Non-Markovian Majority-Vote model

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Non-Markovian dynamics pervades human activity and social networks and it induces memory effects and burstiness in a wide range of processes including inter-event time distributions, duration of interactions in temporal networks and human mobility. Here we propose a non-Markovian Majority-Vote model (NMMV) that introduces non-Markovian effects in the standard (Markovian) Majority-Vote model (SMV). The SMV model is one of the simplest two-state stochastic models for studying opinion dynamics, and displays a continuous order-disorder phase transition at a critical noise. In the NMMV model we assume that the probability that an agent changes state is not only dependent on the majority state of his neighbors but it also depends on his age, i.e. how long the agent has been in his current state. The NMMV model has two regimes: the aging regime implies that the probability that an agent changes state is decreasing with his age, while in the anti-aging regime the probability that an agent changes state is increasing with his age. Interestingly, we find that the critical noise at which we observe the order-disorder phase transition is a non-monotonic function of the rate $\beta$ of the aging (anti-aging) process. In particular the critical noise in the aging regime displays a maximum as a function of $\beta$ while in the anti-aging regime displays a minimum. This implies that the aging/anti-aging dynamics can retard/anticipate the transition and that there is an optimal rate $\beta$ for maximally perturbing the value of the critical noise. The analytical results obtained in the framework of the heterogeneous mean-field approach are validated by extensive numerical simulations on a large variety of network topologies.

I. INTRODUCTION

Many natural, social and technological phenomena can be well described by stochastic binary-state models formed by a large number of interacting agents. Depending on the application, various types of dynamical rules determining the stochastic switch of the states of the agents can be considered. This framework includes very well known processes, such as the Ising model, the voter model and the susceptible-infected-susceptible model, that have been used to model magnetic materials, opinion formation, and epidemic spreading, among others. Strikingly, extensions or modifications for the models can lead in a variety of cases to dynamical behaviors drastically different from the original ones. For example, the presence of non-trivial structure in the interacting patterns such as heavy-tailed degree distribution, mesoscopic structures, multilayer structures, and time delay, non-homogeneous interevent intervals or a fraction of zealots, can induce significant change in the dynamics. Moreover relevant effect can be obtained also changing the dynamical rules by introducing of more than two states, time delay, non-homogeneous interevent intervals or a fraction of zealots.

The Majority-Vote (MV) model is a simple non-equilibrium Ising-like system with up-down symmetry that presents an order-disorder phase transition at a critical value of noise. The MV model is also one of the paradigmatic model for studying opinion dynamics, and it has been extensively studied in regular lattices, random graphs, and in complex networks including small-world networks, scale-free networks, modular networks, complete graphs, and spatially embedded networks. Some extensions were also proposed, such as multi-state MV models, inertial effect, frustration due to anticonformists, and cooperation in multilayer structures.

Most of stochastic binary-state models are based on a memoryless Markovian assumption, which implies that the switching rates from one state to the other depend only on the present state of the system. One of important properties of Markovian processes is that the inter-event time intervals follow an exponential distribution and the number of events in a given time interval follows a Poissonian distribution. The Markovian assumption facilitates theoretical analysis of models. However, there is growing evidence that human activity follows a non-Markovian dynamics driven by memory effects. Non-Markovian bursty dynamics characterized by heavy tail inter-event time distributions is ubiquitous in human activities, and strongly affects the duration of interactions in temporal networks. Memory effects have also shown to be essential to model human mobility and random walks over complex networks. Therefore, the Markovian assumption provides only an approximate picture of the real world.

In recent years, there is an increasing interest in un-
nderstanding the role of non-Markovian effects in stochastic binary-state models, from the theoretical \cite{65, 70} and from the numerical \cite{71, 72} perspective as well.

In the context of the voter model and of the noisy voter model, non-Markovian effects have been introduced by assuming that the switching probability between states depends on the age of the agent, i.e., how long an agent has been in its current state \cite{73, 74}. The induced effects of this non-Markovian dynamics are also called aging effects when the switching probability decreases with the agent’s age and anti-aging effects when the switching probability increases with the agent’s age. In Ref. \cite{23}, Stark et al. reported an aging-induced counterintuitive phenomenon in the voter model. They showed that the transition probability between two opposite states decreases with age, but the time to reach a macroscopically ordered state can be accelerated. In Ref. \cite{22}, Peralta et al. studied systematically the aging version of voter model at the mean-field level, and they showed that the model reaches consensus or gets trapped in a frozen state depending on the specific form between transition probability and age. They also considered the anti-aging case when the transition probability is an increasing function of age. For the latter case, the model always reaches a steady state with coexistence of two states. In Refs. \cite{70, 77} it has been shown that the aging effects in the noisy-voter model can alter the shape of phase transition. In the absence of aging, the model shows a finite-size discontinuous transition between ordered and disordered phases. When the aging is involved, the transition becomes of a second order well defined in the thermodynamic limit. Moreover, recently Peralta et al. in Ref. \cite{78} proved that the non-Markovian noisy-voter model can be approximately reduced to a non-linear noisy-voter model which is Markovian.

In the present work, we reveal the role of non-Markovian dynamics in the Majority-Vote model providing results that enrich the understanding provided by the works above summarized. We propose the non-Markovian Majority-Vote (NMMV) model by incorporating non-Markovian dynamics in the Majority-Vote model. In the NMMV model the transition probability between states is dependent on the majority state of the agent’s neighbors. In particular the transition probability depends on the so-called noise parameter $f$ that allows state switches that are not aligned with the majority state of the neighbors of the agent. Moreover in the NMMV model the transition probability also depends on the agent’s age. Specifically the NMMV model includes two regimes: the aging regime in which the probability of a state switch decreases with the agent’s age and an anti-aging regime in which the probability of a state switch increases with the agent’s age. We indicate with $\beta$ rate of change of the transition probability with age. The model displays a phase transition as a function of the noise parameter $f$: for $f < f_c^{NMMV}$ the NMMV model is in the ordered phase, i.e., the network displays a clear majority state, for $f \geq f_c^{NMMV}$ the NMMV is in the disordered phase where overall, in the entire network, there is no majority state. We show that the non-Markovian dynamics strongly affects the value of the critical noise $f_c^{NMMV}$. In particular in the aging regime the non-Markovian dynamics retards the transition with respect to the standard Majority-Vote model (SMV) and the critical noise $f_c^{NMMV}$ in the NMMV model is larger or equal to the critical noise $f_c^{SMV}$ in the SMV model, i.e., $f_c^{NMMV} \geq f_c^{SMV}$. In the anti-aging regime, instead, the relation between the critical noise in the NMMV model and in the SMV model are reversed, i.e., $f_c^{NMMV} \leq f_c^{SMV}$. Interestingly, by solving the model in the framework of an heterogeneous mean-field approach, we can derive analytically the non-monotonic dependence of the critical noise $f_c^{NMMV}$ on the rate $\beta$. In the aging regime, the critical noise displays a maximum at a non-zero but finite value of $\beta$. In the anti-aging regime, a minimum of the critical noise as a function of $\beta$ shows up. This means that the non-Markovian dynamics can be used to retard or anticipate the transition and to maximally perturb the critical noise $f_c^{NMMV}$ in the typical time-scale of the aging/anti-aging dynamics does not need to be too-fast or too-slow.

The theoretical predictions are in well agreement with the reported extensive simulations of the model.

The paper is structured as follows: in Sec. III we define the NMMV model; in Sec. IV we present the analytic solution of the model obtained in the framework of the heterogeneous mean-field approach; in Sec. V we characterize the critical properties of the model including the analytical expression of the critical noise, and its dependence on the rate $\beta$; in Sec. VI we compare the analytic predictions to the simulation results; finally in Sec. VII we provide the conclusions.

**II. MAJORITY-VOTE MODEL WITH NON-MARKOVIAN SWITCHING OF STATES**

In this section we introduce the non-Markovian Majority-Vote model which differs from the standard Majority-Vote model \cite{23} by introducing a non-Markovian mechanism for the switching of states. Therefore in the NMMV model the agents have a probability of switching states that depends on their age, i.e., for how they have been in their current state.

We consider a population of $N$ agents defined on a static network topology. Each agent $i$ with $i = 1, \cdots, N$ is located on a node $i$ of the network. Each agent is assigned two dynamical variables: a binary variable $\sigma_i = \pm 1$ (his state) describing the agent’s opinion/vote and a variable $a_i$ (his age) indicating how long the agent has not changed his state. Initially the states $\{\sigma_i\}$ are randomly assigned to the agents and the variables $\{a_i\}$ are initialized by setting $a_i = 0$ for every agent $i$ of the network. At each time step, an agent $i$ is chosen at random and his state is switched with probability $w_i$ which implements the non-Markovian Majority Vote
process. Thus with probability \( w_i \), the agent \( i \) switches state and the age of agent \( i \) is reset to zero, \( i.e. \)

\[
\begin{align*}
\sigma_i &\rightarrow -\sigma_i, \\
a_i &\rightarrow 0. 
\end{align*}
\]  

(1)

Otherwise, nothing happens except for the age increased by one, \( i.e. \)

\[
a_i \rightarrow a_i + 1. 
\]  

(2)

In both cases the time is updated according to

\[
t \rightarrow t + \Delta t, 
\]  

(3)

with \( \Delta t = 1/N \). The richness of the model reside on the definition of the switching probability \( w_i \), given by

\[
w_i = \nu w_i^{SMV}, 
\]  

(4)

where \( 0 \leq \nu_i \leq 1 \), called the activation probability, is a function of the age \( a_i \) of agent \( i \) and where \( w_i^{SMV} \) is the switching probability in the SMV model, \( i.e. \) it is independent of the age variable. The contribution \( w_i^{SMV} \) to the switching probability \( w_i \) of the agent \( i \) depends on the majority state of \( i \)’s neighborhood and on a parameter \( f \) called the noise intensity. If the state \( \sigma_i \) of the agent is opposite to the majority state of his neighbors \( w_i^{SMV} \) contributes to the switching probability to the majority state by a term \( 1 - f \). If the state \( \sigma_i \) of the agent is the same as the majority state of his neighbors \( w_i^{SMV} \) contributes to the switching probability to the majority state by a term \( f \). If there is no clear majority of the agent \( i \)’s neighbors, \( i.e. \) half of the neighbors have state \( \sigma_j = +1 \) and half of the neighbors have state \( \sigma_j = -1 \), then \( w_i^{SMV} = 1/2 \). Therefore, \( w_i^{SMV} \) can be expressed as

\[
w_i^{SMV} = \frac{1}{2} \left[ 1 - (1 - 2f) \sigma_i S\left( \sum_{j \in N_i} \sigma_j \right) \right], 
\]  

(5)

where \( N_i \) denotes the set of neighbors of agent \( i \), and \( S(x) = \text{sgn}(x) \) if \( x \neq 0 \) and \( S(0) = 0 \) indicates the majority state of his neighborhood.

The NMMV model reduces to the SMV model in the case in which we consider a trivial choice of \( \nu_i \), \( i.e. \) \( \nu_i = 1 \) for all agent \( i \). In this case, as \( f \) increases, the model undergoes a continuous order-disorder phase transition at a critical value of noise intensity \( f = f_c^{SMV} \).

However in a number of real scenarios for social and opinion networks it has been shown that non-Markovian effects are relevant. In order to capture this non-Markovian dynamics we consider the following choice of the term \( \nu_i = \nu(a_i) \),

\[
\nu(a) = (\nu_0 - \nu_\infty)e^{-\beta a} + \nu_\infty, 
\]  

(6)

where \( \nu_0 = \nu(0) \) and \( \nu_\infty = \lim_{a \to \infty} \nu(a) \). The non-Markovian contribution \( \nu(a) \) to the switching probability is parameterized by the parameter \( \beta > 0 \). Note that \( \beta \) characterizes the rate of exponential change of \( \nu \) as a function of \( a \). Obviously, in the limits of \( \beta \to 0 \) and \( \beta \to \infty \), all the agents have the same fixed value of activity, \( \nu \equiv \nu_0 \) and \( \nu \equiv \nu_\infty \), and the dynamics is thus equivalent to the SMV model with the time scaled by a factor \( \nu_0^{-1} \) and \( \nu_\infty^{-1} \), respectively.

We distinguish two different regimes of the dynamics:

(i) Aging regime. For \( \nu_0 > \nu_\infty \), \( \nu(a) \) decays exponentially with \( a \), implying that the longer an agent is in a given state, the more difficult is for him to change state.

(ii) Anti-aging regime. For \( \nu_0 < \nu_\infty \), \( \nu(a) \) increases exponentially with \( a \). Such an case can be interpreted as “rejuvenating” dynamics where agents become more prone to change state as they are longer on a given state.

Without loss of generality, set equal to one the maximum between \( \nu_0 \) and \( \nu_\infty \), \( i.e. \) we put \( \max\{\nu_0, \nu_\infty\} = 1 \). Moreover to avoid trivial frozen states of the dynamics, the minimum between \( \nu_0 \) and \( \nu_\infty \) is set to be larger than zero, \( i.e. \), \( \min\{\nu_0, \nu_\infty\} > 0 \).

III. HETEROGENEOUS MEAN-FIELD SOLUTION OF THE MODEL

In order to capture the phase diagram of the NMMV model on a random network with given degree distribution \( P(k) \), we solve the model using the heterogeneous mean-field approach. Therefore we assume that the probability that an agent \( i \) is in a given state depends exclusively on his degree \( k \) and his age \( a \) and we denote by \( x_{k,a}^\pm \) the probability that an agent of degree \( k \) with age \( a \) is in the state \#1 ±. It follows that the probability \( x_k^\pm \) of an agent of degree \( k \) in the state \#1 ±, is given by

\[
x_k^\pm = \sum_{a=0}^{\infty} x_{k,a}^\pm. 
\]  

(7)

In order to solve the dynamical equations of the NMMV model in the heterogeneous mean-field approximation we also need to evaluate the switching probability \( w_{k,a}^\pm \) of an agent of degree \( k \) and age \( a \). Let us define \( \tilde{x}^\pm \) the probability that by following a link we reach a node in state \#1 ±, given by

\[
\tilde{x}^\pm = \sum_k \frac{kP(k)}{\langle k \rangle} x_k^\pm = \sum_k \frac{kP(k)}{\langle k \rangle} \sum_{a=0}^{\infty} x_{k,a}^\pm. 
\]  

(8)

For a node of degree \( k \), the probability that the majority state among his neighborhoods is \#1 ± is given by the binomial distribution,

\[
\psi_k(\tilde{x}^\pm) = \sum_{n=\lfloor k/2 \rfloor}^{k} \left( 1 - \frac{1}{2} \delta_{n,k/2} \right) C_k^n (\tilde{x}^\pm)^n (1 - \tilde{x}^\pm)^{k-n}, 
\]  

(9)
where \( \lceil \cdot \rceil \) is the ceiling function, \( \delta_{r,s} \) is the Kronecker symbol, and \( C^p_k = k!/[n! (k-n)!] \) are the binomial coefficients. According to Eq. (4), we can write down the switching probability \( w_{k,a}^\pm \) of an agent of state \( \pm 1 \) with degree \( k \) and age \( a \) as

\[
w_{k,a}^\pm = \nu(a+1) \Psi_k(\bar{x}^\pm),
\]

where \( \nu(a+1) \) is given by Eq. (6), and \( \Psi_k(\bar{x}^\pm) \) is the flipping probability of an agent of state \( \pm 1 \) without the aging effect \( [36] \), i.e.

\[
\Psi_k(\bar{x}^\pm) = (1 - f) \left[ 1 - \psi_k(\bar{x}^\pm) \right] + f \psi_k(\bar{x}^\pm).
\]

The dynamical equations that determine the time evolution of the probabilities \( x_{k,a}^\pm \) are a function of the switching probabilities \( w_{k,a}^\pm \). These equations can be deduced by observing that at each time step one of the following four possible events occurs.

(i) An agent in state +1 having degree \( k \) and age \( a \) is chosen and his state is flipped. The rate at which \( x_{k,a}^+ \) decreases and \( x_{k,0}^- \) increases due to this process is \( w_{k,a}^- \).

(ii) An agent in state +1 having degree \( k \) and age \( a \) is chosen but his state is not flipped. The rate at which \( x_{k,a}^+ \) decreases and \( x_{k,a+1}^- \) increases due to this process is \( x_{k,a}(1 - w_{k,a}^+ \).

(iii) An agent in state −1 having degree \( k \) and age \( a \) is chosen and the state is flipped. The rate at which \( x_{k,a}^- \) decreases and \( x_{k,0}^+ \) increases due to this process is \( x_{k,a}^- \).

(iv) An agent in state −1 having degree \( k \) and age \( a \) is chosen but the state is not flipped. The rate at which \( x_{k,a}^- \) decreases and \( x_{k,a+1}^+ \) increases due to this process is \( x_{k,a}(1 - w_{k,a}^-) \).

Accordingly, the rate equations for \( x_{k,a}^\pm \) read

\[
\frac{dx_{k,0}^+}{dt} = \sum_{a=0}^{\infty} x_{k,a}^- w_{k,a}^+ - x_{k,0}^+;
\]

\[
\frac{dx_{k,a}^+}{dt} = x_{k,a-1}^+ (1 - w_{k,a-1}^+) - x_{k,a}^-; \quad a \geq 1,
\]

\[
\frac{dx_{k,0}^-}{dt} = \sum_{a=0}^{\infty} x_{k,a}^+ w_{k,a}^- - x_{k,0}^-;
\]

\[
\frac{dx_{k,a}^-}{dt} = x_{k,a-1}^- (1 - w_{k,a-1}^-) - x_{k,a}^+; \quad a \geq 1.
\]

In stationary state, by setting the time derivative of \( x_{k,a}^\pm \) equal to zero, we obtain that the probabilities \( x_{k,a}^\pm \) obey

\[
x_{k,0}^+ = \sum_{a=0}^{\infty} x_{k,a}^- w_{k,a}^-;
\]

\[
x_{k,a}^+ = x_{k,a-1}^+ (1 - w_{k,a-1}^+) - x_{k,a}^-; \quad a \geq 1,
\]

\[
x_{k,0}^- = \sum_{a=0}^{\infty} x_{k,a}^+ w_{k,a}^+;
\]

\[
x_{k,a}^- = x_{k,a-1}^- (1 - w_{k,a-1}^-); \quad a \geq 1.
\]

Using Eq. (17) and Eq. (18), and summing \( x_{k,a}^+ \) over the values of \( a \) greater or equal to one we get

\[
x_{k,0}^+ = x_{k,0}^+.
\]

This condition is a necessary condition for stationarity. In fact at stationarity the probability that a node is in a state (and reset their age to 1) changes is

\[
F_k(\bar{x}) = \sum_{a=0}^{\infty} x_{k,a}^+.
\]

In terms of Eq. (17) and Eq. (18), \( x_{k,a}^+ \) for \( a \geq 1 \) can be computed in a recursive way, and then are expressed by

\[
x_{k,a}^+ = x_{k,0}^+ F_k(\bar{x}); \quad a \geq 1,
\]

where for convenience we have introduced the function \( F_{k,a} \) given by

\[
F_{k,a} = \prod_{j=0}^{a-1} \left( 1 - w_{k,j}^+ \bar{x}^\pm \right).
\]

Substituting Eq. (21) into the definition \( x_{k}^\pm = \sum_{a=0}^{\infty} x_{k,a}^\pm \), we have

\[
x_{k}^\pm = x_{k,0}^+ F_k(\bar{x}^\pm),
\]

with

\[
F_k(\bar{x}^\pm) = 1 + \sum_{a=0}^{\infty} F_{k,a} (\bar{x}^\pm).
\]

In order to find \( x_{k}^+ \) we note that by using Eq. (20), we can express the ratio \( x_{k}^+ / x_{k}^- \) as

\[
\frac{x_{k}^+}{x_{k}} = \frac{x_{k,0}^+ F_k(\bar{x}^+)}{x_{k,0}^- F_k(\bar{x}^-)} = \frac{F_k(\bar{x}^+)}{F_k(\bar{x}^-)}.
\]

Substituting \( \bar{x}^- \) with 1 − \( \bar{x}^+ \) in Eq. (25), we then obtain

\[
x_{k}^+ = \frac{F_k(\bar{x}^+)}{F_k(\bar{x}^+ + F_k(1 - \bar{x}^+)).
\]
Finally by using Eq. (26) in the left-hand side of Eq. (8), we find the self-consistent equation of $\tilde{x}^+$

$$\tilde{x}^+ = \sum_k kP(k) \frac{F_k(\tilde{x}^+)}{F_k(\tilde{x}^+)} + F_k(1 - \tilde{x}^+).$$

This equation can be solved numerically by finding $\tilde{x}^+$ by iterating Eq. (27) starting from an initial value of $\tilde{x}^+ \neq 1/2$. Once $\tilde{x}^+$ is found, we can calculate $x^+_k$ by using Eq. (26). This allow us to find the average magnetization per node by

$$m = \sum_k P(k) (x^+_k - x^-_k) = \sum_k P(k) (2x^+_k - 1).$$

This theoretical treatment of the model provides predictions that can be compared to simulation results revealing the critical properties of the NMMV model. In particular the main features of the steady state configurations can be described by plotting $m$ as a function of $f$ for different values of $\beta$.

In Fig. 1(a), we report such results when for $\nu_0 > \nu_\infty$ when the non-Markovian dynamics is in the aging regime. Here we have used regular random networks (RR) whose degree distribution follows a delta function, $P(k) = \delta(k - \langle k \rangle)$ with $\langle k \rangle = 20$ and network size $N = 10^4$. Direct simulation results are compared to theoretical predictions finding excellent agreement (see Fig. 1). The order parameter $|m|$ shows a continuous second-order phase transition as noise intensity $f$ varies, similar to the SMV model. The transition point, i.e., the critical value of noise intensity $f_c$, depends on the value of $\beta$. In the aging regime, as $\beta$ increases, $f_c$ displays a maximum at $\beta = \beta_{\text{aging}}^{\text{max}}$. In the anti-aging regime ($\nu_0 < \nu_\infty$) $f_c$ shows again a non-monotonic behavior but instead of displaying a maximum as a function of $\beta$ (like in presence of the aging dynamics) it displays a minimum at $\beta = \beta_{\text{anti-aging}}^{\text{min}}$ (see Fig. 1(b)).

\section{IV. THE PHASE DIAGRAM}

\subsection{A. The critical noise}

In this paragraph we will use the heterogeneous mean-field approach to derive the expression for the critical noise $f_c^{\text{NMMV}}$ in the NMMV model. First of all, we notice that $\tilde{x}^+ = 1/2$, is always a solution of Eq. (27). This state corresponds to the disordered phase where the state of each agent is totally random. Such a trivial solution loses its stability when the noise intensity is less than a critical value, i.e., $f < f_c$. According to linear stability analysis, the critical noise $f_c$ can be found by imposing that the derivative of the right-hand side of Eq. (27) with respect to $\tilde{x}^+$ calculated for $\tilde{x}^+ = 1/2$ is equal to one, i.e., $f_c$ satisfies

$$\sum_k kP(k) \frac{F_k' \left( \frac{1}{2} \right)}{2F_k \left( \frac{1}{2} \right)} = 1.$$  

At $\tilde{x}^+ = 1/2$, $\nu_k$ and also $\Psi_k$ are independent of $k$. In particular we have $\Psi_k \left( \frac{1}{2} \right) = \frac{1}{2}$ for all value of $k$. Therefore using Eq. (24), this implies that also $F_k \left( \frac{1}{2} \right)$ is independent of $k$ and is given by

$$F \left( \frac{1}{2} \right) = 1 + \sum_{a=1}^{\infty} F_a \left( \frac{1}{2} \right),$$

with

$$F_a \left( \frac{1}{2} \right) = \prod_{j=1}^{a} \left( 1 - \frac{1}{2} \nu_j \right),$$

(note that here we have omitted the subscript $k$ in the expression of $F_k \left( \frac{1}{2} \right)$ and $F_{k,a} \left( \frac{1}{2} \right)$ as they do not depend on $k$.) After some simple algebra, we can express $F_k' \left( \frac{1}{2} \right)$ as

$$F_k' \left( \frac{1}{2} \right) = -\Psi_k \left( \frac{1}{2} \right) \sum_{a=1}^{\infty} F_a \left( \frac{1}{2} \right) \sum_{j=1}^{a} \frac{\nu_j}{1 - \frac{1}{2} \nu_j}.$$  

With

$$\Psi_k \left( \frac{1}{2} \right) = (2f - 1) \psi_k' \left( \frac{1}{2} \right),$$

and

$$\psi_k' \left( \frac{1}{2} \right) = 2^{1-k} kC_{k-1}^{(k-1)/2}.$$  

Substituting Eqs. (30), (31) into Eq. (29), we obtain the critical noise $f_c^{\text{NMMV}}$ in the NMMV model,

$$f_c^{\text{NMMV}} = \frac{1}{2} - G (\beta; \nu_0, \nu_\infty) \frac{\langle k \rangle}{\sum_k k^2 P(k) 2^{1-k} C_{k-1}^{(k-1)/2}}.$$
where
\[ G(\beta; \nu_0, \nu_\infty) = \frac{F(\frac{1}{2})}{\sum_{a=1}^{\infty} f_a(\frac{1}{2}) \sum_{j=1}^{\infty} \nu_j j^{-\frac{1}{2}}}. \] (36)

Using Stirling’s approximation for large \( k \), \( C^{[k-1]/2} \approx 2^{k-1}/\sqrt{k\pi/2} \), Eq.(36) can be simplified to
\[ f_c^{\text{NMMV}} = \frac{1}{2} - G(\beta; \nu_0, \nu_\infty) \sqrt{\frac{\pi}{2}} \langle k \rangle^{3/2}. \] (37)

where \( \langle \ldots \rangle \) denotes the average over the degree distribution \( P(k) \). The critical noise \( f_c^{\text{NMMV}} \) dependence on the non-Markovian dynamics is fully captured by the function \( G(\beta; \nu_0, \nu_\infty) \), which can be considered as an function of \( \beta \) for any given value of the parameters \( \nu_0 \) and \( \nu_\infty \). We distinguish two main regimes:

(i) For \( \nu_0 > \nu_\infty \) \( G(\beta; \nu_0, \nu_\infty) \) captures the dependence of \( f_c \) on \( \beta \) in the aging regime;

(ii) For \( \nu_0 < \nu_\infty \) it captures the dependence of \( f_c \) on \( \beta \) in the anti-aging regime.

When the aging effects are not taken into account, \( \nu(a) \equiv \nu \), \( G(\beta; \nu, \nu_\infty) = \frac{1}{2}, \) and Eq.(36) thus reduces to the expression of the critical noise \( f_c \) in the SMV model (36),
\[ f_c^{\text{SMV}} = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\pi}{2}} \langle k \rangle^{3/2}. \] (38)

B. The function \( G(\beta; \nu_0, \nu_\infty) \)

As noted before the function \( G(\beta; \nu_0, \nu_\infty) \) captures all the dependence of the critical noise \( f_c^{\text{NMMV}} \) on the non-Markovian dynamics. In particular from \( (37) \) and \( (38) \) we deduce that the function \( G(\beta; \nu_0, \nu_\infty) \) characterizes the relation between the critical noise in NMMV model and in the SMV model. In fact we have
\[ 2G(\beta; \nu_0, \nu_\infty) = \frac{1}{2} - \frac{f_c^{\text{NMMV}}}{f_c^{\text{SMV}}}. \] (39)

The numerical solution of Eq.(39) reveals that the function \( G(\beta; \nu_0, \nu_\infty) \) displays a non-monotonic behavior as a function of \( \beta \) when \( \nu_0 \) and \( \nu_\infty \) are fixed to a constant value. In particular the function \( G(\beta; \nu_0, \nu_\infty) \) displays a minimum as a function of \( \beta \) in the aging regime and a maximum in the anti-aging regime (see Fig.2). In the limit \( \beta \to 0 \) or \( \beta \to \infty \), we obtain \( G(\beta; \nu_0, \nu_\infty) \to 1/2 \) indicating the marginal role of the non-Markovian dynamics, i.e. using Eq.(39) \( f_c^{\text{NMMV}} \to f_c^{\text{SMV}}. \) Since the critical noise \( f_c^{\text{NMMV}} \) depends on \( \beta \) only through the function \( G(\beta; \nu_0, \nu_\infty) \) in the aging regime, the minimum of \( G(\beta; \nu_0, \nu_\infty) \) is achieved for \( \beta = \beta^{\text{aging}}_m \), corresponding to the maximum of \( f_c^{\text{NMMV}} \); conversely in the anti-aging regime the maximum of \( G(\beta; \nu_0, \nu_\infty) \) is achieved for \( \beta = \beta^{\text{anti-aging}}_m \). Let us indicate with \( \Delta G_m \) the maximal deviation of the function \( G \) from its asymptotic value 1/2 achieved in the limit \( \beta \to 0 \) and \( \beta \to \infty \), i.e.
\[ \Delta G_m = \frac{1}{2} - G(\beta_m; \nu_0, \nu_\infty). \] (40)

Specifically let us indicate with \( \Delta G^{\text{aging}}_m \) the values obtained in the aging regime and with \( \Delta G^{\text{anti-aging}}_m \) the values obtained in the anti-aging regime. In Fig.3 we show the dependence of \( \Delta G^{\text{aging}}_m, \Delta G^{\text{anti-aging}}_m \) and \( \Delta G^{\text{anti-aging}}_m \) as a function of \( \nu_0, \nu_\infty \) for fixed value of \( \nu_0, \nu_\infty = 1 \). As \( \nu_0, \nu_\infty \) increases, both \( \Delta G^{\text{aging}}_m \) and \( \Delta G^{\text{anti-aging}}_m \) increase and they approach each other as \( \nu_0, \nu_\infty \) approached max \( \nu_0, \nu_\infty \). The values of \( \Delta G^{\text{aging}}_m \) and \( \Delta G^{\text{anti-aging}}_m \) both decrease with increasing \( \nu_0, \nu_\infty \) and go to zero in the limit of \( \nu_0, \nu_\infty \) \( \to 1 \). While the definition of \( G \) given by \( (39) \) is valid for arbitrary functions \( \nu(a) \) the investigation performed in this paragraph is obtained starting from the expression for \( \nu(a) \) given by \( (39) \). Interestingly these results do not change significantly if other functional forms of \( \nu(a) \) are considered. In particular we have considered the linear function
\[ \nu(a) = \begin{cases} \beta (\nu_\infty - \nu_0) a + \nu_0, & a < 1/\beta, \\ \nu_\infty, & a \geq 1/\beta, \end{cases} \]
the rational function
\[ \nu(a) = \frac{\nu_\infty a + \nu_0/\beta}{a + 1/\beta}, \]
and the expression
\[ \nu(a) = (\nu_0 - \nu_\infty) (1 + a)^{-\beta} + \nu_0. \]
including a power-law dependence on the age \( a \). We have studied the function \( G(\beta; \nu_0, \nu_\infty) \) for all these kernels, and we have found that qualitatively the results are unchanged with respect to the results obtained for the exponential kernel.

Therefore, the values of \( \Delta G^{\text{aging}}_m \) and \( \Delta G^{\text{anti-aging}}_m \) characterize the maximal difference between \( f_c^{\text{NMMV}} \) and \( f_c^{\text{SMV}} \) for the aging rule and anti-aging rule, respectively. We note that since \( G \) is independent of the topology of the underlying network, \( \beta^{\text{aging}}_m, \beta^{\text{anti-aging}}_m \) at which \( f_c^{\text{NMMV}} \) is maximized (minimized), is not affected by the network topology.

V. COMPARISON WITH NUMERICAL RESULTS

In this section we compare the results obtained analytically using the heterogeneous mean-field approximation with extensive numerical results on different network topologies.

We have considered three different random networks generated using the configuration model [79]:
the anti-aging regime (for \( \nu_0 > \nu_\infty \)) and in the aging regime (for \( \nu_0 < \nu_\infty \)). All curves are obtained for \( \max \{ \nu_0, \nu_\infty \} = 1 \).

(a) regular random networks (RR) with degree distribution \( P(k) = \delta(k - \bar{k}) \);

(b) Erdős-Rényi networks (ER) with degree distribution \( P(k) = e^{-(\nu_0 - \nu_\infty)} k^\nu_0 / k! \);

(c) scale-free networks (SF) with degree distribution \( P(k) \sim k^{-\gamma} \).

In order to numerically determine the critical noise \( f_c^{NMMV} \), we calculated the Binder’s fourth-order cumulant \( U \), defined as

\[
U = 1 - \frac{1}{3} \left[ \frac{m^4}{m^2} \right],
\]

where \( m = \sum_1^N \sigma_i / N \) is the average magnetization per node, \( \tau \) denotes the time averages taken in the stationary regime, and \( [ \cdot ] \) indicates the averages over different network configurations. The critical noise \( f_c^{NMMV} \) is obtained by detecting the point \( f = f_c^{NMMV} \) where the curves \( U = U(f) \) obtained for different network sizes \( N \), intercept each other. In Fig.4 we show \( f_c^{NMMV} \) as a function of \( \beta \) for the aging and anti-aging regime for the three considered network models finding excellent agreement with the theoretical predictions. As predicted by the mean-field theory, the critical noise \( f_c^{NMMV} \) shows a non-monotonic dependence on \( \beta \) in both regimes. In the aging regime, there exists an optimal value of \( \beta \) in which \( f_c^{NMMV} \) is maximized in the anti-aging regime instead \( f_c^{NMMV} \) displays a minimum as a function of \( \beta \). The optimal \( \beta \) for the two regimes are independent on the network degree distribution as predicted by the heterogeneous mean-field solution. In Fig.4, we report the results obtained by setting \( \max \{ \nu_0, \nu_\infty \} = 1 \) and \( \min \{ \nu_0, \nu_\infty \} = 0.05 \). We have also performed simulation with some other values of \( \min \{ \nu_0, \nu_\infty \} \) and found that the non-monotonous behavior of \( f_c^{NMMV} \) is qualita-
FIG. 5: The critical noise $f_c^{\text{NMMV}}$ is plotted as a function of $\beta$ in the aging and anti-aging regime for a 2d square lattices (panel a) and in 3d cubic lattices (panel b). All results are obtained setting $\max \{\nu_0, \nu_\infty\} = 1$ and $\min \{\nu_0, \nu_\infty\} = 0.05$.

tively the same. Quantitatively, and in agreement with the theoretical expectations, as $\min \{\nu_0, \nu_\infty\}$ increases, the optimal $\beta$ in the aging and anti-aging regime shift to larger value and become closer to each other. Furthermore, the maximum (minimum) of $f_c^{\text{NMMV}}$ critical noise becomes less pronounced as $\min \{\nu_0, \nu_\infty\}$ increases.

Finally, we investigated the NMMV model also on two-dimensional and three-dimensional regular lattices, which are network topologies for which the heterogeneous mean-field approximation is not valid. The results are shown in Fig. 5. For two-dimensional lattices, the critical noise shows a maximum $f_c^{\text{NMMV}} \approx 0.3$ at $\beta^{\text{aging}} \approx 0.2$ in the aging regime and a minimum $f_c^{\text{NMMV}} \approx 0.008$ at $\beta^{\text{anti-aging}} \approx 0.01$ in the anti-aging regime. For three-dimensional lattices, the critical noise shows a maximum $f_c^{\text{NMMV}} \approx 0.36$ at $\beta^{\text{aging}} \approx 0.2$ in the regime regime and a minimum $f_c^{\text{NMMV}} \approx 0.05$ at $\beta^{\text{anti-aging}} \approx 0.01$ in the anti-aging regime. In the limits of $\beta \to 0$ and $\beta \to \infty$, the critical noise tend respectively to 0.075 and 0.18 in two-dimensional and three-dimensional lattices, consistent to the results valid for the SMV model [81]. This result shows evidently that also in situations in which we are far from the conditions necessary for the application of the heterogeneous approximation we observe a non-monotonic dependence of the critical noise $f_c^{\text{NMMV}}$ of the NMMV model on $\beta$ revealing that the observed phenomenology is universal, i.e. it is independent of the network topology.

VI. CONCLUSION

In this work we have introduced the non-Markovian Majority-Vote (NMMV) model that differs from the standard Majority-Vote (SMV) model as it includes memory effects. In fact in the NMMV model the probability that an agent switches state (activation probability) is not only dependent on the majority state of its neighbours as for the SMV model, but it is also age-dependent, i.e. depends on how long a agent has been in the same state (his age).

We distinguish two regime of the NMMV model: the aging regime in which the activation probability is a decreasing function of the agent’s age, and the anti-aging regime in which the activation probability is an increasing function of the age of the agent. The NMMV model displays a phase transition as a function of the noise $\beta$ determining the probability that an agent switches to the minority state of its neighbors. For $f < f_c^{\text{NMMV}}$ the NMMV model is in an ordered phase and displays an overall majority state, for $f \geq f_c^{\text{NMMV}}$ the model is in a disordered phase in which there half of the agents are in one state and half of the agents are in the other state.

By analytically solving the model using the heterogeneous mean-field approach and by performing extensive numerical simulations, we reveal how the non-Markovian dynamics affects the critical noise $f_c^{\text{NMMV}}$.

Interestingly, we show that the critical noise $f_c^{\text{NMMV}}$ in the NMMV model exhibits a non-monotonic dependence on the rate $\beta$ at which the activation probability changes with age. In particular we found two opposite behaviors in the aging and in the anti-aging regimes. In the aging regime, the critical noise $f_c^{\text{NMMV}}$ displays a maximum as a function of $\beta$ in the anti-aging regime instead $f_c^{\text{NMMV}}$ displays a minimum as a function of $\beta$.

These results indicate that in the aging regime the non-Markovian dynamics retards the transition, and in the anti-aging dynamics it anticipates the transition. Interestingly the most significant effect of the non-Markovian dynamics is achieved at a finite and non-zero value of the rate $\beta$, indicating that the aging/anti-aging dynamics needs to have a characteristic time-scale that is neither too fast or too slow.

Finally, this work highlights the importance of non-Markovian dynamics in determining the phase diagram of the NMMV model and we hope that it will stimulate interest in further investigations of the effect of memory and non-Markovian dynamics in critical phenomena defined on networks.

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