COLIMITS OF CROSSED MODULES IN MODIFIED CATEGORIES OF INTEREST

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ABSTRACT. In this paper, we give the constructions of the coequalizer and coproduct objects for the category of crossed modules, in a modified category of interest (MCI). In other words, we prove that the corresponding category is finitely cocomplete.

1. Introduction. The notions of category of interest [18] and groups with operations [19] are date back to Higgins [15]. They both aim to unify various algebraic structures and their properties. Precisely the notion of groups with operations is given as a relaxed version of category of interest. Therefore, groups with operations do not capture some algebraic structures which categories of interest do – clearly, every category of interest is a group with operation as well. Although many well-known algebraic categories (such as groups, vector spaces, associative algebras, Lie algebras, etc.) are the essential examples of categories of interest, there are some others which are not. For instance, the categories of cat¹-Lie (associative, Leibniz, etc.) algebras are not categories of interest.

At this point, a new and more general type of this notion is introduced in [4] which is called a modified category of interest. It satisfies all axioms of the former notion except one, which is replaced by a new and modified one. According to this definition, every category of interest becomes a modified category of interest. Further examples of modified categories of interest are those, which are equivalent to the categories of crossed modules in the categories of groups, associative algebras, commutative algebras, dialgebras, Lie algebras, Leibniz algebras, etc.

A crossed module of groups [21] ∂: E → G is given by a group homomorphism together with a group action ⊳ of G on E satisfying the following relations (for all e, f ∈ E and g ∈ G):

\[ \partial(g \triangleright e) = g + \partial(e) - g, \quad \partial(e) \triangleright f = e + f - e. \]

Crossed modules are used for modeling homotopy systems of connected CW-complexes, and also for the classification of algebraic 2-types [17]. On the other hand, the category of crossed modules is also equivalent to the category of cat¹-groups [16] as well as to the categories of interest in the sense of [10, 11].

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definition of crossed module is adapted to modified categories of interest in [4] that unifies all crossed module structures of the algebraic structures we mentioned above. It is strongly recommended to see [6, 7] for a very detailed survey of crossed modules and related structures. Some categorical properties of crossed modules are examined in [1, 2, 3, 5, 8, 12, 13, 14, 20] for various algebraic structures. In fact, some of them are examples of modified categories of interest.

As the modified category of interest is the unification of many well-known algebraic structures and their properties, it is natural to ask whether it is possible to unify some categorical properties of crossed modules via modified categories of interest. In this context, constructions of limits (of crossed modules) in a modified category of interest are given in [13] that yields the completeness of the corresponding category. Following that study, in this paper, we prove that a category of crossed modules in modified categories of interest is (finitely) cocomplete, namely, it has all finite colimits.

2. Preliminaries. We recall some notions from [4] that will be used in the sequel.

Definition 2.1. Let \( C \) be a category of groups with a set of operations \( \Omega \) and with a set of identities \( E \), such that \( E \) includes the group identities and the following conditions hold. If \( \Omega_* \) is the set of \( i \)-ary operations in \( \Omega \), then:

(a) \( \Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \);
(b) the group operations (written additively : 0, −, +) are elements of \( \Omega_0, \Omega_1 \) and \( \Omega_2 \) respectively. Let \( \Omega'_2 = \Omega_2 \setminus \{+\} \), \( \Omega'_1 = \Omega_1 \setminus \{−\} \). Assume that if \( * \in \Omega_2 \), then \( \Omega'_2 \) contains \( *^o \) defined by \( x *^o y = y * x \) and assume \( \Omega_0 = \{0\} \);
(c) for each \( * \in \Omega'_2 \), \( E \) includes the identity \( x * (y + z) = x * y + x * z \);
(d) for each \( \omega \in \Omega'_1 \) and \( * \in \Omega'_2 \), \( E \) includes the identities \( \omega(x + y) = \omega(x) + \omega(y) \) and either the identity \( \omega(x * y) = \omega(x) * \omega(y) \) or the identity \( \omega(x * y) = \omega(x) * y \).

Denote by \( \Omega'_1S \) the subset of those elements in \( \Omega'_1 \), which satisfy the identity \( \omega(x * y) = \omega(x) * y \), and by \( \Omega'_2 \) all other unary operations, i.e. those which satisfy the first identity from (d).

Let \( C \) be an object of \( \mathbb{C} \) and \( x_1, x_2, x_3 \in C \):
(e) \( x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1 \), for each \( * \in \Omega'_2 \);
(f) For each ordered pair \((*, \overline{\tau}) \in \Omega'_2 \times \Omega'_2 \) there is a word \( W \) such that

\[
(x_1 * x_2) \overline{\tau} x_3 = W(x_1 x_2 x_3, x_1 x_3 x_2, x_2 x_3 x_1, (x_3 x_2) x_1, \ldots)
\]

for all ordered pairs \((*, \overline{\tau}) \in \Omega'_2 \times \Omega'_2 \).

Remark 1. Let us fix an arbitrary modified category of interest \( \mathbb{C} \) throughout this section.

Definition 2.2. Let \( A, B \) be two objects of \( \mathbb{C} \). A morphism in \( \mathbb{C} \) is a map \( f : A \rightarrow B \) commutes with all possible \( w \in \Omega'_1 \), such that

\[
f(a + a') = f(a) + f(a'),
f(a * a') = f(a) * f(a'),
\]

for all \( a, a' \in A, * \in \Omega'_2 \).
Example 2.3. The categories of groups, (commutative) algebras, modules over a ring, vector spaces, Lie algebras, Leibniz algebras, dialgebras are well-known examples of modified categories of interest.

However, there exist other well-known algebraic categories that are not modified categories of interest. For instance, the categories of Leibniz-Rinehart algebras, Hopf algebras, racks (or quandles), etc.

As we underlined in the introduction, the following are the essential examples of modified categories of interest (which are not categories of interest), and they were the main motivation to define modified categories of interest.

Example 2.4. The categories of cat\(^1\)-(commutative) algebras, cat\(^1\)-Lie algebras and cat\(^1\)-Leibniz algebras are also modified categories of interest.

Definition 2.5. Let \(B\) be an object of \(C\). A subobject of \(B\) is called an ideal if it is the kernel of some morphism. In other words, \(A\) is an ideal of \(B\) if \(A\) is a normal subgroup of \(B\), and \(a \ast b \in A\), for all \(a \in A\), \(b \in B\) and \(\ast \in \Omega_2\).

Definition 2.6. Let \(A, B\) be two objects of \(C\). An extension of \(B\) by \(A\) is a sequence

\[
0 \rightarrow A \xrightarrow{i} E \xrightarrow{p} B \rightarrow 0
\]

where \(p\) is surjective and \(i\) is the kernel of \(p\). We say that an extension is split if there exists a morphism \(s: B \rightarrow E\) such that \(ps = 1_B\).

Definition 2.7. The split extension induces a set of actions of \(B\) on \(A\) corresponding to the operations in \(C\) with being

\[
b \cdot a = s(b) + a - s(b),
\]

\[
b \ast a = s(b) \ast a,
\]

for all \(b \in B\), \(a \in A\) and \(\ast \in \Omega_2\).

Actions defined by the previous equations are called derived actions of \(B\) on \(A\).

Remark that we use the notation “\(\ast\)” to denote both the star operation and the star action.

Definition 2.8. Given an action of \(B\) on \(A\), the semi-direct product \(A \rtimes B\) is a universal algebra, whose underlying set is \(A \times B\), and the operations are defined by

\[
\omega(a, b) = (\omega(a), \omega(b)),
\]

\[
(a', b') + (a, b) = (a' + b' \cdot a, b' + b),
\]

\[
(a', b') \ast (a, b) = (a' \ast a + a' \ast b + b' \ast a, b' \ast b),
\]

for all \(a, a' \in A\), \(b, b' \in B\), \(\ast \in \Omega_2\).

Remark that, an action of \(B\) on \(A\) is a derived action, if and only if, \(A \rtimes B\) is an object of \(C\).

Theorem 2.9. Denote a general category of groups with operations of a modified category of interest \(C\) by \(C_G\). A set of actions of \(B\) on \(A\) in \(C_G\) is a set of derived actions, if and only if, it satisfies the following conditions:

1. \(0 \cdot a = a\),
2. \(b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2\),
3. \((b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)\),
4. \(b \ast (a_1 + a_2) = b \ast a_1 + b \ast a_2\),
5. \((b_1 + b_2) \ast a = b_1 \ast a + b_2 \ast a\).
6. \((b_1 * b_2) \cdot (a_1 * a_2) = a_1 * a_2\),  
7. \((b_1 * b_2) \cdot (a * b) = a * b\),  
8. \(a_1 * (b \cdot a_2) = a_1 * a_2\),  
9. \(b * (b_1 \cdot a) = b * a,\)  
10. \(\omega(b \cdot a) = \omega(b) \cdot \omega(a)\),  
11. \(\omega(a * b) = \omega(a) * b = a * \omega(b)\) for any \(\omega \in \Omega^1_\mathbb{S}\), and \(\omega(a * b) = \omega(a) * \omega(b)\) for any \(\omega \in \Omega^\prime\).  
12. \(x * y + z * t = z * t + x * y\),  
for each \(\omega \in \Omega^1_\mathbb{S}, \ * \in \Omega^2_\mathbb{S}, \ b, \ b_1, \ b_2 \in B, \ a, a_1, a_2 \in A\) and for \(x, y, z, t \in A \cup B\) whenever each side of 12 makes sense.

**Definition 2.10.** A “crossed module” \((C_1, C_0, \partial)\) in \(\mathbb{C}\) is given by a morphism \(\partial: C_1 \to C_0\) with a derived action of \(C_0\) on \(C_1\) such that

\[
\begin{align*}
\partial(c_0 \cdot c_1) &= c_0 + \partial(c_1) - c_0, \\
\partial(c_0 * c_1) &= c_0 * \partial(c_1),
\end{align*}
\]

\[
\begin{align*}
\partial(c_1) \cdot c'_1 &= c_1 + c'_1 - c_1, \\
\partial(c_1) * c'_1 &= c_1 * c'_1,
\end{align*}
\]

for all \(c_0 \in C_0, c_1, c'_1 \in C_1, \ * \in \Omega^2_\mathbb{S}\). Without the second condition, we call it a precrossed module.

A morphism between (pre)crossed modules \((C_1, C_0, \partial) \to (C'_1, C'_0, \partial')\) is a pair \((\mu_1, \mu_0)\) of morphisms \(\mu_0: C_0 \to C'_0\) and \(\mu_1: C_1 \to C'_1\), such that the diagram

\[
\begin{array}{ccc}
C_1 & \xrightarrow{\partial} & C_0 \\
\mu_1 \downarrow & & \downarrow \mu_0 \\
C'_1 & \xrightarrow{\partial'} & C'_0
\end{array}
\]

commutes and also

\[
\begin{align*}
\mu_1(c_0 \cdot c_1) &= \mu_0(c_0) \cdot \mu_1(c_1), \\
\mu_1(c_0 * c_1) &= \mu_0(c_0) * \mu_1(c_1),
\end{align*}
\]

for all \(c_0 \in C_0, c_1 \in C_1\) and \(* \in \Omega^2_\mathbb{S}\).

We denote the category of crossed modules by XMod, and similarly, of precrossed modules by PXMod.

The following two are the characteristic examples of crossed modules in any modified category of interest \(\mathbb{C}\).

**Example 2.11.** Let \(B\) be an object of \(\mathbb{C}\) and \(A\) is an ideal of \(B\). Then, the inclusion map \(A \to B\) becomes a crossed module where the action is defined via conjugation, namely

\[
b \cdot a = b + a - b, \\
b * a = b * a,
\]

for all \(a \in A\) and \(b \in B\).

**Example 2.12.** Let \(B\) be an object of \(\mathbb{C}\). Then, we have a natural crossed module \(0 \to B\) with the trivial action. More generally, if \(A\) is an abelian object (i.e. \(x + y = y + x\) and \(x * y = 0\), for all \(x, y \in A\) and \(* \in \Omega^2_\mathbb{S}\)), then the zero map \(A \to B\) defines a crossed module with any derived action, for all \(B\).
Considering Example 2.3, the following well-known crossed module definitions are particular examples of crossed modules in a modified category of interest.

**Example 2.13.** A crossed module of groups \([5]\) is a group homomorphism \(\partial : E \to G\), together with a group action \(\triangleright\) of \(G\) on \(E\) such that

\[
X1) \quad \partial(g \triangleright e) = g + \partial(e) - g,
\]

\[
X2) \quad \partial(e) \triangleright f = e + f - e,
\]

for all \(e, f \in E\) and \(g \in G\).

**Example 2.14.** A dialgebra crossed module \([9]\) is a dialgebra homomorphism \(\partial : D_1 \to D_0\) with a dialgebra action (via four bilinear maps) of \(D_0\) on \(D_1\), such that

\[
X1) \quad \partial(d_1 \triangleright_2 d_1') = d_1 + d_1' - \partial(d_1),
\]

\[
\partial(d_1 \triangleright_3 d_1') = d_1 + d_1' - \partial(d_1),
\]

\[
\partial(d_1 \triangleleft_1 d_0) = \partial(d_1) - d_0,
\]

\[
\partial(d_1 \triangleleft_2 d_0) = \partial(d_1) - d_0,
\]

\[
X2) \quad \partial(d_1) \triangleright_2 d_1' = d_1 + d_1' - \partial(d_1),
\]

\[
\partial(d_1) \triangleright_3 d_1' = d_1 + d_1' - \partial(d_1),
\]

\[
\partial(d_1) \triangleleft_1 d_0 = \partial(d_1) - d_0,
\]

\[
\partial(d_1) \triangleleft_2 d_0 = \partial(d_1) - d_0,
\]

for all \(d_1, d_1' \in D_1\), \(d_0 \in D_0\).

**Example 2.15.** A Lie algebra crossed module \([9]\) is a Lie algebra homomorphism \(\partial : \mathfrak{e} \to \mathfrak{g}\), together with a Lie algebra action \(\triangleright\) of \(\mathfrak{g}\) on \(\mathfrak{e}\) such that

\[
X1) \quad \partial([g \triangleright e]) = [g, \partial(e)],
\]

\[
X2) \quad \partial(e) \triangleright f = [e, f],
\]

for all \(e, f \in \mathfrak{e}\) and \(g \in \mathfrak{g}\).

3. Colimits in \(\text{XMod}/C_0\). From now on, \(\mathbb{C}\) will be a fixed MCI where \(\mathbb{E}\) includes the identity \(x + y = y + x\). Remark that Lie algebras, (commutative) associative algebras, dialgebras are all examples of \(\mathbb{C}\).

**Definition 3.1.** Consider the subcategory \(\text{XMod}/C_0\) of crossed modules with a fixed codomain \(C_0\). Its objects will be called crossed \(C_0\)-modules, and the morphism between \((E, \partial_E)\) and \((D, \partial_D)\) is defined by a morphism\(^1\) \(\mu : E \to D\) such that the following diagram commutes

\[
\begin{array}{ccc}
E & \xrightarrow{\partial_E} & C_0 \\
\mu \downarrow & & \downarrow \partial_D \\
D
\end{array}
\]

and also

\[
\mu(c_0 \cdot e) = c_0 \cdot \mu(e),
\]

\[
\mu(c_0 \ast e) = c_0 \ast \mu(e),
\]

for all \(c_0 \in C_0\), \(e \in E\), \(* \in \Omega_2\).

A crossed \(C_0\)-module \(\partial : E \to C_0\) will be denoted by \((E, \partial_E)\) for short.

\(^1\)In full, it is a tuple \((\mu, \text{id}_{\mathbb{C}_0})\).
Proposition 1. Let $\mu, \mu' : (E, \partial_E) \to (D, \partial_D)$ be two crossed $C_0$-module morphisms. Then,

$$I = \{ \mu(e) - \mu'(e) \mid e \in E \}$$

is an ideal of $D$.

Proof. For all $d \in D$, we have

$$d + \mu(e) - \mu'(e) - d = d + \mu(e) - d + \mu'(e) - d$$

$$= \partial_D(d) \cdot \mu(e) + \partial_D(d) \cdot \mu'(e)$$

$$= \mu(d \cdot e) + \mu'(d \cdot (-e))$$

$$= \mu(d \cdot e) - \mu'(d \cdot e) \in I,$$

and

$$d * (\mu(e) - \mu'(e)) = d * \mu(e) - d * \mu'(e)$$

$$= \partial_D(d) * \mu(e) - \partial_D(d) * \mu'(e)$$

$$= \mu(d \cdot e) - \mu'(d \cdot e) \in I,$$

from which $I$ becomes an ideal of $D$. \qed

Proposition 2. Let $I$ be the ideal given in Proposition 1. Then $(I, \partial_D)$ is a crossed $C_0$-module with the action induced from that of $C_0$ on $D$.

Proof. We only prove that the action of $C_0$ on $I$ is well-defined. Let $c \in C_0$ and $\mu(e) - \mu'(e) \in I$. Then we have

$$c \cdot (\mu(e) - \mu'(e)) = c \cdot \mu(e) - c \cdot \mu(e)$$

$$= \mu(c \cdot e) - \mu'(c \cdot e) \in I,$$

and, similarly,

$$c * (\mu(e) - \mu'(e)) = \mu(c * e) - \mu'(c * e),$$

that completes the proof. \qed

Theorem 3.2. Any pair of parallel morphisms $\mu, \mu' : (E, \partial_E) \to (D, \partial_D)$ has a coequalizer.

Proof. Consider the diagram

$$
\begin{array}{ccc}
(E, \partial_E) & \xrightarrow{\mu} & (D, \partial_D) \\
\mu' \downarrow & & \downarrow p \\
(D/I, \partial_D) & \xrightarrow{\overline{\mu}} & (F, \partial_F)
\end{array}
$$

We obviously have $p \mu = p \mu'$ since

$$p(\mu(e)) = \mu(e) + I$$

$$= \mu'(e) - \mu'(e) + [\mu(e) + I]$$

$$= \mu'(e) + I$$

$$= p(\mu'(e)),$$

for all $e \in E$.

Let $q : (D, \partial_D) \to (F, \partial_F)$ be a morphism such that $q \mu = q \mu'$. Define

$$q' : (D/I, \partial_D) \xrightarrow{id + I} (F, \partial_F)$$

where $q(d)$. \qed
Then we have
\[ \partial_F(q'(d + I)) = \partial_D(q(d)) \]
\[ = \partial_D(d) \]
\[ = \overline{\partial}_D(p(d)) \]
\[ = \overline{\partial}_D(d + I), \]
and
\[ q'(c \cdot (d + I)) = q'(c \cdot d + I) \]
\[ = q(c \cdot d) \]
\[ = c \cdot q(d) \]
\[ = c \cdot (q'(d + I)), \]
and, similarly,
\[ q'(c \ast (d + I)) = c \ast q'((d + I)), \]
for all \( c \in C_0 \) and \( d + I \in D/I \). Hence, we obtain that
\[ q' : (D/I, \overline{\partial}_D) \rightarrow (F, \partial_F) \]
is a crossed \( C_0 \)-module morphism.
Moreover, let \( h : (D/I, \overline{\partial}_D) \rightarrow (F, \partial_F) \) be a crossed \( C_0 \)-module morphism such that \( h \circ p = q \). We have
\[ h(d + I) = h(p(d)) \]
\[ = q(d) \]
\[ = q'(p(d)) \]
\[ = q'(d + I), \]
for all \( d + I \in D/I \), that proves \( q' \) is unique and completes the proof. \( \square \)

**Proposition 3.** Let \( (D, \partial_D) \) and \( (E, \partial_E) \) be two crossed \( C_0 \)-modules. Then, the set of actions of \( D \) on \( E \) defined via \( \partial_D \) is a set of derived actions.

**Proof.** We show that the set of actions defined by
\[ d \cdot e = \partial_D(d) \cdot e, \]
\[ d \ast e = \partial_D(d) \ast e, \]
satisfies the conditions in Theorem 2.9, as follows:
2) For all \( d \in D \), \( e, e' \in E \), we have
\[ d \cdot (e + e') = \partial_D(d) \cdot (e + e') \]
\[ = \partial_D(d) \cdot e + \partial_D(d) \cdot e' \]
\[ = d \cdot e + d \cdot e'. \]
11) For all \( d \in D \), \( e \in E \) and \( w \in \Omega_1^1 \), we have
\[ w(d \ast e) = w(\partial_D(d) \ast e) \]
\[ = w(\partial_D(d)) \ast e \]
\[ = \partial_D(w(d)) \ast e \]
\[ = w(d) \ast e, \]
and the other conditions follow immediately.

**Proposition 4.** With the assumptions in Theorem 3.2, $C_0$ acts on $E \times D$ componentwise, i.e.

\[
c \cdot (e, d) = (c \cdot e, c \cdot d),
\]
\[
c * (e, d) = (c * e, c * d),
\]

for all $c \in C_0$ and $(e, d) \in E \times D$.

**Proof.** Since $C_0$ acts on $D$ and $E$, we have the split extensions

\[
0 \rightarrow D \rightarrow X \xrightarrow{s_D} C_0 \rightarrow 0
\]

and

\[
0 \rightarrow E \rightarrow X' \xrightarrow{s_E} C_0 \rightarrow 0
\]

Consequently, we have the following split extension

\[
0 \rightarrow E \times D \rightarrow X' \times X \xrightarrow{(s_E, s_D)} C_0 \rightarrow 0
\]

with $(s_E, s_D)(c) = (s_E(c), s_D(c))$, from which we get the derived actions

\[
c \cdot (e, d) = (c \cdot e, c \cdot d),
\]
\[
c * (e, d) = (c * e, c * d),
\]

for all $c \in C_0$ and $(e, d) \in E \times D$.

**Proposition 5.** Let $(D, \partial_D)$, $(E, \partial_E)$ be two crossed $C_0$-modules. Then

\[
\partial : E \times D \rightarrow C_0
\]

\[
(e, d) \mapsto \partial_E(e) + \partial_D(d)
\]

is a precrossed $C_0$-module.

**Proof.** First of all, $\partial$ is a morphism in $C$ since

\[
\partial((e, d) + (e', d')) = \partial(e + d \cdot e', d + d')
\]
\[
= \partial_E(e + d \cdot e') + \partial_D(d + d')
\]
\[
= \partial_E(e) + \partial_E(d \cdot e') + \partial_D(d + d')
\]
\[
= \partial_E(e) + \partial_E(\partial_D(d) \cdot e') + \partial_D(d + d')
\]
\[
= \partial_E(e) + \partial_D(d) + \partial_E(e') - \partial_D(d) + \partial_D(d) + \partial_D(d')
\]
\[
= \partial_E(e) + \partial_D(d) + \partial_E(e') + \partial_D(d')
\]
\[
= \partial(e, d) + \partial(e', d'),
\]

and

\[
\partial((e, d) * (e', d'))
\]
\[
= \partial(e * e' + d * e' + e * d', d * d')
\]
\[
= \partial_E(e * e' + d * e' + e * d') + \partial_D(d * d')
\]
\[
= \partial_E(e * e') + \partial_E(d * e') + \partial_E(e * d') + \partial_D(d * d')
\]
Then, ideal containing the set \( C \) is a crossed \( C \)-module.

Since the crossed module conditions are already satisfied, we only need to prove that \((E \rtimes D, \partial)\) is a precrossed \( C_0 \)-module. 

\[
\partial(c \cdot (e, d)) = \partial(c \cdot e, c \cdot d) = \partial_E(c \cdot e) + \partial_D(c \cdot d) = c + \partial_E(e) - c + \partial_D(d) - c = c + \partial_E(e) + \partial_D(d) - c = c + \partial(e, d) - c,
\]

and

\[
\partial(c \ast (e, d)) = \partial(c \ast e, c \ast d) = \partial_E(c \ast e) + \partial_D(c \ast d) = c \ast \partial_E(e) + c \ast \partial_D(d) = c \ast (\partial_E(e) + \partial_D(d)) = c \ast \partial(c, d),
\]

for all \( c \in C_0 \) and \( (e, d) \in E \rtimes D \), which proves that \((E \rtimes D, \partial)\) is a precrossed \( C_0 \)-module. \(\square\)

**Proposition 6.** For a given precrossed \( C_0 \)-module \( \partial : A \to C_0 \), let \( P \) be the smallest ideal containing the set 

\[
\{ (\partial(a) \ast a') - a \ast a', (\partial(a) \cdot a') - a' | a, a' \in A \}.
\]

Then, 

\[
\overline{\partial} : A/P \to C_0, \quad a + P \mapsto \partial(a)
\]

is a crossed \( C_0 \)-module with the action 

\[
c \cdot (a + P) = c \cdot a + P
\]

\[
c \ast (a + P) = c \ast a + P
\]

for all \( c \in C_0 \), \( (a + P) \in A/P \).

**Proof.** Since the crossed module conditions are already satisfied, we only need to prove that the action of \( C_0 \) on \( A/P \) is well-defined.

Let \( p = \partial(a) \cdot a' - a' \in P \). Then we have 

\[
c \cdot p = c \cdot (\partial(a) \cdot a' - a') = c \cdot (\partial(a) \cdot a') - c \cdot a' = (c + \partial(a)) \cdot a' - c \cdot a' = (c + \partial(a) - c) \cdot (c \cdot a') - c \cdot a' = \partial(c \cdot a) \cdot (c \cdot a') - c \cdot a' \in P,
\]

for all \( (e, d), (e', d') \in E \rtimes D \).
and similarly, by Proposition (3.10) in [4], we have
\[ c \cdot (\partial(a) \ast a' - a \ast a') = \partial(a) \ast a' - a \ast a', \]
for all \( c \in C_0, * \in \Omega_2'. \)

Remark 2. We have the functor
\[(\ )^{cr} : PXMod \to XMod\]
which assigns the crossed module \( \bar{\partial} : A/P \to C_0 \) for a given precrossed module \( \partial : A \to C_0. \)

Theorem 3.3. \( \left( ((E \times D)/P, \bar{\partial}), i_E, i_D \right) \) is the coproduct of crossed \( C_0 \)-modules \((E, \partial_E)\) and \((D, \partial_D)\) where
\[ i_E(e) = (e, 0) + P, \quad i_D(d) = (0, d) + P, \]
for all \( e \in E, d \in D. \)

Proof. Consider the diagram

\[
\begin{array}{ccc}
(E, \partial_E) & \xrightarrow{i_E} & (E \times D/P, \bar{\partial}) \\
J_E & \searrow & \downarrow J_D \\
(X, \partial_X) & \searrow & \downarrow J_D \\
 & \downarrow J_E & \\
(D, \partial_D) & \xrightarrow{i_D} & (E \times D/P, \bar{\partial})
\end{array}
\]

in XMod/C_0. Define
\[ h : E \times D/P \to X \]
\[ \overline{(e, d)} \mapsto J_E(e) + J_D(d). \]

First of all, \( h \) is a morphism in \( C \) since
\[
h((e, d) + (e', d')) = h(e + d \cdot e', d + d')
\]
\[ = J_E(e + \partial_D(d) \cdot e') + J_D(d + d')
\]
\[ = J_E(e) + J_E(\partial_D(d) \cdot e') + J_D(d + d')
\]
\[ = J_E(e) + \partial_D(d) \cdot J_E(e') + J_D(d + d')
\]
\[ = J_E(e) + (\partial_X J_D(d)) \cdot J_E(e') + J_D(d + d')
\]
\[ = J_E(e) + J_D(d) + J_E(e') - J_D(d) + J_D(d) + J_D(d')
\]
\[ = J_E(e) + J_D(d) + J_E(e') + J_D(d')
\]
\[ = h(e, d) + h(e', d'), \]

\[ = \]
and
\[ h(\bar{e}, \bar{d}) * h(\bar{e}', \bar{d}') = h(\bar{e} * e' + d * e' + e * d', d * d') \]
\[ = J_E(e * e') + J_E(d * e') + J_D(e * d') + J_D(d * d') \]
\[ = J_E(e * e') + J_E(\partial_D(d) * e') + J_E(e * \partial_D(d')) + J_D(d * d') \]
\[ = J_E(e * e') + \partial_D(d) * J_E(e') + J_E(e) * \partial_D(d') + J_D(d * d') \]
\[ = (J_E(e) + J_D(d)) * J_E(e') + (J_E(e) + J_D(d)) * J_D(d') \]
\[ = h(\bar{e}, \bar{d}) + h(\bar{e}', \bar{d}'), \]
for all \((\bar{e}, \bar{d}), (\bar{e}', \bar{d}') \in E \times D/P.\)

Since \(J_E, J_D\) are morphisms in XMod/C_0, we have
\[ \partial_X h(\bar{e}, \bar{d}) = \partial_X (J_E(e) + J_D(d)) \]
\[ = \partial_X J_E(e) + \partial_X J_D(d) \]
\[ = \partial_E(e) + \partial_D(d) \]
\[ = \partial (\bar{e}, \bar{d}), \]
and
\[ h(c \cdot (\bar{e}, \bar{d})) = h(c \cdot e, c \cdot d) \]
\[ = J_E(c \cdot e) + J_D(c \cdot d) \]
\[ = c \cdot J_E(e) + c \cdot J_D(d) \]
\[ = c \cdot (J_E(e) + J_D(d)) \]
\[ = c \cdot h(\bar{e}, \bar{d}), \]
for all \(c \in C_0, (\bar{e}, \bar{d}) \in E \times D/P,\) that yields \(h\) is a crossed \(C_0\)-module morphism. On the other hand, it is easy to prove that \(h\) is unique. \(\square\)

**Remark 3.** A category \(D\) is said to be (finitely) “cocomplete” if it has all (finite) colimits. On the other hand, the cocompleteness can be characterized in several ways as follows. For a category \(D\), the following are equivalent:

- \(D\) is finitely cocomplete.
- \(D\) has coequalizers and coproducts.
- \(D\) has pushouts and the initial object.

Consequently, recalling Theorems 3.2 and 3.3, we already proved the following:

**Corollary 1.** The category XMod/C_0 is (finitely) cocomplete.
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