On the equivalence of fractional-order Sobolev semi-norms

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Abstract

We present various results on the equivalence and mapping properties under affine transformations of fractional-order Sobolev norms and semi-norms of orders between zero and one. Main results are mutual estimates of the three semi-norms of Sobolev-Slobodeckij, interpolation and quotient space types. In particular, we show that the former two are uniformly equivalent under affine mappings that ensure shape regularity of the domains under consideration.

Key words: fractional-order Sobolev spaces, semi-norms, Poincaré-Friedrichs' inequality

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1 Introduction

Sobolev norms and semi-norms play a central role in the numerical analysis of discretization methods for partial differential equations. For instance, standard finite element error analysis is essentially a combination of the Bramble-Hilbert lemma and transformation properties of Sobolev (semi-) norms. These properties are also central to the area of preconditioners for (and based on) variational methods. More precisely, arguments based on finite dimensions of local spaces are inherently connected with scaling arguments to keep dimensions bounded. Norms are usually not scalable. That is, the corresponding equivalence numbers behave differently with respect to a scaling parameter like the diameter $D_O$ of the domain $O$ when the domain under consideration is transformed by an affine map that maintains shape regularity (i.e., the ratio of $D_O$ and the “inner diameter” of $O$ is bounded). This can be usually fixed only when essential boundary conditions are present. An example is using the $H^1$-semi-norm as norm in $H^1_0$. More generally, semi-norms have better scaling properties: usually they can be defined so that equivalence numbers are of the same order with respect to $D_O$ under shape-regular affine transformations of the domain.

Whereas properties of Sobolev (semi-) norms under smooth transformations or simple scalings are straightforward as long as their orders are integer, things are getting more complicated
for fractional-order Sobolev norms. Such norms appear, e.g., in a natural way when considering
boundary integral equations of the first kind \cite{14,12} or when studying the regularity of elliptic
problems in non-convex polygonal domains \cite{10}. There are different ways to define fractional-
order Sobolev norms and they all have advantages and disadvantages (standard references are
\cite{13,11}). Different norm variants are known to be equivalent. But dependence of the equivalence
constants on the order and the domain are more involved.

In this paper we analyze the equivalence of different variants of fractional-order semi-norms
of positive orders bounded by one. The use of semi-norms is essential to guarantee scaling
properties and we don’t know of any publication that analyzes their equivalence.

The rest of the paper is organized as follows. In Section 2 we collect all definitions and tech-
nical results. In Section 2.1 we recall two definitions of norms and define three different semi-norms:
one of the Sobolev-Slobodeckij type, one by interpolation, and one of a quotient space type. Sec-
tion 2.2 is devoted to basic equivalence estimates. In particular, we present Poincaré-Friedrichs’
inequalities for the Sobolev-Slobodeckij and interpolation semi-norms (Propositions 2.2 and 2.6).
A direct proof in the case of the Sobolev-Slobodeckij semi-norm is cited from Faermann \cite{9}. An
indirect proof for the interpolation semi-norm is standard, and is given for completeness. Affine
transformation properties of norms (also given for completeness) and semi-norms are analyzed in
Section 2.3. Eventually, in Section 3 we combine the intermediate results to show the equivalence
of all the semi-norms under consideration, with explicit equivalence numbers depending on the
domain via its transformation from a reference domain (Theorems 3.1–3.3). In Theorem 3.4 we
resume the results in a form that is appropriate for affine maps that maintain shape-regularity
of the domain. In particular, it shows (i) the uniform (with respect to $D_O$) equivalence of the
Sobolev-Slobodeckij and the interpolation semi-norms, (ii) that the Sobolev-Slobodeckij and
quotient space semi-norms are uniformly equivalent as long as the diameter of the domain is
bounded from above, and (iii) that the interpolation and quotient space semi-norms can be
uniformly bounded mutually in one direction depending on whether the diameter of the domain
is bounded from above or from below. Finally, in Corollary 3.5 we collect the scaling properties
of all the norms and semi-norms studied in this paper that have this property.

2 Sobolev norms

In this section we recall definitions of several Sobolev (semi-) norms and collect technical results
that are needed to prove our main results in Section 3 or which are interesting in its own.

Throughout the paper, $O \subset \mathbb{R}^n$ denotes a generic bounded connected Lipschitz domain. We
consider the usual $L^2(O)$- and $H^1(O)$-norms with notations $\| \cdot \|_{0,O}$ and $\| \cdot \|_{1,O}$, respectively and
the $H^1(O)$-semi-norm $\| \cdot \|_{1,O}$. Here and in the following, in all types of norms, the underlying
domain of definition $O$ will be occasionally dropped from the notation when not being ambiguous.

2.1 Fractional-order norms and semi-norms

There are several ways to define Sobolev norms. We use the ones defined by a double integral
(Sobolev-Slobodeckij) and by interpolation. For the latter we use the so-called real K-method,
For $0 < s < 1$, the interpolation norm in the fractional-order Sobolev space $H^s(\mathcal{O})$ is defined by

$$
\|v\|_{[L^2(\mathcal{O}), H^1(\mathcal{O})], s} := \left( \int_0^\infty t^{-2s} \inf_{v_0 + v_1} \left( \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + t^2 \|v_1\|^2_{H^1(\mathcal{O})} \right) \frac{dt}{t} \right)^{1/2}.
$$

Here and in the following, the notation $\inf_{v_0 + v_1} \left( \|v_0\|^2_{H^0(\mathcal{O})} + t^2 \|v_1\|^2_{H^1(\mathcal{O})} \right)$ implies that the infimum is taken over $v_0 \in L^2(\mathcal{O})$ and $v_1 \in H^1(\mathcal{O})$, or corresponding spaces as indicated by the respective norms.

We also define the interpolation space

$$
\tilde{H}^s(\mathcal{O}) = [L^2(\mathcal{O}), H^1(\mathcal{O})], s
$$

with corresponding notation for the norm. The notation $\tilde{H}^s$ is used by Grisvard and is common in the boundary element literature, whereas the notation $H^s_{00} = \tilde{H}^s$ is used by Lions and Magenes and is common in the finite element literature.

The Sobolev-Slobodeckij variant of these norms is defined (for $0 < s < 1$) by

$$
\|v\|_{H^s(\mathcal{O})} := \|v\|_{s, \mathcal{O}} := \left( \|v\|^2_{L^2(\mathcal{O})} + \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{1/2},
$$

$$
\|v\|_{\tilde{H}^s(\mathcal{O})} := \|v\|_{\tilde{s}, \mathcal{O}} := \left( \|v\|^2_{H^s(\mathcal{O})} + \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|v(x) - v(y)|^2}{\text{dist}(x, \partial \mathcal{O})^{s+1}} \, dx \, dy \right)^{1/2}
$$

(2.1)

The corresponding semi-norms are

$$
|v|_{[L^2(\mathcal{O}), H^1(\mathcal{O})], s} := |v|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} := \left( \int_0^\infty t^{-2s} \inf_{v_0 + v_1} \left( \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + t^2 \|v_1\|^2_{H^1(\mathcal{O})} \right) \frac{dt}{t} \right)^{1/2}
$$

and

$$
|v|_{H^s(\mathcal{O})} := |v|_{s, \mathcal{O}} := \left( \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{1/2}.
$$

Additionally, it is useful to define the semi-norm of quotient space type

$$
|v|_{s, \mathcal{O}, \inf} := \|v\|_{H^s(\mathcal{O})/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|v + c\|_{s, \mathcal{O}}.
$$

### 2.2 Equivalence of semi-norms on a fixed domain

The aim of this section is to study equivalences of the semi-norms previously defined, on a fixed domain. Together with mapping properties (provided in Section 2.3), these estimates are needed to prove our main results in Section 3. Proofs are based on a standard norm equivalence and specific Poincaré-Friedrichs’ inequalities, which are also recalled here.
It is well known that for Lipschitz domains different definitions of Sobolev norms are equivalent. However, equivalence constants depend usually on the order and the domain under consideration. In particular, for a bounded Lipschitz domain $\mathcal{O}$, the norms $\| \cdot \|_{s,\mathcal{O}}$ and $\| \cdot \|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s}$ are equivalent for $0 < s < 1$, cf. [13, 10, 14]. Such equivalences are shown by corresponding equivalences on $\mathbb{R}^n$ and the use of appropriate extension operators, cf. [4], see also [5] for non-Lipschitz domains. In particular, the norms previously defined are uniformly equivalent for $s$ in a closed subset of $(0, 1)$, see [11].

Here, for the norms, we don’t elaborate on the dependence of the equivalence constants on $s$ and $\mathcal{O}$. We rather give them specific names to be used in estimates to follow.

**Proposition 2.1** (equivalence of norms). For a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^n$ and for given $s \in (0, 1)$ there exist constants $k(s, \mathcal{O}), K(s, \mathcal{O}) > 0$ such that

$$k(s, \mathcal{O}) \| v \|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \leq \| v \|_{s, \mathcal{O}} \leq K(s, \mathcal{O}) \| v \|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \quad \forall v \in H^s(\mathcal{O}).$$

For a proof see, e.g., [14].

It is well known that, on bounded Lipschitz domains, lower-order norms can be bounded by higher-order semi-norms plus finite rank terms. Such estimates are referred to as Poincaré-Friedrichs’ inequalities. For integer-order norms there are direct proofs with explicit constants (depending on orders and domains) [16, Théorème 1.3] and attention has received finding best constants and deriving improved weighted estimates, see, e.g., [17, 18] and [6], respectively. We need such a Poincaré-Friedrichs’ inequality for fractional-order norms on bounded domains (for unbounded domains, see [15]), and refer to [9, Lemma 3.4] for a proof. This proof is given for two dimensions but immediately extends to the general case.

**Proposition 2.2** (Poincaré-Friedrichs inequality, Sobolev-Slobodeckij semi-norm). Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain, and $s \in (0, 1)$. Then there holds

$$\| v \|_{s, \mathcal{O}} \leq C_{PF, SS}(s, \mathcal{O}) \left( | v |_{s, \mathcal{O}} + \left| \int_{\mathcal{O}} v \right| \right) \quad \forall v \in H^s(\mathcal{O})$$

with

$$C_{PF, SS}(s, \mathcal{O}) = \left| \mathcal{O} \right|^{-1/2} \max\{1, 2^{-1/2} D_{\mathcal{O}}^{n/2+s}\}.$$  

Here, $|\mathcal{O}|$ denotes the area of $\mathcal{O}$ and, as mentioned in the introduction, $D_{\mathcal{O}}$ is its diameter.

**Lemma 2.3.** Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain. Then there holds

$$| v |_{s, \mathcal{O}}^2 \leq | v |_{s, \mathcal{O}, \inf}^2 + \inf_{c \in \mathbb{R}} \| v + c \|_{0, \mathcal{O}}^2 \leq (1 + C_{PF, SS}^2) \| v \|_{s, \mathcal{O}}^2$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0, 1)$. Here, $C_{PF, SS} = C_{PF, SS}(s, \mathcal{O})$ is the number from Proposition 2.2.

**Proof.** By definition of $| \cdot |_{s, \mathcal{O}}$ there holds for any $c \in \mathbb{R}$ and any $v \in H^s(\mathcal{O})$ (we now drop $\mathcal{O}$ from the notation)

$$| v |_s = | v + c |_s.$$
Therefore
\[ |v|_s \leq \inf_{c \in \mathbb{R}} \|v + c\|_s = |v|_{s, \inf} \]
which is the first assertion. By the initial argument and the definition of the Sobolev-Slobodeckij norm one also finds that
\[ |v|_{s, \inf}^2 = \inf_{c \in \mathbb{R}} \|v + c\|_s^2 = \inf_{c \in \mathbb{R}} \|v + c\|_0^2 + |v|^2_s. \]
This is the second assertion.

The last relation and the Poincaré-Friedrichs’ inequality (Proposition 2.2) lead to
\[ |v|_{s, \inf}^2 \leq C_{PF, SS}^2 \inf_{c \in \mathbb{R}} \left( |v|_s + \left| \int_\Omega (v + c) \right| \right)^2 + |v|^2_s = (1 + C_{PF, SS}) |v|^2_s. \]
This finishes the proof.

\[ \square \]

**Lemma 2.4.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain. There holds
\[ k^2 |v|_{L^2(\Omega), H^1(\Omega), s}^2 \leq |v|_{s, \Omega, \inf}^2 \leq 3K^2 |v|_{L^2(\Omega), H^1(\Omega), s}^2 + \frac{K^2}{s(1 - s)} \inf_{c \in \mathbb{R}} \|v + c\|_0^2 \]
for any \( v \in H^s(\Omega) \) and \( s \in (0, 1) \). Here, \( k = k(s, \Omega) \) and \( K = K(s, \Omega) \) are the numbers from Proposition 2.1.

**Proof.** Let \( v \in H^s(\Omega) \), and let \( c_0, c_1 \) denote generic constants. For any \( t > 0 \) there holds
\[ \inf_{v = v_0 + v_1} \left( \|v_0\|_0^2 + t^2 |v_1|^2_1 \right) = \inf_{v = v_0 + c_0 + v_1 + c_1} \left( \|v_0 + c_0\|_0^2 + t^2 |v_1|^2_1 \right) \]
\[ = \inf_{c_1, v = c_0 + v_0 + v_1} \left( \|v_0\|_0^2 + t^2 |v_1|^2_1 \right), \]
that is
\[ \inf_{v = v_0 + v_1} \left( \|v_0\|_0^2 + t^2 |v_1|^2_1 \right) = \inf_{v + c = v_0 + v_1} \left( \|v_0\|_0^2 + t^2 |v_1|^2_1 \right) \]
\[ \leq \inf_{c \in \mathbb{R}} \inf_{v + c = v_0 + v_1} \left( \|v_0\|_0^2 + t^2 |v_1|^2_1 \right). \]
(Recall that our convention for the notation \( \inf_{v = v_0 + v_1} \left( \|v_0\|_0^2 + t^2 |v_1|^2_1 \right) \) implies that the infimum is taken with respect to \( v_0 \in L^2(\Omega) \) and \( v_1 \in H^1(\Omega) \).) We conclude that
\[ |v|_{L^2, H^1, s}^2 = \int_0^\infty t^{-2s} \inf_{v = v_0 + v_1} \left( \|v_0\|_0^2 + t^2 |v_1|^2_1 \right) \frac{dt}{t} \]
\[ \leq \inf_{c \in \mathbb{R}} \int_0^\infty t^{-2s} \inf_{v + c = v_0 + v_1} \left( \|v_0\|_0^2 + t^2 |v_1|^2_1 \right) \frac{dt}{t} = \inf_{c \in \mathbb{R}} \|v + c\|_{L^2, H^1, s}^2. \]
By Proposition 2.1

\[ \inf_{c \in \mathbb{R}} \| v + c \|^2_{L^2, H^1} \leq K^{-2} \inf_{c \in \mathbb{R}} \| v + c \|^2_{L^2, H^1, s} = k^{-2} |v|_{s, \inf}^2, \]

so that the first assertion follows.

By definition and using Proposition 2.1 there holds

\[ |v|_{s, \inf}^2 = \inf_{c \in \mathbb{R}} \| v + c \|^2_s \leq K^2 \inf_{c \in \mathbb{R}} \| v + c \|^2_{L^2, H^1, s} \]

\[ = K^2 \inf_{c \in \mathbb{R}} \int_0^\infty t^{-2s} \inf_{v + c = v_0 + v_1} \left( \| v_0 \|^2_0 + t^2 \| v_1 \|^2_0 + t^2 |v_1|_1^2 \right) \frac{dt}{t}. \tag{2.2} \]

We bound the integrand separately for \( t < 1 \) and \( t \geq 1 \).

For \( t < 1 \) we use the representation \( v + c = v_0 + v_1 \) to bound

\[ \| v_0 \|^2_0 + t^2 \| v_1 \|^2_0 + t^2 |v_1|_1^2 \leq \| v_0 \|^2_0 + 2t^2 (\| v + c \|^2_0 + \| v_0 \|^2_0) + t^2 |v_1|_1^2 \]

\[ \leq 3\| v_0 \|^2_0 + 2t^2 \| v + c \|^2_0 + t^2 |v_1|_1^2. \]

If \( t \geq 1 \) then we select \( v_0 := v + c \) to conclude that

\[ \inf_{v + c = v_0 + v_1} \left( \| v_0 \|^2_0 + t^2 \| v_1 \|^2_0 + t^2 |v_1|_1^2 \right) \leq \| v + c \|^2_0. \]

Together this yields

\[ \int_0^\infty t^{-2s} \inf_{v + c = v_0 + v_1} \left( \| v_0 \|^2_0 + t^2 \| v_1 \|^2_0 + t^2 |v_1|_1^2 \right) \frac{dt}{t} \]

\[ \leq \int_0^1 t^{-2s} \inf_{v + c = v_0 + v_1} \left( 3\| v_0 \|^2_0 + 2t^2 \| v + c \|^2_0 + t^2 |v_1|_1^2 \right) \frac{dt}{t} + \int_1^\infty t^{-2s} \| v + c \|^2_0 \frac{dt}{t} \]

\[ = \int_0^1 t^{-2s} \inf_{v + c = v_0 + v_1} \left( 3\| v_0 \|^2_0 + t^2 |v_1|_1^2 \right) \frac{dt}{t} + \| v + c \|^2_0 \left( \int_0^1 2t^{1-2s} dt + \int_1^\infty t^{-1-2s} dt \right) \]

\[ \leq 3|v|_{L^2, H^1, s}^2 + \frac{1}{s(1-s)} \| v + c \|^2_0. \tag{2.3} \]

Therefore, recalling (2.2), we obtain

\[ |v|_{s, \inf}^2 \leq 3K^2 |v|_{L^2, H^1, s}^2 + \frac{K^2}{s(1-s)} \inf_{c \in \mathbb{R}} \| v + c \|^2_0, \]

which is the second assertion. \( \square \)

From the proof of the previous lemma one can conclude that the semi-norm \( | \cdot |_{L^2(O), H^1(O), s} \) is indeed the principal part of a norm in \( H^1(O) \). This will be useful to deduce a Poincaré-Friedrichs inequality with this semi-norm. First let us specify what we mean by the semi-norm being principal part of a norm.

\[ \sqrt{ } \]

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Corollary 2.5. Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There holds
\[
\|v\|_{s,\mathcal{O}}^2 \leq \frac{K^2}{s(1-s)}\|v\|_{0,\mathcal{O}}^2 + 3K^2\|v\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2
\]
for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $K = K(s,\mathcal{O})$ is the number from Proposition 2.1.

Proof. This is a combination of the second bound from Proposition 2.1 and (2.3) with $c = 0$. \qed

We are now ready to prove a second Poincaré-Friedrichs inequality.

Proposition 2.6 (Poincaré-Friedrichs inequality, interpolation semi-norm). Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain, and $s \in (0,1)$. Then there exists a constant $C_{PF,1} > 0$, depending on $\mathcal{O}$ and $s$, such that
\[
\|v\|_{0,\mathcal{O}} \leq C_{PF,1}(s,\mathcal{O})\left(\|v\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} + \int_{\mathcal{O}} |v| \right) \quad \forall v \in H^s(\mathcal{O}).
\]

Proof. Assume that the inequality is not true. Then there is a sequence $(v_j) \subset H^s(\mathcal{O})$ such that
\[
\|v_j\|_{0,\mathcal{O}} = 1, \quad |v_j|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} + \int_{\mathcal{O}} v_j \to 0 \quad (j \to \infty).
\]
Therefore, by Corollary 2.5 $(v_j)$ is bounded in $H^s(\mathcal{O})$ with respect to the Sobolev-Slobodeckij norm. Then, by Rellich’s theorem (see [14, Theorem 3.27]) there is a convergent subsequence (again denoted by $(v_j)$) in $L^2(\mathcal{O})$. Since $|v_j|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \to 0$ this sequence is Cauchy and with limit $v$ in $H^s(\mathcal{O})$. It holds $|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} = 0$ so that $v$ is constant. Furthermore, since $\int_{\mathcal{O}} v = 0$ and $\mathcal{O}$ is connected we conclude that $v = 0$, a contradiction to $\|v_j\|_{0,\mathcal{O}} = 1$. \qed

With the help of Proposition 2.6 we can now turn the estimate by Lemma 2.4 into a semi-norm equivalence.

Lemma 2.7. Let $\mathcal{O} \subset \mathbb{R}^n$ be a connected bounded Lipschitz domain. There holds
\[
k^2|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 \leq |v|_{s,\mathcal{O},inf}^2 \leq K^2\left(3 + \frac{C_{PF,1}^2}{s(1-s)}\right)|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2
\]
for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $k = k(s,\mathcal{O}), \quad K = K(s,\mathcal{O})$ are the numbers from Proposition 2.1 and $C_{PF,1} = C_{PF,1}(s,\mathcal{O})$ is the number from Proposition 2.6.

Proof. The lower bound is the one from Lemma 2.4. The upper bound is a combination of the upper bound from the same lemma and the Poincaré-Friedrichs’ inequality from Proposition 2.6. To this end note that the infimum $\inf_{c \in \mathbb{R}} \|v + c\|_{0,\mathcal{O}}$ is achieved by the same constant $c$ that eliminates the integral in the bound of the Poincaré-Friedrichs’ inequality for $v + c$. \qed

Meanwhile we have accumulated quite some parameters in the semi-norm estimates that depend on the order $s$ and the domain $\mathcal{O}$ under consideration. Our goal is to show equivalence of semi-norms which is uniform for a family of affinely transformed domains. We therefore study transformation properties of semi-norms in the following section. In this way, parameters from this section enter final results only via their values on a reference domain.
2.3 Transformation properties of norms and semi-norms

Obviously, both norms in $H^s(\mathcal{O})$ defined previously, $\| \cdot \|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s}$ and $\| \cdot \|_{s, \mathcal{O}}$, are not scalable. This could be achieved by weighting the $L^2(\mathcal{O})$-contributions according to the diameter of $\mathcal{O}$, for instance, cf. [8]. Of course, in this way one does not obtain uniformly equivalent norms (of un-weighted and weighted variants) under transformation of the domain.

This is different for the norm in $\tilde{H}^s(\mathcal{O})$. It can be easily fixed (to be scalable) by using that the semi-norm $| \cdot |_{1, \mathcal{O}}$ is a norm in $H^1_0(\mathcal{O})$, and re-defining

$$\|v\|_{[L^2(\mathcal{O}), H^1_0(\mathcal{O})], s} := \left( \int_0^\infty t^{-2s} \inf_{v=v_0+v_1, v_1 \in H^1_0(\mathcal{O})} \left( \|v_0\|_{0, \mathcal{O}}^2 + t^2 |v_1|_{1, \mathcal{O}}^2 \right) \frac{dt}{t} \right)^{1/2}$$

in the case of interpolation. In the case of the Sobolev-Slobodeckij norm one can ensure scalability by re-defining

$$\|v\|_{\tilde{H}^s(\mathcal{O})} := \|v\|_{\sim, s, \mathcal{O}} := \left( |v|^2_{\tilde{H}^s(\mathcal{O})} + \|v(x)\|_{\text{dist}(x, \partial \mathcal{O})^s}^2 \right)^{1/2}$$

since the last term guarantees positivity. In the following we will make use of these re-defined norms.

For a domain $\hat{\mathcal{O}} \subset \mathbb{R}^n$ we denote by $\mathcal{O} = F(\hat{\mathcal{O}})$ the affinely transformed domain

$$\mathcal{O} := \{ F\hat{x}; \hat{x} \in \hat{\mathcal{O}} \} \quad \text{with} \quad F\hat{x} = x_0 + B\hat{x}, \quad x_0 \in \mathbb{R}^n, \quad B \in \mathbb{R}^{n \times n}. \quad (2.4)$$

Here, $B$ is assumed to be invertible. Correspondingly, for a given real function $v$ defined on $\mathcal{O}$,

$$\hat{v} : \hat{\mathcal{O}} \to \mathbb{R} \quad \hat{x} \mapsto v(F\hat{x})$$

is the function transformed onto $\hat{\mathcal{O}}$.

**Lemma 2.8** (transformation properties of norms). Let $\hat{\mathcal{O}} \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\mathcal{O}$ be the affinely transformed domain defined by $(2.4)$. Then there hold the transformation properties

$$| \det B| \|B\|^{-2s} \|\hat{v}\|^2_{L^2(\hat{\mathcal{O}}), H^1_0(\hat{\mathcal{O}}), s} \leq \|v\|^2_{L^2(\mathcal{O}), H^1_0(\mathcal{O}), s} \leq | \det B| \|B^{-1}\|^{2s} \|\hat{v}\|^2_{L^2(\hat{\mathcal{O}}), H^1_0(\hat{\mathcal{O}}), s}, \quad (2.5)$$

$$| \det B| \|B\|^{-2s} \min\{ | \det B| \|B\|^{-n}, 1 \} \|\hat{v}\|^2_{\sim, s, \hat{\mathcal{O}}} \leq \|v\|^2_{\sim, s, \mathcal{O}} \leq | \det B| \|B^{-1}\|^{2s} \max\{ | \det B| \|B^{-1}\|^{-n}, 1 \} \|\hat{v}\|^2_{\sim, s, \hat{\mathcal{O}}} \quad (2.6)$$

for any $\hat{v} \in \tilde{H}^s(\hat{\mathcal{O}})$ and $s \in (0, 1)$. 


Proof. For the interpolation norm and \( \hat{O} \subseteq \mathbb{R}^1 \) being a cubes, this property (with an unspecified equivalence constant) has been shown in [11]. It is simply the scaling properties of the \( L^2 \) and \( H^1 \) norms together with the exactness of the K-method of interpolation (employed here). The proof generalizes to affine mappings in a straightforward way as follows. In Euclidean norm one has \( \| \nabla v(x) \| \leq B^{-1} \| \nabla \hat{v}(\hat{x}) \| \) so that the following relations are immediate,

\[
\| v \|_{L^2(\hat{O})}^2 = | \det B | \| \hat{v} \|_{L^2(\hat{O})}^2, \quad \| v \|_{H^1(\hat{O})}^2 \leq | \det B | \| B^{-1} \|_{L^2}^2 \| \hat{v} \|_{H^1(\hat{O})}^2.
\]

Then, with transformation \( r = \| B^{-1} \|_t \), we deduce that

\[
\| v \|_{L^2(\hat{O}), H^1(\hat{O}), s}^2 = \int_0^\infty t^{-2s} \inf_{v = v_0 + v_1, v_1 \in H^1(\hat{O})} \left( \| v_0 \|_{0, \hat{O}}^2 + t^2 \| v_1 \|_{1, \hat{O}}^2 \right) dt \leq | \det B | \int_0^\infty t^{-2s} \inf_{\hat{v} = \hat{v}_0 + \hat{v}_1, \hat{v}_1 \in H^1(\hat{O})} \left( \| \hat{v}_0 \|_{0, \hat{O}}^2 + t^2 \| B^{-1} \|_{L^2}^2 \| \hat{v}_1 \|_{1, \hat{O}}^2 \right) dt
\]

\[
= | \det B | \int_0^\infty (\| B^{-1} \|_{-1} r)^{-2s} \inf_{\hat{v} = \hat{v}_0 + \hat{v}_1, \hat{v}_1 \in H^1(\hat{O})} \left( \| \hat{v}_0 \|_{0, \hat{O}}^2 + r^2 \| \hat{v}_1 \|_{1, \hat{O}}^2 \right) dr
\]

\[
= | \det B | \| B^{-1} \|_{-2s} \| \hat{v} \|_{L^2(\hat{O}), H^1(\hat{O}), s}^2.
\]

This proves the upper bound in (2.5). The lower bound is verified by using the inverse transformation \( F^{-1} \) with matrix \( B^{-1} \).

The transformation property of the second norm is obtained similarly, see also [7, page 461] for the term of the double integral.

\[
\| v \|_{\infty, s, \hat{O}}^2 = \int_\hat{O} \int_\hat{O} |v(x) - v(y)|^2 |x - y|^{-n+2s} dx dy + \int_\hat{O} \left( \frac{v(x)}{\operatorname{dist}(x, \partial \hat{O})^s} \right)^2 dx
\]

\[
\leq | \det B |^2 \int_\hat{O} \int_\hat{O} |\hat{v}(\hat{x}) - \hat{v}(\hat{y})|^2 |\hat{x} - \hat{y}|^{-n+2s} d\hat{x} d\hat{y} + | \det B | \int_\hat{O} \left( \frac{\hat{v}(\hat{x})}{\| B^{-1} \|_{-s} \operatorname{dist}(\hat{x}, \partial \hat{O})^s} \right)^2 d\hat{x}
\]

\[
\leq | \det B | \| B^{-1} \|_{-2s} \max \{ | \det B | \| B^{-1} \|_{n, 1} \} \| \hat{v} \|_{\infty, s, \hat{O}}^2.
\]

This is the upper bound in (2.6). Analogously one finds that

\[
\| \hat{v} \|_{\infty, s, \hat{O}}^2 \leq | \det B^{-1} | \| B \|_{-2s} \max \{ | \det B^{-1} | \| B \|_{n, 1} \} \| v \|_{\infty, s, \hat{O}}^2.
\]

This proves the lower bound in (2.6).

Lemma 2.9 (transformation properties of semi-norms). Let \( \hat{O} \subseteq \mathbb{R}^n \) be a bounded Lipschitz domain and let \( O \) be the affinely transformed domain defined by (2.4). Then there hold the transformation properties

\[
| \det B | \| B \|_{-2s} \| \hat{v} \|_{L^2(\hat{O}), H^1(\hat{O}), s}^2 \leq | \det B | \| B^{-1} \|_{2s} \| \hat{v} \|_{L^2(\hat{O}), H^1(\hat{O}), s}^2
\]

\[
(2.7)
\]

\[
| \det B | \| B \|_{-n-2s} \| \hat{v} \|_{s, \hat{O}}^2 \leq | \det B | \| B^{-1} \|_{n+2s} \| \hat{v} \|_{s, \hat{O}}^2
\]

\[
(2.8)
\]

for any \( \hat{v} \in H^s(\hat{O}) \) and \( s \in (0, 1) \).
Proof. The proof is basically identical to the one of Lemma 2.8.

The third semi-norm, $|\cdot|_{s,\Omega,\inf}$, behaves under affine transformations as follows.

**Lemma 2.10.** Let $\hat{\Omega} \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\Omega$ be the affinely transformed domain defined by (2.4). Then there hold the transformation properties

\[
|\det B|^2 \|B\|^{-n-2s} \hat{v}^2_{s,\hat{\Omega}} + |\det B| \inf_{c \in \mathbb{R}} \|\hat{v} + c\|_{0,\hat{\Omega}}^2 \leq |v|^2_{s,\Omega,\inf}
\]

\[
\leq |\det B|^2 \|B^{-1}\|^{n+2s} \hat{v}^2_{s,\hat{\Omega}} + |\det B| \inf_{c \in \mathbb{R}} \|\hat{v} + c\|_{0,\hat{\Omega}}^2
\]

for any $\hat{v} \in H^s(\hat{\Omega})$ and $s \in (0,1)$.

Proof. This result is immediate from the representation of the semi-norm given in Lemma 2.3 and the transformation properties of the $|\cdot|_s$-semi-norm by Lemma 2.9 and of the $L^2$-norm.

3 Main results

We are now ready to state and prove our main results on certain equivalences of fractional-order Sobolev semi-norms. We use the notation (2.4) from Section 2.3 for affine transformations. In particular, we assume that the domain $\Omega$ under consideration is the affine image of a bounded Lipschitz domain $\hat{\Omega}$. The following results specify how equivalence constants depend on the affine map. At the end of this section we conclude the equivalence of some semi-norms which is uniform for a family of so-called shape regular domains (Theorem 3.4) and some scaling properties (Corollary 3.5). These results are of importance for the approximation theory of piecewise polynomial spaces in fractional-order Sobolev spaces.

The first theorem shows the equivalence of the semi-norms $|\cdot|_{L^2(\Omega),H^1(\Omega),s}$ and $|\cdot|_{s,\Omega}$.

**Theorem 3.1.** Let $\hat{\Omega} \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain and let $\Omega$ be the affinely transformed domain defined by (2.4). Then there hold the following relations.

(i) $|v|^2_{s,\Omega} \leq \det B \|B^{-1}\|^{n+2s} \|B\|^{2s} K(s,\hat{\Omega})^2 \left(3 + \frac{C_{PF,1}(s,\hat{\Omega})^2}{s(1-s)}\right) |v|^2_{L^2(\Omega),H^1(\Omega),s}$

for any $v \in H^s(\Omega)$ and $s \in (0,1)$ with $K(s,\hat{\Omega})$ from Proposition 2.1 and $C_{PF,1}(s,\hat{\Omega})$ from Proposition 2.6.

(ii) $|v|^2_{L^2(\Omega),H^1(\Omega),s} \leq \det B \|B^{-1}\|^{n+2s} \|B^{-1}\|^{2s} k(s,\hat{\Omega})^{-2} \left(1 + C_{PF,SS}(s,\hat{\Omega})^2\right) |v|^2_{s,\Omega}$

for any $v \in H^s(\Omega)$ and $s \in (0,1)$ with $k(s,\hat{\Omega})$ from Proposition 2.7 and $C_{PF,SS}(s,\hat{\Omega})$ from Proposition 2.8.
Proof. On a fixed domain $\hat{\mathcal{O}}$ we obtain, by combining Lemmas 2.3 and 2.7 the equivalence of semi-norms:

$$|\hat{\nu}|_{s,\hat{\mathcal{O}},\inf}^2 \leq K(s,\hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,1}(s,\hat{\mathcal{O}})^2}{s(1-s)}\right)|\hat{\nu}|_{L^2(\hat{\mathcal{O}}, H^1(\hat{\mathcal{O}}), s)}^2$$

and

$$|\hat{\nu}|_{L^2(\hat{\mathcal{O}}, H^1(\hat{\mathcal{O}}), s)}^2 \leq k(s,\hat{\mathcal{O}})^{-2} |\hat{\nu}|_{s,\hat{\mathcal{O}},\inf}^2 \leq k(s,\hat{\mathcal{O}})^{-2} \left(1 + C_{PF,SS}(s,\hat{\mathcal{O}})^2\right)|\hat{\nu}|_{s,\hat{\mathcal{O}}}^2.$$  

The first assertion of the theorem then follows by combining (3.9) with the transformation properties of the semi-norms by Lemma 2.9

$$|v|_{s,\mathcal{O}}^2 \leq |\det B|^2 \|B^{-1}\|^{n+2s} |\hat{\nu}|_{s,\hat{\mathcal{O}}}^2$$

$$\leq |\det B|^2 \|B^{-1}\|^{n+2s} K(s,\hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,1}(s,\hat{\mathcal{O}})^2}{s(1-s)}\right)|\hat{\nu}|_{L^2(\hat{\mathcal{O}}, H^1(\hat{\mathcal{O}}), s)}^2$$

$$\leq |\det B| \|B^{-1}\|^{n+2s} \|B\|^{2s} K(s,\hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,1}(s,\hat{\mathcal{O}})^2}{s(1-s)}\right)|v|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s}^2.$$  

The second assertion of the theorem is proved by a combination of (3.10) with the transformation properties by Lemma 2.9.

The next two theorems study the other pairs of semi-norms for equivalence in combination with affine maps, $(|\cdot|_{s,\mathcal{O}}, |\cdot|_{s,\mathcal{O},\inf})$ and $(|\cdot|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s}, |\cdot|_{s,\mathcal{O},\inf})$.

**Theorem 3.2.** Let $\hat{\mathcal{O}} \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain and let $\mathcal{O}$ be the affinely transformed domain defined by (2.3). Then there hold the following relations.

(i) $|v|_{s,\mathcal{O}} \leq |v|_{s,\mathcal{O},\inf} \quad \forall v \in H^s(\mathcal{O}), \forall s \in (0, 1),$

(ii) $|v|_{s,\mathcal{O},\inf}^2 \leq \left(1 + |\det B|^{-1} \|B\|^{n+2s} C_{PF,SS}(s,\hat{\mathcal{O}})^2\right)|v|_{s,\mathcal{O}}^2 \quad \forall v \in H^s(\mathcal{O}), \forall s \in (0, 1)$

with $C_{PF,SS}(s,\hat{\mathcal{O}})$ being the number from Proposition 2.2.

**Proof.** Assertion (i) is a repetition of the first estimate in Lemma 2.3.

To show the second assertion we use Proposition 2.2 and Lemma 2.9 to deduce that

$$\inf_{c \in \mathbb{R}} \|v + c\|_{0,\hat{\mathcal{O}}}^2 = |\det B| \inf_{c \in \mathbb{R}} \|\hat{\nu} + c\|_{0,\hat{\mathcal{O}}}^2 \leq |\det B| C_{PF,SS}(s,\hat{\mathcal{O}})^2 |\hat{\nu}|_{s,\hat{\mathcal{O}}}^2$$

$$\leq |\det B|^{-1} \|B\|^{n+2s} C_{PF,SS}(s,\hat{\mathcal{O}})^2 |v|_{s,\mathcal{O}}^2.$$

The assertion then follows by the definition of the semi-norm $|\cdot|_{s,\mathcal{O},\inf}$.
**Theorem 3.3.** Let $\hat{\mathcal{O}} \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain and let $\mathcal{O}$ be the affinely transformed domain defined by (2.4). Then there hold the following relations.

(i) $$\|v\|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \leq \|B^{-1}\|^{2s} \max\{|\det B|^{-1}\|B\|^{n+2s}, 1\} \ k(s, \hat{\mathcal{O}})^{-2} \|v\|_{s, \mathcal{O}, \inf}^2$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$ with $k(s, \hat{\mathcal{O}})$ from Proposition 2.7.

(ii) $$\|v\|_{s, \mathcal{O}, \inf}^2 \leq \|B\|^{2s} \max\{|\det B| \|B^{-1}\|^{n+2s}, 1\} \ K(s, \hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,1}(s, \hat{\mathcal{O}})^2}{s(1-s)}\right) \|v\|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s}^2$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$ with $K(s, \hat{\mathcal{O}})$ from Proposition 2.7 and $C_{PF,1}(s, \hat{\mathcal{O}})$ from Proposition 2.6.

**Proof.** By Lemmas 2.9, 2.7, and 2.10 we obtain

$$\|v\|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \leq |\det B| \|B^{-1}\|^{2s} |\hat{v}|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s}^2 \leq |\det B| \|B^{-1}\|^{2s} k(s, \hat{\mathcal{O}})^{-2} |\hat{v}|_{s, \mathcal{O}, \inf}^2$$

$$\leq |\det B| \|B^{-1}\|^{2s} k(s, \hat{\mathcal{O}})^{-2} \left(|\det B|^{-2}\|B\|^{n+2s} |v|_{s, \mathcal{O}}^2 + |\det B|^{-1} \inf_{c \in \mathbb{R}} \|v + c\|_{0, \mathcal{O}}^2\right)$$

$$\leq \|B^{-1}\|^{2s} \max\{|\det B|^{-1}\|B\|^{n+2s}, 1\} k(s, \hat{\mathcal{O}})^{-2} \|v\|_{s, \mathcal{O}, \inf}^2.$$

This is the first assertion. The second one follows analogously by the same lemmas:

$$\|v\|_{s, \mathcal{O}, \inf}^2 \leq |\det B| |\hat{v}|_{s, \mathcal{O}}^2 + |\det B| \inf_{c \in \mathbb{R}} \|\hat{v} + c\|_{0, \mathcal{O}}^2$$

$$\leq |\det B| \max\{|\det B| |\hat{v}|_{s, \mathcal{O}}, 1\} \|\hat{v}\|_{s, \mathcal{O}, \inf}^2$$

$$\leq |\det B| \max\{|\det B| |\hat{v}|_{s, \mathcal{O}}, 1\} \ K(s, \hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,1}(s, \hat{\mathcal{O}})^2}{s(1-s)}\right) |\hat{v}|_{s, \mathcal{O}, \inf}^2$$

$$\leq \max\{|\det B| |\hat{v}|_{s, \mathcal{O}}, 1\} \|B\|^{2s} K(s, \hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,1}(s, \hat{\mathcal{O}})^2}{s(1-s)}\right) |v|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s}^2.$$

\[\square\]

We end this section with establishing uniform equivalence of the semi-norms $|\cdot|_{s, \mathcal{O}}$ and $|\cdot|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s}$ for shape-regular domains. Three of the four remaining bounds for other combinations of semi-norms are uniform under further restrictions on the diameter of the domain.

Let us introduce some notation. We consider a bounded, connected Lipschitz domain $\hat{\mathcal{O}} \subset \mathbb{R}^n$ and maps of $\hat{\mathcal{O}}$ onto domains $\mathcal{O}$ where the ratio $\rho_{\mathcal{O}} := D_{\mathcal{O}}/d_{\mathcal{O}}$ is controlled. Here, $D_{\mathcal{O}}$ denotes the diameter of $\mathcal{O}$ and $d_{\mathcal{O}}$ is the supremum of the diameters of all balls contained in $\mathcal{O}$. In the case of finite elements (or convex polygons) boundedness of $\rho$ is referred to as shape regularity of $\mathcal{O}$. Also, when defining $d_{\mathcal{O}}$ with balls with respect to which $\mathcal{O}$ is star-shaped, then $\rho_{\mathcal{O}}$ is referred to as chunkiness parameter.
Using the notation \[ \text{cf., e.g., [3]} \] there holds
\[
\|B\| \leq \frac{D_\Omega}{d_\Omega} = \frac{D_\hat{\Omega}}{D_\Omega} \rho_\Omega, \quad \|B^{-1}\| \leq \frac{D_\Omega}{D_\hat{\Omega}} = \frac{D_\Omega}{D_\hat{\Omega}} \rho_\hat{\Omega}, \quad \|B\| \|B^{-1}\| \leq \rho_\Omega \rho_\hat{\Omega},
\] (3.11)
cf., e.g., [3]. Furthermore, we conclude that
\[
|\det B| = \frac{|\Omega|}{|\hat{\Omega}|} \leq \frac{D_\Omega^n}{D_\hat{\Omega}^n} \rho_\Omega^n \rho_\hat{\Omega}^{-1} \leq \frac{D_\Omega^n}{D_\hat{\Omega}^n} \rho_\Omega^n \rho_\hat{\Omega}^{-1} \quad \text{(3.12)}
\]
With this notation, the results of Theorems 3.1, 3.2 imply the following.

**Theorem 3.4.** Let \( \Omega \) be the affine map of a bounded connected Lipschitz domain \( \hat{\Omega} \subset \mathbb{R}^n \), cf. (2.4).

(i) The semi-norms \(| \cdot |_{s,\Omega} \) and \(| \cdot |_{L^2(\Omega),H^1(\Omega),s} \) are uniformly equivalent for a family of shape-regular domains \( \Omega \):
\[
|v|^2_{s,\Omega} \leq \rho_\Omega^n \rho_\hat{\Omega}^{n+2s} K(s, \hat{\Omega})^2 \left( 1 + \frac{C_{PF,1}(s, \hat{\Omega})^2}{s(1-s)} \right) |v|^2_{L^2(\Omega),H^1(\Omega),s},
\]
\[
|v|^2_{L^2(\Omega),H^1(\Omega),s} \leq \rho_\Omega^n \rho_\hat{\Omega}^{n+2s} k(s, \hat{\Omega})^{-2} \left( 1 + C_{PF,SS}(s, \hat{\Omega})^2 \right) |v|^2_{s,\Omega}
\]
for any \( v \in H^2(\Omega) \) and \( s \in (0,1) \). Here, \( k(s, \hat{\Omega}), K(s, \hat{\Omega}) \) are the numbers from Proposition 2.4 and \( C_{PF,SS}(s, \hat{\Omega}), C_{PF,1}(s, \hat{\Omega}) \) are as in Propositions 2.2, 2.6, respectively.

(ii) The semi-norms \(| \cdot |_{s,\Omega} \) and \(| \cdot |_{s,\Omega,inf} \) are uniformly equivalent for a family of uniformly bounded, shape-regular domains \( \Omega \):
\[
|v|_{s,\Omega} \leq |v|_{s,\Omega,inf},
\]
\[
|v|^2_{s,\Omega,inf} \leq \left( 1 + \frac{D_\Omega^n}{D_\hat{\Omega}^{n+2s}} \rho_\Omega^n \rho_\hat{\Omega}^{n+2s} \right) |v|^2_{s,\Omega}
\]
for any \( v \in H^2(\Omega) \) and \( s \in (0,1) \). Here, \( C_{PF,SS}(s, \hat{\Omega}) \) is the number from Proposition 2.2.

(iii) a) For a family of shape-regular domains \( \Omega \) whose diameters are bounded from below by a positive constant, the semi-norm \(| \cdot |_{L^2(\Omega),H^1(\Omega),s} \) is uniformly bounded by \(| \cdot |_{s,\Omega,inf} \):
\[
|v|^2_{L^2(\Omega),H^1(\Omega),s} \leq \max\{\rho_\Omega^n \rho_\hat{\Omega}^{-2s}, D_\Omega^{-2s} D_\hat{\Omega}^{2s}\} \rho_\hat{\Omega}^2 k(s, \hat{\Omega})^{-2} |v|^2_{s,\Omega,inf}
\]
for any \( v \in H^2(\Omega) \) and \( s \in (0,1) \).

b) For a family of uniformly bounded, shape-regular domains \( \Omega \), the semi-norm \(| \cdot |_{s,\Omega,inf} \) is uniformly bounded by \(| \cdot |_{L^2(\Omega),H^1(\Omega),s} \):
\[
|v|^2_{s,\Omega,inf} \leq \max\{\rho_\Omega^n \rho_\hat{\Omega}^{n+2s}, D_\Omega^{-2s} D_\hat{\Omega}^{2s}\} \rho_\hat{\Omega}^2 K(s, \hat{\Omega})^2 \left( 1 + \frac{C_{PF,1}(s, \hat{\Omega})^2}{s(1-s)} \right) |v|^2_{L^2(\Omega),H^1(\Omega),s}
\]
for any \( v \in H^s(\mathcal{O}) \) and \( s \in (0, 1) \).

Here, \( k(s, \hat{\mathcal{O}}) \), \( K(s, \hat{\mathcal{O}}) \) are the parameters from Proposition 2.1, and \( C_{PF,1}(s, \hat{\mathcal{O}}) \) is the number from Proposition 2.6.

Proof. The assertions (i)–(iii) are a combination of Theorems 3.1–3.3 respectively, with the bounds provided by (3.11), (3.12).

The uniform equivalence of the semi-norms \( \| \cdot \|_{s, \mathcal{O}} \) and \( \| \cdot \|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \) for shape-regular domains is based on what one calls their scaling property. It means that both semi-norms for functions on a domain \( \mathcal{O} \) are uniformly equivalent to the respective semi-norm of the affinely transformed functions onto a fixed domain \( \hat{\mathcal{O}} \), when one of the semi-norms is multiplied by an appropriate number (it is a power of the diameter of \( \mathcal{O} \)). This property applies also to the norms \( \| \cdot \|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \) and \( \| \cdot \|_{\sim, s, \mathcal{O}} \), cf. Lemma 2.8. Scaling properties are relevant for the error analysis of piecewise polynomial approximations. We formulate the result as a corollary to Lemmas 2.8 and 2.9.

**Corollary 3.5.** The norms \( \| \cdot \|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \), \( \| \cdot \|_{\sim, s, \mathcal{O}} \) and semi-norms \( \| \cdot \|_{s, \mathcal{O}} \), \( \| \cdot \|_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \) are scalable of order \( D_{\hat{\mathcal{O}}}^{-2s} \):

\[
D_{\hat{\mathcal{O}}}^{n-2s} \rho_{\hat{\mathcal{O}}}^{-n} D_{\hat{\mathcal{O}}}^{2s-n} \rho_{\hat{\mathcal{O}}}^{-2s} \| \hat{v} \|^2_{L^2(\hat{\mathcal{O}}), H^1(\hat{\mathcal{O}}), s} \leq \| v \|^2_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \leq D_{\hat{\mathcal{O}}}^{n-2s} \rho_{\hat{\mathcal{O}}}^{2s-2n} \rho_{\hat{\mathcal{O}}}^{-n} \| \hat{v} \|^2_{L^2(\hat{\mathcal{O}}), H^1(\hat{\mathcal{O}}), s},
\]

\[
D_{\hat{\mathcal{O}}}^{n-2s} \rho_{\hat{\mathcal{O}}}^{-n} D_{\hat{\mathcal{O}}}^{2s-n} \rho_{\hat{\mathcal{O}}}^{-2s} \min\{\rho_{\hat{\mathcal{O}}}^{-n} \rho_{\hat{\mathcal{O}}}^{-1} \} \| \hat{v} \|^2_{\sim, s, \hat{\mathcal{O}}} \leq \| v \|^2_{\sim, s, \mathcal{O}} \leq D_{\hat{\mathcal{O}}}^{n-2s} \rho_{\hat{\mathcal{O}}}^{2s-2n} \rho_{\hat{\mathcal{O}}}^{-n} \max\{\rho_{\hat{\mathcal{O}}}^{n} \rho_{\hat{\mathcal{O}}}^{-1} \} \| \hat{v} \|^2_{\sim, s, \hat{\mathcal{O}}}
\]

for any \( v \in \tilde{H}^s(\mathcal{O}) \) and \( s \in (0, 1) \), and

\[
D_{\hat{\mathcal{O}}}^{n-2s} \rho_{\hat{\mathcal{O}}}^{-n} D_{\hat{\mathcal{O}}}^{2s-n} \rho_{\hat{\mathcal{O}}}^{-2s} \| v \|^2_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \leq \| v \|^2_{L^2(\mathcal{O}), H^1(\mathcal{O}), s} \leq D_{\hat{\mathcal{O}}}^{n-2s} \rho_{\hat{\mathcal{O}}}^{2s-2n} \rho_{\hat{\mathcal{O}}}^{-n} \| v \|^2_{L^2(\mathcal{O}), H^1(\mathcal{O}), s},
\]

\[
D_{\hat{\mathcal{O}}}^{n-2s} \rho_{\hat{\mathcal{O}}}^{-n} D_{\hat{\mathcal{O}}}^{2s-n} \rho_{\hat{\mathcal{O}}}^{-2s} \| v \|^2_{s, \mathcal{O}} \leq \| v \|^2_{s, \mathcal{O}} \leq D_{\hat{\mathcal{O}}}^{n-2s} \rho_{\hat{\mathcal{O}}}^{n+2s} D_{\hat{\mathcal{O}}}^{s-n} \rho_{\hat{\mathcal{O}}}^{-n} \| v \|^2_{s, \hat{\mathcal{O}}}
\]

for any \( v \in H^s(\mathcal{O}) \) and \( s \in (0, 1) \).

Proof. The bounds are a combination of Lemmas 2.8 and 2.9 with (3.11), (3.12).

**Remark 3.6.** The estimate by Theorem 3.4 (iii) a) breaks down when \( D_{\hat{\mathcal{O}}} \to 0 \). In fact, for a family of scaled domains \( \mathcal{O}_h \) with \( D_{\mathcal{O}_h} = h \) and a non-constant function \( v \) scaled to a family \( \{v_h\} \) of functions on \( \{\mathcal{O}_h\} \), \( \|v_h\|^2_{L^2(\mathcal{O}_h), H^1(\mathcal{O}_h), s} \simeq h^{n-2s} \) by Corollary 3.5 whereas \( \|v_h\|_{s, \mathcal{O}_h, \inf} \geq \inf_{c \in \mathbb{R}} \|v_h - c\|^2_{\mathcal{O}_h} \simeq h^n \). Therefore, the dependence on \( D_{\hat{\mathcal{O}}} \) like \( D_{\hat{\mathcal{O}}}^{-2s} \) of the upper bound in Theorem 3.4 (iii) a) is optimal.

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