Counterexamples to the comparison principle in the special Lagrangian potential equation

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Abstract. For each \( k = 0, \ldots, n \) we construct a continuous phase \( f_k \), with \( f_k(0) = (n - 2k)\frac{\pi}{2} \), and viscosity sub- and supersolutions \( v_k, u_k \), of the elliptic PDE \( \sum_{i=1}^{n} \arctan(\lambda_i(\mathcal{H}w)) = f_k(x) \) such that \( v_k - u_k \) has an isolated maximum at the origin.

It has been an open question whether the comparison principle would hold in this second order equation for arbitrary continuous phases \( f: \mathbb{R}^n \supseteq \Omega \to (-n\pi/2, n\pi/2) \). Our examples show it does not.

1. Introduction

The special Lagrangian potential operator is the mapping \( F: S^n \to \mathbb{R} \) given by

\[
F(X) := \sum_{i=1}^{n} \arctan(\lambda_i(X))
\]

where \( \lambda_1(X) \leq \cdots \leq \lambda_n(X) \) are the eigenvalues of the symmetric \( n \times n \) matrix \( X \). The corresponding equation

\[
F(\mathcal{H}w) = f(x)
\]

in \( \Omega \subseteq \mathbb{R}^n \), including the autonomous version

\[
F(\mathcal{H}w) = \theta,
\]

has attained much interest since it was introduced in [HL82]. For a right-hand side constant \( \theta \in (-n\pi/2, n\pi/2) \) the solutions of (1.2) have a nice geometrical interpretation. The graph of the gradient \( \nabla w \) in \( \Omega \times \mathbb{R}^n \) is a special Lagrangian manifold, i.e., it is a Lagrangian manifold of minimal area. See [HL21] and the references therein.
Recently, [CP21] were able to prove the comparison principle for (1.1) when $f$ is continuous and avoids the special phase values
\[ \theta_k := (n - 2k) \frac{\pi}{2}, \quad k = 1, \ldots, n - 1. \]

In Section 3 we show that their proof can be somewhat simplified by applying a result in [Bru22] valid for equations on the generic form (1.1). However, our main purpose of this short note is to demonstrate how the comparison principle may fail when $\theta_k \in f(\Omega)$. Interestingly, the comparison principle is valid in (1.2) for all $\theta \in \mathbb{R}$. This follows immediately from the facts that $X \leq Y$ implies $\lambda_i(X) \leq \lambda_i(Y)$, $\lambda_i(X + \tau I) = \lambda_i(X) + \tau$, and that $\arctan$ is strictly increasing. Thus, $F$ is elliptic and $F(X + \tau I) > F(X)$ for all $X \in \mathcal{S}^n$, $\tau > 0$. See for example Proposition 2.6 in [Bru22].

In our construction of the counterexamples, we shall take advantage of a couple of symmetries in $F$. Firstly, as $F(X)$ only depends on the eigenvalues of $X$, we have $F(QXQ^T) = F(X)$ for every orthogonal matrix $Q$. Secondly, since $\arctan$ is odd – and since $\lambda_i(-X) = -\lambda_{n-i+1}(X)$ and the different eigenvalues are treated equally by $F$ – it follows that $F$ is odd as well. Moreover, we shall make use of the $n \times n$ exchange matrix
\[ J := \begin{bmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}. \]

It is the corresponding matrix to the reverse order permutation on the set $\{1, 2, \ldots, n\}$. Obviously, $J$ is symmetric and orthogonal.

2. The counterexamples

For $k = 0, \ldots, n$, define $v_k, u_k \in C(\mathbb{R}^n)$ as
\[
v_k(x) := \frac{1}{4} - \sum_{i=1}^{k} |x_i| + \sum_{i=k+1}^{n} \frac{1}{2} |x_i|^{3/2},
\]
\[
u_k(x) := -v_{n-k}(Jx),
\]
and let $f_k \in C(\mathbb{R}^n)$ be the continuous extension of
\[
f_k(x) := -\sum_{i=1}^{k} \arctan \left( \frac{3}{8} |x_i|^{-1/2} \right) + \sum_{i=k+1}^{n} \arctan \left( \frac{3}{8} |x_i|^{-1/2} \right), \quad x_i \neq 0.
\]

Observe that
\[
f_k(0) = -k \frac{\pi}{2} + (n - k) \frac{\pi}{2} = \theta_k.
\]

**Proposition 2.1.** The functions $v_k$ and $u_k$ are, respectively, viscosity sub- and supersolutions of the equation
\[
\sum_{i=1}^{n} \arctan (\lambda_i(Hw)) = f_k(x) \quad \text{in } \mathbb{R}^n.
\]
Proof. Away from the axes, \( v_k \) is smooth with Hessian matrix
\[
\mathcal{H}v_k(x) = \frac{3}{8} \text{diag}(0, \ldots, 0, |x_{k+1}|^{-1/2}, \ldots, |x_n|^{-1/2})
\]
and, clearly, \( F(\mathcal{H}v_k(x)) = 0 + \sum_{i=k+1}^n \arctan \left( \frac{3}{8} |x_i|^{-1/2} \right) \geq f_k(x) \).

Let \( \phi \) be a \( C^2 \) test function touching \( v_k \) from above at a point \( x^* \in \mathbb{R}^n \). We may assume that \( x_i^* \neq 0 \) for all \( i > k \) since no touching is possible otherwise. Thus \( e_i^T \mathcal{H}\phi(x^*) e_i \geq \frac{3}{8} |x_i^*|^{-1/2}, \ i = k+1, \ldots, n \), and the top \( n-k \) eigenvalues of \( \mathcal{H}\phi(x^*) \) are larger than \( \frac{3}{8} |x_i^*|^{-1/2} \) (respectively, in some order). Likewise, for each \( i \leq k \) with \( x_i^* \neq 0 \) we have \( e_i^T \mathcal{H}\phi(x^*) e_i \geq 0 \) and an additional eigenvalue of \( \mathcal{H}\phi(x^*) \) is non-negative. The remaining second order directional derivatives of \( \phi \) at \( x^* \) may be arbitrarily negative, providing no bound on the smallest eigenvalues, but that does not matter since
\[
F(\mathcal{H}\phi(x^*)) = \sum_{i=1}^n \arctan \left( \lambda_i(\mathcal{H}\phi(x^*)) \right)
\]
\[
\geq \sum_{i \leq k, x_i^*=0} \frac{-\pi}{2} + \sum_{i \leq k, x_i^* \neq 0} 0 + \sum_{i > k} c(x_i^*), \quad c(t) := \arctan \left( \frac{3}{8} |t|^{-1/2} \right),
\]
\[
\geq \sum_{i \leq k, x_i^*=0} \frac{-\pi}{2} - \sum_{i \leq k, x_i^* \neq 0} c(x_i^*) + \sum_{i > k} c(x_i^*),
\]
\[
= f_k(x^*).
\]
This shows that \( v_k \) is a subsolution of (2.1) in \( \mathbb{R}^n \) for all \( k = 0, \ldots, n \).

In order to prove that \( u_k(x) = -v_{n-k}(Jx) \) is a supersolution, we first note that
\[
-f_{n-k}(Jx) = \sum_{i=1}^{n-k} c((Jx)_i) - \sum_{i=n-k+1}^n c((Jx)_i)
\]
\[
= -\sum_{i=n-k+1}^n c(x_{n+1-i}) + \sum_{i=1}^{n-k} c(x_{n+1-i})
\]
\[
= -\sum_{j=1}^k c(x_j) + \sum_{j=k+1}^n c(x_j), \quad j := n+1-i,
\]
\[
= f_k(x).
\]
Now, let \( \psi \) be a test function touching \( u_k \) from below at \( x^* \in \mathbb{R}^n \). Then \( \phi(x) := -\psi(Jx) \) touches \( v_{n-k} \) from above at \( Jx^* \) because \( \phi(Jx) = -\psi(x) \geq -u_k(x) = v_{n-k}(Jx) \) for \( Jx \) close to \( Jx^* \), and \( \phi(Jx^*) = -\psi(x^*) = -u_k(x^*) = v_{n-k}(Jx^*) \). Thus, \( F(\mathcal{H}\phi(Jx^*)) \geq f_{n-k}(Jx^*) \). Therefore, since \( F \) is odd, rotationally invariant, and \( \mathcal{H}\psi(x) = -J\mathcal{H}\phi(Jx)J \),
\[
F(\mathcal{H}\psi(x^*)) = -F(\mathcal{H}\phi(Jx^*)) \leq -f_{n-k}(Jx^*) = f_k(x^*)
\]
as claimed. \( \square \)
We obviously have \( v_k(0) - u_k(0) = 1/2 > 0 \). In order to create a counterexample to the comparison principle it only remains to observe that
\[
-u_k(x) = v_{n-k}(Jx)
\]
\[
= \frac{1}{4} - \sum_{i=1}^{n-k} |x_{n+1-i}| + \sum_{i=n-k+1}^{n} \frac{1}{2} |x_{n+1-i}|^{3/2}
\]
\[
= \frac{1}{4} - \sum_{j=k+1}^{n} |x_j| + \sum_{j=1}^{k} \frac{1}{2} |x_j|^{3/2}
\]
and
\[
v_k(x) - u_k(x) = \frac{1}{2} + \sum_{i=1}^{n} \frac{1}{2} |x_i|^{3/2} - |x_i|
\]
\[
= \frac{1}{2} + \sum_{i=1}^{n} \frac{1}{2} |x_i| \left( |x_i|^{1/2} - 2 \right)
\]

independently of \( k \). If we take the domain to be the unit ball in the infinity-norm,
\[
\Omega := \{ x \in \mathbb{R}^n : |x_i| < 1 \},
\]

it follows that \( v_k(x) - u_k(x) \leq 0 \) whenever \( |x|_\infty \leq 1 \) and when there is at least one index \( j \) such that \( |x_j| = 1 \). That is, for \( x \in \partial \Omega \).

**Remark 2.1.** There is nothing special about the exponent 3/2 in the sub- and supersolutions. If we adjust the phase accordingly, any number strictly between 1 and 2 would do.

**Remark 2.2.** The subsolutions \( v_k \) are not supersolutions and the supersolutions \( u_k \) are not subsolutions. Thus, the question of uniqueness of *solutions* in the Dirichlet problem is still open.

The ideas behind the above constructions are all contained, and therefore best illustrated, by the case \( n = 2 \), \( k = 1 \). Then also \( n-k = 1 \) and, dropping the subscript 1 yields,
\[
v(x, y) = \frac{1}{4} - |x| + \frac{1}{2} |y|^{3/2},
\]
\[
u(x, y) = -v(y, x) = -\frac{1}{4} - \frac{1}{2} |x|^{3/2} + |y|,
\]
with phase
\[
f(x, y) = \begin{cases} 
- \arctan\left(\frac{3}{8} |x|^{-1/2}\right) + \arctan\left(\frac{3}{8} |y|^{-1/2}\right), & x \neq 0, y \neq 0, \\
-\pi/2 + \arctan\left(\frac{3}{8} |y|^{-1/2}\right), & x = 0, y \neq 0, \\
- \arctan\left(\frac{3}{8} |x|^{-1/2}\right) + \pi/2, & x \neq 0, y = 0, \\
0, & x = 0, y = 0.
\end{cases}
\]

In addition to the difference
\[
v(x, y) - u(x, y) = \frac{1}{2} + \frac{1}{2} |x| \left( |x|^{1/2} - 2 \right) + \frac{1}{2} |y| \left( |y|^{1/2} - 2 \right)
\]
the graph of these functions over the square
\[ \Omega := \{ (x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1 \} \]
are shown in Figure 1.

(A) \( v(x, y) = \frac{1}{4} - |x| + \frac{1}{2}|y|^{3/2} \).
(B) \( u(x, y) = -v(y, x) \).

(c) The difference \( v - u \) has a strict interior maximum in \( \Omega \).
(d) The phase \( f \) is continuous, but not differentiable on the axes.

**Figure 1.** The case \( n = 2, k = 1 \) in the square \( \Omega \).

3. An alternative proof of the comparison principle when the phase does not attain the special values

Theorem 6.18 [CP21] establish the comparison principle for the equation
\[
\sum_{i=1}^{n} \arctan(\lambda_i(\mathcal{H}w)) = f(x)
\]
in every open and bounded $\Omega \subseteq \mathbb{R}$ whenever $f \in C(\Omega)$, $\theta_n < f < \theta_0$, and

$$\theta_k := (n - 2k) \frac{\pi}{2} \notin f(\Omega), \quad k = 1, \ldots, n - 1.$$  

The main idea in our proof, as conducted in Example 2.2 in [Bru22], is the same as in [CP21]. Namely, to reach a contradiction when, for each $i = 1, \ldots, n$, $|\lambda_i(X_j)| \to \infty$ as $j \to \infty$ for some sequence $X_j \in S^n$. Our contribution is to show how this follows almost immediately from a general result for equations on the form $F(\mathcal{H}w) = f(x)$. For convenience, we reproduce the proof below.

Note that the pathological situation when $f$ takes values outside the interval $[\theta_n, \theta_0]$ comes for free: If, say $f > \theta_0$ somewhere in $\Omega$, the comparison principle vacuously holds since the equation will have no subsolutions. On the other hand, the case $\theta_0 \in f(\Omega)$, which is covered by our counterexample, is not pathological. For example, one can easily confirm that $w(x) = |x|^{3/2}$ is a viscosity solution to the equation in $\mathbb{R}^n$ when the right-hand side is the continuous extension of $f(x) := F(\mathcal{H}w(x))$.

Assume that

$$\theta_k \notin f(\Omega) \subseteq [\theta_n, \theta_0] \quad (3.1)$$

for all $k = 0, \ldots, n$. Proposition 2.7 in [Bru22] states that the comparison principle will hold if whenever $X_j \in S^n$ is a sequence such that $\lim_{j \to \infty} F(X_j) = \theta \in f(\Omega)$, then

$$\liminf_{j \to \infty} F(X_j + \tau I) > \theta$$

for every $\tau > 0$. Suppose to the contrary that this is not true. Then there are numbers $\theta \in f(\Omega)$ and $\tau > 0$, and a sequence $X_j \in S^n$ with $F(X_j) \to \theta$, such that

$$0 = \lim_{j \to \infty} F(X_j + \tau I) - F(X_j)$$

$$= \lim_{j \to \infty} \sum_{i=1}^{n} \arctan(\lambda_i(X_j) + \tau) - \arctan(\lambda_i(X_j)) \geq 0,$$

which – since arctan is strictly increasing – is possible only if each $\lambda_i(X_j)$ is unbounded as $j \to \infty$. There is thus a subsequence (still indexed by $j$) such that either $\lambda_i(X_j) \to +\infty$ or $\lambda_i(X_j) \to -\infty$. But this is a contradiction of (3.1) as

$$f(\Omega) \ni \theta = \lim_{j \to \infty} F(X_j) = \lim_{j \to \infty} \sum_{i=1}^{n} \arctan(\lambda_i(X_j)) = \sum_{i=1}^{n} \pm \frac{\pi}{2} = \theta_k$$

for some $k = 0, \ldots, n$.

References

[Bru22] Karl K. Brustad. On the comparison principle for second order elliptic equations without first and zeroth order terms (preprint). arxiv: 2008.08399. To appear in Nonlinear Differential Equation and Applications, 2022.
Marco Cirant and Kevin R. Payne. Comparison principles for viscosity solutions of elliptic branches of fully nonlinear equations independent of the gradient. *Math. Eng.*, 3(4):Paper No. 030, 45, 2021.

Reese Harvey and H. Blaine Lawson, Jr. Calibrated geometries. *Acta Math.*, 148:47–157, 1982.

F. Reese Harvey and H. Blaine Lawson, Jr. Pseudoconvexity for the special Lagrangian potential equation. *Calc. Var. Partial Differential Equations*, 60(1):Paper No. 6, 37, 2021.

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