AN UPPER BOUND FOR THE LOWER CENTRAL SERIES QUOTIENTS OF A FREE ASSOCIATIVE ALGEBRA

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Abstract. Feigin and Shoikhet conjectured in [FS] that successive quotients $B_m(A_n)$ of the lower central series filtration of a free associative algebra $A_n$ have polynomial growth. In this paper we give a proof of this conjecture, using the structure of a representation of $W_n$, the Lie algebra of polynomial vector fields on $\mathbb{C}^n$, on $B_m(A_n)$ which was defined in [FS]. Moreover, we show that the number of squares in a Young diagram $D$ corresponding to an irreducible $W_n$-module in the Jordan-Holder series of $B_m(A_n)$ is bounded above by the integer $(m-1)^2 + 2\frac{m(m+1)}{2}(m-1)$, which allows us to confirm the structure of $B_3(A_3)$ conjectured in [FS].

1. Introduction

Let $A := A_n$ be the free associative algebra with $n$ generators over $\mathbb{C}$. Consider its lower central series as a Lie algebra, i.e., the Lie ideals $L_m(A) \subset A$ defined recursively by $L_1(A) = A$, $L_i(A) = [A, L_{i-1}(A)]$, and the corresponding associated graded Lie algebra $B(A) := \bigoplus_{i \geq 1} B_i(A)$, where $B_i(A) = L_i(A)/L_{i+1}(A)$.

Giving each of the generators of $A$ degree 1, we define a grading on $A$ and hence on each of the spaces $B_i(A)$. It is an interesting (and, in general, unsolved) problem to determine the Hilbert series of $B_i(A)$ for each $i$ with respect to this grading. For $i = 1$, this is easy since $B_1(A)$ is the space of cyclic words in $n$ letters. It particular, one can easily see that $B_1(A)$ has exponential growth. So at first sight one might expect that the spaces $B_i(A)$ for $i > 1$ also have exponential growth. However, computer experiments performed by Eric Rains in 2005 suggested that, to the contrary, the spaces $B_i(A)$ for $i > 1$ should have polynomial growth.

This phenomenon was studied systematically by Feigin and Shoikhet in the paper [FS]. The first important observation of Feigin and Shoikhet is that the ideal $Z$ of $A[[A, A], A]A$ in $B_1(A) = A/[A, A]$ is central in the Lie algebra $B(A)$, and therefore, it is natural to define the space $\bar{B}_1(A) = B_1(A)/Z$ and the graded Lie algebra $\bar{B}(A) = \bar{B}_1(A) \oplus \bigoplus_{i \geq 2} B_i(A)$. Their second important observation is that the graded Lie algebra $\bar{B}(A)$ carries a natural grading-preserving action of the Lie algebra $W_n$ of polynomial vector fields in $n$ variables (while there is no such action on $B_1(A)$), and the $W_n$-modules $\bar{B}_1(A)$ and $B_i(A)$, $i \geq 2$, admit a Jordan-Hölder series whose simple composition factors are irreducible $W_n$-modules $\mathcal{F}_D$ of tensor fields corresponding to Young diagrams $D$ with at most $n$ rows.

Feigin and Shoikhet in [FS] determined the exact structure of $\bar{B}_1(A)$ and $\bar{B}_2(A)$ as representations of $W_n$, showing that they have polynomial growth, and conjectured that for any $m$, $B_m(A)$ has polynomial growth, i.e., the dimension $B_m(A_n)[\ell]$, the degree $\ell$ part of $B_m(A_n)$, grows like $c_{mn} \ell^{n-1}$, where $c_{mn}$ is a constant.
In this paper we give a proof of this conjecture. In fact, we show that the \(W_n\)-modules \(B_m(A_n)\) have finite length, which implies the conjecture. More specifically, we give an explicit upper bound on the number of squares in a Young diagram \(D\) if the corresponding tensor field module \(F_D\) over \(W_n\) occurs as a composition factor in \(B_m(A_n)\). This bound not only implies that \(B_m(A_n)\) has finite length, but also, when combined with the computation by Eric Rains (see [FS]), allows us to confirm the conjectural structure of \(B_3(A_3)\) (as conjectured in [FS], it turns out to be isomorphic to \(F_{2,1,0}\)).

Remark 1. In this paper we work over the ground field \(\mathbb{C}\) following [FS], but the discussion carries over without changes to an arbitrary field of characteristic zero.

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2. Results

2.1. Tensor field modules over \(W_n\). The simple \(W_n\)-modules \(F_D\) are defined in the following way. Let \(D\) be a Young diagram with at most \(n\) rows. Then \(F_D\) is the irreducible \(W_n\)-submodule of the space of polynomial tensor fields on \(\mathbb{C}^n\) of type \(D\). In other words, we take a Young diagram \(D\) and the corresponding irreducible \(\mathfrak{gl}(n,\mathbb{C})\)-module \(F_D\). Then we extend the action of \(\mathfrak{gl}(n,\mathbb{C})\) on \(F_D\) to the action of \(W_n\), the subalgebra of \(W_n\) of vector fields vanishing at the origin, so that quadratic and higher vector fields act by zero. Then, if \(D\) has more than one column, we let \(F_D = \text{Hom}_{U(W_n)}(U(W_n), F_D)\) be the \(W_n\)-module coinduced from a \(\mathfrak{gl}(n,\mathbb{C})\)-module \(F_D\), which is known to be irreducible. If \(D\) has one column, we take \(F_D\) to be the irreducible \(W_n\)-submodule of the coinduced module \(\text{Hom}_{U(W_n)}(U(W_n), F_D)\) (which is known to be unique in this case).

It is known that if \(D\) has more than one column, then \(F_D\) is the whole space of tensor fields, otherwise (if \(D\) has one column of \(k\) squares) \(F_D\) is the space of closed polynomial \(k\)-forms on \(\mathbb{C}^n\). If \((k_1,\ldots,k_n)\) is a partition (possibly ending with some zeroes) corresponding to a Young diagram \(D\), we will write \(F_{(k_1,\ldots,k_n)}\) to denote the module \(F_D\). For references about modules \(F_D\), see [FF] or [F]; for reference about Schur modules \(F_D\), see [Ful].

2.2. The main result. The main result of this paper is the following theorem, proved in the next section.

Theorem 2. For \(m \geq 3\), \(n \geq 2\) and \(F_D\) in the Jordan-Hölder series of \(B_m(A_n)\), we have the following estimate on the size (i.e., the number of squares) of the Young diagram \(D\):

\[
|D| \leq (m - 1)^2 + 2\left\lfloor \frac{n - 2}{2} \right\rfloor (m - 1)
\]

(where \(\lfloor x \rfloor\) denotes the integer part of \(x\)).

Corollary 3. For all \(m \geq 2\), \(n \geq 2\), \(B_m(A_n)\) has finite length as a \(W_n\)-module. In particular, \(\dim B_m(A_n)\) has finite length as a \(W_n\)-module. In particular, \(\dim B_m(A_n) < \infty\), where \(c_{mn} > 0\) is a constant.

Proof. It suffices to consider the case \(m \geq 3\), as the case \(m = 2\) is considered in [FS]. The first statement follows from Theorem 2 and the fact that \(\dim B_m(A_n) < \infty\). The second statement follows from the first statement.

Corollary 4. \(B_3(A_3) = F_{2,1,0}\).
Proof. According to the MAGMA computation by Eric Rains (see [FS], formula (17) for $H_3(u, t)$), the characters of both sides agree up to degree 6 inclusively (see [FS], section 4.2). So if in addition to $F_{2, 1, 0}$, $B_3(A_3)$ had included another $F_D$, then $D$ would have to have at least 7 squares. But according to Theorem 2 $D$ can have at most 4 squares. Thus, there is no additional constituents, and we are done.

Remark 5. In a similar way, one can rederive the results of [DKM] on the structure of $B_3(A_2)$ and $B_4(A_2)$. Namely, for $B_4(A_2)$, the bound for the size of $D$ given by Theorem 2 is 9, while the Hilbert series of $B_4(A_2)$ is computed by Eric Rains using MAGMA up to degree 9 (see [FS]).

3. Proof of Theorem 2

3.1. The map $\psi$ and its kernel. Consider the polynomial rings $\mathcal{O}_{mn} := \mathbb{C}[x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m}$. Below we write $A := A_n$. Let $\Omega^{even}(\mathbb{C}^n)$ be the space of even polynomial differential forms, and set $\Omega^{mn} := \Omega^{even}(\mathbb{C}^n)^{\otimes m}$. As an $\mathcal{O}_{mn}$-module, $\Omega^{mn} \cong \mathcal{O}_{mn} \otimes \Lambda^{mn}$, where $\Lambda^{mn} := \Lambda^{even}(\mathbb{C}^n)^{\otimes m}$.

Recall that by a result of [FS], we have a surjective map $\xi : \Omega^{even}(\mathbb{C}^n) \to B_1(A)$, which descends to an isomorphism $\Omega^{even}(\mathbb{C}^n)/\Omega^{exact}(\mathbb{C}^n) \cong B_1(A)$, which we will also denote by $\xi$. For any $m \geq 2$, this isomorphism gives rise to a map $\psi : \Omega^{mn} \to B_m(A)$ defined as a composition of two maps

$$\Omega^{mn} = \Omega^{even}(\mathbb{C}^n)^{\otimes m} \to B_1(A)^{\otimes m} \to B_m(A),$$

$\omega_1 \otimes \cdots \otimes \omega_m \mapsto b_1 \otimes \cdots \otimes b_m \mapsto [(b_1, b_2), \ldots, b_m]$, where $b_i := \xi(\omega_i)$. The map $\psi$ is surjective, since by the results of [FS], the direct sum $B_1(A) \oplus \bigoplus_{m \geq 2} B_m(A)$ is a graded Lie algebra generated in degree 1.

Recall also from [FS] that we have a natural isomorphism $\eta : \Omega^{even}_{\geq 0}(\mathbb{C}^n) \to B_2(A)$, and upon identification by $\xi, \eta$, the bracket map $B_1(A)^{\otimes 2} \to B_2(A)$ reduces to the map $\omega_1 \otimes \omega_2 \mapsto d\omega_1 \wedge d\omega_2$.

Therefore, we can factor the map $\psi$ as

$$\Omega^{even}(\mathbb{C}^n)^{\otimes 2} \otimes \Omega^{even}(\mathbb{C}^n)^{\otimes (m-2)} \to B_2(A) \otimes B_1(A)^{\otimes (m-2)} \to B_m(A)$$

using the map $\eta \circ \phi : \Omega^{even}(\mathbb{C}^n)^{\otimes 2} \to B_2(A)$, where $\phi : \omega_1 \otimes \omega_2 \mapsto d\omega_1 \wedge d\omega_2$.

Lemma 1. For any $i, j \in [1, n]$ and forms $\omega_1, \omega_2 \in \Omega^{even}(\mathbb{C}^n)$, the element $(x_{i1} - x_{i2})(x_{j1} - x_{j2})\omega_1 \otimes \omega_2$ is in the kernel of the map $\phi$. Consequently, the submodule $(x_{i1} - x_{i2})(x_{j1} - x_{j2})\Omega^{mn}$ is in the kernel of $\psi$.

Proof. We have

$$\phi((x_{i1} - x_{i2})\omega_1 \otimes \omega_2) = \{d(x_{i1}\omega_1) \wedge d\omega_2 - d\omega_1 \wedge d(x_{i2}\omega_2)\}|_{x_{i1} = x_{i2} = x_1}$$

$$= dx_i \wedge \omega_1 \wedge d\omega_2 + x_1 d\omega_1 \wedge d\omega_2 - d\omega_1 \wedge dx_i \wedge \omega_2 - d\omega_1 \wedge x_1 d\omega_2$$

(since $\omega_1$ is even)

$$= dx_i \wedge (\omega_1 \wedge d\omega_2 + d\omega_1 \wedge \omega_2).$$

$$= dx_i \wedge d(\omega_1 \wedge \omega_2).$$

Using this, we compute

$$\phi((x_{i1} - x_{i2})(x_{j1} - x_{j2})\omega_1 \otimes \omega_2) =$$

$$= \phi((x_{j1} - x_{j2})(x_{i1}\omega_1 \otimes \omega_2) - (x_{j1} - x_{j2})(\omega_1 \otimes x_{i2}\omega_2))$$

$$= dx_j \wedge d(x_{i1}\omega_1 \wedge \omega_2) - dx_j \wedge d(\omega_1 \wedge x_{i2}\omega_2) = 0.$$
Therefore, $\phi$ maps $(x_{i1} - x_{i2})(x_{j1} - x_{j2})\Omega^{even}(\mathbb{C}^n) \otimes 2$ to zero. In particular, this implies that $(x_{i1} - x_{i2})(x_{j1} - x_{j2})\Omega^{mn}$ belongs to the kernel of the map $\psi$. \hfill \Box

**Lemma 2.** Let $L$ be a Lie algebra, and $b_i \in L$, $i = 1, \ldots, m$. For any $k = 1, \ldots, m$, the bracket $[[b_1, b_2], \ldots, b_m]$ is a linear combination of the brackets of the form $[[b_k, b_1], \ldots, b_{m-1}]$ where $(l_1, \ldots, l_{m-1})$ is a permutation of $[1, m] \setminus \{k\}$.

**Proof.** This lemma is well known, but we will give a proof, as it is very short. The proof is by induction in $k$. Indeed, for $k = 1, 2$ the statement is true. To go from $k - 1$ to $k$, we notice that by the Jacobi identity

$$[[b_1, b_2], \ldots, b_{k-1}, b_k] = [[b_1, b_2], \ldots, b_k]_{k-1} + [[b_1, b_2], \ldots, b_k].$$

Putting $b_{k-1} := b_k$ we have that the first bracket on the RHS can be expressed as a linear combination of brackets of the form $[[b_k, b_1], \ldots, b_{k-1}]$ by the induction assumption. Similarly, in the second bracket on the RHS we put $b_{k+1} := [b_{k-1}, b_k]$ and use the induction assumption to express the second bracket as a combination of brackets of the form $[[b_k, b_{k-1}], \ldots]$. Bracketing the LHS and RHS with $b_{k+1}, \ldots, b_m$, we obtain the desired result. \hfill \Box

Let $I$ be the ideal in $\Omega^{mn}$ generated by the polynomials

$$\prod_{s=1}^{k-1} (x_{i_s} - x_{i_k})(x_{j_s} - x_{j_k}).$$

where $2 \leq k \leq m$ and arbitrary $i_s, j_s \in [1, n]$ for $s \in [1, k]$.

**Lemma 3.** The submodule $I\Omega^{mn}$ of $\Omega^{mn}$ is $W_n$-invariant, and is contained in the kernel of the map $\psi$.

**Proof.** First we show that $I\Omega^{mn}$ is stable under the action of of $W_n$ on $\Omega^{mn}$. For this, it suffices to show that the ideal $I$ is $W_n$-invariant. Let $f \frac{\partial}{\partial y_m}$ be some vector field in $W_n$. Then we have:

$$f \frac{\partial}{\partial y_m} x_{ij} = \begin{cases} 0 & \text{if } k \neq j \\ f_j & \text{if } m = i \end{cases}$$

where $f_j$ denotes $f$ where instead of the variables $y_j$ we substitute $x_{i_j}$. Therefore,

$$\left(f \frac{\partial}{\partial y_m}\right) \prod_{s=1}^{k-1} (x_{i_s} - x_{i_k})(x_{j_s} - x_{j_k})$$

$$= \prod_{s=1}^{k-1} (x_{i_s} - x_{i_k})(x_{j_s} - x_{j_k}) \sum_{s=1}^{k-1} \left(\frac{f_s - f_k}{x_{i_s} - x_{i_k}} + \frac{f_s - f_k}{x_{j_s} - x_{j_k}}\right).$$

The last factor is clearly a polynomial, so the right hand side belongs to $I$. This implies that $I$ is $W_n$-invariant, as desired.

Next we prove that the element $\prod_{s=1}^{k-1} (x_{i_s} - x_{i_k})(x_{j_s} - x_{j_k})\omega_1 \otimes \cdots \otimes \omega_m$ goes to zero under the map $\psi$. Let $b_i = \xi(\omega_i)$. By Lemma 2 in $B_m(A)$ we have

$$[[b_1, b_2], \ldots, b_m] = \sum_{\sigma} c_{\sigma} [b_{\sigma(1)}, \ldots, b_{\sigma(m-1)}],$$

where $\sigma$ is a bijection from $[1, m - 1]$ to $[1, m] \setminus \{k\}$. Therefore, we have that

$$\omega_1 \otimes \cdots \otimes \omega_m - \sum_{\sigma} c_{\sigma} (\omega_k \otimes \omega_{\sigma(1)}) \otimes \cdots \otimes \omega_{\sigma(m-1)}$$
is in the kernel of $\psi$.

But by Lemma 5, $\Omega_{mn}$ is in the kernel of $\psi$.

Therefore, $\Omega_{mn}$ is in the kernel of $\psi$. \hfill $\square$

Set \( r := 2\left\lfloor \frac{n-2}{2} \right\rfloor (m - 1) \). Note that the space $\Lambda(\mathbb{C}^n)$ is equipped with the natural grading by rank of exterior forms; hence so is the space $\Lambda_{mn}$. Consider the space $\Lambda_{mn}$ spanned by homogeneous elements of degree \( r \) in $\Lambda_{mn}$.

**Lemma 4.** The submodule $\mathcal{O}_{mn} \otimes \Lambda_{mn}$ of $\Omega_{mn}$ is $W_n$-invariant, and is contained in the kernel of $\psi$.

**Proof.** Since vector fields act as Lie derivative on the components of $\Omega_{mn}$, they leave rank unchanged, so $\mathcal{O}_{mn} \otimes \Lambda_{mn}$ is stable under the $W_n$-action on $\Omega_{mn}$.

Now we show that this submodule is annihilated by $\psi$. Elements of $\psi(\mathcal{O}_{mn} \otimes \Lambda_{mn})$ are linear combinations of elements of the form $[b_1, b_2, \ldots, b_m]$, where $\sum \text{rk} \psi_i > 2\left\lfloor \frac{n-2}{2} \right\rfloor (m - 1)$. By Lemma 2 for every $k \in [1, m]$ we have $[b_1, b_2, \ldots, b_m] = \sum_{\sigma} c_{\sigma} [b_k, b_{\sigma}(1)] \cdots b_{\sigma}(m-1)$.

Pick $k$ such that $\omega_k$ has maximal rank among $\omega_1, \ldots, \omega_m$. Then $\text{rk} \omega_k + \text{rk} \omega_{\sigma(1)} > \frac{1}{m-1} \sum \text{rk} \omega_i > 2\left\lfloor \frac{n-2}{2} \right\rfloor$. But $\text{rk} \omega_k + \text{rk} \omega_{i_1}$ is even so $\text{rk} \omega_k + \text{rk} \omega_{i_1} \geq 2\left\lfloor \frac{n-2}{2} \right\rfloor + 2$. So we have

$$\text{rk}(d\omega_k \wedge d\omega_{\sigma(1)}) = \text{rk} \omega_k + \text{rk} \omega_{\sigma(1)} + 2 = 2\left(\frac{n-2}{2} + 2\right) > n.$$ 

Thus $d\omega_k \wedge d\omega_{\sigma(1)} = 0$ in $\Omega^{even}(\mathbb{C}^n)$.

But $[b_k, b_{\sigma(1)}] = \eta(d\omega_k \wedge d\omega_{\sigma(1)})$ in $B_2(A)$. So $[b_k, b_{\sigma(1)}] = 0$, and thus we have $[b_1, b_2, \ldots, b_m] = 0$. Therefore the submodule $\mathcal{O}_{mn} \otimes \Lambda_{mn}$ is in the kernel of $\psi$. \hfill $\square$

3.2. **The structure of $\mathcal{O}_{mn}/I$ as a graded space.** Note that the algebra $\mathcal{O}_{mn}$ has a natural grading, which assigns degree 1 to each generator, and $I$ is a graded ideal. Thus, $\mathcal{O}_{mn}/I$ is a graded algebra.

**Lemma 5.** We have an isomorphism of graded vector spaces

$$\mathcal{O}_{mn}/I \cong \mathbb{C}[y_1, \ldots, y_n] \otimes \bigotimes_{k=2}^{m} \text{Sym}^{2k-3}(\mathbb{C} + \sum_{i=1}^{n} \mathbb{C} y_i),$$

where $y_i$ have degree 1.

**Proof.** Denote by $\text{in}(I)$ the set of initial (highest) terms of polynomials in $I$ with respect to the lexicographic monomial order $x_{ij} > x_{kl}$ iff $j > l$ or $j = l$ and $i > k$.

By a theorem about Gröbner bases (Theorem 15.3 in [1]) monomials not in $\text{in}(I)$ form a vector space basis of $\mathcal{O}_{mn}/I$. By the form of the generators of $I$ we see that $\text{in}(I)$ is generated by the monomials $\prod_{i=1}^{2k-2} x_{i,k}$, where $k \geq 2$, $i_k \in [1, n]$ is arbitrary. So monomials not in $\text{in}(I)$ are precisely the monomials in $\mathbb{C}[x_{11}, \ldots, x_{nn}] \otimes \bigotimes_{k=2}^{m} \text{Sym}^{2k-3}(\mathbb{C} + \sum_{l=1}^{n} \mathbb{C} x_{lk})$. \hfill $\square$

3.3. **Proof of Theorem 2** For any graded space $M$ define its character (the Hilbert series of $M$) $\text{char} M = \sum_{a} \dim M[a] t^a$, where $M[a]$ is the $a^\text{th}$ graded piece of $M$. For instance, $\text{char} \mathcal{F} = \sum_{t \in \mathbb{C}} |\psi(t)|$. If $P_D(t)$ is a polynomial of degree $n$ if $D$ has one column, and $P_D(t) = N_D t^{D}$ if $D$ has more than one column, where $N_D$ is the dimension of the irreducible representation of $GL_n$ corresponding to $D$. 


The space $\Omega^{mn} = \mathcal{O}_{mn} \otimes \Lambda^{mn}$ has a natural grading coming from the grading on the factors, and we have a surjective map of graded vector spaces $\psi : \Omega^{mn} \to B_m(A_n)$. By Lemma \[3\] $I\Omega^{mn} = I(\mathcal{O}_{mn} \otimes \Lambda^{mn})$ is in the kernel of $\psi$, and by Lemma \[4\] $\mathcal{O}_{mn} \otimes \Lambda^{mn}_{> r}$ is in the kernel of $\psi$. So we have a surjective map

$$(\mathcal{O}_{mn} \otimes \Lambda^{mn})/(I\mathcal{O}_{mn} \otimes \Lambda^{mn} + \mathcal{O}_{mn} \otimes \Lambda^{mn}_{> r}) \to B_m(A_n).$$

Thus we have a surjective map

$$\hat{\psi} : (\mathcal{O}_{mn}/I) \otimes (\Lambda^{mn}/\Lambda^{mn}_{> r}) \to B_m(A_n),$$

which is a homomorphism of $W_n$-modules. We also notice that the space $\Lambda^{mn}/\Lambda^{mn}_{> r}$ is naturally identified with $\Lambda^{mn}_{\leq r}$.

Let $\mu(D)$ be the multiplicity of $\mathcal{F}_D$ in $(\mathcal{O}_{mn}/I) \otimes (\Lambda^{mn}/\Lambda^{mn}_{> r})$, and $\nu(D)$ be the multiplicity of $\mathcal{F}_D$ in $B_m(A_n)$. Since $\hat{\psi}$ is surjective, $\nu(D) \leq \mu(D)$. Also, we have

$$(1) \quad \sum_D \mu(D) \text{char}\mathcal{F}_D = \text{char}(\mathcal{O}_{mn}/I)\text{char}(\Lambda^{mn}_{\leq r}).$$

From Lemma \[5\] we have

$$\text{char}\mathcal{O}_{mn}/I = \text{char}\mathbb{C}[y_1, \ldots, y_n] \prod_{k=2}^{m} \text{char}\text{Sym}^{2k-3}(\mathbb{C} + \sum_{i=1}^{n} \mathbb{C}y_i)$$

$$= \frac{1}{(1-t)^n} \prod_{k=2}^{m} \left( \sum_{a=0}^{k} \binom{a+n-1}{n} t^a \right)$$

$$= \frac{1}{(1-t)^n} (c_0 t^{(m-1)^2} + \text{l.o.t.})$$

where $c_0 \neq 0$ is a constant and l.o.t. denote monomials of smaller degree than $(m - 1)^2$. The character of $\Lambda^{mn}_{\leq r}$ is a polynomial of degree $r$, $C_0 t^r + \text{l.o.t.}$

Thus, multiplying (1) by $(1-t)^n$, we obtain

$$\sum_D \mu(D) P_D(t) = Q(t),$$

where $Q(t)$ is a polynomial of degree $(m - 1)^2 + r$.

Now, for $m \geq 3, n \geq 2$

$$n \leq 2(n-1) \leq 4[n/2] = 4 + 4[(n-2)/2] \leq (m-1)^2 + r.$$

So, we get

$$\sum_D \mu(D) N_D |D|^r = Q_*(t),$$

where $Q_*$ is another polynomial of degree $(m - 1)^2 + r$. Hence $\mu(D) = 0$ for $|D| > (m - 1)^2 + r$ and hence $\nu(D) = 0$ for $|D| > (m - 1)^2 + r$, as desired.

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