Positivity results for Weyl’s pseudo-differential calculus on the Wiener space

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Abstract
This paper deals with positivity properties for a pseudo-differential calculus, generalizing Weyl’s classical quantization, and set on an infinite dimensional configuration space, the Wiener space. In this frame, we show that a positive symbol does not, in general, give a positive operator. In order to measure the nonpositivity, we establish a Gårding’s inequality, which holds for the symbol classes at hand. Nevertheless, for symbols with radial aspects, additional assumptions ensure the positivity of the associated operator.

Keywords Pseudo-differential calculus · Symbol classes · Positivity · Gårding’s inequality · Wiener spaces · Stochastic extensions

Mathematics Subject Classification 35S99 · 28C20

1 Introduction and main results

This paper comes after a series of articles defining a pseudo-differential calculus on the Wiener space, which is an infinite dimensional measure space. This calculus, constructed and developed in [1–4], generalizes Weyl’s well-known calculus. The aim of the construction was to treat problems from mathematical physics, in which an unpredictable and arbitrarily high number of particles may appear, for examples photons. This explains the necessity of replacing the finite dimensional phase and configuration spaces by infinite dimensional spaces. An argument in favour of pseudo-differential calculus is that it allows working with functions (the symbols) rather than
dealing with operators. This makes Weyl’s calculus on the Wiener space, sometimes, preferable to the use of the Fock space (which is linked with the Wiener space anyway).

The preceding articles contain different constructions of the calculus, L^2-boundedness properties, the definition of two different symbol classes. Tools and results analogous to the finite dimensional results exist, such as a Beals characterization [2, 3], composition results [5], in the shape of semiclassical asymptotic expansions in powers of a small parameter h. One of the constructions relies on the notion of Wigner function, as in the finite dimensional case. Parallel calculation are available, like the Anti-Wick calculus (associating an operator with a symbol) and the Wick calculus (associating a function with an operator). A special kind of heat operators links these calculi together [2, 4, 5]. The Wick calculus is defined thanks to a family of coherent states, which are elements of a space of square summable functions defined on the Wiener space. The coherent spaces have a counterpart in the Fock space.

In this paper, we add to this theory by proving positivity and nonpositivity results. Some answer natural questions: if the symbol is nonnegative, what can we say about the operator? Is Flandrin’s conjecture (about integrals of Wigner functions on convex sets) valid? Proposition 11 establishes that, under particular conditions, a positive symbol gives a positive operator. Under more general conditions, a positive symbol gives an operator which is not too negative (Gårding’s inequality, Proposition 12).

The paper is organized as follows. Before stating the main results, the introduction gives the necessary definitions about Weyl’s pseudo-differential calculus on the Wiener space. In particular, we recall the definition of the Wiener space, which consists of a couple \((\mathcal{H}, B)\), in which \(\mathcal{H}\) is a Hilbert space and \(B\), a normed probability space \(B\) containing \(\mathcal{H}\). One calls \(B\) a Wiener extension of \(\mathcal{H}\) and the couple \((\mathcal{H}, B)\) replaces the configuration space \(\mathbb{R}^n\). This is followed by the constructions of the calculus, one using Wigner functions, the other one relying on symbol classes. A L^2-boundedness result is recalled after. To keep this first section as light as possible, other notions are delayed until Sect. 4, in which they are ingredients of the proofs.

Section 2 contains some complements about stochastic extensions in particular cases, mainly for the so-called cylindrical functions. Stochastic extensions are the main means of turning a function defined on \(\mathcal{H}\) (resp. \(\mathcal{H}^2\)) into a function defined on its Wiener extension \(B\) (resp. \(B^2\)), when it is possible. It is a probabilistic definition, the topological usual extensions or restrictions generally failing to operate. This section gives the existence and an explicit expression for stochastic extensions in the case, for example, of Wigner functions.

Section 3 gives the first results about positivity. They are presented together since the arguments are rather similar. They intensely use the Wigner functions, either finite dimensional or not, and a decomposition on a Hilbert basis (of \(L^2(B)\)) consisting of Hermite functions. Section 2 is mainly exploited here, since stochastic extensions of cylindrical functions allow the transposition of finite dimensional results in an infinite dimensional frame. For example, Weyl’s calculus on the Wiener space is no more positive than its finite dimensional predecessor and model. The Flandrin conjecture does not hold either (as was proved in finite dimension in [6]). We give a result about positivity in the case of radial symbols or of tensor products of radial symbols too.

Finally, Sect. 4 proves Gårding’s inequality for this calculus. It is more or less independent of Sects. 2 and 3 and relies on notions and results from [1] which are recalled.
here, like partial heat operators and hybrid Weyl-Anti-Wick operators. Although shorter than Sect. 3, this part may probably find more applications in the sequel.

Conventions and notations. In this work we denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra of a normed space $X$ and by $\mathcal{F}(X)$ the set of all finite dimensional subspaces of $X$. On a (finite dimensional) Euclidean space $E$, we denote by $\mu_{E,s}$ the Gaussian measure of variance $s > 0$. If $E = \mathbb{R}^n$, $\mu_{\mathbb{R}^n,s} = (2\pi s)^{-n/2} e^{-\frac{|y|^2}{2s}} dy$, where $dy$ is the Lebesgue measure.

We have tried to denote differently the functions defined on the Hilbert spaces $\mathcal{H}, \mathcal{H}^2$ and the functions defined on the Wiener extensions $B$ or $B^2$. The latter wear a tilde or are indexed by $B$. Similarly, notions relative to two different spaces, for example relative either to $\mathcal{H}$ or to $B$, are indexed by $\mathcal{H}$ or $B$. This is the case for Wigner functions.

The remaining paragraphs of this section now recall the notions which allow us to state the results.

1.1 The Wiener space

Most of the notions recalled here are defined in [7] (chap. 1 par. 4), which sums up the construction of the articles [8–11]. Stochastic extensions appear in [12] too.

Let $\mathcal{H}$ be a real, separable, infinite dimensional Hilbert space, with norm $||$ or $|.|_\mathcal{H}$ associated with the scalar product $\cdot$. It would be practical to endow $\mathcal{H}$ with a measure similar to the Lebesgue measure or a Gaussian measure, for example. This is not directly possible and one needs to enlarge $\mathcal{H}$ in a convenient way.

The first step is to define a protomeasure on particular subsets of $\mathcal{H}$ called cylinders. Let $E \in \mathcal{F}(\mathcal{H})$ be a finite dimensional subspace of $\mathcal{H}$, let $P_E$ be the orthogonal projection of $\mathcal{H}$ on $E$. A cylinder based on $E$ is the inverse image, by $P_E$, of a Borel set $A$ of $E$. Explicitly, the cylinder defined by

$$C = \{ x \in \mathcal{H} \mid P_E(x) \in A \}$$

has a protomeasure given by

$$\mu_{\mathcal{H},s}(C) = \int_A e^{-\frac{|y|^2}{2s}} (2\pi s)^{-\frac{\dim(E)}{2}} dy = \int_A d\mu_{E,s}(y). \tag{1}$$

We have implicitly chosen an orthonormal basis on $E$, $dy$ is the Lebesgue measure on $E$ and the positive parameter $s$ represents the variance. The real nonnegative number $\mu_{\mathcal{H},s}(C)$ does not depend on the space $E$ on which the cylinder is based (nor on the choice of the basis). Similarly, a cylindrical or tame function on $\mathcal{H}$ is a function $f$ which can be written as $f = \varphi \circ P_E$ for a given $E \in \mathcal{F}(\mathcal{H})$ and a function $\varphi$ defined on $E$. In this case, $f$ is said to be based on $E$. One may think of cylindrical functions as depending on a finite number of variables.

The protomeasure (1) is finitely additive on the set of cylinders but it is not $\sigma$-additive (unless one restricts oneself to cylinders based on a fixed $E$). Therefore, it can’t be extended as a measure on the $\sigma$-algebra generated by the cylinders. To obtain
a measure, one introduces a new and larger set containing $\mathcal{H}$. Let $\| \cdot \|$ be another norm on $\mathcal{H}$, satisfying the condition below, classically called *measurability*:

$$\forall \varepsilon > 0, \exists E_\varepsilon \in \mathcal{F}(\mathcal{H}) : \forall F \in \mathcal{F}(\mathcal{H}), \ F \perp E_\varepsilon,$$

$$\mu_{\mathcal{H},s}([x \in \mathcal{H} : \|P_F(x)\| > \varepsilon]) < \varepsilon. \quad (2)$$

For this new norm, all $d$-dimensional subspaces $F$ of $\mathcal{H}$ are not on the same level. With the original norm, the cylinder $\{x \in \mathcal{H} : |P_F(x)| > \varepsilon\}$ has a protomeasure $\mu_{\mathcal{H},s}$ which depends only on $\dim(F)$ and $\varepsilon$, but not on the situation of $F$ with respect to a space $E_\varepsilon$.

One denotes by $B$ the completion of $\mathcal{H}$ with respect to $\| \cdot \|$. It is called a *Wiener extension* of $\mathcal{H}$; it depends on the choice of the norm $\| \cdot \|$. The dual space of $B$ is called $B'$ and $\mathcal{H}$ is identified with its dual space. This gives the sequence of inclusions:

$$B' \subset \mathcal{H} \subset B,$$

where each space is a dense subset of the following one and the inclusions are continuous. The couple $(\mathcal{H}, B)$ (norm and inclusions remaining implicit) is called an *abstract Wiener space*. The new norm on $\mathcal{H}$ is not necessarily associated with a scalar product and even if it were, it would not be equivalent to the first one.

One may then define a measure on the cylinders of the Wiener extension $B$. For $y_1, \ldots, y_n \in B'$ and $A$ a Borel set of $\mathbb{R}^n$, set:

$$\mu_{\mathcal{H},s}([x \in B : ((y_i, x)_{i \leq n}) \in A]) = \mu_{\mathcal{H},s}([x \in \mathcal{H} : (y_i \cdot x)_{i \leq n} \in A]). \quad (3)$$

This expression is, formally, analogous to the definition of the protomeasure, but it gives a real probability measure on the $\sigma$-algebra generated by the cylinders of $B$. As a consequence of the separability of $\mathcal{H}$ and of $B$, this $\sigma$-algebra is the Borel $\sigma$-algebra of $B$. Note that $\mathcal{H}$ is dense in $(B, \| \cdot \|)$ but negligible for $\mu_{B,s}$. Even if we do not use the following fact in the paper, we may mention that the symmetrized Fock space $\mathcal{F}_s(\mathcal{H})$ is isometric to any $L^2(B, \mu_{B,s})$.

We now define fundamental simple functions which will play an important part in this paper. They replace the first degree monomials in the finite dimensional theory and appear, for example, in Sect. 3.1, as elementary components of a Hilbert basis of a space $L^2(B)$. Right now they allow one to define the notion of stochastic extension defined, for example, in [12].

An element $a$ of $B'$ can be seen in three different ways. Of course, $a$ is a linear continuous form on $B$. Since $B' \subset \mathcal{H}$, the element $a$ gives a linear form from $\mathcal{H}$ to $\mathbb{R}$, identified with the form $x \mapsto x \cdot a$ defined on $\mathcal{H}$. The point is that, since $B$ is endowed with the $\sigma$-algebra $\mathcal{B}(B)$ and the probability measure $\mu_{B,s}$, this element $a$ is also a *random variable* on $(B, \mathcal{B}(B), \mu_{B,s})$. To stress the difference of status, we denote the random variable by $\ell_a$. Definition (3) of the measure implies that $\ell_a$ has the Gaussian distribution $\mathcal{N}(0, \sigma^2 = s|a|^2_{\mathcal{H}})$ and its norm in $L^2(B, \mu_{B,s})$ is equal to $\sqrt{s}|a|_{\mathcal{H}}$. By
density (of $B'$ in $\mathcal{H}$) we obtain an application from $\mathcal{H}$ into $L^2(B, \mu_{B,s})$:

$$\ell : \mathcal{H} \rightarrow L^2(B, \mu_{B,s}), \quad a \mapsto \ell_a. \quad (4)$$

Remark that $\frac{1}{\sqrt{s}}\ell$ is isometric. The set of all $\ell_a, a \in \mathcal{H}$ is a Gaussian Hilbert space in the sense of [13].

If $a$ belongs to $\mathcal{H} \setminus B'$, $\ell_a$ is only a random variable and there is no reason why it should be linear. Strictly speaking, it is defined only almost everywhere. Still, it satisfies the relationships $\ell_a(x + y) = \ell_a(x) + a \cdot y$ for $y \in \mathcal{H}$ and $\ell_a(-x) = -\ell_a(x)$. Sometimes we will write $\ell_{a + ib}$ instead of $\ell_a + i\ell_b$, for $a, b \in \mathcal{H}$ (recall that $\mathcal{H}$ is a real space).

One generalizes the projections, replacing the scalar product with an element $a$ by the corresponding function $\ell_a$. Precisely, if $(u_j)_{j \leq \dim(E)}$ is a Hilbert basis of $E \in \mathcal{F}(\mathcal{H})$, the orthogonal projection $P_E$ (from $\mathcal{H}$ to $E$) and its extension $\tilde{\pi}_E$ (from $B$ to $E$) are written below:

$$\forall x \in \mathcal{H}, \quad P_E(x) = \sum_{j=1}^{\dim(E)} (x \cdot u_j)u_j, \quad \forall y \in B, \quad \tilde{\pi}_E(y) = \sum_{j=1}^{\dim(E)} \ell_{u_j}(y)u_j. \quad (5)$$

The operators $\tilde{\pi}_E$ are defined on $B$ almost everywhere, whereas the orthogonal projections are defined on $\mathcal{H}$. If $E \in \mathcal{F}(B')$, $\tilde{\pi}_E$ is defined everywhere on $B$ (and not just “almost everywhere”) and linear, since $\ell_{u_j} = u_j \in B'$ for all $j$.

The generalized projections allow one to extend, in a certain sense, functions initially defined on $\mathcal{H}$:

**Definition 1** Let $(\mathcal{H}, B)$ be an abstract Wiener space. Let $s$ be a positive real number.

A function $f$ defined on $\mathcal{H}$ is said to admit a stochastic extension $\tilde{f} \in L^p(B, \mu_{B,s})$ in the sense of $L^p(B, \mu_{B,s})$ (1 $\leq p < \infty$) if, for every increasing sequence $(E_n)_{n \in \mathbb{N}}$ in $\mathcal{F}(\mathcal{H})$, whose union is dense in $\mathcal{H}$, the functions $f \circ \tilde{\pi}_{E_n}$ are in $L^p(B, \mu_{B,s})$ and if the sequence $f \circ \tilde{\pi}_{E_n}$ satisfies

$$f \circ \tilde{\pi}_{E_n} \quad \longrightarrow \quad \tilde{f} \quad \text{in} \quad L^p(B, \mu_{B,s}).$$

The original notion of Gross and Ramer required a convergence in probability, which is implied by the $L^p$ convergences above. In general, a stochastic extension is not a continuity extension, though it is sometimes the case (Theorem 6.3 [7]). In the same way, restricting to $\mathcal{H}$ a function defined on $B$ does not always make sense, since $\mathcal{H}$ is negligible for all measures $\mu_{B,s}, s > 0$.

A function defined on $\mathcal{H}$ has not necessarily a stochastic extension, or it may have an extension which is useless. For example, $x \in \mathcal{H} \mapsto |x|^2$ or $x \mapsto e^{-|x|^2}$ have no stochastic extension. The first example justifies the introduction of the notion of a measurable norm (2), see [7], Chap.1, Sec. 4. The function $f : x \mapsto e^{-|x|^2}$ admits, as an extension, the null function on $B$. 
1.2 Weyl’s calculus in the Wiener space

In this section we recall the main definitions pertaining to the infinite dimensional Weyl calculus. Since there are two spaces $\mathcal{H}$ and $B$ and two different constructions, we felt that a short guideline stressing the main features could be useful.

The first construction, which is the less technical one, associates, with a symbol $\tilde{F}$ defined on $B^2$, a quadratic or bilinear form $Q_W^h(\tilde{F})$, which applies to a couple of convenient cylindrical functions (see Definition 2 for the meaning of convenient, Definition 5 for the form $Q_W^h(\tilde{F})$). This form is defined by an integral on $B^2$ and corresponds to the following expression in the finite dimensional situation:

$$\langle Op_{Weyl,cl}^h(F)u, v\rangle_{L^2(\mathbb{R}^n, dx)} = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} F(z, \zeta) H_{cl}^h(u, v)(z, \zeta) dz d\zeta. \quad (6)$$

Here, $H_{cl}^h(u, v)$ is the finite dimensional Wigner function of the couple $(u, v)$ and the superscript “cl” stands for “classical”.

The second construction introduces symbol classes, as in the finite dimensional Weyl calculus. The symbols belonging to these classes are defined on $\mathcal{H}^2$ and satisfy regularity conditions, their partial derivatives or Fréchet differential are bounded in a precise way (Definitions 6, 8). In this frame, one associates, with a symbol $F$ defined on $B^2$, an operator $Op_W^h(F)$, which is linear and bounded on a space $L^2(B)$. The operator is not defined directly, it is the limit of a Cauchy sequence of rather complicated hybrid operators, which are recalled later on in Sect. 4 but are not necessary right now to state the result. The convergence takes place in the space of the operators bounded on a $L^2(B)$. Contrary to the first construction, $Op_W^h(F)u$ has no integral expression generalizing

$$\langle Op_{Weyl,cl}^h(F)(u), (\tilde{F})u\rangle_{L^2(\mathbb{R}^n, dx)} = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{i\frac{xy}{2h} - \frac{y^2}{2h}} F(\frac{x-y}{2}, \xi) u(y) dy d\xi. \quad (7)$$

When the operator is applied to (convenient) cylindrical functions, there is a link with the form $Q_W^h(\tilde{F})$, where $\tilde{F}$ is a stochastic extension of $F$.

These notions are taken from [1, 4, 14] and are now recalled at length.

First construction

Let $E$ be a $d$-dimensional Euclidean space, identified with $\mathbb{R}^d$ by the choice of an orthonormal basis. For $h > 0$, one defines an isometric isomorphism between $L^2(E, \mu_{\mathbb{R}^d, h/2})$ and $L^2(E, dy)$, setting

$$\forall y \in E, \quad \gamma_{E,h/2} f(y) = (\pi h)^{-d/4} e^{-\frac{|y|^2}{2h}} f(y). \quad (8)$$

The space of the “test” functions, to which the quadratic form will be applied, is given by the following definition.

**Definition 2** Let $E$ belong to $\mathcal{F}(B')$. 
• One denotes by \( S_{E,h/2} \) the space of all functions \( \varphi \) defined on \( E \), such that \( \gamma_{E,h/2} \varphi \) is rapidly decreasing.
• One denotes by \( D_{E,h/2} \) the set of all functions \( \tilde{f} \) defined on \( B \) and based on \( E \), that is to say, of the form \( \tilde{f} = \varphi \circ \tilde{\pi}_E \), with \( \varphi \in S_{E,h/2} \) and \( \tilde{\pi}_E \) as in (5).
• One then sets

\[ D_{B',h/2} = \bigcup_{E \in \mathcal{F}(B')} D_{E,h/2}, \quad D_{\mathcal{H},h/2} = \bigcup_{E \in \mathcal{F}(B\mathcal{H})} D_{E,h/2}. \]

The spaces \( D_{B',h/2} \) and \( D_{\mathcal{H},h/2} \) are dense in \( L^2(B, \mu_{B,h/2}) \). The functions in \( D_{B',h/2} \) and \( D_{\mathcal{H},h/2} \) are defined on \( B \) but depend only on a finite number of variables. The slightly unnatural parameter \( h/2 \) comes from [1], where the Segal-Bargman transformation links \( L^2(B, \mu_{B,h/2}) \) and \( L^2(B^2, \mu_{B,h}) \). Such changes of variance are unavoidable, see for example Proposition 4 and Definition 10 below.

The Wigner functions, which are Gaussian Wigner functions in this paper, are defined differently according to whether the test functions are defined on \( E \) or on \( B \). The relationships between these functions are specified in Proposition 20. The definition of \( W_{h,E} \) (for test functions on \( E \)) is inspired by the classical definition of a Wigner function, taking into account the fact that the measure is Gaussian.

**Definition 3** Let \( E \) be in \( \mathcal{F}(\mathcal{H}) \), let \( \hat{f}, \hat{g} \) be in \( S_{E,h/2} \). The Wigner function of the couple \((\hat{f}, \hat{g})\) is defined on \( E^2 \) by: for all \((z, \zeta)\) in \( E^2 \),

\[ W_{h,E}(\hat{f}, \hat{g})(z, \zeta) = e^{i|\zeta|^2/h} \int_E e^{-2i\zeta \cdot t/h} \hat{f}(t) \hat{g}(z-t) e^{-|t|^2/h} (\pi h)^{-d/2} dt. \quad (9) \]

Suppose that \( \tilde{f} \) and \( \tilde{g} \) are defined on \( B \) and satisfy \( \tilde{f} = \tilde{f} \circ \tilde{\pi}_E, \tilde{g} = \tilde{g} \circ \tilde{\pi}_E \). Then the Wigner function of \( \tilde{f} \) and \( \tilde{g} \) is defined on \( B^2 \) by: for all \((z, \zeta)\) in \( B^2 \),

\[ W_{h,B}(\tilde{f}, \tilde{g})(z, \zeta) = W_{h,E}(\hat{f}, \hat{g})(\pi E(z), \pi E(\zeta)) \]

\[ = e^{i|\pi E(\zeta)|^2/2} \int_E e^{-2i\pi E(\zeta) \cdot t} \hat{f}(\pi E(z) + t) \hat{g}(\pi E(z) - t) d\mu_{E,h/2}(t). \]

We need here the extended projections \( \tilde{\pi}_E \). What we denote here by \( W_{h,E}(\hat{f}, \hat{g}) \) would have been called \( H^Gauss_h(\hat{f}, \hat{g}) \) in [1], formula (9), but we wish to indicate on which space the test functions \( \hat{f}, \hat{g} \) are defined.

In the second case, the Wigner function of \((\tilde{f}, \tilde{g})\) does not depend on the space \( E \) on which \( \tilde{f} \) and \( \tilde{g} \) are based. We may state some of its properties, which hold for \( \tilde{f}, \tilde{g} \) in \( D_{\mathcal{H},h/2} \) or \( D_{B',h/2} \). The result below is taken from [1] (Proposition 4.8).

**Proposition 4** For all \( \tilde{f}, \tilde{g} \) in \( D_{\mathcal{H},h/2} \), the Wigner function \( W_{h,B}(\tilde{f}, \tilde{g}) \) belongs to \( L^1(B^2, \mu_{B^2,h/2}) \). The operator associating, with all functions \( \tilde{f}, \tilde{g} \) of \( D_{\mathcal{H},h/2} \), their Wigner function \( W_{h,B}(\tilde{f}, \tilde{g}) \), extends uniquely as a continuous bilinear map from \( L^2(B, \mu_{B,h/2}) \times L^2(B, \mu_{B,h/2}) \) in \( L^2(B^2, \mu_{B^2,h/4}) \), with norm \( \leq 1 \).
Now, we can give the definition of the quadratic form, which is the first construction mentioned above.

**Definition 5** Let $\tilde{F}$ be a bounded Borel function on $B^2$. One defines $Q_{h}^{Weyl}(\tilde{F})$ by its action on $D^2_{B',h/2}$:

$$\forall (\tilde{f}, \tilde{g}) \in D^2_{B',h/2}, \quad Q_{h}^{Weyl}(\tilde{F})(\tilde{f}, \tilde{g}) = \int_{B^2} \tilde{F}(Z)W_{h,B}(\tilde{f}, \tilde{g})(Z)d\mu_{B^2,h/2}(Z).$$

(10)

If $\tilde{F}$ is not bounded, but if there exists an integer $m \geq 0$ such that

$$N_m(\tilde{F}) := \sup_{Y \in H^2} \|\tau_Y \tilde{F}\|_{L^1(B^2,\mu_{B^2,h/2})} < +\infty,$$

(11)

then $Q_{h}^{Weyl}(\tilde{F})$ can be defined as above.

Condition (11) means that the translates of $\tilde{F}$ by vectors $Y$ belonging to $H^2$ still belong to $L^1$ and that their norms depend polynomially at most on the translation vector $Y$. If this vector were not in $H^2$, the initial measure and the measure obtained by translation would be mutually orthogonal.

**Remarks** This construction is a first step towards the construction of an operator. But it is useful in itself, since it allows one to consider unbounded symbols like polynomials in the functions $\ell_a$. For example, the symbol $\varphi_{a,b} : (x, \xi) \mapsto \ell_a(x) + \ell_b(\xi)$ gives a multiplication by a monomial and a differentiation operator, as in the finite dimensional case.

The normalization of the Wigner function $W_{h,E}$ has be chosen in order to recover the classical Weyl calculus in the case when the symbols only depend on a finite number of variables. But, since we need a Gaussian measure, a Gaussian factor appears in (9).

**Second construction**

A multiindex $\alpha$ is an element of $\mathbb{N}^d$ when the dimension $d$ is finite. Otherwise, it is a mapping from $\mathbb{N}$ or $\mathbb{N}^*$ into $\mathbb{N}$ with a finite number of nonzero coordinates. In both cases, one calls depth of the multiindex the maximum of its coordinates, $\max_{j \in \{1, \ldots, d\}} \alpha_j$ or $\max_{j \in \mathbb{N}^*} \alpha_j$.

The first symbol class has features recalling the conditions of the Calderon-Vaillancourt Theorem and will be named after this result.

**Definition 6** Let $(\mathcal{H}, B)$ be an abstract Wiener space. Let $B = (e_j)_{j \in \mathbb{N}^*}$ be a Hilbert basis of $\mathcal{H}$, each vector belonging to $B'$. For $j \geq 1$, let $u_j = (e_j, 0)$ and $v_j = (0, e_j)$. Let $m$ be a nonnegative integer and $\varepsilon = (\varepsilon_j)_{j \in \mathbb{N}^*}$ be a family of nonnegative real numbers. One denotes by $S_m(B, \varepsilon)$ the set of bounded continuous functions $F : \mathcal{H}^2 \to \mathbb{C}$ satisfying the following condition. There exists $M \in \mathbb{R}^+$ such that, for any multiindices $\alpha, \beta$ of depth $m$, the following derivative
\[ \partial^\alpha_u \partial^\beta_v F = \left[ \prod_{j \in \mathbb{N}^*} \partial^\alpha_j u_j \partial^\beta_j v_j \right] F \] (12)

is well defined, continuous on \( \mathcal{H}^2 \) and satisfies, for every \((x, \xi)\) in \( \mathcal{H}^2 \)

\[ \left| \left[ \prod_{j \in \mathbb{N}^*} \partial^\alpha_j u_j \partial^\beta_j v_j \right] F(x, \xi) \right| \leq M \prod_{j \in \mathbb{N}^*} \varepsilon_\alpha^\alpha_j + \beta_j^\beta_j . \] (13)

Denote by \( \| F \| = \| F \|_{S_m(B, \varepsilon)} \) the smallest constant \( M \) for which (13) holds. With this norm \( \| S_m(B, \varepsilon) \|_{S_m(B, \varepsilon)} \) is a Banach space.

We then recall the existence and boundedness result ([1], Th. 1.4).

**Theorem 7** Let \((\mathcal{H}, B)\) be an abstract Wiener space and let \( h \) be a positive number. Let \((e_j)_{j \in \mathbb{N}^*}\) be a Hilbert basis of \( \mathcal{H} \), each vector belonging to \( B' \). Let \( F \) be a function on \( \mathcal{H}^2 \) satisfying the following two hypotheses:

- it belongs to the class \( S_2(B, \varepsilon) \), where \( \varepsilon = (\varepsilon_j)_{j \in \mathbb{N}^*} \) is a square summable family of nonnegative real numbers;
- it has a stochastic extension \( \tilde{F} \) in the \( L^2 \) sense with respect to both measures \( \mu_{B^2, h} \) and \( \mu_{B^2, h/2} \) (see Definition 1).

Then there exists an operator, denoted by \( Op^{Weyl}_h(F) \), bounded in \( L^2(B, \mu_{B, h/2}) \), such that, for all \( \tilde{f} \) and \( \tilde{g} \) in \( D_{B', h/2} \)

\[ < Op^{Weyl}_h(F) \tilde{f}, \tilde{g} > = Q^{Weyl}_h(\tilde{F})(\tilde{f}, \tilde{g}), \] (14)

where the right hand side is defined by Definition 5. Moreover, if \( h \) is in \((0, 1]\):

\[ \| Op^{Weyl}_h(F) \|_{\mathcal{L}(L^2(B, \mu_{B, h/2}))} \leq \| F \|_{S_2(B, \varepsilon)} \prod_{j \in \mathbb{N}^*} \left( 1 + 81 \pi h S_\varepsilon^2 e_j^2 \right), \] (15)

where

\[ S_\varepsilon = \sup_{j \in \mathbb{N}^*} \max(1, e_j^2) . \] (16)

We need the basis to belong to \( B' \) because of a decomposition result stated in Sect. 4, hence the use of the space \( D_{B', h/2} \).

In the more restrictive case when the sequence \((\varepsilon_j)_{j \in \mathbb{N}^*}\) is summable, a function \( F \) belonging to the Calderón-Vaillancourt class \( S_1(B, \varepsilon) \) admits a stochastic extension in \( L^q(B^2, \mu_{B^2, h}) \) for all \( h > 0 \) and all \( q \in [1, +\infty[ \) ([4], Proposition 3.1). Moreover, there exists a function \( \tilde{F} \) which is the stochastic extension of \( F \) for all \( h > 0 \) and all \( q \in [1, \infty[ \).

Let us now define the second symbol class.
**Definition 8** Let \( A \) be a linear, selfadjoint, nonnegative, trace class application on a Hilbert space \( \mathcal{H}^2 \). For all \((x, \xi) \in \mathcal{H}^2\), one sets \( Q_A(x, \xi) = \langle A(x, \xi), (x, \xi) \rangle \), where \( \langle \rangle \) denotes the scalar product in \( \mathcal{H}^2 \). Let \( S(Q_A) = S(Q_A, \mathcal{H}^2) \) be the class of all functions \( F \in C^\infty(\mathcal{H}^2) \) such that there exists \( C(F) > 0 \) satisfying:

\[
\forall (x, \xi) \in \mathcal{H}^2, \quad |F(x, \xi)| \leq C(F),
\]

\[
\forall m \in \mathbb{N}^*, \forall (x, \xi) \in \mathcal{H}^2, \forall (U_1, \ldots, U_m) \in (\mathcal{H}^2)^m,
\]

\[
|(d^m F)(x, \xi)(U_1, \ldots, U_m)| \leq C(F) \prod_{j=1}^{m} Q_A(U_j)^{\frac{1}{2}}.
\]

The smallest constant \( C(F) \) such that (17) holds is denoted by \( \|F\|_{Q_A} \).

One checks that \( S(Q_A) \), endowed with the norm \( \|Q_A\| \), is a Banach space.

The class \( S(Q_A) \) is more restrictive than the preceding class. Indeed, \( S(Q_A) \subset S_\infty(B, \varepsilon) \) for any orthonormal basis \( B = (e_j) \) of \( \mathcal{H} \), with \( \varepsilon_j = \max(Q_A(e_j, 0)^{1/2}, Q_A(0, e_j)^{1/2}) \) and \( \|F\|_{S_\infty(B, \varepsilon)} \leq \|F\|_{S_\infty(B, \varepsilon)} = \|F\|_{Q_A} \). This sequence \( \varepsilon \) is only square summable. But functions in a class \( S(Q_A) \) admit a stochastic extension in \( L^p(B^2, \mu_{B^2}) \) for all \( p \in [1, \infty[ \) and \( s > 0 \) for orthogonality reasons (see [4], Proposition 3.9). This justifies that Theorem 7 holds for a symbol \( F \in S(Q_A) \).

**Remark** This class was introduced to relax an assumption in [15]. Moreover, its properties make the construction of the operator easier, because one can use the Anti-Wick calculus - which exists in the infinite dimensional frame - as a transition. The results stated in the next section are proved for the Calderón-Vaillancourt classes of Definition 6 and are therefore valid for the classes of Definition 8.

### 1.3 Main results

The first results of this section, Propositions 9 and 10, extend, to the Weyl calculus on the Wiener space, results already known in the finite dimensional case. We show that an operator with a positive symbol is not necessarily positive, exactly as in the case of Weyl’s classical calculus. The other result concerns Wigner functions and the Flandrin conjecture. Recalled in Part 3.4, this conjecture has been recently invalidated in dimension 1, for the configuration space \( \mathbb{R} \), in [6, 16]. Using these articles, we show that it does not hold either when the configuration space is the Wiener space.

The third result generalizes the paper [17] concerning operators with a radial symbol, in the finite dimensional case: if the symbol is radial, positive and satisfies further assumptions, the operator is positive. This property holds for the infinite dimensional calculus too.

The fourth result is probably the most satisfactory, since it is Gårding’s inequality in the infinite dimensional frame. As in the classical case, if the symbol \( F \) is positive, we can’t say that the operator \( \text{Op}_h^{Weyl}(F) \) is positive. But the loss of positivity is quantified in Proposition 12.
Proposition 9 The Weyl calculus recalled in Part 1.2 is not positive. There exists a positive symbol $F$ belonging to the Calderón-Vaillancourt classes of Definition 6 and a test function $\tilde{g} \in L^2(B, \mu_{B,h/2})$ such that

$$(Op^W_p(F)\tilde{g}, \tilde{g})_{L^2(B, \mu_{B,h/2})} < 0.$$ 

Now the result about Flandrin’s conjecture:

Proposition 10 Take $e_1 \in B'$ with norm 1. For $a > 0$ or $a = +\infty$, define $\tilde{F}_a$ on $B^2$ by

$$\tilde{F}_a(z, \xi) = 1_{[0,a]}(\ell e_1(\xi))1_{[0,2\pi ha]}(\ell e_1(z)).$$

For $a = \infty$, the indicator functions are $1_{\mathbb{R}^+}$.

Then, for $a > 0$ sufficiently large or $a = \infty$, there exists a cylindrical function $\tilde{v}_a \in L^2(B, \mu_{B,h/2})$ such that

$$Q^W_p(\tilde{F}_a)(\tilde{v}_a, \tilde{v}_a) = \int_{B^2} \tilde{F}_a(z, \xi)W_{B, h}(\tilde{v}_a, \tilde{v}_a)(z, \xi)\,d\mu_{B^2, h/2}(z, \xi) > \|\tilde{v}_a\|_{L^2(B, \mu_{B,h/2})}^2.$$ 

Observe that the function $F$ considered here does not belong to a symbol class, since it is not defined on $\mathcal{H}^2$ and it is not even continuous. Therefore, there is no operator associated with it.

For commodity reasons, the positivity result for symbols with radial properties is stated below in an expurgated form. A more general (and hence more technical) version, Proposition 25, is proved in Part 3.5.

Proposition 11 Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth, increasing, function. Assume that $\Phi$ is such that the radial function, defined on $\mathbb{R}^{2d}$ by

$$(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d) \mapsto \Phi \left( \sum_{j=1}^d (|x_j|^2 + |\xi_j|^2) \right)$$

is smooth too.

Let $(e_1, \ldots, e_d)$ be an orthonormal family of $\mathcal{H}$, with all vectors in $B'$.

Set

$$\forall (x, \xi) \in \mathcal{H}^2, F(x, \xi) = \Phi \left( \sum_{j=1}^d (|x \cdot e_j|^2 + |\xi \cdot e_j|^2) \right),$$

$$\forall (z, \xi) \in B^2, \tilde{F}(z, \xi) = \Phi \left( \sum_{j=1}^d (|\ell e_j(x)|^2 + |\ell e_j(\xi)|^2) \right).$$
with the applications $\ell$ defined by 4. Theorem 7 allows one to associate, with $F$, an operator $Op_W(F)$, bounded on $L^2(B, \mu_{B,h/2})$. Moreover, $Op_W(F)$ has the following properties:

- For all $\tilde{f} \in L^2(B, \mu_{B,h/2})$,

$$\langle Op_W(F)\tilde{f}, \tilde{f}\rangle_{L^2(B,\mu_{B,h/2})} \geq \left(\frac{1}{h}\int_0^\infty \Phi(t)e^{-t/h} \, dt\right) \|\tilde{f}\|_{L^2(B,\mu_{B,h/2})}^2.$$

- If, moreover, $\tilde{f}$ belongs to $D_{B',h/2}$, the expression above is given by the quadratic form:

$$\langle Op_W(F)\tilde{f}, \tilde{f}\rangle_{L^2(B,\mu_{B,h/2})} = \int_{B^2} \tilde{F}(z, \xi) W_{h,B}(\tilde{f}, \tilde{f})(z, \xi) \, d\mu_{B,h/2}(z, \xi).$$

Under these assumptions, the positivity of the symbol $F$ implies the positivity of $\Phi$ and hence the positivity of the operator.

We now state the Gårding inequality, which holds for both symbol classes. Its proof is not based on a composition result, as will be seen in Sect. 4.

**Proposition 12** Let $B = (e_j)_{j \geq 1}$ be an orthonormal basis of $H$, with all vectors in $B'$. Let $\varepsilon = (\varepsilon_j)_{j \geq 1}$ be a square summable sequence of positive real numbers. Set $S_\varepsilon = \sup_{j \in \mathbb{N}^*} \max(1, \varepsilon_j^2)$.

Let $F \in S_2(B, \varepsilon)$ be nonnegative on $H$. Suppose that $F$ has a stochastic extension $\tilde{F}$ for $L^2(B^2, \mu_{B^2,h/2})$ and $L^2(B^2, \mu_{B^2,h})$ (this is the case if $\varepsilon$ is summable).

Then, for every function $\tilde{f}$ in $L^2(B, \mu_{B,h/2})$, the following inequality holds:

$$\langle Op_W(F)\tilde{f}, \tilde{f}\rangle \geq -\|F\|_{S_2(B,\varepsilon)} \sum_{j \geq 1} \lambda_j \prod_{s \geq 1} (1 + \lambda_s) \|\tilde{f}\|^2, \quad (18)$$

with $\lambda_j = 81\pi h S_\varepsilon \varepsilon_j^2$ and with the norm and scalar product of $L^2(B, \mu_{B,h/2})$.

Since the symbol classes defined by a quadratic form of Definition 8 are included in the Calderón-Vaillancourt classes and since they have stochastic extensions in $L^2$ for both measures, this Gårding inequality holds for them too:

**Corollary 13** Let $F$ belong to $S(Q_A, H^2)$ for a linear, selfadjoint, nonnegative, trace class application $A$ defined on $H^2$.

Then for every function $\tilde{f}$ in $L^2(B, \mu_{B,h/2})$,

$$\langle Op_W(F)\tilde{f}, \tilde{f}\rangle \geq -\|F\|_{Q_A} \sum_{j \geq 1} \lambda_j \prod_{s \geq 1} (1 + \lambda_s) \|\tilde{f}\|^2, \quad (19)$$

where $\lambda_j = 81\pi h S_\varepsilon \varepsilon_j^2$ and $S_\varepsilon = \sup_{j \in \mathbb{N}^*} \max(1, Q_A(e_j, 0), Q_A(0, e_j))$. The norm and scalar product are on $L^2(B, \mu_{B,h/2})$. 
2 Explicit stochastic extensions of cylindrical functions

This section deals with stochastic extensions of cylindrical functions, which are functions defined on $\mathcal{H}$ and depending on a finite number of scalar products with elements of $\mathcal{H}$ (as recalled in Part 1.1). We prove that, in many cases, one just needs to replace the scalar products by the corresponding functions $\ell$ defined in (4). This has been proved for polynomial functions of scalar products in [4].

We first restate (for further reference) a lemma which already appeared in a previous article. It concerns the functions $\ell$ themselves, functions which replace the monomials in the finite dimensional case. We then give results for cylindrical functions designed to play different parts in the calculus: test functions, symbols or Wigner functions.

Lemma 14 For every $a \in \mathcal{H}$, the scalar product function $\mathcal{H} \to \mathbb{R}, x \mapsto a \cdot x$ admits, as a stochastic extension in $L^p(B, \mu_{B,s})$, the function $\ell_a$. This holds for all $p \in [1, +\infty[$ and $s > 0$.

This means that, if $(E_n)_{n \in \mathbb{N}}$ in an increasing sequence of $\mathcal{F}(\mathcal{H})$, with union dense in $\mathcal{H}$, the sequence of random variables $(\ell_a - \ell_{E_n(a)})_{n \in \mathbb{N}}$ converges to 0 in $L^p(B, \mu_{B,s})$ (for all $p \in [1, +\infty[$ and $s > 0$). Hence it converges in $\mu_{B,s}$ probability too.

Proof For $E \in \mathcal{F}(\mathcal{H})$, one checks that

$$a \cdot \widetilde{\pi}_E = \ell_{E(a)}.$$ (20)

Indeed, if $(e_1, \ldots, e_d)$ is an orthonormal basis of $E$ for the scalar product of $\mathcal{H}$,

$$\forall x \in B, \quad a \cdot \widetilde{\pi}_E(x) = a \cdot \sum_{i=1}^d \ell_{e_i}(x)e_i = \sum_{i=1}^d \ell_{e_i}(x)a \cdot e_i = \ell_{\sum_{i=1}^d (a \cdot e_i)e_i}(x)$$

by linearity of $a \mapsto \ell_a$. One then recognizes the orthogonal projection on $E$.

Let $(E_n)_{n \in \mathbb{N}}$ be an increasing sequence of $\mathcal{F}(\mathcal{H})$, with union dense in $\mathcal{H}$. For $1 \leq p < +\infty$, the formula above and Definition 1 of stochastic extensions lead us to consider $\|\ell_a - \ell_{E_n(a)}\|_{L^p(B, \mu_{B,s})}$. We may write that

$$\|\ell_a - \ell_{E_n(a)}\|_{L^p(B, \mu_{B,s})} = C_{p,s}|a - P_{E_n(a)}(a)|,$$

with $C_{p,s} = \sqrt{2}\pi^{-1/2}p \Gamma\left(\frac{p+1}{2}\right)^{1/p}$. This is a consequence of the transfer theorem, which brings us back to the following equality involving an integral on $\mathbb{R}$: $\forall b \in \mathcal{H}\setminus\{0\}$,

$$\int_B |b|^p d\mu_{B,s} = \int_{\mathbb{R}} \frac{|x|^p}{(2\pi s|b|^2)^{1/2}} e^{-\frac{x^2}{2s|b|^2}} dx = \frac{(2s)^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{p+1}{2}\right) |b|^p. \quad (21)$$

Hence $\|\ell_a - \ell_{E_n(a)}\|_{L^p(B, \mu_{B,s})}$ converges to 0, which, in turn, implies the convergence in probability. □

We now turn to more general regular cylindrical functions on $\mathcal{H}^2$. Under regularity and decay assumptions, they belong to a Càlderon-Vaillancourt symbol class
(of Definition 6). Even if this gives the existence of the stochastic extension when the sequence \( \varepsilon \) is summable, the extension is not necessarily explicit. In the case of cylindrical functions, one may be more precise.

**Lemma 15** Let \( \tilde{F} \) be a bounded, \( C^m \) function on \( \mathbb{R}^{2d} \), with bounded partial derivatives of all orders (smaller than \( m \)). Let \( (e_n)_{n \in \mathbb{N}^*} \) be a Hilbert basis of \( \mathcal{H} \), with elements in \( B' \).

Let the functions \( F \) and \( \tilde{F} \) be defined, respectively, on \( \mathcal{H}^2 \) and \( B^2 \) by:

\[
\forall (x, \xi) \in \mathcal{H}^2, \quad F(x, \xi) = \tilde{F}(e_1 \cdot x, \ldots, e_d \cdot x, e_1 \cdot \xi, \ldots, e_d \cdot \xi),
\]

\[
\forall (z, \zeta) \in B^2, \quad \tilde{F}(z, \zeta) = \tilde{F}(\ell_e(z), \ldots, \ell_{ed}(z), \ell_e(\zeta), \ldots, \ell_{ed}(\zeta)).
\]

Then \( F \) belongs to the symbol class \( S_m(B, \varepsilon) \) for the sequence \( \varepsilon = (n^{-2})_{n \geq 1} \).

The function \( F \) admits \( \tilde{F} \) as a stochastic extension in \( L^p(B^2, \mu_{B^2,s}) \) for all finite \( p \geq 1 \) and all \( s > 0 \).

**Proof** Let \( \alpha \) and \( \beta \) be two multiindices of depth \( m \). If \( \alpha \) or \( \beta \) has a nontrivial component for an index \( j > d \), inequality (13) holds because its left side is equal to 0. Otherwise, there is a finite number of inequalities to satisfy and it suffices to set

\[
M = \sup_{\alpha, \beta} \left( \prod_{j \leq d} j^{2(\alpha_j + \beta_j)} \sup_{(x, \xi) \in \mathbb{R}^{2d}} \left| \prod_{j=1}^d \frac{\partial^{\alpha_j} \partial^{\beta_j}}{\partial x_j^{\alpha_j} \partial \xi_j^{\beta_j}} \tilde{F}(x, \xi) \right| \right).
\]

The supremum above is taken on all multiindices of depth \( m \) with all components equal to 0 for \( j \geq d \).

Since the sequence \( \varepsilon \) is summable, \( F \) has a stochastic extension (temporarily denoted by \( F^* \)) in \( L^p(B^2, \mu_{B^2,s}) \) for all \( p \in [1, \infty] \) and all \( s > 0 \), according to Proposition 3.1 of [4]. We will prove that it coincides with \( \tilde{F} \).

Let \( (E_n)_{n \in \mathbb{N}^*} \) be an increasing sequence of \( \mathcal{F}(\mathcal{H}) \), with union dense in \( \mathcal{H} \). Set \( \tilde{F}_n(z, \zeta) = F(\tilde{\pi}_{E_n}(z), \tilde{\pi}_{E_n}(\zeta)) \), for \( z, \zeta \in B \). Since \( (\tilde{F}_n)_{n \in \mathbb{N}^*} \) converges to \( F^* \) in \( L^p(B^2, \mu_{B^2,s}) \), a subsequence \( (\tilde{F}_{\psi(n)}) \) converges \( \mu_{B^2,s} \)-almost surely to \( F^* \).

According to the definition of \( F \), one may write

\[
\tilde{F}_{\psi(n)}(z, \zeta) = \tilde{F}(e_1 \cdot \tilde{\pi}_{E_{\psi(n)}}(z), \ldots, e_d \cdot \tilde{\pi}_{E_{\psi(n)}}(z), e_1 \cdot \tilde{\pi}_{E_{\psi(n)}}(\zeta), \ldots, e_d \cdot \tilde{\pi}_{E_{\psi(n)}}(\zeta))
\]

Lemma 14 gives the convergence in \( \mu_{B,s} \) probability of \( e_j \cdot \tilde{\pi}_{E_{\psi(n)}}(\cdot) = \ell_{E_{\psi(n)}(e_j)} \) to \( \ell_{e_j} \). Extracting a further subsequence and using the continuity of \( \tilde{F} \), we get that

\[
\tilde{F}_{\psi(n)}(z, \zeta) \longrightarrow \tilde{F}(\ell_{e_1}(z), \ldots, \ell_{ed}(z), \ell_{e_1}(\zeta), \ldots, \ell_{ed}(\zeta)) = \tilde{F}(z, \zeta)
\]

\( \mu_{B^2,s} \)-almost surely. Hence, the functions \( F^* \) and \( \tilde{F} \) are almost surely equal for every measure \( \mu_{B^2,s} \), which proves Lemma 15.

We now prove that the functions of \( D_{\mathcal{H}, h/2} \), defined in Definition 2 are stochastic extensions of functions cylindrical on \( \mathcal{H} \). But we may state a slightly more general result, namely:
Theorem 16 Let $s > 0$. Let $p \geq 1$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be continuous and such that the function

$$x \mapsto e^{-|x|^2/2p^{s}} \varphi(x) P(x)$$

is bounded for every polynomial $P$.

Let $E \in \mathcal{F}(\mathcal{H})$ have dimension $d$ and an orthonormal basis $(e_1, \ldots, e_d)$.

Define $f$ on $\mathcal{H}$ and $\tilde{f}$ on $\mathcal{B}$ setting:

$$\forall x \in \mathcal{H}, \ f(x) = \varphi((x \cdot e_i)_{1 \leq i \leq d}) \quad \text{and} \quad \forall y \in \mathcal{B}, \ \tilde{f}(y) = \varphi((\ell_{e_i}(y))_{1 \leq i \leq d}).$$

Then $\tilde{f}$ is the stochastic extension of $f$ in $L^p(B, \mu_{B,s})$.

This result has the following important corollaries:

Corollary 17 Let $E \in \mathcal{F}(\mathcal{H})$, let $s > 0$. Let $(e_1, \ldots, e_d)$ be an orthonormal basis of $E$.

If $f$, defined on $\mathcal{H}$, has the form $f(x) = \varphi((x \cdot e_i)_{1 \leq i \leq d})$ with $\varphi_{\mathbb{R}^d} \varphi$ rapidly decreasing (which means that $f \in S_{E,s}$), then $f$ has a stochastic extension in $L^2(B, \mu_{B,s})$, which is the function $\tilde{f}$ defined on $\mathcal{B}$ by $\tilde{f}(y) = \varphi((\ell_{e_i}(y))_{1 \leq i \leq d})$.

This is true because, if $f$ satisfies the above conditions, $x \mapsto e^{-|x|^2/4s} \varphi(x)$ is rapidly decreasing. The exponent $p$ of Theorem 16 is equal to 2, the transformations $\gamma$ are defined in (8).

This proves that the functions belonging to $D_{\mathcal{H},s}$ are indeed stochastic extensions of functions defined thanks to the same $\varphi$, scalar products with $a \in \mathcal{H}$ replacing the functions $\ell_a$.

Corollary 18 Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be continuous, compactly supported, let $E \in \mathcal{F}(\mathcal{H})$ have dimension $d$ and an orthonormal basis $(e_1, \ldots, e_d)$.

Define $f$ on $\mathcal{H}$ and $\tilde{f}$ on $\mathcal{B}$ setting:

$$\forall x \in \mathcal{H}, \ f(x) = \varphi((x \cdot e_i)_{1 \leq i \leq d}), \quad \forall y \in \mathcal{B}, \ \tilde{f}(y) = \varphi((\ell_{e_i}(y))_{1 \leq i \leq d}).$$

Then, for all $p \geq 1$ and $s > 0$, $\tilde{f}$ is the stochastic extension of $f$ in $L^p(B, \mu_{B,s})$.

This result holds because, if $\varphi$ is continuous and compactly supported, the function defined in (22) for a polynomial $P$ is bounded for all $p \in [1, +\infty[$ and $s > 0$.

Proof of Theorem 16 The $d$-uple $(\ell_{e_1}, \ldots, \ell_{e_d})$ has distribution $\mathcal{N}(0, s I_d)$. Hence,

$$\int_B |\tilde{f}(x)|^p \, d\mu_{B,s}(x) = \int_{\mathbb{R}^d} |\varphi(y)|^p (2\pi s)^{-d/2} e^{-|y|^2/2s} \, dy.$$

Condition (22) on $\varphi$ ensures that the integral is finite, which shows that $\tilde{f} \in L^p(B, \mu_{B,s})$.

Recall that, in a measured space $(X, \mathcal{T}, \mu)$, if $p \in [1, \infty]$ and if $(f_n)_{n \in \mathbb{N}}$ is a sequence of $L^p(\mu)$, converging almost everywhere to $f \in L^p(\mu)$, then

$$\lim_{n \to \infty} \|f - f_n\|_p = 0 \iff \lim_{n \to \infty} \|f_n\|_p = \|f\|_p.$$  

(23)
(See, for example, [18]). Let us then take an increasing sequence of $\mathcal{F}(\mathcal{H})$, $(E_n)_{n \in \mathbb{N}}$ with union dense in $\mathcal{H}$. Formula (20) implies that, $\mu_{B,s}$-almost everywhere on $B$,

$$P_E(\tilde{\pi}_{E_n}(x)) = \sum_{i=1}^{d} \ell_{P_{E_n}(e_i)}(x)e_i,$$

where $P_E$ is the orthogonal projection on $E$, defined by (5).

According to Lemma 14, for $1 \leq i \leq d$, $\ell_{P_{E_n}(e_i)}$ converges to $\ell_{e_i}$ in $\mu_{B,s}$ probability.

Take a subsequence of $(E_n)$ indexed by $\psi(n)$. There exists a further subsequence, indexed by $\psi(\zeta(n))$, such that, for all $i \leq d$, $\ell_{e_i} - \ell_{P_{E}\psi(\zeta(n))}(e_i)$ converges $\mu_{B,s}$-almost everywhere to 0. By the definition of $f$ and by formula (20), for every $y \in B$,

$$f(\tilde{\pi}_{E\psi(\zeta(n))}(y)) = \varphi((\tilde{\pi}_{E\psi(\zeta(n))}(y) \cdot e_i)_{i \leq d}) = \varphi((\ell_{P_{E\psi(\zeta(n))}(e_i)}(y))_{i \leq d}).$$

Since $\varphi$ is continuous, we get that

$$f \circ \tilde{\pi}_{E\psi(\zeta(n))} \longrightarrow \varphi((\ell_{e_i})_{i \leq d}) = \tilde{f} \quad \mu_{B,s} \ p.s.$$

We now must check that $\|\varphi((\ell_{P_{E\psi(\zeta(n))}(e_i)})_{i \leq d})\|_{L^p(B,\mu_{B,s})}$ converges to $\|\tilde{f}\|_{L^p(B,\mu_{B,s})}$.

The $d$-uple $(\ell_{P_{E\psi(\zeta(n))}(e_i)})_{i \leq d})$ is normally distributed, with 0 means and covariance matrix $K_n$ equal to

$$K_n = (((\ell_{P_{E\psi(\zeta(n))}(e_i)}, \ell_{P_{E\psi(\zeta(n))}(e_j)})_{L^2(B,\mu_{B,s})})_{i,j \leq d} = s(P_{E\psi(\zeta(n))}(e_i) \cdot P_{E\psi(\zeta(n))}(e_j))_{i,j \leq d},$$

where $\cdot$ is the scalar product of $\mathcal{H}$. When $n$ goes to infinity, the coordinates of $K_n$ converge to $s e_i \cdot e_j$ and $K_n$ itself converges to $s I_d$. For sufficiently large $n$, $K_n$ is then invertible, hence $(\ell_{P_{E\psi(\zeta(n))}(e_i)})_{i \leq d}$ admits a density and

$$\|f \circ \tilde{\pi}_{E\psi(\zeta(n))}\|_{L^p(B,\mu_{B,s})}^p = \int_B |\varphi((\ell_{P_{E\psi(\zeta(n))}(e_i)})_{i \leq d})|^p \ d\mu_{B,s} = \int_{\mathbb{R}^d} |\varphi(y)|^p \frac{1}{(2\pi)^{d/2}\sqrt{\det(K_n)}} e^{-\frac{1}{2}<y,K_n^{-1}y> } \ dy.$$ 

We know that, for all $y \in \mathbb{R}^d$,

$$s < y, K_n^{-1} y > \geq < y, y > = |y|^2$$

with the scalar product and the Euclidean norm on $\mathbb{R}^d$. This fact will be proved in Lemma 19 below. Moreover, since $\det(K_n)$ converges to $s^d$, its inverse is bounded independently of $n$ for $n$ large enough. All this allows using the dominated convergence
Theorem, since $K_n^{-1}$ converges to $s^{-1}I_d$. One deduces that

$$\|f \circ \tilde{\pi}_{E_{\psi(n)}}\|_{L^p(B, \mu_{B,s})} \longrightarrow \int_{\mathbb{R}^d} \frac{|\varphi(y)|^p}{(2\pi s)^{d/2}} e^{-\frac{1}{2s} \langle y, y \rangle} dy = \|\tilde{f}\|_{L^p(B, \mu_{B,s})}^p.$$  

Then, for this subsequence, (23) yields that

$$\|f \circ \tilde{\pi}_{E_{\psi(n)}} - \tilde{f}\|_{L^p(B, \mu_{B,s})} \longrightarrow 0.$$

A proof by contradiction then ensures that $\|f \circ \tilde{\pi}_{E_n} - \tilde{f}\|_{L^p(B, \mu_{B,s})}$ itself converges to 0, which achieves the proof of Theorem 16. \(\square\)

We now give the result about the inverse of $K_n$.

**Lemma 19** Let $K_n = s(P_{E_n(e_i)} \cdot P_{E_n(e_j)})_{i,j \leq d}$ be the covariance matrix appearing in the preceding proof. For all $y \in \mathbb{R}^d$, 

$$s < y, K_n^{-1} y > = < y, y > = |y|^2.$$  

**Proof** For $n$ large enough, $K_n$ is invertible. According to its definition, it is definite positive and its inverse has the same property. Using their square roots gives $< y, K_n^{-1} y > = < K_n^{-1/2} y, K_n^{-1/2} y >$. Setting $x = K_n^{-1/2} y$ we get

$$\forall y \in \mathbb{R}^d, s \langle y, K_n^{-1} y \rangle \geq \langle y, y \rangle \iff \forall x \in \mathbb{R}^d, \langle x, x \rangle \geq \frac{1}{s} \langle K_n^{1/2} x, K_n^{1/2} x \rangle \iff \forall x \in \mathbb{R}^d, \langle x, x \rangle \geq \frac{1}{s} \langle x, K_n x \rangle.$$  

Since

$$\langle x, K_n x \rangle = s \sum_{i,j} x_i x_j P_{E_n}(e_i) \cdot P_{E_n}(e_j) = s \sum_i x_i P_{E_n}(e_i) \cdot \sum_j x_j P_{E_n}(e_j),$$

one obtains

$$\langle x, K_n x \rangle = s |P_{E_n}\left(\sum_i x_i e_i\right)|^2 \leq s \sum_i x_i e_i \|e_i\|^2 = s \langle x, x \rangle.$$  

This proves the inequality and achieves the proof of Lemma 19. \(\square\)

We now apply the extension results to the Wigner functions. Let $E \in \mathcal{F}(H)$. According to Definition 3, for $\hat{f}, \hat{g} \in \mathcal{S}_{E,h/2}$, $W_{h,E}(\hat{f}, \hat{g})(z, \xi)$ is given, for all $(z, \xi)$ in $E^2$, by

$$W_{h,E}(\hat{f}, \hat{g})(z, \xi) = e^{|\xi|^2/h} \int_E e^{-2i \xi \cdot t/h} \hat{f}(z + t) \hat{g}(z - t) e^{-|t|^2/h} (\pi h)^{-d/2} dt.$$
Define functions $f, g$ on $\mathcal{H}$ by $f = \hat{f} \circ P_E$, $g = \hat{g} \circ P_E$, let $\tilde{f}, \tilde{g}$ be defined on $B$ by $\tilde{f} = \hat{f} \circ \tilde{\pi}_E$, $\tilde{g} = \hat{g} \circ \tilde{\pi}_E$. Corollary 17 says that $\tilde{f}$ is the stochastic extension of $f$ in $L^2(B, \mu_{B,h/2})$.

According to Definition 3 again, the function $\tilde{\mathcal{W}}(\tilde{f}, \tilde{g})$ is given by

$$\forall (z, \zeta) \in B^2, \quad \tilde{\mathcal{W}}(\tilde{f}, \tilde{g})(z, \zeta) = \mathcal{W}_{E}(\hat{f}, \hat{g})(\tilde{\pi}_E(z), \tilde{\pi}_E(\zeta)).$$

We now set

$$\forall (z, \zeta) \in \mathcal{H}^2, \quad \mathcal{W}_{H}(f, g)(z, \zeta) = \mathcal{W}_{E}(\hat{f}, \hat{g})(P_E(z), P_E(\zeta)).$$

One sees that, to get $\tilde{\mathcal{W}}(\tilde{f}, \tilde{g})$ when one knows $\mathcal{W}_{H}(f, g)$, it suffices to replace the scalar products (in the projections $P_E$) by the corresponding functions $\ell$.

This implies the following result.

**Corollary 20** With the preceding notations, $\tilde{\mathcal{W}}(\tilde{f}, \tilde{g})$ is the stochastic extension of $\mathcal{W}_{H}(f, g)$ in $L^2(B^2, \mu_{B,h/4})$.

**Proof of Corollary 20** We express the Wigner function thanks to $\gamma_{E,h/2,\hat{f}}$ and $\gamma_{E,h/2,\hat{g}}$, which are rapidly decreasing. This yields

$$\mathcal{W}_{E}(\hat{f}, \hat{g})(z, \zeta) = e^{(|z|^2 + |\zeta|^2)/h} \int_E e^{-2i\zeta \cdot t/h} \gamma_{E,h/2,\hat{f}}(z + t) \gamma_{E,h/2,\hat{g}}(z - t) \, dt.$$  

The integral factor above is rapidly decreasing in $(z, \zeta)$ (see, for example, [6] for a recent reference). Applying Corollary 17 with $s = h/4$ for the “target” space $B^2$ proves that $\mathcal{W}_{H}(f, g)$ has a stochastic extension in $L^2(B^2, \mu_{B^2,h/4})$, which is $\tilde{\mathcal{W}}(\tilde{f}, \tilde{g})$. 

\[\square\]

### 3 Around positivity

This section extends, to the infinite dimension, positivity or nonpositivity results already known in the finite dimensional case. The main tools are the finite dimensional Hermite functions, their Wigner functions, recalled in the first part below, their stochastic extensions, studied in the preceding Sect. 2.

The point is that results in finite dimension can be transposed to cylindrical functions (resp. cylindrical symbols), officially defined on an infinite dimensional space, but in fact depending on $d$ (resp. $2d$) variables.

#### 3.1 The Wigner functions of the Hermite functions - finite dimensional case

We first fix the normalization for the Hermite functions in dimension 1. We give a Hilbert basis of $L^2(\mathbb{R}, \mu_{\mathbb{R},h/2})$, to make the transition with the choices made in the infinite dimensional case easier. The definition of the Hermite functions and some of their properties are followed by the computation of their Wigner functions. The
formulas are given explicitly, for the sake of clarity. Since they are classical (up to normalization) and may be found, with their proofs, in [13, 19], the arguments are only sketched here.

At the end of this part, a technical lemma allowing integrations by parts is proved. It will be applied in Part 3.5 about radial symbols.

One denotes by $\mu_{\mathbb{R}, h/2}$ the Gaussian measure with density $(\pi h)^{-1/2} e^{-x^2/h}$ with respect to the Lebesgue measure. One sets $\tau(x) = e^{-x^2}$. For every real number $x$, one sets

$$\psi_{-1}(x) = 0, \quad \forall j \geq 0, \quad \psi_j(x) = \frac{(-1)^j}{\sqrt{j!}} \left( \frac{h}{2} \right)^{j/2} e^{x^2/2h} \tau^{(j)}(x).$$

(25)

The classical relations then have this form:

$$\forall x \in \mathbb{R}, \quad \forall j \geq 0, \quad \psi_{j+1}(x) = \frac{-1}{\sqrt{j+1}} \sqrt{\frac{h}{2}} \left( \left( \frac{d}{dx} - \frac{2}{h} x \right) \psi_j \right)(x)$$

creation

$$\forall j \geq 1, \quad \psi_j(x) = \sqrt{\frac{2}{h}} \frac{x}{\sqrt{j}} \psi_{j-1}(x) - \sqrt{\frac{j-1}{j}} \psi_{j-2}(x)$$

recurrence

$$\forall j \geq 1, \quad \frac{d \psi_j}{dx} = \sqrt{j} \sqrt{\frac{2}{h}} \psi_{j-1}$$

annihilation

(26)

With $\psi_{-1}(x) = 0$, the first Hermite functions are, explicitly,

$$\psi_0(x) = 1 \quad \psi_1(x) = \sqrt{\frac{2}{h}} x$$

$$\psi_2(x) = \frac{1}{\sqrt{2}} \left( \frac{2}{h} x^2 - 1 \right) \psi_3(x) = \frac{1}{\sqrt{6}} \left( \left( \frac{2}{h} \right)^{3/2} x^3 - 3 \left( \frac{2}{h} \right)^{1/2} x \right)$$

(27)

The recurrence formula proves that $\psi_j$ is a polynomial with degree $j$ exactly and leading coefficient $\left( \frac{2}{h} \right)^{j/2} \frac{1}{\sqrt{j!}}$. The family $(\psi_j)_{j \geq 0}$ is an orthonormal family of $L^2(\mathbb{R}, \mu_{\mathbb{R}, h/2})$.

Computation of the Wigner functions of the Hermite functions in dimension 1

Direct computations and the definition formula (9) for $E = \mathbb{R}$ lead to the following results

$$W_{h, \mathbb{R}}(\psi_0, \psi_0)(x, \xi) = 1, \quad W_{h, \mathbb{R}}(\psi_0, \psi_1)(x, \xi) = \sqrt{\frac{2}{h}}(x + i\xi),$$

$$W_{h, \mathbb{R}}(\psi_1, \psi_1)(x, \xi) = -1 + \frac{2}{h}(x^2 + \xi^2).$$

(28)
For more general orders, one needs to introduce the Bargman kernel $K_v$ associated with a complex number $v$ and defined below: for every $v \in \mathbb{C}$ and every $x \in \mathbb{R}$, set

$$K_v(x) = \sum_{j=0}^{\infty} \psi_j(x) \frac{v^j}{\sqrt{j!}} = e^{xv\sqrt{2h}} - v^2/2.$$  \hspace{1cm} (29)

The second equality comes from Taylor’s Formula. For a fixed $v \in \mathbb{C}$, the convergence takes place in $L^2(\mathbb{R}, \mu_{\mathbb{R}, h/2})$ because the sequence of the coefficients of the $\psi_j$ is square summable. For a fixed real number $x$, the convergence is uniform in every compact subset of $\mathbb{C}$, because the convergence radius (in the variable $v$) is infinite.

One checks that the following Wigner function of two Bargman kernels is equal to

$$W_{h, \mathbb{R}}(K_u, K_\xi)(x, \xi) = \exp \left( -uv + \sqrt{\frac{2}{h}}x(u + v) + i \sqrt{\frac{2}{h}}\xi(v - u) \right).$$  \hspace{1cm} (30)

Exchanging discrete sums and integration and identifying the coefficient of the term $u^j v^k$, one proves that, for all real numbers $x, \xi$ and all integers $j, k$:

$$W_{h, \mathbb{R}}(\psi_j, \psi_k)(x, \xi) = \sum_{q = \max(0, k-j)}^{k} \frac{(-1)^{q-k}}{(k-q)!} \left( \frac{2}{h} \right)^{(j-k+2q)/2} \sqrt{j!k!(x - i\xi)^{j-k+q}(x + i\xi)^q} \frac{1}{(j-k+q)!q!}.$$  \hspace{1cm} (31)

Hence we can express $W_{h, \mathbb{R}}$ in terms of the Laguerre polynomials:

$$W_{h, \mathbb{R}}(\psi_j, \psi_k)(x, \xi) = \begin{cases} \sqrt{\frac{j!}{k!}}(x + i\xi)^{k-j}(-1)^j \left( \frac{2}{h} \right)^{(k-j)/2} L_j^{(k-j)} \left( \frac{2}{h} \left( x^2 + \xi^2 \right) \right) & \text{if } j \leq k, \\ \sqrt{\frac{j!}{k!}}(x - i\xi)^{j-k}(-1)^k \left( \frac{2}{h} \right)^{(j-k)/2} L_k^{(j-k)} \left( \frac{2}{h} \left( x^2 + \xi^2 \right) \right) & \text{if } j \geq k. \end{cases}$$  \hspace{1cm} (32)

Up to normalization, these results and their proofs can be found in [19], Theorem 1.105. The Laguerre polynomials are defined in [19, 20] by:

$$L_k^{(\alpha)}(x) = \sum_{m=0}^{k} \frac{(k + \alpha)!}{(k - m)! (\alpha + m)!} \frac{(-x)^m}{m!}.$$  \hspace{1cm} (33)

The leading coefficient of $L_k^{(\alpha)}$ in (33) is $\frac{(-1)^k}{k!}$. The Laguerre polynomials $(L_k^{(\alpha)})_k$ are orthogonal with respect to the measure $x^\alpha e^{-x} \, dx$ over $\mathbb{R}^+$, but not orthonormal. They are normalized by the choice of their leading coefficient.

The last result in dimension 1 is an integration by parts lemma, which applies to the radial functions of a section below, or more generally to functions cancelled by the differential operator $[x \partial_\xi - \xi \partial_x]$. 
Lemma 21 Let $P : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function, such that $P$ and $P'$ are at most polynomially increasing.

Let $s \in \mathbb{N}$, let $n \in \mathbb{N}^*$. Let $\varepsilon = 1$ or $-1$.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a $C^n$ function, at most polynomially increasing, along with its partial derivatives.

Then

$$(-si\varepsilon)^n \int_{\mathbb{R}^2} F(x, \xi)(x + i\varepsilon\xi)^s P(x^2 + \xi^2) e^{-\frac{x^2 + \xi^2}{\hbar}} \, dx d\xi = \int_{\mathbb{R}^2} ([x \partial_\xi - \xi \partial_x]^n F)(x, \xi)(x + i\varepsilon\xi)^s P(x^2 + \xi^2) e^{-\frac{x^2 + \xi^2}{\hbar}} \, dx d\xi.$$  \hfill (34)

Proof We start from the right hand-side.

In the case when $s \neq 0$, the properties of $F$, $P$ and the presence of the exponential function allow one to integrate by parts. The operator $[x \partial_\xi - \xi \partial_x]$ cancels radial functions. Hence, the only term that does not vanish comes from $-[x \partial_\xi - \xi \partial_x](x + i\varepsilon\xi)^s = (-i\varepsilon x)(x + i\varepsilon\xi)^s$. In particular, no derivative of $P$ remains.

If $s = 0$, the equality has the form $0 = 0$. Indeed, $s = 0$ is a factor of the left term and an integration by parts cancels the right term, since $[x \partial_\xi - \xi \partial_x]$ applies only to radial functions. \hfill \qed

We can use this lemma and the expression of $W(\psi_j, \psi_k)$ to recover the classical equalities in the finite dimensional case:

$$\forall (j, k) \in \mathbb{N}^2, \int_{\mathbb{R}^2} W(\psi_j, \psi_k)(x, \xi) \, d\mu_{\mathbb{R}^2, \hbar/2}(x, \xi) = \delta_{j, k}.$$  \hfill (35)

Let us denote $\int_{\mathbb{R}^2} W(\psi_j, \psi_k)(x, \xi) \, d\mu_{\mathbb{R}^2, \hbar/2}(x, \xi)$ by $w_{j, k}$.

If $j < k$, one may write

$$w_{j, k} = C \int_{\mathbb{R}^2} (x + i\xi)^{k-j} L_j^{(k-j)} \left( \frac{2}{\hbar} \left( x^2 + \xi^2 \right) \right) e^{-\frac{x^2 + \xi^2}{\hbar}} \, dx d\xi,$$

for a real constant $C$. Then applying the lemma with $F = 1$ and $n = 1$ proves that the integral term is zero.

If $j = k$ we obtain

$$w_{j, j} = \frac{(-1)^j}{\pi \hbar} \int_{\mathbb{R}^2} L_j^{(0)} \left( \frac{2}{\hbar} \left( x^2 + \xi^2 \right) \right) e^{-\frac{x^2 + \xi^2}{\hbar}} \, dx d\xi.$$

A polar change of variables and the expression of $L_j^{(0)}$ then give

$$w_{j, j} = (-1)^j \int_{\mathbb{R}^+} \sum_{m=0}^{j} \frac{j!}{(j-m)!(m!)^2} (-2u)^m e^{-u} \, du = 1,$$
using the fact that \( \int_0^\infty u^m e^{-u} \, du = m! \).

The interpretation is that

\[
\int_\mathbb{R}^2 W(\psi_j, \psi_k)(x, \xi) \, d\mu_{\mathbb{R}^2, h/2}(x, \xi) = \langle Op_{Weyl}^{h}(1)\psi_j, \psi_k \rangle = \langle \psi_j, \psi_k \rangle,
\]

because \( Op_{Weyl}^{h}(1) \) is the identity operator.

### 3.2 Decompositions over a Hilbert basis of \( L^2(B, \mu_{B,h/2}) \)

Let \( \mathcal{H} \) be a real, separable and infinite dimensional Hilbert space, with an orthonormal basis \((e_i)_{i \in \mathbb{N}^*}\). Let \( B \) be a Wiener extension of \( \mathcal{H} \), endowed with the measure \( \mu_{B,h/2} \).

For a multiindex \( \alpha \) we set

\[
\forall y \in B, \quad \psi^B_\alpha(y) = \prod_{j \in \mathbb{N}^*} \psi_{\alpha_j}(\ell_{e_j}(y)). \tag{36}
\]

Although it runs over \( \mathbb{N}^* \), this product is finite since the \( \alpha_j \) are all 0 but for a finite number and the factor \( \psi_{\alpha_j} \) is equal to 1 if \( \alpha_j = 0 \). The same formula defines Hermite functions of finite dimension \( d \geq 1 \). When \( \alpha \) runs over all multiindices, the family of the \( \psi^B_\alpha \) is an orthonormal family of \( L^2(B, \mu_{B,h/2}) \) (see [13], Th. 2.6 and 3.21).

We get

**Lemma 22** For all multiindices \( \alpha \) and \( \beta \),

\[
\forall (z, \xi) \in B^2, \quad W_{h,B}(\psi^B_\alpha, \psi^B_\beta)(z, \xi) = \prod_{j \in \mathbb{N}^*} W_h(\psi_{\alpha_j}, \psi_{\beta_j})(\ell_{e_j}(z), \ell_{e_j}(\xi)),
\]

with a mock infinite product once again.

**Proof** We may write \( W_{h,B}(\psi^B_\alpha, \psi^B_\beta)(z, \xi) \) as a finite dimensional integral, using the distribution of the random vector \((\ell_{e_i})_i\), the index running on all indices such that \( \alpha_i \) or \( \beta_i \) is not equal to 0. This integral splits into integrals over \( \mathbb{R}^2 \), since the distribution is normal with diagonal covariance matrix and the integrated function itself is a product.

When we use this lemma later on, we will express each Wigner function of dimension 1 in terms of the Laguerre polynomials, according to (32).

Recall that Formula (10) defines a bilinear form \( Q_{Weyl}^h(\tilde{F}) \) associated with a function \( \tilde{F} \), defined on \( B^2 \). In certain cases, it may be linked with an operator. In view of a decomposition on the orthonormal basis \( (\psi^B_\alpha)_\alpha \), we compute the quantities \( Q_{Weyl}^h(\tilde{F})(\psi^B_\alpha, \psi^B_\beta) \). For two multiindices \( \alpha, \beta \), we consequently set

\[
I_{\alpha,\beta}(\tilde{F}) := \int_{B^2} \tilde{F}(z, \xi) W_{h,B}(\psi^B_\alpha, \psi^B_\beta)(z, \xi) \, d\mu_{B^2, h/2}(z, \xi) = Q_{Weyl}^h(\tilde{F})(\psi^B_\alpha, \psi^B_\beta). \tag{37}
\]
We address the case \( \alpha \neq \beta \) just below. We treat the case when \( \alpha = \beta \), under stronger conditions, in Proposition 26.

**Proposition 23** Suppose that \( \tilde{F} \) is cylindrical, based on \( E = \text{Vect}(e_1, \ldots, e_d) \subset B' \) and has the expression

\[
\forall (z, \xi) \in B^2, \quad \tilde{F}(z, \xi) = \tilde{F}(\ell_{e_1}(z), \ldots, \ell_{e_d}(z), \ell_{e_1}(\xi), \ldots, \ell_{e_d}(\xi)),
\]

with \( \tilde{F} \) smooth \((C^\infty)\) on \( \mathbb{R}^{2d} \), increasing at most polynomially, along with all its partial derivatives.

Consider \( I_{\alpha, \beta}(\tilde{F}) \) for \( \alpha \neq \beta \).

If \( \alpha_j \neq \beta_j \) for an index \( j > d \), then \( I_{\alpha, \beta}(\tilde{F}) = 0 \).

If, for all \( j > d \), \( \alpha_j = \beta_j \), take \( j \leq d \) such that \( \alpha_j \neq \beta_j \). For any differentiation order \( n \), one may write:

\[
I_{\alpha, \beta}(\tilde{F}) = \frac{i^n}{(\beta_j - \alpha_j)^n} \int_{\mathbb{R}^{2d}} (x_j \partial_{\xi_j} - \xi_j \partial_{x_j})^n \tilde{F}(x, \xi) \prod_{l=1}^d W_{h, \mathbb{R}}(\psi_{\alpha_l}, \psi_{\beta_l})(x_l, \xi_l) \mu_{\mathbb{R}^d, h}(x, \xi).
\]

In particular, if \( (x_j \partial_{\xi_j} - \xi_j \partial_{x_j}) \tilde{F} \) vanishes for all \( j \), \( I_{\alpha, \beta}(\tilde{F}) = 0 \) if \( \alpha \neq \beta \).

The proof of Proposition 23 is a consequence of the following lemma and of Lemma 21 in dimension 1.

**Lemma 24** Under the conditions and with the notations of Proposition 23 above, if, for all \( j > d \), \( \alpha_j = \beta_j \), then the integral \( I_{\alpha, \beta}(\tilde{F}) \) satisfies:

\[
I_{\alpha, \beta}(\tilde{F}) = \int_{\mathbb{R}^{2d}} \tilde{F}(x, \xi) \prod_{j=1}^d W_{h, \mathbb{R}}(\psi_{\alpha_j}, \psi_{\beta_j})(x_j, \xi_j) e^{-\frac{1}{\hbar}(|x|^2 + |\xi|^2)} \frac{dx d\xi}{(\pi \hbar)^d}.
\]

If there exists \( j > d \) such that \( \alpha_j \neq \beta_j \), then \( I_{\alpha, \beta} = 0 \).

**Proof of the lemma** Let \( n \) be the largest index for which \( \alpha_n \) or \( \beta_n \) is not equal to 0.

Suppose \( n \leq d \). In this case, \( \alpha_j = \beta_j = 0 \) for all \( j > d \). The measure on \( B \) (and on \( B^2 \)) decomposes as a product of Gaussian measures on \( E \times E^\perp \), with \( E^\perp = \{ x \in B, \ W u \in E, \ u(x) = 0 \} \), as in [12]. This decomposition requires that \( E \) be a subset of \( B' \) and not a more general subset of \( \mathcal{H} \). Every element \( z \) of \( B \) writes uniquely as \( z = z_E + z_{\perp} \), with \( z_E = \sum_{i=1}^d e_i(z) e_i \in E \) and \( u(z_{\perp}) = \ell_u(z_{\perp}) = 0 \) for all \( u \in E \).

Then, thanks to Lemma 22 and to Fubini’s Theorem, we get, by Formula (37)

\[
I_{\alpha, \beta}(\tilde{F}) = \int_{E^2} \tilde{F}(\ell_{e_1}(z_E), \ldots, \ell_{e_d}(z_E), \ell_{e_1}(\xi_E), \ldots, \ell_{e_d}(\xi_E)) \times \prod_{j=1}^d W_{h, \mathbb{R}}(\psi_{\alpha_j}, \psi_{\beta_j})(\ell_{e_j}(z_E), \ell_{e_j}(\xi_E)) \mu_{E^2, \frac{\hbar}{2}} \int_{(E^\perp)^2} 1 d\mu_{(E^\perp)^2, \frac{\hbar}{2}}.
\]
The last integral is equal to 1, the first one is equal to
\[ \int_{\mathbb{R}^{2d}} \tilde{F}(x, \xi) \prod_{k=1}^{d} W_{h, \mathbb{R}}(\psi_{\alpha_k}, \psi_{\beta_k})(x, \xi) \, d\mu_{\mathbb{R}^{2d}, h/2}(x, \xi), \]
which gives the result.

Now suppose \( n > d \). Let \( G = \text{Vect}(e_{d+1}, \ldots, e_n) \). We decompose \( B \) in the product of \( B = E \times G \times (E \oplus G)^\perp \). The integral \( I_{\alpha, \beta}(\tilde{F}) \) is a product of three integral factors.

- The last one is an integral on \( ((E \oplus G)^\perp)^2 \) and it is equal to 1 as in the preceding case.
- The first one is an integral on \( E^2 \) and has the same shape as in the preceding case.
- The integral in the middle is an integral on \( G^2 \), in which the integrated function is only expressed thanks to Wigner functions. Indeed, \( \tilde{F} \) and \( \tilde{F} \) do not depend on the variables in \( G \). This integral is then equal to a product of factors like
  \[ \int_{\mathbb{R}^2} W(\psi_{\alpha_s}, \psi_{\beta_s})(x, \xi) \, d\mu_{\mathbb{R}} \]
  which are equal to \( \delta_{\alpha_s, \beta_s} \) by (35).

If there exists \( j > d \) such that \( \alpha_j \neq \beta_j \), the product \( \prod_{d+1}^{n} \delta_{\alpha_s, \beta_s} \) is equal to 0 since this \( j \leq n \). If not, the integral “in the middle” is equal to 1 and there just remains the first factor.

This achieves the proof of Lemma 24.

Proof of Proposition 23 When there is an index \( j > d \) for which \( \alpha_j \neq \beta_j \), the proposition is a direct consequence of Lemma 24.

Otherwise, take \( j \leq d \) such that \( \alpha_j \neq \beta_j \). The Wigner function corresponding to this index \( j \) writes
\[
W_{h, \mathbb{R}}(\psi_{\alpha_j}, \psi_{\beta_j})(x_j, \xi_j) = C_j(x_j + \varepsilon i \xi_j)^{|\alpha_j - \beta_j|}L_{\min(\alpha_j, \beta_j)}^{(2|\alpha_j - \beta_j|)} \left( \frac{2}{h} x_j^2 + \xi_j^2 \right),
\]
with \( \varepsilon = -\text{sgn}(\alpha_j - \beta_j) \), and the constant \( C_j \) given by (32). Set \( s = |\alpha_j - \beta_j| \).

Applying Lemma 21 with a differentiation order \( n \in \mathbb{N}^* \) to the integral on \( (x_j, \xi_j) \)
in the integral
\[
\int_{\mathbb{R}^{2d}} \tilde{F}(x, \xi) \prod_{k=1}^{d} W_{h, \mathbb{R}}(\psi_{\alpha_k}, \psi_{\beta_k})(x, \xi) \, d\mu_{\mathbb{R}^{2d}, h/2}(x, \xi)
\]
gives the result. This achieves the proof of Proposition 23.

3.3 Non positivity of the calculus

This short paragraph contains the proof of Proposition 9, which states a non positivity result analogous to the well-known result in the finite dimensional case: an operator with positive symbol is not necessarily positive.

The proof strongly relies on the finite dimensional situation. We choose a symbol and a test function which give the result for the phase space \( \mathbb{R}^2 \). We then build a cylindrical symbol and a cylindrical test function adapted to our purpose. The computations
are then exactly the same as in the finite dimensional case, for integrating cylindrical functions gives rise to finite dimensional integrals.

For \( a \in B' \) different from 0 and \( v > 0 \), one defines \( F \) on \( H^2 \) by

\[
\forall (x, \xi) \in H^2, \quad F(x, \xi) = e^{-v(\langle a, x \rangle^2 + \langle a, \xi \rangle^2)}.
\]

Set \( e_1 = a/|a| \) and let \((e_j)_{j \geq 1}\) be a Hilbert basis of \( H \), consisting of elements of \( B' \) and beginning with \( e_1 \). Let \( \tilde{F} \) be the function defined on \( \mathbb{R}^2 \) by \( \tilde{F}(x, y) = e^{-v|a|^2(x^2 + y^2)} \). It is rapidly decreasing, hence \( y \in \mathbb{R}^2, \tilde{F} \) is rapidly decreasing too for any variance parameter \( s > 0 \). According to Corollary 17, the function \( F \) admits, as a stochastic extension, the function \( \tilde{F} \) given by

\[
\tilde{F}(y, \eta) = e^{-v(\ell_a(y)^2 + \ell_a(\eta)^2)}, \quad (y, \eta) \in B^2.
\]

This holds for any variance \( s \).

On the other hand, according to Lemma 15, \( F \in S_m(B, \varepsilon) \) for every \( m \in \mathbb{N} \), with respect to the sequence \( \varepsilon = \left( \frac{1}{n^2} \right)_{n \geq 1} \). Taking \( m = 2 \) one may, by Theorem 7, associate, with \( F \), an operator \( OP_W^{\text{Weyl}}(F) \) which is bounded on \( L^2(B, \mu_{B,h/2}) \), for any \( h > 0 \). Moreover, since \( \ell_a \in D_{B',h/2} \), (14) shows that the operator satisfies:

\[
\langle OP_W^{\text{Weyl}}(F)\ell_a, \ell_a \rangle_{L^2(B,\mu_{B,h/2})} = Q_h^{\text{Weyl}}(\tilde{F})(\ell_a, \ell_a) = \int_{B^2} \tilde{F}(y, \eta)W_{h,B}(\ell_a, \ell_a)(y, \eta) \, d\mu_{B^2,h/2}(y, \eta).
\]

Since \( \ell_a = |a|\sqrt{\frac{h}{2}}\psi_1(\ell_{e_1}) \) by (27), formula (28) gives \( W_{h,R}(\psi_1, \psi_1) \). By Definition 3, we obtain

\[
W_{h,B}(\ell_a, \ell_a)(y, \eta) = |a|^2 \frac{h}{2} W_{h,R}(\psi_1, \psi_1)(\ell_{e_1}(y), \ell_{e_1}(\eta))
\]

\[
= |a|^2 \left( \frac{h}{2} + (\ell_{e_1}(y)^2 + (\ell_{e_1}(\eta)^2) \right).
\]

Consequently, we may write that

\[
Q_h^{\text{Weyl}}(\tilde{F})(\ell_a, \ell_a) = |a|^2 \int_{B^2} e^{-v|a|^2(\ell_{e_1}(y)^2 + (\ell_{e_1}(\eta)^2)} \left( \ell_{e_1}(y)^2 + (\ell_{e_1}(\eta)^2) - \frac{h}{2} \right) \, d\mu_{B^2,h/2}(y, \eta).
\]

The random vector \((\ell_{e_1}(y), \ell_{e_1}(\eta))\) is normally distributed with distribution \( \mathcal{N}(0, \frac{h}{2} I_2) \). Therefore

\[
Q_h^{\text{Weyl}}(\tilde{F})(\ell_a, \ell_a) = \int_{\mathbb{R}^2} e^{-v|a|^2(u^2 + v^2)} |a|^2 \left( u^2 + v^2 - \frac{h}{2} \right) e^{-(u^2 + v^2)/h} \, du \, dv / \pi h.
\]
A polar decomposition then gives
\[
\langle Op^\text{Weyl} h(F)\ell_a, \ell_a \rangle_{L^2(B,\mu_{B,h/2})} = Q^\text{Weyl} h(\tilde{F})(\ell_a, \ell_a) = \frac{h|a|^2(1 - hv|a|^2)}{2(1 + hv|a|^2)^2}.
\]

Since this expression is negative for sufficiently large \( \nu > 0 \), the operator \( Op^\text{Weyl} h(F) \) is not positive. This concludes the proof of Proposition 9. \( \square \)

### 3.4 The Flandrin conjecture for infinite dimensional Wigner functions

This conjecture, emitted by Flandrin in 1988 in the article [21] (section 5), concerns maximization and localization of a signal in time and frequency. The question, which remained open a long time, is to know whether it is true that, for any convex and bounded set \( C \) of \( \mathbb{R}^{2n} \) and any rapidly decreasing function \( u \), the Wigner function of \( (u, u) \), normalized by

\[
W(u, v)(x, \xi) = \int_{\mathbb{R}} e^{-2i\pi z \xi} u\left(x + \frac{z}{2}\right) \bar{v}\left(x - \frac{z}{2}\right) dz,
\]

satisfies

\[
\int_C W(u, u)(x, \xi) dx d\xi \leq \|u\|_{L^2(\mathbb{R}, dx)}^2.
\]

The result is true for dimension 2 disks and Euclidean balls in more general dimension. In [6] (Theorem 1.2 or p 31 of the article), the authors consider the convex set \([0, a]^2\) or \((\mathbb{R}^+)^2\) (in this case, we agree that \( a = \infty \)). They prove that, for \( a > 0 \) sufficiently large or infinite, there exists, to the contrary, a rapidly decreasing function \( u_a \) such that

\[
\int_{[0,a]^2} W(u_a, u_a)(x, \xi) dx d\xi > \|u_a\|_{L^2(\mathbb{R}, dx)}^2.
\]

The proof is complex and intricate. The function \( u_a \) is not explicit, for instance, and has no reason to be “simple”.

This beautiful result is easy to “translate” to the infinite dimensional case. As in the preceding paragraph, we derive, from the finite dimensional symbol and test function, a cylindrical symbol and a cylindrical test function.

We now turn to the proof of Proposition 10. Let \( e_1 \in B' \) such that \( |e_1|_{\mathcal{H}} = 1 \). Let \( a \) be infinite or sufficiently large for (40) to hold. Define \( \tilde{v}_a \) by

\[
\forall z \in B, \quad \tilde{v}_a(z) = v_a(\ell_{e_1}(z)), \quad \text{with} \quad \gamma_{\mathbb{R}, h/2} v_a = u_a,
\]
which means explicitly that $\forall x \in \mathbb{R},\ v_a(x) = (\pi h)^{1/4} e^{x^2/2h} u_a(x)$.

Recall that $\tilde{F}_a$ is given by

$$
\tilde{F}_a(z, \xi) = 1_{[0, a[}((\ell_{e_1}(z))\)1_{[0, 2\pi h a[}((\ell_{e_1}(\xi))
$$

in Proposition 10. We have to prove that

$$
Q^W_h(\tilde{F}_a)(\tilde{v}_a, \tilde{v}_a) = \int_{B^2} \tilde{F}_a(z, \xi) W_{B,h}(\tilde{v}_a, \tilde{v}_a)(z, \xi) \ d\mu_{B^2, h/2}(z, \xi)
$$

$$
> \|\tilde{v}_a\|^2_{L^2(B, \mu_{B,h/2})}.
$$

Remark that the use of the quadratic form (10) is licit because $\tilde{F}_a$ is a bounded Borel function.

Since $u_a \in \mathcal{S}(\mathbb{R}), \tilde{v}_a \in \mathcal{D}_{E,h/2}$ with $E = \text{Vect}(e_1)$ and $W_{h,B}(\tilde{v}_a, \tilde{v}_a)(z, \xi) = W_{h,E}(v_a, v_a)(\ell_{e_1}(z), \ell_{e_1}(\xi))$ by Definition 3. This gives

$$
Q^W_h(\tilde{F}_a)(\tilde{v}_a, \tilde{v}_a)
$$

$$
= \int_{\mathbb{R}^2} 1_{[0, a[}(\ell_{e_1}(z))1_{[0, 2\pi h a[}(\ell_{e_1}(\xi)) W_{h,\mathbb{R}}(v_a, v_a)(\ell_{e_1}(z), \ell_{e_1}(\xi)) \ d\mu_{B^2, h/2}(z, \xi)
$$

$$
= \int_{\mathbb{R}^2} 1_{[0, a[}(x)1_{[0, 2\pi h a[}(\xi) W_{h,\mathbb{R}}(v_a, v_a)(x, \xi) e^{-\frac{1}{\pi}(x^2+\xi^2)} \frac{1}{\pi h} \ dx \ d\xi
$$

because the random vector $(\ell_{e_1}(z), \ell_{e_1}(\xi))$ has the normal distribution $\mathcal{N}(0, \frac{h}{2} I_2)$.

Now, one can check that, for two functions $u$ and $v$ defined on $\mathbb{R}$ and such that $\gamma_{\mathbb{R}, h/2} u, \gamma_{\mathbb{R}, h/2} v$ are rapidly decreasing, the different Wigner functions are linked by

$$
e^{-\frac{1}{\pi}(x^2+\xi^2)} W_{h,\mathbb{R}}(u, v)(x, \xi) = \frac{1}{2} W(\gamma_{\mathbb{R}, h/2} u, \gamma_{\mathbb{R}, h/2} v)(x, \frac{\xi}{2\pi h}).
$$

Hence

$$
Q^W_h(\tilde{F}_a)(\tilde{v}_a, \tilde{v}_a)
$$

$$
= \int_{\mathbb{R}^2} 1_{[0, a[}(x)1_{[0, 2\pi h a[}(\xi) W(\gamma_{\mathbb{R}, h/2} u_a, \gamma_{\mathbb{R}, h/2} v_a)(x, \frac{\xi}{2\pi h}) \frac{1}{2\pi h} dxd\xi.
$$

It remains to set $\eta = \frac{\xi}{2\pi h}$ and to exploit the results of [6] recalled above to obtain that

$$
Q^W_h(\tilde{F}_a)(\tilde{v}_a, \tilde{v}_a) > \|u_a\|_{L^2(\mathbb{R}, dx)} = \|v_a\|^2_{L^2(\mathbb{R}, \mu_{B,h/2})} = \|\tilde{v}_a\|^2_{L^2(B, \mu_{B,h/2})}.
$$

This achieves the proof of Proposition 10.
3.5 Positivity for a symbol with radial properties

In this part, we aim at proving Proposition 25 below, which generalizes Proposition 11 stated in the introduction. This result concerns a tensor product of cylindrical radial functions and not a strictly radial function.

Let us first state the hypotheses.

**H1** The symbol $F$ is cylindrical.

Let $\tilde{F}$ be smooth on $\mathbb{R}^{2d}$, bounded as well as all its partial derivatives of any order. Let $(e_1, \ldots, e_d)$ be an orthonormal family of $\mathcal{H}$ with vectors in $B'$ and let $E = \text{Vect}(e_1, \ldots, e_d)$.

One defines $\tilde{F}$ on $B^2$ and $F$ on $\tilde{\mathcal{H}_2}$, as in (38), by

$$\forall (x, \xi) \in \tilde{\mathcal{H}_2}, \quad F(x, \xi) = \tilde{F}(e_1 \cdot x, \ldots, e_d \cdot x, e_1 \cdot \xi, \ldots, e_d \cdot \xi),$$

$$\forall (z, \zeta) \in B^2, \quad \tilde{F}(z, \zeta) = \tilde{F}(\ell e_1(z), \ldots, \ell e_d(z), \ell e_1(\zeta), \ldots, \ell e_d(\zeta)).$$

(41)

**H2** The symbol $F$ is a tensor product of radial functions.

Precisely, let $\{1, \ldots, d\}$ split into $s$ pairwise disjoint parts $D_j$ with cardinal $d_j > 0$, $1 \leq j \leq s$ (with $\sum_{j=1}^{s} d_j = d$). Set $d_0 = 0$ and suppose that

$$\{1, \ldots, d\} = \bigcup_{j=1}^{s} D_j, \quad \text{with} \quad D_j = \{d_0 + \cdots + d_{j-1} + 1, \ldots, d_0 + \cdots + d_{j-1} + d_j\}.$$  

Denote by $x_{D_j}$ the variable corresponding to the coordinates indexed by $D_j$, and adopt the same conventions for the dual variable $\xi$ and the couples $(x, \xi)$.

Suppose that

$$\tilde{F}(x, \xi) = \prod_{j=1}^{s} \tilde{F}_j(x_{D_j}, \xi_{D_j}), \quad \text{with} \quad \tilde{F}_j(x_{D_j}, \xi_{D_j}) = \Phi_j(|x_{D_j}|^2 + |\xi_{D_j}|^2),$$

where the $\Phi_j$ are smooth on $\mathbb{R}^+$, satisfy $\Phi_j' \geq 0$ and are such that the $\tilde{F}_j$ are smooth, bounded and with bounded derivatives of arbitrary order. Here, $|| \cdot ||$ denotes the Euclidean norm on $\mathbb{R}^{d_j}$.

Under these conditions, the following result holds.

**Proposition 25** Suppose that $\tilde{F}$ and $\tilde{F}$ satisfy conditions H1 and H2.

There exists an operator $\text{Op}_{h}^{Weyl}(F)$ associated with $F$ and bounded on $L^2(B, \mu_{B,h}/2)$.

Let $\tilde{f} \in L^2(B, \mu_{B,h}/2)$. Then

$$\langle \text{Op}_{h}^{Weyl}(F) \tilde{f}, \tilde{f} \rangle_{L^2(B, \mu_{B,h}/2)} \geq \prod_{j=1}^{s} \left( \frac{1}{h} \int_{0}^{\infty} \Phi_j(t)e^{-t/h} \, dt \right) \| \tilde{f} \|^2_{L^2(B, \mu_{B,h}/2)}.$$
Moreover, if $\tilde{f}$ is in $\mathcal{D}_{B',h/2}$, this expression can be written with the quadratic form:

$$\langle O_{h}^{\text{Weyl}}(F)\tilde{f}, \tilde{f} \rangle_{L^{2}(B,\mu_{B},h/2)} = \int_{B^{2}} \hat{F}(z,\zeta)W_{h,B}(\tilde{f},\tilde{f})(z,\zeta)\,d\mu_{B^{2},h/2}(z,\zeta).$$

An important argument in the proof of this proposition is the decomposition of the test function $\tilde{f}$ on the basis of the $\psi_{B}^{\alpha}$ defined in (36). Then we are brought back to integrals like

$$I_{\alpha,\beta}(\hat{F}) = \int_{B^{2}} \hat{F}(z,\zeta)W_{h,B}(\psi_{B}^{\alpha},\psi_{B}^{\beta})(z,\zeta)\,d\mu_{B^{2},h/2}(z,\zeta).$$

The study of such integrals began in Proposition 23, which implies that, in the present case, $I_{\alpha,\beta}(\hat{F}) = 0$ if $\alpha \neq \beta$. It remains to study the terms for which $\alpha = \beta$.

To that aim, we require a stronger condition, in order to apply the result established in [17]: namely, the functions $\hat{F}_{j}$ and their derivatives are supposed to be bounded (see H2).

In the article [17], the fact that the symbol is radial implies, as in Proposition 23, that the oblique terms (for $\alpha \neq \beta$) vanish. But it is crucial too for the terms for which $\alpha = \beta$, because it allows the use of a radial change of variables. Nevertheless, under the present weaker hypotheses of tensorization, it is possible to apply the result of [17] to the individual factors associated with the parts $D_{j}$. Hence the result in [17] could be slightly generalized (already in the finite dimensional frame) to tensor products of radial functions.

Proposition 25 relies on the following result, which deals with a couple of functions of the basis $(\psi_{B}^{\alpha})_{\alpha}$, in view of the said decomposition.

**Proposition 26** Under the preceding hypotheses and with the notations above, for every multiindex $\alpha$,

$$\int_{B^{2}} \hat{F}(z,\zeta)W_{h,B}(\psi_{B}^{\alpha},\psi_{B}^{\alpha})(z,\zeta)\,d\mu_{B^{2},h/2}(z,\zeta) \geq \prod_{j=1}^{s} \left( \frac{1}{\hbar} \int_{0}^{\infty} \Phi_{j}(t)e^{-t/\hbar}\,dt \right),$$

where $s$ is the number of radial factors in the tensor product.

In particular, if the $\Phi_{j}$ are nonnegative, this expression is nonnegative. But the nonnegativity of the integral is sufficient.

If $\alpha \neq \beta$, then

$$\int_{B^{2}} \hat{F}(z,\zeta)W_{h,B}(\psi_{B}^{\alpha},\psi_{B}^{\beta})(z,\zeta)\,d\mu_{B^{2},h/2}(z,\zeta) = 0.$$

**Proof of Proposition 26** Using Lemma 24 and the fact that the Wigner function is itself a tensor product, one gets

$$I_{\alpha,\alpha}(\hat{F}) = \prod_{j=1}^{s} \int_{\mathbb{R}^{2d_{j}}} \hat{F}_{j}(x_{D_{j}},\xi_{D_{j}}) \prod_{l \in D_{j}} W_{h,l}(\psi_{\alpha_{l}},\psi_{\alpha_{l}})(x_{l},\xi_{l}) e^{-\frac{1}{\hbar}(|x_{D_{j}}|^{2}+|\xi_{D_{j}}|^{2})} \frac{dx_{D_{j}}d\xi_{D_{j}}}{(\pi\hbar)^{d_{j}}}. $$
We then apply Theorem 1.1 of [17] to every integral factor of the product. The relationship between the Wigner function of the present paper and the one of [17] given by

\[ W_{h,\mathbb{R}^d}(u, v)(x, \xi) = \frac{1}{2d} e^{\frac{1}{h}(|x|^2 + |\xi|^2)} H_{h}(\gamma_{\mathbb{R}^d,h/2}u, \gamma_{\mathbb{R}^d,h/2}v)(x, \xi). \]

Therefore

\[
\int_{\mathbb{R}^{2d_j}} \tilde{F}_j(x_{D_j}, \xi_{D_j}) \left( \prod_{l \in D_j} W_{h,\mathbb{R}}(\psi_{\alpha_l}, \psi_{\alpha_l})(x_l, \xi_l) \right) e^{-\frac{1}{h}(|x_{D_j}|^2 + |\xi_{D_j}|^2)} \frac{dx_{D_j}d\xi_{D_j}}{(\pi h)^{d_j}}
\]

\[ = \int_{\mathbb{R}^{2d_j}} \tilde{F}_j(x_{D_j}, \xi_{D_j}) H_{h}(f_j, f_j)(x_{D_j}, \xi_{D_j}) \frac{dx_{D_j}d\xi_{D_j}}{(2\pi h)^{d_j}}, \]

where the function \( f_j \) is given by \( f_j = \gamma_{\mathbb{R}^d,h/2}(\prod_{l \in D_j} \psi_{\alpha_l}). \)

Theorem 1.1 of [17] then states that

\[
\int_{\mathbb{R}^{2d_j}} \tilde{F}_j(x_{D_j}, \xi_{D_j}) H_{h}(f_j, f_j)(x_{D_j}, \xi_{D_j}) \frac{dx_{D_j}d\xi_{D_j}}{(2\pi h)^{d_j}} \geq \frac{1}{h} \int_0^{\infty} \Phi_j(t)e^{-t/h} \ dt \| f_j \|^2_{L^2(\mathbb{R}^{d_j}, dx)}.
\]

Note that the norm of \( f_j \) is the norm in a \( L^2 \) space for the Lebesgue measure.

Now, \( \gamma_{\mathbb{R}^d,h/2} \) is an isometric isomorphism between \( L^2(\mathbb{R}^d, \mu_{\mathbb{R}^d,h/2}) \) and \( L^2(\mathbb{R}^d, dy) \) (for the Lebesgue measure), hence \( f_j \) has the same norm as \( \prod_{l \in D_j} \psi_{\alpha_l} \) in \( L^2(\mathbb{R}^d, \mu_{\mathbb{R}^d,h/2}) \), which is 1.

This gives the first result of the proposition.

If the multiindices are different, one applies Proposition 23, which proves that the expression is equal to 0.

This concludes the proof of Proposition 26.

\[ \square \]

**Proof of Proposition 25** Recall that, for all \( (x, \xi) \in \mathcal{H}^2 \), \( F(x, \xi) = \tilde{F}(e_1 \cdot x, \ldots, e_d \cdot x, e_1 \cdot \xi, \ldots, e_d \cdot \xi) \).

Since the function \( \tilde{F} \) is smooth, bounded and with bounded derivatives, the function \( F \) above is in \( S_m(B, \varepsilon) \) for arbitrary large \( m \geq 2 \), with \( \varepsilon = \left( \frac{1}{n^2} \right)_{n \geq 1} \) by Lemma 15.

Its stochastic extension in \( L^p(B^2, \mu_{B,s}) \) is \( \tilde{F} \), defined by (41), for any \( p \in [1, \infty[ \) and any \( s > 0 \).

According to Theorem 7, there exists an operator \( OP_{h}^{Weyl}(F) \) which is bounded on \( L^2(\mathbb{R}^d, \mu_{B,h/2}) \). It satisfies, for all couple \( (\tilde{f}, \tilde{g}) \) of \( \mathcal{D}_B',h/2 \)

\[
\langle OP_{h}^{Weyl}(F) \tilde{f}, \tilde{g} \rangle_{L^2(B,\mu_{B,h/2})} = \int_{B^2} \tilde{F}(z, \xi) W_{h,B}(\tilde{f}, \tilde{g})(z, \xi) \ d\mu_{B,h/2}. \]
Let \( \tilde{f} \in L^2(B, \mu_{B,h/2}) \) and decompose \( \tilde{f} \) on the basis of the \( \psi^B_\alpha \). Set \( f_\alpha = \langle \tilde{f}, \psi^B_\alpha \rangle_{L^2(B,\mu_{B,h/2})} \). Owing to the continuity of the operator, we obtain

\[
\langle Op_{Weyl}^h (F) \tilde{f}, \tilde{f} \rangle_{L^2(B,\mu_{B,h/2})} = \sum_{\alpha,\beta} \langle Op_{Weyl}^h (F) \psi^B_\alpha, \psi^B_\beta \rangle_{L^2(B,\mu_{B,h/2})} f_\alpha \overline{f_\beta}.
\]

The functions \( \psi^B_\alpha \) are in \( D_{B,h/2} \), hence the factors \( \langle Op_{Weyl}^h (F) \psi^B_\alpha, \psi^B_\beta \rangle_{L^2(B,\mu_{B,h/2})} \) can be expressed with the Wigner function. For \( \alpha \neq \beta \) we saw that the factor is 0 (Proposition 23 and 25). It remains

\[
\langle Op_{Weyl}^h (F) \tilde{f}, \tilde{f} \rangle_{L^2(B,\mu_{B,h/2})} = \sum_{\alpha} f_\alpha \overline{f_\alpha} \times \int_{B^2} \tilde{F}(z, \zeta) W_{h,B}(\psi^B_\alpha, \psi^B_\alpha)(z, \zeta) \, d\mu_{B^2,h/2}(z, \zeta).
\]

Now, all the integrals for \( \alpha = \beta \) are greater than \( \prod_{j=1}^{s} \left( \frac{1}{h} \int_{0}^{\infty} \Phi_j(t)e^{-t/h} \, dt \right) \), by Proposition 26. We then get

\[
\langle Op_{Weyl}^h (F) \tilde{f}, \tilde{f} \rangle_{L^2(B,\mu_{B,h/2})} \geq \prod_{j=1}^{s} \left( \frac{1}{h} \int_{0}^{\infty} \Phi_j(t)e^{-t/h} \, dt \right) \sum_{\alpha} |f_\alpha|^2 \geq \sum_{j=1}^{s} \left( \frac{1}{h} \int_{0}^{\infty} \Phi_j(t)e^{-t/h} \, dt \right) \| \tilde{f} \|_{L^2(B,\mu_{B,h/2})}^2.
\]

This concludes the proof of Proposition 25. \( \square \)

### 4 Gårding’s inequality

We must recall the definitions and results which have not taken place in the introduction. In particular, we define the partial heat operators which act on functions defined on the Wiener space as in [12, 22] and the hybrid quadratic forms associated with a symbol \( F \) in \( S_m(B, \varepsilon) \) and a subspace \( E \) of \( \mathcal{F}(B') \). These notions allow us to give the decomposition (48) of the symbol \( F \). The decomposition was a step in the construction of the operator \( Op_{Weyl}^h (F) \) in [1]. Here it is useful to isolate a term which is nonnegative.

#### 4.1 Heat operators, hybrid and Anti-Wick quadratic forms

The following notions and results are taken from [1] (Sects. 2 and 3).

Let \( E \in \mathcal{F}(B') \) and denote by \( E^\perp \) the following subspace of \( B \):

\[
E^\perp = \{ x \in B, \forall u \in E, u(x) = 0 \}.
\]
According to [12], \((E^\perp \cap \mathcal{H}, E^\perp)\) is a Wiener space and, for all \(h > 0\), \(\mu_{B,h} = \mu_{E,h} \otimes \mu_{E^\perp,h}\). The same decomposition is valid for \(B^2\) and already appears in the proof of Lemma 24. We denote by \(X_E, X_{E^\perp}\) the variables of \(E^2\) and \((E^\perp)^2\) which appear below, in the integrals.

For \(E \in \mathcal{F}(B')\) one defines, for \(t > 0\), the following partial heat operators. For every Borel, bounded function \(\tilde{F}\) defined on \(B^2\), we set

\[
\forall X \in B^2, \quad H_{E,t}^+(\tilde{F})(X) = \int_{E^2} \tilde{F}(X + Y_E) \, d\mu_{E^2,t}(Y_E) \tag{42}
\]

\[
\forall X \in B^2, \quad H_{E,t}^{-}(\tilde{F})(X) = \int_{E^\perp t} \tilde{F}(X + Y_{E^\perp}) \, d\mu_{E^\perp,t}(Y_{E^\perp}).
\]

The functions obtained are Borel functions, bounded on \(B^2\).

Let \(E_1 \subset E_2 \in \mathcal{F}(B')\) and let \(S \subset E_2\) be the orthogonal complement of \(E_1\) in \(E_2\). The heat operators satisfy

\[
\overline{H_{E_1,t}} = \overline{H_{E,t}^{-}} \cdot \overline{H_{S,t}}. \tag{43}
\]

For a given, bounded, Borel function \(\tilde{F}\) defined on \(B^2\) and a subset \(E \in \mathcal{F}(B')\), let us define, as in [1] (Definition 2.2), a hybrid quadratic form \(Q_{h}^{hyb,E}(\tilde{F})\) on \(\mathcal{D}_{B',h/2}^2\), setting, for \(\tilde{f}, \tilde{g}\) in \(\mathcal{D}_{B',h/2}^2\):

\[
Q_{h}^{hyb,E}(\tilde{F})(\tilde{f}, \tilde{g}) = Q_{h}^{Weyl}(H_{E^\perp,h/2}^+(\tilde{F}))(\tilde{f}, \tilde{g}). \tag{44}
\]

Recall that \(Q_{h}^{Weyl}\) is defined by (10).

If \(E_1 \subset E_2 \in \mathcal{F}(B')\) and if \(S \subset E_2\) is the orthogonal complement of \(E_1\) in \(E_2\), the hybrid forms satisfy the relationship

\[
Q_{h}^{hyb,E_1}(\tilde{F}) = Q_{h}^{hyb,E_2}(H_{S,h/2}^+(\tilde{F})). \tag{45}
\]

In the particular case when \(E = \{0\}, E^\perp = B\) and the heat operator is the classical heat operator (classical in the theory of Wiener spaces: [22]). It allows one to define a quadratic form \(Q_{h}^{AW}(\tilde{F})\) on \(\mathcal{D}_{B',h/2}^2\). This is done in Formula (23) in [1]:

\[
\forall (\tilde{f}, \tilde{g}) \in \mathcal{D}_{B',h/2}^2, \quad Q_{h}^{AW}(\tilde{F})(\tilde{f}, \tilde{g}) = Q_{h}^{Weyl}(H_{B,h/2}^+(\tilde{F}))(\tilde{f}, \tilde{g}). \tag{46}
\]

In the finite dimensional case, the quadratic form defined this way is linked with the Anti-Wick operator with symbol \(F\), hence the denomination of the form introduced above. According to Corollary 4.12 of [1], if \(\tilde{F}\) is nonnegative, so is \(Q_{h}^{AW}(\tilde{F})\), exactly as for the finite dimensional Anti-Wick calculus:

\[
\forall \tilde{f} \in \mathcal{D}_{B',h/2}^2, \quad Q_{h}^{AW}(\tilde{F})(\tilde{f}, \tilde{f}) \geq 0.
\]
It remains to introduce operators which give a decomposition of identity. For \( j \geq 1 \) we set \( D_j = \text{Vect}((e_j, 0), (0, e_j)) \in \mathcal{F}(B'^2) \). Then, for any finite subset \( J \) of \( \mathbb{N}^* \), we set:

\[
\widetilde{T}_{J,h} = \prod_{j \in J} (I - \widetilde{H}_{D_j,h/2}), \quad \widetilde{S}_{J,h} = \prod_{j \in J} \widetilde{H}_{D_j,h/2},
\]

agreement that, for \( J = \emptyset \), the operator is \( I \).

### 4.2 Proof of Gårding’s inequality

This proof follows, on the one hand the arguments of [1] which, in this infinite dimensional frame, define \( O_{h}^{\text{Weyl}}(F) \) and, on the other hand, the method used in [23] to establish a Gårding’s inequality, in which the constant does not depend on the dimension. We highlight a term, corresponding to the Anti-Wick quadratic form, which thus has positivity properties.

Let \( \Lambda \) be a finite subset of \( \mathbb{N}^* \). Set \( E(\Lambda) = \text{Vect}(e_j, j \in \Lambda) \). One checks that

\[
\tilde{F} = \sum_{J \subset \Lambda} \tilde{T}_{J,h} \tilde{S}_{\Lambda \setminus J,h} \tilde{F}.
\]

Hence

\[
Q_{h}^{\text{hyb}, E(\Lambda)}(\tilde{F}) = \sum_{J \subset \Lambda} Q_{h}^{\text{hyb}, E(\Lambda)}(\tilde{T}_{J,h} \tilde{S}_{\Lambda \setminus J,h} \tilde{F}).
\]

One isolates the term corresponding to the empty set \( J = \emptyset \), for which \( \tilde{T}_{\emptyset,h} = I \). This term is

\[
Q_{h}^{\text{hyb}, E(\Lambda)}(\tilde{S}_{\Lambda,h} \tilde{F}) = Q_{h}^{\text{hyb}, E(\Lambda)}(H_{E(\Lambda),h/2} \tilde{F})
\]

\[
= Q_{h}^{\text{hyb},[0]}(\tilde{F}) \quad \text{by (45) with } E_2 = S = E(\Lambda)
\]

\[
= Q_{h}^{\text{Weyl}}(H_{B,h/2} \tilde{F}) \quad \text{by (44)}
\]

\[
= Q_{h}^{\text{AW}}(\tilde{F}) \quad \text{by (46)}.
\]

Let \( \tilde{f} \in \mathcal{D}_{B',h/2} \). Since \( Q_{h}^{\text{AW}}(\tilde{F})(\tilde{f}, \tilde{f}) \geq 0 \),

\[
Q_{h}^{\text{hyb}, E(\Lambda)}(\tilde{F})(\tilde{f}, \tilde{f}) \geq \sum_{J \subset \Lambda, J \neq \emptyset} Q_{h}^{\text{hyb}, E(\Lambda)}(\tilde{T}_{J,h} \tilde{S}_{\Lambda \setminus J,h} \tilde{F})(\tilde{f}, \tilde{f}) .
\]
Now,
\[ Q_h^{hyb,E(\Lambda)}(\overline{T_{J,h}S_{\Lambda \setminus J,h}F})(\tilde{f}, \tilde{f}) = Q_h^{hyb,E(J)}(\overline{T_{J,h}F})(\tilde{f}, \tilde{f}) \]
according to (45) with \( E_2 = E(\Lambda), S = E(\Lambda \setminus J) \) and \( E_1 = E(J) \).

Therefore, we obtain
\[ Q_h^{hyb,E(\Lambda)}(\tilde{F})(\tilde{f}, \tilde{f}) \geq \sum_{J \subset \Lambda, J \neq \emptyset} Q_h^{hyb,E(J)}(\overline{T_{J,h}F})(\tilde{f}, \tilde{f}) \]
\[ \geq - \sum_{J \subset \Lambda, J \neq \emptyset} |Q_h^{hyb,E(J)}(\overline{T_{J,h}F})(\tilde{f}, \tilde{f})| \]
\[ \geq -M \sum_{J \subset \Lambda, J \neq \emptyset} (81\pi h S_\varepsilon)^{|J|} \prod_{j \in J} \varepsilon_j^2 \| \tilde{f} \|_{L^2(B, \mu_{B,h/2})}^2 \]
using Formula (35) in Proposition 3.1 of [1], where \( M = \| F \|_{S_2(B, \varepsilon)} \) and \( S_\varepsilon \) is defined in Proposition 12. Set \( \lambda_j = 81\pi h S_\varepsilon \varepsilon_j^2 \). The sequence \((\lambda_n)_{n \geq 1}\) is summable, since \( \varepsilon \) is square summable. One may check that
\[ \sum_{J \subset \Lambda, J \neq \emptyset} \prod_{j \in J} \lambda_j \leq \sum_{j \in \Lambda} \lambda_j \prod_{s \in \Lambda} (1 + \lambda_s), \]
(this is formula (20) of [23]). Hence, for every finite subset \( \Lambda \) of \( \mathbb{N}^* \), under the conditions of Proposition 12, one can write, for \( \tilde{f} \in D_{B', h/2} \)
\[ Q_h^{hyb,E(\Lambda)}(\tilde{F})(\tilde{f}, \tilde{f}) \geq -\| F \|_{S_2(B, \varepsilon)} \sum_{j \in \Lambda} \lambda_j \prod_{s \in \Lambda} (1 + \lambda_s) \| \tilde{f} \|_{L^2(B, \mu_{B,h/2})}^2. \]  

(49)

Let now \((\Lambda_n)_{n \in \mathbb{N}^*}\) be an increasing sequence of finite subsets of \( \mathbb{N}^* \), with union equal to \( \mathbb{N}^* \). With every \( n \), Proposition 3.2 of [1] associates an operator \( O_{P_h^{hyb,E(\Lambda_n)}}(\tilde{F}) \), bounded in \( L^2(B, \mu_{B,h/2}) \) and satisfying
\[ \forall (\tilde{f}, \tilde{g}) \in D_{B', h/2}^2, \quad Q_h^{hyb,E(\Lambda_n)}(\tilde{F})(\tilde{f}, \tilde{g}) = \langle O_{P_h^{hyb,E(\Lambda_n)}}(\tilde{F}) \tilde{f}, \tilde{g} \rangle_{L^2(B, \mu_{B,h/2})}. \]
Moreover, the sequence of these operators is a Cauchy sequence in the space \( \mathcal{L}(L^2(B, \mu_{B,h/2})) \) and converges to the operator \( O_{P_h^{Weyl}}(F) \). Originally, it is what defines \( O_{P_h^{Weyl}}(F) \) in [1]. Using (49) with \( \Lambda = \Lambda_n \), we get
\[ O_{P_h^{hyb,E(\Lambda_n)}}(\tilde{F})(\tilde{f}, \tilde{f}) \geq -\| F \|_{S_2(B, \varepsilon)} \sum_{j \in \Lambda_n} \lambda_j \prod_{s \in \Lambda_n} (1 + \lambda_s) \| \tilde{f} \|_{L^2(B, \mu_{B,h/2})}^2. \]
Letting \( n \) go to infinity yields

\[
\text{Op}_h^{\text{Weyl}}(F)(\tilde{f}, \tilde{f}) \geq -\|F\|_{\mathcal{S}_2(B_\varepsilon)} \sum_{j \geq 0} \lambda_j \prod_{s \geq 0} (1 + \lambda_s) \|\tilde{f}\|^2_{L^2(B, \mu_{B, h/2})},
\]

since the sequence \( (\lambda_n)_{n \geq 1} \) is summable.

By density of \( \mathcal{D}_{B', h/2} \) in \( L^2(B, \mu_{B, h/2}) \) and because of the continuity of \( \text{Op}_h^{\text{Weyl}}(F) \), this inequality still holds for an arbitrary \( \tilde{f} \in L^2(B, \mu_{B, h/2}) \). This concludes the proof of Proposition 12. \( \square \)

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**Declarations**

**Conflict of interest**  The author declares no conflict of interest.

**References**

1. Amour, L., Jager, L., Nourrigat, J.: On bounded Weyl pseudodifferential operators in Wiener spaces. J. Funct. Anal. **269**, 2747–2812 (2015)
2. Amour, L., Jager, L., Nourrigat, J.: Infinite dimensional semiclassical analysis and applications to a model in nuclear magnetic resonance. J. Math. Phys. **60**(7), 071503 (2019). https://doi.org/10.1063/1.5094396
3. Amour, L., Lascar, R., Nourrigat, J.: Beals characterization of pseudodifferential operators in Wiener spaces. Appl. Math. Res. Express **2017**(1), 242–270 (2017)
4. Jager, L.: Stochastic extensions in Wiener space and heat operator. Annales mathématiques Blaise Pascal **28**, 157–198 (2021)
5. Amour, L., Jager, L., Nourrigat, J.: Composition of states and observables in Fock space. Reviews. Math. Phys. **32**(5), 20500129 (2020). https://doi.org/10.1142/S0129055X20500129
6. Delourme, B., Duyckaerts, T., Lerner, N.: On integrals over a convex set of the Wigner distribution. J. Fourier Anal. Appl. **26**(1), 6 (2020)
7. Kuo, H.-H.: Gaussian Measures in Banach Spaces. Springer, Berlin (1975)
8. Gross, L.: Measurable functions on Hilbert space. Trans. Amer. Math. Soc. **105**, 372–390 (1962)
9. Gross, L.: Abstract Wiener spaces. Proceedings of 5th Berkeley Symposium on Mathematical Statistics and Probability, 2, 31–42 (1965)
10. Gross, L.: Potential theory on Hilbert space. J. Funct. Anal. **1**, 123–181 (1967)
11. Gross, L.: Abstract Wiener measure and infinite dimensional potential theory. In: Springer (ed.) Lectures in Modern Analysis and Applications, II., vol. Lecture Notes in Mathematics 140, pp. 161–174. Springer, Berlin (1970)
12. Ramer, R.: On nonlinear transformations of Gaussian measures. J. Funct. Anal. **15**, 166–187 (1974)
13. Janson, S.: Gaussian Hilbert Spaces. Cambridge University Press, Cambridge Tracts in Math, Cambridge (1997)
14. Jager, L.: Pseudodifferential operators in infinite dimensional spaces?: a survey of recent results. Rev. Roum. Math. Pures Appl. **64**(2–3), 251–282 (2019)
15. Amour, L., Lascar, R., Nourrigat, J.: Weyl calculus in QED 1, the unitary group. J. Math. Phys. **58**(1), 242–270 (2017)
16. Lerner: Integrating the Wigner distribution on subsets of the phase space, a survey. Preprint at arxiv:2102.08090 (2022)
17. Amour, L., Jager, L., Nourrigat, J.: Lower bounds for pseudodifferential operators with a radial symbol. J. Math. Pures Appl. **103**, 1157–1162 (2015)
18. Briane, M., Pagès, G.: Théor. L’intégr. Vuibert, Paris (1998)
19. Folland, G.B.: Harmonic Analysis in Phase Space. Annals of Mathematics Studies. Princeton University Press, Princeton (1989)
20. Magnus, W., Oberhettinger, F., Soni, R.P.: Formulas and Theorems for the Special Functions of Mathematical Physics. Springer, New York (1966)
21. Flandrin, P.: Maximal signal energy concentration in a time-frequency domain. Proc IEEE Int. Conf. Acoust 4(1), 2176–2179 (1988)
22. Hall, B.C.: The heat operator in infinite dimensions. In: Sengupta, A.N., Sundar, P. (eds.) Infinite Dimensional Stochastic Analysis, pp. 161–174. World Scientific, New Jersey (2008)
23. Lascar, R., Nourrigat, J.: Gårding inequality in large dimension. Israel J. Math. 200, 79–84 (2014). https://doi.org/10.1007/s11856-014-0008-4

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