THE HASSE PRINCIPLE FOR RANDOM FANO HYPERSURFACES

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ABSTRACT. It is known that the Brauer–Manin obstruction to the Hasse principle is vacuous for smooth Fano hypersurfaces of dimension at least 3 over any number field. Moreover, for such varieties it follows from a general conjecture of Colliot-Thélène that the Brauer–Manin obstruction to the Hasse principle should be the only one, so that the Hasse principle is expected to hold. Working over the field of rational numbers and ordering Fano hypersurfaces of fixed degree and dimension by height, we prove that almost every such hypersurface satisfies the Hasse principle provided that the dimension is at least 3. This proves a conjecture of Poonen and Voloch in every case except for cubic surfaces.

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1. Introduction

Let \( d, n \geq 2 \) be such that \( n \geq d \) and let \( N_{d,n} = \binom{n+d}{d} \) be the number of monomials of degree \( d \) in \( n+1 \) variables. Ordering monomials lexicographically, degree \( d \) hypersurfaces in \( \mathbb{P}^n \) that are defined over \( \mathbb{Q} \) are parametrized by \( \mathbb{V}_{d,n} = \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}) \). It follows from the assumption \( n \geq d \) that a generic element of \( \mathbb{V}_{d,n} \) is a smooth Fano hypersurface.

We shall order elements of \( \mathbb{V}_{d,n} \) using the usual exponential height on projective space. With this in mind, for any \( N \geq 1 \), let \( \mathbb{Z}_{\text{prim}}^N \) be the set of \( (c_1, \ldots, c_N) \in \mathbb{Z}^N \) such that \( \gcd(c_1, \ldots, c_N) = 1 \) and let \( \| \cdot \| \) be the Euclidean norm in \( \mathbb{R}^N \). The height of \( V \in \mathbb{V}_{d,n} \) is then defined to be \( \| a_V \| \) where \( a_V \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \) denotes any of the two primitive coefficient vectors associated to \( V \). Moreover, for any \( A \geq 1 \), we let

\[ \mathbb{V}_{d,n}(A) = \{ V \in \mathbb{V}_{d,n} : \| a_V \| \leq A \}. \]

The primary goal of this article is to investigate the asymptotic behaviour of the quantity

\[ \varrho_{d,n}(A) = \frac{\# \{ V \in \mathbb{V}_{d,n}(A) : V(\mathbb{Q}) \neq \emptyset \}}{\# \mathbb{V}_{d,n}(A)}, \]

as \( A \to \infty \). The ratio \( \varrho_{d,n}(A) \) is the proportion of degree \( d \) hypersurfaces in \( \mathbb{P}^n \) which are defined over \( \mathbb{Q} \), have height at most \( A \), and admit a rational point.

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For any $V \in V_{d,n}$, we let $V(\mathbb{A}_\mathbb{Q})$ denote the set of adèles of $V$. We introduce the set $V_{d,n}^{\text{loc}}$ of elements of $V_{d,n}$ that are everywhere locally soluble, that is

$$V_{d,n}^{\text{loc}} = \{ V \in V_{d,n} : V(\mathbb{A}_\mathbb{Q}) \neq \emptyset \}.$$  

We also let $V_{d,n}^{\text{loc}}(A) = V_{d,n}^{\text{loc}} \cap V_{d,n}(A), \quad (1.1)$

and we denote the density of the set $V_{d,n}^{\text{loc}}$ by

$$\varrho_{d,n}^{\text{loc}} = \lim_{A \to \infty} \frac{\# V_{d,n}^{\text{loc}}(A)}{\# V_{d,n}(A)},$$

whenever this limit exists. In the case $(d, n) = (2, 2)$, work of Serre [18, Exemple 4] shows that a typical rational plane conic is not everywhere locally soluble, that is $\varrho_{2,2}^{\text{loc}} = 0$. (1.2)

Note that a far-reaching interpretation of this phenomenon can be found in recent work of Loughran [15]. If $(d, n) \neq (2, 2)$, Poonen and Voloch prove [16, Theorem 3.6] that $\varrho_{d,n}^{\text{loc}}$ exists, is equal to a product of local densities and moreover

$$\varrho_{d,n}^{\text{loc}} > 0. \quad (1.3)$$

Put another way, the proportion of degree $d$ hypersurfaces in $\mathbb{P}^n$ defined over $\mathbb{Q}$, which are everywhere locally soluble, exists and is positive. Furthermore, Poonen and Voloch conjecture [16, Conjecture 2.2.(ii)] that $\varrho_{d,n}(A)$ tends to a limit as $A \to \infty$ and

$$\lim_{A \to \infty} \varrho_{d,n}(A) = \varrho_{d,n}^{\text{loc}}. \quad (1.4)$$

They check [16, Proposition 3.4] that their prediction follows from Colliot-Thélène’s conjecture [11] that for smooth, proper, geometrically integral and rationally connected varieties, the Brauer–Manin obstruction to the Hasse principle is the only one. Indeed, Colliot-Thélène shows in an appendix to their work [16, Corollary A.2] that there is no Brauer–Manin obstruction when $n \geq 4$. The remaining case $(d, n) = (3, 3)$ of cubic surfaces relies on a result of Swinnerton-Dyer [19] asserting that the Brauer–Manin obstruction is vacuous when the action of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the 27 lines is the full Weyl group $W(E_6)$. The equality (1.4) then follows from Hilbert’s irreducibility theorem.

We remark that in the case of general type hypersurfaces, that is when $d > n + 1$, Poonen and Voloch [16, Conjecture 2.2.(i)] conjecture that the ratio $\varrho_{d,n}(A)$ should approach 0 as $A \to \infty$, but this range of variables lies outside the scope of the present investigation.

The expectation (1.4) holds if either $d = 2$ or $n \geq (d - 1)2^d$, as it follows respectively from the Hasse–Minkowski theorem and the celebrated work of Birch [4]. In addition, we also note that in the setting of diagonal hypersurfaces, Brüdern and Dietmann have confirmed in [7, Theorem 1.3] that the analogue of the equality (1.4) holds under the assumption $n > 3d$.

The following is our main result and only leaves open the case of cubic surfaces in the Poonen–Voloch conjecture for Fano hypersurfaces.

**Theorem 1.1.** Let $d \geq 2$ and $n \geq d$ with $(d, n) \neq (3, 3)$. Then we have

$$\lim_{A \to \infty} \varrho_{d,n}(A) = \varrho_{d,n}^{\text{loc}}.$$
In other words, in the Fano range \( n \geq d \) and when degree \( d \) hypersurfaces in \( \mathbb{P}^n \) that are defined over \( \mathbb{Q} \) are ordered by height, 100\% of these hypersurfaces satisfy the Hasse principle provided that \( (d, n) \neq (3,3) \). In fact, it transpires from Propositions 2.3 and 2.4 that

\[
\frac{\# \{ V \in \mathcal{V}_{d,n}^\text{loc}(A) : V(\mathbb{Q}) = \emptyset \}}{\# \mathcal{V}_{d,n}(A)} \ll \frac{1}{(\log A)^{1/48n}}.
\]

We have not made any effort to optimise the exponent of \( \log A \) in this upper bound.

Unfortunately, in the case \( (d, n) = (3,3) \) our understanding of the geometry of the lattices involved in our work does not allow us to establish the equality (1.4). However we can still show that a positive proportion of cubic surfaces have a rational point. Indeed, it suffices to consider the more stringent constraint in which the rational points are restricted to lie in one of the coordinate hyperplanes. This reduces the analysis to the case of plane cubic curves and we can therefore appeal to the work of Bhargava [3, Theorem 2] to conclude. Recalling the lower bound (1.3), we see that we have the following corollary of Theorem 1.1.

**Corollary 1.2.** Let \( d \geq 2 \) and \( n \geq d \) with \( (d, n) \neq (2,2) \). Then we have

\[
\liminf_{A \to \infty} \varrho_{d,n}(A) > 0.
\]

Corollary 1.2 states that, putting aside the particular case of plane conics, a positive proportion of Fano hypersurfaces of fixed degree and dimension admit a rational point.

Our methods actually allow us to prove a much stronger result than Theorem 1.1. Indeed, as stated in Theorem 2.2, we are able to estimate in an optimal way the smallest height of a rational point on a hypersurface for 100\% of everywhere locally soluble hypersurfaces.

The proof of Theorem 2.2 relies upon various arguments coming from the geometry of numbers, together with a careful study of local densities. To establish results such as Theorem 2.2, it is customary to prove that the number of rational points of bounded height on a hypersurface is on average well-approximated by an adequate quantity, which is traditionally taken to be the main term in the asymptotic formula predicted by the Hardy–Littlewood circle method. Unfortunately in our setting this object is rather complicated to analyse and we shall replace it by a carefully chosen localised counting function, which is designed to approximate the Hardy–Littlewood expectation and yet remain amenable to analysis via the geometry of numbers.

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2. Roadmap of the proof

Our purpose in this section is to describe our strategy for proving Theorem 1.1. Associated to any Fano hypersurface \( V \in \mathcal{V}_{d,n} \) is the anticanonical height function \( H : V(\mathbb{Q}) \to \mathbb{R}_{>0} \) metrised by the Euclidean norm \( \| \cdot \| \) in \( \mathbb{R}^{n+1} \). Thus, for \( x \in V(\mathbb{Q}) \) we choose \( x = (x_0, \ldots, x_n) \in \mathbb{Z}_{\text{prim}}^{n+1} \) such that \( x = (x_0 : \cdots : x_n) \) and we set
\[
H(x) = \|x\|^{n+1-d}. \tag{2.1}
\]
This allows us to define the counting function
\[
N_V(B) = \# \{ x \in V(\mathbb{Q}) : H(x) \leq B \}. \tag{2.2}
\]
To tackle Theorem 1.1 we will first show that \( N_V(B) \) is on average well-approximated by a certain localised counting function, and we will then prove that the localised counting function is only rarely smaller than its expected value.

Given \( N \geq 1 \) and \( c \in \mathbb{Z}^N \), of special importance in our work is the integral lattice
\[
\Lambda_c = \{ y \in \mathbb{Z}^N : \langle c, y \rangle = 0 \}, \tag{2.3}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the usual Euclidean inner product in \( \mathbb{R}^N \). In addition, it will be very convenient to introduce the following notation.

**Definition 2.1.** Given \( d, n \geq 1 \), we let \( \nu_{d,n} : \mathbb{R}^{n+1} \to \mathbb{R}^{N_{d,n}} \) denote the Veronese embedding, defined by listing all the monomials of degree \( d \) in \( n+1 \) variables using the lexicographical ordering.

We see that
\[
N_V(B) = \frac{1}{2} \sum_{x \in \Xi_{d,n}(B)} 1, \tag{2.4}
\]
where \( a_V \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \) denotes any of the two primitive coefficient vectors associated to \( V \), and where we have set
\[
\Xi_{d,n}(B) = \left\{ x \in \mathbb{Z}_{\text{prim}}^{n+1} : \|x\| \leq B^{1/(n+1-d)} \right\}. \tag{2.5}
\]
Manin’s conjecture [13] gives a precise prediction for the asymptotic behaviour of \( N_V(B) \) as \( B \to \infty \) for Fano hypersurfaces \( V \in \mathcal{V}_{d,n} \). We remark that most \( V \in \mathcal{V}_{d,n} \) do not possess accumulating thin subsets and have Picard group isomorphic to \( \mathbb{Z} \). Thus \( N_V(B) \) is expected to grow linearly in terms of \( B \) whenever \( V(\mathbb{Q}) \) is Zariski dense in \( V \).

The localised counting function we work with is chosen to mimic the main term in this expected asymptotic formula.

For any \( N \geq 1 \), any real \( \gamma > 0 \) and \( \nu \in \mathbb{R}^N \), we introduce the region
\[
C^{(\gamma)}_\nu = \left\{ t \in \mathbb{R}^N : |\langle \nu, t \rangle| \leq \frac{\|\nu\| \cdot \|t\|}{2\gamma} \right\}, \tag{2.6}
\]
and for any \( Q \geq 1 \) and \( c \in \mathbb{Z}^N \), we define the lattice
\[
\Lambda^{(Q)}_c = \{ y \in \mathbb{Z}^N : \langle c, y \rangle \equiv 0 \mod Q \}. \tag{2.7}
\]
Furthermore, we set
\[
\alpha = \log B, \tag{2.8}
\]
and
\[
W = \prod_{p \leq w} p^{[\log w / \log p] + 1}, \tag{2.9}
\]
where
\[
w = \frac{\log B}{\log \log B}. \tag{2.10}
\]
Our localised counting function is then defined as
\[
N^{loc}_{V}(B) = \frac{1}{2} \cdot \frac{\alpha W}{\|a_{V}\|} \sum_{x \in \xi_{d,n}(B)} \frac{1}{\|v_{d,n}(x)\|}.
\] (2.11)

The main contribution to \( \log W \) comes from the primes \( p \in (w^{1/2}, w) \). Therefore, an application of the prime number theorem reveals that \( \log W \sim 3w \), which implies in particular
\[
W \ll B^{4/\log \log B}.
\] (2.12)

We thus see that \( \alpha \) and \( W \) both tend to infinity rather slowly with respect to \( B \). This fact together with the observation that \( W \) becomes more and more divisible as \( B \) grows will turn out to be crucial in our argument.

Our methods not only allow us to prove that 100\% of the everywhere locally soluble Fano hypersurfaces \( V \in \mathcal{V}_{d,n} \) admit a rational point, but we actually obtain an upper bound for the smallest height of a rational point on \( V \). Recall the definition (2.1) of the anticanonical height \( H \). For any \( V \in \mathcal{V}_{d,n} \), it is convenient to define
\[
\mathfrak{M}(V) = \begin{cases} 
\min_{x \in V(\mathbb{Q})} H(x), & \text{if } V(\mathbb{Q}) \neq \emptyset, \\
\infty, & \text{if } V(\mathbb{Q}) = \emptyset.
\end{cases}
\]

The interested reader is invited to refer to the introduction of [14] for a survey of works on the quantity \( \mathfrak{M}(V) \) in the setting of Fano hypersurfaces.

Theorem 1.1 is an immediate consequence of the following result.

**Theorem 2.2.** Let \( d \geq 2 \) and \( n \geq d \) with \( (d, n) \neq (3, 3) \). Let \( \psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) be such that \( \psi(u)/u \to \infty \) as \( u \to \infty \). Then we have
\[
\lim_{A \to \infty} \frac{\# \{ V \in \mathcal{V}_{d,n}(A) : \mathfrak{M}(V) \leq \psi(\|a_{V}\|) \}}{\# \mathcal{V}_{d,n}(A)} = \varrho^{loc}_{d,n}.
\]

Combining Theorem 2.2 with work of the second author [14, Theorem 1], we deduce that if \( (d, n) \notin \{ (2, 2), (3, 3) \} \) and if \( \xi : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) is any function satisfying \( \xi(u) \to \infty \) as \( u \to \infty \), then for 100\% of the everywhere locally soluble Fano hypersurfaces \( V \in \mathcal{V}_{d,n} \) we have the optimal inequalities
\[
\frac{1}{\xi(\|a_{V}\|)} \leq \frac{\mathfrak{M}(V)}{\|a_{V}\|} \leq \xi(\|a_{V}\|).
\]

In the case \( d = 2 \), Cassels’ celebrated bound [8] states that for any hypersurface \( V \in \mathcal{V}^{loc}_{2,n} \), we have
\[
\mathfrak{M}(V) \ll \|a_{V}\|^{n(n-1)/2},
\] (2.13)

where the implied constant depends at most on \( n \). Note that the exponent is known to be optimal thanks to Kneser’s example [9]. Theorem 2.2 shows that for typical quadratic hypersurfaces \( V \in \mathcal{V}^{loc}_{2,n} \), Cassels’ upper bound (2.13) is very far from the truth as soon as \( n \geq 3 \).

Heuristically we expect the counting functions \( N_{V}(B) \) and \( N^{loc}_{V}(B) \) to be of exact order \( B/A \), for generic \( V \in \mathcal{V}_{d,n}(A) \). Our first result shows that when the ratio \( B/A \) tends to \( \infty \) sufficiently slowly with respect to \( A \) then it is rare for \( N_{V}(B) \) not to be well-approximated by \( N^{loc}_{V}(B) \), as \( V \) runs over the set \( \mathcal{V}_{d,n}(A) \). We stress that here and throughout Sections 4 and 5, all the implied constants depend at most on \( d \) and \( n \) unless specified otherwise.
Proposition 2.3. Let $d \geq 2$ and $n \geq d$ with $(d, n) \notin \{(2, 2), (3, 3)\}$. Let $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{>1}$ be such that $\phi(A) \leq (\log A)^{1/2}$. Then we have

$$\frac{1}{\# V_{d,n}(A)} \cdot \# \left\{ V \in V_{d,n}(A) : \left| N_{V}(A\phi(A)) - N_{V}^{\text{loc}}(A\phi(A)) \right| > \phi(A)^{2/3} \right\} \ll \frac{1}{\phi(A)^{1/3}}.$$  

Proposition 2.3 will follow directly from Proposition 4.1, which provides a sharp upper bound for the variance

$$\sum_{V \in V_{d,n}(A)} \left( N_{V}(A\phi(A)) - N_{V}^{\text{loc}}(A\phi(A)) \right)^{2}.$$  

In order to prove Proposition 4.1 we will start in Section 3 by gathering a series of tools coming from the geometry of numbers. Section 4 will then be devoted to the proof of Proposition 4.1.

Our second major ingredient in the proof of Theorem 2.2 states that if the ratio $B/A$ tends to $\infty$ as $A$ tends to $\infty$ and satisfies a mild upper bound, then it is rare for the localised counting function $N_{V}^{\text{loc}}(B)$ to be smaller than its expected size, as $V$ runs over the set $V_{d,n}(A)$.

Proposition 2.4. Let $d \geq 2$ and $n \geq d$ with $(d, n) \neq (2, 2)$. Let $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{>1}$ be such that $\phi(A) \leq A^{3/n}$. Then we have

$$\frac{1}{\# V_{d,n}^{\text{loc}}(A)} \cdot \# \left\{ V \in V_{d,n}^{\text{loc}}(A) : N_{V}^{\text{loc}}(A\phi(A)) \leq \phi(A)^{2/3} \right\} \ll \frac{1}{\phi(A)^{1/2n}}.$$  

Proposition 2.4 will be established in Section 5. The proof will consist in checking that certain non-Archimedean and Archimedean factors that are hidden in our localised counting function $N_{V}^{\text{loc}}(A\phi(A))$ are rarely small as $V$ runs over $V_{d,n}^{\text{loc}}(A)$, as stated in Propositions 5.2 and 5.3. It may be worth noting that Section 5 is independent from Sections 3 and 4.

We now proceed to prove that Theorem 2.2 follows from Propositions 2.3 and 2.4.

Proof of Theorem 2.2. The case $(d, n) = (2, 2)$ of plane conics is a direct consequence of the equality (1.2) so we assume that $(d, n) \neq (2, 2)$. We set

$$\mathcal{P}_{d,n}(A) = \# \left\{ V \in V_{d,n}^{\text{loc}}(A) : \mathfrak{M}(V) > \psi(||a_{V}||) \right\},$$

and we observe that our goal is to prove that

$$\lim_{A \to \infty} \frac{\mathcal{P}_{d,n}(A)}{\# V_{d,n}(A)} = 0.$$  

Let $\eta \in (0, 1)$. Since we clearly have $V_{d,n}^{\text{loc}}(\eta A) \subset V_{d,n}(\eta A)$ and $\# V_{d,n}(\eta A) \ll \eta^{N_{d,n}} A^{N_{d,n}}$, we see that

$$\mathcal{P}_{d,n}(A) = \# \left\{ V \in V_{d,n}^{\text{loc}}(A) : ||a_{V}|| > \eta A \mathfrak{M}(V) > \psi(||a_{V}||) \right\} + O \left( \eta^{N_{d,n}} A^{N_{d,n}} \right).$$

By assumption if $A$ is large enough then for any $u \geq \eta A$ we have $\psi(u) \geq u/\eta^{2}$ and thus $\psi(u) \geq A/\eta$. We deduce that

$$\mathcal{P}_{d,n}(A) \ll \# \left\{ V \in V_{d,n}^{\text{loc}}(A) : \mathfrak{M}(V) > \frac{A}{\eta} \right\} + \eta^{N_{d,n}} A^{N_{d,n}}.$$  

Next, we note that if a hypersurface $V \in V_{d,n}^{\text{loc}}(A)$ satisfies the lower bound $\mathfrak{M}(V) > A/\eta$ then $N_{V}(A/\eta) = 0$, which implies that we have either

$$\left| N_{V}\left( \frac{A}{\eta} \right) - N_{V}^{\text{loc}}\left( \frac{A}{\eta} \right) \right| > \frac{1}{\eta^{2/3}}.$$  

(2.16)
or
\[
N^\text{loc}_V \left( \frac{A}{\eta} \right) \leq \frac{1}{\eta^{2/3}}. \tag{2.17}
\]

As a result, taking \( \phi(A) = 1/\eta \) in Propositions 2.3 and 2.4 to bound the number of \( V \in V^\text{loc}_{d,n}(A) \) satisfying either the lower bound (2.16) or the upper bound (2.17), we derive
\[
\frac{1}{\#V_{d,n}(A)} \cdot \# \left\{ V \in V^\text{loc}_{d,n}(A) : \mathfrak{m}(V) > \frac{A}{\eta} \right\} \ll \eta^{1/3} + \eta^{1/24n} \frac{\#V^\text{loc}_{d,n}(A)}{\#V_{d,n}(A)} 
\ll \eta^{1/24n}. \tag{2.18}
\]

Note that we have used the fact that \( \eta \in (0, 1) \). We now remark that we have the lower bound
\[
\#V_{d,n}(A) \gg A^{N_{d,n}}. \tag{2.19}
\]

Hence, putting together the upper bounds (2.15) and (2.18) we obtain
\[
\limsup_{A \to \infty} \frac{\mathcal{P}_{d,n}(A)}{\#V_{d,n}(A)} \ll \eta^{1/24n}.
\]

Since this upper bound holds for any \( \eta \in (0, 1) \) we see that the equality (2.14) follows, which completes the proof of Theorem 2.2.

\[\square\]

3. Tools from the geometry of numbers

Our goal in this section is to gather all the geometry of numbers results that we will need to establish Proposition 2.3. In Section 3.1 we start by recalling classical facts about lattices in \( \mathbb{R}^N \) and we prove a series of lattice point counting estimates. We will then calculate the determinants of certain lattices in Section 3.2. Our next task in Section 3.3 will be to investigate the size of the successive minima of the key lattices \( \Lambda_{d,n}(x) \) and \( \Lambda_{d,n}(x) \cap \Lambda_{d,n}(y) \) for given linearly independent vectors \( x, y \in \mathbb{Z}^N \).

In Section 3.4 we will then turn to estimating the typical size of some key quantities uncovered in Section 3.3. Finally, in Section 3.5 we will collect some further results that we will require to handle the specific case of quartic threefolds.

3.1. Basic facts and lattice point counting estimates. Let \( N \geq 1 \). A lattice \( \Lambda \subset \mathbb{R}^N \) is a discrete subgroup of \( \mathbb{R}^N \). The dimension of the subspace \( \text{Span}_{\mathbb{R}}(\Lambda) \) is called the rank of \( \Lambda \). If \( \Lambda \) is a rank \( R \) lattice and if \( (b_1, \ldots, b_R) \) is any basis of \( \Lambda \) then the determinant of \( \Lambda \) is the \( R \)-dimensional volume of the fundamental parallelepiped spanned by \( (b_1, \ldots, b_R) \), and is thus given by
\[
\det(\Lambda) = \sqrt{\det(B^T B)}, \tag{3.1}
\]

where \( B \) is the \( N \times R \) matrix whose columns are the vectors \( b_1, \ldots, b_R \).

Let \( \Lambda \subset \mathbb{R}^N \) be a lattice and \( \Gamma \) be a sublattice of \( \Lambda \) such that \( \text{Span}_{\mathbb{R}}(\Gamma) \cap \Lambda = \Gamma \), which is equivalent to saying that any basis of \( \Gamma \) can be extended into a basis of \( \Lambda \). Letting \( \pi : \mathbb{R}^N \to \text{Span}_{\mathbb{R}}(\Gamma) ^\perp \) denote the orthogonal projection on \( \text{Span}_{\mathbb{R}}(\Gamma) ^\perp \), we define the quotient lattice of \( \Lambda \) by \( \Gamma \) by
\[
\Lambda/\Gamma = \pi(\Lambda).
\]

It is clear that the rank of \( \Lambda \) is equal to the sum of the ranks of \( \Gamma \) and \( \Lambda/\Gamma \). Moreover, the calculation of the determinant of \( \Lambda \) using a basis of \( \Lambda \) extending a basis of \( \Gamma \) yields the equality
\[
\det(\Lambda/\Gamma) = \frac{\det(\Lambda)}{\det(\Gamma)}. \tag{3.2}
\]
In addition, a lattice $\Lambda \subset \mathbb{R}^N$ is said to be integral if $\Lambda \subset \mathbb{Z}^N$. Moreover, an integral lattice $\Lambda$ of rank $R$ is said to be primitive if it is not properly contained in another integral lattice of rank $R$, that is if $\text{Span}_\mathbb{R}(\Lambda) \cap \mathbb{Z}^N = \Lambda$. Given an integral lattice $\Lambda \subset \mathbb{Z}^N$, the lattice $\Lambda^\perp$ orthogonal to $\Lambda$ is defined by
\[ \Lambda^\perp = \{ a \in \mathbb{Z}^N : \forall z \in \Lambda \langle a, z \rangle = 0 \}. \]
It is clear that $\Lambda^\perp$ is a primitive lattice of rank $N - R$, and we see that $(\Lambda^\perp)^\perp = \Lambda$ if and only if $\Lambda$ is a primitive lattice. Furthermore, if $\Lambda$ is primitive then (see for example [17, Corollary of Lemma 1]) we have
\[ \det(\Lambda^\perp) = \det(\Lambda). \quad (3.3) \]

The dual lattice $\Lambda^*$ of a lattice $\Lambda \subset \mathbb{R}^N$ is defined by
\[ \Lambda^* = \{ a \in \text{Span}_\mathbb{R}(\Lambda) : \forall z \in \Lambda \langle a, z \rangle \in \mathbb{Z} \}. \]
It is easy to check (see for instance [10, Chapter I, Lemma 5]) that the lattices $\Lambda^*$ and $\Lambda$ have equal rank, and moreover
\[ \det(\Lambda^*) = \frac{1}{\det(\Lambda)}. \quad (3.4) \]

The following result is due to Schmidt [17, Lemma 1].

**Lemma 3.1.** Let $N \geq 1$ and let $\Lambda \subset \mathbb{Z}^N$ be a primitive lattice. We have
\[ (\Lambda^\perp)^\ast = \mathbb{Z}^N / \Lambda. \]

For any $u > 0$ we let
\[ B_N(u) = \{ y \in \mathbb{R}^N : ||y|| \leq u \} \]
be the closed Euclidean ball of radius $u$ in $\mathbb{R}^N$. We now introduce notation for the successive minima of a lattice.

**Definition 3.2.** Let $N \geq 1$ and $R \in \{1, \ldots, N\}$. Given a lattice $\Lambda \subset \mathbb{R}^N$ of rank $R$, the successive minima $\lambda_1(\Lambda), \ldots, \lambda_R(\Lambda)$ of $\Lambda$ with respect to the unit ball $B_N(1)$ are defined for $i \in \{1, \ldots, R\}$ by
\[ \lambda_i(\Lambda) = \inf \{ u \in \mathbb{R}_{>0} : \dim(\text{Span}_\mathbb{R}(\Lambda \cap B_N(u))) \geq i \}. \]

We clearly have $\lambda_1(\Lambda) \leq \cdots \leq \lambda_R(\Lambda)$ and moreover Minkowski’s second theorem (see for example [10, Chapter VIII, Theorem V]) states that
\[ \det(\Lambda) \leq \lambda_1(\Lambda) \cdots \lambda_R(\Lambda) \ll \det(\Lambda), \quad (3.5) \]
where the implied constant depends at most on $R$.

We now record a result which relates the successive minima of a lattice and those of its dual. The following version is due to Banaszczyk [1, Theorem 2.1].

**Lemma 3.3.** Let $N \geq 1$ and $R \in \{1, \ldots, N\}$. Let $\Lambda \subset \mathbb{R}^N$ be a lattice of rank $R$. For any $i \in \{1, \ldots, R\}$, we have
\[ \lambda_i(\Lambda) \leq \frac{R}{\lambda_{R-i+1}(\Lambda^*)}. \]

We use the convention that empty products and empty summations are respectively equal to 1 and 0. Recall the definition (2.6) of the region $C^{(\gamma)}_v$ for given $\gamma > 0$ and $v \in \mathbb{R}^N$. The following lattice point counting result will prove pivotal in our work.
Lemma 3.4. Let $N \geq 2$ and $R \in \{1, \ldots, N\}$. Let $\Lambda \subset \mathbb{R}^N$ be a lattice of rank $R$. Let $I \in \{1, \ldots, N-1\}$ and $v_1, \ldots, v_I \in \mathbb{R}^N$. Let also $\gamma > 0$. Define
\[
\mathcal{R}_{v_1,\ldots,v_I}(T,\gamma) = \mathcal{B}_N(T) \cap \mathcal{C}_{v_1}^{(\gamma)} \cap \cdots \cap \mathcal{C}_{v_I}^{(\gamma)},
\]
and
\[
\mathcal{V}_{v_1,\ldots,v_I}(\Lambda;\gamma) = \text{vol} (\text{Span}_{\mathbb{R}}(\Lambda) \cap \mathcal{R}_{v_1,\ldots,v_I}(1,\gamma)).
\]
Let $Y \geq \lambda_R(\Lambda)$. For $T \geq Y$, we have
\[
# (\Lambda \cap \mathcal{R}_{v_1,\ldots,v_I}(T,\gamma)) = \frac{T^R}{\det(\Lambda)} \mathcal{V}_{v_1,\ldots,v_I}(\Lambda;\gamma) + O \left( \frac{Y}{T} \right),
\]
where the implied constant depends at most on $R$.

Proof. Let $O$ be an $N \times N$ orthogonal matrix mapping $\text{Span}_{\mathbb{R}}(\Lambda)$ to the subspace of $\mathbb{R}^N$ spanned by the first $R$ coordinates, which we simply denote by $\mathbb{R}^R$. Let $\Gamma = O \cdot \Lambda$ and $\mathcal{T}_{v_1,\ldots,v_I}(T,\gamma) = O \cdot \mathcal{R}_{v_1,\ldots,v_I}(T,\gamma)$. We clearly have
\[
# (\Lambda \cap \mathcal{R}_{v_1,\ldots,v_I}(T,\gamma)) = # (\Gamma \cap \mathcal{T}_{v_1,\ldots,v_I}(T,\gamma)).
\]
The region $\mathcal{T}_{v_1,\ldots,v_I}(T,\gamma)$ is a semi-algebraic set and is thus definable in an o-minimal structure. Hence, it follows from work of Barroero and Widmer [2, Theorem 1.3] that
\[
# (\Gamma \cap \mathcal{T}_{v_1,\ldots,v_I}(T,\gamma)) = \frac{\text{vol}(\mathbb{R}^R \cap \mathcal{T}_{v_1,\ldots,v_I}(T,\gamma))}{\det(\Gamma)} + O \left( \sum_{i=1}^{R} \frac{T^{R-i}}{\lambda_1(\Gamma) \cdots \lambda_{R-i}(\Gamma)} \right).
\]
Since the matrix $O$ is orthogonal we have $\det(\Gamma) = \det(\Lambda)$ and also $\lambda_i(\Gamma) = \lambda_i(\Lambda)$ for any $i \in \{1, \ldots, R\}$, and moreover
\[
\text{vol}(\mathbb{R}^R \cap \mathcal{T}_{v_1,\ldots,v_I}(T,\gamma)) = T^R \mathcal{V}_{v_1,\ldots,v_I}(\Lambda;\gamma).
\]
We thus obtain
\[
# (\Lambda \cap \mathcal{R}_{v_1,\ldots,v_I}(T,\gamma)) = \frac{T^R}{\det(\Lambda)} \mathcal{V}_{v_1,\ldots,v_I}(\Lambda;\gamma) + O \left( \sum_{i=1}^{R} \frac{T^{R-i}}{\lambda_1(\Lambda) \cdots \lambda_{R-i}(\Lambda)} \right).
\]
We deduce from Minkowski’s theorem (3.5) that
\[
# (\Lambda \cap \mathcal{R}_{v_1,\ldots,v_I}(T,\gamma)) = \frac{T^R}{\det(\Lambda)} \mathcal{V}_{v_1,\ldots,v_I}(\Lambda;\gamma) + O \left( \sum_{i=1}^{R} \frac{\lambda_{R-i+1}(\Lambda) \cdots \lambda_R(\Lambda)}{T^i} \right).
\]
By assumption we have $\lambda_i(\Lambda) \leq Y$ for any $i \in \{1, \ldots, R\}$. We deduce that
\[
# (\Lambda \cap \mathcal{R}_{v_1,\ldots,v_I}(T,\gamma)) = \frac{T^R}{\det(\Lambda)} \mathcal{V}_{v_1,\ldots,v_I}(\Lambda;\gamma) + O \left( \sum_{i=1}^{R} \frac{Y^i}{T^i} \right),
\]
which completes the proof since $T \geq Y$. \hfill $\square$

Our arguments will make intensive use of the following classical result.

Lemma 3.5. Let $N \geq 1$ and $R \in \{1, \ldots, N\}$. Let $\Lambda \subset \mathbb{R}^N$ be a lattice of rank $R$. For $T \geq 1$, we have
\[
# (\Lambda \cap \mathcal{B}_N(T)) \ll \sum_{i=0}^{R} \frac{T^{R-i}}{\lambda_1(\Lambda) \cdots \lambda_{R-i}(\Lambda)}.
\]
In particular, if $i_0 \in \{1, \ldots, R\}$ then for $T \geq \lambda_{i_0}(\Lambda)$, we have
\[
# (\Lambda \cap \mathcal{B}_N(T)) \ll \frac{T^R}{\lambda_1(\Lambda) \cdots \lambda_{i_0-1}(\Lambda) \lambda_{i_0}(\Lambda)^{R-i_0+1}}.
\]
Moreover, the implied constants depend at most on $R$. 

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Lemma 3.6. Let $M > T \leq \lambda_{i_0}$ for any $i \in \{i_0, \ldots, R\}$ we deduce
\[
\# (\Lambda \cap B_N(T)) \ll \sum_{i=0}^{R-i_0} \frac{T^R - i}{\lambda_1(\Lambda) \cdots \lambda_{i_0-1}(\Lambda) \lambda_{i_0}(\Lambda)^{R-i-i_0+1}} + \sum_{i=R-i_0+1}^R \frac{T^R - i}{\lambda_1(\Lambda) \cdots \lambda_{R-i}(\Lambda)}.
\]
The assumption $T \geq \lambda_{i_0}(\Lambda)$ thus completes the proof. \hfill \Box

Lemma 3.4 is satisfactory when the successive minima of the lattice $\Lambda$ are small. Alternatively, if the successive minima of $\Lambda$ are known to be large then we can use Lemma 3.5 to derive a useful upper bound for the cardinality of the set $\Lambda \cap B_N(T)$.

Lemma 3.6. Let $N \geq 1$ and $R \in \{1, \ldots, N\}$. Let $\Lambda \subset \mathbb{R}^N$ be a lattice of rank $R$. Let $M > 0$ be such that $M < \lambda_1(\Lambda)$ and let $Y \geq \lambda_R(\Lambda)$. For any $R_0 \in \{0, \ldots, R-1\}$ and $T \leq Y$, we have
\[
\# ((\Lambda \setminus \{0\}) \cap B_N(T)) \ll \frac{T^{R-R_0} Y^{R_0}}{\det(\Lambda)} + \left(\frac{T}{M}\right)^{R-R_0-1}.
\]
Further, let $j_0 \in \{0, \ldots, R-1\}$ and $J \geq M$ be such that $J < \lambda_{j_0+1}(\Lambda)$. For any $R_0 \in \{0, \ldots, R-1-j_0\}$ and $T \leq Y$, we have
\[
\# ((\Lambda \setminus \{0\}) \cap B_N(T)) \ll \frac{T^{R-R_0} Y^{R_0}}{\det(\Lambda)} + \left(\frac{T}{J}\right)^{j_0} \left(\left(\frac{T}{J}\right)^{R-R_0-j_0} + 1\right).
\]
Moreover, the implied constants depend at most on $R$.

Proof. By assumption if $T \leq M$ then $(\Lambda \setminus \{0\}) \cap B_N(T) = \emptyset$ so both upper bounds trivially hold and we can assume that $T > M$. It is now clear that the first upper bound follows from the second by taking $j_0 = 0$ and $J = M$. The estimate (3.5) and Lemma 3.5 imply that
\[
\# (\Lambda \cap B_N(T)) \ll \frac{T^R}{\det(\Lambda)} \sum_{i=0}^{R_0} \frac{\lambda_{R-i}(\Lambda) \cdots \lambda_R(\Lambda)}{T^i} + \sum_{i=R_0+1}^R \frac{T^{R-i}}{\lambda_1(\Lambda) \cdots \lambda_{R-i}(\Lambda)}.
\]
Since by assumption we have $\lambda_i(\Lambda) \leq Y$ for any $i \in \{1, \ldots, R\}$ and $T \leq Y$, we see that
\[
\sum_{i=0}^{R_0} \frac{\lambda_{R-i}(\Lambda) \cdots \lambda_R(\Lambda)}{T^i} \ll \left(\frac{Y}{T}\right)^{R_0}.
\]
Moreover, our assumptions also imply that $\lambda_i(\Lambda) > M$ for any $i \in \{1, \ldots, j_0\}$ and $\lambda_i(\Lambda) > J$ for any $i \in \{j_0+1, \ldots, R\}$. We thus have
\[
\sum_{i=R_0+1}^R \frac{T^{R-i}}{\lambda_1(\Lambda) \cdots \lambda_{R-i}(\Lambda)} \ll \sum_{i=R_0+1}^{R-j_0} \frac{T^{R-i}}{M^{j_0} J^{R-i-j_0}} + \sum_{i=R-j_0}^R \frac{T^{R-i}}{M^{R-i}}.
\]
Since $T > M$ we obtain
\[
\# (\Lambda \cap B_N(T)) \ll \frac{T^{R-R_0} Y^{R_0}}{\det(\Lambda)} + \left(\frac{J}{M}\right)^{j_0} \sum_{i=R_0+1}^{R-j_0} \left(\frac{T}{J}\right)^{R-i} + \left(\frac{T}{M}\right)^{j_0}.
\]
If \( R_0 = R - 1 - j_0 \) then the summation in the right-hand side is empty so in this case the proof is complete. If \( R_0 \leq R - 2 - j_0 \) we get

\[
\# (\Lambda \cap B_N(T)) \ll \frac{T^{R-R_0}y_{R_0}}{\det(\Lambda)} \left( \frac{J}{M} \right)^{j_0} \left( \frac{T}{J} \right)^{R-R_0-1} + \left( \frac{T}{J} \right)^{j_0+1} + \left( \frac{T}{J} \right)^{j_0},
\]

which completes the proof on noting that \( R - R_0 - 1 - j_0 \geq 1 \). \( \square \)

3.2. The determinant of certain lattices. In this section we establish formulae for the determinants of several lattices. The following notation will be very useful.

**Definition 3.7.** Let \( N \geq 1 \) and \( k \in \{1, \ldots, N\} \). Given linearly independent vectors \( c_1, \ldots, c_k \in \mathbb{Z}^N \) we let \( \mathcal{G}(c_1, \ldots, c_k) \) denote the greatest common divisor of the \( k \times k \) minors of the \( N \times k \) matrix whose columns are the vectors \( c_1, \ldots, c_k \).

Recall the respective definitions (2.3) and (2.7) of the lattices \( \Lambda_c \) and \( \Lambda_d^{(Q)} \), for given \( c \in \mathbb{Z}^N \) and \( Q \geq 1 \). The following lemma can be found in work of the second author [14, Lemma 4].

**Lemma 3.8.** Let \( N \geq 1 \) and \( k \in \{1, \ldots, N-1\} \). Let also \( c_1, \ldots, c_k \in \mathbb{Z}^N \) be linearly independent vectors. We have

\[
\det (\Lambda_{c_1} \cap \cdots \cap \Lambda_{c_k}) = \frac{\det (\mathbb{Z}c_1 \oplus \cdots \oplus \mathbb{Z}c_k)}{\mathcal{G}(c_1, \ldots, c_k)}.
\]

The next result provides us with formulae for the determinants of two further lattices involved in our work.

**Lemma 3.9.** Let \( N \geq 2 \) and \( Q \geq 1 \). Let \( c, d \in \mathbb{Z}^N_{\text{prim}} \) be two linearly independent vectors. We have

\[
\det \left( \Lambda_c^{(Q)} \cap \Lambda_d^{(Q)} \right) = \frac{Q^2}{\gcd(\mathcal{G}(c, d), Q)},
\]

and

\[
\det \left( \Lambda_c \cap \Lambda_d^{(Q)} \right) = ||c|| \cdot \frac{Q}{\gcd(\mathcal{G}(c, d), Q)}.
\]

**Proof.** Since \( c, d \in \mathbb{Z}^N_{\text{prim}} \) are linearly independent, the Smith form theorem implies that there exist \( f \in \mathbb{Z} \) and \( T \in \text{GL}_N(\mathbb{Z}) \) such that

\[
\begin{pmatrix} c^T \\ d^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ f & \mathcal{G}(c, d) & 0 & \cdots & 0 \end{pmatrix}.
\]

By definition we have

\[
\Lambda_c^{(Q)} \cap \Lambda_d^{(Q)} = \left\{ y \in \mathbb{Z}^N : \begin{pmatrix} c^T \\ d^T \end{pmatrix} y \equiv 0 \mod Q \right\}.
\]

We thus deduce that

\[
\Lambda_c^{(Q)} \cap \Lambda_d^{(Q)} = T \cdot \left\{ z \in \mathbb{Z}^N : \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ f & \mathcal{G}(c, d) & 0 & \cdots & 0 \end{pmatrix} z \equiv 0 \mod Q \right\},
\]

(3.7)

and the first part of the lemma follows since \( |\det(T)| = 1 \).

Furthermore, we have

\[
\Lambda_c \cap \Lambda_d^{(Q)} = \left\{ y \in \mathbb{Z}^N : \begin{pmatrix} c^T \\ d^T \end{pmatrix} y \in \mathbb{Z}^Q \right\}.
\]
so we see that
\[
\Lambda_c \cap \Lambda^{(Q)}_d = T \cdot \left\{ z \in \mathbb{Z}^N : \begin{pmatrix} 1 & 0 & \cdots & 0 \\ f & G(c, d) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} z \in \begin{pmatrix} 0 \\ Q \mathbb{Z} \end{pmatrix} \right\}.
\]

Therefore, a basis of the lattice \( \Lambda_c \cap \Lambda^{(Q)}_d \) is given by
\[
\left( \frac{Q}{\gcd(G(c, d), Q)} \cdot T_{e_2}, T_{e_3}, \ldots, T_{e_N} \right),
\]
where \((e_1, \ldots, e_N)\) denotes the canonical basis of \( \mathbb{R}^N \). Using the definition (3.1), we obtain
\[
\det \left( \Lambda_c \cap \Lambda^{(Q)}_d \right) = \frac{Q}{\gcd(G(c, d), Q)} \cdot \sqrt{\det(S)},
\]
where
\[
S = \left( e_i^T T^T T_{e_j} \right)_{i,j=2,\ldots,N}.
\]
The matrix \( S \) is formed by removing from \( T^T T \) its first line and its first column. Therefore, \( \det(S) \) is the cofactor of index \((1, 1)\) of \( T^T T \) and is thus equal to the entry of index \((1, 1)\) of the matrix \((T^T T)^{-1}\) since \( \det(T^T T) = 1 \). In other words, we have
\[
\det(S) = e_1^T (T^T T)^{-1} e_1.
\]
But the equality (3.6) gives \( e^T = e_1^T T^{-1} \), so that \( \det(S) = ||e||^2 \), which completes the proof.

3.3. Bounding the successive minima of the key lattices. Recall that for \( d, n \geq 1 \) the Veronese embedding \( \nu_{d,n} : \mathbb{R}^{n+1} \to \mathbb{R}^{N_{d,n}} \) was introduced in Definition 2.1. Given two linearly independent vectors \( x, y \in \mathbb{Z}_{prim}^{n+1} \), the lattices \( \Lambda_{\nu_{d,n}(x)} \) and \( \Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)} \) respectively have rank \( N_{d,n} - 1 \) and \( N_{d,n} - 2 \). These two lattices play a pivotal role in our arguments and this section is concerned with bounding the size of their successive minima, thereby aligning us for an efficient application of Lemma 3.4. It is convenient to introduce the following notation.

**Definition 3.10.** Let \( d, n \geq 1 \) and let \( X = (X_0, \ldots, X_n) \). We let \( \mathbb{R}[X]^{(d)} \) denote the vector space of homogeneous polynomials of degree \( d \) in \( n+1 \) variables and we let \( \omega_d : \mathbb{R}[X]^{(d)} \to \mathbb{R}^{N_{d,n}} \) be the isomorphism defined using the lexicographical ordering. We also let \( \mathcal{M}_{d,n} \) denote the set of monomials of degree \( d \) in \( n+1 \) variables.

We start by stating an elementary result that we will use repeatedly in our arguments.

**Lemma 3.11.** Let \( d, n \geq 1 \). Let \( x, y \in \mathbb{Z}_{prim}^{n+1} \) be two linearly independent vectors. Then
\[
\mathcal{G}(\nu_{d,n}(x), \nu_{d,n}(y)) = \mathcal{G}(x, y).
\]

**Proof.** First, it is not hard to check that \( \mathcal{G}(x, y) \mid \mathcal{G}(\nu_{d,n}(x), \nu_{d,n}(y)) \). Indeed, if \( q \) is a positive integer such that \( x_i y_j \equiv x_{i'} y_{j'} \mod q \) for any \( i, j \in \{0, \ldots, n\} \), then we clearly have \( P(x)Q(y) \equiv P(y)Q(x) \mod q \) for any \( P, Q \in \mathcal{M}_{d,n} \). Second, let \( p \) be a prime divisor of \( \mathcal{G}(\nu_{d,n}(x), \nu_{d,n}(y)) \) and let us show that the \( p \)-adic valuation of \( \mathcal{G}(\nu_{d,n}(x), \nu_{d,n}(y)) \) is at most the \( p \)-adic valuation of \( x_{k} y_{k} - x_{\ell} y_{\ell} \) for any \( k, \ell \in \{0, \ldots, n\} \). By definition of \( \mathcal{G}(\nu_{d,n}(x), \nu_{d,n}(y)) \) we have
\[
\mathcal{G}(\nu_{d,n}(x), \nu_{d,n}(y)) \mid x_{j}^{d-1} y_{j}^{d-1} (x_{k} y_{k} - x_{\ell} y_{\ell}),
\]
for any \( j, k, \ell \in \{0, \ldots, n\} \). Therefore, it suffices to check that there exists \( j \in \{0, \ldots, n\} \) such that \( p \nmid x_{j} y_{j} \). Otherwise, since \( x \) and \( y \) are primitive vectors, there would exist distinct indices \( j_0, j_1 \in \{0, \ldots, n\} \) such that \( p \nmid x_{j_0} y_{j_0} \) and \( p \mid x_{j_1} \). It would follow that \( p \nmid x_{j_0}^{d-j_0} (x_{j_1} y_{j_1} - x_{j_0} y_{j_0}) \), which would contradict the fact that \( p \mid \mathcal{G}(\nu_{d,n}(x), \nu_{d,n}(y)) \). This completes the proof. \( \square \)
Our work will make crucial use of the following notion.

**Definition 3.12.** Let \( n \geq 1 \) and let \( x, y \in \mathbb{Z}^{n+1} \) be two linearly independent vectors. For any \( r \in \{2, \ldots, n+1\} \) let \( \varrho_r(x) \) be the minimum determinant of a rank \( r \) sublattice of \( \mathbb{Z}^{n+1} \) containing \( x \), and we let \( \varrho_n(x, y) \) be the minimum determinant of a rank \( r \) sublattice of \( \mathbb{Z}^{n+1} \) containing \( x \) and \( y \).

Let \( \mathcal{L}(x, y) \) denote the unique primitive lattice of \( \mathbb{Z}^{n+1} \) of rank 2 containing \( x \) and \( y \), that is

\[
\mathcal{L}(x, y) = (\mathbb{R}x \oplus \mathbb{R}y) \cap \mathbb{Z}^{n+1}.
\]

By definition we have

\[
det(\mathcal{L}(x, y)) = \varrho_2(x, y).
\]

The next result provides us with a formula for this determinant.

**Lemma 3.13.** Let \( n \geq 1 \) and let \( x, y \in \mathbb{Z}^{n+1} \) be two linearly independent vectors. We have

\[
\varrho_2(x, y) = \frac{(||x||^2||y||^2 - \langle x, y \rangle^2)^{1/2}}{G(x, y)}.
\]

**Proof.** We note that \( \mathcal{L}(x, y) = \Lambda_x \cap \Lambda_y \). Therefore, since the lattice \( \mathcal{L}(x, y) \) is primitive, the equality \( (3.3) \) gives \( \varrho_2(x, y) = det(\Lambda_x \cap \Lambda_y) \). An application of Lemma 3.8 thus yields

\[
\varrho_2(x, y) = \frac{\det(\mathbb{Z}x \oplus \mathbb{Z}y)}{G(x, y)}.
\]

It follows from the definition \( (3.1) \) of the determinant of a lattice that

\[
\det(\mathbb{Z}x \oplus \mathbb{Z}y)^2 = ||x||^2||y||^2 - \langle x, y \rangle^2,
\]

which completes the proof. \( \Box \)

Given an integer \( R \geq 1 \) and a lattice \( \Lambda \) of rank \( R \), we recall that for \( i \in \{1, \ldots, R\} \) we have defined \( \lambda_i(\Lambda) \) to be the \( i \)-th successive minimum of \( \Lambda \), as stated in Definition 3.2. We are now ready to reveal the main results of this section. The following result sharpens work of the second author \([14, \text{Lemma } 5]\), a saving that is key for our application.

**Lemma 3.14.** Let \( d, n \geq 1 \) and let \( x \in \mathbb{Z}_{prim}^{n+1} \). We have

\[
\lambda_{N_{d,n}-1}(\Lambda_{\nu_{d,n}}(x)) \leq n ||x|| \frac{\varrho_2(x)}{G(x)}.
\]

**Proof.** We start by dealing with the case \( d = 1 \) and we note that \( N_{1,n} = n + 1 \) and \( \nu_{1,n}(x) = x \). We aim to apply Lemma 3.3 and we thus let \( a \in \Lambda^*_{x} \) be a non-zero vector. The lattice \( \mathbb{Z}x \) is primitive so we deduce from Lemma 3.1 that \( \Lambda^*_{x} = \mathbb{Z}^{n+1}/\mathbb{Z}x \). The vector \( a \) can thus be written as \( a = b + t \) for some \( b \in \mathbb{Z}^{n+1} \) and \( t \in \mathbb{R}x \). Since \( \langle a, x \rangle = 0 \) and \( a \) is non-zero, the vectors \( b \) and \( x \) are linearly independent and therefore the integral lattice \( \mathbb{Z}b \oplus \mathbb{Z}x \) has rank 2. It follows that \( \varrho_2(x) \leq \det(\mathbb{Z}b \oplus \mathbb{Z}x) \). Since \( t \in \mathbb{R}x \) we have \( \det(\mathbb{Z}b \oplus \mathbb{Z}x) = \det(\mathbb{Z}a \oplus \mathbb{Z}x) \) and we thus see that \( \varrho_2(x) \leq ||a|| \cdot ||x|| \). This gives

\[
\lambda_1(\Lambda^*_{x}) \geq \frac{\varrho_2(x)}{||x||}.
\]

An application of Lemma 3.3 thus completes the proof in the case \( d = 1 \).

Assume now that \( d \geq 2 \). By the case \( d = 1 \), we can pick \( n \) linearly independent vectors \( a_1, \ldots, a_n \) of the lattice \( \Lambda_x \) in the ball \( B_{n+1}(n||x||/\varrho_2(x)) \). For any \( P \in \mathcal{M}_{d-1,n} \) we let \( \Theta_P : \mathbb{R}^{n+1} \to \mathbb{R}^{N_{d,n}} \) be the linear map defined for \( c \in \mathbb{R}^{n+1} \) by

\[
\Theta_P(c) = \omega_d(P(X) \cdot \langle c, X \rangle),
\]
and we note that by definition \((\Theta_P(c), \nu_{d,n}(X)) = P(X) \cdot (c, X)\). For any \(i \in \{1, \ldots, n\}\) we have \((a_i, x) = 0\) and it follows that \(\Theta_P(a_i) \in \Lambda_{\nu_{d,n}(X)}\). Also, we see that for any \(c \in \mathbb{R}^{n+1}\) we have \(||\Theta_P(c)|| = ||c||\) and thus \(\Theta_P(a_i) \in B_{N_{d,n}}(n||x||/d_2(x))\) for any \(i \in \{1, \ldots, n\}\). Therefore, in order to complete the proof it suffices to check that there are \(N_{d,n} - 1\) linearly independent vectors in the set

\[
\left\{ \Theta_P(a_i) : P \in \mathbb{M}_{d-1,n} \right\}
\]

But the real subspace spanned in \(\mathbb{R}^{N_{d,n}}\) by this set of vectors contains in particular the image under \(\omega_d\) of the set

\[
\text{Span}_\mathbb{R} \left( \left\{ P(X) \cdot (x_i X_j - x_j X_i) : P \in \mathbb{M}_{d-1,n}, i, j \in \{0, \ldots, n\} \right\} \right) = \left\{ Q \in \mathbb{R}[X]^{(d)} : Q(x) = 0 \right\}.
\]

Note that we have used the fact that \(\Lambda_x = \mathbb{R}a_1 \oplus \cdots \oplus \mathbb{R}a_n\). This implies that

\[
\text{Span}_\mathbb{R} \left( \left\{ \Theta_P(a_i) : P \in \mathbb{M}_{d-1,n}, i \in \{1, \ldots, n\} \right\} \right) = (\mathbb{R}\nu_{d,n}(x))^\perp,
\]

which finishes the proof.

The proof of our second result follows a similar strategy, but is much more challenging.

**Lemma 3.15.** Let \(d, n \geq 2\) and let \(x, y \in \mathbb{Z}_{\text{prim}}^{n+1}\) be two linearly independent vectors. We have

\[
\lambda_{N_{d,n}-2} \left( \Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)} \right) \leq 3n^2 \max \left\{ \frac{d_2(x, y)}{|x| \cdot |y|}, \frac{d_3(x, y)}{d_2(x, y)^2} \right\}.
\]

Given two linearly independent vectors \(x, y \in \mathbb{Z}_{\text{prim}}^{n+1}\), the proof of Lemma 3.15 depends on a close analysis of the primitive lattices

\[
Q_2(x, y) = (\mathbb{R}\nu_{2,n}(x) \oplus \mathbb{R}\nu_{2,n}(y)) \cap \mathbb{Z}^{N_{2,n}},
\]

and

\[
Q_3(x, y) = (\mathbb{R}\nu_{2,n}(x) \oplus \mathbb{R}\nu_{2,n}(y) \oplus \mathbb{R}\nu_{2,n}(x + y)) \cap \mathbb{Z}^{N_{2,n}}.
\]

We note that for given \(i, j \in \{0, \ldots, n\}\), a straightforward calculation yields

\[
\det \begin{pmatrix}
  x_i^2 & y_i^2 & (x_i + y_i)^2 \\
  x_i x_j & y_i y_j & (x_i + y_i)(x_j + y_j) \\
  x_j^2 & y_j^2 & (x_j + y_j)^2
\end{pmatrix} = (x_j y_i - x_i y_j)^3. \tag{3.10}
\]

If the vectors \(x, y\) are linearly independent then there exist \(i_0, j_0 \in \{0, \ldots, n\}\) such that \(x_{j_0} y_{i_0} - x_{i_0} y_{j_0} \neq 0\) so we deduce that the vectors \(\nu_{2,n}(x), \nu_{2,n}(y),\) and \(\nu_{2,n}(x + y)\) are linearly independent. It follows that the lattices \(Q_2(x, y)\) and \(Q_3(x, y)\) respectively have rank 2 and 3 and thus the quotient lattice \(Q_3(x, y)/Q_2(x, y)\) has rank 1. The following result is concerned with the determinant of this lattice.

**Lemma 3.16.** Let \(n \geq 2\) and let \(x, y \in \mathbb{Z}_{\text{prim}}^{n+1}\) be two linearly independent vectors. Then

\[
\det (Q_3(x, y)/Q_2(x, y)) \geq \frac{1}{3} \cdot \frac{d_2(x, y)^2}{|x| \cdot |y|}.
\]

**Proof.** In order to achieve our goal, we employ the identity (3.2), giving

\[
\det (Q_3(x, y)/Q_2(x, y)) = \frac{\det (Q_3(x, y))}{\det (Q_2(x, y))}. \tag{3.11}
\]

We start by proving an upper bound for the determinant of \(Q_2(x, y)\). Since the lattice \(Q_2(x, y)\) is primitive, we see that we have

\[
Q_2(x, y) = (\Lambda_{\nu_{2,n}(x)} \cap \Lambda_{\nu_{2,n}(y)})^\perp. \tag{3.12}
\]
Letting $Q_2(x, y)$, which implies in particular that

$$\det(Q_2(x, y)) = \det(\Lambda_{\nu_2, n}(x) \cap \Lambda_{\nu_2, n}(y)),\$$

and Lemma 3.8 thus gives

$$\det(Q_2(x, y)) = \frac{\det(Z_{\nu_2, n}(x) + Z_{\nu_2, n}(y))}{G(\nu_2, n(x), \nu_2, n(y))}. \quad (3.13)$$

Recalling the definition (3.1) of the determinant of a lattice, we see that

$$\det(Z_{\nu_2, n}(x) + Z_{\nu_2, n}(y))^2 = ||\nu_2, n(x)||^2||\nu_2, n(y)||^2 - \langle \nu_2, n(x), \nu_2, n(y) \rangle^2.$$

This can be rewritten as

$$\det(Z_{\nu_2, n}(x) + Z_{\nu_2, n}(y))^2 = \frac{1}{2} \sum_{P_1, P_2 \in \mathbb{Z}_n} (P_1(x)P_2(y) - P_2(x)P_1(y))^2,$$

which implies in particular that

$$\det(Z_{\nu_2, n}(x) + Z_{\nu_2, n}(y))^2 \leq \frac{1}{2} \sum_{i_1, j_1, i_2, j_2 = 0}^n (x_{i_1}x_{j_1}y_{i_2}y_{j_2} - x_{i_2}x_{j_2}y_{i_1}y_{j_1})^2.$$

Expanding the square, this leads to

$$\det(Z_{\nu_2, n}(x) + Z_{\nu_2, n}(y))^2 \leq ||x||^4||y||^4 - \langle x, y \rangle^4.$$  

Recalling the equality (3.13) and using Lemma 3.11, we deduce

$$\det(Q_2(x, y)) \leq \left(\frac{||x||^4||y||^4 - \langle x, y \rangle^4}{G(x, y)}\right)^{1/2}. \quad (3.14)$$

We now follow a similar approach to prove a lower bound for the determinant of $Q_3(x, y)$. Since the lattice $Q_3(x, y)$ is primitive we have

$$Q_3(x, y) = (\Lambda_{\nu_2, n}(x) \cap \Lambda_{\nu_2, n}(y) \cap \Lambda_{\nu_2, n}(x+y))^2.$$  

The equality (3.3) thus gives

$$\det(Q_3(x, y)) = \det(\Lambda_{\nu_2, n}(x) \cap \Lambda_{\nu_2, n}(y) \cap \Lambda_{\nu_2, n}(x+y)).$$  

Using Lemma 3.8 we deduce

$$\det(Q_3(x, y)) = \frac{\det(Z_{\nu_2, n}(x) + Z_{\nu_2, n}(y) + Z_{\nu_2, n}(x+y))}{G(\nu_2, n(x), \nu_2, n(y), \nu_2, n(x+y))}. \quad (3.15)$$

Applying the definition (3.1), we see that the square of the determinant of the lattice $Z_{\nu_2, n}(x) + Z_{\nu_2, n}(y) + Z_{\nu_2, n}(x+y)$ is equal to

$$\det\left(\begin{array}{ccc} ||\nu_2, n(x)||^2 & \langle \nu_2, n(x), \nu_2, n(y) \rangle & \langle \nu_2, n(x), \nu_2, n(x+y) \rangle \\
\langle \nu_2, n(x), \nu_2, n(y) \rangle & ||\nu_2, n(y)||^2 & \langle \nu_2, n(y), \nu_2, n(x+y) \rangle \\
\langle \nu_2, n(x), \nu_2, n(x+y) \rangle & \langle \nu_2, n(y), \nu_2, n(x+y) \rangle & ||\nu_2, n(x+y)||^2 \end{array}\right).$$

Letting $S_3$ denote the permutation group of the set $\{1, 2, 3\}$, the calculation of this determinant shows that it can be rewritten as

$$\frac{1}{6} \sum_{P_1, P_2, P_3 \in \mathbb{Z}_n} \left(\sum_{\sigma \in S_3} \text{sgn}(\sigma)P_\sigma(1)(x)P_\sigma(2)(y)P_\sigma(3)(x+y)\right)^2.$$

Therefore, we note that it is in particular bounded below by

$$\frac{1}{48} \sum_{i_1, j_1, i_2, j_2, i_3, j_3 = 0}^n \left(\sum_{\sigma \in S_3} \text{sgn}(\sigma)x_{i_{\sigma(1)}}x_{j_{\sigma(1)}}y_{i_{\sigma(2)}}y_{j_{\sigma(2)}}(x_{j_{\sigma(3)}} + y_{j_{\sigma(3)}})(x_{j_{\sigma(3)}} + y_{j_{\sigma(3)}})\right)^2.$$
Expanding the square, a straightforward calculation shows that this quantity equals 
\[ \frac{1}{8} \left( \|x\|^4 \|y\|^4 \|x + y\|^4 + 2 \langle x, y \rangle^2 \langle x, x + y \rangle^2 \langle y, x + y \rangle \right. \]
\[ - \left. (x, y)^4 \|x + y\|^4 - (x, x + y)^4 \|y\|^4 - (y, x + y)^4 \|x\|^4 \right) = \frac{1}{4} \left( \|x\|^2 \|y\|^2 - (x, y)^2 \right)^3. \]

We have thus obtained
\[ \det(\mathbb{Z} \nu_{2,n}(x) \oplus \mathbb{Z} \nu_{2,n}(y) \oplus \mathbb{Z} \nu_{2,n}(x + y))^2 \geq \frac{1}{4} \left( \|x\|^2 \|y\|^2 - (x, y)^2 \right)^3. \]

In addition, the identity (3.10) shows that
\[ G(\nu_{2,n}(x), \nu_{2,n}(y), \nu_{2,n}(x + y)) \leq G(x, y)^3. \]

Therefore, recalling the equality (3.15) we eventually derive
\[ \det(Q_3(x, y)) \geq \frac{1}{2} \cdot \frac{\|x\|^2 \|y\|^2 - (x, y)^2}{\sqrt{2}} G(x, y)^3. \]

Putting together the lower bound (3.11) and the upper bound (3.14), we obtain
\[ \det(Q_3(x, y)/Q_2(x, y)) \geq \frac{1}{2} \cdot \frac{\|x\|^2 \|y\|^2 - (x, y)^2}{\sqrt{2} \cdot \|x\| \cdot \|y\|}. \]

Noticing that the Cauchy–Schwarz inequality gives
\[ \left( \|x\|^2 \|y\|^2 + (x, y)^2 \right)^{1/2} \leq \sqrt{2} \cdot \|x\| \cdot \|y\| \]
and \( 2\sqrt{2} \leq 3 \), we see that an application of Lemma 3.13 completes the proof. \( \square \)

We are now ready to furnish the proof of Lemma 3.15.

**Proof of Lemma 3.15.** We start by proving the result in the case \( d = 2 \). Recalling the identity (3.12) and aiming to apply Lemma 3.3 we let \( a \in (Q_2(x, y)^{\perp})^* \) be a non-zero vector. The lattice \( Q_2(x, y) \) is primitive so it follows from Lemma 3.1 that \( (Q_2(x, y)^{\perp})^* = \mathbb{Z}^{N_{2,n}} / Q_2(x, y) \). We thus deduce the existence of vectors \( b \in \mathbb{Z}^{N_{2,n}} \) and \( t \in \text{Span}_\mathbb{R}(Q_2(x, y)) \) such that \( a = b + t \).

We are going to distinguish two cases depending on whether the vector \( a \) belongs to \( \text{Span}_\mathbb{R}(Q_3(x, y)) \) or not. In the former case we see that \( b \in Q_3(x, y) \). Therefore, if \( \pi : \mathbb{R}^{N_{2,n}} \to \text{Span}_\mathbb{R}(Q_2(x, y))^{\perp} \) denotes the orthogonal projection on \( \text{Span}_\mathbb{R}(Q_2(x, y))^{\perp} \) then \( a = \pi(a) = \pi(b) \) is a non-zero vector in the quotient lattice \( Q_3(x, y)/Q_2(x, y) \).

Since this lattice has rank 1 an application of Lemma 3.16 shows that, in the case where \( a \in \text{Span}_\mathbb{R}(Q_3(x, y)) \), we have
\[ \lambda_1 \left( \left( Q_2(x, y)^{\perp} \right)^* \right) \geq \frac{1}{3} \cdot \frac{Q_2(x, y)^2}{\|x\| \cdot \|y\|}. \] \[ (3.16) \]

We now deal with the case where \( a \notin \text{Span}_\mathbb{R}(Q_3(x, y)) \). Indexing the coordinates of \( \mathbb{R}^{N_{2,n}} \) using the lexicographical ordering, we introduce a symmetric bilinear form \( \sigma : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{N_{2,n}} \) by defining the coordinate of index \((i, j)\) of \( \sigma(u, v) \) as being equal to
\[
\left\{ \begin{array}{ll}
u_i v_i, & \text{if } i = j, \\
u_i v_j + u_j v_i, & \text{if } i \neq j,
\end{array} \right.
\]
for any \( i \in \{0, \ldots, n\}, j \in \{i, \ldots, n\} \) and \((u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \). Recall the definition (3.8) of the lattice \( L(x, y) \). Our next task is to check that
\[ \text{Span}_\mathbb{R} \left( \sigma \left( \mathbb{Z}^{n+1} \times (L(x, y)^{\perp}) \right) \right) = \text{Span}_\mathbb{R}(Q_3(x, y))^{\perp}. \] \[ (3.17) \]
We start by noting that for any $z \in \mathbb{Z}^{n+1}$, we have
\[
\langle \nu_{2,n}(z), \sigma(u,v) \rangle = \sum_{i=0}^{n} z_i^2 u_i v_i + \sum_{i=0}^{n} \sum_{j=i+1}^{n} z_i z_j (u_i v_j + u_j v_i)
\]
\[
= \sum_{i=0}^{n} z_i u_i \left( \sum_{j=i}^{n} z_j v_j + \sum_{j=i+1}^{n} z_j u_j v_i \right)
\]
\[
= z_0 u_0 (z,v) + \sum_{i=1}^{n} z_i u_i \left( \sum_{j=i}^{n} z_j v_j + \sum_{j=0}^{i-1} z_j v_j \right)
\]
\[
= \langle z, u \rangle \cdot \langle z, v \rangle.
\]

It follows that $\sigma(u,v) \in \text{Span}_R (Q_3(x,y)) \perp$ when $(u,v) \in \mathbb{Z}^{n+1} \times \mathcal{L}(x,y) \perp$. In addition, by examining the values of $\sigma$ at pairs of vectors of the canonical basis of $\mathbb{R}^{n+1}$, we get
\[
\text{Span}_R (\sigma (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})) = \mathbb{R}^{N_2,n}.
\]

Moreover, it is clear that any $v \in \mathbb{R}^{n+1}$ can uniquely be written as $v = s x + t y + z$ for some $s, t \in \mathbb{R}$ and $z \in \text{Span}_R (\mathcal{L}(x,y) \perp)$. We thus have
\[
\sigma (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) = \left\{ \sum_{s_1,t_1,s_2,t_2} (z_1, z_2) : s_1, t_1, s_2, t_2 \in \mathbb{R}, z_1, z_2 \in \text{Span}_R (\mathcal{L}(x,y) \perp) \right\},
\]
where
\[
\sum_{s_1,t_1,s_2,t_2} (z_1, z_2) = s_1 s_2 \sigma(x,x) + (s_1 t_2 + s_2 t_1) \sigma(x,y) + t_1 t_2 \sigma(y,y) + \sigma(s_1 x + t_1 y, z_2) + \sigma(s_2 x + t_2 y, z_1) + \sigma(z_1, z_2).
\]

Using the bilinearity of $\sigma$ we immediately deduce that for any $s_1, t_1, s_2, t_2 \in \mathbb{R}$ and any $z_1, z_2 \in \text{Span}_R (\mathcal{L}(x,y) \perp)$ the vectors $\sigma(s_1 x + t_1 y, z_2), \sigma(s_2 x + t_2 y, z_1)$ and $\sigma(z_1, z_2)$ all belong to the vector space $\text{Span}_R (\sigma (\mathbb{Z}^{n+1} \times \mathcal{L}(x,y) \perp))$. We thus derive the lower bound
\[
\dim (\text{Span}_R (\sigma (\mathbb{Z}^{n+1} \times \mathcal{L}(x,y) \perp))) \geq N_2,n - 3,
\]
which finishes the proof of the identity (3.17) since we have
\[
\dim (\text{Span}_R (Q_3(x,y) \perp)) = N_2,n - 3.
\]

Recall that we are treating the case where $a \notin \text{Span}_R (Q_3(x,y))$ and let $(e_0, \ldots, e_n)$ be the canonical basis of $\mathbb{Z}^{n+1}$. If we had $(a, \sigma(e_i, v)) = 0$ for every $i \in \{0, \ldots, n\}$ and every $v \in \mathcal{L}(x,y) \perp$ then we would have $(a, w) = 0$ for any $w \in \text{Span}_R (\sigma (\mathbb{Z}^{n+1} \times \mathcal{L}(x,y) \perp))$ and it would follow from the equality (3.17) that $(a, Q_3(x,y)) = 0$. We may thus assume that there exist $i \in \{0, \ldots, n\}$ and $v \in \mathcal{L}(x,y) \perp$ such that $\langle a, \sigma(e_i, v) \rangle \neq 0$.

We now define a linear map $r_i : \mathbb{R}^{N_2,n} \rightarrow \mathbb{R}^{n+1}$ by using the lexicographical ordering of coordinates in $\mathbb{R}^{N_2,n}$. For any $(c_{0,0}, \ldots, c_{n,n}) \in \mathbb{R}^{N_2,n}$, we set
\[
r_i(c_{0,0}, \ldots, c_{n,n}) = (c_{0,i}, \ldots, c_{i,i}, c_{i,i+1}, \ldots, c_{n,n}).
\]

We recall that the vector $a$ can be written as $a = b + t$ for some $b \in \mathbb{Z}^{N_2,n}$ and $t \in \text{Span}_R (Q_2(x,y))$. We note that $r_i(\nu_{2,n}(x)) = x_i x$ and $r_i(\nu_{2,n}(y)) = y_i y$ and thus $r_i(t) \in \text{Span}_R (\mathcal{L}(x,y))$. Since $v \in \mathcal{L}(x,y) \perp$ we deduce that $\langle r_i(b), v \rangle = \langle r_i(a), v \rangle$. Furthermore, by definition of the map $r_i$ we see that
\[
\langle r_i(a), v \rangle = \langle a, \sigma(e_i, v) \rangle \neq 0,
\]
and it follows that the integral lattice $Z r_i(b) \oplus \mathcal{L}(x,y)$ has rank 3. Since this lattice contains $x$ and $y$ we have
\[
\mathcal{D}_3(x,y) \leq \det (Z r_i(b) \oplus \mathcal{L}(x,y)).
\]
Using the fact that \( r_i(t) \in \text{Span}_\mathbb{R}(L(x, y)) \) we see that

\[
\det (\mathbb{Z}r_i(b) \oplus L(x, y)) = \det (\mathbb{Z}r_i(a) \oplus L(x, y)).
\]

Recalling the equality (3.9) and using the upper bound \( ||r_i(a)|| \leq ||a|| \) we obtain

\[
\partial_3(x, y) \leq ||a|| \cdot \partial_2(x, y).
\]

This eventually shows that, in the case where \( a \notin \text{Span}_\mathbb{R}(Q_3(x, y)) \), we have

\[
\lambda_1 \left( \left( Q_2(x, y)^\perp \right)^* \right) \geq \frac{\partial_3(x, y)}{\partial_2(x, y)}.
\]

(3.18)

Recalling the equality (3.12) and combining the lower bounds (3.16) and (3.18) we deduce that

\[
\lambda_1 \left( (\Lambda_{\nu_2,n}(x) \cap \Lambda_{\nu_2,n}(y))^* \right) \geq \frac{1}{3} \min \left\{ \frac{\partial_2(x, y)}{\partial_2(x, y)^2} \right\}.
\]

Applying Lemma 3.3 and using the fact that \( N_{2,n} - 2 \leq n^2 \) for any \( n \geq 1 \) completes the proof in the case \( d = 2 \).

Assume now that \( d \geq 3 \). By the case \( d = 2 \), we can pick \( N_{2,n} - 2 \) linearly independent vectors \( a_1, \ldots, a_{N_{2,n}-2} \) of the lattice \( \Lambda_{\nu_2,n}(x) \cap \Lambda_{\nu_2,n}(y) \) in the ball \( B_{N_{2,n}}(\mu(x, y)) \) where \( \mu(x, y) \) denotes the right-hand side of the upper bound stated in Lemma 3.15. For any \( P \in \mathcal{M}_{d-2,n} \) we let \( \Psi_P : \mathbb{R}^{N_{2,n}} \rightarrow \mathbb{R}^{N_{d,n}} \) be the linear map defined for \( c \in \mathbb{R}^{N_{2,n}} \) by

\[
\Psi_P(c) = \omega_d \cdot \langle P(X), c, \nu_{d,n}(X) \rangle,
\]

and we note that \( \langle \Psi_P(c), \nu_{d,n}(X) \rangle = P(X) \cdot \langle c, \nu_{d,n}(X) \rangle \). For any \( i \in \{1, \ldots, N_{2,n}-2\} \) we have \( \langle a_i, \nu_{d,n}(X) \rangle = 0 \) and \( \langle a_i, \nu_{d,n}(Y) \rangle = 0 \) so it follows that \( \Psi_P(a_i) \in \Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)} \).

Also, for any \( c \in \mathbb{R}^{N_{2,n}} \) we have \( ||\Psi_P(c)|| = ||c|| \) and thus \( \Psi_P(a_i) \in B_{N_{d,n}}(\mu(x, y)) \) for any \( i \in \{1, \ldots, N_{2,n}-2\} \). Therefore, in order to complete the proof it suffices to check that there are \( N_{d,n} - 2 \) linearly independent vectors in the set

\[
\left\{ \Psi_P(a_i) : P \in \mathcal{M}_{d-2,n}, i \in \{1, \ldots, N_{2,n}-2\} \right\}.
\]

The real subspace spanned in \( \mathbb{R}^{N_{d,n}} \) by this set of vectors contains in particular the image under \( \omega_d \) of the set

\[
\text{Span}_\mathbb{R} \left( \left\{ P(X) : L^{x,y}_{i,j,k}(X) : P \in \mathcal{M}_{d-2,n}, i,j,k \in \{0, \ldots, n\} \right\} \right) = \left\{ Q \in \mathbb{R}[X]^{(d)} : Q(x) = 0, Q(y) = 0 \right\},
\]

where

\[
L^{x,y}_{i,j,k}(X) = (x_j y_k - x_k y_j) X_i + (x_k y_i - x_i y_k) X_j + (x_i y_j - x_j y_i) X_k.
\]

Note that we have used the fact that \( \Lambda_{\nu_2,n}(x) \cap \Lambda_{\nu_2,n}(y) = \mathbb{R} a_1 \oplus \cdots \oplus \mathbb{R} a_{N_{2,n} - 2} \). We eventually deduce that

\[
\text{Span}_\mathbb{R} \left( \left\{ \Psi_P(a_i) : P \in \mathcal{M}_{d-2,n}, i \in \{1, \ldots, N_{2,n}-2\} \right\} \right) = (\mathbb{R} \nu_{d,n}(x) \oplus \mathbb{R} \nu_{d,n}(y))^\perp,
\]

which finishes the proof. \( \square \)
3.4. On the typical size of some key quantities. Given two linearly independent vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{Z}^{n+1} \), recall that the quantities \( \mathcal{C}_r(\mathbf{x}) \) and \( \mathcal{C}_r(\mathbf{x}, \mathbf{y}) \) were introduced in Definition 3.12 and note that the values \( \mathcal{C}_r(\mathbf{x}) \) and \( \mathcal{C}_r(\mathbf{x}, \mathbf{y}) \) are necessarily attained by primitive lattices. In addition, for any \( r \in \{2, \ldots, n+1\} \) we have the trivial upper bounds

\[
\mathcal{C}_r(\mathbf{x}) \leq ||\mathbf{x}||, \tag{3.19}
\]

and

\[
\mathcal{C}_r(\mathbf{x}, \mathbf{y}) \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||. \tag{3.20}
\]

We shall need to understand how often one can improve upon these upper bounds as the vectors \( \mathbf{x} \) and \( \mathbf{y} \) run over \( \mathbb{Z}^{n+1} \).

It is well-known that the successive minima of a random lattice are expected to have equal order of magnitude. In order to exploit this fact we will require an upper bound for the number of primitive lattices of given rank and whose successive minima are constrained to lie in dyadic intervals. The following notation will thus be very useful.

**Definition 3.17.** Let \( n \geq 2 \) and \( r \in \{1, \ldots, n+1\} \). Given \( s_1, \ldots, s_r \geq 1 \), we let \( S_{r,n}(s_1, \ldots, s_r) \) denote the set of primitive lattices \( L \subset \mathbb{Z}^{n+1} \) of rank \( r \) and such that \( \lambda_j(L) \in (s_j/2, s_j] \) for any \( j \in \{1, \ldots, r\} \).

We shall prove the following result.

**Lemma 3.18.** Let \( n \geq 2 \) and \( r \in \{1, \ldots, n+1\} \). For \( s_1, \ldots, s_r \geq 1 \), we have

\[
\#S_{r,n}(s_1, \ldots, s_r) \ll s_1^{n+r} s_2^{n+r-2} \cdots s_r^{n-r+2},
\]

where the implied constant depends at most on \( n \).

**Proof.** We proceed by induction on the integer \( r \) and we start by noting that the case \( r = 1 \) follows from the observation that

\[
\#S_{1,n}(s_1) \leq \#(\mathbb{Z}^{n+1} \cap B_{n+1}(s_1)).
\]

We now assume that the result holds for some integer \( r - 1 \in \{1, \ldots, n\} \). Given \( L_r \in S_{r,n}(s_1, \ldots, s_r) \), for each \( j \in \{1, \ldots, r-1\} \) we pick \( \mathbf{b}_j \in L \) such that \( ||\mathbf{b}_j|| = \lambda_j(L) \) and we introduce the primitive lattice

\[
L_r^{-1} = (\mathbb{R}\mathbf{b}_1 \oplus \cdots \oplus \mathbb{R}\mathbf{b}_{r-1}) \cap \mathbb{Z}^{n+1}.
\]

Note that \( L_r^{-1} \) depends on \( L \) but may also depend on our choice of \( \mathbf{b}_1, \ldots, \mathbf{b}_{r-1} \). Since \( L_r^{-1} \in S_{r-1,n}(s_1, \ldots, s_{r-1}) \), we deduce that

\[
\#S_{r,n}(s_1, \ldots, s_r) \leq \sum_{L_{r-1} \in S_{r-1,n}(s_1, \ldots, s_{r-1})} \# \left\{ L_r \in S_{r,n}(s_1, \ldots, s_r) : L_r^{-1} = L_{r-1} \right\}.
\]

The estimates (3.5) imply that \( \det(L_{r-1}) \ll s_1 \cdots s_{r-1} \) and \( \det(L_r)/\det(L_{r-1}) \ll s_r \) for any \( L_{r-1} \in S_{r-1,n}(s_1, \ldots, s_{r-1}) \) and \( L_r \in S_{r,n}(s_1, \ldots, s_r) \). It thus follows from the work of Schmidt [17, Lemma 6] (with \( i = 1 \)) that

\[
\# \left\{ L_r \in S_{r,n}(s_1, \ldots, s_r) : L_r^{-1} = L_{r-1} \right\} \ll s_r^{n-r+2} s_1 \cdots s_{r-1}.
\]

Hence we obtain

\[
\#S_{r,n}(s_1, \ldots, s_r) \ll s_r^{n-r+2} s_1 \cdots s_{r-1} \cdot \#S_{r-1,n}(s_1, \ldots, s_{r-1}).
\]

An application of the induction hypothesis completes the proof. \( \square \)
For given $X, Y, \Delta \geq 1$, let
\[
\ell_{r,n}(X; \Delta) = \# \left\{ x \in \mathbb{Z}^{n+1} : 0 < ||x|| \leq X, \lambda_r(x) \leq \Delta \right\},
\]
and
\[
\ell_{r,n}(X, Y; \Delta) = \# \left\{ (x, y) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} : ||x|| \leq X, ||y|| \leq Y, \lambda_r(x, y) \leq \Delta \right\}.
\]

We begin by analysing the first of these quantities.

**Lemma 3.19.** Let $n \geq 2$ and $r \in \{2, \ldots, n+1\}$. For $X, \Delta \geq 1$, we have
\[
\ell_{r,n}(X; \Delta) \ll X^{r} \Delta^{n} \log \Delta,
\]
where the implied constant depends at most on $n$.

**Proof.** We clearly have
\[
\ell_{r,n}(X; \Delta) \ll \sum_{x \in \mathbb{Z}^{n+1} \atop 0 < ||x|| \leq X} \sum_{s_1 \leq \cdots \leq s_r} \sum_{L \in S_{r,n}(s_1, \ldots, s_r)} 1,
\]
where the summations over $s_1, \ldots, s_r$ are over dyadic intervals. Moreover, we note that if $L \subset \mathbb{Z}^{n+1}$ is a lattice containing a non-zero vector $x$ then $\lambda_1(L) \leq ||x||$. We thus have
\[
\ell_{r,n}(X; \Delta) \ll \sum_{s_2 \leq \cdots \leq s_r} \sum_{s_1 \leq X} \sum_{L \in S_{r,n}(s_1, \ldots, s_r)} \# (L \cap B_{n+1}(X)),
\]
since $X \geq s_1$, Lemma 3.5 gives
\[
\sum_{L \in S_{r,n}(s_1, \ldots, s_r)} \# (L \cap B_{n+1}(X)) \ll \frac{X^{r}}{s_1}, \# S_{r,n}(s_1, \ldots, s_r).
\]
An application of Lemma 3.18 thus yields
\[
\ell_{r,n}(X; \Delta) \ll X^{r} \sum_{s_2 \leq \cdots \leq s_r} \sum_{s_1 \leq \Delta/s_2 \cdots s_r} s_1^{n} s_2^{n+r-2} \ldots s_r^{n-r+2}.
\]
Summing over $s_1$ we obtain
\[
\ell_{r,n}(X; \Delta) \ll X^{r} \Delta^{n} \sum_{s_2 \leq \cdots \leq s_r} s_2^{r-2} s_3^{r-4} \cdots s_r^{r-2},
\]
which completes the proof on summing over the remaining variables.

We now handle the quantity $\ell_{r,n}(X, Y; \Delta)$ and prove the following result.

**Lemma 3.20.** Let $n \geq 2$ and $r \in \{2, \ldots, n+1\}$. For $X, Y, \Delta \geq 1$, we have
\[
\ell_{r,n}(X, Y; \Delta) \ll X^{r} Y^{r} \Delta^{n-1} (\log \Delta)^{2 \min\{r-2, 1\}},
\]
where the implied constant depends at most on $n$.

**Proof.** We proceed as in the proof of Lemma 3.19 but here we note that if $L \subset \mathbb{Z}^{n+1}$ is a lattice containing two linearly independent vectors $x$ and $y$ then
\[
\lambda_1(L) \leq \min \{ ||x||, ||y|| \},
\]
and
\[
\lambda_2(L) \leq \max \{ ||x||, ||y|| \}.
\]
We can assume by symmetry that $Y \geq X$. We thus obtain
\[
\ell_{r,n}(X,Y; \Delta) \ll \sum_{s_2 \leq \ldots \leq s_r} \sum_{s_1 \leq \min\{s_2,X\}} \sum_{s_1 \leq \cdots \leq s_r \ll \Delta} \# (L \cap B_{n+1}(X)) \# (L \cap B_{n+1}(Y)),
\]
where the summations over $s_1, \ldots, s_r$ are over dyadic intervals.

We first treat the case $r \geq 3$. We apply twice Lemma 3.5 using the inequalities $X \geq s_1$ and $Y \geq s_2$. It follows that
\[
\sum_{L \in S_r(n(s_1, \ldots, s_r))} \# (L \cap B_{n+1}(X)) \# (L \cap B_{n+1}(Y)) \ll \frac{X^r}{s_1} \cdot \frac{Y^r}{s_1 s_2} \cdot \# S_{r,n}(s_1, \ldots, s_r).
\]
Invoking Lemma 3.18, we thus deduce
\[
\ell_{r,n}(X,Y; \Delta) \ll X^r Y^r \sum_{s_2 \leq \ldots \leq s_r} \sum_{s_1 \leq \Delta/s_2 \cdots s_r} s_1^{n-1} s_2^{n-1-r} s_3^{n-r-4} s_4^{n-r-6} \cdots s_r^{n-r+2}.
\]
The summation over $s_1$ leads to
\[
\ell_{r,n}(X,Y; \Delta) \ll X^r Y^r \Delta^{n-1} \sum_{s_2 \leq \ldots \leq s_r \ll \Delta} s_3^{r-3} s_4^{r-5} \cdots s_r^{r+3}.
\]
Summing over the remaining variables completes the proof in the case $r \geq 3$.

In the case $r = 2$, using the inequalities $X \geq s_1$ and $Y \geq s_2 \geq s_1$ we see that Lemma 3.5 gives
\[
\sum_{L \in S_2(n(s_1, s_2))} \# (L \cap B_{n+1}(X)) \# (L \cap B_{n+1}(Y)) \ll \left( \frac{X^2}{s_1 s_2} + \frac{X}{s_1} \right) \frac{Y^2}{s_1 s_2} \cdot \# S_{2,n}(s_1, s_2).
\]
Therefore, it follows from Lemma 3.18 that
\[
\ell_{2,n}(X,Y; \Delta) \ll X^2 Y^2 \sum_{s_1 \leq s_2} s_1^2 s_2^{n-2} + X Y^2 \sum_{s_1 \leq \Delta} s_1^{n-1}.
\]
Summing over $s_2$ we derive
\[
\ell_{2,n}(X,Y; \Delta) \ll X^2 Y^2 \Delta^{n-2} \sum_{s_1 \ll \Delta^{1/2}} s_1^2 + X Y^2 \Delta^{n-1} \sum_{s_1 \leq \Delta},
\]
which finishes the proof in the case $r = 2$ on summing over $s_1$. \hfill \Box

3.5. Handling the case of quartic threefolds. In the hardest case $(d,n) = (4,4)$ of Theorem 1.1, we will struggle to handle the contribution from choices of linearly independent vectors $x, y \in \mathbb{Z}_\text{prim}^5$ which produce particularly short vectors in the lattice $\Lambda_{4,4}(x) \cap \Lambda_{4,4}(y)$ and which happen to lie in a lattice of rank 3 with small determinant. We shall deal with this issue by showing that such vectors are very rare.

Note that for any linearly independent vectors $x, y \in \mathbb{Z}^5$ the lattice $\Lambda_{4,4}(x) \cap \Lambda_{4,4}(y)$ has rank 68 since $N_{4,4} = 70$. For given $Z, \Delta, M \geq 1$ and $j \in \{1, \ldots, 68\}$ we define
\[
\ell^{(j)}(Z; \Delta, M) = \# \left\{ (x, y) \in \mathbb{Z}^5 \times \mathbb{Z}^5 : \begin{array}{c}
\dim (\text{Span}_2(\{x, y\})) = 2 \\
||x||, ||y|| \leq Z \\
\delta_3(x, y) \leq \Delta \\
\lambda_j(\Lambda_{4,4}(x) \cap \Lambda_{4,4}(y)) \leq M
\end{array} \right\}.
\]
(3.23)

Lemma 3.20 immediately shows that for any $j \in \{1, \ldots, 68\}$ we have
\[
\ell^{(j)}(Z; \Delta, M) \ll Z^6 \Delta^{3(\log \Delta)^2}.
\]
(3.24)
We now focus on the case $j = 1$ and we shall obtain the following result, which improves on this upper bound when $M$ is small.
Lemma 3.21. Let $\varepsilon > 0$. For $Z, \Delta, M \geq 1$, we have

$$f^{(1)}(Z; \Delta, M) \ll M^{40} Z^4 \Delta \left( M^{30} Z^2 \Delta^\varepsilon + M^{30} \Delta^{s/3} + Z^2 \Delta \right) (\log \Delta)^2,$$

where the implied constant depends at most on $\varepsilon$.

Proof. Recall that the set $S_{3,4}(s_1, s_2, s_3)$ was introduced in Definition 3.17. In a similar way as in the proofs of Lemmas 3.19 and 3.20, we start by noting that

$$f^{(1)}(Z; \Delta, M) \ll \sum_{c \in \mathbb{Z}^{N_{4,4}}} \sum_{\substack{s_1 \leq s_2 \leq \max\{Z, s_3\} \quad L \in S_{3,4}(s_1, s_2, s_3)}} \sum_{\ell \in S_{3,4}(s_1, s_2, s_3)} N_c(Z; L)^2,$$

where the summations over $s_1, s_2$ and $s_3$ are over dyadic intervals and

$$N_c(Z; L) = \# \{ x \in L \cap B_5(Z) : \langle \nu_{4,4}(x), c \rangle = 0 \}.$$

Given a non-zero vector $c \in \mathbb{Z}^{N_{4,4}}$, we let

$$S_{3,4}(s_1, s_2, s_3; c) = \{ L \in S_{3,4}(s_1, s_2, s_3) : \text{Span}_c(L) \subset \{ x \in \mathbb{R}^5 : \langle \nu_{4,4}(x), c \rangle = 0 \} \},$$

and we also let $T_{3,4}(s_1, s_2, s_3; c)$ be the complement of $S_{3,4}(s_1, s_2, s_3; c)$ in $S_{3,4}(s_1, s_2, s_3)$. We first handle the contribution from lattices belonging to the set $S_{3,4}(s_1, s_2, s_3; c)$. We note that if $L \in S_{3,4}(s_1, s_2, s_3; c)$ then

$$N_c(Z; L) = \# (L \cap B_5(Z)),$$

so the inequality $Z \geq s_2$ gives

$$\sum_{L \in S_{3,4}(s_1, s_2, s_3; c)} N_c(Z; L)^2 \ll \left( \frac{Z}{s_1} \cdot \frac{Z}{s_2} \left( \frac{Z}{s_3} + 1 \right) \right)^2 \# S_{3,4}(s_1, s_2, s_3; c).$$

In addition, for any lattice $L \in S_{3,4}(s_1, s_2, s_3; c)$ we can use [12, Lemma 5] to pick a basis $(b_1, b_2, b_3)$ of $L$ such that for any $j \in \{1, 2, 3\}$, we have

$$\lambda_j(L) \leq ||b_j|| \ll \lambda_j(L).$$

We deduce that there exists an absolute constant $C > 0$ such that

$$\# S_{3,4}(s_1, s_2, s_3; c) \ll \prod_{j=1}^{3} \# \{ b_j \in B_5(Cs_j) : \langle \nu_{4,4}(b_j), c \rangle = 0 \}.$$

Since $c$ is a non-zero vector we trivially have

$$\# S_{3,4}(s_1, s_2, s_3; c) \ll s_1^4 s_2^4 s_3^4,$$

where the implied constant is independent of $c$. Furthermore, if the quartic form $\langle \nu_{4,4}(u), c \rangle$ is irreducible over $\mathbb{Q}$ then we can appeal to work of Broberg and Salberger [5, Theorem 1]. It follows in this case that for any $\varepsilon > 0$ we have

$$\# S_{3,4}(s_1, s_2, s_3; c) \ll s_1^{3+\varepsilon} s_2^{3+\varepsilon} s_3^{3+\varepsilon},$$

where the implied constant may depend on $\varepsilon$ but, crucially, not on $c$. In addition, if the form $\langle \nu_{4,4}(u), c \rangle$ is reducible over $\mathbb{Q}$ but irreducible over $\mathbb{Q}$ then we see that the set $\{ b \in \mathbb{Z}^5 : \langle \nu_{4,4}(b), c \rangle = 0 \}$ lies on an affine subvariety of codimension at least 2, and a trivial estimate directly yields the upper bound (3.28) with $\varepsilon = 0$. Recalling the upper bound (3.27) we thus see that

$$\sum_{L \in S_{3,4}(s_1, s_2, s_3; c)} N_c(Z; L)^2 \ll Z^6 \frac{s_1^6}{s_2^2} \frac{s_2^6}{s_3^2} + \frac{Z^4}{s_1^2 s_2^2} \# S_{3,4}(s_1, s_2, s_3; c).$$
where
\[ \vartheta_c = \begin{cases} 1 + \varepsilon, & \text{if the form } \langle \nu_{4,4}(\mathbf{u}), \mathbf{c} \rangle \text{ is irreducible over } \mathbb{Q}, \\ 2, & \text{otherwise}. \end{cases} \]

We now deal with the contribution from lattices belonging to the set \( T_{3,4}(s_1, s_2, s_3; \mathbf{c}) \). Given \( L \in T_{3,4}(s_1, s_2, s_3; \mathbf{c}) \) we again use [12, Lemma 5] to select a basis \((b_1, b_2, b_3)\) of \( L \) with the property that if \( \mathbf{x} \in L \) is given by \( \mathbf{x} = t_1b_1 + t_2b_2 + t_3b_3 \) for some \((t_1, t_2, t_3) \in \mathbb{Z}^3\) then \( t_j \ll ||\mathbf{x}||/s_j \) for any \( j \in \{1, 2, 3\} \). We thus have
\[ N_c(Z; L) \ll \# \left\{ (t_1, t_2, t_3) \in \mathbb{Z}^3 : t_j \ll Z/s_j, \ j \in \{1, 2, 3\} \right\} \langle \nu_{4,4}(t_1b_1 + t_2b_2 + t_3b_3), \mathbf{c} \rangle = 0 \right\}. \tag{3.30} \]

Since \( L \in T_{3,4}(s_1, s_2, s_3; \mathbf{c}) \) the polynomial function \( \langle \nu_{4,4}(u_1b_1 + u_2b_2 + u_3b_3), \mathbf{c} \rangle \) is not identically equal to 0 as \((u_1, u_2, u_3)\) runs over \( \mathbb{R}^3 \). It follows that there exists \( j \in \{1, 2, 3\} \) such that the coordinate \( t_j \) of the elements of the set in the right-hand side of the upper bound (3.30) can assume at most 4 values when the two other coordinates are fixed. Therefore, since \( Z \geq s_2 \) we have
\[ N_c(Z; L) \ll \frac{Z}{s_1} \cdot \frac{Z}{s_2}, \tag{3.31} \]
where the implies constant does not depend on \( c \). This trivially yields
\[ \sum_{L \in T_{3,4}(s_1, s_2, s_3; \mathbf{c})} N_c(Z; L)^2 \ll \frac{Z^4}{s_1^2 s_2^2} \cdot \# T_{3,4}(s_1, s_2, s_3; \mathbf{c}). \tag{3.32} \]

Combining the upper bounds (3.29) and (3.32) we deduce that
\[ \sum_{L \in S_{3,4}(s_1, s_2, s_3)} N_c(Z; L)^2 \ll Z^6 s_{1\varepsilon} s_{2\varepsilon} s_{3\varepsilon} + \frac{Z^4}{s_1^2 s_2^2} \cdot \# S_{3,4}(s_1, s_2, s_3). \]

Recall that \( N_{4,4} = 70 \). Using the upper bound (3.25) and Lemma 3.18 we thus derive
\[ \ell^{(1)}(Z; \Delta, M) \ll M^{70} \sum_{\substack{s_1 \leq s_2 \leq s_3 \leq \Delta \\text{satisfying } ||\mathbf{c}|| \leq M \text{ and } \text{the form } \langle \nu_{4,4}(\mathbf{u}), \mathbf{c} \rangle \text{ is reducible over } \mathbb{Q} \text{ is trivially bounded by an absolute constant times} \}} Z^{\delta} s_1^{\gamma_1} s_2^{\gamma_2} s_3^{\gamma_3} + M^{40} Z^6 \sum_{\substack{s_1 \leq s_2 \leq s_3 \leq \Delta \\text{satisfying } ||\mathbf{c}|| \leq M \text{ and } \text{the form } \langle \nu_{4,4}(\mathbf{u}), \mathbf{c} \rangle \text{ is reducible over } \mathbb{Q} \text{ is trivially bounded by an absolute constant times} \}} s_1^2 s_2^2 s_3^2. \]

Note that we have used the fact that the number of \( \mathbf{c} \in \mathbb{Z}^{N_{4,4}} \) satisfying \( ||\mathbf{c}|| \leq M \) and such that the form \( \langle \nu_{4,4}(\mathbf{u}), \mathbf{c} \rangle \) is reducible over \( \mathbb{Q} \) is trivially bounded by an absolute constant times
\[ M_{\text{max}}(\ell_1^{(1)} + \ell_2^{(1)} + \ell_3^{(1)}) = M^{40}. \]

We finally remark that it follows from the inequalities \( s_1 \leq s_2 \leq s_3 \) and \( s_1 s_2 s_3 \ll \Delta \) that \( s_1^2 s_2^2 s_3^2 \ll \Delta^{11/3} \), which finishes the proof.

We now combine Lemmas 3.20 and 3.21 to deduce an upper bound which will prove to be very convenient in the proof of Lemma 4.9.

**Lemma 3.22.** For \( Z, \Delta, M \geq 1 \), we have
\[ \ell^{(1)}(Z; \Delta, M) \ll Z^{4\delta} \Delta^{2} \left( Z^{3/2} + \Delta^{3/2} \right) (\log \Delta)^2 \min \{ \Delta, M^{53} \}, \]
where the implied constant is absolute.

**Proof.** If \( \Delta \leq M^{53} \) then the upper bound (3.24) provides the desired result. In the case where \( \Delta > M^{53} \), we see that \( M^{30} Z^2 \Delta^\varepsilon \leq Z^2 \Delta \) if \( \varepsilon = 1/3 \), say. Therefore, using the upper bound (3.24) and Lemma 3.21 we deduce that
\[ \ell^{(1)}(Z; \Delta, M) \ll Z^{4\delta} \Delta^{2} (\log \Delta)^2 \min \left\{ Z^2 \Delta^2, M^{40} \Delta \left( M^{30} \Delta^{5/3} + Z^2 \right) \right\}. \]
First, if \( Z^2 > M^{30} \Delta^{5/3} \) then we obtain
\[
\ell^{(1)}(Z; \Delta, M) \ll Z^6 \Delta^2 (\log \Delta)^2 M^{40},
\]
which is satisfactory. Next, if \( Z^2 \leq M^{30} \Delta^{5/3} \) we get
\[
\ell^{(1)}(Z; \Delta, M) \ll Z^4 \Delta^3 (\log \Delta)^2 \min \left\{ Z^2, M^{70} \Delta^{2/3} \right\}.
\]
Using the inequality
\[
\min \left\{ Z^2, M^{70} \Delta^{2/3} \right\} \leq (Z^2)^{1/4} \left( M^{70} \Delta^{2/3} \right)^{3/4},
\]
we derive
\[
\ell^{(1)}(Z; \Delta, M) \ll Z^{9/2} \Delta^{7/2} (\log \Delta)^2 M^{105/2},
\]
which completes the proof. \( \square \)

In the worst situation, that is when \( M \) and \( \Delta \) are both small, we see that Lemma 3.22 is not much stronger than the upper bound (3.24). In this case we will thus require a different argument. Given a rank 3 lattice \( L \subset \mathbb{Z}^5 \) we define the lattice
\[
\mathcal{V}(L) = \text{Span}_{\mathbb{Q}}(\nu_{4,4}(L)) \cap \mathbb{Z}^{N_{4,4}},
\]
and we note that the dimension of the subspace \( \text{Span}_{\mathbb{Q}}(\nu_{4,4}(L)) \) is equal to \( (6)_4 = 15 \) and therefore \( \mathcal{V}(L) \) has rank 15. We shall prove the following result, which states that the determinant of the lattice \( \mathcal{V}(L) \) can be controlled in terms of the determinant of \( L \).

**Lemma 3.23.** Let \( L \subset \mathbb{Z}^5 \) be a lattice of rank 3. Then
\[
\det(\mathcal{V}(L)) \ll \det(L)^{20},
\]
where the implied constant is absolute.

**Proof.** We start by using [12, Lemma 5] to select a basis \( (b_1, b_2, b_3) \) of \( L \) satisfying
\[
||b_1|| \cdot ||b_2|| \cdot ||b_3|| \ll \det(L). \quad (3.34)
\]
By definition we have
\[
\mathcal{V}(L) = \text{Span}_{\mathbb{Q}}(\{\nu_{4,4}(\ell_1 b_1 + \ell_2 b_2 + \ell_3 b_3) : (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3\}) \cap \mathbb{Z}^{N_{4,4}}.
\]
Letting \( (b_{j,0}, \ldots, b_{j,4}) \) be the coordinates of the vector \( b_j \) for any \( j \in \{1, 2, 3\} \), we see that the vector \( \nu_{4,4}(X_1 b_1 + X_2 b_2 + X_3 b_3) \) has coordinates
\[
\prod_{j=1}^{4} (X_1 b_{1,j} + X_2 b_{2,j} + X_3 b_{3,j}),
\]
that are indexed by the \( (i_1, \ldots, i_4) \in \mathbb{Z}^4 \) satisfying \( 0 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq 4 \). As a result, by using the polynomial identity
\[
\nu_{4,4}(X_1 b_1 + X_2 b_2 + X_3 b_3) = \sum_{0 \leq e_1 + e_2 + e_3 \leq 4} X_1^{e_1} X_2^{e_2} X_3^{e_3} v_{e_1, e_2, e_3},
\]
we define 15 vectors \( v_{e_1, e_2, e_3} \in \mathbb{Z}^{N_{4,4}} \) indexed by the \( (e_1, e_2, e_3) \in \{0, \ldots, 4\}^3 \) such that \( e_1 + e_2 + e_3 = 4 \), and we note that
\[
||v_{e_1, e_2, e_3}|| \ll ||b_1||^{e_1} ||b_2||^{e_2} ||b_3||^{e_3}. \quad (3.35)
\]
Since the lattice \( \mathcal{V}(L) \) has rank 15, we see that the 15 vectors \( v_{e_1, e_2, e_3} \) are linearly independent. Therefore, the lattice
\[
\bigoplus_{0 \leq e_1 + e_2 + e_3 \leq 4} \mathbb{Z} v_{e_1, e_2, e_3}
\]
is a sublattice of $\mathcal{V}(L)$ of finite index, whence
\[
\det(\mathcal{V}(L)) \leq \prod_{0 \leq e_1, e_2, e_3 \leq 4 \atop e_1 + e_2 + e_3 = 4} ||\mathbf{v}_{e_1, e_2, e_3}||.
\]
Using the upper bound (3.35) and the identity
\[
\sum_{0 \leq e, f, g \leq 4 \atop e + f + g = 4} e = 20,
\]
we eventually obtain
\[
\det(\mathcal{V}(L)) \ll ||\mathbf{b}_1||^{20}||\mathbf{b}_2||^{20}||\mathbf{b}_3||^{20},
\]
which completes the proof on recalling the upper bound (3.34).

Recall the definition (3.23) of $\ell(j)(Z; \Delta, M)$ for a given integer $j \in \{1, \ldots, 68\}$. We now use Lemma 3.23 to derive an upper bound for the quantity $\ell(56)(Z; \Delta, M)$.

**Lemma 3.24.** For $Z, \Delta, J \geq 1$, we have
\[
\ell(56)(Z; \Delta, J) \ll Z^4 \Delta^{24} J^{15},
\]
where the implied constant is absolute.

**Proof.** We start by noting that
\[
\ell(56)(Z; \Delta, J) \ll \sum_{0 \leq s_1, s_2 \leq \max\{Z, s_3\} \atop s_1 s_2 s_3 \ll \Delta} \sum_{L \in S_{1,4}(s_1, s_2, s_3)} \#\mathcal{L}(Z, J; L),
\]
where
\[
\mathcal{L}(Z, J; L) = \left\{ (x, y) \in (L \cap B_5(Z))^2 : \dim(\operatorname{Span}_{\mathbb{R}}(\{x, y\})) = 2, \lambda_{56}(\Lambda_{v_{4,4}(x)} \cap \Lambda_{v_{4,4}(y)}) \leq J \right\}.
\]
Let $(x, y) \in \mathcal{L}(Z, J; L)$. Recall the definition (3.33) of the lattice $\mathcal{V}(L)$ and recall that its rank is equal to 15. The orthogonal lattice $\mathcal{V}(L)^\perp$ has rank 70 − 15 = 55 so the inequality $\lambda_{56}(\Lambda_{v_{4,4}(x)} \cap \Lambda_{v_{4,4}(y)}) \leq J$ implies that there exists $a \in \Lambda_{v_{4,4}(x)} \cap \Lambda_{v_{4,4}(y)}$ such that $||a|| \leq J$ and $a \not\in \mathcal{V}(L)^\perp$. Furthermore, by definition the lattice $\mathcal{V}(L)$ is primitive so $(\mathcal{V}(L)^\perp)^\perp = \mathcal{V}(L)$ and we deduce from Lemma 3.1 that $\mathcal{V}(L)^* = \mathbb{Z}^{70}/\mathcal{V}(L)^\perp$. Hence, if $\pi : \mathbb{R}^{70} \to \operatorname{Span}_{\mathbb{R}}(\mathcal{V}(L))$ denotes the orthogonal projection on $\operatorname{Span}_{\mathbb{R}}(\mathcal{V}(L))$ then the vector $b = \pi(a)$ belongs to $\mathcal{V}(L)^*$. In addition, we have $||b|| \leq ||a|| \leq J$ and $b$ is non-zero since $a \notin \operatorname{Span}_{\mathbb{R}}(\mathcal{V}(L))^\perp$. Finally we note that
\[
\langle b, v_{4,4}(x) \rangle = \langle b, v_{4,4}(y) \rangle = 0.
\]
Indeed, by definition of $b$ we have $a - b \in \operatorname{Span}_{\mathbb{R}}(\mathcal{V}(L))^\perp$, which implies that
\[
\langle a - b, v_{4,4}(x) \rangle = \langle a - b, v_{4,4}(y) \rangle = 0,
\]
since $x, y \in L$. Moreover we also have $a \in \Lambda_{v_{4,4}(x)} \cap \Lambda_{v_{4,4}(y)}$ so the equalities (3.36) follow. As a result, recalling the definition (3.26) of $N_h(Z; L)$ we deduce that
\[
\ell(56)(Z; \Delta, J) \ll \sum_{0 \leq s_1, s_2 \leq \max\{Z, s_3\} \atop s_1 s_2 s_3 \ll \Delta} \sum_{L \in S_{1,4}(s_1, s_2, s_3)} \sum_{b \in \mathcal{V}(L)^*} N_b(Z; L)^2.
\]
If we had $\operatorname{Span}_{\mathbb{R}}(L) \subset \{ x \in \mathbb{R}^5 : \langle v_{4,4}(x), b \rangle = 0 \}$ then it would follow that $\langle b, b \rangle = 0$ since $b \in \operatorname{Span}_{\mathbb{R}}(\mathcal{V}(L))$. This is impossible as $b$ is non-zero. Therefore, since we have
Therefore, it follows from Lemma 3.5 and the estimates (3.5) that
\[ \lambda \leq \text{Lemma 3.3 thus implies that we also have by estimating the volume of certain regions in } \]
\[ \text{then devoted to the proof of Proposition 4.1. With this goal i n mind we will start variance upper bound, as stated in Proposition 4.1. The rema inder of Section 4 is} \]
\[ \text{imply that } \text{s} \text{ which completes the proof on noting that the inequalities} \]
\[ \text{(3.38) and (3.39), we get} \]
\[ \text{Using again Minkowski’s estimates (3.5) and putting togeth er the upper bounds (3.37),} \]
\[ \text{An application of Lemma 3.18 eventually gives} \]
\[ \text{An application of Lemma 3.18 eventually gives} \]
\[ \text{which completes the proof on noting that the inequalities } s_1 \leq s_2 \leq s_3 \text{ and } s_1 s_2 s_3 \leq \Delta \text{i} \]
\[ \text{4. THE GLOBAL AND LOCALISED COUNTING FUNCTIONS ARE RARELY APART} \]
\[ \text{In Section 4.1 we check that Proposition 2.3 is a direct consequence of a certain variance upper bound, as stated in Proposition 4.1. The remainder of Section 4 is then devoted to the proof of Proposition 4.1. With this goal in mind we will start by estimating the volume of certain regions in } \mathbb{R}^N \text{ in Section 4.2. In Section 4.3 we will then produce tight bounds for the average of the inverse of the determinant of the lattice } \Lambda_{V_d,n}(x) \cup \Lambda_{V_d,n}(y), \text{ as one varies the linearly independent vectors } x, y \in \mathbb{Z}^{n+1}. \]
\[ \text{Our next task in Section 4.4 will be to prove an upper bound for the first moment of the counting function } N_V(B). \text{ In Section 4.5 we will turn to proving estimates for second moments involving both } N_V(B) \text{ and our localised counting function } N_{Vloc}(B). \text{ We will finally combine all these estimates to prove Proposition 4.1 in Section 4.6.} \]
\[ \text{4.1. The key variance upper bound. Recall the respective definitions (2.2) and (2.11) of our two counting functions } N_V(B) \text{ and } N_{Vloc}(B). \text{ The following result is the culmination of our work in Section 4.} \]
\[ \text{Proposition 4.1. Let } d \geq 2 \text{ and } n \geq d \text{ with } (d,n) \notin \{(2,2),(3,3)\}. \text{ Assume that } B/(\log B)^{1/2} \leq A \leq B^2. \text{ Then we have} \]
\[ \frac{1}{\#V_d,n(A)} \sum_{V \in V_d,n(A)} \left( N_V(B) - N_{Vloc}(B) \right)^2 \leq \frac{B}{A}. \]
We now proceed to prove that Proposition 2.3 follows from Proposition 4.1.

**Proof of Proposition 2.3.** It is convenient to set

\[ \mathcal{L}_\phi(A) = \frac{1}{\#V_{d,n}(A)} \cdot \# \left\{ V \in V_{d,n}(A) : \left| V \right| (A) - N_{V}^{\text{loc}}(A) \phi(A) \right\} \, . \]

We observe that

\[ \mathcal{L}_\phi(A) \leq \frac{1}{\#V_{d,n}(A)} \sum_{V \in V_{d,n}(A)} \left( \frac{N_{V}(A) - N_{V}^{\text{loc}}(A)}{\phi(A)^{2/3}} \right)^{2} \, . \]

By assumption we have \( \phi(A) \leq (\log A)^{1/2} \) so we are in position to apply Proposition 4.1, which immediately completes the proof of Proposition 2.3. \( \square \)

### 4.2. Volume estimates.

For any \( N \geq 1 \) we let \( V_{N} \) denote the volume of the unit ball \( B_{N}(1) \) in \( \mathbb{R}^{N} \). For \( w, z \in \mathbb{R}^{N} \), we introduce the \((N - 1)\)-dimensional volume

\[ I(w, z) = \text{vol} \left( \left\{ t \in (\mathbb{R}w)^{\perp} : \left| (z, t) \right| \leq \| t \| \leq 1 \right\} \right) \, , \tag{4.1} \]

and we put

\[ \delta_{w, z} = \| w \|^{2} \| z \|^{2} - \langle w, z \rangle^{2} \, . \]

We start by proving the following result.

**Lemma 4.2.** Let \( N \geq 3 \) and let \( w, z \in \mathbb{R}^{N} \) be two linearly independent vectors. Then

\[ I(w, z) = 2^{N-2} \frac{N-2}{N-1} V_{N-2} \frac{\| w \|}{\delta_{w, z}^{1/2}} \left( 1 + O \left( \frac{1}{\delta_{w, z}^{1/2}} \right) \right) \, , \]

where the implied constant depends at most on \( N \).

**Proof.** Let \( (g_{3}, \ldots, g_{N}) \) be an orthonormal basis of \((\mathbb{R}w \oplus \mathbb{R}z)^{\perp}\) and set

\[ g_{2} = \frac{\| w \|^{2} z - \langle w, z \rangle w}{\delta_{w, z}^{1/2} \cdot \| w \|} \, . \]

We have \( g_{2} \in \mathbb{R}w \oplus \mathbb{R}z, \langle g_{2}, w \rangle = 0 \) and \( \| g_{2} \| = 1 \), so the family \((g_{2}, \ldots, g_{N})\) is an orthonormal basis of \((\mathbb{R}w)^{\perp}\). It follows that

\[ I(w, z) = \text{vol} \left( \left\{ (u_{2}, u) \in \mathbb{R}^{N-1} : \delta_{w, z} \frac{u_{2}^{2}}{\| w \|^{2}} \leq u_{2}^{2} + \| u \|^{2} \leq 1 \right\} \right) \, . \]

If \( \| w \|^{2} \geq \delta_{w, z} \) then \( I(w, z) = V_{N-1} \) and the claimed estimate holds. We now handle the case where \( \| w \|^{2} < \delta_{w, z} \). Integrating over \( u \) we obtain

\[ I(w, z) = 2V_{N-2} \int_{0}^{\| w \|^{2} / \delta_{w, z}^{1/2}} \left( 1 - u_{2}^{2} \right)^{(N-2)/2} \left( \frac{\delta_{w, z}}{\| w \|^{2}} \right)^{(N-2)/2} \frac{u_{2}^{2} - 1}{u_{2}^{2} - 1} \, du_{2} \]

\[ = 2V_{N-2} \int_{0}^{\| w \|^{2} / \delta_{w, z}^{1/2}} \left( 1 - \left( \frac{\delta_{w, z}}{\| w \|^{2}} \right)^{(N-2)/2} u_{2}^{2} \right) \left( 1 + O \left( \frac{\| w \|^{2}}{\delta_{w, z}^{1/2}} \right) \right) \, du_{2} \]

\[ = \frac{2}{N-1} V_{N-2} \frac{\| w \|}{\delta_{w, z}^{1/2}} \left( 1 + O \left( \frac{\| w \|^{2}}{\delta_{w, z}^{1/2}} \right) \right) \, , \]

which completes the proof. \( \square \)

Next, for \( w, z \in \mathbb{R}^{N} \), we let

\[ J(w, z) = \text{vol} \left( \left\{ t \in \mathbb{R}^{N} : \langle w, t \rangle, \langle z, t \rangle \leq \| t \| \leq 1 \right\} \right) \, . \tag{4.2} \]

We shall establish the following estimate for this quantity.
Lemma 4.3. Let $N \geq 3$ and let $w, z \in \mathbb{R}^N$ be two linearly independent vectors. Then

$$J(w, z) = 4 \frac{N-2}{N} V_{N-2} \frac{1}{\delta_{w,z}} \left(1 + O \left( \min \left\{ 1, \frac{(||w|| + ||z||)^2}{\delta_{w,z}} \right\} \right) \right),$$

where the implied constant depends at most on $N$.

Proof. Let $(f_3, \ldots, f_N)$ be an orthonormal basis of $(\mathbb{R}w \oplus \mathbb{R}z)^\perp$ and let $C$ be the $N \times N$ matrix

$$C = \begin{pmatrix} ||z||^2 w - (w, z)z & -(w, z)w + ||w||^2 z \\ \delta_{w,z} & \delta_{w,z} \end{pmatrix} f_3 \ldots f_N.$$ 

Letting $0$ be the zero vector of size $N-2$ and $I_{N-2}$ be the identity matrix of size $N-2$, we note that $C^T C$ is a block diagonal matrix given by

$$C^T C = \frac{1}{\delta_{w,z}} \begin{pmatrix} ||z||^2 & - (w, z) & 0^T \\ - (w, z) & ||w||^2 & 0 \\ 0 & 0 & \delta_{w,z} I_{N-2} \end{pmatrix}.$$ 

This yields in particular $|\text{det}(C)| = 1/\delta_{w,z}$. Making the change of variables $t = Cu$ and letting $v$ be the vector whose coordinates are the $N-2$ final coordinates of $u$, we find that

$$J(w, z) = \frac{1}{\delta_{w,z}} \cdot \text{vol} \left( \{ (u_1, u_2, v) \in \mathbb{R}^N : \max \left\{ u_1^2, u_2^2 \right\} \leq q_{w,z}(u_1, u_2) + ||v||^2 \leq 1 \} \right),$$

where we have introduced the positive definite quadratic form

$$q_{w,z}(u_1, u_2) = \frac{||u_1 z - u_2 w||^2}{\delta_{w.z}}.$$ 

We see that $J(w, z) \ll 1/\delta_{w,z}^{1/2}$, so if $(||w|| + ||z||)^2 \geq \delta_{w,z}$ then the claimed estimate holds. We now deal with the case where $(||w|| + ||z||)^2 < \delta_{w,z}$. We first note that if $(u_1, u_2) \in \mathbb{R}^2$ satisfies the conditions $u_1^2, u_2^2 \leq 1$ then

$$q_{w,z}(u_1, u_2) \leq \frac{(||w|| + ||z||)^2}{\delta_{w,z}}.$$ 

We thus deduce that

$$\frac{K^{(1)}(w, z)}{\delta_{w,z}^{1/2}} \leq J(w, z) \leq \frac{K^{(2)}(w, z)}{\delta_{w,z}^{1/2}},$$

where

$$K^{(1)}(w, z) = \text{vol} \left( \{ (u_1, u_2, v) \in \mathbb{R}^N : \max \left\{ u_1^2, u_2^2 \right\} \leq ||v||^2 \leq 1 - \frac{(||w|| + ||z||)^2}{\delta_{w,z}} \} \right),$$

and

$$K^{(2)}(w, z) = \text{vol} \left( \{ (u_1, u_2, v) \in \mathbb{R}^N : \max \left\{ u_1^2, u_2^2 \right\} \leq \frac{(||w|| + ||z||)^2}{\delta_{w,z}} \leq ||v||^2 \leq 1 \} \right).$$

Integrating over $u_1$ and $u_2$ we easily obtain that for any $i \in \{1, 2\}$ we have

$$K^{(i)}(w, z) = 4 \left( \int_{B_{N-2}(1)} ||v||^2 dv \right) \left( 1 + O \left( \frac{(||w|| + ||z||)^2}{\delta_{w,z}} \right) \right),$$

Recalling the inequalities (4.3) we see that

$$J(w, z) = 4 \frac{1}{\delta_{w,z}^{1/2}} \left( \int_{B_{N-2}(1)} ||v||^2 dv \right) \left( 1 + O \left( \frac{(||w|| + ||z||)^2}{\delta_{w,z}} \right) \right).$$
Finally, a straightforward calculation involving spherical coordinates shows that
\[
\int_{B_{N-2}(1)} ||v||^2 dv = \frac{N - 2}{N} V_{N-2},
\]
which completes the proof. □

4.3. Inverses of lattice determinants on average. Let \(x, y \in \mathbb{Z}^{n+1}_{\text{prim}}\) be two linearly independent vectors. The lattice \(\Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)}\) features heavily in our work and we shall need to be able to control the inverse of its determinant on average. Recall that the quantities \(G(x, y)\) and \(\mathcal{O}_2(x, y)\) were respectively introduced in Definitions 3.7 and 3.12. It follows from Lemmas 3.8 and 3.11 that
\[
\det(\Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)}) = \frac{\det(\mathbb{Z}_{\nu_{d,n}(x)} \oplus \mathbb{Z}_{\nu_{d,n}(y)})}{G(x, y)}. \quad (4.4)
\]
We start by proving the following pointwise bounds.

**Lemma 4.4.** Let \(d, n \geq 2\) and let \(x, y \in \mathbb{Z}^{n+1}_{\text{prim}}\) be two linearly independent vectors. We have
\[
\mathcal{O}_2(x, y) \cdot ||x||^{d-1}||y||^{d-1} \ll \det(\Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)}) \ll ||x||^d||y||^d.
\]

**Proof.** Since \(G(x, y) \geq 1\), the equality (4.4) implies that
\[
\det(\Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)}) \leq ||\nu_{d,n}(x)|| \cdot ||\nu_{d,n}(y)||,
\]
and the claimed upper bound follows.

Recall that the set \(\mathcal{M}_{d,n}\) of monomials of degree \(d\) in \(n + 1\) variables was introduced in Definition 3.10. By the definition (3.1) of the determinant of a lattice we have
\[
\det(\mathbb{Z}_{\nu_{d,n}(x)} \oplus \mathbb{Z}_{\nu_{d,n}(y)}) = ||\nu_{d,n}(x)||^2 ||\nu_{d,n}(y)||^2 - \langle \nu_{d,n}(x), \nu_{d,n}(y) \rangle^2, \quad (4.5)
\]
which can be rewritten as
\[
\det(\mathbb{Z}_{\nu_{d,n}(x)} \oplus \mathbb{Z}_{\nu_{d,n}(y)}) = \frac{1}{2} \sum_{P_1, P_2 \in \mathcal{M}_{d,n}} (P_1(x)P_2(y) - P_2(x)P_1(y))^2.
\]
We thus see that
\[
\det(\mathbb{Z}_{\nu_{d,n}(x)} \oplus \mathbb{Z}_{\nu_{d,n}(y)}) \geq \frac{1}{4} \sum_{i_1, j_1, i_2, j_2 = 0}^n \left( x_{i_1} x_{j_1} - y_{i_2} y_{j_2} \right)^2.
\]
It follows that
\[
\det(\mathbb{Z}_{\nu_{d,n}(x)} \oplus \mathbb{Z}_{\nu_{d,n}(y)}) \geq \frac{1}{2} \left( ||x||^2 ||y||^2 ||\psi_d(x)||^2 ||\psi_d(y)||^2 - \langle x, y \rangle^2 (\psi_d(x), \psi_d(y))^2 \right),
\]
where the map \(\psi_d : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}\) is defined for \(z \in \mathbb{R}^{n+1}\) by
\[
\psi_d(z) = \left( z^{d-1}_0, \ldots, z^{d-1}_n \right).
\]
The Cauchy–Schwarz inequality
\[
(\langle \psi_d(x), \psi_d(y) \rangle)^2 \leq ||\psi_d(x)||^2 ||\psi_d(y)||^2
\]
and the lower bounds \(||\psi_d(x)|| \gg ||x||^{d-1}\) and \(||\psi_d(y)|| \gg ||y||^{d-1}\) eventually yield
\[
\det(\mathbb{Z}_{\nu_{d,n}(x)} \oplus \mathbb{Z}_{\nu_{d,n}(y)}) \geq \left( ||x||^2 ||y||^2 - \langle x, y \rangle^2 \right) \cdot ||x||^{2(d-1)} ||y||^{2(d-1)}.
\]
An application of Lemma 3.13 completes the proof. □
We define  
\[ E_{d,n}(B) = \sum_{(x,y) \in \Omega_{d,n}(B)} \frac{1}{\det(\Lambda_{d,n}(x) \cap \Lambda_{d,n}(y))}, \]  
(4.6)  
where  
\[ \Omega_{d,n}(B) = \left\{ (x,y) \in \mathbb{Z}_{\text{prim}}^{n+1} \times \mathbb{Z}_{\text{prim}}^{n+1} : \|x\|,\|y\| \leq B^{1/(n+1-d)} \right\}, \]  
(4.7)  
and we shall prove the following sharp upper and lower bounds.

**Lemma 4.5.** Let \( d \geq 2 \) and \( n \geq d \) with \((d,n) \neq (2,2)\). We have  
\[ B^2 \ll E_{d,n}(B) \ll B^2. \]

**Proof.** The upper bound in Lemma 4.4 yields  
\[ E_{d,n}(B) \gg \sum_{(x,y) \in \Omega_{d,n}(B)} \frac{1}{\|x\|^d \|y\|^d}, \]  
which immediately gives \( E_{d,n}(B) \gg B^2 \). Recall the definition (3.22) of the quantity \( \ell_{2,n}(X,Y; \Delta_2) \). We proceed to break the sizes of \( \|x\|,\|y\|\) and \( \Delta_2(x,y) \) into dyadic intervals. Recalling that we have the upper bound (3.20) and using the lower bound in Lemma 4.4 we get  
\[ E_{d,n}(B) \ll \sum_{X,Y \ll B^{1/(n+1-d)}} \sum_{\Delta_2 \ll XY} \frac{1}{\Delta_2(XY)^{d-1}} \cdot \ell_{2,n}(X,Y; \Delta_2). \]  
Applying Lemma 3.20 we deduce that  
\[ E_{d,n}(B) \ll \sum_{X,Y \ll B^{1/(n+1-d)}} \sum_{\Delta_2 \ll XY} \frac{1}{\Delta_2(XY)^{d-3}} \cdot \Delta_2^{n-2} \]  
\[ \ll \sum_{X,Y \ll B^{1/(n+1-d)}} (XY)^{n+1-d}, \]  
since \( n \geq 3 \). The upper bound \( E_{d,n}(B) \ll B^2 \) follows, which completes the proof. \( \square \)

Recall the respective definitions (2.8), (2.9) and (2.10) of \( \alpha, W \) and \( w \). We let \( \text{rad}(W) \) denote the radical of the integer \( W \), that is  
\[ \text{rad}(W) = \prod_{p \leq w} p. \]  
(4.8)  
Given two linearly independent vectors \( x,y \in \mathbb{Z}^{n+1} \) we put  
\[ \Delta(x,y) = \frac{\|\nu_{d,n}(x)\| \cdot \|\nu_{d,n}(y)\|}{\det(\mathbb{Z}\nu_{d,n}(x) \oplus \mathbb{Z}\nu_{d,n}(y))}, \]  
(4.9)  
and  
\[ \mathcal{E}_{x,y}(B) = \min \left\{ 1, \frac{\Delta(x,y)^2}{\alpha^2} \right\} + 1_{\nu(x,y)|W/\text{rad}(W)}. \]  
(4.10)  
Bearing this in mind, we let  
\[ F_{d,n}(B) = \sum_{(x,y) \in \Omega_{d,n}(B)} \frac{\mathcal{E}_{x,y}(B)}{\det(\Lambda_{d,n}(x) \cap \Lambda_{d,n}(y))}, \]  
(4.11)  
and we shall seek a saving over the trivial upper bound \( F_{d,n}(B) \ll B^2 \) that follows from taking \( \mathcal{E}_{x,y}(B) \leq 2 \) and applying Lemma 4.5.
Lemma 4.6. Let \( d \geq 2 \) and \( n \geq d \) with \((d, n) \neq (2, 2)\). We have

\[
F_{d,n}(B) \ll \frac{B^2}{(\log B)^{1/2}}.
\]

Proof. We let

\[
F_{d,n}^{(1)}(B) = \sum_{(x,y) \in \Omega_{d,n}(B)} \frac{1}{\det(\Lambda_{d,n}(x) \cap \Lambda_{d,n}(y))} \cdot \min\left\{ 1, \frac{\Delta(x,y)^2}{\alpha^2} \right\},
\]

and

\[
F_{d,n}^{(2)}(B) = \sum_{(x,y) \in \Omega_{d,n}(B)} \frac{1}{\det(\Lambda_{d,n}(x) \cap \Lambda_{d,n}(y))} \cdot \mathbf{1}_{\mathcal{G}(x,y)|W/\rad(W)},
\]

so that

\[
F_{d,n}(B) = F_{d,n}^{(1)}(B) + F_{d,n}^{(2)}(B).
\]

We start by proving an upper bound for the sum \( F_{d,n}^{(1)}(B) \). It follows from the equality (4.4) and the lower bound in Lemma 4.4 that for any linearly independent vectors \( x, y \in \mathbb{Z}_{\text{prim}}^{n+1} \), we have

\[
\Delta(x,y) \ll \frac{|x| \cdot |y|}{\delta_2(x,y)}.
\]

Recall the definition (3.22) of the quantity \( \ell_{2,n}(X,Y;\Delta_2) \). Breaking the sizes of \( |x|, |y| \) and \( \delta_2(x,y) \) into dyadic intervals and using again Lemma 4.4 we deduce that

\[
F_{d,n}^{(1)}(B) \ll \sum_{X,Y \leq B^{1/(n+1-d)}} \sum_{\Delta_2 \leq XY} \frac{1}{\Delta_2^{d-1}} \left( \min\left\{ 1, \frac{(XY)^2}{\Delta_2^{n-5/2}} \right\} \ell_{2,n}(X,Y;\Delta_2) \right).
\]

Writing that

\[
\min\left\{ 1, \frac{(XY)^2}{\Delta_2^{n-5/2}} \right\} \leq \frac{(XY)^{1/2}}{\Delta_2^{n/2}},
\]

appealing to Lemma 3.20 and using the fact that \( n \geq 3 \), we obtain

\[
F_{d,n}^{(1)}(B) \ll \frac{1}{\alpha^{1/2}} \sum_{X,Y \leq B^{1/(n+1-d)}} \frac{1}{(XY)^{d-7/2}} \sum_{\Delta_2 \leq XY} \Delta_2^{n/2} \]

\[
\ll \frac{1}{\alpha^{1/2}} \sum_{X,Y \leq B^{1/(n+1-d)}} (XY)^{n+1-d} \]

\[
\ll \frac{B^2}{\alpha^{1/2}}.
\]

We now consider the sum \( F_{d,n}^{(2)}(B) \). For any prime number \( p \) and any \( m \geq 1 \), we let \( v_p(m) \) denote the \( p \)-adic valuation of \( m \). Suppose that \( m \) is a positive integer which does not divide \( W/\rad(W) \). In view of the definition (2.9) of \( W \) this means that either there is a prime \( p > w \) which divides \( m \), or else there is a prime \( p \leq w \) such that

\[
v_p(m) > v_p\left( \frac{W}{\rad(W)} \right) = \left\lceil \frac{\log w}{\log p} \right\rceil \geq \frac{\log w}{\log p}.
\]

In either case we deduce that \( m > w \). As a result, Lemma 3.13 shows that for any linearly independent vectors \( x, y \in \mathbb{Z}_{\text{prim}}^{n+1} \) such that \( \mathcal{G}(x,y) \nmid W/\rad(W) \), we have

\[
\delta_2(x,y) \leq \frac{|x| \cdot |y|}{w}.
\]
Breaking the sizes of \( ||x||, ||y|| \) and \( \vartheta_2(x, y) \) into dyadic intervals and using Lemma 4.4, we thus see that
\[
F_{d,n}^{(2)}(B) \ll \sum_{X,Y \ll B^{1/(n+1-d)}} \sum_{\Delta_2 \ll XY/w} \frac{1}{\Delta_2(XY)^{d-1}} \cdot \ell_{2,n}(X,Y; \Delta_2).
\]

Applying Lemma 3.20 and using the fact that \( n \geq 3 \) we derive
\[
F_{d,n}^{(2)}(B) \ll \sum_{X,Y \ll B^{1/(n+1-d)}} \sum_{\Delta_2 \ll XY/w} \frac{1}{w^{n-2}} \Delta_2^{n-2} \ll \sum_{X,Y \ll B^{1/(n+1-d)}} (XY)^{n+1-d} \ll \sum_{X,Y \ll B^{1/(n+1-d)}} \frac{1}{w^{n-2}}. \tag{4.14}
\]

Recalling the respective definitions \( (2.8) \) and \( (2.10) \) of \( \alpha \) and \( w \), we see that combining the equality \( (4.12) \) and the upper bounds \( (4.13) \) and \( (4.14) \) completes the proof. \( \Box \)

4.4. A first moment bound. Recall the definition \( (2.2) \) of the counting function \( N_V(B) \). A result of the second author [14, Theorem 3] implies in particular that for \( A \geq B^{1/(n+1-d)} \), we have
\[
\frac{1}{\# \mathcal{V}_{d,n}(A)} \sum_{V \in \mathcal{V}_{d,n}(A)} N_V(B) \ll \frac{B}{A}.
\]

Unfortunately, we will require the allowable range of \( A \) to be greater in the critical case \( n = d \). We shall achieve this by using our work from Section 3. In fact, although we elect not to do so here, it would be straightforward to obtain an asymptotic formula that improves upon [14, Theorem 3].

Given an integer \( R \geq 1 \) and a lattice \( \Lambda \) of rank \( R \), recall that the successive minima \( \lambda_1(\Lambda), \ldots, \lambda_R(\Lambda) \) of \( \Lambda \) were introduced in Definition 3.2. We will employ this notation without further notice throughout Sections 4.4 and 4.5.

**Lemma 4.7.** Let \( d \geq 2 \) and \( n \geq d \). Assume that \( A \geq B^{4/5} \). Then we have
\[
\frac{1}{\# \mathcal{V}_{d,n}(A)} \sum_{V \in \mathcal{V}_{d,n}(A)} N_V(B) \ll \frac{B}{A}.
\]

**Proof.** To begin with we clearly have
\[
\sum_{V \in \mathcal{V}_{d,n}(A)} N_V(B) = \sum_{x \in \mathbb{Z}^{n+1}} \# \{ V \in \mathcal{V}_{d,n}(A) : x \in V(\mathbb{Q}) \}.
\]

Recall the definition \( (2.5) \) of the set \( \Xi_{d,n}(B) \). We see that
\[
\sum_{V \in \mathcal{V}_{d,n}(A)} N_V(B) = \frac{1}{4} \sum_{x \in \Xi_{d,n}(B)} \# \left( \Lambda_{\vartheta_2(x)} \cap \mathbb{Z}_{\text{prim}}^{n+1} \cap B_{N_{d,n}(A)} \right).
\]

Recall that the quantity \( \vartheta_2(x) \) was introduced in Definition 3.12. For \( x \in \mathbb{Z}_{\text{prim}}^{n+1} \), we define
\[
\mu(x) = \frac{||x||}{\vartheta_2(x)}, \tag{4.15}
\]
and we note that Lemma 3.14 states that
\[
\lambda_{N_{d,n}-1} \left( \Lambda_{\vartheta_2(x)} \right) \leq \mu(x). \tag{4.16}
\]
It is convenient to set
\[ M^{(1)}_{d,n}(A, B) = \frac{1}{4} \sum_{x \in \Xi_{d,n}(B)} \# \left( \Lambda_{\nu_{d,n}(x)} \cap \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap B_{N_{d,n}}(A) \right), \]
and
\[ M^{(2)}_{d,n}(A, B) = \sum_{V \in \mathbb{V}_{d,n}(A)} N_{V}(B) - M^{(1)}_{d,n}(A, B). \tag{4.17} \]

We first deal with the sum \( M^{(1)}_{d,n}(A, B) \). We have the inequalities (4.16) and \( A \geq \mu(x) \) so we are in position to apply Lemma 3.4 with \( I = 1 \) and \( \gamma = 1/2 \), say. This allows us to derive the upper bound
\[ \# \left( \Lambda_{\nu_{d,n}(x)} \cap \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap B_{N_{d,n}}(A) \right) \ll \frac{A^{N_{d,n} - 1}}{\det(\Lambda_{\nu_{d,n}(x)})}. \]

Moreover Lemma 3.8 gives \( \det(\Lambda_{\nu_{d,n}(x)}) = ||\nu_{d,n}(x)|| \gg ||x||^{d} \) so we immediately deduce that
\[ M^{(1)}_{d,n}(A, B) \ll A^{N_{d,n} - 1} B. \tag{4.18} \]

We now handle the sum \( M^{(2)}_{d,n}(A, B) \). We have the inequalities (4.16) and \( A < \mu(x) \) so we can apply the first part of Lemma 3.6 with \( M = 1/2 \) and \( R_{0} = n - 1 \). This yields
\[ \# \left( \Lambda_{\nu_{d,n}(x)} \cap \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap B_{N_{d,n}}(A) \right) \ll \frac{A^{N_{d,n} - n}}{||x||^{d}} \left( \frac{||x||}{\delta_{2}(x)} \right)^{n-1} + A^{N_{d,n} - n - 1}. \]

Recall the definition (3.21) of the quantity \( \ell_{2,n}(X; \Delta_{0}) \). Breaking the sizes of \( ||x|| \) and \( \delta_{2}(x) \) into dyadic intervals we see that we have
\[ M^{(2)}_{d,n}(A, B) \ll A^{N_{d,n} - 1} \sum_{X \ll B^{1/(n+1-d)}} \sum_{\Delta_{0} \ll X/A} \left( \frac{X^{n-d-1}}{A^{n-1}\Delta_{0}^{n-1}} + \frac{1}{A^{n}} \right) \ell_{2,n}(X; \Delta_{0}). \]

It follows from Lemma 3.19 that
\[ M^{(2)}_{d,n}(A, B) \ll A^{N_{d,n} - 1} \left( \log B \right) \sum_{X \ll B^{1/(n+1-d)}} \sum_{\Delta_{0} \ll X/A} \left( \frac{X^{n+1-d}\Delta_{0}}{A^{n-1}} + \frac{X^{2}\Delta_{0}^{n}}{A^{n}} \right) \]
\[ \ll A^{N_{d,n} - 1} \left( \log B \right) \sum_{X \ll B^{1/(n+1-d)}} \left( \frac{X^{n+2-d}}{A^{n}} + \frac{X^{n+2}}{A^{2n}} \right) \]
\[ \ll A^{N_{d,n} - 1} \left( \frac{B^{1+1/(n+1-d)}}{A^{n}} + \frac{B^{(n+2)/(n+1-d)}}{A^{2n}} \right) \log B. \]

As a result, we see that the assumption \( A \geq B^{4/5} \) implies in particular that for any \( d \geq 2 \) and \( n \geq d \) we have
\[ M^{(2)}_{d,n}(A, B) \ll A^{N_{d,n} - 1} B. \tag{4.19} \]

We complete the proof by putting together the equality (4.17) and the upper bounds (4.18) and (4.19), and by using the lower bound (2.19). \( \square \)
4.5. Second moment estimates. We now turn to the proof of three second moment estimates. We start by setting some notation. Recall the expression (2.4) for the global counting function \( N_V(B) \) and the definition (2.11) of the localised counting function \( N_{V}^{\text{loc}}(B) \). We introduce the second moment of \( N_V(B) \) with its diagonal contribution removed, that is
\[
D_{d,n}^{d}(A,B) = \sum_{V \in V_{d,n}(A)} N_V(B)^2 - \sum_{V \in V_{d,n}(A)} N_V(B).
\]
Similarly, we also define the mixed moment and the second moment of \( N_{V}^{\text{loc}}(B) \) with their respective diagonal contributions removed. We thus set
\[
D_{d,n}^{\text{mix}}(A,B) = \sum_{V \in V_{d,n}(A)} N_V(B)N_{V}^{\text{loc}}(B) - \sum_{V \in V_{d,n}(A)} \Delta_{V}^{\text{mix}}(B),
\]
where
\[
\Delta_{V}^{\text{mix}}(B) = \frac{1}{2} \left\| \frac{\alpha W}{\|a_V\|^2} \sum_{x \in \Xi_{d,n}(B)} \frac{1}{\|\nu_{d,n}(x)\|^2} \right\|
\]
and
\[
D_{d,n}^{\text{loc}}(A,B) = \sum_{V \in V_{d,n}(A)} N_{V}^{\text{loc}}(B)^2 - \sum_{V \in V_{d,n}(A)} \Delta_{V}^{\text{loc}}(B),
\]
where
\[
\Delta_{V}^{\text{loc}}(B) = \frac{1}{2} \left\| \frac{\alpha^2 W^2}{\|a_V\|^2} \sum_{x \in \Xi_{d,n}(B)} \frac{1}{\|\nu_{d,n}(x)\|^2} \right\|^2.
\]
We shall use our work in Section 3 to establish asymptotic formulae for the quantities \( D_{d,n}(A,B), D_{d,n}^{\text{mix}}(A,B) \) and \( D_{d,n}^{\text{loc}}(A,B) \).

For any lattice \( \Lambda \subset \mathbb{Z}^{N_{d,n}} \), any bounded region \( R \subset \mathbb{R}^{N_{d,n}} \) and any integer \( k \geq 0 \), it is convenient to set
\[
S_k^{s}(\Lambda; R) = \sum_{\substack{a \in \Lambda \cap \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap R}} \frac{1}{\|a\|^k},
\]
and
\[
S_k(\Lambda; R) = \sum_{\substack{a \in (\Lambda \cap \mathbb{Z}) \cap R}} \frac{1}{\|a\|^k}.
\]
In addition, we recall that \( V_N \) denotes the volume of the unit ball \( B_N(1) \) in \( \mathbb{R}^N \) for any \( N \geq 1 \). Moreover, for any \( d,n \geq 2 \) we set
\[
\iota_{d,n} = \frac{V_{N_{d,n}-2}}{8 \zeta(N_{d,n} - 2)}.
\]
Finally, recall the definition (4.6) of the quantity \( E_{d,n}(B) \).

We start by handling the quantity \( D_{d,n}(A,B) \) in the case \((d,n) \neq (4,4)\).

\textbf{Lemma 4.8.} Let \( d \geq 2 \) and \( n \geq d \) with \((d,n) \notin \{(2,2), (3,3), (4,4)\}\). Assume that \( A \geq B/(\log B) \). Then we have
\[
D_{d,n}(A,B) = \iota_{d,n} A^{N_{d,n}-2} E_{d,n}(B) \left( 1 + O \left( \frac{(\log A)^{9/2}}{A^{1/2}} \right) \right).
\]
We then set and we note that Lemma 3.15 states that

\[
V_n \ni x \quad \text{and} \quad \nu \geq 2
\]

Therefore, using Lemma 4.5 we deduce that

\[
D_{d,n}(A, B) = \frac{1}{8} \sum_{(x, y) \in \Omega_d, n(B)} S^*_0(\Gamma_{x, y}; B_{N_{d,n}}(A)).
\]

Recall the definition (4.7) of the set \(\Omega_{d,n}(B)\). We start by noting that

\[
D_{d,n}(A, B) = \frac{1}{8} \sum_{(x, y) \in \Omega_d, n(B)} S^*_0(\Gamma_{x, y}; B_{N_{d,n}}(A)).
\]

Recall that the quantities \(d_2(x, y)\) and \(d_3(x, y)\) were introduced in Definition 3.12. Given two linearly independent vectors \(x, y \in \mathbb{Z}^{n+1}_{\text{prim}}\), we define

\[
\mu(x, y) = 3n^2 \max \left\{ \frac{d_2(x, y)}{d_3(x, y)}, \frac{||x|| \cdot ||y||}{d_2(x, y)^2} \right\},
\]

and we note that Lemma 3.15 states that

\[
\lambda_{N_{d,n}}(\Gamma_{x, y}) \leq \mu(x, y).
\]

We then set

\[
\Sigma_{d,n}^{(1)} (A, B) = \frac{1}{8} \sum_{(x, y) \in \Omega_{d,n}(B)} S^*_0(\Gamma_{x, y}; B_{N_{d,n}}(A)),
\]

and

\[
\Sigma_{d,n}^{(2)} (A, B) = D_{d,n}(A, B) - \Sigma_{d,n}^{(1)} (A, B).
\]

We first handle the sum \(\Sigma_{d,n}^{(1)} (A, B)\). The lattice \(\Gamma_{x, y}\) is primitive and we have the inequalities (4.28) and \(A \geq \mu(x, y)\) so we are in position to apply [14, Lemma 3]. We deduce that

\[
S^*_0(\Gamma_{x, y}; B_{N_{d,n}}(A)) = 8t_{d,n} A^{N_{d,n} - 2} \det(\Gamma_{x, y}) \left( 1 + O \left( \frac{\mu(x, y)}{A} \right) \right) + O(A \log B).
\]

Using the upper bound in Lemma 4.4, the trivial lower bound \(\mu(x, y) \geq 1\), and the assumption \(B \ll A \log A\) it is easy to check that for any \((x, y) \in \Omega_{d,n}(B)\) we have

\[
\frac{A^{N_{d,n} - 3} \mu(x, y)}{\det(\Gamma_{x, y})} \gg A \log B.
\]

We thus obtain

\[
\Sigma_{d,n}^{(1)} (A, B) = t_{d,n} A^{N_{d,n} - 2} \sum_{(x, y) \in \Omega_{d,n}(B)} \frac{1}{\det(\Gamma_{x, y})} \left( 1 + O \left( \frac{\mu(x, y)}{A} \right) \right).
\]

Note that we have the obvious inequality

\[
\sum_{(x, y) \in \Omega_{d,n}(B)} \frac{1}{\det(\Gamma_{x, y})} \leq \frac{1}{A} \sum_{(x, y) \in \Omega_{d,n}(B)} \frac{\mu(x, y)}{\det(\Gamma_{x, y})}.
\]

Therefore, using Lemma 4.5 we deduce that

\[
\Sigma_{d,n}^{(1)} (A, B) = t_{d,n} A^{N_{d,n} - 2} E_{d,n}(B) \left( 1 + O \left( \varepsilon_{d,n}^{(1)}(A, B) \right) \right),
\]

(4.30)
where
\[
\mathcal{E}_{d,n}^{(1)}(A, B) = \frac{1}{AB^2} \sum_{(x,y) \in \Omega_{d,n}(B)} \frac{\mu(x, y)}{\det(\Gamma_{x,y})}.
\]

We shall handle the quantity \( \mathcal{E}_{d,n}^{(1)}(A, B) \) by breaking the sizes of \( ||x||, ||y||, \partial_2(x, y) \) and \( \partial_3(x, y) \) into dyadic intervals. Recall that we have the upper bound (3.20). Moreover, we note that the inequality \( \partial_2(x, y) \geq \partial_3(x, y) \) implies that
\[
\mu(x, y) \leq 3n^2 \frac{||x|| \cdot ||y||}{\partial_3(x, y)}.
\]

Using the lower bound in Lemma 4.4, we thus deduce
\[
\frac{\mu(x, y)}{\det(\Gamma_{x,y})} \ll \frac{1}{\partial_2(x, y) \cdot \partial_3(x, y) \cdot ||x||^{d-2} ||y||^{d-2}}.
\]

Recalling the definition (3.22) of the quantity \( \ell_{r,n}(X, Y; \Delta_r) \), we see that we have the upper bound
\[
\mathcal{E}_{d,n}^{(1)}(A, B) \ll \frac{1}{AB^2} \sum_{X, Y \ll B/(n+1-d)} \Delta_2 \Delta_3 \sum_{X, Y \ll X Y} \frac{1}{\Delta_2 \Delta_3 (XY)^{d-2}} \cdot \min_{r \in \{2, 3\}} \ell_{r,n}(X, Y; \Delta_r).
\]

It follows from Lemma 3.20 that
\[
\min_{r \in \{2, 3\}} \ell_{r,n}(X, Y; \Delta_r) \ll (\log B)^2 (XY)^{2} \min \{ \Delta_2^{-1}, \Delta_3^{-1} XY \}.
\]

We now make use of the inequality
\[
\min \{ \Delta_2^{-1}, \Delta_3^{-1} XY \} \leq (\Delta_2^{-1})^{1-1/(n-1)} (\Delta_3^{-1} XY)^{1/(n-1)}.
\]

This yields
\[
\mathcal{E}_{d,n}^{(1)}(A, B) \ll \frac{(\log B)^2}{AB^2} \sum_{X, Y \ll B/(n+1-d)} \sum_{\Delta_2 \Delta_3 \ll XY} \frac{\Delta_2^{-3}}{(XY)^{d-2-1/(n-1)}}
\]
\[
\ll \frac{(\log B)^4}{AB^2} \sum_{X, Y \ll B/(n+1-d)} (XY)^{n+1-d+1/(n-1)}
\]
\[
\ll \frac{(\log B)^4 B^{2/(n+1-d)}}{A}.
\] (4.31)

Since \( (d, n) \notin \{ (2, 2), (3, 3), (4, 4) \} \) we have \( (n-1)(n+1-d) \geq 4 \), so the assumption \( B \ll A \log A \) gives
\[
\mathcal{E}_{d,n}^{(1)}(A, B) \ll \frac{(\log A)^{9/2}}{A^{1/2}}.
\]

Recalling the estimate (4.30), we see that we have obtained
\[
\Sigma_{d,n}^{(1)}(A, B) = \ell_{d,n} A^{N_{d,n} - 2} E_{d,n}(B) \left( 1 + O \left( \frac{(\log A)^{9/2}}{A^{1/2}} \right) \right).
\] (4.32)

We now handle the sum \( \Sigma_{d,n}^{(2)}(A, B) \). We start by noting that the trivial upper bound \( \mu(x, y) \leq 3n^2 ||x|| \cdot ||y|| \) and the condition \( A < \mu(x, y) \) together give \( A < 3n^2 B^{2/(n+1-d)} \). Therefore, the assumption \( B \ll A \log A \) implies that we have either \( n = d+1 \) or \( n = d \). In addition, we have the inequalities (4.28) and \( A < \mu(x, y) \) so we can apply the first part of Lemma 3.6 with \( M = 1/2 \). This yields
\[
\mathcal{S}^0_0(\Gamma_{x,y}; B_{N_{d,n}}(A)) \ll \min_{R_0 \in \{0, \ldots, N_{d,n} - 3\}} \left( \frac{A^{N_{d,n} - 2 - R_0} \mu(x, y)^{R_0}}{\det(\Gamma_{x,y})} + A^{N_{d,n} - 3 - R_0} \right).
\]
As a result, appealing to the lower bound in Lemma 4.4 and to Lemma 4.5 we see that
\[
\Sigma^{(2)}_{d,n}(A, B) \ll A^{N_{d,n} - 2} E_{d,n}(B) E^{(2)}_{d,n}(A, B),
\]
(4.33)
where
\[
E^{(2)}_{d,n}(A, B) = \frac{1}{B^2} \sum_{\substack{(x, y) \in \Omega_{d,n}(B) \\ \mu(x, y) > A}} \min_{R_0 \in \{0, \ldots, N_{d,n} - 3\}} \left( \frac{\mu(x, y) R_0}{A R_0 \mathfrak{d}_2(x, y) \cdot ||x|| / ||y||} \right). + \frac{1}{A R_0 + 1}.
\]
We let \( F_{d,n}(A, B) \) be the contribution to \( E^{(2)}_{d,n}(A, B) \) coming from the \((x, y) \in \Omega_{d,n}(B)\)
satisfying
\[
\frac{\mathfrak{d}_2(x, y)}{\mathfrak{d}_3(x, y)} \leq \frac{||x|| \cdot ||y||}{\mathfrak{d}_2(x, y)^2},
\]
(4.34)
and we also let
\[
\mathcal{I}_{d,n}(A, B) = E^{(2)}_{d,n}(A, B) - F_{d,n}(A, B).
\]
(4.35)
We first handle the quantity \( F_{d,n}(A, B) \). Recalling the definition (4.27) of \( \mu(x, y) \), we see that
\[
F_{d,n}(A, B) \ll \frac{1}{B^2} \sum_{\substack{(x, y) \in \Omega_{d,n}(B) \\ \mathfrak{d}_2(x, y) < 3n^2 ||x|| ||y|| / A}} \min_{R_0 \in \{0, \ldots, N_{d,n} - 3\}} \left( \frac{B^{2(n-d) / (n+1-d)}}{A^{n+1} \Delta_2} + \frac{1}{A^n} \right).
\]
Choosing \( R_0 = n - 1 \) and breaking the size of \( \mathfrak{d}_2(x, y) \) into dyadic intervals we get
\[
F_{d,n}(A, B) \ll \frac{1}{A^n} \sum_{\Delta_2 \ll B^{1 / (n+1-d) / A^{1/2}}} \left( \frac{B^{2(n-d) / (n+1-d)}}{A^{n+1} \Delta_2} + \frac{1}{A^n} \right).
\]
\[
\ll \frac{B^{2(n+1-d) / A^{n+1}}} {A^{(3n-1)/2}} + \frac{B^{2 / (n+1-d) / A^{n+1}}} {A^{(3n-1)/2}}.
\]
Recall that we have either \( n = d + 1 \) or \( n = d \). For any \( n \geq 3 \), we have proved that
\[
F_{n-1,n}(A, B) \ll \frac{B}{A^{n-1}} + \frac{B^{(n-1) / 2}}{A^{(3n-1)/2}},
\]
and
\[
F_{n,n}(A, B) \ll \frac{B^2}{A^{n-1}} + \frac{B^{n+1}}{A^{(3n-1)/2}}.
\]
(4.36)
Therefore, using the assumptions \((d, n) \notin \{(2, 2), (3, 3), (4, 4)\}\) and \( B \ll A \log A \) we conclude that
\[
F_{d,n}(A, B) \ll \frac{(\log A)^6}{A}.
\]
(4.37)
We finally deal with the quantity \( \mathcal{I}_{d,n}(A, B) \). We have
\[
\mathcal{I}_{d,n}(A, B) \leq \frac{1}{B^2} \sum_{\substack{(x, y) \in \Omega_{d,n}(B) \\ \mathfrak{d}_3(x, y) < 3n^2 ||x|| ||y|| / A}} \min_{R_0 \in \{0, \ldots, N_{d,n} - 3\}} \left( \frac{||x|| R_0 \mathfrak{d}_3(x, y) + 1}{A R_0 + 1} \right).
\]
Note that we have restricted the minimum to $R_0 \geq d$ and we have then applied the upper bound (3.20) with $r = 2$. Breaking the size of $\partial_3(x, y)$ into dyadic intervals we see that

$$I_{d,n}(A, B) \ll \frac{1}{B^2} \sum_{\Delta_3 \ll B^{2/(n+1-d)}/A} \left( \min_{R_0 \in \{d, \ldots, N_{d,n} - 3\}} \left( \frac{B(2R_0 - 2d)/(n+1-d)}{A R_0 \Delta_3 R_0^d} + \frac{1}{A R_0 + 1} \right) \right) \times \ell_{3,n} \left( B^{1/(n+1-d)}, B^{1/(n+1-d)}; \Delta_3 \right).$$

Using Lemma 3.20 we get

$$I_{d,n}(A, B) \ll (\log B)^2 \sum_{\Delta_3 \ll B^{2/(n+1-d)/A}} \min_{R_0 \in \{d, \ldots, N_{d,n} - 3\}} \left( \frac{B(2R_0 - 2n+4)/(n+1-d)}{A R_0 \Delta_3 R_0^d} + \frac{B \Delta_3^{n-1}}{A n+1} \right),$$

where the minimum is over $R_0 \in \{d, \ldots, N_{d,n} - 3\}$. We recall that we have either $n = d + 1$ or $n = d$ and we first handle the case $n = d + 1$. Choosing $R_0 = n - 1$ we deduce that

$$I_{n-1,n}(A, B) \ll (\log B)^2 \sum_{\Delta_3 \ll B/A} \left( \frac{B}{A n+1} + \frac{B \Delta_3^{n-1}}{A n} \right).$$

Using the assumption $B \ll A \log A$ we conclude that for any $n \geq 3$ we have

$$I_{n-1,n}(A, B) \ll \frac{(\log A)^4}{A^{n-2}}. \tag{4.38}$$

We now treat the case $n = d$. The assumption $B \ll A \log A$ yields

$$I_{n,n}(A, B) \ll (\log A)^2 N_{n,n} - 2n \sum_{\Delta_3 \ll A(\log A)^2} \min_{R_0 \in \{d, \ldots, N_{n,n} - 3\}} \left( \frac{A R_0 - 2n+4}{\Delta_3 R_0^d} + \frac{\Delta_3^{n-1}}{A R_0 - d} \right).$$

Changing $R_0$ in $R_0 + 1$ we see that the right-hand side can grow at most by $(\log A)^2$ when $n$ increases by 1. For any $n \geq 5$ we thus have

$$I_{n,n}(A, B) \ll (\log A)^2 N_{n,n} - 10 \sum_{\Delta_3 \ll A(\log A)^2} \min_{R_0 \in \{5, \ldots, N_{5,5} - 3\}} \left( \frac{A R_0 - 6}{\Delta_3 R_0^d} + \frac{\Delta_3^{4}}{A R_0 - 3} \right).$$

Taking successively $R_0 = 5, \ldots, R_0 = 8$ as $\Delta_3$ increases we obtain

$$I_{n,n}(A, B) \ll (\log A)^2 N_{n,n} - 10 \sum_{A^{1/3} \leq \Delta_3 \leq A^{1/3}} \left( \frac{1}{A \Delta_3^3 + A^2} + \sum_{A^{1/3} \leq \Delta_3 \leq A^{1/7}} \left( \frac{1}{\Delta_3^3} + \frac{\Delta_3^4}{A^3} \right) \right) + \sum_{A^{4/7} \leq \Delta_3 \leq A^{1/4}} \left( \frac{A^2}{\Delta_3^3} + \frac{\Delta_3^4}{A^3} \right) + \sum_{A^{3/4} \leq \Delta_3 \ll A(\log A)^2} \left( \frac{A^2}{\Delta_3^3} + \frac{\Delta_3^4}{A^3} \right).$$

It thus follows that for any $n \geq 5$ we have

$$I_{n,n}(A, B) \ll \frac{(\log A)^2 N_{n,n} - 10}{A^{2/3}}. \tag{4.39}$$

Combining the equality (4.35) with the upper bounds (4.37), (4.38) and (4.39) we deduce that

$$\mathcal{E}_{d,n}^{(2)}(A, B) \ll \frac{(\log A)^2 N_{n,n} - 10}{A^{2/3}}.$$

Recalling the upper bound (4.33) we conclude that

$$\Sigma_{d,n}^{(2)}(A, B) \ll A^{N_{d,n} - 2} E_{d,n}(B) \cdot \frac{(\log A)^2 N_{n,n} - 10}{A^{2/3}}. \tag{4.40}$$
Putting together the equality (4.29), the estimate (4.32) and the upper bound (4.40) completes the proof.

We now use our work in Section 3.5 to handle separately the quantity \( D_{4,4}(A, B) \). Recall the definition (4.26) of \( \iota_{4,4} \).

**Lemma 4.9.** Assume that \( A \geq B/(\log B) \). Then we have

\[
D_{4,4}(A, B) = \iota_{4,4} A^{N_{4,4} - 2} E_{4,4}(B) \left( 1 + O \left( \frac{1}{A^{1/21}} \right) \right).
\]

**Proof.** Combining the equality (4.29), the estimate (4.30) and the upper bound (4.31), and using the assumption \( B \ll A \log A \) we obtain

\[
D_{4,4}(A, B) = \iota_{4,4} A^{N_{4,4} - 2} E_{4,4}(B) \left( 1 + O \left( \frac{1}{A^{1/3}} \right) \right) + \sum_{2}^{(2)}(A, B).
\]

(4.41)

Let \( \delta \in (0, 1/15) \) to be selected in due course. Let \( \mathcal{J}_{>\delta}(A, B) \) denote the contribution to \( \Sigma_{4,4}^{(2)}(A, B) \) from the \((x, y) \in \Omega_{4,4}(B) \) satisfying \( \delta_3(x, y) > A^\delta \) and set

\[
\mathcal{J}_{<\delta}(A, B) = \mathcal{J}_{4,4}^{(2)}(A, B) - \mathcal{J}_{>\delta}(A, B).
\]

We first handle the quantity \( \mathcal{J}_{>\delta}(A, B) \). We are going to use the fact that the first successive minimum of the lattice \( \Gamma_{x,y} \) is usually quite large as \((x, y) \) runs over \( \Omega_{4,4}(B) \). We have the inequalities (4.28) and \( A < \mu(x, y) \) so we can apply the first part of Lemma 3.6 with \( M = \lambda_1(\Gamma_{x,y}) - 1/2 \). We deduce

\[
S_0^1(\Omega_{x,y}; B_{N_{4,4}}(A)) \ll \min_{R_0 \in \{1, ..., N_{4,4} - 3\}} \left( \frac{A^{N_{4,4} - 2 - R_0} \mu(x, y) R_0}{\det(\Gamma_{x,y})} + \left( \frac{A}{\lambda_1(\Gamma_{x,y})} \right)^{N_{4,4} - 3 - R_0} \right).
\]

Noting that \( N_{4,4} = 70 \) and appealing to the lower bound in Lemma 4.4 and to Lemma 4.5 we see that

\[
\mathcal{J}_{>\delta}(A, B) \ll A^{N_{4,4} - 2} E_{4,4}(B) \mathcal{E}_{>\delta}(A, B),
\]

where

\[
\mathcal{E}_{>\delta}(A, B) = \frac{1}{B^2} \sum_{(x, y) \in \Omega_{4,4}(B) \atop \mu(x, y) > A \atop \delta_3(x, y) > A^\delta} \min_{R_0} \left( \frac{\mu(x, y) R_0}{A R_0 \delta_2(x, y) \cdot ||x||^3 ||y||^3} + \frac{1}{A^{R_0 + 1} \lambda_1(\Gamma_{x,y})^{67 - R_0}} \right),
\]

where the minimum is over \( R_0 \in \{0, ..., 67\} \). Since \( \lambda_1(\Gamma_{x,y}) \geq 1 \) the contribution to \( \mathcal{E}_{>\delta}(A, B) \) coming from the \((x, y) \in \Omega_{4,4}(B) \) satisfying the inequality (4.34) is at most \( \mathcal{F}_{4,4}(A, B) \). Using the upper bound (4.36) we deduce that

\[
\mathcal{J}_{>\delta}(A, B) \ll A^{N_{4,4} - 2} E_{4,4}(B) \left( \mathcal{F}_{>\delta}(A, B) + \frac{(\log A)^5}{A^{1/2}} \right),
\]

(4.43)

where

\[
\mathcal{F}_{>\delta}(A, B) = \frac{1}{B^2} \sum_{(x, y) \in \Omega_{4,4}(B) \atop A^\delta < \delta_3(x, y) < ||x|| ||y|| / A} \min_{R_0} \left( \frac{||x||^{R_0 - 4} ||y||^{R_0 - 4}}{A R_0 \delta_3(x, y)^{R_0}} + \frac{1}{A^{R_0 + 1} \lambda_1(\Gamma_{x,y})^{67 - R_0}} \right),
\]

where the minimum is over \( R_0 \in \{1, ..., 67\} \). Note that we have restricted the minimum to \( R_0 \geq 1 \) and we have then applied the upper bound (3.20) with \( r = 2 \). Since the lattice \( \Gamma_{x,y} \) has rank 68 it follows from Minkowski’s estimate (3.5) and the upper bound in Lemma 4.4 that

\[
\lambda_1(\Gamma_{x,y}) \ll ||x||^{1/17} ||y||^{1/17}.
\]
Recall the definition (3.23) of the quantity $\ell^{(1)}(B; \Delta_3, M)$. Breaking the sizes of $\Delta_3(x, y)$ and $\lambda_1(\Gamma_{x,y})$ into dyadic intervals we see that

$$\mathcal{F}_\delta(A, B) \ll \frac{1}{B^2} \sum_{A^4 \ll A^2 \ll A(\log A)^2} \left( \min_{R_0 \in \{4, \ldots, 67\}} \left( \min_{R_0 \in \{4, \ldots, 67\}} \left( \frac{B^2 R_0^{-8} + \frac{1}{A R_0 + 1 M 67^{-R_0}}}{\Delta_3} \right) \right) \right) \ell^{(1)}(B; \Delta_3, M),$$

where the minimum is over $R_0 \in \{4, \ldots, 67\}$. Using Lemma 3.22 and the assumption $B/(\log B) \leq A$ we get

$$\ell^{(1)}(B; \Delta_3, M) \ll B^{9/2} \Delta_3^2 A^{3/2} (\log A)^5 \min \{ \Delta_3, M^{53} \}.$$

It follows that

$$\mathcal{F}_\delta(A, B) \ll (\log A)^{134} \mathcal{G}_\delta(A), \quad (4.44)$$

where

$$\mathcal{G}_\delta(A) = \sum_{A^4 \ll A^2 \ll A(\log A)^2} \left( \min_{R_0 \in \{4, \ldots, 67\}} \left( \frac{A R_0^{-4} + \frac{\Delta_3^2}{A R_0^{-2} - M 67^{-R_0}}}{\Delta_3} \right) \right) \min \{ \Delta_3, M^{53} \}.$$

Taking successively $R_0 = 4, \ldots, R_0 = 6$ as $\Delta_3$ increases we obtain

$$\mathcal{G}_\delta(A) \ll \sum_{A^4 \ll A^2 \ll A(\log A)^2} \left( \frac{1}{\Delta_3} + \frac{\Delta_3^2 \min \{ \Delta_3, M^{53} \}}{A M^{63}} \right)
+ \sum_{A^{2/5} M^{63/5} < \Delta_3 \ll A^{2/3} M^{31/3}} \left( \frac{A \min \{ \Delta_3, M^{53} \}}{\Delta_3} + \frac{\Delta_3^2 \min \{ \Delta_3, M^{53} \}}{A^2 M^{62}} \right)
+ \sum_{A^{2/3} M^{31/3} < \Delta_3 < A(\log A)^2} \left( \frac{A^2 \min \{ \Delta_3, M^{53} \}}{\Delta_3} + \frac{\Delta_3^2}{A^3 M^{58}} \right).$$

The summation over $\Delta_3$ leads to

$$\mathcal{G}_\delta(A) \ll \sum_{M \ll A^{1/8}} \left( \frac{1}{A^6} + \min \left\{ \frac{A^{1/5}}{M^{126/5}}, \frac{M^{76/5}}{A^{1/5}} \right\} + \min \left\{ \frac{1}{M^{31}}, \frac{M^{35/3}}{A^{2/3}} \right\} + (\log A)^4 \right)
\ll \frac{\log A}{A^6} + \sum_{M \ll A^{1/101}} \frac{M^{76/5}}{A^{1/5}} + \sum_{M > A^{1/101}} \frac{A^{1/5}}{M^{126/5}} + \sum_{M \ll A^{1/64}} \frac{A^{35/3}}{A^{2/3}} + \sum_{M > A^{1/64}} \frac{1}{M^{31}}$$
\ll \frac{\log A}{A^6} + \frac{A^{5/101}}{A^5}.$$

Recalling the upper bounds (4.43) and (4.44), we see that we have proved

$$\mathcal{F}_\delta(A, B) \ll A^{N_{4,4} - 2} E_{4,4}(B) \left( \frac{(\log A)^{135}}{A^6} + \frac{(\log A)^{134}}{A^{5/101}} \right). \quad (4.45)$$

We now handle the quantity $\mathcal{S}_\delta(\Gamma_3, |B|)$. We are going to take advantage of the fact that it is rare for the lattice $\Gamma_{x,y}$ to have 56 small successive minima as $(x, y)$ runs over $\Omega_{4,4}(B)$. We have the inequalities (4.28) and $A < \mu(x, y)$ so we can make use of the second part of Lemma 3.6 with $M = 1/2$, $j_0 = 55$ and $J = \lambda_{56}(\Gamma_{x,y}) - 1/2$. We obtain

$$\mathcal{S}_\delta(\Gamma_{x,y}; |B|) \ll \min_{R_0 \in \{0, \ldots, 12\}} \left( \frac{A^{N_{4,4} - 2} R_0 \mu(x, y)}{\det(\Gamma_{x,y})} + \frac{A^{67 - R_0}}{\lambda_{56}(\Gamma_{x,y})^{12 - R_0}} \right) + A^{55}.$$
The lower bound in Lemma 4.4 and Lemma 4.5 thus give
\[
\mathcal{F}_{\leq \delta}(A, B) \ll A^{N_{4,4} - 2} E_{4,4}(B) \left( \mathcal{E}_{\leq \delta}(A, B) + \frac{B^8}{A^{1/2}} \right),
\] (4.46)
where
\[
\mathcal{E}_{\leq \delta}(A, B) = \frac{1}{B^2} \sum_{(x,y) \in \Omega_{4,4}(B)} \min_{R_0 \geq 1} \left( \frac{\mu(x,y)}{A^{R_0} \mathfrak{d}_3(x,y) \cdot ||x||^4 ||y||^4} + \frac{1}{A^{R_0 + 1} \lambda_{56}(\Gamma_{x,y})^{12 - R_0}} \right),
\]
and where the minimum is over $R_0 \in \{0, \ldots, 12\}$. We use again the observation that the trivial lower bound $\lambda_{56}(\Gamma_{x,y}) \geq 1$ implies that the contribution to $\mathcal{E}_{\leq \delta}(A, B)$ coming from the $(x, y) \in \Omega_{4,4}(B)$ satisfying the inequality (4.34) is at most $\mathcal{F}_{4,4}(A, B)$. Hence the upper bounds (4.36) and (4.46) and the assumption $B \ll A \log A$ give
\[
\mathcal{F}_{\leq \delta}(A, B) \ll A^{N_{4,4} - 2} E_{4,4}(B) \left( \mathcal{F}_{\leq \delta}(A, B) + \frac{(\log A)^5}{A^{1/2}} \right),
\] (4.47)
where
\[
\mathcal{F}_{\leq \delta}(A, B) = \frac{1}{B^2} \sum_{(x,y) \in \Omega_{4,4}(B)} \min_{R_0 \in \{1, \ldots, 12\}} \left( \frac{||x||^{R_0 - 4} ||y||^{R_0 - 4}}{A^{R_0} \mathfrak{d}_3(x,y) \cdot R_0} + \frac{1}{A^{R_0 + 1} \lambda_{56}(\Gamma_{x,y})^{12 - R_0}} \right).
\]
Note that we have used the upper bound (3.20) with $r = 2$ after restricting the minimum to $R_0 \geq 1$. Since the lattice $\Gamma_{x,y}$ has rank 68 it follows from Minkowski’s estimate (3.5) and the upper bound in Lemma 4.4 that
\[
\lambda_{56}(\Gamma_{x,y}) \ll ||x||^{4/13} ||y||^{4/13}.
\]
Recall the respective definitions (3.22) and (3.23) of the quantities $\ell_{3,4}(X,Y; \Delta_3)$ and $\ell_{56}(B; \Delta_3, J)$. Breaking the sizes of $||x||$, $||y||$, $\mathfrak{d}_3(x,y)$ and $\lambda_{56}(\Gamma_{x,y})$ into dyadic intervals we see that
\[
\mathcal{F}_{\leq \delta}(A, B) \ll \frac{1}{B^2} \sum_{X,Y \leq B} \sum_{\Delta_3 \leq A^4} \left( \min_{R_0 \in \{1, \ldots, 12\}} \left( \frac{(XY)^{R_0 - 4}}{A^{R_0} \Delta_3^{R_0}} + \frac{1}{A^{R_0 + 1} J^{12 - R_0}} \right) \right)
\]
\[
\times \min \left\{ \ell_{3,4}(X,Y; \Delta_3), \ell_{56}(B; \Delta_3, J) \right\}.
\]
Applying Lemmas 3.20 and 3.24 and using the assumption $B \ll A \log A$ we deduce that
\[
\mathcal{F}_{\leq \delta}(A, B) \ll (\log A)^6 \sum_{X,Y \leq B} \sum_{\Delta_3 \leq A^4} \left( \min_{R_0 \in \{1, \ldots, 12\}} \left( \frac{(XY)^{R_0 - 1}}{A^{R_0} B^2 \Delta_3^{R_0 - 3}} + \frac{\Delta_3^{3} \min \{ A^2, \Delta_3^{21} J^{15} \}}{A^{R_0 - 1} J^{12 - R_0}} \right) \right).
\]
We take $R_0 = 3$ and make use of the inequality
\[
\min \{ A^2, \Delta_3^{21} J^{15} \} \leq (A^2)^{2/5} (\Delta_3^{21} J^{15})^{3/5}.
\]
We thus derive
\[
\mathcal{F}_{\leq \delta}(A) \ll (\log A)^6 \sum_{X, Y < B} \sum_{J < (A \log A)^{8/13}} \left( \frac{(XY)^2}{A^3 B^2} + \frac{\Delta_{3}^{78/5}}{A^{6/5}} \right)
\]
\[
\ll (\log A)^{10} \left( \frac{1}{A} + \frac{1}{A^{6/5 - 78/5}} \right).
\]
Recalling the upper bound (4.47) we see that this completes the proof.

Recalling the estimate (4.41), we see that we have obtained
\[
\mathcal{F}_{\leq \delta}(A, B) \ll A^{N_{d,n} - 2} E_{4,4}(B) \left( \frac{(\log A)^{5}}{A^{1/2}} + \frac{(\log A)^{10}}{A^{6/5 - 78/5}} \right).
\]
(4.48)

Putting together the equality (4.42) and the upper bounds (4.45) and (4.48) and choosing for instance \( \delta = 1/20 \) we deduce that
\[
\Sigma^{(2)}_{4,4}(A, B) \ll A^{N_{d,n} - 2} E_{4,4}(B) \cdot \frac{(\log A)^{134}}{A^{5/101}}.
\]
Recalling the estimate (4.41), we see that this completes the proof. \( \square \)

Recall the definition (4.26) of \( t_{d,n} \). Our next task is to establish an estimate for the quantity \( D_{d,n}^{\text{mix}}(A, B) \) defined in (4.20).

**Lemma 4.10.** Let \( d \geq 2 \) and \( n \geq d \) with \( (d, n) \neq (2, 2) \). Assume that \( B^{5/6} \leq A \leq B^{2} \). Then we have
\[
D_{d,n}^{\text{mix}}(A, B) = t_{d,n} A^{N_{d,n} - 2} E_{d,n}(B) \left( 1 + O \left( \frac{1}{(\log A)^{1/2}} \right) \right).
\]

**Proof.** Recall that the definitions of the lattices \( \Lambda_{\nu_{d,n}(x)} \) and \( \Lambda_{\nu_{d,n}(y)}^{(W)} \) and the region \( C_{\nu_{d,n}(y)}^{(\alpha)} \) were respectively given in (2.3), (2.7) and (2.6). We introduce the lattice
\[
\Gamma_{x,y}^{\text{mix}}(W) = \Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)}^{(W)},
\]
and the region
\[
T_{y}^{\text{mix}}(A, \alpha) = B_{N_{d,n}}^{(A)} \cap C_{\nu_{d,n}(y)}^{(\alpha)}.
\]
Recall the respective definitions (4.24) and (4.25) of the sums \( S_{k}^{x}(\Lambda; \mathcal{R}) \) and \( S_{k}(\Lambda; \mathcal{R}) \) for any given lattice \( \Lambda \subset \mathbb{Z}^{N_{d,n}} \), any bounded region \( \mathcal{R} \subset \mathbb{R}^{N_{d,n}} \) and any integer \( k \geq 0 \). Recall also the definition (4.7) of the set \( \Omega_{d,n}(B) \). We see that
\[
D_{d,n}^{\text{mix}}(A, B) = \frac{\alpha W}{8} \sum_{(x,y) \in \Omega_{d,n}(B)} S_{1}^{x} \left( \Gamma_{x,y}^{\text{mix}}(W); T_{y}^{\text{mix}}(A, \alpha) \right) / ||\nu_{d,n}(y)||.
\]
Recall the definition (4.15) of the quantity \( \mu(x) \) and set
\[
\Sigma_{1}^{\text{mix}}(A, B) = \frac{\alpha W}{8} \sum_{(x,y) \in \Omega_{d,n}(B)} S_{1}^{x} \left( \Gamma_{x,y}^{\text{mix}}(W); T_{y}^{\text{mix}}(A, \alpha) \right) / ||\nu_{d,n}(y)||,
\]
and
\[
\Sigma_{2}^{\text{mix}}(A, B) = D_{d,n}^{\text{mix}}(A, B) - \Sigma_{1}^{\text{mix}}(A, B).
\]
(4.50)

We start by dealing with the sum \( \Sigma_{1}^{\text{mix}}(A, B) \). A Möbius inversion gives
\[
S_{1}^{x} \left( \Gamma_{x,y}^{\text{mix}}(W); T_{y}^{\text{mix}}(A, \alpha) \right) = \sum_{\ell \leq A} \frac{\mu(\ell)}{\ell} S_{1}^{x} \left( \Gamma_{x,y} \left( \frac{W}{\gcd(\ell, W)} \right); T_{y}^{\text{mix}} \left( \frac{A}{\ell}, \alpha \right) \right).
\]
For any real \( u \geq 1 \), it is clear that we have
\[
S_1 \left( \Gamma_{x,y}^{\text{mix}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_y^{\text{mix}} (u, \alpha) \right) \leq S_1 \left( \Lambda_{u,d,n}(x); B_{N_{d,n}}(u) \right).
\]

Breaking the size of \( a \) into dyadic intervals, we see that
\[
S_1 \left( \Lambda_{u,d,n}(x); B_{N_{d,n}}(u) \right) \ll \sum_{U \ll u} \frac{1}{U} S_0 \left( \Lambda_{u,d,n}(x); B_{N_{d,n}}(U) \right).
\]

Recall that we have the inequality (4.16). Therefore, we see that if \( U \geq \mu(x) \) then we are in position to apply Lemma 3.4 with \( I = 1 \) and \( \gamma = 1/2 \), say. On the other hand, if \( U < \mu(x) \) then we can apply the first part of Lemma 3.6 with \( M = 1/2 \) and \( R_0 = N_{d,n} - 2 \). It follows that
\[
S_1 \left( \Lambda_{u,d,n}(x); B_{N_{d,n}}(u) \right) \ll \sum_{U \ll u} \frac{1}{U} \left( \frac{U^{N_{d,n}-1}}{||x||^d} + \frac{U \mu(x)^{N_{d,n}-2}}{||x||^d} + 1 \right).
\]

Note that we have used the fact that Lemma 3.8 gives \( \det(\Lambda_{u,d,n}(x)) = ||\mu_{d,n}(x)|| \gg ||x||^d \).

As a result, for \( u \geq \mu(x) \) we obtain
\[
S_1 \left( \Gamma_{x,y}^{\text{mix}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_y^{\text{mix}} (u, \alpha) \right) \ll \frac{u^{N_{d,n}-2}}{||x||^d} \log u + 1. \tag{4.51}
\]

Moreover, writing that
\[
S_0 \left( \Gamma_{x,y}^{\text{mix}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_y^{\text{mix}} (u, \alpha) \right) \leq S_0 \left( \Lambda_{u,d,n}(x); B_{N_{d,n}}(u) \right),
\]

and applying Lemma 3.4 with \( I = 1 \) and \( \gamma = 1/2 \), we deduce that for \( u \geq \mu(x) \) we have
\[
S_0 \left( \Gamma_{x,y}^{\text{mix}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_y^{\text{mix}} (u, \alpha) \right) \ll \frac{u^{N_{d,n}-1}}{||x||^d}. \tag{4.52}
\]

Using the upper bound (4.51) with \( u = W \mu(x) \), we see that
\[
\sum_{A/W \mu(x) < \ell \leq A} \frac{\mu(\ell)}{\ell} S_1 \left( \Gamma_{x,y}^{\text{mix}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_y^{\text{mix}} \left( \frac{A}{\ell}, \alpha \right) \right) \ll \left( \frac{(W \mu(x))^{N_{d,n}-2}}{||x||^d} \right) + 1 \times (\log A)^2.
\]

Therefore, using the upper bound (4.51) once again with \( u = W \mu(x) \) we get
\[
S^*_1 \left( \Gamma_{x,y}^{\text{mix}}(W); \mathcal{T}_y^{\text{mix}}(A, \alpha) \right) = \sum_{\ell \leq A/W \mu(x)} \frac{\mu(\ell)}{\ell} S_{x,y}^{\text{mix}}(A, B; \ell) \
+ O \left( \left( \frac{(W \mu(x))^{N_{d,n}-2}}{||x||^d} \right) + 1 \right) (\log A)^2, \tag{4.53}
\]

where
\[
S_{x,y}^{\text{mix}}(A, B; \ell) = S_1 \left( \Gamma_{x,y}^{\text{mix}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_y^{\text{mix}} \left( \frac{A}{\ell}, \alpha \right) \right) \times \mathcal{T}_y^{\text{mix}} \left( W \mu(x), \alpha \right).
\]

Next, an application of partial summation yields
\[
S_{x,y}^{\text{mix}}(A, B; \ell) = \frac{\ell}{A} S_0 \left( \Gamma_{x,y}^{\text{mix}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_y^{\text{mix}} \left( \frac{A}{\ell}, \alpha \right) \right) \
+ \int_{W \mu(x)}^{A/\ell} S_0 \left( \Gamma_{x,y}^{\text{mix}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_y^{\text{mix}}(t, \alpha) \right) dt + O \left( \frac{(W \mu(x))^{N_{d,n}-2}}{||x||^d} \right),
\]

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Note that we have used the upper bound (4.52) with \( u = W\mu(x) \). In addition, Lemma 3.14 states that the ball \( B_{N_d,n}(\mu(x)) \) contains \( N_{d,n} - 1 \) linearly independent vectors of the lattice \( \Lambda_{\nu_{d,n}}(x) \). Therefore, multiplying these vectors by \( W \) we deduce that

\[
\lambda_{N_d,n-1} \left( \Gamma_{\text{mix}}^{x,y} \left( \frac{W}{\gcd(\ell, W)} \right) \right) \leq W\mu(x).
\]

Lemma 3.4 thus implies that for any \( t \in [W\mu(x), A/\ell] \), we have

\[
S_0 \left( \Gamma_{\text{mix}}^{x,y} \left( \frac{W}{\gcd(\ell, W)} \right); T_y^{\text{mix}}(t, \alpha) \right) = \ell^{N_{d,n}-1} \det \left( \Gamma_{\text{mix}}^{x,y} \left( \frac{W}{\gcd(\ell, W)} \right) \right)^{-1}
\times \left( \vol(\Sigma_{x,y}(\alpha)) + O \left( \frac{W\mu(x)}{t} \right) \right),
\]
where we have set

\[
\Sigma_{x,y}(\alpha) = \text{Span}_R \left( \Lambda_{\nu_{d,n}}(x) \right) \cap T_y^{\text{mix}}(1, \alpha).
\]

It follows that

\[
S_{x,y}^{\text{mix}}(A, B; \ell) = \frac{N_{d,n} - 1}{N_{d,n} - 2} A^{N_{d,n} - 2} \cdot \det \left( \Gamma_{\text{mix}}^{x,y} \left( \frac{W}{\gcd(\ell, W)} \right) \right)^{-1}
\times \left( \vol(\Sigma_{x,y}(\alpha)) + O \left( \frac{W\mu(x)}{A} \right) \right) + O \left( \frac{(W\mu(x))^{N_{d,n} - 2}}{||x||^d} \right). \tag{4.54}
\]

Recalling the definition (4.1) of \( I(w, z) \), we note that we have

\[
\vol(\Sigma_{x,y}(\alpha)) = I \left( 2\alpha \frac{\nu_{d,n}(x)}{||\nu_{d,n}(x)||}, 2\alpha \frac{\nu_{d,n}(y)}{||\nu_{d,n}(y)||} \right).
\]

Recall the definition (4.9) of the quantity \( \Delta(x, y) \). Using Lemma 4.2 and the equality (4.5) we see that

\[
\vol(\Sigma_{x,y}(\alpha)) = \frac{N_{d,n} - 2}{N_{d,n} - 1} V_{N_{d,n} - 2} \frac{\Delta(x, y)}{\alpha} \left( 1 + O \left( \min \left\{ 1, \frac{\Delta(x, y)^2}{\alpha^2} \right\} \right) \right). \tag{4.55}
\]

Moreover, Lemmas 3.9 and 3.11 give

\[
\det \left( \Gamma_{\text{mix}}^{x,y} \left( \frac{W}{\gcd(\ell, W)} \right) \right) = W ||\nu_{d,n}(x)|| \cdot \gcd \left( \frac{G(x, y)}{\gcd(\ell, W)} \right)^{-1}.
\]

Recall the definition (4.8) of the radical of the integer \( W \). We note that if \( \ell \) is a squarefree integer then

\[
\gcd \left( \frac{G(x, y)}{\gcd(\ell, W)} \right) = G(x, y) \left( 1 + O \left( 1_{G(x,y)||W/\gcd(\ell, W)} \right) \right). \tag{4.56}
\]

Hence

\[
\det \left( \Gamma_{\text{mix}}^{x,y} \left( \frac{W}{\gcd(\ell, W)} \right) \right) = W ||\nu_{d,n}(x)|| \cdot \gcd(\ell, W) G(x, y) \left( 1 + O \left( 1_{G(x,y)||W/\gcd(\ell, W)} \right) \right). \tag{4.57}
\]

Recall the definition (4.10) of the quantity \( E_{x,y}(B) \). Combining the estimates (4.54), (4.55) and (4.57) and using the lower bound \( \Delta(x, y) \geq 1 \), we obtain

\[
S_{x,y}^{\text{mix}}(A, B; \ell) = V_{N_{d,n} - 2} \frac{A^{N_{d,n} - 2} \gcd(\ell, W) \Delta(x, y) G(x, y)}{\alpha W \cdot \ell^{N_{d,n} - 2} \cdot ||\nu_{d,n}(x)||}
\times \left( 1 + O \left( E_{x,y}(B) + \frac{\alpha W\mu(x)}{A} \right) \right) + O \left( \frac{(W\mu(x))^{N_{d,n} - 2}}{||x||^d} \right).
\]

Recall the definition (2.8) of \( \alpha \) and the upper bound (2.12) for \( W \) and note that the assumption \( B \leq A^{6/5} \) gives \( \alpha \ll \log A \) and \( W \ll A^{5/\log \log A} \). Using these facts and
the inequalities $\Delta(x, y), G(x, y) \geq 1$ and $W\mu(x) \leq A$, we see that the estimate (4.53) implies in particular that

$$S_1^+ (\Gamma_{x,y}^{\text{mix}}(W); \mathcal{Y}_y^{\text{mix}}(A, \alpha)) = V_{d,n}^{N_d,n-2} \frac{\Delta(x, y) \mathcal{G}(x, y)}{\alpha W \| \nu_{d,n}(x) \|} \times \left( \sum_{\ell \leq A/W \mu(x)} \mu(\ell) \frac{\gcd(\ell, W)}{\ell^{N_d,n-1}} + O \left( \frac{w}{\ell^{N_d,n-3}} \right) \right).$$

Now it is clear that

$$\sum_{\ell \leq A/W \mu(x)} \mu(\ell) \frac{\gcd(\ell, W)}{\ell^{N_d,n-1}} = \sum_{\ell \leq w} \mu(\ell) \frac{\gcd(\ell, W)}{\ell^{N_d,n-1}} + O \left( \left( \frac{W\mu(x)}{A} \right)^{N_d,n-3} + \frac{1}{\ell^{N_d,n-3}} \right).$$

But the definition (2.9) of $\gcd(\ell, W)$ shows that for any squarefree integer $\ell \leq w$ we have $\gcd(\ell, W) = \ell$. We thus get

$$\sum_{\ell \leq A/W \mu(x)} \mu(\ell) \frac{\gcd(\ell, W)}{\ell^{N_d,n-1}} = \frac{1}{\zeta(N_d,n - 2)} + O \left( \left( \frac{W\mu(x)}{A} \right)^{N_d,n-3} + \frac{1}{\ell^{N_d,n-3}} \right). (4.58)$$

Finally, we note that the equality (4.4) yields

$$\frac{\Delta(x, y) \mathcal{G}(x, y)}{\| \nu_{d,n}(x) \| \cdot \| \nu_{d,n}(y) \|} = \frac{1}{\det(\Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)})}. (4.59)$$

Recall that the quantity $F_{d,n}(B)$ was defined in (4.11). It follows from Lemma 4.5 and the equality (4.49) that

$$\Sigma_1^{\text{mix}}(A, B) = \ell_{d,n} A^{N_d,n-2} E_{d,n}(B) \left( 1 + O \left( \frac{1}{\ell_{d,n}^{N_d,n-3}} + \frac{F_{d,n}(B)}{B^2} + \frac{G_{d,n}(B)}{B^2} \right) \right), (4.60)$$

where we have set

$$G_{d,n}(B) = \frac{1}{A^{3/4}} \sum_{(x, y) \in \Omega_{d,n}(B)} \frac{\mu(x)}{\det(\Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)})}.$$ 

Note that we have used the obvious fact that

$$\sum_{(x, y) \in \Omega_{d,n}(B)} \frac{1}{\det(\Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)})} \leq \frac{W}{A} \sum_{\mu(x) > A/W} \frac{\mu(x)}{\det(\Lambda_{\nu_{d,n}(x)} \cap \Lambda_{\nu_{d,n}(y)})} \ll G_{d,n}(B).$$

An application of Lemma 4.4 gives

$$G_{d,n}(B) \ll \frac{1}{A^{3/4}} \sum_{(x, y) \in \Omega_{d,n}(B)} \frac{1}{\varphi_2(x) \varphi_2(x, y) \cdot |x|^{d-2} |y|^{d-1}}.$$ 

Recall the respective definitions (3.21) and (3.22) of the quantities $\ell_{2,n}(X; \Delta_0)$ and $\ell_{2,n}(X, Y; \Delta_2)$. We proceed to break the sizes of $|x|, |y|, \varphi_2(x)$ and $\varphi_2(x, y)$ into dyadic intervals. Recalling that we have the upper bounds (3.19) and (3.20) we deduce that

$$G_{d,n}(B) \ll \frac{1}{A^{3/4}} \sum_{X, Y \ll B^{1/(n+1-d)}} \sum_{\Delta_0 \ll X} \sum_{\Delta_2 \ll XY} \min \left\{ Y^{n+1} \ell_{2,n}(X; \Delta_0), \ell_{2,n}(X, Y; \Delta_2) \right\} \Delta_0 \Delta_2 \Delta_{X}^{d-2} \Delta_{Y}^{d-1}.$$
Applying Lemmas 3.19 and 3.20 we see that
\[
\min \{ Y^{n+1} \ell_2, Y; \Delta_0), \ell_2, Y; \Delta_2 \} \ll (\log X) \min \{ X^2 Y^{n+1} \Delta_0^n, (XY)^2 \Delta_2^{n-1} \}
\ll (\log X) (X^2 Y^{n+1} \Delta_0^n)^{1/n} ((XY)^2 \Delta_2^{n-1})^{1-1/n}.
\]
We thus derive
\[
G_{d,n}(B) \ll \frac{\log B}{A^{3/4}} \sum_{X, Y \ll B^{1/(n+1-d)}} \sum_{\Delta_0 \ll X} \sum_{\Delta_2 \ll XY} \frac{\Delta_2^{n-3+1/n}}{X^{d-4} Y^{d-4+1/n}}
\ll (\log B)^2 \frac{A^{5/4}}{B^{2+1/n(n+1-d)}}.
\]
Using the fact that \( n(n+1-d) \geq 3 \) and the assumption \( B \leq A^{6/5} \) we obtain in particular
\[
G_{d,n}(B) \ll B^2 A^{1/3}.
\] (4.61)
Note that the definition (2.10) of \( w \) implies that \( w \Delta_2 \ll (\log B)^{1/2} \). Therefore, combining the estimate (4.60), Lemma 4.6 and the upper bound (4.61), and using the assumption \( A \leq B^2 \) we deduce that
\[
\Sigma_1^{\text{mix}}(A, B) = i_{d,n} A^{N_{d,n}-2} E_{d,n}(B) \left( 1 + O \left( \frac{1}{(\log A)^{1/2}} \right) \right).
\] (4.62)
We now deal with the quantity \( \Sigma_2^{\text{mix}}(A, B) \). Recall the definition (2.5) of the set \( \Xi_{d,n}(B) \). We start by noting that we trivially have
\[
\Sigma_2^{\text{mix}}(A, B) \leq \frac{\alpha W}{8} \sum_{(x, y) \in \Xi_{d,n}(B)} \frac{S^*_1(A_{\nu_{d,n}(x)}; B_{N_{d,n}(A)})}{||\nu_{d,n}(y)||} \ll \alpha W B \sum_{x \in \Xi_{d,n}(B)} \# \left( \Lambda_{\nu_{d,n}(x)} \cap B_{N_{d,n}(A)} \right).
\]
Lemma 3.14 implies in particular that
\[
\lambda_{N_{d,n}-1} \left( \Lambda_{\nu_{d,n}(x)} \right) \leq W \mu(x).
\]
Moreover we have \( A < W \mu(x) \) so we can apply the first part of Lemma 3.6 with \( M = 1/2 \) and \( R_0 = n - 1 \). This yields
\[
\# \left( \Lambda_{\nu_{d,n}(x)} \cap B_{N_{d,n}(A)} \right) \ll \frac{A^{N_{d,n}-n} (W \mu(x))^{n-1}}{\det(\Lambda_{\nu_{d,n}(x)})} + A^{N_{d,n}-n-1} \ll W^{-1} A^{N_{d,n}-n} ||x||^{n-d-1} a_2(x)^{n-1} + A^{N_{d,n}-n-1}.
\]
Note that we have used the fact that Lemma 3.8 gives \( \det(\Lambda_{\nu_{d,n}(x)}) = ||\nu_{d,n}(x)|| \gg ||x||^{d} \). Breaking the sizes of \( ||x|| \) and \( a_2(x) \) into dyadic intervals we see that we have
\[
\Sigma_2^{\text{mix}}(A, B) \ll \alpha W^n A^{N_{d,n}-2} B \sum_{X \ll X^{1/(n+1-d)}} \frac{X^{n-d-1}}{A^{n-2} \Delta_0^{n-1}} \left( \frac{1}{A^{n-1}} \right) \ell_2, Y; \Delta_0).
\]
It follows from Lemma 3.19 that

\[
\Sigma_{2}^{\text{mix}}(A, B) \ll \alpha W^{n} A^{N_{d,n}} B(\log B) \sum_{X \ll B^{1/(n+1-d)}} \sum_{\Delta_{0} \ll WX/A} \left( \frac{X^{n+1-d} \Delta_{0}}{A^{n-2}} + \frac{X^{2} \Delta_{0}^{n}}{A^{n-1}} \right)
\]

\[
\ll \alpha W^{n} A^{N_{d,n}} B(\log B) \sum_{X \ll B^{1/(n+1-d)}} \left( \frac{WX^{n+2-d}}{A^{n-1}} + \frac{WnX^{n+2}}{A^{2n-1}} \right)
\]

\[
\ll \alpha W^{2n} A^{N_{d,n}} B^{2} \left( \frac{B^{1/(n+1-d)}}{A^{n-1}} + \frac{B^{(d+1)/(n+1-d)}}{A^{2n-1}} \right) \log B.
\]

Since \((d+1)/(n + 1 - d) \leq n + 1\), the assumption \(B \leq A^{6/5}\) gives

\[
\frac{B^{(d+1)/(n+1-d)}}{A^{2n-1}} \leq \frac{1}{A^{(4n-11)/5}}.
\]

Therefore, using the upper bounds \(\alpha \ll \log A\) and \(W \ll A^{5/\log\log A}\) and the fact that \(n \geq 3\), we see that Lemma 4.5 implies in particular that

\[
\Sigma_{2}^{\text{mix}}(A, B) \ll A^{N_{d,n} - 2} E_{d,n}(B) \cdot \frac{1}{A^{1/10}}.
\]  

(4.63)

Putting together the equality (4.50), the estimate (4.62) and the upper bound (4.63) completes the proof. \(\square\)

Recall the definition (4.26) of \(\iota_{d,n}\). Our final task in this section is to prove an estimate for the quantity \(D_{d,n}^{\text{loc}}(A, B)\) defined in (4.22).

**Lemma 4.11.** Let \(d \geq 2\) and \(n \geq d\) with \((d, n) \neq (2, 2)\). Assume that \(B^{1/2} \leq A \leq B^{2}\). Then we have

\[
D_{d,n}^{\text{loc}}(A, B) = \iota_{d,n} A^{N_{d,n} - 2} E_{d,n}(B) \left( 1 + O \left( \frac{1}{(\log A)^{1/2}} \right) \right).
\]

**Proof.** Recall that the definitions of the lattice \(\Lambda_{\nu_{d,n}}^{(W)}\) and the region \(C_{\nu_{d,n}}^{(\alpha)}\) were respectively given in (2.7) and (2.6). We define the lattice

\[
\Gamma_{x,y}^{\text{loc}}(W) = \Lambda_{\nu_{d,n}}^{(W)} \cap \Lambda_{\nu_{d,n}}^{(W)},
\]

and the region

\[
T_{x,y}^{\text{loc}}(A, \alpha) = B_{N_{d,n}}(A) \cap C_{\nu_{d,n}}^{(\alpha)} \cap C_{\nu_{d,n}}^{(\alpha)}.
\]

Recall that the definitions of the sums \(S_{k}^{\text{loc}}(\Lambda; \mathcal{R})\) and \(S_{k}(\Lambda; \mathcal{R})\) for any given lattice \(\Lambda \subset \mathbb{Z}^{N_{d,n}}\), any bounded region \(\mathcal{R} \subset \mathbb{R}^{N_{d,n}}\) and any integer \(k \geq 0\), were respectively given in (4.24) and (4.25). Recalling the definition (4.7) of the set \(\Omega_{d,n}(B)\) we see that

\[
D_{d,n}^{\text{loc}}(A, B) = \frac{\alpha^{2} W^{2}}{8} \sum_{(x,y) \in \Omega_{d,n}(B)} \frac{S_{2}^{\text{loc}}(\Gamma_{x,y}^{\text{loc}}(W); T_{x,y}^{\text{loc}}(A, \alpha))}{\|\nu_{d,n}(x)\| \cdot \|\nu_{d,n}(y)\|}.
\]

(4.64)

A Möbius inversion gives

\[
S_{2}^{\text{loc}}(\Gamma_{x,y}^{\text{loc}}(W); T_{x,y}^{\text{loc}}(A, \alpha)) = \sum_{\ell \leq A} \mu(\ell) \frac{\ell}{\ell^{2}} S_{2}^{\text{loc}}(\Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right); T_{x,y}^{\text{loc}} \left( \frac{A}{\ell}, \alpha \right)).
\]

For any integer \(k \geq 0\) and any real \(u \geq 1\) we clearly have

\[
S_{k} \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right); T_{x,y}^{\text{loc}}(u, \alpha) \right) \leq S_{k} \left( \mathbb{Z}^{N_{d,n}}; B_{N_{d,n}}(u) \right).
\]
It thus follows that if \( k \in \{0, \ldots, N_{d,n} - 1\} \) then
\[
S_k \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_{x,y}^{\text{loc}} (u, \alpha) \right) \leq u^{N_{d,n} - k}.
\] (4.65)

Using the upper bound (4.65) with \( k = 2 \) and \( u = A/\ell \) we deduce that
\[
\sum_{\ell > A/W} \frac{\mu(\ell)}{\ell^2} S_2 \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_{x,y}^{\text{loc}} \left( \frac{A}{\ell}, \alpha \right) \right) \leq \frac{W^{s N_{d,n} - 1}}{A}.
\]

Therefore, using the upper bound (4.65) once again with \( k = 2 \) and \( u = W \) we get
\[
S_2^s \left( \Gamma_{x,y}^{\text{loc}} (W); \mathcal{T}_{x,y}^{\text{loc}} (A, \alpha) \right) = \sum_{\ell \leq A/W} \frac{\mu(\ell)}{\ell^2} S_{x,y}^{\text{loc}} (A, B; \ell) + O \left( W^{N_{d,n} - 2} \right),
\] (4.66)

where
\[
S_{x,y}^{\text{loc}} (A, B; \ell) = S_2 \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_{x,y}^{\text{loc}} \left( \frac{A}{\ell}, \alpha \right) \right) - \mathcal{T}_{x,y}^{\text{loc}} (W, \alpha).
\]

Next, an application of partial summation yields
\[
S_{x,y}^{\text{loc}} (A, B; \ell) = \frac{\ell^2}{A^2} S_0 \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_{x,y}^{\text{loc}} \left( \frac{A}{\ell}, \alpha \right) \right)
+ 2 \int_{W}^{A/\ell} S_0 \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_{x,y}^{\text{loc}} (t, \alpha) \right) \frac{dt}{t^3} + O (W^{N_{d,n} - 2}).
\]

Note that we have made use of the upper bound (4.65) with \( k = 0 \) and \( u = W \). In addition, it is clear that we have
\[
\lambda_{N_{d,n}} \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right) \right) \leq W.
\]

Therefore, Lemma 3.4 shows that for any \( t \in [W, A/\ell] \), we have
\[
S_0 \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right); \mathcal{T}_{x,y}^{\text{loc}} (t, \alpha) \right) = t^{N_{d,n}} \det \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right) \right)^{-1}
\times \left( \text{vol} \left( \mathcal{T}_{x,y}^{\text{loc}} (1, \alpha) \right) + O \left( \frac{W}{A} \right) \right).
\]

We thus derive
\[
S_{x,y}^{\text{loc}} (A, B; \ell) = \frac{N_{d,n}}{N_{d,n} - 2} \frac{A^{N_{d,n} - 2}}{\ell^{N_{d,n} - 2}} \cdot \det \left( \Gamma_{x,y}^{\text{loc}} \left( \frac{W}{\gcd(\ell, W)} \right) \right)^{-1}
\times \left( \text{vol} \left( \mathcal{T}_{x,y}^{\text{loc}} (1, \alpha) \right) + O \left( \frac{W}{A} \right) \right) + O (W^{N_{d,n} - 2}).
\] (4.67)

Recalling the definition (4.2) of \( J (w, z) \), we see that we have
\[
\text{vol} \left( \mathcal{T}_{x,y}^{\text{loc}} (1, \alpha) \right) = \mathcal{J} \left( 2 \alpha \frac{\nu_{d,n}(x)}{\|\nu_{d,n}(x)\|}, 2 \alpha \frac{\nu_{d,n}(y)}{\|\nu_{d,n}(y)\|} \right).
\]

Recall the definition (4.9) of the quantity \( \Delta(x, y) \). Using Lemma 4.3 and the equality (4.5) we deduce that
\[
\text{vol} \left( \mathcal{T}_{x,y}^{\text{loc}} (1, \alpha) \right) = \frac{N_{d,n} - 2}{N_{d,n}} \frac{\Delta(x, y)}{\alpha^2} \left( 1 + O \left( \min \left\{ 1, \frac{\Delta(x, y)^2}{\alpha^2} \right\} \right) \right).
\] (4.68)
In addition, Lemmas 3.9 and 3.11 and the estimate (4.56) give
\[
\det \left( \Gamma_{x,y} \left( \frac{W}{\gcd(\ell, W)} \right) \right) = \frac{W^2}{\gcd(\ell, W)^2} \cdot \gcd \left( G(x, y), \frac{W}{\gcd(\ell, W)} \right)^{-1} = \frac{W^2}{\gcd(\ell, W)^2} \left( 1 + O\left( 1_{G(x, y) \mid W/\rad(W)} \right) \right). \tag{4.69}
\]
Recall the definition (4.10) of the quantity \( \mathcal{E}_{x,y}(B) \). Putting together the estimates (4.67), (4.68) and (4.69) and using the lower bound \( \Delta(x, y) \geq 1 \), we obtain
\[
S_{x,y}^\infty(A, B; \ell) = V_{N^d,n-2,2} \frac{A^N_{d,n-2}}{\alpha^2 W^2} \frac{\gcd(\ell, W)^2}{\ell N^d,n-2,2} \Delta(x, y) G(x, y)
\times \left( 1 + O \left( \mathcal{E}_{x,y}(B) + \frac{\alpha^2 W \ell}{A} \right) \right) + O \left( W N^d,n-2 \right).
\]
Recall the upper bound (2.12) for \( W \) and note that the assumption \( B \leq A^2 \) gives \( W \ll A^{8/\log \log A} \). Therefore, using the inequalities \( \Delta(x, y), G(x, y) \geq 1 \), we deduce from the estimate (4.66) that
\[
S_{x,y}^\infty \left( \Gamma_{x,y}(W); T_{x,y}^\infty(A, \alpha) \right) = V_{N^d,n-2,2} \frac{A^N_{d,n-2}}{\alpha^2 W^2} \Delta(x, y) G(x, y)
\times \left( \sum_{\ell \leq A/W} \mu(\ell) \frac{\gcd(\ell, W)^2}{\ell N^d,n} + O \left( \mathcal{E}_{x,y}(B) + \frac{\alpha^2 W}{A} \right) \right).
\]
Arguing as in the proof of the estimate (4.58), we see that
\[
\sum_{\ell \leq A/W} \mu(\ell) \frac{\gcd(\ell, W)^2}{\ell N^d,n} = \frac{1}{\zeta(N^d,n-2)} + O \left( \left( \frac{W}{A} \right)^{N^d,n-3} + \frac{1}{w^{N^d,n-3}} \right).
\]
Recall the definition (4.11) of the quantity \( F_{d,n}(B) \). We remark that the respective definitions (2.8) and (2.10) of \( \alpha \) and \( w \) show that \( w^{N^d,n-3} \ll A/\alpha^2 W \). As a result, Lemma 4.5 and the equalities (4.59) and (4.64) yield
\[
D_{d,n}^\infty(A, B) = t_{d,n} A^{N^d,n-2} E_{d,n}(B) \left( 1 + O \left( \frac{1}{w^{N^d,n-3}} + \frac{F_{d,n}(B)}{B^2} \right) \right).
\]
We complete the proof by applying Lemma 4.6 and by using the assumption \( A \leq B^2 \) and the fact that \( w^{N^d,n-3} \gg (\log A)^{1/2} \).

4.6. Proof of the key variance upper bound. We now combine the tools developed in Sections 4.4 and 4.5 in order to establish Proposition 4.1.

Proof of Proposition 4.1. Recall the respective definitions (4.21) and (4.23) of the two quantities \( \Delta_{V}^{\text{mix}}(B) \) and \( \Delta_{V}^{\text{loc}}(B) \). It is convenient to set
\[
K(A, B) = \sum_{V \in V_{d,n}(A)} \left( N_{V}(B) + \Delta_{V}^{\text{mix}}(B) + \Delta_{V}^{\text{loc}}(B) \right).
\]
Expanding the square, we see that
\[
\sum_{V \in V_{d,n}(A)} \left( N_{V}(B) - \Delta_{V}^{\text{loc}}(B) \right)^2 = D_{d,n}^\infty(A, B) - 2 D_{d,n}^\text{mix}(A, B) + D_{d,n}^\text{loc}(A, B) + O(K(A, B)).
\]
It follows from the lower bound (2.19) and Lemmas 4.5, 4.8, 4.9, 4.10 and 4.11 that
\[
\frac{1}{\# V_{d,n}(A)} \left( D_{d,n}^\infty(A, B) - 2 D_{d,n}^\text{mix}(A, B) + D_{d,n}^\text{loc}(A, B) \right) \ll \frac{B^2}{A^2} \cdot \frac{1}{(\log A)^{1/2}}.
\]
We thus derive
\[
\frac{1}{\# V_{d,n}(A)} \sum_{V \in V_{d,n}(A)} \left( N_V(B) - N_V^{\text{loc}}(B) \right)^2 \ll \frac{B^2}{A^2} \cdot \frac{1}{(\log A)^{1/2}} + \frac{K(A, B)}{\# V_{d,n}(A)},
\]  
(4.70)
Recall the definition (2.8) of \( \alpha \) and the upper bound (2.12) for \( W \). We trivially have
\[
\Delta_{\text{mix}}^V(B) \leq \alpha W \frac{||a(||}{||v||} N_V(B).
\]
Hence, using partial summation it follows from Lemma 4.7 that
\[
\frac{1}{\# V_{d,n}(A)} \sum_{V \in V_{d,n}(A)} \Delta_{\text{mix}}^V(B) \ll \frac{B^{1+5/\log \log B}}{A^2}.
\]  
(4.71)
Moreover, using the trivial upper bound
\[
\Delta_{\text{loc}}^V(B) \ll \frac{\alpha^2 W^2}{||a(||^2} \sum_{x \in \mathbb{Z}^d_n(B)} \frac{1}{||v_{d,n}(x)||},
\]
we obtain
\[
\frac{1}{\# V_{d,n}(A)} \sum_{V \in V_{d,n}(A)} \Delta_{\text{loc}}^V(B) \ll \frac{B^{1+9/\log \log B}}{A^2}.
\]  
(4.72)
Applying Lemma 4.7 and using the upper bounds (4.71) and (4.72) together with the assumption \( B \ll A(\log A)^{1/2} \), we derive
\[
\frac{K(A, B)}{\# V_{d,n}(A)} \ll \frac{B}{A}.
\]  
(4.73)
Putting together the upper bounds (4.70) and (4.73), we immediately see that the assumption \( B \ll A(\log A)^{1/2} \) allows us to complete the proof. \( \square \)

5. THE LOCALISED COUNTING FUNCTION IS RARELY SMALL

In Section 5.1 we start by introducing certain non-Archimedean and Archimedean factors that will arise throughout the dissection of our localised counting function \( N_{V}^{\text{loc}}(B) \). We then check that Proposition 2.4 follows from upper bounds for the number of \( V \in V_{d,n}(A) \) at which one of these two factors is exceptionally small, as stated in Propositions 5.2 and 5.3. We finally turn to the proofs of Propositions 5.2 and 5.3 in Sections 5.2 and 5.3, respectively.

5.1. The local factors. Given \( N \geq 1 \), recall the definition (2.6) of the region \( C_{v}^{(\gamma)} \) for any real \( \gamma > 0 \) and any \( v \in \mathbb{R}^N \). For \( a \in \mathbb{R}^{N_{d,n}} \) and \( \gamma > 0 \), we introduce the Archimedean factor
\[
\tau(a; \gamma) = \gamma \cdot \text{vol} \left( \left\{ u \in \mathcal{B}_{n+1}(1) : a \in C_{v}^{(\gamma)}(u) \right\} \right).
\]  
(5.1)
In addition, given \( Q \geq 1 \) and \( b \in (\mathbb{Z}/Q\mathbb{Z})^N \) we let \( \text{gcd}(Q,b) \) denote the greatest common divisor of \( Q \) and the coordinates of the vector \( b \). In analogy with the Archimedean setting it is convenient for our purpose to define
\[
\mathfrak{A}_{N}(Q) = \left\{ b \in (\mathbb{Z}/Q\mathbb{Z})^N : \text{gcd}(Q,b) = 1 \right\}.
\]  
(5.2)
Recall the definition (2.7) of the lattice \( \Lambda_{c}^{Q} \) for given \( Q \geq 1 \) and \( c \in \mathbb{Z}^N \). For \( a \in \mathbb{Z}^{N_{d,n}} \) and \( Q \geq 1 \), we introduce the non-Archimedean factor
\[
\sigma(a; Q) = \frac{1}{Q^n} \cdot \# \left\{ b \in \mathfrak{A}_{n+1}(Q) : a \in \Lambda_{c}^{Q}(b) \right\}.
\]  
(5.3)
We note that for any vector \( \mathbf{b} \in \mathcal{R}_{n+1}(Q) \) the lattice \( \Lambda_{\nu_{d,n}}^{(Q)}(\mathbf{b}) \) is well-defined.

Recall the respective definitions (2.8) and (2.9) of the quantity \( \alpha \) and the integer \( W \). Given \( V \in \mathcal{V}_{d,n} \) we set

\[
\mathfrak{A}_V(B) = \tau(\alpha_V; \alpha), \quad (5.4)
\]

and

\[
\mathfrak{S}_V(B) = \sigma(\alpha_V; W). \quad (5.5)
\]

One may check that \( \mathfrak{A}_V(B) \) converges to the usual singular integral for the problem at hand as \( B \) tends to \( \infty \). Similarly, using the Chinese remainder theorem it is possible to show that \( \mathfrak{S}_V(B) \) converges to the singular series as \( B \) tends to \( \infty \). We shall use neither of these facts in our work, however.

Recall the definition (2.11) of our localised counting function \( N_{\nu}^{\text{loc}}(B) \). We prove the following upper bound for the product of the local factors.

**Lemma 5.1.** Let \( d \geq 2 \) and \( n \geq d \). For any \( V \in \mathcal{V}_{d,n}(A) \), we have

\[
\mathfrak{S}_V(B) \cdot \mathfrak{A}_V(B) \ll \frac{A}{B} N_{\nu}^{\text{loc}}(B) + \frac{1}{B^{1/n}}. \quad (5.6)
\]

**Proof.** Recall the definition (2.5) of the set \( \Xi_{d,n}(B) \). We start by noting that for any \( \mathbf{x} \in \Xi_{d,n}(B) \) we have \( ||\nu_{d,n}(\mathbf{x})|| \ll B^{d/(n+1-d)} \). Using the fact that \( ||\alpha_V|| \leq A \) for \( V \in \mathcal{V}_{d,n}(A) \), we deduce that

\[
\sum_{x \in \Xi_{d,n}(B)} 1 \ll \frac{AB^{d/(n+1-d)}}{\alpha W} N_{\nu}^{\text{loc}}(B). \quad (5.6)
\]

Breaking the summation into residue classes modulo \( W \), we obtain

\[
\sum_{x \in \Xi_{d,n}(B)} 1 = \sum_{\mathbf{b} \in \mathcal{R}_{n+1}(W)} \sum_{\mathbf{a}_V \in \Lambda_{\nu_{d,n}}^{(W)}} \# \left\{ \mathbf{x} \in \Xi_{d,n}(B) : \mathbf{x} \equiv \mathbf{b} \mod W, \mathbf{a}_V \in \mathcal{C}_{\nu_{d,n}}^{(W)}(x) \right\}. \quad (5.7)
\]

We proceed to use a Möbius inversion to handle the condition that the vectors \( \mathbf{x} \) are primitive. Note that for any non-zero real number \( t \) and any vector \( \mathbf{z} \in \mathbb{Z}^{n+1} \), we have

\[
\mathcal{C}_{\nu_{d,n}}^{(W)}(\mathbf{z}) = \mathcal{C}_{\nu_{d,n}}^{(W)}(\mathbf{z}). \quad (5.8)
\]

Given \( \mathbf{b} \in \mathcal{R}_{n+1}(W) \), it follows that

\[
\# \left\{ \mathbf{x} \in \Xi_{d,n}(B) : \mathbf{x} \equiv \mathbf{b} \mod W, \mathbf{a}_V \in \mathcal{C}_{\nu_{d,n}}^{(W)}(x) \right\} = \sum_{k \leq B^{1/(n+1-d)}} \mu(k)M_k(V; B), \quad (5.9)
\]

where

\[
M_k(V; B) = \# \left\{ \mathbf{z} \in \mathbb{Z}^{n+1} \setminus \{0\} : \begin{array}{l} ||\mathbf{z}|| \leq B^{1/(n+1-d)}/k \\ kz \equiv \mathbf{b} \mod W \\ \mathbf{a}_V \in \mathcal{C}_{\nu_{d,n}}^{(W)}(x) \end{array} \right\}. \]

A trivial application of the lattice point counting result [2, Theorem 1.3] shows that

\[
M_k(V; B) = \frac{1}{W^{n+1}} \cdot \text{vol} \left( \left\{ \mathbf{v} \in \mathbb{R}^{n+1} : ||\mathbf{v}|| \leq B^{1/(n+1-d)}/k, \mathbf{a}_V \in \mathcal{C}_{\nu_{d,n}}^{(W)}(\mathbf{v}) \right\} \right) + O \left( \frac{B^n/(n+1-d)}{k^n} \right).
\]
We now sum this upper bound over the vectors \( b \) of the non-Archimedean factor. Combining the equality (5.7) and the upper bound (5.6) and recalling the definition (5.4) of the Archimedean factor, we have

\[
M_k(V; B) = \frac{B^{(n+1)/(n+1-d)}}{W^{n+1}k^{n+1}} \cdot \frac{\tilde{\mathfrak{S}}_V(B)}{\alpha} + O \left( \frac{B^n}{k^n} \right).
\]

In addition, the respective definitions (2.9) and (2.10) of the integer \( W \) and the quantity \( w \) easily yield

\[
\sum_{k \leq B^{(n+1)/(n+1-d)}} \frac{\mu(k)}{k^{n+1}} = 1 + O \left( \frac{1}{w^n} \right).
\]

As a result, we deduce from the equality (5.9) that

\[
\frac{B^{(n+1)/(n+1-d)}}{W^{n+1}} \cdot \frac{\tilde{\mathfrak{S}}_V(B)}{\alpha} \ll \# \left\{ x \in \mathcal{Z}_{d,n}(B) : \begin{array}{c} x \equiv b \mod W \\ a_V \in C_{v_{d,n}}(x) \end{array} \right\} + B^{n/(n+1-d)}.
\]

We now sum this upper bound over the vectors \( b \in \mathcal{G}_{n+1}(W) \) such that \( a_V \in C_{v_{d,n}}(b) \). Combining the equality (5.7) and the upper bound (5.6) and recalling the definition (5.5) of the non-Archimedean factor \( \mathfrak{S}_V(B) \), we derive

\[
B^{(n+1)/(n+1-d)} \cdot \frac{\tilde{\mathfrak{S}}_V(B)}{W} \cdot \frac{\tilde{\mathfrak{J}}_V(B)}{\alpha} \ll \frac{AB^{d/(n+1-d)}}{\alpha W} \cdot N_{V_{d,n}}^\text{loc}(B) + W^{n+1}B^{n/(n+1-d)}.
\]

Recalling the definition (2.8) of \( \alpha \) and the upper bound (2.12) for \( W \), we see that this completes the proof.

Recall the definition (1.1) of the space \( \mathcal{V}^{\text{loc}}_{d,n}(A) \) of hypersurfaces of height at most \( A \) which are everywhere locally soluble. We now state upper bounds for the frequencies of occurrence of particularly small values of the local factors \( \mathfrak{S}_V(B) \) and \( \tilde{\mathfrak{J}}_V(B) \) as \( V \) runs over the set \( \mathcal{V}^{\text{loc}}_{d,n}(A) \), under the assumption that the ratio \( B/A \) does not grow too rapidly as \( A \) tends to \( \infty \). It is worth highlighting the fact that these two results hold without any restriction on \( d \geq 2 \) and \( n \geq 3 \).

The following statement is concerned with the non-Archimedean factor.

**Proposition 5.2.** Let \( d \geq 2 \) and \( n \geq 3 \). Let \( \phi : \mathbb{R}_{>0} \to \mathbb{R}_{>1} \) be such that \( \phi(A) \leq A \) and let \( C > 0 \). Then we have

\[
\frac{1}{\# \mathcal{V}^{\text{loc}}_{d,n}(A)} \cdot \# \left\{ V \in \mathcal{V}^{\text{loc}}_{d,n}(A) : \mathfrak{S}(A \phi(A)) < \frac{C}{\phi(A)^{1/6}} \right\} \ll \frac{1}{\phi(A)^{1/24n}},
\]

where the implied constant may depend on \( C \).

The next result deals with the Archimedean factor.

**Proposition 5.3.** Let \( d \geq 2 \) and \( n \geq 3 \). Let \( \phi : \mathbb{R}_{>0} \to \mathbb{R}_{>1} \) be such that \( \phi(A) \leq A \) and let \( C > 0 \). Then we have

\[
\frac{1}{\# \mathcal{V}^{\text{loc}}_{d,n}(A)} \cdot \# \left\{ V \in \mathcal{V}^{\text{loc}}_{d,n}(A) : \tilde{\mathfrak{J}}(A \phi(A)) < \frac{C}{\phi(A)^{1/6}} \right\} \ll \frac{1}{\phi(A)^{1/6n}},
\]

where the implied constant may depend on \( C \).

Propositions 5.2 and 5.3 will respectively be established in Sections 5.2 and 5.3. We now have everything in place to provide the proof of Proposition 2.4.
Proof of Proposition 2.4. It is convenient to set
\[ \mathcal{D}_\phi(A) = \frac{1}{\# \mathcal{V}_{d,n}^{loc}(A)} \cdot \# \left\{ V \in \mathcal{V}_{d,n}^{loc}(A) : N_{d,n}^{loc}(A \phi(A)) \leq \phi(A)^{2/3} \right\}. \]

Lemma 5.1 and the assumption \( \phi(A) \leq A^{3/n} \) imply that there exists a constant \( c > 0 \) depending at most on \( d \) and \( n \) such that
\[ \mathcal{D}_\phi(A) \leq \frac{1}{\# \mathcal{V}_{d,n}^{loc}(A)} \cdot \# \left\{ V \in \mathcal{V}_{d,n}^{loc}(A) : \mathfrak{S}_V(A \phi(A)) \cdot \mathfrak{I}_V(A \phi(A)) \leq \frac{c^{1/2}}{\phi(A)^{1/6}} \right\}. \]

Therefore, we also have
\[ \mathcal{D}_\phi(A) \leq \frac{1}{\# \mathcal{V}_{d,n}^{loc}(A)} \cdot \# \left\{ V \in \mathcal{V}_{d,n}^{loc}(A) : \min \{ \mathfrak{S}_V(A \phi(A)), \mathfrak{I}_V(A \phi(A)) \} \leq \frac{c^{1/2}}{\phi(A)^{1/6}} \right\}. \]

Our assumption \( \phi(A) \leq A^{3/n} \) allows us to apply Propositions 5.2 and 5.3 to conclude that
\[ \mathcal{D}_\phi(A) \ll \frac{1}{\phi(A)^{1/24n}}, \]
which completes the proof of Proposition 2.4. \( \square \)

5.2. The non-Archimedean factor is rarely small. The purpose of this section is to provide the proof of Proposition 5.2. Given \( N, Q \geq 1 \), recall the respective definitions (5.2) and (5.3) of the set \( \mathfrak{R}_N(Q) \) and the quantity \( \sigma(a; Q) \) for \( a \in \mathbb{Z}_{d,n}^{N,d,n} \). We note that given a prime number \( p \) and \( r \geq 1 \), we clearly have
\[ \# \mathfrak{R}_N(p^r) = p^{rN} \left( 1 - \frac{1}{p^N} \right). \] (5.10)

In addition, it follows from the Chinese remainder theorem that for any \( a \in \mathfrak{R}_{N,d,n}(Q) \) we have
\[ \sigma(a; Q) = \prod_{\mathfrak{p} \mid Q} \sigma(a; \mathfrak{p}), \] (5.11)
where the notation \( \mathfrak{p} \mid Q \) means that \( p \) is a prime number dividing \( Q \) and \( r \) is the \( p \)-adic valuation of \( Q \). The following pair of results will allow us to handle situations in which one of the factors \( \sigma(a; p^r) \) is somewhat large despite the fact that \( \sigma(a; Q) \) is assumed to be small. We shall start by estimating the first moment of the quantity \( \sigma(a; p^r) \) as \( a \) runs over the set \( \mathfrak{R}_{N,d,n}(p^r) \).

Lemma 5.4. Let \( d \geq 2 \) and \( n \geq 3 \). Let also \( p \) be a prime number and \( r \geq 1 \). We have
\[ \frac{1}{\# \mathfrak{R}_{N,d,n}(p^r)} \sum_{a \in \mathfrak{R}_{N,d,n}(p^r)} \sigma(a; p^r) = 1 + O \left( \frac{1}{p^{n+1}} \right). \]

Proof. Inverting the order of summation we obtain
\[ \sum_{a \in \mathfrak{R}_{N,d,n}(p^r)} \sigma(a; p^r) = \frac{1}{p^{rn}} \sum_{b \in \mathfrak{R}_{n+1}(p^r)} \# \left\{ a \in \mathfrak{R}_{N,d,n}(p^r) : a \in \Lambda_{d,n}^{(p^r)}(b) \right\}. \]
Recall the definition (2.7) of the lattice \( \Lambda_{d,n}^{(p^r)}(b) \) and observe that if \( p \nmid b \) then \( p \nmid \nu_{d,n}(b) \). We deduce that
\[ \# \left\{ a \in \mathfrak{R}_{N,d,n}(p^r) : a \in \Lambda_{d,n}^{(p^r)}(b) \right\} = p^{r(N_{d,n}-1)} - p^{(r-1)(N_{d,n}-1)} \]
\[ = p^{r(N_{d,n}-1)} \left( 1 - \frac{1}{p^{N_{d,n}-1}} \right). \]
We thus get
\[
\frac{1}{\#\mathcal{R}_{n,d,n}(p^r)} \sum_{a \in \mathcal{R}_{n,d,n}(p^r)} \sigma(a; p^r) = \frac{\#\mathcal{R}_{n+1}(p^r)}{p^{r(n+1)}} \cdot \frac{\#\mathcal{R}_{n,d,n}(p^r)}{\#\mathcal{R}_{n,d,n}(p^r)} \left( 1 - \frac{1}{p^{N_{d,n}-1}} \right).
\]

We see that two applications of the equality (5.10) allow us to complete the proof. \( \Box \)

We now establish an upper bound for the variance of the quantity \( \sigma(a; p^r) \) as \( a \) runs over the set \( \mathcal{R}_{n,d,n}(p^r) \).

**Lemma 5.5.** Let \( d \geq 2 \) and \( n \geq 3 \). Let also \( p \) be a prime number and \( r \geq 1 \). We have
\[
\frac{1}{\#\mathcal{R}_{n,d,n}(p^r)} \sum_{a \in \mathcal{R}_{n,d,n}(p^r)} (\sigma(a; p^r) - 1)^2 \ll \frac{1}{p^{r-1}}.
\]

**Proof.** We start by estimating the second moment of \( \sigma(a; p^r) \) as \( a \) runs over \( \mathcal{R}_{n,d,n}(p^r) \). We have
\[
\sum_{a \in \mathcal{R}_{n,d,n}(p^r)} \sigma(a; p^r)^2 = \frac{1}{p^{2r+1}} \sum_{b_1, b_2 \in \mathcal{R}_{n+1}(p^r)} \#L(b_1, b_2; p^r), \tag{5.12}
\]
where
\[
L(b_1, b_2; p^r) = \left\{ a \in \mathcal{R}_{n,d,n}(p^r) : a \in \Lambda^{(p^r)}_{\nu_{d,n}(b_1)} \cap \Lambda^{(p^r)}_{\nu_{d,n}(b_2)} \right\}.
\]

We first investigate the cardinality of the set \( L(b_1, b_2; p^r) \) under the assumption that there does not exist \( g \in (\mathbb{Z}/p\mathbb{Z})^\times \) such that \( b_1 = gb_2 \). In this case we select any primitive representatives \( c_1, c_2 \in \mathbb{Z}^{n+1} \) of \( b_1 \) and \( b_2 \) respectively, and we note that \( c_1 \) and \( c_2 \) are linearly independent. Appealing to the equality (3.7) we deduce that there exists \( f \in \mathbb{Z} \) such that the cardinality of the set \( L(b_1, b_2; p^r) \) is equal to
\[
\# \left\{ z \in \mathcal{R}_{n,d,n}(p^r) : \begin{bmatrix} 1 & 0 & \cdots & 0 \\ f & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\}.
\]

Using Lemma 3.11 and the fact that \( \gcd(\mathcal{G}(c_1, c_2), p^r) = \gcd(\mathcal{G}(b_1, b_2), p^r) \), we obtain
\[
\#L(b_1, b_2; p^r) = p^{r(N_{d,n}-2)} \gcd(\mathcal{G}(b_1, b_2), p^r) - p^{(r-1)(N_{d,n}-2)} \gcd(\mathcal{G}(b_1, b_2), p^r-1).
\]

In the case where there exists \( g \in (\mathbb{Z}/p\mathbb{Z})^\times \) such that \( b_1 = gb_2 \), we have
\[
\#L(b_1, b_2; p^r) = \# \left\{ a \in \mathcal{R}_{n,d,n}(p^r) : a \in \Lambda^{(p^r)}_{\nu_{d,n}(b_1)} \right\},
\]
and since \( p \nmid b_1 \) we get
\[
\#L(b_1, b_2; p^r) = p^{r(N_{d,n}-1)} - p^{(r-1)(N_{d,n}-1)}.
\]

As a result, we have proved in particular that in both cases we have
\[
\#L(b_1, b_2; p^r) = p^{r(N_{d,n}-2)} \gcd(\mathcal{G}(b_1, b_2), p^r) \left( 1 + O \left( \frac{1}{p^{N_{d,n}-2}} \right) \right). \tag{5.13}
\]

Next, we note that
\[
\sum_{b_1, b_2 \in \mathcal{R}_{n+1}(p^r)} \gcd(\mathcal{G}(b_1, b_2), p^r) = \#\mathcal{R}_{n+1}(p^r) \cdot \#(a^{(p^r)}) + O \left( \sum_{e=1}^{r} p^e \cdot \#(a^{(p^r)}) \right),
\]
where, for \( e \in \{1, \ldots, r\} \), we have introduced the set
\[
(a^{(p^r)}) = \left\{ (b_1, b_2) \in \mathcal{R}_{n+1}(p^r)^2 : \gcd(\mathcal{G}(b_1, b_2), p^r) = p^e \right\}.
\]

Furthermore, for given \( b_1 \in \mathcal{R}_{n+1}(p^r) \), a little thought reveals that the number of \( b_2 \in \mathcal{R}_{n+1}(p^r) \) such that \( (b_1, b_2) \in (a^{(p^r)}) \) is at most \( p^{(n+1)-en} \). This yields
\[
\#(a^{(p^r)}) \ll p^{2r(n+1)-en}.
\]
Using the equality (5.10), we deduce
\[
\sum_{\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{R}_{n+1}(p^r)} \gcd(G(\mathbf{b}_1, \mathbf{b}_2), p^r) = p^{2r(n+1)} \left(1 + O \left(\frac{1}{p^r} \right)\right).
\] (5.14)

Putting together the equality (5.12) and the estimates (5.13) and (5.14), we derive
\[
\sum_{a \in \mathcal{R}_{N_{d,n}}(p^r)} \sigma(a; p^r)^2 = p^{rN_{d,n}} \left(1 + O \left(\frac{1}{p^r} \right)\right).
\]

In view of the equality (5.10), this can eventually be rewritten as
\[
\frac{1}{\# \mathcal{R}_{N_{d,n}}(p^r)} \sum_{a \in \mathcal{R}_{N_{d,n}}(p^r)} \sigma(a; p^r)^2 = 1 + O \left(\frac{1}{p^r} \right).
\] (5.15)

It is now immediate to check that we may complete the proof by combining Lemma 5.4 and the estimate (5.15). □

Given an integer $Q \geq 1$, for any vector $a \in (\mathbb{Z}/Q\mathbb{Z})_{d,n}$ we let $f_a$ denote the form of degree $d$ in $n+1$ variables which has coefficient vector $a$. The set of vectors $a \in \mathcal{R}_{N_{d,n}}(Q)$ such that the form $f_a$ has a non-trivial point modulo $p^{r_Q}$ for any prime divisor $p$ of $Q$ will play a major role in the proof of Proposition 5.2. For $Q \geq 1$ we thus introduce the set
\[
\mathbb{P}_{d,n}^\text{loc}(Q) = \left\{ a \in \mathcal{R}_{N_{d,n}}(Q) : \forall p^r \mid Q \exists x \in \mathcal{R}_{n+1}(p^r) \ f_a(x) \equiv 0 \mod p^r \right\}.
\] (5.16)

In addition, given a prime number $p$ and integers $r, N \geq 1$ we define the $p$-adic valuation $v_p(\mathbf{v})$ of a vector $\mathbf{v} \in (\mathbb{Z}/p^r\mathbb{Z})^N$ as the largest integer $e \in \{0, \ldots, r\}$ such that we have $\mathbf{v} \equiv 0 \mod p^e$. For $e \in \{0, \ldots, r\}$, we also define the set
\[
\mathcal{R}_{d,n}^{(e)}(p^r) = \left\{ a \in \mathcal{R}_{N_{d,n}}(p^r) : \exists x \in \mathcal{R}_{n+1}(p^r) \ f_a(x) \equiv 0 \mod p^r \ \forall p^e(f_a(x)) = e \right\}.
\] (5.17)

It will be very important for our purpose to note that
\[
\mathbb{P}_{d,n}^\text{loc}(p^r) = \bigcup_{e=0}^{r} \mathcal{R}_{N_{d,n}}^{(e)}(p^r).
\] (5.18)

The following two results provide us with the suitable tools to deal with small values of the quantity $\sigma(a; p^r)$ for $a \in \mathcal{R}_{N_{d,n}}(p^r)$. We start by proving an upper bound for the cardinality of the set $\mathcal{R}_{N_{d,n}}^{(e)}(p^r)$.

**Lemma 5.6.** Let $d \geq 2$ and $n \geq 3$. Let also $p$ be a prime number and $r \geq 1$. For $e \in \{1, \ldots, r\}$, we have
\[
\mathcal{R}_{N_{d,n}}^{(e)}(p^r) \leq 2p^{rN_{d,n}-e}.
\]

**Proof.** For $a \in \mathcal{R}_{N_{d,n}}(p^r)$, we let
\[
\mathcal{D}_a(p^r) = \left\{ x \in \mathcal{R}_{n+1}(p^e) : \ f_a(x) \equiv 0 \mod p^e \ \nabla f_a(x) \equiv 0 \mod p^e \right\},
\]
and we observe that
\[
\# \mathcal{R}_{N_{d,n}}^{(e)}(p^r) \leq \# \left\{ a \in \mathcal{R}_{N_{d,n}}(p^r) : \mathcal{D}_a(p^r) \neq \emptyset \right\}.
\]
Moreover these \( p^{r-1}(p-1) \) vectors are distinct. In this way we see that

\[
\# \mathcal{R}_N^{(e)}(p^e) \leq \sum_{a \in \mathcal{R}_N^{(e)}(p^e)} \frac{\# \mathcal{D}_a(p^e)}{p^{r-1}(p-1)}.
\]

We deduce that

\[
\# \mathcal{R}_N^{(e)}(p^r) \leq 2 \frac{2}{p^{d-1}} \sum_{x \in \mathcal{R}_{n+1}(p^r)} \# \left\{ a \in \mathcal{R}_{N,n}(p^r) : f_a(x) \equiv 0 \mod p^e, \nabla f_a(x) \equiv 0 \mod p^e \right\}.
\] (5.19)

Given \( x \in \mathcal{R}_{n+1}(p^r) \) we assume without loss of generality that \( p \nmid x_0 \). For \( a \in \mathcal{R}_{N,n}(p^r) \) and \( i \in \{0, \ldots, n\} \), we let \( c_a(i) \in \mathbb{Z}/p^r\mathbb{Z} \) be the coordinate of \( a \) corresponding to the monomial \( x_0^{d-1}x_i \) and we set

\[
g_a(x) = f_a(x) - x_0^{d-1} \left( c_a(0)x_0 + c_a(1)x_1 + \cdots + c_a(n)x_n \right).
\] (5.20)

Therefore we have

\[
\nabla f_a(x) = \left( \begin{array}{c}
-x_0^{d-2} \left( dc_a(0)x_0 + (d-1) \left( c_a(1)x_1 + \cdots + c_a(n)x_n \right) \right) \\
- \frac{\partial c_a(i)}{\partial x_0} x_0^{d-1} \\
\vdots \\
- \frac{\partial c_a(n)}{\partial x_0} x_0^{d-1}
\end{array} \right) + \nabla g_a(x).
\] (5.21)

Setting \( c_a = (c_a(0), \ldots, c_a(n)) \) and estimating first the number of \( c_a(0) \in \mathbb{Z}/p^r\mathbb{Z} \), we deduce from the assumption \( p \nmid x_0 \) that the cardinality of the set

\[
\left\{ c_a \in (\mathbb{Z}/p^r\mathbb{Z})^{n+1} : \begin{array}{l}
c_a(0)x_0^{d} \equiv -x_0^{d-1} \left( c_a(1)x_1 + \cdots + c_a(n)x_n \right) - g_a(x) \mod p^e \\
c_a(i)x_0^{d-1} \equiv -\frac{\partial g_a}{\partial x_i}(x) \mod p^e, \ i \in \{1, \ldots, n\}
\end{array} \right\}
\]

is equal to \( p^{r(n+1) - 1} \). We thus derive

\[
\# \left\{ a \in \mathcal{R}_{N,n}(p^r) : f_a(x) \equiv 0 \mod p^e, \nabla f_a(x) \equiv 0 \mod p^e \right\} \leq p^{r(N-d,n-1) + r(n+1) - 1}.
\]

Recalling the upper bound (5.19), we see that an application of the trivial inequality \( \# \mathcal{R}_{n+1}(p^r) \leq p^{r(n+1)} \) completes the proof. \( \square \)

We now establish a lower bound for the quantity \( \sigma(a;p^r) \) for \( a \in \mathcal{R}_{N,n}^{(e)}(p^r) \).

**Lemma 5.7.** Let \( d \geq 2 \) and \( n \geq 3 \). Let also \( p \) be a prime number and \( r \geq 1 \). For \( e \in \{0, \ldots, r\} \) and \( a \in \mathcal{R}_{N,n}^{(e)}(p^r) \), we have

\[
\sigma(a;p^r) \geq \frac{1}{p^{(e+1)n}}.
\]

**Proof.** Since by assumption \( a \in \mathcal{R}_{N,n}^{(e)}(p^r) \) we may select \( x \in \mathcal{R}_{n+1}(p^r) \) satisfying the conditions \( f_a(x) \equiv 0 \mod p^e \) and \( v_p(\nabla f_a(x)) = e \). In the case where \( e = r \) the existence of \( x \) implies that \( \sigma(a;p^r) \geq 1/p^rn \), and the desired lower bound follows. We thus assume that \( e \in \{0, \ldots, r-1\} \) and we note that we trivially have

\[
\sigma(a;p^r) \geq \frac{\# \mathcal{R}_a(p^{e+1};p^r)}{p^rn}.
\]
where, for \( c \in \{1, \ldots, r\} \), we have set
\[
\mathcal{C}_{a,x}(p^r; p^r) = \left\{ b \in \mathcal{R}_{n+1}(p^r) : b \equiv x \mod p^r, \quad f_a(b) \equiv 0 \mod p^r \right\}.
\]

We first handle the case where \( e \in \{\lceil r/2 \rceil, \ldots, r - 1\} \). For any \( b \in \mathcal{R}_{n+1}(p^r) \) such that \( b \equiv x \mod p^{r+1} \), we see that we have
\[
f_a(b) \equiv f_a(x) + (\nabla f_a(x), b - x) \mod p^{2e+2}.
\]
Therefore, the conditions \( f_a(x) \equiv 0 \mod p^r \) and \( v_p(\nabla f_a(x)) = e \) together with the facts that \( b \equiv x \mod p^{r+1} \) and \( r \leq 2e + 1 \) imply that \( f_a(b) \equiv 0 \mod p^r \). It follows that
\[
\mathcal{C}_{a,x}(p^{r+1}; p^r) = \left\{ b \in \mathcal{R}_{n+1}(p^r) : b \equiv x \mod p^{r+1} \right\}.
\]
Hence we have \( \#\mathcal{C}_{a,x}(p^{r+1}; p^r) = p^{r(e-1)(n+1)} \) and thus
\[
\sigma(a; p^r) \geq \frac{p^{r-e-1}}{p^{(e+1)n}},
\]
which is satisfactory since \( r - e - 1 \geq 0 \).

Finally, in the case where \( e \in \{0, \ldots, \lceil r/2 \rceil - 1\} \) we use Hensel’s lemma (in the form of \( [6, \text{Lemma 3.3}] \)) with \( \#\mathcal{C}_{a,x}(p^{r+1}; p^s) = \#\mathcal{R}_e(p^s, 0; p^{r+1}) \) for example) to deduce that for any \( s \geq 2e + 2 \), we have
\[
\frac{\#\mathcal{C}_{a,x}(p^{r+1}; p^s)}{p^{sn}} = \frac{\#\mathcal{C}_{a,x}(p^{r+1}; p^{s-1})}{p^{(s-1)n}}.
\]
Applying this equality \( r - (2e + 1) \) times, we derive
\[
\sigma(a; p^r) \geq \frac{\#\mathcal{C}_{a,x}(p^{r+1}; p^{2e+1})}{p^{(2e+1)n}},
\]
which completes the proof since we have \( \#\mathcal{C}_{a,x}(p^{r+1}; p^{2e+1}) = p^{r(n+1)} \) by the previous case.

We now have the tools at hand to establish Proposition 5.2.

**Proof of Proposition 5.2.** Recall the respective definitions (5.5), (2.9) and (2.10) of the non-Archimedean factor \( \mathfrak{S}_V(B) \), the integer \( W \) and the quantity \( w \), where we take \( B = A\phi(A) \). It is convenient to set
\[
\mathcal{F}_\phi(A) = \frac{1}{\#\mathfrak{V}_{d,n}^\text{loc}(A)} \cdot \# \left\{ V \in \mathfrak{V}_{d,n}^\text{loc}(A) : \mathfrak{S}_V(A\phi(A)) < \frac{C}{\phi(A)^{1/6}} \right\}.
\]
Recall also the definition (5.16) of the set \( \mathfrak{V}_{d,n}^\text{loc}(Q) \) for given \( Q \geq 1 \). Breaking the summation over \( a \in \mathcal{R}_{N_{d,n}}(W) \) into residue classes modulo \( W \), we see that
\[
\mathcal{F}_\phi(A) = \frac{1}{\#\mathfrak{V}_{d,n}^\text{loc}(A)} \sum_{a \in \mathfrak{V}_{d,n}^\text{loc}(W)} \# \left\{ V \in \mathfrak{V}_{d,n}^\text{loc}(A) : a_V \equiv a \mod W \right\}.
\]
The upper bound (2.12) and the assumption \( \phi(A) \leq A \) ensure that \( W \ll A \). We thus have
\[
\# \left\{ V \in \mathfrak{V}_{d,n}^\text{loc}(A) : a_V \equiv a \mod W \right\} \ll \left( \frac{A}{W} \right)^{N_{d,n}}.
\]
Since the lower bound (1.3) implies in particular that \( \#\mathfrak{V}_{d,n}^\text{loc}(A) \gg A^{N_{d,n}} \), we deduce
\[
\mathcal{F}_\phi(A) \ll \frac{1}{W^{N_{d,n}}} \cdot \# \left\{ a \in \mathfrak{V}_{d,n}^\text{loc}(W) : \sigma(a; W) < \frac{C}{\phi(A)^{1/6}} \right\}.
\]
In addition, we remark that it follows from the equality (5.11) that \( \sigma(a;W) > 0 \) whenever \( a \in \mathbb{P}_{d,n}^{\text{loc}}(W) \). We now let \( \kappa \in (0,1/n) \) and we use the standard trick
\[
\# \left\{ a \in \mathbb{P}_{d,n}^{\text{loc}}(W) : \sigma(a;W) < \frac{C}{\phi(A)^{1/\sigma}} \right\} \leq \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(W)} \left( \frac{C}{\phi(A)^{1/\sigma}} \right)^\kappa.
\]
Therefore, we deduce from the equality (5.11) that
\[
\mathcal{F}_\phi(A) \ll \frac{1}{W^{N_d,n}} \phi(A)^{\kappa/6} \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(W)} \prod_{p \mid W} \frac{1}{\sigma(a;p)^\kappa}.
\]
As a result, an application of the Chinese remainder theorem yields
\[
\mathcal{F}_\phi(A) \ll \frac{1}{W^{N_d,n}} \phi(A)^{\kappa/6} \prod_{p \mid W} \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} \frac{1}{\sigma(a;p)^\kappa}.
\]
In order to estimate the sum over \( a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r) \) we need to argue differently depending on whether or not \( \sigma(a;p^r) \) is particularly small. We thus let
\[
\Sigma_{>}(p^r) = \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} \frac{1}{\sigma(a;p^r)^\kappa},
\]
and
\[
\Sigma_{\leq}(p^r) = \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} \frac{1}{\sigma(a;p^r)^\kappa},
\]
so that
\[
\mathcal{F}_\phi(A) \ll \frac{1}{W^{N_d,n}} \phi(A)^{\kappa/6} \prod_{p \mid W} \left( \Sigma_{>}(p^r) + \Sigma_{\leq}(p^r) \right)^r . \tag{5.22}
\]
We start by handling the sum \( \Sigma_{>}(p^r) \). In order to do so, we employ the estimate
\[
\frac{1}{\sigma(a;p^r)^\kappa} = 1 - \kappa \left( \sigma(a;p^r) - 1 \right) + O \left( \left( \sigma(a;p^r) - 1 \right)^2 \right),
\]
where the implied constant depends at most on \( \kappa \). We obtain
\[
\Sigma_{>}(p^r) = \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} 1 - \kappa \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} \left( \sigma(a;p^r) - 1 \right) + O \left( \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} \left( \sigma(a;p^r) - 1 \right)^2 \right)
\]
\[
\leq \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} 1 - \kappa \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} \left( \sigma(a;p^r) - 1 \right) + O \left( \sum_{a \in \mathbb{P}_{d,n}^{\text{loc}}(p^r)} \left( \sigma(a;p^r) - 1 \right)^2 \right).
\]
On appealing to Lemmas 5.4 and 5.5 and to the equality (5.10), we therefore conclude that
\[
\Sigma_{>}(p^r) \leq p^r N_d,n \left( 1 + \frac{1}{p^{\kappa-1}} \right). \tag{5.23}
\]
We now consider the sum \( \Sigma_{\leq}(p^r) \). Recall the definition (5.17) of the set \( \mathcal{R}^{(e)}_{N_d,n}(p^r) \) for given \( e \in \{0, \ldots, r\} \). It follows from the equality (5.18) that
\[
\Sigma_{\leq}(p^r) \leq \sum_{e=0}^r \mathcal{S}^{(e)}(e;p^r), \tag{5.24}
\]
where, for \( e \in \{0, \ldots, r\} \), we have set

\[
S^{(\kappa)}(e; p^r) = \sum_{a \in \mathfrak{N}_{d,n}^{(e)}(p^r) \atop \sigma(a; p^r) \leq 1/2} \frac{1}{\sigma(a; p^r)^{\kappa}}.
\]

We first handle the case where \( e \in \{0, 1\} \). Applying Lemma 5.7, we see that

\[
S^{(\kappa)}(e; p^r) \leq p^{r(e+1)n} \sum_{a \in \mathfrak{N}_{d,n}^{(e)}(p^r) \atop \sigma(a; p^r) \leq 1/2} 1.
\]

Using the fact that \( 1 \leq 4(\sigma(a; p^r) - 1)^2 \) whenever \( \sigma(a; p^r) \leq 1/2 \), we get

\[
S^{(\kappa)}(e; p^r) \leq 4p^{r(e+1)n} \sum_{a \in \mathfrak{N}_{d,n}^{(e)}(p^r)} (\sigma(a; p^r) - 1)^2.
\]

Therefore, Lemma 5.5 gives

\[
S^{(\kappa)}(e; p^r) \ll p^{rN_{d,n} - n + 1 + \kappa(e+1)n},
\]

from which it eventually follows that

\[
S^{(\kappa)}(0; p^r) + S^{(\kappa)}(1; p^r) \ll p^{rN_{d,n} - n + 2 + 3\kappa n}. \quad (5.25)
\]

We now treat the case where \( e \in \{2, \ldots, r\} \). Dropping the condition \( \sigma(a; p^r) \leq 1/2 \) and applying Lemma 5.7 we obtain

\[
S^{(\kappa)}(e; p^r) \leq p^{r(e+1)n} \# \mathfrak{N}_{d,n}^{(e)}(p^r).
\]

Since \( \kappa < 1/n \), we deduce from Lemma 5.6 that

\[
\sum_{e=2}^{r} S^{(\kappa)}(e; p^r) \ll p^{rN_{d,n} - 2 + 3\kappa n}. \quad (5.26)
\]

Combining the inequality (5.24) and the upper bounds (5.25) and (5.26), we see that

\[
\Sigma^{(\kappa)}(p^r) \ll p^{rN_{d,n} \left( \frac{1}{p^{n-1-2\kappa n}} + \frac{1}{p^{2-3\kappa n}} \right)}.
\]

Moreover, we have \( n \geq 3 \) by assumption so the choice \( \kappa = 1/4n \) yields

\[
\Sigma^{(\kappa)}(p^r) \ll p^{rN_{d,n} - 5/4}. \quad (5.27)
\]

Putting together the upper bounds (5.22), (5.23) and (5.27), we see that we have proved that

\[
\mathcal{F}_\phi(A) \ll \frac{1}{\phi(A)^{1/24n}} \prod_{p^r \mid W} \left( 1 + O \left( \frac{1}{p^{5/4}} \right) \right).
\]

The product in the right-hand side is convergent so we see that this completes the proof of Proposition 5.2. \( \square \)
5.3. The Archimedean factor is rarely small. Our goal in this section is to prove Proposition 5.3. We shall follow the traces of our argument in the non-Archimedean setting. To begin with, we recall that the purpose of Lemmas 5.4 and 5.5 was to allow us to handle several non-Archimedean places simultaneously, which is of course irrelevant here so we will not need analogues of these results. However, we will establish direct analogues of Lemmas 5.6 and 5.7, and we will also need some preparatory work in order to apply these results.

For any vector $a \in \mathbb{R}^{N_{d,n}}$ we let $f_a$ denote the form of degree $d$ in $n+1$ variables which has coefficient vector $a$. The set of non-zero vectors $a \in \mathbb{R}^{N_{d,n}}$ such that the form $f_a$ has a non-trivial real point will be of primary importance in the proof of Proposition 5.3. We thus define the set

$$\mathcal{U}_{d,n}(A) = \{a \in \mathbb{R}^{N_{d,n}} \cap \{0\} : \exists x \in \mathbb{S}^n f_a(x) = 0\},$$

where, for given $N \geq 1$, we have introduced the $N$-dimensional hypersphere

$$\mathbb{S}^N = \{x \in \mathbb{R}^{N+1} : ||x|| = 1\}.$$

Our first task will be to establish an upper bound for the number of integral vectors $a \in \mathbb{Z}^{N_{d,n}}$ having norm at most $A$ and lying close to the boundary of the region $\mathcal{U}_{d,n}$. In order to do so, for $a \in \mathbb{Z}^{N_{d,n}}$ we define the neighbourhood

$$N(a) = \{y \in \mathbb{R}^{N_{d,n}} : y - a \in B_{N_{d,n}}(1)\},$$

and we set

$$\mathcal{U}_{d,n}(A) = \{a \in \mathbb{Z}^{N_{d,n}} \cap B_{N_{d,n}}(A) \cap \|\cdot\|_{d,n} : N(a) \not\subset \mathcal{U}_{d,n}\}.$$

Heuristically, given an integer $N \geq 1$ and a real hypersurface embedded in $\mathbb{R}^N$ it is natural to expect that the number of integral vectors of norm at most $A$ and whose distance to the hypersurface is at most 1 should have order of magnitude $A^{N-1}$. The following result shows that these elementary heuristics apply in our setting.

Lemma 5.8. Let $d \geq 2$ and $n \geq 3$. We have

$$\#\mathcal{U}_{d,n}(A) \ll A^{N_{d,n}-1}.$$  

Proof. Given $a \in \mathcal{U}_{d,n}(A)$ we let $b \in N(a) \setminus \mathcal{U}_{d,n}$ and we define

$$M_a = \max\{t \in (0,1] : a + t(b-a) \in \mathcal{U}_{d,n}\}.$$

We also set $c = a + M_a(b-a)$ and we check that for any $x \in \mathbb{S}^n$ satisfying $f_c(x) = 0$ we have $\nabla f_c(x) = 0$. Indeed, for $\rho \in (0,1/A^2)$ and $y \in B_{n+1}(\rho)$ we have

$$f_{c+\rho^2(b-a)}(x+y) = f_{c+\rho^2(b-a)}(x) + (\nabla f_{c+\rho^2(b-a)}(x), y) + O(\rho^3)$$

$$= f_c(x) + (\nabla f_c(x), y) + O(\rho^3)$$

$$= (\nabla f_c(x), y) + O(\rho^3).$$

Let us assume that $\nabla f_c(x) \neq 0$ and let $y_0 \in \mathbb{S}^n$ satisfying $(\nabla f_c(x), y_0) \neq 0$. For $|u| \leq \rho$ we thus have

$$f_{c+\rho^2(b-a)}(x + uy_0) = u \cdot (\nabla f_c(x), y_0) + O(\rho^3).$$

We now see that if $\rho$ is chosen sufficiently small then the intermediate value theorem shows that there exists $u_0 \in \mathbb{R}$ such that $f_{c+\rho^2(b-a)}(x + u_0y_0) = 0$, which contradicts the maximality of $M_a$. We have thus proved that

$$\#\mathcal{U}_{d,n}(A) \ll \# \{a \in \mathbb{Z}^{N_{d,n}} \cap B_{N_{d,n}}(A) : \exists c \in N(a) \exists x \in \mathbb{S}^n \nabla f_c(x) = 0\}.$$
Next, we note that given \( a \in B_{N_{d,n}}(A) \), if \( c \in N(a) \) and \( x \in \mathbb{S}^n \) satisfy \( \nabla f_c(x) = 0 \) then for any \( y \in \mathbb{R}^{n+1} \) such that \( ||y - x|| \leq 1/A \) we have \( ||\nabla f_a(y)|| \ll 1 \). Indeed, the triangle inequality gives

\[
||\nabla f_a(y)|| \leq ||\nabla f_{a-c}(y)|| + ||\nabla f_c(y) - \nabla f_c(x)|| + ||\nabla f_c(x)||
\]

\[
\ll ||a-c|| \cdot ||y||^{d-1} + ||c|| \cdot ||y - x|| \cdot \max(||x||, ||y||)^{d-2}.
\]

We thus get \( ||\nabla f_a(y)|| \ll 1 \) as wished. Since we have in addition

\[
\text{vol} \left( \left\{ y \in \mathbb{R}^{n+1} : ||y - x|| \leq \frac{1}{A} \right\} \right) \gg \frac{1}{A^{n+1}},
\]

it follows that

\[
\# \mathcal{U}_{d,n}(A) \ll A^{n+1} \sum_{a \in \mathbb{Z}^{N_{d,n}} \cap B_{N_{d,n}}(A)} \text{vol} \left( \left\{ y \in \mathbb{R}^{n+1} : 1 - 1/A \leq ||y|| \leq 1 + 1/A \right\} \right)
\]

\[
\ll A^{n+1} \int_{\mathcal{H}_{n+1}(A)} \# \left\{ a \in \mathbb{Z}^{N_{d,n}} \cap B_{N_{d,n}}(A) : ||\nabla f_a(y)|| \ll 1 \right\} dy,
\]

(5.29)

where we have introduced the hyperspherical shell

\[
\mathcal{H}_{n+1}(A) = B_{n+1} \left( 1 + \frac{1}{A} \right) \setminus B_{n+1} \left( 1 - \frac{1}{A} \right).
\]

Given \( y \in \mathcal{H}_{n+1}(A) \) we may clearly assume without loss of generality that \( |y_0| \geq 1/n \). For \( a \in \mathbb{Z}^{N_{d,n}} \cap B_{N_{d,n}}(A) \) and \( i \in \{0, \ldots, n\} \), we let \( c_a^{(i)} \in \mathbb{Z} \) be the coordinate of \( a \) corresponding to the monomial \( x_0^{d-1} x_i \). Recall the definition (5.20) of \( g_a \) and the equality (5.21). Setting \( c_a = \left( c_a^{(0)}, \ldots, c_a^{(n)} \right) \) and estimating first the number of \( c_a^{(0)} \in \mathbb{Z} \), we deduce from the assumption \( |y_0| \geq 1/n \) that

\[
\# \left\{ c_a \in \mathbb{Z}^{n+1} : \right. \begin{aligned}
y_0^{d-2} \left( d c_a^{(0)} y_0 + (d-1) \left( c_a^{(1)} y_1 + \cdots + c_a^{(n)} y_n \right) \right) + \frac{\partial g_a}{\partial x_0}(y) &\ll 1 \\
c_a^{(i)} y_0^{d-1} + \frac{\partial g_a}{\partial x_i}(y) &\ll 1, \quad i \in \{1, \ldots, n\}
\end{aligned} \ll 1.
\]

This yields

\[
\# \left\{ a \in \mathbb{Z}^{N_{d,n}} \cap B_{N_{d,n}}(A) : ||\nabla f_a(y)|| \ll 1 \right\} \ll A^{N_{d,n}-n-1}.
\]

Recalling the upper bound (5.29) and noting that we clearly have

\[
\text{vol} \left( \mathcal{H}_{n+1}(A) \right) \ll \frac{1}{A},
\]

we see that this finishes the proof. \( \square \)

For \( \lambda > 0 \), we introduce the set

\[
B_{N_{d,n}}^{(\lambda)} = \left\{ a \in B_{N_{d,n}}(1) : \exists x \in \mathbb{S}^n \text{ with } f_a(x) = 0 \lambda ||a|| < ||\nabla f_a(x)|| \leq 2\lambda ||a|| \right\}.
\]

(5.30)

It will be crucial for our purpose to note that

\[
B_{N_{d,n}}(1) \cap n_{\text{loc}} = \bigcup_{\ell=1}^{\infty} B_{N_{d,n}}^{(M_{d,n}/2^\ell)},
\]

(5.31)

where we have set

\[
M_{d,n} = \max \left\{ ||\nabla f_a(x)|| : (a, x) \in \mathbb{S}^{N_{d,n}-1} \times \mathbb{S}^n \right\}.
\]

(5.32)

The following result is the Archimedean analogue of Lemma 5.6 and gives an upper bound for the volume of the set \( B_{N_{d,n}}^{(\lambda)}. \)
Lemma 5.9. Let $d \geq 2$ and $n \geq 3$. For $\lambda \in (0, M_{d,n})$, we have
\[ \text{vol} \left( \mathcal{B}_{N,d,n}^{(\lambda)} \right) \ll \lambda^2. \]

Proof. For $a \in B_{N,d,n}(1)$ we let
\[ \mathcal{D}_a(\lambda) = \left\{ x \in \mathbb{S}^n : |f_a(x)| \leq \lambda^2, \|\nabla f_a(x)\| \leq 2\lambda \right\}, \]
and we observe that
\[ \text{vol} \left( \mathcal{B}_{N,d,n}^{(\lambda)} \right) \leq \text{vol} \left( \{ a \in B_{N,d,n}(1) : \mathcal{D}_a(\lambda) \neq \emptyset \} \right). \]

Given $x \in \mathcal{D}_a(\lambda/2)$, it follows from the estimates
\[ f_a(y) = f_a(x) + \langle \nabla f_a(x), y - x \rangle + O \left( \|y - x\|^2 \right), \]
and
\[ \|\nabla f_a(y)\| = \|\nabla f_a(x)\| + O \left( \|y - x\| \right), \]
that there exists an absolute constant $K > 0$ such that if $y \in \mathbb{S}^n$ and $\|x - y\| \leq K\lambda$ then $y \in \mathcal{D}_a(\lambda)$. Since we have
\[ \text{vol} \left( \{ y \in \mathbb{S}^n : \|x - y\| \leq K\lambda \} \right) \gg \lambda^n, \]
we deduce that
\[ \text{vol} \left( \mathcal{B}_{N,d,n}^{(\lambda)} \right) \ll \int_{B_{N,d,n}(1)} \frac{\text{vol} (\mathcal{D}_a(\lambda))}{\lambda^n} \, da. \]

Therefore we have
\[ \text{vol} \left( \mathcal{B}_{N,d,n}^{(\lambda)} \right) \ll \frac{1}{\lambda^n} \int_{\mathbb{S}^n} \text{vol} \left( \left\{ a \in B_{N,d,n}(1) : |f_a(x)| \leq \lambda^2, \|\nabla f_a(x)\| \leq 2\lambda \right\} \right) \, dx. \tag{5.33} \]

Given $x \in \mathbb{S}^n$ we may clearly assume without loss of generality that $|x_0| \geq 1/n$. For $a \in B_{N,d,n}(1)$ and $i \in \{0, \ldots, n\}$, we let $c_a(i) \in \mathbb{R}$ be the coordinate of $a$ corresponding to the monomial $x_0^{d-1}x_i$. Recall the definition (5.20) of $g_a$ and the equality (5.21). Setting $c_a = (c_a(0), \ldots, c_a(n))$ and estimating first the number of $c_a(0) \in \mathbb{R}$, we deduce from the assumption $|x_0| \geq 1/n$ that
\[ \text{vol} \left( \left\{ c_a \in \mathbb{R}^{n+1} : \left| c_a(0)^d + x_0^{d-1} (c_a(1)x_1 + \cdots + c_a(n)x_n) + g_a(x) \right| \leq \lambda^2 \right\} \right) \ll \lambda^{n+2}. \]

It follows that
\[ \text{vol} \left( \left\{ a \in B_{N,d,n}(1) : |f_a(x)| \leq \lambda^2, \|\nabla f_a(x)\| \leq 2\lambda \right\} \right) \ll \lambda^{n+2}. \]

Recalling the equality (5.33) we see that this completes the proof. \hfill \Box

Given $N \geq 1$, recall the definition (5.1) of the quantity $\tau(a; \gamma)$ for $a \in \mathbb{R}^{N,d,n}$ and $\gamma > 0$. The following result is the Archimedean analogue of Lemma 5.7 and provides us with a lower bound for the quantity $\tau(a; \gamma)$ for $a \in \mathcal{B}_{N,d,n}^{(\lambda)}$.

Lemma 5.10. Let $d \geq 2$ and $n \geq 3$. Let also $\gamma > 0$. For $\lambda \in (0, M_{d,n})$ and $a \in \mathcal{B}_{N,d,n}^{(\lambda)}$, we have
\[ \tau(a; \gamma) \gg \lambda^{n+1} \cdot \min \left\{ \gamma, \frac{1}{\lambda^2} \right\}. \]
Proof. We may assume without loss of generality that \(|a| = 1\). In addition, since by assumption \(a \in B_{N_d, n}^{(\lambda)}\) we may select \(x \in \mathbb{R}^{n+1}\) such that \(|x| = 1/2\) and satisfying the conditions \(f_a(x) = 0\) and

\[
\frac{\lambda}{2^d} < ||\nabla f_a(x)|| \leq \frac{\lambda}{2^{d-1}}.
\]

We have

\[
\tau(a; \gamma) = \gamma \cdot \text{vol}\left(\left\{ u \in B_{n+1}(1) : |f_a(u)| \leq \frac{||u_d,n(u)||}{2\gamma}\right\}\right).
\]

Since \(||u_d,n(u)|| \geq ||u||^d\), we see that

\[
\tau(a; \gamma) \geq \gamma \cdot \text{vol}\left(\left\{ u \in B_{n+1}(1) \setminus B_{n+1}\left(\frac{1}{4}\right) : |f_a(u)| \leq \frac{1}{2^{d+1}\gamma}\right\}\right).
\]

We note that if \(||u - x|| \leq \lambda/4M_{d,n}\) then \(1/4 < ||u|| < 3/4\) and thus

\[
\tau(a; \gamma) \geq \gamma \cdot \text{vol}\left(\left\{ v \in B_{n+1}\left(\frac{\lambda}{4M_{d,n}}\right) : |f_a(x + v)| \leq \frac{1}{2^{2d+1}\gamma}\right\}\right).
\]

It follows that

\[
\tau(a; \gamma) \geq \gamma \int_{B_n(\lambda/8M_{d,n})} W_{a,x}(w; \lambda, \gamma)dw,
\]

where \(w = (v_1, \ldots, v_n)\) and

\[
W_{a,x}(w; \lambda, \gamma) = \text{vol}\left(\left\{ v_0 \in [-\lambda/8M_{d,n}, \lambda/8M_{d,n}] : |f_a(x + (v_0, w))| \leq \frac{1}{2^{2d+1}\gamma}\right\}\right).
\]

The inequalities (5.34) imply that we can clearly assume without loss of generality that

\[
\frac{\lambda}{2^d} < \left|\frac{\partial f_a}{\partial x_0}(x)\right| \leq \frac{\lambda}{2^{d-1}}.
\]

We can thus make the change of variables

\[
v_0 = -\left(\frac{\partial f_a}{\partial x_0}(x)\right)^{-1}\left(\frac{\partial f_a}{\partial x_1}(x)v_1 + \cdots + \frac{\partial f_a}{\partial x_n}(x)v_n - w_0\right).
\]

The upper bound (5.34) shows that there exists an absolute constant \(L > 0\) such that if \(w \in B_n(L\lambda)\) and \(|w_0| \leq L\lambda^2\) then \(|v_0| \leq \lambda/8M_{d,n}\). Using the upper bound (5.36) and the assumption \(f_a(x) = 0\) we thus deduce that for \(w \in B_n(L\lambda)\), we have

\[
W_{a,x}(w; \lambda, \gamma) \geq \frac{1}{\lambda} \cdot \text{vol}\left(\left\{ w_0 \in [-L\lambda^2, L\lambda^2] : |w_0 + P_{a,x}(w_0, w)| \leq \frac{1}{2^{2d+1}\gamma}\right\}\right),
\]

where \(P_{a,x}(w_0, w)\) is a polynomial free of constant and linear terms in \((w_0, w)\). As a result, if \(L > 0\) is chosen sufficiently small and if \(w \in B_n(L\lambda)\) then for \(w_0 \in [L\lambda^2/2, L\lambda^2]\) we have

\[
w_0 + P_{a,x}(w_0, w) \geq \frac{w_0}{2},
\]

and for \(w_0 \in [-L\lambda^2, -L\lambda^2/2]\) we have

\[
w_0 + P_{a,x}(w_0, w) \leq -\frac{w_0}{2}.
\]

Given \(w \in B_n(L\lambda)\), it follows from the intermediate value theorem that there exists \(w_{a,x}(w) \in (-L\lambda^2/2, L\lambda^2/2)\) such that

\[
\omega_{a,x}(w) + P_{a,x}(\omega_{a,x}(w), w) = 0.
\]

In addition, if \(L > 0\) is small enough then for \(w \in B_n(L\lambda)\) and \(|w_0| \leq L\lambda^2/2\) we have

\[
\left|\frac{\partial P_{a,x}}{\partial w_0}(w_0, w)\right| \leq 1,
\]
so the mean value inequality gives

$$|w_0 + P_{a,x}(w_0, w)| \leq 2|w_0 - \omega_{a,x}(w)|.$$  

We have thus proved that if $L > 0$ is sufficiently small then for $w \in B_n(L\lambda)$ we have

$$W_{a,x}(w; \lambda, \gamma) \gg \frac{1}{\lambda} \cdot \text{vol } \left( \left\{ w_0 \in [-L\lambda^2/2, L\lambda^2/2] : |w_0 - \omega_{a,x}(w)| \leq \frac{1}{2^{2d+2} \gamma} \right\} \right)$$

$$\gg \frac{1}{\lambda} \cdot \min \left\{ \lambda^2, \frac{1}{\gamma} \right\}.$$  

Recalling the lower bound (5.35) we eventually derive

$$\tau(a; \gamma) \gg \frac{2}{\lambda} \cdot \min \left\{ \lambda^2, \frac{1}{\gamma} \right\} \cdot \text{vol } (B_n(L\lambda)),$$

which completes the proof. $\square$

We are now ready to establish Proposition 5.3.

**Proof of Proposition 5.3.** We follow closely the lines of the proof of Proposition 5.2. Recall the respective definitions (5.4) and (2.8) of the Archimedean factor $\mathfrak{J}_V(B)$ and the quantity $\alpha$, where we take $B = A\phi(A)$. It is convenient to set

$$\mathcal{J}_\phi(A) = \frac{1}{\# \mathcal{Y}_{d,n}(A)} \cdot \# \left\{ V \in \mathcal{V}_{d,n}(A) : \mathfrak{J}_V(A\phi(A)) < \frac{C}{\phi(A)^{1/6}} \right\}.$$  

Recall also the definition (5.28) of the set $\mathcal{Y}_{d,n}$. The lower bound (1.3) yields in particular $\# \mathcal{Y}_{d,n} \gg A^{\mathcal{N}_{d,n}}$. so we see that

$$\mathcal{J}_\phi(A) \ll \frac{1}{A^{\mathcal{N}_{d,n}} \cdot \# \left\{ a \in \mathbb{Z}^{\mathcal{N}_{d,n}} \cap B_{\mathcal{N}_{d,n}}(A) \cap \mathcal{Y}_{d,n} : \tau(a; \alpha) < \frac{C}{\phi(A)^{1/6}} \right\}}.$$  

It follows from Lemma 5.8 that

$$\mathcal{J}_\phi(A) \ll \frac{1}{A^{\mathcal{N}_{d,n}}} \cdot \# \left\{ a \in \mathbb{Z}^{\mathcal{N}_{d,n}} \cap B_{\mathcal{N}_{d,n}}(A) : \mathcal{N}(a) \subset \mathcal{Y}_{d,n} \right\}$$

$$\frac{\mathcal{N}(a) \subset \mathcal{Y}_{d,n}}{\tau(a; \alpha) < \frac{C}{\phi(A)^{1/6}}} \frac{1}{A}. \quad (5.37)$$

We now show that if $a \in \mathbb{R}^{\mathcal{N}_{d,n}}$ satisfies $\|a\| \geq 8\alpha$ then for any $y \in \mathcal{N}(a)$, we have

$$\tau(y; 2\alpha) \leq \frac{2}{\alpha} \cdot \tau(a; \alpha). \quad (5.38)$$

Let $u \in B_{n+1}(1)$ be such that $y \in \mathcal{C}_{\nu_{d,n}(u)}^{(2\alpha)}$, that is

$$\|\nu_{d,n}(u), y\| \leq \|\nu_{d,n}(u), y\| \leq \frac{\|\nu_{d,n}(u), y\|}{4\alpha}.$$  

Since $y \in \mathcal{N}(a)$ the Cauchy–Schwarz inequality gives $|\nu_{d,n}(u), y - a\| \leq \|\nu_{d,n}(u)\|$. We thus see that

$$|\nu_{d,n}(u), a\| \leq \frac{\|\nu_{d,n}(u), y\|}{4\alpha} + \|\nu_{d,n}(u)\|.$$  

We have assumed that $\|a\| \geq 8\alpha$ so we have in particular $\|y\| \leq 3\|a\|/2$. We deduce that

$$|\nu_{d,n}(u), a\| \leq \frac{3\|\nu_{d,n}(u)\| \cdot \|a\|}{8\alpha} + \|\nu_{d,n}(u)\|.$$  

Our assumption $\|a\| \geq 8\alpha$ now yields

$$|\nu_{d,n}(u), a\| \leq \frac{\|\nu_{d,n}(u)\| \cdot \|a\|}{2\alpha},$$

which shows that $a \in \mathcal{C}_{\nu_{d,n}(u)}^{(\alpha)}$, and the upper bound (5.38) follows.
Recalling the upper bound (5.37) we note that the inequality (5.38) gives
\[
\mathcal{I}_\phi(A) \ll \frac{1}{A^{Nd,n}} \cdot \# \left\{ a \in \mathbb{Z}^{Nd,n} \cap (B_{Nd,n}(A) \setminus B_{Nd,n}(8\alpha)) : \begin{aligned}
&N(a) \subseteq I_{\text{loc}}^{d,n} \\
&\tau(a;\alpha) < \frac{C}{\phi(A)^{1/6}} 
\end{aligned} \right\} + \frac{1}{A}
\ll \frac{1}{A^{Nd,n}} \sum_{a \in \mathbb{Z}^{Nd,n} \cap B_{Nd,n}(A)} \text{vol} \left( \left\{ y \in N(a) \cap I_{\text{loc}}^{d,n} : \tau(y;2\alpha) < \frac{2C}{\phi(A)^{1/6}} \right\} \right) + \frac{1}{A}.
\]
Swapping the summation over \( a \) and the integration over \( y \) we obtain
\[
\mathcal{I}_\phi(A) \ll \frac{1}{A^{Nd,n}} \cdot \text{vol} \left( \left\{ y \in B_{Nd,n}(A + 1) \cap I_{\text{loc}}^{d,n} : \tau(y;2\alpha) < \frac{2C}{\phi(A)^{1/6}} \right\} \right) + \frac{1}{A}
\ll \text{vol} \left( \left\{ y \in B_{Nd,n}(1) \cap I_{\text{loc}}^{d,n} : \tau(y;2\alpha) < \frac{2C}{\phi(A)^{1/6}} \right\} \right) + \frac{1}{A}.
\]
We are now in position to make use of a trick analogous to the one used in the non-Archimedean setting. It is clear that \( \tau(y;2\alpha) > 0 \) whenever \( y \in I_{\text{loc}}^{d,n} \). We let \( \kappa \in (0,2/n) \) and we write
\[
\mathcal{I}_\phi(A) \ll \frac{1}{\phi(A)^{\kappa/6}} \int_{B_{Nd,n}(1) \cap I_{\text{loc}}^{d,n}} \frac{dy}{\tau(y;2\alpha)^\kappa} + \frac{1}{A}.
\]
Recall the respective definitions (5.30) and (5.32) of the set \( B_{Nd,n}^{(\lambda)} \), for given \( \lambda > 0 \) and the quantity \( M_{d,n} \). We deduce from the equality (5.31) that
\[
\int_{B_{Nd,n}(1) \cap I_{\text{loc}}^{d,n}} \frac{dy}{\tau(y;2\alpha)^\kappa} \leq \sum_{\ell=1}^{\infty} \int_{B_{Nd,n}(\ell^2)} \frac{dy}{\tau(y;2\alpha)^\kappa}.
\]
It thus follows from Lemma 5.10 that
\[
\int_{B_{Nd,n}(1) \cap I_{\text{loc}}^{d,n}} \frac{dy}{\tau(y;2\alpha)^\kappa} \ll \sum_{\ell=1}^{\infty} \left( 2^{\ell(n+1)} + 2^{\ell(n-1)} \right)^\kappa \text{vol}(B_{Nd,n}^{(\ell^2)}).
\]
Appealing to Lemma 5.9 and taking \( \kappa = 1/n \), we obtain
\[
\int_{B_{Nd,n}(1) \cap I_{\text{loc}}^{d,n}} \frac{dy}{\tau(y;2\alpha)^\kappa} \ll \frac{1}{1/n} \sum_{\ell=1}^{\infty} \frac{1}{2^{\ell(n-1)/n}} + \sum_{\ell=1}^{\infty} \frac{1}{2^{\ell(n+1)/n}}.
\]
Recalling the upper bound (5.39), we therefore conclude that
\[
\mathcal{I}_\phi(A) \ll \frac{1}{\phi(A)^{1/6n}} + \frac{1}{A}.
\]
But by assumption we have \( \phi(A) \leq A \) and we thus see that this completes the proof of Proposition 5.3. \( \square \)

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