STABILITY OF NON-ISOLATED ASYMPTOTIC PROFILES
FOR FAST DIFFUSION

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Abstract. The stability of asymptotic profiles of solutions to the Cauchy-Dirichlet problem for Fast Diffusion Equation (FDE, for short) is discussed. The main result of the present paper is the stability of any asymptotic profiles of least energy. It is noteworthy that this result can cover non-isolated profiles, e.g., those for thin annular domain cases. The method of proof is based on the Lojasiewicz-Simon inequality, which is usually used to prove the convergence of solutions to prescribed limits, as well as a uniform extinction estimate for solutions to FDE. Besides, local minimizers of an energy functional associated with this issue are characterized. Furthermore, the instability of positive radial asymptotic profiles in thin annular domains is also proved by applying the Lojasiewicz-Simon inequality in a different way.

1. Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. We are concerned with the Cauchy-Dirichlet problem for Fast Diffusion Equation (shortly, FDE) of the form

$$
\partial_t \left( |u|^{m-2} u \right) = \Delta u \quad \text{in } \Omega \times (0, \infty), \quad (1.1)
$$

$$
u = 0 \quad \text{on } \partial \Omega \times (0, \infty), \quad (1.2)
$$

$$
|u|_{t=0} = u_0 \quad \text{in } \Omega, \quad (1.3)
$$

where $\partial_t = \partial/\partial t$, under the assumptions that

$$
u_0 \in H^1_0(\Omega), \quad 2 < m < 2^* := \frac{2N}{(N-2)_+}. \quad (1.4)
$$

FDE arises in plasma physics to describe anomalous diffusion of plasma in a Tokamak, a toroidal device to confine plasma by imposing a magnetic field (see [5, 6, 7] and [33]). One of typical features of solutions to (1.1)–(1.3) is the extinction in finite time, namely, every solution vanishes at a finite time (see [35, 8, 19, 28]). Moreover, Berryman and Holland [6] determined the optimal extinction rate of solutions $u = u(x,t)$ vanishing at a finite time $t_\ast = t_\ast(u_0)$ under (1.4). More precisely, it holds that

$$
c_1(t_\ast - t)^{1/(m-2)} \leq \|u(\cdot,t)\|_{H^1_0(\Omega)} \leq c_2(t_\ast - t)^{1/(m-2)} \quad \text{for all } t \geq 0
$$

with $c_1, c_2 > 0$, provided that $u_0 \neq 0$. Here and henceforth, we write $\|u\|_{H^1_0(\Omega)} = \|\nabla u\|_{L^2(\Omega)} = (\int_{\Omega} |\nabla u(x)|^2 \, dx)^{1/2}$. Furthermore, they also proved the existence of asymptotic profiles of vanishing solutions, that is, a nonzero limit of the rescaled solution $(t_\ast - t)^{-1/(m-2)}u(x,t)$ along a sequence $t_\ast \searrow t_\ast$ (see also [30, 20, 36] and [9, 10, 11, 12, 13]).

In order to characterize the asymptotic profile of $u(x,t)$, apply the change of variable,

$$
v(x,s) = (t_\ast - t)^{-1/(m-2)}u(x,t) \quad \text{with } s = \log(t_\ast/(t_\ast - t)).
$$
Then \( v(x,s) \) solves
\[
\begin{align*}
\partial_s \left( |v|^{m-2} v \right) - \Delta v &= \lambda_m |v|^{m-2} v & \text{in } \Omega \times (0, \infty), \\
v &= 0 & \text{on } \partial \Omega \times (0, \infty), \\
v|_{s=0} &= v_0 & \text{in } \Omega.
\end{align*}
\] (1.5) (1.6) (1.7)
with \( \lambda_m := (m-1)/(m-2) > 0 \) and the initial data \( v_0 := t_*(u_0)^{-1/(m-2)}u_0 \). Each asymptotic profile can be regarded as the limit of \( v(x,s) \) along a subsequence \( s_n \not\to \infty \); therefore, profiles are characterized as nontrivial solutions to the stationary problem,
\[
\begin{align*}
-\Delta \phi &= \lambda_m |\phi|^{m-2} \phi & \text{in } \Omega, \\
\phi &= 0 & \text{on } \partial \Omega.
\end{align*}
\] (1.8) (1.9)

On the other hand, each nontrivial solution \( \phi(x) \) of (1.8), (1.9) forms a separable solution \( U(x,t) := (1-t)^{1/(m-2)} \phi(x) \) to (1.1)–(1.3), and then, \( U(x,0) = \phi(x) \), \( t_*(\phi) = 1 \) and \( \phi(x) \) is the asymptotic profile of \( U(x,t) \). Therefore the set of all nontrivial solutions to (1.8), (1.9) coincides with the set of all asymptotic profiles for (1.1)–(1.3). From now on, we denote this set by \( S \).

This paper addresses the stability of asymptotic profiles for FDE, that is, whether or not solutions of (1.8), (1.9) emanating from a small neighborhood (in \( H^1_0(\Omega) \)) of an asymptotic profile \( \phi \in S \) also have the same profile \( \phi \). Such a notion of stability has been formulated in [1] by introducing a dynamical system generated by (1.8), (1.9) in a peculiar phase set
\[
\mathcal{X} := \{ t_*(u_0)^{-1/(m-2)} u_0; u_0 \in H^1_0(\Omega) \setminus \{ 0 \} \},
\]
which is equivalently rewritten by \( \mathcal{X} = \{ v_0 \in H^1_0(\Omega); t_*(v_0) = 1 \} \) (hence \( \mathcal{S} \subseteq \mathcal{X} \)) and homeomorphic to the unit sphere in \( H^1_0(\Omega) \) (see [1] Propositions 6 and 10). More precisely, it is defined as follows:

**Definition 1.1 (Stability and instability of asymptotic profiles [1]).** Let \( \phi \in \mathcal{S} \).

(i) \( \phi \) is said to be stable, if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that any solution \( v \) of (1.5), (1.6) satisfies
\[
\sup_{s \in [0,\infty)} \| v(\cdot,s) - \phi \|_{H^1_0(\Omega)} < \varepsilon,
\]
whenever \( v(\cdot,0) \in \mathcal{X} \) and \( \| v(\cdot,0) - \phi \|_{H^1_0(\Omega)} < \delta \).

(ii) \( \phi \) is said to be unstable, if \( \phi \) is not stable.

(iii) \( \phi \) is said to be asymptotically stable, if \( \phi \) is stable, and moreover, there exists \( \delta_0 > 0 \) such that any solution \( v \) of (1.5), (1.6) satisfies
\[
\lim_{s \not\to \infty} \| v(\cdot,s) - \phi \|_{H^1_0(\Omega)} = 0,
\]
whenever \( v(\cdot,0) \in \mathcal{X} \) and \( \| v(\cdot,0) - \phi \|_{H^1_0(\Omega)} < \delta_0 \).

In [1], some stability criteria are also established for isolated profiles (see Proposition 1.2 below). Here least energy solutions to (1.8), (1.9) mean nontrivial solutions achieving the least energy, that is, the infimum over \( \mathcal{S} \) of the energy functional \( J : H^1_0(\Omega) \to \mathbb{R} \) defined by
\[
J(w) := \frac{1}{2} \int_\Omega |\nabla w(x)|^2 \, dx - \frac{\lambda_m}{m} \int_\Omega |w(x)|^m \, dx \quad \text{for } w \in H^1_0(\Omega)
\]
(see, e.g., [24] for more details). Least energy solutions of (1.8), (1.9) turn out to be sign-definite by strong maximum principle. We also note that \( J \) is an action functional associated with (1.8), (1.9) and also a Lyapunov functional for (1.8)–(1.9).

**Proposition 1.2 (Stability criteria for isolated asymptotic profiles [1]).** The following (i) and (ii) hold true:
(i) Let $\phi$ be a least energy solution to \((1.8), (1.9)\) which is isolated (in $H^1_0(\Omega)$) from all the other least energy solutions. Then $\phi$ is stable (in the sense of Definition 1.1). In addition, if $\phi$ is isolated from all the other sign-definite solutions, $\phi$ is asymptotically stable.

(ii) Sign-changing solutions $\psi$ to \((1.8), (1.9)\) are not asymptotically stable. In addition, if $\psi$ is isolated from nontrivial solutions whose energies are lower than that of $\psi$, then $\psi$ is unstable.

In [1 §4], it is proved that $\mathcal{X}$ forms a separatrix of the dynamical system generated by \((1.5)–(1.7)\) in the whole of the energy space $H^1_0(\Omega)$ to divide its stable and unstable sets. Moreover, it is also pointed out that $\mathcal{X}$ is different from the so-called Nehari manifold $\mathcal{N} := \{w \in H^1_0(\Omega) \setminus \{0\} : \langle J'(w), w \rangle_{H^1_0(\Omega)} = 0\}$ and $\mathcal{X} \cap \mathcal{N} = \mathcal{S}$.

However, these stability criteria cannot cover all situations. For instance, in a thin annular domain case, it is known that least energy solutions form a continuum in $H^1_0(\Omega)$ due to the symmetry breaking of least energy solutions (see Coffman [18] and also [31, 15]) and the invariance of the equation to rotations. So one cannot apply Proposition 1.2 to determine the stability of such non-isolated least energy solutions to \((1.8), (1.9)\) in the sense of Definition 1.1 (cf. see [3]). On the other hand, obviously, they are never asymptotically stable.

The main purpose of the present paper is to prove the stability of all (possibly non-isolated) asymptotic profiles of least energy. A main difficulty apparently stems from the lack of solitary symmetry breaking of least energy solutions (see Coffman [18] and also [31, 15]) and the invariance of the equation to rotations. So one cannot apply Proposition 1.2 to determine the stability of the equation to rotations. Hence one can assume the positivity of the functional $E$., e.g., [37, 29, 26, 23, 22, 25, 27, 16, 17, 24]: there exist constants $\theta$, $\omega$, $\delta > 0$ such that\[|E(v) - E(\psi)|^{1-\theta} \leq \omega \|E'(v)\|_{X^*}^\theta,\]for all $v \in X$ satisfying $\|v - \psi\|_X < \delta$ (cf. there are several variants with different choices of norms). Here the constants $\theta$, $\omega$, $\delta$ may depend on the choice of each critical point $\psi$ of the functional $E$. To prove the convergence of a
flow of a dissipative dynamical system along with $E(\cdot)$ as a Lyapunov energy to a prescribed limit $\phi$, one assigns $\phi$ to the critical point $\psi$ of the Lojasiewicz-Simon inequality, and then investigates the behavior of the flow for sufficiently large time. By contrast, to discuss the (Lyapunov) stability of a stationary point $\phi$ of the system, the limit of each flow (emanating from a neighborhood of $\phi$) is not prescribed. Here we focus on the behavior of the flow near the initial time by assigning the target of stability analysis (i.e., $\phi$) to the critical point $\psi$ of the Lojasiewicz-Simon inequality.

However, another difficulty then arises from the frame of stability analysis. More precisely, in Definition 1.1, the notions of stability are formulated in the energy space $H^1_0(\Omega)$, whose elements may not be uniformly bounded in $\Omega$. On the other hand, due to the nonlinearity of FDE (see, e.g., Lemma 3.3, uniform estimates for solutions of (1.3)–(1.7) will be required to investigate the stability by using the Lojasiewicz-Simon inequality, which is also established in [22] for uniformly bounded functions in a small neighborhood of each solution $\phi$ of (1.8), (1.9) with non-integer power $m > 1$. Therefore we need to compensate the gap between the frame of stability analysis and the validity of the argument based on the Lojasiewicz-Simon inequality. To this end, we shall develop a uniform extinction estimate for (possibly sign-changing) solutions of FDE by utilizing some results of [20] and [21].

Moreover, the Lojasiewicz-Simon inequality will be also applied to prove the instability of asymptotic profiles for FDE. Let us consider the annular domain

$$A_N(a, b) := \{ x \in \mathbb{R}^N : a < |x| < b \}$$

with $0 < a < b < \infty$. As mentioned above, the positive radial asymptotic profile for FDE does not take the least energy, provided that the thickness $(b-a)/a$ of the annulus is sufficiently thin; thereby it is beyond the scope of Proposition 1.2. One may expect that the positive radial profile is unstable (i.e., not stable) in the sense of Definition 1.1. This conjecture was proved only for the two dimensional case, $N = 2$, without providing any quantitative information of the thickness of the annulus in [2], where the restriction on the space dimension $N$ and the lack of quantitative information of the thickness arise from some technical difficulty of spectral analysis of the corresponding linearized operator. The general $N$-dimensional case has been left as an open question (cf. it was proved for general $N$ in [2] that the positive radial profile is not asymptotically stable). In this paper, we shall also prove the instability of the positive radial profile for general spacial dimension $N$ and give an upper bound of the thickness of the annulus by applying the Lojasiewicz-Simon inequality.

**Theorem 1.4** (Instability of positive radial asymptotic profiles in thin annuli). Let $\Omega = A_N(a, b)$ and assume

$$\left( \frac{b}{a} \right)^{(N-3)+} \left( \frac{b-a}{\pi a} \right)^2 < \frac{m-2}{N-1}. \quad (1.10)$$

Then the positive radial solution $\phi$ of (1.8), (1.9) is unstable in the sense of Definition 1.1.

This paper consists of five sections: In Section 2 we prepare several lemmas to be used in a proof of Theorem 1.3. Section 3 is devoted to proving Theorem 1.3. More precisely, we shall prove the stability for all local minimizers of $J$ over $\mathcal{X}$ (see 3.1 below for definition). Since every asymptotic profile of least energy is a (global) minimizer of $J$ over $\mathcal{X}$, Theorem 1.3 will be also obtained as a special case. In Section 4 we discuss a couple of properties of local minimizers of $J$ over $\mathcal{X}$. In particular, we investigate the relation of (local) minimizers of $J$ over $\mathcal{X}$ and those over the so-called Nehari manifold $\mathcal{N}$, which has been vigorously studied in variational analysis of nonlinear elliptic equations. The final section is concerned with the instability of positive radial asymptotic profiles in thin annular domains.

**Notation.** Let $u = u(x,t) : \Omega \times [0, \infty) \to \mathbb{R}$ be a function with space and time variables. Throughout the paper, for each $t \geq 0$ fixed, we simply denote by $u(t)$ the function $u(\cdot,t) : \Omega \to \mathbb{R}$ with only the space variable. We denote by $H^{-1}(\Omega)$ the dual space of $H^1_0(\Omega)$. Moreover,
\[ B_{H^1_0(\Omega)}(\phi; r) \] denotes the open ball in \( H^1_0(\Omega) \) with radius \( r > 0 \) centered at \( \phi \), i.e.,
\[
B_{H^1_0(\Omega)}(\phi; r) := \left\{ \phi \in H^1_0(\Omega) : \|\phi\|_{H^1_0(\Omega)} < r \right\} \quad \text{for } r > 0.
\]
Furthermore, \( C_m \) and \( R(\cdot) \) stand for the best possible constant of the Sobolev-Poincaré inequality below and the corresponding Rayleigh quotient, respectively (see, e.g., [14], [39]). Hence the extinction time (see Appendix for more details).

2. Preliminaries and Lemmas

In this section, we collect preliminary facts and several lemmas.

2.1. Preliminaries. Let us start with recalling the definition of solutions.

Definition 2.1. A function \( u : \Omega \times (0, \infty) \to \mathbb{R} \) is said to be a solution of \([1.1]-[1.3]\), if the following conditions hold true:

- \( u \in L^{\infty}(0, T; H^1_0(\Omega)) \) and \( |u|^{m-2}u \in W^{1, \infty}(0, T; H^{-1}(\Omega)) \) for any \( T > 0 \).
- It holds that
  \[
  \langle \partial_t (|u|^{m-2}u) (t), \phi \rangle_{H^1_0} + \int_{\Omega} \nabla u(x, t) \cdot \nabla \phi(x) \, dx = 0
  \]
  for a.e. \( t \in (0, \infty) \) and \( \phi \in H^1_0(\Omega) \),
  where \( \langle \cdot, \cdot \rangle_{H^1_0} \) denotes the duality pairing between \( H^1_0(\Omega) \) and its dual space \( H^{-1}(\Omega) \).
- \( u(\cdot, 0) = u_0 \) a.e. in \( \Omega \).

Solutions of \([1.5]-[1.7]\) are also defined in an analogous manner. The well-posedness of \([1.1]-[1.3]\) in the sense of Definition 2.1 is well known (see, e.g., [14], [29]). Hence the extinction time \( t_\ast = t_\ast(u_0) \) is uniquely determined for each initial data \( u_0 \). Moreover, one can also ensure that

\[
u \in C([0, T]; L^m(\Omega)) \cap C_w([0, T]; H^1_0(\Omega)) \cap C_+([0, T]; H^1_0(\Omega)) \quad \text{for any } T > 0,
\]
(2.1)

\[
\partial_s (|u|^{m-2}u) \in C_+([0, T]; H^{-1}(\Omega)) \quad \text{for any } T > 0
\]
(2.2)

(see Appendix for more details).

Equation \([1.2]\) can be formulated as a generalized gradient flow in \( H^{-1}(\Omega) \) of the form,

\[
\partial_s (|u|^{m-2}v) (s) = -J'(v(s)) \quad \text{in } H^{-1}(\Omega), \quad s > 0,
\]
(2.3)

where \( J' \) stands for the Fréchet derivative of the energy functional \( J : H^1_0(\Omega) \to \mathbb{R} \). Therefore the following energy inequalities hold true:

\[
\frac{1}{m'} \frac{d}{ds} \|v(s)\|_{L^m(\Omega)}^m + \|v(s)\|_{H^1_0(\Omega)}^2 = \lambda_m \|v(s)\|_{L^m(\Omega)}^m;
\]
(2.4)

\[
\mu_m \left| \partial_s \left( |u|^{(m-2)/2} v \right)(s) \right|_{L^2(\Omega)}^2 + \frac{d}{ds} J(v(s)) \leq 0 \quad \text{for a.e. } s > 0,
\]
(2.5)

where \( m' \) is the Hölder conjugate of \( m \), i.e., \( m' := m/(m - 1) \) and \( \mu_m := 4/(mm') > 0 \) (see, e.g., [4] for the precise derivation of these energy inequalities). In particular, \( s \mapsto J(v(s)) \) is non-increasing.

Define a Rayleigh quotient by
\[
R(w) := \frac{\|w\|_{H^1_0(\Omega)}}{\|w\|_{L^m(\Omega)}} \quad \text{for } w \in H^1_0(\Omega) \setminus \{0\},
\]
associated with the Sobolev-Poincaré inequality
\[
\|w\|_{L^m(\Omega)} \leq C_m \|w\|_{H^1_0(\Omega)} \quad \text{for } w \in H^1_0(\Omega),
\]
(2.6)
provided that $m \in [1, 2^+]$, with the best possible constant $C_m$ which is the supremum of $R(w)^{-1}$ over $w \in H^1_0(\Omega) \setminus \{0\}$. Then the function $t \mapsto R(u(t))$ is non-increasing, and hence, so is the function $s \mapsto R(v(s))$ (see, e.g., [20, 30, 35, 41]).

Finally, we list up properties of the phase set $X$ obtained in [1] for later use.

**Proposition 2.2 (Properties of phase sets, cf. [1])**. The phase set $X$ satisfies the following properties:

(i) If $v_0 \in X$, then $v(s)$ lies on $X$ for any $s \geq 0$.
(ii) If $v_0 \in X$, then there exist $\psi \in S$ and a sequence $s_n \to \infty$ such that $v(s_n) \to \psi$ strongly in $H^1_0(\Omega)$.
(iii) The set $S$ is included in $X$.
(iv) The infimum of $J$ over $X$ coincides with the least energy, i.e., the infimum of $J$ over $S$. Moreover, if $w \in X$ achieves the infimum, then $w$ is a least energy solution of (1.8).
(v) For any $w \in X$, it holds true that $t_* (w) = 1$.
(vi) The set $X$ is sequentially closed in the weak topology of $H^1_0(\Omega)$.

Proofs of (i)–(vi) can be found in [1, Propositions 5–8 and 10].

2.2. **Lemmas**. In this subsection, we shall develop several lemmas for later use. The following lemma provides a uniform estimate for (possibly sign-changing) solutions of the rescaled problem (1.3)–(1.7). To prove this, we shall employ some results of DiBenedetto and Kwong [21] and DiBenedetto, Kwong and Vespri [20].

**Lemma 2.3 (Uniform estimate for rescaled solutions)**. Assume (1.3). Then there exists a constant $C > 0$ depending only on $N, m$ such that for every $s_0 \in (0, \log 2)$ and $v_0 \in X$, the unique solution $v = v(x,s)$ of (1.3)–(1.7) with the initial data $v_0$ satisfies

$$\|v(s)\|_{L^\infty(\Omega)} \leq C (e^{s_0} - 1)^{-\frac{N}{2m}} R(v_0)^{-\frac{4}{m-2}} \quad \text{for all } s \geq s_0$$

with $\kappa := 2N -Nm + 2m > 0$ (by (1.4)).

**Proof**. Let $u$ be a solution of (1.1)–(1.3) with an initial data $u_0 \in H^1_0(\Omega)$. Fix $T > 0$, $R > 0$ and let $\Omega_* \subset \mathbb{R}^N$ be a smooth bounded domain such that

$$\Omega \subset B_R \subset B_{4R} \subset \Omega_*,$$

where $B_r := \{x \in \mathbb{R}^N : |x| < r\}$ for $r > 0$. Moreover, set a nonnegative function $\overline{u}_0 \in H^1_0(\Omega_*)$ by

$$\overline{u}_0(x) = \begin{cases} |u_0(x)| & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\overline{u}$ be the unique weak solution for (1.1)–(1.3) with $\Omega_*$, $\partial \Omega_*$ and $u_0$ replaced by $\Omega_*$, $\partial \Omega_*$ and $\overline{u}_0$, respectively. Then by the positivity result $\overline{u} > 0$ in $\Omega_* \times (0, T)$ due to [20], one particularly observes that

$$\overline{u} > 0 \quad \text{on } \partial \Omega \times (0, T).$$

Hence by comparison principle,

$$u \leq \overline{u} \quad \text{in } \Omega \times (0, T). \quad (2.7)$$

A local $L^\infty$-estimate for nonnegative solutions to FDE (see Theorem 3.1 of [21] with some change of notation, e.g., $u(x,t)$ and $m$ of [21] correspond to $u(x,t)^{m-1}$ and $1/(m-1)$, respectively, of our notation) yields that

$$\sup_{x \in B_R} \overline{u}(x,t) \leq \gamma t^{\frac{N}{m-2}} \sup_{0 < \tau < t} \left( \int_{B_{2R}} \overline{u}(x,\tau)^m \, dx \right)^{\frac{2}{m}} + \gamma \left( \frac{t}{R^2} \right)^{\frac{4}{m-2}} \quad \text{for } t \in (0, T).$$
with some constant $\gamma = \gamma(N, m) > 0$ and $\kappa := 2N - Nm + 2m > 0$ (by (1.4)). Here, by using a standard energy estimate for FDE, one can derive
\[
\sup_{t \geq 0} \| \mathbf{u}(t) \|_{L^m(\Omega)}^m \leq \| \mathbf{u}_0 \|_{L^m(\Omega)}^m = \| u_0 \|_{L^m(\Omega)}^m,
\]
which along with (2.7) and the relation $\Omega \subset B_R$ implies
\[
\sup_{x \in \Omega} u(x, t) \leq \sup_{x \in B_R} \mathbf{u}(x, t) \leq \gamma t^{-\frac{N}{2}} \| u_0 \|_{L^m(\Omega)}^m + (\frac{t}{R^2})^{\frac{m-2}{2}} \quad \text{for } t \in (0, T).
\]
Since $\gamma$ and $\kappa$ are independent of $R$ and $T$, by letting $R, T \to \infty$, we conclude that
\[
\sup_{x \in \Omega} u(x, t) \leq \gamma t^{-\frac{N}{2}} \| u_0 \|_{L^m(\Omega)}^m \quad \text{for all } t > 0.
\]
Repeating the preceding argument with $u$ and $u_0$ replaced by $-u$ and $-u_0$, respectively, we deduce that
\[
\sup_{x \in \Omega} |u(x, t)| \leq \gamma t^{-\frac{N}{2}} \| u_0 \|_{L^m(\Omega)}^m \quad \text{for all } t > 0.
\]
Furthermore, replace $u_0$ by $u(s)$ for $0 < s < t$ to get
\[
\sup_{x \in \Omega} |u(x, t)| \leq \gamma (t-s)^{-\frac{N}{2}} \| u(s) \|_{L^m(\Omega)}^m \quad \text{for all } 0 < s < t < \infty. \tag{2.8}
\]
In particular, let us set $u_0 = v_0 \in \mathcal{X}$. Then $u$ vanishes at $t_*(v_0) = 1$. Moreover, let $t_0 \in (0, 1/2)$ be given by
\[
s_0 = \log \left( \frac{1}{1 - t_0} \right) > 0.
\]
As in [20] Lemma 6.1 (see also [36]), substituting
\[
s = t - \frac{t_0}{1 - t_0} (1 - t) = \frac{t - t_0}{1 - t_0} \in (0, t) \quad \text{for } t \in (t_0, 1)
\]
to (2.8) and employing [1] Proposition 2], one can derive that
\[
\| u(t) \|_{L^\infty(\Omega)} \leq C_0 (1 - t)^{-\frac{N}{2}} \quad \text{for all } t \geq t_0
\]
with a constant $C_0$ given by
\[
C_0 = \tilde{\gamma} \left( \frac{t_0}{1 - t_0} \right)^{-\frac{N}{2}} R(u_0)^{\frac{4m}{4(m-2)}} = \tilde{\gamma} (e^{s_0} - 1)^{-\frac{N}{2}} R(v_0)^{\frac{4m}{4(m-2)}},
\]
where $\tilde{\gamma}$ is a constant depending only on $m, N$. By change of variables, $v(x, s) = (1-t)^{-1/(m-2)} u(x, t)$ and $s = \log(1/(1 - t))$, we find that
\[
\| v(s) \|_{L^\infty(\Omega)} \leq C_0 \quad \text{for all } s \geq s_0.
\]
The proof is completed. \hfill $\Box$

We next exhibit a couple of variational properties of the Rayleigh quotient on the set $\mathcal{X}$.

**Lemma 2.4 (Rayleigh quotient on $\mathcal{X}$).** Assume (1.4). It follows that
\[
\inf_{w \in \mathcal{X}} R(w) = C_m^{-1} > 0, \quad \inf_{w \in \mathcal{X}} \| u \|_{L^m(\Omega)} \geq \left( \lambda_m C_m^2 \right)^{-1/(m-2)} \| w \|_{H_0^m(\Omega)}
\]
for all $w \in \mathcal{X}$. In particular, $R(w) < \infty$ for all $w \in \mathcal{X}$. Moreover, $R(\cdot)$ is continuous on $\mathcal{X}$ in the strong topology of $H_0^m(\Omega)$. 

Proof. Let \( w \in \mathcal{X} \) and recall that \( t_*(w) = 1 \) (see Proposition 2.2). From the estimates from below and above for the extinction time \( t_*(\cdot) \) (see [1] Corollary 1), it follows that
\[
\lambda_m \|w\|^{m-2}_{L^m(\Omega)} R(w)^{-2} \leq 1 \leq \lambda_m C_m^2 \|w\|^{m-2}_{L^m(\Omega)},
\]
which yields that
\[
\frac{1}{\lambda_m C_m^2} \leq \|w\|^{m-2}_{L^m(\Omega)} \quad \text{and} \quad \lambda_m \|w\|^{m-2}_{L^m(\Omega)} \leq R(w)^2.
\]
Hence we observe that \( R(w) \geq C_m^{-1} \) and \( \|w\|_{L^m(\Omega)} \geq (\lambda_m C_m^2)^{-1/(m-2)} > 0 \) for all \( w \in \mathcal{X} \). Moreover, it is known that \( R(\psi) = C_m^{-1} \) for least energy solutions \( \psi \) of \((1.8)\), \((1.9)\) under \( m < 2^* \). It follows that
\[
R(w) = \frac{\|w\|_{H^1_0(\Omega)}}{\|w\|_{L^m(\Omega)}} \leq (\lambda_m C_m^2)^{1/(m-2)} \|w\|_{H^1_0(\Omega)} \quad \text{for all} \ w \in \mathcal{X}.
\]
Moreover, if \( w_n \in \mathcal{X} \) and \( w_n \to w \) strongly in \( H^1_0(\Omega) \) (hence, \( w \in \mathcal{X} \) by Proposition 2.2), then one can derive that \( R(w_n) \to R(w) \), since \( \|w_n\|_{L^m(\Omega)} \) and \( \|w\|_{L^m(\Omega)} \) are not less than \( (\lambda_m C_m^2)^{-1/(m-2)} > 0 \).

Moreover we have:

**Lemma 2.5 (Estimate for solutions on \( \mathcal{X} \)).** Let \( v_0 \in \mathcal{X} \) and let \( v \) be the solution of \((1.5),(1.6)\) for the initial data \( v_0 \). Then it holds that
\[
\sup_{s \geq 0} \|v(s)\|_{H^1_0(\Omega)}^2 \leq 2J(v_0) + \frac{2R(v_0)^{2m/(m-2)}}{m \lambda_m^{2/(m-2)}}.
\]

**Proof.** Note that \( v(s) \) belongs to \( \mathcal{X} \) for all \( s \geq 0 \). Hence, by the proof of Lemma 2.4,
\[
\|v(s)\|_{L^m(\Omega)}^{m-2} \leq \frac{R(v(s))^2}{\lambda_m} \quad \text{for all} \ s \geq 0.
\]
Since \( J(v(\cdot)) \) and \( R(v(\cdot)) \) are nonincreasing, it follows that
\[
\frac{1}{2} \|\nabla v(s)\|_{L^2(\Omega)}^2 = J(v(s)) + \frac{\lambda_m}{m} \|v(s)\|_{L^m(\Omega)}^m \leq J(v_0) + \frac{R(v_0)^{2m/(m-2)}}{m \lambda_m^{2/(m-2)}},
\]
which completes the proof. \( \square \)

We close this section with the continuous dependence of solutions to \((1.5)-(1.7)\) on data.

**Lemma 2.6 (Continuous dependence of solutions on data).** For \( i = 1, 2 \), let \( v_i \) be solutions to \((1.5)-(1.7)\) with initial data \( v_{i0} \in H^1_0(\Omega) \). It then holds true that
\[
\|v_1|^{m-2}v_1(s) - |v_2|^{m-2}v_2(s)\|_{H^{-1}(\Omega)}^2 \leq \|v_{10}|^{m-2}v_{10} - |v_{20}|^{m-2}v_{20}\|_{H^{-1}(\Omega)}^2 e^{2\lambda_m s}
\]
for all \( s \geq 0 \).

This lemma can be proved in a standard way; however, we give a proof for the convenience of the reader.

**Proof.** Subtract equations and test it by \((-\Delta)^{-1}(|v_1|^{m-2}v_1(s) - |v_2|^{m-2}v_2(s))\) to see that
\[
\frac{1}{2} \frac{d}{ds} \|v_1|^{m-2}v_1(s) - |v_2|^{m-2}v_2(s)\|_{H^{-1}(\Omega)}^2 + \int_{\Omega} (v_1(s) - v_2(s)) \left( |v_1|^{m-2}v_1(s) - |v_2|^{m-2}v_2(s) \right) dx
\]
\[
= \lambda_m \|v_1|^{m-2}v_1(s) - |v_2|^{m-2}v_2(s)\|_{H^{-1}(\Omega)}^2.
\]
By using the monotonicity of \( w \mapsto |w|^{m-2}w \) and by applying Gronwall’s inequality, we obtain the desired conclusion. \( \square \)
3. Proof of Theorem 1.3

This section is devoted to a proof of Theorem 1.3. We shall prove the stability for all local minimizers $\phi$ of $J$ over $\mathcal{X}$, i.e., $\phi$ satisfies

$$J(\phi) = \inf \{ J(w) : w \in \mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; r_0) \} \tag{3.1}$$

for some $r_0 > 0$. Obviously, every least energy solution of (1.8), (1.9) is a global minimizer of $J$ over $\mathcal{X}$, since the least energy is the minimum of $J$ over $\mathcal{X}$ and $S \subset \mathcal{X}$ (see Proposition 2.2); hence it always satisfies 3.1 (with $r_0 = \infty$). Moreover, we stress again that $\phi$ is not supposed to be isolated even in the neighborhood $\mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; r_0)$. Hence there might be a sequence of (local) minimizers $w_n \in \mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; r_0) \setminus \{ \phi \}$ converging to $\phi$ in $H^1_0(\Omega)$.

Our result reads,

THEOREM 3.1 (Stability of local minimizers of $J$ over $\mathcal{X}$). Let $\phi > 0$ be a local minimizer of $J$ over $\mathcal{X}$. Then $\phi$ is stable under the flow on $\mathcal{X}$ generated by solutions for (1.5)–(1.7). Hence, in particular, Theorem 1.3 holds true.

One of most crucial points of a proof for Theorem 3.1 is how to control the distance between $\phi$ and the solution $v(s)$ of (1.5)–(1.7) before proceeding to a proof. Let $\phi$ be a local minimizer of $J$ over $\mathcal{X}$ and let $r_0 > 0$ be such that 3.1 is satisfied. Since every local minimizer of $J$ over $\mathcal{X}$ is a sign-definite (nontrivial) solution of (1.8), (1.9) (see Proposition 2.2), we can assume $\phi \geq 0$ without any loss of generality. Moreover, by strong maximum principle and elliptic regularity, one can assure that

$$0 < \phi(x) < L_\phi := \| \phi \|_{L^\infty(\Omega)} + 1 \quad \text{for all } x \in \Omega \quad \text{and} \quad \partial_\nu \phi < 0 \quad \text{on } \partial \Omega. \tag{3.2}$$

Then the following Feireisl-Simondon version (see [22]) of the Lojasiewicz-Simon inequality holds true:

LEMMA 3.2 (Lojasiewicz-Simon inequality [22]). For any $L > L_\phi$, there exist constants $\theta \in (0, 1/2]$, $\omega, \delta_0 > 0$ such that

$$|J(w) - J(\phi)|^{1-\theta} \leq \omega \| J'(w) \|_{H^{-1}(\Omega)}, \tag{3.3}$$

whenever $w \in H^1_0(\Omega)$ satisfies $|w(x)| \leq L$ for a.e. $x \in \Omega$ and $\| w - \phi \|_{H^1_0(\Omega)} < \delta_0$.

This lemma follows from Proposition 6.1 of [22], where the Lojasiewicz-Simon inequality is established for some functional associated with the operator $v \mapsto -\Delta v + F(v)$, by introducing a function $F \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ satisfying $F(s) = -\lambda_m s^{m-2} s$ for all $s \in [-M, M]$ with $M := L + 1$, $F \in C^\infty(0, M)$ and

$$|F^{(n)}(s)| \leq \frac{s^r}{s^n} \quad \text{for all } s \in (0, M), \ n \in \mathbb{N}$$

for some $r > 0$ (cf. see also §5 of [22]). Furthermore, we remark that the positivity (or negativity) of $\phi$ is essentially required; however, the sign of $w$ is not specified in the proof of Proposition 6.1 of [22].

Throughout the rest of this section, let $s_0 \in (0, \log 2)$ be fixed. By Lemma 2.3 one can take

$$C_1 > 0 \quad \text{such that} \quad R(v_0) \leq (\lambda_m C_m^2)^{1/(m-2)} \| \nabla v_0 \|_{L^\infty(\Omega)} \leq C_1 \quad \text{for all } v_0 \in B_{H^1_0(\Omega)}(\phi; r_0) \cap \mathcal{X}. \tag{3.4}$$

By Lemma 2.3 one can take a constant $L = L(s_0, C_1, N, m) > 0$ such that for any $v_0 \in B_{H^1_0(\Omega)}(\phi; r_0) \cap \mathcal{X}$, the unique solution $v = v(x, s)$ of (1.5)–(1.7) satisfies

$$\| v(s) \|_{L^\infty(\Omega)} \leq L \quad \text{for all } s \geq s_0. \tag{3.4}$$
Then we see that and suppose that Then (3.5) holds true for all $s$.
Furthermore, let us recall that $\kappa$ with $C$ to $\gamma$ and denote by Proposition 2.2). Here, we particularly took $L$ larger than $L_\varphi$. Then thanks to the Lojasiewicz-Simon inequality (see Lemma 3.2), there exist constants $\theta \in (0, 1/2)$, $\omega > 0$ such that for any $v_0 \in B_{H^1(\Omega)}(\phi; \delta_0 \wedge r_0) \cap \mathcal{X}$, the solution $v = v(x, s)$ of \textcolor{red}{(1.5) - (1.7)} with the initial data $v_0$ satisfies
\[
(J(v(s)) - J(\phi))^{1-\theta} \leq \omega \|J'(v(s))\|_{H^{-1}(\Omega)}, \tag{3.5}
\]
whenever $\|v(s) - \phi\|_{H^1(\Omega)} \leq \delta_0 \wedge r_0$ and $s \geq s_0$ (hence, (3.4) is satisfied). Here we used the fact by (3.3) that $J(v(s)) - J(\phi) \geq 0$ whenever $v(s) \in B_{H^1(\Omega)}(\phi; r_0)$, since $v(s) \in \mathcal{X}$ for all $s \geq 0$ (see Proposition 2.2).

Let $\delta, \delta'$ be real numbers such that
\[
0 < \delta' < \delta < \delta_0 \wedge r_0.
\]
Take any $v_0 \in \mathcal{X} \cap B_{H^1(\Omega)}(\phi; \delta')$ and denote by $v = v(x, s)$ the solution of \textcolor{red}{(1.5) - (1.7)} with the initial data $v_0$. Then since $v$ belongs to $C_+([0, \infty); H^1(\Omega))$, one can take $s_\delta > 0$ such that $v(s) \in B_{H^1(\Omega)}(\phi; \delta)$ for all $s \in [0, s_\delta)$.

Furthermore, let us recall that $v(s) \in \mathcal{X}$ for any $s \geq 0$ and (3.3) is satisfied. Moreover, suppose that
\[
s_0 < s_\delta. \tag{A1}
\]
Then (3.5) holds true for all $s \in [s_0, s_\delta)$.
Define
\[
H(s) := (J(v(s)) - J(\phi))^{\theta} \geq 0 \quad \text{for} \quad s \in [0, s_\delta)
\]
and suppose that
\[
J(v(s)) - J(\phi) > 0 \quad \text{for all} \quad s \in [0, s_\delta). \tag{A2}
\]
Then we see that
\[
-\frac{d}{ds}H(s) = -\theta (J(v(s)) - J(\phi))^{\theta-1} \frac{d}{ds}J(v(s)) \\
\geq \mu_m \theta (J(v(s)) - J(\phi))^{\theta-1} \left\| \partial_s \left( |v|^{(m-2)/2} v \right)(s) \right\|_{L^2(\Omega)}^2 \quad \text{for a.e.} \quad s \in (0, s_\delta).
\]
Here the last inequality follows from the energy inequality (2.5).

Now, we claim that

**Lemma 3.3.** It holds that
\[
\left\| \partial_s \left( |v|^{m-2} v \right)(s) \right\|_{L^2(\Omega)}^2 \leq \kappa_m \|v(s)\|_{L^\infty(\Omega)}^{m-2} \left\| \partial_s \left( |v|^{(m-2)/2} v \right)(s) \right\|_{L^2(\Omega)}^2 \quad \text{for a.e.} \quad s > s_0
\]
with $\kappa_m := 4(m - 1)^2/m^2 > 0$.

**Proof.** Set $\gamma(\sigma) := |\sigma|^{r-2} \sigma$ for $\sigma \in \mathbb{R}$ and determine $r > 1$ such that
\[
\gamma \left( |\sigma|^{(m-2)/2} \sigma \right) = |\sigma|^{\frac{m-2}{2} - 1} - |\sigma|^{m-2} \sigma \quad \text{for} \quad \sigma \in \mathbb{R}.
\]
Then $r = (3m - 2)/m > 1$. Hence
\[
\int_\Omega \left\| \partial_s \left( |v|^{m-2} v \right) \right\|^2 \, dx = \int_\Omega \left\| \partial_s \gamma \left( |v|^{(m-2)/2} v \right) \right\|^2 \, dx \\
= (r - 1)^2 \int_\Omega \left\| v \right\|_{\infty}^{(m-2)/2} v \right\|^{r-2} \left\| \partial_s \left( |v|^{(m-2)/2} v \right) \right\|^2 \, dx \\
\leq (r - 1)^2 \|v(s)\|_{L^\infty(\Omega)}^{m-2} \left\| \partial_s \left( |v|^{(m-2)/2} v \right)(s) \right\|_{L^2(\Omega)}^2,
\]
which completes the proof. \qed
Recalling that \( v(s) \neq 0 \) by \( 0 \not\in \mathcal{X} \), we find that
\[
- \frac{d}{ds}H(s) \geq \frac{\mu_n \theta}{\kappa_m} \|v(s)\|^{-(m-2)}_{L^\infty(\Omega)} (J(v(s)) - J(\phi))^{\theta-1} \|\partial_s (|v|^{m-2}v)(s)\|_{L^2(\Omega)}^2
\]
for a.e. \( s \in (s_0, s_\delta) \). Since (3.5) holds for all \( s \in [s_0, s_\delta) \), it follows that
\[
- \frac{d}{ds}H(s) \geq \frac{\mu_n \theta}{\kappa_m \omega} \|v(s)\|^{-(m-2)}_{L^\infty(\Omega)} \|J(v(s))\|^{-1}_{H^{-1}(\Omega)} \|\partial_s (|v|^{m-2}v)(s)\|_{L^2(\Omega)}^2
\geq \frac{\mu_n \theta}{\kappa_m \omega C_s^2} \left( \sup_{s \geq s_0} \|v(s)\|_{L^\infty(\Omega)} \right)^{-(m-2)} \|\partial_s (|v|^{m-2}v)(s)\|_{H^{-1}(\Omega)}
\]
for a.e. \( s \in (s_0, s_\delta) \), by noting that
\[
\|\partial_s (|v|^{m-2}v)(s)\|_{L^2(\Omega)} \geq C_{s_0}^{-1} \|\partial_s (|v|^{m-2}v)(s)\|_{H^{-1}(\Omega)} \geq C_{s_0}^{-1} \|J'(v(s))\|_{H^{-1}(\Omega)}
\]
with the best possible constant \( C_2 > 0 \) of (2.6) with \( m = 2 \). Thus we obtain
\[
\left\| \partial_s (|v|^{m-2}v)(s) \right\|_{H^{-1}(\Omega)} \geq \left( \sup_{s \geq s_0} \|v(s)\|_{L^\infty(\Omega)} \right) \left( H(s) - H(s_0) \right) + \|v|^{m-2}v(s_0) - \phi|^{m-1}\|_{H^{-1}(\Omega)}
\leq \frac{\kappa_m \omega C_s^2}{\mu_n \theta} \left( \sup_{s \geq s_0} \|v(s)\|^{m-2}_{L^\infty(\Omega)} \right) H(s_0) + \|v|^{m-2}v(s_0) - \phi|^{m-1}\|_{H^{-1}(\Omega)}
\]
for all \( s \in [s_0, s_\delta) \).

Now, we are ready to prove the stability of \( \phi \).

**Proof of Theorem 3.7** Suppose on the contrary that there exists \( \varepsilon_0 > 0 \) such that for all \( n \in \mathbb{N} \), there exist solutions \( v_n = v_n(x, t) \) of (1.5)–(1.7) satisfying
\[
v_n(0) \in \mathcal{X}, \quad \|v_n(0) - \phi\|_{H^1_0(\Omega)} < \frac{1}{n}, \quad \sup_{s \geq 0} \|v(s) - \phi\|_{H^1_0(\Omega)} \geq \varepsilon_0
\]
(see Definition 1.1). Here we note that
\[
\|v_n(s)|^{m-2}v_n(0) \rightarrow \phi|^{m-1} \quad \text{strongly in } H^{-1}(\Omega),
\]
(3.8)
since the operator \( w \mapsto \|w|^{m-2}w \) is continuous from \( L^m(\Omega) \) to \( L^m(\Omega) \) and \( H^1_0(\Omega) \) (resp., \( L^m(\Omega) \)) is continuously embedded in \( L^m(\Omega) \) (resp., \( H^{-1}(\Omega) \)). Set \( \varepsilon_1 := (\varepsilon_0 \wedge 0 \wedge r_0)/2 > 0 \). Then from the right-continuity of \( s \mapsto v_n(0) \) in the strong topology of \( H^1_0(\Omega) \) on \( [0, \infty) \), for each \( n > \varepsilon_1 \), one can take \( s_0 > 0 \) such that
\[
\|v_n(s_n) - \phi\|_{H^1_0(\Omega)} = \varepsilon_1 \quad \text{and} \quad \|v_n(s) - \phi\|_{H^1_0(\Omega)} < \varepsilon_1 \quad \text{for all } s \in [0, s_n).
\]
Indeed, by the right-continuity of \( s \mapsto v_n(s) \) in the strong topology of \( H^1_0(\Omega) \), we infer that
\[
s_n := \inf \{ s > 0 : \|v_n(s) - \phi\|_{H^1_0(\Omega)} \geq \varepsilon_1 \} \in (0, \infty)
\]
(then there exists a sequence \( \sigma_k \downarrow s_n \) such that \( \|v_n(\sigma_k) - \phi\|_{H^1_0(\Omega)} \geq \varepsilon_1 \) for all \( k \)) and
\[
\|v_n(s_n) - \phi\|_{H^1_0(\Omega)} = \lim_{\sigma_k \searrow s_n} \|v_n(\sigma_k) - \phi\|_{H^1_0(\Omega)} \geq \varepsilon_1
\]
Moreover, \( \|v_n(s_n) - \phi\|_{H^1_0(\Omega)} < \varepsilon_1 \) for all \( s \in [0, s_n) \). Since \( s \mapsto v_n(s) \) is continuous in the weak topology of \( H^1_0(\Omega) \), it holds that
\[
\varepsilon_1 \geq \limsup_{s \searrow s_n} \|v_n(s) - \phi\|_{H^1_0(\Omega)} \geq \liminf_{s \searrow s_n} \|v_n(s) - \phi\|_{H^1_0(\Omega)} \geq \|v_n(s_n) - \phi\|_{H^1_0(\Omega)}.
\]
Thus we obtain \( \|v_n(s_n) - \phi\|_{H^1_0(\Omega)} = \varepsilon_1 \). In particular, \( \{v_n(s_n)\} \) is bounded in \( H^1_0(\Omega) \).
In order to apply \((3.7)\), we shall check the assumptions \((A1)\) and \((A2)\). We first claim that

**Lemma 3.4 (Check of \((A1)\)).** It holds that \(s_n \to \infty\) as \(n \to \infty\). In particular, \(s_n > s_0 \in (0, \log 2)\) for \(n \in \mathbb{N}\) large enough. Moreover, \(v_n(s) \to \phi\) strongly in \(H^1_0(\Omega)\) at each \(s \geq 0\).

*Proof.* Indeed, suppose on the contrary that a subsequence \((s_{n'}\) of \((s_n)\) is bounded, i.e., \(S := \sup \{s_{n'} : n' \in \mathbb{N}\} < \infty\). From now on, we simply write \(n\) instead of \(n'\). Since \(v_n(0) \to \phi\) strongly in \(H^1_0(\Omega)\) and \(\phi\) is a stationary solution, by Lemma 2.4 one can prove that \(|v_n|^{m-2}v_n \to \phi^{m-1}\) strongly in \(C([0, S]; H^{-1}(\Omega))\). Moreover, by Tartar’s inequality,

\[
\omega_m |a - b|^m \leq \left(|a|^{m-2}a - |b|^{m-2}b\right) (a - b)
\]

for some constant \(\omega_m > 0\), we find that

\[
\omega_m \|v_n(s) - \phi\|_{L^m(\Omega)}^m \leq \int_\Omega \left(|v_n|^{m-2}v_n(s) - \phi^{m-1}\right) (v_n(s) - \phi) \, dx
\]

\[
\leq \left\|v_n^{m-2}v_n(s) - \phi^{m-1}\right\|_{H^{-1}(\Omega)} \|v_n(s) - \phi\|_{H^1_0(\Omega)}.
\]

From the boundedness of \((v_n)\) in \(L^\infty(0, S; H^1_0(\Omega))\) (by Lemma 2.4 and the boundedness of \(J(v_n(0))\) and \(R(v_n(0))\)) along with the convergence of \(|v_n|^{m-2}v_n\) in \(C([0, S]; H^{-1}(\Omega))\), it follows that

\[
v_n \to \phi \quad \text{strongly in } C([0, S]; L^m(\Omega)).
\]

By subtraction of equations, we have

\[
\partial_s \left(|v_n|^{m-2}v_n\right)(s) - \Delta (v_n(s) - \phi) = \lambda_m \left(|v_n|^{m-2}v_n(s) - \phi^{m-1}\right) \quad \text{in } H^{-1}(\Omega), \quad s > 0.
\]

Let us formally test it by \(\partial_s v_n(s) = \partial_s (v_n(s) - \phi)\) to get

\[
\mu_m \left\|\partial_s \left(|v_n|^{m-2}v_n\right)(s)\right\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{ds} \|\nabla v_n(s) - \nabla \phi\|_{L^2(\Omega)}^2
\]

\[
\leq \lambda_m \int_\Omega \left(|v_n|^{m-2}v_n(s) - \phi^{m-1}\right) \partial_s v_n(s) \, dx
\]

\[
= \frac{d}{ds} \left(\frac{\lambda_m}{m} \|v_n(s)\|_{L^m(\Omega)}^m - \lambda_m \int_\Omega \phi^{m-1}v_n(s) \, dx\right).
\]

The integration of both sides over \((0, s)\) leads us to see that

\[
\mu_m \int_0^s \left\|\partial_s \left(|v_n|^{m-2}v_n\right)(\sigma)\right\|_{L^2(\Omega)}^2 \, d\sigma + \frac{1}{2} \frac{d}{ds} \|\nabla v_n(s) - \nabla \phi\|_{L^2(\Omega)}^2
\]

\[
\leq \frac{1}{2} \left\|\nabla v_n(0) - \nabla \phi\right\|_{L^2(\Omega)}^2 + \frac{\lambda_m}{m} \|v_n(s)\|_{L^m(\Omega)}^m - \frac{\lambda_m}{m} \|v_n(0)\|_{L^m(\Omega)}^m
\]

\[
- \lambda_m \int_\Omega \phi^{m-1} (v_n(s) - v_n(0)) \, dx,
\]

which can be rigorously derived as in [4]. Thus by virtue of \((3.11)\) one obtains

\[
\frac{1}{2} \sup_{s \in [0, S]} \left\|\nabla v_n(s) - \nabla \phi\right\|_{L^2(\Omega)}^2
\]

\[
\leq \frac{1}{2} \left\|\nabla v_n(0) - \nabla \phi\right\|_{L^2(\Omega)}^2 + \frac{\lambda_m}{m} \sup_{s \in [0, S]} \left\|v_n(s)\right\|_{L^m(\Omega)}^m - \|v_n(0)\|_{L^m(\Omega)}^m
\]

\[
+ \lambda_m \|\phi\|_{L^m(\Omega)}^m \sup_{s \in [0, S]} \left\|v_n(s) - v_n(0)\right\|_{L^m(\Omega)} \to 0.
\]

Therefore \(v_n \to \phi\) strongly in \(L^\infty(0, S; H^1_0(\Omega))\); in particular, we see that

\[
\|v_n(s_n) - \phi\|_{H^1_0(\Omega)} \leq \sup_{s \in [0, S]} \|v_n(s) - \phi\|_{H^1_0(\Omega)} \to 0,
\]

which contradicts the fact that \(\|v_n(s_n) - \phi\|_{H^1_0(\Omega)} = \varepsilon_1 > 0\). Hence \(s_n\) diverges to \(\infty\).
Moreover, repeating the argument above, one can also verify that
\[
\sup_{s \in [0,S]} \|v_n(s) - \phi\|_{H^1_0(\Omega)} \to 0 \quad \text{for any fixed } S > 0.
\] (3.12)

Thus we have proved the lemma. \qed

We next see that

**Lemma 3.5** (Check of (A2)). It holds that \(J(v_n(s)) - J(\phi) > 0\) for all \(s \in [0,s_0)\).

**Proof.** Suppose on the contrary that \(J(v_n(s_0)) = J(\phi)\) for some \(s_0, n \in [0, s_0]\). Then by (2.10),
\[
\mu_m \int_{s_0, n} \| \partial_s \left( |v_n|^{(m-2)/2} v_n \right) (\sigma) \|_{L^2(\Omega)}^2 \, d\sigma + J(v_n(s_0)) \leq J(v_n(s_0)) = J(\phi),
\]
which along with the fact by (3.1) that \(J(\phi) \geq 0\) implies \(\partial_s (|v_n|^{(m-2)/2} v_n) \equiv 0\) a.e. on \(\Omega \times (s_0, s_0)\). Hence \(v_n(s) = v_n(s_0)\) for all \(s \in [s_0, s_0]\). On the other hand, from the fact that \(s_0, n < s_0\), one has \(\|v_n(s_0, n) - \phi\|_{H^1_0(\Omega)} < \varepsilon_1\). Combining these facts, we particularly obtain \(\|v_n(s) - \phi\|_{H^1_0(\Omega)} < \varepsilon_1\), which is a contradiction to the definition of \(s_n\). Thus \(J(v_n(s)) - J(\phi) > 0\) for all \(s \in [0, s_0]\). \qed

Since \(v_n(0) \in B_{H^1_0(\Omega)}(\phi; r_0) \cap \mathcal{X}\), by (3.3) we see that
\[
\sup_{s \geq s_0} \|v_n(s)\|_{L^\infty(\Omega)} \leq L \quad \text{for all } n \in \mathbb{N}. \tag{3.13}
\]

By taking \(n \in \mathbb{N}\) so large that \(s_n > s_0\) (see Lemma 3.3) and using Lemma 3.5, one can employ
\[
\text{3.11}
\]
\[
\text{to obtain}
\]
\[
\| |v_n|^{m-2} v_n(s_0) - \phi^{m-1} \|_{H^{-1}(\Omega)} \leq \frac{\kappa_m \omega C^2_2 \mu_m \theta}{\inf_{s \geq s_0} \|v_n(s)\|_{L^\infty(\Omega)}} \left( J(v_n(s_0)) - J(\phi) \right) + \| |v_n|^{m-2} v_n(s_0) - \phi^{m-1} \|_{H^{-1}(\Omega)}.
\]

which together with (3.13) gives
\[
\| |v_n|^{m-2} v_n(s_0) - \phi^{m-1} \|_{H^{-1}(\Omega)} \leq C \left( J(v_n(s_0)) - J(\phi) \right) + \| |v_n|^{m-2} v_n(s_0) - \phi^{m-1} \|_{H^{-1}(\Omega)}
\]
for some constant \(C \geq 0\) independent of \(n\). Hence, by (3.12), we deduce that
\[
\| |v_n|^{m-2} v_n(s_n) - \phi^{m-1} \|_{H^{-1}(\Omega)} \to 0.
\]

As in (3.10) by Tartar’s inequality, it follows that
\[
v_n(s_n) \to \phi \quad \text{strongly in } L^m(\Omega). \tag{3.14}
\]

Since \((v_n(s_n))\) is bounded in \(H^1_0(\Omega)\) by (3.9), up to a subsequence, \(v_n(s_n) \to \phi\) weakly in \(H^1_0(\Omega)\). Furthermore, we deduce that
\[
\frac{1}{2} \|v_n(s_n)\|^2_{H^1_0(\Omega)} = J(v_n(s_n)) + \frac{\lambda_m}{m} \|v_n(s_n)\|^m_{L^m(\Omega)}
\]
\[
\leq J(v_n(0)) + \frac{\lambda_m}{m} \|v_n(s_n)\|^m_{L^m(\Omega)}
\]
\[
\to J(\phi) + \frac{\lambda_m}{m} \|\phi\|^m_{L^m(\Omega)} = \frac{1}{2} \|\phi\|^2_{H^1_0(\Omega)} \tag{3.13}
\]
from the fact that \(\|v_n(0) - \phi\|_{H^1_0(\Omega)} < 1/n\) as well as the non-increase of the energy \(J(v_n(\cdot))\). Due to the uniform convexity of \(H^1_0(\Omega)\), we also obtain
\[
v_n(s_n) \to \phi \quad \text{strongly in } H^1_0(\Omega).
\]

However, it contradicts the definition of \(s_n\), i.e., \(\|v_n(s_n) - \phi\|_{H^1_0(\Omega)} = \varepsilon_1 > 0\). Consequently, we conclude that \(\phi\) is stable. \qed
4. Local minimizers of $J$ over $\mathcal{X}$

In this section, we are concerned with local minimizers of $J$ over the set $\mathcal{X}$. Let us start with the following proposition, which was already used in Section 3.

**Proposition 4.1.** Let $\phi$ satisfy (3.1). Then $\phi$ is a positive or negative solution of (1.8), (1.9).

**Proof.** Let $v = v(x, s)$ be the solution of (1.3) with the initial data $v(0) = \phi$. Due to the right-continuity of $s \mapsto v(s)$ in $H^1_0(\Omega)$, one can take $s_* \in (0, \infty]$ such that $v(s) \in B_{H^1_0(\Omega)}(\phi; r_0)$ for all $s \in [0, s_*]$. Then from (2.3) along with the fact that

$$J(\phi) = \inf\{J(w) : w \in \mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; r_0)\} \leq J(v(s)) \leq J(v(0)) = J(\phi) \quad \text{for all } s \in [0, s_*),$$

it follows that

$$\int_0^s \left\| \partial_s \left( |v|^{(m-2)/2} v \right) (\sigma) \right\|^2_{L^2(\Omega)} d\sigma = 0 \quad \text{for all } s \in [0, s_*),$$

which implies $v(s) = v(0) = \phi$ for all $s \in [0, s_*]$. Hence we have $s_* = \infty$ and $v(s) \equiv \phi$. Therefore $\phi$ solves (1.8), (1.9). We next prove the positivity (or negativity) of $\phi$. Suppose on the contrary that $\phi$ is sign-changing. Then let $D \subseteq \Omega$ be a nodal domain of $\phi$, a connected component of the set $\{x \in \Omega : \phi(x) \neq 0\}$. As in [1] Proof of Theorem 3, one can define

$$\phi_\mu(x) := \begin{cases} \mu \phi(x) & \text{if } x \in D, \\ \phi(x) & \text{if } x \notin D \end{cases}$$

and observe that $J(c_\mu \phi) < J(\phi)$ for any $\mu > 0$, $\mu \neq 1$ and any $c > 0$. Hence put $v_{0, \mu} := t_*(\phi_\mu)^{-1/(m-2)} \phi_\mu \in \mathcal{X}$. Then

$$J(v_{0, \mu}) < J(\phi) \quad \text{for any } \mu > 0, \neq 1.$$ 

On the other hand, from the continuity of $t_* : H^1_0(\Omega) \to [0, \infty)$ (see [1] Proposition 4) and the fact that $t_*(\phi) = 1$ by $\phi \in \mathcal{X}$, one deduces that

$$v_{0, \mu} \to \phi \quad \text{strongly in } H^1_0(\Omega),$$

whence $v_{0, \mu}$ belongs to $B_{H^1_0(\Omega)}(\phi; r_0)$ for $\mu$ sufficiently close to 1. However, these facts contradict the local minimality of $J$ at $\phi$ over $\mathcal{X}$. Therefore $\phi$ turns out to be nonnegative or nonpositive. Finally, by strong maximum principle, $\phi$ is positive or negative in $\Omega$. \hfill \Box

**Remark 4.2.** In the annular domain case, $\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$ for $0 < a < b < \infty$, as in [2] Proposition 5.3, one may also prove that every local minimizer of $J$ over $\mathcal{X}$ is not radially symmetric under some quantitative assumption on the thickness of the annulus. Indeed, suppose on the contrary that a local minimizer $\phi$ is radially symmetric. Then one can construct $v_{0, \mu} \in \mathcal{X}$ such that $J(v_{0, \mu}) < J(\phi)$ for any $0 < \mu \ll 1$ and $v_{0, \mu} \to \phi$ strongly in $H^1_0(\Omega)$ as $\mu \to 0$, if $a$ and $b$ satisfy $(b/a)^{(N-3)/2}((b-a)/(\pi a))^2 < (m-2)/(N-1)$. Hence these facts yield a contradiction.

Let us next discuss the relation of local minimizers of $J$ over the so-called Nehari manifold,

$$\mathcal{N} := \{w \in H^1_0(\Omega) \setminus \{0\} : \|\nabla w\|^2_{L^2(\Omega)} = \lambda_m \|w\|^m_{L^m(\Omega)}\},$$

and those over $\mathcal{X}$. Emden-Fowler equation (1.8), (1.9) has been well studied in variational analysis, where nontrivial solutions are often characterized as global or local minimizers of the functional $J$ over $\mathcal{N}$. However, the phase set $\mathcal{X}$ is different from $\mathcal{N}$, and their intersection is just $\mathcal{S}$ (see [1] Proposition 10). Hence it is unclear whether or not every local minimizer of $J$ over $\mathcal{N}$ also locally minimizes $J$ over $\mathcal{X}$. The following proposition gives an affirmative answer to this question for isolated local minimizers over $\mathcal{N}$.

**Proposition 4.3.** Let $\phi$ be an isolated local minimizer of $J$ over $\mathcal{N}$, that is, there exists $r_0 > 0$ such that

$$J(\phi) < J(w) \quad \text{for all } w \in (\mathcal{N} \cap B_{H^1_0(\Omega)}(\phi; r_0)) \setminus \{\phi\}. \quad (4.1)$$

Then $\phi$ is also a local minimizer of $J$ over $\mathcal{X}$. 

\hfill \Box
Before proving the proposition above, we note that:

**Lemma 4.4.** For each \( \phi \in S \) the following conditions are equivalent:

(i) There exists \( r_0 > 0 \) such that \( J(\phi) \leq J(w) \) for all \( w \in \mathcal{N} \cap B_{H^1_0(\Omega)}(\phi; r_0) \).

(ii) There exists \( r_0 > 0 \) such that \( R(\phi) \leq R(w) \) for all \( w \in \mathcal{N} \cap B_{H^1_0(\Omega)}(\phi; r_0) \).

(iii) There exists \( r_1 > 0 \) such that \( R(\phi) \leq R(w) \) for all \( w \in \mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; r_1) \).

Here we note that one can take the same \( r_0 \) for (i) and (ii).

**Proof.** We first note that

\[
J(w) = \frac{m - 2}{2m} \lambda_m^{-2/(m-2)} R(w)^{2m/(m-2)} \quad \text{for all } w \in \mathcal{N}.
\]

Hence it is obvious that (i) and (ii) are equivalent with the same choice of \( r_0 > 0 \). So it remains to prove the equivalence between (ii) and (iii). For each \( w \in H^1_0(\Omega) \setminus \{0\} \), set positive constants

\[
x(w) := t_n(w)^{-1/(m-2)}, \quad n(w) := \left( \frac{\|w\|_{H^1_0(\Omega)}^2}{\lambda_m \|w\|_{L^m(\Omega)}} \right)^{1/(m-2)}.
\]

Then it follows that \( x(w) \leq n(w) \), \( x(w)w \in \mathcal{X} \) and \( n(w)w \in \mathcal{N} \) (see [1, Proposition 10]). First, assume (ii). Let \( w \in \mathcal{X} \) be such that \( \|w - \phi\|_{H^1_0(\Omega)} < r_1 \) with \( r_1 > 0 \) which will be determined later. We observe that

\[
|n(w)w - \phi|_{H^1_0(\Omega)} \leq |n(w) - 1| \|w\|_{H^1_0(\Omega)} + \|w - \phi\|_{H^1_0(\Omega)} < |n(w) - 1| \left( \|\phi\|_{H^1_0(\Omega)} + r_1 \right) + r_1.
\]

Since \( n(\cdot) \) is continuous in \( H^1_0(\Omega) \setminus \{0\} \) and \( n(\phi) = 1 \), one can take \( r_1 > 0 \) small enough that the right-hand side of (4.2) is less than \( r_0 \). It follows that

\[
R(\phi) \leq R(n(w)w) = R(w).
\]

Thus (iii) follows. Next assume (iii) and let \( w \in \mathcal{N} \) be such that \( \|w - \phi\|_{H^1_0(\Omega)} < r_0 \) with \( r_0 > 0 \) to be determined. Then one can similarly derive

\[
|x(w)w - \phi|_{H^1_0(\Omega)} < |x(w) - 1| \left( \|\phi\|_{H^1_0(\Omega)} + r_0 \right) + r_0.
\]

So choosing \( r_0 > 0 \) small enough and employing the continuity of \( t_n(\cdot) \) in \( H^1_0(\Omega) \) along with \( t_n(\phi) = 1 \), one deduces that \( \|x(w)w - \phi\|_{H^1_0(\Omega)} < r_1 \). Consequently, (iii) implies

\[
R(\phi) \leq R(x(w)w) = R(w),
\]

whence (ii) follows.

The fact above also holds true for global minimizers (i.e., \( r_0 = r_1 = \infty \)). Moreover, it is known (see Proposition [2,2]) that the set of (global) minimizers of \( J \) over \( \mathcal{X} \) coincides with the set of least energy solutions, which can be also formulated as (global) minimizers of \( J \) over \( \mathcal{N} \) (see, e.g., [10, Chap. 4]).

**Proof of Proposition 4.3.** Since \( \phi \) is the (unique) minimizer of \( J \) over \( \mathcal{N} \cap B_{H^1_0(\Omega)}(\phi; r_0) \) by assumption, due to Lemma 4.4 it holds that

\[
R(\phi) \leq R(w) \quad \text{for all } w \in \mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; r_1)
\]

for some \( r_1 > 0 \). Suppose on the contrary that \( \phi \) is not a local minimizer of \( J \) over \( \mathcal{X} \); then for each \( n \in \mathbb{N} \) we can take \( v_{0,n} \in \mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; 1/n) \) such that

\[
J(v_{0,n}) < J(\phi).
\]

Let \( v_n = v_n(x, s) \) be the solution of (13) with the initial data \( v_n(0) = v_{0,n} \). Then one observes that

\[
R(v_n(s)) \leq R(v_{0,n}), \quad J(v_n(s)) \leq J(v_{0,n}) < J(\phi) \quad \text{for all } s \geq 0.
\]
Since $v_{0,n}$ belongs to $\mathcal{X}$, there is $\psi_n \in S \subset \mathcal{N}$ such that $v_n(s) \to \psi_n$ along a subsequence of $s \to \infty$. Thus we see that $J(\psi_n) < J(\phi)$, which implies $\psi_n \notin B_{H^1_0(\Omega)}(\phi; r_0)$ by assumption. Moreover, it follows that $R(\psi_n) \leq R(v_{0,n})$.

Now, let us take $s_n > 0$ such that

$$
\|v_n(s) - \phi\|_{H^1_0(\Omega)} < \varepsilon \quad \text{for all } s \in [0, s_n), \quad \text{and} \quad \|v_n(s_n) - \phi\|_{H^1_0(\Omega)} = \varepsilon
$$

with $\varepsilon \in (0, r_0 \wedge r_1)$ which will be determined later (cf. see (3.3)). Then since $\mathcal{X}$ is sequentially closed in the weak topology of $H^1_0(\Omega)$ (see Proposition 2.2), there exists $z \in \mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; r_1)$ such that

$$v_n(s_n) \to z \quad \text{weakly in } H^1_0(\Omega) \quad \text{and strongly in } L^m(\Omega).$$

Therefore by Lemma 2.4 we deduce that

$$
\liminf_{n \to \infty} R(v_n(s_n)) = \liminf_{n \to \infty} \frac{\|\nabla v_n(s_n)\|_{L^2(\Omega)}}{\|v_n(s_n)\|_{L^m(\Omega)}} \geq \frac{\|\nabla z\|_{L^2(\Omega)}}{\|z\|_{L^m(\Omega)}} = R(z) \geq R(\phi).
$$

On the other hand, recalling $R(v_n(s_n)) \leq R(v_{0,n})$, one has

$$
\limsup_{n \to \infty} R(v_n(s_n)) \leq \lim_{n \to \infty} R(v_{0,n}) = R(\phi).
$$

Combining these facts, we obtain

$$R(v_n(s_n)) \to R(\phi) \quad \text{and} \quad R(z) = R(\phi).$$

Therefore we see that

$$
\|\nabla v_n(s_n)\|_{L^2(\Omega)} = R(v_n(s_n))\|v_n(s_n)\|_{L^m(\Omega)} \\
\to R(\phi)\|z\|_{L^m(\Omega)} = R(z)\|z\|_{L^m(\Omega)} = \|\nabla z\|_{L^2(\Omega)},
$$

which along with the uniform convexity of $H^1_0(\Omega)$ implies

$$v_n(s_n) \to z \quad \text{strongly in } H^1_0(\Omega).$$

Thus we get

$$\|z - \phi\|_{H^1_0(\Omega)} = \varepsilon.$$

Now, $n(z)z$ belongs to $\mathcal{N}$. We here claim that

$$0 \neq \|n(z)z - \phi\|_{H^1_0(\Omega)} < r_0 \quad (4.4)$$

for sufficiently small $\varepsilon > 0$. Indeed, repeating the same argument as in the proof of Lemma 4.4 we find that $\|n(z)z - \phi\|_{H^1_0(\Omega)} < r_0$ for $\varepsilon > 0$ small enough. On the other hand, recall that $z \neq \phi$ and $z, \phi \in \mathcal{X}$. The ray from the origin through $w \in H^1_0(\Omega) \setminus \{0\}$, i.e., $\{kw \in H^1_0(\Omega) : k > 0\}$, intersects $\mathcal{X}$ (resp., $\mathcal{N}$) only at the single point $x(w)w$ (resp., $n(w)w$) (see [1, Proposition 10]); therefore $z$ and $\phi$ do not lie on the same ray from the origin. Hence one observes that

$$n(z)z \neq n(\phi)z = \phi.$$

Thus we obtain (4.4). Recall $R(n(z)z) = R(z) = R(\phi)$ and note that

$$J(n(z)z) = \frac{m-2}{2m-2}\lambda^2/m - 2(m-2)R(n(z)z)^{2m/(m-2)} = \frac{m-2}{2m-2}\lambda^2/m - 2(m-2)R(\phi)^{2m/(m-2)} = J(\phi).$$

However, these facts yield a contradiction to the assumption (4.1). The proof is completed. □

The inverse relation can be easily proved without imposing any additional assumption.

**Proposition 4.5.** Let $\phi$ satisfy (3.1). Then $\phi$ locally minimizes $J$ over $\mathcal{N}$.
Proof. Assume that $\phi$ satisfies (5.1). As in the proof of Lemma 1.4 one can choose $\delta > 0$ small enough that $x(w)w \in \mathcal{X} \cap B_{H^1_0(\Omega)}(\phi; r_0)$ for all $w \in \mathcal{N} \cap B_{H^1_0(\Omega)}(\phi; \delta)$. Then by assumption, we obtain

$$J(\phi) \leq J(x(w)w) \quad \text{for all} \quad w \in \mathcal{N} \cap B_{H^1_0(\Omega)}(\phi; \delta).$$

On the other hand, by the definition of $\mathcal{N}$, it holds that $J(w) = \sup_{c>0} J(cw)$ for each $w \in \mathcal{N}$. Hence it follows that

$$J(\phi) \leq J(x(w)w) \leq J(w) \quad \text{for all} \quad w \in \mathcal{N} \cap B_{H^1_0(\Omega)}(\phi; \delta).$$

Thus we conclude that $\phi$ is a local minimizer of $J$ over $\mathcal{N}$. \qed

5. Instability of positive radial profiles in thin annular domains

In the final section, we shall apply the Łojasiewicz-Simon inequality to prove the instability of sign-definite asymptotic profiles which do not attain local minima of $J$ over $\mathcal{X}$. Then one can prove Theorem 1.4 that is, the instability of the positive radial asymptotic profile (equivalently, the positive radial solution of (1.8), (1.9)) in the annular domain

$$\Omega = \{x \in \mathbb{R}^N : a < |x| < b\}$$

with $0 < a < b < \infty$ satisfying (1.10), as a corollary.

**Theorem 5.1 (Instability of sign-definite profiles except for local minimizers of $J$ over $\mathcal{X}$).** Let $\phi$ be a positive (or negative) solution of (1.8), (1.9) which does not attain any local minimum of $J$ over $\mathcal{X}$. Then $\phi$ is an unstable asymptotic profile for FDE (in the sense of Definition 1.1).

**Proof.** Let $\phi$ be a positive (or negative) solution of (1.8), (1.9) such that $\phi$ does not attain any local minimum of $J$ over $\mathcal{X}$, that is, there exists a sequence $(v_{0,n})$ in $\mathcal{X}$ such that $J(v_{0,n}) < J(\phi)$ and $v_{0,n} \to \phi$ strongly in $H^1_0(\Omega)$. Then by strong maximum principle and elliptic regularity, $\phi$ also satisfies (3.2); therefore the Łojasiewicz-Simon inequality, i.e., Lemma 3.2, is valid for $\phi$ as well. Since $v_{0,n}$ lies on $\mathcal{X}$, one can take a nontrivial solution $\psi_n$ of (1.8), (1.9) such that the solution $v_{0,n}$ of (1.6), (1.7) with $v_0 = v_{0,n}$ converges to $\psi_n$ strongly in $H^1_0(\Omega)$ along a subsequence of $s \to \infty$. From the non-increase of the energy, one has $J(\psi_n) < J(\phi)$. Now, suppose that $\psi_n$ converges to $\phi$ strongly in $H^1_0(\Omega)$ as $n \to \infty$. Then by utilizing elliptic regularity technique, one can check that $\psi_n$ converges to $\phi$ in $C^2(\Omega)$; in particular, $\|\psi_n\|_{L^\infty(\Omega)} \leq \|\phi\|_{L^\infty(\Omega)} + 1 =: L$ for $n > 0$ large enough. Hence thanks to Lemma 3.2 since $J(\psi_n) = 0$, for sufficiently large $n$, $\psi_n$ must take the same critical value as $\phi$, that is, $J(\psi_n) = J(\phi)$. However, it is a contradiction to the difference of the energy. Therefore $(\psi_n)$ does not converge to $\phi$ in $H^1_0(\Omega)$ as $n \to \infty$.

Hence one can take $\delta_1 > 0$ and a subsequence $(\psi_k)$ of $(\psi_n)$ such that $\|\psi_k - \phi\|_{H^1_0(\Omega)} \geq \delta_1$ for all $k \in \mathbb{N}$. Therefore for each $k \in \mathbb{N}$, the solution $v_{0,k}(s)$ of (1.5), (1.7) for the initial data $v_{0,n,k}$ must go away from the neighborhood $B_{H^1_0(\Omega)}(\phi; \delta_1/2)$ for $s > 0$ sufficiently large. On the other hand, $v_{0,n,k} \to \phi$ strongly in $H^1_0(\Omega)$ as $k \to \infty$. Thus we have proved the instability of $\phi$. \qed

Theorem 1.4 follows from the theorem stated above.

**Proof of Theorem 1.4** Let $\phi$ be the positive radial solution of (1.8), (1.9). Then as in [2], for $\varepsilon > 0$ small enough, one can explicitly construct $v_{0,\varepsilon} \in \mathcal{X}$ such that

$$J(v_{0,\varepsilon}) < J(\phi) \quad \text{and} \quad v_{0,\varepsilon} \to \phi \quad \text{strongly in} \quad H^1_0(\Omega) \quad \text{as} \quad \varepsilon \to 0$$

under the assumption (1.10). This fact yields that $\phi$ is not a local minimizer of $J$ over $\mathcal{X}$; thus the instability of $\phi$ follows from Theorem 5.1. \qed
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Appendix A. Derivation of (2.1) and (2.2)

This appendix is devoted to verifying (2.1) and (2.2). Formally test (1.1) by \( \partial_t u \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H_0^1(\Omega)}^2 \leq 0 \quad \text{for a.e. } t > 0
\]

(it can be justified as in [4]). Then

\[
\sup_{t \geq 0} \|u(t)\|_{H_0^1(\Omega)} \leq \|u_0\|_{H_0^1(\Omega)}.
\]

By using Tartar’s inequality along with the fact that \( |u|^{m-2}u \in C ([0, T]; H^{-1}(\Omega)) \), as in (3.10), we see that

\[
u \in C ([0, T]; L^m(\Omega)).
\]

Moreover, recalling Lemma 8.1 of [32], we deduce that

\[
u \in C_w ([0, T]; H_0^1(\Omega)).
\]

Hence it follows that, for each \( s \in [0, T] \),

\[
\|u(s)\|_{H_0^1(\Omega)} \leq \liminf_{t \to s} \|u(t)\|_{H_0^1(\Omega)}.
\]

Since \( t \mapsto \|u(t)\|_{H_0^1(\Omega)} \) is non-increasing, we have

\[
\lim_{t \searrow s} \|u(t)\|_{H_0^1(\Omega)} = \|u(s)\|_{H_0^1(\Omega)}.
\]

Thus by the uniform convexity of \( H_0^1(\Omega) \),

\[
u(t) \to u(s) \quad \text{strongly in } H_0^1(\Omega) \quad \text{as } t \searrow s,
\]

which implies \( u \in C_+ ([0, T]; H_0^1(\Omega)) \). Finally, by comparison of both sides of (1.1), since \(- \Delta : H_0^1(\Omega) \to H^{-1}(\Omega) \) is an isomorphism, \( \partial_t (|u|^{m-2}u) \) belongs to \( C_+ ([0, T]; H^{-1}(\Omega)) \).

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