Lie Group Structures on Symmetry Groups of Principal Bundles

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Abstract

In this paper we describe how one can obtain Lie group structures on the group of (vertical) bundle automorphisms for a locally convex principal bundle $P$ over the compact manifold $M$. This is done by first considering Lie group structures on the group of vertical bundle automorphisms $\text{Gau}(P)$. Then the full automorphism group $\text{Aut}(P)$ is considered as an extension of the open subgroup $\text{Diff}(M)_P$ of diffeomorphisms of $M$ preserving the equivalence class of $P$ under pull-backs, by the gauge group $\text{Gau}(P)$. We derive explicit conditions for the extensions of these Lie group structures, show the smoothness of some natural actions and relate our results to affine Kac–Moody algebras and groups.

Keywords: infinite-dimensional Lie group; mapping group; gauge group; automorphism group; Kac–Moody group; Kac–Moody algebra

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Introduction

This paper enlarges the scope of infinite-dimensional Lie groups from mapping groups to the group of bundle automorphisms and of vertical bundle automorphisms (shortly called gauge group) of a principal bundle $P$ with compact base $M$. We do this by introducing natural and easy accessible locally convex Lie group structures on these groups, denoted by $\text{Aut}(P)$ and $\text{Gau}(P)$. They are interesting examples of infinite-dimensional Lie groups, since they arise naturally as symmetry groups of gauge field theories. In particular, it is shown that $\text{Aut}(P)$ may also in a Lie theoretic context be interpreted as an extension of (an open subgroup of) $\text{Diff}(M)$ by $\text{Gau}(P)$, such that it describes most naturally gauge field symmetries and space-time symmetries in a unified way. Moreover, for special types of bundles, $\text{Gau}(P)$ may be interpreted as a Kac–Moody group and $\text{Aut}(P)$ as the automorphism group of the corresponding Kac–Moody algebra (cf. Example 2.21). This gives geometric interpretations of these groups and leads to topological information on them.

Lie group topologies on $\text{Gau}(P)$ and $\text{Aut}(P)$ have been considered in the literature, e.g. in [OMYK83, CM85, ACM89, ACMM89, Mi91, KM97], mostly by methods of the Convenient Setting and for finite-dimensional structure groups. These approaches have the common disadvantage that they identify $\text{Gau}(P)$ and $\text{Aut}(P)$ with subsets of larger manifolds of mappings with less structure, making it hard to access the Lie group structure in concrete computations. Unfortunately, the proof given in [KM97, Theorem 42.21] has a serious gap since the chart used in the construction needs to be defined on an $\text{Ad}$-invariant zero neighbourhoods in the Lie algebra of the structure group, which does not exist the generality claimed there. The key advantage
of the approach taken in this paper is that we provide explicit charts for the Lie group structures without any compactness assumptions on the structure group, making it possible to check smoothness conditions directly, as illustrated in case of some natural actions in Proposition 2.15 and Proposition 2.16. In addition, the power of the approach to the Lie group structure on $\text{Gau}(\mathcal{P})$ from Section 1 becomes clear in Section 2, where smoothness conditions for $\text{Gau}(\mathcal{P})$-valued cocycles are explicitly checked. However, it should be emphasised that our interest in these groups is not a gauge-theoretic one but comes from infinite-dimensional Lie theory. This will also become clear from the applications we provide.

We now describe our results in some detail. In the first section we introduce the Lie group structure on $\text{Gau}(\mathcal{P})$, quite similar to the case of mapping groups $C^\infty(M,K)$ as in [PS86] or [Gl02a], but there is a little subtlety. In order to make this approach work we have to impose a technical condition on $\mathcal{P}$, ensuring the compatibility of charts of the structure group $K$ and transition functions of the bundle $\mathcal{P}$. This condition is called “property SUB” and will ensure the constructions to work throughout the whole paper. Our first result is then the following.

**Theorem** (Lie group structure on $\text{Gau}(\mathcal{P})$). Let $\mathcal{P}$ be a smooth principal $K$-bundle over the compact manifold $M$ (possibly with corners). If $\mathcal{P}$ has the property SUB, then the gauge group $\text{Gau}(\mathcal{P}) \cong C^\infty(\mathcal{P},K)^K$ carries a Lie group structure, modelled on $C^\infty(\mathcal{P},K)^K$. If, moreover, $K$ is locally exponential, then $\text{Gau}(\mathcal{P})$ is so.

In the remainder of the section we elaborate on the property SUB and, moreover, show that a broad variety of bundles have this property. Since all prominent classes of infinite-dimensional Lie groups occur as structure groups of bundles having the property SUB it is a natural question whether this is the case for all locally convex Lie groups but this question remains unanswered.

In the second section we enlarge the scope of Lie group structures to the full automorphism group $\text{Aut}(\mathcal{P})$. This is done by considering $\text{Aut}(\mathcal{P})$ as an extension of the open subgroup $\text{Diff}(M)_{\mathcal{P}}$ of diffeomorphisms of $M$ which preserve the equivalence class of $\mathcal{P}$ under pull-backs, by $\text{Gau}(\mathcal{P})$. Using the recently established theory of non-abelian Lie group extensions for locally convex Lie groups from [Ne06a] we obtain our second result.

**Theorem** ($\text{Aut}(\mathcal{P})$ as an extension of $\text{Diff}(M)_{\mathcal{P}}$ by $\text{Gau}(\mathcal{P})$). Let $\mathcal{P}$ be a smooth principal $K$-bundle over the closed compact manifold $M$. If $\mathcal{P}$ has the property SUB, then $\text{Aut}(\mathcal{P})$ carries a Lie group structure such that we have an extension of smooth Lie groups

$$\text{Gau}(\mathcal{P}) \hookrightarrow \text{Aut}(\mathcal{P}) \longrightarrow \text{Diff}(M)_{\mathcal{P}},$$

where $Q : \text{Aut}(\mathcal{P}) \to \text{Diff}(M)$ is the natural homomorphism and $\text{Diff}(M)_{\mathcal{P}}$ is the open subgroup of $\text{Diff}(M)$ preserving the equivalence class of $\mathcal{P}$ under pull-backs.

An interesting thing about this theorem is that it relates two major classes of infinite-dimensional Lie group in a non-trivial way, namely mapping groups, having many ideals from evaluation homomorphisms, and groups of diffeomorphisms, which are perfect (cf. [HT03] and references therein). Moreover, two points are important about the proof of this theorem. The first is that it avoids connections and differential equations and thus also works beyond regularity, taking the property SUB as sole technical requirement. Furthermore, this procedure provides explicit charts for the Lie group structure, making it possible to prove smoothness of natural actions as already mentioned.

In the rest of the section we illustrate the interpretation of $\text{Aut}(\mathcal{P})$ as automorphism group of certain types of Kac–Moody Algebras. As an immediate consequence of the preceding theorem we obtain a long exact homotopy sequence, explaining some of the topological properties of $\text{Aut}(\mathcal{P})$.

1 The Lie group topology on the gauge group

We shall mostly identify the gauge group with the space of $K$-equivariant continuous mappings $C^\infty(\mathcal{P},K)^K$, where $K$ acts on itself by conjugation from the right. This identification allows
us to topologise the gauge group very similar to mapping groups $C^\infty(M, K)$ for compact $M$. Since the compactness of $M$ is the crucial point in the topologisation of mapping groups, we can not take this approach directly, because our structure groups $K$ shall not be compact, even infinite-dimensional.

**Definition 1.1.** If $K$ is a topological group and $\mathcal{P} = (K, \pi : P \to M)$ is a continuous principal $K$-bundle, then we denote by

$$\text{Aut}_c(\mathcal{P}) := \{ f \in \text{Homeo}(P) : \rho_k \circ f = f \circ \rho_k \text{ for all } k \in K \}$$

the group of continuous bundle automorphisms and by

$$\text{Gau}_c(\mathcal{P}) := \{ f \in \text{Aut}_c(\mathcal{P}) : \pi \circ f = \pi \}$$

the group of continuous vertical bundle automorphisms or continuous gauge group. If, in addition, $K$ is a Lie group, $M$ is a manifold with corners and $\mathcal{P}$ is a smooth principal bundle, then we denote by

$$\text{Aut}(\mathcal{P}) := \{ f \in \text{Diff}(P) : \rho_k \circ f = f \circ \rho_k \text{ for all } k \in K \}$$

the the group of smooth bundle automorphisms (or shortly bundle automorphisms). Then each $F \in \text{Aut}(\mathcal{P})$ induces an element $F_M \in \text{Diff}(M)$, given by $F_M(p \cdot k) := F(p) \cdot k$ if we identify $M$ with $P/K$. This yields a homomorphism $Q : \text{Aut}(\mathcal{P}) \to \text{Diff}(M)$, $F \mapsto F_M$ and we denote by $\text{Gau}(\mathcal{P})$ the kernel of $Q$ and by $\text{Diff}(M)_P$ the image of $Q$. Thus

$$\text{Gau}(\mathcal{P}) = \{ f \in \text{Aut}(\mathcal{P}) : \pi \circ f = \pi \},$$

which we call the group of (smooth) vertical bundle automorphisms or shortly the gauge group of $\mathcal{P}$.

**Remark 1.2.** If $\mathcal{P}$ is a smooth principal $K$-bundle and if we denote by

$$C^\infty(P, K)^K := \{ \gamma \in C^\infty(P, K) : \gamma(p \cdot k) = k^{-1} \cdot \gamma(p) \cdot k \text{ for all } p \in P, k \in K \}$$

the group of $K$-equivariant smooth maps from $P$ to $K$, then the map

$$C^\infty(P, K)^K \ni f \mapsto (p \mapsto p \cdot f(p)) \in \text{Gau}(\mathcal{P})$$

is an isomorphism of groups and we will mostly identify $\text{Gau}(\mathcal{P})$ with $C^\infty(P, K)^K$ via this map.

The algebraic counterpart of the gauge group is the gauge algebra. This will serve as the modelling space for the gauge group later on.

**Definition 1.3.** If $\mathcal{P}$ is a smooth principal $K$-bundle, then the space

$$gau(\mathcal{P}) := C^\infty(P, \mathfrak{t})^K := \{ \eta \in C^\infty(P, \mathfrak{t})^K : \eta(p \cdot k) = \text{Ad}(k^{-1}) \cdot \eta(p) \text{ for all } p \in P, k \in K \}$$

is called the gauge algebra of $\mathcal{P}$. We endow it with the subspace topology from $C^\infty(P, \mathfrak{t})$ and with the pointwise Lie bracket.

**Proposition 1.4.** Let $\mathcal{P} = (K, \pi : P \to M)$ be a smooth principal $K$-bundle over the finite-dimensional manifold with corners $M$. If $\mathcal{V} := (V_i, \sigma_i)_{i \in I}$ is a smooth closed trivialising system of $\mathcal{P}$ with transition functions $k_{ij} : V_i \cap V_j \to K$, then we denote

$$g_{\mathcal{V}}(\mathcal{P}) := \left\{ (\eta_i)_{i \in I} \in \prod_{i \in I} C^\infty(V_i, \mathfrak{t}) : \eta_i(m) = \text{Ad}(k_{ij}(m)) \cdot \eta_j(m) \text{ for all } m \in V_i \cap V_j \right\}.$$

If $\mathcal{V}$ denotes the smooth open trivialising system underlying $\mathcal{V}$, then we set

$$g_{\mathcal{V}}(\mathcal{P}) := \left\{ (\eta_i)_{i \in I} \in \prod_{i \in I} C^\infty(V_i, \mathfrak{t}) : \eta_i(m) = \text{Ad}(k_{ij}(m)) \cdot \eta_j(m) \text{ for all } m \in V_i \cap V_j \right\},$$
and we have isomorphisms of topological vector spaces

\[ \mathfrak{gau}(P) = C^\infty(P, \mathfrak{k})^K \cong S(\text{Ad}(P)) \cong \mathfrak{g}_V(P) \cong \mathfrak{g}_T(P). \]

Furthermore, each of these spaces is a locally convex Lie algebra in a natural way and the isomorphisms are isomorphisms of topological Lie algebras.

**Proof.** The last two isomorphisms are provided by Proposition A.15 and Corollary A.16 so we show \( C^\infty(P, \mathfrak{k})^K \cong \mathfrak{g}_T(P). \)

For each \( \eta \in C^\infty(P, \mathfrak{k})^K \) the element \( (\eta_i)_{i \in I} \) with \( \eta_i = \eta \circ \sigma_i \) defines an element of \( \mathfrak{g}_T(P) \) and the map

\[ \psi : C^\infty(P, \mathfrak{k})^K \to \mathfrak{g}_T(P), \quad \eta \mapsto (\eta_i)_{i \in I} \]

is continuous. In fact, \( \sigma_i(m) = \sigma_j(m) \cdot k_{ji}(m) \) for \( m \in V_i \cap V_j \) implies

\[ \eta_i(m) = \eta(\sigma_i(m)) = \eta(\sigma_j(m) \cdot k_{ji}(m)) = \text{Ad}(k_{ji}(m))^{-1} \cdot \eta(\sigma_j(m)) = \text{Ad}(k_{ij}(m)) \cdot \eta_j(m) \]

and thus \( (\eta_i)_{i \in I} \in \mathfrak{g}_T(P) \).

Recall that if \( X \) is a topological space, then a map \( f : X \to C^\infty(V, \mathfrak{k}) \) is continuous if and only if \( x \mapsto d^\kappa f(x) \) is continuous for each \( n \in \mathbb{N}_0 \) (Remark A.9). This implies that \( \psi \) is continuous, because \( d^n \eta_i = d^n \eta \circ T^n \sigma_i \) and pull-backs along continuous maps are continuous.

On the other hand, if \( k_i : \pi^{-1}(V_i) \to K \) is given by \( p = \sigma_i(p) \cdot k_{ij}(p) \) and if \( (\eta_i)_{i \in I} \in \mathfrak{g}_T(P) \), then the map

\[ \eta : P \to \mathfrak{k}, \quad p \mapsto \text{Ad}(k(p))^{-1} \cdot \eta_i(\pi(p)) \quad \text{if} \quad \pi(p) \in V_i \]

is well-defined, smooth and \( K \)-equivariant. Furthermore, \( (\eta_i)_{i \in I} \mapsto \eta \) is an inverse of \( \psi \) and it thus remains to check that it is continuous, i.e.,

\[ \mathfrak{g}_T(P) \ni (\eta_i)_{i \in I} \mapsto d^n \eta \in C(T^n P, \mathfrak{k}) \]

is continuous for all \( n \in \mathbb{N}_0 \). If \( C \subseteq \mathcal{T}^n P \) is compact, then \( (\mathcal{T}^n \pi)(C) \subseteq \mathcal{T}^n M \) is compact and hence it is covered by finitely many \( \mathcal{T}^n V_{i_1}, \ldots, \mathcal{T}^n V_{i_m} \) and thus \( (T^n \pi^{-1}(V_i))_{i = i_1, \ldots, i_m} \) is a finite closed cover of \( C \subseteq \mathcal{T}^n P \). Hence it suffices to show that the map

\[ \mathfrak{g}_T(P) \ni (\eta_i)_{i \in I} \mapsto T^n(\eta_{\pi^{-1}(V_i)}) \in C(T^n \pi^{-1}(V_i), \mathfrak{k}) \]

is continuous for \( n \in \mathbb{N}_0 \) and \( i \in I \) and we may thus w.l.o.g. assume that \( \mathcal{T} \) is trivial. In the trivial case we have \( \eta = \text{Ad}(k^{-1}) \cdot (\eta \circ \pi) \) if \( p \mapsto (\pi(p), k(p)) \) defines a global trivialisation. We shall make the case \( n = 1 \) explicit. The other cases can be treated similarly and since the formulae get quite long we skip them here.

Given any open zero neighbourhood in \( C(T P, \mathfrak{k}) \), which we may assume to be \([C, V]\) with \( C \subseteq \mathcal{T} P \) compact and \( 0 \in V \subseteq \mathfrak{k} \) open, we have to construct an open zero neighbourhood \( O \) in \( C^{\infty}(M, \mathfrak{k}) \) such that \( \varphi(O) \subseteq [C, V] \). For \( \eta' \in C^{\infty}(M, \mathfrak{k}) \) and \( X_p \in C \) we get with Lemma A.19

\[ d(\varphi(\eta'))(X_p) = \text{Ad}(k^{-1}(p)) \cdot d\eta'(T \pi(X_p)) - [\delta'(k)(X_p), \text{Ad}(k^{-1}(p)) \cdot \eta'(\pi(p))]. \]

Since \( \delta'(C) \subseteq \mathfrak{k} \) is compact, there exists an open zero neighbourhood \( V' \subseteq \mathfrak{k} \) such that

\[ \text{Ad}(k^{-1}(p)) \cdot V' + [\delta'(k)(X_p), \text{Ad}(k^{-1}(p)) \cdot V'] \subseteq V \]

for each \( X_p \in C \). Since \( T \pi : T P \to T M \) is continuous, \( T \pi(C) \) is compact and we may set \( O = [T \pi(C), V'] \).

That \( \mathfrak{g}_V(P) \) and \( \mathfrak{g}_T(P) \) are locally convex Lie algebras follows because they are closed subalgebras of \( \prod_{i \in I} C^\infty(V_i, \mathfrak{k}) \) and \( \prod_{i \in I} C^\infty(V_i, \mathfrak{k}) \). Since the isomorphisms

\[ C^\infty(P, \mathfrak{k})^K \cong S(\text{Ad}(P)) \cong \mathfrak{g}_V(P) \cong \mathfrak{g}_T(P). \]

are all isomorphisms of abstract Lie algebras an isomorphisms of locally convex vector spaces, it follows that they are isomorphisms of topological Lie algebras. \( \square \)
Definition 1.5. If \( P \) is a smooth \( K \)-principal bundle with compact base and \( \overline{V} = (V_i, \sigma_i)_{i=1,\ldots,n} \) is a smooth closed trivialising system with corresponding transition functions \( k_{ij} : V_i \cap V_j \to K \), then we denote
\[
G_{\overline{V}}(P) := \left\{ (\gamma_i)_{i=1,\ldots,n} \in \prod_{i=1}^n C^\infty(V_i, K) : \gamma_i(m) = k_{ij}(m)\gamma_jk_{ji}(m) \text{ for all } m \in V_i \cap V_j \right\}
\]
and turn it into a group with respect to pointwise group operations.

Remark 1.6. In the situation of Definition 1.3, the map
\[
\psi : G_{\overline{V}}(P) \to C^\infty(P, K)^K, \quad \psi((\gamma_i)_{i=1,\ldots,n})(p) = k_{\sigma_i}^{-1}(p) \cdot \gamma_i(p) \cdot k_{\sigma_i}(p)
\]
is an isomorphism of abstract groups, where the map on the right hand side is well-defined because \( k_{\sigma_i}(p) = k_{ij}(\pi(p)) \cdot k_{\sigma_j}(p) \) and thus
\[
k_{\sigma_i}^{-1}(p) \cdot \gamma_i(\pi(p)) \cdot k_{\sigma_j}(p) = k_{\sigma_j}(p)^{-1} \cdot k_{ij}(\pi(p)) \cdot \gamma_i(\pi(p)) \cdot k_{ij}(\pi(p)) \cdot k_{\sigma_j}(p) = k_{\sigma_j}(p)^{-1} \cdot \gamma_j(\pi(p)) \cdot k_{\sigma_j}(p).
\]
In particular, this implies that \( \psi((\gamma_i)_{i=1,\ldots,n}) \) is smooth. Since for \( m \in V_i \) the evaluation map \( ev_m : C^\infty(V_i, K) \to K \) is continuous, \( G_{\overline{V}}(P) \) is a closed subgroup of \( \prod_{i=1}^n C^\infty(V_i, K) \).

Since an infinite-dimensional Lie group may possess closed subgroups which are no Lie groups (cf. Exercise III.8.2), the preceding remark does not automatically yield a Lie group structure on \( G_{\overline{V}}(P) \). However, in many situations, it will turn out that \( G_{\overline{V}}(P) \) has a natural Lie group structure.

The following definition encodes the necessary requirement ensuring a Lie group structure on \( G_{\overline{V}}(P) \) that is induced by the natural Lie group structure on \( \prod_{i=1}^n C^\infty(V_i, K) \). Since quite different properties of \( P \) will ensure this requirement it seems to be worth extracting it as a condition on \( P \). The name for this requirement will be justified in Corollary 1.10.

Definition 1.7. If \( P \) is a smooth principal \( K \)-bundle with compact base and \( \overline{V} = (V_i, \sigma_i)_{i=1,\ldots,n} \) is a smooth closed trivialising system, then we say that \( P \) has the property \( \text{SUB} \) with respect to \( \overline{V} \) if there exists a convex centred chart \( \varphi : W \to W' \) of \( K \) such that
\[
\varphi_* : G_{\overline{V}}(P) \cap \prod_{i=1}^n C^\infty(V_i, W) \to g_{\overline{V}}(P) \cap \prod_{i=1}^n C^\infty(V_i, W'), \quad (\gamma_i)_{i=1,\ldots,n} \mapsto (\varphi \circ \gamma_i)_{i=1,\ldots,n}
\]
is bijective. We say that \( P \) has the property \( \text{SUB} \) if \( P \) has this property with respect to some trivialising system.

It should be emphasised that in all relevant cases, known to the author, the bundles have the property \( \text{SUB} \), and it is still unclear, whether there are bundles, which do not have this property (cf. Lemma 1.14 and Remark 1.15). This property now ensures the existence of a natural Lie group structure on \( G_{\overline{V}}(P) \).

Proposition 1.8. a) Let \( P \) be a smooth principal \( K \)-bundle with compact base \( M \), which has the property \( \text{SUB} \) with respect to the smooth closed trivialising system \( \overline{V} \). Then \( \varphi_* \) induces a smooth manifold structure on \( G_{\overline{V}}(P) \cap \prod_{i=1}^n C(V_i, W) \). Furthermore, the conditions i) - iii) of Proposition 1.4 are satisfied such that \( G_{\overline{V}}(P) \) can be turned into a Lie group modelled on \( g_{\overline{V}}(P) \).

b) In the setting of a), the map \( \psi : G_{\overline{V}}(P) \to C^\infty(P, K)^K \) is an isomorphism of topological groups if \( C^\infty(P, K)^K \) is endowed with the subspace topology from \( C^\infty(P, K) \).

c) In the setting of a), we have \( L(G_{\overline{V}}(P)) \cong g_{\overline{V}}(P) \).
Proposition A.4. We show next that in fact, different choices of trivialising systems also lead to Proposition 1.10. Let $f$ be continuous as a composition of a pullback and the map trivialising systems. First, we note that if the covers underlying $g$ are open in $\mathbb{R}^n$ then $f$ is trivialised. So let $V_i$ be an open unit neighbourhood with $V_i \subseteq V$ and $V_i \cap V_j = \emptyset$. Hence

$$(\gamma_i)_{i=1,...,n} \cdot (G(\mathbb{P}) \cap (U_1 \times \cdots \times U_n)) \cdot (\gamma_i^{-1})_{i=1,...,n} \subseteq U$$

and conditions i) – iii) of Proposition [A.4] are satisfied, where the required smoothness properties are consequences of the smoothness of push forwards of mappings between function spaces (cf. [Wo06c Proposition 28 and Corollary 29] and [Gl02b, Section 3.2]).

b) We show that the map $\psi : G(\mathbb{P}) \to C^\infty(P, K)$ from $\prod_i$ is a homeomorphism. Let $\mathcal{P}\mathcal{P} := \mathcal{P}$ be the restricted bundle. Since $T^nV_i$ is closed in $T^nM$, we have that $C^\infty(P, K)$ is homeomorphic to

$$\tilde{G}(\mathbb{P}) := \{(\tilde{\gamma}_i)_{i=1,...,n} \mid \prod_i C^\infty(P_i, K) : \tilde{\gamma}_i(p) = \tilde{\gamma}_j(p) \text{ for all } p \in \pi^{-1}(V_i \cap V_j)\}$$

as in Proposition [A.15]. With respect to this identification, $\psi$ is given by

$$(\gamma_i)_{i=1,...,n} \mapsto (k_{\sigma_i}^{-1} \cdot \gamma_i \circ \pi \cdot k_{\sigma_i})_{i=1,...,n}$$

and it thus suffices to show the assertion for trivial bundles. So let $\sigma : M \to P$ be a global section. The map $C^\infty(M, K) \ni f \mapsto f \circ \pi \in C^\infty(P, K)$ is continuous since

$$C^\infty(M, K) \ni f \mapsto T^k(f \circ \pi) = T^k f \circ T^k \pi = (T^k \pi)_* (T^k f) \in C(T^kP, T^kK)$$

is continuous as a composition of a pullback and the map $f \mapsto T^k f$, which defines the topology on $C^\infty(M, K)$. Since conjugation in $C^\infty(P, K)$ is continuous, it follows that $\varphi$ is continuous. Since the map $f \mapsto f \circ \sigma$ is also continuous (with the same argument), the assertion follows.

c) This follows immediately from $\text{L}(C^\infty(\mathcal{V}, K)) \cong C^\infty(\mathcal{V}, \mathfrak{t})$ (cf. [Gl02b, Section 3.2]).

The next corollary is a mere observation. Since it justifies the name “property SUB”, it is made explicit here.

Corollary 1.9. If $\mathcal{P}$ is a smooth principal $K$-bundle with compact base $M$, having the property SUB with respect to the smooth closed trivialising system $\mathcal{V}$, then $G(\mathbb{P})$ is a closed subgroup of $\prod_i C^\infty(\mathcal{V}_i, K)$, which is a Lie group modelled on $\mathfrak{g}(\mathbb{P})$.

That different choices of charts lead to isomorphic Lie group structures follows directly from Proposition [A.4]. We show next that in fact, different choices of trivialising systems also lead to isomorphic Lie group structures on $\text{Gau}(\mathcal{P})$.

Proposition 1.10. Let $\mathcal{P}$ be a smooth principal $K$-bundle with compact base. If we have two trivialising systems $\mathcal{V} = (\mathcal{V}_i, \sigma_i)_{i=1,...,n}$ and $\mathcal{U} = (\mathcal{U}_j, \tau_j)_{j=1,...,m}$ and $\mathcal{P}$ has the property $\text{SUB}$ with respect to $\mathcal{V}$ and $\mathcal{U}$, then $G(\mathcal{P})$ is isomorphic to $G(\mathcal{P})$ as a Lie group.

Proof. First, we note that if the covers underlying $\mathcal{V}$ and $\mathcal{U}$ are the same, but the sections differ by smooth functions $k_i \in C^\infty(\mathcal{V}_i, K)$, i.e., $\sigma_i = \tau_i \cdot k_i$, then this induces an automorphism of Lie groups

$$G(\mathcal{P}) \to G(\mathcal{P}), \quad (\gamma_i)_{i=1,...,n} \mapsto (k_i^{-1} \cdot \gamma_i \cdot k_i)_{i=1,...,n},$$

because conjugation with $k_i^{-1}$ is an automorphism of $C^\infty(\mathcal{V}_i, K)$.

Since each two open covers have a common refinement it suffices to show the assertion if one cover is a refinement of the other. So let $V_1, \ldots, V_n$ be a refinement of $U_1, \ldots, U_m$ and let $\{1, \ldots, n\} \ni i \mapsto j(i) \in \{1, \ldots, m\}$ be a function with $V_i \subseteq U_{j(i)}$. Since different choices
of sections lead to automorphisms we may assume that $\sigma_i = \sigma_{j(i)}|_{\mathbf{U}_i}$, implying in particular $k_{ij}(m) = k_{j(i)(i)}(m)$. Then the restriction map from Lemma 1.7 yields a smooth homomorphism

$$\psi : G_{\mathbf{U}_i}(\mathcal{P}) \to G_{\mathbf{U}_i}(\mathcal{P}), \quad (\gamma_j)_{j \in J} \mapsto (\gamma_j|_{\mathbf{U}_i})_{i \in I}.$$  

For $\psi^{-1}$ we construct each component $\psi^{-1}_j : G_{\mathbf{U}_i}(\mathcal{P}) \to C^\infty(\mathbf{U}_j, K)$ separately. The condition that $(\psi^{-1}_j)_{j \in J}$ is inverse to $\psi$ is then

$$(2) \quad \psi^{-1}_j((\gamma_i)_{i \in I}|_{\mathbf{U}_i}) = \gamma_i \text{ for all } i \text{ with } j = j(i).$$

Set $I_j := \{i \in I : \mathbf{V}_i \subseteq \mathbf{U}_j\}$ and note that $j(i) = j$ implies $i \in I_j$. Since a change of the sections $\sigma_i$ induces an automorphism on $G_{\mathbf{U}_i}(\mathcal{P})$ we may assume that $\sigma_i = \sigma_{j(i)}|_{\mathbf{U}_i}$ for each $i \in I_j$. Let $x \in \mathbf{U}_j \setminus \bigcup_{i \in I_j} \mathbf{V}_i$. Then $x \in \mathbf{V}_i$ for some $i \in I$ and thus there exists an open neighbourhood $U_x$ of $x$ such that $\mathbf{U}_x$ is a manifold with corners, contained in $\mathbf{U}_j \cap \mathbf{V}_i$. Now finitely many $U_{x_1}, \ldots, U_{x_n}$ cover $\mathbf{U}_j \setminus \bigcup_{i \in I_j} \mathbf{V}_i$ and we set

$$\psi^{-1}_j((\gamma_i)_{i \in I}|_{\mathbf{U}_i}) = \text{glue} \left( (\gamma_i)_{i \in I_j}, \left( (k_{j(i)k}, \cdot \gamma_{i_k}, \cdot k_{i_k})|_{U_{x_k}} \right)_{k = 1, \ldots, n} \right).$$

Then this defines a smooth map by Proposition 1.8 and (2) is satisfied because $j(i) = i$ implies $i \in I_j$. 

We now come to the main result of this section.

**Theorem 1.11 (Lie group structure on $\text{Gau}(\mathcal{P})$).** Let $\mathcal{P}$ be a smooth principal $K$-bundle over the compact manifold $M$ (possibly with corners). If $\mathcal{P}$ has the property SUB, then the gauge group $\text{Gau}(\mathcal{P}) \cong C^\infty(\mathcal{P}, K)^K$ carries a Lie group structure, modelled on $C^\infty(\mathcal{P}, \mathfrak{t})^K$. If, moreover, $K$ is locally exponential, then $\text{Gau}(\mathcal{P})$ is so.

**Proof.** We endow $\text{Gau}(\mathcal{P})$ with the Lie group structure induced from the isomorphisms of groups $\text{Gau}(\mathcal{P}) \cong C^\infty(\mathcal{P}, K)^K \cong G_{\mathbf{U}_i}(\mathcal{P})$ for some smooth closed trivialising system $\mathbf{U}_i$. To show that $\text{Gau}(\mathcal{P})$ is locally exponential if $K$ is so we first show that if $M$ is a compact manifold with corners and $K$ has an exponential function, then

$$(\exp_K)_* : C^\infty(M, \mathfrak{t}) \to C^\infty(M, K), \quad \eta \mapsto \exp_K \circ \eta$$

is an exponential function for $C^\infty(M, K)$. For $x \in \mathfrak{t}$ let $\gamma_x \in C^\infty(\mathbb{R}, K)$ be the solution of the initial value problem $\gamma(0) = e, \gamma'(t) = \gamma(t).x$. Take $\eta \in C^\infty(M, \mathfrak{t})$. Then

$$\Gamma_\eta : \mathbb{R} \to C^\infty(M, K), \quad (t, m) \mapsto \gamma_x(m)(t) = \exp_K(t \cdot \eta(m))$$

is a homomorphism of abstract groups. Furthermore, $\Gamma_\eta$ is smooth, because it is smooth on a zero neighbourhood of $\mathbb{R}$, for the push-forward of the local inverse of $\exp_K$ provide charts on a unit neighbourhood in $C^\infty(M, K)$. Then

$$\delta'(\Gamma_\eta)(t) = \Gamma_\eta(t)^{-1} \cdot \Gamma'(t) = \Gamma_\eta(t)^{-1} \cdot \Gamma_\eta(t) \cdot \eta = \eta,$$

thought of as an equation in the Lie group $T(C^\infty(M, K)) \cong C^\infty(M, \mathfrak{t}) \times C^\infty(M, K)$, shows that $\eta \mapsto \Gamma_\eta(1) = \exp_K \circ \gamma$ is an exponential function for $C^\infty(M, K)$. The proof of the preceding lemma yields immediately that

$$G_{\mathbf{U}_i}(\mathcal{P}) \cap \prod_{i=1}^n C^\infty(\mathbf{V}_i, W_i) \to G_{\mathbf{U}_i}(\mathcal{P}), \quad (\eta_i)_{i=1,\ldots,n} \mapsto (\exp_K \circ \eta_i)_{i=1,\ldots,n}$$

is a diffeomorphism and thus $\text{Gau}(\mathcal{P})$ is locally exponential. 

\qed
It remains to elaborate on the arcane property SUB. First we shall see that this property behaves well with respect to refinements of trivialising systems.

**Lemma 1.12.** Let $\mathcal{P}$ be a smooth principal $K$-bundle with compact base and $\overline{\mathcal{V}} = (\overline{V}_i, \overline{\sigma}_i)_{i=1,\ldots,m}$ be a smooth closed trivialising system of $\mathcal{P}$. If $\overline{U} = (\overline{U}_j, \overline{\tau}_j)_{j=1,\ldots,m}$ is a refinement of $\overline{\mathcal{V}}$, then $\mathcal{P}$ has the property SUB with respect to $\overline{\mathcal{V}}$ if and only if $\mathcal{P}$ has the property SUB with respect to $\overline{U}$.

**Proof.** Let $\{1, \ldots, m\} \ni j \mapsto i(j) \in \{1, \ldots, n\}$ be a map such that $U_j \subseteq V_{i(j)}$ and $\tau_j = \sigma_{i(j)}|_{\overline{U}_j}$. Then we have bijective mappings

$$\psi_G : G_{\overline{\mathcal{V}}}(\mathcal{P}) \to G_{\overline{\mathcal{U}}}(\mathcal{P}), \quad (\gamma_i)_{i=1,\ldots,n} \mapsto (\gamma_{i(j)})_{j=1,\ldots,m}$$

$$\psi_\theta : g_{\overline{\mathcal{V}}}(\mathcal{P}) \to g_{\overline{\mathcal{U}}}(\mathcal{P}), \quad (\eta_i)_{i=1,\ldots,n} \mapsto (\eta_{i(j)})_{j=1,\ldots,m}\,.$$ (cf. Proposition 1.10). Now let $\varphi : W \to W'$ be an arbitrary convex centred chart of $K$ and set

$$Q := G_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^n C(\overline{V}_i, W), \quad \tilde{Q} := G_{\overline{\mathcal{U}}}(\mathcal{P}) \cap \prod_{i=1}^n C(\overline{U}_i, W)$$

$$Q' := g_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^n C(\overline{V}_i, W'), \quad \tilde{Q}' := g_{\overline{\mathcal{U}}}(\mathcal{P}) \cap \prod_{i=1}^n C(\overline{U}_i, W').$$

Then we have $\psi_G(Q) = \tilde{Q}$ and $\psi_\theta(Q') = \tilde{Q}'$ and the assertion follows from the commutative diagram

$$\begin{array}{ccc}
Q & \xrightarrow{\varphi_*} & Q' \\
\downarrow \psi_G & & \downarrow \psi_\theta \\
\tilde{Q} & \xrightarrow{\tilde{\varphi}'} & \tilde{Q}'.
\end{array}$$

The following lemma will be needed later in the proof that bundles with structure group a direct limit Lie group have the property SUB.

**Lemma 1.13.** Let $C$ be a compact Lie group, $M, N$ be smooth finite-dimensional manifolds with $N \subseteq M$ and assume that the inclusion $i : M \to N$ is a smooth immersion. Furthermore, let $C$ act on $N$ from the right such that $M \cdot C = M$ and that $x \in M$ is fix-point. Then $C$ acts on $T_x M$ invariant.

If there exists a $C$-invariant smoothly and $C$-equivariantly contractible relatively compact open neighbourhood $U$ of $x$ and a $C$-equivariant chart $\varphi : U \to \tilde{U} \subseteq T_x M$ with $\varphi(x) = 0$, then there exists a $C$-equivariant chart $\psi : V \to \tilde{V} \subseteq T_x M$ such that $V$ is a $C$-invariant smoothly and $C$-equivariantly contractible relatively compact open neighbourhood of $x$ in $M$, satisfying $V \cap M = U$, $\psi(V) \cap T_x M = \varphi(U)$ and $\psi|_U = \varphi$.

**Proof.** (cf. [G05] Lemma 2.1 for the non-equivariant case) Fix a $C$-invariant metric on $TN$, inducing a $C$-invariant metric on $N$ (c.f. [B72] Section VI.2]). Then $C$ acts on $N$ by isometries. As in [G05] Lemma 2.1 we find a $\sigma$-compact, relatively compact open submanifold $W'$ of $N$ such that $W' \cap \overline{U} = U$, whence $U$ is a closed submanifold of $W'$. We now set $W := \cup_{c \in C} W' \cdot c$. Since $C$ acts by isometries, this is an open $C$-invariant subset of $N$ and we deduce that we still have

$$W \cap \overline{U} = (\cup_{c \in C} W' \cdot c) \cap (\cup_{c \in C} \overline{U} \cdot c) = \cup_{c \in C} \left( (W' \cap \overline{U}) \cdot c \right) = U.$$

By shrinking $W$ if necessary we thus get an open $C$-invariant relatively compact submanifold of $M$ with $U$ as closed submanifold.

By [B72] Theorem VI.2.2, $U$ has an open $C$-invariant tubular neighbourhood in $W$, i.e., there exists a $C$-vector bundle $\xi : E \to U$ and a $C$-equivariant diffeomorphism $\Phi : E \to W$ onto some $C$-invariant open neighbourhood $\Phi(E)$ of $W$ such that the restriction of $\Phi$ to the zero section $U$ is the inclusion of $U$ in $W$. The proof of [B72] Theorem 2.2 shows that $E$ can be
taken to be the normal bundle of \( U \) with the canonical \( C \)-action, which is canonically isomorphic to the \( C \)-invariant subbundle \( TU^\perp \). We will thus identify \( E \) with \( TU^\perp \) from now on.

That \( U \) is smoothly and \( C \)-equivariantly contractible means that there exists a homotopy \( F : [0, 1] \times U \to U \) such that each \( F(t, \cdot) : U \to U \) is smooth and \( C \)-equivariant and that \( F(1, \cdot) \) is the map which is constantly \( x \) and \( F(0, \cdot) = id_U \). Pulling back \( E \) along the smooth and equivariant map \( F(1, \cdot) \) gives the \( C \)-vector bundle \( \text{pr}_1 : U \times T_xU^\perp \to U \), where the action of \( C \) on \( U \) is the one given by assumption and the action of \( C \) on \( T_xU^\perp \) is the one induced from the canonical action of \( C \) on \( T_xM \). By [Wa69, Corollary 2.5], \( F(1, \cdot)^*(TU^\perp) \) and \( F(0, \cdot)^*(TU^\perp) \) are equivalent \( C \)-vector bundles and thus there exists a smooth \( C \)-equivariant bundle equivalence \( \Psi : TU^\perp = F(1, \cdot)^*(TU^\perp) \to U \times T_xU^\perp = F(0, \cdot)^*(TU^\perp) \).

We now define
\[
\psi : V := \Phi(TU^\perp) \to T_xN, \quad y \mapsto \varphi \left( \Psi_1(\Phi^{-1}(y)) \right) + \Psi_2(\Phi^{-1}(y)),
\]
where \( \Psi_1 \) and \( \Psi_2 \) are the components of \( \Psi \). Since \( \Phi \), \( \Psi \) and \( \varphi \) are \( C \)-equivariant so is \( \psi \) and it is a diffeomorphism onto the open subset \( U \times T_xU^\perp \) because \( T_xN = T_xM \oplus T_xU^\perp \). This yields a \( C \)-equivariant chart. Moreover, \( V \) is relatively compact as a subset of \( W \). Furthermore, if we denote by \( \psi_1 \) the \( T_xM \)-component and by \( \psi_2 \) the \( T_xU^\perp \)-component of \( \psi \), then
\[
[0, 1] \times V \to V, \quad (t, y) \mapsto (\psi_1(y), t \cdot \psi_2(y))
\]
defines a smooth \( C \)-equivariant homotopy from \( \text{id}_V \) to the projection \( V \to U \). Composing this homotopy with the smooth \( C \)-equivariant homotopy from \( \text{id}_V \) to the map which is constantly \( x \) yields the asserted homotopy. Since \( V \cap M \) is the zero section in \( TU^\perp |_{U'} \), we have \( V \cap M = U \). Furthermore, we have \( \psi(V) \cap T_xM = \varphi(U) \), because \( \Phi \) and \( \Psi \) restrict to the identity on the zero section. This also implies \( \psi|_{U'} = \varphi \). We thus have checked all requirements from the assertion. \( \square \)

Although it is presently unclear, which bundles have the property \( \text{SUB} \) and which not, we shall now see that \( \mathcal{P} \) has the property \( \text{SUB} \) in many interesting cases, covering large classes of presently known locally convex Lie groups.

**Lemma 1.14.** Let \( \mathcal{P} \) be a smooth principal \( K \)-bundle over the compact manifold with corners \( M \).

a) If \( \mathcal{P} \) is trivial, then there exists a global smooth trivialising system and \( \mathcal{P} \) has the property \( \text{SUB} \) with respect to each such system.

b) If \( K \) is abelian, then \( \mathcal{P} \) has the property \( \text{SUB} \) with respect to each smooth closed trivialising system.

c) If \( K \) is a Banach–Lie group, then \( \mathcal{P} \) has the property \( \text{SUB} \) with respect to each smooth closed trivialising system.

d) If \( K \) is locally exponential, then \( \mathcal{P} \) has the property \( \text{SUB} \) with respect to each smooth closed trivialising system.

e) If \( K \) is a countable direct limit of finite-dimensional Lie groups in the sense of [Gl05], then there exists a smooth closed trivialising system such that the corresponding transition functions take values in a compact subgroup \( C \) of some \( K_i \) and \( \mathcal{P} \) has the property \( \text{SUB} \) with respect to each such system.

**Proof.**

a) If \( \mathcal{P} \) is trivial, then there exists a global section \( \sigma : M \to \mathcal{P} \) and thus \( \mathcal{V} = (M, \sigma) \) is a trivialising system of \( \mathcal{P} \). Then \( \mathcal{C}(\mathcal{P}) = C^\infty(M, K) \) and \( \varphi_* \) is bijective for any convex centred chart \( \varphi : W \to W' \).

b) If \( K \) is abelian, then the conjugation action of \( K \) on itself and the adjoint action of \( K \) on \( \mathfrak{k} \) are trivial. Then a direct verification shows that \( \varphi_* \) is bijective for any trivialising system \( \mathcal{V} \) and any convex centred chart \( \varphi : W \to W' \).
c) If $K$ is a Banach–Lie group, then it is in particular locally exponential (cf. Remark \[A.7\]) and it thus suffices to show d).

d) Let $K$ be locally exponential and $\overline{\mathcal{V}} = (\overline{V}_i, \sigma_i)_{i=1,\ldots,n}$ be a trivialising system. Furthermore, let $W' \subseteq \mathfrak{k}$ be an open zero neighbourhood such that $\exp_K$ restricts to a diffeomorphism on $W'$ and set $W = \exp(W')$ and $\varphi := \exp^{-1} : W \to W'$. Then we have

$$(\gamma_i)_{i=1,\ldots,n} \in G(\mathcal{P}) \cap \bigcap_{i=1}^n C(\overline{V}_i, W) \iff \varphi_*((\gamma_i)_{i=1,\ldots,n}) \in G(\mathcal{P}) \cap \bigcap_{i=1}^n C(\overline{V}_i, W'),$$

because $\exp_K(\Ad(k).x) = k \cdot \exp_K(x) \cdot k^{-1}$ holds for all $k \in K$ and $x \in W'$ (cf. Lemma \[A.6\]). Furthermore, $(\eta_i)_{i=1,\ldots,n} \mapsto (\exp \circ \eta_i)_{i=1,\ldots,n}$ provides an inverse to $\varphi_*$.  

e) Let $K$ be a direct limit of the countable direct system $\mathcal{S} = ((K_i)_{i \in I}, (\lambda_{ij})_{i,j \geq 1})$ of finite-dimensional Lie groups $K_i$ and Lie group morphisms $\lambda_{ij} : K_j \to K_i$ with $\lambda_{ij} \circ \lambda_{jk} = \lambda_{ik}$ if $i \geq j \geq 0$. Then there exists an associated injective quotient system $((K_i)_{i \in I}, (\lambda_{ij})_{i,j \geq 1})$ with $K_i = K_i / N_i$, where $N_i = \bigcup_{j \geq 1} \ker \lambda_{ij}$ and $\lambda_{ij} : K_j \to K_i$ is determined by $\lambda_{ij} \circ \lambda_{jk} = \lambda_{ik} \circ q_j$ for the quotient map $q_i : K_i \to K_i$. In particular, each $\lambda_{ij}$ is an injective immersion.

After passing to a cofinal subsequence of an equivalent direct system (cf. [Gl05, §1.6]), we may without loss of generality assume that $I = N$, that $K_1 \subseteq K_2 \subseteq \ldots$ and that the inclusions are the inclusion maps. Then a chart $\varphi : W \to W'$ of $K$ around $e$ is the direct limit of a sequence of charts $(\varphi_i : W_i \to W'_i)_{i \in \mathbb{N}}$ such that $W = \bigcup_{i \in \mathbb{N}} W_i$, $W' = \bigcup_{i \in \mathbb{N}} W'_i$ and that $\varphi_i | W_i = \varphi_j$ if $i \geq j$ (cf. [Gl05, Theorem 3.1]).

Now, let $\overline{\mathcal{V}} = (\overline{V}_i, \overline{\sigma}_i)_{i=1,\ldots,n}$ be a smooth closed trivialising system of $\mathcal{P}$. Then the corresponding transition functions are defined on the compact subset $\overline{V}_i \cap \overline{V}_j$ and thus take values in a compact subset of $K$. Since each compact subset of $K$ is entirely contained in one of the $K_i$ (cf. [Gl05, Lemma 1.7]), the transition functions take values in some $K_a$. Since each finite-dimensional principal $K_a$-bundle can be reduced to a $C$-bundle, where $C \subseteq K_a$ is the maximal compact subgroup of $K_a$ (cf. [St51, p. 59]), we find smooth mappings $f_i : \overline{V}_i \to K_i$ such that $\overline{\sigma}_i = f_i$ is a smooth closed trivialising system of $\mathcal{P}$ and the corresponding transition functions take values in the compact Lie group $C$.

We now define a chart $\varphi : W \to W'$ satisfying the requirements of Definition \[L.7\]. For $i < a$ let $W_i$ and $W'_i$ be empty. For $i = a$ denote $\mathfrak{t}_a := L(K_a)$ and let $\exp_a : \mathfrak{t}_a \to K_a$ be the exponential function of $K_a$. Now $C$ acts on $K_a$ from the right by conjugation and on $\mathfrak{t}_a$ by the adjoint representation, which is simply the induced action on $T_a K_a$ for the fixed-point $e \in K_a$. By Lemma \[A.6\] we have $\exp_a(\Ad(e).x) = c \cdot \exp_a(x) \cdot c^{-1}$ for each $c \in C$. We choose a $C$-invariant metric on $\mathfrak{t}_a$. Then there exists an $\varepsilon > 0$ such that $\exp_a$ restricts to a diffeomorphism on the open $\varepsilon$-ball $W'_a$ around $0 \in \mathfrak{t}_a$ in the chosen invariant metric. Then

$$\varphi_a := (\exp_a | W'_a)^{-1} : W_a := \exp_a(W'_a) \to W'_a, \quad \exp_a(x) \mapsto x$$

defines an equivariant chart of $K_a$ for the corresponding $C$-actions on $K_a$ and $\mathfrak{t}_a$, and, moreover, we may choose $W'_a$ so that $\exp_a(W'_a)$ is relatively compact in $K_a$. In addition,

$$[0, 1] \times W_a \ni (t, k) \mapsto \exp_a(t \cdot \varphi(k)) \in W_a$$

defines a smooth $C$-equivariant contraction of $W_a$. By Lemma \[L.13\] we may extend $\varphi_a$ to an equivariant chart $\varphi_{a+1} : W_{a+1} \to W'_{a+1}$ with $W_{a+1} \cap K_a = W_a \varphi_{a+1}|W_a = \varphi_a$ such that $W_{a+1}$ is relatively compact in $K_{a+1}$ and smoothly and $G$-equivariantly contractible. Proceeding in this way we define $G$-equivariant charts $\varphi_i$ for $i \geq a$.

This yields a direct limit chart $\varphi := \lim_{a} \varphi_i : W \to W'$ of $K$ for which we have $W = \bigcup_{i \in I} W_i$ and $W' = \bigcup_{i \in I} W'_i$. Since the action of $C$ on $T_a K_i = \mathfrak{t}_a$ is the induced action in each step and the construction yields $\varphi_i(c^{-1} \cdot k \cdot c) = \Ad(c^{-1}) \cdot \varphi_i(k)$ we conclude that we have

$$\varphi^{-1}(\Ad(c^{-1}).x) = c^{-1} \cdot \varphi^{-1}(x) \cdot c \quad \text{for all} \quad x \in W' \quad \text{and} \quad c \in C$$

(note that $\exp$ is not an inverse to $\varphi$ any more). Since the transition functions of the trivialising system $\overline{\mathcal{V}} := (\overline{\sigma}_i, \overline{V}_i)$ take values in $C$, we may proceed as in d) to see that $\varphi_*$ is bijective. \[\square\]
Remark 1.15. The preceding lemma shows that there are different kinds of properties of $P$ that can ensure the property SUB, i.e., topological in case a), algebraical in case b) and geometrical in case d). Case e) is even more remarkable, since it provides examples of principal bundle with the property SUB, whose structure groups are not locally exponential in general (c.f. [Gl05, Remark 4.7]). It thus seems to be hard to find a bundle which does not have this property. However, a more systematic answer to the question which bundles have this property is not available at the moment.

Problem 1.16. Is there a smooth principal $K$-bundle $P$ over a compact base space $M$ which does not have the property SUB?

Lie group structures on the gauge group have already been considered by other authors in similar settings.

Remark 1.17. If the structure group $K$ is the group of diffeomorphisms $\text{Diff}(N)$ of some closed compact manifold $N$, then it does not follow from Lemma 1.14 that $P$ has the property SUB, because $\text{Diff}(N)$ fails to be locally exponential or abelian. However, in this case, $\text{Gau}(P)$ is as a split submanifold of the Lie group $\text{Diff}(P)$, which provides a smooth structure on $\text{Gau}(P)$ [Mi91, Theorem 14.4].

Identifying $\text{Gau}(P)$ with the space of section in the associated bundle $\text{AD}(P)$ for the conjugation action $\text{AD}: K \times K \to K$, [OMYK83, Proposition 6.6] also provides a Lie group structure on $\text{Gau}(P)$.

The advantage of Theorem 1.14 is, that it provides charts for $\text{Gau}(P)$, which allows us to reduce questions on gauge groups to similar question on mapping groups. This correspondence is crucial for all the following considerations.

In the end of the section we provide as approximation result that makes homotopy groups of $\text{Gau}(P)$ accessible in terms of continuous data from the bundle $P$ (cf. [Wo95]).

Remark 1.18. Let $P$ be one of the bundles that occur in Lemma 1.14 with the corresponding closed trivialising system $\overline{V}$ and let $\overline{\text{g}}(\text{P})_c$ (resp. $\overline{\text{G}}(\text{P})_c$) be the continuous counterparts of $\overline{\text{g}}(\text{P})$, which is isomorphic to the space of $K$-equivariant continuous maps $C(P, K)^K$ (resp. $C(P, K)^K$). We endow all spaces of continuous mappings with the compact-open topology.

Then the map

$$\varphi_* : \overline{\text{G}}(\text{P})_c \cap \prod_{i=1}^n C(\overline{V}_i, W) \to \overline{\text{g}}(\text{P})_c \cap \prod_{i=1}^n C(\overline{V}_i, W'), \quad (\gamma_i)_{i=1,\ldots,n} \mapsto (\varphi \circ \gamma_i)_{i=1,\ldots,n}$$

is also bijective, inducing a smooth manifold structure on the left-hand-side, because the right-hand-side is an open subset in a locally convex space. Furthermore, it can be shown exactly as in the smooth case, that this manifold structure endows $\overline{\text{G}}(\text{P})_c$ with a Lie group structure by Proposition 1.4. One could go on and call the requirement that $\varphi_*$ is bijective “continuous property SUB”, but since the next proposition is the sole application of it this seems to be exaggerated.

Lemma 1.19. Let $P$ be a smooth principal $K$-bundle over the compact base $M$, having the property SUB with respect to the smooth closed trivialising system $\overline{V} = (\overline{V}_i, \sigma_i)_{i=1,\ldots,n}$ and let $\varphi : W \to W'$ be the corresponding chart of $K$ (cf. Definition 1.7). If $(\gamma_i)_{i=1,\ldots,n} \in \overline{\text{G}}(\text{P})$ represents an element of $C^\infty(P, K)^K$ (cf. Remark 1.6), which is close to identity, in the sense that $\gamma_i(\overline{V}_i) \subseteq W$, then $(\gamma_i)_{i=1,\ldots,n}$ is homotopic to the constant map $(x \mapsto e)_{i=1,\ldots,n}$.

Proof. Since the map

$$\varphi_* : U := \overline{\text{G}}(\text{P}) \cap \prod_{i=1}^n C^\infty(\overline{V}_i, W) \to \overline{\text{g}}(\text{P}), \quad (\gamma_i)_{i=1,\ldots,n} \mapsto (\varphi \circ \gamma_i)_{i=1,\ldots,n},$$

is a chart of $\overline{\text{G}}(\text{P})$ (cf. Proposition 1.8) and $\varphi_* (U) \subseteq \overline{\text{g}}(\text{P})$ is convex, the map

$$[0,1] \ni t \mapsto \varphi_*^{-1}(t \cdot \varphi_* ((\gamma_i)_{i=1,\ldots,n})) \in \overline{\text{G}}(\text{P})$$

defines the desired homotopy. □
Proposition 1.20. If \( \mathcal{P} \) is one of the bundles that occur in Lemma 1.14, the natural inclusion \( i : \text{Gau}(\mathcal{P}) \to \text{Gau}_c(\mathcal{P}) \) of smooth into continuous gauge transformations is a weak homotopy equivalence, i.e., the induced mappings \( \pi_n(\text{Gau}(\mathcal{P})) \to \pi_n(\text{Gau}_c(\mathcal{P})) \) are isomorphisms of groups for \( n \in \mathbb{N}_0 \).

Proof. We identify \( \text{Gau}(\mathcal{P}) \) with \( C^\infty(\mathcal{P},K)^K \) and \( \text{Gau}_c(\mathcal{P}) \) with \( C(\mathcal{P},K)^K \). To see that \( \pi_n(i) \) is surjective, consider the continuous principal \( K \)-bundle \( \text{pr}^*(\mathcal{P}) \) obtained form \( \mathcal{P} \) by pulling it back along the projection \( \text{pr} : S^n \times M \to M \). Then \( \text{pr}^*(\mathcal{P}) \cong (K, \text{id} \times \pi, S^n \times P, S^n \times M) \), where \( K \) acts trivially on the first factor of \( S^n \times P \). We have with respect to this action \( C(\text{pr}^*(\mathcal{P}),K)^K \cong C(S^n \times P,K)^K \) and \( C^\infty(\text{pr}^*(\mathcal{P}))^K \cong C^\infty(S^n \times P,K)^K \). The isomorphisms \( C(S^n,G_0) \cong C_*(S^n,G_0) \times G_0 = C_*(S^n,G) \times G_0 \), where \( C_*(S^n,G) \) denotes the space of basepoint-preserving maps from \( S^n \) to \( G \), yield \( \pi_0(G) = \pi_0(C_*(S^n,G)) = \pi_0(C(S^n,G_0)) \) for any topological group \( G \). We thus get a map

\[
\pi_n(C^\infty(\mathcal{P},K)^K) = \pi_0(C_*(S^n,C^\infty(\mathcal{P},K)^K)) = \pi_0(C(S^n,C^\infty(\mathcal{P},K)^K_0)) \to \pi_0(C(S^n,C(\mathcal{P},K)^K_0)),
\]

where \( \eta \) is induced by the inclusion \( C^\infty(\mathcal{P},K)^K \to C(\mathcal{P},K)^K \).

If \( f \in C(S^n \times P,K) \) represents an element \( [F] \in \pi_0(C(S^n,C(\mathcal{P},K)^K_0)) \) (recall that we have \( C(\mathcal{P},K)^K \cong G_{v_0}(\mathcal{P}) \subseteq \prod_{i=1}^n C(V_i,K) \) and \( C(S^n,C(V_i,K)) \cong C(S^n \times V_i,K) \)), then there exists \( \tilde{f} \in C^\infty(S^n \times P,K)^K \) which is contained in the same connected component of \( C(S^n \times P,K)^K \) as \( f \) (cf. [Wot06a Theorem 11]). Since \( \tilde{f} \) is in particular smooth in the second argument, it follows that \( \tilde{F} \in C(S^n,C^\infty(\mathcal{P},K)^K) \). Since the connected components and the arc components of \( C(S^n \times P,K)^K \) coincide (since it is a Lie group, cf. Remark 1.18), there exists a path \( \tau : [0,1] \to C(S^n \times P,K)^K_0 \) such that \( t \mapsto \tau(t) \cdot f \) is a path connecting \( f \) and \( \tilde{f} \). Since \( S^n \) is connected it follows that \( C(S^n \times P,K)^K_0 \cong C(S^n,C(\mathcal{P},K)^K_0) \subseteq C(S^n,C(\mathcal{P},K)^K_0) \). Thus \( \tau \) represents a continuous path in \( C(S^n,C(\mathcal{P},K)^K_0) \) connecting \( F \) and \( \tilde{F} \) whence \( [F] = [\tilde{F}] \in \pi_0(C(S^n,C(\mathcal{P},K)^K_0)) \). That \( \pi_n(i) \) is injective follows with Lemma 1.19 as in [Ne02] Theorem A.3.7. \( \square \)

2 The automorphism group as an infinite-dimensional Lie group

In this section we describe the Lie group structure on Aut(\( \mathcal{P} \)) for a principal \( K \)-bundle over a compact manifold \( M \) without boundary, i.e., a closed compact manifold. We will do this using the extension of abstract groups

\[
(3) \quad \text{Gau}(\mathcal{P}) \to \text{Aut}(\mathcal{P}) \overset{Q}{\to} \text{Diff}(M),
\]

where \( \text{Diff}(M)_\mathcal{P} \) is the image of the homomorphism \( Q : \text{Aut}(\mathcal{P}) \to \text{Diff}(M), F \mapsto F_\mathcal{P} \) from Definition 1.14. More precisely, we will construct a Lie group structure on Aut(\( \mathcal{P} \)) that turns \( 3 \) into an extension of Lie groups, i.e., into a locally trivial bundle.

We should advertise in advance that we shall not need regularity assumptions on \( K \) in order to lift diffeomorphisms of \( M \) to bundle automorphisms by lifting vector fields. However, we elaborate shortly on the regularity of \( \text{Gau}(\mathcal{P}) \) and Aut(\( \mathcal{P} \)) in the end of the section.

We shall consider bundles over bases without boundary, i.e., our base manifolds will always be closed compact manifolds. Throughout this section we fix one particular given principal \( K \)-bundle \( \mathcal{P} \) over a closed compact manifold \( M \) and we furthermore assume that \( \mathcal{P} \) has the property SUB.
Definition 2.1. (cf. [Ne06a]) If \( N, \hat{G} \) and \( G \) are Lie groups, then an extension of groups

\[ N \hookrightarrow \hat{G} \twoheadrightarrow G \]

is called an extension of Lie groups if \( N \) is a split Lie subgroup of \( \hat{G} \). That means that \((N, q : \hat{G} \to G)\) is a smooth principal \( N \)-bundle, where \( q : \hat{G} \to G \equiv \hat{G}/N \) is the induced quotient map. We call two extensions \( N \hookrightarrow \hat{G}_1 \twoheadrightarrow G \) and \( N \hookrightarrow \hat{G}_2 \twoheadrightarrow G \) equivalent if there exists a morphism of Lie groups \( \psi : \hat{G}_1 \to \hat{G}_2 \) such that the diagram

\[
\begin{array}{ccc}
N & \longrightarrow & \hat{G}_1 \\
\text{id}_N \downarrow & & \downarrow \psi \\
N & \longrightarrow & \hat{G}_2
\end{array}
\]

commutes.

Remark 2.2. Unless stated otherwise, for the rest of this section we choose and fix one particular smooth closed trivialising system \( \mathcal{V} = (\overline{V_i}, \sigma_i)_{i=1, \ldots, n} \) of \( \mathcal{P} \) such that

- each \( \overline{V_i} \) is a compact manifold with corners diffeomorphic to \([0,1]^{\dim(M)}\),
- \( \mathcal{V} \) is a refinement of a smooth open trivialising system \( \mathcal{U} = (U_i, \tau_i)_{i=1, \ldots, n} \) and we have \( \overline{V_i} \subseteq U_i \) and \( \sigma_i = \tau_i|_{\overline{V_i}} \),
- each \( \overline{\mathcal{V}_i} \) is a compact manifold with corners diffeomorphic to \([0,1]^{\dim(M)}\) and \( \tau_i \) extends to a smooth section \( \tau_i : \overline{U_i} \to P \),
- \( \overline{\mathcal{U}} = (\overline{U_i}, \tau_i)_{i=1, \ldots, n} \) is a refinement of a smooth open trivialising system \( \mathcal{U}' = (U'_i, \tau'_i)_{j=1, \ldots, m} \),
- the values of the transition functions \( k_{ij} : U'_i \cap U'_j \to K \) of \( \mathcal{U}' \) are contained in open subsets \( W_{ij} \) of \( K \), which are diffeomorphic to open zero neighbourhoods of \( \mathfrak{t} \),
- \( \mathcal{P} \) has the property SUB with respect to \( \mathcal{V} \) (and thus with respect to \( \overline{\mathcal{U}} \) by Lemma 1.12).

We choose \( \mathcal{V} \) by starting with an arbitrary smooth closed trivialising system such that \( \mathcal{P} \) has the property SUB with respect to this system. Note that this exists because we assume throughout this section that \( \mathcal{P} \) has the property SUB. Then Lemma A.21 implies that there exists a refinement \( \mathcal{U}' = (U'_j, \tau'_j)_{j=1, \ldots, m} \) such that the transition functions \( k_{ij} : U'_i \cap U'_j \to K \) take values in open subsets \( W_{ij} \) of \( K \), which are diffeomorphic to open convex zero neighbourhoods of \( \mathfrak{t} \). Now each \( x \in M \) has neighbourhoods \( V_x \) and \( U_x \) such that \( \overline{V}_x \subseteq U_x \), \( \overline{V}_x \) and \( U_x \) are diffeomorphic to \([0,1]^{\dim(M)}\) and \( \overline{U}_x \subseteq U_x(\{x\}) \) for some \( j(x) \in \{1, \ldots, m\} \). Then finitely many \( V_{x_1}, \ldots, V_{x_n} \) cover \( M \) and so do \( U_{x_1}, \ldots, U_{x_n} \). Furthermore, the sections \( \tau_j \) restrict to smooth sections on \( V_{x_i}, \overline{V}_x, U_x, U'_x \).

This choice of \( \overline{\mathcal{U}} \) in turn implies that \( k_{ij}|_{\overline{V_i} \cap \overline{V_j}} \) arises as the restriction of some smooth function on \( M \). In fact, if \( \varphi_{ij} : W_{ij} \to W'_{ij} \subseteq \mathfrak{t} \) is a diffeomorphism onto a convex zero neighbourhood and \( f_{ij} \in C^\infty(M, \mathbb{R}) \) is a smooth function with \( f_{ij}|_{\overline{V_i} \cap \overline{V_j}} \equiv 1 \) and \( \text{supp}(f_{ij}) \subseteq U'_i \cap U'_j \), then

\[
m \mapsto \begin{cases}
\varphi_{ij}^{-1}(f_{ij}(m) \cdot \varphi_{ij}(k_{ij}(m))) & \text{if } m \in U'_i \cap U'_j \\
\varphi_{ij}^{-1}(0) & \text{if } m \notin U'_i \cap U'_j
\end{cases}
\]

is a smooth function, because each \( m \in \partial(U'_i \cap U'_j) \) has a neighbourhood on which \( f_{ij} \) vanishes, and this function coincides with \( k_{ij} \) on \( U'_i \cap U'_j \).

Similarly, let \( (\gamma_1, \ldots, \gamma_n) \in C_0(\mathcal{P}) \subseteq \prod_{i=1}^n C_0(\overline{U}_i, K) \) be the local description of some element \( \gamma \in C_0(\mathcal{P}, K)^K \). We will show that each \( \gamma_i|_{\overline{V}_i} \) arises as the restriction of a smooth map on \( M \). In fact, take a diffeomorphism \( \varphi_i : \overline{V}_i \to [0,1]^{\dim(M)} \). Then \( \overline{V}_i \subseteq U_i \) implies that we have \( \varphi_i(\overline{V}_i) \subseteq (0,1]^{\dim(M)} \) and thus there exists an \( \varepsilon > 0 \) such that \( \varphi_i(\overline{V}_i) \subseteq (\varepsilon,1-\varepsilon)^{\dim(M)} \) for all \( i = 1, \ldots, n \). Now let

\[
f : [0,1]^{\dim(M)}\setminus(\varepsilon,1-\varepsilon)^{\dim(M)} \to [\varepsilon,1-\varepsilon]\]
be a map that restricts to the identity on $\partial[\varepsilon, 1 - \varepsilon]^{\dim(M)}$ and collapses $\partial[0, 1]^{\dim(M)}$ to a single point $x_0$. We then set

$$\gamma'_i : M \to K \quad m \mapsto \begin{cases} \gamma_i(m) & \text{if } m \in U_i \text{ and } \varphi_i(m) \in [\varepsilon, 1 - \varepsilon]^{\dim(M)} \text{ and } f(\varphi_i(m)) = x_0, \\ \gamma_i(\varphi^{-1}_i(x_0)) & \text{if } m \notin U_i, \end{cases}$$

and $\gamma'_i$ is well-defined and continuous, because $f(\varphi_i(m)) = \varphi_i(m)$ if $\varphi_i(m) \in [\varepsilon, 1 - \varepsilon]^{\dim(M)}$ and $f(\varphi_i(m)) = x_0$ if $\varphi_i(m) \in \partial[0, 1]^{\dim(M)}$. Since $\gamma'_i$ coincides with $\gamma_i$ on the neighbourhood $\varphi^{-1}_i((\varepsilon, 1 - \varepsilon)^{\dim(M)})$, it thus is smooth on this neighbourhood. Now [Wo06a, Corollary 12], yields a smooth map $\tilde{\gamma}_i$ on $M$ with $\gamma_i|_{\tilde{\gamma}_i} = \tilde{\gamma}_i|_{\gamma_i}$.

We now give the description of a strategy for lifting special diffeomorphisms to bundle automorphisms. This should motivate the procedure of this section.

**Remark 2.3.** Let $U \subseteq M$ be open and trivialising with section $\sigma : U \to P$ and corresponding $k_{\sigma} : \pi^{-1}(U) \to K$, given by $\sigma(\pi(p)) \cdot k_{\sigma}(p) = p$. If $g \in \text{Diff}(M)$ is such that $\text{supp}(g) \subseteq U$, then we may define a smooth bundle automorphism $\tilde{g}$ by

$$\tilde{g}(p) = \begin{cases} \sigma(g(\pi(p))) \cdot k(p) & \text{if } p \in \pi^{-1}(U) \\
 & \text{else}, \end{cases}$$

because each $x \in \partial U$ has a neighbourhood on which $g$ is the identity. Furthermore, one easily verifies $Q(\tilde{g}) = \tilde{g}_M = g$ and $\tilde{g}^{-1} = \tilde{g}^{-1}$, where $Q : \text{Aut}(P) \to \text{Diff}(M)$ is the homomorphism from Definition 1.1.

**Remark 2.4.** Let $M$ be a closed compact manifold with a fixed Riemannian metric $g$ and let $\pi : TM \to M$ be its tangent bundle and $\text{Exp} : TM \to M$ be the exponential mapping of $g$. Then $\pi \times \text{Exp} : TM \to M \times M$, $X_m \mapsto (m, \text{Exp}(X_m))$ restricts to a diffeomorphism on an open neighbourhood $U$ of the zero section in $TM$. We set $O' := \{X \in \mathcal{V}(M) : X(M) \subseteq U\}$ and define

$$\varphi^{-1} : O' \to C^\infty(M, M), \quad \varphi^{-1}(X)(m) = \text{Exp}(X(m))$$

For the following, observe that $\varphi^{-1}(X)(m) = m$ if and only if $X(m) = 0_m$. After shrinking $O'$ to a convex open neighbourhood in the $C^1$-topology, one can also ensure that $\varphi^{-1}(X) \in \text{Diff}(M)$ for all $X \in O'$. Since $\pi \times \text{Exp}$ is bijective on $U$, $\varphi^{-1}$ maps $O'$ bijectively to $O := \varphi^{-1}(O') \subseteq \text{Diff}(M)$ and thus endows $O$ with a smooth manifold structure. Furthermore, it can be shown that in view of Proposition 2.3 this chart actually defines a Lie group structure on $\text{Diff}(M)$ (cf. [Le67], [KM97] Theorem 43.1) or [Go06]. It is even possible to put Lie group structures on $\text{Diff}(M)$ in the case of non-compact manifolds, possibly with corners [Ma80] Theorem 11.11], but we will not go into this generality here.

**Lemma 2.5.** For the open cover $V_1, \ldots, V_n$ of the closed compact manifold $M$ and the open identity neighbourhood $O \subseteq \text{Diff}(M)$ from Remark 2.3, there exist smooth maps

$$s_i : O \to O \circ O^{-1}$$

for $1 \leq i \leq n$ such that $\text{supp}(s_i(g)) \subseteq V_i$ and $s_n(g) \circ \ldots \circ s_1(g) = g$.

**Proof.** (cf. [HT04] Proposition 1) Let $f_1, \ldots, f_n$ be a partition of unity subordinated to the open cover $V_1, \ldots, V_n$ and let $\varphi : O \to \varphi(O) \subseteq \mathcal{V}(M)$ be the chart of $\text{Diff}(M)$ from Remark 2.4. In particular, $\varphi^{-1}(X)(m) = m$ if $X(m) = 0_m$. Since $\varphi(O)$ is convex, we may define $s_i : O \to O \circ O^{-1}$,

$$s_i(g) = \varphi^{-1}((f_n + \ldots + f_i) \cdot \varphi(g)) \circ (\varphi^{-1}((f_n + \ldots + f_{i+1}) \cdot \varphi(g)))^{-1}$$

if $i < n$ and $s_n(g) = \varphi^{-1}(f_n \cdot \varphi(g))$, which are smooth since they are given by a push-forward of the smooth map $\mathbb{R} \times TM \to TM (\lambda, X_m) \mapsto \lambda \cdot X_m$. Furthermore, if $f_i(x) = 0$, then the left and the right factor annihilate each other and thus $\text{supp}(s_i(g)) \subseteq V_i$. \qed
The preceding lemma enables us now to lift elements of $O \subseteq \text{Diff}(M)$ to elements of $\text{Aut}(P)$.

**Definition 2.6.** If $O \subseteq \text{Diff}(M)$ is the open identity neighbourhood from Remark 2.3 and $s_i : O \to O \circ O^{-1}$ are the smooth mappings from Lemma 2.5 then we define

$$(5) \quad S : O \to \text{Aut}(P), \quad g \mapsto S(g) := \tilde{g}_n \circ \ldots \circ \tilde{g}_1,$$

where $\tilde{g}_i$ is the bundle automorphism of $P$ from Remark 2.3. This defines a local section for the homomorphism $Q : \text{Aut}(P) \to \text{Diff}(M)$, $F \mapsto F_M$ from Definition 1.1.

We shall frequently need an explicit description of $S(g)$ in terms of local trivialisations, i.e., how $S(g)(\sigma_i(x))$ can be expressed in terms of $g_j$, $\sigma_j$ and $k_{ij}$.

**Remark 2.7.** Let $x \in V_\iota \subseteq M$ be such that $x \notin V_j$ for $j < i$ and $g_i(x) \notin V_j$ for $j > i$. Then $g_j(x) = x$ for all $j < i$, $g_j(g_i(x)) = g_i(x)$ for all $j > i$ and thus $S(g)(\sigma_i(x)) = \sigma_i(g_i(x)) = \sigma_i(g(x))$.

In general, things are more complicated. The first $\tilde{g}_{ij}$ in (5) that could move $\sigma_i(x)$ is the one for the minimal $j_1$ such that $x \in V_{j_1}$. We then have

$$g_{j_1}(\sigma_i(x)) = \tilde{g}_{j_1}(\sigma_j(x)) \cdot k_{j_1}(x) = \sigma_j(g_{j_1}(x)) \cdot k_{j_1}(x).$$

The next $\tilde{g}_{j_2}$ in (5) that could move $\sigma_{j_1}(\sigma_i(x))$ in turn is the one for the minimal $j_2 > j_1$ such that $g_{j_2}(x) \in V_{j_2}$, and we then have

$$\tilde{g}_{j_2}(\tilde{g}_{j_1}(\sigma_i(x))) = \tilde{g}_{j_2}(g_{j_2} \circ g_{j_1}(x)) \cdot k_{j_2,j_1}(g_{j_1}(x)) \cdot k_{j_1}(x).$$

We eventually get

$$(6) \quad S(g)(\sigma_i(x)) = \sigma_j(g(x)) \cdot k_{j_1,j_2-1}(g_{j_2-1} \circ \ldots \circ g_{j_1}(x)) \cdot \ldots \cdot k_{j_1}(x),$$

where $\{j_1, \ldots, j_\ell\} \subseteq \{1, \ldots, n\}$ is maximal such that $g_{j_\ell-1} \circ \ldots \circ g_{j_1}(x) \in U_{j_\ell} \cap U_{j_\ell-1}$ for $2 \leq p \leq \ell$ and $j_1 < \ldots < j_\ell$.

Note that we cannot write down such a formula using all $j \in \{1, \ldots, n\}$, because the corresponding $k_{ij}$ and $\sigma_j$ would not be defined properly.

Of course, $g$ and $x$ influence the choice of $j_1, \ldots, j_\ell$, but there exist open neighbourhoods $O_g$ of $g$ and $U_x$ of $x$ such that we may use (6) as a formula for all $g' \in O_g$ and $x' \in U_x$. In fact, the action $\text{Diff}(M) \times M \to M$, $g.m = g(m)$ is smooth by Proposition 7.2, and thus in particular continuous. If

$$g_{j_\ell} \circ \ldots \circ g_{j_1}(x) \notin V_j \quad \text{for} \quad 2 \leq p \leq \ell \quad \text{and} \quad j \notin \{j_1, \ldots, j_\ell\}$$

then this is also true for $g'$ and $x'$ in some open neighbourhood of $g$ and $x$. This yields finitely many open neighbourhoods of $g$ and $x$ and we define their intersections to be $O_g$ and $U_x$. Then (6) still holds for $g' \in O_g$ and $x' \in U_x$, because (7) implies $g_{j_\ell} \circ \ldots \circ g_{j_1}(x) = g_{j_\ell} \circ \ldots \circ g_{j_1}(x)$ and (8) implies that $k_{j_\ell,j_\ell-1}$ is defined and satisfies the cocycle condition.

In order to determine a Lie group structure on $\text{Aut}(P)$, the map $S : O \to \text{Aut}(P)$ has to satisfy certain smoothness properties, which will be ensured by the subsequent lemmas.

**Remark 2.8.** If we identify the normal subgroup $\text{Gau}(P) \trianglelefteq \text{Aut}(P)$ with $C^\infty(P,K)^K$ via

$$C^\infty(P,K)^K \to \text{Gau}(P), \quad \gamma \mapsto F_\gamma$$

with $F_\gamma(p) = p \cdot \gamma(p)$, then the conjugation action $c : \text{Aut}(P) \times \text{Gau}(P) \to \text{Gau}(P)$, given by $c(F_\gamma) = F \circ F_\gamma \circ F^{-1}$ changes into

$$c : \text{Aut}(P) \times C^\infty(P,K)^K \to C^\infty(P,K)^K, \quad (F, \gamma) \mapsto \gamma \circ F^{-1}.$$

In fact, this follows from

$$(F \circ F_\gamma \circ F^{-1})(p) = F(F^{-1}(p) \cdot \gamma(F^{-1}(p))) = p \cdot \gamma(F^{-1}(p)) = F(\gamma \circ F^{-1})(p).$$
In the following remarks and lemmas we show the smoothness of the maps \( T, \omega \) and \( \omega_g \), mentioned before.

**Lemma 2.9.** Let \( O \subseteq \text{Diff}(M) \) be the open identity neighbourhood from Remark 2.4 and let \( S: O \to \text{Aut}(P) \) be the map from Definition 2.6. Then we have that for each \( F \in \text{Aut}(P) \) the map \( C^\infty(P, t)^K \to C^\infty(P, f)^K, \eta \mapsto \eta \circ F^{-1} \) is an automorphism of \( C^\infty(P, t)^K \) and the map

\[
t: C^\infty(P, t)^K \times O \to C^\infty(P, f)^K, \quad (\eta, g) \mapsto \eta \circ S(g)^{-1}
\]

is smooth.

**Proof.** That \( \eta \mapsto \eta \circ F^{-1} \) is an element of \( \text{Aut}(C^\infty(P, t)^K) \) follows immediately from the (point-wise) definition of the bracket on \( C^\infty(P, t)^K \). We shall use the previous established isomorphisms \( C^\infty(P, t)^K \cong \mathfrak{g}_C(P) \cong \mathfrak{g}_V(P) \) from Proposition 1.4 and reduce the smoothness of \( t \) to the smoothness of

\[
m: C^\infty(M, t) \times \text{Diff}(M) \to C^\infty(M, f), \quad (\eta, g) \mapsto \eta \circ g^{-1}
\]

from Proposition 6 and to the action of \( g_i^{-1} \) on \( C^\infty(V_i, t) \), because we have no description of what \( g_i^{-1} \) does with \( U_j \) for \( j \neq i \). It clearly suffices to show that the map

\[
t_i: C^\infty(P, t)^K \times \text{Diff}(M) \to C^\infty(P, f)^K \times \text{Diff}(M), \quad (\eta, g) \mapsto (\eta \circ g_i^{-1}, g)
\]

is smooth for each \( 1 \leq i \leq n \), because then \( t = \text{pr}_1 \circ t_n \circ \ldots \circ t_1 \) is smooth. This in turn follows from the smoothness of

\[
(9)
C^\infty(U_i', t) \times \text{Diff}(M) \to C^\infty(U_i, t), \quad (\eta, g) \mapsto \eta \circ g_i^{-1}|_{U_i'},
\]

because this is the local description of \( t_i \). In fact, for each \( j \neq i \) there exists an open subset \( V'_j \) with \( U_j \setminus U_i \subseteq V'_j \subseteq U_j \setminus V_i \), because \( V_i \subseteq U_j \) and \( U_i \) is diffeomorphic to \((0,1)^{\dim(M)}\). Furthermore, we set \( V'_i := U_i' \). Then \((V'_1, \ldots, V'_n)\) is an open cover of \( M \), leading to a refinement \( V' \) of the trivialising system \( U' \) and we have

\[
t_i: \mathfrak{g}_{V'}(P) \times O \to \mathfrak{g}_{V'}(P), \quad ((\eta_1, \ldots, \eta_n), g) \mapsto (\eta_i|_{V'_i}, \ldots, \eta_n|_{V'_i})
\]

because \( \text{supp}(g_i) \subseteq V_i \) and \( V'_j \cap V_i = \emptyset \) if \( j \neq i \). To show that (9) is smooth, choose some \( f_i \in C^\infty(M, \mathbb{R}) \) with \( f_i|_{U_i} \equiv 1 \) and \( \text{supp}(f_i) \subseteq U_i' \). Then

\[
h_i: C^\infty(U'_i, t) \to C^\infty(M, t), \quad \eta \mapsto \left( m \mapsto \begin{cases} f_i(m) \cdot \eta(m) & \text{if } m \in U'_i \\ 0 & \text{if } m \notin U'_i \end{cases} \right)
\]

is smooth by Corollary 1.14 because \( \eta \mapsto f_i|_{U'_i} \cdot \eta \) is linear, continuous and thus smooth. Now we have \( \text{supp}(g_i) \subseteq V_i \subseteq U_i \) and thus \( h_i(\eta) \circ g_i^{-1}|_{U_i} = \eta \circ g_i^{-1}|_{U_i} \) depends smoothly on \( g \) and \( \eta \) by Corollary 1.14. \( \square \)

The following proofs share a common idea. We will always have to show that certain mappings with values in \( C^\infty(P, K) \) are smooth. This can be established by showing that their compositions with the pull-back \((\sigma_i)^*\) of a section \( \sigma_i: V_i \to P \) (then with values in \( C^\infty(V_i, K) \)) are smooth for all \( 1 \leq i \leq n \). As described in Remark 2.7 it will not be possible to write down explicit formulas for these mappings in terms of the transition functions \( k_{ij} \) for all \( x \in V_i \) simultaneously, but we will be able to do so on some open neighbourhood \( U_x \) of \( x \). For different \( x_1 \) and \( x_2 \) these formulas will define the same mapping on \( U_{x_1} \cap U_{x_2} \), because there they define \((\sigma_1^*(S(g))) = S(g) \circ \sigma_i\). By restriction and gluing we will thus be able to reconstruct the original mappings and then see that they depend smoothly on their arguments.

**Lemma 2.10.** If \( O \subseteq \text{Diff}(M) \) is the open identity neighbourhood from Remark 2.4 and if \( S: O \to \text{Aut}(P) \) is the map from Definition 2.6, then for each \( \gamma \in C^\infty(P, K)^K \) the map

\[
O \ni g \mapsto \gamma \circ S(g)^{-1} \in C^\infty(P, K)^K
\]

is smooth.
The automorphism group as an infinite-dimensional Lie group

Proof. It suffices to show that \( \gamma \circ S(g)^{-1} \circ \sigma_i \) depends smoothly on \( g \) for \( 1 \leq i \leq n \). Let \((\gamma_1, \ldots, \gamma_n) \in G_2(\mathcal{P}) \subseteq \prod_{i=1}^{\infty} C^\infty(U_i, K)\) be the local description of \( \gamma \). Fix \( g \in O \) and \( x \in \mathcal{V}_i \). Then Remark 2.7 yields open neighbourhoods \( O_g \) of \( g \) and \( U_x \) of \( x \) (w.l.o.g. such that \( \overline{U}_x \subseteq \mathcal{V}_i \)) and \( U \) is a manifold with corners) such that

\[
\gamma(S(g)^{-1}(\sigma_i(x'))) = \gamma(\sigma_j(g'(x'))) \cdot \kappa_{x,g'}(x') \cdot \sigma_i(\kappa_{x,g'}(x'))
\]

for all \( g' \in O_g \) and \( x' \in \overline{U}_x \). Since we will not vary \( i \) and \( g \) in the sequel, we suppressed the dependence of \( \kappa_{x,g'}(x') \) on \( i \) and \( g \). Note that each \( k_j \) and \( \gamma_i \) can be assumed to be defined on \( M \) (cf. Remark 2.7). Thus, for fixed \( x \), the formula for \( \theta_{x,g'} \) defines a smooth function on \( M \) that depends smoothly on \( g' \), because the action of \( \text{Diff}(M) \) on \( C^\infty(M, K) \) is smooth (cf. [17, Proposition 10.3]).

Furthermore, \( \theta_{x,g'} \) and \( \theta_{\gamma, g'} \) coincide on \( \overline{U}_x \), because both define \( \gamma \circ S(g')^{-1} \circ \sigma_i \) there. Now finitely many \( U_{x_1}, \ldots, U_{x_m} \) cover \( \mathcal{V}_i \), and since the gluing and restriction maps from Lemma A.17 and Proposition A.18 are smooth we have that

\[
\gamma \circ S(g')^{-1} \circ \sigma_i = \text{glue}(\theta_{g_1,g'}, \ldots, \theta_{g_m,g'})
\]

depends smoothly on \( g' \).

\[\Box\]

Lemma 2.11. Let \( O \subseteq \text{Diff}(M) \) be the open identity neighbourhood from Remark 2.4 and let \( S : O \to Aut(\mathcal{P}) \) be the map from Definition 2.4. Then we have that for each \( F \in Aut(\mathcal{P}) \) the map \( c_F : C^\infty(P, K) \to C^\infty(P, K) \), \( \gamma \mapsto \gamma \circ F^{-1} \) is an automorphism of \( C^\infty(P, K) \) and the map

\[
T : C^\infty(P, K) \times O \to C^\infty(P, K), \quad (\gamma, g) \mapsto \gamma \circ S(g)^{-1}
\]

is smooth.

Proof. Since \( \gamma \mapsto \gamma \circ F^{-1} \) is a group homomorphism, it suffices to show that it is smooth on a unit neighborhood. Because the charts on \( C^\infty(P, K) \) are constructed by push-forwards (cf. Proposition 1.8) this follows immediately from the fact that the corresponding automorphism of \( C^\infty(P, \mathfrak{g}) \), given by \( \eta \mapsto \eta \circ F^{-1} \), is continuous and thus smooth. For the same reason, Lemma 2.9 implies that there exists a unit neighbourhood \( U \subseteq C^\infty(P, K) \) such that

\[
U \times O \to C^\infty(P, K), \quad (\gamma, g) \mapsto \gamma \circ S(g)^{-1}
\]

is smooth.

Now for each \( \gamma_0 \in C^\infty(P, K) \) there exists an open neighbourhood \( U_{\gamma_0} \) with \( \gamma_0^{-1} : U_{\gamma_0} \subseteq U \).

Hence

\[
\gamma \circ S(g)^{-1} = (\gamma_0 \cdot \gamma_0^{-1} \cdot \gamma) \circ S(g)^{-1} = (\gamma_0 \circ S(g)^{-1}) \cdot (\gamma_0^{-1} \cdot \gamma) \circ S(g)^{-1},
\]

and the first factor depends smoothly on \( g \) due to Lemma 2.10 and the second factor depends smoothly on \( \gamma \) and \( g \), hence \( \gamma_0^{-1} \cdot \gamma \in U \).

\[\Box\]

Lemma 2.12. If \( O \subseteq \text{Diff}(M) \) is the open identity neighbourhood from Remark 2.4 and if \( S : O \to Aut(\mathcal{P}) \) is the map from Definition 2.4, then

\[
\omega : O \times O \to C^\infty(P, K), \quad (g, g') \mapsto S(g) \circ S(g') \circ S(g \circ g')^{-1}
\]

is smooth. Furthermore, if \( Q : Aut(\mathcal{P}) \to \text{Diff}(M) \), \( F \mapsto F_M \) is the homomorphism from Definition 1.3, then for each \( g \in Q(\text{Diff}(M)) \) there exists an open identity neighbourhood \( O_g \subseteq O \) such that

\[
\omega_g : O_g \to C^\infty(P, K), \quad g' \mapsto F \circ S(g') \circ F^{-1} \circ S(g \circ g')^{-1}
\]

is smooth for any \( F \in Aut(\mathcal{P}) \) with \( F_M = g \).
The automorphism group as an infinite-dimensional Lie group

Proof. First observe that \( \omega(g, g') \) actually is an element of \( C(\mathcal{P}, K)^K \cong \text{Gau}(\mathcal{P}) = \ker(Q) \), because \( Q \) is a homomorphism of groups, \( S \) is a section of \( Q \) and thus

\[
Q(\omega(g, g')) = Q(S(g)) \circ Q(S(g')) \circ Q(g \circ g')^{-1} = \text{id}_M.
\]

To show that \( \omega \) is smooth, we derive an explicit formula for \( \omega(g, g') \circ \sigma_i \in C(\mathcal{V}_i, K) \) that depends smoothly on \( g \) and \( g' \).

Denote \( \tilde{g} := g \circ g' \) for \( g, g' \in O \) and fix \( g, g' \in O, x \in \mathcal{V}_i \). Proceeding as in Remark 2.7 we find \( i_1, \ldots, i_t \) such that

\[
S(\tilde{g})^{-1}(\sigma_i(x)) = \sigma_i(\tilde{g}^{-1}(x)) \cdot k_{i_1i_1}(1) \circ \ldots \circ k_{i_ti_t}(1)(x) \cdot k_{i_1i_t}(x).
\]

Accordingly we find \( i_1', \ldots, i_t' \) for \( g' \) and \( i_1'', \ldots, i_t'' \) for \( g \). We get as in Remark 2.7 open neighbourhoods \( O_g, O_g' \) of \( g, g' \) and \( U_x \) of \( x \) (w.l.o.g. such that \( U_x \subseteq V_i \) is a manifold with corners) such that for \( h \in O_g, h \in O_g' \) and \( x' \in U_x \) we have \( S(h) \cdot S(h') \cdot S(h \cdot h')^{-1}(\sigma_i(x')) = \sigma_i(x') \cdot k_{i_1i_1}(x') \).

Denote by \( \kappa_{x,h,h'}(x') \in K \) the element in brackets on the right hand side, and note that it defines \( \omega(h, h') \circ \sigma(x') \) by Remark 1.2. Since we will not vary \( g \) and \( g' \) in the sequel we suppressed the dependence of \( \kappa_{x,h,h'}(x') \) on them.

Now each \( k_{ij} \) can be assumed to be defined on \( M \) (cf. Remark 2.2). Thus, for fixed \( x \), the formula for \( \kappa_{x,h,h'} \) defines a smooth function on \( M \) that depends smoothly on \( h \) and \( h' \), because the action of \( \text{Diff}(M) \) on \( C(M, K) \) is smooth (cf. Proposition 10.3]). Furthermore, \( \kappa_{x_1,h,h'} \), coincides with \( \kappa_{x_2,h,h'} \) on \( \overline{U_{x_1}} \cap \overline{U_{x_2}} \), because

\[
\sigma_i(x') \cdot \kappa_{x_1,h,h'}(x') = S(h) \circ S(h') \circ S(h \circ h')^{-1}(\sigma_i(x')) = \sigma_i(x') \cdot \kappa_{x_2,h,h'}(x')
\]

for \( x' \in \overline{U_{x_1}} \cap \overline{U_{x_2}} \). Now finitely many \( U_{x_1}, \ldots, U_{x_m} \) cover \( \mathcal{V}_i \) and we thus see that

\[
\omega(h, h') \circ \sigma_i = \text{glue}(\kappa_{x_1,h,h'}, \kappa_{x_2,h,h'}, \ldots, \kappa_{x_m,h,h'})
\]

depends smoothly on \( h \) and \( h' \).

We derive an explicit formula for \( \omega(g')(g' \circ \sigma_i \in C(\mathcal{V}_i, K) \) to show the smoothness of \( \omega_g \). Let \( O_g \subseteq O \) be an open identity neighbourhood with \( g \circ O_g \circ g^{-1} \subseteq O \) and denote \( \overline{g} = g \circ g' \circ g^{-1} \) for \( g' \in O_g \). Fix \( g' \) and \( x \in \mathcal{V}_i \). Proceeding as in Remark 2.7 we find \( j_1, \ldots, j_t \) such that

\[
S(\overline{g})^{-1}(\sigma_i(x)) = \sigma_i(\overline{g}^{-1}(x)) \cdot k_{j_1j_1}(1) \circ \ldots \circ k_{j_tj_t}(1)(x) \cdot k_{j_1j_t}(x).
\]

Furthermore, let \( j_1' \) be minimal such that

\[
(F_M^{-1} \circ S(\overline{g})^{-1})(x) = g^{-1} \circ g^{-1}(x) \in V_{j_1'}
\]

and let \( U_x \) be an open neighbourhood of \( x \) (w.l.o.g. such that \( \overline{U}_x \subseteq \mathcal{V}_i \) is a manifold with corners) such that \( \overline{g}^{-1}(\overline{U}_x) \subseteq V_{j_1} \) and \( g^{-1} \circ g^{-1}(\overline{U}_x) \subseteq V_{j_1'} \). Since \( F_M = g \) and

\[
F^{-1}(\sigma_{j_1'}(\overline{g}^{-1}(x'))) = \sigma_{j_1'}(g^{-1} \circ g^{-1}(x')) \text{ for } x' \in U_x
\]

we have

\[
F^{-1}(\sigma_{j_1'}(\overline{g}^{-1}(x'))) = \sigma_{j_1'}(g^{-1} \circ g^{-1}(x')) \cdot k_{F_x, g'}(x') \text{ for } x' \in U_x,
\]
for some smooth function \( k_{F,x,g'} : U_x \to K \). In fact, we have

\[
k_{F,x,g'}(x) = k_{\sigma_{i_1}} (F^{-1}(\sigma_{j_1}(g^{-1}(x)))).
\]

After possibly shrinking \( U_x \), a construction as in Remark 2.2 shows that \( k_{\sigma_{i_1}} \circ F^{-1} \circ \sigma_{j_1} |_{\overline{U}_x} \)
extends to a smooth function on \( M \). Thus \( k_{F,x,g'} |_{\overline{U}_x} \in C^\infty(\overline{U}_x,K) \) depends smoothly on \( g' \) for fixed \( x \).

Accordingly, we find \( j'_2, \ldots, j'_m \) and a smooth function \( k'_{F,x,g'} : \overline{U}_x \to K \) (possibly after shrinking \( U_x \)), depending smoothly on \( g' \) such that

\[
\omega_g(g')(\sigma_i(x)) = \sigma_i(x) \cdot [k'_{F,x,g'}(x) \cdot k_{j'_1} \cdot \ldots \cdot k_{j'_{m-1}}(g'(x)) \ldots \cdot k_{j'_{m-1}}(g^{-1}(x)) \cdot k_{F,x,g'}(x)]
\]

(13)

\[ \cdot k_{j'_{m-1}}(g'(x)) \ldots \cdot k_{j_1}(x). \]

Denote the element in brackets on the right hand side by \( \kappa_{x,g'} \). Since we will not vary \( F \) and \( g \) in the sequel, we suppressed the dependence of \( \kappa_{x,g'} \) on them. By continuity (cf. Remark 2.7), we find open neighbourhoods \( O_{g'} \) and \( U'_x \) of \( g' \) and \( x \) (w.l.o.g. such that \( \overline{U}'_x \subseteq \overline{V}_x \) is a manifold with corners) such that (13) defines \( \omega_g(h')(\sigma_j(x)) \) for all \( h' \in O_{g'} \) and \( x' \in U'_x \). Then \( \kappa_{x_1,g'} = \kappa_{x_2,g'} \) on \( \overline{U}_{x_1} \cap \overline{U}_{x_2} \), finitely many \( U_{x_1}, \ldots, U_{x_m} \) cover \( \overline{V}_x \) and since the gluing and restriction maps from Lemma A.17 and Proposition A.18 are smooth,

\[
\omega_g(g') \circ \sigma_i = \text{glue}(\kappa_{x_1,g'} |_{\overline{U}_{x_1}}, \ldots, \kappa_{x_m,g'} |_{\overline{U}_{x_m}})
\]

shows that \( \omega_g(g') \circ \sigma_i \) depends smoothly on \( g' \). \( \square \)

Before coming to the main result of this section we give a description of the image of \( \text{Diff}(M)_P := Q(\text{Aut}(P)) \) in terms of \( P \), without referring to \( \text{Aut}(P) \).

**Remark 2.13.** Let \( Q : \text{Aut}(P) \to \text{Diff}(M), F \mapsto F_M \) be the homomorphism from Definition 1.1. If \( g \in \text{Diff}(M)_P \), then there exists an \( F \in \text{Aut}(P) \) that covers \( g \). Hence the commutative diagram

\[
g^*(P) \xrightarrow{g^p} P \xrightarrow{\pi} P \xrightarrow{g^{-1}} P
\]

shows that \( g^*(P) \) is equivalent to \( P \). On the other hand, if \( P \sim g^*(P) \), then the commutative diagram

\[
P \xrightarrow{\sim} g^*(P) \xrightarrow{g^p} P \xrightarrow{\pi} P \xrightarrow{g^{-1}} P
\]

shows that there is an \( F \in \text{Aut}(P) \) covering \( g \). Thus \( \text{Diff}(M)_P \) consists of those diffeomorphisms preserving the equivalence class of \( P \) under pull-backs. This shows also that \( \text{Diff}(M)_P \) is open because homotopic maps yield equivalent bundles. It thus is contained in \( \text{Diff}(M)_0 \).

Note, that it is not possible to say what \( \text{Diff}(M)_P \) is in general, even in the case of bundles over \( M = S^1 \). In fact, we then have \( \pi_0(\text{Diff}(S^1)) \cong \mathbb{Z}_2 \) (cf. [Mi84]), and the component of \( \text{Diff}(S^1) \), which does not contain the identity, are precisely the orientation reversing diffeomorphisms on \( S^1 \). It follows from the description of equivalence classes of principal bundles over \( S^1 \) by \( \pi_0(K) \) that pulling back the bundle along a orientation reversing diffeomorphism inverts a representing element for the bundle in \( K \). Thus we have \( g^*(P_k) \cong P_{k-1} \) for \( g \notin \text{Diff}(S^1)_0 \). If \( \pi_0(K) \cong \mathbb{Z}_2 \), then \( P_k \) and \( P_{k-1} \) are equivalent because \( [k] = [k^{-1}] \) in \( \pi_0(K) \) and thus \( g \in \text{Diff}(S^1)_P \) and \( \text{Diff}(S^1)_P = \text{Diff}(S^1)_0 \). If \( \pi_0(K) \cong \mathbb{Z}_3 \), then \( P_k \) and \( P_{k-1} \) are not equivalent because \( [k] \neq [k^{-1}] \) in \( \pi_0(K) \) and thus \( g \notin \text{Diff}(S^1)_P \) and \( \text{Diff}(S^1)_P = \text{Diff}(S^1)_0 \).
Theorem 2.14 (Aut(\(\mathcal{P}\)) as an extension of Diff(\(M\)) by Gau(\(\mathcal{P}\))). Let \(\mathcal{P}\) be a smooth principal \(K\)-bundle over the closed compact manifold \(M\). If \(\mathcal{P}\) has the property SUB, then Aut(\(\mathcal{P}\)) carries a Lie group structure such that we have an extension of smooth Lie groups

\[
(14) \quad \text{Gau}(\mathcal{P}) \hookrightarrow \text{Aut}(\mathcal{P}) \xrightarrow{\Omega} \text{Diff}(\mathcal{P}),
\]

where \(\Omega : \text{Aut}(\mathcal{P}) \to \text{Diff}(M)\) is the homomorphism from Definition 1.1 and \(\text{Diff}(\mathcal{P})\) is the open subgroup of \(\text{Diff}(M)\) preserving the equivalence class of \(\mathcal{P}\) under pull-backs.

**Proof.** We identify \(\text{Gau}(\mathcal{P})\) with \(C^\infty(P,K)^K\) and extend \(S\) to a (possibly non-continuous) section \(S : \text{Diff}(\mathcal{P}) \subseteq \text{Diff}(M)\) of \(Q\). Now the preceding lemmas show that \((T,\omega)\) is a smooth pair structure [Ne06a, Proposition II.8], which yields the assertion. \(\square\)

**Proposition 2.15.** In the setting of the previous theorem, the natural action

\[
\text{Aut}(\mathcal{P}) \times P \to P, \quad (F,p) \mapsto F(p)
\]

is smooth.

**Proof.** First we note the \(\text{Gau}(\mathcal{P}) \cong C^\infty(P,K)^K\) acts smoothly on \(P\) by \((\gamma,p) \mapsto p \cdot \gamma(p)\). Let \(O \subseteq \text{Diff}(M)\) be the neighbourhood from Remark 2.4 and \(S : O \to \text{Aut}(\mathcal{P})\), \(g \mapsto \tilde{g}_n \circ \ldots \circ \tilde{g}_1\) be the map from Definition 2.6. Then \(\text{Gau}(\mathcal{P}) \circ S(O)\) is an open neighbourhood in \(\text{Aut}(\mathcal{P})\) and it suffices to show that the restriction of the action to this neighbourhood is smooth. Since \(\text{Gau}(\mathcal{P})\) acts smoothly on \(P\), this in turn follows from the smoothness of the map

\[
R : O \times P \to P, \quad (g,p) \mapsto S(g)(p) = \tilde{g}_n \circ \ldots \circ \tilde{g}_1(p).
\]

To check the smoothness of \(R\) it suffices to check that \(r_i : O \times P \to P \times O\), \((g,p) \mapsto (\tilde{g}_i(p),g)\) is smooth, because then \(R = pr_1 \circ r_n \circ \ldots \circ r_1\) is smooth. Now the explicit formula

\[
\tilde{g}_i(p) = \begin{cases} 
\sigma_i(g_i(p)) \cdot k_i(p) & \text{if } p \in \pi^{-1}(U_i) \\
p & \text{if } p \in \pi^{-1}(\overline{U_i})^c
\end{cases}
\]

shows that \(r_i\) is smooth on \((O \times \pi^{-1}(U_i)) \cup (O \times \pi^{-1}(\overline{U_i})^c) = O \times P\). \(\square\)

**Proposition 2.16.** If \(\mathcal{P}\) is a finite-dimensional smooth principal \(K\)-bundle over the closed compact manifold \(M\), then the action

\[
\text{Aut}(\mathcal{P}) \times \Omega^1(P,\xi) \to \Omega^1(P,\xi), \quad F \mapsto (F^{-1})^* A,
\]

is smooth. Since this action preserves the closed subspace \(\text{Conn}(\mathcal{P})\) of connection 1-forms of \(\Omega^1(P,\xi)\), the restricted action

\[
\text{Aut}(\mathcal{P}) \times \text{Conn}(\mathcal{P}) \to \text{Conn}(\mathcal{P}), \quad F \mapsto (F^{-1})^* A
\]

is also smooth.

**Proof.** As in Proposition 2.15 it can be seen that the canonical action \(\text{Aut}(P) \times TP \to TP\), \(F \cdot X_p = TF(X_p)\) is smooth. Since \(P\) is assumed to be finite-dimensional and the topology on \(\Omega^1(P,\xi)\) is the induced topology from \(C^\infty(TP,\xi)\), the assertion now follows from [Gol06, Proposition 6.4]. \(\square\)

**Remark 2.17.** Of course, the Lie group structure on \(\text{Aut}(\mathcal{P})\) from Theorem 2.14 depends on the choice of \(S\) and thus on the choice of the chart \(\varphi : O \to \mathcal{V}(M)\) from Remark 2.1 and the choice of the trivialising system from Remark 2.2 and the choice of the partition of unity chosen in the proof of Lemma 2.3.

However, different choices lead to isomorphic Lie group structures on \(\text{Aut}(\mathcal{P})\) and, moreover to equivalent extensions. To show this we show that \(\text{id}_{\text{Aut}(\mathcal{P})}\) is smooth when choosing two different trivialising systems \(\overline{\mathcal{V}} = (\overline{V}_i,\sigma_i)_{i=1,\ldots,n}\) and \(\overline{\mathcal{V}} = (\overline{V}_j,\tau_j)_{j=1,\ldots,m}\).
Denote by $S : O \to \text{Aut}(P)$ and $S' : O \to \text{Aut}(P)$ the corresponding sections of $Q$. Since

$$\text{Gau}(P) \circ S(O) = Q^{-1}(O) \circ \text{Gau}(P) \circ S'(O)$$

is an open unit neighbourhood and $\text{id}_{\text{Aut}(P)}$ is an isomorphism of abstract groups, it suffices to show that the restriction of $\text{id}_{\text{Aut}(P)}$ to $Q^{-1}(O)$ is smooth. Now the smooth structure on $Q^{-1}(O)$ induced from $S$ and $S'$ is given by requiring

$$Q^{-1}(O) \ni F \mapsto (F \circ S(F_M)^{-1}, F_M) \in \text{Gau}(P) \times \text{Diff}(M)$$
$$Q^{-1}(O) \ni F \mapsto (F \circ S'(F_M)^{-1}, F_M) \in \text{Gau}(P) \times \text{Diff}(M)$$

to be diffeomorphisms and we thus have to show that

$$O \ni g \mapsto S(g) \circ S'(g)^{-1} \in \text{Gau}(P)$$

is smooth. By deriving explicit formulae for $S(g) \circ S'(g)^{-1}(\sigma(x))$ on a neighbourhood $U_x$ of $x \in V$, and $O_g \ni g \in O$ this follows exactly as in Lemma 2.12.

We have not mentioned regularity so far since it is not needed to obtain the preceding results. However, it is an important concept and we shall elaborate on it now.

**Proposition 2.18.** Let $P$ be a smooth principal $K$-bundle over the compact manifold with corners $M$. If $K$ is regular, then so is $\text{Gau}(P)$ and, furthermore, if $M$ is closed, then $\text{Aut}(P)$ is also regular.

**Proof.** The second assertion follows from the first, because extensions of regular Lie groups by regular ones are themselves regular [OMYK83 Theorem 5.4] (cf. [Ne06b Theorem V.1.8]).

Let $\mathcal{V} = (\mathcal{V}_i, \sigma)$ be a smooth closed trivialising system such that $P$ has the property SUB with respect to $\mathcal{V}$. We shall use the regularity of $C^\infty(\mathcal{V}_i, K)$ to obtain the regularity of $\text{Gau}(P)$.

If $\xi : [0, 1] \to \text{gau}(P)$ is smooth, then this determines smooth maps $\xi_i : [0, 1] \to C^\infty(\mathcal{V}_i, t)$, satisfying $\xi_i(t)(m) = \text{Ad}(k_j(m)) \cdot \xi_j(t)(m)$. By regularity of $C^\infty(\mathcal{V}_i, K)$ this determines smooth maps $\gamma_{\xi,i} : [0, 1] \to C^\infty(\mathcal{V}_i, K)$.

By uniqueness of solutions of differential equations we see that the mappings $t \mapsto \gamma_{\xi,i}(t)$ and $t \mapsto k_{ij} \cdot \gamma_{\xi,j}(t) \cdot k_{ji}$ have to coincide, ensuring $\gamma_{\xi,i}(t)(m) = k_{ij}(m) \cdot \gamma_{\xi,j}(t)(m) \cdot k_{ji}(m)$ for all $t \in [0, 1]$ and $m \in \mathcal{V}_i \cap \mathcal{V}_j$. Thus $[0, 1] \ni t \mapsto (\gamma_{\xi,i}(t))_{i=1, \ldots, n} \in G_P(P)$ is a solution of the corresponding initial value problem and the desired properties follows from the regularity of $C^\infty(\mathcal{V}_i, K)$. 

**Remark 2.19.** A Lie group structure on $\text{Aut}(P)$ has been considered in [ACMMMS9] in the convenient setting, and the interest in $\text{Aut}(P)$ as a symmetry group coupling the gauge symmetry of Yang-Mills theories and the Diff($M$)-invariance of general relativity is emphasised. Moreover, it is also shown that $\text{Gau}(P)$ is a split Lie subgroup of $\text{Aut}(P)$, that

$$\text{Gau}(P) \hookrightarrow \text{Aut}(P) \to \text{Diff}(M)$$

is an exact sequence of Lie groups and that the action $\text{Aut}(P) \times P \to P$ is smooth. However, the Lie group structure is constructed out of quite general arguments allowing to give the space $\text{Hom}(P, P)$ of bundle morphisms a smooth structure and then to consider $\text{Aut}(P)$ as an open subset of $\text{Hom}(P, P)$.

The approach taken in this section is somehow different, since the Lie group structure on $\text{Aut}(P)$ is constructed by foot and the construction provides explicit charts given by charts of $G(P)$ and Diff($M$).

**Remark 2.20.** The approach to the Lie group structure in this section used detailed knowledge on the chart $\varphi : O \to \mathcal{V}(M)$ of the Lie group Diff($M$) from Remark 2.4. We used this when decomposing a diffeomorphism into a product of diffeomorphisms with support in some trivialising subset of $M$. The fact that we needed was that for a diffeomorphism $g \in O$ we have $g(m) = m$ if the vector field $\varphi(g)$ vanishes in $m$. This should also be true for the charts on Diff($M$) for compact manifolds with corners and thus the procedure of this section should carry over to bundles over manifolds with corners.
Example 2.21 \((\text{Aut}(C_k^\infty(S^1, \mathfrak{t})))\). Let \(K\) be a simple finite-dimensional Lie group, \(K_0\) be compact and simply connected and \(P_k\) be a smooth principal \(K\)-bundle over \(S^1\), uniquely determined up to equivalence by \([k] \in \pi_0(K)\). Identifying the twisted loop algebra

\[ C_k^\infty(S^1, \mathfrak{t}) := \{ \eta \in C^\infty(\mathbb{R}, \mathfrak{t}) : \eta(x + n) = \text{Ad}(k)^{-n} \cdot \eta(x) \text{ for all } x \in \mathbb{R}, n \in \mathbb{Z} \}, \]

with the gauge algebra of the flat principal bundle \(P_k\), we get a smooth action of \(\text{Aut}(P_k)\) on \(C_k^\infty(S^1, \mathfrak{t})\), which can also be lifted to the twisted loop group \(C_k^\infty(S^1, K)\), the affine Kac–Moody algebra \(C_k^\infty(S^1, \mathfrak{t})\) and to the affine Kac–Moody group \(C_k^\infty(S^1, K)\) [We06a]. Various results (cf. [Le80, Theorem 16]) assert that each automorphism of \(C_k^\infty(S^1, \mathfrak{t})\) arises in this way and we thus have a geometric description of \(\text{Aut}(C_k^\infty(S^1, \mathfrak{t})) \cong \text{Aut}(C_k^\infty(S^1, K)_0) \cong \text{Aut}(P_k)\) for \(C_k^\infty(S^1, K)_0\) is simply connected. Furthermore, this also leads to topological information on \(\text{Aut}(C_k^\infty(S^1, \mathfrak{t}))\), since we get a long exact homotopy sequence

\[
\cdots \to \pi_{n+1}(\text{Diff}(S^1)) \xrightarrow{\delta_{n+1}} \pi_n(C_k^\infty(S^1, K)) \to \pi_n(\text{Aut}(P_k)) \to \pi_n(\text{Diff}(S^1)) \xrightarrow{\delta_n} \pi_{n-1}(C_k^\infty(S^1, K)) \to \cdots
\]

induced by the locally trivial bundle \(\text{Gau}(P_k) \to \text{Aut}(P_k) \xrightarrow{q} \text{Diff}(S^1)\) and the isomorphisms \(\text{Gau}(P_k) \cong C_k^\infty(S^1, K)\) and \(\text{Aut}(P_k) \cong \text{Aut}(C_k^\infty(S^1, \mathfrak{t}))\). E.g., in combination with

\[
\pi_n(\text{Diff}(S^1)) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n = 0 \\ \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}
\]

(cf. [Mi81]), one obtains information on \(\pi_n(\text{Aut}(P_k))\). In fact, consider the exact sequence

\[
0 \to \pi_1(C_k^\infty(S^1, K)) \to \pi_1(\text{Aut}(P_k)) \to \pi_1(\text{Diff}(M)) \xrightarrow{\delta_1} \pi_0(C_k^\infty(S^1, K))
\]

\[
\cong \pi_0(\text{Aut}(P_k)) \xrightarrow{\pi_0(q)} \pi_0(\text{Diff}(S^1)\text{Pr}_k)
\]

induced by \((15)\) and \((16)\). Since \(\pi_1(C_k^\infty(S^1, K))\) vanishes, this implies \(\pi_1(\text{Aut}(P_k)) \cong \mathbb{Z}\). A generator of \(\pi_1(\text{Diff}(S^1))\) is \(\text{id}_{S^1}\), which lifts to a generator of \(\pi_1(\text{Aut}(P_k))\). Thus the connecting homomorphism \(\delta_1\) vanishes. The argument from Remark 2.13 shows precisely that \(\pi_0(\text{Diff}(S^1)\text{Pr}_k) \cong \mathbb{Z}_2\) if and only if \(k^2 \in K_0\) and that \(\pi_0(q)\) is surjective. We thus end up with an exact sequence

\[
\text{Fix}_{\pi_0(K)}([k]) \to \pi_0(\text{Aut}(P_k)) \to \begin{cases} \mathbb{Z}_2 & \text{if } k^2 \in K_0 \\ 1 & \text{else} \end{cases}
\]

Since \((16)\) implies that \(\text{Diff}(S^1)\) is a \(K(1, \mathbb{Z})\), we also have \(\pi_n(\text{Aut}(P_k)) \cong \pi_n(C_k^\infty(S^1, K))\) for \(n \geq 2\).

Remark 2.22. The description of \(\text{Aut}(C_k^\infty(S^1, \mathfrak{t}))\) in Example 2.21 should arise out of a general principle, describing the automorphism group for gauge algebras of flat bundles, i.e., of bundles of the form

\[
P_\varphi = \tilde{M} \times K / \sim \text{ where } (m, k) \sim (m \cdot d, \varphi^{-1}(d) \cdot k).
\]

Here \(\varphi : \pi_1(M) \to K\) is a homomorphism and \(\tilde{M}\) is the simply connected cover of \(M\), on which \(\pi_1(M)\) acts canonically. Then

\[
\text{gau}(P) \cong C_\varphi^\infty(M, \mathfrak{t}) := \{ \eta \in C^\infty(\tilde{M}, \mathfrak{t}) : \eta(m \cdot d) = \text{Ad}(\varphi(d))^{-1} \cdot \eta(m) \}.
\]

and this description should allow to reconstruct gauge transformations and diffeomorphisms out of the ideals of \(C_\varphi^\infty(M, \mathfrak{t})\) (cf. [Le80]).
Problem 2.23. (cf. [Ne06b, Problem IX.5]) Let $\mathcal{P}_x$ be a (flat) principal $K$-bundle over the closed compact manifold $M$. Determine the automorphism group $\text{Aut}(\text{gau}(\mathcal{P}))$. In which cases does it coincide with $\text{Aut}(\mathcal{P})$ (the main point here is the surjectivity of the canonical map $\text{Aut}(\mathcal{P}) \to \text{Aut}(\text{gau}(\mathcal{P}))$).

Remark 2.24. In some special cases, the extension $\text{gau}(\mathcal{P}) \hookrightarrow \text{Aut}(\mathcal{P}) \to \text{Diff}(M)_\mathcal{P}$ from Theorem 2.14 splits. This is the case for trivial bundles and for bundles with abelian structure group $K$, but also for frame bundles, since we then have a natural homomorphism $\text{Diff}(M) \to \text{gau}(\mathcal{P})$, $g \mapsto dg$. However, it would be desirable to have a characterisation of the bundles, for which this extension splits.

Problem 2.25. (cf. [Ne06b, Problem V.5]) Find a characterisation of those principal $K$-bundles $\mathcal{P}$ for which the extension $\text{gau}(\mathcal{P}) \hookrightarrow \text{Aut}(\mathcal{P}) \to \text{Diff}(M)_\mathcal{P}$ splits on the group level.

A Appendix: Differential calculus on spaces of mappings

Definition A.1. Let $X$ and $Y$ be a locally convex spaces and $U \subseteq X$ be open. Then $f : U \to Y$ is differentiable or $C^1$ if it is continuous, for each $v \in X$ the differential quotient

$$df(x)_v := \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

exists and if the map $df : U \times X \to Y$ is continuous. If $n > 1$ we inductively define $f$ to be $C^n$ if it is $C^1$ and $df$ is $C^{n-1}$ and to be $C^\infty$ or smooth if it is $C^n$. We say that $f$ is $C^\infty$ or smooth if $f$ is $C^n$ for all $n \in \mathbb{N}_0$. We denote the corresponding spaces of maps by $C^n(U, Y)$ and $C^\infty(U, Y)$.

Definition A.2. Let $X$ and $Y$ be locally convex spaces, and let $U \subseteq X$ be a set with dense interior. Then $f : U \to Y$ is differentiable or $C^1$ if it is continuous, $f_{\text{int}} := f|_{\text{int}(U)}$ is $C^1$ and the map

$$d(f_{\text{int}}) : \text{int}(U) \times X \to Y, \ (x, v) \mapsto d(f_{\text{int}})(x)_v$$

extends to a continuous map on $U \times X$, which is called the differential $df$ of $f$. If $n > 1$ we inductively define $f$ to be $C^n$ if it is $C^1$ and $df$ is $C^{n-1}$. We say that $f$ is $C^\infty$ or smooth if $f$ is $C^n$ for all $n \in \mathbb{N}_0$. We denote the corresponding spaces of maps by $C^n(U, Y)$ and $C^\infty(U, Y)$.

Definition A.3. From the definition above, the notion of a Lie group is clear. It is a group which is a smooth manifold modelled on a locally convex space such that the group operations are smooth. Moreover, the notion of a finite-dimensional manifold with corners is clear, i.e., a smooth manifold modelled on $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$ (cf. [Mi80] and [Wo06c]).

Proposition A.4. Let $G$ be a group with a locally convex manifold structure on some subset $U \subseteq G$ with $e \in U$. Furthermore, assume that there exists $V \subseteq U$ open such that $e \in V$, $VV \subseteq U$, $V = V^{-1}$ and

\begin{itemize}
  \item[i)] $V \times V \to U$, $(g, h) \mapsto gh$ is smooth,
  \item[ii)] $V \to V$, $g \mapsto g^{-1}$ is smooth,
  \item[iii)] for all $g \in G$, there exists an open unit neighbourhood $W \subseteq U$ such that $g^{-1}Wg \subseteq U$ and the map $W \to U$, $h \mapsto g^{-1}hg$ is smooth.
\end{itemize}

Then there exists a unique locally convex manifold structure on $G$ which turns $G$ into a Lie group, such that $V$ is an open submanifold of $G$.

Definition A.5. Let $G$ be a locally convex Lie group. The group $G$ is said to have an exponential function if for each $x \in g$ the initial value problem

$$\gamma(0) = e, \quad \gamma'(t) = T\lambda_{\gamma(t)}(e)x$$

has a solution $\gamma_x \in C^\infty(\mathbb{R}, G)$ and the function
\[
\exp_G : \mathfrak{g} \to G, \quad x \mapsto \gamma_x(1)
\]
is smooth. Furthermore, if there exists a zero neighbourhood $W \subseteq \mathfrak{g}$ such that $\exp_{G|W}$ is a diffeomorphism onto some open unit neighbourhood of $G$, then $G$ is said to be \textit{locally exponential}.

**Lemma A.6.** If $G$ and $G'$ are locally convex Lie groups with exponential function, then for each morphism $\alpha : G \to G'$ of Lie groups and the induced morphism $d\alpha(e) : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras, the diagram
\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & G' \\
\exp_G & \uparrow & \exp_{G'} \\
\mathfrak{g} & \xrightarrow{d\alpha(e)} & \mathfrak{g}'
\end{array}
\]
commutes.

**Remark A.7.** The Fundamental Theorem of Calculus for locally convex spaces (cf. [Gl02a, Theorem 1.5]) yields that a locally convex Lie group $G$ can have at most one exponential function (cf. [Sc06b, Lemma II.3.5]). If $G$ is a Banach-Lie group (i.e., $\mathfrak{g}$ is a Banach space), then $G$ is locally exponential due to the existence of solutions of differential equations, their smooth dependence on initial values ([La99, Chapter IV]) and the Inverse Mapping Theorem for Banach spaces ([La99, Theorem I.5.2]. In particular, each finite-dimensional Lie group is locally exponential.

**Definition A.8.** If $X$ is a Hausdorff space and $Y$ is a topological spaces, then the \textit{compact-open topology} on the space of continuous functions is defined as the topology generated by the sets of the form
\[
[C, W] := \{ f \in C(X, Y) : f(C) \subseteq W \},
\]
where $C$ runs over all compact subsets of $X$ and $W$ runs over all open subsets of $Y$. We write $C(X, Y)_c$ for the space $C(X, Y)$ endowed with the compact-open topology.

If $G$ is a topological group, then $C(X, G)$ is a group with respect to pointwise group operation. Furthermore, the topology of compact convergence coincides with the compact-open topology ([Bo89a, Theorem X.3.4.2]) and thus $C(X, G)_c$ is again a topological group. A basis of unit neighbourhoods of this topology is given by $[C, W]$, where $C$ runs over all compact subsets of $X$ and $W$ runs over all open unit neighbourhoods of $G$. If $X$ itself is compact, then this basis is already given by $[X, W]$, where $W$ runs over all unit neighbourhoods of $G$.

If $Y$ is a locally convex space, then $C(X, Y)$ is a vector space with respect to pointwise operations. The preceding discussion implies that addition is continuous and scalar multiplication is also continuous. Since its topology is induced by the seminorms
\[
p_C : C(X, Y) \to \mathbb{K}, \quad f \mapsto \sup_{x \in C} \{ p(f(x)) \},
\]
where $C$ runs over all compact subsets of $X$ and $p$ runs over all seminorms, defining the topology on $Y$, we see that $C(X, Y)_c$ is again locally convex.

If $M$ and $N$ are manifolds with corners as in [Wo06c], then every smooth map $f : M \to N$ defines a sequence of continuous map $T^n f : T^n M \to T^n N$ on the iterated tangent bundles. We thus obtain an inclusion
\[
C^\infty(M, N) \hookrightarrow \prod_{n=0}^\infty C(T^n M, T^n N)_c, \quad f \mapsto (T^n f)_{n \in \mathbb{N}}
\]
and we define the $C^\infty$-\textit{topology} on $C^\infty(M, N)$ to be the initial topology induced from this inclusion. For a locally convex space $Y$ we thus get a locally convex vector topology on $C^\infty(M, Y)$.

If $E = (Y, \xi : E \to X)$ is a continuous vector bundle and $S_c(E)$ is the set of continuous sections, then we have an inclusion $S_c(E) \hookrightarrow C(X, E)$ and we thus obtain a topology on $S_c(E)$. If $E$ is also smooth, then we have an inclusion $S(E) \hookrightarrow C^\infty(M, E)$, inducing a topology $S(E)$, which we also call $C^\infty$-topology.
Remark A.9. If $M$ is a manifold with corners and $Y$ is a locally convex space, then we can describe the $C^\infty$-topology on $C^\infty(M,Y)$ alternatively as the initial topology with respect to the inclusion

$$C^\infty(M,Y) \hookrightarrow \prod_{n=0}^\infty C(T^n M,Y), \quad f \mapsto (d^n f)_{n\in\mathbb{N}},$$

where $d^n f = \text{pr}_n \circ T^n f$. In fact, we have $Tf = (f, df)$ and we can inductively write $T^n f$ in terms of $d^l f$ for $l \leq n$. This implies for a map into $C^\infty(M,Y)$ that its composition with each $d^n$ is continuous if and only if its composition with all $T^n$ is continuous. Because the initial topology is characterised by this property, the topologies coincide.

Definition A.10. If $E = (Y, \xi : E \to M)$ is a smooth vector bundle and $p \in \mathbb{N}_0$, then an $E$-valued $p$-form on $M$ is a function $\omega$ which associates to each $m \in M$ a $p$-linear alternating map $\omega_m : (T_m M)^p \to E_m$ such that in local coordinates the map

$$(m, X_1, m, \ldots, X_p, m) \mapsto \omega_m(X_1, m, \ldots, X_p, m)$$

is smooth. We denote by

$$\Omega^p(M, E) := \{\omega : \bigcup_{m \in M} (T_m M)^p \to E : \omega \text{ is a } E \text{-valued } p \text{-form on } M\}$$

the space of $E$-valued $p$-forms on $M$ which has a canonical vector space structure induced from pointwise operations.

Remark A.11. If $E = (Y, \xi : E \to M)$ is a smooth vector bundle over the finite-dimensional manifold $M$, then each $E$-valued $p$-form maps vector fields $X_1, \ldots, X_p$ to a smooth section $\omega(X_1, \ldots, X_p) := \omega \circ (X_1 \times \cdots \times X_p)$ in $S(E)$, which is $C^\infty(M, \mathbb{R})$-linear by definition. Conversely, any alternating $C^\infty(M)$-linear map $\omega : \mathcal{V}^p(M) \to S(E)$ determines uniquely an element of $\Omega^p(M, E)$ by setting

$$\omega_m(X_1, \ldots, X_n, m) := \omega(\tilde{X}_1, \ldots, \tilde{X}_p)(m),$$

where $\tilde{X}_i$ is an extension of $X_{i, m}$ to a smooth vector field. That $\omega_m(X_1, m, \ldots, X_p, m)$ does not depend on the choice of this extension follows from the $C^\infty(M, \mathbb{R})$-linearity of $\omega$, if one expands different choices in terms of basis vector fields. Note that the assumption on $M$ to be finite-dimensional is crucial for this argument.

We now consider the continuity properties of some very basic maps, i.e., restriction maps and gluing maps. These maps we shall encounter often in the sequel.

Lemma A.12. If $E$ is a smooth vector bundle over $M$ and $U \subseteq M$ is open and $E|_U = E|_U$, is the restricted vector bundle, then the restriction map $\text{res}_U : S(E) \to S(E|_U)$, $\sigma \mapsto \sigma|_U$, is continuous. If, moreover, $Y$ is a manifold with corners, then the restriction map $\text{res}_Y : S(E) \to S(E|_Y)$, $\sigma \mapsto \sigma|_Y$, is continuous.

Proposition A.13. If $E$ is a smooth vector bundle over the finite-dimensional manifold with corners $M$ and $S(E)$ is the vector space of smooth sections with pointwise operations, then the $C^\infty$-topology is a locally convex vector topology on $S(E)$. Furthermore, if $(U_i)_{i \in I}$ is an open cover of $M$ such that each $\overline{U}_i$ is a manifold with corners and $E_i := E|_{\overline{U}_i}$ denotes the restricted bundle, then the $C^\infty$-topology on $S(E)$ is initial with respect to

$$\text{res} : S(E) \to \prod_{i \in I} S(E_i), \quad \sigma \mapsto (\sigma|_{\overline{U}_i})_{i \in I}. \quad (17)$$

Corollary A.14. The restriction maps $\text{res}_U$ and $\text{res}_Y$ from Lemma A.12 are smooth.
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**Proposition A.15.** If $E$ is a smooth vector bundle over the finite-dimensional manifold with corners $M$, $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of $M$ such that each $\overline{U}_i$ is a manifold with corners and $E_i := E|_{\overline{U}_i}$ denotes the restricted bundle, then

$$S_{\mathcal{U}}(E) = \{ (\sigma_i)_{i \in I} \in \bigoplus_{i \in I} S(E_i) : \sigma_i(x) = \sigma_j(x) \text{ for all } x \in \overline{U}_i \cap \overline{U}_j \}$$

is a closed subspace of $\bigoplus_{i \in I} S(E_i)$ and the gluing map

$$\text{glue} : S_{\mathcal{U}}(E) \to S(E), \quad \text{glue}(\sigma_i)_{i \in I})(x) = \sigma_i(x) \text{ if } x \in \overline{U}_i$$

is inverse to the restriction map $\text{res}$.

**Corollary A.16.** If $E$ is a smooth vector bundle over the finite-dimensional manifold with corners $M$, $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of $M$ and $E_i := E|_{U_i}$ denotes the restricted bundle, then

$$S_{U_i}(E) = \{ (\sigma_i)_{i \in I} \in \bigoplus_{i \in I} S(E_i) : \sigma_i(x) = \sigma_j(x) \text{ for all } x \in U_i \cap U_j \}$$

is a closed subspace of $\bigoplus_{i \in I} S(E_i)$ and the gluing map

$$\text{glue} : S_{U_i}(E) \to S(E), \quad \text{glue}(\sigma_i)_{i \in I})(x) = \sigma_i(x) \text{ if } x \in U_i$$

is inverse to the restriction map.

**Lemma A.17.** If $M$ is a compact manifold with corners, $K$ is a Lie group and $\overline{U} \subseteq M$ is a manifold with corners, then the restriction

$$\text{res} : C^\infty(M, K) \to C^\infty(\overline{U}, K), \quad \gamma \mapsto \gamma|_{\overline{U}}$$

is a smooth homomorphism of Lie groups.

**Proposition A.18.** Let $K$ be a Lie group, $M$ be a compact manifold with corners with an open cover $\mathcal{U} = (V_1, \ldots, V_n)$ such that $\overline{\mathcal{U}} = (\overline{V}_1, \ldots, \overline{V}_n)$ is a cover by manifolds with corners. Then

$$G_{\overline{\mathcal{U}}} := \{ (\gamma_1, \ldots, \gamma_n) \in \prod_{i=1}^n C^\infty(\overline{V}_i, K) : \gamma_i(x) = \gamma_j(x) \text{ for all } x \in V_i \cap V_j \}$$

is a closed subgroup of $\prod_{i=1}^n C^\infty(\overline{V}_i, K)$, which is a Lie group modelled on the closed subspace

$$\mathfrak{g}_{\overline{\mathcal{U}}} := \{ (\eta_1, \ldots, \eta_n) \in \prod_{i=1}^n C^\infty(\overline{V}_i, \mathfrak{g}) : \eta_i(x) = \eta_j(x) \text{ for all } x \in V_i \cap V_j \}$$

of $\bigoplus_{i=1}^n C^\infty(\overline{V}_i, \mathfrak{g})$ and the gluing map

$$\text{glue} : G_{\overline{\mathcal{U}}} \to C^\infty(M, K), \quad \text{glue}(\gamma_1, \ldots, \gamma_n) = \gamma_i(x) \text{ if } x \in \overline{V}_i$$

is an isomorphism of Lie groups.

**Lemma A.19.** Let $M$ be a smooth locally convex manifold with corners, $G$ be a locally convex Lie group and $\lambda : G \times Y \to Y$ be a smooth linear action on the locally convex space $Y$. If $h : M \to G$ and $f : M \to Y$ are smooth, then we have

$$d(\lambda(h) f) . X_m = \lambda(h) . (df . X_m) + \dot{\lambda} (\text{Ad}(h) . \delta f(h) . X_m) . (\lambda(h(m)) f(m))$$

with $\lambda(h^{-1}) f : M \to E$, $m \mapsto \lambda(h(m)^{-1}) f(m)$. If $\lambda = \text{Ad}$ is the adjoint action of $G$ on $\mathfrak{g}$, then we have

$$d (\text{Ad}(h) f) . X_m = \text{Ad}(h) . (df . X_m) + \text{Ad}(h) . [\delta f(h) . X_m, f(m)]$$
Definition A.20. Let $\mathcal{P} = (K, \pi : P \to X)$ be a continuous principal $K$-bundle. If $(U_i)_{i \in I}$ is an open cover of $X$ by trivialising neighbourhoods and $(\sigma_i : U_i \to P)_{i \in I}$ is a collection of continuous sections, then the collection $\mathcal{U} = (U_i, \sigma_i)_{i \in I}$ is called a continuous open trivialising system of $\mathcal{P}$.

If $(\mathcal{U}_i)_{i \in I}$ is a closed cover of $X$ by trivialising sets and $(\sigma_i : U_i \to P)_{i \in I}$ is a collection of continuous sections, then the collection $\overline{\mathcal{U}} = (\overline{U}_i, \sigma_i)_{i \in I}$ is called a continuous closed trivialising system of $\mathcal{P}$.

If $\mathcal{U} = (U_i, \sigma_i)_{i \in I}$ and $\mathcal{V} = (V_j, \tau_j)_{j \in J}$ are two continuous open trivialising systems of $\mathcal{P}$, then $\mathcal{V}$ is a refinement of $\mathcal{U}$ if there exists a map $J \ni j \mapsto i(j) \in I$ such that $V_j \subseteq U_{i(j)}$ and $\tau_j = \sigma_{i(j)}|_{V_j}$, i.e., $(V_j)_{j \in J}$ is a refinement of $(U_i)_{i \in I}$ and the sections $\tau_j$ are obtained from the section $\sigma_i$ by restrictions.

If $\mathcal{U} = (U_i, \sigma_i)_{i \in I}$ is a continuous open trivialising system and $\overline{\mathcal{V}} = (\overline{V}_j, \tau_j)_{j \in J}$ is a continuous closed trivialising system, then $\overline{\mathcal{V}}$ is a refinement of $\mathcal{U}$ if there exists a map $I \ni i \mapsto j(i) \in J$ such that $\overline{V}_j \subseteq U_{i(j)}$ and $\tau_j = \sigma_{i(j)}|_{\overline{V}_j}$ and vice versa.

Furthermore, if $\mathcal{P}$ is a smooth principal $K$-bundle over $M$, then a smooth open trivialising system $\mathcal{U}$ of $\mathcal{P}$ consists of an open cover $(U_i)_{i \in I}$ and smooth sections $\sigma_i : U_i \to P$. If each $U_i$ is also a manifold with corners and the section $\sigma_i$ can be extended to smooth sections $\sigma_i : \overline{U}_i \to P$, then $\overline{\mathcal{U}} = (\overline{U}_i, \sigma_i)_{i \in I}$ is called a smooth closed trivialising system of $\mathcal{P}$. In this case, $\mathcal{U}$ is called the trivialising system underlying $\overline{\mathcal{U}}$.

For each kind of trivialising system, the sections define continuous maps $k_{ij} : U_i \cap U_j \to K$ (respectively $k_{ij} : \overline{U}_i \cap \overline{U}_j \to K$ in the case of a closed trivialising system) by

$$k_{ij}(x) = k_{ij}(\sigma_i(x)) \quad \text{or equivalently} \quad \sigma_i(x) \cdot k_{ij}(x) = k_{ij}(x),$$

called transition functions.

Lemma A.21. Let $X$ be a compact space, $K$ be topological group and $(O_l)_{l \in L}$ be an open cover of $K$. If $\mathcal{P}$ is a continuous principal $K$-bundle over $X$, then for each continuous open trivialising system $\mathcal{U} = (U_i, \sigma_i)_{i = 1, \ldots, n}$ there exists a refinement $\mathcal{V} = (V_s, \tau_s)_{s = 1, \ldots, r}$ such that for each transition function $k_{st} : V_s \cap V_t \to K$ of $\mathcal{V}$ we have $k_{st}(V_s \cap V_t) \subseteq O_\ell$ for some $\ell \in L$.

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