A NOTE ON A GEOGRAPHY PROBLEM IN KNOT FLOER HOMOLOGY

SUBHANKAR DEY

Abstract. We prove that knot Floer homology of a certain class of knots is non-trivial in next-to-top Alexander grading. This gives a partial affirmative answer to a question posed by Baldwin and Vela-Vick which asks if the same is true for all non-trivial knots in $S^3$.

1. Introduction

Knot Floer homology was defined by Ozsváth and Szabó in [14], and independently by Jacob Rasumussen in [19]. Given a knot $K$ in $S^3$, this invariant assigns a bi-graded vector space, denoted $\widehat{HF}(S^3, K)$. One of the two gradings on $\widehat{HF}$ is the homological grading, or Maslov grading, and the other is the Alexander grading:

$$\widehat{HF}(S^3, K) = \bigoplus_{m,j \in \mathbb{Z}} \widehat{HF}_{m}(Y, K, j)$$

where $m$ denotes the Maslov grading and $j$ denotes the Alexander grading.

In [15], Ozsváth and Szabó proved that knot Floer homology of any knot $K \subset S^3$ is supported between Alexander gradings $-g(K)$ to $g(K)$, where $g(K)$ is the Seifert genus of the knot, and that knot Floer homology is non-trivial in Alexander gradings $\pm g(K)$. They also proved in [16] that for fibered knots, $\widehat{HF}$ has dimension 1 in these extremal Alexander gradings. Conversely, Ghiggini and Ni proved in [7] and [20] respectively for genus one knots and in general, that a knot $K \subset S^3$ is fibered if $\widehat{HF}(S^3, K, g(K))$ has dimension 1. For a particular class of fibered knots, more can be said. A rational homology 3-sphere $Y$ is called an $L$-space if $\dim \widehat{H}(Y) = |H_1(Y)|$. A knot in $S^3$ is called an $L$-space knot if it admits a non-trivial surgery resulting in an $L$-space. As proved by Ozsváth and Szabó in [17], there are strong restrictions on knot Floer homology of $L$-space knots, combining those restriction with Seifert genus and fiberedness detection imply that $L$-space knots are fibered. Combined with the following theorem due to Hedden and Watson, these restrictions imply that if $K$ is an $L$-space knot of genus $g > 0$, then its knot Floer homology in next-to-top Alexander grading is non-trivial.

Theorem 1 ([9], Theorem 7). Suppose $K \subset S^3$ is a knot of genus $g > 1$. If $\tau(K) = g$ and $\widehat{HF}_{-1}(S^3, K, g) = 0$, then $\widehat{HF}_{g-1}(S^3, K, g - 1) \neq 0$.

This work is supported in part by a Simons Foundation grant No. 519352.
More recently, Baldwin and Vela-Vick generalized this non-vanishing result to null-homologous fibered knots in arbitrary closed oriented 3-manifolds.

**Theorem 2** ([2], Theorem 1.1). Let $Y$ be a closed oriented 3-manifold, $K \subset Y$ a fibered knot of genus $g > 0$, and $\Sigma$ a genus-$g$ Seifert surface for $K$. Then $\widehat{HF}_K(Y,K,[\Sigma],g-1)$ is non-zero.

Further they asked the following question:

**Question 3** ([2], Question 1.11). Is the knot Floer homology of every knot in $S^3$ of positive genus nontrivial in its next-to-top Alexander grading?

The result of Baldwin and Vela-Vick has been recently generalized by Yi Ni in [21] for knots in $S^3$ where he proves that

**Theorem 4.** Let $K \subset S^3$ is a knot of genus $g$. If $\widehat{HF}_K(K,g)$ is supported in a single Maslov grading $d_0$, then

$$\text{rank}(\widehat{HF}_{d_0-1}(K,g-1)) \geq \text{rank}(\widehat{HF}_K(K,g))$$

As a result of an attempt to answer Question 3, we prove the following:

**Theorem 5.** Let $K$ be a non-trivial knot in $S^3$ and suppose that there exists a Legendrian representative $L_K$ of $K$ with respect to some contact structure on $S^3$ such that the Legendrian knot invariant $\widehat{L}(L_K)$ as defined in [10] is non-zero. Then either

- $\widehat{HF}_K(S^3,K,A(\alpha_{\widehat{L}}(L_K))-1) \neq 0$, or
- $\widehat{HF}_K(S^3,K,A(\alpha_{\widehat{L}}(L_K))+1) \neq 0$,

where $A(\alpha_{\widehat{L}}(K))$ denotes the Alexander grading of the class $\alpha_{\widehat{L}}(L_K) \in \widehat{HF}_K(-S^3,K)$ that defines the invariant $\widehat{L}(L_K)$.

The above theorem has the following immediate corollary.

**Corollary 6.** If there exists some Legendrian representative $L_K$ of $K$ with respect to some contact structure in $S^3$ with non-vanishing $\widehat{L}(L_K)$ such that the Alexander grading of the class $\alpha_{\widehat{L}}(L_K)$ is $\pm g(K)$, then $\widehat{HF}_K(S^3,K,g(K)-1) \neq 0$.

In addition, one might say something about the invariant $\widehat{L}$ of Legendrian representatives of a given knot:

**Corollary 7.** For a knot $K \subset S^3$, if there are no integer $j \in [-g(K),g(K))$ such that $\widehat{HF}_K(S^3,K,j)$ and $\widehat{HF}_K(S^3,K,j+1)$ are both non-trivial, then all Legendrian representatives $L_K$ of $K$ in the standard tight contact $S^3$ have vanishing $\widehat{L}(L_K)$.

Note that if the answer to Question 3 is affirmative for all knots in $S^3$, then the condition of this corollary will cease to exist.

As an application, we show that Question 3 has an affirmative answer for a certain subset of quasi-positive knots.

**Theorem 8.** If $K$ is a quasi-positive knot, then either
A NOTE ON A GEOGRAPHY PROBLEM IN KNOT FLOER HOMOLOGY

• $\widehat{HF}(S^3, K, \tau(K) - 1) \neq 0$ or
• $\widehat{HF}(S^3, K, \tau(K) + 1) \neq 0$

In particular, if $K$ is a quasi-positive knot and $\tau(K) = g(K)$, then $\widehat{HF}(S^3, K, g(K) - 1) \neq 0$, where $g(K)$ is the Seifert genus of the knot $K \subset S^3$.

Recall that a quasi-positive knot is by definition the closure of a quasi-positive braid. A braid $\sigma$ is called quasi-positive if it is a product of conjugates of the standard generators of the braid group $B_n$, i.e. $\sigma = \prod w_i \sigma_i w_i^{-1}$, where $w_i \in B_n$ and $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are the standard generators of $B_n$. If one can take $w_i$ to be of the form

$$\sigma_{i,j} = (\sigma_1 \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_1 \cdots \sigma_{j-2})^{-1}$$

then the resulting braid is called strongly quasi-positive and its closure a strongly quasi-positive knot. Hedden proves in [8] that a fibered knot in $S^3$ is strongly quasi-positive if and only if $\tau(K) = g(K)$. L-space knots provide examples of such knots. Also strongly quasi-positivity of a fibered knot $K$ is equivalent to the open book decomposition associated to $(F, K)$ inducing the unique tight contact structure in $S^3$. Thus it implies that a part of [2, Theorem 1.1] can also be recovered using Theorem 8.

As shown by Baker and Motegi, there is a strongly quasi-positive knot $K$ which is not fibered but has $\tau(K) = g(K)$ (see [3, Example 4.2]). Therefore, Theorem 8 implies that $\widehat{HF}(S^3, K, g(K) - 1) \neq 0$ for that knot. In comparison, [2, Theorem 1.1] cannot be applied to $K$ since $K$ is not fibered, and there is no obvious way to apply either of Theorem 4 from [21] and [9, Theorem 7] since the Maslov grading of all the elements of $\widehat{HF}(S^3, K, g(K))$ is hard to compute. As a conclusion, Theorem 8 provides a new tool to answer Question 3 affirmatively for such knots.

Organization. In Section 2, we prove the Theorem 5 and corollary 6 after briefly discussing LOSS invariant and its settings. Our proof of Theorem 5 is motivated by the proof of Theorem 1.1 in [2]. Finally, we study applications of Theorem 5 to certain families of knots and prove Theorem 8 in Section 3.

Acknowledgements: The author is grateful to his advisor Professor Çağatay Kutluhan for his constant support and careful feedback on earlier drafts of this note. The author would like to specially thank Professor Lenhard Ng, Professor Matthew Hedden and Professor Yi Ni for invaluable conversations and feedback.

2. Proof of Theorem 5

For the proof of the Theorem 5, we will be making use of the Legendrian knot invariant or LOSS invariant from [10]. First we very briefly describe the setup that the invariant is defined.

Given a knot $K \subset S^3$ and a fixed contact structure $\xi$ in $S^3$, one starts with a Legendrian representative $L_K$ of $K$ with respect to that contact structure $\xi$. In [10, Proposition 2.4], Lisca-Ozsváth-Stipsicz-Szabó proved that given a Legendrian knot $L$ in a closed, contact 3-manifold $(Y, \xi)$, there always exists an open book decomposition compatible with $\xi$, with connected binding, containing $L$ on a page $S$ such that the contact framing of $L$ is equal
to the framing induced on $L$ by $S$. Also the open book can be chosen such a way that $L$ is homologically essential in $S$. Using that proposition, an open book decomposition $(S, h_\phi)$ compatible with the contact structure $(S^3, \xi)$ such that $L_K$ lies on one of its pages $S = S \times \{1\}$ can be obtained. This open book decomposition $(S, h_\phi)$ is used to find an appropriate doubly pointed Heegaard diagram for $L_K \subset S^3$. Specifically, one starts with finding a properly embedded arc $a_1$ in $S$ intersecting $L_K$ only once transversely. The orientation on $a_1$ is given in such a way that the orientation of the boundary of the disk agrees with the natural orientation of a meridian of $L_K$ and thus $a_1$ is called a half-meridian.

A basis $A = \{a_1, \ldots, a_g\}$ of properly embedded pairwise disjoint arcs in $S$ is constructed such that it makes a basis of $H_1(S, \partial S)$. Then one finds $\{b_i\}_i$ by doing a small isotopy on $\{a_i\}_i$ which is shifting the endpoints of $a_i$ along the orientation of $\partial \Sigma_g$. This is done such that there is a unique intersection point of $a_i$ and $b_i$. Then a basepoint $w$ is placed in the region swept out by the isotopy and depending on the chosen orientation of $L_K$, another basepoint $z$ is placed accordingly in one of the two places. Figure 1 shows the possible cases. This produces a doubly pointed Heegaard diagram $(S, \alpha, \beta, w, z)$ for $L_K \subset S^3$, where $\alpha_i = (a_i \times \{-1\} \cup a_i \times \{1\})/\sim$, $\beta_i = (b_i \times \{1\} \cup h_\phi(b_i) \times \{-1\})/\sim$.

Note that the single intersection point $c = (a_i \cap b_i)$ on $S_1 \subset \Sigma$ (see Figure 2) is an element in both $\CFK(-S^3, K)$ and $\CFK^-(S^3, K)$. It can also be observed that the placement of the basepoint $z$ makes it a cycle. Indeed there is no pseudo-holomorphic Whitney disk $\psi \in \pi_2(c, y)$ connecting $c$ and another intersection point $y$ such that $n_z(\psi) = 0$. Hence it defines an element in both $\HF(-S^3, K)$ and $\HF(-S^3, K)$. Proposition 3.3 from [10] shows that it is invariant upon the choice of open books. In the proposition, Lisca-Ozsváth-Stipsicz-Szabó describe an $\F[U]$-module isomorphism for $\HF$ and an $\F$-module isomorphism between two such choices of open books compatible with $(S^3, \xi, L)$ (endowed with adapted bases and basepoints adapted to $L_K$). The said isomorphisms send the distinguished intersection point for one such open book to the distinguished intersection point for another open book. Thus the homology class of $c$ in $\HF^-(S^3, L_K)$ and in $\HF(-S^3, K)$ is defined to be the Legendrian invariant of $L_K \subset (S^3, \xi)$ and is denoted by $L(L_K)$ and $\hat{L}(L_K)$, respectively.

Notice that the said homology class of $c$ is the contact invariant associated to that specific contact structure $\xi$ in $-S^3$, as defined by Ozsváth-Szabó in [16] as the Heegaard Floer Contact Invariant.

![Figure 1. Two possible choices of planting basepoints, depending on the orientation of $L$](image-url)
Baldwin and Vela-Vick in [2] used non-right veering property (by Honda-Kazez-Matić in [5] and in [6]) of the monodromy $\phi$ of the open book structure and non-triviality of the contact class $[c]$ to find an element in $\widehat{HF}(S^3, -K, 1 - g)$ of which $[c]$ lives in the boundary. In other words, Baldwin and Vela-Vick find a Whitney disk connecting $[c]$ which has only one $z$ basepoint. Thus proving that $\dim(\widehat{HF}(S^3, K, g-1)) \neq 0$ (for subtle details see [2]).

In the context of the statement of Theorem 5, given a knot $K \subset S^3$ we assume that there exists a contact structure $\xi'$ in $S^3$ such that there is a Legendrian representative of $K$ with respect to $\xi'$, say $L_K$, such that $\mathcal{L}(L_K) \neq 0$. Then we start with an open book decomposition $(S_t, h_{\phi})$ of $(S^3, \xi')$ such that $L_K$ is an homologically essential closed curve on one its pages. Now we start by finding a half-meridian of $L_k$, call $a_1 \subset S_1$.

Then we argue that it suffices to assume that the monodromy of the concerned open book is not right-veering. This is because if we assume that $(S_t, h_{\phi})$ is right-veering, then we can consider $(S_t, h_{\phi}^{-1})$ instead, which is an open book decomposition of $(-S^3, K)$. Now by the symmetry of knot Floer homology under orientation reversal of ambient manifolds (See [14, Section 3]) we have,

$$\dim \widehat{HF}(S^3, K, -l) = \dim \widehat{HF}(S^3, K, l) = \dim \widehat{HF}(S^3, -K, l) = \dim \widehat{HF}(-S^3, -K, -l)$$

we can still look to prove both Theorem 5 and corollary 6 in that case. Indeed since we are only concerned about showing that the dimension is non-zero, we can choose $K \subset -S^3$ instead. Note that the monodromy for this case can not be identity since the induced ambient closed manifold will then be $S^2 \times S^1$ and not $S^3$, which is the only case we are considering here.

A basis of arcs $A$ such that $L \cap a_k = \emptyset, k \geq 2$ and $a_1$ intersects $L$ at one transverse point, called an adapted basis of $(S, L_K)$. Given an adapted basis, there is an analogue of handle-slide operations, which can transform $A$ to another adapted basis for $(S, L)$. This is

\[\text{Figure 2. A page of the open book where the blue curve which is a Legendrian copy of } K \text{ sits and the intersection points indicate LOSS invariant}\]
called admissible arc slides, in which \( \{a_i, a_j\} \mapsto \{a_i + a_j, a_j\} \) such that \( j \neq 1 \). Here \( a_1 + a_2 \) is the isotopy class (relative to endpoints) of the union \( a_1 \cup \tau \cup a_2 \).

We start by finding a properly embedded non-separating arc \( a_2 \) such that, after possible isotopy, \( h_\phi \) sends \( a_2 \) to the left of one of its endpoints. Observe that we can make an admissible arc slide here: \( \{a_1, a_2\} \mapsto \{a_1 + a_2, a_2\} \). Figure 3 shows such an operation. After this, we complete the basis \( \{a_1', a_2, \ldots, a_k\} \) of properly embedded pairwise disjoint arcs to obtain a basis of \( H_1(S_1, \partial S_1) \) such that \( a_1' \) is both a half-meridian and a non-right veering arc.

**Lemma 9.** There is a bigon bounded by \( \alpha_1', \beta_1 \), say \( \psi \) such that \( n_w(\psi) = 1, n_z(\psi) = 0 \) with vertices at intersection points \( c_1, d_1 \).

**Proof.** Up to changing orientation of \( L_K \), one can observe that due to the non-right veering property of the arc \( a_1 \), there exists a bigon with distinguished intersection points as its vertices, \( c_1 \in S_1 \) and \( d_1 \in S_{-1} \), bounded by \( \alpha_1, \beta_1 \) such that the number of \( w \) and \( z \) multiplicities in the bigon is 1 and 0, respectively (cf. Figure [4]). Thus there exists a Whitney disk \( \psi \in \pi_2(c, d) \), where \( c = \{c_1, c_2, \ldots, c_k\} \), \( d = \{d_1, c_2, \ldots, c_k\} \) such that \( n_w(\psi) = 1, n_z(\psi) = 0 \). Hence the claim. \( \square \)

Recall that the knot Floer complex \((\text{CFK}^-, \partial^-)\) is a chain complex where \( \text{CFK}^- \) is a free \( \mathbb{F}[U] \)-module generated by the intersection points of \( \mathcal{T}_\alpha, \mathcal{T}_\beta \). Here \( \partial^- \) is given by the formula:

\[
\partial^- x = \sum_{\{y \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta\}} \sum_{\phi \in \pi_2(x, y), n_z(\phi) = 0, \mu(\phi) = 1} \# \mathcal{M}(\phi) \cdot U^{n_w(\phi)} \cdot y
\]

where \( \pi_2(x, y) \) is the homotopy class of Whitney disks from \( x \) to \( y \), \( \mu(\phi) \) is the Maslov index of \( \phi \), the integers \( n_w(\phi) \) and \( n_z(\phi) \) are respectively the intersections of \( \phi \) with \( \{z\} \times \text{Sym}^{g-1}(S) \) and \( \{w\} \times \text{Sym}^{g-1}(S) \), and \( \mathcal{M}(\phi) \) is the moduli space of all pseudo-holomorphic representatives of \( \phi \) modulo conformal automorphisms of the domain. See [13], [14] for a detailed discussions of Whitney disks and technicalities involving them.

Lemma 9 suggests that there exist a horizontal boundary arrow between \( c \) and \( d \) in the knot Floer complex of \( K \). Rigorously, \( \partial^- d = U \cdot c \) in \( \text{HFK}^-(-S^3, K) \). \((C\{j = 0\}, \partial^-)\) is a chain complex which has a natural filtration coming from the \( U \)-multiplication in \( \text{CFK}^\infty \). Indeed the fact that \((C\{j = 0\}, \partial^-)\) is a chain complex, can be seen by reversing the role of
A NOTE ON A GEOGRAPHY PROBLEM IN KNOT FLOER HOMOLOGY

Figure 4. A Whitney disk connecting $c_1$ and $d_1$

$z$ and $w$ in $(\hat{CFK}, \hat{\partial} K)$ (where $\hat{\partial} K$ can be obtained from $\partial K$ by setting $U = 0$). We denote the $U$-filtration by $\mathcal{F}_U$. Also the fact that there is no other element in the boundary of $d$ other than $c$ follows from the observation that there is no pseudo-holomorphic Whitney disk connecting $c$ which has zero $z$ multiplicity.

Now let there exist a homogenous element $e$ such that $\partial K e = d + U^m \cdot f(m > 0)$, where $d, e$ are in the same associated graded complex with respect to $\mathcal{F}_U$. Then $\partial K \circ \partial K = 0$ implies that $m = 1$ and $\partial K (f) = c$. Which then implies that $f$ and $e$ stays in the same associated graded complex, a contradiction to the non-triviality of $c$ or $\hat{L}(L_K)$, which was our assumption for Theorem 5. This proves the theorem.  

**Proof of Corollary 6.** The proof follows from the property of LOSS invariant under orientation reversal (cf. [10]) and the above proof. If for $K \subset S^3$, there exists a Legendrian representative of $K$, $L_K$ with respect to some contact structure $\xi \subset S^3$ such that $\hat{L}(L_K) \neq 0$ and $A(\hat{L}(L_K)) = -g(K)$, then we can follow the above proof and see that $HF K(-S^3, m(K), g - 1) \cong HF K(S^3, K, g - 1) \neq 0$. Now if there is a knot $K \subset S^3$ such that for some contact structure $\xi'$ in $S^3$ there exists a Legendrian representative with respect to $\xi'$, $L_K$, such that $\hat{L}(L_K) \neq 0$ and $A(\hat{L}(L_K)) = g(K)$, then one can look at the mirror of $K$, $m(K) \subset S^3$ instead. It is because in that case there is a Legendrian representative of $m(K)$ with respect to $\xi$, $L_m(K)$, such that $A(\hat{L}(L_m(K))) = -g(K) = -g(m(K))$ and $\hat{L}(L_m(K)) \neq 0$. 

□
Figure 5. Here c indicates the LOSS invariant $L(L_K)$

3. Applications

In this section, we apply Theorem 5 to some specific family of knots. First, we prove Theorem 8 which is an application of Theorem 5 to certain quasi-positive knots.

Proof of Theorem 8. In [1] Baldwin- Vela-Vick -Vértesi proved the equivalence of the grid invariant $\hat{\theta}(T)$ for transverse knots in $S^3$, defined by Ozsváth-Szabó-Thurston in [18] and the LOSS invariant for transverse knots. Note that for a given knot $K \subset (S^3, \xi)$ one can find a transverse copy of $K$, denoted by $T_K$, which is transverse to the contact structure $\xi$. Also transverse knots can be approximated by Legendrian knots, upto negative stabilization. The properties [10, Theorem 1.6] of $\hat{\theta}(T)$ and $L(T)$ under stabilization and connected sum makes sure that one can define the LOSS invariant for transverse knots as well (see [10, Theorem 1.5]). Precisely, if $L$ is a Legendrian approximation of the transverse knot $T$, then $\hat{\theta}(T) := \hat{\theta}(L)$. Now

$$\hat{\theta}(T) \in \widehat{HFK}(S^3, K, \frac{sl(T) + 1}{2})$$

If $T$ is a quasi-positive transverse knot, then $\tau(T) = \frac{sl(T) + 1}{2}$ and $\hat{\theta}(T) \neq 0$ (ref. [12, Proposition 3.7]). Hence if $\tau(K) = g(K)$, then by the equivalence of Baldwin-Vela-Vick-Vértesi from [1] and using corollary 6, one gets the statement of the theorem. □

As it is mentioned in the introduction, Baker and Motegi construct in [3, Section 4] a non-trivial band some of two strongly quasi-positive fibered knots, $T_{2,3}$ and $T_{2,3}^{2,1}$ ($(2,1)$-cable of $T_{2,3}$) to find a strongly quasi-positive non-fibered knot $K = T_{2,3}#_\beta T_{2,3}^{2,1}$. They find a strongly quasi-positive braid diagram of $K$ and show that it is prime. Then using the [3, Theorem 1.1] they conclude that $K$ is not fibered. Using result by Miyazaki from [11] that
non-trivial band sum of two knots is ribbon concordant to the connected sum of those knots and the additive property of concordance invariant $\tau$ under connected sum operation, we can see that $\tau(K) = \tau(T_{2,3}) + \tau(T_{2,3}^{2,1}) = g(K)$. Hence $K$ provides a non-fibered example of (strongly) quasi-positive knot which satisfy the assumption of corollary 6. More examples can be constructed by taking connecting sum of $K$ and a strongly quasi-positive fibered knot.

Note that one can also use the main theorem of [2] and [22] to infer the non-triviality of the next to top dimensional knot Floer homology of the previous example. Precisely since a fibered knot $(T_{2,3}, T_{2,3}^{2,1})$ of the same genus is ribbon concordant to the said knot $K$, [22, Theorem 1.1] implies that knot Floer homology of the fibered knot at the Alexander grading $g(K) - 1$ sits inside the next to top knot Floer homology of $K$. Then [2, Theorem 1.1] implies that it is non-trivial. Also note that there is no immediate way to use [9, Theorem 7] to this example as it is not straightforward to find the top dimensional $\hat{HFK}$ of $K$.

Also using the non-triviality of the grid invariant of a transverse knot and the proof of [4, Proposition 5.2], one can prove that in certain cases $K_{p,q}$ ($p \geq 2$) also satisfies the hypothesis of corollary 8. In particular if $\hat{L}(L_K)$ is the element that generates $H_2(i = 0)$ (or the generator of the free part of $HFK^-(S^3, K)$, see [8]) and $\tau(K) = -A(\hat{L}(T_K)) = -g(K)$ (note that by the property of LOSS invariant [10, Theorem 1.2] the contact structure in question have to be tight in this case). Hence for such $K$, all $(p, q)$ cables of $K$ also have non-trivial knot Floer homology at the next-to-top Alexander grading, where $p \geq 2$.

□

Remark. Note that we can use $L(L_K)$ instead of $\hat{L}(L_K)$ in Theorem 5 and corollary 6. Recall that $L(L_K) \in HFK^-(S^3, K)$ such that $\hat{L}(L_K) \neq 0$ when the $U$-filtration of $L(L_K)$ is 0. Now if the $U$-filtration of $L(L_K)$ is $n$, then $U^n \cdot L(L_K) \in HFK^-(S^3, K, n)$ cf. Figure 5. Using this description and the proof of corollary 6 can be used to show that the statement of the corollary 6 stays true if the $U$-filtration of $L(L_K)$ is $-g$.

References

1. John Baldwin, David Shea Vela-Vick & Vera Vértesi. On the equivalence of Legendrian and transverse invariants in knot Floer homology. Geometry & Topology 17: 925-974 (2013)
2. John Baldwin and David Shea Vela-Vick. A Note on the knot Floer homology and Fibered knots. Algebraic & Geometric Topology 18: 3669-3960 (2018).
3. Kenneth L. Baker & Kimihiko Motegi. Tight fibered knots and band sums. Mathematische Zeitschrift, 286(3), 1357-1365
4. A.Chakroborty. Transverse and Legendrian invariants of cables in combinatorial link Floer homology. arXiv:1903.12256v2.
5. K. Honda, W. H. Kazez & Gordana Matić. Right-veering diffeomorphisms of compact surfaces with boundary. Invent. Math. 169 (2007), no. 2, 427-449, Zbl 1167.57008. MR 2318562 (2008e:57028)
6. K.Honda, W. H.Kazez & G. Matić. On the contact class in Heegaard Floer homology. J. Differential Geom. 83 (2009), no. 2, 289-311, Zbl 1186.53098. MR 2577470 (2011f:57050)
7. P.Ghiggini. Knot Floer homology detects genus-one fibred knots. Amer. J. Math. 130 (2008), no. 5, 1151-1169
8. M.Hedden. Notions of positivity and the Ozsváth-Szabó concordance invariant. Journal of Knot Theory and its Ramifications, 19 (2010), no. 5, 617-629.
9. M.Hedden & L.Watson. On the geography and botany of knot Floer homology. Selecta Mathematica. New Series, 24 (2018), no. 2, 997-1037.
10. P. Lisca, P. S. Ozsváth, A. Stipsicz & Z. Szabó. *Heegaard Floer invariants of Legendrian knots in contact three-manifolds*. Journal of The European Mathematical Society 11:6 pp. 1307-1363, 57 p. (2009).
11. K. Miyazaki. *Band sums are ribbon concordant to the connected sum*. Proceedings of the American Mathematical Society Volume 126, Number 11, November 1998, Pages 3401-3406.
12. Olga Plamenevskaya. *Braid monodromy, orderings, and transverse invariants*. Algebr. Geom. Topol. 18 (2018), 3691-3718.
13. P. S. Ozsváth and Z. Szabó. *Holomorphic disks and topological invariants for closed three-manifolds*. Annals of Mathematics, Pages 1027-1158 from Volume 159 (2004), Issue 3.
14. P. S. Ozsváth and Z. Szabó. *Holomorphic disks and knot invariants*. Adv. Math. 186 (2004), no. 1, 58-116.
15. P. S. Ozsváth and Z. Szabó. *Holomorphic disks and genus bounds*. Geom. Topol. 8 (2004) 311-334.
16. P. S. Ozsváth and Z. Szabó. *Heegaard Floer homology and contact structures*. Duke Math. J. 129 (2005), no. 1, 39-61.
17. P. S. Ozsváth and Z. Szabó. *On knot Floer homology and lens space surgeries*. Topology, 44(2005), no. 6, 1281-1300.
18. P. S. Ozsváth, Z. Szabó and Dylan P. Thurston. *Legendrian knots, transverse knots and combinatorial Floer homology*. Geometry and Topology, 12 (2008), 941-980.
19. Jacob A. Rasmussen. *Floer homology and knot complements*. Ph.D. thesis, Harvard University, 2003, arXiv:math/0509499.
20. Yi Ni. *Knot Floer homology detects fibered knots*. Invent. Math. 170 (2007), no. 3, 577-608.
21. Yi Ni. *The next-to-top term in knot Floer homology*. (tentative title) In preparation
22. I. Zemke. *Knot Floer homology obstructs ribbon concordance* arXiv:1902.04050

Department of Mathematics, University at Buffalo

Email address: subhanka@buffalo.edu