FAILURE OF THE BRAUER–MANIN PRINCIPLE FOR A SIMPLY CONNECTED FOURFOLD OVER A GLOBAL FUNCTION FIELD, VIA ORBIFOLD MORDELL

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ABSTRACT. Almost one decade ago, Poonen constructed the first examples of algebraic varieties over global fields for which Skorobogatov’s étale Brauer–Manin obstruction does not explain the failure of the Hasse principle. By now, several constructions are known, but they all share common geometric features such as large fundamental groups.

In this paper, we construct simply connected fourfolds over global fields of positive characteristic for which the Brauer–Manin machinery fails. Contrary to earlier work in this direction, our construction does not rely on major conjectures. Instead, we establish a new diophantine result of independent interest: a Mordell-type theorem for Campana’s “geometric orbifolds” over function fields of positive characteristic. Along the way, we also construct the first example of simply connected surface of general type over a global field with a non-empty, but non-Zariski dense set of rational points.

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1. INTRODUCTION

1.1. Insufficiency of the Brauer–Manin obstruction. Let $X$ be a smooth, projective variety over a global field $K$ with adèle ring $\mathbb{A}_K$. To decide whether the variety $X$ has a $K$-rational point, the Brauer–Manin obstruction and its refinements are often very useful tools. A conjecture of Colliot-Thélène asserts that if $X$ is rationally connected and if $K$ is a number field, then the Brauer–Manin obstruction to the existence of a rational point is the only one. In other words: if the Brauer–Manin set $X(\mathbb{A}_K)^{Br}$ is non-empty, then $X(K)$ is non-empty. This conjecture is now known in many special cases, including many geometrically rational surfaces and many homogeneous spaces of linear algebraic groups. On the other side of the geometric spectrum, we know very little. For example, we do not have any examples of smooth hypersurfaces of general type in $\mathbb{P}^n_Q$, where $n \geq 4$, for which the Hasse principle fails, even though one should expect such examples to abound. An interesting line of recent research has tackled the problem of constructing varieties for which one can prove that the Brauer–Manin machinery fails.

1.2. Main result. This paper addresses the challenge to construct the first example of a simply connected variety over a global field for which the Brauer–Manin obstruction does not explain the failure of the Hasse principle. The following is our main result.

**Theorem 1.1.** For every sufficiently large prime $p$, there exist a global function field $K$ of characteristic $p$ and a smooth, projective, geometrically integral fourfold $Z$ over $K$ such that $\pi_1^\text{et}(Z_{\overline{\mathbb{Q}}}) = 0$, $Z(\mathbb{A}_K)^{Br} \neq \emptyset$ and $Z(K) = \emptyset$.

**Remark 1.2.** Theorem 1.1 is no abstract existence result. The construction carried out in Section 12 is quite explicit.

**Remark 1.3.** To the best of our knowledge, Theorem 1.1 provides the first example of a simply connected variety over a global field where Brauer–Manin fails. Over number fields, simply connected examples have been constructed conditionally.
(assuming the Bombieri–Lang conjectures) by Sarnak–Wang [SW95] and Poonen [Poo01], and conditionally (assuming the abc conjecture) by Smeets [Sme17, §4].

1.2.1. Earlier results. The first example of a smooth, projective variety over a number field for which the Brauer–Manin obstruction does not explain the failure of the Hasse principle was a bi-elliptic surface constructed by Skorobogatov, [Sko99, §2]. For this surface, however, the failure of the Hasse principle could be explained by the (finer) étale Brauer–Manin obstruction [Sko99, §3]. Next, Poonen [Poo10a] found the first examples of varieties for which this finer obstruction fails as well. His examples are threefolds fibred over a curve of genus at least one. Soon after, Harpaz–Skorobogatov [HS14] constructed surfaces with this property, Colliot-Thélène–Pál–Skorobogatov [CTPS16] found examples of quadric bundles, and Smeets [Sme17, §3] came up with the first examples with trivial Albanese variety.

1.3. A simply connected surface with a non-empty, but non-Zariski dense set of rational points. The proof of Theorem 1.1 builds on an idea that goes back to the work of Poonen [Poo10a]: to construct a simply connected example over a field $K$, it suffices to construct a simply connected $K$-surface $S$, equipped with a fibration $f : S \to \mathbb{P}^1_K$, such that only finitely many (but more than zero) fibres of $f$ have $K$-rational points. The following theorem claims the existence of $S$ abstractly. The construction given in Section 11 is however really quite explicit.

**Theorem 1.4.** For every sufficiently large prime $p$, there exist a global function field $K$ of characteristic $p$ and a smooth, projective, geometrically integral and geometrically simply connected $K$-surface $Y$ of general type, equipped with a dominant morphism $\pi : Y \to \mathbb{P}^1_K$, such that $\pi(Y(K))$ is finite and non-empty.

To reach this goal, we adapt a cunning strategy devised by Campana in [Cam05]. He constructed simply connected surfaces of general type over $\mathbb{Q}$ fibred over $\mathbb{P}^1_\mathbb{Q}$ with a so-called “orbifold base of general type”, and argued that his “orbifold Mordell conjecture” would imply non-Zariski density of the set of rational points on the surface.

We make his work unconditional in positive characteristic. To achieve this, we build on a construction of Stoppino [Sto11] which is simpler than the one used by Campana in [Cam05, §5], and more easily transportable to positive characteristic. We combine this construction with a new diophantine ingredient: a version the Mordell conjecture for Campana’s “geometric orbifolds” over global function fields. The orbifolds considered by Campana are not stacks, but simply pairs consisting of a smooth variety and a certain type of $\mathbb{Q}$-divisor. We call these $C$-pairs, see Section 1.4 and Definition 2.7 for more details.

**Remark 1.5.** It is a major open problem to construct such an $X$ over $\mathbb{Q}$ for which $X(\mathbb{Q})$ is both non-empty and not Zariski dense, see [Poo10b, Rem. 1.4]. To the best of our knowledge, Theorem 1.4 yields the very first example of a simply connected surface over a global field $K$ with $X(K) \neq \emptyset$ for which one can verify the non-Zariski density of the set rational points unconditionally, in the direction of the
Bombieri–Lang conjecture. Our methods are, however, restricted to global fields of positive characteristic. As will be clear from our proof of Theorem 1.4, the very same statement holds with $K$ replaced by a function field of characteristic zero, what was already known to Campana. After the present work was made public, Carlo Gasbarri explained to us in private communication how to construct explicit examples of smooth projective varieties of arbitrary positive dimension, defined over a function field of characteristic zero, which are simply connected and whose sets of rational points are not Zariski dense.

**Remark 1.6.** In fact, the construction used in Section 11 to prove Theorem 1.4 immediately yields a slightly stronger statement: for the field $K$ and surface $Y$ constructed in Section 11, we know that the set $\pi(Y(L))$ remains finite for any finite, separable field extension $L/K$.

1.4. **An orbifold version of the Mordell conjecture.** Let $k$ be an algebraically closed field, and let $K$ be the function field of a $k$-curve. The classical Mordell conjecture over function fields, proven by Grauert and Manin in characteristic zero and by Samuel in positive characteristic, states that if $X$ is a smooth, projective curve of genus $g(X) \geq 2$ over $K$, then $X$ has finitely many $K$-rational points, provided that it is non-isotrivial. In the orbifold setting, the curve $X$ can be of arbitrary genus, but comes equipped with $Q$-divisor $D$, with coefficients of the form $1 - \frac{1}{m}$. We refer the reader to Section 2.3 for a precise definition of the "$C$-pair" $(X, D)$. Given a sufficiently nice integral model of $(X, D)$, one can consider the set of $C$-integral points. One should think about these as interpolating between two classical notions: $K$-rational points on $X$ on the one hand, and integral points on a model of $(X, \lceil D \rceil)$ on the other hand. Campana proposed a version of Mordell’s conjecture for $C$-pairs over global fields of characteristic zero. We prove such a statement unconditionally over global fields of positive characteristic, where the formulation becomes slightly more involved. Stated in rather loose terms, our orbifold Mordell-type theorem says the following.

**Theorem 1.7** (Mordell-type theorem for $C$-integral points (= Theorem 3.12)). Let $K$ be the function field of a curve defined over an algebraically closed field $k$. Let $(X, D)$ be a one–dimensional $C$-pair of general type over $K$, with non-vanishing Kodaira–Spencer class. Then the set of $C$-integral points on any integral model of $(X, D)$ is finite.

**Remark 1.8.** The assumption that $(X, D)$ is of general type simply means that the degree of $K_X + D$ is positive; this is an analogue of the assumption "$g(X) \geq 2$" in the classical Mordell-type theorem. The analogue of the condition on isotriviality is slightly more subtle. One could declare a $C$-pair $(X, D)$ to be "non-isotrivial" if the associated "logarithmic" pair $(X, \lceil D \rceil)$ is non-isotrivial. Unlike in the classical setting, this does however not suffice to guarantee finiteness of the set of $C$-integral points, see Section 3.4.1. We impose the stronger condition that the Kodaira–Spencer class associated to $(X, \lceil D \rceil)$ does not vanish, and it turns out that this condition does in fact suffice to guarantee finiteness of the set of $C$-integral points on any integral model of the $C$-pair $(X, D)$.
Remark 1.9. Theorem 1.7 can be thought as a generalisation of the function field versions of the classical theorems of Mordell and Siegel, valid in arbitrary characteristic. We want to stress that the case where $k = \mathbb{C}$ has already been treated by Campana [Cam05, §3]; we give an alternative treatment, but the real novelties lie in positive characteristic. Campana also conjectured such a statement for number fields, see [Cam05, §4]. This conjecture is wide open, but it is known to follow from the abc conjecture thanks to observations of Colliot-Thélène and Abramovich (see the exposition in [Sme17, Appendix]).

1.5. **Height bounds.** Like other proofs of Mordell-type theorems, the proof of Theorem 1.7, relies on height bounds and on rigidity results for $\mathbb{C}$-integral points; these are formulated in Theorems 3.7 and 3.10, respectively. Establishing the relevant height bounds is the main difficulty of this paper. The proof combines ideas pioneered by Grauert, Vojta and Kim [Gra65, Voj91, Kim97] that often allow one to restrict one’s attention to integral points that are tangent to suitable foliations, with ideas of Campana-Păun [CP15] who construct foliations on suitable “adapted” covers of the original space $X$.

Two main difficulties arise and need to be overcome. To begin, we need to construct and discuss covering spaces and foliations in positive characteristic, where covers might well be inseparable or wildly ramified, and where the discussion of foliations becomes substantially more involved when compared with the characteristic zero case. Next, the comparison of height functions on $X$ and heights on an adapted cover turns out to be a second major issue. One of the main technical insights of this paper, hidden in Claim 7.10 of Section 7 is the observation that a lift of a $\mathbb{C}$-integral point from $X$ to an adapted cover is often quite singular, and that the singularities improve the height bounds on the cover by exactly as much as is necessary to establish bounds on the original space $X$. We hope that this technique might be of interest for others.

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2. **Notation and global assumptions**

2.1. **Global assumptions.** Throughout, the letter $k$ will always denote an algebraically closed field of arbitrary characteristic.

2.2. **Varieties and pairs.** We follow notation and conventions of Hartshorne’s book [Har77]. In particular, varieties are always assumed to be irreducible.
Definition 2.1 (Curves and surfaces). Let $k$ be an algebraically closed field. A $k$-curve is a quasi-projective $k$-variety of dimension one that is smooth over $\text{Spec } k$. Analogously for $k$-surfaces.

Definition 2.2 (Pairs). Let $k$ be an algebraically closed field. A $k$-pair is a pair $(X, D)$, consisting of a normal, quasi-projective $k$-variety $X$ and a Weil $Q$-divisor $D = \delta_1 \cdot D_1 + \cdots + \delta_d \cdot D_d$ on $X$, with coefficients $\delta_i$ in the set $[0, 1] \cap \mathbb{Q}$.

Notation 2.3 (Round-up, round-down and fractional part). In the setting of Definition 2.2. We denote the round-up and round-down of $D$ as $[D] = \sum_{i=1}^d \lceil \delta_i \rceil \cdot D_i$ and $\lfloor D \rfloor = \sum_{i=1}^d \lfloor \delta_i \rfloor \cdot D_i$. The fractional part of $D$ will be written as $\{D\} := D - [D]$.

Notation 2.4 (Intersection of boundary components). In the setting of Definition 2.2, if $I \subseteq \{1, \ldots, d\}$ is any non-empty subset, consider the scheme-theoretic intersection $D_I := \cap_{i \in I} \text{supp } D_i$. If $I$ is empty, set $D_I := X$.

The notion of relatively snc divisors has been used in the literature, but its definition has not been discussed much. For the reader’s convenience, we reproduce the definition given in [Keb13, § 3.1].

Definition 2.5 (SNC morphism, relatively snc divisor, [VZ02, Def. 2.1]). Let $k$ be an algebraically closed field, let $(X, D)$ be a $k$-pair and let $\varphi : X \to Y$ be a surjective morphism of quasi-projective $k$-varieties. We say that $D$ is relatively snc, or that $\varphi$ is an snc morphism of the $k$-pair $(X, D)$ if for any set $I$ with $D_I \neq \emptyset$, all restricted morphisms $\varphi|_{D_I} : D_I \to Y$ are smooth of relative dimension $\dim X - \dim Y - |I|$.

Definition 2.6 (k-SNC). Let $k$ be an algebraically closed field. A $k$-pair $(X, D)$ is called snc if the morphism to $\text{Spec } k$ is an snc morphism.

2.3. $C$-pairs. The core notion of this paper is that of a “$C$-pair”. These pairs were introduced under the name “orbifolds géométriques” by Campana and feature prominently in his work. We briefly recall the main definition and refer the reader to one of the many survey papers, including [Cam04, Abr09, CKT16], for a more detailed introduction to Campana’s ideas.

Definition 2.7 (Pairs and $C$-pairs). Let $(X, D)$ be a $k$-pair, as in Definition 2.2. The pair $(X, D)$ is called a $C$-pair if the coefficients $\delta_i$ are contained in the set
\[
\left\{1 - \frac{1}{m_i} \mid m_i \in \mathbb{N}^+\right\} \cup \{1\}.
\]

We follow the convention that $1 - \frac{1}{m_i} = 1$ and write $\delta_i = 1 - \frac{1}{m_i}$ with $m_i \in \mathbb{N}^+ \cup \{\infty\}$. We refer to the numbers $m_i$ as $C$-multiplicities.

Definition 2.8 (C-curve). In the setting of Definition 2.7, assume that the $k$-pair $(X, D)$ is snc. A $C$-curve is a $k$-curve $T$ and a morphism of $k$-varieties $\gamma : T \to X$ such that $\text{Image } \gamma \not\subseteq \text{supp } D$ and such that the following conditions hold for every index $i$.

(2.8.1) If $m_i = \infty$, then $\gamma^* D_i = 0$.
(2.8.2) If $m_i < \infty$, then $\gamma^* D_i \geq m_i \cdot \text{supp } \gamma^* D_i$. 
Remark 2.9. Roughly speaking, C-curves avoid all integral components of D. At points of intersection with one of the remaining $D_i$, the local intersection of every branch of T with $D_i$ is at least $m_i$.

2.4. Projectivised bundles. Like earlier papers on the subject, we follow Grauert’s approach, [Gra65], and study curves on X by looking at their natural lifts to the projectivised bundle $\mathbb{P}(\Omega^1_X)$. The following setting fixes assumptions and notation.

Setting 2.10 (Projectivized vector bundles). Let $k$ be an algebraically closed field. Given a smooth $k$-variety X and a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules, consider the projectivisation $\mathbb{P}(\mathcal{E})$ together with the natural projection morphism $\pi : \mathbb{P} \to X$ and the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}/X}(1) \to \pi^* \mathcal{E} \xrightarrow{\tau} \mathcal{O}_\mathbb{P}(1) \to 0.$$ (2.10.1)

The construction satisfies a number of universal properties, and the “tautological sheaf morphism” $\tau$ appears prominently in all of them. For the reader’s convenience, we recall three standard facts.

Fact 2.11 (Universal property of projectivized bundles – Quotients of $\mathcal{E}$). In Setting 2.10, if $\varphi : Z \to X$ is any morphism, then to give a morphism $\Phi : Z \to \mathbb{P}$ that makes following diagram commute,

$$\begin{array}{ccc}
Z & \xrightarrow{\varphi} & X \\
\downarrow{\Phi} & & \downarrow{\pi} \\
\mathbb{P} & \xrightarrow{\pi} & \mathbb{P} \\
\end{array}$$ (2.11.1)

is equivalent to give an invertible quotient of $\varphi^* \mathcal{E}$. The relation between the morphisms $\Phi$ and quotients is described as follows.

(2.11.2) Given morphism $\Phi$, then the associated quotient is given by the pull-back $\Phi^* \tau$, which maps $\varphi^* \mathcal{E} = \Phi^* \pi^* \mathcal{E}$ to $\Phi^* \mathcal{O}_\mathbb{P}(1)$.

(2.11.3) Given quotient $q : \varphi^* \mathcal{E} \to \mathcal{Q}$, then the associated morphism $\Phi$ gives an isomorphism of sheaves $\mathcal{Q} \cong \Phi^* \mathcal{Q}$ (1) that identifies $q$ and $\Phi^* \tau$. □

Fact 2.12 (Relatively ample hypersurfaces in $\mathbb{P}$ – Subsheaves of $\text{Sym} \mathcal{E}$). In Setting 2.10, to give a relatively ample Cartier divisor $H \in \text{Div}(\mathbb{P})$ of relative degree $M \in \mathbb{N}^+$, it is equivalent to give an invertible subsheaf $\mathcal{L} \subseteq \text{Sym}^M \mathcal{E}$. The relation between subsheaves and divisors is described as follows.

(2.12.1) Given a divisor $H \in \text{Div}(\mathbb{P})$ of relative degree $M \in \mathbb{N}^+$, observe that there exists an invertible $\mathcal{L} \in \text{Pic}(X)$ such that $\pi^* \mathcal{L} \cong \mathcal{O}_\mathbb{P}(M) \otimes \mathcal{I}_H \subseteq \mathcal{O}_\mathbb{P}(M)$. Push-down to X, in order to obtain the inclusion $\mathcal{L} \subseteq \pi_* \mathcal{O}_\mathbb{P}(M) = \text{Sym}^M \mathcal{E}$. □

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1We follow Grothendieck’s convention, where $\mathbb{P}_X(\mathcal{E}) := \text{Proj}(\otimes d \text{Sym}^d \mathcal{E})$. 


(2.12.2) Given an invertible $\mathcal{L} \subseteq \text{Sym}^M \mathcal{E}$, then the following composed morphism of invertibles on $\mathbb{P}$ is non-trivial:

$$\pi^* \mathcal{L} \longrightarrow \pi^* \text{Sym}^M \mathcal{E} \xrightarrow{\text{Sym}^M \tau} \mathcal{O}_{\mathbb{P}}(M),$$

The associated vanishing locus is a Cartier divisor $H \subseteq \mathbb{P}$ of relative degree $M$. \hfill \square

The constructions described in Fact 2.11 and 2.12 are of course related.

**Fact 2.13** (Relation between Facts 2.11 and 2.12). In Setting 2.10, assume we are given a diagram as in (2.11.1), with associated quotient $q : \phi^* \mathcal{E} \to \mathcal{D}$, as well as a relatively ample Cartier divisor $H \in \text{Div}(\mathbb{P})$ of relative degree $M \in \mathbb{N}^+$, with associated subsheaf $\mathcal{L} \subseteq \text{Sym}^M \mathcal{E}$. Abusing notation slightly, the symbol $H$ will also be used to denote the associated complete intersection subscheme of $\mathbb{P}$. Then, the morphism $\Phi$ factors via $H \subseteq \mathbb{P}$ if and only if the following composed sheaf morphism vanishes identically:

$$\phi^* \mathcal{L} \to \phi^* \text{Sym}^M \mathcal{E} = \Phi^* \pi^* \text{Sym}^M \mathcal{E} \to \Phi^* \mathcal{O}_{\mathbb{P}}(M) \equiv \mathcal{D}^\otimes M. \hfill \square$$

**Part I. The Mordell conjecture for integral points on orbifolds**

3. Main results

This present section formulates the main results of Part I of this paper: orbifold analogues of the classical theorems on boundedness and rigidity for algebraic points over function fields, as well as an orbifold version of the Mordell conjecture. Proofs are given in Sections 4–10 below.

**3.1. Setup.** To begin, we specify the setup that is used throughout Part I in some detail and fix notation. The central setting is that of a surface pair $(X, D)$ fibred over a curve $S$, where the fibration is assumed to be an snc morphism, away from a finite set $\Delta$ of exceptional points in $S$.

**Setting 3.1** (Surface pair fibred over a curve). Let $k$ be an algebraically closed field of characteristic $p \geq 0$. Let $\phi : X \to S$ be a surjective morphism of smooth, projective $k$-varieties, where $X$ is of dimension two and $S$ is of dimension one. Assume that $\phi$ has connected fibres. Let $D = \sum_{i=1}^d \delta_i \cdot D_i$ be a $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is an snc $\mathbb{C}$-pair, and let $m_i \in \mathbb{N} \cup \{\infty\}$ denote the $\mathbb{C}$-multiplicities of $D_i$. Finally, let $S^\circ \subseteq S$ be a dense open set such that the restriction $\phi^\circ := \phi|_{X^\circ}$ is an snc morphism for the pair $(X^\circ, D^\circ)$, where $X^\circ := f^{-1}(S^\circ)$ and $D^\circ := D \cap X^\circ$. We denote the generic point of $S$ by $\eta$ and write $X_\eta$ for the generic fibre. Write $\Delta := S \setminus S^\circ$ and view $\Delta$ as a reduced subscheme of $S$.

**Notation 3.2**. Throughout the text, we consider index sets

$$\text{log} := \{i : m_i = \infty\} \quad \text{fract} := \{i : m_i < \infty\}$$

$$\text{wild} := \{i \in \text{fract} : p \neq 0 \text{ and } p|m_i\} \quad \text{tame} := \text{fract} \setminus \text{wild}$$
The figure shows a $C$-integral point in Setting 3.1, with a boundary divisor of the form $D = \frac{1}{2} \cdot (D_1 + D_2)$. The set $\Delta \subset S$ contains two points where $\varphi$ fails to be an snc morphism. In applications, the set $\Delta$ might also contain points where $\varphi$ is an snc morphism, but where $\gamma$ fails to be a $C$ curve.

Figure 3.1. $C$-integral points

and the associated reduced sub-divisors of $\text{supp} D$,

$$
D_{\log} := \lfloor D \rfloor, \quad D_{\text{frac}} := \lceil D \rceil - \lfloor D \rfloor, \\
D_{\text{wild}} := \sum_{i \in \text{wild}} D_i, \quad D_{\text{tame}} := \sum_{i \in \text{tame}} D_i.
$$

In addition, we write $\{ D \} := D - \lfloor D \rfloor$ for the fractional part of $D$.

Remark 3.3 (SNC morphism). The assumption that $\varphi^\circ$ is an snc morphism implies that the induced morphism $D^\circ \to S^\circ$ is étale. In particular, every component of $D^\circ$ is smooth over $\text{Spec } k$. The assumption also implies that there exists an exact sequence of logarithmic differentials, as follows:

$$
0 \longrightarrow (\varphi^\circ)^* \Omega^1_{S^\circ} \longrightarrow \Omega^1_{X^\circ}(\log [D^\circ]) \longrightarrow \omega_{X^\circ/S^\circ}(\log [D^\circ]) \longrightarrow 0.
$$

As pointed out in the introduction, we are mainly concerned with $C$-integral points. These are parameterised curves $T \to X$ that are not contained in $\text{supp } D$, that dominate $S$, and that intersect the components $D_i$ of $D$ locally with $C$-multiplicity $m_i$ or more, at least away from the exceptional set $\Delta \subset S$. Figure 3.1 illustrates the somewhat technical definition.

Definition 3.4 (C-integral points). In Setting 3.1, a $C$-integral point is a morphism $\gamma : T \to X$, where $T$ is a smooth, projective $k$-curve satisfying the following conditions.

1. The curve $T$ dominates $S$. In particular, $T^\circ := \gamma^{-1}(X^\circ)$ is not empty.
2. The restriction $\gamma^\circ : T^\circ \to X^\circ$ is a $C$-curve for the pair $(X^\circ, D^\circ)$.
3. The induced morphism $\gamma : T \to \text{Image}(\gamma)$ is birational.
If the divisor $D$ is empty, we refer to $C$-integral points simply as algebraic points.

Remark 3.5 (Discussion of $C$-integral points). Item (3.4.2) implies in particular that $\Sigma_T := \text{Image}(\gamma)$, which is a one-dimensional $k$-subvariety of $X$, is not contained in $\text{supp} D$. Item (3.4.3) implies that the pull-back map of differentials, $d\gamma : \gamma^* \Omega^1_X \to \omega_T$ is not the zero map. The following diagrams summarise the objects and morphisms of Definition 3.4.

[Diagram]

As in the classical setting, the two main invariants associated with $C$-integral points are its height and its discriminant. The definitions below are identical to those found in the literature. They need no adjustment to work in the orbifold case.

**Definition 3.6 (Height and discriminant).** In Setting 3.1, let $B$ be any $\mathbb{Q}$-divisor on $X$. Given a $C$-integral point $\gamma : T \to X$ as in Definition 3.4, define the height $h_B(\gamma)$ and the discriminant $\delta(\gamma)$ as

$$h_B(\gamma) := \deg \gamma^* B \quad \text{and} \quad \delta(\gamma) := \deg \omega_T \frac{[\gamma \circ S]}{[T : S]}.$$

If $\mathcal{B}$ is an invertible sheaf on $X$, we define $h_{\mathcal{B}}(\gamma)$ in the obvious fashion. We simply write $h(\gamma)$ for the somewhat lengthy symbol $h_{K_X\eta + D}(\gamma)$. In order to avoid awkward case-by-case definition, we define $h_{\mathcal{B}}(\gamma) = 0$ for all $\gamma$ if $\mathcal{B}$ is the zero-sheaf.

3.2. **Geometric height inequalities for $C$-integral points.** The following geometric height inequality for $C$-integral points is the main result of Part I in this paper.

**Theorem 3.7 (Geometric height inequalities for $C$-integral points).** In Setting 3.1, assume the following.

(3.7.1) The degree $d := \deg_{X_{\eta}} (K_X + D)$ is strictly positive.

(3.7.2) Sequence (3.3.1) does not split when restricted to the generic fibre $X_{\eta}$.

Then, a height inequality of the following form holds for all $C$-integral points $\gamma$:

$$h(\gamma) \leq \text{const}^+ \cdot \delta(\gamma) + O \left( \sqrt{h(\gamma)} \right).$$

Theorem 3.7 is in fact a simple corollary of the following more precise result, which generalises works of Vojta, Kim and others. A proof is given in Sections 4–7 below.

**Theorem 3.8 (Geometric height inequalities for $C$-integral points).** In the setting of Theorem 3.7, assume that the characteristic $p$ is zero, or that none of the finite $C$-multiplicities is a multiple of $p^2$. Write $d' := d \cdot \text{lcm} \{ m_i \mid m_i \neq \infty \}$. Then, given any
number \( \epsilon \in \mathbb{Q}^+ \), a height inequality of the following form holds for all \( C \)-integral points \( \gamma \):

\[
h(\gamma) \leq \max\{d', 2 + \epsilon\} \cdot \delta(\gamma) + O\left(\sqrt{h(\gamma)}\right).
\]

3.2.1. **Explanation of Assumption (3.7.2).** In case where \( \text{char}(k) \neq 0 \) and \( D \neq 0 \), Assumption (3.7.2) can be interpreted in terms of the field of definition for the affine curve \( C_{\eta} := (X \setminus \operatorname{supp} D)_{\eta} \) over \( \text{Spec} k(\eta) \), as follows. The affine curve \( C_{\eta} \) is defined over \( k(\eta)^{\text{char}(k)} \) if and only if the exact sequence (3.3.1) splits, see [Vol91, Lem. 1]. In particular, if \( X = S \times \mathbb{P}^1 \) and \( \operatorname{supp} D \) is the union of sections, then the exact sequence (3.3.1) splits if and only if the cross ratios of any 4 sections in \( D \) lie in \( k(\eta)^{\text{char}(k)} \), see [KTV00, Sect. 7].

3.2.2. **Necessity of Assumption (3.7.2).** In case where \( \text{char}(k) = 0 \), set \( X := S \times \mathbb{P}^1 \) and let \( \operatorname{supp} D \) be a union of graphs of inseparable morphisms \( S \to \mathbb{P}^1 \), taken with \( C \)-multiplicities that are less than the characteristic. Sequence (3.3.1) will then split on \( X_{\eta} \). In this case, no matter whether \( \deg_{X_{\eta}}(K_X + D) \) is positive or not, the graph of every inseparable morphism \( S \to \mathbb{P}^1 \) is either contained in \( \operatorname{supp} D \), or is a \( C \)-integral point. Therefore, without Assumption (3.7.2), the height of a \( C \)-integral point cannot be bounded in terms of the discriminant. At the same time, we do not claim that our assumptions are optimal. It is an interesting problem to determine optimal assumptions in order to guarantee that a height inequality of the form presented in Theorem 3.7 holds true.

3.2.3. **Improved height bounds in characteristic zero.** This paper is mainly concerned with height bounds for \( C \)-integral points over fields of finite characteristic. Still, we would like to remark that if \( \text{char}(k) = 0 \), then the height bound of Theorem 3.8 can easily be improved to

\[
(3.9.1) \quad h(\gamma) \leq (2 + \epsilon) \cdot \delta(\gamma) + O\left(\sqrt{h(\gamma)}\right).
\]

To keep the paper readable, we chose to not discuss the characteristic at every single step of the proof. Instead, we refer the reader to Sections 4.4 and 7.1.1 where the improvements in case \( \text{char}(k) = 0 \) are briefly explained.

3.2.4. **Sharpness of Theorem 3.8 and relation to earlier results in case \( \text{char}(k) = 0 \).** In the classical setting where \( D = 0 \) and \( \text{char}(k) = 0 \), the height bound (3.9.1) is due to Vojta, [Voj91, Thm. 0.2], and is known to be nonoptimal. Independent works by McQuillan and Yamanoi that are specific to characteristic zero, [McQ13, Yam04], allow to replace the constant \( 2 + \epsilon \) by \( 1 + \epsilon \). This was previously conjectured by Vojta, see [Voj91, Conj. 0.1] and the survey [Gas09]. If desired, the results of McQuillan and Yamanoi may be applied to further improve Theorem 3.8 in case where \( \text{char}(k) = 0 \).

3.2.5. **Sharpness of Theorem 3.8 and relation to earlier results in case \( \text{char}(k) \neq 0 \).** The situation in positive characteristic is less well understood. In case where \( D = 0 \) and \( \text{char}(k) \neq 0 \), the height bound of Theorem 3.8 is due to Kim, [Kim97, Thm. 1]. Voloch has shown by way of example that one cannot hope for a better constant...
than \( \max\{d, 2 + \varepsilon\} \) without further assumptions, [Kim97, p. 45–46]. Nevertheless, more stringent constraints on the behaviour of the Kodaira-Spencer maps do yield better bounds, see [Kim97, Thm. 2] and [KTV00, Claim 2.2]. In the orbifold setting and in positive characteristic, it is presently unclear to us if stronger assumptions on the Kodaira-Spencer of the pair \((X^\circ, [D^\circ])\) might lead to a better constant. We will return to the subject in Section 7.1.2.

It is conceivable that the assumption “none of the \(\mathcal{C}\)-multiplicities \(m_i\) is a multiple of \(p^2\)” is not necessary and that a more general statement can be proven if one is willing to replace the Artin-Schreier covers that we use in Section 5 by more complicated Artin-Schreier-Witt covers, cf. Remark 5.4 on page 22. Since this paper is rather long already and since Theorem 3.8 suffices for our applications, we have chosen to avoid a detailed analysis of the algebra and combinatorics involved and to leave this problem until later.

### 3.3. Rigidity theorem for \(\mathcal{C}\)-integral points.

As a second step towards our Mordell-type theorem for \(\mathcal{C}\)-integral points, we show that \(\mathcal{C}\)-integral points do not deform. The following theorem makes this statement precise.

**Theorem 3.10** (Rigidity theorem for \(\mathcal{C}\)-integral points). In Setting 3.1, assume the following.

1. The degree \(d := \deg_{X_\eta}(K_X + D)\) is strictly positive.
2. Sequence \((3.3.1)\) does not split when restricted to the generic fibre \(X_\eta\).

If \(T\) is any smooth, projective \(k\)-curve over \(S\), if \(H\) is any \(k\)-variety and if \(\gamma : T \times H \to X\) is any family of \(S\)-morphisms whose individual members \((\gamma_h : T \to X)_{h \in H(k)}\) are \(\mathcal{C}\)-integral points, then \(\gamma\) is constant.

**Remark 3.11.** The assumption that \(T\) is a \(k\)-curve over \(S\) means that \(T\) comes equipped with a surjective morphism \(\zeta : T \to S\). The assumption that \(\gamma\) is a family of \(S\)-morphisms means that \(\varphi \circ \gamma_h = \zeta\), for every \(h \in H(k)\). These assumptions are essential for the rigidity as, otherwise, finiteness would be impossible for curves with infinite automorphism group. The conclusion “\(\gamma\) is constant” asserts that \(\gamma_{h_1} = \gamma_{h_2}\), for all \(h_1, h_2 \in H(k)\).

### 3.4. The Mordell conjecture for \(\mathcal{C}\)-integral points.

As a fairly immediate consequence of height bounds and of rigidity, we obtain the following Mordell-type theorem, asserting the finiteness of \(\mathcal{C}\)-integral points.

**Theorem 3.12** (Mordell-type theorem for \(\mathcal{C}\)-integral points). In Setting 3.1, assume the following.

1. The degree \(d := \deg_{X_\eta}(K_X + D)\) is strictly positive.
2. Sequence \((3.3.1)\) does not split when restricted to the generic fibre \(X_\eta\).

If \(T\) is any smooth, projective \(k\)-curve over \(S\), then the number of \(\mathcal{C}\)-integral points \(\gamma : T \to X\) over \(S\) is finite.

Theorem 3.12 will be shown in Section 10, starting from Page 48 below.
3.4.1. **Necessity of Assumption (3.12.2).** The discussion in Section 3.2.2 shows that without Assumption (3.12.2), finiteness of $C$-integral points does not hold in general.

3.4.2. **Earlier results.** In the classical situation, where $D = 0$, Theorem 3.12 was proven in characteristic zero by Grauert and Manin independently, [Man63, Gra65]. See also [Col90], which translates Manin’s proof to modern language and fixes a gap in it. In positive characteristic, but still in the classical situation, the Theorem is due to Samuel [Sam66a, Sam66b]. An alternative approach in characteristic zero was laid down by Parshin in [Par68], and was later generalised to arbitrary characteristic by Szpiro [Szp81, Sect. 8, Cor. 1]. In positive characteristic, there is yet another approach, due to Voloch [Vol91]. For orbifolds in characteristic zero, Theorem 3.12 is due to Campana [Cam05, Thm. 3.8]. To the best of our knowledge, Theorem 3.12 is new for orbifolds in positive characteristic.

4. **Geometric height bounds in a generalised logarithmic setting**

Theorem 4.2 below is the technical core of Part I of this paper. It generalises the height bounds found by Kim, [Kim97] and will be used in Section 7 to prove the height inequalities that were stated in Section 3.2 above. More specifically, we establish height bounds for $C$-integral points on $(X, D)$ by applying the following Theorem 4.2 to a suitable adapted cover of $X$ and taking the sheaf of “adapted differentials” for $\mathcal{A}$. For clarity of exposition, we specify our precise setting first.

**Setting 4.1.** In Setting 3.1, assume that the divisor $D$ is reduced. In other words, assume that all coefficients $\delta_i$ are equal to 1. Also, assume that there exists a sequence of inclusions $\varphi^* \Omega^1_X \subseteq \mathcal{A} \subseteq \Omega^1_X (\log D)$ where $\mathcal{A}$ is locally free of rank two and where $\mathcal{A}/\varphi^* \Omega^1_S$ is invertible over $X^\circ$. Setting $\mathcal{B} := (\mathcal{A}/\varphi^* \Omega^1_S)^{**}$, we obtain a complex of sheaves on $X$,

\[
\varphi^* \Omega^1_S \xrightarrow{d\varphi} \mathcal{A} \xrightarrow{\mathcal{B}},
\]

and an exact sequence of locally free sheaves on $X^\circ$.

(4.1.1)  
\[
0 \xrightarrow{} (\varphi^* \Omega^1_S)_{\langle S \rangle} \xrightarrow{d\varphi^*} \mathcal{A}|_{X^\circ} \xrightarrow{} \mathcal{B}|_{X^\circ} \xrightarrow{} 0.
\]

We follow the notation introduced in Definition 3.4 concerning $C$-integral points $\gamma : T \to X$ and write $dy_\mathcal{A}$ for the composed morphism,

(4.1.2)  
\[
\gamma^* \mathcal{A} \to \gamma^* \Omega^1_X (\log D) \xrightarrow{dy} \omega_T (\log (\gamma^* D)_{\text{red}}).
\]

**Theorem 4.2** (Height bounds for subsheaves of $\omega_{X/S}$). In Setting 4.1, assume that the degree $d := \deg_{X_\eta} \mathcal{B}$ is positive and that Sequence (4.1.1) does not split when restricted to $X_\eta$. Then, given any number $\epsilon \in \mathbb{Q}^+$, a height inequality of the following form will hold for all $C$-integral points $\gamma : T \to X$,

(4.2.1)  
\[
h_{\mathcal{B}}(\gamma) \leq \max \{d, 2 + \epsilon\} \cdot \frac{\deg_T \text{Image } dy_{\mathcal{B}}}{[T : S]} + O \left(\sqrt{h_{\mathcal{B}}(\gamma)}\right).
\]
Notation 4.3. To avoid overly verbose notation, we write \( \omega_T(\log \gamma^*D) \) instead of the more correct \( \omega_T(\log(\gamma^*D)_{\text{red}}) \) throughout the rest of the present Section 4.

Remark 4.4 (Relation to Theorem 3.8). No matter what sheaf \( \mathcal{A} \) is chosen, the assumption that \( \gamma^* \) is a C-curve, as spelled out in (3.4.2), implies that \( \text{supp} \gamma^*D \) lies over \( \Delta = S \setminus S^o \). In particular, \( \text{deg}_T(\gamma^*D)_{\text{red}} \leq [T : S] \cdot \#\Delta \) and

\[
\frac{\text{deg}_T \text{ Image } d_{Y,\mathcal{A}}}{[T : S]} \leq \frac{\text{deg}_T \omega_T(\log \gamma^*D)}{[T : S]} \leq \delta(\gamma) + \#\Delta
\]

Theorem 4.2 therefore implies Theorem 3.8 in the special case where \( D \) is reduced.

Remark 4.5 (Relation to results in the literature). In the special case where \( D = 0 \) and \( \mathcal{A} = \Omega^1_X \), Theorem 4.2 replaces the number \( \text{deg} \omega_T \) which appears in Kim’s work [Kim97, Thms. 1 and 2] by \( \text{deg}_T \text{ Image}(d_{Y,\mathcal{A}}) \), which gets smaller in comparison to \( \text{deg}_T \omega_T \) the more singular the curve \( \Sigma_T \) is. This improvement will be of critical importance for our applications. In fact, the whole proof of Theorem 3.8 hinges on this observation.

The proof of Theorem 4.2 follows ideas of [Kim97, Kim00] and extends them to our generalised setting. We present a detailed and self-contained proof in the rest of Section 4.

4.1. Preparation for the proof: elementary fact. The proof of Theorem 4.2 considers the following situation more than once.

Situation 4.6. Let \( k \) be an algebraically closed field. Let \( \psi : Z \to B \) be a surjective morphism with connected fibres between smooth, projective \( k \)-varieties, where \( Z \) is of dimension two, and \( B \) of dimension one. We will also consider a chain \( 0 \subset \mathcal{F} \subset \mathcal{A} \) of coherent sheaves on \( Z \), where \( \mathcal{F} \) is invertible, where \( \mathcal{A} \) is locally free of rank two, and where the quotient by \( \mathcal{D} := \mathcal{A}/\mathcal{F} \) is torsion free.

Reminder 4.7 (Saturated subsheaf). The assumption that \( \mathcal{D} \) is torsion free is often formulated by saying that \( \mathcal{F} \) is a saturated subsheaf of \( \mathcal{A} \). We will use this language in the sequel and refer to [HL10, Sect. 1.1] for a more detailed discussion of saturation.

Lemma 4.8. In Situation 4.6, there exists an invertible sheaf \( \mathcal{G} \subseteq \mathcal{D} \) such that the support of the quotient \( \mathcal{D}/\mathcal{G} \) does not dominate \( B \). If \( Z_\eta \) denotes the generic \( \psi \)-fibre, this implies in particular that \( \mathcal{G}|_{Z_\eta} \equiv \mathcal{D}|_{Z_\eta} \).

Proof. The quotient \( \mathcal{D} \) is torsion free and of rank one, but need not be locally free. However, since \( Z \) is regular, its reflexive hull \( \mathcal{D}^{**} \) will be invertible, [Har80, Cor. 1.4 or Prop. 1.9]. Also, recall from [HL10, Ex. 1.1.16], that there exists a dimension-zero (and hence finite) subscheme \( A \subseteq Z \) such that the natural inclusion \( \mathcal{D} \hookrightarrow \mathcal{D}^{**} \) identifies \( \mathcal{D} \) with \( \mathcal{J}_A \otimes \mathcal{D}^{**} \). Consider the image \( C := \psi(A) \), equipped with its natural structure as a reduced subscheme of \( B \). If \( N \gg 0 \), then \( \mathcal{J}_{\psi^{-1}(C)}^{\otimes N} \subseteq \mathcal{J}_A \), and we obtain the following chain of inclusions which finishes the proof:

\[
\mathcal{G} := \mathcal{J}_{\psi^{-1}(C)}^{\otimes N} \otimes \mathcal{D}^{**} \subseteq \mathcal{D} \subseteq \mathcal{D}^{**}.
\]

\( \Box \)
Lemma 4.8 will be used in the following form.

Observation 4.9. In Situation 4.6, let \( \mathcal{G} \subseteq \mathcal{D} \) be one of the subsheaves given by Lemma 4.8. If \( \gamma : T \to Z \) is a morphism from a smooth, projective \( k \)-curve \( T \) such that the induced map \( T \to B \) is dominant, if \( \mathcal{L} \in \text{Pic}(T) \) and if \( \rho : \gamma^*\mathcal{G} \to \mathcal{L} \) is any non-trivial morphism such that the composition \( \gamma^*\mathcal{F} \to \gamma^*\mathcal{G} \to \mathcal{L} \) vanishes, then there exists a non-trivial morphism \( \gamma^*\mathcal{G} \to \text{Image}(\rho) \), and we obtain a diagram as follows.

\[
\begin{array}{ccc}
\gamma^*\mathcal{F} & \xrightarrow{\text{composition vanishes by assumption}} & \text{Image}(\rho) \\
\downarrow & & \downarrow \text{map exists} \\
\gamma^*\mathcal{G} & \xrightarrow{\text{generically an isomorphism}} & \gamma^*\mathcal{D} \\
\end{array}
\]

4.2. Preparation for the proof: height bounds for curves tangent to a foliation. To prepare for the proof of Theorem 4.2, we will first prove a height bound for \( \mathcal{C} \)-integral points \( \gamma \) tangent to a given foliation, or which satisfy a "Pfaffian equation" in the language of [Kim97]. While such bounds are easy to obtain in characteristic zero, where tangent curves always form a bounded family, the proof below is somewhat more involved. The following observation will be used.

Observation 4.10. In the setting of Theorem 4.2, assume that \( k \) is algebraically closed. If \( \gamma : T \to X \) is a \( \mathcal{C} \)-integral point, then \( d\gamma_{\mathcal{G}} \) can vanish identically, but only if the image curve \( \gamma(T) \) is contained in the support of \( \Omega^1_X(\log D)/\mathcal{A} \). To see why, recall from Item (3.4.3) of Definition 3.4 that \( \gamma \) maps \( T \) birationally onto its image. The differential

\[
\gamma^*\Omega^1_X(\log D) \xrightarrow{d\gamma} \omega_T(\log(\gamma^*D)_{\text{red}})
\]

will therefore not vanish. The composed map \( d\gamma_{\mathcal{G}} \) of (4.1.2) will then likewise not vanish if \( \gamma(T) \) intersects the open set of \( X \) where the sheaves \( \mathcal{A} \) and \( \Omega^1_X(\log D) \) agree. If \( d\gamma_{\mathcal{G}} \) does vanish, then the image \( \gamma(T) \) must therefore be contained in the closed set where \( \mathcal{A} \) and \( \Omega^1_X(\log D) \) differ; this is however exactly the support of the quotient \( \Omega^1_X(\log D)/\mathcal{A} \).

It follows that there are at most finitely many \( \mathcal{C} \)-integral points \( \gamma \) where \( d\gamma_{\mathcal{G}} = 0 \), up to reparametrisation of the morphism \( \gamma : T \to X \). So, there exists a number \( \text{const}_1 \in \mathbb{R} \) such that \( h_{\mathcal{G}}(\gamma) \leq \text{const}_1 \) for all \( \mathcal{C} \)-integral points \( \gamma \) with \( d\gamma_{\mathcal{G}} = 0 \).\hfill \Box

Notation 4.11 (Foliation). Let us adopt the setting of Theorem 4.2. By an \( \mathcal{A} \)-foliation on \( X \), we mean a saturated, invertible subsheaf \( \mathcal{F} \subseteq \mathcal{D} \). A \( \mathcal{C} \)-integral point \( \gamma \) is said to be tangent to the \( \mathcal{A} \)-foliation \( \mathcal{F} \) if the following composed map \( d\gamma_{\mathcal{F}} \) vanishes identically:

\[
\gamma^*\mathcal{F} \to \gamma^*\mathcal{A} \xrightarrow{d\gamma_{\mathcal{F}}} \omega_T(\log \gamma^*D).
\]
Proposition 4.12 (Height bound for curves tangent to foliation, compare [Kim97, p. 53]). If, in the setting of Theorem 4.2, $\mathcal{F} \subseteq \mathcal{A}$ is an $\mathcal{A}$-foliation on $X$, then a height inequality of the following form will hold for all $C$-integral points $y : T \to X$ that are tangent to $\mathcal{F}$:

$$h_{\mathcal{A}}(y) \leq d \cdot \frac{\deg_T \text{Image} \, d\gamma_{\mathcal{A}}}{[T : S]} + O\left(\sqrt{h_{\mathcal{A}}(y)}\right).$$

Proof. The generic fibre $X_\eta$ is a smooth $k(\eta)$-curve, the sheaf $\mathcal{A}|_{X_\eta}$ is a vector bundle on that curve, and Sequence (4.1.1) presents $\mathcal{A}|_{X_\eta}$ as an extension of the trivial line bundle $\varphi^* \Omega^1_{\mathcal{A}|_{X_\eta}} \cong \mathcal{O}|_{X_\eta}$ and the ample line bundle $\mathcal{B}|_{X_\eta}$. This extension is not split by assumption. Therefore, any invertible quotient of $\mathcal{A}|_{X_\eta}$ will be of positive degree.

To apply this observation, consider the cokernel $\mathcal{D} := \mathcal{A}/\mathcal{F}$. Let $\mathcal{G} \subseteq \mathcal{D}$ be the invertible subsheaf given by Lemma 4.8, so that $\mathcal{G}|_{X_\eta} = \mathcal{D}|_{X_\eta}$. The sheaf $\mathcal{G}|_{X_\eta}$ is then an invertible quotient of $\mathcal{A}|_{X_\eta}$, so that $\deg_{X_\eta} \mathcal{G} > 0$. Néron’s theorem, [Ser89, Thm. 2.11], now yields the following inequality for all $y$:

$$(4.12.1) \quad h_{\mathcal{A}}(y) \leq \left(\deg_{X_\eta} \mathcal{G}\right) \cdot h_{\mathcal{A}}(y) + O(1) \leq \left(\deg_{X_\eta} \mathcal{B}\right) \cdot h_{\mathcal{A}}(y) + O\left(\sqrt{h_{\mathcal{A}}(y)}\right).$$

To end the proof, Observation 4.10 implies that it suffices to establish the inequality $\deg_T \gamma^* \mathcal{G} \leq \deg_T \text{Image}(d\gamma_{\mathcal{A}})$ for all $y$ whose associated morphism $d\gamma_{\mathcal{A}}$ does not vanish identically. This, however, follows because Observation 4.9 yields the existence of a non-trivial morphism $\gamma^* \mathcal{G} \to \text{Image}(d\gamma_{\mathcal{A}}) \subseteq \omega_T(\log \gamma^* D)$. □

4.3. Proof of Theorem 4.2. In this paragraph we adopt the setting of Theorem 4.2 and the notation introduced in Section 4.2 above, and we assume that a number $c \in \mathbb{Q}^+$ is given. Let $n_c$ be the smallest positive integer such that $n_c \cdot c$ is integral. For the reader’s convenience, we subdivided the proof of Theorem 4.2 into a number of relatively independent steps.

Step 1, the projectivisation of $\mathcal{A}$. Continuing along the ideas of Grauert, Vojta and Kim, [Gra65, Voj91, Kim97], we consider the projectivisation $\mathbb{P} := \mathbb{P}_X(\mathcal{A})$ together with the natural projection morphism $\pi : \mathbb{P} \to X$ and the standard “Euler sequence”

$$(4.13.1) \quad 0 \to \Omega^1_{\mathbb{P}/X}(1) \to \pi^* \mathcal{A} \to \mathcal{O}(1) \to 0.$$  

If $y : T \to X$ is a $C$-integral point, the image of the morphism $d\gamma_{\mathcal{A}}$ is torsion free, hence either zero or invertible. If non-zero, we have seen in Fact 2.11 that it defines a lifting to $\mathbb{P}$, which will always be denoted by $\Gamma$,
The lifting has the property that $\Gamma^* \tau \equiv dY_{\mathcal{A}}$, where $\tau$ is the morphism that appears in the Euler sequence (4.13.1) above. In particular, the following holds.

(4.13.2) We have $\Gamma^* \mathcal{O}(1) \equiv \text{Image}(dY_{\mathcal{A}}) \subseteq \omega_T(\log \gamma^* D)$.

(4.13.3) The composition
\[
\Gamma^* \Omega^1_{P/X}(1) \to \Gamma^* \pi^* \mathcal{A} = \gamma^* \mathcal{A} \xrightarrow{dY_{\mathcal{A}}} \omega_T(\log \gamma^* D)
\]
vanishes.

**Step 2, symmetric differentials, degenerate and nondegenerate points.**

Next, we adapt some arguments from [Kim97, Sect. 3] to our setting.

**Claim 4.14.** If $m$ is a sufficiently large positive integer, then there exists an ample invertible sheaf $\mathcal{H}_m \in \text{Pic}(S)$ and a non-trivial morphism
\[
\sigma : \phi^*(\mathcal{H}^N) \otimes \mathcal{B}^m \rightarrow \text{Sym}(2 + \varepsilon) \mathcal{A}.
\]

**Proof of Claim 4.14.** Writing $N := (2 + \varepsilon) \cdot m$, for brevity and
\[
\mathcal{H} \mathcal{O}_m := \mathcal{H} \mathcal{O}_m(\mathcal{B}^m, \text{Sym}^N \mathcal{A}),
\]
it suffices to show that the push-forward sheaf $\phi_{\ast} \mathcal{H} \mathcal{O}_m$ is not the zero sheaf, for sufficiently large $m$. Choose a closed point $s \in S^0$ with fibre $X_s$. Standard computations show that
\[
\text{rank } \mathcal{H} \mathcal{O}_m = (2 + \varepsilon) \cdot m + 1
\]
\[
\deg \mathcal{H} \mathcal{O}_m|_{X_s} = \deg \left( (\mathcal{B}^m) \otimes \text{Sym}^N \mathcal{A} \right) \big|_{X_s}
\]
\[
= \deg \left( (\mathcal{B}^m) \otimes \text{Sym}^N \phi^* \Omega^1_\mathcal{L} \otimes \mathcal{B} \right) \big|_{X_s}
\]
\[
= \deg \left( (\mathcal{B}^m) \otimes \bigoplus_{i=0}^N \mathcal{B}^i \right) \big|_{X_s}
\]
\[
= \deg(\phi^* \Omega^1_\mathcal{L}) \big|_{X_s} = 0
\]

By Riemann-Roch, the holomorphic Euler characteristic of $\mathcal{H} \mathcal{O}_m|_{X_s}$ grows quadratically in $m$:
\[
\chi(\mathcal{H} \mathcal{O}_m|_{X_s}) = \deg \mathcal{H} \mathcal{O}_m|_{X_s} + (1 - g(X_s)) \cdot \text{rank } \mathcal{H} \mathcal{O}_m|_{X_s}
\]
\[
= \text{const}^* \cdot m^2 + (\text{lower order terms in } m).
\]

It follows that the vector bundles $\mathcal{H} \mathcal{O}_m|_{X_s}$ have sections when $m$ is sufficiently large, and hence that $\phi_{\ast} \mathcal{H} \mathcal{O}_m$ is indeed non-zero. \hfill \Box

We fix a positive integer number $m$, an ample line bundle $\mathcal{H}_m \in \text{Pic}(S)$ and a non-trivial morphism $\sigma : \phi^*(\mathcal{H}^N) \otimes \mathcal{B}^m \rightarrow \text{Sym}(2 + \varepsilon) \mathcal{A}$, and maintain this choice for the remainder of the present proof. For brevity of notation, write
\[
M := mn \quad \text{and} \quad L := \phi^*(\mathcal{H}^N) \otimes \mathcal{B}^M.
\]
The sheaf morphism $\sigma$ identifies $L$ with a subsheaf of $\text{Sym}(2 + \varepsilon) \mathcal{A}$. As we have seen in Fact 2.12, this defines a divisor $H \in \text{Div}(\mathbb{P})$.
Notation 4.15 (Degenerate points). Let $\gamma$ be a $C$-integral point with $d_{\gamma,\sigma} \neq 0$. Following Kim, we call $\gamma$ degenerate with respect to $\sigma$ if $\Gamma$ factors via $\text{supp} \, H$. If $Y \subseteq \text{supp} \, H$ is an irreducible component, we say that $\gamma$ is degenerate with respect to $Y$ if $\Gamma$ factors via $Y$.

Let $\gamma : T \to X$ be a $C$-integral point with $d_{\gamma,\sigma} \neq 0$. Fact 2.13 asserts that $\gamma$ is degenerate with respect to $\sigma$ if and only if the composed map

$$
\gamma^* \mathcal{L} \xrightarrow{\gamma^* \sigma} \gamma^* \text{Sym}^{(2+\epsilon)M} \xrightarrow{\text{Sym}^{(2+\epsilon)M} d_{\gamma,\sigma}} \omega_T(\log \gamma^* D)^{\otimes (2+\epsilon)M}
$$

vanishes identically. Nondegenerate points therefore satisfy the following height bound, which allows us to concentrate on degenerate points for the remainder of the proof.

Observation 4.16. If a $C$-integral point $\gamma : T \to X$ is nondegenerate with respect to $\sigma$, then

$$
h_B(\gamma) \leq (2+\epsilon) \cdot \frac{\deg \text{Image} \, d_{\gamma,\sigma}}{[T : S]} + \frac{(2+\epsilon)M + 1}{M} \cdot \deg_s \mathcal{H}_m.
$$

□ (Observation 4.16)

Step 3, height bounds for degenerate points – setup and simplifications.

Since the divisor $H \in \text{Div}(\mathbb{P})$ has only finitely many components, it suffices to prove our height bounds for algebraic points that are degenerate with respect to any one given irreducible component $Y \subseteq \text{supp} \, H$. We fix one component $Y$ for the remainder of the proof. If $\pi(Y)$ is a proper subset of $X$, then there are (up to reparametrisation) at most finitely $C$-integral points that are degenerate with respect to $Y$, and there is nothing to prove. We will therefore assume the following.

Assumption w.l.o.g. 4.17. The component $Y$ dominates $X$.

To set the stage for the next steps in the proof, we use resolution of singularities in dimension two, [Lip78], in order to find $k$-smooth varieties $\widetilde{Y}$, $\widehat{Y}$, $\widehat{S}$ and a diagram,

\[ Y \xrightarrow{\gamma} X \]

\[ \gamma^* \mathcal{L} \xrightarrow{\gamma^* \sigma} \gamma^* \text{Sym}^{(2+\epsilon)M} \xrightarrow{\text{Sym}^{(2+\epsilon)M} d_{\gamma,\sigma}} \omega_T(\log \gamma^* D)^{\otimes (2+\epsilon)M} \]

where $\text{supp} \, \beta^* D \subseteq \widehat{Y}$ and $\text{supp} \, \alpha^* \beta^* D \subseteq \widetilde{Y}$ are strict normal crossings divisors.

Notation 4.18. Write $\widehat{\eta}$ for the generic point of $\widehat{S}$ and $\widetilde{\eta}$ for the generic fibre.

Observation and Notation 4.19. If $\gamma$ is degenerate with respect to $Y$ and if $\Gamma(T)$ is not contained in the set of fundamental points of the birational map $\rho^{-1}$, then
the morphism $\Gamma$ factorises via $\tilde{Y}$. In other words, there exist morphisms $\tilde{\Gamma}$, $\hat{\Gamma}$ that make the following diagram commute:

\[
\begin{array}{cccccc}
T & \to & T & \to & T & \to & T \\
\downarrow^{\tilde{\Gamma}} & & \downarrow^{\hat{\Gamma}} & & \downarrow^{\gamma} & & \downarrow^{\Gamma} \\
\tilde{Y} & \xrightarrow{\alpha} & \hat{Y} & \xrightarrow{\beta} & X & \xrightarrow{\pi} & Y. \\
\end{array}
\]

We say these points are liftable to $\tilde{Y}$, or lift to $\hat{Y}$.

The fundamental points of $\rho^{-1}$ form a proper subset of $Y$ that does not dominate $X$. In particular, there are (up to reparametrisation) at most finitely many $C$-integral points $y$ that are degenerate with respect to $Y$, but do not lift to $\tilde{Y}$. This yields the following claim, which allows us to concentrate on liftable curves for the remainder of the proof.

**Claim 4.20.** An inequality of the form $h_{\mathcal{A}}(y) \leq O(1)$ holds for all $C$-integral points $y$ that are degenerate with respect to $Y$ but do not lift to $\tilde{Y}$. \(\square\) (Claim 4.20)

**Step 4, height bounds for degenerate points – generalised foliation.** We will see in this step that the surface $\hat{Y}$ carries a generalised foliation, to which almost all liftings $\tilde{\Gamma}$ are tangent. The arguments are similar in spirit to those of Section 4.2 above. The existence of the foliation comes out of the following claim.

**Claim 4.21.** There exists a saturated, invertible sheaf $\hat{\mathcal{F}} \subseteq \beta^* \mathcal{A}$ such that the composed map

\[
(4.21.1) \quad \hat{\Gamma}^* \hat{\mathcal{F}} \to \hat{\Gamma}^* \beta^* \mathcal{A} = \gamma^* \mathcal{A} \xrightarrow{d\mathcal{A}} \omega_T(\log \gamma^* D)
\]

vanishes, up to reparametrisation for almost all $Y$-degenerate $C$-integral points $y$ that are liftable to $\tilde{Y}$.

**Proof of Claim 4.21.** The construction depends on whether the restricted morphism $\pi|_Y : Y \to X$ is separable or not.

**Construction in the case where $\pi|_Y$ is separable.** If $\pi|_Y$ is separable and $\alpha$ is therefore an isomorphism, the discussion of the Euler sequence (Item (4.13.3) on page 17) allows one to choose $\hat{\mathcal{F}}$ as the saturation of $\hat{\mathcal{F}}'$ inside $\beta^* \mathcal{A}$, where

\[
\hat{\mathcal{F}}' := \text{Image}((\alpha^{-1})^* \rho^* (\Omega^1_{\tilde{Y}/X}(1)|_Y) \to (\alpha^{-1})^* \rho^* \pi^* \mathcal{A}) \subseteq \beta^* \mathcal{A}.
\]

As a reflexive sheaf of rank one on a regular, two-dimensional scheme, $\hat{\mathcal{F}}$ will then be invertible, as required.
Construction in the case where $\pi|_Y$ is inseparable. In this case, we follow [Kim00] and recall that the natural morphism $\Omega^1_X(\log D) \to \alpha_*\Omega^1_Y$ has a non-trivial kernel: it suffices to check this at the generic points of $\hat{Y}$ and $\bar{Y}$, where this is a well-known property of Kähler differentials for inseparable field extensions, cf. [Liu02, Sect. 6.1, Exerc. 1.6]. The same statement therefore holds for the associated morphism of logarithmic differentials. To be more precise, we use separability of $\beta$ to consider the inclusions

$$
\beta^*\mathcal{A} \subseteq \beta^*\Omega^1_X(\log D) \subseteq \Omega^1_Y(\log(\beta^*D)_{\text{red}}),
$$

and use these to view $\beta^*\mathcal{A}$ as a subsheaf of $\Omega^1_Y(\log(\beta^*D)_{\text{red}})$. Then, we choose an invertible subsheaf

$$
\mathcal{F}' \subseteq \ker\left(\Omega^1_Y(\log(\beta^*D)_{\text{red}}) \to \alpha_*\Omega^1_Y(\log(\alpha^*\beta^*D)_{\text{red}})\right) \cap \beta^*\mathcal{A}
$$

and let $\mathcal{F}$ again be the saturation of $\mathcal{F}'$ inside $\beta^*\mathcal{A}$. Since $\hat{T}$ factors via $\Gamma$, it is clear that the composed map $\hat{T}:\mathcal{F}' \to \omega_T(\log \gamma^*D)$ vanishes. It is then clear that the composed map (4.21.1) vanishes, except perhaps if $\hat{T}(T)$ is contained in the support of $\mathcal{F}/\mathcal{F}'$. There are, however, up to reparametrisation only finitely many $C$-integral points with that property.

For the remainder of the proof, we choose one saturated, invertible sheaf $\mathcal{F} \subseteq \beta^*\mathcal{A}$ as given by Claim 4.21. Consider the quotient $\mathcal{D} := \beta^*\mathcal{A}/\mathcal{F}$, apply Lemma 4.8 to the morphism $\mathcal{D}$ and let $\mathcal{G} \subseteq \mathcal{D}$ be one of the invertible sheaves given by that lemma. Following [Kim97, Sect. 4], we will now prove the desired height bound under the assumption that a certain numerical inequality holds.

Claim 4.22. If $(2 + \epsilon) \cdot \deg_{\gamma}\mathcal{G} \geq \deg_{\gamma}\beta^*\mathcal{B}$, then a height inequality of the form (4.2.1) holds for all $C$-integral points $\gamma$ that are degenerate with respect to $Y$.

Proof of Claim 4.22. In view of Claim 4.20, we are interested in $C$-integral points $\gamma$ that are degenerate with respect to $Y$ and liftable to $\bar{Y}$. For all such $\gamma$, Néron’s theorem yields the inequalities

$$
h_\mathcal{B}(\gamma) = \frac{1}{[S : S]} \cdot h_{\beta^*\mathcal{B}}(\hat{T}) \leq \frac{1}{[S : S]} \cdot \frac{\deg_{\gamma}\beta^*\mathcal{B}}{\deg_{\gamma}\mathcal{G}} \cdot h_{\gamma}(\hat{T}) + O\left(\sqrt{h_{\beta^*\mathcal{B}}(\hat{T})}\right)
$$

$$
\leq \frac{1}{[S : S]} \cdot (2 + \epsilon) \cdot h_{\gamma}(\hat{T}) + O(h_{\mathcal{B}}(\gamma)).
$$

But then, Claim 4.21 asserts that the composed map (4.21.1) vanishes, up to reparametrisation for all but finitely many $C$-integral points $\gamma$. This in turn implies by Observation 4.9 that these $\gamma$ satisfy

$$
\frac{1}{[S : S]} \cdot h_{\gamma}(\hat{T}) \leq \frac{1}{[S : S]} \cdot \frac{\deg_{\mathcal{B}}\text{Image }d_\gamma}{[T : S]} = \frac{\deg_{\mathcal{B}}\text{Image }d_\gamma}{[T : S]}.
$$
Hence, the claim. □ (Claim 4.22)

**Step 5, height bounds for degenerate points – end of proof.** Claim 4.22 finishes the proof of Theorem 4.2 in one special case. We will therefore assume for the remainder of the proof that the assumption of Claim 4.22 does not hold.

**Assumption w.l.o.g. 4.23.** We have $(2 + \varepsilon) \cdot \deg_{\gamma}^{\beta^{*}} G < \deg_{\gamma}^{\beta^{*}} B$.

A short computation using Assumption 4.23 shows that $\beta^{*} A|\gamma$ is slope-unstable. The fact that $\beta$ is separable immediately implies that the morphism $\gamma : Y_{\eta} \to X_{\eta}$ is separable, too. In particular, [Miy87, Prop. 3.2] implies that the sheaf $A|X_{\eta}$ is slope-unstable. In other words, there exists a saturated (hence reflexive, hence invertible) subsheaf $F \subseteq A$ such that

$$\deg_{X_{\eta}} F > \frac{1}{2} \cdot \deg_{X_{\eta}} A = \frac{1}{2} \cdot \deg_{X_{\eta}} B. \quad (4.23.1)$$

Recalling from Proposition 4.12 that a height bound has already been established for all points $\gamma$ tangent to the $A$-foliation $F$, it remains to consider those points $\gamma$ for which the associated morphism

$$\gamma^{*} F \to \gamma^{*} \mathcal{A} \xrightarrow{d_{\mathcal{A}}} \omega_{T}(\log \gamma^{*} D)$$

does not vanish identically, so that $[T : S] \cdot h_{\mathcal{A}}(\gamma) \leq \deg_{T} \operatorname{Image} d_{\mathcal{A}}$. But then again Néron’s theorem and Inequality (4.23.1) give a height inequality of the form

$$\h_{\mathcal{A}}(\bullet) \leq 2 \cdot h_{\mathcal{A}}(\bullet) + O\left(\sqrt{\h_{\mathcal{A}}(\bullet)}\right).$$

This finishes the proof of Theorem 4.2. □

4.4. **Improved height bounds in characteristic zero.** If $\text{char}(k) = 0$, then Jouanolou’s theorem implies that the family of curves on $X$ with general member irreducible and with every member tangent to a given foliation is always bounded, [Jou78, Ghy00]. As a result, we may replace the complicated height estimate of Proposition 4.12 by the much stronger estimate $h_{\mathcal{A}}(\gamma) \leq O(1)$. With this improvement, the proof of Theorem 4.2, which uses Proposition 4.12 only in its last step, goes through without many changes, and allows us to replace Inequality (4.2.1) of Theorem 4.2 by the stronger bound

$$h_{\mathcal{A}}(\gamma) \leq (2 + \varepsilon) \cdot \frac{\deg_{T} \operatorname{Image} d_{\mathcal{A}}}{[T : S]} + O\left(\sqrt{h_{\mathcal{A}}(\gamma)}\right). \quad (4.23.2)$$

5. **Covering constructions**

Following ideas that originate from the work of Campana [Cam11] and Campana-Păun, [CP15], we show Theorem 3.8 by passing to a strongly adapted cover. For varieties over $\mathbb{C}$, adapted covers are discussed in Campana’s work, but also in the survey article [CKT16] or [JK11, Sect. 2.4]. In the present setup, where the characteristic might be positive, more care needs to be taken. The following claim summarises the relevant properties.
Proposition 5.1 (Existence of a strongly adapted cover). In Setting 3.1, assume that the characteristic $p := \text{char}(k)$ is either zero, or else that none of the $C$-multiplicities $m_i$ is a multiple of $p^2$. Then, there exists a smooth, projective $k$-variety $\hat{X}$, a generically finite and separable surjection $\hat{c} : \hat{X} \to X$ and a dense open set $S^\circ \subseteq S^0$ with preimages $\hat{X}^\circ \subseteq \hat{X}$ and $X^\circ \subseteq X$ forming a commutative diagram as follows,

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{c}} & X \\
\downarrow & & \downarrow \\
\hat{X}^\circ & \xrightarrow{\hat{c}^\circ} & X^\circ \\
\uparrow & & \uparrow \\
\hat{S}^\circ & \xrightarrow{\hat{\varphi}^\circ} & \hat{S}^\circ \\
\end{array}
\]

such that the following holds.

(5.1.1) We have $[\hat{X}^\circ : X^\circ] = \text{lcm} \{m_i | m_i \neq \infty\}$.

(5.1.2) If $p = 0$, then $\hat{c}$ is the identity morphism. If $p > 0$, then $\deg \hat{c} \in \{1, p\}$, and $\deg \hat{c}$ is coprime to $p$.

(5.1.3) The morphism $\hat{\varphi}^\circ$ is snc for the pair $(\hat{X}^\circ, (c^\circ)^*D)$.

(5.1.4) If $\hat{R}^\circ \subseteq \hat{X}^\circ$ is an irreducible component of the ramification divisor for $c^\circ$, then $\hat{R}^\circ$ is either contained in $\text{supp}(c^\circ)^*D$, or disjoint from that support.

(5.1.5) If $i$ is any index for which $m_i < \infty$, then all irreducible components $\Gamma$ of $(c^\circ)^*D_i$ have multiplicity $\text{mult}_{\Gamma}(c^\circ)^*D_i = m_i$.

(5.1.6) If $i$ is any index with $m_i = \infty$, then all irreducible components $\Gamma$ of $(c^\circ)^*D_i$ have multiplicity $\text{mult}_{\Gamma}(c^\circ)^*D_i = 1$.

Remark 5.2 (Smoothness). Item (5.1.3) implies that $\hat{X}^\circ$ is smooth over $\text{Spec} \ k$, that the components of $(c^\circ)^*D$ are smooth, and that no two of them intersect.

Remark 5.3 (Strongly adapted covers). In the language of earlier papers, Items (5.1.5) and (5.1.6) are summarised by saying that the morphism $c^\circ$ is strongly adapted with respect to $(X^\circ, D|_{X^\circ})$, see for instance [CKT16, Sect. 2.6].

Remark 5.4 (Optimality and generalisations). It is conceivable that the assumption “none of the $C$-multiplicities $m_i$ is a multiple of $p^2$” of Proposition 5.1 is not necessary and that a more general statement can be proven if one is willing to replace the Artin-Schreier covers (that we discuss below) by more complicated Artin-Schreier-Witt covers.

5.1. Notation. The setup of Proposition 5.1 will be maintained throughout Sections 6 and Section 7, and the following notation will be used.
Notation 5.5 (Strict transforms, restriction of divisors to open sets). Setting as in Proposition 5.1. If $H \in \text{Div}(X)$ is any reduced divisor, let $\tilde{H} \in \text{Div}(\tilde{X})$ be the largest divisor in $(c^*H)_{\text{red}}$ with the property that every component $\tilde{H}'$ of $\tilde{H}$ is generically finite over the image component of $H$. By minor abuse of notation, we refer to $\tilde{H}$ as the strict transform. Ditto for $\tilde{H}$. Restrictions of divisors $A \in \mathbb{Q}\text{Div}(X)$ to the open set $X^\circ$ will as always be denoted by $A^\circ$, ditto for divisors on $\tilde{X}$ and $\tilde{X}$. Since no confusion is likely to arise, we abuse Notation 5.5 slightly, by writing $\tilde{D}_{\text{fract}}^\circ$ instead of the more correct, but somewhat clumsy $D_{\text{fract}}^\circ$. Ditto for other divisors, and ditto for divisors on $\tilde{X}^\circ$.

5.2. Artin–Schreier covers. The proof of Proposition 5.1 uses Artin–Schreier covers which we briefly recall here.

Construction 5.6 (Artin-Schreier cover). Let $K$ be a field of positive characteristic $p$, not necessarily perfect or algebraically closed. Let $C$ be a proper, smooth curve over $K$ and let $f \in K(C)$ be a non-constant rational function on $C$ which is regular on $U \subseteq C$ and has poles along $D = C \setminus U$. Assume that $f$ is not of the form $g^p - g$ for some $g \in K(C)$. Consider the following curve over $\bar{U}$:

$$A_U = \{(c, y) \in U \times \mathbb{A}_K^1 \mid y^p - y = f\}.$$  

The classical theory of Artin-Schreier coverings (see [Lan02, Chapt. VI, Thm. 6.4] or [Bos18, Sect. 4.10]) says that $A_U$ is irreducible, and the projection $A_U \rightarrow U$ has degree $p$, is Galois and étale. To wit, identifying $\mathbb{F}_p$ with the prime field of $K$, the group $(\mathbb{F}_p, +)$ acts on $A_U$ by

$$\mathbb{F}_p : (c, y) \mapsto (c, y + n).$$

Then $A_U \rightarrow U$ extends to a cyclic degree-$p$ cover $A \rightarrow C$, where $A$ is the normal, proper $K$-curve constructed as follows: take the Zariski closure of $A_U$ in $C \times \mathbb{A}_K^1$, and normalise.

Proposition 5.7. In the setting of Construction 5.6, assume that the following holds.

(5.7.1) All reduced, irreducible components of $D$ are étale over Spec $K$.
(5.7.2) The pole orders of $f$ along the components of $D$ are prime to $p$.

Then, the following will hold.

(5.7.3) The curve $A$ is smooth over $K$.
(5.7.4) The morphism $A \rightarrow C$ is étale over $U$ and totally (wildly) ramified over $D$.

Remark 5.8. This result is well-known if $K$ is perfect, but caution is advised when $K$ is imperfect (and normality does not necessarily imply geometric normality).

Proof of Proposition 5.7. Since the problem at hand is local on $C$, we may assume that $D$ consists of a unique closed point $c$, with residue field $\kappa(c)$ separable over $K$.

Let $a$ be a point of $A$ lying over $c$ and view $y$ as a rational function on $A$. Then, $y$ has a pole at $a$. Writing $\text{mult}_c$ and $\text{mult}_a$ for the valuations induced by $c$ and $a$ on $K(C)$ and $K(A)$, respectively, we find that

$$p \text{ mult}_a(y) = \text{mult}_a(y^p) = \text{mult}_a(y^p - y) = \text{mult}_a(f) = e_{a/c} \text{ mult}_c(f).$$
where $e_{a/c}$ denotes the ramification index of the extension $\mathcal{O}_{C,c} \subseteq \mathcal{O}_{A,a}$. Since $\text{mult}_c(f)$ is prime to $p$ by assumption, we have $p \mid e_{a/c}$. Since

$$e_{a/c} \leq \deg(A \to C) = p,$$

we obtain that $e_{a/c} = p$. It follows that $a$ is the only point of $A$ lying over $c$ and that $\mathcal{O}_{C,c} \subseteq \mathcal{O}_{A,a}$ is a totally ramified extension of discrete valuation rings. This proves (5.7.4).

For (5.7.3), it remains to prove that $A$ is smooth over $K$ at the closed point $a$. Indeed, $A_U \to U$ is étale and $U$ is smooth over $K$, so the only remaining point is $a$. Recall from [Sta21, Tag 0385, Lem. 33.12.2 and 33.12.6] that it is sufficient to check that if $L/K$ is a purely inseparable field extension, then $A_L$ is regular at $a_L$.

Since the extension $\kappa(c)/K$ is separable, we know that $\mathcal{O}_{C,c} \otimes_K L$ is again a discrete valuation ring and the extension $\mathcal{O}_{C,c} \subseteq \mathcal{O}_{C,a} \otimes_K L$ is weakly unramified, in the sense of [Sta21, Tag 0EXQ, Def. 15.104.1]. It then follows from [Sta21, Tag 09ER] that

$$\mathcal{O}_{A,a} \otimes_K L = \mathcal{O}_{A,a} \otimes \mathcal{O}_{C,c} (\mathcal{O}_{C,c} \otimes_K L)$$

is again a discrete valuation ring, which is what we wanted to prove. □

### 5.3. Taking roots out of a section

For the reader’s convenience, we will briefly recall (in our special setting) the construction of cyclic covers by “taking roots out of a section”. We refer the reader to [EV92, Sect. 3.5] for a thorough, more general discussion and to [Vie95, Lem. 2.3] for a summary.

**Construction 5.9** (Taking the $N^{th}$ root out of a divisor). Let $k$ be an algebraically closed field of arbitrary characteristic $p$. Let $X$ be a smooth $k$-variety, equipped with an effective divisor $B = \sum a_i \cdot B_i$ with snc support. Assume that at least one of the numbers $a_i$ equals one. Further, let $N \in \mathbb{N}$ be a number; if $p > 0$ assume that $N$ is prime to $p$. Finally, let $\mathcal{L} \in \text{Pic}(X)$ be any invertible sheaf whose $N^{th}$ tensor power is $\mathcal{L}^\otimes N \cong \mathcal{O}_X(B)$.

Since $B$ is effective, there exists a section $\sigma \in H^0(X, \mathcal{O}_X(B))$ whose zero-divisor is exactly $B$. The dual of $\sigma$ gives a morphism $\sigma^* : (\mathcal{L}^\otimes N) \to \mathcal{O}_X$ and equips the direct sum

$$\mathcal{M} := \bigoplus_{i=0}^{N-1} (\mathcal{L}^\otimes i)^*$$

with the structure of a sheaf of $\mathcal{O}_X$-algebras. We consider the associated space $\text{Spec} \mathcal{M}$ and write $\overline{X}$ for its normalisation. We call the natural morphism $\gamma : \overline{X} \to X$ the covering obtained by taking the $N^{th}$ root out of $B$.

**Remark 5.10** (Alternate description). In more geometric terms, the space $\text{Spec} \mathcal{M}$ of Construction 5.9 can also be described as follows. The total spaces of the relevant sheaves, $|\mathcal{L}|$ and $|\mathcal{L}^\otimes N|$ are closely related: the group of $N^{th}$ unit roots acts on $|\mathcal{L}|$ by homotheties in fibre direction, and the space $|\mathcal{L}^\otimes N|$ is the quotient of $|\mathcal{L}|$ under this action. If $\Sigma \subseteq |\mathcal{L}^\otimes N|$ denotes the graph of the section $\sigma$, then $\text{Spec} \mathcal{M}$ is exactly the preimage of $\Sigma$ under the quotient map.

We summarise the main properties of Construction 5.9 in brief.
**Proposition 5.11** (Properties of the covering construction). Assume the setting of Construction 5.9. Then, the following holds.

(5.11.1) The space $\overline{X}$ is irreducible.

(5.11.2) The morphism $\gamma$ is finite, separable of degree $N$, Galois with cyclic Galois group, and étale away from $\text{supp} B$.

(5.11.3) If $i$ is any index and if $\Gamma \subset \text{supp} \gamma^* B_i$ is any prime divisor, then $\Gamma$ appears in $\gamma^* B_i$ with multiplicity

$$\text{mult}_\Gamma \gamma^* B_i = \frac{N}{\gcd(N, a_i)}.$$

(5.11.4) If $Y \subset X$ is any smooth curve that intersects the snc divisor $B$ transversely, and if $Y$ intersects at least one component of $B$ that has multiplicity one, then $\overline{Y} := \gamma^{-1}(Y)$ is normal and $\gamma|_{\overline{Y}} : \overline{Y} \to Y$ is the covering obtained by taking the $N^{\text{th}}$ root out of $B|_Y$.

**Proof.** Items (5.11.1) is [Vie95, Lem. 2.3.d], using the assumption that at least one of the numbers $a_i$ equals one. Items (5.11.2) and (5.11.3) are [Vie95, Lem. 2.3.c] and e]. For Item (5.11.4), recall from [EV92, Claim 3.8 and 3.12] that the algebra structure of $\mathcal{A}$ extends to an algebra structure on the larger sheaf $\mathcal{B}$, and that $\text{Spec} \mathcal{B} = \sum_{i=0}^{N-1} \left( \mathcal{L}^\otimes i \otimes \mathcal{O}_X \left( \left\lfloor \frac{i}{N} \cdot B \right\rfloor \right) \right)^*$, and that $\text{Spec} \mathcal{B}|_Y$ is, again by [EV92, Claim 3.8 and 3.12], itself normal and in fact the covering obtained by taking the $N^{\text{th}}$ root out of $B|_Y$.

5.4. **Proof of Proposition 5.1.** Assuming Setting 3.1, we will construct the diagram of Proposition 5.1 step-by-step and then prove that the construction has all the desired properties. Recall from Setting 3.1 that $D = \sum_{i=1}^d \frac{m_i - 1}{m_i} : D_i$, where $m_i \in \mathbb{N} \cup \{\infty\}$ denote the $C$-multiplicities of $D$. Write

$$\text{fract} := \{i : m_i < \infty\} \quad \text{and} \quad n := \text{lcm} \{m_i \mid i \in \text{fract}\}.$$

**Step 1: Construction of $\overline{X}$.** For any index $i \in \text{fract}$, set

$$m'_i := \begin{cases} m_i & \text{if } p = 0, \\ \text{prime-to}-p\text{-part of } m_i & \text{otherwise}. \end{cases}$$

Write $N := \text{lcm} \{m'_i \mid i \in \text{fract}\}$. Choose a very ample line bundle $\mathcal{L} \in \text{Pic}(X)$ such that

$$\mathcal{H} := \mathcal{L}^\otimes N \otimes \mathcal{O}_X \left( - \sum_{i \in \text{fract}} \frac{N}{m'_i} \cdot D_i \right) \in \text{Pic}(X)$$
is still very ample, and choose a general element of the linear system, $H \in |\mathcal{H}|$. By Bertini’s theorem, $H$ is an ample prime divisor that is not contained in $\text{supp} \ D$, the morphism $\text{supp} \ H \to S$ is separable and the divisor $H + D$ has snc support. By construction, there exists an isomorphism of line bundle, $\mathcal{L} \otimes \mathcal{N} \cong O_X(B)$, where

$$B := H + \sum_{i \in \text{frac}} \frac{N}{m_i} \cdot D_i \in \text{Div}(X).$$

Finally, let $c : \overline{X} \to X$ be the covering obtained by taking the $N$th root out of $B$, as summarised in Construction 5.9 on page 24. Let $\tilde{X}$ be a desingularisation of $\overline{X}$ that is isomorphic over $\overline{X}_{\text{reg}}$. Desingularisation of this kind exist, because $\overline{X}$ has dimension two, cf. [CJS20, Thm. 1.2] for a convenient reference. Let $\tilde{c} : \tilde{X} \to X$ and $\tilde{\varphi} : \tilde{X} \to S$ be the composed morphisms. We summarise the main properties of our construction.

**Observation 5.12.** The degree of $\tilde{c}$ is given as

$$[\tilde{X} : X] = N = \begin{cases} n & \text{if } p = 0 \text{ or none of the finite } m_i \text{ are multiples of } p, \\ n/p & \text{otherwise.} \end{cases}$$

**Claim 5.13.** There exists a dense open subset of $S$ over which $\tilde{\varphi}$ is smooth.

**Proof of Claim 5.13.** Let $s \in S$ be a general, closed point and consider the scheme-theoretic fibres $X_s, \overline{X}_s$ and $\tilde{X}_s$. We need to show that $\tilde{X}_s$ is smooth. But since the resolution morphism $\tilde{X} \to \overline{X}$ is an isomorphism away from a finite subset of $\overline{X}$, we find that the fibres $\overline{X}_s$ and $\tilde{X}_s$ agree, and it suffices to show that $\overline{X}_s$ is smooth. But then, we have seen in Item (5.11.4) of Proposition 5.11 that the covering $c|_{\overline{X}_s} : \overline{X}_s \to X_s$ is the covering obtained by taking the $N$th root out of $B|_{X_s}$. In particular, $\overline{X}_s$ is an irreducible and normal $k$-curve, hence smooth over $\text{Spec} \ k$ because $k$ is perfect. \hfill \Box (Claim 5.13)

**Claim 5.14.** There exists a dense open subset of $S$ over which $\tilde{\varphi}$ is an snc morphism for the pair $\tilde{(X, \tilde{D} + \tilde{H})}$, where $\tilde{D}$ and $\tilde{H}$ are the strict transforms introduced in Notation 5.5.

**Proof of Claim 5.14.** If $k$ is of characteristic zero, this is clear by generic smoothness. If the characteristic of $k$ is finite, we already know that $\tilde{\varphi}$ is generically smooth. The covering degree $N = \deg \tilde{c}$ is coprime to the characteristic by assumption. It follows that the restriction of $\tilde{c}$ to any component of $\tilde{c}^*(\tilde{D} + \tilde{H})$ is separable, and hence that the restriction of $\tilde{c}$ to any component of $\tilde{D} + \tilde{H}$ is separable. \hfill \Box (Claim 5.14)

**Step 2: Construction of $\tilde{X}$.** Let $\tilde{D}'$ denote the reduced divisor on $\tilde{X}$, given as the strict transforms of those divisors whose $C$-multiplicities $m_i$ are finite multiples of $p$,

$$\tilde{D}' := \sum_{i \in \text{frac}} (1 - \delta_{m_i, m'_i}) \cdot \tilde{D}_i,$$

where

$$\delta_{m_i, m'_i} := \begin{cases} 1 & \text{if } m_i = m'_i \\
0 & \text{if } m_i \neq m'_i. \end{cases}$$
If $\tilde{D}' = 0$, set $\tilde{X} := \tilde{X}$, and let $\tilde{c} : \tilde{X} \to X$ be the identity morphism. Otherwise, write $K := k(S)$ for the function field of $S$. We have seen in Claim 5.13 that the generic fibre $\tilde{X}_\eta$ is then a $K$-smooth, projective $K$-curve. Choose a rational function $f \in K(\tilde{X}_\eta)$ that has poles along $\tilde{D}'_\eta$ of order prime to $p$, and no other poles elsewhere. The existence of such a function is a straightforward consequence of the Riemann–Roch theorem. Construction 5.6 will then give a cover $\tilde{X}_\eta \to \tilde{X}_\eta$.

As before, let $\tilde{\varphi} : \tilde{X} \to S$ and $\tilde{c} : \tilde{X} \to X$ be the composed morphism. We summarise the main properties of our construction.

**Observation 5.15.** By choice of $\tilde{D}'$, the degree of $\tilde{c}$ is given as

$$[\tilde{X} : X] = \begin{cases} 1 & \text{if } p = 0 \text{ or none of the finite } m_i \text{ are multiples of } p, \\ p & \text{otherwise.} \end{cases}$$

**Observation 5.16.** It follows from Claim 5.14 and from Proposition 5.7 that there exists a dense open subset of $S$ over which $\tilde{\varphi}$ is an snc morphism for the pair $(\tilde{X}, D + H)$.

To end Step 2, choose one dense open subset $S^{oo} \subset S$ over which $\tilde{\varphi}$ is an snc morphism and maintain this choice throughout. We follow the notation from the diagram of Proposition 5.1 to denote the preimages of $S^{oo}$ and to denote the restrictions of the relevant morphisms.

**Step 3: End of proof.** We need to check that our construction satisfies all properties spelled out in Proposition 5.1. We go through the list.

**The morphisms $\tilde{\zeta}^{oo}$ and $\tilde{c}^{oo}$ are Galois:** It is clear from construction that $\tilde{\zeta}^{oo}$ and $\tilde{c}^{oo}$ are Galois. But $\tilde{\zeta}^{oo}$ and $\tilde{c}^{oo}$ agree over the open subset $S^{oo}$ where $\tilde{\varphi}$ is smooth and where the resolution morphism is therefore an isomorphism.

**Properties (5.1.1) and (5.1.2):** Follows from Observations 5.12 and 5.15.

**Property (5.1.3):** Follows by choice of $S^{oo}$ and Observation 5.16.

**Property (5.1.4):** Recall from Item (5.11.2) of Proposition 5.11 and from Item (5.7.4) of Proposition 5.7 that the ramification loci relate to the strict transforms of $D$ and $H$ as follows,

$$\text{supp Ramification } \tilde{c}^{oo} \subseteq \tilde{B}^{oo} \subseteq \tilde{D}^{oo} \cup \tilde{H}^{oo}$$

$$\text{supp Ramification } \tilde{c}^{oo} = \text{supp}(\tilde{c}^{oo})^{\ast} \tilde{D}' \subseteq \tilde{D}^{oo}$$

and therefore

$$\text{supp Ramification } c^{oo} \subseteq \tilde{D}^{oo} \cup \tilde{H}^{oo}$$

Property (5.1.4) thus follows from the choice of $S^{oo}$ and Observation 5.16.

**Property (5.1.5):** If $i$ is any index for which $m_i < \infty$, it follows from Item (5.11.3) of Proposition 5.11 that every irreducible component $\tilde{\Gamma}$ of $\tilde{D}_i$...
has multiplicity equal to
\[
\text{mult}_F \tilde{c}^* D_i = \frac{N}{\gcd(N, N/ m'_i)} = \frac{N}{N/ m'_i} = m'_i.
\]

Next, it follows from Item (5.7.4) of Proposition 5.7 that every irreducible component \( \Gamma \) of \( D_i \) has multiplicity equal to
\[
\text{mult}_F \tilde{c}^* \tilde{\Gamma} = \begin{cases} 
1 & \text{if } m_i = m'_i, \\
 p & \text{if } m_i \neq m'_i \text{ (and hence } m_i = m'_i \cdot p). 
\end{cases}
\]

Property (5.1.5) then follows.

**Property (5.1.6):** If \( i \) is any index for which \( m_i = \infty \), it follows from Item (5.11.2) of Proposition 5.11 that every \( \tilde{c} \) is étale over the generic point of \( D_i \). Likewise, if \( \tilde{\Gamma} \) is any irreducible component of the strict transform \( \tilde{D}_i \), then \( \tilde{\Gamma} \) is not contained in \( \tilde{D}' \), and Item (5.7.4) of Proposition 5.7 shows that \( \tilde{c} \) is étale over the generic point of \( \tilde{\Gamma} \). Property (5.1.6) follows.

In summary, we checked that Properties (5.1.1)–(5.1.6) all hold. This finishes the proof of Proposition 5.1. \( \Box \)

6. The sheaf of adapted differentials

6.1. **Construction.** We maintain the setting and assumptions of Proposition 5.1 in this section. Following ideas of Campana, we will consider a sheaf on \( \tilde{X}^{\log} \) called sheaf of adapted differentials, and written as \( \tilde{\Omega}^1_{\tilde{X}^{\log}}(\log \tilde{D}^\log) \). In characteristic zero, this sheaf is introduced and discussed at great length in [CKT16, Sect. 3], though the definition given there looks a little different from ours. On a technical level, the sheaf of adapted differentials is defined as follows.

**Definition 6.1.** Maintaining the setting of Proposition 5.1, and using Notation 5.5 on page 22 for strict transforms, we define the sheaf of adapted differentials as

\[
(6.1.1) \quad \tilde{\Omega}^1_{\tilde{X}^{\log}}(\log \tilde{D}^\log) := \left( \mathcal{F}_{\tilde{D}^{\log}} \otimes (c^{\log})^* \tilde{\Omega}^1_{\tilde{X}}(\log \lceil D \rceil) \right) + (\psi^{\log})^* \Omega_{S^{\log}}^1,
\]

where the sum is the sum of subsheaves in \( (c^{\log})^* \tilde{\Omega}^1_{\tilde{X}}(\log \lceil D \rceil) \). Its dual is called sheaf of adapted tangents and will be denoted by \( \tilde{\mathcal{T}}_{\tilde{X}^{\log}}(\log \tilde{D}^\log) \).

To justify the notation \( \tilde{\Omega}^1_{\tilde{X}^{\log}}(\log \tilde{D}^\log) \) in Definition 6.1, observe that sections in \( (c^{\log})^* \tilde{\Omega}^1_{\tilde{X}}(\log \lceil D \rceil) \) are differential forms with logarithmic poles along the support of \( (c^{\log})^* \lceil D \rceil \). Taking the tensor product with the ideal sheaf \( \mathcal{F}_{\tilde{D}^{\log}} \), however cancels some of these poles, so that sections in the tensor product are differential forms, with logarithmic poles along the support of \( (c^{\log})^* D^\log \) only. This observation will become important in Proposition 6.2 on the next page.
6.2. **Main properties.** The following three propositions summarise the main properties of \( \Omega^1_{X^{\text{red}}} (\log D_{\text{log}}) \) that will be relevant in the sequel. While Proposition 6.2 and 6.3 are rather elementary, the proof of Proposition 6.6 requires some effort and is not nearly as straightforward as one might wish.

**Proposition 6.2 (Containment in \( \Omega^1_{X^{\text{red}}} (\log D_{\text{log}}) \)).** Maintaining the setting of Proposition 5.1 and the notation introduced in this section, \( \Omega^1_{X^{\text{red}}} (\log D_{\text{log}}) \) is a subsheaf of \( \Omega^1_{X^{\text{red}}} (\log D_{\text{log}}) \). More precisely, there exists a commutative diagram of injective sheaf morphisms,

\[
\begin{array}{ccc}
\widehat{\Omega}^1_{X^{\text{red}}} (\log D_{\text{log}}) & \xrightarrow{\iota_1} & (\epsilon^*)^\circ \Omega^1_{X} (\log [D]) \\
& \downarrow & \downarrow \\
\Omega^1_{X^{\text{red}}} (\log D_{\text{log}}) & \xrightarrow{\iota_2} & \Omega_{X^{\text{red}}} (\log (c^*D)_{\text{red}})
\end{array}
\]

where \( \iota_1 \) and \( \iota_2 \) are the obvious inclusions, and where \( \iota_2 \) is the standard pull-back map for logarithmic differential forms.

**Proof.** The restriction of \( \iota_2 \circ \iota_1 \) to each of the two summands in \( (6.1.1) \) factorises via \( \iota_2 \).

**Proposition 6.3 (Quotient by \( (\varphi^*)^\circ \Omega^1_S \)).** Maintaining the setting of Proposition 5.1 and the notation introduced in this section, the sheaf \( \widehat{\Omega}^1_{X^{\text{red}}} (\log D_{\text{log}}) \) contains \( (\varphi^*)^\circ \Omega^1_S \), and the quotient is isomorphic to \( \mathcal{J}_{\text{fact}} (\epsilon^*)^\circ \omega_{X/S} (\log [D]) \). In particular, \( \widehat{\Omega}^1_{X^{\text{red}}} (\log D_{\text{log}}) \) is an extension of two locally frees, hence locally free.

**Proof.** Immediate from \( (6.1.1) \).

**Notation 6.4 (Sequence of relative adapted differentials).** We refer to the quotient sequence

\[
(6.4.1) \quad 0 \to (\varphi^*)^\circ \Omega^1_S \to \widehat{\Omega}^1_{X^{\text{red}}} (\log D_{\text{log}}) \to \mathcal{J}_{\text{fact}} (\epsilon^*)^\circ \omega_{X/S} (\log [D]) \to 0
\]

as the sequence of relative adapted differentials.

**Remark 6.5.** Items \( (5.1.5) \) and \( (5.1.6) \) of Proposition 5.1 immediately imply that the restriction of the quotient to the generic fibre of \( \varphi \) is an invertible sheaf of degree

\[
\deg_{X_{\text{gen}}} \left( \mathcal{J}_{\text{fact}} \otimes (\epsilon^*)^\circ \omega_{X/S} (\log [D]) \right) = [X^\circ : X^\circ] \cdot \deg_{X_{\text{gen}}} (K_X + D).
\]

In order to apply the construction of adapted differentials to the problem of finding geometric height inequalities for \( C \)-integral points, we need to relate the splitting behaviour of Sequence \( (3.3.1) \) to that of \( (6.4.1) \). The following proposition compares the two.

**Proposition 6.6 (Splitting of the sequence of relative adapted differentials).** Maintaining the setting of Proposition 5.1 and the notation introduced in this section, assume that
(6.6.1) the degree $d := \deg_{\chi}(K_X + D)$ is strictly positive, and
(6.6.2) the sequence of relative adapted differentials, Sequence (6.4.1), splits when restricted to the generic fibre $\tilde{X}_\eta$.

Then, Sequence (3.3.1) splits when restricted to the generic fibre $X_\eta$.

Proposition 6.6 is shown in the subsequent Sections 6.3 and 6.4.

6.3. Preparation for the proof of Proposition 6.6. We aim to relate Sequences (6.4.1) and (3.3.1) via equivariant push-forward. We refer the reader to [Gro57, Sect. 5.1] for an overview of elementary facts concerning $G$-sheaves and their $G$-invariant push forwards; see also [GKKP11, App. A] and references therein. The following elementary lemma turns out to be key.

Lemma 6.7. Let $k$ be an algebraically closed field, and let $\lambda : A \to B$ be a finite Galois morphism between smooth $k$-varieties, with Galois group $G$. Let $\Delta_B$ be a reduced divisor on $B$ and consider a $G$-invariant divisor $\Delta_A$ on $A$ with the following properties.

(6.7.1) The divisors $\Delta_A$ and $\lambda^*\Delta_B - \Delta_A$ are effective, so $0 \leq \Delta_A \leq \lambda^*\Delta_B$.
(6.7.2) Everywhere along the support of $\lambda^*\Delta_B$, the divisor $\Delta_A$ is strictly smaller than $\lambda^*\Delta_B$. In other words, $\text{supp}(\lambda^*\Delta_B - \Delta_A) = \text{supp} \lambda^*\Delta_B$.

Equip $\mathcal{O}_A(\Delta_A)$ with the obvious structure of a $G$-subsheaf of $\lambda^*\mathcal{O}_B(\Delta_B)$. If $\mathcal{E}$ is any locally free sheaf of $\mathcal{O}_B$-modules, then the canonical morphism
\[ \mathcal{E} \hookrightarrow \lambda_* (\mathcal{O}_A(\Delta_A) \otimes \lambda^*\mathcal{E})^G \]
is an isomorphism.

Remark 6.8. The morphism $\lambda$ of Lemma 6.7 is Galois, which is to say that $B$ is the quotient variety for the $G$-action on $A$. In particular, regular functions on $G$-invariant open subsets of $A$ come from $B$ if and only if they are $G$-invariant. The natural morphism $\mathcal{O}_B \to \lambda_*(\mathcal{O}_A)^G$ is therefore an isomorphism, and then so are the natural morphisms $\mathcal{F} \to \lambda_*(\lambda^*\mathcal{F})^G$, for all locally free sheaves $\mathcal{F}$ on $B$.

Proof of Lemma 6.7. The problem is local on $B$ and respects direct sums. We may therefore assume without loss of generality that $\mathcal{E} = \mathcal{O}_B$. We have inclusions of $G$-sheaves on $X$:
\[ \lambda^*\mathcal{O}_B \subseteq \mathcal{O}_A(\Delta_A) \subseteq \lambda^*\mathcal{O}_B(\Delta_B). \]

Using the fact that equivariant push-forward is left-exact, [Gro57, p. 197f], we obtain a commutative diagram as follows:
\[ \begin{array}{ccc}
\mathcal{O}_B^G & \xrightarrow{\text{inclusion}} & \mathcal{O}_B(\Delta_B) \\
\downarrow \text{natl. morphism, } n_1 & & \downarrow \text{natl. morphism, } n_1 \\
\lambda_*(\lambda^*\mathcal{O}_B)^G & \xrightarrow{\alpha} & \lambda_*(\mathcal{O}_A(\Delta_A))^G \xrightarrow{\beta} \lambda_*(\lambda^*\mathcal{O}_B(\Delta_B))^G.
\end{array} \]

With the identifications indicated by the vertical arrows, a section in $\lambda_*(\mathcal{O}_A(\Delta_A))^G$ is seen as a rational function $f$ on $B$, satisfying the following properties.
The function \( f \) has at most simple poles along \( \Delta_B \), and no poles elsewhere.

Along any component \( \delta \subset \text{supp} \lambda^* \Delta_B \), the pole order of the pull-back \( \lambda^* f \) is required to satisfy

\[
poleOrder_\delta \lambda^* f \leq \text{mult}_\delta \Delta_A < \text{mult}_\delta \lambda^* \Delta_B.
\]

Such a function is necessarily regular, which shows that the composition \( \alpha \circ n_1 \) is an isomorphism, as required. \( \square \)

### 6.4. Proof of Proposition 6.6

We assume that the sequence of relative adapted differentials, Sequence (6.4.1), splits when restricted to the generic fibre \( \hat{X}_\eta \). Equivalently, there exists an open subset \( S^{\infty}_o \subseteq S^\infty \) such that (6.4.1) splits over the preimage of \( S^{\infty}_o \). To keep the text readable, we assume that \( S^{\infty}_o = S^\infty \), so that there exists a morphism

\[
s : \mathcal{E} \otimes (c^{\infty})^* \omega_{X/S}(\log [D]) \to \hat{\Omega}_X^1 (\log \hat{D}_\log)
\]

that splits Sequence (6.4.1).

**Observation 6.9.** Remark 6.5 and Assumption (6.6.1) guarantee that Image(s) equals the maximal destabilising subsheaf of \( \hat{\Omega}_X^1 (\log \hat{D}_\log) \) on \( \hat{X}_\eta \). It follows that the splitting morphism \( s \) is unique.

#### Step 1: Embedding the sequence of relative adapted differentials

By definition, the sheaf of adapted differentials contains \( \mathcal{E} \otimes (c^{\infty})^* \Omega_X^1(\log [D]) \) and is contained in \( (c^{\infty})^* \Omega_X^1(\log [D]) \). As a consequence, we find that the sequence of relative adapted differentials, whose splitting behaviour needs to be understood, is sandwiched between two sequences that relate to the splitting behaviour of Sequence (3.3.1).

**Claim 6.10** (Embedding the sequence of relative adapted differentials). The sequence of relative adapted differentials fits into a commutative diagram of sheaf morphisms on \( \hat{X}^{\infty} \) with exact rows, as follows:

\[
\begin{array}{cccccc}
(\tilde{\phi}^{\infty})^* \Omega_{\hat{S}}^1 & \longrightarrow & (c^{\infty})^* \Omega_X^1(\log [D]) & \longrightarrow & (c^{\infty})^* \omega_{X/S}(\log [D]) \\
\text{\( \tilde{\alpha} \)} & \downarrow & \text{\( \tilde{\alpha} \)-splitting} & & \\
(\tilde{\phi}^{\infty})^* \Omega_{\hat{S}}^1 & \longrightarrow & \hat{\Omega}_X^1(\log \hat{D}_\log) & \longrightarrow & \hat{D}_\frac{}{\text{fract}} \otimes (c^{\infty})^* \omega_{X/S}(\log [D]) \\
& & & & \\
\mathcal{E} \otimes (\tilde{\phi}^{\infty})^* \Omega_{\hat{S}}^1 & \longrightarrow & \mathcal{E} \otimes (c^{\infty})^* \Omega_X^1(\log [D]) & \longrightarrow & \mathcal{E} \otimes (c^{\infty})^* \omega_{X/S}(\log [D])
\end{array}
\]

All sheaves that appear in the diagram carry natural structures of \( \hat{G} \)-sheaves, and all morphisms are morphisms of \( \hat{G} \)-sheaves.

**Proof of Claim 6.10.** All assertions are clear from the construction, except perhaps the \( \hat{G} \)-equivariance of the splitting morphism \( \tilde{s} \). The equivariance of \( \tilde{s} \) follows from the uniqueness pointed out in Observation 6.9. \( \square \) (Claim 6.10)
**Notation 6.11.** To keep this proof readable, we denote the entries in the diagram of Claim 6.10 by

\[ \begin{array}{c}
\hat{A} & \hat{B} & \hat{C} \\
\hat{D} & \hat{E} & \hat{F} \\
\hat{G} & \hat{H} & \hat{I}
\end{array} \]

**Step 2: Dualisation.** To see how the sequence of relative adapted differentials relates to Sequence (3.3.1), one might be tempted to consider \( \hat{A} \)-invariant push-forward of the diagram from Claim 6.10 at this point. This will, however, not give the sheaves we are interested in. Instead, we need to dualise first.

**Claim 6.12 (Dualisation).** Dualising the diagram of Claim 6.10, we obtain a commutative diagram with exact rows of \( \hat{G} \)-sheaves as follows:

\[
\begin{array}{c}
(\hat{\phi}^\circ)^* \mathcal{K} \leftarrow (\hat{\phi}^\circ)^* \mathcal{K}_{X/S}(-\log[D]) \leftarrow (\hat{\phi}^\circ)^* \mathcal{K}_{X}(-\log[D])
\end{array}
\]

As a subsheaf of \( \hat{H}^* = \left( (\hat{\phi}^\circ)^* \mathcal{K}_{X}(-\log[D]) \right) (\hat{D}_{\text{fract}}^\circ) \),
the sheaf \( \hat{E}^* = \hat{\mathcal{K}}_{X/S}(-\log[D]) (\hat{D}_{\text{fract}}^\circ) \)
is described as

\[ (6.12.1) \quad \hat{E}^* = \ker \left( \hat{H}^* \to \hat{\mathcal{G}}^*/\hat{D}^* \right). \]

**Proof of Claim 6.12.** Only the last line of the claim needs to be shown. To begin, it follows directly from the definition of the sheaf of adapted differentials in (6.1.1) that we have an exact sequence of locally free sheaves

\[ 0 \to \hat{G} \to \hat{D}^* \to \hat{E} \to 0. \]

Dualising, this give a commutative diagram with exact rows,

\[
\begin{array}{c}
0 \to 0 \to \hat{D}^* \xrightarrow{\text{Id}} \hat{D}^* \to 0 \\
0 \to \hat{E}^* \xrightarrow{\hat{\beta}^\circ \oplus (-\hat{\varphi}^\circ)} \hat{D}^* \oplus \hat{H}^* \xrightarrow{\hat{\beta}^\circ \pi_1 - \hat{\varphi}^\circ \pi_2} \hat{G}^* \to 0.
\end{array}
\]

and the assertion then follows from the snake lemma. \( \square \) (Claim 6.12)
Step 3: $\widehat{G}$-invariant push-forward. We consider the $\widehat{G}$-invariant push-forward of the diagram found above. In other words, we apply the (left-exact) functor $(\mathcal{C}^\sigma)_* \circ (\cdot)^{\widehat{G}}$ to all sheaves and sheaf morphisms involved. For convenience of notation, write
\[ \mathcal{F}_{X^\circ}(-\log \bar{D}^\circ_{\log}) := (\mathcal{C}^\sigma)_* \left( \mathcal{F}_{X^\circ}(-\log \bar{D}^\circ_{\log}) \right)^{\widehat{G}}. \]

Claim 6.13 ($\widehat{G}$-invariant push-forward). The $\widehat{G}$-invariant push-forward of the diagram in Claim 6.12 is a commutative diagram with exact rows that reads as follows:

\[
\begin{array}{cccccc}
(\mathcal{C}^\sigma)_* \mathcal{F}_S & \rightarrow & (\mathcal{C}^\sigma)_* \mathcal{F}_X(-\log[D]) & \rightarrow & (\mathcal{C}^\sigma)_* \mathcal{F}_{X/S}(-\log[D]) \\
\end{array}
\]

Claim 6.14 ($\widehat{G}$-sheaves in the diagram of Claim 6.13). All sheaves that appear in the diagram of Claim 6.13 carry natural structures of $\widehat{G}$-sheaves, and all morphisms except possibly $\tilde{s}$ are morphisms of $\widehat{G}$-sheaves.

Proof of Claim 6.14. Exception for $\mathcal{F}_{X^\circ}(-\log \bar{D}^\circ_{\log})$ it is clear that all sheaves in the diagram of Claim 6.13 are $\widehat{G}$-sheaves, and exception for the morphisms pointing to/from $\mathcal{F}_{X^\circ}(-\log \bar{D}^\circ_{\log})$, all morphisms are morphisms of $\widehat{G}$-sheaves. To prove the assertion, it will therefore suffice to show that the sheaf $\mathcal{F}_{X^\circ}(-\log \bar{D}^\circ_{\log})$ is stable under the action of $\widehat{G}$, as a subsheaf of the $\widehat{G}$-sheaf $((\mathcal{C}^\sigma)_* \mathcal{F}_X(-\log[D]))(\bar{D}^\circ_{\text{tame}})$. 

Proof of Claim 6.13. The identification of the sheaves in the top row is clear by Remark 6.8. As for the identification of the other sheaves, observe that
\[
\left[ (\mathcal{C}^\sigma)_* \mathcal{F}_X(-\log[D]) \right](\bar{D}^\circ_{\text{fract}}) = \left[ (\mathcal{C}^\sigma)_* \left[ \left( (\mathcal{C}^\sigma)_* \mathcal{F}_X(-\log[D]) \right)(\bar{D}^\circ_{\text{tame}}) \right] \right](\bar{D}^\circ_{\text{wild}}).
\]

The identification
\[
(\mathcal{C}^\sigma)_* \left[ \left( (\mathcal{C}^\sigma)_* \mathcal{F}_X(-\log[D]) \right)(\bar{D}^\circ_{\text{fract}}) \right]^{\widehat{G}} = (\mathcal{C}^\sigma)_* \left[ \left( (\mathcal{C}^\sigma)_* \mathcal{F}_X(-\log[D]) \right)(\bar{D}^\circ_{\text{tame}}) \right]^{\widehat{G}}
\]
is thus an immediate consequence of Lemma 6.7; ditto for the identifications of all the other sheaves. It remains to prove surjectivity of the horizontal arrows towards the left column. For the top arrow, this is clear. Since the bottom row equals the top row tensored with the locally free $\mathcal{O}_X(\bar{D}^\circ_{\text{tame}})$, surjectivity is also clear for the horizontal arrow in the bottom row. Surjectivity of the middle arrow follows from the commutativity of the upper left square. \qed
However, using that the fact that the $\tilde{G}$-invariant push-forward functor $(\tilde{c}^{\circ})_* (\bullet)^{\tilde{G}}$ is left-exact, we know from Equation (6.12.1) of Claim 6.12 that

$$(\tilde{c}^{\circ})_* (\tilde{E})^{\tilde{G}} = \ker ((\tilde{c}^{\circ})_* (\tilde{H})^{G}) \to (\tilde{c}^{\circ})_* ((\tilde{G})^{G}) \bigg/ (\tilde{c}^{\circ})_* ((\tilde{D})^{G})^{\tilde{G}}$$

The desired stability under the action of $\tilde{G}$ follows from the observation that all morphisms to the right of the equality sign are morphisms of $\tilde{G}$-sheaves.

\textbf{Step 4: $\tilde{G}$-invariant push-forward.} In our discussion of the $\tilde{G}$-invariant push-forward we used the fact that the splitting $\tilde{s}$ in the diagram of Claim 6.10 was unique for numerical reasons, hence $\tilde{G}$-invariant. These arguments do not necessarily apply to the splitting $\hat{s}$, and we do not see why it should be $\hat{G}$-invariant in general. Using the special situation at hand, we can however always find another $\hat{G}$-invariant splitting.

\textbf{Claim 6.15 ($\hat{G}$-invariant splitting).} There exists a $\hat{G}$-invariant morphism

$$\hat{s}': \mathcal{F}_{X^{\circ}} (- \log \hat{D}_{\log}^{\circ}) \to \left( (\tilde{c}^{\circ})_* \mathcal{F}_{X/S} (- \log [D]) \right) (\hat{D}_{\lame}^{\circ})$$

that splits the middle row in the diagram of Claim 6.13.

\textbf{Proof of Claim 6.15.} The order of the group $\hat{G}$ equals the degree $[X^{\circ} : X^{\circ}]$ and is therefore coprime to the characteristic. Setting $\hat{s}' := \frac{1}{\#G} \cdot \sum_{g \in \hat{G}} g^* \hat{s}$ therefore yields the desired $\hat{G}$-invariant splitting. □ (Claim 6.15)

\textbf{Notation 6.16.} Again we abuse notation slightly and assume without loss of generality that $\hat{s} = \hat{s}'$, so that all morphisms in Diagram 6.13 are in fact morphisms of $\hat{G}$-sheaves.

\textbf{Claim 6.17 ($\hat{G}$-invariant splitting).} The $\hat{G}$-invariant push-forward of the diagram in Claim 6.13 is a commutative diagram with exact rows that reads as follows:

$$\begin{align*}
\mathcal{F}_{\hat{S}^{\circ}} & \longrightarrow \mathcal{F}_{X^{\circ}} (- \log [D]) \longrightarrow \mathcal{F}_{X^{\circ}/S} (- \log [D]) \\
\mathcal{F}_{\hat{S}^{\circ}} & \longrightarrow (\tilde{c}^{\circ})_* \left( \mathcal{F}_{X^{\circ}} (- \log \hat{D}_{\log}^{\circ}) \right)^{\tilde{G}} \longrightarrow \mathcal{F}_{X^{\circ}/S} (- \log [D]) \\
\mathcal{F}_{\hat{S}^{\circ}} & \longrightarrow \mathcal{F}_{X^{\circ}} (- \log [D]) \longrightarrow \mathcal{F}_{X^{\circ}/S} (- \log [D]).
\end{align*}$$

\textbf{Proof of Claim 6.17.} The identifications of the sheaves follow again from Lemma 6.7. Surjectivity of the leftmost horizontal arrows follows as in the proof of Claim 6.13. □ (Claim 6.17)

\textbf{Step 5: End of proof.} Observing that the top and bottom rows in the diagram of Claim 6.17 agree with Sequence (3.3.1), we found the desired splitting. Proposition 6.6 is thus shown. □
7. Geometric height bounds — proof of Theorem 3.8

We work in the setting of Theorem 3.8 and assume that a number \( \epsilon \in \mathbb{Q}^+ \) is given. We apply Proposition 5.1 and use the notation introduced there as well as in Sections 5.1, 6.1 and 6.2. In particular, we consider the sequence of relative adapted differentials

\[
\begin{align*}
(7.0.1) \quad 0 &\to (\tilde{\phi}^{\infty})^* \Omega_S^1 \to \tilde{\Omega}_{X^{\infty}}^1 (\log \tilde{D}_\log^{\infty}) \to \mathcal{J} \tilde{D}_\text{fract}^\infty \otimes (c^{\infty})^* \omega_X/S (\log [D]) \to 0.
\end{align*}
\]

**Observation 7.1 (Splitting of Sequence 7.0.1).** Using Assumptions (3.7.1) and (3.7.2), Proposition 6.6 implies that Sequence (7.0.1) does not split when restricted to \( \tilde{X}_\eta \).

**Step 1, extension of sheaves to \( \tilde{X} \).** We aim to apply Theorem 4.2 to the surface \( \tilde{X} \). To this end, we need to extend \( \tilde{\Omega}_{X^{\infty}}^1 (\log \tilde{D}_\log^{\infty}) \) to a sheaf \( \mathcal{A} \) that is defined on all of \( \tilde{X} \).

**Construction 7.2.** Recalling from Proposition 6.2 that there exists a natural diagram of inclusions,

\[
\begin{array}{ccc}
\tilde{\Omega}_{X^{\infty}}^1 (\log \tilde{D}_\log^{\infty}) & \xrightarrow{t_0} & (c^{\infty})^* \Omega_X^1 (\log [D]) \\
& \overset{t_2}{\longrightarrow} & \Omega_X^1 (\log (c^* D)_{\text{red}}) \\
& \overset{t_1}{\longrightarrow} & \tilde{\Omega}_{X^{\infty}}^1 (\log \tilde{D}_\log^{\infty}).
\end{array}
\]

we consider the extended morphisms

\[
\begin{align*}
t_2 : \quad & c^* \Omega_X^1 (\log [D]) \to \Omega_X^1 (\log (c^* D)_{\text{red}}) \\
t_3 : \quad & \Omega_X^1 (\log \tilde{D}_\log^{\infty}) \to \Omega_X^1 (\log (c^* D)_{\text{red}})
\end{align*}
\]

and let

\[
\mathcal{A} \subseteq \text{Image}(t_2) \cap \text{Image}(t_3) \subseteq \Omega_X^1 (\log (c^* D)_{\text{red}})
\]

be the largest subsheaf whose restriction to \( \tilde{X}^{\infty} \) agrees with \( \text{Image}(t_2 \circ t_3^{\infty}) \).

**Claim 7.3.** The sheaf \( \mathcal{A} \) is locally free and contains \( \tilde{\phi}^* \Omega_S^1 \). Writing \( \mathcal{B} := (\mathcal{A} \bigg/ \tilde{\phi}^* \Omega_S^1)^{**} \), we have

\[
(7.3.1) \quad \deg_{\tilde{X}_\eta} \mathcal{B} = [\tilde{X} : X] \cdot \deg_{X_\eta} (K_X + D).
\]

**Proof of Claim 7.3.** Both claims about \( \mathcal{A} \) are consequences of its definition as “the largest subsheaf ...”. First, it follows that \( \mathcal{A} \) is reflexive, and hence locally free since \( \tilde{X} \) is smooth of dimension two, [Har80, Cor. 1.4]. Second, the inclusion \((\tilde{\phi}^{\infty})^* \Omega_S^1 \subseteq \tilde{\Omega}_{X^{\infty}}^1 (\log \tilde{D}_\log^{\infty})\) implies that \( \mathcal{A} \) must contain \( \tilde{\phi}^* \Omega_S^1 \). The description of \( \mathcal{B} \) follows from Proposition 6.3 and Remark 6.5. \( \Box \) (Claim 7.3)

Claim 7.3 yields a complex of sheaves, \( \tilde{\phi}^* \Omega_S^1 \to \mathcal{A} \to \mathcal{B} \) that agrees over \( \tilde{X}^{\infty} \) with Sequence (7.0.1), the sequence of relative adapted differentials. The description of \( \mathcal{B} \) given in Claim 7.3 has two immediate consequences which we note for future reference.
Consequence 7.4. We have inequalities of height functions for \( \widehat{X}/S \), applicable to all \( C \)-integral points \( \gamma : \widehat{T} \to \widehat{X} \), namely
\[
h_{\mathcal{B}}(\bullet) - \text{const} \leq h_{c^*(K_{X/S} + D)}(\bullet) \leq h_{\mathcal{B}}(\bullet) + \text{const}.
\]
In other words, \( h_{c^*(K_{X/S} + D)} = h_{\mathcal{B}} + O(1) \).

Proof of Consequence 7.4. Properties (5.1.5) and (5.1.6) from Proposition 5.1 imply the equality of divisors
\[
(c^{\infty})^*[D] - \widehat{D}^{\infty}_{\text{fract}} = (c^{\infty})^*D.
\]
The restriction of \( \mathcal{B} \) to \( \widehat{X}^{\infty} \) is therefore isomorphic to \( \mathcal{O}_{\widehat{X}^{\infty}}(c^*(K_{X/S} + D)) \). Standard arguments, cf. [Kim97, Sect. 2], will then give the relation between the two height functions. \( \square \)

Consequence 7.5. We can express the number \( d' \), which appears in the formulation of Theorem 3.8 as
\[
d' \overset{(5.1.1)}{=} [\widehat{X} : X] \cdot \deg_{\mathcal{X}_q}(K_{X/S} + D) = \deg_{\mathcal{X}_q}(\mathcal{B}) > 0.
\]

The positivity of \( \deg_{\mathcal{X}_q}(\mathcal{B}) \) and the non-splitting of the sequence of relative adapted differentials pointed out in Observation 7.1 allow us to apply Theorem 4.2 to the pair \( (\widehat{X}, \widehat{D}_{\log}) \) over \( S \). In a nutshell, the following height inequality holds for all \( \widehat{\gamma} : \widehat{T} \to \widehat{X} \) that are \( C \)-integral points for the pair \( (\widehat{X}, \widehat{D}_{\log}) \),
\[
(7.6.1) \quad h_{\mathcal{B}}(\widehat{\gamma}) \overset{\text{Thm. 4.2}}{\leq} \max \left\{ \deg_{\mathcal{X}_q}(\mathcal{B}), 2 + \varepsilon \right\} \cdot \frac{\deg_{\widehat{T}}\text{Image } d_{\mathcal{Y},\mathcal{A}}}{[\widehat{T} : S]} + O\left(\sqrt{h_{\mathcal{B}}(\widehat{\gamma})}\right),
\]
where, as before, \( d_{\mathcal{Y},\mathcal{A}} \) is the composed map
\[
\widehat{\gamma}^*: \mathcal{O}^1_{\widehat{X}}(\log \widehat{D}_{\log}) \to \omega_{\widehat{T}}(\log(\widehat{\gamma}^* \widehat{D}_{\log})_{\text{red}}).
\]

Step 2, orbifold integral points on \( X \) and algebraic points on \( \widehat{X}/S \). Given a \( C \)-integral point on \( X \), we aim to bound its height using (7.6.1), by considering its preimage in \( \widehat{X} \). The following notation will be used.

Setting and Notation 7.7. Given a \( C \)-integral point \( \gamma : T \to \Sigma_T \subset X \) of the pair \( (X, D) \), we consider the preimage \( c^{-1}\Sigma_T \subset \widehat{X} \) and choose a component \( \Sigma_{\widehat{T}} \subset c^{-1}\Sigma_T \) that dominates \( \Sigma_T \). Denoting its normalisation by \( \widehat{\gamma} : \widehat{T} \to \Sigma_{\widehat{T}} \), we obtain a commutative diagram as follows:
\[
(7.7.1)
\]
Since \( \Sigma_{\widehat{T}} \) is not contained in \( \text{supp} \hat{D} \) by assumption, we may consider the map \( d_{\mathcal{Y},\mathcal{A}} \) defined above.
Remark 7.8 (The morphism $\alpha$ is nearly always separable). Recall the assumption that the morphism $\gamma$ of Setting 7.7 is birational onto its image and hence generically étale onto its image. By base change, [Mil80, Prop. 3.3.c], the same will hold for $\tilde{\gamma}$. The morphism $c$ is separable and generically étale as well. Unless $\text{img } \gamma$ is contained in the closed set of $X$ over which $c$ fails to be étale, the composition $c \circ \tilde{\gamma}$ thus is generically étale onto its image, [Mil80, Prop. 3.3.b], and then so is $\alpha$, [Mil80, Cor. 3.6]. In particular, there are (up to reparametrisation) at most finitely many $C$-integral points $y$ whose associated morphisms $\alpha : \hat{T} \rightarrow T$ are inseparable.

We aim to understand the image of the map $d\alpha : \alpha^* \omega_T \rightarrow \omega_{\hat{T}}$ and to compare it to other sheaves of differentials on $\hat{T}$. The following computation will be key.

Computation 7.9 (Local computation). In Setting 7.7, assume that the morphism $\alpha$ is separable and let $p_T \in \hat{T}$ be any given closed point, with image $p_T := \alpha(p_T) \in T$. Choose uniformising parameters $\tilde{t} \in \mathcal{O}_{\hat{T}, p_T}$ and $t \in \mathcal{O}_{T, p_T}$ and write $a := \text{ord}_{p_T} \alpha^* t$, so that $\alpha^* t = \tilde{t}^a \cdot \tilde{u}$, where $\tilde{u} \in \mathcal{O}_{\hat{T}, p_T}$ is a unit. Then

$$d\alpha(dt) = a \cdot \tilde{u} \cdot \tilde{t}^{a-1} d\tilde{t} + \tilde{t}^a du$$

and

$$d\alpha(d \log t) = a \cdot d \log \tilde{t} + d \log \tilde{u}.$$

Since $\alpha$ is separable by assumption, both $d\alpha(dt)$ and $d\alpha(d \log t)$ are non-zero. In particular, either $p$ does not divide $a$ or $d\tilde{u} \neq 0$. In any case, observing that $d\tilde{u}$ and $d \log \tilde{u}$ either vanish simultaneously or differ only by a unit, we find that $\tilde{t}^a \cdot d\alpha(d \log t)$ is a section in $\text{Image}(d\alpha)$, and more generally that

$$(7.9.1) \quad \mathcal{J}_{p_T}^a \cdot \text{Image}(d\alpha_{\log} : \alpha^* \omega_T(\log p_T) \rightarrow \omega_{\hat{T}}(\log p_T))$$

$$\subseteq \text{Image}(d\alpha : \alpha^* \omega_T \rightarrow \omega_{\hat{T}}).$$

Observe that the number $a$ is bounded from above by the degree of $c$, that is, $a \leq [\hat{X} : X]$.

Claim 7.10. In Setting 7.7, assume that $\alpha$ is separable. Then, the following sequence of inclusions holds over $S^{\infty}$,

$$(7.10.1) \quad \text{Image } \tilde{d}\gamma_* \subseteq \text{Image}(d\alpha : \alpha^* \omega_T \rightarrow \omega_{\hat{T}}) \subseteq \omega_{\hat{T}}.$$ 

Proof of Claim 7.10. We will prove Inequality (7.10.1) locally, in the neighbourhood of any given closed point $p_{\hat{T}} \in \hat{T}$ lying over $S^{\infty}$. We use the notation introduced in Computation 7.9 and write

$$p_{\hat{X}} := \gamma(p_{\hat{T}}), \quad p_X := \gamma(p_T).$$

The assumption that $\gamma$ is a $C$-integral point immediately implies that $p_{\hat{X}}$ is not contained in the support of $\hat{D}_{\log}$. In particular, we see that $\text{Image } \tilde{d}\gamma_* \subseteq \omega_{\hat{T}}$ near $p_{\hat{T}}$. If $p_X$ is not contained in $\text{supp } D$, Inclusion (7.10.1) follows easily: by definition of $\hat{\Omega}_{X^{\infty}}^1(\log \hat{D}_{\log}^{\infty})$, one sees that $\omega_* = c^* \Omega_X^1 \subseteq \Omega_{\hat{X}}^1$ near $p_{\hat{X}}$. Inclusion (7.10.1) will then follow immediately from the chain rule for taking derivatives.

For the remainder of the proof, we consider the case where $p_X$ is contained in $\text{supp } D$. The assumption that $\phi^{\infty}$ is snc for the pair $(X^{\infty}, D^{\infty})$ implies that $p_X$ is
contained in a unique component of \( D \), say \( D_i \subseteq D \); the coefficient \( m_i \) is then finite. Choose local systems of parameters as follows.

- Choose parameters \( x, y \in \mathcal{O}_X, p_x \) so that \( D_i = \{ y = 0 \} \).
- Choose parameters \( \tilde{x}, \tilde{y} \in \mathcal{O}_{\tilde{X}, p_{\tilde{x}}} \) so that \( \tilde{x} = c^*(x) \) and \( \tilde{y}^m = (\text{unit}) \cdot c^*(y) \).

Near \( p_{\tilde{T}} \), the sheaf \( \mathcal{A} \) equals the sheaf \( \tilde{\Omega}_X^1(\log \tilde{D}_{\log}^0) \) of adapted differentials. Recall from Definition 6.1 that \( \mathcal{A} \) is a subsheaf of \( c^*\Omega_X^1(\log[D]) \), and can be generated as follows,

\[
\mathcal{A}_{p_{\tilde{T}}} = \langle c^*dx, \tilde{y} \cdot c^*d\log y \rangle_{p_{\tilde{T}}} \subseteq \langle c^*\Omega_X^1(\log[D]) \rangle_{p_{\tilde{T}}}.
\]

To prove Inclusion (7.10.1), it will therefore suffice to show that the following two logarithmic forms, which are \textit{a priori} sections in \( \omega_{\tilde{T}}(\log((\gamma \circ \alpha)^* [D])_{\text{red}}) \), are in fact contained in the smaller sheaf image (\( da \)):

\[
\begin{align*}
(d\tilde{y} \circ dc)(dx) &= da(dy^*x) \quad \text{and} \\
\tilde{d}(\tilde{y} \cdot dc(d\log y)) &= \tilde{y}^* \tilde{y} \cdot da(d\log y^*y).
\end{align*}
\]

(7.10.2)

Since there is nothing to show for \( da(dy^*x) \), we concentrate on the second form. There, the desired inclusion will follow from (7.9.1) once we show that \( \text{ord}_{p_{\tilde{T}}} \tilde{y}^* \tilde{y} \geq a \). But then,

\[
\begin{align*}
\text{ord}_{p_{\tilde{T}}} \tilde{y}^* \tilde{y} &= m_i^{-1} \cdot \text{ord}_{p_{\tilde{T}}} \tilde{y}^* c^*y \quad \text{Choice of } \tilde{y} \\
&= m_i^{-1} \cdot \text{ord}_{p_{\tilde{T}}} \tilde{y}^* y^* y \quad \text{Diagram (7.7.1)} \\
&= m_i^{-1} \cdot a \cdot \text{ord}_{p_{\tilde{T}}} (\tilde{y}^* y)
\end{align*}
\]

and the claim follows once we recall that \( \gamma \) is a \( C \)-integral point, which implies in particular that \( \text{ord}_{p_{\tilde{T}}} \gamma^* y \geq m_i \). In summary, we have seen that the rational forms mentioned in (7.10.2) are both contained in Image (\( da \)), which finishes the proof.

\[\square\] (Claim 7.10)

Claim 7.11. In Setting 7.7, there exists a number const \( \in \mathbb{N} \) such that the following inequality holds for every \( C \)-integral point \( \gamma : \tilde{T} \to \Sigma_T \) and for every choice of a preimage component \( \Sigma_{\tilde{T}} \subset c^{-1} \Sigma_T \) that dominates \( \Sigma_T \):

\[
\frac{\deg_{\tilde{T}} \text{Image } d \tilde{y}_\alpha}{[\tilde{T} : S]} \leq \frac{\deg_{\tilde{T}} \text{Image } da}{[\tilde{T} : S]} + \text{const} = \delta(\gamma) + \text{const}.
\]

(7.11.1)

Proof of Claim 7.11. Recalling from Remark 7.8 that the morphism \( \alpha \) is nearly always separable, it suffices to consider the separable case only. The equality in (7.11.1) follows by a simple computation, using that \( \alpha \) is separable, so \( \deg_{\tilde{T}} \text{Image } da = \deg_{\tilde{T}} \alpha^* \omega_T \).

The inequality in (7.11.1) would clearly follow if the Inclusion (7.10.1) would hold for all points of \( \tilde{T} \). This may, however, not always be the case. If \( p_{\tilde{T}} \in \tilde{T} \) is a closed point where the Inclusion (7.10.1) fails, then we know from Claim 7.10 that \( p_{\tilde{T}} \) cannot lie over \( S^{oo} \). In particular, there are at most \([\tilde{T} : S] \cdot \#(S \setminus S^{oo}) \) such
points in \( \widehat{T} \). How badly can Inequality (7.10.1) fail at the point \( p_{\widehat{T}} \)? By definition, we know that \( \mathcal{A} \subseteq \Omega^1_X(\log(c^*D)_{\text{red}}) \), which gives us an inclusion
\[
\text{Image } d\widetilde{\gamma}_{|\mathcal{A}} \subseteq \text{Image} \left( \frac{d(\gamma \circ \alpha) : c^*\Omega^1_X(\log[D])}{\omega_{\widehat{T}}(\log((\gamma \circ \alpha)^*D)_{\text{red}})} \right).
\]

Locally near \( p_{\widehat{T}} \), Computation 7.9 shows
\[
\mathcal{J}_{p_{\widehat{T}}} \cdot \text{Image } d\widetilde{\gamma}_{|\mathcal{A}} \subseteq \mathcal{J}_{p_{\widehat{T}}} \cdot A \subseteq \text{Image} (d\alpha : \alpha^*\omega_T \to \omega_{\widehat{T}}).
\]

Recalling that \( a \leq [\widehat{X} : X] \), we can therefore finish the proof by setting
\[
\text{const} := [\widehat{X} : X] \cdot \#(S \setminus S^\infty).
\]

\( \square \) (Claim 7.11)

**Step 3, end of proof.** Combining the results obtained so far, we find that the following sequence of inequalities holds for all orbifold integral points \( \gamma \) in \( X \), and all choices of preimage components. As before, we use the notation introduced in 7.7 above.

\[
\begin{align*}
    h(\gamma) &= h_{c^*(K_X + S^*D)}(\gamma) \leq h_{\mathcal{F}}(\gamma) + \text{const} \quad \text{Consequence 7.4} \\
    &\leq \max\{d', 2 + \varepsilon\} \cdot \frac{\deg \text{Image } d\widetilde{\gamma}_{|\mathcal{A}}}{[\widehat{T} : S]} + O\left(\sqrt{h_{\mathcal{F}}(\gamma)}\right) \quad \text{Inequality (7.6.1)} \\
    &\leq \max\{d', 2 + \varepsilon\} \cdot \delta(\gamma) + O\left(\sqrt{h_{\mathcal{F}}(\gamma)}\right) \quad \text{Claim 7.11} \\
    &= \max\{d', 2 + \varepsilon\} \cdot \delta(\gamma) + O\left(\sqrt{h(\gamma)}\right) \quad \text{Consequence 7.4}
\end{align*}
\]

This ends the proof of Theorem 3.8. \( \square \)

### 7.1. Notes on the proof.

#### 7.1.1. Improved height bounds in characteristic zero.
If \( \text{char}(k) = 0 \), recall from Section 4.4 that the height bound of Theorem 4.2 can be improved. In fact, replacing (7.6.1) by the improved bound (4.23.2) when we apply Theorem 4.2 in Step 4 of our proof, we obtain the improved result claimed in Section 3.2.3 above.

#### 7.1.2. Sharpness of Theorem 3.8 and the Kodaira-Spencer map.
As promised on Page 11 at the end of Section 3.2.5, we add a few words concerning the sharpness of the bound given by Theorem 3.8. The bounds obtained in [Kim97, Thm. 2] and [KTV00, Claim 2.2], rely on a control of the degree of the restriction of the foliation \( \mathcal{G} \) to \( \widehat{X}_\eta \), appearing in the proof of Proposition 4.12. More specifically, better bounds for \( \deg_{X_\eta} \mathcal{G} \) allow us to extract improved height bounds from Inequality (4.12.1). We do not know if one can replace the number
\[
d' = \deg_{X_\eta}(K_X + D) \cdot \text{lcm} \{m_i | m_i \neq \infty\} = \deg_{X_\eta}(K_X + D) \cdot \deg(c)
\]
by the smaller number \( d = \deg_{X_\eta}(K_X + D) \). For that, one would need to show that \( \deg_{X_\eta} G \geq [\tilde{X} : X] \) when \( G \) is the foliation on \( \tilde{X} \) tangent to infinitely many degenerate \( \tilde{X}/S \) algebraic points. We could not find a way to prove this stronger inequality, without imposing further conditions on the logarithmic Kodaira-Spencer map of the pair \((X, [D])\). 

8. Geometric height bounds — proof of Theorem 3.7

We aim to apply Theorem 3.8 to a pair \((X, D')\), where the divisor \(D'\) is obtained from \(D\) by lowering the coefficients. To be precise, we set

\[
D' := \sum_{i \in \text{fract}} \frac{m_i' - 1}{m_i'} \cdot D_i + D_{\log},
\]

where

\[
m_i' := \begin{cases} m_i & \text{if } p^2 \nmid m_i \\ 3 & \text{if } p = 2 \text{ and } p^2|m_i \\ \text{largest prime factor of } m_i & \text{otherwise.} \end{cases}
\]

We assume in Item (3.7.2) of Theorem 3.7 that Sequence (3.3.1) does not split when restricted to the generic fibre \(X_\eta\). Recalling that the sequence always splits if \(X_\eta = \mathbb{P}^1\) and \(D\) contains less than four points, the assumptions of Theorem 3.7 imply, by means of elementary computations, that the pair \((X, D')\) satisfies each of the assertions below.

- (8.0.1) The pair \((X, D')\) is a \(C\)-pair. We have \(D' \leq D\).
- (8.0.2) None of the numbers \((m_i)_{i \in \text{fract}}\) is a multiple of \(p^2\).
- (8.0.3) We have \(\deg_{X_\eta}(K_X + D') \geq \frac{1}{6}\).
- (8.0.4) We have \(\text{supp } D' = \text{supp } D\). The analogue of Sequence (3.3.1) for the pair \((X, D')\) does not split when restricted to the generic fibre \(X_\eta\).
- (8.0.5) We have \(\lcm\{m_i' | i \in \text{fract}\} \leq 3 \cdot \lcm\{m_i | i \in \text{fract}\}\).

Item (8.0.1) guarantees that any \(C\)-integral point of the pair \((X, D)\) is automatically \(C\)-integral for the pair \((X, D')\). Items (8.0.2) — (8.0.4) guarantee that the pair \((X, D')\) satisfies all assumptions made in Theorem 3.8. Now, given any number \(\epsilon > 1\) and choosing

\[
d := \deg_{X_\eta}(K_X + D) \quad d_{K_X + D'} := \deg_{X_\eta}(K_X + D')
\]

\[
d' := d \cdot \lcm\{m_i | i \in \text{fract}\} \quad d'_{K_X + D'} := d_{K_X + D'} \cdot \lcm\{m_i' | i \in \text{fract}\},
\]

and applying Néron’s theorem, [Ser89, Thm. 2.11], we obtain the following

\[
d'_{K_X + D'} \leq d \cdot \lcm\{m_i' | i \in \text{fract}\} \leq 3 \cdot d' \quad (8.0.1) \text{ and } (8.0.5)
\]
and
\[ h_{K_x+D}(y) \leq \frac{d}{d_{K_x+D'}} h_{K_x+D'}(y) + O \left( \sqrt{h_{K_x+D}(y)} \right) \quad \text{Néron’s theorem} \]
\[ \leq 6d \cdot h_{K_x+D}(y) + O \left( \sqrt{h_{K_x+D}(y)} \right) \quad \text{Item (8.0.3)} \]
\[ \leq 6d \cdot \max\{d_{K_x+D'}, 2 + \epsilon\} \cdot \delta(y) + O \left( \sqrt{h_{K_x+D}(y)} \right) \quad \text{Theorem 3.8 for } (X, D') \]
\[ \leq 6d \cdot \max\{3d', 2 + \epsilon\} \cdot \delta(y) + O \left( \sqrt{h_{K_x+D}(y)} \right) \]

Theorem 3.7 is thus shown. \( \square \)

9. RIGIDITY THEOREM FOR \( C \)-INTEGRAL POINTS — PROOF OF THEOREM 3.10

Our proof of Theorem 3.10 depends on the characteristic. In contrast to the proofs of Theorems 3.7 and 3.8, where inseparability is the main source of difficulties, here inseparability greatly simplifies the argument. No matter what the characteristic, we argue by contradiction and assume that there exists a smooth, projective \( k \)-curve \( T \) over \( S \), a smooth, quasi-projective \( k \)-curve \( H' \) and a non-constant morphism \( \gamma' : T \times H' \to X \) such that the induced morphisms \( \gamma' : T \to X \) are \( S \)-morphisms and \( C \)-integral points over \( S \), for all \( h \in H'(k) \).

**Remark 9.1.** Since \( X \) is a surface and since the morphisms \( \gamma'_h \) are \( S \)-morphisms, the assumption “non-constant” immediately implies that \( \gamma' \) is dominant.

**Notation 9.2.** Take \( H \) to be the unique compactification of \( H' \) to a smooth, projective \( k \)-curve. The morphism \( \gamma' \) extends to a rational map \( \gamma : T \times H \dasharrow X \), whose set of fundamental points is finite. The following diagrams summarise the situation.

\[
\begin{array}{c}
\xymatrix{
T \ar[r]^-{\gamma'} & X \\
T \ar[u]^{\pi_1} \ar[r]_{\gamma} & X \\
S \ar[u]_{\pi_2} \ar[ur]_{\varphi} & \\
}
\end{array}
\] (9.2.1)

**Remark 9.3.** Since its indeterminacy locus is finite, \( \gamma \) restricts to rational maps \( \gamma_h : T \dasharrow X \), for every \( h \in H(k) \). But since \( T \) is a \( k \)-smooth curve, these maps are in fact morphisms.

**Assumption w.l.o.g. 9.4.** Shrinking \( S^0 \), we may assume that \( \gamma \) is well-defined over \( S^0 \) and that for every \( h \in H(k) \), the morphism \( \gamma_h : T \to X \) is either a \( C \)-integral point, or that its image is completely contained in the support of \( D \).

9.1. **Proof of Theorem 3.10 in characteristic zero.** In this section, we prove Theorem 3.10 under the assumption that the algebraically closed field \( k \) has characteristic zero. In particular, the morphism \( \zeta \) is separable and generically étale.

Our goal is to construct a splitting of Sequence (3.3.1) over the generic fibre \( X_\eta \) of \( \varphi \), contradicting Assumption (3.10.2).

**Assumption w.l.o.g. 9.5.** Shrinking \( S^0 \), we assume that \( \zeta \) is étale over \( S^0 \).
Step 1. Splitting of $\gamma^* \Omega_X^1$. We begin with a discussion of $\gamma^* \Omega_X^1$. Over $S^0$, where $\gamma$ is well-defined and $\zeta$ is étale, we show that the pull-back of the sequence of relative differentials for the smooth morphism $\varphi$ splits. In fact, a splitting over $S^0$,

\begin{equation}
\begin{array}{ccc}
0 & \rightarrow & y^* \varphi^* \Omega_S^1 \\
\downarrow{a} & & \downarrow{b} \\
y^* \Omega_X^1 & \rightarrow & y^* \omega_X/S \\
\downarrow{c} & & \rightarrow \\
0
\end{array}
\end{equation}

is easily constructed by taking $a$ to be the composition of the following morphisms,

$$y^* \Omega_X^1 \xrightarrow{dy} \Omega^1_{T \times H} \xrightarrow{=} \pi_1^* \Omega^1_T \oplus \pi_2^* \Omega^1_H \xrightarrow{\text{projection}} \pi_1^* \Omega^1_T \xrightarrow{=} \text{over } S^0 \ y^* \varphi^* \Omega_S^1.$$ 

The assumption that $\zeta$ is étale guarantees that the morphisms $y_h$ are immersive over $S^0$, because they remain immersive when composed with $\varphi$. This guarantees that the concatenation of the first three arrows, $y^* \Omega_X^1 \rightarrow \pi_1^* \Omega_T^1$ is surjective. It follows that $a$ is surjective. We can then take $b$ as the inverse of the restriction of $c$ to $\ker(a)$. To make use of this splitting, we need to relate it to the Sequence (3.3.1), which involves log differentials rather than differentials. The following claim will turn out to be key.

Claim 9.6. Over $S^0$, we have $\text{Image}(b) \subseteq \ker(y^* \Omega_X^1 \rightarrow y^* \Omega^1_{\supp D})$.

Proof of Claim 9.6. Since $c$ is surjective, we may as well prove that

$$\text{Image}(b \circ c : y^* \Omega_X^1 \rightarrow y^* \Omega^1_{\supp D}) \subseteq \ker(y^* \Omega_X^1 \rightarrow y^* \Omega^1_{\supp D}).$$

This can be shown locally, near any given point $\tilde{p} \in (\supp y^* D) \cap T^0 \times H$. In fact, given $\tilde{p}$, write $\tilde{x} = y(\tilde{p})$ and choose a local equation $f \in \mathcal{O}_{X,\tilde{x}}$ for $\supp D$, and a uniformising parameter $s \in \mathcal{O}_{S,\varphi(\tilde{x})}$. Recall from Setting 3.1 that $(X, D)$ is relatively snc over $S^0$. Near $\tilde{x}$, the sheaf $\Omega^1_X$ is therefore generated by the differential forms $df$ and $d\varphi(ds)$. Near $\tilde{p}$, the pull-back sheaf $y^* \Omega^1_X$ is then generated by the pull-back sections $y^* df$ and $y^* d\varphi(ds)$. To prove the claim, it will then suffice to show that

\begin{equation}
(b \circ c)(y^* df) \in \ker(y^* \Omega_X^1 \rightarrow y^* \Omega^1_{\supp D}) \quad \text{and} \quad (9.6.1)
\end{equation}

\begin{equation}
(b \circ c)(y^* d\varphi(ds)) \in \ker(y^* \Omega_X^1 \rightarrow y^* \Omega^1_{\supp D}). \quad (9.6.2)
\end{equation}

Inclusion (9.6.2) follows because $y^* d\varphi(ds) \in \ker(c)$. Inclusion (9.6.1) requires a little more thought. To begin, observe that $df|_{\supp D} = 0 \in \Omega^1_{\supp D}$, which implies that

\begin{equation}
(9.6.3) \quad y^* df \in \ker(y^* \Omega_X^1 \rightarrow y^* \Omega^1_{\supp D}).
\end{equation}

Secondly, recall from Assumption 9.4 that the divisor $\gamma^*([D])$ intersects all horizontal curves $T \times \{h\}$ with multiplicity at least 2, or else contains these curves in its support. Either way, it then follows from the construction of $a$ that near $\tilde{p}$, the section $a(y^* df)$ vanishes along the support of $y^* D$. By construction of $b$, this means that near $\tilde{p}$, the sections $y^* df$ and $(b \circ c)(y^* df)$ agree along $\supp y^* D$. Inclusion (9.6.3) will then show that the section of $y^* \Omega^1_{\supp D}$ that is induced by $(b \circ c)(y^* df)$, vanishes there, as desired. \hfill \Box (Claim 9.6)
Step 2. Splitting of $\gamma^*\Omega_X^1(\log[D])$. As a next step, we claim that Sequence (3.3.1) splits over $S^\circ$, once one pulls it back via $\gamma$. The following commutative diagram with exact rows and columns summarises the situation, and relates Sequence (3.3.1) to Sequence (9.5.1) discussed in the previous step.

$$
\begin{array}{c}
\xymatrix{
\gamma^*\mathcal{J}_{[D]} \otimes \phi^*\Omega_S^1 \ar[r]^-{\gamma^* d\phi} & \gamma^*\Omega_S^1 \ar[r]^-{a} & \gamma^*\Omega_X^1 \ar[r]^-{c} & \gamma^*\omega_{X/S} \\
\gamma^*\Omega^1_{\supp D} \ar[r]^-{\text{isomorphic}} & \gamma^*\Omega^1_{\supp D} \ar[r]^-{\beta} & 0
}
\end{array}
$$

Explanation 9.7. The first row in the diagram equals Sequence (3.3.1) twisted with the ideal sheaf $\mathcal{J}_{[D]}$ and pulled back via $\gamma$. The second row equals Sequence (9.5.1). The middle column is a standard description of logarithmic differentials, as discussed for instance in [EV92, Prop. 2.3(c) on p. 13].

We have seen in Claim 9.6 that the composed map $\beta \circ b : \gamma^*\omega_{X/S} \to \gamma^*\Omega^1_{\supp D}$ vanishes. As a consequence, we obtain a morphism $\gamma^*\omega_{X/S} \to \gamma^*\left(\mathcal{J}_{[D]} \otimes \Omega_X^1(\log[D])\right)$, and hence a splitting of the top row. Since a sequence of coherent sheaves splits if and only if it splits after tensoring with a locally free sheaf, we can summarise our results so far as follows.

Claim 9.8. The $\gamma$-pull-back of Sequence (3.3.1),

$$
(9.8.1) \quad 0 \to \gamma^*\phi^*\Omega_S^1 \to \gamma^*\Omega_X^1(\log[D]) \to \gamma^*\omega_{X/S}(\log[D]) \to 0
$$

splits over $T^\circ$. \qed

Step 3. Splitting of Sequence (3.3.1). We will now show that Sequence (3.3.1) itself splits over the generic fibre $X_\eta$ of $\phi$. Recall that $\eta$ denotes the generic point of $S$, and denote the generic point of $T$ by $\eta_T$. To begin, observe that

$$
\deg_{\eta_T \times H} \gamma^*\phi^*\Omega_S^1 = 0 \\
\deg_{\eta_T \times H} \gamma^*\omega_{X/S}(\log[D]) > 0
$$

Assumption (3.10.1).

It follows that the sheaf $\gamma^*\Omega_X^1(\log[D])|_{\eta_T \times H}$ is unstable. Recalling that the pull-back of a semistable sheaf under separable morphisms of curves remains semistable, [Miy87, Prop. 3.2], we infer that $\Omega_X^1(\log[D])|_{X_\eta}$ is likewise unstable. Its maximal destabilising subsheaf is then a saturated, invertible subsheaf $\mathcal{A} \subset \Omega_X^1(\log[D])|_{X_\eta}$, of degree

$$
(9.8.2) \quad \frac{1}{2} \deg_{X_\eta} \omega_{X/S}(\log[D]) < \deg_{X_\eta} \mathcal{A} \leq \deg_{X_\eta} \omega_{X/S}(\log[D]). \quad \text{Sequence (9.8.1)}
$$

Claim 9.9. The second inequality in (9.8.2) is indeed an equality.
Proof of Claim 9.9. Writing $B$ for the quotient of $\Omega^1_X(\log [D])|_{X_\eta}$ by $A$, we obtain an exact sequence of locally free sheaves on $X_\eta$,

$$0 \to A \to \Omega^1_X(\log [D])|_{X_\eta} \to B \to 0.$$ 

Assuming that we have strict inequalities, we obtain degree bounds

$$0 < \deg_{\text{EX}_\eta} A < \deg_{\text{EX}_\eta} \omega_{X/S}(\log [D])$$

$$0 < \deg_{\text{EX}_\eta} B < \deg_{\text{EX}_\eta} \omega_{X/S}(\log [D])$$

In particular, any morphism from $\gamma^* \omega_{X/S}(\log [D])|_{\eta \times H}$ to the $\gamma$-pull-back of either $A$ or $B$ vanishes for degree reasons, contradicting Claim 9.8. □ (Claim 9.9)

Claim 9.9 implies in particular that the composed morphism

$$A \to \Omega^1_X(\log [D])|_{X_\eta} \to \omega_{X/S}(\log [D])|_{X_\eta}$$

is an isomorphism and that Sequence (3.3.1) splits over $X_\eta$. This contradicts Assumption (3.10.2) and therefore ends the proof of Theorem 3.10 in characteristic zero. □

9.2. Proof of Theorem 3.10 over fields of positive characteristic. In this section, we prove Theorem 3.10 under the assumption that $k$ is an algebraically closed field of positive characteristic. The strategy consists in deducing from Diagram (9.2.1) the existence of an infinite sequence of $C$-integral points over $S$ with bounded discriminant and unbounded height, contradicting the geometric height inequalities for $C$-integral points provided by Theorem 3.7.

Step 1. Simplification. Given any number $m \in \mathbb{N}^\geq 2$, set

$$D^m := D - \frac{1}{m} \cdot D_{\text{log}}.$$ 

Observe that $\text{supp } D = \text{supp } D^m$, that $(X, D^m)$ a $C$-pair and that all $C$-integral points of $(X, D)$ are automatically $C$-integral points of $(X, D^m)$. Choosing $m$ sufficiently large, Assumption (3.10.1) of Theorem 3.10 is still satisfied for the pair $(X, D^m)$. Replacing $D$ by $D^m$, we are thus safe to make the following assumption.

Assumption w.l.o.g. 9.10. The divisor $D$ has no component with $C$-multiplicity equal to $\infty$. In other words, $D_{\text{log}} = 0$.

Step 2. Iterated Frobenius morphisms. Given any number $n \in \mathbb{N}$, let $F_n : H_n \to H$ be the $n$th iterated $k$-linear Frobenius morphism, as discussed in [Har77, IV, Rem. 2.4.1] or [Sta21, Tag 0CC9]. The $H_n$ are then $k$-algebraic curves, with genus $g(H_n) = g(H)$, cf. [Har77, IV, Prop. 2.5].

Choose divisors $A_H \in \text{Div}(H)$ and $A_T \in \text{Div}(T)$ with degrees $\deg A_H \geq 2g(\bullet) + 1$. For every number $n$, set $A_{H_n} := p^{-n} \cdot F_n A_H \in \text{Div}(H_n)$, which is a divisor of degree $\deg A_H A_{H_n} = \deg A_H$. By [Har77, IV, Cor. 3.2.b], this assumption implies that the divisors $A_H$, $A_T$ and $A_{H_n}$ are all very ample, and then by [Gro65, Prop. 4.4.10.iv] so are the divisors

$$A_{T \times H_n} := \pi_1^* A_T \otimes \pi_2^* A_{H_n} \in \text{Div}(T \times H_n), \quad \text{for every } n.$$
For every number $n$, let $C_n \in |A_{T \times H_n}|$ be a general section. We summarise some of its main properties.

(9.10.1) Following [Kle74, Cor. 12] or [Fle77, Satz 5.2], the scheme $C_n$ is a regular curve, hence smooth over the perfect field $k$.

(9.10.2) The curve $C_n$ avoids the (finite) indeterminacy locus of the composed morphism $\gamma \circ (\text{Id}_T \times F_n) : T \times H_n \rightarrow X$.

(9.10.3) The natural morphism of curves $C_n \rightarrow T$ is separable, since the very ample linear system $|A_{T \times H_n}|$ separates tangents and the projection map $T \times H_n \rightarrow T$ is smooth of relative dimension one.

(9.10.4) By construction, the degree of $C_n$ over $S$ equals

$$[C_n : S] = [C_n : T] \cdot [T : S] = \deg_H A_H \cdot [T : S]$$

and is therefore independent of $n$.

**Notation 9.11.** The following diagram extends (9.2.1), summarises the situation, and fixes the notation used to denote the relevant morphisms.

\[
\begin{array}{ccccccccc}
C_n & \xrightarrow{\iota_n} & T \times H_n & \xrightarrow{\eta_n = \text{Id}_T \times F_n} & T \times H & \xrightarrow{\gamma} & X \\
\text{inclusion} & & & & & & & & \\
\psi_n & & & & & & & \downarrow & T \\
\end{array}
\]

By minor abuse of notation, we denote the projections from $T \times H_n$ by $\pi_\bullet$, as it will always be clear from the context what number $n$ is meant.

**Step 3. The curves $\gamma_n$ as $C$-integral points.** We will show that the curves $\gamma_n$ are $C$-integral points for the family $\varphi : X \rightarrow S$. For the reader’s convenience, we prove birationality first.

**Claim 9.12.** Given any number $n \in \mathbb{N}$, the morphism $\gamma_n$ is generically injective.

**Proof of Claim 9.12.** The very ample linear system $|A_{T \times H_n}|$ separates points. In particular, if $p_1 \in T \times H_n$ is a general point with set-theoretic fibre

$$(\gamma \circ \eta_n)^{-1}(\gamma \circ \eta_n)(p_1) = \{p_1, \ldots, p_t\} \subset T \times H_n,$$

then there are sections $\sigma_2, \ldots, \sigma_t \in H^0(T \times H_n, \mathcal{O}_{T \times H_n}(A_{T \times H_n}))$ where $\sigma_1(p_1) = 0$ and $\sigma_i(p_i) \neq 0$ for every $2 \leq i \leq t$. Since the algebraically closed base field $k$ is infinite, there exists a $k$-linear combination $\sigma \in H^0(T \times H_n, \mathcal{O}_{T \times H_n}(A_{T \times H_n}))$ that vanishes at $p_1$, but not at any of the remaining points $p_2, \ldots, p_t$. If $C \in |A_{T \times H_n}|$ is general among the elements of the linear system that contain $p_1$, then $C$ will not contain any of the points $p_2, \ldots, p_t$, and the $\gamma_n$ maps the curve $C$ generically injectively onto its image. The claim follows. □ (Claim 9.12)

**Claim 9.13.** Given any number $n \in \mathbb{N}$, the morphism $\gamma_n$ maps the curve $C_n$ birationally onto its image.
Proof of Claim 9.13. In view of Claim 9.12, it remains to show that the morphism \( C_n \to y_n(C_n) \) is separable, or equivalently, that the pull-back map of differential forms, \( d\gamma_n : \gamma_n^* \Omega^1_X \to \Omega^1_{C_n} \), does not vanish identically. The proof relies on two observations.

- First, recalling that the morphisms \( \gamma'_h : T \to X \) are assumed to be \( C \)-integral points over \( S \) for all closed points \( h \in H' \), the following composed morphism between invertible sheaves on \( T \times H \setminus \text{indet. locus of } \gamma \) does not vanish identically:

\[
y^* \Omega^1_X \xrightarrow{dy} \Omega^1_{T \times H} \xrightarrow{=} \pi^*_1 \Omega^1_T \oplus \pi^*_2 \Omega^1_H \xrightarrow{\text{projection}} \pi^*_1 \Omega^1_T.
\]

By general choice, the pull-back of this map to \( C_n \), denoted by \( \alpha : \gamma_n^* \Omega^1_X \to \gamma_n^* \pi^*_1 \Omega^1_T \), will then likewise not vanish.

- Second, recall from (9.10.3) that the map \( \pi_1 \circ \iota_n : C_n \to T \) is separable. The composed morphism between invertibles on \( C_n \),

\[
t^*_n \pi_1 \Omega^1_T \xrightarrow{\text{Id} \oplus 0} t^*_n \pi^*_1 \Omega^1_T \oplus t^*_n \pi^*_2 \Omega^1_{H_n} \xrightarrow{=} t^*_n \Omega^1_{T \times H_n} \xrightarrow{d_{\iota_n}} \Omega^1_{C_n}
\]

will thus again not vanish identically.

With these two observations in place, the map \( d\gamma_n \) can now be rewritten as

\[
\begin{array}{ccc}
y^*_n \Omega^1_X & \xrightarrow{dy} & \psi^*_n \Omega^1_{T \times H} \\
\xrightarrow{\alpha \oplus \text{other}} & \xrightarrow{=} & t^*_n \Omega^1_{T \times H_n} \\
\xrightarrow{\psi^*_n \pi^*_1 \Omega^1_T \oplus \psi^*_n \pi^*_2 \Omega^1_H} & \xrightarrow{\text{Isom.} \oplus 0} & t^*_n \pi^*_1 \Omega^1_T \oplus t^*_n \pi^*_2 \Omega^1_{H_n} \\
\xrightarrow{\beta \oplus \text{other}} & \xrightarrow{=} & \Omega^1_{C_n}
\end{array}
\]

Recalling that \( \psi^*_n \pi^*_1 \Omega^1_T \), \( t^*_n \pi^*_1 \Omega^1_T \), and \( \Omega^1_{C_n} \), are invertible, so that any composition of non-vanishing morphisms is itself non-vanishing, a look at the bottom row will convince the reader. \( \square \) (Claim 9.13)

Claim 9.14. For every sufficiently large number \( n \), the curve \( y_n \) is a \( C \)-integral point for the family \( \varphi : X \to S \), in the standard sense of Definition 3.4.

Proof of Claim 9.14. Recall from Assumption 9.10 that no component of the divisor \( D \) has \( C \)-multiplicity equal to \( \infty \). We will show the morphisms \( y_n \) are \( C \)-integral points whenever

\[(9.14.1) \quad p^n \geq \max \{ m_i \mid 1 \leq i \leq d \}.\]

Assuming that one such \( n \) is given, we need to show that \( y_n \) satisfies the conditions spelled out in Definition 3.4 on page 9. Condition (3.4.1) ("the curve \( C_n \) dominates \( S' \)) is clear by construction, and Condition (3.4.3) ("the morphism \( y_n \) is birational onto is image") has been verified in Claim 9.13 above. It remains to verify Condition (3.4.2): the obvious restriction of \( y \) to the preimages of \( S' \), which we write as \( y^*_n : C^*_n \to X^* \), is a \( C \)-curve for the pair \( (X^*, D^*) \).
To this end, let $x \in C_n^\circ$ be any closed point such that $\gamma_n(x)$ belongs to the support of $D^\circ$. Using the assumption that $D^\circ$ is relatively snc over $S^\circ$, observe that the point $\gamma_n(x)$ is then contained in exactly one component of $D$. Write $m$ for the $C$-multiplicity of that component and recall from Assumption 9.10 that $m < \infty$. To show Condition (3.4.2), we need to show that the multiplicity of $D$ at the point $x$ is at least $m$, that is, $\mult_x \gamma_n^* D \geq m$. This will be done by means of a local computation. Write $(a, b) := (\eta_n \circ t_n)(x) \in T \times H$ and choose uniformising parameters $t \in \mathcal{O}_{T,a}$ and $h \in \mathcal{O}_{H,b}$. Abusing notation, we use the symbols $t$ and $h$ to also denote the associated elements of the local ring $\mathcal{O}_{T \times H, (a, b)}$. We claim that the following inclusion of ideals holds true:

\[(9.14.2) \quad \mathcal{J}_{\gamma^* D} \subset \mathcal{J}_{\gamma^* D \cap (T \times \{b\})} \supseteq (h, t^m) \quad \text{in} \quad \mathcal{O}_{T \times H, (a, b)}.
\]

In fact, if $b \in H'$ and if $\gamma_n$ is not $C$-integral, then Inclusion (9.14.2) follows from the assumption that $\gamma_b : T \to X$ is a $C$-integral point. If on the other hand $b \notin H'$, then Inclusion (9.14.2) follows from Assumption 9.4, which asserts that $(T \times \{b\}) \subseteq \text{supp} \gamma^* D$. But now we are done: observing that

$$\Ord_x (\eta_n \circ t_n)^* h \geq p^n \geq m \quad \text{and} \quad \Ord_x (\eta_n \circ t_n)^* t^m \geq m,$$

the claim follows directly from (9.14.2). \qed (Claim 9.14)

**Step 4. Conclusion.** We conclude by showing that the family $\gamma_n$ of $C$-integral points has bounded discriminant but unbounded height, contradicting Theorem 3.7.

**Claim 9.15.** The discriminants of the $(\gamma_n)_{n \in \mathbb{N}}$ are constant.

**Proof of Claim 9.15.** Given $n \in \mathbb{N}$, choose closed points $t \in T(k)$ and $h_n \in H_n(k)$, with fibres $F_t$ and $F_{h_n}$ in $T \times H_n$. We have equalities of numerical classes,

\[
\begin{align*}
[C_n] &\equiv \text{num} (\deg_T A_T) \cdot [F_t] + \deg_H A_H \cdot [F_{h_n}] \\
[K_{T \times H_n}] &\equiv \text{num} (2 \cdot g(T) - 2) \cdot [F_t] + (2 \cdot g(H) - 2) \cdot [F_{h_n}]
\end{align*}
\]

and therefore

$$\deg_{C_n} \omega_{C_n} = [C_n] \cdot [C_n + K_{T \times H_n}] = \text{const}^* \in \mathbb{N}^+.$$

Item (9.10.4) will thus conclude the proof. \qed (Claim 9.15)

**Claim 9.16.** The heights of the $(\gamma_n)_{n \in \mathbb{N}}$ are unbounded. More precisely, we have

$$\lim_{n \to \infty} h(\gamma_n) = \lim_{n \to \infty} \deg_{C_n} \gamma_n^*(K_{X/S} + D) / [C_n : S] = \infty.$$

**Proof of Claim 9.16.** As before, choose $t \in T(k)$ and $h \in H(k)$, with fibres $F_t$ and $F_h$ in $T \times H$ and write

$$[\gamma^*(K_{X/S} + D)] \equiv \text{num} a \cdot [F_t] + b \cdot [F_h]$$

\[\text{failure of the Brauer–Manin principle for a simply connected fourfold}\]
where \(a\) and \(b\) are rational numbers. Assumption (3.10.1) guarantees that \(b > 0\). But then,

\[
\deg_{C_a} Y_n(K_{X/S} + D) = [A_{T \times H_a}] \cdot \left[ \eta_n^{-1} (K_{X/S} + D) \right] = (\deg_T A_T) \cdot p^n \cdot b + (\deg_H A_H) \cdot a
\]

As before, Item (9.10.4) concludes the proof. \(\square\) (Claim 9.16)

To sum up, assuming that there exists a positive family of \(C\)-integral points, we found a sequence of \(C\)-integral points of bounded discriminant but unbounded height. This contradicts the height bound found in Theorem 3.7 above and therefore ends the proof of Theorem 3.10 in the last remaining case, when the characteristic of the base field is positive. \(\square\)

10. The Mordell conjecture for \(C\)-integral points — proof of Theorem 3.12

With Theorems 3.7 ("Height bound") and 3.10 ("Rigidity") at our disposal, Theorem 3.12 follows quickly, adapting standard arguments to our setting. We argue by contradiction: maintaining the setting and assumptions of Theorem 3.12, we assume that there exists a smooth \(k\)-curve \(T\) over \(S\) and an infinite number of \(C\)-integral points \(y \in \text{Hom}_S(T, X)(k)\).

10.1. Step 1: the set of \(C\)-integral points. The following claim asserts that the set of \(C\)-integral points,

\[
\text{Hom}_S(T, X, C) := \{ y \in \text{Hom}_S(T, X)(k) \mid y \text{ is } C\text{-integral} \},
\]

is locally closed. The proof uses little but the definition of "\(C\)-integral" and the standard fact that effective Cartier divisors on the smooth curve \(T\) are parameterised by the Hilbert scheme, as discussed in [Sta21, Sect. 0B9C] or [Kol96, I, Thm. 1.13].

Claim 10.1. The set \(\text{Hom}_S(T, X, C)\) is a locally closed subset of \(\text{Hom}_S(T, X)(k)\).

Proof of Claim 10.1. Considering one component of \(D\) at a time, we may assume without loss of generality that the divisor \(D \in \text{Div}(X)\) is irreducible, so \(D = m_1 \cdot D_1\) with \(m_1 \in \mathbb{N}^+ \cup \{\infty\}\). Write \(T := T^\circ \cup \{t_1, \ldots, t_d\}\). Now, given any component \(H \subseteq \text{Hom}_S(T, X)_{\text{red}}\), we need to show that the set

\[
H_C := \{ y \in H(k) \mid y \text{ is } C\text{-integral} \}
\]

is locally closed in \(H(k)\). To begin, remark that

\[
H_C \subseteq \{ y \in H \mid \text{Image}(y) \not\subseteq \text{supp } D \} := H^\circ,
\]

where \(H^\circ \subseteq H\) is open. We will show that \(H_C\) is a closed subset of \(H^\circ(k)\).

Next, choose any morphism \(y \in H(k)\) and set \(b_H := \deg_T y^* D_1\). This number is independent of the choice of \(y\). Denoting the universal morphism by \(u^o : H^\circ \times T \to X\), the pull-back \((u^o)^* D_1\) is a relative effective Cartier divisor for the family \(H^\circ \times T \to T\); we refer the reader to [Sta21, Section 056P] for the definition of "relative effective Cartier" and to [Sta21, Lem. 062Y] for the criterion used here.
By [Sta21, Section 0B9C], this divisor yields a morphism \( \nu : H^o \to \text{Hilb}^{b_H}_{T/k} \). We aim to describe \( H_C \) in terms of this morphism. To this end, we consider sequences of numbers as follows,

\[
M := \left\{ (n_1, \ldots, n_{a+b_H}) \in \mathbb{N}^{a+b_H} \ \middle| \ \sum_{j} n_j = b_H \text{ and for all } i > a, \right. \\
\left. \text{we have either } n_i = 0 \text{ or } n_i > m_1 \right\}.
\]

Here, the inequality \( n_i > m_1 \) is understood to be never satisfied when \( m_1 = \infty \). If any sequence \( \vec{n} = (n_1, \ldots, n_{a+b_H}) \in M \) is given, we consider the relative effective Cartier divisor on \( \pi_2 \times \cdots \times \pi_{a+b_H+1} : T^{x(a+b_H+1)} \to T^{x(a+b_H)} \) given as

\[
D_{\vec{n}} := \sum_{i=1}^{a+b_H} n_i \cdot \left\{ (t_0, \ldots, t_{a+b_H}) \in T^{x(a+b_H+1)} \mid t_0 = t_i \right\}.
\]

As before, \( D_{\vec{n}} \) defines a morphism \( u_{\vec{n}} : T^{x(a+b_H)} \to \text{Hilb}^{b_H}_{T/k} \). The morphism \( u_{\vec{n}} \) is proper; its image is therefore closed and then so is the union

\[
C \text{ Hilb}^{b_H}_{T/k} := \bigcup_{\vec{n} \in M} \text{Image}(u_{\vec{n}}).
\]

Returning to the original problem of describing \( H_C \), it follows immediately from the construction that a point \( \gamma \in H(k) \) is in \( H_C \) if and only if \( \nu(\gamma) \in C \text{ Hilb}^{b_H}_{T/k} \).

The set \( H_C \) is therefore closed in \( H^o(k) \). \( \square \) (Claim 10.1)

10.2. **Step 2: boundedness, end of proof.** Theorem 3.7 implies that the height of all \( C \)-integral points \( \gamma : T \to X \) is bounded:

\[
\exists \text{ const}^+ : \forall \gamma \in \text{Hom}^i_T(T, X, C) : \deg_T(y^*(K_{X/S} + D)) < \text{ const}^+.
\]

Using that \( K_{X/S} + D \) is relatively ample and that \( \text{Hom}^i_S(T, X) \) is an open subscheme of \( \text{Hilb}_{T \times X/S} \), we find finitely many irreducible components of \( \text{Hom}^i_S(T, X) \) that contain all of \( \text{Hom}^i_S(T, X, C) \). In particular, there exists one irreducible component of \( \text{Hom}^i_S(T, X, C) \) that contains infinitely many points and must therefore be of positive dimension. Eventually, this allows us to find a quasi-projective curve \( C^o \subseteq \text{Hom}^i_S(T, X) \) and an \( S \)-morphism \( \gamma^o : C^o \times T \to X \) such that the morphisms \( \gamma^c : T \to X \) are \( C \)-integral, for all \( c \in C^o(k) \). Theorem 3.10, however, asserts that no such morphism can possibly exist. We obtain a contradiction, which ends the proof of Theorem 3.12. \( \square \)

**Part II. Insufficiency of the Brauer–Manin obstruction for a simply connected fourfold**

11. **Fibrations of general type — proof of Theorem 1.4**

The goal of this section is to prove Theorem 1.4. The case of number fields has been treated by Campana in [Cam05, §5] conditionally on the abc conjecture. Our construction proceeds along similar lines – we use our orbifold Mordell-type theorem in positive characteristic as a substitute for the abc conjecture, together
with a delicate construction of genus-two fibrations on certain simply connected surfaces with orbifold base of general type, due to Stoppino [Sto11]. Her construction is simpler than the one used by Campana in [Cam05, §5], and therefore has the advantage of being more easily transportable to positive characteristic.

11.1. **The orbifold base.** For fibrations between algebraic varieties over the complex numbers, Campana defines the notion of orbifold base in [Cam04, §1.1.4, §1.2.1]. Let us recall the definition.

**Definition 11.1** (Orbifold base). Let $k$ be an arbitrary field and let $\pi : Y \to X$ be a surjective morphism of smooth, quasi-projective $k$-varieties. Assume that $\pi$ is smooth over an open subset of $X$. For each $P$ in $X^{(1)}$, the set of points of codimension $1$ on $X$, we define $m_P$ as the minimum of the set of multiplicities of the irreducible components of the fibre $\pi^{-1}(P)$. We then set

$$\Delta_\pi := \sum_{P \in X^{(1)}} \left( 1 - \frac{1}{m_P} \right) \cdot \overline{P},$$

where $\overline{P}$ denotes the Zariski closure of $P$. We refer to $(X, \Delta_\pi)$ as the orbifold base of $\pi$.

**Remark 11.2.** The support of $\Delta_\pi$ in Definition 11.1 need not be SNC.

**Remark 11.3.** Definition 11.1 considers only codimension-one points of $X$ and ignores the behaviour of $f$ over points of higher codimension. While more elaborate definitions of “orbifold base” have been suggested, the simple version described above is good enough for our purposes.

**Lemma 11.4** (Orbifold base and $C$-curves). In the setting of Definition 11.1, assume that $k$ is algebraically closed. Let $D \leq \Delta_\pi$ be any $\mathbb{Q}$-divisor on $X$, such that $(X, D)$ forms an snc $C$-pair. If $T$ is a smooth, quasi-projective $k$-curve equipped with a morphism $\gamma : T \to Y$, then either $\text{Image}(\pi \circ \gamma) \subset \text{supp} D$, or $\pi \circ \gamma : T \to X$ is a $C$-curve for the $C$-pair $(X, D)$.

**Proof.** If suffices to consider the case where the support of $D$ is irreducible. Let us assume that $\text{Image}(\pi \circ \gamma) \not\subset \text{supp} D$, and let $t \in T(k)$ be such that $(\pi \circ \gamma)(t) \in \text{supp} D$. To show that $\pi \circ \gamma$ is a $C$-curve, we need to check that

$$\text{mult}_t(\pi \circ \gamma)^*[D] \geq \min(\text{multiplicities of irreducible components of } \pi^*[D]).$$

This is however immediate from the definition of orbifold base once we observe that

$$\text{mult}_t(\pi \circ \gamma)^*[D] = \text{mult}_t \gamma^*(\pi^*[D]).$$

11.2. **Multiple fibres.** In order to guarantee that the variety constructed in Section 11.3 is geometrically simply connected, we use an elementary criterion which generalises [Cam05, Lem. 5.8]. The following definition will be used.

**Definition 11.5** (Multiple fibres). Let $k$ be an algebraically closed field. Let $\pi : X \to C$ be a projective, surjective $k$-morphism from a normal $k$-variety to a smooth $k$-curve. Assume that $\pi$ has connected fibres. Given $P \in C(k)$, say that $\pi$ has a
multiple fibre over $P$ if there exist an integer $m \geq 2$ and a Weil divisor $D$ on $X$ such that $\pi^* P = m \cdot D$.

**Lemma 11.6** (Multiple fibres and simple connectedness). In the setting of Definition 11.5, fix a base point $x \in X(k)$. Assume that $\pi$ does not have any multiple fibres and that at least one fibre of $\pi$ is simply connected. Then, the natural homomorphism

$$\pi^\text{ét}(X, x) \to \pi^\text{ét}_1(C, \pi(x))$$

is an isomorphism.

**Proof.** Let $f : X' \to X$ be a finite, étale cover. We have to show that $f$ is the base change along $\pi$ of a finite étale cover of the base curve $C$. Taking the Stein factorisation of $\pi \circ f$, we obtain the commutative diagram,

$$
\begin{array}{ccc}
X' & \xrightarrow{f, \text{étale}} & X \\
\downarrow \pi', \text{connected fibres} & & \downarrow \pi \\
C' & \xrightarrow{g, \text{finite}} & C.
\end{array}
$$

Since $\pi$ has a simply connected fibre, we see that $g$ must be étale above at least one point of $C$, and hence also generically. We claim that $g$ is étale everywhere. If fact, if $c' \in C'(k)$ is any closed point with image $c \in C$, then the coefficients of $\pi^* c \in \text{Div}(X)$ are coprime; since $f$ is étale, so are the coefficients found in any connected component of $f^* \pi^* c$. On the other hand, if $B'$ is any connected component of $(\pi')^* g^* c = f^* \pi^* c$ that maps to $c'$, then all coefficients in $B'$ are multiples of $\text{mult}_{\mathcal{O}} g' c$. Therefore, $\text{mult}_{\mathcal{O}} g' c = 1$, so $g$ is étale at $c'$. It is now easy to see that the natural morphism $X' \to X \times_C C'$ is an isomorphism, as required. $\blacksquare$

11.3. **Proof of Theorem 1.4.** We begin directly with the construction of the example. A second step will show that the example does indeed satisfy all required properties. Throughout, we view $\mathbb{P}^1$ as an extension of $\mathbb{A}^1$ and write $t$ instead of $[t : 1]$ and $\infty$ instead of $[1 : 0]$.

**Step 1. Construction.** We start by constructing a smooth, projective $\mathbb{Q}$-surface $F_\mathbb{Q}$ and a $\mathbb{Q}$-morphism $g_\mathbb{Q} : F_\mathbb{Q} \to \mathbb{P}^1_\mathbb{Q}$ with the following properties.

(11.7.1) The $\mathbb{Q}$-surface $F_\mathbb{Q}$ is smooth, of general type, and $g_\mathbb{Q}$ is smooth over an open subset of $\mathbb{P}^1_\mathbb{Q}$.

(11.7.2) The natural morphism $\mathcal{O}_{\mathbb{P}^1_\mathbb{Q}} \to (g_\mathbb{Q})_* \mathcal{O}_{F_\mathbb{Q}}$ is an isomorphism.

(11.7.3) No geometric fibre of $g_\mathbb{Q}$ is multiple.

(11.7.4) The fibre $g_\mathbb{Q}^{-1}(1 : 1)$ has a $\mathbb{Q}$-rational point $x_\mathbb{Q}$.

(11.7.5) The geometric fibres of $g_\mathbb{Q}$ over $[0 : 1]$ and $[1 : 0]$ are supported on trees of smooth rational curves with transverse intersections, and each of the irreducible components has multiplicity at least two.

The construction outlined below is due to Stoppino. We refer the reader to Stoppino’s paper [Sto11], in particular to [Sto11, Fig. 2 as well as Thm. 3.1, Rem. 3.6] for full details, proofs and instructive sample computations.
Construction 11.8 (Special case of Stoppino’s construction with “fibres of type 1”). Fix the rational number $\alpha := -\frac{72}{216}$ and consider the following curve in $\mathbb{P}^1_Q \times \mathbb{P}^1_Q$,

$$B := \left\{ ([x : y], [t : s]) \in \mathbb{P}^1_Q \times \mathbb{P}^1_Q : st \cdot (t^2 \cdot x^6 + \alpha \cdot st \cdot x^3 y^3 + s^2 \cdot y^6) = 0 \right\}.$$  

Let $Y_Q$ be the canonical desingularisation of the double covering of $\mathbb{P}^1_Q \times \mathbb{P}^1_Q$ ramified over $B$. As shown in [Sto11, Section 2 and proof of Thm. 3.1], the relative minimal model of the composition

$$Y_Q \to \mathbb{P}^1_Q \times \mathbb{P}^1_Q \xrightarrow{\text{2nd projection}} \mathbb{P}^1_Q$$

is a smooth surface $Z_Q$, and the natural morphism $f_Q : Z_Q \to \mathbb{P}^1_Q$ is a fibration satisfying Properties (11.7.2), (11.7.3), and (11.7.5).

The specific numerical value of $\alpha$ implies the following properties for the $f_Q$-fibres over $[\pm 1 : 1]$:

(11.9.1) The point $([2 : 3], [1 : 1]) \in \mathbb{P}^1_Q \times \mathbb{P}^1_Q$ lies on the curve $B$. This implies in particular that the $f_Q$-fibre over $[1 : 1]$ contains a $Q$-rational point. In other words, the fibration $f_Q : Z_Q \to \mathbb{P}^1_Q$ satisfies Property (11.7.4) as well.

(11.9.2) The curves $B$ and $\mathbb{P}^1_Q \times \{[\pm 1 : 1]\}$ intersect transversely $\mathbb{P}^1_Q \times \mathbb{P}^1_Q$. In particular, we find that the $f_Q$-fibres over $[\pm 1 : 1]$ are smooth, so that $f_Q$ is smooth over neighbourhoods of the points $[\pm 1 : 1]$.

To construct a surface $F_Q$ and a $Q$-morphism $g_Q : F_Q \to \mathbb{P}^1_Q$ which also satisfy Property (11.7.1), consider the $Q$-morphism

$$\gamma : \mathbb{P}^1_Q \to \mathbb{P}^1_Q, \quad [t : s] \mapsto \left[ (t + s)^3 + (t - s)^3 : (t + s)^3 - (t - s)^3 \right],$$

and perform the base change

$$\begin{array}{ccc}
F_Q & \xrightarrow{\gamma} & Z_Q \\
\downarrow g_Q & & \downarrow f_Q \\
\mathbb{P}^1_Q & \xrightarrow{\gamma} & \mathbb{P}^1_Q.
\end{array}$$

We check that $g_Q : F_Q \to \mathbb{P}^1_Q$ satisfies all required properties.

**Property (11.7.1):** The morphism $\gamma$ is totally ramified over the points $[\pm 1 : 1]$ and étale elsewhere. Smoothness of $Z_Q$ and Item (11.9.2) will therefore imply that $F_Q$ is smooth, and that $f_Q$ is smooth near $\gamma^{-1}([\pm 1 : 1])$. We refer the reader to [Sto11, Prop. 3.2] or [Cam05, Prop. 1.7] for the fact that $F_Q$ is of general type.

**Property (11.7.2):** This follows from flatness-and-base change, since $\gamma$ is flat and since $\partial_{\mathbb{P}^1_Q} \rightarrow (f_Q)_{\partial Z_Q}$ is isomorphic.

**Property (11.7.3):** This holds because the geometric fibres of $f_Q$ and $g_Q$ agree.

**Property (11.7.4):** Follows from (11.9.1), given that $\gamma$ is totally ramified over $[1 : 1]$ and that $\gamma([1 : 1]) = [1 : 1]$. 
**Property (11.7.5):** This property holds because it holds for $g_{Q}$, and because

$$\gamma([0:1]) = [0:1] \quad \text{and} \quad \gamma([1:0]) = [1:0].$$

**Step 1a. Reduction mod $p$.** The fibration $g_{Q} : F_{Q} \to \mathbb{P}^{1}_{Q}$ described above can be defined over a suitable localisation of the ring of integers. Let us therefore choose a proper model $\tilde{g} : \mathcal{F} \to \mathbb{P}^{1}_{\mathbb{Z}_{T}}$, where $T$ is a finite set of primes containing 2 and $\mathbb{F}_{p}$. In analogy, we consider the morphism $g : F \to \mathbb{P}^{1}_{\mathbb{F}_{p}}$ and the $\mathbb{F}_{p}$-point $x \in \tilde{g}^{-1}(1)$. The following properties will then hold if $p$ is sufficiently large:

1. (11.10.1) The natural morphism $\mathcal{O}_{\mathbb{P}^{1}_{\mathbb{F}_{p}}} \to g_{*}\mathcal{O}_{F}$ is an isomorphism; this follows from (11.7.2) and [Gro66, Prop. 9.4.4]. In particular, $g$ has geometrically connected fibres, by [GW10, Thm. 12.69 on p. 348].
2. (11.10.2) No geometric fibre of $g$ is multiple: this follows from (11.7.3) and the fact that multiplicities "spread out well over an open on the base", as follows for example from the arguments in [LSS20, Lem. 3.12] and [Gvi20, Lem. 3.9].
3. (11.10.3) The geometric fibres of $g$ over 0 and $\infty$ are simply connected and each irreducible component of these fibres has multiplicity at least two: this is a consequence of (11.7.5) and the classical specialisation theorem for the étale fundamental group, see [Gro71, Exp. X, Cor 2.4].

By [Gro67, Prop. 17.7.11], we may choose a dense open set $1 \in S^{c} \subseteq \mathbb{P}^{1}_{\mathbb{F}_{p}} \setminus \{0, \infty\}$ over which the morphism $g$ is smooth.

**Step 1b. Products and covers.** Spreading out and using a suitable version of the cyclic covering trick – which features already in the proof of [Sto11, Thm. 3.1], and in Campana’s construction [Cam05, §5] – we can now construct a commutative diagram of morphisms between $\mathbb{F}_{p}$-varieties as follows,

\begin{equation}
\begin{array}{c}
Y^{c} \xrightarrow{f} X^{c} \xrightarrow{\varphi^{c}} S^{c} \\
\mathbb{P}^{1}_{\mathbb{F}_{p}} \times \mathbb{P}^{1}_{\mathbb{F}_{p}} \xrightarrow{g \times \text{Id}} \mathbb{P}^{1}_{\mathbb{F}_{p}} \times S^{c} \xrightarrow{pr_{2}} S^{c}.
\end{array}
\end{equation}

Set $X^{c} := \mathbb{P}^{1}_{\mathbb{F}_{p}} \times S^{c}$ and let $\varphi^{c}$ be the projection to the second factor. Writing

$$B_{t} := \{(t) \times S^{c}\} \quad \text{and} \quad B_{\Lambda} := \{(t,s) \in \mathbb{P}^{1}_{\mathbb{F}_{p}} \times S^{c} \mid t = s\},$$

the $\mathbb{F}_{p}$-morphism $h$ is the (separable!) triple cover which ramifies totally over the curves $B_{t}$ and $B_{\Lambda}$,

$$h : X^{c} \to \mathbb{P}^{1}_{\mathbb{F}_{p}} \times S^{c}, \quad h(x,t) := \left(\frac{(x-t)^{3}+t(x-1)^{3}}{(x-t)^{3}+(x-1)^{3}}, t\right).$$
Take $Y^\circ$ as the fibre product, and $\sigma^\circ$ as the fibre product of the following morphisms,
\begin{align*}
\alpha : S^\circ \to X^\circ = \mathbb{P}^1_{F_p} \times S^\circ, & \quad \alpha(t) := (1, t) \\
\beta : S^\circ \to F \times S^\circ, & \quad \beta(t) := (t, t).
\end{align*}

Finally, set $D^\circ := \frac{1}{2} \cdot h^*(B_0 + B_\infty)$.

**Step 1c. Summary.** The main properties of this construction are summarised as follows.

(11.11.1) Since $F$ is smooth over $\mathbb{F}_p$, [Gro67, Prop. 17.7.11], we find that $\varphi^\circ \circ \pi^\circ : Y^\circ \to S^\circ$ is smooth. The geometric fibres of $\pi^\circ$ are those of $g$, and therefore geometrically connected by (11.10.1). The morphism $\pi^\circ$ is smooth over an open subset of $X^\circ$.

(11.11.2) The choice $S^\circ$ guarantees that $h$ is étale along $B_0$ and $B_\infty$. The pair $(X^\circ, D^\circ)$ is therefore relatively snc over $S^\circ$. The Sequence (3.3.1) does not split when restricted to the generic fibre $X_\eta$ of $\varphi^\circ$. Indeed, the existence of such splitting is equivalent to the existence of a vector field of the form
\[
v = \frac{\partial}{\partial t} + (a(t) + b(t)x + c(t)x^2) \frac{\partial}{\partial x}
\]
which is tangent to $\text{supp} \ D^\circ$. In other words, the derivation of $\mathbb{F}_p(t)[x]$ determined by $v$ must preserve the ideal generated by $n(x, t) = (x - t)^3 + t \cdot (x - 1)^3$, which is the numerator of the first component of $h(x, t)$, as well as the ideal generated by $d(x, t) = (x - t)^3 + (x - 1)^3$, which is the denominator of the first component of $h(x, t)$. An elementary computation shows that the ideal generated by $d(x, t)$ is preserved by $v$ if, and only if,
\[
(a(t), b(t), c(t)) = \left( \frac{-1}{t - 1}, \frac{1}{t - 1}, 0 \right).
\]
Likewise, the ideal generated by $n(x, t)$ is preserved by $v$ if, and only if,
\[
(a(t), b(t), c(t)) = \left( \frac{-2}{3(t - 1)}, \frac{2t - 1}{3t(t - 1)}, \frac{1}{3t(t - 1)} \right).
\]
Therefore, no vector field $v$ of the form above is tangent to $\text{supp} \ D^\circ$. We conclude that Sequence (3.3.1) does not split, as claimed.

(11.11.3) Using that $X_\eta^\circ$ is rational and that $\text{supp} \ D^\circ \to S^\circ$ is six-to-one, we find that the degree of $K_{X^\circ} + D^\circ$ on the generic fibre $X_\eta$ of $\varphi^\circ$ equals one.

(11.11.4) We claim that the support of $(\pi^\circ)^*D^\circ$ is isomorphic to the product of $\text{supp} \ D^\circ$ with a geometrically simply connected curve, and that every component $(\pi^\circ)^*[D^\circ]$ has multiplicity at least 2. This follows from (11.10.3) since
\[
(\pi^\circ)^*[D] = (\pi^\circ)^*h^*(B_0 + B_\infty) = f^* (g \times \text{Id})^*(B_0 + B_\infty)
= f^* \left( g^* \left( \{0\} + \{\infty\} \right) \times S^\circ \right).
\]
(11.11.5) If $E^o \subset X^o$ is any prime divisor which dominates $S^o$, then the coefficients of $(\pi^o)^*E^o$ are coprime. This is again a consequence of (11.10.2), using the fact that the fibres of $\pi^o$ are those of $g$.

Step 2. Verification of properties. Writing $\eta$ for the generic point of $S^o$, with residue field $K := \mathbb{F}_p(t)$, we view $Y_\eta$ is a smooth projective surface over $K$, equipped with a surjection $\pi^o_\eta : Y_\eta \to X^o_\eta = \mathbb{P}^1_K$. It remains to show that the surface $Y_\eta$ is of general type, geometrically simply connected, and that $\pi^o_\eta(Y_\eta(K))$ is finite and not empty.

Step 2a. General type. Since $F_Q$ is of general type over $Q$, the surface $F_{\mathbb{F}_p}$ is of general type over $\mathbb{F}_p$, and $(F \times S^o)_\eta$ is of general type over $K$. Since $h$ is separable, so is the induced morphism of $K$-varieties, $f_\eta : Y_\eta \to (F \times S^o)_\eta$. As a separable cover of a surface of general type, $Y_\eta$ is then itself of general type.

Step 2b. Simple connectedness. Writing $\overline{K}$ for an algebraic closure of $K$, we need to show that $Y^o_{\overline{K}}$ is simply connected. Item (11.11.1) equips us with a proper map $\pi^o_{\overline{K}} : Y^o_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ with connected fibres. Better still, Item (11.11.4) asserts that $\pi^o_{\overline{K}}$ admits at least one simply connected fibre, while Item (11.11.5) asserts that $\pi^o_{\overline{K}}$ has no multiple fibre. Lemma 11.6 therefore applies to show the simple connectedness.

Step 2c. Rational points. The existence of $\sigma^o$ shows that $Y^o(K)$ is not empty. To prove finiteness, we pass to the algebraic closure, $k := \overline{\mathbb{F}}_p$, consider the sequence of morphisms over $k$,

$$
\xymatrix{ Y^o_k \ar[r]^{\pi^o_k, \text{dominant}} & X^o_k \ar[r]^{\varphi^o_k, \text{dominant}} & S^o_k \subseteq \mathbb{P}^1_k,}
$$

and use the Mordell-type theorem for $C$-integral points, Theorem 3.12, to show the stronger statement that $\pi^o_k(Y^o_k(k(t))) \subseteq X^o_k(k(t))$ is finite. To this end, choose compactifications,

$$
\xymatrix{ Y_k \ar[r]^{\pi_k, \text{dominant}} & X_k \ar[r]^{\varphi_k, \text{dominant}} & S_k = \mathbb{P}^1_k,
$$

such that $X_k$ is $k$-smooth and $(X_k, D_k)$ is snc, where $D_k$ is the Zariski-closure of the divisor $D^o_k$. Recalling that smoothness and non-vanishing of the Kodaira- Spencer map remain invariant when passing from $\mathbb{F}_p$ to $k$, we find that the morphism $\varphi_k : X_k \to S_k$ and the pair $(X_k, D_k)$ satisfy all assumptions made in Theorem 3.12. In particular, there are at most finitely many $C$-integral points $S_k \to X_k$. To conclude, we need to check that given any $k(t)$-valued point $y \in Y^o_k(k(t))$, or equivalently any section $\gamma : S_k \to Y_k$, then either $(\pi \circ \gamma)$ is $C$-integral, or $\text{Image}(\pi \circ \gamma) \subseteq \text{supp} D$. Recalling from (11.11.1) that $\pi^o_k$ is smooth over an open of $X^o_k$, this follows from Lemma 11.4; hence we are done. □
12. Construction of the example — proof of Theorem 1.1

The goal of this paragraph is to prove Theorem 1.1. We will explain how to construct examples of simply connected fourfolds over global fields for which the failure of the local-global principle is not explained by a Brauer–Manin obstruction. We will mimic the construction presented in [Sme17, Prop. 3.2]. While most of the arguments given there carry over to our setting, the construction in [Sme17] is done over number fields. A few adjustments are required.

12.1. Comparison of Brauer groups in conic bundles. The following preliminary lemma extends [CTPS16, Prop. 2.2.(i)] to positive characteristic. For the sake of simplicity, we restrict ourselves to the case where the characteristic is odd. We do expect, however, that the result should remain true even in characteristic two.

Lemma 12.1 (Comparison of Brauer groups in conic bundles). Let $B$ be a smooth, projective, geometrically integral variety over a field $K$ of characteristic different from two. Let $f : W \to B$ be a conic bundle, i.e., a surjective, flat morphism from a smooth, projective, geometrically integral $K$-variety $W$ to $B$, the generic fibre of which is a conic. Assume that there exists a codimension-one point $P$ on $B$ such that for any other codimension-one point $Q \neq P$ on $B$, the fibre $f^{-1}(Q)$ is a smooth conic. Then, the induced morphism $f^* : Br(B) \to Br(W)$ is surjective.

Proof. The argument is based on the proof of [CTPS16, Prop. 2.2.(i)], with some modifications needed to deal with the typical subtleties in positive characteristic. Let $\eta = \text{Spec} K(B)$ be the generic point of $B$, the generic fibre $W_\eta$ is then a smooth conic over $K(B)$. The following diagram summarises the pull-back morphisms between the Brauer groups that will be relevant for us,

$$
\begin{array}{ccc}
\text{Br}(B) & \xrightarrow{\iota_\eta} & \text{Br}(\eta) \\
\downarrow f^* & & \downarrow f^*_\eta, \text{surjective} \\
\text{Br}(W) & \xrightarrow{\iota_{W_\eta}} & \text{Br}(W_\eta).
\end{array}
$$

Fix a prime $\ell$, possibly equal to the characteristic $p$ of $K$, and let $\alpha \in \text{Br}(W)[\ell^\infty]$. We need to show that there exists a class $\beta \in \text{Br}(B)$ with $f^*(\beta) = \alpha$. If $\ell$ is prime to $p$ (in particular, if $\ell = 2$) then the argument from [CTPS16, Prop. 2.2.(i)], which uses residues, works verbatim. Let us therefore assume that $\ell = p$. Set $\alpha_\eta := \iota_{W_\eta}^* \alpha$. Since $f^*_\eta$ is surjective, there exists an element $\beta_\eta \in \text{Br}(\eta)$ which maps to $\alpha_\eta$. We claim that $\beta_\eta \in \text{Image} \iota_\eta$. A well-known purity result for the Brauer group, [Gab93, 2.5], implies that it suffices to prove the following: if $Q \in B$ is a point of codimension 1 on the base, inducing a discrete valuation on $K(B)$ with valuation ring $R = \mathcal{O}_{B,Q}$, then $\beta_\eta \in \text{Br}(R)$.

Assume that one such $Q$ is given, and write $v$ for the associated valuation on $K(B)$. The discussion in [CT11, §3.5] implies that the smooth $K(B)$-conic $W_\eta$ admits a diagonal model over $R$ given by an equation of the form $x^2 - ay^2 - bz^2 = 0$, where $a, b \in R^*$ and $\text{gcd}(a, b, \ell) = 1$. Since $f^*_\eta$ is surjective, the image of $\beta_\eta$ under $f^*_\eta$ is an element $\beta_\eta^* \in \text{Br}(W)$.
where \( a \in \mathbb{R}_Q^\times \), and \( v(b) \in \{0,1\} \). In fact, if \( Q \) is not the point \( P \) from the statement of the lemma, then both \( a \) and \( b \) can be chosen in \( \mathbb{R}_Q^\times \). In any case, the ring \( R' = R[\sqrt{a}] \) is a quadratic unramified extension of \( R \), and if \( \eta' \) denotes the generic point of \( \text{Spec} R' \), then \( W_\eta \) acquires an \( \eta' \)-point – equivalently, \( W_R \) acquires an \( R' \)-point. Since \( \beta_\eta \) maps to \( \alpha_\eta \), the restriction \( \beta_\eta \in \Br(W_\eta) \) maps to \( \alpha_\eta \in \Br(W_\eta') \). But \( \alpha_\eta \in \Br(W_R) \) since \( a \in \Br(W) \); evaluating on the \( R' \)-point mentioned above shows that \( \beta_\eta \in \Br(R' \}. \) Corestricting from \( R' \) to \( R \) (see [CTS21, §3.8]) then shows that \( 2\beta \in \Br(R) \). Since \( p \neq 2 \), we conclude that \( \beta \in \Br(R) \), as required. \( \square \)

12.2. Families of Châtelet surfaces. The second main ingredient in our proof of Theorem 1.1 is the following explicit construction of families of Châtelet surfaces. The idea of using Châtelet surfaces to address the problem at hand goes back to Poonen.

**Proposition 12.2** (Families of Châtelet surfaces). Let \( K \) be a global field of odd characteristic. Let \( Y \) be a smooth, projective, geometrically integral \( K \)-variety. Assume there exists a dominant, proper morphism \( f : Y \to \mathbb{P}_K^1 \) with geometrically integral generic fibre, such that \( f(Y(K)) \) is finite. Then, there exists a family of Châtelet surfaces \( g : S \to \mathbb{P}_K^1 \) such that the fibre product \( Z := Y \times_{\mathbb{P}_K^1} S \) is a smooth, projective and geometrically integral \( K \)-variety for which \( Z(K) = \emptyset \), even though \( Z(\mathbb{A}_K)_{\text{ét,Br}} \neq \emptyset \).

**Proof.** Let us recall the construction from [Sme17, Prop. 3.2]. Let \( g : \mathbb{P}_K^1 \to \mathbb{P}_K^1 \) be a morphism which maps \( f(Y(K)) \) to \( \{\infty\} \). We can then find an integer \( n \gg 0 \) and a section \( s \in \Gamma(\mathbb{P}_K^1 \times_K \mathbb{P}_K^1, \mathcal{O}(n,2)^{\otimes 2}) \), such that the following conditions are all satisfied:

1. (12.2.1) The vanishing locus of \( s \) is a smooth, proper, geometrically integral \( K \)-curve \( C \).
2. (12.2.2) The composition \( C \to \mathbb{P}_K^1 \times_K \mathbb{P}_K^1 \to \mathbb{P}_K^1 \) is étale at points where the composed morphism \( h := g \circ f \) is not smooth.
3. (12.2.3) For suitable \( a \in K \), the equation \( y^2 - ax^2 = s \) defines a conic bundle \( S \to \mathbb{P}_K^1 \times_K \mathbb{P}_K^1 \) such that its restriction \( \overline{S}_\infty := S|_{\{\infty\} \times \mathbb{P}_K^1} \) is a Châtelet surface with \( \overline{S}_\infty(K) = \emptyset \), even though \( \overline{S}_\infty(\mathbb{A}_K) \neq \emptyset \).

Over number fields, the existence of a Châtelet surface \( \overline{S}_\infty \), as in Item (12.2.3) follows from [Poo09, Prop. 5.1]. Over global function fields of odd characteristic, the existence of such an \( \overline{S}_\infty \) is proven in [Poo09, §11]. Once such a surface \( \overline{S}_\infty \) has been chosen, the proof of [Sme17, Prop. 3.2] (heavily based on [Poo09, § 6 and Lem. 7.1]) yields the existence of \( s \) as above, and the composition

\[ \overline{S} \to \mathbb{P}_K^1 \times_K \mathbb{P}_K^1 \to \mathbb{P}_K^1 \]

is a family of Châtelet surfaces. Let \( g : S \to \mathbb{P}_K^1 \) be the pullback of this family along \( g \); we claim that this yields the desired family for the statement of Proposition 12.2.

**Claim 12.3.** Writing \( Z := Y \times_{\mathbb{P}_K^1} S \), the morphism \( Z \to Y \) induces an equivalence of between the categories of finite étale covers, \( \text{Ét}(Y) \to \text{Ét}(Z) \). If, moreover, \( G \)
is a finite étale group scheme over $K$ and if $Z' \to Z$ denotes any right $G$-torsor, there exists a right $G$-torsor $Y' \to Y$ such that $Z' \to Z$ and $Z \times_Y Y' \to Z$ are isomorphic as $G$-torsors.

**Proof of Claim 12.3.** Both statements follow from the fact that all geometric fibres of $Z \to Y$ are reduced and have trivial étale fundamental group. For more details, we refer to the proofs of [Poo10a, Lem. 8.1] and [CTPS16, Prop. 2.3]; the arguments given there are purely algebraic and work in any characteristic. □ (Claim 12.3)

The arguments in the proof of [Sme17, Prop. 3.2] now work verbatim to show that the family of surfaces $g : S \to \mathbb{P}^1_K$ constructed above satisfies all requirements. To be precise, the geometric properties of $Z$ are a consequence of (12.2.1) and (12.2.2). Next, $Z(K) = \emptyset$ follows from the fact all fibres of $Z \to Y$ over $K$-rational points of $Y$ are isomorphic to $\widetilde{S}_{\infty}$, which satisfies $\widetilde{S}_{\infty}(K) = \emptyset$ by (12.2.3). Finally, $Z(\mathbb{A}_K)^{\text{ét,Br}} \neq \emptyset$ is a consequence of Lemma 12.1 and the fact that $\widetilde{S}_{\infty}(\mathbb{A}_K) \neq \emptyset$, again by (12.2.3).

**Remark 12.4.** As in [Sme17, Rem. 3.3], one sees that $\pi_1^{\text{ét}}(Z) \cong \pi_1^{\text{ét}}(Y)$ in the situation of Proposition 12.2: this follows from the arguments given in [Sme17] since conics, whether they are smooth or singular, are geometrically simply connected in any characteristic.

**Remark 12.5.** The proof of Proposition 12.2 starts by constructing a section $s$ of the bundle $\mathcal{O}(n,2)^{\otimes 2}$. If the characteristic of $K$ is equal to two, one can still find $s$ such that the surface defined by the modified equation $y^2 + yz + az^2 = s$ satisfies Properties (12.2.1)–(12.2.3). The existence of a suitable Châtelet surface $\widetilde{S}_{\infty}$ in characteristic two follows from a result of Viray, [Vir10, Thm. 1.1]. We chose to restrict ourselves to odd characteristic because our previous result, Lemma 12.1, has been stated only in this setup.

### 12.3. Proof of Theorem 1.1

Theorem 1.4 yields the existence of a global field $K$ and a smooth, projective, geometrically simply connected $K$-surface $S$, which comes equipped with a dominant, proper morphism $f : S \to \mathbb{P}^1_K$, such that $f(S(K))$ is finite and non-empty. Proposition 12.2 then yields a smooth, projective fourfold over $K$ such that $Z(K) = \emptyset$, whereas $Z(\mathbb{A}_K)^{\text{ét,Br}} \neq \emptyset$. We have seen in Remark 12.4 that $Z$ is geometrically simply connected, so that $Z(\mathbb{A}_K)^{\text{ét,Br}} = Z(\mathbb{A}_K)^{\text{Br}}$. Theorem 1.1 is therefore shown. □

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