A SUPERDIMENSION FORMULA FOR $\mathfrak{gl}(m|n)$-MODULES

MICHAEL CHMUTOV, RACHEL KARPMAN AND SHIFRA REIF

Abstract. We give a formula for the superdimension of a finite-dimensional simple $\mathfrak{gl}(m|n)$-module using the Su-Zhang character formula. As a corollary, we obtain a simple algebraic proof of a conjecture of Kac-Wakimoto for $\mathfrak{gl}(m|n)$, namely, a simple module has nonzero superdimension if and only if it has maximal degree of atypicality. This conjecture was proven originally by Serganova using the Duflo-Serganova associated variety.

1. Introduction

The superdimension of a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ is defined to be

$$\text{sdim } V := \dim V_0 - \dim V_1.$$ 

It was conjectured in 1994 by Kac and Wakimoto that the superdimension of a finite-dimensional simple module of a basic Lie superalgebra $\mathfrak{g}$ is nonzero if and only if the atypicality is maximal [KW]. This conjecture was supported by a theorem stating that the evaluation of the so-called Bernstein-Leites (super)character is nonzero exactly under this condition. At the time, there was no character formula for Lie superalgebras, and it was not known precisely how the Bernstein-Leites character was related to the actual character of the module. The conjecture was finally proved by Serganova but without giving a formula for the superdimension [S3].

Major progress has been made since then on the character theory for basic Lie superalgebras. In 1996, Serganova gave a general character formula for finite dimensional irreducible representations of $\mathfrak{gl}(m|n)$ in terms of generalized Kazhdan-Lusztig polynomials [S1, S2]. Brundan, in 2003, gave an explicit algorithm for computing these generalized Kazhdan-Lusztig polynomials [B]. In 2007, Su and Zhang used Brundan’s algorithm to prove a character formula which consists of a finite alternating sum of Bernstein-Leites characters (see Theorem 2).

In this paper we use the Su-Zhang character formula to give a formula for the superdimension of a simple finite dimensional $\mathfrak{gl}(m|n)$-module $L(\Lambda)$ with highest weight $\Lambda$. When $\Lambda$ is of maximal atypicality, the formula consists of a product of two positive terms. The first term, denoted by $s_\Lambda$, is the maximal number of monomials that can appear in a Kazhdan-Lusztig polynomial $K_{\Lambda, \mu}$ for any weight $\mu$. As shown by Su and Zhang, this number can be computed easily using Brundan’s algorithm (see Equation (3.1)). The second term is equal to the dimension of a simple module of a Lie algebra isomorphic to $\mathfrak{gl}(|m-n|)$. Using the dimension formula for simple Lie algebras, we obtain:

Theorem 1. Let $\Gamma_\Lambda$ be a maximal set of isotropic roots which are mutually orthogonal and orthogonal to $\Lambda + \rho$ and let $M_\Lambda^+$ be the set of even positive roots of $\mathfrak{g}$ orthogonal to $\Gamma_\Lambda$. Then

$$|\text{sdim } L(\Lambda)| = \left\{ \begin{array}{ll} s_\Lambda \prod_{\alpha \in M_\Lambda^+} \frac{\langle \Lambda + \rho, \alpha^\vee \rangle}{\langle \Lambda + \rho - \rho_\Lambda^{\alpha^\vee}, \alpha^\vee \rangle} & \text{if } \Lambda \text{ of maximal atypicality} \\ 0 & \text{otherwise} \end{array} \right.$$ 

where $\rho_\Lambda^{\alpha} = \frac{1}{2} \sum_{\alpha \in M_\Lambda^+} \alpha$.

It would be interesting to give a geometric proof for Theorem 1 and extend the result to representations of other types of Lie superalgebras.
2. Preliminaries

Let $\mathfrak{g}$ denote the Lie superalgebra $\mathfrak{gl}(m|n)$, and without loss of generality assume $m \geq n$. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$. We use the standard notation for the odd and even roots, namely

$$\Delta_0 = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq m\} \cup \{\delta_i - \delta_j \mid 1 \leq i \neq j \leq n\}$$

$$\Delta_1 = \{\epsilon_i - \delta_j, \delta_j - \epsilon_i \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

We normalize the bilinear form on $\mathfrak{h}^*$ so that for all $i, j$ we have $(\epsilon_i, \epsilon_j) = \delta_{i,j}$, $(\epsilon_i, \delta_j) = -\delta_{i,j}$, and $(\epsilon_i, \delta_j) = 0$ where $\delta_{i,j}$ is the Kronecker delta function.

We fix our choice of simple roots to be the standard one, that is

$$\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}.$$

Let $\Delta_0^+$ and $\Delta_1^+$ be the corresponding sets of even and odd positive roots, respectively. Let $Q^+$ denote the set of positive roots of $\mathfrak{g}$. Let $p$ denote the parity function on the roots of $\mathfrak{g}$, and extend $p$ to $Q^+$ in the natural way. We shall use the standard partial order on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. Let $p = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha$. For $\mu \in \mathfrak{h}^*$, we say that $\mu$ is dominant (resp. strictly dominant) if $\frac{2(\mu, \alpha)}{(\alpha, \alpha)} \geq 0$ (resp. $\frac{2(\mu, \alpha)}{(\alpha, \alpha)} > 0$) for all $\alpha \in \Delta_0^+$. Let $R$ and $\tilde{R}$ be the Weyl denominator and superdenominator, respectively, that is

$$R = \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha})}$$

and

$$\tilde{R} = \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})}.$$

A root of a Lie superalgebra is isotropic if it is orthogonal to itself. The isotropic roots of $\mathfrak{gl}(m|n)$ are precisely the odd roots. For a Lie superalgebra $\mathfrak{a}$, we define the defect of $\mathfrak{a}$, denoted $\text{def } \mathfrak{a}$, to be the size of a maximal set of isotropic positive roots which are mutually orthogonal. For $\mathfrak{g} = \mathfrak{gl}(m|n)$, with $m \geq n$, we have $\text{def } \mathfrak{g} = n$.

Let $L(\Lambda)$ be a simple finite-dimensional representation of highest weight $\Lambda$. Note that $\Lambda$ is a dominant weight, and since we chose the standard set of simple roots, $\Lambda + \rho$ is strictly dominant. Let $\Gamma_\Lambda$ be a maximal set of isotropic roots, which are orthogonal to each other and to $\Lambda + \rho$. Since $\Lambda + \rho$ is strictly dominant, this set is unique. The atypicality of $\Lambda$ is defined to be $r = |\Gamma_\Lambda|$. We set $M_\Lambda$ to be the set of even roots of $\mathfrak{g}$ orthogonal to $\Gamma_\Lambda$, and let $\mathfrak{g}_\Lambda$ be the Lie algebra with root system $M_\Lambda$. Note that if $r = n$, $\mathfrak{g}_\Lambda \cong \mathfrak{gl}(m-n)$. Denote $M_\Lambda^+ := M_\Lambda \cap \Delta_0^+$, $\rho_\Lambda^0 = \frac{1}{2} \sum_{\alpha \in M_\Lambda^+} \alpha$ and $R_\Lambda := \prod_{\alpha \in M_\Lambda^+} (1 - e^{-\alpha})$. We denote the simple $\mathfrak{g}_\Lambda$ module of highest weight $\mu$ by $L_\Lambda(\mu)$. We use the same notation for a weight $\lambda \in \mathfrak{h}$ and its restriction to $\mathfrak{g}_\Lambda \cap \mathfrak{h}$.

Given a weight space decomposition $L(\Lambda) = \bigoplus_{\mu \in Q^+} L_{\Lambda-\mu}$, the character and supercharacter of $L(\Lambda)$ are given by

$$\text{ch } L(\Lambda) = \sum_{\mu \in Q^+} (\dim L_{\Lambda-\mu}) e^{\Lambda-\mu}, \quad \text{sch } L(\Lambda) = \sum_{\mu \in Q^+} (-1)^{p(\mu)} (\dim L_{\Lambda-\mu}) e^{\Lambda-\mu}.$$ 

Note that these characters yield functions on $\mathfrak{h}$, defined by $e^\lambda(h) = e^{\Lambda(h)}$ for $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$. For $f$ a function on $\mathfrak{h}^*$, let $f|_0$ denote evaluation at 0. We have

$$\text{ch } L(\Lambda)|_0 = \dim L(\Lambda) \quad \text{and} \quad \text{sch } L(\Lambda)|_0 = s \dim L(\Lambda).$$

The Weyl group $W$ acts on the space $\mathcal{E}$ of rational functions in $e^\lambda$, $\lambda \in \mathfrak{h}$, by $w e^\lambda = e^{w\lambda}$. Let $\ell$ denote the length function of $W$. For $W' \subseteq W$ and $X \in \mathcal{E}$, we denote

$$\mathcal{F}_{W'}(X) = \sum_{w \in W'} (-1)^{\ell(w)} w(X).$$
3. Proof of Theorem 1

The Su-Zhang character formula gives the character of a finite-dimensional irreducible g-module \( L(\Lambda) \). We use this formula to derive a formula for the supercharacter, which we evaluate at zero to find the superdimension of \( L(\Lambda) \).

3.1. The Su-Zhang Character Formula. To state the Su-Zhang formula, we need additional notation. In particular, we shall introduce two subsets of the Weyl group \( W \) of \( g \), denoted \( S_\Lambda \) and \( C_r \).

We denote the elements of \( \Gamma_\Lambda \) by \( \{\beta_1, \ldots, \beta_r\} \) where \( \beta_k = \epsilon_k - \delta_{jk} \), and \( j_1 < j_2 < \cdots < j_r \). Note that this notation imposes an order on \( \Gamma_\Lambda \). We embed \( \text{Sym}_r \) in the Weyl group \( W \) of \( g \) by sending the transposition \( (k, \ell) \in \text{Sym}_r \) to the product of reflections \( s_{\epsilon_k} s_{\epsilon_\ell} s_{\delta_{jk}} s_{\delta_{\ell k}} \). Thus \( (k, \ell) \) maps to an element of \( W \) which interchanges \( \beta_k \) and \( \beta_\ell \). Note that for \( \mu \in \Lambda - \mathbb{Z} \Gamma_\Lambda \) and \( \sigma \in \text{Sym}_r \), we have \( \sigma(\mu + \rho) \in \Lambda + \rho - \mathbb{Z} \Gamma_\Lambda \).

To define \( S_\Lambda \), we recall the weight diagram construction introduced in [BS]. We write

\[
\Lambda + \rho = \sum_{i=1}^{m} a_i \epsilon_i - \sum_{j=1}^{n} b_j \delta_j.
\]

To construct the weight diagram of \( \Lambda + \rho \), we assign a symbol to each integer \( k \), according to the rule

\[
\begin{align*}
> & \quad \text{if } k \in \{a_1, \ldots, a_m\}, \ k \not\in \{b_1, \ldots, b_n\} \\
< & \quad \text{if } k \not\in \{a_1, \ldots, a_m\}, \ k \in \{b_1, \ldots, b_n\} \\
\times & \quad \text{if } k \in \{a_1, \ldots, a_m\}, \ k \in \{b_1, \ldots, b_n\} \\
\circ & \quad \text{if } k \not\in \{a_1, \ldots, a_m\}, \ k \not\in \{b_1, \ldots, b_n\}
\end{align*}
\]

We number the \( \times \)'s from left to right. For example, if

\[
\Lambda + \rho = 6\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + 2\epsilon_4 + \epsilon_5 - \delta_1 - 3\delta_2 - 6\delta_3 - 8\delta_4,
\]

then the corresponding weight diagram is

\[
\begin{array}{cccccccc}
\cdots & \times_1 & > & \times_2 & \circ & > & \times_3 & \circ & < & \cdots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
\]

For \( k \leq \ell \), we say that \( \beta_k \) and \( \beta_\ell \) are strongly connected if for every \( k < i \leq \ell \), the number of entries \( \circ \) between \( \times_k \) and \( \times_i \) in the weight diagram for \( \Lambda + \rho \) is less than or equal to the number of entries \( \times \) between \( \times_k \) and \( \times_i \). In the example above, \( \beta_1 \) and \( \beta_2 \) are strongly connected, as are \( \beta_1 \) and \( \beta_3 \); however, \( \beta_2 \) and \( \beta_3 \) are not strongly connected. For each \( 1 \leq s \leq r \), let \( \max^A_s \) be the largest \( t \leq r \) such that \( \beta_s \) and \( \beta_t \) are strongly connected.

**Definition 1.** Let

\[
S^A := \{\sigma \in \text{Sym}_r \mid \sigma^{-1}(s) < \sigma^{-1}(t) \text{ if } s < t \text{ and } \beta_s \text{ and } \beta_t \text{ are strongly connected}\}.
\]

By [SZ1] 3.18, we obtain

\[
(3.1) \quad s_\Lambda := |S_\Lambda| = \frac{r!}{\prod_{s=1}^{r} (\max^A_s - s + 1)}.
\]

**Definition 2.** Let \( C_r \) be the set of cyclic permutations of order \( r \), that is, all permutations of the form

\[
\pi = (1, \ldots, i_1)(i_1 + 1, \ldots, i_1 + i_2) \cdots (i_1 + \ldots + i_{t-1} + 1, i_1 + \ldots + i_{t-1} + 2, \ldots, r)
\]
where \( i_1, \ldots, i_t \in \mathbb{N} \) and \( i_1 + \ldots + i_t = r \). For \( \pi \in C_r \), we define

\[
\binom{r}{\pi} = \frac{r!}{i_1!i_2!\cdots i_t!}.
\]

We define the operation \( \dagger \) for \( \lambda \in \Lambda + \rho - N \Gamma_\Lambda \) by setting \( \mu = \lambda \dagger \) to be the maximal weight in \( \Lambda + \rho - N \Gamma_\Lambda \) where the coefficients of \( \delta_{j_k} \) in \( \mu \) are weakly increasing or, equivalently, the coefficients of \( \epsilon_{j_k} \) in \( \mu \) are weakly decreasing (with the notation of [SZ1], \( \lambda \dagger = (\lambda + \rho)_{\dagger} - \rho \)).

For \( \lambda, \mu \in \Lambda + \rho - Z \Gamma_\Lambda \), let \( \lambda - \mu = \sum_{i=1}^{r} c_i \beta_i \). We define \( |\lambda - \mu| \) by \( |\lambda - \mu| := \sum_{i=1}^{r} |c_i| \).

With the notation above, we have:

**Theorem 2. [SZ1] Theorem 4.9**] The character of a finite dimensional simple \( g \)-module with highest weight \( \Lambda \) is given by

\[
\text{ch} \ L(\Lambda) = \sum_{\sigma \in S^\Lambda, \pi \in C_r} \frac{1}{r!} \binom{r}{\pi} (-1)^{\lambda + \rho - \pi(\sigma(\lambda + \rho))_{\dagger}} \cdot e^{-\rho} R^{-1} \cdot F_{W} \left( \frac{e^{\pi(\sigma(\lambda + \rho))_{\dagger}}}{\prod_{\beta \in \Gamma_\Lambda} (1 + e^{-\beta})} \right)
\]

We transform this character formula to a formula for the supercharacter. For \( \nu \in \mathfrak{h}^* \) let

\[
\chi(\nu) := e^{-\rho} R^{-1} \cdot F_{W} \left( \frac{e^{\nu}}{\prod_{\beta \in \Gamma_\Lambda} (1 + e^{-\beta})} \right).
\]

Expanding each term \( \frac{1}{1 - e^{-\beta}} \) as a geometric series, and changing signs as appropriate, we obtain

\[
\text{sch} \ L(\Lambda) = \sum_{\sigma \in S^\Lambda, \pi \in C_r} \frac{1}{r!} \binom{r}{\pi} (-1)^{\ell(\sigma)} \chi(\pi(\sigma(\lambda + \rho))_{\dagger})_{\dagger}.
\]

### 3.2. Evaluation.

We now compute the superdimension of \( L(\Lambda) \) by evaluating the formula for the supercharacter in (3.2). We first show that many of the terms evaluate to the same number.

**Lemma 1.** For any \( \mu \in \Lambda - Z \Gamma_\Lambda \) with \( \mu + \rho \) \( M_\Lambda \) -dominant, \( \chi(\lambda + \rho)|_{0} = \chi(\mu + \rho)|_{0} \).

**Proof.** Let \( W_\Lambda \) be the subgroup of \( W \) generated by roots from \( M_\Lambda \) and let \( W_1 \) be a set of left coset representatives, so that \( W = W_1 W_\Lambda \). We have

\[
\chi(\mu + \rho) = e^{-\rho} R^{-1} \cdot F_{W_1} \left( \frac{F_{W_\Lambda}(e^{\mu + \rho})}{\prod_{\beta \in \Gamma_\Lambda} (1 + e^{-\beta})} \right).
\]

Since \( \Lambda \) is dominant, \( \mu + \rho - \rho_\Lambda^0 \) is \( M_\Lambda \) -dominant, the Weyl character formula implies

\[
F_{W_\Lambda}(e^{\mu + \rho}) = e^{\rho_\Lambda} R_\Lambda \cdot \text{ch} \ L_\Lambda(\mu + \rho - \rho_\Lambda^0).
\]

Since \( e^{-\rho} R^{-1} \) is \( W_1 \) -anti-invariant, we have

\[
\chi(\mu + \rho) = \sum_{w \in W_1} w \left( e^{-\rho} R^{-1} \cdot e^{\rho_\Lambda} R_\Lambda \cdot \text{ch} \ L_\Lambda(\mu + \rho - \rho_\Lambda^0) \right).
\]

The number of zeros minus the number of poles of the term \( e^{-\rho} R^{-1} \cdot e^{\rho_\Lambda} R_\Lambda \cdot \prod_{\beta \in \Gamma_\Lambda} (1 - e^{-\beta})^{-1} \) at 0 is \( (m-r)(n-r) \). Indeed \( |\Delta_0^+| = \frac{n(n-1)+m(m-1)}{2} \), \( |\Delta_0^-| = mn \), \( |M_\Lambda^+| = \frac{(m-r)(m-r-1)+2(n-r)(n-r-1)}{2} \), and \( |\Gamma_\Lambda| = r \). Since \( (m-r)(n-r) \geq 0 \), we can evaluate \( \chi(\mu + \rho) \) term by term, that is

\[
\chi(\mu + \rho)|_{0} = \sum_{w \in W_1} w \left( e^{-\rho} R^{-1} \cdot e^{\rho_\Lambda} R_\Lambda \right) \cdot \text{ch} \ L_\Lambda(\mu + \rho - \rho_\Lambda^0)|_{0}.
\]
A SUPERDIMENSION FORMULA FOR $\mathfrak{gl}(m|n)$-MODULES

\[ \text{Since } \Lambda - \mu \text{ is orthogonal to } M_{\Lambda}, \text{ we get that} \]
\[ \text{ch } L_{\Lambda}(\Lambda + \rho - \rho_{\Lambda}^0) = e^{\Lambda - \mu} \text{ch } L_{\Lambda}(\mu + \rho - \rho_{\Lambda}^0). \]

Since $e^{\Lambda - \mu}$ is orthogonal to $M_{\Lambda}$, we have that
\[ \text{ch } L_{\Lambda}(\Lambda + \rho - \rho_{\Lambda}^0) = e^{\Lambda - \mu} \text{ch } L_{\Lambda}(\mu + \rho - \rho_{\Lambda}^0). \]

We use the following theorem of Kac and Wakimoto to compute $\chi(\Lambda + \rho)|_0$ (note with the notation of [KW], $\chi(\Lambda + \rho) = j_{\Lambda} \text{ch}_{\Lambda}$).

**Theorem 3.** [KW, Theorem 3.3] One has
\[ |\chi(\Lambda + \rho)|_0| \begin{cases} n! \dim L_{\Lambda}(\Lambda + \rho - \rho_{\Lambda}^0) & r = \text{def } g \\ 0 & \text{otherwise.} \end{cases} \]

Since $(\pi(\sigma(\Lambda + \rho))|_0)$ is contained in $\Lambda + \rho - Z_{\Gamma_{\Lambda}}$ for every $\pi \in C_r$ and $\sigma \in S_\Lambda$, Lemma 1 implies that each term $\chi((\pi(\sigma(\Lambda + \rho))|_0)|_0$ is equal to the constant $\chi(\Lambda + \rho)|_0$. Hence, if $r \neq \text{def } g$, the formula evaluates to 0, completing the proof for this case. If $r = \text{def } g = n$, we have
\[ \text{sdim } L(\Lambda) = \pm \sum_{\sigma \in S_\Lambda, \pi \in C_r} \frac{1}{r!} \bigg( \frac{r}{\pi} \bigg) (-1)^{\ell(\pi)} n! \dim L_{\Lambda}(\Lambda + \rho - \rho_{\Lambda}^0). \]

By the dimension formula for simple Lie algebras we have
\[ \dim L_{\Lambda}(\Lambda + \rho - \rho_{\Lambda}^0) = \prod_{\alpha \in M_{\Lambda}^+} \frac{\langle \Lambda + \rho, \alpha^\vee \rangle \langle \Lambda + \rho - \rho_{\Lambda}^0, \alpha^\vee \rangle}{\langle \Lambda + \rho, \alpha^\vee \rangle}. \]

To complete the proof of the Theorem it remains to prove the following lemma.

**Lemma 2.** For $r > 0$, we have
\[ \sum_{\pi \in C_r} \bigg( \frac{r}{\pi} \bigg) (-1)^{\ell(\pi)} = 1. \]

**Proof.** The parity of a permutation $\pi \in \text{Sym}_r$ is $r$ plus the number of cycles of $\pi$. Splitting the sum on the left hand side based on the number of cycles we perform the following calculation using generating functions, where $[x^r]$ is the operator that takes the coefficient of $x^r$ of a power series.

\[ \sum_{\pi \in C_r} \bigg( \frac{r}{\pi} \bigg) (-1)^{\ell(\pi)} = \sum_{t \geq 1} \frac{r!}{r_1! \ldots r_t!} (-1)^{r+t} \]
\[ = (-1)^r r! \sum_{t \geq 1} [x^r] (1 - e^x)^t \]
\[ = (-1)^r r! [x^r] \left( \sum_{t \geq 1} (1 - e^x)^t \right) \]
\[ = (-1)^r r! [x^r] \left( \frac{1 - e^x}{1 - (1 - e^x)} \right) \]
\[ = (-1)^r r! [x^r] (e^{-x} - 1) \]
\[ = 1 \]

4. Examples

Let us illustrate our formula with a few examples. We use the vector notation for the weights of $\mathfrak{g}$, namely
\[ (a_1, \ldots, a_m \mid b_1, \ldots, b_n) := \sum_{i=1}^m a_i \epsilon_i - \sum_{i=1}^n b_i \delta_i. \]
Note that $a_i = b_j$ means that $\epsilon_i - \delta_j \in \Gamma_A$. Shifting the highest weight of a $\mathfrak{g}$-module by $\text{str} := (1, \ldots, 1|1, \ldots, 1)$ does not change the superdimension of the module. Similarly, shifting the highest weight of a $\mathfrak{g}_A$-module by $\text{tr} := \sum_{i \in M} \epsilon_i, M = \{ i \mid \epsilon_i - \delta_j \notin \Gamma(\Lambda) \forall j \}$ does not change the dimension. Thus, the computations below are done up to a multiple of $\text{str}$ and $\text{tr}$.

4.1. **The trivial representation.** The highest weight of the trivial representation is $\Lambda = 0$, so

$$\Lambda + \rho = (m, m-1, \ldots, 2, 1|1, 2, \ldots, n)$$

and $s_A = 1$. We have that $M_A = \{ \epsilon_i - \epsilon_j \mid 1 < i, j \leq m - n \}$ and $\rho - \rho_A^0 = 0$. Thus, our formula gives that the superdimension of the trivial representation is equal to the dimension of the trivial representation of $\mathfrak{g}_A$ which is 1 as desired.

4.2. **The natural representation.** Let $V$ be the natural representation of $\mathfrak{g}$. Let us show that our formula gives $|s\dim(V)| = m - n$.

The highest weight of $V$ is $\Lambda = \epsilon_1$, so we have $\Lambda + \rho = (m + 1, m - 1, \ldots, 1|1, 2, \ldots, n)$, and $s_A = 1$. Then the atypicality is $n$ for $m > n$, and $n - 1$ for $m = n$. In the latter case, we have $r \neq n$, and so our formula gives 0 for the superdimension, as desired.

For $m > n$, $M_A = \{ \epsilon_i - \epsilon_j \mid i, j \leq m - n \}$, and we have $\Lambda + \rho - \rho_A^1 = \epsilon_1$. However, this is the highest weight of the natural representation of $M_A \cong \mathfrak{g}((m-n))$, and we get $|s\dim V| = \dim L_{Lm}(\epsilon_1) = m - n$.

4.3. **The adjoint representation.** Let $V$ be the irreducible component of the adjoint representation of $\mathfrak{g}$ corresponding to $\mathfrak{sl}(m|n)$ for $m > n$, and to $\mathfrak{psl}(n|n)$ for $m = n$. Then

$$|s\dim(V)| = m^2 + n^2 - 2mn - 1 - \delta_{mn}$$

The highest weight of $V$ is the highest root $\Lambda = \epsilon_1 - \epsilon_n$ and

$$\Lambda + \rho = (m + 1, m - 1, \ldots, 2, 1|1, 2, \ldots, n - 1, n + 1).$$

For $m = n$, $s_A = 2$ and $M_A = \emptyset$. We obtain $|s\dim(V)| = 2$. For $m = n + 1$, we have $|\Gamma_A| = n - 1$. So $r < \text{def} \mathfrak{g}$, and our formula gives 0. Finally, for $m > n + 1$, $s_A = 1$ and

$$M_A = \{ \epsilon_i - \epsilon_j \mid i, j \in \{ 1, 2, \ldots, m - n - 1 \} \cup \{ m - n + 1 \} \}.$$

Thus $\Lambda + \rho - \rho_A^0 = \epsilon_1 - \epsilon_{m-n+1}$ which is the highest weight of the adjoint representation of $\mathfrak{g}_A$ and we get that $|s\dim(V)| = (m - n)^2 - 1$ as required.

**References**

[B] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$*, J. Amer. Math. Soc. 16 (2003), no. 1, 185–231.

[BS] J. Brundan, C. Stroppel, *Highest weight categories arising from Khovanov’s diagram algebra I: Cellularity*, Mosc. Math. J. 11 (2011), no. 4, 685–722

[KW] V.G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, Lie theory and geometry, edited by J.-L. Brylinski et al., Progr. Math. 123, Birkhuser, Boston, MA, (1994) 415–456.

[S1] V. Serganova, *Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra $\mathfrak{gl}(m|n)$*, Selecta Math. (N.S.) 2 (1996), no. 4, 607–651.

[S2] V. Serganova, *Characters of irreducible representations of simple Lie superalgebras*, Proceedings of the International Congress of Mathematicians, vol. II, 1998, Berlin, Doc. Math., J. Deutsch. Math.-Verein. (1998) 583–593.

[S3] Vera Serganova, *On the superdimension of an irreducible representation of a basic classical Lie superalgebra*, Supersymmetry in mathematics and physics, Lecture Notes in Math., vol. 2027, Springer, Heidelberg, 2011, pp. 253-273. MR 2906346 (2012m:17014).

[SZ1] Y. Su, R.B. Zhang, *Character and dimension formulae for general linear superalgebra*, Advances in Mathematics 211 (2007) 1–33.