The purity phenomenon for symmetric separated set-systems

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Abstract

Let \( n \) be a positive integer. A collection \( S \) of subsets of \([n] = \{1, \ldots, n\}\) is called symmetric if \( X \in S \) implies \( X^* \in S \), where \( X^* := \{i \in [n]: n-i+1 \notin X\} \). As the main results of this paper, one shows that in each of the three types of separation relations: strong, weak and chord ones, the following “purity phenomenon” takes place: all inclusion-wise maximal symmetric separated collections in \( 2^{[n]} \) have the same cardinality. These give “symmetric versions” of well-known results on the purity of usual strongly, weakly and chord separated collections of subsets of \([n]\), and in the case of weak separation, this extends a result due to Karpman on the purity of symmetric weakly separated collections in \( \binom{[n]}{n/2} \) for \( n \) even.

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1 Introduction

Let \( n \) be a positive integer and let \([n]\) denote the set \{1, 2, \ldots, n\}. In this paper, we deal with three known types of relations on subsets of \([n]\), called strong, weak and chord separation ones. In the definitions below, for subsets \( A, B \subseteq [n] \), we write \( A < B \) if the maximal element \( \max(A) \) of \( A \) is smaller than the minimal element \( \min(B) \) of \( B \) (letting \( \max(\emptyset) := -\infty \) and \( \min(\emptyset) := \infty \)). When \( B - A \neq \emptyset \), we say that \( A \) surrounds \( B \) if \( \min(A - B) < \min(B - A) \) and \( \max(A - B) > \max(B - A) \) (where \( A - B \) denotes the set difference \( \{i: A \ni i \notin B\} \)).

Definitions. Sets \( A, B \subseteq [n] \) are called strongly separated (from each other) if there are no three elements \( i < j < k \) of \([n]\) such that one of \( A - B \) and \( B - A \) contains \( i, k \), and

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the other contains \(j\) (equivalently, either \(A - B < B - A\) or \(B - A < A - B\) or \(A = B\)). Sets \(A, B \subseteq [n]\) are called chord separated if there are no four elements \(i < j < k < \ell\) of \([n]\) such that one of \(A - B\) and \(B - A\) contains \(i, k\), and the other contains \(j, \ell\). Chord separated sets \(A, B \subseteq [n]\) are called weakly separated if the following additional condition holds: if \(A\) surrounds \(B\) then \(|A| \leq |B|\), and if \(B\) surrounds \(A\) then \(|B| \leq |A|\) (where \(|A'|\) is the number of elements in \(A'\)). Accordingly, a collection \(\mathcal{A} \subseteq 2^{[n]}\) of subsets of \([n]\) is called strongly (weakly, chord) separated if any two members of \(\mathcal{A}\) are strongly (resp. weakly, chord) separated.

The notions of strong and weak separations were introduced by Leclerc and Zelevinsky [10], and the notion of chord separation by Galashin [6]. The first two notions appeared in [10] in connection with the problem of characterizing quasi-commuting flag minors of a quantum matrix. (In particular, one shows there that in the quantized coordinate ring \(O_q(\mathcal{M}_{m,n}(\mathbb{K}))\) of \(m \times n\) matrices over a field \(\mathbb{K}\), where \(q \in \mathbb{K}^*\), two flag minors \([I]\) and \([J]\) quasi-commute, i.e., satisfy the equality \([J][I] = q^c[I][J]\) for some integer \(c\), if and only if their column sets \(I, J \subseteq [n]\) are weakly separated.) For a discussion on this and wider relations between the weak/strong separation and quantum minors, see also [1, Sect. 8]).

For brevity we will refer to strongly, weakly, and chord separated collections as \(s\)-, \(w\)-, and \(c\)-collections, respectively. The sets of such collections in \(2^{[n]}\) are denoted by \(S_n\), \(W_n\), and \(C_n\), respectively. As is shown in [10],

\[
(1.1) \text{ the maximal possible sizes of strongly and weakly separated collections in } 2^{[n]} \text{ are the same and equal to } s_n := \binom{n}{2} + \binom{n}{1} + \binom{n}{0} \left(= \frac{1}{2} n(n+1)+1\right).
\]

When dealing with one or another set (class) \(C\) of collections in \(2^{[n]}\), one says that \(C\) is pure if any maximal by inclusion collection in it is maximal by size (viz. number of members). Leclerc and Zelevinsky showed in [10] that the set \(S_n\) is pure and conjectured that \(W_n\) is pure as well. This was affirmatively answered in [2], by proving that

\[
(1.2) \text{ any w-collection in } 2^{[n]} \text{ can be extended to a w-collection of size } s_n.
\]

(One application of the purity of \(W_n\) mentioned in [10] concerns the dual canonical basis in \(O_q(\mathcal{M}_{m,n}(\mathbb{K}))\) containing all quasi-commuting monomials.) For other interesting classes of w-collections with the purity behavior, see [11, 12, 3].) The corresponding purity result for chord separation was obtained by Galashin [6]:

\[
(1.3) \text{ any c-collection in } 2^{[n]} \text{ can be extended to a c-collection of size } c_n := \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + \binom{n}{0}.
\]

Recently Karpman [9] revealed the purity for a special class of symmetric w-collections, and we now outline this result.

It will be convenient to us to interpret the elements of the ground set \([n]\) as colors. Consider the following relations on \([n]\) and \(2^{[n]}\):

\[
(1.4) \text{ (i) for } i \in [n], \text{ define } i^c := n + 1 - i \text{ (the “complementary color” to } i);\]

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(ii) for $A \subseteq [n]$, define $\overline{A} := [n] - A$ (the “complementary set” to $A$);

(iii) for $A \subseteq [n]$, define $A^* := \{i^o : i \in \overline{A}\}$.

Clearly (i) and (ii) are involutions: $(i^o)^o = i$ and $\overline{A} = A$. And (iii) is viewed as the composition of these two involutions (which commute); so it is an involution as well: $(A^*)^* = A$. Involution (iii), which is of most interest to us in this paper, was introduced by Karpman [9] in the special case when $n$ is even and $|A| = n/2$. (In that case, the author treats the so-called symmetric plabic graphs, which are closely related to the corresponding Lagrangian Grassmannian, the space of maximal isotropic subspaces with respect to a symplectic form.)

For convenience, we will refer to involution (iii) as the $K$-involution and use the following definition for corresponding set-systems.

**Definition.** A collection $S \subseteq 2^{[n]}$ is called symmetric if it is closed under the $K$-involution, i.e., $A \in S$ implies $A^* \in S$.

Using a technique of plabic tilings and relying on the purity of the set of $w$-collections in a “discrete Grassmannian” $\binom{[n]}{m} = \{A \subseteq [n] : |A| = m\}$ with $m \in [n]$ (cf. [11]), Karpman showed the following

**Theorem 1.1** ([9]) For $n$ even, all inclusion-wise maximal symmetric $w$-collections in $\binom{[n]}{n/2}$ have the same cardinality, which is equal to $n^2/4 + 1$.

(This coincides with the maximum cardinality when the symmetry condition is discarded.) We give the following generalization of that result.

**Theorem 1.2** For any $n \in \mathbb{Z}_{>0}$, all inclusion-wise maximal symmetric $w$-collections in $2^{[n]}$ have the same cardinality. When $n$ is even, it is equal to $s_n$. When $n$ is odd, it is equal to $s_n - (n - 1)/2$.

**Remark 1.** Theorem 1.2 implies Theorem 1.1. Moreover, one can show a sharper result, as follows. For an even $n$ and an integer $k$ such that $0 \leq k < n/2$, let $\Lambda_{n,k}$ denote the union of sets $\binom{[n]}{i}$ over $n/2 - k \leq i \leq n/2 + k$. We assert that: (i) all inclusion-wise maximal symmetric $w$-collections $S$ in $\Lambda_{n,k}$ have the same size (which gives Theorem 1.1 when $k = 0$). To see this, given a symmetric $w$-collection $S$ in $\Lambda_{n,k}$, extend it to the collection $W \subseteq 2^{[n]}$ by adding the set $I$ of all intervals $I \subseteq [n]$ of size $\geq n/2 + k$ and the set $J$ of all co-intervals $J \subseteq [n]$ of size $\leq n/2 - k$ (including the empty co-interval $\emptyset$). Hereinafter, an **interval** in $[n]$ is meant to be a set of the form $\{a, a+1, \ldots, b\} \subseteq [n]$, denoted as $[a..b]$ (in particular, $[n] = [1..n]$), and a **co-interval** is the complement $\overline{I} = [n] - I$ of an interval $I$. One can check that: (i) each interval $I$ is weakly separated from any set $A \subseteq [n]$ with $|A| \leq |I|$; symmetrically, each co-interval $J$ is weakly separated from any $B \subseteq [n]$ with $|B| \geq |J|$; (ii) for any $I \in I$, $I^*$ is a co-interval in $J$, a vice versa; (iii) any set $A \in 2^{[n]} - \overline{I}$ with $|A| > n/2 + k$ is not weakly separated from some $I \in \overline{I}$; symmetrically, any $B \in 2^{[n]} - J$ with $|B| < n/2 - k$ is not weakly separated from some $J \in \overline{J}$. These properties imply that $S$ as above is inclusion-wise maximal in $\Lambda_{n,k}$ if and only if $W := S \cup \overline{I} \cup J$ is inclusion-wise maximal.

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in $2^{[n]}$, whence assertion (*) follows from Theorem 1.2. Also, given $n, k$, we can easily express $|W|$ via $|S|$, and back.

The proof of Theorem 1.2 relies on (1.2) and is based on a geometric approach: it attracts a machinery of combined tilings, or combies for short, which are certain planar polyhedral complexes on two-dimensional zonogons introduced and studied in [3]. It turns out that when $n$ is even, there is a natural bijection between the maximal symmetric w-collections in $2^{[n]}$ and the so-called “symmetric combies” on the zonogon $Z(n, 2)$ (this is analogous to the existence of a bijection between usual maximal w-collections and combies, see [3, Theorems 3.4,3.5]).

As a by-product of our method of proof of Theorem 1.2 we obtain a similar purity result for the strong separation:

(1.5) all inclusion-wise maximal symmetric s-collections in $2^{[n]}$ have the same cardinality; it is equal to $s_n$ when $n$ is even, and $s_n - (n - 1)/2$ when $n$ is odd.

The next group of results of this paper concerns symmetric chord separated collections. Our main theorem in this direction is as follows.

**Theorem 1.3** For any $n \in \mathbb{Z}_{>0}$, all inclusion-wise maximal symmetric c-collections in $2^{[n]}$ have the same cardinality $c_n$.

(Note that this can be regarded as a generalization of Theorem 1.1 as well since within any domain of the form $([n], k)$ the notions of weak and chord separations coincide.)

An important ingredient of the proof of this theorem is the geometric characterization of maximal chord separated collections in $2^{[n]}$ in terms of fine zonotopal tilings, or cubillages, of 3-dimensional cyclic zonotopes $Z(n, 3)$, due to Galashin [6] (the term “cubillage” that we prefer to use in this paper appeared in [8]). We establish a symmetric analog of that nice property (valid for both even and odd cases of $n$): any maximal symmetric c-collection in $2^{[n]}$ can be expressed by the vertex set of a symmetric cubillage on $Z(n, 3)$.

This paper is organized as follows. Section 2 contains basic definitions and reviews some known facts. In particular, it explains the notions of combined tilings, or combies, and fine zonotopal tilings, or cubillages (in the 3-dimensional case), and recalls basic results on them needed to us. Section 3 deals with the even color case of symmetric weakly separated collections and proves Theorem 1.2 for $n$ even. The odd color case of this theorem is studied in Section 4. Section 5 is devoted to the even color case of symmetric chord separated collections, giving the proof of Theorem 1.3 for $n$ even. The odd case of this theorem is shown in the concluding Section 6. This section finishes with a slightly sharper version of Theorem 1.3 (in Remark 6). Also we add two more results in the ends of Sections 5 and 6 (Theorems 5.1 and 6.2 which are devoted to geometric constructions related to embeddings of maximal symmetric w-collections in maximal c-collections.)

It should be noted that the above purity results do not remain true for “higher” symmetric separation. Recall that sets $A, B \subseteq [n]$ are called (strongly) $k$-separated if
there are no \( k + 2 \) elements \( i_1 < i_2 < \cdots < i_{k+2} \) of \([n]\) such that the elements with odd indexes belong to one, while those with even indexes to the other set among \( A - B \) and \( B - A \). In particular, chord separated sets are just 2-separated ones. As is shown in \cite{7}, when \( k \geq 3 \), a maximal by inclusion \( k \)-separated collection in \( 2^{[n]} \) need not be maximal by size. (In fact, a counterexample to the purity with \( k = 3 \) given there can be adjusted to the symmetric 3-separation as well.)

Surprisingly, the maximal by size symmetric \( k \)-separated collections in \( 2^{[n]} \) possess nice structural and geometric properties; they are systematically studied in \cite{5} in the context of higher Bruhat orders of types B and C (where some open questions and conjectures are raised as well). One important property among those is that such collections for \( n,k \) even can be connected by use of symmetric local mutations (or “flips”) yielding a poset structure with one minimal and one maximal elements. More about symmetric flips will appear in a forthcoming paper.

2 Preliminaries

In this section we give additional definitions and notation and review some facts about combined tilings and cubillages needed for the proofs of Theorems \cite{12} and \cite{13}.

- For an edge \( e \) of a directed graph \( G \) without parallel edges, we write \( e = (u, v) \) if \( e \) connects vertices \( u \) and \( v \) and is directed (or “going”) from \( u \) to \( v \). A path in \( G \) is a sequence \( P = (v_0, e_1, v_1, \ldots, e_k, v_k) \) in which each \( e_i \) is an edge connecting vertices \( v_{i-1} \) and \( v_i \). It is called a directed path if each edge \( e_i \) is directed from \( v_{i-1} \) to \( v_i \). When it is not confusing, we may write \( P = v_0v_1 \ldots v_k \) (using notation via vertices).

- Let \( n \) be a positive integer. Define \( m := \lfloor n/2 \rfloor \); then \( n = 2m \) if \( n \) is even, and \( n = 2m + 1 \) if \( n \) is odd. Instead of colors \( 1, 2, \ldots, n \) (forming \([n]\)), it will be often more convenient to deal with the set of “symmetric colors” \(-i, i \) for \( i = 1, \ldots, m \), to which we also add color 0 when \( n \) is odd. This gives the symmetrized color sets

\[
\{−m, \ldots, −1, 0, 1, \ldots, m\} \text{ denoted as } [-m.m] \text{ when } n \text{ is odd, and } \\
\{−m, \ldots, −1, 1, \ldots, m\} \text{ denoted as } [-m.m^-] \text{ when } n \text{ is even.}
\]

So \( i^o = −i \) for each color \( i \), and the only self-symmetric color is 0 (when \( n \) is odd).

- For \( A \subseteq [n] \) and \( p = 0, 1, 2 \), define \( \Pi_p(A) \) to be the set of symmetric color pairs \( \{i, i^o\} \) in \([n]\) such that \( |A \cap \{i, i^o\}| = p \). We say that the pairs in \( \Pi_0(A) \), \( \Pi_1(A) \), and \( \Pi_2(A) \) are, respectively, poor, ordinary, and full for \( A \). (Note that if \( n \) is odd and \( i = m + 1 \), then the “middle” pair \( \{i, i^o = i\} \) is regarded as either poor or full in \( A \).) In these terms, we observe a useful relationship between symmetric sets \( A \) and \( A^* \):

\[
\Pi_0(A) = \Pi_2(A^*), \Pi_2(A) = \Pi_0(A^*), \text{ and } \Pi_1(A) = \Pi_1(A^*); \text{ moreover, the ordinary pairs are stable, in the sense that if } i \in A \not\in i^o \text{ then } i \in A^* \not\in i^o.
\]

Indeed, \( i \in A \not\in i^o \) implies \( i \notin \overline{A} \supseteq i^o \), whence \( i \in A^* \not\in i^o \). And \( i, i^o \notin A \) implies \( i, i^o \in \overline{A}, \) whence \( i, i^o \in A^* \).

When dealing with colors in the symmetrized form as above, the sets \( \Pi_p(A) \) are defined accordingly.
For a symmetric collection $S \subseteq 2^n$ and $h = 0, 1, \ldots, n$, define the $h$-th level of $S$ as

$$S_h := \{ A \in S : |A| = h \}.$$ 

Then $S_h$ consists of the sets $A \in S$ with $\Pi_1(A) + \frac{1}{2}\Pi_2(A) = h$, and (2.2) implies

$$S_h = (S_{n-h})^*,$$

where we extend the operator $*$ to collections in $2^n$ in a natural way.

The next two subsections review the constructions of combined tilings and cubillages, which are the key objects in our proofs of Theorems 1.2 and 1.3 respectively.

### 2.1 Zonogon and combies.

Let $\Xi$ be a set of $n$ vectors $\xi_i = (x_i, y_i) \in \mathbb{R}^2$ such that

(2.3) $x_1 < \cdots < x_n$ and $y_i = 1 - \delta_i$, $i = 1, \ldots, n$,

where each $\delta_i$ is a sufficiently small positive real. In addition, we assume that

(2.4) (i) $\Xi$ satisfies the strict concavity condition: for any $i < j < k$, there exist $\lambda, \lambda' \in \mathbb{R}_{>0}$ such that $\lambda + \lambda' > 1$ and $\xi_j = \lambda \xi_i + \lambda' \xi_k$; and

(ii) the vectors in $\Xi$ are $\mathbb{Z}_2$-independent, i.e., all 0,1-combinations of these vectors are different.

An example is illustrated in the picture; here $n = 5$, $(x_1, \ldots, x_5) = (-2, -1, 0, 1, 2)$ and $y_i = 1 - x_i^2/12$.

![Example of a zonogon](image)

The zonogon generated by $\Xi$ is the $2n$-gon being the Minkowski sum of segments $[0, \xi_i]$, $i = 1, \ldots, n$, i.e., the set

$$Z = Z(\Xi) := \{ \lambda_1 \xi_1 + \cdots + \lambda_n \xi_n : \lambda_i \in \mathbb{R}, \ 0 \leq \lambda_i \leq 1, \ i = 1, \ldots, n \}.$$ 

When the choice of $\Xi$ is not important to us (subject to (2.3),(2.4)), we may denote $Z$ as $Z(n, 2)$. Each subset $X \subseteq [n]$ is identified with the point $\sum_{i \in X} \xi_i$ in $Z$ (due to (2.4)(ii), different subsets are identified with different points).

Besides $\xi_1, \ldots, \xi_n$, we use the vectors $\epsilon_{ij} := \xi_j - \xi_i$ for $1 \leq i < j \leq n$.

A combined tiling, or a combi for short, is a subdivision $K$ of $Z = Z(\Xi)$ into convex polygons specified below and called tiles. Any two intersecting tiles share a common vertex or edge, and each edge of the boundary of $Z$ belongs to exactly one tile.
We associate to \( K \) the planar graph \((V_K, E_K)\) whose vertex set \( V_K \) and edge set \( E_K \) are formed by the vertices and edges occurring in tiles. Each vertex is (a point identified with) a subset of \([n]\). And each edge is a line segment viewed as a parallel transfer of either \( \xi_i \) or \( \epsilon_{ij} \) for some \( i < j \). In the former case, it is called an edge of type or color \( i \), or an \( i \)-edge, and in the latter case, an edge of type \( ij \), or an \( ij \)-edge. An \( i \)-edge (\( ij \)-edge) is directed according to the direction of \( \xi_i \) (resp. \( \epsilon_{ij} \)), and \((V_K, E_K)\) is the corresponding directed graph. In particular, the left boundary of \( K \) (and of \( Z \)) is the directed path \( v_0v_1 \ldots v_n \) in which each vertex \( v_i \) represents the interval \([i]\) (and the edge from \( v_{i-1} \) to \( v_i \) has color \( i \)). And the right boundary is the directed path \( v'_0v'_1 \ldots v'_n \) in which \( v'_i \) represents the interval \([n+1-i..n]\).

In what follows, for disjoint subsets \( A \) and \( \{a, \ldots, b\} \) of \([n]\), we will use the abbreviated notation \( Aa \ldots b \) for \( A \cup \{a, \ldots, b\} \), and write \( A - c \) for \( A - \{c\} \) when \( c \in A \).

There are three sorts of tiles in a combi \( K \): \( \Delta \)-tiles, \( \nabla \)-tiles, and lenses.

I. A \( \Delta \)-tile (\( \nabla \)-tile) is a triangle with vertices \( A, B, C \subseteq [n] \) and edges \((B, A), (C, A), (B, C)\) (resp. \((A, C), (A, B), (B, C)\)) of types \( j, k \) and \( ik \), respectively, where \( i < j \). We denote this tile as \( \Delta(A|BC) \) (resp. \( \nabla(A|BC) \)). See the left and middle fragments of the picture.

II. In a lens \( \lambda \), the boundary is formed by two directed paths \( U_\lambda \) and \( L_\lambda \), with at least two edges in each, having the same beginning vertex \( \ell_\lambda \) and the same end vertex \( r_\lambda \); see the right fragment of the above picture. The upper boundary \( U_\lambda = (v_0, e_1, v_1, \ldots, e_p, v_p) \) is such that \( v_0 = \ell_\lambda, v_p = r_\lambda \), and \( v_k = X_{i_k} \) for \( k = 0, \ldots, p \), where \( p \geq 2, X \subseteq [n] \) and \( i_0 < i_1 < \cdots < i_p \) (so \( k \)-th edge \( e_k \) is of type \( i_{k-1}i_k \)). And the lower boundary \( L_\lambda = (u_0, e'_1, u_1, \ldots, e'_q, u_q) \) is such that \( u_0 = \ell_\lambda, u_q = r_\lambda \), and \( u_m = Y - j_m \) for \( m = 0, \ldots, q \), where \( q \geq 2, Y \subseteq [n] \) and \( j_0 > j_1 > \cdots > j_q \) (so \( m \)-th edge \( e'_m \) is of type \( j_{m-1}j_m \)). Then \( Y = Xi_{j_0}i_{j_q} = Xi_{j_0}i_{j_p} \), implying \( i_0 = j_q \) and \( i_p = j_0 \). Note that \( X \) as well as \( Y \) need not be a vertex in \( K \). Due to the concavity condition (2.4)(i), \( \lambda \) is a convex polygon of which vertices are exactly the vertices of \( U_\lambda \cup L_\lambda \).

Besides, we will deal with two derivatives of combies (cf. [4 Sect. 6.3]).

A. A quasi-combi \( K \) differs from a combi by the condition that in each lens \( \lambda \), either the upper boundary \( U_\lambda \) or the lower boundary \( L_\lambda \) (not both) can consist of only one edge; we refer to \( \lambda \) as a lower semi-lens in the former case, and as an upper semi-lens in the latter case. If, in addition, no two upper semi-lenses can share an edge, and similarly for the lower semi-lenses, then we say that a quasi-combi \( K \) is fine. Typically, a fine quasi-combi is produced from a combi by subdividing each lens \( \lambda \) of the latter into two semi-lenses of different types (lower and upper ones) along the segment \([\ell_\lambda, r_\lambda]\). See the left fragment of the picture.
B. The second derivative is of most use in this paper. It was introduced in [4] under the name of a fully triangulated quasi-combi, that we will abbreviate as an ftq-combi. In an ftq-combi $K$, all lenses are semi-lenses and, moreover, they are triangles. So an upper semi-lens is a triangle $U = U(ABC)$ formed by three vertices $A, B, C$ and three directed edges of types $ij, jk, ik$ such that $i < j < k$. The vertices are expressed as $A = X_i, B = X_j$ and $C = X_k$ for some $X \subseteq [n]$, called the root of $U$ (which is not necessarily a vertex of $K$). And a lower semi-lens is a triangle $L = L(A'B'C')$ formed by vertices $A', B', C'$ and directed edges of types $j'k', i'j', i'k'$ such that $i' < j' < k'$. The vertices are viewed as $A' = Y - k', B' = Y - j'$ and $C' = Y - i'$ for some $Y \subseteq [n]$, called the root of $L$. See the middle and right fragments of the above picture. Typically, an ftq-combi is produced from a combi by subdividing each lens into one lower and one upper semi-lenses (forming a fine quasi-combi as in A) and then subdividing the former into lower triangles, and the latter into upper ones. Conversely, starting from an ftq-combi, if we choose, step by step, a pair of semi-lenses that share an edge and have the same type and replace them by their union, then we eventually obtain a fine quasi-combi (which preserves the set of vertices and does not depend on the choice of pairs in the process).

The picture below illustrates fragments of a combi (left) and an ftq-combi (right); here lenses ($\lambda$ and $\lambda'$) and triangular semi-lenses are drawn bold.

Remark 2. In the definition of a combi given in [3], the generators $\xi_i$ are assumed to have equal euclidean lengths. However, taking generators subject to (2.3) does not affect, in essence, the structure of combies, as well as results on them, and we may vary generators, with a due care, when needed. To simplify visualizations, it is convenient to think of edges of type $i$ as “almost vertical”, while of those of type $ij$ as “almost horizontal” (since the values $\delta_i$ in (2.3) are small). Note that any rhombus tiling turns into a combi without lenses in a natural way: each rhombus is subdivided into two “semi-rhombi” $\Delta$ and $\nabla$ by drawing the “almost horizontal” diagonal in it. Note that from axioms (2.3),(2.4)(i) it follows that
(2.5) all vectors $\xi_1, \ldots, \xi_n$ and $\epsilon_{ij}, 1 \leq i < j \leq n$, are different.

In particular, this implies that if some $\Delta$-tile and $\nabla$-tile share an “almost horizontal” edge (i.e. they are of the form $\Delta(A|BC)$ and $\nabla(A'|B'C')$ with $BC = B'C'$) then their union is a parallelogram. As a consequence, any ftq-combi without semi-lenses is equivalent to a rhombus tiling (and vice versa).

The central result on combies shown in [3] (which in turn relies on the purity of $W_n$ shown in [2]) is that there is a one-to-one correspondence between the set $K$ on $Z(n, 2)$ and the set $W_n$ of maximal w-collections $W$ in $2^{[n]}$; it is given by $K \mapsto V_K =: W$. As a consequence, (2.6) for any ftq-combi $K$ on $Z(n, 2)$, the set $V_K$ of vertices (regarded as subsets of $[n]$) forms a maximal w-collection in $2^{[n]}$, and conversely, any w-collection in $2^{[n]}$ is representable by the vertex set $V_K$ of some ftq-combi $K$.

Next, in order to handle symmetric ws-collections, it is convenient to assume that the set $\Xi$ of generating vectors $\xi = (x_i, y_i)$ is symmetric, in the sense that:

(2.7) $x_i = -x_i^\ast$ and $y_i = y_i^\ast$ for each $i \in [n]$

cf. (2.3). Note that in this case we may assume that conditions (2.4)(i),(ii) continue to hold. (To provide this, we first assign $Z_2$-independent numbers $x_i$ for $i = 1, \ldots, m = [n/2]$ so that $x_1 < \cdots < x_m < 0$, and accordingly define $x_m+1, \ldots, x_n$ by symmetry (where $x_m+1 = 0$ if $n$ is odd). Then assign symmetric $y_1, \ldots, y_n$ (with $y_i = y_i^\ast$) so as to satisfy the concavity condition (2.4)(i). Then slightly perturbing the values $y_i$, if needed, we ensure that all 0,1,2-combinations of the numbers $y_1, \ldots, y_m$ are different. One can see that the resulting vectors $\xi_1, \ldots, \xi_n$ are $Z_2$-independent, yielding (2.4)(ii).)

When $n$ is even, (2.7) implies that the zonogon $Z := Z(\Xi)$ admits the reflection with respect to the horizontal line

$$M := \{(x, y) \in Z : y = y_1 + \cdots + y_{n/2}\},$$

(2.8) called the middle line of $Z$. Moreover, we observe that

(2.9) for any $A \subseteq [n]$, the sets $A$ and $A^\ast$ are symmetric w.r.t. $M$, or $M$-symmetric for short, which means that their corresponding points $(x_A, y_A)$ and $(x_{A^\ast}, y_{A^\ast})$ in $Z$ satisfy $x_A = x_{A^\ast}$ and $y_A - y_M = y_M^M - y_{A^\ast}$, where $y_M := y_1 + \cdots + y_{n/2}$.

(In particular, $\emptyset$ is $M$-symmetric to $[n]$, and $\{i, i^\ast\}$ is $M$-symmetric to $[n] - \{i, i^\ast\}$ for each $i \in [n]$.) Clearly if $A \subseteq [n]$ lies on $M$, then $|A| = n/2$, implying that the amounts of poor and full pairs in $A$ are equal: $|\Pi_0(A)| = |\Pi_2(A)|$. Moreover, in view of (2.2) and (2.4)(ii), for any $A \subseteq [n]$, the points $(x_A, y_A)$ and $(x_{A^\ast}, y_{A^\ast})$ coincide if and only if $\Pi_0(A) = \Pi_2(A) = \emptyset$. This gives the useful property that

(2.10) all sets $A \subseteq [n]$ with $\Pi_0(A) = \Pi_2(A) = \emptyset$ and only these are contained in $M$.
In other words, $M$ contains merely self-symmetric sets $A = A^*$ and all these.

Also the middle line $M$ enables us to define an important class of ftq-combies (extending the notion of $M$-symmetry to subsets of points in $Z$ in a natural way).

**Definition.** Let $n$ be even. An ftq-combi $K$ on $Z$ is called symmetric if for any tile of $K$, its $M$-symmetric tile belongs to $K$ as well. In particular, $V_K$ is symmetric.

We shall see in Sect. 3 that such ftq-combi do exist, and moreover, they just give rise to all maximal symmetric w-collections in $2^n$. On the other hand, no “symmetric ftq-combi” can be devised when $n$ is odd, as we explain in Sect. 4.

### 2.2 Cubillages on cyclic zonotopes of dimension 3.

Cubillages arising when we deal with chord separated collections live within a 3-dimensional $n$-colored cyclic zonotope. To define the latter, consider a set $\Theta$ of $n$ vectors $\theta_i = (t_i, 1, \phi(t_i)) \in \mathbb{R}^3$, $i = 1, \ldots, n$, such that

$$t_1 < t_2 < \cdots < t_n \text{ and } \phi(t) \text{ is a strictly convex function; for example, } \phi(t) = t^2.$$  

(2.11)

(Note that for our purposes, for a vector $v = (a, b, c) \in \mathbb{R}^3$, it is more convenient to interpret $b$ as the vertical coordinate (height), $a$ as the left-to-right coordinate, and $c$ as the depth of $v$. So all vectors in $\Theta$ have the unit height.) An example with $n = 5$ is illustrated in the picture (where $z_i = x_i^2$ and $x_i = -x_{6-i}$).

The zonotope $Z(\Theta)$ generated by $\Theta$ is the Minkowski sum of line segments $[0, \theta_i]$, $i = 1, \ldots, n$. Then a fine zonotopal tiling, or a cubillage, in terminology of [8], is (the polyhedral complex determined by) a subdivision $Q$ of $Z(\Theta)$ into 3-dimensional paralleloptopes such that: any two intersecting ones share a common face, and each face of the boundary of $Z(\Theta)$ is entirely contained in some of these paralleloptopes. For brevity, we refer to these paralleloptopes as cubes, and to $Q$ as a cubillage.

Note that the choice of one or another cyclic configuration $\Xi$ (subject to (2.11)) is not important to us in essence, and we usually write $Z(n, 3)$ rather than $Z(\Theta)$, referring to it as the (cyclic 3-dimensional) zonotope with $n$ colors.

Like the case of zonogons and combies, each vertex $v$ of a cubillage $Q$ (i.e., a vertex of some cube in it) is viewed as $\sum_{i \in X} \theta_i$ for some subset $X \subseteq [n]$, and we identify such $v$ and $X$. The set of vertices of $Q$ (as subsets of $[n]$) is called the spectrum of $Q$ and denoted as $V_Q$. One shows that $|V_Q|$ is equal to $c_n$ as in [13], and an important result due to Galashin establishes a relation of cubillages to chord separation.

**Theorem 2.1** ([6]) The correspondence $Q \mapsto V_Q$ gives a bijection between the set of cubillages $Q$ on $Z(n, 3)$ and the set $C_n$ of maximal $c$-collections in $2^n$. 

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For a closed subset $U$ of points in $Z = Z(n, 3)$, the front (rear) side of $U$, denoted as $U^f$ (resp. $U^r$), is defined to be the set of points $(a, b, c) \in U$ such that $c \leq c'$ (resp. $c \geq c'$) among all $(a', b', c') \in U$ with $(a', b') = (a, b)$, i.e., consisting of the points of $U$ with locally minimal (resp. maximal) depths. In particular, $Z^f$ ($Z^r$) denotes the front (rear) side of the entire zonotope $Z$; it is well-known that the vertices occurring in $Z^f$ ($Z^r$) are exactly the intervals (resp. co-intervals) in $[n]$.

Next, to handle symmetric $c$-collections, we will deal with a symmetric set $\Theta$ of generating vectors $\theta_i = (t_i, 1, \phi(t_i))$, which means that

$$t_i = -t_i^\circ \text{ and } \phi(t_i) = \phi(-t_i) \text{ for each } i \in [n],$$

and consider the corresponding symmetric zonotope $Z(\Theta)$. It turns out that, in contrast to the situation when symmetric (ftq)-combies exist only for $n$ even, symmetric cubillages on a symmetric cyclic zonotope do exist in both even and odd cases, as we shall see in Sects. 5 and 6.

### 3 Maximal symmetric w-collections: even case

In this section, we throughout assume that $n$ is even. Our goal is to prove Theorem 1.2 in this case. We consider a symmetric zonogon $Z = Z(\Xi) \simeq Z(n, 2)$, and an important role is played by the middle line $M$ in $Z$ (defined in (2.8)).

We know (cf. (2.10)) that all points $A \subset [n]$ lying on $M$ are self-symmetric, have size $n/2$, and admit only ordinary pairs. Consider two distinct points $A, B$ in $M$. Since $|A| = |B|$, the symmetric difference $A \triangle B (= (A - B) \cup (B - A))$ has size at least 2. This is strengthened as follows (this will be used in the next section):

(3.1) if $A, B$ are self-symmetric and $|A \triangle B| = 2$, then $A \triangle B = \{i, i^\circ\}$ for some $i \in [n]$. Indeed, if $A - B = \{i\}$, $B - A = \{i\}$ and $j \neq i^\circ$, then $j^\circ \in A - B$ (since each of $A, B$ contains exactly one elements of $\{j, j^\circ\}$). But then $|A \triangle B| \geq 3$, a contradiction.

Next we prove the theorem (with $n$ even) as follows. Let $C$ be a maximal by inclusion symmetric w-collection in $2^{[n]}$ and suppose, for a contradiction, that $|C| < s_n$. Extend $C$ to a maximal (non-symmetric) w-collection $W \subset 2^{[n]}$. Then $|W| = s_n$, and in view of (2.6), there exists an ftq-combi $K$ on $Z$ whose vertex set $V_K$ is exactly $W$.

Let $R = (R_0, R_1, \ldots, R_q)$ be the sequence of vertices of $K$ occurring in $M$ and ordered from left to right. Note that $R$ is nonempty, since it contains the vertex $[n/2]$ of the left boundary of $Z$, which is just $R_0$ (and the vertex $[(n/2 + 1) \ldots n]$ of the right boundary of $Z$, which is $R_q$). Consider two possible cases. (It should be noted that an idea of our analysis in items II and III below is borrowed from Karpman’s work [9].)

**Case 1**: Assume that the middle line $M$ of $Z$ is covered by edges of $K$. Then (by the planarity and the construction of ftq-combies) for each $p = 1, \ldots, q$, the pair $e_p = (R_{p-1}, R_p)$ forms an edge of $K$. Moreover, $M$ separates the tiles of $K$ into two subsets $T$ and $T'$, where the former (latter) consists of the tiles lying in the half of $Z$ below (resp. above) $M$. The intersection of these halves is just $M$, and they are
M-symmetric to each other. Now for each tile $\tau \in \mathcal{T}$, take the $M$-symmetric triangle $\tau^*$. Then the set $\mathcal{T}''$ of such tiles gives a subdivision of the half of $Z$ above $M$, and combining $\mathcal{T}$ and $\mathcal{T}''$ (which have the same set of edges within $M$), we obtain a symmetric ftq-combi $\tilde{K}$ on $Z$. Note also that if $A \in C$ is a vertex in $\mathcal{T}'$, then $A$ is a vertex of $\mathcal{T}''$ as well (since the symmetric set $A^*$ must be a vertex in $\mathcal{T}$). Thus, $V_{\tilde{K}}$ is a symmetric w-collection of size $s_n$ including $C$, contradicting the maximality of $C$.

Case 2: Now assume that for some $1 \leq p \leq q$, the segment $\sigma$ of $M$ between the points $R_{p-1}$ and $R_p$ is not an edge of $K$. Then there is a tile $\tau$ of $K$ with vertices $A, B, C$ such that one vertex, $A$ say, is $R_{p-1}$, and $\sigma$ meets the edge connecting $B$ and $C$ at an interior point. Let for definiteness the point $B$ lies above $M$, and $C$ below $M$. Our aim is to show that this is not the case.

A priori, $\tau$ can be one of the following shapes: $\Delta$-tile, $\nabla$-tile, upper semi-lens, or lower semi-lens (defined in Sect. 2.1). For reasons of symmetry, it suffices to consider the cases when $\tau$ is either a $\nabla$-tile or an upper semi-lens. In the former case $\tau$ is viewed as $\nabla(C|AB)$, while the latter case falls into two subcases depending on the location of the vertex $A$; namely, $\tau$ is either $U(ABC)$ or $U(CAB)$. So we have to consider three situations; they are illustrated in the picture (from left to right) and described in items I, II, III below. In order to analyze these situations, we use two auxiliary assertions.

(3.2) If $S \subset [n]$ is such that each of $\Pi_0(S)$ and $\Pi_2(S)$ consists of at most one pair, then $S$ and $S^*$ are weakly separated.

Indeed, we have $\Pi_2(S) = \{S - S^*\}$ and $\Pi_0(S) = \{S^* - S\}$ (cf. (2.2)); so the assertion is immediate when some of $\Pi_0(S)$ and $\Pi_2(S)$ is empty. And if $|\Pi_0(S)| = |\Pi_2(S)| = 1$, then $|S - S^*| = |S^* - S| = 2$, whence $S$ and $S^*$ have the same size and one of them surrounds the other, again yielding the assertion.

(3.3) If sets $S, T \subseteq [n]$ are weakly separated, then so are $S^*$ and $T^*$.

Indeed, obviously, the weak separation of $S, T$ implies that of $\overline{S}$ and $\overline{T}$, and that of $\{i^*: i \in S\}$ and $\{i^*: i \in T\}$. Then the assertion follows from the fact that the $K$-involution is the combination of the two involutions as in (1.4)(i),(ii).

As a consequence of (3.3),

(3.4) In a set $S \subseteq [n]$ is weakly separated from a symmetric w-collection $C \subset 2^{[n]}$ and from the set $S^*$, then $S^*$ is weakly separated from $C$ as well, implying that $C \cup \{S, S^*\}$ is a symmetric w-collection.
I. We first consider the case $\tau = \nabla(C|AB)$. Then $|A| = |B| = m$ and $|C| = m - 1$, where $m = n/2$, and $A, B$ are expressed as $A = Ca$ and $B =Cb$ for some $a, b \in [n]$. Obviously, $a < b$. Moreover, since $A$ lies on $M$, while $B$ above $M$, we have $y_a = y_{a^o} < y_b = y_{b^o}$. Then, by the the concavity condition (2.4)(i), the pair $\{a, a^o\}$ surrounds $\{b, b^o\}$, and therefore

\[(3.5) \text{ either } a < b < b^o < a^o \text{ or } a < b^o < b < a^o.\]

Note that the relations $A = Ca$, $B = Cb$ and $\Pi_0(A) = \Pi_2(A) = \emptyset$ give

$$C \cap \{a, a^o, b, b^o\} = \{b^o\} \quad \text{and} \quad B \cap \{a, a^o, b, b^o\} = \{b, b^o\}. $$

Then $B^* \cap \{a, a^o, b, b^o\} = \{a, a^o\}$ (in view of $\Pi_1(B^*) = \Pi_{2-1}(B)$). It follows that $C - B^*$ contains the element $b^o$, whereas $B^* - C$ contains the elements $a, a^o$ surrounding $b^o$ (by (3.5)). This together with $|B^*| = m > |C|$ implies that $B^*$ and $C$ are not weakly separated. Therefore, $B^*$ is not in $\mathcal{W}$. On the other hand, since $\Pi_0(B)$ consists of the only pair $\{a, a^o\}$, and $\Pi_0(B)$ of the only pair $\{b, b^o\}$, the sets $B, B^*$ are weakly separated from each other (by (3.2)). Also $B$ is weakly separated from $C$ (since $\{B\} \cup C \subseteq V_K = \mathcal{W}$). But then $C \cup \{B, B^*\}$ is a symmetric $w$-collection (by (3.4)), and now the maximality of $C$ implies $B, B^* \in C$, contrary to $B^* \notin \mathcal{W}$.

II. Next we consider the case $\tau = U(ABC)$. Then $|A| = |B| = |C|$. Also $A = Xa$, $B = Xb$ and $C = Xc$ for some $X \subset [n]$ and $a, b, c \in [n]$. Obviously, $a < b < c$. Moreover, since $A$ lies on $M$, $B$ above $M$, and $C$ below $M$, it follows from (2.4)(i) that $\{c, c^o\}$ surrounds $\{a, a^o\}$, and the latter surrounds $\{b, b^o\}$. This is possible only if

\[(3.6) \text{ either } c^o < a < b < b^o < a^o < c \text{ or } c^o < a < b^o < b < a^o < c.\]

Let $D := \{a, a^o, b, b^o, c, c^o\}$. Then $\Pi_0(A) = \Pi_2(A) = \emptyset$ and $a \in A \neq b, c$ imply $X \cap D = \{b^o, c^o\}$, whence

$$B \cap D = \{b, b^o, c^o\} \quad \text{and} \quad C \cap D = \{b^o, c, c^o\}. $$

It follows that $B^* \cap D = \{a, a^o, c^o\}$. Then $C - B^*$ contains $b^o, c$ and $B^* - C$ contains $a, a^o$, implying that $B^*$ and $C$ are not weakly separated since $a < b^o < a^o < c$ (cf. (3.6)). On the other hand, one can see that $\Pi_0(B) = \{\{a, a^o\}\}$ and $\Pi_2(B) = \{\{b, b^o\}\}$; therefore, $B$ and $B^*$ are weakly separated, by (3.2). As in the previous case, we obtain that $C \cup \{B, B^*\}$ is symmetric and weakly separated, yielding $B, B^* \in C$ (by the maximality of $C$), contrary to $B^* \notin \mathcal{W}$.

III. Finally, consider the case $\tau = U(CAB)$. Then $A = Xa$, $B = Xb$ and $C = Xc$ for some $X \subset [n]$ and elements $c < a < b$. Arguing as above, we observe that

\[(3.7) \text{ either } c < a < b < b^o < a^o < c^o \text{ or } c < a < b^o < b < a^o < c^o,\]

and for $D := \{a, a^o, b, b^o, c, c^o\}$, we have

$$B \cap D = \{b, b^o, c^o\}, \quad B^* \cap D = \{a, a^o, c^o\} \quad \text{and} \quad C \cap D = \{b^o, c, c^o\}. $$

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Then \( C - B^* \) contains \( b^\circ, c \) and \( B^* - C \) contains \( a, a^\circ \), whence \( B^* \) and \( C \) are not weakly separated, in view of \( c < a < b^\circ < a^\circ \) (cf. (5.17)). On the other hand, \( C \cup \{B, B^*\} \) is symmetric and weakly separated, yielding \( B, B^* \in C \).

This completes the proof of Theorem 1.2 when \( n \) is even.

**Remark 3.** Assertion (1.5) on the purity of symmetric strongly separated collections in \( 2^{[n]} \) with \( n \) even is proved in a similar way (and even simpler, using an observation that if \( A \subseteq [n] \) is strongly separated from \( A^* \), then at least one of \( \Pi_0(A) \) and \( \Pi_2(A) \) must be empty). On this way, given a maximal by inclusion symmetric \( s \)-collection \( A \subset 2^{[n]} \), we extend it to a maximal \( s \)-collection \( S \) and take the combi \( K \) without lenses (equivalent to a rhombus tiling) with \( V_K = S \). The situation when the middle line \( M \) is not fully covered by edges of \( K \) is again impossible (now an analysis of the only case \( \tau = \nabla(C|AB) \) is sufficient, repeating part I of the above proof). And when \( M \) is covered by edges of \( K \), we replace the subcombi of \( K \) above \( M \) in a due way (as described in Case 1 of the proof), obtaining a symmetric combi without lenses (viz. rhombus tiling) whose vertex set includes \( A \). This gives (1.5) in the even case:

\[(3.8) \text{ when } n \text{ is even, all inclusion-wise maximal symmetric s-collections in } 2^{[n]} \text{ have the same cardinality, which is equal to } s_n.\]

We finish this section with one more assertion that will be used in the next section:

\[(3.9) \text{ for } n \text{ even, if an ftq-combi } K \text{ on the symmetric zonogon } Z(n,2) \text{ has a path } P \text{ covering the middle line } M, \text{ then this path contains exactly } n/2 \text{ edges.}\]

Indeed, let \( R_0, R_1, \ldots, R_q \) be the sequence of vertices of \( P \), and let \( e_p \) denote the edge from \( R_{p-1} \) to \( R_p \). Then \( |R_{p-1} \Delta R_p| = 2 \), and by (3.1), \( e_p \) is congruent to the vector \( e_{ii^\circ} = \xi_{ii^\circ} - \xi_i \) for some \( i \in [n/2] \). The sum of these vectors over \( M \) is equal to the difference of \( R_q \) and \( R_0 \) (regarded as vectors), namely, \( \sum_{i=n/2+1}^{n} \xi_i - \sum_{i=1}^{n/2} \xi_i \). This is just equal to \( \sum (\xi_{ii^\circ} : i \in [n/2]) \), whence the result easily follows.

### 4 Maximal symmetric \( w \)-collections: odd case

In this section we prove Theorem 1.2 when \( n \) is odd, \( n = 2m + 1 \). It is convenient to deal with the set of colors in the symmetrized form, using notation \([-m..m] = \{-m, \ldots, -1, 0, 1, \ldots, m\}\) introduced in Sect. 2. The proof is based on a reduction to the above result with an even number of colors, namely, \([-m..m]^-\) (= \( \{-m, \ldots, -1, 1, \ldots, m\} \)).

Let \( C \) be a symmetric \( w \)-collection in \( 2^{-m..m} \). Consider the partition \( C = C' \sqcup C'' \), where

\[
C' := \{ A \in C : 0 \notin A \} \quad \text{and} \quad C'' := \{ A \in C : 0 \in A \}.
\]

Note that for any \( A \subseteq [-m..m] \), \( |A| + |A^*| = 2m + 1 \) and exactly one of \( A, A^* \) contains the element 0 (in view of \( 0^\circ = 0 \)). So for \( A \in C' \), its symmetric set \( A^* \) belongs to \( C'' \),
and vice versa. Also $A$ and $A^*$ coincide on the ordinary pairs $\{i, i^0\}$ (i.e., such that $i \neq 0$ and $|\{i, i^0\} \cap A| = 1$), and are complementary on the poor and full pairs:

$$\Pi_0(A^*) = \Pi_2(A) \quad \text{and} \quad \Pi_2(A^*) = \Pi_0(A);$$

cf. (2.2) (recall that the “pair” $\{0, 0^0\}$ is regarded as poor or full). Observe that

(4.1) $|A| \leq m$ if $A \in C'$, and $|A| \geq m + 1$ if $A \in C''$.

Indeed, let $A \in C'$; then $0 \in A^* - A$. If $A$ has no full pair, then $|A| < |A^*|$ follows from the fact that the ordinary pairs for $A$ and $A^*$ are the same. And if $A$ contains a full pair, then this pair belong to $A - A^*$ and surrounds the element 0. Since $A, A^*$ are weakly separated, we have $|A| \leq |A^*|$ (and this inequality is strict since $|A| + |A^*| = 2m + 1$).

In particular, $|A| = m$ and $|A^*| = m + 1$ if $\Pi_2(A) = \emptyset$ and $\Pi_0(A) = \{\{0, 0^0\}\}$; in this case, we say that the pair $\{A, A^*\}$ is squeezed (an analog of self-symmetric sets for $n$ even).

Now form the collections $D', D'', D$ of subsets of $[-m..m]^-$ as

$$D' := C', \quad D'' := \{A - 0: A \in C''\}, \quad \text{and} \quad D := D' \cup D''.$$

For convenience, we will denote the K-involution on sets in $[-m..m]^-$ with symbol $\sharp$ (to differ from the K-involution $\ast$ for $[-m..m]$). We observe that

(4.2) the collection $D$ is $\sharp$-symmetric (i.e., symmetric w.r.t. $\sharp$) and weakly separated.

Indeed, $C$ is partitioned into symmetric pairs $\{A, A^*\}$, where $A \in C'$ and $A^* \in C''$. Then $A \in D'$ and $A^* - 0 \in D''$. One can see that $A^* - 0$ is just $A^0$. Therefore, $D$ is $\sharp$-symmetric. Next, let $A, B \in D$. Obviously, $A, B$ are weakly separated if both are either in $D'$ or in $D''$. So assume that $A \in D'$ and $B \in D''$. Then $A \in C'$ and $B0 \in C''$. Since $C$ is a w-collection and $|A| \leq m < |B0|$ (by (4.1)), either $A$ and $B0$ are strongly separated, or they are weakly separated and $A$ surrounds $B0$. In the former case, $A$ and $B$ are strongly separated, while in the latter case, the inequality $|A| \leq |B|$ ensures that $A$ and $B$ are weakly separated.

We call $D$ the contraction of $C$, and call the operation of getting rid of color 0 as above the contraction operation on $C$. A converse operation handles a symmetric w-collection $D$ in $2^{-m..m}^-$ and lifts it to a collection $E$ in $2^{-m..m}$, as follows.

Represent the sets $A \in D$ as points in the symmetric zonogon $Z = Z(\Xi)$ with

$$\Xi = \{\xi_{-m}, \ldots, \xi_{-1}, \xi_1, \ldots, \xi_m\}$$

and define $D'$ ($D''$) to consist of those points $A \in D$ that lie below the middle line $M$ or on $M$ (resp. above $M$ or on $M$). They determine the collections $E', E''$ in $2^{-m..m}$ by

$$E' := D' \quad \text{and} \quad E'' := \{A0: A \in D''\}.$$

In particular, if $A \in D$ is self-symmetric (viz. lies on $M$), then $A \in D' \cap D''$ and $A$ determines the squeezed pair $\{A, A^* = A0\}$ in $[-m..m]$.

We call $E := E' \cup E''$ the expansion of $D$ using color 0. The following properties are valid:
(4.3) $E$ is symmetric and weakly separated;

(4.4) if $C \subset 2^{[-m..m]}$ is a symmetric w-collection, $D$ is the contraction of $C$, and $E$ is the expansion of $D$ using color 0, then $E = C$.

We check these properties and simultaneously finish the proof of the theorem by using the geometric construction from the previous section.

More precisely, we extend the contraction $D$ of $C$ to a maximal $z$-symmetric w-collection $W$ in $2^{[-m..m]}$ and take a $z$-symmetric ftq-combi $K$ on the zonogon $Z = Z(\Xi)$ such that $V_K = W$. Let $D', D''$ be defined as above. Then $D'$ represents a subset of vertices of $K$ in the lower half $Z_{\text{low}}$ of $Z$ (up to $M$), and $D''$ is the set $M$-symmetric to $D'$, which lies in the upper half $Z_{\text{up}}$ of $Z$.

Let $K_{\text{low}}$ and $K_{\text{up}}$ be the parts (subcomplexes) of $K$ contained in $Z_{\text{low}}$ and $Z_{\text{up}}$, respectively. Then $K_{\text{low}} \cap K_{\text{up}}$ gives a directed path $P$ on $M$ consisting of $m + 1$ vertices and $m$ edges, say, $P = R_0R_1 \cdots R_m$ (cf. (3.9)). Moreover, by (3.1), each edge $e_p := (R_{p-1}, R_p)$ is of type $ii^\circ$ for some $-m \leq i \leq -1$.

Now consider the larger zonogon $Z' = Z(\Xi')$, where $\Xi'$ is obtained by adding to $\Xi$ the vertical vector $\xi_0 = (0, y_0)$. Equivalently, $Z'$ is formed by splitting $Z$ along $M$, keeping the part $Z_{\text{low}}$ of $Z$, moving the part $Z_{\text{up}}$ by $y_0$ units in the vertical direction, and filling the gap between $Z_{\text{low}}$ and $Z_{\text{up}} + \xi_0$ by the rectangle $F$ congruent to $M \times \xi_0$. Accordingly, the part $K_{\text{up}}$ of $K$ is moved by $\xi_0$, thus transforming each vertex $A$ of $K_{\text{up}}$ into $A0$, and we subdivide $F$ into the sequence of rectangles $F_1, \ldots, F_m$, where $F_p$ is congruent to $e_p \times \xi_0$. This results in a “pseudo-combi” $K'$, with the natural involution on the vertices which brings each $A \in V_K$ to its $M'$-symmetric vertex $A'$, where $M'$ is the updated middle line $M'$ with $y^{M'} = y^M + y_0 / 2$. Clearly $A' = A$, and the converse transformation $Z' \mapsto Z$ returns the ftq-combi $K$.

Finally, we can transform $K'$ into a correct, though not symmetric, ftq-combi $K''$ on $Z'$, by subdividing each rectangle $F_p$ into four triangles. Namely, using the fact that the edge $e_p$ of $P$ has type $ii^\circ$ for some $-m \leq i \leq -1$, the devised triangles are viewed as (using notation from Sect. 2.1)

$$
\nabla(R_{p-1} R_{p-1}^* v_p), \quad \nabla(R_p v_p R_p^*), \quad \Delta(v_p | R_{p-1} R_p), \quad L(R_{p-1} v_p R_p^*), \quad \Delta(R_{p-1} v_p R_p^*),
$$

(4.5)

where $v_p$ is the corresponding point (viz. subset of $[-m..m]$) in the interior of $F_p$ (so that the edge $(R_{p-1}, v_p)$ has color $i^\circ$, and $(R_p, v_p)$ color $i$). See the picture.

As a result, the initial collection $C$ is included in $W' := V_{K''} - \{v_1, \ldots, v_m\}$ and coincides with the expansion $E$ of $D$, yielding (4.3) and (4.4). The collection $W'$ in
$2^{-m..m}$ is symmetric and weakly separated. It has size $|V_{K''}| - m = s_{2m+1} - m$, and the result follows.

**Remark 4.** The above method can be applied (in a simpler form) to strongly separated collections for $n$ odd. Namely, for a symmetric $s$-collection $C \subset 2^{-m..m}$, we form the collections $C', C'', D', D'', E', E''$ as described above. (Note that properties (4.1) and (4.2) easily follow from the fact that $\Pi_2(A) = \emptyset$ for each $A \in C'$, which is provided by the strong separation of $C$.) Extending the contraction $D = D' \cup D''$ of $C$ to a maximal symmetric $s$-collection $S$ in $2^{-m..m}$, we take the corresponding symmetric rhombus tiling $T$ on $Z$ with $V_T = S$. Acting as above, we move the “upper” parts of $Z$ and $T$ (lying above $M$) by the vector $\xi_0$ and for each vertex $R_i$ on $M$, add the vertical edge $u_i$ connecting $R_i$ and $R_i + \xi_0$. The difference with the above construction for f7q-combies is that, instead of replicating the horizontal edges $(R_{i-1}, R_i)$ lying on $M$, we now simply split each symmetric rhombus between $R_{i-1}$ and $R_i$ into two triangles and move the upper one (of $\Delta$ type) by $\xi_0$. This together with the vertical edges $u_{i-1}$ and $u_i$ produces a symmetric hexagon. As a result, we obtain a “pseudo” rhombus tiling $T'$ in which the middle tiles are formed by symmetric hexagons, not rhombi. The transformation $T \mapsto T'$ is illustrated in the picture where $m = 2$. Note that $T'$ can be transformed into a rhombus tiling (which is not symmetric) by subdividing each middle hexagon into three rhombi (by one of two possible ways); such a subdivision is shown by dotted lines in the picture. Note that the subdivision adds $m$ new vertices.

As a result, we obtain assertion (4.6) in the odd case, namely:

(4.6) when $n$ is odd, all inclusion-wise maximal symmetric $s$-collections in $2^n$ have the same cardinality, equal to $s_n - (n-1)/2$.

5 Maximal symmetric $c$-collections: even case

In this section we prove Theorem 1.3 when the number $n$ of colors is even, $n = 2m$. Our method of proof uses a reduction to symmetric weakly separated collections and combies. We will deal with the set of colors given in the symmetrized form, namely, $[-m..m]$ (which means $\{-m, \ldots, -1, 1, \ldots, m\}$).

Let $C$ be a maximal symmetric chord separated collection in $2^{-m..m}$. We have to show that $|C| = c_{2m}$ (where $c$ is defined in (1.3)). Suppose, for a contradiction, that this is not so: $|C| < c_{2m}$. Extend $C$ to a (non-symmetric) $c$-collection $\mathcal{F} \subseteq 2^{-m..m}$.
of size \( c_{2m} \) and represent \( F \) as the spectrum (vertex set) \( V_Q \) of a cubillage \( Q \) in the corresponding zonotope; such \( F \) and \( Q \) exist by Galashin’s result (Theorem 2.1).

More precisely, we consider the zonotope \( Z = Z(\Theta) \) generated by the set \( \Theta \) of vectors \( \theta_i = (t_i, 1, \phi(t_i)) \), \( i \in [-m..m] \), subject to (2.11). Recall that the second coordinate of a point in \( \mathbb{R}^3 \) is thought of as the height of this point. From the symmetry conditions on \( \Theta \) (namely, \( i^\circ = -i \), \( t_i = t_i = -t_i \), and \( \phi(t_i) = \phi(t_i) \)) it follows that:

(a) \( Z \) is centrally symmetric w.r.t. the point \( \zeta_Z \) that is the half sum of vectors in \( \Theta \), called the center of \( Z \) (i.e., \( \zeta_Z = (0, m, \phi(t_1) + \cdots + \phi(t_m)) \)); more precisely, \( v = (a, b, c) \in Z \) implies \( \omega(v) := 2\zeta_Z - (a, b, c) \in Z \); and

(b) \( Z \) is symmetric w.r.t. the vertical plane \( \{ (a, b, c) \in \mathbb{R}^3 : a = 0 \} \); namely, \( v = (a, b, c) \in Z \) implies \( \nu(v) := (-a, b, c) \in Z \).

Let \( L \) be the line segment in \( Z \) going through the center \( \zeta_Z \) orthogonal to the plane as in (b). Then \( L \) connects the vertices of \( Z \) representing the sets (intervals) \( [-m..-1] \) and \( [1..m] \); we call \( L \) the axis of \( Z \). The involutions \( \omega \) and \( \nu \) as in (a) and (b) commute, and taking their composition, we have one more involution \( \mu \) on \( Z \); it preserves the first coordinate and gives a symmetry w.r.t. \( L \). A nice property of \( \mu \) (which is easy to verify) is that

(5.1) \( \mu \) sends each set \( A \subseteq [-m..m]^- \) (regarded as a point) to the symmetric set \( A^\ast \); in particular, \( A \) lies on the axis \( L \) if and only if \( A \) is self-symmetric: \( A = A^\ast \).

So \( \mu \) can be thought of as a linear extension of the \( K \)-involution on \( 2^{[-m..m]} \); we call it the geometric \( K \)-involution. (Note also that \( \mu \) swaps the front side \( Z^fr \) and the rear side \( Z^{rear} \) of \( Z \).)

One more object important to us is the section \( \Omega \) of \( Z \) by the horizontal plane \( H_m := \{(a, b, c) \in \mathbb{R}^3 : b = m\} \); it contains the center \( \zeta_Z \) and axis \( L \) and we call it the plate in \( Z \). This \( \Omega \) divides \( Z \) into two halves \( Z^{low} \) and \( Z^{up} \), which lie below and above \( \Omega \), respectively, and intersect by \( \Omega \). The symmetry \( \mu \) swaps \( Z^{low} \) and \( Z^{up} \).

In its turn, the axis \( L \) divides the plate (disk) \( \Omega \) into two halves, symmetric to each other by \( \mu \), and the boundary of \( \Omega \) is partitioned into two piece-wise linear paths \( \Omega^{fr} \) and \( \Omega^{rear} \) connecting the vertices \( [-m..-1] \) and \( [1..m] \), where the former lies in \( Z^{fr} \), and the latter in \( Z^{rear} \).

In fact, we wish to transform the cubillage \( Q \) as above into a symmetric cubillage \( Q' \) keeping the collection \( C \) in its spectrum, where we call a cubillage on \( Z \) symmetric if it is stable under \( \mu \). Then \( V_{Q'} \) is symmetric, whence \( V_{Q'} = C \), and we are done.

The task of constructing the desired \( Q' \) is reduced to handling certain weakly separated collections and ftq-combies. This relies on a method involving weak membranes in fragmented cubillages developed in [H Sect. 6]. We now interrupt our description for a while to briefly review the notions and constructions needed to us.

**Fragmentation and weak membranes.** For an arbitrary \( n \), let \( Q \) be a cubillage on the zonotope \( Z \simeq Z(n, 3) \) generated by vectors \( \theta_1, \ldots, \theta_n \) as in (2.11). A cube \( C \) in \( Q \) may be denoted as \( (X|T) \), where \( X \subseteq [n] \) is the lowest vertex, and \( T \) the triple of edge colors, \( i < j < k \) say, in \( C \).
By the fragmentation of $Q$ we mean the complex $Q^\equiv$ obtained by cutting $Q$ by the horizontal planes $H_\ell := \{(a,b,c) \in \mathbb{R}^3: b = \ell\}$ for $\ell = 1, \ldots, n-1$. These planes subdivide each cube $C = (X|T)$ into 3 pieces $C^\equiv_1, C^\equiv_2, C^\equiv_3$, where $C^\equiv_h = C_h$ is the portion of $C$ between $H_{|X|+h-1}$ and $H_{|X|+h}$. So $C^\equiv_1$ is a simplex with the bottom vertex $X$, $C^\equiv_2$ is a simplex with the top vertex $X \cup T$, and $C^\equiv_3$ is an octahedron; see the picture (where the objects are slightly slanted).

\begin{align*}
\text{A facet of $Q^\equiv$ is meant to be a facet of any fragment $C^\equiv_h$ of a cube $C$ in $Q$. This is a triangle of one of two sorts: either a horizontal triangle (section) $S_h(C) := C \cap H_{|X|+h}$, $h = 1, 2$, or a half of the parallelogram forming a face of $C$, that we conditionally call a \textit{“vertical”} triangle. Note that the triple of vertices of a horizontal facet $F$ are of the form either $X_i, X_j, X_k$ or $Y - i, Y - j, Y - k$ for some $X, Y \subseteq [n]$ and $i < j < k$, and $i, j, k$ are just the colors of the cube containing $F$ as a section. This implies that (5.2) any horizontal facet $F$ in $Q^\equiv$ determines both fragments sharing $F$, as well as the cube separated by $F$ (among all cubillages whose fragmentation contains $F$).}

(This need not be so for a vertical facet.) Consider the projection $\pi^\rho : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

\begin{equation}
(5.3) \ (a,b,c) \mapsto (x,y), \text{ where } x := a \text{ and } y := b - \rho c
\end{equation}

for a sufficiently small real $\rho > 0$. Observe that $\pi^\rho$ maps the generators $\theta_1, \ldots, \theta_n$ of the zonotope $Z = Z(\Theta) \simeq Z(n,3)$ to generators $\xi_1, \ldots, \xi_n$ (respectively) as in Sect. 2.4 we may assume, w.l.o.g., that the latter generators satisfy (2.3), (2.4), and therefore, the image by $\pi^\rho$ of $Z$ is the zonogon $Z' = Z(\Xi) \simeq Z(n,2)$ (here the concavity condition (2.4)(i) is provided by the convexity of $\phi$ in (2.11) and the relation $\rho > 0$). Note that for points $(a,b,c), (a,b,c') \in \mathbb{R}^3$ if $c < c'$, then their images $(x,y)$ and $(x',y')$ (respectively) satisfy $x = x'$ and $y > y'$. This implies that under the projection $\pi^\rho$, each horizontal facet of $Q^\equiv$ becomes “fully seen” (slightly from behind), and therefore forms a non-degenerate triangle in $Z'$.

\textbf{Definition.} A 2-dimensional subcomplex (\textit{“surface”}) $N$ in $Q^\equiv$ is called a \textit{weak membrane}, or a \textit{w-membrane} for short, of $Q$ if $\pi^\rho$ bijectively projects $N$ onto $Z'$. When $N$ has only vertical triangles (but no horizontal ones), $N$ is called a \textit{strong membrane}.

Additional explanations: Let $N$ contain a facet $F$ of a fragment $C^\equiv_h$ of a cube $C = (X|T = (i < j < k))$. Then: (a) if $F$ is the section $S_1(C)$ (resp. $S_2(C)$), then $\pi^\rho(F)$ is an upper (resp. lower) semi-lens with edges of types $ij, jk, ik$; and (b) if $F$ is the lower (upper) half of a facet (\textit{“rhombus”}) of $C$, then $\pi^\rho(F)$ is a $\nabla$-tile (resp. $\Delta$-tile) with the same edge colors as those of $F$. 

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As is explained in [4, Sects. 6.3,7], w-membranes are closely related to ftq-combies:

\[(5.4) \text{ for a cubillage } Q \text{ on } Z \text{ and a w-membrane } N \text{ in } Q^=, \text{ the projection } \pi^\rho(N) \text{ (regarded as a complex) forms an ftq-combi on the zonogon } Z', \text{ and conversely, for any ftq-combi } K \text{ on } Z', \text{ there exist a cubillage } Q \text{ on } Z \text{ and a w-membrane } N \text{ in } Q^= \text{ such that } \pi^\rho(N) = K.\]

Particular cases of membranes are the front side \(Z^{\text{fr}}\) and the rear side \(Z^{\text{rear}}\) of \(Z\) (see Sect. 2.1), regarded as complexes in which each facet (“rhombus”) is subdivided (fragmented) into two halves; both membranes are strong.

The picture below illustrates the simplest case when the cubillage consists of only one cube \(C\), i.e., \(n = 3\) and \(Z = C\); here there are four w-membranes \(N_1 \simeq C^{\text{fr}}, N_2, N_3, N_4 \simeq C^{\text{rear}},\) and the horizontal facets in \(N_2\) and \(N_3\) are dark.

Next we return to the initial cubillage \(Q\) with \(C \subset V_Q\) on the symmetric zonotope \(Z = Z(\Theta)\) with \(n = 2m\) colors. In its fragmentation \(Q^=\), we distinguish the particular weak membrane \(N^\circ\) as follows:

\[(5.5) N^\circ = Z^{\text{fr}}_+ \cup Q_\Omega \cup Z^{\text{rear}}_- \text{, where } Q_\Omega \text{ is formed by the facets of } Q^= \text{ lying on the plate } \Omega; \text{ } Z^{\text{fr}}_+ \text{ is the part of the front side } Z^{\text{fr}} \text{ of } Z \text{ contained in the upper half } Z^\up, \text{ and } Z^{\text{rear}}_- \text{ is the part of the rear side } Z^{\text{rear}} \text{ contained in the lower half } Z^\low.\]

This \(N^\circ\) is indeed a w-membrane (taking into account that the plate \(\Omega\) becomes “fully seen” under \(\pi^\rho\), whence the projection \(\pi^\rho\) is injective on \(N^\circ\); the fact that \(\pi^\rho(N^\circ) = Z'\) is easy). Also one can see that \(Z^{\text{fr}}_+\) is symmetric to \(Z^{\text{rear}}_-\). The pieces (disks) \(Z^{\text{fr}}_+\) and \(Q_\Omega\) share the path \(\Omega^{\text{fr}}\) (connecting the vertices \([-m..-1]\) and \([1..m]\) and lying on \(Z^{\text{fr}}\)), while \(Q_\Omega\) and \(Z^{\text{rear}}_-\) share the rear boundary path \(\Omega^{\text{rear}}\) of \(\Omega\). Then \(K := \pi^\rho(N^\circ)\) is an ftq-combi on \(Z' \simeq Z(2m, 2)\) in which the part \(\pi^\rho(Z^{\text{fr}}_+)\) is symmetric to \(\pi^\rho(Z^{\text{rear}}_-)\), and the image by \(\pi^\rho\) of the axis \(L\) of \(Z\) is just the middle line \(M\) of \(Z'\).

In the rest of the proof we rely on the simple fact analogous to (3.3) for the weak separation: if \(A, B \subseteq [-m..m]^-\) are chord separated, then so are \(A^*, B^*\). This implies:

\[(5.6) \text{ for the maximal symmetric } c\text{-collection } C \text{ as above, if } A \in 2[-m..m]^\perp \text{ is chord separated from } C \text{ and from } A^*, \text{ then } A, A^* \in C.\]

Indeed, one can see that \(A^*\) is chord separated from \(C\) (using the symmetry of \(C\)). Then \(C \cup \{A, A^*\}\) is chord separated, and the maximality of \(C\) implies \(A, A^* \in C\).

Next we proceed as follows. By reasonings in Sect. 3, the middle line \(M\) in \(K = \pi^\rho(N^\circ)\) is covered by edges of \(K\), and combining the part \(K^\low\) of \(K\) below \(M\) with its
$M$-symmetric complex $(K^{\text{low}})^*$ (where $*$ stands for the symmetry in $Z'$), we obtain a correct symmetric ftq-combi $K'$ on $Z'$. (Note that the part $\pi^n(Z'_1)$ of $K^{\text{low}}$ is already symmetric to the part $\pi'(Z'_1)_{\text{rear}}$ of $K^{\text{up}}$; so $K'$ and $K$ coincide within these parts.) Each set $A \in V_{K^{\text{low}}}$ belongs to the spectrum $V_Q$, and therefore it is chord separated from $C$. Also the sets $A, A^*$ are chord separated (since both belong to the same ftq-combi $K'$, whence they are even weakly separated). Then $A, A^* \in \mathcal{C}$, by (5.6). It follows that

(5.7) the set of vertices of $Q$ lying on the plate $\Omega$, i.e., $V_{Q_{\Omega}}$, is self-symmetric (w.r.t. $L$) and belongs to $\mathcal{C}$, implying that the vertex set of $K$ is symmetric.

**Remark 5.** Note that a priori (5.7) does not guarantee that the subcomplex $Q_{\Omega}$ itself (or the set of triangles it is) is self-symmetric. (The reason is that in a quasi-combi, a semi-lens with more than three vertices can be triangulated in several different ways.) If this were so, then the proof could be finished as follows. Take the part $Q^\equiv_1$ of $Q^\equiv$ below $\Omega$ (which subdivides $Z^{\text{low}}$) and replace the part of $Q^\equiv$ above $\Omega$ by the complex symmetric to $Q^\equiv_1$. These two complexes coincide within the plane $\Omega$ (due to the symmetry of $Q_{\Omega}$), and we can conclude that their union gives the correct fragmentation of some symmetric cubillage $Q'$ on $Z$. (Here we rely on the fact, due to (5.2), that any horizontal triangle in $Q_{\Omega}$ uniquely determines the cube having this triangle as a section.) Then $V_{Q'} \supseteq \mathcal{C}$ would imply the result.

We argue in a different way, using a trick of *perturbing* the plate $\Omega$ described below.

To slightly simplify our description, we identify the w-membrane $N^\circ$ and its image by $\pi^n$, the ftq-combi $K = \pi^n(N^\circ)$. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be the graph whose vertices are the semi-lenses (viz. horizontal triangles) on $\Omega$ and where two semi-lenses are connected by edge in $\Gamma$ if they share an edge and are of the same type, i.e., both are lower or both are upper ones. Let $\Phi$ be a connected component of $\Gamma$ formed by *upper* semi-lenses. Considering semi-lenses sharing edges, we can conclude that all semi-lenses in $\Phi$ have the same root $X \subset [-m..m]^-$, i.e., each vertex occurring in a semi-lens there is of the form $X i$ with $i \in [-m..m]^-$ (see Sect. 2.1). Each triangle $\tau \in \Phi$, having edges of types $ij, jk, ik$ for $i < j < k$ say, is the section $S_h(C)$ at level $h = 1$ of some cube $C = C(\tau)$ in $Q$; more precisely, $C$ is of the form $(X \mid T = \{i, j, k\})$, i.e., $C$ has the lowest point at the vertex $X$ in $Q$, and edges of colors $i, j, k$. Then $\tau$ is the horizontal facet of the fragment $C^\equiv_1$ in $Q^\equiv$ with the bottom vertex $X$ of height $|X| = m - 1$ in $Z$.

The union of triangles $\tau \in \Phi$, denoted by $\Sigma_\Phi$, forms a (possibly big) upper semi-lens. This is nothing else than an upper semi-lens in the corresponding fine quasi-combi $\tilde{K}$ related to $K$. In its turn, the union of fragments $C^\equiv_1$ over the cubes $C = C(\tau)$ for $\tau \in \Phi$ forms the convex truncated cone $D_\Phi$ with the bottom vertex $X$ and the upper (horizontal) side $\Sigma_\Phi$.

Symmetrically, for each connected component $\Psi$ formed by *lower* semi-lenses in $\Gamma$, all semi-lenses have the same “upper” root $Y$ (i.e., their vertices are expressed as $Y - i$), and each triangle $\sigma \in \Psi$ is the section $S_2(C)$ at level 2 of some cube $C = C(\sigma)$ with the top vertex $Y$ (which is of height $|Y| = m + 1$ in $Z$). Then the union $\Sigma_\Psi := \cup(\sigma \in \Psi)$ is a lower semi-lens as well (which is a tile in the fine quasi-combi $\tilde{K}$). And the union $F_\Psi$ of the upper fragments (simplexes) $C^\equiv_3(\sigma)$ over $\sigma \in \Psi$ forms the truncated cone with the top vertex $Y$ and the lower (horizontal) side $\Sigma_\Psi$.

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Note that the fine quasi-combi $\tilde{K}$ related to the ftq-combi $K$ is symmetric (since it is determined by the vertex set $V_K$, which is symmetric, by (5.7)). This implies that the involution $\mu$ on $\Omega$ sends each upper semi-lens of $\tilde{K}$ to a lower one (and vice versa), which in turn implies that the corresponding lower and upper cones are symmetric to each other: $F_\Psi = \mu(D_\Phi)$. The picture below illustrates such cones when $|\Phi| = 2$.

![Diagram of cones](image)

Using the above constructions, we perturb the plate $\Omega$ as follows. For each component $\Phi$ of upper type in $\Gamma$, the interior of the big semi-lens $U_\Phi$ is replaced by the side surface of the cone $D_\Phi$ (as though squeezing out a “pit” in place of this semi-lens). Similarly, for each component $\Psi$ of lower type, the interior of $L_\Psi$ is replaced by the side surface of the cone $F_\Psi$ (making a “peak”). Let $\tilde{\Omega}$ denote the resulting surface (which has the same boundary as in $\Omega$ but consists of a gathering of pits and peaks, without horizontal pieces at all). Then the above-mentioned correspondence between the lower and upper cones implies that the perturbed $\tilde{\Omega}$ is symmetric w.r.t. the axis $L$.

Now take the sub-cubillage $Q'$ of $Q$ formed by all cubes whose bottom vertex is of height at least $m - 1$. Then $\tilde{\Omega}$ is exactly the lower boundary of $Q'$. The symmetry of $\tilde{\Omega}$ implies that the cubillage $Q''$ symmetric to $Q'$ has $\tilde{\Omega}$ as the upper boundary. As a result, $Q' \cup Q''$ is a correct symmetric cubillage on $Z$ containing $C$.

This contradicts the assumption that $C$ is maximal and $|C| < c_{2m}$, completing the proof of Theorem 1.3 for $n$ even.

As a consequence of the above proof, any maximal $c$-collection $C$ in $2^{[−m..m]}$ is representable as the spectrum $V_Q$ of a symmetric cubillage $Q$ on $Z(2m,3)$ (which is determined by $C$, due to Theorem 2.1). In particular, this implies that (5.8) the subcomplex (facet structure) $Q_\Omega$ of $Q^=$ on the plate $\Omega$ is symmetric.

We finish this section with one more observation. As is shown in [4] (and quoted in (5.4)), for any maximal $w$-collection $W \subseteq 2^{[n]}$, there exist a cubillage $Q$ and a $w$-membrane $N$ in its fragmentation $Q^=$ such that $V_N = W$ (moreover, one shows there how to construct such $Q$ and $N$ efficiently). In particular, $V_Q$ gives a maximal $c$-collection including $W$. A symmetric counterpart of such a relation between $w$- and $c$-collections is as follows.

**Theorem 5.1** For $n$ even, any maximal symmetric $w$-collection $W \subseteq 2^{[n]}$ can be represented as the spectrum (vertex set) of a symmetric $w$-membrane $N$ in the fragmentation.
of some symmetric cubillage $Q$ on $Z(n, 3)$ (and such $Q$ and $N$ can be constructed efficiently). In particular, $V_Q$ is a maximal symmetric $c$-collection including $W$. Moreover, if $K$ is an arbitrary symmetric ftq-combi with $V_K = W$, then $Q$ and $N$ can be chosen so that $K = \pi^\rho(N)$.

**Proof** For a maximal symmetric $w$-collection $W \subseteq 2^{[n]}$ with $n$ even, take a symmetric ftq-combi $K$ with $V_K = W$ (which exists and can be constructed as described in Case 1 of the proof of Theorem 1.2 in Sect. 3). This $K$ is the projection by $\pi^\rho$ of a $w$-membrane $N$ in some cubillage $Q'$ on $Z(n, 3)$; see (5.4). The symmetry of $K$ implies that $N$ (regarded as a complex) is stable under the involution $\mu$ on $Z$. This and the fact that the front side $Z^fr$ and the rear side $Z^{rear}$ of $Z$ are symmetric to each other (by $\mu$) imply that the part $Z'$ of $Z$ between $Z^fr$ and $N$ is symmetric to the part $Z''$ between $N$ and $Z^{rear}$. Now take the sub-complex $B$ of the fragmentation $\mathcal{Q}' \equiv \mathcal{Q}$ lying in $Z'$. Then $\mu(B)$ gives a subdivision of $Z''$ respecting $N$. In view of (5.2), any pair of fragments of $B$ and $\mu(B)$ sharing a horizontal facet in $N$ must belong to the same cube. Using this, one can conclude that the union of $B$ and $\mu(B)$ constitutes a well-defined fragmentation of a symmetric cubillage $Q$ on $Z$ containing $N$, whence the theorem follows.

| 6 Maximal symmetric $c$-collections: odd case |

In this section we prove Theorem 1.3 when the number $n$ of colors is odd, $n = 2m + 1$. We assume that the set of colors is given in the symmetrized form, namely, as $[-m..m]$ (i.e., $\{-m, \ldots, -1, 0, 1, \ldots, m\}$). Our goal is to show that any symmetric chord separated collection $\mathcal{D} \subset 2^{-m..m}$ is included in the spectrum $V_Q$ of a symmetric cubillage $Q$ on the symmetric zonotope $Z(n, 3)$, whence the theorem follows from Galashin’s results ((1.3) and Theorem 2.1). We will use a reduction to the even case as in Sect. 5.

First of all we partition $\mathcal{D}$ as $\mathcal{D}' \sqcup \mathcal{D}''$, where

$$\mathcal{D}' := \{A \in \mathcal{D} : 0 \notin A\} \quad \text{and} \quad \mathcal{D}'' := \{A \in \mathcal{D} : 0 \in A\}.$$  

By the symmetry of $\mathcal{D}$, the map $A \mapsto A^*0$ gives a bijection between $\mathcal{D}'$ and $\mathcal{D}''$.

Define the collections $\mathcal{C}', \mathcal{C}'', \mathcal{C}$ in $2^{-m..m}$ (where, as before, $[-m..m]$ means $[-m..m] \setminus \{0\}$) by

$$\mathcal{C}' := \mathcal{D}', \quad \mathcal{C}'' := \{A - 0 : A \in \mathcal{D}''\}, \quad \text{and} \quad \mathcal{C} := \mathcal{C}' \cup \mathcal{C}''. $$

In what follows, we will use symbol $*$ for the $K$-involution in $2^{-m..m}$, and $\sharp$ for that in $2^{-m..m}$, and similarly for the geometric versions of these involutions. The symmetry of $\mathcal{D}$ implies that of $\mathcal{C}$. Also it is not difficult to see that (6.1) $\mathcal{C}$ is chord separated;

we leave this, as well as verifications of assertions (6.3) and (6.5) below, to the reader as an exercise.

As is shown in the proof of Theorem 1.3 for the even case in the previous section, $\mathcal{C}$ is extendable to the spectrum $V_Q$ of a symmetric cubillage $Q$ on the zonotope $Z:_=
$Z(\Theta) \simeq Z(2m, 3)$ (where, as before, $\Theta$ is a symmetric set of generators $\theta_i$, $i \in [-m..m]$−, as in (2.11), (2.12)). We are going to expand $Q$ to a cubillage $\tilde{Q}$ on the zonotope $Z \simeq Z(2m + 1, 3)$ generated by $\Theta$ plus the vector $\theta_0$ which, w.l.o.g., can be defined as the vertical vector $(0, 1, 0)$. Based on properties of cubillages (see [3, Sect. 3.3]), an expansion of a cubillage to a larger one with an additional color is performed by use of a certain strong membrane.

More precisely, in our case, by a strong membrane related to color 0, or a $0$-membrane, we mean a (closed two-dimensional) subcomplex $N$ of $Q$ such that the projection $\pi_0 : \mathbb{R}^3 \to \mathbb{R}^2$ parallel to the vector $\theta_0$ is injective on $N$ (regarded as a set of points), and $\pi_0(N) = \pi_0(Z_-)$. It divides $Z_-$ into two halves $Z_{\text{low}}(N)$ and $Z_{\text{up}}(N)$ consisting of the points lying below and above $N$ (including $N$ itself), respectively. Two additional conditions on $N$ are needed to us, namely:

(6.2) (i) $N$ is symmetric, i.e., $N^\sharp = N$; and

(ii) the vertices of $C'$ lie in $Z_{\text{low}}(N)$, while those of $C''$ in $Z_{\text{up}}(N)$;

such an $N$ is called feasible.

The expansion operation on $Q$ using $N$ acts as follows: it splits $Q$ into two closed parts (subcomplexes) $Q'$ and $Q''$ lying in $Z_{\text{low}}(N)$ and $Z_{\text{up}}(N)$, respectively; the part $Q'$ is preserved, while $Q''$ moves by $\theta_0$, and the gap between $Q'$ and $Q'' + \theta_0$ is filled by new cubes, each being the Minkowski sum of a facet (“rhombus”) of $Q$ contained in $N$ and the segment $[0, \theta_0]$. The resulting complex is called the expansion of $Q$ by color 0 using $N$. One easily shows that

(6.3) if $Q$ is symmetric and $N$ is feasible, then the expansion of $Q$ by color 0 using $N$ is a symmetric cubillage on $Z(2m + 1, 3)$ whose spectrum includes $\mathcal{D}$.

In light of (6.3), we will attempt to show that a feasible membrane in an appropriate $Q$ does exist (whence the result will immediately follow).

Our programme to accomplish this task is a bit tricky. We first lift $\mathcal{D}$ to a symmetric collection $\mathcal{E}$ in $2^{[-m..m]}$, where $[-m..m]$+ is $[-m..m]$− to which two symmetric colors 0′ and 0″ are added, namely, $\{-m, \ldots, -1, 0', 0'', 1, \ldots, m\}$. Second, we include $\mathcal{E}$ in the spectrum of a symmetric cubillage $R$ on the zonotope $Z_+ \simeq Z(2m + 2, 3)$ generated by $\Theta$ plus two vectors $\theta_{0r}, \theta_{0r}'$ viewed as $\theta_{0r} := (-\varepsilon, 1, 0)$ and $\theta_{0r} := (\varepsilon, 1, 0)$ for a small real $\varepsilon > 0$. Third, we reduce $R$ by eliminating colors 0′ and 0″ so as to obtain a symmetric cubillage $\tilde{Q}$ on $Z_- \simeq (2m, 3)$ equipped with two particular 0-membranes $N'$ and $N''$ which are symmetric to each other. Fourth, updating $N'$ and $N''$ symmetrically, step by step, we eventually turn these into the desired symmetric membrane $N$ in $Q$.

Next we describe these stages more carefully, using symbol $\bullet$ to denote the symmetry in $2^{[-m..m]}$ (and the K-involution in $Z_+$). One can see that for $X \subseteq [-m..m]$+, if $X$ contains none of 0′, 0″, then $X^\bullet = X^3 0'0''$; if $X$ contains 0′ but not 0″ (resp. 0″ but not 0′), then $X^\bullet$ is $(X - 0')^2 0'$ (resp. $(X - 0'')^2 0''$); and if $X$ contains both 0′, 0″, then $X^\bullet = (X - \{0', 0''\})^2$.

Define the collections $\mathcal{E}', \mathcal{E}''$, $\mathcal{E}$ in $2^{[-m..m]}$ by

$$\mathcal{E}' := \mathcal{D}', \quad \mathcal{E}'' := \{(X - 0) 0'0'': X \in \mathcal{D}''\}, \quad \mathcal{E} := \mathcal{E}' \cup \mathcal{E}''.$$  \quad (6.4)
The symmetry of \( \mathcal{D} \) implies that of \( \mathcal{E} \), and a routine verification shows that

(6.5) \( \mathcal{E} \) is chord separated.

In view of \([6.5]\) and since \(2m + 2\) is even, there exists a symmetric cubillage \( R \) on \( \mathbb{Z}_+ \simeq \mathbb{Z}(2m + 2, 3) \) such that \( \mathcal{E} \subset V_R \). Let \( \mathcal{P}' (\mathcal{P}''') \) be the set of cubes \( C = (X | T) \) whose type \( T \) contains color \( 0' \) (resp. \( 0'' \)); following terminology in \([3] \) Sect. 3.3, we refer to this set (or its induced subcomplex in \( R \)) as the \( 0' \text{-} \text{pie} \) (resp. \( 0'' \text{-} \text{pie} \)) in \( R \). By a simple, but important, property of pies, \( \mathcal{P}' \) is representable as the (Minkowski) sum of a \( 0' \)-membrane \( N' \) and the segment \([0, \theta_{0'}] \). Similarly, \( \mathcal{P}''' \) is the sum of a \( 0'' \)-membrane \( N'' \) and \([0, \theta_{0''}] \).

One can check that for a cube \( C = F + [0, \theta_0] \) in \( \mathcal{P}' \), where \( F \) is a facet in \( N' \), its symmetric cube \( C^* \) in \( R \) is viewed as \((F0')^* + [0, \theta_{0'}] \), and therefore, \((F0')^* \) belongs to \( N'' \), and \( C^* \) to \( \mathcal{P}''' \). And similarly for cubes in \( \mathcal{P}''' \). This and the identity \(((F0')^*0'')^* = F \) (where \( F \) is in \( N' \)) imply that

(6.6) the pies \( \mathcal{P}' \) and \( \mathcal{P}''' \) are symmetric to each other, the \( 0' \)-membrane \( N' \) is symmetric to the \( 0'' \)-membrane \( N'' + \theta_{0''} \), and similarly, \( N'' \) is symmetric to \( N' + \theta_{0'} \).

Now consider the collections \( \mathcal{E}' \) and \( \mathcal{E}'' \) as in \([6.4]\). Since the sets in \( \mathcal{E}' \) (in \( \mathcal{E}'' \)) do not contain colors \( 0', 0'' \) (resp. contain both \( 0', 0'' \)), the vertices of \( R \) representing these sets are located (non-strictly) below (resp. above) \( \mathcal{P}' \cup \mathcal{P}''' \). We further need to use the contraction operations on pies of cubillages, which are converse, in a sense, to the expansion operations on membranes; for details, see \([4] \) Sect. 3.3.

The contraction operation applied to \( \mathcal{P}' \) removes from \( R \) the cubes lying between the membranes \( N' \) and \( N' + \theta_{0'} \) and moves the part of \( R \) above \( N' + \theta_{0'} \) by \(- \theta_{0'} \), so that the images of these membranes become glued together. The contraction operation applied to \( \mathcal{P}''' \) (or to the image of \( \mathcal{P}''' \) after contracting \( \mathcal{P}' \)) acts similarly. These two operations commute, and applying both (in any order), we obtain a symmetric cubillage \( Q \) on \( Z_- \) and two membranes \( \tilde{N}' \) and \( \tilde{N}'' \) in it, which are the images of \( N', N'' \) and can be simultaneously thought of as \( 0 \)-membranes since the vectors \( \theta_0, \theta_{0'}, \theta_{0''} \) are close to each other. Moreover, since the vertices in \( \mathcal{E}' \) contain none of colors \( 0', 0'' \) (and therefore lie below the pies \( \mathcal{P}', \mathcal{P}''' \)), whereas those in \( \mathcal{E}'' \) contain both \( 0', 0'' \) (and lie above these pies), we observe, by comparing \( C' \) with \( \mathcal{E}' \), and \( C'' \) with \( \mathcal{E}'' \), that

(6.7) \( C' \) lies in the region \( Z_-^\text{low}(\tilde{N}') \cap Z_-^\text{low}(\tilde{N}'') \) (i.e., below both \( \tilde{N}', \tilde{N}'' \)), while \( C'' \) lies in \( Z_-^\text{up}(\tilde{N}') \cap Z_-^\text{up}(\tilde{N}'') \) (i.e., above both \( \tilde{N}', \tilde{N}'' \)).

Next, for a closed subset \( U \) of points in \( Z_- \), define \( U_0^\text{fr} \) (resp. \( U_0^\text{rear} \)) to be the set of points \( u \in U \) “seen from below” (resp. “seen from above”), i.e., such that \( U \) has no point of the form \( u - \delta \theta_0 \) (resp. \( u + \delta \theta_0 \)) with \( \delta > 0 \); we call this the \( 0 \text{-} \text{front} \) (resp. \( 0 \text{-} \text{rear} \)) side of \( U \). It is not difficult to show (cf. \([4] \) Sect. 3.4]) that any set \( S \) of cubes in a cubillage on \( Z_- \) has a minimal (maximal) element \( C \), in the sense that there is no \( C' \in S \) such that \( C_0^\text{fr} \) and \( C_0^\text{rear} \) (resp. \( C_0^\text{rear} \) and \( C_0^\text{fr} \)) share a facet. (In other words, there is no cube in \( S \) before (resp. after) \( C \) in the direction \( \theta_0 \).

Define
\[ H' := (\tilde{N}' \cup \tilde{N}'')_0^\text{fr} \quad \text{and} \quad H'' := (\tilde{N}' \cup \tilde{N}'')_0^\text{rear}. \]
One can see that $H', H''$ are 0-membranes in $Z_+$, $H''$ is symmetric to $H'$ and lies (non-strictly) above $H'$, and in view of (6.7), the collection $C'$ lies below $H'$, while $C''$ does above $H''$. Now the result is provided by the following

**Lemma 6.1** There exists a symmetric 0-membrane $H$ lying between $H'$ and $H''$.

**Proof** If $H' = H''$, we are done. So assume that the set $S$ of cubes of $Q$ filling the gap $\omega$ between $H'$ and $H''$ is nonempty.

Let $C$ be a maximal cube in $S$ (in the sense explained above). Then the 0-rear side $C^0_{\text{rear}}$ of $C$ is entirely contained in $H''$ (for if a facet $F$ of $C^0_{\text{rear}}$ is not in $H''$, then there is a cube $C'$ with $F \subset C^0_{\text{fr}}$, and this $C'$ lies in $\omega$ and is greater than $C$). It follows that replacing in $H''$ the side (disk) $C^0_{\text{rear}}$ by $C^0_{\text{fr}}$, we obtain a correct 0-membrane $\hat{H}''$ lying in $\omega$. By the symmetry of $\omega$, the cube $C^0$ symmetric to $C$ lies in $\omega$ as well, and its 0-front side is entirely contained in $H'$. Let $\hat{H}'$ be the 0-membrane obtained by replacing in $H'$ the 0-front side of $C^0$ by its 0-rear side.

An important fact is that the cubes $C$ and $C^0$ are different. This is so because if $C$ has type $\{i, j, k\}$ (the edge colors in $C$), then, by symmetry, $C^0$ has type $\{i^0, j^0, k^0\}$, and $\{i, j, k\} = \{i^0, j^0, k^0\}$ is impossible since $[-m..m]$ has no color symmetric to itself. The transformation $(H', H'') \mapsto (\hat{H}', \hat{H}'')$ is illustrated in the picture (where for simplicity we give two-dimensional projections).

So the new 0-membranes $\hat{H}'$ and $\hat{H}''$ are symmetric to each other, the former lies (non-strictly) below the latter, and the gap between them is smaller than $\omega$. Repeating the procedure, step by step, we eventually obtain a pair for which the gap vanishes, yielding the desired $H$, as required.

This completes the proof of Theorem 1.3 for the case of odd colors.

**Remark 6.** For an integer $n > 0$, let us say that $J \subseteq [0..n]$ is symmetric if $k \in J$ implies $n - k \in J$. Let $\Lambda_n(J)$ be the collection of sets $\binom{[n]}{k}$ over $k \in J$. From Theorem 1.3 one can obtain a slightly sharper purity result, namely:

(6.8) for a fixed symmetric $J \subseteq [0..n]$, all inclusion-wise maximal symmetric chord separated collections in $\Lambda_n(J)$ have the same size.

Indeed, consider such a collection $C$ and extend it to a maximal symmetric c-collection $S$ in $2^{[n]}$. From the equality $|S| = c_n$ (by Theorem 1.3) in follows that for each $k \in [0..n]$, the number of sets $A \in S$ with $|A| = k$ is equal to exactly $k(n - k) + 1$. Also the symmetry of $J$ implies that for each $k \in J$, the subcollection $C \cap \binom{[n]}{k}$ coincides with $S \cap \binom{[n]}{k}$, and the result follows.

In conclusion of this paper, we give an analog of Theorem 5.1 for $n$ odd. Recall that in this case the maximal size of a symmetric $w$-collection $W$ in $2^{[n]}$ is $(n - 1)/2$ units less
that for each section Q the correct extended fragmentation of a symmetric cubillage

One can realize that each quadruple Q in the extended fragmentation of C

Then the middle vertical section S

Remark 4); so the non-symmetry of \( \tilde{\pi} \) on Fviding each

is subdivided into rectangles F

can be transformed into a correct (non-symmetric) ftq-combi on the

is the rectangle spanning the vertices X_1, X_0, X_0, X_0. In addition, we assume that such facets can appear only in the middle level of Q, namely, when |X| = m - 1, where n = 2m + 1. We refer to the corresponding subcomplexes in the extended fragmentation of Q constructed in this way (and respecting the projection \( \pi^\rho \)) as extraordinary weak membranes.

**Theorem 6.2** For n odd, any maximal symmetric w-collection \( \mathcal{W} \subseteq 2^{[n]} \) can be represented as the spectrum of a symmetric extraordinary weak membrane N in the extended fragmentation of some symmetric cubillage Q on Z(n, 3) (and such Q and N can be constructed efficiently). In particular, \( V_Q \) is a maximal symmetric c-collection including \( \mathcal{W} \).

**Proof** For \( n = 2m + 1 \), let \( \mathcal{W} \) be a symmetric w-collection in \( 2^{[n]} \) of size \( s_n - m \). As is explained in Sect. 4, in the symmetric zonogon Z generated by vectors \( \xi_{-m}, \ldots, \xi_0, \xi_1, \ldots, \xi_m \) as in (2.27), one can construct a tiling \( K \) on Z, called a pseudo-combi, such that: (a) \( V_K = \mathcal{W} \); (b) the sub-tiling \( K^{\text{low}} \) of \( K \) below the horizontal line \( M \) (cf. (2.8)) is symmetric to the sub-tiling \( K^{\text{up}} \) of \( K \) above the line \( M^+ = M + \xi_0 \); (c) combining \( K^{\text{low}} \) with \( K^{\text{up}} \) shifted by \( -\xi_0 \) results in a symmetric ftq-combi on the reduced 2m-colored zonogon; and (d) the rectangular gap \( F \) between \( M \) and \( M^+ \) in Z is subdivided into rectangles \( F_1, \ldots, F_m \).

This \( K \) can be transformed into a correct (non-symmetric) ftq-combi \( \tilde{K} \) by subdi-

viding each \( F_p \) into the quadruple \( T_p \) of triangles as is (4.3) (see also the picture above

Remark 4); so the non-symmetry of \( \tilde{K} \) concerns merely the part \( F \). Take a cubillage \( \tilde{Q} \) on \( Z' \simeq Z(n, 3) \) and a w-membrane \( \tilde{N} \) in \( \tilde{Q} \) such that \( \tilde{K} = \pi^\rho(\tilde{N}) \) (existing by (4.4)). One can realize that each quadruple \( T_p \) in \( \tilde{K} \) corresponds, in the fragmentation \( \tilde{Q}^\rho \) of \( \tilde{Q} \), to the rear side \( D(p)^{\text{rear}} \) of the octahedral (middle) fragment \( D(p) \) of some cube \( C(p) \) of \( \tilde{Q} \), i.e., \( \pi^\rho \) maps the facets of \( D(p) \) to the triangles of \( T_p \). (This cube has colors \( i < 0 < i^\rho \), where \( F_p \) is of the form \( e_p \times \xi_0 \) and the horizontal edge \( e_p \) has type \( ii^\rho \).) Then the middle vertical section \( S_p \) of \( C(p)^{\pi^\rho} \) corresponds to the rectangle \( F_p \) in \( \tilde{K} \), namely, \( \pi^\rho(S_p) = F_p \).

Now for each \( p = 1, \ldots, m \), replace in \( \tilde{N} \) the part \( D(p)^{\text{rear}} \) by \( S_p \). This gives an extraordinary weak membrane \( N \) which is bijective (under \( \pi^\rho \)) to the pseudo-combi \( \tilde{K} \). In the extended fragmentation of \( Q' \), take the sub-complex \( G \) lying between \( Z'^{\text{fr}} \) and \( N \), and form its symmetric complex \( G' \). The facts that \( N \) is self-symmetric and \( Z'^{\text{fr}} \) is symmetric to \( Z'^{\text{rear}} \) imply that \( G' \) is a subdivision of the part of \( Z' \) between \( N \) and \( Z'^{\text{rear}} \). Then the union of \( G \) and \( G' \) contains \( N \) (as a sub-complex) and constitutes the correct extended fragmentation of a symmetric cubillage \( Q \) (taking into account that for each section \( S(p) \), the fragments in \( G \) and \( G' \) sharing \( S(P) \) are symmetric to
each other and their union forms the octahedral fragment of a symmetric cube). This implies the theorem.

**Remark 7.** A similar result is valid for the strong separation, namely: for $n$ odd, any maximal symmetric $s$-collection $S \subseteq 2^{[n]}$ can be represented as the spectrum of a symmetric extraordinary strong membrane $N$ in the sparse fragmentation of some symmetric cubillage $Q$ on $Z(n, 3)$. Here the sparse fragmentation concerns transformations of merely self-symmetric cubes $C = (X | T)$, i.e., such that $|X| = m - 1$ (where $n = 2m + 1$) and $T = (i < 0 < i^\circ)$ for $i \in [-m.. - 1]$, under which $C$ is subdivided into two parts (symmetric to each other) sharing the vertical section (rectangle) $S(C)$ spanning the vertices $Xi, Xi^\circ, Xi0, X0i^\circ$. Accordingly, an extraordinary strong membrane differs from a usual strong membrane by allowing to use as facets such sections $S(C)$ (and the other facets are halves (vertical triangles) of facets of cubes, as usual). To obtain the assertion, we take the symmetric non-standard tiling $T$ with $V_T = S$ on $Z(n, 2)$ as described in Remark 4. In the construction of the desired $Q$ and $N$, each central hexagon $H$ in $T$ should be the projection (by $\pi_\rho$) of the union of three facets in the sparse fragmentation of a self-symmetric cube $C$ in $Q$ (of which one is the rectangular section $S(C)$). We leave details to the reader.

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