An Infinite Number of Commuting Quantum $\hat{W}_\infty$ Charges in the $SL(2, R)/U(1)$ Coset Model

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Abstract

The conformal non-compact $SL(2, R)/U(1)$ coset model in two dimensions has been recently shown to embody a nonlinear $\hat{W}_\infty$ current algebra, consisting of currents of spin $\geq 2$ including the energy-momentum tensor. In this letter we explicitly construct an infinite set of commuting quantum $\hat{W}_\infty$ charges in the model with $k = 1$. These commuting quantum charges generate a set of infinitely many compatible flows (quantum KP flows), which maintain the nonlinear $\hat{W}_\infty$ current algebra invariant.
In the last a few years the exploitation of infinite dimensional algebras, such as Virasoro algebra [1], Kac-Moody current algebras [2] and extended conformal $W_N$ algebras [3], has played a crucial role in solving several large classes of two-dimensional quantum conformal field theories, e.g. the minimal models [4], Wess-Zumino-Witten models [5] and compact coset models [6]. Very recently a new nonlinear current algebra, called the $\hat{W}_\infty$ algebra [7], is shown to hide in the 2d non-compact $SL(2,R)/U(1)$ coset model both at the classical [8] and the quantum [9,8] levels. This algebra is generated by currents of all spin $s \geq 2$, including the energy-momentum tensor, and can be viewed as a generalization of Zamolochikov’s nonlinear $W_N$ algebra [3] which is generated by currents of spin $s$ from 2 to $N$. At the classical level, the $\hat{W}_\infty$ algebra is shown [8] to be a nonlinear, centerless deformation of the linear $W_\infty$ algebra [10,11], and isomorphic to the second Hamiltonian structure of the KP hierarchy [12] proposed by Dickey [13]. This isomorphism immediately suggests the existence of infinitely many charges that are in involution in the classical $SL(2,R)/U(1)$ coset model [8].

However, after quantizing the model the $\hat{W}_\infty$ current algebra, like the $W_N$ algebras, undergoes nontrivial quantum deformation and receives intriguing quantum corrections: The current-current commutators acquire not only non-vanishing central terms, but also additional linear and nonlinear terms which have non-trivial dependence on the level $k$ of the coset model. Despite these complications, it has been conjectured in ref. [9,8] that there exists an infinite set of commuting quantum $\hat{W}_\infty$ charges in the quantized model. This letter is devoted to showing the validity of this conjecture, by explicit construction of the commuting quantum charges in the model with level $k = 1$.

Let us start with a brief review for the classical $\hat{W}_\infty$ current algebra [8] in the $SL(2,R)_k/U(1)$ model. Essentially this is a two-boson realization of the $\hat{W}_\infty$ algebra. Consider only the holomorphic part. The $SL(2,R)_k$ current algebra is known [14] to have a three-boson realization as follows:

$$J_{\pm} = \frac{k}{2} e^{\pm \sqrt{k} \phi_3} (\phi_1' \mp i\phi_2') e^{\pm \sqrt{k} \phi_1}, \quad J_3 = -\sqrt{\frac{k}{2}} \phi_3'. \quad (1)$$

To take the coset, one simply restricts the $U(1)$ current $J_3 = 0$ or the boson $\phi_3 = 0$. Thus one is left with $J_{\pm}$ which are nothing but the bosonized para-fermion currents:

$$J_+ = j e^{\phi + \bar{\phi}}, \quad J_- = j e^{-\phi - \bar{\phi}}. \quad (2)$$
Here $\phi = (1/\sqrt{2})(\phi_1 + i\phi_2)$ and $\bar{\phi} = (1/\sqrt{2})(\phi_1 - i\phi_2)$, and their currents $\bar{j}(z) = \bar{\phi}'(z)$, $j(z) = \phi'(z)$ satisfy the Poisson brackets

$$\{\bar{j}(z), \bar{j}(w)\} = \{j(z), j(w)\} = 0,$$
$$\{j(z), \bar{j}(w)\} = \partial_z \delta(z - w).$$

(3)

At the classical level, we can set the level $k$ equal to 1 without loss of generality. The classical $\hat{W}_\infty$ currents, denoted as $u_r(z)$ ($r = 0, 1, 2, \cdots$) with spin $s = r+2$, are generated by the following (bi-local) product expansion of the coset currents

$$J_+(z)J_-(z') = \sum_{r=0}^{\infty} u_r(z) \frac{(z-z')^r}{r!}.$$

(4)

It has been shown in ref.[8] that these currents $u_r(z)$ are formed only from the currents $j(z), \bar{j}(z)$ and their derivatives, containing no fields $\phi$ or $\bar{\phi}$ themselves; and their Poisson brackets satisfy the $\hat{W}_\infty$ algebra [7]:

$$\{u_r(z), u_s(w)\} = k_{rs}(z) \delta(z - w)$$

(5)

with $k_{rs}(z)$ explicitly given by

$$k_{rs}^{(2)} = \sum_{l=0}^{s+1} \binom{s+1}{l} D^l u_{r+s+1-l} - \sum_{l=0}^{r+1} \binom{r+1}{l} u_{r+s+1-l} (-D)^l$$
$$- \sum_{l=0}^{r-1} \sum_{k=0}^{s-1} (-1)^{r-l} \binom{r}{l} \binom{s}{k} u_l D^{r+s-l-k-1} u_k$$
$$+ \sum_{l=0}^{\infty} \sum_{t=l+1}^{l+s} (-1)^{l} \binom{t-s-1}{l}$$
$$- \sum_{l=r+1}^{r+s} \sum_{k=0}^{t-r-1} (-1)^{l+k} \binom{t-k-1}{l-k} \binom{s}{k} u_{t-l-1} D^l u_{r+s-t}.$$ 

(6)

As shown in ref.[7], on one hand, this Poisson bracket algebra can be viewed as a nonlinear deformation of $W_\infty$; on the other hand, it is isomorphic to the second Hamiltonian structure of the KP hierarchy proposed by Dickey [13].

In the context of the KP hierarchy, it is the coefficient functions $u_r(z)$ in the KP pseudo-differential operator

$$L = D + \sum_{r=0}^{\infty} u_r(z) D^{-r-1}, \quad D \equiv \partial/\partial z,$$

(7)
which have the same Poisson brackets (5) and (6). These Poisson brackets give rise to the second KP Hamiltonian structure, in the sense that the original KP hierarchy
\[ \frac{\partial L}{\partial t_m} = [(L^m)_+, L] \quad (m = 1, 2, 3, \ldots) \]  
(8)
can be put into the Hamiltonian form
\[ \frac{\partial u_r(z)}{\partial t_m} = \{u_r(z), \oint_0 H_{m+1}(w)dw\}, \]  
(9)with the Hamiltonian functions defined by
\[ H_{m+1} = \frac{1}{m} \text{Res} L^m, \]  
(10)where the residue Res means the coefficient of the $D^{-1}$ term. These KP Hamiltonian functions give rise to an infinite set of conserved charges in involution [13]:
\[ \{\oint H_m(z)dz, \oint H_n(w)dw\} = 0. \]  
(11)

As shown in ref.[8], the above two-boson realization (4) of the $\hat{W}_\infty$ currents is equivalent to expressing the KP operator $L$ directly in terms of $\bar{j}$ and $j$ as follows:
\[ L = D + \bar{j} \frac{1}{D - (j + j)} j \equiv D + jD^{-1}j + \bar{j}D^{-1}(j + j)D^{-1}j + \cdots. \]  
(12)

Since the $\hat{W}_\infty$ algebra for the currents $u_r$ is isomorphic to the second KP Hamiltonian structure, by the same algebraic manipulations as in the KP case we infer that in the classical $SL(2, R)/U(1)$ model there exists an infinite set of commuting $\hat{W}_\infty$ charges, given by the integral of the same set of $H_{m+1}$ in eq. (10) with the $u_r(z)$ in $L$ identified with those constructed from the generating function (4).

We now proceed to the quantized $SL(2, R)/U(1)$ model and attack the problem of constructing an infinite set of commuting quantum charges which are a deformation of the above classical charges. This problem is a difficult one since in the present case quantization will introduce quite involved quantum corrections to the current algebra.

To start we note in the first place that compared to their classical expression (2), the two boson prescription for the coset currents $J_\pm$ receives additional terms:
\[
J_+(p, z) = \frac{1}{2}[(1 + \sqrt{1 - 2p})\bar{j} + (1 - \sqrt{1 - 2p})j]e^{\sqrt{p}(\bar{\phi} + \phi)},
\]
\[
J_-(p, z) = \frac{1}{2}[(1 - \sqrt{1 - 2p})\bar{j} + (1 + \sqrt{1 - 2p})j]e^{-\sqrt{p}(\bar{\phi} + \phi)}.
\]  
(13)
Here for convenience we use $p = k^{-1}$ as the deformation parameter, so that the classical limit is $p \to 0$ after rescaling $\phi \to \phi/\sqrt{p}$, $J_\pm \to J_\pm/\sqrt{p}$ (and $u_r \to u_r/p$). (Recall that in the quantum theory, $p$ actually represents $\hbar$ and the field $\phi$ has the dimension $\sqrt{\hbar}$ so does each current.) As usual, the currents $J_\pm$ are operators so that the right-hand side of eq.(13) should be normal-ordered. But from now on we will always suppress the notation for normal ordering. Furthermore, the quantum $\hat{W}_\infty$ currents $u_r(z;p)$ are generated by the following operator product expansion (OPE)

$$J_+(p, z)J_-(p, z') = e^{-2p}\{e^{-2} + \sum_{r=0}^{\infty} u_r(p, z) \frac{\epsilon^r}{r!}\}$$

where $\epsilon \equiv z - z'$. This OPE can be viewed as the quantum deformation of the classical product expansion (4). By substituting eq.(13) into the left-hand side, it is easy to see that each current $u_r$ on the right-hand side will acquire many new terms, which are highly nonlinear in $j(z)$ and $\bar{j}(z)$ and have nontrivial $p$-dependence.

As usual the complete structure of the quantum $\hat{W}_\infty(p)$ algebra can be easily read off from the OPE’s of the quantum currents $u_r(z;p)u_s(w;p)$. For the latter we extract them from the OPE of the following four $SL(2, R)/U(1)$ coset currents:

$$(J_+(z;p)J_-(z - z';p))(J_+(w;p)J_-(w - w';p)).$$

The first few quantum currents and their OPE’s obtained in this way have been explicitly presented in our previous paper [8]. In contrast to the classical case, a general expression in the quantized model (with generic value of $p$) for the commuting Hamiltonians $H_m$ as elegant as eq.(10) seems beyond our reach. However, very fortunately we have been able to find a general formula for the commuting quantum charges in the model with level $k = 1$.

When $k=1$ or $p=1$, the computation of the OPE’s in eqs.(14) and (15) becomes rather simplified. The first a few $\hat{W}_\infty$ currents $u_r(z)$ read

$$u_0 = -\bar{j}j - \frac{i}{2}(\bar{j}^3 - j'),$$

$$u_1 = -\frac{1}{2}(\bar{j}^2 j + j \bar{j}^2) + \frac{1}{6}(\bar{j}^3 + j^3) - \frac{i}{2}(\bar{j}^3 - j^3) - \frac{1}{12}(1 - 3i)\bar{j}^{(3)},$$

$$-\frac{1}{12}(1 + 3i)j^{(3)},$$
\[ u_2 = \frac{1}{2} j^2 j' + \frac{1}{4} (j^4 + j^4) + \tilde{j} j (\tilde{j}' + j') - \frac{1}{2} (1 + i) (\tilde{j}^2 - j^2) \tilde{j}' \\
+ \frac{1}{2} (1 - i) (\tilde{j}^2 - j^2) j' - \frac{1}{2} j j'' - \frac{1}{2} j' j + \frac{1}{4} (1 - 2i) \tilde{j}' \\
- \frac{1}{4} (1 + 2i) j^2 + \frac{i}{2} (\tilde{j} j'' - j j'') + \frac{1}{12} (1 - 2i) j'' + \frac{1}{12} (1 + 2i) j'''. \quad (16) \]

What makes the \( p = 1 \) case manipulable is that without too much difficulty from the OPE (15) we can extract the crucial \((z - w)^{-1}\) terms in the following OPE's:

\[ u_0(z) u_s(w) \sim -\frac{1}{z - w} u'_s(w) + \text{terms in other powers of } \frac{1}{z - w}, \]

\[ u_1(z) u_s(w) \sim -\frac{1}{z - w} \sum_{l=1}^{s} (-1)^l \binom{s}{l} u_0^{(l)} u_{s-l} + \frac{u_{s+1}}{(s + 1)} + \frac{(-1)^s u_0^{(s+2)}}{(s + 1)(s + 2)}(w) + \text{terms in other powers of } \frac{1}{z - w} \quad (17) \]

where \( u^{(l)} \equiv \partial^l u \). Note that the terms in other powers of \((z - w)^{-1}\) do not contribute to the commutators between charges after double integration. Eq.(17) are essential to our following construction of commuting \( \hat{W}_\infty \) charges.

Before proceeding, let us recall some definitions for the local (normal-ordered) product of several local operators. First the local product \((AB)(z)\) of two local operators \(A(z)\) and \(B(z)\) is defined by

\[ (AB)(z) = \oint_z \frac{A(w)B(z)}{w - z} dw \quad (18) \]

in which the small contour of integration encircles \( z \). This local product is non-commutative, i.e. \((AB)(z) \neq (BA)(z)\); however they differ from each other only by total derivatives:

\[ \oint_0 (AB - BA)(z) dz = 0. \quad (19) \]

According to the definition (18), the operator product of \(C(z)\) with the local product \((AB)(w)\) is given by

\[ C(z)(AB)(w) = ((C(z)A(w))B(w)) + (A(w)(C(z)B(w))). \quad (20) \]

The multiple local product of several operators are generally non-associative, e.g.

\[ (A(BC)) - (B(AC)) = ((AB)C) - ((BA)C). \quad (21) \]
Finally we define the symmetric local product of $N$ local operators to be the totally symmetrized sum of their multiple local products taken from the left:

$$\langle A_1 A_2 \cdots A_N \rangle = \frac{1}{N!} \sum_{P\{i\}} (\cdots ((A_{i_1} A_{i_2}) A_{i_3}) \cdots A_{i_N})$$

(22)

where $P\{i\}$ denotes the summation over all possible permutations.

Now we turn to construct an infinite set of mutually commuting quantum $\hat{W}_\infty$ charges (with $p = 1$). Let us assign a degree 1 to $\partial_z$ and $r + 2$ to $u_r$ and assume that the $m$-th quantum charge density $H_m(z)$ is homogeneous and of degree $m$ and is led by a term linear in the highest-spin current $u_{m-2}$ with unit coefficient. Therefore the most general form for $H_m(z)$ is

$$H_m(z) = \sum_{l \in \{i,a\}} \sum_{\{i,a\}} C_{i_1^{a_1} i_2^{a_2} \cdots i_l^{a_l}}(m) (\cdots (u_{i_1}^{(a_1)} u_{i_2}^{(a_2)}) \cdots u_{i_l}^{(a_l)})(z), \quad m = 2, 3, \ldots$$

(23)

where $l = 1, 2, \cdots, \lfloor m/2 \rfloor$ is the number of currents $u_r$ in the product (the maximal value of $l$ being the integral part of $m/2$); $\{i, a\}$ stands for the set of all possible indices satisfying $i_1 + i_2 + \cdots + i_l + a_1 + a_2 + \cdots + a_l = m - 2l$ and $C_{i_1^{a_1} i_2^{a_2} \cdots i_l^{a_l}}(m)$ are constant coefficients. We want to determine these coefficients so that the corresponding charges $Q_m \equiv \oint H_m(z) dz$ commute with each other:

$$[\oint H_n(z) dz, \oint H_m(w) dw] = 0.$$  

(24)

Especially, for the $n = 2$ case, the current $H_2$, by definition, is nothing but the energy-momentum tensor $u_0$. It is easy to see from the first equation of (17) and eq.(20) that the commutativity with $Q_2$, i.e.

$$[\oint H_2(z) dz, \oint H_m(w) dw] = 0,$$

(25)

is always satisfied. The first set of nontrivial equations in eq.(24) start with $n = 3$:

$$[\oint H_3(z) dz, \oint H_m(w) dw] = 0.$$  

(26)

Here $H_3$ can only be $u_1$ plus a derivative of $u_0$; the latter does not contribute to the charge $Q_3$. In the following we will show that all the charge density $H_m(z)$ modulo total derivatives, and thus all charges $Q_m$, are completely determined by eq.(26) alone.
Both amusingly and amazingly the so-determined charges $Q_m$ automatically commute with each other; in other words, eqs.(24) are automatically satisfied by the solution to eq.(26).

First let us consider how to solve eq.(26). A straightforward calculation, making extensive use of the second equation of (17), determines uniquely the first seven charges from eq.(26), giving

\[ Q_2 = \oint u_0(z)dz, \quad Q_3 = \oint u_1(z)dz, \]
\[ Q_4 = \oint (u_2 - u_0u_0)(z)dz, \quad Q_5 = \oint (u_3 - 6u_0u_1)(z)dz, \]
\[ Q_6 = \oint (u_4 - 12u_0u_2 - 12u_1u_1 + 8(u_0u_0)u_0)(z)dz, \]
\[ Q_7 = \oint (u_5 - 20u_0u_3 - 60u_1u_2 + 60(u_0u_0)u_1 + 60(u_0u_1)u_0)(z)dz, \]
\[ Q_8 = \oint (u_6 - 30u_0u_4 - 120u_1u_3 - 90u_2u_2 + 180(u_0u_0)u_2 + 180(u_0u_2)u_0 + 360(u_1u_0)u_1 + 360(u_1u_1)u_0 - 180((u_0u_0)u_0)(z)dz. \] (27)

In principle, the explicit construction of charges that satisfy eq.(26) may successively continue to higher orders with increasing labor and effort. However, by inspecting eq.(27), it is amusing to observe the following simple and nice features: (a) No term containing any derivative of the currents $u_r$ appears in these charges at all. (b) We note this is true only when all local products of currents are chosen to start from the left; if they had started from the right, terms with derivatives of $u_r$ would appear. (We do not feel fully understand this phenomenon.) (c) For every charge in eq.(27), the coefficients in the right-hand side conspire to result in totally symmetric multiple products of currents involved, after applying eqs.(19) and (21). Generalizing these empirical rules, we are led to the ansatz that the infinite set of charges we are looking for to solve eq.(26) are of the simple form

\[ Q_m = \oint \sum_l \sum_{\{i\}} C_{i_1i_2...i_l}(m)\langle u_{i_1}u_{i_2}...u_{i_l}\rangle(z)dz \] (28)

where $l = 1, 2, ..., \lfloor m/2 \rfloor$ is the number of currents in the symmetrized product, which we call the level of the term. (Do not confuse it with the level of the model.) For given $l$, the summation is over all partitions $\{i_k\}$ satisfying $i_1 + i_2 + ... + i_l = m - 2l$ and
\[ i_1 = i_2 = \cdots = i_d_1 < i_d_1 + 1 = \cdots = i_d_1 + d_2 < \cdots = i_d_1 + d_2 + \cdots d_k (= 1); \] here \( d \)'s denote the degeneracies in the \( i \)'s. We have been able to find an elegant formula for the coefficients that nicely summarizes our lower-order results (27):

\[ C_{i_1i_2 \cdots i_l}(m) = \frac{(-1)^{l-1}(l-1)!}{d_1!d_2! \cdots d_k!i_1!i_2! \cdots i_l!}. \] (29)

The coefficient of the leading linear term \( u_{m-2} \) is unity as desired. One feels the success of this formula should not be accidental. Indeed eq.(29) provides a unique solution to eq.(26) for arbitrary \( m \). Here we give a sketchy description of the proof.

Using eq.(17) and performing one integration, we rewrite eq.(26) as

\[ \oint_0 \sum_l \sum_{\{i\}} C_{i_1i_2 \cdots i_l}(m) \sum_{a=1}^l \langle u_{i_1} \cdots u_{i_{a-1}} \sum_{k=1}^{i_a} (-1)^k \binom{i_a}{k} u^{(k)}_{0} u_{i_{a-k}} \rangle \frac{u'_{i_a+1}}{(i_a+1)} + \frac{(-1)^{i_a} u_{0}^{(i_a+2)}}{(i_a+1)(i_a+2)} u_{i_{a+1}} \cdots u_{i_l} \rangle(z) dz = 0. \] (30)

Note that each term has only one derivative on one of the currents. To verify eq.(29), one needs to show that all the terms in eq.(30) with the same order of derivative on one of the \( u \)'s must cancel each other. In particular, collecting the terms with a first-order derivative on one of the currents we want to verify that

\[ \oint_0 \sum_l \sum_{\{i\}} C_{i_1i_2 \cdots i_l}(m) \sum_{a=1}^l \langle u_{i_1} \cdots u_{i_{a-1}} [-i_a (u'_{0} u_{i_{a-1}}) + \frac{u'_{i_a+1}}{(i_a+1)} u_{i_{a+1}} \cdots u_{i_l}] \rangle(z) dz = 0. \] (31)

At first sight how to handle the local product \( (u'_{0} u_{i_{a-1}}) \) in the first term in the middle of a symmetrized product seems to be a big problem. Fortunately, by applying repeatedly the relations (19) and (21), we have been able to prove the following lemma:

\[ \oint_0 \langle (A_0 A_1) \cdots A_i \cdots A_N \rangle = \oint_0 \langle A_0 A_1 \cdots A_i \cdots A_N \rangle \] (32)

where the left-hand side is symmetrized with respect to the \( N \) indices \( (1, 2, \cdots, N) \) and the right-hand side the \( N + 1 \) indices \( (0, 1, 2, \cdots, N) \). Using this to rearrange the factors in eq.(31) and balancing the number of factors, we have for each \( l \):

\[ \oint_0 \sum_{\{i\}_{l+1}} C_{d_1d_2 \cdots d_k}(m) \sum_{a=1}^k \frac{d_a}{i_a+1} u^{d_{i_1}}_{i_a} \cdots u^{d_{i_{a-1}}}_{i_{a+1}} u_{i_{a+1}}^{d_{i_{a+1}}} \cdots u_{i_k}^{d_k}(z) dz = \oint_0 \sum_{\{j\}_l} C_{j_1j_2 \cdots j_n}(m) \sum_{a=1}^n c_{a_j} u^{c_1}_{j_a} \cdots u^{c_{a-1}}_{j_{a-1}} u^{c_a-1}_{j_{a-1}} u^{c_{a+1}}_{j_{a+1}} \cdots u^{c_n}_{j_n}(z) dz \] (33)
where the $C$’s on the left-hand side are level-$(l + 1)$ coefficients and those on the right-hand side level-$l$ ones: $\sum d_a = l + 1$, $\sum d_a i_a = m - 2(l + 1)$, and $\sum c_b = l$, $\sum c_a j_b = m - 2l$, with $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_n$.

We are going to prove the validity of this equation by induction for both the level $l$ and the first index $i_1$. For terms on the left-hand side having derivative on the highest-spin current $u_{i_k}$, we do integration by parts and turn such terms into those containing no derivative on $u_{i_k}$. On the left-hand side there appears a term with the first index $i_1 = 0$ and of the form $u_0 u_{i_1}^{d_1-1} u_{i_2}^{d_2} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-1} u_{i_{k+1}}$, with coefficient

$$- \frac{d_1 d_k}{i_k + 1} C_{d_1 i_1 d_2 \cdots d_k} (m). \quad \text{(34)}$$

Terms of the same form on the right-hand side have the coefficients

$$C_{d_1 i_1 d_2 \cdots d_k} (m) + (i_2 + 1) C_{d_1 i_2 d_3 \cdots d_k} (i_1 + 1) (m) + \cdots + (i_{k-1} + 1) C_{d_1 i_{k-1} d_k} (i_1 + 1) (m)$$

$$+ 2(i_k + 1) C_{d_1 i_k} (i_1 + 2) (m) + (i_k + 2) C_{d_1 d_2 \cdots d_k} (i_1 + 2) (m)$$

$$= \frac{(-1)^{l-1}(m-2)!}{(d_1 - 1)! d_2! \cdots d_k! (d_k - 1)! (i_2)! (i_k)! (i_1)! (i_{k-1})! (i_{k+1})!}. \quad \text{(35)}$$

Here we have assumed the validity of eq. (29) for level $l$. The equality between (34) and (35) requires exactly the validity of eq. (29) with $i_1 = 0$ at level $l + 1$. Furthermore, we need to verify that after the above-mentioned integration by parts, all the terms having derivative on $u_i$ with $i \neq 0$ on the left-hand side of eq. (33) cancel each other. For example, consider the terms of the form $u_0 u_{i_1}^{d_1-1} u_{i_2}^{d_2} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-1} u_{i_{k+1}}$. Their coefficients are

$$- \frac{d_1 d_k}{i_k + 1} C_{d_1 i_1 d_2 \cdots d_k} (m) + \frac{1}{i_1} C_{i_1 - 1 d_1 d_2 \cdots d_{k-1} d_k} (i_1 + 1) (m). \quad \text{(36)}$$

Assuming eq. (29) is true for the level-$(l + 1)$ coefficient with the first index $i_1 - 1$, then the vanishing of (36) yields the correct level-$(l + 1)$ coefficient $C_{d_1 i_1 d_2 \cdots d_k} (m)$ with the first index $i_1 (\neq 0)$. Similarly we have checked the cancellation of all other terms, particularly those having derivative on higher-spin currents $u_{i_a}$ (with $i_a \neq 0, i_k$) on the left-hand side of eq. (33). Thus, the validity of eq. (31) or (33) is verified with the coefficient $C$’s given by eq. (29).
With eq.(31) verified, eq.(30) reduces to
\[ \oint_0 \sum_{\{i\}} \sum_l C_{i_1i_2\ldots i_l}(m) \sum_{a=1}^l \langle u_{i_1} \cdots u_{i_{a-1}} [\sum_{k=2}^{i_a} (-1)^k \binom{i_a}{k} (u_0^{(k)} u_{i_a-k}) + (-1)^i u_0^{(i+2)} \frac{i_a}{(i_a+1)(i_a+2)} \rangle i_{a+1} \cdots i_l \rangle (z) dz = 0. \] (37)

Again we need to separate out terms having the same order \( k \) for the derivative on one of the currents and show the cancellation among them. This can be done in a way very similar to above discussion, and we leave such details to a longer publication [15]. It is not hard to convert the above verification into an inductive determination of the expression (29) for the coefficient \( C \)'s in the ansatz (28), starting from the normalized coefficient at level \( l = 1 \). If we had set the coefficient (of the leading linear term \( u_{m-2} \)) to be zero, then all other coefficients in eq.(30) should vanish.

Thus we have proved that eq.(29) uniquely solves eq.(30) or eq.(26) under the ansatz (28). Furthermore, we have been able to prove that this solution is actually the only solution to the commutativity equation (26) under the very general assumption (23). In fact let us add to this solution a term \( \oint_0 p_m(z) dz \), with \( p_m(z) \) arbitrary homogeneous polynomial of \( u_r \) and their derivatives of total degree \( m \), and require
\[ [Q_3, \oint_0 p_m(z) dz] = 0. \] (38)

(Obviously, \( p_3 = 0 \) up to total derivatives.) By an inductive argument similar to what we have before, the only solution containing at least one derivative of the currents is
\[ \oint_0 p_m(z) dz = 0. \] (39)

Now we need to prove that the commutativity (26) of \( Q_m \)'s with \( Q_3 \) will guarantee their mutual commutativity (24). This might look a bit too strong, but a similar situation happened for the infinite set of commuting charges for quantum KdV equation [16]. In fact this is a consequence of the Jacobi identities
\[ [Q_3, [Q_m, Q_n]] + [Q_m, [Q_n, Q_3]] + [Q_n, [Q_3, Q_m]] = 0. \] (40)

It follows immediately from eq.(26) that
\[ [Q_3, [Q_m, Q_n]] = 0. \] (41)
Here we note that both $Q_m$ and $Q_n$ are homogeneous and of degree $m - 1$ and $n - 1$ respectively. Besides the OPE’s in the $\hat{W}_\infty$ algebra are homogeneous, so is the commutator $[Q_m, Q_n]$ with degree $m + n - 2$. Thus, $[Q_m, Q_n]$ must be an integral of something which is of the general form (23). The above-proved uniqueness of the homogeneous solution (28) plus (29) assures us that in view of eq.(41), the commutator $[Q_m, Q_n]$ must be proportional to $Q_{m+n-1}$ up to a constant factor:

$$[Q_m, Q_n] = c Q_{m+n-1}. \tag{42}$$

We note that on the left-hand side the charge density $H_{m+n-1}$ is led by the linear term $u_{m+n-3}$ and does not involve any term containing derivatives of currents. On the other hand, as a general feature of the $\hat{W}_\infty$ algebra (14)-(15), the commutator between densities $H_m$ and $H_n$, led by $u_{m-2}$ and $u_{n-2}$ respectively, does not give rise to the desired $u_{m+n-3}$ term or any term with no derivatives on currents. Therefore, the constant $c$ in eq.(42) must be zero, yielding eq.(24). (As a check, we have explicitly verified that the charges $Q_m$ in eq.(27) are really commuting with each other up to $m = 5$.)

We emphasize that in the above proof starting from eq.(23), besides the OPE’s between the $\hat{W}_\infty$ currents $u_r$, nowhere we have used the two-boson realization (14) or (16) of these currents.

In the present $SL(2, R)/U(1)$ model at level $k=1$, the existence of an infinite set of commuting quantum charges $Q_m$ gives rise to a huge infinite-dimensional symmetry in this model. In fact with the help of these charges we can generate an infinite dimensional compatible flow of the basic bosonic currents $j(z)$ and $\bar{j}(z)$ as follows:

$$\frac{\partial j}{\partial t_m} = [j, Q_m], \quad \frac{\partial \bar{j}}{\partial t_m} = [\bar{j}, Q_m]. \tag{43}$$

It is easy to see that the fundamental OPE’s between $j$ and $\bar{j}$

$$\bar{j}(z)j(w) \sim \frac{1}{(z - w)^2},$$

$$\bar{j}(z)\bar{j}(w) \sim j(z)j(w) \sim 0 \tag{44}$$

are invariant under these flows. For example, for the OPE $\bar{j}j$, we have (at $t_m = 0$)

$$\frac{\partial (\bar{j}j)}{\partial t_m} = [\bar{j}, Q_m]j + j[\bar{j}, Q_m] = [\bar{j}j, Q_m] \sim \frac{1}{(z - w)^2}, Q_m = 0. \tag{45}$$

\footnote{This is obvious from the classical $\hat{W}_\infty$ algebra (5-6) and it appears to survive quantization.}
Moreover, the flows of $j$ and $\bar{j}$ induce similar flows of the composite $\hat{W}_\infty$ currents

$$\frac{\partial u_r}{\partial t_m} = [u_r, Q_m],$$

(46)

under which the $\hat{W}_\infty$ current algebra for $u_r$ is invariant as well. We point out that the commuting charges $Q_m = \oint H_m(z)dz$ are constant along the flows they generate:

$$\frac{\partial Q_n}{\partial t_m} = [Q_n, Q_{m+1}] = 0.$$ 

(47)

Thus the flows (46) they generate are compatible:

$$\frac{\partial^2 u_r}{\partial t_m \partial n} = \frac{\partial^2 u_r}{\partial t_n \partial m}.$$ 

(48)

And eq.(47) implies the integrability of the flows (46).

We observe that the charges we construct in this paper are simple integrals, $\oint H_m(z)dz$, of the Hamiltonian currents $H_m$. Though our spin-2 current $H_2(z)$ is indeed the energy-momentum tensor of the model, however our charge $Q_2$, given by eq.(27), is the charge $L_{-1}$ in usual notation, rather than the usual energy (or the Hamiltonian) $L_0$. Therefore what we have obtained is an infinite set of mutually commuting quantum charges containing $L_{-1}$. Though they are not conserved charges in the model (since they do not commute with $L_0$), they are still interesting charges in that they generate a huge symmetry maintaining the basic current algebra for $j$ and $\bar{j}$ and the composite $\hat{W}_\infty$ current algebra invariant. We speculate that when we go from the conformal model to string theory, these symmetries would turn into an infinite set of compatible $\hat{W}_\infty$ constraints.

To conclude, three more remarks are in order. First, we expect the existence of an infinite set of commuting quantum $\hat{W}_\infty$ charges which include the Hamiltonian $L_0$ of the model. Secondly, it should be possible to construct a Hamiltonian (out of two bosons) for which the set of charges we have constructed in this letter become genuine conserved charges; and we expect such Hamiltonian may represent a perturbed conformal but integrable field theory. We hope to address these problems in future publications.

Finally, the similarity of eq.(46) with the classical KP flows (9) suggests that the former can be viewed as a quantum version or quantum deformation of the latter, so that the flows (46) are eligible to be called as quantum KP flows. The justification needs to go beyond the $k = 1$ case, and is left to another publication of ours [15].
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