The Continuous Graph FFT

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Abstract

The discrete Fourier transform and the FFT algorithm are extended from the circle to continuous graphs with equal edge lengths.

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1 Introduction

The discrete Fourier transform (DFT) and algorithms for its efficient computation (FFT) enjoy an enormous range of applications. One might roughly divide such applications into the analysis of data which is sampled in time or in space. Applications involving spatially sampled data are basic in numerical analysis, since constant coefficient partial differential equations and some of their discrete analogs, including the heat, wave, Schrödinger, and beam equations in one space dimension, have exact solutions in Fourier series. Other applications of the DFT involving the analysis of spatially sampled data include noise removal, data compression, data interpolation, and approximation of functions. These applications often take advantage of the tight and explicit linkage binding the harmonic analysis of uniformly sampled data with the harmonic analysis of periodic functions defined on $\mathbb{R}$.

Many aspects of Fourier analysis can be developed through the spectral theory of the second derivative operator $-D^2 = -d^2/dx^2$ acting on the Hilbert space of square integrable functions with period 1. An orthogonal basis of eigenfunctions is given by $\exp(2\pi ikx)$. If $N$ uniformly spaced samples $x_n = n/N$ for $n = 0, \ldots, N - 1$ are considered, the restriction of the exponential functions to the sample points $\exp\left(2\pi ikn/N\right)$, $k, n = 0, \ldots, N - 1$, again provides an orthogonal basis for functions on the set $\{x_n\}$. These sampled exponential functions are also eigenfunctions for 'local' operators such as $\Delta_N$, which acts by

$$\Delta_N f(x_n) = f(x_n) - \frac{1}{2}[f(x_{n-1} + f(x_{n+1})],$$

index arithmetic being carried out modulo $N$. The fact that efficient FFT algorithms are available for sampled data is not a consequence of abstract spectral theory, but relies on the exponential form of the eigenfunctions and the arithmetic progression of the frequencies $k$.

Our discussion now shifts to graphs and their refinements, with the circle and its uniform sampling serving as an example. Discrete graphs with $N_V < \infty$ vertices and $N_E$ edges have a well developed spectral theory [7] based on the adjacency or Laplace operators acting on functions defined on the vertex set $\mathcal{V}$. While these ideas play a role here, the lead actor is a continuous graph,
also known as a metric graph or, when a differential operator is emphasized, a quantum graph. Here the edges of a graph are identified with intervals $[a_n, b_n] \subset \mathbb{R}$, functions are elements of the Hilbert space $\oplus_n L^2[a_n, b_n]$, and differential operators such as $-D^2$ provide a basis for both physical modeling and harmonic analysis. (The terminology for such ‘continuous graphs’ is not settled, with the terms topological graphs or networks also used.)

The natural and technological worlds offer numerous opportunities for graph modeling, where applications such as those mentioned above can be considered with data sampled from continuous graphs. These include the sampling of populations along river systems, and the description of traffic density on road networks. Biological systems transport nutrients, waste, heat, and pressure waves through vascular networks, and electrochemical signals through natural neural networks. Elaborate networks are manufactured in microelectronics, and for microfluidic laboratories on chips.

Despite a rapidly growing literature on ‘quantum graphs’ [4, 8, 9], and some work [2] generalizing the classical theory of Fourier series to functions defined on a continuous graph, there is a very limited literature considering the harmonic analysis of sampled data for these geometric objects. The papers [13, 14] consider sampling on continuous graphs with virtually no length restrictions on the edges. In contrast, this work exploits the features appearing when graphs have equal length edges, and obtains the following conclusion: every finite continuous graph with edges of equal length admits a family of DFTs closely analogous to that of the circle, and an FFT algorithm for their efficient computation.

To unify the presentation a bit, we focus on simple discrete graphs, without loops or multiple edges between vertices, whose vertices have degree at least two. Given a continuous version of a nonsimple graph without boundary vertices, one can insert additional vertices of degree 2 to reduce to the simple case. This insertion of ‘invisible’ vertices can also be used to reduce graphs whose edge lengths are integer multiples of a common value to the equal length case. Algorithmically, this reduction increases the size of the discrete graph spectral problem whose solution is an important component of the DFT.

There are four subsequent sections in the paper. The second section starts with a review of differential operators on continuous graphs. This leads to a more detailed discussion of the spectral theory of the standard Laplace differential operator on continuous graphs with equal length edges. Most of the material in this section was previously known [3, 11], although Theorem
2.9 appears to be new. The third section explores the linked spectral theory of Laplacians for continuous graphs and their uniformly sampled subgraphs. The fourth section shows that efficient algorithms for Fourier analysis are available. The final section presents a simple example where many of the computations can be done 'by hand'.

2 Spectra of continuous graph Laplacians

2.1 Continuous graphs and the standard Laplacian

In this work a discrete graph $\mathcal{G}$ will be finite and simple, with a vertexset $V$ having $N_V$ points, and a set $\mathcal{E}$ of $N_E$ edges. Each edge $e$ has a positive length $l_e$. It is convenient to number the edges. For $n = 1, \ldots, N_E$ the edge $e_n$ is identified with a real interval $[a_n, b_n]$ of length $l_n$. The resulting topological graph is also denoted $\mathcal{G}$. In an obvious fashion one may extend the standard metric and Lebesgue measure from the edges to $\mathcal{G}$.

The identification of graph edges and intervals allows us to define the Hilbert space $L^2(\mathcal{G}) = \bigoplus_{n=1}^{N_E} L^2[a_n, b_n]$ with the inner product

$$\langle f, g \rangle = \int_{\mathcal{G}} f(x)\overline{g(x)} \, dx, \quad f_n : [a_n, b_n] \rightarrow \mathbb{C}.$$ 

We will employ the standard vertex conditions, requiring continuity at the vertices,

$$\lim_{x \in e(i) \rightarrow v} f(x) = \lim_{x \in e(j) \rightarrow v} f(x), \quad e(i), e(j) \sim v, \quad \text{(2.1)}$$

and the derivative condition

$$\sum_{e_n \sim v} \partial_{e_n} f_n(v) = 0. \quad \text{(2.2)}$$

Here the derivative $\partial_{e_n} f_n(v)$ is $f_n'(a_n)$, respectively $-f_n'(b_n)$ if $a_n$, respectively $b_n$, is identified with $v$.

The standard vertex conditions are used to define the Laplacian $\mathcal{L} f = -f''$ of the continuous graph $\mathcal{G}$. Let $\mathcal{D}_{\max}$ denote the set of functions $f \in L^2(\mathcal{G})$ with $f'$ absolutely continuous on each $e_n$, and $f'' \in L^2(\mathcal{G})$. The domain
$D$ of $L$ is then the set of $f \in D_{\text{max}}$ satisfying the standard conditions (2.1) and (2.2). From the classical theory of ordinary differential operators [12, p. S123] one knows that $L$ is self adjoint with compact resolvent. This operator is nonnegative, with 0 being a simple eigenvalue if $G$ is connected. Writing eigenvalues with multiplicity, the spectrum is thus a sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, and there is an orthonormal basis of eigenfunctions.

We now impose the additional requirement that all edges of $G$ have equal length. In this situation the operator $L$ will be denoted $\Delta_\infty$. Initially the edge lengths are taken to be 1, but the rescaling $\xi = x/L$ converts the system
\[ D_x^2 Y = \lambda Y \]

to
\[ D_\xi^2 Y = \mu Y, \quad 0 \leq \xi \leq 1, \quad \mu = L^2 \lambda. \quad (2.3) \]

Values of eigenfunctions at graph vertices are not effected, and derivatives are scaled by $L$. Thus eigenfunctions with eigenvalue $\lambda$ for the standard Laplacian on the graph with edge lengths $L$ are taken to eigenfunctions with eigenvalue $L^2 \lambda$ for the standard Laplacian on the graph with edge lengths 1.

The spectrum of $\Delta_\infty$ is tightly linked to the discrete graph $G$ and the spectrum of the discrete Laplacian of combinatorial graph theory. Let $E(\lambda)$ denote the eigenspace of $\Delta_\infty$ for the eigenvalue $\lambda$. The connections between $E(\lambda)$ and the discrete graph $G$ arise in two ways, depending on whether or not $\lambda \in \{n^2 \pi^2 \mid n = 1, 2, 3, \ldots\}$. Before exploring this dichotomy, we note the following result.

**Proposition 2.1.** If $\omega > 0$,
\[ \dim E(\omega^2) = \dim E([\omega + 2n\pi]^2), \quad n = 1, 2, 3, \ldots. \]

**Proof.** Suppose $n \geq 1$ and $Y$ is an eigenfunction with eigenvalue $\omega^2$. On each edge
\[ y(x) = A \cos(\omega x) + B \sin(\omega x). \]

The function
\[ y_1(x) = A \cos([\omega + 2n\pi]x) + B \sin([\omega + 2n\pi]x) \]
then satisfies the equation,
\[ -y_1'' = [\omega + 2n\pi]^2 y_1, \]
and at the endpoints of the edge we have
\[ y(0) = y_1(0), \quad y(1) = y_1(1). \]
The endpoint derivatives are

\[ y'(0) = \omega B, \quad y'(1) = \omega [B \cos(\omega) - A \sin(\omega)] \]

and

\[ y'_1(0) = [\omega + 2n\pi] B, \quad y'_1(1) = [\omega + 2n\pi] [B \cos(\omega) - A \sin(\omega)], \]

so

\[ y'_1(0) = \frac{\omega + 2n\pi}{\omega} y'(0), \quad y'_1(1) = \frac{\omega + 2n\pi}{\omega} y'(1). \]

Thus \( y_1 \) satisfies the same interior vertex conditions that \( y \) does. Moreover, the linear map taking \( y \rightarrow y_1 \) is one-to-one, as is the analogous map from \( E([\omega + 2n\pi]^2) \) to \( E(\omega^2) \).

\[ \square \]

### 2.2 Role of the discrete Laplacian

Given a vertex \( v \in \mathcal{G} \), let \( u_1, \ldots, u_{\text{deg}(v)} \) be the vertices adjacent to \( v \). A discrete graph carries a number of linear operators acting on the vertex space \( \mathbb{H} \) of functions \( f : \mathcal{V} \rightarrow \mathbb{C} \), including the adjacency operator

\[ Af(v) = \sum_{i=1}^{\text{deg}(v)} f(u_i), \]

and the degree operator

\[ Tf(v) = \text{deg}(v) f(v). \]

Define the operator \( \Delta_1 \) by

\[ \Delta_1 f(v) = f(v) - T^{-1} Af(v). \]

\( \Delta_1 \) is similar to the much studied Laplacian [7, p. 3],

\[ I - T^{-1/2} AT^{-1/2}. \]

The distinction of the cases \( \lambda \in \{n^2\pi^2\} \) is related to the following fact.

**Lemma 2.2.** Fix \( \lambda \in \mathbb{C} \), and consider the vector space of solutions of \(-y'' = \lambda y\) on the interval \([0, 1]\). The linear function taking \( y(x) \) to \((y(0), y(1))\) is an isomorphism if and only if \( \lambda \notin \{n^2\pi^2 \mid n = 1, 2, 3, \ldots \} \).
Proof. For \( \lambda \notin \{n^2 \pi^2\} \), the formula
\[
y(x, \lambda) = y(0) \cos(\omega x) + [y(1) - y(0) \cos(\omega)] \frac{\sin(\omega x)}{\sin(\omega)}
\] (2.4)
shows that the map is surjective. On the other hand, if \( \lambda \in \{n^2 \pi^2\} \), then \( y(0) = 0 \) implies \( y(1) = 0 \), since \( y(x) = B \sin(n \pi x) \).

As an immediate consequence we have the following result for graphs.

Lemma 2.3. Suppose the edges of \( G \) have length 1, and \( \lambda \notin \{n^2 \pi^2 \mid n = 1, 2, 3, \ldots\} \). Let \( y : G \to \mathbb{C} \) be continuous, and satisfy \(-y'' = \lambda y\) on the edges. If \( y(v) = 0 \) at all vertices of \( G \), then \( y(x) = 0 \) for all \( x \in G \).

Theorem 2.4. Suppose \( \lambda \notin \{n^2 \pi^2 \mid n = 1, 2, 3, \ldots\} \) and \( y \) is an eigenfunction for \( \Delta_\infty \). If \( v \) has adjacent vertices \( u_1, \ldots, u_{\deg(v)} \), then
\[
\cos(\omega) y(v) = \frac{1}{\deg(v)} \sum_{i=1}^{\deg(v)} y(u_i). \tag{2.5}
\]
Proof. In local coordinates identifying each \( u_i \) with 0, (2.4) for \( y_i \) on the edges \( e_i = (u_i, v) \) gives
\[
y_i'(v) = -\omega \sin(\omega) y_i(u_i) + [y_i(v) - y_i(u_i) \cos(\omega)] \frac{\omega \cos(\omega)}{\sin(\omega)}. \tag{2.6}
\]
Summing over \( i \), the derivative condition at \( v \) then gives
\[
0 = \sum_i y_i'(v) = -\omega \sin(\omega) \sum_i y_i(u_i) + \frac{\omega \cos(\omega)}{\sin(\omega)} \sum_i [y_i(v) - y_i(u_i) \cos(\omega)].
\]
Using the continuity of \( y \) at \( v \) and elementary manipulations gives (2.5). \( \square \)

(2.5) is clearly an eigenvalue equation, with eigenvalue \( \cos(\omega) \), for the linear operator \( T^{-1}A \) acting on the space of (real or complex valued) functions on the vertex set. With this background established, we are ready to relate the spectra of \( \Delta_1 \) and \( \Delta_\infty \).

Theorem 2.5. If \( \lambda \notin \{n^2 \pi^2 \mid n = 0, 1, 2, \ldots\} \), then \( \lambda \) is an eigenvalue of \( \Delta_\infty \) if and only if \( 1 - \cos(\omega) = 1 - \cos(\sqrt{\lambda}) \) is an eigenvalue of \( \Delta_1 \), with the same geometric multiplicity.
Proof. Since $\Delta_1 = I - T^{-1}A$, we may work with $T^{-1}A$. Suppose first that
$y(x, \lambda)$ is an eigenfunction of $\Delta_\infty$ satisfying the given vertex conditions. The-
orem 2.4 shows that the (linear) evaluation map taking $y : \mathcal{G} \to \mathbb{C}$ to $y : \mathcal{V} \to \mathbb{C}$ takes eigenfunctions to solutions of (2.5). By Lemma 2.3 the
kernel of this map is the zero function, so the map is injective.

Suppose conversely that $y : \mathcal{V} \to \mathbb{C}$ satisfies

$$T^{-1}Ay(v) = \mu y(v), \quad |\mu| < 1.$$ 

Pick $\lambda \in \cos^{-1}(\mu)$. By Lemma 2.2 the function $y : \mathcal{V} \to \mathbb{C}$ extends to a
unique continuous function $y(x, \lambda) : \mathcal{G} \to \mathbb{C}$ satisfying $-y'' = \lambda y$ on each
edge.

In local coordinates identifying $v$ with 0 for each edge $e_i = (v, u_i)$ incident
on $v$, this extended function satisfies (2.6). Summing gives

$$\sum_i y_i'(v) = \frac{\omega}{\sin(\omega)}[-\sin^2(\omega) - \cos^2(\omega)]\sum_i y_i(u_i) + \sum_i y_i(v)\frac{\omega \cos(\omega)}{\sin(\omega)}$$

$$= -\frac{\omega}{\sin(\omega)}\sum_i y_i(u_i) + \deg(v)y(v)\frac{\omega \cos(\omega)}{\sin(\omega)}.$$ 

The vertex values satisfy (2.5), so

$$\sum_i y_i'(v) = -\frac{\omega}{\sin(\omega)}\deg(v)\cos(\omega)y(v) + \deg(v)y(v)\frac{\omega \cos(\omega)}{\sin(\omega)} = 0.$$ 

Thus the extended functions are eigenfunctions of $\Delta_\infty$ satisfying the standard vertex conditions. Since the extension map is linear, and the kernel is the
zero function, this map is also injective.

\[\square\]

2.3 Eigenfunctions at $n^2\pi^2$

Now we turn to eigenvalues $\lambda \in \{n^2\pi^2\}$. First recall [7, p. 7] that for both $\Delta_1$
and $\Delta_\infty$ 0 is an eigenvalue whose eigenspace is spanned by functions which
are constant on connected components of $\mathcal{G}$.

For $n \geq 1$ these eigenspaces for $\Delta_\infty$ also have a combinatorial interpreta-
tion, closely related to the cycles in $\mathcal{G}$. If $C$ is a cycle, and $x$ is distance along
the cycle starting at some selected vertex, then the function $\sin(2n\pi x)$ is an
eigenfunction of $\Delta_\infty$. Similarly, if $C$ is an even cycle, then $\sin(n\pi x)$ gives a
similar eigenfunction. We can make the following observation.
Lemma 2.6. Suppose $G$ is connected, and $\psi$ is an eigenfunction of $\Delta_\infty$ with eigenvalue $\lambda = n^2 \pi^2$ for $n \geq 1$. If $\psi$ vanishes at any vertex, then $\psi$ vanishes at all vertices.

Proof. Suppose $\psi(v) = 0$ for some vertex $v$. On any edge incident on $v$, the eigenfunction is a linear combination

$$\psi(x) = A \cos(n\pi x) + B \sin(n\pi x), \quad v \simeq 0,$$

and clearly $A = 0$. Since all edge lengths are 1, at all adjacent vertices $w$, we then have $\psi(w) = B \sin(n\pi) = 0$. By continuity of $\psi$ and connectivity of the graph, $\psi$ vanishes at all vertices.

The next result explores the existence of eigenfunctions vanishing at no vertices.

Lemma 2.7. If $\lambda = (2n\pi)^2$, for $n = 1, 2, 3, \ldots$, then $\Delta_\infty$ has an eigenfunction vanishing at no vertices.

If $\lambda = (2n - 1)^2 \pi^2$, $n = 1, 2, 3, \ldots$, then $\Delta_\infty$ has an eigenfunction vanishing at no vertices if and only if $G$ is bipartite.

Proof. If $\lambda = (2n\pi)^2$, then the desired eigenfunction is simply $\cos(2n\pi x)$ in local coordinates on each edge.

Suppose $G$ is bipartite, with the two classes of vertices labelled 0 and 1. Pick local coordinates on each edge consistent with the vertex class labels, and define the eigenfunction to be $\cos([2n - 1]\pi x)$.

Suppose conversely that for some $\lambda = (2n - 1)^2 \pi^2$, there is an eigenfunction $\psi$ vanishing at no vertex. Label the vertices $v$ according to the sign of $\psi(v)$. In local coordinates for an edge,

$$\psi(x) = a \cos([2n - 1]\pi x) + b \sin([2n - 1]\pi x), \quad a \neq 0.$$

Then if $w$ is a vertex adjacent to $v$ we see that $\psi(w) = -\psi(v)$, showing that vertices are only adjacent to vertices of opposite sign, and $G$ is bipartite.

In addition to the vertex space mentioned above, an edge space may be constructed using the edges of a graph as a basis (we assume the field is $\mathbb{R}$ or $\mathbb{C}$). The edge space has the cycle subspace $Z_0(G)$ generated by cycles, with dimension $N_E - N_V + 1$ [5, pp. 51–58] or [10, pp. 23–28]. Let $E_0(n^2 \pi^2) \subset E(n^2 \pi^2)$ be those eigenfunctions of $\Delta_\infty$ vanishing at the vertices.
Theorem 2.8. \[ \dim Z_0(G) = \dim E_0(4n^2\pi^2). \]

Proof. It suffices to prove the result for a connected graph. Suppose \( Z_0(G) \) has dimension \( M \). Picking a spanning tree \( T \) for \( G \), there is [5, p. 53] a basis of cycles \( C_1, \ldots, C_M \) such that each \( C_j \) contains an edge \( e_j \notin T \), with \( e_j \) not contained in any other \( C_i \). Fix \( n \in \{1, 2, 3, \ldots\} \) and construct eigenfunctions \( f_j = \sin(2\pi nx) \) on the edges of \( C_j \), and 0 on all other edges. Here \( x \) denotes distance along the cycle starting at some selected vertex. If a linear combination \( \sum a_i f_i \) is 0, then for \( x \in e_j \)

\[
0 = \sum_{i=1}^{M} a_i f_i(x) = a_j f_j(x), \quad j = 1, \ldots, M.
\]

There are \( x \in e_j \) where \( f_j(x) \neq 0 \), so \( a_j = 0 \) and the functions \( f_i \) are independent. This shows \( \dim E_0(4n^2\pi^2) \geq \dim Z_0(G) \).

Now suppose that \( \psi \in E_0(4n^2\pi^2) \). After subtracting a linear combination \( \sum a_i f_i \) we may assume that \( \psi \) vanishes on all edges not in the spanning tree \( T \). For every boundary vertex \( v \) of \( T \), \( \psi \) vanishes identically on all but one edge of \( G \) incident on \( v \), and by the vertex conditions it then vanishes on all edges incident on \( v \). Continuing away from the boundary of the spanning tree, we see that \( \psi \) is the 0 function. \( \square \)

Figure 2.1: Bowtie graph

The proof of Theorem 2.8 provides a combinatorial basis construction for \( E_0(4n^2\pi^2) \). A similar construction using even cycles will provide independent elements of \( E_0((2n-1)^2\pi^2) \), but these may not form a complete set. Consider the bowtie graph in Figure 2.1, whose cycles have length 3. An eigenfunction \( \psi \) with eigenvalue \( \pi^2 \) can be constructed by letting \( \psi(x) = 2\sin(\pi x) \) on the
middle edge. The function $\psi$ then continues as $-\sin(\pi x)$ on the adjacent four edges, and $\sin(\pi x)$ on the remaining two edges. Notice that on each edge the function $\psi$ is an integer multiple of $\sin(\pi x)$.

A generalization of the even cycles will be used for a combinatorial construction of $E_0((2n - 1)^2\pi^2)$. Let $Z_1$ denote the set of functions $f : \mathcal{E} \to \mathbb{C}$ such that

$$\sum_{e \sim v} f(e) = 0, \quad v \in \mathcal{V}.$$ 

**Theorem 2.9.** The linear map taking $f \in Z_1$ to $g(x)$ defined by

$$g_e(x) = f(e) \sin((2n - 1)\pi x),$$

is an isomorphism from $Z_1$ onto $E_0((2n - 1)^2\pi^2)$. The subspace $Z_1$ has an integral basis.

**Proof.** Since $\sin((2n-1)\pi x) = \sin((2n-1)\pi(1-x))$, the edge orientation does not affect the definition, so $g$ is well defined. Clearly $g(x)$ satisfies the eigenvalue equation and vanishes at each vertex. The condition $\sum_{e \sim v} f(e) = 0$ for all $v \in \mathcal{V}$ gives the derivative condition, so $g \in E_0((2n - 1)^2\pi^2)$. Moreover the map is one to one.

Suppose $\psi(x) \in E_0((2n - 1)^2\pi^2)$. Then $\psi_e(x) = a_e \sin((2n - 1)\pi x)$ on each edge, and because $\psi$ satisfies the derivative conditions we have

$$\sum_{e \sim v} a_e = 0, \quad v \in \mathcal{V}.$$ 

Define $f(e) = a_e$ to get a linear map from $E_0((2n - 1)^2\pi^2)$ to $Z_1$, which is also one to one. This establishes the isomorphism.

To see that $Z_1$ has an integral basis, let $e_n$ be a numbering of the edges of $\mathcal{G}$, and let $x_n = f(e_n)$. The set of functions $Z_1$ is then given by the set of $x_1, \ldots, x_{N_\mathcal{E}}$ satisfying the $N_\mathcal{V}$ equations

$$\sum_{e_n \sim v} x_n = 0.$$ 

This is a system of linear homogeneous equations whose coefficient matrix consists of ones and zeros. Reduction by Gaussian elimination shows that the set of solutions has a rational basis, and so an integral basis.

$\square$
3 Graph refinements

Now we introduce the notion of graph refinement for graphs whose edge lengths are 1. Let the original combinatorial graph be denoted $G_1$, with the operator $\Delta_1$ acting on the vertex space $\mathbb{H}_1$. For each integer $N > 1$ we will define a graph $G_N$ with vertex space $\mathbb{H}_N$ and operator $\Delta_N : \mathbb{H}_N \to \mathbb{H}_N$ by subdividing each edge $e \in G_1$ into $N$ edges. Pick local coordinates identifying $e$ with $[0, 1]$, and labelling $0 = x_0$ and $1 = x_N$. For $n = 1, \ldots, N - 1$, introduce points $x_n = n/N$ in the local coordinates. The new graph $G_N$ with $I = N V + (N - 1) N E$ vertices will have the vertex set consisting of the vertices of $G_1$, together with the new vertices $x_n$, for each edge $e$ of $G_1$. The vertex $x_n$ is adjacent to $x_{n-1}$ and $x_{n+1}$ for $n = 1, \ldots, N - 1$. If $v$ is a vertex in $G_1$, and the local coordinates for edges incident on $v$ are chosen so $v$ is identified with 0 on each edge, then $v$ is adjacent in $G_N$ with the vertices $x_1$ for each of the incident edges. An example is illustrated in Figure 3.1.

![Figure 3.1: Refinement of a graph](image)

An inner product on the vertex space $\mathbb{H}_N$ is defined by

$$\langle f, g \rangle_N = \frac{1}{W} \sum_v \deg(v) f(v) g(v), \quad W = \sum_{v \in G_N} \deg(v) = 2 NN_E.$$

Using the above identifications of edges of $G_N$ with subintervals of $[0, 1]$, we may also construct the corresponding continuous graph having edges of length $1/N$. These continuous graphs may all be identified with the continuous graph $G_\infty$ corresponding to $G_1$. In particular they will share the inner product

$$\langle f, g \rangle_\infty = \frac{1}{N E} \int_{G_\infty} f \overline{g}.$$
where $N_\mathcal{G}$ is the number of edges in $\mathcal{G}_1$. As a continuous graph, the standard vertex conditions hold at all vertices of $\mathcal{G}_N$. Although appearing different in definition, the continuous graph Laplacian $\Delta_\infty$ has not changed as we pass from $\mathcal{G}_1$ to $\mathcal{G}_N$.

Let $\Delta_N$ denote the following operator on the vertex space of $\mathcal{G}_N$,

$$\Delta_N = N^2(I - T^{-1}A).$$

The eigenspaces of these operators with eigenvalue $\lambda$ will be denoted $E_N(\lambda)$, while $E_\infty(\lambda)$ will denote an eigenspace for $\Delta_\infty$. By (2.3) and Theorem 2.5, the mapping $N^2(1 - \cos(\sqrt{\lambda}/N))$ carries eigenvalues of $\Delta_\infty$ to eigenvalues of $\Delta_N$ if $\lambda/N^2 \notin \{n^2\pi^2\}$. That is, the eigenvalues of the normalized adjacency operator are $\cos(\sqrt{\lambda}/N)$ whenever $\lambda \notin \{(Nn\pi)^2\}$ is an eigenvalue for $\Delta_\infty$. Notice the eigenvalues of $\Delta_N$ are approximately $\lambda/2$ when $\lambda$ is small compared to $N$.

**Lemma 3.1.** For $1 \leq N < \infty$ the operator $\Delta_N$ is self adjoint on the vertex space with the inner product $\langle f, g \rangle_N$.

**Proof.** It suffices to check the normalized adjacency operator $T^{-1}A$. \hfill \Box

Let $E_p(n^2\pi^2)$ denote the subspace spanned by eigenfunctions of $\Delta_\infty$ having the form $\cos(n\pi x)$ on each edge (so not vanishing at the vertices). Let $S_N \subset L^2(\mathcal{G}_\infty)$ denote the subspace

$$S_N = \text{span}\{E_p(N^2\pi^2), E_\infty(\lambda), 0 \leq \lambda < N^2\pi^2\}.$$

Here is preliminary result describing the restriction of eigenfunctions of $\Delta_\infty$ to the vertices of $\mathcal{G}_N$.

**Proposition 3.2.** The restriction map $R_N : S_N \to \mathcal{H}_N$ is an bijection. For $0 \leq \lambda < N^2\pi^2$ this map takes distinct orthogonal eigenspaces $E_\infty(\lambda)$ of $\Delta_\infty$ onto distinct orthogonal eigenspaces $E_N(N^2(1 - \cos(\sqrt{\lambda}/N)))$ of $\Delta_N$, and $R_N$ takes $E_p(N^2\pi^2)$ onto $E_N(2N^2)$.

**Proof.** By Theorem 2.5 and (2.3), $R_N$ is a bijection from the eigenspace $E_\infty(\lambda)$ of $\Delta_\infty$ to the eigenspace $E_N(N^2(1 - \cos(\sqrt{\lambda}/N)))$ of $\Delta_N$. The 0 eigenspaces for $\Delta_\infty$ and $\Delta_N$ are just the functions which are constant on the connected components of the respective graphs, so $R_N$ is a bijection from $E_\infty(0)$ to $E_N(0)$. From the proof of Lemma 2.7 we also see that $R_N$ is a bijection from $E_p(N^2\pi^2)$ to $E_N(2N^2)$.
Since $\Delta_\infty$ and $\Delta_N$ are self-adjoint on their respective function spaces, and since $\cos(t)$ is strictly decreasing on $(0, \pi)$, distinct eigenspaces of $\Delta_\infty$, which are orthogonal, are mapped to distinct eigenspaces of $\Delta_N$, which are orthogonal and span $H_N$.

The restriction map $R_N : S_N \to \mathbb{H}_N$ also has noteworthy features on the individual eigenspaces $E_\infty(\lambda)$ of $\Delta_\infty$. Before stating the results, we start with a simple observation relating the $H_N$ inner product and the sums appearing in the trapezoidal rule for integration. Recalling that $W = \sum_v \deg(v) = 2NN_\varepsilon$, the inner product for $G_N$ satisfies the identity

$$W(f, g)_N = \sum_v \deg(v)f(v)\overline{g(v)}$$

$$= \sum_{e \in G_1} [f_e(x_0)\overline{g_e(x_0)} + f_e(x_N)\overline{g_e(x_N)} + 2 \sum_{n=1}^{N-1} f_e(x_n)\overline{g_e(x_n)}].$$

These last sums are just the trapezoidal rule sums used for integrals over the edges of $G_\infty$. With this motivation, if the continuous linear functional $T_N : C[0, 1] \to \mathbb{C}$ is defined by

$$T_N(f) = \frac{1}{2N} [f(x_0) + f(x_N) + 2 \sum_{n=1}^{N-1} f(x_n)],$$

then

$$\langle R_N f, R_N g \rangle_N = \frac{1}{N_\varepsilon} \sum_{e \in E} T_N(f_e\overline{g_e}).$$

The following identities are also useful. First

$$\int_0^1 e^{2i\omega x} = \frac{\exp(2i\omega) - 1}{2i\omega} = e^{i\omega}\frac{\sin(\omega)}{\omega}.$$  

Then, for $0 < \omega < N$, a geometric series computation gives

$$2NT_N(\exp(i\omega x)) = \exp(i\omega) - 1 + 2 \sum_{n=0}^{N-1} \exp(i\omega n/N)$$

$$= \exp(i\omega) - 1 + 2 \frac{1 - \exp(i\omega)}{1 - \exp(i\omega/N)} = (1 - \exp(i\omega))\frac{1 + \exp(i\omega/N)}{1 - \exp(i\omega/N)},$$

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or

\[ T_N(e^{i\omega x}) = M_0\left(\frac{\omega}{2N}\right) \int_0^1 e^{i\omega x}, \quad M_0(z) = M_0(-z) = z \cot(z). \quad (3.2) \]

These identities will be useful for comparing the inner products on \( L^2(G_{\infty}) \) and \( \mathbb{H}_N \). The cases \( \lambda = k^2\pi^2 \) with \( 0 \leq k < N \) are considered first.

**Theorem 3.3.** Suppose \( f, g \in E_\infty(\lambda) \), with \( 0 \leq \lambda \leq N^2\pi^2 \). If \( \lambda = k^2\pi^2 \) for an integer \( k \) with \( 0 \leq k < N \), then

\[ \langle f, g \rangle_\infty = \langle R_N f, R_N g \rangle_N. \quad (3.3) \]

If \( f, g \in E_p(N^2\pi^2) \), then

\[ \langle f, g \rangle_\infty = \frac{1}{2} \langle R_N f, R_N g \rangle_N. \quad (3.4) \]

**Proof.** Suppose \( f, g \in E_\infty(\lambda) \), with \( 0 \leq \lambda < N^2\pi^2 \). On each edge \( e \) we have

\[ f_e = \alpha_e e^{i\omega x} + \beta_e e^{-i\omega x}, \quad g_e = \gamma_e e^{i\omega x} + \delta_e e^{-i\omega x}, \quad \omega^2 = \lambda, \quad (3.5) \]

\[ f_e g_e = \alpha_e \gamma_e + \beta_e \delta_e + \alpha_e \delta_e e^{2i\omega x} + \beta_e \gamma_e e^{-2i\omega x}. \]

If \( \lambda = k^2\pi^2 \) for an integer \( k \) with \( 0 \leq k < N \), then

\[ T_N(e^{2\pi ikx}) = \int_0^1 e^{2\pi ikx} = 0. \]

Using (3.1) and \( T_N(1) = 1 \), it follows that

\[ \langle f, g \rangle_\infty = \frac{1}{N_e} \int_{g_\infty} f \overline{g} = \frac{1}{N_e} \sum_e [\alpha_e \gamma_e + \beta_e \delta_e] \]

\[ = \frac{1}{N_e} \sum_e T_N(\alpha_e \gamma_e + \beta_e \delta_e) = \langle R_N f, R_N g \rangle_N, \]

which is (3.3). Similar calculations handle the case \( \lambda = 0 \).

Suppose \( f, g \in E_p(N^2\pi^2) \), a one dimensional space. For \( f_e = g_e = \cos(N\pi x) \), we have

\[ \frac{1}{N_e} \int_{G_\infty} \cos^2(N\pi x) = 1/2, \]

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but
\[
\langle R_N f, R_N g \rangle_N = \frac{1}{N_E} \sum_{e \in \mathcal{E}} T_N(f_e g_e) = \frac{1}{N_E} \sum_{e \in \mathcal{E}} 1 = 1,
\]
giving (3.4).

For general \( \omega \) a variation of a classical trapezoidal rule estimate appears.

**Theorem 3.4.** Suppose \( f, g \in E_\infty(\lambda) \), with \( 0 < \lambda < N^2 \pi^2 \). Then for \( \omega \geq \omega_0 > 0 \),
\[
\left| \langle R_N f, R_N g \rangle_N - \langle f, g \rangle \right| = O(|\omega|^4/N^4) \| f \| \| g \|. \tag{3.6}
\]

**Proof.** The argument starts with a variation of standard error estimates \([1, \text{p. 285}], [p. 358-369][6]\) for the trapezoidal rule for integration. If \( \phi = \exp(i \omega x) \) then
\[
\frac{\phi'(1) - \phi'(0)}{12N^2} = \frac{1}{12N^2} \int_0^1 \phi'' = -\frac{\omega^2}{12N^2} \int_0^1 e^{i \omega x}.
\]
Putting this together with (3.2) yields
\[
\int_0^1 e^{i \omega x} - T_N(e^{i \omega x}) = -\frac{\phi'(1) - \phi'(0)}{12N^2} + M_1(\frac{\omega}{2N}) \int_0^1 e^{i \omega x},
\]
with
\[
M_1(z) = 1 - z \cot(z) - \frac{z^2}{3}.
\]
A Taylor expansion gives
\[
M_1(z) = O(z^4),
\]
and the function \( M_1(z) = M_1(-z) \) is analytic for \( |z| < \pi \).

On each edge we have the representation (3.5). Since \( f \) and \( g \) are in the domain of \( \Delta_\infty \), the function \( \psi = f\mathcal{G} \) satisfies the vertex conditions (2.1) and (2.2). Thus
\[
\sum_e [\psi_e'(1) - \psi_e'(0)] = -\sum_{v \in \mathcal{V}} \sum_{e \sim v} \partial_v \psi_e(v) = 0,
\]
and
\[
N_E [\langle f, g \rangle - \langle R_N f, R_N g \rangle_N] = \sum_e \left[ \int_0^1 \psi_e - T_N(\psi_e) \right]
\]
\[
= -\sum_e \left[ \frac{\psi_e'(1) - \psi_e'(0)}{12N^2} \right] + M_1(\frac{\omega}{N}) \left[ \int_0^1 e^{2i \omega x} \sum_e \alpha_e \bar{\delta}_e + \int_0^1 e^{-2i \omega x} \sum_e \beta_e \gamma_e \right].
\]
\[
M_1\left(\frac{\omega}{N}\right) \left[ \int_0^1 e^{2i\omega x} \sum_e \alpha_e \overline{\delta_e} + \int_0^1 e^{-2i\omega x} \sum_e \beta_e \overline{\gamma_e} \right]
\]

Using \(2|\alpha_e||\beta_e| \leq |\alpha_e|^2 + |\beta_e|^2\), we find

\[
\int_0^1 |f_e|^2 = |\alpha_e|^2 + |\beta_e|^2 + \alpha \overline{\beta} e^{i\omega \sin(\omega)} + \beta \overline{\alpha} e^{-i\omega \sin(\omega)}
\]

\[
\geq (1 - \frac{\sin(\omega)}{\omega})(|\alpha_e|^2 + |\beta_e|^2).
\]

This gives the desired result

\[
N_E \left| \langle R_N f, R_N g \rangle - \langle f, g \rangle \right|
\]

\[
\leq \left| M_1\left(\frac{\omega}{N}\right)(1 - \frac{\sin(\omega)}{\omega})^{-1} \sum_e \left( \int_0^1 |f_e|^2 \right)^{1/2} \left( \int_0^1 |g_e|^2 \right)^{1/2} \right|
\]

\[
\leq \left| M_1\left(\frac{\omega}{N}\right)(1 - \frac{\sin(\omega)}{\omega})^{-1} \left( \int_{\mathcal{G}_\infty} |f|^2 \right)^{1/2} \left( \int_{\mathcal{G}_\infty} |g|^2 \right)^{1/2} \right|.
\]

\[\square\]

4 The FFT

In this section we turn to an algorithm for the efficient computation of the Fourier transform and its inverse transform on the spaces \(\mathbb{H}_N\).

\[
\begin{array}{cccccccc}
0 & \pi & 2\pi & 3\pi & 4\pi & 5\pi & \sqrt{\lambda} \\
\end{array}
\]

Figure 4.1: Square root of the spectrum of \(\Delta_\infty\).

Recall some of the structures associated with the spectrum of \(\Delta_\infty\). The square roots \(\sqrt{\lambda_k}\) of the positive eigenvalues of \(\Delta_\infty\) and the eigenspaces \(E(\lambda_k)\) exhibit the '2\(\pi\)-periodicity' discussed in Proposition 2.1 and illustrated in Figure 4.1. Let \(0 < \omega_{0,1} < \omega_{0,2} < \cdots \leq 2\pi\) denote the square roots of the \(K\) distinct eigenvalues \(\lambda_k\) of \(\Delta_\infty\) with \(0 < \lambda_k \leq (2\pi)^2\). For each \(\omega_{0,k}\)
pick an orthonormal basis \( \Psi(\omega_{0,k}) \) for \( E(\omega^2_{0,k}) \), with basis functions \( \Psi_j(\omega_{0,k}) \), \( j = 1, \ldots, \dim E(\omega^2_{0,k}) \).

\[
\begin{array}{cccc}
\omega_{0,k} & \omega_{1,k} & \omega_{2,k} & \omega_{3,k} \\
2\pi & \bullet & \bullet & \bullet & \bullet \\
\pi & \bullet & \bullet & \bullet & \bullet \\
0 & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

Figure 4.2: Eigenfunction maps for \( \Delta_\infty \).

For \( m = 1, 2, 3, \ldots \), let \( \omega_{m,k} = \omega_{0,k} + 2m\pi \). From each basis \( \Psi(\omega_{0,k}) \) we may produce bases \( \Psi(\omega_{m,k}) \) for \( E(\omega^2_{m,k}) \) by the method of Proposition 2.1. Recall that on each edge \( e \) a function \( \Psi_j(\omega_{0,k}) \) has the form

\[
\Psi_j = A_e \exp(i\omega_{0,k}x) + B_e \exp(-i\omega_{0,k}x), \quad 0 \leq x \leq 1.
\]

The set of functions \( \Psi_j(\omega_{m,k}) \) whose values on \( e \) are

\[
\Psi_j(\omega_{m,k}) = A_e \exp(i[\omega_{0,k} + 2m\pi]x) + B_e \exp(-i[\omega_{0,k} + 2m\pi]x), \quad (4.1)
\]

then give a basis \( \Psi(\omega_{m,k}) \) for \( E([\omega_{0,k} + 2m\pi]^2) \). This basis need not be orthonormal. Figure 4.2 indicates the relations among the bases \( \Psi(\omega_{m,k}) \).

**Proposition 4.1.** If \( \omega_{0,k} \in \{\pi, 2\pi\} \), then the functions \( \Psi_j(\omega_{m,k}) \) are orthonormal, as are the functions \( R_N \Psi_j(\omega_{m,k}) \in \mathbb{H}_N \) for \( 0 \leq \omega_{m,k} < N\pi \).

**Proof.** Suppose \( \Psi_i(\omega_{m,k}) \in \Psi(\omega_{m,k}) \) is represented on the edge \( e \) by

\[
\Psi_i(\omega_{m,k}) = C_e \exp(i[\omega_{0,k} + 2m\pi]x) + D_e \exp(-i[\omega_{0,k} + 2m\pi]x), \quad 0 \leq x \leq 1,
\]

and (4.1) describes \( \Psi_j \). With \( \omega_{0,k} = \pi, 2\pi \),

\[
\int_0^1 \exp(\pm 2i[\omega_{0,k} + 2m\pi]x) = 0,
\]

so

\[
\sum_e \int_0^1 \Psi_j(\omega_{m,k}) \overline{\Psi_i(\omega_{m,k})} = \sum_e [A_e \overline{C_e} + B_e \overline{D_e}] = \sum_e \int_0^1 \Psi_j(\omega_{0,k}) \overline{\Psi_i(\omega_{0,k})}.
\]

The extension to \( \mathbb{H}_N \) follows from Theorem 3.3. \( \Box \)
Let $I_0 = \dim E(0)$, $I_k = \dim E(\omega_{0,k}^2)$, and let
\[ \Phi_j(\omega_{m,k}) = R_N \Psi_j(\omega_{m,k}) \]
be the restriction of $\Psi_j(\omega_{m,k})$ to the vertices of $G_N$. The bases $\Phi_j(\omega_{k,m})$ will be used for efficient Fourier transform algorithms \[1\] p. 182 \[6\] p. 383. Define the discrete Fourier transforms (DFT) $\mathcal{F}_N : \mathbb{H}_N \rightarrow \mathbb{C}^{I_0} \oplus_{m,k} \mathbb{C}^{I_k}$ for the continuous graph $G_\infty$ by taking the inner product of a function $f \in \mathbb{H}_N$ with the bases $\Phi_j(\omega_{k,m})$,
\[ \mathcal{F}_N(f) = \{ \langle f, \Phi_j(\omega_{m,k}) \rangle_N \}, \quad \omega_{m,k} \leq N\pi. \] (4.2)
The condition $\omega_{m,k} \leq N\pi$ amounts to $m = 0, \ldots, N/2 - 1$ when $N$ is even, which will hold in the cases of interest below. Except for the cases noted in Proposition 4.1 the functions of $\Phi_j(\omega_{m,k})$ may not be orthonormal, so the Fourier transform will not be an isometry from $\Phi_j(\omega_{m,k})$ to $\mathbb{C}^{I_k}$ if the usual inner product is used on the range. We consider a modified inner product.

Suppose $B(\omega_{m,k}) = (b_{ij})$ is a matrix taking the basis $\Phi_j(\omega_{m,k})$ to an orthonormal basis
\[ \eta_i = \sum_j b_{ij} \Phi_j(\omega_{m,k}) \] (4.3)
for the same eigenspace in $\mathbb{H}_N$. (Such a matrix may be obtained by the Gram-Schmidt process.) For $f$ in the span of $\Phi_j(\omega_{m,k})$, the map $f \rightarrow \{ \langle f, \eta_i \rangle_N \}$ is an isometry to $\mathbb{C}^{I_k}$ with the usual inner product $X \cdot Y = \sum x_j y_j$. Expressing this in the original basis,
\[ \|f\|^2 = \sum_i |\langle f, \eta_i \rangle_N|^2 = \sum_i |\langle f, \sum_j b_{ij} \Phi_j \rangle_N|^2 = \sum_i |\sum_j b_{ij} \langle f, \Phi_j \rangle_N|^2 \]
\[ = \overline{B} \cdot X \cdot B, \quad X = (\langle f, \Phi_1 \rangle_N, \ldots, \langle f, \Phi_{I_k} \rangle_N)^T. \]

For $f = \sum_j c_j \Phi_j(\omega_{m,k})$ in the span of $\Phi_j(\omega_{m,k})$, the coefficients $c_j$ may be recovered from the DFT values $\langle f, \Phi_j \rangle_N$. Starting from
\[ f = \sum_i a_i \eta_i, \quad a_i = \langle f, \eta_i \rangle_N, \]
and using (4.3) we find
\[ f = \sum_i \sum_j b_{ij} \langle f, \Phi_j \rangle_N \sum_l b_{il} \Phi_l = \sum_i \sum_j \sum_l b_{ij} b_{il} \langle f, \Phi_j \rangle_N \Phi_l \]
\[= \sum_i \sum_j (B^* B)_{jl}(f, \Phi_j) \Phi_l,\]
or
\[c_l = \sum_j (B^* B)_{jl}(f, \Phi_j) \Phi_l, \quad (4.4)\]

Calling \(BX \cdot BX\) the \(B(\omega_{m,k})\) inner product on \(C^I_k\), and noting \(\Phi(\omega_{m,k})\) is a basis for the \(\mu_{m,k}\) eigenspace of \(\Delta_N\), we obtain the first part of the next result.

**Theorem 4.2.** The Fourier transform \(\mathcal{F}_N : \mathbb{H}_N \rightarrow \mathbb{C}^{I_0} \oplus_{m,k} \mathbb{C}^{I_k}\) satisfies
\[\mathcal{F}_N(\Delta_N f) = \{\mu_{m,k} \mathcal{F}(f)_{m,k}\},\]
and is an isometry if \(\mathbb{C}^{I_k}_{m}\) has the \(B(\omega_{m,k})\) inner product.

If \(N\) is a power of 2, then \(\mathcal{F}_N(f)\) can be computed in time \(O(N \log_2(N))\).

**Proof.** The inner products of (4.2) may be split into trapezoidal sum s for the edges as in (3.1),
\[\mathcal{F}_N(f) = \frac{1}{N_e} \sum_e T_N(f_e \overline{\Phi_j(\omega_{m,k})}).\]
Using the representation (4.1) we have
\[T_N(f_e A_e \exp(i[\omega_{0,k} + 2m\pi]x) + B_e \exp(-i[\omega_{0,k} + 2m\pi]x))\]
\[= A_e T_N(\exp(-i\omega_{0,k}x)f_e \exp(-2m\pi ix) + B_e T_N(\exp(i\omega_{0,k}x)f_e \exp(2m\pi ix)).\]
If \(N\) is a power of 2, \(e\) and \(k\) are fixed, and \(m = 0, \ldots, \frac{N}{2} - 1\), then the sequence \(T_N(\exp(\pm i\omega_{0,k}x)f_e(x) \exp(\pm 2m\pi ix))\) differs trivially from the conventional discrete Fourier transform of the sampled data \(f_e(x_n)\), which may be computed [1, p. 182] [6, p. 383] in time \(O(N \log_2(N))\). Since the number of edges \(e\) and indices \(k\) is fixed, we obtain the result.

It remains to consider the inverse transform. Since the matrix \(B^* B\) of (4.4) depends only on the graph and functions \(\Phi_j(\omega_{m,k})\), these matrices are assumed to be precomputed. Suppose the DFT sequence (4.2) of a function \(f\) is given. Since the computations in (4.4) are limited to sequence blocks...
$C^I_k$, the coefficients $c_j(\omega_{m,k})$ for all $\omega_{m,k}$ may be computed in time $O(N)$. It remains to compute

$$f(x_n) = \sum_{m,k} c_j(\omega_{m,k})\Phi_j(\omega_{m,k})(x_n).$$

Fixing the edge $e$, the primitive frequency index $k$, and the basis index $j$, and taking advantage of (4.1), the remaining sum has the form

$$f_{e,k,j}(x_n) = A_e \exp(i\omega_{0,k}n/N) \sum_m c_j(\omega_{m,k}) \exp(i2m\pi n/N)$$

$$+ B_e \exp(-i\omega_{0,k}n/N) \sum_m c_j(\omega_{m,k}) \exp(-i2m\pi n/N).$$

These sums may be computed with a conventional FFT. Since the number of edges $e$ and primitive frequency indices $k$ are constant, and the range of basis indices $j$ is bounded independent of $m$, the next result is obtained.

**Theorem 4.3.** If $N$ is a power of 2, then the inverse DFT can be computed in time $O(N\log_2(N))$.

5 A family of examples

A relatively simple family of examples is provided by the complete bipartite graphs $K(m,2)$ on $m,2$ vertices. The $m = 4$ case is portrayed in Figure 5.1. Discussion of these graphs is facilitated if the vertices are colored, the 2 vertices of degree $m$ being red and the remaining $m$ vertices blue.

Suppose $v$ is a vertex, and $\{u_i\}$ are the vertices adjacent to $v$. Then the operator $\Delta_1$ on the $m+2$ dimensional vertex space is given by

$$\Delta_1 f(v) = f(v) - \frac{1}{\text{deg}(v)} \sum_{i=1}^{\text{deg}(v)} f(u_i).$$

Eigenvalues $\mu$ for $\Delta_1$ are 0, 1, 2, with 1 having multiplicity $m$. The eigenfunctions $\psi$ for $\mu = 0, 2$ are

$$\mu_0 = 0, \quad \psi_0(v) = 1,$$

$$\mu_{m+1} = 2, \quad \psi_{m+1}(v) = \begin{cases} 1, & v \text{ red} \\ -1, & v \text{ blue} \end{cases}$$
Next, consider the eigenfunctions for $\mu = 1$. A function $f(v)$ will satisfy $\Delta_1 f(v) = f(v)$ if and only if $\sum_{i=1}^{\text{deg}(v)} f(u_i) = 0$ for every vertex $v$. Number the red vertices $r_1$ and $r_2$, and the two blue vertices $b_1, \ldots, b_m$. For $j = 1, \ldots, m - 1$, independent eigenfunctions are

$$\psi_j(v) = \begin{cases} 1, & v = b_j \\ -1, & v = b_{j+1} \\ 0, & \text{otherwise} \end{cases}.$$ 

There is one additional independent eigenfunction

$$\psi_m(v) = \begin{cases} 1, & v = r_1 \\ -1, & v = r_2 \\ 0, & \text{otherwise} \end{cases}.$$ 

To establish the independence of $\psi_1, \ldots, \psi_m$, note that if

$$\sum_{j=1}^{m} c_j \psi_j(v) = 0,$$

then $c_m = 0$ by evaluation at $r_1$, and evaluation at the points $b_j$ leads to equations

$$c_1 = 0, c_1 = c_2, \ldots, c_{m-1} = c_m.$$ 

Moving to the continuous graph, identify each edge with $[0, 1]$ so the blue vertex is identified with 0. Consider the eigenvalues $\lambda$ and corresponding eigenfunctions $\Psi$ for $\Delta_\infty$, focusing on $\sqrt{\lambda} \in [0, 2\pi]$. Of course we have
\( \lambda_0 = 0 \) with the constant eigenfunction \( \Psi(0) = 1 \). From \( \mu_j = 1 \) we obtain eigenvalues
\[
\sqrt{\lambda} = \cos^{-1}(1 - \mu_j) = \cos^{-1}(0) = \pi/2, 3\pi/2.
\]
That is, \( \lambda = (\pi/2)^2 \) and \( \lambda = (3\pi/2)^2 \) are eigenvalues, each with multiplicity \( m \). The corresponding eigenfunctions \( \Psi_j(\pi/2) \) and \( \Psi_j(3\pi/2) \) are obtained by extrapolation as in Lemma 2.2 from the vertex values of \( \psi_j \).

This graph [5, p. 53] has \( m - 1 \) independent cycles, all of which are even. One basis of cycles \( C_i \) is given by the vertex sequence \( r_1, b_i, r_2, b_{i+1}, r_1 \). Each \( C_i \) supports eigenfunctions \( \Psi_i(\pi) \) and \( \Psi_i(2\pi) \) which are respectively \( \pm \sin(\pi x) \) and \( \pm \sin(2\pi x) \) on the edges of the cycle.

Finally, there are two additional independent eigenfunctions, \( \Psi_m(\pi) \) having values \( \cos(\pi x) \) and \( \Psi_m(2\pi) \) with values \( \cos(2\pi x) \) on the edges of \( G \). This gives a total of
\[
2N_E = 4m = 2m + 2(m - 1) + 2
\]
independent eigenfunctions for \( \Delta_\infty \) with eigenvalues \( 0 < \lambda \leq 2\pi \).

This graph has an obvious involution in which \( r_1 \) and \( r_2 \) are interchanged, and we may split the eigenspaces into even and odd subspaces with respect to the involution. Notice that the eigenfunctions \( \Psi_m(\pi) \), \( \Psi_m(2\pi) \), \( \Psi_i(\pi/2) \) and \( \Psi_i(3\pi/2) \) for \( i = 1, \ldots, m - 1 \) are even while \( \Psi_m(\pi/2) \), \( \Psi_m(3\pi/2) \), \( \Psi_i(\pi) \) and \( \Psi_i(2\pi) \) for \( i = 1, \ldots, m - 1 \) are odd. By restricting to the even, respectively odd, subspaces we may treat the case of a star graph with \( m \) edges and boundary vertices with the conditions \( f'_e(0) = 0 \), respectively \( f_e(0) = 0 \).
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