Bayesian inference for a single factor copula stochastic volatility model using Hamiltonian Monte Carlo

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Abstract

Single factor models are used in finance to model the joint behaviour of stocks. The dependence is commonly modeled with a multivariate normal distribution. Krupskii and Joe (2013) provide a copula based extension. This single factor copula requires the specification of bivariate linking copulas. Resulting joint models can accommodate symmetric or asymmetric tail dependence. For modeling multivariate time series we propose a single factor copula model together with stochastic volatility margins. For this model we develop joint Bayesian inference using Hamiltonian Monte Carlo (HMC) within Gibbs sampling. The Bayesian approach allows for high dimensional parameter spaces as they are present here in addition to uncertainty quantification through credible intervals. Furthermore we avoid the two step approach for margins and dependence in copula models as followed by Schamberger et al (2017). In a first simulation study the performance of HMC is compared to the Markov Chain Monte Carlo (MCMC) approach developed by Schamberger et al (2017) for the copula part. It is shown that HMC considerably outperforms this approach in terms of effective sample size, MSE and observed coverage probabilities. In a second simulation study satisfactory small sample performance is seen for the full HMC within Gibbs procedure. The approach is illustrated for a portfolio of financial assets with respect to one day ahead value at risk forecasts. We provide comparison to a two step estimation procedure of the proposed model and to relevant benchmark models: a model with dynamic linear models for the margins and a single factor copula for the dependence proposed by Schamberger et al (2017) and a multivariate factor stochastic volatility model proposed by Kastner et al (2017). Our proposed approach shows superior performance.

Keywords: factor copula, stochastic volatility model, Hamiltonian Monte Carlo, value at risk

1 Introduction

Multivariate time series models are employed to model the joint behaviour of stocks. It is important to understand the dependence among these financial assets since it has high influence on the performance and the risk associated with a corresponding portfolio (Embrechts et al (2002), Donnelly and Embrechts (2010)). Vine copulas (Aas et al (2009), Bedford and Cooke (2001)) have proven a useful tool to facilitate complicated dependence structures (Brechmann and Czado (2013), Aas (2016), Fink et al (2017), Nikoloulopoulos et al (2012), Nagler et al (2018)). A vine copula model consists of \( \frac{d(d-2)}{2} \) pair copulas, where \( d \) is the number of assets. So the number of parameters grows quadratically with \( d \). Krupskii and Joe (2013) proposed the factor copula model, where the number of parameters grows only linearly in \( d \). This model can be seen as a generalization of the Gaussian factor model. The factor copula model provides much more flexibility as it is made up of different pair copulas that can be chosen arbitrarily. Thus it covers a broad range of dependence structures that can accommodate symmetric as well as asymmetric tail dependence.

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One way to construct multivariate time series models is to combine a univariate time series model for the margins with a dependence model such as the factor copula. Univariate time series models for financial data need to account for typical characteristics like time varying volatility and volatility clustering. Popular examples of such models include generalized autoregressive conditional heteroskedasticity (GARCH) models (Engle (1982), Bollerslev (1986)), the more recently developed generalized autoregressive score (GAS) models (Creal et al (2013)) and stochastic volatility (SV) models (Kim et al (1998)). Using the classification of Cox et al (1981) GARCH and GAS models are observation driven models, whereas the SV model is a parameter driven model. In observation driven models volatility is modeled deterministically through the observed past and as such results cannot be transferred to other data sets following the same data generating process. Inference for these observation driven models is often easier since evaluation of the likelihood is straightforward. Inference for SV models is more involved since likelihood evaluation requires high dimensional integration. But efficient MCMC algorithms have been developed (Kastner and Frühwirth-Schnatter (2014)). In the SV model volatility is modeled as latent variables that follow an autoregressive process of order 1. This representation has compared favorably to GARCH specifications in several data sets (Yu (2002), Chan and Grant (2016)).

We propose a copula based stochastic volatility model. The marginals follow a stochastic volatility model and the dependence is modeled through a single factor copula. In contrast to other factor stochastic volatility models as proposed by Kastner et al (2017) or Han (2003) we only allow for one factor and dependence parameters remain constant. But we do not assume that conditional on the volatilities the observed data is multivariate normal or Student t distributed. Here we provide more flexibility through the choice of different pair copula families.

The single factor copula model has also been deployed by Schamberger et al (2017) who use dynamic linear models (West et al (1985)) as marginals and by Krupskii and Joe (2013) who use GARCH models as marginals. As it is common in copula modeling, Schamberger et al (2017) and Krupskii and Joe (2013) both use a two step approach for estimation. They first estimate marginal parameters and base on these estimates they infer the dependence parameters. Tan et al (2018) provide full Bayesian inference for a single factor copula based model, but their marginal models have only few parameters and the proposals for MCMC are built using independence among components. However, for SV margins we need to estimate all T log volatilities, where T denotes the length of the time series. Thus we have more than T parameters to estimate per margin. The more sophisticated marginal models for financial data are difficult to handle within a full Bayesian approach. But nevertheless we overcome the in copula modeling commonly used two step approach and provide full Bayesian inference. For this we use Hamiltonian Monte Carlo (HMC) (Duane et al (1987), Neal et al (2011)) within Gibbs sampling. In HMC information of the gradient of the log posterior density is used to propose new states which leads to an efficient sampling procedure.

In a first simulation study we compare HMC for the copula part with the MCMC approach of Schamberger et al (2017) who use adaptive rejection Metropolis within Gibbs sampling (Gilks et al (1995)). HMC shows superior performance in terms of effective sample size, MSE and observed coverage probabilities. A second simulation study reveals gratifying small sample performance of the full HMC within Gibbs procedure. In the last part we demonstrate the usefulness of the proposed single factor copula stochastic volatility model with one day ahead value at risk prediction for financial data involving seven stocks. In contrast to a two step approach, no point estimates of model parameters are necessary to obtain VaR forecasts. Within our full Bayesian approach a VaR forecast is obtained as an empirical quantile of simulations from the predictive distribution.

2 Hamiltonian Monte Carlo

Hamiltonian dynamics describe the time evolution of a physical system through differential equations. In Hamiltonian Monte Carlo (HMC) the posterior density is related to the energy function of a physical system. This makes it possible to propose new states in the sampling process by evolving appropriate differential equations. New states are chosen utilizing information of the gradient of the log posterior density, which can lead to more efficient sampling
procedures. Therefore HMC has become popular. For example Hartmann and Ehlers (2017) demonstrate how to estimate parameters of generalized extreme value distributions with HMC, while Pakman and Paninski (2014) use HMC to sample from truncated multivariate Gaussian distributions. Especially with the development of the probabilistic programming language STAN by Carpenter et al (2016) its popularity is growing. STAN allows easy model specification and deploys the No-U-Turn sampler of Hoffman and Gelman (2014). This extension of HMC automatically and adaptively selects the tuning parameters. In this section we provide a short introduction to HMC based on Neal et al (2011). We start with the introduction of the Hamiltonian dynamics.

Hamiltonian dynamics

We consider a position vector \( q \in \mathbb{R}^d \) with associated momentum vector \( p \in \mathbb{R}^d \) at time \( t \). Their change over time is described through the function \( H(p, q) \), the Hamiltonian, which satisfies the following differential equations:

\[
\frac{dq_i}{dt} = \frac{dH}{dp_i}, \quad i = 1, \ldots, d.
\]

(1)

Here \( H \) represents the total energy of the system. In HMC, it is assumed that \( H \) can be expressed as

\[
H(p, q) = U(q) + K(p) = U(q) + p^t M^{-1} p / 2,
\]

(2)

where \( U(q) \) is called the potential energy and \( K(p) \) the kinetic energy. Further \( M \) is a symmetric positive definite mass matrix, which is usually assumed to be diagonal. The Hamiltonian dynamics, specified in (1), can therefore be rewritten as

\[
\frac{dq_i}{dt} = (M^{-1} p)_i, \quad \frac{dp_i}{dt} = - \frac{dU}{dq_i}, i = 1, \ldots, d.
\]

(3)

Leapfrog method

Since it is usually not possible to solve the system of differential equations given in (3) analytically we need to find iterative approximations. Therefore we use the Leapfrog method, where the state one step ahead of time \( t \) with step size \( \epsilon \), i.e. the state at time \( t + \epsilon \), is approximated by

\[
p_i(t + \epsilon/2) = p_i(t) - \frac{\epsilon}{2} \frac{dU}{dq_i}(q(t))
\]

\[
q_i(t + \epsilon) = q_i(t) + \frac{p_i(t + \epsilon/2)}{m_i}
\]

\[
p_i(t + \epsilon) = p_i(t + \epsilon/2) - \frac{\epsilon}{2} \frac{dU}{dq_i}(q(t + \epsilon)), \text{ for } i = 1, \ldots d.
\]

Canonical distribution

To use Hamiltonian dynamics within MCMC sampling we need to relate the energy function to a probability distribution. Therefore we utilize the canonical distribution \( P(x) \) associated with a general energy function \( E(x) \) with state \( x \) defined through the density

\[
p(x) := \frac{1}{Z} \exp(-E(x)/T).
\]

Here \( T \) is the temperature of the system and \( Z \) the normalizing constant needed to satisfy the density constraint. So the Hamiltonian \( H(p, q) \) specified in (2) defines a probability density given by

\[
p(q, p) = \frac{1}{Z} \exp(-H(p, q)/T) = \frac{1}{Z} \exp(-U(q)/T) \exp(-K(p)/T),
\]

where \( q \) and \( p \) are independent. In the following we assume \( T = 1 \).
Sampling with Hamiltonian Monte Carlo

In HMC we specify the corresponding energy function of \( q \) and \( p \), i.e. the Hamiltonian, and sample from the corresponding canonical distribution of \( q \) and \( p \). In a Bayesian setup we identify \( q \) as our parameters of interest and \( p \) are auxiliary variables. Therefore we set

\[
U(q) := -\log(\pi(q)\ell(q|D)),
\]

where \( \pi(q) \) is the prior density and \( \ell(q|D) \) the likelihood function for the given data \( D \). Therefore the canonical distribution of \( q \) corresponds to the posterior distribution of \( q \), when \( T = 1 \).

Since \( K(p) = p^T M^{-1} p / 2 \), it holds that the auxiliary parameter vector \( p \) is multivariate normal distributed with zero mean vector and covariance matrix \( M \). Sampling is then done in the following way.

1. Sample \( p \) from the normal distribution with zero mean vector and covariance matrix \( M \).
2. Metropolis update: Start with the current state \((q, p)\) and simulate \( L \) steps of Hamiltonian dynamics with step size \( \epsilon \) using the Leapfrog method. Obtain \((q', p')\) and accept this proposal with Metropolis acceptance probability

\[
\min(1, \exp(-H(q', p') + H(q, p))) = \min\left(1, \frac{\pi(q')\ell(q'|D)\exp(p' M^{-1} p'/2)}{\pi(q)\ell(q|D)\exp(p M^{-1} p/2)}\right).
\]

### 3 Bayesian inference for the stochastic volatility model

The proposed model in this paper builds on the stochastic volatility model which we review in more detail. According to Kim et al. [1998], the stochastic volatility (SV) model with parameters \( \mu \in \mathbb{R}, \phi \in (-1, 1), \sigma \in (0, \infty) \), \( h_{t:T} = (h_1, \ldots, h_T) \in \mathbb{R}^T \) is given by

\begin{align*}
Z_t &= \exp\left(\frac{h_t}{2}\right) \epsilon_t, \quad t = 1, \ldots, T, \\
h_t &= \mu + \phi(h_{t-1} - \mu) + \sigma \eta_t, \quad t = 2, \ldots, T,
\end{align*}

(4)

where \( h_1 \sim N(\mu, \frac{\sigma^2}{1 - \phi^2}) \) and \( \epsilon_t, \eta_t \sim N(0, 1) \) independently, for \( t = 1, \ldots, T \). The vector \( h_{1:T} \) represents the latent log variances, which follow an AR(1) process. This AR(1) process has mean parameter \( \mu \), persistence parameter \( \phi \) and standard deviation parameter \( \sigma \).

Kastner and Frühwirth-Schnatter [2014] develop an MCMC algorithm for this model which uses the ancillarity-sufficiency interweaving strategy proposed by Yu and Meng [2011]. This strategy leads to an efficient MCMC sampling procedure as they show in a simulation study. The sampler is implemented in the \texttt{R} package \texttt{stochvol} (see Kastner [2016a]). We now discuss the prior densities as implemented in Kastner [2016a] since we also utilize them later.

#### Prior densities

Kastner [2016a] chooses the following priors for \( \mu, \phi \) and \( \sigma \)

\[
\mu \sim N(0, 100), \quad \phi + \frac{1}{2} \sim \text{Beta}(5, 1.5), \quad \sigma^2 \sim \chi_1^2.
\]

(5)

This implies that \( \text{E}(\phi) = 0.54 \), \( \text{Var}(\phi) = 0.09 \) and \( \text{E}(\sigma^2) = 1 \), \( \text{Var}(\sigma^2) = 2 \).

The prior density of \((\mu, \phi, \sigma, h_{1:T})\) is therefore given by

\[
\pi_{SV}(\mu, \phi, \sigma, h_{1:T}) = f(h_{1:T}|\mu, \phi, \sigma) f(\mu, \phi, \sigma) \\
= \varphi(h_1|\mu, \frac{\sigma^2}{1 - \phi^2}) \prod_{t=2}^T \varphi(h_t|\mu + \phi(h_{t-1} - \mu), \sigma^2) \pi(\mu) \pi(\phi) \pi(\sigma),
\]

(6)

where \( \varphi(\cdot|m, s^2) \) denotes a univariate normal density with mean \( m \) and variance \( s^2 \) and \( \pi(\cdot) \) denotes the corresponding prior density as specified in (5).
Posterior density

The posterior density for data $z_{1:T} = (z_1, \ldots, z_T)^T$ is proportional to

$$f(\mu, \phi, \sigma, h_{1:T}|z_{1:T}) \propto f(z_{1:T}|\mu, \phi, \sigma, h_{1:T})\pi_{SV}(\mu, \phi, \sigma, h_{1:T})$$

$$= \prod_{t=1}^{T} [\varphi(z_t|0, \exp(h_t))] \pi_{SV}(\mu, \phi, \sigma, h_{1:T}).$$

4 Bayesian inference for single factor copulas

4.1 Model specification

We discuss the single factor copula model as a special case of the $p$ factor copula model according to [Krupskii and Joe (2013)]. We are given $d$ uniform(0,1) distributed variables $U_1, \ldots, U_d$ together with a uniform(0,1) distributed latent factor $W$. In the single factor copula model we assume that given $W$, the variables $U_i$ and $U_j$ are independent, i.e.

$$U_i \perp \perp U_j | W,$$

for $i \neq j, i, j \in \{1, \ldots, d\}$. This implies that the joint density of $U_{1:d} = (U_1, \ldots, U_d)^T$ can be written as

$$c_{U_{1:d}}(u_{1:d}) = \int_0^1 \prod_{j=1}^d c_j(W(u_j)|w)dw = \int_0^1 \prod_{j=1}^d c_j(u_j, w)dw,$$

where $c_j$ is the density of $C_j$, the copula of $(U_j, W)$. The copulas $C_1, \ldots, C_d$ are called linking copulas as they link each of the observed copula variables $u_j$ to the latent factor $w$.

For inference we use one parametric copula families, i.e. we equip each linking copula density with a corresponding parameter $\theta_j$, and (7) becomes

$$c_{U_{1:d}}(u_{1:d}; \theta_{1:d}) = \int_0^1 \prod_{j=1}^d c_j(u_j, w; \theta_j)dw.$$

As it is common in Bayesian statistics we treat the latent variable $W = w$ as a parameter. The joint density of $U_{1:d}$, denoted by $c_{U_{1:d}}$, given the parameters $(\theta_{1:d}, w)$ is obtained as

$$c_{U_{1:d}}(u_{1:d}; \theta_{1:d}, w) = \prod_{j=1}^d c_j(u_j, w; \theta_j).$$

Since the latent variable $W$ is random for each observation vector $(u_{t1}, \ldots, u_{td})^T$, we have $T$ latent parameters $w_{1:T} = (w_1, \ldots, w_T)^T$ for $T$ time points. The likelihood of the parameters $(\theta_{1:d}, w_{1:T})$ given $T$ independent observations $U_{1:T,1:d} = (u_{tj})_{t=1, \ldots, T, j=1, \ldots, d}$ is therefore

$$\ell(\theta_{1:d}, w_{1:T}|U_{1:T,1:d}) = \prod_{t=1}^{T} \prod_{j=1}^{d} c_j(u_{tj}, w_t; \theta_j).$$

4.2 Bayesian inference

So far, Bayesian inference for the single factor copula model was addressed by [Schamberger et al (2017)] and [Tan et al (2015)]. Both approaches use Gibbs sampling where one can exploit the fact that the factors $w_1, \ldots, w_T$ are independent given the copula parameters $\theta_1, \ldots, \theta_d$ and vice versa. We demonstrate how HMC can be used for the single factor copula model. Sampling with HMC is slower since it requires several evaluations of the gradient of the log posterior density. However with HMC there is no blocking involved and we update the whole parameter vector, with well chosen proposals obtained from the Leapfrog approximation, at once. We expect more accurate samples since this sampler suffers less from the dependence between factors and copula parameters. To support this statement we compare HMC to adaptive rejection
Metropolis sampling within Gibbs sampling (ARMGS) (Gils et al. (1995)). ARMGS is the sampler that worked best among several samplers that have been investigated by Schamberger et al. (2017).

In Section 5 we provide a sampler for the single factor copula SV model. This sampler updates two types of parameters: Dependence parameters and marginal parameters. The HMC method developed in this section is used for the dependence parameters. However, most of the computational cost of the sampler in Section 5 comes from the marginal parameters. This is why we care less about speed but more about accuracy in this section.

**Parametrization**

Since HMC operates on unconstrained parameters we need to provide parameter transformations to remove the constraints present in our problem. For many one parametric copula families there is a one to one correspondence between the copula parameter $\theta$ and Kendall’s $\tau$. So we can write $\tau(\theta)$, i.e. Kendall’s $\tau$ can be considered as a function of the copula parameter. Furthermore we restrict the Kendall’s $\tau$ values to be in $(0,1)$ to avoid problems that might occur due to multimodal posterior distributions. This is not a severe restriction for applications since $\tau_{U_1, U_2} = -\tau_{\tilde{U}_1, \tilde{U}_2}$, So we can replace $U_2$ by $1-U_2$ if we want to model negative dependence between $U_1$ and $U_2$. The latent factors $w_{1:T}$ are also in $(0,1)$. To transform parameters on the $(0,1)$ scale to the unconstrained scale the logit function is a common choice. We use the following parameter transformations for the copula parameters $\theta_{1:d}$ and the latent factors $w_{1:T}$

$$
\eta_j = \ln \left( \frac{\tau(\theta_j)}{1-\tau(\theta_j)} \right), \quad v_t = \ln \left( \frac{w_t}{1-w_t} \right),
$$

and obtain unconstrained parameters $\eta_j, v_t \in \mathbb{R}$ for $j = 1, \ldots, d, t = 1, \ldots, T$.

**Prior densities**

We specify the prior distributions for $(\eta_{1:d}, v_{1:T})$ such that the distributions implied for the corresponding Kendall’s $\tau(\theta_j)$ and for $w_t$ are independently uniform on the interval $(0,1)$. Applying the density transformation law this implies that the factor copula (FC) prior density can be expressed as

$$
\pi_{FC}(\eta_{1:d}, v_{1:T}) = \prod_{t=1}^{T} \pi_u(v_t) \prod_{j=1}^{d} \pi_u(\eta_j),
$$

where $\pi_u(x) = (1 + \exp(-x))^{-2} \exp(-x), x \in \mathbb{R}$.

**Posterior density**

With these choices in (8) and (10) the posterior density is proportional to

$$
f(\eta_{1:d}, v_{1:T}|U_{1:T,1:d}) \propto l(\theta_{1:d}, w_{1:T}|U_{1:T,1:d}) \cdot \pi_{FC}(\eta_{1:d}, v_{1:T}),
$$

where $\theta_j$ and $w_t$ are functions of $\eta_j$ and $v_t$ respectively. Therefore the log posterior density is, up to an additive constant, given by

$$
\mathcal{L}(\eta_{1:d}, v_{1:T}|U_{1:T,1:d}) \propto \sum_{t=1}^{T} \sum_{j=1}^{d} \ln(c_j(u_{t,d}, w_t; \theta_j)) + \sum_{t=1}^{T} \ln(\pi_u(v_t)) + \sum_{j=1}^{d} \ln(\pi_u(\eta_j)).
$$

$\tau_{U_1, U_2} = P((U_1 - \tilde{U}_1)((1 - U_2) - (1 - \tilde{U}_2)) > 0) - P((U_1 - \tilde{U}_1)((1 - U_2) - (1 - \tilde{U}_2)) < 0) = P((U_1 - \tilde{U}_1)(U_2 - \tilde{U}_2) < 0) - P((U_1 - \tilde{U}_1)(U_2 - \tilde{U}_2) > 0) = -\tau_{\tilde{U}_1, \tilde{U}_2}$, where $(\tilde{U}_1, \tilde{U}_2)$ is an independent copy of $(U_1, U_2)$. 

6
Sampling with HMC

Derivatives of the log posterior density with respect to all parameters are determined to perform Leapfrog approximations (see Appendix 8.1). With this at hand, HMC can be implemented as any Metropolis Hastings sampler. To finally run the algorithm we need to set hyper parameters: the Leapfrog stepsize $\epsilon$, the number of Leapfrog steps $L$ and the mass matrix $M$. Choosing $\epsilon$ and $L$ is not easy since good choices of these parameters can vary depending on different regions of the state space. [Neal et al. (2011)] suggest to randomly select $\epsilon$ and $L$ from a set of values that may be appropriate for different regions. This is the approach that we follow. For our simulation study we have seen that choosing $\epsilon$ uniformly between 0 and 0.2 and choosing $L$ uniformly between 0 and 40 leads to reasonable mixing as measured by the effective sample size (Gelman et al. (2014), page 286). The mass matrix $M$ is set equal to the identity matrix. The MCMC procedure is implemented in R using the R package Rcpp by [Eddelbuettel et al. (2011)] which allows the integration of C++. Effective sample sizes are calculated with the R package coda by [Plummer et al. (2008)].

4.3 Simulation study

To compare our approach we conduct the same simulation study as in [Schamberger et al. (2017)]. The simulation study investigates three scenarios, where each scenario contains 100 from the single factor copula model simulated data sets. The three scenarios are characterized by the values of Kendall’s $\tau$ of the linking copulas and are denoted by the low $\tau$, the high $\tau$ and the mixed $\tau$ scenario. The Kendall’s $\tau$ values are shown in Table 1. For each simulated data we set $T = 200$ and $d = 5$. As linking copulas only Gumbel copulas are considered. Table 2 shows the results of the simulation study and compares them to the results obtained by [Schamberger et al. (2017)] using adaptive rejection Metropolis sampling within Gibbs sampling (ARMGS). We use the posterior mode to estimate parameters. The corresponding error statistics (e.g. mean absolute deviation (MAD), mean squared error (MSE)) for each parameter is obtained from 100 replications. Then, e.g. the MSE for $\tau$ in Table 2 is computed as the average of MSE for $\tau_1, \ldots, \tau_5$. We see that a more accurate credible interval, a lower mean absolute deviation and a lower mean squared error is achieved in most cases by HMC compared to ARMGS. Furthermore the effective sample size per minute is much higher for HMC. Table 3 shows the results of the simulation study in more detail, i.e. we do not average over values of $\tau_1, \ldots, \tau_5$ and $w_1, \ldots, w_{200}$. It is noticeable that mixing is worse for higher values of Kendall’s $\tau$ in every scenario, whereas it is most extreme in the mixed $\tau$ scenario. This was also observed for ARMGS (see [Schamberger et al. (2017)] Table 9 in the appendix).

| C1 | C2 | C3 | C4 | C5 |
|----|----|----|----|----|
| low $\tau$ | 0.10 | 0.12 | 0.15 | 0.18 | 0.20 |
| high $\tau$ | 0.50 | 0.57 | 0.65 | 0.73 | 0.80 |
| mixed $\tau$ | 0.10 | 0.28 | 0.45 | 0.62 | 0.80 |

Table 1: Kendall’s tau values for the linking copulas $C_1, \ldots, C_5$ in the three scenarios.
Table 2: Comparison of the AMRGS and HMC method in terms of mean absolute deviation (MAD), mean squared error (MSE), effective sample size per minute (ESS/min) and observed coverage probability of the credible intervals (C.I.).

|          | Low τ |          |          |          |          |          |          |
|----------|------|----------|----------|----------|----------|----------|----------|
| τ        | ARMGS| HMC      | ARMGS    | HMC      |
|          | MAD  | 0.1088   | 0.0564   | 0.2808   | 0.2158   |
|          | MSE  | 0.0314   | 0.0059   | 0.1248   | 0.0716   |
|          | ESS/min | 6 | 92 | 20 | 246 |
|          | 90% C.I. | 0.91 | 0.94 | 0.84 | 0.88 |
|          | 95% C.I. | 0.96 | 0.98 | 0.91 | 0.94 |
| High τ  | MAD  | 0.0292   | 0.0201   | 0.0709   | 0.0502   |
|          | MSE  | 0.0014   | 0.0007   | 0.0095   | 0.0046   |
|          | ESS/min | 24 | 268 | 44 | 278 |
|          | 90% C.I. | 0.89 | 0.90 | 0.89 | 0.91 |
|          | 95% C.I. | 0.95 | 0.94 | 0.95 | 0.95 |
| Mixed τ | MAD  | 0.0509   | 0.0340   | 0.0828   | 0.0684   |
|          | MSE  | 0.0043   | 0.0019   | 0.0132   | 0.0082   |
|          | ESS/min | 21 | 132 | 24 | 268 |
|          | 90% C.I. | 0.87 | 0.89 | 0.79 | 0.85 |
|          | 95% C.I. | 0.93 | 0.93 | 0.88 | 0.93 |

Table 3: Detailed simulation results for the HMC method. We show the estimated mean absolute deviation (MAD), mean squared error (MSE), effective sample size per minute (ESS/min) and observed coverage probability of the credible intervals (C.I.) for τ₁, ..., τ₅ and five selected latent variables wₜ, t = 10, 50, 100, 150, 190.

5 The single factor copula stochastic volatility model

5.1 Model specification

We propose a multivariate dynamic model where each marginal follows a stochastic volatility model and the dependence between the marginals is captured by a single factor copula, the single factor copula stochastic volatility (factor copula SV) model. In particular for t = 1, ..., T, j = 1, ..., d we assume that

\[ Z_{tj} = \exp \left( \frac{h_{tj}}{2} \right) \epsilon_{tj} \]

\[ h_{tj} = \mu_j + \phi_j (h_{t-1j} - \mu_j) + \sigma_j \eta_{tj}, \]

where \( h_{tj} \sim N(\mu_j, \sigma^2_j / \tau^2_j) \) and \( \eta_{tj} \sim N(0,1) \) i.i.d. holds. In addition we assume that there exists a latent factor \( w_t \sim \text{unif}(0,1) \) i.i.d. holds. At each time t that the conditional density of the error vector at time t \( \epsilon_t = (\epsilon_{t1},...,\epsilon_{td})^T \) given the latent factor \( w_t \) is given by

\[
g_{\epsilon_t | w_t}(\epsilon_t | w_t) = \prod_{j=1}^{d} \left( \tilde{c}_j(\Phi(\epsilon_{tj}), w_t; \theta_j, \varphi(\epsilon_{tj})) \right). \tag{12} \]
In particular $\epsilon_{tj} \sim N(0, 1)$ for any $t$ and $j$. Here $c_j(\cdot, \cdot; \theta_j)$ is a bivariate copula density with parameter $\theta_j$. The unconditional density of $\epsilon_t$ is then given by

$$g_\epsilon(\epsilon_t) = \int_{(0,1)} \prod_{j=1}^d c_j(\Phi(\epsilon_{tj}), w_t; \theta_j) dw_t \prod_{j=1}^d \phi(\epsilon_{tj}). \quad (13)$$

Furthermore we assume that $(\epsilon_t, w_t)$ are independent of $(\epsilon_k, w_k)$ for $t \neq k$. To shorten notation we use the following abbreviations:

- $Z = (z_{tj})_{t=1,\ldots,T,j=1,\ldots,d}$ the matrix of observations,
- $E = (\epsilon_{tj})_{t=1,\ldots,T,j=1,\ldots,d}$ the matrix of errors,
- $\mu = (\mu_j)_{j=1,\ldots,d}$ the vector of means of the marginal stochastic volatility models,
- $\phi = (\phi_j)_{j=1,\ldots,d}$ the vector of persistence parameters of the marginal stochastic volatility models,
- $\sigma = (\sigma_j)_{j=1,\ldots,d}$ the vector of standard deviations of the marginal stochastic volatility models,
- $H = (h_{tj})_{t=1,\ldots,T,j=1,\ldots,d}$ the matrix of log volatilities,
- $h_{j} = (h_{tj})_{t=1,\ldots,T}$ the vector of log volatilities of the $j$-th marginal,
- $w = (w_t)_{t=1,\ldots,T}$ the vector of latent factors,
- $\theta = (\theta_j)_{j=1,\ldots,d}$ the vector copula parameters.

For the special case where $c_j, j = 1, \ldots, d$ are Gaussian copula densities, Krupskii and Joe (2013) show that the errors $\epsilon_{tj}$ specified in (13) allow for the following stochastic representation

$$\epsilon_{tj} = \theta_j v_t + \sqrt{1 - \theta_j^2} \xi_{tj},$$

where $v_t$ and $\xi_{tj} \sim N(0, 1)$ independently. Therefore we obtain the following additive structure

$$Z_{tj} = \theta_j \exp\left(\frac{h_{tj}}{2}\right) v_t + \exp\left(\frac{h_{tj}}{2}\right) \sqrt{1 - \theta_j^2} \xi_{tj}. \quad (14)$$

This implies a time dynamic covariance matrix with elements

$$\text{cov}(Z_{tj}, Z_{tk}) = \theta_j \theta_k \exp\left(\frac{h_{tj}}{2}\right) \exp\left(\frac{h_{tk}}{2}\right) \quad \text{for } j \neq k.$$ 

The correlation matrix however remains constant as time evolves and its off-diagonal elements are given by

$$\text{cor}(Z_{tj}, Z_{tk}) = \theta_j \theta_k \quad \text{for } j \neq k.$$ 

The additive structure in (14) shows remembrance to other multivariate factor stochastic volatility models (see Chib et al (2006), Kastner et al (2017)). This can be seen by considering the following reparametrization

$$h'_{tj} := h_{tj} + \log(1 - \theta_j^2), \quad \lambda_j := \frac{\theta_j}{\sqrt{1 - \theta_j^2}},$$

which implies the following representation of (14)

$$Z_{tj} = \lambda_j \exp\left(\frac{h'_{tj}}{2}\right) v_t + \exp\left(\frac{h'_{tj}}{2}\right) \xi_{tj}. \quad (15)$$

Here $h'_{tj}$ is an AR(1) process with mean $\mu_j + \log(1 - \theta_j^2)$, persistence parameter $\phi_j$ and standard deviation parameter $\sigma_j$. 

9
For comparison, the model of Kastner et al. (2017) with one factor is given by

$$Z_{tj} = \lambda_j \exp\left(\frac{h_{td+1}'}{2}\right) \epsilon_t + \exp\left(\frac{h_{tj}'}{2}\right) \xi_t,$$

with one additional latent AR(1) process $h_{td+1}', t = 1, \ldots, T$. This implies time varying correlations given by

$$\text{cor}(Z_{tj}, Z_{tk}) = \frac{\lambda_j \lambda_k \exp(h_{td+1}')}{\sqrt{\lambda_j^2 \exp(h_{td+1}') + \exp(h_{tj}')}}$$

for $j \neq k$.

Dividing $Z_{tj}$ by $\exp(h_{tj}')$ in (15) we recognize the structure of a standard factor model for

$$Z_{tj}' := \frac{Z_{tj}}{\exp\left(\frac{h_{tj}'}{2}\right)}$$

given by

$$Z_{tj}' = \lambda_j \epsilon_t + \xi_t,$$

with factor loadings $\lambda_1, \ldots, \lambda_d$ and factor $\epsilon_t$. In representation (16) the variance of $\xi_t$ is restricted to 1 whereas in the standard factor model (see e.g. Lopes and West (2004)) it is usually modeled through an additional variance parameter. Since the variance of $\epsilon_{tj}$ is already determined ($\epsilon_{tj} \sim N(0, 1)$) we have this additional restriction compared to factor models with flexible marginal variance. Note that $Z_{tj}$ still has flexible variance and the restriction for $\epsilon_{tj}$ is necessary to ensure identifiability.

If all copula families are Gaussian other multivariate factor stochastic volatility models provide generalizations by allowing for more factors and for a time varying correlation. We provide generalization with respect to the error distribution. The choice of different pair copula families provides a flexible modeling approach and our model can accommodate features that can not be modeled with a multivariate normal distribution as e.g. symmetric or asymmetric tail dependence.

Schamberger et al. (2017) also use factor copulas to model dependence among financial assets. Their approach differs to our approach in the choice of the marginal model. They use dynamic linear models. Secondly they perform a two step estimation approach, whereas we provide full Bayesian inference.

### 5.2 Bayesian inference

In the following we develop a full Bayesian approach for the proposed model. We use a block Gibbs sampler to sample from the posterior distribution. We use $d$ blocks for the marginal parameters $(\mu_j, \phi_j, \sigma_j, h_{.j}), j = 1, \ldots, d$ and one block for the dependence parameters $(\theta, \omega)$. Sampling from the full conditionals is done with HMC. Conditioning the copula parameters on the marginal parameters we are in the single factor copula framework of Section 4. We have seen that HMC provides an efficient way to sample the dependence parameters. Conditioned on the copula parameters the marginal parameters corresponding to different dimensions are independent. Each dimension can be considered as a generalized stochastic volatility model, where the distribution of the errors is determined by the corresponding linking copula. Sampling from the posterior distribution is more involved than in the Gaussian case. In the Gaussian case one can use an approximation of a mixture of normal distributions and rewrite the observation equation $Z_{tj} = \exp(h_{tj}') \epsilon_{tj}$ as a linear, conditionally Gaussian state space model (Omori et al. (2007), Kastner and Frühwirth-Schnatter (2014)). This is not possible in our case and therefore HMC, which has already shown good performance for the copula part and only requires derivation of the derivatives, is our method of choice.

**Prior densities**

The prior density is chosen as the product of priors used for the marginal stochastic volatility models and for the single factor copula model, i.e.

$$\pi_j(\mu, \phi, \sigma, H, \theta, \omega) = \prod_{j=1}^d \pi_{SV}(\mu_j, \phi_j, \sigma_j, h_{.j}) \pi_{FC}(\theta, \omega),$$

(17)
where \( \pi_{SV}(\cdot) \) and \( \pi_{FC}(\cdot) \) are specified in (8) and (10), respectively.

**Posterior density**

The independence of \((\varepsilon_t, w_t)\) and \((\varepsilon_k, w_k)\) for \( t \neq k \) implies that the conditional distribution of the errors given the latent factors is

\[
f(E|w) = \prod_{t=1}^{T} g_{t|w}(\varepsilon_t | w_t).
\]

Using the density transformation rule the likelihood of parameters \((\mu, \phi, \sigma, H, \theta, w)\) given the observation matrix \(Z\) is obtained as

\[
\ell(\mu, \phi, \sigma, H, \theta, w|Z) = \prod_{t=1}^{T} \left[ g_{t|w} \left( \frac{z_{t1}}{\exp(\frac{h_{t1}}{2})}, \ldots, \frac{z_{td}}{\exp(\frac{h_{td}}{2})} | w_t \right) \prod_{j=1}^{d} \frac{1}{\exp\left(\frac{h_{tj}}{2}\right)} \right].
\]

Therefore the posterior density is proportional to

\[
f(\mu, \phi, \sigma, H, \theta, w|Z) \propto \ell(\mu, \phi, \sigma, H, \theta, w|Z) \pi_{SV}(\mu, \phi, \sigma, H, \theta, w).
\]

**Sampling the marginal parameters**

The conditional density we need to sample from is given by

\[
f(\mu_j, \phi_j, \sigma_j, h_{-j}|Z, \mu_{-j}, \phi_{-j}, \sigma_{-j}, H_{-j}, \theta, w) \propto \ell(\mu, \phi, \sigma, H, \theta, w|Z) \pi_{SV}(\mu, \phi, \sigma, H, \theta, w)
\]

\[
\propto \prod_{t=1}^{T} \left[ c_j \left( \Phi \left( \frac{z_{tj}}{\exp(\frac{h_{tj}}{2})}, w_t; \theta_j \right) \right) \varphi \left( \frac{z_{tj}}{\exp(\frac{h_{tj}}{2})} \right) \frac{1}{\exp(\frac{h_{tj}}{2})} \right] \pi_{SV}(\mu_j, \phi_j, \sigma_j, h_{-j}).
\]

Here the abbreviation \( x_{-j} \) refers to the vector \((x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)^T\) with the \(j\)-th component removed. We sample from this density with HMC as will be outlined below.

**Parametrization** As in Section 4 we need to provide parametrizations such that resulting parameters are unconstrained. In particular we use the following transformations

\[
\xi_j = F_Z^{-1}(\phi_j)
\]

\[
s_j = \ln(\sigma_j)
\]

\[
g_{tj} = \frac{(h_{1j} - \mu_j) \sqrt{1 - \sigma_j^2}}{\sigma_j}
\]

\[
g_{tj} = \frac{h_{tj} - \mu - \phi_j (h_{t-1j} - \mu)}{\sigma_j}, t = 2, \ldots, T,
\]

where \( F_Z(x) = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) \) is Fisher’s Z transformation. The transformation for \( h_{-j} \) was proposed by the Stan Team (2015) and implies that \( g_{tj}\mu_j, \phi_j, \sigma_j \sim N(0, I_T) \), where \( I_T \) denotes the \( T \)-dimensional identity matrix.

**Prior densities** The prior densities for \( \mu_j, \phi_j \) and \( \sigma_j^2 \) in (17) imply the following prior densities for \( \mu_j, \xi_j \) and \( s_j \).

- We have that \( \mu_j \sim N(0, \sigma_j^2) \). So the prior density for \( \mu_j \) is up to a constant given by

\[
\pi_{\mu}(x) \propto \exp\left(-\frac{x^2}{2\sigma_j^2}\right).
\]
• We have that \( \frac{\phi_j + 1}{2} \sim \text{Beta}(a, b) \). So the density of \( \phi_j \) is given by

\[
f_\phi(x) = f_{\text{Beta}}(\frac{x + 1}{2}),
\]

This implies that the prior density of \( \xi_j \) is

\[
\pi_\xi(x) = \left| \frac{d}{d\xi} F^{-1}_Z(\xi) \right| = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \left( \frac{F^{-1}_Z(\xi) + 1}{2} \right)^{a-1} \left( 1 - \frac{F^{-1}_Z(\xi) + 1}{2} \right)^{b-1} \left( 1 - (F^{-1}_Z(\xi))^2 \right).
\]

• We have that \( \sigma_j^2 \sim \chi^2_1 \), i.e.

\[
f_\sigma(x) = 2xf_{\chi^2_1}(x^2).
\]

So the prior density for \( s_j \) is given by

\[
\pi_s(x) = f_\sigma(\exp(x)) \exp(x)
= 2 \exp(x) \exp(x) \frac{1}{\sqrt{2\Gamma(\frac{1}{2})}} \exp(-x) \exp(-\frac{\exp(2x)}{2})
= 2 \frac{1}{\sqrt{2\Gamma(\frac{1}{2})}} \exp(x) \exp(-\frac{\exp(2x)}{2}).
\]

• The log density of the joint prior of the parameters \( \mu_j, \xi_j, s_j \) and \( g_j \) is, up to an additive constant, given by

\[
\ln(\pi_{SV2}(\mu_j, \xi_j, s_j, g_j)) \propto \ln(\pi_\mu(\mu_j)) + \ln(\pi_\xi(\xi_j)) + \ln(\pi_s(s_j)) - \frac{1}{2} \sum_{t=1}^{T} g_{ij}^2.
\]

**Posterior density**

The log posterior density we need to sample from is, up to an additive constant, given by

\[
\mathcal{L}(\mu_j, \xi_j, s_j, g_j | Z, \theta, w) \propto \sum_{t=1}^{T} \left[ \ln \left( c \left( \Phi \left( \frac{z_{ij}}{\exp\left(\frac{h_{ij}}{T} \right)} \right), w_t; \theta_j \right) \right) + \ln \left( \varphi \left( \frac{z_{ij}}{\exp\left(\frac{h_{ij}}{T} \right)} \right) \right) - \frac{h_{ij}}{2} \right]
+ \ln(\pi_{SV2}(\mu_j, \xi_j, s_j, g_j)).
\]

The necessary derivatives of this log posterior are derived (see Appendix 8.2) for the Leapfrog approximations and then sampling of the marginal parameters is straightforward.

**Sampling the copula parameters**

The conditional density we need to sample from for the dependence parameters is proportional to

\[
f(\theta, w | Z, \mu, \phi, \sigma, H) \propto \ell(Z | \mu, \phi, \sigma, H, \theta, w) \pi_j(\mu, \phi, \sigma, H, \theta, w)
\propto \ell(Z | \mu, \phi, \sigma, H, \theta, w) \pi_{FC}(\theta, w)
\propto \prod_{t=1}^{T} \prod_{j=1}^{d} c_j \left( \Phi \left( \frac{z_{ij}}{\exp\left(\frac{h_{ij}}{T} \right)} \right), w_t; \theta_j \right) \pi_{FC}(\theta, w).
\]

To sample from this density we use the HMC approach described in Section 4.
5.3 Simulation study

We conduct a simulation study to evaluate the small sample performance of the proposed joint HMC sampler. We consider 5 different scenarios as specified in Table 4. We choose rather high values for the marginal persistence parameter $\phi$ and moderate to high values for the dependence parameter Kendall’s $\tau$. These choices roughly correspond to what we expect to see in financial data. For each scenario we simulate 100 data sets from the model introduced in Section 5. The proposed MCMC sampler with HMC updates is then applied to the simulated data. The sampler is run for 2000 iterations, where the first 500 are discarded for burn in. The simulation results are summarized in Table 5 for Scenario 4 and in Tables 7 and 8 in the appendix for the other scenarios. The results suggest that the method performs well. For all parameters we obtain reasonable MSE and ESS values. But we also observe notable differences. For example we see that the ESS decreases from $\tau_1$ up to $\tau_5$. This is in line with our findings in Section 4.3 where we have seen that mixing is worse for higher Kendall’s $\tau$ values. We can also observe differences with respect to observed coverage probability of credible intervals. For the lower values of the marginal persistence parameter ($\phi_1$, $\phi_2$) coverage probabilities are very high suggesting a broad posterior distribution. For a high persistence parameter ($\phi_5$) the observed coverage probabilities are lower. Figures 1 and 2 show estimated posterior densities and trace plots of one MCMC run in Scenario 4. These Figures suggest that we achieve proper mixing. Furthermore we see that posterior densities of $\phi_1$ and $\phi_2$ are more dispersed.

$$
\mu_{sim} = (-6, -6, -7, -7, -8)
$$

$$
\phi_{sim} = (0.7, 0.7, 0.8, 0.8, 0.9)
$$

$$
\sigma_{sim} = (0.2, 0.2, 0.3, 0.3, 0.4)
$$

$$
\tau_{sim} = (0.4, 0.5, 0.5, 0.6, 0.7)
$$

Table 4: Parameter specification for the different scenarios in the simulation study.

| Scenario | $d$ | $T$ | $\mu_{sim}$ | $\phi_{sim}$ | $\sigma_{sim}$ | $\tau_{sim}$ | family       |
|----------|----|-----|-------------|--------------|--------------|--------------|-------------|
| 1        | 5  | 500 | $\mu$      | $\phi_{sim}$ | $\sigma_{sim}$ | $\tau_{sim}$ | Gaussian    |
| 2        | 5  | 500 | $\mu_{sim}$ | $\phi_{sim}$ | $\sigma_{sim}$ | $\tau_{sim}$ | t ($\nu = 4$) |
| 3        | 5  | 500 | $\mu_{sim}$ | $\phi_{sim}$ | $\sigma_{sim}$ | $\tau_{sim}$ | t ($\nu = 6$) |
| 4        | 5  | 500 | $\mu_{sim}$ | $\phi_{sim}$ | $\sigma_{sim}$ | $\tau_{sim}$ | Gumbel     |
| 5        | 10 | 500 | ($\mu_{sim}$, $\mu_{sim}$) | ($\phi_{sim}$, $\phi_{sim}$) | ($\sigma_{sim}$, $\sigma_{sim}$) | ($\tau_{sim}$, $\tau_{sim}$) | Gumbel     |

Table 5: MSE estimated using the posterior mode, observed coverage probability of the credible intervals (C.I.) and effective samples size calculated from 1500 posterior draws for selected parameters (Scenario 4).
Figure 1: Kernel density estimates of the posterior density of selected parameters of a single data set from Scenario 4. The true parameter value is added in red.
Figure 2: Trace plots of selected parameters of a single data set from Scenario 4. The true parameter value is added in red.

6 Application

We illustrate our approach with one day ahead value at risk (VaR) prediction for a portfolio consisting of several stocks. These predictions can be obtained from simulations of the predictive distribution. As before, $Z$ is the data matrix containing $T$ observations of the $d$ stocks. We need to sample from the predictive distribution of the log returns at time $T+1$, $Z_{T+1} = (Z_{T+1,1}, \ldots, Z_{T+1,d})$, given $Z$. We obtain simulations from the joint density

$$f(z_{T+1}, h_{T+1}, H, \mu, \phi, \sigma, \theta, w_{T+1}, w | Z),$$

with the following steps:

- Simulate $H, \mu, \phi, \sigma, \theta, w$ from $f(H, \mu, \phi, \sigma, \theta, w | Z)$ with our sampler developed in Section 5.
- Simulate $w_{T+1} \sim \text{unif}(0, 1)$.
- Obtain simulations from $(Z_{T+1}, h_{T+1} | H, \mu, \phi, \sigma, \theta, w_{T+1}, w, Z)$ with Gibbs sampling. For the full conditional
\[
f(z_{T+1} \mid h_{T+1}, H, \mu, \phi, \sigma, \theta, w_{T+1}, w, Z) \propto \prod_{j=1}^{d} \left[ c_j \left( \frac{z_{T+1j}}{\exp \left( h_{T+1j} \right)} \right)^{\frac{z_{T+1j}}{\exp \left( h_{T+1j} \right)}} \right] \phi \left( \frac{z_{T+1j}}{\exp \left( h_{T+1j} \right)} \right) \exp \left( \frac{1}{2} \exp \left( h_{T+1j} \right) \right)
\]

we simulate \(u_j\) from \(C_j(\cdot \mid w_{T+1}; \theta_j)\) and set \(z_{T+1j} = \Phi^{-1}(u_j \mid 0, \exp \left( h_{T+1j} \right))\) for \(j = 1, \ldots, d\). For the full conditional

\[
f(h_{T+1} \mid z_{T+1}, H, \mu, \phi, \sigma, \theta, w_{T+1}, w, Z) \propto \prod_{j=1}^{d} \left[ c_j \left( \frac{z_{T+1j}}{\exp \left( h_{T+1j} \right)} \right)^{\frac{z_{T+1j}}{\exp \left( h_{T+1j} \right)}} \right] \phi \left( \frac{z_{T+1j}}{\exp \left( h_{T+1j} \right)} \right) \exp \left( \frac{1}{2} \exp \left( h_{T+1j} \right) \right)
\]

we use HMC.

We consider a equally weighted portfolio consisting of 7 stocks from German companies (BASF, Bayer, Fresenius Medical Care, Fresenius SE, Linde, Merck, K+S). Since all companies are chosen from the chemical/pharmaceutical/medical industry we assume that a model with one factor is suitable to capture the dependence structure. Our data, obtained from Yahoo Finance (https://finance.yahoo.com), contains daily log returns of these stocks from 2012 to 2017. We use 250 days as trading period, which corresponds to data of approximately one year. The first predicted value is the first trading day in January 2013. The same 250 days training period is used for the remaining daily one day ahead predictions from January 2013 until December 2017. Thus we simulate from the predictive distribution as described above with \(T = 250\) for each trading day in the period from January 2013 to December 2017. We then calculate the portfolio value from the simulations, and take the corresponding quantile to obtain the VaR prediction. We choose the same VaR level of 90% as in Schamberger et al (2017). In the MCMC procedure 2000 iterations are sampled where the first 500 are discarded for burn in. For the linking copulas different choices are considered: Gauss copulas, Student t copulas with 4 and 6 degrees of freedom, Gumbel and survival Gumbel copulas. With these choices we cover a range of different tail dependence structures: no tail dependence (Gauss), symmetric tail dependence (t) and asymmetric tail dependence (Gumbel and survival Gumbel). We choose the same copula for all linking copulas within one setup. Predicting the 90% VaR for each trading day in five years results in 1268 90% VaR predictions. The log portfolio value and corresponding 90% VaR predictions are visualized in Figure 3. We observe that the one day ahead VaR forecast adapts to changes in the volatility.

![Figure 3: Observed log return of the portfolio and the logarithm of the estimated one day ahead 90% VaR of the portfolio in blue.](image-url)
To benchmark the proposed model (factor copula SV (fc SV)) we repeated the procedure for VaR prediction with two other models: marginal dynamic linear models combined with single factor copulas (fc dlm) estimated with a two step procedure as proposed by [Schamberger et al (2017)] and a multivariate factor stochastic volatility model with dynamic factors (df Gaussian SV) as proposed by [Kastner et al (2017)]. The df Gaussian SV model is here restricted to one factor. Furthermore we compare the proposed approach to a two step estimation of the factor copula SV model (fc SV (two step)). In this two step approach we obtain simulations from the predictive distribution of the log returns at time $T + 1$, $Z_{T+1}$, given $Z$ as follows:

- Estimate a SV model for each margin separately and obtain posterior mode estimates for the latent log variances denoted by $\hat{h}_{tj}$ for $t = 1, \ldots, T$, $j = 1, \ldots, d$.
- Use the probability integral transform to obtain data on the $(0,1)$ scale, $U_{tj} := \Phi \left( Z_{tj} \cdot \exp \left( -\frac{\hat{h}_{tj}}{2} \right) \right)$.
- For the data $U_{tj}, t = 1, \ldots, T, j = 1, \ldots, d$, we fit a single factor copula model with HMC as described in Section 4 and obtain posterior mode estimates of the copula parameters denoted by $\hat{\theta}_1, \ldots, \hat{\theta}_d$.
- For each margin, simulate from the predictive distribution of the log variances at time $T + 1$, i.e. from $h_{T+1,j} | z_1, j, \ldots, z_T, j$, and obtain the posterior mode estimates $\hat{h}_{T+1,j}$ for $j = 1, \ldots, d$.
- Simulate $u_{1}^{sim}, \ldots, u_{d}^{sim}$ from the single factor copula with parameters $\hat{\theta}_1, \ldots, \hat{\theta}_d$.
- Set $z_{T+1,j} = \Phi^{-1} \left( u_{j}^{sim} | 0, \exp \left( \hat{h}_{T+1,j} \right) \right)$ for $j = 1, \ldots, d$.

In comparison to this two step procedure our full Bayesian approach does not require any point estimates of model parameters.

Standard measures to compare the predictive accuracy between different models are the continuous ranked probability score (Gneiting and Raftery (2007)) or log predictive scores as used in Kastner (2016b). These scores evaluate the overall performance. But we are interested in the VaR, a quantile of the predictive distribution, which is only one specific aspect. Therefore we use the rate of VaR violations, which is commonly used to compare VaR forecasts, as in Schamberger et al (2017). The VaR violation rates for the different models are shown in Table 8.4. From an optimal VaR measure at level $p$ we would expect that there are $1 - p$ VaR violations. From Table 8.4 we see that the rate of VaR violations is closest to the optimal value of 10% for the factor copula SV model with Student t linking copulas with 4 degrees of freedom. This model also performs best with respect to Christoffersen’s conditional coverage test (Christoffersen (2012), Chapter 8) which is commonly used for VaR backtesting in a frequentist perspective (see Appendix 8.4). In particular we conclude that the preferred model in this scenario is a factor copula SV model with a tail dependent linking copula given by the Student t copula, demonstrating the usefulness of the proposed model.

| Model          | fc SV | fc dlm | df Gaussian SV | fc SV (two step) |
|----------------|-------|--------|----------------|------------------|
| Gaussian       | 8.83% | 8.68%  | 8.99%          | 9.70%            |
| t, df=4        | 10.02%| 8.6%   | 10.33%         |                  |
| t, df=6        | 9.15% | 8.6%   | 10.09%         |                  |
| Gumbel         | 9.15% | 9.46%  | 10.80%         |                  |
| Survival Gumbel| 7.81% | 8.36%  | 9.54%          |                  |

Table 6: The rate of 90% VaR violations of the portfolio return for the different models.

7 Conclusion

We propose a single factor copula SV model, a combination of the SV model and factor copulas. Dependence and marginal parameters are estimated jointly within a Bayesian approach, avoiding a two step estimation procedure which is commonly used for copula models. The proposed model can be seen as one way to extend factor SV models that rely on Gaussian dependence.
to more complex dependence structures. The necessity of such models was illustrated with one day ahead value at risk prediction. In the application our stocks were chosen such that one factor is suitable to describe dependencies. However this might not be appropriate for different portfolios and the extension of the proposed model to multiple factors will be subject to future research. This extension to multiple factors could exploit the partition of different stocks into sectors as in the structured factor copula model proposed by Krupskii and Joe (2015). Another extension could allow for time varying dependence parameters.

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References

Aas K (2016) Pair-copula constructions for financial applications: A review. Econometrics 4(4):43
Aas K, Czado C, Frigessi A, Bakken H (2009) Pair-copula constructions of multiple dependence. Insurance: Mathematics and economics 44(2):182–198
Bedford T, Cooke RM (2001) Probability density decomposition for conditionally dependent random variables modeled by vines. Annals of Mathematics and Artificial intelligence 32(1-4):245–268
Bollerslev T (1986) Generalized autoregressive conditional heteroskedasticity. Journal of econometrics 31(3):307–327
Brechmann EC, Czado C (2013) Risk management with high-dimensional vine copulas: An analysis of the euro stoxx 50. Statistics & Risk Modeling 30(4):307–342
Carpenter B, Gelman A, Hoffman M, Lee D, Goodrich B, Betancourt M, Brubaker MA, Guo J, Li P, Riddell A (2016) Stan: A probabilistic programming language. Journal of Statistical Software 20
Chan JC, Grant AL (2016) Modeling energy price dynamics: Garch versus stochastic volatility. Energy Economics 54:182–189
Chib S, Nardari F, Shephard N (2006) Analysis of high dimensional multivariate stochastic volatility models. Journal of Econometrics 134(2):341–371
Christoffersen PF (2012) Elements of financial risk management. Academic Press
Cox DR, Gudmundsson G, Lindgren G, Bondesson L, Harsaae E, Laake P, Juselius K, Lauritzen SL (1981) Statistical analysis of time series: Some recent developments [with discussion and reply]. Scandinavian Journal of Statistics pp 93–115
Creal D, Koopman SJ, Lucas A (2013) Generalized autoregressive score models with applications. Journal of Applied Econometrics 28(5):777–795
Donnelly C, Embrechts P (2010) The devil is in the tails: actuarial mathematics and the subprime mortgage crisis. ASTIN Bulletin: The Journal of the IAA 40(1):1–33
Duane S, Kennedy AD, Pendleton BJ, Roweth D (1987) Hybrid monte carlo. Physics letters B 195(2):216–222
Eddelbuettel D, François R, Allaire J, Ushey K, Kou Q, Russel N, Chambers J, Bates D (2011) Rcpp: Seamless r and c++ integration. Journal of Statistical Software 40(8):1–18
Embrechts P, McNeil A, Straumann D (2002) Correlation and dependence in risk management: properties and pitfalls. Risk management: value at risk and beyond

Engle RF (1982) Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. Econometrica: Journal of the Econometric Society pp 987–1007

Fink H, Klimova Y, Czado C, Stöber J (2017) Regime switching vine copula models for global equity and volatility indices. Econometrics 5(1):3

Gelman A, Carlin JB, Stern HS, Dunson DB, Vehtari A, Rubin DB (2014) Bayesian data analysis, vol 2. CRC press Boca Raton, FL

Gilks WR, Best N, Tan K (1995) Adaptive rejection metropolis sampling within gibbs sampling. Applied Statistics pp 455–472

Gneiting T, Raftery AE (2007) Strictly proper scoring rules, prediction, and estimation. Journal of the American Statistical Association 102(477):359–378

Han Y (2003) Asset allocation with a high dimensional latent factor model. Olin School of Business Washington University in St Louis

Hartmann M, Ehlers RS (2017) Bayesian inference for generalized extreme value distributions via hamiltonian monte carlo. Communications in Statistics-Simulation and Computation pp 1–18

Hoffman MD, Gelman A (2014) The no-u-turn sampler: adaptively setting path lengths in hamiltonian monte carlo. Journal of Machine Learning Research 15(1):1593–1623

Kastner G (2016a) Dealing with stochastic volatility in time series using the r package stochvol. Journal of Statistical software 69(5):1–30

Kastner G (2016b) Sparse bayesian time-varying covariance estimation in many dimensions. arXiv preprint arXiv:160808468

Kastner G, Frühwirth-Schnatter S (2014) Ancillarity-sufficiency interweaving strategy (asis) for boosting mcmc estimation of stochastic volatility models. Computational Statistics & Data Analysis 76:408–423

Kastner G, Frühwirth-Schnatter S, Lopes HF (2017) Efficient bayesian inference for multivariate factor stochastic volatility models. Journal of Computational and Graphical Statistics (just-accepted)

Kim S, Shephard N, Chib S (1998) Stochastic volatility: likelihood inference and comparison with arch models. The review of economic studies 65(3):361–393

Krupskii P, Joe H (2013) Factor copula models for multivariate data. Journal of Multivariate Analysis 120:85–101

Krupskii P, Joe H (2015) Structured factor copula models: Theory, inference and computation. Journal of Multivariate Analysis 138:53–73

Lopes HF, West M (2004) Bayesian model assessment in factor analysis. Statistica Sinica pp 41–67

Nagler T, Bumann C, Czado C (2018) Model selection in sparse high-dimensional vine copula models with application to portfolio risk. arXiv preprint arXiv:180109739

Neal RM, et al (2011) Mcmc using hamiltonian dynamics. Handbook of Markov Chain Monte Carlo 2:113–162

Nikoloulopoulos AK, Joe H, Li H (2012) Vine copulas with asymmetric tail dependence and applications to financial return data. Computational Statistics & Data Analysis 56(11):3659–3673
8 Appendix

8.1 Derivatives for HMC for the single factor copula model

The derivatives of the log posterior density with respect to the parameters $\eta_j$ and $v_t$ are given by

\[
\frac{d}{d\eta_j} \mathcal{L}(\eta_{1:d}, v_{1:T} | U_{1:T,1:d}) = \sum_{j=1}^{d} \sum_{t=1}^{T} \frac{d}{d\eta_j} \ln(c_d(U_{t,j}, w_{t}; \theta_j)) + \frac{d}{d\eta_j} \ln(\pi_{FC}(\eta_{1:d}, v_{1:T}))
\]

\[
= \sum_{t=1}^{T} \frac{d}{d\theta_j} \ln(c_j(U_{t,j'}, w_{t}; \theta_j')) \frac{d\theta_j'}{d\eta_j} + \frac{d}{d\eta_j} \ln(\pi_{FC}(\eta_{1:d}, v_{1:T}))
\]

and

\[
\frac{d}{dv_t} \mathcal{L}(\eta_{1:d}, v_{1:T} | U_{1:T,1:d}) = \sum_{j=1}^{d} \sum_{t=1}^{T} \frac{d}{dv_t} \ln(c_d(U_{t,j}, w_{t}; \theta_j)) + \frac{d}{dv_t} \ln(\pi_{FC}(\eta_{1:d}, v_{1:T}))
\]

\[
= \sum_{j=1}^{d} \frac{d}{dv_t} \ln(c_d(U_{t',j}, w_{t'; \theta_j}) \frac{dv_{t'}}{dv_t} + \frac{d}{dv_t} \ln(\pi_{FC}(\eta_{1:d}, v_{1:T}))
\]
The components of the derivative of the log posterior density are derived in the following.

**Derivatives of log copula densities**

For all considered copula families [Schepsmeier and Stöber (2014)] calculate the derivatives of the copula density with respect to the copula parameter $\theta_j$ and with respect to the argument $w_t$. Based on their results the derivatives of the log copula density are easily derived. The derivatives are also implemented in the R package VineCopula by [Schepsmeier et al. (2012)].

**Derivatives of the parameter transformation**

We consider derivatives of the parameter transformation, i.e. $\frac{d\theta_j}{d\eta_j}$ and $\frac{dw_t}{dv_t}$. In this part we suppress the indices $j'$ and $t'$. We have that

$$w = (1 + \exp(-v))^{-1},$$

and the derivative is given by

$$\frac{dw}{dv} = (1 + \exp(-v))^{-2} \exp(-v).$$

Now we address the derivative $\frac{d\theta}{d\eta}$. The parameter $\eta$ was chosen to be the logit transform of the corresponding Kendall’s $\tau$ and so $\tau$ can be written as

$$\tau = (1 + \exp(-\eta))^{-1},$$

with corresponding derivative

$$\frac{d\tau}{d\eta} = (1 + \exp(-\eta))^{-2} \exp(-\eta).$$

The copula parameter $\theta$ is a function of Kendall’s $\tau$ and dependent on the copula family considered we obtain the following derivatives.

- **Gauss and Student t copula**

  $$\theta = \sin\left(\frac{1}{2} \pi \tau\right)$$

  $$\frac{d\theta}{d\eta} = \frac{d\theta}{d\tau} \frac{d\tau}{d\eta}$$

  $$= \frac{1}{2} \pi \cos\left(\frac{1}{2} \pi \tau\right) \frac{d\tau}{d\eta}$$

  $$= \frac{1}{2} \pi \cos\left(\frac{1}{2} \pi (1 + \exp(-\eta))^{-1}\right) (1 + \exp(-\eta))^{-2} \exp(-\eta)$$

- **Clayton copula**

  $$\theta = \frac{2\tau}{1 - \tau}$$

  $$= \frac{2}{\tau^{-1} - 1}$$

  $$= \frac{2}{1 + \exp(-\eta) - 1}$$

  $$= 2 \exp(\eta)$$

  $$\frac{d\theta}{d\eta} = 2 \exp(\eta)$$
• Gumbel copula

$$\theta = (1 - \tau)^{-1}$$

$$\frac{d\theta}{d\eta} = \frac{d\theta}{d\tau} \frac{d\tau}{d\eta}$$

$$= (1 - \tau)^{-2} \frac{d\tau}{d\eta}$$

$$= (1 - [1 + \exp(-\eta)]^{-1})^{-2}[1 + \exp(-\eta)]^{-2}\exp(-\eta)$$

$$= [1 + \exp(-\eta) - 1]^{-2}\exp(-\eta)$$

$$= \exp(-\eta)^{-2}\exp(-\eta)$$

$$= \exp(\eta)$$

**Derivatives of the log prior distribution**

The derivative of the log prior density $\pi_u$ is given by

$$\frac{d}{dx} \ln(\pi_u(x)) = \frac{d}{dx}(-2\ln(1 + \exp(-x)) - x) = 2(1 + \exp(x))^{-1} - 1.$$

### 8.2 Derivatives for HMC for the stochastic volatility model

We need to calculate derivatives of the function

$$L_{\text{SV}}(\mu, \xi, s, g, \theta, w) = \sum_{i=1}^T \left[ \ln \left( c \left( \Phi \left( \frac{Z_{ij}}{\exp(\frac{h_i}{2})} \right), w_i; \theta \right) \right) + \ln \left( \frac{Z_{ij}}{\exp(\frac{h_i}{2})} \right) - \frac{h_i}{2} \right]$$

where $\propto$ refers to an additive constant. To shorten notation we omit the index $j$ in the following and consider the function

$$L_2(\mu, \xi, s, g_1:T | Z, \theta, w) = \sum_{t=1}^T \left[ \ln \left( c(\Phi(\frac{Z_t}{\exp(\frac{h_t}{2})}), w_t; \theta) \right) + \ln \left( \frac{Z_t}{\exp(\frac{h_t}{2})} \right) - \frac{h_t}{2} \right]$$

We define

$$\Omega(h_{1:T}) = \sum_{i=1}^T \left( \ln \left( c(\Phi(\frac{Z_t}{\exp(\frac{h_t}{2})}), w_t; \theta) \right) + \ln \left( \frac{Z_t}{\exp(\frac{h_t}{2})} \right) - \frac{h_t}{2} \right),$$

and the derivatives can be expressed as

- \( \frac{d}{d\mu} L_2(\mu, \xi, s, g_1:T) = \frac{d \Omega(h_{1:T})}{dh_{1:T}} \frac{d h_{1:T}}{d\mu} + \frac{d}{d\mu} \ln(\pi_{SV2}(\mu, \xi, s, g_1:T)) \)
- \( \frac{d}{d\xi} L_2(\mu, \xi, s, g_1:T) = \frac{d \Omega(h_{1:T})}{dh_{1:T}} \frac{d h_{1:T}}{d\xi} + \frac{d}{d\xi} \ln(\pi_{SV2}(\mu, \xi, s, g_1:T)) \)
- \( \frac{d}{ds} L_2(\mu, \xi, s, g_1:T) = \frac{d \Omega(h_{1:T})}{dh_{1:T}} \frac{d h_{1:T}}{ds} + \frac{d}{ds} \ln(\pi_{SV2}(\mu, \xi, s, g_1:T)) \)
- \( \frac{d}{dg_i} L_2(\mu, \xi, s, g_1:T) = \frac{d \Omega(h_{1:T})}{dh_{1:T}} J + \frac{d}{dg_i} \ln(\pi_{SV2}(\mu, \xi, s, g_1:T)) \),

where $J \in \mathbb{R}^{T \times T}$ denotes the corresponding Jacobian matrix, i.e. $J_{ij} = \frac{d h_i}{d g_j}$. The derivatives are calculated in the following.

- \( \frac{d}{d\mu} \Omega(h_{1:T}) = \frac{d}{d\mu} \ln(c(x, w_i; \theta)) \bigg|_{x = \Phi(\frac{Z_i}{\exp(\frac{h_i}{2})})} \varphi(\frac{Z_i}{\exp(\frac{h_i}{2})}) \frac{Z_i}{\exp(\frac{h_i}{2})} \left( \frac{1}{2} + \frac{Z_i^2}{\exp(h_i)} - \frac{1}{2} \right) \)
• We have that $h_1 = \frac{g_1 \sigma}{\sqrt{1-\phi^2}} + \mu$, $h_t = g_t \sigma + \mu + \phi(h_{t-1} - \mu)$, $t = 2, \ldots, T$ and obtain

$$\frac{d}{d\mu} h_1 = 1 \quad \frac{d}{d\mu} h_t = 1 - \phi + \phi \frac{d}{d\mu} h_{t-1}, t = 2, \ldots, T$$

$$\frac{d}{d\phi} h_1 = \frac{g_1 \sigma}{\sqrt{1-\phi^2}} \quad \frac{d}{d\phi} h_t = h_{t-1} - \mu + \phi \frac{d}{d\phi} h_{t-1}, t = 2, \ldots, T$$

$$\frac{d}{dg_1} h_1 = \phi^{t-1} \frac{\sigma}{\sqrt{1-\phi^2}}, t = 1, \ldots, T \quad \frac{d}{dg_j} h_{1:t} = \phi^{t-j} \sigma_{1:t}^j, t = 1, \ldots, T, j = 2, \ldots, T$$

• $\frac{d\phi}{dx} = 1 - F^{-1}(\xi)^2$, $\frac{d\sigma}{dx} = \exp(s)$

• $\frac{d}{d\mu} \ln(\pi_{SV2}(\mu, \xi, s, g_{1:T})) = -\frac{\mu^2}{\sigma^2}$

• $\frac{d}{d\xi} \ln(\pi_{SV2}(\mu, \xi, s, g_{1:T})) = (a - 1) \frac{(1-F_Z^{-1}(\xi)^2)}{(F_Z^{-1}(\xi)+1)} - (b - 1)(1+F_Z^{-1}(\xi)) - 2F_Z^{-1}(\xi)(1-F_Z^{-1}(\xi)^2)$

where $a = 5$ and $b = 1.5$ are the parameters of the beta distribution.

• $\frac{d}{ds} \ln(\pi_{SV2}(\mu, \xi, s, g_{1:T})) = 1 - \exp(2s)$

• $\frac{d}{dg_{1:T}} \ln(\pi_{SV2}(\mu, \xi, s, g_{1:T})) = -g_{1:T}$
8.3 Results of the simulation study

| Scenario | µ_1 | µ_2 | µ_3 | µ_4 | µ_5 | φ_1 | φ_2 | φ_3 | φ_4 | φ_5 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| MSE      | 0.0057 | 0.0047 | 0.0085 | 0.0090 | 0.0273 | 0.0511 | 0.0389 | 0.0213 | 0.0204 | 0.0024 |
| C.I. 95% | 0.92 | 0.94 | 0.91 | 0.92 | 0.94 | 0.99 | 0.98 | 0.85 | 0.87 | 0.84 |
| C.I. 90% | 0.96 | 0.98 | 0.95 | 0.97 | 0.97 | 1.00 | 1.00 | 0.96 | 0.95 | 0.94 |
| ESS      | 512 | 305 | 461 | 295 | 561 | 555 | 518 | 354 | 317 | 360 |

| MSE      | 0.0012 | 0.0010 | 0.0012 | 0.0090 | 0.0052 | 0.0068 | 0.0067 | 0.0066 | 0.0085 | 0.0080 |
| C.I. 90% | 0.97 | 0.97 | 0.90 | 0.95 | 0.88 | 0.92 | 0.87 | 0.89 | 0.93 | 0.89 |
| C.I. 95% | 0.99 | 0.99 | 0.94 | 0.96 | 0.95 | 0.96 | 0.94 | 0.95 | 0.96 | 0.94 |
| ESS      | 305 | 272 | 334 | 270 | 200 | 307 | 238 | 239 | 176 | 97 |

| MSE      | 0.0067 | 0.0065 | 0.1772 | 0.1689 | 0.2622 | 0.0138 |
| C.I. 90% | 0.88 | 0.89 | 0.89 | 0.91 | 0.92 | 0.86 |
| C.I. 95% | 0.95 | 0.92 | 0.96 | 0.96 | 0.93 |
| ESS      | 1081 | 1082 | 1065 | 1045 | 878 | 383 |

| MSE      | 0.0055 | 0.0058 | 0.0125 | 0.0100 | 0.0268 | 0.0448 | 0.0492 | 0.0154 | 0.0135 | 0.0016 |
| C.I. 90% | 0.92 | 0.93 | 0.90 | 0.90 | 0.97 | 0.99 | 1.00 | 0.90 | 0.91 | 0.91 |
| C.I. 95% | 0.94 | 0.94 | 0.93 | 0.94 | 0.94 | 0.94 |
| ESS      | 438 | 285 | 450 | 340 | 614 | 554 | 536 | 351 | 315 | 367 |

| MSE      | 0.0056 | 0.0047 | 0.0077 | 0.0035 | 0.0035 | 0.0011 | 0.0012 | 0.0012 | 0.0011 | 0.0011 |
| C.I. 90% | 0.98 | 0.98 | 0.93 | 0.94 | 0.94 | 0.94 | 0.94 | 0.94 | 0.94 | 0.94 |
| C.I. 95% | 0.99 | 1.00 | 0.97 | 0.96 | 0.97 | 0.91 | 0.86 | 0.92 | 0.88 | 0.89 |
| ESS      | 309 | 269 | 313 | 263 | 234 | 430 | 321 | 223 | 133 |

| MSE      | 0.0012 | 0.0041 | 0.1884 | 0.1679 | 0.2522 | 0.0157 |
| C.I. 90% | 0.88 | 0.86 | 0.87 | 0.86 | 0.89 | 0.90 |
| C.I. 95% | 0.91 | 0.92 | 0.93 | 0.94 | 0.91 | 0.93 |
| ESS      | 1068 | 1086 | 1027 | 1017 | 907 | 364 |

| MSE      | 0.0058 | 0.0046 | 0.0099 | 0.0085 | 0.0257 | 0.0501 | 0.0596 | 0.0224 | 0.0194 | 0.0025 |
| C.I. 90% | 0.92 | 0.92 | 0.91 | 0.92 | 0.96 | 0.99 | 0.97 | 0.90 | 0.96 | 0.91 |
| C.I. 95% | 0.96 | 0.96 | 0.97 | 0.97 | 0.97 | 0.99 | 1.00 | 0.97 | 0.96 | 0.91 |
| ESS      | 338 | 234 | 392 | 288 | 547 | 504 | 501 | 336 | 292 | 311 |

| MSE      | 0.0103 | 0.0093 | 0.1119 | 0.0789 | 0.0554 | 0.0009 | 0.0009 | 0.0007 | 0.0007 | 0.0007 |
| C.I. 90% | 0.97 | 0.99 | 0.88 | 0.93 | 0.91 | 0.91 | 0.86 | 0.90 | 0.92 | 0.90 |
| C.I. 95% | 0.98 | 0.99 | 0.94 | 0.96 | 0.95 | 0.92 | 0.91 | 0.96 | 0.95 | 0.94 |
| ESS      | 274 | 250 | 302 | 242 | 203 | 403 | 280 | 293 | 202 | 110 |

| MSE      | 0.0639 | 0.0631 | 0.1783 | 0.1672 | 0.2551 | 0.0149 |
| C.I. 90% | 0.91 | 0.91 | 0.90 | 0.90 | 0.91 | 0.88 |
| C.I. 95% | 0.96 | 0.96 | 0.96 | 0.96 | 0.96 | 0.94 |
| ESS      | 1099 | 1039 | 1050 | 1027 | 876 | 375 |

Table 7: MSE estimated using the posterior mode, observed coverage probability of the credible intervals (C.I.) and effective samples size calculated from 1500 posterior draws for selected parameters (Scenarios 1-3).
Table 8: MSE estimated using the posterior mode, observed coverage probability of the credible intervals (C.I.) and effective samples size calculated from 1500 posterior draws (Scenario 5).

8.4 Frequentist assessment

From an optimal value at risk measure at level $p$ we would expect that there are $1 - p$ VaR violations and that VaR violations occur independently which constitutes the null hypotheses of Christoffersen’s conditional coverage test. The test is formulated as a likelihood ratio test and in our case high $p$-values are preferred.

Table 9: $p$-value for Christoffersen’s conditional coverage test for the VaR of the portfolio for the different models.