Hamiltonian multiform description of an integrable hierarchy

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Abstract

Motivated by the notion of Lagrangian multiforms, which provide a Lagrangian formulation of integrability, and by results of the authors on the role of covariant Hamiltonian formalism for integrable field theories, we propose the notion of Hamiltonian multiforms for integrable 1 + 1-dimensional field theories. They provide the Hamiltonian counterpart of Lagrangian multiforms and encapsulate in a single object an arbitrary number of flows within an integrable hierarchy. For a given hierarchy, taking a Lagrangian multiform as starting point, we provide a systematic construction of a Hamiltonian multiform based on a generalisation of techniques of covariant Hamiltonian field theory. This also produces two other important objects: a symplectic multiform and the related multi-time Poisson bracket. They reduce to a multisymplectic form and the related covariant Poisson bracket if we restrict our attention to a single flow in the hierarchy. Our framework offers an alternative approach to define and derive conservation laws for a hierarchy. We illustrate our results on three examples: the potential Korteweg-de Vries hierarchy, the sine-Gordon hierarchy (in light cone coordinates) and the Ablowitz-Kaup-Newell-Segur hierarchy.

1 Introduction

The objects and results presented in this paper, to be detailed below, come from the confluence of several new ideas that have emerged in the theory of integrable systems in recent years. The first idea, introduced in 2009 by Lobb and Nijhoff [1] is the notion of Lagrangian multiforms. The motivation was to address the completely open problem of characterising integrability of (partial) differential (or difference) equations purely from a variational/Lagrangian point of view. Despite the well known and fundamental interplay between Lagrangian and Hamiltonian formalism in classical and quantum physics, when it comes to integrable systems, one can only observe that the Hamiltonian approach has been the overwhelming favourite, mainly (but not fully) because of the extraordinary success of the canonical quantization procedure. This was carried out via the classical $r$-matrix approach [2, 3] which leads to the quantum $R$-matrix approach [2, 4], both of which have given unifying frameworks for dealing with integrable systems and led to their own fully fledged research areas in (Poisson) geometry and quantum groups. Initially developed in the realm of fully discrete integrable systems, Lagrangian multiforms provide a framework whereby the notion of multidimensional consistency [5, 6], which captures the analog of the commutativity of Hamiltonian flows known in continuous integrable systems, is encapsulated in a generalised variational principle. The latter contains the standard Euler-Lagrange equations for the various equations forming an integrable hierarchy as well as additional equations, originally called corner equations which can be interpreted as determining the allowed integrable Lagrangians themselves. The set of all these equations is now called multiform Euler-Lagrange equations. The original work of Lobb and Nijhoff [1] stimulated a wealth of subsequent developments, first in the discrete realm, see e.g. [7, 8, 9, 10, 11, 12], then progressively into the continuous realm for finite dimensional systems, see e.g. [13, 14] and 1 + 1-dimensional field theories, see e.g. [15], up to more recent developments in continuous field theory, see e.g. [16, 17, 18, 19, 20], including the first example in 2 + 1-dimensions [19].

Given that our focus is on 1 + 1-dimensional field theories in this paper, let us present briefly the main ingredients of the theory of Lagrangian multiforms in this context. The starting point is to consider a two-form

$$\mathcal{L}[u] = \sum_{i<j=1}^{n} L_{ij}[u] dx^i \wedge dx^j, \quad n > 2,$$

where for each $i, j$, $L_{ij}[u]$ is a function of a field $u$ depending on the $n$ independent variables $x_1, \ldots, x_n$ (the “times” of the hierarchy) and of the derivatives of $u$ with respect to these variables up to some

\footnote{We only consider a single scalar field $u$ at this stage for simplicity of exposition but multicomponent fields are...}
finite order. We used the notation $dx^i = dx^1 \wedge dx^i$ and the convention $L_{ij}[u] = -L_{ji}[u]$. For convenience in this paper, we assume that the $L_{ij}[u]$ do not depend explicitly on the independent variables. Associated to this two-form is an action

$$S[u, \sigma] = \int_\sigma \mathcal{L}[u],$$

(1.2)
or rather a collection of actions, labelled by a 2-dimensional surface $\sigma$ in $\mathbb{R}^n$. At this stage, it is worthwhile noting that the standard variational approach to a field theory with two independent variables $x_1, x_2$ would consider a volume form $\mathcal{L}[u] = L[u]dx^1 \wedge dx^2$, with Lagrangian density $L[u]$, and simply an action $S[u] = \int L[u]dx^1 \wedge dx^2$. The novelty is in considering a 2-form in a larger space as well as an action labelled by a surface into this larger space. The (generalised) equations of motion, called multiform Euler-Lagrange equations, are then obtained by postulating a (generalised) variational principle: we look for critical points of the action $S[u, \sigma]$ with respect to variations of the field $u$ for fixed but arbitrary $\sigma$ as well as with respect to variation of surface of integration $\sigma$. It can be shown [21, 19] that the first requirement translates into the equation $d\delta \mathcal{L} = 0$ where we have used the two operator $\delta$ and $d$ arising in the variational bicomplex formalism (see below for a recap).

The second requirement gives us the closure relation on the equations of motion, i.e. the fact that on the equations of motion $d\mathcal{L} = 0$. We thus take the following definition:

**Definition 1.1.** The horizontal 2-form (1.1) is a Lagrangian multiform if $\delta d \mathcal{L} = 0$ implies $d\mathcal{L} = 0$.

Note that the requirement of imposing the closure relation on the equations of motion to define a Lagrangian multiform was an important feature of the original work of Lobb and Nijhoff. It has been dropped in some subsequent works by other authors and the related terminology is “pluri-Lagrangian” in that context. However, in the recent work [22], it is shown that this property is the Lagrangian counterpart of having Hamiltonian functions in involution. Our results here shed some more light on this connection between Lagrangian multiforms and Hamiltonians in involution, in the form of Theorem 2.4. This clarifies the role of the closure relation to capture integrability in a Lagrangian framework.

The second idea, introduced by the authors in [23] in the context of 1 + 1-dimensional integrable field theories, is to use ideas from covariant Hamiltonian field theory, whose origins can be traced to the early work of de Donder and Weyl [24, 25], in conjunction with the $r$-matrix formalism. The latter had been confined to the standard Hamiltonian formalism since its introduction, thus breaking the natural symmetry between the independent space-time variables. The work [23] had been motivated by earlier results [26, 27, 28] which showed a surprising spacetime duality in the classical $r$-matrix structure of a field theory. The origin of this duality was later explained in [29] in the case of the Ablowitz-kaup-newell-segur (AKNS) hierarchy [30]. In [23], building on results and formalisms due for instance to Kanatchikov [31] and Dickey [32], we were able to construct a covariant Poisson bracket which possesses the classical $r$-matrix structure when evaluated on the natural Lax form of the theory. It is important to stress that the results were obtained from a single Lagrangian corresponding to the given 1 + 1-dimensional theory at hand, i.e. from a standard Lagrangian volume form. This allowed us to construct a multisymplectic form which in turn gave us access to the desired covariant Poisson bracket and $r$-matrix structure. We were also able to obtain the zero curvature representation, typical of integrable field theories, as a covariant Hamilton equation for the Lax form.

In view of this account, two natural questions arise:

1. What happens to the construction of the multisymplectic form and the covariant Poisson bracket if we use a Lagrangian multiform instead of a standard Lagrangian (volume form) as a starting point?

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*Note that we restrict this space to be of finite dimension $n$ here whereas strictly speaking, for an integrable field hierarchy one should let $n \to \infty$. The number $n$ corresponds to the number of commuting flows with respect to $x_1, \ldots, x_n$ that we incorporate in the Lagrangian multiform. Our pragmatic approach is to consider $n$ fixed but arbitrary.

*The literature on this topic is vast and forms an entire community in its own right. It cannot be included here but we kindly refer the interested reader to the introduction of [23] where an effort was made to point to key references, at least from the point of view of our work.*
2. Provided the previous construction can be implemented, can we generalise the results of [23] about the $r$-matrix structure of the covariant Poisson bracket to the structure that generalises this covariant Poisson bracket? In other words, can we extend the derivation of the $r$-matrix to a multi-time Poisson bracket that would appear when dealing with a whole integrable hierarchy?

In the present paper, we investigate in detail the first question and leave the second question for future work [33]. Specifically, in the context of $1 + 1$-dimensional integrable field theories, our main results are as follows:

- We introduce a Hamiltonian multiform $\mathcal{H} = \sum_{i<j=1}^{n} H_{ij} dx^{ij}$, which is naturally associated to any given Lagrangian multiform. This requires to adapt techniques and notions from covariant Hamiltonian field theory and multisymplectic geometry that were conveniently cast into a purely algebraic framework in [32]. We then prove the central result that $d\mathcal{H} = -2 dL$ on the multiform Euler-Lagrange equations.

- Alongside the Hamiltonian multiform associated to a Lagrangian multiform, our approach produces a generalisation of the multisymplectic form that is canonically associated to a standard Lagrangian. We call it symplectic multiform for reasons that will be elaborated upon in the text. It naturally incorporates into a single object each symplectic form related to each Lagrangian $L_{ij}$ in the Lagrangian multiform. The symplectic multiform encapsulates the motion under the flows of with respect to the $n$ independent variables $x_1, \ldots, x_n$ on our space of variables which forms the analog of the covariant phase space usually associated to first order field theories.

- Equipped with the symplectic multiform, we are able to define a multi-time Poisson bracket. It naturally incorporates certain single time Poisson brackets, which can be assembled naturally into pairs of dual Poisson brackets that were originally observed to possess the same $r$-matrix structure in [26, 27, 28]. Our multi-time Poisson bracket reproduces the covariant Poisson bracket in the case of $n = 2$ independent (spacetime) variables, as considered for instance in [23].

- We use these results to derive conservation laws traditionally signalling integrability in a field theory and whose construction has been the object of numerous studies based on fundamental ideas such as bi-Hamiltonian structures, recursion operators and Lax pairs, see e.g. [34]. Our method relies only on the elements introduced in our approach. This is implemented on the examples of the potential Korteweg-de Vries (pKdV) and AKNS Hamiltonian multiforms.

The paper is organised as follows. In Section 2 we review the essential ingredients of the variational bicomplex as presented in [32] in an algebraic language, instead of the original geometric approach, see e.g. [35, 36]. We use this framework to define and develop the theory of Hamiltonian multiforms, starting from a Lagrangian multiform. We then show how conservation laws fit into this context. In Sections 3, 4 and 5, we illustrate the various constructions on the examples respectively of the pKdV hierarchy, of the sine-Gordon (sG) hierarchy in light-cone coordinates, and the AKNS hierarchy.

2 Hamiltonian multiform, symplectic multiform and multi-time Poisson brackets

In this section, we first review essential notions and notations for our purposes in Sections 2.1 and 2.2. By presenting what is well known for a field theory associated to a Lagrangian volume form, i.e. covariant Hamiltonian, multisymplectic form and covariant Poisson bracket, it will be easier to appreciate the novelty brought in by the transition to the Lagrangian multiform framework, despite several of the defining relations looking superficially the same. In particular, in analogy with the three fundamental objects just mentioned, we will introduce in Sections 2.3 and 2.4 the notion of Hamiltonian multiform, symplectic multiform and multi-time Poisson brackets together with their basic properties.
2.1 Elements of variational calculus with the variational bicomplex

The intuition behind the variational bicomplex formalism for a field theory can be summarised as follows. Let $M$ be the (spacetime) manifold with local coordinates $x^i, i = 1, \ldots, n$. The manifold $M$ is viewed as the base manifold in a fibered manifold $\pi : E \to M$ whose sections represent the fields of the theory. The variational bicomplex is a double complex of differential forms defined on the infinite jet bundle of $\pi : E \to M$. One introduces vertical and horizontal differentials $\delta$ and $d$ which satisfy

$$d^2 = 0 = \delta^2, \quad d\delta = -\delta d,$$  

(2.1)

so that the operator $d + \delta$ satisfies $(d + \delta)^2 = 0$.

We now follow [32] for a more detailed exposition of what we need in this paper. For convenience, we will only consider theories whose Lagrangian do not depend explicitly on the independent variables $x_i$. Let $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$. Consider the differential algebra with the commuting derivations $\partial_i, i = 1, \ldots, n$ generated by the commuting variables$^4$ $u_k^{(i)}, k = 1, \ldots, N, (i) = (i_1, \ldots, i_n)$ being a multi-index, quotiented by the relations

$$\partial_i u_k^{(i)} = u_k^{(i)+e_j},$$

(2.2)

where $e_j = (0, 0, \ldots, 1, 0, \ldots, 0)$ only has 1 in position $j$. We simply denote $u_k^{(0, \ldots, 0)}$ by $u_k$, the fields of the theory which would be the local fibre coordinates mentioned above. We denote this differential algebra by $\mathcal{A}$. We will need the notation

$$\partial^{(i)} = \partial^{(i)}_1 \partial^{(i)}_2 \ldots \partial^{(i)}_n.$$  

(2.3)

We consider the spaces $\mathcal{A}^{(p,q)}$, $p, q \geq 0$ of finite sums of the following form

$$\omega = \sum_{(i),(k),(j)} f^{(i)}_{(k),(j)} \delta u_{k_1}^{(i)} \wedge \ldots \wedge \delta u_{k_p}^{(i)} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}, \quad f^{(i)}_{(k),(j)} \in \mathcal{A}$$

(2.4)

which are called $(p,q)$-forms. In other words, $\mathcal{A}^{(p,q)}$ is the space linearly generated by the basis elements $\delta u_{k_1}^{(i)} \wedge \ldots \wedge \delta u_{k_p}^{(i)} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}$ over $\mathcal{A}$, where $\wedge$ denotes the usual exterior product. We define the operations $d : \mathcal{A}^{(p,q)} \to \mathcal{A}^{(p,q+1)}$ and $\delta : \mathcal{A}^{(p,q)} \to \mathcal{A}^{(p+1,q)}$ as follows. They are graded derivations

$$d(\omega^{(p_1,q_1)} \wedge \omega^{(p_2,q_2)}) = dw^{(p_1,q_1)} \wedge \omega^{(p_2,q_2)} + (-1)^{p_1+q_1} \omega^{(p_1,q_1)} \wedge dw^{(p_2,q_2)},$$

(2.5a)

$$\delta(\omega^{(p_1,q_1)} \wedge \omega^{(p_2,q_2)}) = \delta \omega^{(p_1,q_1)} \wedge \omega^{(p_2,q_2)} + (-1)^{p_1+q_1} \omega^{(p_1,q_1)} \wedge \delta \omega^{(p_2,q_2)},$$

(2.5b)

and on the generators, they satisfy

$$df = \sum \partial_i f dx^i = \sum \frac{\partial f}{\partial u_k^{(i)}} u_k^{(i)+e_j} dx^j, \quad f \in \mathcal{A},$$

(2.6a)

$$\delta f = \sum \frac{\partial f}{\partial u_k^{(i)}} \delta u_k^{(i)}, \quad f \in \mathcal{A},$$

(2.6b)

$$\delta(dx^i) = \delta(\delta u_k^{(i)}) = d(dx^i) = 0,$$  

(2.6c)

$$d(\delta u_k^{(i)}) = -\delta dx^i,$$  

(2.6d)

This determines the action of $d$ and $\delta$ on any form as in (2.4). As a consequence, one can show that

$$d^2 = 0 = \delta^2, \quad d\delta = -\delta d.$$

For our purpose, it is sufficient to take the following (simplified) definition for the variational bicomplex: it is the space $\mathcal{A}^* = \bigoplus_{p,q} \mathcal{A}^{(p,q)}$ equipped with the two derivation $d$ and $\delta$. Due to the geometrical interpretation of these derivations, $d$ is called horizontal derivation while $\delta$ is called vertical derivation. Note that the direct sum over $q$ is finite and runs from 0 (scalars) to $n$ (volume horizontal forms) whereas the sum over $p$ runs from 0 to infinity. Of course, each form in $\mathcal{A}^*$ only contains a finite sum of elements of the form (2.4) for certain values of $p$ and $q$. The bicomplex $\mathcal{A}^*$ generates an associated complex $\mathcal{A}^{(r)} = \bigoplus_{p+q=r} \mathcal{A}^{(p,q)}$ and derivation $d + \delta$. It is proved that both the horizontal sequence and the vertical sequence are exact, see e.g. [32].

$^4$Recall that for the most part in this paper, we will only consider a scalar field, $N = 1$, except for the AKNS example.
Dual to the notion of forms is the notion of vector fields. We consider the dual space of vector fields \( \mathcal{T} \mathcal{A} \) to the space of one-forms \( \mathcal{A}^{(1)} \) with elements of the form
\[
\xi = \sum_{k,i} \xi_{k,i} \partial_{u_k} \delta i + \sum_i \xi_i \partial_i .
\] (2.7)
The interior product with a form is obtained in the usual graded way together with the rule
\[
\partial_i \delta u_k = \delta i, \quad \partial_{u_k} \delta u_k^{(i)} = \delta k \delta (i) .
\] (2.8)
where \( \delta (i) = \prod_k \delta i_k \). For instance, with \( i \neq j \) and \( (i) \neq (j) \) or \( k \neq l \),
\[
\partial_i \delta (u_k^{(i)} \wedge \delta u_k^{(j)} \wedge \delta x^j) = -\delta u_k^{(i)} \wedge dx^j,
\] (2.9a)
\[
\partial_{u_k} \delta (u_k^{(i)} \wedge \delta u_k^{(j)} \wedge \delta x^m) = -\delta u_k^{(j)} \wedge dx^m .
\] (2.9b)
In particular, we will need the following vertical vector fields
\[
\tilde{\partial}_j = \sum_{k,(i)} u_k^{(j) + e_i} \partial \delta u_k^{(j)} .
\] (2.10)
Let us also introduce the notation \( \partial'_k \) by \( \partial_k = \partial'_k + \tilde{\partial}_k \). If \( f \in \mathcal{A} \) does not depend explicitly on the space-time variables then \( \partial'_k f = \tilde{\partial}_k f \). In addition to the vector fields \( (2.7) \), in general calculations in the variational bicomplex also require the use of multivector fields of the form \( \xi_1 \wedge \cdots \wedge \xi_r \) where each \( \xi_i \) is of the form \( (2.7) \). In this paper, we will mostly need those multivector fields that are linear combination of \( \partial'_k \) and \( \partial_k \) with coefficients in \( \mathcal{A} \) and we may simply call them vector fields as the context should not lead to any confusion. The following example shows the rule for the interior product of such a multivector field, with \( (i) \neq (j) \) or \( k \neq l \),
\[
(\partial_{u_k}^{(i)} \wedge \partial_k) \delta (u_k^{(j)} \wedge \delta u_k^{(i)} \wedge dx^m) = \partial_{u_k}^{(i)} \delta (\partial u_k^{(j)} \wedge \delta u_k^{(i)} \wedge dx^m)) = -\delta u_k^{(j)} \delta u_k^{(i)} .
\]
Finally, we will need the following useful identity, cf [32, Corollary 19.2.11].
\[
\tilde{\partial}_k = \delta \tilde{\partial}_i + \tilde{\partial}_i \delta .
\] (2.11)

2.2 The multisymplectic approach to a PDE

Equipped with the above basic elements of the variational bicomplex, we now recall how to describe a partial differential equation admitting a Lagrangian formulation into a covariant Hamiltonian formulation. This serves as a basis to introduce known results and objects, in particular the multisymplectic form. What is reviewed here will be helpful to identify the novel ingredients in the rest of this paper.

Recall that we focus on two-dimensional field theories so our starting point is a Lagrangian volume 2-form
\[
\lambda = L dx^1 \wedge dx^2 .
\] (2.12)
\( L \) is the Lagrangian density and depends on the fields \( u^k \) \( k = 1, \ldots , N \) and their derivatives with respect to \( x^1 \) and \( x^2 \). We use the variational bicomplex described in the previous section with \( n = 2 \). It is known that there exist unique elements \( A_k \in \mathcal{A} \) such that
\[
\delta \lambda = \sum_k A_k \delta u_k \wedge dx^1 \wedge dx^2 - d\Omega^{(1)}
\] (2.13)
where \( \Omega^{(1)} \in \mathcal{A}^{(1,1)}/d\mathcal{A}^{(1,0)} \) is only determined up to a total horizontal derivative. The coefficients \( A_k \) are denoted \( \frac{\partial L}{\partial u^k} \) and are the variational derivatives with respect to \( u_k \). One then obtains the Euler-Lagrange equations by setting \( \frac{\partial L}{\partial u^k} = 0 \) for every \( k \). \( \Omega^{(1)} \) is obtained using the property \( \delta d + d\delta = 0 \) in \( \delta \lambda \) as much as possible. Of course, the content of this result is simply the local analog of the standard integration by parts procedure used when varying the action \( \int \mathcal{L} \). In the latter, the boundary term \( \int d\Omega^{(1)} \) is usually discarded. To understand the role played by \( \Omega^{(1)} \), we remark the following
facts. For a classical finite-dimensional Lagrangian system, this is (the pull-back to the tangent bundle of) the canonical one form \( \frac{\partial}{\partial \dot{q}} \delta q \), and one can obtain the symplectic form by taking its \( \delta \)-differential. Similarly, in the case of field theories where \( L \) is taken to be a volume form, the form is \( \Omega^{(1)} = \omega^{(1)}_1 \wedge dx^1 + \omega^{(1)}_2 \wedge dx^2 \) where \( \omega^{(1)}_1 \) and \( \omega^{(1)}_2 \) each have a similar structure to the canonical one form of the finite dimensional case. It contains the usual symplectic structure \( -\omega^{(1)}_1 \) (if we consider \( x_2 \) as our time) but also the dual structure \( \omega^{(1)}_2 \) (which would correspond to performing the Legendre transform when choosing \( x_1 \) as the time variable). To summarize, for a field theory, \( \Omega^{(1)} \) realises the Legendre transform simultaneously with respect to all independent variables.

The next step is to define the covariant Hamiltonian as

\[
H = -\Lambda + \sum_{j=1}^{2} dx^j \wedge \widetilde{\partial}_j \Omega^{(1)}. \tag{2.14}
\]

and the multisymplectic form \( \Omega \in \mathcal{A}^{(2,1)} \) as

\[
\Omega = \delta \Omega^{(1)}. \tag{2.15}
\]

One obtains the covariant Hamilton equations as

\[
\delta H = \sum_{j=1}^{n} dx^j \wedge \widetilde{\partial}_j \Omega, \tag{2.16}
\]

which are equivalent to the to the Euler-Lagrange equation, as they should. In general, let us note that if a PDE involves \( n \) independent variables and admits a Lagrangian description, \( \Lambda \) and \( H \) are volume \( n \)-forms, \( \Omega^{(1)} \in \mathcal{A}^{(1,n-1)} \) and \( \Omega \in \mathcal{A}^{(2,n-1)} \).

Equipped with a multisymplectic form we can consider the definition of a covariant Poisson bracket, following for instance Kanatchikov [31]. We stress that the definition of a covariant Poisson bracket from a multisymplectic form, in a way that mimics the situation in classical mechanics, has been part of a rich activity since the early proposals. In particular, the Jacobi identity is a delicate issue, as well as the need to restrict to certain forms, called Hamiltonian, as we explain below. We refer the reader to [37] for a detailed account. For our purpose, we will simply use Kanatchikov’s ideas and adapt them to our purposes. The results of [23] show that, at least in our context, this leads to a satisfactory covariant Poisson bracket satisfying the Jacobi identity, thanks to the fact that the latter translates into the classical Yang-Baxter equation for the classical \( r \)-matrix.

We need to restrict our attention to the a special class of forms called Hamiltonian. We take the following definition which is sufficient for our purposes: a horizontal form \( F \) is said to be Hamiltonian if there exists a (multi)vector field \( \xi_F \) such that \( \xi_F \cdot \Omega = dF \). Contrary to the usual symplectic case, the property of Hamiltonianicity is quite restrictive in the multisymplectic case.

For two Hamiltonian forms \( P \) and \( Q \), of (horizontal) degree respectively \( r \) and \( s \), we can define their covariant Poisson bracket as

\[
\{ | P, Q | \}_c = (-1)^r \xi_P \wedge \xi_Q \cdot \Omega. \tag{2.17}
\]

We now state the following fact, which was only obtained explicitly on examples in [23], but for which no general proof was given.

**Proposition 2.1.** If the covariant Hamiltonian density \( h = *^{-1} H \) is a Hamiltonian form, then we have for any Hamiltonian 1-form \( F \)

\[
dF = \{ | h, F | \}_c dx^1 \wedge dx^2. \tag{2.18}
\]

This is of course the multisymplectic analog of the well-known equation in Hamiltonian mechanics \( dF = \{ H, F \} dt \) giving the time evolution of a function \( F \) on the phase space under the Hamiltonian flow of \( H \). In this paper, we will give this statement and a proof in the more general setting of Section 2.4, from which the above can be recovered by setting \( n = 2 \).
2.3 The multiform Hamilton equations and the symplectic multiform

The main observation at the basis of this paper is that the objects and results reviewed in Section 2.2 can be extended to a Lagrangian multiform system

\[ \mathcal{L}[u] = \sum_{i<j}^n L_{ij}[u] \, dx^j, \quad n > 2. \quad (2.19) \]

The first step is to construct the generalisation of the form \( \Omega^{(1)} \) in (2.13). The following result from [21, Proposition 6.3], which we reproduce here with a little change of notation, shows that this can be done in principle.

**Proposition 2.2.** The field \( u \) is a critical point of \( S[u, \sigma] = \int_{\sigma} \mathcal{L}[u] \) if and only if there exists a form \( \Omega^{(1)} \) such that in a

\[ \delta \mathcal{L}[u] = -d\Omega^{(1)}. \quad (2.20) \]

In practice, compared to the case of (2.13), when \( \mathcal{L}[u] \) is a Lagrangian multiform, one has more freedom in the “integration by parts” steps, leading to more freedom in what terms contribute to the analog of the \( d\Omega^{(1)} \) term and which ones contribute to the analog of the \( A_k = \frac{\partial L}{\partial \partial x^k} \) terms. There is however a simple guide to ensure we obtain the desired structures: given \( \mathcal{L}[u] \), we know the corresponding system of multiform Euler-Lagrange equations. Therefore to compute \( \Omega^{(1)} \) from \( \mathcal{L}[u] \) we give the following prescription (writing \( \mathcal{L} \) for \( \mathcal{L}[u] \) for conciseness).

1. Compute \( \delta \mathcal{L} \),
2. Compute the multiform E-L equations as \( d\delta \mathcal{L} = 0 \),
3. Apply \( d\delta + d = 0 \) on \( \delta \mathcal{L} \) to recognize a \( d \)-differential, obtaining an equation of the form

\[ \delta \mathcal{L} = A(\mathcal{L}) + dB, \quad (2.21) \]

where \( A(\mathcal{L}) \in A^{(1,2)} \) and \( B \in A^{(1,1)} \),
4. Repeat until \( A(\mathcal{L}) = 0 \) is equivalent to \( d\delta \mathcal{L} = 0 \). Then one sets

\[ A(\mathcal{L}) \equiv E(\mathcal{L}), \quad B \equiv -\Omega^{(1)} \quad (2.22) \]

Of course this prescription still leaves the freedom which is analogous to the freedom of adding a total horizontal derivative to \( \Omega^{(1)} \) in the standard case. Indeed, suppose that \( \Omega^{(1)} \) and \( \Omega^{(1)} \) are such that

\[ d\delta \mathcal{L} = 0 \iff \delta \mathcal{L} = -d\Omega^{(1)} \quad \text{and} \quad d\delta \mathcal{L} = 0 \iff \delta \mathcal{L} = -d\Omega^{(1)}. \quad (2.23) \]

Then on the equations \( d\delta \mathcal{L} = 0 \) we have \( d(\Omega^{(1)} - \Omega^{(1)}) = d\Omega^{(1)} - d\Omega^{(1)} = \delta \mathcal{L} - \delta \mathcal{L} = 0 \). So \( \Omega^{(1)} - \Omega^{(1)} = dF \) for some \( F \) on the equations of motion, as the variational bicomplex remains horizontally exact on the equations of motions (except for horizontal forms), cf. [32, Proposition 19.6.9]. We will illustrate this on the example of the potential KdV example in Section 3. Equipped with \( \Omega^{(1)} \), we can perform a Legendre-like transform and define:

**Definition 2.3 (Hamiltonian multiform).** The **Hamiltonian multiform** is defined by

\[ \mathcal{H} = -\mathcal{L} + \sum_{j=1}^n dx^j \wedge \delta_{ij} \Omega^{(1)}. \quad (2.24) \]

As announced, this definition looks very similar to the definition of the covariant Hamiltonian in (2.14). However note that the sum involves \( n \) terms here (the number of independent variables included in the Lagrangian multiform) and that \( \mathcal{H} \) has the form \( \mathcal{H} = \sum_{i<j} H_{ij} dx^i \wedge dx^j \) and is in \( A^{(0,2)} \), like \( \mathcal{L} \). \( \mathcal{H} \) plays the role of the covariant Hamiltonian form in the multiform context. In fact, if we make the right choices on \( \Omega^{(1)} \) we notice that the \( H_{ij} \) are the covariant Hamiltonian densities related to the Lagrangians \( L_{ij} \), which describes the \( j \)-th level of the hierarchy.

We can easily see that there is a relation between the \( d \)-differential of \( \mathcal{H} \) and the one of \( \mathcal{L} \). The next result is important and connects the closure relation (or absence thereof) in the Lagrangian multiform to the Hamiltonian multiform formalism.
Theorem 2.4. \(d\mathcal{H} = -2d\mathcal{L}\) modulo the multiform E-L equations.

Proof. We start from the definition of \(\mathcal{H}\):

\[
d\mathcal{H} = -d\mathcal{L} + d \left( \sum_{j=1}^{n} dx^j \wedge \tilde{\partial}_j \Omega^{(1)} \right) = -d\mathcal{L} - \sum_{j=1}^{n} dx^j \wedge d\tilde{\partial}_j \Omega^{(1)} = -d\mathcal{L} + \sum_{j=1}^{n} dx^j \wedge \tilde{\partial}_j d\Omega^{(1)} \quad (2.25)
\]

where we used \(d\tilde{\partial}_j + \tilde{\partial}_j d = 0\). Now we use the equation \(\delta \mathcal{L} = -d\Omega^{(1)}\) to obtain

\[
d\mathcal{H} = -d\mathcal{L} - \sum_{j=1}^{n} dx^j \wedge (\tilde{\partial}_j - \delta \tilde{\partial}_j) \mathcal{L} = -d\mathcal{L} - \sum_{j=1}^{n} dx^j \wedge \tilde{\partial}_j \mathcal{L} = -2d\mathcal{L}. \quad (2.26)
\]

In the last line we used the property \(\tilde{\partial}_j = \delta \tilde{\partial}_j + d\), and the fact that \(\mathcal{L}\) is purely horizontal and does not depend explicitly on the space-time variables.

We remark that in [17, 22] the closure of a pluri-Lagrangian form \(\mathcal{L}\) was linked to the involution of the single-time Hamiltonians. In the particular case where the Hamiltonian multiform is a Hamiltonian form in the sense defined below, we expect Theorem 2.4 to provide a general framework in which to recast these results (with appropriate modifications for the examples in 0 + 1 dimensions presented in [17, 22]). This point is left for future investigation. Recalling that a Lagrangian multiform is defined to satisfy the closure relation on the equations of motion, we obtain:

**Corollary 2.5** (Closedness of \(\mathcal{H}\)). The Hamiltonian multiform is horizontally closed on the multiform E-L equations \(d\mathcal{H} = 0\). In other words, \(\mathcal{H}\) satisfies the closure relation.

These results justify our terminology *Hamiltonian multiform* since we have the closure relation for \(\mathcal{H}\) if and only if it holds for \(\mathcal{L}\). This corollary is the multiform equivalent of the well known fact in finite-dimensional mechanics that the Hamiltonian is a conserved quantity \(\frac{d\mathcal{H}}{dt} = 0\) (recall that we do not include explicit dependence on the independent variables here).

We are now in a position to introduce the multiform analog of the multisymplectic form (2.15), again denoting it by \(\Omega\).

**Definition 2.6.** The *symplectic multiform* is defined as \(\Omega = \delta \Omega^{(1)}\).

The reader will hopefully forgive us for the choice of terminology, very similar to multisymplectic form. Another candidate, polysymplectic form, is already in use in the literature. We could not simply keep multisymplectic form for our new object since, although both objects are derived in a similar fashion and play a similar role in the theory, they are quite different in structure. Indeed, recall that the multisymplectic form for a theory with \(n\) independent variables would be in \(A^{(2,n-1)}\) whereas our symplectic multiform is in \(A^{(2,1)}\) so they only coincide in the case where \(n = 2\) (for our case of 1 + 1 field theories). The symplectic multiform is of the form

\[
\Omega = \sum_{j=1}^{n} \omega_j \wedge dx^j, \quad \omega_j \in A^{(2,0)}, \quad j = 1, \ldots, n. \quad (2.27)
\]

The following corollary gives support for our terminology as it is reminiscent of the fact that a symplectic form \(\omega\) is closed in classical mechanics.

**Corollary 2.7.** The symplectic multiform is horizontally closed on the multiform E-L equations

\[
\delta d\mathcal{L} = 0 \implies d\Omega = 0. \quad (2.28)
\]

Proof. The equations are equivalent to \(\delta \mathcal{L} = -d\Omega^{(1)}\), so

\[
0 = \delta^2 \mathcal{L} = -d\delta \Omega^{(1)} = d\Omega^{(1)} = d\Omega. \quad (2.29)
\]
The symplectic multiform $\Omega$ achieves an important unification of the various (standard and dual) symplectic structures appearing in an integrable hierarchy, as originally observed in [28]. When $x_1$ is chosen to be the $x$ variable and $x_j, j \geq 2$ to be the higher times $t_j$ of the hierarchy then $\omega_1$ represents (up to a sign) the usual symplectic form, while each $\omega_j, j \neq 1$ represents the dual symplectic form related to the time $t_j$. For each $j \geq 2$, the multisymplectic form $\Omega$ which would be obtained by considering the Lagrangian $L_1$ as a standalone Lagrangian, as in Section 2.2, is simply obtained by taking $\omega_1 \wedge dx^1 + \omega_j \wedge dx^j$.

We now use the symplectic multiform to obtain the multiform Hamilton equations.

**Proposition 2.8** (multiform Hamilton equations). The multiform Euler-Lagrange equations for the Lagrangian multiform $\mathcal{L}$ are equivalent to

$$\delta H = \sum_{j=1}^{n} dx^j \wedge \widetilde{\partial}_j \Omega. \tag{2.30}$$

Proof. The proof is a simple adaptation of the similar result obtained in [32, Chapter 19] to the multiform case. From the definition of $H$ we get

$$\delta H = -\delta \mathcal{L} - \sum_{j=1}^{n} dx^j \wedge \delta \widetilde{\partial}_j \Omega^{(1)}. \tag{2.31}$$

Thanks to Proposition 2.2 the equations of motion are equivalent to

$$\delta H = d\Omega^{(1)} - \sum_{j=1}^{n} dx^j \wedge \partial_j \Omega^{(1)} = \sum_{j=1}^{n} dx^j \wedge (\partial_j + \tilde{\partial}_j, \delta + \tilde{\partial}_j, \delta) \Omega^{(1)} - \sum_{j=1}^{n} dx^j \wedge \delta \partial_j, \Omega^{(1)}. \tag{2.32}$$

$\Omega^{(1)}$ does not depend explicitly on the space-time variables so $\partial_j \Omega^{(1)} = 0$. The result is obtained by cancellation.

### 2.4 Multi-time Poisson brackets and conservation laws

Continuing with the inspiration given by covariant Hamiltonian field, the next step is to try to construct a Poisson bracket related to our symplectic multiform and investigate how the multiform Hamilton equations can be cast into Poisson Bracket form. Similarly to the situation reviewed at the end of Section 2.2, this can only be done for a restricted class of forms, called Hamiltonian forms. For convenience, we restrict our attention to horizontal forms as this is sufficient for our purposes.

**Definition 2.9** (Hamiltonian forms). We will say that a horizontal form $P$ is Hamiltonian if there exists a (multi)vector field $\xi_P$ such that

$$\xi_P \cdot \Omega = \delta P. \tag{2.33}$$

$\xi_P$ is called the Hamiltonian vector field related to $P$.

**Proposition 2.10.** $P$ can be a Hamiltonian form only if either $P \in \mathcal{A}$ or $P \in \mathcal{A}^{(0,1)}$.

Proof. The proof follows from a simple counting argument. Suppose $P \in \mathcal{A}^{(0,s)}$. Then, since $\Omega \in \mathcal{A}^{(2,1)}$, in order for a $(p,q)$-vector field $\xi_P$ to exist such that

$$\xi_P \cdot \Omega = \delta P \tag{2.34}$$

then necessarily $2 - p = 1$ and $1 - q = s$. So $p = 1$ and $q = 1 - s \geq 0$, and therefore $s$ can only be 0 or 1.

We now produce a statement that is similar to [23, Proposition 2], but for the multiform case. The proof is easily obtained as an extension. We will use this result systematically without quoting it in our examples below.

Let us denote by $\mathcal{S}_\Omega$ the set of basis elements $\delta u^{(i)}_l$ that appear explicitly the symplectic multiform. It is a finite set since $\Omega$ is derived from $\mathcal{L}$ which is assumed to depend on $u^{(i)}_l$ with $|i| \leq m$ for some
\( m \) (finite jet dependence). Hence, we can assume some ordering on \( S_{\Omega} \) such that we can label the \( \delta u^{(i)}_j \)'s as \( \delta v_j, j = 1, \ldots, |S_{\Omega}| \). We then write
\[
\Omega = \sum_{k=1}^{n} \sum_{i<j}^{I_k} \omega^{ij}_k \delta v_i \wedge \delta v_j \wedge dx^k
\]  
(2.35)
where \( I_k \subseteq \{1, \ldots, |S_{\Omega}|\} \) for each \( k = 1, \ldots, n \). Note that each \( \omega^{ij}_k \in A \) so has a dependence on the local coordinates \( u^{(j)}_m \) which we do not show explicitly.

**Proposition 2.11** (Necessary form of a Hamiltonian one-form.) Suppose \( F = \sum_{k=1}^{n} F_k dx^k \), where \( F_k \in A \), is a Hamiltonian form for the multisymplectic form (2.35). Then, for each \( k = 1, \ldots, n \), \( F_k \) can only depend (at most) on \( v_j, j \in I_k \).

We can now define the multi-time Poisson brackets for Hamiltonian forms, in analogy with the covariant Poisson bracket.

**Definition 2.12** (multi-time Poisson brackets). For two Hamiltonian forms \( P \) and \( Q \), of degree respectively \( r \) and \( s \), we define their multi-time Poisson bracket as
\[
\{ |P, Q| \} = (-1)^r \xi_P \lrcorner \delta Q.
\]  
(2.36)
This definition is formally the same as the one given by Kanatchikov, cf (2.17), but we stress that since the degree of the symplectic multiform is \( (2,1) \) (for every \( n \)) is different from the degree of the multisymplectic form, which is \( (2,n-1) \) in general, then the resulting degree of the Poisson bracket of two horizontal forms will be different. In particular, we see that the multi-time Poisson bracket of two horizontal 1-forms is still a horizontal 1-form. The two brackets coincide when \( n = 2 \).

As mentioned before for the covariant Poisson bracket, our definition may lead to issues regarding the Jacobi identity for instance. However, in the spirit of [23], we will investigate this further in [33] in connection with the \( r \)-matrix structure of the multi-time Poisson bracket whereby the Jacobi identity translates into the classical Yang-Baxter equation.

**Theorem 2.13.** On the equations of motion
\[
dF = \xi_F \lrcorner \delta H
\]  
(2.37)
for any Hamiltonian 1-form that does not depend on the independent variables.

**Proof.** Using (2.30) and the antisymmetry of \( \Omega \) we have
\[
\xi_F \lrcorner \delta H = \xi_F \sum_{j=1}^{n} dx^j \wedge \tilde{\partial}_j \lrcorner \Omega = -\sum_{j=1}^{n} dx^j \wedge \xi_F \lrcorner \partial_j \lrcorner \Omega = \sum_{j=1}^{n} dx^j \wedge \tilde{\partial}_j \lrcorner \xi_F \lrcorner \Omega.
\]  
(2.38)
Since \( \xi_F \lrcorner \Omega = \delta F \) we obtain
\[
\xi_F \lrcorner \delta H = \sum_{j=1}^{n} dx^j \wedge \tilde{\partial}_j \lrcorner \delta F.
\]  
(2.39)
Using the property \( \tilde{\partial}_j \lrcorner = \tilde{\partial}_j - \delta \tilde{\partial}_j \),
\[
\xi_F \lrcorner \delta H = \sum_{j=1}^{n} dx^j \wedge \tilde{\partial}_j F - \sum_{j=1}^{n} dx^j \wedge \delta \tilde{\partial}_j F.
\]  
(2.40)
Since $F$ is purely horizontal $\tilde{\partial}_j F = 0$, and since it does not depend explicitly on the space-time variables $\tilde{\partial}_j F = \partial_j F$, so that

$$\xi_F \delta H = \sum_{j=1}^{n} dx^j \wedge \partial_j F = dF. \quad (2.41)$$

If the components $H_{ij}$ of $H$ are Hamiltonian 0-forms, then the previous proposition leads to:

**Corollary 2.14.** On the equations of motion

$$dF = \sum_{i<j=1}^{n} \{ H_{ij}, F \} dx^{ij}. \quad (2.42)$$

for any Hamiltonian 1-form that does not depend on the independent variables.

Proof.

$$dF = \xi_F \delta H = \sum_{i<j=1}^{n} \xi_F \delta H_{ij} \wedge dx^{ij} = - \sum_{i<j=1}^{n} \{ F, H_{ij} \} dx^{ij} = \sum_{i<j=1}^{n} \{ H_{ij}, F \} dx^{ij}. \tag{2.43}$$

This is a generalisation of the usual Hamilton equations in Poisson Bracket form for classical finite-dimensional mechanics $\dot{f} = \{ H, f \}$. In our context, this result turns out to be useful in relation to conservation laws within an integrable hierarchy. Indeed, if $F$ is a 1-form, we have

$$dF = \sum_{j=1}^{n} dx^j \wedge \partial_j F = \sum_{i,j=1}^{n} \partial_i F_j dx^i \wedge dx^j = \sum_{i<j=1}^{n} (\partial_i F_j - \partial_j F_i) dx^i \wedge dx^j \quad (2.43)$$

which means that, in fact if $dF = 0$ on the equations of motion, then

$$\partial_i F_j = \partial_j F_i, \quad \forall i \neq j. \quad (2.44)$$

This suggests the following

**Definition 2.15.** We say that a Hamiltonian 1-form $F$ is a conservation law if $dF = 0$ on the equations of motion.

It is then immediate from Proposition 2.13 that

**Corollary 2.16.** $F$ is a conservation law if and only if

$$\xi_F \delta H = 0 \quad (2.45)$$

This is clearly an extension of the concept of first integral in classical mechanics. As we will show on some examples below, the very definition of a Hamiltonian form being a conservation law can lead to its explicit form. This is a rather elegant byproduct of our approach.

We now address the relationship between the multi-time Poisson bracket that we just defined and the single-time Poisson brackets that can be derived from the single Lagrangians $L_{ij}$ using the usual construction. Starting from the decomposition (2.27), for each $i = 1, \ldots, n$, it is natural to want to define the $i$-th Poisson bracket of two 0-forms $f, g \in A$ as

$$\{ f, g \}_i := -\xi_f^i \delta g, \quad \text{where} \quad \xi_f^i \omega_i = \delta f. \quad (2.46)$$

We remark that there is no sum on the $i$ index. Compared to the standard finite-dimensional case, let us note that this definition requires some care as in general, we cannot guarantee that each $\omega_i$ is non degenerate (see e.g. the KdV example). Therefore, in the above definition we need to do two things. Viewing $\omega_i$ as a linear map from vertical vector fields to vertical 1-forms, we restrict our attention to 0-forms $f$ such that $\delta f$ is in the image of $\omega_i$. In other words, we consider $f$ such that there exists a (vertical) vector field $\xi_f^i$ which satisfies $\delta f = \xi_f^i \omega_i$. In that case, we say that $f$ is Hamiltonian with respect to $\omega_i$. We also remedy the possible non trivial kernel by working modulo it, hence obtaining a non degenerate map, which we keep denoting $\omega_i$, on equivalence classes of vertical vector fields. This has no effect on the above definition of $\{ f, g \}_i$, where $f$ and $g$ are two Hamiltonian 0-forms with respect to $\omega_i$. We work with this understanding in the rest of the paper.
Theorem 2.17 (Decomposition of the multi-time Poisson Bracket). Let $F = \sum_{i=1}^{n} F_i dx^i$ be a Hamiltonian 1-form, then for $i = 1, \ldots, n$, $F_i$ is Hamiltonian with respect to $\omega_i$. Let $G = \sum_{i=1}^{n} G_i dx^i$ be another Hamiltonian 1-form, then the following splitting of the multi-time Poisson bracket holds:

$$\{ [F,G] \} = \sum_{i=1}^{n} \{ F_i, G_i \} dx^i.$$  

(2.47)

Proof. On the one hand, by definition

$$\delta F = \sum_{i=1}^{n} \delta F_i \wedge dx^i,$$

and on the other hand, since $F$ is Hamiltonian

$$\delta F = \xi_F \sum_{i=1}^{n} \omega_i \wedge dx^i = \sum_{i=1}^{n} \xi_{F, \omega_i} \wedge dx^i,$$

(2.48)

hence $\delta F_i = \xi_F,\omega_i$, so $F_i$ is Hamiltonian with respect to $\omega_i$ for each $i = 1, \ldots, n$ and we can take $\xi_{F_i} = \xi_F$ for all $i = 1, \ldots, n$. Note that this gives an idea of how restrictive it is for $F$ to be Hamiltonian. Next, consider the following chain of equalities

$$\{ [F,G] \} = -\xi_{F,\omega} G = -\xi_{F,\omega}(\sum_{i=1}^{n} \delta G_i \wedge dx^i) = -\xi_{F,\omega}(\sum_{i=1}^{n} \xi_{G_i,\omega_i} \wedge dx^i)$$

$$= \sum_{i=1}^{n} \xi_{G_i,\omega_i} \wedge dx^i = \sum_{i=1}^{n} \xi_{G_i,\omega_i} \wedge dx^i = \sum_{i=1}^{n} \{ F_i, G_i \} dx^i,$$

which concludes the proof. 

This is the generalization to an arbitrary number $n$ of flows in an integrable hierarchy of the splitting theorem that was obtained in [23] on examples. Theorem 2.17 provides a general proof, independent of examples, and reproduces the result of [23] in the particular case $n = 2$. This theorem describes the relationship between our multi-time Poisson bracket $\{ \cdot , \}$, encapsulating an arbitrary number of flows in the hierarchy, and the usual and dual single-time Poisson brackets $\{ \cdot , \}_i$, which are related to each flow separately.

3 Example: (potential) KdV hierarchy

In the following we will see the example of the KdV hierarchy with respect to its first two times, so in usual hierarchy notations, we would have $x_1 = x$, $x_2 = t_2$ and $x_3 = t_3$ (if one consider the KdV alone, $t_3$ is simply the time $t$). In fact, we consider the potential form of the KdV hierarchy which is the appropriate form for a Lagrangian formulation. It is known that for KdV hierarchy the even flows are trivial $v_{2k} = 0 \forall k$, so we will also treat the less trivial case of the first two odd times $x_1 = x$, $x_3 = t_3$ and $x_5 = t_5$. We use the Lagrangians multiforms presented in [21].

3.1 Times 1,2 and 3

3.1.1 Multiform Euler-Lagrange equations

We write the Hamiltonian formulation of the first two levels of the (potential) KdV hierarchy, described by the Lagrangian multiform $\mathcal{L} = L_{12} dx^{12} + L_{23} dx^{23} + L_{13} dx^{13}$, where

$$L_{12} = v_1 v_2,$$

(3.1a)

$$L_{23} = -3 v_1^2 v_2 - v_1 v_{112} + v_{11} v_{12} - v_{111} v_2,$$

(3.1b)

$$L_{13} = -2 v_1^3 - v_1 v_{111} + v_1 v_3.$$

(3.1c)
We see that
\[ \frac{\Omega}{\Omega_a} \]
and differential consequences: in particular we have the potential KdV from \( v_{13} = (v_3)_1 = v_{1111} + 6v_1v_{11} \).

### 3.1.2 The symplectic multiform

We are now going to show the procedure to obtain the symplectic multiform from \( \mathcal{L} \) and (3.1). We start by computing the \( \delta \)-differential of the Lagrangian multiform:

\[
\delta \mathcal{L} = v_1 \delta v_2 \wedge dx^{12} + v_2 \delta v_1 \wedge dx^{12} + v_1 \delta v_1 \wedge dx^{23} + v_2 \delta v_3 \wedge dx^{23} + v_1 \delta v_2 \wedge dx^{23} + v_2 \delta v_1 \wedge dx^{23} + v_1 \delta v_3 \wedge dx^{13} + v_2 \delta v_2 \wedge dx^{13} + v_1 \delta v_2 \wedge dx^{13}.
\]

We now use the property \( \delta \delta = -\delta \delta \) on some of the terms to obtain the desired expression \( \delta \mathcal{L} = E(\mathcal{L}) - d\Omega^{\mathcal{L}} \), where \( E(\mathcal{L}) \) is equivalent to (3.1). The reader can verify the following identities

\[
v_1 \delta v_2 \wedge dx^{12} = -v_1 \delta v_2 \wedge dx^{12} - v_1 \delta v_1 \wedge dx^{13} - v_1 \delta v_3 \wedge dx^{13} - v_1 \delta v_2 \wedge dx^{13} - d(-v_1 \delta v \wedge dx^1),
\]

\[
v_2 \delta v_1 \wedge dx^{12} = -v_2 \delta v_1 \wedge dx^{12} - v_2 \delta v_2 \wedge dx^{23} - v_2 \delta v_3 \wedge dx^{23} - v_2 \delta v_1 \wedge dx^{23} - d(v_2 \delta v \wedge dx^2),
\]

\[
(v_3 - v_{111} - 6v_1^2) \delta v_1 \wedge dx^{13} = -(v_3 - v_{111} - 6v_1^2) \delta v \wedge dx^{13} - (v_3 - 6v_1^2) \delta v \wedge dx^{23},
\]

\[
-d(-v_1 \delta v_1 \wedge dx^3 + v_1 \delta v_1 \wedge dx^3 - v_1 \delta v_1 \wedge dx^3).
\]

Using these identities in \( \delta \mathcal{L} \) we get

\[
\delta \mathcal{L} = -2v_1 \delta v \wedge dx^{12} + (-2v_{13} + 2v_{1111} + 12v_1v_{11}) \delta v \wedge dx^{13} + 2v_{12} \delta v_1 \wedge dx^{23} + (-6v_1v_2 - 2v_{112}) \delta v_1 \wedge dx^{23} + (-v_3 + v_{111} + 3v_1^2) \delta v_2 \wedge dx^{23} + v_2 \delta v_3 \wedge dx^{23} + v_1 \delta v_1 \wedge dx^{23} + v_2 \delta v_2 \wedge dx^{23} + v_1 \delta v_3 \wedge dx^{13} + v_2 \delta v_2 \wedge dx^{13} - d(-v_1 \delta v \wedge dx^1 + v_2 \delta v \wedge dx^2) + (v_3 - 6v_1^2) \delta v \wedge dx^3 + v_1 \delta v_1 \wedge dx^3 - v_1 \delta v_1 \wedge dx^3)
\]

\[\equiv E(\mathcal{L}) - d\Omega^{\mathcal{L}}\]

if we define \( \Omega^{\mathcal{L}} = -\delta v_1 \wedge dx^1 + \delta v_2 \wedge dx^2 + (v_3 - 6v_1^2) \delta v \wedge dx^3 + v_1 \delta v_1 \wedge dx^3 - v_1 \delta v_1 \wedge dx^3. \)

We see that \( E(\mathcal{L}) = \delta \mathcal{L} + d\Omega^{\mathcal{L}} = 0 \) is equivalent to the equations (3.1) and differential consequences. The symplectic multiform is then

\[
\Omega = -\delta v_1 \wedge \delta v \wedge dx^1 + \delta v_2 \wedge \delta v \wedge dx^2 + \delta v_3 \wedge \delta v \wedge dx^3 - 2\delta v_{111} \wedge \delta v \wedge dx^3 - 12\delta v_1 \wedge \delta v \wedge dx^3 + 2\delta v_1 \wedge \delta v_1 \wedge dx^3 + \delta v_{11} \wedge dx^3.
\]

### 3.1.3 The Hamiltonian multiform

We can now compute the Hamiltonian multiform \( \mathcal{H} = \sum_{i \leq j} H_{ij} dx^{ij} \), using

\[
H_{ij} = \partial_i \omega_j^{(1)} - \partial_j \omega_i^{(1)} - L_{ij}
\]

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to find

\[ H_{12} = v_1 v_2, \]  
\[ H_{23} = -3v_1^2 v_2 - v_{111} v_2 \]  
\[ H_{13} = v_1 v_3 - 4v_1^2 + v_{111}^2 - 2v_1 v_{111}. \]  

The multiform Hamiltonian equations are obtained as

- \( \delta H_{12} = \tilde{\theta}_{2, \omega_1} - \tilde{\theta}_{1, \omega_2} : \)
  \[ v_1 \delta v_2 + v_2 \delta v_1 = -v_{12} \delta v + v_2 \delta v_1 - v_{12} \delta v + v_1 \delta v_2 \implies v_{12} = 0. \]  

- \( \delta H_{23} = \tilde{\theta}_{3, \omega_2} - \tilde{\theta}_{2, \omega_3} : \)
  \[ -3v_1^2 \delta v_2 - 6v_1 v_2 \delta v_1 - v_{111} \delta v_2 - v_2 \delta v_{111} = v_{23} \delta v - v_3 \delta v_2 - v_{23} \delta v + v_2 \delta v_3 + 2v_{112} \delta v - 2v_2 \delta v_{111} + 12v_1 v_{12} \delta v \]
  \[ -12v_1 v_2 \delta v_1 - 2v_{112} \delta v_1 + 2v_{12} \delta v_{11} \]

which implies the following system of equations

\[ v_2 = 0, \]  
\[ v_{12} = 0, \]  
\[ v_3 - 3v_1^2 = v_{111} = 0, \]  
\[ v_{112} + 6v_1 v_{12} = 0, \]

which implies \( v_{13} - v_{111} - 6v_1 v_{11} = 0. \)

This system of equations is equivalent to (3.1) as expected.

### 3.1.4 Hamiltonian forms

We now describe Hamiltonian forms for this case. A 1-form \( Q = Q_1(v, v_1) dx^1 + Q_2(v, v_2) dx^2 + Q_3(v, v_1, v_3, v_{11}, v_{111}) dx^3 \) for the symplectic multiform \( \Omega \) is Hamiltonian if and only if

\[ \frac{\partial Q_1}{\partial v_1} = -\frac{\partial Q_2}{\partial v_2} = -\frac{\partial Q_3}{\partial v_{11}} = \frac{1}{2} \frac{\partial Q_3}{\partial v_{111}}, \]
\[ \frac{\partial Q_1}{\partial v} = 1 \frac{\partial Q_3}{\partial v_1}. \]  

Its related Hamiltonian vector field is

\[ \xi_Q = \frac{\partial Q_1}{\partial v_1} \partial_v - \frac{\partial Q_1}{\partial v} \partial_{v_1} + \frac{\partial Q_2}{\partial v} \partial_{v_2} + \left( \frac{\partial Q_3}{\partial v} - 6v_1 \frac{\partial Q_3}{\partial v_{111}} \right) \partial_{v_3} + \left( \frac{1}{2} \frac{\partial Q_3}{\partial v_1} - 3v_1 \frac{\partial Q_3}{\partial v_{1111}} \right) \partial_{v_{111}}. \]  

This can be proved as followed: one takes a generic vector field

\[ \xi_Q = A \partial_v + B \partial_{v_1} + C \partial_{v_2} + D \partial_{v_3} + E \partial_{v_{11}} + D \partial_{v_{111}}, \]

and determines the coefficients comparing the right and left hand-side of

\[ \xi_Q \cdot \Omega = \delta Q. \]  

This translates into constraints on the derivatives of \( Q \) with respect to the field and its derivatives, and determines the coefficients of the vector field.

Here we verify that for any Hamiltonian 1-form \( Q \) and modulo the equations of motion

\[ dQ = \xi_Q \cdot \delta H, \]  

or, more explicitly
\[ \partial_1 Q_2 - \partial_2 Q_1 = \xi Q_1 \partial H_{12}, \text{ which means} \]

\[
\frac{\partial Q_2}{\partial v_1} v_1 + \frac{\partial Q_2}{\partial v_2} v_{12} - \frac{\partial Q_1}{\partial v_1} v_2 - \frac{\partial Q_1}{\partial v_2} v_{12} = - \frac{\partial Q_1}{\partial v_1} \partial H_{12} + \frac{\partial Q_2}{\partial v_2} \partial H_{12} = \frac{\partial Q_1}{\partial v_2} v_1 + \frac{\partial Q_2}{\partial v_1} v_1
\]

\[ (3.20) \]

\[ \implies -2v_{12} \frac{\partial Q_1}{\partial v_1} = 0. \]

\[ \partial_2 Q_3 - \partial_3 Q_2 = \xi Q_1 \partial H_{23}, \text{ which means} \]

\[
\frac{\partial Q_3}{\partial v_2} v_2 + \frac{\partial Q_3}{\partial v_3} v_{23} + \frac{\partial Q_3}{\partial v_1} v_{12} + \frac{\partial Q_3}{\partial v_{11}} v_{112} - \frac{\partial Q_2}{\partial v_3} v_3 - \frac{\partial Q_2}{\partial v_2} v_{23}
\]

\[ = - \frac{\partial Q_1}{\partial v_1} \partial H_{23} + \frac{\partial Q_2}{\partial v_2} \partial H_{23} \]

\[ = 6v_1 v_2 \frac{\partial Q_1}{\partial v_2} - (3v_1^2 + v_{111}) \frac{\partial Q_2}{\partial v_2}, \]

which again is

\[ v_2 \frac{\partial Q_3}{\partial v_1} + v_{12} \frac{\partial Q_3}{\partial v_3} + (2v_{12} - 6v_1 v_2) \frac{\partial Q_1}{\partial v_1} + (-v_3^2 + v_{111}) \frac{\partial Q_2}{\partial v_2} = 0. \]

\[ (3.22) \]

\[ \partial_1 Q_3 - \partial_3 Q_1 = \xi Q_1 \partial H_{13}, \text{ which means} \]

\[
\frac{\partial Q_3}{\partial v_1} v_1 + \frac{\partial Q_1}{\partial v_3} v_{111} + \frac{\partial Q_3}{\partial v_3} v_{113} + \frac{\partial Q_3}{\partial v_1} v_{111} - \frac{\partial Q_1}{\partial v_1} v_3 - \frac{\partial Q_3}{\partial v_1} v_{113}
\]

\[ = - \frac{\partial Q_1}{\partial v_1} \partial H_{13} + \frac{\partial Q_2}{\partial v_1} \partial H_{13} + \left( \frac{1}{2} \frac{\partial Q_3}{\partial v_1} - 3v_1 \frac{\partial Q_3}{\partial v_{111}} \right) \partial H_{13} \]

\[ = \left( 12v_1^2 + 2v_{111} - v_3 \right) \frac{\partial Q_1}{\partial v_1} + v_3 \frac{\partial Q_3}{\partial v_3} - 6v_1^2 \frac{\partial Q_3}{\partial v_{111}} + \frac{\partial Q_1}{\partial v_{111}} - 6v_1 v_{111} \frac{\partial Q_1}{\partial v_{111}} \]

\[ = (12v_1^2 + 2v_{111} - v_3 - 6v_1^2) \frac{\partial Q_1}{\partial v_1} + v_3 \frac{\partial Q_3}{\partial v_3} - 6v_1^2 \frac{\partial Q_3}{\partial v_{111}} + \frac{\partial Q_1}{\partial v_{111}} - 6v_1 v_{111} \frac{\partial Q_1}{\partial v_{111}} \]

\[ = (12v_1^2 + 2v_{111} - v_3) \frac{\partial Q_1}{\partial v_1} + v_3 \frac{\partial Q_3}{\partial v_3} - 6v_1^2 \frac{\partial Q_3}{\partial v_{111}} + \frac{\partial Q_1}{\partial v_{111}} - 6v_1 v_{111} \frac{\partial Q_1}{\partial v_{111}} \]

which again is \((2v_3 - 2v_{111} - 12v_1 v_{111}) \frac{\partial Q_1}{\partial v_{111}} = 0. \)

### 3.1.5 Conservation Laws

We can now find a conservation law for the Lagrangian multiform \( \mathcal{L} \), i.e. a Hamiltonian 1-form

\[ F = F_1(v_1)v_1 dx^1 + F_2(v_2) dx^2 + F_3(v_1, v_2, v_3, v_{11}, v_{111}) dx^3 \]

such that \( \xi F \partial H = \xi F \mathcal{H} = 0: \)

- \( \xi F H_{12} = 0 \) means that \( \frac{\partial F}{\partial v_2} v_2 + \frac{\partial F}{\partial v_1} v_1 = 0. \) Since \( \frac{\partial F}{\partial v_2} = - \frac{\partial F}{\partial v_1} \), necessarily \( F_1 = a(v) v_1 + b(v) \) and \( F_2 = -a(v) v_2 + c(v). \) The condition above then translates to

\[ -a'(v) v_1 v_2 - b'(v) v_2 - a'(v) v_1 v_2 + c'(v) v_1 = 0 \implies a'(v) = b'(v) = c'(v) = 0. \]

We will set \( a = 1 \), and \( b = c = 0 \), so we have \( F_1 = v_1 \) and \( F_2 = -v_2. \)

- \( \xi F H_{23} = 6v_1 v_2 \frac{\partial F}{\partial v_2} - (3v_1^2 + v_{111}) \frac{\partial F}{\partial v_2} = 0 \) automatically.

- Because of the Hamiltonianity constraint we have that \( F_3 = -v_3 + 2v_{111} + d(v, v_1). \) Now we solve for \( d \) the equation \( \xi F H_{13} = (12v_1^2 + 2v_{111} - v_3) \frac{\partial d}{\partial v_1} + v_1 \frac{\partial d}{\partial v_3} - 6v_1^2 \frac{\partial d}{\partial v_{111}} + v_1 \frac{\partial d}{\partial v_{111}} - 6v_1 v_{111} \frac{\partial d}{\partial v_{111}} = 0. \)

This implies

\[ \frac{\partial d}{\partial v_1} = 0, \quad \frac{\partial d}{\partial v_1} = 12v_1, \implies d = -6v_1^2. \]

The conservation law is then

\[ F = v_1 dx^1 - v_2 dx^2 + (v_3 + 6v_{111}) dx^3. \]

In fact its differential \( dF \) is

\[ v_{12} dx^{21} + v_{13} dx^{31} - v_{12} dx^{12} - v_{23} dx^{32} + (v_{13} + 2v_{111} + 12v_1 v_{11}) dx^{13} \]

\[ + (v_{23} + 2v_{112} + 12v_1 v_{12}) dx^{23} = \]

\[ -2v_{12} dx^{12} + (-v_{13} + 2v_{111} + 12v_1 v_{11}) dx^{13} + (2v_{112} + 12v_1 v_{12}) dx^{23} \]

which vanishes on the equations of motion.
Another symplectic multiform

We now mention how to compute another symplectic multiform (and its related Hamiltonian multiform). One can perform an equivalent computation to the one above, making different choices as to what to apply $\delta d = -d\delta$ on, and obtain

$$\tilde{\Omega}^{(1)} = -v_1 \delta v \wedge dx^1 + \frac{v_2}{2} \delta v \wedge dx^2 + \frac{1}{2} (v_3 - 9v_1^2 - 3v_{111}) \delta v \wedge dx^3 + v_{11} \delta v_1 \wedge dx^3 - v_1 \delta v_{11} \wedge dx^3. \quad (3.28)$$

It is easy to check that both $\delta \Lambda + d\tilde{\Omega}^{(1)} = 0$ and $d((\Omega^{(1)} - \tilde{\Omega}^{(1)}) = 0$ are equivalent to (3.1). We then define

$$\tilde{\Omega} = -\delta v_1 \wedge \delta v \wedge dx^1 + \frac{1}{2} \delta v_2 \wedge \delta v \wedge dx^2$$
$$+ \frac{1}{2} \delta v_3 \wedge \delta v \wedge dx^3 - 9v_1 \delta v_1 \wedge \delta v \wedge dx^3 - \frac{3}{2} \delta v_{111} \wedge \delta v \wedge dx^3 + 2\delta v_{11} \wedge \delta v_1 \wedge dx^3. \quad (3.29)$$

The coefficients of Hamiltonian multiform $\tilde{\mathcal{H}} = \tilde{H}_{12} dx^{12} + \tilde{H}_{23} dx^{23} + \tilde{H}_{13} dx^{13}$ are

$$\tilde{H}_{12} = \frac{1}{2} v_1 v_2, \quad (3.30a)$$
$$\tilde{H}_{23} = -\frac{3}{2} v_1^2 v_2 - \frac{1}{2} v_2 v_{111}, \quad (3.30b)$$
$$\tilde{H}_{13} = \frac{1}{2} v_1 v_3 + v_1^2 v_3 - \frac{5}{2} v_1 v_{111} \quad (3.30c)$$

and the multiform Hamilton equations for $\tilde{\mathcal{H}}$ and $\tilde{\Omega}$ bring the same set of equations as expected.

3.2 Times 1,3 and 5

3.2.1 The symplectic and Hamiltonian multiform

In the previous section we considered the times 1, 2 and 3 of (potential) KdV hierarchy. We can also describe the odd-time flows 1, 3 and 5, using the Lagrangian multiform $\mathcal{L} = L_{13} dx^{13} + L_{15} dx^{15} + L_{35} dx^{35}$, where

$$L_{13} = -2v_1^3 + v_1 v_3 - v_1 v_{111}, \quad (3.31a)$$
$$L_{15} = -5v_1^4 + 10v_1 v_2^2 + v_1 v_5 - v_{11}^2, \quad (3.31b)$$
$$L_{35} = 6v_1^5 - 10v_1^3 v_3 + 20v_1^3 v_{111} - 15v_1^2 v_{111} + 3v_2^2 v_3 + 3v_2^2 v_{1111} - 10v_1 v_3 v_{111}$$
$$+ 20v_1 v_{11113} - 12v_1 v_{111} v_{111} + 6v_1 v_{111}^2 - 5v_2 v_{111}^2 + 7v_2 v_{111} + v_1 v_{115}$$
$$- v_3 v_{11111} + v_3 v_{1111} - v_1 v_{115} + 2v_1 v_{1111} - 2v_1 v_{1113} + v_1 v_{11111} - v_{1111}^2. \quad (3.31c)$$

The multiform E-L equations are equivalent to

$$v_3 = v_{111} + 3v_1^2, \quad v_5 = v_{1111} + 10v_1^3 + 5v_1^2 + 10v_1 v_{111}. \quad (3.32)$$

and differential consequences. If we define the form $\Omega^{(1)}$ to be

$$\Omega^{(1)} = -v_1 \delta v \wedge dx^1 + (v_3 - 2v_{111} - 6v_1^2) \delta v \wedge dx^3 + v_{11} \delta v_1 \wedge dx^3 - v_1 \delta v_{11} \wedge dx^3$$
$$+ (v_5 - 20v_1^3 - 20v_1 v_{111} - 10v_1^2 - 2v_{11111}) \delta v \wedge dx^5$$
$$+ (20v_1 v_{11} + 2v_{1111}) \delta v_1 \wedge dx^5 - 2v_{111} \delta v_{11} \wedge dx^5, \quad (3.33)$$

one can check that $\delta \Lambda + d\Omega^{(1)} = 0$ are equivalent to (3.32). The symplectic multiform is then

$$\Omega = \omega_1 \wedge dx^1 + \omega_3 \wedge dx^3 + \omega_5 \wedge dx^5, \quad \text{where}$$

$$\omega_1 = \delta v \wedge \delta v_1, \quad (3.34a)$$
$$\omega_3 = \delta v_3 \wedge \delta v - 2\delta v_{11} \wedge \delta v + 2\delta v_{111} \wedge \delta v_1 - 12v_1 \delta v_1 \wedge \delta v, \quad (3.34b)$$
$$\omega_5 = \delta v_5 \wedge \delta v + (60v_1^2 + 20v_{111}) \delta v \wedge \delta v_1 - 20v_1 \delta v_{111} \wedge \delta v - 20v_{11} \delta v_{11} \wedge \delta v$$
$$- 2\delta v_{1111} \wedge \delta v + 20v_1 \delta v_{11} \wedge \delta v_1 + 2\delta v_{111} \wedge \delta v_1 - 2\delta v_{11} \wedge \delta v_1. \quad (3.34c)$$
The Hamiltonian multiform is obtained in the usual way and reads

\[ H_{13} = v_1 v_3 + v_1^2 - 2v_1 v_{111} - 4v_1^3, \]  
\[ H_{15} = v_1 v_5 - 15v_1^3 - 20v_1^2 v_{111} - 2v_{11111} v_1 + 2v_{1111} v_1 - v_1^2, \]  
\[ H_{35} = -10v_1^2 v_3 - 10v_1 v_{1111} v_3 - 5v_1^2 v_3 - v_{1111} v_3 + v_{111} v_5 + 3v_1^2 v_5 - 6v_1^5 - 20v_1^3 v_{111} 
+ 15v_1^2 v_{111} - 3v_1^2 v_{11111} + 12v_1 v_{111111} - 6v_1^2 v_{111} - 7v_1^2 v_{11111} - v_{1111111} + v_{11111}. \]

One can then proceed in a similar way to the 123-times case and verify the validity of the multiform Hamilton equations.

### 3.2.2 Hamiltonian forms and conservation laws

We obtain that a 1-form

\[ F = F_1(v, v_1) dx^1 + F_3(v, v_1, v_3, v_{11}, v_{111}) dx^3 + F_5(v, v_1, v_3, v_{11}, v_{111}, v_{11111}) dx^5 \]  

is Hamiltonian if and only if

\[ \frac{\partial F_1}{\partial v} = \frac{1}{2} \frac{\partial F_3}{\partial v_1} = \frac{1}{2} \frac{\partial F_5}{\partial v_{11}} = - \frac{\partial F_3}{\partial v_3} = - \frac{\partial F_5}{\partial v_5}, \]  

\[ \frac{\partial F_1}{\partial v} = \frac{1}{2} \frac{\partial F_3}{\partial v_1} = \frac{1}{2} \frac{\partial F_5}{\partial v_{11}} = \frac{\partial F_3}{\partial v_1} = \frac{\partial F_5}{\partial v_{11}}. \]

Its related Hamiltonian vector field is

\[ \xi_F = \frac{\partial F_1}{\partial v} \partial v - \frac{\partial F_1}{\partial v_1} \partial v_1 + \frac{\partial F_3}{\partial v_1} \partial v_1 + 10v_1 \frac{\partial F_3}{\partial v_{11}} + 4v_1 \frac{\partial F_3}{\partial v_{11}} + \frac{\partial F_5}{\partial v_1} \partial v_1 \]
\[ + \left( \frac{1}{2} \frac{\partial F_3}{\partial v_1} - 3v_1 \frac{\partial F_3}{\partial v_{11}} - 10v_1 + 10v_1 \frac{\partial F_3}{\partial v_{11}} + 70v_1^2 - 10v_1^4 \frac{\partial F_3}{\partial v_{11}} \right) \partial v_{11} \]
\[ + \left( \frac{1}{2} \frac{\partial F_5}{\partial v_1} - 3v_1 \frac{\partial F_5}{\partial v_{11}} + 35v_1^2 - 5v_{11111} \frac{\partial F_3}{\partial v_{11111}} \right) \partial v_{11111}. \]

From the equations (3.37) one can obtain a conservation law:

\[ F = v_1 dx^1 + (-v_3 + 2v_{111} + 6v_1^2) dx^3 + (-v_5 + 2v_{11111} + 20v_1 v_{111} + 5v_1^2 + 10v_1^3) dx^5. \]

### 4 Example: sine-Gordon hierarchy in light-cone coordinates

In this section we will show another example, i.e. the first two levels of the sine-Gordon hierarchy in light-cone coordinates.

#### 4.1 Multiform Euler-Lagrange equations

A Lagrangian multiform for this set of equations has been obtained for example in [19] and is

\[ \mathcal{L} = L_{12} dx^{12} + L_{13} dx^{13} + L_{23} dx^{23}, \]

where

\[ L_{12} = \frac{1}{2} u_1 u_2 - \cos u, \]
\[ L_{13} = \frac{1}{2} u_1 u_3 + \frac{1}{2} u_1^2 u_2 - \frac{1}{8} u_1^4, \]
\[ L_{23} = -\frac{1}{2} u_2 u_3 + u_{11} u_{12} - u_{11} \sin u + \frac{1}{2} u_1^2 \cos u. \]

The multiform E-L equations \( d \mathcal{L} = 0 \) are equivalent to

\[ u_{12} - \sin u = 0, \quad u_3 - \frac{1}{2} u_1^3 - u_{111} = 0 \]

and differential consequences.
4.2 The symplectic and Hamiltonian multiform

An equivalent computation to the ones above yields the form $\Omega^{(1)}$ as

$$\Omega^{(1)} = -\frac{1}{2} u_1 \delta u \wedge dx^1 + \frac{1}{2} u_2 \delta u \wedge dx^2 - \left(\frac{u_{111}}{2} + \frac{u_3}{4}\right) \delta u \wedge dx^3 + u_{11} \delta u_1 \wedge dx^3. \tag{4.3}$$

The $\delta$-differential of $\Omega^{(1)}$ is the symplectic multiform $\Omega = \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2 + \omega_3 \wedge dx^3$, with

$$\omega_1 = \frac{1}{2} \delta u \wedge \delta u_1, \tag{4.4a}$$

$$\omega_2 = \frac{1}{2} \delta u_2 \wedge \delta u, \tag{4.4b}$$

$$\omega_3 = -\frac{1}{2} \delta u_{111} \wedge \delta u - \frac{3u_1^2}{4} \delta u_1 \wedge \delta u + \delta u_{11} \wedge \delta u_1. \tag{4.4c}$$

The Hamiltonian multiform $H = H_{12} dx^{12} + H_{13} dx^{13} + H_{23} dx^{23}$ is computed as

$$H_{12} = \frac{1}{2} u_1 u_2 + \cos u, \quad H_{13} = -\frac{1}{2} u_1 u_{111} + \frac{1}{2} u_1^2 - \frac{1}{8} u_1^4, \quad H_{23} = -\frac{1}{2} u_2 u_{11} - \frac{1}{4} u_1^2 u_2 + u_{11} \sin u - \frac{1}{2} u_1^2 \cos u. \tag{4.5a-b-c}$$

The multiform Hamilton equations are obtained as $\delta H = \sum_{j=1}^{3} dx^j \wedge \tilde{\partial}_j \Omega$ and are equivalent to the generalised Euler-Lagrange equations, as required.

4.3 Hamiltonian forms and multi-time Poisson brackets

One can then investigate the presence of Hamiltonian forms:

- A 0-form $H(u, u_1, u_2, u_{11}, u_{111})$ is always Hamiltonian, with Hamiltonian vector field

$$\xi_H = \left(\frac{\partial H}{\partial u_1} - 3u_1^2 \frac{\partial H}{\partial u_{111}}\right) \partial u \wedge \partial u_1 - 2 \frac{\partial H}{\partial u_2} \partial u \wedge \partial u_2 + 2 \frac{\partial H}{\partial u_{111}} \partial u \wedge \partial u_{11} + \left(\frac{3}{2} u_1^2 \frac{\partial H}{\partial u_1} - 2 \frac{\partial H}{\partial u_1}\right) \partial u_1 \wedge \partial u_1 - \frac{\partial H}{\partial u_{111}} \partial u_{11} \wedge \partial u_{11}. \tag{4.6}$$

We remark that $\xi_H$ is not unique;

- A 1-form $P = P_1 dx^1 + P_2 dx^2 + P_3 dx^3$ is Hamiltonian if $P_1 = P_1(u, u_1)$, $P_2 = P_2(u, u_2)$, $P_3 = P_3(u, u_1, u_{111}, u_{1111})$, and

$$\frac{\partial P_3}{\partial u_{111}} = 2 \frac{\partial P_1}{\partial u} \quad \frac{\partial P_3}{\partial u_{1111}} = - \frac{\partial P_2}{\partial u}, \tag{4.7}$$

and its related vector field is

$$\xi_P = 2 \frac{\partial P_1}{\partial u} \partial u - 2 \frac{\partial P_1}{\partial u} \partial u_1 + 2 \frac{\partial P_2}{\partial u} \partial u_2 + \left(\frac{\partial P_3}{\partial u_1} - \frac{3}{2} u_1^2 \frac{\partial P_3}{\partial u_{111}}\right) \partial u_{111} + \left(\frac{3}{2} u_1^2 \frac{\partial P_3}{\partial u_1} - \frac{3}{2} \frac{\partial P_3}{\partial u}\right) \partial u_{1111}. \tag{4.8}$$

- The only Hamiltonian 2-forms or 3-forms are the constant ones.

For such forms we can define the multi-time Poisson brackets. The Poisson bracket between an Hamiltonian 0-form $H$ and an Hamiltonian 1-form $P = P_1 dx^1 + P_2 dx^2 + P_3 dx^3$ is $\xi_P H$, therefore

$$\{[H, P]\} = 2 \frac{\partial P_1}{\partial u_1} \frac{\partial H}{\partial u_{11}} - 2 \frac{\partial P_1}{\partial u} \frac{\partial H}{\partial u_{111}} + 2 \frac{\partial P_3}{\partial u_1} \frac{\partial H}{\partial u_{1111}} - 2 \frac{\partial P_3}{\partial u} \frac{\partial H}{\partial u_{11111}}, \tag{4.9}$$

For such forms we can define the multi-time Poisson brackets. The Poisson bracket between an Hamiltonian 0-form $H$ and an Hamiltonian 1-form $P = P_1 dx^1 + P_2 dx^2 + P_3 dx^3$ is $\xi_P H$, therefore

$$\{[H, P]\} = 2 \frac{\partial P_1}{\partial u_1} \frac{\partial H}{\partial u_{11}} - 2 \frac{\partial P_1}{\partial u} \frac{\partial H}{\partial u_{111}} + 2 \frac{\partial P_3}{\partial u_1} \frac{\partial H}{\partial u_{1111}} - 2 \frac{\partial P_3}{\partial u} \frac{\partial H}{\partial u_{11111}}.$$
Thus, the sG example points to a need to extend our approach to conservation laws beyond Hamiltonian 1-forms. Then, on the equations of motion, one checks that
\[ \{P, Q\} = \{P_1, Q_1\} dx^1 + \{P_2, Q_2\} dx^2 + \{P_3, Q_3\} dx^3, \] (4.10)
where
\[ \{P_1, Q_1\} = 2\frac{\partial P_1}{\partial u} \frac{\partial Q_1}{\partial u_11} - 2\frac{\partial P_1}{\partial u_1} \frac{\partial Q_1}{\partial u}, \] (4.11a)
\[ \{P_2, Q_2\} = 2\frac{\partial P_2}{\partial u} \frac{\partial Q_2}{\partial u_11} - 2\frac{\partial P_2}{\partial u_1} \frac{\partial Q_2}{\partial u_2}, \] (4.11b)
\[ \{P_3, Q_3\} = 2\frac{\partial P_3}{\partial u} \frac{\partial Q_3}{\partial u_11} - 2\frac{\partial P_3}{\partial u_1} \frac{\partial Q_3}{\partial u_11} - \frac{3}{2} u_1^2 \frac{\partial P_3}{\partial u_{111}} \frac{\partial Q_3}{\partial u_{111}} - \frac{3}{2} u_1^2 \frac{\partial P_3}{\partial u_{111}} \frac{\partial Q_3}{\partial u_{111}}. \] (4.11c)

Contrary to the pKdV example, (and AKNS example below), for the sG we were not able to find a Hamiltonian 1-form producing conservation laws in the sense of Definition 2.15. However, it is possible to find a non-Hamiltonian 1-form \( F = F_1 dx^1 + F_2 dx^2 + F_3 dx^3 \) that is closed on the equations of motion, as follows:
\[ F_1 = \frac{1}{2} u_1^2, \quad F_2 = -\cos u, \quad F_3 = \frac{3}{8} u_1^4 + u_1 u_{111} - \frac{1}{2} u_1^2. \] (4.12)

Then, on the equations of motion, one checks that
\[ \partial_1 F_2 = \partial_2 F_1, \quad \partial_1 F_3 = \partial_3 F_1, \quad \partial_2 F_3 = \partial_3 F_2. \] (4.13)

Thus, the sG example points to a need to extend our approach to conservation laws beyond Hamiltonian forms.

5 Example: AKNS hierarchy

Our last example deals with the AKNS hierarchy. For this example, we include one more time compared to previous example, to remind the reader that in principle we can keep adding more times in a multiform, corresponding to adding more and more flows in the hierarchy. As becomes clear in this example, the explicit expression soon become cumbersome though.

5.1 Multiform Euler-Lagrange equations

We start from the Lagrangian multiform found in [38]
\[ \mathcal{L} = L_{12} dx^{12} + L_{13} dx^{13} + L_{14} dx^{14} + L_{23} dx^{23} + L_{24} dx^{24} + L_{34} dx^{34}, \]
where
\[ L_{12} = \frac{1}{2} (r_2 - q_2) + \frac{i}{2} q_1 r_1 + \frac{i}{2} q^2 r^2, \] (5.1a)
\[ L_{13} = \frac{1}{2} (r_3 - q_3) - \frac{1}{8} (r_{11} q_{11} - q_{11} r_{11}) - \frac{3}{8} q r (r_1 - q_1), \] (5.1b)
\[ L_{14} = \frac{1}{2} (q_4 - r_4) - \frac{5 i}{16} q r (q_{11} + r_{11}) + \frac{3 i}{16} (q^2 r^2 + q^2 r^2) - \frac{i}{4} q r q_1 r_1 + \frac{i}{8} q_1 r_{11} + \frac{i}{4} q^3 r^3, \] (5.1c)
\[ L_{23} = \frac{1}{4} (q_2 r_{11} - r_{21} q_1) - \frac{i}{8} (q_3 r_1 + r_3 q_1) + \frac{1}{8} (q_1 r_{12} - r_1 q_{12}) + \frac{3 i}{8} r_{11} r_{11} + \frac{i}{4} q r (q_{11} + r_{11}) - \frac{i}{8} (q_1 - q_1)^2 - \frac{i}{2} q^3 r^3. \] (5.1d)
\[
L_{24} = \frac{3}{8} q^2 r^2 (r_1 - q r_1) - \frac{i}{16} (q^2 r_1 r_2 + r^2 q_1 q_2) - \frac{5i}{16} q r (q r_12 + q r_12) - \frac{1}{8} q r (q r_{11} - q r_{11}) \\
- \frac{1}{8} (q^2 r_1 r_1 - r^2 q_1 q_1) - \frac{1}{8} q r_1 (q r_1 - q r_1) + \frac{1}{8} q r (q r_{11} - q r_{11}) + \frac{3i}{8} q r (q r_12 + r_12 q_2) \\
- \frac{i}{8} (q_{111} r_2 + r_{111} q_2) + \frac{i}{16} (q_{1111} r_1 - r_{1111} q_1) + \frac{i}{8} (q_{111} r_12 + r_{111} q_12) - \frac{i}{2} (q r_4 + r_4 q_1).
\]

\[
L_{34} = \frac{i}{8} (q_1 r_{13} + r_1 q_{13}) - \frac{i}{8} (q_{111} r_3 + r_{111} q_3) - \frac{i}{32} q_{111} r_{111} + \frac{i}{32} (q^2 r_{11} + r^2 q_{11}) \\
+ \frac{i}{16} q^2 r_1^2 + \frac{3}{8} q r_1 (q r_1 - q r_1) + \frac{9i}{32} q^4 r^2 (q r_{11} + q r_{11}) - \frac{i}{16} (q^2 r_1 r_3 + r_3 q_1) \\
- \frac{5i}{16} q r (q_{13} + r_{13}) + \frac{i}{4} (q_{111} q_4 - r_{111} q_4) + \frac{3i}{16} q r (q_{111} r_{13} + r_{111} q_{13}) + \frac{i}{16} q r (q_{1111} r_{11}) \\
- \frac{i}{16} q_1 (q r_{11} + r_{11} q_1) - \frac{15i}{16} q^2 r_1^2 q_1 + \frac{3i}{8} q r (q r_{13} + r_{13} q_3) - \frac{1}{8} (q r_{14} - r q_{14}),
\]

The corresponding multiform Euler-Lagrange equations \(\delta dL = 0\) produce the familiar first three levels of the AKNS hierarchy

\[
q_2 - \frac{i}{2} q_{11} + i q^2 r = 0, \quad r_2 + \frac{i}{2} r_{11} - i q r^2 = 0,
\]

\[
q_3 + \frac{1}{4} q_{111} - \frac{3}{2} q r q_1 = 0, \quad r_3 + \frac{1}{4} r_{111} - \frac{3}{2} q r r_1 = 0,
\]

\[
q_4 = -\frac{i}{8} q_{111} - \frac{3i}{4} q^2 r^2 + \frac{i}{4} q^2 r_{11} + \frac{i}{2} q r q_{11} + i q r q_1 + \frac{3i}{4} q r^2 r,
\]

\[
r_4 = \frac{i}{8} q_{111} + \frac{3i}{4} q^2 r^2 - \frac{i}{4} q^2 r_{11} - \frac{i}{2} q r q_{11} - i q r q_1 - \frac{3i}{4} q r^2 r.
\]

### 5.2 The symplectic and Hamiltonian multiforms

As done in the previous two examples, the computation of the form \(\Omega^{(1)}\) from \(dL\) gives

\[
\Omega^{(1)} = \left( -\frac{1}{2} \delta q + \frac{1}{2} q \delta r \right) \wedge dx + \left( \frac{i}{2} q_1 \delta r + \frac{i}{2} r_1 \delta q \right) \wedge dx^2
\]

\[
+ \left( \frac{1}{4} q_{11} - \frac{3}{8} q r^2 \right) \delta q + \left( -\frac{1}{4} q_{11} + \frac{3}{8} q^2 r \right) q r - \frac{1}{8} q_{11} \delta q_1 + \frac{1}{8} q_1 \delta r_1 \right) \wedge dx^3
\]

\[
+ \left( \frac{i}{8} r_{111} - \frac{i}{16} q r_{111} + \frac{3i}{8} q r q_{11} \right) \delta q + \left( -\frac{i}{8} q_{111} - \frac{i}{16} q^2 r_1 + \frac{3i}{8} q r q_1 \right) \delta r
\]

\[
+ \left( \frac{i}{8} r_{111} - \frac{5i}{16} q r^2 \right) \delta q_1 + \left( \frac{i}{8} q_{111} - \frac{5i}{16} q^2 r \right) \delta r_1 \right) \wedge dx^4.
\]

and its \(\delta\)-differential is the symplectic multiform

\[
\Omega = \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2 + \omega_3 \wedge dx^3 + \omega_4 \wedge dx^4,
\]

where

\[
\omega_1 = \delta q \wedge \delta r,
\]

\[
\omega_2 = \frac{i}{2} (\delta q_1 \wedge \delta r + \delta r_1 \wedge \delta q),
\]

\[
\omega_3 = \frac{1}{4} (\delta r_{11} \wedge \delta q - \delta q_{11} \wedge \delta r) + \frac{1}{4} \delta q_1 \wedge \delta r_1 + \frac{3i}{2} q r \delta q \wedge \delta r,
\]

\[
\omega_4 = -\frac{i}{8} \delta r_{111} \wedge \delta q - \frac{i}{8} \delta q_{111} \wedge \delta r + \frac{i}{4} \delta q_1 \wedge \delta q - \frac{i}{4} q^2 r_1 \wedge \delta r
\]

\[
+ i q r \delta q_1 \wedge \delta r + i q r \delta r_1 \wedge \delta q + \frac{i}{2} (q r - q r_1) \delta q \wedge \delta r.
\]
The Hamiltonian multiform $\mathcal{H} = H_{12} \, dx^{12} + H_{13} \, dx^{13} + H_{14} \, dx^{14} + H_{23} \, dx^{23} + H_{24} \, dx^{24} + H_{34} \, dx^{34}$ can now be computed and brings

$$H_{12} = \frac{i}{2} (q_1 r_1 - q^2 r^2),$$

$$H_{13} = \frac{1}{4} (r_1 q_1 - q_1 r_1)$$

$$H_{14} = \frac{i}{8} (q_1^2 r^2 + q^2 r_1^2) + \frac{i}{8} (q_1 r_1 - q_1 r_{111}) + \frac{i}{8} q_1 r_{11} + \frac{i}{2} q r_1 q_1 - \frac{i}{4} q^3 r^3$$

$$H_{23} = \frac{i}{8} q_1 r_{11} - \frac{i q r_1}{4} (q_1 r_1 + q_1 r) + \frac{i}{8} (q r_1 - q r)^2 + \frac{i}{4} q^3 r^3.$$  

$$H_{24} = \frac{q r}{8} (r q_{111} - q r_{111}) + \frac{3}{8} q^2 r^2 (q r_1 - q r_1) + \frac{q_1 r_1}{8} (r q_1 - q r_1)$$

$$H_{34} = \frac{i}{16} (q_1^2 r_1 r_3 + r^2 q_1 q_3) + \frac{i}{32} q_1 q_{11} r_{11} - \frac{i}{32} (q_1^2 r_1^2 + q^2 r_1^2) - \frac{i}{32} q^2 r_1^2$$

$$- 3 i q^4 + \frac{3 i}{16} q^2 r_1^2 (q r_1 + q r_1) - \frac{3 i}{16} q r (r q_{111} + r_1 q_1)$$

$$- \frac{i}{16} q r q_{111} + \frac{i}{16} q_1 r_1 (q r_{111} + q r_1) + \frac{15 i}{16} q^2 r^2 q_1 r_1.$$

The multiform Hamiltonian equations are obtained as $\delta \mathcal{H} = \sum_{j=1}^{4} dx^j \wedge \tilde{J}_j \Omega$. One checks with a direct computation that they indeed reproduce the set of equations (5.2a)-(5.2d). We remark that $H_{12}$ and $H_{13}$ are the covariant Hamiltonian densities of respectively the NLS equations and the modified KdV equation already obtained in [23].

### 5.3 Hamiltonian forms and multi-time Poisson brackets

We have the following facts

- Any 0-form $H$ is Hamiltonian;
- A 1-form

$$F = F_1 (q, r) \, dx^1 + F_2 (q, r, q_1, r_1) \, dx^2 + F_3 (q, r, q_1, r_1, q_{11}, r_{11}) \, dx^3$$

$$+ F_4 (q, r, q_1, r_1, q_{11}, r_{11}, q_{111}, r_{111}) \, dx^4$$

is Hamiltonian if

$$\frac{\partial F_1}{\partial r} = 2 i \frac{\partial F_2}{\partial r} = - i \frac{\partial F_3}{\partial q} = - 8 i \frac{\partial F_4}{\partial q},$$

$$\frac{\partial F_1}{\partial q} = - 2 i \frac{\partial F_2}{\partial q} = - i \frac{\partial F_3}{\partial q_{11}} = 8 i \frac{\partial F_4}{\partial q_{11}},$$

$$\frac{\partial F_2}{\partial r} = 2 i \frac{\partial F_3}{\partial r} = - 4 \frac{\partial F_4}{\partial q},$$

$$\frac{\partial F_2}{\partial q} = - 2 i \frac{\partial F_3}{\partial q} = - 4 \frac{\partial F_4}{\partial q_{11}},$$

$$\frac{\partial F_3}{\partial r} = - i \frac{\partial F_4}{\partial r} = \frac{i}{2} \frac{\partial F_1}{\partial q} + \frac{i}{4} \frac{\partial F_2}{\partial q},$$

$$\frac{\partial F_3}{\partial q} = \frac{i}{2} \frac{\partial F_1}{\partial q} + \frac{i}{4} \frac{\partial F_2}{\partial q} = \frac{i}{4} \frac{\partial F_4}{\partial r},$$

and its related vector field is

$$\xi_F = \frac{\partial F_1}{\partial r} \frac{\partial}{\partial q} - \frac{\partial F_1}{\partial q} \frac{\partial}{\partial r} - 2 i \frac{\partial F_2}{\partial q} \frac{\partial}{\partial q} - 2 i \frac{\partial F_2}{\partial q} \frac{\partial}{\partial r}$$

$$- 4 \left( 6 q r \frac{\partial F_3}{\partial q_{11}} + \frac{\partial F_3}{\partial r} \right) \frac{\partial}{\partial q_{11}} + 4 \left( 6 q r \frac{\partial F_3}{\partial q_{11}} + \frac{\partial F_3}{\partial q_{11}} \right) \frac{\partial}{\partial q_{11}}$$

$$+ 8 i \left( \frac{\partial F_4}{\partial r} + 2 q^2 \frac{\partial F_4}{\partial q_{11}} + 8 q r \frac{\partial F_4}{\partial q_{11}} + 4 (q r_1 - r q_1) \frac{\partial F_4}{\partial q_{111}} \right) \frac{\partial}{\partial q_{111}}$$

$$+ 8 i \left( \frac{\partial F_4}{\partial r} + 2 q^2 \frac{\partial F_4}{\partial q_{11}} + 8 q r \frac{\partial F_4}{\partial q_{11}} - 4 (q r_1 - r q_1) \frac{\partial F_4}{\partial q_{111}} \right) \frac{\partial}{\partial q_{111}}.$$
We can write the general expression of a Hamiltonian 1-form, given the first coefficient $F_1(q, r)$. Since \( \frac{\partial F_1}{\partial r} = 2i\frac{\partial F_2}{\partial q} \) and \( \frac{\partial F_1}{\partial q} = -2i\frac{\partial F_2}{\partial r} \), we need

\[
F_2 = \frac{i}{2} \left( \frac{\partial F_1}{\partial q} q_1 - \frac{\partial F_1}{\partial r} r_1 \right) + a(q, r). \quad (5.10)
\]

\( a(q, r) \) is a term left to determine. Then, since \( \frac{\partial F_2}{\partial q} = -\frac{1}{4} \frac{\partial F_3}{\partial q} \) and \( \frac{\partial F_2}{\partial r} = -\frac{1}{4} \frac{\partial F_3}{\partial r} \), we have

\[
F_3 = -\frac{1}{4} \frac{\partial F_3}{\partial q} q_{11} - \frac{1}{4} \frac{\partial F_3}{\partial r} r_{11} + (\ldots)(q, r, q_1, r_1). \quad (5.11)
\]

Then we use the fact that \( \frac{\partial F_3}{\partial q} = 2i\frac{\partial F_4}{\partial q} \) and \( \frac{\partial F_3}{\partial r} = -2i\frac{\partial F_4}{\partial r} \) to obtain

\[
\frac{\partial F_3}{\partial q} = \frac{1}{4} \left( \frac{\partial^2 F_1}{\partial q \partial r} r_1 - \frac{\partial^2 F_1}{\partial q^2} q_1 \right) + i \frac{\partial a}{\partial q}, \quad \frac{\partial F_3}{\partial r} = \frac{1}{4} \left( \frac{\partial^2 F_1}{\partial q \partial r} q_1 - \frac{\partial^2 F_1}{\partial r^2} r_1 \right) + i \frac{\partial a}{\partial r}. \quad (5.12)
\]

we then use partial integration and find

\[
F_3 = -\frac{1}{4} \frac{\partial F_3}{\partial q} q_{11} - \frac{1}{4} \frac{\partial F_3}{\partial r} r_{11} - \frac{1}{8} \left( \frac{\partial^2 F_1}{\partial q \partial r} r_1 + \frac{\partial^2 F_1}{\partial q^2} q_1 - 2\frac{\partial^2 F_1}{\partial q \partial r} q_1 r_1 \right) + i \frac{\partial a}{\partial q} - \frac{1}{2} \frac{\partial a}{\partial r}. \quad (5.13)
\]

where \( b(q, r) \) is another term left to determine. Similarly we can compute the fourth coefficient \( F_4 \), which is

\[
F_4 = \frac{i}{8} \left( \frac{\partial F_1}{\partial r} r_{11} - \frac{\partial F_1}{\partial q} q_{11} + \frac{\partial^2 F_1}{\partial r^2} r_1 r_1 - \frac{\partial^2 F_1}{\partial q^2} q_1 q_1 + \frac{\partial^2 F_1}{\partial q \partial r} q_1 r_1 \right) + i \frac{\partial a}{\partial q} - \frac{1}{2} \frac{\partial a}{\partial r}. \quad (5.14)
\]

The only Hamiltonian 2-forms or 3-forms are the constant ones.

For Hamiltonian forms we can define the multi-time Poisson brackets. The Poisson bracket between a 0-form \( H(q, r, q_1, r_1, q_{11}, r_{11}, q_{111}, r_{111}) \) and an Hamiltonian 1-form \( P = P_1 dx^1 + P_2 dx^2 + P_3 dx^3 + P_4 dx^4 \) is \( \xi_P H \):
\[ \{P_3, Q_3\}_3 = 4 \left( \frac{\partial P_3}{\partial r_{11}} \frac{\partial Q_3}{\partial q} - \frac{\partial P_3}{\partial q} \frac{\partial Q_3}{\partial r_{11}} - \frac{\partial P_3}{\partial q_{11}} \frac{\partial Q_3}{\partial r} + \frac{\partial P_3}{\partial r} \frac{\partial Q_3}{\partial q_{11}} \right), \]

\[ \{P_4, Q_4\}_4 = -8i \left( \frac{\partial P_4}{\partial q_{11}} \frac{\partial Q_4}{\partial r} - \frac{\partial P_4}{\partial q} \frac{\partial Q_4}{\partial q_{11}} \right) + 8i \left( \frac{\partial P_4}{\partial q} \frac{\partial Q_4}{\partial q_{11}} - \frac{\partial P_4}{\partial q_{11}} \frac{\partial Q_4}{\partial q} \right) - 8i \left( \frac{\partial P_4}{\partial q_{11}} \frac{\partial Q_4}{\partial r_{11}} - \frac{\partial P_4}{\partial q} \frac{\partial Q_4}{\partial q_{11}} \right) + 8i \left( \frac{\partial P_4}{\partial q_{11}} \frac{\partial Q_4}{\partial r_{11}} - \frac{\partial P_4}{\partial q} \frac{\partial Q_4}{\partial q_{11}} \right) \]

\[ + 64iqr \left( \frac{\partial P_4}{\partial q_{111}} \frac{\partial Q_4}{\partial r_{111}} - \frac{\partial P_4}{\partial q} \frac{\partial Q_4}{\partial q_{111}} \right) - 64iqr \left( \frac{\partial P_4}{\partial q_{111}} \frac{\partial Q_4}{\partial r_{111}} - \frac{\partial P_4}{\partial q} \frac{\partial Q_4}{\partial q_{111}} \right) \]

\[ + 16iqr^2 \left( \frac{\partial P_4}{\partial q_{1111}} \frac{\partial Q_4}{\partial r_{1111}} - \frac{\partial P_4}{\partial q} \frac{\partial Q_4}{\partial q_{1111}} \right) + 16iqr^2 \left( \frac{\partial P_4}{\partial q_{1111}} \frac{\partial Q_4}{\partial r_{1111}} - \frac{\partial P_4}{\partial q} \frac{\partial Q_4}{\partial q_{1111}} \right) \]

\[ + 32i(qr - qr) \left( \frac{\partial P_4}{\partial q_{1111}} \frac{\partial Q_4}{\partial q_{1111}} - \frac{\partial P_4}{\partial q_{1111}} \frac{\partial Q_4}{\partial q_{1111}} \right) \]

Using this we can read the single-time Poisson brackets: \( \{ , \} \) is the usual equal-time Poisson bracket of the AKNS hierarchy, which in the traditional infinite dimensional setting provide the first structure (in the sense of bi-Hamiltonian theory) for the whole hierarchy, while \( \{ , \} \) are the dual Poisson Bracket of respectively the NLS and mKdV which can be found in [28]. We remark the presence of a minus sign in front of \( dx^4 \), which is just so \( \{ , \} \) reproduces the usual single-time Poisson Brackets with the right sign. It remains an open question to relate our findings with the traditional theory of bi-Hamiltonian structures à la Magri [39].

### 5.4 Conservation laws

Since the coefficients of the Hamiltonian multiform are Hamiltonian, the multiform Hamilton equations in a Poisson bracket form are

\[ df = \xi_F \delta H = \sum_{i<j=1}^4 \langle [H_{ij}, F] \rangle dx^j \]

for any Hamiltonian 1-form \( F = F_1 dx^1 + F_2 dx^2 + F_3 dx^3 + F_4 dx^4 \). We can also find the first conservation laws for the AKNS hierarchy, i.e. \( F \) is a conservation law if and only if

\[ \langle [H_{ij}, F] \rangle = 0 \quad \forall i < j. \]

We can solve the latter equation in the space of Hamiltonian forms (see previous section for the general expression of the coefficients) to find a conservation law. From \((i, j) = (1, 2)\) we get

\[ [H_{12}, F] = -iqr \frac{\partial F_1}{\partial q} + iq^2 \frac{\partial F_1}{\partial q} + i \frac{\partial^2 F_1}{2 \partial q^2} q^2 - \frac{i}{2} \frac{\partial^2 F_1}{\partial q^2} q^2 + \frac{\partial}{\partial r} q - \frac{\partial}{\partial r} q = 0. \]

This translates into \( r \frac{\partial F_1}{\partial q} = q \frac{\partial F_1}{\partial q} \) and \( \frac{\partial F_1}{\partial q} = 0 \), and therefore \( F_1 = qr \), and \( \frac{\partial}{\partial q} = \frac{\partial}{\partial q} = 0 \), so therefore \( a \) is constant, which we set to zero. The coefficients become then

\[ F_1 = qr, \quad F_2 = \frac{i}{2} (qr - qr), \quad F_3 = -\frac{1}{4} q_{11} - \frac{1}{4} qr_{11} + \frac{1}{4} q_{11} + b(q, r), \]

\[ F_4 = \frac{i}{8} (qr_{11} + q_{11} r_{11} - q_{11} r_{11}) + \frac{i}{2} \left( \frac{\partial b}{\partial q} q_1 - \frac{\partial b}{\partial q} r_1 \right) + c(q, r) \]

with \( b \) and \( c \) left to determine. From \((i, j) = (1, 3)\) we get

\[ [H_{13}, F] = -\frac{3}{2} (qr^2 q_1 + q^2 r_1) + q_1 \frac{\partial b}{\partial q} r_1 + r_1 \frac{\partial b}{\partial q} = 0, \]

and therefore \( b = \frac{3}{2} q^2 r_1^2 \). The fourth coefficient becomes then \( F_4 = \frac{i}{8} (qr_{11} - qr_{11}) + \frac{i}{2} (q_{11} r_1 - q_{11} r_1) + \frac{3}{2} q^2 (q_1 r - qr_1) + c(q, r) \). It can be verified by looking at the coefficient \((1, 4)\) that we have a
conservation law when \( c = 0 \). The conservation law is then

\[
F = qr \, dx^1 + \frac{i}{2} (qr - r_1 q) \, dx^2 + \frac{1}{4} (3q^2 r^2 + qr_1 - q_1 r - r_{11} q) \, dx^3 \\
+ \left( \frac{3i}{8} (qr_{11} - rq_{111}) + \frac{i}{8} (q_{11} r_1 - q_1 r_{11}) + \frac{3i}{4} qr (q_1 r - qr_1) \right) \, dx^4.
\]  

(5.23)

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