Darboux polynomial matrices: the classical massive Thirring model as a study case

A Degasperis

Dipartimento di Fisica, Sapienza Università di Roma, Rome, Italy

E-mail: antonio.degasperis@uniroma1.infn.it

Received 25 November 2014, revised 19 March 2015
Accepted for publication 7 April 2015
Published 21 May 2015

Abstract
One way of constructing explicit expressions of solutions of integrable systems of partial differential equations goes via the Darboux method. This requires the construction of Darboux matrices. Here we introduce a novel algorithm to obtain such matrices in polynomial form. Our method is illustrated by applying it to the classical massive Thirring model, and by combining it with the Dihedral group of symmetries possessed by this model.

Keywords: integrable PDE, nonlinear waves, soliton solutions

1. Introduction

Several systems of coupled nonlinear partial differential equations (PDEs) have been proved to be both integrable and good (though approximate) models of a variety of wave propagation phenomena. The mathematical property of integrability, together with the broad range of physical contexts in which integrable models find their application, are the main ingredients of soliton theory. This subject is almost half-century old and has motivated such a vast production of results that we limit ourselves to quote only those references which are strictly related to our present work. In soliton theory a particularly fruitful method of constructing special wave solutions (mainly soliton solutions) is that introduced by Darboux in 1882 while investigating ordinary differential equations (ODEs). If applied to the Lax pair of matrix ODEs, this method yields explicit expressions of parametric families of soliton solutions (see f.i. [1]) in both the isospectral evolution case (which is the one considered here) and non-isospectral evolution case. The basic point of this method is the covariance (form-invariance) of a linear homogeneous matrix ODE of the form

\[ \Psi_\xi = A(\zeta, \xi) \Psi \]  

(1)

with respect to a linear transformation (Darboux transformation, DT) of the dependent variable \( \Psi \to \Psi' = D(\zeta, \xi) \Psi \). Thus the covariance requirement is
with additional conditions as specified here below. The real variable $\xi$ is the independent one, the subscript standing for differentiation, the coefficient $A(\zeta, \xi)$ as well as the transformed coefficient $A'(\zeta, \xi)$ (and therefore also $\Psi, \Psi'$, and the DT matrix $D(\zeta, \xi)$) are $M \times M$ matrices. Moreover the matrix $A(\zeta, \xi)$ is assumed to have a rational dependence on the (so-called spectral) complex parameter $\zeta$, namely

$$A(\zeta, \xi) = \sum_{j=0}^{n} A_j(\xi) \zeta^j + \sum_{k=1}^{l} \frac{Q_k(\xi)}{\zeta - \alpha_k} \quad (3)$$

and the Darboux matrix $D(\zeta, \xi)$ is required to maintain this form, namely

$$A'(\zeta, \xi) = \sum_{j=0}^{n} A'_j(\xi) \zeta^j + \sum_{k=1}^{l} \frac{Q'_k(\xi)}{\zeta - \alpha_k} \quad (4)$$

Finally, and most crucially, the Darboux matrix itself is asked to have a rational dependence on $\zeta$,

$$D(\zeta, \xi) = \sum_{j=0}^{N} D^{(j)}(\xi) \zeta^j + \sum_{k=1}^{L} \frac{R_k(\xi)}{\zeta - \beta_k} \quad (5)$$

As a matter of fact, this rational structure of the Darboux matrix can be changed, without affecting the action of the DT itself, by multiplying the matrix $D(\zeta, \xi)$ by an arbitrary scalar $\xi$-independent function $f(\zeta)$. This way the matrix (5), according to personal taste, may be given equally well a polynomial expression of degree $N + L$ or a simple-pole expression with $N + L$ singularities, or any intermediate form like (5). While dealing with concrete cases, computational tasks aimed to construct $D(\zeta, \xi)$ may suggest the form of this matrix which seems to be more convenient. By and large, the pole expansion is naturally close to the dressing method [2], or to the inverse spectral technique (see f.i. [3]), and, for an instructive review of the Darboux method in this connection, see also [5]. As for the polynomial form, its use is perhaps less common and its formulation for an arbitrary degree of $D(\zeta, \xi)$ can be found for instance in [6, 7] (see also [1, 5]).

The purpose of the following section is introducing the Lagrange form of the Darboux matrix which is alternative to the power expansion representation. In fact the choice of the representation is a general issue concerning any matrix valued polynomial and has no relation with the Lax pair of ODEs. Combining this representation with the Lax pair is the main content of the next section 3. Our treatment there will be focused on the (classical) massive Thirring model (MTM) [8] and confined, as an illustration of our method, to the simplest DT associated with the Lax pair. Our interest in this integrable model is not only motivated by its field theoretical role as a nonlinear Dirac spinor system (which is the relativistic counterpart of the nonlinear Schrödinger equation), but also by the fact that it may be considered close enough to propagation equations of two coupled optical modes in nonlinear periodic media [9, 10]. The construction of the $N$-soliton solution, over both the vacuum [11] and the continuous wave background [12], have been done by the dressing method (say by the simple-pole formula of the Darboux matrix). Thus the expression of these solutions is already available (see also [13]) and it is not our primary concern. Here we rather focus on the construction of the lowest degree polynomial Darboux matrix to point out the novel features of our technique which takes advantage of a matrix version of the Lagrange interpolation method (for a similar observation in a more elementary context see [14]). While doing this we also combine our treatment with the symmetries of the MTM which are related to the dihedral
symmetry group $D_2$ of the corresponding Lax pair (for the dihedral group as reduction group see [15, 16]). Finally, in section 4, a few comments and conclusions are briefly reported.

2. Lagrange form of polynomial matrices

To the purpose of introducing a new method of constructing polynomial Darboux matrices, we preliminarily consider in this section the Lagrange interpolation algorithm as applied to matrix valued polynomials. Let $\{\zeta_1, \zeta_2, \ldots, \zeta_N\}$ be a given set of $N$ complex numbers such that $\zeta_j \neq \zeta_k$ if $j \neq k$. In the space of $(N - 1)$-degree polynomials of $\zeta$ the Lagrange basis is defined as

$$L_j^{(N)}(\zeta) = \frac{Q_j^{(N)}(\zeta)}{Q_j^{(N)}(\zeta_j)}, \quad j = 1, \ldots, N, \quad Q_j^{(N)}(\zeta) = \prod_{k \neq j} (\zeta - \zeta_k). \quad (6)$$

The $N$ Lagrange polynomials $L_j^{(N)}(\zeta)$ are such that

$$L_j^{(N)}(\zeta_k) = \delta_{jk}, \quad j, k = 1, \ldots, N, \quad (7)$$

where the right-hand side is the Kronecker delta. We note here that the coefficients $C_{jk}^{(N)}$ of $L_j^{(N)}(\zeta)$,

$$L_j^{(N)}(\zeta) = \sum_{k=1}^{N} C_{jk(N)}^{k-1}, \quad (8)$$

are the matrix elements of the inverse $C^{(N)}$ of the $N \times N$ Vandermonde matrix $V^{(N)}(\zeta_1, \ldots, \zeta_N)$, whose entries are $V_{jk}^{(N)} = \zeta_j^{k-1}$, namely $C^{(N)} = V^{(N)-1}$. In fact, relations between Darboux matrices and Vandermonde, or Vandermonde-like matrices, have been already pointed out [7]. Here we do not have to use the explicit expression of the inverse of the Vandermonde matrix, see f.i. [17], with the exception for a few of them in section 3. Our main use of the Lagrange polynomials (8) is the expression

$$D(\zeta) = \sum_{j=1}^{N} L_j^{(N)}(\zeta_j)D_j, \quad (9)$$

of the $M \times M$ matrix valued polynomial $D(\zeta)$ of degree $N - 1$ whose values $D_j$ at $N$ given points $\zeta_j$ of the complex plane, $D(\zeta_j) = D_j$, are given. For future reference, we observe that if the polynomial $D(\zeta)$ is of degree $S$ (as we assume from now on), with $S < N - 1$, then the expression (9) still holds provided the given matrices $D_j$, $j = 1, \ldots, N$ satisfy the $N - S - 1$ conditions

$$\sum_{j=1}^{N} C_{jk(N)}^{(N)}D_j = 0, \quad k = S + 1, \ldots, N - 1. \quad (10)$$

It is obvious that these conditions are identically satisfied if the matrix $D_j$ were set a priori as the value of the given polynomial $D(\zeta)$ at $\zeta = \zeta_j$. Instead, if $D_j$ is chosen by a different criterium, as we do below, these conditions (10) become significant and crucial to our method. This observation is relevant to the next step that is assigning the $n$ points $\zeta_j$ as the roots, which are assumed to be simple, of the polynomial

$$P(\zeta) = \det[D(\zeta)], \quad P'(\zeta_j) = 0. \quad (11)$$
Thus \( N = MS \) and the expression (9) is the \((MS - 1)\)-degree Lagrange form of the \(S\)-degree polynomial \(D(\zeta)\). This implies that the given matrices \(D_j\) have to satisfy the \(S(M - 1) - 1\) conditions (10) with \(N = MS\). Since, by construction, \(\det D_j = 0\), for each \(j\) there exists a \(M\)-dimensional non-vanishing vector \(z_j\) such that

\[
D_j z_j = 0, \quad j = 1, ..., MS. \tag{12}
\]

We further assume that \(z_j\) is simple, namely that \(\ker [D_j]\) is a one-dimensional subspace. Associated with each vector \(z_j\) we introduce a basis of \(M - 1\) vectors, \(\{y_j^{(1)}, ..., y_j^{(M-1)}\}\) of the subspace which is orthogonal to \(z_j\), say

\[
y_j^{(a)} z_j = 0, \quad a = 1, ..., M - 1. \tag{13}
\]

Hereafter any complex vector \(v\) is treated as a one-column matrix and therefore its Hermitian conjugate \(v^\dagger\) is a one-row matrix. The standard scalar product of two vectors \(y\) and \(v\) is \(y^\dagger v\) while the dyadic product \(y v^\dagger\) is a square matrix. With this notation, any matrix \(D_j\) can be parametrized as

\[
D_j = \sum_{a=1}^{M-1} w_j^{(a)} y_j^{(a)^\dagger}, \quad j = 1, ..., MS, \tag{14}
\]

since this expression of \(D_j\) satisfies (12) for any choice of the \(M - 1\) linearly independent arbitrary vectors \(w_j^{(1)}, ..., w_j^{(M-1)}\). Note that, if, for any \(j\), the vectors \(z_j\) and \(y_j^{(1)}, ..., y_j^{(M-1)}\) are fixed, then the matrix \(D_j\) is parametrized, see (14), by \(M(M - 1)\) complex numbers, namely the components of the \(M - 1 \times M\)-dimensional vectors \(w_j^{(a)}\).

At this point it is worth noticing that we have obtained a particular Lagrange representation (9) of a \(S\)-degree polynomial matrix \(D(\zeta)\) by choosing the \(N = MS\) points \(\zeta_j\) according to the prescription (11), and by consequently expressing the values \(D(\zeta_j) = D_j\) as given by (14). On the other hand the polynomial \(D(\zeta)\) has the standard power representation

\[
D(\zeta) = \sum_{k=0}^{S} D^{(k)} \zeta^k, \tag{15}
\]

whose \(S + 1\) coefficients \(D^{(k)}\) are \(M \times M\) matrices. These two equivalent representations of the same polynomial matrix \(D(\zeta)\) imply a relation between the matrices \(D_j\), see (14), and the coefficients \(D^{(k)}\), see (15). This relation follows from (8), (9) and (15) and reads

\[
D^{(k)} = \sum_{j=1}^{MS} C^{(MS)}_{k+1} D_j, \quad k = 0, ..., S. \tag{16}
\]

In addition the matrices \(D_j\) have to satisfy the \(MS - S - 1\) relations (10) with \(N = MS\). It should be stressed that the connection formula (16) requires only the knowledge of the coefficients \(C^{(MS)}_{k+1}\) which are available as known entries of the inverse Vandermonde matrix [17]. This is the main scheme to arrive at a Lagrange form of a \(M \times M\) matrix polynomial \(D(\zeta)\) of degree \(S\) for given \(MS > S + 1\) roots \(\zeta_1, ..., \zeta_{MS}\) of \(\det D(\zeta)\) and corresponding given eigenvectors (see (12)) \(z_1, ..., z_{MS}\). On the technical side, the choice of the \(M - 1\) vectors \(y_j^{(a)}\), see (14), which are orthogonal to \(z_j\) for each given \(j\), is more conveniently made while dealing with specific applications.
3. The MTM model

Here we compute, by the Lagrange representation method, the explicit expression of the Darboux matrix associated with the Lax pair of the MTM model. For the sake of simplicity we produce the explicit expression of the lowest degree Darboux matrix polynomial which is compatible with the group of symmetries of this model. The construction of higher degree Darboux matrices requires more computational efforts and it is not considered here. In laboratory coordinates \( x \) (space) and \( t \) (time) the MTM equations are

\[
i(u_t - cu_x) + \mu v = \frac{1}{\mu} |v|^2 u, \quad i(v_t + cv_x) + \mu u = \frac{1}{\mu} |u|^2 v. \tag{17}
\]

Here \( u = u(x, t) \) and \( v = v(x, t) \) are the complex dependent variables, while \( \mu \) is a constant real parameter and \( c \) is a constant characteristic velocity. Although, by rescaling \( x, t, u, v \), one could set \( c = \mu = 1 \) we prefer to keep these parameters as arbitrary. As for the integrability and relativistic spinor formulation of (17), see [8, 11]. We note that in relativistic field theory the parameter \( \mu \) plays the role of the mass while in optics is related to the medium periodic (f.i. Bragg grating) constant [10]. We also note that, like for the nonlinear Schrödinger equation to which the MTM equations (17) reduce in the non-relativistic limit, the nonlinearity is cubic and describes only cross-interaction (the addition of a cubic self-interaction term destroys both integrability and relativistic invariance).

In order to write down the Lax pair, we find it more convenient to use the light-cone coordinates \( \xi = (ct + x)/(2c), \eta = (ct - x)/(2c) \). In these coordinates the MTM equations (17), which become

\[
iu_\eta + \mu v = \frac{1}{\mu} |v|^2 u, \quad iv_\xi + \mu u = \frac{1}{\mu} |u|^2 v, \tag{18}
\]

follow from the condition that the two ODEs

\[
\Psi_\xi = A(\zeta) \Psi, \quad \Psi_\eta = B(\zeta) \Psi, \tag{19}
\]

be compatible with each other. The pair of matrices \( A(\zeta) \) and \( B(\zeta) \), as well as the solution \( \Psi \) of the Lax equations (19), depend on the spectral complex parameter \( \zeta \) and on the coordinates \( \xi, \eta \). All these matrices are \( 2 \times 2 \), and \( A(\zeta) \) and \( B(\zeta) \) take the rational expression

\[
A(\zeta) = \frac{i\mu}{2} \zeta \sigma_3 + \zeta U + \frac{i}{2\mu} |u|^2 \sigma_3, \quad B(\zeta) = \frac{i\mu}{2} \zeta^{-2} \sigma_3 + \zeta^{-1} V + \frac{i}{2\mu} |v|^2 \sigma_3, \tag{20}
\]

where \( \sigma_3 \) is one of the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{21}
\]

In these expressions the matrices \( U = U(\xi, \eta) \) and \( V = V(\xi, \eta) \) are Hermitian and off-diagonal,

\[
U = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix}, \tag{22}
\]

whose entries are a solution of the MTM equations (18) (an asterisk stands for complex conjugation). Because of the very special expressions (20), it is clear that the MTM equations (18) are a reduction of a larger system of equations [12, 13] containing more field functions rather than just two, i.e. \( u, v \). Indeed the reduction conditions can be obtained via the reduction method based on automorphic matrix valued functions [15] as applied to the
dihedral group $D_2$. This can be easily realized by looking first at the representation of $D_2$ whose four elements are generated by two reflections on the complex plane, namely

$$g_0(\zeta) = \zeta, \quad g_1(\zeta) = \zeta^*, \quad g_2(\zeta) = -\zeta^*, \quad g_3(\zeta) = -\zeta,$$  \hspace{1cm} (23)

where $\zeta$ is any non-vanishing strictly complex number so that the rectangle \{\zeta, \zeta^*, -\zeta^*, -\zeta\} does not degenerate into a segment. An automorphic function $f(\zeta)$ on the complex plane is a meromorphic function which satisfies the symmetry conditions on the group orbit (alias the rectangle (23))

$$G_j \cdot f \left( g_j(\zeta) \right) = f(\zeta), \quad j = 0, 1, 2, 3,$$  \hspace{1cm} (24)

where the four operators $G_j$, which are a representation of $D_2$, are chosen to our purpose as $G_0 \cdot z = G_1 \cdot z = z, \quad G_1 \cdot z = G_2 \cdot z = z^*$. Next we consider an automorphic matrix valued function $F(\zeta)$ as meromorphic and satisfying the symmetry relations $S_j[F(\zeta)] = F(\zeta), \quad j = 0, 1, 2, 3$. Here the four operators $S_j$ are the following representation of $D_2$

$$S_j[F(\zeta)] = \sigma_j G_j \cdot F \left( g_j(\zeta) \right) \sigma_j^{-1}, \quad j = 0, 1, 2, 3,$$  \hspace{1cm} (25)

where, in addition to the Pauli matrices (21), we have introduced the unit matrix $\sigma_0 = 1$. It is plain that the operators $g_j, G_j, \sigma_j$ are in involution, and that therefore

$$S_j \cdot S_j = 1.$$  \hspace{1cm} (26)

Since both matrices $A(\zeta), B(\zeta)$ in the Lax equations (19) are automorphic,

$$S_j[A(\zeta)] = A(\zeta), \quad S_j[B(\zeta)] = B(\zeta), \quad j = 0, 1, 2, 3,$$  \hspace{1cm} (27)

the following observations turn out to be useful:

**Remark 1.** If the matrix $F(\zeta)$ depends on a variable $y$, then

$$S_j \left[ F_j(\zeta) \right] = \left( S_j[F(\zeta)] \right)_y, \quad j = 0, 1, 2, 3.$$  \hspace{1cm} (28)

**Remark 2.** For any two matrices $F_1(\zeta)$ and $F_2(\zeta)$

$$S_j \left[ F_1(\zeta)F_2(\zeta) \right] = S_j \left[ F_1(\zeta) \right] S_j \left[ F_2(\zeta) \right], \quad j = 0, 1, 2, 3.$$  \hspace{1cm} (29)

**Proposition 1.** If $\Psi(\zeta)$ is a fundamental matrix solution of the Lax equations (19), then

$$S_j[\Psi(\zeta)] = \Psi(\zeta) \Gamma_j(\zeta), \quad j = 0, 1, 2, 3, \quad \Gamma_{ij} = \Gamma_{ji} = 0, \quad \det \Gamma_j \neq 0.$$  \hspace{1cm} (30)

While remarks 1 and 2 are straight consequences of the definition (25), proposition 1 follows from remarks 1 and 2 as applied to the Lax equations (19) together with the invariance property (27). Moreover the involution property (26) implies $\Gamma_j S_j[\Gamma_j] = 1$ and therefore $\Gamma_j \neq 0$. Thus, if the solution $\Psi(\zeta)$ is fundamental (say $\det \Psi(\zeta) \neq 0$) then also the solution $S_j[\Psi(\zeta)]$ is fundamental.

**Proposition 2.** Any solution $\Psi(\zeta)$ of the Lax equations (19) is such that the matrix $\sigma_3 \Psi^*(\zeta^*) \sigma_3 \Psi(\zeta)$ is $\zeta$- and $\eta$-independent.
This conclusion is obtained by conveniently introducing the additional operator
\[ S[F(\zeta)] = -\sigma_3 F^\dagger (\zeta) \sigma_3, \] (31)
and by noticing that its action on traceless matrices (TrF = 0) coincides with that of \( S_1 \), namely \( S[F(\zeta)] = S_1[F(\zeta)] \). Then we note that the two matrices \( A(\zeta), B(\zeta) \) are traceless (see (20)) and that therefore \( (S[\Psi(\zeta)])^{-1} \) and \( \Psi(\zeta) \) are both solutions of the same Lax equations (19). This implies our claim.

Let us consider now the DT \( \Psi^{(1)} \rightarrow \Psi^{(2)} \) characterized by the Darboux matrix \( D(\zeta) \) as follows
\[ \Psi^{(2)}(\zeta) = D(\zeta) \Psi^{(1)}(\zeta), \] (32)
with the requirement that the Lax pair (19) transforms itself in covariant manner. This means that \( \Psi^{(i)}, i = 1, 2, \) is a matrix solution of (19) with \( A(\zeta) = A^{(i)}(\zeta) \) and \( B(\zeta) = B^{(i)}(\zeta) \), where \( A^{(i)}(\zeta) \) and \( B^{(i)}(\zeta) \) have the same expression (20) with \( u, v \) replaced by \( u^{(i)}, v^{(i)} \). As a consequence \( (u^{(1)}, v^{(1)}) \) and \( (u^{(2)}, v^{(2)}) \) are two different solutions of the MTM equations (18), and their relation with each other is induced by the DT (32) itself (see below). On the other hand the Darboux matrix \( D(\zeta) \) depends on the coordinates \( \xi, \eta \) according to the two compatible differential equations
\[ D_{\xi} + DA^{(1)} - A^{(2)}D = 0, \quad D_{\eta} + DB^{(1)} - B^{(2)}D = 0, \] (33)
with the property that its determinant, \( \det D(\zeta) \), is \( \xi, \eta \)-independent, as it follows from the very definition (32) and from Liouville’s formula as applied to the lax pair (19). For future reference, we note here that, if \( \hat{D}(\zeta) \) is a particular solution of these ODEs (33), then the general solution has the expression \( D(\zeta) = \hat{D}(\zeta) \Gamma(\zeta) \Psi^{(1)}(\zeta)^{-1} \), where \( \Gamma(\zeta) \) is an arbitrary constant (i.e. \( \xi \) - and \( \eta \)-independent) matrix. This implies, in full analogy with proposition 1, that

Remark 3. if \( D(\zeta) \) is a solution of the pair of equations (33) then
\[ S_j[\hat{D}(\zeta)] = D(\zeta) \Psi^{(1)} \Gamma_j(\zeta) \Psi^{(1)}^{-1}, \quad j = 0, 1, 2, 3, \]
\[ \Gamma_{j \xi} = \Gamma_{j \eta} = 0, \quad \det \Gamma_{j} \neq 0. \] (34)
We now assume that the Darboux matrix is polynomial in \( \zeta \), and observe that

Remark 4. if \( D(\zeta) \) has a polynomial dependence on \( \zeta \) then also \( S_j[D(\zeta)] \) does with the same degree.

This follows from the linearity of the transformation (25) and from remark 2 which implies that \( S_j[\zeta^nC] = \zeta^n S_j[C] \) if \( j = 0, 1 \) and \( S_j[\zeta^2C] = (-1)^n \zeta^n S_j[C] \) if \( j = 2, 3 \). Before going further, we observe that any polynomial Darboux matrix is defined by (32) only modulo a multiplication by an arbitrary scalar and \( (\xi, \eta) \)-independent polynomial function of \( \zeta \). Such factor does not affect the construction of solutions of the MTM equations and, therefore, from now on we assume that no scalar polynomial factor of non-vanishing degree can be factored out. Or, in other words, we assume that for no value of \( \zeta, D(\zeta) = 0 \).

Theorem 1. If the Darboux matrix \( D(\zeta) \) is polynomial (see the observation above), and its determinant has only strictly complex simple roots, then its degree is even and \( D(\zeta) \) is
automorphic, namely

\[ S_j[D(\zeta)] = D(\zeta), \quad j = 0, 1, 2, 3. \]  

(35)

In order to prove this statement we first note that the matrix \( R_j \equiv \Psi^{(1)} j(\zeta) \Psi^{(1)}^{-1} = D(\zeta)^{-1} S_j[D(\zeta)] \), see (34), is rational in \( \zeta \) (see remark 4) and satisfies the two ODEs \( R_{j'} = [A^{(1)}, R_j] \), \( R_{j0} = [B^{(1)}, R_j] \). Next we observe that the cofactor matrix \( D^* (\zeta) = \det D(\zeta)D(\zeta)^{-1} \) is itself polynomial in \( \zeta \). Therefore the matrix \( P_j = D^* (\zeta) S_j[D(\zeta)] = \det D(\zeta) R_j \) is a polynomial solution of the same ODEs as for \( R_j \), namely

\[ P_{j'} = \left[ A^{(1)}, P_j \right], \quad P_{j0} = \left[ B^{(1)}, P_j \right], \]  

(36)

since \( \det D(\zeta) \) is \( \xi, \eta \)-independent. It now remains to verify by direct computation that the (strong) condition that \( P_j(\xi, \eta, \zeta) \) be polynomial implies that such solution of these ODEs (36) exists if and only if \( P_j \) is scalar and \( \xi, \eta \)-independent, say a constant arbitrary polynomial in \( \zeta \) times the unit matrix. Though strait, the computations are omitted here as they are fairly long. They go by inserting into the equations (36) the power expansion of \( P_j \), together with the given expression of the matrices \( A^{(1)} \) and \( B^{(1)} \), see (20). Then, by solving the resulting recurrence equations for the coefficients of \( P_j \), one finally proves this claim. Next we rewrite (34) as

\[ S_j[D(\zeta)] = R_j D(\zeta), \quad j = 0, 1, 2, 3, \]  

(37)

where \( R_0 = 1 \) and \( R_j(\zeta), j = 1, 2, 3, \) are rational functions in the complex \( \zeta \)-plane. This matrix relation (37) implies that

\[ S_j[\det D(\zeta)] = \frac{P_j^2}{\det D(\zeta)}, \quad j = 1, 2, 3. \]  

(38)

Since the left-hand side of this equation is polynomial, it cannot have poles with the consequence that necessarily \( P_j = \phi_j \det D \), say \( R_j = \phi_j \), for some constant factor \( \phi_j \). Thus the symmetry relations (37) reduce to

\[ S_j[D(\zeta)] = \phi_j D(\zeta), \quad j = 0, 1, 2, 3. \]  

(39)

We further note that the values of the four scalar factors \( \phi_j, j = 0, \ldots, 3 \) are constrained by the group properties of the transformation operators \( S_j \), namely \( S_0 = 1 \), the involution (26) and the composition rule \( S_j \cdot S_k = S_m \) for any permutation \((j, k, m)\) of \((1, 2, 3)\) (note that \( S_j[\phi_k] = G_j \cdot \phi_k \), and the property (29)). This way one arrives at the general expressions \( \phi_0 = 1, \phi_1 = \exp(i\theta), \phi_2 = e \exp(i\theta), \phi_3 = e \) with \( e^2 = 1 \) and \( \theta \) being an arbitrary real phase. Arriving at the final result (35) amounts to prove that \( \phi_1 = 1 \), see (39). To this aim we first show that, with no loss of generality, \( \theta \) can be set to vanish. In fact, if we change the matrix \( D(\zeta) \) by a constant complex factor \( \alpha, D(\zeta) \rightarrow \tilde{D}(\zeta) = \alpha D(\zeta) \), then the corresponding factors \( \phi_j \), see (39), transform as \( \tilde{\phi}_j = (\alpha/G_j \cdot \alpha) \phi_j \). Thus it is sufficient to choose \( \alpha = \exp(-i\theta/2) \) to arrive at the new Darboux matrix whose corresponding \( \phi_j \) factors reduce to \( \phi_0 = 1, \phi_1 = 1, \phi_2 = e, \phi_3 = e \) (where, for the sake of simplicity, we have dropped the ‘hat’ notation). As a byproduct, \( \phi_j^2 = 1 \) with the consequence that the determinant \( P(\zeta) = \det D(\zeta) \) is automorphic

\[ G_j \cdot P(\hat{g}_j(\zeta)) = P(\zeta), \quad j = 0, 1, 2, 3. \]  

(40)

This follows by taking the determinant of both sides of the equation (39) or directly from (38). It is plain from (24) that, if \( \hat{\chi} \) is a root of an automorphic polynomial, then all four points \( \hat{g}_j(\chi) \)
of its associated $D_j$ orbit are roots. Since we assume (here and in the following) that all roots are strictly complex (say no rectangle $g_j(\zeta)$ is degenerate) and simple, the roots of the polynomial $\det(D_j(\zeta))$ come in quadruplets with the following representation

$$ \prod_{n=1}^L (\zeta - \chi_n)(\zeta - \chi_n^*)(\zeta + \chi_n)(\zeta + \chi_n^*). \quad (41) $$

Thus the polynomial $P(\zeta)$ has degree $4L$ since its roots are the vertices of $L$ rectangles. Moreover the Darboux matrix $D_j(\zeta)$, which is $2 \times 2$, has degree $2L$.

$$ D_j(\zeta) = \sum_{k=0}^{2L} D^{(k)}_j \xi^k. \quad (42) $$

While the monodic form (41) of the polynomial $P(\zeta)$ is proved below by the next theorem 2, we prove now that $\epsilon = 1$. The starting observation is that by inserting the power representation (15) in the first of the equations (33), and by looking at the highest power $\zeta^{2L+2}$ (see (20)), one finds that the coefficients $D_j^{(2L)}$ commutes with $\sigma_1, [\sigma_1, D_j^{(2L)}] = 0$. Then looking at the highest power $\zeta^{2L}$ of the equality $S_jD_j(\zeta) = \epsilon D_j(\zeta)$ leads to the equation $D_j^{(2L)} = \epsilon D_j^{(2L)}$ and therefore $\epsilon = 1$. This completes the proof.

**Theorem 2.**

$$ \sigma_1 D_j^T(\zeta^*) \sigma_1 D_j(\zeta) = P(\zeta) \mathbf{1}. \quad (43) $$

Since $(S_jD_j)^{-1}$ and $D_j$ are solutions of the same ODEs (33), by the same arguments used to prove proposition 3, the product $S_jD_j$ is polynomial in $\zeta$ and solution of the equations (36). Therefore, because of theorem 1, this product has to be scalar and $\zeta, \eta$-independent. More explicitly, because of the definition (31), we conclude that indeed $\sigma_1 D_j^T(\zeta^*) \sigma_1 D_j(\zeta)$ which can be written as $\sigma_1 (D_j^T(\zeta^*))^T \sigma_1 D_j(\zeta)$, is a scalar and constant polynomial in $\zeta$. We now use the property (35), in particular $D_j^T(\zeta^*) = \sigma_1 D_j(\zeta) \sigma_1$ to obtain the equality $\sigma_1 D_j^T(\zeta^*) \sigma_1 D_j(\zeta) = \sigma_1 D_j^T(\zeta) \sigma_2 D_j(\zeta)$. The identity $\sigma_2 M^T \sigma_2 = \det(M) \cdot 1$, which is valid for any $2 \times 2$ matrix $M$, concludes our proof of (43). Finally we prove also the monodic property of $P(\zeta)$. This comes from the very equality (43) together with the expression of the coefficient of the highest power of the left-hand side of (43), namely $\sigma_1 D_j^{(2L)} \sigma_2 D_j^{(2L)}$. Now, in the process of proving theorem 1, we have also pointed out that $[\sigma_1, D_j^{(2L)}] = 0$. Thus the highest power coefficient of $P(\zeta)$ is certainly positive, $D_j^{(2L)}D_j^{(2L)} > 0$ and therefore it suffices normalizing the Darboux matrix $D_j(\zeta)$ by multiplication by a real number to arrive at the monodic polynomial (41).

At each root $g_j(\chi_n)$ of $P(\zeta)$ the matrix $D_j^{(n)} = D_j(g_j(\chi_n))$ has the eigenvector $z_j^{(n)}$ corresponding to the vanishing eigenvalue

$$ D_j^{(n)} z_j^{(n)} = 0, \quad j = 0, 1, 2, 3, \quad n = 1, ..., L. \quad (44) $$

Because of the symmetry relations (39), for each quadruplet $g_j(\chi_n)$ associated with the root $\chi_n$ the corresponding four matrices $D_j^{(n)}$ are related to each other. In fact, as a straight consequence of (39) for $\zeta = \chi_n$ together with the definition (25), we obtain the following relations within each quadruplet (i.e. for fixed $n$)
\[ D_j^{(n)} = \sigma_j G_j \cdot (D_0^{(n)}) \sigma_j, \quad j = 0, 1, 2, 3. \] (45)

By assuming that the roots \( \chi_n \) and the corresponding eigenvectors \( z_n \) are given, we proceed to consider the Lagrange representation of the Darboux matrix \( D(\zeta) \) according to the prescription given in the previous section 2. Moreover, in order to illustrate this construction of \( D(\zeta) \) in the simplest possible way, we treat the case in which only one quadruplet is given, namely \( L = 1 \) with the notation: \( \chi_1 = \chi, \ z_1^{(1)} = z_j \). Thus, since the rank \( M \) and degree \( S \) of the Darboux matrix \( D(\zeta) \) is 2, the Lagrange form (9), which in the present context reads

\[ D(\zeta) = \sum_{j=1}^{4} \frac{L_{j}^{(4)}(\zeta)D_{j-1}}{1}, \quad D_j = D\left(g_j(\chi)\right), \] (46)

is a four-point representation. Here the definition (6) applies with \( \zeta_j = g_{j-1}(\chi_j), j = 1, ..., 4 \), or, more explicitly, \( \zeta_1 = \chi, \ z_2 = \chi^* \), \( \zeta_3 = -\chi^*, \ z_4 = -\chi \). As pointed out in section 2, comparing the degree of the left-hand side with that of the right-hand side of this equation (46) yields just one condition, see (10) with \( S = 2 \) and \( N = 4 \), on the matrices \( D_j \), which is

\[ \sum_{j=1}^{4} C_{j}^{(4)}D_{j-1} = 0. \] (47)

Next we go to the expression (14) of the four matrices \( D_j \), which takes the dyadic form\[ D_j = w_j y_j^T. \] Here the vector \( y_j \), which is orthogonal to the eigenvector \( z_j \) according to the prescription (13), may be chosen as \[ y_j = \sigma z_j^*. \] (48)

Since we have to deal only with the matrix \( D_0 \) to perform our computing because of the symmetry relation (45), it remains to find the unknown vector \( w_0 \). To this aim, the starting point is the condition (47), which explicitly reads

\[ C_{i}^{(4)}D_0 + \sum_{j=1}^{4} C_{i}^{(4)}\sigma_j D_j^* \sigma_j = 0, \] (49)

together with the expressions \( C_{i}^{(4)} = C_{i}^{(4)*} = [2\chi (\chi^2 - \chi^{*2})]^{-1} \), \( C_{24}^{(4)} = C_{34}^{(4)} = C_{44}^{(4)*} \), and the final result is

\[ w_0 = \rho \chi z_0^*, \] (50)

where the normalizing factor \( \rho \) is real and positive, but still to be found. The details of this computation, and of similar ones in the following, are omitted as lengthy and simple. Next, the Lagrange form (46) of \( D(\zeta) \) and its power expansion (42) (for \( L = 1 \)) imply the expressions (see also (16) for \( M = S = 2 \))

\[ D^{(k)} = \sum_{j=1}^{4} C_{j}^{(4)}\sigma_j G_j \cdot (D_0) \sigma_j, \quad k = 0, 1, 2, \] (51)

where the Lagrange coefficients (see (8)) take the following expressions

\[ C_{13}^{(4)} = C_{23}^{(4)} = [2(\chi^2 - \chi^{*2})]^{-1}, \ C_{23}^{(4)} = C_{23}^{(4)*}, \]

\[ C_{34}^{(4)} = C_{34}^{(4)*}, \ C_{34}^{(4)} = -C_{34}^{(4)*} = -2(\chi^2 - \chi^{*2})^{-1}, \ C_{24}^{(4)} = C_{24}^{(4)*} = -C_{24}^{(4)*} = -2(\chi^2 - \chi^{*2})^{-1}, \ C_{24}^{(4)} = C_{24}^{(4)*} = -C_{24}^{(4)*} = -2(\chi^2 - \chi^{*2})^{-1}, \] (51)

The upshot of these computations is given by the following expressions
Here \([\{\cdot, \cdot\}\] and \([\cdot, \cdot]\) stand for the anticommutator and, respectively, for the commutator. We also note that, as it follows from (43), the matrix coefficient \(D^{(2)}\) turns out to be diagonal and unitary, \(D^{(2)} D^{(2)\dagger} = I\), while the coefficient \(D^{(1)}\) is off-diagonal and Hermitian, \(D^{(1)\dagger} = D^{(1)}\).

By a further computational effort, the matrix coefficients \(D^{(2)}\) and \(D^{(1)}\) can be given the explicit expressions

\[
D^{(2)} = \rho \frac{\chi^* \{z_0 \chi^* \} + \chi \sigma_3 \{z_0 \chi^* \}}{2 \text{Im} \chi^2} \sigma_3, \quad \epsilon^{\phi} = \frac{\chi^* \{z_0 \chi^* \} + \chi \sigma_3 \{z_0 \chi^* \}}{\chi^* \{z_0 \chi^* \} + \chi \sigma_3 \{z_0 \chi^* \}}, \quad z_0 = \left( \frac{z_{01}}{z_{02}} \right). 
\]

\[
D^{(1)} = \begin{pmatrix} 0 & \delta \phi \\ \delta & 0 \end{pmatrix}, \quad \delta = -\rho \sigma_3 \sigma_3 z_{02}, 
\]

where the function \(\phi = \phi(\zeta, \eta)\) is real. We are now in the position to compute the value of the normalizing factor \(\rho\), see (50). Indeed from the unitarity property of \(D^{(2)}\) it follows that

\[
\rho = \frac{2 \text{Im} \chi^2}{\chi^* \{z_0 \chi^* \} + \chi \sigma_3 \{z_0 \chi^* \}}.
\]

Here we have conveniently, and with no loss of generality, set \(\chi\) in the first quadrant of the complex plane, say \(\text{Re} \chi > 0\) and \(\text{Im} \chi > 0\).

The DT transformation (32), which acts on the solution \(\Psi^{(1)}\) of the Lax pair, obviously induces the transformation \((u^{(1)}, v^{(1)}) \rightarrow (u^{(2)}, v^{(2)})\) on the corresponding solutions of the MTM equations. This transformation is derived from the ODEs (33) for the Darboux matrix itself \(D(\zeta, \eta, \zeta)\) by looking at the coefficient of the power \(\zeta^3\) in the first equation and at the coefficient of the power \(\zeta^{-1}\) in the second equation. This simple derivation leads to the following matrix transformation

\[
\begin{align*}
U^{(2)} &= D^{(2)} U^{(1)} D^{(2)\dagger}, \\
V^{(2)} &= D^{(2)} V^{(1)} D^{(2)\dagger}.
\end{align*}
\]

or, more explicitly (see (22)),

\[
\begin{align*}
u^{(2)} &= e^{-i\phi} \left( e^{-i \delta} u^{(1)} + i \mu \delta \right), \\
u^{(2)} &= e^{i \phi} \left( e^{i \delta} v^{(1)} + i \frac{\mu}{\kappa} \delta \right).
\end{align*}
\]

This concludes the algebraic side of the construction of the Darboux transformation. At last we turn our attention to the differential part by deriving the \(\zeta\)- and \(\eta\)-dependence of the vector \(z_0\). To this purpose we consider the ODEs (33) for \(\zeta = \chi\), each term being applied to the eigenvector \(z_0\), see (44) with \(j = 0\), \(n = 1\). As a consequence \(D_0(-z_0 + A^{(1)}(\chi)z_0) = 0\) and \(D_0(-z_0 + B^{(1)}(\chi)z_0) = 0\), and therefore, without any loss of generality, the vector \(z_0\) can be characterized as a solution of the Lax pair corresponding to the spectral parameter \(\zeta = \chi\) and to the solution \((u^{(1)}, v^{(1)})\) of the MTM. Thus its expression is known...
where only the constant vector $\gamma$ is left arbitrary. Indeed the solution $(u^{(1)}(\eta, \chi), v^{(1)}(\eta, \chi))$, together with its corresponding matrix $\Psi^{(1)}(\zeta, \eta, \chi)$, are considered as known. This is the case if the known solution is the vanishing one (vacuum). In this case the transformation (57) yields the one-soliton solution [11]. If instead the known solution $u^{(1)}$, $v^{(1)}$ is a continuous wave, then the new solution $u^{(2)}$, $v^{(2)}$ describes one soliton propagating over such a background wave [12]. Quite recently [18] the explicit expression of one rogue wave, the relativistic analogue of the Peregrine soliton, has been derived by means of the formulae given in this section. This is the first example of solutions of the MTM which feature a rational dependence on both coordinates.

4. Conclusions

Darboux transformations have been recognized as a useful tool to obtain the expression of solutions describing soliton propagation over an arbitrary known background. In this spirit we have addressed the issue of the effective construction of Darboux matrices in polynomial form rather than in pole expansion as suggested by the dressing approach. To this aim we have presented a novel algorithmic way by expressing a Darboux matrix in Lagrange representation (9) rather than in standard power form (15). A good side of our technique is that it requires computations in the algebra of matrices with no need to deal explicitly with specific matrix entries and large linear systems. As a difference, and advantage, with respect to other competitive approaches based on polynomial Darboux matrices, such as [5, 6], we note that also the matrix coefficient of the highest power of the Darboux matrix $D(\zeta)$ is found by purely algebraic computation rather than evaluated by separate means. It should be also recognized that the Lagrange representation, in addition of being well suited to deal with the way $D(\zeta)$ depend on the known solution of the Lax pair (see (14), (49) and (50)), it also serves the purpose to take into account the symmetries of the associated Lax pair as it deals with the values of $D(\zeta)$ on the group orbit (see (45) and (51)). We deem all the main features of our way of constructing Darboux matrices are well illustrated by the MTM study case of section 3. The technical problem which has not been treated here is the extension of our method to construct N-fold Darboux transformations. This generalization to higher degree Darboux matrices will be given in a subsequent paper, with particular focus on the class of rational solutions whose first member has been discussed in [18].

References

[1] Matveev V B and Salle M A 1991 Darboux Transformations and Solitons (Series in Nonlinear Dynamics) (Berlin: Springer)
[2] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1980 Theory of Solitons: The Inverse Problem Method (Moscow: Nauka) (in Russian) Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 Theory of Solitons: The Inverse Problem Method (New York: Plenum) 1984 (Engl. transl.)
[3] Calogero F and Degasperis A 1982 Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations vol 1 (Amsterdam: North-Holland)
[4] Cieśliński J L 1995 An algebraic method to construct the Darboux matrix J. Math. Phys. 36 5670–706
[5] Cieśliński J L 2009 Algebraic construction of the Darboux matrix revisited J. Phys. A: Math. Theor. 42 404003
[6] Neugebauer G and Meinel R 1984 General N-soliton solution of the AKNS class on arbitrary background Phys. Lett. A 100 467
[7] Steudel H, Meinel R and Neugebauer G 1997 Vandermonde-like determinants and N-fold Darboux/Backlund transformations J. Math. Phys. 38 4692
[8] Mikhailov A V 1976 Integrability of the two-dimensional Thirring model JETP Lett. 23 320
[9] Aceves A B and Wabnitz S 1989 Self-induced transparency solitons in nonlinear refractive periodic media Phys. Lett. A 141 37
[10] Miri M-A, Aceves A B, Kottos T, Kovannis V and Christodoulides D N 2012 Bragg solitons in nonlinear PT-symmetric periodic potentials Phys. Rev. A 86 033801
[11] Kuznetsov E A and Mikhailov A V 1977 On the complete integrability of the two-dimensional classical-Thirring model Theor. Math. Phys. 30 193
[12] Barashenkov I V, Getmanov B S and Kovtun V E 1993 The unified approach to integrable relativistic equations: Soliton solutions over nonvanishing bakcground1 J. Math. Phys. 34 3039
[13] David D, Harnad J and Shnider S 1984 Multi-soliton solutions to the Thirring model trough the reduction method Lett. Math. Phys. 8 27
[14] Cieśliński J L and Biernacki W 2005 A new approach to the Darboux-Bäcklund transformation versus the standard dressing method J. Math. Phys. 38 9491–501
[15] Lombardo S and Mikhailov A V 2004 Reductions of integrable equations: dihedral group J. Phys. A: Math. Gen. 37 7727
[16] Mikhailov A V, Papamikos G and Wang J P 2014 Darboux transformation with dihedral reduction group J. Math. Phys. 55 113507
[17] Knuth D E 1973 The Art of Computer Programming vol 1 (Reading, MA: Addison-Wesley)
[18] Degasperis A, Wabnitz S and Aceves A B 2015 Phys. Lett. A 379 1067–70