STRUCTURE OF IRREDUCIBLE HOMOMORPHISMS TO/FROM FREE MODULES

SAEED NASSEH AND RYO TAKAHASHI

Abstract. The primary goal of this paper is to investigate the structure of irreducible monomorphisms to and irreducible epimorphisms from finitely generated free modules over a noetherian local ring. Then we show that over such a ring, self-vanishing of Ext and Tor for a finitely generated module admitting such an irreducible homomorphism forces the ring to be regular.

1. Introduction

Convention. In this paper, \((R, \mathfrak{m}, k)\) is a commutative noetherian local ring and all modules are finitely generated.

A homomorphism \(f: M \to N\) of \(R\)-modules is called irreducible if \(f\) is neither a split monomorphism nor a split epimorphism, and for every factorization \(M \xrightarrow{g} L \xrightarrow{h} N\) of \(f\) we have \(g\) is a split monomorphism or \(h\) is a split epimorphism. Irreducible homomorphisms are used in the theory of Auslander-Reiten sequences which was established in [4] and play a central role in representation theory of artin algebras. (Excellent references on these topics are [5, 15, 25].)

In this paper we investigate the structure of irreducible monomorphisms to and irreducible epimorphisms from free modules over a commutative noetherian local ring. Section 3 deals with the case where we have an irreducible monomorphism to a free module. Our main result in this section, stated next, is proven in [5, 11] and [5, 6].

Theorem A. The following assertions hold for the local ring \(R\).

(a) If \(I\) is a non-zero proper ideal of \(R\) that is a direct summand of \(\mathfrak{m}\), then the inclusion map \(I \to R\) is an irreducible homomorphism.

(b) Assume that \(R\) is Henselian, and let \(\phi: M \to F\) be an irreducible monomorphism of \(R\)-modules with \(F\) free such that \(\text{Im}(\phi) \subseteq \mathfrak{m}F\). Then the following hold.

\(\text{(b1)}\) The \(R\)-module \(M\) is isomorphic to a direct summand of \(\mathfrak{m}\).

\(\text{(b2)}\) If \(M\) is indecomposable, then for every surjection \(\pi: F \to R\) there exists a split monomorphism \(\eta: M \to \mathfrak{m}\) such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & F \\
\eta \downarrow & & \downarrow \pi \\
\mathfrak{m} & \xrightarrow{\theta} & R
\end{array}
\]

2010 Mathematics Subject Classification. 13C10, 13D05, 13D07, 13H05.

Key words and phrases. Auslander-Reiten Conjecture, Ext-vanishing, Injective dimension, Irreducible homomorphism, Projective dimension, Regular ring, Tor-vanishing.

Takahashi was partly supported by JSPS Grants-in-Aid for Scientific Research 16K05098.
commutes, where $\theta$ stands for the inclusion map.

Section 4 is devoted to the case where we have an irreducible epimorphism from a free module. We prove our main result in this section, stated next, in 4.1. In this theorem, $\text{Soc} R$ denotes the socle of $R$.

**Theorem B.** Let $\phi: F \to M$ be an irreducible epimorphism of $R$-modules with $F$ free. Then the following assertions hold.

(a) The kernel of $\phi$ is isomorphic to $k$.

(b) Assume that $\text{End}_R(M)$ is a local ring, and let $\iota: R \to F$ be a split monomorphism. Then the following hold.

(b1) The composition $\phi \iota: R \to M$ is irreducible and hence, it is either surjective or injective.

(b2) Suppose that $\phi \iota$ is surjective. Then $R$ has type one, $F$ has rank one, and there is a commutative diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\phi} & M \\
\downarrow & & \downarrow \\
R & \xrightarrow{\pi} & R/\text{Soc} R \\
\iota & & \rho
\end{array}
\]

such that $\iota$ and $\rho$ are isomorphisms and $\pi$ is the natural surjection.

Our motivation for the main result in Section 5 comes from the Auslander-Reiten Conjecture [3] that originates in representation theory of artin algebras. This section deals with this conjecture and also with a Tor version of it when the module admits irreducible homomorphisms described in Theorems A and B; see Theorem 5.1.

## 2. Basic properties

This section contains some results that will be used in the subsequent sections. The next result is a part of [5] Lemma 5.1] in which $R$ is assumed to be an artin algebra. Here we give the proof (with no such assumption on $R$) for the reader’s convenience.

**Proposition 2.1.** Let $f: M \to N$ be an irreducible homomorphism of $R$-modules. Then $f$ is either surjective or injective.

**Proof.** The map $f$ has a factorization $M \xrightarrow{g} \text{Im} f \xrightarrow{h} N$, where $g$ is the surjection induced by $f$ and $h$ is the inclusion map. Since $f$ is irreducible, either $g$ is a split monomorphism or $h$ is a split epimorphism. In the first case, $g$ is an isomorphism, which means that $f$ is injective. In the second case, $h$ is an isomorphism, which means that $f$ is surjective. \qed

**Lemma 2.2.** Let $f: M \to N$ be a homomorphism of $R$-modules. Then the following are equivalent:

(i) $f$ is irreducible.

(ii) $\left( \begin{array}{cc} f & 0 \\ 0 & 1 \end{array} \right): M \oplus X \to N \oplus X$ is irreducible for all $R$-modules $X$.

(iii) $\left( \begin{array}{cc} f & 0 \\ 0 & 1 \end{array} \right): M \oplus X \to N \oplus X$ is irreducible for some $R$-module $X$. 

\textbf{Proof.} It is straightforward to see that \( f \) is neither a split monomorphism nor a split epimorphism if and only if \( \begin{pmatrix} 0 & f \\ 0 & 1 \end{pmatrix} \) is neither a split monomorphism nor a split epimorphism.

For an arbitrary \( R \)-module \( X \) consider a factorization
\[
M \oplus X \xrightarrow{(\alpha \beta)} L \xrightarrow{(f)} N \oplus X
\]
of \( \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \). Then \( \gamma \alpha = f \), \( \gamma \beta = 0 \), \( \delta \alpha = 0 \), and \( \delta \beta = 1 \). If \( f \) is irreducible, then either \( \alpha \) is a split monomorphism or \( \gamma \) is a split epimorphism.

If \( \alpha \) is a split monomorphism, then there is a homomorphism \( \varepsilon : L \to M \) such that \( \varepsilon \alpha = 1 \) and we have
\[
\begin{pmatrix} 1 & -\varepsilon \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & -\varepsilon \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Therefore, \( \begin{pmatrix} \alpha & \beta \end{pmatrix} \) is a split monomorphism.

If \( \gamma \) is a split epimorphism, then there is a homomorphism \( \zeta : N \to L \) such that \( \gamma \zeta = 1 \) and we have
\[
\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} \varepsilon & \beta \\ -\delta \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\delta \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Therefore, \( \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \) is a split epimorphism. Thus, \( \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \) is irreducible. This proves (i) \( \implies \) (ii).

To show (iii) \( \implies \) (i), let \( M \xrightarrow{a} L \xrightarrow{b} N \) be a factorization of \( f \), and let \( X \) be an \( R \)-module such that \( \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} : M \oplus X \to N \oplus X \) is irreducible. We then have
\[
\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ba & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.
\]
Then either \( \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \) is a split epimorphism or \( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \) is a split monomorphism. It is straightforward to see that either \( b \) is a split epimorphism or \( a \) is a split monomorphism. Therefore, \( f \) is irreducible. \( \square \)

Let \( n \geq 1 \) be an integer. In the next lemma, \( \text{diag}(a_1, a_2, \ldots, a_n) \) denotes the square matrix with \( a_1, a_2, \ldots, a_n \) on the main diagonal and zero everywhere else.

\textbf{Lemma 2.3.} (a) For \( 1 \leq i \leq n \) let \( f_i : M_i \to N \) be a homomorphism of \( R \)-modules, and assume that \( \text{End}_R(N) \) is a local ring. If
\[
(f_1 \quad f_2 \quad \ldots \quad f_n) : M_1 \oplus \cdots \oplus M_n \to N
\]
is irreducible, then for all \( 1 \leq i \leq n \) the \( R \)-homomorphism \( f_i \) is irreducible.

(b) For \( 1 \leq i \leq n \) let \( h_i : M \to N_i \) be a homomorphism of \( R \)-modules, and assume that \( \text{End}_R(M) \) is a local ring. If
\[
(h_1 \quad h_2 \quad \ldots \quad h_n)^{\text{tr}} : M \to N_1 \oplus \cdots \oplus N_n
\]
is irreducible, then for all \( 1 \leq i \leq n \) the \( R \)-homomorphism \( h_i \) is irreducible.

\textbf{Proof.} We only prove the first assertion; the second one is shown dually. Also, we only prove that \( f_1 \) is irreducible, as the irreducibility of the other \( f_i \) follow similarly.

First, suppose that \( f_1 \) is a split epimorphism. Then there is \( g : N \to M_1 \) such that \( f_1 g = 1 \), and we have \( (f_1 \quad f_2 \quad \ldots \quad f_n) (g \quad 0 \quad \ldots \quad 0)^{\text{tr}} = 1 \). This implies \( (f_1 \quad f_2 \quad \ldots \quad f_n) \) is a split epimorphism, which contradicts the assumption that it is irreducible. Hence, \( f_1 \) is not a split epimorphism.
Next, suppose that $f_1$ is a split monomorphism. Since $\text{End}(N)$ is local, $N$ is indecomposable.\footnote{Note that we do not need Henselian property here.} Hence, $f_1: M_1 \to N$ becomes an isomorphism, and we have $(f_1 \ f_2 \ \ldots \ f_n) (f_1^{-1} \ 0 \ \ldots \ 0)^{tr} = 1$. This implies $(f_1 \ f_2 \ \ldots \ f_n)$ is a split epimorphism, which contradicts the assumption that it is irreducible. Hence, $f_1$ is not a split monomorphism.

Now let $M_1 \xrightarrow{\alpha} X \xrightarrow{\beta} N$ be a factorization of $f_1$. We then have

$$(f_1 \ f_2 \ \ldots \ f_n) = (\beta \alpha \ f_2 \ \ldots \ f_n) = (\beta \ f_2 \ \ldots \ f_n) \cdot \text{diag}(\alpha, 1, \ldots, 1).$$

By assumption, either $(\beta \ f_2 \ \ldots \ f_n)$ is a split epimorphism or $\text{diag}(\alpha, 1, \ldots, 1)$ is a split monomorphism. In the latter case, we easily see that $\alpha$ is a split monomorphism. In the former case, we find homomorphisms $c \in \text{Hom}_R(N, X)$ and $g_i \in \text{Hom}_R(N, M_i)$ for $2 \leq i \leq n$ such that the equality

$$\beta c + f_2 g_2 + \cdots + f_n g_n = 1$$

holds in $\text{End}_R(N)$. Assume that $f_2 g_2$ is a unit of $\text{End}_R(N)$. Then $f_2$ is a split epimorphism, so there is a map $d: N \to M_2$ such that $f_2 d = 1$. Hence, we have $(f_1 \ f_2 \ \ldots \ f_n) (0 \ d \ 0 \ \ldots \ 0)^{tr} = 1$, which says that $(f_1 \ f_2 \ \ldots \ f_n)$ is a split epimorphism, contrary to the assumption that it is irreducible. Therefore, $f_2 g_2$ is not a unit of $\text{End}_R(N)$. Similarly, we can show that $f_3 g_3, \ldots, f_n g_n$ are not units of $\text{End}_R(N)$. Since $\text{End}_R(N)$ is a local ring, from equation (2.3.1) we conclude that $\beta c$ is a unit. Hence, $\beta$ is a split epimorphism. Thus $f_1$ is irreducible. \hfill $\square$

For an $R$-module $M$ and for a positive integer $n$, by $M^{\oplus n}$ we denote the direct sum $\bigoplus_{i=1}^{n} M$.

**Lemma 2.4.** Let $f: M \to N$ be a homomorphism of $R$-modules, and let $n$ be a positive integer. If $f^{\oplus n}: M^{\oplus n} \to N^{\oplus n}$ is irreducible, then $n = 1$.

**Proof.** Suppose $n \geq 2$. Then $f^{\oplus n}: M^{\oplus n} \to N^{\oplus n}$ has a factorization

$$M^{\oplus n} \xrightarrow{\text{diag}(f, 1, \ldots, 1)} N \oplus M^{\oplus n-1} \xrightarrow{\text{diag}(1, f, \ldots, f)} N^{\oplus n}.$$

Since $f^{\oplus n}$ is irreducible, we conclude that either $\text{diag}(f, 1, \ldots, 1)$ is a split monomorphism or $\text{diag}(1, f, \ldots, f)$ is a split epimorphism. If $\text{diag}(f, 1, \ldots, 1)$ is a split monomorphism (resp. $\text{diag}(1, f, \ldots, f)$ is a split epimorphism), then so is $f$, and so is $f^{\oplus n}$, contrary to its irreducibility. Thus, we must have $n = 1$. \hfill $\square$

### 3. Irreducible Monomorphisms to Free Modules

In this section we provide the proof of Theorem A.

3.1 (Proof of Theorem A Part (a)). Let $\theta: I \to R$ be the inclusion map. Note that $\theta$ is neither a split monomorphism nor a split epimorphism.

Case 1: $I = m$. Suppose that there is a factorization $m \xrightarrow{\alpha} M \xrightarrow{\beta} R \xrightarrow{\gamma} 0$ of $\theta$. Since $\theta$ is injective, so is $\alpha$ and we have a commutative diagram

$$\begin{array}{ccc}
0 & \xrightarrow{0} & m \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} 0 \\
0 & \xrightarrow{0} & m \xrightarrow{\theta} R \xrightarrow{\pi} k \xrightarrow{0}
\end{array}$$

with exact rows.
If $\gamma \neq 0$, then $\gamma$ is a surjection, so $\beta$ is also a surjection. Hence, $\beta$ is a split epimorphism.

If $\gamma = 0$, then $\pi \beta = 0$, and there exists a homomorphism $\delta : M \rightarrow m$ such that $\beta = \theta \delta$. Since $\theta$ is injective, we have $\delta \alpha = 1$, whence $\alpha$ is a split monomorphism. Consequently, $\theta$ is an irreducible homomorphism.

Case 2: General case. We can take a proper ideal $J$ of $R$ such that $m \sim I \oplus J$.

Let $I \xrightarrow{\alpha} M \xrightarrow{\beta} R$ be a factorization of $\theta$, and denote by $\theta'$ the inclusion map $J \rightarrow R$. Then we have a factorization

$$I \oplus J \xrightarrow{\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}} M \oplus J \xrightarrow{\begin{pmatrix} \beta' \\ \theta' \end{pmatrix}} R$$

of the map $\begin{pmatrix} \theta & \theta' \end{pmatrix} : I \oplus J \rightarrow R$, which is exactly the inclusion map $m \rightarrow R$. It follows from Case 1 that either $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ is a split monomorphism or $\begin{pmatrix} \beta' & \theta' \end{pmatrix}$ is a split epimorphism.

If $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ is a split monomorphism, then we easily see that $\alpha$ is a split monomorphism. If $\begin{pmatrix} \beta' & \theta' \end{pmatrix}$ is a split epimorphism, then we can find elements $x \in M$ and $y \in J$ such that $\beta(x) + \theta'(y) = 1$. As $\theta'(y) = y$ is an element of the maximal ideal $m$, the element $\beta(x)$ is a unit of $R$. Now define $\beta' : R \rightarrow M$ by $\beta'(r) = r(\beta(x))^{-1}x$ for every $r \in R$. It follows then that $\beta \beta' = 1$. Hence, $\beta$ is a split epimorphism and therefore, $\theta$ is irreducible, as desired.

Recall that two homomorphisms $h : X \rightarrow Y$ and $h' : X' \rightarrow Y'$ of $R$-modules are called equivalent if there exist isomorphisms $p : X \rightarrow X'$ and $q : Y \rightarrow Y'$ such that $qh = h'p$. (It is easy to see that irreducibility is preserved by equivalence.)

The next lemma enables us to replace an arbitrary monomorphism $M \rightarrow F$ of $R$-modules, where $F$ is free, with one whose image is contained in $mF$.

**Lemma 3.2.** Let $\phi : M \rightarrow R^{\oplus m}$ be a monomorphism, where $M$ is an $R$-module and $m \geq 0$ is an integer. Then there exist an integer $n$ with $0 \leq n \leq m$, an $R$-module $N$, and a monomorphism $g : N \rightarrow R^{\oplus n}$ such that $\text{Im}(g) \subseteq mR^{\oplus n}$ and such that $\phi$ is equivalent to the map

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} : N \oplus R^{\oplus m-n} \rightarrow R^{\oplus n} \oplus R^{\oplus m-n}$$

where $1$ denotes the identity map of $R^{\oplus m-n}$. In this situation, $\phi$ is irreducible if and only if so is $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ if and only if so is $g$. 
Proof. There is a commutative diagram

```
0 0
↓  
0  N  g  R^\oplus n  \psi'  C  0
\|  \|  \|  \|
\|  \|  \|  
0  M  \phi  R^\oplus m  \psi  C  0
\|  \|  \|  
\|  \|  \|
R^\oplus m-n  R^\oplus m-n
\|  \|  
0  0
```

with exact rows and exact columns such that \( \psi' \) is minimal, that is, \( \ker(\psi') \subseteq \mathfrak{m}R^\oplus n \), for some integer \( n \). (Note that here \( M \cong N \oplus R^\oplus m-n \) such that \( N \) does not have any free direct summand.) Therefore, \( \text{Im}(g) \subseteq \mathfrak{m}R^\oplus n \). Note that the R-module homomorphism \( \phi \) is equivalent to the map

\[
\begin{pmatrix}
g & 0 \\
0 & 1
\end{pmatrix} : N \oplus R^\oplus m-n \to R^\oplus n \oplus R^\oplus m-n.
\]

Finally, the fact that \( \phi \) is irreducible if and only if so is \( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \) if and only if so is \( g \) follows from Lemma 2.2. \( \square \)

**Remark 3.3.** By (the end of) Lemma 3.2, the irreducibility of \( \phi \) is equivalent to the irreducibility of \( g \). Thus, replacing \( \phi \) with \( g \), we will work with irreducible maps \( \phi : M \to R^\oplus m \) such that \( \text{Im}(\phi) \subseteq \mathfrak{m}R^\oplus m \) in the rest of this section.

**Proposition 3.4.** Let \((R, xR)\) be a discrete valuation ring, and let \( \theta : xR \to R \) be the inclusion map. Let \( \phi : M \to F \) be a monomorphism of R-modules with \( F \) free such that \( \text{Im}(\phi) \subseteq \mathfrak{m}F \). Then the following are equivalent:

(i) \( \phi \) is irreducible.

(ii) There is a commutative diagram

```
M  \phi  F
\|  \|
\|  \|
\|  \|
xR  \theta  R
```

such that \( \eta \) and \( \pi \) are isomorphisms.

**Proof.** (ii) \( \Rightarrow \) (i): Assume that there is a commutative diagram

```
M  \phi  F
\|  \|
\|  \|
\|  \|
xR  \theta  R
```

such that \( \eta \) and \( \pi \) are isomorphisms. It follows from Theorem A(a) that \( \theta \) is irreducible. The above commutative diagram shows that \( \phi \) is also irreducible.
(i) $\implies$ (ii): Since $R$ has global dimension one, $M$ is a free $R$-module. Let $m := \text{rank}_R F$ and $n := \text{rank}_R M$. As $\phi$ is injective, we have $m \geq n$.

Let $A$ be a representation matrix of $\phi$, which is an $m \times n$ matrix over $R$. Since $\psi$ is minimal, each component of $A$ is an element of $xR$, whence there is another $m \times n$ matrix $B$ such that $A = xB$. Hence, there is a factorization $R^{\oplus n} \xrightarrow{x} R^{\oplus n} \xrightarrow{B} R^{\oplus m}$ of $A$. Irreducibility of $\phi$ implies that either $R^{\oplus n} \xrightarrow{x} R^{\oplus m}$ is a split monomorphism or $R^{\oplus n} \xrightarrow{B} R^{\oplus m}$ is a split epimorphism. In the former case, we see that $R^{\oplus n} \xrightarrow{x} R^{\oplus m}$ is also a split monomorphism, which is impossible. Hence, $R^{\oplus n} \xrightarrow{B} R^{\oplus m}$ is a split epimorphism. In particular, this says that $n \geq m$. Therefore, we have $m = n$, and $R^{\oplus n} \xrightarrow{B} R^{\oplus m}$ is an isomorphism by [18, Theorem 2.4]. It follows that there is a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & F \\
\downarrow{\lambda} & & \downarrow{\pi} \\
R^{\oplus n} & \xrightarrow{x} & R^{\oplus n}
\end{array}
\]

such that $\lambda$ and $\pi$ are isomorphisms. In view of Lemma 2.4 we have $n = 1$, and get a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & F \\
\downarrow{\eta} & & \downarrow{\pi} \\
xR & \xrightarrow{a} & R
\end{array}
\]

where $\eta$ is the composition $M \xrightarrow{\lambda} R \xrightarrow{x} R$. This completes the proof. \hfill $\square$

**Remark 3.5.** (a) Let $0 \to L \xrightarrow{f} M \xrightarrow{\partial} N \to 0$ be a split exact sequence of $R$-modules. Then there is an isomorphism $h: M \to L \oplus N$ such that $g = (0 \ 1)h$. Indeed, there is a homomorphism $\ell: M \to L$ such that $\ell f = 1_L$. Set $h = (\ell \ g)$.

(b) Let $F$ be a free $R$-module of rank $r$ and $a: F \to R^{\oplus r}$ be an isomorphism. Let $\pi: F \to R$ be an arbitrary surjection and set $b := \pi a^{-1}$. Suppose that $p = (0 \ldots 0 \ 1): R^{\oplus r} \to R$ is the $r$-th projection. Since $b$ is a split epimorphism, applying part (a) to $M = R^{\oplus r}$ and $N = R$ with $q = b$ we obtain an automorphism $c: R^{\oplus r} \to R^{\oplus r}$ such that $pc = b$. Setting $q := ca$ we have a commutative diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\pi} & R \\
\downarrow{q} & & \downarrow{p} \\
R^{\oplus r} & \xrightarrow{p} & R
\end{array}
\]

of $R$-modules in which $q$ is an isomorphism. This shows that $\pi$ and $p$ are equivalent.

**3.6 (Proof of Theorem A Part (b)).** We prove the theorem step by step.
Step 1: Fix a proper submodule $D$ of $C := \text{Coker}(\phi)$ and consider the pull-back diagram:

Since $\phi$ is irreducible, either $\alpha$ is a split monomorphism or $\beta$ is a split epimorphism. If $\beta$ is a split epimorphism, then $\beta$ must be an isomorphism, which implies $C = D$. This is a contradiction because we assumed that $D$ is a proper submodule of $C$. Hence, $\alpha$ has to be a split monomorphism.

Step 2: Let $D$ be the submodule $mC$ of $C$. Since $\psi$ is minimal, the dimension of the $k$-vector space $C/mC$ is equal to $r := \text{rank}_R F$. From the middle column we observe that $E \cong m^{\oplus r}$. By Step 1, $M$ is isomorphic to a direct summand of $m^{\oplus r}$.

Step 3: Let $D$ be a maximal submodule of $C$, that is, $C/D \cong k$. Then it is observed that $E$ is isomorphic to $R^{\oplus r-1} \oplus m$. Hence, $M$ is isomorphic to a direct summand of $R^{\oplus r-1} \oplus m$.

Step 4: Suppose that $R$ is isomorphic to a direct summand of $m$. Then it follows from [9, Corollary 1.3] that $R$ is regular. Hence, $R$ is a domain which forces $m$ to be indecomposable. Therefore, $m$ is isomorphic to $R$, which means that $R$ is a discrete valuation ring. In light of Proposition 3.4, we have now both of the conclusions (b1) and (b2) in case that $R$ is isomorphic to a direct summand of $m$.

From now on, we assume that $R$ is not isomorphic to a direct summand of $m$.

Step 5: Since $R$ is assumed to be Henselian, we can apply the Krull-Schmidt theorem. (See [25, Proposition 1.18].) It follows from Step 2 that $M$ does not contain a non-zero free summand; note here that $R$ is indecomposable as an $R$-module. By Step 3 we see that $M$ is isomorphic to a direct summand of $m$. This shows Part (b1) of the theorem.

Step 6: To prove Part (b2), suppose that $M$ is indecomposable. As $R$ is Henselian, $\text{End}_R(M)$ is a local ring. For each $1 \leq i \leq r$ let

$$p_i = (0 \ldots 0 1 0 \ldots 0) : R^{\oplus r} \to R$$

be the $i$-th projection (the $i$-th entry of $p_i$ is 1). As we see in Remark 3.5(b), there is an isomorphism $q: F \to R^{\oplus r}$ such that the diagram

is commutative.
Now for each \(1 \leq i \leq r\) set \(h_i := p_i q \phi\) and note that \(h_r = \pi \phi\). We have 
\(q \phi = (h_1, h_2, \ldots, h_r)^t r\) and \(q \phi\) is irreducible. Lemma 2.3 then implies that the composition \(h_r = \pi \phi\) is irreducible. In particular, it is not an epimorphism, so its image is contained in \(m\). Thus, \(\pi \phi\) has a factorization \(M \xrightarrow{\eta} m \xrightarrow{\theta} R\). Irreducibility of \(\pi \phi\) implies that \(\eta\) is a split monomorphism. We now have a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & F \\
\downarrow{\eta} & & \downarrow{\pi} \\
m & \xrightarrow{\theta} & R
\end{array}
\]

and the proof is completed. \(\square\)

**Remark 3.7.** A natural question to ask is the following: Let \(I\) be an ideal of \(R\), and for \(1 \leq i \leq n\) let \(f_i: I \to R\) be an irreducible monomorphism. Let 
\(f = (f_1, f_2, \ldots, f_n)^t r: I \to R^{\oplus n}\). When is \(f\) irreducible?

Note that under the above assumptions, the map \(f\) is not necessarily irreducible. For example, if \(n = 2\) and \(f_2 = af_1\) for some unit element \(a \in R\), then there is a factorization

\[
\begin{array}{ccc}
R & \xrightarrow{(1)} & R^{\oplus 2} \\
\downarrow{f_1} & & \downarrow{(f_1)} \\
I & \xrightarrow{(f_1, f_2)} & R^{\oplus 2}
\end{array}
\]

which shows that \((f_1): I \to R^{\oplus 2}\) is not irreducible.

4. Irreducible epimorphisms from free modules

This section is entirely devoted to the proof of Theorem B. The proof of Part (a) is also given in [17]. However, we include it for the convenience of the reader.

**4.1 (Proof of Theorem B).** (a) If \(M\) is free, then \(\phi\) splits, which is a contradiction. Hence, \(M\) is not free, and therefore \(\text{Ext}_R^1(M, k) \neq 0\). Let \(0 \to k \to L \xrightarrow{h} M \to 0\) be a non-split short exact sequence of finitely generated \(R\)-modules in \(\text{Ext}_R^1(M, k)\). Since \(F\) is free there exists a homomorphism \(g: F \to L\) such that \(hg = \phi\). Since \(\phi\) is irreducible and \(h\) is not a split epimorphism, \(g\) is a split monomorphism. Hence, there exist a finitely generated \(R\)-module \(L'\) such that \(L \cong F \oplus L'\) and a
commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker}(\phi) & \rightarrow & F & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \phi & & \downarrow & & 0 \\
0 & \rightarrow & k & \rightarrow & L & \rightarrow & M & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Coker}(t) & \rightarrow & L' & & & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

in which Ker(\(\phi\)) \(\neq 0\). Since, \(\text{dim}_k \text{Ker}(\phi) + \text{dim}_k \text{Coker}(\phi) = \text{dim}_k k = 1\), we conclude that \(\text{dim}_k \text{Coker}(t) = 0\). Hence, \(\text{Coker}(t) = 0\) and Ker(\(\phi\)) = \(k\), as desired.

(b1) The irreducibility of \(\phi_1\) follows from Lemma 2.3. Hence, by Proposition 2.1 it is either surjective or injective.

(b2) By Theorem B(a) there is an exact sequence \(0 \rightarrow k \rightarrow R \phi_1 \rightarrow M \rightarrow 0\). Let \(x = f(1)\). Note that \(x\) is a non-zero element of Soc \(R\). Let \(y\) be any element of Soc \(R\). Then \(g: k \rightarrow R\) given by \(g(1) = y\) is a monomorphism. Considering the push-out diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & k & \rightarrow & R & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \phi_1 & & \downarrow & & 0 \\
0 & \rightarrow & R & \rightarrow & E & \rightarrow & M & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Coker}(t) & \rightarrow & L' & & & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

since \(\phi_1\) is irreducible, either \(e\) is a split epimorphism or \(m\) is a split monomorphism.

If \(e\) is a split epimorphism, then \(h\) is a split monomorphism and there is a homomorphism \(p: E \rightarrow R\) such that \(ph = 1\). Hence, we have \(g = phg = pmf\). Note that \(pm\) is an endomorphism of \(R\). Setting \(a = pm(1) \in R\), we get

\[y = g(1) = pmf(1) = pm(x) = ax.\]

It follows then that Soc \(R = (x)\).

If \(m\) is a split monomorphism, then there is a homomorphism \(q: E \rightarrow R\) such that \(qm = 1\), and we have \(f = qmf = qbh\). Similarly as above, setting \(b = qh(1)\), we get \(x = by\). Since \(x\) is non-zero and \(y\) is a socle element, \(b\) must be a unit of \(R\). Hence, \(y = b^{-1}x\) and Soc \(R = (x)\).

So, if \(e\) is a split epimorphism or \(m\) is a split monomorphism, then we have Soc \(R = (x) \cong k\), and \(R\) has type one. There is an isomorphism \(\rho: R/\text{Soc}R \rightarrow M\) such that the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\phi_1} & M \\
\downarrow & & \downarrow \rho \\
R & \xrightarrow{\pi} & R/\text{Soc}R
\end{array}
\]
commutes. Using Theorem B(a), we obtain a commutative diagram

0 \rightarrow k \xrightarrow{f} R \xrightarrow{\pi} R/\text{Soc } R \xrightarrow{\sim} 0
\downarrow \gamma \downarrow \downarrow \downarrow \downarrow \gamma
0 \rightarrow k \xrightarrow{\iota} F \xrightarrow{\phi} M \xrightarrow{\rho} 0

with exact rows. Since \iota is injective, so is \gamma, and hence \gamma is an isomorphism. Five Lemma then shows that \iota is also an isomorphism, whence \textit{F} has rank one. \hfill \Box

\textbf{Remark 4.2.} We work in the setting of Theorem B. By Theorem B(b1), the map \phi \iota is either surjective or injective. As we see in Theorem B(b2), in case that \phi \iota is surjective, one can conclude that \textit{R} has type one, \textit{F} has rank one, and there is a commutative diagram

F \xrightarrow{\phi} M
\downarrow \iota \downarrow \rho
R \xrightarrow{\pi} R/\text{Soc } R

such that \iota and \rho are isomorphisms and \pi is the natural surjection.

Now, a natural question to ask is the following:

What can one conclude if in Theorem B(b2) we replace the assumption “\phi \iota is surjective” with “\phi \iota is injective for all split monomorphisms \iota: \textit{R} \rightarrow \textit{F}”? Note that the assumption “\phi \iota is injective for all split monomorphisms \iota: \textit{R} \rightarrow \textit{F}” is equivalent to saying that when we regard \phi: \textit{F} \rightarrow \textit{M} as a surjection

(\phi_1 \phi_2 \ldots \phi_m) : R^{\oplus m} \rightarrow \textit{M},

all the components \phi_i: \textit{R} \rightarrow \textit{M} are injective.

5. \textbf{Irreducible homomorphisms and vanishing of \textit{(co)}homology}

A commutative version of the Auslander-Reiten Conjecture \cite{3} states that if \textit{M} is an \textit{R}-module with Ext^i_R(\textit{M}, \textit{M} \oplus \textit{R}) = 0 for all \textit{i} \geq 1, then \textit{M} is free. This conjecture has been proven affirmatively in some special cases; see for instance \cite{10, 11, 13, 19, 21, 22}. The next theorem deals with this conjecture and also with a Tor version of it when the \textit{R}-module \textit{M} admits irreducible homomorphisms described in Theorems A and B.

\textbf{Theorem 5.1.} Let \textit{M} be an \textit{R}-module and consider the following conditions:

(i) Tor^i_R(\textit{M}, \textit{M}) = 0 for \textit{i} \gg 0
(ii) Ext^i_R(\textit{M}, \textit{M}) = 0 for \textit{i} \gg 0
(iii) pd_{\textit{R}} \textit{M} < \infty
(iv) id_{\textit{R}} \textit{M} < \infty
(v) \textit{R} is a regular ring.

Then (i)-(v) are equivalent under any of the following two conditions:
(a) \textit{m} is indecomposable and there exists an irreducible monomorphism \textit{M} \rightarrow \textit{F}, where \textit{F} is free;
(b) there exists an irreducible epimorphism \textit{F} \rightarrow \textit{M}, where \textit{F} is free.

Moreover, under the equivalent conditions (i)-(v) in these cases we have

\text{pd}_{\textit{R}} \textit{M} = \min\{n \in \mathbb{Z} \mid \text{Ext}^n_{\textit{R}}(\textit{M}, \textit{M}) \neq 0 \text{ and } \text{Ext}^i_{\textit{R}}(\textit{M}, \textit{M}) = 0 \text{ for all } \textit{i} > \textit{n}\}. \hfill (*)
Note that under the equivalent conditions in Theorem 5.1 the equality (ii) follows from [3] Proposition 2.4. To prove Parts (a) and (b), we need the following proposition.

**Proposition 5.2.** Let \( R^\oplus n \to M \) be an irreducible epimorphism of \( R \)-modules, where \( n \) is a positive integer. If \( \text{id}_R M < \infty \), then \( R \) is a field.

**Proof.** Since \( M \) is non-zero, the “Bass Conjecture” Theorem (see for instance [2] 9.6.2 and 9.6.4 (ii)]) shows that \( R \) is Cohen-Macaulay. By Theorem B Part (a) there is an exact sequence \( 0 \to k \to R^\oplus n \to M \to 0 \). In particular, this says that depth \( R = 0 \). Hence, \( R \) is artinian and \( M \) is injective. Thus, \( M \) is isomorphic to a direct sum of copies of the injective hull \( E \) of \( k \), say \( E^\oplus n \), where \( n \) is a positive integer. The exact sequence \( 0 \to k \to R^\oplus m \to E^\oplus n \to 0 \) then shows that \( m \ell = 1 + n \ell \), where \( \ell = \ell_R(R) = \ell_R(E) \). (Here \( \ell_R \) denotes the length.) We now have \((m-n)\ell = 1\), which implies that \( \ell = 1 \). This means that \( R \) is a field. \( \Box \)

**5.3.** (Proof of Theorem 5.1). (a) Assume that \( M \) is indecomposable and \( f: M \to F \) is an irreducible monomorphism of \( R \)-modules, where \( F \) is free. Note that conditions (i)-(v) and our assumptions in Part (a) of Theorem 5.1 are preserved under completion. So, we replace \( R \) by its completion in \( m \)-adic topology and assume that \( R \) is complete (hence, \( R \) is Henselian).

To show that conditions (i)-(v) are equivalent, it suffices to prove that each of (i) and (ii) implies (v). For this, let \( g: N \to R^\oplus n \) be the map obtained by removing from \( f \) the identity map of a maximal direct summand, as in Lemma 5.2 Then the \( R \)-homomorphism \( g \) is irreducible and \( N \) is isomorphic to a direct summand of \( M \) by Theorem A(b1). Since \( N \) is non-zero and \( m \) is indecomposable, \( N \cong m \).

(i) \( \implies \) (v) From our Tor-vanishing assumption we have \( \text{Tor}^i_R(N,N) = 0 \) for all \( i \gg 0 \). Hence, \( \text{Tor}^i_R(m,m) = 0 \) for all \( i \gg 0 \). This implies that \( \text{Tor}^i_R(k,k) = 0 \) for all \( i \gg 0 \), and therefore, \( R \) is regular.

(ii) \( \implies \) (v) From our Ext-vanishing assumption we have \( \text{Ext}^i_R(N,N) = 0 \) for all \( i \gg 0 \). Hence, \( \text{Ext}^i_R(m,m) = 0 \) for all \( i \gg 0 \). It follows then that \( \text{Ext}^i_R(k,m) = 0 \) for all \( i \gg 0 \), which implies that \( \text{id}_R m < \infty \). Therefore, \( R \) is regular, as desired. (See Levin and Vasconcelos [14 Theorem 1.1].)

(b) Assume that there exists an irreducible epimorphism \( F \to M \), where \( F \) is free. Using the short exact sequence \( 0 \to k \to F \to M \to 0 \) from Theorem B(a), we see that (i) implies (v) and (ii) implies (iv). Hence, it suffices to show that (iv) implies (v). This follows from Proposition 5.2 \( \Box \)

**Corollary 5.4.** Let depth \( R \geq 2 \), and assume that there exists an irreducible monomorphism \( M \to F \), where \( F \) is free. Then all of the conditions (i)-(v) from Theorem 5.1 are equivalent.

**Proof.** By [24] Corollary 3.3, we know that \( m \) is indecomposable. Now the assertion follows from Theorem 5.1 Part (a). \( \Box \)

We conclude this section by proving the following result and one of its consequences. The proof is given in [27] below. In this theorem, \( \text{G-dim} \) stands for the Gorenstein dimension of Auslander and Bridger [2].

**Theorem 5.5.** Let \( M \) be an \( R \)-module. If \( m \) is decomposable and \( \text{G-dim}_R M < \infty \), then the following are equivalent.

(i) \( \text{Tor}^i_R(M,M) = 0 \) for \( i \gg 0 \)
(ii) \( \operatorname{Ext}^i_R(M, M) = 0 \) for \( i \gg 0 \)

(iii) \( \operatorname{pd}_R M < \infty \).

The following lemma is from [6, Theorem 6.3]. (See also [1, Corollary 4.4].)

**Lemma 5.6.** Assume that \( R \cong Q/(q) \) where \( Q \) is a regular local ring with \( q \in Q \). Then \( \operatorname{Ext}^i_R(M, M) = 0 \) for all \( i \gg 0 \) if and only if \( \operatorname{pd}_R M < \infty \).

5.7 (Proof of Theorem 5.5). We prove that if (i) or (ii) holds, then \( \operatorname{pd}_R M < \infty \). Note that we can replace \( R \) by its completion in the \( m \)-adic topology and assume that \( R \) is complete.

Assume on the contrary that \( \operatorname{pd}_R M = \infty \). Then by [23, Theorem A], the ring \( R \) is Gorenstein and is isomorphic to \( Q/(q) \), where \( Q \) is a regular local ring with \( q \in Q \). Thus, if \( \operatorname{Tor}^R_i(M, M) = 0 \) for all \( i \gg 0 \), it follows from [12, Theorem 1.9] that \( \operatorname{pd}_R M < \infty \), which is a contradiction.

Also, if \( \operatorname{Ext}^i_R(M, M) = 0 \) for all \( i \gg 0 \), then by Lemma 5.6 we have \( \operatorname{pd}_R M < \infty \), that is again a contradiction.

Hence, under the above assumptions we must have \( \operatorname{pd}_R M < \infty \), as desired. □

**Remark 5.8.** After this paper was submitted, the authors were able to prove the equivalence of (i) and (iii) in Theorem 5.5 without assuming that \( M \) has finite \( G \)-dimension. The proof uses the notion of fiber products; see [20].

Following [16], a finitely generated indecomposable \( R \)-module \( M \) is called IG-projective if \( \operatorname{Gdim}_R M = 0 \) and if \( M \) admits either an irreducible epimorphism \( F \to M \) or an irreducible monomorphism \( M \to F \), in which \( F \) is \( R \)-free. An example of IG-projective modules is the module \( A/(X) \) over the local ring \( A = \mathbb{K}[X]/(X^2) \) in which \( \mathbb{K} \) is a field. Note that \( A/(X) \) is not \( A \)-free.

As an immediate corollary of Theorems 5.1 and 5.5 we obtain the following result which is [17, Theorem 1.1].

**Theorem 5.9.** Let \( M \) be an IG-projective \( R \)-module. Then \( \operatorname{Ext}^i_R(M, M) = 0 \) for all \( i \geq 1 \) if and only if \( M \) is projective.

**Acknowledgments**

We are grateful to the referee for reading the paper very carefully and for giving many valuable suggestions that improved the presentation of the paper significantly.

**References**

1. T. Araya and Y. Yoshino, *Remarks on a depth formula, a grade inequality and a conjecture of Auslander*, Comm. Algebra **26** (1998), no. 11, 3793–3806.
2. M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc., **94** (1969).
3. M. Auslander and I. Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. **52** (1975), 69-74.
4. M. Auslander and I. Reiten, *Representation theory of Artin algebras III*, Comm. Algebra **3** (1975), 239–294.
5. M. Auslander, I. Reiten, and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge Univ. Press, Cambridge, 1995. xiv+423 pp.
6. L. L. Avramov, S. B. Iyengar, S. Nasseh, and S. Sather-Wagstaff, *Persistence of homology over commutative noetherian rings*, in preparation.
7. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, **39**, Cambridge University Press, Cambridge, 1993. xii+403 pp.
8. K. Diveris, *Finitistic extension degree*, Algebr. Represent. Theory, **17** (2014), 495–506.
9. S. P. Dutta, *Syzygies and homological conjectures*, Commutative algebra (Berkeley, CA, 1987), 139–156, Math. Sci. Res. Inst. Publ., 15, Springer, New York, 1989.
10. C. Huneke and G. Leuschke, *On a conjecture of Auslander and Reiten*, J. Algebra 275 (2004), no. 2, 781–790.
11. C. Huneke, L. Şega and A. Vraciu, *Vanishing of Ext and Tor over some Cohen-Macaulay local rings*, Ill. J. Math. 48 (2004), no. 1, 295–317.
12. C. Huneke and R. Wiegand, *Tensor products of modules, rigidity and local cohomology*, Math. Scand., 81 (1997), no. 2, 161–183.
13. D. Jorgensen and L. Şega, *Nonvanishing cohomology and classes of Gorenstein rings*, Adv. Math. 188 (2004), no. 2, 470–490.
14. G. Levin and W. V. Vasconcelos, *Homological dimensions and Macaulay rings*, Pacific J. Math., 25 (1968), no. 2, 315–323.
15. G. J. Leuschke and R. Wiegand, *Cohen-Macaulay representations*, Mathematical Surveys and Monographs, 181, American Mathematical Society, Providence, RI, 2012. xviii+367 pp.
16. R. Luo, *IG-projective modules*, J. Pure Appl. Algebra, 218 (2014), 252–255.
17. R. Luo and D. Jian, *On the Gorenstein projective conjecture: IG-projective modules*, J. Algebra Appl., 15 (2016), no. 6, 1650117, 11 pp.
18. H. Matsumura, *Commutative ring theory*, Translated from the Japanese by M. Reid, Second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989.
19. S. Nasseh and S. Sather-Wagstaff, *Vanishing of Ext and Tor over fiber products*, Proc. Amer. Math. Soc., to appear.
20. S. Nasseh and R. Takahashi, *Local rings with quasi-decomposable maximal ideal*, preprint (2017), arXiv:1704.00719.
21. S. Nasseh and Y. Yoshino, *On Ext-indices of ring extensions*, J. Pure Appl. Algebra 213 (2009), no. 7, 1216–1223.
22. L. Şega, *Vanishing of cohomology over Gorenstein rings of small codimension*, Proc. Amer. Math. Soc. 131 (2003), no. 8, 2313–2323.
23. R. Takahashi, *Direct summands of syzygy modules of the residue class field*, Nagoya Math. J., 189 (2008), 1–25.
24. R. Takahashi, *Syzygy modules with semidualizing or G-projective summands*, J. Algebra, 295 (2006), 179–194.
25. Y. Yoshino, *Cohen-Macaulay Modules over Cohen-Macaulay Rings*, in: London Math. Soc. Lecture Note Ser., vol. 146, Cambridge Univ. Press, Cambridge, 1990.

DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460, USA
E-mail address: snasseh@georgiasouthern.edu
URL: https://cosm.georgiasouthern.edu/math/saeed.nasseh

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FIROCHO, CHIKUSAKU, NAGOYA, AICHI 464-8602, JAPAN
E-mail address: takahashi@math.nagoya-u.ac.jp
URL: http://www.math.nagoya-u.ac.jp/~takahashi/