Average-case reconstruction for the deletion channel: subpolynomially many traces suffice

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Abstract

The deletion channel takes as input a bit string \( x \in \{0,1\}^n \), and deletes each bit independently with probability \( q \), yielding a shorter string. The trace reconstruction problem is to recover an unknown string \( x \) from many independent outputs (called “traces”) of the deletion channel applied to \( x \).

We show that if \( x \) is drawn uniformly at random and \( q < 1/2 \), then \( e^{O(\log^{1/2} n)} \) traces suffice to reconstruct \( x \) with high probability. The previous best bound, established in 2008 by Holenstein, Mitzenmacher, Panigrahy, and Wieder [5], uses \( n^{O(1)} \) traces and only applies for \( q \) less than a smaller threshold (it seems that \( q < 0.07 \) is needed).

Our algorithm combines several ideas: 1) an alignment scheme for “greedily” fitting the output of the deletion channel as a subsequence of the input; 2) a version of the idea of “anchoring” used in [5]; and 3) complex analysis techniques from recent work of Nazarov and Peres [9] and De, O’Donnell, and Servedio [3].

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1 Introduction

The deletion channel takes as input a bit string $x \in \{0, 1\}^n$. Each bit of $x$ is (independently of other bits) retained with probability $p := 1 - q$. The channel then outputs the concatenation of the retained bits; such an output is called a trace. Suppose that the input $x$ is unknown. The trace reconstruction problem asks the following: how many i.i.d. traces from the deletion channel do we need to observe in order to determine $x$ with high probability?

There are two basic variants of this problem, which we will call the “worst case” and “average case”. In the worst case variant, the problem is to provide bounds that hold uniformly over all possible input strings $x$. The average case variant supposes that the input is chosen uniformly at random. In particular, we are allowed to ignore some “hard-to-reconstruct” inputs, as long as they comprise a small fraction of all $2^n$ possible inputs. In this paper, we study the average case. Our main result is the following.

Theorem 1.1. Suppose $q < \frac{1}{2}$, and let $X \in \{0, 1\}^n$ be an unknown bit string of length $n$ chosen uniformly at random. There is a constant $C_q$ depending only on $q$ such that it is possible to reconstruct $X$ with probability at least $1 - \frac{C_q}{n}$ using at most $\exp(C_q \sqrt{\log n})$ independent samples from the deletion channel with deletion probability $q$ applied to $X$.

1.1 Related work

The study of trace reconstruction for the deletion channel seems to have been initiated by Batu, Kannan, Khanna and McGregor [1], who were motivated by multiple sequence alignment problems in computational biology. We focus on the regime where the deletion probability $q$ is held constant as $n$ grows.

Previously, the best bound in the average case was due to Holenstein, Mitzenmacher, Panigrahy and Wieder [5], who gave an algorithm for reconstructing random inputs using polynomially many traces when $q$ is less than some small threshold $c$. Theorem 1.1 improves on this result in two ways: the number of traces is subpolynomial, and we extend the range of allowed $q$ to the interval $(0, 1/2)$.

In [5] it is also shown that $e^{O(n^{1/3} \log n)}$ traces suffice for reconstruction with high probability with worst case input. This was recently improved by Nazarov-Peres [9] and De-O’Donnell-Servedio [3] (simultaneously and independently) to $e^{O(n^{1/3})}$. Their techniques, which we use in Section 4, play an important role in our proofs.

The question of whether the above bounds are optimal remains open. The best lower bounds known are of order $\log^2 n$ (McGregor, Price and Vorotnikova [7]) in the average case and order $n$ in the worst case [11].

Other settings for trace reconstruction include the case when $q \to 0$ [11], when insertions and substitutions are allowed as well as deletions [6, 10], or when the

\[1\] The threshold $c$ is not given explicitly in [5]. It seems that by optimizing their methods we cannot achieve $c > 0.07$.\]
strings are taken over an alphabet whose size grows with $n$ ([7]). For a more comprehensive review of the literature, we refer readers to the introduction of [3] or the survey of Mitzenmacher [8].

1.2 Outline of approach

Let us give a high-level description of the algorithm used to prove Theorem 1.1. Suppose that we have already reconstructed the first $k$ bits of $X$, and we consider a new trace $\tilde{X}$. Roughly speaking, our goal is to carry out the following steps:

**Alignment:** Find some suitable index $m$ slightly less than $k$, and try to (approximately) identify the position $\ell$ in $\tilde{X}$ that corresponds to the $m$-th position of $X$. This occurs in two stages (see Figure 1.1):

- **Initial alignment:** Find a position $\ell_0$ in $\tilde{X}$ whose corresponding position in $X$ is known to be about $O(\log n)$ places ahead of $m$.
- **Refined alignment:** Consider a specific substring $w$ of $X$ located at $m$ and having length $O(\log^{1/2} n)$. Look for $w$ to occur in $\tilde{X}$ within $O(\log n)$ characters following position $\ell_0$, and take $\ell$ to be the last position of this occurrence of $w$.

**Reconstruction:** Use the bits of $\tilde{X}$ after $\ell$ as a trace of the bits of $X$ after $m$. From these “traces”, we reconstruct at least $k + 1 - m$ bits of $X$ starting from position $m$, which in particular includes the $(k + 1)$-th bit of $X$.

We can repeat the above procedure for each $k$. In each iteration, the number of traces needed will be $e^{O(\sqrt{\log n})}$. Moreover, these traces may be reused for each iteration, because we will ultimately bound the probability of failure by a union bound.

1.2.1 Initial alignment step

The initial alignment step is based on fitting $\tilde{X}$ as a subsequence of $X$ following a “greedy algorithm”. Let $X_i$ and $\tilde{X}_i$ denote the $i$-th bits of $X$ and $\tilde{X}$, respectively. We
Figure 1.2: Illustration of the greedy algorithm used in the initial alignment step. Here, $X = 11000110$ and $\tilde{X} = 1010$. Gray arrows point from the positions in $X$ that were retained to their corresponding positions in $\tilde{X}$. Red arrows indicate the associations produced by our algorithm (i.e. $X_1$ goes to $X_1$, $X_2$ goes to $X_3$, $X_3$ goes to $X_6$, $X_4$ goes to $X_8$).

associate $\tilde{X}_1$ to the first bit in $X$ that matches $\tilde{X}_1$, then associate $\tilde{X}_2$ to the next bit in $X$ that matches $\tilde{X}_2$, and so on (see Figure 1.2). This gives the “first possible” occurrence of $\tilde{X}$ as a subsequence of $X$, but does not necessarily reflect the true alignment of $\tilde{X}$ to $X$. However, when $q < 1/2$ and $X$ is random, it turns out that this greedy alignment actually matches the true one to within $O(\log n)$ (stated precisely in Lemma 2.2).

Let us briefly describe why this is the case. Suppose that the position assigned by our greedy algorithm lags behind the true position. Looking at the next bit in the trace, the true position should advance by $\frac{1}{1-q} < 2$ places in expectation. However, since the bits of $X$ are uniformly random, the position for the greedy algorithm should advance like a geometric random variable with mean 2, thereby “catching up”.

The same greedy matching idea was also considered by Mitzenmacher (see Section 3 of [8]) in the slightly different context of decoding for the deletion channel. Lemma 2.2 is a variant of Theorem 3.2 in [8]. However, many details are omitted in [8], so we provide a self-contained proof in Section 2.

1.2.2 Refined alignment step

For the refined alignment, we take an approach similar to the use of “anchors” in [5]. We again rely on the randomness of $X$ and the assumption $q < 1/2$. Consider a substring $w$ of length $a \approx \log^{1/2} n$ which contains the $m$-th bit of $X$. (In the language of [5], $w$ is our “anchor”.)

With probability $p^a$, the string $w$ appears in our trace because none of its bits were deleted. There is also a chance that this exact sequence just happens to appear after deletions to another part of the input. However, because $X$ is random, the latter scenario only happens with probability $2^{-a} \ll p^a$. Thus, when we see $w$ in our trace, it most likely came from near position $m$ of $X$ (we discard traces if we do not see $w$), thereby aligning to within $O(\log^{1/2} n)$.

We remark here that the above discussion sweeps under the rug a few considerations about how to avoid accumulation of many small probabilities of error. In particular, note that the error probabilities involved during the refined alignment step are like
\( e^{-O(\log^{1/2} n)} \), which is not small enough to union bound over the whole string.

For example, a problem may arise if we have another copy of \( w \) appearing in \( X \) that is only \( O(\log n) \) positions away from \( m \). In that case, appearances of \( w \) in \( \tilde{X} \) might come from either copy of \( w \) in \( X \), and it would be hard to distinguish the two scenarios.

Recall, however, that we have allowed ourselves some flexibility in the choice of \( m \). Note that the initial alignment step means that we only need to worry about what \( X \) looks like within distance \( O(\log n) \) from the location \( m \). We look at \( O(\log^{1/2} n) \) possible locations of \( m \) which are spaced \( O(\log n) \) apart, and we argue that with high probability, at least one of these locations (and the corresponding choice of \( w \)) behaves in the desired way.

### 1.2.3 Reconstruction step

For the reconstruction step, we analyze bit statistics using methods based on those of [9] and [3]. However, two adaptations are needed for our setting. First, our reconstruction step only needs to recover a small number of bits, not the full string. The statement we need is roughly that \( e^{O(r^{1/3})} \) traces are enough to recover the first \( r \) bits of an unknown string, which we apply with \( r = O(\log^{3/2} n) \).

Second, since our alignment is not perfect, we must allow some random shifts of the input string. The amount of shifting we can tolerate is relatively small, which explains the need for accurate alignment. The issue of calculating bit statistics with random shifts also appears in [5], although our techniques for handling this are rather different from theirs.

These two adaptations can be carried out by small modifications to the relevant proofs in [9] and [3], which are based on bounds for Littlewood polynomials on arcs of the unit circle.

### 1.3 Notation

We will use boldface to denote bit strings, while the values of their bits are non-bolded and subscripted by indices; for example, \( x = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n \). Let \(|x| = n \) denote the length of \( x \), and let \( x^{a:b} \) denote the substring \( (x_a, x_{a+1}, \ldots, x_b) \). For brevity, we also write \( x^{a:|x|} = x^a:|x| \) for the suffix of \( x \) starting at \( x_a \).

Next, we introduce notation for describing the deletion channel. For a given parameter \( p \in (0, 1) \), let \( D_p^*(x) \) denote the distribution over pairs \((t, \tilde{x})\) of sequences defined as follows: \( t = (t_1, t_2, \ldots, t_m) \) is the random sequence of indices of \( x \) which are retained by the deletion channel applied to \( x \) with deletion probability \( q = 1 - p \), and \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \) is given by \( \tilde{x}_i = x_{t_i} \). Note that the length \( m = |t| \) is random.

In some cases, we are only interested in the final output \( \tilde{x} \) and not in \( t \). Thus, we also introduce the notation \( D_p(x) \) for the marginal distribution of \( D_p^*(x) \) over the strings \( \tilde{x} \). We will sometimes use the notation \( \mathbb{P}_x(\cdot) \) to emphasize that the string going through the deletion channel is \( x \).
At some point, we will want to use $t$ to associate several indices at once in $\tilde{x}$ to their counterparts in $x$, or vice versa. Consider sets $S \subseteq \{1, 2, \ldots, |x|\}$ and $\tilde{S} \subseteq \{1, 2, \ldots, |\tilde{x}|\}$. Then, we use the notation

$$t(\tilde{S}) := \{t_s : s \in \tilde{S}\} \quad \text{and} \quad t^{-1}(S) := \{s : t_s \in S\},$$

which matches the usual notation for images/preimages if $t$ is regarded as a map from indices in $\tilde{x}$ to indices in $x$.

Finally, in addition to the standard notation $O(\cdot)$ and $\Omega(\cdot)$, we also use $O_p(\cdot)$ and $\Omega_p(\cdot)$ in cases where the implied constant may depend on $p$ but nothing else.

### 1.4 Organization of the paper

The rest of the paper is organized as follows. In Sections 2 and 3, we prove the lemmas needed to for the initial and refined alignment steps, respectively. In Section 4, we prove the lemmas needed for the reconstruction step. Finally, in Section 5, we pull together all the ingredients to prove Theorem 1.1.

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### 2 Alignment by greedy matching

Suppose we have a string $x \in \{0, 1\}^n$ and a sample $(t, \tilde{x}) \sim D_p^*(x)$. Given only $x$ and $\tilde{x}$, it is not in general possible to infer uniquely what $t$ is. However, we may obtain an approximation using a “greedy algorithm” as described in Section 1.2.

To state things precisely, consider any two bit strings $x$ and $y$. We define a sequence $\left(g_k(y, x)\right)_{k=1}^{|y|}$ as follows:

- Define $g_1(y, x)$ to be the least index such that $x_{g_1(y, x)} = y_1$. If no bits in $x$ are equal to $y_1$, we set $g_1(y, x) = \infty$.

- For $k < |y|$, define inductively $g_{k+1}(y, x)$ to be the least index greater than $g_k(y, x)$ for which $x_{g_{k+1}(y, x)} = y_{k+1}$. If no bits in $x$ after the $g_k(y, x)$-th position are equal to $y_{k+1}$, we set $g_{k+1}(y, x) = \infty$. (Note that in particular if $g_k(y, x) = \infty$, then $g_{k+1}(y, x) = \infty$).

We are primarily interested in the case where $y = \tilde{x}$, where $\tilde{x}$ is a trace drawn from $D_p(x)$. In this situation, $g_k(\tilde{x}, x)$ represents the “earliest possible” place in $x$ that the $k$-th bit of $\tilde{x}$ could have come from. For an illustration, we refer back to Figure 1.2. In that picture, we have $g_1(\tilde{x}, x) = 1$, $g_2(\tilde{x}, x) = 3$, $g_3(\tilde{x}, x) = 6$, and $g_4(\tilde{x}, x) = 8$. 

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One may check by a straightforward induction that \( g_k(\bar{x}, x) \leq t_k \) for all \( 1 \leq k \leq |\bar{x}| \). (This means that \( g_k(\bar{x}, x) \) is never \( \infty \); the possibility of having \( g_k(y, x) = \infty \) doesn’t come into play until the proof of Lemma 5.4.) We will show that for retention probability \( p > \frac{1}{2} \) and \( x \) drawn uniformly at random, \( g_k(\bar{x}, x) \) is usually not much less than \( t_k \). The following definition makes this precise.

**Definition 2.1.** Consider a sequence \( x \in \{0, 1\}^n \), and take \((t, \bar{x}) \sim D_p^*(x)\). We say that \( x \) is \((\alpha, \beta)\)-trackable if

\[
\mathbb{P}_x \left( \max_{1 \leq k \leq |t|} \left( t_k - g_k(\bar{x}, x) \right) \geq \lambda \right) \leq e^{-\frac{\lambda^2 n}{p}}.
\]

The main result of this section is the following lemma.

**Lemma 2.2.** Suppose \( p > \frac{1}{2} \), and let \( X \in \{0, 1\}^n \) be a uniformly random string of \( n \) bits. There exists \( C_p > 0 \) depending only on \( p \) such that

\[
\mathbb{P} (X \text{ is } (C_p \log n, C_p)\text{-trackable}) \geq 1 - O_p \left( \frac{1}{n} \right).
\]

Lemma 2.2 is implied by Theorem 3.2 of [8]. However, many details are omitted there, so we devote the rest of this section to proving Lemma 2.2 formally. We use the same general approach, except that it is more natural for us to focus on the quantity \( t_k - g_k(\bar{X}, X) \) rather than a slightly different quantity considered in [8]. The starting point is a conditional independence property similar to Lemma 3.3 of [8].

**Lemma 2.3.** Let \( X \in \{0, 1\}^n \) be drawn uniformly at random, and suppose \((t, \bar{X}) \sim D_p^*(X)\). Then, for any integer \( k \geq 1 \), conditioned on the event \(|t| \geq k\) and the values of \( t_1, t_2, \ldots, t_k \) and \( g_1(\bar{X}, X), g_2(\bar{X}, X), \ldots, g_k(\bar{X}, X) \),

the bits \( X_{g_k(\bar{X}, X)+1}, X_{g_k(\bar{X}, X)+2}, \ldots, X_n \) are i.i.d. uniformly distributed.

**Remark 2.4.** The above lemma also applies when \( X \) is an infinite sequence of i.i.d. uniform bits. In this case, the conclusion is that all of \((X_i)_{i=g_k(\bar{X}, X)+1}^{\infty}\) are i.i.d. uniform.

**Proof.** We first condition on \( t \); this conditioning will stay in effect for the remainder of the proof. Note that all of the \( X_i \) are still i.i.d. uniform, since the \( t_i \) depend only on which bits are deleted and not on the values of the bits themselves. Since we have conditioned on \( t \), we may regard \( g_i(\bar{X}, X) \) as a deterministic function of \( X \). Therefore, for brevity we will write \( g_i(X) = g_i(\bar{X}, X) \).

Next, fix any sequence \( S \) of integers \( s_1, s_2, \ldots, s_k \) where \( s_1 < s_2 < \cdots < s_k \) and \( s_i \leq t_i \) for each \( i \). We say a bit string \( z \) is \( S\)-compatible if \( g_i(z) = s_i \) for each \( i \), and let \( E_S \) be the event that \( X \) is \( S\)-compatible.

Consider any two strings \( w, w' \in \{0, 1\}^{n-s_k} \) which differ in a single bit. We will give a bijection between \( S\)-compatible realizations of \( X \) that end in \( w \) and those that
end in \( \mathbf{w}' \). This is enough to establish the lemma, since by repeated application, it shows that any two strings for \( X^{(s_k+1)} \) are equally likely conditioned on \( E_S \), and this holds for arbitrary \( S \).

To carry out the bijection, for any index \( j \) with \( 1 \leq j \leq k \), we define its influencing set to be the set

\[
I_j = \{ t : s_{j-1} < t \leq s_j \},
\]

with the convention \( s_0 = 0 \). Informally, it is the set of all indices \( t \) where the value of \( X_t \) had some effect on the value of \( g_j(\mathbf{X}) \) (which is equal to \( s_j \) if \( \mathbf{X} \) is \( S \)-compatible).

For any two indices \( i \) and \( j \) with \( 1 \leq i, j \leq k \), we say \( i \) directly influences \( j \) if \( t_i \in I_j \). Note that because \( s_j \leq t_j \), we see that if \( i \) influences \( j \), then \( i \leq j \) with equality if and only if \( s_i = t_i \). We say that \( i \) influences \( j \) if there is a chain of direct influences from \( i \) to \( j \) (i.e. there exist \( c_1, c_2, \ldots, c_N \) such that \( c_1 = i \), \( c_N = j \), and \( c_\alpha \) directly influences \( c_{\alpha+1} \) for \( \alpha = 1, 2, \ldots, N - 1 \)).

Suppose now that we have a \( S \)-compatible sequence \( \mathbf{z} \) that ends in \( \mathbf{w} \). We will describe a way to modify \( \mathbf{z} \) so that it remains \( S \)-compatible but ends in \( \mathbf{w}' \). Let \( \ell \) be the index at which \( w_\ell \neq w'_\ell \). First, suppose that \( s_k + \ell \neq t_i \) for any \( i \leq k \). Then, we may simply flip the \( (s_k + \ell) \)-th bit of \( \mathbf{z} \) to obtain a \( S \)-compatible sequence ending in \( \mathbf{w}' \).

Otherwise, \( s_k + \ell = t_m \) for some \( m \leq k \). Define the sets

\[
U = \{ m \} \cup \{ j : j \text{ influences } m \} \quad \text{and} \quad V = \{ t_m \} \cup \left( \bigcup_{j \in U} I_j \right).
\]

We claim that by flipping all the bits of \( \mathbf{z} \) at positions in \( V \), the resulting sequence \( \mathbf{z}' \) ends in \( \mathbf{w}' \) and is \( S \)-compatible. The first claim follows from the fact that \( I_j \subseteq \{ 1, 2, \ldots, s_k \} \) for all \( j \leq k \), so the only bit flipped after position \( s_k \) is the bit at position \( t_m = s_k + \ell \).

To show \( S \)-compatibility, we show by induction on \( j \) that \( g_j(\mathbf{z}') = s_j \) for each \( j \), where the base case \( j = 0 \) is established by the convention \( g_0(\mathbf{z}') = s_0 = 0 \). For the inductive step, suppose that \( g_i(\mathbf{z}') = s_i \) for each \( i < j \). We consider two cases.

**Case** \( j \in U \). By the definition of \( U \), either \( j = m \) or there exists \( j' \in U \) for which \( t_j \in I_{j'} \). In either case, we see that \( t_j \in V \). We also have by definition that \( I_j \subseteq V \). By \( S \)-compatibility of \( \mathbf{z} \), the condition \( g_j(\mathbf{z}) = s_j \) says that \( s_j \) is the first position after \( g_{j-1}(\mathbf{z}) = s_{j-1} \) having the same value as position \( t_j \). In other words, \( s_j \) is the unique position in \( I_j \) with the same value as position \( t_j \).

The bits at positions \( t_j \) and elements of \( I_j \) are all flipped for \( \mathbf{z}' \), so the same property holds in \( \mathbf{z}' \). Since \( g_{j-1}(\mathbf{z}') = s_{j-1} \) by the inductive hypothesis, we have \( g_j(\mathbf{z}') = s_j \) as well.

**Case** \( j \notin U \). Note that \( t_m > s_k \), so \( t_m \notin I_j \). Since \( j \notin U \), it follows that \( I_j \) is disjoint from \( V \). Note that if \( t_j \in I_{j'} \) for some \( j' \in U \), then \( j \) directly influences \( j' \).
and hence influences \( m \), but this contradicts \( j \notin U \). Also, clearly \( t_j \neq t_m \) since \( j \neq m \). Thus, \( t_j \notin V \).

We see that none of the bits at positions \( t_j \) or elements of \( I_j \) are flipped for \( z' \), so by the same argument as in the previous case, we conclude that \( g_j(z') = s_j \).

This completes the induction, showing that \( z' \) indeed ends in \( w' \) and is \( S \)-compatible. Furthermore, observe that the set \( V \) depends only on \( S \), and so we may symmetrically recover \( z \) from \( z' \) by the same transformation. Thus, this gives a bijection from \( S \)-compatible sequences ending in \( w \) to those ending in \( w' \), completing the proof. \( \square \)

The next two lemmas describe how closely \( g_k \) tracks \( t_k \). To avoid boundary issues, it is convenient to state them for infinite bit sequences.

**Lemma 2.5.** Let \( X \) be an infinite sequence of i.i.d. uniform bits, and let \((t, \tilde{X}) \sim D_p^r(X) \). Define \( d_k = t_k - g_k(\tilde{X}, X) \).

Then, \( d_{k+1} - d_k \) is independent of \( d_1, d_2, \ldots, d_k \) and has the same law as \( \max(G_p - G_{1/2} - d_k) \), where \( G_p \) and \( G_{1/2} \) are independent geometrics with parameters \( p \) and \( \frac{1}{2} \), respectively.

**Proof.** For brevity, write \( g_k = g_k(\tilde{X}, X) \). We condition on \( t_i \) and \( g_i \) for \( 1 \leq i \leq k \). By Lemma 2.3, the bits \((X_i)_{i=2k+1}^\infty \) are i.i.d. uniform even after this conditioning. Next, we sample \( t_{k+1} \), which we may write as \( t_{k+1} = t_k + G_p \) since each bit is retained independently with probability \( p \). We then examine the bits

\[
X_{g_k+1}, X_{g_k+2}, \ldots, X_{t_{k+1}},
\]

which are still i.i.d. uniformly distributed. Recall that \( g_{k+1} \) is defined to be the earliest position of these bits where the value matches \( \tilde{X}_{k+1} = X'_{t_{k+1}} \). Each of the above bits has a \( \frac{1}{2} \) chance of being a match except for the last one, which is guaranteed to match.

Thus, \( g_{k+1} \) may be written as \( \min(g_k + G_{1/2}, t_{k+1}) \). Consequently,

\[
d_{k+1} - d_k = (t_{k+1} - t_k) - (g_{k+1} - g_k) = G_p - \min(G_{1/2}, t_{k+1} - g_k)
\]

\[
= \max(G_p - G_{1/2}, G_p + g_k - t_{k+1}) = \max(G_p - G_{1/2}, -d_k),
\]

as desired. \( \square \)

**Lemma 2.6.** Suppose \( p > \frac{1}{2} \). Let \( X \) be an infinite sequence of i.i.d. uniform bits, and let \((t, \tilde{X}) \sim D_p^r(x) \). Define \( d_k = t_k - g_k(\tilde{X}, X) \).

Then, there exist positive constants \( c_p \) and \( C_p \) depending only on \( p \) such that for each \( k \), we have

\[
P(d_k \geq \lambda) \leq C_p e^{-c_p \lambda}.
\]

**Proof.** Let \( G_p \) and \( G_{1/2} \) be independent geometrics with parameters \( p \) and \( \frac{1}{2} \), as in Lemma 2.5. Consider the function \( f(x) = \frac{px}{1-(1-p)x} \cdot \frac{1}{2x-1} \), which satisfies \( f(1) = 1 \) and \( f'(1) = \frac{1}{p} - 2 < 0 \). Thus, we may take \( \alpha \in (1,1/p) \) to be a constant so that
We will show that $\mathbb{E}(\alpha^d_k)$ is bounded above uniformly in $k$, from which the result immediately follows by Markov’s inequality.

We proceed by induction. For the base case, note that $t_1$ has the distribution of $G_0$, and we chose $\alpha < 1/p$, so $\mathbb{E}(\alpha^{G_0})$ is finite. Since $d_1 \leq t_1$, it follows that $\mathbb{E}(\alpha^{d_1})$ is also finite.

For the inductive step, define $\gamma = f(\alpha) < 1$, and let $M$ be a large enough integer so that

$$1 + 2(\alpha - 1) \left(\frac{1}{2\alpha}\right)^M \leq \frac{1}{\sqrt{\gamma}}.$$ 

Note that we have the formulas

$$\mathbb{E}(\alpha^{G_0}) = \sum_{k=1}^{\infty} (1-p)^{k-1} \alpha^k = \frac{p\alpha}{1-(1-p)\alpha}$$

$$= \mathbb{E}(\alpha^{-\min(G_{1/2},M)}) = 2^{-M} \alpha^{-M} + \sum_{k=1}^{M} 2^{-k} \alpha^{-k} = \left(\frac{1}{2\alpha}\right)^M + \frac{1 - \left(\frac{1}{2\alpha}\right)^M}{1 - \frac{1}{2\alpha}}$$

$$= \frac{1 + 2(\alpha - 1) \left(\frac{1}{2\alpha}\right)^M}{2\alpha - 1} \leq \frac{1}{\sqrt{\gamma}(2\alpha - 1)}.$$ 

These calculations allow us to bound two conditional expectations, depending on whether $d_k \geq M$. By Lemma 2.5, we have

$$\mathbb{E}(\alpha^{d_{k+1}} | d_k \geq M) \leq \mathbb{E}(\alpha^{\max(G_0-G_{1/2},-M)}) \mathbb{E}(\alpha^{d_k} | d_k \geq M)$$

$$\leq \mathbb{E}(\alpha^{G_0}) \mathbb{E}(\alpha^{-\min(G_{1/2},M)}) \mathbb{E}(\alpha^{d_k} | d_k \geq M)$$

$$\leq \frac{p\alpha}{1-(1-p)\alpha} \cdot \frac{1}{\sqrt{\gamma}(2\alpha - 1)} \cdot \mathbb{E}(\alpha^{d_k} | d_k \geq M)$$

$$\leq \sqrt{\gamma} \cdot \mathbb{E}(\alpha^{d_k} | d_k \geq M)$$

$$\mathbb{E}(\alpha^{d_{k+1}} | d_k < M) \leq \mathbb{E}(\alpha^{M+G_0}) = \frac{p\alpha^{M+1}}{1-(1-p)\alpha} \leq 8\alpha^M.$$ 

Together, these imply that

$$\mathbb{E}(\alpha^{d_{k+1}}) \leq \sqrt{\gamma} \cdot \mathbb{E}(\alpha^{d_k}) + 8\alpha^M.$$ 

Recall that $\alpha$, $\gamma$, and $M$ are all constants that depend only on $p$, and $\gamma < 1$. Hence, $\mathbb{E}(\alpha^{d_k})$ is bounded above uniformly in $k$, completing the proof.

We are finally ready to prove Lemma 2.2.

**Proof of Lemma 2.2** For a given string $z$ of $n$ bits and $(t_z, \bar{z}) \sim \mathcal{D}_p^*(z)$, write

$$d(z) = \max_{1 \leq k \leq |t_z|} (t_{z,k} - g_k(\bar{z}, z)) \quad \text{and} \quad r_\lambda(z) = \mathbb{P}_z(d(z) \geq \lambda).$$
We apply Lemma 2.6 to the sequence $X$, where we may think of $X$ as the first $n$ bits of an infinite sequence of i.i.d. uniform bits. Union bounding over all indices $1 \leq k \leq n$, we have

$$
\mathbb{E}[r_\lambda(X)] \leq n \cdot C_{1,p} \cdot e^{-c_{1,p} \lambda},
$$

where $C_{1,p}$ and $c_{1,p}$ are constants depending only on $p$. Consequently,

$$
\mathbb{P}(r_\lambda(X) \geq e^{-c_{1,p} \lambda/2}) \leq n \cdot C_{1,p} \cdot e^{-c_{1,p} \lambda/2}. \quad (1)
$$

Define the event

$$
E = \bigcap_{\lambda=2\lceil \log n \rceil}^{\infty} \left\{ r_{2\lambda/c_{1,p}}(X) \leq e^{-\lambda} \right\}.
$$

Then, a union bound using (1) gives

$$
\mathbb{P}(E) \geq 1 - n \cdot C_{1,p} \sum_{\lambda=2\lceil \log n \rceil}^{\infty} e^{-\lambda} \geq 1 - \frac{C_{2,p}}{n}, \quad (2)
$$

where $C_{2,p}$ is another constant depending only on $p$.

Meanwhile, on the event $E$, consider any $t > \frac{2}{c_{1,p}}(2\lceil \log n \rceil + 1)$. Let $t' = \lceil \frac{c_{1,p} t}{2} \rceil$. Since $t' \geq 2\lceil \log n \rceil$, we have

$$
\mathbb{P}(d(X) \geq t) \leq \mathbb{P}\left(d(X) \geq \frac{2t'}{c_{1,p}}\right) = r_{2t'/c_{1,p}}(X) \leq e^{-t'}
\leq e^{-c_{1,p} t/2 + 1}. \quad (3)
$$

Combining (2) and (3), we conclude that

$$
\mathbb{P}(X \text{ is } (C_p \log n, C_p)\text{-trackable}) \geq 1 - \frac{C_p}{n}
$$

for a sufficiently large constant $C_p$. \qed

3 Alignment by seeing a particular sequence

In this section, we develop the tools for our second alignment strategy based on looking for a particular sequence of consecutive bits. The strategy follows the same main idea as the use of “anchors” in [5]. However, our analysis is more precise. We first establish some terminology and notation.

**Definition 3.1.** For any two bit strings $w$ and $y$, we say that $w$ occurs in $y$ if there is some index $j$ such that $y_{j+i-1} = w_i$ for $i = 1, 2, \ldots, |w|$. We use the following notation to describe occurrences:
• Ind\(_w(y)\) denotes the first index at which \(w\) occurs in \(y\) (i.e. the smallest possible \(j\) as above), or \(\infty\) if \(w\) does not occur in \(y\).

• Whenever \(\text{Ind}_w(y) < \infty\),

\[
\text{IndSet}_w(y) := \{j : \text{Ind}_w(y) \leq j < \text{Ind}_w(w) + |w|\}
\]

denotes the set of all the indices in \(y\) corresponding to the occurrence of \(w\) in \(y\).

In later sections, we will be interested in occurrences of \(w\) within a particular substring \(y^{i:j}\) of \(y\). However, we still want to work with indices based on position in \(y\) rather than in \(y^{i:j}\). In these cases, we use the notation

• \(\text{Ind}_{i:j}^w(y) := \text{Ind}_w(y^{i:j}) + i - 1\).

• \(\text{IndSet}_{i:j}^w(y) := \{k : \text{Ind}_{i:j}^w(y) \leq k < \text{Ind}_{i:j}^w(w) + |w|\}\).

Suppose that \(x\) is a string of length \(2n\), and \(w = x^{(n-a+1):(n+a)}\) is a substring in the middle of \(x\). Now, suppose we observe a trace \(\tilde{x} \sim \mathcal{D}_p(x)\), and we see that \(w\) occurs in \(\tilde{x}\). We would like to say that in this case the bits in \(\tilde{x}\) corresponding to the occurrence of \(w\) likely came from the occurrence of \(w\) in \(x\) (or at least, some of them did). Not all strings \(x\) have this property, but as we will see shortly, it turns out that typical ones do. We formalize the property in the following definition.

**Definition 3.2.** Suppose \(p > \frac{1}{2}\), let \(x \in \{0, 1\}^{2n}\), and take \((t, \tilde{x}) \sim \mathcal{D}_p^*(x)\). Consider a positive integer \(a \leq n\) and positive real \(\gamma < 1\), and write \(w = x^{(n-a+1):(n+a)}\). We say that \(x\) is \((a, \gamma)\)-distinguishable if

\[
P_x\left(\text{Ind}_w(\tilde{x}) < \infty \quad \text{and} \quad t(\text{IndSet}_w(\tilde{x})) \cap [n-a, n+a] = \emptyset\right) \leq \gamma^a \cdot p^{2a}.
\]

**Remark 3.3.** It is always possible for \(w\) to occur in \(\tilde{x}\) if each of the positions \(n-a+1\) through \(n+a\) in \(x\) are retained. This happens with probability \(p^{2a}\). The bound on the probability in the above definition is given in the form \(\gamma^a \cdot p^{2a}\) to highlight that it should be smaller than \(p^{2a}\) by a factor that is exponential in \(a\).

The main result of this section is that random sequences are likely to be distinguishable.

**Lemma 3.4.** Suppose \(p > \frac{1}{2}\), and suppose \(X \in \{0, 1\}^{2n}\) is chosen uniformly at random. Then, there exist \(\gamma_p < 1\) and \(c_p > 0\) depending only on \(p\) such that

\[
P\left(X \text{ is } (\lceil n^{1/2} \rceil, \gamma_p)\text{-distinguishable}\right) \geq 1 - e^{-c_p n^{1/2}}.
\]
Proof. Let \( a = \left\lceil n^{1/2} \right\rceil \), let \( \mathbf{w} = \mathbf{X}^{(n-a+1):(n+a)} \), and take \((\mathbf{t}, \mathbf{X}) \sim \mathcal{D}_p^*(\mathbf{X})\). Let
\[
J = \mathbf{t}^{-1}(\lfloor 1, 2n \rfloor \setminus [n-a, n+a]) = \{ j : 1 \leq j \leq \lceil \mathbf{X} \rceil, \ t_j \notin [n-a, n+a] \}
\]
denote the set of indices of \( \mathbf{X} \) which did not come from the middle \( 2a \) positions of \( \mathbf{X} \). Define the event
\[
E = \left\{ \text{Ind}_{\mathbf{w}}(\mathbf{X}) < \infty \quad \text{and} \quad \mathbf{t}(\text{IndSet}_{\mathbf{w}}(\mathbf{X})) \subseteq J \right\},
\]
which is the relevant event for \((a, \gamma_p)\)-distinguishability.

Let us condition on the middle \( 2a \) bits of \( \mathbf{X} \) (i.e. the bits that form \( \mathbf{w} \)) as well as on \( \mathbf{t} \). The key observation is that \((\widetilde{\mathbf{X}}_j)_{j \in J}\) are still i.i.d. uniform after our conditioning. Now, if \( \mathbf{w} \) occurs in \( \mathbf{X} \), but \( \mathbf{t}(\text{IndSet}_{\mathbf{w}}(\mathbf{X})) \subseteq J \), then it means that \( \mathbf{w} \) occurs in the sequence \((\widetilde{\mathbf{X}}_j)_{j \in J}\). However, since the \((\widetilde{\mathbf{X}}_j)_{j \in J}\) are i.i.d., in each possible position this only happens with probability \( 2^{-|\mathbf{w}|} = 2^{-2a} \). Union bounding over at most \( 2n \) positions yields
\[
\mathbb{P}(E) \leq 2n \cdot 2^{-2a},
\]
where we have also taken the expectation over our initial conditioning on the middle \( 2a \) bits and \( \mathbf{t} \).

The above probability is with respect to simultaneously two sources of randomness: the random choice of \( \mathbf{X} \) and the random choice of the deletions. To highlight this, recall the notation \( \mathbb{P}_x \) for the probability over the randomness of the deletion channel for a given input string \( x \).

Take \( \gamma_p = (2p)^{-1/2} < 1 \). By Markov’s inequality,
\[
\mathbb{P}(\mathbb{P}_x(E) \geq \gamma_p^a \cdot p^{2a}) \leq \gamma_p^a \cdot p^{-2a} \cdot \mathbb{E}(\mathbb{P}_x(E)) = \gamma_p^a \cdot p^{-2a} \cdot \mathbb{P}(E) \\
\leq 2n \cdot \gamma_p^{3a} = e^{-\Omega_p(n^{1/2})},
\]
which yields \((a, \gamma_p)\)-distinguishability with the desired probability. \( \square \)

We conclude the section by establishing a consequence of \([(n^{1/2}], \gamma_p)\)-distinguishability that is more convenient to work with than Definition 3.2.

**Lemma 3.5.** Suppose \( p > \frac{1}{2} \), let \( a = \lceil n^{1/2} \rceil \), and suppose \( x \in \{0, 1\}^{2n} \) is \((a, \gamma_p)\)-distinguishable for some constant \( \gamma_p < 1 \) depending only on \( p \). Consider \((\mathbf{t}, \mathbf{X}) \sim \mathcal{D}_p^*(\mathbf{x})\). Then,
\[
\mathbb{P}_x\left( \text{Ind}_{\mathbf{w}}(\mathbf{X}) < \infty \quad \text{and} \quad \mathbf{t}(\text{IndSet}_{\mathbf{w}}(\mathbf{X})) \not\subseteq [n-10a, n+10a] \right) \leq e^{-\Omega_p(a) \cdot p^{2a}}.
\]

**Proof.** The main idea is that if the set \( \mathbf{t}(\text{IndSet}_{\mathbf{w}}(\mathbf{X})) \) intersects the interval \([n-a, n+a]\), then it is unlikely to stretch out very far from that interval.

Let
\[
E_1 = \left\{ \text{Ind}_{\mathbf{w}}(\mathbf{X}) < \infty \quad \text{and} \quad \mathbf{t}(\text{IndSet}_{\mathbf{w}}(\mathbf{X})) \cap [n-a, n+a] = \emptyset \right\},
\]

so that \((a, \gamma_p)\)-distinguishability ensures \(P_x(E_1) \leq \gamma_p^a p^{2a}\).

Next, let
\[
E_2 = \left\{ \text{more than } 7a \text{ deletions occurred among some } 9a \text{ consecutive positions in } x \right\}.
\]

By a standard Chernoff bound (see, e.g., [4]) and union bounding over all blocks of \(9a\) bits in \(x\), we have
\[
P_x(E_2) \leq 2n \cdot P(\text{Binom}(9a, 1/2) > 7a) \leq 2n \cdot e^{-\frac{25a^2}{18n}}.
\]

Finally, let
\[
E_3 = \left\{ \text{Ind}_w(\tilde{x}) < \infty \text{ and } t(\text{IndSet}_w(\tilde{x})) \not\subseteq [n - 10a, n + 10a] \right\},
\]

which is the event of interest for the lemma. Suppose now that \(E_1\) holds but not \(E_3\), i.e. \(t(\text{IndSet}_w(\tilde{x}))\) is not disjoint from \([n - a, n + a]\) but is also not contained within \([n - 10a, n + 10a]\). Then \(t(\text{IndSet}_w(\tilde{x}))\) must have two elements which are at least \(9a\) apart, so that \(E_2\) holds. Thus, we find that
\[
P_x(E_3) \leq P_x(E_1) + P_x(E_2) \leq e^{-\Omega_p(a)} \cdot p^{2a}.
\]

\[\square\]

### 4 Reconstruction from approximate alignment

In this section, we adapt the trace reconstruction methods of [9] and [3] to a setting where the input string also undergoes a random shift. The main result of this section is the following lemma.

**Lemma 4.1.** Let \(k, n, \text{ and } N\) be positive integers with \(k < n < N\). Let \(x, x' \in \{0, 1\}^N\) be two strings whose first \(k\) digits are identical but whose first \(n\) digits are not. Let \(S\) be a random variable taking integer values between 0 and \(k - 1\).

Suppose the following conditions are satisfied:
\[
E[|S - ES|] \leq n^{1/3}, \quad k \leq n^{2/3}.
\]
Then, for some constant \(C_p\) depending only \(p\), there exists an index \(j \leq C_p n\) such that if \(\tilde{x} \sim D_p(x^{(S+1)})\) and \(\tilde{x}' \sim D_p((x')^{(S+1)})\), then
\[
|P_x(\tilde{x}_j = 1) - P_{x'}(\tilde{x}'_j = 1)| \geq \exp \left(-C_p n^{1/3}\right).
\]

The first ingredient in the proof of this lemma is a polynomial identity, which is analogous to Lemma 2.1 in [9] or Section 4 in [3], but accounts for possible shifts to the input sequence.
Lemma 4.2. Let \( n \) and \( k \) be positive integers with \( k \leq n \). Let \( \mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \) be a sequence of real numbers whose first \( k \) elements are zero. Let \( S \) be a random variable taking integer values between \( 0 \) and \( k - 1 \), with \( \mathbb{P}(S = i) = \beta_i \).

Let \( \tilde{\mathbf{a}} \sim \mathcal{D}_p(\mathbf{a}^{(S+1):}) \), and pad \( \tilde{\mathbf{a}} \) with zeroes to the right. Then,

\[
\mathbb{E} \left[ \sum_{j \geq 0} \tilde{a}_j w^j \right] = p \left( \sum_{s=0}^{k-1} \beta_s (pw + q)^{-s} \right) \left( \sum_{j=0}^{n-1} a_j (pw + q)^j \right).
\]

Proof. Let \( \ell \) be any integer with \( k \leq \ell \leq n - 1 \). By linearity, it suffices to show the result for \( \mathbf{a} \) having all zeroes except \( a_\ell = 1 \). We now restrict to this case.

Let us condition on \( S = s \) and analyze for each \( j \) the probability \( \mathbb{P}(\tilde{a}_j = 1 \mid S = s) \) that the single non-zero entry \( a_\ell \) gets shifted to position \( j \) without being deleted. Clearly, if \( j > \ell - s \), then \( \mathbb{P}(\tilde{a}_j = 1 \mid S = s) = 0 \). Otherwise, the probability must account for the retention of \( a_\ell \) and the retention of exactly \( j \) of the first \( \ell - s \) entries of \( \mathbf{a}^{(S+1):} \). Note that the condition \( k \leq \ell \) ensures that \( \ell - s > 0 \). Thus for \( j \leq \ell - s \),

\[
\mathbb{P}(\tilde{a}_j = 1 \mid S = s) = p \cdot \left( \frac{\ell - s}{j} \right) p^j q^{\ell - s - j},
\]

so that

\[
\mathbb{E} \left( \sum_{j \geq 0} \tilde{a}_j w^j \mid S = s \right) = p \sum_{j=0}^{\ell - s} \left( \frac{\ell - s}{j} \right) p^j q^{\ell - s - j} w^j = p \cdot (pw + q)^{\ell - s}.
\]

Taking the expectation over \( S \), we conclude that

\[
\mathbb{E} \left( \sum_{j \geq 0} \tilde{a}_j w^j \right) = \sum_{s=0}^{\ell-1} \beta_s \mathbb{E} \left( \sum_{j \geq 0} \tilde{a}_j w^j \mid S = s \right) = p \sum_{s=0}^{k-1} \beta_s (pw + q)^{\ell - s}
\]

\[
= p \left( \sum_{s=0}^{k-1} \beta_s (pw + q)^{-s} \right) (pw + q)^\ell,
\]

which completes the proof. \( \square \)

As in [9] and [3], we also use the following Littlewood-type estimate of Borwein and Erdélyi.

Lemma 4.3 (Borwein and Erdélyi, special case of Corollary 3.2 in [2]). There exists a finite constant \( C \) such that the following holds. Let \( A(z) \) be a polynomial with coefficients in \([-1, 1]\) and \( A(0) = 1 \). Denote by \( \gamma_L \) the arc \( \{e^{i\theta} : -1/L \leq \theta \leq 1/L\} \). Then \( \max_{z \in \gamma_L} |A(z)| \geq e^{-CL} \).

We now carry out the proof of Lemma 4.1 using these two ingredients.
Proof of Lemma 4.4 For a fixed value of \( p \), clearly it is enough to prove the statement for sufficiently large \( n \). We will assume implicitly at various points that \( n \) is sufficiently large.

Write \( \beta_j = \mathbb{P}(S = j) \), let \( a_j = x_{j+1} - x'_{j+1} \), and let \( a = (a_j)_{j=0}^{n-1} \). Define the polynomials

\[
P(z) = \sum_{j=0}^{k-1} \beta_j z^j, \quad Q(z) = \sum_{j=0}^{n-1} a_j z^j, \quad \text{and} \quad A(z) = p \cdot P(z^{-1})Q(z).
\]

Let \( \ell \) be the smallest index for which \( a_\ell \neq 0 \); note that by our hypotheses, \( \ell \leq n \).

Define \( \tilde{Q}(z) = \frac{1}{z} Q(z) \), so that \( |\tilde{Q}(0)| = 1 \).

For convenience, let \( L = n^{1/3} \), and define \( \rho = 1 - 1/L^2 \). Applying Lemma 4.3 to the function \( \tilde{Q}(\rho z) \), there exists \( z_0 = e^{i\theta} \) with \( -\frac{p}{10L} \leq \theta \leq \frac{p}{10L} \) and \( |\tilde{Q}(\rho z_0)| \geq e^{-CL/p} \).

We next lower bound \( |P(\rho^{-1}z_0^{-1})| \). Let \( \tilde{P}(z) = z^{-ES}P(z) \), which is an analytic function on the right half-plane. For all \( z \) in the right half-plane satisfying \( 1 \leq |z| \leq \rho^{-1} \), differentiating \( \tilde{P} \) gives

\[
|\tilde{P}'(z)| = \left| \sum_{j=0}^{k-1} (j - ES) \beta_j z^{j-ES-1} \right| \leq \sum_{j=0}^{k-1} |j - ES| \cdot |z|^{j-ES-1}
\]

\[
\leq \rho^{-k} \cdot \mathbb{E}||S - ES|| \leq \rho^{-k} L \leq e^{1/2k} \cdot L \leq 4L,
\]

where we have used \( \mathbb{E}||S - ES|| \leq L \) and \( k \leq L^2 \). Also,

\[
|\rho^{-1}z_0^{-1} - 1| = \rho^{-1}|1 - \rho z_0| \leq |z_0 - 1| + \rho^{-1}(1 - \rho)
\]

\[
\leq \frac{p}{10L} + \frac{2}{L^2} \leq \frac{p}{8L}.
\]

Consequently,

\[
|P(\rho^{-1}z_0^{-1})| = \rho^{-ES} |\tilde{P}(\rho^{-1}z_0^{-1})| \geq 1 - |\tilde{P}(\rho^{-1}z_0^{-1}) - 1|
\]

\[
= 1 - \left| \int_{1}^{\rho^{-1}z_0^{-1}} \tilde{P}'(z) \, dz \right| \geq 1 - |\rho^{-1}z_0^{-1} - 1| \cdot 4L
\]

\[
\geq 1 - \frac{p}{2} \geq \frac{1}{2}
\]

Thus,

\[
|A(\rho z_0)| = p \cdot |P(\rho^{-1}z_0^{-1})| \cdot \rho^\ell \cdot |\tilde{Q}(\rho z_0)| \geq \frac{p}{2} \cdot e^{1/2k} \cdot \frac{CL}{p} \geq e^{(C+2)L/p}.
\]
Next, define $w = 1 + \frac{\rho_0 - 1}{p}$, so that $\rho_0 = pw + q$. We have that

$$|w|^2 = 1 + \frac{2}{p}(\rho \cdot \text{Re}(z_0) - 1) + \frac{1}{p^2}|\rho_0 - 1|^2$$

$$\leq 1 + \frac{2}{p} (\rho - 1) + \frac{p^2}{p^2} |\rho^{-1}z_0^{-1} - 1|^2$$

$$\leq 1 - \frac{2}{L^2} + \frac{1}{64L^2} \leq \rho.$$

Let $\tilde{a} \sim D_p(a^{(S+1)}).$ By Lemma 4.2

$$\left| \mathbb{E} \left[ \sum_{j \geq 0} \tilde{a}_j w^j \right] \right| = |A(\rho z_0)| \geq e^{-\frac{(C+2)L}{p}}.$$

Now, take $C_p$ to be an integer larger than $\frac{C_p+4}{p}$. Note that

$$\left| \sum_{j=C_p n}^{\infty} \mathbb{E}[\tilde{a}_j w^j] \right| \leq \sum_{j=C_p n}^{\infty} \rho^j \leq L^2 \rho_{C_p n} \leq \frac{1}{2} \cdot e^{-\frac{(C+2)L}{p}}.$$

Hence,

$$\left| \mathbb{E} \left[ \sum_{j=0}^{C_p n-1} \tilde{a}_j w^j \right] \right| \geq \frac{1}{2} \cdot e^{-\frac{(C+2)L}{p}} \geq e^{-(C_p-1)L},$$

and therefore, we must have for some $j$ with $0 \leq j \leq C_p n - 1$ that

$$\left| \mathbb{P}(\tilde{x}_j = 1) - \mathbb{P}(\tilde{x}'_j = 1) \right| = |\mathbb{E}\tilde{a}_j| \geq |\mathbb{E}\tilde{a}_j w^j|$$

$$\geq \frac{1}{C_p n} e^{-(C_p-1)L} \geq e^{-C_p L},$$

as desired. \qed

5 Proof of Theorem 1.1

Throughout this section, we fix a deletion probability $q < \frac{1}{2}$ (and hence a retention probability $p > \frac{1}{2}$). In addition, all of our inequalities are meant to apply for $n$ sufficiently large (i.e. larger than a constant depending only on $p$).

Let $C_p$ be the larger of the two constants in Lemmas 2.2 and 4.1 and let $c_p$ be the constant in Lemma 3.4. We define the following integers:

$$M = \left[ C_p \log n \right], \quad K_1 = 40M, \quad K_0 = \left[ K_1^{1/2} \right], \quad K_2 = \left[ \frac{10}{c_p} K_1^{1/2} \log n \right].$$
It is helpful to keep in mind that $K_0 = \Theta_p(\log^{1/2} n)$, $K_1 = \Theta_p(\log n)$, and $K_2 = \Theta_p(\log^{3/2} n)$.

Recall the high-level strategy of the proof from Section 1.2: we align traces against what we have reconstructed so far, and then we use bit statistics to reconstruct additional bits. The alignment step in particular relies on the input $X$ having certain special properties which don’t hold for all strings but do hold for “most”. We encapsulate these properties in the following definition.

**Definition 5.1.** Let $\gamma_p < 1$ be the constant from Lemma 3.4. We say that a string $x \in \{0, 1\}^n$ is **good** if the following conditions are satisfied:

(i). $x$ is $(M, C_p)$-trackable,

(ii). there is no run of $M$ consecutive identical bits in $x$,

(iii). among any $K_2$ consecutive bits of $x$, there is a block of $2K_1$ of them that is $(K_0, \gamma_p)$-distinguishable.

As the next lemma shows, a random string is good with high probability.

**Lemma 5.2.** Let $X \in \{0, 1\}^n$ be drawn uniformly at random. Then,

$$\Pr(X \text{ is good}) \geq 1 - O_p \left( \frac{1}{n} \right).$$

**Proof.** It suffices to show that each condition in 5.1 holds with probability at least $1 - O_p \left( \frac{1}{n} \right)$. For condition (i), this is immediate by Lemma 2.2.

To establish condition (ii), note that the probability for $M$ i.i.d. uniform bits to be identical is $2^{1 - M}$. Union bounding over all blocks of $M$ consecutive bits in $X$, we find that (ii) holds with probability at least $1 - n \cdot 2^{1 - M} \geq 1 - O \left( \frac{1}{n} \right)$.

Finally, for condition (iii), note that any $K_2$ consecutive bits contain at least $\lfloor K_2/2K_1 \rfloor \geq \frac{2\log n}{c_p\sqrt{K_1}}$ disjoint blocks of size $2K_1$. By Lemma 3.4, the probability that a single block fails to be $(K_0, \gamma_p)$-distinguishable is at most $e^{-c_p\sqrt{K_1}}$. Thus, the probability that none of these blocks is $(K_0, \gamma_p)$-distinguishable is at most

$$\exp \left( -\frac{2\log n}{c_p\sqrt{K_1}} \cdot c_p\sqrt{K_1} \right) = \frac{1}{n^2}.$$

Union bounding over at most $n$ possible blocks of $K_2$ consecutive bits shows that condition (iii) also holds with probability at least $1 - O \left( \frac{1}{n} \right)$. \hfill $\square$

### 5.1 Alignment

Suppose $x$ is a bit string that we know, and let $m \leq |x|$ be some position in $x$. Suppose that we also have a sample $\tilde{x}$ from the deletion channel applied to $x$ (or some longer string having $x$ as a prefix). As described in Section 1.2, we would like to identify (with high probability) a bit of $\tilde{x}$ that was originally positioned near the $m$-th bit of $x$. This motivates the following definition.
Definition 5.3. An alignment rule is a function $L$ which takes as input a bit string $x$, an index $m \leq |x|$, and another bit string $y$. It outputs a value $L(x, m, y) \in \{1, 2, \ldots, |y| - 1, |y|, \infty\}$.

In addition, we require that $L$ satisfy the following adaptedness property with respect to $y$: whenever $L(x, m, y) < \infty$, for any other string $y'$ identical to $y$ in their first $L(x, m, y)$ bits, we have $L(x, m, y') = L(x, m, y)$.

Let us explain the conceptual meaning of this definition. Although it is not strictly required for the definition, we emphasize that for our purposes, $y$ will be a sample from the deletion channel applied to a string whose prefix is $x$. The idea is that bits near the $m$-th position of $x$ should end up near the $L(x, m, y)$-th position in $y$ after going through the deletion channel; in this way, the position $m$ in $x$ is “aligned” with position $L(x, m, y)$ in $y$. When $L(x, m, y) = \infty$, it means that the rule cannot reliably locate which bits of $y$ came from around the $m$-th position of $x$.

The adaptedness condition says that an alignment rule must proceed by examining the bits of $y$ in order one by one, and either outputting the current position or giving up and outputting $\infty$. In particular, we do not allow alignment rules to look ahead in the string $y$ before deciding whether a previous position should be the output. The purpose of this requirement is to ensure that the deletion pattern after our alignment position is independent of the alignment itself.

The next lemma constructs a particular alignment rule that has good quantitative bounds on the quality of the alignment.

Lemma 5.4. Let $k$ be a given integer with $K_2 \leq k \leq n/2$, and let $x_0 \in \{0, 1\}^k$ be a string of length $k$. Then, there exists an index $m$ with $k - K_2 + K_1 \leq m \leq k - K_1$ and an alignment rule $L$ with the following property:

For any good sequence $x \in \{0, 1\}^n$ with $x_0$ as a prefix, taking $(t, \bar{x}) \sim D^*_p(x)$, we have

(i). $\mathbb{P}_x(L(x_0, m, \bar{x}) < \infty) \geq \frac{1}{2} p^{2K_0}$

(ii). $\mathbb{P}_x(|t_{L(x_0, m, \bar{x})} - m| \geq K_1 \mid L(x_0, m, \bar{x}) < \infty) \leq n^{-\Omega(1)}$

(iii). $\mathbb{P}_x(|t_{L(x_0, m, \bar{x})} - m| \geq 10K_0 \mid L(x_0, m, \bar{x}) < \infty) \leq e^{-\Omega_p(K_0)}$.

Informally speaking, the properties in the above lemma should be interpreted as saying that (i) the alignment succeeds with some not-too-small probability; (ii) it is extremely likely to align within $K_1$ of the correct position; and (iii) it usually aligns within $10K_0$. Before giving the proof, we first establish an auxiliary lemma.

Lemma 5.5. Suppose $x \in \{0, 1\}^n$ is a good sequence, and suppose $(t, \bar{x}) \sim D^*_p(x)$. Consider any $k \leq n/2$, and let $\ell$ be the smallest index such that $g_\ell(\bar{x}, x) \geq k$. Then,

$$\mathbb{P}_x\left( \ell \text{ exists and } k \leq t_\ell \leq k + 4M \right) \geq 1 - \frac{1}{n}$$

for all sufficiently large $n$. 19
Proof. Let $n' = \left\lceil \frac{5}{6} \cdot pn \right\rceil$. If $\ell$ does not exist, it means that $g_{|\tilde{x}|}(\tilde{x}, \mathbf{x}) < k$ (or $\tilde{x}$ is empty). This can be bounded by

$$
P_x(\ell \text{ does not exist}) \leq P_x(|\tilde{x}| < n') + P_x(g_{n'}(\tilde{x}, \mathbf{x}) < n/2) \\
\leq P_x(|\tilde{x}| < n') + P_x(t_{n'} - g_{n'}(\tilde{x}, \mathbf{x}) \geq n/6 + P_x(t_{n'} < 2n/3). \tag{5}$$

Note that $|\tilde{x}|$ is distributed as Binom($n, p$), so $P_x(|\tilde{x}| < n') = e^{-\Omega(n)}$. The second term in (5) is at most $e^{-\Omega(n)}$ because $\mathbf{x}$ was assumed to be $(M, C_p)$-trackable. Finally, if $t_{n'} < 2n/3$, it means that at least $5pn/6$ out of the first $2n/3$ bits were retained, which also occurs with probability at most $e^{-\Omega(n)}$. Thus, all three probabilities in (5) are exponentially small in $n$, so

$$P_x(\ell \text{ does not exist}) \leq \frac{1}{n^2} \tag{6}$$

for large enough $n$.

We now work under the assumption that $\ell$ exists. We always have

$$t_\ell \geq g_\ell(\tilde{x}, \mathbf{x}) \geq k \tag{7}$$

Since $\mathbf{x}$ does not have more than $M$ consecutive identical bits, by the minimality of $\ell$, we must have $g_\ell(\tilde{x}, \mathbf{x}) \leq k + M$. By $(M, C_p)$-trackability of $\mathbf{x}$, we have

$$P_x(t_\ell > k + 4M) \leq P_x(t_\ell - g_\ell(\tilde{x}, \mathbf{x}) > 3M) \leq \frac{1}{n^2}. \tag{8}$$

Combining (6), (7), and (8) completes the proof. \hfill \Box

Proof of Lemma 5.4. If $\mathbf{x}_0$ is not a prefix of any good sequence, then there is nothing to prove. Otherwise, because $\mathbf{x}_0$ is a prefix of a good sequence, there must exist $m$ with $k - K_2 + K_1 \leq m \leq k - K_1$ such that $\mathbf{x}_0^{(m-K_1+1):(m+K_1)}$ is $(K_0, \gamma_p)$-distinguishable. We choose such an $m$ and let $\mathbf{w} = \mathbf{x}_0^{(m-K_0+1):(m+K_0)}$ (see Figure 5.1).

Now, suppose $\mathbf{x}$ is any good sequence having $\mathbf{x}_0$ as a prefix, and take $(t, \tilde{x}) \sim D_p^\mathbf{w}(\mathbf{x})$. Roughly speaking, our alignment rule will be to first use Lemma 5.5 to identify an index $\ell_0$ in $\tilde{x}$ such that $t_{\ell_0}$ is slightly smaller than $m$. Then, we will look for an occurrence of $\mathbf{w}$ in $\tilde{x}$ shortly after position $\ell_0$. If such an occurrence exists, we output the position of the last bit of the occurrence. If not, we output $\infty$.

To specify the alignment rule precisely, consider any string $\mathbf{y}$, and define the statement

$$P_0(\mathbf{y}) = \text{“there exists } \ell \text{ such that } \infty > g_\ell(\mathbf{y}, \mathbf{x}_0) \geq m - 8M” \tag{2}$$

Whenever $P_0(\mathbf{y})$ holds, take $\ell_0(\mathbf{y})$ to be the smallest such $\ell$. Then, define

$$P_1(\mathbf{y}) = P_0(\mathbf{y}) \land \text{“Ind}^\mathbf{w}_{\ell_0(\mathbf{y}):\ell_0(\mathbf{y})+16M}(\mathbf{y}) < \infty”.$$
We then define our alignment rule to be

$$\mathcal{L}(x_0, m, y) = \begin{cases} \text{Ind}_w^{\ell_0(y): (\ell_0(y)+16M)}(y) + 2K_0 - 1 & \text{if } P_1(y) \text{ holds} \\ \infty & \text{otherwise.} \end{cases}$$

Note that this satisfies the adaptedness requirement for alignment rules. We will specifically apply the above definition with $y = \tilde{x}$, so it is convenient to define the events

$$E_0 = \{P_0(\tilde{x}) \text{ holds}\}, \quad E_1 = \{P_1(\tilde{x}) \text{ holds}\},$$

and we abbreviate $\ell_0 = \ell_0(\tilde{x})$.

Next, we establish properties (i), (ii), and (iii). In what follows, the reader may find it helpful to refer to Figure 5.2. Define

$$F_0 = E_0 \cap \{m - 8M \leq t_{\ell_0} \leq m - 4M\}.$$ 

By Lemma 5.5, we have

$$\Pr_x(F_0) \geq 1 - \frac{1}{n} \quad (9)$$

We note a subtlety in our use of the lemma: the event $E_0$ concerns existence of $g_{t_0}(\tilde{x}, x_0)$, while Lemma 5.5 concerns existence of $g_{t}(\tilde{x}, x)$. However, as long as $t_{\ell_0} \leq m - 4M$, the relevant indices are all less than $k$, so there is no difference between using $x_0$ and using $x$, and the lemma still applies.

We can now lower bound $\Pr_x(\mathcal{L}(x_0, m, \tilde{x}) < \infty) = \Pr_x(E_1)$. Conditioned on $F_0$, it is always possible for $E_1$ to occur by retaining all the bits in positions $m - K_0 + 1$ through $m + K_0$ in $x$. Thus,

$$\Pr_x(\mathcal{L}(x_0, m, \tilde{x}) < \infty) = \Pr_x(E_1) \geq \Pr_x(F_0) \cdot p^{2K_0} \geq \frac{1}{2} p^{2K_0},$$

establishing property (i).

To show property (ii), consider the event

$$F_1 = \{t_{\ell_0+16M} \leq m + K_1\}.$$
Figure 5.2: A possible configuration for \( x_0, m, \) and \( \tilde{x} \). In the diagram above, events \( F_0, F_1, \) and \( E_1 \) all hold.

Note that if \( F_0 \) and \( F_1^c \) both occur, then it means that fewer than \( 16M \) bits were retained among the positions in \( x \) between \( m - 4M \) and \( m + K_1 \). There are \( K_1 + 4M = 44M \) such positions, so

\[
P_x(F_0 \cap F_1^c) \leq P_x(\text{Binom}(44M, p) < 16M) \leq e^{-\Omega(M)} = n^{-\Omega(1)}.
\]  

(10)

If \( F_0, F_1, \) and \( E_1 \) all occur, then we have

\[
t_{\mathcal{L}(x, m, \tilde{x})} \geq t_0 \geq m - 8M \geq m - K_1
\]

\[
t_{\mathcal{L}(x, m, \tilde{x})} \leq t_{0+16M} \leq m + K_1.
\]

Thus,

\[
P_x\left( t_{\mathcal{L}(x, m, \tilde{x})} - m \leq K_1 \mid E_1 \right) \geq P_x(F_0 \cap F_1 \mid E_1) \geq 1 - \frac{P_x(F_0 \cap F_1^c) + P_x(F_0 \cap F_1^c)}{P_x(E_1)}
\]

\[
\geq 1 - n^{-\Omega(1)},
\]

establishing property (ii).

Finally, we show property (iii). Let

\[
I = t^{-1}(\{m - K_1 + 1, m - K_1 + 2, \ldots, m + K_1\})
\]

be the set of indices in \( \tilde{x} \) which “came from” \( x_0^{(m-K_1+1):(m+K_1)} \). Note that we can regard \( (\tilde{x}_i)_{i \in I} \) in increasing order as being drawn from \( D_{\mathcal{L}}(x_0^{(m-K_1+1):(m+K_1)}) \). Consider the event

\[
F_2 = E_1 \cap \left\{ t(\text{IndSet}_{\mathcal{L}}^\ell_0(\ell_0+16M)(\tilde{x})) \subseteq [m - 10K_0, m + 10K_0] \right\}.
\]

Note that we have the implication

\[
t(\text{IndSet}_{\mathcal{L}}^\ell_0(\ell_0+16M)(\tilde{x})) \not\subseteq [m - 10K_0, m + 10K_0] \quad \text{and} \quad [\ell_0 : (\ell_0 + 16M)] \subseteq I
\]
\[ \implies t \left( \text{IndSet}_w^I(\tilde{x}) \right) \not\subseteq [m - 10K_0, m + 10K_0], \]

which means
\[ \mathbb{P}_x (F_2^c \cap (F_0 \cap F_1 \cap E_1)) \leq \mathbb{P}_x \left( t \left( \text{IndSet}_w^I(\tilde{x}) \right) \not\subseteq [m - 10K_0, m + 10K_0] \right) \leq e^{-\Omega_p(K_0)} \cdot p^{2K_0}, \quad (11) \]

where the last inequality follows from the fact that \( x_0^{0:(m-K_0+1):(m+K_1)} \) is \((K_0, \gamma_p)\)-distinguishable combined with Lemma \[3.5\].

Recall from property (i) that \( \mathbb{P}_x(E_1) \geq \frac{1}{2}p^{2K_0} \). We conclude that
\[ \mathbb{P}_x \left( \left| t_{\tilde{L}(\tilde{x})} - m \right| \leq 10K_0 \bigg| E_1 \right) \geq \frac{\mathbb{P}_x(F_0 \cap F_2 \mid E_1)}{\mathbb{P}_x(E_1)} \geq 1 - \frac{\mathbb{P}_x(F_0^c) + \mathbb{P}_x(E_1 \cap F_0 \cap F_1^c) + \mathbb{P}_x(E_1 \cap F_0 \cap F_1 \cap F_2^c)}{\mathbb{P}_x(E_1)} \geq 1 - n^{-\Omega(1)} - n^{-\Omega(1)} - e^{-\Omega_p(K_0)} = 1 - e^{-\Omega_p(K_0)}, \]

where we have used \[9\], \[10\], and \[11\] to bound the numerator appearing in the second line. This proves (iii). \[\square\]

5.2 Reconstruction

The following lemma provides a template for how we will reconstruct bits.

**Lemma 5.6.** Consider integers \( k_1 \) and \( k_2 \) with \( k_1 < k_2 \), and let \( S \subseteq \{0,1\}^{k_2} \) be a known set of length-\( k_2 \) bit strings. Suppose that we have a number \( \epsilon > 0 \) and a family of statistics \( b_j : S \rightarrow \mathbb{R} \) for \( 1 \leq j \leq k_2 \) which satisfies the following property: for any two strings \( w, w' \in S \) whose first \( k_1 \) bits are not identical, there exists an index \( j_{w,w'} \) such that \( |b_{j_{w,w'}}(w) - b_{j_{w,w'}}(w')| > \epsilon \).

Let \( z \in S \) be an unknown string, and suppose that we observe estimates \( \hat{b}_j \) \( j=1 \) such that \( |\hat{b}_j - b_j(z)| < \epsilon/2 \) for each \( j \). Then, we can determine the first \( k_1 \) bits of \( z \).

**Proof.** For any two strings \( w, w' \in S \) whose first \( k_1 \) bits are not identical, we say that \( w \) beats \( w' \) if \( \hat{b}_{j_{w,w'}} \) is closer to \( b_{j_{w,w'}}(w) \) than to \( b_{j_{w,w'}}(w') \). We say \( w \) is dominant if it beats all other strings \( w' \in S \) that do not share its first \( k_1 \) bits.

Our hypotheses imply that \( z \) is dominant. Moreover, any two dominant strings must share their first \( k_1 \) bits. Thus, we may recover the first \( k_1 \) bits of \( z \) as the first \( k_1 \) bits of any dominant string. \( \square \)

We now apply the template in two lemmas. The first lemma shows how to reconstruct the initial \( K_2 \) bits, and the second lemma shows how to reconstruct additional bits once we have already reconstructed a long enough prefix of \( x \).
Lemma 5.7. Let \( x \in \{0, 1\}^n \) be a good sequence. There is a constant \( C'_p \) depending only on \( p \) such that \( N = \lceil \exp \left( C'_p \sqrt{\log n} \right) \rceil \) independent samples from \( D_p(x) \) are sufficient to recover the first \( K_2 \) bits of \( x \) with probability at least \( 1 - \frac{1}{n} \) for all sufficiently large \( n \).

Proof. Let \( \tilde{x}_1, \ldots, \tilde{x}_N \) be the sampled traces. For each \( j \leq n \), let

\[
\tilde{x}_j^{\text{avg}} = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_{i,j}
\]

be the average of the bits of the \( \tilde{x}_i \) at position \( j \), where \( \tilde{x}_i \) are padded to the right with zeroes.

We will apply Lemma 5.6 with \( k_1 = K_2 \) and \( k_2 = n \). We consider statistics \( b_j(z) \) equal to the expected value of the \( j \)-th bit of a string drawn from \( D_p(z) \). By Lemma 4.1 we may take \( \epsilon = e^{-O_p \left( k_2^{1/4} \right) } = e^{-O_p \left( \log \frac{1}{n} \right) } \).

Choose \( C'_p \) sufficiently large so that \( \epsilon^2 N \geq e^{\sqrt{\log n}} \). Noting that \( \mathbb{E}[\tilde{x}_j^{\text{avg}}] = b_j(x) \), by a Chernoff bound we have

\[
\mathbb{P}_x (|\tilde{x}_j^{\text{avg}} - b_j(x)| > \epsilon/2) \leq e^{-\frac{\epsilon^2 N}{2}} \leq \frac{1}{n^2}
\]

for all large enough \( n \). Thus, letting

\[ E = \{ |\tilde{x}_j^{\text{avg}} - b_j(x)| \leq \epsilon/2 \text{ for each } j \} \]

and union bounding over all \( 1 \leq j \leq n \), we have \( \mathbb{P}_x(E) \geq 1 - \frac{1}{n} \).

Using \( \tilde{b}_j = \tilde{x}_j^{\text{avg}} \) as our estimates, Lemma 5.6 asserts that we can recover the first \( K_2 \) bits of \( x \) on the event \( E \), which proves the desired statement. \( \square \)

Lemma 5.8. Let \( n \) be a positive integer, and let \( k \) be an integer with \( K_2 \leq k \leq n/2 \). There is a constant \( C'_p \) depending only on \( p \) such that the following holds:

Consider a good sequence \( x \in \{0, 1\}^n \), and suppose that \( N := \lceil \exp \left( C'_p \sqrt{\log n} \right) \rceil \) i.i.d. samples \( \tilde{x}_1, \ldots, \tilde{x}_N \) are drawn from \( D_p(x) \). Then, whenever \( n \) is sufficiently large, seeing only the first \( k \) bits of \( x \) and the traces \( \tilde{x}_1, \ldots, \tilde{x}_N \) is sufficient to recover the \((k + 1)\)-th bit of \( x \) with probability at least \( 1 - \frac{1}{n^2} \).

Proof. Let \( m \) and \( \mathcal{L} \) be the index and alignment rule given by Lemma 5.4 where we take \( x_0 = x^{1:k} \) to be the first \( k \) bits of \( x \) (which we have been given). Let us consider a single trace \( \tilde{x} \sim D_p(x) \). For brevity, write \( \ell = \ell(\tilde{x}) = \mathcal{L}(x_0, m, \tilde{x}) \).

We say that \( \tilde{x} \) is a usable trace if \( \ell < \infty \). Let \( E \) denote the event that \( \tilde{x} \) is usable, and let

\[
E' = E \cap \{ m - K_1 \leq t_\ell \leq m + K_1 \}
\]

\[
E'' = E \cap \{ m - 10K_0 \leq t_\ell \leq m + 10K_0 \}
\]
Lemma 5.4 ensures that $\mathbb{P}_x(E' \mid E) \geq 1 - n^{-\Omega(1)}$ and $\mathbb{P}_x(E'' \mid E) \geq 1 - e^{-\Omega_p(K_0)}$, which taken together imply that

$$
\mathbb{P}_x(E'' \mid E') \geq 1 - e^{-\Omega_p(K_0)} = 1 - e^{-\Omega_p(\log^{1/2} n)}.
$$

(12)

Let $H = m - K_1$, and let $\Delta$ be a random variable having the same distribution as $t_f - H$ conditioned on $E'$. The reason for defining $\Delta$ in this particular way will be made clearer shortly. For now, let us take note of several properties of $\Delta$:

- $\Delta$ is an integer between 0 and $2K_1$.
- The distribution of $\Delta$ can be calculated just by looking at $x_0$ (in particular, it does not depend on bits of $x$ after the $k$-th one).
- By (12), it is straightforward to deduce that $E \Delta = m + O_p(K_0)$ and $E[|\Delta - E\Delta|] = O_p(K_0)$.

Define $K_3 = \left\lceil C''_p \log^{3/2} n \right\rceil$, where $C''_p$ is a large enough constant to ensure that

$$
E[|\Delta - E\Delta|] \leq K_3^{1/3}, \quad 2K_1 \leq K_3^{2/3}, \quad \text{and} \quad K_3 > K_2.
$$

Our goal will be to distinguish the true suffix $x^{(H+1)}$ from other possible suffixes via Lemma 5.6, where we take $(k_1, k_2) = (K_3, n - H)$. Here, the set $S$ is taken to be all strings of length $n - H$ having $x^{(H+1):k}$ as a prefix. By reconstructing the first $K_3$ bits of $x^{(H+1)}$, we will have in particular reconstructed $x_{k+1}$, since $H + K_3 = m - K_1 + K_3 > k + 1$.

The statistics we use are, for any $z \in S$,

$$
b_j(z) := \text{expected value of the } j\text{-th bit of a string drawn from } D_p(z^{(\Delta+1)}),
$$

and we take $\epsilon = e^{-C_p K_3^{1/3}}$ (it may be helpful to keep in mind that $\epsilon = e^{-\Theta_p(\log^{1/2} n)}$).

Note that we are able to compute these quantities $b_j(z)$ since we are able to compute the distribution of $\Delta$.

Let us first verify the property required of the $b_j$ and $\epsilon$ in Lemma 5.6. Consider any two strings $w, w' \in S$ that do not agree in their first $K_3$ bits. We apply Lemma 1.1 to these strings with $(k, n, S) = (2K_1, K_3, \Delta)$. To check the hypotheses of the lemma, note that by the definition of $S$ and the assumption $m \leq k - K_1$, $w$ and $w'$ agree in their first $k - H = k + K_1 - m \geq 2K_1$ bits, as required. We also recall that by the way we defined $K_3$, the conditions

$$
E[|\Delta - E\Delta|] = O_p(K_0) \leq K_3^{1/3}, \quad 2K_1 \leq K_3^{2/3}
$$

3It should be noted that the probability $\mathbb{P}_x(E)$ of having a usable trace does depend on later bits of $x$. However, the additional constraint $m - K_1 \leq t_f \leq m + K_1$ combined with the adaptedness property of $\mathcal{L}$ removes this dependence.
are satisfied. Thus, Lemma 4.1 tells us that there exists an index $j_{w,w'}$ for which

$$|b_j(w) - b_j(w')| \geq e^{-C_kK_3^{1/3}} = \epsilon,$$

establishing that our choice of $b_j$ and $\epsilon$ are suitable for use in Lemma 5.6.

Unfortunately, we cannot directly observe samples with the law of $D_p(x^{(H+1+\Delta)})$ in order to estimate $b_j(x^{(H+1)})$. However, a usable trace $\tilde{x}$ allows us to sample from this distribution approximately. The fact that $\mathcal{L}$ is adapted to $\tilde{x}$ (as required in Definition 5.3) means that if we condition on $t_{\ell}(\tilde{x}) = h$ for some index $h$, the string $\tilde{x}^{(\ell+1)}$ has the same distribution as $D_p(x^{(h+1)})$. Thus, the definition of $\Delta$ ensures that, conditioned on the event $E'$, $\tilde{x}^{(\ell+1)}$ has exactly the law of $D_p(x^{(h+1)})$.

As long as $\tilde{x}$ is usable, we define $\hat{b}_j(\tilde{x}) := \tilde{x}_{\ell+j}$ and $\bar{b}_j = \mathbb{E}(\hat{b}_j(\tilde{x}) | E)$. The above discussion implies that

$$|\bar{b}_j - b_j(x^{(H+1)})| \leq \mathbb{P}_x(E'^c | E) \leq n^{-\Omega(1)} \leq \epsilon/4,$$  \hfill (13)

where the bound on $\mathbb{P}_x(E'^c | E)$ comes from Lemma 5.4.

Averaging over our $N$ traces $\tilde{x}_1, \ldots, \tilde{x}_N$ will then give us a fairly good estimate on $b_j(x^{(H+1)})$. Choose $C'_p$ large enough so that the following hold:

$$N \geq 64p^{-6K_0} \implies \frac{1}{2}p^{2K_0} N \geq 2N^{2/3} \hfill (14)$$

$$N \geq \epsilon^{-3}e^{\sqrt{\log n}} \implies N^{2/3} \epsilon^2 = e^{\Omega(\sqrt{\log n})}. \hfill (15)$$

Let $M$ be the number of usable traces. Since our alignment rule ensures that the probability of being usable is at least $\frac{1}{2}p^{2K_0}$, it follows by a Chernoff bound and (14) that

$$\mathbb{P}_x(M < N^{2/3}) \leq \exp\left(-\frac{2N^{4/3}}{N}\right) = \exp\left(-2\epsilon^{\Omega(\sqrt{\log n})}\right) \leq \frac{1}{n^3}.$$

Define

$$\hat{b}_{avg}^j = \frac{1}{M} \sum_{\tilde{x}_i \text{ is usable}} \hat{b}_j(\tilde{x}_i).$$

By another Chernoff bound and (15),

$$\mathbb{P}_x\left( |\hat{b}_{avg}^j - \bar{b}_j| > \epsilon/4 \right) \leq \mathbb{P}_x(M < N^{2/3}) + \exp\left(-\frac{N^{2/3} \epsilon^2}{8}\right) \leq \frac{1}{n^3} + \exp\left(-\epsilon^{\Omega(\sqrt{\log n})}\right) \leq \frac{1}{n^2}. \hfill (16)$$

Combining (13) and (16), we conclude that

$$\mathbb{P}_x\left( |\hat{b}_{avg}^j - b_j(x^{(H+1)})| < \epsilon/2 \text{ for all } j \leq n \right) \geq 1 - \frac{1}{n^2}.$$  \hfill (17)

Thus, with probability at least $1 - \frac{1}{n^2}$, the conclusion of Lemma 5.6 allows us to determine the first $K_3$ bits of $x^{(H+1)}$. As noted earlier, this includes the $(k+1)$-th bit of $x$, as desired. \hfill \square
5.3 Completing the proof

We are finally ready to prove Theorem 1.1, which is mostly a matter of combining Lemmas 5.7 and 5.8.

**Proof of Theorem 1.1** We sample $N = \lceil \exp\left(C'_p \sqrt{\log n}\right) \rceil$ traces, where $C'_p$ is large enough so that Lemmas 5.7 and 5.8 apply.

We first condition on a realization $X = x$, and suppose that $x$ is good. We will construct a string $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$. Let $E_k$ denote the event that $\hat{x}$ matches $x$ in the first $k$ bits. We construct the first $K_2$ bits of $\hat{x}$ using Lemma 5.7 which yields

$$\mathbb{P}_x(E_{K_2}) \geq 1 - \frac{1}{n}. \quad (17)$$

Next, consider any $k$ with $K_2 \leq k \leq n/2$, and suppose we have constructed $\hat{x}_1, \ldots, \hat{x}_k$ already. We apply the algorithm of Lemma 5.8 and set $\hat{x}_{k+1}$ to its output. Although we do not have access to the first $k$ bits of $x$, we use the first $k$ bits of $\hat{x}$ instead. As long as $E_k$ holds, this will give us the correct value for $\hat{x}_{k+1}$ with probability at least $1 - \frac{1}{n^2}$. Thus,

$$\mathbb{P}_x(E_{k+1}) \geq \mathbb{P}_x(E_k) - \frac{1}{n^2}. \quad (18)$$

Using (17) followed by repeated applications of (18), we find that

$$\mathbb{P}_x(E_{\lceil n/2 \rceil}) \geq 1 - \frac{1}{n}.$$ 

By symmetry, we can repeat the same procedure in reverse to reconstruct the last $\lfloor n/2 \rfloor$ bits of $x$. Accounting for both the forward and reverse steps, the probability of failure is at most $\frac{2}{n}$.

The final possible mode of failure is if $x$ is not good. However, by Lemma 5.2, this only happens with probability at most $\frac{1}{n}$. In total, we can reconstruct $X$ with probability at least $1 - \frac{3}{n}$. Moreover, we have only used $N = e^{O_p(\sqrt{\log n})}$ traces. This completes the proof. □

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