CONTRACTIONS OF INTEGRABLE EQUATIONS

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Abstract
The contraction is applied to obtaining of integrable systems associated with nonsemisimple algebras. The effect of contraction is splitting off some components from initial system without loss of integrability.

1 Introduction
The Lie algebraic approach to integrable systems is well known. There are many models associated with simple finite and infinite algebras. The step toward general position algebras was made by Lesnov and Saveliev. They used nonsemisimple algebras and the contraction procedure. Here we investigate an action of the contraction procedure on integrable equations. All examples are related to Toda and KdV systems. We consider the contractions of finite and affine algebras and \( \mathbb{Z}_2 \)-graded contractions of the Virasoro algebra.

2 Contraction

2.1 Finite algebra
Let \( G_1 = (V, \{.,.\}_1) \) and \( G_2 = (V, \{.,.\}_2) \) be two Lie algebras constructed on the vector space \( V \). \( G_2 \) is a contraction of \( G_1 \) if there exists a family \( \Phi_\varepsilon \), \( \varepsilon \in (0, 1) \), of invertible linear transformations of \( V \) so that \( \lim_{\varepsilon \to 0} \Phi_\varepsilon^{-1} \{\Phi_\varepsilon x, \Phi_\varepsilon y\}_1 = \{x, y\}_2 \), \( \forall x, y \in V \). One can also say that \( G_1 \) is a deformation of \( G_2 \). Let \( G_1 \), be a simple Lie algebra of rank \( r \), equipped with an invariant scalar product denoted (.,.). We choose a Cartan subalgebra with an orthonormal basis \( H_i \) and a set of simple roots \( \alpha \)

\[
\begin{align*}
[H_i, H_i]_1 &= 0, \quad [H_i, E_{\pm\alpha}]_1 = \pm \alpha_i E_{\pm\alpha}, \quad [E_\alpha, E_{-\alpha}]_1 = H_\alpha, \quad \text{(1)}
\end{align*}
\]
where $H_\alpha = (E_\alpha, E_{-\alpha}) \sum \alpha_i H_i$. Let $R_c$ denote a subset of simple roots. If we choose $\Phi_\varepsilon : E_\alpha \to \varepsilon E_\alpha$ for $\alpha \in R_c$, then for $\mathcal{G}_2$ the last commutators in (1) are

$$[E_\alpha, E_{-\alpha}]_2 = H_\alpha \quad \alpha \notin R_c, \quad [E_\alpha, E_{-\alpha}]_2 = 0 \quad \alpha \in R_c. \quad (2)$$

### 2.2 Infinite algebra

Let $h_i, e_i, f_i, i = 0, 1, \ldots, r$ are generators of the affine Kac-Moody algebra $\hat{\mathcal{G}}$ with Cartan matrix $\hat{k}_{ij} = \alpha_j(h_i)$ and commutators

$$[h_i, h_j]_1 = 0,$$

$$[h_i, e_j]_1 = \alpha_j(h_i)e_j, \quad [h_i, f_j]_1 = -\alpha_j(h_i)f_j,$$

$$ad^{1-k_{ij}}e_ie_j = 0, \quad ad^{1-k_{ij}}f_if_j = 0,$$

$$[e_i, f_j]_1 = \delta_{ij}h_i. \quad (3)$$

If we choose $\Phi_\varepsilon : e_p \to \varepsilon e_p$ for some $p$, then for $\hat{\mathcal{G}}_2$ the last commutator in (3) is equal zero for $i = p$.

### 2.3 Virasoro algebra

Let us split the generators of the Virasoro algebra $Vir_c$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad (4)$$

into two sets, even $L_0 = 2A_0 - \frac{c}{3}, L_{2n} = 2A_p$ and odd $L_{2n+1} = 2B_p$. In $A_p, B_p$ language (4) looked as

$$[A_p, A_q] = (p-q)A_{p+q} + \frac{2c}{12}(p^2-1)\delta_{p+q,0},$$

$$[A_p, B_q] = (p-q)A_{p+q} + \frac{2c}{12}(p+\frac{1}{2})(p+\frac{3}{2})\delta_{p+q,0}.$$  \quad (5)

There is a $Vir_{2c}$ subalgebra in $Vir_c$: $Vir_c = Vir_{2c} + something$

This is $Z_2$ graded $Vir_c$. Three $Z_2$-grading contractions are given by the matrix $\varepsilon_1, \varepsilon_2, \varepsilon_3$. We will use only $\varepsilon_1: \varepsilon_{11} = \varepsilon_{12} = \varepsilon_{21} = 1, \varepsilon_{22} = 0$. In this case the last commutator in (5) is equal zero.

### 3 Toda Chain

#### 3.1 1D Toda chain

The Toda chain $\Gamma_i$ is the system with $r$ degrees of freedom, phase space coordinates $(q_i, p_i)$ $i = 1, \ldots, r$, Poisson bracket $\{p_i, q_j\} = \delta_{ij}$, and the Hamiltonian

$$h = \frac{1}{2} \sum_i p_i^2 + \sum_i \exp(\alpha \cdot q_i). \quad (6)$$
where \((\alpha \cdot q) = \sum_i \alpha_i q_i\). The equations of motion
\[
\begin{aligned}
\dot{q}_i & = p_i, \\
\dot{p}_i & = -\sum_i \alpha_i \exp(\alpha \cdot q)
\end{aligned}
\] (7)

admit a Lax pair representation
\[
\dot{L} = [L, M],
\]
\[
L = \frac{1}{2} \sum_i p_i H_i + \sum_R n_\alpha e^{\frac{1}{2}(\alpha \cdot q)}(E_\alpha + E_{-\alpha}),
\]
\[
M = \sum_R n_\alpha e^{\frac{1}{2}(\alpha \cdot q)}(E_\alpha - E_{-\alpha}),
\]
\[
n_\alpha = \frac{1}{4(E_\alpha, E_{-\alpha})}.
\] (8)

where \(H_i, E_{\pm} \) and \([,]_1\) are generators and Lie bracket of \(G_1\). Let \(G_2\) be a contraction of \(G_1\) defined in previous section. We call the system (8) in algebra \(G_2\) the contracted Toda chain. The equations of motion, Hamiltonian and Lax pair representation are
\[
\begin{aligned}
\dot{p}_i & = -\sum_R R_{R_c} \alpha_i \exp(\alpha \cdot q) \\
\dot{q}_i & = p_i,
\end{aligned}
\]
\[
h = \sum_i \frac{1}{2} p_i^2 + \sum_R \exp(\alpha \cdot q),
\]
\[
\dot{L} = [L, M]_2.
\] (9)

The \(sl(2)\) Toda contracted chain is a free particle. The \(sl(3)\) case leads to the elimination of some interacting terms in Hamiltonian.

### 3.2 2D Toda chain

The equation of motion is zero curvature condition
\[
\left[ \frac{\partial}{\partial z_+} + A_+, \frac{\partial}{\partial z_-} + A_- \right] = 0,
\]
\[
A_\pm = (h u_\pm) + (E_\pm f_\pm) = \sum_{i=1}^r (h_i u_i^\pm + E_i f_i^\pm).
\] (10)

For \(\rho_i = \ln f_i^+ f_i^-\) we have in \(G_1\) algebra
\[
\frac{\partial^2 \rho_i}{\partial z_+ \partial z_-} = \sum_{i=1}^r k_{ji} e^{\rho_j},
\] (11)

where \(k_{ij}\) is \(r \times r\) Cartan matrix of \(G_1\). Let \(G_2\) be a contraction of \(G_1\). We call the system (11) in \(G_2\) algebra the contracted 2D Toda chain. The equations of motion are
\[
\frac{\partial^2 \rho_i}{\partial z_+ \partial z_-} = \sum_{i=1}^r k'_{ji} e^{\rho_j},
\] (12)
where several columns of Cartan matrix corresponding to contracted roots \( \alpha \in R_c \) are vanished. Let \( R_c = \alpha_i \), then the equations are reduced to

\[
\frac{\partial^2 \rho_i}{\partial z_+ \partial z_-} = F(\rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_r), \\
\frac{\partial^2 \rho_l}{\partial z_+ \partial z_-} = \sum_{j=1}^{r} k_{ij} \rho_j, \quad l, j \neq i.
\] (13)

The contracted component \( \rho_i \) has no self-action and split off from the other components, because there is no \( \rho_i \) in r.h.s. of (13). Therefore, we may consider the contraction of 2D Toda chain as the transition between chain with Cartan matrix \( k \) and chain with matrix \( \tilde{k} \), where \( (r-1) \times (r-1) \) matrix \( \tilde{k} \) is obtained by crossing out of \( i \)-th column and \( i \)-th row from \( k \). The equations of contracted system looked as

\[
\frac{\partial^2 \rho_{\text{contr}}}{\partial z_+ \partial z_-} = F(\rho_1, \ldots, \rho_{r-1}), \\
\frac{\partial^2 \rho_{\text{contr}}}{\partial z_+ \partial z_-} = \sum_{j=1}^{r-1} \tilde{k}_{ij} \rho_j, 
\] (14)

where \( \rho_{\text{contr}} \) is contracted component and \( l, j = 1, 2, \ldots, r-1 \). All these transitions may be classified in terms of Dynkin diagrams using the following property: if the lines connecting any two points are severed, the resulting diagram is a Dynkin diagram. Thus, \( \tilde{k} \) remains a Cartan matrix.

For example, there are three contractions of \( D_4 \) chain:

\[
D_4 \rightarrow A_3 + \rho_{\text{contr}}, \\
D_4 \rightarrow D_3 + \rho_{\text{contr}}, \\
D_4 \rightarrow A_1 + A_1 + A_1 + \rho_{\text{contr}}.
\] (15)

### 3.3 2D affine Toda system

The base formulas for the affine Toda chain are given in [4]. The equations of motion is zero curvature condition

\[
[L, \bar{L}]_1 = 0, \quad L = e^{-\psi} \partial_x e^\psi + \Lambda, \quad \bar{L} = \partial_t + e^{-\psi} \bar{\Lambda} e^\psi, \\
\frac{\partial^2 \psi}{\partial x \partial t} = \sum_{i=0}^{r} h_i e^{\alpha_i(\psi)}, \\
\psi = \sum_{i=0}^{r} \psi_i h_i, \quad \Lambda = \sum_{i=0}^{r} e_i, \quad \bar{\Lambda} = \sum_{i=0}^{r} f_i,
\] (16)

where \( h_i, e_i, f_i, i = 0, 1, \ldots, r \) are generators of the affine Kac-Moody algebra \( \hat{G}_1 \) with Cartan matrix \( \hat{k}_{ij} = \alpha_j(h_i) \). We call the system (16) in algebra
the contracted affine Toda chain. The Lax par representation and equations of motion are

$$[L, \bar{L}]_2 = 0,$$

$$\frac{\partial^2 \psi_i}{\partial x \partial t} = \exp \sum_{i=0}^{r} \hat{k}_{ji} \psi_j \quad i \neq p,$$

$$\frac{\partial^2 \psi_p}{\partial x \partial t} = 0,$$

(17)

where \( p \)-th column of Cartan matrix corresponding to contracted field \( \psi_p \) is not arise in equations. Let \( \phi_j = \psi_j - \psi_p \) for \( j \neq p \) then for \( \hat{A}_n \) (17) looked as

$$\frac{\partial^2 \phi_i}{\partial x \partial t} = \exp \sum_{i=0}^{r} \tilde{k}_{ji} \phi_j, \quad \frac{\partial^2 \psi_{\text{contr}}}{\partial x \partial t} = 0,$$

(18)

where matrix \( \tilde{k} \) is obtained by crossing out of \( p \)-th column and \( p \)-th row from \( \hat{k} \). The affine Cartan matrix of \( \hat{A}_n \) algebra reduced after these crossing to Cartan matrix of finite algebra. Thus, the affine Toda chain reduced to usual Toda chain. For example, there are two contractions of \( \hat{A}_3 \) chain:

$$\hat{A}_3 \rightarrow A_3 + \psi_{\text{contr}},$$

$$\hat{A}_3 \rightarrow D_3 + \psi_{\text{contr}}.$$

(19)

There is a contraction of \( \hat{A}_2 \) chain, \( \hat{A}_2 \rightarrow A_1 + \psi_{\text{contr}} \), which described a transition from the Sine-Gordon to the Liouville equation.

Thus, contractions give a method of transitions between the different Toda models.

4 KdV equation

4.1 The AKNS hierarchy

One of the approach to the contraction of the KdV equation is based on AKNS method of reproducing of integrable equations. It is easily shown for \( sl(2) \) AKNS hierarchy that after contraction to \( isl(2) \) all equations are reduced to linear equations.

4.2 The second Poisson bracket structure

The another approach to the contraction of the KdV equation is based on \( Z_2 \)-graded contraction of the Virasoro algebra. The KdV equation can be written as

$$u_t = (\partial^3 + 2u \partial + 2 \partial u) \left( \frac{\delta H}{\delta u} \right) = \Theta \left( \frac{\delta H}{\delta u} \right),$$

(20)

where \( H = I_2[u] = \int \frac{u^2}{2} dx \), \( \delta \) denotes a variational derivative. The associated Poisson bracket for functionals \( F[u] \) and \( G[u] \) are

$$\{F[u], G[u]\} = \int \frac{\delta F}{\delta u} \Theta \frac{\delta G}{\delta u} dx.$$  

(21)
The conserved quantities $I_n[u]$ of KdV equation are in involution with respect to this bracket. For $u(x)$ coordinates Poisson bracket looked as

$$\{u(x), u(y)\} = \Theta \delta(x - y). \quad (22)$$

Considering equating (22) in the Fourier space, one is lead to the Virasoro algebra

$$\{L_n, L_m\} = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m, 0}. \quad (23)$$

After rewriting in $Z_2$ grading language (sect.2.3), and returning from the Fourier to configuration space we have two kinds of coordinates $v(x)$, $w(x)$ and a new $2 \times 2$ matrix $\Theta$–operator. It is easily shown that after the $\varepsilon_1$ contraction of the Virasoro algebra, $\tilde{\Theta}$–operator looked as

$$\tilde{\Theta}_{\text{contr}} = \begin{pmatrix} \partial^3 + 2v\partial + 2\partial v & 2w\partial + 2\partial w + i\omega/2 \\ 2w\partial + 2\partial w - i\omega/2 & 0 \end{pmatrix}. \quad (24)$$

We can define the contracted Poisson bracket

$$\{F[v, w], G[v, w]\}_{\text{contr}} = \int \nabla F[v, w] \tilde{\Theta}_{\text{contr}} \nabla G[v, w] dx, \quad (25)$$

where $\nabla$ denotes variational gradient. This bracket has at least one set of the conserved quantities $I_n[v]$ in involution, because there is a Virasoro subalgebra in contracted Virasoro algebra. Put $H[v, w] = I_2[v]$ as the Hamiltonian than we have two equations

$$v_t = \{v, H\}_{\text{contr}} = \partial^3 v + 6vv_x,$$

$$w_t = \{w, H\}_{\text{contr}} = 2vwx + 4vw^x - \frac{i}{2}vw. \quad (26)$$

Thus, after contraction we have again the KdV equation for $v(x)$ and some equation for $w(x)$.

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