Hamiltonian flows on null curves

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Abstract

The local motion of a null curve in Minkowski 3-space induces an evolution equation for its Lorentz invariant curvature. Special motions are constructed whose induced evolution equations are the members of the Korteweg–de Vries (KdV) hierarchy. The null curves which move under the KdV flow without changing shape are proven to be the trajectories of a certain particle model on null curves described by a Lagrangian linear in the curvature. In addition, we show that the curvature of a null curve which evolves by similarities can be computed in terms of the solutions of the second Painlevé equation.

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1. Introduction

Many completely integrable nonlinear PDEs (soliton equations) describe the evolution of differential invariants associated with curves moving in a homogeneous space ([2, 5, 6, 8, 13, 14, 17, 20, 21, 29, 35]). Some of these motions share the property that the curves which evolve by congruences of the ambient space have both a variational and a Hamiltonian description: as extremals of a geometric variational problem defined by the conserved integrals of the corresponding soliton equation and as solutions of an integrable (finite dimensional) contact Hamiltonian system. Examples include the modified Korteweg–de Vries (mKdV) equation describing the (geodesic) curvature evolution induced by a local motion of curves in two-dimensional space-forms ([13, 26]). See also [15, 22, 23] for other examples.

In this paper, we investigate the local motion of null curves in Minkowski 3-space and find local motions inducing the Korteweg–de Vries (KdV) hierarchy of equations. (The motion of spacelike and timelike curves in Minkowski 3-space has recently been related to integrable equations from the AKNS hierarchy in [7].) Our approach is similar in spirit to that used...
in [13], where integrable equations from the mKdV hierarchy are related to local motions of curves in the plane. We then provide a variational description of null curves which move by Lorentzian rigid motions under the KdV flow and show that they solve a finite dimensional integrable Hamiltonian system. In this regard, motivations are provided by recent studies on relativistic particle models on null curves ([9, 12, 19, 31]).

Let $\mathbb{R}^{2,1}$ be Minkowski 3-space and $\gamma \subset \mathbb{R}^{2,1}$ a null curve parametrized by the natural (pseudo-arc) parameter $s$ which normalizes the derivative of its tangent vector field. It is known that in general $\gamma$ is uniquely determined up to Lorentz transformations by a Lorentzian invariant function $\kappa_\gamma(s)$, called the curvature of $\gamma$ ([1, 4, 9, 16]). We show (cf theorem 1) that the local motion of a null curve in Minkowski 3-space induces a local evolution equation for its curvature of the form

$$\frac{\partial \kappa}{\partial t} = DD^{-1} p[\kappa],$$

(1.1)

where $D$ is the total derivative operator with respect to $s$, $\mathcal{D} = D^3 + 4\kappa D + 2\kappa s$ and $p[\kappa] = p(\kappa, \kappa s, \ldots)$ is a differential polynomial such that $\kappa s p$ is a total derivative.Interestingly, the right-hand side of (1.1) is expressed in terms of the operators which define the bi-Hamiltonian structure of the KdV equation ([32]). For a particular sequence of differential functions $p$, we show that (1.1) coincide with the equations of the KdV hierarchy, hence providing a new geometric interpretation of the KdV flows (cf theorem 2).

We then discuss the motion of null curves corresponding to the travelling wave solutions of the KdV equation. Such curves evolve under the KdV flow by Lorentzian rigid motions, retaining their shape. Theorem 3 shows that they coincide with the critical points of the variational problem on null curves defined by the Lorentz invariant functional

$$\mathcal{L}_\lambda(\gamma) = \int_\gamma (2\kappa_\gamma + \lambda) \, ds, \quad \lambda \in \mathbb{R}.$$  

(1.2)

Functionals of this type have been considered in the literature as action functionals of natural geometrical particle models for null trajectories in three- and four-dimensional spacetimes of constant curvature (cf [3, 9, 19, 30, 31, 34] and the references there). The integration of the worldlines can be achieved by quadratures and the explicit formulae of the natural parametrizations can be given in terms of Weierstrass elliptic functions [12]. The main point in the integration of the extremal curves is the existence of a Lax pair encoding the Euler–Lagrange equations of (1.2). Remarkably, the Lax pair can be directly deduced from the invariance of the trajectory with respect to the KdV dynamics. Finally, in theorem 4, we show that if the shape of a null curve evolves by similarities under the KdV flow then the curvature function can be integrated in terms of the solutions of the second Painlevé equation. This provides a geometric interpretation of the similarity reduction of the KdV equation investigated in [18].

2. Preliminaries

In this section we summarize some background material, referring to [32, 33] for a complete exposition.

Let $J_h(\mathbb{R}, \mathbb{R}) = \mathbb{R} \times \mathbb{R}^{h+1}$ denote the space of $h$th order jets of smooth $\mathbb{R}$-valued functions $u$ of one independent variable $s$ with coordinates $s, u(0), u(1), \ldots, u(h)$. The jet space $J_h(\mathbb{R}, \mathbb{R})$ is endowed with the contact system generated by the 1-forms

$$\zeta_j = du(j-1) - u(j) \, ds, \quad j = 1, \ldots, h$$

The study can actually be extended to null curves in three-dimensional Lorentzian space forms.
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and independent condition $ds$. The projective limit of the sequence
\[ \cdots \rightarrow J_h(\mathbb{R}, \mathbb{R}) \rightarrow J_{h-1}(\mathbb{R}, \mathbb{R}) \rightarrow \cdots \rightarrow J_1(\mathbb{R}, \mathbb{R}) \rightarrow J_0(\mathbb{R}, \mathbb{R}) \]  
(2.1)
is the total jet space of $\mathbb{R}$-valued smooth functions of one independent variable. It is denoted by $J(\mathbb{R}, \mathbb{R})$. If $u : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function, its prolongation of order $h$, $j_h(u) : I \rightarrow J_h(\mathbb{R}, \mathbb{R})$, is the integral curve of the contact system given by
\[ j_h(u) : s \mapsto \left( s, u \left|_s, \frac{du}{ds}, \ldots, \frac{d^h u}{ds^h} \right. \right). \]

A smooth map $w : J(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ is said a polynomial differential function (differential polynomial) of order $h$ if there exists a polynomial $w \in \mathbb{R}[x_0, \ldots, x_h]$ such that
\[ w(u) = w(u_0, u_1, \ldots, u_h), \]
for each $u = (s, u_0, u_1, \ldots, u_h) \in J(\mathbb{R}, \mathbb{R})$. For a polynomial differential function $w$, we will write $w[u]$ to recall that $w$ depends on $u$ and the derivatives of $u$. The algebra of polynomial differential functions, $J[u]$, is endowed with a derivation, called the total derivative, defined by
\[ D_w = \sum_{p=0}^{\infty} \frac{\partial w}{\partial u(p)} u(p+1). \]

A differential function $w \in J[u]$ is a total derivative if there exists $p \in J[u]$ such that $w = D(p)$. The ‘primitive’ $p$ is unique up to an additive constant. By $\int w \, ds$ we denote the unique primitive of $w$ which vanishes at $u = 0$.

On $J[u]$ there is a natural differential operator $E$, the Euler operator, defined by
\[ E(w) = \sum_{t=0}^{\infty} (-1)^t D^t \left( \frac{\partial w}{\partial u(t)} \right). \]
Note that $E(w)$ is the gradient of the functional $\mathcal{W}$ defined by
\[ \mathcal{W} : u \mapsto \int w[u] \, ds \]
and that it coincides with the variational derivative $\delta \mathcal{W}/\delta u$. We also recall a few more basic facts:

• $w \in J[u]$ is a total derivative if and only if $E(w) = 0$;
• $u$ is an extremal of the functional $\mathcal{W}$ if and only if $E(w)[u] = 0$;
• for each $w \in J[u]$, $u_1, E(w)$ is a total derivative (cf [33, theorem 7.36]);
• $w(u) = \int_0^1 E(w)[\epsilon u] \cdot u \, d\epsilon$.

Next, consider the operator $D : J[u] \rightarrow J[u]$, defined by
\[ D = D^3 + 4u(0)D + 2u(1). \]  
(2.2)
According to [11] (cf also [25, 32]), there exist two sequences $\{g_n\}$ and $\{p_n\}$ of polynomial differential functions satisfying the Lenard recursion formula:
\[ Dg_n = Dg_{n-1}, \quad g_0 = E(p_0), \quad n = 1, 2, \ldots, \]  
(2.3)
where
\[ g_0 = \frac{1}{2}, \quad p_0 = \frac{1}{2}u(0). \]
The first $g_n$’s are
\[ g_1 = u, \quad g_2 = 3u^2 + u_{ss}, \quad g_3 = 10u^3 + 10u_{ss}u + 5u_s^2 + u_{sss}. \]
The Lenard recursion formula leads to the two Hamiltonian representations of the KdV hierarchy of integrable evolution equations, namely
\[ u_t = -D g_n [u] = -D g_{n-1} [u], \quad n = 1, 2, \ldots, \] (2.4)
where the right-hand sides depend only on \( u \) and its derivatives with respect to \( s \). The first member of the hierarchy is the wave equation \( u_t + u_s = 0 \), while the second is the KdV equation in the form
\[ u_t = -6uu_s - u_{sss}. \]
The next member is the fifth order equation
\[ u_t = -30usu - 10usss u - 20ususs - u_5 s. \]

Remark 1. The \( g_n \)'s are the gradients of the Gardner–Kruskal–Miura \([11]\) sequence of conserved functionals
\[ P_n(u) = \int p_n [u] \, ds \]
of the KdV hierarchy. The first three of them are
\[ P_0 = \int \frac{u}{2} \, ds, \quad P_1 = \int \frac{u^2}{2} \, ds, \quad P_2 = \int \left( u^3 - u_s^2 \right) \, ds. \]

3. Null curves in Minkowski 3-space

3.1. Null curves and frames
Let \( \mathbb{R}^{2,1} \) denote affine Minkowski 3-space with the Lorentzian inner product
\[ \langle x, y \rangle = -(x^1 y^3 + x^3 y^1) + x^2 y^2 = x^i g_{ij} y^j. \]
We fix an orientation on \( \mathbb{R}^{2,1} \) by requiring that the standard basis \( (e_1, e_2, e_3) \) is positive, and fix a time-orientation by saying that a timelike or null (lightlike) vector \( x \neq 0 \) is future-directed if \( \langle x, e_1 + e_3 \rangle < 0 \). Let \( \mathbb{E}(2, 1) = \mathbb{R}^{2,1} \rtimes \text{SO}_0(2, 1) \) denote the restricted Poincaré group, i.e. the group of isometries of \( \mathbb{R}^{2,1} \) preserving the given orientations. The elements of \( \mathbb{E}(2, 1) \) can be viewed as affine frames \( (x, \alpha) \) consisting of a point \( x \in \mathbb{R}^{2,1} \) and a positive basis \( \alpha = (a_1, a_2, a_3) \) such that \( a_1, a_3 \) are future-directed null vectors and
\[ \langle a_i, a_j \rangle = g_{ij}, \quad i, j \in \{1, 2, 3\}. \] (3.1)
We will think of \( \mathbb{E}(2, 1) \) as the closed subgroup of \( \text{GL}(4, \mathbb{R}) \) whose elements are of the form
\[ X(x, \alpha) = \begin{pmatrix} 1 & 0 \\ x & a \end{pmatrix}, \]
where \( (x, \alpha) \in \mathbb{E}(2, 1) \). Correspondingly, the Lie algebra \( \mathfrak{e}(2, 1) \) of \( \mathbb{E}(2, 1) \) is the subalgebra of \( \mathfrak{gl}(4, \mathbb{R}) \) of all \( 4 \times 4 \) matrices of the form
\[ X(q, v) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q & v^2 & v^3 & 0 \\ q^2 & v^3 & 0 & v^1 \\ q^3 & 0 & v^1 & -v^2 \end{pmatrix}. \]

A null curve in \( \mathbb{R}^{2,1} \) is a regular, smooth parametrized curve \( \gamma : I \to \mathbb{R}^{2,1} \), defined on some interval \( I \subset \mathbb{R} \), such that the velocity vector \( \gamma'(t) \) is a future-directed null vector, for
each \( t \in I \). Since \( \gamma'(t) \) is null, \( \langle \gamma'(t), \gamma''(t) \rangle = 0 \), and \( \gamma'' \) has to be spacelike or proportional to \( \gamma' \). Away from flex points\(^4\), the differential 1-form

\[
\omega_\gamma = \| \gamma''(t) \|^{1/2} dt
\]

is never zero, and is invariant under changes of parameter and the action of the Poincaré group. The integral of \( \omega_\gamma \) can then be used to introduce a natural parameter (or pseudo-arc parameter) \( s \), intrinsically defined by \( \gamma \), such that

\[
\| \gamma''(s) \|^{1/2} = 1.
\]

The natural parameter \( s \) is fixed up to an additive constant. From now on we will consider null curves without flex points, parametrized by the natural parameter.

To any null curve \( \gamma(s) \), we associate the Frenet frame

\[
F = (\gamma; t, n, b) : I \rightarrow \mathbb{E}(2,1),
\]

(3.2)
defined by

\[
t(s) = \gamma'(s), \quad n(s) = \gamma''(s), \quad b(s) = \gamma'''(s) + \frac{1}{2} \|\gamma'''(s)\|^2 \gamma'(s),
\]

for every \( s \in I \). The orthonormality conditions (3.1) are readily verified by observing that the derivative of \( \langle \gamma', \gamma'' \rangle = 0 \) yields \( \langle \gamma', \gamma''' \rangle = -1 \). The Frenet frame satisfies the Frenet–Serret system

\[
\gamma' = t, \quad t' = n, \quad n' = 2\kappa t + b, \quad b' = -2\kappa n,
\]

(3.3)
where the function

\[
\kappa(s) := \frac{1}{2} \|\gamma'''(s)\|^2, \quad s \in I
\]

is called the curvature of \( \gamma \). Equivalently, (3.3) can be written in the form

\[
F' = F \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -2\kappa & 0 \\ 0 & 1 & 0 & -2\kappa \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

(3.4)

Remark 2. Let \( M = J^3_3(\mathbb{R}, \mathbb{R}^{2,1}) \) be the space of third order jets of null curves \( \gamma : \mathbb{R} \rightarrow \mathbb{R}^{2,1} \) parametrized by the natural parameter. The space \( M \) is the \( \mathbb{E}(2,1) \)-invariant submanifold of \( J^3(\mathbb{R}, \mathbb{R}^{2,1}) \) defined by

\[
\dot{x} \in \mathcal{L}^*, \quad \|\vec{x}\| = 1, \quad \langle \dot{x}, \vec{x} \rangle = \langle \vec{x}, \vec{x} \rangle = 0, \quad \dot{x} \wedge \vec{x} \wedge \vec{x} \neq 0,
\]

where \( \mathcal{L}^* \subset \mathbb{R}^{2,1} \) is the future-directed lightcone. Then

\[
\rho : M \ni (P, \dot{x}, \vec{x}, \vec{x}) \mapsto (P; \dot{x}, \vec{x}, \vec{x} + \frac{1}{2} \|\vec{x}\|^2 \dot{x}) \in \mathbb{E}(2,1)
\]
is a moving frame map in the sense of Fels–Olver [10]. Note that the Frenet frame along a null curve \( \gamma \) is given by \( F = \rho \circ j_3(\gamma) \).

3.2. Tangent vectors

Let \( \mathcal{M} \) denote the space of null curves in \( \mathbb{R}^{2,1} \) without flex points, parametrized by the natural parameter \( s \), and complete, i.e. defined on all \( \mathbb{R} \).

\(^4\) \( \gamma(t) \) is a flex point if \( \gamma'(t) \wedge \gamma''(t) = 0 \).
Lemma 1. Let

\[ V_\gamma = p_1 t + p_2 n + p_3 b \]

be a vector field along \( \gamma \in \mathcal{M} \). Then \( V_\gamma \) is tangent to \( \mathcal{M} \) if and only if

\[
\begin{aligned}
    p_2 &= -p_3', \\
    p_1 &= \frac{1}{2} p_3'' + \int_0^s \kappa'(u) p_3(u) \, du + \text{cost.}
\end{aligned}
\] (3.5)

Proof. Let \( \Gamma(s, t) \) be a variation of \( \gamma(s) \) through null curves and assume that \( \Gamma(s, 0) = \gamma(s) \). Let

\[
F(\cdot, t) = (\Gamma; t_\Gamma, n_\Gamma, b_\Gamma)(\cdot, t) : \mathbb{R} \to E(2, 1)
\]
denote the Frenet frame for each null curve \( \Gamma(\cdot, t) \). Let \( \kappa_\Gamma(\cdot, t) \) be the curvature function of \( \Gamma(\cdot, t) \). If we set \( \Theta = F^{-1}dF \), then

\[
\Theta = K(s, t) ds + P(s, t) dt
\]

for \( \epsilon(2, 1) \)-valued functions

\[
K = \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    1 & 0 & -2\kappa_\Gamma & 0 \\
    0 & 1 & 0 & -2\kappa_\Gamma \\
    0 & 0 & 1 & 0
\end{pmatrix}, \quad P = \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    p_1 & p_5 & p_6 & 0 \\
    p_2 & p_4 & 0 & p_6 \\
    p_3 & 0 & p_4 & -p_5
\end{pmatrix},
\]
such that

\[
d\Theta + \Theta \wedge \Theta = 0 \quad (3.7)
\]
is satisfied. The equation (3.7) can be rewritten in the form

\[
\frac{\partial K}{\partial t} - \frac{\partial P}{\partial s} = [K, P],
\]

which computed at \( t = 0 \) yields

\[
\begin{aligned}
    p_2 &= -p_3', \quad (3.8) \\
    p_1 &= \frac{1}{2} p_3'' + \int_0^s \kappa'(u) p_3(u) \, du + \text{cost}, \quad (3.9) \\
    p_4 &= -p_3'' - 2\kappa_\Gamma p_3 + p_1, \quad (3.10) \\
    p_5 &= p_1' + 2\kappa_\Gamma p_3', \quad (3.11) \\
    p_6 &= p_5' - 2\kappa_\Gamma p_4 \quad (3.12)
\end{aligned}
\]

and

\[
\frac{\partial \kappa_\Gamma}{\partial t}(s, 0) = -\frac{1}{2} p_6' - \kappa_\gamma p_5, \quad (3.13)
\]

where \( \kappa_\gamma = \kappa_\Gamma(\cdot, 0) \). In particular, we have that the infinitesimal variation of \( \Gamma(s, t) \) at \( t = 0 \) is

\[
\frac{\partial \Gamma}{\partial t}(s, 0) = p_1(s)t_\gamma + p_2(s)n_\gamma + p_3(s)b_\gamma.
\]

Conversely, any tangent vector arises as an infinitesimal variation. In fact, let \( V_\gamma \) be a tangent vector at \( \gamma \). Let \( p_4 \), \( p_5 \) and \( p_6 \) be the functions determined by \( p_1 \), \( p_2 \), \( p_3 \) through the equations (3.8)–(3.12), and consider the function

\[
c = -\frac{1}{2} p_6' - \kappa_\gamma p_5.
\]
Under these hypotheses, we have that
\[ \frac{\partial P}{\partial s} = C - [K_\gamma, P], \]
where
\[ C = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -2c & 0 \\
0 & 0 & 0 & -2c \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad K_\gamma = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & -2\kappa_\gamma & 0 \\
0 & 1 & 0 & -2\kappa_\gamma \\
0 & 0 & 1 & 0 \\
\end{pmatrix}. \]

Next, define
\[ K(s, t) := K_\gamma(s) + tC(s) \]
and consider the differential equation for \( P = P(s, t) \) given by
\[ \frac{\partial P}{\partial s} = C - [K_\gamma, P] - t [C, P]. \]
Let \( P(s, t) \) be the solution with initial condition \( P(0, t) = p_0(t) \) and consider the \( C(2, 1) \)-valued 1-form
\[ K(s, t)ds + P(s, t)dt. \]
By construction, this form satisfies the Maurer–Cartan equation, which implies the existence of an \( E(2, 1) \)-valued map
\[ F = (\Gamma; t, n, b) \]
such that \( F^{-1}F = K(s, t)ds + P(s, t)dt \). As a consequence, \( \Gamma \) represents a variation through null curves whose infinitesimal variation is \( (\partial \Gamma/\partial t)(s, 0) = V_\gamma(p) \).

**Remark 3.** A solution of (3.5) is uniquely given by prescribing arbitrarily the function \( p_3 \) and a constant. This gives a canonical trivialization of the tangent bundle \( T\mathcal{M}, \)

\[ T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R} \times C^\infty(\mathbb{R}, \mathbb{R}). \]

### 4. Local motion of null curves and evolution equations

#### 4.1. Local motion and vector fields

An *invariant local motion of null curves* is an integral curve of a *local vector field* on \( \mathcal{M}, \) that is a section of \( T\mathcal{M} \) of the form
\[ V : \mathcal{M} \ni \gamma \mapsto V_\gamma = p_1[k_\gamma]t_\gamma + p_2[k_\gamma]n_\gamma + p_3[k_\gamma]b_\gamma, \quad (4.1) \]
where \( p_1, p_2, p_3 \) are polynomial differential functions satisfying
\[ E(u_{(1)}p_3) = 0 \quad (4.2) \]
and
\[ p_1 = \frac{1}{2}D^2p_3 + \int u_{(1)}p_3 + \text{cost}, \quad p_2 = -Dp_3. \quad (4.3) \]

**Remark 4.** Note that, according to lemma 1, (4.2) and (4.3) are necessary and sufficient conditions for \( V_\gamma \) being tangent to \( \mathcal{M} \) at \( \gamma \), for each \( \gamma \in \mathcal{M}. \) Moreover, a local vector field is completely determined by a differential function \( p_3 \) such that \( E(u_{(1)}p_3) = 0 \) and a constant. Henceforth, such a constant will be assumed to be zero. From (4.1), it follows that a local motion of null curves is a solution \( \Gamma(s, t) \) of the flow equation
\[ \frac{\partial \Gamma}{\partial t} = p_1[k_\Gamma]t_\Gamma + p_2[k_\Gamma]n_\Gamma + p_3[k_\Gamma]b_\Gamma. \quad (4.4) \]
We can now state the following.

**Theorem 1.** Suppose $\Gamma(s, t)$ is the local motion associated with a local vector field $V$. Then the evolution of the curvature is governed by

\[
\frac{\partial \kappa}{\partial t} = -\frac{1}{4} DD^{-1} D p_3 [\kappa],
\]

where $p_3$ is a polynomial differential function such that $u(1)_1 p_3$ is a total derivative.

Conversely, let $\kappa(s, t)$ be a solution of (4.5) and let $\gamma$ be the null curve with curvature $\kappa(s, 0)$. Then there exists a unique local motion $\Gamma(s, t)$ such that $\Gamma(s, 0) = \gamma$ which corresponds to the local vector field determined by $p_3$.

**Proof.** As above, the flow $\Gamma$ of the local vector field $V$ lifts to a map $F: \mathbb{R}^2 \rightarrow \mathbb{E}(2, 1)$ and there exist $e(2, 1)$-valued polynomial differential functions

\[
\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -2u(0) & 0 \\ 0 & 1 & 0 & -2u(0) \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p_1 & p_5 & p_6 & 0 \\ p_2 & p_4 & 0 & p_6 \\ p_3 & 0 & p_4 & -p_5 \end{pmatrix}
\]

so that the $e(2, 1)$-valued 1-form

\[
\Theta = F^{-1} dF = \mathcal{R}[\kappa] ds + \mathcal{P}[\kappa] dt,
\]

satisfies the Maurer–Cartan equation $d\Theta + \Theta \wedge \Theta = 0$. Writing out this equation yields

\[
\begin{align*}
p_1 &= \frac{1}{2} D^2 p_3 + \int u(1)_1 p_3, \\
p_2 &= -D p_3, \\
p_4 &= -\frac{1}{2} D^2 p_3 - 2u(0)_1 p_3 + \int u(1)_1 p_3, \\
p_5 &= D p_1 + 2u(0)_0 D p_3, \\
p_6 &= D p_5 - 2u(0)_0 p_4
\end{align*}
\]

and

\[
\frac{\partial \kappa}{\partial t} = \frac{1}{4} D p_4.
\]

From the third equation of (4.7), and the hypothesis that $E(u(1)_1 p_3) = 0$, it follows that

\[
D p_4 = -\frac{1}{2} D p_3,
\]

and hence (4.8) can be written in the form

\[
\frac{\partial \kappa}{\partial t} = -\frac{1}{4} D D^{-1} D p_3.
\]

Conversely, if $\kappa: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a solution to equation (4.5), then the $e(2, 1)$-valued 1-form $\Theta$ defined as in (4.6) satisfies the Maurer–Cartan equation. Thus, by the Cartan–Darboux theorem, there exists a map

\[
F = (\Gamma; \alpha): \mathbb{R}^2 \rightarrow \mathbb{E}(2, 1),
\]

unique up to left multiplication by an element of $\mathbb{E}(2, 1)$, such that $F^{-1} dF = \Theta$. Consequently, $\Gamma: \mathbb{R}^2 \rightarrow \mathbb{R}^{2,1}$ defines the flow of a local vector field. \( \square \)

**Remark 5.** The curvature evolution of a local vector field is given by a local evolution equation, that is

\[
\frac{\partial \kappa}{\partial t} = \epsilon[\kappa],
\]

where $\epsilon \in J[u]$ is a polynomial differential function. Remarkably, the right-hand side of (4.5) is expressed in terms of the two differential operators defining the bi-Hamiltonian structure of the KdV equation (cf [32]).
5. Dynamics of null curves and the KdV hierarchy

5.1. Motion by integrable evolution equations

We now show that the evolution of the curvature induced by certain motions is completely integrable. This is the content of the following.

**Theorem 2.** The sequence of polynomial differential functions

\[ p_3^{[n]} = 4g_{n-2}, \quad n \geq 2 \]  

(5.1)
defines a hierarchy of local motions whose curvature evolution equations are the members of the KdV hierarchy.

**Proof.** Since any \( p_3^{[n]} \) in the sequence is the variational derivative of some local functional, \( \kappa(t) p_3^{[n]} \) is a total derivative (cf [33], theorem 7.36), or equivalently \( E(\kappa(t), p_3) = 0 \). Therefore, \( p_3^{[n]} \) determines a local vector field. The result now follows from the curvature equation (4.5) and the Lenard recursion formula (2.3) for the \( g_n \)'s. For \( n = 2 \),

\[ \kappa_2 = -\frac{1}{2} D^{-1} D^4 g_0 = -Dg_1 = -6uu_s - u_{sss}, \]

which is the KdV equation, the second member of the KdV hierarchy. For \( n > 2 \),

\[ \kappa_n = -\frac{1}{2} D^{-1} D^4 g_{n-2} = -Dg_{n-1} = -Dg_n, \]

which is the \( n \)th member of the KdV hierarchy.

\( \square \)

**Remark 6.** Let \( \mathcal{M} \simeq \mathbb{E}(2, 1) \times C^\infty(\mathbb{R}, \mathbb{R}) \) be the space of null curves parametrized by the natural parameter and defined on the whole \( \mathbb{R} \). Assume, for simplicity, that \( C^\infty(\mathbb{R}, \mathbb{R}) \) is the space of rapidly decreasing smooth functions or of smooth periodic functions of period 2\( \pi \). Consider the space \( \hat{\mathcal{M}} \simeq \mathcal{M}/\mathbb{E}(2, 1) \simeq C^\infty \) of 'geometric' null curves (the space of curvature functions). If we think of a tangent vector \( v_\epsilon \in T_\epsilon \mathcal{M} \) as defined by a smooth function \( c \) and an element of the Lie algebra \( \mathfrak{e}(2, 1) \), so that \( T_\epsilon \mathcal{M} \simeq \mathfrak{e}(2, 1) \times C^\infty \), it is clear that two tangent vectors \( v_\epsilon, \tilde{v}_\epsilon \in T_\epsilon \mathcal{M} \) descend to the same element of \( T_\epsilon \hat{\mathcal{M}} \) if and only if \( c = \tilde{c} \). This allows us to define a symplectic 2-form \( \Omega \) on \( \hat{\mathcal{M}} \) by

\[ \Omega_\epsilon(c, \tilde{c}) = -\frac{1}{2} \int_{\mathbb{R}} (c D_\epsilon^{-1} \tilde{c} - \tilde{c} D_\epsilon^{-1} c) \, ds. \]

If \( \mathcal{F} = \int F(\kappa, \kappa_s, \kappa_{ss}, \ldots) ds \) is a local functional on \( C^\infty \), then

\[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} F(c + \epsilon \tilde{c}) = \left( \frac{\delta \mathcal{F}}{\delta c}(c, \tilde{c}) \right) \Omega \left( \frac{\delta \mathcal{F}}{\delta \kappa}(c, \tilde{c}), \frac{\delta \mathcal{F}}{\delta \kappa_s}(c, \tilde{c}), \ldots \right), \]

where \( (, \ldots) \) denotes the \( L^2 \) inner product and \( \delta \mathcal{F}/\delta \kappa \) is the variational derivative of \( \mathcal{F} \). The symplectic gradient (Hamiltonian vector field) of \( \mathcal{F} \) is given by \( \mathcal{D}(\delta \mathcal{F}/\delta \kappa) \), which defines the Hamiltonian flow \( \partial \kappa/\partial t = \mathcal{D}(\delta \mathcal{F}/\delta \kappa) \). The corresponding Poisson bracket is given by

\[ \{ \mathcal{F}_1, \mathcal{F}_2 \} = \Omega \left( \frac{\delta \mathcal{F}_1}{\delta \kappa}, \frac{\delta \mathcal{F}_2}{\delta \kappa} \right) = \left( \int \frac{\delta \mathcal{F}_1}{\delta \kappa}, \frac{\delta \mathcal{F}_1}{\delta \kappa_s}, \frac{\delta \mathcal{F}_2}{\delta \kappa_s}, \ldots \right). \]

A similar construction can be given also for the operator \( \mathcal{D} \).

A local vector field on \( \mathcal{M} \) is said Hamiltonian with respect to a differential operator \( \mathcal{J} \) if its curvature equation can be written in the form

\[ \frac{\partial \kappa}{\partial t} = \mathcal{J} \frac{\delta \mathcal{F}}{\delta \kappa}[\kappa], \]

(5.2)
where $\delta F / \delta \kappa$ is the variational derivative of $F = \int F(\kappa, \kappa_s, \kappa_{ss}, \ldots) ds$ and $J$ determines the Poisson bracket $\{ F_1, F_2 \} = \int (\delta F_1 / \delta \kappa) J (\delta F_2 / \delta \kappa) ds$ on the space of functionals.

Note that the local vector fields of theorem 2 are Hamiltonian with respect to the two Hamiltonian structures $D$ and $D'$, which coincide with the canonical Hamiltonian structures of the KdV.

5.2. Evolution by congruences and null worldlines

A null curve that moves without changing its shape (by Lorentz rigid motion) under the KdV flow

$$\kappa_t + \kappa_{sss} + 6\kappa\kappa_s = 0$$

is said a congruence curve. Congruence curves correspond to the travelling wave solutions of the KdV equation. If $\kappa(s, t) = f(s - \lambda t)$ is a travelling wave solution of (5.3), then

$$f'''' + 6ff' - \lambda f' = 0.$$  \hspace{1cm} (5.4)

Integrating twice, we obtain

$$(h')^2 = 4h^3 - g_2h - g_3,$$

for real constants $g_2$ and $g_3$, where

$$h = -\frac{1}{2} \left( f - \frac{\lambda}{6} \right).$$

Thus $f$ can be expressed in terms of the Weierstrass $\wp$-function with invariants $g_2$ and $g_3$. Quasi-periodic congruence curves may occur only if the polynomial $p(x) = 4x^3 - g_2x - g_3$ has three distinct real roots, i.e.

$$27g_2^2 - g_3^2 < 0.$$ 

In this case, the periodic solution of (5.4) is

$$f(s) = -2\wp(s + \omega_1 + g_2, g_3) + \frac{\lambda}{6}, \quad s \in \Re,$$

where $\omega_1, \omega_3$ are the primitive half-periods of $\wp(\cdot; g_2, g_3)$ (cf [24]).

Now, the third order ODE (5.4) coincides with the Euler–Lagrange equation of the action functional on null curves defined by

$$\int (2\kappa + \lambda) ds$$  \hspace{1cm} (5.5)

(cf [9, 12]). Thus, congruence curves are the worldlines of the relativistic particle model defined by (5.5). (For more details on the physical models associated with action functionals of the type above see [19, 27, 28, 31, 34].)

More interestingly, let $\kappa$ be a solution of (5.4) and let $\mathfrak{R}$ and $\mathfrak{P}$ be as in (4.6), where $\mathfrak{P}$ is computed for $p^{[2]} = 4g_0 = 2$, using (4.7). If we set

$$\Omega_\lambda := [\mathfrak{P[\kappa]} + \lambda \mathfrak{R[\kappa]}],$$

then from the Maurer–Cartan equation of $\Theta = \mathfrak{R} ds + \mathfrak{P} dr$, which in turn is equivalent to

$$\frac{\partial \mathfrak{P[\kappa]}}{\partial s} - \frac{\partial \mathfrak{R[\kappa]}}{\partial t} = [\mathfrak{P}, \mathfrak{R}][\kappa],$$

it follows that

$$\frac{\partial \Omega_\lambda}{\partial s} = [\Omega_\lambda, \mathfrak{R}].$$  \hspace{1cm} (5.6)
Thus, $\mathcal{L}_\lambda$ and $\mathcal{R}$ form a Lax pair for the variational problem (5.5). Equation (5.6) means that the linear endomorphism

$$\mu = F \cdot \mathcal{L}_\lambda \cdot F^{-1}$$

is constant along the solutions. From this it follows that the worldlines can be obtained by quadratures and expressed in terms of Weierstrass $\sigma$, $\zeta$ and $\wp$ functions (cf [12]).

Summarizing, we can state the following.

**Theorem 3.** The congruence solutions of the flow generated by $\mathfrak{p}_3^{[2]}$ coincide with the worldlines of the particle model defined by (5.5). In particular, the congruence curves are integrable by quadrature.

**Remark 7.** More generally, a local motion is by congruences if the curves of the motion do not change their shapes during the evolution. This means that the curvature of the motion is a travelling wave solution of equation (4.5), i.e. $\kappa(s, t) = f(s - \lambda t)$, for some constant $\lambda$, and $f$ is a solution to

$$4 \lambda f' - D D^{-1} Dp_3[f] = 0. \quad (5.7)$$

In this case, if we set

$$\mathcal{L}_\lambda := \mathfrak{P}[f] + \lambda \mathcal{R}[f],$$

then the Maurer–Cartan equation of $\Theta = \mathcal{R} ds + \mathfrak{P} dr$ can be written in Lax form

$$(\mathcal{L}_\lambda)' = [\mathcal{L}_\lambda, \mathcal{R}].$$

Unlike the local motion discussed in theorem 3, in general the Lax formulation does not imply the integration by quadratures of congruence curves.

### 5.3. Evolution by similarities

Consider a null curve $\gamma : I \to \mathbb{R}^{2,1}$ parametrized by the natural parameter, and let $\kappa_{\gamma}$ be its curvature. Then

$$\bar{\gamma} : s \in \sqrt{r}I \to r\gamma \left( \frac{s}{\sqrt{r}} \right) \in \mathbb{R}^{2,1}$$

is the natural parametrization of $r\gamma$ and its curvature is given by

$$\bar{\kappa}_{\gamma}(s) = \frac{1}{r} \kappa_{\gamma} \left( \frac{s}{\sqrt{r}} \right).$$

Thus, solutions of (5.3) which corresponds to curves whose shapes evolves by similarities under the KdV flow are in the form

$$\kappa(s, t) = \frac{1}{r(t)} \kappa_{\gamma} \left( \frac{s}{\sqrt{r(t)}} \right),$$

where $r$ is a positive smooth function. Setting $x = s/\sqrt{r(t)}$ we have

$$2\kappa'''_{\gamma}(x) - x \sqrt{r(t)} \kappa_{\gamma}(x) + 2\kappa_{\gamma}(x) \left( 6\kappa_{\gamma}'(x) - \sqrt{r(t)} \kappa_{\gamma}(x) \right) = 0.$$
The resulting third order ODE for $\kappa_\gamma$ is
\[
\kappa_\gamma''' + 6\kappa_\gamma'\kappa_\gamma - \frac{a}{3}(\kappa_\gamma' + 2\kappa_\gamma) = 0.
\] (5.8)

Without loss of generality we may assume $a = 1$. Thus, (5.8) can be integrated by setting
\[
\kappa_\gamma = v' - v^2,
\]
where $v$ is a solution of the second Painlevé equation
\[
v''' - 2v^3 + xv + c = 0.
\]

This explains the geometrical origin of the similarity reduction of the KdV equation considered in [18]. We have proved the following.

**Theorem 4.** The curvature of the similarity solutions corresponding to the flow generated by $p^{[2]}_3$ can be integrated by means of the solutions of the second Painlevé equation.

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**References**

[1] Bonnor W B 1969 Null curves in a Minkowski space-time *Tensor (N.S.)* 20 229–42
[2] Calini A, Ivey T and Mari-Beffa G 2009 Remarks on KdV-type flows on star-shaped curves *Physica D* 238 788–97
[3] Capovilla R, Guven J and Rojas E 2006 Null Frenet–Serret dynamics *Gen. Rel. Grav.* 38 689–98
[4] Castagnino M 1964 Sulle formule di Frenet–Serret per le curve nulle di una $V_4$ riemanniana a metrica iperbolica normale *Rend. Mat. Appl.* 23 438–61
[5] Chou K S and Qu C 2002 Integrable equations arising from motions of plane curves *Physica D* 162 9–33
[6] Chou K S and Qu C 2003 Integrable equations arising from motions of plane curves. II *J. Nonlinear Sci.* 13 487–517
[7] Ding Q and Inoguchi J 2004 Schrödinger flows, binormal motions for curves and the second AKNS-hierarchies *Chaos Solitons Fractals* 21 669–77
[8] Doliwa A and Santini P M 1994 An elementary geometric characterisation of the integrable motions of a curve *Phys. Lett. A* 185 373–84
[9] Fernández A, Giménez A and Lucas P 2002 Geometrical particle models on 3D null curves *Phys. Lett. B* 543 311–7 (arXiv: hep-th/0205284)
[10] Fels M and Olver P J 1999 Moving coframes. II: Regularization and theoretical foundations *Acta Appl. Math.* 55 127–208
[11] Gardner C S, Greene J M, Kruskal M D and Miura R M 1974 Korteweg–de Vries equation and generalizations: VI. Methods for exact solutions *Commun. Pure Appl. Math.* 27 97–133
[12] Grant J D E and Musso E 2004 Coisotropic variational problems *J. Geom. Phys.* 50 303–38 (arXiv: math.DG/0307216)
[13] Goldstein R E and Petch D M 1991 The Korteweg–de Vries hierarchy as dynamics of closed curves in the plane *Phys. Rev. Lett.* 67 3203–6
[14] Hasimoto H 1972 A soliton on a vortex filament *J. Fluid Mech.* 51 477–85
[15] Huang R and Singer D A 2002 A new flow on starlike curves in $\mathbb{R}^3$ *Proc. Am. Math. Soc.* 130 2725–35
[16] Inoguchi J and Lee S 2008 Null curves in Minkowski 3-space *Int. Electron. J. Geom.* 1 40–83
[17] Ivey T A 2001 Integrable geometric evolution equations for curves *Contemp. Math.* 285 71–84
[18] Joshi N 2004 The second Painlevé hierarchy and the stationary KdV hierarchy *Publ. Res. Inst. Math. Sci.* 40 1039–61
[19] Kuznetsov Y A and Plyushchay M S 1994 (2+1)-dimensional models of relativistic particles with curvature and torsion *J. Math. Phys.* 35 2772–8
[20] Lamb G L 1977 Solitons on moving space curves *J. Math. Phys.* 18 1654–61
[21] Langer J L and Perline R 1998 Curve motion inducing modified Korteweg–de Vries systems Phys. Lett. A \textbf{239} 36–40
[22] Langer J L and Singer D A 1996 Lagrangian aspects of the Kirchhoff elastic rod SIAM Rev. \textbf{38} 605–18
[23] Langer J L and Singer D 1994 Liouville integrability of geometric variational problems Comment. Math. Helv. \textbf{69} 272–80
[24] Lawden D F 1989 Elliptic Functions and Applications (Series in Applied Mathematical Science vol 80) (New York: Springer)
[25] Lax P D 1976 Almost periodic solutions of the KdV equation SIAM Rev. \textbf{18} 351–75
[26] Musso E 2009 An experimental study of Goldstein–Petrich curves Rend. Semin. Mat. Univ. Politech. Torino \textbf{67} 407–26 (special issue in memory of A Sanini)
[27] Musso E and Nicolodi L 2007 Closed trajectories of a particle model on null curves in anti-de Sitter 3-space Class. Quantum Grav. \textbf{24} 5401–11
[28] Musso E and Nicolodi L 2008 Reduction for constrained variational problems on 3-dimensional null curves, SIAM J. Control Optim. \textbf{47} 1399–414
[29] Nakayama K, Segur H and Wadati M 1992 Integrability and the motion of curves Phys. Rev. Lett. \textbf{69} 2603–6
[30] Nesterenko V V, Feoli A and Scarpetta G 1996 Complete integrability for Lagrangians dependent on acceleration in a spacetime of constant curvature Class. Quantum Grav. \textbf{13} 1201–11
[31] Nersessian A, Manvelyan R and Müller-Kirsten H J W 2000 Particle with torsion on 3d null-curves Nucl. Phys. B \textbf{88} 381–4 (arXiv: hep-th/9912061)
[32] Olver P J 1993 Applications of Lie Group to Differential Equations (Graduate Texts in Mathematics vol 107) (New York: Springer)
[33] Olver P J 1995 Equivalence, Invariants, and Symmetry (New York: Cambridge University Press)
[34] Psarski R D 1986 Field theory of paths with a curvature-dependent term Phys. Rev. D \textbf{34} 670–3
[35] Pinkall U 1995 Hamiltonian flows on the space of star-shaped curves Results Math. \textbf{27} 328–32