Estimating the Parameters of the Two-Parameter Rayleigh Distribution Based on Adaptive Type II Progressive Hybrid Censored Data with Competing Risks

Shuhan Liu and Wenhao Gui *

Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China; 17271186@bjtu.edu.cn
* Correspondence: whgui@bjtu.edu.cn

Received: 11 September 2020; Accepted: 8 October 2020; Published: 15 October 2020

Abstract: This paper attempts to estimate the parameters for the two-parameter Rayleigh distribution based on adaptive Type II progressive hybrid censored data with competing risks. Firstly, the maximum likelihood function and the maximum likelihood estimators are derived before the existence and uniqueness of the latter are proven. Further, Bayesian estimators are considered under symmetric and asymmetric loss functions, that is the squared error loss function, the LINEX loss function, and the general entropy loss function. As the Bayesian estimators cannot be obtained explicitly, the Lindley method is applied to compute the approximate Bayesian estimates. Finally, a simulation study is conducted, and a real dataset is analyzed for illustrative purposes.

Keywords: adaptive Type II progressive hybrid censoring; two-parameter Rayleigh distribution; competing risks; maximum likelihood estimation; Bayesian estimation; symmetric and asymmetric loss functions

1. Introduction

1.1. Two-Parameter Rayleigh Distribution

The Rayleigh distribution, originally proposed by Lord Rayleigh in the field of acoustics, is one of the most important distributions when dealing with skewed data. It is closely related to many commonly-used distributions, such as the Weibull distribution, the chi-squared distribution, and the extreme value distribution. Since it was proposed, it has been widely used by engineers and physicists for modeling radiation, synthetic aperture radar images, and other related phenomena.

In this paper, we will direct our attention to the two-parameter Rayleigh distribution. The probability density function (pdf) of the two-parameter Rayleigh distribution is:

\[ f(x|\lambda, \mu) = 2\lambda(x - \mu)e^{-\lambda(x-\mu)^2}; \quad x \geq \mu, \lambda > 0, \]

where \( \lambda \) is the scale parameter and \( \mu \) is the location parameter. The corresponding cumulative distribution function (cdf) is:

\[ F(x|\lambda, \mu) = 1 - e^{-\lambda(x-\mu)^2}. \]

Compared with the Rayleigh distribution, the two-parameter Rayleigh distribution is more flexible due to the introduction of a location parameter \( \mu \). When the location parameter \( \mu \) is equal to zero, the two-parameter Rayleigh distribution degrades to the Rayleigh distribution. This kind of flexibility is especially important when the lifetimes of the units do not begin from zero. For example, a specific disease may just strike people over 20 years old. To analyze the lifetimes of these patients,
a traditional Rayleigh distribution may not provide a satisfactory result. Besides, the two-parameter Rayleigh distribution is always unimodal and has an increasing hazard rate function, making it very appropriate for modeling the lifetime distribution of the units that rapidly age with time. The plots of the pdf and cdf of the two-parameter Rayleigh distribution can be found in Figure 1.

![Figure 1. pdf and cdf of the two-parameter Rayleigh distribution.](image)

Recently, the two-parameter Rayleigh distribution has attracted much attention. Reference [1] estimated the parameters of the two-parameter Rayleigh distribution based on maximum likelihood estimation, moment estimation, L-moments estimation, percentile based estimation, least squares estimation, and Bayesian estimation. Reference [2] considered the inference of the parameters for the two-parameter Rayleigh distribution based on progressive Type II censoring. Reference [3] proposed the exact confidence intervals for the parameters of the two-parameter Rayleigh distribution and developed the exact predictive intervals for the future upper record values. Reference [4] studied confidence intervals for the parameters of the two-parameter Rayleigh distribution, solved the constrained optimization problems, and analyzed three numerical examples for illustrative purpose. Reference [5] made the prediction from a two-parameter Rayleigh distribution based on doubly censored sample, derived the predictive distributions by adopting the Bayesian approach, and studied a numerical example to compare the effects of the hyperparameters. Though the point estimation, interval estimation, and prediction for the two-parameter Rayleigh distribution have been well studied based on both complete data and censored data, scholars have not paid much attention to the estimation based on censored competing risks data, which is the theme of our paper.

1.2. Adaptive Type II Progressive Hybrid Censoring

In a life-testing experiment, sometimes it is hard to obtain complete and independent data, which leads us to the field of censored data. The most well-known censoring schemes are Type I and Type II censoring schemes. Under the Type I censoring scheme, the experiment is stopped when a pre-fixed time (denoted as \( T \)) has been reached, and under the Type II censoring scheme, the experiment is stopped when a specific number (denoted as \( m \)) of failures have been observed.
In order to further reduce the experimental time and cost, a mixture of the Type I and Type II censoring scheme, which is also known as the hybrid censoring scheme, was proposed. Under the hybrid censoring scheme, the experimenter stops the experiment either when the $m$-th failure occurs or when the ideal experimental time $T$ is reached. Additionally, intermediate removals are desirable; namely, sometimes units are required to be removed from the experiment at points other than the terminal point, and accordingly, progressive censoring schemes are needed. A brief description of the progressive hybrid censoring scheme is as follows. Suppose that $n$ identical units are put in an experiment. The failure time of the unit is denoted as $X = (X_1, X_2, \ldots, X_m)$, and the censoring scheme is defined as $R = (R_1, R_2, \ldots, R_m)$, which satisfies $n - m = \sum_{i=1}^{m} R_i$. When the first unit fails at time $X_1$, $R_1$ units are randomly removed from the experiment, and when the second unit fails at $X_2$, $R_2$ units are randomly removed. This process continues until either the $m$-th failure occurs at $X_m$ or the pre-fixed ideal experimental time $T$ is reached and then all the remained units are removed. Suppose $J$ units fail before the ideal experimental time $T$, then the terminal time $T^* = \min\{X_m, T\}$ and the terminal removal $R^* = n - J - \sum_{i=1}^{J} R_i$.

However, one drawback of progressive hybrid censoring scheme is that the accuracy of the obtained estimators can be quite low on the assumption that only a few units (or even zero) fail before the terminal time $T^*$. Recently, an adjustment of the Type II progressive hybrid censoring scheme was proposed by Reference [6] in order to increase the efficiency of statistical analysis and save the total time in the meantime. As we mentioned before, under the progressive Type II hybrid censoring scheme, units are not allowed to survive over the ideal experimental time $T$, while under the adaptive Type II progressive hybrid censoring scheme, units are allowed to exceed $T$. More specifically, when $X_m < T$, the experiment stops at time $X_m$, and a regular progressive Type II censoring with the censoring scheme $R = (R_1, R_2, \ldots, R_m)$ is obtained. When $X_m > T$, we do not stop the experiment at time $T$, but adjust the units removed from the experiment after the pre-fixed ideal experimental time $T$ by setting $R_{j+1} = R_{j+2} = \cdots = R_{m-1} = 0$, $R_m = n - m - \sum_{i=1}^{j} R_i$ to speed up the experiment, where $j$ is the number of units that fail before $T$, namely $j = \max\{j|X_j < T\}$. The corresponding updated censoring scheme is $R = (R_1, R_2, \ldots, R_j, 0, \ldots, 0, n - m - \sum_{i=1}^{j} R_i)$. One can change the value of $T$ as a compromise between a shorter experimental time and a higher accuracy. A graphic representation of adaptive Type II progressive hybrid censoring is presented as follows.

**Case I** $X_m < T$

![Graphic representation of Case I](image)

**Case II** $X_m \geq T$

![Graphic representation of Case II](image)

Since adaptive the Type II progressive hybrid censoring scheme was proposed, many interesting and important research works have been done in this field. Reference [7] studied the estimation of the log-normal distribution under adaptive Type II progressive hybrid censoring and Type II progressive hybrid censoring. The maximum likelihood estimators and the approximate maximum likelihood estimators are derived based on both schemes. Reference [8] estimated the parameters of the inverted
1.3. Competing Risks

In this subsection, we discuss the competing risks model, which was raised from reliability theory for practical purposes. In an experiment, a unit may fail due to many risks, and each of the risks alone can cause the failure of the unit. In this situation, the risks are called competing risks. Recently, the competing risks model has been widely applied in the field of censored data. Reference [12] considered the Rayleigh distribution based on progressive Type II censored competing risks data. Both the maximum likelihood estimation and the Bayesian estimation were studied, and the asymptotic confidence intervals and Bayesian credible intervals were constructed as well. Reference [13] studied a constant-stress accelerated life test based on progressive Type II hybrid censoring with competing risks. Reference [14] considered the inference of the unknown parameters of the Weibull distribution under adaptive Type II progressive hybrid censoring scheme can be found in References [9–11].

In this article, we focus on the point estimation of the parameters for the two-parameter Rayleigh distribution under adaptive Type II progressive hybrid censored data using the maximum likelihood method and the Bayesian method. More research based on adaptive Type II censoring scheme can be found in References [9–11].
2. Maximum Likelihood Estimation

In this section, the traditional frequentists’ method is applied to estimate the parameters. Suppose that \( n \) units are placed in an experiment. Let \( x_1 < x_2 < \cdots < x_m \) denote the adaptive Type II progressive hybrid censored failure times, \( \delta_i (i = 1, \ldots, m) \) denote the failure cause of each unit, \( T \) denote the ideal experimental time, and \( J \) denote the last failure order before time \( T \). The adjusted censoring scheme is presented as \( R = (R_1, R_2, \ldots, R_J, 0, \ldots, 0, R_m = n - m - \sum_{i=1}^{J} R_i) \). Then, the maximum likelihood function can be expressed as:

\[
L \propto \prod_{i=1}^{m} \{f_1(x_i)[1 - F_2(x_i)]\}^{\delta_i} \prod_{i=1}^{m} \{f_2(x_i)[1 - F_2(x_i)]\}^{I(\delta_i = 2)} \times \prod_{i=1}^{J} [1 - F(x_i)]^{R_i} \times [1 - F(x_m)]^{R^*},
\]

where \( I(\delta_i = j) = 1 \) when \( \delta_i = j, I(\delta_i = j) = 0 \) when \( \delta_i \neq j \), \( R^* = n - m - \sum_{i=1}^{J} R_i \) \( (i = 1, 2, \ldots, m, j = 1, 2) \).

In order to further simplify the expression, we need to rearrange the data according to the failure cause. For each cause, the data are still sorted as ordered. Then, the data can be rearranged as:

\[
x = ((x_{(1)}, R_{(1)}, \delta_{(1)} = 1), \ldots, (x_{(n_{(1)} - 1)}, R_{(n_{(1)} - 1)}, \delta_{(n_{(1)} - 1)} = 1), (x_{(n_{(1)} + 1)}, R_{(n_{(1)} + 1)}, \delta_{(n_{(1)} + 1)} = 2), \ldots, (x_{(n_{(n_{(2)} - 1)} + 1)}, R_{(n_{(n_{(2)} - 1)} + 1)}, \delta_{(n_{(n_{(2)} - 1)} + 1)} = 2)),
\]

where \( n_1 = \sum_{i=1}^{n_2} I(\delta_i = 1) \) and \( n_2 = \sum_{i=1}^{m} I(\delta_i = 2) \). In other words, \( n_1 \) is the number of failures due to the first risk, and \( n_2 \) is the number of failures due to the second risk. Thus, \( n_1 + n_2 = m \).

Besides, \( x_m \) should be greater than \( \max(\mu_1, \mu_2) \). On the contrary, if (without loss of generality) \( x_m \leq \mu_1 \), there will be no failures caused by the first risk, and thus, \( \mu_1 \) and \( \lambda_1 \) cannot be estimated. For sorted data, the maximum likelihood function can be rewritten as:

\[
L \propto \prod_{i=1}^{n_1} f_1(x_{(i)})[1 - F_2(x_{(i)})] \prod_{i=n_1+1}^{m} f_2(x_{(i)})[1 - F_2(x_{(i)})] \prod_{i=1}^{J} [1 - F(x_i)]^{R_i} \times [1 - F(x_m)]^{R^*},
\]

\[
\propto \lambda_1^{n_1} \lambda_2^{n_2} \prod_{i=1}^{n_1} (x_{(i)} - \mu_1) \prod_{i=n_1+1}^{m} (x_{(i)} - \mu_2)
\times e^{-\lambda_1 \sum_{i=1}^{n_2} (x_{(i)} - \mu_1)^2 + \sum_{i=n_{(1)}+1}^{m} (x_{(i)} - \mu_1)^2 I(x_{(i)} > \mu_1) + \sum_{i=1}^{J} R_i (x_i - \mu_2)^2 I(x_i > \mu_2) + R^* (x_m - \mu_2)^2}.
\]

Then, the corresponding log-likelihood function without constants is:

\[
I = n_1 \ln \lambda_1 + n_2 \ln \lambda_2 + \sum_{i=1}^{n_1} \ln (x_{(i)} - \mu_1) + \sum_{i=n_1+1}^{m} (x_{(i)} - \mu_2) - \lambda_1 \sum_{i=1}^{n_1} (x_{(i)} - \mu_1)^2
\]
\[
+ \sum_{i=1}^{m} (x_{(i)} - \mu_1)^2 I(x_{(i)} > \mu_1) + \sum_{i=n_{(1)}+1}^{m} R_i (x_i - \mu_2)^2 I(x_i > \mu_2) + R^* (x_m - \mu_2)^2 - \lambda_2]
\]
\[
\sum_{i=1}^{n_1} (x_{(i)} - \mu_2)^2 I(x_{(i)} > \mu_2) + \sum_{i=n_{(1)}+1}^{m} (x_{(i)} - \mu_2)^2 + \sum_{i=1}^{J} R_i (x_i - \mu_2)^2 I(x_i > \mu_2) + R^* (x_m - \mu_2)^2]. \tag{1}
\]

In order to compute the MLEs, we take the first derivatives of Equation (1) with respect to \( \lambda_1, \mu_1, \lambda_2, \) and \( \mu_2 \) and assign them to be zeros. The equations obtained are:

\[
\frac{\partial I}{\partial \lambda_1} = \frac{n_1}{\lambda_1} - \sum_{i=1}^{n_1} (x_{(i)} - \mu_1)^2 + \sum_{i=n_1+1}^{m} (x_{(i)} - \mu_1)^2 I(x_{(i)} > \mu_1)
\]
\[
+ \sum_{i=1}^{J} R_i (x_i - \mu_2)^2 I(x_i > \mu_2) + R^* (x_m - \mu_2)^2 = 0, \tag{2}
\]
\[
\begin{align*}
\frac{\partial l}{\partial \lambda_2} &= \frac{n_2}{\lambda_2} - \sum_{i=1}^{n_1} (x(i) - \mu_2)^2 I\{x(i) > \mu_2\} + \sum_{i=n_1+1}^{m} (x(i) - \mu_2)^2 \\
&\quad + \sum_{i=1}^{f} R_i(x_i - \mu_2)^2 I\{x_i > \mu_2\} + R^*(x_m - \mu_2)^2 = 0, \\
\frac{\partial l}{\partial \mu_1} &= \sum_{i=1}^{n_1} \frac{1}{\mu_1 - x(i)} - 2\lambda_1 \sum_{i=1}^{n_1} (x(i) - \mu_1)^2 + \sum_{i=n_1+1}^{m} (x(i) - \mu_1)^2 I\{x(i) > \mu_1\} \\
&\quad + \sum_{i=1}^{f} R_i(x_i - \mu_1)^2 I\{x_i > \mu_1\} + R^*(x_m - \mu_1)^2 = 0, \\
\frac{\partial l}{\partial \mu_2} &= \sum_{i=n_1+1}^{m} \frac{1}{\mu_2 - x(i)} - 2\lambda_2 \sum_{i=1}^{n_1} (x(i) - \mu_2)^2 I\{x(i) > \mu_2\} + \sum_{i=n_1+1}^{m} (x(i) - \mu_2)^2 \\
&\quad + \sum_{i=1}^{f} R_i(x_i - \mu_2)^2 I\{x_i > \mu_2\} + R^*(x_m - \mu_2)^2 = 0.
\end{align*}
\]

**Theorem 1.** Suppose the failure time of the competing risks follows the two-parameter Rayleigh distribution under adaptive Type II progressive hybrid censoring. Then, for \(0 < \mu_1 < x(1)\), \(0 < \mu_2 < x(n_1+1)\), \(\lambda_1 > 0\), and \(\lambda_2 > 0\), the MLEs for the parameters \(\lambda_1\), \(\mu_1\), \(\lambda_2\) and \(\mu_2\) exist and are unique in the domain.

**Proof.** Taking the second partial derivatives of the log-likelihood function, we obtained results are:

\[
\begin{align*}
\frac{\partial^2 l}{\partial \lambda_1^2} &= -\frac{n_1}{\lambda_1^2} < 0, \\
\frac{\partial^2 l}{\partial \lambda_2^2} &= -\frac{n_2}{\lambda_2^2} < 0, \\
\frac{\partial^2 l}{\partial \mu_1^2} &= -\sum_{i=1}^{n_1} \left(\frac{1}{(\mu_1 - x(i))^2}\right) - 2\lambda_1 \left[\sum_{i=n_1+1}^{m} I\{x(i) > \mu_1\}\right] + \sum_{i=1}^{f} R_i I\{x_i > \mu_1\} + R^* \] < 0, \\
\frac{\partial^2 l}{\partial \mu_2^2} &= -\sum_{i=n_1+1}^{m} \left(\frac{1}{(\mu_2 - x(i))^2}\right) - 2\lambda_2 \left[\sum_{i=n_1+1}^{m} I\{x(i) > \mu_2\}\right] + \sum_{i=1}^{f} R_i I\{x_i > \mu_2\} + R^* \] < 0,
\end{align*}
\]

Accordingly, the log-likelihood function is in fact a strictly concave function with respect to the unknown parameters. Besides, based on Equations (2)-(5), when \(\mu_j\) is fixed, we have:

\[
\begin{align*}
\lim_{\lambda_j \to +\infty} \frac{\partial l}{\partial \lambda_j} < 0, &\quad \lim_{\lambda_j \to 0} \frac{\partial l}{\partial \lambda_j} = +\infty,
\end{align*}
\]

and when \(\lambda_j\) is fixed,

\[
\begin{align*}
\lim_{\mu_j \to x(1)} \frac{\partial l}{\partial \mu_j} \to -\infty, &\quad \lim_{\mu_j \to x(n_1+1)} \frac{\partial l}{\partial \mu_j} \to -\infty, \quad \lim_{\mu_j \to +\infty} \frac{\partial l}{\partial \mu_j} \to +\infty, \quad \lim_{\mu_j \to 0} \frac{\partial l}{\partial \mu_j} = c_j,
\end{align*}
\]

where \(c_1 = -\sum_{i=1}^{n_1} \frac{1}{x(i)} + 2\lambda_1 \sum_{i=1}^{n_1} x(i) + \sum_{i=n_1+1}^{m} x(i) I\{x(i) > \mu_1\} + \sum_{i=1}^{f} R_i x_i I\{x_i > \mu_1\} + R^* x_m\) and \(c_2 = -\sum_{i=n_1+1}^{m} \frac{1}{x(i)} + 2\lambda_2 \sum_{i=1}^{n_1} x(i) I\{x(i) > \mu_2\} + \sum_{i=n_1+1}^{m} x(i) I\{x(i) > \mu_2\} + R^* x_m\).

Therefore, Equations (2)-(5) have unique roots \((\mu_1^*, \mu_2^*, \lambda_1^*, \lambda_2^*)\) for 

\([\mu_1, \mu_2, \lambda_1, \lambda_2] \in (-\infty, x(1)) \times (-\infty, x(n_1+1)) \times (0, +\infty) \times (0, +\infty)\). Firstly, assume \(c_1 > 0\) and \(c_2 > 0\), then the MLEs of the unknown parameters in the domain \((0, x(1)) \times (0, x(n_1+1)) \times (0, +\infty) \times (0, +\infty)\).
are exactly the uniroots \((\mu_1^*, \mu_2^*, \lambda_1^*, \lambda_2^*)\). Secondly, if \(c_j < 0\), the MLEs for \(\mu_j\) are equal to zero, and the MLEs for \(\lambda_j\) are still \(\lambda_j^*\) \((j = 1, 2)\). That is because when \(c_j < 0\), we can deduce that \(\frac{\partial L}{\partial \mu_j} < 0\) in the domain; namely, the log-likelihood function decreases with the increasing of \(\mu_j\). In order to acquire the maximum of the log-likelihood function, we need to set \(\mu_j = 0\). In other words, the MLE of \(\mu_j\) is equal to zero. In either case, the MLEs of the unknown parameters exist and are unique in the domain. 

Hence, the maximum likelihood estimates can be computed by solving the roots of Equations (2)–(5). As the roots of the equations cannot be expressed in closed form, some numerical methods, the Newton–Raphson iterative method, for instance, can be applied to acquire the approximate roots.

3. Bayesian Estimation

In this section, Bayesian estimation of the unknown parameters is studied based on several loss functions. Unlike the frequentists’ method, this relatively new yet efficient method is an effort to combine the observed data with prior information.

3.1. Prior Distribution

Reference [2] pointed out that even for complete data, all the elements for the expected Fisher information matrix are not finite, and thus, Jeffrey’s prior does not exist. Considering the domain for the unknown parameters, which is \((\mu_1, \mu_2, \lambda_1, \lambda_2) \in (0, x(1)) \times (0, x(n_1 + 1)) \times (0, +\infty) \times (0, +\infty)\), we apply the same priors as Reference [2] by assuming the priors are independent, \(\lambda_1\) follows a gamma prior, and \(\mu_j\) has a uniform prior in the domain. The priors are respectively:

\[
\pi(\lambda_1 | a_1, a_2) \propto \lambda_1^{a_1-1}e^{-a_2\lambda_1}, \quad a_1, a_2 > 0,
\]

\[
\pi(\lambda_2 | b_1, b_2) \propto \lambda_2^{b_1-1}e^{-b_2\lambda_2}, \quad b_1, b_2 > 0,
\]

\[
\pi(\mu_1) \propto 1, \quad \mu_1 \in (0, x(1)),
\]

\[
\pi(\mu_2) \propto 1, \quad \mu_2 \in (0, x(n_1+1)).
\]

where \(a_1, a_2, b_1, b_2\) are the non-negative hyperparameters of the prior distribution containing the information of \(\lambda_1\) and \(\lambda_2\).

Therefore, the prior distribution \((\pi_0)\) and the joint posterior distribution \((\pi_1)\) are given.

\[
\pi_0(\lambda_1, \lambda_2, \mu_1, \mu_2) \propto \lambda_1^{a_1-1}\lambda_2^{b_1-1}e^{-a_2\lambda_1}e^{-b_2\lambda_2},
\]

\[
\pi_1(\lambda_1, \lambda_2, \mu_1, \mu_2) \propto \lambda_1^{a_1}+b_1-1\lambda_2^{b_2}+1 \prod_{i=1}^{m}(x(i) - \mu_1) \prod_{i=n_1+1}^{m_1}(x(i) - \mu_2)
\]

\[
\times e^{-\lambda_1[(a_1+1)\sum_{i=1}^{m_1}(x(i) - \mu_1)^2+\sum_{i=n_1+1}^{m}(x(i) - \mu_1)^2 I(x(i) > \mu_1)]+\sum_{i=1}^{m_1}R_i(x_i-\mu_1)^2 I(x_i > \mu_1)+R^*(x_m-\mu_1)^2}
\]

\[
\times e^{-\lambda_2[(b_2+1)\sum_{i=n_1+1}^{m}(x(i) - \mu_2)^2+\sum_{i=1}^{m_1}(x(i) - \mu_2)^2 I(x(i) > \mu_2)+\sum_{i=1}^{m_1}R_i(x_i-\mu_2)^2 I(x_i > \mu_2)+R^*(x_m-\mu_2)^2]}.
\]

3.2. Loss Functions

This subsection will shed light on Bayesian estimates under symmetric and asymmetric loss functions. Three different loss functions are applied, namely the squared error loss function, the LINEX loss function, and the general entropy loss function. The most commonly-used symmetrical loss function is the squared error loss function and is defined as \(L_1(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2\), where \(\theta\) is the unknown parameter and \(\hat{\theta}\) is an estimate of \(\theta\). The Bayesian estimate under the squared error loss function can be expressed as:

\[
\hat{\theta}_L = E_{\theta}(\theta|X) = \frac{\int_0^{x(1)} \int_0^{x(n_1+1)} \int_0^{+\infty} \int_0^{+\infty} \theta \times L \times \pi_0 d\mu_1 d\mu_2 d\lambda_1 d\lambda_2 \times \pi_0 d\mu_1 d\mu_2 d\lambda_1 d\lambda_2}{\int_0^{x(1)} \int_0^{x(n_1+1)} \int_0^{+\infty} \int_0^{+\infty} L \times \pi_0 d\mu_1 d\mu_2 d\lambda_1 d\lambda_2}.
\]
One limitation of the squared error loss function is its symmetry. A well-known asymmetric loss function is the LINEX loss function, which is \( L_2(\theta, \bar{\theta}) = \frac{1}{2} \theta (\bar{\theta} - \theta) - h(\bar{\theta} - \theta) - 1 \). The sign of \( h \) represents the direction of asymmetry, and its magnitude reflects the degree of its asymmetry. For \( h < 0 \), a negative error has a more serious effect, and for \( h > 0 \), the effect of a positive error is more serious.

The Bayesian estimate of \( \theta \) under the LINEX loss function is:

\[
\hat{\theta}_L = -\frac{1}{h} \ln \left[ E_\theta (e^{-h\bar{\theta}} | X) \right] = -\frac{1}{h} \ln \left[ \int_0^{\infty} \int_0^{\infty} e^{-h\theta} \times \Delta \times \pi_0 d\mu_1 d\mu_2 d\lambda_1 d\lambda_2 \right].
\]

Another famous asymmetric loss function is the general entropy loss function, which is \( L_3(\theta, \bar{\theta}) = (\frac{d}{\bar{\theta}})^q - q \ln (\frac{d}{\bar{\theta}}) - 1 \). When \( q > 0 \), an overestimation has a more serious influence, while when \( q < 0 \), an underestimation is more serious. In this case, the Bayesian estimate can be written as:

\[
\hat{\theta}_E = \left[ E_{\theta}(\theta^{-q} | X) \right]^{-\frac{1}{q}} = \left[ \int_0^{\infty} \int_0^{\infty} \theta^{-q} \times \Delta \times \pi_0 d\mu_1 d\mu_2 d\lambda_1 d\lambda_2 \right]^{-\frac{1}{q}}.
\]

### 3.3. Lindley Method

Clearly, the Bayesian estimates of \( \theta \), namely \( \hat{\theta}_L, \hat{\theta}_E \), and \( \hat{\theta}_E \), are in the form of a ratio of two integrals and cannot be computed easily. A very commonly-used approximation method to solve the problem is the Lindley approximation method (proposed by Reference [15]). The Lindley method was proposed to evaluate the ratio of two integrals and requires the third derivatives of the likelihood function or the posterior density. The rationale of the Lindley method and an outline of the procedure can be found in Reference [16]. Here, suppose \( \theta = (\lambda_1, \lambda_2, \mu_1, \mu_2) \), \( u(\theta) \) is a function of \( \theta \), \( u^*(\theta) \) is the estimate of \( u(\theta) \), and \( \rho(\theta) = \ln(\pi_0(\theta)) \), then the Lindley approximation can be presented as:

\[
u^*(\theta) = u(\hat{\theta}) + \frac{1}{2} \sum_{ij} \left[ u_{ij}(\hat{\theta}) + 2u_i(\hat{\theta})\rho_j(\hat{\theta}) \right] \sigma_{ij} + \frac{1}{2} \sum_{ijk} l_{ijk}(\hat{\theta})u_{ij}(\hat{\theta})\sigma_{kl}.
\]

where \( l_{ijk} = \frac{\partial^3 u}{\partial \theta_i \partial \theta_j \partial \theta_k}, u_{ij} = \frac{\partial^2 u}{\partial \theta_i \partial \theta_j}, u_i = \frac{\partial u}{\partial \theta_i}, \) and \( \rho_j = \frac{\partial \rho(\theta)}{\partial \theta_j} \), \( \sigma_{ij} \) denotes the \((i,j)\)-th element of the inverse of the negative Hessian matrix of the log-likelihood function, and \( \hat{\theta} \) denotes the MLE of \( \theta \). For brevity, the results for each term in the formula are not presented in that all the terms are acquired by basic computation and most results are obtained by derivation. One may also use the Deriv package in the R software to take the derivatives.

By means of the Lindley approximation method, it is feasible to compute the estimates for the unknown parameters under the squared error, the LINEX, and the general entropy loss functions.

### 4. Simulation

In this section, simulation experiments are conducted to test the performance of our estimates. First, we need to generate an adaptive Type II progressive hybrid censored sample with two competing risks from the two-parameter Rayleigh distribution. The algorithm for generating the adaptive Type II hybrid censored data can be found in Reference [8]. We make a basic adaptation of the algorithm just as Reference [17] did in their paper to generate adaptive Type II progressive hybrid censored data with competing risks. The algorithm to generate the simulated data can be found in Algorithm 1.

In order to conduct simulation studies, we set \( \lambda_1 = 0.3, \mu_1 = 0.8, \lambda_2 = 0.5, \) and \( \mu_2 = 1.0 \). For informative priors, the hyperparameters are assigned as \( a_1 = 3, a_2 = 10, b_1 = 2, \) and \( b_2 = 4 \), and for comparison, the non-informative prior is also considered by setting the hyperparameters as \( a_1 = a_2 = b_1 = b_2 = 0 \). For the LINEX loss function, we set \( h = 1 \), and for the general entropy loss function, we set \( q = 1 \). The two selected censoring schemes are:
When \( \mu \lambda \) when \( (S) \), the LINEX (L) and the general entropy loss function (E). The tables can be found at the end of the article. For Bayesian estimates, the Bayesian estimates with informative priors are more satisfactory than those with non-informative priors, and the Bayesian estimates under the squared error loss function are slightly better than those under the LINEX loss function. For most censoring schemes, the performance of Bayesian estimates with informative priors is better compared with that of the MLEs. Besides, the Bayesian estimates with non-informative priors perform better than the MLEs when \( \lambda_1 \) is considered; yet, this advantage no longer stands when it comes to other parameters.

### Algorithm 1: Generating adaptive Type II progressive hybrid censored data with two competing risks from the two-parameter Rayleigh distribution.

1. Generate Type II progressive censored sample of size \( n \) with a censoring scheme \( R = (R_1, R_2, \ldots, R_m) \) for the two-parameter Rayleigh distribution:
   
   (a) Generate \( m \) independent variables from the uniform distribution \( U(0,1) \) as \( U_1, U_2, \ldots, U_m \).
   
   (b) Set \( V_i = U_i (\sum_{j=1}^{m-1} R_j \cdot (i = 1, 2, \ldots, m) \).
   
   (c) Set \( W_i = 1 - V_m V_{m-1} \ldots V_{m-j+1}, (i = 1, 2, \ldots, m) \).
   
   (d) For given \( \lambda_1, \mu_1, \lambda_2, \mu_2 \), let \( X_i = F^{-1}(W_i), (i = 1, 2, \ldots, m) \). Then, \( X = (X_1, X_2, \ldots, X_m) \) are progressive Type II censored samples from the two-parameter Rayleigh distribution.

2. Set \( X_{J+1}, \ldots, X_m \) as the first \( m - J - 1 \) order statistics from a distribution \( \frac{f(x)}{1-F(x_{J+1})} \) with sample size \( n - J - 1 - \sum_{i=1}^{J} R_i \).

3. Generate the factors of the failures. For \( X_i \leq \mu_2 \), set \( \delta_i = 1 \), and for \( X_i > \mu_2 \), get \( \delta_i \) from the Bernoulli distribution with \( p_i \), which is the conditional probability that a unit fails at \( x_i \) due to the first cause given that a unit fails at \( x_i \). \( p_i \) can be computed from the following formula:

\[
p_i = \frac{f_1(x_i)F_2(x_i)}{f_1(x_i)F_2(x_i) + f_2(x_i)F_1(x_i)}.
\]

The simulation results for \( \lambda_1, \mu_1, \lambda_2, \) and \( \mu_2 \) are provided respectively in Tables 1–4. For each value in the tables, the simulation was repeated at least 2000 times. For each censoring scheme, the biases and the mean squared errors (MSEs) serve as criteria to compare the performance of the maximum likelihood estimates and the Bayesian estimates. The Bayesian estimates are acquired using the Lindley approximation method based on three different loss functions, including the squared error (S), the LINEX (L) and the general entropy loss function (E). The tables can be found at the end of the article.

From the tables, one can see that in general, the MSEs of the estimates decrease with the increasing of \( m \) when \( n \) and \( T \) are fixed. When \( n \) and \( m \) are fixed, no specific pattern is observed with the increasing of \( T \). This is understandable because for some censoring schemes, the observed data may remain unchanged with the increasing of \( T \). For Bayesian estimates, the Bayesian estimates with informative priors are more satisfactory than those with non-informative priors, and the Bayesian estimates under the LINEX loss function perform better than those under other loss functions considering \( \lambda_1 \) and \( \lambda_2 \). When \( \mu_1 \) and \( \mu_2 \) are considered, one can see that Bayesian estimates under the squared error loss function are slightly better than those under the LINEX loss function. For most censoring schemes, the performance of Bayesian estimates with informative priors is better compared with that of the MLEs. Besides, the Bayesian estimates with non-informative priors perform better than the MLEs when \( \lambda_1 \) is considered; yet, this advantage no longer stands when it comes to other parameters.
Table 1. The results for $\lambda_1$.

| $(n,m,T)$ | Sch | $\hat{\lambda}_1$ Bias | $\hat{\lambda}_1$ MSE | $\hat{\lambda}_{1S}$ Bias | $\hat{\lambda}_{1S}$ MSE | $\hat{\lambda}_{1L}$ Bias | $\hat{\lambda}_{1L}$ MSE | $\hat{\lambda}_{1F}$ Bias | $\hat{\lambda}_{1F}$ MSE |
|-----------|-----|-------------------------|------------------------|---------------------------|--------------------------|---------------------------|--------------------------|---------------------------|--------------------------|
| (30,25,4) | I   | 0.0843 0.0417           | 0.0396 0.0299          | −0.0569 0.1299            | 0.0281 0.025             | −0.0352 0.0043           | 0.0034 0.023             | −0.0334 0.0066           |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.0822 0.0343           | 0.0366 0.0245          | −0.0459 0.015             | 0.0252 0.021             | −0.0361 0.0045           | 0.0002 0.0191           | −0.0348 0.0066           |                          |
| (30,25,2) | I   | 0.0749 0.0332           | 0.032 0.0243           | −0.0423 0.0085            | 0.0219 0.0211            | −0.0349 0.0041           | −0.0025 0.0207           | −0.0364 0.0099           |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.0771 0.0317           | 0.0315 0.0226          | −0.0422 0.0092            | 0.021 0.0197             | −0.0356 0.004             | −0.004 0.018             | −0.0364 0.0096           |                          |
| (30,20,2) | I   | 0.0894 0.0494           | 0.0376 0.0349          | −0.088 0.0857             | 0.0236 0.0288            | −0.0566 0.0065           | −0.0039 0.0268           | −0.0472 0.0074           |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.1018 0.0529           | 0.0434 0.0357          | −0.111 0.1742             | 0.0283 0.0291            | −0.0672 0.0125           | 0.001 0.0266           | −0.0503 0.0143           |                          |
| (50,40,4) | I   | 0.0416 0.0141           | 0.0131 0.0111          | −0.0077 0.0037            | 0.008 0.0104             | −0.0104 0.0039           | −0.0093 0.0099           | −0.0217 0.0054           |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.0505 0.0171           | 0.0208 0.0134          | −0.0056 0.0038            | 0.0154 0.0123            | −0.0074 0.004             | −0.0016 0.0118           | −0.0182 0.0056           |                          |
| (50,40,2) | I   | 0.0433 0.0139           | 0.0149 0.0109          | −0.0063 0.0036            | 0.0097 0.0102            | −0.0092 0.0038           | −0.0078 0.0097           | −0.021 0.0053            |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.0448 0.0142           | 0.0161 0.0112          | −0.0054 0.0037            | 0.0109 0.0105            | −0.0083 0.0038           | −0.0066 0.0099           | −0.002 0.0053            |                          |
| (50,30,2) | I   | 0.0575 0.0221           | 0.0228 0.0169          | −0.0176 0.005             | 0.0156 0.0152            | −0.0175 0.0038           | −0.0051 0.0142           | −0.0264 0.0061           |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.0578 0.0238           | 0.0212 0.0168          | −0.0342 0.1274            | 0.0098 0.0197            | −0.0254 0.0093           | −0.0091 0.0152           | −0.0311 0.0069           |                          |
| (80,70,4) | I   | 0.025 0.0072            | 0.0068 0.0061          | 0.0004 0.0038             | 0.0042 0.0059            | −0.0018 0.0037           | −0.0066 0.0057           | −0.0114 0.0041           |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.0268 0.0072           | 0.0089 0.0063          | 0.0024 0.0043             | 0.0065 0.0064            | −0.0006 0.0037           | −0.0009 0.0224           | −0.0107 0.0042           |                          |
| (80,70,2) | I   | 0.0208 0.0065           | 0.0092 0.0475          | 0.0035 0.0428             | 0.0002 0.0053            | −0.0049 0.0034           | −0.0108 0.0054           | −0.0149 0.0039           |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.0291 0.0068           | 0.0104 0.0057          | 0.0033 0.0035             | 0.0077 0.0055            | 0.001 0.0034            | −0.0032 0.0052           | −0.0086 0.0037           |                          |
| (80,60,2) | I   | 0.0289 0.0077           | 0.0085 0.0065          | 0.0004 0.0037             | 0.0055 0.0062            | −0.0021 0.0037           | −0.0065 0.006            | −0.0124 0.0041           |                          |
|           |     |                         |                        |                           |                          |                           |                          |                          |                          |
|           | II  | 0.0288 0.0078           | 0.0083 0.0065          | 0.0001 0.0037             | 0.0052 0.0062            | −0.0026 0.0036           | −0.0007 0.006            | −0.0131 0.0041           |                          |
Table 2. The results for $\mu_1$.

| $(n,m,T)$ | Sch | $\hat{\mu}_1$ Bias | $\hat{\mu}_1$ MSE | $\hat{\mu}_{1S}$ Bias | $\hat{\mu}_{1S}$ MSE | $\hat{\mu}_{1L}$ Bias | $\hat{\mu}_{1L}$ MSE | $\hat{\mu}_{1E}$ Bias | $\hat{\mu}_{1E}$ MSE |
|-----------|-----|---------------------|-------------------|----------------------|---------------------|----------------------|----------------------|----------------------|----------------------|
|           |     | Non-Informative | Informative | Non-Informative | Informative | Non-Informative | Informative | Non-Informative | Informative |
| (30,25,4) | I   | 0.0860 0.0422 | -0.0252 0.0436 | -0.0398 0.0257 | -0.0319 0.0447 | -0.0446 0.0282 | -0.0344 0.0435 | -0.0503 0.0324 |
|           | II  | 0.0895 0.0436 | -0.0222 0.0457 | -0.0356 0.0262 | -0.0278 0.0460 | -0.0405 0.0286 | -0.0310 0.0435 | -0.0464 0.0328 |
| (30,25,2) | I   | 0.0751 0.0492 | -0.0341 0.0455 | -0.0488 0.0484 | -0.0380 0.0399 | -0.0535 0.0424 | -0.0473 0.0544 | -0.0634 0.0963 |
|           | II  | 0.0895 0.0382 | -0.0225 0.0379 | -0.0338 0.0232 | -0.0286 0.0388 | -0.0390 0.0254 | -0.0319 0.0387 | -0.0447 0.0288 |
| (30,20,2) | I   | 0.0818 0.0452 | -0.0384 0.0481 | -0.0536 0.0249 | -0.0445 0.0499 | -0.0578 0.0281 | -0.0465 0.0456 | -0.0633 0.0325 |
|           | II  | 0.0976 0.0492 | -0.0246 0.0489 | -0.0497 0.0619 | -0.0278 0.0472 | -0.0485 0.0376 | -0.0295 0.0464 | -0.0485 0.0288 |
| (50,40,4) | I   | 0.0509 0.0193 | -0.0222 0.0197 | -0.0253 0.0150 | -0.0271 0.0205 | -0.0298 0.0157 | -0.0319 0.0209 | -0.0361 0.0176 |
|           | II  | 0.0642 0.0224 | -0.0073 0.0205 | -0.0144 0.0171 | -0.0102 0.0225 | -0.0178 0.0170 | -0.0173 0.0224 | -0.0193 0.0258 |
| (50,40,2) | I   | 0.0629 0.0208 | -0.0098 0.0190 | -0.0143 0.0146 | -0.0146 0.0194 | -0.0187 0.0153 | -0.0194 0.0202 | -0.0239 0.0165 |
|           | II  | 0.0548 0.0200 | -0.0171 0.0195 | -0.0214 0.0151 | -0.0218 0.0201 | -0.0258 0.0159 | -0.0267 0.0210 | -0.0311 0.0174 |
| (50,30,2) | I   | 0.0626 0.0241 | -0.0163 0.0216 | -0.0251 0.0217 | -0.0190 0.0232 | -0.0286 0.0197 | -0.0247 0.0222 | -0.0322 0.0180 |
|           | II  | 0.0535 0.0213 | -0.0333 0.0472 | -0.0403 0.0707 | -0.0342 0.0266 | -0.0390 0.0215 | -0.0373 0.0246 | -0.0426 0.0195 |
| (80,70,4) | I   | 0.0336 0.0113 | -0.0142 0.0112 | -0.0155 0.0097 | -0.0174 0.0115 | -0.0186 0.0100 | -0.0213 0.0119 | -0.0227 0.0107 |
|           | II  | 0.0362 0.0113 | -0.0101 0.0109 | -0.0111 0.0111 | -0.0129 0.0110 | -0.0153 0.0095 | -0.0192 0.0130 | -0.0218 0.0165 |
| (80,70,2) | I   | 0.0337 0.0105 | 0.0014 0.2513 | 0.0001 0.2365 | -0.0173 0.0103 | -0.0181 0.0092 | -0.0163 0.0420 | -0.0225 0.0103 |
|           | II  | 0.0400 0.0115 | -0.0075 0.0108 | -0.0095 0.0094 | -0.0105 0.0110 | -0.0124 0.0096 | -0.0142 0.0113 | -0.0162 0.0101 |
| (80,60,2) | I   | 0.0391 0.0112 | -0.0120 0.0112 | -0.0136 0.0096 | -0.0151 0.0113 | -0.0166 0.0097 | -0.0187 0.0115 | -0.0204 0.0102 |
|           | II  | 0.0363 0.0104 | -0.0132 0.0100 | -0.0150 0.0086 | -0.0163 0.0102 | -0.0180 0.0089 | -0.0200 0.0107 | -0.0218 0.0094 |
Table 3. The results for $\hat{\lambda}_2$.

| $(n,m,T)$ | Sch | $\hat{\lambda}_2$ Bias | $\hat{\lambda}_2$ MSE | $\hat{\lambda}_2S$ Bias | $\hat{\lambda}_2S$ MSE | $\hat{\lambda}_2L$ Bias | $\hat{\lambda}_2L$ MSE | $\hat{\lambda}_2E$ Bias | $\hat{\lambda}_2E$ MSE |
|-----------|-----|------------------------|----------------------|------------------------|----------------------|------------------------|----------------------|------------------------|----------------------|
|           |     | Non-Informative Bias   | Non-Informative MSE  | Informative Bias       | Informative MSE      | Non-Informative Bias   | Non-Informative MSE  | Informative Bias       | Informative MSE      |
| (30,25,4) | I   | 0.1202 0.0779          | 0.06 0.0576          | -0.0062 0.0132        | 0.0365 0.0469        | -0.0121 0.0156        | 0.0065 0.0456        | -0.0292 0.0219        |
|           | II  | 0.1131 0.0703          | 0.0516 0.05           | -0.0067 0.013          | 0.0302 0.0413        | -0.0134 0.0147        | 0.0015 0.0412        | -0.0299 0.0202        |
| (30,25,2) | I   | 0.1256 0.0813          | 0.0647 0.0588        | -0.0063 0.0246        | 0.0401 0.0468        | -0.0092 0.0151        | 0.0117 0.0455        | -0.0251 0.0207        |
|           | II  | 0.1432 0.0915          | 0.0808 0.0666        | 0.0022 0.0154         | 0.0564 0.0561        | 0.0002 0.017          | 0.0277 0.0535        | -0.0141 0.0231        |
| (30,20,2) | I   | 0.168 | 0.124       | 0.0922 0.0852        | -0.0343 0.0281        | 0.0591 0.0649        | -0.0225 0.0146        | 0.0265 0.0625        | -0.0309 0.0229        |
|           | II  | 0.1621 | 0.1266     | 0.0835 0.088         | -0.0349 0.0295        | 0.0517 0.0685        | -0.0236 0.0172        | 0.0207 0.0672        | -0.0336 0.0253        |
| (50,40,4) | I   | 0.0813 0.0379          | 0.0414 0.0291        | 0.0188 0.0148         | 0.0292 0.0258        | 0.0098 0.0144        | 0.0083 0.0245        | -0.0072 0.0158        |
|           | II  | 0.0827 0.0363          | 0.0427 0.0281        | 0.0213 0.0155         | 0.0304 0.025         | 0.0118 0.0147        | 0.0093 0.024         | -0.0057 0.0162        |
| (50,40,2) | I   | 0.0831 0.0378          | 0.0433 0.0291        | 0.0212 0.0155         | 0.0312 0.026         | 0.012 0.0149        | 0.0102 0.0246        | -0.0052 0.0161        |
|           | II  | 0.0621 0.0291          | 0.0233 0.0232        | 0.0069 0.0132         | 0.0121 0.021         | -0.0022 0.0128       | -0.0085 0.0205       | -0.02 0.0142         |
| (50,30,2) | I   | 0.1101 0.0663          | 0.0599 0.0503        | 0.0125 0.0184         | 0.0417 0.0423        | 0.0054 0.0176        | 0.0159 0.0404        | -0.0119 0.021         |
|           | II  | 0.0863 0.0514          | 0.0326 0.0383        | -0.0032 0.0157        | 0.0159 0.0335        | -0.0131 0.0162       | -0.0093 0.0326       | -0.0311 0.0199        |
| (80,70,4) | I   | 0.0405 0.0153          | 0.0155 0.0129        | 0.01 0.0099          | 0.0096 0.0122        | 0.0046 0.0096        | -0.0033 0.0119       | -0.0077 0.0097        |
|           | II  | 0.0458 0.0151          | 0.0217 0.0133        | 0.016 0.0109          | 0.0149 0.0117        | 0.0094 0.0092        | 0.0011 0.0122       | -0.0036 0.0099        |
| (80,70,2) | I   | 0.0407 0.0148          | 0.0155 0.0124        | 0.0101 0.0097        | 0.0097 0.0117        | 0.0047 0.0093        | -0.0031 0.0114       | -0.0074 0.0094        |
|           | II  | 0.0494 0.0157          | 0.024 0.0129         | 0.0179 0.01          | 0.0181 0.0122        | 0.0124 0.0096        | 0.0053 0.0117       | 0.0003 0.0095         |
| (80,60,2) | I   | 0.0574 0.0198          | 0.0291 0.0159        | 0.0198 0.0112        | 0.0218 0.0147        | 0.0133 0.0106        | 0.0068 0.0139        | -0.0004 0.0107        |
|           | II  | 0.0529 0.0182          | 0.0243 0.0146        | 0.0163 0.0107        | 0.0173 0.0136        | 0.0099 0.0102        | 0.0027 0.0131        | -0.0036 0.0103        |
Table 4. The results for $\mu_2$.

| $(n,m,T)$ | Sch | $\mu_2$ Bias | $\mu_2$ MSE | $\mu_2$ Bias | $\mu_2$ MSE | $\mu_2L$ Bias | $\mu_2L$ MSE | $\hat{\mu}_2$ Bias | $\hat{\mu}_2$ MSE | $\mu$ Bias | $\mu$ MSE | $\mu$ Bias | $\mu$ MSE |
|-----------|-----|--------------|-------------|--------------|-------------|--------------|-------------|--------------|-------------|-----------|---------|-----------|---------|
| (30,25,4) | I   | 0.0771       | 0.0302      | 0.0019       | 0.0273      | 0.0112       | 0.0039      | 0.0077       | 0.0268      | 0.0158    | 0.214   | 0.012    | 0.284   | 0.0202   |
|           | II  | 0.0642       | 0.0234      | 0.0213       | 0.0208      | 0.0218       | 0.0168      | 0.0018       | 0.0021      | 0.025     | 0.0172  | 0.0215  | 0.0229  | 0.0283   |
| (30,25,2) | I   | 0.0762       | 0.0276      | 0.0034       | 0.0224      | 0.0131       | 0.0075      | 0.0085       | 0.0227      | 0.0172    | 0.0183  | 0.0117  | 0.0232  | 0.0203   |
|           | II  | 0.0771       | 0.0261      | 0.0005       | 0.0228      | 0.0094       | 0.0173      | 0.0039       | 0.0231      | 0.013     | 0.0181  | 0.0068  | 0.0235  | 0.0159   |
| (30,20,2) | I   | 0.0926       | 0.0326      | 0.002        | 0.0256      | 0.0143       | 0.0183      | 0.0032       | 0.0259      | 0.0179    | 0.0194  | 0.0064  | 0.0263  | 0.0206   |
|           | II  | 0.0739       | 0.024       | 0.0104       | 0.0214      | 0.0215       | 0.016       | 0.0145       | 0.0217      | 0.0247    | 0.0167  | 0.017   | 0.0221  | 0.0271   |
| (50,40,4) | I   | 0.0538       | 0.0141      | 0.0002       | 0.0117      | 0.0036       | 0.0102      | 0.0031       | 0.0118      | 0.0068    | 0.0104  | 0.0059  | 0.0121  | 0.0094   |
|           | II  | 0.0492       | 0.013       | 0.0008       | 0.0152      | 0.0032       | 0.0119      | 0.0042       | 0.011       | 0.0074   | 0.0098  | 0.0075  | 0.0123  | 0.0107   |
| (50,40,2) | I   | 0.0552       | 0.0148      | 0.0018       | 0.0123      | 0.002        | 0.0106      | 0.0016       | 0.0125      | 0.0052    | 0.0108  | 0.0043  | 0.0127  | 0.0079   |
|           | II  | 0.045        | 0.0113      | 0.0067       | 0.0101      | 0.009        | 0.0089      | 0.0097       | 0.0102      | 0.012      | 0.0091  | 0.0122  | 0.0109  | 0.0144   |
| (50,30,2) | I   | 0.067        | 0.0206      | 0.0037       | 0.0166      | 0.0032       | 0.0136      | 0.0003       | 0.0067      | 0.0138    | 0.0033  | 0.017   | 0.0236   | 0.0105   |
|           | II  | 0.0432       | 0.012       | 0.0155       | 0.0119      | 0.0184       | 0.0097      | 0.0184       | 0.0121      | 0.0212    | 0.0101  | 0.0206  | 0.0236  | 0.0105   |
| (80,70,4) | I   | 0.0332       | 0.0073      | 0.002        | 0.0064      | 0.003        | 0.006       | 0.0042       | 0.0064      | 0.0051    | 0.006   | 0.0065  | 0.0069  | 0.0061   |
|           | II  | 0.0329       | 0.0069      | 0.0002       | 0.0007      | 0.0069       | 0.0027      | 0.006        | 0.0038      | 0.0057    | 0.0087  | 0.0247  | 0.0085  | 0.0152   |
| (80,70,2) | I   | 0.0326       | 0.0074      | 0.0027       | 0.0066      | 0.0037       | 0.0061      | 0.005        | 0.0066      | 0.0062    | 0.0066  | 0.0067  | 0.0075  | 0.0063   |
|           | II  | 0.0335       | 0.0068      | 0.0003       | 0.0059      | 0.0015       | 0.0055      | 0.0022       | 0.006       | 0.0034    | 0.0056  | 0.0039  | 0.006   | 0.0051   |
| (80,60,2) | I   | 0.0395       | 0.0085      | 0.001        | 0.0072      | 0.0001       | 0.0066      | 0.0014       | 0.0072      | 0.0033    | 0.0067  | 0.0034  | 0.0073  | 0.0052   |
|           | II  | 0.0314       | 0.0063      | 0.0042       | 0.0057      | 0.0056       | 0.0053      | 0.0061       | 0.0058      | 0.0075    | 0.0054  | 0.0078  | 0.0059  | 0.0091   |
5. Data Analysis

In this section, a real dataset is analyzed for illustrative purposes. The dataset was originally from [18] and further analyzed by [17]. The data were collected from a lifetime experiment for small electrical appliances. This set of data contains the failure time and the failure mode for each unit. There are 18 modes in the sample, while only seven failure modes appear in the test. Among these seven modes, Mode 9 is the main concern. Therefore, Failure Mode 9 is considered as the first competing risk, and the other failure modes are considered as the second failure causes. The original data (denoted as $Z$) can be found in [17]. As the data contain some very large values, we made a transformation of the data by $X = Z^{0.1}$ for computational convenience.

In order to test whether the two-parameter Rayleigh distribution can be used to analyze the dataset, Kolmogorov–Smirnov (K-S) tests were conducted. Since the K-S distances and the associated $p$-values are $(0.17529, 0.6119)$ and $(0.2082, 0.4333)$ for Cause 1 and Cause 2, respectively, we cannot reject the null hypothesis that the two-parameter Rayleigh distribution provides a reasonable fit.

The maximum likelihood function for complete data (denoted as $L_c$) can be written as:

$$L_c \propto \prod_{i=1}^{n_1} f_1(x_{1i})[1 - F_2(x_{1i})] \prod_{i=n_1+1}^{m} f_2(x_{2i})[1 - F_1(x_{2i})],$$

$$\propto \lambda_1^{n_1} \lambda_2^{m-n_1} \prod_{i=1}^{n_1} (x_{1i} - \mu_1) \prod_{i=n_1+1}^{m} (x_{2i} - \mu_2) \times e^{-\lambda_1 \sum_{i=1}^{n_1} (x_{1i} - \mu_1)^2 + \sum_{i=n_1+1}^{m} (x_{2i} - \mu_2)^2 I(x_{2i} > \mu_1)}$$

$$\times e^{-\lambda_2 \sum_{i=n_1+1}^{m} (x_{2i} - \mu_2)^2 I(x_{2i} > \mu_2) + \sum_{i=n_1+1}^{m} (x_{2i} - \mu_2)^2},$$

which is a special case for the maximum likelihood function based on censored data. We can compute the MLEs for the complete data to be $(\lambda_1, \mu_1, \lambda_2, \mu_2) = (14.1375, 2.0100, 0.4932, 1.1743).

Clearly, the original data contain 33 values, namely $n = 33$. We take $m = 27$ and $T = 1.5$. The two applied censoring schemes are the same as those in Section 4, and we still set $h = 1$ for the LINEX loss function and $q = 1$ for the general entropy loss function. For the Bayesian estimation, a non-informative prior is applied in that we have no information about the prior. The generated data can be found in Table 5. The estimates based on the censored data can be found in Table 6. In Table 6, the values in the upper cell denote the obtained results based on Data I, and the values in the lower cell correspond to the results based on Data II. The values of the proposed estimates for a specific parameter are rather close to each other, and the estimates based on censored data are rather close to the MLEs based on complete data.

| $(n,m,T)$ | Sch | Data and Corresponding Risk Causes |
|-----------|-----|----------------------------------|
| (37, 30, 0.5) | I | (1.2710, 2), (1.4269, 2), (1.6713, 2), (1.7853, 2), (1.8117, 2), (1.9275, 2), (1.9867, 2), (2.0263, 1), (2.0905, 2), (2.1303, 1), (2.1612, 1), (2.1711, 2), (2.1778, 1), (2.1911, 1), (2.1923, 1), (2.1926, 1), (2.2031, 2), (2.2038, 2), (2.2141, 1), (2.2295, 2), (2.2313, 1), (2.2351, 1), (2.2424, 1), (2.2618, 1), (2.3101, 1), (2.4010, 1) |
| | II | (1.2710, 2), (1.4269, 2), (1.4758, 2), (1.6713, 2), (1.7853, 2), (1.9275, 2), (1.9867, 2), (2.0073, 2), (2.0263, 1), (2.0905, 2), (2.1303, 1), (2.1374, 1), (2.1711, 2), (2.1778, 1), (2.1824, 2), (2.1842, 1), (2.1911, 1), (2.1923, 1), (2.1926, 1), (2.2038, 2), (2.2085, 2), (2.2141, 1), (2.2295, 2), (2.2313, 1), (2.2351, 1), (2.2424, 1), (2.2601, 1) |
Table 6. Results for the datasets (S represents the Bayesian estimates under squared error loss function, L the Bayesian estimates under the LINEX loss function, and E the Bayesian estimates under the general entropy loss function).

| Data I   | MLE  | S    | L    | E    |
|----------|------|------|------|------|
| $\lambda_1$ | 13.2035 | 11.4226 | 10.7848 | 10.7186 |
| $\mu_1$     | 2.0062  | 1.9863 | 1.9863 | 1.9863  |
| $\lambda_2$ | 0.5235  | 0.4487 | 0.4362 | 0.4146  |
| $\mu_2$     | 1.1631  | 1.0766 | 1.0755 | 1.0751  |
| Data II    | MLE  | S    | L    | E    |
| $\lambda_1$ | 13.9304 | 12.2914 | 11.4543 | 11.4838 |
| $\mu_1$     | 2.0109  | 1.9957 | 1.9957 | 1.9957  |
| $\lambda_2$ | 0.6213  | 0.5051 | 0.4894 | 0.4722  |
| $\mu_2$     | 1.1719  | 1.0750 | 1.0750 | 1.0754  |

6. Conclusions

In this paper, we considered the classical and the Bayesian inference of adaptive Type II progressive hybrid censored competing risks data for the two-parameter Rayleigh distribution. The maximum likelihood estimators are derived, and the existence and uniqueness of the MLEs are proven. The Bayesian estimates are obtained using the Lindley approximation based on both symmetric and asymmetric loss functions, including the squared error loss function, the LINEX loss function, and the general entropy loss function.

Furthermore, simulation studies are conducted to compare the performance of the estimates with respect to biases and MSEs. In Bayesian estimation, the Bayesian estimates with informative priors are much more satisfactory than those with non-informative priors. When $\lambda_1$ and $\lambda_2$ are considered, Bayesian estimates under the LINEX loss function are much better than those under other loss functions, yet when $\mu_1$ and $\mu_2$ are considered, estimates based on the squared error loss function are slightly better than those based on the LINEX loss function. Besides, although MLEs compete quiet well with the Bayesian estimates with non-informative priors, Bayesian estimates with informative priors are superior to MLEs. On the whole, the Bayesian estimates with informative priors under the LINEX loss function are recommended.

Though we only consider two risk causes in this paper, the methods can easily be extended to more competing risk causes. Binomial removals can also be considered in further studies. Research on adaptive Type II progressive hybrid censored data with competing risks and binomial removals is still of great potential.

Author Contributions: Investigation, S.L.; Supervision, W.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by Project 202110004002 which was supported by National Training Program of Innovation and Entrepreneurship for Undergraduates.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dey, S.; Dey, T.; Kundu, D. Two-Parameter Rayleigh Distribution: Different Methods of Estimation. *Am. J. Math. Manag. Sci.* 2014, 33, 55–74. [CrossRef]
2. Dey, T.; Dey, S.; Kundu, D. On Progressively Type-II Censored Two-parameter Rayleigh Distribution. *Commun. Stat. Simul. Comput.* 2014, 45, 438–455. [CrossRef]
3. Seo, J.I.; Jeon, J.W.; Kang, S.B. Exact Interval Inference for the Two-Parameter Rayleigh Distribution Based on the Upper Record Values. *J. Probab. Stat.* 2016, 2016, 8246390. [CrossRef]
4. Asgharzadeh, A.; Fernández, A.; Abdi, M. Confidence sets for the two-parameter Rayleigh distribution under progressive censoring. *Appl. Math. Model.* 2017, 47, 656–667. [CrossRef]
5. Khan, H.M.R.; Provost, S.B.; Singh, A. Predictive Inference from a Two-Parameter Rayleigh Life Model Given a Doubly Censored Sample. *Commun. Stat. Theory Methods* 2010, 39, 1237–1246. [CrossRef]
6. Ng, H.K.T.; Kundu, D.; Chan, P.S. Statistical analysis of exponential lifetimes under an adaptive Type-II progressive censoring scheme. *Nav. Res. Logist.* 2009, 56, 687–698. [CrossRef]
7. Hemmati, F.; Khorram, E. Statistical Analysis of the Log-Normal Distribution under Type-II Progressive Hybrid Censoring Schemes. *Commun. Stat. Simul. Comput.* 2013, 42, 52–75. [CrossRef]
8. Panahi, H.; Moradi, N. Estimation of the inverted exponentiated Rayleigh Distribution Based on Adaptive Type II Progressive Hybrid Censored Sample. *J. Comput. Appl. Math.* 2020, 364, 112345. [CrossRef]
9. Nassar, M.; Abo-Kasem, O. Estimation of the inverse Weibull parameters under adaptive type-II progressive hybrid censoring scheme. *J. Comput. Appl. Math.* 2017, 315, 228–239. [CrossRef]
10. Sobhi, M.M.A.; Soliman, A.A. Estimation for the exponentiated Weibull model with adaptive Type-II progressive censored schemes. *Appl. Math. Model.* 2016, 40, 1180–1192. [CrossRef]
11. Nassar, M.; Abo-Kasem, O.; Zhang, C.; Dey, S. Analysis of Weibull Distribution Under Adaptive Type-II Progressive Hybrid Censoring Scheme. *J. Indian Soc. Probab. Stat.* 2018, 19, 25–65. [CrossRef]
12. Liao, H.; Gui, W. Statistical Inference of the Rayleigh Distribution Based on Progressively Type II Censored Competing Risks Data. *Symmetry* 2019, 11, 898. [CrossRef]
13. Shi, Y.; Jin, L.; Wei, C.; Yue, H. Constant-Stress Accelerated Life Test with Competing Risks under Progressive Type-II Hybrid Censoring. *Adv. Mater. Res.* 2013, 712–715, 2080–2083. [CrossRef]
14. Wang, L. Inference for Weibull Competing Risks Data Under Generalized Progressive Hybrid Censoring. *IEEE Trans. Reliab.* 2018, 67, 998–1007. [CrossRef]
15. Lindley, D.V. Approximate Bayesian methods. *Trab. Estadística Investig. Oper.* 1980, 31, 223–245. [CrossRef]
16. Howlader, H.A.; Hossain, A.M. Bayesian survival estimation of Pareto distribution of the second kind based on failure-censored data. *Comput. Stat. Data Anal.* 2002, 38, 301–314. [CrossRef]
17. Nie, J.; Gui, W. Parameter Estimation of Lindley Distribution Based on Progressive Type-II Censored Competing Risks Data with Binomial Removals. *Mathematics* 2019, 7, 646. [CrossRef]
18. Lawless, J.F. Statistical Models and Methods for Lifetime Data. 2011. Available online: http://www.ru.ac.bd/wp-content/uploads/sites/25/2019/03/403_02_Lawless_Statistical-Models-and-Methods-for-Lifetime-Data-Second-Edition.pdf (accessed on 5 May 2020).

**Publisher’s Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.