Liouville term for neutrinos: flavor structure and wave interpretation

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Abstract. Neutrino production, absorption, transport, and flavor evolution in astrophysical environments is described by a kinetic equation \( D\varrho = -i[H, \varrho] + C[\varrho] \). Its basic elements are generalized occupation numbers \( \varrho \), matrices in flavor space, that depend on time \( t \), space \( x \), and momentum \( p \). The commutator expression encodes flavor conversion in terms of a matrix \( H \) of oscillation frequencies, whereas \( C[\varrho] \) represents source and sink terms as well as collisions. The Liouville operator on the left hand side involves linear derivatives in \( t, x \) and \( p \). The simplified expression \( D = \partial_t + \hat{p} \cdot \partial_x \) for ultra-relativistic neutrinos was recently questioned in that flavor-dependent velocities should appear instead of the unit vector \( \hat{p} \). Moreover, a new damping term was postulated as a result. We here derive the full flavor-dependent velocity structure of the Liouville term although it appears to cause only higher-order corrections. Moreover, we argue that on the scale of the neutrino oscillation length, the kinetic equation can be seen as a first-order wave equation.

Keywords: neutrino theory, supernova neutrinos

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1 Introduction

Neutrino flavor oscillations [1] are one example for the propagation of a multi-component wave \( \psi = (\psi_1, \ldots, \psi_N) \), where the \( N \) components obey different dispersion relations. Other examples include the polarization components of electromagnetic waves, notably the Faraday effect, or more speculative oscillations between photons and hidden photons [2] or between photons and axion-like particles [3]. Typically one considers the evolution of polarization or flavor from a source along a trajectory to a detector. The dispersion relations usually depend on the frequency of the radiation so that the expected result depends on the source spectrum and the detector energy resolution. If \( s \) is a coordinate along the beam, for every frequency \( \omega \) one needs to solve an equation of the form\(^1\),

\[
i \partial_s \psi = H \psi, \tag{1.1}
\]

where \( H \) is a Hermitean \( N \times N \) matrix. In the propagation basis, where \( H \) is diagonal, its entries are the wave numbers of the \( N \) branches of the dispersion relation with frequency \( \omega \). Solving equation (1.1) can be complicated when \( H \) varies as a function of \( s \). In particular, if different branches of the dispersion relation cross (“avoided level crossing”) one can get

\(^1\)In the context of neutrino flavor oscillations, the coordinate \( s \) is usually interpreted as “time of propagation” along the beam and the equation is interpreted as a Schrödinger equation for the flavor content of a given neutrino, whereas in the context of the astrophysical Faraday effect one always interprets \( s \) as a distance along the beam. The physical result is of course the same.

\(^2\)We use units in which the vacuum speed of light and Planck’s constant are unity, \( c = \hbar = 1 \).
complete flavor conversion\footnote{The terminology of “flavor oscillations” is actually a bit of a misnomer in this context \cite{6}.} even for a small mixing angle — the celebrated MSW effect \cite{4, 5}. It can also arise for photons in the astrophysical Faraday context \cite{7, 8}.

A more sophisticated approach is needed for a class of problems where neutrinos scatter many times after production. One generic example is sterile-neutrino production in the early universe by oscillations and collisions \cite{9, 10} by what has come to be called the Dodelson-Widrow mechanism \cite{11}. On the most elementary level we may think of a single-particle neutrino state with a flavor content initially described by the amplitudes $\psi$. Subsequent flavor-dependent collisions decohere the flavor content as it gets entangled with the environment and we need to switch to a density-matrix description in the form $\rho_{ij} = \psi_j^\ast \psi_i$ with $i,j = 1, \ldots, N$. The oscillation part of the evolution corresponding to equation (1.1) is the usual commutator expression $\partial_t \rho = -i[H, \rho]$ that applies both to pure and mixed states.

The decoherence part of the evolution has been formulated in different ways \cite{12–14}. The key point is that in the interaction basis, the off-diagonal elements of $\rho$ are damped if the scattering amplitudes for the different flavors are different. A compact way to express this behavior is in terms of Lindblad operators \cite{15, 16}, so overall we have \cite{17, 18}

$$\partial_t \rho = -i[H, \rho] - \{L, [L, \rho]\}, \quad (1.2)$$

where $L$ is a Hermitean $N \times N$ matrix. Notice that the commutator structure preserves $\text{tr} \rho = 1$, i.e., our single neutrino is not absorbed in this model, only flavor coherence is damped.\footnote{In reference \cite{18} an ensemble of neutrinos was considered with a Gaussian distribution of oscillation frequencies. Even without a Lindblad term, the average $\rho$ matrix loses its off-diagonal terms because of dephasing of different neutrinos and it was argued that this effect was equivalent to the loss of coherence by a Lindblad term. However, apparent decoherence caused by dephasing is reversible, for example by a detector with sufficient energy resolution, and also does not increase the von Neumann entropy of the ensemble. Therefore, one should always carefully distinguish between “kinematical decoherence” caused by dephasing of many neutrinos or different Fourier components of a single-neutrino wavepacket, and “dynamical decoherence” caused by irreversible entanglement with the environment.}

Moreover, the von Neumann entropy $-\text{tr} (\rho \log \rho)$ increases monotonically thanks to $L$ being self-adjoint. In the context of active-sterile oscillations, $H$ is diagonal in the propagation basis whereas $L$ is diagonal in the interaction basis, so $H$ and $L$ do not commute. The final asymptotic $\rho$ matrix which commutes with both $H$ and $L$ would be proportional to the unit matrix, representing flavor equilibrium.

The damping of flavor coherence arises more directly if one considers a kinetic equation for the neutrino distribution. It is standard to use occupation numbers $f_p$ to describe the evolution of a quantum field in terms of a Boltzmann kinetic equation. In a seminal paper, Dolgov \cite{9} extended this description to mixed neutrinos in terms of matrices $\varrho_p$ which are generalized occupation numbers.\footnote{We use the symbol $\varrho$ for generalized occupation numbers (“matrix of densities”) in contrast to $\rho$ as in equation (1.2) which is a single-particle density matrix. In practice, they differ by normalization with $\text{tr} \rho = 1$, whereas $0 \le \text{tr} \varrho_p \le N$. Moreover, $\text{tr} \varrho_p$ changes under the kinetic evolution by source and sink terms.} It is assumed that the different $p$ components of the quantum field decohere quickly (“molecular chaos”), i.e., correlations between different $p$ modes are ignored, whereas flavor coherence survives on the relevant time scale and is followed explicitly in terms of the off-diagonal $\varrho_p$-components. For neutrinos in the early universe, Dolgov’s equation is \cite{9, 10}

$$\partial_t \varrho_p - H_p \varrho_p|_{\partial_p} \varrho_p = -i[H_p, \varrho_p] + C[\varrho_p^{\prime}, \bar{\varrho_p}^{\prime}], \quad (1.3)$$

where $H$ is the Hubble expansion parameter and we have assumed isotropy. The crucial new ingredient is the collision term, where scattering on electrons and positrons as well as pair
processes $\epsilon^+ \epsilon^- \leftrightarrow \nu \bar{\nu}$ were explicitly included. There is an equation of type (1.3) for every neutrino mode $\varrho_p$ and anti-neutrino mode $\bar{\varrho}_p$. They are all coupled by the collision term. Key for the approach to equilibrium are scattering amplitudes that distinguish between different flavors and thus introduce a nontrivial flavor structure of the collision term [9, 10].

These and similar discussions [19] explicitly ignore degeneracy effects and can be seen as describing single-neutrino states. However, in compact astrophysical objects such as core-collapse supernovae or neutron-star mergers, neutrinos can be degenerate. By considering flavor mixing among quantum fields instead of wave functions one finds that the oscillation term in equation (1.3) remains unchanged, in particular when $\text{tr} \varrho_p > 1$, i.e., a given mode is occupied by more than one neutrino [20]. Likewise, the formal appearance of the collision term remains the same, yet each process such as elastic scattering and pair or $\beta$ processes now include initial-state occupation numbers and/or final-state Pauli-blocking factors in terms of $\varrho_p$ matrices. Therefore, the collision term shows a nontrivial flavor structure beyond flavor-dependent scattering amplitudes [21–25].

In a toy model of a simplified collision term which does not couple different momentum modes, one can show explicitly the emergence of the Lindblad structure of equation (1.2) in the form of a double commutator for the damping of flavor coherence [21]. In general, of course, flavor coherence is not damped separately for each $\varrho_p$ and actually can temporarily increase for some range of modes. It is noteworthy, however, that the kinetic equation reproduces the expected evolution of the appropriate thermodynamic potential, assuming the collision term arises from the interaction with a thermal background medium that can exchange energy and lepton number with the neutrino gas [22]. In other words, flavor decoherence of the ensemble follows from the kinetic equation without further ado, a point that does not seem to be controversial in the literature.

Thus far we have focussed on flavor conversion driven by neutrino masses. However, neutrino masses unavoidably couple positive with negative helicity states, for example implying small neutrino electromagnetic dipole and transition moments [26]. Moreover, if the background medium is not isotropic it induces transitions between helicity states which, for Majorana neutrinos, effectively implies transitions between neutrinos and antineutrinos. Such effects can be included by considering larger $\varrho_p$ matrices that may include sterile neutrinos (Dirac case) or neutrinos and antineutrinos in a single $\varrho_p$ matrix (Majorana case) [25–32]. On the other hand, if we take the background medium to be isotropic, angular momentum conservation precludes such effects which anyway always seem to be negligibly small in practice. For simplicity we will thus ignore neutrino helicity conversion.

Somewhat surprisingly, it is not the collision or oscillation terms, but the Liouville term that has aroused some doubts in the recent literature [33]. If we consider unmixed particles such as electrons, and if we include weak inhomogeneities, one uses space-dependent occupation numbers $f_{l,x,p}$. This construction makes physical sense when the spatial variation is on scales much larger than a typical radiation wavelength so that the uncertainty relation between the non-commuting variables $x$ and $p$ does not impose a serious limitation. Using such Wigner functions allows one to calculate averages in terms of classical phase-space integrations instead of invoking rigorous quantum-mechanical expectation values [39–41].

The kinetic equation is then of the form$^6$ $\partial_t f_{l,x,p} + \mathbf{v} \cdot \partial_x f_{l,x,p} - \dot{\mathbf{p}} \cdot \partial_p f_{l,x,p} = C[f]$. The last term on the left hand side (lhs) represents momentum changes by coherent external forces that are not part of the microscopic collision term. This includes cosmic expansion as in Dolgov’s equation (1.3), but could also include gravitational redshift or deflection in

$^6$We use $\partial_x$ to denote the gradient with regard to $x$. It is identical with the vector $\mathbf{\nabla}_x$. 

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the supernova context [42]. For neutrinos, the background medium causes a modification of the refractive index and thus of $H$, whereas coherent forces by medium gradients are usually neglected and not even mentioned. Of course, a neutrino produced off-center in the Sun or in a supernova core suffers refractive deflection unless it moves radially, but this effect is extremely small compared with gravitational deflection. Likewise, neutrino diffraction e.g. by the roughness of the Earth surface plays no practical role. For neutrinos, only gravitational effects seem to be of any relevance for the momentum drift term.

The doubts voiced in reference [33] actually concern the drift term in coordinate space, where $v$ is the velocity corresponding to momentum $p$, i.e., $v = p/E_p$ where $E_p^2 = p^2 + m^2$ for unmixed particles. In the absence of collisions or coherent forces, $\partial_t \psi_{t,x,p} + v \cdot \partial_x \psi_{t,x,p} = 0$ simply represents conservation of particles with momentum $p$ (flux conservation). In the context of mixed neutrinos, often the ultra-relativistic limit was invoked to write flux conservation in the form $\partial_t \psi_{t,x,p} + \hat{p} \cdot \partial_x \psi_{t,x,p} = 0$, where $\hat{p}$ is a unit vector in the direction of $p$ [23, 34, 35]. In other words, propagation with the speed of light was assumed for all $\psi_p$ components. Non-vanishing masses were only included in the oscillation matrix $H$.

While ignoring neutrino masses everywhere except in $H$ is probably a good approximation in practice, we recall that the kinetic equation derived by two of us a long time ago actually stated the Liouville equation explicitly in the form [22]

$$\partial_t \psi + \frac{i}{2} \{ \partial_x \psi, \partial_p H \} - \frac{i}{2} \{ \partial_p \psi, \partial_x H \} = -i [H, \psi] + C[\psi], \tag{1.4}$$

where both $\psi$ and $H$ depend on $t$, $x$ and $p$ and $\{ \cdot, \cdot \}$ is an anti-commutator. Notice that the Hermitian matrix $H_{t,x,p}$ in diagonal form gives us the energies of quanta with momentum $p$ in the medium which has properties that can depend on $t$ and $x$. If the matter effect contained in $H$ is the usual electroweak potential and as such does not depend on $p$, the only part of $H$ that depends on $p$ is the neutrino kinetic energy. In this case $V_p = \partial_p H_{t,x,p}$ is a matrix of velocities which is diagonal in the mass basis and has $v_i = p_i/(p^2 + m_i^2)^{1/2}$, appropriate for momentum $p$ on the diagonal. So we may interpret $\frac{1}{2} \{ \psi, V \}$ as a matrix of neutrino fluxes and, in the absence of coherent forces or collisions, equation (1.4) is simply flux conservation in the form $\partial_t \psi_{t,x,p} + \frac{i}{2} \partial_x \{ V_p, \psi_{t,x,p} \} = 0$. For unmixed neutrinos this is flux conservation in the usual sense for every separate mass eigenstate.

While the full Liouville term was stated in reference [22], we acknowledge that the derivation was limited to the statement “as one can easily show . . . ” — the main focus at that time was the collision term and its impact on flavor evolution. In a more formal covariant derivation, Yamada [24] found the same structure (cf. his equation 35) but then turned quickly to the ultra-relativistic limit (cf. text after equation 53). The same structure is also borne out by references [25] (cf. for example their equations 163–166), but was not spelled out or discussed from a phenomenological perspective. Reference [35] probably implies the same results, although in their more phenomenological section they immediately turn to the ultra-relativistic approximation (cf. their equation 73). On the other hand, the phenomenological derivation of the Liouville term in reference [34] does not address the flavor structure and applies only in the ultra-relativistic limit.\footnote{The oscillation equations found here by two methods are of the form $p^\nu \frac{\partial}{\partial x} \psi_{t,x,p} = -\frac{i}{2} [M^2, \psi_{t,x,p}]$, where the meaning of $p^\nu$ remains at first unspecified. Later the ultra-relativistic limit is taken so that $p^\nu = (|p|, p)$.}

This situation motivates us to return to this topic and provide in section 2 a derivation of the Liouville term of equation (1.4) with as little theoretical overhead as possible. In section 3 we work out an explicit two-flavor example when keeping the flavor structure of the Liouville term. We conclude with a summary and discussion in section 4.
2 Spatial transport of mixed neutrinos

2.1 Derivation of the Liouville term

2.1.1 Two-point correlators

On the level of a kinetic treatment our goal is to understand the space-time evolution of the neutrino mean field in the form of the $\varrho_{t,x,p}$ matrices which are expectation values of field bilinears. The information contained in $\varrho_{t,x,p}$ is sufficient for all common questions arising in flavor oscillation physics. First, they provide the refractive effect of neutrinos on other neutrinos. Second, $\varrho_{t,x,p}$ allows us to calculate local interaction rates either in astrophysical environments or in laboratory detectors. Field bilinears are the lowest-order field correlators in the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [29] and as such the lowest-order terms of a systematic perturbative expansion. Of course, there can be physical circumstances where higher-order correlators are important, although in astrophysical or laboratory neutrino physics no such cases seem to have emerged.

Some authors prefer to study flavor oscillations in terms of individual neutrinos propagating as wavepackets [33, 36–38], a philosophy that explicitly goes beyond the mean-field level. However, as long as we only ask mean-field questions (neutrino-neutrino refraction or local interaction rates) this treatment does not produce new results. Moreover, it requires unavailable information about the production of the assumed wavepackets for which only back-of-the-envelope estimates exist. Therefore, while a kinetic treatment in terms of $\varrho_{t,x,p}$ matrices is of course not a complete description of the fluctuating neutrino gas in a supernova or of the neutrino flux from a laboratory source, this treatment is complete on the level of those questions that are addressed in present-day neutrino physics.

The starting point for deriving the kinetic equation is the Dirac quantum field\(^8\) $\hat{\psi}_i(t,x)$ that destroys a neutrino or creates an antineutrino of flavor $i$ at time $t$ and location $x$. This field and its conjugate are expanded in spatial Fourier modes $p$ in terms of Dirac spinors and neutrino and antineutrino destruction and creation operators $\hat{a}_i(h,p,t)$, $\hat{a}_i^\dagger(h,p,t)$ and $\hat{b}_i(h,p,t)$, where $h$ is the helicity. For propagation in an isotropic medium we only consider negative-helicity neutrinos and positive-helicity antineutrinos. Moreover, we here focus on the advection part of the kinetic equation\(^9\) so that neutrinos and antineutrinos are not coupled, for example, by pair processes. We thus consider a simplified model that includes only neutrino destruction and creation operators $\hat{a}_i(p,t)$ and $\hat{a}_i^\dagger(p,t)$ fulfilling the equal-time anti-commutation relation $\{\hat{a}_i(p,t),\hat{a}_j^\dagger(p',t)\} = (2\pi)^3\delta^3(p-p')\delta_{ij}$. The advection part of the Liouville equation for antineutrinos is the same except for a well-known sign change in the refractive term. After dismissing the entire Dirac structure we could also consider bosons and use commutation relations for the destruction and creation operators instead.

In the mean-field approximation, the system is described by expectation values of field bilinears of the type $\hat{\psi}_i^\dagger(t,x)\hat{\psi}_j(t,x)$. On the level of the Fourier components we thus require expectation values of expressions such as

$$\hat{D}_{ij}(p,p',t) = \hat{a}_j^\dagger(p',t) \hat{a}_i(p,t).$$

As discussed earlier [22], we dismiss fast-varying bilinears of the type $\hat{a}_i^\dagger(p',t)\hat{a}_i^\dagger(p,t)$ and also mixed bilinears between neutrinos and antineutrinos, although in a non-isotropic medium,
neutrino-antineutrino pair correlations can be relevant after all [30, 31]. If the medium is homogeneous, the expectation value of every observable constructed from the fields $\hat{\psi}_i$ and $\hat{\psi}_j^\dagger$ is independent of location, implying that the expectation value of $D_{ij}(p, p', t)$ contributes only at equal momenta. Therefore, the mean field of a homogeneous neutrino gas is completely characterized by dimensionless $N \times N$ “matrices of densities” $\varrho_{p, t}$ given by

$$\langle \hat{\alpha}_j^\dagger(p', t) \hat{\alpha}_i(p, t) \rangle = (2\pi)^d \delta^d(p - p') \langle \varrho_{p, t} \rangle_{ij}.$$  

The diagonal entries of $\varrho_{p, t}$ are the usual occupation numbers of different flavors.

### 2.1.2 Wigner transformation

If the neutrino gas is inhomogeneous it is described by matrices $\varrho_{t, x, p}$ that also depend on location. Such a quasi-probability distribution in phase space makes sense when inhomogeneities are weak, i.e., spatial variations are on scales much larger than a typical neutrino wavelength. To arrive at this construction we make use of the Wigner transformation [39–41] and its inverse. For a function $F(k, k')$ of two momentum variables it is

$$\tilde{F}(x, p) = \int \frac{d^3 \Delta}{(2\pi)^3} e^{i \Delta \cdot x} F\left(p - \frac{\Delta}{2}, p + \frac{\Delta}{2}\right), \quad (2.2a)$$

$$F(k, k') = \int d^3 x e^{-i(k' - k) \cdot x} \tilde{F}\left(x, \frac{k + k'}{2}\right). \quad (2.2b)$$

With $\Delta = k' - k$ and $p = \frac{1}{2} (k' + k)$ the inverse transformation can also be written as

$$F\left(p - \frac{\Delta}{2}, p + \frac{\Delta}{2}\right) = \int d^3 x e^{-i \Delta \cdot x} \tilde{F}(x, p). \quad (2.3)$$

Such transformations are motivated in situations when $F(k, k')$ has most of its power near $k = k'$ so that it makes sense to use an average momentum. Moreover, in this case $\tilde{F}(x, p)$ varies slowly as a function of $x$. However, the Wigner transformation is a general mathematical operation that is not limited to these assumptions.

Next we apply the Wigner transformation to the second-quantized correlator $\tilde{D}(k, k', t)$ defined in equation (2.1),

$$\hat{\varrho}_{ij}(t, x, p) = \int \frac{d^3 \Delta}{(2\pi)^3} e^{i \Delta \cdot x} \hat{\alpha}_j^\dagger\left(p - \frac{\Delta}{2}, t\right) \hat{\alpha}_i\left(p + \frac{\Delta}{2}, t\right). \quad (2.4)$$

Both $\tilde{D}(k, k', t)$ and $\hat{\varrho}(t, x, p)$ carry the same information. The mean field of the neutrino gas is finally characterized by $\varrho_{t, x, p} = \langle \hat{\varrho}_{t, x, p} \rangle$, playing the role of space-varying occupation-number matrices.

### 2.1.3 Equations of motion

To derive equations of motion we begin with Heisenberg’s equation for an operator $\hat{A}$ in the form $i \partial_t \hat{A} = [\hat{A}, \hat{H}]$, where $\hat{H}$ is the Hamiltonian. We write it in the form

$$\hat{H} = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \hat{\alpha}_i^\dagger(p) H_{ij}(p, p') \hat{\alpha}_j(p'), \quad (2.5)$$

where $H_{ij}$ is a matrix of numbers. Here and henceforth we no longer show the dependence on time explicitly. A summation over repeated flavor indices $i, j = 1, \ldots, N$ is implied. This bilinear form does not include neutrino-neutrino refraction that we leave out for simplicity.
If the background medium is homogeneous, the matrix of energies depends only on one momentum and is of the form $H(p, p') = (2\pi)^3 \delta^3(p-p') H_0(p)$. In particular, in the mass basis this includes the energies $(p^2 + m^2)/2$ on the diagonal. On this level, the Hamiltonian $\hat{H}$ simply represents a collection of quantum harmonic oscillators. In addition, there are refractive energy shifts that are diagonal in the interaction basis. In the propagation basis, $\hat{H}$ still represents a collection of quantum harmonic oscillators. In general, however, the medium is not homogeneous so that $\hat{H}(p, p')$ can deflect neutrinos, i.e., destroy one with momentum $p'$ and create one with $p$. This process is not a microscopic collision, but rather a refractive deflection by weak inhomogeneities.

The evolution of the annihilation operator following from Heisenberg’s equation is given by $i\partial_t \hat{a}_i(p) = [\hat{a}_i(p), \hat{H}(p, p')] = \int d^3 p''/(2\pi)^3 \ H_{ik}(p', p) \ \hat{a}_k(p'')$. With this result we can evaluate Heisenberg’s equation for $\hat{\rho}(x, p)$ and find

$$i\partial_t \hat{\rho}_{ij}(x, p) = \int \frac{d^3 \Delta}{(2\pi)^3} \ \frac{d^3 p'}{(2\pi)^3} \ e^{i\Delta \cdot x} \ [H_{ik}(p + \Delta/2, p') \ \hat{a}_j\left(p - \Delta/2\right) \ \hat{a}_k(p') - \hat{a}_k(p') \ \hat{a}_i\left(p + \Delta/2\right) H_{kj}\left(p - \Delta/2, p'\right)]. \tag{2.6}$$

We next introduce the variables $\Delta_1$ and $\Delta_2$ that are defined by $p' = p + \frac{1}{2}(\Delta_1 - \Delta_2)$ and $\Delta = \Delta_1 + \Delta_2$, leading to the more symmetric expression

$$i\partial_t \hat{\rho}_{ij}(x, p) = \int \frac{d^3 \Delta_1}{(2\pi)^3} \ \frac{d^3 \Delta_2}{(2\pi)^3} \ e^{i(\Delta_1 + \Delta_2) \cdot x} \ [H_{ik}\left(p + \frac{\Delta_2}{2}, p_1 - \frac{\Delta_2}{2}\right) \ \hat{a}_j\left(p_2 - \frac{\Delta_1}{2}\right) \ \hat{a}_k\left(p_2 + \frac{\Delta_1}{2}\right) - \hat{a}_k\left(p_1 - \frac{\Delta_2}{2}\right) \ \hat{a}_i\left(p_1 + \frac{\Delta_2}{2}\right) H_{kj}\left(p_2 - \frac{\Delta_1}{2}, p_2 + \frac{\Delta_1}{2}\right)] \tag{2.7},$$

where we have used the notation $p_1 = p + \frac{1}{2}\Delta_1$ and $p_2 = p - \frac{1}{2}\Delta_2$. Notice that the integrals over $d^3\Delta_{1,2}$ cannot be evaluated to produced Wigner transforms because $\Delta_{1,2}$ is also hidden in $p_{1,2}$. However, under the integral we can substitute for each factor the inverse Wigner transformation in the form of equation (2.3) and find

$$i\partial_t \hat{\rho}_{x,p} = \int \frac{d^3 \Delta_1}{(2\pi)^3} \ \frac{d^3 \Delta_2}{(2\pi)^3} \ e^{-i\Delta_1 \cdot (x_1 - x) - i\Delta_2 \cdot (x_2 - x)} \ [H_{x_2,p_1} \ \hat{\rho}_{x_1,p_2} - \hat{\rho}_{x_2,p_1} H_{x_1,p_2}]. \tag{2.8}$$

where we have used $x_{1,2}$ as the conjugate variables to $\Delta_{1,2}$.

To obtain the argument $p$ instead of $p_{1,2}$ we use the shift operator in the form $F(k+q) = e^{q \cdot \partial_k} F(k)$. This construction implies e.g. $H(x_2, p_1) = e^{\frac{i}{2} \Delta_1 \cdot \partial_p} H(x_2, p)$ and overall we find

$$i\partial_t \hat{\rho}_{x,p} = \int \frac{d^3 \Delta_1}{(2\pi)^3} \ \frac{d^3 \Delta_2}{(2\pi)^3} \ e^{-i\Delta_1 \cdot (x_1 - x + \frac{1}{2} \partial_p) - i\Delta_2 \cdot (x_2 - x - \frac{1}{2} \partial_p)} \ \hat{\rho}_{x_1,p} - \hat{\rho}_{x_2,p} \ e^{-i\Delta_1 \cdot (x_1 - x + \frac{1}{2} \partial_p) - i\Delta_2 \cdot (x_2 - x - \frac{1}{2} \partial_p)} \ H_{x_1,p}. \tag{2.9}$$

where $\frac{1}{2} \partial_p$ means that the differential operator is to be applied to the expression left of it. Using the representation of the delta function $\delta^{(3)}(x) = \int d^3 \Delta e^{i \Delta \cdot x}/(2\pi)^3$ it is now straight-
forward to evaluate the integrals and, for example, the first term in square brackets becomes $H(x + \frac{i}{\hbar} \partial_p, p) \delta(x - \frac{i}{\hbar} \partial_p, p)$. The differential operator in the argument of one matrix is to be applied to the other matrix. A more elegant way to express this structure is found by using once more the shift operator to lift the deviation from $x$ in the arguments to an exponential,

$$
i \partial_t \varrho_{x,p} = H_{x,p} e^{\frac{i}{\hbar} (\partial_x \partial_p - \partial_p \partial_x)} \varrho_{x,p} - \varrho_{x,p} e^{\frac{i}{\hbar} (\partial_x \partial_p - \partial_p \partial_x)} H_{x,p}.$$  

(2.10)

An equation equivalent to this result was first derived by Moyal [43] with two minor differences. We here use matrices in flavor space as opposed to scalar functions and our matrix $\varrho$ is a second-quantized operator as opposed to a purely quantum-mechanical setting. We also note that if we were to keep Planck's constant $\hbar$ explicitly, it would cancel out in the Liouville term, consistent with its classical nature.

Actually the advection term can also be found more directly from equation (2.8). We can take the expectation value $\varrho_{x,p} \rightarrow \varrho_{x,p}$ already in that equation, assume that $H_{x,p}$ and $\varrho_{x,p}$ vary only slowly as a function of their arguments we may expand equation (2.10) to lowest order, providing the advection part of equation (1.4), i.e. the Liouville term, now for the nontrivial matrix structure in flavor space, and the refractive term. It is interesting to note that if one would keep $\hbar$ explicitly, it would cancel out in the Liouville term, consistent with its classical nature.

2.1.4 Mean field

Up to this point we have not made any approximations in that equation (2.10) follows from Heisenberg’s equation for $\varrho$ under the Hamiltonian $\hat{H}$ defined in equation (2.5). Next we take the expectation value of $\varrho$ so that we substitute $\varrho_{x,p} \rightarrow \varrho_{x,p}$. If we assume that $H_{x,p}$ and $\varrho_{x,p}$ vary only slowly as a function of their arguments we may expand equation (2.10) to lowest order, providing the advection part of equation (1.4), i.e. the Liouville term, now for the nontrivial matrix structure in flavor space, and the refractive term. It is interesting to note that if one would keep $\hbar$ explicitly, it would cancel out in the Liouville term, consistent with its classical nature.

Note that if we were to keep Planck’s constant $\hbar$, it would multiply the l.h.s. of equation (2.10) as well as the exponents on the right hand side (rhs).
However, the question if a quantity like $\langle \hat{f}_{x,p} \rangle$ is non-negative appears to be moot for the kinetic equation (1.4). This is a closed set of differential equations where the $\varrho$ matrices are not found from taking expectation values of underlying quantum operators, but the local occupations of modes are filled or depleted by source, sink and collision terms as well as free streaming. We are not aware that this kinetic equation could produce pathological solutions such as negative occupation numbers. In other words, we have formulated the kinetic equation in terms of Wigner functions, but the solutions of this equation are independent of the true underlying quantum system. There is no guarantee that the kinetic equation produces the same results that would be found by solving the full quantum system, but the kinetic equation itself appears to be well behaved. Of course, we expect that the solution of the kinetic equation agrees with the full system unless we consider scales where the uncertainty relation is important, but on such scales one would not use the kinetic equation anyway, at least not in the context of neutrino transport and flavor oscillations.

2.2 Particle transport or wave equation?

The mean-field description in terms of $\varrho_{t,x,p}$ naturally obeys a partial differential equation with independent derivatives in all arguments. Some of the doubts about this structure [33] are apparently related to picturing the evolution of $\varrho_{t,x,p}$ as describing a single neutrino on a trajectory so that the time and space variables are said to be related by a classical trajectory of the type $x = v t$. Related to this doubt is apparently the question of the connection between a partial differential operator on the l.h.s. of the kinetic equation with an ordinary differential equation of the form (1.1) for flavor oscillations. Somewhat in reverse, Cardall [34] started from the picture of flavor oscillations along a trajectory of the form (1.1) and argued in two different ways on how to arrive at a Liouville equation $(\partial_t + \hat{p} \cdot \partial_x)\varrho_{t,x,p}$, where $t$ and $x$ are independent variables.

One simple heuristic way to understand the appearance of the Liouville operator is to ignore the collision part of the kinetic equation as in references [33, 34] and focus on flavor oscillations alone. However, in this case we do not need matrices of densities and can formulate the problem on the level of wave amplitudes as in equation (1.1). Flavor oscillations and similar phenomena arise from the interference of wave components with different dispersion relations and as such are wave phenomena. If we ignore issues of helicity or particle-antiparticle oscillations for neutrinos we may ignore the Dirac structure and in vacuum each field component obeys the Klein-Gordon equation $(\partial_t^2 - \partial_x^2)\psi_i = -m_i^2 \psi_i$ for $i = 1, \ldots, N$. For one space dimension this is

$$ (\partial_t - \partial_x)(\partial_t + \partial_x)\psi = -M^2 \psi, \quad (2.11) $$

where $M^2 = \text{diag}(m_1^2, \ldots, m_N^2)$ in the mass basis. Next we consider a plane-wave solution of the form $e^{-i(E t - p \cdot x)}$, which however involves a different $p_i$ for every $m_i$ for a common $E$. However, in the ultra-relativistic limit $E \approx |p|$ and we can linearize the Klein-Gordon equation if we observe that for such plane waves $(\partial_t - \partial_x) \rightarrow -i(E + p_x) \approx -2iE$. In the second factor this approximation would not be possible because what appears is the difference between $E$ and $p_x$ that would vanish in the same approximation. Essentially by separating the scales between the flavor oscillation length and the neutrino wavelength we arrive at

$$ (\partial_t + \partial_x)\psi = -\frac{M^2}{2E} \psi. \quad (2.12) $$
Of course, we can also consider the complex conjugate equation and combine them in the usual way as an equation for the density matrix $\rho_{ij} = \psi_j^* \psi_i$ in the form

$$\left( \partial_t + \mathbf{p} \cdot \partial_x \right) \rho = -i[H, \rho],$$

where $H = \frac{M^2}{2E}$

(2.13)

and we have restored a general direction $\mathbf{p}$ of propagation.

The practical meaning of equation (2.13) depends on initial and/or boundary conditions. If we consider a homogeneous situation without spatial gradients, we are back to a simple Schrödinger equation of the form $\partial_t \rho = -i[H, \rho]$. If we consider a stationary source, nothing depends on time and we need to solve $\partial_x \rho = -i[H, \rho]$ along the beam. Either way, such an equation only applies to monochromatic waves because $H$ is only defined for a specific $E \approx |\mathbf{p}|$.

Of course one can also consider wavepackets, but then an equation of the form (2.12) needs to be solved for every Fourier component. Therefore, a wavepacket has $\rho$ matrices involving wave amplitudes with different $E$ values, not only those with equal $E$. However, in all practical oscillation experiments we take the average over many measured neutrinos (each of which may have been emitted as a wavepacket) and in this average the phase relations between different Fourier components of individual wavepackets are lost. Therefore, such an ensemble average requires only the occupation numbers $\rho_\mathbf{p}$ of the beam, not the phase relations between Fourier components encoded in wavepackets [46].

So what we make of equations (2.12) and (2.13) depends on the specific physical circumstances and on the questions we wish to address. Either way, these equations can be seen as wave equations in the approximation of ultra-relativistic neutrinos. They do not require or motivate an interpretation in terms of point-like particles moving along classical trajectories. In particular, in the stationary-source example, there is no need to think of a neutrino with momentum $\mathbf{p}$ to exist at some precise location $\mathbf{x}$ in violation of Heisenberg’s uncertainty relation. All we need is a boundary condition at some location, not a localization of the wave itself. The physical nature of the boundary condition is not part of the wave equation or of the Liouville operator.

In astrophysics, a kinetic equation of unmixed particles is used, for example, as the basis for treating radiative transport by photons or neutrinos. Particle fluxes are driven by gradients of temperature or lepton number. The mean free path of the particles and the relevant gradients are large compared with the radiation wavelength. Then the wave nature of the radiation is irrelevant and one may think of the kinetic equation as describing classical particles. However, contrary to the doubts voiced in reference [33], the kinetic equation is not an equation for point-like classical particles. Rather, we only use it on scales where the distinction between waves and particles is irrelevant. The kinetic equation does not assume that neutrinos are localized within phase space to better than allowed by the uncertainty relation. The function $\rho_{\mathbf{x}, \mathbf{p}}$ does not give us the flavor content of a specific neutrino that would be localized precisely at $(\mathbf{x}, \mathbf{p})$ in phase space. The mean field $\rho_{\mathbf{x}, \mathbf{p}}$ does not describe the localization and flavor content of individual particles. Rather it describes the expectation values of occupation number operators in flavor space which themselves depend on location and momentum.

Once we include flavor oscillations the wave nature of the underlying radiation becomes apparent and we can interpret the advection part of the kinetic equation as a first-order wave equation as argued earlier. Indeed, the derivation shown earlier is entirely quantum physical. It is only the final step of taking expectation values and the subsequent Taylor expansion where small-scale information is lost.
The interpretation of the advection part of the kinetic equation (1.4) as a wave equation is crucial in the context of self-induced flavor conversion by neutrino-neutrino refraction [47, 48]. In this situation the neutrino mean field acts back on itself through the oscillation term, i.e., \( H \) depends on the collection of \( \rho \) matrices. Without this effect, flavor conversion is a purely kinematical phenomenon that arises from the interference of independently propagating waves that do not know about each other. Neutrino-neutrino refraction causes these “flavor waves” to become dynamical and we obtain a first-order wave equation with propagating and/or run-away solutions [49, 50].

3 Flavor-dependent Liouville term: phenomenological consequences

3.1 Matrices of velocities

To develop some phenomenological understanding of the flavor-dependent Liouville term in the kinetic equation (1.4) we ignore the momentum drift term caused by external forces. It appears to be dominated by gravitational effects in all practical cases. We also ignore the collision term and thus only worry about the advection part without external forces.

We recall that in an isotropic dispersive medium, a wave with frequency \( \omega \) and wavevector \( k = |k| \) has group velocity \( \partial \omega / \partial k \) and phase velocity \( \omega / k \). In this sense \( V_p = \partial p H \) is a matrix of group velocities whereas \( H \), after dividing by \( |p| \), is a matrix of phase velocities. Flavor oscillations are an interference effect between waves with different dispersion relations and so it comes as no surprise that the oscillation term involves phase velocities.

Particles with mass have energy \( E = (p^2 + m^2)^{1/2} \) so that the group velocity is \( \partial E / \partial p = p/E \approx 1 - m^2/2p^2 \) with \( p = |p| \). Of course, this is the usual particle velocity and thus subluminal. The phase velocity \( E/p \approx 1 + m^2/2p^2 \), on the other hand, is superluminal. Therefore, in the ultra-relativistic limit, the two velocities deviate from the speed of light by the same amount in opposite directions.

For mixed neutrinos, the group velocities appear on the l.h.s. of the kinetic equation, the phase velocities on the r.h.s. In practice we always consider ultra-relativistic neutrinos, so all group and phase velocities are very close to the speed of light. However, the phase velocities appear in a commutator, i.e., it is the difference between phase velocities that causes flavor oscillations, whereas the group velocities appear in an anti-commutator. Therefore, to lowest order in the small deviation from the speed of light we may use \( V_p \approx \hat{p} \), an approximation that was always used in the literature.\(^{11}\)

In reference [33] the impact of having different group velocities was studied in the context of a Liouville equation for wavepackets and a new damping term was found. It was attributed to the effect of wavepacket separation which leads to the loss of flavor coherence. Of course, this effect is not new and has been studied many times in the context of wavepacket discussions of flavor oscillations. This loss of flavor coherence must be interpreted in the sense of kinematical decoherence and as such is included in the advection part of the kinetic equation (1.4), whereas dynamical decoherence from the entanglement with the environment is caused by the collision term as mentioned earlier. After some distance of propagation, the off-diagonal elements of \( \rho_{\nu,\nu'} \) vary fast as a function of \( |p| \) so that a detector with insufficient energy resolution can no longer see the oscillatory pattern. However, this apparent loss of flavor coherence is related to the detector properties and should not be part of the neutrino equation of motion.

\(^{11}\)Instead of \( \hat{p} \) one often used \( v \) to denote a unit vector in the direction of \( p \). However, in reference [33] and in our further discussion, \( v \) stands for the average velocity of two neutrino mass eigenstates.
3.2 Stationary source

While the deviation of the neutrino group velocities from the speed of light is a higher-order effect, it is still interesting to consider a simple example for the impact of the full $V_p$ matrix on flavor oscillations. To this end we consider a stationary situation (nothing depends on time), i.e., a stationary neutrino source and we ask for the flavor content as a function of distance from the source. So we consider the equation

$$\frac{1}{2} \{ V_p, \nabla_x \varrho_{p,x} \} = -i [ H_p, \varrho_{p,x} ] ,$$

where $V_p = \nabla_p H_p$.

(3.1)

The background medium is taken to be homogeneous, isotropic and stationary and we ignore neutrino-neutrino refraction. Therefore, in the mass basis the matrix of velocities is simply

$V_p = \text{diag}(v_1, \ldots, v_N)$ with $v_i = p/(p^2 + m_i^2)^{1/2}$.

One first observation concerns the conservation of particles, often stated as conservation of $\text{tr}(\varrho_p)$. Indeed the r.h.s. of equation (3.1) is traceless due to its commutator structure. If we consider a time-dependent situation with $\partial_t \varrho_{p,t}$ on the l.h.s. we see that indeed $\text{tr}(\varrho_{p,t})$ is conserved. However, in our case it is the trace of the flux matrix $\frac{1}{2} \{ V_p, \varrho_{p,x} \}$ which is conserved, in agreement with physical intuition. If a source produces neutrinos at a given rate, the flux through a surface surrounding the source is stationary. As neutrinos with different mass propagate at different speeds, the local neutrino density outside of the source depends on the velocity. Slower-moving neutrinos take a longer time to cover the distance between the source and detector and so their density must be larger.

3.3 Two flavors

Henceforth it is understood that $V$ and $H$ depend on $p$ and $\varrho$ on $p$ and $x$, so we no longer show these variables explicitly. Moreover, we turn to a two-flavor system and write the velocity matrix in the mass basis in the form

$$V = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} = v \sigma_0 + \delta v \sigma_3 ,$$

(3.2)

where $\sigma_j (j = 0, \ldots, 3)$ are Pauli matrices with $\sigma_0$ the $2 \times 2$ unit matrix. Moreover, we use $v = (v_1 + v_2)/2$ and $\delta v = v_1 - v_2$. Thus we need to solve the equation

$$v \cdot \nabla \varrho + \frac{\delta v}{4} \cdot \{ \nabla \varrho, \sigma_3 \} = -i [ H, \varrho ] .$$

(3.3)

Without loss of generality we consider a one-dimensional system evolving in the $x$-direction and we use the notation $\varrho' = \partial_x \varrho$. Therefore, we need to solve

$$\varrho' + \delta v \left\{ \varrho', \frac{\sigma_3}{2} \right\} = -i \left[ \frac{H}{v}, \varrho \right] ,$$

(3.4)

where $v = (v_1 + v_2)/2$ and $\delta v = (v_1 - v_2)/(v_1 + v_2)$. The appearance of $H/v$ on the r.h.s. is understood because with $v = |p|/E$ we notice that $H/v$ is something like a matrix of wave numbers, which is appropriate for the phase evolution along a beam.

The meaning of this equation becomes more transparent if we write it in terms of polarization vectors $B$ and $P$ defined by

$$\frac{H}{v} = \sum_{j=1}^3 B_j \sigma_j$$

and

$$\varrho = \sum_{j=0}^3 P_j \sigma_j .$$

(3.5)
Notice that we write $H/v$ in traceless form because it always appears in a commutator, whereas for $\rho$ we include the trace in terms of $P_0$. Equation (3.4) then takes the form

$$P_0' + \delta_v P_3' = 0 \quad \text{and} \quad P' + \delta_v P_0' e_3 = B \times P,$$

where $e_3$ is the unit vector in the mass direction in flavor space. Sticking the first equation into the second provides

$$P_0' = -\delta_v P_3' \quad \text{and} \quad P' - \delta_v^2 P_0' e_3 = B \times P.$$

Therefore, the three components of $P$ obey a closed set of differential equations

$$P_1' = B_2 P_3 - B_3 P_2, \quad P_2' = B_3 P_1 - B_1 P_3, \quad \text{and} \quad P_3' = \frac{B_1 P_2 - B_2 P_1}{1 - \delta_v^2}.$$  (3.8)

This equation is simplified with the notation $\tilde{P}_3 = P_3\sqrt{1 - \delta_v^2}$ and $\tilde{B}_{1,2} = B_{1,2}/\sqrt{1 - \delta_v^2}$, whereas for the other components the symbols with or without tilde are the same. Then the equation of motion reads

$$\tilde{P}' = \tilde{B} \times \tilde{P}.$$  (3.9)

Therefore, the evolution is an ordinary precession of an abstract polarization vector $\tilde{P}$ around an abstract magnetic field $\tilde{B}$. The evolution is perfectly periodic — there is no damping.

It is the length of $P$ that is conserved, not the length of $\tilde{P}$. On the other hand, if we prepare the system in an eigenstate of $H$ (propagation eigenstate) then initially $P \propto B$ which also implies that initially $\tilde{P} \propto \tilde{B}$. So there are no oscillations along the beam in the same way as there would be no oscillations in the corresponding time-dependent problem.

### 3.4 Interpretation

In the simplest case of vacuum oscillations, the mass basis is identical with the propagation basis and with the basis where the velocity matrix is diagonal. In the two-flavor context of the previous section this implies $B_{1,2} = 0$ and thus $\tilde{B}_{1,2} = 0$, i.e., the polarization vectors precess around the mass direction in flavor space. As a consequence, $P_3' = 0$ and thus $P_0' = 0$, i.e., the projection of the polarization vector on the mass direction is conserved and thus also $P_0$. Therefore, in this case $\text{tr}(\rho)$ remains conserved: along the beam both the neutrino flux and the neutrino density remain the same.

This result makes physical sense. At the source we produce some coherent combination of mass eigenstates which then propagate independently. The probability for any mass eigenstate along the beam remains constant, only their relative phases evolve, implying oscillations of interaction eigenstates. Of course, the flux ratio of the mass eigenstates is not the same as their density ratio, but both remain constant along the beam.

In a dispersive medium, the group velocity of a wave can be a complicated expression. In our case the background medium produces a simple potential, i.e., the same shift of energy for all $p$ so that for a given $p$ the group velocities of the particles are the same with or without the potential. However, the matrix of particle velocities is not diagonal in the same basis as $H$ which is proportional to the matrix of phase velocities. Therefore, along the beam we not only have oscillations between interaction eigenstates but also oscillations between mass eigenstates and thus between eigenstates of velocity. Therefore, the overall particle density along the beam cannot be the same if the overall particle flux is conserved. Therefore, it makes physical sense that the evolution of $P$, which describes the particle density, is not a simple precession and that $\text{tr}(\rho)$, represented by $P_0$, varies along the beam.
We finally notice, e.g. from equation (3.7), that the flavor dependence of neutrino velocities causes modifications of the order of $\delta^2_v \ll 1$, i.e., of the order of $(m_1^2 - m_2^2)^2/(2|p|)^4$. Therefore, these corrections are of higher order compared with flavor oscillation effects as argued earlier.

4 Conclusion

We have derived the advection part of the kinetic equation (1.4) using only the most elementary ingredients of field theory. This derivation fills a gap left in our earlier paper [22] and is complementary to more recent derivations based on a more advanced formalism. Our step-by-step derivation should be accessible to anyone interested in flavor oscillations.

The matrix of neutrino velocities appearing in the Liouville operator implies a conceptually interesting deviation from the usual picture of flavor oscillations. As the neutrino flux is conserved along the beam, periodic modulations of the neutrino density can obtain. However, the corrections are of higher order in the small deviation of neutrino velocities from the speed of light, the dominant term being the usual commutator expression causing oscillations. Therefore, in situations of practical interest we can use the standard approach of assuming the speed of light for neutrinos everywhere except in the matrix $H$.

While the Liouville operator is identical to the transport part of a kinetic equation for classical particles, we have argued that the advection part of equation (1.4) can be seen as a linearized first-order wave equation, in particular for the flavor degree of freedom. Such an interpretation becomes crucial in the presence of neutrino-neutrino refraction when this equation can be seen as a dynamical equation for “flavor waves” with their own dispersion relation [49, 50]. With this interpretation, equation (1.4) may ultimately lead to a better understanding of neutrino flavor evolution in core-collapse supernovae or neutron-star mergers.

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A Husimi distribution

Here we derive the Liouville equation from quantum mechanics in yet another way, using the Husimi transformation [44]. For simplicity we restrict ourselves to one flavor. The smearing in location and momentum space represented by the Husimi distribution can be defined as

$$F(x, p) \equiv \frac{1}{(2\pi \eta \sigma)^3} \int d^3x' d^3p' f(x', p') \exp \left[ -\frac{(x - x')^2}{2\eta^2} - \frac{(p - p')^2}{2\sigma^2} \right], \quad (A.1)$$

where $\eta$ and $\sigma$ are the length and momentum scales, respectively, over which the Wigner distribution $f(x, p)$ is smeared and here and in the following we again suppress time dependencies. To express the momentum integral in terms of a spatial integral it is useful to
rewrite the second-quantized Wigner distribution corresponding to equation (2.4) in terms of a spatial integral,

$$\hat{f}(x, p) = \int \frac{d^3 \Delta x}{(2\pi \hbar)^3} e^{i p \cdot \Delta x / \hbar} \hat{\psi}^\dagger \left( x + \frac{\Delta x}{2} \right) \hat{\psi} \left( x - \frac{\Delta x}{2} \right),$$  \hspace{1cm} (A.2)

where we have kept $\hbar$ explicit and the spatial wave function operator $\hat{\psi}(x)$ is related to the destruction operator $\hat{a}(p)$ by

$$\hat{\psi}(x) = \int \frac{d^3 \mathbf{p}}{(2\pi \hbar)^{3/2}} e^{-i \mathbf{p} \cdot x / \hbar} \hat{a}(p).$$  \hspace{1cm} (A.3)

Choosing $\sigma = \hbar/(2\eta)$, a straightforward calculation then allows us to express the operator valued equivalent of equation (A.1) as

$$\hat{F}(x, p) = \frac{1}{(2\pi \eta)^{3/2}} \int d^3 x_1 d^3 x_2 \hat{\psi}^\dagger(x_1) \hat{\psi}(x_2) \exp \left[ -\frac{(x-x_1)^2 + (x-x_2)^2}{4\eta^2} + \frac{i \mathbf{p} \cdot (x_1-x_2)}{\hbar} \right].$$  \hspace{1cm} (A.4)

For the time evolution of the quantum field $\hat{\psi}$ we now make the ansatz

$$i \hbar \partial_t \hat{\psi} = \mathbf{v} \cdot \mathbf{p} \hat{\psi} + V(x) \hat{\psi},$$  \hspace{1cm} (A.5)

leading to $\partial_t \hat{\psi} = -\mathbf{v} \cdot \partial_x \hat{\psi} - iV(x) \hat{\psi} / \hbar$. The obvious identity $\mathbf{v} \cdot (\partial_{x_1} + \partial_{x_2}) \exp[\cdots] = -\mathbf{v} \cdot \partial_x \exp[\cdots]$ and partial integration imply that the time derivative $\partial_t \hat{F}(x, p)$ is given by the same integral as in equation (A.4) with the additional operator $-\mathbf{v} \cdot \partial_x + i[V(x_1) - V(x_2)] / \hbar$ acting on the exponential. To lowest order $i [V(x_1) - V(x_2)] \exp[\cdots] / \hbar = \partial_x V(x) \partial_p \exp[\cdots]$ so that, after taking expectation values, one arrives at

$$\partial_t F = -\mathbf{v} \cdot \partial_x F + \partial_x V \partial_p F + O(\hbar).$$  \hspace{1cm} (A.6)

To zeroth order in $\hbar$ this is the standard Liouville equation and corresponds to equation (1.4) for only one flavor and thus absence of oscillations.

We note that for non-relativistic matter the Husimi transformation is also used in the context of simulating the dynamical evolution of dark matter distributions. There the Husimi transformation relates the Schrödinger-Poisson equation for a wave function in three spatial dimensions subject to a gravitational potential to the classical description by six dimensional phase space distributions governed by Liouville equations, also known as collisionless Vlasov equations [51–53]. Their momentum integrated version leads to the classical equations of hydrodynamics, i.e., the continuity and Euler equations, which in turn are related to the Schrödinger equation by a so-called Madelung transformation [54]. In these contexts, classical behavior emerges on length scales large compared to the de Broglie wavelength $\hbar / p$. The use of Schrödinger-like equations instead of classical equations of motion can have practical advantages in dark matter simulations: one has to deal only with three space coordinates instead of six phase-space coordinates or a large number of particles in $N$–body simulations. Furthermore, singularities that can develop in solutions of the classical Liouville or hydrodynamics equations, for example at caustics and shocks, are smoothed out by a finite de Broglie wavelength.
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