GALOIS EXTENSIONS OVER COMMUTATIVE AND NON-COMMUTATIVE BASE

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Abstract

This paper is a written form of a talk. It gives a review of various notions of Galois (and in particular cleft) extensions. Extensions by coalgebras, bialgebras and Hopf algebras (over a commutative base ring) and by corings, bialgebroids and Hopf algebroids (over a non-commutative base algebra) are systematically recalled and compared.

In the first version of this paper, the journal version of [15, Theorem 2.6] was heavily used, in two respects. First, it was applied to establish an isomorphism between the comodule categories of two constituent bialgebroids in a Hopf algebroid. Second, it was used to construct a Morita context for any bicomodule for a coring extension. Regrettably, it turned out that the proof of [15, Theorem 2.6] contains an unjustified step. Therefore, our derived results are not expected to hold at the stated level of generality either. In the revised version we make the necessary corrections in both respects. In doing so, we obtain a corrected version of [5, Theorem 4.2] as well, whose original proof contains a very similar error to [15, Theorem 2.6].

Introduction

The history of Hopf Galois extensions is nearly 40 years long, as it can be traced back to [21]. Since then it is subject to a study of always renewing interest. There are several reasons of this interest. First of all, the algebraic structure is very rich. It has strong relations with the problem of ring extensions. It is connected to (co-)module theory and a descent problem. On the other hand, Hopf Galois theory unifies various situations in an elegant manner. It is capable to describe e.g. classical Galois extensions of fields or strongly group-graded algebras. Another application of fundamental importance comes from non-commutative differential geometry. From this latter point of view, a (faithfully flat) Hopf Galois extension is interpreted as a (dual form of a) non-commutative principal bundle.

Although the theory of Hopf Galois extensions was very fruitful, the appearance of non-fitting examples forced it to be generalized. Generalizations have been made in two different directions. In one of them the coacting Hopf algebra (or bialgebra) was replaced by a coalgebra. Since this change results in loosing the monoidality of the category of comodules, the notion of a comodule algebra is no longer available. Still, the essential features of the theory turned out to be possible to maintain. The notion of a Galois extension by a coalgebra appeared first
in [16] and then in [17]. The most general definition, which will be used in this paper, can be found in the paper [16] by Brzeziński and Hajac.

In another direction of generalization, initiated by Kadison and Szlachányi in [27], the coacting bialgebra was replaced by a bialgebroid over a non-commutative base algebra. Although in this case monoidality of the category of comodules persists, non-commutativity of the base algebra results in a conceptually new situation. While modules over a commutative ring form a symmetrical monoidal category, the monoidal category of bimodules for any algebra is not symmetrical.

The two directions of generalization can be unified in the framework of extensions by corings. The aim of the current paper is to review all listed notions of Galois extensions. We would like to show that – after finding the proper, categorically well established notions – the theories over commutative and non-commutative bases are reassuringly parallel.

All Galois extensions in this paper are defined via bijectivity of a certain canonical map. In the literature one can find a further generalization, called a weak Galois extension, where the canonical map is required to be only a split monomorphism [18, 37.9]. Weak Galois extensions by coalgebras include Galois extensions by weak Hopf algebras in [19]. The issue of weak Galois extensions is not considered in this paper.

The paper is organized as follows. In Section 1 we fix the notations, and recall the basic notions, used later on. In Section 2 definitions of the various Galois extensions are recalled. We start in Section 2.1 with the most classical and best understood case of a Hopf Galois extension. Then in Section 2.2 we show how it fits the more general case of a Galois extension by a coalgebra. These two sections deal with definitions of Galois extensions over commutative base. In the case of a non-commutative base, we proceed in a converse order. We start in Section 2.3 with recalling the most general instance of a Galois extension by a coring. It is quite easily derived from the particular case of a coalgebra. In Section 2.4 we define a Galois extension by a bialgebroid as a special Galois extension by the underlying coring. Characterization is built on the monoidality of the category of comodules of a bialgebroid. Finally, in Section 2.5 the coacting bialgebroid is specialized to a Hopf algebroid. Here the complications are caused by the presence of two bialgebroid structures, whose roles are clarified. All sections are completed by examples.

An important (and relatively simple) class of Galois extensions is provided by cleft extensions. In Section 3 the various notions of cleft extensions are recalled. The order of the cases revisited follows the order in Section 2. We present a unifying picture of most cleft extensions, the one of cleft bicomodules, developed by Vercruysse and the author in [10]. It is shown that any cleft extension can be characterized as a Galois extension with additional normal basis property.

Under mild further assumptions, a Strong Structure Theorem is proven for cleft extensions. Cleft extensions by Hopf algebras or Hopf algebroids are characterized as crossed products. It has to be emphasized that this paper is of a review type. Very few new results are presented. However, we hope that the way, they are collected from various resources and re-organized here, give some new insight. Since the results presented in the paper are already published, all proofs given here are sketchy. On the other hand, a big emphasis is put on giving precise references to the original publications.

Parts of the first version of these notes (those concerning Galois extensions with Hopf algebroids and those about cleft bicomodules) heavily relied on the journal version of [15, Theorem 2.6]. Regrettably, a few years after publishing these notes it turned out that the proof of [15, Theorem 2.6] contains an unjustified step. Although we are not aware of any counterexamples, this gap forces us to revise our work, as we had to revise references [8] and [10]. Note similar errors
also in [6, Proposition 3.1] and [5, Theorem 4.2].

There are two most important changes compared to the first version. First, in the revised form of Section 2.5 we describe the Galois theory of Hopf algebroids by distinguishing between comodules of the two constituent bialgebroids. Second, we have to face the fact that the results in [10] are capable to handle cleft bicomodules only for so called pure coring extensions. While this unifies cleft extensions by coalgebras (hence in particular by Hopf algebras) and by corings, it is not known if all cleft extensions by Hopf algebroids fit this framework. In Section 3.6 we propose a treatment of cleft extensions by Hopf algebroids. Similarly to [10], a key role is played by an appropriate Morita context. However, in general, this Morita context is not known to be associated to a coring extension. Applying this Morita context, for a cleft extension by an arbitrary Hopf algebroid, we prove a Strong Structure Theorem (Theorem 3.27). This yields in particular a corrected version of [5, Theorem 4.2], whose original version is not known to be true because of an unjustified step in the proof, see Remark 3.28.

Using informality of the arXiv, as an experiment, for the convenience of the readers of the first version, we write changes compared to the first version in blue.

1 Preliminaries

Throughout the paper we work over a commutative associative unital ring $k$.

The term $k$-algebra (or sometimes only algebra) means a $k$-module $A$ equipped with a $k$-linear associative multiplication $\mu: A \otimes_k A \to A$, with $k$-linear unit $\eta: k \to A$. On elements of $A$ multiplication is denoted by juxtaposition. The unit element is denoted by $1 \in A$. We add labels, write $\mu_A$, $\eta_A$ or $1_A$, when the algebra $A$ needs to be specified.

The term $k$-coalgebra (or sometimes only coalgebra) means a $k$-module $C$ equipped with a $k$-linear coassociative comultiplication $\Delta: C \to C \otimes_k C$, with $k$-linear counit $\varepsilon : C \to k$. On elements $c$ of $C$, Sweedler’s index notation is used for comultiplication: $\Delta(c) = c_{(1)} \otimes_k c_{(2)}$ – implicit summation understood.

A right module of a $k$-algebra $A$ is a $k$-module $M$ equipped with a $k$-linear associative and unital action $M \otimes_k A \to M$. On elements an algebra action is denoted by juxtaposition. An $A$-module map $M \to M'$ is a $k$-module map which is compatible with the $A$-actions. The category of right $A$-modules is denoted by $\mathcal{M}_A$. For its hom sets the notation $\text{Hom}_A(-,-)$ is used. Left $A$-modules are defined symmetrically. Their category is denoted by $\mathcal{M}^A$, and the hom sets by $A\text{Hom}(-,-)$. For two $k$-algebras $A$ and $B$, an $A$-$B$ bimodule is a left $A$-module and right $B$-module, such that the left $A$-action $A \otimes_k M \to M$ is a right $B$-module map (equivalently, the right $B$-action $M \otimes_k B \to M$ is a left $A$-module map). An $A$-$B$ bimodule map is a map of left $A$-modules and right $B$-modules. The category of $A$-$B$ bimodules is denoted by $A\mathcal{M}_B$ and the hom sets by $A\text{Hom}_B(-,-)$.

A right comodule of a $k$-coalgebra $C$ is a $k$-module $M$ equipped with a $k$-linear coassociative and counital coaction $\Delta: M \to M \otimes_k C$. On elements a Sweedler type index notation is used for a coaction: $\Delta(m) = m_{(0)} \otimes_k m_{(1)}$ – implicit summation understood. A $C$-comodule map $M \to M'$ is a $k$-module map which is compatible with the $C$-coactions. The category of right $C$-comodules is denoted by $\mathcal{M}^C$. For its hom sets the notation $\text{Hom}^C(-,-)$ is used. Left $C$-comodules are defined symmetrically. Their category is denoted by $C\mathcal{M}$, and the hom sets by $C\text{Hom}(-,-)$. For two $k$-coalgebras $C$ and $D$, a $C$-$D$ bicomodule is a left $C$-comodule and right $D$-comodule $M$, such that the left $C$-coaction $C \otimes_k M \to M$ is a right $D$-comodule map (equivalently, the right $D$-coaction $M \otimes_k D \to M$ is a left $C$-comodule map). A $C$-$D$ bicomodule
map is a map of left C-comodules and right D-comodules. The category of C-D bicomodules is denoted by \( C \mathcal{M}^D \) and the hom sets by \( C \mathcal{H}om^D(\cdot, \cdot) \).

We work also with bicomodules of a mixed type. For a k-algebra \( A \) and a k-coalgebra \( C \), an A-C bicomodule is a left \( A \)-module and right \( C \)-comodule \( M \), such that the left \( A \)-action \( A \otimes_k M \rightarrow M \) is a right \( C \)-comodule map (equivalently, the right \( C \)-coaction \( M \rightarrow M \otimes_k C \) is a left \( A \)-module map). An A-C bicomodule map is a map of left \( A \)-modules and right \( C \)-comodules. The category of A-C bicomodules is denoted by \( A \mathcal{M}^C \) and the hom sets by \( A \mathcal{H}om^C(\cdot, \cdot) \).

For any algebra \( R \), the category \( R \mathcal{M}R \) of \( R-R \) bimodules is monoidal. Monoidal product is given by the \( R \)-module tensor product, with monoidal unit the regular bimodule. Hence the notion of a \( k \)-algebra (i.e. a monoid in the monoidal category of \( k \)-modules) can be extended to an arbitrary (non-commutative) base algebra \( R \), as follows. An \( R \)-ring is a monoid in \( R \mathcal{M}R \). By definition, it means an \( R-R \) bimodule \( A \) equipped with an \( R-R \) bilinear associative multiplication \( \mu : A \otimes_R A \rightarrow A \), with \( R-R \) bilinear unit \( \eta : R \rightarrow A \). Note that an \( R \)-ring \((A, \mu, \eta)\) can be characterized equivalently by a \( k \)-algebra structure in \( A \) and a \( k \)-algebra map \( \eta : R \rightarrow A \). A right module of an \( R \)-ring \( A \) is defined as an algebra for the monad \( \eta \otimes_A \cdot : \mathcal{M}R \rightarrow \mathcal{M}R \). This notion coincides with the one of a module for the respective \( k \)-algebra \( A \).

Later in the paper we will be particularly interested in rings over a base algebra \( R \otimes_k R^{op} \), i.e. the tensor product of an algebra \( R \) and its opposite \( R^{op} \), which is the same \( k \)-module \( R \) with opposite multiplication. In this case the unit map \( \eta : R \otimes_k R^{op} \rightarrow A \) can be equivalently given by its restrictions \( s := \eta(- \otimes_k 1_R) : R \rightarrow A \) and \( t := \eta(1_R \otimes_k -) : R^{op} \rightarrow A \), with commuting ranges in \( A \). The algebra maps \( s \) and \( t \) are called the source and target maps, respectively. Thus an \( R \otimes_k R^{op} \)-ring is given by a triple \((A, s, t)\), consisting of a \( k \)-algebra \( A \) and \( k \)-algebra maps \( s : R \rightarrow A \) and \( t : R^{op} \rightarrow A \), with commuting ranges in \( A \).

Just as the notion of a \( k \)-algebra can be generalized to an \( R \)-ring, also that of a \( k \)-coalgebra can be generalized to an \( R \)-co-ring, for an arbitrary (non-commutative) base algebra \( R \). By definition, an \( R \)-co-ring is a comonoid in \( R \mathcal{M}R \). It means an \( R-R \) bimodule \( C \) equipped with an \( R-R \) coassociative comultiplication \( \Delta : C \rightarrow C \otimes_R C \), with \( R-R \) counital coaction \( \varepsilon : C \rightarrow R \). Analogously to the coalgebra case, on elements \( c \) of \( C \) Sweedler’s index notation is used for comultiplication: \( \Delta(c) = c(1) \otimes_R c(2) \) – implicit summation understood. A right comodule of an \( R \)-co-ring \( C \) is defined as a coalgebra for the comonad \( \varepsilon \otimes_R C : \mathcal{M}C \rightarrow \mathcal{M}C \). It means a right \( R \)-module \( M \) equipped with a right \( R \)-linear coassociative and counital coaction \( M \rightarrow M \otimes_R C \). On elements a Sweedler type index notation is used for a coaction of a coring: we write \( m \mapsto m_{[0]} \otimes_R m_{[1]} \) – implicit summation understood. A right \( C \)-comodule map \( M \rightarrow M' \) is a right \( R \)-module map which is compatible with the \( C \)-coactions. The category of right \( C \)-comodules is denoted by \( \mathcal{M}C \). For its hom sets the notation \( \text{Hom}^C(\cdot, \cdot) \) is used. Left comodules and bicomodules are defined analogously to the coalgebra case and also notations are analogous.

An element \( g \) of an \( R \)-co-ring \( C \) is said to be grouplike if \( \Delta(g) = g \otimes_R g \) and \( \varepsilon(g) = 1_R \). Recall a bijective correspondence \( g \mapsto (r \mapsto gr) \) between grouplike elements \( g \) in \( C \) and right \( C \)-coactions in \( R \), cf. [14] Lemma 5.1.

A \( k \)-algebra map \( \phi : R \rightarrow R' \) induces an \( R-R \) bimodule structure in any \( R' \)-\( R' \) bimodule. What is more, it induces a canonical epimorphism \( \omega_\phi : M \otimes_R N \rightarrow M \otimes_{R'} N \), for any \( R' \)-\( R' \) bimodules \( M \) and \( N \). A map from an \( R \)-ring \((A, \mu, \eta)\) to an \( R' \)-ring \((A', \mu', \eta')\) consists of a \( k \)-algebra map \( \phi : R \rightarrow R' \) and an \( R-R \) bimodule map \( \Phi : A \rightarrow A' \) (where the \( R-R \) bimodule structure of \( A' \) is induced by \( \phi \)), such that

\[
\Phi \circ \eta = \eta' \circ \phi, \quad \Phi \circ \mu = \mu' \circ \omega_\phi \circ (\Phi \otimes \Phi).
\]

Note that \( \Phi \) is necessarily a \( k \)-algebra map with respect to the canonical \( k \)-algebra structures of
A and $A'$.

Dually, a map from an $R$-coring $(C, \Delta, \varepsilon)$ to an $R'$-coring $(C', \Delta', \varepsilon')$ consists of a $k$-algebra map $\phi : R \to R'$ and an $R$-$R$ bimodule map $\Phi : C \to C'$ (where the $R$-$R$ bimodule structure of $C'$ is induced by $\phi$), such that

$$
\varepsilon' \circ \Phi = \phi \circ \varepsilon, \quad \Delta' \circ \Phi = \omega \phi \circ (\Phi \otimes R) \circ \Delta.
$$

The notion of a coring extension was introduced in [15, Definition 2.1], as follows. An $R$-coring $D$ is a right extension of an $A$-coring $C$ if $C$ is a $C$-$D$ bicomodule, with left $C$-coaction provided by the coproduct and some right $D$-coaction. Consider an $R$-coring $(D, \Delta_D, \varepsilon_D)$, which is a right extension of an $A$-coring $(C, \Delta_C, \varepsilon_C)$. If the equalizer

$$
\begin{array}{ccc}
M & \overset{\rho}{\longrightarrow} & M \otimes_A C \\
& \downarrow{\rho \otimes_A C} & \downarrow{M \otimes_A \Delta_C} \\
& M \otimes_A C \otimes_A C
\end{array}
$$

in $\mathcal{M}_R$ is $D \otimes_R D$-pure, i.e. it is preserved by the functor $- \otimes_R D \otimes_R D : \mathcal{M}_R \to \mathcal{M}_R$, for any right $C$-comodule $(M, \rho)$, then we say that $D$ is a pure coring extension of $C$. By [18, 22.3] and its Erratum, by taking cotensor products over $C$, a pure coring extension $D$ of $C$ induces a $k$-linear functor $U = - \otimes_C C : M^C \to M^D$ that commutes with the forgetful functors $M^C \to M_k$ and $M^D \to M_k$, cf. the arXiv version of [15, Theorem 2.6].

### 2 Definitions and examples

In this section various notions of Galois extensions are reviewed. We start with the most classical notion of a Hopf Galois extension. The definition is formulated in such a way which is appropriate for generalizations. Generalizations are made in two directions. First, the bialgebra symmetry in a Hopf Galois extension is weakened to a coalgebra – still over a commutative base. Then the base ring is allowed to be non-commutative, so Galois extensions by corings are introduced. It is understood then how the particular case of a Galois extension by a bialgebroid is obtained. Finally we study Hopf algebroid Galois extensions, first of all the roles of the two constituent bialgebroids.

#### 2.1 Hopf Galois extensions

Galois extensions of non-commutative algebras by a Hopf algebra have been introduced in [21] and [28], generalizing Galois extensions of commutative rings by groups. Hopf Galois extensions unify several structures (including strongly group graded algebras), studied independently earlier. Beyond an algebraic importance, Hopf Galois extensions are relevant also from the (non-commutative) geometric point of view. A Hopf Galois extension (if it is faithfully flat) can be interpreted as a (dual version of a) non-commutative principal bundle.

**Definition 2.1.** A bialgebra over a commutative ring $k$ is a $k$-module $H$, together with a $k$-algebra structure $(H, \mu, \eta)$ and a $k$-coalgebra structure $(H, \Delta, \varepsilon)$ such that the counit $\varepsilon : H \to k$ and the coproduct $\Delta : H \to H \otimes_k H$ are $k$-algebra homomorphisms with respect to the tensor product algebra structure of $H \otimes_k H$. Equivalently, the unit $\eta : k \to H$ and the product $\mu : H \otimes_k H \to H$ are $k$-coalgebra homomorphisms with respect to the tensor product coalgebra structure of $H \otimes_k H$. 
A morphism of bialgebras is an algebra and coalgebra map.

A bialgebra $H$ is a Hopf algebra if there exists a $k$-module map $S : H \to H$, called the antipode, such that

$$
\mu \circ (S \otimes H) \circ \Delta = \eta \circ \varepsilon = \mu \circ (H \otimes S) \circ \Delta.
$$

Since both the coproduct and the counit of a bialgebra $H$ are unital maps, the ground ring $k$ is an $H$-comodule via the unit map $k \to H$. Furthermore, since both the coproduct and the counit are multiplicative, the $k$-module tensor product of two right $H$-comodules $M$ and $N$ is an $H$-comodule, with the so-called diagonal coaction $m \otimes_k n \mapsto m_{[0]} \otimes_k n_{[0]} \otimes_k m_{[1]} n_{[1]}$. Since the coherence natural transformations in $\mathcal{M}_k$ turn out to be $H$-comodule maps with respect to these coactions, the following theorem holds.

**Theorem 2.2.** For a $k$-bialgebra $H$, the category of (left or right) comodules is a monoidal category, with a strict monoidal forgetful functor to $\mathcal{M}_k$.

Theorem 2.2 allows us to introduce a structure known as an algebra extension by a bialgebra.

**Definition 2.3.** A right comodule algebra of a $k$-bialgebra $H$ is a monoid in the monoidal category of right $H$-comodules. That is, an algebra and right $H$-comodule $A$, whose multiplication and unit maps are right $H$-comodule maps. Equivalently, $A$ is an algebra and right $H$-comodule such that the coaction $\rho^A : A \to A \otimes_k H$ is a $k$-algebra map, with respect to the tensor product algebra structure in $A \otimes_k H$.

The coinvariants of a right $H$-comodule algebra $A$ are the elements of $A^{coH} \equiv \{ b \in A \mid \rho^A(ba) = b\rho^A(a), \quad \forall a \in A \}$. Clearly, $A^{coH}$ is a $k$-subalgebra of $A$. We say that $A$ is a (right) extension of $A^{coH}$ by $H$.

Note that, by the existence of a unit element $1_H$ in a $k$-bialgebra $H$, coinvariants of a right comodule algebra $A$ (with coaction $\rho^A$) can be equivalently described as those elements $b \in A$ for which $\rho^A(b) = b \otimes_k 1_H$.

Let $H$ be a $k$-bialgebra, $A$ a right $H$-comodule algebra and $B := A^{coH}$. The right $H$-coaction $\rho^A$ in $A$ can be used to introduce a canonical map

$$
\text{can} : A \otimes_k A \to A \otimes_k H \quad a \otimes a' \mapsto a\rho^A(a').
$$

**Definition 2.4.** A $k$-algebra extension $B \subseteq A$ by a bialgebra $H$ is said to be a Hopf Galois extension (or $H$-Galois extension) provided that the canonical map (3) is bijective.

An important comment has to be made at this point. Although the structure introduced in Definition 2.4 is called a Hopf Galois extension $B \subseteq A$ (by a $k$-bialgebra $H$), the involved bialgebra $H$ is not required to be a Hopf algebra. However, in the most interesting case when $A$ is a faithfully flat $k$-module, $H$ can be proven to be a Hopf algebra, cf. [31, Theorem].

**Example 2.5.** (1) Let $H$ be a bialgebra with coproduct $\Delta$ and counit $\varepsilon$. The algebra underlying $H$ is a right $H$-comodule algebra, with coaction provided by $\Delta$. Coinvariants are multiples of the unit element. If $H$ is a Hopf algebra with antipode $S$, then the canonical map

$$
\text{can} : H \otimes_k H \to H \otimes_k H, \quad h \otimes h' \mapsto hh'(1) \otimes hh'(2)
$$

is bijective, with inverse $h \otimes_k h' \mapsto hS(h''(1)) \otimes_k h''(2)$. This proves that $k \subseteq H$ is an $H$-Galois extension. Conversely, if the canonical map is bijective, then $H$ is a Hopf algebra with antipode $(H \otimes_k \varepsilon) \circ \text{can}^{-1}(1_H \otimes_k -)$. 

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(2) Galois extensions of fields. Let \( G \) be a finite group acting by automorphisms on a field \( F \). That is, there is a group homomorphism \( G \to \text{Aut}(F) \), \( g \mapsto \alpha_g \). For any subfield \( k \) of \( F \), \( F \) is a right comodule algebra of \( k(G) \), the Hopf algebra of \( k \)-linear functions on \( G \). The coaction is given by \( a \mapsto \sum_{g \in G} \alpha_g(a) \otimes \delta_g \), where the function \( \delta_g \in k(G) \) takes the value \( 1_k \) on \( g \in G \) and 0 everywhere else. Denote \( B := F^{\text{cok}(G)} \). Then \( F \) is a \( k(G) \)-Galois extension of \( B \). That is, the canonical map
\[
F \otimes F \to F \otimes_k k(G), \quad a \otimes a' \mapsto \sum_{g \in G} a \alpha_g(a') \otimes \delta_g
\]
is bijective [22 Example 6.4.3].

(3) Let \( A := \bigoplus_{g \in G} A_k \) be a \( k \)-algebra graded by a finite group \( G \). \( A \) has a natural structure of a comodule algebra of the group Hopf algebra \( kG \), with coaction induced by the map \( a_k \mapsto a_g \otimes k \), on \( a_g \in A_k \). The \( kG \)-coinvariants of \( A \) are the elements of \( A_{1G} \), the component at the unit element \( 1_G \) of \( G \). It is straightforward to see that the canonical map \( A \otimes_{A_{1G}} A \to A \otimes_k kG \) is bijective if and only if \( A_g A_h = A_{gh} \), for all \( g, h \in G \), that is \( A \) is strongly graded.

2.2 Galois extensions by coalgebras

Comodules of any coalgebra \( C \) over a commutative ring \( k \) do not form a monoidal category. Hence one can not speak about comodule algebras. Still, as it was observed in [34], [17] and [16], there is a sensible notion of a Galois extension of algebras by a coalgebra.

**Definition 2.6.** Consider a \( k \)-coalgebra \( C \) and a \( k \)-algebra \( A \) which is a right \( C \)-comodule via some coaction \( \rho^A : A \to A \otimes_k C \). The coinvariants of \( A \) are the elements of the \( k \)-subalgebra \( A^{\text{coC}} = \{ b \in A \mid \rho^A(ba) = b \rho^A(a), \ \forall a \in A \} \). We say that \( A \) is a (right) extension of \( A^{\text{coC}} \) by \( C \).

Let \( B \subseteq A \) be an algebra extension by a coalgebra \( C \). One can use the same formula (3) to define a canonical map in terms of the \( C \)-coaction \( \rho^A \) in \( A \),
\[
\text{can} : A \otimes B \to A \otimes_k C \quad a \otimes b \mapsto a \otimes \rho^A(b).
\]

**Definition 2.7.** A \( k \)-algebra extension \( B \subseteq A \) by a coalgebra \( C \) is said to be a coalgebra Galois extension (or \( C \)-Galois extension) provided that the canonical map (4) is bijective.

**Remark 2.8.** In light of Definition 2.7, a Hopf Galois extension \( B \subseteq A \) by a bialgebra \( H \) in Definition 2.4 is the same as a Galois extension by the coalgebra underlying \( H \), such that in addition \( A \) is a right \( H \)-comodule algebra.

**Example 2.9.** Extending [34] Example 3.6 (and thus Example 2.5(1)), coalgebra Galois extensions can be constructed as in [18] 34.2. Examples of this class are called quantum homogeneous spaces. Let \( H \) be a Hopf algebra, with coproduct \( \Delta \), counit \( \varepsilon \) and antipode \( S \). Assume that \( H \) is a flat module of the base ring \( k \). Let \( A \) be subalgebra and a left coideal in \( H \). Denote by \( A^+ \) the augmentation ideal, i.e. the intersection of \( A \) with the kernel of \( \varepsilon \). The quotient of \( H \) with respect to the coideal and right ideal \( A^+ H \) is a coalgebra and right \( H \)-module \( \overline{H} \). The canonical epimorphism \( \pi : H \to \overline{H} \) induces a right \( \overline{H} \)-coaction \( (H \otimes_k \pi) \circ \Delta \) on \( H \). Denote the \( \overline{H} \)-coinvariant subalgebra of \( H \) by \( B \). Clearly, \( A \subseteq B \). Hence the map
\[
H \otimes \overline{H} \to H \otimes_k B, \quad h \otimes \pi(h') \mapsto h S(h'(1)) \otimes h'(2)
\]
is well defined and yields the inverse of the canonical map

\[ H \otimes H \to H \otimes \overline{H}, \quad h \otimes h' \mapsto hh'_1(1) \otimes \pi(h'_2). \]

This proves that \( B \subseteq H \) is a Galois extension by \( \overline{H} \). Geometrically interesting examples of this kind are provided by Hopf fibrations over the Podleś quantum spheres \([11], [30]\).

### 2.3 Galois extensions by corings

If trying to understand what should be called a Galois extension by a coring, one faces a similar situation as is the coalgebra case: comodules of an arbitrary coring \( C \) over an algebra \( R \) do not form a monoidal category. As the problem, also the answer is analogous.

**Definition 2.10.** For a \( k \)-algebra \( R \), consider an \( R \)-coring \( C \) and an \( R \)-ring \( A \) which is a right \( C \)-comodule via some coaction \( \rho^A : A \to A \otimes_R C \). Require the right \( R \)-actions on \( A \), corresponding to the \( R \)-ring, and to the \( C \)-comodule structures, to be the same. The coinvariants of \( A \) are the elements of the \( k \)-subalgebra \( A^{coC} = \{ b \in A \mid \rho^A(ba) = b \rho^A(a), \ \forall a \in A \} \). The \( k \)-algebra \( A \) is said to be a (right) extension of \( A^{coC} \) by \( C \).

It should be emphasized that though \( A^{coC} \) in Definition 2.10 is a \( k \)-subalgebra of \( A \), it is not an \( R \)-subring (not even a one-sided \( R \)-submodule) in general.

Let \( B \subseteq A \) be a \( k \)-algebra extension by an \( R \)-coring \( C \). Analogously to (4), one defines a canonical map in terms of the \( C \)-coaction \( \rho^A \) in \( A 

\[ \text{can} : A \otimes_B A \to A \otimes C, \quad a \otimes a' \mapsto a \rho^A(a'). \] (5)

**Definition 2.11.** A \( k \)-algebra extension \( B \subseteq A \) by an \( R \)-coring \( C \) is said to be a coring Galois extension (or \( C \)-Galois extension) provided that the canonical map (5) is bijective.

**Example 2.12.** In fact, in view of \([18, 28.6]\), any algebra extension \( B \subseteq A \), in which \( A \) is a faithfully flat left or right \( B \)-module, is a Galois extension by a coring, cf. a comment following \([18, 28.19]\). Indeed, the obvious \( A \)-\( A \) bimodule \( C := A \otimes_B A \) is an \( A \)-coring, with coproduct and counit

\[ \Delta : C \to C \otimes C, \quad a \otimes a' \mapsto (a \otimes 1_A) \otimes (1_A \otimes a'), \quad \text{and} \quad \varepsilon : C \to A, \quad a \otimes a' \mapsto aa'. \]

By the existence of a grouplike element \( 1_A \otimes_B 1_A \) in \( C \), \( A \) is a right \( C \)-comodule via the coaction \( A \to C, \ a \mapsto 1_A \otimes_B a \). The coinvariants are those elements \( b \in A \) for which \( b \otimes_B 1_A = 1_A \otimes_B b \). Hence, by the faithful flatness assumption made, \( A^{coC} = B \). The canonical map is then the identity map \( A \otimes_B A \), which is obviously bijective.

### 2.4 Bialgebroid Galois extensions

In Remark \([2,3]\) we characterized Hopf Galois extensions as Galois extensions \( B \subseteq A \) by a constituent coalgebra in a bialgebra \( H \) such that \( A \) is in addition an \( H \)-comodule algebra. The aim of the current section is to obtain an analogous description of Galois extensions by a bialgebroid \( \mathcal{H} \), replacing the bialgebra \( H \) in Section \([2,2]\). In order to achieve this goal, a proper notion of a ‘bialgebra over a non-commutative algebra \( R \)’ is needed – such that the category of comodules is monoidal.
In Section 2.2 we could easily repeat considerations in Section 2.1 by replacing $k$-coalgebras – i.e. comonoids in the monoidal category $\mathcal{M}_k$ of modules over a commutative ring $k$ – by $R$-corings – i.e. comonoids in the monoidal category $R\mathcal{M}_R$ of bimodules over a non-commutative algebra $R$. There is no such simple way to generalize the notion of a bialgebra to a non-commutative base algebra $R$, by the following reason. While the monoidal category $\mathcal{M}_k$ is also symmetrical, not the bimodule category $R\mathcal{M}_R$. One can not consider bimonoids (bialgebras) in $R\mathcal{M}_R$. In order to define a non-commutative base analogue of the notion of a bialgebra, more sophisticated ideas are needed. The right definition was proposed in [35] and independently in [29].

**Definition 2.13.** A (right) bialgebroid over a $k$-algebra $R$ consists of an $R \otimes_k R^{\text{op}}$-ring structure $(H, s, t)$ and an $R$-coring structure $(H, \Delta, \varepsilon)$ on the same $k$-module $H$, such that the following compatibility axioms hold.

(i) The $R$-$R$ bimodule structure of the $R$-coring $H$ is related to the $R \otimes_k R^{\text{op}}$-ring structure via

$$rh' = hs(r')t(r) \quad \text{for } r, r' \in R \quad h \in H.$$  

(ii) The coproduct $\Delta$ corestricts to a morphism of $R \otimes_k R^{\text{op}}$-rings

$$H \to H \times_R H \equiv \{ \sum_i h_i \otimes h_i' \mid \sum_i s(r)h_i \otimes h_i' = \sum_i h_i \otimes t(r) h_i', \quad \forall r \in R \},$$

where $H \times_R H$ is an $R \otimes_k R^{\text{op}}$-ring via factorwise multiplication and unit map $R \otimes_k R^{\text{op}} \to H \times_R H$, $r \otimes r' \mapsto t(r') \otimes_R s(r)$.

(iii) The counit determines a morphism of $R \otimes_k R^{\text{op}}$-rings

$$H \to \text{End}_k(R)^{\text{op}}, \quad h \mapsto \varepsilon(s(-)h),$$

where $\text{End}_k(R)^{\text{op}}$ is an $R \otimes_k R^{\text{op}}$-ring via multiplication given by opposite composition of endomorphisms and unit map $R \otimes_k R^{\text{op}} \to \text{End}_k(R)^{\text{op}}$, $r \otimes r' \mapsto r'(-)r$.

A morphism from an $R$-bialgebroid $\mathcal{H}$ to an $R'$-bialgebroid $\mathcal{H}'$ is a pair consisting of a $k$-algebra map $\Phi : R \to R'$ and an $R \otimes_k R^{\text{op}}-R \otimes_k R^{\text{op}}$ bimodule map $\Phi : H \to H'$, such that $(\Phi \otimes_k \Phi^{\text{op}} : R \otimes_k R^{\text{op}} \to R' \otimes_k R^{\text{op}}, \Phi : H \to H')$ is a map from an $R \otimes_k R^{\text{op}}$-ring to an $R' \otimes_k R^{\text{op}}$-ring and $(\Phi : R \to R', \Phi : H \to H')$ is a map from an $R$-coring to an $R'$-coring.

A new feature of Definition 2.13 compared to Definition 2.11 is that here the monoid (ring) and comonoid (coring) structures are defined in different categories. This results in the quite involved form of the compatibility axioms. In particular, the $R$-module tensor product $H \otimes_R H$ is not a monoid in any category, hence the coproduct itself can not be required to be a ring homomorphism. It has to be corestricted to the so called Takeuchi product $H \times_R H$, which is an $R \otimes_k R^{\text{op}}$-ring, indeed. Similarly, it is not the counit itself which is a ring homomorphism, but the related map in axiom (iii).

Changing the comultiplication in a bialgebra to the opposite one (and leaving the algebra structure unmodified) we obtain another bialgebra. If working over a non-commutative base algebra $R$, one can replace the $R$-coring $(H, \Delta, \varepsilon)$ in a right $R$-bialgebroid with the co-opposite $R^{\text{op}}$-coring. Together with the $R^{\text{op}} \otimes_k R$-ring $(H, t, s)$ (roles of $s$ and $t$ are interchanged!) they form a right $R^{\text{op}}$-bialgebroid.
Similarly, changing the multiplication in a bialgebra to the opposite one (and leaving the coalgebra structure unmodified) we obtain another bialgebra. Obviously, Definition 2.13 is not invariant under the change of the $R \otimes k R^{op}$-ring structure $(H, s, t)$ to $(H^{op}, t, s)$ (and leaving the coring structure unmodified). The $R \otimes k R^{op}$-ring $(H^{op}, t, s)$ and the $R$-coring $(H, \Delta, \varepsilon)$ satisfy symmetrical versions of the axioms in Definition 2.13. The structure in Definition 2.13 is usually termed a right $R$-bialgebroid and the opposite structure is called a left $R$-bialgebroid. For more details we refer to [27].

A most important feature of a bialgebroid for our application is formulated in following Theorem 2.14. Comodules of an $R$-bialgebroid are meant to be comodules of the constituent $R$-coring. Theorem 2.14 below was proven first in [32] Proposition 5.6, using an apparently more restrictive but in fact equivalent definition of a comodule. The same version of Theorem 2.14 presented here can be found in Section 2.2 of [6] or [2, Proposition 1.1].

**Theorem 2.14.** For a right $R$-bialgebroid $\mathcal{H}$, the category of right comodules is a monoidal category, with a strict monoidal forgetful functor to $r\mathcal{M}_R$.

**Proof.** (Sketch.) Let $\mathcal{H}$ be a right $R$-bialgebroid with $R \otimes k R^{op}$-ring structure $(H, s, t)$ and $R$-coring structure $(H, \Delta, \varepsilon)$. By the right $R$-linearity and unitality of the counit and the coproduct, the base ring $R$ is a right comodule via the source map $s : R \rightarrow H$. A right $\mathcal{H}$-comodule $M$ (with coaction $m \mapsto m^{[0]} \otimes_R m^{[1]}$) is a priori only a right $R$-module. Let us introduce a left $R$-action

$$rm := m^{[0]} \varepsilon(s(r)m^{[1]}), \quad \text{for } r \in R, m \in M. \quad (6)$$

On checks that $M$ becomes an $R$-$R$ bimodule in this way, and any $H$-comodule map becomes $R$-$R$ linear. Thus we have a forgetful functor $\mathcal{M}^{\mathcal{H}} \rightarrow r\mathcal{M}_R$. What is more, (6) implies that, for any $m \in M$ and $r \in R$,

$$rm^{[0]} \otimes_R m^{[1]} = m^{[0]} \otimes_R t(r)m^{[1]}.$$ 

Hence the $R$-module tensor product of two right $\mathcal{H}$-comodules $M$ and $N$ can be made a right $\mathcal{H}$-comodule with the so called diagonal coaction

$$m \otimes_R n \mapsto m^{[0]} \otimes_R n^{[0]} \otimes_R m^{[1]} n^{[1]}. \quad (7)$$

Coassociativity and counitality of the coaction (7) easily follow by Definition 2.13. The proof is completed by checking the right $\mathcal{H}$-comodule map property of the coherence natural transformations in $r\mathcal{M}_R$ with respect to the coactions above. □

Let $\mathcal{H}$ be a right bialgebroid. The category of right $\mathcal{H}$-comodules, and the category of left comodules for the co-opposite right bialgebroid $\mathcal{H}^{op}$, are monoidally isomorphic. The category of right comodules for the opposite left bialgebroid $\mathcal{H}^{op}$ is anti-monoidally isomorphic to $\mathcal{M}^{\mathcal{H}}$. Thus (using a bijective correspondence between left and right bialgebroid structures on a $k$-module $H$, given by switching the order of multiplication), we conclude by Theorem 2.14 that the category of right comodules of a left $R$-bialgebroid is monoidal, with a strict monoidal forgetful functor to $R^{op}\mathcal{M}_{R^{op}}$.

The important message of Theorem 2.14 is that there is a sensible notion of a comodule algebra of a right bialgebroid $\mathcal{H}$. Namely, a right $\mathcal{H}$-comodule algebra is a monoid in $\mathcal{M}^{\mathcal{H}}$. This means an $R$-ring and right $\mathcal{H}$-comodule $A$ (with one and the same right $R$-module structure), whose multiplication and unit maps are right $\mathcal{H}$-comodule maps. Equivalently, $A$ is an $R$-ring and right $\mathcal{H}$-comodule such that the coaction $\rho^A$ corestricts to a map of $R$-rings

$$A \rightarrow A \times_R H \equiv \{ \sum_i a_i \otimes_R h_i \in A \otimes_R H \mid \sum_i ra_i \otimes_R h_i = \sum_i a_i \otimes_R t(r)h_i, \quad \forall r \in R \}. \quad (8)$$
The Takeuchi product $A \times_R H$ is an $R$-ring with factorwise multiplication and unit map $r \mapsto 1_A \otimes_R s(r)$. Thus, in analogy with Remark 2.8 we impose the following definition.

**Definition 2.15.** A right Galois extension $B \subseteq A$ by a right $R$-bialgebroid $\mathcal{H}$ is defined as a Galois extension by the $R$-coring underlying $\mathcal{H}$, such that in addition $A$ is a right $\mathcal{H}$-comodule algebra.

**Remark 2.16.** Consider a right $R$-bialgebroid $\mathcal{H}$, with structure maps denoted as in Definition 2.13 and a right $\mathcal{H}$-comodule algebra $A$. Denote by $\eta : R \to A$ the unit of the $R$-ring $A$. By the left $B$-linearity and the unitality of the $\mathcal{H}$-coaction on $A$, for any element $b \in B := A^{co\mathcal{H}}$, $b^{[0]} \otimes_R b^{[1]} = b \otimes_R 1_H$. By the right $R$-linearity and the unitality of the $\mathcal{H}$-coaction on $A$, for any element $r \in R$, $\eta(r)^{[0]} \otimes_R \eta(r)^{[1]} = 1_A \otimes_R s(r)$. Thus by the $\mathcal{H}$-colinearity of the multiplication in $A$,

$$(b \eta(r))^{[0]} \otimes_R (b \eta(r))^{[1]} = b \otimes_R s(r) = (\eta(r)b)^{[0]} \otimes_R (\eta(r)b)^{[1]}.$$  

Applying $A \otimes_R \epsilon$ to both sides, we conclude that the subalgebra $B$ of $A$ commutes with the range of $\eta$.

**Example 2.17.** The depth 2 property of an extension of arbitrary algebras $B \subseteq A$ was introduced in [27], generalizing depth 2 extensions of $C^*$-algebras. By definition, an algebra extension $B \subseteq A$ is right depth 2 if there exists a finite integer $n$ such that $(A \otimes_B A)^{+} \cong \oplus^n A$, as $A$-$B$ bimodules. By [27] Theorem 5.2 (see also [26] Theorem 2.1), for a right depth 2 algebra extension $B \subseteq A$, the centralizer $H$ of $B$ in the obvious $B$-$B$ bimodule $A \otimes_B A$ has a right bialgebroid structure $\mathcal{H}$ over $R$, where $R$ is the commutant of $B$ in $A$. Its total algebra $H$ is a finitely generated and projective left $R$-module. What is more, if the algebra extension $B \subseteq A$ is also balanced, i.e. the endomorphism algebra of $A$ as a left $\text{End}_B(A)$-module is equal to $B$, then $B \subseteq A$ is a right $\mathcal{H}$-Galois extension. This example is a very general one. By [26] Theorem 2.1 (see also [2] Theorem 3.7), any Galois extension by a finitely generated and projective bialgebroid arises in this way.

After Definition 2.4 we recalled an observation in [31] that a bialgebra, which admits a faithfully flat Hopf Galois extension, is a Hopf algebra. Although the definition of a Hopf algebroid will be presented only in forthcoming Section 2.5 let us anticipate here that no analogous result is known about bialgebroids. As a matter of fact, the following was proven in [25] Lemma 4.1.21]. Let $\mathcal{H}$ be a right $R$-bialgebroid and $B \subseteq A$ a right $\mathcal{H}$-Galois extension such that $A$ is a faithfully flat left $R$-module. Then the total algebra $\tilde{H}$ in $\mathcal{H}$ is a right $\mathcal{H}$-Galois extension of the base algebra $R^{op}$ (via the target map). That is to say, $\mathcal{H}$ is a right $\times_R$-Hopf algebra in the terminology of [33]. However, this fact does not seem to imply the existence of a Hopf algebroid structure in $\mathcal{H}$.

### 2.5 Hopf algebroid Galois extensions

As it is well known, the antipode of a $k$-Hopf algebra $H$ is a bialgebra map from $H$ to a Hopf algebra on the same $k$-module $H$, with opposite multiplication and co-opposite comultiplication. We have seen in Section 2.4 that the opposite multiplication in a bialgebroid does not satisfy the same axioms the original product does. In fact, the opposite of a left bialgebroid is a right bialgebroid and vice versa. Thus if the antipode in a Hopf algebroid $\mathcal{H}$ is expected to be a bialgebroid map between $\mathcal{H}$ and a Hopf algebroid with opposite multiplication and co-opposite comultiplication, then one has to start with two bialgebroid structures in $\mathcal{H}$, a left and a right
one. The following definition fulfilling this requirement was proposed in [9], where the antipode was required to be bijective. The definition was extended by relaxing the requirement about bijectivity of the antipode in [5]. The set of axioms was reduced slightly further in [8] Remark 2.1.

**Definition 2.18.** A Hopf algebroid \( \mathcal{H} \) consists of a left bialgebroid \( \mathcal{H}_L \) over a base algebra \( L \) and a right bialgebroid \( \mathcal{H}_R \) over a base algebra \( R \) on the same total algebra \( H \), together with a \( k \)-module map \( S : H \to H \), called the antipode. Denote the \( L \otimes_k L^{op} \)-ring structure in \( \mathcal{H}_L \) by \( (H, s_L, t_L) \) and the \( L \)-coring structure by \( (H, \Delta_L, \varepsilon_L) \). Analogously, denote the \( R \otimes_k R^{op} \)-ring structure in \( \mathcal{H}_R \) by \( (H, s_R, t_R) \) and the \( R \)-coring structure by \( (H, \Delta_R, \varepsilon_R) \). Denote the multiplication in \( H \) (as an \( L \)-ring or as an \( R \)-ring) by \( \mu \). The compatibility axioms are

(i) The source and target maps satisfy the conditions

\[
sl \circ \varepsilon_l \circ tr = tr, \quad tl \circ \varepsilon_l \circ sl = sl, \quad sr \circ \varepsilon_r \circ tl = tl, \quad tr \circ \varepsilon_r \circ sl = sl.
\]

(ii) The two coproducts are compatible in the sense that

\[
(\Delta_L \otimes_k H) \circ \Delta_R = (H \otimes_k \Delta_R) \circ \Delta_L, \quad (\Delta_R \otimes_k H) \circ \Delta_L = (H \otimes_k \Delta_L) \circ \Delta_R.
\]

(iii) The antipode is an \( R-L \) bimodule map. That is,

\[
S(t_l(l)htr(r)) = s_r(r)S(h)sl(l), \quad \text{for } r \in R, l \in L, h \in H.
\]

(iv) The antipode axioms are

\[
\mu \circ (S \otimes_k H) \circ \Delta_L = s_r \circ \varepsilon_r, \quad \mu \circ (H \otimes_k S) \circ \Delta_R = s_l \circ \varepsilon_L.
\]

Throughout these notes the structure maps of a Hopf algebroid \( \mathcal{H} \) will be denoted as in Definition 2.18. For the two coproducts \( \Delta_L \) and \( \Delta_R \) we systematically use two versions of Sweedler’s index notation: we write \( \Delta_L(h) = h_{(1)} \otimes R h_{(2)} \) (with lower indices) and \( \Delta_R(h) = R h_{(1)} \otimes R h_{(2)} \) (with upper indices), for \( h \in H \). In both cases implicit summation is understood. The following consequences of the Hopf algebroid axioms in Definition 2.18 were observed in [5] Proposition 2.3.

**Remark 2.19.** (1) The base algebras \( L \) and \( R \) are anti-isomorphic, via the map \( \varepsilon_R \circ s_L : L \to R \) (or \( \varepsilon_R \circ t_L : L \to R \)), with inverse \( \varepsilon_L \circ t_R : R \to L \) (or \( \varepsilon_L \circ s_R : R \to L \)).

(2) The pair \( (\varepsilon_L \circ s_R : R \to L^{op}, S : H \to H^{op}) \) is a morphism of right bialgebroids, from \( \mathcal{H}_R \) to the opposite-co-opposite of \( \mathcal{H}_L \) and \( (\varepsilon_R \circ s_L : L \to R^{op}, S : H \to H^{op}) \) is a morphism of left bialgebroids, from \( \mathcal{H}_L \) to the opposite-co-opposite of \( \mathcal{H}_R \).

Since a Hopf algebroid \( \mathcal{H} \) comprises two bialgebroid structures \( \mathcal{H}_L \) and \( \mathcal{H}_R \), there are in general two different notions of their comodules. As it turns out, the right definition of an \( \mathcal{H} \)-comodule comprises both structures. The following definition was proposed in [6] Definition 3.2] and [2, Section 2.2].

**Definition 2.20.** A right comodule of a Hopf algebroid \( \mathcal{H} \) is a right \( L \)-module as well as a right \( R \)-module \( M \), together with a right coaction \( \rho_R : M \to M \otimes_R H \) of the constituent right bialgebroid \( \mathcal{H}_R \) and a right coaction \( \rho_L : M \to M \otimes_L H \) of the constituent left bialgebroid \( \mathcal{H}_L \).
such that $\rho_R$ is an $\mathcal{H}_L$-comodule map and $\rho_L$ is an $\mathcal{H}_R$-comodule map. Explicitly, $\rho_R$ is a right $L$-module map, $\rho_L$ is a right $R$-module map and

$$
(M \otimes \Delta_L) \circ \rho_R = (\rho_R \otimes H) \circ \rho_L \quad \text{and} \quad (M \otimes \Delta_R) \circ \rho_L = (\rho_L \otimes H) \circ \rho_R.
$$

(8)

Morphisms of $\mathcal{H}$-comodules are meant to be $\mathcal{H}_R$-comodule maps as well as $\mathcal{H}_L$-comodule maps. The category of right $\mathcal{H}$-comodules is denoted by $\mathcal{M}^{\mathcal{H}}$.

In the sequel we fix the following notation. For a Hopf algebroid $\mathcal{H}$, with constituent right bialgebroid $\mathcal{H}_R$ and left bialgebroid $\mathcal{H}_L$, and a right $\mathcal{H}$-comodule $M$, for $m \in M$ we write $m \mapsto m_0 \otimes_R m_1$ and $m \mapsto m_0 \otimes_L m_1$ for the $\mathcal{H}_R$- and $\mathcal{H}_L$-coactions related by (8). In both cases implicit summation is understood.

**Proposition 2.21.** For a Hopf algebroid $\mathcal{H}$, with constituent right bialgebroid $\mathcal{H}_R$ and left bialgebroid $\mathcal{H}_L$, the forgetful functor $\mathcal{M}^{\mathcal{H}} \to \mathcal{M}^{\mathcal{H}_L}$ is fully faithful.

**Proof.** The forgetful functor $\mathcal{M}^{\mathcal{H}} \to \mathcal{M}^{\mathcal{H}_L}$ is obviously faithful. In order to see that it is also full, consider two $\mathcal{H}$-comodules $M$ and $M'$, and an $\mathcal{H}_R$-comodule map $f : M \to M'$. That is, a right $R$-module map $f$, such that, for all $m \in M$,

$$
f(m)_{[0]} \otimes f(m)_{[1]} = f(m_{[0]}) \otimes f(m_{[1]}).
$$

(9)

By definition, $M$ is a right $L$-module and by the right $L$-linearity of the $\mathcal{H}_R$-coaction, $ml = m_{[0]}e_R(l_{[0]}m_{[1]})$. Similarly, $M'$ is a right $L$-module and $f$ is clearly right $L$-linear. Regard $H$ as a left $L$-module via $\alpha_L$ and as a left $R$-module via $\alpha_R$. Consider the well defined map

$$
\Phi_{M'} : M' \otimes_R H \to M' \otimes_L H, \quad m' \otimes h \mapsto m'_{[0]} \otimes m'_{[1]} S(h).
$$

(10)

Applying $\Phi_{M'}$ to both sides of (9) and using the right $\mathcal{H}_R$-colinearity of the $\mathcal{H}_L$-coaction on $M'$ and one of the antipode axioms to simplify the left hand side, we obtain the identity

$$
f(m)_{[0]} \otimes 1_H = f(m_{[0]})_{[0]} \otimes f(m_{[1]}), \quad m \in M.
$$

(11)

for all $m \in M$. By the right $L$-linearity of $f$, (11) can be used to compute

$$
f(m_{[0]})_{[0]} \otimes m_{[1]} = f(m_{[0]})_{[0]} \otimes f(m_{[1]}) = f(m_{[0]})_{[0]} \otimes f(m_{[1]}) = f(m_{[0]})_{[0]} \otimes f(m_{[1]}),
$$

for all $m \in M$. In the first equality we applied (11). The second equality follows by the right $\mathcal{H}_L$-colinearity of the $\mathcal{H}_R$-coaction on $M$. In the third equality we used one of the antipode axioms in a Hopf algebroid. The penultimate equality follows by the right $R$-linearity of the $\mathcal{H}_L$-coaction on $M'$. The last equality follows by the right $R$-linearity of $f$ and the counitality of the $\mathcal{H}_R$-coaction on $M$. This proves that $f$ is right $\mathcal{H}_L$-colinear, hence it is a morphism in $\text{Hom}^{\mathcal{H}_L}(M, M')$, as stated.

As a simple consequence of Proposition 2.21, we obtain a Corrigendum, Proposition 3].
Corollary 2.22. Let $\mathcal{H}$ be a Hopf algebra and $M$ be a right $\mathcal{H}$-comodule. Then any coinvariant of the $\mathcal{H}_R$-comodule $M$ is coinvariant also for the $\mathcal{H}_L$-comodule $M$. If moreover the antipode of $\mathcal{H}$ is bijective then coinvariants of the $\mathcal{H}_R$-comodule $M$ and the $\mathcal{H}_L$-comodule $M$ coincide.

Proof. The base algebra $R$ of the constituent right bialgebra $\mathcal{H}_R$ is a right $\mathcal{H}_R$-comodule via the right regular $R$-action and the coaction $R \to R \otimes_R H \cong H$, $r \mapsto 1_R \otimes_R s_R(r) \cong s_R(r)$ (determined by the grouplike element $1_H$). Moreover, $R$ is also a right comodule of the constituent left $L$-bialgebra $\mathcal{H}_L$, with right $L$-action $R \otimes L \to R$, $r \otimes l \mapsto e_R(t_L(l)) r$ and coaction $R \to R \otimes_L H \cong H$, $r \mapsto 1_R \otimes_L s_R(r) \cong s_R(r)$. With these structures $R$ is a right $\mathcal{H}$-comodule. Similarly, $L$ is a right $\mathcal{H}$-comodule via the right regular $L$-action and $\mathcal{H}_L$-coaction $L \to L \otimes_R H \cong H$, $l \mapsto 1_L \otimes_R t_L(l) \cong t_L(l)$, the right $\mathcal{H}$-comodule of $\mathcal{H}_R$-comodule $L \to L \otimes_R H \cong H$, $l \mapsto 1_L \otimes_R t_L(l) \cong t_L(l)$. The map $e_R \circ t_L : L \to R$ is an isomorphism of $\mathcal{H}$-comodules with the inverse $e_L \circ s_R$.

By [18 28.4], for any right $\mathcal{H}$-comodule $M$ there are isomorphisms $M^{co\mathcal{H}_R} \cong \text{Hom}_{\mathcal{H}_R}^0(R, M)$ and $M^{co\mathcal{H}_L} \cong \text{Hom}_{\mathcal{H}_L}^0(L, M)$. Therefore, the following sequence of isomorphisms and inclusions holds.

\[
M^{co\mathcal{H}_R} \cong \text{Hom}_{\mathcal{H}_R}^0(R, M) \subset \text{Hom}_{\mathcal{H}_L}^0(L, M) \cong M^{co\mathcal{H}_L}.
\]

The inclusion in the second step follows by Proposition 2.21. If the antipode is bijective then the same reasoning can be applied to the opposite Hopf algebra to conclude that also $M^{co\mathcal{H}_L} \subset M^{co\mathcal{H}_R}$.

In order to have a meaningful notion of a comodule algebra of a Hopf algebra $\mathcal{H}$, the category of $\mathcal{H}$-comodules has to be monoidal. The following was proven in [8 Corrigendum, Theorem 6].

Theorem 2.23. For any Hopf algebra $\mathcal{H}$, $\mathcal{M}^{\mathcal{H}}$ is a monoidal category. Moreover, there are strict monoidal forgetful functors rendering commutative the following diagram:

\[
\begin{array}{ccc}
\mathcal{M}^{\mathcal{H}} & \rightarrow & \mathcal{M}^{\mathcal{H}_R} \\
\downarrow & & \downarrow \\
\mathcal{M}^{\mathcal{H}_L} & \rightarrow & R \mathcal{M}_R.
\end{array}
\]

Proof. The functor on the right hand side appeared already in Theorem 2.14. Let us explain first what is meant by the functor in the bottom row. A right $\mathcal{H}_L$-comodule $N$ is a priori a right $L$-module, and it is made an $L$-bimodule via the left action $In := n_{(0)} e_L(n_{(1)} t_L(l))$, for $l \in L$ and $n \in N$, cf. a symmetrical form of (6). The functor in the bottom row takes the $\mathcal{H}_L$-comodule $N$ to the $R$-$R$ bimodule $N$, with actions

\[
r \triangleright n \triangleleft r' := e_L(s_R(r')) n e_L(s_R(r)) \equiv n_{(0)} e_L(s_R(r) n_{(1)} s_R(r')), \quad \text{for } r, r' \in R, n \in N. \quad (12)
\]

In order to see commutativity of the diagram, take a right $\mathcal{H}$-comodule $M$. Composing the functor on the left hand side with the functor in the bottom row, it takes $M$ to an $R$-$R$ bimodule with actions in (12). Applying to $M$ the functor in the top row and the functor on the right hand side, we obtain the $R$-$R$ bimodule $M$ with actions

\[
rmr' = m^{(0)} e_R(s_R(r) m^{(1)}) r' = m^{(0)} e_R(s_R(r) m^{(1)} s_R(r')), \quad \text{for } r, r' \in R, m \in M. \quad (13)
\]
By definition, the \( H_L \)-coaction on \( M \) is right \( R \)-linear (with respect to the action on \( M \) denoted by juxtaposition). Therefore,

\[
(rmr')|_0 \otimes (rmr')|_1 = m|_0 \otimes m|_1 s_R \left( \varepsilon_R(s_R(r)m|_1 s_R(r')) \right) = m|_0 \otimes m|_1 s_R \left( \varepsilon_R(s_R(r)m|_1 s_R(r')) \right) = m|_0 \otimes s_R(r)m|_1 s_R(r').
\]

The first equality follows by \( \text{(13)} \) and the right \( R \)-linearity of the \( H_L \)-coaction on \( M \). The second equality follows by the right \( H_R \)-co-linearity of the \( H_L \)-coaction on \( M \). By multiplicativity, right \( R \)-linearity and unitarity of \( \Delta_R \), \( \Delta_R(s_R(r)) = 1_H \otimes_R s_R(r) \). In the last equality we used this identity and multiplicativity of \( \Delta_R \). Applying \( M \otimes_L \varepsilon_L \) to both sides, we conclude that

\[
rmr' = m|_0 \otimes m|_1 s_R(r) \equiv r \triangleright m \triangleleft r'.
\]

This proves commutativity of the diagram in the theorem.

Strict monoidality of the functors on the right hand side and in the bottom row follows by Theorem 2.14 (and its application to the opposite of the bialgebroid \( H_L \)). In order to see strict monoidality of the remaining two functors, recall that by Theorem 2.14 (applied to \( H_R \) and the opposite of \( H_L \)), the \( R \)-module tensor product of any two \( H \)-comodules is an \( H_R \)-comodule and an \( H_L \)-comodule, via the diagonal coactions, cf. \( \text{(1)} \). Compatibility of these coactions in the sense of Definition 2.20 is checked as follows. For \( M, N \in M^H, m \in M \) and \( n \in N \),

\[
(m \otimes n)|_0 \otimes (m \otimes n)|_1 (1) \otimes (m \otimes n)|_1 (2) = (m|_0 \otimes n|_0 \otimes m|_1 \otimes n|_1 (1) \otimes m|_1 \otimes n|_1 (2)). \tag{14}
\]

Moreover, for any \( m \in M \) and \( k \in H \), there is a well defined (i.e. \( R \)-balanced and \( L \)-balanced) map

\[
N \otimes_R H \otimes_L H \to (M \otimes_R N) \otimes_R H \otimes_L H,
\]

where we used that the range of the coproduct of \( H \) lies within the Takeuchi product \( H \times_L H \). Composing it with the equal maps \( N \to N \otimes_R H \otimes_L H, n \to n|_0 \otimes n|_1 (1) \otimes n|_1 (2) = n|_0 \otimes_R n|_0 \otimes_L n|_1 (1) \otimes n|_1 (2) \), we conclude that the right hand side of \( \text{(14)} \) is equal to \( (m|_0 \otimes_R n|_0 \otimes_R m|_1 \otimes_R m|_1 \otimes_L n|_1 (1) \otimes_L n|_1 (2)) \).

Similarly, since the range of the \( H_R \)-coaction on \( N \) lies within the Takeuchi product \( N \times_R H \), for any \( n \in N \) and \( k \in H \) there is a well defined map

\[
M \otimes_R H \otimes_L H \to (M \otimes_R N) \otimes_R H \otimes_L H,
\]

where we used that the range of the coproduct of \( H \) lies within the Takeuchi product \( H \times_L H \). Composing it with the equal maps \( M \to M \otimes_R H \otimes_L H, m \to m|_0 \otimes_R m|_1 \otimes_R m|_1 \otimes_R m|_1 \otimes_L m|_1 \), we conclude that the right hand side of \( \text{(14)} \) is equal also to

\[
(m|_0 \otimes_R n|_0 \otimes_R m|_0 \otimes_R n|_0 \otimes_R m|_0 \otimes_R m|_0 \otimes_R m|_0 \otimes_R m|_0 \otimes_R m|_1 \otimes_L n|_1 (1) \otimes_L n|_1 (1) \otimes_L m|_1 n|_1 (1) \).
\]

The other compatibility relation in Definition 2.20 is checked by similar steps. Recall from the proof of Corollary 2.22 that \( R(\cong L) \) is a right \( H \)-comodule as well. Finally, the \( R \)-module tensor product of \( H \)-comodule maps is an \( H_R \)-comodule map and an \( H_L \)-comodule map by Theorem 2.14. Thus it is an \( H \)-comodule map. By Theorem 2.14 also the coherence natural transformations in \( R \mathcal{M}_R \) are \( H_R \)- and \( H_L \)-comodule maps, so \( H \)-comodule maps, what completes the proof.

In light of Theorem 2.23 comodule algebras of a Hopf algebroid are defined as follows.
**Definition 2.24.** A right comodule algebra of a Hopf algebroid \( \mathcal{H} \) is a monoid in the monoidal category \( \mathcal{M}^{3l} \) of right \( \mathcal{H} \)-comodules. Explicitly, this means an \( R \)-ring and right \( \mathcal{H} \)-comodule \( A \), such that the unit map \( R \rightarrow A \) and the multiplication map \( A \otimes_R A \rightarrow A \) are right \( \mathcal{H} \)-comodule maps. Using the notations \( a \mapsto a[0] \otimes_R a^*[1] \) and \( a \mapsto a[0] \otimes_L a^*[1] \) for the \( \mathcal{H}_R \)- and \( \mathcal{H}_L \)-coactions, respectively, \( \mathcal{H} \)-colinearity of the unit and the multiplication means the following identities, for all \( a, a' \in A \):

\[
\begin{align*}
1_A^{[0]} \otimes_R 1_A^{[1]} &= 1_A \otimes_R 1_H, \\
1_A^{[0]} \otimes_L 1_A^{[1]} &= 1_A \otimes_L 1_H,
\end{align*}
\]

\[
(\alpha a')^{[0]} \otimes_R (\alpha a')^{[1]} = a^{[0]} \alpha a^{[0]} \otimes_R a^{[1]} a'^{[1]},
\]

\[
(\alpha a')^{[0]} \otimes_L (\alpha a')^{[1]} = a^{[0]} \alpha a^{[0]} \otimes_L a^{[1]} a'^{[1]}.
\]

**Definition 2.25.** For an \( \mathcal{H} \)-comodule algebra \( A \) and \( B := A^{\otimes 3l} \), we say that \( B \subseteq A \) is an \( \mathcal{H} \)-extension.

The related \( \mathcal{H}_R \)-, and \( \mathcal{H}_L \)-coactions in an \( \mathcal{H} \)-extension \( B \subseteq A \) determine two canonical maps,

\[
\begin{align*}
can_R : A \otimes_R A &\rightarrow A \otimes_R H, & a \otimes a' &\mapsto \alpha a^{[0]} \otimes a^{[1]} & \text{and} \\
can_L : A \otimes_L A &\rightarrow A \otimes_L H, & a \otimes a' &\mapsto a[0] \alpha a'^{[1]} a[1].
\end{align*}
\]

Bijectivity of the maps in (15) implies that \( B \subseteq A \) is a Galois extension by the right bialgebroid \( \mathcal{H}_R \), and the left bialgebroid \( \mathcal{H}_L \), respectively. It is not known in general if bijectivity of one implies bijectivity of the other. A partial result is given in following [6 Lemma 3.3].

**Proposition 2.26.** For an algebra extension \( B \subseteq A \) by a Hopf algebroid \( \mathcal{H} \) with a bijective antipode \( S \), the canonical map \( \text{can}_L \) in (15) is bijective if and only if \( \text{can}_R \) is bijective.

**Proof.** (Sketch.) By the bijectivity of the antipode, for any right \( \mathcal{H} \)-comodule \( M \), the map \( \Phi_M \) in (10) is bijective, with the inverse \( m \otimes_L h \mapsto m[0] \otimes_R S^{-1}(h)m[1] \). The canonical maps in (15) are related by \( \Phi_A \circ \text{can}_R = \text{can}_L \), what proves the claim.

The Galois theory of a Hopf algebroid \( \mathcal{H} \) is greatly simplified whenever the category of \( \mathcal{H} \)-comodules is isomorphic to the categories of comodules of the constituent left and right bialgebroids: In this case \( \mathcal{M}^{3l} \) can be described as a category of an appropriate coring. The following was obtained in [3, Corrigendum, Theorem 4].

**Theorem 2.27.** Consider a Hopf algebroid \( \mathcal{H} \), with structure maps denoted as in Definition 2.18 and the forgetful functors \( F_L : \mathcal{M}^{3l} \rightarrow \mathcal{M}_L \), \( F_R : \mathcal{M}^{3l} \rightarrow \mathcal{M}_R \), \( G_L : \mathcal{M}^{3l} \rightarrow \mathcal{M}^{3l} \), and \( G_R : \mathcal{M}^{3l} \rightarrow \mathcal{M}^{3l} \).

1. If the equalizer

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi_R} & M \otimes_R H \\
\downarrow \varphi_R & & \downarrow \varphi_R \otimes_R H \\
M & \xrightarrow{\varphi_R \otimes_R H} & M \otimes_R H \otimes_R H
\end{array}
\]

in \( \mathcal{M}_L \) is \( H \otimes_L H \)-pure, i.e., it is preserved by the functor \( - \otimes_L H \otimes_L H : \mathcal{M}_L \rightarrow \mathcal{M}_L \), for any right \( \mathcal{H}_R \)-comodule \((M, \varphi_R)\), then there exists a functor \( U : \mathcal{M}^{3l} \rightarrow \mathcal{M}^{3l} \), such that \( F_L \circ U = F_R \) and \( U \circ G_R = G_L \).

2. If the equalizer

\[
\begin{array}{ccc}
N & \xrightarrow{\varphi_L} & N \otimes_L H \\
\downarrow \varphi_L & & \downarrow \varphi_L \otimes_L H \\
N & \xrightarrow{\varphi_L \otimes_L H} & N \otimes_L H \otimes_L H
\end{array}
\]

in \( \mathcal{M}_R \) is \( H \otimes_R H \)-pure, i.e., it is preserved by the functor \( - \otimes_R H \otimes_R H : \mathcal{M}_R \rightarrow \mathcal{M}_R \), for any right \( \mathcal{H}_L \)-comodule \((N, \varphi_L)\), then there exists a functor \( V : \mathcal{M}^{3l} \rightarrow \mathcal{M}^{3l} \), such that \( F_R \circ V = F_L \) and \( V \circ G_L = G_R \). In particular, \( G_L \) is full.
(3) If both purity assumptions in parts (1) and (2) hold, then the forgetful functors $G_R : \mathcal{M}^{\mathcal{H}} \to \mathcal{M}^{\mathcal{H}_R}$ and $G_L : \mathcal{M}^{\mathcal{H}} \to \mathcal{M}^{\mathcal{H}_L}$ are isomorphisms, hence $U$ and $V$ are inverse isomorphisms.

Proof. (1) Recall that (16) defines the $\mathcal{H}_R$-cotensor product $M \Box_{\mathcal{H}_R} H \cong M$. By axiom (ii) in Definition 2.18 $H$ is an $\mathcal{H}_R$-$\mathcal{H}_L$ bicomodule, with left coaction $\Delta_R$ and right coaction $\Delta_L$. Thus in light of [18, 22.3] and its Erratum, we can define a desired functor $U := - \Box_{\mathcal{H}_R} H$. Clearly, it satisfies $F_L \circ U = F_R$. For an $\mathcal{H}$-comodule $(M, \rho_L, \rho_R)$, the coaction on the $\mathcal{H}_L$-comodule $U(G_R(M, \rho_L, \rho_R)) = U(M, \rho_R)$ is given by

$$M \xrightarrow{\rho_R} M \Box_{\mathcal{H}_R} H \xrightarrow{M \Box_{\mathcal{H}_R} \Delta_L} M \Box_{\mathcal{H}_R} (H \otimes H) \xrightarrow{\sim} (M \Box_{\mathcal{H}_R} H) \otimes H \xrightarrow{M \otimes_{\mathcal{H}_R} \epsilon \otimes H} M \otimes H, \quad (18)$$

where in the third step we used that since the equalizer (16) is $H \otimes L$-pure, it is in particular $H$-pure. Using that $\rho_R$ is a right $\mathcal{H}_L$-comodule map and counitality of $\rho_R$, we conclude that (18) is equal to $\rho_L$. Hence $U \circ G_R = G_L$. Note that this yields an alternative proof of fully faithfulness of $G_R$. Indeed, this proves that for any two $\mathcal{H}$-comodules $M$ and $M'$, and any $\mathcal{H}_R$-comodule map $f : M \to M'$, $U(f) = f$ is an $\mathcal{H}_L$-comodule map hence an $\mathcal{H}$-comodule map. Part (2) is proven symmetrically.

(3) For the functor $U$ in part (1) and a right $\mathcal{H}_R$-comodule $(M, \rho_R)$, denote $U(M, \rho_R) := (M, \rho_L)$. With this notation, define a functor $G_R^{-1} : \mathcal{M}^{\mathcal{H}_R} \to \mathcal{M}^{\mathcal{H}}$, with object map $(M, \rho_R) \mapsto (M, \rho_R, \rho_L)$, and acting on the morphisms as the identity map. Being coassociative, $\rho_R$ is an $\mathcal{H}_L$-comodule map, so by part (1), $U(\rho_R) = \rho_R$ is an $\mathcal{H}_L$-comodule map. Symmetrically, by part (2), $V(\rho_L) = \rho_L$ is an $\mathcal{H}_R$-comodule map. So $G_R^{-1}$ is a well defined functor. One easily checks that it is the inverse of $G_R$.

In a symmetrical way, in terms of the functor $V(N, \rho_L) := (N, \rho_R)$ in part (2), one constructs $G_L^{-1}$ with object map $(N, \rho_L) \mapsto (N, \rho_L, \rho_R)$, and acting on the morphisms as the identity map. The identities $G_L \circ G_R^{-1} = U$ and $G_R \circ G_L^{-1} = V$ prove that $U$ and $V$ are mutually inverse isomorphisms, as stated. \(\square\)

**Definition 2.28.** Hopf algebroids for that the purity assumptions in Theorem 2.27 (3) hold, are termed *pure* Hopf algebroids.

All known examples of Hopf algebroids are pure, cf. [8, Corrigendum, Example 5].

**Example 2.29.** (1) Example 2.25 (1) can be extended as follows. Consider a Hopf algebroid $\mathcal{H}$ and use the notations introduced in and after Definition 2.18. The coproduct $\Delta_R$ in $\mathcal{H}_R$ equips the total algebra $H$ with a right $\mathcal{H}_R$-comodule algebra structure and the coproduct $\Delta_L$ in $\mathcal{H}_L$ equips $H$ with a right $\mathcal{H}_L$-comodule algebra structure. In this way $H$ becomes an $\mathcal{H}$-comodule algebra. $\mathcal{H}_R$-coinvariants are those elements $h \in H$, for which $h^{(1)} \otimes_R h^{(2)} = h \otimes_R 1_H$, i.e. the image of $R^{op}$ under $t_R$. $\mathcal{H}_L$-coinvariants are those elements $h \in H$, for which $h^{(1)} \otimes_L h^{(2)} = h \otimes_L 1_H$, i.e. elements of $t_R(R^{op}) = s_L(L)$. The canonical map

$$H \otimes_L H \to H \otimes_R H, \quad h \otimes_L h' \mapsto hh^{(1)} \otimes_R h'^{(2)}$$

is bijective, with the inverse $h \otimes_R h' \mapsto hS(h^{(1)}') \otimes_L h'^{(2)}$. That is, the algebra extension $L \subseteq H$, given by $s_L$ (equivalently, the algebra extension $R^{op} \subseteq H$, given by $t_R$) is a Galois extension by $\mathcal{H}_R$. In other words, a constituent rightbialgebroid $\mathcal{H}_R$ in a Hopf algebroid $\mathcal{H}$ provides an example of a (right) $\times_R$-Hopf algebra, in the sense of [33].

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The other canonical map
\[ H \otimes L H \rightarrow H \otimes L H, \quad h \otimes h' \mapsto h(1) h' \otimes h(2) \]
(mind the different \(L\)-actions in the domain and the codomain!) is bijective provided that \(S\) is bijective. In this case the inverse is given by \( h' \otimes L h \mapsto h(2) \otimes L S^{-1}(h(1)) h' \).

(2) Consider a right depth 2 and balanced algebra extension \( B \subseteq A \), as in Example \( \text{2.17} \). We have seen in Example \( \text{2.17} \) that \( B \subseteq A \) is a Galois extension by a right bialgebroid \( \mathcal{H}_R \). Assume that the extension \( B \subseteq A \) is also Frobenius (i.e. \( A \) is a finitely generated and projective right \( B \)-module and \( \operatorname{Hom}_B(A, B) \cong A \) as \( B \)-\( A \) bimodules). In this situation, the right bialgebroid \( \mathcal{H} \) was proven to be a constituent right bialgebroid in a Hopf algebroid \( \mathcal{H} \) in \( \text{[9, Section 3]} \). The Hopf algebroid \( \mathcal{H} \) is finitely generated and projective as a left \( L \)-module and as a left \( R \)-module. This implies that \( \mathcal{H} \) is a pure Hopf algebroid, hence a right \( \mathcal{H}_R \)-comodule algebra \( A \) is also a right \( \mathcal{H} \)-comodule algebra. Moreover, the antipode of \( \mathcal{H} \) is bijective, hence the \( \mathcal{H}_R \)-Galois extension \( B \subseteq A \) is also \( \mathcal{H}_L \)-Galois by Proposition \( \text{2.26} \).

3 Cleft extensions

In this section a particular class of Galois extensions, so called cleft extensions will be studied. These are the simplest and best understood examples of Galois extensions and also the closest ones to the classical problem of Galois extensions of fields.

Similarly to Section \( \text{2} \), we start with reviewing the most classical case of a cleft extension by a Hopf algebra in \( \text{[24]} \) and \( \text{[4]} \). A cleft extension by a coalgebra was introduced in \( \text{[17]} \) and \( \text{[13]} \) and studied further in \( \text{[1]} \) and \( \text{[20]} \). In all these papers the analysis is based on the study of an entwining structure associated to a Galois extension. Here we present an equivalent description of a cleft extension by a coalgebra, which avoids using entwining structures. Instead, we make use of the coring extension behind. The advantage of this approach, developed in \( \text{[10]} \), is that it provides a uniform approach to many kinds of cleft extensions, including cleft extensions by pure Hopf algebroids, where the coring extension in question does not come from an entwining structure.

3.1 Cleft extensions by Hopf algebras

The notion of a cleft extension by a Hopf algebra emerged already in papers by Doi and Sweedler, but it became relevant by results in \( \text{[24]} \) and \( \text{[4]} \). Note that, for a \( k \)-algebra \((A, \mu, \eta)\) and a \( k \)-coalgebra \((C, \Delta, \varepsilon)\), the \( k \)-module \( \operatorname{Hom}_k(C, A) \) is a \( k \)-algebra via the convolution product
\[(f, g) \mapsto \mu \circ (f \otimes g) \circ \Delta, \quad \text{for } f, g \in \operatorname{Hom}_k(C, A),\]
and unit element \( \eta \circ \varepsilon \).

**Definition 3.1.** An algebra extension \( B \subseteq A \) by a Hopf algebra \( H \) is said to be cleft provided that there exists a convolution invertible right \( H \)-comodule map \( j : H \rightarrow A \), called a cleaving map.

By antipode axioms \( \text{(2)} \) in a Hopf algebra \( H \), the antipode is convolution inverse of the \((H\)-colinear) identity map \( H \). Thus any \( k \)-Hopf algebra \( H \) is an \( H \)-cleft extension of \( k \) (via the unit map).

Following \( \text{[24, Theorem 9]} \) explains in what sense cleft extensions are distinguished Hopf Galois extensions.
**Theorem 3.2.** An algebra extension $B \subseteq A$ by a Hopf algebra $H$ is cleft if and only if it is an $H$-Galois extension and the normal basis property holds, i.e. $A \cong B \otimes_k H$ as left $B$-modules right $H$-comodules.

The construction of a crossed product with a bialgebra $H$ (with coproduct $\Delta(h) = h^{(1)} \otimes_k h^{(2)}$ and counit $\varepsilon$) was introduced in [3], as follows.

**Definition 3.3.** A $k$-bialgebra $H$ measures a $k$-algebra $B$ if there exists a $k$-module map $\cdot : H \otimes_k B \to B$, such that $h \cdot 1_B = \varepsilon(h)1_B$ and $h \cdot (bb') = (h^{(1)} \cdot b)(h^{(2)} \cdot b')$, for $h \in H$, $b, b' \in B$.

A $B$-valued 2-cocycle on $H$ is a $k$-module map $\sigma : H \otimes_k H \to B$, such that $\sigma(1_H, h) = \varepsilon(1_H)1_B = \sigma(h, 1_H)$, for $h \in H$, and

$$(h^{(1)} \cdot \sigma(k^{(1)}, m^{(1)}))\sigma(h^{(2)}, k^{(2)}m^{(2)}) = \sigma(h^{(1)}, k^{(1)})\sigma(h^{(2)}k^{(2)}, m), \quad \text{for } h, k, m \in H.$$ 

The $H$-measured algebra $B$ is a $\sigma$-twisted $H$-module if in addition

$$(h^{(1)} \cdot (k^{(1)} \cdot b))\sigma(h^{(2)}, k^{(2)}) = \sigma(h^{(1)}, k^{(1)})(h^{(2)}k^{(2)} \cdot b), \quad \text{for } b \in B, h, k \in H.$$ 

**Proposition 3.4.** Consider a $k$-bialgebra $H$ and an $H$-measured $k$-algebra $B$. Let $\sigma : H \otimes_k H \to B$ be a $k$-module map. The $k$-module $B \otimes_k H$ is an algebra, with multiplication

$$(b \# h')(b' \# h') = b(h^{(1)} \cdot b')\sigma(h^{(2)}, h^{(1)}')\# h(3)h^{(2)}$$

and unit element $1_B \# 1_H$, if and only if $\sigma$ is a $B$-valued 2-cocycle on $H$ and $B$ is a $\sigma$-twisted $H$-module. This algebra is called the crossed product of $B$ with $H$, with respect to the cocycle $\sigma$. It is denoted by $B\#_\sigma H$.

It is straightforward to see that a crossed product algebra $B\#_\sigma H$ is a right $H$-comodule algebra, with coaction given in terms of the coproduct $\Delta$ in $H$, as $B \otimes_k \Delta$. What is more, $B \subseteq B\#_\sigma H$ is an extension by $H$. It is most natural to ask what $H$-extensions arise as crossed products. One implication in forthcoming Theorem 3.5 was proven first in [24, Theorem 11]. Other implication was proven in [3, Theorem 1.18]. Since for a $k$-bialgebra $H$ also $H \otimes_k H$ is a $k$-bialgebra, convolution invertibility of a $B$-valued 2-cocycle $\sigma$ on $H$ is understood in the convolution algebra $\text{Hom}_k(H \otimes_k H, B)$.

**Theorem 3.5.** An algebra extension $B \subseteq A$ by a Hopf algebra $H$ is a cleft extension if and only if $A$ is isomorphic to $B\#_\sigma H$, as a left $B$-module and right $H$-comodule algebra, for some convolution invertible $B$-valued $2$-cocycle $\sigma$ on $H$.

Another important aspect of cleft extensions is that they provide examples of Hopf Galois extensions $B \subseteq A$, beyond the case when $A$ is a faithfully flat $B$-module, when a Strong Structure Theorem holds. Recall that, for a $k$-bialgebra $H$ and its right comodule algebra $A$, right-right relative Hopf modules are right modules for the monoid $A$ in $M^H$. That is, right $A$-modules and right $H$-comodules $M$, such that the $A$-action $M \otimes_k A \to M$ is a right $H$-comodule map with respect to the diagonal $H$-coaction in $M \otimes_k A$. Equivalently, the $H$-coaction $M \to M \otimes_k H$ is a right $A$-module map with respect to the right $A$-action in $M \otimes_k H$ given by the $H$-coaction in $A$. Clearly, for any right $B := A^{coH}$-module $N$, the tensor product $N \otimes B$ inherits a relative Hopf module structure of $A$.

**Theorem 3.6.** For a cleft extension $B \subseteq A$ by a Hopf algebra $H$, the category of right $B$-modules is equivalent to the category of right-right $(H, A)$-relative Hopf modules, via the induction functor $- \otimes_B A : \mathcal{M}_B \to \mathcal{M}_A^H$. 

3.2 Cleft bicomodules for pure coring extensions

The aim of the current section is to reformulate Definition [3.1] of a cleft extension by a Hopf algebra, using the Morita theory of pure coring extensions developed in [10] (see the corrected versions). This will allow us to place the results in Theorem 3.2 and Theorem 3.6 into a broader context. More importantly, it will provide us with a tool of generalizations in later sections.

Observe first that, for a right comodule algebra \( A \) of a \( k \)-bialgebra \( H \), the tensor product \( C := A \otimes_k H \) is an \( A \)-coring. The \( A \)-\( A \) bimodule structure is given in terms of the right \( H \)-coaction in \( A \), \( p^A : a \mapsto a[0] \otimes_k a[1] \), as

\[
a_1(a \otimes h) a_2 := a_1 a a_2[0] \otimes_k a_2[1], \quad \text{for } a_1, a_2 \in A, \ a \otimes h \in A \otimes_k H.
\]

The coproduct is \( A \otimes_k \Delta : A \otimes_k H \rightarrow A \otimes_k H \otimes_k H \cong (A \otimes_k H) \otimes_k (A \otimes_k H) \), determined by the coproduct \( \Delta \) in \( H \), and the counit is \( A \otimes_k \varepsilon : A \otimes_k H \rightarrow A \), coming from the counit \( \varepsilon \) in \( H \). What is more, \( C \) is a right \( H \)-comodule via the coaction \( A \otimes_k \Delta : C \rightarrow C \otimes_k H \). The coproduct in \( C \) is a right \( H \)-comodule map, that is, (the constituent coalgebra in) \( H \) is a right extension of \( C \). Moreover, since for any right \( C \)-comodule \( M \) the equalizer \( \{ \} \) in \( \mathcal{M}_k \) is split by the map \( M \otimes_k H \otimes_k \varepsilon : M \otimes_k A \otimes_k \varepsilon \otimes_k H \otimes_k H \rightarrow M \otimes_k C \otimes_k H \), it is preserved by any functor of domain \( \mathcal{M}_k \). Therefore, \( H \) is pure right extension of \( C \). Via the right regular \( A \)-action and coaction \( p^A : A \rightarrow A \otimes_k H \cong A \otimes_k A \), \( A \) is a right \( C \)-comodule.

By the above motivation, we turn to a study of any \( L \)-coring \( D \), which is a pure right extension of an \( A \)-coring \( C \), and an \( L \)-\( C \) bimodule \( \Sigma \). In [10] Proposition 3.1] to any such bimodule \( \Sigma \) a Morita context \( \mathbb{M}(\Sigma) \) was associated. In order to write it up explicitly, recall that by the purity assumption, \( \Sigma \cong \Sigma \otimes C \otimes C \) is also a right \( D \)-comodule. Put \( T := \text{End}^C(\Sigma) \). It is an \( L \)-ring. Introduce the following index notations. For the coproduct in \( C \), write \( c \mapsto c^{(1)} \otimes_A c^{(2)} \). For the coproduct in \( D \), write \( d \mapsto d^{(1)} \otimes_L d^{(2)} \). For the \( C \)-coaction in \( \Sigma \) write \( x \mapsto x^{[0]} \otimes_A x^{[1]} \) and for the corresponding \( D \)-coaction in \( \Sigma \) write \( x \mapsto x^{[0]} \otimes_L x^{[1]} \). In each case implicit summation is understood. Then

\[
\mathbb{M}(\Sigma) = (L \text{Hom}_L(D,T) , \ C \text{End}^D(C)^{op} , L \text{Hom}^D(D,\Sigma) , \ \tilde{Q} , \ \hat{\bullet} , \ \hat{\circ}), \tag{19}
\]

where

\[
\tilde{Q} : = \{ \ q \in A \text{Hom}_L(C,\text{Hom}_A(\Sigma,A)) \mid c^{(1)} q(c^{(2)})(x) = q(c)(x^{[0]} x^{[1]}), \quad \forall x \in \Sigma, c \in C \ \}.
\]

The algebra structures, bimodule structures and connecting homomorphisms are given by the following formulae.

\[
(vv')(d) = v(d_{[1]}) v'(d_{[2]})
\]

\[
(uu')(c) = u'(u(c))
\]

\[
(vp)(d) = v(d_{[1]}) (p(d_{[2]}))
\]

\[
(pu)(d) = p(d)^{[0]} \varepsilon_C (u(p(d)^{[1]}))
\]

\[
(qv)(c) = q(c^{[0]}) v(c^{[1]})
\]

\[
(uq)(c) = q(u(c))
\]

\[
(q \bullet p)(c) = c^{(1)} q(c^{(2)})(0) (p(c^{(2)})(1)) \equiv q(c^{[0]}) (p(c^{[1]}))^{[0]}) p(c^{[1]}))^{[1]}
\]

\[
(p \circ q)(d) = p(d)^{[0]} q(p(d)^{[1]})(-),
\]
for \( v, v' \in L\text{Hom}_L(\mathcal{D}, T), u, u' \in C\text{End}^D(\mathcal{C}), p \in L\text{Hom}_D(\mathcal{D}, \Sigma), q \in \tilde{Q}, d \in \mathcal{D} \) and \( c \in \mathcal{C} \).

As it is explained in [10, Proposition 3.1], inspite of its involved form, the Morita context \( \mathbb{M}(\Sigma) \) has a very simple origin. For any pure coring extension \( \mathcal{D} \) of \( \mathcal{C} \), there is a functor \( U := - \square_C : \mathcal{M} \mathcal{C} \rightarrow \mathcal{M}^D \). An \( L : \mathcal{C} \) bimodule \( \Sigma \) determines another functor \( V := \text{Hom}_C(\Sigma, -) \otimes_L \mathcal{D} : \mathcal{M} \mathcal{C} \rightarrow \mathcal{M}^D \). In terms of natural transformations \( \text{Nat}(-, -) \) between these functors, \( \mathbb{M}(\Sigma) \) is isomorphic to the Morita context

\[
( \text{Nat}(V, V), \text{Nat}(U, U), \text{Nat}(V, U), \text{Nat}(U, V), \bullet, \circ ),
\]

where all algebra and bimodule structures and also the connecting maps are given by opposite composition of natural transformations.

Let us compute the Morita context \( \mathbb{M}(A) \) in our motivating example, coming from an algebra extension \( B \subseteq A \) by a \( k \)-Hopf algebra \( H \). We claim that it is isomorphic to a sub-Morita context of a (degenerate) Morita context, in which both algebras and both bimodules are equal (as \( k \)-modules) to \( \text{Hom}_k(H, A) \), and all algebra structures, bimodule structures and connecting homomorphisms are given by the convolution product. Indeed, in the current case the \( k \)-algebra \( L \) reduces to \( k \) and the coring \( \mathcal{D} \) reduces to the \( k \)-coalgebra in \( H \). The role of the bimodule \( \Sigma \) is played by \( A \) and the endomorphism algebra \( T \) is isomorphic to the covariant subalgebra \( B = A^{coH} \). The \( A \)-coring \( \Sigma \) is equal to \( A \otimes_k H \). The isomorphism \( \text{AHom}(A \otimes_k H, A) \cong \text{Hom}_k(H, A) \) induces an isomorphism

\[
\tilde{Q} \cong \{ \tilde{q} \in \text{Hom}_k(H, A) : \tilde{q}(h(2))_{[0]} \otimes h(1) \tilde{q}(h(2))_{[1]} = \tilde{q}(h) \otimes 1_H, \ \forall h \in H \}
\equiv \{ \tilde{q} \in \text{Hom}_k(H, A) : \tilde{q}(h)_{[0]} \otimes \tilde{q}(h)_{[1]} = \tilde{q}(h(2)) \otimes \text{S}(h(1)), \ \forall h \in H \}.
\]

Thus we conclude that \( \tilde{Q} \) is isomorphic to \( \text{Hom}^H(H^w, A) \), where \( H^w \) is the \( k \)-module \( H \), considered to be a right \( H \)-comodule via the twisted coaction \( h \mapsto h(2) \otimes_k \text{S}(h(1)). \)

The monomorphism

\[
C\text{End}^H(A \otimes_k H) \hookrightarrow \text{AEnd}^H(A \otimes H) \cong \text{Hom}_k(H, A), \quad u \mapsto \tilde{u} := (A \otimes_k \varepsilon) \circ u \circ (1_A \otimes_k -)
\]

establishes an isomorphism (with inverse \( \tilde{u} \mapsto (a \otimes_k h \mapsto a\tilde{u}(h(1)) \otimes_k h(2)) \)) from \( C\text{End}^H(A \otimes H) \) to

\[
X := \{ \tilde{u} \in \text{Hom}_k(H, A) : \tilde{u}(h(2))_{[0]} \otimes h(1) \tilde{u}(h(2))_{[1]} = \tilde{u}(h(1)) \otimes_k h(2), \ \forall h \in H \}
\equiv \{ \tilde{u} \in \text{Hom}_k(H, A) : \tilde{u}(h)_{[0]} \otimes \tilde{u}(h)_{[1]} = \tilde{u}(h(2)) \otimes_k \text{S}(h(1))h(3), \ \forall h \in H \}.
\]

So \( \mathbb{M}(A) \) is isomorphic to the Morita context

\[
(\text{Hom}_k(H, B), X, \text{Hom}^H(H, A), \text{Hom}^H(H^w, A), \bullet, \circ').
\]

A monomorphism of Morita contexts is given by the obvious inclusions \( \text{Hom}_k(H, B) \hookrightarrow \text{Hom}_k(H, A), X \mapsto \text{Hom}_k(H, A), \text{Hom}^H(H, A) \hookrightarrow \text{Hom}_k(H, A), \text{Hom}^H(H^w, A) \hookrightarrow \text{Hom}_k(H, A). \)

Since it is well known (cf. [23, Lemma 3.2]) that the convolution inverse of a right \( H \)-comodule map \( j : H \rightarrow A \), if it exists, belongs to \( \text{Hom}^H(H^w, A) \), we obtained the following reformulation of Definition 3.1.

**Proposition 3.7.** An algebra extension \( B \subseteq A \) by a Hopf algebra \( H \) is cleft if and only if there exist elements \( j \in \text{Hom}^H(H, A) \) and \( \tilde{j} \in \text{Hom}^H(H^w, A) \) in the two bimodules in the Morita context [27] which are mapped by the connecting maps \( \bullet' \) and \( \circ' \) to the unit elements of the respective algebras in the Morita context.
Motivated by Proposition 3.7, we impose following [10, Definition 5.1].

**Definition 3.8.** Let an $L$-coring $D$ be a pure right extension of an $A$-coring $C$. An $L$-$C$ bicomodule $\Sigma$ is said to be cleft provided that there exist elements $j \in \mathcal{L}\text{Hom}^D(D, \Sigma)$ and $\tilde{j} \in \tilde{Q}$ in the two bimodules in the associated Morita context $\mathcal{M}(\Sigma)$ in (19), such that $j \circ \tilde{j}$ is equal to the unit element in the convolution algebra $\mathcal{L}\text{Hom}^D(D, T)$ and $\tilde{j} \circ j$ is equal to the unit element in the other algebra $\mathcal{C}\text{End}^D(C)^{op}$ in the Morita context $\mathcal{M}(\Sigma)$. We say that $j$ and $\tilde{j}$ are mutual inverses in $\mathcal{M}(\Sigma)$.

In the forthcoming sections we will define cleft extensions $B \subseteq A$ by coalgebras, corings and pure Hopf algebroids, by finding an appropriate coring extension and requiring $A$ to be a cleft bicomodule for it in the sense of Definition 3.8.

In the rest of this section we recall some results from [10] about cleft bicomodules, extending Theorem 3.2 and Theorem 3.6.

**Theorem 3.9.** Consider an $L$-coring $D$ which is a pure right extension of an $A$-coring $C$. For an $L$-$C$ bicomodule $\Sigma$, put $T := \text{End}^C(\Sigma)$. The bicomodule $\Sigma$ is cleft if and only if the following hold.

1. The natural transformation of functors $\mathcal{M}_A \rightarrow \mathcal{M}^C$

$$\text{can} : \text{Hom}_A(\Sigma, -) \otimes \Sigma \rightarrow - \otimes \mathcal{C}, \quad \text{can}_N : \phi_N \otimes x \mapsto \phi_N(x^{[0]}_A \otimes x^{[1]}_A)$$

is a natural isomorphism.

2. The normal basis property holds, i.e. $\Sigma \cong T \otimes_L D$ as left $T$-modules right $D$-comodules.

**Proof.** (Sketch.) Let us use the index notations introduced in the paragraph preceding (19). If $\Sigma$ is a cleft bicomodule then the inverse of the natural transformation (22) can be constructed in terms of the mutually inverse elements $j$ and $\tilde{j}$ in the two bimodules of the Morita context (19):

$$\text{can}_N^{-1} : N \otimes A \mathcal{C} \rightarrow \text{Hom}_A(\Sigma, N) \otimes \Sigma, \quad n \otimes c \mapsto n \tilde{j}(c^{[0]}_A) \otimes (\text{can}_A^{-1}) (\tilde{j}(c^{[1]}_A)).$$

This proves property (1). In order to verify property (2), a left $T$-module right $D$-comodule isomorphism $\kappa : \Sigma \rightarrow T \otimes_L D$ is constructed as $x \mapsto x^{[0]}_A \tilde{j}(x^{[1]}_A) \otimes (-) \otimes_L x^{[1]}_A$, with the inverse $t \otimes_L d \mapsto (j(d))$. Conversely, if (22) is a natural isomorphism and $\kappa : \Sigma \rightarrow T \otimes_L D$ is an isomorphism of left $T$-modules and right $D$-comodules, then the required mutually inverse elements in the two bimodules of the Morita context (19) are given by

$$j := \kappa^{-1}(1_T \otimes -) \quad \text{and} \quad \tilde{j} := [\text{Hom}_A(\Sigma, A) \otimes (T \otimes \varepsilon_D) \circ \kappa] \circ \text{can}_A^{-1},$$

respectively, where $\varepsilon_D$ is the counit in the coring $D$. For more details we refer to [10, Theorem 3.6].

A right $C$-comodule $\Sigma$, for which (22) is a natural isomorphism, was termed a Galois comodule in [16]. Note that if $\Sigma$ is a finitely generated and projective right $A$-module then (22) is a natural isomorphism if and only if $\text{can}_A$ is bijective. Hence a right comodule algebra $A$ of a Hopf algebra $H$ is a Galois comodule for the canonical $A$-coring $A \otimes_k H$ if and only if $A^{coH} \subseteq A$ is an $H$-Galois extension. Thus Theorem 3.9 extends Theorem 3.2.
Theorem 3.10. Consider an L-coring \( D \) which is a pure right extension of an A-coring \( C \). Take a cleft L-C bimodule \( \Sigma \) and put \( T := \text{End}^L(\Sigma) \). If there exist finite sets, \( \{v_i\} \subseteq \text{LHom}_L(D,T) \) and \( \{d_i\} \subseteq D \) satisfying \( \sum_i v_i(d_i) = 1_T \), then the category of right T-modules is equivalent to the category of right C-comodules, via the induction functor \( - \otimes_T \Sigma : \mathcal{M}_T \to \mathcal{M}_C \).

Proof. (Sketch.) The induction functor \( - \otimes_T \Sigma : \mathcal{M}_T \to \mathcal{M}_C \) possesses a right adjoint, the functor \( \text{Hom}^C(\Sigma,-) \). One can choose the counit \( v \) in the two bimodules of the Morita context (19):

\[
n_M^{-1}(m) := m_0^{-1}(0) j(m_0^{-1})(- \otimes j(m_1)) \equiv \text{can}^{-1}_M(m_0^{-1} \otimes m_1),
\]

where in the last expression the natural isomorphism (22) appears (cf. Theorem 3.9).

The unit of the adjunction is

\[
u_N : N \to \text{Hom}^C(\Sigma,N \otimes \Sigma), \quad n \mapsto (x \mapsto x \otimes x),
\]

for any right \( T \)-module \( N \). For more details we refer to Theorem 4.1, Theorem 2.6 and Proposition 4.3 in [10].

If \( B \subseteq A \) is a cleft extension by a Hopf algebra \( H \), then there exist one-element sets satisfying the condition in Theorem 3.10. Indeed, for \( v \) one can choose the counit in \( H \) and for \( d \) the unit element in \( H \). Thus Theorem 3.10 extends Theorem 3.6.

3.3 Cleft extensions by coalgebras

Consider a coalgebra \( (C,\Delta,\varepsilon) \) over a commutative ring \( k \). The cleft property of a \( C \)-extension \( B \subseteq A \) appeared first in a more restricted form in [17] (and was related to a crossed product with a coalgebra in [12]). The definition we use here was introduced in [13], and studied further in [11] and [20]. The aim of this section is to reformulate the definition of the cleft property of a \( C \)-extension \( B \subseteq A \) in the spirit of Definition 3.8. That is, we want to see that a \( C \)-extension \( B \subseteq A \) is cleft if and only if \( B \) is a cleft bimodule for an appropriate pure coring extension. With the experience of Hopf algebra cleft extensions in mind, the obvious candidate is the coalgebra \( C \) to be a pure right extension of an \( A \)-coring \( A \otimes_k C \). Note however, that \( A \otimes_k C \) is not an \( A \)-coring without further assumptions. By [14] Proposition 2.2, \( C := A \otimes_k C \) is an \( A \)-coring (with the left regular \( A \)-module of the first factor, coproduct \( A \otimes_k \Delta \) and counit \( A \otimes_k \varepsilon \)) if and only if the algebra \( A \) and the coalgebra \( C \) are entwined. Furthermore, in this case, \( A \) is an entwined module (i.e. a right \( C \)-comodule) if and only if the (given) \( C \)-coaction on \( A \) is a right \( A \)-module map \( A \to C \). In light of [13] Proposition 2.3 (1) \( \Leftrightarrow \) (2), the following is an equivalent formulation of the definition of a cleft coalgebra extension in [13] p. 293. 

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**Definition 3.11.** Let \((C, \Delta, \varepsilon)\) be a \(k\)-coalgebra and \(B \subseteq A\) be an algebra extension by \(C\). The \(C\)-extension \(B \subseteq A\) is said to be \(C\)-cleft provided that the following properties hold.

1. \(C := A \otimes_k C\) is an \(A\)-coring, with coproduct \(A \otimes \Delta \subseteq C\) and counit \(A \otimes \varepsilon \subseteq C\), the left regular \(A\)-module structure of the first factor and a right \(A\)-module structure such that the \(C\)-coaction in \(A\) is a right \(A\)-module map \(A \to C\).

2. There exists a convolution invertible right \(C\)-comodule map \(j : C \to A\), the so called clefting map.

Following \([10, Proposition 6.1]\) reveals the relation of Definition 3.11 to Definition 3.8.

**Proposition 3.12.** Let \((C, \Delta, \varepsilon)\) be a \(k\)-coalgebra and \(B \subseteq A\) an extension of algebras. The algebra extension \(B \subseteq A\) is \(C\)-cleft if and only if the following conditions hold.

1. \(C := A \otimes_k C\) is an \(A\)-coring (with the left regular \(A\)-module of the first factor, coproduct \(A \otimes \Delta \subseteq C\) and counit \(A \otimes \varepsilon \subseteq C\), hence \(C\) is a pure right extension of \(C\).

2. The right regular \(A\)-module extends to a cleft bicomodule for the coring extension \(C\) of \(C\).

3. \(B = A^\text{coC}\).

**Proof.** (Sketch.) \(A\) is a right \(C\)-comodule if and only if it is a right \(C\)-comodule such that the coaction is a right \(A\)-module map \(A \to C\). By definition, \(B = A^\text{coC}\) if and only if \(B \subseteq A\) is an extension by \(C\). Analogously to \([21]\), the Morita context \(M(A)\), corresponding to the \(k\)-\(C\) bicomodule \(A\) via \([19]\), is isomorphic to

\[
(\text{Hom}_k(C, B), X, \text{Hom}^C(C, A), \tilde{Q}', \diamond', \diamond'),
\]

where

\[
\tilde{Q}' = \{ \tilde{q} \in \text{Hom}_k(C, A) \mid (1_A \otimes c_{(1)})\tilde{q}(c_{(2)}) = \tilde{q}(c)1_A[0] \otimes 1_A[1], \quad \forall c \in C \} \quad \text{and}
\]

\[
X' = \{ \tilde{u} \in \text{Hom}_k(C, A) \mid (1_A \otimes c_{(1)})\tilde{u}(c_{(2)}) = \tilde{u}(c_{(1)}) \otimes c_{(2)}, \quad \forall c \in C \},
\]

and all algebra and bimodule structures and also the connecting maps \(\diamond'\) and \(\diamond'\) are given by the convolution product. Thus the claim follows by \([1, Lemma 4.7.1]\), stating that the convolution inverse of \(j \in \text{Hom}^C(C, A)\), if it exists, belongs to \(\tilde{Q}'\).

By application of Theorem 3.9, we recover a characterization of \(C\)-cleft extensions in the last paragraph of Section 4 in \([13, Theorem 4.9.1.\Rightarrow.3.\) or \([20, Theorem 4.5.1.\Rightarrow.3.\)]

**Theorem 3.13.** An algebra extension \(B \subseteq A\) by a \(k\)-coalgebra \(C\) is cleft if and only if it is a \(C\)-Galois extension and the normal basis property holds, i.e. \(A \cong B \otimes_k C\) as left \(B\)-modules right \(C\)-comodules.

Application of Theorem 3.10 yields the following.

**Theorem 3.14.** Consider a \(k\)-coalgebra \(C\) and a \(C\)-cleft algebra extension \(B \subseteq A\). If there exist finite sets, \(\{v_i\} \subseteq \text{Hom}_k(C, B)\) and \(\{d_i\} \subseteq C\) satisfying \(\sum_i v_i(d_i) = 1_B\), then the category of right \(B\)-modules is equivalent to the category of right \(C\)-comodules for the \(A\)-coacting \(C := A \otimes_k C\), via the induction functor \(- \otimes_B A : M_B \to M^C\).

Consider a \(k\)-coalgebra \((C, \Delta, \varepsilon)\) and an algebra extension \(B \subseteq A\) by \(C\). If there exists a grouplike element in \(C\) then it is mapped by the \(k\)-module map \(\varepsilon(-)1_B : C \to B\) to \(1_B\). Hence Theorem 3.14 extends \([1, Theorem 4.9.1.\Rightarrow.2.\) or \([20, Theorem 4.5.1.\Rightarrow.2.\)]. However, it goes beyond the quoted theorems. The premises of Theorem 3.14 clearly hold whenever the counit of \(C\) is surjective, e.g. if \(C\) is a faithfully flat \(k\)-module (cf. \([1, Theorem 4.9.1.\Rightarrow.5.\)).
3.4 Cleft extensions by corings

Motivated by Proposition 3.14 and Proposition 3.12, we propose the following definition of a cleft extension by a coring.

Definition 3.15. Let \((\mathcal{D}, \Delta, \varepsilon)\) be an \(R\)-coring and \(B \subseteq A\) an extension of algebras. The algebra extension \(B \subseteq A\) is said to be \(\mathcal{D}\)-cleft provided that the following conditions hold.

(1) \(C := A \otimes_R \mathcal{D}\) is an \(A\)-coring (with the left regular \(A\)-module structure of the first factor, coproduct \(A \otimes_R \Delta\) and counit \(A \otimes_R \varepsilon\)), hence the coring \(\mathcal{D}\) is a pure right extension of \(C\).

(2) The right regular \(A\)-module extends to a cleft bicomodule for the coring extension \(\mathcal{D}\) of \(C\).

(3) \(B = A^{\text{corg}}\).

Note that, for an \(R\)-ring \((A, \mu, \eta)\) and an \(R\)-coring \((\mathcal{D}, \Delta, \varepsilon)\), the \(k\)-module \(\text{rHom}_R(\mathcal{D}, A)\) is a \(k\)-algebra via the convolution product

\[
(f, g) \mapsto \mu \circ (f \otimes g) \circ \Delta,
\]
and unit element \(\eta \circ \varepsilon\). In parallel to Proposition 3.12, following [10, Proposition 6.4] characterises those algebra extensions by an \(R\)-coring \(\mathcal{D}\) which are \(\mathcal{D}\)-cleft.

Proposition 3.16. Let \((\mathcal{D}, \Delta, \varepsilon)\) be an \(R\)-coring and \(B \subseteq A\) be an algebra extension by \(\mathcal{D}\). The \(\mathcal{D}\)-extension \(B \subseteq A\) is \(\mathcal{D}\)-cleft (with respect to the given \(\mathcal{D}\)-comodule structure of \(A\)) if and only if the following conditions hold.

(1) \(C := A \otimes_R \mathcal{D}\) is an \(A\)-coring, with coproduct \(A \otimes_R \Delta\) and counit \(A \otimes_R \varepsilon\), the left regular \(A\)-module structure of the first factor and a right \(A\)-module structure such that the \(\mathcal{D}\)-coaction in \(A\) is a right \(A\)-module map \(A \to C\).

(2) \(B\) is an \(R\)-subring of \(A\).

(3) There exists a convolution invertible left \(R\)-module, right \(\mathcal{D}\)-comodule map \(j : \mathcal{D} \to A\).

Theorem 3.9 implies the following theorem.

Theorem 3.17. An algebra extension \(B \subseteq A\) by an \(R\)-coring \(\mathcal{D}\) is cleft if and only if it is a \(\mathcal{D}\)-Galois extension and the normal basis property holds, i.e. \(A \cong B \otimes_R \mathcal{D}\) as left \(B\)-modules right \(\mathcal{D}\)-comodules.

The following theorem is a consequence of Theorem 3.10.

Theorem 3.18. Consider an \(R\)-coring \(\mathcal{D}\) and a \(\mathcal{D}\)-cleft algebra extension \(B \subseteq A\). If there exist finite sets, \(\{v_i\} \subseteq \text{rHom}_R(\mathcal{D}, B)\) and \(\{d_i\} \subseteq \mathcal{D}\) satisfying \(\sum_i v_i(d_i) = 1_B\), then the category of right \(B\)-modules is equivalent to the category of right comodules for the \(A\)-coring \(C := A \otimes_R \mathcal{D}\), via the induction functor \(- \otimes_B A : M_B \to M^C\).

An algebra extension \(B \subseteq A\) by a right \(R\)-bialgebroid \(\mathcal{H}\) is said to be \(\mathcal{H}\)-cleft provided that it is a cleft extension by the coring underlying \(\mathcal{H}\). Via the canonical isomorphism \(M \otimes_R \mathcal{H} \cong M \otimes_A A \otimes_R \mathcal{H}\), for any right \(A\)-module \(M\), right comodules for the \(A\)-coring \(A \otimes_R \mathcal{H}\) are identified with right modules for the monoid \(A\) in \(\mathcal{M}_A^R\). They are called (right-right) relative Hopf modules. Since the unit element in the algebra \(\mathcal{H}\) is grouplike, Theorem 3.18 implies that, for an \(\mathcal{H}\)-cleft algebra extension \(B \subseteq A\), the functor \(- \otimes_B A\) is an equivalence from the category of right \(B\)-modules to the category \(\mathcal{M}_A^R\) of right-right relative Hopf modules.
3.5 Cleft extensions by pure Hopf algebroids

As we have seen in Section 2.5, a Hopf algebroid incorporates two coring (and bialgebroid) structures. Along the lines in Section 3.4, one can consider cleft extensions by either one of them. However, it turns out that there is a third, more useful notion of a cleft extension by a Hopf algebroid. It is more useful in the following sense. First, it is the notion for which – analogously to the case of Hopf algebras – the total algebra of a Hopf algebroid \( \mathcal{H} \) is an \( \mathcal{H} \)-cleft extension of the base algebra. Second, this definition of a cleft extension by a Hopf algebroid allows one to extend Theorem 3.5 to algebra extensions by Hopf algebroids. This definition was proposed in [8, Corrigendum].

Definition 3.19. Let \( \mathcal{H} \) be any (not necessarily pure) Hopf algebroid, with structure maps denoted as in Definition 2.18. We say that an \( \mathcal{H} \)-extension \( B \subseteq A \) is cleft provided that the following properties are obeyed.

1. In addition to its \( R \)-ring structure \( \eta_R : R \to A \), \( A \) is also an \( L \)-ring, with some unit map \( \eta_L : L \to A \).
2. \( B \) is an \( L \)-subring of \( A \).
3. There exist morphisms \( j \in L \text{Hom}_\mathcal{H}(H, A) \) and \( \tilde{j} \in R \text{Hom}_L(H, A) \), such that
   \[
   \mu_R \circ (j \otimes \tilde{j}) \circ \Delta_R = \eta_L \circ \epsilon_L, \quad \text{and} \quad \mu_L \circ (\tilde{j} \otimes j) \circ \Delta_L = \eta_R \circ \epsilon_R,
   \]
   where \( \mu_L \) and \( \mu_R \) denote the multiplications in \( A \), as an \( L \)-ring and as an \( R \)-ring, respectively. The module structures in \( H \) are determined by the respective coring structures and the module structures in \( A \) are determined by the respective ring structures (see the proof of Proposition 3.21 below).

We divide our study of cleft extensions by Hopf algebroids into two parts. In this section we restrict to pure Hopf algebroids. Under this restriction, we can apply the methods and results from Section 3.2. Treatment of the general case, i.e. cleft extensions by arbitrary Hopf algebroids, requiring some new technics, is left to next Section 3.6.

Consider a Hopf algebroid \( \mathcal{H} \) with constituent left bialgebroid \( \mathcal{H}_L \) over a base algebra \( L \), right bialgebroid \( \mathcal{H}_R \) over a base algebra \( R \), and antipode \( S \). Take a right \( \mathcal{H} \)-comodule algebra \( A \) with \( \mathcal{H}_R \)-coaction \( a \mapsto a^{[0]} \otimes_R a^{[1]} \) and \( \mathcal{H}_L \)-coaction \( a \mapsto a^{[0]} \otimes_L a^{[1]} \), related via (8). The comodule algebra \( A \) determines an \( A \)-coring \( C_R := A \otimes_R H \), with \( A \)-\( A \) bimodule structure
   \[
   a_1(a \otimes h)a_2 = a_1a_2^{[0]} \otimes_R h a_2^{[1]}, \quad \text{for } a_1, a_2 \in A, \ a \otimes h \in A \otimes_R H.
   \]

Using the notations in Definition 2.18 the coproduct in \( C_R \) is \( A \otimes_R \Delta_R \) and the counit is \( A \otimes_R \epsilon_R \). Obviously, by coassociativity of \( \Delta_R \), \( C_R \) is a \( C_R \)-\( \mathcal{H}_R \) bicomodule, via the left \( C_R \)-coaction provided by the coproduct in \( C_R \) and right \( \mathcal{H}_R \)-coaction \( A \otimes_R \Delta_R \). That is to say, the \( R \)-coring \( (H, \Delta_R, \epsilon_R) \) is a right extension of \( C_R \). But we have more: by the Hopf algebroid axiom (ii) in Definition 2.18 \( C_R \) is also a \( C_R \)-\( \mathcal{H}_L \) bicomodule, via the left \( C_R \)-coaction provided by the coproduct in \( C_R \) and right \( \mathcal{H}_L \)-coaction \( A \otimes_R \Delta_L \). Thus also the \( L \)-coring \( (H, \Delta_L, \epsilon_L) \) underlying \( \mathcal{H}_L \) is a right extension of \( C_R \).

Lemma 3.20. Consider a Hopf algebroid \( \mathcal{H} \), with structure maps denoted as in Definition 2.18 and a right \( \mathcal{H} \)-comodule algebra \( A \). If \( \mathcal{H} \) is a pure Hopf algebroid then (the coring underlying) \( \mathcal{H}_L \) is a pure extension of the \( A \)-coring \( C_R := A \otimes_R H \).
\textbf{Proof.} By the isomorphism
\begin{equation}
W \otimes_A C_R \cong W \otimes_R H,
\end{equation}
for any right $A$-module $W$, right comodules for $C_R$ are identified with right $A$-modules and right $H_R$-comodules $M$, such that the $H_R$-coaction $M \to M \otimes_R H$ is a right $A$-module map with respect to the $A$-action of the codomain arising from the isomorphism (24).

If $\mathcal{H}$ is a pure Hopf algebroid then in particular the equalizer (16) in $\mathcal{M}_L$ is $H \otimes_L H$-pure, for any $C_R$-comodule $M$. By the isomorphism (24), this means $H \otimes_L H$-purity of the equalizer
\begin{equation*}
\begin{array}{c}
M \\
\bigm\to
M \otimes_A C_R \\
\bigm\to
M_A \otimes A_C R \otimes A C_R
\end{array}
\end{equation*}
in $\mathcal{M}_L$, that is, purity of the coring extension in the claim. \hfill \Box

Analogously to Proposition 3.7 one proves following [10, Proposition 6.6]. It tells us that, for a pure Hopf algebroid $\mathcal{H}$ and an algebra extension $B \subseteq A$ by $\mathcal{H}$, the right $C_R$-comodule $A$ extends to a cleft bicomodule for the coring extension $\mathcal{H}_L$ of $C_R$, introduced above, if and only if $B \subseteq A$ is an $\mathcal{H}$-cleft extension in the sense of [8, Definition 3.5] (recalled in Definition 2.18 below).

Recall that a right comodule algebra $A$ of a Hopf algebroid $\mathcal{H}$ is in particular an $R$-ring, with unit map $\eta_R : R \to A$, over the base algebra $R$ of the constituent right bialgebroid in $\mathcal{H}$.

\textbf{Proposition 3.21.} Consider a pure Hopf algebroid $\mathcal{H}$ with constituent left bialgebroid $\mathcal{H}_L$ over a base algebra $L$ and right bialgebroid $\mathcal{H}_R$ over a base algebra $R$, and an algebra extension $B \subseteq A$ by $\mathcal{H}$. The right $C_R := A \otimes_R H$-comodule $A$ extends to a cleft bicomodule for the coring extension $\mathcal{H}_L$ of $C_R$ if and only if $B \subseteq A$ is an $\mathcal{H}$-cleft extension.

\textbf{Proof.} (Sketch.) Since $\mathcal{H}$ is a pure Hopf algebroid, $A$ is a right $\mathcal{H}$-comodule algebra if and only if $A$ is a right $\mathcal{H}_R$-comodule algebra, in which case it is a right $C_R$-comodule as well. The right $C_R$-comodule $A$ extends to an $L$-$C_R$ bicomodule if and only if it is also a left $L$-module such that the left $L$-action is a right $A$-module map and a right $H_R$-comodule map. The left $L$-action is a right $A$-module map if and only if there exists an algebra map $\eta_L : L \to A$ in terms of which the left action by $l \in L$ on $A$ is given by left multiplication by $\eta_L(l)$. The left $L$-action is a right $H_R$-comodule map if and only if $\eta_L(l) \in B$, for all $l \in L$. Hence properties (1) and (2) in Definition 3.12 are equivalent to $A$ being an $L$-$C_R$ bicomodule.

Analogously to (21), the Morita context $\mathcal{M}(A)$, corresponding the the $L$-$C_R$ bicomodule $A$ via (19), is isomorphic to
\begin{equation}
\begin{aligned}
\left( \_L \text{Hom}_{L}(H, B) , X , \_L \text{Hom}^{\mathcal{H}}(H, A) , L_{\mathcal{H}} \text{Hom}^{\mathcal{H}}(H^w, A) ; \smallcirc , \smalltriangleleft \right).
\end{aligned}
\end{equation}

The (co)module structures in (25) need to be explained. Let us use the notations introduced in, and after Definition 2.18, and in the paragraph after Definition 2.20. An element $v \in \_L \text{Hom}_{L}(H, B)$ is a bimodule map with respect to the actions
\begin{equation}
v(s_l(l)t_L(l')h) = \eta_L(h)v(h)\eta_L(l'), \quad \text{for } l, l' \in L, \ h \in H.
\end{equation}

As a $k$-module,
\begin{equation}
X = \{ \ w \in \_R \text{Hom}_{R}(H, A) \mid w(h^{(2)})^{[0]} \otimes h^{(1)} \tilde{w}(h^{(2)})^{[1]} = \tilde{w}(h^{(1)}) \otimes h^{(2)}, \ \forall h \in H \},
\end{equation}
where $\tilde{w} \in \_R \text{Hom}_{R}(H, A)$ is a bimodule map with respect to the actions
\begin{equation}
\tilde{w}(hs_R(r)t_R(r')) = \eta_R(r')\tilde{w}(h)\eta_R(r), \quad \text{for } r, r' \in R, \ h \in H.
\end{equation}

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In \( L\text{Hom}^3(H, A) \), \( H \) is a left \( L \)-module via \( s_L \), and \( A \) is a left \( L \)-module via \( \eta_L \). Note that the coproducts in \( \mathcal{H} \) are left \( L \)-module maps and (since \( B \) is an \( L \)-ring by definition) so are the \( \mathcal{H} \)-coactions in \( A \). Thus both \( H \) and \( A \) are \( L - \mathcal{H} \) bicomodules. Since an element \( p \in L\text{Hom}^3(H, A) \) is a comodule map with respect to the regular \( \mathcal{H} \)-coactions, it is \( R-R \) bilinear in the sense that \( p(s_L(r)h_R(r')) = \eta_L(r)p(h)\eta_R(r') \), for \( h \in H \) and \( r, r' \in R \). In particular, \( p \) is an \( L-R \) bimodule map with respect to the actions

\[
p(s_L(l)h_R(r)) = \eta_L(l)p(h)\eta_R(r), \quad \text{for } h \in H, \ l \in L, \ r \in R.
\]

In \( L^{op}\text{Hom}^3(H^{tw}, A) \), \( H^{tw} \) is the same \( k \)-module as \( H \). It is considered to be a left \( L^{op} \)-module via \( t_L \), and \( A \) is a left \( L^{op} \)-module via right multiplication by \( \eta_L \). \( H^{tw} \) is a right \( R \)-module via \( t_R \), a right \( L \)-module via \( t_R \circ \epsilon_R \circ t_L \), and a right \( \mathcal{H} \)-comodule via the twisted \( \mathcal{H}_R \)-coaction \( h \mapsto h(2) \otimes_R S(h(1)) \) and \( \mathcal{H}_L \)-coaction \( h \mapsto h(2) \otimes_L S(h(1)) \). Note that both the twisted coactions in \( H^{tw} \) and the \( \mathcal{H} \)-coactions in \( A \) are left \( L^{op} \)-module maps, thus both \( H^{tw} \) and \( A \) are \( L^{op} - \mathcal{H} \) bicomodules. An element \( q \in L^{op}\text{Hom}^3(H^{tw}, A) \) is \( R-R \) bilinear in the sense that \( q(t_R(r)ht_R(r')) = \eta_R(r')q(h)\eta_R(r) \), for \( h \in H \) and \( r, r' \in R \). In particular, \( q \) is an \( R-L \) bimodule map with respect to the actions

\[
q(t_L(l)ht_R(r)) = \eta_R(r)q(h)\eta_L(l), \quad \text{for } h \in H, \ l \in L, \ r \in R.
\]

With the bimodule structures (26), (28), (29) and (30) in mind, in the Morita context (25) all algebra and bimodule structures and also the connecting maps \( \dagger \) and \( \dagger' \) can be written as a generalized convolution product in [8, Section 3]:

\[
(f, g) \mapsto \mu_Q \circ (f \otimes g) \circ \Delta_Q, \quad \text{for } P, Q, S \in \{L, R\}, \ f \in p\text{Hom}_Q(H, A), \ g \in q\text{Hom}_S(H, A),
\]

where \( \mu_Q \) denotes multiplication in (the \( Q \)-ring) \( A \). Thus the equivalence of the clef property of the \( L-C_R \) bicomodule \( A \) and property (3) in Definition 3.19 follows by [8, Lemmata 3.7 and 3.8], stating that the (generalized) convolution inverse of \( j \in L\text{Hom}^3(H, A) \), if it exists, belongs to \( L^{op}\text{Hom}^3(H^{tw}, A) \).

In the same way as Definition 3.19 of a cleft extension by a Hopf algebroid \( \mathcal{H} \) combines the two bialgebroids in \( \mathcal{H} \), so does the following analogue of Theorem 3.2, which is a consequence of Theorem 3.9. It was proven for arbitrary Hopf algebroids in [8, Theorem 3.12] (see Theorem 3.26).

**Theorem 3.22.** Consider a pure Hopf algebroid \( \mathcal{H} \) with constituent left bialgebroid \( \mathcal{H}_L \) over a base algebra \( L \) and right bialgebroid \( \mathcal{H}_R \) over a base algebra \( R \). An algebra extension \( B \subset A \) by \( \mathcal{H} \) is clef if and only if the following properties hold.

1. The extension \( B \subset A \) is \( \mathcal{H}_R \)-Galois.
2. The normal basis property holds, i.e. \( A \cong B \otimes_L H \), as \( L \)-\( B \)-modules and \( R \)-\( H \)-comodules.

Since the unit element of the algebra underlying a Hopf algebroid \( \mathcal{H} \) is grouplike, Theorem 3.10 implies the following Strong Structure Theorem. Recall that, for a right \( \mathcal{H} \)-comodule algebra \( A \), a right-right \( (\mathcal{H}, A) \)-relative Hopf module is defined as a right module of the monoid \( A \) in \( M^3_{\mathcal{H}} \).

**Theorem 3.23.** For a cleft algebra extension \( B \subset A \) by a pure Hopf algebroid \( \mathcal{H} \), the category of right \( B \)-modules is equivalent to the category \( M^3_{\mathcal{H}} \) of right-right relative Hopf modules, via the induction functor \(- \otimes_B A : M_B \to M^3_{\mathcal{H}} \).

A generalization of Theorem 3.23 for any Hopf algebroid is given in Theorem 3.27.
3.6 Cleft extensions by arbitrary Hopf algebroids

In Section 3.5 we applied the theory of cleft bicomodules of pure coring extensions to study cleft extensions by pure Hopf algebroids. In Definition 3.19 the cleft property of algebra extensions by arbitrary Hopf algebroids was defined. The aim of the current section is to show that also the definition of any Hopf algebroid cleft extension in Definition 3.19 is equivalent to the existence of mutually inverse elements in the Morita context (25), which is perfectly meaningful for any Hopf algebroid. Although derivation of the the Morita context (25) in the general case below is very similar to the methods in the previous sections, it is not known to correspond to any coring extension. In the final part of the section we describe cleft extensions of arbitrary Hopf algebroids as crossed products with invertible cocycles, extending in this way Theorem 3.5.

For an arbitrary Hopf algebroid \( \mathcal{H} \) and a right \( \mathcal{H} \)-comodule algebra \( A \), the coring extension in Lemma 3.20 is not known to be pure. Hence a cotensor-functor \( - \otimes_{\mathcal{C}_R} \mathcal{C}_R : \mathcal{M}^{\mathcal{C}_R} \rightarrow \mathcal{M}^{\mathcal{C}_L} \) is not available to derive the Morita context used in the previous section. However, if we assume that there is an algebra map \( \eta_L \) from the base algebra \( L \) of the constituent left bialgebroid \( \mathcal{H}_L \) to \( B := A^{co\mathcal{H}_R} \subseteq A \), then we are equipped with two functors from the category \( \mathcal{M}^{\mathcal{H}_R} \) of right \( A \)-modules in \( \mathcal{M}^{\mathcal{H}_L} \) to \( \mathcal{M}^{\mathcal{H}_L} \). The forgetful functor \( U \) and \( V \) differ by trivial isomorphisms from the respective functors \( - \otimes_{\mathcal{C}_R} \mathcal{C}_R \) and \( \text{Hom}_{\mathcal{C}_R}(A, -) \otimes_{\mathcal{C}_R} \mathcal{H} : \mathcal{M}^{\mathcal{C}_R} \rightarrow \mathcal{M}^{\mathcal{C}_L} \). The following proposition is a detailed version of an observation in [Corrigendum].

**Proposition 3.24.** Let \( \mathcal{H} \) be a Hopf algebroid with structure maps denoted as in Definition 2.18 and \( B \subseteq A \) be an \( \mathcal{H} \)-extension. Assume that there exists an algebra map \( \eta_L : L \rightarrow B = A^{co\mathcal{H}_R} \) and consider the above functors \( U \) and \( V \) from \( \mathcal{M}^{\mathcal{H}_R} \) to \( \mathcal{M}^{\mathcal{H}_L} \). The corresponding Morita context (25) is isomorphic to the Morita context (25).

**Proof.** The four maps establishing the stated isomorphism of Morita contexts are very similar to those in the proof of [Corrigendum]. Proposition 3.1:

- \( \alpha_1 : L \text{Hom}_L(H, B) \rightarrow \text{Nat}(V, V), \)
  \( v \mapsto [\Phi_M : M^{co\mathcal{H}_R} \otimes_{\mathcal{L}} H \rightarrow M^{co\mathcal{H}_R} \otimes_{\mathcal{L}} H; \)
  \( n \otimes_L h \mapsto nv(h_1(1)) \otimes_L h_2(2)]; \)

- \( \alpha_1^{-1} : \text{Nat}(V, V) \rightarrow L \text{Hom}_L(H, B), \)
  \( \Phi \mapsto (B \otimes_{\mathcal{L}} \epsilon_L)(\Phi_A(1_B \otimes L -)); \)

- \( \alpha_2 : X \rightarrow \text{Nat}(U, U), \)
  \( u \mapsto [\Xi_M^L : M \rightarrow M; m \mapsto [0]_u(m[1])]; \)

- \( \alpha_3 : L \text{Hom}^{\mathcal{H}}(H, A) \rightarrow \text{Nat}(V, U), \)
  \( p \mapsto [\Theta^L_M : M^{co\mathcal{H}_R} \otimes_{\mathcal{L}} H \rightarrow M; n \otimes_L h \mapsto np(h)]; \)

- \( \alpha_4 : L^{op} \text{Hom}^{\mathcal{H}}(H^{tw}, A) \rightarrow \text{Nat}(U, V), \)
  \( q \mapsto [\Omega^L_M^q : M \rightarrow M^{co\mathcal{H}_R} \otimes_{\mathcal{L}} H; m \mapsto [0]_q(m[0]) \otimes_{\mathcal{L}} m[1]]; \)

- \( \alpha_4^{-1} : \text{Nat}(U, V) \rightarrow L^{op} \text{Hom}^{\mathcal{H}}(H^{tw}, A), \)
  \( \Omega \mapsto (A \otimes_{\mathcal{L}} \epsilon_L)(\Omega_A \otimes_{\mathcal{R}}(1_A \otimes_{\mathcal{R}} -)). \)

By the \( L-L \) bilinearity of \( v \), \( \Phi_M^v \) is a well defined map \( M^{co\mathcal{H}_R} \otimes_{\mathcal{L}} H \rightarrow M \otimes_{\mathcal{L}} H \). Since \( M^{co\mathcal{H}_R} \) is a right \( B \)-module and the range of \( v \) is in \( B \), the range of \( \Phi_M^v \) is in \( M^{co\mathcal{H}_R} \otimes_{\mathcal{L}} H \) as needed. By the right \( \mathcal{H} \)-colinearity of the coproduct \( \Delta_L \) in \( \mathcal{H}_L \) and \( \Phi_M^v \) is right \( \mathcal{H} \)-colinear. Naturality of
In the converse direction, for $\Phi \in \text{Nat}(V,V)$, $\Phi_A$ is right $H$-colinear hence in particular right $L$-linear. Since also the $H_L$-counit $\varepsilon_L$ is right $L$-linear, so is $\alpha_1^{-1}(\Phi)$. In order to see that $\alpha_1^{-1}(\Phi)$ is also left $L$-linear, note that for any $M \in \mathcal{M}_A^{H_L}$ and $n \in M^{co^2H_L}$, the map

$$A \rightarrow M, \quad a \mapsto na \quad (31)$$

is a morphism in $\mathcal{M}_A^{H_L}$. Thus by naturality of $\Phi$,

$$\Phi_M(nb \otimes h) = n\Phi_A(b \otimes h), \quad \text{for } n \in M^{co^2H_L}, \ b \otimes h \in B \otimes_L H. \quad (32)$$

Applying it to $M = A$, we conclude that $\Phi_A$ is left $B$-linear thus in particular left $L$-linear. This proves the left $L$-linearity of $\alpha_1^{-1}(\Phi)$. By the right $L$-linearity of $v \in L\text{Hom}_L(H,B)$ and the counitality of $\Delta_L$, for all $h \in H$,

$$(\alpha_1^{-1} \circ \alpha_1)(v)(h) = v(h(1))\varepsilon_L(h(2)) = v(h).$$

In order to see that $\alpha_1^{-1}$ is also the right inverse of $\alpha_1$, use the right $H_L$-colinearity of $\Phi_A$, i.e. the identity $(B \otimes_L \Delta_L) \circ \Phi_A = (\Phi_A \otimes_L H) \circ (B \otimes_L \Delta_L)$, (in the second equality) and (32) (in the last equality), for $n \otimes \Delta_L \in M^{co^2H_L} \otimes_L H$:

$$(\alpha_1 \circ \alpha_1^{-1})(\Phi)(n \otimes h) = n(B \otimes_L \varepsilon_L)((\Phi_A(1_B \otimes_L h(1))) \otimes h(2)) = n\Phi_A(1_B \otimes_L h) = \Phi_M(n \otimes h).$$

By the left $R$-linearity of $u \in X$, $\Xi^u_M$ is a well defined map $M \rightarrow M$. In order to check the $H_R$-colinearity of $\Xi^u_M$, use the $H_R$-colinearity of the $A$-action on $M$ and the defining condition (27) of $X$:

$$\Xi_M^u(m)[0] \otimes R \Xi_M^u(m)[1] = m[0][0]u(m[2]) [0] \otimes_R m[1][0]u(m[2])[1] = m[0][0]u(m[1]) \otimes_R m[2] = \Xi_M^u(m)[0] \otimes_R m[1],$$

for all $m \in M$. We conclude by Proposition 2.21 that $\Xi^u_M$ is right $H$-colinear. Naturality of $\Xi^u$ is obvious. In order to check that $\alpha_2$ is a ring map, compute for $m \in M, u, u' \in X$,

$$\Xi_M^{\eta_R \circ \Xi_R}(m) = m[0][0] \eta_R(\varepsilon_R(m[1])) = m; \quad (\Xi_M^u \circ \Xi_M^{u'})(m) = m[0][0]u'(m[2])[0]u(m[1])u'(m[2])[1] = m[0][0]u(m[1])u(m[2]) = m[0][0]u'(m[1]) = \Xi_M^{u'u}(m),$$

where in the first equality of the last line we used (27).

In the converse direction, for $\Xi \in \text{Nat}(U,U)$, $\Xi_A \otimes_R H$ is right $H$-colinear hence in particular right $R$-linear. Since also the $H_R$-counit $\varepsilon_R$ is right $R$-linear, so is $\alpha_2^{-1}(\Xi)$. For any right $A$-module $W$, $W \otimes_R H$ is an object in $\mathcal{M}_A^{H_L}$, via the $H$-comodule structure of the second factor (both coproducts $\Delta_L$ and $\Delta_R$ are left $R$-linear) and the right $A$-action $(w \otimes_R h)a = wa[0] \otimes_R ha[1]$. Moreover, for any $w \in W$, the map

$$A \otimes_R H \rightarrow W \otimes_R H, \quad a \otimes_R h \mapsto wa \otimes_R h \quad (33)$$
is a morphism in $\mathcal{M}_{A}^{2\mathcal{H}}$. Thus by the naturality of $\Xi$,

$$\Xi_{W \otimes_{R} H}(wa \otimes_{R} h) = w\Xi_{A \otimes_{R} H}(a \otimes_{R} h), \quad \text{for } w \in W, \ a \otimes_{R} h \in A \otimes_{R} H. \quad (34)$$

Applying it to $W = A$, we conclude that $\Xi_{A \otimes_{R} H}$ is left $A$-linear hence in particular left $R$-linear. This proves the left $R$-linearity of $\alpha_{2}^{-1}(\Xi)$. In order to see that $\alpha_{2}^{-1}(\Xi)$ belongs to $X$, observe that for any $M \in \mathcal{M}_{A}^{2\mathcal{H}}$, the $\mathcal{H}_{R}$-coaction $M \rightarrow M \otimes_{R} H$, $m \mapsto m^{[0]} \otimes_{R} m^{[1]}$ is a morphism in $\mathcal{M}_{A}^{2\mathcal{H}}$. Thus by the naturality of $\Xi$,

$$\Xi_{M}(m^{[0]} \otimes_{R} \Xi_{M}(m^{[1]}) = \Xi_{M \otimes_{R} H}(m^{[0]} \otimes_{R} m^{[1]}), \quad \text{for } m \in M. \quad (35)$$

Applying (34) for $W = A \otimes_{R} H$ (in the second equality), (35) for $M = A \otimes_{R} H$ (in the third equality) and the right $\mathcal{H}_{R}$-co linearity of $\Xi_{A \otimes_{R} H}$ (in the last equality), one computes for $h \in H$,

$$\alpha_{2}^{-1}(\Xi)(h^{(2)})^{[0]} \otimes_{R} h^{(1)} \alpha_{2}^{-1}(\Xi)(h^{(2)})^{[1]} = (1_{A} \otimes_{R} h^{(1)})(A \otimes_{R} \varepsilon_{R})(\Xi_{A \otimes_{R} H}(1_{A} \otimes_{R} h^{(2)})) = (A \otimes_{R} H \varepsilon_{R})(\Xi_{A \otimes_{R} H \otimes_{R} H}(1_{A} \otimes_{R} h^{(2)})) = ((A \otimes_{R} H \varepsilon_{R}) \circ (A \otimes_{R} \Delta_{R}))(\Xi_{A \otimes_{R} H}(1_{A} \otimes_{R} h)) = (A \otimes_{R} \varepsilon_{R} \otimes_{R} H) \circ (A \otimes_{R} \Delta_{R}))(\Xi_{A \otimes_{R} H}(1_{A} \otimes_{R} h)) = \alpha_{2}^{-1}(\Xi)(1_{A} \otimes_{R} h^{(1)}) \otimes_{R} h^{(2)}. \quad (27)$$

Making use of (27), the right $R$-linearity of $u \in X$ and the counitality of $\Delta_{R}$, we see that for all $h \in H$,

$$(\alpha_{2}^{-1} \circ \alpha_{2})(u)(h) = u(h^{(2)})^{[0]} \eta_{R}(\varepsilon_{R}(h^{(1)})(u(h^{(2)})^{[1]})) = u(h^{(1)}) \eta_{R}(\varepsilon_{R}(h^{(2)})) = u(h).$$

Using (34) (in the second equality) and (35) (in the third equality) one checks that for $\Xi \in \text{Nat}(U, U)$ and $m \in M$, also

$$(\alpha_{2} \circ \alpha_{2}^{-1})(\Xi)(m) = (M \otimes_{R} \varepsilon_{R})(m^{[0]} \Xi_{A \otimes_{R} H}(1_{A} \otimes_{R} m^{[1]})) = (M \otimes_{R} \varepsilon_{R})(\Xi_{M \otimes_{R} H}(m^{[0]} \otimes_{R} m^{[1]})) = \Xi_{M}(m^{[0]} \eta_{R}(\varepsilon_{R}(\Xi_{M}(m^{[1]})))) = \Xi_{M}(m). \quad (36)$$

By the left $L$-linearity of $p \in L_{H}(H, A)$, $\Theta_{M}^{p}$ is a well defined map. By the right $\mathcal{H}$-co linearity of $p$, $\Theta_{M}^{p}$ is $\mathcal{H}$-colinear. Naturality of $\Theta$ is obvious. Compatibility of $\alpha_{3}$ with the bimodule structures is checked by the following simple computations, for $p \in L_{H}(H, A)$, $v \in L_{H}(H, B)$, $u \in X$ and $n \otimes_{L} h \in M^{\text{co}3\mathcal{H}} \otimes_{L} H$.

$$(\Theta_{M}^{p} \circ \Phi_{M}^{v})(n \otimes_{L} h) = mv(h_{(1)})p(h_{(2)}) = n(p(h_{(1)})) = np(h_{(1)})u(h_{(2)}) = n(pu)(h) = \Theta_{M}^{pu}(n \otimes_{L} h).$$

In the converse direction, using that (31) is a morphism in $\mathcal{M}_{A}^{2\mathcal{H}}$, the naturality of $\Theta \in \text{Nat}(V, U)$ implies that

$$\Theta_{M}(nb \otimes_{L} h) = n\Theta_{A}(b \otimes_{L} h), \quad \text{for } n \in M^{\text{co}3\mathcal{H}}, b \otimes_{L} h \in B \otimes_{L} H. \quad (36)$$

In particular, $\Theta_{A}$ is left $B$-linear hence in particular left $L$-linear. This proves that $\alpha_{3}^{-1}(\Theta)$ is left $L$-linear. By the right $\mathcal{H}$-colinearity of $\Theta_{A}$, $\alpha_{3}^{-1}(\Theta)$ is right $\mathcal{H}$-colinear. It is an immediate
In the second and penultimate equalities of both computations we used the right $H_R$-colinearity of the $A$-action on $M$. In the second equality we used the right $H_R$-colinearity of $q$. In the third equality we used the right $H_R$-colinearity of the $H_R$-coproduct $\Delta_R$. The fourth equality follows by the second one of the antipode axioms (iv) in Definition 2.18. In the penultimate equality we applied the Hopf algebraoid axiom $t_R \circ \epsilon_R \circ s_L = s_L$, cf. Definition 2.18 (i). The same Hopf algebraoid axiom $t_R \circ \epsilon_R \circ s_L = s_L$ is applied again in the last equality, together with the right $H$-linearity of $q$ and the counitality of $\Delta_R$. By the right $H$-colinearity of the $H_L$-coaction on $M$, $\Omega_M$ is right $H$-colinear. Naturality of $\Omega_M$ is obvious. The compatibility of $\alpha_4$ with the bimodule structures is checked by the following computations, for $v \in L \text{Hom}_L(H,B)$, $u \in Q$, $q \in L \text{Hom}^q(H^q,A)$ and $m \in M$:

\[
\begin{align*}
(\Phi_M^q \circ \Omega_M^q)(m) &= m^{[0]} q(m^{[1]}) v(m^{[1]}) \otimes m^{[2]} = m^{[0]} q(m^{[1]}) v(m^{[1]}) \otimes m^{[1]}(3) \\
&= m^{[0]} q(v(m^{[1]}) \otimes m^{[1]}(2)) = m^{[0]} q(v(m^{[1]}) \otimes m^{[1]}) = \Omega_M^q(m); \\
(\Omega_M^q \circ \Xi_M^q)(m) &= m^{[0]} q(m^{[1]}) \otimes m^{[1]}(3) = m^{[0]} q(m^{[1]}) \otimes m^{[1]}(1) \\
&= m^{[0]} q(m^{[1]}) \otimes m^{[1]}(2) = m^{[0]} q(m^{[1]}) \otimes m^{[1]}(1) = \Omega_M^q(m).
\end{align*}
\]

In the second and penultimate equalities of both computations we used the right $H_R$-colinearity of the right $H_R$-coaction on $M$. In the third equality of the second computation we used the right $H_R$-colinearity of the $A$-action on $M$ and coassociativity of the $H_R$-coaction. In the fourth equality (27) has been used and in the fifth equality we made use of the right $H_L$-colinearity of $\Delta_R$.

In the definition of $\alpha_4^{-1}$ the isomorphism

\[
W \to (W \otimes H)^{co\mathfrak{H}_R}, \quad w \mapsto w \otimes 1_H; \quad \quad (W \otimes H)^{co\mathfrak{H}_R} \to W, \quad \sum_l w_l \otimes h_l \mapsto \sum_l w_l \eta_R(\epsilon_R(h_l))
\]

is implicitly used, for $W \in \mathcal{M}_A$. This isomorphism induces a right $B$-action on $(W \otimes H)^{co\mathfrak{H}_R}$, as $(\sum_l w_l \otimes h_l) b = \sum_l w_l b \otimes h_l$ (meaningful in light of Remark 2.16). In particular, since $B$ is an $L$-ring, (38) is a right $L$-module isomorphism. For $\Omega \in \text{Nat}(U,V)$, the map $\Omega_{A \otimes_R H}$ is right
$\mathcal{H}$-colinear hence in particular right $L$-linear. Since also $\varepsilon_L$ is right $L$-linear, so is $\alpha_4^{-1}(\Omega)$. For any $r \in R$, the map $A \otimes_R H \to A \otimes_R H$, $a \otimes_R h \mapsto a \otimes_R t_R(r)h$ is a morphism in $\mathcal{M}_A^{\mathcal{H}}$. Hence it follows by the naturality of $\Omega$ that $\alpha_4^{-1}(\Omega)$ is right $R$-linear. Since the right $\mathcal{H}_R$-coaction on any $M \in \mathcal{M}_A^{\mathcal{H}}$ is a morphism in $\mathcal{M}_A^{\mathcal{H}}$, naturality of $\Omega$ implies

$$\Omega_{M \otimes_R H}(m[0] \otimes_R m[1]) = \Omega_M(m) \in M \otimes_L H, \quad \text{for } m \in M,$$

where the isomorphism (38) is implicitly used. Applying $M \otimes_L \varepsilon_L$ to both sides and using the isomorphism (33), it follows that

$$(M \otimes_R H)^{\mathcal{H}} \otimes_L L \left(\Omega_{M \otimes_R H}(m[0] \otimes_R m[1])\right) = (M \otimes_L \varepsilon_L)(\Omega_M(m)) \otimes_L 1_H, \quad \text{for } m \in M,$$

as elements of $(M \otimes_R H)^{\mathcal{H}} \subseteq M \otimes H$. Moreover, using that (33) is a morphism in $\mathcal{M}_A^{\mathcal{H}}$ and the naturality of $\Omega$, it follows that for any right $A$-module $W$, $\Omega_{W \otimes_R H}(wa \otimes h) = w\Omega_{A \otimes_R H}(a \otimes h), \quad \text{for } w \in W, \ a \otimes h \in A \otimes_R H.$ (41)

Combining (40) for $M = A \otimes_R H$, and (41) for $W = A \otimes_R H$, we obtain that

$$\alpha_4^{-1}(\Omega)(h) \otimes_R 1_H = \alpha_4^{-1}(\Omega)(h^{(2)})[0] \otimes_R h^{(1)} \alpha_4^{-1}(\Omega)(h^{(2)})[1].$$

(42)

By the right $R$-linearity of $\alpha_4^{-1}(\Omega)$ and the Hopf algebroid axiom $t_R \circ \varepsilon_R \circ s_L = s_L$, it follows that for $l \in L$ and $h \in H$, $\alpha_4^{-1}(\Omega)(s_L(l)h) \otimes_R 1_H = \alpha_4^{-1}(\Omega)(h) \otimes_R s_L(l)$. Thus the expression $\alpha_4^{-1}(\Omega)(h^{(2)}) \otimes_L S(h^{(1)})$ is meaningful, and with identity (42) at hand, it satisfies

$$\alpha_4^{-1}(\Omega)(h^{(2)}) \otimes_R S(h^{(1)}) = \alpha_4^{-1}(\Omega)(h^{(2)})[0] \otimes_R S(h^{(1)})[1] \alpha_4^{-1}(\Omega)(h^{(2)})[1] = \alpha_4^{-1}(\Omega)(h^{(2)})[0] \otimes_R \alpha_4^{-1}(\Omega)(h^{(2)})[1].$$

In the first equality we used (42) and in the second one we used the right $\mathcal{H}_R$-colinearity of $\Delta_L$. The third equality follows by the first one of the antipode axioms in Definition 2.18 iv). In the penultimate equality we used that by the right $R$-linearity and unitality the $\mathcal{H}_R$-coaction on $A$, it follows that $\eta_R(r)[0] \otimes_R \eta_R(r)[1] = 1_A \otimes_R s_R(r)$, for any $r \in R$, and that the multiplication in $A$ is a right $\mathcal{H}_R$-comodule map. The last equality follows by noting that by (41) the map $\Omega_{A \otimes_R H}$ is left $A$-linear, hence $\alpha_4^{-1}(\Omega)$ is left $R$-linear in the sense that, for $r \in R$ and $h \in H$,

$$\eta_R(r) \alpha_4^{-1}(\Omega)(h) = \left(\alpha_4^{-1}(\Omega)(h^{(2)})[0] \otimes_R S(h^{(1)})[1] \alpha_4^{-1}(\Omega)(h^{(2)})[1]] = \left(\eta_R(\varepsilon_R(h^{(1)})) \alpha_4^{-1}(\Omega)(h^{(2)})[0] \otimes_R \alpha_4^{-1}(\Omega)(h^{(2)})[1].$$

This proves that $\alpha_4^{-1}(\Omega)$ is right $\mathcal{H}_R$-colinear, hence right $\mathcal{H}$-colinear by Proposition 2.21. By the right $\mathcal{H}_R$-colinearity of $q \in L^{op}\text{Hom}_R(\mathcal{H}, H)$, the right $\mathcal{H}_R$-colinearity of $\Delta_L$, the second antipode axiom in Definition 2.18 iv), the Hopf algebroid axiom $s_L = t_R \circ \varepsilon_R \circ s_L$ in Definition 2.18 i), the left $R$-linearity of $q$ and counitality of $\Delta_L$, for any $h \in H$,
That is, using the isomorphism (28), we conclude that \( \Omega^\ell_{L \otimes_R H} (1_A \otimes_R h) = s(h) \otimes_R h \). With this identity at hand, by the right \( L \)-linearity of \( q \) and countability of the coproduct in \( \mathcal{H}_L \), it follows that \( (\alpha_4^1 \circ \alpha_4)(q) = q \). On the other hand, by (41), (39) and the right \( \mathcal{H}_L \)-colinearity of \( \Omega_M \), for \( \Omega \in \text{Nat}(U, V) \) and \( m \in M \),

\[
(\alpha_4 \circ \alpha_4^{-1})(\Omega)_M(m) = m_0[0] (A \otimes \varepsilon \Lambda \otimes \varepsilon \Lambda) (\Omega) \otimes m_1[1] = (M \otimes \varepsilon \Lambda)(\Omega)(m_0[0] \otimes m_0[1]) \otimes m_1[1] = (M \otimes \varepsilon \Lambda \otimes H)(\Omega)(m_0(0)) \otimes m_1[1] = ((M \otimes \varepsilon \Lambda \otimes H) \circ (M \otimes \Delta_L)) (\Omega)(m) = \Omega_M(m).
\]

It remains to check the compatibility of the constructed isomorphisms with the connecting maps of both Morita contexts. For \( p \in \text{Hom}^{\mathcal{H}}(H, A) \) and \( q \in L^\text{op} \text{Hom}^{\mathcal{H}}(H^w, A) \), \( m \in M \) and \( n \otimes L h \in M^{co\mathcal{H}_L \otimes_L H} \),

\[
(\Theta^\ell_M \circ \Omega^\ell_M)(m) = m_0[0] q(m_0[1]) p(m_1[1]) = m_0[0] q(p(m_1[1])) p(m_1[2]) = m_0[0] q(p^L) p(m_1[1]) = \varepsilon_M p^L p(m) ;
\]

\[
(\Omega^\ell_M \circ \Theta^\ell_M)(n \otimes h) = np(m)[0] q(p(m)[1]) \otimes p(m)[1] = np(p_0(1)) q(h(1))[0] \otimes h(2) = n(p_0 \otimes h)(h(1)) \otimes h(2) = \Phi_M^{p_0 \otimes h}. \]

This completes the proof. \( \square \)

Proposition 3.24 justifies that (20) is a well defined Morita context, for any Hopf algebroid \( \mathcal{H} \) and right \( \mathcal{H} \)-comodule algebra \( A \) such that there is an algebra map from the base algebra \( L \) of the constituent left bialgebroid \( \mathcal{H}_L \) to the \( \mathcal{H}_R \)-coinvariant subalgebra \( B \) of \( A \). In light of [8, Lemmata 3.7 and 3.8], the existence of mutually inverse elements in the isomorphic Morita contexts in Proposition 3.24 is equivalent to the cleft property of the \( \mathcal{H} \)-extension \( B \subseteq A \) in the sense of Definition 3.19.

**Example 3.25.** Proposition 3.24 implies in particular that, for any Hopf algebroid \( \mathcal{H} \) with constituent left bialgebroid \( \mathcal{H}_L \) over a base algebra \( L \) and right bialgebroid \( \mathcal{H}_R \) over a base algebra \( R \), the \( \mathcal{H} \)-extension \( L \subseteq H \), given by the source map \( S_L \) in \( \mathcal{H}_L \) (equivalently, the extension \( R^\text{op} \subseteq H \), given by the target map \( t_R \) in \( \mathcal{H}_R \)), is \( \mathcal{H} \)-cleft. Indeed, the unit of the \( R \)-ring \( H \) is the source map \( S_L \) in \( \mathcal{H}_L \). \( H \) is an \( L \)-ring via the source map \( S_L \) in \( \mathcal{H}_L \), and the coinvariants \( L \cong \{ s_L(l) \mid l \in L \} \cong \{ t_R(r) \mid r \in R \} \) (cf. Example 2.29 (1)) form an \( L \)-subring by axiom (i) in Definition 2.18. Mutually inverse elements in the corresponding Morita context (25) are provided by \( j \), the identity map \( H \) of the total algebra, and \( \widetilde{j} \), the antipode \( S \) in \( \mathcal{H} \), cf. axiom (iv) in Definition 2.18.

A generalization of Theorem 3.22 to arbitrary Hopf algebroids was obtained in [8, Theorem 3.12]:

**Theorem 3.26.** For any Hopf algebroid \( \mathcal{H} \), an \( \mathcal{H} \)-extension \( B \subseteq A \) is \( \mathcal{H} \)-cleft if and only if the following assertions hold.

1. The canonical map \( \text{can}_B \) in (15) is bijective.
2. There exists a left \( B \)-linear and right \( \mathcal{H} \)-colinear isomorphism \( A \cong B \otimes_L H \) (where \( B \otimes_L H \) is a left \( B \)-module via the first factor and a right \( \mathcal{H} \)-comodule via the second factor).
Proof. (Sketch.) Assume first that $B \subseteq A$ is an $\mathcal{H}$-cleft extension, i.e. there exist mutually inverse elements $j \in L\text{Hom}^\mathcal{H}(H, A)$ and $\tilde{j} \in L^{op}\text{Hom}^\mathcal{H}(H^{tw}, A)$ in the Morita context (25).

The inverse of $\text{can}_R$ is given by

$$\text{can}_R^{-1}(a \otimes h) = a\tilde{j}(h(1)) \otimes j(h(2)).$$

By the $R$-$L$ bilinearity of $\tilde{j}$ (cf. (30)) and the left $L$-linearity of $j$ (cf. (29)), there is a left $R$-module map $H \to A \otimes_L A$, $h \mapsto j(h(1)) \otimes_L j(h(2))$. Composing it with the canonical (left $R$-linear) epimorphism $\otimes_L A \to A \otimes_B A$, we conclude that the map $\text{can}_R^{-1}$ is well defined.

A left $B$-linear and right $\mathcal{H}$-colinear isomorphism is given by the mutually inverse maps

$$\kappa : A \to B \otimes_L H, \ a \mapsto a_0^{[0]} \tilde{j}(a_0^{[1]}) \otimes a_1^{[1]} \quad \text{and} \quad \kappa^{-1} : B \otimes_L H \to A, \ b \otimes_L h \mapsto b j(h).$$

Recall from (37) that $\kappa$ has the required range.

Conversely, assume that $\text{can}_R$ is bijective and there exists a left $B$-linear and right $\mathcal{H}$-colinear isomorphism $\kappa : A \to B \otimes_L H$. Then mutually inverse elements $j \in L\text{Hom}^\mathcal{H}(H, A)$ and $\tilde{j} \in L^{op}\text{Hom}^\mathcal{H}(H^{tw}, A)$ in the Morita context (25) are given by

$$j := \kappa^{-1}(1_A \otimes_B -) \quad \text{and} \quad \tilde{j} := (A \otimes_B (B \otimes_L \varepsilon_L) \circ \kappa)(\text{can}_R^{-1}(1_A \otimes_B -)).$$

Theorem 3.23 extends to arbitrary Hopf algebroids as follows, see [10, Corrigendum].

**Theorem 3.27.** Let $\mathcal{H}$ be a Hopf algebroid and $B \subseteq A$ be an $\mathcal{H}$-cleft extension. Then there is an equivalence functor $- \otimes_B A : \mathcal{M}_B \to \mathcal{M}_A^{\mathcal{H}}$.

**Proof.** Consider a cleaving map $j \in L\text{Hom}^\mathcal{H}(H, A)$ and its inverse $\tilde{j} \in L^{op}\text{Hom}^\mathcal{H}(H^{tw}, A)$ in the Morita context (25).

For any right $B$-module $W$, $W \otimes_B A$ is an object in $\mathcal{M}_A^{\mathcal{H}}$ via the $A$-action and the $\mathcal{H}$-coactions on the second factor. The resulting functor $- \otimes_B A : \mathcal{M}_B \to \mathcal{M}_A^{\mathcal{H}}$ is left adjoint of the $\mathcal{H}_R$-coinvariants functor $(-)^{\mathcal{H}_R} : \mathcal{M}_A^{\mathcal{H}} \to \mathcal{M}_B$, where the $B$-action on $M^{\mathcal{H}_R}$ is induced by the $A$-action on $M \in \mathcal{M}_A^{\mathcal{H}}$. The counit and the unit of the adjunction are given by

$$c_M : M^{\mathcal{H}_R} \otimes_B A \to M, \quad (n \otimes_B a) \mapsto na, \quad \text{for } M \in \mathcal{M}_A^{\mathcal{H}};$$

$$e_W : W \to (W \otimes_B A)^{\mathcal{H}_R}, \quad w \mapsto w \otimes_B 1_A, \quad \text{for } W \in \mathcal{M}_B.$$

The map $c_M$ is a morphism in $\mathcal{M}_A^{\mathcal{H}}$ by Corollary 2.22. We prove that $- \otimes_B A : \mathcal{M}_B \to \mathcal{M}_A^{\mathcal{H}}$ is an equivalence by constructing the inverse of the above natural transformations.

The inverse of $c_M$ is given by the map

$$c_M^{-1}(m) = m_0^{[0]} \text{can}_R^{-1}(1_A \otimes_B m^{[1]}) = m_0^{[0]} \tilde{j}(m_0^{[1]}) \otimes j(m^{[1]}),$$

where we used that $\text{can}_R$ is an isomorphism, cf. Theorem 3.26. The third equality follows by the right $\mathcal{H}_L$-colinearity of the $\mathcal{H}_R$-coaction on $M$. By (37), the range of $c_M^{-1}$ is in $M^{\mathcal{H}_R} \otimes_B A$, as needed. For $m \in M$, and $n \otimes_B a \in M^{\mathcal{H}_R} \otimes_B A$,

$$(c_M \circ c_M^{-1})(m) = m_0^{[0]} \tilde{j}(m_0^{[1]}) j(m^{[1]}), \quad (c_M^{-1} \circ c_M)(n \otimes_B a) = na_0^{[0]} \text{can}_R^{-1}(1_A \otimes_B a^{[1]}) = n \text{can}_R^{-1}(a_0^{[0]} \otimes B a^{[1]}) = n \otimes_B a.$$
The inverse of $e_W$ is given by
\[
e_W^{-1}(\sum_i w_i \otimes a_i) = \sum_i w_i a_i^0 j_i(a_i^1) j(1_H).
\]

By (37), $a_i^0 j_i(a_i^1)$ belongs to $B$, for any $a \in A$, and by the $\mathcal{H}_R$-co-linearity of $j$ and the unitality of the right $\mathcal{H}_R$-coproduct $\Delta_R$, the element $j(1_H)$ belongs to $B$ as well. Hence the expression of $e_W^{-1}$ is meaningful. Obviously, $(e_W^{-1} \circ e_W)(w) = w$, for all $w \in W$. To check that $e_W^{-1}$ is also the right inverse of $e_W$, note that for $\sum_i w_i \otimes_B a_i \in (W \otimes_B A)^{\text{cop}}$, the identity $\sum_i w_i \otimes_B a_i^0 \otimes_R a_i^1 = \sum_i w_i \otimes_B a_i \otimes_R 1_H$ holds, which implies
\[
(e_W \circ e_W^{-1})(\sum_i w_i \otimes_B a_i) = \sum_i w_i a_i^0 j_i(a_i^1) j(1_H) \otimes 1_H = \sum_i w_i \otimes_B a_i^0 j(a_i^1) j(1_H) = \sum_i w_i \otimes_B a_i.
\]

\[\square\]

**Remark 3.28.** Since by Example 3.25 the total algebra of any Hopf algebroid $\mathcal{H}$ is an $\mathcal{H}$-cleft extension of the base algebra $L$, Theorem 3.27 implies, in particular, that for any Hopf algebroid $\mathcal{H}$, there is an equivalence functor $- \otimes_L H : \mathcal{M}_L \to \mathcal{M}_H$. This yields a corrected version of the Fundamental Theorem of Hopf modules, [5, Theorem 4.2].

Regrettably, the proof of the journal version of [5] Theorem 4.2 turned out to be incorrect: When checking that the to-be-inverse of the counit of the adjunction has the required range, some ill-defined maps are used. Hence in the given (stronger form) [5] Theorem 4.2 is not justified. The journal version of [5] Theorem 4.2 becomes justified only under some further purity assumption, see the revised arXiv version.

It remains to extend Theorem 3.5 to algebra extensions by a Hopf algebroid. This requires first to understand the construction of a crossed product with a bialgebroid. The following notions, introduced in [8] Definitions 4.1 and 4.2, extend Definition 3.3. For the coproduct in a left $L$-bialgebroid $\mathcal{H}$, evaluated on an element $h$, the index notation $\Delta(h) = h_{(1)} \otimes_L h_{(2)}$ is used.

**Definition 3.29.** A left $L$-bialgebroid $\mathcal{H}$, with $L \otimes_k L^{op}$-ring structure $(H, s, t)$ and $L$-coring structure $(H, \Delta, e)$, measures an $L$-ring $B$ with unit map $\iota : L \to B$ if there exists a $k$-module map $\cdot : H \otimes_k B \to B$, the so called measuring, such that, for $h \in H$, $l \in L$ and $b, b' \in B$,

1. $h \cdot 1_B = 1 \circ e(h) 1_B$,
2. $(t(l))h \cdot b = (h \cdot b) \iota(l)$ and $(s(l))h \cdot b = \iota(l)(h \cdot b)$,
3. $h \cdot (bb') = (h_{(1)} \cdot b)(h_{(2)} \cdot b')$.

A $B$-valued 2-cocycle on $H$ is a $k$-module map $\sigma : H \otimes_L H \to B$ (where the right and left $L^{op}$-module structures in $H$ are given via $t$), such that, for $h, k, m \in H$ and $l \in L$,

4. $\sigma(s(l)h, k) = \iota(l)\sigma(h, k)$ and $\sigma(t(l)h, k) = \sigma(h, k)\iota(l),$
5. $\iota(h_{(1)} \cdot l)\sigma(h_{(2)}, k) = \sigma(h, s(l)k),$
6. $\iota(h_{(1)}, h) = 1 \circ e(1_H) = \sigma(h, 1_H),$
7. $(\sigma(h_{(1)}, k_{(1)} m_{(1)}))\sigma(h_{(2)}, k_{(2)} m_{(2)}) = \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}, k_{(2)}, m)$.

The $\mathcal{H}$-measured $L$-ring $B$ is a $\sigma$-twisted $\mathcal{H}$-module if in addition, for $b \in B$ and $h, k \in H$,

8. $1_H \cdot b = b$,
9. $\iota(h_{(1)}, k_{(1)} \cdot b)\sigma(h_{(2)}, k_{(2)}) = \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}, k_{(2)} \cdot b)$.

Note that condition (3) makes sense in view of (2). Conditions (7) and (9) make sense in view of (2), (4) and (5).
Similarly to Proposition \[\text{3.4}\] the reader can easily verify following \[\text{8 Proposition 4.3}\].

**Proposition 3.30.** Consider a left \(L\)-bialgebroid \(\mathcal{H}\) and an \(\mathcal{H}\)-measured \(L\)-ring \(B\). Let \(\sigma : H \otimes_{L_{op}} H \to B\) be a \(k\)-module map satisfying properties (4) and (5) in Definition \[\text{3.29}\]. Take the \(L\)-module tensor product \(B \otimes_{L} H\), where \(H\) is a left \(L\)-module via the source map. The \(k\)-module \(B \otimes_{L} H\) is a \(k\)-algebra with multiplication

\[
(b \# h)(b' \# h') = b(h_{1} \cdot b')\sigma(h_{2}, h_{1}'\# h_{2}')
\]

and unit element \(1_{B} \# 1_{H}\), if and only if \(\sigma\) is a \(B\)-valued 2-cocycle on \(H\) and \(B\) is a \(\sigma\)-twisted \(\mathcal{H}\)-module. This algebra is called the crossed product of \(B\) with \(\mathcal{H}\) with respect to the cocycle \(\sigma\). It is denoted by \(B\#_{\sigma}\mathcal{H}\).

Similarly to the case of a crossed product with a bialgebra, a crossed product algebra \(B\#_{\sigma}\mathcal{H}_{L}\) of a left \(L\)-bialgebroid \(\mathcal{H}_{L}\) is a right \(\mathcal{H}_{L}\)-comodule algebra. The coaction is given in terms of the coproduct \(\Delta_{L}\) in \(\mathcal{H}\) as \(B \otimes_{L} \Delta_{L}\). What is more, \(B \subseteq B\#_{\sigma}\mathcal{H}_{L}\) is an extension by \(\mathcal{H}_{L}\). If \(\mathcal{H}_{L}\) is a constituent left bialgebroid in a Hopf algebra, then \(B \subseteq B\#_{\sigma}\mathcal{H}_{L}\) is an \(\mathcal{H}\)-extension, with respect to the \(\mathcal{H}\)-comodule structure on \(B\#_{\sigma}\mathcal{H}_{L}\) induced by the two coproducts in \(\mathcal{H}\).

Analogously to Proposition \[\text{3.4}\], the reader can easily verify following \[\text{8 Proposition 4.3}\].

**Definition 3.31.** Let \(\mathcal{H}\) be a left \(L\)-bialgebroid, with \(L \otimes_{k} L_{op}\)-ring structure \((H, s, t)\) and \(L\)-coring structure \((H, \Delta, \varepsilon)\), and let \(t : L \to B\) be an \(L\)-ring, measured by \(\mathcal{H}\). A \(B\)-valued 2-cocycle \(\sigma\) on \(\mathcal{H}\) is **invertible** if there exists a \(k\)-linear map \(\tilde{\sigma} : H \otimes_{L} H \to B\) (where the \(L\)-module structures on \(H\) are given by \(s\)), satisfying for all \(h, k \in H\) and \(l \in L\),

1. \(\tilde{\sigma}(s(l)h, k) = t(l)\tilde{\sigma}(h, k)\) and \(\tilde{\sigma}(t(l)h, k) = \tilde{\sigma}(h, k)t(l)\),
2. \(\tilde{\sigma}(h_{1}, k_{1})(h_{2}, t(l)) = \tilde{\sigma}(h_{1}, k_{1})t(l)\),
3. \(\tilde{\sigma}(h_{1}(1), k_{1}(1))\tilde{\sigma}(h_{1}(2), k_{2}) = h \cdot (k \cdot 1_{B})\) and \(\tilde{\sigma}(h_{1}(1), k_{1}(1))\sigma(h_{2}(1), k_{2}(2)) = hk \cdot 1_{B}\).

Conditions in (3) make sense in view of (1) and (2). A map \(\tilde{\sigma}\) is called an **inverse** of \(\sigma\).

**Theorem 3.32.** An algebra extension \(B \subseteq A\) by a Hopf algebroid \(\mathcal{H}\) is a cleft extension if and only if \(A\) is isomorphic, as a left \(B\)-module and right \(\mathcal{H}\)-comodule algebra, to \(B\#_{\sigma}\mathcal{H}_{L}\), a crossed product of \(B\) with the constituent left bialgebroid \(\mathcal{H}_{L}\) in \(\mathcal{H}\) with respect to some invertible \(B\)-valued 2-cocycle \(\sigma\) on \(\mathcal{H}\).

**Proof.** (Sketch.) For the two coproducts \(\Delta_{L}\) and \(\Delta_{R}\) in \(\mathcal{H}\) we use the two versions of Sweedler’s index notation introduced after Definition \[\text{2.18}\]. In order to show that the \(\mathcal{H}\)-extension \(B \subseteq B\#_{\sigma}\mathcal{H}_{L}\) is cleft, one constructs the inverse of the \(L\)-\(\mathcal{H}\) bicomodule map \(j : H \to B\#_{\sigma}\mathcal{H}_{L}\), \(h \mapsto 1_{B} \# h\) in the Morita context \[\text{25}\]. It has the explicit form

\[
\tilde{j}(h) := \tilde{\sigma}(S(h_{1}(1)), h_{1}(2)) \# S(h_{1}(1))
\]

In the converse direction, starting with an \(\mathcal{H}\)-cleft extension \(B \subseteq A\), one constructs a measuring of \(\mathcal{H}_{L}\) on \(B\) and an invertible \(B\)-valued 2-cocycle on \(\mathcal{H}_{L}\) in terms of the cleaving map \(j\) and its inverse \(\tilde{j}\) in the Morita context \[\text{25}\]. Explicitly,

\[
h \cdot b := j(h_{1})b \tilde{j}(h_{1}(2)), \quad \sigma(h, k) := j(h_{1})j(k_{1})\tilde{j}(h_{2}(2))\tilde{j}(h_{2}(2)).
\]
The 2-cocycle $\sigma$ is proven to be invertible by constructing an inverse

$$\tilde{\sigma}(h,k) = j(h(1)k(1))\tilde{j}(k(2))\tilde{j}(h(2)).$$

An isomorphism $A \rightarrow B \# \mathcal{H}_F$ of left $B$-modules and right $\mathcal{H}$-comodule algebras is given by the map $\kappa$ in the proof of Theorem 3.26.

For more details we refer to [8] Theorems 4.11 and 4.12.

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