BSDEs with monotone generator and two irregular reflecting barriers

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Abstract
We consider BSDEs with two reflecting irregular barriers. We give necessary and sufficient conditions for existence and uniqueness of $L^p$ solutions for equations with generators monotone with respect to $y$ and Lipschitz continuous with respect to $z$, and with data in $L^p$ spaces for $p \geq 1$. We also prove that the solutions can be approximated via penalization method.

1 Introduction
Nonlinear reflected BSDEs with one continuous barrier were introduced in [7] as a generalization of the notion of nonlinear BSDEs considered for the first time in [26]. At the same time in [6] a nonlinear reflected BSDE with two continuous barriers were introduced. Already in these initial papers it was shown that reflected BSDEs have natural connections with the optimal stopping problem, mixed control problem, viscosity solutions of the obstacle problem for PDEs and Dynkin games. In subsequent works on reflected BSDEs these connections were used successfully to investigation of the problems mentioned above and to the mixed game problem, risk-sensitive optimal problem, switching problem and other optimality problems (see, e.g., [9, 13, 15, 16, 29]). The connections with the obstacle problem for PDEs allowed to give new existence results and numerical schemes for solutions of PDEs and appeared powerful in investigation of homogenization problems and regularity properties of solutions of PDEs (see, e.g., [2, 3, 10, 16, 17, 18, 23, 25]).

Let $B$ be a standard $d$-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t\}$ be the standard augmentation of the filtration generated by $B$. Suppose we are given two real progressively measurable processes $U, L$ on $[0, T]$ such that $L \leq U$, an $\mathcal{F}_T$-measurable random variable $\xi$ such that $L_T \leq \xi \leq U_T$ and $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $f(\cdot, y, z)$ is progressively measurable. Let us recall that if $U, L$ are continuous then by a solution of the reflected BSDE with generator

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f, terminal condition \( \xi \) and barriers \( L, U \) we mean a triple \((Y, Z, R)\) of progressively measurable processes such that \( t \mapsto f(\cdot, Y_t, Z_t) \in L^1(0, T), P(\int_0^T |Z_t|^2 \, dt < \infty) = 1 \) and \( R \) is a continuous process finite variation such that \( R_0 = 0 \) and

\[
\begin{align*}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T dR_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T], \\
L_t \leq Y_t \leq U_t, \quad t \in [0, T], \\
\int_0^T (Y_t - L_t) \, dR_t^+ &= \int_0^T (U_t - Y_t) \, dR_t^- = 0,
\end{align*}
\tag{1.1}
\]

where \( R^-, R^+ \) are increasing processes such that \( R = R^+ - R^- \).

Because of many applications of reflected BSDEs many attempts have been made to relax the assumptions on the data considered in the pioneering papers \cite{6, 7}, i.e. continuity of \( U, L \), linear growth of \( f \) and Lipschitz-continuity of \( f \) with respect to \( y, z \), square-integrability of the data. Among the attempts one can distinguish three main directions. First of all, many papers are devoted to proving existence of solutions under weaker assumptions on regularity of the generator. For instance, in \cite{3, 8, 12, 20} generators having quadratic growth with respect to \( z \) are considered. In \cite{21, 32} it is assumed that \( f \) is continuous and monotone with respect to \( y \) and satisfies the general growth condition, in \cite{22, 24} it is only required that \( f \) is continuous with respect to \( y, z \) and satisfies the linear growth condition, and in \cite{33} the generator is left-continuous and monotone with respect to \( y \). In the second group of papers less regular barriers are considered. In \cite{9, 11} the barriers are càdlàg whereas in \cite{28, 29} they are merely measurable, square-integrable and satisfy the so-called Mokobodzki condition which roughly speaking says that between barriers there exists a semimartingale having some integrability properties. It is worth mentioning that in \cite{11} (see also \cite{10}) the Mokobodzki condition is replaced by the following one: \( L_t < U_t, L_{t-} < U_{t-}, t \in [0, T] \).

In the third group existence and uniqueness results for reflecting BSDEs with data that are not square-integrable are proved (see \cite{19, 31} for results for equations with data in \( \mathbb{L}^p \) with \( p \in [1, 2] \) and \cite{1, 14} for the case where \( p \in (1, 2) \)). Finally, let us note that to our knowledge at present there are only few papers, all on equations with one reflecting barrier, that cover two of the three cases described above (see \cite{1, 19, 31}) and there is no paper which covers all the three.

In the present paper we consider reflected BSDEs with data in \( \mathbb{L}^p \) spaces, \( p \in [1, 2] \), and with two merely measurable barriers satisfying the generalized Mokobodzki condition. Regarding the generator, we focus attention on its dependence on the variable \( y \). We assume that \( f \) is monotone with respect to \( y \), Lipschitz continuous with respect to \( z \) and satisfies a very general growth condition with respect to \( y \) which is weaker then the so-called general growth condition considered in \cite{5} in case of nonreflected BSDEs and in \cite{19, 31} in case of BSDEs with one continuous reflecting barrier. This growth condition has the form

\[
\forall r > 0 \quad \sup_{|y| \leq r} |f(\cdot, y, 0) - f(\cdot, 0, 0)| \in \mathbb{L}^1(0, T).
\]

It seems to be the best possible growth condition on \( f \) with respect to \( y \); it is widely used in the theory of partial differential equations (see \cite{4} and the references given there). Motivated by future applications to PDEs, we consider reflected BSDEs more
general than (1.1). Suppose that in addition to \(\xi, f, U, L\) we are given a progressively measurable càdlàg process \(V\) such that \(V_0 = 0\). The problem is to find a triple \((Y, Z, R)\) of progressively measurable processes such that \(Z, f(\cdot, Y, Z)\) have the same integrability properties as in case of equation (1.1) and \(R\) is a càdlàg process of finite variation such that \(R_0 = 0\) and

\[
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dV_s + \int_t^T dR_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T], \\
L_t &\leq Y_t \leq U_t \quad \text{for a.e. } t \in [0, T], \\
\int_0^T (Y_t - \hat{L}_t) \, dR_t^+ - \int_0^T (\hat{U}_t - Y_t) \, dR_t^- &= 0
\end{aligned}
\]  

(1.2)

for any progressively measurable càdlàg processes \(\hat{L}, \hat{U}\) such that \(L_t \leq \hat{L}_t \leq Y_t \leq \hat{U}_t \leq U_t\) for a.e. \(t \in [0, T]\).

In the paper we give existence, uniqueness and comparison results for equations of the form (1.2). We also provide Snell’s envelope representation and prove that for every \(p \in [1, 2)\) solutions of (1.2) can be approximated by the penalization method.

In [28] it is shown that in case \(p = 2\) there exists a solution of BSDE with two reflecting barriers for Lipschitz continuous generators satisfying the linear growth condition. The remarkable feature of the theory of \(\mathbb{L}^p\) solutions of equations with monotone generators is the fact that conditions ensuring existence of solutions of nonreflecting BSDEs together with the Mokobodzki condition are insufficient for existence of solutions of reflected BSDEs in the given class of integrability. To get existence we introduce a generalized Mokobodzki condition which contrary to the standard one also relates the growth of the generator with that of the barriers. In fact, we formulate two versions of the generalized condition: one for \(p > 1\) and another one for \(p = 1\). One of our main results says that under some minimal assumptions ensuring existence of solutions of nonreflected BSDE (see [5, Theorem 4.2, Theorem 6.3]) the generalized Mokobodzki condition is necessary and sufficient for existence of a solution of reflected BSDE in a given class of integrability. In the proof of that result we use among others things the method of supersolutions applied earlier in [28] and the technique of stopping times used in [10, 19, 31].

In the last part of the paper we answer the question what happens if despite the fact that we consider reflected BSDEs with monotone generator we only assume the standard Mokobodzki condition. From the comments given in the preceding paragraph it follows that in that case in general we can not expect existence of \(\mathbb{L}^p\) solutions. Nevertheless, we show that there always exists a unique solution of (1.2). It may happen, however, that some of its components are nonintegrable for every \(p > 0\).

The paper is organized as follows. In Section 2 we provide basic notation used in the paper and we prove Itô’s formula for càdlàg processes and the function \(x \to |x|^p, p \in [1, 2)\). In Section 3 we prove existence, comparison results a priori estimates for solutions of (1.2). In Section 4 we prove some properties of supersolutions of (1.2) and some useful lemmas required in proofs of existence of solutions of reflected BSDEs. In particular, we prove a generalization of the monotone convergence theorem for BSDEs, Snell’s envelope representation of supersolutions and a lemma on behavior of their jumps. In Section 5 we prove existence and uniqueness results for BSDEs with one reflecting barrier whereas in Section 6 for equations with two barriers. In Section 7 we consider the case of nonintegrable solutions.
2 Preliminary results

Let $B = \{B_t, t \geq 0\}$ be a standard $d$-dimensional Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ is the augmented filtration generated by $B$. In the whole paper all notions whose definitions are related to some filtration are understood with respect to the filtration $\mathcal{F}$.

Given a stochastic process $X$ on $[0, T]$ with values in $\mathbb{R}^n$ we set $X^*_t = \sup_{0 \leq s \leq t} |X_s|$, $t \in [0, T]$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. By $\mathcal{S}$ (resp. $\mathcal{D}$) we denote the set of all progressively measurable continuous (resp. càdlàg) processes. For $p > 0$ we denote by $\mathcal{S}^p$ (resp. $\mathcal{D}^p$) the set of all processes $X \in \mathcal{S}$ (resp. $\mathcal{D}$) such that

$$ E \sup_{t \in [0, T]} |X_t|^p < \infty. $$

$\mathcal{M}$ is the set of all progressively measurable processes $X$ such that

$$ P(\int_0^T |X_t|^2 \, dt < \infty) = 1 $$

and for $p > 0$, $\mathcal{M}^p$ is the set of all processes $X \in \mathcal{M}$ such that

$$ E(\int_0^T |X_t|^2 \, dt)^{p/2} < \infty. $$

For $p, q > 0$, $\mathbb{L}^{p,q}(\mathcal{F})$ denotes the set of all progressively measurable processes $X$ such that

$$ E(\int_0^T |X_t|^p \, dt)^{q/(1 + 1/p)} < \infty. $$

For brevity we denote $\mathbb{L}^{p,p}(\mathcal{F})$ by $\mathbb{L}^p(\mathcal{F})$.

For a given measurable space $(X, \mu, \mathcal{G})$ we denote by $\mathbb{L}^p(X, \mu, \mathcal{G})$ the set of all $\mathcal{G}$-measurable real valued functions $f$ such that $\int_X |f|^p \, d\mu(x) < \infty$. If it is clear from the context which measure and $\sigma$-field is considered we omit them in the notation.

$\mathcal{M}_c$ (resp. $\mathcal{M}^{loc}_c$) is the set of all continuous martingales (resp. local martingales) and $\mathcal{M}^p_c$, $p \geq 1$, is the set of all martingales $M \in \mathcal{M}_c$ such that $E((M)_T)^{p/2} < \infty$. $\mathcal{V}_c$ (resp. $\mathcal{V}^+_c$) is the set of all continuous progressively measurable processes of finite variation (resp. increasing processes) such that $V_0 = 0$ and $\mathcal{V}^p_c$ (resp. $\mathcal{V}^{+p}_c$) is the set of all processes $V \in \mathcal{V}_c$ (resp. $V \in \mathcal{V}^+_c$) such that $E|V|_T^p < \infty$. $\mathcal{V}$ (resp. $\mathcal{V}^+$) is the set of all càdlàg progressively measurable processes of finite variation (resp. increasing processes) such that $V_0 = 0$ and $\mathcal{V}^p$ (resp. $\mathcal{V}^{+p}$) is the set of all processes $V \in \mathcal{V}$ (resp. $V \in \mathcal{V}^+$) such that $E|V|_T^p < \infty$. $\mathcal{H}^p = \mathcal{M}^p_c + \mathcal{V}^p$, $\mathcal{H}^+_c = \mathcal{M}^+_c + \mathcal{V}^+_c$. For a given process $V \in \mathcal{V}$ by $dV$ we denote the random measure generated by its trajectories.

By $\mathcal{T}$ we denote the set of all stopping times with values in $[0, T]$ and by $\mathcal{T}_t$ the set of all stopping times with values in $[t, T]$. We say that a sequence $\{\tau_k\} \subset \mathcal{T}$ is stationary if

$$ P(\liminf_{k \to +\infty} \{\tau_k = T\}) = 1. $$

For a given measurable process $Y$ of class (D) we denote

$$ ||Y||_1 = \sup \{E|Y_\tau|, \tau \in \mathcal{T}\}. $$
For $X \in \mathcal{D}$ we set $X_{t-} = \lim_{s \uparrow t} X_s$ and $\Delta X_t = X_t - X_{t-}$ with the convention that $X_{0-} = 0$. Let $\{X^n\} \subset \mathcal{D}$, $X \in \mathcal{D}$. We say that $X^n \to X$ in ucp if $\sup_{t \in [0,T]} |X^n_t - X_t| \to 0$ in probability $P$.

In the whole paper all equalities and inequalities and other relations between random elements are understood to hold $P$-a.s.. To avoid ambiguity we stress that writing $X_t = Y_t$, $t \in [0,T]$ we mean that $X_t = Y_t$, $t \in [0,T]$, $P$-a.s., while writing $X_t = Y_t$ for a.e. (resp. for every) $t \in [0,T]$ we mean that $X_t = Y_t$, $P$-a.s. for a.e. (resp. for every) $t \in [0,T]$. We also adopt the convention that $\int_a^b = \int_{(a,b)}$.

$T_k(x) = \min\{k, \max\{-k, x\}\}$, $x \in \mathbb{R}$, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ and
\[
\dot{x} = \text{sgn}(x), \quad \text{sgn}(x) = \begin{cases} x & \text{if } x \neq 0, \\ 0 & \text{otherwise} \end{cases}, \quad x \in \mathbb{R}^d.
\]

One of our basic tools will be the following Itô-Tanaka formula. The formula is probably known, but we do not have a reference.

**Proposition 2.1.** Let $p \geq 1$ and let $X$ be a progressively measurable process of the form
\[
X_t = X_0 + \int_0^t dK_s + \int_0^t Z_s dB_s, \quad t \in [0,T],
\]
where $K \in \mathcal{V}$ and $Z \in \mathcal{M}$. Then
\[
|X_t|^p - |X_0|^p = \int_0^t |X_{s-}|^{p-1}\dot{X}_{s-} dK_s + \frac{1}{2} p(p-1)\mathbf{1}_{X_s \neq 0}|X_s|^{p-2}|Z_s|^2 ds + \int_0^t |X_s|^{p-1}\dot{X}_s \Delta X_s,
\]
where $L \in \mathcal{V}^+_c$ and
\[
I_t(p) = \sum_{0<s\leq t} \Delta|X_s|^p - \sum_{0<s\leq t} p|X_{s-}|^{p-1}\dot{X}_{s-} \Delta X_s
\]
is a nonnegative increasing pure jump process.

**Proof.** Write $u^p_\varepsilon(x) = (|x|^2 + \varepsilon^2)^{p/2}$, $x \in \mathbb{R}$. Then
\[
\frac{d}{dx} u^p_\varepsilon(x) = pu^{p-2}_\varepsilon(x)x, \quad \frac{d^2}{dx^2} u^p_\varepsilon(x) = pu^{p-2}_\varepsilon(x) + p(p-2)u^{p-4}_\varepsilon(x)x^2
\]
for $x \in \mathbb{R}$. By Itô’s formula,
\[
u^p_\varepsilon(X_t) - u^p_\varepsilon(X_0) = \int_0^t \frac{d}{dx} u^p_\varepsilon(X_{s-}) dX_s + \frac{1}{2} \int_0^t \frac{d^2}{dx^2} u^p_\varepsilon(X_s) d[X, X]_c^\varepsilon + \sum_{0<s\leq t} \{\Delta u^p_\varepsilon(X_s) - \frac{d}{dx} u^p_\varepsilon(X_{s-}) \Delta X_s\}
\]
\[
= \int_0^t pu^{p-2}_\varepsilon(X_{s-}) X_{s-} dK_s + \int_0^t pu^{p-2}_\varepsilon(X_{s-}) X_{s-} Z_s dB_s + \frac{1}{2} \int_0^t pu^{p-2}_\varepsilon(X_s) + p(p-2)u^{p-4}_\varepsilon(X_s) X_s^2 Z_s^2 ds + \sum_{0<s\leq t} \{\Delta u^p_\varepsilon(X_s) - pu^{p-2}_\varepsilon(X_{s-}) X_{s-} \Delta X_s\}.
\]
Observe that \( u^p_\varepsilon \rightarrow |\cdot|^p \) uniformly on compact subsets of \( \mathbb{R} \). Hence
\[
u^p_\varepsilon(X) - u^p_\varepsilon(X_0) \rightarrow |X|^p - |X_0|^p \quad \text{in ucp.} \quad (2.4)
\]
By elementary computation, for \( \varepsilon \leq 1 \) we have
\[
|\Delta u^p_\varepsilon(X_t)| \leq \left| \frac{du^p_\varepsilon}{dx}(X^*_t) \right| |\Delta X_t| \leq (|X^*_t|^2 + 1)^{p/2} |\Delta X_t| \quad (2.5)
\]
Since \( pu^{p-2}_\varepsilon(x)x \rightarrow p|x|^{p-1}x \) for \( x \in \mathbb{R} \) and, by (2.1), \( \sum_{0<s\leq t} |\Delta X_s| \leq |K|_t \), applying the Lebesgue dominated convergence theorem we see that as \( \varepsilon \rightarrow 0^+ \) then P-a.s.,
\[
\sum_{0<s\leq t} \{ \Delta u^p_\varepsilon(X_s) - pu^{p-2}_\varepsilon(X_s-)|X_s-\Delta X_s| \} \rightarrow \sum_{0<s\leq t} \{ |X_s|^p - p|X_{s-}|^{p-1} \hat{X}_s - \Delta X_s \} \quad (2.6)
\]
for \( t \in [0,T] \). Using once again the second inequality in (2.5) and the Lebesgue dominated convergence theorem we conclude that
\[
\int_0^t pu^{p-2}_\varepsilon(X_s-)X_s- dK_s \rightarrow \int_0^t p|X_s-|^{p-1} \hat{X}_s- dK_s, \quad t \in [0,T] \quad (2.7)
\]
P-a.s. and
\[
\int_0^t pu^{p-2}_\varepsilon(X_s)X_s Z_s dB_s \rightarrow \int_0^t p|X_s| \hat{X}_s Z_s dB_s \quad \text{in ucp} \quad (2.8)
\]
as \( \varepsilon \rightarrow 0^+ \). For every \( q \in \mathbb{R} \) and \( x \in \mathbb{R} \),
\[
u^q_\varepsilon(x)|x|^2 = (|x|^2 + \varepsilon^2)^{q/2}|x|^2 = (|x|^2 + \varepsilon^2)^{q/2}(|x|^2 + \varepsilon^2) - \varepsilon^2 \nu^q_\varepsilon(x) = \nu^{q+2}_\varepsilon(x) - \varepsilon^2 \nu^q_\varepsilon(x).
\]
Hence
\[
S_t(\varepsilon) \equiv \frac{1}{2} \int_0^t (pu^{p-2}_\varepsilon(X_s) + p(p-2)u^{p-4}_\varepsilon(X_s)X_s^2 Z_s^2) \, ds
\]
\[
= \frac{1}{2} \int_0^t p(u^{p-4}_\varepsilon(X_s)|X_s|^2 + \varepsilon^2 u^{p-4}_\varepsilon(X_s)|Z_s|^2) \, ds
\]
\[
+ \frac{1}{2} \int_0^t p(p-2)u^{p-4}_\varepsilon(X_s)|X_s|^2 |Z_s|^2 \, ds
\]
\[
= \frac{1}{2} \int_0^t p(p-1)u^{p-4}_\varepsilon(X_s)|X_s|^2 |Z_s|^2 \, ds + \frac{1}{2} \int_0^t \varepsilon^2 u^{p-4}_\varepsilon(X_s)|Z_s|^2 \, ds,
\]
that is
\[
S_t(\varepsilon) = \frac{1}{2} \int_0^t p(p-1)u^{p-4}_\varepsilon(X_s)|X_s|^2 |Z_s|^2 \, ds + L_t(\varepsilon), \quad t \in [0,T], \quad (2.9)
\]
where \( L_t(\varepsilon) \equiv \frac{1}{2} \int_0^t \varepsilon^2 u^{p-4}_\varepsilon(X_s)|Z_s|^2 \, ds \). Since \( \frac{|X_s|}{u_\varepsilon(X_s)} \uparrow 1_{\{X_s \neq 0\}} \), applying the monotone convergence theorem gives
\[
\int_0^t u^{p-4}_\varepsilon(X_s)|X_s|^2 |Z_s|^2 \, ds = \int_0^t \left( \frac{|X_s|}{u_\varepsilon(X_s)} \right)^{4-p} |X_s|^{p-2} |Z_s|^2 1_{\{X_s \neq 0\}} \, ds
\]
\[
\rightarrow \int_0^1 1_{\{X_s \neq 0\}} |X_s|^{p-2} |Z_s|^2 \, ds \quad \text{in ucp.} \quad (2.10)
\]
Putting (2.3)–(2.10) together we get
\[ |X_t|^p - |X_0|^p = p \int_0^t |X_{s-}|^{p-1} \hat{X}_{s-} \, dK_s + p \int_0^t |X_s|^{p-1} \hat{X}_s \, dB_s \]
\[ + \frac{1}{2} p(p-1) \mathbf{1}_{X_s \neq 0} |X_s|^{p-2} |Z_s|^2 \, ds + L_t(p) + I_t(p) \]
with \( L(p) \in \mathcal{V}^+ \) defined by \( L_t(p) = \lim_{\varepsilon \to 0^+} L_{t}^\varepsilon(p), \ t \in [0, T] \). An elementary computation analogous to that in the proof of [5, Lemma 2.2] shows that in fact \( L_t(p) = L_t(p) \mathbf{1}_{\{p=1\}} \). Therefore putting \( L_t = L_t(1), \ t \in [0, T] \) we get (2.2). Finally, comparing Itô’s formula proved in [30, Section 2] with formula (2.2) with \( p = 1 \) shows that \( L \) is a symmetric local time at zero of the process \( X \). In particular, \( L \) is continuous, and the proof is complete. \( \square \)

3 Existence and uniqueness of solutions of BSDEs

In this section we study existence and uniqueness of solutions of equations of the form
\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dV_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T], \] (3.1)
where \( V \in \mathcal{V} \). In what follows it will be convenient to denote equations of the form (3.1) by BSDE(\( \xi, f + dV \)).

With formula (2.2) at hand, to prove existence and uniqueness of solutions of (3.1) it suffices to repeat step by step, with some obvious changes, the proofs of corresponding results from [5] obtained for multidimensional equations of the form (3.1) with \( V = 0 \). In [5], however, to prove existence of solutions of (3.1) the authors use some important results from other papers proved in case \( V = 0 \). Instead of repeating arguments from all these papers and from [5] we decided to take advantage of the fact that we are concerned with one dimensional equations and provide new proof which is based only on results (or slightly modified results) obtained in [5].

Let \( p \geq 1 \). We will need the following hypotheses.

(H1) \( E |\xi|^p + E(\int_0^T |f(t, 0, 0)| \, dt)^p + E(\int_0^T d|V_s|)^p < \infty \).

(H2) There exists \( \lambda > 0 \) such that \( |f(t, y, z) - f(t, y, z')| \leq \lambda |z - z'| \) for every \( t \in [0, T], y \in \mathbb{R}, z, z' \in \mathbb{R}^d \).

(H3) There exists \( \mu \in \mathbb{R} \) such that \( (f(t, y, z) - f(t, y', z))(y - y') \leq \mu(y - y')^2 \) for every \( t \in [0, T], y \in \mathbb{R}, z, z' \in \mathbb{R}^d \).

(H4) For every \( (t, z) \in [0, T] \times \mathbb{R}^d \) the mapping \( \mathbb{R} \ni y \mapsto f(t, y, z) \) is continuous.

(H5) For every \( r > 0 \) the mapping \( [0, T] \ni t \mapsto \sup_{|y| \leq r} |f(t, y, 0) - f(t, 0, 0)| \) belongs to \( L^1(0, T) \).

(A) There exist \( \mu \in \mathbb{R} \) and \( \lambda \geq 0 \) such that
\[ \hat{y} f(t, y, z) \leq f_t + \mu |y| + \lambda |z|, \]
for every \( (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \), where \( \hat{y} = \mathbf{1}_{\{y \neq 0\}} \frac{y}{|y|} \) and \( f_t \) is a nonnegative progressively measurable process.
(Z) There exist $\alpha \in (0, 1)$, $\gamma \geq 0$ and nonnegative process $g \in L^1(\mathcal{F})$ such that

$$|f(t, y, z) - f(t, y, 0)| \leq \gamma (g_t + |y| + |z|)^\alpha$$

for every $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$.

We begin with proving comparison result for BSDEs of the form (3.1).

**Proposition 3.1.** Let $(Y^i, Z^i)$, $i = 1, 2$, be a solution of BSDE$(\xi^i, f^i + dV^i)$. Assume that $(Y^1 - Y^2)^+ \in \mathcal{D}^q$ for some $q > 1$. If $\xi^1 \leq \xi^2$, $dV^1 \leq dV^2$, $f^1, f^2$ satisfy (H3) and either

$$f^2 \text{ satisfies (H2), } 1_{\{Y^1 > Y^2\}}(f^1(t, Y^1_t, Z^1_t) - f^2(t, Y^1_t, Z^1_t)) \leq 0 \quad (3.2)$$

for a.e. $t \in [0, T]$ or

$$f^1 \text{ satisfies (H2), } 1_{\{Y^1 > Y^2\}}(f^1(t, Y^1_t, Z^1_t) - f^2(t, Y^1_t, Z^1_t)) \leq 0 \quad (3.3)$$

for a.e. $t \in [0, T]$ then $Y^1_t \leq Y^2_t$, $t \in [0, T]$.

**Proof.** Without loss of generality we may assume that $\mu \leq 0$. Assume that (3.2) is satisfied and let $p \in (1, q)$. Then by the Itô-Tanaka formula (see [30, Chapter IV, Section 7]) and Proposition 2.1, for every stopping time $\tau \in \mathcal{T}$ we have

$$|(Y^1_{\tau \land T} - Y^2_{\tau \land T})|^p + \frac{p(p-1)}{2} \int_{\tau \land T}^T 1_{\{Y^1_s \neq Y^2_s\}} |(Y^1_s - Y^2_s)^+|^p - 2|Z^1_s - Z^2_s|^2 ds$$

$$\leq |(Y^1_\tau - Y^2_\tau)^+|^p + p \int_{\tau \land T}^T |(Y^1_s - Y^2_s)^+|^p - 1(f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)) ds$$

$$+ p \int_{\tau \land T}^T |(Y^1_s - Y^2_s)^+|^p - 1 (dV^1_s - dV^2_s)$$

$$- p \int_{\tau \land T}^T |(Y^1_s - Y^2_s)^+|^p - 1 (Z^1_s - Z^2_s) dB_s. \quad (3.4)$$

By the assumptions,

$$\int_{\tau \land T}^T |(Y^1_{s-} - Y^2_{s-})^+|^p - 1 (dV^1_s - dV^2_s) \leq 0.$$

Moreover, using (3.2), (H2), (H3) one can check that for a.e. $t \in [0, T]$,

$$1_{\{Y^1 > Y^2\}}(f^1(t, Y^1_t, Z^1_t) - f^2(t, Y^1_t, Z^1_t)) \leq \lambda 1_{\{Y^1 > Y^2\}}|Z^1_t - Z^2_t|.$$
Therefore from (3.4) it follows that for a.e. $t \in [0, T]$, 

\[
|Y_{t \wedge \tau}^1 - Y_{t \wedge \tau}^2|^p + \frac{p(p-1)}{2} \int_{t \wedge \tau}^\tau 1_{\{Y_s^1 \neq Y_s^2\}} |(Y_s^1 - Y_s^2)^+|^p - 1 |Z_s^1 - Z_s^2|^2 ds \\
\leq |(Y_{t \wedge \tau}^1 - Y_{t \wedge \tau}^2)|^p + p\lambda \int_{t \wedge \tau}^\tau |(Y_s^1 - Y_s^2)^+|^p ds \\
\leq |(Y_{t \wedge \tau}^1 - Y_{t \wedge \tau}^2)|^p + \frac{p\lambda^2}{p-1} \int_{t \wedge \tau}^\tau |(Y_s^1 - Y_s^2)^+|^p ds \\
+ \frac{p(p-1)}{4} \int_{t \wedge \tau}^\tau |(Y_s^1 - Y_s^2)^+|^p - 1 |Z_s^1 - Z_s^2|^2 ds \\
- p \int_{t \wedge \tau}^\tau |(Y_s^1 - Y_s^2)^+|^p - 1 |Z_s^1 - Z_s^2|^2 dB_s.
\]  

(3.5)

Set $\tau_k = \inf\{t \in [0, T]; \int_0^t |(Y_s^1 - Y_s^2)^+|^2(p-1) |Z_s^1 - Z_s^2|^2 ds \geq k\} \land T$. Then from (3.5) with $\tau = \tau_k$ we obtain

\[
E|(Y_t^1 - Y_t^2)^+|^p \leq E|(Y_{\tau_k}^1 - Y_{\tau_k}^2)^+|^p + \frac{p\lambda^2}{p-1} E \int_{t \wedge \tau_k}^\tau |(Y_s^1 - Y_s^2)^+|^p ds.
\]

Since $(Y^1 - Y^2)^+ \in D^q$ for some $q > 1$, the process $(Y^1 - Y^2)^+$ is of class (D). Therefore letting $k \to +\infty$ in the above inequality and using the fact that $\xi^1 \leq \xi^2$ we get

\[
E|(Y_t^1 - Y_t^2)^+|^p \leq \frac{p\lambda^2}{p-1} E \int_t^T |(Y_s^1 - Y_s^2)^+|^p ds, \quad t \in [0, T],
\]

so the desired result follows by Gronwall’s lemma. Since the proof in case (3.3) is satisfied is analogous, we omit it. \hfill \Box

**Corollary 3.2.** Assume (Z). Let $(Y^i, Z^i)$, $i = 1, 2$, be a solution of BSDE($\xi^i, f^i + dB^i$) such that $(Y^i, Z^i) \in L^q(F) \otimes L^q(F)$ for some $q > \alpha$. If $(Y^1 - Y^2)^+$ is of class (D), $\xi^1 \leq \xi^2$, $dV^1 \leq dV^2$, $f^1, f^2$ satisfy (H2) and (3.2) or (3.3) is satisfied then $Y_t^1 \leq Y_t^2$, $t \in [0, T]$.

**Proof.** We only consider the case where (3.2) is satisfied. As usual, without loss of generality we may assume that $\mu \leq 0$. Due to Proposition 3.1 it suffices to show that $(Y^1 - Y^2)^+ \in D^p$ for some $p > 1$. By the Itô-Tanaka formula and the inequality $dV^1 \leq dV^2$, for every stopping time $\tau \in \mathcal{T}$,

\[
(Y_{t \wedge \tau}^1 - Y_{t \wedge \tau}^2)^+ \leq (Y_t^1 - Y_t^2)^+ + \int_{t \wedge \tau}^\tau 1_{\{Y_s^1 > Y_s^2\}} (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) ds \\
- \int_{t \wedge \tau}^\tau 1_{\{Y_s^1 > Y_s^2\}} (Z_s^1 - Z_s^2) dB_s.
\]  

(3.6)

Write

\[
I_t = 1_{\{Y_t^1 > Y_t^2\}} (f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2)).
\]  

(3.7)
By (3.2),
\[ I_t = 1_{\{Y^1_t > Y^2_t\}} \{f^1(t, Y^1_t, Z^1_t) - f^2(t, Y^1_t, Z^1_t) + f^2(t, Y^1_t, Z^1_t) - f^2(t, Y^2_t, Z^2_t)\} \]
\[ \leq 1_{\{Y^1_t > Y^2_t\}}(f^2(t, Y^1_t, Z^1_t) - f^2(t, Y^2_t, Z^2_t)) \]
and by monotonicity of $f^2$ with respect to $y$,
\[ 1_{\{Y^1_t > Y^2_t\}}(f^2(t, Y^1_t, Z^1_t) - f^2(s, Y^2_t, Z^2_t)) = 1_{\{Y^1_t > Y^2_t\}}(f^2(t, Y^1_t, Z^1_t) - f^2(t, Y^1_t, 0)) \
+ 1_{\{Y^1_t > Y^2_t\}}(f^2(t, Y^1_t, 0) - f^2(t, Y^2_t, 0)) + 1_{\{Y^1_t > Y^2_t\}}(f^2(t, Y^2_t, 0) - f^2(s, Y^2_t, Z^2_t)) \]
\[ \leq 1_{\{Y^1_t > Y^2_t\}}(f^2(t, Y^1_t, Z^1_t) - f^2(t, Y_1^1, 0)) + 1_{\{Y^1_t > Y^2_t\}}(f^2(t, Y^2_t, 0) - f^2(t, Y^2_t, Z^2_t)) \]
Using assumption (Z) we conclude from the above inequality that
\[ I_t \leq 2\gamma(g_t + |Y^1_t| + |Z^1_t| + |Y^2_t| + |Z^2_t|)^\alpha \]
for a.e. $t \in [0, T]$. Therefore taking the conditional expectation with respect to $\mathcal{F}_t$ of both sides of (3.6) with $\tau$ replaced by $\tau_k = \inf\{t \in [0, T]; \int_0^t |Z^1_s - Z^2_s|^2 ds \geq k\} \land T$, letting $k \to +\infty$ and using the fact that $(Y^1 - Y^2)^+$ is of class (D) we obtain
\[ (Y^1_t - Y^2)^+ \leq 2\gamma E^{\mathcal{F}_t}\left\{ \int_0^T (g_t + |Y^1_s| + |Z^1_s| + |Y^2_s| + |Z^2_s|)^\alpha \, ds \right\} \]
Using now the assumptions of the corollary and applying Doob’s inequality gives the desired result.

**Remark 3.3.** Observe that if $f$ does not depend on $z$ then in Corollary 3.2 assumption (Z) and the assumptions that $(Y^1, Z^1) \in L^q(\mathcal{F}) \otimes L^q(\mathcal{F})$ for some $q > \alpha$ are superfluous. This follows from the fact that $I_t$ defined by (3.7) is less or equal to zero if $f$ does not depend on $z$.

The proofs of the following lemma and proposition are analogous to those of Lemma 3.1 and Proposition 3.2 in [5], the only difference being in the fact that we use the Itô’s formula proved in Proposition 2.1 instead of Itô’s formula proved in [5].

**Lemma 3.4.** Let assumption (A) hold and let $(Y, Z)$ be a solution of BSDE($\xi, f + dV$) with $f, V$ such that
\[ E\left( \int_0^T f_s \, ds \right)^p + E|V|^p_T < \infty \quad (3.8) \]
for some $p > 0$. If $Y \in D^p$ then $Z \in M^p$ and there exists $C$ depending only on $p$ such that for every $a \geq \mu + \lambda^2$,
\[ E\int_0^T e^{2\alpha s} |Z_s|^2 \, ds^{p/2} \leq CE \left( \sup_{t \leq T} e^{\lambda t} |Y_t|^p + \left( \int_0^T e^{\alpha s} f_s \, ds \right)^p + \left( \int_0^T e^{\alpha s} d|V|_s \right)^p \right) \]

**Proposition 3.5.** Let assumption (A) hold and let $(Y, Z)$ be a solution of BSDE($\xi, f + dV$) with $f, V$ satisfying (3.8) for some $p > 1$. If $Y \in D^p$ then there exists $C$ depending only on $p$ such that for every $a \geq \mu + \lambda^2/[1 \land (p - 1)]$,
\[ E\sup_{t \leq T} e^{\lambda t} |Y_t|^p + E\left( \int_0^T e^{2\alpha s} |Z_s|^2 \, ds \right)^{p/2} \]
\[ \leq CE \left( e^{\lambda T} |\xi|^p + \left( \int_0^T e^{\alpha s} f_s \, ds \right)^p + \left( \int_0^T e^{\alpha s} d|V|_s \right)^p \right) . \]
Proposition 3.6. (i) Let \( p > 1 \) and let \((Y, Z) \in \mathcal{D}^p \otimes M^p\) be a solution of \( \text{BSDE}(\xi, f + dV) \) with \( \xi, f, V \) satisfying (H1)–(H3). Then there exists \( C \) depending only on \( \mu^+, \lambda, T, p \) such that

\[
E \left( \int_0^T |f(s, Y_s, Z_s)|^p ds \right) 
\leq CE \left( |Y_T|^p + \left( \int_0^T |Z_s|^2 ds \right)^{p/2} + \left( \int_0^T |f(s, 0, 0)| ds \right)^p + |V_T|^p \right).
\]

(ii) Let \( p = 1 \) and let \((Y, Z)\) such that \((Y, Z) \in \mathcal{D}^q \otimes M^q\) for \( q \in (0, 1) \) and \( Y \) is of class (D) be a solution of \( \text{BSDE}(\xi, f + dV) \) with \( \xi, f, V \) satisfying (H1)–(H3), \( (Z) \). Then there exists \( C \) depending only on \( \mu^+, \lambda, T \) such that

\[
E \int_0^T |f(s, Y_s, Z_s)| ds \leq CE \left( \|Y\|_1 + \gamma \int_0^T (g_t + |Y_s| + |Z_s|)^\alpha ds 
+ \int_0^T |f(s, 0, 0)| ds + |V_T| \right).
\]

Proof. By Itô’s formula,

\[
-\int_0^T \dot{Y}_s f(s, Y_s, Z_s) ds \leq |Y_t| - |Y_0| + \int_0^T \dot{Y}_s dV_s - \int_0^T \dot{Y}_s dB_s. \tag{3.9}
\]

Since by (H3), \(-\dot{Y}_s f(s, Y_s, 0) - \mu Y_s \geq \dot{Y}_s f(s, 0, 0)\), we have

\[-\dot{Y}_s f(s, Y_s, Z_s) + \dot{Y}_s (f(s, Y_s, Z_s) - f(s, Y_s, 0)) - \dot{Y}_s f(s, 0, 0) + \mu \dot{Y}_s Y_s \geq 0.\]

Hence, by (3.9),

\[
\int_0^t |f(s, Y_s, Z_s)| ds \leq |Y_t| + |V_t| - \int_0^t \dot{Y}_s dV_s + 2\int_0^t |f(s, Y_s, Z_s) - f(s, Y_s, 0)| ds 
+ 2\int_0^t |f(s, 0, 0)| ds + 2\mu \int_0^t |Y_s| ds 
\leq |Y_t| + |V_t| - \int_0^t \dot{Y}_s dV_s + 2\int_0^t |Z_s| ds + 2\int_0^t |f(s, 0, 0)| ds 
+ 2\mu \int_0^t |Y_s| ds,
\]

from which one can easily get (i). To prove (ii) we use assumption (Z) to estimate the integral involving \( |f(s, Y_s, Z_s) - f(s, Y_s, 0)|. \) We then get

\[
\int_0^t |f(s, Y_s, Z_s)| ds \leq |Y_t| + |V_t| - \int_0^t \dot{Y}_s dV_s + 2\gamma \int_0^t (g_t + |Y_s| + |Z_s|)^\alpha ds 
+ 2\int_0^t |f(s, 0, 0)| ds + 2\mu \int_0^t |Y_s| ds,
\]

from which (ii) immediately follows. \( \square \)
Theorem 3.7. Let $p > 1$. Under assumptions (H2), (H3) there exists at most one solution $(Y, Z) \in \mathcal{D}^p \otimes M^p$ of BSDE$(\xi, f + dV)$.

Proof. Follows from Proposition 3.1. \hfill \Box

Theorem 3.8. Let $p > 1$ and (H1)–(H5) hold. Then there exists a unique solution $(Y, Z) \in \mathcal{D}^p \otimes M^p$ of BSDE$(\xi, f + dV)$.

Proof. Without loss of generality we may assume that $\mu \leq 0$. Let us assume that $f$ is bounded. By the representation property of the Brownian filtration there exists a unique process $(\tilde{Y}, \tilde{Z}) \in \mathcal{D}^p \otimes M^p$ such that

$$
\tilde{Y}_t = \int_t^T dV_s - \int_t^T \tilde{Z}_s dB_s, \quad t \in [0, T].
$$

Put $\tilde{f}(t, y, z) = f(t, y + \tilde{Y}_t, z + \tilde{Z}_t)$ and observe that the data $(\xi, \tilde{f})$ satisfy assumptions (H1)–(H5). Therefore by [5, Theorem 4.2] there exists a unique solution $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^p \otimes M^p$ of the BSDE

$$
\tilde{Y}_t = \xi + \int_t^T \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s) \, ds - \int_t^T \tilde{Z}_s \, dB_s, \quad t \in [0, T].
$$

Clearly the pair $(\tilde{Y} + \tilde{Y}, \tilde{Z} + \tilde{Z})$ is a unique solution of BSDE$(\xi, f + dV)$.

Now suppose that $f$ is bounded from below. Write $\bar{f}(s, Y_s, Z_s) = f(s, Y_s + \bar{Y}_s, Z_s + \bar{Z}_s)$ and observe that the data $(\xi, \bar{f})$ satisfy assumptions (H1)–(H5). Therefore by [5, Theorem 4.2] there exists a unique solution $(\bar{Y}, \bar{Z}) \in \mathcal{S}^p \otimes M^p$ of the BSDE

$$
\bar{Y}_t = \xi + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s) \, ds - \int_t^T \bar{Z}_s \, dB_s, \quad t \in [0, T].
$$

Moreover, by Proposition 3.5, there exists $C > 0$ not depending on $n$ such that

$$
E \sup_{0 \leq t \leq T} |Y^n_t|^p + E(\int_0^T |Z^n_s|^2 \, ds)^{p/2} \leq C. \quad (3.11)
$$

Hence, by Proposition 3.6,

$$
\sup_{n \in \mathbb{N}} E\left(\int_0^T |f_n(s, Y^n_s, Z^n_s)| \, ds\right)^p < \infty. \quad (3.12)
$$

By Itô’s formula,

$$
|Y^n_t - Y^n_t|^p + \frac{1}{2}p(p-1) \int_t^T |Y^n_s - Y^n_t|^p |Z^n_s - Z^n_t|^2 \, ds
$$

$$
= p \int_t^T (f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^n_s, Z^n_s)) |Y^n_s - Y^m_t|^p \, ds
$$

$$
+ p \int_t^T (Z^n_s - Z^n_t) |Y^n_s - Y^m_t|^p \, dB_s, \quad t \in [0, T]. \quad (3.13)
$$
By the Burkholder-Davis-Gundy inequality, (H2) and Hölder’s inequality,

\[
E \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^p \leq \left( E \left( \int_0^T |Z^n_s|^2 \, ds \right)^{p/2} + E \left( \int_0^T |Z^m_s|^2 \, ds \right)^{p/2} \right)^{1/p}
\times \left( E \left( \int_0^T |Y^n_s - Y^m_s|^{2(p-1)} \, ds \right)^{p/(p-1)} \right)^{(p-1)/p}
+ \left( E \left( \int_0^T |f_n(s, Y^n_s, 0) - f_m(s, Y^m_s, 0)| \, ds \right)^p \right)^{(p-1)/p}
\times \left( E \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^p \right)^{(p-1)/p}.
\]  

(3.14)

By (3.10) and (3.11),

\[
\lim_{n,m \to +\infty} E \left( \int_0^T |Y^n_s - Y^m_s|^{2(p-1)} \, ds \right)^{p/(p-1)} = 0.
\]  

(3.15)

Therefore by (3.11) the first term on the right-hand side of (3.14) converges to zero. By monotonicity of \( f_n, f \) with respect to \( y \) and monotonicity of the sequence \( \{Y^n\} \),

\[
f_1(t, Y^1_t, 0) \leq f_n(t, Y^n_t, 0) \leq f(t, Y^1_t, 0).
\]

Therefore from (H4), (H5), (3.10), (3.11), (3.12) it follows that

\[
E \left( \int_0^T |f_n(s, Y^n_s, 0) - f_m(s, Y^m_s, 0)| \, ds \right)^{p'} \to 0,
\]  

(3.16)

for every \( p' < p \). Without loss of generality we may assume that (3.16) holds true for \( p \) in place of \( p' \), which when combined with (3.11) implies convergence to zero of the second term on the right-hand side of inequality (3.14). Consequently, \( Y \in D^p \) and

\[
E \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^p \to 0.
\]  

(3.17)

Since

\[
E \left( \int_0^T (f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s))(Y^n_s - Y^m_s) \, ds \right)^{p/2}
\leq \left( E \left( \int_0^T |f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s)| \, ds \right)^p \right)^{1/2} \left( E \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^p \right)^{1/2},
\]

we conclude from (3.12), (3.13) with \( p = 2 \) and (3.17) that

\[
\lim_{n,m \to +\infty} E \left( \int_0^T |Z^n_s - Z^m_s|^2 \, ds \right)^{p/2} = 0.
\]  

(3.18)

Therefore there exists a process \( Z \in M^p \) such that (3.18) holds with \( Z \) in place of \( Z^m \). From this and (3.16) we conclude that

\[
\lim_{n \to +\infty} E \left( \int_0^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| \, ds \right)^p = 0,
\]
which together with (3.17), (3.18) shows that \((Y, Z)\) is a solution of BSDE\((\xi, f + dV)\).

Finally, in the general case, we approximate \(f\) by the sequence \(\{f_n\}\), where \(f_n = f \vee (-n)\), \(n \in \mathbb{N}\). By what has already been proved for each \(n\) there exists a unique solution \((Y^n, Z^n)\) in \(D^p \otimes M^p\) of BSDE\((\xi, f_n + dV)\). Repeating arguments from the proof of the previous step shows that \((Y^n, Z^n)\) converges in \(S^p \otimes M^p\) to the unique solution of BSDE\((\xi, f + dV)\).

\[\square\]

**Theorem 3.9.** If \(p = 1\) and (H2), (H3), (Z) are satisfied then there exists at most one solution of BSDE\((\xi, f + dV)\) such that \(Y\) is of class (D) and \(Z \in \bigcup_{p>1} M^p\).

**Proof.** Follows from Corollary 3.2. \[\square\]

**Proposition 3.10.** Assume that (H1)–(H5) hold with \(p = 1\) and \(f\) does not depend on \(z\). Then there exists a solution \((Y, Z)\) of BSDE\((\xi, f + dV)\) such that \(Y\) is of class (D) and \((Y, Z) \in \bigcap_{p<1} D^p \otimes M^p\).

**Proof.** Standard arguments show that without loss of generality we may assume that \(\mu \leq 0\). Set

\[\xi^n = T_n(\xi), \quad f_n(t, y) = f(t, y) - f(t, 0) + T_n(f(t, 0)), \quad V^n_t = \int_0^t 1_{\{|V|^s \leq n\}} dV_s.\]

By Theorem 3.8, for every \(n \in \mathbb{N}\) there exists a solution \((Y^n, Z^n)\) in \(D^2 \otimes M^2\) of BSDE\((\xi^n, f_n + dV^n)\). Let \(m \geq n\). Write \(\delta Y = Y^m - Y^n\), \(\delta Z = Z^m - Z^n\), \(\delta \xi = \xi^m - \xi^n\) and

\[\tau_k = \inf\{t \in [0, T]; \int_0^t |\delta Z_s|^2 ds > k\} \wedge T.\]

By the Itô-Tanaka formula,

\[|\delta Y_{t \wedge \tau_k}| \leq |\delta Y_{\tau_k}| + \int_{\tau_k \wedge t}^{\tau_k} \hat{\text{sgn}}(\delta Y_s)(f_m(s, Y^m_s) - f_n(s, Y^n_s))ds + \int_{\tau_k \wedge t}^{\tau_k} \hat{\text{sgn}}(\delta Y_s) d(V^m_s - V^n_s) + \int_{\tau_k \wedge t}^{\tau_k} \hat{\text{sgn}}(\delta Y_s)\delta Z_s dB_s \leq |\delta Y_{\tau_k}| + \int_{\tau_k \wedge t}^{\tau_k} |f_m(s, Y^m_s) - f_n(s, Y^n_s)|ds + \int_{\tau_k \wedge t}^{\tau_k} \hat{\text{sgn}}(\delta Y_s)\delta Z_s dB_s + \int_{\tau_k \wedge t}^{\tau_k} d|V^m_s - V^n_s| + \int_{\tau_k \wedge t}^{\tau_k} \hat{\text{sgn}}(\delta Y_s)\delta Z_s dB_s, \quad t \in [0, T],\]

the last inequality being a consequence of monotonicity of \(f_n\) with respect to \(y\). Conditioning with respect to \(\mathcal{F}_t\), using the fact that \(\delta Y\) is of class (D) and the definitions of \(f_n\), \(\xi^n\), \(V^n\) we conclude from the above inequality that

\[|\delta Y_t| \leq E^{\mathcal{F}_t}[|\xi| 1_{\{|\xi| > n\}} + \int_0^T |f(s, 0)| 1_{\{|f(s, 0)| > n\}} ds + \int_0^T 1_{\{|V|^s > n\}} d|V|^s).\]

Now repeating step by step the arguments following Eq. (12) in the proof of [5, Proposition 6.4] we get the existence result. \[\square\]
Theorem 3.11. Assume that $p = 1$ and (H1)-(H5), (Z) are satisfied. Then there exists a solution $(Y, Z)$ of BSDE$(\xi, f + dV)$ such that $Y$ is of class (D) and $(Y, Z) \in \bigcap_{\beta < 1} \mathcal{D}^\beta \otimes \mathcal{M}^\beta$.

Proof. Without loss of generality we may assume that $\mu \leq 0$. Let $(Y^0, Z^0) = (0, 0)$. By Proposition 3.10 we can define recursively the sequence $\{(Y^n, Z^n)\}$ by putting

$$Y_{t+1}^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) \, ds - \int_t^T dV_s - \int_t^T Z_{s+1}^n \, dB_s, \quad t \in [0, T]. \quad (3.19)$$

Since

$$Y_{t+1}^n - Y_t^n = \int_t^T (f(s, Y_{s+1}^n, Z_s^n) - f(s, Y_s^n, Z_{s+1}^n)) \, ds - \int_t^T (Z_{s+1}^n - Z_s^n) \, dB_s$$

for $t \in [0, T]$, repeating step by step the proof of Theorem 6.3 in [5] shows that $(Y^n, Z^n)$ converges to some process $(Y, Z)$ belonging to $\mathcal{D}^q \otimes \mathcal{M}^q$ for $q \in (0, 1)$ and that $Y^n \to Y$ in the norm $\| \cdot \|_1$. Therefore passing to the limit in (3.19) in ucp topology we see that $(Y, Z)$ is a solution of BSDE$(\xi, f + dV)$. \qed

4 Supersolutions of BSDEs

In this section we investigate supersolutions of BSDEs. In particular we provide a priori estimates for supersolutions, Snell envelope representation result for minimal supersolutions and explicit formula for its jumps. Moreover, we prove some useful technical lemmas which generalize known results on monotone convergence of solutions of BSDEs and regularity properties of monotone limits of supersolutions. The results on supersolutions play a pivotal role in the study of reflected RBSDEs because one can regard solution of reflected BSDE with one barrier as a minimal supersolution of some BSDE and view solution of reflected BSDE with two barriers as a minimal supersolution of some BSDE of the form (3.1).

Let us fix a process $V \in \mathcal{V}$.

Definition. We say that a pair of processes $(Y, Z)$ is a supersolution (resp. subsolution) of BSDE$(\xi, f + dV)$ if

a) $Z \in M, \ t \mapsto f(t, Y_t, Z_t) \in L^1(0, T)$,

b) There exists a process $K \in \mathcal{V}^+$ (resp. $K \in \mathcal{V}^-$) such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T].$$

Suppose that $(Y, Z)$ is a supersolution of some BSDE with data $(\xi, f, V)$. In the rest of this section $K$ stands for the increasing càdlàg process such that $K_0 = 0$ and the above equation is satisfied.

The following Lemma 4.1, Lemma 4.2 and Proposition 4.3 were proved in [19] (Lemma 3.1, Lemma 3.2 and Proposition 3.4, respectively) in the case where the measure $dV$ is absolutely continuous with respect to the Lebesgue measure. Using the Itô-Tanaka formula proved in Proposition 2.1 one can prove these results for general $V$ by the same method as in [19].
Lemma 4.1. Let \((Y, Z)\) be a supersolution of BSDE\((\xi, f + dV)\). Assume that (H3) is satisfied, there exists a progressively measurable process \(X\) such that \(X_t \geq Y_t, t \in [0, T]\) and the mappings \([0, T] \ni t \rightarrow X_t^+, [0, T] \ni t \rightarrow f^{-}(t, X_t, 0)\) belong to \(L^1(0, T)\), \(P\)-a.s.

(i) If (H2) is satisfied then for every \(\tau \in \mathcal{T}\) and \(a \geq \mu, \int_0^\tau e^{at}dK_t \leq |e^{\alpha T}Y_\tau| + |Y_0| + \int_0^\tau e^{as}Z_adB_s + \lambda \int_0^\tau e^{as}|Z_s|ds + \int_0^\tau e^{as}(f^-(s, X_s, 0)ds + dV^-) + \int_0^\tau \alpha^+ e^{as}X_s^+ds.

(ii) If (Z) is satisfied then for every \(\tau \in \mathcal{T}\) and \(a \geq \mu, \int_0^\tau e^{at}dK_t \leq |e^{\alpha T}Y_\tau| + |Y_0| + \int_0^\tau e^{as}Z_adB_s + \gamma \int_0^\tau e^{as}(g_s + |Y_s| + |Z_s|)^\alpha ds + \int_0^\tau e^{as}(f^-(s, X_s, 0)ds + dV^-) + \int_0^\tau \alpha^+ e^{as}X_s^+ds.

Lemma 4.2. Let \((Y, Z)\) be a supersolution of BSDE\((\xi, f + dV)\). If (A) is satisfied and for some \(p > 0, Y \in \mathcal{D}^p\), (H1) is satisfied and

\[E(\int_0^T X_s^+ds)^p + E\left(\int_0^T f^-(s, X_s, 0)ds\right)^p < \infty\]

for some progressively measurable process \(X\) such that \(X_t \geq Y_t, t \in [0, T]\), then \(Z \in \mathcal{M}^p\) and there exists \(C\) depending only on \(\lambda, p, T\) such that for every \(a \geq \mu + \lambda^2\),

\[E\left(\int_0^T e^{2as}|Z_s|^2ds\right)^{p/2} + \left(\int_0^T e^{as}dK_s\right)^p \leq CE\left(\sup_{t \leq T} e^{apt}|Y_t|^p + \left(\int_0^T e^{as}|f(s, 0, 0)|ds\right)^p + \left(\int_0^T e^{as}ds\right)^p + \left(\int_0^T e^{as}f^-(s, X_s, 0)ds\right)^p + \left(\int_0^T a^+ e^{as}X_s^+ds\right)^p\right).

Proposition 4.3. Assume (A) and let \((Y, Z)\) be a supersolution of BSDE\((\xi, f + dV)\). If for some \(p > 1, Y \in \mathcal{D}^p\), (H1) is satisfied and

\[E(\int_0^T f^-(s, X_s, 0)ds)^p < \infty\]  \hspace{1cm} (4.1)

for some \(X \in \mathcal{D}^p\) such that \(X_t \geq Y_t, t \in [0, T]\), then there exists \(C\) depending only on \(\lambda, p, T\) such that for every \(a \geq \mu + \lambda^2/[1 \wedge (p - 1)]\) and every \(\tau \in \mathcal{T}\),

\[E\sup_{t \leq \tau} e^{apt}|Y_t|^p + E(\int_0^\tau e^{2as}|Z_s|^2ds)^{p/2} + E\left(\int_0^\tau e^{as}dK_s\right)^p \leq CE\left(e^{apt}|Y_\tau|^p + \left(\int_0^\tau e^{as}|f(s, 0, 0)|ds\right)^p + \left(\int_0^\tau e^{as}d|V_s|\right)^p + \sup_{t \leq \tau} e^{at}X_t^+ |^p + \left(\int_0^\tau e^{as}f^-(s, X_s, 0)ds\right)^p + \left(\int_0^\tau a^+ e^{as}X_s^+ds\right)^p\right).

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Assume additionally that \( f \) does not depend on \( z \). Then if (H1) and (4.1) are satisfied with \( p = 1 \) and \( X, Y \) are of class \((D)\) then for every \( a \geq \mu\),

\[
\|e^{ao}Y\|_1 + E \int_0^T e^{ao} dK_s \leq E\left(e^{aT}\|\xi\| + \int_0^T e^{ao} f(s,0) ds \right) + \int_0^T e^{ao} d|V|_s + \int_0^T e^{ao} f(s,X_s) ds + \int_0^T a^+ e^{ao} X_s^+ ds + \|e^{a^+}X^+\|_1.
\]

**Lemma 4.4.** Let \((Y, Z)\) be a supersolution of BSDE\((\xi, f + dV)\). Assume that

(a) \( f \) does not depend on \( y, z \), (H1) with \( p = 1 \) is satisfied.

(b) \( Y^n \not\geq Y_t, t \in [0,T], Y^n, Y \) are of class \((D), Y_t \geq L_t \) for a.e. \( t \in [0,T], dV^n \leq dV, \)

where

\[
Y^n_t = \xi + \int_t^T f(s) ds + \int_t^T dV^n_s + \int_t^T n(Y^n_s - L_s)^- ds - \int_t^T Z^n_s dB_s, \quad t \in [0,T].
\]

Then \((Y, Z)\) is the smallest supersolution of BSDE\((\xi, f + dV)\) such that \( L_t \leq Y_t \) for a.e. \( t \in [0, T] \) and \( Y \) is of class \((D)\).

**Proof.** Let \((\hat{Y}, \hat{Z})\) be a supersolution of BSDE\((\xi, f + dV)\) such that \( \hat{Y} \) is of class \((D)\) and \( L_t \leq \hat{Y}_t \) for a.e. \( t \in [0,T] \). Then there exists \( \hat{K} \in \mathcal{V}^{+,1} \) such that

\[
\hat{Y}_t = \xi + \int_t^T f(s) ds + \int_t^T dV_s + \int_t^T d\hat{K}_s - \int_t^T \hat{Z}_s dB_s, \quad t \in [0,T].
\]

Since \( \hat{Y}_t \geq L_t \) for a.e. \( t \in [0,T] \), we have

\[
\hat{Y}_t = \xi + \int_t^T f(s) ds + \int_t^T dV_s + \int_t^T d\hat{K}_s + \int_t^T n(\hat{Y}_s - L_s)^- ds - \int_t^T \hat{Z}_s dB_s, \quad t \in [0,T].
\]

By Corollary 3.2, \( Y^n_t \leq \hat{Y}_t, t \in [0,T] \), and consequently \( Y_t \leq \hat{Y}_t, t \in [0,T]. \)

**Corollary 4.5.** Let \( Y \) be the process of Lemma 4.4 and let assumptions of Lemma 4.4 hold. Then for every \( \hat{L} \in \mathcal{D} \) such that \( L_t \leq \hat{L}_t \leq Y_t \) for a.e. \( t \in [0,T], \)

\[
Y_t = \text{ess sup}_{\tau \in T_t} E\left(\int_t^\tau f(s) ds + \int_t^\tau dV_s + \hat{L}_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_t\right), \quad t \in [0,T].
\]

**Proof.** It suffices to observe that from Lemma 4.4 it follows that for every \( \hat{L} \in \mathcal{D} \) such that \( L_t \leq \hat{L}_t \leq Y_t \) for a.e. \( t \in [0,T] \) the process \( \hat{Y}_t \equiv Y_t + \int_0^t f(s) ds + \int_0^t dV_s \) is the smallest supermartingale majorizing \( \hat{L}_t \equiv \hat{L}_t + \int_0^t f(s) ds + \int_0^t dV_s \) such that \( \hat{Y}_T = \xi + \int_0^T f(s) ds + \int_0^T dV_s. \)

**Lemma 4.6.** Assume that (H1) holds with \( p = 1 \), \( L \in \mathcal{D} \) is of class \((D)\) and \( Y \in \mathcal{D} \) is of the form

\[
Y_t = \text{ess sup}_{\tau \in T_t} E\left(\int_t^\tau f(s) ds + \int_t^\tau dV_s + L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_t\right), \quad t \in [0,T].
\]

Then

\[
Y_{t-} = L_{t-} \vee (Y_t + \Delta V_t), \quad t \in (0,T].
\]
Proof. Let us fix \( t \in (0, T] \). By the properties of the Snell envelope, for every \( s \in [0, t) \),

\[
Y_s = \text{ess sup}_{s \leq \tau \leq t} E\left( \int_s^\tau f(s) \, ds + \int_s^\tau dV_r + L_\tau 1_{\{\tau < t\}} + Y_t 1_{\{\tau = t\} |\mathcal{F}_s} \right).
\]

Letting \( s \to t^- \) we get the desired result. \( \square \)

**Lemma 4.7.** Let \( Y \) be a nonnegative supermartingale of class (D). Then there exists a stationary sequence \( \{\tau_k\} \subseteq \mathcal{T} \) of stopping times such that \( Y_{\tau_k} \leq Y_0 \lor k, k \in \mathbb{N} \).

**Proof.** Since \( Y \) is of class (D), there exists \( K \in \mathcal{V}^{+1} \) such that

\[
Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad t \in [0, T].
\]

Let \( k, l > 0 \) and let \( Y_{k,l}^I = Y_T \land k, Y_{l}^I = K_T \land l, t \in [0, T] \). By Theorem 3.8, there exists a solution \((Y^I, \bar{Z}^I) \in \mathcal{D}^2 \otimes \mathcal{M}^2\) of BSDE\((Y^I_{T}, dK^I)\). One can check that \( Y_{k,l}^I \uparrow Y_t^I, t \in [0, T] \). Since we can regard \((Y^I, \bar{Z}^I)\) as a solution of RBSDE\((Y^I_{T}, 0, Y^I)\) (for the definition of the last equation see Section 5), it follows from [27, Theorem 4.2] that \( Y_{k,l} \uparrow Y_t^I, \quad t \in [0, T], \) where \((Y_{k,l}, Z_{k,l}) \in \mathcal{S}^2 \otimes \mathcal{M}^2\) is a solution of the BSDE

\[
\bar{Y}_{k,l}^I = Y_T^I + \int_t^T n(Y_{k,l}^I - Y^I_t) \, ds - \int_t^T \bar{Z}_{k,l}^I \, dB_s, \quad t \in [0, T].
\]

Put \( \tau_k = \inf\{t \in [0, T], Y_t > k\} \land T \). The sequence \( \{\tau_k\} \) is stationary. Moreover, \( \bar{Y}_{k,l}^I \uparrow Y_{\tau_k}^I \leq k \) on \( \{\tau_k > 0\} \). Hence \( \bar{Y}_{k,l}^I \leq k \lor Y_0 \), so letting \( n \to +\infty \) and then \( l \to +\infty \) we get the desired result. \( \square \)

**Lemma 4.8.** If \( Y \in \mathcal{V}^1 + \mathcal{M}^{loc} \) is of class (D) then there exist a stationary sequence \( \{\tau_k\} \subseteq \mathcal{T} \) and a sequence of constants \( \{c_k\} \subseteq \mathbb{R}, c_k = c(k, Y_0) \), such that \( Y_{\tau_k}^* \leq c_k \) for \( k \in \mathbb{N} \).

**Proof.** By the representation property of Brownian filtration and the assumptions of the lemma there exists \( Z \in \mathcal{M}, V \in \mathcal{V}^1 \) such that

\[
Y_t = Y_T + \int_t^T dV_s - \int_t^T Z_s dB_s, \quad t \in [0, T]
\]

and \( Y_T \in \mathcal{L}^1 (\mathcal{F}_T) \). By Proposition 3.10 there exist unique solutions \((Y^1, Z^1), (Y^2, Z^2) \in \bigcap_{q<1} \mathcal{D}^q \otimes \mathcal{M}^q\) of BSDE\((Y^1_{T}, dV^+)\) and BSDE\((Y^2_{T}, dV^-)\), respectively, such that \( Y^1, Y^2 \) are of class (D). Since \( Y^1, Y^2 \) are nonnegative supermartingales, it follows from Lemma 4.7 that there exist a stationary sequence \( \{\tau_k\} \subseteq \mathcal{T} \) and a sequence \( \{b_k\} \subseteq \mathbb{R} \) such that \( Y_{\tau_k}^{*,*} + Y_{\tau_k}^{*,*} \leq b_k \). Therefore the result follows from the fact that \( Y = Y^1 - Y^2 \). \( \square \)

**Lemma 4.9.** Let \( p \geq 1 \) and let \((Y, Z)\) be a supersolution of BSDE\((\xi, f + dV)\). Assume that

(a) \( \xi, f, L, V \) satisfy assumptions (H1)–(H5), \( L^+ \in \mathcal{L}^{\infty,p}(\mathcal{F}), (Y, Z) \in \mathcal{D}^p \otimes \mathcal{M}^p, E(\int_0^T |f(s, Y_s, Z_s)| \, ds)^p < \infty \) in case \( p > 1 \) or, in case \( p = 1 \), \( \xi, f, L, V \) satisfy (H1)–(H5) and (Z), \( L^+, Y \) are of class (D), \( (Y, Z) \in \mathcal{D}^q \otimes \mathcal{M}^q \) for every \( q \in (0, 1) \) and \( E\int_0^T |f(s, Y_s, Z_s)| \, ds < \infty \).
(b) $V^n \in \mathcal{V}, dV^n \leq dV, V^n \not\geq V_t, t \in [0, T], |V^n|_T \leq |V|_T, Y^n \in \mathcal{D}^p$ (in case $p = 1$, $Y^n \in \mathcal{D}^q, q \in (0, 1)$, $Y^n$ is of class (D)) and $Y^n_t \not\geq Y_t, t \in [0, T], Z^n \rightarrow Z$ in $\mathbb{L}^1(0, T)$ in probability $P$, where

$$Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s) \, ds + \int_t^T dV^n_s + \int_t^T n(Y^n_s - L_s)^- \, ds - \int_t^T Z^n_s \, dB_s, \quad t \in [0, T].$$

Then with the notation $h(t) = f(t, Y_t, Z_t)$, we have

(i) $(Y, Z)$ is the smallest supersolution of BSDE($\xi, f + dV$) such that $L_t \leq Y_t$ for a.e. $t \in [0, T]$ in the class of all processes $(Y, Z)$ from $\mathcal{D}^p \otimes M^p$ such that $E(\int_0^T |f(s, Y_s, Z_s)| \, ds)^p < \infty$ in case $p > 1$ (in the class of all processes $(Y, Z)$ in $\mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ such that $Y$ is of class (D) and $E(\int_0^T |f(s, Y_s, Z_s)| \, ds < \infty$ in case $p = 1$).

(ii) $(Y, Z)$ is the smallest supersolution of the linear BSDE($\xi, h + dV$) such that $L_t \leq Y_t$ for a.e. $t \in [0, T]$ in the class of all processes $(Y, Z)$ such that $Y \in \mathcal{D}^p$ in case $p > 1$ (in the class of processes $(Y, Z)$ such that $Y$ is of class (D) in case $p = 1$).

(iii) For every $\hat{L} \in \mathcal{D}$ such that $L_t \leq \hat{L}_t \leq Y_t$ for a.e. $t \in [0, T],$

$$Y_t = \text{ess sup}_{\tau \in T} E\left( \int_t^\tau f(s, Y_s, Z_s) \, ds + \int_t^\tau dV_s + \hat{L}_s 1_{\{\tau < T\}} + \hat{\xi}_s 1_{\{\tau = T\}} |\mathcal{F}_t \right), \quad t \in [0, T].$$

(iv) For every $\hat{L} \in \mathcal{D}$ such that $L_t \leq \hat{L}_t \leq Y_t$ for a.e. $t \in [0, T],$

$$Y_{t-} = \hat{L}_{t-} \vee (Y_t + \Delta V_t), \quad t \in (0, T). \quad (4.2)$$

Proof. For fixed $p \geq 0$ by Lemma 3.4 and Proposition 3.6, $Z^n \in M^p$ if $p > 1$, and if $p = 1$ then $Z^n \in M^q$, $q \in (0, 1)$ and $E(\int_0^T |f(s, Y^n_s, Z^n_s)| \, ds)^p < \infty$.

(i) Suppose that $p = 1$. Let $(\hat{Y}, \hat{Z})$ be a supersolution of BSDE($\xi, f + dV$) such that $\hat{Y}_t \geq L_t$ for a.e. $t \in [0, T], \hat{Y}$ is of class (D), $(\hat{Y}, \hat{Z}) \in \mathcal{D}^q \otimes M^q, q \in (0, 1)$ and $E(\int_0^T |f(s, \hat{Y}_s, \hat{Z}_s)| \, ds < \infty$. Then there exists $\hat{K} \in \mathcal{V}^+$ such that

$$\hat{Y}_t = \xi + \int_t^T f(s, \hat{Y}_s, \hat{Z}_s) \, ds + \int_t^T dV_s + \int_t^T n(\hat{Y}_s - L_s)^- \, ds + \int_t^T d\hat{K}_s - \int_t^T \hat{Z}_s \, dB_s$$

for $t \in [0, T]$ since $\int_t^T (\hat{Y}_s - L_s)^- \, ds = 0$. Hence, by Corollary 3.2, $Y^n_t \leq \hat{Y}_t, t \in [0, T]$, which implies that $Y_t \leq \hat{Y}_t, t \in [0, T]$. The proof of (i) in case of $p > 1$ is similar, so we omit it.

(ii) Let $p = 1$. From Theorem 3.11 it follows that there exists a unique solution $(\hat{Y}^n, \hat{Z}^n)$ of the BSDE

$$\hat{Y}^n_t = \xi + \int_t^T h(s) \, ds + \int_t^T dV^n_s + \int_t^T n(\hat{Y}^n_s - L_s)^- \, ds - \int_t^T \hat{Z}^n_s \, dB_s, \quad t \in [0, T]$$

such that $\hat{Y}^n$ is of class (D) and $(\hat{Y}^n, \hat{Z}^n) \in \mathcal{D}^q \otimes M^q$ for all $q \in (0, 1)$. Observe that by Corollary 3.2,

$$\hat{Y}^n_t \leq Y_t, \quad t \in [0, T]. \quad (4.3)$$
By Lemma 4.8 there exist a stationary sequence \( \{\tau_k\} \subset T \) and constants \( c_k \) such that
\[
\int_0^{\tau_k} |h(s)| \, ds + Y_{\tau_k}^{+,*} + \bar{Y}_{\tau_k}^{1,*} + \int_0^{\tau_k} d|V_s| \leq c_k
\]
for \( k \in \mathbb{N} \). Therefore from (4.3) and Proposition 4.3 it follows that for every \( k \in \mathbb{N} \),
\[
E \sup_{t \leq \tau_k} |\bar{Y}_t^n|^2 + E \int_0^{\tau_k} |\bar{Z}_s^n|^2 \, ds + E(\int_0^{T_k} dK_s^n)^2 \\
\leq C E \left( \int_0^{\tau_k} |h(s)| \, ds \right)^2 + E(\int_0^{T_k} d|V_s|^2 + |Y_{\tau_k}^{+,*}|^2 + |\bar{Y}_{\tau_k}^{1,*}|^2) \leq c_k.
\]
Applying [28, Theorem 3.1] in each interval \([0, \tau_k]\) and using stationarity of the sequence \( \{\tau_k\} \) we conclude that the process \( \bar{Y}_t \equiv \sup_{n \geq 1} \bar{Y}_t^n, t \in [0, T] \), is càdlàg and there exist processes \( \bar{K} \in \mathcal{V}^+ \) and \( \bar{Z} \in M \) such that
\[
\bar{Y}_t = \xi + \int_t^T h(s) \, ds + \int_t^T dV_s + \int_t^T d\bar{K}_s - \int_t^T \bar{Z}_s \, dB_s.
\]
From the above formula and integrability of \( \bar{Y}, \xi, h, V \) it follows immediately that \( \bar{K} \in \mathcal{V}^{+,1} \), whereas from Lemma 4.2 it follows that \( \bar{Z} \in M^q, q \in (0, 1) \). In view of Lemma 4.4 to complete the proof it suffices to show that \( \bar{Y}_t = Y_t, t \in [0, T] \). To this end, let us observe that by Itô’s formula and monotonicity of the mapping \( x \mapsto (x - L_t)^- \), for every \( \tau \in T \) we have
\[
|Y_{t\wedge \tau}^n - \bar{Y}_{t\wedge \tau}^n| \leq |Y_t^n - \bar{Y}_t^n| + \int_{t\wedge \tau}^T (f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)) \text{sgn}(Y_s^n - \bar{Y}_s^n) \, ds \\
+ n \int_{t\wedge \tau}^T \text{sgn}(Y_s^n - \bar{Y}_s^n)((Y_s^n - L_s)^- - (\bar{Y}_s^n - L_s)^-) \, ds \\
+ \int_{t\wedge \tau}^T (Z_s^n - \bar{Z}_s^n) \text{sgn}(Y_s^n - \bar{Y}_s^n) \, dB_s \\
\leq |Y_t^n - \bar{Y}_t^n| + \int_{t\wedge \tau}^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| \, ds \\
+ \int_{t\wedge \tau}^T (Z_s^n - \bar{Z}_s^n) \text{sgn}(Y_s^n - \bar{Y}_s^n) \, dB_s.
\]
Therefore taking expectation of both sides of the above inequality with \( \tau \) replaced by \( \tau_k \equiv \inf\{t \in [0, T], \int_0^t |Z_s^n - \bar{Z}_s^n|^2 \, ds > k\} \wedge T \), letting \( k \to +\infty \) and using the fact that \( Y_n - \bar{Y}_n \) is of class (D) we obtain
\[
E|Y_t^n - \bar{Y}_t^n| \leq E \int_0^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| \, ds, \quad t \in [0, T]. \tag{4.4}
\]
We now show that the right-hand side of (4.4) converges to zero. To this end, let us first observe that by the assumptions on the convergence of the sequence \( \{(Y^n, Z^n)\} \), (H2), (H4), (H5) and the Lebesgue dominated convergence theorem,
\[
\int_0^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| \, ds \to 0 \tag{4.5}
\]
and the same is true with $Z$ in place of $Z^n$ and $Y^n$ in place of $Y$. Let us note that (4.4), (4.5) hold true for $p > 1$ as well. The proof of (4.4), (4.5) for $p > 1$ is analogous to the above proof for $p = 1$, the only difference being in the fact that in case $p > 1$ assumption (Z) is not needed and the processes $(Y^n, Z^n), (Y, Z)$ considered above belong to $D^p \otimes M^p$. If $p > 1$ then (4.5) implies convergence of the right-hand side of (4.4) to zero because by Proposition 4.3 with $X = Y$ and Proposition 3.6, 

$$
\sup_{n \geq 1} E\left( \int_0^T |f(s, Y^n_s, Z^n_s)| \, ds \right)^p < \infty.
$$

If $p = 1$ then by (H3), monotonicity of the sequence \{$Y^n$\} and (Z),

$$
f(s, Y_s, Z_s) \leq f(s, Y^n_s, Z_s) \leq f(s, Y^1_s, Z_s) \leq |f(s, Y^1_s, Z^1_s)| + 2\gamma(g_s + |Y^1_s| + |Z_s| + |Z^1_s|)^\alpha.
$$

Hence, by (a) and the remark at the beginning of the proof,

$$
E\int_0^T |f(s, Y^1_s, Z_s)| \, ds < \infty,
$$

which when combined with the fact that $h \in L^1(F)$ allows us to apply the Lebesgue dominated convergence theorem to get

$$
E\int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y^n_s, Z_s)| \, ds \to 0.
$$

(4.6)

By (Z) we also have

$$
|f(s, Y^n_s, Z^n_s) - f(s, Y^n_s, Z_s)| \leq 2\gamma(g_s + |Y^n_s| + |Z^n_s| + |Z_s|)^\alpha.
$$

Therefore by Lemma 4.2 applied to $(Y^n, Z^n)$ (with $X = Y$) there exists $p > 1$ such that

$$
\sup_{n \geq 1} E\left( \int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y^n_s, Z_s)| \, ds \right)^p < \infty,
$$

which when combined with (4.5) gives

$$
E\int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y^n_s, Z_s)| \, ds \to 0.
$$

(4.7)

By (4.6), (4.7) the right-hand side of (4.4) converges to zero. Therefore $\bar{Y}_t = Y_t$, $t \in [0, T]$, and the proof of (ii) is complete.

Assertion (iii) follows from (ii) and Corollary 4.5, whereas (iv) follows from (iii) and Lemma 4.6.

We close this section with very useful theorem on monotone convergence of semi-martingales. The theorem generalizes [28, Theorem 3.1] (see also [27, Theorem 2.1]). In the proof we will need the following two lemmas.
Then if \( \text{Lemma 4.11.} \) \( \{(Y^n, X^n, K^n, A^n)\} \) is a sequence of progressively measurable processes such that

\[ Y^n_t = -K^n_t + A^n_t + X^n_t, \quad t \in [0, T] \]

for \( n \in \mathbb{N} \) and

(a) \( A^n, K^n \in \mathcal{V}^{+,1} \),

(b) \( \{dA^n\} \) is increasing, \( A^n_t \rightarrow A_t, \ t \in [0, T], \ EA_T < \infty \),

(c) \( Y^n_t \not\subset Y_t, \ t \in [0, T], \ Y^1, Y \) are of class (D), \( Y_0 = 0 \),

(d) There exists a càdlàg process \( X \) of class (D) such that \( X_0 = 0 \) and for some subsequence \( \{n'\} \), \( X^n_{t'} \rightarrow X_\tau \) weakly in \( L^1(\mathcal{F}_T) \) for every \( \tau \in \mathcal{T} \).

Then \( Y \in \mathcal{D}, \ A \in \mathcal{V}^+ \), there exists \( K \in \mathcal{V}^+ \) such that \( K^n_{t'} \rightarrow K_\tau \) weakly in \( L^1(\mathcal{F}_T) \) for every \( \tau \in \mathcal{T} \) and

\[ Y_t = -K_t + A_t + X_t, \quad t \in [0, T]. \]

Proof. Put \( K_t = A_t + X_t - Y_t, \ t \in [0, T] \). By (c), \( Y^n_{t'} \rightarrow Y_\tau \) weakly in \( L^1(\mathcal{F}_T) \) for every \( \tau \in \mathcal{T} \), and hence, by (b)-(d), \( K^n_{t'} \rightarrow K_\tau \) weakly in \( L^1(\mathcal{F}_T) \) for every \( \tau \in \mathcal{T} \). Since \( K^n_\sigma \leq K^n_\tau \), for any \( \sigma, \tau \in \mathcal{T} \) such that \( \sigma \leq \tau \), it follows that \( K_\sigma \leq K_\tau \), hence that \( K \) is increasing. Finally, by Lemmas 3.1 and 3.2 [28], \( A, K, Y \in \mathcal{D} \). \( \square \)

Lemma 4.11. Assume (H2)-(H5). Let \( L^n, L \in \mathcal{V}, \ g_n, g, \bar{f} \in L^1(\mathcal{F}) \) and let \( (Y^n, Z^n), (Y, Z) \in \mathcal{D} \otimes M \) be processes such that \( t \mapsto f(t, Y^n_t, Z^n_t), \ t \mapsto f(t, Y_t, Z_t) \in L^1(0, T) \) and

\[ Y^n_t = Y^n_0 - \int_0^t g_n(s) \, ds - \int_0^t f(s, Y^n_s, Z^n_s) \, ds - \int_0^t dB^n_s + \int_0^t Z^n_s \, dB_s, \quad t \in [0, T], \]

\[ Y_t = Y_0 - \int_0^t g(s) \, ds - \int_0^t \bar{f}(s) \, ds - \int_0^t dB_s + \int_0^t Z_s \, dB_s, \quad t \in [0, T]. \]

Then if

(a) \( E \sup_{n \geq 0} (L^n)_T + E \int_0^T |f(s, 0, 0)| \, ds < \infty \),

(b) \( \lim \inf_{n \to \infty} \int_0^T (Y_s - Y^n_s) \, dB^n_s \geq 0 \) for every \( \sigma, \tau \in \mathcal{T}, \ \sigma \leq \tau \),

(c) There exists \( C \in \mathcal{V}^{1,+} \) such that \( |\Delta (Y_t - Y^n_t)| \leq |\Delta C_t|, \ t \in [0, T] \),

(d) There exist \( \underline{y}, \bar{y} \in \mathcal{V}^{1,+} + \mathcal{M}_{loc} \) of class (D) such that

\[ \underline{y} \leq Y_t \leq \bar{y}, \ t \in [0, T], \quad E \int_0^T f^+(s, \underline{y}_s, 0) \, ds + E \int_0^T f^-(s, \bar{y}_s, 0) \, ds < \infty, \]

(e) There exists \( h \in L^1(\mathcal{F}) \) such that \( |g_n(s)| \leq h(s) \) for a.e. \( s \in [0, T] \),

(f) \( Y^n_t \rightarrow Y_t, \ t \in [0, T] \),

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then

\[ Z^n \to Z, \quad \lambda \otimes P\text{-a.e.,} \quad \int_0^t |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| \, ds \to 0 \quad \text{in ucp} \quad (4.8) \]

and there exists a stationary sequence \( \{\tau_k\} \subset \mathcal{T} \) such that for every \( k \in \mathbb{N} \) and \( p \in (0, 2) \),

\[ E \int_0^{\tau_k} |Z^n_s - Z_s|^p \, ds \to 0, \quad (4.9) \]

and if \( |\Delta C_t| = 0, \ t \in [0, T] \), then the above convergence holds for \( p = 2 \), too. If additionally \( g_n \to g \) weakly in \( L^1(F) \) and \( L^n \to L \) weakly in \( L^1(\mathcal{F}_\tau) \) for every \( \tau \in \mathcal{T} \), then \( f(s) = f(s, Y_s, Z_s) \) for a.e. \( s \in [0, T] \).

Proof. Step 1. (Reduction to the study of stopped processes) Assume that \( \{\tau_k\} \subset \mathcal{T} \) is a stationary sequence. Write \( Y_t^{n,k} = Y_{t\wedge \tau_k}^n, L_t^{n,k} = L_{t\wedge \tau_k}^n, Z_t^{n,k} = Z_{t\wedge \tau_k}^n \), \( g_t^{n,k} = f(\cdot, Y_t^{n,k}, Z_t^{n,k}) \mathbf{1}_{[0,\tau_k]}, Y_t^k = Y_{t\wedge \tau_k}^k, L_t^k = L_{t\wedge \tau_k}^k, Z^k = Z_{t\wedge \tau_k}^k \) and \( \eta^k = f(0,0,0) \mathbf{1}_{[0,\tau_k]} \). Then

\[ Y_t^{n,k} = Y_0^{n,k} - \int_0^t g_t^{n,k}(s) \, ds - \int_0^t f_t(s, Y_s^{n,k}, Z_s^{n,k}) \, ds - \int_0^t dL_s^{n,k} + \int_0^t Z_s^{n,k} \, dB_s \]

and

\[ Y_t^k = Y_0^k - \int_0^t g_t^k(s) \, ds - \int_0^t f_t(s) \, ds - \int_0^t dL_s^k + \int_0^t Z_s^k \, dB_s \]

for \( t \in [0, T] \). Since \( \{\tau_k\} \) is stationary, it follows from the above that it suffices to prove the lemma for stopped processes \((Y^k, Z^k)\).

Step 2. (Localization) By Lemma 4.8 there exist a stationary sequence \( \{\delta_k^1\} \subset \mathcal{T} \) and constants \( c_k \) such that \( g_t^{n,k} \leq c_k \) and \( g_t^{n,k} \leq c_k \). Let \( D \) be a càdlàg version of the process \( D_t = \sup_{n \geq 0} (L^n)_t^+, \ t \in [0, T] \). Then \( (L^n)_t^+ \leq D_t, \ t \in [0, T] \) and by assumption (a), \( ED_T < \infty \). Therefore by Lemma 4.8 there exist a stationary sequence \( \{\delta_k^2\} \subset \mathcal{T} \) and constants \( b_k \) such that \( \bar{D}_t \leq b_k \). Put \( \tau_k = \min\{\delta_k^1, \delta_k^2, \delta_k^3\} \), where

\[ \delta_k^3 = \inf\{t \in [0, T] : \int_0^t |f(s, 0, 0)| \, ds + \int_0^t |h(s)| \, ds \]

\[ + \int_0^t f^-(s, y_s, 0) \, ds + \int_0^t f^+(s, y_s, 0) \, ds > k\} \land T. \]

From Proposition 3.6, Proposition 4.3 and the definition of \( \tau_k \) it follows that there exists \( C \) not depending on \( n \) such that

\[ E \left( \int_0^{\tau_k} |Z^n_s|^2 \, ds + \left( \int_0^{\tau_k} d[L^n_s]^2 \right) + \left( \int_0^{\tau_k} |g_n(s)| \, ds \right)^2 + \left( \int_0^{\tau_k} |f(s, Y^n_s, Z^n_s)| \, ds \right)^2 \right) \leq C. \]

Similarly, by Lemma 4.8 there exist a stationary sequence \( \{\delta_k^4\} \subset \mathcal{T} \) and constants \( a_k \) such that \( L_{\delta_k^4} \leq a_k \), and furthermore, putting \( \tau'_k = \min\{\delta_k^1, \delta_k^4, \delta_k^5\} \), where \( \delta_k^5 \) is defined
as $\delta_k$ but with $f(s,0,0)$ replaced by $\bar{f}(s)$, we conclude from Propositions 3.6 and 4.3 that
\[
E \left( \int_0^{T} |Z_s|^2 \, ds + (\int_0^{T} d|L|_s)^2 + (\int_0^{T} |g(s)| \, ds)^2 + (\int_0^{T} |\bar{f}(s)| \, ds)^2 \right) < \infty.
\]

Step 3. By Step 1 and Step 2 we may assume that there exists $C$ not depending on $n$ such that
\[
E \left( \int_0^{T} |Z^n_s|^2 \, ds + (\int_0^{T} d|L^n|_s)^2 + (\int_0^{T} |g_n(s)| \, ds)^2 + (\int_0^{T} |f(s, Y^n_s, Z^n_s)| \, ds)^2 \right) \leq C. \tag{4.10}
\]
and $y, \overline{y}$ are bounded, $h, \bar{f}, g, f(\cdot, 0, 0), f^+(\cdot, \overline{y}, 0), f^- (\cdot, \overline{y}, 0) \in \mathbb{L}^{1,2}(\mathcal{F}), Z \in M^2, L, C \in \mathcal{Y}^2$. We also assume that $\mu \leq 0$. By (d), (f), (H4), (H5) and the Lebesgue dominated convergence theorem, we get
\[
\int_0^{T} |f(s, Y^n_s, 0) - f(s, Y_s, 0)| \, ds \to 0 \tag{4.11}
\]
By (d) and (H3),
\[
-f^-(s, \overline{y}, 0) \leq f(s, Y^n_s, 0) \leq f^+(s, \overline{y}, 0),
\]
Therefore using (4.11) and the Lebesgue dominated convergence theorem we get
\[
\lim_{n \to +\infty} E \left( \int_0^{T} |f(s, Y^n_s, 0) - f(s, Y_s, 0)| \, ds \right)^2 = 0. \tag{4.12}
\]
Observe that by (H2) and (4.10), $f(s, Y^n_s, Z^n_s) = f(s, Y^n_s, 0) + \eta^n(s)$ for some $\{\eta^n\} \subset \mathbb{L}^2(\mathcal{F})$ such that
\[
\sup_{n \geq 0} E \int_0^{T} |\eta^n(s)|^2 \, ds < \infty. \tag{4.13}
\]
We have
\[
E \int_0^{T} |f(s, Y^n_s, Z^n_s) - \bar{f}(s)||Y^n_s - Y_s| \, ds
\leq E \int_0^{T} |f(s, Y^n_s, 0) - f(s, Y_s, 0)||Y^n_s - Y_s| \, ds + E \int_0^{T} |f(s, Y_s, 0)||Y^n_s - Y_s| \, ds
+ (E \int_0^{T} |\eta^n(s)|^2 \, ds)^{1/2}(E \int_0^{T} |Y^n_s - Y_s|^2 \, ds)^{1/2} + E \int_0^{T} |\bar{f}(s)||Y^n_s - Y_s| \, ds.
\]
Therefore from (d), (f), (4.12), (4.13) and the fact that $y, \overline{y}$ are bounded we get
\[
E \int_0^{T} |f(s, Y^n_s, Z^n_s) - \bar{f}(s)||Y^n_s - Y_s| \, ds \to 0. \tag{4.14}
\]
Since $y, \overline{y}$ are bounded and $g \in \mathbb{L}^1(\mathcal{F})$, it follows from (d)–(f) that
\[
E \int_0^{T} |g(s) - g_n(s)||Y^n_s - Y_s| \, ds \leq E \int_0^{T} (|h(s)| + |g(s)|)|Y^n_s - Y_s| \, ds \to 0. \tag{4.15}
\]
Let \( \sigma, \tau \in \mathcal{T}, \sigma \leq \tau \). Then by Itô’s formula and (c),

\[
E \int_{\tau}^{\sigma} |Z_s^n - Z_s^n|^2 \, ds \leq E|Y_\tau - Y_\sigma^n|^2 + 2 \int_{\sigma}^{\tau} |g(s) - g^n(s)||Y_s - Y_s^n| \, ds \\
+ E \int_{\sigma}^{\tau} |f(s, Y_s, Z_s) - f(s, Y_s^n, Z_s^n)||Y_s - Y_s^n| \, ds \\
+ 2 \int_{\sigma}^{\tau} (Y_s - Y_s^n) \, d(L_s - L_s^n) + 2 \sum_{\sigma < t \leq \tau} |\Delta C_t|^2. \tag{4.16}
\]

Therefore from (b), (d), (f), (4.14), (4.15) and boundedness of \( y, \overline{y} \) it may be concluded that for any \( N \in \mathbb{N} \) and any \( \sigma_1, \tau_1, \ldots, \sigma_N, \tau_N \in \mathcal{T} \) such that \( \sigma_k \leq \tau_k \) for \( k = 1, \ldots, N \) we have

\[
\limsup_{n \to +\infty} \sum_{k=1}^{N} E \int_{\tau_k}^{\sigma_k} |Z_s^n - Z_s^n|^2 \, ds \leq \sum_{k=1}^{N} E \sum_{\sigma_k < t \leq \tau_k} |\Delta C_t|^2. \tag{4.17}
\]

It suffices now to repeat arguments following Eq. 2.10 in [27, Theorem 2.1] to show that \( Z^n \to Z, \lambda \otimes P \)-a.e.. In view of (4.10) this implies that \( E \int_{0}^{T} |Z_s^n - Z_s| \, ds \to 0 \), which when combined with (4.12) and (H2) yields (4.8). Next, by (4.10) and pointwise convergence of \( Z^n, E \int_{0}^{T} |Z_s^n - Z_s|^p \, ds \to 0 \) for every \( p \in (0, 2) \), and moreover, if \( |\Delta C_t| = 0, t \in [0, T] \), then by (4.17) with \( N = 1, \sigma_1 = 0, \tau_1 = T \) it follows that \( E \int_{0}^{T} |Z_s^n - Z_s|^2 \, ds \to 0 \). Thus, (4.9) is satisfied, because we consider processes \( Y^n, Z^n \) stopped at \( \tau_k \) (see Step 1). Finally, if \( \{g_n\}, \{L^n\} \) satisfy the additional assumptions then \( \int_{0}^{T} f(s, Y_s^n, Z_s^n) \, ds \to \int_{0}^{T} \overline{f}(s) \, ds \) weakly in \( L^1(\mathcal{F}_T) \), which when combined with (4.8) implies that \( \overline{f}(s) = f(s, Y_s, Z_s) \) for a.e. \( s \in [0, T] \).

We are now ready to prove the main result on monotone convergence of semimartingales.

**Theorem 4.12.** Let (H2)–(H5) be satisfied, \( (Y^n, Z^n) \in \mathcal{D} \otimes M, A^n, K^n \in \mathcal{V}^+, t \mapsto f(t, Y^n_t, Z^n_t) \in L^1(0, T) \) and

\[
Y^n_t = Y^n_0 - \int_{0}^{t} g_n(s) \, ds - \int_{0}^{t} f(s, Y^n_s, Z^n_s) \, ds - \int_{0}^{t} dK^n_s + \int_{0}^{t} dA^n_s + \int_{0}^{t} Z^n_s \, dB_s
\]

for \( t \in [0, T] \). Assume that

(a) \( dA^n \leq dA^{n+1}, n \in \mathbb{N}, \sup_{n \geq 0} EA^n_T < \infty \),

(b) \( \liminf_{n \to +\infty} \int_{\sigma}^{\tau} (Y_s^n - Y_s) \, d(K^n_s - A^n_s) \geq 0 \) for every \( \sigma, \tau \in \mathcal{T}, \sigma \leq \tau \),

(c) There exists \( C \in \mathcal{V}^{1+} \) such that \( |\Delta K^n_t| \leq |\Delta C_t|, t \in [0, T] \),

(d) There exist processes \( y, \overline{y} \in \mathcal{V}^1 + \mathcal{M}_{loc} \) of class (D) such that

\[
E \int_{0}^{T} f^+(s, y_s, 0) \, ds + E \int_{0}^{T} f^-(s, y_s, 0) \, ds < \infty, \quad \overline{y}_t \leq Y^n_t \leq y_t, \quad t \in [0, T],
\]

(e) \( E \int_{0}^{T} |f(s, 0, 0)| \, ds < \infty \) and \( |g_n(s)| \leq h(s) \) for a.e. \( s \in [0, T] \) for some \( h \in L^1(\mathcal{F}) \),

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(f) \( Y^n_t \not\rightarrow Y_t, t \in [0, T] \).

Then \( Y \in D \), there exist \( K \in \mathcal{V}^+, A \in \mathcal{V}^{1,+}, Z \in M, g \in L^1(F) \) such that

\[
Y_t = Y_0 - \int_0^t g(s) \, ds - \int_0^t f(s, Y_s, Z_s) \, ds - \int_0^t dK_s + \int_0^t dA_s + \int_0^t Z_s \, dB_s, \quad t \in [0, T]
\]

and

\[
Z^n \rightarrow Z, \quad \lambda \otimes P \text{-a.e.,} \quad \int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| \, ds \rightarrow 0 \quad \text{in ucp.}
\]

Moreover, there exists a stationary sequence \( \{\tau_k\} \subset T \) such that for every \( p \in (0, 2) \),

\[
E \int_0^{\tau_k} |Z^n_s - Z_s|^p \, ds \rightarrow 0,
\]

and if \( |\Delta C_t| + |\Delta K_t| = 0, t \in [0, T] \) then the above convergence holds for \( p = 2 \), too.

**Proof.** First of all let us note that if there exists a stationary sequence \( \{\tau_k\} \subset T \) such that assertions of the theorem hold on the interval \([0, \tau_k]\) for every \( k \in \mathbb{N} \) then they hold on \([0, T]\) as well (see Step 1 of the proof of Lemma 4.11). Let \( A_t = \sup_{n \geq 1} A^n_t, t \in [0, T] \).

By (a) and Lemma 4.10, \( A \in \mathcal{V}^{1,+} \). By Lemma 4.8 there exist a stationary sequence \( \{\delta_k\} \subset T \) and constants \( c_k \) such that \( A \geq c_k, k \in \mathbb{N} \). Let \( \delta_k \) be defined as in the proof of Lemma 4.11. Then the sequence \( \{\tau_k\} \subset T \), where \( \tau_k = \min\{\delta_k, \delta_k^2, \delta_k^3\} \), is stationary. By the remark at the beginning of the proof and the definition of \( \tau_k \) we may assume that \( y, y \) are bounded, \( A \in \mathcal{V}^{2,+}, f(\cdot, 0, 0), h, f^+(\cdot, y, 0), f^-(\cdot, y, 0) \in L^{1,2}(F) \) and, by Propositions 3.6 and 4.3, that there exists \( C > 0 \) not depending on \( n \) such that

\[
E \int_0^T |Z^n_s|^2 \, ds + E(\int_0^T dK^n_s)^2 + E(\int_0^T |f(s, Y^n_s, Z^n_s)| \, ds)^2 \leq C. \quad (4.18)
\]

We aim to apply Lemma 4.10 to the process \( Y^n \). To do this we have to show that up to a subsequence, \( \{f(\cdot, Y^n, Z^n)\} \) is weakly convergent in \( L^1(F) \). Since by (4.18) and (H2),

\[
f(s, Y^n_s, Z^n_s) = f(s, Y^n_s, 0) + \eta^n(s) \text{ for some sequence } \{\eta^n\} \subset L^{1,2}(F)
\]

such that \( \sup_{n \geq 0} E \int_0^T |\eta^n(s)|^2 \, ds < \infty \), it suffices to prove the desired convergence for the sequence \( \{f(\cdot, Y^n, 0)\} \). Let us observe that by (c), (e), (H4), (H5) and the Lebesgue dominated convergence theorem,

\[
\int_0^T |f(s, Y^n_s, 0) - f(s, Y_s, 0)| \, ds \rightarrow 0.
\]

By (c) and (H3),

\[
-f^-(s, y_s, 0) \leq f(s, Y^n_s, 0) \leq f^+(s, y_s, 0).
\]

Since \( f^+(\cdot, y, 0), f^-(\cdot, y, 0) \in L^{1,2}(F) \), it follows from the above and the Lebesgue dominated convergence theorem that

\[
E \int_0^T |f(s, Y^n_s, 0) - f(s, Y_s, 0)| \, ds \rightarrow 0.
\]
Let us denote by $\eta$ the weak limit of $\{\eta^n\}$ in $L^2(F)$, by $g$ the weak limit of $\{g_n\}$ in $L^1(F)$ and by $Z$ the weak limit of $\{Z^n\}$ in $L^2(F)$. Then by Lemma 4.10, $Y \in \mathcal{D}$ and there exists $K \in \mathcal{V}^+$ such that

$$Y_t = Y_0 - \int_0^t g(s) \, ds - \int_0^t f(s, Y_s, 0) \, ds - \int_0^t \eta(s) \, ds - \int_0^t dK_s + \int_0^t dA_s + \int_0^t Z_s \, dB_s.$$

Applying now Lemma 4.11 we prove the theorem except for the last assertion. Finally, using monotonicity of $\{Y^n\}, \{dA^n\}$ one can show inequality (4.16) with $|\Delta K^n_\tau| + |\Delta K_\tau|$ in place of $|\Delta C_\tau|$, which proves the last assertion. $\Box$

5 BSDEs with one reflecting barrier

In this section we prove existence and uniqueness of solutions of reflected BSDEs with one irregular barrier and data in $L^p$ with $p \in [1, 2]$. We also prove that the solutions can be approximated by penalization method and give a comparison result.

We will need the following additional hypotheses.

(H6) $L, U$ are progressively measurable processes, $L_t \leq U_t$ for a.e. $t \in [0, T]$.

(H7) There exists a semimartingale $X \in \mathcal{H}^p$ such that $L_t \leq X_t$ for a.e. $t \in [0, T]$ and $E(\int_0^T f^-(s, X_s, 0) \, ds)^p < \infty$.

(H7*) There exists a semimartingale $X$ of class (D) such that $X \in \mathcal{V}^1 + \mathcal{M}^2$ for every $q \in (0, 1)$, $L_t \leq X_t, t \in [0, T]$ and $E \int_0^T f^-(s, X_s, 0) \, ds < \infty$.

**Definition.** We say that a triple $(Y, Z, K)$ of progressively measurable processes is a solution of $\text{RBSDE}(\xi, f + dV, L)$ if

(a) $K \in \mathcal{V}^+$,

(b) $Z \in M$ and the mapping $[0, T] \ni t \mapsto f(t, Y_t, Z_t)$ belongs to $L^1(0, T)$, P-a.s.,

(c) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T]$,

(d) $L_t \leq Y_t$ for a.e. $t \in [0, T], \int_0^T (Y_{t-} - \hat{L}_{t-}) \, dK_t = 0$ for every $\hat{L} \in \mathcal{D}$ such that $L_t \leq \hat{L}_t \leq Y_t, P$-a.s. for a.e. $t \in [0, T]$.

Uniqueness of solutions of BSDEs follows from the following comparison results.

**Proposition 5.1.** Assume (H2). Let $(Y^i, Z^i, K^i)$ be a solution of $\text{RBSDE}(\xi^i, f^i + dV^i, L^i), i = 1, 2$. If $(Y^1 - Y^2)^+ \in \mathcal{D}^q$ for some $q > 1$, $\xi^1 \leq \xi^2$, $dV^1 \leq dV^2$, $L^1_t \leq L^2_t$ for a.e. $t \in [0, T]$ and either (3.2) or (3.3) is satisfied then $Y^1_t \leq Y^2_t, t \in [0, T]$.

**Proof.** Without loss of generality we may assume that $\mu \leq 0$. Assume that (3.2) is satisfied. By the Itô-Tanaka formula and Proposition 2.1, for any $p \in (1, q)$ and $\tau \in \mathcal{T}$
we have
\[
\begin{align*}
(Y_{1,T}^1 - Y_{1,T}^2)^+ &+ p\frac{p-1}{2} \int_{t\wedge T}^T 1_{\{Y_s^1 \neq Y_s^2\}} |(Y_s^1 - Y_s^2)^+| p^{-2} |Z_s^1 - Z_s^2|^2 \, ds \\
&\leq |(Y_{1,T}^1 - Y_{1,T}^2)^+| + p \int_{t\wedge T}^T |(Y_s^1 - Y_s^2)^+| p^{-1} (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) \, ds \\
&\quad + p \int_{t\wedge T}^T |(Y_s^1 - Y_s^2)^+| p^{-1} (dV_s^1 - dV_s^2) + p \int_{t\wedge T}^T |(Y_s^1 - Y_s^2)^+| p^{-1} (dK_s^1 - dK_s^2) \\
&\quad - p \int_{t\wedge T}^T |(Y_s^1 - Y_s^2)^+| p^{-1} (Z_s^1 - Z_s^2) \, dB_s.
\end{align*}
\]

Since \( L_t^1 \leq L_t^2 \) for a.e. \( t \in [0,T] \), \( L_t^1 \leq Y_t^1 \wedge Y_t^2 \leq Y_t^1 \) for a.e. \( t \in [0,T] \). By monotonicity of the function \( x \mapsto p|x|p^{-1}x \) and property (d) of the definition of a solution of the reflected BSDE,
\[
\int_{t\wedge T}^T |(Y_s^1 - Y_s^2)^+| p^{-1} (dK_s^1 - dK_s^2) \leq \int_{t\wedge T}^T |(Y_s^1 - Y_s^2)^+| p^{-1} dK_s^1
\]
\[
= \int_{t\wedge T}^T |(Y_s^1 - Y_s^2)^+| p^{-1} \text{sgn}[(Y_s^1 - Y_s^2)^+] \, dK_s^1
\]
\[
\leq \int_{t\wedge T}^T |(Y_s^1 - Y_s^2)^+| p^{-1} \text{sgn}[(Y_s^1 - Y_s^2)^+] \, dK_s^1 = 0,
\]
the last equality being a consequence of the fact that
\[
\int_{t\wedge T}^T \text{sgn}[(Y_s^1 - Y_s^2)^+] \, dK_s^1
\]
\[
= \int_{t\wedge T}^T 1_{\{Y_s^1 \neq Y_s^2\}} |Y_s^1 - Y_s^2|^1 \, d(\int_0^s (Y_u^1 - Y_u^2) \, dK_u^1).
\]
The rest of the proof runs as the proof of Proposition 3.1 (see the reasoning following (3.4)).

**Corollary 5.2.** Assume (H2), (Z). Let \((Y^i, Z^i, K^i)\), be a solution of RBSDE\((\xi^i, f^i + dV^i, L^i)\) such that \((Y^i, Z^i) \in L^q(\mathcal{F}) \otimes L^q(\mathcal{F})\) for some \( q > \alpha \), \( i = 1, 2 \). If \((Y_t^1 - Y_t^2)^+ \) is of class \((D)\), \( \xi^1 \leq \xi^2 \), \( dV^1 \leq dV^2 \), \( L^1_t \leq L^2_t \) for a.e. \( t \in [0,T] \) and (3.2) or (3.3) is satisfied then \( Y_t^1 \leq Y_t^2 \), \( t \in [0,T] \).

**Proof.** As in the proof of Proposition 5.1 one can reduce the problem to the case of nonreflected BSDEs and then use Corollary 3.2 to conclude the result.

**Remark 5.3.** Since the proof of Proposition 5.1 and Corollary 5.2 is based on the proof of Proposition 3.1, arguments from Remark 3.3 show that if \( f \) does not depend on \( z \) then assumption \((Z)\) and the assumptions that \((Y^i, Z^i) \in L^q(\mathcal{F}) \otimes L^q(\mathcal{F})\) for some \( q > \alpha \) are superfluous in Corollary 5.2.

For an arbitrary sequence \( \{x_n\} \) of elements of some linear space we set
\[
\sigma_k(\{x_n\}) = \frac{x_1 + x_2 + \ldots + x_k}{k}, \quad k \geq 0.
\]
Theorem 5.4. Let assumptions (H2), (H3) hold. Then there exists at most one solution \((Y, Z, K)\) of RBSDE\((\xi, f + dV, L)\) such that \(Y \in D^p\) for some \(p > 1\).

Proof. Follows immediately from Proposition 5.1. \(\square\)

Theorem 5.5. Let assumptions (H2), (H3), (Z) hold. Then there exists at most one solution \((Y, Z, K)\) of RBSDE\((\xi, f + dV, L)\) such that \(Y\) is of class \((D)\) and \(Z \in \bigcup_{\beta > \alpha} M^\beta\).

Proof. Follows immediately from Corollary 5.2. \(\square\)

We prove existence of solutions separately for data in \(L^p\) with \(p > 1\) and for data in \(L^1\).

Theorem 5.6. Let \(p > 1\).

(i) Assume (H1)–(H6). Then there exists a solution \((Y, Z, K) \in D^p \otimes M^p \otimes V^{+,p}\) of RBSDE\((\xi, f + dV, L)\) iff (H7) is satisfied.

(ii) Assume (H1)–(H7). For \(n \in \mathbb{N}\) let \((Y^n, Z^n) \in D^p \otimes M^p\) be a solution of the BSDE

\[
Y^n_t = \xi_n + \int_t^T f(s, Y^n_s, Z^n_s) \, ds + \int_t^T dV_s + n \int_t^T (Y^n_s - L_s)^- \, ds - \int_t^T Z^n_s \, dB_s, \quad t \in [0, T],
\]

and let \(\xi_1 \in L^p(\mathcal{F}_T), \xi_n \nearrow \xi\). Then

\[
Y^n_t \nearrow Y_t, \quad t \in [0, T], \quad Z^n \rightarrow Z, \ \lambda \otimes P\text{-a.s.},
\]

for every \(r \in [1, p), q \in [1, 2),\)

\[
E\left(\int_0^T |Z^n_s - Z_s|^q \, ds\right)^{r/q} \rightarrow 0,
\]

and for every stopping time \(\tau \in \mathcal{T},\)

\[
K^n_\tau \rightarrow K_\tau \text{ weakly in } L^p(\mathcal{F}_T),
\]

where

\[
K^n_\tau = n \int_0^\tau (Y^n_s - L_s)^- \, ds.
\]

Proof. Necessity. Assume that there exists a solution \((Y, Z, K) \in D^p \otimes M^p \otimes V^{+,p}\) of RBSDE\((\xi, f + dV, L)\). Then by Proposition 3.6, \(E(\int_0^T |f(s, Y_s, Z_s)| \, ds)^p < \infty\), which implies that \(Y \in H^p\). Moreover, by the definition of a solution of RBSDE\((\xi, f, L)\), \(Y_t \geq L_t\) for a.e. \(t \in [0, T]\), i.e. (H7) is satisfied with \(X = Y\).

Sufficiency. First of all let us note that by Theorem 3.8 there exists a unique solution \((Y^n, Z^n) \in D^p \otimes M^p\) of (5.1). Using standard change of variable we may reduce the proof to the case \(\mu \leq 0\). Therefore in what follows we assume that \(\mu \leq 0\). For simplicity of notation we will also assume that \(\xi_n = \xi, n \geq 0\). The proof in the general case only requires some obvious changes.

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Step 1. We will show that there exists a supersolution \((\bar{X}, \bar{Z}) \in \mathcal{D}^p \otimes M^p\) of BSDE\((\xi, f + dV)\) such that
\[
X_t \leq \bar{X}_t, \quad Y^n_t \leq \bar{X}_t, \quad t \in [0, T] \tag{5.6}
\]
and
\[
E(\int_0^T |f(s, \bar{X}_s, \bar{Z}_s)| \, ds)^p < \infty. \tag{5.7}
\]
Since \(X \in \mathcal{H}^p\) and the Brownian filtration has the representation property, there exist \(C \in \mathcal{V}^p, H \in M^p\) such that
\[
X_t = X_T - \int_t^T dC_s - \int_t^T H_s \, dB_s, \quad t \in [0, T].
\]
The above equation can be rewritten in the form
\[
X_t = X_T + \int_t^T f(s, X_s, H_s) \, ds + \int_t^T dV_s - \int_t^T (f^+(s, X_s, H_s) \, ds + dC^+_s + dV^+_s)
+ \int_t^T (f^-(s, X_s, H_s) \, ds + dC^-_s + dV^-_s) - \int_t^T H_s \, dB_s, \quad t \in [0, T].
\]
By (H7) and (H2), \(E(\int_0^T f^-(s, X_s, H_s) \, ds)^p < \infty\). By Theorem 3.8 there exists a solution \((\bar{X}, \bar{Z}) \in \mathcal{D}^p \otimes M^p\) of the BSDE
\[
\bar{X}_t = X_T \vee \xi + \int_t^T f(s, \bar{X}_s, \bar{Z}_s) \, ds + \int_t^T dV_s
+ \int_t^T (f^-(s, X_s, H_s) \, ds + dC^-_s + dV^-_s) - \int_t^T \bar{Z}_s \, dB_s, \quad t \in [0, T].
\]
Moreover, by Proposition 3.6,
\[
E \left( \int_0^T |f(s, \bar{X}_s, \bar{Z}_s)| \, ds \right)^p < \infty,
\]
and by Proposition 3.1,
\[
X_t \leq \bar{X}_t, \quad t \in [0, T].
\]
Of course, \((\bar{X}, \bar{Z})\) is a supersolution of BSDE\((\xi, f + dV)\). Since \(L_t \leq X_t\) for a.e. \(t \in [0, T]\), it follows from the last estimate that \(L_t \leq \bar{X}_t\) for a.e. \(t \in [0, T]\). Therefore
\[
\bar{X}_t = X_T \vee \xi + \int_t^T f(s, \bar{X}_s, \bar{Z}_s) \, ds + \int_t^T dV_s + \int_t^T (f^-(s, X_s, H_s) \, ds + dC^-_s + dV^-_s)
+ n \int_t^T (\bar{X}_s - L_s)^- \, ds - \int_t^T \bar{Z}_s \, dB_s, \quad t \in [0, T].
\]
Hence, by Proposition 3.1, \(Y^n_t \leq \bar{X}_t, t \in [0, T]\), which completes the proof of Step 1.

Step 2. By Proposition 3.1, \(Y^n_t \leq Y^{n+1}_t, t \in [0, T]\) for every \(n \in \mathbb{N}\). Therefore setting \(Y_t = \sup_{n \geq 1} Y^n_t, t \in [0, T]\), we have
\[
Y^n_t \nearrow Y_t, \quad t \in [0, T]. \tag{5.8}
\]
Since the assumptions of Theorem 4.12 are satisfied with \( y = \bar{X} \) and \( \bar{y} = Y^1 \), there exists \( Z \in M \) such that
\[
Z^n \to Z, \quad \lambda \otimes P\text{-a.e.} \quad (5.9)
\]
and there exists \( K \in \mathcal{V}^+ \) such that
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T].
\]
By (5.6), (5.7) and Propositions 4.3 and 3.6, there exists \( C \) not depending on \( n \) such that
\[
E \sup_{t \leq T} |Y^n_t|^p + E(\int_0^T |Z^n_s|^2 \, ds)^{p/2} + E(\int_0^T |f(s, Y^n_s, Z^n_s)| \, ds)^p \leq C. \quad (5.10)
\]
From (5.8)–(5.10) it follows that \( Y \in D^p, Z \in M^p, f(\cdot, Y, Z) \in L^{1,p}(\mathcal{F}), K \in \mathcal{V}^{p,+} \) and (5.3), (5.4) hold true.

Step 3. In the last part of the proof we show that \( Y \) majorizes \( L \) and \( K \) satisfies the minimality condition. Let \( C \) be the constant appearing on the right-hand side of (5.10). From (5.1) one can easily deduce that there exists \( c_p \) depending only on \( p \) such that
\[
E(\int_0^T (Y^n_s - L_s) \, ds)^p \leq c_p C n^{-p},
\]
which when combined with (5.8) implies that
\[
L_t \leq Y_t \quad \text{for a.e. } t \in [0, T].
\]
From (5.10) and the fact that the space \( L^{2,p}(\mathcal{F}) \) has the Banach-Saks property we conclude that there exists a subsequence (still denoted by \( \{n\} \)) such that
\[
\sigma_n(\{ \int_0^T Z^n_s \, dB_s \}) \to \int_0^T Z_s \, dB_s \quad \text{in ucp.}
\]
By Theorem 4.12,
\[
\int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| \, ds \to 0 \quad \text{in ucp.} \quad (5.11)
\]
By (5.8) and (5.11),
\[
\sigma_n(\{ \int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| \, ds \}) \to 0 \quad \text{in ucp,} \quad \sigma_n(\{Y^n_t\}) \to Y_t, \quad t \in [0, T],
\]
and hence
\[
\sigma_n(\{K^n_t\}) \to K_t, \quad t \in [0, T]. \quad (5.12)
\]
Let \( \hat{L} \in \mathcal{D} \) be an arbitrary process such that \( L_t \leq \hat{L}_t \leq Y_t \) for a.e. \( t \in [0, T] \) and let \( \{\tau_k\} \) be an increasing sequence of successive jumps of process \( V, K \) (with the convention that \( \tau_0 \equiv 0 \)). Since \( Y^n \) is continuous on \((\tau_k, \tau_{k+1})\), it follows from Dini’s theorem that
for every $k \in \mathbb{N}$, $Y^n(\omega) \to Y(\omega)$ uniformly on compact sets in $(\tau_k(\omega), \tau_{k+1}(\omega))$ for a.e. $\omega \in \Omega$. Therefore by (5.12) and Helly’s theorem,

$$
\int_{(\tau_k, \tau_{k+1})} (Y_s^{[n/\ell]} - \hat{\ell}_s) \, d\sigma_n(\{K_s^n\}) \to \int_{(\tau_k, \tau_{k+1})} (Y_s - \hat{\ell}_{s-}) \, dK_s \quad (5.13)
$$

for every $k, l \in \mathbb{N}$. On the other hand,

$$
(Y_s^{[n/\ell]} - \hat{\ell}_{s-}) d\sigma_n(\{K_s^n\}) = \frac{1}{n} \sum_{k=1}^{n} k(Y_s^{[n/\ell]} - \hat{\ell}_{s-})(Y_s^{k} - L_s)^- \, ds
$$

$$
= \frac{1}{n} \sum_{k<n/\ell} k(Y_s^{[n/\ell]} - \hat{\ell}_{s-})(Y_s^{k} - L_s)^- \, ds + \frac{1}{n} \sum_{[n/\ell] \leq k \leq n} k(Y_s^{[n/\ell]} - \hat{\ell}_{s-})(Y_s^{k} - L_s)^- \, ds
$$

$$
\leq \frac{2}{n} |Y_T^*| \sum_{k<n/\ell} k(Y_s^{k} - L_s)^- \, ds + \frac{1}{n} \sum_{[n/\ell] \leq k \leq n} k(Y_s^{k} - \hat{\ell}_{s})(Y_s^{k} - \hat{\ell}_{s})^- \, ds
$$

$$
\leq \frac{2}{n} |Y_T^*| \sum_{k<n/\ell} k(Y_s^{k} - L_s)^- \, ds.
$$

Hence

$$
(Y_s^{[n/\ell]} - \hat{\ell}_{s-}) d\sigma_n(\{K_s^n\}) \leq 2|Y_T^*|[n/\ell] \frac{1}{n} d\sigma_n(\{K_s^n\}).
$$

By the above, (5.13) and Helly’s theorem,

$$
0 \leq \int_{(\tau_k, \tau_{k+1})} (Y_s - \hat{\ell}_{s-}) \, dK_s \leq 2|Y_T^*| \frac{1}{T} \int_0^T \, dK_s
$$

for $k, l \in \mathbb{N}$, which implies that for every $k \in \mathbb{N}$,

$$
\int_{(\tau_k, \tau_{k+1})} (Y_s - \hat{\ell}_{s-}) \, dK_s = 0. \quad (5.14)
$$

What is left is to show that

$$
\sum_{0 < t \leq T} (Y_t - \hat{\ell}_t) \Delta K_t = 0. \quad (5.15)
$$

But (5.15) is an immediate consequence of Lemma 4.9. Indeed, if $\Delta K_t > 0$ then $-\Delta Y_t - \Delta V_t = \Delta K_t > 0$, which implies that $Y_t + \Delta V_t < Y_t$. By the last inequality and (4.2), $Y_t = \hat{L}_{t-}$. Thus, for every $t \in (0, T]$, if $\Delta K_t > 0$ then $Y_t = \hat{L}_{t-}$, which forces (5.15). By (5.14) and (5.15),

$$
\int_0^T (Y_t - \hat{L}_{t-}) \, dK_t = 0. \quad (5.16)
$$

Since (5.16) holds true for any process $\hat{L} \in \mathcal{D}$ such that $L_t \leq \hat{L}_t \leq Y_t$ for a.e. $t \in [0, T]$, the process $K$ satisfies the minimality condition.

**Theorem 5.7.** Let $p = 1$. 

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(i) Assume (H1)–(H6), (Z). Then there exists a solution \((Y, Z, K)\) of RBSDE\((\xi, f + dV, L)\) such that \((Y, Z, K) \in \mathcal{D}^q \otimes M^p \otimes \mathcal{V}^{+,1}\) for every \(q \in (0,1)\) and \(Y\) is of class (D) iff (H7*) is satisfied.

(ii) Assume (H1)–(H6), (H7*), (Z). For \(n \in \mathbb{N}\) let \((Y^n, Z^n)\) be a solution of (5.1) with \(\xi_n\) such that \(\xi_1 \in L^p(F_T), \xi_n \rightarrow^a \xi, (Y^n, Z^n) \in \bigcap_{q < 1} \mathcal{D}^q \otimes M^p\) and \(Y^n\) is of class (D). Then (5.2) holds true and there exists a stationary sequence \(\{\tau_k\}\) of stopping times such that for any \(q \in [1,2], p > 1,\)

\[E(\int_0^{\tau_k} |Z^n_s - Z_s|^q ds)^p \rightarrow 0 \quad (5.17)\]

and for every stopping time \(\tau \in \mathcal{T},\)

\[K^n_{\tau \wedge \tau_k} \rightarrow K_{\tau \wedge \tau_k} \quad \text{weakly in } L^p(F_T), \quad (5.18)\]

where \(K^n\) is defined by (5.5).

**Proof.** (i) Necessity. Follows from Proposition 3.6 by the same method as in the proof of Theorem 5.6.

Sufficiency. In much the same way as in the proof of Theorem 5.6 one can show that there exists a supersolution \((\overline{X}, \overline{Z})\) of BSDE\((\xi, f + dV, L)\) such that \((\overline{X}, \overline{Z}) \in \mathcal{D}^q \otimes M^q\) for \(q \in (0,1)\), \(\overline{X}\) is of class (D), (5.6), (5.7) with \(p = 1\) are satisfied and \(Y^n_t \leq Y^{n+1}_t, t \in [0,T]\). The only difference in the proof lies in the fact that we replace the space \(\mathcal{D}^p \otimes M^p\) by the space of processes \((Y, Z) \in \mathcal{D}^q \otimes M^q, q \in (0,1)\), such that \(Y\) is of class (D), we replace \(\mathcal{V}^{+p}\) by \(\mathcal{V}^{+1}\) and we use Theorem 3.11 instead of Theorem 3.8, and Corollary 3.2 instead of Proposition 3.1. By Lemma 4.8, there exist a stationary sequence \(\{\delta^1_k\}\) \(\in \mathcal{T}\) and constants \(c_k\) such that \(\overline{X}_{\delta^1_k} + |\overline{V}_{\delta^1_k}| + Y_{\delta^1_k} \leq c_k\). Let \(\tau_k = \delta^1_k \wedge \delta^2_k\), where

\[\delta^2_k = \inf\{t \in [0,T]; \int_0^t |f(s, 0, 0)| ds + \int_0^t |f(s, \overline{X}_s, \overline{Z}_s)| ds > k\} \wedge T.\]

The sequence \(\{\tau_k\}\) is stationary. Observe that the data \((Y^n, f, V, L)\) satisfy the assumptions of Theorem 5.6 on each interval \([0, \tau_k]\) for every \(p > 1\). Using Theorem 5.6 and stationarity of the sequence \(\{\tau_k\}\) shows that there exists a triple \((Y, Z, K) \in \mathcal{D} \otimes M \otimes \mathcal{V}^+\) such that \(t \mapsto f(t, Y_t, Z_t) \in L^1(0,T)\), (5.2), (5.17), (5.18) hold true,

\[Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad t \in [0,T]\]

and \(Y_t \geq L_t\) for a.e. \(t \in [0,T]\), \(\int_0^T (Y_{t-} - \tilde{L}_{t-}) dK_t = 0\) for every \(\tilde{L} \in \mathcal{D}\) such that \(L_t \leq \tilde{L}_t \leq Y_t\) for a.e. \(t \in [0,T]\). The proof is completed by showing integrability properties of \(Y, Z, K\). Integrability of \(Y\) follows from (5.6), monotonicity of the sequence \(\{Y^n\}\) and the fact that the processes \(Y^1, \overline{X}\) belong to \(\mathcal{D}^q\) for \(q \in (0,1)\) and are of class (D). By integrability of \(Y\) and Lemma 4.2, \(Z \in M^q, q \in (0,1)\). Let us set

\[\tau_k = \inf\{t \in [0,T]; \int_0^t |Z_s|^2 ds \geq k\} \wedge T.\]
Then
\[ K_{\tau_k} = Y_0 - Y_{\tau_k} - \int_0^{\tau_k} f(s, Y_s, Z_s) \, ds + \int_0^{\tau_k} dB_s. \]

Since \( Y \) is of class \((D)\), using Fatou’s lemma, \((H2)\), \((Z)\) and the fact that \( Y_t \leq \bar{X}_t \), \( t \in [0, T] \), we conclude from the above equality that
\[ E K_T \leq E Y_0^+ + E \xi^- + E \int_0^T f^- (s, \bar{X}_s, 0) \, ds + \gamma E \int_0^T (g_s + |Y_s| + |Z_s|)^\alpha \, ds. \]

Hence \( K \in \mathcal{V}^{+, 1} \) by \((5.7)\) with \( p = 1 \) and integrability of \((Y, Z)\). \(\square\)

### 6 BSDEs with two reflecting barriers

In this section we generalize results of Section 5 to the case of BSDEs with two irregular reflecting barriers and data in \( \mathbb{L}^p \) with \( p \in [1, 2) \).

The following natural hypotheses on the barriers generalize the so-called Mokobodzki condition.

(H8) There exists \( X \in \mathcal{H}^p \) such that \( L_t \leq X_t \leq U_t \) for a.e. \( t \in [0, T] \) and
\[ E \left( \int_0^T |f(s, X_s, 0)| \, ds \right)^p < \infty. \]

(H8*) There exists a semimartingale \( X \) of class \((D)\) such that \( X \in \mathcal{V}^1 + \mathcal{M}_q^0 \) for every \( q \in (0, 1) \), \( L_t \leq X_t \leq U_t \) for a.e. \( t \in [0, T] \) and
\[ E \int_0^T |f(s, X_s, 0)| \, ds < \infty. \]

In the sequel the abbreviation \( \text{RBSDE} \) stands for reflected BSDE with lower obstacle and \( \text{RBSDE} \) stands for reflected BSDE with upper obstacle.

**Definition.** We say that a triple \((Y, Z, R)\) of progressively measurable processes is a solution of \( \text{RBSDE}(\xi, f + dV, L, U) \) if
\begin{enumerate}[(a)]  
  
  (a) \( R \in \mathcal{V} \),  
  
  (b) \( Z \in \mathcal{M} \), the mapping \([0, T] \ni t \mapsto f(t, Y_t, Z_t)\) belongs to \( \mathbb{L}^1(0, T) \), \( P\text{-a.s.} \),  
  
  (c) \( Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dV_s + \int_t^T dR_s - \int_t^T Z_s \, dB_s \), \( t \in [0, T] \),  
  
  (d) \( L_t \leq Y_t \leq U_t \) for a.e. \( t \in [0, T] \) and
\[ \int_0^T (Y_t - Y_{t-} - \check{L}_t) \, dR_t = \int_0^T (Y_t - Y_{t-} - \check{L}_t) \, dR_{t-} = 0 \]  
\end{enumerate}

for every \( \check{L}, \check{U} \in \mathcal{D} \) such that \( L_t \leq \check{L}_t \leq Y_t \leq \check{U}_t \leq U_t \) for a.e. \( t \in [0, T] \).
Proposition 6.1. Assume (H2). Let \((Y^i, Z^i, R^i)\) be a solution of RBSDE\((\zeta^i, f^i + dV^i, L^i, U^i)\), \(i = 1, 2\). If \((Y^1 - Y^2)^+ \in \mathcal{D}^q\) for some \(q > 1\), \(\xi^1 \leq \xi^2\), \(dV^1 \leq dV^2\), \(L^1_t \leq L^2_t, U^1_t \leq U^2_t\) for a.e. \(t \in [0, T]\) and either (3.2) or (3.3) is satisfied then \(Y^1_t \leq Y^2_t\), \(t \in [0, T]\).

Proof. Without loss of generality we may assume that \(\mu \leq 0\). Let us fix \(p \in (1, q)\) and assume that (3.2) is satisfied. By the Itô-Tanaka formula and Proposition 2.1, for every \(\tau \in \mathcal{T}\),

\[
\begin{align*}
|Y^1_{t\wedge \tau} - Y^2_{t\wedge \tau} + p &\frac{(p-1)}{2} \int_{t\wedge \tau}^{\tau} 1_{\{Y^1_s \neq Y^2_s\}} |(Y^1_s - Y^2_s)^{+}|^{p-2} |Z^1_s - Z^2_s|^2 \, ds \\
&\leq |(Y^1_{\tau} - Y^2_{\tau})^{+}|^p + p \int_{t\wedge \tau}^{\tau} |(Y^1_s - Y^2_s)^{+}|^{p-1} \left( f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s) \right) \, ds \\
&\quad + p \int_{t\wedge \tau}^{\tau} |(Y^1_s - Y^2_s)^{+}|^{p-1} \left( dV^1_s - dV^2_s \right) + p \int_{t\wedge \tau}^{\tau} |(Y^1_s - Y^2_s)^{+}|^{p-1} \left( dR^1_s - dR^2_s \right) \\
&\quad - p \int_{t\wedge \tau}^{\tau} |(Y^1_s - Y^2_s)^{+}|^{p-1} \left( Z^1_s - Z^2_s \right) \, dB_s.
\end{align*}
\]

Since \(L^1_t \leq L^2_t, U^1_t \leq U^2_t\) for a.e. \(t \in [0, T]\), \(L^1_t \leq Y^1_t \land Y^2_t \leq Y^1_t, Y^2_t \leq Y^1_t \lor Y^2_t \leq U^2_t\) for a.e. \(t \in [0, T]\). By monotonicity of the function \(x \mapsto p|x|^{p-1}x\) and property (d) of the definition of a solution of a reflected BSDE,

\[
\begin{align*}
\int_{t\wedge \tau}^{\tau} |(Y^1_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}|^{p-1} \left( dR^1_s - dR^2_s \right) \leq \int_{t\wedge \tau}^{\tau} |(Y^1_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}|^{p-1} d(R^1_{s\wedge \tau} + R^2_{s\wedge \tau}) \\
&= \int_{t\wedge \tau}^{\tau} |(Y^1_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}|^{p-1} \text{sgn}[(Y^1_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}] \, dR^1_{s\wedge \tau} \\
&\quad + \int_{t\wedge \tau}^{\tau} |(Y^1_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}|^{p-1} \text{sgn}[(Y^1_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}] \, dR^2_{s\wedge \tau} \\
&\leq \int_{t\wedge \tau}^{\tau} |(Y^1_{s\wedge \tau} - Y^1_{s\wedge \tau} \land Y^2_{s\wedge \tau})^{+}|^{p-1} \text{sgn}[(Y^1_{s\wedge \tau} - Y^1_{s\wedge \tau} \land Y^2_{s\wedge \tau})^{+}] \, dR^1_{s\wedge \tau} \\
&\quad + \int_{t\wedge \tau}^{\tau} |(Y^1_{s\wedge \tau} \lor Y^2_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}|^{p-1} \text{sgn}[(Y^1_{s\wedge \tau} \lor Y^2_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}] \, dR^2_{s\wedge \tau} = 0,
\end{align*}
\]

the last equality being a consequence of the fact that

\[
\int_{t\wedge \tau}^{\tau} \text{sgn}[(Y^1_{s\wedge \tau} - Y^1_{s\wedge \tau} \land Y^2_{s\wedge \tau})^{+}] \, dR^1_{s\wedge \tau}
= \int_{t\wedge \tau}^{\tau} 1_{\{Y^1_{s\wedge \tau} \neq Y^1_{s\wedge \tau} \land Y^2_{s\wedge \tau}\}} |Y^1_{s\wedge \tau} - Y^1_{s\wedge \tau} \land Y^2_{s\wedge \tau}|^{-1} d\left( \int_0^s (Y^1_{r\wedge \tau} - Y^1_{r\wedge \tau} \land Y^2_{r\wedge \tau}) \, dR^1_{r\wedge \tau} \right)
\]

and

\[
\int_{t\wedge \tau}^{\tau} \text{sgn}[(Y^1_{s\wedge \tau} \lor Y^2_{s\wedge \tau} - Y^2_{s\wedge \tau})^{+}] \, dR^2_{s\wedge \tau}
= \int_{t\wedge \tau}^{\tau} 1_{\{Y^1_{s\wedge \tau} \lor Y^2_{s\wedge \tau} \neq Y^2_{s\wedge \tau}\}} |Y^1_{s\wedge \tau} \lor Y^2_{s\wedge \tau} - Y^2_{s\wedge \tau}|^{-1} d\left( \int_0^s (Y^1_{r\wedge \tau} \lor Y^2_{r\wedge \tau} - Y^2_{r\wedge \tau}) \, dR^2_{r\wedge \tau} \right).
\]

The rest of the proof runs as the proof of Proposition 3.1 (see the reasoning following (3.4)). \(\square\)
Corollary 6.2. Assume that (H2), (Z) are satisfied. For \( i = 1, 2 \) let \((Y^i, Z^i, R^i)\) be a solution of \( \text{RBSDE}(\xi^i, f^i + dV^i, L^i, U^i) \) such that \((Y^i, Z^i) \in L^q(\mathcal{F}) \otimes L^q(\mathcal{F})\) for some \( q > \alpha \). If \((Y^1 - Y^2)^+\) is of class (D), \( \xi^1 \leq \xi^2, dV^1 \leq dV^2, L^1_t \leq L^2_t, U^1_t \leq U^2_t \) for a.e. \( t \in [0, T] \) and (3.2) or (3.3) is satisfied then \( Y^1_t \leq Y^2_t, t \in [0, T] \).

Proof. The proof is analogous to that of Proposition 6.1, the only difference being in the fact that in the latter part of the proof we now refer to the proof of Corollary 3.2 instead of the proof of Proposition 3.1.

Theorem 6.3. Assume (H2), (H3). Then there exists at most one solution \((Y, Z, R)\) of \( \text{RBSDE}(\xi, f + dV, L, U) \) such that \( Y \in \mathcal{D}^p \) for some \( p > 1 \).

Proof. Follows immediately from Proposition 6.1.

Theorem 6.4. Assume (H2), (H3), (Z). Ten there exists at most one solution \((Y, Z, R)\) of \( \text{RBSDE}(\xi, f + dV, L, U) \) such that \( Y \) is of class (D) and \( Z \in \bigcup_{\beta > \alpha} M^\beta \).

Proof. Follows immediately from Corollary 6.2.

Theorem 6.5. Let \( p > 1 \). Assume that (H1)–(H6) are satisfied.

(i) There exists a solution \((Y, Z, R) \in \mathcal{D}^p \otimes M^p \otimes \mathcal{V}^p\) of \( \text{RBSDE}(\xi, f + dV, L, U) \) iff (H8) is satisfied.

(ii) Let \((Y^{n,n}, Z^{n,n}) \in \mathcal{D}^p \otimes M^p\) be a solution of the BSDE

\[
Y^{n,n}_t = \xi_n + \int_t^T f(s, Y^{n,n}_s, Z^{n,n}_s) \, ds + \int_t^T dV_s
+ n \int_t^T (Y^{n,n}_s - L_s)^- \, ds - n \int_t^T (Y^{n,n}_s - U_s)^+ \, ds - \int_t^T Z^{n,n}_s \, dB_s, \quad t \in [0, T]
\]

with \( \xi_n \) such that there exist \( \xi^1_n, \xi^2_n \in L^p(\mathcal{F}_T) \) with the property that \( \xi^1_n \leq \xi_n \leq \xi^2_n \), \( \xi^1_n \searrow \xi, \xi^2_n \nearrow \xi \). Then \( Y^{n,n}_t \to Y_t, \ t \in [0, T], \ Z^{n,n} \to Z, \ \lambda \otimes P - \text{a.e.} \)

and for every \( q \in [1, 2), r \in [1, p), \)

\[
E\left(\int_0^T |Z^{n,n}_s - Z_s|^q \, ds\right)^{r/q} \to 0. \quad (6.2)
\]

(iii) Let \((\bar{Y}^n, \bar{Z}^n, \bar{A}^n) \in \mathcal{D}^p \otimes M^p \otimes \mathcal{V}^{+p}\) be a solution of \( \overline{\text{RBSDE}}(\bar{\xi}_n, \bar{f}_n + dV, U) \) with

\[
\bar{f}_n(t, y, z) = f(t, y, z) + n(y - L_t)^-
\]

and \( \bar{\xi}_n \in L^p(\mathcal{F}_T) \) such that \( \bar{\xi}_n \searrow \xi \). Then \( \bar{Y}^n_t \searrow Y_t, \ t \in [0, T], \ \bar{Z}^n \to Z, \ \lambda \otimes P - \text{a.e.} \)

and

\[
d\bar{A}^n \leq d\bar{A}^{n+1}, n \in \mathbb{N}, \ \bar{A}_t^n \nearrow R_t, \ t \in [0, T],
\]
for every $q \in [1, 2)$, $r \in [1, p)$,
\[
E\left(\int_0^T |\bar{Z}_s^n - Z_s|^q ds\right)^{r/q} \to 0,
\]
and for every $\tau \in \mathcal{T}$,
\[
\bar{K}_\tau^n \to R_\tau^+ \quad \text{weakly in } L^p(\mathcal{F}_\tau),
\]
where $\bar{K}_\tau^n = n \int_0^\tau (\bar{Y}_s^n - L_s)^- ds$.

(iv) Let $(\bar{Y}_m^n, \bar{Z}_m^n, \bar{K}_m^n) \in \mathcal{D}^p \otimes M^p \otimes \mathcal{V}^{+,p}$ be a solution of \textit{RBSDE}(\xi_m, \bar{F}_m + dV, L) with
\[
\bar{F}_m(t, y, z) = f(t, y, z) - m(y - U_t)^+
\]
and $\xi_m \in L^p(\mathcal{F}_T)$ such that $\xi_m \nabla \xi$. Then
\[
\bar{Y}_m^n \nabla Y_t, \quad t \in [0, T], \quad \bar{Z}_m^n \to Z, \quad \lambda \otimes P\text{-a.e.}
\]
and
\[
dK_m^n \leq dK_m^{n+1}, \quad m \in \mathbb{N}, \quad K_m^n \nearrow R_t^+, \quad t \in [0, T],
\]
for every $q \in [1, 2)$, $r \in [1, p)$,
\[
E\left(\int_0^T |\bar{Z}_s^n - Z_s|^q ds\right)^{r/q} \to 0,
\]
and for every $\tau \in \mathcal{T}$,
\[
\bar{A}_\tau^m \to R_\tau^- \quad \text{weakly in } L^p(\mathcal{F}_\tau),
\]
where $\bar{A}_\tau^m = \int_0^\tau m(\bar{Y}_s^n - U_s)^+ ds$.

(v) If $L, U, V$ are continuous and $L_T \leq \xi \leq U_T$ then as $n, m \to +\infty$,
\[
E\sup_{0 \leq t \leq T} |A_t^n - R_t^-|^p + E\sup_{0 \leq t \leq T} |\bar{K}_t^n - R_t^+|^p \to 0,
\]
\[
E\sup_{0 \leq t \leq T} |\bar{Y}_t^n - Y_t|^p + E\sup_{0 \leq t \leq T} |Y_t^m - Y_t|^p + E\sup_{0 \leq t \leq T} |Y_t^{n,m} - Y_t|^p \to 0
\]
and
\[
E\left(\int_0^T |\bar{Z}_s^{n,m} - Z_s|^2 ds\right)^{p/2} + E\left(\int_0^T |\bar{Z}_s^n - Z_s|^2 ds\right)^{p/2}
\]
\[
+ E\left(\int_0^T |\bar{Z}_s^n - Z_s|^2 ds\right)^{p/2} \to 0. \quad (6.3)
\]

(vi) For every $\hat{U}, \hat{L} \in \mathcal{D}$ such that $L_t \leq \hat{L}_t \leq Y_t \leq \hat{U}_t \leq U_t$ for a.e. $t \in [0, T]$,
\[
\Delta R_t^+ = (\hat{L}_t - Y_t - \Delta V_t)^+, \quad \Delta R_t^- = (Y_t - \hat{U}_t - \Delta V_t)^+, \quad t \in (0, T].
\]
Proof. Necessity. Let \((Y, Z, R) \in D^p \otimes M^p \otimes V^p\) be a solution of RBSDE\((\xi, f + dV, L, U)\). Then \(L_t \leq Y_t \leq U_t\) for a.e. \(t \in [0, T]\) and by Proposition 3.6,

\[
E\left(\int_0^T |f(s, Y_s, Z_s)| \, ds\right)^p < \infty,
\]

which implies that \(Y \in H^p\). Thus, (H8) is satisfied with \(X = Y\).

Sufficiency. To shorten notation we give the proof under the assumption that \(\bar{\xi} = \xi\) and \(\bar{X} = X\). The proof in the general case is analogous. It only requires some obvious changes. By Theorem 3.8, for every \(n, m \in \mathbb{N}\) there exists a unique solution \((Y^{n,m}, Z^{n,m}) \in D^p \otimes M^p\) of the BSDE

\[
Y_t^{n,m} = \xi + \int_t^T f(s, Y_s^{n,m}, Z_s^{n,m}) \, ds + \int_t^T dV_s + n \int_t^T (Y_s^{n,m} - L_s)^- \, ds
- m \int_t^T (Y_s^{n,m} - U_s)^+ \, ds - \int_t^T Z_s^{n,m} \, dB_s, \quad t \in [0, T].
\]  

(6.4)

Set

\[
A_t^{n,m} = m \int_0^t (Y_s^{n,m} - U_s)^+ \, ds, \quad K_t^{n,m} = n \int_0^t (Y^{n,m} - L_s)^- \, ds, \quad t \in [0, T].
\]  

(6.5)

Step 1. We first show that

\[
\sup_{n,m \geq 1} \left( E\left(\int_0^T dK_t^{n,m}\right)^p + E\left(\int_0^T dA_t^{n,m}\right)^p\right) < \infty.
\]  

(6.6)

Since \(X \in H^p\) and the Brownian filtration has the representation property, there exist \(C \in V^p, H \in M^p\) such that

\[
X_t = X_0 - \int_0^t dC_s - \int_0^t H_s dB_s.
\]

The above formula may be rewritten in the form

\[
X_t = X_T + \int_t^T f(s, X_s, H_s) \, ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T dA_s - \int_t^T H_s dB_s,
\]

where

\[
K_t = \int_0^t (f^{-}(s, X_s, H_s) \, ds + dC_s^- + dV_s^-), \quad A_t = \int_0^t (f^{+}(s, X_s, H_s) \, ds + dC_s^+ + dV_s^+).
\]

By Theorem 3.8, for every \(m \in \mathbb{N}\) there exists a solution \((\bar{X}^m, H^m) \in D^p \otimes M^p\) of the BSDE

\[
\bar{X}_t^m = X_T \vee \xi + \int_t^T f(s, \bar{X}_s^m, \bar{H}_s^m) \, ds + \int_t^T dV_s + \int_t^T dK_s'
- \int_t^T m(\bar{X}_s^m - U_s)^+ \, ds - \int_t^T \bar{H}_s^m dB_s, \quad t \in [0, T].
\]  

(6.7)
Since $L_t \leq X_t \leq U_t$ for a.e. $t \in [0, T]$, we have
\[
X_t = X_T + \int_t^T f(s, X_s, Z_s) \, ds + \int_t^T dV_s - m \int_t^T (X_s - U_s)^+ \, ds
+ \int_t^T dK'_s - \int_t^T dA'_s - \int_t^T H_s dB_s, \quad t \in [0, T].
\]

Hence, by Proposition 3.1, $X^m_t \geq X_t$ for a.e. $t \in [0, T]$, which implies that $X^m_t \geq L_t$ for a.e. $t \in [0, T]$. Consequently,
\[
\bar{X}^m_t = X_T \vee \xi + \int_t^T f(s, \bar{X}^m_s, \bar{H}^m_s) \, ds + \int_t^T dV_s + \int_t^T dK'_s
+ n \int_t^T (\bar{X}^m_s - L_s)^- \, ds - m \int_t^T (\bar{X}^m_s - U_s)^+ \, ds - \int_t^T \bar{H}^m_s dB_s, \quad t \in [0, T].
\]

Applying once again Proposition 3.1 we see that $\bar{X}^m_t \geq Y^{n,m}_t$, $t \in [0, T]$, for every $n, m \in \mathbb{N}$. Thus,
\[
dA^{n,m} = m(Y^{n,m}_s - U_s)^+ \, ds \leq m(\bar{X}^m_s - U_s)^+ \, ds, \quad n, m \in \mathbb{N}. \tag{6.8}
\]

Observe now that $(-\bar{X}^m, -\bar{H}^m)$ is a supersolution of BSDE $(-\xi \vee X_T, \bar{f} - dV - dK')$ with
\[
\bar{f}(t, y, z) = -f(t, -y, -z).
\]

Since $\bar{K}' \in \mathcal{Y}^p$, $-\bar{X}^m_t \leq -X_t$, $t \in [0, T]$, $X \in \mathcal{D}^p$ and $E(\int_0^T \bar{f}^-(s, -X_s, 0) \, ds)^p < \infty$, it follows from Proposition 4.3 that there exists $C > 0$ not depending on $n$ such that $E(\int_0^T dA^{n,m}_s)^p \leq C$. The same conclusion can be drawn for $K^{n,m}$. To see this it suffices to consider a solution $(\bar{X}^n, \bar{H}^n) \in \mathcal{D}^p \otimes M^p$ of the BSDE
\[
\bar{X}^n_t = X_T \wedge \xi + \int_t^T f(s, \bar{X}^n_s, \bar{H}^n_s) \, ds + \int_t^T dV_s
+ \int_t^T n(\bar{X}^n_s - L_s)^- \, ds - \int_t^T dA'_s - \int_t^T \bar{H}^n_s dB_s, \quad t \in [0, T] \tag{6.9}
\]

and then repeat (with some obvious changes) arguments following (6.7).

Step 2. We will show that there exists a triple $(Y, Z, R) \in \mathcal{D}^p \otimes M^p \otimes \mathcal{Y}^p$ which is in some sense a limit of triple $(\bar{Y}^n, \bar{Z}^n, \bar{K}^n - \bar{A}^n)$ and
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dV_s + \int_t^T dR_s - \int_t^T Z_s dB_s, \quad t \in [0, T]. \tag{6.10}
\]

Let us first observe that
\[
dA^n \leq d\bar{A}^{n+1}, \quad n \in \mathbb{N}. \tag{6.11}
\]

Indeed, by Theorem 5.6, for every $t \in [0, T]$, $\bar{A}^n_t, \bar{A}^{n+1}_t$ are weak limits in $L^p(\mathcal{F}_T)$ of
\[
\{A^{n,m}_t\} \text{ and } \{A^{n+1,m}_t\}, \text{ respectively. This implies (6.11) because by Proposition 3.1,}
Y^{n,m}_t \leq Y^{n+1,m}_t, \quad t \in [0, T], \text{ for every } n, m \in \mathbb{N}, \text{ and consequently}
\]
\[
dA^{n,m} = m(Y^{n,m}_t - U_t)^+ \, dt \leq m(Y^{n+1,m}_t - U_t)^+ \, dt = dA^{n+1,m}. 
\]

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Set

\[ A_t = \sup_{n \geq 1} \bar{A}^n_t, \quad t \in [0, T]. \]  \hfill (6.12)

By (6.6), (6.11) and [27, Lemma 2.2], \( A \in V^{+, p} \). Now observe that by Proposition 3.1 and Theorem 5.6,

\[ \bar{y}_t \leq Y^{n,m}_t \leq \bar{y}_t, \quad t \in [0, T], \quad n, m \in \mathbb{N}, \]  \hfill (6.13)

where \((\bar{y}, \bar{z}, \bar{k}) \in D^p \otimes M^p \otimes V^{+, p}\) (resp. \((y, z, k) \in D^p \otimes M^p \otimes V^{+, p}\)) is a solution of \( \bar{\text{RBSDE}}(\xi, f + dV, U) \) (resp. \( \text{RBSDE}(\xi, f + dV, L) \)). By Theorem 5.6,

\[ Y^{n,m}_t \leq \bar{Y}^n_t, \quad t \in [0, T], \quad n, m \in \mathbb{N}. \]  \hfill (6.14)

Hence

\[ \bar{y}_t \leq \bar{Y}^n_t \leq y_t, \quad t \in [0, T], \quad n \in \mathbb{N}. \]  \hfill (6.15)

By Proposition 5.1, for every \( n \in \mathbb{N} \), \( \bar{Y}^n_t \leq \bar{Y}^{n+1}_t \) for \( t \in [0, T] \). Therefore setting \( Y_t = \sup_{n \geq 1} \bar{Y}^n_t, \quad t \in [0, T] \), we see that

\[ Y_t = \lim_{n \to +\infty} \bar{Y}^n_t, \quad t \in [0, T]. \]  \hfill (6.16)

From Step 1 and what has already been proved in Step 2 we conclude that assumptions (a), (c)-(f) of Theorem 4.12 are satisfied. Let \( \sigma, \tau \in \mathcal{T} \) be such that \( \sigma \leq \tau \). Then by (6.11), (6.12), (6.15) and (6.16),

\[
\int_\sigma^\tau (Y_s - \bar{Y}^n_s) d(\bar{K}^n_s - \bar{A}^n_s + V_s) \geq - \int_\sigma^\tau (Y_s - \bar{Y}^n_s) d(\bar{A}^n_s - V_s) \\
\geq - \int_\sigma^\tau (Y_s - \bar{Y}^n_s) d(A_s - V_s) \to 0.
\]

This shows that assumption (b) of Theorem 4.12 is satisfied as well. By Theorem 4.12 and (6.15), (6.6) there exists a quadruple \((Y, Z, K, A) \in D^p \otimes M \otimes V^{+, p} \otimes V^{+, p}\) such that

\[ \bar{Z}^n \to Z, \quad \lambda \otimes P\text{-a.e.} \]  \hfill (6.17)

and (6.10) is satisfied with \( R = K - A \). Using (6.6), Proposition 3.5 and Proposition 3.6 we conclude from (6.4) that there exists \( C \) not depending on \( n, m \) such that

\[
E(\int_0^T |f(s, Y^{n,m}_s, Z^{n,m}_s)| ds)^p + E(\int_0^T |Z^{n,m}_s|^2 ds)^{p/2} \leq C.
\]  \hfill (6.18)

From this we deduce that

\[ \bar{K}^n_t \to K_\tau \quad \text{weakly in } L^p(\mathcal{F}_T) \]  \hfill (6.19)

for every \( \tau \in \mathcal{T}, \ f(\cdot, Y, Z) \in L^{1,p}(\mathcal{F}), \ Z \in M^p \) and (6.2) holds true for every \( q \in [1, 2], \ r \in [1, p] \).

Step 3. We will show that \( K, A \) satisfy the minimality condition. From (6.4), (6.6) and (6.18) it may be concluded that there exists \( C \) not depending on \( n, m \) such that

\[
E(\int_0^T (Y^{n,m}_s - L_s)^- ds)^p + E(\int_0^T (Y^{n,m}_s - U_s)^+ ds)^p \leq C(n^{-p} + m^{-p}).
\]
By the above inequality, (6.14) and (6.16),

\[ L_t \leq Y_t \leq U_t \quad \text{for a.e. } t \in [0, T]. \] (6.20)

Let us fix \( U \in \mathcal{D} \) such that \( Y_t \leq U_t \leq U_t \) for a.e. \( t \in [0, T] \). By (6.11) and (6.12), \( d\bar{A}^n \rightarrow dA \) in the variation norm. Hence

\[
\int_0^T (\bar{U}_{s^-} - Y_{s^-})d\bar{A}_s^n \rightarrow \int_0^T (\bar{U}_{s^-} - Y_{s^-})dA_s.
\]

On the other hand,

\[
0 \leq \int_0^T (\bar{U}_{s^-} - Y_{s^-})d\bar{A}_s^n \leq \int_0^T (\bar{U}_{s^-} - \bar{Y}_{s^-})d\bar{A}_s^n = 0,
\]

because the triple \((\bar{Y}^n, \bar{Z}^n, \bar{A}^n)\) is a solution of \( \bar{\text{RBSDE}}(\xi, \bar{f}_n + dV, \bar{U}) \). Consequently,

\[
\int_0^T (\bar{U}_{s^-} - Y_{s^-})dA_s = 0, \quad (6.21)
\]

i.e. \( A \) satisfies the minimality condition. Suppose now that \( \Delta A_t > 0 \). Since \( d\bar{A}_s^n \rightarrow dA \) in the variation norm, there exists \( n_0 \in \mathbb{N} \) such that \( \Delta \bar{A}_t^n > 0 \) for every \( n \geq n_0 \). Since \( \bar{A}^n \) satisfies the minimality condition, \( \Delta \bar{A}_t^n = \bar{Y}_{t^n} - \bar{Y}_{t^n} + \Delta V_t = \bar{Y}_{t^n} - \bar{U}_{t^n} + \Delta V_t \), so using once again the fact that \( d\bar{A}_s^n \rightarrow dA \) we get

\[
\Delta A_t = Y_t - \bar{U}_{t^-} + \Delta V_t. \quad (6.22)
\]

Since \( A \) satisfies the minimality condition, \( Y_t = \bar{U}_{t^-} \). Hence

\[
Y_t - \bar{U}_{t^-} = \Delta Y_t = -\Delta K_t - \Delta V_t + \Delta A_t = -\Delta K_t + Y_t - \bar{U}_{t^-},
\]

which forces \( \Delta K_t = 0 \). Thus, we have shown that for every \( t \in [0, T] \), if \( \Delta K_t > 0 \) then \( \Delta A_t = 0 \). Let us fix \( \hat{L} \in \mathcal{D} \) such that \( L_t \leq \hat{L}_t \leq Y_t \) for a.e. \( t \in [0, T] \). By Lemma 4.9,

\[
Y_{t^-} = \hat{L}_{t^-} \lor (Y_t + \Delta V_t - \Delta A_t), \quad t \in (0, T]. \quad (6.23)
\]

Suppose that \( \Delta K_t > 0 \). Then \( Y_{t^-} = \hat{L}_{t^-} \lor (Y_t + \Delta V_t) \) since \( \Delta A_t = 0 \). On the other hand, \( \Delta K_t = -\Delta Y_t - \Delta V_t \), which implies that \( Y_{t^-} > Y_t + \Delta V_t \). The last inequality when combined with (6.23) shows that \( Y_{t^-} > \hat{L}_{t^-} \). Thus, for every \( t \in (0, T] \), if \( \Delta K_t > 0 \) then \( Y_{t^-} = \hat{L}_{t^-} \). Consequently,

\[
\sum_{0 < t \leq T} (Y_{t^-} - \hat{L}_{t^-}) \Delta K_t = 0. \quad (6.24)
\]

Let \( \{\tau_k\} \) be an increasing sequence of successive jumps of \( K, A, V \) with the convention that \( \tau_0 \equiv 0 \). Then by Dini’s theorem, \( Y^n \rightarrow Y \) uniformly on compact subsets of \( (\tau_k, \tau_{k+1}) \) for each \( k \in \mathbb{N} \). Since \( \{\bar{A}^n\} \) converges pointwise, in much the same way as in Step 3 of the proof of Theorem 5.6 one can show that for every \( k \in \mathbb{N} \),

\[
\int_{(\tau_k, \tau_{k+1})} (Y_{t^-} - \hat{L}_{t^-}) dK_t = 0.
\]
Combining this with (6.24) we get

\[ \int_0^T (Y_{t-} - \hat{L}_{t-}) \, dK_t = 0, \tag{6.25} \]

which proves that \( K \) satisfies the minimality condition.

Step 4. We will show (i). Write \( R_t = K_t - A_t \). Since \( K, A \in \mathcal{V}^{+,p} \), it follows from the minimality property of the Jordan decomposition of signed measures that \( dR^+ \leq dK \), \( dR^- \leq dA \). From this and (6.21), (6.25) we get (6.1), which when combined with (6.10), (6.20) and integrability properties of \( Y, Z, K, A \) proved in Step 2 shows that the triple \( (Y, Z, R) \in \mathcal{D}^p \otimes M^p \otimes \mathcal{V}^p \) is a solution of RBSDE\( (\xi, f + dV, L, U) \).

Step 5. We will show (iii) and (iv). By what has already been proved in Steps 1-3, assertion (iii) will be proved once we prove that \( A_t = R_t^- \), \( K_t = R_t^+ \), \( t \in [0,T] \). To prove these equalities let us first note that by the same method as in the proof of (6.8), but with the process \( X \) replaced by \( Y \), the processes \( K', A' \) replaced by \( R^+, R^- \) and \( (\bar{X}^m, \bar{H}^m) \) replaced by the solution \( (\tilde{Y}^m, \tilde{Z}^m, \tilde{A}^m) \) of the BSDE

\[
\tilde{Y}_t^m = \xi + \int_t^T f(s, \tilde{Y}_s^m, \tilde{Z}_s^m) \, ds + \int_t^T dV_s + \int_t^T dR_s^+ - m \int_t^T (\tilde{Y}_s^m - U_s) \, ds - \int_t^T \tilde{Z}_s^m \, dB_s, \quad t \in [0,T]
\]

one can show that

\[ dA_{t}^{n,m} \leq d\tilde{A}_{t}^{m}, \quad n, m \in \mathbb{N}, \tag{6.26} \]

where \( \tilde{A}_{t}^{m} = m \int_0^t (\tilde{Y}_s^m - U_s) \, ds, t \in [0,T] \). By Theorems 5.4 and 5.6, the sequence \( \{(\tilde{Y}^m, \tilde{Z}^m, \tilde{A}^m)\} \) converges in the sense of Theorem 5.6 to \( (Y, Z, R^-) \). Since the arithmetic mean preserves inequalities, without loss of generality we may assume that

\[ A_{t}^{n,m} \to \bar{A}_{t}^{n}, \quad \tilde{A}_{t}^{m} \to R_t^-, \quad t \in [0,T] \]

(see the reasoning in Step 3 of the proof of Theorem 5.6). Therefore from (6.11), (6.12), and (6.26) it follows that \( dA \leq dR^- \). By the minimality property of the Jordan decomposition, \( R_t^- = A_t, t \in [0,T] \), and consequently \( R_t^+ = K_t, t \in [0,T] \), which completes the proof of (iii). Applying similar arguments to \( \{(Y^m, Z^m, K^m, A^m)\} \) proves (iv).

Step 6. We will show (ii). By Proposition 3.1,

\[ \tilde{Y}_t^m \leq Y_t^{n,m} \leq Y_t^n, \quad t \in [0,T]. \tag{6.27} \]

By (6.27) and (iii), (iv),

\[ Y_t^{n,m} \to Y_t, \quad t \in [0,T]. \tag{6.28} \]

Therefore the assumptions (a), (c)-(f) of Lemma 4.11 are satisfied. By (6.13), (6.20),
(6.28) for every $\sigma, \tau \in T$ such that $\sigma \leq \tau$,

$$
\int_{\sigma}^{\tau} (Y_s - Y_s^{n,n}) d(K^{n,n} - A^{n,n}) + V)_s = n \int_{\sigma}^{\tau} (Y_s - Y_s^{n,n})(Y_s^{n,n} - L_s)^{-} \, ds \\
- n \int_{\sigma}^{\tau} (Y_s - Y_s^{n,n})(Y_s^{n,n} - U_s)^{+} \, ds + \int_{\sigma}^{\tau} (Y_s - Y_s^{n,n}) \, dV_s \\
\geq n \int_{\sigma}^{\tau} (L_s - Y_s^{n,n})(Y_s^{n,n} - L_s)^{-} \, ds - n \int_{\sigma}^{\tau} (U_s - Y_s^{n,n})(Y_s^{n,n} - U_s)^{+} \, ds \\
+ \int_{\sigma}^{\tau} (Y_s - Y_s^{n,n}) \, dV_s \geq \int_{\sigma}^{\tau} (Y_s - Y_s^{n,n}) \, dV_s \to 0.
$$

This shows that assumption (b) of Lemma 4.11 is satisfied as well. Consequently, $Z^{n,n} \to Z, \lambda \otimes P$-a.e., which together with (6.18) implies (6.2).

Step 7. We will show (v) and (vi). Observe that (vi) follows immediately from (6.21), (6.22), (6.25) and the equalities $K = R^{+} A = R^{-}$ proved in Step 5. From (vi) and the fact that $V$ is continuous it follows that the processes $K, A, Y$ are continuous. By Theorem 5.6 the processes $\bar{Y}^{n}, \bar{K}^{n}, \underline{Y}^{m}, \underline{A}^{m}$ are continuous as well. Using Dini’s theorem, integrability of $\bar{Y}^{n}, \bar{K}^{n}, \underline{Y}^{m}, \underline{A}^{m}, A, Y, (6.27)$ and the Lebesgue dominated convergence theorem shows the desired convergence of the sequences $\{Y^{n}\}, \{\bar{K}^{n}\}, \{\underline{Y}^{m}\}, \{\underline{A}^{m}\}$. To prove (6.3) let us first observe that by Itô’s formula, (H2) and (H3),

$$
\int_{0}^{T} |Z_{t}^{n,n} - Z_{t}|^{2} \, dt \leq 2 \lambda \int_{0}^{T} |Y_{t}^{n,n} - Y_{t}| |Z_{t}^{n,n} - Z_{t}| \, dt + 2 \int_{0}^{T} |Y_{t}^{n,n} - Y_{t}| \, dR_{t}^{+} \\
+ 2 \int_{0}^{T} |Y_{t}^{n,n} - Y_{t}| \, dR_{t}^{-} + \sup_{0 \leq t \leq T} \left| \int_{t}^{T} (Z_{s}^{n} - Z_{s}^{m}) (Y_{s}^{n} - Y_{s}^{m}) \, dB_s \right|.
$$

Hence

$$
E \left( \int_{0}^{T} |Z_{t}^{n,n} - Z_{t}|^{2} \, dt \right)^{p/2} \leq C(p, \lambda) \left( E \sup_{0 \leq t \leq T} |Y_{t}^{n,n} - Y_{t}|^{p} \\
+ (E \sup_{0 \leq t \leq T} |Y_{t}^{n,n} - Y_{t}|^{p})^{1/2} (E|R_{T}^{+}|^{p})^{1/2} + (E \sup_{0 \leq t \leq T} |Y_{t}^{n,n} - Y_{t}|^{p})^{1/2} (E|R_{T}^{-}|^{p})^{1/2} \right),
$$

which together with uniform convergence of $\{Y^{n,n}\}$ implies (6.3). Since the proof of the other convergences in (v) is similar, we omit it. $\Box$

**Theorem 6.6.** Assume that (H1)–(H6), (Z) are satisfied with $p = 1$.

(i) There exists a solution $(Y, Z, R) \in \mathcal{D}^{q} \otimes M^{q} \otimes \mathcal{V}^{q}, q \in (0, 1)$, such that $Y$ is of class (D) iff (H8*) is satisfied.

(ii) Let $(Y^{n,n}, Z^{n,n}) \in \mathcal{D}^{q} \otimes M^{q}, q \in (0, 1)$, such that $Y^{n,n}$ is of class (D) be a solution of the BSDE

$$
Y_{t}^{n,n} = \xi + \int_{t}^{T} f(s, Y_{s}^{n,n}, Z_{s}^{n,n}) \, ds + \int_{t}^{T} dV_{s} + n \int_{t}^{T} (Y_{s}^{n,n} - L_{s})^{-} \, ds \\
- n \int_{t}^{T} (Y_{s}^{n,n} - U_{s})^{+} \, ds - \int_{t}^{T} Z_{s}^{n,n} \, dB_{s}, \quad t \in [0, T].
$$
Then

\[ Y_t^{n,n} \to Y_t, \quad t \in [0,T], \quad Z_t^{n,n} \to Z, \quad \lambda \otimes P\text{-a.e.} \]

and there exists a stationary sequence \( \{\tau_k\} \subset \mathcal{T} \) such that for every \( q \in [1, 2) \), \( r > 1 \),

\[ E(\int_{0}^{\tau_k} |Z_s^{n,n} - Z_s|^q \, ds)^r \to 0. \]

(iii) Let \((\bar{Y}^n, \bar{Z}^n, \bar{A}^n) \in \mathcal{D}^q \otimes M^q \otimes \mathcal{V}^{+1}, q \in (0, 1)\), such that \( \bar{Y}^n \) is of class (D) be a solution of \( \bar{RBSDE}(\xi, \bar{f}_n + dV, U) \) with

\[ \bar{f}_n(t, y, z) = f(t, y, z) + n(y - L_t). \]

Then

\[ \bar{Y}_t^n \nearrow Y_t, \quad t \in [0, T], \quad \bar{Z}_t^n \to Z, \quad \lambda \otimes P\text{-a.e.}, \]

\[ d\bar{A}_t^n \leq d\bar{A}_t^{n+1}, \quad n \in \mathbb{N}, \quad \bar{A}_t^n \nearrow R_t^+, \quad t \in [0,T] \]

and there exists a stationary sequence \( \{\tau_k\} \subset \mathcal{T} \) such that for every \( \tau \in \mathcal{T} \) and \( q \in [1, 2) \), \( r > 1 \),

\[ E(\int_{0}^{\tau_k} |\bar{Z}_s^n - Z_s|^q \, ds)^r \to 0, \quad \bar{K}_t^{n,\tau} \to R_{\tau,\tau}^+ \quad \text{weakly in } \mathbb{L}^r(\mathcal{F}_T), \]

where \( \bar{K}_t^n = n\int_{0}^{t} (\bar{Y}_s^n - L_s)^- \, ds \).

(iv) Let \((\bar{Y}_t^m, \bar{Z}_t^m, \bar{K}_t^m) \in \mathcal{D}^q \otimes M^q \otimes \mathcal{V}^{+1}, q \in (0, 1)\), such that \( \bar{Y}_t^m \) is of class (D) be a solution of \( \bar{RBSDE}(\xi, \bar{f}_m + dV, L) \) with

\[ \bar{f}_m(t, y, z) = f(t, y, z) - m(y - U_t)^+. \]

Then

\[ \bar{Y}_t^m \nearrow Y_t, \quad t \in [0, T], \quad \bar{Z}_t^m \to Z, \quad \lambda \otimes P\text{-a.e.}, \]

\[ d\bar{K}_t^m \leq d\bar{K}_t^{m+1}, \quad m \in \mathbb{N}, \quad \bar{K}_t^m \nearrow R_t^+, \quad t \in [0,T] \]

and there exists a stationary sequence \( \{\tau_k\} \subset \mathcal{T} \) such that for every \( \tau \in \mathcal{T} \) and \( q \in [1, 2) \), \( r > 1 \),

\[ E(\int_{0}^{\tau_k} |\bar{Z}_s^m - Z_s|^q \, ds)^r \to 0, \quad \bar{A}_t^m \to R_{\tau,\tau}^- \quad \text{weakly in } \mathbb{L}^r(\mathcal{F}_T), \]

where \( \bar{A}_t^m = m\int_{0}^{t} (\bar{Y}_s^m - U_s)^+ \, ds \).

(v) If \( L, U, V \) are continuous and \( L_T \leq \xi \leq U_T \) then there exists a stationary sequence \( \{\tau_k\} \subset \mathcal{T} \) such that for every \( p > 1 \),

\[ E \sup_{0 \leq t \leq \tau_k} |\bar{A}_t^m - R_t^-|^p + E \sup_{0 \leq t \leq \tau_k} |\bar{K}_t^n - R_t^+|^p \to 0, \]

\[ E \sup_{0 \leq t \leq \tau_k} |\bar{Y}_t^n - Y_t|^p + E \sup_{0 \leq t \leq \tau_k} |\bar{Y}_t^m - Y_t|^p + E \sup_{0 \leq t \leq \tau_k} |Y_t^{n,n} - Y_t|^p \to 0 \]
and
\[ E\left(\int_0^{\tau_k} |Z_{s}^{m,n} - Z_{s}^{m}|^2 ds\right)^{p/2} + E\left(\int_0^{\tau_k} |Z_{s}^{m,n} - Z_{s}^{m}|^2 ds\right)^{p/2} + E\left(\int_0^{\tau_k} |Z_{s}^{m,n} - Z_{s}^{m}|^2 ds\right)^{p/2} \rightarrow 0. \]

Moreover,
\[ E \sup_{0 \leq t \leq T} |\Delta_{m} - R_{t, \tau_{k}}^{-}| + E \sup_{0 \leq t \leq T} |\tilde{K}_{m} - R_{t, \tau_{k}}^{+}| \rightarrow 0, \quad (6.29) \]

and for every \( q \in (0, 1) \),
\[ E \sup_{0 \leq t \leq T} |\tilde{Y}_{t}^{n} - Y_{t}^{n}|^q + E \sup_{0 \leq t \leq T} |\tilde{Y}_{t}^{m} - Y_{t}^{m}|^q + E \sup_{0 \leq t \leq T} |Y_{t}^{n,n} - Y_{t}^{n}|^q \rightarrow 0. \quad (6.30) \]

(vi) For every \( \hat{U}, \hat{L} \in \mathcal{D} \) such that \( \hat{L}_t \leq Y_t \leq \hat{U}_t \) for \( t \in [0, T] \),
\[ \Delta R_{t}^{+} = (\hat{L}_t - Y_t - \Delta V_t)^+, \quad \Delta R_{t}^{-} = (Y_t - \hat{U}_t + \Delta V_t)^+, \quad t \in (0, T]. \]

**Proof.** First let us note that by Theorem 5.7 there exist solutions \( (Y^{n,m}, Z^{n,m}), (\tilde{Y}^{n}, \tilde{Z}^{n}, \tilde{A}^{n}), (\tilde{Y}^{m}, \tilde{Z}^{m}, \tilde{K}^{m}) \) of equations of assertions (ii)–(iv) of the theorem having the required integrability properties. By Theorem 3.11 there exists a solution \( (\tilde{X}^{m}, \tilde{H}^{m}) \) of (6.7) and a solution \( (\tilde{X}^{n}, \tilde{H}^{n}) \) of (6.9) such that \( \tilde{X}^{m}, \tilde{H}^{m}, (\tilde{X}^{n}, \tilde{H}^{n}) \in \mathcal{D}^q \otimes M^q \) for \( q \in (0, 1) \) and \( \tilde{X}^{m}, \tilde{X}^{n} \) are of class (D). Using this in much the same way as in Step 1 of the proof of Theorem 6.5 we show that
\[ \sup_{n,m \geq 1} \left( E \int_0^T dA_{s}^{n,m} + E \int_0^T dK_{s}^{n,m} \right) < \infty, \quad (6.31) \]
where \( A^{n,m}, K^{n,m} \) are defined by (6.5). By Corollary 3.2,
\[ \tilde{y}_t \leq Y_{t}^{n,m} \leq y_t, \quad t \in [0, T], \quad (6.32) \]
where \( Y_{t}^{n,m} \) is defined by (6.4) and \( (\tilde{y}, \tilde{z}, \tilde{k}), (y, z, k) \in \mathcal{D}^q \otimes M^q \otimes \mathcal{V}^{+,1} \) are solutions of \( \tilde{RBSDE}(\xi, f + dV, U) \) and \( \tilde{RBSDE}(\xi, f + dV, L) \), respectively, such that \( \tilde{y}, y \) are of class (D). Since \( X \in \mathcal{V}^1 \) and \( H \in M^q \), there exist \( C \in \mathcal{V}^1 \) and \( \tilde{H} \in M^q \) such that
\[ X_t = X_0 - \int_0^t dC_s - \int_0^t \tilde{H}_s dB_s, \quad t \in [0, T]. \]

Let
\[ \delta_k^1 = \inf \{ t \in [0, T], \int_0^t |f(s, 0, 0)| ds + \int_0^t |f(s, X_s, 0)| ds > k \} \wedge T. \]

By Lemma 4.8 there exist a stationary sequence \( \{\delta_k^2\} \subset \mathcal{T} \) and constants \( c_k \) such that \( |V|_{\delta_k^1} + \tilde{y}_{\delta_k^1} + y_{\delta_k^2} + |C| \delta_{k} \leq c_k \) for \( k \in \mathbb{N} \). Let us put \( \tau_k = \delta_k^1 \wedge \delta_k^2 \) and observe that by the definition of \( \tau_k \), the data \( (\tilde{Y}_{\tau_k}^{n}, f, V, L, U) \) satisfy the assumptions of Theorem 6.5 on the interval \( [0, \tau_k] \) for every \( p > 1 \). Therefore on each interval \( [0, \tau_k] \) the sequence \( \{ (Y^{n}, Z^{n}, \tilde{K}^{n}, \tilde{A}^{n}) \} \) converges in the sense of Theorem 6.5 to the unique solution \( (Y^k, Z^k, R^{k,+}, R^{k,-}) \) of \( RBSDE(\xi^k, f + dV, L, U) \) with \( \xi^k = \lim_{n \to +\infty} Y_{\tau_k}^{n} \). By
stationarity of the sequence \( \{ \tau_k \} \), there exists a triple \( (Y, Z, R) \in \mathcal{D} \otimes M \otimes \mathcal{V} \) such that 
\[
t \mapsto f(t, Y_t, Z_t) \in \mathbb{L}^1(0, T),
\]
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dV_s + \int_t^T dR_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T],
\]
and
\[
L_t \leq Y_t \leq U_t \quad \text{for a.e. } t \in [0, T]
\]
for every \( \hat{L}, \hat{U} \in \mathcal{D} \) such that \( L_t \leq \hat{L}_t \leq Y_t \leq \hat{U}_t \leq U_t \) for a.e. \( t \in [0, T] \). Actually, by (6.31) and (6.32), \( R, Y \in \mathcal{D}^q \) for \( q \in (0, 1) \), which implies that \( Z \in M^q \) by Lemma 3.4. All the desired in (iv) and (v) convergences of the sequence \( \{(\bar{Y}^n, \bar{Z}^n, \bar{K}^n, \bar{A}^n)\} \), except for the case \( q = 1 \) in (iv) and \( q \in (0, 1) \) in (v), and all the desired properties of the limits follow now from Theorem 6.5 and stationarity of \( \{\tau_k\} \). If \( L, U, V \) are continuous and \( L_T \leq \xi \leq U_T \) then by Theorem 6.5 and stationarity of \( \{\tau_k\} \), the processes \( Y, R \) are continuous. Therefore using the fact that \( Y \in S^q, q \in (0, 1), R^+ \in \mathcal{V}_c^{+,1} \), the monotone character of the convergence of \( \{\bar{Y}^n\} \) and \( \{\bar{K}^n\} \), Dini’s theorem and the Lebesgue dominated convergence theorem one can show the required convergence in assertion (iv) for \( q = 1 \) and in assertion (v) for \( q \in (0, 1) \). The same reasoning may be applied to the quadruple \( (Y^m, Z^m, K^m, A^m) \). Finally, since
\[
\bar{Y}^n_t \leq Y^{n,n}_t \leq Y^n_t, \quad t \in [0, T],
\]
the data \( (Y^{n,n}_t, f, V, L, U) \) satisfy the assumptions of Theorem 6.5 on \([0, \tau_k]\). Therefore the required in (v) convergences of \( \{Y^{n,n}\}, \{Z^{n,n}\} \) follow from the above inequality and Theorem 6.5. \( \square \)

7 Nonintegrable solutions of reflected BSDEs

In this section we consider reflected BSDEs with monotone generator and barriers satisfying only the standard Mokobodzki condition. In the case of BSDEs with one reflecting barrier this means that we assume (H1)–(H6) ((H1)–(H6) and (Z) in case \( p = 1 \)) and that \( L \in \mathbb{L}^{\infty,p}(\mathcal{F}) \) (\( L \) is of class (D) in case \( p = 1 \)). In the case of two barriers this means that we assume (H1)–(H6) ((H1)–(H6) and (Z) in case \( p = 1 \)) and the standard Mokobodzki condition (M) ((M*) in case \( p = 1 \)) formulated later on. Theorems 5.6, 5.7, 6.5 and 6.6 say that in general we can not expect existence of \( \mathbb{L}^p \) solutions. Nevertheless we show that under the standard Mokobodzki condition there exist solutions having weaker integrability properties. In fact, it may happen that some components of the solution are not in \( \mathbb{L}^p \) for any \( p > 0 \) (see [19, Example 7.3]).

We begin with BSDEs with one reflecting barrier.

**Theorem 7.1.** Let \( p \geq 1 \). Assume that (H1)–(H6) are satisfied and \( L \in \mathbb{L}^p(\mathcal{F}) \) in case \( p > 1 \) and that (H1)–(H6), (Z) are satisfied and \( L \) is of class (D) if \( p = 1 \). Then there exists a solution \((Y, Z, K)\) of RBSDE(\( \xi, f + dV, L \)) such that \((Y, Z, K) \in \mathcal{D}^p \otimes M \otimes \mathcal{V}^+ \) if \( p > 1 \) and \((Y, Z, K) \in \mathcal{D}^q \otimes M \otimes \mathcal{V}^+, q \in (0, 1) \), \( Y \) is of class (D) if \( p = 1 \), and all the statements of assertion (ii) of Theorem 5.7 hold true.
Proof. In the proof the basic text relates to the case where \( p > 1 \); the statements in parentheses relate to the case \( p = 1 \). Let \((\tilde{Y}^n, \tilde{Z}^n) \in D^p \otimes M^p \) (resp. \((\tilde{Y}^n, \tilde{Z}^n) \in D^q \otimes M^q, q \in (0, 1)\), such that \( \tilde{Y}^n \) of class (D)) be a solution of the BSDE

\[
\tilde{Y}_t^n = \xi + \int_t^T f^+(s, \tilde{Y}^n_s, \tilde{Z}^n_s) \, ds + \int_t^T n(\tilde{Y}^n - L_s)^- \, ds + \int_t^T dV_s - \int_t^T \tilde{Z}^n_s \, dB_s, \quad t \in [0, T].
\]

By Proposition 3.1 (resp. Corollary 3.2), \( \tilde{Y}_t^n \leq \tilde{Y}_{t+1}^n \) and \( Y_t^n \leq \tilde{Y}_t^n, t \in [0, T] \), for \( n \in \mathbb{N} \). Consequently,

\[
Y_1^t \leq Y_t^n \leq \tilde{Y}_t^n, \quad t \in [0, T], \quad n \in \mathbb{N}.
\]

Let us observe that the data \((\xi, f^+, L)\) satisfy assumptions (H1)–(H7) (resp. (H1)–(H6), (H7*)) with \( X = R(L) \), where \( R(L) \) is a c\'adl\'ag version of Snell’s envelope of the process \( L \). By Theorem 5.6 (resp. Theorem 5.7), \( \tilde{Y}_t^n \rightarrow \tilde{Y}_t \), \( t \in [0, T] \), where \( \tilde{Y} \in D^p \) (resp. \( \tilde{Y} \in D^q, q \in (0, 1) \), \( \tilde{Y} \) is of class (D)) is the first component of the solution of RBSDE(\( \xi, f^+ + dV, L \)). Hence

\[
Y_1^t \leq Y_t^n \leq Y_t^n, \quad t \in [0, T], \quad n \in \mathbb{N}.
\]

By Lemma 4.8 there exist a stationary sequence \( \{\delta_k^1\} \subset T \) and constants \( c_k \) such that

\[
|V|_{\delta_k^1} + Y_{\delta_k^1}^1 + \tilde{Y}_{\delta_k^1} \leq c_k, \quad k \in \mathbb{N}.
\]

Put \( \tau_k = \delta_k^1 \wedge \delta_k^2 \), where

\[
\delta_k^2 = \inf\{t \in [0, T]; X_t^* + \int_0^t f^-(s, X_s, 0) \, ds > k\} \wedge T.
\]

Since for every \( p > 1 \) the data \((Y_t^n, f, L)\) satisfy the assumptions of Theorem 5.6 on each interval \([0, \tau_k]\), the theorem follows.

Remark 7.2. If \( p > 1 \) then by Theorem 5.4 the solution of Theorem 7.1 is unique in the class \( D^p \otimes M \otimes \mathcal{V}^+ \). We do not know whether in general the solution is unique in case \( p = 1 \). However, if \( p = 1 \), then by Remark 5.3, the solution is unique in its class if \( f \) does not depend on \( z \).

Each of the following conditions is called the Mokobodzki condition.

(M) There exists \( X \in \mathcal{H}^p \) such that \( L_t \leq X_t \leq U_t \) for a.e. \( t \in [0, T] \).

\( (M^*) \) There exists \( X \in \mathcal{V}^1 + \mathcal{M}_c^q, q \in (0, 1) \), such that \( X \) is of class (D) and \( L_t \leq X_t \leq U_t \) for a.e. \( t \in [0, T] \).

Theorem 7.3. Let \( p > 1 \) and let assumptions (H1)–(H6) and (M) hold.

(i) There exists a solution \((Y, Z, R)\) of RBSDE(\( \xi, f + dV, L, U \)) such that \((Y, Z, R) \in D^p \otimes M \otimes \mathcal{V}\).
(ii) Let \((Y^{n,m}, Z^{n,m}) \in D^p \otimes M^p\) be a solution of (6.4) and let the triples \((Y^n, \bar{Z}^n, \bar{A}^n), (Y^m, Z^m, K^m) \in D^p \otimes M \otimes V^+\) be solutions of RBSDE \((\xi, f_n + dV, U)\) and \(RBSDE(\xi, f_m + dV, L)\) of Theorem 6.6. Then assertions (ii)–(vi) of Theorem 6.6 apart from (6.29), (6.30) hold true.

**Proof.** Existence of solutions \((Y^{n,m}, Z^{n,m})\) and \((\bar{Y}^n, \bar{Z}^n, \bar{A}^n), (\bar{Y}^m, \bar{Z}^m, K^m)\) follow from Theorem 7.1 and Theorem 3.8. Let \(\bar{y}, \bar{y}\) be the first components of the solutions \(\bar{Y}, \bar{Z}\) of Theorem 7.4. To some process \((\bar{Y}, \bar{Z})\) of class (D) and \(L\). Similarly, if we denote by \((\bar{Y}^n, \bar{Z}^n, \bar{A}^n)\) a solution of (6.4) and we define \(K^n, m\) by (6.5) then \(\{(Y^{n,m}, Z^{n,m}, A^{n,m})\}\) converges, in the sense of Theorem 5.7, to some process \((\bar{Y}^n, \bar{Z}^n, \bar{A}^n)\) of class (D). Similarly, if we denote by \((Y^{n,m}, Z^{n,m})\) a solution of (6.4) and we define \(K^n, m\) of class (D). As we shall see in the proof of the following theorem the sequences \(\{(Y^p, Z^p, A^p)\}\) defined this way are monotone.

**Theorem 7.4.** Let \(p = 1\) and let assumptions (H1)–(H6), (Z) and (M) hold.

(i) There exists a solution \((Y, Z, R)\) of RBSDE \((\xi, f + dV, L, U)\) such that \((Y, Z, R) \in D^q \otimes M \otimes V\) for \(q \in (0, 1)\) and \(Y\) is of class (D).
(ii) Let \((\bar{Y}^n, \bar{Z}^n, \bar{A}^n), (Y^m, Z^m, K^m)\) denote processes defined in the paragraph preceding the theorem. Then assertions (ii)–(vi) of Theorem 6.6 apart from (6.29), (6.30) hold true.

Proof. The proof runs as the proof Theorem 7.3 apart from the fact that we use Theorem 3.11 instead of Theorem 3.8 and Corollary 3.2 instead of Proposition 3.1, and now we consider \(\bar{y}, y \in \mathcal{D}^q, q \in (0, 1)\), such that \(\bar{y}, y\) are of class (D) and are limits of penalizations for \(\overline{\text{RBSDE}}(\xi, f + dV, U)\) and \(\overline{\text{RBSDE}}(\xi, f + dV, L)\), respectively. The only additional fact we have to prove is the monotonicity of the sequences \(\{\bar{Y}^n\}, \{Y^m\}\). But the monotonicity follows immediately from the fact that
\[
\bar{Y}^n_t = \lim_{m \to +\infty} Y^{n,m}_t, \quad Y^m_t = \lim_{n \to +\infty} Y^{n,m}_t, \quad t \in [0, T],
\]
and by Corollary 3.2, \(Y^{n,m}_t \leq Y^{n+1,m}_t, Y^{n,m}_t \geq Y^{n,m+1}_t, t \in [0, T], n \in \mathbb{N}\). □

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