Marginal Fermi liquid behaviour in the 
\(d = 2\) Hubbard model with cut-off

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Abstract 1. We consider the half-filled Hubbard model with a cut-off forbidding momenta close to the angles of the square shaped Fermi surface. By Renormalization Group methods we find a convergent expansion for the Schwinger function up to exponentially small temperatures.

We prove that the system is not a Fermi liquid, but on the contrary it behaves like a Marginal Fermi liquid, a behaviour observed in the normal phase of high \(T_c\) superconductors.

1. Main results

1.1 Motivations. The notion of Fermi liquids, introduced by Landau, refers to a wide class of interacting fermionic systems whose thermodynamic properties (like the specific heat or the resistivity) are qualitatively the same of a gas of non interacting fermions. While there is an enormous number of metals having Fermi liquid behaviour, in recent times new materials has been found whose properties are qualitatively different. In particular the high-temperature superconducting materials (so anisotropic to be considered essentially bidimensional) in their normal phase have a non Fermi liquid behaviour, in striking contrast with previously known superconductors, which are Fermi liquids above the critical temperature. While in Fermi liquids the wave function renormalization \(Z\) is \(Z = 1 + O(\lambda^2)\), where \(\lambda\) is the strength of the interaction, in such metals it was found \(Z \simeq 1 + O(\lambda^2 \log T)\) for temperatures \(T\) above the critical temperature, see [VLSAR] (see also [VNS] for a review); metals behaving in this way were called Marginal Fermi liquids. Such results stimulated an intense theoretical research. It was found by a perturbative analysis, see for instance [AGD] or [Sh], that in a system of weakly interacting fermions in \(d = 2\) \(Z\) is essentially temperature independent, at least for circular or ”almost” circular Fermi surfaces. Despite doubts appeared about the reliability of results obtained by perturbative expansions [A], such results were indeed confirmed recently by rigorous Renormalization group methods. It was proved in [FMRT] and [DR] that indeed a weakly interacting Fermi system with a circular Fermi surface is a Fermi liquid, up to exponentially small temperatures. Such result was extended in [BGM] to all possible weakly interacting \(d = 2\) fermionic systems with symmetric, smooth and convex Fermi surfaces, up to exponentially small temperatures. These results cannot be obtained by dimensional power counting arguments as such arguments give a bound \(|Z - 1| \leq C\lambda^2|\log T|\) from which one cannot distinguish Fermi or non Fermi liquid behaviour; for obtaining \(Z = 1 + O(\lambda^2)\) one has instead to use delicate volume improvements.
in the integrals expressing $Z$, based on the geometrical constraints to which the momenta close to the Fermi surface (assumed convex, regular and symmetric) are subjected.

As Fermi liquid behaviour is found in systems with symmetric, smooth and convex Fermi surfaces, in order to find non Fermi liquid behaviour one has to relax some of such conditions. It was pointed out, see for instance [VR] and [ZYD], that the presence in the Fermi surface of flat regions in opposite sides could produce a non Fermi liquid behaviour; flat regions are indeed present in the Fermi surfaces of high $T_c$ superconductors [S]. The simplest model exhibiting a Fermi surface with flat pieces is the half-filled Hubbard model, describing a system of spinning $d = 2$ fermions with local interaction and dispersion relation given by

$$
\varepsilon(k_x, k_y) = \cos k_x + \cos k_y.
$$

The Fermi surface is the set of momenta such that $\varepsilon(k_x, k_y) = 0$ and it is a square with corners $(\pm \pi, 0)$ and $(0, \pm \pi)$. However this model has the complicating feature of vanishing Fermi velocity at the points $(\pm \pi, 0)$ and $(0, \pm \pi)$ i.e. at the corners of the Fermi surface; this originates to the so called Van Hove singularities in the density of states. In order to investigate the possible non Fermi liquid behaviour of interacting fermions with a Fermi surface with flat pieces, independently from the presence of Van Hove singularities, one can introduce in the half filled Hubbard model a cut-off forbidding momenta near the corners of the Fermi surface. The half filled Hubbard model with cut-off (or the essentially equivalent, but slightly simpler, problem of fermions with the linearized Hove singularities, one can introduce in the half filled Hubbard model a cut-off forbidding states. In order to investigate the possible non Fermi liquid behaviour of interacting fermions with a Fermi surface with flat pieces, independently from the presence of Van Hove singularities, one can introduce in the half filled Hubbard model a cut-off forbidding momenta near the corners of the Fermi surface. The half filled Hubbard model with cut-off (or the essentially equivalent, but slightly simpler, problem of fermions with the linearized dispersion relation $\varepsilon(k_x, k_y) = |k_x| + |k_y| - \pi$ has been extensively studied in literature, see for instance [M], [L], [ZYD],[VR],[FSW], [DAD], [FSL]. The cut-off is somewhat artificially introduced but the idea is that the model, at least for some values of the parameters, belongs to the same university class of models with "almost" squared and smooth Fermi surface, like the anisotropic Hubbard models [Sh], the Hubbard model with nearest and next to nearest neighbor interaction [M], or the half-filled Hubbard model close to half filling.

Aim of this paper is to compute in a rigorous way the asymptotic behaviour of the Schwinger functions of the half filled Hubbard model with cut-off up to exponentially small temperatures. We will show that such a system is indeed a Marginal Fermi liquid, and our result furnishes indeed the first example rigorously established of such behaviour in $d = 2$.

For our convenience, we will consider new variables $k_+ = \frac{k_x + k_y}{2}$ and $k_- = \frac{k_x - k_y}{2}$ so that the dispersion relation of the half-filled Hubbard model is given by

$$
\varepsilon(k_+, k_-) = 2 \cos k_+ \cos k_-
$$

and the Fermi surface is the set $k_+ = \pm \frac{\pi}{2}$ or $k_- = \pm \frac{\pi}{2}$.

1.2 The model. Given a square $[0, L]^2 \subset \mathbb{R}^2$, the inverse temperature $\beta$ and the (large) integer $M$, we introduce in $\Lambda = [0, L]^2 \times [0, \beta]$ a lattice $\Lambda_M$, whose sites are given by the space-time points $\mathbf{x} = (x_0, x_+, x_-)$ with $(x_+, x_-) \in \mathbb{Z}^2$ and $x_0 = n_0 \beta / M$, $n_0 = 0, 1, \ldots, M - 1$. We also consider the set $\mathcal{D}$ of space-time momenta $\mathbf{k} = (k_0, k_+, k_-) \equiv (k, \vec{k})$, with $k_\pm = \frac{2\pi n_\pm}{L}$, $(n_+, n_-) \in \mathbb{Z}^2$, $[-L/2] \leq n_\pm \leq [L - 1/2]$; $k_0 = \frac{2\pi n_0}{L}$, $n_0 = 0, 1, \ldots, M - 1$. With each $\mathbf{k} \in \mathcal{D}$ we associate four Grassmanian variables $\psi_{\mathbf{k}, s}, \varepsilon, s \in \{+, -\}; \varepsilon$ is the spin. The lattice $\Lambda_M$ is introduced only for technical reasons so that the number of Grassmann variables is finite, and eventually the (essentially trivial) limit $M \to \infty$ is taken. We introduce also a linear functional $P(d\psi)$ on the Grassmanian algebra generated by the variables $\psi_{\mathbf{k}, \varepsilon}^s$, such that

$$
\int P(d\psi)\psi_{\mathbf{k}_1, s_1, \varepsilon_1}^+\psi_{\mathbf{k}_2, s_2}^- = L^2 \beta \delta_{s_1, s_2}\delta_{\mathbf{k}_1, \mathbf{k}_2}\hat{g}(\mathbf{k}_1),
$$

where $g(\mathbf{k})$ is defined by

$$
\hat{g}(\mathbf{k}) = \frac{\chi(\mathbf{k})}{-ik_0 + 2 \cos k_+ \cos k_-}
$$

where $\chi(\mathbf{k})$ is a cut-off function

$$
\chi(\mathbf{k}) = H(a_0^2 \sin^2 k_+)C_0^{-1}(\mathbf{k}) + H(a_0^2 \sin^2 k_-)C_0^{-1}(\mathbf{k})
$$
where
\[ C_0^{-1}(k) = H(\sqrt{k_0^2 + 4 \cos^2(k_+ \cos^2(k_-)}) \] (1.5)
and, if \( \gamma > 1 \) and \( a_0 \geq \sqrt{2} \)
\[ H(t) = \begin{cases} 1 & \text{if } |t| < \gamma^{-1} \\ 0 & \text{if } |t| > 1 \end{cases} \] (1.6)

The function \( C_0^{-1}(k) \) acts as an ultraviolet cut-off forcing the momenta \( \vec{k} \) to be not too far from the Fermi surface, and \( k_0 \) not too large; the cut-off on \( k_0 \) is imposed only for technical convenience and it could be easily removed. The functions \( H(a_0^2 \sin^2 k_\pm) \) forbids momenta near the corners of the Fermi surface i.e. the points \((\pm \pi/2, \pm \pi/2)\). The Grassmanian field \( \psi_x^\varepsilon \) is defined by
\[ \psi_{x,s}^\varepsilon = \frac{1}{L^2 \beta} \sum_{k \in \mathcal{D}} \psi_{k,s}^\varepsilon e^{\pm ik \cdot x} . \] (1.7)

The “Gaussian measure” \( P(d\psi) \) has a simple representation in terms of the “Lebesgue Grassmanian measure”
\[ D\psi = \prod_{k \in \mathcal{D}, s = \pm} \prod_{i} d\psi_{k,s}^+, d\psi_{k,s}^- , \] (1.8)
defined as the linear functional on the Grassmanian algebra, such that, given a monomial \( Q(\psi^-, \psi^+) \) in the variables \( \psi_{k,s}^-, \psi_{k,s}^+ \), its value is 0, except in the case \( Q(\psi^-, \psi^+) = \prod_{k,s} \psi_{k,s}^-, \psi_{k,s}^+ \), up to a permutation of the variable, in which case its value is 1. Finally \( \prod_{k \in \mathcal{D}, s = \pm} \) means a product over the \( k \) such that \( \chi(k) > 0 \). We define
\[ P(d\psi) = N^{-1}D\psi \cdot \exp[-\frac{1}{L^2 \beta} \sum_{k \in \mathcal{D}, s = \pm} \chi^{-1}(k)(-i k_0 + 2 \cos k_+ \cos k_-) \hat{\psi}_{k,s}^+ \hat{\psi}_{k,s}^-] , \] (1.9)
with \( N \) is a renormalization constant and again \( \sum_{k} \) means a sum over \( k \) such that \( \chi(k) > 0 \).

The two point \( Schwinger \) function is defined by the following \( Grassman \) functional integral
\[ S(x - y) = \lim_{L \to \infty} \lim_{M \to \infty} \frac{\int P(d\psi) e^{-iV(\psi)\psi_{x,s}^+ \psi_{y,s}^-}}{\int P(d\psi) e^{-iV(\psi)}} , \] (1.10)
where, if we use \( \int dx \) as a shorthand for \( \frac{\beta}{M} \sum_{x \in \Lambda_M} \)
\[ V(\psi) = \lambda \sum_s \int dx \psi_{x,s}^+ \psi_{x,s}^- \psi_{x,-s}^+ \psi_{x,-s}^- . \] (1.11)

We call \( \hat{S}(k) \) the Fourier transform of \( S(x - y) \).

### 1.3 Main Theorem.
Our main results are summarized by the following Theorem, which will be proved in the following sections.

**Theorem.** Given \( a_0 \) large enough, there exist two positive constants \( \varepsilon \) and \( \bar{c} \) such that, for all \( |\lambda| \leq \varepsilon \) and \( T' \geq \exp\{-|\bar{c}|\lambda^{-1}\} \), for all \( k \in \mathcal{D} \) such that \( \frac{\beta}{M} \leq \sqrt{k_0^2 + 4 \cos^2 k_+ \cos^2 k_-} \leq \frac{3\pi}{2T} \) and \( H(a_0^2 \sin^2 k_\pm) = 1 \) then
\[ \hat{S}(k) = \frac{(k_0^2 + 4 \cos^2 k_+ \cos^2 k_-)^{n(k_-)}}{-i k_0 + 2 \cos k_+ \cos k_-} (1 + \lambda^2 A_I(k)) \] (1.12)
and for \( k \in \mathcal{D} \) such that \( \frac{\beta}{M} \leq \sqrt{k_0^2 + 4 \cos^2 k_+ \cos^2 k_-} \leq \frac{3\pi}{2T} \) and \( H(a_0^2 \sin^2 k_+ \pm) = 1 \) then
\[ \hat{S}(k) = \frac{(k_0^2 + 4 \cos^2 k_+ \cos^2 k_-)^{n(k_+)}}{-i k_0 + 2 \cos k_+ \cos k_-} (1 + \lambda^2 A_{II}(k)) \] (1.13)
where \(|A_i(k)| \leq c\), where \(c > 0\) is a constant, and \(\eta(k_\pm) = a(k_\pm)\lambda^2 + O(\lambda^3)\) is a critical index expressed by a convergent series with \(a(k_\pm) \geq 0\) a not identically vanishing smooth function.

1.4 Remarks. The above theorem describes the behaviour of the two point Schwinger function up to exponentially small temperatures, \(i.e. T \geq \exp\left(-\tilde{c}|\lambda|^{-1}\right)\); the constant \(\tilde{c}\) is essentially given by the second order terms of the perturbative expansion. A straightforward consequence of (1.12), (1.13) is that the wave function renormalization is \(Z = 1 + O(\lambda^2 \log \beta)\), which means that the half-filled Hubbard model with cut-off is a marginal Fermi liquid up to exponentially small temperatures. From (1.12), (1.13) we see that the behaviour of the Schwinger function close to the Fermi surface is anomalous and described by critical indices which are functions of the projection of the momentum on the Fermi surface. Critical indices which are momentum dependent were found for the same model also in [FSL] by heuristic bosonization methods. The presence of the critical indices makes the Schwinger function quite similar to the one for \(d = 1\) interacting spinless fermionic systems, characterized by Luttinger liquid behaviour (see for instance [A]). However an important difference is that the critical exponent \(\eta\) in a Luttinger liquid is a number, while here is a function of the momenta. Another crucial difference is that in a Luttinger liquid \(S(k) \simeq \tilde{g}(k)|k|^\eta\), with \(\eta = a\lambda^2 + O(\lambda^3)\) up to \(T = 0\); hence a Luttinger liquid is a Marginal Fermi liquid for high enough temperatures but not all the marginal Fermi liquids are Luttinger liquids.

The paper is organized in the following way. In §2 we implement Renormalization Group ideas by writing the Grassman integration in (1.10) as the product of many integrations at different scales. The integration of a single scale leads to new effective interactions, and the renormalization consists in subtracting from the kernels of the effective interaction, which are not dimensionally irrelevant, of the effective interaction their value computed at the Fermi surface. One obtains an expansion for the Schwinger functions as power series of a set of running couplings functions (depending from the momentum on the Fermi surface and the scale). In §3 we prove that this series is convergent if the running coupling functions are small enough; the convergence radius is finite and temperature independent, and this means that the theory is renormalizable. In the proof of convergence one uses the Gram-Hadamard inequality. In §4 we show that the running coupling functions obey to a recursive set of integral equation, called Beta function, and we show that the running coupling functions remain small up to exponentially small temperatures \(T \geq \exp\left(-\tilde{c}|\lambda|^{-1}\right)\). Moreover we show that the wave function renormalization has an anomalous flow, with a non vanishing exponent (contrary to what happens for instance in the case of circular Fermi surfaces), and this essentially concludes the proof of the Theorem. It would be possible to use our beta function to detect (at least numerically) the main instabilities of the system at very low temperatures. At the moment, this kind of numerical analysis was done for this model only in [ZYD] in the parquet approximations, with no control on higher orders which are simply neglected. Finally in §5 we compare the Marginal Fermi liquid behaviour we find in this model with the Luttinger liquid behaviour, and we discuss briefly what happens in the Hubbard model with cut-off close to half filling.

It is very likely that the half-filled Hubbard model with cut-off can work as a paradigm for a large class of systems, in which the Fermi surface is flat or almost flat but there are no Van Hove singularities. Marginal Fermi liquid behaviour can be surely found in the Hubbard model with cut-off and close to half-filling, up to temperatures above the inverse of the radius of curvature of the Fermi surface. Another model in which one could possibly find Marginal Fermi liquid behaviour is the anisotropic Hubbard model introduced in [Sh] with dispersion relation \(\cos k_1 + t \cos k_2\), with \(t = 1 + \varepsilon\). Such model has a Fermi surface with no van Hove singularities and four "almost" flat and parallel pieces, and one can expect \(Z = 1 + O(\lambda^2 \log(|\varepsilon|) \log \beta)\) for \(\beta \leq O(\min[\varepsilon^{-1}, \exp(|\tilde{c}|\lambda)])\). Another interesting question is the possibility of Marginal Fermi liquid behaviour in the Hubbard model close to half-filling.

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(with no cut-off). At half-filling it is believed $Z \simeq 1 + O(\lambda^2 \log^2 \beta)$, so a different behaviour with respect to Marginal Fermi liquid behaviour. A renormalization group analysis for this problem was begun in [R], and it was proved the convergence of the series not containing subgraphs with two external lines for $T \geq \exp\{-|c_0|\lambda\}^{2/3}$.

2. Renormalization Group analysis

2.1 The scale decomposition. As the spin index will play no role in the following analysis (on the contrary it is expected to have an important role at lower temperatures) we simply omit it. The cut-off function $\chi(k)$ defined in (1.4) has a support in the $k$ space which is given by four disconnected regions, each one containing only one flat side of the Fermi surface. It is natural then to write each Grassman variable as a sum of four independent Grassman variables, with momentum $\tilde{k}$ having value in one of the four disconnected regions; each field will be labeled by a couple of indices, $\sigma = I, II$ and $\omega = \pm 1$, so that each field has spatial momenta with values in the region containing $(\omega p_F, 0)$ if $\sigma = I$ or $(0, \omega p_F)$ if $\sigma = II$. We write then Grassman integration as

$$ \int P(d\psi)F(\psi) = \int \prod_{\sigma=I,II} \prod_{\omega=\pm 1} P_{\sigma,\omega}(d\psi)F(\sum_{\sigma=I,II} \sum_{\omega=\pm 1} \psi_{\sigma,\omega}) \quad (2.1) $$

where $F$ is any monomial, $\omega = \pm 1$ and

$$ \int P_{I,\omega}(d\psi) \hat{\psi}_{I,\omega, k_0+\omega p_F, I}^{\omega} \hat{\psi}_{I,\omega', k_0+\omega' p_F, I}^{\omega'} = \delta_{\omega,\omega'} \delta_{k_0,k_0'} H(a_0^2 \sin^2 k_0) \frac{C^{-1}_\omega(k_0, k_0', k_0, k_0')}{i k_0 + 2 \omega \sin k_0' \cos k_0} \quad (2.2) $$

$$ \int P_{II,\omega}(d\psi) \hat{\psi}_{II,\omega, k_0+\omega p_F, II}^{\omega} \hat{\psi}_{II,\omega', k_0+\omega' p_F, II}^{\omega'} = \delta_{k_0,k_0'} \delta_{\omega,\omega'} H(a_0^2 \sin^2 k_0) \frac{C^{-1}_\omega(k_0, k_0', k_0, k_0')}{i k_0 + 2 \omega \sin k_0' \cos k_0} \quad (2.3) $$

where

$$ C^{-1}_\omega(k_0, k_0', k_0, k_0') = \theta(\omega k_0' + p_F) H(\sqrt{k_0^2 + 4 \sin^2 k_0' \cos k_0}) \quad (2.4) $$

$$ C^{-1}_\omega(k_0, k_0', k_0, k_0') = \theta(\omega k_0' + p_F) H(\sqrt{k_0^2 + 4 \sin^2 k_0' \cos k_0}) \quad (2.5) $$

and $p_{F,\sigma}$ is defined such that $p_{F, I} = (\bar{\pi}, 0)$ and $p_{F, II} = (0, \bar{\pi})$; moreover $p_F = \frac{\pi}{2}$ and $\bar{k} = \bar{k} + \omega p_{F, \sigma}$ ( $\bar{k}$ is the momentum measured from the Fermi surface).

It is convenient, for clarity reasons, to start by studying the “free energy” of the model, defined as

$$ -\frac{1}{L^2 \beta} \log \int P(d\psi) e^{-\mathcal{V}} \quad (2.6) $$

where, calling with a slight abuse of notation $\hat{\psi}_{\sigma,\omega, k' + \omega p_{F, \sigma}} \equiv \hat{\psi}_{\sigma,\omega, k'}$, $\mathcal{V}$ is equal to

$$ \lambda \sum_{\omega_1, \ldots, \omega_4} \sum_{\sigma_1, \ldots, \sigma_4} \sum_{I,II} \int dk_1 \cdots dk_4 \delta(\sum_{i=1}^4 \epsilon_i(k_i + \omega_i p_{F, \sigma_i})) \hat{\psi}_{\sigma_1, \omega_1, k_1}^{\omega_1} \hat{\psi}_{\sigma_2, \omega_2, k_2}^{\omega_2} \hat{\psi}_{\sigma_3, \omega_3, k_3}^{\omega_3} \hat{\psi}_{\sigma_4, \omega_4, k_4}^{\omega_4} \quad (2.7) $$

where $\int dk = \frac{1}{L^2 \beta} \sum_{k}$ and $\delta(k - k') = L^2 \beta \delta(k - k')$.

We will evaluate the Grassman integral (2.6) by a multiscale analysis based on (Wilsonian) Renormalization Group ideas. The starting point is the following decomposition of the cut-off functions (2.4), (2.5)

$$ H(\sqrt{k_0^2 + 4 \cos^2 \bar{k}_0 \sin^2 k_0'}) = \sum_{k = -\infty}^0 \hat{f}_k(\sqrt{k^2_0 + 4 \cos^2 \bar{k}_0 \sin^2 k_0'}) \equiv \sum_{k = -\infty}^0 f_k(k_0, k_0', \bar{k}_0) \quad (2.8) $$
with \( \tilde{f}(t) = H(\gamma^{-kt}) - H(\gamma^{-(k+1)t}) \) is a smooth compact support function, with support \( \gamma^{k-1} \leq |t| \leq \gamma^{k+1} \); moreover:

a) \( \tilde{k}_\sigma = k_- \) if \( \sigma = I \) and \( \tilde{k}_\sigma = k_+ \) if \( \sigma = II \); \( \tilde{k}_\sigma \) is the projection of \( \tilde{k} \) in the direction parallel to the Fermi surface.

b) \( \tilde{k}'_\sigma = k'_+ \) if \( \sigma = I \) and \( \tilde{k}'_\sigma = k'_- \) if \( \sigma = II \); \( \tilde{k}'_\sigma + \omega p_{F,\sigma} \) is the projection of \( \tilde{k} \) in the direction normal to the Fermi surface.

For each \( \sigma \), the function \( f_k(k_0, k'_\sigma, \tilde{k}_\sigma) \) has a support in two regions of thickness \( O(\gamma^h) \) around each flat side of the Fermi surface, at a distance \( O(\gamma^h) \) from it. We will assume \( L = \infty \) for simplicity and it follows that there is a \( h_\beta = O(\log \beta) \) such that \( f_k = 0 \) for \( k < h_\beta \), while \( f_k \) is not identically vanishing for \( k \geq h_\beta \).

The integration of (2.6) will be done iteratively integrating out the fields with momenta closer and closer to the Fermi surface. We will prove by induction that it is possible to define a sequence of functions \( Z_h(k'_\sigma, \omega) \) and a sequence of effective potentials \( V^{(h)} \) such that

\[
\int P_1(d\psi)P_{II}(d\psi)e^{-V} = e^{-\frac{1}{2}Z_h(k'_\sigma, \omega)} \int P_{Z_h}(d\psi^{(\leq h)})P_{Z_h,II}(d\psi^{(\leq h)})e^{-V^{(h)}(\sqrt{Z_h}(\leq h))}. \tag{2.9}
\]

where \( E_h \) is a constant and \( \sqrt{Z_h}(\leq h) \) equal to

\[
\langle \sqrt{Z_h(k'_1, \omega)} \psi_{I,1,k'}^{(\leq h)}, \sqrt{Z_h(k'_1, \omega)} \psi_{I,1,k'}^{(\leq h)} \rangle = \sum_{n=1}^{\infty} f(k_0, k'_\sigma, \tilde{k}_\sigma).
\tag{2.10}
\]

The \( \theta \)-function in (2.11) can be omitted by the definition of the variables \( k'_\sigma \).

We define \( \tilde{k}_\sigma, \omega = (\tilde{\sigma}, \omega p_{F, \sigma}, k_-) \) if \( \sigma = I \) and \( \tilde{k}'_\sigma, \omega = (\tilde{\sigma}, k_+, \omega p_{F, \sigma}) \) if \( \sigma = II \); moreover \( \tilde{k}_\sigma, \omega = (\tilde{\sigma}, 0, k_-) \) if \( \sigma = I \) and \( \tilde{k}'_\sigma, \omega = (\tilde{\sigma}, k_+, 0) \) if \( \sigma = II \); moreover we call \( \tilde{k}'_\sigma, \omega = (-\tilde{\sigma}, 0, k_-) \) if \( \sigma = I \) and \( \tilde{k}'_\sigma, \omega = (-\tilde{\sigma}, k_+, 0) \) if \( \sigma = II \).

If \( \varepsilon = \pm \)

\[
V^{(h)}(\psi^{(\leq h)}) = \sum_{n=1}^{\infty} \sum_{\omega_1 \ldots \omega_{2n}} \sum_{\sigma_1 \ldots \sigma_{2n}} \sum_{\varepsilon_1 \ldots \varepsilon_{2n}} \int d\psi^{(\leq h)}_i \prod_{i=1}^{2n} \psi_{\sigma_i, \omega_i, k'_i}^{(h)} \tilde{W}^{(h)}_{2n}(k'_1, \ldots, k'_{2n-1}) \tag{2.13}
\]

where

\[
\tilde{W}^{(h)}_{2n}(k_1 \ldots k_{2n-1}) = \tilde{W}^{(h)}_{2n}(k'_1 + \omega_1 p_{F, \sigma_1} \ldots k_{2n-1} + \omega_{2n-1} p_{F, \sigma_{2n-1}}) = \tilde{W}^{(h)}_{2n}(k'_1 \ldots k'_{2n-1}) \tag{2.14}
\]

2.2 The renormalization procedure. Let us show that (2.9) is true for \( h = 1 \), assuming that it is true for \( h \). We define an \( \mathcal{L} \) operator acting linearly on the kernels of the effective potential (2.13):

1) \( \mathcal{L} \tilde{W}^{(h)}_{2n} = 0 \) if \( n \geq 2 \)

2) If \( n = 1 \)

\[
\mathcal{L} \tilde{W}^{(h)}(k') = \frac{1}{2} [W^{(h)}(\tilde{k}'_\sigma, \omega) + \tilde{W}^{(h)}(\tilde{k}'_\sigma, \omega)] + k_0 \partial_{k_0} \tilde{W}^{(h)}(\tilde{k}'_\sigma, \omega) + \sin k'_\sigma \partial_{k'_\sigma} \tilde{W}^{(h)}(\tilde{k}'_\sigma, \omega) \tag{2.15}
\]
where \(\partial_{k_0}\) means the discrete derivative and \(\partial_{\sigma} = \partial_{k_+}\) is \(\sigma = I\) and \(\partial_{\bar{\sigma}} = \partial_{k_-}\) is \(\sigma = II\). We will prove in §4 that \([\hat{W}^h_2(\tilde{k}_{\sigma,\omega}^*) + \hat{W}^h_2(\tilde{k}_{\sigma,\omega}^*)] = 0\).

3) If \(n = 2\)

\[
\mathcal{L}\hat{W}^h_4(k_1', k_2', k_3') = \hat{W}^h_4(\tilde{k}_{\sigma_1,\omega_1}', \tilde{k}_{\sigma_2,\omega_2}', \tilde{k}_{\sigma_3,\omega_3}')
\]

(2.16)

Calling \(\partial_{\bar{\sigma}}\hat{W}^h_2(\tilde{k}_{\bar{\sigma},\omega}) = -ia_h(\tilde{k}_{\bar{\sigma},\omega})\), \(\partial_{\bar{\sigma}}\hat{W}^h_2(\tilde{k}_{\bar{\sigma},\omega}) = 2\omega \cos k_\bar{\sigma} z_h(\tilde{k}_{\bar{\sigma},\omega})\) and

\[
l_h(\tilde{k}_{\sigma_1,\omega_1}', \tilde{k}_{\sigma_2,\omega_2}', \tilde{k}_{\sigma_3,\omega_3}') = \hat{W}^h_4(\tilde{k}_{\sigma_1,\omega_1}', \tilde{k}_{\sigma_2,\omega_2}', \tilde{k}_{\sigma_3,\omega_3}')
\]

(2.17)

we can write

\[
\mathcal{L}^h = \sum_{\sigma = I, II} \int d\tilde{k}'[z_h(\tilde{k}_{\sigma,\omega})2\omega \cos k_\sigma \sin k_\sigma - ik_0a_h(\tilde{k}_{\sigma,\omega})] \tilde{\psi}^+(\leq h) \tilde{\psi}^-(\leq h) + \sum_{\{\}, \{\}} \int d\tilde{k}'_{1,2,3} h_l(\tilde{k}_{\sigma_1,\omega_1}', \tilde{k}_{\sigma_2,\omega_2}', \tilde{k}_{\sigma_3,\omega_3}') \tilde{\psi}^+(\leq h) \tilde{\psi}^+(\leq h) \tilde{\psi}^-(\leq h) \tilde{\psi}^-(\leq h) \delta(\sum_1 \varepsilon_i(k_i' + p_{F,\sigma}))
\]

(2.18)

We write the r.h.s. of (2.9) as

\[
\int P_{I,\bar{\cal Z}_h}(d\psi^{(\leq h)}) \int P_{II,\bar{\cal Z}_h}(d\psi^{(\leq h)}) e^{-\mathcal{L}^h(\sqrt{Z_h}\psi^{(\leq h)})} = \mathcal{R}^h(\sqrt{Z_h}\psi^{(\leq h)})
\]

(2.19)

with \(\mathcal{R} = 1 - \mathcal{L}\).

2.3 Remark 1. The non trivial action of \(\mathcal{R}\) on the kernel with \(n = 2\) can be written as

\[
\mathcal{R}\hat{W}^h_4(k_1', k_2', k_3') = [\hat{W}^h_4(k_1', k_2', k_3') - \hat{W}^h_4(\tilde{k}_{\sigma_1,\omega_1}', k_2', k_3')]
\]

\[
+ [\hat{W}^h_4(\tilde{k}_{\sigma_1,\omega_1}', k_2', k_3') - \hat{W}^h_4(\tilde{k}_{\sigma_2,\omega_2}', k_2', k_3')]
\]

\[
+ [\hat{W}^h_4(\tilde{k}_{\sigma_1,\omega_1}', \tilde{k}_{\sigma_2,\omega_2}', k_3') - \hat{W}^h_4(\tilde{k}_{\sigma_1,\omega_1}', k_2', \tilde{k}_{\sigma_3,\omega_3}')]
\]

(2.20)

The first addend can be written as, if \(\sigma_1 = I\) (say), in the limit \(L \to \infty\)

\[
(k_{0,1} - \frac{\pi}{\beta}) \int_0^L dt \partial_{k_{0,1}} \hat{W}^h_4(\frac{\pi}{\beta} + t(k_{0,1} - \frac{\pi}{\beta}), k_1', k_2', k_3') +
\]

\[
k_{1,1}' \int_0^L dt \partial_{k_{1,1}'} \hat{W}^h_4(0, t(k_{1,1}' + k_{1,1}'), k_2', k_3')
\]

(2.21)

The factors \(k_{0,1} - \pi/\beta\) and \(k_{1,1}'\) are \(O(\gamma^h)\), for the compact support properties of the propagator associated to \(\psi^{(\leq h)}_{I,\omega_1,k}'\), with \(h' \leq h\), while the derivatives are dimensionally \(O(\gamma^{-h+1})\); hence the effect of \(\mathcal{R}\) is to produce a factor \(\gamma^{h'-h-1} \leq 1\). Similar considerations can be done for the other addenda and for the action of \(\mathcal{R}\) on the \(n = 1\) terms.

Remark 2. From (2.16) we see that the effect of the \(\mathcal{L}\) operation is to replace in \(W^h_2(k)\) the momentum \(\tilde{k}\) with its projection on the closest flat side of the Fermi surface. Hence the fact that the propagator is singular over an extended region (the Fermi surface) and not simply in a point has the effect that the renormalization point cannot be fixed but it must be left moving on the Fermi surface.

2.4 The anomalous integration. In order to integrate the field \(\psi^{(h)}\) we can write

\[
\int P_{I,\bar{\cal Z}_h}(d\psi^{(\leq h)}) \int P_{II,\bar{\cal Z}_h}(d\psi^{(\leq h)}) e^{-\mathcal{L}^h(\sqrt{Z_h}\psi^{(\leq h)}) - \mathcal{R}^h(\sqrt{Z_h}\psi^{(\leq h)})} =
\]

(2.22)
\[ \int P_{1, Z_{h-1}}(d\psi^{(\leq h)}) \int P_{11, Z_{h-1}}(d\psi^{(\leq h)}) e^{-\mathcal{L}V^h(\sqrt{Z_{h-1}\psi^{(\leq h)}}) - R\mathcal{V}^h(\sqrt{Z_{h-1}\psi^{(\leq h)}})} \]

where \( P_{\sigma, Z_{h-1}}(d\psi^{(\leq h)}) \) is the fermionic integration with propagator

\[ \frac{1}{Z_{h-1}(k')} \frac{H(a_0^2 \sin^2 \tilde{k}_\sigma) C_{h-1}^{-1}(k_0, k', \tilde{k}_\sigma)}{-i k_0 + 2 \omega \cos k_\sigma \sin k'_\sigma} \]  

and

\[ Z_{h-1}(k') = Z_h(\tilde{k}'_{\sigma,\omega}) \left[ 1 + H(a_0^2 \sin^2 \tilde{k}_\sigma) C_{h}^{-1}(k_0, k', \tilde{k}_\sigma) a_h(\tilde{k}'_{\sigma,\omega}) \right] \]

Moreover

\[ \mathcal{L}^h = \mathcal{L}^h - \sum_{\sigma=1,11} \int dk' z_h(\tilde{k}'_{\omega,\sigma}) [2\omega \cos \tilde{k}_\sigma \sin k'_\sigma - i k_0] \psi^+_{\omega,\sigma}^h \psi^-_{\omega,\sigma}^h. \]

We rescale the fields by rewriting the r.h.s. of (2.22) as

\[ \int P_{1, Z_{h-1}}(d\psi^{(\leq h)}) \int P_{11, Z_{h-1}}(d\psi^{(\leq h)}) e^{-\mathcal{L}V^h(\sqrt{Z_{h-1}\psi^{(\leq h)}}) - R\mathcal{V}^h(\sqrt{Z_{h-1}\psi^{(\leq h)}})} \]

where

\[ \mathcal{L}^h = \int dk \sum_{\sigma=1,11} \left[ \delta_h(\tilde{k}'_{\sigma,\omega}) 2\omega \cos \tilde{k}_\sigma \sin k'_\sigma \right] \psi^+_{\omega,\sigma} \psi^-_{\omega,\sigma} + \sum_{\sigma_1,\ldots,\sigma_4=1,11} \int dk'_1 \ldots dk'_4 \]

\[ \lambda_h(\tilde{k}'_{\omega,\sigma_1,\omega_2}, \tilde{k}'_{\omega_2,\omega_3}, \tilde{k}'_{\omega_3,\omega_4}) \psi^+_{\omega_1,\sigma_1} \psi^+_{\omega_2,\sigma_2} \psi^-_{\omega_3,\sigma_3} \psi^-_{\omega_4,\sigma_4} \delta^4(\sum_i \varepsilon_i) (k'_i + \bar{F}_F) \]

and

\[ \delta_h(\tilde{k}'_{\omega,\sigma}) = \frac{Z_h(\tilde{k}'_{\omega,\sigma})}{Z_{h-1}(k'_\sigma,\omega)} (z_h(\tilde{k}'_{\omega,\sigma}) - a_h(\tilde{k}'_{\sigma,\omega})) \]

\[ \lambda_h(\tilde{k}'_{\sigma_1,\omega_1}, \tilde{k}'_{\sigma_2,\omega_2}, \tilde{k}'_{\omega_3,\omega_3}, k'_\sigma) = \left[ \prod_{i=1}^4 \frac{Z_h(k'_\sigma,\omega)}{Z_{h-1}(k'_\sigma,\omega)} \right] \lambda_h(\tilde{k}'_{\sigma_1,\omega_1}, \tilde{k}'_{\sigma_2,\omega_2}, \tilde{k}'_{\sigma_3,\omega_3}) \]

We will call \( \delta_h \) and \( \lambda_h \) running coupling functions; the above procedure allow to write a recursive equation for them, see § 5.

Then we write

\[ \int P_{1, Z_{h-1}}(d\psi^{(\leq h-1)}) \int P_{11, Z_{h-1}}(d\psi^{(\leq h-1)}) \int P_{1, Z_{h-1}}(d\psi^{(h)}) \int P_{11, Z_{h-1}}(d\psi^{(h)}) e^{-\mathcal{L}V^h(\sqrt{Z_{h-1}\psi^{(\leq h-1)}}) - R\mathcal{V}^h(\sqrt{Z_{h-1}\psi^{(\leq h-1)}})} \]

and the propagator of \( P_{\sigma, Z_{h-1}}(d\psi) \) is

\[ \delta^h_{\omega,\sigma}(k') = H(a_0^2 \sin^2 \tilde{k}_\sigma) \frac{1}{Z_{h-1}(k'_\sigma,\omega)} \frac{f_h(k_0, k'_\sigma, \tilde{k}_\sigma)}{-i k_0 + 2 \omega \cos k_\sigma \sin k'_\sigma} \]

and

\[ f_h(k_0, k'_\sigma, \tilde{k}_\sigma) = \frac{C_{h-1}(k_0, k'_\sigma, \tilde{k}_\sigma)}{Z_{h-1}(k'_\sigma,\omega)} - \frac{C_{h-1}^{-1}(k_0, k'_\sigma, \tilde{k}_\sigma)}{Z_{h-1}(k'_\sigma,\omega)} \]

with \( H(a_0^2 \sin^2 \tilde{k}_\sigma)f_h(k_0, k'_\sigma, \tilde{k}_\sigma) \) having the same support that \( H(a_0^2 \sin^2 \tilde{k}_\sigma)f_h(k_0, k'_\sigma, \tilde{k}_\sigma) \).

We integrate then the field \( \psi^h \) and we get

\[ e^{-L^2 \beta E_{h-1}} \int P_{1, Z_{h-1}}(d\psi^{(\leq h-1)}) \int P_{11, Z_{h-1}}(d\psi^{(\leq h-1)}) e^{-\mathcal{V}^{h-1}(\sqrt{Z_{h-1}\psi^{(\leq h-1)}})} \]

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and the procedure can be iterated.

We will see in the following section that if the running coupling functions are small

$$\sup_{k \geq h} \sup_{k \geq h} \left| \delta_k (k^0_{n,\omega}) \right| \leq 2|\lambda| \quad \sup_{k \geq h} \frac{Z_{k-1} (k^0_{n,\omega})}{Z_k (k^0_{n,\omega})} \leq e^{2|\lambda|}$$

then the effective potential is given by a convergent series. In §4 we will show that up to exponentially small temperatures this is indeed true.

3. Analyticity of the effective potential

3.1 Coordinate representation. It is convenient to perform bounds to introduce the variables $\psi^{\tilde{\sigma}}_{x,\omega,\epsilon} = e^{i\epsilon \omega \tilde{p}_{F,\sigma} x} \psi^{\tilde{\sigma}}_{x,\omega,\epsilon}$, or more explicitly

$$\psi^{\tilde{\sigma}}_{x,\omega,1} = e^{i \epsilon \omega \tilde{p}_{F} x} \psi^{\tilde{\sigma}}_{x,\omega,1} \quad \psi^{\tilde{\sigma}}_{x,\omega,II} = e^{i \epsilon \omega \tilde{p}_{F} x} \psi^{\tilde{\sigma}}_{x,\omega,II}$$

and the propagators of such fields is

$$\tilde{g}^{h}_{\omega,\sigma}(x-y) = \int \frac{dk}{Z_{h-1}(k_{\sigma})} e^{-ik(x-y)} h(k_{\sigma}) e^{-ik_{\sigma}} 2 \sin k_{\sigma}$$

It is easy to prove, by integration by parts, that for any integer $N$, for $L \to \infty$

$$|\partial_{x_0}^{n_0} \partial_{x_+}^{n_+} \partial_{x_-}^{n_-} \tilde{g}^{h}_{\omega,\sigma}(x-y)| \leq \frac{C_{n_0, n_+, n_-}}{1 + [\gamma^h |d(x_0 - y_0)| + \gamma^h |x_+ - y_+| + |x_- - y_-|]^N}$$

$$|\partial_{x_0}^{n_0} \partial_{x_+}^{n_+} \partial_{x_-}^{n_-} \tilde{g}^{h}_{I,\omega,\sigma}(x-y)| \leq \frac{C_{n_0, n_+, n_-}}{1 + [\gamma^h |d(x_0 - y_0)| + |x_+ - y_+| + \gamma^h |x_- - y_-|]^N}$$

where $d(x_0) = \frac{\tilde{p}}{2} \sin \frac{\tilde{p}_{F}}{2}.$

Proof. The above formula can be derived by integration by parts; note that, if instance $\sigma = I$

$$\partial_{k_-} \frac{1}{-ik_0 + 2\omega \sin k_{+}^f \cos k_{-}} \frac{1}{(-ik_0 + 2\omega \sin k_{+}^f \cos k_{-})^2} 2\omega \sin k_{+}^f \sin k_{-}$$

which is $O(g^{-h})$; in the same way the $n$-th derivative with respect to $k_{-}$ is still $O(g^{-h})$. On the other hand $\partial_{k_0} \partial_{k_+}^{n_+}$ is bounded by $g^{-h-n_0 h - n_+}$, finally the integration gives a volume factor $\gamma^{2h}$. 

We define

$$W^{h}_{2n}(x_1, ..., x_{2n}) = \frac{1}{(L^2 \beta)^{2n}} \sum_{k'_1, ..., k'_{2n}} e^{-\sum_{i=1}^{2n} \epsilon_i k'_i x_i} W^{h}_{2n}(k'_1, ..., k'_{2n-1}) \delta(\sum_{i} \epsilon_i (k'_i + \omega_i \tilde{p}_{F,\sigma_i}))$$

Hence (2.13) can be written as

$$\begin{align*}
\psi^{(h)}(\psi^{(h)}) &= \sum_{n=1}^{\infty} \sum_{\omega_1, ..., \omega_{2n}} \sum_{\sigma_1, ..., \sigma_{2n}} \sum_{\epsilon_1, ..., \epsilon_{2n}} \int d\bar{x}_1 ... d\bar{x}_{2n} \int \left[ \prod_{i=1}^{2n} \frac{W^{(h)}_{2n}(\bar{x}_1, ..., \bar{x}_{2n})}{W^{(h)}_{2n}(x_1, ..., x_{2n})} \right] \end{align*}$$
We now discuss the action of the operator $\mathcal{L}$ and $\mathcal{R} = 1 - \mathcal{L}$ on the effective potential in the $x$-space representation. Noting that from (3.6), if $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4 = +$

$$W^h_i(x_1, x_2, x_3, x_4) = e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} W_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$ (3.8)

we can write the action of $\mathcal{R}$ (2.16) as

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$

$$\mathcal{R} \int \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} \tilde{\psi}_{i,x_i,\sigma_i,\omega_i} e^{i\tilde{e}_4(x_1+pF, x_1) + i\rho_2 x_1} \tilde{W}_i^h(x_1-x_4, x_2-x_4, x_3-x_4)$$
where

\[ T_{x_1, x_2, \omega} = \psi_{x_1, x_2, \omega} - \psi_{x_1, x_2} \left( x_0, 2 - x_0, 1 \right) \partial_{x_0} \psi_{x_1, x_2, \omega} - \left( x_2, 2 - x_2, 1, \omega \right) \partial_{x_2} \psi_{x_1, x_2, \omega} \]

(3.15)

\[ \partial_{x_0} = \partial_{x_+} \text{ if } \sigma = I \text{ and } \partial_{x_2} = \partial_{x_-} \text{ if } \sigma = II. \]

In this case the "gain" produced by the \( R \) operation is \( \gamma^{-2(h-h')} \).

We can write the local part of the effective potential (2.18) in the following way

\[ \mathcal{L}V^h = \sum_{\sigma_1, \sigma_2} \int dx_1 dx_2 [\omega \delta_{h, \omega} ((\hat{x}_1 - \hat{x}_2)_{\sigma_1})] e^{i \int \hat{x}_2 (\omega \hat{p}_{\sigma_1} - \omega \hat{p}_{\sigma_2})} \psi_{\omega, \sigma_1, x_1} \partial_{\sigma_2} \psi_{\omega, \sigma_2, x_2} + \]

\[ + \sum_{\sigma_1, \ldots, \sigma_4 = I, II} \int dx_1 dx_2 dx_3 dx_4 \lambda_{h, \omega_1, \ldots, \omega_4} ((\hat{x}_1 - \hat{x}_4)_{\sigma_1}, (\hat{x}_2 - \hat{x}_4)_{\sigma_2}, (\hat{x}_3 - \hat{x}_4)_{\sigma_3}) e^{i \int \hat{x}_4 (\omega \hat{p}_{\sigma_1} + \omega \hat{p}_{\sigma_2} - \omega \hat{p}_{\sigma_3} - \omega \hat{p}_{\sigma_4})} \psi_{x_1, x_2, x_3, x_4}^+ \psi_{x_1, x_2, x_3, x_4} \]

(3.16)

where \( \delta_h \) is the Fourier transform of \( \delta_{h, \omega} \) with respect to \( \hat{k}_\sigma \) and \( (\hat{x}_i - \hat{x}_j)_\sigma = x_{-i} - x_{-j} \) if \( \sigma = I \) and \( (\hat{x}_i - \hat{x}_j)_\sigma = x_{+i} - x_{+j} \) if \( \sigma = II \); moreover \( \hat{x}_\sigma = x_+ \) if \( \sigma = I \) and \( \hat{x}_\sigma = x_- \) if \( \sigma = II \).

### 3.2 Tree expansion

By using iteratively the “single scale expansion” we can write the effective potential \( V^{(h)}(\psi^{(\leq h)}) \), for \( h \leq 0 \), in terms of a tree expansion. For a tutorial introduction to the tree formalism we will refer to the review [GM].

![Tree diagram](image)

We need some definitions and notations.

1) Let us consider the family of trees which can be constructed by joining a point \( r \), the root, with an ordered set of \( n \geq 1 \) points, the end points of the unlabeled tree (see Fig. 1), so that \( r \) is not a branching point. \( n \) will be called the order of the unlabeled tree and the branching points will be called the non trivial vertices. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol \( < \) to denote the partial order.

Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with \( n \) end-points is bounded by \( 4^n \).

We shall consider also the labeled trees (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label \( h \leq -1 \) with the root and we denote \( T_{h, n} \) the corresponding set of labeled trees with \( n \) endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in \( [h, 1] \), and we represent any tree \( \tau \in T_{h, n} \) so that, if \( v \) is an endpoint or a non trivial vertex, it is contained in a vertical line with index \( h_v > h \), to be called the scale of \( v \), while the root is on the line with index \( h \). There is the constraint that, if \( v \) is an endpoint, \( h_v > h + 1 \).
The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of \( \tau \) will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if \( v_1 \) and \( v_2 \) are two vertices and \( v_1 < v_2 \), then \( h_{v_1} < h_{v_2} \). We will call \( s_v \) the number of subtrees coming out from \( v \).

Moreover, there is only one vertex immediately following the root, which will be denoted \( v_0 \) and can not be an endpoint; its scale is \( h + 1 \).

3) To each end-point of scale \(+1\) we associate \( V \) (1.11). With each endpoint \( v \) of scale \( h_v \leq 0 \) we associate one of the two terms appearing in (3.16), with coupling \( \lambda_{h_v} \) or \( \delta_{h_v} \).

Moreover, we impose the constraint that, if \( v \) is an endpoint and \( h_v \leq 0 \), \( h_v = h_{v'} + 1 \), if \( v' \) is the non trivial vertex immediately preceding \( v \).

4) We introduce a *field label* \( f \) to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint \( v \) will be called \( I_v \). Analogously, if \( v \) is not an endpoint, we shall call \( I_v \) the set of field labels associated with the endpoints following the vertex \( v \); \( x(f), \varepsilon(f) \) and \( \omega(f) \) will denote the space-time point, the \( \varepsilon \) index and the \( \omega \) index, respectively, of the field variable with label \( f \).

If \( h_v \leq 0 \), one of the field variables belonging to \( I_v \) carries also a derivative \( \partial_\sigma \) if the corresponding local term is of type \( \delta \), see (3.16). Hence we can associate with each field label \( f \) an integer \( m(f) \in \{0,1\} \), denoting the order of the derivative.

If \( h \leq -1 \), the effective potential can be written in the following way:

\[
\Psi^{(h)}(\psi^{(\leq h)}) + L \beta E_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) ,
\]

(3.17)

where, if \( v_0 \) is the first vertex of \( \tau \) and \( \tau_1, \ldots, \tau_s \) \((s = s_{v_0})\) are the subtrees of \( \tau \) with root \( v_0 \), \( \mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) \) is defined inductively by the relation

\[
\mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}[\mathcal{V}^{(h+1)}(\tau_1, \psi^{(\leq h+1)}); \ldots; \mathcal{V}^{(h+1)}(\tau_s, \psi^{(\leq h+1)})] ,
\]

(3.18)

and \( \mathcal{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)}) \)

a) is equal to \( \mathcal{R} \mathcal{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)}) \) if the subtree \( \tau_i \) is not trivial, with \( \mathcal{R} \) defined as acting on kernels according to (3.9) and its analogous for \( n = 1 \);

b) if \( \tau_i \) is trivial and \( h < -1 \), it is equal to \( \mathcal{E} \mathcal{V}^{(h+1)}(\psi^{(\leq h+1)}) \) (3.16) or, if \( h = -1 \), to \( \mathcal{V} \).

\( \mathcal{E}^T_{h+1} \) denotes the truncated expectation with respect to the measure \( \prod_{\sigma = I,II} P_{\sigma, \mathcal{Z}_h}(d\psi^{(h+1)}) \), that is

\[
\mathcal{E}^T_{h+1}(X_1; \ldots; X_p) \equiv \frac{\partial^p}{\partial \lambda_1 \ldots \partial \lambda_p} \log \int \prod_{\sigma = I,II} P_{\sigma, \mathcal{Z}_h}(d\psi^{(h+1)}) e^{\lambda_1 X_1 + \ldots + \lambda_p X_p} \bigg|_{\lambda_i = 0} .
\]

(3.19)

We write (3.18) in a more explicit way. If \( h = -1 \), the r.h.s. of (3.18) can be written in the following way. Given \( \tau \in \mathcal{T}_{-1,n} \), there are \( n \) endpoints of scale \( 1 \) and only another one vertex, \( v_0 \), of scale \( 0 \); let us call \( v_1, \ldots, v_n \) the endpoints. We choose, in any set \( I_v \), a subset \( Q_{v_i} \) and we define \( P_{v_0} = \cup_i Q_{v_i} \). We have

\[
\Psi^{(-1)}(\tau, \psi^{(\leq -1)}) = \sum_{P_{v_0}} \Psi^{(-1)}(\tau, P_{v_0}) ,
\]

(3.20)

\[
\Psi^{(-1)}(\tau, P_{v_0}) = \int d\xi \psi^{\leq -1}(P_{v_0}) K^{(0)}_{\tau, P_{v_0}}(\xi) ,
\]

(3.21)
\( K^{(0)}_{\tau,P_v} (x_{v_0}) = \frac{1}{n!} \mathcal{E}^T_0 [\tilde{\psi}^{(0)}(P_v \setminus Q_{v_1}), \ldots, \tilde{\psi}^{(0)}(P_{v_n} \setminus Q_{v_n})] \prod_{i=1}^{n} K^{(1)}_{v_i}(x_{v_i}) , \) (3.22)

where we use the definitions
\[ \tilde{\psi}^{(\leq h)}(P_v) = \prod_{f \in P_v} \partial_{\sigma(f)}^{m(f)} \psi^{(\leq h)}(\varepsilon(f)) , \quad h \leq -1 , \] (3.23)
\[ \tilde{\psi}^{(0)}(P_v) = \prod_{f \in P_v} ^{\chi(f)} , \] (3.24)
\[ K^{(1)}_{v_i}(x_{v_i}) = e^{\sum_{f \in I_{v_i}} \varepsilon_f(x) \omega(f) \bar{\nu}_{P_v}(f)} \chi \quad x_{v_i} = x \] (3.25)

It is not hard to see that, by iterating the previous procedure, one gets for \( \mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) \), for any \( \tau \in \mathcal{T}_{h,n} \), the representation described below. We associate with any vertex \( v \) of the tree a subset \( P_v \) of \( I_v \), the *external fields of \( v \). These subsets must satisfy various constraints. First of all, if \( v \) is not an endpoint and \( v_1, \ldots, v_{s_v} \) are the vertices immediately following it, then \( P_v \subseteq \bigcup_i P_{v_i} \); if \( v \) is an endpoint, \( P_v = I_v \). We shall denote \( Q_{v} \), the intersection of \( P_v \) and \( P_v \); this definition implies that \( P_v = \bigcup_i Q_{v_i} \). The subsets \( P_{v_i} \setminus Q_{v_i} \), whose union \( \mathcal{I}_v \) will be made, by definition, of the *internal fields of \( v \), have to be non empty, if \( s_v > 1 \). Moreover, we associate with any \( f \in \mathcal{I}_v \) a scale label \( h(f) = h_v \). Given \( \tau \in \mathcal{T}_{h,n} \), there are many possible choices of the subsets \( P_v, \ v \in \tau \), compatible with all the constraints; we shall denote \( \mathcal{P}_\tau \) the family of all these choices and \( \mathbf{P} \) the elements of \( \mathcal{P}_\tau \).

Then we can write
\[ \mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}) . \] (3.26)

\( \mathcal{V}^{(h)}(\tau, \mathbf{P}) \) can be represented as
\[ \mathcal{V}^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \tilde{\psi}^{(\leq h)}(P_{v_0}) K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0}) , \] (3.27)
with \( K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0}) \) defined inductively (recall that \( h_{v_0} = h+1 \)) by the equation, valid for any \( v \in \tau \) which is not an endpoint,
\[ K^{(h_v)}_{v}(\mathbf{x}_{v}) = \frac{1}{s_v} \prod_{i=1}^{s_v} [K^{(h_{v_i})}_{v_i}(\mathbf{x}_{v_i})] \mathcal{E}^T_{h_v} [\tilde{\psi}^{(h_v)}(P_{v_1} \setminus Q_{v_1}), \ldots, \tilde{\psi}^{(h_v)}(P_{v_{s_v}} \setminus Q_{v_{s_v}})] , \] (3.28)
where \( \tilde{\psi}^{(h_v)}(P_v) \) is defined as in (3.23), with \( (h_v) \) in place of \( (\leq h_v) \), if \( h_v \leq -1 \), while, if \( h_v = 0 \), it is defined as in (3.24).

Moreover, if \( v \) is an endpoint and \( h_v = 0 \), \( K^{(1)}_{v}(\mathbf{x}_{v}) \) is given by (3.25), otherwise, see (3.16)
\[ K^{(h_v)}_{v}(\mathbf{x}_{v}) = \begin{cases} l_{h_{v_i}}(x_1, x_2, x_3, x_4) & \text{if } v \text{ is of type } \lambda, \\ d_{h_{v_i}}(x_1, x_2) & \text{if } v \text{ is of type } z, \end{cases} \] (3.29)
where
\[ l_{h_{v_i}}(x_1, x_2, x_3, x_4) = e^{i \bar{x} \varepsilon_1(x_1) \tilde{\psi}^{(z)}(x_1)} e^{\Omega_{\bar{x}}(x_1)} \chi_{\Lambda_{x_i}}(x_1, x_2, x_3, x_4) \]
\[ d_{h_{v_i}}(x_1, x_2) = e^{i \bar{x} \varepsilon_1(x_1) \tilde{\psi}^{(z)}(x_1)} e^{\Omega_{\bar{x}}(x_1)} \chi_{\Lambda_{x_i}}(x_1, x_2) \]

If \( v \) is not an endpoint,
\[ K^{(h_v+1)}_{v}(\mathbf{x}_{v}) = \mathcal{R} K^{(h_v+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v}) , \] (3.30)
where \( \tau_i \) is the subtree of \( \tau \) starting from \( v \) and passing through \( v_i \) (hence with root the vertex immediately preceding \( v \)), \( \mathbf{P}^{(i)} \) and is the restrictions to \( \tau_i \) of \( \mathbf{P} \). The action of \( \mathcal{R} \) is defined using the representation (3.9) of the \( \mathcal{R} \) operation.

(3.26) is not the final form of our expansion, since we further decompose \( \mathcal{V}^{(h)}(\tau, \mathbf{P}) \), by using the following representation of the truncated expectation in the r.h.s. of (3.28). Let
us put \( s = s_v, P_i \equiv P_v \setminus Q_v \); moreover we order in an arbitrary way the sets \( P_i^\pm \equiv \{ f \in P, \sigma(f) = \pm \} \), we call \( f_{ij} \) their elements and we define \( x^{(i)} = \cup_{f \in P_i^+} x(f), y^{(i)} = \cup_{f \in P_i^-} x(f) \). Note that \( |P_i^+| = \sum_{i=1}^n |P_i^+| \equiv n \), otherwise the truncated expectation vanishes. A couple \( l = (f_{ij}^-, f_{ij}^+) \equiv (f_{ij}^-, f_{ij}^+) \) will be called a line joining the fields with labels \( f_{ij}^-, f_{ij}^+ \) and sector indices \( \omega = \omega(f_{ij}^-), \omega = \omega(f_{ij}^+) \) and connecting the points \( x_l \equiv x_{ij} \) and \( y_l \equiv y_{ij} \), the endpoints of \( l \). Moreover, we shall put \( m_l = m(f_{ij}^-) + m(f_{ij}^-) \) and, if \( \omega_l^- = \omega_l^+ \), \( \omega_l \equiv \omega_l^- = \omega_l^+ \). A similar definition is repeated for \( \sigma \). Then, it is well known (see [Le], [BM], [GM] for example) that, up to a sign, if \( s > 1 \),

\[
E_l^{(s)}(P_1, \ldots, P_s) = \prod_{l \in T} \delta_{\omega_l} (x_l - y_l) \delta_{\omega_l} (x_l - y_l) \delta_{\omega_l} \delta_{\omega_l} \int dP_T(t) \det G^{h,T}(t)
\]  

(3.31)

where \( T \) is a set of lines forming an anchored tree graph between the clusters of points \( x^{(i)} \cup y^{(i)} \), that is \( T \) is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover \( t = \{ t_{i,i'} \in [0,1], 1 \leq i, i' \leq s \} \), \( dP_T(t) \) is a probability measure with support on a set of \( m \) such that \( t_{i,i'} = u_i \cdot u_{i'} \) for some family of vectors \( u_i \in \mathbb{R}^2 \) of unit norm. Finally \( G^{h,T}(t) \) is a \((n-s + 1) \times (n-s + 1)\) matrix, whose elements are given by \( G_{i,i',j,j'}^{h,T} = t_{i,i'} \delta_{\omega_l}(f_{ij}^-) \delta_{\omega_l}(f_{ij}^-) g_{ij}(x_l - y_{i'} \cdot y_{i'}') \delta_{\omega_l} \delta_{\omega_l} \delta_{\omega_l} \delta_{\omega_l} \) with \( (f_{ij}^-, f_{ij}^+) \) not belonging to \( T \).

In the following we shall use (3.31) even for \( s = 1 \), when \( T \) is empty, by interpreting the r.h.s. as equal to 1, if \( |P_1| = 0 \), otherwise as equal to \( \det G = E^{h}_l \). If we apply the expansion (3.31) in each non trivial vertex of \( \tau \), we get an expression of the form

\[
V^{(h)}(\tau, P) = \sum_{T \in T} \int dX_{x_{\tau}} \mathcal{W}^{(h)}(P_{m})W^{(h)}_{\tau, P, T}(x_{m}) \equiv \sum_{T \in T} V^{(h)}(\tau, P, T),
\]  

(3.32)

where \( T \) is a special family of graphs on the set of points \( x_{m} \), obtained by putting together an anchored tree graph \( T_{x} \) for each non non trivial vertex \( v \). Note that any graph \( T \in T \) becomes a tree graph on \( x_{m} \), if one identifies all the points in the sets \( x_{v} \), for any vertex \( v \) which is also an endpoint.

We are writing the \( R \) operation as acting on the kernels, according to (3.9) and its analogous for \( n = 1 \). Such representation for the \( R \) operation is however not suitable to “gain” the convergence factor \( \gamma^{-3(h-h')} \) or \( \gamma^{-2(h-h')} \), for which is much more convenient representation of \( R \) in (3.10), (3.14). However if we write simply all the \( R \) operations as in (3.10), (3.14) one gets possibly factors \( (x_i - x_j)^{\alpha n} \) with \( \alpha_n = O(n) \), which when integrated give \( O(n!) \) terms. One has to proceed in a more subtle way starting from the vertices of \( \tau \) closest to the root from which the \( R \) operation is non trivial, and writing \( R \) as in (3.10),(3.14) leaving all the other \( R \) operation as in (3.1). One distributes the “zero” along a path connecting a family of end points, and from (3.9) \( (x_i - x_j)R\hat{W}_h = (x_i - x_j)\hat{W}_h \), if \( x_i, x_j \) are two coordinates of \( \hat{W}_h \) and \( R\hat{W}_h \) is the term in square brackets in the l.h.s. of (3.9); an analogous property holds for \( R\hat{W}_h \). There are same technical complications in implementing this idea, which are discussed in [BM] (see also [BoM]), §3.2, §3.3 for a different model, but the adapting of such argument to the present case is straightforward. We obtain, in the \( L \rightarrow \infty \) limit

\[
V^{(h)}(\tau, P, T, \alpha) = \sum_{T \in T} \sum_{\alpha \in A_T} \int dX_{x_{\tau}} \mathcal{W}^{(h)}(P_{m})W^{(h)}_{\tau, P, T, \alpha}(x_{m})
\]  

\[
\cdot \left\{ \prod_{f \in P_{m}} \left[ \left( \hat{p}_f - \omega(f) \right) \delta^{(h)}(x_{f}, \omega(f)) \right] \right\} ,
\]  

(3.33)
where

\[
W_{\tau,\mathbf{p},T,\alpha}(\mathbf{x}_{v_0}) = \left[ \prod_{v \text{ not e.p.}} \left( \frac{Z_{h_v}/Z_{h_v-1}}{|P_v|/2} \right) \right] \cdot \left[ \prod_{i=1}^{n} d_{j_0}^{(v_i^*)} (\mathbf{x}_i, y_i) K_{v_i}^{(h_v)} (\mathbf{x}_{v_i^*}) \right] \left[ \prod_{v \text{ not e.p.}} \frac{1}{s_v} \int dP_{T_v}(\mathbf{t}_v) \right] \cdot \det G_{\alpha}^{h_v,T_v}(\mathbf{t}_v) \left[ \prod_{l \in T_v} \frac{\partial q_{\alpha}(f^l_{j^l})}{\partial q_{\alpha}(f^l_{j^l})} \left( d_{j_0}^{(l)} \right)(\mathbf{x}_i, y_l) \frac{\partial m_{ij}^{(h_v)}}{\partial \sigma_\beta^{(h_v)}} (x_i, y_l) \right],
\]

(3.34)

where:

1) \( P \) is the set of \( \{ P_v \} \);
2) \( T \) is the set of the tree graphs on \( \mathbf{x}_{v_0} \), obtained by putting together an anchored tree graph \( T_v \) for each non trivial vertex \( v \);
3) \( A_T \) is a set of indices which allows to distinguish the different terms produced by the non trivial \( \mathcal{R} \) operations and the iterative decomposition of the zeros; \( v_1^*, \ldots, v_n^* \) are the endpoints of \( \tau \), \( f_{j^-}^l \) and \( f_{j^+}^l \) are the labels of the two fields forming the line \( l \), “e.p.” is an abbreviation of “endpoint”.
4) \( G_{\alpha}^{h_v,T_v}(\mathbf{t}_v) \) is obtained from the matrix \( G_{\alpha}^{h_v,T_v}(\mathbf{t}_v) \), associated with the vertex \( v \) and \( T_v \), by substituting \( G_{\alpha}^{h_v,T_v}(\mathbf{t}_v) = t_{v,i',\nu} \frac{\partial m_{ij}^{(h_v)}}{\partial \sigma_\beta^{(h_v)}} (x_{ij}, y_{i'}) \) with

\[
G_{\alpha,ij,i',j'}^{h_v,T_v} = t_{v,i',\nu} \frac{\partial q_{\alpha}(f_{j^l}^l)}{\partial q_{\alpha}(f_{j^l}^l)} \left( d_{j_0}^{(l)} \right)(\mathbf{x}_i, y_l) \frac{\partial m_{ij}^{(h_v)}}{\partial \sigma_\beta^{(h_v)}} (x_{ij}, y_{i'}) \cdot
\]

(3.35)

5) \( \partial_j^q \), \( q = 0, 1, 2 \), are discrete derivatives or operators dimensionally equivalent to derivatives, due to the presence of the lattice and the fact that \( \beta \) is finite, see [BM] §3. Morever \( \partial_j^q \) denotes the identity and \( j = 0, +, - \). According to (3.13), (3.15) if \( \alpha(f) = I \) then in \( \partial_j^q(f) \) one has \( j(f) = 0, + \) and if \( \alpha(f) = II \) then \( j(f) = 0, - \).
6) \( d_0(\mathbf{x}_i - y_i) = \frac{\beta}{4} \sin \frac{\beta}{2} (x_{0i} - y_{0i}) \) and \( d_0(\mathbf{x}_i - y_i) = (x_{i,l} - y_{i,l}) \), \( i = \pm \) are the ”zeros” produced by the \( \mathcal{R} \) operation, see (3.13),(3.15). Finally by construction \( b_0(l) \leq 2 \).
7) The factors \( \frac{Z_{h_v}}{Z_{h_v-1}} \) are functions of the coordinates, and such dependence is not explicitly written.

Of course the coefficients \( b_\alpha \) and \( q_\alpha \) are not independent, and, by the definition of \( \mathcal{R} \) (see the discussion after (3.13)) it holds for any \( \alpha \in A_T \), the following inequality

\[
\left[ \prod_{f \in I_{\alpha}} \gamma_{h_{\alpha}(f)}q_{\alpha}(f) \right] \left[ \prod_{l \in T_{\alpha}} \gamma_{h_{\alpha}(l)}(h_{\alpha}(l)) \right] \leq \prod_{v \text{ not e.p.}} \gamma_{-z(P_v)},
\]

(3.36)

where \( h_{\alpha}(f) = h_{v_0} - 1 \) if \( f \in P_{v_0} \), otherwise it is the scale of the vertex where the field with label \( f \) is contracted; \( h_{\alpha}(l) = h_v \), if \( l \in T_v \) and

\[
z(P_v) = \begin{cases} 1 & \text{if } |P_v| = 4, \\ 2 & \text{if } |P_v| = 2, \\ 0 & \text{otherwise}. \end{cases}
\]

(3.37)

It holds

\[
\left| \det G_{\alpha}^{h_v,T_v}(\mathbf{t}_v) \right| \leq C \sum_{v_0}^{n} |P_v|^{-2s_v} \cdot \gamma_{h_{\alpha}^{\tau}} \cdot \gamma_{-z(P_v)} \cdot \frac{h_{\alpha}}{h_{\alpha}(l)h_{\alpha}(l)} \cdot \frac{h_{\alpha}}{h_{\alpha}(l)h_{\alpha}(l)} \cdot \gamma_{-z(P_v)} .
\]

(3.38)

This follows from the well known Gram-Hadamard inequality, see also [Le],[BM],[GM], stating that, if \( M \) is a square matrix with elements \( M_{ij} \) of the form \( M_{ij} = < A_i, B_j > \), where \( A_i, B_j \) are vectors in a Hilbert space with scalar product \( < \cdot, \cdot > \), then

\[
| \det M | \leq \prod_{i} \| A_i \| \cdot \| B_i \| .
\]

(3.39)
where $\| \cdot \|$ is the norm induced by the scalar product.

In our case it can be shown that

$$C_{A,B} = \int \frac{1}{|k|} |\phi_f(\omega)|^2 |\phi_t(\omega)|^2 \, d\omega,$$

where $\phi_f(\omega)$ and $\phi_t(\omega)$ are the Fourier transforms of $\phi_f$ and $\phi_t$, respectively.

For instance $A$ and $B$ can be chosen as:

$$A = -i \int \frac{1}{|k|} |\phi_f(\omega)|^2 |\phi_t(\omega)|^2 \, d\omega,$$

and from (3.39) we easily get (3.38).

By using (3.34) and (3.38) we get, assuming (2.33)

$$\int d\omega \left| W_{T,P,T} \right|^2 \leq C^n J_{T,P,T} \prod_{v \text{ not e.p.}} \left[ \sum_{i=1}^{s_v} |P_i| - |P_v| - 2(s_v - 1) \right] \gamma^{|h_v| \sum_{i=1}^{s_v} [n_a(f_i^+ + q_a(f_i^-)]} \right],$$

where

$$J_{T,P,T} = \int \prod_{v \text{ not e.p.}} \left[ \left| d_{\omega_i}(\omega_i) \right| \right] \prod_{i=1}^{n} \left| d_{\omega_i}(\omega_i) \right| \left| \phi_f(\omega) \right|^2 \left| \phi_t(\omega) \right|^2 \, d\omega,$$

and

$$\int d\omega \left| W_{T,P,T} \right|^2 \leq C^n J_{T,P,T} \prod_{v \text{ not e.p.}} \left[ \sum_{i=1}^{s_v} |P_i| - |P_v| - 2(s_v - 1) \right] \gamma^{|h_v| \sum_{i=1}^{s_v} [n_a(f_i^+ + q_a(f_i^-)]} \right],$$

In [BM], [BoM] it is proved that

$$d(x_v) = dx \prod_{l \in T} dr_l,$$

where $r_l = x_l(t_l) - y_l(s_l)$ and $x_l(t_l), y_l(s_l)$ are interpolated points, see (3.13), (3.15), and $x$ is an arbitrary point of $x_v$. By using (3.3), (3.4) we bound dimensionally each propagator, each derivative and each zero and we find

$$J_{T,P,T} \leq C^{n} \prod_{v \text{ not e.p.}} \left[ \sum_{i=1}^{s_v} |P_i| - |P_v| - 2(s_v - 1) \right] \gamma^{|h_v| \sum_{i=1}^{s_v} [n_a(f_i^+ + q_a(f_i^-)]} \right],$$

We find then

$$\int d\omega \left| W_{T,P,T} \right|^2 \leq C^n J_{T,P,T} \prod_{v \text{ not e.p.}} \left[ \sum_{i=1}^{s_v} |P_i| - |P_v| - 2(s_v - 1) \right] \gamma^{|h_v| \sum_{i=1}^{s_v} [n_a(f_i^+ + q_a(f_i^-)]} \right],$$

$$\int d\omega \left| W_{T,P,T} \right|^2 \leq C^n J_{T,P,T} \prod_{v \text{ not e.p.}} \left[ \sum_{i=1}^{s_v} |P_i| - |P_v| - 2(s_v - 1) \right] \gamma^{|h_v| \sum_{i=1}^{s_v} [n_a(f_i^+ + q_a(f_i^-)]} \right].$$
\[ C^nL^2|\lambda|^n\gamma^{-hD(P_{v0})} \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_{v_i}|_\gamma - 2 + \frac{|P_{v_i}|}{2} + z(P_{v_i})} \right\} \]  

(3.47)

where \( D(P_{v0}) = -2 + m \). The sum over \( t, P, T, \alpha \) is standard and we refer to [BM], §3.15; at the end the following theorem is proved.

**Theorem.** Let \( h > h_\beta \geq 0 \). If (2.33) holds then there exists a constant \( c_0 \) such that

\[ \sum_{\tau \in T_{h,n}} \sum_{P_{v_0} = 2} \int d\mathbf{x}_{v_0} |W_{\tau,P,T,\alpha}(\mathbf{x}_{v_0})| \leq L^2 h^{-hD(P_{v0})}(c_0 \lambda)^n, \]

(3.48)

where

\[ D(P_{v0}) = -2 + m. \]

(3.49)

### 4. The flow of running coupling functions

**4.1 Lemma** It holds that \( \sum_{k=\pm} \hat{W}^h_2(\eta \frac{\pi}{2}, \pm \frac{\pi}{2}, k) = \sum_{k=\pm} \hat{W}^h_2(\eta \frac{\pi}{2}, k_+, \pm \frac{\pi}{2}) = 0 \)

**Proof** - We can compute \( \hat{W}^h_2(\mathbf{k}) \) also by a "single scale" integration; in fact \( \hat{W}^h_2(\mathbf{k}) \) is the kernel of the term \( \psi_k^{(+h)} \psi_k^{(-h)} \) on \( \mathcal{V}^{h} \), defined by

\[ e^{-V^{h}} = \int P(d\psi^{[h,0]}_0)e^{-V(\psi^{[h,0]}_0 + \psi^{(-h)})} \]

(4.1)

where \( P(d\psi^{[h,0]}_0) \) is the fermionic integration with propagator

\[ g^{[h,0]}(k_0, k_+, k_-) = \frac{\chi_{h,0}(k)}{-ik_0 + 2 \cos k_+ \cos k_-} \]

(4.2)

where

\[ \chi_{h,0}(k) = [H(a_0^2 \sin^2 k_+) + H(a_0^2 \sin^2 k_-)]C^{-1}_h(\sqrt{k_0^2 + 4 \cos^2 k_+ \cos^2 k_-}) \]

(4.3)

with \( C^{-1}_h = \sum_{k=0}^{h} f_k \). We can write \( W^h_2(\eta \frac{\pi}{2}, \frac{\pi}{2}, k_-) \) as sum over Feynman graphs (see for instance [GM]) and each Feynman diagram can be written as

\[ \frac{1}{L^2} \sum_{k_1} \cdots \frac{1}{L^2} \sum_{k_n} g^{[h,0]}(k_1) \cdots g^{[h,0]}(k_m) \prod_{i=1}^{m} \sigma_i^{1} k_i + \sigma_i^{2} k \]

(4.4)

where \( n + m \) is an odd number, \( \sigma_i^{1}, \sigma_i^{2} = 0, 1, -1 \), \( \sigma_i^{1} + \sigma_i^{2} \) is an odd integer and \( k = (\eta \frac{\pi}{2}, \frac{\pi}{2}, k_2) \). In order to write (4.4) we consider a spanning tree \( T \) formed by propagators connecting all the vertices of the graphs. We will call the propagators not belonging to \( T \) loop lines and we write the momenta of the propagators of \( T \) as a linear combination of the momenta of the loop propagators and of the external momentum. We perform the shift \( k_{+,i} \rightarrow k'_{+,i} + \frac{\pi}{2} \), and the summation domain is not changed by periodicity. The loop propagators become

\[ g^{[h,0]}(k') = \frac{\chi_{h,0}(k')}{-ik_0 + 2 \sin k'_+ \cos k_-} \]

(4.5)
with
\[ \bar{\chi}_{h,0}(k^\prime) = [H(a_0^2 \cos^2 k^\prime_+ + H(a_0^2 \sin^2 k^-)] C_h^{-1}(\sqrt{k_0^2 + 4 \sin^2 k_+ \cos^2 k_-}) \] (4.6)

Of course \( g^{[h,0]}(k) \) is odd in the exchange \( k_0, k_+ , k_- \rightarrow -k_0, -k_+, k_- \). On the other hand the momenta of the propagators belonging to \( T \) becomes

\[ \sum_{i=1}^{m} \sigma_i^j k_i + \sigma^j k = (\sum_{i=1}^{m} \sigma_i^j k_0,i + \frac{\pi}{\beta} \sum_{i=1}^{m} \sigma_i^j k_+^i,i) + \left( \sum_{i=1}^{m} \sigma_i^j + \sigma^j \right) \frac{\pi}{2} \sum_{i=1}^{m} \sigma_i^j k_-^i,i + \sigma^j k_- \] (4.7)

with \( \sum_{i=1}^{m} \sigma_i^j + \sigma^j \) an odd integer; hence the propagators belonging to \( T \) have the form

\[ \bar{\chi}_{h,0}(k^\prime) = -i \sum_{i} \sigma_i k_0,i + \frac{\pi}{\beta} + 2 [(-1)^k \sin(\sum_{i} \sigma_i k_0^i,i)] \cos(\sum_{i} \sigma_i k_-^i,i + \sigma k_-) \] (4.8)

and

\[ \bar{\chi}_{h,0}(k^\prime) = [H(a_0^2 \cos^2(\sum_{i} \sigma_i k_0^i,i) + H(a_0^2 \sin^2(\sum_{i} \sigma_i k_-^i,i + k_-))] C_h^{-1}(\sqrt{(\sum_{i} \sigma_i k_0^i,i)^2 + 4 \sin^2(\sum_{i} \sigma_i k_0^i,i) \cos^2(\sum_{i} \sigma_i k_-^i,i + \sigma k_-)}) \] (4.9)

Hence by performing the change of variables \( k_0, k_+^i, k_- \rightarrow -k_0, -k_+^i, k_- \) we find

\[ W_2^h(\frac{\pi}{\beta}, \frac{\pi}{2}, k_-) = -W_2^h(-\frac{\pi}{\beta}, \frac{\pi}{2}, k_-) \] (4.10)

### 4.2 Finite temperature flow.

The multiscale analysis defined above has the effect that the running coupling functions \( \delta_h(k_{\sigma,\omega}) \), \( Z_h(k_{\sigma,\omega}) \) and \( \lambda_h(k_{\sigma_1,\omega_1}^i, k_{\sigma_2,\omega_2}^i, k_{\sigma_3,\omega_3}^i) \) verify a recursive relation of the form

\[ \delta_{h-1}(k_{\sigma,\omega}^i) = \delta_h(k_{\sigma,\omega}^i) + \beta_0^h(k_{\sigma,\omega}^i) \]

\[ \frac{Z_{h-1}(k_{\sigma,\omega}^i)}{Z_h(k_{\sigma,\omega}^i)} = 1 + \beta_0^h(k_{\sigma,\omega}^i) \] (4.11)

\[ \lambda_{h-1}(k_{\sigma_1,\omega_1}^i, k_{\sigma_2,\omega_2}^i, k_{\sigma_3,\omega_3}^i) = \lambda_h(k_{\sigma_1,\omega_1}^i, k_{\sigma_2,\omega_2}^i, k_{\sigma_3,\omega_3}^i) + \beta_0^h(k_{\sigma_1,\omega_1}^i, k_{\sigma_2,\omega_2}^i, k_{\sigma_3,\omega_3}^i) \]

It is quite easy to prove that, at temperature not too low, indeed (2.33) hold. The proof is done by induction assuming that (2.33) holds for \( h \) and proving that it holds also for \( h-1 \), if \( h - 1 \geq h_\beta \) and \( \beta \leq \exp(-c^\gamma) \), where \( c^\gamma \) is a suitable constant. In fact iterating for instance the last of (4.11) we find

\[ \lambda_{h-1}(k_{\sigma_1,\omega_1}^i, k_{\sigma_2,\omega_2}^i, k_{\sigma_3,\omega_3}^i) = \lambda + \sum_{k=h+1}^{0} \beta_0^h(k_{\sigma_1,\omega_1}^i, k_{\sigma_2,\omega_2}^i, k_{\sigma_3,\omega_3}^i) \] (4.12)

and from (2.33) and (3.48) we find, for \( |\lambda| \leq \frac{c^2_1}{2} \)

\[ \sup_{\{k^i\}, \{\sigma \}} \beta_0^h(k_{\sigma_1,\omega_1}^i, k_{\sigma_2,\omega_2}^i, k_{\sigma_3,\omega_3}^i) \leq 2c_2^\gamma \lambda^2 \]

if \( c^2_2 > 0 \) is a bound for the norm of the second order contribution to \( \lambda_h \). Hence

\[ \sup_{\{k\}, \{\sigma \}} |\lambda_{h-1}(k_{\sigma_1,\omega_1}^i, k_{\sigma_2,\omega_2}^i, k_{\sigma_3,\omega_3}^i)| \leq |\lambda| + |h|2c_2^\gamma \lambda^2 \] (4.13)
that and this is equivalent to show that there exists a non vanishing function $g$ over the immediately checked by dimensional considerations and applying the derivative in (4.15) one has to check that indeed the critical index $\eta$ is nonvanishing, the correction will be surely smaller for $\langle h \rangle$ large enough. Hence the dominant correction will be surely smaller at least for $\sigma, \omega > 0$ systems of interacting fermions.

4.3 Flow of the wave function renormalization. To complete the proof of the main Theorem one has to check that indeed the critical index $\eta(k_+)$ or $\eta(k_-)$ are non identically vanishing, and this is equivalent to show that there exists a non vanishing function $a(k^\prime_{\sigma,\omega}) > 0$ such that

$$e^{-\frac{a(k^\prime_{\sigma,\omega})}{\lambda^2 h}} \leq Z_h(k^\prime_{\sigma,\omega}) \leq e^{-\frac{\lambda^2 h}{a(k^\prime_{\sigma,\omega})}}$$

(4.14)

From the fact that $\beta^h_{\xi}(k^\prime_{\sigma,\omega}) = \sum_{n=2}^\infty \beta^h_{\xi(n)}(k^\prime_{\sigma,\omega})$ with $|\beta^h_{\xi(n)}(k^\prime_{\sigma,\omega})| \leq c_0^0|\xi|_n$, as a consequence of (3.49) and (2.33), it is sufficient to find an upper and lower bound for $\beta^h_{\xi(2)}$. From an explicit computation one finds

$$2\omega \cos \tilde{k}_{\sigma} \beta^h_{\xi(2)} = 24 \sum_{\omega_1,\omega_2,\omega_3} \sum_{\sigma_1,\sigma_2,\sigma_3} \frac{\partial}{\partial k_0} \left[ \int dk_1 dk_2 dk_3 \tilde{g}^{h}_{\sigma_2,\omega_3}(k_2) \tilde{g}^{h}_{\sigma_1,\omega_1}(k_1) \delta(k + k_3 - k_1 - k_2) \lambda_h(k_\sigma, k_3, \sigma_1, k_1, \sigma_1, k_\sigma, k_2) \right]_{k = \tilde{k}_{\sigma,\omega}}$$

where $\tilde{g}^{h}_{\sigma,\omega} = \sum_{k=\tilde{k}_{\sigma,\omega}} g^{h}_{\sigma,\omega}$. As the dependence from the momenta of $\lambda_h$ is quite complex, it is convenient to replace in the above integral $\lambda_h$ with $\lambda$; if the integral so obtained is nonvanishing, the correction will be surely smaller for $T \geq e^{-\frac{\tilde{c}(\lambda)}{\beta}}$ for a suitable $\tilde{c}$, as $\lambda_h = \lambda + O(\lambda^2 \log \beta)$ from (4.13). We can choose $\sigma = I$ for definiteness (the analysis for $\sigma = II$ is identical), and we can distinguish two kind of contributions in the sum over $\sigma_1, \sigma_2, \sigma_3$: one in which all the propagators are $g_I$ and the other such that there is at least a propagator $g_{II}$. The estimate of this second contribution is $O(\lambda^2 \gamma^h)$, as it can be immediately checked by dimensional considerations and applying the derivative in (4.15) over the $g_{II}$ propagators (one can always do that). We can further simplify the expression we have to compute noting that

$$\tilde{g}^{h}_{I,\omega}(x - y) = \int dk e^{-ikx} H(a_0^2 \sin^2 k_-) \tilde{f}^{h}(k_0, k_+) \tilde{g}^{h}_{I,\omega}(x - y)$$

(4.16)

with

$$|\tilde{g}^{h}_{I,\omega}(x - y)| \leq a_0^2 \frac{C_n \gamma^h}{1 + |\gamma^h| |x_0 - y_0| + \gamma^h |x_- - y_+| + |x_+ - y_-|}$$

(4.17)

i.e. similar to (3.3) with an extra $a_0^2$. We can replace in (4.15) the propagators $\tilde{g}^{h}_{I,\omega}$ with the first addend in the r.h.s. of (4.16); if such term will be given by a nonvanishing constant, the correction will be surely smaller at least for $a_0$ large enough. Hence the dominant contribution to (4.15) is given by

$$\sum_{\omega_1,\omega_2,\omega_3} \int dk_{-1} dk_{-3} H(a_0^2 \sin^2 k_{-1}) H(a_0^2 \sin^2 k_{-3}) H(a_0^2 \sin^2 (-k_{-1} + k_{-3} + k_+)) A$$

(4.18)

with

$$A = \sum_{\omega_1,\omega_2,\omega_3} \int dk_{0,1} dk_{0,3} \int dk_{+,1} dk_{+,3} \frac{f^{h}(k_{0,1}, k_{+,1})}{-ik_{0,1} + \omega_1 k_{+,1}}$$

$$\frac{f^{h}(k_{0,3}, k_{+,3})}{-ik_{0,3} + \omega_2 k_{+,3}} \partial_{k_+} \frac{f^{h}(-k_{0,1} + k_{0,3} + k_0, -k_{+,1} + k_{+,3} + k_+)}{-i(-k_{0,1} + k_{0,3} + k_0) + 2\omega_2(-k_{+,1} + k_{+,3} + k_+)}$$

(4.19)

with $\omega = \omega_1 + \omega_2 - \omega_3$; it is easy to check that this term is indeed non vanishing. Note also that $A$ is the first non trivial contribution to the critical index $\eta$ of the Schwinger function of a $d = 1$ systems of interacting fermions.

4.4 Schwinger functions. We will note repeat here the analysis of the Schwinger functions at the temperature scale, as one can proceed as in the $d = 1$ to obtain an expansion for the
Schwinger function once that the expansion for the effective potential is understood; see for instance [GM]. We only remark that the $A_I$ and $A_{II}$ in (1.12) and (1.13) are indeed $O(\lambda^2)$ as a consequence of

$$\int dk_0dk'_+dk_-g^h_I(k_0,k'_+ + \omega p_F, k_-) = 0 \quad \int dk_0dk'_+dk_-g^h_II(k_0,k'_+ + \omega p_F) = 0 \quad (4.20)$$

5. Conclusions

5.1 Marginal Fermi liquids and Luttinger liquids. We can compare the behaviour of the half filled Hubbard model with cut-off with other models. We have found that the wave function renormalization has an anomalous flow up to exponentially small temperatures, $Z_{h-1}/Z_h = 1 + O(\lambda^2)$, see (4.14); in the case of circular fermi surfaces one finds instead, see [DR], for $|\lambda| \leq \varepsilon$

$$\frac{Z_{h-1}}{Z_h} = 1 + O(\varepsilon^2 \gamma^{\frac{h}{2}}) \quad (5.1)$$

which means that $Z_h = 1 + O(\lambda^2)$, up to exponentially small temperatures; the factor $\gamma^{h/2}$ in the r.h.s. of (5.1) is an improvement with respect to a power counting bound and is found by using a volume improvement based on the geometrical constraints to which the momenta close to the Fermi surface are subjected. An equation similar to (5.1) holds also for any symmetric smooth Fermi surfaces with non vanishing curvature; a proof can be obtained by combining the results of [BGM] with Appendix 2 of [DR].

The similarity of the equation for $Z_h$ with its analogous for one dimensional systems may suggest that the behaviour of the half-filled Hubbard model with cut-off up to zero temperature is similar to the one of a system of spinless interaction fermions in $d=1$ (the so called Luttinger liquid behaviour). However this is false; in a Luttinger liquid in fact one has that

$$\lambda_{h-1} = \lambda_h + O(\varepsilon^2 \gamma^{\frac{h}{2}}) \quad (5.2)$$

a property known as vanishing of Beta function. One can easily check that this cancellation is not present in the half-filled Hubbard model with cut-off; in fact the dominant second order contribution to $\lambda_h(k_{1,I}, k_{-1,I}, k_{1,I})$ containing only $\sigma = I$ internal lines is

$$\int dk^i f^h(k_0, k_+)f^{\geq h}(k_0, k_+\frac{1}{k_0^2 + k_+^2}H(a_0^2 \sin^2 k_-)$$

$$[H(a_0^2 \sin^2(k_{1,-} + k_{2,-} - k_-)\lambda\lambda - H(a_0^2 \sin^2(k_{3,-} - k_{2,-} + k_-)\lambda\lambda] \quad (5.3)$$

where the dependence from $k$ of the $\lambda_h$ has not been explicitated. It is clear then that even at the second order the flow of $\lambda_h$ is quite complex, and we plan to analyze it in a future work, in order to understand the leading instabilities. Replacing $H$ with 1 and having $\lambda_h$ not momentum dependent one recovers the $d=1$ situation in which the beta function is vanishing. The theory resembles the theory of $d=1$ Fermi systems in which each particle has an extra degree of freedom, the component of the momentum parallel to the flat Fermi surface, playing the role of a ”continuous” spin index; and it is known that in $d=1$ even a spin $\frac{1}{2}$ index can destroy the Luttinger liquid behaviour.

5.2 Marginal Fermi liquid behaviour close to half filling. A similar analysis can be performed in the case of the Hubbard model with cut-off close to half-filling ($\mu = -\varepsilon$ with $\varepsilon$ small and positive); in such a case the Fermi surface is convex and with finite radius of curvature but still resembles a square with non flat sides and rounded corners. The propagator has the form $\frac{\chi(k)}{-k_0 + 2 \cos k_+ \cos k_- - \varepsilon}$ and it is easy to verify that, if $\beta < C \min[\frac{1}{\rho}]$ where $\rho$ is the radius of curvature of the Fermi surface, the bounds (3.3), (3.4) for the single scale
propagator $g^R_{\sigma\omega}$ still holds; the reason is that, up to temperatures greater than the inverse of the curvature radius, the bounds are insensitive to the fact that the sides of the Fermi surface are not perfectly flat. One can repeat all the analysis of the preceding sections and it is found that the Schwinger functions behave like (1.12), (1.12) for $\lambda$ small enough and $\beta < C \min[\min\{\rho\}, e^{(\bar{c}|\lambda|^{-1})}]$; in other words marginal Fermi liquid behaviour is still found close to half filling, up to such temperatures.

On the other hand at lower temperatures, for $\min[\rho] \leq \beta \leq e^{(\bar{c}|\lambda|^{-1})}$ (of course assuming $\min[\rho] \leq e^{(\bar{c}|\lambda|^{-1})}$) one can apply the results of [BGM] (valid for any convex symmetric and regular Fermi surface) so finding $Z = 1 + C_\rho \lambda^2 + O(\lambda^3)$ where $C_\rho$ is a constant which is very large for small $\epsilon$ (and diverging at half filling $\epsilon = 0$). Hence, depending on the values of the parameters, one can have, in the low temperature region and before the critical temperature, two possibilities: the first is to have only marginal Fermi liquid behaviour $Z = 1 + O(\lambda^2 \log \beta)$, and the second is to have marginal Fermi liquid behaviour up to temperatures $O(\rho^{-1})$ and then Fermi liquid behaviour up to the critical temperature.

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