Degenerations of Curves in Projective Space and the Maximal Rank Conjecture

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Abstract

In this note, we give an overview of a new technique for studying Brill–Noether curves in projective space via degeneration. In particular, we give a roadmap to the proof of the Maximal Rank Conjecture.

1 Introduction

The technique of degeneration to a reducible curve has enabled the proof of many results in the theory of algebraic curves. These results include the Brill–Noether theorem, proven by Griffiths and Harris [3], Gieseker [2], Kleiman and Laksov [6], and others, which describes the space of maps from a general curve to projective space: If \( C \) is a general curve of genus \( g \), it states that there exists a nondegenerate degree \( d \) map \( C \to \mathbb{P}^r \) if and only if the Brill–Noether number \( \rho(d, g, r) \) is nonnegative, where

\[
\rho(d, g, r) := (r + 1)d - rg - r(r + 1).
\]

Moreover, in this case, there exists a unique component \( \overline{M}_g^{(\mathbb{P}^r, d)} \) of Kontsevich’s space of stable maps \( \overline{M}_g(\mathbb{P}^r, d) \) that both dominates the moduli space of curves \( \overline{M}_g \) and whose general member is nondegenerate. We call curves in this component of the space of stable maps Brill–Noether curves (BN-curves).

These results have been extended in various ways. For example, Sernesi argues by degeneration to produce components of \( \overline{M}_g(\mathbb{P}^r, d) \) whose image in \( \overline{M}_g \) is of the expected dimension when the Brill–Noether number is negative [15].

However, in this note we focus on the question: How can we study the geometry in projective space of the general BN-curve via degeneration? Our goal here is to give an overview of a series of papers by the author and others [1, 7, 8, 9, 10, 11, 14, 16] which give rise to a general technique for studying BN-curves via degeneration. This technique is then applied in [13] (in conjunction with results on hyperplane sections of BN-curves obtained in [12]) to give a proof of the Maximal Rank Conjecture, a conjecture made originally by Severi in 1915 [4] which determines the Hilbert function of a general BN-curve:

**Conjecture 1.1** (Maximal Rank Conjecture). If \( C \subset \mathbb{P}^r \) is a general BN-curve (\( r \geq 3 \)), the restriction maps

\[
H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(\mathcal{O}_C(k))
\]
are of maximal rank (i.e. either injective or surjective).

Equivalently, the dimension of the space of polynomials of degree $k$ which vanish on $C$ is given by

$$
\begin{cases}
\binom{r+k}{k} - (kd + 1 - g) & \text{if } kd + 1 - g \leq \binom{r+k}{k} \text{ and } k \geq 2; \\
0 & \text{otherwise}.
\end{cases}
$$

The Maximal Rank Conjecture can also be reformulated cohomologically: From the long exact sequence in cohomology arising from the short exact sequence of sheaves

$$
0 \to \mathcal{I}_{C \subset \mathbb{P}^r}(k) \to \mathcal{O}_{\mathbb{P}^r}(k) \to \mathcal{O}_C(k) \to 0,
$$

we see that $C$ satisfies "maximal rank for polynomials of degree $k"$ if and only if

$$
H^0(\mathcal{I}_{C \subset \mathbb{P}^r}(k)) = 0 \text{ or } H^1(\mathcal{I}_{C \subset \mathbb{P}^r}(k)) = 0.
$$

Many special cases of the maximal rank conjecture have been previously studied, using an approach originally due to Hirschowitz: Degeneration to a reducible curve $C' \cup C''$ with $C''$ contained in a hypersurface $S$ of degree $n$ (typically a quadric if $r = 3$ and a hyperplane if $r \geq 4$), and $C'$ transverse to $S$:

![Diagram](image)

In this case, from the long exact sequence in cohomology arising from the short exact sequence of sheaves

$$
0 \to \mathcal{I}_{C' \subset \mathbb{P}^r}(k-n) \to \mathcal{I}_{C' \cup C'' \subset \mathbb{P}^r}(k) \to \mathcal{I}_{C'' \cup (C' \cap S) \subset S}(k) \to 0,
$$

we conclude that to show $H^i(\mathcal{I}_{C' \cup C'' \subset \mathbb{P}^r}(k)) = 0$ as desired, it suffices to show

$$
H^i(\mathcal{I}_{C' \subset \mathbb{P}^r}(k-n)) = H^i(\mathcal{I}_{C'' \cup (C' \cap S) \subset S}(k)) = 0.
$$

One can thus hope to argue by induction on $r$ (if $S \simeq \mathbb{P}^{r-1}$ is a hyperplane) and $k$. However, three fundamental difficulties have limited this approach to special cases:

1. We need a uniform way to construct the degenerations $C' \cup C''$. Previous methods were ingenious, but relatively ad-hoc, and hence not generalizable.

2. It is not possible to always find such reducible curves at which the fiber dimension of the map $\overline{M_g}(\mathbb{P}^r, d) \to \overline{M}_g$ at $[C' \cup C'']$ is $\rho(d, g, r) + \dim \text{Aut} \mathbb{P}^r$. We therefore need some other way to see that such curves are BN-curves.
3. This approach relates maximal rank for $C' \cup C''$ to maximal rank for $C'$ and maximal rank for $C'' \cup (C' \cap S)$. But $C'' \cup (C' \cap S)$ is not a curve, so we need a stronger inductive hypothesis.

Even worse, $C'$ and $C''$ must satisfy various incidence conditions, so $C''$ and $C' \cap S$ are not independently general and there is no nice description of $C'' \cup (C' \cap S)$ that doesn’t reference the entire reducible curve $C' \cup C''$.

We begin by discussing the first two difficulties (in Sections 2 and 3), which arise whenever we wish to study the geometry of general BN-curves via degeneration. Namely, we show the existence of such degenerations of BN-curves can be reduced to the existence of integer solutions to certain systems of inequalities.

Then we discuss the third difficulty (in Section 4), which is specific to the proof of the Maximal Rank Conjecture.

Finally (in Section 5), we describe a method for proving the existence of integer solutions to the type of systems of inequalities that appear when applying this method to the maximal rank conjecture.

## 2 The Uniform Construction of Reducible Curves

We construct our desired reducible curves via the following method. First, we fix a finite set of points $\Gamma$ which is general in $\mathbb{P}^r$, or general in a hyperplane (or other hypersurface of small degree) $H \subset \mathbb{P}^r$. Then, we find BN-curves $C' \subset \mathbb{P}^r$, and $C'' \subset \mathbb{P}^r$; or $C' \subset \mathbb{P}^r$ transverse to $H$, and $C'' \subset H$ — both passing through $\Gamma$:

![Diagram of C', C'', and Gamma](image)

Taking their union then gives a reducible curve $C = C' \cup C''$ as desired.

To carry out this construction, we need to answer the questions: When does there exist a BN-curve of given degree $d$ and genus $g$ passing through a set of $n$ general points in $\mathbb{P}^r$? If some of these points are constrained to lie in a hypersurface of small degree (usually a hyperplane), can we find such a BN-curve transverse to this hypersurface? We will not be able to answer these questions, but we will get close enough for our needs.

For this first question, we are asking when the natural map $\pi: \overline{M}_{g,n}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^n$ is dominant. The natural conjecture is that $\pi$ is dominant if and only if the dimensions allow it:

$$(r + 1)d - (r - 3)(g - 1) + n = \dim \overline{M}_{g,n}(\mathbb{P}^r, d) \geq \dim(\mathbb{P}^r)^n = rn,$$

or upon rearrangement, if and only if

$$n \leq \left[ \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1} \right]. \quad (1)$$
This condition can also be expressed as $\chi(N_C(-p_1 - p_2 - \cdots - p_n)) \geq 0$, where $N_C$ denotes the normal bundle of $C$, and $p_1, p_2, \ldots, p_n \in C$ are the $n$ marked points. Since the obstruction to smoothness of $\pi$ lies in $H^1(N_C(-p_1 - p_2 - \cdots - p_n))$, this follows in turn from the following property for the normal bundle $N_C$:

**Definition 2.1.** We say that a vector bundle $E$ on an irreducible curve $C$ satisfies interpolation if for a general effective Cartier divisor $D \subset C$ of every nonnegative degree, either

$$H^0(E(-D)) = 0 \quad \text{or} \quad H^1(E(-D)) = 0.$$  

The property of interpolation for normal bundles is studied in the following sequence of papers:

**A.** In joint work with Atanasov and Yang [1], we show that the normal bundle of a general nonspecial BN-curve (i.e. one with $d \geq g + r$) satisfies interpolation except in exactly three cases: $(d, g, r) \in \{(5, 2, 3), (6, 2, 4), (7, 2, 5)\}$. Even though the normal bundle of a general BN-curve of degree 6 and genus 2 in $\mathbb{P}^4$ does not satisfy interpolation, it turns out that such curves can still pass through the expected number of general points. We conclude that a general nonspecial BN-curve passes through $n$ general points if and only if (1) holds, with exactly two exceptions: $(d, g, r) \in \{(5, 2, 3), (7, 2, 5)\}$.

The argument is by inductive degeneration of $C$ to a reducible nodal curve $X \cup Y$. Older results of Hartshorne and Hirschowitz [5] give geometric descriptions of $N_{X\cup Y}|_X$ and $N_{X\cup Y}|_Y$; however, to describe $N_{X\cup Y}$, one needs a compatible description of the gluing data $N_{X\cup Y}|_X|_{X \cap Y} \simeq N_{X\cup Y}|_Y|_{X \cap Y}$, which is quite difficult in general.

The key new insight is to study line subbundles of the normal bundle obtained by saturating the images of vertical tangent spaces of projection maps. These enable us to give an essentially complete geometric description of the gluing data when $Y$ is a line.

**B.** In [7], we study the intersection of a general BN-curve of degree $d$ and genus $g$ in $\mathbb{P}^r$ with a hypersurface $S$ of degree $n$. An easy dimension count (plus a tiny bit more work when $r = 2$) implies that there are only five pairs $(r, n)$ where this intersection could be, with the exception of finitely many $(d, g)$ pairs, a collection of $dn$ general points on $S$.

The main result of this paper is that conversely, in each of these five cases, the intersection is indeed general with finitely many exceptions:

(a) The intersection of a plane curve with a line yields a general $d$-tuple of points on the line, always;

(b) The intersection of a plane curve with a conic, always;

(c) The intersection of a space curve with a quadric, with six exceptions:

$$(d, g) \in \{(4, 1), (5, 2), (6, 2), (6, 4), (7, 5), (8, 6)\};$$

(d) The intersection of a space curve with a plane, with one exception:

$$(d, g) = (6, 4)$$
(e) The intersection of a curve in $\mathbb{P}^4$ with a hyperplane, with three exceptions:

$$(d, g) \in \{(8, 5), (9, 6), (10, 7)\}.$$ 

In each of these exceptions, a complete description of the intersection is given.

These statements can be reduced to statements about the cohomology of twists of normal bundles of general BN-curves, namely that $H^1(N_C(-n)) = 0$; as in [A] these are approached by inductive degeneration. However, unlike in [A] we do not know of a compatible description of the gluing data.

The key new idea here is that when one of the curves is contained in a hyperplane (or other hypersurface of small degree), and certain stringent numerical constraints are satisfied, the required properties of $N_{X \cup Y}$ can be reduced to properties of $N_X$ and $N_Y$ that (essentially) do not depend upon the gluing data.

C. In [16], Vogt shows that the normal bundle of a general BN space curve satisfies interpolation except in exactly two cases: $(d, g) \in \{(5, 2), (6, 4)\}$.

The argument proceeds by noting that for $C$ a space curve, $H^1(N_C(-2)) = 0$ implies $N_C$ satisfies interpolation. Using [13] it thus remains to show $N_C$ satisfies interpolation when $(d, g) \in \{(4, 1), (6, 2), (7, 5), (8, 6)\}$. The cases $(d, g) \in \{(4, 1), (6, 2)\}$ are done in [A] so it remains to show $N_C$ satisfies interpolation when $(d, g) \in \{(7, 5), (8, 6)\}$.

In these cases, degeneration to a reducible curve is difficult, and new techniques are needed. For curves of degree 7 and genus 5, which are projections of canonical curves in $\mathbb{P}^4$ from a point on the curve, Vogt finds and analyzes a description of the normal bundle exact sequence associated to the projection, which is compatible with the description of a canonical curve in $\mathbb{P}^4$ as the complete intersection of a net of quadrics. For curves of degree 8 and genus 6, Vogt degenerates to a smooth curve lying on a cubic surface with 3 ordinary double points.

D. In joint work with Vogt [14], we show that, for $C$ a general BN-curve in $\mathbb{P}^4$, the normal bundle $N_C$ (respectively the twist $N_C(-1)$) satisfies interpolation, except in exactly 1 case: $(d, g) = (6, 2)$ (respectively exactly 4 cases: $(d, g) \in \{(6, 2), (8, 5), (9, 6), (10, 7)\}$).

Interpolation for $N_C(-1)$ implies that $C$ can pass through $n$ points which are general subject to the constraint that $d$ of them lie in a transverse hyperplane, and subject to (II).

Unlike in $\mathbb{P}^r$ for $r \geq 5$ — where general curves have only 1- and 2- secant lines — (most) curves in $\mathbb{P}^4$ have trisecant lines; the techniques of [A] for inductively degenerating to reducible curves one component of which is a line can thus be applied here in greater generality. Combining this with methods of [B] we devise an inductive argument to prove interpolation for $N_C$ and $N_C(-1)$.

E. Finally, in [8], we deduce “bounded-error approximations” which are valid for BN-curves of arbitrary degree and genus, in a projective space of arbitrary dimension.

For example, we show that a BN-curve of degree $d$ and genus $g$ in $\mathbb{P}^r$ passes through $n$ general points if

$$n \leq \left\lceil \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1} \right\rceil - 3$$
(compare to (1)), and gives similar statements when some of the points are constrained to lie in a hyperplane.

The proof is by degeneration to reducible BN-curves whose components fall into a case which has already been analyzed in one of the above papers; this degeneration is studied using methods introduced in [3].

These results let us build the desired reducible curves for the maximal rank conjecture simply by showing that there are integers (representing the degrees and genera of the components) satisfying certain systems of inequalities — a problem considered in Section 5.

### 3 The Incorrect Fiber Dimension

The union described in the previous section can be constructed in the space of stable maps: Writing $C' \cup_{\Gamma} C''$ for the curve obtained from $C'$ and $C''$ by gluing along $\Gamma$, we obtain a map $f: C' \cup_{\Gamma} C'' \to \mathbb{P}^r$ (which may not be an immersion). Conditions under which such unions are BN-curves are studied in the following sequence of papers:

**I.** In [9], we show $T_{\mathbb{P}^r}|_C$ satisfies interpolation, where $C \subset \mathbb{P}^r$ is a general BN-curve. We also give results for the twist $T_{\mathbb{P}^r}|_C(-1)$.

  This implies an analog of the question considered in Section 2, for maps from fixed curves with fixed marked points which must be sent to the specified points in $\mathbb{P}^r$.

  As with the papers on interpolation for normal bundles, the argument is via degeneration — but for $T_{\mathbb{P}^r}|_C$, the gluing data is easy to understand.

**II.** In [10], we study this construction of reducible curves $f: C' \cup_{\Gamma} C'' \to \mathbb{P}^r$ in the regime where both components are nonspecial (as well as some other special cases).

  First we leverage the results of I to calculate the fiber dimension of the map from the space of stable maps to the moduli space of curves at certain reducible curves, thereby showing they are BN-curves.

  Then we show such reducible curves are BN-curves (subject to some mild conditions), even when the fiber dimension is wrong, by showing that they lie in the same component as another curve which we know is a BN-curve by calculation of the fiber dimension. Rather than finding an irreducible curve in the space of maps, the key insight here is to draw a "broken arc" (iteratively specialize and then deform) in the space of stable maps, connecting these two points of the moduli space:

\[
\begin{array}{c}
\text{want:} \\
\text{is BN-curve}
\end{array}
\quad
\begin{array}{c}
\text{know:} \\
\text{is BN-curve}
\end{array}
\]

\[
M_g(\mathbb{P}^r, d)
\]
Provided we check the specializations are to smooth points of the space of stable maps, this shows our given such reducible curve is in the same component as the other curve, and is thus a BN-curve as desired.

These arcs are constructed by further specializing one of the components, say \( C' \), to a reducible curve \( C'_1 \cup D'_1 \); this results in a specialization of \( C' \cup C'' \) given by

\[
(C'_1 \cup D'_1) \cup C'' = C'_1 \cup (D'_1 \cup C'').
\]

We then deform \( D'_1 \cup C'' \) to a smooth curve \( C''_1 \). Finally, we iterate this procedure, alternating between components (next we would specialize \( C''_1 \) — to a different reducible curve, not back to \( D'_1 \cup C'' \)).

Note that even if \( C' \) and \( C'' \) do not meet at any additional point not in \( \Gamma \), and have distinct tangent directions at the points of \( \Gamma \) — so that \( f \) is the natural immersion of the scheme-theoretic union — this broken arc may still not make sense in the Hilbert scheme compactification, so it is important to work in the space of stable maps even in this case.

III. Finally, in [11], we repeat the analysis in [11] to study this construction of reducible curves in the regime where such reducible curves can be constructed using the approximate results on interpolation discussed in the Section 2 (as well as some other special cases). This is a separate paper from [11] since results of [11] are needed in the proof of many of the results on interpolation discussed in Section 2 while those results on interpolation are needed for [11].
4 The Hyperplane Section

Our study of subschemes of the form $C'' \cup (C' \cap S)$ which arise in the inductive argument is divided as follows:

1. In [12], we show by degeneration that the union of hyperplane sections

   $$(C_1 \cup C_2 \cup \cdots \cup C_n) \cap H$$

of independently general BN-curves $C_1, C_2, \ldots, C_n$ satisfies maximal rank for polynomials of degree $k$, unless $k = 2$ and some $C_i$ is special.

   In low dimensions, we furthermore show that if $X \subset H$ and its hyperplane section $X \cap H'$ satisfy maximal rank, then subject to mild conditions, so does the union $X \cup (C \cap H)$ of $X$ with the hyperplane section of an independently general BN-curve $C$. The proofs of these statements depend crucially on results of [7] discussed earlier.

2. In [13], one of the key steps is to study conditions under which $C'$ can be further specialized so that its hyperplane section becomes independent from $C''$.

   As an analogy, consider a set of 1 black point and 5 white points in the plane, which are general subject to the condition that they lie on a conic. The black and white points are not independent — i.e. there is no description of what the white points can be that doesn’t reference the position of the black point. However, we can specialize the conic to the union of two lines, such that the black point and 1 white point lie on one line, while 4 white points lie on the other line:

   ![Diagram of a conic and its specialization to two lines]

   After specialization, the black and white points become independent: The white points specialize to a set of 5 points which are general subject to the constraint that 4 of them are collinear — a description that doesn’t reference the position of the black point.

   In our setting, we further specialize $C'$ to a reducible curve $C'_1 \cup C'_2$, such that

   $$[C'_2 \cap H] \times [C'_1 \cap C'_2] \in \text{Sym}^{\deg C'_2} \mathbb{P}^r \times \text{Sym}^\#(C'_1 \cap C'_2) H$$

   is general. This induces a specialization of $C' \cup C''$:

   ![Diagram of the specialization of $C' \cup C''$]

   $$\left[4\right]$$
Via the method of Hirschowitz, we reduce maximal rank for \( C' \cup C'' \) to maximal rank for \( C' \), and maximal rank for \( C'' \cup (C' \cap H) \), which in turn reduces to maximal rank for \( C'' \cup (C'_1 \cap H) \cup (C'_2 \cap H) \). Under our assumption (2), \( C'', C'_1 \cap H, \) and \( C'_2 \cap H \) are an independently general BN-curve, hyperplane section of a BN-curve, and set of points. The upshot is that we can then argue by induction on the following stronger hypothesis. (Note that taking \( n = \epsilon = 0 \) recovers the maximal rank conjecture.)

**Theorem 4.1.** Fix an inclusion \( \mathbb{P}^r \subset \mathbb{P}^{r+1} \) (for \( r \geq 3 \)), and let \( k \) be a positive integer. Let \( C \subset \mathbb{P}^r \) be a general BN-curve or a general degenerate rational curve of degree \( 0 < d < r \). Let \( D_1, D_2, \ldots, D_n \subset \mathbb{P}^{r+1} \) be independently general BN-curves, which are required to be nonspecial if \( k = 2 \) and \( r \geq 4 \). Let \( p_1, p_2, \ldots, p_\ell \in \mathbb{P}^r \) be a general set of points. Then any subset of

\[
T := C \cup ((D_1 \cup D_2 \cup \cdots \cup D_n) \cap \mathbb{P}^r) \cup \{p_1, p_2, \ldots, p_\ell\} \subset \mathbb{P}^r
\]

which contains \( C \) satisfies maximal rank for polynomials of degree \( k \).

**5 Integer Solutions to Systems of Inequalities**

Applying the techniques of previous sections, we show in [13] that the maximal rank conjecture may be reduced to several instances of the following problem: Given integers (e.g. \( r, k, d, g, \ldots \)) satisfying a certain system of inequalities (e.g. \( \rho(d, g, r) \geq 0, \ldots \)), show that there are either additional integers (e.g. \( d', g', d'', g'', \ldots \)) satisfying a first additional system of inequalities, or other additional integers (e.g. \( d'_1, g'_1, d'_2, g'_2, d'', g'', \ldots \)) satisfying a second additional system of inequalities.

Crucially, all the inequalities arising in the proof of the maximal rank conjecture are linear in all the variables except \( r \) and \( k \), with coefficients that are polynomials in \( r, k, \) and the binomial coefficient \( \binom{r+k}{k} \). For fixed \( r \) and \( k \), these systems of inequalities describe compact convex polyhedra.

These computations can be approached in three steps:

1. First, we eliminate the additional variables one by one, using the following fact: There exists a real number, respectively integer, \( n \) satisfying the inequalities

\[
n \leq \frac{a_i}{b_i} \quad \text{and} \quad n \geq \frac{c_j}{d_j}
\]

if, for each \((i, j)\),

\[
a_i d_j - b_i c_j \geq 0 \quad \text{respectively} \quad a_i d_j - b_i c_j \geq (b_i - 1)(d_j - 1).
\]

2. Then we reduce the given problem to checking positivity of polynomials in \( r, k, \) and \( \binom{r+k}{k} \), using the following fact: Let \( P \subset \mathbb{R}^n \) be a compact convex polyhedron, and \( C_1, C_2 \subset \mathbb{R}^n \) be convex sets. Then \( P \subset C_1 \cup C_2 \) if and only if:

(a) Every vertex of \( P \) is contained in either \( C_1 \) or \( C_2 \); and
Every edge of \( P \) joining a vertex not contained in \( C_1 \) to a vertex not contained in \( C_2 \) meets \( C_1 \cap C_2 \).

3. Finally, we verify positivity of these polynomials, using the following fact: A polynomial \( P(r,k) \) in two variables \( r \) and \( k \) is positive for all \( r \geq r_0 \) and \( k \geq k_0 \), provided that:

   (a) Every monomial on the outside of the Newton polygon has positive coefficient.
   (b) The leading coefficient with respect to \( r \) is positive for \( k \geq k_0 \).
   (c) The leading coefficient with respect to \( k \) is positive for \( r \geq r_0 \).
   (d) The polynomial \( P(r_0,k) \) is positive for \( k \geq k_0 \).
   (e) The value of \( k_0 \) exceeds all branch points of the projection onto the \( k \)-axis of \( P(r,k) = 0 \).

Sometimes this method may fail. For example, when \( r = 17 \) and \( k = 4 \), there is a vertex of a polyhedron \( P \) appearing in the second step — corresponding to the value of \( d \) and \( g \) for which \( \rho(d,g,17) = 0 \) and the maximal rank map is expected to be an isomorphism — which is not contained in either convex set. However, this vertex has non-integer values of \( d \) and \( g \), and is only barely not contained in either convex set; brute force search shows that, in this case, the desired existence of additional integers holds for every integral \((d,g)\). (In particular, we see that this proof of the maximal rank conjecture barely works; with only slightly worse approximate results on interpolation, it would fail.)

Proofs of the above facts, as well as computer code implementing this method (combined with brute-force search where it fails), are given in Appendix E of [13].

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