Coherent states for Hamiltonians generated by supersymmetry

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Abstract Coherent states are derived for one-dimensional systems generated by supersymmetry from an initial Hamiltonian with a purely discrete spectrum for which the levels depend analytically on their subindex. It is shown that the algebra of the initial system is inherited by its SUSY partners in the subspace associated to the isospectral part or the spectrum. The technique is applied to the harmonic oscillator, infinite well and trigonometric Pöschl-Teller potentials.

1 Introduction

The great interest in the study of coherent states (CS) stems from the beautiful properties that the so-called standard ones have, which are a natural consequence of the huge symmetry supplied by the Heisenberg-Weyl algebra ruling the harmonic oscillator. Indeed, these characteristics suggested Glauber to model light by means of standard coherent states [1], which was a breakthrough in the development of quantum optics, one of most successful branches of the physics of 20-th century (see, e.g. [2–7]).

Among the several definitions available in the literature for general systems, algebraically the most important ones are those which define the CS either as eigenstates of annihilation operators or as resulting of a 'displacement' operator acting onto certain extremal state. In order to derive the CS following the first definition, one has to identify the appropriate algebra ruling the system Hamiltonian, and to find then the annihilation and creation operators suitable to perform the construction. Since typically the resulting algebra is not linear, it is usual to call them nonlinear coherent states [8–16].

For Hamiltonians $H_k$ generated by supersymmetric quantum mechanics (SUSY QM) [17–28], the CS analysis has been focussed mainly on the SUSY partners of the harmonic oscillator [29–34] (see however [35, 36]). The key ingredient in the approach introduced in [29,34] is to construct a natural pair of annihilation and creation operators of $H_k$ simply as products of intertwining and standard annihilation and creation operators. An important conclusion of these works was that the natural algebra ruling the SUSY partner Hamiltonians of the oscillator is a polynomial deformation of the Heisenberg-Weyl algebra.
For the SUSY partners of a general initial potential, an appropriate algebraic treatment of the corresponding Hamiltonian $H_0$, ensuring a right identification of the annihilation and creation operators, had not been realized. However, for a set of one-dimensional Hamiltonians with a purely discrete spectrum for which the levels depend analytically on their index, an *intrinsic* algebra has been identified recently, allowing to calculate in a simple way the corresponding CS [37]. Let us notice that this intrinsic algebra is in general nonlinear. One of the results of the present paper is to show that such algebraic structures can be linearized: one can associate to those systems the Heisenberg-Weyl algebra. Consequently, an additional set of CS will be constructed, their explicit expressions containing small variations from the standard harmonic oscillator CS.

It is remarkable that [37] draws as well the attention to the main subject of this paper, namely, the CS analysis for the SUSY partners of arbitrary potentials in the spirit of [29, 34]. In this context several novel results will be found, e.g., we will show that the nonlinear and linear algebras of $H_0$ are inherited by its SUSY partners $H_k$ in the subspace associated to the isospectral part of the spectrum. In addition, we will find a natural algebra for which the generators are products of annihilation and creation operators of $H_0$ times the intertwiners of $H_0$ and $H_k$, thus generalizing the previous results for the harmonic oscillator [29, 34]. The corresponding CS will be built up for the several algebras of $H_k$ we are going to study. Our procedure will be illustrated with the harmonic oscillator, infinite well and trigonometric Pöschl-Teller potentials. The results for the SUSY partners of the infinite well and trigonometric Pöschl-Teller potentials, as far as we know, are new.

Let us observe that for specific potentials, like trigonometric Pöschl-Teller, Morse and others, there are alternative methods of construction of CS which employ the symmetry of the differential equations related to $H_0$ (see e.g. [38]). However, to implement the SUSY transformations departing from such treatments seems involved, as compared with the technique which will be presented in this paper (based on [37]).

In the next section the initial Hamiltonian we deal with as well as its related algebras will be studied. The CS analysis for the several algebras of $H_0$ is the subject of section 3. A brief overview of SUSY QM as a technique for generating solvable potentials from a given initial one will be presented in section 4. In section 5, a pair of nonlinear algebras ruling the SUSY partner potentials will be discussed, while in section 6 we will explore the corresponding linear structure. The CS construction for the several algebras associated to the SUSY partner potentials will be performed in section 7. In section 8 our general results will be illustrated with some examples. Finally, in section 9 we close the paper with our conclusions.

### 2 Algebraic structures of the initial Hamiltonian $H_0$

Let us suppose that the initial system is described by a Hermitian Schrödinger Hamiltonian

$$H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + V_0(x),$$

(1)
whose eigenvectors and eigenvalues satisfy:

\[ H_0 |\psi_n\rangle = E_n |\psi_n\rangle, \quad E_0 < E_1 < E_2 < \ldots \]  

(2)

We assume that there is an analytic dependence, defined by a certain function \( E(n) \), of the eigenvalues with the index labeling them, namely,

\[ E_n \equiv E(n), \]  

(3)

and the eigenvectors satisfy the standard orthonormality and completeness relationships

\[ \langle \psi_m | \psi_n \rangle = \delta_{mn}, \quad \sum_{m=0}^{\infty} |\psi_m\rangle\langle \psi_m| = 1, \]  

(4)

where the symbol 1 in any operator expression of this paper represents the identity operator. There will be different forms of \( E(n) \) according to the system under study, for instance, for the harmonic oscillator it will be a linear function of \( n \), for an infinite square well it will be quadratic, etc. This function defines an intrinsic algebra which will be next discussed.

### 2.1 Intrinsic nonlinear algebra of \( H_0 \)

Let us define a pair of annihilation and creation operators \( a_0^\pm \) by

\[ a_0^- |\psi_n\rangle = r_T(n) |\psi_{n-1}\rangle, \quad a_0^+ |\psi_n\rangle = \bar{r}_T(n+1) |\psi_{n+1}\rangle, \]  

\[ r_T(n) = e^{i\alpha(E_n - E_{n-1})} \sqrt{E_n - E_0}, \quad \alpha \in \mathbb{R}, \]  

(5)

(6)

such that their product becomes:

\[ a_0^+ a_0^- = H_0 - E_0. \]  

(7)

The number operator \( N_0 \) is now introduced with the properties:

\[ N_0 |\psi_n\rangle = n |\psi_n\rangle, \quad [N_0, a_0^\pm] = \pm a_0^\pm. \]  

(8)

As a consequence, two equations which will be widely used along this work are obtained:

\[ a_0^\pm g(N_0) = g(N_0 \mp 1) a_0^\pm, \]  

(9)

\( g(x) \) being a real arbitrary non-singular function for \( x \in \mathbb{Z}^+ \). Combining Eqs. (2,5-8), it turns out that the intrinsic algebra of the system is characterized by the relationships:

\[ H_0 = E(N_0), \quad a_0^+ a_0^- = E(N_0) - E_0, \quad a_0^- a_0^+ = E(N_0 + 1) - E_0, \]  

(10)

\[ [a_0^-, a_0^+] = E(N_0 + 1) - E(N_0) \equiv f(N_0), \]  

(11)

\[ [H_0, a_0^\pm] = \pm a_0^\pm f(N_0 - 1/2 \pm 1/2) = \pm f(N_0 - 1/2 \mp 1/2) a_0^\pm. \]  

(12)

We will see below that this is not the only algebra of \( H_0 \) which can be defined.

Let us notice that we can express \( a_0^\pm \) in the form

\[ a_0^- = r_T(N_0 + 1) \sum_{m=0}^{\infty} |\psi_m\rangle\langle \psi_{m+1}|, \quad a_0^+ = \bar{r}_T(N_0) \sum_{m=0}^{\infty} |\psi_{m+1}\rangle\langle \psi_m|, \]  

(13)

where each term in both summations is a Hubbard operator \[39-41\]. Hence, throughout this paper we will call these decompositions Hubbard representations.
2.2 Linear algebra of $H_0$

The intrinsic algebra (8,10,12) admits a linearizing procedure, i.e., one can build up new annihilation and creation operators satisfying the standard oscillator algebra (29,34). Let us construct them in the form:

$$a_{0c}^- = b(N_0) a_0^- = a_0^- b(N_0 - 1), \quad a_{0c}^+ = a_0^+ b(N_0) = b(N_0 - 1) a_0^+, \quad (14)$$

$b(x)$ being a real non-singular function for $x \in \mathbb{Z}^+$ to be determined. Suppose that the action of $a_{0c}^\pm$ onto the eigenvectors of $H_0$, up to the same phase factors as in (5,6), is equal to the oscillator one, namely:

$$a_{0c}^- |\psi_n\rangle = r_L(n) |\psi_{n+1}\rangle, \quad a_{0c}^+ |\psi_n\rangle = \bar{r}_L(n + 1) |\psi_{n+1}\rangle, \quad (15)$$

$$r_L(n) = e^{i\alpha f(n-1)} \sqrt{n}. \quad (16)$$

On the other hand, the expressions for $a_{0c}^\pm$ given in (14) and the use of (5) lead to:

$$a_{0c}^- |\psi_n\rangle = b(n-1) r_I(n) |\psi_{n+1}\rangle, \quad a_{0c}^+ |\psi_n\rangle = b(n) \bar{r}_I(n + 1) |\psi_{n+1}\rangle. \quad (17)$$

By comparing (15) with (17) we get:

$$b(n) = \frac{\bar{r}_L(n + 1)}{r_I(n + 1)} = \frac{r_L(n + 1)}{r_I(n + 1)} = \sqrt{\frac{n+1}{E(n+1) - E_0}}. \quad (18)$$

Making use of (13,14,13), the Hubbard representation of $a_{0c}^\pm$ is obtained:

$$a_{0c}^- = r_L(N_0 + 1) \sum_{m=0}^{\infty} |\psi_m\rangle \langle \psi_{m+1}|, \quad a_{0c}^+ = \bar{r}_L(N_0) \sum_{m=0}^{\infty} |\psi_{m+1}\rangle \langle \psi_m|, \quad (19)$$

which, up to the exponential factors of $r_L$, is equal to the oscillator one. Let us notice that, as a consequence of (19), we get $a_{0c}^\pm g(N_0) = g(N_0 + 1) a_{0c}^\pm$. Thus, the set $\{N_0, a_{0c}^-, a_{0c}^+\}$ satisfies the oscillator algebra:

$$[N_0, a_{0c}^\pm] = \pm a_{0c}^\pm, \quad a_{0c}^+ a_{0c}^- = N_0, \quad a_{0c}^- a_{0c}^+ = N_0 + 1, \quad [a_{0c}^-, a_{0c}^+] = 1. \quad (20)$$

However, the commutator of $H_0$ with $a_{0c}^\pm$ remains the same as for $a_0^\pm$ (see Eq. (12)).

2.3 General deformation of the intrinsic algebra of $H_0$

Since it will be used later, it is worth to mention that the previous linearization is a particular case of a general deformation of the intrinsic algebra defined by Eqs. (8,10,12) for $N_0, a_0^-, a_0^+$. In this procedure, new annihilation and creation operators $a^- = \beta(N_0) a_0^-$, $a^+ = a_0^+ \beta(N_0)$, are constructed such that:

$$[N_0, a^\pm] = \pm a^\pm, \quad a^+ a^- = \bar{E}(N_0), \quad a^- a^+ = \bar{E}(N_0 + 1), \quad (21)$$

$$[a^-, a^+] = \bar{E}(N_0 + 1) - \bar{E}(N_0) = \bar{f}(N_0), \quad (22)$$
where \( \tilde{E}(N_0) \) and \( \tilde{E}(N_0 + 1) \) are positive definite operators and \( \beta(x) \) is a real non-singular function for \( x \in \mathbb{Z}^+ \) to be adjusted according to the chosen \( \tilde{E}(N_0) \). It is clear that different choices of \( \tilde{E}(N_0) \) lead to different deformations:

\[
\tilde{E}(N_0) = \beta^2(N_0 - 1)[E(N_0) - E_0] \quad \Rightarrow \quad \beta(N_0) = \sqrt{\frac{\tilde{E}(N_0 + 1)}{E(N_0 + 1) - E_0}}.
\] (23)

In particular, in the previous section we were interested in a deformation simplifying maximally the original algebra. It can be here recovered by the choice \( \tilde{E}(N_0) = N_0 \), and by using (14,18,23), it turns out that \( \beta(x) = b(x), a^\pm = a^\pm_0, \tilde{f}(N_0) = 1. \)

3 Coherent states of \( H_0 \)

Once some algebras ruling our system have been identified, let us look for the associated CS. We will derive them as eigenstates of the several annihilation operators defined previously.

3.1 Intrinsic nonlinear coherent states of \( H_0 \)

In the first place, let us analyze the CS \( |z, \alpha_0 \rangle \) which are eigenstates of the annihilation operator of the intrinsic algebra:

\[
a_0^0 |z, \alpha_0 \rangle = z|z, \alpha_0 \rangle, \quad z \in \mathbb{C}.
\] (24)

By expanding \( |z, \alpha_0 \rangle \) in the basis of eigenstates of \( H_0 \) and following the standard procedure to determine the expansion coefficients, it turns out that:

\[
|z, \alpha_0 \rangle = \left( \sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m} \right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{\rho_m}} |\psi_m \rangle,
\] (25)

\[
\rho_m = \begin{cases} 
1 & \text{if } m=0, \\
(E_m - E_0) \ldots (E_1 - E_0) & \text{if } m > 0.
\end{cases}
\] (26)

It is important to seek now if the intrinsic nonlinear CS (25) form a complete set, i.e., if they satisfy

\[
\int |z, \alpha_0 \rangle \langle z, \alpha_0 | d\mu(z) = 1.
\] (27)

Let us express the positive definite measure \( d\mu(z) \) in the form:

\[
d\mu(z) = \frac{1}{\pi} \left( \sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m} \right) \rho(|z|^2) d^2z,
\] (28)

\( \rho(y) \) being a function to be determined. Working in polar coordinates and making the change of variable \( y = |z|^2 \), it is straightforward to show that \( \rho(y) \) must satisfy:

\[
\int_0^\infty y^m \rho(y) dy = \rho_m, \quad m = 0, 1, \ldots
\] (29)
The moment problem \((29)\), in which we look for a positive definite function \(\rho(y)\) with the given \(m\)-th order moments \(\rho_m\), often arises in the literature when a completeness relationship of kind \((27)\) is to be proven \([29, 34, 42–44]\). The generic answer is nowadays known: \(\rho(y)\) is the inverse Mellin transform of \(\rho_m\) \([34]\). However, for each particular system this calculation has to be performed explicitly, which is not always easy (see e.g. \([29]\)).

The expression \((27)\) guarantees that any state of the system can be expanded in terms of CS. In particular, this can be done for an arbitrary CS \(|z', \alpha\rangle_0\):

\[
|z', \alpha\rangle_0 = \int |z, \alpha\rangle_0 \, 0\langle z, \alpha|z', \alpha\rangle_0 \, d\mu(z),
\]

where the reproducing kernel \(0\langle z, \alpha|z', \alpha\rangle_0\) is expressed as:

\[
0\langle z, \alpha|z', \alpha\rangle_0 = \left( \sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m} \right)^{-\frac{1}{2}} \left( \sum_{m=0}^{\infty} \frac{|z'|^{2m}}{\rho_m} \right)^{-\frac{1}{2}} \left( \sum_{m=0}^{\infty} (\bar{z}z')^m \rho_m \right). \tag{31}
\]

Let us notice that the eigenvalue \(z = 0\) of \(a^{-0}_{\bar{z}}\) is non-degenerated since:

\[
|z = 0, \alpha\rangle_0 = |\psi_0\rangle. \tag{32}
\]

Another important property of the intrinsic nonlinear CS \(|z, \alpha\rangle_0\), which is due to the phase choice of Eqs. \((5-6)\), is that they evolve coherently in time:

\[
U_0(t)|z, \alpha\rangle_0 = e^{-itE_0}|z, \alpha + t\rangle_0, \tag{33}
\]

\(U_0(t) = \exp(-itH_0)\) being the evolution operator associated to \(H_0\).

### 3.2 Linear coherent states of \(H_0\)

Let us study the CS which are eigenstates of the linear annihilation operator of \(H_0\):

\[
a^{-0}_{\bar{z}}|z, \alpha\rangle_{0_{\bar{z}}} = z|z, \alpha\rangle_{0_{\bar{z}}}, \quad z \in \mathbb{C}. \tag{34}
\]

Hence:

\[
|z, \alpha\rangle_{0_{\bar{z}}} = e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{m!}} |\psi_m\rangle. \tag{35}
\]

Up to the phases involving \(\alpha\), they have the form of the standard harmonic oscillator CS.

Contrasting with the difficulty to find a positive definite measure ensuring the completeness of the non-linear CS \((25)\), now the problem is already solved:

\[
\frac{1}{\pi} \int |z, \alpha\rangle_{0_{\bar{z}}} \, 0_{\bar{z}}\langle z, \alpha| \, d^2 z = 1, \tag{36}
\]

i.e., the measure is the standard one, \(d^2 z/\pi\). Thus, an arbitrary linear CS \(|z', \alpha\rangle_{0_{\bar{z}}}\) admits a non-trivial decomposition in terms of \(|z, \alpha\rangle_{0_{\bar{z}}}\):

\[
|z', \alpha\rangle_{0_{\bar{z}}} = \frac{1}{\pi} \int |z, \alpha\rangle_{0_{\bar{z}}} \, 0_{\bar{z}}\langle z, \alpha|z', \alpha\rangle_{0_{\bar{z}}} \, d^2 z, \tag{37}
\]
where the reproducing kernel is equal to the oscillator one:

$$0_L \langle z, \alpha | z', \alpha \rangle _{0_L} = \exp \left( -\frac{|z|^2}{2} + \bar{z}z' - \frac{|z'|^2}{2} \right). \quad (38)$$

The only eigenstate of $H_0$ which is as well a linear CS (35) is again the ground state:

$$|z = 0, \alpha \rangle _{0_L} = |\psi_0 \rangle . \quad (39)$$

Since $[a^-_{0_L}, a^+_{0_L}] = 1$, the linear CS also result from acting a ‘displacement’ operator onto $|\psi_0 \rangle$:

$$|z, \alpha \rangle _{0_L} = D_L(z)|\psi_0 \rangle = \exp(za^+_{0_L} - \bar{z}a^{-}_{0_L})|\psi_0 \rangle. \quad (40)$$

4 The SUSY partner Hamiltonians $H_k$

Let us discuss in the first place some generalities of the SUSY partner Hamiltonians $H_k$,

$$H_k = -\frac{1}{2} \frac{d^2}{dx^2} + V_k(x), \quad (41)$$

generated from $H_0$ through a $k$-th order differential intertwining operator $B_k^+$ [33],

$$H_k B_k^+ = B_k^+ H_0 \quad \Leftrightarrow \quad H_0 B_k = B_k H_k. \quad (42)$$

The potential $V_k(x)$ reads:

$$V_k(x) = V_0(x) - \sum_{i=1}^{k} \alpha'_i(x, \epsilon_i), \quad (43)$$

where, in case that the $k$ factorization energies $\epsilon_i, \ i = 1, \ldots, k$ are all different, $\alpha_i(x, \epsilon_i)$ is obtained from a recursive (Bäcklund) formula:

$$\alpha_i(x, \epsilon_i) = -\alpha_{i-1}(x, \epsilon_{i-1}) - \frac{2(\epsilon_i - \epsilon_{i-1})}{\alpha_{i-1}(x, \epsilon_i) - \alpha_{i-1}(x, \epsilon_{i-1})}, \quad i = 2, \ldots, k, \quad (44)$$

and $\alpha_1(x, \epsilon_i)$ are solutions of the following Riccati equation:

$$\alpha'_1(x, \epsilon_i) + \alpha^2_1(x, \epsilon_i) = 2[V_0(x) - \epsilon_i], \quad i = 1, \ldots, k. \quad (45)$$

This is equivalent to the initial stationary Schrödinger equation for the factorization energies $\epsilon_i$, as can be seen from the change $\alpha_1(x, \epsilon_i) = u'_i(x)/u_i(x)$:

$$-\frac{1}{2} u''_i + V_0(x) u_i = \epsilon_i u_i. \quad (46)$$

In terms of the transformation functions $u_i$, the new potential in (43) becomes:

$$V_k(x) = V_0(x) - \{\ln[W(u_1, \ldots, u_k)]\}'' \quad, \quad (47)$$
We define first a number operator \( H \) with respect to \( \text{Sp}(H) \) which will be useful later. Since the positions of the new levels \( n, m \) one might think that some algebraic properties of \( H \) where \( n, m \) the initial ones are inherited by \( \text{Sp}(H_0) \), but without deleting any of the original levels of \( H_0 \), i.e.,

\[
\text{Sp}(H_k) = \{ \epsilon_1, \ldots, \epsilon_q, E_0, E_1, \ldots \}, \quad q \leq s.
\]

This means that \( p \equiv s - q \) factorization energies \( \epsilon_{q+j} \) coincide with \( p \) energy levels \( E_{m_j} \) of \( H_0 \), i.e., \( \epsilon_{q+j} = E_{m_j} \), \( j = 1, \ldots, p \), \( m_j < m_{j+1} \), and thus \( B_k^+|\psi_{m_j}\rangle = 0 \). The eigenstates \( |\theta_n\rangle \) of \( H_k \) which are associated to the remaining energies \( E_n \), \( n \neq m_j \), are obtained from the initial ones \( |\psi_n\rangle \) and vice versa through the intertwining operators \( B_k^+ \) and \( B_k \), namely:

\[
|\theta_n\rangle = \frac{B_k^+|\psi_n\rangle}{\sqrt{\prod_{i=1}^{k} (E_n - \epsilon_i)}}, \quad |\psi_n\rangle = \frac{B_k|\theta_n\rangle}{\sqrt{\prod_{i=1}^{k} (E_n - \epsilon_i)}}.
\]

It is convenient to extend now the definition of \( |\theta_n\rangle \) for \( n = m_j \) in the way:

\[
|\theta_{m_j}\rangle \equiv |\theta_{q+j}\rangle, \quad j = 1, \ldots, p.
\]

Summarizing all this information, the eigenstates \( |\theta_{\epsilon_i}\rangle \), \( |\theta_n\rangle \) of \( H_k \) obey:

\[
\text{Sp}(H_k) = \{ \epsilon_1, \ldots, \epsilon_q, E_0, E_1, \ldots \}, \quad q \leq s,
\]

\[
|\theta_{\epsilon_i}\rangle = E_n|\theta_n\rangle, \quad H_k|\theta_{\epsilon_i}\rangle = \epsilon_i|\theta_{\epsilon_i}\rangle, \quad \langle \theta_{\epsilon_i}|\theta_n\rangle = 0, \quad \langle \theta_{m}|\theta_{\epsilon_i}\rangle = \delta_{mn}, \quad \langle \theta_{\epsilon_i}|\theta_{\epsilon_j}\rangle = \delta_{ij},
\]

\[
\sum_{i=1}^{s} |\theta_{\epsilon_i}\rangle\langle \theta_{\epsilon_i}| + \sum_{m=0}^{\infty} |\theta_{m}\rangle\langle \theta_{m}| = \sum_{i=1}^{q} |\theta_{\epsilon_i}\rangle\langle \theta_{\epsilon_i}| + \sum_{m=0}^{\infty} |\theta_{m}\rangle\langle \theta_{m}| = 1,
\]

where \( n, m = 0, 1, \ldots \), \( i, j = 1, \ldots, q \), \( \sum_m \) is the sum over \( m = 0, 1, \ldots \) except by \( m_j \), \( j = 1, \ldots, p \), and the identity operator has been expanded in two alternative ways which will be useful later. Since the positions of the new levels \( \epsilon_i \), \( i = 1, \ldots, q \) are arbitrary, one might think that some algebraic properties of \( H_0 \) are inherited by \( H_k \) on the subspace spanned by the \( |\theta_n\rangle \), \( n = 0, 1, \ldots \). Keeping this in mind, let us analyze some interesting algebras of the SUSY partner Hamiltonians \( H_k \).

## 5 Nonlinear algebras of \( H_k \)

We define first a number operator \( N_k \) by its action onto the eigenstates of \( H_k \):

\[
N_k|\theta_n\rangle = n|\theta_n\rangle, \quad N_k|\theta_{\epsilon_i}\rangle = 0, \quad n = 0, 1, \ldots \quad i = 1, \ldots, q.
\]
Notice that this definition is more natural than a previous one, introduced as the “generalized number operator” for the SUSY partners of the oscillator (compare with Eq. (3.4) of [34]).

Let us study next two pairs of annihilation and creation operators of \( H_k \) (and \( N_k \)) as well as their corresponding nonlinear algebras.

### 5.1 Natural algebra of \( H_k \)

Here we will obtain annihilation and creation operators of \( H_k \) following a 3-steps construction previously introduced for the SUSY partner Hamiltonians of the harmonic oscillator [29][34][47]. Thus, starting from the intrinsic operators \( a_{0 \pm} \) of \( H_0 \) and the intertwining ones \( B_k, B_k^+ \) of (42), a pair of natural annihilation and creation operators \( a_{kN}^{\pm} \) of \( H_k \) is built up:

\[
a_{kN}^{\pm} = B_k a_{0 \pm} B_k.
\]

(56)

Since \( B_k |\theta_{i_j}\rangle = 0, i = 1, \ldots, s \), one can find the action of \( a_{kN}^{\pm} \) onto the basis of eigenvectors of \( H_k \) (and \( N_k \)) by using (5,50):

\[
a_{kN}^{-} |\theta_n\rangle = r_{N}(n) |\theta_{n-1}\rangle, \quad a_{kN}^{+} |\theta_n\rangle = \bar{r}_{N}(n+1) |\theta_{n+1}\rangle, \quad n = 0, 1, \ldots
\]

(57)

\[
r_{N}(n) = \left\{ \prod_{i=1}^k [E(n) - \epsilon_i][E(n-1) - \epsilon_i] \right\}^{1/2} r_{I}(n).
\]

(59)

Notice that \( r_{N}(m_j) = 0, j = 1, \ldots, p \), which is consistent with \( B_k |\theta_{m_j}\rangle = a_{kN}^{-} |\theta_{m_j}\rangle = 0 \).

From these expressions one can find the Hubbard representation for \( a_{kN}^{\pm} \):

\[
a_{kN}^{-} = r_{N}(N_k + 1) \sum_{m=0}^{\infty} |\theta_m\rangle \langle \theta_{m+1}|, \quad a_{kN}^{+} = \bar{r}_{N}(N_k) \sum_{m=0}^{\infty} |\theta_{m+1}\rangle \langle \theta_m|.
\]

(60)

Making use of \( a_{kN}^{\pm} g(N_k) = g(N_k+1)a_{kN}^{\pm} \) for an arbitrary regular function \( g(x), x \in \mathbb{Z}^+ \), one can show that:

\[
[a_{kN}^{-}, a_{kN}^{+}] = [\bar{r}_{N}(N_k + 1)r_{N}(N_k + 1) - \bar{r}_{N}(N_k)r_{N}(N_k)] \sum_{m=0}^{\infty} |\theta_m\rangle \langle \theta_m|.
\]

(61)

### 5.2 Intrinsic algebra of \( H_k \)

It is interesting to observe that simpler annihilation and creation operators for \( H_k \) can be constructed, proceeding by analogy with [13]. Thus, we define the intrinsic annihilation and creation operators \( a_k^{\pm} \) of \( H_k \) as follows:

\[
a_{k}^{-} = r_{I}(N_k + 1) \sum_{m=0}^{\infty} |\theta_m\rangle \langle \theta_{m+1}|, \quad a_{k}^{+} = \bar{r}_{I}(N_k) \sum_{m=0}^{\infty} |\theta_{m+1}\rangle \langle \theta_m|,
\]

(62)
where \( r_\mathcal{I}(n) \) is given in (60). It can be checked that \( a_k^\pm |\theta_{\epsilon_i}\rangle = 0 \), \( i = 1, \ldots, q \), and:

\[
\begin{align*}
 a_k^- |\theta_n\rangle &= r_\mathcal{I}(n) |\theta_{n-1}\rangle, & a_k^+ |\theta_n\rangle &= r_\mathcal{I}(n+1) |\theta_{n+1}\rangle, \\
 a_k^+ a_k^- |\theta_n\rangle &= (E_n - E_0) |\theta_n\rangle, & a_k^- a_k^+ |\theta_n\rangle &= (E_{n+1} - E_0) |\theta_n\rangle.
\end{align*}
\]

(63)

Thus, the commutator between \( a_k^\pm \) is similar to that for the intrinsic algebra of \( H_0 \) on the subspace spanned by \( \{|\theta_n\rangle, \ n = 0, 1, \ldots \} \):

\[
[a_k^-, a_k^+] = f(N_k) \sum_{m=0}^{\infty} |\theta_m\rangle \langle \theta_m|.
\]

(65)

We would like to seek next if there is any connection between the initial and final number operators \( N_0 \) and \( N_k \). After some simple manipulations, it can be shown that:

\[
N_k = C_k^+ N_0 C_k + \sum_{j=1}^{p} m_j |\theta_{m_j}\rangle \langle \theta_{m_j}| \iff N_k \sum_{m=0}^{\infty} |\theta_m\rangle \langle \theta_m| = C_k^+ N_0 C_k,
\]

(66)

\[
C_k = \frac{1}{\sqrt{\prod_{i=1}^{k} [E(N_0) - \epsilon_i]}} B_k, \quad C_k^+ = \frac{1}{\sqrt{\prod_{i=1}^{k} [E(N_k) - \epsilon_i]}} B_k^+.
\]

(67)

\( C_k, C_k^+ \) being modified intertwining operators inverse to each other when acting on the eigenstates of the isospectral part which are not used as seeds in the SUSY procedure, i.e.,

\[
C_k |\theta_n\rangle = |\psi_n\rangle, \quad C_k^+ |\psi_n\rangle = |\theta_n\rangle, \quad \mathbb{Z}^+ \ni n \neq m_j, \ j = 1, \ldots, p,
\]

(68)

but in general they are not invertible in the full Hilbert space \( \mathcal{L}^2(\mathbb{R}) \) since \( C_k |\theta_{\epsilon_i}\rangle = C_k |\theta_{m_j}\rangle = C_k^+ |\psi_{m_j}\rangle = 0, \ i = 1, \ldots, q, \ j = 1, \ldots, p \). From these expressions one can check that

\[
a_k^\pm \sum_{m} |\theta_m\rangle \langle \theta_m| = C_k^+ a_k^\pm C_k.
\]

(69)

By using Eqs. (68,69) one recovers (63). Moreover, it turns out that

\[
a_k^+ a_k^- = [E(N_k) - E_0] = [H_k - E_0] \sum_{m=0}^{\infty} |\theta_m\rangle \langle \theta_m|.
\]

(70)

The RHS of the expressions (69) for the intrinsic operators \( a_k^\pm \) consist of a 3-steps action, similar to the natural ones \( a_{kN}^\pm \) of (60). The difference is that the new intertwiners \( C_k, C_k^+ \) transform the states \( |\theta_n\rangle \leftrightarrow |\psi_n\rangle, \ \mathbb{Z}^+ \ni n \neq m_j, \ j = 1, \ldots, p \), without changing the norm (compare (68) with (60)). This explains why the intrinsic algebra generated by \( \{N_k, a_k^-, a_k^+\} \) is simpler than the natural one obtained from \( \{N_k, a_{kN}^-, a_{kN}^+\} \). In addition, the intrinsic algebra is a deformation of the natural one and vice versa (remember section 2.3). Indeed, by comparing (60) with (62) one can show that:

\[
a_{kN}^- = \frac{r_N(N_k + 1)}{r_\mathcal{I}(N_k + 1)} a_k^- \quad \text{and} \quad a_{kN}^+ = \frac{r_N(N_k)}{r_\mathcal{I}(N_k)} a_k^+ \quad \text{and} \quad a_{kN}^+ a_{kN}^- = [E(N_k) - E_0] \left[ \frac{r_N(N_k)}{r_\mathcal{I}(N_k)} \right]^2
\]

(71)

We will see next another deformation of the intrinsic algebra generated by \( \{N_k, a_k^-, a_k^+\} \).
6 Linear algebra of $H_k$

Let us introduce now a new pair of annihilation and creation operators for $H_k$, such that their action onto the $|\theta_n\rangle$’s is similar to the oscillator one (see (15-16)):

$$a_{k_L}^-|\theta_n\rangle = r_L(n)|\theta_{n-1}\rangle, \quad a_{k_L}^+|\theta_n\rangle = \bar{r}_L(n+1)|\theta_{n+1}\rangle,$$
$$a_{k_L}^\pm|\epsilon_i\rangle = 0, \quad i = 1, \ldots, q.$$

In the Hubbard representation we have:

$$a_{k_L}^- = r_L(N_k + 1)\sum_{m=0}^{\infty} |\theta_m\rangle\langle\theta_{m+1}|, \quad a_{k_L}^+ = \bar{r}_L(N_k)\sum_{m=0}^{\infty} |\theta_{m+1}\rangle\langle\theta_m|. \quad (72)$$

It is simple to show that:

$$[N_k, a_{k_L}^\pm] = \pm a_{k_L}^\pm, \quad [a_{k_L}^-, a_{k_L}^+] = \sum_{m=0}^{\infty} |\theta_m\rangle\langle\theta_m|. \quad (73)$$

One can also find that:

$$\sum_{m=0}^{\infty} |\theta_m\rangle\langle\theta_m| = C_k a_{0_L}^+ C_k. \quad (74)$$

By comparing (72) with (62), it is seen that the linear annihilation and creation operators $a_{k_L}^\pm$ are deformations of the intrinsic ones $a_k^\pm$ to get a simpler algebra, namely:

$$a_{k_L}^- = \frac{r_L(N_k + 1)}{r_I(N_k + 1)} a_k^-, \quad a_{k_L}^+ = \frac{r_L(N_k)}{r_I(N_k)} a_k^+, \quad a_{k_L}^+ a_{k_L}^- = N_k. \quad (75)$$

7 Coherent states of $H_k$

Let us construct three sets (in general non-equivalent) of CS as eigenstates of $a_{k_L}^\pm$, $a_{k_L}^-$, $a_{k_L}^+$. According to the algebra involved, they will be called natural, intrinsic and linear CS respectively. It will be seen that some differences with respect to the CS of $H_0$ arise.

7.1 Natural nonlinear coherent states of $H_k$

We build up first the natural nonlinear coherent states $|z, \alpha\rangle_{k_N}$ which are eigenstates of $a_{k_N}^\pm$. Their expansion in terms of eigenstates of $H_k$ read:

$$|z, \alpha\rangle_{k_N} = \sum_{i=1}^{q} \gamma_{\epsilon_i}|\theta_{\epsilon_i}\rangle + \sum_{m=0}^{\infty} \gamma_m|\theta_m\rangle. \quad (76)$$

From the CS definition and making use of (77-58), we get $\gamma_{\epsilon_i} = 0$, $i = 1, \ldots, q$, and

$$r_{N}(m)\gamma_m = z\gamma_{m-1}, \quad m = 1, 2, \ldots \quad (77)$$
According to our SUSY treatment, $\epsilon_s = E_{m_p}$ is the largest eigenvalue of $H_k$, of the part isospectral to $H_0$, for which $B_k |\theta_{m_p}\rangle = a_+^{k_N} |\theta_{m_p}\rangle = 0$. Moreover, since $B_k^+ |\psi_{m_p}\rangle = 0$ it turns out that $a_-^{k_N} |\theta_{m_p+1}\rangle = 0$, i.e., $r_N(m_p+1) = 0$, and by using (77) this implies that $\gamma_{m_p} = 0$. By iterating down this equation we arrive at $\gamma_m = 0$, $m = 0, \ldots, m_p$. Eq. (77) can be used again to express $\gamma_{m+m_p+1}, m > 0$, in terms of $\gamma_{m_p+1}$:

$$\gamma_{m+m_p+1} = E_{m_p} \frac{\sum_{i=0}^\infty \frac{|z|^{2m}}{\rho_m}}{E_{m_p} \frac{m}{r_N(m+p+1)r_N(m+p+2)r_N(m+p+3) \cdots} \gamma_{m_p+1}, m > 0. \quad (78)$$

By using the normalization condition and asking for $\gamma_{m_p+1} \in \mathbb{R^+}$, we finally obtain:

$$|z, \alpha\rangle_{k_N} = \left( \sum_{m=0}^\infty \frac{|z|^{2m}}{\rho_m} \right)^{-\frac{1}{2}} \sum_{m=0}^\infty \rho_m e^{-i\alpha(E_{m+m_p+1} - E_{m_p+1})} \frac{z^m}{\sqrt{\rho_m}} |\theta_{m_p+1}\rangle, \quad (79)$$

where $\rho_0 = 1$ and, for $m > 0$,

$$\tilde{\rho}_m = \frac{\rho_{m+m_p+1}}{\rho_{m_p+1}} \prod_{i=1}^{k_N} (E_{m+m_p+1} - \epsilon_i)(E_{m+m_p} - \epsilon_i)^2 \cdots (E_{m_p+2} - \epsilon_i)^2 (E_{m_p+1} - \epsilon_i), \quad (80)$$

with $\rho_m$ given by (26).

An important difference of $|z, \alpha\rangle_{k_N}$ with respect to the sets of CS of $H_0$ is that the completeness relationship now has to include the eigenstates of $H_k$ which are missing in the expansion (79), i.e.,

$$\sum_{i=1}^q |\theta_{e_i}\rangle \langle \theta_{e_i}| + \sum_{m=0}^{m_p} |\theta_m\rangle \langle \theta_m| + \int |z, \alpha\rangle_{k_N} |z, \alpha\rangle_{k_N} \langle z, \alpha| d\tilde{\mu}(z) = 1. \quad (81)$$

A similar procedure as for the CS of $H_0$ leads to:

$$d\tilde{\mu}(z) = \frac{1}{\pi} \left( \sum_{m=0}^\infty \frac{|z|^{2m}}{\rho_m} \right) \tilde{\rho}(|z|^2) d^2z, \quad (82)$$

$\tilde{\rho}(y)$ satisfying a moment problem more complicated than the initial one (compare $\rho_m$ of (26) with $\rho_m$ of (50)):

$$\int_0^\infty y^m \tilde{\rho}(y) dy = \tilde{\rho}_m, m \geq 0. \quad (83)$$

Another relevant difference is that, since $B_k |\theta_{e_i}\rangle = a_+^{k_N} |\theta_{e_i}\rangle = 0$, $i = 1, \ldots, q$, $B_k |\theta_{m_j}\rangle = a_-^{k_N} |\theta_{m_j}+1\rangle = 0$, $a_-^{k_N} |\theta_{m_j}\rangle = 0$, $j = 1, \ldots, p$, and $a_-^{k_N} |\theta_{0}\rangle = 0$, then the degeneracy of the eigenvalue $z = 0$ of $a_-^{k_N}$ can be any integer in the set $\{s+1, \ldots, s+p+1\}$, depending on the positions of the levels $E_{m_j}$, $j = 1, \ldots, p$. However, once again by the phase choice of Eq. (84), the natural CS $|z, \alpha\rangle_{k_N}$ of (79) evolve coherently in time:

$$U_k(t)|z, \alpha\rangle_{k_N} = e^{-itE_{m_p+1}|z, \alpha\rangle_{k_N} + t}, \quad (84)$$

$U_k(t) = \exp(-itH_k)$ being the evolution operator associated to $H_k$. This property also will be valid for the other CS of $H_k$ which will be further derived.
Let us remark that some properties of the natural nonlinear CS of $H_k$ were studied previously for the SUSY partners of the harmonic oscillator [29,34]. To compare with the case discussed in [34], let us restrict ourselves to SUSY transformations for which the seeds are just nonphysical eigenfunctions of $H_0$, i.e., take $p = 0$ and $q = s \leq k$. Now the only eigenstate of $H_k$ for the part of the spectrum isospectral to $H_0$ which is annihilated by $a^i_{kN}$ is $|\theta_0\rangle$, and thus the CS expansion (79) should start from this state. This is achieved by defining $n_{q=0} = -1$: with this choice and taking the harmonic oscillator energy levels in the CS of (79) one arrives to the CS of Eq. (5.14) in [34].

7.2 Intrinsic nonlinear coherent states of $H_k$

Let us analyze next the intrinsic nonlinear CS $|z,\alpha\rangle_k$ which are eigenstates of $a^-_k$. A similar procedure as before leads to

$$|z,\alpha\rangle_k = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m-E_0)} \frac{z^m}{\sqrt{\rho_m}} |\theta_m\rangle. \quad (85)$$

This expansion is also obtained from the intrinsic nonlinear CS $|z,\alpha\rangle_0$ of $H_0$ and vice versa by the change $|\psi_n\rangle \leftrightarrow |\theta_n\rangle$ (compare Eqs. (25) and (85)). Thus, the completeness relationship is automatically satisfied,

$$\sum_{i=1}^{q} |\theta_{\epsilon_i}\rangle\langle\theta_{\epsilon_i}| + \int |z,\alpha\rangle_k \langle z,\alpha| d\mu(z) = 1, \quad (86)$$

where $d\mu(z)$ is given by equations (28,29). This is a simplification with respect to the natural nonlinear CS $|z,\alpha\rangle_{kN}$ of (79,80). After some simple manipulations we also arrive at

$$|z,\alpha\rangle_k = C^+_k |z,\alpha\rangle_0 + \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}} \sum_{j=1}^{p} e^{-i\alpha(E_{m_j}-E_0)} \frac{z^{m_j}}{\sqrt{\rho_{m_j}}} \langle \theta_{m_j}|. \quad (87)$$

Since $a^-_k |\theta_{\epsilon_i}\rangle = 0$, $i = 1,\ldots,q$ and taking into account that

$$|z = 0,\alpha\rangle_k = |\theta_0\rangle, \quad (88)$$

it turns out that the eigenvalue $z = 0$ of $a^-_k$ is $(q + 1)$-th degenerated.

7.3 Linear coherent states of $H_k$

Let us consider the linear CS which are eigenstates of $a^-_{k\ell}$. Since the algebra of $a^\pm_{k\ell}$ acting onto $\text{Span}\{|\theta_n\rangle, n = 0,1,\ldots\}$ is equal to that of $a^\pm_{0\ell}$ acting onto $\text{Span}\{|\psi_n\rangle, n = 0,1,\ldots\}$, it can be shown that:

$$|z,\alpha\rangle_{k\ell} = e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m-E_0)} \frac{z^m}{\sqrt{m!}} |\theta_m\rangle. \quad (89)$$
This expression is also obtained from the corresponding one for $|z, \alpha\rangle_0$ and vice versa by the mapping $|\psi_m\rangle \leftrightarrow |\theta_m\rangle$ (compare (35) and (89)). Thus, the completeness relationship is identified in a simple way:

$$\sum_{i=1}^{q} |\theta_{\epsilon_i}\rangle \langle \theta_{\epsilon_i}| + \frac{1}{\pi} \int |z, \alpha\rangle_0 \langle z, \alpha| d^2z = 1.$$  

(90)

Moreover, it turns out that:

$$|z, \alpha\rangle_k = C_k^+ |z, \alpha\rangle_0 + e^{-|z|^2} \sum_{j=1}^{p} e^{-i\alpha(E_{m_j} - E_0)} \frac{z^{m_j}}{\sqrt{m_j!}} |\theta_{m_j}\rangle.$$  

(91)

The eigenvalue $z = 0$ of $a^-_k$ is $(q + 1)$-th degenerated, a property discovered for the first time for the SUSY partners of the harmonic oscillator \[29, 34\]. It can also be found that

$$|z, \alpha\rangle_k = D_k |\theta_0\rangle = \exp(z a^+_k - \bar{z} a^-_k) |\theta_0\rangle.$$  

(92)

8 Examples

We will apply the previous techniques to some examples: the harmonic oscillator, infinite square well and trigonometric Pöschl-Teller potentials. For each system we will use a different kind of SUSY transformation, depending on how many physical eigenstates $|\theta_{\epsilon_i}\rangle$ of $H_k$ which are annihilated by $B_k$ have energies different from the ones of $H_0$. Thus, for the harmonic oscillator we will study the general situation with $q \neq 0, \ p \neq 0$, while for the infinite square well the strictly isospectral case with $q = 0, \ p = s$ will be explored. For the Pöschl-Teller potential the $s$ levels $\epsilon_i$ will be different from the ones of $H_0$ (i.e. for $q = s, \ p = 0$).

8.1 The harmonic oscillator

Let us consider the harmonic oscillator potential:

$$V_0(x) = \frac{x^2}{2}.$$  

(93)

The normalized eigenfunctions and eigenvalues of $H_0$ are given by:

$$\psi_n(x) = \langle x|\psi_n\rangle = \frac{e^{-x^2} H_n(x)}{\sqrt{\sqrt{\pi} 2^n n!}}, \ E(n) \equiv E_n = n + \frac{1}{2}, \ n = 0, 1, \ldots$$  

(94)

where $H_n(x)$ are the Hermite polynomials. Since $E(n)$ is linear in $n$, it is simple to show that $f(N_0) = 1$. Thus, after dropping some unimportant global phases, the intrinsic algebra reduces to the Heisenberg-Weyl one, as it was expected. This implies that the corresponding CS become as well the canonical ones (take $\alpha = 0$ in the formulae of sections 2.1 and 3.2).
8.1.1 The SUSY partners $H_k$.

Let us study the $k$-th order SUSY partners of the harmonic oscillator. In order to implement the transformation, we look for the general solution $u(x)$ of the stationary Schrödinger equation (46) with the oscillator potential (93) for an arbitrary factorization energy $\epsilon$. Up to a constant factor we obtain:

$$u(x) = e^{-\frac{x^2}{2}} \left[ _1F_1 \left( \frac{1}{4} - \frac{\epsilon}{2}, \frac{3}{2}; x^2 \right) + 2\mu x \Gamma \left( \frac{3}{4} - \frac{\epsilon}{2}, \frac{1}{2} \right) _1F_1 \left( \frac{3}{4} - \frac{\epsilon}{2}, \frac{3}{2}; x^2 \right) \right],$$

where $_1F_1(a; b; y)$ is the confluent hypergeometric function and $u(x)$ is nodeless for $\epsilon < 1/2, |\mu| < 1$. By using this expression to specify the seed solution $s$, the associated Wronskian can be calculated, which automatically leads to the new potential and the corresponding energy eigenstates.

8.1.2 Algebraic structures of $H_k$.

The annihilation and creation operators for the several algebras of $H_k$, in terms of the intrinsic ones $a_\pm^k$, are given by Eqs. (71, 75), where:

$$r_X(n) = \prod_{i=1}^k (n - \epsilon_i - \frac{1}{2}) (n - \epsilon_i + \frac{1}{2})^{\frac{1}{2}}, \quad r_L(n) = 1.$$  \hspace{1cm} (96)

Up to a global phase factor, the intrinsic operators $a_\pm^k$ are those of (62) with $r_L(n) = \sqrt{n}$, i.e., we recover the Heisenberg-Weyl algebra onto $\text{Span}\{|\theta_n\rangle, n = 0, 1, \ldots \}$.

8.1.3 Coherent states of $H_k$.

In order to find the natural nonlinear CS of $H_k$, we determine first the coefficients $\tilde{\rho}_m$ of (79, 80):

$$\tilde{\rho}_m = (m_p + 2)_m \prod_{i=1}^k \left( m_p - \epsilon_i - \frac{3}{2} \right)_m \left( m_p - \epsilon_i + \frac{5}{2} \right)_m, \quad m \geq 0,$$

with the Pochhammer symbol given by $(b)_m = \Gamma(b + m)/\Gamma(b)$. Hence we get:

$$|z, \alpha\rangle_{kN} = \frac{1}{\sqrt{1F_{2k+1}(1; m_p + 2, \ldots, m_p - \epsilon_i + \frac{3}{2}, m_p - \epsilon_i + \frac{5}{2}; |z|^2)}} \times \sum_{m=0}^{\infty} \frac{z^m \theta_{m+m_p+1}}{(m_p + 2)_m \prod_{i=1}^k \sqrt{(m_p - \epsilon_i + \frac{3}{2})_m (m_p - \epsilon_i + \frac{5}{2})_m}},$$

where $pF_q$ is a generalized hypergeometric function defined by:

$$pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m x^m}{(b_1)_m \cdots (b_q)_m m!}.$$
It is clear that the moment problem (83) with the $\tilde{\rho}_m$ of (97) is more involved than the already solved initial one, and it can be worked out once the factorization energies $\epsilon_i$ are specified. Indeed, a few solutions for some SUSY transformations have been derived elsewhere [29, 34].

For the intrinsic nonlinear and linear CS of $H_k$, both expressions are the same and coincide with the canonical expansion, which arises from (35) for $\alpha = 0$ with the change $|\psi_m\rangle \rightarrow |\theta_m\rangle$.

In particular, we illustrate the SUSY partner potential $\tilde{V}_3(x)$ generated from a third-order transformation with $k = 3, q = p = 1$. The seeds $u_1, u_2, u_3$, correspond to the solution (95) with $\epsilon_1 = -3/2$ for $u_1$, the ground state eigenfunction $\psi_0(x)$ of (94) with $\epsilon_2 = E_0 = 1/2$ for $u_2$, and a generalized eigenfunction of second order associated to $\epsilon_3 = \epsilon_2$ for $u_3$ such that $(H_0 - \epsilon_2)u_3 = u_2 \Rightarrow (H_0 - \epsilon_2)^2u_3 = 0$, its nontrivial part given by [46]:

$$u_3 = e^{-\frac{x^2}{2\pi}} \left[ \pi w_0 \text{Erfi}(x) + x^2 F_2\left(1, 1, \frac{3}{2}, 2; x^2\right) \right]. \quad (100)$$

The new potential is obtained from (47), with the Wronskian expressed as:

$$W(u_1, u_2, u_3) = \frac{e^{-\frac{x^2}{2\pi}}}{\sqrt{\pi}} \left\{ -2x + 4\pi w_0 \mu x e^{2x^2} + \sqrt{\pi} e^{x^2} \left[ 4w_0 - \mu - 2\mu x^2 \right. \\
+ \left. \left(1 + 2\sqrt{\pi} (\mu + 2w_0) xe^{x^2} - 2x^2\right) \text{Erf}(x) \right] + 2\pi xe^{2x^2} [\text{Erf}(x)]^2 \right\} \quad (101)$$

This Wronskian is nodeless for $|\mu| < 1$ and $|w_0| > 1/2$. A member of the family of potentials (47) is shown in Fig. 1 for $\mu = 0.99$ and $w_0 = 0.51$. The spectrum of the Hamiltonian $H_3$ is $\{\epsilon_1 = -3/2, E_n = n + 1/2, n = 0, 1, \ldots \}$. 

Figure 1: Third-order SUSY partner potential $V_3(x)$ (black curve) of the oscillator (gray curve) obtained by composing a confluent second-order transformation with seed the ground state of $H_0$ ($w_0 = 0.51$) and a first-order one with $\epsilon_1 = -3/2 (\mu = 0.99)$. The net result is the ‘creation’ of an energy level at $\epsilon_1$ for $H_3$. 

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8.2 The infinite well potential

In dimensionless units, the infinite well potential we shall study reads:

\[ V_0(x) = \begin{cases} \infty & \text{for } x = 0, \pi \\ 0 & \text{for } 0 < x < \pi. \end{cases} \]  \hspace{1cm} (102)

The eigenfunctions and eigenvalues are well known:

\[ \psi_n(x) = \sqrt{\frac{2}{\pi}} \sin\left((n + 1)x\right), \quad E_n = E(n) = \frac{(n + 1)^2}{2}, \quad n = 0, 1, \ldots \]  \hspace{1cm} (103)

8.2.1 Intrinsic algebra of \( H_0 \).

It is determined by the operator function

\[ E(N_0) = \frac{(N_0 + 1)^2}{2} = H_0, \]  \hspace{1cm} (104)

leading thus to the following structure function:

\[ f(N_0) = E(N_0 + 1) - E(N_0) = N_0 + \frac{3}{2}. \]  \hspace{1cm} (105)

The Hubbard representation for the intrinsic operators \( a_0^\pm \) is given by (13), where now:

\[ r_\pm(n) = e^{i\alpha(n + \frac{1}{2})} \sqrt{\frac{n(n + 2)}{2}}. \]  \hspace{1cm} (106)

The operator set \( \{N_0, a_0^-, a_0^+\} \) satisfies then the commutation relationships:

\[ [N_0, a_0^\pm] = \pm a_0^\pm, \quad [a_0^-, a_0^+] = N_0 + \frac{3}{2}, \]  \hspace{1cm} (107)

which, after redefining the number operator as \( \tilde{N}_0 = N_0 + \frac{3}{2} \), reduce to the su(1,1) algebra.

8.2.2 Linear algebra of \( H_0 \).

The linear operators \( a_{0c}^\pm \), expressed as deformations of the intrinsic ones \( a_0^\pm \), acquire the form:

\[ a_{0c}^- = \sqrt{\frac{2}{N_0 + 3}} a_0^- , \quad a_{0c}^+ = a_0^+ \sqrt{\frac{2}{N_0 + 3}} , \quad a_{0c}^+ a_{0c}^- = N_0. \]  \hspace{1cm} (108)

By construction, their action onto the eigenstates of \( H_0 \) is the standard one (up to some phase factors).
8.2.3 Coherent states of $H_0$.

The intrinsic nonlinear and linear CS of $H_0$ become:

$$\langle z, \alpha \rangle_0 = \left[ _0F_1(3; 2|z|^2) \right]^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\frac{z^2}{2}m(m+2)} \frac{2^{m+1}}{m!(m+2)!} \frac{z^m}{m!} |\psi_m\rangle, \quad (109)$$

$$\langle z, \alpha \rangle_0^L = e^{-|z|^2} \sum_{m=0}^{\infty} e^{-i\frac{z^2}{2}m(m+2)} \frac{z^m}{\sqrt{m!}} |\psi_m\rangle. \quad (110)$$

The completeness of the intrinsic nonlinear CS (109) is ensured since the moment problem (29) with $\rho_m = m! (m+2)!/2^{m+1}$ admits the positive definite solution

$$\rho(y) = 4yK_2\left(2\sqrt{2y}\right), \quad (111)$$

$K_2(y)$ being a modified Bessel function of second kind. Hence, the measure (28) reads:

$$d\mu(z) = \frac{4|z|^2}{\pi} _0F_1(3; 2|z|^2) K_2(2\sqrt{2|z|}) d^2z. \quad (112)$$

The reproducing kernel (11) acquires the form:

$$\langle z, \alpha | z', \alpha \rangle_0 = \left[ _0F_1(3; 2|z|^2) _0F_1(3; 2|z'|^2) \right]^{-\frac{1}{2}} _0F_1(3; 2\bar{z}z'). \quad (113)$$

On the other hand, for the linear CS (110) directly apply the formulae of section 3.2, in particular the completeness relationship (36) and the reproducing kernel (38).

8.2.4 The SUSY partners $H_k$.

For generating the $k$-th order SUSY partners of the infinite well potential, we employ isospectral transformations which do not create new levels. This implies that $q = 0$, $p = s \leq k$, and there are $p$ levels of $H_0$, $\epsilon_j = E_{m_j} = (m_j + 1)^2/2$, $j = 1, \ldots, p$, whose physical eigenstates $|\psi_{m_j}\rangle$ are annihilated by $B_k^+$ and will be used as seeds to implement the procedure.

8.2.5 Algebraic structures of $H_k$.

The natural and linear annihilation and creation operators of $H_k$, in terms of the intrinsic ones $a_k^\pm$, are written in Eqs. (71,75), where:

$$\frac{r_{\mathcal{L}}(n)}{r_{\mathcal{I}}(n)} = 2^{-k} \prod_{i=1}^{k} \sqrt{|n^2 - 2\epsilon_i|(n + 1)^2 - 2\epsilon_i}, \quad \frac{r_{\mathcal{I}}(n)}{r_{\mathcal{L}}(n)} = \sqrt{\frac{2}{n+2}}. \quad (114)$$

The intrinsic operators are given in Eq. (62) with $r_{\mathcal{I}}(n)$ given by (106).
8.2.6 Coherent states of $H_k$.

The coefficients $\tilde{\rho}_m$ in (79,80), required to find the natural nonlinear CS $|z, \alpha\rangle_{k_N}$, take the form:

$$\tilde{\rho}_m = \frac{(m_p + 2)_m(m_p + 4)_m}{2^m(2k+1)} \prod_{i=1}^{k} (m_p - \sqrt{2}\epsilon_i + 2)_m(m_p - \sqrt{2}\epsilon_i + 3)_m \times (m_p + \sqrt{2}\epsilon_i + 2)_m(m_p + \sqrt{2}\epsilon_i + 3)_m, \quad m \geq 0. \quad (115)$$

Therefore:

$$|z, \alpha\rangle_{k_N} = \frac{1}{\sqrt{W_k}} \sum_{m=0}^{\infty} e^{-\frac{1}{2}\sum_{m=0}^{m+m+4} \sqrt{2m(2k+1)} z^m |\theta_{m+m+1}|}$$

The moment problem (83) with the $\tilde{\rho}_m$ of (115) can be worked out once the factorization energies $\epsilon_1, \ldots, \epsilon_k$ are specified. These quantities determine as well the degeneracy of the eigenvalue $z = 0$ of $a_{k_N}$, which can take a value in the set $\{p+1, \ldots, 2p+1\}$.

The intrinsic nonlinear and linear CS of $H_k$ are obtained from (109) and (110) respectively by the replacement $|\psi_m\rangle \rightarrow |\theta_m\rangle$.

For illustrating some isospectral SUSY partners of the infinite well (102), we employ a confluent second-order transformation involving one physical eigenfunction of $H_0$, i.e., we take $k = 2$, $\epsilon_1 = \epsilon_2 = E_{m_1} = (m_1 + 1)^2/2$. We need to evaluate the Wronskian of two generalized eigenfunctions $u_1, u_2$ of $H_0$: $u_1$ is the standard physical eigenfunction $\psi_{m_1}(x)$ of (103) obeying $(H_0 - \epsilon_1)u_1 = 0$, but $u_2$ is a second-order generalized eigenfunction such that $(H_0 - \epsilon_1)u_2 = u_1 \Rightarrow (H_0 - \epsilon_1)^2 u_2 = 0$. The expression for $u_2$ is:

$$u_2(x) = -\frac{(\pi w_0 + x)}{\sqrt{2\pi(m_1 + 1)}} \cos[(m_1 + 1)x]. \quad (117)$$

This allows to evaluate the Wronskian $W(u_1, u_2)$, and then the new potential,

$$V_2(x) = \left\{ \begin{array}{ll} \infty & \text{for } x = 0, \pi \\ \frac{16(m_1+1)^2 \sin((m_1+1)x) \sin((m_1+1)x)(\cos[(m_1+1)x])}{\sin(2(m_1+1)x) - 2(m_1+1)(w_0 + x))} & \text{for } 0 < x < \pi, \end{array} \right. \quad (118)$$

which is non-singular for $x \in (0, \pi)$ if $w_0 > 0$ or $w_0 < -1$. An example of these potentials is shown in Fig. 2 for $m_1 = 1$, $w_0 = 0.1$ (black curve), where it is drawn in gray the infinite well (102).
Figure 2: Second-order SUSY partner potential $V_2(x)$ (black curve) isospectral to the infinite well (gray line) obtained by a confluent second-order transformation involving the eigenfunction of the first excited state of $H_0$ and $w_0 = 0.1$.

8.3 The trigonometric Pöschl-Teller potential

In appropriate units the trigonometric Pöschl-Teller potential can be written:

$$V_0(x) = \frac{\nu(\nu - 1)}{2 \cos^2(x)}, \quad \nu > 1. \quad (119)$$

The energy eigenstates $\psi_n(x)$ are expressed in terms of Gegenbauer polynomials $C_n^\nu(y)$ while the eigenvalues are quadratic in $n$ [12, 48]:

$$\psi_n(x) = \left[ \frac{n!(n + \nu)\Gamma(\nu)\Gamma(2\nu)}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2}) \Gamma(n + 2\nu)} \right]^{1/2} \cos^\nu(x) C_n^\nu(\sin(x)),$$

$$E_n = E(n) = \frac{(n + \nu)^2}{2}, \quad n = 0, 1, 2, \ldots \quad (120)$$

8.3.1 Intrinsic algebra of $H_0$.

It is defined by:

$$E(N_0) = \frac{(N_0 + \nu)^2}{2} = H_0, \quad (121)$$

giving place to the following structure function:

$$f(N_0) = E(N_0 + 1) - E(N_0) = N_0 + \nu + \frac{1}{2}. \quad (122)$$

The Hubbard representation for the intrinsic operators $a_0^{\pm}$ is given again by [13] with:

$$r_\pm(n) = e^{i\alpha(n + \nu - \frac{1}{2})} \sqrt{\frac{n(n + 2\nu)}{2}}. \quad (123)$$

The operator set $\{N_0, a_0^-, a_0^+\}$ satisfies the commutation relationships:

$$[N_0, a_0^\pm] = \pm a_0^\pm, \quad [a_0^-, a_0^+] = N_0 + \nu + \frac{1}{2}. \quad (124)$$
which, redefining the number operator as \( \tilde{N}_0 = N_0 + \nu + \frac{1}{2} \), reduce to the \( \text{su}(1,1) \) algebra.

### 8.3.2 Linear algebra of \( H_0 \).

The linear annihilation and creation operators \( a^{\pm}_{0,c} \) can be expressed as deformations of the intrinsic ones \( a^{\pm}_0 \):

\[
a^{\pm}_{0,c} = \sqrt{\frac{2}{N_0 + 2\nu + 1}} a^{\pm}_0, \quad a^{\pm}_0 = a^{\pm}_{0,c} \sqrt{\frac{2}{N_0 + 2\nu + 1}}, \quad a^{\pm}_{0,c} a^{\pm}_{0,c} = N_0.
\]  

(125)

Once again, by construction they act on the eigenstates of \( H_0 \) in a standard way (up to some phase factors).

### 8.3.3 Coherent states of \( H_0 \).

The intrinsic nonlinear and linear CS become now:

\[
|z, \alpha\rangle_0 = \left[ {}_0F_1(2\nu + 1; 2|z|^2) \right]^{\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha \frac{2}{m}(m+2\nu)} \sqrt{\frac{2m}{m!(2\nu + 1)_m}} z^m |\psi_m\rangle,
\]

(126)

\[
|z, \alpha\rangle_{0,c} = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} e^{-i\alpha \frac{2}{m}(m+2\nu)} \frac{z^m}{\sqrt{m!}} |\psi_m\rangle.
\]

(127)

The set of intrinsic nonlinear CS (126) is complete since the moment problem (29) with

\[
\rho_m = \frac{m!(2\nu + 1)_m}{2m}
\]

(128)

can be simply solved, with a positive definite function \( \rho(y) \) given by:

\[
\rho(y) = \frac{2^{\nu+2}y^{\nu}}{\Gamma(2\nu + 1)} K_{2\nu}(2\sqrt{2y}).
\]

(129)

Hence, the invariant measure (28) becomes:

\[
d\mu(z) = \frac{2^{\nu+2}|z|^{2\nu}}{\pi \Gamma(2\nu + 1)} {}_0F_1(2\nu + 1; 2|z|^2) K_{2\nu}(2\sqrt{2|z|}) d^2z.
\]

(130)

The reproducing kernel (31) reads:

\[
\langle z, \alpha | z', \alpha' \rangle_0 = \left[ {}_0F_1(2\nu + 1; 2|z|^2) {}_0F_1(2\nu + 1; 2|z'|^2) \right]^{-\frac{1}{2}} {}_0F_1(2\nu + 1; 2|\bar{z}z'|).
\]

(131)

For the linear CS (127) of \( H_0 \) all formulae of section 3.2 become the same, so we skipped them, as we did for the infinite well potential (102).

### 8.3.4 The SUSY partners \( H_k \).

For generating the \( k \)-th order SUSY partners of the Pöschl-Teller potential (119), we use transformations involving just seed solutions associated to non-physical factorization
energies $e_i, i = 1, \ldots, k,$ of $H_0,$ $q$ of them becoming physical levels of $H_k.$ The general mathematical eigenfunction $u(x)$ of $H_0$ for arbitrary $\epsilon$ is given by:

$$u(x) = \cos^\nu(x) \left[ 2 F_1 \left( \frac{\nu}{2} - \sqrt{\frac{\epsilon}{2}}, \frac{\nu}{2} + \sqrt{\frac{\epsilon}{2}}; \sin^2(x) \right) + \mu \sin(x) 2 F_1 \left( \frac{\nu}{2} + \sqrt{\frac{\epsilon}{2}}, \frac{\nu}{2} - \sqrt{\frac{\epsilon}{2}} + \frac{3}{2}; \sin^2(x) \right) \right].$$

(132)

This expression supplies any seed solution involved in the Wronskian of the transformation, which leads to the potential $V_k(x)$ as well as the eigenstates of $H_k.$

### 8.3.5 Algebraic structures of $H_k$.

The annihilation and creation operators for the natural and linear algebras of $H_k$ are written in Eqs. (71,75) with:

$$r^N_{\xi}(n) = 2^{k-1} \prod_{i=1}^{k} \sqrt{[(n+\nu-1)^2 - 2\epsilon_i]((n+\nu)^2 - 2\epsilon_i)}, \quad r^L_{\xi}(n) = \sqrt{\frac{2}{n+2\nu}}.$$  

(133)

The intrinsic operators are given in Eq. (62) with $r^I_{\xi}(n)$ given by (123).

### 8.3.6 Coherent states of $H_k$.

The coefficients $\tilde{\rho}_m$ of (79,80) required to find the natural nonlinear CS of $H_k$ are now:

$$\tilde{\rho}_m = \frac{m!(2\nu+1)_m}{2(2k+1)!} \prod_{i=1}^{k} (\nu - \sqrt{2\epsilon_i})_m (\nu - \sqrt{2\epsilon_i} + 1)_m (\nu + \sqrt{2\epsilon_i})_m (\nu + \sqrt{2\epsilon_i} + 1)_m,$$  

(134)

where $m \geq 0.$ Therefore:

$$|z, \alpha\rangle_{k_N} = \frac{1}{\sqrt{\text{F}_{4k+1}(2\nu+1, \ldots, \nu-\sqrt{2\epsilon_1}, \nu+\sqrt{2\epsilon_1}, \nu+\sqrt{2\epsilon_1}+1, \ldots, 2^{2k+1})}} \sum_{m=0}^{\infty} \frac{e^{-\frac{\alpha m}{2\nu}} \sqrt{2m(2k+1)}}{m! \sqrt{\text{F}_{4k+1}(2\nu+1, \ldots, \nu-\sqrt{2\epsilon_1}, \nu+\sqrt{2\epsilon_1}, \nu+\sqrt{2\epsilon_1}+1, \ldots, 2^{2k+1})}} \langle 0 | \theta_m \rangle_{k_N}.$$  

(135)

The moment problem (83) with the $\tilde{\rho}_m$ of (134) can be worked out once $\epsilon_1, \ldots, \epsilon_k$ are specified. However, the degeneracy of the eigenvalue $z = 0$ of $a_{k_N}$ is $q + 1.$

The intrinsic nonlinear and linear CS of $H_k$ are obtained from the corresponding ones of $H_0$ (see (126,127)) by the replacement $|\psi_m\rangle \rightarrow |\theta_m\rangle.$

As an illustration, a first-order SUSY transformation which ‘creates’ a new level at $\epsilon$ for $H_1$ is taken (for $k = q = 1, p = 0$). The ‘Wronskian’ is directly the solution $u(x)$ of (132); with this input for $\mu = 1.9, \epsilon = 3/2 < E_0 = 9/2$ we have drawn in Fig. 3 the SUSY partner potential (black curve) of the Pöschl-Teller potential with $\nu = 3$ (gray curve).
Figure 3: First-order SUSY partner potential $V_1(x)$ (black curve) of the Pöschl-Teller potential with $\nu = 3$ (gray curve) obtained by using as seed the $u(x)$ of (132) with $\mu = 1.9$, $\epsilon = 3/2 < E_0 = 9/2$. The new potential has an additional level at $\epsilon$.

9 Conclusions

In this paper we have derived coherent states for Hamiltonians $H_k$ attained from a given initial one through the higher-order SUSY QM. We have shown here, and previously for the harmonic oscillator [29, 34], that it is important to determine the algebraic structures ruling those potentials. It turns out that the intrinsic and linear algebras of the initial Hamiltonian are inherited by its corresponding SUSY partners in the subspace associated to the isospectral part of the spectrum. Moreover, we have discussed an interesting additional algebra of $H_k$ (the so-called natural) generalizing the one which was first introduced for the SUSY partners of the harmonic oscillator [29, 34]. We have shown as well that the natural and intrinsic algebras are deformations from each other, and our analysis shows that the natural is more involved than the intrinsic one. On the other hand, the linear algebra we have studied is a deformation simplifying at maximum the intrinsic structure of our systems. It is worth to notice that, up to this moment, the last procedure has been elaborated at a purely algebraic level, and it has been implemented to map somehow the original system into the harmonic oscillator. This suggests a class of problems which could be addressed in the future, in particular, it would be important to analyze the consequences of this linearization at a differential level. This is a quite interesting problem which, as far as we know, is open.

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