Nuclear energy density functional from chiral pion-nucleon
dynamics: Isovector spin-orbit terms

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Abstract

We extend a recent calculation of the nuclear energy density functional in the systematic framework of chiral perturbation theory by computing the isovector spin-orbit terms:

\[
(\vec{\nabla} \rho_p - \vec{\nabla} \rho_n) \cdot (\vec{J}_p - \vec{J}_n) G_{so}(k_f) + (\vec{J}_p - \vec{J}_n)^2 G_J(k_f).
\]

The calculation includes the one-pion exchange Fock diagram and the iterated one-pion exchange Hartree and Fock diagrams. From these few leading order contributions in the small momentum expansion one obtains already a good equation of state of isospin-symmetric nuclear matter. We find that the parameterfree results for the (density-dependent) strength functions \(G_{so}(k_f)\) and \(G_J(k_f)\) agree fairly well with that of phenomenological Skyrme forces for densities \(\rho > \rho_0/10\). At very low densities a strong variation of the strength functions \(G_{so}(k_f)\) and \(G_J(k_f)\) with density sets in. This has to do with chiral singularities \(m_\pi^{-1}\) and the presence of two competing small mass scales \(k_f\) and \(m_\pi\). The novel density dependencies of \(G_{so}(k_f)\) and \(G_J(k_f)\) as predicted by our parameterfree (leading order) calculation should be examined in nuclear structure calculations.

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Among the various phenomenological interactions that have been used extensively in the description of nuclei, the Skyrme force [1] has gained much popularity because of its analytical simplicity and its ability to reproduce nuclear properties over the whole periodic table within the self-consistent Hartree-Fock approximation. Several Skyrme parameterizations have been tailored to account for single-particle spectra [2], giant monopole resonances [3] or fission barriers of heavy nuclei [4]. Recently, a new Skyrme force which also reproduces the equation of state of pure neutron matter up to neutron star densities, \(\rho_n \simeq 1.5 \text{ fm}^{-3}\), has been proposed in ref.[5] for the study of nuclei far from stability. A microscopic interpretation of the various parameters entering the effective Skyrme forces is generally put aside. Sometimes the energy density functional is just parameterized without reference to any effective (zero-range) NN-interaction [6].

Another widely and successfully used approach to nuclear structure calculations are relativistic mean-field models [7, 8]. In these models the nucleus is described as a collection of independent Dirac-particles moving in self-consistently generated scalar and vector mean-fields. The footprints of relativity become visible through the large nuclear spin-orbit interaction which emerges in that framework naturally from the interplay of the two strong and counter-acting (scalar and vector) mean-fields. The corresponding many-body calculations are usually carried out in the Hartree approximation, ignoring the negative-energy Dirac-sea. For a recent review on self-consistent mean-field models for nuclear structure, see ref.[6]. In that article the relationship between the relativistic mean-field models and the Skyrme phenomenology is also discussed.
The first necessary conditions to be fulfilled by any phenomenological nucleon-nucleon interaction come from the (few empirically known) properties of infinite nuclear matter. These are the saturation density $\rho_0 = 2k_f^3/3\pi^2$, the binding energy per particle $-\bar{E}(k_f)$ and the compression modulus $K = k_f^3\bar{E}''(k_f)$ of isospin-symmetric nuclear matter as well as the asymmetry energy $A(k_f)$. In a recent work [9], we have applied the systematic framework of chiral perturbation theory to the nuclear matter many-body problem. In this calculation the contributions to the energy per particle, $\bar{E}(k_f)$, originate exclusively from one- and two-pion exchange between nucleons and they are ordered in powers of the Fermi-momentum $k_f$ (modulo functions of $k_f/m_\pi$). It has been demonstrated in ref.[9] that the empirical saturation point $(\rho_0 \simeq 0.17\text{fm}^{-3}, \bar{E}(k_f) \simeq -15.3\text{MeV})$ and the nuclear matter compressibility $K \simeq 255\text{MeV}$ can be well reproduced at order $O(k_f^4)$ in the small momentum expansion with just one single momentum cut-off scale of $\Lambda \simeq 0.65\text{GeV}$ which parameterizes the short-range NN-dynamics necessary for nuclear binding. Most surprisingly, the prediction for the asymmetry energy, $A(k_f) = 33.8\text{MeV}$ [9], is in good agreement with its empirical value. In fact very similar results for nuclear matter can be obtained already at order $O(k_f^2)$ in the small momentum expansion (by dropping the relativistic $1/M^2$-correction to $1\pi$-exchange and the irreducible $2\pi$-exchange of order $O(k_f^4)$) with a somewhat reduced cut-off scale of $\Lambda \simeq 0.61\text{GeV}$ (for detailed results see ref.[10]).

Given the fact that many properties of nuclear matter can be well described by chiral $\pi N$-dynamics treated perturbatively up to three-loop order it is natural to consider in a further step the energy density functional relevant for inhomogeneous many-nucleon systems (i.e. finite nuclei). Such an extension to inhomogeneous many-nucleon systems can be done with the help of the density matrix-expansion of Negele and Vautherin [11]. The bilocal density-matrix (given by a sum over the occupied energy eigenfunctions) is expanded in relative and center-of-mass coordinates with expansion coefficients determined by purely local quantities: nucleon density $\rho(\vec{r})$, kinetic energy density $\tau(\vec{r})$ and spin-orbit density $\vec{J}(\vec{r})$. The Fourier-transform of the (so expanded) density-matrix defines in momentum-space a "medium-insertion" $\Gamma(\vec{p}, \vec{q})$ for the inhomogeneous many-nucleon system which then allows to calculate diagrammatically the nuclear energy density functional.

In a recent work [12] we have considered the isospin-symmetric case of equal (even) proton and neutron number. The corresponding energy density functional takes the general form: $\mathcal{E}[\rho, \tau, \vec{J}] = \rho\bar{E}(k_f) + [\tau - 3\rho k_f^2/5]/2\bar{M}^*(\rho) + (\nabla\rho)^2 F_\nabla(k_f) + \nabla^2 \rho \cdot \vec{J} F_{so}(k_f) + \bar{J}^2 F_J(k_f)$. We have found that the effective nucleon mass $\bar{M}^*(\rho)$ deviates at most by $\pm 15\%$ from its free space value $M$, with $0.89M < \bar{M}^*(\rho) < M$ for $\rho < 0.11\text{fm}^{-3}$ and $\bar{M}^*(\rho) > M$ for higher densities $\rho < \rho_0 = 0.174\text{fm}^{-3}$. Interestingly, a recent large scale fit of (almost two thousand) nuclide masses by Pearson et al. [13] finds a similarly enhanced effective nucleon mass: $\bar{M}^*(\rho_0) \simeq 1.05M$. The strength of the $(\nabla\rho)^2$-term, $F_\nabla(k_f)$, is (at saturation density) comparable to that of phenomenological Skyrme forces. The magnitude of $F_J(k_f)$ accompanying the squared spin-orbit density $\bar{J}^2$ comes out considerably larger. Both quantities increase strongly as the nucleon density $\rho$ tends to zero. This has to do with the explicit presence of the small mass scale $m_\pi = 135\text{MeV}$ which amounts to about half of the Fermi momentum $k_f$ in equilibrated nuclear matter. The strength of the isoscalar nuclear spin-orbit interaction, $F_{so}(k_f)$, as given by iterated $1\pi$-exchange is (at saturation density) about half as large as the corresponding empirical value $\sim 90\text{MeVfm}^5$, however, with the wrong negative sign. This isoscalar spin-orbit interaction is not a relativistic effect but proportional to the large nucleon $M$. From that result it becomes clear that perturbative chiral pion-nucleon dynamics (alone) cannot account for the mechanisms underlying the empirical isoscalar nuclear spin-orbit force, whereas relativistic scalar-vector mean-field models [7, 8] give a successful phenomenology of it.
The purpose of the present work is to calculate using the same framework as in ref.[12] the isovector spin-orbit interactions generated by one- and two-pion exchange in order to reveal the underlying isospin dependence. Note, for example, that the Skyrme force gives rise to isoscalar and isovector spin-orbit interactions with a fixed ratio 3:1. In particular, this restrictive feature of the Skyrme energy density functional has been made responsible for the less accurate description of isotope shifts in the Pb region [14] in comparison to relativistic mean-field calculations. Naturally, one expects that the finite-range character of the two-pion exchange interaction will lift such restrictive features of the zero-range Skyrme force.

Let us begin with writing down the explicit form of the isovector spin-orbit terms in the nuclear energy density functional:

\[ \mathcal{E}[\rho_p, \rho_n, J_p, J_n] = (\nabla \rho_p - \nabla \rho_n) \cdot (\vec{J}_p - \vec{J}_n) G_{so}(k_f) + (\vec{J}_p - \vec{J}_n)^2 G_J(k_f) + \ldots \quad (1) \]

Here,

\[ \rho_{p,n}(\vec{r}) = \frac{k_{p,n}^2(\vec{r})}{3\pi^2} = \sum_{\alpha \in \text{occ}} \bar{\Psi}_{p,n}^{(\alpha)}(\vec{r})^\dagger \bar{\Psi}_{p,n}^{(\alpha)}(\vec{r}), \]

are the local proton and neutron densities written in terms of the corresponding (local) proton and neutron Fermi-momenta \( k_{p,n}(\vec{r}) \), and expressed as sums over the occupied single-particle orbitals \( \bar{\Psi}_{p,n}^{(\alpha)}(\vec{r}) \). The spin-orbit densities of protons and neutrons are defined similarly:

\[ \vec{J}_{p,n}(\vec{r}) = \sum_{\alpha \in \text{occ}} \bar{\Psi}_{p,n}^{(\alpha)}(\vec{r}) i \vec{\sigma} \times \nabla \bar{\Psi}_{p,n}^{(\alpha)}(\vec{r}). \]

Furthermore, \( G_{so}(k_f) \) and \( G_J(k_f) \) denote the associated strength functions. In Skyrme parameterizations [2, 3, 4, 5, 13] these are just constants, \( G_{so}(k_f) = W_0/4 \) and \( G_J(k_f) = (t_1 - t_2)/32 \), whereas in our calculation their explicit density dependence originates from the finite-range character of the \( 1\pi \)- and \( 2\pi \)-exchange interaction.

The starting point for the construction of an explicit nuclear energy density functional \( \mathcal{E}[\rho_p, \rho_n, \vec{J}_p, \vec{J}_n] \) is the bi-local density-matrix as given by a sum over the occupied energy eigenfunctions: \( \sum_{\alpha \in \text{occ}} \bar{\Psi}_{p,n}^{(\alpha)}(\vec{r} - \vec{a}/2) \Psi_{p,n}^{(\alpha)}(\vec{r} + \vec{a}/2) \). According to Negele and Vautherin [11] it can be expanded in relative and center-of-mass coordinates, \( \vec{a} \) and \( \vec{r} \), with expansion coefficients determined by local quantities (nucleon density, kinetic energy density, spin-orbit density). As outlined in section 2 of ref.[12] the Fourier-transform of the (so expanded) density-matrix defines in momentum-space a medium-insertion \( \Gamma(\vec{p}, \vec{q}) \) for the inhomogeneous many-nucleon system. It is straightforward to generalize this construction to the isospin-asymmetric situation of different proton and neutron local densities \( \rho_{p,n}(\vec{r}) \) and \( \vec{J}_{p,n}(\vec{r}) \). We display here only that part of the medium-insertion \( \Gamma(\vec{p}, \vec{q}) \) which is actually relevant for the diagrammatic calculation of the isovector spin-orbit terms eq.(1):

\[ \Gamma(\vec{p}, \vec{q}) = \int d^3r e^{-i\vec{q} \cdot \vec{r}} \left\{ \frac{1 + \tau_3}{2} \theta(k_p - |\vec{p}|) + \frac{1 - \tau_3}{2} \theta(k_n - |\vec{p}|) \right. \\
+ \frac{\pi^2}{4k_f^4} \left[ \delta(k_f - |\vec{p}|) - k_f \delta'(k_f - |\vec{p}|) \right] \tau_3 \vec{\sigma} \cdot [\vec{p} \times (\vec{J}_p - \vec{J}_n)] + \ldots \right\}. \quad (4) \]

When working to quadratic order in deviations from isospin symmetry (proton-neutron differences) it is sufficient to use an average Fermi-momentum \( k_f \) in the prefactor of the isovector spin-orbit density \( \vec{J}_p - \vec{J}_n \). The double line in the left picture of Fig.1 symbolizes this medium insertion together with the assignment of the out- and in-going nucleon momenta \( \vec{p} \pm \vec{q}/2 \). The momentum transfer \( \vec{q} \) is provided by the Fourier components of the inhomogeneous (matter) distributions \( \rho_{p,n}(\vec{r}) \) and \( \vec{J}_{p,n}(\vec{r}) \).
Fig. 1: Left: The double line symbolizes the medium insertion defined by eq.(4). Next are shown: The two-loop one-pion exchange Fock-diagram and the three-loop iterated one-pion exchange Hartree- and Fock-diagrams. Their combinatoric factors are $1/2$, $1/4$ and $1/4$, in the order shown.

Now we turn to the analytical evaluation of the pion-exchange diagrams shown in Fig. 1. We give for each diagram only the final result omitting all technical details related to extensive algebraic manipulations and solving elementary integrals. A collection of the relevant "master integrals" can be found in the appendix of ref.[12]. We obtain from the $1\pi$-exchange Fock-diagram with two medium insertions:

$$G_J(k_f) = \frac{g_A^2}{(8m_\pi f_\pi)^2} \left\{ -\frac{10 + 24u^2}{3(1 + 4u^2)^2} - \frac{1}{6u^2} \ln(1 + 4u^2) \right\}, \quad u = \frac{k_f}{m_\pi}. \quad (5)$$

This contribution to $G_J(k_f)$ is just $-1/3$ of the contribution to the isoscalar strength function $F_J(k_f)$ (see eq.(10) in ref.[12]) as a simple consequence of the isospin-trace $\text{tr}[\tau_a(\bar{J}_s + \tau_3J_v)\tau_a(\bar{J}_s + \tau_3J_v)] = 6J_s^2 - 2J_v^2$. There is no contribution of the $1\pi$-exchange Fock-diagram to the strength function $G_{so}(k_f)$ since the spin-trace with one insertion proportional to $\vec{\sigma} \cdot (\vec{p} \times \vec{J}_v)$ vanishes. We note also that the $1\pi$-exchange Hartree diagram (not shown in Fig.1) vanishes identically either because of a zero spin-trace or because of the momentum integral $\int d^3p \vec{p} \theta(k_{p,n} - |\vec{p}|) = 0$. The iterated one-pion exchange Hartree diagram with two medium insertions leads to the result:

$$G_{so}(k_f) = \frac{g_A^4 M}{\pi m_\pi (4f_\pi)^4} \left\{ \frac{1}{u^2} \ln(1 + 4u^2) - \frac{8}{3(1 + 4u^2)} \right\}, \quad (6)$$

which is $-2/3$ of the contribution to the isoscalar spin-orbit strength $F_{so}(k_f)$ (see eq.(13) in ref.[12]). Let us briefly explain the mechanism which generates the strength function $G_{so}(k_f)$. The exchanged pion-pair transfers the momentum $\vec{q}$ between the left and the right nucleon ring and this momentum $\vec{q}$ enters also the pseudovector $\pi N$-interaction vertices. The isovector spin-orbit strength $G_{so}(k_f)$ arises from the spin-trace $\text{tr}[\vec{\sigma} \cdot (\vec{l} + \vec{q}/2) \vec{\sigma} \cdot (\vec{l} - \vec{q}/2) \vec{\sigma} \cdot (\vec{p} \times \vec{J}_v)] = 2i(\vec{q} \times \vec{l}).(\vec{p} \times \vec{J}_v)$ where $\vec{q}$ gets converted to $\vec{\nabla}k_p - \vec{\nabla}k_n \simeq \pi^2(\vec{\nabla}\rho_p - \vec{\nabla}\rho_n)/k_p^2$ by Fourier transformation. From the iterated one-pion exchange Fock diagram with two medium insertions we obtain:

$$G_{so}(k_f) = \frac{5g_A^4 M}{6\pi m_\pi (4f_\pi)^4} \left\{ \frac{\arctan 2u}{u(1 + 2u^2)} - \frac{3 + 4u^2}{u(1 + 2u^2)} \arctan u \right\} + \frac{1}{2u^2} \ln \frac{1 + u^2}{1 + 4u^2} + \int_0^u dx \frac{\arctan 2x - \arctan x}{u^2(1 + 2x^2)} \right\}, \quad (7)$$
\[ G_J(k_f) = \frac{5g_A^4M}{3\pi^2m_\pi(8f_\pi)^4} \left\{ \frac{2(5 + 8u^2)}{u(1 + 2u^2)} \arctan u - \frac{2\arctan 2u}{u(1 + 2u^2)} + \frac{1}{1 + u^2} \right. \\
+ \left. \frac{1}{u^2} \ln \frac{1 + 4u^2}{1 + u^2} + \frac{2}{u^2} \int_0^u d\tau \arctan \tau - \arctan 2\tau \right\} . \] (8)

One notices a relative factor of \(-5/3\) in comparison to the contributions to the isoscalar strength functions \(F_{so,J}(k_f)\) (see eqs.(16,17) in ref.[12]) which comes from the isospin-trace \(\text{tr}[\tau_\lambda(J_s + \tau_3J_\nu)\tau_\lambda(J_s + \tau_3J_\nu)\tau_\lambda] = 10J_\nu^2 - 5J_\nu^2\) of that diagram. In case of the iterated one-pion exchange Hartree diagram with three medium insertions one has to evaluate nine-dimensional principal value integrals over the product of three Fermi spheres. We find the following contribution to the isovector spin-orbit strength:

\[ G_{so}(k_f) = \frac{4g_A^4Mu^{-6}}{3\pi^2m_\pi(4f_\pi)^4} \int_0^u d\tau \int_0^1 dy \left\{ \frac{2us[2s^2y^2(s - s') + u^2(2s' - s)]}{(1 + s^2)(u^2 - x^2y^2)} \right. \\
+ \left. \frac{u(3u^2 - 5x^2y^2)}{2 - x^2y^2} - 4xyH \right\} \left[ 3\arctan s - \frac{3s + 2s^3}{1 + s^2} \right] + \frac{Hs^3}{(1 + s^2)^3} \left[ (8xy - 5s - s^3)s^2 - 2xys'(7s + 3s^3) \right. \\
+ \left. s^2xy(11 + 7s^2) + (s + s^3)(2xy - s)(s'' - s') \right\} . \] (9)

with the auxiliary functions \(H = \ln(u + xy) - \ln(u - xy)\) and \(s = xy + \sqrt{u^2 - x^2 + y^2}^2\). The partial derivatives of \(s\) are abbreviated by \(s' = u\partial s/\partial u\) and \(s'' = u^2\partial^2 s/\partial u^2\). In comparison to the contribution to the isoscalar spin-orbit strength \(F_{so}(k_f)\) (see eq.(20) in ref.[12]) relative isospin factors of \(\pm 2/3\) have occurred in various subparts. The contribution of the iterated one-pion exchange Hartree diagram with three medium insertions to the strength function \(G_J(k_f)\) reads:

\[ G_J(k_f) = \frac{g_A^4M}{3\pi^2m_\pi(4f_\pi)^4} \left\{ \frac{2u^2}{u(1 + 2u^2)^3} - \frac{1 + 2u^2}{4u^3} \ln(1 + 4u^2) \right. \\
+ \left. \frac{8u^3y^2}{(1 + 4u^2y^2)^4} \left[ 5 - (30 + 32u^2)y^2 \right. \right. \\
\left. + (35 + 24u^2 - 24u^4)y^3 + (56u^2y^4 + 48u^4y^8) \ln \frac{1 + y^2}{1 - y^2} \right\} . \] (10)

It is obtained if only if both insertions proportional to \(\vec{\sigma} \cdot (\vec{p}_{1,2} \times \vec{J}_\nu)\) stand under a single spin-trace and this feature explains also the relative isospin factor \(-1/3\) in comparison to the contribution to \(F_J(k_f)\) written in eq.(21) of ref.[12]. The iterated one-pion exchange Fock diagram with three medium insertions is most tedious to evaluate. It is advisable to split the contributions to the strength functions \(G_{so,J}(k_f)\) into ”factorizable” and ”non-factorizable” parts. These two pieces are distinguished by whether the nucleon propagator in the denominator can be canceled or not by terms from the product of \(\pi N\)-interaction vertices in the numerator. We find the following ”factorizable” contributions:

\[ G_{so}(k_f) = \frac{g_A^4Mu^{-3}}{3\pi^2m_\pi(4f_\pi)^4} \left\{ \frac{(1 - 2u^2)(2u^4 - 4u^2 - 1)}{u(1 + 2u^2)(1 + 4u^2)} \ln(1 + 4u^2) - \frac{1 + 2u^2}{32u^2} \ln^2(1 + 4u^2) \right. \\
\left. + \frac{u^2(11 + 52u^2)}{2(1 + 4u^2)} + \frac{u(5 + 39u^2 + 64u^4)}{(1 + u^2)(1 + 4u^2)} \arctan 2u + 5 \int_0^u dx \left\{ -3u^3x^{-2} \right. \right. \\
\left. + \left[ 4x^2 + 1 + u^2 - 3x^{-2}(1 + u^2)^2 \right] uL^2 + \left[ 6(u^2 + u^4)x^{-2} + 2 \right. \right. \right. \\
\left. \right. \]
with the auxiliary function:

$$L = \frac{1}{4x} \ln \frac{1 + (u + x)^2}{1 + (u - x)^2}.$$  (13)

The non-factorizable contributions from the iterated one-pion exchange Fock diagram with three medium insertions (stemming from nine-dimensional principal value integrals over the product of three Fermi spheres) read on the other hand:

$$G_{so}(k_f) = \frac{g_A^4 M}{3\pi^2 m_\pi(4f_\pi)^4} \left\{ \int\int_{-1}^{1} dy \int_{-1}^{1} dz \frac{y z \theta(y^2 + z^2 - 1)}{|yz|\sqrt{y^2 + z^2 - 1}} \left[ 16y^2 z \theta(y)\theta(z) \frac{1 + 2u^2y^2}{(1 + 4u^2y^2)^2} \right] \times \left( 2uz - \arctan 2uz \right) + \frac{u^3 z (2z^2 - 1)}{(1 + 4u^2y^2)(1 + 4u^2z^2)} \right\} + \int_0^u dx \frac{5u - 8x^2 st' t''}{2(1 + s^2)^2(1 + t^2)} \times \left[ (s + s^3)(s'' - s')(st + sz - txy) + s'^2((3s + s^3)(t + xz) - 2txy) \right],$$  (14)

$$G_{J}(k_f) = \frac{g_A^4 M}{3\pi^2 m_\pi(4f_\pi)^4} \left\{ \int\int_{-1}^{1} dy \int_{-1}^{1} dz \frac{y z \theta(y^2 + z^2 - 1)}{|yz|\sqrt{y^2 + z^2 - 1}} \left[ y^2 \theta(y)\theta(z) \frac{4u^2 z^2 - \ln(1 + 4u^2 z^2)}{u(1 + 4u^2y^2)^2} \right] \times \frac{9 + 4u^2(5 + 2y^2) + 16u^4(y^2 + y^4)}{u(1 + 4u^2y^2)^3} + \frac{16u^3(3 + 4u^2y^2)z(2z^2 - 1)}{(1 + 4u^2y^2)^2(1 + 4u^2z^2)} \right\} + \int_0^u dx \frac{5x^4 s^2 t^2 (y^2 + z^2 - 1)}{4u^4 (1 + s^2)^2(1 + t^2)^2} \left[ (s + s^3)(s'' - s') + (3 + s^2)s'^2 \right] \times \left[ (t + t^3)(t'' - t') + (3 + t^2)t'' \right],$$  (15)

with the auxiliary function $t = xz + \sqrt{u^2 - x^2 + x^2z^2}$ and its partial derivatives $t' = u\partial t/\partial u$ and $t'' = u^2\partial^2 t/\partial u^2$. In comparison to the contributions to the isoscalar strength functions $F_{so,J}(k_f)$ (see eqs.(25,26,29,30) in ref.[12]) there occur relative isospin factors of $-1/3$ and $-5/3$ in various subparts. Let us add some general power counting considerations for the nuclear energy density functional $\mathcal{E}[\rho_p,\rho_n,\vec{J}_p,\vec{J}_n]$. Counting the Fermi momenta $k_{p,n,f}$, the pion mass $m_\pi$ and a spatial gradient $\nabla$ collectively as small momenta one deduces from eqs.(2,3) that the nucleon densities $\rho_{p,n}(\vec{r})$ and the spin-orbit densities $\vec{J}_{p,n}(\vec{r})$ are quantities of third and fourth order in small momenta, respectively. With these counting rules the contributions from $1\pi$-exchange to the nuclear energy density functional $\mathcal{E}[\rho_p,\rho_n,\vec{J}_p,\vec{J}_n]$ are of sixth order in small momenta while all contributions from iterated $1\pi$-exchange are of seventh order. Concerning NN-interactions induced by pion-exchange the isovector spin-orbit terms presented here are in fact complete up-to-and-including seventh order in small momenta.
Next we turn to the numerical results. We use the physical parameters: \( M = 939 \text{ MeV} \) (nucleon mass), \( m_\pi = 135 \text{ MeV} \) (neutral pion mass), \( f_\pi = 92.4 \text{ MeV} \) (pion decay constant) and \( g_A = 1.3 \) (equivalent to a \( \pi N \)-coupling constant \( g_\pi^N = g_A M / f_\pi = 13.2 \)). The full line in Fig. 2 shows the result of iterated 1\( \pi \)-exchange for the strength function \( G_{so}(k_f) \) belonging to the isovector spin-orbit coupling term \( (\vec{\nabla} \rho_p - \vec{\nabla} \rho_n) \cdot (\vec{J}_p - \vec{J}_n) \) versus the nucleon density \( \rho = 2k_f^3/3\pi^2 \). For comparison we have drawn the constant values \( G_{so}(k_f) = W_0/4 \) of the three Skyrme forces Sly [5], MSk [13] and SkP [15] (horizontal dashed lines). In the case of Sly and MSk we have performed averages over the various parameter sets Sly4-7 and MSk1-6. One observes a fair agreement of our parameterfree prediction with these empirical values. In contrast to the isoscalar spin-orbit strength \( F_{so}(k_f) \) (see Fig. 4 in ref.[12]) iterated 1\( \pi \)-exchange gives now the correct positive sign. The density dependence of the isovector spin-orbit strength \( G_{so}(k_f) \) is moderate for densities \( \rho > \rho_0/10 \). A rapid decrease sets however in when \( \rho \) tends to zero.

\[ \begin{align*}
\text{Fig. 2: The strength function } G_{so}(k_f) \text{ related to the isovector spin-orbit term } (\vec{\nabla} \rho_p - \vec{\nabla} \rho_n) \cdot (\vec{J}_p - \vec{J}_n) \text{ in the nuclear energy density functional versus the nucleon density } \\
\rho = 2k_f^3/3\pi^2. \text{ The three horizontal dashed lines show the constant values } G_{so}(k_f) = W_0/4 \text{ of the Skyrme forces Sly [5], MSk [13] and SkP [15].}
\end{align*} \]

Finally, we show in Fig. 3 the strength function \( G_J(k_f) \) accompanying the squared isovector spin-orbit density \((\vec{J}_p - \vec{J}_n)^2\) in the nuclear energy density functional versus the nucleon density \( \rho = 2k_f^3/3\pi^2 \). For comparison we have drawn the constant values \( G_J(k_f) = (t_1 - t_2)/32 \) of the three Skyrme forces Sly [5], SkP [5] and MSk [13], (dashed lines). One observes that our parameterfree prediction for the strength function \( G_J(k_f) \) lies well within the band spanned by these empirical values. Again, the density dependence of \( G_J(k_f) \) is moderate for densities \( \rho > \rho_0/5 \) and a rapid decrease sets in when \( \rho \) tends to zero. Although not visible each (full) curve in Figs. 2,3, approaches a finite (negative) value at \( \rho = 0 \). One can analytically derive
the following low density limits:

\[
G_{\text{so}}(0) = -\frac{g_A^4 M}{3\pi m_\pi (4f_\pi)^4} = -33.8\text{MeVfm}^5, \tag{16}
\]

\[
G_J(0) = -\frac{g_A^2}{(4m_\pi f_\pi)^2}\left[-1 + \frac{15g_A^2 M m_\pi}{256\pi f_\pi^2}\right] = -108.0\text{MeVfm}^5, \tag{17}
\]

to which only the diagrams with two medium insertions contribute. The large numbers in eqs.(16,17) arise from negative powers of the pion mass \(m_\pi\) (so-called chiral singularities). The most singular \(m_\pi^{-2}\)-term can be traced back to the \(1\pi\)-exchange Fock diagram. Note also the relation between the isovector and isoscalar spin-orbit strengths at zero density, \(G_{\text{so}}(0) = F_{\text{so}}(0)/3\) (for \(F_{\text{so}}(0)\) see eq.(42) in ref.[12]). This is a necessary consistency check on our diagrammatic calculation. At extremely low densities \((k_f << m_\pi/2)\) even the pion-exchange interaction becomes effectively short-ranged and therefore the constraint \(G_{\text{so}}(k_f) = F_{\text{so}}(k_f)/3\) known from the (zero-range) Skyrme spin-orbit force must hold.

In this context is important to keep in mind that if pionic degrees of freedom are treated explicitly in the nuclear matter problem the low density limit is realized only at extremely low densities \(k_f << m_\pi/2\). Often, the opposite limit where the pion mass \(m_\pi\) can be neglected against the Fermi momentum \(k_f\) is already applicable at the moderate densities relevant for conventional nuclear physics. This is exemplified here by the approximate density dependence \(G_{\text{so},J}(k_f) \sim k_f^{-1}\) (see Figs. 2,3). Such a \(\rho^{-1/3}\)-behavior becomes exact in the chiral limit \(m_\pi = 0\) as can be deduced by simple mass dimension counting of the iterated \(1\pi\)-exchange diagrams (the basic argument is that \(M/f_\pi^4 k_f\) has the correct unit of MeVfm\(^5\)).

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**Fig. 3:** The strength function \(G_J(k_f)\) accompanying the squared isovector spin-orbit density \((\vec{J}_p - \vec{J}_n)^2\) in the nuclear energy density functional versus the nucleon density \(\rho = 2k_f^3/3\pi^2\). The three horizontal dashed lines show the constant values \(G_J(k_f) = (t_1 - t_2)/32\) of the Skyrme forces Sly [5], SkP [15] and MSk [13].
In summary we extended in this work our recent calculation of the nuclear energy density functional \[12\] in the framework of chiral perturbation by computing the isovector spin-orbit terms: \((\vec{\nabla}\rho_p - \vec{\nabla}\rho_n) \cdot (\vec{J}_p - \vec{J}_n) G_{so}(k_f) + (\vec{J}_p - \vec{J}_n)^2 G_J(k_f)\). Our calculation includes the 1π-exchange Fock diagram and the iterated 1π-exchange Hartree and Fock diagrams. These few leading order contributions in the small momentum expansion give already a good equation of state of isospin symmetric infinite nuclear matter \[9, 10\]. The step to inhomogeneous many-nucleon systems is done with the help of the density-matrix expansion of Negele and Vautherin \[11\]. The specific isospin-structures of the 1π- and 2π-exchange interaction show up through relative isospin-factors \(-1/3, \pm 2/3\) and \(-5/3\) of various subcontributions. We find that the parameterfree results for the (density-dependent) strength functions \(G_{so}(k_f)\) and \(G_J(k_f)\) agree fairly well with that of phenomenological Skyrme forces for densities \(\rho > \rho_0/10\). At very low densities a strong variation of the strength functions \(G_{so}(k_f)\) and \(G_J(k_f)\) with density sets in. This has to do with chiral singularities \(m_\pi^{-1}\) and the presence of two competing small mass scales \(k_f\) and \(m_\pi\). The novel density dependencies of \(G_{so}(k_f)\) and \(G_J(k_f)\) as predicted by our parameterfree (leading order) calculation should be examined in nuclear structure calculations.

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