Numerical Evidence for Continuity of Mean Field and Finite Dimensional Spin Glasses

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We study numerically a disordered model that interpolates among the Sherrington-Kirkpatrick mean field model and the three dimensional Edwards-Anderson spin glass. We find that averages over the disorder of powers of the overlap and of the full \( P(q) \) are smooth, and do not show any discontinuity. Different lattice sizes are used to provide evidence for a smooth behavior of disorder averages in the thermodynamic limit. Quantities defined on a given realization of the disorder show a chaotic behavior. Our results support the validity of a Replica Symmetry Breaking description of finite dimensional models.

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The relation among the mean field spin glass model (SK) and its finite dimensional, realistic counterpart (EA), is a debated and interesting subject. If on one side the Parisi replica symmetry breaking (RSB) solution of the mean field theory is getting a stronger acceptance even on a rigorous ground, on the other side the relation of RSB theory with finite dimensional models is less clear.

Numerical simulations (see and references therein) seem to confirm the validity of the Parisi picture in 3 and 4 spatial dimensions, but the droplet model approach suggests a different behavior of finite dimensional models. Recent analytical, rigorous and heuristic work can be seen as hinting for potential troubles in the application of RSB to finite dimensional models, or as confirming the suitability of such a description.

If the droplet model description applies to finite dimensional spin glasses the mean field model (that we assume as described by the Parisi RSB Ansatz) has to be a singular case: at variance with the usual Ising model the case of \( D = \infty \) would be different in nature from the case of any finite dimensionality. All the exceptional features that appear in the description of the mean field model solution would disappear for any large but finite dimension \( D \). Typical equilibrium configurations of the SK and the EA models would have to be very different. On the contrary validity of the RSB ansatz in the finite dimensional case would imply that phenomena like the existence of a phase transition in non-zero magnetic field, and some form of ultrametric organization of the states appear also for finite dimensional models.

Here we analyze numerically this issue, and show that there are no discontinuities in the typical behavior of the system (clearly the behavior of a single system turns to be, as expected, very discontinuous, while average over the disorder, i.e. the quenched state, does not show any abrupt discontinuity). There are many contexts where this result can be relevant. For example Guerra has shown rigorously the validity of relations like

\[
E(q_{1,2}^2 q_{2,3}^2) = \frac{1}{2} E(q_{1,2}^4) + \frac{1}{2} E(q_{1,2}^2)^2, \tag{1}
\]

and

\[
E(q_{1,2}^2 q_{3,4}^2) = \frac{1}{3} E(q_{1,2}^4) + \frac{2}{3} E(q_{1,2}^2)^2, \tag{2}
\]

where \( q_{a,b} \) if the overlap \( \sum_i \sigma_i^a \sigma_i^b \) among replica \( a \) and replica \( b \) of the system, and \( E(\cdot) \) denotes the expectation value over the thermal noise and over the quenched disorder. The proof holds for mean field, and, under a continuity hypothesis, for finite dimensional models. Numerically one finds that in 3D relations like are satisfied at better than one part over thousand (with \( E(q^2) \) different from zero in the spin glass phase). This is one of the reasons for which checking the continuity of the limit of finite dimensionality is an important task.

Another relevant example where a weak perturbation is used to derive important relations among equilibrium observables and dynamic quantities that can be measured in experiments is reference. Here we will analyze a system where the perturbation does not need to be small: we want to understand the nature of the relation among the mean field model SK and the finite dimensional EA Ising spin glass. In our case starting from a mean field interaction involving with \( p > 2 \) spin would not give a smooth limit to a two spin interaction three dimensional model, since the mean field starting point would be a one step RSB \( P(q) \).

We have defined and studied a model that interpolates from mean field to 3D. The Hamiltonian of our model is a function of the parameter \( \epsilon \), which ranges from zero to one:

\[
H_\epsilon \equiv (1 - \epsilon) H_{3D} + \epsilon H_{SK}. \tag{3}
\]

All the couplings \( J_{i,j} \) are binary, and take with probability one half the values \( \pm 1 \). The three dimensional structure of the system is given by the term

\[
H_{3D} \equiv \sum_{\langle i,j \rangle} \sigma_i J_{i,j}^{(1)} \sigma_j, \tag{4}
\]
where the sum runs over first neighboring sites on a simple cubic lattice in three dimensions, with $N = L^3$ sites. The mean-field, Sherrington Kirkpatrick like contribution is given by the term

$$H_{\text{SK}} = \frac{1}{N} \sum_{i,j} \sigma_i J^{(2)}_{ij} \sigma_j ,$$

(5)

where the sum runs over all site couples of the lattice. The quenched couplings $\{J^{(1)}, J^{(2)}\}$ are independent. For $\epsilon = 0$ we recover the 3D EA spin glass, while for $\epsilon = 1$ we recover the SK mean field model.

We have focused our analysis on the probability distribution of the overlap, $P^\epsilon(q)$, as a function of $\epsilon$, averaged over the quenched disorder, and on the $P^\epsilon_j(q)$ for fixed realization of the disorder $\{J^{(1)}, J^{(2)}\}$. We have also analyzed in detail the second, fourth and sixth momentum $q^2$, $q^4$ and $q^6$.

As we will discuss our results clearly show that $P^\epsilon(q)$ is a smooth function of $\epsilon$ in all the range of $\epsilon$ ranging from zero to one, for, say $T \simeq 0.7T_c$. On the contrary for a given disorder sample the shape of $P^\epsilon_j(q)$ depends dramatically on $\epsilon$. The expectation values of the first even powers of the overlap all have a smooth dependence over $\epsilon$. Larger lattices show a behavior consistent with this picture, and we do not see any singular behavior developing. In the clear limits of a numerical simulation we can state are observing a smooth transition from the mean field to the finite dimensional model.

The Parallel Tempering Monte Carlo approach (for a review see for example [14]) has been crucial for thermalizing our systems down to $T \simeq 0.7T_c$. Since $T_c = 1$ for the mean field model and $T_c \simeq 1.11$ [15] is a reasonable estimate for the 3D system with binary couplings we have analyzed our data as a function of $\epsilon$ at fixed $T = 0.7$. Renormalizing the $T$ value as a function of $\epsilon$ could have been in principle a more exact procedure: we have decided not to do so not to add a further degree of freedom (the scaling of the temperature) that would have in itself potentially added a chaotic behavior, without adding much to the interpretation of the result. We are looking for smoothness of the observables (on the state averaged over the disorder) and not for precise extrapolations, so we can afford ignoring the small renormalization of $T_c$ (that we could only control approximately in any case).

All the data we will present in the following will be at $T = 0.7$.

This numerical simulation is expensive, since for all $\epsilon$ values we pay the price ($N^2$) of a mean field simulation (this is only the scaling of the number of operations needed, and does not include an estimate for the increment of the needed computer time due to critical slowing down, so that it has to be read as a lower bound to the increase of the computer time): the time taken from a lattice sweep scales like the volume squared instead than like the volume.

At $L = 8$ we have been able to analyze 150 disorder samples, each with 11 $\epsilon$ values (from 0 to 1.0 with steps of 0.1), while for $L = 12$ we have 11 disorder samples with the same $\epsilon$ values. As we have said this is not a small computational effort because of the factor volume added by the mean field interaction term and the factor eleven for the different $\epsilon$ values that multiply the total computer time. We have used a multispin coded program (for the long range part of the interaction). The simulations have taken a few years of a nowadays typical workstation.

The Parallel Tempering Monte Carlo updating scheme worked effectively at all our $\epsilon$ values for both $L = 8$ and $L = 12$. For $L = 8$ we have used 13 copies of the system, with temperatures starting with 0.7 and increasing with steps of 0.05 up to 1.3, while at $L = 12$ we have used 26 copies with the same minimum value of $T$, up to $T = 1.95$ with the same step. We ran 100.000 lattice sweeps (and tempering updates) for each disorder realization and $\epsilon$ value both at $L = 8$ and at $L = 12$. At all values of the parameters we have discarded the first half of the Monte Carlo sweeps for sake of thermalization. At each new $\epsilon$ value we restarted the system from a random configuration, even when running the same disorder sample (in order to avoid any bias toward continuity). All thermalization tests [14] were satisfactory: the $T$ swap acceptance ratio was included among 0.3 and 0.7, and each copy of the system had eventually spent time in all allowed $T$ values.

We have used the two different lattice sizes ($L = 8$ and $L = 12$) to get control over the finite size scaling of our results. It turns out that all the features we have revealed and we will discuss now are stable when increasing the lattice size, and make us confident we are observing an asymptotic behavior. The $L = 12$ results have a larger error than the $L = 8$ ones (since in this case the $N^2$ scaling of the computer time needed for the simulation has limited us to 11 samples), but they confirm the behavior we are observing at $L = 8$.

In figure (1) we show the average probability distribution of the overlap $q$, $P(q)$, for the $L = 8$ lattice, as a function of $\epsilon$ (and $q$), at $T = 0.7$. The $\epsilon$ axis goes from zero to one, the overlap axis from $-1$ to $1$. The behavior of the function $P^\epsilon(q)$ as a function of $\epsilon$ is very smooth. It is important to notice that the value of $P(q = 0)$ at $\epsilon = 0$ is largely different from zero (in a definitely significant way: this is very well known, see for example [14] and references therein). Continuity of $P^\epsilon(q)$ as a function of $\epsilon$ seems very clear from figure (1). For $L = 12$ we get a very similar picture (see also the value of $q^2(\epsilon)$ that we will show later on), and things stay as smooth as they are in figure (1). The non trivial probability distribution that one finds in the mean field approach survives smoothly down to the finite dimensional, realistic system.

We do not expect continuity for a given realization of the quenched disorder: the mean field picture would suggest that the typical equilibrium configurations are highly unstable under variations of $\epsilon$. Our results confirm this point of view in a very nice way. We show in figures (2) and (3) two $P^\epsilon_j(q)$ for two typical realizations of the quenched disorder. In both figures one sees that the
shape of \( P_\epsilon^J(q) \) varies completely with \( \epsilon \). In figure (2), the mean field model is in a configuration that carries all the weight close to \( q = 0 \). Already at \( \epsilon = 0.9 \) the system is completely different, and \( P_\epsilon^J(q) \) has two maxima. Notice that the level of symmetry of these \( P_\epsilon^J(q) \) is a good check of the level of thermalization of our Monte Carlo tempered runs. The pure 3D system has in this case a broad distribution, with two clear, small maxima, and weight also at \( q \simeq 0 \).

In the sample of figure (3) instead the \( \epsilon = 1 \) mean field model shows two broad maxima. Again the evolution is chaotic, down to the \( \epsilon = 0 \) 3D model where there are four sharp maxima. We want to notice that, if any, the structure of the individual \( P_\epsilon^J(q) \) becomes more complex when approaching the pure 3D model: structures with many sharp peaks appear on the larger lattices more in the 3D model than in the mean field one. One reason for that is probably connected to the fact that finite size effects are larger in the mean field model. Recently a detailed discussion of models that show a “modified droplet model” structure has been reported in [16]. This is very interesting since provides a rigorous construction of models with a complex structure of states, but looks different from what we find here. \( P_\epsilon^J(q) \) like the one we observe at \( \epsilon = 0 \) in figure (3) would not appear in a “modified droplet model” structure: our results hint that we are dealing with a complete RSB like structure of states.

In figure (4) we show \( E(q^2) \) as a function of \( \epsilon \) for \( L = 8 \) and \( L = 12 \). The two lines are the best fit of the data to a third order polynomial. The most part of this (smooth) dependence over \( \epsilon \) can be related to the renormalization of \( T \). The \( L = 12 \) data show an even smaller dependence over \( \epsilon \) than the \( L = 8 \) data. We have also analyzed the fourth and the sixth momentum of the overlap \( q \), that have the same smooth behavior.

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FIG. 1. $P^*(q)$ as a function of $\epsilon$ (going from 0 to 1) and $q$ (from $-1$ to $+1$). $L = 8$, $T = 0.7$, disorder average taken over 150 realizations.

FIG. 2. As in figure [1], but $P_J(q)$ for one realization of the quenched disorder.

FIG. 3. As in figure [1], but $P_J(q)$ for a second realization of the quenched disorder.

FIG. 4. $E(q^2)$ as a function of $\epsilon$ for $L = 8$ and $L = 12$. 

$P(q)$

$PJ(q)$

$E(q^2)$