Framed cohomological Hall algebras and stable envelopes I

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Abstract

There are multiple conjectures relating the cohomological Hall algebras (CoHAs) of certain substacks of the moduli stack of representations of a quiver $Q$ to the Yangian $Y^Q_{MO}$ by Maulik-Okounkov, whose construction is based on the notion of stable envelopes of Nakajima varieties. In this article, we introduce the cohomological Hall algebra of the moduli stack of framed representations of a quiver $Q$ (framed CoHA) and we show that the equivariant cohomology of the disjoint union of the Nakajima varieties $M_Q(v,w)$ for all dimension vectors $v$ and framing vectors $w$ has a canonical structure of subalgebra of the framed CoHA. Restricted to this subalgebra, the algebra multiplication is identified with the stable envelope map. As a corollary, we deduce an explicit inductive formula to compute stable envelopes in terms of tautological classes.

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1 Introduction

1.1 Overview

The purpose of this article is to establish a connection between the theory of stable envelopes and the world of cohomological Hall algebras.

Stable envelopes, developed by Okounkov and its coauthors in the last decade \cite{19,20,21}, are the geometric answer to the problem of constructing a quantum group associated with an arbitrary quiver and of studying the monodromy of the associated quantum differential and difference equations. The study of stable envelopes of Nakajima varieties is particularly important, not only because these are symplectic resolutions, for which stable envelopes enjoy nicer properties, but also because they bridge the gap between the abstract theory of stable envelopes and the Lie-theoretic realm of quantum groups. Indeed, a choice of Nakajima variety is first and foremost a choice of a quiver, and its associated stable envelopes allow to produce R-matrices for the corresponding quantum group.

Cohomological Hall algebras (CoHAs) associated with a quiver for a generalized cohomology theory were first introduced by Kontsevich and Soibelman \cite{12} in relation to Donaldson-Thomas theory and the algebra of BPS states and were later related to Nakajima varieties, whose cohomologies are modules over the CoHA \cite{28,32}.

Essentially, CoHAs are built from the cohomology of stacks of quiver representations and the algebra multiplication encodes extensions. In this article, we introduce a variant of Kontsevich-Soibelman’s CoHA that involves moduli stacks of framed representations, and we argue that this algebra governs stable envelopes of Nakajima varieties. More precisely, we realize the cohomology of the disjoint union of all possible Nakajima varieties associated to a quiver $Q$ as a subalgebra of the framed CoHA and we show that on this subalgebra CoHA’s multiplication reduces to stable envelopes.

The general philosophy of the geometric representation theory of CoHAs is that taking framings corresponds to studying modules over the (unframed) CoHA. As a consequence, one can interpret the results of this article as evidence of the fact that the modules themselves, considered altogether, admit an algebra structure, which we call framed CoHA, and that this algebra contains the algebra of stable envelopes of Nakajima varieties.

1.1.1 Framed cohomological Hall algebras

The acronym CoHA hints at a large class of algebras that are constructed from quiver representations and a given cohomology theory, possibly with some extra data such as some potential. The conceptual starting point of this article is one of the most elementary instances of CoHA, namely Kontsevich-Soibelman’s CoHA associated with a doubled quiver $(\overline{Q}, I)$ without potential. Its underlying vector space is simply defined as

$$\mathcal{H}_{\overline{Q}} = \bigoplus_{v \in \mathbb{N}^I} \mathcal{H}_{\overline{Q}}(v)$$

$$\mathcal{H}_{\overline{Q}}(v) = H([T^*\text{Rep}(v)/G_v])$$

where $[T^*\text{Rep}(v)/G_v]$ is the moduli stack of representations of the path algebra $\mathcal{C}\overline{Q}$ with dimension vector $v \in \mathbb{N}^I$, and the multiplication $m = q \circ p^*$ is obtained using the stack $Z$ parametrizing pairs of representations $x \subset y$ of dimension $v_1$ and $v_1 + v_2$ and the natural maps

$$[T^*\text{Rep}(v_1)/G_{v_1}] \times [T^*\text{Rep}(v_2)/G_{v_2}] \xrightarrow{p} Z \xrightarrow{q} [T^*\text{Rep}(v_1 + v_2)/G_{v_1+v_2}].$$

With this construction in mind, we introduce a framed CoHA

$$\mathcal{H}_{\overline{Q}^\text{fr}} = \bigoplus_{v, w \in \mathbb{N}^I} \mathcal{H}_{\overline{Q}^\text{fr}}(v, w)$$

$$\mathcal{H}_{\overline{Q}^\text{fr}}(v, w) = H_{T_w}([T^*\text{Rep}(v, w)/G_v])$$

by replacing representations $T^*\text{Rep}(v)$ with framed representations $T^*\text{Rep}(v, w)$ and ordinary cohomology with equivariant cohomology with respect to the residual action of the torus $T_w$ acting on the
framing and rescaling the cotangent directions. The multiplication is defined with a correspondence analogous to the previous one, and hence consists of a collection of maps

$$H^{Q_{fr}}(v_1, w_1) \otimes_k H^{Q_{fr}}(v_2, w_2) \to H^{Q_{fr}}(v_1 + v_2, w_1 + w_2)$$  \hspace{1cm} (2)

In this way, the unframed CoHA $H^Q$ can be naturally identified with the subalgebra of $H^{Q_{fr}}$ with trivial framing $w = 0$.

1.1.2 Stable envelopes of Nakajima varieties

An important class of symplectic varieties associated with framed representations of a quiver is given by Nakajima quiver varieties\cite{16}\cite{15}\cite{17}\cite{9}. These are defined as the GIT-symplectic reduction of the cotangent bundle $T^*\text{Rep}(v, w)$ with respect to the Hamiltonian action of $G_v$. This means that, if we denote the moment map by

$$\mu_{v,w} : T^*\text{Rep}(v, w) \to g_v^*,$$

then we can describe a Nakajima variety as

$$M(v, w) = \mu_{v,-1}^{-1}(0)^{ss}/G_v.$$

By construction, $M(v, w)$ comes with a torus $T_w$ acting on the framing and rescaling the symplectic form. Stable envelopes of Nakajima quiver varieties are certain axiomatically defined maps

$$\text{Stab}_\xi : H_{T_w}(M(v, w)^A) \to H_{T_w}(M(v, w))$$

depending on the choice of a subtorus $A \subset T_w$ fixing the symplectic form of $M(v, w)$ and on the extra parameter $\xi$, called chamber. Remarkably, each fixed component of $M(v, w)^A$ is itself a product of Nakajima varieties and, for an appropriate choice of $A$, it takes the form

$$M(v_1, w_1) \times M(v_2, w_2)$$

with $v_1 + v_2 = v$ and $w_1 + w_2 = w$. In this particular case, the stable envelope becomes a map

$$\text{Stab}_\xi : H_{T_{w_1}}(M(v_1, w_1)) \otimes_k H_{T_{w_2}}(M(v_2, w_2)) \to H_{T_{w_1+w_2}}(M(v_1 + v_2, w_1 + w_2))$$  \hspace{1cm} (3)

1.1.3 Framed preprojective semistable CoHA

The apparent similarity with (2) suggests tentatively defining an algebra structure on

$$H^{Q_{fr}} = \bigoplus_{v,w \in N^I} H^{Q_{fr}}(v, w) \hspace{1cm} H^{Q_{fr}}(v, w) = H_{T_w}(M(v, w))$$

via the stable envelopes maps (3). This construction indeed defines a graded associative algebra structure on $H^{Q_{fr}}$, which we call the semistable preprojective framed cohomological Hall algebra (framed CoHA). The choice of the name can be partially understood by observing that the vector space $H^{Q_{fr}}$ consists of cohomologies of Nakajima varieties, which are exactly moduli spaces of framed semistable representations of the preprojective path algebra $CQ_{fr}/J_\mu$, but of course, more is needed to justify this definition. Namely, one needs to show that the multiplication defined by stable envelopes has something to do with CoHA’s multiplication. This is indeed the main goal of the article.

To accomplish this, we exploit Aganagic and Okounkov’s construction \cite{2} of a map

$$\psi : H_{T_w}(M(v, w)) \to H_{T_w}([T^*\text{Rep}(v, w)/G_v])$$

which is, up to a multiplicative factor, a section of the tautological pullback

$$j^* : H_{T_w}([T^*\text{Rep}(v, w)/G_v]) \to H_{T_w}(M(v, w))$$
associated with the inclusion
\[ j : \mathcal{M}(v, w) = \mu^{-1}(0) \text{ss}/G_v \hookrightarrow [T^*\text{Rep}(v, w)/G_v]. \]

Considering all these maps together, one gets a map \( \psi : \overline{\mathcal{H}}^{\mathfrak{T} \mu} \to \mathcal{H}^{\mathfrak{T} \mu}. \) In the main theorem of this article, we show:

**Theorem (5.6).** The map \( \psi : \overline{\mathcal{H}}^{\mathfrak{T} \mu} \to \mathcal{H}^{\mathfrak{T} \mu} \) is an injective morphism of graded algebras.

The subscript \( \tau \) in the notation refers to the fact that the algebra structure on \( \mathcal{H}^{\mathfrak{T} \mu} \) is appropriately twisted, although in a naive and geometrically meaningful way.

In words, the theorem above says that the map \( \psi \) realizes the cohomology of Nakajima varieties as a subalgebra of the framed CoHA \( \mathcal{H}^{\mathfrak{T} \mu} \), and on this subalgebra CoHA’s multiplication is identified with taking stable envelopes.

1.1.4

An interesting application of the previous theorem is the production of an explicit inductive formula for the stable envelopes of Nakajima varieties. The map \( \psi \) allows giving a tautological presentation of the stable envelopes by setting
\[
\text{Stab}^\psi_e := \psi \circ \text{Stab}_{\frac{1}{e(h_{\mathfrak{g}_v})}}.
\]

This map has as target a localization of the tautological ring \( H_{T^w}(\mathcal{M}(v, w)/G_v) \) and it restricts to \( \text{Stab}^\psi_e \) on \( \mathcal{M}(v, w) \subset [T^*\text{Rep}(v, w)/G_v] \).

Applying Theorem 5.6, we show that for every fixed component \( F \subset \mathcal{M}(v, w)^A \) and chamber \( \mathfrak{c} \), there are \( k = \dim(A) \) decompositions \( v = v_1 + v_2, w = w_1 + w_2 \), fixed components \( F_1 \subset \mathcal{M}(v_1, w_1)^A_1 \), \( F_2 \subset \mathcal{M}(v_2, w_2)^A_2 \) and chambers \( \mathfrak{c}_1, \mathfrak{c}_2 \) such that \( A = A_1 \times A_2 \) and
\[
\text{Stab}^\psi_e (F) = \text{Shuffle} \left( \frac{e(T^*\text{Rep}(v, w)[-1])\text{Stab}^\psi_{\mathfrak{c}_1}(F_1)\text{Stab}^\psi_{\mathfrak{c}_2}(F_2)}{e(g_v[-1])e(h_{\mathfrak{g}_v}[-1])} \right).
\]

All the classes in the formula have an explicit tautological presentation. By induction on the dimension of \( A \), this formula gives an explicit presentation of the stable envelopes of Nakajima varieties for the action of a torus \( A \subset A_w \).

This formula can be seen as the cohomological limit of the main result of [6], where an analogous formula is presented in the elliptic setting.

1.2 Further directions

This article can be seen as the starting point of a larger project aimed at establishing a connection between the geometric theory of quantum groups initiated with the introduction of stable envelopes and cohomological Hall algebras. We conclude this introduction with a brief discussion of prospective directions of development of the author’s current work.

1.2.1 K-theory and elliptic cohomology

First of all, the results of the present paper will be extended to K-theory and elliptic cohomology in the companion paper [5]. While the results in K-theory are very close to the cohomological ones and can be essentially guessed from those of this article, the elliptic setting requires a more sophisticated approach. This is partly due to features intrinsic to elliptic cohomology, and partly to the fact that the elliptic stable envelopes depend on extra parameters, called dynamical or Kähler parameters. In view of this, the algebras of this article will be replaced by certain algebra objects in an appropriate monoidal category, whose tensor product takes into account of some shifts of the dynamical parameters. Because of that, this construction can be seen as a dynamical version of the sheafified elliptic cohomological Hall algebra of Yang and Zhao [31].
1.2.2 Cohomological Hall algebras and stable envelopes of bow varieties

The elliptic stable envelope of a Nakajima variety $X$ can be interpreted as the transformation matrix that relates the vertex functions of $X$, which are certain enumerative invariants that provide $q$-holonomic modules for a wide class of difference equations such as the $q$-KZ equations \cite{1}, with the vertex functions of its 3d mirror dual $X^!$. As a consequence, stable envelopes can be seen as an adequate topological invariant to test 3d mirror symmetry. Some positive results have already been obtained in \cite{24}, but only in the fortunate case when the 3d-dual of the Nakajima variety $X$ is itself a Nakajima variety. Indeed, the class of Nakajima varieties is not closed under mirror symmetry, so to work in full generality one needs to consider a larger class, namely bow varieties, which have been introduced in the context of enumerative geometry in \cite{23}. In joint work with Richárd Rimányi \cite{7}, it will be shown that all the main ideas of this paper, namely stable envelopes, cohomological Hall algebras, and their interplay can be extended to bow varieties. Moreover, these results will be combined with the neat combinatorial properties of these varieties to prove fusion rules for $R$-matrices and 3d mirror symmetry of stable envelopes.

1.2.3 Preprojective CoHA and Yangians

Going back to the cohomological level, another version of CoHA can be related to those discussed in this article. Namely, one can consider the preprojective version of (1) by setting

\[
\mathcal{P}^{\mathcal{Q}} = \bigoplus_{v \in \mathbb{N}^I} \mathcal{P}^{\mathcal{Q}}(v), \quad \mathcal{P}^{\mathcal{Q}}(v) = H^{BM}(\{\mu_{v,w}(0)/G_v\}),
\]

where $H^{BM}$ stands for Borel-Moore homology. Davison conjectured in \cite{8} that an appropriate subalgebra of $\mathcal{P}^{\mathcal{Q}}$ is isomorphic to the positive half of the Maulik-Okounkov Yangian $Y^{Q,+}_{MO}$. Other results in this direction have been obtained by Yang and Zhao \cite{32}, who showed that a certain subalgebra of $\mathcal{P}^{\mathcal{Q}}$ is isomorphic to the positive half of Drinfel’d Yangians of type ADE, and by Schiffmann and Vasserot \cite{26}, who built a map between $Y^{Q,+}_{MO}$ and an algebra closely related to $\mathcal{P}^{\mathcal{Q}}$ that is conjecturally an isomorphism.

As shown in \cite{32}, $\mathcal{P}^{\mathcal{Q}}$ acts on cohomology of Nakajima varieties by maps

\[
\mathcal{P}^{\mathcal{Q}}(v_1) \otimes k \mathbb{H}^{\mathcal{Q}}(v_1,w) \rightarrow \mathbb{H}^{\mathcal{Q}}(v_1 + v_2, w),
\]

and there is also a canonical morphism of algebras $\mathcal{P}^{\mathcal{Q}} \rightarrow \mathcal{H}^{\mathcal{Q}}$ associated with the inclusion

\[
[\mu_{v,w}^{-1}(0)/G_v] \subset [T^*\text{Rep}(v,w)/G_v],
\]

which induces an action of $\mathcal{P}^{\mathcal{Q}}$ on $\mathcal{H}^{\mathcal{Q}}$. We expect that all these actions are compatible with the morphism $\psi : \mathbb{H}^{\mathcal{Q}} \rightarrow \mathcal{H}^{\mathcal{Q}}$ in the obvious way. A positive answer to this expectation would mean that the $\mathcal{P}^{\mathcal{Q}}$-module structure of the cohomology of Nakajima varieties is exactly the one induced on $\mathbb{H}^{\mathcal{Q}}$ by stable envelopes, and hence would allow building a bridge between $\mathcal{P}^{\mathcal{Q}}$ and Maulik-Okounkov Yangian for an arbitrary quiver $Q$.

1.2.4 Elliptic quantum groups

Of course, preprojective CoHAs can be also constructed in K-theory and elliptic cohomology \cite{22},\cite{31}. On the other hand, a definition of the elliptic quantum group of an arbitrary quiver is missing in the literature. As a consequence, an identification of the preprojective CoHA with the positive half of quantum groups in cohomology and K-theory could be used to define the elliptic quantum group itself as a Drinfel’d double of the elliptic preprojective CoHA.
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2 Set-up

2.1 Cohomology functors

2.1.1

Let $G$ be an algebraic group. The cohomology functor that we consider in this article is equivariant cohomology

$$H_G(-) : G\text{-Spaces} \to \text{Alg}_{H_G(*)},$$

taken with complex coefficients. Given a $G$-space $X$, its $G$-equivariant cohomology $H_G(X)$ is defined as the ordinary cohomology of the space

$$X_G = X \times EG/G,$$

where $EG \to BG$ is the classifying space of $G$. By construction, it is an $H_G(*) = H(BG)$-algebra.

2.1.2

In this article, we only consider the case when $G$ is either a torus $T$, a parabolic subgroup $P$ of $GL(n)$, or $GL(n)$ itself. In these cases, $EG$ and $BG$ admit explicit representatives. The infinite Grassmannian

$$\text{Gr}(n, \infty) = \lim_{n \to \infty} \text{Gr}(n, \mathbb{C}^n)$$

is a model for the classifying space of $BGL(v)$ with universal bundle given by its tautological bundle of rank $n$. Similarly, a model for the classifying space of a $n$-dimensional torus $T \cong (\mathbb{C}^\times)^n$ is $(\mathbb{P}^\infty)^n$, with $\mathbb{P}^\infty = \text{Gr}(1, \infty)$, and the universal bundle $ET$ is the direct sum of the tautological line bundles living on each copy of $\mathbb{P}^\infty$. As a consequence, we have

$$H_T(*) = \mathbb{C}[t] \quad H_{GL(n)}(*) = \mathbb{C}[t]^W,$$

where $t$ is the Lie algebra of $T$ and $W$ is the Weyl group of $GL(n)$. Let now $P$ be a parabolic subgroup of $GL(n)$ containing the Levi subgroup $L \cong \prod_i GL(n_i)$. Its classifying space fits in the Cartesian squares

$$\begin{array}{ccc}
ET & \to & EP \\
\downarrow & & \downarrow \\
BL = \prod_i \text{Gr}(n_i, \infty) & \to & BP \\
\downarrow & & \downarrow \\
BGL(n) = \text{Gr}(n, \infty) & \to & BGL(n)
\end{array}$$

Additionally, notice that $BP$ is homotopy equivalent to $BL$ and a fibration over $BGL(n)$ by partial flags $GL(n)/P$.

2.1.3

Beyond the standard pullback in cohomology, there are also pushforward maps

$$f_* : H_G(X) \to H_G(Y)$$

associated with a proper equivariant morphism between smooth varieties. Their construction is standard and is obtained through Poincaré duality in Borel-Moore homology.
2.1.4
In the article, we will also consider the cohomology of quotients stacks \([X/G]\), which is the algebraic counterpart of equivariant cohomology. As discussed in [3], the cohomology of \([X/G]\) coincides with the equivariant cohomology of the prequotient:

\[ H([X/G]) = H_G(X). \]

More generally, if we are given an action of \(G \times H\) on \(X\), we can define the \(H\)-equivariant cohomology of \([X/G]\) as

\[ H_H([X/G]) = H([X/G \times H]). \]

2.2 Stacks of quiver representations

2.2.1
Let \((Q, I)\) be a quiver, i.e. an oriented graph, with finite vertex set \(I\), where loops and multiple edges are allowed. The quiver data is encoded in the adjacency matrix \(Q = \{Q_{ij}\}\), where

\[ Q_{ij} = \{\text{number of edges from } i \text{ to } j\}. \]

By abuse of notation, we identify the set of oriented edges with the matrix \(Q\). There are two natural “head” and “tail” maps \(h, t: Q \to I\) that associate to an oriented edge \(e \in Q\) its head and tail vertices \(h(e)\) and \(t(e)\) respectively.

A representation of a quiver \((Q, I)\) is an assignment of a vector space \(V_i\) to every vertex \(i \in I\) and a linear map \(x_{h(e), t(e)} \in \text{Hom}(V_{t(e)}, V_{h(e)})\) to every edge \(e \in Q\). The space of representations \(\text{Rep}(v)\) is

\[ \text{Rep}(v) = \bigoplus_{e \in Q} \text{Hom}(V_{t(e)}, V_{h(e)}), \]

where \(V_i = \mathbb{C}^{v_i}\).

2.2.2
Given a quiver \((Q, I)\), we also introduce the space of framed representations

\[ \text{Rep}(v, w) \cong \text{Rep}(v) \oplus \bigoplus_{i \in I} \text{Hom}(V_i, W_i) \]

with dimension

\[ v_i = \dim(V_i) \quad w_i = \dim(W_i), \quad v, w \in \mathbb{N}^I. \]

Equivalently, this can be seen as the space of representations of the framed quiver \((Q_{fr}, I_{fr})\), which is the quiver whose set of vertices \(I_{fr}\) consists of two copies of \(I\) and whose adjacency matrix is

\[ Q_{fr} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}. \]

Also notice that the cotangent bundle \(T^*\text{Rep}(v, w)\) can be identified, via trace pairing, with the representations of the framed doubled quiver \((Q_{fr}, T_{fr})\), defined by \(T_{fr} = I_{fr}\) and \(Q_{fr} = Q_{fr} + Q_{fr}^T\).

Finally, we remark that \(\text{Rep}(v, 0)\) is naturally identified with \(\text{Rep}(v)\) and, similarly, \(T^*\text{Rep}(v, 0)\) is identified with \(T^*\text{Rep}(v)\), the space of representations of the (unframed) doubled quiver \((Q, T) = (Q + Q^T, I)\).
2.2.3

The groups

\[ G_v := \prod_{i \in I} \text{GL}(v_i) \quad G_w := \prod_{i \in I} \text{GL}(w_i) \]

act naturally on on \( T^*\text{Rep}(v, w) \) by change of basis. There is also an action of \( \mathbb{C}_h^\times \) rescaling the cotangent directions of \( T^*\text{Rep}(v, w) \), and we denote the corresponding weight by \( h^{-1} \). This choice of sign, although somewhat unnatural, allows us to match the conventions of other works on which this paper is based. Overall, we get an action of \( G_v \times G_w \times \mathbb{C}_h^\times \) on \( T^*\text{Rep}(v, w) \).

Notice that the vector space \( T^*\text{Rep}(v, w) \) admits a canonical symplectic structure, which is preserved by the action of \( G_v \times T_w \) and is shifted by \( \mathbb{C}_h^\times \) with weight \( h \).

**Definition 2.1.** The moduli space \( \mathcal{R}(v, w) \) is the quotient stack \([T^*\text{Rep}(v, w)]/G_v\).

Since we will never consider the stack \([\text{Rep}(v, w)/G_v]\), no confusion should arise from the notation.

Notice that we only take the quotient with respect to \( G_v \), so \( \mathcal{R}(v, w) \) admits a residual action by \( G_w \times \mathbb{C}_h^\times \). We will be actually interested in the action of a maximal torus \( T_w \subset G_w \times \mathbb{C}_h^\times \). We factor it as \( T_w = A_w \times \mathbb{C}_h^\times \) where \( A_w = \prod_{i \in I} A_{w_i} \) is a maximal torus in \( G_w \).

2.3 Nakajima quiver varieties

2.3.1

By \( \mathcal{M}_{Q, \theta}(v, w) \) we denote the Nakajima variety associated to a quiver \((Q, I)\), dimension vectors \( w, v \in \mathbb{N}^I \) and stability condition \( \theta \in \mathbb{Z}^I \cong \text{Char}(G_v) \), cf. [9][15][16][17]. Namely, \( \mathcal{M}_{Q, \theta}(v, w) \) is realized as the GIT symplectic reduction of the space of representations \( T^*\text{Rep}(v, w) \) by the Hamiltonian action of \( G_v \):

\[ \mathcal{M}_{Q, \theta}(v, w) := T^*\text{Rep}(v, w)/\!/\!/^\theta G_v. \]

Equivalently, \( \mathcal{M}_{Q, \theta}(v, w) \) is the quotient

\[ \mu^{-1}_{v, w}(0)^{\theta-ss}/G_v, \]

where

\[ \mu_{v, w} : T^*\text{Rep}(v, w) \to g_v^* \]

is the moment map of the \( G_v \)-action and \( \mu^{-1}_{v, w}(0)^{\theta-ss} \) is the open subset of \( \theta \)-semistable representations in \( \mu_{v, w}^{-1}(0) \). Since the action of \( T_w = A_w \times \mathbb{C}_h^\times \) commutes with the one of \( G_v \), it descends to an action on \( \mathcal{M}_{\theta}(v, w) \). In particular, the subgroup \( \mathbb{C}_h^\times \) rescales the symplectic form while all the other actions preserve it.

2.3.2

A striking feature of Nakajima varieties is that the fixed components of the action of a subtorus of \( A_w \) are still quiver varieties [13, Section 2.4.1].

**Proposition 2.2.** Let \( \mathbb{C}^x \) be a one-dimensional subtorus of \( A_w \) acting with weight one on the subspaces \( W_i^{(1)} \subset W_i \), and trivially on their complements \( W_i^{(2)} \). Then

\[ \mathcal{M}(v, w)^{\mathbb{C}^x} \cong \bigsqcup_{v_1 + v_2 = v} \mathcal{M}(v_1, w_1) \times \mathcal{M}(v_2, w_2), \]

where \( w_1 \) and \( w_2 \) are the dimension vectors of \( W^{(1)} \) and \( W^{(2)} \) and the isomorphism is induced by the diagonal inclusion

\[ T^*\text{Rep}(v, w) \times T^*\text{Rep}(v, w) \hookrightarrow T^*\text{Rep}(v, w). \]

By iteration, one gets a decomposition of the connected components of \( \mathcal{M}(v, w)^{A_w} \) as a product of Nakajima varieties.
2.3.3

A representation $U$ of $G_v \times T_w$ allows defining the vector bundle

$$\mu_{v,w}^{-1}(0)^{\theta-ss} \times U/G_v$$

over $X$, which, by abuse of notation, we denote in the same way. In particular, this gives tautological bundles $V_i$ and $W_i$ for all $i \in I$.

Kirwan surjectivity, proved by McGerty and Nevins in [14], states that the restriction map

$$H^*_w \times G_v(\mu_{v,w}^{-1}(0)) \to H^*_w \times G_v(\mu_{v,w}^{-1}(0)^{\theta-ss}) = H^*_w(M(v,w))$$

is surjective, and, as a consequence, the cohomology ring $H^*_w(M(v,w))$ is generated as a $H^*_h(*)$-algebra by the Chern roots of the tautological bundles $V_i$ and $W_i$.

3 Framed CoHA

3.1 The basic construction

3.1.1

Fix a quiver $(Q, I)$ and consider once again a pair of decompositions $v = v_1 + v_2$ and $w = w_1 + w_2$ and the inclusion

$$T^*\text{Rep}(v_1, w_1) \times T^*\text{Rep}(v_2, w_2) \hookrightarrow T^*\text{Rep}(v, w).\quad (5)$$

For $j = 1, 2$, set

$$V^{(j)} = \bigoplus_{i \in I} V_i^{(j)} \quad W^{(j)} = \bigoplus_{i \in I} W_i^{(j)}$$

where $v_j = (\dim V_i^{(j)})_{i \in I}$ and $w_j = (\dim W_i^{(j)})_{i \in I}$. Interpreting the decompositions above as the introduction of an additional grading

$$V = V^{(1)}[0] \oplus V^{(2)}[1] \quad W = W^{(1)}[0] \oplus W^{(2)}[1],$$

we can identify the image of (5) with the degree zero subspace in the induced grading

$$T^*\text{Rep}(v, w) = T^*\text{Rep}(v, w)[-1] \oplus T^*\text{Rep}(v, w)[0] \oplus T^*\text{Rep}(v, w)[1].$$

The space

$$Z := T^*\text{Rep}(v, w)[0] \oplus T^*\text{Rep}(v, w)[1]$$

naturally lives on a Lagrangian correspondence

$$T^*\text{Rep}(v_1, w_1) \times T^*\text{Rep}(v_2, w_2) \hookrightarrow Z \hookrightarrow T^*\text{Rep}(v, w)\quad (6)$$

where the first map is projection and the second one is inclusion. By Lagrangian here we mean that $Z$ can be seen as a Lagrangian subspace of the symplectic space

$$(T^*\text{Rep}(v_1, w_1) \times T^*\text{Rep}(v_2, w_2)) \times T^*\text{Rep}(v, w)$$

with symplectic form given by the sum of the standard symplectic form of the first factor and the opposite of the standard symplectic form of the second one.

Remark 3.1. The previous grading is equivalent to an action of a one-dimensional torus $\mathbb{C}^\times$ acting on the various spaces with the weights prescribed by their degrees. From this point of view, the zero-degree subspace

$$T^*\text{Rep}(v, w)[0] \cong T^*\text{Rep}(v_1, w_1) \times T^*\text{Rep}(v_2, w_2)$$

is just the fixed locus of the action of $\mathbb{C}^\times$ on $T^*\text{Rep}(v, w)$ and $Z$ is the attracting set of $T^*\text{Rep}(v, w)[0].$
3.1.2

It is important to notice that

\[ p : Z \to T^*\text{Rep}(v_1, w_1) \times T^*\text{Rep}(v_2, w_2) \]

is a vector bundle whose fibers \( Z(x_1, x_2) \) can be identified with the \( \text{Ext} \) groups

\[ \text{Ext}^1(x_1, x_2). \]

Here we see the elements \( x_1 \) and \( x_2 \) as modules over the path algebra of the framed doubled quiver \((Q_{fr}, I_{fr})\) and the \( \text{Ext} \)-group is taken in this category.

3.1.3

The subspace \( Z \subset T^*\text{Rep}(v, w) \) is preserved by the action of

\[ P = \prod_{i \in I} P_i \quad P_i = \begin{pmatrix} G_{v_i} & * \\ 0 & G_{v_i} \end{pmatrix} \]

i.e. the product of the parabolic subgroups preserving \( V_i^{(1)} \subset V_i \). Passing to quotients, we get a correspondence

\[ \mathcal{R}(v_1, w_1) \times \mathcal{R}(v_2, w_2) \xleftarrow{p} [Z/P] \xrightarrow{q} \mathcal{R}(v, w) \]  

(7)

Following the general construction of Kontsevich and Soibelman, we will define an algebra via pullback-pushforward along these maps.

3.1.4

For later reference, we record some formulas for certain \( \text{K} \)-theory classes of \( \mathcal{R}(v, w) \) and \( \mathcal{M}(v, w) \).

Firstly, the tangent class

\[ T\mathcal{R}(v, w) \in K_{T_w}(\mathcal{R}(v, w)) = K_{T_w} \times G_v(*) \]

can be written as

\[ T^*\text{Rep}(v, w) - g_v. \]

The inclusion (5) descends a \( T_w \)-equivariant closed immersion of stacks

\[ \mathcal{R}(v_1, w_1) \times \mathcal{R}(v_2, w_2) \hookrightarrow \mathcal{R}(v, w). \]

whose normal class

\[ \mathcal{N} = T\mathcal{R}(v, w) - T\mathcal{R}(v_1, w_1) - T\mathcal{R}(v_2, w_2) \in K_{T_w}(\mathcal{R}(v_1, w_1) \times \mathcal{R}(v_2, w_2)) \]

can be rewritten as \( \mathcal{N} = \mathcal{N}[1] + \mathcal{N}[-1] \), where

\[ \mathcal{N}[1] = T^*\text{Rep}(v, w)[1] - g_v[1] \quad \mathcal{N}[-1] = T^*\text{Rep}(v, w)[-1] - g_v[-1]. \]

Similarly, we have

\[ T\mathcal{M}(v, w) = T^*\text{Rep}(v, w) - g_v - h g_v \in K_{T_w}(\mathcal{M}(v, w)) \]

and the normal class of

\[ \mathcal{M}(v_1, w_1) \times \mathcal{M}(v_2, w_2) \hookrightarrow \mathcal{M}(v, w) \]

is \( N = N[1] + N[-1] \), where

\[ N[1] = T^*\text{Rep}(v, w)[1] - g_v[1] - h g_v[1] \quad N[-1] = T^*\text{Rep}(v, w)[-1] - g_v[-1] - h g_v[-1]. \]
3.2 Framed Cohomological Hall Algebras

3.2.1

From now on, we denote the $C^*_\mathbb{C}$-equivariant cohomology of the point by $k$. This means that $k$ is the polynomial ring $\mathbb{C}[h]$. Let now $(Q, I)$ be a quiver. We define a $\mathbb{N}^I \times \mathbb{N}^I$-graded $k$-module
\[
\mathcal{H}^{Q,I}_k = \bigoplus_{v,w \in \mathbb{N}^I} \mathcal{H}^{Q,I}_k(v, w)
\]
by setting
\[
\mathcal{H}^{Q,I}_k(v, w) := H_{T^w}(\mathcal{R}(v, w)).
\]
As a $k$-module, $\mathcal{H}^{Q,I}_k(v, w)$ is just the ring
\[
k[a_w \times s_v]_{W^w,v},
\]
where $a_w$ and $s_v$ are the Lie algebras of $A_w \subset T_w$ and of some maximal torus $S_v \subset G_v$ respectively. Whenever convenient, we will drop the reference to the quiver by simply writing $\mathcal{H}$ in place of $\mathcal{H}^{Q,I}_k$.

Clearly, instead of defining $\mathcal{H}(v, w)$ as the $T_w$-equivaraint cohomology of $\mathcal{R}(v, w) = [T^*\text{Rep}(v, w)/G_v]$, we could have taken the non-equivariant cohomology of the stack $[T^*\text{Rep}(v, w)/(G_v \times T_w)]$. However, we believe that the former is the right perspective, especially when it comes to study the relation between stable envelopes and CoHAs.

Following Kontsevich and Soibelman [12], we define a multiplication map
\[
m : \mathcal{H} \otimes_k \mathcal{H} \to \mathcal{H}
\]
that turns $\mathcal{H}$ into a $\mathbb{N}^I \times \mathbb{N}^I$-graded $k$-algebra. Each graded component
\[
m : \mathcal{H}(v_1, w_1) \otimes_k \mathcal{H}(v_2, w_2) \to \mathcal{H}(v_1 + v_2, w_1 + w_2)
\]
is defined by means of correspondence (7). Explicitly, $m$ is defined as the composition of the following maps:

1. the Künneth isomorphism
\[
H^*_{T^w}(\mathcal{R}(v_1, w_1)) \otimes_k H^*_{T^w}(\mathcal{R}(v_2, w_2)) \cong H^*_{T^w}(\mathcal{R}(v_1, w_1) \times \mathcal{R}(v_2, w_2)),
\]
2. the pullback
\[
p^* : H^*_{T^w}(\mathcal{R}(v_1, w_1) \times \mathcal{R}(v_2, w_2)) \to H_{T^w([Z/P])},
\]
3. the pushforward
\[
q_* : H_{T^w([Z/P])} \to H_{T^w}(\mathcal{R}(v_1 + v_2, w_1 + w_2)).
\]

The proof of [12, Theorem 1] extends to our framed setting and gives the following

**Theorem 3.2.** The map $m$ defines an $\mathbb{N}^I \times \mathbb{N}^I$-graded associative unital $k$-algebra structure on $\mathcal{H}^{Q,I}_k$ with unit element given by $1 \in k = \mathcal{H}^{Q,I}_k(0, 0)$.

**Definition 3.3.** We call the associative unital $\mathbb{N}^I \times \mathbb{N}^I$-graded algebra $(\mathcal{H}^{Q,I}_k, m)$ the framed doubled cohomological Hall algebra associated with the quiver $(Q, I)$.

Notice that, seen as a $k$-module, $\mathcal{H}^{Q,I}_k$ does not depend on the edges of $Q$, which are encoded in the multiplication map $m$ nonetheless.

**Remark 3.4.** The (unframed) CoHA associated with the doubled quiver $(\overline{Q}, \overline{T})$ defined by Kontsevich-Soibelman is recovered as the $w = 0$ subalgebra of our framed CoHA $\mathcal{H}^{Q,I}_k$. Indeed, the $k$-submodule with trivial $w$-grading is preserved by the multiplication, which reduces to the one defined in [12, Section 2.2]. On the other hand, the algebra $\mathcal{H}^{Q,I}_k$ should not be confused with Kontsevich-Soibelman cohomological Hall algebra associated with the framed doubled quiver $(\overline{Q}_{fr}, \overline{T}_{fr})$. Indeed, the graded component of the latter are identified with $H^*_{G_v \times T_w \times C^*_h}(T^*\text{Rep}(v, w))$ rather than $H^*_{G_v \times T_w \times C^*_h}(T^*\text{Rep}(v, w))$, and also the multiplication is affected by the change of group.
3.2.2

To relate $\mathcal{H}_{fr}$ with stable envelopes, we need to twist the multiplication

$$m : \mathcal{H}(v_1, w_1) \otimes_{k} \mathcal{H}(v_2, w_2) \to \mathcal{H}(v_1 + v_2, w_1 + w_2).$$

This is done as follows: multiplication by the class

$$e(h_{\mathfrak{g}_v}[1]) \in H^*_T(\mathcal{R}(v_1, w_1) \times \mathcal{R}(v_2, w_2))$$

gives an endomorphism that can be inserted between the maps at points 1. and 2. in the definition of $m$ to get a twisted multiplication $m_\tau$. It is easy to see that $m_\tau$ also defines unital associative $k$-algebra structure on the $k$-module $\mathcal{H}_{fr}^{J_\mu}$. To distinguish the twisted algebra structure on $\mathcal{H}_{fr}^{J_\mu}$ without explicitly referring to the multiplication map $m_\tau$, we add a subscript to the $k$-module itself: $\mathcal{H}_{fr}^{J_\mu}$. 

**Definition 3.5.** We call the associative unital $\mathbb{N} \times \mathbb{N}$-graded $k$-algebra $(\mathcal{H}_{fr}^{J_\mu}, m_\tau)$ the twisted framed doubled cohomological Hall algebra associated with the quiver $(Q, I)$. 

**Remark 3.6.** The introduction of the twist is not new in the literature. Indeed, Yang and Zhao introduced in [32] the same twist to define a canonical morphism between the cohomological Hall algebra associated with the preprojective path algebra $CQ/J_\mu$ and the twisted version of Kontsevich-Soibelman CoHA for the doubled quiver $\overline{Q}$. Here $J_\mu$ is the ideal associated with the moment map of the action of $G_v$ on $T^*\text{Rep}(v)$. Following their construction, one could define a framed version of their preprojective cohomological Hall algebra together with an algebra morphism to $\mathcal{H}_{fr}^{J_\mu}$. 

3.3 **Shuffle description for the product**

The framed CoHA admits a very explicit description under the identifications

$$\mathcal{H}(v, w) = k[a_w \times s_v]^{W_{G_v}}.$$

An element $f \in \mathcal{H}(v, w)$ is simply a polynomial $f(a, s)$, symmetric in each set of Chern roots associated to the groups $\text{GL}(v_1) \subset G_v$. 

Fix decompositions $v = v_1 + v_2$ and $w = w_1 + w_2$ and view both $\mathcal{H}(v, w)$ and $\mathcal{H}(v_1, w_1) \otimes_{k} \mathcal{H}(v_2, w_2)$ as subalgebras of

$$k[a_w \times s_v] = k[a_{w_1} \times s_{v_1}] \otimes_{k} k[a_{w_2} \times s_{v_2}]$$

Let

$$\text{Shuffle} : k[a_{w_1} \times s_{v_1}]^{W_{G_{v_1}}} \otimes_{k} k[a_{w_2} \times s_{v_2}]^{W_{G_{v_2}}} \to k[a_w \times s_v]^{W_{G_v}}$$

be the symmetrization map by shuffles and recall the definitions of the classes in section 3.1.4. 

**Theorem 3.7.** The product $m(f_1, f_2)$ of two elements $f_1(a_1, s_1) \in \mathcal{H}(v_1, w_1)$ and $f_2(a_2, s_2) \in \mathcal{H}(v_2, w_2)$ is given by

$$\text{Shuffle} \left( \frac{e(T^*\text{Rep}(v, w)[-1])f_1f_2}{e(h_{\mathfrak{g}_v}[1])} \right).$$

Similarly, the twisted product $m_\tau(f_1, f_2)$ is given by

$$\text{Shuffle} \left( \frac{e(T^*\text{Rep}(v, w)[-1])e(h_{\mathfrak{g}_v}[1])f_1f_2}{e(h_{\mathfrak{g}_v}[1])} \right).$$

Notice that the kernel $e(T^*\text{Rep}(v, w)[-1])/e(h_{\mathfrak{g}_v}[1])$ of the multiplication map $m$ coincides with $e(\mathfrak{Q}[-1])$, the Euler class of the negative part of the normal bundle defined in 3.1.4.
Proof. We decompose the correspondence (7) giving rise to the multiplication map \( m \) as follows:

\[
\begin{array}{ccc}
Z/G_{v_1} \times G_{v_2} & \overset{\rho}{\rightarrow} & Z/P \\
\mathcal{R}(v_1, w_1) \times \mathcal{R}(v_2, w_2) & \overset{p_1}{\leftarrow} & \mathcal{R}(v, w) \\
\mathcal{R}(v, w) / P & \overset{\pi}{\rightarrow} & [T^* \text{Rep}(v, w) / P] \\
\end{array}
\]

All the maps are the obvious ones. In particular, \( p_1 \) is induced by the projection \( Z \rightarrow T^* \text{Rep}(v_1, w_1) \times T^* \text{Rep}(v_2, w_2) \) and \( p_2 \) is an affine bundle, so \( p^* = (p_2^*)^{-1} \circ p_1^* \) is just the identity map in the tautological presentation. The map \( i \) is induced by the inclusion \( Z \hookrightarrow T^* \text{Rep}(v, w) \) so \( i_* \) is multiplication by \( e(T^* \text{Rep}(v, w)[-1]) \), the Euler class of its normal bundle. Finally, the map \( \pi_* \) is modelled by the map

\[
\tilde{\pi} : T^* \text{Rep}(v, w)_P \rightarrow T^* \text{Rep}(v, w)_{G_v},
\]

which is a fibration by partial flags \( G_v / P \), and hence \( \text{Ker}(d\tilde{\pi}) = g_v[-1] \). Thus

\[
\pi_*(f) = \tilde{\pi}_*(f) = \text{Shuffle} \left( \frac{f}{e(g_v[-1])} \right).
\]

Altogether,

\[
m(f_1, f_2) = q_* \circ p^*(f_1 \otimes f_2) = \pi_* \circ i_* \circ (p_2^*)^{-1} \circ p_1^*(f_1 \otimes f_2) = \text{Shuffle} \left( \frac{e(T^* \text{Rep}(v, w)[-1]) f_1 f_2}{e(g_v[-1])} \right),
\]

as claimed.

Finally, since \( m_{\tau} \) is defined as the composition of \( m \) with cup product by \( e(hg_v[1]) \), also its formula follows.

3.4 CoHA via abelian quotients

3.4.1 Choose a maximal torus \( S \subset G_v \) with Lie algebra \( s \). In this section, we introduce an abelianized version of the CoHA using \( S \)-equivariant rather than \( G_v \)-equivariant cohomology.

Let \( B \) be a Borel subgroup containing \( S \) with Lie algebra \( b = s \oplus n \) and consider the following diagram

\[
\begin{array}{ccc}
[T^* \text{Rep}(v, w) / S] & \overset{i_*}{\rightarrow} & [T^* \text{Rep}(v, w) \oplus hn / S] \\
\downarrow Q & & \downarrow i_* \\
[T^* \text{Rep}(v, w) / B] & \overset{i}{\rightarrow} & [T^* \text{Rep}(v, w) \oplus hs^\perp / S] \\
\end{array}
\]

Here, \( Q \) is an affine bundle with fiber \( B / S \) and \( q \) is a fibration by flags \( G_v / B \). As usual, the label \( h \) indicates the presence of an action of \( C_b^\times \) with weight one.

All the maps are clearly \( T_w \)-equivariant. Passing to cohomology, notice that

- The map \( q \) is proper, and the pushforward \( q_* \) is surjective.
- The pullback \( Q^* \) is an isomorphism.
An argument similar to that of Theorem \textit{H}.
The maps \textit{Proposition 3.8.} simply given by cup product by the class \( \tau \) where

\[
\text{Remark that the need to add the factor } N^{2,1} \text{ is the chosen maximal torus of } G_v \supset G_{v_1} \times G_{v_2} \text{ and, similarly, } B_1 \times B_2 \text{ is the subgroup of } B \text{ such that }
\]

\[
b = b_1 \oplus b_2 \oplus g_v[1],
\]

or, equivalently, such that

\[
n = n_1 \oplus n_2 \oplus g_v[1].
\]

We now introduce the k-module

\[
\mathcal{H}_{ab,\tau} = \bigoplus_{v,w \in \mathbb{N}^I} \mathcal{H}_{ab,\tau}(v,w)
\]

by setting

\[
\mathcal{H}_{ab,\tau}(v,w) = H_{T_w}([T^*\Rep(v,w) \oplus hs^\perp/S])
\]

The goal of this section is to promote this module to a \( \mathbb{N}^I \times \mathbb{N}^I \) graded algebra and to recover the twisted CoHA \( \mathcal{H}_\tau \) from \( \mathcal{H}_{ab,\tau} \). For similar constructions in a different framework, see [8][22]. We remark that the need to add the factor \( s^\tau = n \oplus n' \) and to introduce the horizontal maps in the diagram (8) is due to the fact that we want to reproduce \( \mathcal{H}_\tau \) rather than the untwisted \( \mathcal{H} \).

\textbf{3.4.2}

The analog of the correspondence (6) now is

\[
(T^*\Rep(v_1,w_1) \oplus hs^\perp_1) \times (T^*\Rep(v_2,w_2) \oplus hs^\perp_2) \xrightarrow{Z_S} T^*\Rep(v,w) \oplus hs^\perp
\]

where

\[
Z_S = Z \oplus hs^\perp_1 \oplus hs^\perp_2.
\]

Quotienting by \( S = S_1 \times S_2 \), we get

\[
[T^*\Rep(v_1,w_1) \oplus hs^\perp_1/S_1] \times [T^*\Rep(v_2,w_2) \oplus hs^\perp_2/S_2] \xrightarrow{ps} [Z_S/S] \xrightarrow{qs} [T^*\Rep(v,w) \oplus hs^\perp/S].
\]

We now define multiplication maps

\[
m_{ab,\tau} : \mathcal{H}_{ab,\tau}(v_1,w_1) \otimes_k \mathcal{H}_{ab,\tau}(v_2,w_2) \to \mathcal{H}_{ab,\tau}(v,w)
\]

as the composition of the following morphisms

1. the K"unneth isomorphism

\[
\mathcal{H}_{ab,\tau}(v_1,w_1) \otimes_k \mathcal{H}_{ab,\tau}(v_2,w_2) \cong H^*_w([T^*\Rep(v_1,w_1) \oplus hs^\perp_1/S_1] \times [T^*\Rep(v_2,w_2) \oplus hs^\perp_2/S_2]),
\]

2. the pullback

\[
p^*_S : H^*_w([T^*\Rep(v_1,w_1) \oplus hs^\perp_1/S_1] \times [T^*\Rep(v_2,w_2) \oplus hs^\perp_2/S_2]) \to H_{T_w}([Z_S/S]),
\]

3. the pushforward

\[
(qs)_* : H_{T_w}([Z_S/S]) \to H_{T_w}([T^*\Rep(v,w) \oplus hs^\perp/S]) = \mathcal{H}_{ab,\tau}(v,w).
\]

An argument similar to that of Theorem 3.7, but simpler, shows that the multiplication map \( m_{ab,\tau} \) is simply given by cup product by the class

\[
e(T^*\Rep(v,w)[-1])e(hg_v[-1])e(hg_v[1])
\]

(9)

\textbf{Proposition 3.8.} The maps \( m_{ab,\tau} \) define an \( \mathbb{N}^I \times \mathbb{N}^I \)-graded associative unital k-algebra structure on \( \mathcal{H}_{ab,\tau} \) with unit element given by \( 1 \in k = \mathcal{H}(0,0) \).
We now relate \( \mathcal{H}_r \) with \( \mathcal{H}_{ab,r} \). For simplicity, let us denote
\[
\mathfrak{R}_b := [T^* \text{Rep}(v \otimes w) \oplus hn/S].
\]

**Theorem 3.9.** The multiplication map \( m_{ab,r} \) uniquely factors through the following commutative diagram
\[
\begin{array}{ccc}
H^*_T (\mathfrak{R}_{b_1}) & \otimes & H^*_T (\mathfrak{R}_{b_2}) \\
\downarrow & & \downarrow m_{ab,r} \\
H^*_T (\mathfrak{R}_b) & \rightarrow & H_{ab,r}(v, w)
\end{array}
\]

Denote the dashed arrow by \( m_{ab,r}^h \). Then the following diagram
\[
\begin{array}{ccc}
\mathcal{H}_r(v_1, w_1) & \otimes & \mathcal{H}_r(v_2, w_2) \\
\downarrow m_r & & \downarrow H_{ab,r}(v, w) \\
\mathcal{H}_r(v, w) & \leftarrow & H_{ab,r}(v, w)
\end{array}
\]
is also commutative.

**Proof.** We claim that the dashed arrow of the first diagram is given by the correspondence
\[
[T^* \text{Rep}(v_1, w_1) \oplus hn_1/S_1] \times [T^* \text{Rep}(v_2, w_2) \oplus hn_2/S_2] \xrightarrow{\text{correspondence}} [Z_b/S] \rightarrow [T^* \text{Rep}(v, w) \oplus hn/S]
\]
with
\[
Z_b = Z \oplus hn_1 \oplus hn_2.
\]
It fits in the following pair of Cartesian squares
\[
\begin{array}{ccc}
[T^* \text{Rep}(v_1, w_1) \oplus hn_1/S_1] \times [T^* \text{Rep}(v_2, w_2) \oplus hn_2/S_2] & \xrightarrow{\text{correspondence}} & [Z_b/S] \\
\downarrow (i_{1,-}) \times (i_{2,-}) & & \downarrow i \\
[T^* \text{Rep}(v_1, w_1) \oplus hS^+_1/S_1] \times [T^* \text{Rep}(v_2, w_2) \oplus hS^+_2/S_2] & \xrightarrow{\text{correspondence}} & [Z_S/S] \\
\end{array}
\]
where the central arrow \( i \) is induced by the inclusion map
\[
Z_b = Z \oplus hn_1 \oplus hn_2 \rightarrow Z \oplus hS^+_1 \oplus hS^+_2.
\]

By compatibility of pullback and pushforward on Cartesian squares, we deduce that
\[
(q_S)_* \circ \rho^*_S \circ ((i_{1,-})_* \otimes (i_{2,-})_*) = (q_S)_* \circ i_* \circ \rho^*_b = (i_-)_* \circ (q_b)_* \circ \rho^*_b,
\]
as claimed. Notice that this implies that the map \( m_{ab,r}^h \) is the cup product operation by
\[
e(T^* \text{Rep}(v, w)[-1])e(hq_b[1]).
\]
We now prove commutativity of the second diagram. Firstly, notice that the pullbacks \( (i_{1,+})^* \) and \( (i_{1,+})^* \otimes (i_{2,+})^* \) are isomorphisms, and the composition \( (i_{1,+})^* \circ m_{ab,r}^h \circ ((i_{1,+})^* \otimes (i_{2,+})^*)^{-1} \) is given by \( (q_S)_* \circ (\bar{\rho}_S)_* \circ e(hq_b[1]) \), where \( \bar{\rho}_S \) and \( \bar{q}_S \) are maps fitting in the correspondence
\[
[T^* \text{Rep}(v_1, w_1)/S_1] \times [T^* \text{Rep}(v_2, w_2)/S_2] \xrightarrow{\text{correspondence}} [Z/S] \rightarrow [T^* \text{Rep}(v, w)/S].
\]
Since $m_\tau$ is by definition $m \circ e(h_{g_\nu}[1])$, to complete the proof it suffices to show that the following diagram

\[
\begin{array}{c}
\mathcal{H}_\tau(v_1, w_1) \otimes_k \mathcal{H}_\tau(v_2, w_2) (q_1 \times q_2)_* \circ (Q_1 \times Q_2)^{-1} \rightarrow H^*_\tau(v_1, w_1) \otimes_k H^*_\tau(v_2, w_2) ([T^*\text{Rep}(v_1, w_1)/S_1]) \otimes_k H^*_\tau(v_2, w_2) ([T^*\text{Rep}(v_2, w_2)/S_2]) \\
\downarrow m = p \circ \rho_* \\
\mathcal{H}_\tau(v, w) \leftarrow \mathcal{H}_\tau(v, w) (q_1 \circ (Q^*)^{-1}) \rightarrow H^*_\tau([T^*\text{Rep}(v, w)/S])
\end{array}
\]

is commutative. Consider the commutative diagram

\[
\begin{array}{c}
[T^*\text{Rep}(v_1, w_1)/S_1] \times [T^*\text{Rep}(v_2, w_2)/S_2] \xrightarrow{\rho_*} [Z/S] \xrightarrow{\tilde{q}_S} [T^*\text{Rep}(v, w)/S] \\
\downarrow Q_1 \times Q_2 \\
[T^*\text{Rep}(v_1, w_1)/B_1] \times [T^*\text{Rep}(v_2, w_2)/B_2] \xrightarrow{\tilde{q}} [T^*\text{Rep}(v, w)/B] \\
\downarrow q_1 \times q_2 \\
[T^*\text{Rep}(v_1, w_1)/G_{v_1}] \times [T^*\text{Rep}(v_2, w_2)/G_{v_2}] \xrightarrow{\rho} [Z/P] \xrightarrow{q} [T^*\text{Rep}(v, w)/G_v]
\end{array}
\]

where the unlabelled maps are the obvious ones. The squares in the lower left and upper right are Cartesian, so once again commutativity of diagram (10) follows from compatibility of pushforward and pullback on Cartesian squares. Details are left to the reader.

\[\square\]

4 Cohomological stable envelopes

4.1 Chambers, attracting sets and polarization

Let $X$ be a smooth quasi-projective symplectic variety with an action a torus $T$ rescaling the symplectic form with weight $h$ and let $A \subset \ker(h)$. In this work, we will be interested in the case when

\[X = T^*\text{Rep}(v, w)//\!/^\theta H\]

where $H$ is either $G_v$ or a maximal torus $S \subset G_v$ thereof.

4.1.1 Definition 4.1 ([13]). A chamber $\mathcal{C}$ is a connected component of $\text{Cochar}(A) \otimes Z \mathbb{R} \setminus \Delta$, where $\Delta$ is the hyperplane arrangement determined by the $A$-weights of the normal bundles of $X^A$. One says that a weight $\chi \in \text{Char}(A)$ is attracting (resp. repelling) with respect to $\mathcal{C}$ if $\lim_{t \to 0} \chi \circ \sigma(t) = 0$ (resp. if $\lim_{t \to 0} \chi \circ \sigma(t) = \infty$) for any $\sigma \in \mathcal{C}$. The trivial character is called the $A$-fixed character.

Restricted to some fixed component $F \subset X^A$, any $A$-equivariant $K$-theoretic class $V$ of $X$ decomposes according to the characters of $A$ in attracting, $A$-fixed and repelling classes

\[V|_F = V|_{F,+} + V|_{F,0} + V|_{F,-}\]

For instance, we have that

\[TX|_F = TX|_{F,+} + TF + TX|_{F,-}\]

where $N^+_F = TX|_{F,+}$ and $N^-_F = TX|_{F,-}$ are the attracting and repelling parts of the normal bundle $N_F$ to $F$ in $X$. If $X$ is a Nakajima variety and $A = A_w$, these geometrically-defined chambers coincide with the Lie-theoretic chambers. Indeed, the $A_w$-weights of the normal bundles of the $A_w$-fixed components of a Nakajima variety $X$ are the roots of the group $\text{GL}(\sum_j w_j)$, and hence the hyperplane arrangement $\Delta$ coincides with the one that defines the Lie-theoretic chambers of a maximal torus of $\text{GL}(\sum_j w_j)$.
4.1.2
Let \( F \) be a fixed component in \( X^A \). The attracting set
\[
\text{Att}_\mathcal{C}(F) = \{ x \in X \mid \lim_{t \to 0} \sigma(t) \cdot x \in F, \text{ for some } \sigma \in \mathcal{C} \}
\]
is well defined, i.e. independent of the choice of \( \sigma \in \mathcal{C} \), and it is also an affine bundle over \( F \). This follows from the standard results in [4]. Notice in particular that \( T(\text{Att}_\mathcal{C}(F))|_F = N_F^+ \).

We also recall the following key definition:

**Definition 4.2.** The full attracting set \( \text{Att}_f^\mathcal{C}(F) \) is the minimal closed subset of \( X \) containing \( F \) and closed under taking \( \text{Att}_\mathcal{C}(\cdot) \).

4.1.3
The choice of a chamber determines a partial ordering on the set of fixed components of \( X^A \) by taking the transitive closure of the relation
\[
\text{Att}_\mathcal{C}(F) \cap F' \neq \emptyset \quad \Rightarrow \quad F \geq F'.
\]
See [13, Section 3.2.3] for a proof of the fact that this is really a partial order.

4.2 Stable envelopes

We now recall Maulik-Okounkov’s definition of stable envelopes.

4.2.1
Since \( A \) acts trivially on \( X^A \), we have
\[
H_T(X^A) \cong H_{T/A}(X^A) \otimes H_A(*),
\]
where \( H_A(*) \) is the ring of polynomials in equivariant parameters in \( A \). Although the isomorphism above is not canonical, different isomorphisms give the same filtration of \( H_T(X^A) \) by the degree \( \deg_A \) in the equivariant parameters.

With this observation, we can recall the definition of cohomological stable envelope.

**Theorem 4.3** ([13], Theorem 3.3.4). Let \( \mathcal{C} \) be a chamber for the action of \( A \) on \( X \). There exist a unique map of \( H_T(*) \)-modules
\[
\text{Stab}_\mathcal{C} : H_T(X^A) \to H_T(X)
\]
such that

1. For every fixed component \( F \subset X^A \), then
\[
\text{Stab}_\mathcal{C}(F)|_F = e(N_F^-),
\]
where \( \text{Stab}_\mathcal{C}(F) \) is the restriction of \( \text{Stab}_\mathcal{C} \) to \( H_T(F) \subset H_T(X^A) \).

2. \( \text{Stab}_\mathcal{C}(F) \) is supported on \( \text{Att}_f^\mathcal{C}(F) \), i.e.
\[
\text{Stab}_\mathcal{C}(F)|_U = 0,
\]
where \( U = X \setminus \text{Att}_f^\mathcal{C}(F) \).

3. \( \deg_A(\text{Stab}_\mathcal{C}(F)(\gamma)|_{F'}) < \frac{1}{2} \text{codim}_X(F') \) for all \( \gamma \in H_{T/A}(F) \) and \( F' < F \).
4.2.2

The next lemma is a key result for the application of stable envelopes in the geometric representation theory of Yangians, see [13], and it also lies at the heart of the interpretation of stable envelopes in terms of CoHAs that we shall give.

**Lemma 4.4 ([13])**. Let \( C \) be a chamber for the action of \( A \) on \( X \) and let \( C' \subset C \) be a face of some dimension. Let \( A' \) be torus whose Lie algebra is the span of \( C' \) in the Lie algebra of \( A \) and let \( C/C' \) be the projection of \( C \) on the Lie algebra of \( A/A' \). Then the following diagram

\[
\begin{array}{ccc}
H_T(X^A) \quad \xrightarrow{\text{Stab}_C} \quad H_T(X) \\
\downarrow \quad \quad \quad \downarrow \\
H_T(X^{A'}) \quad \xrightarrow{\text{Stab}_{C'}} \quad H_T(X^{A'})
\end{array}
\]

is commutative.

4.3 Stable envelopes of Nakajima varieties: R-matrices

4.3.1

Let now \( X \) be a Nakajima variety \( X = \mathcal{M}(v, w) \) equipped with the action of the torus \( T = T_w \). A decomposition \( w = w_1 + w_2 + \ldots w_k \) gives a homomorphism

\[
A = \{(a_1, a_2, \ldots, a_k) \mid a_i \in \mathbb{C}^\times \} \hookrightarrow A_w.
\]

By iteration of Proposition 2.2, we get

\[
H_{T_w}(\mathcal{M}(v, w)^A) = \bigoplus_{\sum_{j+1} v_j = v} \bigotimes_{j=1}^k H_{T_{w_j}}(\mathcal{M}(v_j, w_j)).
\]

(11)

and hence the stable envelope is a collection of maps

\[
H_{T_{w_1}}(\mathcal{M}(v_1, w_1)) \otimes_k \cdots \otimes_k H_{T_{w_k}}(\mathcal{M}(v_k, w_k)) \to H_{T_w}(\mathcal{M}(v, w)).
\]

4.3.2

For this torus action, there are \( k! \) possible chambers, namely

\[
\mathcal{C}_\sigma = \{ a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(k)} \} \quad \sigma \in S_k.
\]

The change of basis matrix relating stable envelopes for two different chambers \( \mathcal{C}_\sigma \) and \( \mathcal{C}_\tau \), namely the map

\[
R_{\mathcal{C}_\tau, \mathcal{C}_\sigma} := \text{Stab}_{\mathcal{C}_\tau}^{-1} \circ \text{Stab}_{\mathcal{C}_\sigma} \in H_{T_w}(X^A) \otimes \mathbb{C}^{(\text{Lie}(A))}
\]

is called R-matrix and is a fundamental object is the geometric representation theory of Yangians developed by Maulik and Okounkov [13]. In particular, when \( k = 3 \) different factorizations of \( R_{\mathcal{C}_\tau, \mathcal{C}_\sigma} \) provide solutions of the classical Yang-Baxter equation with spectral parameters.

4.4 Stable envelopes of hypertoric varieties: explicit formulas

4.4.1

If we consider the GIT-symplectic reduction of \( T^* \text{Rep}(v, w) \) by a maximal torus \( S \subset G_v \) with stability condition \( \theta \in \text{Char}(S) \)

\[
\mathcal{M}_S(v, w) := T^* R(v, w)////_{\theta} S = \mu_{S}^{-1}(0)^{\theta-ss}/S,
\]

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we produce a hypertoric variety rather than a Nakajima quiver variety. Like quiver varieties, they come equipped with an action of $T_w$ and, for a generic stability condition, they are smooth symplectic varieties, so it makes sense to talk about their stable envelopes. Moreover, also for hypertoric varieties, Kirwan surjectivity holds \[10\], and hence

$$H_{T_w}(\mathcal{M}_S) \cong k[a_w \times s]/I,$$

where $I$ is some ideal. For an explicit description of the generators of $I$, see \[10\].

Remarkably, stable envelopes of hypertoric varieties admit a very explicit description in terms of tautological classes. This description was originally developed in full generality by Shenfeld \[29\], but for our interests, it suffices to have an explicit formula for the stable envelope map of the form

$$\text{Stab}_{\{a < 0\}} : H_{T_w}(\mathcal{M}_{S_1}(v_1, w_1) \times \mathcal{M}_{S_2}(v_2, w_2)) \to H_{T_w}(\mathcal{M}_S(v_1 + v_2, w_1 + w_2))$$

Here $S = S_1 \times S_2$ and the torus action considered is the one of the one-dimensional torus $\mathbb{C}^\times \subset A_w$ acting with weight one on $W_1$ and trivially elsewhere. Notice that the cohomology

$$H_{T_w}(\mathcal{M}_{S_1}(v_1, w_1) \times \mathcal{M}_{S_2}(v_2, w_2))$$

is isomorphic to

$$k[a_{w_1} \times s_1]/I_1 \otimes_k k[a_{w_2} \times s_2]/I_2 \cong k[a_w \times s]/(I_1 + I_2)$$

hence both the domain and the codomain of the stable envelope map $\text{Stab}_{\{a < 0\}}$ are quotients of the same ring $k[a_w \times s]$.

**Proposition 4.5.** Let $p_\gamma \in k[a_w \times s]$ be a polynomial representing a class $\gamma \in H_{T_w}(\mathcal{M}_{S_1}(v_1, w_1) \times \mathcal{M}_{S_2}(v_2, w_2))$. Then the polynomial

$$e(T^*\text{Rep}(v, w)[-1])p_\gamma$$

represents the class $\text{Stab}_{\{a < 0\}}(\gamma)$.

For the proof, see \[29\]. The fact that this assignment gives a well defined map can be also seen as follows: by equivariant formality, the $H_{T_w}(\ast)$-module \[12\] is free\(^1\), so one can pick a set of polynomials $\{p_\gamma\}_i$ representing a basis $\{\gamma_i\}_i$ of $H_{T_w}(\mathcal{M}_{S_1}(v_1, w_1) \times \mathcal{M}_{S_2}(v_2, w_2))$ and use the assignment

$$\gamma_i \mapsto e(T^*\text{Rep}(v, w)[-1])p_{\gamma_i}$$

to define a $H_{T_w}(\ast)$-linear map to $H_{T_w}(\mathcal{M}_S(v_1 + v_2, w_1 + w_2))$. Since this map satisfies the axioms of a stable envelope (see \[29\]), by uniqueness of the latter it is actually independent of the choice of the basis and of its polynomial representatives.

### 4.5 Abelianization of stable envelopes

#### 4.5.1

In this section, we recall abelianization of stable envelopes, which is a procedure that relates the stable envelopes of a Nakajima variety

$$\mathcal{M}(v, w) = T^*R(v, w)//G_v = \mu_v^{-1}(0)^{G_v-ss}/G$$

with the ones of the abelian quotient

$$\mathcal{M}_S(v, w) := T^*R(v, w)//S = \mu_S^{-1}(0)^{S-ss}/S.$$  

\(^1\)Indeed, hypertoric varieties are GKM, which is a stronger condition than equivariant formality, see \[10\].
In the following, we assume that the stability condition \( \theta \in \text{Char}(G_v) \subset \text{Char}(S) \), which determines the \((\theta, G_v)\)-semistable locus \( \mu_{v,w}^{-1}(0)^{G_v-ss} \) and \((\theta, S)\)-semistable locus \( \mu_S^{-1}(0)^{S-ss} \) respectively, is chosen generically, in such a way that both the varieties above are smooth. Abelianization of stable envelopes was first developed by Shenfeld [29] for singular cohomology and then reproduced in elliptic cohomology by Aganagic and Okounkov in [1]. The following version is the cohomological limit of abelianization in elliptic cohomology. Let \( B \) be a Borel subgroup of \( G_v \) containing \( S \), and let \( \mathfrak{s} \subset \mathfrak{b} \subset \mathfrak{g}_v \) be the corresponding Lie algebras. Since \( \mu_S \) is the projection of \( \mu_{v,w} \) to \( \mathfrak{s}^* \), which we identify with \( \mathfrak{s} \) via trace pairing, we have

\[
\mathcal{M}_S(v, w) = \mu_{v,w}^{-1}(\mathfrak{s}^\perp)^{S-ss} / S,
\]

where \( \mathfrak{s}^\perp = \mathfrak{n} \oplus \mathfrak{n}^\vee \) is the subspace of \( \mathfrak{g}_v \) vanishing on \( \mathfrak{s} \). Consider the diagram

\[
\begin{array}{c}
\mu_{v,w}^{-1}(0)^{G_v-ss} / S \\
\downarrow \Pi \\
\mu_{v,w}^{-1}(0)^{G_v-ss} / B \\
\downarrow \pi \\
\mu_{v,w}^{-1}(0)^{G_v-ss} / G
\end{array}
\]

(13)

Notice the similarity with the abelianization diagram for the CoHA (8). Indeed, the latter was inspired by diagram (13).

Notice that \( \Pi \) is an affine bundle with fiber \( B / S \), \( \pi \) is a fibration by flags \( G_v / B \) and \( j_- \) is a closed embedding.

4.5.2

All the maps in the diagram are \( T_w \)-equivariant, so the diagram above restricts to fixed components of the action of some torus \( A \subset T_w \). Let \( \pi^A, \Pi^A, \) etc. be the restrictions of the maps above. Passing to cohomology, notice that

- The maps \( \pi \) and \( \pi^A \) are proper, and the pushforwards \( \pi_* \) and \( \pi_*^A \) are surjective.
- The pullbacks \( \Pi^* \) and \( (\Pi^A)^* \) are isomorphisms.

Fix some component \( F \subset \mathcal{M}(v, w)^A \) and a chamber \( \mathcal{C} \) for the action of \( A \). Let \( F'_B \) be the (unique) \( A \)-fixed component in \( \pi^{-1}(F) \) whose normal weights in \( \pi^{-1}(F) \) are non-attracting, i.e.

\[
\text{Ker}(d\pi) \big|_{F_B, +} = 0.
\]

Overall, we get a diagram

\[
\begin{array}{c}
(\Pi^A)^{-1}(F_B) \\
\downarrow \Pi^A \\
F_B \\
\downarrow \pi^A \\
F
\end{array}
\]

whose vertices are \( A \)-fixed components.
4.5.3

We can now review the statement of the abelianization theorem.

**Theorem 4.6 ([1]).** The composition $\text{Stab}_G(F_S) \circ (j^A)_*$ factors through $(j^-)_*$, i.e. there exists a map making the diagram

$$
\begin{array}{ccc}
H_{T_w}(F^*_S) & \xrightarrow{(j^A)_*} & H_{T_w}(F) \\
\downarrow & & \downarrow \\
H_{T_w}(\mu^{-1}(\mathfrak{b}^\perp)S-ss/S) & \xrightarrow{(j^-)_*,} & H_{T_w}(\mathcal{M}_S(v, w))
\end{array}
$$

commute. Denote a choice of dashed arrow by $\text{Stab}^b_G(F_S)$. Then the following diagram also commutes

$$
\begin{array}{ccc}
H_{T_w}(F) & \xleftarrow{\ (\pi^A)_* \circ ((\Pi^A)^*)^{-1} \circ (j^A)_* \ } & H_{T_w}(F^*_S) \\
\downarrow & & \downarrow \\
H_{T_w}(\mathcal{M}(v, w)) & \xleftarrow{\pi_* (\Pi^*)^{-1} \circ (j^+_*) \ } & H_{T_w}(\mu^{-1}(\mathfrak{b}^\perp)S-ss/S)
\end{array}
$$

Since each of the maps in the composition $(\pi^A)_* \circ ((\Pi^A)^*)^{-1} \circ (j^A)_*$ is surjective, the previous theorem expresses the stable envelopes of the Nakajima variety $\mathcal{M}(v, w)$ in terms of those of the hypertoric variety $\mathcal{M}_S(v, w)$.

**Remark 4.7.** Actually, in the original reference [1] Theorem 4.6 is presented in a slightly different way. Indeed, the composition $(\Pi^*)^{-1} \circ (j^+_*)$ is replaced with pullback by a single map

$$\tilde{j}_+: \mu_{v, w}^{-1}(0)^G_{v-ss}/B \to \mu_{v, w}^{-1}(\mathfrak{b}^\perp)^{S-ss}/S,$$

which is the canonical map obtained from the $C^\infty$-isomorphism of $G/B \cong U/U \cap S$ bundles

$$\mu_{v, w}^{-1}(0)^G_{v-ss}/B \cong \mu_{v, w}^{-1}(0) \cap \mu_{R}^{-1}(\eta)/(U \cap S) \tag{14}$$

and the inclusion

$$\mu_{v, w}^{-1}(0) \cap \mu_{R}^{-1}(\eta)/(U \cap S) \hookrightarrow \mu_{v, w}^{-1}(\mathfrak{b}^\perp)^{S-ss}/S.$$

Here $U$ is a compact form of $G$, and the real moment map $\mu_{R}^{-1}(\eta)$ plays the role of the stability condition [15]. Similarly, also the composition $((\Pi^A)^*)^{-1} \circ (j^A)_*$ is replaced with $(\tilde{j}_+^A)_*$. Despite appearances, it is easy to see that $(\tilde{j}_+^A)_* = (\Pi^*)^{-1} \circ (j^+_*)$ and $(\Pi^A)^* \circ (j^A)_* = (\tilde{j}^A_+)_*$ under the identification (14). Indeed, the cohomology of $\mu_{v, w}^{-1}(0)^G_{v-ss}/B$ is generated by the Chern roots of the tautological bundles, and both $(\tilde{j}_+^A)_*$ and $(\Pi^*)^{-1} \circ (j^+_*)$ send Chern roots to Chern roots, hence they are the same map. This argument proves the first equality; the proof of the second one is similar. Therefore, the statement of the theorem above differs from the original one [1] only in the notation. However, in this article, we prefer to avoid using the maps $\tilde{j}_+$ and $\tilde{j}_+^A$ because they are not algebraic.

5 Framed semistable CoHA via stable envelopes

5.1 The framed semistable preprojective CoHA

5.1.1

Fix a quiver $Q$ and consider the graded $k$-module

$$\mathbb{H}^{\mathfrak{fr}} := \bigoplus_{v, w \in \mathbb{N}^I} \mathbb{H}^{\mathfrak{fr}}(v, w),$$
where
\[ H^\mathcal{T}_a(v, w) := H_{T_a}(\mathcal{M}_Q(v, w)). \]
As usual, we will drop the reference to the quiver whenever this is clear from the context.

We now define a graded multiplication on \( \mathbb{H} \) via stable envelopes
\[ \text{Stab}_{(a_1 < a_2)} : \mathbb{H}(v_1, w_1) \otimes_k \mathbb{H}(v_2, w_2) \to \mathbb{H}(v_1 + v_2, w_1 + w_2) \]

**Theorem 5.1.** The previous multiplication map defines an associative unital \( \mathbb{N}^L \times \mathbb{N}^L \)-graded \( k \)-algebra structure on \( \mathbb{H} \) with unit given by the element \( 1 \in k = \mathbb{H}_{0,0} \).

**Proof.** Let \( v = v_1 + v_2 + v_3 \) and \( w = w_1 + w_2 + w_3 \) and consider the stable envelope map
\[ \text{Stab}_{(a_1 < a_2 < a_3)} : \mathbb{H}(v_1, w_1) \otimes_k \mathbb{H}(v_2, w_2) \otimes_k \mathbb{H}(v_3, w_3) \to \mathbb{H}(v, w). \]
Applying Lemma 4.4 twice with \( \mathcal{E} = \{ a_1 < a_2 < a_3 \} \) and \( \mathcal{E}/\mathcal{E}' \) equal either to \( \{ a_1 < a_2 \} \) or to \( \{ a_2 < a_3 \} \), we get exactly the commutative diagram

That proves the claim.

**Definition 5.2.** We call the associative unital \( \mathbb{N}^L \times \mathbb{N}^L \)-graded \( k \)-algebra \( (\mathbb{H}^\mathcal{T}_a, \text{Stab}) \) the framed semistable preprojective cohomological Hall algebra of \( Q \).

**Remark 5.3.** Unlike in \( \mathcal{H} \), some graded components \( \mathbb{H}_{v,w} \) of \( \mathbb{H} \) are zero. This is a consequence of the fact that, because of the stability condition, some Nakajima varieties \( \mathcal{M}(v, w) \) are empty. However, the Nakajima variety \( \mathcal{M}(0,0) \) is a singleton for every quiver \( Q \) and stability condition, and hence \( \mathbb{H}(0,0) \) is always equal to \( k \).

**Remark 5.4.** One could naively try to define a CoHA structure on \( \mathbb{H}^\mathcal{T}_a \) by means of a correspondence of the form
\[ (\mu_{v_1,w_1}^{-1}(0))^{ss} \times (\mu_{v_2,w_2}^{-1}(0))^{ss} \overset{\text{-----}}{\longrightarrow} (Z \cap \mu_{v,w}^{-1}(0))^{ss} \overset{\text{-----}}{\longleftarrow} (\mu_{v,w}^{-1}(0))^{ss} \]
which is the restriction of (6) to the appropriate zero loci and semistable loci. However, it is easy to check that the dashed map is in general not well defined, because it does not respect semistability. Substituting \( (Z \cap \mu_{v,w}^{-1}(0))^{ss} \) with the open subset for which the dashed map is well defined does not solve the problem, because then the inclusion inside \( (\mu_{v,w}^{-1}(0))^{ss} \) is not proper anymore, and hence it does not descend to a pushforward in cohomology. Stable envelopes can be interpreted as a refined correspondence, obtained by iterated corrections on lower strata, that guarantees both the existence of a dashed map and the properness of the solid arrow (the corrections come with multiplicities, so in general the variety \( (Z \cap \mu_{v,w}^{-1}(0))^{ss} \) is replaced with a Borel-Moore cycle, i.e. a collection of varieties with multiplicities). As a consequence, one can look at \( \text{Stab}_E \) as the natural substitute of the correspondence giving rise to a CoHA structure on the cohomology of Nakajima varieties.
5.2 Framed CoHA vs. framed semistable preprojective CoHA

5.2.1 In this section, we review Aganagic-Okounkov’s construction of a map\(^2\)

\[ \psi : H^*(v, w) \to H^*(v, w) \]

that is, up to a multiplicative factor, a section of the restriction map

\[ j^* : H^*_T(\mathcal{R}(v, w)) \to H^*_T(M(v, w)) \]

associated with the inclusion

\[ j : M(v, w) = \mu_{v, w}^{-1}(0)G_v^{-ss} \hookrightarrow \mathcal{T}^*R(v, w)/G_v = \mathcal{R}(v, w). \]

Although this construction was performed in K-theory in \(^2\), it holds in ordinary cohomology as well.

The basic idea is to enlarge the space \(\mu_{v, w}^{-1}(0)G_v^{-ss}\) in such a way that it hosts a copy of \(T^*\text{Rep}(v, w)\).

To this end, consider the space of representations \(T^*R(v, w + v)\), which we visualize vertex-wise as

\[
\begin{array}{ccc}
V_i & \xrightarrow{\phi_i} & V_i \\
\downarrow{\psi_i} & & \downarrow{\psi_i} \\
W_i & \xrightarrow{\phi_i} & W_i
\end{array}
\]

Notice that extending the framing produces an extra action of a copy of \(G_v\) on the framing, which we denote by \(G'_v\). Define

\[
\mu_{v, w+v}^{-1}(0)^{iso} \subset \mu_{v, w+v}^{-1}(0)
\]

as the open subset of \(\mu_{v, w+v}^{-1}(0)\) where for every \(i \in I\) either \(\phi_i\) or \(\psi_i\) is an isomorphism, in such a way that we have a factorization

\[
\mu_{v, w+v}^{-1}(0)^{iso} \subset \mu_{v, w+v}^{-1}(0)^{G_v^{-ss}} \subset \mu_{v, w+v}^{-1}(0).
\]

For simplicity, we assume from now on that the stability condition is such that \(\phi_i\) is an isomorphism for every \(i \in I\). It is straightforward how to modify the construction below to deal with the most general case.

5.2.2 Choose an isomorphism \(\phi_i\) for all \(i \in I\) and consider the \(G_v \times T_w\)-equivariant inclusion

\[
\iota_{\phi} : T^*R(v, w) \to \mu_{v, w+v}^{-1}(0)^{iso} \quad (x, y, i, j) \to (x, y, i, j, \phi(x, y, i, j))
\]

where \(\chi_i(x, y, i, j) = -\mu_{v, w}(x, y, i, j) \circ \phi_i^{-1}\). Notice that this formula for \(\chi\) is forced by the moment map condition on the codomain. To get rid of the dependence on the isomorphisms \(\phi_i\), we compose \(\iota_{\phi}\) with the quotient by \(G'_v\):

\[
T^*R(v, w) \to \mu_{v, w+v}^{-1}(0)^{iso}/G'_v.
\]

Quotienting further by \(G_v\), we get a \(T_w\)-equivariant map

\[ \iota : \mathcal{R}(v, w) \to [\mathcal{M}(v, w + v)/G'_v]. \]

\(^2\)AO denoted this map by \(s\). The letter \(s\) being overused in this paper, we substitute it with \(\psi\).
Passing to cohomology, we get a map
\[ \iota^*: H_{T_w}([\mathcal{M}(v, w + v)/G'_v]) \rightarrow H_{T_w}(\mathcal{R}(v, w)). \]
Notice that this map factors as
\[ H_{T_w}([\mathcal{M}(v, w + v)/G'_v]) \xrightarrow{\iota^*} H_{T_w}(\mathcal{R}(v, w)) \]
\[ \xrightarrow{\text{res}} H_{T_w \times G'_v}(\mu^{-1}_{v, w + v}(0)^{G_v - ss}) \xrightarrow{\iota^*_\phi} H_{T_w \times G_v}(T^* R(v, w)) \]
Because of the \( G'_v \) action, the dependence of the bottom horizontal map on \( \phi \) is only apparent. Moreover, since by definition of \( \mu^{-1}_{v, w + v}(0)^{iso} \) the maps \( \phi_i \) as in (15) are isomorphisms, the Chern roots of \( G_v \) are identified with those of \( G'_v \) in
\[ H_{T_w \times G_v \times G'_v}(\mu^{-1}_{v, w + v}(0)^{iso}). \]

5.2.4

To complete the construction of \( \psi \) we need a map
\[ H_{T_w}(\mathcal{M}(v, w)) \rightarrow H_{T_w \times G'_v}(\mathcal{M}(v, w + v)). \] (16)
Once again this map is provided by stable envelopes. First of all, notice that the trivial identity
\[ \mathcal{M}(v, w) = \mathcal{M}(v, w) \times \mathcal{M}(0, v) \]
allows us to think of \( \mathcal{M}(v, w) \) as a fixed component of \( \mathcal{M}(v, w + v) \) by the action of the one dimensional subtorus \( A \subset A_{w + v} \) acting on \( W_i \) with weight one and trivially elsewhere. Now consider the stable envelope map
\[ \text{Stab}_{\{a < 0\}} : H_{T_w \times G'_v}(\mathcal{M}(v, w) \times \mathcal{M}(0, v)) \rightarrow H_{T_w \times G'_v}(\mathcal{M}(v, w + v)). \] (17)
Notice that although we have defined stable envelopes for the action of some torus \( T \) on \( \mathcal{M}(v, w) \), the correspondence \( \text{Stab}_{\{a < 0\}} \) actually gives a map even in \( T_w \times G'_v \) equivariant cohomology. This follows from the explicit construction of \( \text{Stab}_{\{a < 0\}} \) as a Lagrangian correspondence, and the fact that the action of \( A \) commutes with the one of \( T_w \times G'_v \).

Finally, composing (17) with
\[ H_{T_w}(\mathcal{M}(v, w)) \hookrightarrow H_{T_w \times G'_v}(\mathcal{M}(v, w)), \]
we get the desired map (16).

5.2.5

Putting it all together, we have built a map
\[ \psi : H_{T_w}(\mathcal{M}(v, w)) \rightarrow H_{T_w \times G'_v}(\mathcal{M}(v, w + v)) = H_{T_w}([\mathcal{M}(v, w + v)/G'_v]) \rightarrow H_{T_w}(\mathcal{R}(v, w)). \]
It enjoys the following properties:

**Lemma 5.5 ([2])**. The map \( \psi \) is supported on \( [\mu^{-1}_{v, w}(0)/G_v] \). Moreover, the restriction \( j^*(\psi(\alpha)) \) is divisible by \( e(hg_v) \) and
\[ \alpha = j^*(\psi(\alpha))/e(hg_v). \]
5.2.6

We can now state the main theorem of this article.

**Theorem 5.6.** The map $\psi : \mathcal{H}_{Q^I} \to \mathcal{H}_{Q^I}^\tau$ is an injective morphism of $\mathbb{N}^I \times \mathbb{N}^I$-graded $k$-algebras.

In words, the theorem above says that the map $\psi$ realizes the equivariant cohomology of Nakajima varieties as a subalgebra of the framed CoHA, and on this subalgebra CoHA’s multiplication is identified with taking stable envelopes.

5.3 Proof of Theorem 5.6: Step one

First of all, notice that injectivity follows at once from Lemma 5.5, so it remains to prove that $\psi$ is a morphism of graded algebras. The proof can be conceptually divided into two parts. In the first part, we prove an “abelianized” version of the statement and in the second we apply Theorem 3.9 and Theorem 4.6 to reduce the claim to its abelianized version.

5.3.1

Define

$$\mathbb{H}_{ab} := \bigoplus_{v,w \in \mathbb{N}^I} \mathbb{H}_{ab}(v, w)$$

where

$$\mathbb{H}_{ab}(v, w) := H_{T^w}(\mathcal{M}_S(v, w)).$$

With the same arguments of section 5.1, one proves that the stable envelope maps

$$\text{Stab}_{\{a_1 < a_2\}} : \mathbb{H}_{ab}(v_1, w_1) \otimes_k \mathbb{H}_{ab}(v_2, w_2) \to \mathbb{H}_{ab}(v_1 + v_2, w_1 + w_2)$$

give $\mathbb{H}_{ab}$ the structure of a $\mathbb{N}^I \times \mathbb{N}^I$-graded associative $k$-algebra.

5.3.2

We now mimic the construction of section 5.2.2 to define an abelianized version of $\psi$, i.e. a map

$$\psi_{ab} : \mathbb{H}_{ab} \to \mathcal{H}_{ab, \tau}$$

Let $S \subset G_v$ be a maximal torus. As in section 5.2.2, choose an isomorphism $\phi_i$ for all $i \in I$ and consider $S \times T_w$-equivariant inclusion

$$\iota_{ab, \phi} : T^w R(v, w) \oplus h_{\mathfrak{g}^+} \to \mu_{v+w+v}^{-1}(\mathfrak{g}^+)^{iso}$$

by

$$(x, y, i, j, v) \mapsto (x, y, i, j, \phi, \chi(x, y, i, j) + v \phi^{-1})$$

where $\chi_i(x, y, i, j) = -\mu_{v,w}(x, y, i, j) \circ \phi_i^{-1}$.

To get rid of the dependence by the isomorphisms $\phi_i$, we compose $\iota_{ab, \phi}$ with the quotient by $G'_v$ to get a map

$$\iota_{ab} : T^w R(v, w) \oplus h_{\mathfrak{g}^+} \to \mu_{v+w+v}^{-1}(\mathfrak{g}^+)^{iso}/G'_v.$$

Quotienting further by $S$, we get a $T_w$-equivariant map

$$\iota_{ab} : [T^w \text{Rep}(v, w) \oplus h_{\mathfrak{g}^+}/S] \to [\mathcal{M}_S(v, w + v)/G'_v],$$

which we denote in the same way.
5.3.3

Passing to cohomology, we get a map

\[ t_{ab} : H_{T_w}(\{M_S(v, w + v)/G'_v\}) \to H_{T_w}(\{T^*\text{Rep}(v, w) \oplus h\mathfrak{s}^+ / S\}) \]

that, composed with the top horizontal map of the diagram

\[
\begin{array}{ccc}
H_{T_w}(M_S(v, w)) & \longrightarrow & H_{T_w}(\{M_S(v, w + v)/G'_v\}) \\
\downarrow & & \downarrow \\
H_{T_w \times G'_v}(M_S(v, w) \times X_S(0, v)) & \xrightarrow{\text{Stab}_{(a < 0)}} & H_{T_w \times G'_v}(M_S(v, w + v))
\end{array}
\]

gives the sought-after map

\[ \psi_{ab} : \mathbb{H}_{ab}(v, w) = H_{T_w}(M_S(v, w)) \to H_{T_w}(\{T^*\text{Rep}(v, w) \oplus h\mathfrak{s}^+ / S\}) = \mathcal{H}_{ab, \tau}(v, w) \]

Lemma 5.7. The map \( \psi_{ab} \) is an algebra morphism.

**Proof.** Let \( v_1 + v_2 = w \) and \( w_1 + w_2 = w \). We have to show that the diagram

\[
\begin{array}{ccc}
\mathbb{H}_{ab}(v_1, w_2) \otimes_k \mathbb{H}_{ab}(v_2, w_2) & \xrightarrow{\psi_{ab} \otimes \psi_{ab}} & \mathcal{H}_{ab, \tau}(v_1, w_1) \otimes_k \mathcal{H}_{ab, \tau}(v_2, w_2) \\
\downarrow & & \downarrow \mathcal{m}_{ab, \tau} \\
\mathbb{H}_{ab}(v, w) & \xrightarrow{\psi_{ab}} & \mathcal{H}_{ab, \tau}(v, w)
\end{array}
\]

commutes. Let \( \gamma = \gamma_1 \otimes \gamma_2 \in \mathbb{H}_{ab}(v_1, w_2) \otimes_k \mathbb{H}_{ab}(v_2, w_2) \) and let \( p_\gamma = p_{\gamma_1} \otimes p_{\gamma_2} \) be a tautological presentation. Applying Proposition 4.5 and the definition of \( \psi_{ab} \) we get

\[
\psi_{ab} \circ \text{Stab}_{\{a < 0\}}(\gamma) = \psi_{ab}(e(T^*\text{Rep}(v, w)[-1])p_\gamma)
\]

Similarly, applying Proposition 4.5 and equation 9 we get

\[
\mathcal{m}_{ab, \tau} \circ (\psi_{ab} \otimes \psi_{ab})(\gamma_1 \otimes \gamma_2) = \mathcal{m}_{ab, \tau}(e(\text{hHom}(V'_1, V_1))p_{\gamma_1} \otimes e(\text{hHom}(V'_2, V_2))p_{\gamma_2})
\]

Since \( g_v = g_{v_1} \oplus g_{v_2} \oplus g_v[-1] \oplus g_v[1] \) and \( p_\gamma = p_{\gamma_1}p_{\gamma_2} \), the result follows.

\[ \square \]

5.4 Proof of Theorem 5.6: Step two

5.4.1

We can now complete the proof by combining abelianization of stable envelopes and abelianization of CoHA. Applying the first part of Theorem 3.9 and Theorem 4.6, one gets the commutative diagram

\[
\begin{array}{ccc}
\mathbb{H}_{ab}(v_1, w_2) \otimes_k \mathbb{H}_{ab}(v_2, w_2) & \xrightarrow{\psi_{ab} \otimes \psi_{ab}} & \mathcal{H}_{ab, \tau}(v_1, w_1) \otimes_k \mathcal{H}_{ab, \tau}(v_2, w_2) \\
\downarrow & & \downarrow \mathcal{m}_{ab, \tau} \\
H_{T_w}(M_{b_1}) \otimes_k H_{T_w}(M_{b_2}) & \xrightarrow{\mathcal{H}_{T_w}(\mathbb{R}_{b_1}) \otimes_k \mathcal{H}_{T_w}(\mathbb{R}_{b_2})} & \mathcal{H}_{ab, \tau}(v, w) \\
\downarrow & & \downarrow \psi_{ab} \\
H_{T_w}(M_b) & \xrightarrow{\mathbb{H}_{ab}(v, w)} & H_{T_w}(\mathbb{R}_b)
\end{array}
\]

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where $\mathcal{M}_b := \mu^{-1}(b^{\perp})^{S_{ss}}/S_l$. We now construct horizontal maps that fill the two missing edges of the cube above.

**Lemma 5.8.** There exist dashed maps

\[
\begin{array}{ccccccccc}
\mathbb{H}_{ab}(v_1, w_2) \otimes_k \mathbb{H}_{ab}(v_2, w_2) & \xrightarrow{\psi_{ab} \otimes \psi_{ab}} & \mathcal{H}_{ab,\tau}(v_1, w_1) \otimes_k \mathcal{H}_{ab,\tau}(v_2, w_2) \\
H_{Tw_1}(\mathcal{M}_{b_1}) \otimes_k H_{Tw_2}(\mathcal{M}_{b_2}) & \xrightarrow{\text{[part 1]}} & H_{Tw_1}^*(\mathcal{R}_{b_1}) \otimes_k H_{Tw_2}^*(\mathcal{R}_{b_2}) \\
\text{Stab}_{\mathcal{S}}^1(\mathcal{M}_{b_1} \times \mathcal{M}_{b_2}) & \xrightarrow{m_{ab,\tau}} & \mathcal{H}_{ab,\tau}(v, w) \\
\mathcal{H}_{Tw}(\mathcal{M}_b) & \xrightarrow{\text{[part 2]}} & H_{Tw}(\mathcal{R}_b)
\end{array}
\]

that make the diagram commute.

**Proof.** We begin with the lower edge. Unraveling the definition of $\psi_{ab}$, one sees that we need to find a dashed arrow that makes the following diagram commute

\[
\begin{array}{ccccccccc}
H_{Tw}(\mathcal{M}_S(v, w) \mathcal{S}_{(a < 0)}^1\mathcal{H}_{Tw \times G_v'}(\mathcal{M}_S(v, w + v)) & \xrightarrow{i_{ab}} & H_{Tw}(\mathcal{M}_S(v, w + v))/G_v' \\
H_{Tw}(\mathcal{M}_b) & \xrightarrow{\text{[part 1]}} & \mathcal{H}_{ab,\tau}(v, w)
\end{array}
\]

The fiber squares\(^3\)

\[
\begin{array}{ccccccccc}
T^* R(v, w) \oplus h_n & \xrightarrow{i_{ab,\psi}} & \mu_{v, w + v}^{-1}(b^{\perp})^{iso} \xrightarrow{\mu_{v, w + v}^{-1}(b^{\perp})^{iso}/G_v'} & \mu_{v, w + v}^{-1}(b^{\perp})^{iso}/G_v' \\
& \xrightarrow{j_{\tau}} & & \mu_{v, w + v}^{-1}(b^{\perp})^{S_{ss}}/G_v' \\
T^* R(v, w) \oplus h_b & \xrightarrow{i_{ab,\psi}} & \mu_{v, w + v}^{-1}(s^{\perp})^{iso} \xrightarrow{\mu_{v, w + v}^{-1}(s^{\perp})^{iso}/G_v'} & \mu_{v, w + v}^{-1}(s^{\perp})^{S_{ss}}/G_v'
\end{array}
\]

(19)

gives the following commutative diagram when passing to $S \times T_w$-equivariant cohomology

\[
\begin{array}{ccccccccc}
H_{Tw \times G_v'}(\mathcal{M}_S(v, w + v)) & \xrightarrow{\text{[part 1]}} & H_{Tw}(\mathcal{M}_S(v, w + v))/G_v' \\
& \xrightarrow{\text{[part 2]}} & \mathcal{H}_{ab,\tau}(v, w) \\
H_{Tw \times G_v'}([\mu_{v, w + v}^{-1}(b^{\perp})^{S_{ss}}/S]) & \xrightarrow{\text{[part 1]}} & H_{Tw}(\mathcal{M}_S(v, w + v))/G_v' \\
& \xrightarrow{\text{[part 2]}} & \mathcal{H}_{ab,\tau}(v, w)
\end{array}
\]

(18)

The tilde in the map

\[
\tilde{j} : \mu_{v, w + v}^{-1}(b^{\perp})^{S_{ss}} \to \mu_{v, w + v}^{-1}(s^{\perp})^{S_{ss}}
\]

is chosen to distinguish it from

\[
j : \mu_{v, w}^{-1}(b^{\perp})^{S_{ss}} \to \mu_{v, w}^{-1}(s^{\perp})^{S_{ss}}.
\]

Applying again the first part of Theorem 4.6, now with $F_S = \mathcal{M}_S(v, w) = \mathcal{M}_S(v, w) \times \mathcal{M}_S(0, w)$, we get the commutative square

\[
\begin{array}{ccccccccc}
H_{Tw}(\mathcal{M}_S(v, w)) \mathcal{S}_{(a < 0)}^1\mathcal{H}_{Tw \times G_v'}(\mathcal{M}_S(v, w + v)) & \xrightarrow{i_{ab}} & H_{Tw}(\mathcal{M}_S(v, w + v)) \\
& \xrightarrow{\text{[part 1]}} & H_{Tw}(\mathcal{M}_S(v, w + v))/G_v' \\
H_{Tw}(\mathcal{M}_b) & \xrightarrow{\text{[part 2]}} & H_{Tw \times G_v'}([\mu_{v, w + v}^{-1}(b^{\perp})^{S_{ss}}/S])
\end{array}
\]

\(^3\)Notice that the central square is Cartesian because the action of $G_v'$ is free on the $iso$-locus.

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that, glued with the previous diagram along \((\tilde{j}_-)\), completes the construction of the dashed arrow in diagram (18). The construction of the other edge of the cube is completely analogous and consists in performing the construction above twice, once for each side of the tensor products. Overall, we have shown that all the sides of the cube commute, except from the outermost face (i.e. the one where both the dashed arrows lie). However, commutativity of this last face follows from the following lemma together with the observation that the map \((i_-)_*\) is injective because it is given by multiplication by the Euler class of the normal bundle to the inclusion \(i_-\) and its source is an integral domain.

**Lemma 5.9.** Consider the following diagram of modules:

\[
\begin{array}{ccc}
\hat{A} & \xrightarrow{\delta} & \hat{B} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A & \xrightarrow{g} & B \\
\downarrow{f} & & \downarrow{h} \\
C & \xrightarrow{k} & D \\
\downarrow{\gamma} & & \downarrow{\delta} \\
\end{array}
\]

Assume that all the faces commute except the outermost and that the map \(\delta\) is injective. Then also the outermost face commute.

**Proof.** By injectivity of \(\delta\), it suffices to check that \(\delta \circ k \circ f = \delta \circ h \circ g\). Indeed, we have

\[
\delta \circ h \circ g = \hat{h} \circ \beta \circ g = \hat{h} \circ \hat{g} \circ \alpha = \hat{k} \circ \hat{f} \circ \alpha = \hat{k} \circ \gamma \circ f = \delta \circ k \circ f.
\]

5.4.2

Applying now the second parts of Theorem 3.9 and Theorem 4.6, we get the solid diagram

\[
\begin{array}{ccc}
H_{T_v^w}(M_{b_1}) \otimes_k H_{T_{w_2}}(M_{b_2}) & \longrightarrow & H_{T_{w_1}}(\mathcal{R}_{b_1}) \otimes_k H_{T_{w_2}}(\mathcal{R}_{b_2}) \\
\oplus \psi \psi & & \oplus \psi \psi \\
\mathcal{H}(v_1, w_1) \otimes_k \mathcal{H}(v_2, w_2) & \xrightarrow{\pi \circ (\Pi^*)^{-1} \circ (j_+)^*} & \mathcal{H}(v_2, w_2) \\
\mathcal{H}(v_1, w_1) \otimes_k \mathcal{H}(v_2, w_2) & \xleftarrow{\pi \circ (\Pi^*)^{-1} \circ (j_+)^*} & \mathcal{H}(v_2, w_2) \\
\xleftarrow{\psi} & & \xleftarrow{\psi} \\
H_{T_{T_w}}(M_0) & \longrightarrow & H_{T_{T_w}}(\mathcal{R}_0) \\
\text{Stab}_{\{a_1 < a_2\}} & \xrightarrow{m^b_{ab, r}} & \text{Stab}_{\{a_1 < a_2\}} \\
\mathcal{H}(v, w) & \xleftarrow{\psi} & \mathcal{H}(v, w) \\
\end{array}
\]

while the dashed arrows are those constructed in the previous section. Commutativity of the outermost face is our final goal. The top left oblique map is \((\pi_1 \times \pi_2)_* \circ ((\Pi_1 \times \Pi_2)^*)^{-1} \circ (j_{1, +} \times j_{2, +})^*\) and the top right oblique map is \((q_1 \times q_2)_* \circ ((Q_1 \times Q_2)^*)^{-1} \circ (i_{1, +} \times i_{2, +})^*\). By construction, the two lateral sides and the back of the diagram commute. Moreover, the oblique arrows are surjective, so to complete the proof it suffices to show that the two remaining faces, the top and the bottom of the cube, are commutative. Therefore, the following lemma concludes the proof.

**Lemma 5.10.** The top and bottom faces of the cubic diagram above commute.
Proof. As for the previous lemma, the top face is just two copies of the bottom face tensored together, so it suffices to prove the commutativity of the bottom face. Recall that the dashed arrow in the bottom face is built from diagram (19), on which we now build up the following commutative diagram

\[
\begin{array}{ccc}
T^* R(v, w) \oplus \text{hn} & \xrightarrow{\iota_b \phi} & \mu_{v, w+v}(b^+)_{\text{iso}}^{-1} \mu_{v, w+v}(b^+)_{\text{iso}} / G_v' \\
& j_+ \searrow & \downarrow j_+ \\
T^* R(v, w) & \xrightarrow{\iota_b} & \mu_{v, w+v}(b^+)_{\text{iso}}^{-1} \mu_{v, w+v}(b^+)_{\text{iso}} / G_v' \\
\end{array}
\]

Passing to $S \times T_w$-equivariant cohomology, we get the commutative diagram

\[
\begin{array}{ccc}
H_{T_w \times G_v}(\mu_{v, w+v}(b^+)_{S-ss} / S) & \xrightarrow{\iota_b} & H_{T_w}(\mu_{v, w+v}(b^+)_{S-ss} / S \times G_v') \\
\| j_+ \downarrow & & \downarrow j_+ \|
\end{array}
\]

Taking quotients by $B$ and $G_v$ we can enlarge the previous commutative diagram as follows

\[
\begin{array}{ccc}
H_{T_w \times G_v}(\mu_{v, w+v}(b^+)_{S-ss} / S) & \xrightarrow{\iota_b} & H_{T_w}(\mu_{v, w+v}(b^+)_{S-ss} / S \times G_v') \\
\| j_+ \downarrow & & \downarrow j_+ \|
\end{array}
\]

Notice that the right side of the square consists of the maps appearing in the lower right side of diagram (20). Similarly, the maps on the left side of the square are the ones appearing in the diagram

\[
\begin{array}{ccc}
H_{T_w \times G_v}(\mathcal{M}(v, w)) & \xrightarrow{\pi_v \circ (\Pi^*)^{-1} \circ j_+^*} & H_{T_w \times G_v}(\mathcal{M}_b) \\
\| \text{Stab}_{(a<0)} \downarrow & & \downarrow \text{Stab}_{b_{(a<0)}} \|
\end{array}
\]

As before, this diagram is the second abelianization diagram for the fixed component $\mathcal{M}(v, w) = \mathcal{M}(v, w) \times \mathcal{M}(0, v) \subset \mathcal{M}(v, w + v)$, see Theorem 4.6. Combining the last diagram with the outer
The two unlabelled horizontal maps are simply the canonical change of group maps. The commuting outer frame of this diagram was exactly what we had to prove that commutes. Therefore we are done.

5.5 Application: Shuffle product of stable envelopes

5.5.1

One of the main reasons for Aganagic and Okounkov to define the map

\[ \psi : H_{T_w}(\mathcal{M}(v, w)) \to H_{T_w}(\mathfrak{R}(v, w)) = \mathbb{K}[a_w \times s]^W_{G_v} \]

was to give a systematic way to produce tautological presentations of stable envelopes

\[ \text{Stab} : H_{T_w}(\mathcal{M}(v, w)^A) \to H_{T_w}(\mathcal{M}(v, w)). \]

Indeed, by Lemma 5.5 the assignment

\[ \text{Stab}^\psi : H_{T_w}(\mathcal{M}(v, w)^A) \to H_{T_w}(\mathcal{R}(v, w)) = \mathbb{K}[a_w \times s]^W_{G_v} \]

defines a function

\[ \text{Stab}^\psi : H_{T_w}(\mathcal{M}(v, w)^A) \to H_{T_w}(\mathfrak{R}(v, w))_{\text{loc}} = \mathbb{K}[a_w \times s]^W_{G_v} \]

that, restricted to \( \mathcal{M}(v, w) \subset \mathfrak{R}(v, w) \), is an integral class, i.e. defined in the non-localized ring, and it coincides with \( \text{Stab}^\varepsilon \).

5.5.2

With this preliminary observation, Theorem 5.6 can be applied to obtain explicit inductive formulas for the stable envelopes of a Nakajima variety \( \mathcal{M}(v, w) \).

As in section 4.3, consider a fixed component \( F \) for the action of a subtorus

\[ A = \{(a_1, a_2, \ldots, a_k) \mid a_i \in \mathbb{C}^\times\} \hookrightarrow A_w \]

associated to a decomposition \( w = w_1 + w_2 + \ldots w_k \) and a chamber

\[ \mathcal{C}_\sigma = \{a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(k)}\} \quad \sigma \in S_k. \]
A partition \( k = k_1 + k_2 \) give rise to a decomposition

\[
F = F_1 \times F_2
\]

where \( F_1 \) is a fixed component of a Nakajima variety \( M(v_1, w_1) \) with respect of the action of the torus

\[
A_1 = \{(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(k_1)}) \mid a_i \in \mathbb{C}^x \} \hookrightarrow A_{w_1}
\]

and similarly \( F_2 \) is a fixed component of another variety \( M(v_2, w_2) \) with respect of the action of

\[
A_2 = \{(a_{\sigma(k_1+1)}, a_{\sigma(k_1+2)}, \ldots, a_{\sigma(k)}) \mid a_i \in \mathbb{C}^x \} \hookrightarrow A_{w_2}
\]

By construction, \( A = A_1 \times A_2 \) and we get the following commutative diagram

\[
\begin{array}{c}
F_1 \times F_2 \longrightarrow F \\
\downarrow \quad \quad \downarrow \\
M(v_1, w_1) \times M(v_2, w_2) \longrightarrow M(v, w)
\end{array}
\]

where all the maps are \( A_w \) equivariant. As usual, the product \( M(v_1, w_1) \times M(v_2, w_2) \) can be also seen as a fixed component of \( M(v, w) \), this time with respect to the action of a two dimensional torus inside \( A \). Let us now introduce the chambers

\[
\mathcal{C}_\sigma^1 := \{a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(k_1)}\} \quad \mathcal{C}_\sigma^2 := \{a_{\sigma(k_1+1)} < a_{\sigma(k_1+2)} < \cdots < a_{\sigma(k)}\}
\]

for the actions of \( A_1 \) and \( A_2 \) on \( M(v_1, w_1) \) and \( M(v_2, w_2) \) respectively.

**Theorem 5.11.** With the notation above, we have

\[
\text{Stab}^\psi_{\mathcal{C}_\sigma}(F) = \text{Shuffle} \left( \frac{e(T^*\text{Rep}(v, w)[-1])\text{Stab}^\psi_{\mathcal{C}_\sigma^1}(F_1)\text{Stab}^\psi_{\mathcal{C}_\sigma^2}(F_2)}{e(g_v[-1])e(h_{g_v}[-1])} \right)
\]

Notice that the kernel \( e(T^*\text{Rep}(v, w)[-1])/(e(g_v[-1])e(h_{g_v}[-1])) \) of the shuffle formula above coincides with \( e(N[-1]) \), the Euler class of the negative part of the normal bundle defined in section 3.1.4.

**Proof.** Consider the diagram

\[
\begin{array}{c}
HT_{w_1}(F_1) \otimes_k HT_{w_2}(F_2) = HT_w(F) \\
\downarrow \quad \quad \downarrow \\
H_{T_{w_1}}(M(v_1, w_1)) \otimes_k H_{T_{w_2}}(M(v_2, w_2)) \longrightarrow \text{Stab}_{\mathcal{C}_\sigma^1}(F_1) \otimes \text{Stab}_{\mathcal{C}_\sigma^2}(F_2) \longrightarrow HT_{w_1}(M(v_1, w_1)) \otimes_k HT_{w_2}(M(v_2, w_2)) \\
\downarrow \quad \quad \downarrow \\
H_{T_{w_1}}(\mathfrak{R}(v_1, w_1)) \otimes_k H_{T_{w_2}}(\mathfrak{R}(v_2, w_2)) \longrightarrow \text{mr} \longrightarrow HT_{w_1}(\mathfrak{R}(v, w))
\end{array}
\]

Here \( \text{Stab} \) stands for \( \text{Stab}_{\mathcal{C}_\sigma^1, \mathcal{C}_\sigma^2} \). Commutativity of the central square is exactly the statement of Theorem 5.6 while commutativity of the upper triangle follows from Lemma 4.4. Therefore, applying
Theorem 3.7 we get

\[ \text{Stab}_\psi^\infty = \frac{\psi \circ \text{Stab}_\epsilon}{e(hg_v)} \]

\[ = \frac{m_r(\psi \circ \text{Stab}_\epsilon^1(F_1) \otimes \psi \circ \text{Stab}_\epsilon^2(F_2))}{e(hg_v)} \]

\[ = \frac{1}{e(hg_v)} \text{Shuffle} \left( \frac{e(T^*\text{Rep}(v, w)[-1])e(hg_v[1])}{e(g_v[-1])} \frac{\psi \circ \text{Stab}_\epsilon^1(F_1)(\psi \circ \text{Stab}_\epsilon^2(F_2))}{e(hg_v[1])} \right) \]

\[ = \text{Shuffle} \left( \frac{e(T^*\text{Rep}(v, w)[-1])}{e(g_v[-1])} \frac{\psi \circ \text{Stab}_\epsilon^1(F_1)(\psi \circ \text{Stab}_\epsilon^2(F_2))}{e(hg_v[-1])} \frac{\psi \circ \text{Stab}_\epsilon^1(F_1)(\psi \circ \text{Stab}_\epsilon^2(F_2))}{e(hg_v[-1])} \right). \]

In the penultimate step we used the fact that \( e(hg_v) \) is symmetric to move it inside the shuffle, and then the factorization \( e(hg_v) = e(hg_v[1])e(hg_v[-1])e(hg_v[1])e(hg_v[1]) \) associated to the decomposition \( g_v = g_v \oplus g_v \oplus g_v[1] \oplus g_v[-1] \) to simplify it with the class \( e(hg_v[1]) \) in the numerator. \( \square \)
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