Height zeta functions of toric bundles over flag varieties

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1 Introduction

1.1 Let $X$ be a nonsingular projective algebraic variety over a number field $F$. Let $\mathcal{L} = (L, (\| \cdot \|_v)_v)$ be a metrized line bundle on $X$, i.e., a line bundle $L$ together with a family of $v$-adic metrics, where $v$ runs over the set $\text{Val}(F)$ of places of $F$. Associated to $\mathcal{L}$ there is a height function

$$H_{\mathcal{L}} : X(F) \to \mathbb{R}_{>0}$$

on the set $X(F)$ of $F$-rational points of $X$ (cf. [24, 18] for the definitions of $v$-adic metric, metrized line bundle and height function). For appropriate subvarieties $U \subset X$ and line bundles $L$ we have

$$N_U(\mathcal{L}, H) := \{ x \in U(F) \mid H_{\mathcal{L}}(x) \leq H \} < \infty$$

for all $H$ (e.g., this holds for any $U$ if $L$ is ample). We are interested in the asymptotic behavior of this counting function as $H \to \infty$. It is expected that the behavior of such asymptotics can be described in geometric terms ([1, 9]).
Let
\[ \Lambda_{\text{eff}}(X) := \sum_{H^0(X,L) \not= 0} \mathbb{R}_{\geq 0}[L] \subset \text{Pic}(X)_\mathbb{R} \]
be the closed cone in \( \text{Pic}(X)_\mathbb{R} \) generated by the classes of effective divisors ([L] denotes the class of the line bundle \( L \) in \( \text{Pic}(X) \)). Let \( L \) be a line bundle on \( X \) such that \([L]\) lies in the interior of \( \Lambda_{\text{eff}}(X) \). Define
\[ a(L) := \inf \{ a \in \mathbb{R} \mid a[L] + [K_X] \in \Lambda_{\text{eff}}(X) \}, \]
where \( K_X \) denotes the canonical line bundle on \( X \). Assume that \( \Lambda_{\text{eff}}(X) \) is a finitely generated polyhedral cone. For \( L \) as above we let \( b(L) \) be the codimension of the minimal face of \( \Lambda_{\text{eff}}(X) \) which contains \( a(L)[L] + [K_X] \).

By a Tauberian theorem (cf. [8], Théorème III), the asymptotic behavior of \( N_U(\mathcal{L}, H) \) can be determined if one has enough information about the height zeta function
\[ Z_U(\mathcal{L}, s) := \sum_{x \in U(F)} H_{\mathcal{L}}(x)^{-s}. \]
More precisely, suppose that \( Z_U(\mathcal{L}, s) \) converges for \( \text{Re}(s) \gg 0 \), that it has an abscissa of convergence \( a > 0 \) and that it can be continued meromorphically to a half-space beyond the abscissa of convergence. Suppose further that there is a pole of order \( b \) at \( s = a \) and that there are no other poles in this half-space. Then
\[ N_U(\mathcal{L}, H) = c H^a (\log H)^{b-1} (1 + o(1)) \]
for \( H \to \infty \) and
\[ c = \frac{1}{(b-1)!a} \lim_{s \to a} (s - a)^b Z_U(\mathcal{L}, s). \]
It is conjectured that for appropriate \( U \) and \( \mathcal{L} \) one has \( a = a(L) \) and \( b = b(L) \) (cf. [1, 9]). Moreover, there is a conjectural framework how to interpret the constant \( c \) (cf. [18, 3]). There are examples which demonstrate that this geometric “prediction” of the asymptotic cannot hold in complete generality, even for smooth Fano varieties (cf. [8]). Our goal is to show that the conjectures do hold for a class of varieties closely related to linear algebraic groups. Our results are a natural extension of corresponding results for flag varieties (cf. [9]) and toric varieties (cf. [3, 4]). We proceed to describe the class of varieties under consideration.
1.2 Let $G$ be a semi-simple simply connected split algebraic group over $F$ and $P \subset G$ an $F$-rational parabolic subgroup of $G$. Let $T$ be a split algebraic torus over $F$ and $X$ a projective nonsingular equivariant compactification of $T$. A homomorphism $\eta : P \to T$ gives rise to an action of $P$ on $X \times G$ and the quotient $Y := (X \times G)/P$ is again a nonsingular projective variety over $F$. There is a canonical morphism $\pi : Y \to W := P \setminus G$ such that $Y$ becomes a locally trivial fiber bundle over $W$ with fiber $X$. Corresponding to a character $\lambda \in X^\ast(P)$ there is a line bundle $L_\lambda$ on $W$ and the assignment $\lambda \mapsto L_\lambda$ gives an isomorphism $X^\ast(P) \to \text{Pic}(W)$.

The toric variety $X$ can be described combinatorially by a fan $\Sigma$ in the dual space of the space of characters $X^\ast(T)_R$. Let $PL(\Sigma)$ be the group of $\Sigma$-piecewise linear integral functions on the dual space of $X^\ast(T)_R$. Any $\varphi \in PL(\Sigma)$ defines a line bundle $L_\varphi$ on $X$ which is equipped with a canonical $T$-linearization and we get an isomorphism $PL(\Sigma) \simeq \text{Pic}^T(X)$. There is a canonical exact sequence

$$0 \to X^\ast(T) \to PL(\Sigma) \to \text{Pic}(X) \to 0.$$  

The $T$-linearization of $L_\varphi$ allows us to define a line bundle $L_\varphi^Y$ on $Y$ and this gives a homomorphism $PL(\Sigma) \to \text{Pic}(Y)$. One can show that there is an exact sequence

$$(1) \quad 0 \to X^\ast(T) \to PL(\Sigma) \oplus X^\ast(P) \to \text{Pic}(Y) \to 0.$$  

Denote by $Y^o := (T \times G)/P$ the open subvariety of $Y$ obtained as the twist of $T$ with $W$.

1.3 By means of a maximal compact subgroup in the adelic group $G(A)$ we can introduce metrics on the line bundles $L_\lambda$. The corresponding height zeta functions are Eisenstein series:

$$\sum_{w \in W(F)} H_{\mathcal{L}_\lambda}(w)^{-s} = E_\mathcal{P}^G(s\lambda - \rho_P, 1_G).$$  

On the other hand, for any $\varphi \in PL(\Sigma)$ there is a function

$$H_\Sigma(\cdot, \varphi) : T(A) \to \mathbb{R}_{>0}$$  

such that $H_\Sigma(x, \varphi)^{-1}$ is the height of $x \in T(F)$ with respect to a metrization $\mathcal{L}_\varphi$ of $L_\varphi$. This metrization induces a metrization $\mathcal{L}_\varphi^Y$ of the line bundle $L_\varphi^Y$ on $Y$.  

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Let \((x, \gamma) \in T(F) \times G(F)\) and let \(y\) be the image of \((x, \gamma)\) in \(Y(F)\). Then there is a \(p_\gamma \in P(A)\) such that

\[
H_{\xi y}(y) = H_\Sigma(x \eta(p_\gamma), \varphi)^{-1}.
\]

Hence we may write formally

\[
Z_Y(\mathcal{L}_\xi^Y \otimes \pi^* L_\lambda, s) = \sum_{\gamma \in P(F) \backslash G(F)} \sum_{x \in T(F)} H_\Sigma(x \eta(p_\gamma), s \varphi).
\]

Now we apply Poisson’s summation formula for the torus and get

\[
\sum_{x \in T(F)} H_\Sigma(x \eta(p_\gamma), s \varphi) = \int_{(T(A)/T(F))^*} \hat{H}_\Sigma(\chi, s \varphi) \chi(\eta(p_\gamma))^{-1} d\chi,
\]

where \(\hat{H}_\Sigma(\cdot, s \varphi)\) denotes the Fourier transform of \(H_\Sigma(\cdot, s \varphi)\) and \((T(A)/T(F))^*\) is the group of unitary characters of \(T(A)\) which are trivial on \(T(F)\) equipped with the orthogonal measure \(d\chi\). Actually, it is sufficient to consider only those characters which are trivial on the maximal compact subgroup \(K_T\) of \(T(A)\), because the function \(H_\Sigma(\cdot, \varphi)\) is invariant under \(K_T\). The expression (3) can now be put into (2), and after interchanging summation and integration the result is

\[
Z_Y(\mathcal{L}_\xi^Y \otimes \pi^* L_\lambda, s) = \int_{(T(A)/T(F)K_T)^*} \hat{H}_\Sigma(\chi, s \varphi) E_{\xi}^G(s \lambda - \rho_P, (\chi \circ \eta)^{-1}) d\chi.
\]

where \(E_{\xi}^G(s \lambda - \rho_P, \xi) = E_{\xi}^G(s \lambda - \rho_P, \xi, 1_G)\) is the Eisenstein series twisted by a character \(\xi\) of \(P(A)\). This is the starting point for the investigation of the height zeta function. To get an expression which is more suited for our study we decompose the group of characters \((T(A)/T(F)K_T)^*\) into a continuous and a discrete part, i.e.,

\[
(T(A)/T(F)K_T)^* = X^*(T)R \oplus \mathcal{U}_T,
\]

where \(X^*(T)R\) is the continuous part and \(\mathcal{U}_T\) is the discrete part. The right-hand side of (4) is accordingly

\[
\int_{X^*(T)R} \left\{ \sum_{\chi \in \mathcal{U}_T} \hat{H}_\Sigma(\chi, s \varphi + ix) E_{\xi}^G(s \lambda - \rho_P - i\eta(x), (\chi \circ \eta)^{-1}) \right\} dx.
\]

Recall that we would like to show that this function which is defined for \(\Re(s) \gg 0\) (assuming that \((\varphi, \lambda)\) is contained in a convex open cone) can be continued meromorphically
beyond the abscissa of convergence. To achieve this we need more information on the function under the integral sign in (3). First we have to determine the singularities of 

\((\varphi, \lambda) \mapsto \hat{H}_{\Sigma}(\chi, \varphi) E_P^G(\lambda - \rho_P, (\chi \circ \eta)^{-1})\)

near the cone of absolute convergence. This is possible because \(\hat{H}_{\Sigma}\) can be calculated rather explicitly and it is not so difficult to determine the singular hyperplanes of the Eisenstein series with characters. The next step consists in an iterated application of Cauchy’s residue formula to the integral over the real vector space \(X^*(T)_\mathbb{R}\). This can be done only if one knows that

\[
\sum_{\chi \in U_T} \hat{H}_{\Sigma}(\chi, s\varphi + ix) E_P^G(s\lambda - \rho_P - i\tilde{\eta}(x), (\chi \circ \eta)^{-1})
\]

satisfies some growth conditions when \(x \in X^*(T)_\mathbb{R}\) tends to infinity. This is true for the function \(x \mapsto \hat{H}_{\Sigma}(\chi, s\varphi + ix)\) thanks to the explicit expression mentioned above. The absolute value of the Eisenstein series \(E_P^G(s\lambda - \rho_P - i\tilde{\eta}(x), (\chi \circ \eta)^{-1})\) will in general increase for \(x \to \infty\) if \(\text{Re}(s)\lambda - \rho_P\) is not contained in the cone of absolute convergence. However, if \(\text{Re}(s)\lambda - \rho_P\) is sufficiently close to the boundary of that cone, this increasing behavior is absorbed by the decreasing behavior of \(\hat{H}_{\Sigma}(\chi, s\varphi + ix)\).

Therefore, we may apply Cauchy’s residue theorem and show that (4) can be continued meromorphically to a larger half-space and that there are no poles (in \(s\)) with non-zero imaginary part.

The Tauberian Theorem can now be used to prove asymptotic formulas for the counting function \(N_{Y^\infty}(L_{\varphi}^Y \otimes \pi^*L_{\lambda}, H)\) provided that one knows the order of the pole of the height zeta function. This problem can be reduced to the question whether the “leading term” of the Laurent series of (3) does not vanish. That this is indeed so will be shown in section 6.

1.4 We have restricted ourself to the case of split tori and split groups because this simplifies some technical details. The general case can be treated similarly.

We consider these results as an important step towards an understanding of the arithmetic of spherical varieties. For example, choosing \(P = B\) a Borel subgroup, \(T = B/U\) where \(U\) is the unipotent radical of \(B\) and \(\eta : B \to T\) the natural projection, we obtain an equivariant compactification of \(U\backslash G\), a horospherical variety.
We close this introduction with a brief description of the remaining sections. Section 2 recalls the relevant facts we need concerning generalized flag varieties, i.e., description of line bundles on $W = P\backslash G$, the cone of effective divisors in $\text{Pic}(W)_R$, metrization of line bundles, height zeta functions. The exposition is based entirely on the paper [9].

The next section contains the corresponding facts for toric varieties. It is a summary of a part of [2]. We give the explicit calculation of the Fourier transform $\hat{H}_\Sigma(\cdot, \varphi)$ and show that Poisson’s summation formula can be used to give an expression of the height zeta function $Z_T(\mathcal{L}_\varphi, s)$.

In section 4 we introduce twisted products, discuss line bundles on these, the Picard group (cf. [1]), metrizations of line bundles etc. It ends with the formula (1) for the height zeta function $Z_{Y^0}(\mathcal{L}_\varphi^Y \otimes \pi^*\mathcal{L}_\lambda, s)$ in the domain of absolute convergence.

The first part of section 5 explains the method for the proof that the height zeta function can be continued meromorphically to a half-space beyond the abscissa of absolute convergence. Moreover, we state a theorem which gives a description of the coefficient of the Laurent series at the pole in question. This coefficient will be the leading one, provided that it does not vanish. One can relate the coefficient to arithmetic and geometric invariants of the pair $(U, \mathcal{L})$ but we decided not to pursue this, since there are detailed expositions of all the necessary arguments in [18, 2, 6].

These two theorems (meromorphic continuation of certain integrals and the description of the coefficient) will be proved in a more general context in section 7. The second part of section 5 contains the proof that the hypothesis of these theorems are fulfilled in our case. It ends with the main theorem on the asymptotic behavior of the counting function $N_{Y^0}(\mathcal{L}, H)$, assuming that the coefficient of the Laurent series mentioned above does not vanish. Section 6 is devoted to the proof of this fact. In the last section we prove some statements on Eisenstein series (well-known to the experts) which are used in section 5.

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Some notations. In this paper $F$ always denotes a fixed algebraic number field. The set of places of $F$ will be denoted by $\text{Val}(F)$ and the subset of archimedean places by $\text{Val}_\infty(F)$. We shall write $v \mid \infty$ if $v \in \text{Val}_\infty(F)$ and $v \nmid \infty$ if $v \notin \text{Val}_\infty(F)$. For any place $v$ of $F$ we denote by $F_v$ the completion of $F$ at $v$ and by $\mathcal{O}_v$ the ring of $v$-adic integers (for $v \nmid \infty$). The local absolute value $| \cdot |_v$ on $F_v$ is the multiplier of the Haar measure,
i.e., \( d(ax) = |a|_v dx_v \) for some Haar measure \( dx_v \) on \( F_v \). Let \( q_v \) be the cardinality of the residue field of \( F_v \) for non-archimedean valuations and put \( q_v = e \) for archimedean valuations. We denote by \( A \) the adele ring of \( F \). For any algebraic group \( G \) over \( F \) we denote by \( X^*(G) \) the group of (algebraic) characters which are defined over \( F \).

## 2 Generalized flag varieties

### 2.1 Let \( G \) be a semi-simple simply connected linear algebraic group which is defined and split over \( F \). We fix a Borel subgroup \( P_0 \) over \( F \) and a Levi decomposition \( P_0 = S_0U_0 \) with a maximal \( F \)-rational torus \( S_0 \) of \( G \). Denote by \( P \) a standard (i.e., containing \( P_0 \)) parabolic subgroup and by \( W = P\backslash G \) the corresponding flag variety. The quotient morphism \( G \rightarrow W \) will be denoted by \( \pi_W \). Any character \( \lambda \in X^*(P) \) defines a line bundle \( L_\lambda \) on \( W \) by

\[
\Gamma(U, L_\lambda) := \{ f \in \mathcal{O}_G(\pi^{-1}_W(U)) \mid f(pg) = \lambda(p)^{-1}f(g) \ \forall g \in \pi^{-1}_W(U), p \in P \}.
\]

The assignment \( \lambda \mapsto L_\lambda \) gives an isomorphism (because \( G \) is assumed to be simply connected)

\[ X^*(P) \rightarrow \text{Pic}(W) \]

(cf. [21], Prop. 6.10). The anti-canonical line bundle \( \omega_W^\vee \) corresponds to \( 2\rho_P \) (the sum of roots of \( S_0 \) occurring in the unipotent radical of \( P \)).

### 2.2 These line bundles will be metrized as follows. Choose a maximal compact subgroup \( K_G = \prod_v K_{G,v} \subset G(A) \) (\( K_{G,v} \subset G(F_v) \)), such that the Iwasawa decomposition

\[ G(A) = P_0(A)K_G \]

holds. Let \( v \in \text{Val}(F) \) and \( w \in W(F_v) \). Choose \( k \in K_{G,v} \) which is mapped to \( w \) by \( \pi_W \). For any local section \( s \) of \( L_\lambda \) at \( w \) we define

\[ \|w^*s\|_w := |s(k)|_v. \]
This gives a $v$-adic norm $\| \cdot \|_w : w^*L_\lambda \to \mathbb{R}$ and we see that the family $\| \cdot \|_v := (\| \cdot \|_w)_{w \in W(F)}$ is a $v$-adic metric on $L_\lambda$. The family $(\| \cdot \|_v)_{v \in \text{Val}(F)}$ will then be an adelic metric on $L_\lambda$ (cf. [18] for $\lambda = 2\rho_P$ and [24] for the definitions of “$v$-adic metric” and “adelic metric”). The metrized line bundle $(L_\lambda, (\| \cdot \|_v)_v)$ will be denoted by $L_\lambda$.

2.3 Define a map
\[ H_P = H_{P,K_G} : G(\mathbb{A}) \to \text{Hom}_\mathbb{C}(X^*(P)_\mathbb{C}, \mathbb{C}) \]
by $\langle \lambda, H_P(g) \rangle = \log(\prod_v |\lambda(p_v)|_v)$ for $g = pk$ with $p = (p_v)_v \in P(\mathbb{A}), k \in K_G$ and $\lambda \in X^*(P)$. For $w = \pi_W(\gamma) \in W(F)$ and $\gamma \in G(F)$ a simple computation ([4]) shows that
\[ H_{\mathcal{L}_\lambda}(w) = e^{-\langle \lambda, H_P(\gamma) \rangle}. \]
The height zeta function
\[ Z_W(\mathcal{L}_\lambda, s) = \sum_{w \in W(F)} H_{\mathcal{L}_\lambda}(w)^{-s} \]
is therefore an Eisenstein series
\[ E_P^G(s\lambda - \rho_P, 1_G) = \sum_{\gamma \in P(F) \backslash G(F)} e^{\langle s\lambda, H_P(\gamma) \rangle}. \]
To describe the domain of absolute convergence of this series we let $\Delta_0$ be the basis of positive roots of the root system $\Phi(S_0, G)$ which is determined by $P_0$. For any $\alpha \in \Delta_0$ denote by $\hat{\alpha}$ the corresponding coroot. For $\lambda \in X^*(P) = X^*(S_0)$ we define $\langle \lambda, \alpha \rangle$ by $\langle \lambda \circ \hat{\alpha}, t \rangle = t^{\langle \lambda, \alpha \rangle}$ and extend this linearly in $\lambda$ to $X^*(P_0)_\mathbb{C}$. Restriction of characters defines an inclusion $X^*(P) \to X^*(P_0)$. Let
\[ \Delta_0^P = \{ \alpha \in \Delta_0 | \langle \cdot, \alpha \rangle \text{ vanishes on } X^*(P) \}, \quad \Delta_P = \Delta_0 - \Delta_0^P. \]
Put
\[ X^*(P)^+ = \{ \lambda \in X^*(P)_\mathbb{R} | \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta_P \}. \]
By [10], Théorème 3, the Eisenstein series
\[ E_P^G(\lambda, g) = \sum_{\gamma \in P(F) \backslash G(F)} e^{\langle \lambda + \rho_P, H_P(\gamma g) \rangle} \]
converges absolutely for $\text{Re}(\lambda) - \rho_P$ in $X^*(P)^+$ and it can be meromorphically continued to $X^*(P)_\mathbb{C}$ (cf. [10], IV, 1.8). The closure of the image of $X^*(P)^+$ in $\text{Pic}(W)_\mathbb{R}$ is the cone $\Lambda_{\text{eff}}(W)$ generated by the effective divisors on $W$ ([13], II, 2.6).
3 Toric varieties

3.1 Let $T$ be a split algebraic torus of dimension $d$ over $F$. We put $M = X^*(T)$ and $N = \text{Hom}(M, \mathbb{Z})$. Let $\Sigma$ be a complete regular fan in $N_{\mathbb{R}}$ such that the corresponding smooth toric variety $X = X_\Sigma$ is projective (cf. [2, 17]). The variety $X$ is covered by affine open sets

$$U_\sigma = \text{Spec}(F[M \cap \tilde{\sigma}]),$$

where $\sigma$ runs through $\Sigma$ and $\tilde{\sigma}$ is the dual cone

$$\tilde{\sigma} = \{ m \in M_{\mathbb{R}} | n(m) \geq 0 \ \forall \ n \in \sigma \}.$$  

Denote by $PL(\Sigma)$ the group of $\Sigma$-piecewise linear integral functions on $N_{\mathbb{R}}$. By definition, a function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$ belongs to $PL(\Sigma)$ if and only if $\varphi(N) \in \mathbb{Z}$ and the restriction of $\varphi$ to every $\sigma \in \Sigma$ is the restriction to $\sigma$ of a linear function on $N_{\mathbb{R}}$. For $\varphi \in PL(\Sigma)$ and every $d$-dimensional cone $\sigma \in \Sigma$ there exists a unique $m_{\varphi,\sigma} \in M$ such that for all $n \in \sigma$ we have

$$\varphi(n) = n(m_{\varphi,\sigma}).$$

Fixing for any $\sigma \in \Sigma$ a $d$-dimensional cone $\sigma'$ containing $\sigma$ we put

$$m_{\varphi,\sigma} = m_{\varphi,\sigma'}.$$

To any $\varphi \in PL(\Sigma)$ we associate an invertible sheaf $L_\varphi$ on $X$ as the subsheaf of rational functions on $X$ generated over $U_\sigma$ by $\frac{1}{m_{\varphi,\sigma}}$, considered as a rational function on $X$ ($L_\varphi$ does not depend on the choice made above). The assignment $\varphi \mapsto L_\varphi$ gives an exact sequence

$$0 \to M \to PL(\Sigma) \to \text{Pic}(X) \to 0$$

(cf. [7], Corollary 2.5).

Denote by $\theta : X \times T \to X$ the action of $T$ on $X$ and by $p_1 : X \times T \to X$ the projection onto the first factor. The induced $T$-action on the sheaf of rational functions restricts to any subsheaf $L_\varphi$, i.e., there is a canonical $T$-linearization

$$\theta_\varphi : \theta^*L_\varphi \to p_1^*L_\varphi$$

(cf. [15], Ch. 1, § 3, for the notion of a $T$-linearization). In section four we will always consider $L_\varphi$ not merely as a line bundle on $X$ but as a $T$-linearized line bundle with this
$T$-linearization. In this sense $PL(\Sigma)$ is isomorphic to the group $\text{Pic}^T(X)$ of isomorphism classes of $T$-linearized line bundles on $X$.

Let $\Sigma_1 \subset N$ be the set of primitive integral generators of the one-dimensional cones in $\Sigma$ and put

$$PL(\Sigma)^+ := \{ \varphi \in PL(\Sigma)_R \mid \varphi(e) > 0 \text{ for all } e \in \Sigma_1 \}.$$ 

It is well-known (cf. [20], [2] Prop. 1.2.11), that the cone of effective divisors $\Lambda_{\text{eff}}(X) \subset \text{Pic}(X)_R$ is the closure of the image of $PL(\Sigma)^+$ under the projection $PL(\Sigma)_R \to \text{Pic}(X)_R$. Further, the anti-canonical line bundle on $X$ is isomorphic to $L_{\varphi_{\Sigma}}$, where $\varphi_{\Sigma}(e) = 1$ for all $e \in \Sigma_1$ (cf. [2], Prop. 1.2.12).

3.2 We shall introduce an adelic metric on the line bundle $L_{\varphi}$ as follows. For $\sigma \in \Sigma$ and $v \in \text{Val}(F)$ define

$$K_{\sigma,v} := \{ x \in U_\sigma(F_v) \mid |m(x)|_v \leq 1 \ \forall \ m \in \sigma \cap M \}.$$ 

These subsets cover $X(F_v)$ and we put for $x \in K_{\sigma,v}$ and any local section $s$ of $L_{\varphi}$ at $x$

$$\|x^*s\|_x := |s(x)m_{\varphi,\sigma}(x)|_v.$$ 

The family $\| \cdot \|_v = (\| \cdot \|_x)_{x \in X(F_v)}$ is then a $v$-adic metric on $L_{\varphi}$ and $L_{\varphi} = (L_{\varphi}, (\| \cdot \|_v)_v)$ is a metrization of $L_{\varphi}$. Let $K_{T,v} \subset T(F_v)$ be the maximal compact subgroup. Assigning to $x \in T(F_v)$ the map

$$M \to \mathbb{Z} \ (\text{resp. } \mathbb{R} \text{ if } v|\infty),$$

$$m \mapsto -\log(|m(x)|_v)/\log(q_v),$$

(where $q_v$ is the order of the residue field of $F_v$ for non-archimedean valuations and $\log(q_v) = 1$ for archimedean valuations) we get an isomorphism $T(F_v)/K_{T,v} \to N$ (resp. $N_R$ if $v|\infty$). We will denote by $\overline{x}$ the image of $x \in T(F_v)$ in $N$ (resp. $N_R$). For $\varphi \in PL(\Sigma)_C$ define

$$H_{\Sigma,v}(\cdot, \varphi) : T(F_v) \to C,$$

$$H_{\Sigma,v}(x, \varphi) := e^{-\varphi(\overline{x})\log(q_v)}.$$ 

The corresponding global function $H_{\Sigma}(\cdot, \varphi) : T(A) \to C$,

$$H_{\Sigma}(x, \varphi) := \prod_v H_{\Sigma,v}(x_v, \varphi),$$

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is well defined since for almost all \( v \) the local component \( x_v \) belongs to \( K_{T,v} \). The functions \( H_{\Sigma,v}(\cdot, \varphi) \), \( \varphi \in PL(\Sigma) \), are related to the \( v \)-adic metric on \( L_\varphi \) by the identity

(7) \[ H_{\Sigma,v}(x, \varphi) = \| x^s \varphi \|_x, \quad (x \in T(F_v)), \]

where \( s_\varphi \in H^0(T, L_\varphi) \) is the constant function 1. In particular, for every \( x \in T(F) \) we have

\[ H_{L_\varphi}(x) = H_{\Sigma}(x, \varphi)^{-1}. \]

3.3 Let \( K_T = \prod_v K_{T,v} \subset T(A) \), and denote by

\[ \mathcal{A}_T = (T(A)/T(F)K_T)^* \]

the group of unitary characters of \( T(A) \) which are trivial on the closed subgroup \( T(F)K_T \). For \( m \in M \) we obtain characters \( \chi^m \) defined by

\[ \chi^m(x) := e^{i \log(|m(x)|_A)}. \]

This gives an embedding \( M_R \to \mathcal{A}_T \). For any archimedean place \( v \) and \( \chi \in \mathcal{A}_T \) there is an \( m_v = m_v(\chi) \in M_R \) such that \( \chi_v(x_v) = e^{-\alpha_v(m_v)} \) for all \( x_v \in T(F_v) \). We get a homomorphism

(8) \[ \mathcal{A}_T \to M_{R,\infty} = \oplus_{v|\infty} M_R, \]

\[ \chi \mapsto m_\infty(\chi) = (m_v(\chi))_{v|\infty}. \]

Define \( T(A)^1 \) to be the kernel of all maps \( T(A) \to R_{>0}, x \mapsto |m(x)|_A \), for \( m \in M \), and put

\[ U_T = (T(A)^1/T(F)K_T)^*. \]

The choice of a projection \( G_m(A) \to G_m(A)^1 \) induces by means of an isomorphism \( T \sim G_m(F)^1 \) a splitting of the exact sequence

\[ 1 \to T(A)^1 \to T(A) \to T(A)/T(A)^1 \to 1. \]

This gives decompositions

(9) \[ \mathcal{A}_T = M_R \oplus U_T \]

and

\[ M_{R,\infty} = M_R \oplus M_{R,\infty}^1. \]
where $M_{1,R,\infty}$ is the minimal $R$-subspace of $M_{R,\infty}$ containing the image of $U_T$ under the map (8). From now on we fix such a (non-canonical) splitting. By Dirichlet’s unit theorem, the image of $U_T \to M_{1,R,\infty}$ is a lattice of maximal rank. Its kernel is isomorphic to the character group of $Cl_F^d$, where $Cl_F$ is the ideal class group of $F$.

For finite $v$ we let $dx_v$ be the Haar measure on $T(F_v)$ giving $K_{T,v}$ the volume one. For archimedean $v$ we take on $T(F_v)/K_{T,v}$ the pull-back of the Lebesgue measure on $N_R$ (normalized by the lattice $N$) and on $K_{T,v}$ the Haar measure with total mass one. The product measure gives an invariant measure $dx = \prod_v dx_v$.

3.4 We will denote by $S^1$ the unit circle. For a character $\chi : T(F_v) \to S^1$ we define the Fourier transform of $H_{\Sigma,v}(\cdot, \varphi)$ by

$$\hat{H}_{\Sigma,v}(\chi, \varphi) = \int_{T(F_v)} H_{\Sigma,v}(x_v, \varphi)\chi(x_v)dx_v.$$ 

If $\chi$ is not trivial on $K_{T,v}$ then $\hat{H}_{\Sigma,v}(\chi, \varphi) = 0$ (assuming the convergence of the integral). We will show that these integrals do exists if $\text{Re}(\varphi)$ is in $PL(\Sigma)^+$. Let $v$ be an archimedean place of $F$. Any $d$-dimensional cone $\sigma \in \Sigma$ is simplicial (since $\Sigma$ is regular) and it is generated by the set $\sigma \cap \Sigma_1$. Let $\chi$ be unramified, i.e., $\chi(x) = e^{-ix(m)}$ with some $m \in M_R$. Then we get

$$\hat{H}_{\Sigma,v}(\chi, \varphi) = \sum_{\text{dim } \sigma = d} \int_{\sigma} e^{-(\varphi(n) + im(m))}dn = \sum_{\text{dim } \sigma = d} \prod_{e \in \sigma \cap \Sigma_1} \frac{1}{\varphi(e) + ie(m)}.$$

To give the result for finite places we define rational functions $R_\sigma$ in variables $u_e, e \in \Sigma_1$, for any $\sigma \in \Sigma$ by

$$R_\sigma((u_e)_e) = \prod_{e \in \sigma \cap \Sigma_1} \frac{u_e}{1 - u_e},$$

and put

$$R_\Sigma((u_e)_e) = \sum_{\sigma \in \Sigma} R_\sigma((u_e)_e),$$

$$Q_\Sigma((u_e)_e) = R_\Sigma((u_e)_e) \prod_{e \in \Sigma_1} (1 - u_e).$$
Although elementary, it is a very important observation that the polynomial $Q_\Sigma - 1$ is a sum of monomials of degree not less than two (cf. [3], Prop. 2.2.3).

Let $\chi$ be an unramified unitary character of $T(F_v)$ and let $\text{Re}(\varphi)$ be in $PL(\Sigma)^+$. Then we can calculate

$$
\hat{H}_{\Sigma,v}(\chi, \varphi) = \int_{T(F_v)} H_{\Sigma,v}(x_v, \varphi) \chi(x_v) dx_v = \sum_{n \in \mathbb{N}} e^{-\varphi(n) \log(q_v)} \chi(n)
$$

$$
= \sum_{\sigma \in \Sigma} \sum_{n \in \sigma \cap \mathbb{N}} q_v^{-\varphi(n)} \chi(n)
= \sum_{\sigma \in \Sigma} R_\sigma ((\chi(e) q_v^{-\varphi(e)})_e)
= Q_\Sigma ((\chi(e) q_v^{-\varphi(e)})_e) \prod_{e \in \Sigma_1} (1 - \chi(e) q_v^{-\varphi(e)})^{-1}.
$$

(Here we denoted by $\sigma^o$ the relative interior of the cone $\sigma$.)

Any $e \in \Sigma_1$ induces a homomorphism $F[M] \to F[Z]$ and hence a morphism of tori $G_m \to T$. For any character $\chi \in A_T$ we denote by $\chi_e$ the Hecke character

$$
G_m(A) \longrightarrow T(A) \xrightarrow{\chi} S^1
$$

thus obtained. The finite part of the Hecke $L$-function with character $\chi_e$ is by definition

$$
L_f(\chi_e, s) = \prod_{v \mid \infty} (1 - \chi_e(\pi_v) q_v^s)^{-1}
$$

and this product converges for $\text{Re}(s) > 1$ (here $\pi_v$ denotes a local uniformizing element).

By (10) and (11) we know that the global Fourier transform

$$
\hat{H}_\Sigma(\chi, \varphi) = \int_{T(A)} H_\Sigma(x, \varphi) \chi(x) dx
$$

exists (i.e., the integral on the right converges absolutely) if $\text{Re}(\varphi)$ is contained in $\varphi_\Sigma + PL(\Sigma)^+$, because

$$
\prod_{v \mid \infty} Q_\Sigma ((\chi_v(e) q_v^{-\varphi(e)})_e)
$$

is an absolutely convergent Euler product for $\text{Re}(\varphi(e)) > 1/2$ (for all $e \in \Sigma_1$) and hence is bounded for $\text{Re}(\varphi)$ in any compact subset in $\frac{1}{2} \varphi_\Sigma + PL(\Sigma)^+$ (by some constant depending only on this subset).
Proposition 3.4  The series
\[ \sum_{x \in T(F)} H_{\Sigma}(xt, \varphi) \]
converges absolutely and uniformly for \((\text{Re}(\varphi), t)\) contained in any compact subset of \((\varphi_{\Sigma} + PL(\Sigma)^+) \times T(A)\).

Proof. Let \(K\) be a compact subset of \(\varphi_{\Sigma} + PL(\Sigma)^+\) and let \(C_v \subset T(F_v)\) (for every \(v \in \text{Val}(F)\)) be a compact subset, equal to \(K_{T,v}\) for almost all \(v\). Since any \(\varphi \in PL(\Sigma)_{\text{c}}\) is a continuous piecewise linear function (with respect to a finite subdivision of \(N_{\mathbb{R}}\) into simplicial cones) there exists a constant \(c_v \geq 1\) (depending on \(K\) and \(C_v\)) such that for all \(\varphi\) with \(\text{Re}(\varphi) \in K\), \(x_v \in T(F_v)\) and \(t_v \in C_v\) we have
\[
1 \leq \left| \frac{H_{\Sigma,v}(x_v t_v, \varphi)}{H_{\Sigma,v}(x_v, \varphi)} \right| = \frac{H_{\Sigma,v}(x_v t_v, \text{Re}(\varphi))}{H_{\Sigma,v}(x_v, \text{Re}(\varphi))} \leq c_v.
\]
If \(C_v = K_{T,v}\) we may assume \(c_v = 1\). Put \(c = \prod_v c_v\). For all \(\varphi\) with \(\text{Re}(\varphi) \in K\) and \(t \in C := \prod_v C_v\) we can estimate
\[
\left| \sum_{x \in T(F)} H_{\Sigma}(xt, \varphi) \right| \leq c \sum_{x \in T(F)} H_{\Sigma}(x, \text{Re}(\varphi)).
\]
Let \(S\) be a finite set of places containing \(\text{Val}_{\infty}(F)\) and let \(U_v \subset T(F_v)\) be a relatively compact open subset of \(T(F_v)\) for each \(v \in S\), such that for all \(x_1 \neq x_2 \in T(F)\)
\[
x_1 U \cap x_2 U = \emptyset,
\]
where \(U = \prod_{v \in S} U_v \prod_{v \notin S} K_{T,v}\). By the preceding argument, there exists a \(c' > 0\) such that for all \(\varphi \in K\), \(x \in T(F)\) and \(u \in U\)
\[
H_{\Sigma}(x, \varphi) \leq c' H_{\Sigma}(xu, \varphi).
\]
Therefore,
\[
\sum_{x \in T(F)} H_{\Sigma}(x, \varphi) \leq \frac{c'}{\text{vol}(U)} \sum_{x \in T(F)} \int_U H_{\Sigma}(xu, \varphi) du
\leq \frac{c'}{\text{vol}(U)} \int_{T(A)} H_{\Sigma}(x, \varphi) dx < \infty
\]
by the discussion above. From the explicit expression for the integral (cf. (12)) we derive the uniform convergence in \(\varphi\) on \(K\). \(\square\)
3.5 The aim is to apply Poisson's summation formula to the height zeta function. It remains to show that \( \hat{H}(\cdot, \varphi) \) is absolutely integrable over \( \mathcal{A}_T \). For \( \chi \in \mathcal{A}_T \) and \( \text{Re}(\varphi) \) contained in \( \frac{1}{2} \varphi_\Sigma + PL(\Sigma)^+ \) we put

\[
\zeta_\Sigma(\chi, \varphi) := \prod_{v \mid \infty} \hat{H}_\Sigma, v(\chi_v, \varphi) \prod_{v \mid \infty} Q_\Sigma((\chi_v(e)q_v^{-\varphi(v)}(e))_e).
\]

**Lemma 3.5** Let \( K \) be a compact subset of \( PL(\Sigma)_C \) such that for all \( \varphi \in K \) and \( e \in \Sigma_1 \)

\[
\text{Re}(\varphi(e)) > \frac{1}{2}.
\]

Then there is a constant \( c = c(K) \) such that for all \( \varphi \in K, \chi \in \mathcal{A}_T \) and \( m \in M_\mathbb{R} \) we have

\[
|\zeta_\Sigma(\chi, \varphi + im)| \leq c \prod_{v \mid \infty} \left\{ \sum_{\dim \sigma = d} \prod_{e \in \sigma \cap \Sigma_1} \frac{1}{(1 + |e(m + m_v(\chi))|)^{1 + 1/d}} \right\}.
\]

**Proof.** For \( K \) as above there exists a \( c' > 0 \) such that for all \( \chi \in \mathcal{A}_T \) and \( m \in M_\mathbb{R} \) one has

\[
\left| \prod_{v \mid \infty} Q_\Sigma((\chi_v(e)q_v^{-\varphi(v) + ie(m)})_e) \right| \leq c'
\]

for all \( \varphi \in K \) (see the argument before Proposition 3.4. By \cite{2}, Prop. 2.3.2, for all \( v \mid \infty \) there is a constant \( c_v \) such that for all \( \varphi \in K, \chi \in \mathcal{A}_T \) and \( m \in M_\mathbb{R} \)

\[
|\hat{H}_{\Sigma, v}(\chi_v, \varphi + im)| \leq c_v \sum_{\dim \sigma = d} \prod_{e \in \sigma \cap \Sigma_1} \frac{1}{(1 + |e(m + m_v(\chi))|)^{1 + 1/d}}.
\]

Putting \( c = c' \prod_{v \mid \infty} c_v \) we get the result. \( \square \)

For \( \text{Re}(\varphi) \) contained in \( \varphi_\Sigma + PL(\Sigma)^+ \) we can write

\[
\hat{H}_\Sigma(\chi, \varphi) = \zeta_\Sigma(\chi, \varphi) \prod_{e \in \Sigma_1} L_f(\chi_e, \varphi(e)).
\]
By the preceding lemma, we see that \( \hat{H}_\Sigma(\cdot, \varphi) \) is absolutely integrable over \( A_T \). For \( t \in T(A) \) we have
\[
\int_{T(A_F)} H_\Sigma(xt, \varphi)\chi(x)dx = \chi^{-1}(t)\hat{H}_\Sigma(\chi, \varphi).
\]
Hence we can apply Poisson’s summation formula (together with (9)) and obtain
\[
\sum_{x \in T(F)} H_\Sigma(xt, \varphi) = \mu_T \int_{\mathbb{M}_\mathbb{R}} \left\{ \sum_{\chi \in \mathcal{U} T} \hat{H}_\Sigma(\chi, \varphi + im)(\chi^{m}(t))^{-1} \right\} dm,
\]
where the Lebesgue measure \( dm \) on \( \mathbb{M}_\mathbb{R} \) is normalized by \( \mu_T = \frac{1}{(2\pi \kappa) d} \), \( \kappa = \frac{c_l F \cdot R_F}{w_F} \) with \( c_l F \) the class number, \( R_F \) the regulator and \( w_F \) the number of roots of unity in \( F \).

Note that \( \hat{H}_\Sigma(\chi \cdot \chi^m, \varphi) = \hat{H}_\Sigma(\chi, \varphi + im) \).

3.6 In section 5 we need uniform estimates for \( L \)-functions in a neighborhood of the line \( \text{Re}(s) = 1 \). For any unramified character \( \chi : G_m(A)/G_m(F) \to S^1 \) and any archimedean place \( v \) there exists a \( \tau_v \in \mathbb{R} \) such that \( \chi_v(x_v) = |x_v|^{\tau_v} \) for all \( x_v \in G_m(F_v) \). We put
\[
\chi_\infty = (\tau_v)_{v|\infty} \in \mathbb{R}^{\text{Val}_\infty(F)} \quad \text{and} \quad \|\chi_\infty\| = \max_{v|\infty} |\tau_v|.
\]
We will use the following theorem of Rademacher ([19], Theorems 4,5), which rests on the Phragmén-Lindelöf principle.

**Theorem 3.6** For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) and a constant \( c(\epsilon) > 0 \) such that for all \( s \) with \( \text{Re}(s) > 1 - \delta \) and all unramified Hecke characters which are non-trivial on \( G_m(A) \) one has
\[
|L_f(\chi, s)| \leq c(\epsilon)(1 + |\text{Im}(s)| + \|\chi_\infty\|)^\epsilon.
\]
For the trivial character \( \chi = 1 \) one has
\[
|L_f(1, s)| \leq c(\epsilon) \left| \frac{1 + s}{1 - s} \right| (1 + |\text{Im}(s)|)^\epsilon.
\]
4 Twisted products

4.1 Let $G, P, W = P\backslash G$ etc. be as in section 2 and $T, \Sigma, X = X_{\Sigma}$ etc. be as in section 3. Let $\eta : P \to T$ be a homomorphism. Then $P$ acts from the right on $X \times G$ by

$$(x, g) \cdot p := (x\eta(p), p^{-1}g).$$

Since $\pi_W : G \to W$ is locally trivial, the quotient

$$Y = X \times^P G := (X \times G)/P$$

exists as a variety over $F$. Moreover, the projection $X \times G \to G$ induces a morphism $\pi : Y \to W$ and $Y$ becomes a locally trivial fiber bundle over $W$ with fiber $X$ (compare \cite{[13]}, I.5.16). Hence, by the properties of $X$ (non-singular, projective), we see that $Y$ is a non-singular projective variety over $F$ ("projectivity" requires a short argument, cf. \cite{[23]}).

The quotient morphism $X \times G \to Y$ will be denoted by $\pi_Y$. Let $\varphi \in PL(\Sigma)$ and let $L_{\varphi}$ be the invertible sheaf on $X$ defined in section 3.1. Denote by $L_{\varphi}$ the corresponding $G_a$-bundle over $X$, i.e., $L_{\varphi} = V(L_\varphi^\vee) = V(L_{-\varphi})$ (with the notation of \cite{[11]}, II, Exercise 5.18). The canonical $T$-linearization

$$\theta_{-\varphi} : \theta^*L_{-\varphi} \to p_1^*L_{-\varphi}$$

induces an action $L_{\varphi} \times T \to L_{\varphi}$ of $T$ on $L_{\varphi}$ which is compatible with the action of $T$ on $X$. The twisted product $L_{\varphi} \times^P G$ will then be a $G_a$-bundle over $Y$ and we define $L_Y^\varphi$ to be the sheaf of local sections of $L_{\varphi} \times^P G$ over $Y$. Note that $L_Y^\varphi$ (and even its isomorphism class in Pic($Y$)) depends on the fixed $T$-linearization $\theta_{-\varphi}$. In fact, for $\varphi \in PL(\Sigma)$ and $m \in M$ we have

$$L_{\varphi+m}^Y \simeq L_{\varphi}^Y \otimes \pi^*L_m$$

Embedding $M$ in $PL(\Sigma) \oplus X^*(P)$ by $m \mapsto (m, -m \circ \eta)$ we see that $M$ is contained in the kernel of the homomorphism

$$\psi : PL(\Sigma) \times X^*(P) \to Pic(Y),$$

$$(\varphi, \lambda) \mapsto \text{isomorphism class of } L_{\varphi}^Y \otimes \pi^*L_\lambda.$$
In the following proposition we collect all relevant facts about the geometry of twisted products which we will need in the sequel.

**Proposition 4.2**  

a) The sequence  
\[0 \to M \to PL(\Sigma) \oplus X^*(P) \to \text{Pic}(Y) \to 0\]  
is exact.

b) The cone of effective divisors \(\Lambda_{\text{eff}}(Y) \subset \text{Pic}(Y)\) is the image of the closure of  
\[PL(\Sigma)^+ \times X^*(P)^+ \subset PL(\Sigma)_R \oplus X^*(P)_R.\]

c) The anti-canonical line bundle \(\omega_Y^\vee\) is isomorphic to \(L_Y \varphi \otimes \pi^* L_{2\rho}\).

**Proof.**  
a) By [21], Proposition 6.10, there is an exact sequence  
\[F[X \times G]^*/F^* \to X^*(P) \to \text{Pic}(Y) \to \text{Pic}(X \times G).\]

Denote by \(\pi_X : X \times G \to X\) the canonical projection. Let \(L\) be an invertible sheaf on \(Y\). Then  
\[\pi_X^* L \simeq \pi_X^* L_{\varphi}\]  
(for some \(\varphi \in PL(\Sigma)\)) because \(\text{Pic}(X \times G) = \text{Pic}(X) \oplus \text{Pic}(G)\) (cf. [21], Lemme 6.6 (i) and Lemme 6.9 (iv)). Note that \(\pi_X^* L_{\varphi} \simeq \pi_X^* L_{\varphi}\), so that  
\[\pi_X^* (L \otimes L_{\varphi})\]

is trivial. Hence there exists a character \(\lambda\) of \(P\) such that \(L \otimes L_{\varphi}\) is isomorphic to \(\pi^* L_{\lambda}\) (the map \(X^*(P) \to \text{Pic}(Y)\) factorizes \(X^*(P) \to \text{Pic}(W) \to \text{Pic}(Y)\)). This shows surjectivity.

Suppose now that for \(\varphi \in PL(\Sigma)\) and \(\lambda \in X^*(P)\) the sheaf \(L_{\varphi} \otimes \pi^* L_{\lambda}\) is trivial on \(Y\). Then \(\pi_X^* (L_{\varphi}^Y \otimes \pi^* L_{\lambda}) \simeq \pi_X^* L_{\lambda}\) is trivial on \(X \times G\), therefore \(L_{\varphi} \simeq \mathcal{O}_X, \varphi = m \in M\) and \(L_{\varphi}^Y \otimes \pi^* L_{\lambda} = \pi^* L_{\lambda+m \circ \eta}\). By Rosenlicht’s theorem,

\[F[X \times G]^*/F^* = F[X]^*/F^* \oplus F[G]^*/F^* = X^*(G) = 0,\]

therefore, the map \(X^*(P) \to \text{Pic}(Y)\) is injective, hence \(\lambda + m \circ \eta = 0\) and \((\varphi, \lambda)\) is in the image of \(M \to PL(\Sigma) \oplus X^*(P)\).
b) For \( \varphi \in PL(\Sigma) \) denote by \( \square_\varphi \) the set of all \( m \in M \) such that for all \( n \in N_R \)
\[
\varphi(n) + n(m) \geq 0.
\]
By [17], Lemma 2.3, \( \square_\varphi \) is a basis for \( H^0(X, L_\varphi) \) (note the different sign conventions). It is easy to see that
\[
\pi_*(L_\varphi^Y \otimes \pi^*L_\lambda) \simeq \bigoplus_{m \in \square_\varphi} L_{-m\eta + \lambda}
\]
has a non-zero global section, hence (cf. section 2.3) there is a non-zero global section, hence (cf. section 2.3) there is a non-zero global section, hence (cf. section 2.3) there is a non-zero global section, hence (cf. section 2.3) there is a non-zero global section, hence (cf. section 2.3) there is a non-zero global section, hence (cf. section 2.3) there is a non-zero global section, hence (cf. section 2.3) there is a non-zero global section, hence (cf. section 2.3) there is a non-zero global section,

\[ 
\pi_*(L_\varphi^Y \otimes \pi^*L_\lambda) \simeq \bigoplus_{m \in \square_\varphi} L_{-m\eta + \lambda}
\]

Suppose \( L_\varphi^Y \otimes \pi^*L_\lambda \) has a non-zero global section. Then
\[
\pi_*(L_\varphi^Y \otimes \pi^*L_\lambda) \simeq \bigoplus_{m \in \square_\varphi} L_{-m\eta + \lambda}
\]
has a non-zero global section, hence (cf. section 2.3) there is a \( m' \in \square_\varphi \) such that \(-m' \circ \eta + \lambda\) is contained in the closure of \( X^+(P)^+ \). Putting \( \lambda' = -m' \circ \eta + \lambda, \varphi' = \varphi + m' \) we have \( L_{\varphi'}^Y \otimes \pi^*L_{\lambda'} \simeq L_\varphi^Y \otimes \pi^*L_\lambda \) and \( (\varphi', \lambda') \) is contained in the closure of \( PL(\Sigma)^+ \times X^+(P)^+ \).

On the other hand, if \( (\varphi, \lambda) \in PL(\Sigma) \oplus X^+(P) \) is contained in the closure of \( PL(\Sigma)^+ \times X^+(P)^+ \) then the trivial character corresponds to a global section of \( L_\varphi \). Hence
\[
\pi_*(L_\varphi^Y \otimes \pi^*L_\lambda) = L_\lambda \oplus \bigoplus_{m \in \square_\varphi - \{0\}} L_{-m\eta + \lambda}
\]
and \( H^0(W, L_\lambda) \neq \{0\} \), i.e., \( L_\varphi^Y \otimes \pi^*L_\lambda \) has a non-zero global section.

c) Note first that the exact sequence
\[
0 \to \pi^*\Omega_W \to \Omega_Y \to \Omega_{Y/W} \to 0
\]
splits, and therefore \( \omega_Y \simeq (\Lambda^d\Omega_{Y/W}) \otimes \pi^*\omega_W \). Since \( \omega_W \simeq L_{-2\rho} \) it remains to show that \( (\Lambda^d\Omega_{Y/W})^\vee \simeq L_{\varphi_\Sigma}^Y \). Let \( \mathcal{J}_{Y/W} \) be the ideal sheaf of the image of the diagonal morphism \( \Delta_{Y/W} : Y \to Y \times_W Y \).

But \( Y \times_W Y \) is canonically isomorphic to \( (X \times X) \times^P G \) and \( \Delta_{Y/W}(Y) \) is just \( \Delta_X(X) \times^P G \).

Hence we see that \( (\mathcal{J}_{Y/W} / \mathcal{J}_X^2)^\vee \) is the sheaf of local sections of \( \mathcal{V}(\mathcal{J}_X / \mathcal{J}_X^2)^\times^P G \), where \( \mathcal{J}_X \) is the ideal sheaf of \( \Delta_X(X) \subset X \times X \). Pulling back to \( Y \) and taking the \( d \)-th exterior power we get
\[
\mathcal{V}(\Lambda^d\Omega_{Y/W}) \simeq \mathcal{V}(\Lambda^d\Omega_X)^\times^P G \simeq \mathcal{V}(\omega_X)^\times^P G.
\]
The canonical \( T \)-linearization of \( \omega_X \) (induced by the action of \( T \) on rational functions) corresponds to the \( T \)-linearization \( \theta_{\varphi_\Sigma} \) of \( L_{-\varphi_\Sigma} \simeq \omega_X \), i.e.,
\[
L_{\varphi_\Sigma} \times^P G \simeq \mathcal{V}(\omega_X)^\times^P G
\]
and we get \( L_{\varphi_\Sigma}^Y \simeq (\Lambda^d\Omega_{Y/W})^\vee \).\( \square \)
4.3 We are going to introduce an adelic metric on the sheaves $L^y_{\varphi}$. A section of $L^y_{\varphi}$ over an open subset $U \subset Y$ can be identified with a $P$-equivariant morphism $s : \pi^{-1}_y(U) \to L_{\varphi}$ over $X$, i.e.,

$$s(x\eta(p), p^{-1}g) = s(x, g) \cdot \eta(p).$$

Let $v$ be a place of $F$ and let $y \in Y(F_v)$ be the image of $(x, k) \in X(F_v) \times G(F_v)$ with $k \in K_{G,v}$. Let $s : \pi^{-1}_y(U) \to L_{\varphi}$ be a local section of $L^y_{\varphi}$ over $U \subset Y$ with $y \in U(F_v)$. Define

$$\| \cdot \|_y : y^*L^y_{\varphi} \to \mathbb{R}$$

by

$$\|y^*s\|_y = \|s \circ (x, k)\|_x.$$ 

Then $\| \cdot \|_v = (\| \cdot \|_{y \in Y(F_v)})$ is a $v$-adic metric on $L^y_{\varphi}$ and $L^Y_{\varphi} = (L^y_{\varphi}, (\| \cdot \|_v)_v)$ is a metrization of $L^y_{\varphi}$ (cf. [23]). Let

$$Y^o = T \times^P G \hookrightarrow X \times^P G = Y$$

be the twisted product of $T$ with $W$. Over $Y^o$ there is a canonical section of $L^y_{\varphi}$, namely

$$s^y_{\varphi} : \pi^{-1}(Y^o) = T \times G \to L_{\varphi},$$

$s^y_{\varphi}(x, g) = s_{\varphi}(x)$, where $s_{\varphi} \in H^0(T, L_{\varphi})$ corresponds to the constant function 1. Let $y = \pi_W(x, g) \in Y(F_v)$ where $g = pk$ with $p \in P(F_v)$ and $k \in K_{G,v}$. Then

$$\|y^*s^y_{\varphi}\|_y = \|s^y_{\varphi} \circ (x\eta(p), k)\|_{x\eta(p)}$$

$$= \|(x\eta(p))^*s^y_{\varphi}\|_{x\eta(p)} = e^{-\varphi(x\eta(p)) \log(y_v)}$$

(by (7)). Globally, for $y \in Y^o(F), y = \pi_Y(x, \gamma), x \in T(F), \gamma \in G(F), \gamma = p_\gamma k_\gamma$ and $p_\gamma, k_\gamma$ as above, in $P(A), K_{G}$, respectively, we get

$$(16) \quad H_{L^y_{\varphi}}(y) = \prod_v \|y_v^*s^y_{\varphi}\|_{y_v}^{-1} = H_{\Sigma}(x\eta(p), -\varphi).$$

4.4 Let $\xi : P(A)/P(F) \to S^1$ be an unramified character, i.e., $\xi$ is trivial on $P(A) \cap K_G$. Using the Iwasawa decomposition we get a well defined function

$$\phi_{\xi} : G(A) \to S^1,$$
\[ \phi_\xi(g) = \xi(p), \]
if \( g = pk \) as above. We denote by
\[ E^G_P(\lambda, \xi, g) = \sum_{\gamma \in P(F) \backslash G(F)} \phi_\xi(\gamma g) e^{(\lambda + \rho_P, H_P(\gamma g))} \]
the corresponding Eisenstein series and we put \( E^G_P(\lambda, \xi) = E^G_P(\lambda, \xi, 1_G) \). This series converges absolutely for \( \text{Re}(\lambda) \) contained in the cone \( \rho_P + X^*(P)^+ \) (cf. (2.3)).

A character \( \chi \in A_T \) induces a character \( \chi \eta = \chi \circ \eta : P(A) / P(F) \to S^1 \). We denote by \( \hat{\eta} : X^*(T)_R \to X^*(P)_R \) the map on characters induced by \( \eta \).

**Proposition 4.4** Let \( L \) be a line bundle on \( Y \) such that its class is contained in the interior of the cone \( \Lambda_{\text{eff}}(Y) \). Let \((\varphi, \lambda) \) be in \( PL(\Sigma)^+ \times X^*(P)^+ \) with \( \psi(\varphi, \lambda) = [L] \). There is a metrization \( \mathcal{L} \) of \( L \) such that for all \( s \) with \( \text{Re}(s) (\varphi, \lambda) \in (\varphi, \Sigma, 2\rho_P + PL(\Sigma)^+ \times X^*(P)^+) \)
the series
\[ Z_{Y^o}(\mathcal{L}, s) = \sum_{y \in Y^o(F)} H_\mathcal{L}(y)^{-s} \]
converges absolutely. Moreover, for these \( s \)
\[ Z_{Y^o}(\mathcal{L}, s) = \mu_T \int_{M_R} \left\{ \sum_{\chi \in i_\ell_{\gamma}} \hat{H}_\Sigma(\chi, s\varphi + im) E^G_P(s\lambda - \rho_P - i\hat{\eta}(m), \chi^{-1}_{\eta}) \right\} dm, \]
where the sum and integral on the right converge absolutely too.

**Proof.** Let \((\varphi', \lambda') \in PL(\Sigma)^+ \times X^*(P)^+ \) such that there is an isomorphism \( L \simeq L^\lambda_{\varphi'} \oplus \pi^* L_{\lambda'} \). Denote by \( \mathcal{L} \) the metrization of \( L \) which is the pullback of \( L^\lambda_{\varphi'} \oplus \pi^* L_{\lambda'} \) via this isomorphism. Let \( m \in M_R \) such that
\[(\varphi, \lambda) = (\varphi' + m, -\hat{\eta}(m) + \lambda') \]
is contained in \( PL(\Sigma)^+ \times X^*(P)^+ \). By (16), we have for any \( y \in Y^o(F), y = \pi_Y(x, \gamma) \) with \( x \in T(F), \gamma \in G(F) \) and \( \gamma = p_{\gamma, \gamma} \)
\[ H_\mathcal{L}(y) = H_{L^\lambda_{\varphi'} \oplus \pi^* L_{\lambda'}}(y) = e^{-\langle \lambda', H_P(\gamma) \rangle} H_\Sigma(x \eta(p_{\gamma}), -\varphi') = e^{-\langle \lambda - \varphi, H_P(\gamma) \rangle} H_\Sigma(x \eta(p_{\gamma}), -(\varphi' + m)) = e^{-\langle \lambda, H_P(\gamma) \rangle} H_\Sigma(x \eta(p_{\gamma}), -\varphi). \]
We consider $s = u + iv \in \mathbb{C}$ such that $u \cdot \varphi$ is contained in the shifted cone $\varphi_\Sigma + PL(\Sigma)^+$ and $u \cdot \lambda$ is contained in the cone $2\rho_P + X^*(P)^+$. Then
\[ \sum_{x \in T(F)} H_\Sigma(x\eta(p_\gamma), u\varphi) \]
converges by Proposition 3.4 and is equal to
\[ \mu_T \int_{MR} \left\{ \sum_{\chi \in U_T} \hat{H}(\chi \chi^m, u\varphi) \chi^m(\eta(p_\gamma))^{-1} \right\} dm \]
(cf. (13)). Moreover, $\hat{H}(\cdot, u\varphi)$ is absolutely convergent on $\mathcal{A}_T$ and therefore
\[ \sum_{x \in T(F)} H_\Sigma(x\eta(p_\gamma), u\varphi) \leq \mu_T \int_{MR} \left\{ \sum_{\chi \in U_T} \left| \hat{H}(\chi \chi^m, u\varphi) \right| \right\} dm \]
is bounded by some constant $c$ (which is independent of $\eta(p_\gamma)$). Thus we may calculate
\[ \sum_{y \in Y_o(F)} \left| H_L(y)^{-s} \right| = \sum_{\gamma \in P(F) \setminus G(F)} e^{(u\lambda, H_P(\gamma))} \sum_{x \in T(F)} H_\Sigma(x\eta(p_\gamma), u\varphi) \]
\[ \leq c \sum_{\gamma \in P(F) \setminus G(F)} e^{(u\lambda, H_P(\gamma))}. \]
This shows the first assertion. Since

\[ \mu_T \int_{MR} \left\{ \sum_{\chi \in U_T} \left| \hat{H}(\chi \chi^m, u\varphi) \right| \right\} dm \]
converges, we can interchange the summation and integration and get
\[ Z_{Y_o}(L, s) = \sum_{\gamma \in P(F) \setminus G(F)} e^{(s\lambda, H_P(\gamma))} \mu_T \int_{MR} \left\{ \sum_{\chi \in U_T} \hat{H}_\Sigma(\chi \chi^m, s\varphi)(\chi \chi^m)^{-1}(\eta(p_\gamma)) \right\} dm \]
\[ = \mu_T \int_{MR} \left\{ \sum_{\chi \in U_T} \hat{H}_\Sigma(\chi, s\varphi + im)E_P^G(s\lambda - \rho_P - i\bar{\eta}(m), \chi^{-1}) \right\} dm. \]

\[ \square \]

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5  Meromorphic continuation

5.1  The proposition in section 4.4 gives an expression of the height zeta function (for
the open subset \( Y^o \subset Y \)) which we will use to determine the asymptotic behavior of the
counting function \( N_{Y^o}(L, H) \) (cf. sec. [3]) by applying a Tauberian theorem.

The first thing to do is to show that \( Z_{Y^o}(L, s) \) can be continued meromorphically to a
half-space beyond the abscissa of convergence and that there is no pole on this line with
non-zero imaginary part. Then it remains to prove that this abscissa is at \( \text{Re}(s) = a(L) \)
and to determine the order of the pole in \( s = a(L) \). We will see that this order is \( b(L) \).

The method which we will explain now consists in an iterated application of Cauchy’s
residue theorem. The proofs will be given in section 7.

5.2  Let \( E \) be a finite dimensional vector space over \( \mathbb{R} \) and \( E_C \) its complexification. Let
\( V \subseteq E \) be a subspace and let \( l_1, \ldots, l_m \in E^\vee = \text{Hom}_\mathbb{R}(E, \mathbb{R}) \) be linearly independent linear
forms. Put \( H_j = \ker(l_j) \) for \( j = 1, \ldots, m \).

Let \( B \subseteq E \) be an open and convex neighborhood of \( 0 \) such that for all \( x \in B \) and
\( j = 1, \ldots, m \) we have \( l_j(x) > -1 \). Let \( T_B = B + iE \subseteq E_C \) be the tube domain over \( B \)
and denote by \( \mathcal{M}(T_B) \) the set of meromorphic functions on \( T_B \). We consider meromorphic
functions \( f \in \mathcal{M}(T_B) \) with the following properties: The function

\[
g(z) = f(z) \prod_{j=1}^{m} \frac{l_j(z)}{l_j(z) + 1}
\]

is holomorphic in \( T_B \) and there is a sufficient function \( c : V \to \mathbb{R}_{\geq 0} \) such that for all compacts \( K \subseteq T_B \), all \( z \in K \) and all \( v \in V \) we have the estimate

\[
|g(z + iv)| \leq \kappa(K)c(v).
\]

(Cf. section 7.3 for a precise definition of a sufficient function. In particular, such a
sufficient function is absolutely integrable over any subspace \( U \subset V \).) In this case we call \( f \) distinguished with respect to the data \( (V; l_1, \ldots, l_m) \).

Let \( C \) be a connected component of \( B - \bigcup_{j=1}^{m} H_j \). By the conditions on \( g \) the integral

\[
\tilde{f}_C(z) := \frac{1}{(2\pi)^{\nu}} \int_V f(z + iv)dv
\]

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(\nu = \dim V \text{ and } dv \text{ is a fixed Lebesgue measure on } V) \text{ converges for every } z \in T_C \text{ and } \tilde{f}_C \text{ is a holomorphic function on } T_C.

**Theorem 5.2** There is an open neighborhood \( \tilde{B} \) containing \( C \), and linear forms \( \tilde{l}_1, \ldots, \tilde{l}_\tilde{m} \) which vanish on \( V \) such that

\[
z \mapsto \tilde{f}_C(z) \prod_{j=1}^{\tilde{m}} \tilde{l}_j(z)
\]

has a holomorphic continuation to \( T_{\tilde{B}} \). Moreover, for all \( j \in \{1, \ldots, \tilde{m}\} \) we have \( \text{Ker}(\tilde{l}_j) \cap C = \emptyset \).

We shall give the proof of this theorem in sections 7.3 and 7.4.

5.3 Put \( E^{(0)} = \cap_{j=1}^{m} \text{Ker}(l_j) \) and \( E_0 = E/E^{(0)} \). Let \( \pi_0 : E \to E_0 \) be the canonical projection and suppose \( V \cap E^{(0)} = \{0\} \). Let

\[
E_0^+ = \{ x \in E_0 \mid l_j(x) \geq 0 \text{ for all } j = 1, \ldots, m \}
\]

and let \( \psi_0 : E_0 \to P := E_0^+ / \pi_0(V) \) be the canonical projection. We want to assume that \( \pi_0(V) \cap E_0^+ = \{0\} \), so that \( \Lambda := \psi_0(E_0^+) \) is a strictly convex polyhedral cone. Let \( dy \) be the Lebesgue measure on \( E_0^\vee \) normalized by the lattice \( \oplus_{j=1}^{m} \mathbb{Z} l_j \). Let \( A \subset V \) be a lattice and let \( dv \) be the measure on \( V \) normalized by \( A \). On \( V^\vee \) we have the Lebesgue measure \( dy' \) normalized by \( A^\vee \) and a section of the projection \( E_0^\vee \to V^\vee \) gives a measure \( dy'' \) on \( P^\vee \) with \( dy = dy' dy'' \).

Define the \( \mathcal{X} \)-function of the cone \( \Lambda \) by

\[
\mathcal{X}_{\Lambda}(x) = \int_{\Lambda^\vee} e^{-y''(x)} dy''
\]

for all \( x \in P_C \) with \( \text{Re}(x) \) contained in the interior of \( \Lambda \) (cf. section 7.1).

Let \( B \subset E \) be as above and let \( f \in \mathcal{M}(T_B) \) be a distinguished function with respect to \( (V; l_1, \ldots, l_m) \). Put

\[
g(z) = f(z) \prod_{j=1}^{m} \frac{l_j(z)}{l_j(z) + 1},
\]

\[
B^+ = B \cap \{ x \in E \mid l_j(x) \geq 0, \text{ for all } j = 1, \ldots, m \},
\]

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\[ \tilde{f}_{B^+}(z) = \frac{1}{(2\pi)^\nu} \int_V f(z + iv)dv \]

(for \( z \in T_B \)). The function \( \tilde{f}_{B^+} : T_{B^+} \to \mathbb{C} \) is holomorphic and has a meromorphic continuation to a neighborhood of \( 0 \in E_C \). In section 7.5 we will prove the following theorem.

**Theorem 5.3** For \( x_0 \in B^+ \) we have

\[ \lim_{s \to 0} s^{m-\nu} \tilde{f}_{B^+}(sx_0) = g(0) \chi\Lambda(\psi_0(x_0)). \]

5.4 In this section we make some preparations in order to apply the general setting of 5.2. Let \( L \) be a line bundle on \( Y \) such that its class in \( \text{Pic}(Y) \) lies in the interior of \( \Lambda_{\text{eff}}(Y) \).

By the definition of \( a(L) \) (cf. section 1 and Proposition 4.2),

\[ a(L)[L] - \psi(\varphi_\Sigma, 2\rho_P) \in \Lambda(L) \]

where \( \Lambda(L) \) is the minimal face of \( \Lambda_{\text{eff}}(Y) \) containing \( a(L)[L] - \psi(\varphi_\Sigma, 2\rho_P) \). Define \( \varphi_e \in PL(\Sigma) \) (for \( e \in \Sigma_1 \)) by \( \varphi_e(e') = \delta_{ee'} \), for all \( e' \in \Sigma_1 \) and put

\[ \Sigma'_1 := \{ e \in \Sigma_1 \mid \psi(\varphi_e, 0) \in \Lambda(L) \}. \]

Let \( P' \subset G \) be the standard parabolic subgroup with

\[ \Delta_{P'} = \{ \alpha \in \Delta_P \mid \psi(0, \varpi_\alpha) \in \Lambda(L) \}, \]

where \( \langle \varpi_\alpha, \beta \rangle = \delta_{\alpha\beta} \) for all \( \alpha, \beta \in \Delta_0 \). Let

\[ (\varphi_L, \lambda_L) \in (\sum_{e \in \Sigma'_1} \mathbb{R}_{>0} \varphi_e) \times (\sum_{\alpha \in \Delta_{P'}} \mathbb{R}_{>0} \varpi_\alpha) \]

such that \( \psi(\varphi_L, \lambda_L) = a(L)[L] - \psi(\varphi_\Sigma, 2\rho_P) \). Then

\[ \hat{L} := \frac{1}{a(L)} (\varphi_\Sigma + \varphi_L, 2\rho_P + \lambda_L) \]

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is mapped onto \([L]\) by \(\psi\). Denote by
\[
(17) \quad h_L(\varphi, \lambda) := \prod_{e \in \Sigma_1 - \Sigma'_1} \frac{\varphi(e)}{\varphi(e) + 1} \prod_{\alpha \in \Delta_P - \Delta'_P} \frac{\langle \lambda, \alpha \rangle}{\langle \lambda, \alpha \rangle + 1}
\]
and put
\[
\tilde{\varphi} = \varphi + \varphi_L + \varphi \quad \text{and} \quad \tilde{\lambda} = \lambda + \rho_P + \lambda_L.
\]
¿From now on we will denote by \(K \subset G(A)\) the maximal compact subgroup defined in section 8.2.

**Lemma 5.4** There exists a convex open neighborhood \(B\) of \(0\) in \(PL(\Sigma)_R \oplus X^*(P)_R\) with the following property: For any compact subset \(K \subset T_B\) there is a constant \(c = c(K) > 0\) such that for all \((\varphi, \lambda) \in K, \chi \in U_T\) and \(m \in M_R\) we have
\[
\left| \hat{H}_\Sigma(\chi, \tilde{\varphi} + im)E_P^G(\tilde{\lambda} - i\tilde{\eta}(m), \chi^{-1})h_L(\varphi + im, \lambda - i\tilde{\eta}(m)) \right|
\]
\[
\leq c \prod_{v | \infty} \left\{ \sum_{\dim \sigma = d} \prod_{e \in \sigma \cap \Sigma_1} \frac{1}{(1 + |e(m + m_e(\chi))|^1 + 1/2d)} \right\}.
\]

**Proof.** Write as in (12)
\[
\hat{H}_\Sigma(\chi, \tilde{\varphi} + im) = \zeta(\chi, \tilde{\varphi} + im) \prod_{e \in \Sigma_1} L_f(\chi_e, 1 + (\varphi + \varphi_L + im)(e)).
\]

For \(\text{Re}(\varphi)\) sufficiently small and \(e \in \Sigma'_1\) we have
\[
\text{Re}(\varphi(e)) + \varphi_L(e) \geq \frac{1}{2} \varphi_L(e) > 0.
\]

Hence
\[
\left| \prod_{e \in \Sigma'_1} L_f(\chi_e, 1 + (\varphi + \varphi_L + im)(e)) \right|
\]
is bounded for \(\text{Re}(\varphi)\) sufficiently small. If \(e \in \Sigma_1 - \Sigma'_1\) then \(\varphi_L(e) = 0\). By the estimates of Rademacher (cf. Theorem 3.6), we have for \(\chi_e \neq 1\)
\[
|L_f(\chi_e, 1 + (\varphi + im)(e))| \leq c_e(1 + |m(e)| + \|\chi_e\|_\infty)^e
\]

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for $\text{Re}(\varphi(e)) > -\delta$ and $\varphi$ in a compact set ($\delta$ depends on $\epsilon$, $c_\epsilon$ depends on this compact subset). If $\chi_e = 1$ (abusing notations we will denote from now on the trivial character by $1$) then

$$\frac{(\varphi + im)(e)}{(\varphi + im)(e) + 1} |L_f(1, 1 + (\varphi + im)(e))| \leq c_\epsilon (1 + |m(e)|)^\epsilon$$

Now we use Proposition 8.7 concerning estimates for Eisenstein series. This proposition tells us that there is for given $\epsilon > 0$ an open neighborhood of $0$ in $X^*(P)_R$ such that for $\text{Re}(\lambda)$ contained in this neighborhood

$$\prod_{\alpha \in \Delta_P} \frac{\langle \lambda + \lambda_L - i\eta(m), \alpha \rangle}{\langle \lambda + \lambda_L - i\eta(m), \alpha \rangle + 1} E^G_P(\tilde{\lambda} - i\tilde{\eta}(m), \chi_\eta^{-1}) \leq c_1 (1 + \|\text{Im}(\lambda) + \eta(m)\| + \|\chi_\eta^{-1}\|_\infty)^\epsilon.$$  

(For the definition of $(\cdots)_\infty$ and the norms see section 8.5.) If we let $\lambda$ vary in a compact subset in the tube domain over this neighborhood then there is a constant $c_2 \geq c_1$ such that

$$c_1 (1 + \|\text{Im}(\lambda) + \eta(m)\| + \|\chi_\eta^{-1}\|_\infty)^\epsilon \leq c_2 (1 + \|\eta(m)\| + \|\chi_\eta^{-1}\|_\infty)^\epsilon$$

For $\text{Re}(\lambda)$ sufficiently small and $\alpha \in \Delta_{P'}$ we have

$$\langle \text{Re}(\lambda) + \lambda_L, \alpha \rangle \geq \frac{1}{2} \langle \lambda_L, \alpha \rangle > 0.$$  

Therefore, there are $c_3, c_4 > 0$ such that for all such $\lambda$ and $m \in M_R$ we have

$$c_3 \leq \prod_{\alpha \in \Delta_{P'}} \frac{\langle \lambda + \lambda_L - i\eta(m), \alpha \rangle}{\langle \lambda + \lambda_L - i\eta(m), \alpha \rangle + 1} \leq c_4$$

Putting everything together, we can conclude that there is a neighborhood $B$ of $0$ in $PL(\Sigma)_R \oplus X^*(P)_R$ such that for $(\varphi, \lambda)$ in a compact subset $K$ of the tube domain over $B$ we have

$$\prod_{\varepsilon \in \Sigma_1} L_f(\chi_e, 1 + (\varphi + \varphi_L + im)(e)) \prod_{\varepsilon \in \Sigma_1 - \Sigma_1'} \frac{(\varphi + im)(e)}{(\varphi + im)(e) + 1}$$

$$\times \left| E^G_P(\tilde{\lambda} - i\tilde{\eta}(m), \chi_\eta^{-1}) \prod_{\alpha \in \Delta_{P'} - \Delta_{P'}} \frac{\langle \lambda - i\eta(m), \alpha \rangle}{\langle \lambda - i\eta(m), \alpha \rangle + 1} \right| \leq c(K) (1 + \|m + m_\infty(\chi)\|)^\epsilon,$$  

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where $\| \cdot \|$ is a norm on $M_{R,\infty}$. On the other hand, by Lemma 3.5, we have
\[|\zeta_\Sigma(\chi, \tilde{\varphi} + im)| \leq c''(K) \prod_{v|\infty} \left\{ \sum_{\dim \sigma = d} \prod_{e \in \sigma \cap \Sigma_1} \frac{1}{(1 + |e(m + m(\chi))|)^{1+1/d}} \right\} \]

Now we may choose $\epsilon$ and $c_5$ such that
\[ (1 + \|m + m(\chi)\|)^\epsilon \leq c_5 \prod_{v|\infty} \prod_{e \in \sigma_v \cap \Sigma} (1 + |e(m + m(\chi))|)^{1/2d} \]
for any system $(\sigma_v)_{v|\infty}$ of $d$-dimensional cones. This gives the claimed estimate. \qed

5.5 To begin with, we let
\[ M'_R := \{ m \in M_R \mid e(m) = 0 \forall e \in \Sigma_1 - \Sigma'_1 \text{ and } \langle \tilde{\eta}(m), \alpha \rangle = 0 \forall \alpha \in \Delta_P - \Delta_{P'} \}, \]
then $M'_R = M' \otimes R$ and $M'' = M/M'$ is torsion free. Put $d' = \text{rank}(M')$, $d'' = \text{rank}(M'')$.

The connection with sections 5.2 and 5.3 is as follows:
\[ E = (PL(\Sigma) \oplus X^*(P)_R)/M'_R, \]
\[ V = M'' \otimes R, A = M'', \nu = d'', \]
the set of linear forms $l_1, ..., l_m$ is given as follows
\begin{align*}
(18) & \quad (\varphi, \lambda) + M'_R \mapsto \varphi(e), e \in \Sigma_1 - \Sigma'_1, \\
(19) & \quad (\varphi, \lambda) + M'_R \mapsto \langle \lambda, \alpha \rangle, \alpha \in \Delta_P - \Delta_{P'}.
\end{align*}

The measure $dv = dm''$ on $V = M''_R$ is normalized by $M''$, $dm = dm'dm''$, where $dm$ (resp. $dm'$) is the Lebesgue measure on $M_R$ (resp. $M'_R$) normalized by $M$ (resp. $M'$).

Fix a convex open neighborhood of $0$ in $PL(\Sigma)_R \oplus X^*(P)_R$ for which Lemma 5.4 is valid. Denote by $B$ the image of this neighborhood in $E$. This is an open and convex neighborhood of $0$. Using Lemma 5.4 we see that
\[ g(\varphi, \lambda) = \mu_B \int_{M'_R} \frac{1}{K_d} \left\{ \sum_{\chi \in L_P} \hat{H}_\Sigma(\chi, \tilde{\varphi} + im')E^G_P(\tilde{\lambda} - i\tilde{\eta}(m'), \chi^{-1})h_L(\varphi, \lambda) \right\} dm' \]

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is a holomorphic function on $T_B$ (here $\mu_{T'} = 1/(2\pi \kappa)^d$). (We use the invariance of $g$ under $iM'_R$ and Cauchy-Riemann differential equations to check that $g$ is actually a function on $T_B$.) Hence,

$$f(\varphi, \lambda) := g(\varphi, \lambda)h_L(\varphi, \lambda)^{-1}$$

is a meromorphic function on $T_B$.

Let $E^{(0)}$ be the common kernel of all maps $\{18, 19\}$. Note that there is an exact sequence

$$0 \to M'_R \to E^{(0)} \to \langle \Lambda(L) \rangle \to 0$$

which implies

$$b(L) = \text{codim } \Lambda(L) = m - d''$$

where $m = \#(\Sigma_1 - \Sigma'_1) + \#(\Delta_P - \Delta_P')$.

By construction, $M''_R \cap E^{(0)} = \{\textbf{0}\}$. Let $dy$ be the Lebesgue measure on $E_0'$ normalized by the lattice generated by the linear forms $\{18, 19\}$. Denote by $E'_0$ the closed simplicial cone in $E_0$ defined by these linear forms, and by $\pi_0 : E \to E_0$ the canonical projection. It is easily seen that $\pi_0(M''_R) \cap E'_0 = \{\textbf{0}\}$ (using the exact sequence above). Let

$$\psi_0 : E_0 \to P := E_0/\pi_0(M''_R)$$

be the canonical projection and put

$$\Lambda = \psi_0(E'_0),$$

$$B^+ = B \cap \{(\varphi, \lambda) \in E \mid \varphi(e) > 0 \ \forall \ e \in \Sigma_1 - \Sigma'_1, \langle \lambda, \alpha \rangle > 0 \ \forall \ \alpha \in \Delta_P - \Delta_P'\}.$$ 

By the following theorem the function $f \in \mathcal{M}(T_B)$ is distinguished with respect to $M''_R$ and the set of linear forms $\{18, 19\}$. Therefore, we can define $\tilde{f}_{B^+} : T_{B^+} \to \mathbb{C}$ by

$$\tilde{f}_{B^+}(z) = \frac{1}{(2\pi)^{d''}} \int_{M''_R} f(z + i(m'', -\hat{\eta}(m''))) \, dm''.$$ 

**Theorem 5.5**  

a) $f$ is a distinguished function with respect to $M''_R$ and the set of linear forms $\{18, 19\}$.

b) There exist an open neighborhood $\tilde{B}$ of $0$ containing $B^+$ and linear forms $\tilde{l}_1, \ldots, \tilde{l}_{\tilde{m}}$ which vanish on $M''_R$ such that

$$\tilde{f}_{B^+}(z) \prod_{j=1}^{\tilde{m}} \tilde{l}_j(z)$$

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has a holomorphic continuation to $T_B$ and $g(0) \neq 0$.

Proof. a) Define $c_0 : M_{R, \infty} \to \mathbb{R}_{\geq 0}$ by

$$c_0((m_\nu)_\nu) = \prod_v \left( \sum_{\dim \sigma = d} \prod_{e \in \sigma \cap \Sigma_1} \frac{1}{1 + |e(m_\nu)|^{1+1/2d}} \right).$$

Let $\mathcal{F} \subset M_{R, \infty}^1$ be the cube spanned by a basis of the image of $U_T$ in $M_{R, \infty}^1$. Let $c' > 0$ such that for all $m_\infty(\chi) \in M_{R, \infty}^1 (\chi \in U_T)$ and all $m^1 \in \mathcal{F}$

$$c_0(m_\infty(\chi)) \leq c' c_0(m_\infty(\chi) + m^1).$$

Let $dm^1$ be the Lebesgue measure on $M_{R, \infty}^1$ normalized by the image of $U_T$. Then for $m \in M_R$

$$\int_{M_R'} \left\{ \sum_{\chi \in U_T} c_0(m_\infty(\chi) + m^1 + m) \right\} dm' \leq c(m \mod M'R'),$$

where $c : M'R' \to \mathbb{R}_{\geq 0}$ is defined by

$$c(m'') := c' c' \int_{M_R'} \int_{M_{R, \infty}} c_0(m^1 + m'' + m^1) dm^1 dm'.$$

By Lemma 5.4, for any compact subset $K$ of $T_B$ there is a $c(K) > 0$ such that

$$|g(z + im'')| \leq c(m'')$$

for all $z \in K$ and $m'' \in M_{R}'$. Obviously, $c$ can be integrated over any subspace $U$ of $M''_{R}$.

It remains to show that for any $m'' \in M''_{R} - U$ one has

$$\lim_{\tau \to \pm \infty} \int_U c(\tau m'' + u) du = 0.$$

This exercise will be left to the reader.

b) The first part concerning the meromorphic continuation and singularities of $\tilde{f}_{B^+}$ is the content of Theorem 5.2. The relation

$$\lim_{s \to h(L)} \tilde{f}_{B^+}(s\tilde{L}) = g(0)\chi_L(\psi_0(\tilde{L}))$$

is satisfied by Theorem 5.3 and (20). It will be shown in Section 6 that $g(0) \neq 0$. \hfill \Box
The main theorem of our paper is:

**Theorem 5.6** Let \( L \) be a line bundle on \( Y \) which lies in the interior of the cone of effective divisors. Then there exists a metrization \( \mathcal{L} \) of \( L \) with the following properties:

a) The height zeta function

\[
Z_{Y^o}(\mathcal{L}, s) = \sum_{y \in Y^o(F)} H_{\mathcal{L}}(y)^{-s}
\]

is holomorphic for \( \Re(s) > a(L) \) and it can be continued meromorphically to a half-space \( \Re(s) > a(L) - \delta \) for some \( \delta > 0 \). In this half-space it has a pole of order \( b(L) \) at \( a(L) \) and no other poles.

b) For the counting function one has the following asymptotic relation

\[
N_{Y^o}(\mathcal{L}, H) = c(\mathcal{L}) H^{a(L)} (\log H)^{b(L)-1} (1 + o(1))
\]

for \( H \to \infty \) with some constant \( c(\mathcal{L}) > 0 \).

**Proof.** a) By construction, \( \hat{L} = \frac{1}{a(L)}(\varphi_\Sigma + \varphi_L, 2\lambda_P + \lambda_L) \) is mapped onto \([L]\) by \( \psi \). Hence \( Z_{Y^o}(\mathcal{L}, s) \) converges absolutely for \( \Re(s) > a(L) \), where \( \mathcal{L} \) is the metrization mentioned in Proposition 4.4. By the same proposition,

\[
Z_{Y^o}(\mathcal{L}, s + a(L)) = \mu_T \int_{M_R} \{ \sum_{\chi \in U_T} f_L(\chi, im) \} dm
\]

where

\[
f_L(\chi, im) := \hat{H}_\Sigma(\chi, \frac{s}{a(L)}(\varphi_\Sigma + \varphi_L) + \varphi_\Sigma + \varphi_L + im) E^G_{\rho_P}(\frac{s}{a(L)}(2\rho_P + \lambda_L) + \rho_P + \lambda_L - i\eta(m), \chi^{-1})
\]

for all \( s \) with \( \Re(s) > 0 \). However, this is just

\[
\frac{1}{(2\pi)^d} \int_{B_T^+} f \left(s\hat{L} + i(m'', -\eta(m''))\right) dm'' = \tilde{f}_{B^+}(s\hat{L})
\]

with \( f, B^+ \) and \( \tilde{f}_{B^+} \) introduced in the preceding section. By Theorem 5.5, \( \tilde{f}_{B^+} \) extends to a meromorphic function on a tube domain over a neighborhood of \( 0 \) and in this tube domain
the only singularities are the hyperplanes defined over \( \mathbb{R} \). Hence there is a \( \delta > 0 \) such that \( Z_{\gamma^\alpha}(L, s + a(L)) \) extends to a meromorphic function in the half-space \( \text{Re}(s) > -\delta \) and the only possible pole is in \( s = 0 \) and its order is exactly \( b(L) \) (Theorems 5.3 and 5.5).

b) This result follows from a Tauberian theorem (cf. [8], Théorème III or [22], Problem 14.1 (in the constant stated there the factor \( \frac{1}{k_0} \) is missing)).

\[ \square \]

6 Non-vanishing of asymptotic constants

6.1 This section is devoted to the proof of the non-vanishing of \( g(0) \) claimed in Theorem 5.5. All notations are as in sections 5.2-5.5. The function \( g(\varphi, \lambda) \) which has been defined in 5.5 is given by

\[
g(\varphi, \lambda) = \frac{\mu_{\kappa}^P}{\kappa^d} \int_{M'} \left\{ \sum_{\chi \in U_T} \hat{H}_\Sigma(\chi, \tilde{\varphi} + im') E_P^G \left( \tilde{\lambda} - i\tilde{\eta}(m'), \chi^{-1}_\eta \right) h_L(\varphi, \lambda) \right\} dm,
\]

where \( \tilde{\varphi}, \tilde{\lambda} \) have been defined in 5.4. The function \( h_L(\varphi, \lambda) \) was defined in 5.4:

\[
h_L(\varphi, \lambda) = \prod_{e \in \Sigma_1 - \Sigma'_1} \frac{\varphi(e)}{\varphi(e) + 1} \prod_{\alpha \in \Delta_P - \Delta_P'} \frac{\langle \lambda, \alpha \rangle}{\langle \lambda, \alpha \rangle + 1}.
\]

The uniform convergency of the integral above in any compact subset of \( T_B \) (cf. Lemma 5.4) allows us to compute the limit

\[
\lim_{(\varphi, \lambda) \to 0} \prod_{e \in \Sigma_1 - \Sigma'_1} \varphi(e) \prod_{\alpha \in \Delta_P - \Delta_P'} \langle \lambda, \alpha \rangle \hat{H}_\Sigma(\chi, \tilde{\varphi} + im') E_P^G \left( \tilde{\lambda} - i\tilde{\eta}(m'), \chi^{-1}_\eta \right)
\]

first and then to integrate. We shall show that this limit vanishes if there are \( e \in \Sigma_1 - \Sigma'_1 \) with \( \chi_e \neq 1 \) or \( \alpha \in \Delta_P - \Delta_P' \) with \( \chi_\eta \circ \tilde{\alpha} \neq 1 \). Therefore, we may consider only \( \chi \in U'_T \) where

\[
U'_T := \{ \chi \in U_T \mid \chi_e = 1 \forall e \in \Sigma_1 - \Sigma'_1, \ \chi_\eta \circ \tilde{\alpha} = 1 \forall \alpha \in \Delta_P - \Delta_P' \}.
\]

Let \( \eta' : P' \to T \) be the uniquely defined homomorphism such that for all \( \alpha \in \Delta_P \) we have \( \eta' \circ \tilde{\alpha} = \eta \circ \tilde{\alpha} \).
Lemma 6.1

\begin{equation}
(21) \quad g(0) = \lim_{\varphi \to 0, \varphi \in PL(\Sigma)^+} \prod_{e \in \Sigma_1 - \Sigma'_1} \varphi(e) \cdot \frac{\mu T'}{\kappa^P'} \cdot \frac{c_P'}{c_P} \int_{M'_R} \left\{ \sum_{\chi \in U'_T} \hat{H}_\Sigma(\chi, \tilde{\varphi} + \imath m') \right. \\
\times E^G_{P'} \left( \lambda_L + \rho_{P'} - i\tilde{\eta}'(m'), (\chi_\eta)^{-1} \right) \left. \right\} \, dm'
\end{equation}

(cf. 8.4 for the definition of $c_{P'}$ and $c_P$).

Proof. Recall that (cf. (12))

\[
\hat{H}_\Sigma(\chi, \tilde{\varphi} + \imath m') = \zeta_\Sigma(\chi, \tilde{\varphi} + \imath m') \prod_{e \in \Sigma_1} L_f(\chi_e, 1 + \varphi(e) + \varphi_L(e) + \imath e(m'))
\]

and that $\zeta_\Sigma(\chi, \tilde{\varphi} + \imath m')$ is regular for $\varphi$ in a tube domain over a neighborhood of $0$ (Lemma 3.5). For $e \in \Sigma'_1$ we have $\varphi_L(e) > 0$, hence we see that the function

$$L_f(\chi_e, 1 + \varphi(e) + \varphi_L(e) + \imath e(m'))$$

is holomorphic for $\text{Re}(\varphi)$ in a neighborhood of $0$. Let $e \in \Sigma_1 - \Sigma'_1$. If $\chi_e \neq 1$ then the restriction of $\chi_e$ to $G_m(A)^1$ is non-trivial (by our construction of the embedding $U_T \to A_T$, cf. 3.3), hence

$$\varphi \mapsto L_f(\chi_e, 1 + \varphi(e))$$

is an entire function and $\varphi(e)L_f(\chi_e, 1 + \varphi(e))$ tends to $0$ as $\varphi \to 0$.

For $\alpha \in \Delta_{P'}$ we have $\langle \lambda_L, \alpha \rangle > 0$, hence $\lambda_L$ is contained in $X^*(P')^+$. Let $\alpha \in \Delta_P - \Delta_{P'}$. If $\chi_\eta \circ \tilde{\alpha} \neq 1$ then $\chi_\eta \circ \tilde{\alpha}$ restricted to $G_m(A)^1$ is non-trivial and therefore

$$\prod_{\alpha \in \Delta_P - \Delta_{P'}} \langle \lambda, \alpha \rangle E^G_P \left( \lambda + \rho_P + \lambda_L - i\tilde{\eta}(m'), \chi_\eta^{-1} \right)$$

vanishes as $\lambda \to 0$ (cf. Proposition 8.3). We have shown that it suffices to take the sum over all $\chi \in U'_T$. To complete the proof, note that for $\chi \in U'_T$ we have (cf. Proposition (8.4))

$$\lim_{\lambda \to 0} \prod_{\alpha \in \Delta_P - \Delta_{P'}} \langle \lambda, \alpha \rangle E^G_P \left( \tilde{\lambda} - i\tilde{\eta}(m'), \chi_\eta^{-1} \right) = \frac{c_{P'}}{c_P} E^G_{P'} \left( \lambda_L + \rho_{P'} - i\tilde{\eta}'(m'), (\chi'_\eta)^{-1} \right).$$

□

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6.2 By the absolute and uniform convergence of
\[
\int_{M_{\mathbb{R}}'} \left\{ \sum_{\chi \in \mathcal{U}_{\mathbb{R}}'} \hat{H}_\Sigma(\chi, \tilde{\varphi} + im') \prod_{e \in \Sigma_1 - \Sigma'_1} \varphi(e) \right\} \, dm'
\]
(cf. Lemma 3.5 and Theorem 3.6) and the convergence of
\[
\sum_{\gamma \in P'(F) \setminus G(F)} e^{\langle \lambda_L + 2\rho_{p'}, H_{p'}(\gamma) \rangle}
\]
we may change summation and integration in (21) and get for all \( \varphi \in PL(\Sigma)^+ \)
\[
\frac{\mu_T'}{K_{H'}} \cdot \frac{c_{\rho'}}{c_P} \int_{M_{\mathbb{R}}'} \sum_{\chi \in \mathcal{U}_{\mathbb{R}}'} \hat{H}_\Sigma(\chi, \tilde{\varphi} + im') E_{p'}^G \left( \lambda_L + \rho_{p'} - i \eta'(m') (\chi_{\eta'})^{-1} \right) \, dm'
\]
\[
= \frac{c_{\rho'}}{c_P K_{H'}} \sum_{\gamma \in P'(F) \setminus G(F)} e^{\langle \lambda_L + 2\rho_{p'}, H_{p'}(\gamma) \rangle} \mu_T' \int_{M_{\mathbb{R}}'} \sum_{\chi \in \mathcal{U}_{\mathbb{R}}'} \hat{H}_\Sigma(\chi, \tilde{\varphi} + im') (\chi^{-m'} \chi_{\eta'})^{-1} (\eta'(p'_\gamma)) \, dm'
\]
where \( \gamma = p'_\gamma k_\gamma \) as above. Let \( I \) be the image of the homomorphism
\[
\prod_{e \in \Sigma_1 - \Sigma'_1} G_m(A) \times \prod_{\alpha \in \Delta_{\rho} - \Delta_{p'}} G_m(A) \to T(A)
\]
induced by \( M \to Z^{\Sigma_1 - \Sigma'_1} \oplus Z^{\Delta_{\rho} - \Delta_{p'}} \),
\[
m \mapsto \left( (e(m))_{e \in \Sigma_1 - \Sigma'_1}, (\langle -m \circ \eta, \alpha \rangle)_{\alpha \in \Delta_{\rho} - \Delta_{p'}} \right).
\]
Then \( M_{\mathbb{R}}' \oplus \mathcal{U}_{\mathbb{R}}' \) is precisely the set of characters \( T(A) \to S^1 \) which are trivial on \( T(F)K_T I \).
Put \( T' = \text{Spec}(F[M']) \) and \( T'' = \text{Spec}(F[M'']) \). Then there is an exact sequence
\[
1 \to T'' \to T' \to T' \to 1.
\]
Note that \( I \subset T''(A) \). Denote by \( K_{T'} \) (resp. \( K_{T''} \)) the maximal compact subgroup of \( T'(A) \) (resp. of \( T''(A) \)). The linear forms
\[
m \mapsto e(m), \quad e \in \Sigma_1 - \Sigma'_1,
\]
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when considered as functions on $M''$, generate a sublattice of finite index in $N'' = \text{Hom}(M'', \mathbb{Z})$. This shows that there is a $q > 0$ such that the image of the $q$-th power homomorphism $T''(A) \rightarrow T''(A), t \mapsto t^q$, is contained in $I$. If $v$ is any archimedean place of $F$ the connected component of one in $T''(F_v)$ is therefore contained in $I$. Consequently,

$$
T''(F) \cdot \prod_{v|\infty} T''(F_v) \prod_{v|\infty} K_{T''[v]} \subset T''(F)K_{T''} \cdot I
$$

and the left hand side is of finite index in $T''(A)$. Put

$$
\mathcal{A}'_T = \{ \chi \in \mathcal{A}_T \mid \chi = 1 \text{ on } T(F)K_T T''(A) \}.
$$

We observe that

$$
\mathcal{A}'_T \simeq \mathcal{A}'_T = (T'(A)/T'(F)K_{T'})^* \subset M'_R \oplus \mathcal{U}_T.
$$

We denote by

$$
\mathcal{I} = (M'_R \oplus \mathcal{U}_T)/\mathcal{A}'_T
$$

and by $\iota$ the order of $\mathcal{I}$. Put $\mathcal{U}_T = \mathcal{U}_T \cap \mathcal{A}'_T$ (then $\mathcal{A}'_T = M'_R \oplus \mathcal{U}_T$). Thus we may write

$$
\int_{M'_R} \left\{ \sum_{\chi \in \mathcal{U}_T'} \hat{H}_\Sigma(\chi, \tilde{\varphi} + im') (\chi m')^{-1}(\eta(p'_\gamma)) \right\} \, dm'
$$

$$
= \sum_{\chi \in \mathcal{I}} \int_{M'_R} \left\{ \sum_{\chi' \in \mathcal{U}_T'} \hat{H}_\Sigma(\chi' \chi m', \tilde{\varphi})(\chi' \chi m')^{-1}(\eta'(p'_\gamma)) \right\} \, dm'
$$

For $\chi \in M'_R \oplus \mathcal{U}_T$ and $x \in T'(A)$ we consider the function

$$
x \mapsto \int_{T''(A)} H_\Sigma(x t \eta'(p'_\gamma), \tilde{\varphi}) \chi(t) \, dt
$$

(the Haar measure $dt$ on $T''(A)$ is defined as $dx$ on $T(A)$, cf. 3.3). The same argument as in the proof of Proposition 3.4 shows that this function is absolutely integrable over $T'(F)$ if $\varphi \in PL(\Sigma)^+$. The Fourier transform for $\chi' \in \mathcal{A}'_T$

$$
\int_{T'(A)} \left( \int_{T''(A)} H_\Sigma(x t \eta'(p'_\gamma), \tilde{\varphi}) \chi(t) \, dt \right) \chi'(x) \, dx
$$

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is absolutely integrable over $A_T'$. Using Poisson’s summation formula twice we get
\[
\sum_{\chi \in I} \mu_{T'} \int_{\mathcal{M}_T'} \left\{ \sum_{\chi' \in \mathcal{U}_{T'}} \hat{H}_\Sigma(\chi' \chi, \bar{\varphi}) (\chi' \chi)^{-1} (\eta'(p')) \right\} \, dm'
= \sum_{\chi \in I} \sum_{x \in T'(F)} \int_{T''(A)} H_\Sigma(x \eta'(p', \bar{\varphi})) \chi(t) \, dt = \sum_{x \in T'(F)} t \int_{T''(F) \mathcal{K}_{T'}} H_\Sigma(x \eta'(p', \bar{\varphi})) \, dt.
\]
Now we collect all the terms together.

Lemma 6.2. The constant $g(0)$ is equal to
\[
\frac{\mu_{P'/\mathcal{U}_P}}{\kappa_{d'}} \sum_{\gamma \in P'(F) \setminus G(F)} e^{(\lambda L + 2\rho_{P'}, H_{P'}(\gamma))} \times
\lim_{\varphi \to 0, \varphi \in P_L(\Sigma)^+} \prod_{e \in \Sigma_1 - \Sigma'_1} \varphi(e) \sum_{x \in T'(F)} t \int_{T''(F) \mathcal{K}_{T'}} H_\Sigma(x \eta'(p', \bar{\varphi})) \, dt.
\]

Lemma 6.3. The limit
\[
\lim_{\varphi \to 0, \varphi \in P_L(\Sigma)^+} \prod_{e \in \Sigma_1 - \Sigma'_1} \varphi(e) \int_{T''(A)} H_\Sigma(t, \bar{\varphi}) \, dt
\]
exists and is positive.

Proof. Consider the embedding $N''_R \to N_R$ and let
\[
\Sigma'' := \{ \sigma \cap N''_R \mid \sigma \in \Sigma \}.
\]
This is a complete fan in $N''_R$ which consists of rational polyhedral cones, but which is not necessary a regular fan. We can obtain a regular fan by subdivision of the cones into regular ones (cf. [14], ch. I, §2, Theorem 11). This gives us a complete regular fan $\Sigma''$ such that any cone in $\Sigma''$ is contained in a cone of $\Sigma''$. Denote by $\Sigma'_1$ the set of primitive integral generators of the one-dimensional cones in $\Sigma''$. Computing the integral as in section 3.4 we get
\[
\int_{T''(A)} H_\Sigma(t, \bar{\varphi}) \, dt = \zeta_{\Sigma''}(1, \bar{\varphi}) \prod_{\bar{e} \in \Sigma''_1} L_f(1, \bar{\varphi} (\bar{e}))
\]
(cf. (12)), where $\zeta_{\Sigma^\prime}(1, \tilde{\varphi})$ is regular in a neighborhood of $\varphi = 0$ and positive for $\varphi = 0$. Let $\tilde{e} \in N$ and $\sigma \in \Sigma'$ be a cone containing $\tilde{e}$. Write $\tilde{e} = \sum_{e \in \sigma \cap \Sigma_1} t_e \cdot e \ (t_e \in \mathbb{Z}_{\geq 0})$. Suppose

$$1 = (\varphi_{\Sigma} + \varphi_L)(\tilde{e}) = \sum_{e \in \sigma \cap \Sigma_1} t_e (1 + \varphi_L(e)).$$

Then $t_e = 0$ for all $e \in \sigma \cap \Sigma'_1$ because $\varphi_L \in \sum_{e \in \Sigma_1} \mathbb{R}_{>0} \varphi_e$ (cf. 5.4). Hence,

$$\tilde{e} = \sum_{e \in \sigma \cap \Sigma_1 - \Sigma'_1} t_e \cdot e$$

and $\varphi_{\Sigma}(\tilde{e}) = 1$ implies $\tilde{e} \in \Sigma_1 - \Sigma'_1$. Therefore,

$$\lim_{\varphi \to 0, \varphi \in PL(\Sigma)^+} \prod_{e \in \Sigma_1 - \Sigma'_1} \varphi(e) \prod_{\tilde{e} \in \Sigma''_1} L_f(1, \tilde{\varphi}(\tilde{e}))$$

$$= \left\{ \prod_{e \in \Sigma_1 - \Sigma'_1} \lim_{\varphi \to 0, \varphi \in PL(\Sigma)^+} \varphi(e) L_f(1, 1 + \varphi(e)) \right\} \prod_{\tilde{e} \in \Sigma''_1 - (\Sigma_1 - \Sigma'_1)} L_f(1, (\varphi_{\Sigma} + \varphi_L)(\tilde{e}))$$

and this is a positive real number. □

In theorem 5.5 we claimed the non-vanishing of $g(0)$. We are now in the position to prove

**Corollary 6.4**

$$g(0) > 0.$$  

**Proof.** By Lemma 6.2 it is enough to show that

$$\lim_{\varphi \to 0, \varphi \in PL(\Sigma)^+} \prod_{e \in \Sigma_1 - \Sigma'_1} \varphi(e) \int_{T''(F)K_{T''} \cdot I} H_{\Sigma}(t, \tilde{\varphi})dt$$

is positive. Let $t_1, ..., t_\nu \in T(A)$ such that

$$T''(A) = \bigcup_{j=1}^{\nu} t_j T''(F)K_{T''} \cdot I.$$  

Then there exists a constant $c > 0$ such that for all $t \in T''(A)$ and $j = 1, ..., \nu$ we have

$$H_{\Sigma}(tt_j, \tilde{\varphi}) \leq \frac{c}{\nu} H_{\Sigma}(t, \tilde{\varphi}).$$
Hence we can estimate

\[ \int_{T''(A)} H_\Sigma(t, \varphi) dt = \sum_{j=1}^{\nu} \int_{T''(F)K_{T''I}} H_\Sigma(tt_j, \tilde{\varphi}) dt \]
\[ \leq c \int_{T''(F)K_{T''I}} H_\Sigma(t, \tilde{\varphi}) dt. \]

Lemma 6.3 allows us to conclude that the limit above is indeed positive. \qed

7 Technical theorems

7.1 Let \((A, V, \Lambda)\) be a triple consisting of a free abelian group \(A\) of rank \(d\), a \(d\)-dimensional real vector space \(V := A \otimes \mathbb{R}\) containing \(A\) as a sublattice of maximal rank, and a closed strongly convex polyhedral \(d\)-dimensional cone \(\Lambda \subset A_{\mathbb{R}}\) such that \(\Lambda \cap -\Lambda = \{0\}\). Denote by \(\Lambda^\circ\) the interior of \(\Lambda\). Let \((A^\vee, V^\vee, \Lambda^\vee)\) be the triple consisting of the dual abelian group \(A^\vee = \text{Hom}(A, \mathbb{Z})\), the dual real vector space \(V^\vee = \text{Hom}(V, \mathbb{R})\) and the dual cone \(\Lambda^\vee \subset V^\vee\). We normalize the Haar measure \(dy\) on \(V^\vee\) by the condition:

\[ \text{Vol}(V^\vee/A^\vee) = 1. \]

We denote by \(\chi_\Lambda(v)\) the set-theoretic characteristic function of the cone \(\Lambda\) and by \(\chi_\Lambda^\vee(v)\) the Laplace transform of the set-theoretic characteristic function of the dual cone

\[ \chi_\Lambda(v) = \int_{V^\vee} \chi_{\Lambda^\vee}(y) e^{-\langle v, y \rangle} dy = \int_{\Lambda^\vee} e^{-\langle v, y \rangle} dy, \]

where \(\text{Re}(v) \in \Lambda^\circ\) (for these \(v\) the integral converges absolutely).

Consider a complete regular fan \(\Sigma\) on \(V\), that is, a subdivision of the real space \(V\) into a finite set of convex rational simplicial cones, satisfying certain conditions (see [4], 1.2). Denote by \(\Sigma_1\) the set of primitive generators of one dimensional cones in \(\Sigma\). Denote by \(\text{PL}(\Sigma)_{\mathbb{R}}\) the vector space of real valued piecewise linear functions on \(V\) and by \(\text{PL}(\Sigma)_{\mathbb{C}}\) its complexification.

**Proposition 7.1** ([4], Prop. 2.3.2, p. 614) For any compact \(K \subset \text{PL}(\Sigma)_{\mathbb{C}}\) with the property that \(\text{Re}(\varphi(v)) > 0\) for all \(\varphi \in K\) and \(v \neq 0\) there exists a constant \(\kappa(K)\) such that for all \(\varphi \in K\) and all \(y \in V^\vee\) the following inequality holds:

\[ \left| \int_V e^{-\varphi(v) - i\langle v, y \rangle} dv \right| \leq \kappa(K) \sum_{\dim \sigma = d} \frac{1}{\prod_{e \in \sigma} (1 + |\langle e, y \rangle|)^{1+1/d}}. \]
7.2 Let $H \subset V$ be a hyperplane with $H \cap \Lambda = \{0\}$. Let $y_0 \in V^\vee$ with $H = \text{Ker}(y_0)$, such that for all $x \in \Lambda : y_0(x) \geq 0$. Then $y_0$ is in the interior of $\Lambda^\vee \subset V^\vee$. Let $x_0 \in \Lambda^\circ$ and let

$$H' = \{ y' \in V^\vee \mid y'(x_0) = 0 \}.$$  

We have $V^\vee = H' \oplus \mathbb{R}y_0$. Define $\varphi : H' \to \mathbb{R}$ by

$$\varphi(y') = \min \{ t \mid y' + ty_0 \in \Lambda^\vee \}.$$  

The function $\varphi$ is piecewise linear with respect to a complete fan of $H'$. Taking a subdivision, if necessary, we may assume it to be regular.

**Proposition 7.2** The function $\mathcal{X}_\Lambda(u)$ is absolutely integrable over any linear subspace $U \subset H$.

**Proof.** For $h \in H$ we have

$$\mathcal{X}_\Lambda(x_0 + ih) = \int_{\Lambda^\vee} e^{-y(x_0 + ih)} dy = \int_{H'} \int_{\varphi(y')}^{\infty} e^{-(y'+ty_0)(x_0+ih)} dt dy'$$

$$= \int_{H'} e^{-\varphi(y')} e^{-iy'(h)} dy'$$

Therefore, $h \mapsto \mathcal{X}_\Lambda(x_0 + ih)$ is the Fourier transform of the function $y' \mapsto e^{-\varphi(y')}$ on $H' \simeq H^\vee$. The statement follows now from 7.1. \qed

7.3 The rest of this section is devoted to the proof of the meromorphic continuation of certain functions which are holomorphic in tube domains over convex finitely generated polyhedral cones. In section 5 we have already introduced the terminology and explained how this technical theorem is applied to height zeta functions.

Let $E$ be a finite dimensional vector space over $\mathbb{R}$ and $E_\mathbb{C}$ its complexification. Let $V \subset E$ be a subspace. We will call a function $c : V \to \mathbb{R}_{\geq 0}$ sufficient if it satisfies the following conditions:

(i) For any subspace $U \subset V$ and any $v \in V$ the function $U \to \mathbb{R}$ defined by $u \to c(v+u)$ is measurable on $U$ and the integral

$$c_U(v) := \int_U c(v + u) du$$

is always finite ($du$ is a Lebesgue measure on $U$).
(ii) For any subspace $U \subset V$ and every $v \in V - U$ we have

$$\lim_{\tau \to \pm \infty} c_U(\tau \cdot v) = 0.$$ 

Let $l_1, \ldots, l_m \in E^\vee = \text{Hom}_\mathbb{R}(E, \mathbb{R})$ be linearly independent linear forms. Put $H_j = \text{Ker}(l_j)$ for $j = 1, \ldots, m$. Let $B \subset E$ be an open and convex neighborhood of $0$, such that for all $x \in B$ and all $j = 1, \ldots, m$ we have $l_j(x) > -1$. We denote by $T_B := B + iE \subset E_\mathbb{C}$ the complex tube domain over $B$. We denote by $\mathcal{M}(T_B)$ the set of meromorphic functions on $T_B$.

A meromorphic function $f \in \mathcal{M}(T_B)$ will be called distinguished with respect to the data $(V; l_1, \ldots, l_m)$ if it satisfies the following conditions:

(i) The function

$$g(z) := f(z) \prod_{j=1}^{m} \frac{l_j(z)}{l_j(z) + 1}$$

is holomorphic in $T_B$.

(ii) There exists a sufficient function $c : V \to \mathbb{R}_{\geq 0}$ such that for any compact $K \subset T_B$ there is a constant $\kappa(K) \geq 0$ such that for all $z \in K$ and all $v \in V$ we have

$$|g(z + iv)| \leq \kappa(K)c(v).$$

Let $C$ be a connected component of $B - \bigcup_{j=1}^{m} H_j$ and $T_C$ a tube domain over $C$. We will consider the following integral:

$$\tilde{f}_C(z) := \frac{1}{(2\pi)^d} \int_V f(z + iv)dv.$$  

Here we denoted by $d = \dim V$ and by $dv$ a fixed Lebesgue measure on $V$.

**Proposition 7.3** Assume that $f$ is an distinguished function with respect to $(V; l_1, \ldots, l_m)$. Then the following holds:

a) $\tilde{f}_C : T_C \to \mathbb{C}$ is a holomorphic function.

b) There exist an open and convex neighborhood $\tilde{B}$ of $0$, containing $C$, and linear forms $\tilde{l}_1, \ldots, \tilde{l}_\tilde{m}$, which vanish on $V$, such that

$$z \to \tilde{f}_C(z) \prod_{j=1}^{\tilde{m}} \tilde{l}_j(z)$$

has a holomorphic continuation to $T_{\tilde{B}}$.  

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Proof. a) Let $K \subset T_C$ be a compact subset and let $\kappa(K) \geq 0$ be a real number such that for all $z \in K$ and all $v \in V$ we have $|g(z + iv)| \leq \kappa(K)c(v)$. Since $K$ is a compact and $C$ doesn’t intersect any of the hyperplanes $H_j$ there exist real numbers $c_j \geq 0$ for any $j = 1, ..., m$, such that for all $z \in K$ and $v \in V$ the following inequalities hold
\[
\left| \frac{l_j(z + iv) + 1}{l_j(z + iv)} \right| \leq c_j.
\]
Therefore, for $z \in K$ and $v \in V$ we have
\[
|f(z + iv)| \leq c_1 \cdots c_m \kappa(K)c(v).
\]
It follows that on every compact $K \subset T_C$ the integral converges absolutely and uniformly to a holomorphic function $\tilde{f}_C$.

b) The proof proceeds by induction on $d = \dim V$. For $d = 0$ there is nothing to prove. Assume that $d \geq 1$ and let $v_0 \in V - \{0\}$ be a vector such that both $v_0, -v_0 \in B$. We define $B_1 \subset B$ as the set of all vectors $x \in B$ which satisfy the following two conditions: the vector $x \pm v_0 \in B$ and $\left| \frac{l_j(x)}{l_j(v_0)} \right| \leq \frac{1}{2}$ for all $j \in \{1, ..., m\}$ with $l_j(v_0) \neq 0$. The set $B_1$ is a convex open neighborhood of $0 \in E$. Fix a vector $x_0 \in C$. Without loss of generality we can assume that
\[
\{1, ..., m_0\} := \{j \in \{1, ..., m\} | l_j(v_0)l_j(x_0) < 0\}
\]
with $0 \leq m_0 \leq m$. For $j \in \{1, ..., m_0\}, k \in \{1, ..., \hat{j}, ..., m\}$ we define
\[
l_{j,k}(x) := l_k(x) - l_j(x)\frac{l_k(v_0)}{l_j(v_0)}, \quad H_{j,k} := \text{Ker}(l_{j,k}) \subset E.
\]
For all $j \in \{1, ..., m_0\}$ we have that $(l_{j,k})_{1 \leq k \leq m, k \neq j}$ is a set of linearly independent linear forms on $E$. Moreover, for all $x \in B_1$ and $j \in \{1, ..., m_0\}$ we have
\[
x - \frac{l_j(x)}{l_j(v_0)}v_0 = \left(1 - \frac{l_j(x)}{l_j(v_0)}\right)x + \frac{l_j(x)}{l_j(v_0)}(x - v_0) \in B,
\]
in the case that $\frac{l_j(x)}{l_j(v_0)} \geq 0$ and, similarly, in the case that $\frac{l_j(x)}{l_j(v_0)} < 0$
\[
x - \frac{l_j(x)}{l_j(v_0)}v_0 = \left(1 + \frac{l_j(x)}{l_j(v_0)}\right)x + \left(-\frac{l_j(x)}{l_j(v_0)}\right)(x + v_0) \in B.
\]
Therefore, for all \( x \in B_1 \) and \( j \in \{1, \ldots, m_0\} \), \( k \in \{1, \ldots, \hat{j}, \ldots, m\} \) we have

\[
l_{j,k}(x) = l_k \left( x - \frac{l_j(x)}{l_j(v_0)} v_0 \right) > -1.
\]

Let \( C_1 \) be a connected component of

\[
B_1 - \left( \bigcup_{j=1}^{m_0} \bigcup_{1 \leq k \leq m, k \neq j} H_{j,k} \cup \bigcup_{j=1}^m H_j \right),
\]

which is contained in \( C \). For \( z \in T_C \) we define

\[
h_C(z) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(z + i\tau v_0) d\tau = \frac{1}{2\pi i} \int_{\text{Re}(\lambda)=0} f(z + \lambda v_0) d\lambda.
\]

As in (i) one shows that \( h_C \) is a holomorphic function on \( T_C \). For \( x \in B_1 \) and \( \lambda \in [0, 1] \) we have

\[
x + \lambda v_0 = (1 - \lambda)x + \lambda(x + v_0) \in B.
\]

If for some \( z = x + iy \in T_{C_1} \) \((x \in C_1)\) and \( \lambda \in [0, 1] + i\mathbb{R}, j \in \{1, \ldots, m\} \) we have

\[
l_j(z + \lambda v_0) = 0,
\]

then it follows that \( l_j(x) + \text{Re}(\lambda)l_j(v_0) = 0 \), and therefore, \( l_j(x)l_j(v_0) < 0 \) (since \( l_j(x) \) has the same sign as \( l_j(x_0) \)). Consequently, \( j \in \{1, \ldots, m_0\} \).

For \( z \in T_{C_1} \) and \( j \in \{1, \ldots, m_0\} \) we put

\[
\lambda_j(z) := -\frac{l_j(z)}{l_j(v_0)}.
\]

By our assumptions, we have \( 0 < \text{Re}(\lambda_j(z)) < \frac{1}{2} \). From \( \lambda_j(z) = \lambda_{j'}(z) \), with \( j, j' \in \{1, \ldots, m_0\} \) and \( j \neq j' \) it follows now that

\[
l_j' \left( z - \frac{l_j(z)}{l_j(v_0)} v_0 \right) = 0.
\]

In particular, we have \( l_{j,j'}(\text{Re}(z)) = 0 \). This is not possible, because \( z \in T_{C_1} \).

Assume now that \( x \in B_1 \). We have, assuming that \( l_k(v_0) \neq 0 \), that

\[
|l_k(x + v_0)| = |l_k(v_0)| \cdot \left| \frac{l_k(x)}{l_k(v_0)} + 1 \right| \geq |l_k(v_0)| \cdot \left| -\frac{1}{2} + 1 \right| = \frac{1}{2} |l_k(v_0)|.
\]
If \( l_k(v_0) = 0 \) then we have \( l_k(x + v_0) = l_k(x) \).

Fix a \( z \in T_{C_1} \). Then there exist some numbers \( c_1(z) \geq 0, c_2(z) \geq 0 \) such that for all \( \lambda \in [0,1], \tau \in \mathbb{R} \) with \( |\tau| \geq c_1(z) \) we have

\[
|f(z + \lambda v_0 + i\tau v_0)| = |g(z + \lambda v_0 + i\tau v_0)| \cdot \prod_{j=1}^{m} \left( 1 + \frac{1}{l_j(z + \lambda v_0 + i\tau v_0)} \right) \leq c_2(z) \kappa(z + [0,1]v_0)c(\tau v_0).
\]

For any \( z \in T_{C_1} \) we have therefore

\[
h_C(z) = -\sum_{j=1}^{m_0} \text{Res}_{\lambda = \lambda_j(z)}(\lambda \rightarrow f(z + \lambda v_0)) + \frac{1}{2\pi i} \int_{\text{Re}(\lambda) = 1} f(z + \lambda v_0) d\lambda.
\]

Since \( \lambda_j(z) \neq \lambda_{j'}(z) \) for \( j, j' \in \{1, ..., m_0\}, j \neq j', z \in T_{C_1} \), we have

\[
\text{Res}_{\lambda = \lambda_j(z)}(\lambda \rightarrow f(z + \lambda v_0)) = \lim_{\lambda \to \lambda_j(z)} \frac{l_j(z + \lambda v_0) + 1}{l_j(z + \lambda v_0)} g(z + \lambda v_0) = \frac{1}{l_j(v_0)} g \left( z - \frac{l_j(z)}{l_j(v_0)} v_0 \right) \prod_{1 \leq k \leq m, k \neq j} \frac{l_j,k(z) + 1}{l_j,k(z)}.
\]

Put now for \( j \in \{1, ..., m_0\}, z \in T_{B_1} \)

\[
f_j(z) := -\frac{1}{l_j(v_0)} g \left( z - \frac{l_j(z)}{l_j(v_0)} v_0 \right) \prod_{1 \leq k \leq m, k \neq j} \frac{l_j,k(z) + 1}{l_j,k(z)}
\]

and

\[
g_j(z) := -\frac{1}{l_j(v_0)} \cdot g \left( z - \frac{l_j(z)}{l_j(v_0)} v_0 \right).
\]

Let \( V_1 \subset V \) be a hyperplane which does not contain \( v_0 \). We want to show that the function \( f_j(z) \) is distinguished with respect to \( (V_1; (l_{j,k})_{1 \leq k \leq m, k \neq j}) \). The function \( f_j \) is meromorphic on \( T_{B_1} \). Also, for all \( x \in B_1 \) and all \( k \in \{1, ..., \bar{j}, ..., m\} \) we have \( l_j,k(x) > -1 \). Further, we have that

\[
g_j(z) = f_j(z) \cdot \prod_{1 \leq k \leq m, k \neq j} \frac{l_j,k(z)}{l_j,k(z) + 1}
\]
is a holomorphic function on $T_{B_1}$.

Let $K_1 \subset T_{B_1} \subset T_B$ be a compact, and let

$$K(j) := \{ z - \frac{l_j(z)}{l_j(v_0)} v_0 | z \in K_1 \}.$$  

This is a compact subset in $T_B$. Put

$$\kappa_j(K_1) = \frac{1}{|l_j(v_0)|} \kappa(K(j)),$$

where $\kappa(K(j))$ is a constant such that $|g(z + iv)| \leq \kappa(K(j))c(v)$ for all $z \in K(j)$ and all $v \in V$. For $v_1 \in V_1$ we put

$$c_j(v_1) = c \left( v_1 - \frac{l_j(v_1)}{l_j(v_0)} v_0 \right).$$

Then for all $z \in K_1$ and $v \in V_1$ we have

$$|g_j(z + iv_1)| \leq \kappa_j(K_1)c_j(v_1).$$

Moreover, for any subspace $U_1 \subset V_1$ and all $v_1 \in V_1$ the function $U_1 \to \mathbb{R}$, $u_1 \to c_j(u_1 + v_1)$ is measurable and we have

$$c_j(U_1)(v_1) := \int_{U_1} c_j(v_1 + u_1) du_1 < \infty.$$  

For all $v_1 \in V_1 - U_1$ we have

$$\lim_{\tau \to \pm \infty} c_j(U_1)(\tau v_1) = 0.$$  

This shows that $f_j$ is distinguished with respect to $(V_1; (l_{j,k})_{1 \leq k \leq m, k \neq j})$.

For $z \in T_{B_1}$ we put

$$f_0(z) := \frac{1}{2\pi i} \int_{\Re(\lambda) = 1} f(z + \lambda v_0) d\lambda.$$  

If $l_k(v_0) \neq 0$ we have (as above) for all $x \in B_1$ the following inequality

$$|l_k(x + v_0)| \geq \frac{1}{2} |l_k(v_0)|.$$  

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Therefore, we conclude that the function
\[
g_0(z) := f_0(z) \prod_{1 \leq k \leq m, l_k(v_0) \neq 0} \frac{l_k(z)}{l_k(z) + 1}
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \prod_{1 \leq k \leq m, l_k(v_0) \neq 0} \frac{l_k(z + v_0 + i\tau v_0) + 1}{l_k(z + v_0 + i\tau v_0)} g(z + v_0 + i\tau v_0) d\tau
\]
is holomorphic in $T_{B_1}$.

Further, we have for $z \in K_1$ (with $K_1 \subset T_{B_1}$ a compact) and $v_1 \in V_1$ the inequality
\[
|g_0(z + iv_1)| \leq \kappa_0(K_1)c_0(v_1),
\]
where $\kappa_0(K_1)$ is some suitable constant and
\[
c_0(v_1) := \int_{-\infty}^{+\infty} c(v_1 + \tau v_0) d\tau.
\]

Again, for any subspace $U_1 \subset V_1$ and any $u_1 \in V_1$ we have that the map $U_1 \rightarrow \mathbb{R}$ given by $u_1 \rightarrow c_0(v_1 + u_1)$ is measurable, and that
\[
c_{0,U_1}(v_1) = \int_{U_1} c_0(v_1 + u_1) du_1 < \infty.
\]

For all $v_1 \in V_1 - U_1$ we have
\[
\lim_{\tau \rightarrow \pm \infty} c_{0,U_1}(\tau v_1) = 0.
\]

Therefore, $f_0$ is distinguished with respect to $(V_1; (l_k)_{1 \leq k \leq m, l_k(v_0) \neq 0})$.

The Cauchy-Riemann equations imply that $g_0$ is invariant under $Cv_0$, that is, for all $z_1, z_2 \in T_{B_1}$ with $z_1 - z_2 \in Cv_0$ we have $g_0(z_1) = g_0(z_2)$. We see that $f_0$ is also invariant under $Cv_0$ (in this sense), as well as $f_1, ..., f_{m_0}$ (this can be seen from the explicit representation of these functions). For $z \in T_{C_1}$ we have
\[
h_C(z) = f_0(z) + \sum_{j=1}^{m_0} f_j(z) = \sum_{j=0}^{m_0} f_j(z).
\]

Moreover, for such $z$ we have
\[
\tilde{f}_C(z) = \frac{1}{(2\pi)^d-1} \int_{V_1} h_C(z + iv_1) dv_1 = \sum_{j=0}^{m_0} \frac{1}{(2\pi)^d-1} \int_{V_1} f_j(z + iv_1) dv_1,
\]

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where \( dv_1 dv = dv \) (and \( \text{Vol}_{dv}(\{ \lambda v_0 \mid \lambda \in [0,1] \}) = 1 \)).

By our induction hypothesis, there exists an open and convex neighborhood \( B' \) of \( 0 \) in \( E \) and linear forms \( \tilde{l}_1, \ldots, \tilde{l}_{\tilde{m}} \), which vanish on \( V \), such that

\[
\tilde{f}_C(z) \prod_{j=1}^{\tilde{m}} \tilde{l}_j(z)
\]

has a holomorphic continuation to \( T_{B'} \). (Strictly speaking, from the induction hypothesis it follows only that the linear forms \( \tilde{l}_1, \ldots, \tilde{l}_{\tilde{m}} \in E^\vee \) vanish on \( V \). But since the functions \( f_0, \ldots, f_{m_0} \) “live” already on a tube domain in \( (E/Rv_0)_C \), it follows that the linear forms are also \( Rv_0 \)-invariant.)

Now we notice that \( \tilde{f}_C(z) \prod_{j=1}^{\tilde{m}} \tilde{l}_j(z) \) is holomorphic on \( T_C \). Let \( \tilde{B} \) be the convex hull of \( B' \cup C \). Then we have that

\[
\tilde{f}_C(z) \prod_{j=1}^{\tilde{m}} \tilde{l}_j(z)
\]

is holomorphic on \( T_{\tilde{B}} \) (cf. [12], Theorem 2.5.10).

\[\square\]

7.4 Let \( E, V, l_1, \ldots, l_m \) and \( B \subset \{ x \in E \mid l_j(x) > -1 \ \forall \ j = 1, \ldots, m \} \) be as above. Let \( C \) be a connected component of \( B - \bigcup_{j=1}^{m} H_j \). Let \( f \in \mathcal{M}(T_B) \) be an distinguished function with respect to \( (V; l_1, \ldots, l_m) \).

**Proposition 7.4** There exist an open convex neighborhood \( \tilde{B} \) of \( 0 \) in \( E \) containing \( C \) and linear forms \( \tilde{l}_1, \ldots, \tilde{l}_{\tilde{m}} \in E^\vee \) vanishing on \( V \) such that

- a) for all \( j \in \{ 1, \ldots, \tilde{m} \} \) we have \( \text{Ker}(\tilde{l}_j) \cap C = \emptyset \)
- b) \( \tilde{f}_C(z) \prod_{j=1}^{\tilde{m}} \tilde{l}_j(z) \) has a holomorphic continuation to \( T_{\tilde{B}} \).

**Proof.** By the proposition above, there exist linear forms \( \tilde{l}_1, \ldots, \tilde{l}_{\tilde{m}} \) such that \( V \subset \cap_{j=1}^{\tilde{m}} \text{Ker}(\tilde{l}_j) \) and \( \tilde{f}_C(z) \prod_{j=1}^{\tilde{m}} \tilde{l}_j(z) \) has a holomorphic continuation to a tube domain \( T_{\tilde{B}} \) over a convex open neighborhood \( \tilde{B} \) of \( 0 \in E \) containing \( C \). Suppose that there exist an \( x_0 \in C \) and a \( j_0 \in \{ 1, \ldots, \tilde{m} \} \) such that \( \tilde{l}_{j_0}(x_0) = 0 \). Then the function

\[
\tilde{f}_C(z) \prod_{j \neq j_0} \tilde{l}_j(z)
\]

is still holomorphic in \( T_{\tilde{B}} \) with \( \tilde{B}' = (B - \text{Ker}(\tilde{l}_{j_0})) \cup C \). It is easy to see that \( \tilde{B}' \) is connected. The convex hull of \( \tilde{B}' \) is equal to \( \tilde{B} \). Therefore, already the function

\[
\tilde{f}_C(z) \prod_{j \neq j_0} \tilde{l}_j(z)
\]

is holomorphic in \( T_{\tilde{B}} \).
is holomorphic on $T_B$ (cf. [12], loc. cit.).

7.5 As above, let $E$ be a finite dimensional vector space over $\mathbb{R}$ and let $l_1, ..., l_m$ be linearly independent linear forms on $E$. Put $H_j := \text{Ker}(l_j)$, for $j = 1, ..., m$, $E^{(0)} = \bigcap_{j=1}^m H_j$ and $E_0 = E/E^{(0)}$. Let $\pi_0 : E \to E_0$ be the canonical projection. Let $V \subset E$ be a subspace with $V \cap E^{(0)} = \{0\}$, such that $\pi_0|_V : V \to E_0$ is an injective map. Let

$$E_0^+ := \{ x \in E_0 | l_j(x) \geq 0, \ j = 1, ..., m \}$$

and let $\psi : E_0 \to P := E_0/\pi_0(V)$ be the canonical projection. We want to assume that $\pi_0(V) \cap E_0^+ = \{0\}$, so that $\Lambda := \psi(E_0^+)$ is a strictly convex polyhedral cone.

Let $dy$ be the Haar measure on $E_0^+$ normalized by $\text{Vol}(E_0^+/\oplus_{j=1}^m \mathbb{Z}l_j) = 1$. Let $A \subset V$ be a lattice, and let $dv$ be a measure on $V$ normalized by $\text{Vol}(V/A) = 1$. On $V$ we have a measure $dy'$ normalized by $A^\vee$ and a section of the projection $E_0^+ \to V^\vee$ gives a measure $dy''$ on $P^\vee$ with $dy = dy'dy''$.

Let $B \subset E$ be an open and convex neighborhood of $0$, such that for all $x \in B$ and $j \in \{1, ..., m\}$ we have $l_j(x) > -1$. Let $f \in \mathcal{M}(T_B)$ be a meromorphic function in the tube domain over $B$ which is distinguished with respect to $(V; l_1, ..., l_m)$. Put

$$B^+ = B \cap \{ x \in E | l_j(x) > 0, \ j = 1, ..., m \},$$

$$\tilde{f}_{B^+}(z) = \frac{1}{(2\pi)^d} \int_V f(z + iv)dv$$

where $d = \text{dim } V$.

By 7.3, the function $\tilde{f}_{B^+} : T_{B^+} \to \mathbb{C}$ is holomorphic and it has a meromorphic continuation to a neighborhood of $0 \in E_0$. Put

$$g(z) = f(z) \prod_{j=1}^m \frac{l_j(z)}{l_j(z) + 1}.$$  

**Proposition 7.5** For $x_0 \in B^+$ we have

$$\lim_{s \to 0} s^{m-d} \tilde{f}_{B^+}(sx_0) = g(0) \chi(\psi(x_0)).$$

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Proof. For \( j \in J := \{1, \ldots, m\} \) we define
\[
H_{j,+} := \{ v \in V \mid l_j(v) = 1 \}, \\
H_{j,-} := \{ v \in V \mid l_j(v) = -1 \}.
\]
Let \( \mathcal{C} \) be the set of connected components of \( V - \bigcup_{j=1}^{m} (H_{j,+} \cup H_{j,-}) \). For a \( C \in \mathcal{C} \) we put
\[
J_C := \{ j \in J \mid \|l_j(v)\| < 1 \text{ for all } v \in C \}
\]
and
\[
V^C := V \cap \bigcap_{j \in J_C} H_j.
\]
Denote by \( V_C \) the complement to \( V^C \) in \( V \) and let \( \pi_C : V = V_C \oplus V^C \rightarrow V_C \) be the projection. Since the map \( V_C \rightarrow \mathbb{R}^{J_C}, v \rightarrow (l_j(v))_{j \in J_C} \) is injective we see that \( \pi_C(C) \) is a bounded open subset of \( V_C \). For \( v_1 \in \pi_C(C) \) we put
\[
C(v_1) := \{ v' \in V^C \mid v_1 + v' \in C \}.
\]
The set \( C(v_1) \) is a convex open subset of \( V^C \). Let \( dv_1, dv' \) be measures on \( V_C \) (resp. on \( V^C \)), with \( dv_1dv' = dv \). For all \( s \in (0, 1] \) we have
\[
sm^{-d} \int_C f(sx_0 + iv)v dv = sm^{-d} \int_{\pi_C(C)} \int_{C(v_1)} f(sx_0 + iv_1 + iv')dv'dv_1
\]
\[
= sm^{-dC} \int_{\frac{1}{2}\pi_C(C)} \int_{C(sv_1)} f(sx_0 + isv_1 + iv')dv'dv_1
\]
\[
= \int_V \chi_{C_s}(v)sm^{-dC} f_{C,s}(v)dv.
\]
Here we denoted by \( d^C := \dim V^C \) and by
\[
C_s := \{ v = v_1 + v' \in V_C \oplus V^C \mid sv_1 + v' \in C \}
\]
and by \( \chi_{C_s} \) the set-theoretic characteristic function of \( C_s \). We have put for any \( s \in (0, 1] \) and any \( v = v_1 + v' \in V \)
\[
f_{C,s}(v) := f(sx_0 + isv_1 + iv').
\]
The set
\[
K_C := \{ sx_0 + iv_1 \mid s \in [0, 1], v_1 \in \pi_C(C) \}
\]
is contained in $T_B$ and is compact. Further, there exist $c', c'' \geq 0$ such that for all $s \in [0, 1]$ and all $v = v_1 + v' \in C$ we have
\[
\left| \prod_{j \in J_C} (l_j(sx_0 + iv) + 1) \prod_{j \notin J_C} \frac{l_j(sx_0 + iv) + 1}{l_j(sx_0 + iv)} \right| \leq c',
\]
\[
1 \leq c'' \cdot \prod_{j \notin J_C} \frac{1}{l_j(x_0 + iv_1)}.
\]

Therefore, for $s \in (0, 1]$ and $v \in V$ we have
\[
|\chi_{C_s}(v)s^{m-d_C}f_{C,s}(v)| = |\chi_{C_s}(v)g(sx_0 + isv_1 + iv')s^{m-d_C} \prod_{j \in J_C} (1 + \frac{1}{l_j(sx_0 + isv_1 + iv')})|
\]
\[
\leq c'\|K_C\|c(v')
\]
\[
\leq c'\|K_C\| \prod_{j \notin J_C} \frac{1}{l_j(x_0 + iv_1)} c(v'),
\]
\[
\leq c'\|K_C\| \prod_{j \notin J_C} \frac{1}{l_j(x_0 + iv_1)} c(v'),
\]

since $m - d_C - \#J_C \geq 0$. (The constant $\|K_C\|$ and the function $c : V \to \mathbb{R}_{\geq 0}$ were introduced above.)

The $\mathcal{X}$-function corresponding to the cone $E_0^+ \subset E_0$ (and the measure $dy$) is given by
\[
\mathcal{X}_{E_0^+}^s(x_0 + iv_1) = \prod_{j=1}^m \frac{1}{l_j(x_0 + iv_1)}.
\]
Since the map from $V_C$ to $E_0$ is injective and since $\pi_0(V_C) \cap E_0^+ = \{0\}$ we know that the function $v_1 \mapsto \mathcal{X}_{E_0^+}^s(x_0 + iv_1)$ is absolutely integrable over $V_C$ (by 7.2). Therefore,
\[
v = v_1 + v' \mapsto c'\|K_C\| \prod_{j \notin J_C} \frac{1}{l_j(x_0 + iv_1)} c(v'),
\]
\[
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\]
is integrable over $V$.

For a fixed $v \in V$ we consider the limit

$$\lim_{s \to 0} \chi_{C,s}(v) s^{m-d_C} f_{C,s}(v).$$

The estimate above shows that this limit is 0 if $m - d_C - \#J_C > 0$. Therefore, we assume that $m = d_C + \#J_C$. Then the map $V^C \to \mathbb{R}^{J-J_C}$ is an isomorphism. Since $\pi_0(V) \cap E_0^+ = \{0\}$, it follows that $J_C = J$. There exists exactly one $C \in \mathcal{C}$ with $J_C = J$ and we denote it by $C_0$. This $C_0$ contains 0 and for all sufficiently small $s > 0$ we have $s \cdot v \in C_0$, and therefore, $v \in C_0$.

Moreover, we have

$$\lim_{s \to 0} s^m f_{C,s}(v) = \lim_{s \to 0} s^m g(sx_0 + isv) \prod_{j=1}^m \frac{l_j(sx_0 + isv) + 1}{l_j(sx_0 + isv)} = g(0) \prod_{j=1}^m \frac{1}{l_j(x_0 + iv)}.$$ 

Using the theorem of dominated convergence (Lebesgue’s theorem), we obtain

$$\lim_{s \to 0} s^{m-d} \tilde{f}_{B+}(sx_0) = \lim_{s \to 0} \sum_{C \in \mathcal{C}} \frac{1}{(2\pi i)^d} \int_V \chi_{C,s}(v) s^{m-d} f_{C,s}(v) dv = \frac{1}{(2\pi i)^d} g(0) \int_V \prod_{j=1}^m \frac{1}{l_j(x_0 + iv)} dv = g(0) \chi_\Lambda(\psi(x_0)).$$

\[\Box\]

8 Some statements on Eisenstein series

8.1 Let $G$ be a semi-simple simply connected algebraic group which is defined and split over $F$. Fix a Borel subgroup $P_0$ (defined over $F$) and a Levi decomposition $P_0 = S_0U_0$, where $S_0$ is a maximal $F$-rational torus of $G$. Denote by $g$ (resp. $a_0$) the Lie algebra of $G$ (resp. $S_0$).
We are going to define a certain maximal compact subgroup $K_G \subset G(A)$. This maximal compact subgroup will have the advantage that the constant term of Eisenstein series, more precisely, certain intertwining operators, can be calculated explicitly, uniformly with respect to all places of $F$. In general, i.e., for an arbitrary maximal compact subgroup, there will be some places where such an explicit expression is not available. In principle, this should cause no problems. Any statement in this section should be valid for an arbitrary maximal compact subgroup.

8.2 Let $\Phi = \Phi(G, S_0)$ be the root system of $G$ with respect to $S_0$. We denote by $\Delta_0$ the basis of simple roots determined by $P_0$. For $\alpha \in \Phi$ let $$g_\alpha := \{X \in g \mid [H, X] = \alpha(H)X\}$$ be the corresponding root space.

Let $((H_\alpha)_{\alpha \in \Delta_0}, (X_\alpha)_{\alpha \in \Phi})$ be the Chevalley basis of $g$. In particular, this means that $$g_\alpha = FX_\alpha \ (\alpha \in \Phi), \quad [X_\alpha, X_{-\alpha}] = H_\alpha \ (\alpha \in \Delta_0),$$ $$a_0 = \oplus_{\alpha \in \Delta_0} FH_\alpha.$$ Put $$g_Q = \sum_{\alpha \in \Delta_0} QH_\alpha + \sum_{\alpha \in \Phi} QX_\alpha \subset g,$$ This is a $\mathbb{Q}$-structure for $g$ and for any $v \in \text{Val}(F)$ the Lie algebra of $G(F_v)$ is $g \otimes \mathbb{Q} F_v$. We put $$k := \oplus_{\alpha \in \Phi^+} R(X_\alpha - X_{-\alpha}),$$ $$p := \bigoplus_{\alpha \in \Delta_0} RH_\alpha \oplus \bigoplus_{\alpha \in \Phi^+} R(X_\alpha + X_{-\alpha}),$$ where $\Phi^+$ is the set of positive roots of $\Phi$ determined by $\Delta_0$. Then $k \oplus p$ is a Cartan decomposition of $gQ \otimes_R \mathbb{Q} C$, $g_c := k \oplus ip \subset gQ \otimes_R \mathbb{Q} C$ is a compact form of $gQ \otimes_R \mathbb{Q} C$ and $gQ \otimes_R \mathbb{Q} C = g_c \oplus ig_c$ is a Cartan decomposition of $gQ \otimes_R \mathbb{Q} C$.

For any complex place $v$ of $F$ we define $K_v$ to be $\langle \exp(g_c) \rangle \subset G(F_v)$ (identifying $F_v$ with $C$ via a corresponding embedding $F \hookrightarrow C$). If $v$ is a real place of $F$ we define $K_v = G(F_v) \cap \langle \exp(g_c) \rangle$ (identifying $F_v(\sqrt{-1}) \simeq C$ via a corresponding embedding $F \hookrightarrow \mathbb{R}$). In this case, $K_v$ contains $\langle \exp(k) \rangle$. In both cases $K_v$ is a maximal compact subgroup of $G(F_v)$.
Now let $v$ be a finite place of $F$ and let $K_v$ be the stabilizer of the lattice
\[ \sum_{\alpha \in \Delta_0} O_v \cdot H_\alpha + \sum_{\alpha \in \Phi} O_v \cdot X_\alpha \subset g \otimes_F F_v. \]

By [7, sec. 3, Example 2], $K_v$ is a maximal compact subgroup of $G(F_v)$. In any case, the Iwasawa decomposition $G(F_v) = P_0(F_v)K_v$ holds (for non-archimedean $v$, cf. [7], loc. cit.). Then $K_G = \prod_v K_v$ is a maximal compact subgroup of $G(A)$ and $G(A) = P_0(A)K_G$.

8.3 As in section 2.3 we defined for any standard parabolic subgroup $P \subset G$
\[ H_P = H_{P,K_G} : G(A) \to \text{Hom}_C(X^*(P)_C, C) \]
by $\langle \lambda, H_P(g) \rangle = \log(\prod_v |\lambda(p_v)|_v)$ for $\lambda \in X^*(P)$ and $g = pk$, $p = (p_v)_v \in P(A), k \in K_G$.

The restriction of $H_{P_0}$ to $S_0(A)$ is a homomorphism, its kernel will be denoted by $S_0(A)^1$. The choice of a projection $G_{m}(A) \to G_{m}(A)^1$ induces by means of an isomorphism $S_0(A) \to S_0(A)^1$ and this in turn gives an embedding
\[ U_0 := (S_0(A)^1/S_0(F)(S_0(A) \cap K_G))^* \hookrightarrow (S_0(A)/S_0(F)(S_0(A) \cap K_G))^*. \]

Let $(\varpi_\alpha)_{\alpha \in \Delta_0}$ be the basis of $X^*(S_0)$ which is determined by $\langle \varpi_\alpha, \beta \rangle = \delta_{\alpha\beta}$ for all $\alpha, \beta \in \Delta_0$. Let $P \subset G$ be a standard parabolic subgroup. Then $\varpi_\alpha$ for all $\alpha \in \Delta_P$ lifts to a character of $P$ and $(\varpi_\alpha)_{\alpha \in \Delta_P}$ is a basis of $X^*(P)$. Put
\[ U_P := \{ \chi \mid \chi = \chi_0 \circ \prod_{\alpha \in \Delta_P} \hat{\alpha} \circ \varpi_\alpha \text{ with } \chi_0 \in U_0 \}. \]

Any $\chi \in U_P$ is a character of $P(A)/P(F)(P(A) \cap K_G)$. Define
\[ \phi_\chi : G(A) \to S^1 \]
by $\phi_\chi(pk) = \chi(p)$ for $p \in P(A), k \in K_G$. The Eisenstein series
\[ E_P^G(\lambda, \chi, g) = \sum_{g \in P(F) \setminus G(F)} \phi_\chi(\gamma g)e^{\langle \lambda + \rho_P, H_P(\gamma g) \rangle} \]
converges absolutely and uniformly for $\text{Re}(\lambda)$ contained in any compact subset of the open cone $\rho_P + X^*(P)^+$ (cf. [10], Théorème III) and can be continued meromorphically to the whole of $X^*(P)_C$. 

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For the Eisenstein series corresponding to $P_0$ a proof is given in [16], chapitre IV. In section 8.4 we will give an explicit expression for the Eisenstein series $E^G_{P_0}$, with $P \neq P_0$ as an iterated residue of $E^G_{P_0}$ which shows the claimed meromorphy on $X^\ast(P)_{\text{C}}$.

Let $\chi \in U_0$. The constant term of $E^G_{P_0}(\lambda, \chi)$ along $P = LU$ is by definition

$$E^G_{P_0}(\lambda, \chi)_P(g) = \int_{U(F)\backslash U(A)} E^G_{P_0}(\lambda, \chi, u g) du,$$

where the Haar measure on $U(A)$ is normalized such that $U(F)\backslash U(A)$ gets volume one. It is an elementary calculation to show that for any parabolic subgroup $P \noteq P_0$ the constant term $E^G_{P_0}(\lambda, \chi)_P$ is orthogonal to all cusp forms in $A_0(L(F)U(A)\backslash G(A))$ (cf. [16], I.2.18, for the definition of this space). More precisely, for any parabolic subgroup $P \noteq P_0$ the cuspidal component of $E^G_{P_0}(\lambda, \chi)$ along $P$ vanishes (cf. [16], I.3.5, for the definition of “cuspidal component”).

By Lemme I.4.10 in [16], the singularities of the Eisenstein series $E^G_{P_0}(\lambda, \chi)$ and the singularities of $E^G_{P_0}(\lambda, \chi)_{P_0}$ coincide. Let

$$W = \text{Norm}_{G(F)}(S_0(F)) / S_0(F)$$

be the Weyl group of $G$ with respect to $S_0$. For any $w \in W$ we normalize the Haar measures such that

$$\int_{(U_0(F)\cap wU_0(F)w^{-1})\backslash(U_0(A)\cap wU_0(A)w^{-1})} du = 1$$

and on $(U_0(A) \cap wU_0(A)w^{-1})\backslash U_0(A)$ we take the quotient measure. Using Bruhat’s decomposition $G(F) = \bigcup_{w \in W} P_0(F)w^{-1}P_0(F)$ we can calculate

$$\int_{U_0(F)\backslash U_0(A)} E^G_{P_0}(\lambda, \chi, u g) du = \sum_{w \in W} c(w, \lambda, \chi) \phi_{w\chi}(g) e^{(w\lambda + \rho_0, H_{P_0}(g))},$$

where $\rho_0 = \rho_{P_0}$, $(w\chi)(t) = \chi(w^{-1}tu)$ for all $t \in S_0(A)$, and the functions $c(w, \lambda, \chi)$ are given by

$$c(w, \lambda, \chi) := \int_{(U_0(A)\cap wU_0(A)w^{-1})\backslash U_0(A)} \phi_{\chi}(w^{-1}) e^{(\lambda + \rho_0, H_{P_0}(w^{-1}u))} du$$

(cf. [16], Prop. II.1.7). They satisfy functional equations:

$$E^G_{P_0}(\lambda, \chi, g) = c(w, \lambda, \chi) E^G_{P_0}(w\lambda, w\chi, g)$$
\[ c(w', w, \lambda, \chi) = c(w, w\lambda, w\chi)c(w, \lambda, \chi) \]

(cf. [16], Théorème IV.1.10). Therefore, it suffices to calculate \( c(w, \lambda, \chi) \) for \( \alpha \in \Delta_0 \) (\( w_a \) corresponds to the reflection along \( \alpha \)).

Put \( S_\alpha = \text{Ker}(\alpha)^0 \subset S_0 \) and \( G_\alpha = Z_G(S_\alpha) \). The Lie algebra of \( G_\alpha \) is \( a_0 \oplus g_{-\alpha} \oplus g_\alpha \). There is a homomorphism \( \varphi_\alpha : SL_2,F \rightarrow DG_\alpha (= \text{derived group of } G_\alpha) \) such that \( d\varphi_\alpha \) maps the matrices
\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
to \( X_\alpha, H_\alpha, X_{-\alpha} \), respectively.

On \( A \) we take the measure \( dx \) that is described in Tate’s thesis (then \( \text{Vol}(F \setminus A) = 1 \)). We have
\[
c(w, \lambda, \chi) = \int_A \phi_\chi(\varphi_a(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix})) \exp \left( \langle \lambda + \rho_0, H_{\rho_0}(\varphi_a(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix})) \rangle \right) dx.
\]

It is an exercise to compute this integral. The result is
\[
c(w, \lambda, \chi) = \frac{L(\chi \circ \alpha, \langle \lambda, \alpha \rangle)}{L(\chi \circ \alpha, 1 + \langle \lambda, \alpha \rangle)}.
\]
The Hecke \( L \)-functions are defined as follows. Let \( \chi : G_m(A)/G_m(F) \rightarrow S^1 \) be an unramified character. For any finite place \( v \) we put
\[
L_v(\chi_v, s) = (1 - \chi_v(\pi_v)|\pi_v|^s)^{-1}
\]
and
\[
L_f(\chi, s) = \prod_{v \mid \infty} L_v(\chi_v, s).
\]

For any archimedean place \( v \) there is a \( \tau_v \in \mathbb{R} \) such that \( \chi_v(x_v) = |x_v|^i\tau_v \) for all \( x_v \in F_v^* \). Then
\[
L_v(\chi_v, s) := \begin{cases} 
\pi^{-\frac{(s+i\tau_v)}{2}} \Gamma(\frac{s+i\tau_v}{2}), & \text{if } v \text{ is real} \\
(2\pi)^{-(s+i\tau_v)} \Gamma(s + i\tau_v), & \text{if } v \text{ is complex}
\end{cases}
\]
We define the complete Hecke \( L \)-function by
\[
L(\chi, s) = D^{s/2}L_\infty(\chi, s)L_f(c, s),
\]
where \( L_\infty(\chi, s) = \prod_{v|\infty} L_v(\chi_v, s) \) and \( D = D(F/\mathbb{Q}) \) is the absolute value of the discriminant of \( F/\mathbb{Q} \). If the restriction of \( \chi \) to \( \mathbb{G}_m(A)^1 \) is not trivial then \( L(\chi, s) \) is an entire function. If \( \chi = 1 \) then \( L(\chi, s) \) has exactly two poles of order one at \( s = 1 \) and \( s = 0 \). To state the functional equation we let \((\pi_v^{d_v})\) (with \( d_v \geq 0 \) and \( d_v = 0 \) for almost all \( v \)) be the local discriminant of \( F_v \) over the completion of \( \mathbb{Q} \in F_v \) (for non-archimedean places \( v \)). Put \( \delta = (\delta_v) \in \mathbb{G}_m(A) \) with \( \delta_v = 1 \) for all archimedean places and \( \delta_v = \pi_v^{d_v} \) for all non-archimedean places. Then

\[
L(\chi, s) = \chi(\delta)L(\chi^{-1}, 1 - s).
\]

Using the functional equations (23) and (25) we get

\[
c(w, \lambda, \chi) = \prod_{\alpha > 0, \; w, \lambda < 0} \frac{L(\chi \circ \check{\alpha}, \langle \lambda, \alpha \rangle)}{L(\chi \circ \check{\alpha}, 1 + \langle \lambda, \alpha \rangle)}.
\]

**Proposition 8.3** Let \( \Delta_0(\chi) \) be the set of \( \alpha \in \Delta_0 \) such that \( \chi \circ \check{\alpha} = 1 \). Then

\[
\prod_{\alpha \in \Delta_0(\chi)} \langle \lambda, \alpha \rangle E_G^{G_0}(\lambda + \rho_0, \chi)
\]

has a holomorphic continuation to the tube domain over \(-\frac{1}{2}\rho_0 + X^*(P_0)^+\).

*Proof.* For \( \alpha \in \Phi^+ - \Delta_0 \) we have \( \langle \rho_0, \alpha \rangle \geq 2 \) and therefore

\[
c(w_\alpha, \lambda + \rho_0, \chi) = \frac{L(\chi \circ \check{\alpha}, \langle \lambda + \rho_0, \alpha \rangle)}{L(\chi \circ \check{\alpha}, 1 + \langle \lambda + \rho_0, \alpha \rangle)}
\]

is holomorphic in this domain. If \( \alpha \in \Delta_0 - \Delta_0(\chi) \) then \( \chi \circ \check{\alpha} \) restricted to \( \mathbb{G}_m(A)^1 \) is nontrivial, hence \( c(w_\alpha, \lambda + \rho_0, \chi) \) is holomorphic in this domain too. For \( \alpha \in \Delta_0(\chi) \) the function

\[
\langle \lambda, \alpha \rangle L(\chi \circ \check{\alpha}, \langle \lambda + \rho_0, \alpha \rangle) = \langle \lambda, \alpha \rangle L(1, 1 + \langle \lambda, \alpha \rangle)
\]

is also holomorphic in this domain. This shows the holomorphy of

\[
\prod_{\alpha \in \Delta_0(\chi)} \langle \lambda, \alpha \rangle E_G^{G_0}(\lambda + \rho_0, \chi) \phi_{\rho_0}(g)
\]

\[
= \sum_{w \in W} \prod_{\alpha \in \Delta_0(\chi)} \langle \lambda, \alpha \rangle c(w, \lambda + \rho_0, \chi) e^{(w(\lambda + \rho_0) + \rho_0, H_{\rho_0}(g))}
\]

for \( \text{Re}(\lambda) \) contained in \(-\frac{1}{2}\rho_0 + X^*(P_0)^+\). By [16], Lemme I.4.10, we conclude that the same holds for \( \prod_{\alpha \in \Delta_0(\chi)} E_G^{G_0}(\lambda + \rho_0, \chi) \).

\[56\]
8.4 Let $P = LU$ be a standard parabolic subgroup and $\chi \in U_P$. For $\lambda \in X^*(P)_C$ with $\text{Re}(\lambda)$ contained in the interior of $X^*(P)^+$ and $\vartheta$ contained in $X^*(P_0)^+$ we have

$$E^G_{P_0}(\vartheta + \lambda + \rho_0, \chi, g) = \sum_{\gamma \in P(F) \setminus G(F)} \phi_{\chi}(\gamma g) e^{\lambda, H_P(\gamma g)} \sum_{\delta \in (L \cap P_0)(F) \setminus L(F)} e^{(\vartheta + 2\rho_0, H_{P_0}(\delta \gamma g))}.$$  

Let $w_L$ be the longest element of the Weyl group of $L$ (with respect to $S_0$) and define

$$c_P := \lim_{\vartheta \to 0, \vartheta \in X^*(P_0)^+} \left( \prod_{\alpha \in \Delta^P_0} \langle \vartheta, \alpha \rangle \right) c(w_L, \vartheta + \rho_0, 1).$$

By (27) this limit exists and is a positive real number.

**Proposition 8.4**  

a)  

$$\lim_{\vartheta \to 0, \vartheta \in X^*(P_0)^+} \prod_{\alpha \in \Delta^P_0} \langle \vartheta, \alpha \rangle E^G_{P_0}(\vartheta + \lambda + \rho_0, \chi, g) = c_P E^G_{P}(\lambda + \rho_P, \chi, g).$$

b) Let $P' = L'U'$ be a standard parabolic subgroup containing $P$ and suppose that $\chi \circ \check{\alpha} = 1$ for all $\alpha \in \Delta^P_0$. Then $\chi \in U_{P'}$ and for all $\lambda \in X^*(P')_C$ we have

$$\lim_{\vartheta \to 0, \vartheta \in X^*(P)^+} \prod_{\alpha \in \Delta^P_0 - \Delta^{P'}} \langle \vartheta, \alpha \rangle E^G_{P}(\vartheta + \lambda + \rho_P, \chi, g) = c_{P'} E^G_{P'}(\lambda + \rho_{P'}, \chi, g).$$

**Proof.** a) The proof rests on the fact that a measurable function of moderate growth on $L(F) \setminus L(A)$ for which all cuspidal components vanish vanishes almost everywhere. (For a proof cf. [16], Prop. I.3.4.) We claim that

$$\lim_{\vartheta \to 0, \vartheta \in X^*(P_0)^+} \prod_{\alpha \in \Delta^P_0} \langle \vartheta, \alpha \rangle \left\{ \sum_{\delta \in (L \cap P_0)(F) \setminus L(F)} e^{(\vartheta + 2\rho_0, H_{P_0}(\delta \gamma g))} \right\} = c_P e^{(2\rho_P, H_P(\gamma g))}.$$  

In fact, the cuspidal components of both sides along all non-minimal standard parabolic subgroups of $L$ vanish. To compare the constant terms along $P_0 \cap L$ we can use (22) (for $L$ instead of $G$ and $P_0 \cap L$ instead of $P_0$) and the explicit expression of the functions $c(w, \vartheta + \rho_0, 1)$ to get the identity stated above (note that $w_L\rho_0 + \rho_0 = 2\rho_P$).
b) Write $\chi = \chi_0 \cdot \prod_{\alpha \in \Delta_P} \alpha \circ \varpi_\alpha$ with $\chi_0 \in U_{P'}$. For $\alpha \in \Delta_P - \Delta_{P'}$, we have $1 = \chi \circ \check{\alpha} = \chi_0 \circ \check{\alpha}$. Thus $\chi = \chi_0 \circ \prod_{\alpha \in \Delta_P - \Delta_{P'}} \alpha \circ \varpi_\alpha \in U_{P'}$. Using a) we get

$$\lim_{\vartheta \to 0, \vartheta \in X^*(P) +} \prod_{\alpha \in \Delta} \langle \vartheta, \alpha \rangle E_P^G(\vartheta + \lambda + \rho_P, \chi, g)$$

$$= \frac{1}{c_P} \lim_{\vartheta \to 0, \vartheta \in X^*(P)} \prod_{\alpha \in \Delta_0} \langle \vartheta, \alpha \rangle E_P^G(\vartheta + \lambda + \rho_0, \chi, g) = \frac{c_{P'}}{c_P} E_{P'}^G(\chi + \rho_{P'}, \chi, g).$$

\[ \square \]

8.5 For $\chi \in (S_0(A)/S_0(F) \cap K_G)^*$ and $v \in \text{Val}_\infty(F)$ there is a character $\lambda_v = \lambda_v(\chi) \in X^*(S_0)_R$ such that for all $x \in S_0(F_v)$

$$\chi(x) = e^{i \log(|(\lambda_v(x))|_v)}.$$ 

This gives a homomorphism

$$(S_0(A)/S_0(F) \cap K_G)^* \to X^*(S_0)_R^{\text{Val}_\infty(F)}$$

$$\lambda \mapsto \chi_\infty = (\lambda_v(\chi))_{v|\infty}$$

which has a finite kernel. The image of $U_0$ under this map is a lattice of rank

$$(\#\text{Val}_\infty(F) - 1) \dim S_0.$$ 

Fix a norm $\|\cdot\|$ on $X^*(S_0)_R$ and denote by the same symbol the induced maximum norm on $X^*(S_0)_R^{\text{Val}_\infty(F)}$ (i.e., $\|(\lambda_v)_{v|\infty}\| = \max_{v|\infty} \|\lambda_v\|$). Let $a, b > 0$ and put

$$B_{a,b} := \{ \lambda \in X^*(P_0)_R \mid -\lambda + \frac{a}{2} \rho_0 \in X^*(P_0)^+, \; \lambda + \frac{b}{2} \rho_0 \in X^*(P_0)^+ \}.$$ 

This is a bounded convex open neighborhood of $0$ in $X^*(P_0)_R$. Note that if $\lambda \in B_{a,b}$ then $w_0 \lambda + \frac{a}{2} \rho_0 \in X^*(P_0)^+$, where $w_0$ is the longest element of $W$.

Fix an $A > 0$ such that

$$\text{Re}(\langle \lambda + s\rho_0, \alpha \rangle((\lambda + s\rho_0, \alpha) - 1)) + A \geq 1$$

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for all \( \lambda \in X^*(P_0)_C \) with \( \text{Re}(\lambda) \in B_{a,b}, -1 - a \leq \text{Re}(s) \leq 1 + b \) and \( \alpha \in \Phi^+ \). Fix \( \lambda \in X^*(P_0)_C \) with \( \text{Re}(\lambda) \in B_{a,b} \) and \( \chi \in U_0 \). Denote by \( \Phi^+(\chi) \) the set of all positive roots \( \alpha \) such that \( \chi \circ \alpha = 1 \). Then

\[
f_{\lambda, \chi}(s, g) := \prod_{\alpha \in \Phi^+(\chi)} \frac{\langle \lambda + s\rho_0, \alpha \rangle (\langle \lambda + s\rho_0, \alpha \rangle - 1)}{\langle \lambda + s\rho_0, \alpha \rangle (\langle \lambda + s\rho_0, \alpha \rangle - 1) + A} \prod_{\alpha > 0} L_f(\chi \circ \alpha, 1 + \langle \lambda + s\rho_0, \alpha \rangle) E^G_{P_0}(\lambda + s\rho_0, \chi, g)
\]

is for any fixed \( s \) in this strip an automorphic form on \( G(F) \backslash G(A) \). Indeed, we observe that all cuspidal components of \( f_{\lambda, c}(s, \cdot) \) along non-minimal standard parabolic subgroups vanish. Now we can use (22) and the explicit formulas for the functions \( L_{\lambda, c} \) to the function

\[
f_{\lambda, c}(s, \cdot) := \prod_{\alpha \in \Phi^+(\chi)} \frac{\langle \lambda - (1 + a + it)\rho_0, \alpha \rangle (\langle \lambda - (1 + a + it)\rho_0, \alpha \rangle - 1)}{\langle \lambda - (1 + a + it)\rho_0, \alpha \rangle (\langle \lambda - (1 + a + it)\rho_0, \alpha \rangle - 1) + A} \prod_{\alpha > 0} L_f((w_0\chi) \circ \alpha, 1 + \langle w_0\lambda + (1 + a + it)\rho_0, \alpha \rangle)
\]

\[
\times \prod_{\alpha > 0} \frac{\langle (w_0\chi) \circ \alpha, 1 + \langle w_0\lambda + (1 + a + it)\rho_0, \alpha \rangle \rangle}{\langle (w_0\chi) \circ \alpha, 1 - \langle w_0\lambda + (1 + a + it)\rho_0, \alpha \rangle \rangle} E^G_{P_0}(w_0\lambda + (1 + a + it)\rho_0, w_0\chi, g).
\]

Note that \( L_{\infty}(\cdot, \cdot) \) is a product of \( \Gamma \)-functions. Using the functional equation of the \( \Gamma \)-function we can derive the following estimate: There is \( c > 0 \) depending only on \( a \) and \( b \) such that for \( \text{Re}(\lambda) \in B_{a,b} \) and \( \chi \in U_0 \) we have

\[
|f_{\lambda, \chi}(-1 - a - it, g)| \leq c E^G_{P_0}(\text{Re}(w_0\lambda) + (1 + a)\rho_0, g) \times (1 + \|\text{Im}(\lambda)\| + \|\chi_\infty\|)^{\delta_{\mu}} |1 + it|^{\delta_{\mu}}.
\]

where \( \delta_{\mu} := \mu(2 + a + b) \) and \( \mu > 0 \) depends only on \( F \) and \( G \). Moreover, assuming \( c \) to be big enough, we have also

\[
|f_{\lambda, \chi}(1 + b + it, g)| \leq c E^G_{P_0}(\text{Re}(\lambda) + (1 + b)\rho_0, g).
\]
The proof of the following lemma was suggested to us by J. Franke.

**Lemma 8.5** For \( \text{Re}(\lambda) \in B_{a,b}, \chi \in U_0 \) and \(-1 - a \leq \sigma \leq 1 + b\) the following estimate holds:

\[
|f_{\lambda,\chi}(\sigma + it, g)| \leq cE^G_{P_0}(\text{Re}(\lambda) + (1 + b)\rho_0, g)^{\frac{a + 3 + b}{2a + 2b}} \times \left\{ E^G_{P_0}(\text{Re}(w_0\lambda) + (1 + a)\rho_0, g)(1 + \|\text{Im}(\lambda)\| + \|\chi\|_\infty)^{\delta_\nu}|2 + a + \sigma + it|^{\delta_\nu}\right\}^{\frac{a + b}{2a + 2b}}.
\]

**Proof.** This follows immediately from Theorem 2 in [19] once we have shown that for \(-1 - a \leq \sigma \leq 1 + b\) we have

\[
|f_{\lambda,\chi}(\sigma + it, g)| \leq c_1e^{ct^2}
\]

for some \(c_1, c_2 > 0\).

By (28) and (29), the function

\[
s \mapsto e^{s^2}f_{\lambda,\chi}(s, g)
\]

can be integrated over the lines \(\text{Re}(s) = -1 - a - \epsilon\) and \(\text{Re}(s) = 1 + b + \epsilon\) for some \(\epsilon > 0\). We claim that for all \(s\) with \(-1 - a \leq \text{Re}(s) \leq 1 + b\)

\[
e^{s^2}f_{\lambda,\chi}(s, g)
\]

\[
= -\frac{1}{2\pi i} \int_{\text{Re}(z)=-1-a-\epsilon} e^{z^2} f_{\lambda,\chi}(z, g) \frac{dz}{z - s} + \frac{1}{2\pi i} \int_{\text{Re}(z)=1+b+\epsilon} e^{z^2} f_{\lambda,\chi}(z, g) \frac{dz}{z - s}.
\]

Denote the right-hand side by \(h(s, g)\). This is a measurable function of moderate growth on \(G(F) \setminus G(A)\) (cf. (28) and (29)). All cuspidal components of \(h(s, \cdot)\) along non-minimal standard parabolic subgroups vanish. The same is true for the left-hand side.

It remains to compare the constant terms of both sides along \(P_0\). By the absolute and uniform convergence of the integrals over the vertical lines we see that the constant term of \(h(s, \cdot)\) along \(P_0\) is

\[
-\frac{1}{2\pi i} \int_{\text{Re}(z)=-1-a-\epsilon} g_{\lambda,\chi}(z, g) \frac{dz}{z - s} + \frac{1}{2\pi i} \int_{\text{Re}(z)=1+b+\epsilon} g_{\lambda,\chi}(z, g) \frac{dz}{z - s}.
\]

where

\[
g_{\lambda,\chi}(z, g) := \int_{U_0(F) \setminus U(A)} e^{z^2} f_{\lambda,\chi}(z, ug) du.
\]
The explicit expression of the constant term of $E^G_{P_0}$ along $P_0$ in (22) and uniform estimates for $L$-functions as in [19], Theorem 5 (but for a larger strip), allow us to conclude that (22) is just the constant term of $e^{2\lambda\chi}(s, \cdot)$ along $P_0$. Thus, by Proposition I.3.4 in [10], we have established (31). From (31) it follows that $|e^{2\lambda\chi}(s, \cdot)|$ is bounded by some constant in the strip $-1 - a \leq \Re(s) \leq 1 + b$ and this in turn implies (30). \hfill \Box

Proposition 8.6 Let $a > 0$. For any $\epsilon > 0$ there exist constants $b, c > 0$ such that for all $\lambda \in X^*(P_0)_C$ with $\Re(\lambda) \in B_{a,b}$ and $\chi \in U_0$ we have

$$\left| \prod_{\alpha \in \Delta_0(\chi)} \frac{\langle \lambda, \alpha \rangle}{\langle \lambda, \alpha \rangle + 1} E^G_{P_0}(\lambda + \rho_0, \chi) \right| \leq c(1 + \|\Im(\lambda)\| + \|\chi_\infty\|^\epsilon).$$

Proof. Note that if $\alpha \in \Phi^+ - \Delta_0$ then $\langle \rho_0, \alpha \rangle \geq 2$ and hence $\langle \lambda + \rho_0, \alpha \rangle(\langle \lambda + \rho_0, \alpha \rangle - 1)$ does not vanish for $\Re(\lambda) \in B_{a,b}$ and $b > 0$ sufficiently small. For such $b$ there is a constant $c_1$ such that

$$\left| \prod_{\alpha \in \Delta_0(\chi)} \frac{\langle \lambda, \alpha \rangle}{\langle \lambda, \alpha \rangle + 1} E^G_{P_0}(\lambda + \rho_0, \chi) \right| \leq c \left| f_{\lambda,\chi}(1,1_G) \right|. $$

Now we use the estimate for $|f_{\lambda,\chi}(1,1_G)|$ in Lemma 8.5 and require that $\mu b \leq \epsilon$. This gives the desired result. \hfill \Box

Proposition 8.7 Let $P$ be a standard parabolic subgroup of $G$. Let $a, \epsilon > 0$. Then there exist $b, c > 0$ such that for all $\chi \in \mathcal{U}_P$ and $\lambda \in X^*(P)_C$ with $-\Re(\lambda) + \frac{a}{2}\rho_0 \in X^*(P)^+$ and $\Re(\lambda) + \frac{b}{2}\rho_0 \in X^*(P)^+$ we have

$$\left| \prod_{\alpha \in \Delta_P(\chi)} \frac{\langle \lambda, \alpha \rangle}{\langle \lambda, \alpha \rangle + 1} E^G_P(\lambda + \rho_P, \chi) \right| \leq c(1 + \|\Im(\lambda)\| + \|\chi_\infty\|^\epsilon),$$

where $\Delta_P(\chi) = \Delta_0(\chi) \cap \Delta_P$.

Proof. By the preceding proposition, there exist $b, c' > 0$ such that for all $\lambda' \in X^*(P_0)_C$ with $\Re(\lambda') \in B_{a,b}$ we have

$$\left| \prod_{\alpha \in \Delta_0(\chi)} \frac{\langle \lambda', \alpha \rangle}{\langle \lambda', \alpha \rangle + 1} E^G_{P_0}(\lambda' + \rho_0, \chi) \right| \leq c'(1 + \|\Im(\lambda')\| + \|\chi_\infty\|)^\epsilon.$$
Note that $\chi \circ \check{\alpha} = 1$ for all $\alpha \in \Delta^P_0$ and hence $\Delta^P_0(\chi) = \Delta_P(\chi) \cup \Delta^P_0$. Now let $\lambda \in X^*(P)_C$ be as in the proposition, i.e., $-\text{Re}(\lambda) + \frac{a}{2}\rho_P \in X^*(P)^+$ and $\text{Re}(\lambda) + \frac{b}{2}\rho_0 \in X^*(P)^+$. Then for all sufficiently small $\vartheta \in X^*(P_0)^+$ we have

$$-(\vartheta + \text{Re}(\lambda)) + \frac{a}{2}\rho_0 \in X^*(P_0)^+$$

$$\vartheta + \text{Re}(\lambda) + \frac{b}{2}\rho_0 \in X^*(P_0)^+,$$

i.e., $\vartheta + \text{Re}(\lambda) \in B_{a,b}$. Hence for those $\vartheta$ we have

$$\left| \prod_{\alpha \in \Delta_0(\chi)} \frac{\langle \vartheta + \lambda, \alpha \rangle}{\langle \vartheta + \lambda, \alpha \rangle + 1} E_{P_0}^G(\vartheta + \lambda + \rho_0, \chi) \right| \leq c'(1 + \|\text{Im}(\lambda)\| + \|\chi_\infty\|)^{\epsilon}.$$  

Letting $\vartheta$ tend to $0$ and using Proposition 8.4 we can conclude that

$$\left| \prod_{\alpha \in \Delta_P(\chi)} \frac{\langle \lambda, \alpha \rangle}{\langle \lambda, \alpha \rangle + 1} E_P^G(\lambda + \rho_P, \chi) \right| \leq \frac{c'}{c_P} (1 + \|\text{Im}(\lambda)\| + \|\chi_\infty\|)^{\epsilon}. \quad \square$$

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