A characterization of bad approximability*

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Abstract

We show that badly approximable vectors are exactly those that cannot, for any inhomogeneous parameter, be inhomogeneously approximated at every monotone divergent rate. This implies in particular that Kurzweil’s theorem cannot be restricted to any points in the inhomogeneous part. Our results generalize to weighted approximations, and to higher irrationality exponents.

Keywords: Diophantine approximation, badly approximable vectors, Kurzweil’s theorem, inhomogeneous approximation, shrinking target

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1. Informal introduction

One of the most fundamental objects of study in classical and modern Diophantine approximation is the set of \textit{badly approximable vectors} in \( \mathbb{R}^d \), that is, the set

\[ \mathcal{B}A \overset{\text{def}}{=} \{ x \in \mathbb{R}^d : \inf_{n \in \mathbb{N}} n \| nx \|^d > 0 \}, \]

where \( \| \cdot \| \) denotes sup-norm distance to \( \mathbb{Z}^d \). The set \( \mathcal{B}A \) has several well-known and important properties, the most often-cited perhaps being that it has zero Lebesgue measure while having full Hausdorff dimension. Research into the structure of \( \mathcal{B}A \) and its generalizations continues to this day.

This paper is inspired by a question regarding Kurzweil’s theorem [16], a classical result that has received renewed attention in the past decade [7–9, 11, 13, 20, 22]. Kurzweil’s theorem is stated in section 2, but for the moment it is worth mentioning that it gives an alternate characterization of bad approximability; it is remarkable because it uses inhomogeneous approximations, that is, expressions of the form \( \| nx + y \| \) where \( x, y \in \mathbb{R}^d \), to define \( \mathcal{B}A \), a set whose standard definition uses only homogeneous approximations, meaning expressions of

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the form $\|nx\|$. The question is: Does Kurzweil’s characterization of $\text{BA}$ persist if we restrict inhomogeneous parameters $y$ to lie on a submanifold of $\mathbb{R}^d$?

The question itself fits into a long-running tradition of determining to what extent classical results in Diophantine approximation survive restriction to subsets of $\mathbb{R}^d$. For more instances of this tradition, see [1–5, 10, 14, 15, 18, 19, 21, 23]. Notice that the results in this list often depend on non-degeneracy (a curvature condition) of the manifold to which we are restricting, and in the degenerate cases (say, affine subspaces) results tend to depend on the Diophantine properties of the subspace. One may be tempted to venture a guess: that the answer to the above question is ‘yes’ if we restrict to non-degenerate manifolds of $\mathbb{R}^d$, or if we restrict to degenerate manifolds satisfying some Diophantine-type condition.

In this note we consider the extreme degenerate case of singletons (connected zero-dimensional manifolds). Here, the answer is: No, the opposite is true. When one tries to restrict to points in the inhomogeneous part, one finds instead a new characterization of bad approximability which is intuitively the opposite of Kurzweil’s theorem. In particular, the degenerate part of the above guess is totally wrong for points.

In [11] and [22], Harrap and Tseng generalize Kurzweil’s theorem to weighted bad approximability and to bad approximability with respect to higher irrationality exponents, respectively. The results in this paper have analogous generalizations.

### 2. Statement of results

Let $d \in \mathbb{N}$. For a function $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$, let

$$
W(\psi) \overset{\text{def}}{=} \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : (\exists \infty n \in \mathbb{N}) \|nx + y\| < \psi(n)\}
$$

denote the set of inhomogeneously $\psi$-approximable pairs in $\mathbb{R}^{2d}$. Let $\mathcal{D}$ denote the set of all non-increasing $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $\sum_n \psi(n)^d$ diverges, and let

$$
\Pi \overset{\text{def}}{=} \bigcap_{\psi \in \mathcal{D}} W(\psi).
$$

Denote by $\pi$ the projection $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ to the first copy of $\mathbb{R}^d$. That is, $\pi(x, y) = x$. Let

$$
V(\psi) \overset{\text{def}}{=} \{x \in \mathbb{R}^d : (\text{a.e.} \ y \in \mathbb{R}^d)(\exists \infty n \in \mathbb{N}) \|nx + y\| < \psi(n)\}.
$$

Finally, let $\text{WA}$ denote the complement of $\text{BA}$, that is, the well approximable vectors.

**Kurzweil’s theorem (1955).**

$$
\text{BA} = \bigcap_{\psi \in \mathcal{D}} V(\psi).
$$

**Remark.** Note that the doubly metric inhomogeneous Khintchine theorem [6] asserts that $W(\psi)$ has full Lebesgue measure in $\mathbb{R}^d \times \mathbb{R}^d$ for every $\psi \in \mathcal{D}$, hence, $V(\psi)$ has full measure in $\mathbb{R}^d$ by Fubini’s theorem.

The following question asks whether Kurzweil’s theorem can be restricted to subsets of the inhomogeneous $\mathbb{R}^d$-component. It was originally asked by Velani in the context of non-degenerate manifolds, where it is sensible to think the answer might be ‘yes’. We phrase the question for general subsets of $\mathbb{R}^d$, allowing that in this generality it is not expected that the answer will always be affirmative.
Let $M \subset \mathbb{R}^d$ be some subset (say, an affine subspace, or any manifold, or a fractal) supporting a measure $\mu_M$. Can we restrict Kurzweil’s theorem to $M$ in the inhomogeneous part? That is, denoting

$$V^M(\psi) \overset{\text{def}}{=} \{ x \in \mathbb{R}^d : (\mu_M\text{-a.e.}\, y \in \mathbb{R}^d)(\exists n \in \mathbb{N})\, \|nx + y\| < \psi(n) \}$$

and

$$\Theta(M) \overset{\text{def}}{=} \bigcap_{\psi \in D} V^M(\psi),$$

do we have $BA = \Theta(M)$ as in Kurzweil’s theorem? And if not, what do we have instead?

Our main finding is that in the case that $M = \{y\}$ is a single point with the Dirac measure $\mu = \delta_y$, the answer is ‘no’. Not only does the set $\Theta(y)$ always fail to coincide with $BA$, it never contains any badly approximable vectors at all. Furthermore, exclusion from $\Theta(y)$ for all $y \in \mathbb{R}^d$ characterizes bad approximability. We prove the following theorem.

**Theorem 2.1.**

$$WA = \pi(\Pi) \quad \text{hence} \quad BA = \mathbb{R}^d \setminus \pi(\Pi).$$

**Remark.** Denoting

$$\Phi(x) \overset{\text{def}}{=} \{ y \in \mathbb{R}^d : (x, y) \in \Pi \},$$

the theorem can be restated as

$$x \in BA \iff \Phi(x) = \emptyset. \quad \text{(Theorem 2.1)}$$

From the proof it is evident that $\Phi(x)$ is uncountable if $x \in WA \setminus Q^d$. We conjecture that $\Phi(x)$ is countable if $x \in Q^d$, as is the case when $d = 1$. (See theorem A.1.)

**Remark.** Building on the previous remark, it should be noted that $\Phi(x)$ and $\Theta(y)$ are ‘vertical’ and ‘horizontal’ fibers (respectively) of $\Pi$. Specifically, viewing $\Pi$ as a subset of $\mathbb{R}^d \times \mathbb{R}^d$, we see that $\Phi(x)$ is the projection of $\Pi \cap (\{x\} \times \mathbb{R}^d)$ to the second $\mathbb{R}^d$ factor, and $\Theta(y)$ is the projection of $\Pi \cap (\mathbb{R}^d \times \{y\})$ to the first $\mathbb{R}^d$ factor. Putting it another way, we have

$$\Pi = \bigcup_{x \in \mathbb{R}^d} (\{x\} \times \Phi(x)) = \bigcup_{y \in \mathbb{R}^d} (\Theta(y) \times \{y\}).$$

This point of view is especially apparent in section 4.

Note that by the Doubly Metric Inhomogeneous Khintchine theorem [6], the set $W(\psi)$ has full Lebesgue measure for every $\psi \in D$. But $\Pi$ is an uncountable intersection, so it is not guaranteed that it will also have full measure, or that it is even measurable. We prove the following.

**Theorem 2.2.** The set $\Pi$ is Lebesgue measurable and has measure 0. In fact, for every $x \in \mathbb{R}^d$, the set $\Phi(x) \subset \mathbb{R}^d$ has Lebesgue measure 0.
This theorem implies that for almost every \( y \in \mathbb{R}^d \), the measure of \( \Theta(y) \) is 0. In lemma 4.5 we show that \( \Theta(y) \cap [0,1]^d \) always has measure 0 or 1. We ask: Does there exist \( y \in \mathbb{R}^d \) for which \( \Theta(y) \) has full measure?

Finally, we extend these results to the cases of weighted approximation (section 5) and higher irrationality exponents (section 6). These results appear as theorems 5.2, 5.3, 6.2 and 6.3.

**Remark 2.3.** Before moving on to the proofs of the main results, we should draw attention to a subtlety that arises in the passage from homogeneous to inhomogeneous approximations. Namely, that \( \|nx + y\| \) is generally not equal to \( \|(-n)x + y\| \). This means that in any discussion about inhomogeneous approximations, one must choose whether one wants to allow integer values for \( n \) or only consider positive integer values. We have chosen to consider only positive integer values of \( n \), (so, for example, our sets \( W(\psi) \) are not invariant under the maps \((x,y) \mapsto (-x,y)\) and \((x,y) \mapsto (x,-y)\), though they are invariant under the map \((x,y) \mapsto (-x,-y)\).) This is consistent with the convention used in Kurzweil’s work [16].

### 3. Proof of theorem 2.1

For \( x, y \in \mathbb{R}^d, \ell \in \mathbb{Z}, \) and \( d \in \mathbb{N} \), let
\[
S_\ell(x,y) \overset{\text{def}}{=} \sum_{n \geq \ell} \min_{\ell \leq m \leq n} \|mx + y\|^d .
\]
These sums will play an essential role here, especially through the use of lemma 3.1. Note that, even though we have defined \( S_\ell \) for any \( \ell \in \mathbb{Z} \), we will really only need to work with \( \ell \in \mathbb{N} \) (see remark 2.3). Still, there is no harm in stating some of our simple facts about \( S_\ell \) — namely lemmas 3.2, A.3 and A.4 — for any \( \ell \in \mathbb{Z} \).

The following lemma shows that membership in \( \Pi \) is equivalent to convergence of \( S_\ell \).

**Lemma 3.1.**
\[
(x,y) \in \Pi \iff S_\ell(x,y) < \infty \quad (\forall \ell \in \mathbb{N}) .
\]

**Proof.** Assume there exists some \( \ell \in \mathbb{N} \) such that \( S_\ell(x,y) = \infty \), and define the function \( \psi : \mathbb{N} \to \mathbb{R}_{\geq 0} \) to be non-increasing and to satisfy
\[
\psi(n) = \min_{\ell \leq m \leq n} \|mx + y\| \quad \text{for all} \quad n \geq \ell .
\]
Since \( S_\ell(x,y) = \infty \), we have \( \psi \in \mathcal{D} \). But \((x,y) \notin W(\psi)\) because \( \|nx + y\| < \psi(n) \) cannot be satisfied when \( n \geq \ell \), so \((x,y) \notin \Pi \).

On the other hand, suppose \( S_\ell(x,y) < \infty \) for every \( \ell \in \mathbb{N} \), and let \( \psi \in \mathcal{D} \). Now, let \( \ell \in \mathbb{N} \) be given. Since \( S_\ell(x,y) \) converges while \( \sum_n \psi(n)^d \) diverges, we may choose some \( n \geq \ell \) such that
\[
\psi(n) > \min_{\ell \leq m \leq n} \|mx + y\| .
\]
(There are infinitely many such \( n \).) This and the fact that \( \psi \) is non-increasing imply that there is some \( m \geq \ell \)— specifically, the \( m \) realizing the minimum in (1) — for which.
\begin{equation}
\|mx + y\| < \psi(m).
\end{equation}

We have just shown that given any \(\ell \in \mathbb{N}\), there is some \(m \geq \ell\) satisfying (2). This implies that there are infinitely many \(m \in \mathbb{N}\) such that (2) holds. Therefore, \((x, y) \in W(\psi)\). Since \(\psi \in \mathcal{D}\) was arbitrary, we have \((x, y) \in \Pi\).

The following lemma implies that if \(S_\ell(x, y) = \infty\), then \(S_{\ell+1}(x, y) = \infty\).

\textbf{Lemma 3.2.} \textit{For any \(\ell \in \mathbb{Z}\), we have \(S_\ell(x, y) \leq \|\ell x + y\|^d + S_{\ell+1}(x, y)\).}

\textbf{Proof.} We have

\[
S_\ell(x, y) = \sum_{n \geq \ell, \ell \leq m \leq n} \min_{\ell \leq m \leq n} \|mx + y\|^d = \|\ell x + y\|^d + \sum_{n \geq \ell+1} \min_{\ell+1 \leq m \leq n} \|mx + y\|^d
\]

\[
\leq \|\ell x + y\|^d + \sum_{n \geq \ell+1} \min_{\ell+1 \leq m \leq n} \|mx + y\|^d = \|\ell x + y\|^d + S_{\ell+1}(x, y)
\]

as needed.

\textbf{Remark (some observations regarding \(S_\ell(x, y)\)).} Lemma 3.2 implies that if \(S_\ell(x, y)\) is ever infinite, then it will remain infinite if we increase \(\ell\). Much more is true. If there are no integers \(n\) for which \(\|nx + y\| = 0\), then \(S_\ell(x, y)\) is either finite for all \(\ell \in \mathbb{Z}\), or infinite for all \(\ell \in \mathbb{Z}\). This is because in this case the sums defining \(S_\ell(x, y)\) for different \(\ell \in \mathbb{Z}\) always have the same tail. On the other hand, if there is some integer \(n_0\) such that \(\|n_0x + y\| = 0\), then \(S_\ell(x, y)\) is either finite for all \(\ell > n_0\), or infinite for all \(\ell \geq n_0\).

We may conclude from this remark that, even though the criterion in lemma 3.1 seems to require checking infinitely many different sums \(S_\ell(x, y)\), we really only have to check one such sum for every given \(x, y \in \mathbb{R}^d\). For example, if \(\|nx + y\| \neq 0\) for all \(n \in \mathbb{N}\), then

\[
(x, y) \in \Pi \iff S_1(x, y) < \infty.
\]

Similarly, if there exists some \(\ell \in \mathbb{N}\) such that \(\|nx + y\| \neq 0\) for all \(n \geq \ell\), then it suffices to examine \(S_\ell(x, y)\). Part of the proof of theorem 3.4 (below) is phrased with reference to such an integer \(\ell\).

The following lemma is a simple consequence of bad approximability.

\textbf{Lemma 3.3.} \textit{Suppose \(x \in \mathcal{B}\mathcal{A}\). Then there exists a positive constant \(c := c(x)\) such that for any \(\varepsilon > 0\),

\[
\min\{n \in \mathbb{N} : \|nx\| < \varepsilon\} \geq \frac{c}{\varepsilon^d}.
\]

\textbf{Proof.} Let

\[
c = \inf n \ \|nx\|^d.
\]

Since \(x \in \mathcal{B}\mathcal{A}\), we know that \(c > 0\). Now, let \(\varepsilon > 0\), and suppose that \(\|nx\| < \varepsilon\). Then \(c \leq n \ \|nx\|^d < n\varepsilon^d\), which implies the lemma.

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We are now able to state a proof of theorem 2.1. We will actually prove the following alternative statement.

**Theorem 3.4 (Theorem 2.1).**

\[ x \in BA \iff \Phi(x) = \emptyset. \]

**Proof.** Let \( x \in BA \), and \( y \in \mathbb{R}^d \). Since \( x \not\in \mathbb{Q}^d \), we are guaranteed that there exists some \( \ell \in \mathbb{N} \) such that \( \|nx + y\| \neq 0 \) for all \( n \geq \ell \). (In fact, for any \( y \not\in \mathbb{Z}^x + \mathbb{Z} \), we may take \( \ell = 1 \).) Suppose \( t \geq \ell \) is a ‘best inhomogeneous approximation after \( \ell \)’ in that

\[ \|tx + y\| < \|t'x + y\| \quad \text{for all} \quad \ell \leq t' < t. \]

(Note that other authors may define ‘best inhomogeneous approximation’ differently, allowing for \( t \in \mathbb{Z} \) and comparing \( |t'| < |t| \). See remark 2.3.) At the next best inhomogeneous approximation \( \tilde{t}x+y \), we must have \( \tilde{t}x+y \) lying within \( 2\|tx + y\| \) of \( tx + y \), in the sup-norm of the torus \( \mathbb{R}^d/\mathbb{Z}^d \). This means that

\[ \|(\tilde{t} - t)x\| < 2\|tx + y\|. \]

So we will have

\[ \tilde{t} - t \geq \frac{c}{2^d\|tx + y\|^d}, \quad (3) \]

by lemma 3.3.

Now, recall that

\[ S_{\ell}(x,y) = \sum_{n \geq \ell} \min_{\ell \leq m \leq n} \|mx + y\|^d = \sum_k \|t_kx + y\|^d (t_{k+1} - t_k), \]

where \( \{t_k\} \) is the sequence of best inhomogeneous approximations after \( \ell \). By (3), this is bounded below by \( \sum_k c/2^d \), which diverges. By lemma 3.1, \( y \not\in \Phi(x) \), and we are done because \( y \) was arbitrary.

Now let \( x \in WA \). If \( x \in \mathbb{Q}^d \), then it is easy to see that \( \Phi(x) \neq \emptyset \), so assume otherwise. Since \( x \in WA \), there is a sequence \( \{n_k\}_{k \geq 0} \) of natural numbers with

\[ \sum_k n_k \|n_kx\|^d < \infty \]

and

\[ \sum_{k \geq K} \|n_kx\| \ll \|n_Kx\| \quad (\forall K \in \mathbb{N}). \]

(The notation \( \ll \) here means that there is some constant \( C > 0 \) such that \( \sum_{k \geq K} \|n_kx\| \leq C \|n_Kx\| \) holds for all \( K \in \mathbb{N} \).) Define

\[ -y = \sum_{k \geq 0} (n_kx - a_k), \]

where \( a_k \in \mathbb{Z}^d \) is such that \( \|n_kx - a_k\|_{\infty} = \|n_kx\| \), and let

\[ N_K = \sum_{k=0}^{K-1} n_k. \]
Notice then that
\[ \|N_K x + y\| = \left\| \sum_{k > K} (a_k - n_k x) \right\| \leq \sum_{k > K} \|n_k x\|. \] 
(6)

Now, let \( \ell \in \mathbb{N} \), and let \( K_\ell \in \mathbb{N} \) be such that \( N_{K_\ell} \geq \ell \). We will show that \( S_{N_{K_\ell}}(x, y) < \infty \), and this will imply, by lemma 3.2, that \( S_\ell(x, y) < \infty \). Note that
\[
S_{N_{K_\ell}}(x, y) \leq \sum_{k > K_\ell} \|N_K x + y\|^d (N_{K+1} - N_K)
\]
\[
\leq \sum_{k > K_\ell} \left( \sum_{k > K} \|n_k x\|^d \right) n_K
\]
\[
\approx \sum_{k > K_\ell} n_K \|n_k x\|^d,
\]
which converges, by (4). So, by lemma 3.2, \( S_\ell(x, y) < \infty \). Since \( \ell \) was arbitrary, lemma 3.1 tells us that \( y \in \Phi(x) \), hence \( \Phi(x) \neq \emptyset \).

From this proof, it is clear that if \( x \in \mathbf{W} \setminus \mathbf{Q}^d \), then \( \Phi(x) \) is uncountable. What if \( x \in \mathbf{Q}^d \)? Theorem A.1 in the appendix tells us that in the case \( d = 1 \), we have that \( \Phi(x) \) is countable whenever \( x \in \mathbf{Q} \). In fact in this case we have that
\[
\Phi(x) = \{ y \in \mathbf{Q} : (\mathbb{Z} x + y) \cap \mathbb{Z} \neq \emptyset \}.
\]
(Proposition A.2)

The proof uses continued fractions, which are not available in higher dimensions. Still, we expect an analogous statement to be true in general.

Many questions remain. Here is another: is \( \Theta(y) \) always non-empty?

4. Proof of theorem 2.2

Lemma 4.1. We may write
\[ \Pi = \bigcap_{\psi \in D'} W(\psi) \]
where \( D' \subset D \) consists of the functions in \( D \) that only take values that are reciprocals of natural numbers.

Proof. To any \( \psi \in D \) we associate the sequence \( k := \{ k_n \} \in \mathbb{N}^\mathbb{N} \) defined by
\[
k_n - 1 < \frac{1}{\psi(n)} \leq k_n,
\]
and we define \( \psi_k \) by
\[ \psi_k(n) = \frac{1}{k_n}. \]
That is, for each \( n \) we replace \( \psi(n) \) with the largest rational number \( 1/k \) \((k \in \mathbb{N}) \) such that \( 1/k \leq \psi(n) \). Clearly, we have \( \psi_k(n) \leq \psi(n) \) for every \( n \in \mathbb{N} \), and therefore \( W(\psi_k) \subseteq W(\psi) \).
We only have to show that $\psi_k \in D$. But it is obvious that $\psi_k$ is non-increasing, so what we must show is that $\sum_n \psi_k(n)^d$ diverges. In the first case, if $k_n$ does not diverge to infinity, it is therefore bounded, and so $\psi_k$ is bounded below by some positive number, hence the series diverges. In the second case, if $k_n \uparrow \infty$, then we have $k_n > 1$ for every sufficiently large $n$. For these $n$, we have $1/k_n \leq \psi(n) < 1/(k_n - 1)$. Divergence of $\sum_n \psi(n)^d$ implies the divergence of $\sum_n 1/(k_n - 1)^d$, which implies divergence of $\sum_n 1/k_n^d = \sum_n \psi_k(n)^d$, so $\psi_k \in D$ as claimed. This proves the lemma.

In view of lemma 4.1, we write
\[
\Pi = \bigcap_{k \in D} W(k)
\]
where $D \subset \mathbb{N}^\mathbb{N}$ denotes the non-decreasing sequences whose reciprocals to $d$th powers form a divergent series, and $W(k) := W(\psi_k)$.

In our proof that $\Pi, \Theta(y), \Phi(x)$ are measurable, we will use some terminology from descriptive set theory. Suppose $X$ is the set $[0,1]^d$ or $[0,1]^{2d}$. A subset of $X$ is called analytic if it is the projection of a Borel set in $\mathbb{N}^\mathbb{N} \times X$. A subset of $X$ is called coanalytic if it is the complement of an analytic set. We will use the following basic facts:

- Coanalytic sets are measurable.
- The projection of an analytic set in $\mathbb{N}^\mathbb{N} \times X$ is an analytic set in $X$.
- Borel sets are exactly those which are both analytic and coanalytic.

These and more are found in [12, 17].

**Lemma 4.2.** The sets $\Pi, \Theta(y), \Phi(x)$ are coanalytic, hence measurable.

**Proof.** We will prove that $\Pi^c$ is analytic. Notice that $\Pi^c$ is the projection from $\mathbb{N}^\mathbb{N} \times [0,1]^{2d}$ to $[0,1]^{2d}$ of the set
\[
\mathcal{W}^c = \{ (k,x,y) : k \in D, (x,y) \notin W(k) \}.
\]
Since analytic sets are closed under projections, it is enough to show that $\mathcal{W}^c$ is analytic. In fact, we will see that it is Borel.

Notice that
\[
\mathcal{W}^c = (D \times [0,1]^{2d}) \cap \{ (k,x,y) : (x,y) \notin W(k) \}.
\]
The set $D \times [0,1]^{2d}$ is Borel. And the set $\{(k,x,y) : (x,y) \notin W(k)\}$ can be expressed as
\[
\bigcup_{m=1}^\infty \bigcap_{n=m}^\infty \{ (k,x,y) : \|nx + y\| \geq 1/k(n) \}
\]
\[
= \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty \bigcap_{\ell=1}^\infty \{ (k,x,y) : k(n) = \ell \text{ and } \|nx + y\| \geq 1/\ell \}.
\]
The set of all $(k,x,y) \in \mathbb{N}^\mathbb{N} \times [0,1]^{2d}$ such that $k(n) = \ell$ is Borel, and so is the set of all $(k,x,y) \in \mathbb{N}^\mathbb{N} \times [0,1]^{2d}$ such that $\|nx + y\| \geq 1/\ell$. Since Borel sets are closed under countable unions and intersections, we are done.

The same argument works for $\Theta(y)$ and $\Phi(x)$. □
Now that we know the sets of interest to us are measurable, we can proceed with calculating their measures.

**Theorem 4.3 (Theorem 2.2).** The set \( \Pi \) is Lebesgue measurable, and for every \( x \in \mathbb{R}^d \), the set \( \Phi(x) \) has Lebesgue measure 0.

**Proof.** The first part of the theorem—that \( \Pi \) is measurable—has just been proved as theorem 4.2, so let us move on to the next part.

Let \( x \in \mathbb{R}^d \). Either the set \( \{nx \pmod{1}\}_{n \geq 0} \) is dense in \([0,1]^d\), or its closure is a proper subtorus \( T \subset [0,1]^d \). In the latter case, we see that if \( y \notin T \), then \( \|nx + y\| \) is uniformly bounded below, and therefore, if \( \psi \in \mathcal{D} \) has the property that \( \psi(n) \to 0 \) as \( n \to \infty \), then \((x,y) \notin W(\psi)\). Hence \((x,y) \notin \Pi\). This shows that \( \Phi(x) \subset T \), so has Lebesgue measure 0.

Now suppose that \( \{nx \pmod{1}\}_{n \geq 0} \) is dense in \([0,1]^d\). In this case, the translation \( y \mapsto y + x \pmod{1} \) is an ergodic transformation of \([0,1]^d\). Notice that if \( \|n + 1)x + y\| < \psi(n + 1) \), then we have

\[
\|nx + (y + x)\| < \psi(n + 1) \leq \psi(n).
\]

This shows that \( \Phi(x) + x \subseteq \Phi(x) \). That is, \( \Phi(x) \cap [0,1]^d \) is an invariant measurable (by lemma 4.2) set for the toral translation by \( x \). Therefore, \( \Phi(x) \cap [0,1]^d \) has measure 0 or 1.

Now, recall that Kurzweil’s theorem asserts that

\[
\bigcap_{\psi \in \mathcal{D}} \{x : W_x(\psi) \text{ has full measure} \} = \mathbb{BA},
\]

where

\[
W_x(\psi) \overset{\text{def}}{=} \{y \in \mathbb{R}^d : (\exists n \in \mathbb{N}) \|nx + y\| < \psi(n)\}.
\]

This implies that for any \( x \in \mathbb{BA} \), there is some \( \psi \in \mathcal{D} \) such that

\[
\text{Leb} \left( W_x(\psi) \cap [0,1]^d \right) < 1,
\]

which in turn implies that

\[
\text{Leb} \left( \Phi(x) \cap [0,1]^d \right) < 1
\]

because

\[
\Phi(x) = \bigcap_{\psi \in \mathcal{D}} W_x(\psi).
\]

The previous paragraph now implies that \( \Phi(x) \cap [0,1]^d \) has measure 0, hence \( \Phi(x) \) does\(^1\).

On the other hand, if \( x \in \mathbb{BA} \), then theorem 3.4 tells us that \( \Phi(x) = \emptyset \), so it has measure 0. \[\square\]

\(^1\) A reviewer has drawn our attention to [16, lemma 12], where it is proved that for any \( x \notin \mathbb{BA} \) there exists \( \psi \in \mathcal{D} \) such that \( \text{Leb}(W_x(\psi)) = 0 \). By citing this, one can write a shorter proof that \( \Phi(x) \) is null.
It follows that $\Pi$ has measure 0, too, and that for almost every $y \in \mathbb{R}^d$, the set $\Theta(y)$ has measure 0. We ask: can $\Theta(y)$ ever have a measure other than 0? We show below that the only other possible measure for $\Theta(y) \cap [0,1]^d$ is 1.

The following lemma states that $\Pi$ is invariant under integer contractions in the ‘homogeneous’ direction, and integer dilations in the ‘inhomogeneous’ direction. We will use it to prove a 0-1 law for the measure of $\Theta(y)$.

Lemma 4.4. We have
\[
\left(\frac{1}{v} \cdot 1_d, \frac{1}{u} \cdot 1_d\right) \Pi \subseteq \Pi
\]
for any $s \in (0,1]$ and $u,v \in \mathbb{N}$.

Proof. We will prove this fiber-wise. That is, we show that $u \cdot \Phi(x) \subseteq \Phi(x)$ and $(1/v) \cdot \Theta(y) \subseteq \Theta(y)$.

Suppose $y \in \Phi(x)$ and let $\psi \in \mathcal{D}$ and $u \in \mathbb{N}$. Define $\tilde{\psi}(n) = \frac{1}{u} \psi(un)$. Notice that $\tilde{\psi}$ is non-increasing, and since $\{un\}_{n \geq 1}$ is an arithmetic sequence and $\psi$ is non-increasing, we therefore have that $\sum \tilde{\psi}(n)^d$ diverges. Therefore $\tilde{\psi} \in \mathcal{D}$, and there are infinitely many solutions $n \in \mathbb{N}$ to $\|nx+y\| < \tilde{\psi}(n)$. But $\|nx+uy\| \leq u \|n/u)x+y\|$, and therefore there are infinitely many $n \in \mathbb{N}$ (all of them being multiples of $u$) satisfying $\|nx+uy\| < u \tilde{\psi}(n/u) = \psi(n)$. This shows that $uy \in \Phi(x)$ and therefore
\[
u \cdot \Phi(x) \subseteq \Phi(x)
\]
for all $u \in \mathbb{N}$.

Suppose $x \in \Theta(y)$ and let $\psi \in \mathcal{D}$ and $v \in \mathbb{N}$. Define $\tilde{\psi}(n) = \psi(vn)$. It is clear that $\tilde{\psi} \in \mathcal{D}$, therefore there are infinitely many solutions $n \in \mathbb{N}$ to $\|nx+y\| < \tilde{\psi}(n)$. Then there are infinitely many solutions $n \in \mathbb{N}$ to $\|nx/v+y\| < \tilde{\psi}(n/v) = \psi(n)$. Since $\psi \in \mathcal{D}$ was arbitrary, $x/v \in \Theta(y)$. We have just shown that
\[
u \cdot \Theta(y) \subseteq \Theta(y)
\]
for all $s \in \mathbb{N}$.

Now a standard argument using the Lebesgue density theorem will prove the desired 0-1 law.

Lemma 4.5. For any $y \in \mathbb{R}^d$, the set $\Theta(y) \cap [0,1]^d$ has measure 0 or 1.

Proof. By lemma 4.4,
\[
u \cdot \Theta(y) \subseteq \Theta(y)
\]
for all $\nu \in \mathbb{N}$. But $\Theta(y)$ is 1-periodic (in all $d$ coordinate directions), and therefore $(1/q) \cdot \Theta(y)$ is $(1/q)$-periodic. It follows that
have the same Lebesgue measure, and since one is contained in the other, we have that

\[
\text{Leb} \left( \left( \frac{1}{q} \cdot \Theta(y) \right) \triangle \Theta(y) \right) = 0,
\]

that is, the symmetric difference is null.

Now, suppose that \( \Theta(y) \) has positive Lebesgue measure and let \( x \in \Theta(y) \) be a density point, meaning in particular that

\[
\text{Leb} \left( x + \left( -\frac{1}{2q}, \frac{1}{2q} \right)^d \cap \Theta(y) \right) \sim \frac{1}{q^d} \quad (q \to \infty).
\]

But by (7),

\[
\text{Leb} \left( x + \left( -\frac{1}{2q}, \frac{1}{2q} \right)^d \cap \Theta(y) \right) = \text{Leb} \left( x + \left( -\frac{1}{2q}, \frac{1}{2q} \right)^d \cap \frac{1}{q} \cdot \Theta(y) \right),
\]

and again by 1-periodicity of \( \Theta(y) \), the measure of the intersection of \( (1/q) \cdot \Theta(y) \) with any \( d \)-cube of side-length \( 1/q \) is equal to \( 1/q^d \) times the measure of the intersection of \( \Theta(y) \) with \([0,1]^d\), so we may conclude that

\[
\text{Leb} \left( x + \left( -\frac{1}{2q}, \frac{1}{2q} \right)^d \cap \Theta(y) \right) = \frac{1}{q^d} \cdot \text{Leb} \left( [0,1]^d \cap \Theta(y) \right).
\]

Finally, (8) and (9) together imply that \([0,1]^d \cap \Theta(y)\) has Lebesgue measure 1, hence \( \Theta(y) \) is full, by periodicity.

Our question above becomes: Does there exist \( y \in \mathbb{R}^d \) for which \( \Theta(y) \) has full measure?

Given that the sets \( \Pi \) and \( \Phi(x) \) have zero Lebesgue measure, it is natural to ask: what are their Hausdorff dimensions? Since \( \pi(\Pi) \) has full measure, it therefore has Hausdorff dimension \( d \), hence we have that the Hausdorff dimension of \( \Pi \) is at least \( d \). What is it exactly? In the \( d = 1 \) case, one can be explicit in the proof of theorem 2.1 using continued fractions, and find \( x \in \mathbb{R} \) such that \( \Phi(x) \) contains Cantor-type sets whose dimensions we can calculate. Perhaps this kind of strategy can be pushed, and even adapted for general \( d \).

5. Weighted approximation

Let \( \mathcal{R} \equiv \{ r \in \mathbb{R}^d : r_i \geq 0, \sum r_i = 1 \} \) be the standard simplex. For \( r = (r_i) \in \mathcal{R} \) and \( x = (x_i) \in \mathbb{R}^d \), denote

\[
\| x \|_r \overset{\text{def}}{=} \left( \max_{1 \leq i \leq d} \| x_i \|^{1/r_i} \right)^{1/d},
\]
and notice that $\| \cdot \|_r$ coincides with $\| \cdot \|$ (in $\mathbb{R}^d$) when $r = (1/d, \ldots, 1/d)$. Define

$$\text{BA}(r) \overset{\text{def}}{=} \{ x \in \mathbb{R}^d : \inf n \|nx\|_r^d > 0 \},$$

the set of $r$-weighted badly approximable vectors. For $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $r \in \mathcal{R}$, let

$$W(r, \psi) \overset{\text{def}}{=} \{ x \in \mathbb{R} : (\exists n \in \mathbb{N}) \|nx\|_r < \psi(n) \}.$$

Let

$$\Pi_r \overset{\text{def}}{=} \bigcap_{\psi \in D} W(r, \psi).$$

Finally, let

$$V(r, \psi) \overset{\text{def}}{=} \{ x \in \mathbb{R}^d : (\exists \infty n \in \mathbb{N}) \|nx\|_r < \psi(n) \}.$$

Kurzweil’s theorem has the following generalization to the weighted case, due to Harrap.

**Theorem 5.1 ([11]).**

$$\forall r \in \mathcal{R} \quad \text{BA}(r) = \bigcap_{\psi \in D} V(r, \psi).$$

We have the following weighted version of theorem 2.1.

**Theorem 5.2.**

$$\forall r \in \mathcal{R} \quad \text{WA}(r) = \pi (\Pi_r) \quad \text{hence} \quad \text{BA}(r) = \mathbb{R}^d \setminus \pi (\Pi_r).$$

**Proof.** All statements in section 3 remain true if all instances of $\| \cdot \|, W(\psi), \text{BA}, \text{WA}, \Phi(x)$ are replaced with $\| \cdot \|_r, W(r, \psi), \text{BA}(r), \text{WA}(r), \Phi_r(x)$, their weighted analogues. \(\square\)

The arguments from section 4 also hold for weighted approximations, giving us the following.

**Theorem 5.3.** For any $r \in \mathcal{R}$ the set $\Pi_r$ is Lebesgue measurable and has measure 0. In fact, for every $x \in \mathbb{R}^d$, the set $\Phi_r(x) \subset \mathbb{R}^d$ has Lebesgue measure 0.

### 6. Higher irrationality exponents

For $\sigma \geq 1/d$, consider the set

$$\sigma \text{BA} \overset{\text{def}}{=} \{ x \in \mathbb{R} : \liminf_{n \to \infty} n^\sigma \|nx\| > 0 \}$$

of $\sigma$-badly approximable vectors. Naturally,

$$\sigma \text{WA} \overset{\text{def}}{=} \mathbb{R} \setminus \sigma \text{BA}$$
is referred to as the set of \( \sigma \)-well approximable vectors. Notice that \( \sigma \mathcal{BA} \) and \( \sigma \mathcal{WA} \) coincide with the usual badly approximable and well approximable vectors when \( \sigma = 1/d \).

Kurzweil’s theorem has been generalized to this setting by Tseng. For \( t > 0 \), let
\[
\mathcal{D}' = \{ \psi' : \psi \in \mathcal{D} \}.
\]
Tseng proves the following.

**Theorem 6.1** ([22]).
\[
(\forall \sigma \geq 1/d) \quad \sigma \mathcal{BA} = \bigcap_{\psi \in \mathcal{D}/\sigma} V(\psi).
\]

Theorems 2.1 and 2.2 also transfer easily to higher irrationality exponents. Let
\[
\sigma \mathcal{I} = \bigcap_{\psi \in \mathcal{D}'/\sigma} W(\psi).
\]
We have

**Theorem 6.2.**
\[
(\forall \sigma \geq 1/d) \quad \sigma \mathcal{WA} = \pi (\sigma \mathcal{I}) \quad \text{hence} \quad \sigma \mathcal{BA} = \mathbb{R}^d \setminus \pi (\sigma \mathcal{I}).
\]

**Remark.** Notice that in theorem 6.2 the intersection is over \( \psi \in \mathcal{D}'/\sigma \), while in theorem 6.1 the intersection is over \( \psi \in \mathcal{D}/\sigma \).

**Proof.** As in the proof of theorem 5.2, we only have to adjust the proof of theorem 2.1. In this case, the main adjustment is to re-define \( S_\ell(x,y) \) as
\[
S_\ell(x,y) \overset{\text{def}}{=} \sum_{n \geq \ell} \min_{m \leq n} \| mx + y \|^{1/\sigma}.
\]
(Notice that if \( \sigma = 1/d \), then we have it as originally defined.) Then the proof of lemma 3.1 is easily adapted to prove a version saying that membership in \( \sigma \mathcal{I} \) is equivalent to convergence of \( S_\ell \), as before. Lemmas 3.2 and 3.3 both remain true if every instance of ‘\( d \)’ is replaced with ‘\( 1/\sigma \)’. And finally the logic of the proof of theorem 2.1 goes through with the new inputs. \( \square \)

Theorem 2.2 generalizes to higher irrationality exponents even more easily, because \( \mathcal{D}'/\sigma \supset \mathcal{D}/\sigma \), and therefore \( \sigma \mathcal{I} \subset \mathcal{I} \).

**Theorem 6.3.** For every \( \sigma \geq 1/d \), the set \( \sigma \mathcal{I} \) is Lebesgue measurable and has measure 0. In fact, all of its ‘vertical’ fibers have \( d \)-dimensional Lebesgue measure 0.

In this case, it would be more informative to study the Hausdorff measures of \( \sigma \mathcal{I} \) and its horizontal and vertical fibers.

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Appendix. The case \( d = 1 \)

There are some things that are easier to prove when \( d = 1 \), but which may still be true in the general case. Chief among them is the following refinement of theorem 2.1.

**Theorem A.1.**

\[
\#\Phi(x) = \begin{cases} 
\aleph_0 & \text{if } x \in \mathbb{Q} \\
2^{\aleph_0} & \text{if } x \in \mathbb{WA} \setminus \mathbb{Q} \\
0 & \text{if } x \in \mathbb{BA}.
\end{cases}
\]

Theorem 2.1 already tells us that \( \Phi(x) = \emptyset \) when \( x \in \mathbb{BA} \), and its proof shows us that \( \Phi(x) \) is uncountable whenever \( x \in \mathbb{WA} \setminus \mathbb{Q} \). It is only left to show that \( \Phi(x) \) is countable when \( x \in \mathbb{Q} \). This follows immediately from the following proposition, which completely describes \( \Theta(y) \) and \( \Phi(x) \) when \( x, y \in \mathbb{Q} \).

**Proposition A.2.**

1. If \( y \in \mathbb{Q} \), then
   \[
   \Theta(y) = \{ x \in \mathbb{Q} : (\mathbb{Z}x + y) \cap \mathbb{Z} \neq \emptyset \}.
   \]
2. If \( x \in \mathbb{Q} \), then
   \[
   \Phi(x) = \{ y \in \mathbb{Q} : (\mathbb{Z}x + y) \cap \mathbb{Z} \neq \emptyset \}.
   \]

In the proof, we will use the next two lemmas about \( S_{\ell} \).

**Lemma A.3.** \( S_{\ell+k}(x, y) = S_{\ell}(x, kx + y) \).

**Proof.** We have

\[
S_{\ell+k}(x, y) = \sum_{n \geq \ell+k} \left( \min_{\ell+k \leq m \leq n} \|(mx + y)\| \right)
\]

\[
= \sum_{n \geq \ell+k} \left( \min_{\ell+k \leq m \leq n} \|(m-k)x + (kx + y)\| \right)
\]

and letting \( \tilde{n} = n - k \) and \( \tilde{m} = m - k \),

\[
= \sum_{\tilde{n} \geq \ell} \left( \min_{\ell \leq \tilde{m} \leq \tilde{n}} \|\tilde{m}x + (kx + y)\| \right)
\]

\[
= S_{\ell}(x, kx + y),
\]

as claimed. \qed

**Lemma A.4.** If \( x \in \mathbb{R} \setminus \mathbb{Q} \) and \( y \in \mathbb{R} \), and \( k \in \mathbb{Z} \) is such that \( kx + y \in \mathbb{Z} \), then

\[
S_{\ell}(x, y) = \begin{cases} 
0 & \text{if } \ell \leq k \\
\infty & \text{if } \ell > k.
\end{cases}
\]
In particular,

\[ S_\ell(x,0) = \begin{cases} 0 & \text{if } \ell \leq 0 \\ \infty & \text{if } \ell > 0 \end{cases} \]

for any irrational \(x\).

**Proof.** The lemma reduces to the \(y = 0\) case after observing that \(S_\ell\) is invariant by integer shifts in both coordinates, and applying lemmas 3.2 and A.3.

Let us therefore prove the ‘in particular’. Let \(\ell \leq 0\). Then

\[ \min_{\ell \leq m \leq n} \|mx\| = 0 \]

for all \(n \geq \ell\), and so \(S_\ell(x,0) = 0\).

Now note that

\[ S_1(x,0) = \sum_{n \geq 1} \left( \min_{1 \leq m \leq n} \|mx\| \right) = \sum_{n \geq 0} \|q_nx\| (q_{n+1} - q_n) \]

where \(q_n\) denotes a convergent of the continued fraction \(x = [a_0; a_1, a_2, \ldots]\). This is bounded by

\[
\sum_{n \geq 0} \|q_nx\| (q_{n+1} - q_n) \geq \sum_{n \geq 0} \frac{q_{n+1} - q_n}{q_{n+1} + q_n} \\
\geq \sum_{n \geq 0} \left( 1 - \frac{2q_n}{q_{n+1} + q_n} \right) \\
> \sum_{n \geq 0} \left( 1 - \frac{2}{(q_{n+1} + 1)} \right), \tag{A.1}
\]

and if there are infinitely many partial quotients greater than 1, then there will be infinitely many terms bounded below by 1/3, and the sum will diverge.

It is only left to treat the case where \(x = [a_0; a_1, \ldots, a_M, \bar{T}]\). Here we can bound (A.1) by

\[
\sum_{n \geq 0} \left( 1 - \frac{2q_n}{q_{n+1} + q_n} \right) \geq \sum_{n \geq M} \left( 1 - \frac{2q_n}{q_{n+1} + q_n} \right) \\
= \sum_{n \geq M} \left( 1 - \frac{2q_n}{q_{n+2}} \right) \\
= \sum_{n \geq M} \left( 1 - 2 \left( \frac{q_n}{q_{n+1}} \right) \left( q_{n+1}/q_{n+2} \right) \right).
\]

Now, since \(\lim_{n \to \infty} \left( \frac{q_n}{q_{n+1}} \right) = \varphi^{-1}\), where \(\varphi\) is the golden ratio, we have that

\[
\left( 1 - 2 \left( \frac{q_n}{q_{n+1}} \right) \left( q_{n+1}/q_{n+2} \right) \right) \to 1 - \frac{2}{\varphi^2},
\]

which is positive, hence the sum diverges.

Now, by lemma 3.2, we have that \(S_\ell(x,0) = \infty\) for all \(\ell \geq 1\). \(\square\)
The following special case of proposition A.2 tells us that the intersections of all sets coming from the divergence case of Khintchine’s theorem is exactly \( Q \). This is an interesting fact in itself.

**Lemma A.5.** Let \( d = 1 \). Then \( \Theta(0) = Q \).

**Proof.** Notice that if \( x \in Q \), then \( (x, 0) \in W(\psi) \) for any \( \psi \in \mathcal{D} \), because we have \( \|nx\| = 0 \) for infinitely many \( n \). Therefore, \( \Theta(0) \subseteq Q \). On the other hand, if \( x \in \mathbb{R} \setminus Q \), then \( S_1(x, 0) = \infty \), by lemma A.4. So \( x \notin \Theta(0) \) by lemma 3.1. Therefore \( \Theta(0) = Q \) as claimed. \( \square \)

**Proof of proposition A.2.** If \( x, y \) are rational and \( \mathbb{Z}x + y \) contains no integers, then the set \( \{nx + y (\text{mod} 1)\}_{n \in \mathbb{Z}} \) is finite, hence the expression \( \|nx + y\| \) is bounded below uniformly over \( n \in \mathbb{Z} \). Therefore \( S_\ell(x, y) = \infty \) for all \( \ell \in \mathbb{N} \), and by lemma 3.1, \( x \notin \Theta(y) \) and \( y \notin \Phi(x) \). (In fact, this argument also shows that \( \Phi(x) \) contains no irrationals if \( x \) is rational.) On the other hand, if \( \mathbb{Z}x + y \) contains integers, then \( \|nx + y\| = 0 \) on an arithmetic sequence of \( n \in \mathbb{N} \), so \( S_\ell(x, y) < \infty \) for all \( \ell \in \mathbb{N} \), and by lemma 3.1, \( x \in \Theta(y) \) and \( y \in \Phi(x) \).

It is only left to prove that if \( y \in Q \), then \( \Theta(y) \) contains no irrationals. Write \( y = plq \) for some \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \). Lemma 4.4 implies that \( \Theta(y) \subseteq \Theta(qy) = \Theta(p) \). And, by periodicity, we have \( \Theta(p) = \Theta(0) \). But lemma A.5 states that \( \Theta(0) = Q \), hence \( \Theta(y) \subseteq Q \). \( \square \)

The following corollary is immediate.

**Corollary A.6.** If \( (x, y) \in \Pi \), then either \( x \) and \( y \) are both rational, or they are both irrational.

**Lemma A.7.** If \( 1, x, y \) are rationally dependent and \( x \) (or \( y \)) is irrational, then \( (x, y) \notin \Pi \).

**Proof.** Suppose \( k_1x + k_2y + k_3 = 0 \). with \( (k_1, k_2, k_3) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\} \). Notice that if either \( x \) or \( y \) is irrational, then they both are. Then, by lemma A.4, \( S_\ell(x, k_2y) = \infty \) for all sufficiently large \( \ell \) and therefore \( k_2y \notin \Phi(x) \), by lemma 3.1. But lemma 4.4 tells us that \( k_2 \cdot \Phi(x) \subseteq \Phi(x) \), therefore \( y \notin \Phi(x) \). The lemma follows. \( \square \)

Lemma A.7 raises the question of whether \( \Pi \) includes all \( (x, y) \) such that \( 1, x, y \) are rationally independent. The answer is ‘no’. For any \( x \), the set of \( y \) such that \( 1, x, y \) are rationally independent has full measure. But by theorem 2.2 the set \( \Phi(x) \) has measure 0.

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