Large deviation principles for renewal-reward processes

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Abstract

We establish a sharp large deviation principle for renewal-reward processes, supposing that each renewal involves a broad-sense reward taking values in a real separable Banach space. In fact, we demonstrate a weak large deviation principle without assuming any exponential moment condition on the law of waiting times and rewards by resorting to a sharp version of Cramér’s theorem. We also exhibit sufficient conditions for exponential tightness of renewal-reward processes, which leads to a full large deviation principle.

Keywords: Large deviations; Cramér’s theorem; Renewal processes; Renewal-reward processes; Banach space valued random variables

Mathematics Subject Classification 2020: 60F10; 60K05; 60K35

1 Main results

Renewal models are widespread tools of probability that find application in Queueing Theory [1], Insurance [2], Finance [3], and Statistical Physics [4] among others. A renewal model describes some event that occurs at the renewal times $T_1, T_2, \ldots$ involving the rewards $X_1, X_2, \ldots$ respectively. If $S_1, S_2, \ldots$ denote the waiting times for a new occurrence of the event, then the renewal time $T_i$ can be expressed for each $i \geq 1$ in terms of the waiting times as $T_i = S_1 + \cdots + S_i$. Through this paper we assume that the waiting time and reward pairs $(S_1, X_1), (S_2, X_2), \ldots$ form an independent and identically distributed sequence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the waiting times taking positive real values and the rewards taking values in a real separable Banach space $\mathcal{X}$ equipped with the Borel $\sigma$-field $\mathcal{B}(\mathcal{X})$. Any dependence between $S_i$ and $X_i$ is allowed and we can suppose without restriction that $\lim_{i \to \infty} T_i(\omega) = +\infty$ for all $\omega \in \Omega$. The cumulative reward by the time $t \geq 0$ is the random variable $W_t := \sum_{i \geq 1} X_i 1_{\{T_i \leq t\}}$, which is measurable because $\mathcal{X}$ is separable [5]. The stochastic process $t \mapsto W_t$ is the so-called renewal-reward process or compound renewal process, which plays an important role in applications [1–4]. The strong law of large numbers holds for a renewal-reward process under the optimal hypotheses $\mathbb{E}[S_1] < +\infty$ and $\mathbb{E}[\|X_1\|] < +\infty$, $\mathbb{E}$ being expectation with respect to the law $\mathbb{P}$, and can be proved by combining standard arguments of renewal theory [4] with the classical strong law of large numbers of Kolmogorov in separable Banach spaces [5]. This paper aims to characterize the fluctuations of the cumulative reward $W_t$ as $t$ goes to infinity by means of large deviation bounds.

1.1 Large deviation bounds

The Cramér’s rate function of waiting time and reward pairs is the function $J$ that maps each $(s, w) \in \mathbb{R} \times \mathcal{X}$ in the extended real number

$$J(s, w) := \sup_{(\zeta, \varphi) \in \mathbb{R} \times \mathcal{X}^*} \left\{ s\zeta + \varphi(w) - \ln \mathbb{E}[e^{s\zeta + \varphi(X_1)}] \right\}.$$
Hereafter $\mathcal{X}^*$ denotes the topological dual of $\mathcal{X}$, which is understood as a Banach space with the norm induced by $\|\cdot\|$. In this paper a special role is played by the function $\inf_{\gamma>0}\{\gamma J((\cdot/\gamma,\cdot/\gamma))\}$, whose lower-semicontinuous regularization $\Upsilon$ associates every $(\beta, w) \in \mathbb{R} \times \mathcal{X}$ with

$$\Upsilon(\beta, w) := \lim_{\delta \downarrow 0} \inf_{s \in (\beta, \beta+\delta)} \inf_{v \in B_{w, \delta} \cap X} \{\gamma J(s/\gamma, v/\gamma)\},$$

$B_{w, \delta} := \{v \in X : \|v - w\| < \delta\}$ being the open ball of center $w$ and radius $\delta$. Setting $\ell_1 := -\liminf_{t \uparrow +\infty} (1/s) \ln P[S_t > s]$ and $\ell_\infty := -\limsup_{t \uparrow +\infty} (1/s) \ln P[S_t > s]$ and observing that $0 \leq \ell_\infty \leq \ell_1 \leq +\infty$, we make use of $\Upsilon$ to build two rate functions $I_i$ and $I_s$ on $\mathcal{X}$ according to the formulas

$$I_i := \begin{cases} \inf_{\beta \in [0,1]} \{\Upsilon(\beta, \cdot) + (1-\beta)\ell_1\} & \text{if } \ell_1 < +\infty, \\ \Upsilon(1, \cdot) & \text{if } \ell_1 = +\infty \end{cases}$$

and

$$I_s := \begin{cases} \inf_{\beta \in [0,1]} \{\Upsilon(\beta, \cdot) + (1-\beta)\ell_s\} & \text{if } \ell_s < +\infty, \\ \Upsilon(1, \cdot) & \text{if } \ell_s = +\infty. \end{cases}$$

The rate functions $I_i$ and $I_s$ enter into a lower large deviation bound and an upper large deviation bound, respectively, as stated by the following theorem which collects the main results of the paper.

**Theorem 1.1.** The following conclusions hold:

(a) the rate functions $I_i$ and $I_s$ are lower semicontinuous and convex;

(b) if $G \subseteq \mathcal{X}$ is open, then

$$\liminf_{t \uparrow +\infty} \frac{1}{t} \ln P \left[ \frac{W_t}{t} \in G \right] \geq -\inf_{w \in G} \{I_i(w)\};$$

(c) if $F \subseteq \mathcal{X}$ is compact, then

$$\limsup_{t \uparrow +\infty} \frac{1}{t} \ln P \left[ \frac{W_t}{t} \in F \right] \leq -\inf_{w \in F} \{I_s(w)\};$$

(d) if $F \in \mathcal{B}(\mathcal{X})$ is open convex, closed convex, or just convex when $\mathcal{X}$ is finite-dimensional, then the bound of part (c) is valid whenever $\ell_s < +\infty$ or $I_s(0) < +\infty$;

(e) if $\mathcal{X}$ is finite-dimensional and $\mathbb{E}[e^{\zeta S_t + \sigma \|X_t\|}] < +\infty$ for some numbers $\zeta \leq 0$ and $\sigma > 0$, then $I_i$ and $I_s$ have compact level sets and the bound of part (c) is valid for any closed set $F$;

(f) if $\mathcal{X}$ is infinite-dimensional and $\mathbb{E}[e^{\sigma S_t + \sigma \|X_t\|}] < +\infty$ for all $\sigma > 0$, then $I_i = I_s = \Upsilon(1, \cdot)\Upsilon(1, \cdot)$ has compact level sets, and the bound of part (c) is valid for any closed set $F$.

Theorem 1.1 is proved in section 2. When $I_i = I_s$ the theorem establishes, through the lower large deviation bound for open sets of part (b) and the upper large deviation bound for compact sets of part (c), a *weak large deviation principle with rate function* $I_i = I_s$ for the renewal-reward process $t \mapsto W_t$. We refer to [R] for the language of large deviation theory. Part (d) states that the upper large deviation bound also holds for open and closed convex sets provided that $\ell_s < +\infty$ or $I_s(0) < +\infty$. It fails in general when $\ell_s = +\infty$ and $I_s(0) = +\infty$, as we shall show in section 2 by means of two examples. We stress that no assumption on the law of waiting time and reward pairs is made to deduce parts (a), (b), (c), and (d) of theorem 1.1. Some assumption is instead necessary for exponential tightness of the distribution of the scaled cumulative reward $W_t/t$, which leads to a full large deviation principle where the large deviation upper bound is valid for all closed sets, and not only for those that are compact. If $I_i = I_s$ and $\mathcal{X}$ is finite-dimensional, then part (e) establishes a
full large deviation principle with good rate function \( I_1 = I_s\) under the exponential moment condition \( \mathbb{E}[e^{\sigma S_1 + \sigma \|X_1\|}] < +\infty \) for some numbers \( \zeta \leq 0 \) and \( \sigma > 0 \). We recall that a rate function is “good” when it has compact level sets. Part (f) states that the same is true when \( X \) is infinite-dimensional and \( \mathbb{E}[e^{\sigma S_1 + \sigma \|X_1\|}] < +\infty \) for all \( \sigma > 0 \). Obviously, we have \( I_1 = I_s\) if \( \ell_1 = \ell_s\) as expected in most applications. We have \( I_1 = I_s\) even if \( \ell_1 > \ell_s\) but rewards are dominated by waiting times according to the following proposition, whose proof is reported in appendix A.

**Proposition 1.1.** Assume that there exists a positive real function \( f \) on \([0, +\infty)\) such that \( \lim_{s \uparrow +\infty} f(s)/s = 0 \) and \( \|X_1\| \leq f(S_1) \) with full probability. Then \( I_1 = I_s = \Upsilon(1, \cdot)\).

### 1.2 Discussion

Large deviation principles (LDPs) for renewal-reward processes have been investigated by many authors over the past decades. Their attention has been focused mostly on rewards taking real values and an almost omnipresent hypothesis of previous works is the Cramér condition on the law of waiting time and reward pairs: \( \mathbb{E}[e^{\sigma S_1 + \sigma \|X_1\|}] < +\infty \) for some number \( \sigma > 0 \).

The simplest example of renewal-reward process has unit rewards and corresponds to the counting renewal process \( t \mapsto N_t := \sum_{i \geq 1} \mathbb{1}_{\{T_i \leq t\}} \). Glynn and Whitt [7] investigated the connection between LDPs of the inverse processes \( t \mapsto N_t \) and \( i \mapsto T_i \), providing a full LDP for \( N_t \) under the Cramér condition. This condition was later relaxed by Duffield and Whitt [11]. Jiang [13] studied the large deviations of the extended counting renewal process \( t \mapsto \sum_{i \geq 1} \mathbb{1}_{\{T_i \leq e^t\}} \) with \( \alpha \in [0, 1) \) under the Cramér condition. Glynn and Whitt [7] and Duffield and Whitt [11], together with Puhalskii and Whitt [10], also investigated the connection between sample-path LDPs of the processes \( t \mapsto N_t \) and \( i \mapsto T_i \) under the Cramér condition.

Starting from sample-path LDPs of inverse and compound processes, Duffy and Rodgers-Lee [12] sketched a full LDP for renewal-reward processes with real rewards by means of the contraction principle under the stringent exponential moment condition \( \mathbb{E}[e^{\sigma S_1 + \sigma \|X_1\|}] < +\infty \) for all \( \sigma > 0 \). Some full LDPs for real renewal-reward processes were later proposed by Macci [8,9] under existence and essentially smoothness of the scaled cumulant generating function, which allow for an application of the Gärtner-Ellis theorem [6]. Essentially smoothness of the scaled cumulant generating function has been recently relaxed by Borovkov and Mogulskii [14,15], which used the Cramér's theorem [6] to establish a full LDP under the Cramér condition. Under this condition, they [16,18] have also obtained sample-path LDPs for real renewal-reward processes.

A different approach based on empirical measures has been considered by Lefevere, Mariani, and Zambotti [19], which have investigated large deviations for the empirical measures of forward and backward recurrence times associated with a renewal process, and have then derived by contraction a full LDP for renewal-rewards processes with rewards determined by the waiting times: \( X_i := f(S_i) \) for each \( i \) with a bounded and continuous real function \( f \). Later, Mariani and Zambotti [20] have developed a renewal version of Sanov’s theorem by studying the empirical law of rewards that take values in a generic Polish space under the hypothesis \( \mathbb{E}[e^{\sigma S_1}] < +\infty \) for all \( \sigma > 0 \). By appealing to the contraction principle, this result could give a full LDP for a renewal-reward process with rewards valued in a separable Banach space provided that the exponential moment condition \( \mathbb{E}[e^{\sigma \|X_1\|}] < +\infty \) is satisfied for all \( \sigma > 0 \) as discussed by Schied [21].

These works leave open the question of whether some LDPs free from exponential moment conditions can be established for renewal-reward processes, in the wake of the sharp version of Cramér’s theorem demonstrated by Bahadur and Zabell [22]. In a recent paper, the author [23] has dropped the Cramér condition in the discrete-time framework, whereby waiting times have a lattice distribution, by establishing a weak LDP for cumulative rewards free from hypotheses. The discrete-time framework is special because allows a super-multiplicativity property of the probability that \( W_t/t \) belongs to a convex set to emerge by conditioning on the event that the time \( t \) is a renewal time. This super-multiplicativity property was the key to get at sharp LDPs in [23]. Unfortunately, the same strategy does
not extend to waiting times with non-lattice distribution since conditioning on the event that a certain time is a renewal time is not a meaningful procedure in this case. The present paper overcomes the difficulty to deal with general waiting times by making a better use of Cramér’s theorem than Borovkov and Mogulskii [14,15]. In fact, starting from the Cramér’s theory for waiting time and reward pairs, here we establish a weak LDP for the renewal-reward process \( t \mapsto W_t \) with no restriction on waiting times and without assuming that the Cramér condition is satisfied. Moreover, when finite-dimensional rewards are considered, we provide a full LDP under the exponential moment condition \( \mathbb{E}[e^{\zeta S_t + \sigma \|X_t\|}] < +\infty \) for some numbers \( \zeta \leq 0 \) and \( \sigma > 0 \), which is weaker than the Cramér condition \( \mathbb{E}[e^{\sigma S_t + \|X_t\|}] < +\infty \) for some \( \sigma > 0 \). For instance, rewards that define macroscopic observables in applications to Statistical Physics [24] are of the order of magnitude of waiting times and always satisfy our weak exponential moment condition, whereas in general they do not fulfill the Cramér condition. But after all is said and done, a super-multiplicativity argument is still the key, as it underlies the sharp version of Cramér’s theorem we have exploited to reach our results.

To conclude, we point out that, at variance with Borovkov and Mogulskii, we propose optimal lower and upper large deviation bounds with possibly different rate functions in order to even address situations where the tail of the waiting time distribution is very oscillating. For instance, a physical renewal model giving rise to two possibly different rate functions has been found by Lefevere, Mariani, and Zambotti [25,26] in the description of a free particle interacting with a heat bath.

2 Proof of theorem 1.1

The proof of theorem 1.1 is organized as follows. In section 2.1 we discuss some properties of \( \Upsilon \) and prove lower semicontinuity and convexity of the rate functions, verifying part (a) of theorem 1.1. Section 2.2 demonstrates the lower large deviation bound for open sets, thus verifying part (b) of theorem 1.1. The upper large deviation bound for convex sets, that is, of \( \Upsilon \) and prove lower semicontinuity and convexity of the rate functions, verifying part (a) of theorem 1.1. Section 2.3 proves part (c) in section 2.4. Finally, the proof of parts (e) and (f) of the theorem 1.1 are reported in section 2.5. The elements of Cramér’s theory for waiting time and reward pairs on which the proof is based are collected in appendix B.

2.1 Rate functions

The function \( \Upsilon \) satisfies the following properties, which will be used in the sequel.

**Lemma 2.1.** The following conclusions hold:

(i) \( \Upsilon \) is lower semicontinuous and convex;

(ii) \( \Upsilon(\beta, w) \geq 0 \) and \( \Upsilon(a\beta, aw) = a\Upsilon(\beta, w) \) for all \( \beta \in \mathbb{R} \), \( w \in \mathcal{X} \), and \( a > 0 \);

(iii) \( \Upsilon(0, 0) = 0 \) and \( \Upsilon(\beta, w) = +\infty \) for all \( \beta < 0 \) and \( w \in \mathcal{X} \);

(iv) for every \( \beta > 0 \) and \( w \in \mathcal{X} \)

\[
\Upsilon(\beta, w) = \lim_{\delta \downarrow 0} \inf_{v \in B_{w, \beta}} \inf_{\gamma > 0} \{\gamma J(\beta/\gamma, v/\gamma)\}.
\]

**Proof.** As \( \Upsilon \) is lower semicontinuous by construction, in order to prove part (i) it suffices to verify convexity. We show that, for every given integer \( k \geq 1 \), the function inf\(_{x \in [1/k,k]} \{\gamma J(x/\gamma, /\gamma)\} \) over \( \mathbb{R} \times \mathcal{X} \) is the convex conjugate of a certain function \( F_k \) on \( \mathbb{R} \times \mathcal{X}^* \). This way, inf\(_{x > 0} \{\gamma J(x/\gamma, /\gamma)\} = \lim_{\gamma \uparrow \infty} \inf_{x \in [1/k,k]} \{\gamma J(x/\gamma, /\gamma)\} \) is convex, and so is \( \Upsilon \). Pick \( s \in \mathbb{R} \) and \( w \in \mathcal{X} \) and denote by \( \mathcal{D} \) the set \( \{(\zeta, \varphi) \in \mathbb{R} \times \mathcal{X}^* : \mathbb{E}[e^{sS_t + \varphi(w)}] < +\infty\} \). The function that associates \( (\zeta, \varphi) \in \mathcal{D} \) with \( \ln \mathbb{E}[e^{sS_t + \varphi(w)}] \) is lower semicontinuous by Fatou’s lemma and convex, so that the real function that maps \( (\gamma, \zeta, \varphi) \in [1/k,k] \times \mathcal{D} \) in
\( \varphi(w) - s\zeta - \gamma \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}] \) is concave and upper semicontinuous with respect to \((\zeta, \varphi)\) for each fixed \( \gamma \in [1/k, k] \) and convex and continuous with respect to \( \gamma \) for each fixed pair \((\zeta, \varphi) \in D\). Then, the compactness of the interval \([1/k, k]\) allows an application of Sion’s minimax theorem to get

\[
\inf_{\gamma \in [1/k, k]} \left\{ \gamma J(s/\gamma, w/\gamma) \right\} = \inf_{\gamma \in [1/k, k]} \sup_{(\zeta, \varphi) \in D} \left\{ s\zeta + \varphi(w) - \gamma \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}]\right\}
\]

\[
= \sup_{(\zeta, \varphi) \in D} \inf_{\gamma \in [1/k, k]} \left\{ s\zeta + \varphi(w) - \gamma \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}]\right\}
\]

\[
= \sup_{(\zeta, \varphi) \in \mathcal{X} \times \mathcal{X}^*} \left\{ s\zeta + \varphi(w) - F_k(\zeta, \varphi) \right\}
\]

with

\[
F_k(\zeta, \varphi) := \max \left\{ (1/k) \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}], k \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}]\right\}
\]

for all \( \zeta \in \mathcal{R} \) and \( \varphi \in \mathcal{X}^* \).

Let us move to part (ii). We have \( J(s, w) \geq 0 \) for all \( s \in \mathbb{R} \) and \( w \in \mathcal{X} \) because \( s\zeta + \varphi(w) - \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}] = 0 \) when \( \zeta = 0 \) and \( \varphi = 0 \). Then, \( Y(\beta, w) \geq 0 \) for all \( \beta \) and \( w \).

The property \( Y(a\beta, aw) = aY(\beta, w) \) for all \( \beta \in \mathbb{R} \), \( w \in \mathcal{X} \), and \( a > 0 \) is immediate.

Regarding part (iii), let us observe that \( \lim_{\zeta \rightarrow -\infty} \{ s\zeta + \varphi(w) - \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}]\} = +\infty \) for each \( s \leq 0 \) and \( \varphi \in \mathcal{X}^* \). This way, \( J(s, w) = +\infty \) for each \( s < 0 \) and \( w \in \mathcal{X} \), which yields \( Y(\beta, w) = +\infty \) for every \( \beta < 0 \) and \( w \in \mathcal{X} \). As far as the equality \( Y(0, 0) = 0 \) is concerned, in the light of part (ii) it remains to demonstrate that \( Y(0, 0) \leq 0 \). The function that maps \((\zeta, \varphi) \in \mathcal{R} \times \mathcal{X}^* \) with \( \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}] \) is proper convex, so that there exist \( s_o \in \mathbb{R} \), \( w_o \in \mathcal{X} \), and a constant \( c \) such that \( \ln \mathbb{E}[e^{\xi_{S_t} + \varphi(X_t)}] \geq s_o\zeta + \varphi(w_o) - c \) for all \( \zeta \) and \( \varphi \) (see [27], theorem 2.2.6). It follows that \( J(s_o, w_o) \leq c < +\infty \). Let \( a \) be a small positive number such that \( a\zeta \in (-1, 1) \) and \( aw_o \in B_{0.1} \). Then, for all \( \delta > 0 \) we find the bound

\[
\inf_{s \in (-\delta, \delta)} \inf_{w \in B_{0.1}} \inf_{\gamma > 0} \gamma J(s/\gamma, v/\gamma) \leq \inf_{\gamma > 0} \gamma J(a\delta s_o/\gamma, a\delta w_o/\gamma) \leq a\delta J(s_o, w_o),
\]

which gives \( Y(0, 0) \leq 0 \) once \( \delta \) is sent to 0.

To conclude, let us prove part (iv). Fix \( \beta > 0 \) and \( w \in \mathcal{X} \). It is clear that

\[
Y(\beta, w) = \lim_{\delta \downarrow 0} \inf_{s \in (\beta - \delta, \beta + \delta)} \inf_{v \in B_{w/(\beta - \delta)}} \gamma J(\beta/\gamma, v/\gamma).
\]

Let us demonstrate the opposite bound. For all \( \delta \in (0, \beta) \) and \( s \in (\beta - \delta, \beta + \delta) \) we have \( s > 0 \) and \( B_{w/\beta} B_{s/\beta} \subseteq B_{w/\beta} B_{s/\beta} \). The latter is due to the fact that if \( s \in (\beta - \delta, \beta + \delta) \) and \( v \in B_{w/\beta} B_{s/\beta} \), then

\[
\|v - w\| \leq \|v - \beta w/\delta\| + \|\beta w/\delta - w\| < \beta\delta/\delta + \|\beta/\delta - 1\|\|w\|
\]

\[
< \beta\delta/(\beta - \delta) + \delta\|w\|/\beta/(\beta - \delta) = (\beta + \|w\|)\|v - w\|/(\beta - \delta).
\]

Thus, recalling that \( J \) is non-negative, for every \( \delta \in (0, \beta) \) we can write

\[
\inf_{s \in (\beta - \delta, \beta + \delta)} \inf_{v \in B_{w/\beta}} \gamma J(\beta/\gamma, v/\gamma) = \inf_{s \in (\beta - \delta, \beta + \delta)} \inf_{v \in B_{w/\beta}} \gamma J(s/\gamma, v/\gamma) \geq (1 - \delta/\beta) \inf_{v \in B_{w/\beta}} \gamma J(\beta/\gamma, v/\gamma).
\]

This inequality shows that

\[
Y(\beta, w) \geq \lim_{\delta \downarrow 0} \inf_{s \in (\beta - \delta, \beta + \delta)} \inf_{v \in B_{w/\beta}} \gamma J(\beta/\gamma, v/\gamma).
\]

We are now in the position to prove part (a) of theorem [11].

**Proposition 2.1.** The rate functions \( I_1 \) and \( I_\delta \) are lower semicontinuous and convex.
Proof. We address the rate function $I$. The same arguments apply to $I_\ell$. If $\ell = +\infty$, then $I_\ell = Y(1, \cdot)$ and lower semicontinuity and convexity of $I_\ell$ immediately follow from part (i) of lemma 2.1. Assume that $\ell_1 < +\infty$. In order to demonstrate lower semicontinuity of $I_\ell$, let us show that the set $F := \{ w \in X : I_\ell(w) \leq \lambda \}$ is closed for any given real number $\lambda$. Let $\{w_k\}_{k \geq 1}$ be a sequence in $F$ converging to a point $w$. We claim that $w \in F$. In fact, by the lower semicontinuity of $Y$ and the compactness of $[0,1]$, for each $k \geq 1$ there exists $\beta_k \in [0,1]$ such that $I_\ell(w_k) = Y(\beta_k, w_k) + (1 - \beta_k)\ell_1$. The compactness of $[0,1]$ also entails that there exists a subsequence $\{\beta_{k_j}\}_{j \geq 1}$ that converges to some number $\beta_0 \in [0,1]$. We have $\lambda \geq I_\ell(w_k_j) = Y(\beta_{k_j}, w_k_j) + (1 - \beta_{k_j})\ell_1$ for all $j \geq 1$, which gives $\lambda \geq Y(\beta_0, w) + (1 - \beta_0)\ell_1 \geq I_\ell(w)$ once $j$ is sent to infinity. Thus, $w \in F$.

As far as convexity of $I_\ell$ is concerned, given $w_1 \in X$ and $w_2 \in X$, let $\beta_1 \in [0,1]$ and $\beta_2 \in [0,1]$ be such that $I_\ell(w_1) = Y(\beta_1, w_1) + (1 - \beta_1)\ell_1$ and $I_\ell(w_2) = Y(\beta_2, w_2) + (1 - \beta_2)\ell_1$. We recall that the existence of $\beta_1$ and $\beta_2$ is guaranteed by the lower semicontinuity of $Y$. This way, if $a_1 \geq 0$ and $a_2 \geq 0$ are two numbers such that $a_1 + a_2 = 1$, then convexity of $Y$ shows that

$$ I_\ell(a_1w_1 + a_2w_2) \leq a_1Y(\beta_1, w_1) + a_2Y(\beta_2, w_2) + (1 - a_1\beta_1 - a_2\beta_2)\ell_1 $$

$$ = a_1I_\ell(w_1) + a_2I_\ell(w_2). $$

2.2 The lower large deviation bound for open sets

The proof of part (b) of theorem 1.1 relies on the following lower bound: for each set $A \in \mathcal{B}(X)$ and integers $1 \leq p < q$

$$ \mathbb{P} \left[ \frac{W_t}{t} \in A \right] \geq \mathbb{P} \left[ \frac{W_t}{t} \in A, T_p \leq t < T_q \right] $$

$$ = \sum_{n=p}^{q-1} \mathbb{P} \left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in A, T_n \leq t < T_{n+1} \right]. \quad (2.1) $$

This lower bound gives the forthcoming lemma, which applies for both $\ell_1 < +\infty$ and $\ell_1 = +\infty$ and demonstrates part (b) of theorem 1.1 directly when $\ell_1 = +\infty$.

**Lemma 2.2.** For every $G \subseteq X$ open and $w \in G$

$$ \liminf_{t \to +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq -Y(1, w). $$

**Proof.** Pick an open set $G$ in $X$. We shall prove that for each point $w \in G$ and real number $\gamma > 0$

$$ \liminf_{t \to +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq -\gamma J(1/\gamma, w/\gamma). \quad (2.2) $$

This bound yields the lemma by optimizing over $\gamma$ and by invoking part (iv) of lemma 2.1.

Fix an arbitrary point $w \in G$ and an arbitrary real number $\gamma > 0$. As $G$ is open, there exists $\delta > 0$ such that $B_{w, 2\delta} \subseteq G$. Since $\mathbb{P}[S_1 > 0] = 1$ there exist a small number $m > 0$ and a large number $M > 0$ with the properties $\mathbb{P}[S_1 \geq m, \|X_1\| \leq M] \geq 1/2$. Let $\epsilon \in (0, 1)$ be such that $(1 + 2M/m + \|w\|)\epsilon < \delta$ and let $t_o > 0$ be such that $\gamma(1 - \epsilon) t_o \geq 1$, $m \leq 2 \epsilon t_o$, $\|w\| < \delta \gamma t_o$, and $\epsilon^2 t_o \geq m + 1/\gamma$. Set $p_1 := \gamma(1 - \epsilon) t_o$ and $q_1 := p_1 + [2 \epsilon t_0/m]$. For $t > t_o$ we have $p_1 \geq 1$ as $\gamma(1 - \epsilon) t > \gamma(1 - \epsilon) t_o \geq 1$ and $p_1 < q_1$ as $2 \epsilon t/m > 2 \epsilon t_0/m \geq 1$. For brevity, we denote by $M_t$ the probability measure that maps a set $A \in \mathcal{B}(X)$ in

$$ M_t[A] := \mathbb{P} \left[ A \left| \min_{p_r < t \leq q_r} \{S_r\} \geq m, \max_{p_r < t \leq q_r} \{\|X_r\|\} \leq M \right. \right]. $$
and we observe that
\[
\mathbb{P}[A] \geq M_t[A] \cdot \mathbb{P} \left[ \min_{p_t < t \leq q_t} \{ S_t \} \geq m, \max_{p_s < t \leq q_s} \{ \|X_t\| \} \leq M \right] \\
= M_t[A] \cdot \mathbb{P} \left[ S_t > m, \|X_t\| \leq M \right] \geq \frac{1}{4e \times m} M_t[A].
\]

Bound (2.1) gives for any \( t > t_o \)
\[
\mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq \frac{q_t - 1}{M} \sum_{n=p_t}^{q_t-1} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^{n} X_t \in B_{w,2\delta}, T_n \leq t < T_{n+1} \right] \\
\geq \frac{1}{4e \times m} M_t \frac{q_t - 1}{M} \sum_{n=p_t}^{q_t-1} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^{n} X_t \in B_{w,2\delta}, T_n \leq t < T_{n+1} \right].
\]

We now notice that the condition \( (1/p_t) \sum_{i=1}^{p_t} X_t \in B_{w,\gamma,\epsilon/\gamma} \) implies \( (1/t) \sum_{i=1}^{n} X_t \in B_{w,2\delta} \) for each \( n \in [p_t, q_t] \) when \( t > t_o \) and \( \max_{p_s < t \leq q_t} \{ \|X_t\| \} \leq M \). In fact, recalling that \( (1+2M/m + \|w\|) \epsilon < \delta \) and that \( \|w\| < \delta t_o \), for \( t > t_o \) we find
\[
\left\| \frac{1}{t} \sum_{i=1}^{n} X_t - w \right\| = \left\| \frac{p_t}{t} \left( \frac{1}{p_t} \sum_{i=1}^{p_t} X_t - \frac{w}{\gamma} \right) + \frac{1}{t} \sum_{i=p_t+1}^{n} X_t + \left( \frac{p_t}{\gamma t} - 1 \right) \left( \frac{w}{\gamma} \right) \right\| \\
< \frac{p_t \epsilon}{\gamma t} + \frac{M(n-p_t)}{t} + \left( \frac{p_t}{\gamma t} - 1 \right) \|w\| \leq (1+2M/m + \|w\|) \epsilon + \frac{\|w\|}{\gamma t_o} < 2\delta.
\]

This argument yields for \( t > t_o \)
\[
\mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq \frac{1}{4e \times m} \sum_{n=p_t}^{q_t-1} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^{p_t} X_t \in B_{w,\gamma,\epsilon/\gamma}, T_n \leq t < T_{n+1} \right] \\
= \frac{1}{4e \times m} M_t \left[ \frac{1}{t} \sum_{i=1}^{p_t} X_t \in B_{w,\gamma,\epsilon/\gamma}, T_{p_t} \leq t < T_{q_t} \right].
\]

The condition \( T_{p_t} < (1/\gamma + \epsilon/\gamma)p_t \) implies \( T_{p_t} < t \). Moreover, under the constraints \( t > t_o \) and \( \min_{p_t < t \leq q_t} \{ S_t \} \geq m \), the condition \( T_{p_t} > (1/\gamma - \epsilon/\gamma)p_t \) entails \( t < T_{q_t} \). Indeed, since \( \epsilon t_o \geq m + 1/\gamma \) by construction we have
\[
T_{q_t} - T_{p_t} = \sum_{i=p_t+1}^{q_t} S_i > (1/\gamma - \epsilon/\gamma)p_t + (q_t - p_t)m \\
> (1/\gamma - \epsilon/\gamma)(1 - \epsilon) / 2\epsilon t/m - 1/m \\
= t + \epsilon t/m - (1 - \epsilon/\gamma) / \epsilon > t + \epsilon^2 t_o - m - 1/\gamma \geq t.
\]

It follows that for \( t > t_o \)
\[
\mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq \frac{1}{4e \times m} M_t \left[ \frac{1}{t} \sum_{i=1}^{p_t} X_t \in B_{w,1/\gamma,\epsilon/\gamma}, (1/\gamma - \epsilon/\gamma)p_t < T_{p_t} < (1/\gamma + \epsilon/\gamma)p_t \right] \\
= \frac{1}{4e \times m} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^{p_t} (S_i, X_t) \in (1/\gamma - \epsilon/\gamma, 1/\gamma + \epsilon/\gamma) \times B_{w,1/\gamma,\epsilon/\gamma} \right],
\]

where the last equality is due to the fact that \( \sum_{i=1}^{p_t} (S_i, X_t) \) is independent of \( \min_{p_s < t \leq q_t} \{ S_t \} \) and \( \max_{p_s < t \leq q_t} \{ \|X_t\| \} \). At this point, part (ii) of proposition (1.3) allows us to conclude that
\[
\lim_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq -\gamma (1 - \epsilon) J(1/\gamma, w/\gamma) - \frac{\epsilon}{m} \ln 4.
\]

We get (2.2) from here since \( \epsilon \in (0,1) \) is any number smaller than \( \delta/(1+2M/m + \|w\|) \).
The next lemma proves part (b) of theorem 1.1 when \( \ell_i < +\infty \).

**Lemma 2.3.** Assume that \( \ell_i < +\infty \). For each \( G \subseteq X \) open, \( w \in G \), and \( \beta \in [0, 1] \)

\[
\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P}\left[ \frac{W_t}{t} \in G \right] \geq -\Upsilon(\beta, w) - (1 - \beta)\ell_i.
\]

**Proof.** The instance \( \beta = 1 \) is solved by lemma 2.2, so that we must tackle the case \( \beta < 1 \). Given an open set \( G \) in \( X \), we prove that for each \( w \in G \) and real numbers \( \gamma > 0 \) and \( s < 1 \)

\[
\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P}\left[ \frac{W_t}{t} \in G \right] \geq -\gamma J(s/\gamma, w/\gamma) - (1 - s)\ell_i. \tag{2.3}
\]

The lemma follows from here by recalling the definition of \( \Upsilon \) and by optimizing over \( \gamma \), \( s \), and \( w \).

Fix \( w \in G \), \( \gamma > 0 \), and \( s < 1 \). If \( s < 0 \), then there is nothing to prove because \( J(s/\gamma, w/\gamma) = +\infty \) as we have seen in the proof of part (iii) of lemma 2.1. Assume that \( s \in [0, 1) \) and pick a small number \( \epsilon > 0 \) such that \( B_{w, \epsilon} \subseteq G \) and \( s + \epsilon \gamma < 1 \). Let \( \delta > 0 \) and \( t_0 > 0 \) be two real numbers satisfying \( \gamma \delta + \|w\|/(\gamma t_0) \leq \epsilon \) and \( \lfloor \gamma t_0 \rfloor \geq 1 \). Set \( p_t := \lfloor t \rfloor \).

For \( t > t_0 \) we have \( 1 \leq p_t \leq \gamma t \) and (2.1) gives

\[
\mathbb{P}\left[ \frac{W_t}{t} \in G \right] \geq \mathbb{P}\left[ \frac{1}{t} \sum_{i=1}^{p_t} X_i \in B_{w, \epsilon}, T_{p_t} \leq t < T_{p_t+1} \right]
\]

\[
\geq \mathbb{P}\left[ \frac{1}{t} \sum_{i=1}^{p_t} X_i \in B_{w, \epsilon}, \gamma \frac{T_{p_t}}{p_t} \leq 1, S_{p_t+1} > t - T_{p_t} \right].
\]

Since \( s + \epsilon \gamma < 1 \), we can write down the bound

\[
\mathbb{P}\left[ \frac{W_t}{t} \in G \right] \geq \mathbb{P}\left[ \frac{1}{t} \sum_{i=1}^{p_t} X_i \in B_{w, \epsilon}, \gamma \frac{T_{p_t}}{p_t} \in (s - \epsilon \gamma, s + \epsilon \gamma), S_{p_t+1} > t - T_{p_t} \right]
\]

\[
\geq \mathbb{P}\left[ \frac{1}{t} \sum_{i=1}^{p_t} X_i \in B_{w, \epsilon}, \gamma \frac{T_{p_t}}{p_t} \in (s/\gamma - \epsilon, s/\gamma + \epsilon), S_{p_t+1} > t - (s/\gamma - \epsilon)p_t \right]
\]

\[
= \mathbb{P}\left[ \frac{1}{t} \sum_{i=1}^{p_t} X_i \in B_{w, \epsilon}, \gamma \frac{T_{p_t}}{p_t} \in (s/\gamma - \epsilon, s/\gamma + \epsilon) \right] \cdot \mathbb{P}\left[ S_1 > t - (s/\gamma - \epsilon)p_t \right].
\]

We now observe that the condition \( (1/p_t) \sum_{i=1}^{p_t} X_i \in B_{w, \gamma, \delta} \) implies \( (1/t) \sum_{i=1}^{p_t} X_i \in B_{w, \epsilon} \) for \( t > t_0 \). Indeed, recalling that \( \gamma \delta + \|w\|/(\gamma t_0) \leq \epsilon \), for \( t > t_0 \) we find

\[
\left\| \frac{1}{t} \sum_{i=1}^{p_t} X_i - w \right\| = \frac{p_t}{t} \left\| \frac{1}{p_t} \sum_{i=1}^{p_t} X_i - w/\gamma + w/\gamma - \frac{t}{p_t} w \right\| < \gamma \delta + \|w\|/(\gamma t) \leq \epsilon.
\]

Then, for \( t > t_0 \) we have

\[
\mathbb{P}\left[ \frac{W_t}{t} \in G \right] \geq \mathbb{P}\left[ \frac{1}{p_t} \sum_{i=1}^{p_t} X_i \in B_{w, \gamma, \delta}, \gamma \frac{T_{p_t}}{p_t} \in (s/\gamma - \epsilon, s/\gamma + \epsilon) \right] \cdot \mathbb{P}\left[ S_1 > t - (s/\gamma - \epsilon)p_t \right]
\]

\[
= \mathbb{P}\left[ \frac{1}{p_t} \sum_{i=1}^{p_t} (S_1, X_i) \in (s/\gamma - \epsilon, s/\gamma + \epsilon) \times B_{w, \gamma, \delta} \right] \cdot \mathbb{P}\left[ S_1 > t - (s/\gamma - \epsilon)p_t \right],
\]

so that part (ii) of proposition 1.1 and \( \liminf_{\sigma \uparrow +\infty} (1/\sigma) \mathbb{P}\left[ S_1 > \sigma \right] =: \ell_i \) yield

\[
\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P}\left[ \frac{W_t}{t} \in G \right] \geq -\gamma J(s/\gamma, w/\gamma) - (1 - s)\ell_i - \gamma \epsilon \ell_i.
\]

Since \( \epsilon \) is any positive number small enough, this bound demonstrates (2.3). \qed
2.3 The upper large deviation bound for convex sets

The starting point to prove some upper large deviation bounds is the following inequality, which holds for every set \( A \in \mathcal{B}(\mathcal{X}) \) and integer \( q > 1 \):

\[
\mathbb{P}\left[ \frac{W_t}{t} \in A \right] \leq \mathbb{P}\left[ \frac{W_{t-q}}{t-q} \in A, T_q > t \right] + \mathbb{P}[T_q \leq t] \\
\leq 1_{\{0 \in A\}}\mathbb{P}[S_1 > t] + \sum_{n=1}^{q-1} \mathbb{P}\left[ \frac{1}{t} \sum_{i=1}^{n} X_i \in A, T_n \leq t < T_{n+1} \right] + \mathbb{P}[T_q \leq t]. \tag{2.4}
\]

We complement this inequality with two lemmas, the first of which controls the small values of the waiting times.

Lemma 2.4. There exists a real number \( \kappa > 0 \) such that for all sufficiently large numbers \( \gamma \) and \( t \)

\[
\mathbb{P}[T_{\gamma t}] \leq \kappa.
\]

Proof. Since \( \mathbb{P}[S_1 > 0] = 1 \), there exists a number \( \eta > 0 \) such that \( \xi := \mathbb{P}[S_1 \geq \eta] > 0 \). Pick three real numbers \( \gamma \geq 3/\eta \xi, t \geq \eta \xi, \) and \( \lambda \geq 0 \) and set \( i := \lceil \gamma t \rceil \) for brevity. Chernoff bound and the equality \( e^{-x} \leq 1 - x + x^2/2 \) valid for all \( z \geq 0 \) allows us to write down the bound

\[
\mathbb{P}[T_i \leq t] \leq e^{\lambda t} \mathbb{E}[e^{-\lambda T_i}] \leq e^{\lambda t} \mathbb{E}[e^{-\lambda S_i \wedge \eta \xi}]^i \\
\leq e^{\lambda t} \left\{ 1 - \lambda \mathbb{E}[S_i \wedge \eta \xi] + \lambda^2 \mathbb{E}[(S_i \wedge \eta \xi)^2]/2 \right\}^i \leq e^{\lambda t} \left\{ 1 - \lambda \eta \xi + \lambda^2 \eta^2 \xi^2/2 \right\}^i.
\]

At this point, the inequality \( 1 + z \leq e^z \) valid for all \( z \in \mathbb{R} \) gives

\[
\mathbb{P}[T_i \leq t] \leq e^{\lambda t - \lambda \eta \xi + \lambda^2 \eta^2 \xi^2/2}. \tag{2.5}
\]

Since \( \gamma \geq 3/\eta \xi \) and \( t \geq \eta \xi \) we have

\[
\eta \xi i - t \geq \eta \xi (\gamma t - 1) - t = \gamma t[\eta \xi - 1/\gamma (1 + \eta \xi/t)] \geq \gamma t[\eta \xi - 2/\gamma] \geq \gamma t \eta \xi/3 > 0.
\]

This way, we can set \( \lambda := (\eta \xi i - t)/(\eta^2 i) \) in (2.5) to get

\[
\mathbb{P}[T_i \leq t] \leq e^{-\frac{(\eta \xi i - t)^2}{2\eta^2 i}} \leq e^{-\frac{\eta^2 \xi^2 i^2}{18i^2 \gamma^2}} = e^{-\frac{\gamma^2 i^2}{18}}.
\]

The second lemma is more technical and is needed to estimate probabilities involving convex sets.

Lemma 2.5. Let \( \alpha < \beta \) be two real numbers and let \( C \subseteq \mathcal{X} \) be open convex, closed convex, or any convex set in \( \mathcal{B}(\mathcal{X}) \) when \( \mathcal{X} \) is finite-dimensional. Then, for all \( n \geq 1 \) and \( t > 0 \)

\[
\ln \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} (S_i, X_i) \in [\alpha, \beta] \times C \right] \leq -t \inf_{(s,w) \in [\alpha, \beta] \times C} \{ \Upsilon(s, w) \}.
\]

Proof. Fix an integer \( n \geq 1 \) and a real number \( t > 0 \). Set \( \alpha_o := t \alpha/n \) and \( \beta_o := t \beta/n \) and denote by \( C_o \) the convex set \( \{tw/n : w \in C\} \in \mathcal{B}(\mathcal{X}) \). Then, \( C_o \) is open or closed if \( C \) is open or closed and lemma 2.3 shows that

\[
\ln \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} (S_i, X_i) \in [\alpha, \beta] \times C \right] = \ln \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} (S_i, X_i) \in [\alpha_o, \beta_o] \times C_o \right] \\
\leq -n \inf_{(s,w) \in [\alpha_o, \beta_o] \times C_o} \{ J(s,w) \} \\
= -t \inf_{(s,w) \in [\alpha, \beta] \times C} \{ (n/t) J(ts/n, tw/n) \}.
\]

On the other hand, we have \( (n/t) J(ts/n, tw/n) \geq \Upsilon(s,w) \) for all \( s \in \mathbb{R} \) and \( w \in \mathcal{X} \) by definition. \( \square \)
We are now in the position to demonstrate part (d) of theorem 1.1. An upper large deviation bound for convex sets comes from the following lemma.

Lemma 2.6. Let \( \ell \leq \ell_s \) be a real number and let \( C \subseteq \mathcal{X} \) be open convex, closed convex, or any convex set in \( \mathcal{B}(\mathcal{X}) \) when \( \mathcal{X} \) is finite-dimensional. Then

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln \Pr \left[ \frac{W_t}{t} \in C \right] \leq - \inf_{w \in C} \inf_{\beta \in [0,1]} \{ Y(\beta, w) + (1 - \beta)\ell \}.
\]

If \( \ell_s < +\infty \), then part (d) of theorem 1.1 follows from this lemma by making the choice \( \ell = \ell_s \). If \( \ell_s = +\infty \) and \( I_s(0) < +\infty \), then part (d) of theorem 1.1 is obtained by taking \( \ell = I_s(0) = Y(1, 0) \). While the former is manifest, the latter is due to parts (i) and (ii) of lemma 2.1, which show that for all \( w \in \mathcal{X} \) and \( \beta \in [0,1] \)

\[
I_s(\beta) := Y(1, w) = Y \left( \frac{2\beta + 2 - 2\beta}{2}, \frac{2w + 0}{2} \right) \leq \frac{1}{2} Y(2, 2w) + \frac{1}{2} Y(2 - 2\beta, 0) = Y(\beta, w) + (1 - \beta)Y(1, 0).
\]

Proof. Pick a real number \( \ell \leq \ell_s \) and notice that \( \inf_{w \in C} \inf_{\beta \in [0,1]} \{ Y(\beta, w) + (1 - \beta)\ell \} > -\infty \) as \( Y \) is non-negative by lemma 2.1. Fix real numbers \( \lambda < \inf_{w \in C} \inf_{\beta \in [0,1]} \{ Y(\beta, w) + (1 - \beta)\ell \} \) and \( \epsilon > 0 \). By lemma 2.2 there exists a large number \( \gamma \) such that \( \Pr [T_{\gamma t} \leq t] \leq e^{-\lambda t} \) for all sufficiently large \( t \). Set \( q_t := \lfloor \gamma t \rfloor \). Then, (2.4) gives for all sufficiently large \( t \)

\[
\Pr \left[ \frac{W_t}{t} \in C \right] \leq 1_{(0<\ell_s)} \Pr [S_1 > t] + \sum_{n=1}^{q_t} \Pr \left[ \frac{1}{t} \sum_{i=1}^{n} X_i \in C, T_n \leq t < T_{n+1} \right] + e^{-\lambda t}.
\]

Since \( -\limsup_{t \to +\infty} (1/t) \ln \Pr [S_1 > t] =: \ell_s \geq \ell \), there exists a constant \( M > 0 \) such that \( \Pr [S_1 > t] \leq Me^{(e-\lambda)t} \) for all \( t > 0 \). Moreover, by definition we have \( \lambda < Y(\beta, w) + (1 - \beta)\ell \) for all \( \beta \in [0,1] \) and \( w \in C \). Recall that \( T(0, 0) = 0 \) by lemma 2.1. Thus, if \( 0 \in C \), then \( \lambda < Y(0, 0) + \ell = \ell \), which shows that \( 1_{(0<\ell_s)} \leq e^{(e-\lambda)t} \) for any \( t > 0 \). It follows that for all sufficiently large \( t \)

\[
\Pr \left[ \frac{W_t}{t} \in C \right] \leq e^{-\lambda t} + Me^{(e-\lambda)t} + \sum_{n=1}^{q_t} \Pr \left[ \frac{1}{t} \sum_{i=1}^{n} X_i \in C, T_n \leq t < T_{n+1} \right].
\]

To address the third term in the r.h.s., pick real numbers \( 0 := \beta_0 < \beta_1 < \cdots < \beta_K := 1 \) such that \( \beta_{k-1} - \beta_{k-1} \leq \epsilon \) for each \( k \). We have

\[
\sum_{n=1}^{q_t} \Pr \left[ \frac{1}{t} \sum_{i=1}^{n} X_i \in C, T_n \leq t < T_{n+1} \right] \leq \sum_{n=1}^{q_t} \sum_{k=1}^{K} \Pr \left[ \frac{1}{t} \sum_{i=1}^{n} X_i \in C, T_n \in [\beta_{k-1}, \beta_k], S_{n+1} > t - T_n \right].
\]

On the other hand, lemma 2.5, together with the fact that \( \lambda < Y(s, w) + (1 - s)\ell \) for all \( s \in [0,1] \) and \( w \in C \), show that for every \( t > 0 \), \( n \geq 1 \), and \( k \leq K \)

\[
\ln \Pr \left[ \frac{1}{t} \sum_{i=1}^{n} (S_i, X_i) \in [\beta_{k-1}, \beta_k] \times C \right] \leq -t \inf_{(s, w) \in [\beta_{k-1}, \beta_k] \times C} \{ Y(s, w) \}
\]

\[
\leq -t \inf_{s \in [\beta_{k-1}, \beta_k]} \{ \lambda - (1 - s)\ell \}
\]

\[
\leq t \{ (1 - \beta_k)\ell + \epsilon|\ell| - \lambda \}.
\]
It follows that for each $t > 0$

$$\sum_{n=1}^{q-1} P\left[ t \sum_{i=1}^{n} X_i \in C, T_n \leq t < T_{n+1} \right] \leq MKq \epsilon e^{(\epsilon + \epsilon|\ell| - \lambda)t} \leq \gamma MK \epsilon e^{(\epsilon + \epsilon|\ell| - \lambda)t}.$$ 

In conclusion, for all sufficiently large $t$ we get

$$P\left[ \frac{W_t}{t} \in C \right] \leq e^{-\lambda t} e^{-\lambda t} + \gamma MK \epsilon e^{(\epsilon + \epsilon|\ell| - \lambda)t},$$

which shows that

$$\limsup_{t \to +\infty} \frac{1}{t} \ln P\left[ \frac{W_t}{t} \in C \right] \leq \epsilon + \epsilon|\ell| - \lambda.$$ 

The lemma is proved by sending $\lambda$ to $\inf_{w \in C} \inf_{\beta \in [0,1]} \{ Y(\beta, w) + (1 - \beta)\ell \}$ and $\epsilon$ to zero. \(\square\)

We conclude the section by showing that the hypotheses $\ell_s < +\infty$ and $L_0(0) < +\infty$ in part (d) of theorem 1.1 can not be relaxed at the same time. Assume that the waiting times satisfy $\lambda > 0$ for all positive $s$, so that $\ell_s = +\infty$. Set $\mathcal{X} := \mathbb{R}^2$ and for each $i$ consider the reward $X_i := (S_i, Z_i)$ with $S_i$ independent of $S_i$ and distributed according to the standard Cauchy law: $P[Z_i \leq z] = 1/2 + (1/\pi) \arctan(z)$ for all $z \in \mathbb{R}$. It is a simple exercise of calculus to show that $L_s(w) = 0$ if $w_1 = 1$ and $L_s(w) = +\infty$ if $w_1 \neq 1$ for all $w := (w_1, w_2) \in \mathcal{X}$, so that in particular we have $L_s(0) = +\infty$. The upper large deviation bound fails for the open convex set $C := \{ w \in \mathcal{X} : w_1 < 1 \}$. Indeed, $\inf_{w \in C} L_s(w) = +\infty$ and for all $t > 0$ we have

$$P\left[ \frac{W_t}{t} \in C \right] = P[S_1 > t] + \sum_{n \geq 1} P[T_n < t < T_{n+1}]$$

$$= P[T_1 > t] + \sum_{n \geq 1} P[T_n \leq t < T_{n+1}] = 1.$$ 

It also fails for the closed convex set $C := \{ w \in \mathcal{X} : w_1 < 1 \text{ and } (1 - w_1)w_2 \geq 1 \}$. Indeed, $\inf_{w \in C} L_s(w) = +\infty$ and

$$\limsup_{t \to +\infty} \frac{1}{t} \ln P\left[ \frac{W_t}{t} \in C \right] = 0 \quad (2.6)$$

as we now demonstrate. Pick a real number $\epsilon \in (0, 1)$ and an integer $N \geq 1$. Let $\mu$ and $\sigma^2$ be the mean and the variance of $S_1$, respectively. Since $(1/N) \sum_{i=1}^{N} Z_i$ is distributed as $Z_1$ by the stability property of the Cauchy law, for every $t > 0$ we have

$$P\left[ \frac{W_t}{t} \in C \right] \geq P\left[ T_N < t < T_{N+1}, 1/N \sum_{i=1}^{N} Z_i \geq \frac{1}{1 - N/t} \right]$$

$$\geq P\left[ t - \sqrt{\epsilon^2 N} < T_N \leq t - \sqrt{\epsilon^2 \sigma^2 N}, S_{N+1} > t - T_N, \sum_{i=1}^{N} Z_i \geq \frac{t^2}{t - T_N} \right]$$

$$\geq P\left[ -\sqrt{\epsilon^2 N} < T_N - t \leq -\sqrt{\epsilon^2 \sigma^2 N} \right] \cdot P\left[ S_1 > \sqrt{\epsilon^2 N} \right] \cdot \left[ \sum_{i=1}^{N} Z_i \geq \frac{t^2}{\sqrt{\epsilon^2 \sigma^2 N}} \right]$$

$$= P\left[ -\sqrt{\epsilon^2 N} < T_N - t \leq -\sqrt{\epsilon^2 \sigma^2 N} \right] \cdot P\left[ \epsilon \sigma Z_1 \geq \frac{t^2}{\sqrt{\epsilon^2 \sigma^2 N}} \right]$$

At this point, by taking $t = \mu N$ and by sending $N$ to infinity we obtain

$$\limsup_{t \to +\infty} \frac{1}{t} \ln P\left[ \frac{W_t}{t} \in C \right] \geq -\frac{\sigma^2}{\mu} \quad (2.7)$$

because $\lim_{N \to \infty} P\left[ -\sqrt{\epsilon^2 \sigma^2 N} < T_N - \mu N \leq -\sqrt{\epsilon^2 \sigma^2 N} \right] \geq \frac{1}{\sqrt{2\pi}} \int_{-\epsilon}^{\epsilon} e^{-\frac{c^2}{2}} dc = 0$ by the central limit theorem and $\lim_{N \to \infty} \mu^2 \sqrt{N} P[\epsilon \sigma Z_1 \geq \mu^2 \sqrt{N}] = \epsilon \sigma / \pi$. The arbitrariness of $\epsilon$ in (2.7) gives (2.6).
2.4 The upper large deviation bound for compact sets

The upper large deviation bound for compact sets is basically due to the following lemma.

**Lemma 2.7.** For each \( w \in \mathcal{X} \) and real number \( \lambda < I_w(w) \) there exists \( \delta > 0 \) such that

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln P \left[ \frac{W_t}{t} \in B_{w,\delta} \right] \leq -\lambda.
\]

**Proof.** Pick \( w \in \mathcal{X} \) and a real number \( \lambda < I_w(w) \). If \( \ell_a < +\infty \), then the lemma is an immediate application of part (d) of theorem 1.1. In fact, by the lower semicontinuity of \( I_a \) there exists \( \delta > 0 \) such that \( I_a(v) \geq \lambda \) for all \( v \in B_{w,\delta} \), and the open ball \( B_{w,\delta} \) is convex.

Assume \( \ell_a = +\infty \). In such case we have \( \lambda < I_a(w) = \Upsilon(1, w) \) and by the lower semicontinuity of \( \Upsilon \) there exists \( \delta \in (0, 1) \) such that \( \Upsilon(s, v) \geq \lambda \) for every \( s \in [1 - \delta, 1] \) and \( v \in B_{w,\delta} \). It follows by lemma 2.5 that for all \( n \geq 1 \) and \( t > 0 \)

\[
\ln \left[ \frac{1}{n} \sum_{i=1}^{n} (S_i, X_i) \in [1 - \delta, 1] \times B_{w,\delta} \right] \leq -t \inf_{\gamma \in [1 - \delta, 1] \times B_{w,\delta}} \{ \Upsilon(s, v) \} \leq -\lambda.
\]

Moreover, lemma 2.4 gives that \( P[T_{\gamma t} \leq t] \leq e^{-\lambda t} \) for all sufficiently large \( t \) and some real number \( \gamma > 0 \). Setting \( \ell_a := [\gamma t] \), bound (2.4) yields for all sufficiently large \( t \)

\[
P \left[ \frac{W_t}{t} \in B_{w,\delta} \right] \leq P[S_1 > t] + \sum_{n=1}^{q-1} \left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in B_{w,\delta}, T_n \leq t < T_{n+1} \right] + e^{-\lambda t}
\]

\[
\leq \gamma t P[S_1 > \delta t] + \sum_{n=1}^{q-1} \left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in B_{w,\delta}, t - S_{n+1} < T_n \leq t, S_{n+1} \leq \delta t \right] + e^{-\lambda t}
\]

\[
\leq \gamma t P[S_1 > \delta t] + \gamma te^{-\lambda t}
\]

This inequality proves the lemma since \( \limsup_{t \to +\infty} (1/t) \ln P[S_1 > t] = -\infty \) as \( \ell_a = +\infty \).

Let us verify part (c) of theorem 1.1. Pick a compact set \( F \) in \( \mathcal{X} \) and a real number \( \lambda < \inf_{w \in F} \{ I_w(w) \} \). Lemma 2.7 guarantees that for each \( w \in F \) there exists \( \delta_w > 0 \) such that

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln P \left[ \frac{W_t}{t} \in B_{w,\delta_w} \right] \leq -\lambda.
\]

Since \( F \) is compact, we can find a finite number of points \( w_1, \ldots, w_K \) in \( F \) such that \( F \subseteq \bigcup_{k=1}^{K} B_{w_k,\delta_w} \). It follows that

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln P \left[ \frac{W_t}{t} \in F \right] \leq \limsup_{t \to +\infty} \frac{1}{t} \ln \sum_{k=1}^{K} P \left[ \frac{W_t}{t} \in B_{w_k,\delta_w} \right] \leq -\lambda,
\]

which yields

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln P \left[ \frac{W_t}{t} \in F \right] \leq - \inf_{w \in F} \{ I_w(w) \}
\]

once \( \lambda \) is sent to \( \inf_{w \in F} \{ I_w(w) \} \).

2.5 The upper large deviation bound for closed sets

The upper large deviation bound can be extended from compact sets to close sets if the probability distribution of \( \frac{W_t}{t} \) is exponential tight (see [6], lemma 1.2.18), namely if for each real number \( \lambda > 0 \) there exists a compact set \( K \) in \( \mathcal{X} \) such that

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln P \left[ \frac{W_t}{t} \not\in K \right] \leq -\lambda.
\]
The following lemma establishes exponential tightness of the scaled cumulant reward when $\mathcal{X}$ has finite dimension and proves part (c) of theorem 1.1.

**Lemma 2.8.** Assume that $\mathcal{X}$ has finite dimension and that there exist numbers $\zeta \leq 0$ and $\sigma > 0$ such that $\mathbb{E}[e^{\zeta S_1 + \sigma \|X_1\|}] < +\infty$. Then, the probability distribution of $W_t/t$ is exponential tight. Moreover, $I_i$ and $I_o$ have compact level sets.

**Proof.** To begin with, let us observe that under the hypotheses of the lemma we have $\lim_{\eta \downarrow 0} \mathbb{E}[e^{\zeta S_1 + \sigma \|X_1\|}] = 0$ by the dominated convergence theorem, so that there exists $\eta \leq 0$ that satisfies $\mathbb{E}[e^{\zeta S_1 + \sigma \|X_t\|}] \leq 1/2$. Let $d$ be the dimension of $\mathcal{X}$, let $\{v_1, \ldots, v_d\}$ be a basis of $\mathcal{X}$, and let $\{\vartheta_1, \ldots, \vartheta_d\} \subset \mathcal{X}^*$ be the dual basis: $\vartheta_k(v_l) = 1$ if $k = l$ and 0 otherwise for all $k$ and $l$. For $k$ ranging from 1 to $d$ set $\varphi_k := \vartheta_k/\|\vartheta_k\|$ and $\varphi_{d+k} := -\varphi_k$. We have $\mathbb{E}[e^{\zeta S_1 + \sigma \varphi_k(X_1)}] \leq \mathbb{E}[e^{\zeta S_1 + \sigma \|X_1\|}] \leq 1/2$ for every $k$. Fix a real number $\lambda > 0$ and introduce the compact set $K := \bigcup_{k=1}^2 \{w \in \mathcal{X} : \varphi_k(w) \leq \rho\}$, where we have set $\rho := (\lambda - \eta)/\sigma > 0$ for brevity. Since 0 does not belong to the complement of $K$, for all $t > 0$

$$
\mathbb{P}\left[ W_{t/n} \notin K \right] = \sum_{n \geq 1} \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] \in K^c, T_n \leq t < T_{n+1} \right].
$$

As $K^c = \bigcup_{k=1}^2 \{w \in \mathcal{X} : \varphi_k(w) > \rho\}$, by making use of the Chernoff bound and by recalling that $\mathbb{E}[e^{\zeta S_1 + \sigma \varphi_k(X_1)}] \leq 1/2$ for any $k$, we obtain for every $t$

$$
\mathbb{P}\left[ W_{t/n} \notin K \right] \leq \sum_{n \geq 1} \sum_{k=1}^2 \mathbb{P}\left[ \sum_{i=1}^n \varphi_k(X_i) > \rho t, T_n \leq t \right]
\leq e^{-\eta t - \sigma \rho t} \sum_{k=1}^2 \sum_{n \geq 1} \mathbb{E}\left[ e^{\zeta S_1 + \sigma \varphi_k(X_1)} \right]^n \leq 2d e^{-\lambda t}.
$$

This inequality proves exponential tightness of the distribution of $W_{t/n}$.

To conclude, let us show that the level sets of $I_i$ and $I_o$ are compact. Regarding $I_i$, compactness of level sets follows by combining the lower large deviation bound of part (b) of theorem 1.1 and the exponential tightness (see [6], lemma 1.2.18). Let us move to $I_o$. To begin with, we observe that for all $s \in \mathbb{R}$, $w \in \mathcal{X}$, and $k \leq 2d$

$$
J(s, w) \geq s \eta + \sigma \varphi_k(w) - \ln \mathbb{E}\left[ e^{\zeta S_1 + \sigma \varphi_k(X_1)} \right] \geq s \eta + \sigma \varphi_k(w)
$$

by definition. It follows that $Y(\beta, w) \geq \beta \eta + \sigma \varphi_k(w)$ for each $\beta \in [0, 1]$, $w \in \mathcal{X}$, and $k \leq 2d$, which yields $L_k(w) \geq \eta + \sigma \varphi_k(w)$ for every $w$ and $k$. Thus, if $w$ is such that $L_k(w) \leq \lambda$ for a given real number $\lambda \geq 0$, then $\varphi_k(w) \leq (\lambda - \eta)/\sigma$ for all $k$. This demonstrates that level set $\{w \in \mathcal{X} : L_k(w) \leq \lambda\}$ is bounded. It is closed by the lower semicontinuity of $I_o$.

The case in which $\mathcal{X}$ has infinite dimension is solved by the next lemma.

**Lemma 2.9.** Assume that $\mathcal{X}$ has infinite dimension and that $\mathbb{E}[e^{\zeta S_1 + \sigma \|X_1\|}] < +\infty$ for all $\sigma > 0$. Then, the probability distribution of $W_{t/n}$ is exponential tight. Moreover, $I_o = +\infty$ and $Y(1, \cdot)$ has compact level sets.

**Proof.** Fix $\lambda > 0$ and set $\epsilon := \lambda/\ln \mathbb{E}[e^{2\lambda S_1}]$, which is positive since $\mathbb{E}[e^{2\lambda S_1}] < +\infty$ by hypothesis. As $\mathbb{E}[e^{\zeta S_1}] < +\infty$ for all $\sigma > 0$, the theory of Cramér in separable Banach spaces tells us that the distribution of $(1/n) \sum_{i=1}^n X_i$ is exponential tight (see [6], exercise 6.2.21). Thus, there exists a compact set $K_o$ in $\mathcal{X}$ such that for all sufficiently large $n$

$$
\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^n X_i \notin K_o \right] \leq e^{-\lambda n/\epsilon}.
$$

Let $C$ be the closure of the convex hull of $\{0\} \cup K_o$, which is compact (see [28], theorem 3.20). Let $\gamma > \epsilon$ be a real number such that $\mathbb{P}[T_{\gamma t} \leq t] \leq e^{-Mt}$ for all $t$ large enough, which exists by lemma 2.2 and set $K := \{w \in \mathcal{X} : \gamma w \in C\}$, which clearly is a compact
set. Set \( p_t := [\epsilon t] \) and \( q_t := [\gamma t] \). For all sufficiently large \( t \) we have \( 1 \leq p_t < q_t \) and \( \mathbb{P}[T_{p_t} > t] \leq \mathbb{E}[e^{2\lambda X}]^{p_t} e^{-2\lambda t} \leq e^{2\lambda X} e^{-2\lambda t} = e^{-\lambda t} \) by the Chernoff bound. Then, for all sufficiently large \( t \) we can write
\[
\mathbb{P} \left[ \frac{W_t}{t} \notin K \right] \leq \mathbb{P}[T_{p_t} > t] + \mathbb{P} \left[ \frac{W_t}{t} \notin K, T_{p_t} \leq t < T_{q_t} \right] + \mathbb{P}[T_{q_t} \leq t] 
\leq 2e^{-\lambda t} + \sum_{n=p_t}^{q_t-1} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^n X_i \notin K \right].
\]

On the other hand, the condition \((1/n) \sum_{i=1}^n X_i \in C\) implies \((n/\gamma t)(1/n) \sum_{i=1}^n X_i \in C\), namely \((1/t) \sum_{i=1}^n X_i \in K\), for \( n < q_t \) as \( C \) is convex and contains the origin. This shows that if \((1/t) \sum_{i=1}^n X_i \notin C\) for \( n < q_t \), then \((1/n) \sum_{i=1}^n X_i \in C\). Thus, by recalling that \( K_0 \subseteq C \) and by invoking (2.8), for all sufficiently large \( t \) we find
\[
\mathbb{P} \left[ \frac{W_t}{t} \notin K \right] \leq 2e^{-\lambda t} + \sum_{n=p_t}^{q_t-1} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^n X_i \notin C \right] \leq 2e^{-\lambda t} + \gamma t e^{-\lambda p_t / \epsilon}.
\]

This bound yields
\[
\limsup_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \notin K \right] \leq -\lambda.
\]

As before, the lower large deviation bound of part (b) of theorem 1.1 and the exponential tightness imply that \( \ell_t \) has compact level sets. On the other hand, since \( e^{s} \mathbb{P}[S_1 > s] \leq \mathbb{E}[e^{sS_1}] < +\infty \) for all \( s > 0 \) we have \( \ell_s = \ell_1 = +\infty \). Thus, \( \Upsilon(1, \cdot) = I_s \).

## A Proof of proposition 1.1

By definition we have \( I_s \leq I_1 \leq \Upsilon(1, \cdot) \). Let us show that \( I_s \geq \Upsilon(1, \cdot) \) under the hypotheses of the proposition, which is nontrivial only when \( \ell_s < +\infty \). Suppose that \( \ell_s < +\infty \). In this case we have \( \mathbb{E}[e^{sS_1 + \varphi(X_1)}] = +\infty \) for all \( \zeta > \ell_s \) and \( \varphi \in X^* \). In fact, given \( \zeta > \ell_s \) and \( \varphi \in X^* \) one can find a real number \( \epsilon > 0 \) such that \( \zeta - \epsilon > \ell_s \geq 0 \) and \( \|\varphi\| f(s) \leq \epsilon s \) for all sufficiently large \( s \). Then, \( \mathbb{E}[e^{sS_1 + \varphi(X_1)}] \geq e^{(\zeta - \epsilon) s} \mathbb{P}[S_1 > t] \) for all sufficiently large \( t \). It follows that \( \mathbb{E}[e^{sS_1 + \varphi(X_1)}] = +\infty \) since \( \zeta - \epsilon > \ell_s \). This way, for all \( \gamma > 0, s \leq 1, \) and \( w \in X \) we find
\[
\gamma J(s/\gamma, w/\gamma) = \sup_{(\zeta, \varphi) \in (-\infty, \ell_s) \times X^*} \left\{ s \zeta + \varphi(w) - \gamma \ln \mathbb{E}[e^{sS_1 + \varphi(X_1)}] \right\} 
\geq \sup_{(\zeta, \varphi) \in (-\infty, \ell_s) \times X^*} \left\{ \zeta \varphi(w) - \gamma \ln \mathbb{E}[e^{sS_1 + \varphi(X_1)}] \right\} + (s - 1) \ell_s 
= \gamma J(1/\gamma, w/\gamma) + (s - 1) \ell_s.
\]

The definition of \( \Upsilon \) and part (iv) of lemma 2.4 then yields \( \Upsilon(\beta, w) \geq \Upsilon(1, w) + (\beta - 1) \ell_s \) for every \( \beta \in [0, 1] \) and \( w \in X \), so that \( I_s \geq \Upsilon(1, \cdot) \).

## B Cramér’s theory for waiting times and rewards

This appendix introduces the basics of Cramér’s theory that are used to prove theorem 1.1. Let the space \( \mathbb{R} \times X \) be endowed with the product topology and the Borel \( \sigma \)-field \( \mathcal{B}(\mathbb{R} \times X) \) and consider the measure \( \mu_n \) over \( \mathcal{B}(\mathbb{R} \times X) \) defined for each integer \( n \geq 1 \) by
\[
\mu_n := \mathbb{P} \left[ \frac{1}{n} \sum_{i=1}^n (S_i, X_i) \in \cdot \right].
\]

Of fundamental importance is the following super-multiplicativity property.
Lemma B.1. Let $C \in \mathcal{B}(\mathbb{R} \times \mathcal{X})$ be convex and let $m \geq 1$ and $n \geq 1$ be two integers. Then, 
$$
\mu_{m+n}(C) \geq \mu_m(C) \cdot \mu_n(C).
$$

Proof. See lemma 6.1.12 of [6].

Super-multiplicativity, which becomes super-additivity once logarithms are taken, makes it possible to describe in general terms the exponential decay with $n$ of the measure $\mu_n$. To this purpose, we denote by $L$ the extended real function over $\mathcal{B}(\mathbb{R} \times \mathcal{X})$ defined by the formula
$$
L := \sup_{n \geq 1} \left\{ \frac{1}{n} \ln \mu_n \right\}.
$$
If $C \in \mathcal{B}(\mathbb{R} \times \mathcal{X})$ is convex, then $\limsup_{n \to \infty} (1/n) \ln \mu_n(C) = L(C)$ due the super-additivity of $\ln \mu_n(C)$. The following lemma improves this result when $C$ is open as well as convex.

Lemma B.2. Let $C \subseteq \mathbb{R} \times \mathcal{X}$ be open and convex. Then, $\lim_{n \to \infty} (1/n) \ln \mu_n(C)$ exists as an extended real number and is equal to $L(C)$.

Proof. See lemma 1.1.14 of [6].

Lemma B.2 prompts one to consider the rate function $J$ that maps any $(s, w) \in \mathbb{R} \times \mathcal{X}$ in the extended real number $J(w)$ defined by
$$
J(s, w) := -\inf_{(s, w) \in \mathcal{X}} \left\{ L(C) : C \subseteq \mathbb{R} \times \mathcal{X} \text{ is open convex and contains $(s, w)$} \right\}.
$$
In fact, the following weak large deviation principle is verified.

Proposition B.1. The following conclusions hold:

(i) the function $J$ is lower semicontinuous and convex;

(ii) $\liminf_{n \to \infty} \frac{1}{n} \ln \mu_n(G) \geq -\inf_{(s, w) \in G} \{ J(s, w) \}$ for each $G \subseteq \mathbb{R} \times \mathcal{X}$ open;

(iii) $\limsup_{n \to \infty} \frac{1}{n} \ln \mu_n(K) \leq -\inf_{(s, w) \in K} \{ J(s, w) \}$ for each $K \subseteq \mathbb{R} \times \mathcal{X}$ compact.

Proof. See lemma 6.1.7 of [6].

The rate function $J$ can be related to the moment generating function of waiting time and reward pairs as follows.

Proposition B.2. For all $(s, w) \in \mathbb{R} \times \mathcal{X}$
$$
J(s, w) = \sup_{(\zeta, \varphi) \in \mathbb{R} \times \mathcal{X}} \left\{ s\zeta + \varphi(w) - \ln \mathbb{E}[e^{s\zeta_1 + \varphi(X_1)}] \right\}.
$$

Proof. See theorem 6.1.3 of [6].

We conclude the appendix with a result about certain convex sets that are met in the proof of theorem 1.1.

Lemma B.3. Let $\alpha < \beta$ be two real numbers and let $C \subseteq \mathcal{X}$ be open convex, closed convex, or any convex set in $\mathcal{B}(\mathcal{X})$ when $\mathcal{X}$ is finite-dimensional. Then, for all $n \geq 1$
$$
\frac{1}{n} \ln \mu_n([\alpha, \beta] \times C) \leq -\inf_{(s, w) \in [\alpha, \beta] \times C} \{ J(s, w) \}.
$$
Proof. Recalling the definition of \( \mathcal{L} \), we show that

\[
\mathcal{L}(\alpha, \beta) \leq \inf_{(s,w) \in [\alpha, \beta] \times C} \{ J(s,w) \}.
\]

Assume \( \mathcal{L}(\alpha, \beta) > -\infty \), otherwise there is nothing to prove, and pick \( \epsilon > 0 \). By definition, there exists an integer \( N \geq 1 \) such that \( \mathcal{L}(\alpha, \beta) \leq (1/N) \ln \mu_N ([\alpha, \beta] \times C) + \epsilon \).

Notice that we must have \( \mu_N ([\alpha, \beta] \times C) > 0 \). Completeness and separability of \( \mathcal{X} \) entail that the measure that associates any \( A \in \mathcal{B}(\mathcal{X}) \) with \( \mu_N ([\alpha, \beta] \times A) \) is tight (see [29], theorem 7.1.7). Consequently, a compact set \( K_o \subseteq C \) can be found so that \( \mu_N ([\alpha, \beta] \times C) \leq \mu_N ([\alpha, \beta] \times K_o) + [1 - \exp(-\epsilon N)] \mu_N ([\alpha, \beta] \times C) \). Thus, \( \mu_N ([\alpha, \beta] \times C) \leq \exp(\epsilon N) \mu_N ([\alpha, \beta] \times K_o) \) and \( \mathcal{L}(\alpha, \beta \times C) \leq (1/N) \ln \mu_N ([\alpha, \beta] \times K_o) + 2\epsilon \) follows. We shall show in a moment that there exists a compact convex set \( K \) with the property that \( K_o \subseteq K \subseteq C \).

Then, using the fact that \( K_o \subseteq K \) we reach the further bound \( \mathcal{L}(\alpha, \beta \times C) \leq (1/N) \ln \mu_N ([\alpha, \beta] \times K) + 2\epsilon \leq \mathcal{L}(\alpha, \beta \times K) + 2\epsilon \). At this point, we observe that on the one hand \( \mathcal{L}(\alpha, \beta \times K) = \limsup_{n \uparrow \infty} (1/n) \ln \mu_n ([\alpha, \beta] \times K) \) by super-additivity as \( K \) is convex, and on the other hand \( \limsup_{n \uparrow \infty} (1/n) \ln \mu_n ([\alpha, \beta] \times K) \leq -\inf_{(s,w) \in [\alpha, \beta] \times K} \{ J(s,w) \} \) by part (ii) of proposition [17] as \( K \) is compact. Thus, \( \mathcal{L}(\alpha, \beta \times K) \leq -\inf_{(s,w) \in [\alpha, \beta] \times K} \{ J(s,w) \} + 2\epsilon \leq -\inf_{(s,w) \in [\alpha, \beta] \times C} \{ J(s,w) \} + 2\epsilon \) because \( K \subseteq C \) and the lemma follows from the arbitrariness of \( \epsilon \).

Let us prove at last that there exists a compact convex set \( K \) with the property that \( K_o \subseteq K \subseteq C \). The hypothesis that the convex set \( C \) is either open or closed when \( X \) if infinite-dimensional comes into play here. Let \( C_o \) be the convex hull of the compact set \( K_o \subseteq C \) and denote the closure of a set \( A \) by \( \text{cl} A \). The set \( C_o \) is convex and compact when \( X \) is finite-dimensional, whereas \( \text{cl} C_o \) is convex and compact even when \( X \) is infinite-dimensional (see [28], theorem 3.20). Clearly, \( K_o \subseteq C_o \subseteq C \). If \( X \) is finite-dimensional, then the problem to find \( K \) is solved by \( K = C_o \). If \( X \) is infinite-dimensional and \( C \) is open, then the problem is solved by \( K = \text{cl} C_o \). Some more effort is needed when \( X \) is infinite-dimensional and \( C \) is open. Assume that \( C \) is open and for each \( w \in C \) let \( \delta_w > 0 \) be such that \( \text{cl} B_{w, \delta_w} \subseteq C \). As \( K_o \) is compact, there exist finitely many points \( w_1, \ldots, w_n \) in \( K_o \) so that \( K_o \subseteq \cup_{i=1}^n B_{w_i, \delta_{w_i}} \). Let \( K \) be the convex hull of \( \cup_{i=1}^n (\text{cl} B_{w_i, \delta_{w_i}} \cap \text{cl} C_o) \), which contains \( K_o \). The set \( K \) is convex and compact since it is the convex hull of the union of the compact convex sets \( \text{cl} B_{w_i, \delta_{w_i}} \cap \text{cl} C_o \). On the other hand, we have \( K_k \subseteq C \) for each \( k = 1(1) n \) because \( \cup_{i=1}^n (\text{cl} B_{w_i, \delta_{w_i}} \cap \text{cl} C_o) \subseteq \cup_{i=1}^n (\text{cl} B_{w_i, \delta_{w_i}} \cap \text{cl} C_o) \subseteq C \). \( \square \)

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