A New Look at the Higgs-Kibble Model

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Dedicated to Wolfhart Zimmermann on the occasion of his 80th birthday

Abstract

An elementary perturbative method of handling the Higgs-Kibble models and deriving their relevant properties, is described. It is based on Wightman field theory and avoids some of the mathematical weaknesses of the standard treatments. The method is exemplified by the abelian case. Its extension to the non-abelian gauge group $SU_2$ is shortly discussed in the last section.

1 Introduction

The spontaneous breaking of gauge invariance as described by the Higgs-Kibble model (henceforth HKM) is an essential ingredient of the electro-weak part of the standard model of elementary particle physics. In the present work we will report on a new, rather elementary, method of deriving the properties of the model, in particular its renormalizability (or lack thereof, see Sect.6).

Our method is entirely perturbative, it consists predominantly in studying the properties of so-called ‘sector graphs’, a simple generalization of Feynman graphs. But the corresponding graph rules are derived in an unconventional way. We do not use path integrals, a not entirely convincing method because of the lack of a solid mathematical underpinning. Nor do we use the canonical formalism with its own weak points, like the dubious status of the canonical commutation relations on account of the non-existence of interacting fields at a sharp time, and the need for introducing and handling constraints. Instead we work with an adaptation of the method introduced in [6] for QED, where many details are found beyond what can be reported here.

We will concentrate on the case of the abelian HKM. The extension of our method, and of its results, to the non-abelian case will, however, be briefly described in the last section. Also, we will work throughout at a formal, non-renormalized level, only getting as far as obtaining the power-counting behavior necessary for establishing renormalizability. This sticking to non-renormalized expressions is not as bad as it sounds. We propose that the theory be renormalized by Zimmermann’s method (known as BPHZ, see [9]), which consists in
subtracting not integrals, but the integrands of the Feynman graphs or, in our case, the sector graphs. And the cancellations between graphs that we need to establish for obtaining renormalizability, also happen for the integrands. Therefore we need only talk about the well defined integrands, and the divergence (before renormalization) of the integration over them need not unduly bother us.

2 The Model

Let us start with a brief reminder of the definition of the HKM. The abelian HKM is a relativistic field theory containing a complex scalar field $\Phi(x)$ and a real vector field $A_\mu(x)$. Its dynamics is specified by the Lagrangian

$$L = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + (\partial_\alpha - igA_\alpha) \Phi^* (\partial^\alpha + igA^\alpha) \Phi + \mu^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2$$

with

$$F_{\alpha\beta}(x) = \partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x).$$

$g, \lambda, \mu,$ are positive real numbers. An important feature of this Lagrangian is the ‘wrong’ sign of the mass term $\mu^2 \Phi^* \Phi$. $L$ is invariant under the gauge transformation

$$\Phi(x) \Rightarrow \exp[ig \vartheta(x)] \Phi(x), \quad A_\mu(x) \Rightarrow A_\mu(x) - \partial_\mu \vartheta(x)$$

for real functions $\vartheta$.

Because of the unconventional mass term, the field equations derived from $L$ possess the non-trivial classical solution of lowest energy

$$\Phi(x) = \Phi^*(x) = \frac{v}{\sqrt{2}}, \quad A_\mu(x) = 0$$

with

$$v = \mu/\sqrt{\lambda} > 0.$$  

Other solutions of the same lowest energy are generated from (1) by applying gauge transformations (3). But they are of no concern to us.

Our perturbative quantum solution consists essentially in a quantum expansion around the real solution (4). We make the ansatz

$$\Phi(x) = \frac{1}{\sqrt{2}} (v + R(x) + i I(x)),$$

where $R$ and $I$ are two real fields. Henceforth we treat $A_\mu, R, I,$ as the fundamental fields of the model, while $\Phi$ is forgotten. With these new fields the solution (4) takes the trivial form

$$A_\mu = R = I = 0.$$  

\footnote{The standard lore about spontaneous symmetry breaking can be found e.g. in \cite{3,8}.}
The gauge transformation (3) can be transcribed into the new field $s$. We will not write the result down since we are not going to use it, apart from the important fact that $F_{\alpha\beta}$ and the ‘Higgs field’

$$\Psi(x) = R(x) + \frac{1}{2v}[R^2(x) + I^2(x)]$$

are gauge invariant.

The Lagrangian (1) can also be transcribed into the new fields. It takes the form

$$L = L_2 + L_3 + L_4,$$

where $L_i$ collects the terms of order $i$ in the fields. A constant term $L_0$ has been dropped as being immaterial. Furthermore, we replace $\lambda, \mu, \ldots$ as parameters of the theory by

$$m = g v = \frac{g \mu}{\sqrt{\lambda}}, \quad M = \sqrt{2} \mu,$$

which denote the masses of the gauge boson and the Higgs particle respectively. They are therefore measurable quantities (barring the need for renormalization), and they will as usual be kept fixed. Perturbation theory amounts then to a power series expansion in the remaining coupling constant $g$. The $L_i$ read

$$L_2 = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{m^2}{2} A_\alpha A^\alpha + m A^\alpha \partial_\alpha I + \frac{1}{2} (\partial_\alpha R \partial^\alpha R - M^2 R^2) + \frac{1}{2} \partial_\alpha I \partial^\alpha I,$$

$$L_3 = -g A_\alpha I \partial_\alpha R + g A^\alpha \partial_\alpha I R + g m A_\alpha A^\alpha R - \frac{g M^2}{2m} R^3 - \frac{g M^2}{2m} R I^2,$$

$$L_4 = \frac{1}{2} g^2 A_\alpha A^\alpha R^2 + \frac{1}{2} g^2 A_\alpha A^\alpha I^2 - \frac{g^2 M^2}{8m^2} R^4 - \frac{g^2 M^2}{8m^2} I^4 - \frac{g^2 M^4}{4m^2} R^2 I^2.$$

$L_2$ will be responsible for the propagators of our graph rules, $L_{int} = L_3 + L_4$ for the vertices. The Higgs field takes the form

$$\Psi(x) = R(x) + \frac{g}{2m} [R^2(x) + I^2(x)].$$

In our method the dynamics is embodied in the field equations rather than in the Lagrangian. They take the form

$$\begin{align*}
(\Box + m^2) A^\mu - \partial^\mu \partial_\nu A^\nu + m \partial^\mu I &= -\frac{\delta L_{int}}{\delta A_\mu} =: \mathcal{R}^\mu(x), \\
-\Box I - m \partial_\nu A^\nu &= -\frac{\delta L_{int}}{\delta I} =: \mathcal{R}_I(x), \\
-(\Box + M^2) R &= -\frac{\delta L_{int}}{\delta R} =: \mathcal{R}_R(x).
\end{align*}$$
As a consequence of the gauge freedom of the theory we note the following fact. Applying the derivation $\partial_\mu$ to the left-hand side of (15) we obtain the left-hand side of (16), up to a constant factor. The equations (15)–(17) can therefore possess solutions only if the consistency condition

$$\mathcal{F} := \partial_\mu R^\mu + m R_I = 0$$

(18)
is satisfied. That this condition is satisfied in our case is essentially a consequence of the field equations having been derived from a Lagrangian. It must, however, be noted that in an explicit verification the field equations must be used. This verification runs as follows. As contribution of $L_3$ to $\mathcal{F}$ we find

$$\mathcal{F}_3 = g I (\Box + M^2) R - g R (\Box I + m \partial_\mu A^\mu) .$$

(19)

Using the field equations (16) and (17) this becomes a polynomial of order 3 in the fields which exactly cancels the $L_4$-contribution

$$\mathcal{F}_4 = -g^2 R^2 \partial_\mu A^\mu - 2 g^2 R R A^\mu - g^2 I^2 \partial_\mu A^\mu - 2 g^2 I \partial_\mu A^\mu - g^2 I A^\mu - 2 g^2 I A^\mu - g^2 M^2 I + g^2 M^2 R^2 I .$$

(20)

This looks at first like a consistency check rather than a proof. It is, however, perfectly acceptable as a proof in perturbation theory.

We will not endeavor to give a general definition of what we understand under a particular gauge of this model. But the following statement is essential. A quantum field theory claiming to be the HKM in a particular gauge must satisfy the field equations (15)–(17). In the following section a particular class of gauges will be constructed.

### 3 Wightman Gauges

Under ‘Wightman gauges’ we understand a class of quantum field theories solving the field equations (15)–(17), and moreover satisfying the Wightman axioms (see [7]), i.e. Poincaré covariance, locality, spectral condition, existence of a vacuum, and the cluster property, with the possible exception of positivity. This last condition can in general not be expected to hold in a gauge theory. Two special cases of Wightman gauges will be of particular interest to us. The first is the ‘unitary’ or ‘physical’ gauge, which allows to specify the physical content of the theory. And the second is the ‘renormalization’ gauge, which is particularly suited for establishing the renormalizability of the physically relevant part of the model.

Our method consists essentially in a recursive solution of the field equations. But the fundamental objects of the approach are the Wightman functions (W-functions), not the field operators themselves, and also not the Green’s functions of the conventional methods. The W-functions are the vacuum expectation values of ordinary (not time ordered) products of field operators. According to
Wightman’s reconstruction theorem \[7\] the theory is fully determined by these W-functions. The field equations applied to any factor in a W-function produce a set of differential equations for these functions. And this set of differential equations we solve recursively.

The resulting expression for a given function \((\Omega, \varphi_1(x_1) \cdots \varphi_n(x_n) \Omega)\), \(\varphi_i\) any of the fundamental fields \(A_\mu, I, R\), in a given order \(g^7\) of perturbation theory can be written as a sum over generalized Feynman graphs called ‘sector graphs’. A sector graph looks at first just like an ordinary Feynman graph not containing any vacuum-vacuum subgraphs. But its vertices are then partitioned into non-overlapping subsets called ‘sectors’, in such a way that each sector contains at most one external point corresponding to one of the fields in \(W\). Lines connecting vertices (including the external points) in the same sector belong to this sector and are called ‘sector lines’. Lines connecting points in different sectors are called ‘cross lines’. The sectors can be of two types, \(T^+\) or \(T^-\). They are numbered such that the sector containing the external point belonging to \(\varphi_i\) carries the number \(i\). It is convenient to alternate the corresponding sectors: sectors with an odd number are \(T^-\), those with an even number \(T^+\), or vice versa. In this case there occur no sectors without external points.

The internal vertices correspond as usual to the terms in \(L_{int}\) as listed in \(12\), \(13\). Their vertex factors are also the conventional ones in \(T^+\) sectors, their complex conjugates in \(T^-\) sectors. E.g. the last term in \(12\) produces a vertex with one \(R\)-line and two \(I\)-lines joining, and with the vertex factor \(\mp g m^{-1} M^2\) in \(T^\pm\) sectors. Note that the \(L_4\)-vertices are of second order in \(g\). A cross line joining the vertex with variable \(u\) in sector \(i\) to the vertex \(v\) in sector \(j\), \(j > i\), carries the ‘cross propagator’

\[ w_{ab}(u - v) = \langle \varphi_a(u), \varphi_b(v) \rangle_0, \tag{21} \]

where \(\langle \varphi_a \varphi_b \rangle_0\) is a free 2-point function (to be specified below), and the indices \(a, b\), signify the field types of the ends of the line in question. A sector line connecting the vertices \(u\) and \(v\) in a \(T^\pm\) sector carries as propagator the time ordered or anti-time ordered function \(\tau^\pm_{ab}(u - v)\) corresponding to the \(w_{ab}\) of \(21\).

With the rules given as yet there holds the Ostendorf theorem \[4\], \[5\], \[6\], stating that the so defined functions \(W_\sigma\) satisfy all Wightman properties with the possible exception of positivity. Hence these rules define a, slightly generalized, Wightman theory.

But we still must satisfy the requirement that these \(W\) solve the interacting field equations \(15\)–\(17\). This problem is easier to handle in \(p\)-space. Therefore we will from now on mainly work in this space, with the Fourier transforms of \(15\)–\(17\). That these equations are satisfied in \(0^{th}\) order in \(g\) is guaranteed by

\(^2\)Positivity of the scalar product is not necessary for the validity of the reconstruction theorem (see Sect. 4.2 of \[6\]).

\(^3\)These subgraphs do not occur in our formulation because we work in the Heisenberg picture. The vacuum graphs are an artifact of the interaction picture.
the condition that the \( w_{ab} \) solve the free field equations. For the following we need to know the \( w_{ab} \) more explicitly. In \( p \)-space we have

\[
\langle \varphi_a(p) \varphi_b(q) \rangle_0 = w_{ab}(p) \delta^4(p + q),
\]

(22)

where this new \( w_{ab} \) is the Fourier transform of the \( w_{ab} \) in (21). The most general solution of the free field equations satisfying all Wightman properties, in particular covariance and locality, is easily found to be

\[
\begin{align*}
    w_{\mu\nu}(p) &= -\omega \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right) \delta^m_+(p) + \frac{1}{m} p_\mu p_\nu T(p) \\
    w_{I\!I}(p) &= m T(p) \\
    w_{\mu I}(p) &= -w_{I\mu}(p) = i p_\mu T(p) \\
    w_{R\!R}(p) &= \alpha \delta^M_+(p) \\
    w_{R\!I}(p) &= \beta \delta^M_+(p), \quad w_{R\!\mu}(p) = -i \frac{\alpha}{m} p_\mu \delta^M_+(p),
\end{align*}
\]

(23)

where \( \delta^m_+(p) = \theta(p_0) \delta(p^2 - m^2) \) is the Dirac measure for the positive mass shell. \( \alpha, \beta, \omega, \) are as yet undetermined real constants, \( T(p) = \theta(p_0) T'(p^2) \) is an arbitrary real invariant function with support in the forward light cone.

The corresponding (anti-)time ordered functions \( \tau^{\pm}(p) \) serving as sector propagators are then also uniquely fixed, provided we restrict ourselves to \( T' \) which tend to 0 for \( p^2 \to \infty \), and that we demand that \( \tau^{\pm} \) should increase for \( p \to \infty \) as weakly as possible. It turns out that the resulting W-functions satisfy the interacting field equations, if the \( \tau^{\pm} \) are propagators in the original sense of the word used in the theory of differential equations. In \( p \)-space this means the following. We write the \( p \)-space form of the field equations (15)–(17) in matrix notation as

\[
C(p) \varphi(p) = R(p).
\]

(24)

Here \( C \) is the \( 6 \times 6 \) coefficient matrix

\[
C = \begin{pmatrix}
    -(p^2 - m^2) \delta^m_+ + p_\mu p_\nu & -i m p^\mu & 0 \\
    i m p_\nu & p^2 & 0 \\
    0 & 0 & p^2 - M^2
\end{pmatrix}.
\]

(25)

The lines are indexed by \( \mu, I, R \), the rows by \( \nu, I, R \), where \( \mu \) and \( \nu \) run over the values \( 0, \ldots, 3 \). \( \varphi \) is a 6-vector with components \( (A^\nu, I, R) \), \( R \) a 6-vector with components \( (R^\mu, R_I, R_R) \). We call the \( 6 \times 6 \) matrix \( P(p) \) a propagator matrix if

\[
C P V = V
\]

(26)

holds for all 6-vectors \( V(p) \) satisfying the consistency condition (18):

\[
- i p_\mu V^\mu + m V_I = 0.
\]

(27)

Then

\[
\varphi = P R
\]

(28)

solves (24). Remember that the \( I \)-line of \( C \) is a linear combination of the \( \mu \)-lines. Hence \( C \) is not invertible and \( P \) cannot be defined as its inverse. Therefore the restriction (27) is necessary.
The sum over our sector graphs solves the field equations of the HKM if
\[ P_{ab}^\pm(p) = \mp 2\pi i \tau_{ab}^\pm(p) \] (29)
constitute a propagator matrix. This is seen by applying the field equations to the propagators of the external graph lines, using that the internal ends of these lines correspond to \( \mathcal{R} \) vertices (see [6], Sect. 9.4, for the QED analogue). External cross propagators do not contribute because they solve the free field equations. It turns out that condition (29) fixes two of the free constants in (23) to be
\[ \omega = \alpha = 1 , \] (30)
while \( \beta \) and the function \( T(p) \) are still free.

From our rules for calculating \( W \)-functions we can also obtain the rules for the fully time ordered functions. At our present formal, non-renormalized, level this is simply done by using the formal definition of time ordering with the help of step functions. The result is a representation as a sum of graphs with only one \( T^+ \) sector containing all external points. The corresponding graph rules are simply the standard Feynman rules. That the Green’s functions thus defined are indeed the time ordered functions of a field theory is of course essential for the applicability of the LSZ reduction formula for the calculation of the \( S \)-matrix.

We will also have occasion to consider functions of the form
\[ \langle \Omega, T^- (\varphi_1(x_1) \cdots \varphi_n(x_n)) T^+ (\psi_1(y_1) \cdots \psi_m(y_m)) \rangle \Omega , \]
where \( \varphi_i \) or \( \psi_j \) stands for any of our fields. These are given by 2-sector graphs with a \( T^- \) sector containing all external \( x_i \) points and a \( T^+ \) sector containing all \( y_j \) points.

4 The Unitary Gauge

The unitary gauge, or U-gauge, is defined as the special Wightman gauge obtained by the choice
\[ \beta = T(p) = 0 . \] (31)
In this gauge we have
\[ I = 0 , \] (32)
it is simply the gauge specified by the ‘gauge condition’ [3,2]. Hence we are left only with the fields \( R \) and \( A_\mu \). The surviving non-vanishing cross propagators are
\[ w_{\mu\nu}(p) = - (g_{\mu\nu} - m^{-2} p_\mu p_\nu) \delta^m_+(p) , \quad w_{RR}(p) = \delta^M_+(p) , \] (33)
and the sector propagators are
\[ \tau_{\mu\nu}^\pm(p) = \mp (i/2\pi) (g_{\mu\nu} - m^{-2} p_\mu p_\nu) (p^2 - m^2 \pm i\epsilon)^{-1} \]
\[ \tau_{RR}^\pm(p) = \pm (i/2\pi) (p^2 - M^2 \pm i\epsilon)^{-1} . \] (34)

4The other W-gauges cannot be characterized in this simple way.
The special interest of this gauge rests on the fact that it might also be called the ‘physical gauge’. The physically relevant objects of a quantum field theory are the observables and the physical states.\(^5\) In an experiment we usually measure expectation values of observables in physical states (meaning states that can actually be prepared in a laboratory). The physical content of a gauge theory must be gauge independent. For the observables this implies that they must be gauge invariant. What it means for states is less easy to characterize. But in the HKM the state space \(\mathcal{V}_U\) of the U-gauge is the obvious candidate for the role as physical state space. This claim rests on two facts. First, the cross propagators are positive. For the \(w_{\mu\nu}\) this means more exactly that they form a positive matrix. This implies that our graph rules define on \(\mathcal{V}_U\) a positive scalar product\(^6\), a necessary requirement for a physical state space.

The second vital point is the following. At first, \(\mathcal{V}_U\) is generated from the vacuum state \(\Omega\) by applying to it polynomials in the fields \(R, A^\mu\), properly integrated over sufficiently smooth test functions. But it turns out that the same state space is also created out of \(\Omega\) by applying polynomials in the gauge invariant fields \(F^\alpha\beta\) and \(\Psi\). This is so because \(A^\mu\) and \(R\) can be expressed as functions of \(F^\alpha\beta\) and \(\Psi\). We see this as follows. The definition (14) of \(\Psi\) becomes in the U-gauge

\[
\Psi = R + \frac{g}{2m} R^2 .
\]

This equation can in principle be solved for \(R\). Of course, square roots of operators are not easy to deal with. But we work in perturbation theory, and here there is no problem. Expand \(R\) in a power series:

\[
R = \sum_{\sigma=0}^\infty R_\sigma g^\sigma ,
\]

and similarly for \(\Psi\), and insert these expansions into (35). We find in zeroth order

\[
R_0 = \Psi_0 ,
\]

in first order

\[
R_1 = \Psi_1 - (2m)^{-1} R_0^2 = \Psi_1 - (2m)^{-1} \Psi_0^2 ,
\]

and so on. The fact that for increasing \(\sigma\) \(R_\sigma\) becomes a polynomial in \(\Psi_\varrho\), \(\varrho \leq \sigma\), of indefinitely increasing order need not worry us, because this expansion will never be used explicitly. Next, from the definition of \(F^\alpha\beta\) and the field equation (15) we obtain

\[
\partial^\alpha F^\alpha\beta = \Box A_\beta - \partial_\beta \partial^\alpha A_\alpha = -m^2 A_\beta + \mathcal{R}_\beta (A_\mu, R) ,
\]

\(^5\)The widely held opinion that the physical content of the theory is fully described by its S-matrix is not tenable. The S-matrix relates states at positive infinite times to states at negative infinite time. But we always measure at finite times. Therefore the S-matrix is, in fact, not measurable.

\(^6\)A formal power series \(Q(g)\) is said to be positive if there exist formal power series \(S_i(g)\) such that \(Q(g) = \sum_i S_i(g)^* S_i(g)\).
or its Fourier transform, hence
\[ m^2 A_\beta = -\partial^a F_{a\beta} + R_\beta (A_\mu, R) . \] (38)

Since \( R_\beta \) contains an explicit factor \( g \), this equation allows again an iterative expansion of \( A_\mu \) in polynomials of \( F_{\mu\nu} \) and \( \Psi \).

Hence \( \mathcal{V}_U \) has an explicitly gauge invariant structure, which fact justifies the claim that it is the physical state space of the HKM.

In this way we seem to have arrived at a nice, clean, identification of the physical content of the HKM. There is, however, a fly in the ointment. The \( p_\mu p_\nu \) term in (34) has a bad behavior at large \( p \), leading to non-renormalizability of the theory in the simple power counting sense. In increasing orders of perturbation theory the individual graphs will have an increasingly bad ultraviolet behavior. The claim that the theory is nevertheless renormalizable amounts to claiming that these bad UV contributions in individual graphs cancel in the sum of all graphs contributing to a specific W-function (or time ordered function) in a given order \( \sigma \) of perturbation theory.

The standard way of handling this problem consists in adding a so-called ‘gauge fixing term’
\[ -\frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \] (39)
to the original Lagrangian of the model. The theory thus obtained is renormalizable in the sense of power counting. But calling (39) a gauge fixing term is highly misleading. The amended \( \alpha \)-Lagrangian does by no means describe a particular gauge of the HKM. It defines a new, different theory, which does not solve the field equations of the HKM. Hence its renormalizability is of no use to our problem, unless it can be established to be in some way physically equivalent to the HKM, in particular to its U-gauge formulation. This necessity does not quite find sufficient attention in the literature.

In any case, if the claimed cancellations between graphs really happen, this ought to be provable inside the HKM. This is the task that we now turn to. There seems to be no easy way to achieve this purpose in the U-gauge. Therefore we introduce in the next section another Wightman gauge better suited to the task.

5 Renormalizability

Particularly suited for our purpose is the \( R \)-gauge (for ‘renormalization gauge’) specified by the choice
\[ \omega = \alpha = 1, \quad \beta = 0, \quad T(p) = -\frac{1}{m} \delta_+(p) \] (40)
in (23), with $\delta_+(p) = \delta_0^+(p)$. The corresponding $T^+$ propagators are

$$
\begin{align*}
\tau_{\mu\nu}^+(p) &= \left( -g_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{p^2 + i\epsilon} \right) \frac{i}{2\pi(p^2 - m^2 + i\epsilon)}, \\
\tau^{+}_{II}(p) &= \frac{i}{2\pi(p^2 + i\epsilon)}, \\
\tau^{+}_{II}(p) &= \frac{p_\mu}{2\pi m (p^2 + i\epsilon)}, \\
\tau^{+}_{RR}(p) &= i\frac{2\pi}{m (p^2 - M^2 + i\epsilon)}.
\end{align*}
$$

(41)

The $\tau^-$ are obtained from $\tau^+$ by the replacements ($i \rightarrow -i$, $p \rightarrow -p$).

Notice that now the propagator $\tau_{\mu\nu}^+$ has a nice, renormalizable, large-$p$ behavior, at the price of introducing the ghost factor $(p^2)^{-1}$. Unfortunately, that does not mean that the theory has become renormalizable. The bad UV behavior has merely been shifted to the mixed $A-I$ propagators. But here the desired cancellations are easier to prove than in the U-gauge.

Before attacking this problem we must decide how the physical content of the model presents itself in the new gauge. Remember that in the U-gauge the physical state space is generated from the vacuum by applying polynomials in the gauge invariant fields $F_{\alpha\beta}$ and $\Psi$. Since the physical content of the theory must be gauge invariant, the same will be true in the R-gauge: the physical state space $V_{ph}$, which is now a proper subspace of the full state space, is generated from the vacuum by applying polynomials in $F_{\alpha\beta}$ and $\Psi$. $V_{ph}$ can again be reconstructed by the Wightman reconstruction theorem from the W-functions of these physical fields only. Only in these ‘physical’ W-functions are we really interested, hence only for them need we prove renormalizability.

The graph representation of the physical fields is clear. To obtain an external $F_{\alpha\beta}$ propagator, simply replace the factor $(-g_{\mu\nu} + p_{\mu}p_{\nu}/p^2)$ of an external $A_{\mu}$ line by $i(p_{\alpha}g_{\beta\nu} - p_{\beta}g_{\alpha\nu})$, the index $\nu$ belonging to the adjacent internal vertex. A $\Psi(p)$ factor is represented as a sum of three terms, an ordinary external $R$-line plus two external 2-prong vertices representing the composite fields $R^2$ and $I^2$ in (14). Both these composite vertices carry the vertex factor $g/(\sqrt{2\pi^3} m)$.\footnote{The Fourier transform of the field $\varphi(x)$ is defined as $\varphi(p) = (2\pi)^{-5/2} \int dx e^{ipx} \varphi(x)$.}

We turn now to the promised proof of the cancellation of UV dangerous terms. The basic idea is the following. Consider a $I-A_{\mu}$ cross propagator

$$
w_{II}(p) = i\frac{p_{\mu}}{m} \delta_+(p),
$$

derived by (22) from the free 2-point function $(I(p) A_{\mu}(q))_0$. The end vertex of the corresponding cross line corresponds to a term in $R^\mu(q)$. Summing over all these terms we obtain (with $q = -p$)

$$
\delta_+(p) \left( -i \frac{q_{\mu}}{m} R^\mu(q) \right) = -\delta_+(p) R_I(q)
$$

(42)

by the consistency condition (18). This means that we can replace the UV nice vertex sum $R^\mu$ by the equally UV nice $R_I$, and the UV bad propagator $w_{II}(p)$ by the UV nice $-w_{II}(p)!$ Unfortunately, in this crude form the argument is
incorrect. The $R^\mu(q)$ vertex in question belongs to, let us say, a $T^+$ sector, which represents a time ordered function of its external vertices, including the $R$ vertex with $R^\mu$ considered a composite external field. But in $x$-space the propagator factor $-iq_\mu$ represents a derivation $\partial_\mu$ acting not only on $R^\mu(x)$ but also on the step functions occurring in the definition of the $T$-product. Hence we must expect that the relevant quantity

$$\frac{-iq_\mu}{m} \tau^+ (R^\mu(q) \cdots) + \tau^+ (R_1(q) \cdots) = \frac{1}{m} \tau^+ (F(q) \cdots)$$

does not vanish but is given as a sum of contact terms. Luckily it turns out that these contact terms are not present if the sector in question contains only gauge invariant external fields. This is established by an explicit study of the graphs in question. Consider first the case that the $R(q)$ vertices are those coming from $L_3$. Then the $F_3$ occurring on the right-hand side is the Fourier transform of the expression (19). Consider a $R$-line with momentum $k$ issuing from the vertex in question. Its denominator $(k^2 - M^2)^{-1}$ is cancelled by the numerator $(k^2 - M^2)$ coming from the first term in (19). Thus this first term leads to an amputation of the adjoining $R$-line, and a corresponding fusion of its two end vertices (internal or external) into a single vertex with more lines. The second term in (19) produces the same effect on $I$- and $A$-lines starting from the $F$ vertex. In this way we obtain a considerable number of fused vertices, among which extensive cancellations occur. And the remaining fused vertices cancel against the $L_4$ terms in $F(q)$. The actual verification of these cancellations is completely elementary but rather lengthy and tedious on account of the large number of different vertices to be considered (see (12), (13), (20)). The remarkable thing is, however, that these cancellations happen locally in the graphs in the immediate neighborhood of the $q$-end of the cross line in question, involving only that end vertex and its nearest neighbors, no matter how large the full sector may be. As a result we can, as proposed, drop our bad $I-A_\mu$ cross line and replace it by the negative of a good $I-I$ line. The same argument, now used for the starting point, applies of course to a $A_\mu-I$ cross propagator. It may also be replaced by the negative of a $I-I$ propagator. By this we end up with two negative $I-I$ propagators for a given position of an appropriate line, plus the positive $I-I$ propagator present from the beginning. The net effect is that we drop the dangerous mixed cross propagators and change the sign of the $I-I$ cross propagators without changing our physical $W$-functions.

In this consideration we have assumed that the internal propagators in the sectors involved still have the original $R$-gauge form, and that the same applies to other cross propagators possibly involved in the cancellations. But the remarkable and lucky fact is that the said cancellations also occur if we have already effected the changes of rules explained above inside the sectors in question and in some of the cross lines, i.e. if we have already dropped there the mixed propagators and changed the signs of the $I-I$ propagators. This enables us to prove the following

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The explicit form of this relation is known as a Ward-Takahashi identity.
Theorem. If in the graph rules of the R-gauge we omit the mixed $A_\mu - I$ lines and change the signs of the $I-I$ propagators, then the resulting Wightman functions and related (partially or fully time ordered) functions of the physical fields $F_{\alpha \beta}$, $\Psi$, remain unchanged.

Notice that the new graph rules arrived at in this way are those of the case $\alpha = 0$ (‘Landau gauge’) of the conventional $L_\alpha$ approach, thus confirming the perturbative validity of that approach. These new graph rules are clearly renormalizable in the sense of power counting. In fact, they are also renormalizable in the stricter sense that the necessary subtractions can be fully absorbed into renormalizations of the masses $m, M$, the coupling constant $g$, and the field normalizations. But the proof of this is quite involved and lies outside the scope of the present work.

The proof of the Theorem is inductive with respect to the order $\sigma$ of perturbation theory. It consists of the following points.

(1) The theorem is correct for $\sigma \leq 2$. This is easily established by explicit calculation.

(2) If the theorem is true for the 2-sector functions $\left( \Omega, T^- (\cdots) T^+ (\cdots) \Omega \right)_\sigma$, then it is true for all $n$-sector functions $\left( \Omega, T^+_1 (\cdots) \cdots T^+_n (\cdots) \Omega \right)_\sigma$, in particular the $W$-functions, with the same fields. This is so because all these functions are in $x$-space boundary values of the same analytic function.\footnote{Strictly speaking this is not true at points where two arguments in the same $T^\pm$ factor coincide. But this is of little concern because it does not happen in the $W$-functions, which are the functions of central interest.}

(3) Amputate the considered functions by multiplying them with $(p^2 - m^2)$ for factors $F_{\alpha \beta}(p)$, $(p^2 - M^2)$ for factors $\Psi(p)$. Then the theorem is true for the full functions if it is true for the amputated ones. This is so because we know precisely how to reconstruct the full functions from the amputated ones.

(4) The theorem is true for the amputated 2-sector functions of order $\sigma$. This is seen by noticing that in the corresponding 2-sector graphs both sectors are of orders $\rho$ with $0 < \rho < \sigma$, so that the inductive hypothesis is applicable to them: the new rules can be used inside these sectors. Then the cross propagators linking them can also be changed to the new form by the arguments related above.

6 The Non-Abelian Case

The methods used for the abelian HKM can be extended to the non-abelian case. In this last section we will briefly describe, without details, this extension and its results in the case of the gauge group $SU_2$. 
The fields of the model are a complex 2-vector $\Phi(x)$ with the scalar components $\phi_1(x)$, $\phi_2(x)$, and a triplet $A_\mu^a(x)$, $\cdots$, $A_\mu^a(x)$, of real vector fields. The Lagrangian is

$$L = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \left[ (\partial_\mu - g A_\mu^a T_a) \Phi \right]^* \left[ (\partial_\mu - g A_\mu^a T_a) \Phi \right] + \mu^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 . \quad (43)$$

Here

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g \varepsilon_{abc} A_\mu^b A_\nu^c , \quad (44)$$

and

$$T_a = -\frac{i}{2} \sigma_a , \quad (45)$$

$\sigma_a$ the Pauli matrices.

$L$ is invariant under the infinitesimal gauge transformations

$$\Phi(x) \Rightarrow (1 + g \vartheta_a(x) T_a) \Phi(x)$$

$$A_\mu^a(x) \Rightarrow A_\mu^a + g \varepsilon_{abc} \vartheta_b(x) A_\mu^c(x) + \partial_\mu \vartheta_a(x) \quad (46)$$

for infinitesimal real functions $\vartheta_a$. In contrast to the abelian case, the field strengths $F_{\mu\nu}^a$ are not gauge invariant.

The corresponding field equations possess the 'vacuum solution'

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix} , \quad A_\mu^a = 0 \ \forall a , \quad v = \frac{\mu}{\sqrt{\lambda}} , \quad (47)$$

which takes over the role of the abelian solution $(4)$. The $\phi_i$ are replaced as fundamental fields by the real scalar fields $R(x)$, $I_a(x)$, defined by the ansatz

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} v + R + i I_3 \\ -I_2 + i I_1 \end{pmatrix} . \quad (48)$$

And, as in the abelian case, we replace the coupling constants $\mu$, $\lambda$, as parameters of the theory by

$$m = \frac{v g}{2} , \quad M = \sqrt{2} \mu , \quad (49)$$

which turn out to be the (unrenormalized) masses of the gauge bosons and the Higgs particle respectively. The field equations of the model look exactly like $(15) - (17)$, except that there are now three $A_\mu^a$-equations and three $I_a$-equations, one for each value of the group index $a$. Correspondingly we get now three consistence conditions:

$$F_a := \partial_\mu R_\mu^a + m R_{Ia} = 0 . \quad (50)$$

We use the summation convention both for Minkowski indices $\mu, \ldots$, and group indices $a, \ldots$. 

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Wightman gauges can be defined and constructed like in the abelian case. We are here not concerned with maximal generality, but need only consider the U- and the R-gauge. The \textit{U-gauge} can again be characterized by the gauge condition \( I_a = 0 \) for all \( a \). Its surviving cross propagators are taken over from
\begin{equation}
  \omega_{\mu\nu}^{ab}(p) = -\delta_{ab} (g^{\mu\nu} - m^{-2} p^\mu p^\nu) \delta^m_+(p), \quad \omega_{RR}(p) = \delta^M_+(p),
\end{equation}
and similarly for the sector propagators. The \textit{R-gauge} is again defined by the propagators (41), where the first three lines hold for \( a-a \) propagators for any value of the group index \( a \), while the mixed \( a-b \) propagators with \( a \neq b \) vanish.

The physical space \( V_{ph} \) is again equated with the state space \( V_U \) of the U-gauge. In order to turn this into a gauge invariant definition also usable in the R-gauge, we must again produce \( V_U \) from the vacuum by applying gauge invariant fields. As one of these fields we use the Higgs field, which is now defined as
\begin{equation}
  \Psi(x) = R(x) + \frac{g}{4m} \left[ R^2(x) + I_a(x) I_a(x) \right].
\end{equation}
But the \( F_{\alpha\beta}^a \) are no longer gauge invariant. However, we can replace them by gauge invariant fields, which we choose to be those introduced by Fröhlich et al.\[2\]. As one of them we define
\begin{equation}
  V^{\mu\nu}_3(x) := \frac{ig^2}{m^2} \Phi^*(x) T_a F_{\alpha\beta}^a(x) \Phi(x),
\end{equation}
where \( \Phi \) is expressed by (48) with \( v = 2m/g \). In the U-gauge this becomes
\begin{equation}
  V_3 = F_3 + \frac{g}{m} R F_3 + \frac{g^2}{4m^2} R^2 F_3.
\end{equation}

\( V^{\mu\nu}_2 \) is defined in the same way, except that the \( T_a \) are replaced by their cyclic permutation \( (T_1 \to T_2, T_2 \to T_3, T_3 \to T_1) \). Repeating this operation we obtain \( V^{\mu\nu}_1 \). By the same kind of arguments as used in Sect. 4 it can be shown that the restrictions to the U-gauge of these \( V_a \), together with \( \Psi \), indeed reproduce \( V_U \).

Hence again, the only W-functions of direct physical relevance are those containing only the physical fields \( \Psi \), \( V_a \), and only the renormalizability of these must be decided. And this is again easiest to achieve in the R-gauge. The method used is the same as in the abelian case. It turns out to be more complicated in its details. The main reason for this is that the simple form \[19\] of \( F_3 \) is replaced by the more complicated expression
\begin{equation}
  F_{a3} = g \varepsilon_{abc} A_{c\nu} \left[ (\Box + m^2) A_\nu^a - \partial^\nu \partial^\mu A_{b\mu} + m \partial^\nu I_b \right] + \frac{g}{2} I_a (\Box + M^2) R - \frac{g}{2} R (\Box I_a + m \partial_\mu A_\mu^a) + \frac{g}{2} \varepsilon_{abc} I_b (\Box I_c + m \partial_\mu A_\mu^c).
\end{equation}
Including this as a sum of composite external vertices in a sector in which the mixed \( A-I \) propagators are already eliminated, we find that

\[ \partial_\mu A_\mu^a = 0, \]
on so that the corresponding terms in (54) can be dropped. But even so the terms in the first line of (54) do not have the desired fusing effect on the adjacent propagators. The factor \((p^2 - m^2)\) of the first term applied to an \( A^\nu - A^\lambda \) propagator produces the ghost term \( p^\nu p^\lambda / (m^2 p^2) \), and the \( p^\nu I_a \) term applied to a \( I-I \) propagator clearly does not remove its singularity at \( p^2 = 0 \). Hence, even if the fusing contributions do cancel like in the abelian case, there remains a non-fusing contribution. But the two offensive terms combine in such a way that they produce a ghost line ending in a new \( F \) vertex, now inside the sector, which fact allows using an inductive procedure leading to a simple result. It turns out that the undesirable non-fusing terms can be removed by the introduction of Faddeev-Popov ghost loops (FP loops)\(^1\). Such a loop is a directed closed loop. Each line carries a propagator

\[ \frac{i}{2\pi(p^2 + i\epsilon)} \]

(in a \( T^+ \) sector) and a group index \( a, \cdots \). The loop contains only 3-line vertices with an \( A^\nu_\nu \) line joining the loop. The vertex factor is

\[ (2\sqrt{2}\pi)^{-1} g\varepsilon_{abc} (p^\nu + q^\nu) \]

with \( p \) the loop momentum leaving the vertex, \( q \) that entering the vertex, and \( a \) and \( b \) the indices of the lines respectively leaving and entering the vertex. And each such ghost loop contributes an extra factor \(-1\).

We might then conjecture the following generalization of the Theorem of Sect. 5 to hold:

Change the graph rules of the R-gauge by omitting the mixed \( A^\mu_\nu-I_a \) propagators and changing the signs of the \( I_a-I_a \) propagators, and by admitting an arbitrary number of FP-loops. This procedure does not change the W-functions and related functions of the physical fields \( V_{\mu\nu}^a, \Psi \).

These conjectured rules are again the rules of the standard formalism in the Landau gauge.

The conjecture would be correct, if the fusing terms of \( F_3 \) did lead to graph-local cancellations in analogy to the abelian case. This turns out to be the case for purely internal cancellations, that is if the end points of the fused lines are internal \( L_{\text{int}} \) vertices. But it is not true in all cases where external vertices (composite fields contributing to \( V_a \)) are involved. Therefore the equality of the physical W-functions in the Landau gauge and the R-gauge, and hence in the HKM in general, cannot be proved. This should not be interpreted as a weakness of our method. There are strong indications that the Landau gauge is indeed not physically equivalent to the HKM, if ‘physical equivalence’ is defined in our sense, not simply as the equality of the S-matrices.

As a result, there exists as yet no convincing proof of the full renormalizability of the non-abelian HKM.
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