = (X_1/X_2)^*
\]

respectively.

Suppose that \( C \) is a simplicial cone in \( X \). Then the pair \( (X, C) \) is isomorphic to the pair \( (\mathbb{R}^n, (\mathbb{R}_{\geq 0})^n) \), where \( n = \dim X \) and \( \mathbb{R}_{\geq 0} \) denotes the set of non-negative real numbers. We will now calculate \( \psi_C \) for the pair \( (\mathbb{R}^n, (\mathbb{R}_{\geq 0})^n) \). Of course we identify \( X^* \) with \( \mathbb{R}^n \) as well. Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \). Write \( I \) for \( \{1, \ldots, n\} \) and then put

\[
I_x = \{ i \in I | x_i \geq 0 \} \\
I_\lambda = \{ i \in I | \lambda_i \geq 0 \}.
\]

**Lemma A.1.** The number \( \psi_C(x, \lambda) \) is 0 unless the subsets \( I_x \) and \( I_\lambda \) of \( I \) are complementary, in which case

\[
\psi_C(x, \lambda) = (-1)^{|I_\lambda|},
\]

where \( |I_\lambda| \) denotes the cardinality of \( I_\lambda \).

Indeed, the faces of \( C \) are indexed by the subsets \( J \subset I \), the face corresponding to \( J \) being

\[
\{ x \in \mathbb{R}^n | x_i \geq 0 \text{ for all } i \in J \text{ and } x_i = 0 \text{ for all } i \in I \setminus J \}.
\]

Thus

\[
\psi_C(x, \lambda) = \sum_J (-1)^{|J|}
\]
where $J$ ranges through all subsets of $I$ such that

$$(I \setminus I_x) \subset J \subset I,$$

and the lemma follows from the elementary fact that for any finite set $S$ the sum

$$\sum_{T \subset S} (-1)^{|T|}$$

is 0 unless $S$ is empty, in which case it is 1.

In the next lemma and corollary it will be convenient to assume that

$$\dim(C) = \dim(C^*) = \dim(X).$$

We refer to the faces of $C$ having codimension 1 in $C$ as facets of $C$, and we refer to the linear spans of the facets of $C$ as the walls of $C$. The walls of $C$ form a finite set of hyperplanes in $X$. We say that a point in $X$ is $C$-regular if it lies on no wall of $C$. We refer to the connected components of the set of $C$-regular points in $X$ as chambers in $X$. Of course the interior of $C$ is one of these chambers. Similarly the walls of $C^*$ divide $X^*$ into chambers. The value of $\psi_C(x, \lambda)$ on $C$-regular $x \in X$ and $C^*$-regular $\lambda \in X^*$ depends only on the chambers in which $x, \lambda$ lie.

**Lemma A.2.** Let $x, x'$ be $C$-regular elements of $X$ and suppose that there is exactly one wall $Y$ of $C$ separating $x$ and $x'$ (in other words $x, x'$ lie in adjacent chambers). Put $C_Y := C \cap Y$, a facet of $C$ which can also be regarded as a closed convex polyhedral cone in $Y$, so that the function $\psi_{C_Y}$ on $Y \times Y^*$ is defined. Assume that $x, C$ lie on the same side of $Y$. Then

$$\psi_C(x, \lambda) - \psi_C(x', \lambda) = \psi_{C_Y}(y, \lambda_Y),$$

where $\lambda_Y \in Y^*$ is the restriction of $\lambda$ to $Y$ and $y \in Y$ is the unique point of $Y$ lying on the line segment joining $x$ and $x'$.

Let $\alpha \in C^*$ be a non-zero element of the 1-dimensional face $(C_Y)^\perp$ of $C^*$. Thus $Y$ is the kernel of the linear form $\alpha$, and $\alpha$ is positive on $x$, negative on $x'$. Since $x, x'$ lie in adjacent chambers, $\beta(x)$ and $\beta(x')$ have the same sign for any non-zero element $\beta$ of any 1-dimensional face of $C^*$ other than $(C_Y)^\perp$, and this sign is the same as that of $\beta(y)$. Therefore

$$\xi_{(F^\perp)^* \times F^*}(x, \lambda) - \xi_{(F^\perp)^* \times F^*}(x', \lambda)$$

is 0 unless $F$ is contained in $Y$ (equivalently, unless $F^\perp$ contains $\alpha$), in which case it is equal to

$$\xi_{G \times H}(y, \lambda_Y)$$

where $G = (F^\perp)^* \cap Y$ and $H$ is the image of $F^*$ under the canonical surjection from $X^*$ to $Y^*$. But the faces of $C_Y$ are precisely the faces $F$ of $C$ contained in $Y$, and, writing
\( F_Y \) to indicate that we are regarding such a face \( F \) as a face of \( C_Y \), we have \( F^*_Y = H \) and

\[
(F^\perp_Y)^* = C_Y + \text{span}(F) = (C + \text{span}(F)) \cap Y = (F^\perp)^* \cap Y = G.
\]

This proves the lemma.

Applying the lemma to \( C^* \) and using (A.1), we obtain the following result.

**Corollary A.3.** Let \( \lambda, \lambda' \) be \( C^* \)-regular elements of \( X \) and suppose that there is exactly one wall \( Z \) of \( C^* \) separating \( \lambda \) and \( \lambda' \). Let \( \omega \in C \) be a non-zero element of the 1-dimensional face \((C^* \cap Z)^\perp\) of \( C \) corresponding to the facet \( C^* \cap Z \) of \( C^* \). Put \( \tilde{X} := X/\mathbb{R} \omega \) and let \( \tilde{C} \) denote the image of \( C \) under the canonical surjection \( X \to \tilde{X} \); note that \( Z \) is the dual vector space to \( \tilde{X} \) and that \( \tilde{C} \) is the closed convex polyhedral cone \((C^* \cap Z)^* \) in \( \tilde{X} \) dual to the cone \( C^* \cap Z \) in \( Z \). In particular the function \( \psi_{\tilde{C}} \) on \( \tilde{X} \times Z \) is defined. Assume that \( \lambda, C^* \) lie on the same side of \( Z \) (equivalently, assume that \( \lambda(\omega) > 0 \) and \( \lambda'(\omega) < 0 \)). Then

\[
\psi_C(x, \lambda) - \psi_C(x, \lambda') = -\psi_{\tilde{C}}(\tilde{x}, \tilde{\lambda})
\]

where \( \tilde{x} \) denotes the image of \( x \) under the canonical surjection \( X \to \tilde{X} \) and \( \tilde{\lambda} \) denotes the unique point of \( Z \) lying on the line segment joining \( \lambda \) and \( \lambda' \).

Now we come to the main point of the appendix, which is to show that if \( C \) decomposes as a union of cones, then \( \psi_C \) decomposes accordingly. For this we need a little preparation. For any closed convex polyhedral cone \( C \) in \( X \) we write \( \hat{C} \) for its relative interior (the interior of \( C \) in \( \text{span}(C) \)). We say that a function \( f : X \to Z \) is **conic** if it can be written as a finite \( Z \)-linear combination of characteristic functions \( \xi_C \) of closed convex polyhedral cones \( C \) in \( X \). Let \( \lambda \) be a non-zero element of \( X^* \), let \( C \) be a closed convex polyhedral cone in \( X \), and put

\[
C_+ = \{ x \in C \mid \lambda(x) \geq 0 \},
C_- = \{ x \in C \mid \lambda(x) \leq 0 \},
C_0 = \{ x \in C \mid \lambda(x) = 0 \}.
\]

Then \( C_+, C_-, C_0 \) are closed convex polyhedral cones in \( X \), and we have the relation

\[
(A.3) \quad \xi_C + \xi_{C_0} - \xi_{C_+} - \xi_{C_-} = 0.
\]

It is not difficult to see that the abelian group \( C(X) \) of conic functions on \( X \) is presented by the generators \( \xi_C \) and the relations (A.3).
The characteristic function $\xi_{\overset{\circ}{C}}$ of the relative interior $\overset{\circ}{C}$ of $C$ is a conic function on $X$. Indeed it can be expressed in the following way in terms of our generators

\begin{equation}
(\xi_{\overset{\circ}{C}}) = (-1)^{\dim(C)} \sum_{F} (-1)^{\dim(F)} \xi_{F},
\end{equation}

where the sum is taken over the set of faces $F$ of $C$. To prove this note that the value of the right-hand side of (A.4) at a point $x \in X$ is 0 unless $x \in C$, in which case it is

\begin{equation}
(-1)^{\dim(C)} \sum_{F} (-1)^{\dim(F)}
\end{equation}

where $F$ now ranges through all faces of $C$ containing $F(x)$, the smallest face of $C$ containing $x$. These faces correspond bijectively to the faces of a new cone, namely $C + \text{span}(F(x))$. Therefore (A.4) follows from the following well-known fact: for any closed convex polyhedral cone $C$

\begin{equation}
\sum_{F} (-1)^{\dim(F)} = \left\{ \begin{array}{ll}
(-1)^{\dim(C)} & \text{if } C \text{ is a linear subspace of } X, \\
0 & \text{otherwise,}
\end{array} \right.
\end{equation}

where the sum is taken over all faces $F$ of $C$ (to prove this fact reduce to the case in which $C \neq \{0\}$ and $C$ contains no non-zero linear subspace and note that in this case $C \setminus \{0\}$ is contractible, hence has Euler characteristic 1; on the other hand this Euler characteristic is equal to

\begin{equation}
- \sum_{F} (-1)^{\dim(F)}
\end{equation}

where $F$ ranges through all faces of $C$ other than $\{0\}$).

We now define a homomorphism $f \mapsto f^*$ from $C(X)$ to itself by putting

\begin{equation}
(\xi_{C})^* := (-1)^{\dim(C)} \xi_{\overset{\circ}{C}}^*.
\end{equation}

Of course we must check that if this definition is applied to the left-hand side of (A.3), we get 0. We may as well assume that $\lambda$ takes both strictly positive and strictly negative values on $C$ (otherwise the relation (A.3) is itself trivial). Then

\begin{align*}
\dim(C_{+}) &= \dim(C_{-}) = \dim(C) \\
\dim(C_{0}) &= \dim(C) - 1
\end{align*}

and $\overset{\circ}{C}$ is the disjoint union of $\overset{\circ}{C}_{+}, \overset{\circ}{C}_{-}$ and $\overset{\circ}{C}_{0}$, which gives what we want.

Next we define a homomorphism $f \mapsto \hat{f}$ from $C(X)$ to $C(X^*)$ by putting

\begin{equation}
(\xi_{C})^* := (-1)^{\dim(X) - \dim(C^*)} \xi_{\overset{\circ}{C}^*}^*.
\end{equation}
Here $\overset{\circ}{C}^*$ denotes the relative interior of the dual cone $C^*$. Again we must check that when we apply this definition to the left-hand side of (A.3), we get 0. We will see in a moment that the operation $*$ is an isomorphism; therefore it is enough to consider instead the operation obtained by following $\wedge$ by $*$ and multiplying by $(-1)^{\dim(X)}$; this operation sends $\xi_C$ to $\xi_{C^*}$. We must show that

$$\xi_{C^*} + \xi_{C_0^*} - \xi_{C_+^*} - \xi_{C_-^*} = 0.$$ 

This is an immediate consequence of the following well-known fact: if $C_1, C_2$ are closed convex cones whose union is convex, then $C_1^* \cup C_2^*$ is also convex and

$$(C_1 \cup C_2)^* = C_1^* \cap C_2^*$$

$$(C_1 \cap C_2)^* = C_1^* \cup C_2^*.$$ 

It is easy to see from the definitions of our two operations that $f^{**} = f$ and $f^\wedge \wedge^* = f$ for all $f \in C(X)$ (for the first equality use (A.4) and for the second use that $C^{**} = C$). It follows that both of our operations are isomorphisms. These operations are best understood by placing them in a more general context (see [KS],[M]), in which $*$ comes from Verdier duality and $\wedge$ from the Fourier-Sato transformation, but this point of view is not needed for what follows.

It is interesting to calculate $(\xi_\overset{\circ}{C})^\wedge$. We claim that

$$(\xi_\overset{\circ}{C})^\wedge = (-1)^{\dim(C)} \xi_{-C^*}.$$ 

We will now give an elementary proof of this consequence of Lemma 3.7.10(ii) of [KS]. Using (A.4), we see that (A.6) is equivalent to the equality

$$\sum_{F \in \mathcal{F}(C)} (-1)^{\dim(F)} \xi_F^\wedge = (\xi_{-C^*})^*,$$

which is in turn equivalent to

$$\sum_{F \in \mathcal{F}(C)} (-1)^{\dim(F)} \xi_{F^*} = (-1)^{\dim(X) - \dim(C^*)} \xi_{-\overset{\circ}{C}^*}.$$ 

Let $\lambda \in X^*$. The value of the left-hand side of (A.7) at $\lambda$ is

$$\sum_{F \in \mathcal{F}_\lambda} (-1)^{\dim(F)}$$

where $\mathcal{F}_\lambda$ denotes the set of faces $F$ of $C$ such that $\lambda$ is non-negative on $F$. Let $C_+$ be the set

$$C_+ := \{x \in C \mid \lambda(x) \geq 0\}.$$
We claim that (A.8) is equal to

\[(A.9) \quad \sum_{F \in \mathcal{F}(C_+)} (-1)^{\dim(F)},\]

If \(\lambda = 0\), the claim is obvious, so we assume that \(\lambda \neq 0\). Then \(\mathcal{F}(C_+)\) contains \(\mathcal{F}_\lambda\). Elements of \(\mathcal{F}(C_+)\) that are not in \(\mathcal{F}_\lambda\) arise in pairs \(F \cap C_+\) and \(F \cap \ker(\lambda)\), one pair for each face \(F\) of \(C\) such that \(\ker(\lambda)\) meets \(\bar{F}\) but does not contain \(\bar{F}\). For such \(F\)

\[\dim(F \cap \ker(\lambda)) = \dim(F \cap C_+) - 1;\]

and therefore each such pair of faces contributes 0 to (A.9), and the claim follows.

It follows from (A.5) (applied to \(C_+\)) that (A.9) is equal to

\[
\begin{cases}
(-1)^{\dim(C_+)} & \text{if } C_+ \text{ is a linear subspace,} \\
0 & \text{otherwise.}
\end{cases}
\]

But \(C_+\) is a linear subspace if and only if \(\lambda \leq 0\) on \(C\) and \(C \cap \ker(\lambda)\) is a linear subspace. If \(\lambda \leq 0\) on \(C\), then \(C \cap \ker(\lambda)\) is a face of \(C\), and therefore \(C \cap \ker(\lambda)\) is a linear subspace if and only if it is equal to the unique minimal face of \(C\). Thus \(C_+\) is a linear subspace if and only if \(\lambda \in -\bar{C}^*\), in which case \(C_+ = C \cap \ker(\lambda)\) is the minimal face of \(C\) and thus has dimension equal to

\[\dim(X) - \dim(C^*).\]

This completes the proof of the equality (A.7).

We say that an integer-valued function on \(X \times X^*\) is \textit{biconic} if it is a finite \(\mathbb{Z}\)-linear combination of characteristic functions \(\xi_{C \times D}\), where \(C\) (respectively, \(D\)) is a closed convex polyhedral cone in \(X\) (respectively, \(X^*\)). Of course the functions \(\psi_C\) considered earlier are biconic functions on \(X \times X^*\). In fact we are now going to generalize \(\psi_C\) by associating to any conic function \(f\) on \(X\) a biconic function \(\psi_f\) on \(X \times X^*\). If \(f\) is the characteristic function \(\xi_C\) of a closed convex polyhedral cone \(C\) we define \(\psi_f\) to be the function \(\psi_C\) defined earlier. Since any conic function \(f\) can be written as a \(\mathbb{Z}\)-linear combination

\[f = \sum_C n_C \xi_C\]

for a finite set of closed convex polyhedral cones \(C\) and integers \(n_C\), we are then forced to define \(\psi_f\) by

\[\psi_f := \sum_C n_C \psi_C.\]

Of course we must show that \(\psi_f\) depends only on \(f\) and not on the way in which we write it as a \(\mathbb{Z}\)-linear combination of characteristic functions of cones. This can be done
directly, but instead we will use the two operations introduced earlier to give a shorter proof. For any biconic function \( \psi \) and any \( \lambda \in X^* \) the function \( \psi(\cdot, \lambda) \) on \( X \) is conic, and therefore we can apply the operation \( * \) to \( \psi \) in the variable \( x \in X \) (holding \( \lambda \) fixed); this operation sends \( \xi_{C \times D} \) to \((-1)^{\dim(C)} \xi_{C \times D} \), which is again biconic. It follows that applying \( * \) to \( \psi \) in the variable \( x \in X \) yields a biconic function. Similarly, applying \( * \) to \( \psi \) in the variable \( \lambda \in X^* \) yields a biconic function. Any biconic function is conic on \( X \times X \) and so we may apply the operation \( \wedge \) on \( X \times X \) to any biconic function \( \psi \) on \( X \times X \). Since the dual of \( X \times X \) is \( X^* \times X \), which we may identify with \( X \times X^* \) by switching the order of the two factors, we may regard \( \wedge \) as an operation taking biconic functions on \( X \times X \) to biconic functions on \( X \times X^* \) (it is clear that \( \psi \) is biconic, not just conic).

For a biconic function \( \psi \) on \( X \times X^* \) we denote by \( \psi' \) the biconic function on \( X \times X^* \) obtained by applying \( * \) in the variable \( \lambda \in X^* \) to the biconic function \( \psi \). We will use the operation \( \psi \mapsto \psi' \) to show that \( f \mapsto \psi_f \) is well-defined. It is enough to show that

\[
\sum_C n_C \psi_C'
\]

depends only on \( f \). We need to find a simple expression for \((\psi_C)\)'\; by definition it is

\[
\sum_F \xi_{\hat{F} \times F^\perp}.
\]

But for any \( x \in \hat{F} \) we have

\[
F^\perp = C^* \cap \{ \lambda \in X^* \mid \lambda(x) = 0 \}.
\]

Therefore \((\psi_C)'(x, \lambda)\) is 0 unless \( \lambda(x) = 0 \), in which case it is equal to

\[
\sum_F \xi_{\hat{F} \times C^*}(x, \lambda) = \xi_{C \times C^*}(x, \lambda).
\]

Our problem has been reduced to the following: for \((x, \lambda) \in X \times X^* \) such that \( \lambda(x) = 0 \) we must show that

\[(A.10) \quad \sum_C n_C \xi_{C \times C^*}(x, \lambda) \]

depends only on \( f \). In fact we will show more: for such \((x, \lambda)\) the expression \((A.10)\) is equal to

\[(A.11) \quad [f^* \xi_{X_+}](x) \]

where \( X_+ \) is the set

\[
X_+ = \{ x \in X \mid \lambda(x) \geq 0 \}.
\]
Since this statement is linear in \( f \), we may assume without loss of generality that \( f = \xi_C \). Then (A.10) is equal to \( \xi_{C \times C^*}(x, \lambda) \), which is in turn equal to
\[
\begin{cases}
\xi_C(x) & \text{if } C \subset X_+,
0 & \text{otherwise}.
\end{cases}
\]
By definition \( f^* \) equals \((-1)^{\dim(C)} \xi_C \) and therefore the product \( f^* \xi_{X_+} \) equals
\[
(-1)^{\dim(C)} \xi_{\hat{C} \cap X_+}.
\]

There are three cases. If \( C \) is contained in \( X_+ \), then \( \hat{C} \cap X_+ = \hat{C} \) and \([f^* \xi_{X_+}]^* = \xi_C \), so that (A.11) agrees with (A.10) in this case. If \( C \) is not contained in \( X_+ \) but is contained in \( X_- \), where
\[ X_- = \{ x \in X \mid \lambda(x) \leq 0 \}, \]
then \( \hat{C} \cap X_+ \) is empty and therefore (A.11) is 0, which again agrees with (A.10). If \( C \) is contained in neither \( X_+ \) nor \( X_- \), then we are in the situation considered during the proof that \( * \) is well-defined, and \( \hat{C} \) is the disjoint union of \( \hat{C}_+ \), \( \hat{C}_- \) and \( \hat{C}_0 \), where \( C_+ = C \cap X_+ \), \( C_- = C \cap X_- \) and \( C_0 = C \cap \ker(\lambda) \). Therefore \( \hat{C} \cap X_+ \) is the disjoint union of \( \hat{C}_+ \) and \( \hat{C}_0 \), and it follows that
\[
[f^* \xi_{X_+}]^* = (-1)^{\dim(C)} [\xi_{\hat{C}_+} + \xi_{\hat{C}_0}]^* = \xi_{C_+} - \xi_{C_0}.
\]
For \( x \) such that \( \lambda(x) = 0 \), we have
\[
(\xi_{C_+} - \xi_{C_0})(x) = \xi_C(x) - \xi_C(x) = 0,
\]
which shows that (A.11) again agrees with (A.10). This concludes the proof that \( \psi_f \) is well-defined.

The following proposition is an easy consequence of the fact that \( \psi_f \) is well-defined.

**Proposition A.4.** Let \( C \) be a closed convex polyhedral cone and suppose that \( \hat{C} \) is the disjoint union of the relative interiors \( \hat{C}_1, \ldots, \hat{C}_r \) of \( r \) closed convex polyhedral cones \( C_1, \ldots, C_r \). Then
\[
\psi_C = \sum_{i=1}^r (-1)^{\dim(C) - \dim(C_i)} \psi_{C_i}.
\]
Indeed our hypothesis is that
\[
\xi_{\hat{C}} = \sum_{i=1}^r \xi_{\hat{C}_i}.
\]
Applying the operation \( \ast \), we find that
\[
(-1)^{\dim(C)} \xi_C = \sum_{i=1}^{r} (-1)^{\dim(C_i)} \xi_{C_i}.
\]

Applying the map \( f \mapsto \psi_f \), we then get the desired equality.

**Proposition A.5.** For any conic function \( f \) on \( X \) the quantity \( \psi_f(x, \lambda) \) vanishes unless \( \lambda(x) \leq 0 \).

The conic function \( f \) can be written as a \( \mathbb{Z} \)-linear combination of characteristic functions of closed convex polyhedral cones \( C \) that are simplicial as cones in their linear spans. Therefore it is enough to prove the proposition in the case that \( f = \xi_C \) for such a cone \( C \). By (A.2) we may assume that \( C \) spans \( X \) and hence that \( C \) is simplicial in \( X \). Then the proposition follows easily from Lemma A.1.

The function \( \psi_C \) is equal to \( \psi_f \) for the conic function \( f = \xi_C \). We can also use the *open* cone \( \overset{\circ}{C} \) to obtain an equally useful variant \( \varphi_{\overset{\circ}{C}} \) of \( \psi_C \) by putting
\[
\varphi_{\overset{\circ}{C}} := \psi_g,
\]
where \( g \) is the conic function \( \xi_{\overset{\circ}{C}} \).

**Lemma A.6.** There is an equality
\[
\varphi_{\overset{\circ}{C}}(x, \lambda) = \sum_{F \in \mathcal{F}} (-1)^{\dim(F)} \xi_{\overset{\circ}{C} + \text{span}(F)}(x) \xi_{F^*}(\lambda).
\]

The cone \( \overset{\circ}{C} + \text{span}(F) \) is the relative interior of the cone \( C + \text{span}(F) \), whose faces are in one-to-one correspondence with the faces \( G \) of \( C \) containing \( F \), via the map \( G \mapsto G + \text{span}(F) \). Applying (A.4) to \( C + \text{span}(F) \), we see that the right-hand side of the equality we are trying to prove is equal to
\[
(A.12) \quad \sum_{F} (-1)^{\dim(F)} \sum_{\{G \in \mathcal{F} | G \ni F \}} (-1)^{\dim(C) - \dim(G)} \xi_{G + \text{span}(F)}(x) \xi_{F^*}(\lambda).
\]

Applying (A.4) to \( C \), we see that
\[
\varphi_{\overset{\circ}{C}} = \sum_{G \in \mathcal{F}} (-1)^{\dim(C) - \dim(G)} \psi_G,
\]
and by writing out the definition of \( \psi_G \), we see that \( \varphi_{\overset{\circ}{C}}(x, \lambda) \) is equal to the expression \((A.12)\). This proves the lemma.
Note that the expression for $\varphi_C(x, \lambda)$ given by Lemma A.6 is almost the same as the expression for $\psi_C(x, \lambda)$ given by its definition; the only difference is that the factor

$$\xi_{C+\text{span}(F)}(x)$$

appearing in the definition of $\psi_C(x, \lambda)$ is replaced by

$$\xi_{\bar{C}+\text{span}(F)}(x)$$

in the expression for $\varphi_C(x, \lambda)$. Since $\bar{C} + \text{span}(F)$ is the relative interior of $C + \text{span}(F)$, we conclude that for any conic function $f$ on $X$, the biconic function $\psi_f^*$ associated to $f^*$ is obtained from $\psi_f$ by applying the operation $*$ to $\psi_f$ in the first variable. Moreover we conclude that

(A.13) \hspace{1cm} \varphi_{\bar{C}}(x, \lambda) = \psi_C(x, \lambda) \text{ if } x \text{ is } C\text{-regular.}

It is no surprise that $\varphi_{\bar{C}}$ behaves just about the same way as $\psi_C$. For example $\varphi_{\bar{C}}$ satisfies the obvious analog of (A.2) (replace $C, \bar{C}$ in (A.2) by their relative interiors). Now suppose that $C$ is a simplicial cone in $X$ and use the same notation as in Lemma A.1. We need one more bit of notation: put

$$\bar{I_x} = \{i \in I \mid x_i > 0\}.$$

Then it is easy to see that $\varphi_{\bar{C}}$ satisfies the following analog of Lemma A.1.

**Lemma A.7.** The number $\varphi_{\bar{C}}(x, \lambda)$ is 0 unless the subsets $\bar{I_x}$ and $I_\lambda$ of $I$ are complementary, in which case

$$\varphi_{\bar{C}}(x, \lambda) = (-1)^{|I_\lambda|},$$

where $|I_\lambda|$ denotes the cardinality of $I_\lambda$.

### B. Combinatorial lemma of Langlands

In this appendix we generalize a combinatorial lemma of Langlands [A1, Lemma 6.3]. Let $X$ be a finite dimensional real vector space, and let $(\cdot, \cdot)$ be a positive definite symmetric bilinear form on $X$, with associated metric

$$d(x, y) = (x - y, x - y)^{1/2}.$$

Let $C$ be a closed convex polyhedral cone in $X$. Let $x \in X$. Since $C$ is closed there exists a point $x_0 \in C$ that is closest to $x$, and since $C$ is convex, the point $x_0 \in C$ is
unique. Again by convexity this closest point can be characterized as the unique point $x_0 \in C$ such that
\[
d(x, x_0) \leq d(x, x_0 + r(y - x_0))
\]
for all $y \in C$ and all real numbers $r \in [0, 1]$. For fixed $y \in C$ the truth of the inequality above for all $r \in [0, 1]$ is equivalent to the inequality
\[(x - x_0, y - x_0) \leq 0.
\]
Thus, since $C$ is a cone, $x_0 \in C$ is characterized by the property that
\[(x - x_0, z) \leq 0
\]
for all $z \in Z$, where $Z$ is the set of all elements in $X$ of the form $y - rx_0$ for some $y \in C$ and some positive real number $r$.

Let $F$ be the unique face of $C$ such that $x_0$ lies in the relative interior $\overset{\circ}{F}$ of $F$. We claim that $Z$ is equal to $C + \text{span}(F)$. Clearly $Z$ is contained in $C + \text{span}(F)$. Moreover, in order to prove the reverse inclusion it is enough to show that $-F$ is contained in $Z$. Let $x_1 \in F$. Since $x_0 \in \overset{\circ}{F}$, there exist $x_2 \in F$ and a positive real number $r$ such that
\[x_1 - x_0 = -r(x_2 - x_0),
\]
which shows that $-x_1 \in Z$, as desired.

We use the inner product $(\cdot, \cdot)$ to identify $X$ with its dual. In particular we now view the dual cone $C^*$ as subset of $X$ itself:
\[C^* = \{x \in X \mid (x, y) \geq 0 \text{ for all } y \in C\}.
\]
As in Appendix A we denote by $F^\perp$ the face $C^* \cap \text{span}(F)^\perp$ of $C^*$ determined by the face $F$ of $C$. Of course $F^\perp$ is equal to
\[(C + \text{span}(F))^*.
\]
We conclude that $x_0 \in \overset{\circ}{F}$ is the point of $C$ closest to $x$ if and only if
\[x - x_0 \in -F^\perp,
\]
in which case $x_0$ is the orthogonal projection of $x$ on $\text{span}(F)$ and $x - x_0$ is the orthogonal projection of $x$ on $\text{span}(F)^\perp$. In particular the set of all points $x \in X$ such that the point $x_0$ in $C$ closest to $x$ lies in $\overset{\circ}{F}$ is equal to
\[\overset{\circ}{F} \oplus (-F^\perp).
\]
We have written $\oplus$ rather than $+$ in order to emphasize that the cones $\circ F$ and $- F^\perp$ lie in the complementary subspaces $\text{span}(F)$ and $\text{span}(F)^\perp$ respectively. Let $\xi_{F \oplus (-F^\perp)}$ denote the characteristic function of the subset $F \oplus (-F^\perp)$ of $X$. We conclude that

\begin{equation}
1 = \sum_{F \in \mathcal{F}} \xi_{F \oplus (-F^\perp)}(x)
\end{equation}

for all $x \in X$, where $\mathcal{F}$ denotes the set of faces of $C$.

The equality (B.1) is a (generalization of) a simple special case of Langlands’s combinatorial lemma. We will now use this special case to prove the general case.

We continue to identify $X$ with its dual, so that the function $\psi_C$ on $X \times X^*$ (see Appendix A) becomes a function on $X \times X$. For any subspace $Y$ of $X$ we denote by $p_Y$ the orthogonal projection map from $X$ onto $Y$. Now we can state our generalization of the combinatorial lemma of Langlands (take $C$ to be simplicial and use Lemma A.1 to recover the usual form of the lemma).

**Lemma B.1.** For $x, y \in X$ there is an equality

\begin{equation}
\sum_{F \in \mathcal{F}} \psi_{C + \text{span}(F)}(-y, -p_{\text{span}(F)^\perp}(x)) \cdot \xi_{F}(p_{\text{span}(F)}(x)) = (-1)^{\dim(C)} \xi_C(y).
\end{equation}

The faces of $C + \text{span}(F)$ are precisely the cones $G + \text{span}(F)$, where $G$ ranges through the set of faces of $C$ containing $F$; note that for such $G$

\[
\text{span}(G + \text{span}(F)) = \text{span}(G)
\]
\[
\dim(G + \text{span}(F)) = \dim(G)
\]
\[
(C + \text{span}(F)) + \text{span}(G + \text{span}(F)) = C + \text{span}(G).
\]

Therefore by the definition of $\psi_{C + \text{span}(F)}$ the left-hand side of the equality in the lemma is equal to

\[
\sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{F}(G) : G \supseteq F} (-1)^{\dim(G)} \xi_{C + \text{span}(G)}(-y) \cdot \xi_{(G + \text{span}(F))^\perp}(x) \cdot \xi_{F}(p_{\text{span}(F)}(x)),
\]

which by interchanging the order of summation we rewrite as

\[
\sum_{G \in \mathcal{F}} (-1)^{\dim(G)} \xi_{C + \text{span}(G)}(-y) \cdot \sum_{F \in \mathcal{F}(G)} \xi_{(G + \text{span}(F))^\perp}(x) \cdot \xi_{F}(p_{\text{span}(F)}(x)),
\]

where $\mathcal{F}(G)$ denotes the set of faces of $G$. The second sum is 1 by (B.1) (applied to $G$). Therefore the double sum reduces to

\[
\sum_{G \in \mathcal{F}} (-1)^{\dim(G)} \xi_{C + \text{span}(G)}(-y).
\]
Applying equality (A.7) to $C^*$, we see that this last expression is equal to 

$$(-1)^{\dim(C)} \xi_{\check{C}}(y).$$

This completes the proof of the lemma.

There are two special cases of Lemma B.1 that are worth noting. First, if $y \in \check{C}$, it is easy to see that Lemma B.1 reduces to the equality (B.1). Second, if $y \in -C$, then Lemma B.1 reduces to the following result.

**Corollary B.2.** There is an equality

$$\sum_{F \in \mathcal{F}} (-1)^{\dim(F^\perp)} \xi_{(F^\perp)^\circ \oplus F} = \begin{cases} (-1)^{\dim(X) - \dim(C)} & \text{if } C \text{ is a linear subspace}, \\ 0 & \text{otherwise}. \end{cases}$$

Indeed, if $x \in C$, then $\xi_{(F^\perp)^*}(x) = 1$ for all $F \in \mathcal{F}$, and therefore

$$\psi_C(x, \lambda) = \sum_{F \in \mathcal{F}} (-1)^{\dim(F)} \xi_{F^*}(\lambda) = (-1)^{\dim(X) - \dim(C^*)} \xi_{\check{C}^*}(-\lambda)$$

(we used the equality (A.7)). Applying this to $C + \text{span}(F)$ instead of $C$, we see that if $y \in -C$, then

$$\psi_{C + \text{span}(F)}(-y, \lambda) = (-1)^{\dim(X) - \dim(F^\perp)} \xi_{(F^\perp)^\circ}(-\lambda).$$

Moreover, if $y \in -C$, then

$$\xi_{\check{C}}(y) = \begin{cases} 1 & \text{if } C \text{ is a linear subspace}, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore Lemma B.1 does reduce to Corollary B.2 when $y \in -C$.

**Corollary B.3.** The sum

$$\sum_{F \in \mathcal{F}} (-1)^{\dim(F)} \xi_{((F^\perp)^*)^\circ \oplus (F^*)^\circ}(x)$$

is 0 unless $C$ is a linear subspace and $x = 0$. Here the dual cone $(F^\perp)^*$ is taken inside the subspace

$$\text{span}(F)^\perp = \text{span}(F^\perp)$$

and the dual cone $F^*$ is taken inside the subspace $\text{span}(F)$.

This corollary can be derived from the previous one by applying the operation $*$ and then the operation $\wedge$. Note that the formula

$$\sum_{\{P_3 : P_3 \supset P_1\}} (-1)^{\dim(A_1/A_3)} \tau_1^3(H) \hat{\tau}_3(H) = 0$$

on p. 940 of [A1] is the special case of this corollary in which the cone $C$ is the closed chamber in $\mathfrak{A}_{M_1}/\mathfrak{A}_G$ determined by $P_1$ (still using Arthur’s notation).
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