UPPER BOUNDS FOR SOME BRILL-NOETHER LOCI OVER A FINITE FIELD

KAMAL KHURI-MAKDISI

ABSTRACT. Let \( C \) be a smooth projective algebraic curve of genus \( g \) over the finite field \( \mathbb{F}_q \). A classical result of H. Martens states that the Brill-Noether locus of line bundles \( \mathcal{L} \) in \( \text{Pic}^d C \) with \( \deg \mathcal{L} = d \) and \( h^0(C, \mathcal{L}) \geq i \) is of dimension at most \( d - 2i + 2 \), under conditions that hold when such an \( \mathcal{L} \) is both effective and special. We show that the number of such \( \mathcal{L} \) that are rational over \( \mathbb{F}_q \) is bounded above by \( K_g q^{d-2i+2} \), with an explicit constant \( K_g \) that grows exponentially with \( g \). Our proof uses the Weil estimates for function fields, and is independent of Martens’ theorem. We apply this bound to give a precise lower bound of the form \( 1 - K'_g/q \) for the probability that a line bundle in \( \text{Pic}^{g+1} C(\mathbb{F}_q) \) is base point free. This gives an effective version over finite fields of the usual statement that a general line bundle of degree \( g+1 \) is base point free. This is applicable to the author’s work on fast Jacobian group arithmetic for typical divisors on curves.

1. Introduction

Let \( C \) be a smooth projective algebraic curve of genus \( g \). A classical object of study is the Brill-Noether locus of \( C \) with parameters \((d, i)\). It is the variety of degree \( d \) line bundles \( \mathcal{L} \) on \( C \) whose space of global sections has dimension at least \( i \):

\[
\{ \mathcal{L} \in \text{Pic}^d C \mid h^0(\mathcal{L}) \geq i \}.
\]

Here \( h^0(\mathcal{L}) = \dim H^0(C, \mathcal{L}) \). For “uninteresting” values of \((d, i)\), Riemann-Roch and other considerations such as Clifford’s theorem on special divisors imply that the above set is either empty or equal to all of \( \text{Pic}^d C \). As we shall recall in Section 2, the “interesting” range is when

\[
0 \leq i - 1 \leq d - i + 1 \leq g - 1.
\]

When (1.2) holds, one has upper and lower bounds on the dimension of the Brill-Noether locus; see for example Chapters IV and V of [ACGH85]. The main aspect that we will consider in this article is the upper bound on this dimension. In this regard, there is a basic theorem of Martens [Mar67], see Theorem IV.5.1 of [ACGH85]. It says that the dimension of the Brill-Noether locus in the interesting range is bounded above by \( d - 2i + 2 \). In fact, Martens’ theorem is somewhat sharper, as it says that the dimension \( d - 2i + 2 \) is attained if and only if the curve \( C \) is hyperelliptic, so the upper bound for a nonhyperelliptic curves is in fact \( d - 2i + 1 \). Sharper bounds are known if one excludes not only hyperelliptic curves, but also other special cases; see for example Mumford’s result in Theorem IV.5.2.
of [ACGHS85]. However, our main interest here is in a bound that applies uniformly for all genus $g$ curves over a finite field.

The main result of this note is a quantitative version of the dimension bound $d - 2i + 2$ over a finite field $\mathbb{F}_q$. We obtain the result, Theorem 2.9 below, that the number of $\mathbb{F}_q$-rational points in the Brill-Noether locus (1.1) is bounded above by $Kd^{d-2i+2}$, where $K = 16^g$ is a constant that depends only on the genus $g$. Applying this result to the $\mathbb{F}_q$-rational points as $a \to \infty$, one recovers Martens’ result, at least when the base field is finite, because a dimension $e$ variety over $\mathbb{F}_q$ has $\Theta(q^{de})$ points over $\mathbb{F}_q$. Our quantitative result also incidentally implies an alternate proof of Clifford’s theorem over a finite field. Our main tool in the proof is the Weil bounds for both zeta and L-functions of the curve $C$. We use the Weil bounds in Theorem 2.7 to bound a larger set $X_d$, defined below in (2.6), that is related to the various Brill-Noether loci with fixed $d$ and varying $i$. Our results do not seem to yield lower bounds; it would be interesting to see if any of the lower bounds on the dimensions on Brill-Noether loci can be proved by counting points over finite fields in some way.

We apply our result on upper bounds for Brill-Noether loci to obtain a precise estimate of the probability that a random element $L \in \text{Pic}^{g+1}C$ is base point free; here the random element is drawn uniformly from $(\text{Pic}^{g+1}C)(\mathbb{F}_q)$. We similarly (and more easily) obtain a precise estimate of the probability that a random element $L \in (\text{Pic}^{g-1}C)(\mathbb{F}_q)$ has $h^0(L) = 0$. For both of these questions, it is easy to see that the dimension of the subvarieties of $\text{Pic}^{g \pm 1}C$ where the desired condition does not hold is at most $g - 1$, from which it follows that the probabilities are at least $1 - A/q$ for a suitable constant $A$. However, finding the precise $A$ from just the dimension count and some kind of estimate of the degree seems elusive. We can moreover use the more precise estimates here to obtain a bound for the probability that a divisor is “typical” for fast Jacobian group arithmetic, in the sense of our preprint [KMT13]. Obtaining that bound was the main motivation for this article.

2. The main result

We first review why (1.2) is the range in which pairs $(d, i)$ can potentially give an interesting Brill-Noether locus (1.1). Let $\mathcal{K}$ denote the canonical bundle on $C$, so that $h^1(\mathcal{K}) = \dim H^1(C, \mathcal{K}) = h^0(\mathcal{K} \otimes L^{-1})$. Then, by Riemann-Roch, $h^0(L) = h^1(L) + d + 1 - g$, whenever $\deg L = d$. Since we always have $h^0(L), h^1(L) \geq 0$, it follows that if either $i \leq 0$ or $i \leq d + 1 - g$, the Brill-Noether locus of (1.1) is all of $\text{Pic}^dC$, hence uninteresting. So the interesting case can only occur when both $i \geq 1$ and $i \geq d + 2 - g$ are satisfied; these are precisely the outer inequalities of (1.2). Moreover, when these outer inequalities hold, we have both $h^0(L) \geq 1$ and $h^1(L) \geq 1$, so the line bundle $L$ corresponds to an effective special divisor; in this setting, Clifford’s theorem (Theorem IV.5.2 of [Har77]) says that $i - 1 \leq d/2$, which gives the middle inequality of (1.2). Actually, our reasoning does not need the inequality arising from Clifford’s theorem. The purist reader may prefer not to assume this inequality, but may rather deduce it from Theorem 2.9 which gives an alternate proof of Clifford’s theorem over a finite field; see Remark 2.11.

Our second preliminary observation is that we can restrict $d$ to the interval $0 \leq d \leq g - 1$ for the purpose of bounding the size of the interesting Brill-Noether loci. First, elementary considerations about degrees of $L$ and $\mathcal{K} \otimes L^{-1}$ tell us that $h^0(L)$ is predicted entirely by $d$, hence uninteresting, except in the range
0 \leq d \leq 2g - 2. Furthermore, we have a bijection between Pic^d C and Pic^{2g-2-d} C sending \( L \) to \( K \otimes L^{-1} \), and this bijection (which is an involution on the set of all line bundles on \( C \)) replaces the Brill-Noether locus for the pair \((d, i)\) with the locus for the pair \((2g - 2 - d, i - d - 1 + g)\), while leaving the quantity \( d - 2i + 2 \) unchanged. This allows us to reduce the interval for \( d \) to half its size. This is indeed the assumption made in Theorem \([2.7]\) and case (i) of Theorem \([2.9]\) below, namely:

\[
0 \leq d \leq g - 1, \quad i \geq 1.
\]

Note that when \((2.1)\) holds, then \( d - i + 1 \leq g - 1 \), so the outer inequalities of \((1.2)\) are both satisfied.

Finally, we include an elementary lemma on the growth of certain functions of \( x \), which we will apply for \( x \) a simple expression in \( g \) or \( g \) as needed in our theorems below.

**Lemma 2.1.**

1. The functions \( x/(x-1)^2 \) and \((1+x^{-1})^2/(1-x^{-1})\) are both decreasing for \( x > 1 \).
2. The function \((1-2^{-x})^x\) is increasing for \( x > 1/\log 2 \approx 1.443 \).
3. The function \([(1+2^{-x})/(1-2^{-x})]^x\) is decreasing for \( x > 1/\log 2 \).

**Proof.** Part (1) is trivial. Part (2) follows from noting that the logarithm of the function in question is \( x \log(1-2^{-x}) = -\sum_{n \geq 1} n 2^{-nx}/n \), and that the functions \( x2^{-nx} \) are all decreasing for \( x > 1/\log 2 \) (consider the derivative of \( x \exp(-ax) \)). Part (3) is similar, using \( x[\log(1+2^{-x}) - \log(1-2^{-x})] = 2\sum_{n \text{ odd}} x2^{-nx}/n \).

We can now get down to the business of our main result. As our proof relies on the Weil bounds for the zeta function and abelian L-functions of \( C \), we begin by fixing some more notation and recalling the statements that we need. For a survey of these results, see Chapter 9 and the appendix of \([Ros02]\).

**Definition 2.2.** We define \( \text{Div}^d C \) to be the set of \( \mathbb{F}_q \)-rational divisors of degree \( d \) on \( C \), and we define \( \text{Eff}^d C \) to be the subset of effective divisors. We write \( J = (\text{Pic}^d C)(\mathbb{F}_q) = (\text{Pic}^d C)/\sim \), where \( D \sim E \) means that \( D \) and \( E \) are linearly equivalent; we write the equivalence class of \( D \) as \([D]\).

Since there is no period-index obstruction over finite fields, the natural map \( \text{Div}^d C \to (\text{Pic}^d C)(\mathbb{F}_q) \) is surjective. Moreover, \( \text{Eff}^d C \to (\text{Pic}^d C)(\mathbb{F}_q) \) is surjective if \( d \geq g \). We also choose once and for all an \( \mathbb{F}_q \)-rational divisor \( D_0 \) of degree 1 (this is possible over a finite field) and use it to identify \((\text{Pic}^d C)(\mathbb{F}_q)\) with \( J \); via \([D] \mapsto [D - dD_0]\). We consider the group \( \tilde{J} \) of characters \( \chi : J \to \mathbb{C}^* \). Due to our identification, we can evaluate \( \chi \) on elements of \((\text{Pic}^d C)(\mathbb{F}_q)\), or, for that matter, on \( \text{Div}^d C \).

**Definition 2.3.** We define \( N_d = |\text{Eff}^d C| \), and for \( \chi \in \tilde{J} \) we define \( N_{d,\chi} = \sum_{D \in \text{Eff}^d C} \chi(D) \). Thus \( N_d = N_{d,1} \) for the trivial character \( \chi = 1 \).

Note in the above definition that \( N_{d,\chi} \) depends on the specific choice of \( D_0 \), but that the effect of changing \( D_0 \) to \( D'_0 \) is to multiply \( N_{d,\chi} \) by the root of unity \( \chi([D_0 - D'_0])^d \). A similar statement holds for the quantities \( \alpha_{i,\chi} \) below. Our main concern is bounds for the absolute value \(|N_{d,\chi}|\), which is not affected by the choice of \( D_0 \).
Theorem 2.4 (A. Weil). For \( \chi \in \hat{J} \), define the L-function \( L_C(T, \chi) \) and, for the trivial character, the zeta-function \( Z_C(T) \) by

\[
L_C(T, \chi) = \sum_{d \geq 0} N_{d, \chi} T^d, \quad Z_C(T) = L_C(T, 1) = \sum_{d \geq 0} N_d T^d.
\]

Then \( L_C(T, \chi) \) is a polynomial for \( \chi \neq 1 \), while \( Z_C(T) \) is a rational function. These have the form

\[
Z_C(T) = \prod_{i=1}^{2g}(1 - \alpha_i T), \quad L_C(T, \chi) = \prod_{i=1}^{2g-2} (1 - \alpha_{i, \chi} T), \text{ for } \chi \neq 1,
\]

with all the \( |\alpha_i| \) and \( |\alpha_{i, \chi}| \) equal to \( q^{1/2} \). Moreover, we have

\[
|J| = \prod_{i=1}^{2g}(1 - \alpha_i), \quad \text{hence } (q^{1/2} - 1)^{2g} \leq |J| \leq (q^{1/2} + 1)^{2g},
\]

\[
|C(F_{q^d})| = q^d + 1 - \sum_{i=1}^{2g} \alpha_i^d.
\]

We deduce the following bounds:

Corollary 2.5. (1) For \( \chi \neq 1 \), we have \( N_{d, \chi} = 0 \) unless \( 0 \leq d \leq 2g - 2 \), in which case we have \( |N_{d, \chi}| \leq (2g-2)q^{d/2} \leq 4^{g-1}q^{d/2} \).

(2) For \( \chi = 1 \), we have \( 0 \leq N_d \leq (\frac{2}{\sqrt{q}})(1 + q^{-1/2})2^g q^d \).

(3) The number \( N_{d, \chi}^{irr} \) of irreducible (i.e., prime) effective divisors of degree \( d \) satisfies \( N_{d, \chi}^{irr} \leq (q^d + 2q^{d/2} + 1)/d \).

Proof. We first prove statement (1). Since \( N_{d, \chi} \) is the coefficient of \( T^d \) in \( L_C(T, \chi) \), one immediately obtains that \( N_{d, \chi} \) is a sum of \( (2g-2) \) terms, each of absolute value \( q^{d/2} \). Moreover, \( (2g-2) \leq (1 + 1)^{2g-2} = 4^{g-1} \). (One can reduce this estimate by a factor of roughly \( q^{1/2} \), but our main concern is the exponential dependence on \( q \).) We now show statement (2). The coefficient of \( T^j \) in \( 1/(1 - T)(1 - qT) \) is the positive number \((q^{j+1} - 1)/(q - 1)\), which is bounded above by \( (\frac{2}{\sqrt{q}})q^{j} \); rewriting \( j = d - k \), the bound becomes \( (\frac{2}{\sqrt{q}})q^{d-k} \). In the other factor of \( Z_C(T) \), which is \( \prod_{\omega^d = 1} (1 - \alpha_i T) \), the coefficient of \( T^k \) is bounded by \( \frac{2^g}{q^k} \). Taking the product with \( 1/(1 - T)(1 - qT) \), we obtain that \( N_d \), being the coefficient of \( T^d \) in \( Z_C(T) \), is bounded by \( N_d \leq (\frac{2^g}{q^k}) \sum_{k=0}^{d} (2^g) q^{d-(k/2)} \), which easily gives the bound in (2) (this works even if \( d > 2g \)). Finally, statement (3) holds because each irreducible effective divisor of degree \( d \) can be regarded as a sum of \( d \) distinct points of \( C(F_{q^d}) \), so we can apply (2.5). \( \square \)

Definition 2.6. Fix \( d \geq 0 \). We define the set \( X_d \subset \text{Eff}^d C \times \text{Eff}^d C \) by

\[
X_d = \{(D, E) \in \text{Eff}^d C \times \text{Eff}^d C \mid D \sim E\}.
\]

One can view \( X_d \) as the set of \( F_q \)-rational points of a subvariety of \( \text{Sym}^d C \times \text{Sym}^d C \).

The set \( X_d \) is closely related to the set of rational functions of degree \( d \) on \( C \), as studied in [Ek01], and our method of bounding \( |X_d(F_q)| \) in Theorem 2.7 below, via a sum over the characters \( \chi \in \hat{J} \), was inspired by that article.
Theorem 2.7. Assume that $0 \leq d \leq g-1$ and $q \geq 5$. The number of points of $X_d$ satisfies the inequality
\begin{equation}
|X_d| \leq 16^g q^d.
\end{equation}

Remark 2.8. The argument below is still valid for $q \leq 4$, but it yields an order of growth of $|X_d|$ that is bounded by $C^g q^d$, for $C$ a constant larger than 16. This is still enough to see that for all $q$, the variety whose set of $F_q$-rational points is $X_d$ has dimension at most $d$.

Proof. We can write $|X_d|$ in terms of a sum over the characters $\chi$ of $J$:
\begin{equation}
|X_d| = \sum_{D, E \in \text{End} C} \frac{1}{|J|} \sum_{\chi \in \hat{J}} \chi(D) \chi^{-1}(E)
\end{equation}

The second term is bounded by $16^{g-1} q^d$, by part (1) of Corollary 2.5, and by the fact that $|J| = |\hat{J}|$. The first term is bounded by $(\frac{1}{q^2})^2 (1+q^{-1/2})^2 (1-q^{-1/2})^2 q^{2d-g}$, by (2.4) and part (2) of Corollary 2.5. Now, as $q$ increases, the quantities $q/(q-1)^2$ and $(1+q^{-1/2})^2/(1-q^{-1/2})^2$ both decrease, by part (1) of Lemma 2.1, so they are bounded by $5/16$ and $(1+1/\sqrt{5})^2/(1-1/\sqrt{5}) < 4$, respectively. Thus the first term is at most $[(5/16)4^g q^{d-g+1} \cdot q^d$, which in turn is bounded by $(5/16)16^g q^d$, because $d \leq g-1$. Combining the two terms completes the proof. \hfill \square

We are now ready for the main result of this article.

Theorem 2.9. Assume that $q \geq 5$, and that either (i) $0 \leq d \leq g-1$ and $i \geq 1$, or (ii) $g-1 \leq d \leq 2g-2$ and $i-d-1+g \geq 1$. Then the cardinality of the Brill-Noether locus $(1.1)$ is bounded by
\begin{equation}
\left| \{ \mathcal{L} \in \text{Pic}^d C(F_q) \mid h^0(\mathcal{L}) \geq i \} \right| \leq 16^g q^{d-2i+2}.
\end{equation}

Proof. As we discussed just before (2.1), there is an involution on the line bundles on $C$ exchanging $\mathcal{L}$ with $\mathcal{K} \otimes \mathcal{L}^{-1}$, and this involution exchanges conditions (i) and (ii) without changing the value of $d-2i+2$. Thus we may assume in the proof that condition (i) holds, which allows us to use Theorem 2.7. Under condition (i), consider the map $f : X_d \to \text{Pic}^d C(F_q)$, given by $f(D, E) = [D] = [E]$. The fibre of $f$ over a point $\mathcal{L}$ with $h^0(\mathcal{L}) = \ell > 0$ is isomorphic to $\mathbb{P}^{\ell-1}(F_q) \times \mathbb{P}^{\ell-1}(F_q)$. If $\ell \geq i$, this fibre has at least $q^{2i-2}$ points, so, combining with our estimate (2.7), we obtain the result. \hfill \square

Remark 2.10. If $i = 1$ above, it is better to bound the cardinality of the Brill-Noether locus by $N_d$, since every $\mathcal{L}$ with $h^0(\mathcal{L}) \geq 1$ is represented by at least one effective divisor. For $q \geq 5$, this gives a bound of $(5/4)(1+1/\sqrt{5})^{2g} \cdot q^d < (5/4)(2.1)^g \cdot q^d$. This is much better than $16^g q^d$ when $g \geq 1$.

Remark 2.11. Theorem 2.9 implies both Clifford’s and Martens’ theorems over a finite field. Namely, replacing $F_q$ with $F_{q^a}$ and letting $a \to \infty$, we obtain that the dimension of the Brill-Noether locus $(1.1)$ is bounded above by $d-2i+2$, recovering Martens’ theorem, and hence is empty whenever $d-2i+2 < 0$, recovering Clifford’s
By part (3) of Corollary 2.5, we can bound (2.10) above by 
\[ \text{condition (i) or condition (ii)} \]
above.

For the next result, note that all line bundles of degree \( g + 1 \) are base point free if \( g \leq 1 \), so we have assumed that \( g \geq 2 \). Also note that the requirement \( q \geq 16^g \) can be weakened, but we are anyhow only interested if our bound on the final probability is less than 1.

**Theorem 2.12.** Assume \( g \geq 2 \) and \( q \geq 16^g \). Then the probability that a uniformly randomly chosen element of \( \text{Pic}^{g+1} C(\mathbb{F}_q) \) is not base point free is at most \( (16^g \cdot g)/q \).

**Proof.** The number of elements of \( \text{Pic}^{g+1} C(\mathbb{F}_q) \) is \( |J| \), which we can bound by \( (2.13) \).

The point is to count the number of line bundles \( \mathcal{L} \in \text{Pic}^{g+1} C(\mathbb{F}_q) \) that are not base point free. All line bundles \( \mathcal{L} \in \text{Pic}^{g+1} C(\mathbb{F}_q) \) have \( h^0(\mathcal{L}) \geq 2 \), but if such a line bundle is not base point free, then there exists an irreducible divisor \( E \) (part of the divisor of base points of \( \mathcal{L} \)) for which \( h^0(\mathcal{L}(-E)) = h^0(\mathcal{L}) \geq 2 \). Such an \( E \) must necessarily have \( \deg E \leq g-1 \), since otherwise we would have \( \deg \mathcal{L}(-E) \leq 1 \), which would force \( h^0(\mathcal{L}(-E)) \leq 1 \). Writing \( \mathcal{L}' = \mathcal{L}(-E) \) (equivalently, \( \mathcal{L} = \mathcal{L}'(E) \)) and \( e = \deg E \), we can thus bound the number of \( \mathcal{L} \in \text{Pic}^{g+1} C(\mathbb{F}_q) \) that are not base point free by the number of pairs \((E, \mathcal{L}')\), where \( E \) is irreducible, \( \deg \mathcal{L}' = g+1-e \), and \( h^0(\mathcal{L}') \geq 2 \), as \( e \) ranges over all of \( \{1, 2, \ldots, g-1\} \). For a given \( e \), there are \( N_e^{\text{irr}} \) choices of \( E \). Also, there are by \( (2.14) \) at most \( 16^g q^{g-1-e} \) choices of \( \mathcal{L}' \). Hence the total number of \( \mathcal{L} \) that are not base point free is at most

\[ (2.10) \sum_{e=2}^{g-1} N_e^{\text{irr}} \cdot 16^g q^{g-1-e}. \]

By part (3) of Corollary 2.5, we can bound \( (2.10) \) above by

\[ (2.11) \sum_{e=1}^{g-1} \frac{1}{e} (q^e + 2gq^{e/2} + 1) 16^g q^{g-1-e} \leq q^{g-1} \cdot 16^g \left[ \sum_{e=1}^{g-1} \frac{1}{e} + 2g \sum_{e=1}^{\infty} q^{-e/2} + \sum_{e=1}^{\infty} q^{-e} \right]. \]

Estimating the harmonic sum, and using that \( q \geq 16^g \geq 256 \), we see that the factor in square brackets is bounded by \( 1 + \log(q-1) - 2g \log(1-1/16) - \log(1-1/256) \approx 1 + \log(q-1) + 0.13q + 0.004 \), which is in turn at most \( 0.7g \) (recall that this is for integers \( g \geq 2 \)). Thus our final probability is at most

\[ (2.12) \frac{q^{g-1} \cdot 16^g \cdot 0.7g}{|J|} \leq \frac{16^g \cdot 0.7g}{q(1-q^{-1/2})2g} \leq \frac{16^g \cdot 0.7g}{q(1-4^{-g})2g} \leq \frac{16^g \cdot g}{q}. \]

The above uses the fact that for \( g \geq 2 \), we have \((1-4^{-g})^{2g} \geq (1-4^{-2})^4 \approx 0.77\), by part (2) of Lemma 2.1.

**Remark 2.13.** If \( g = 2 \) above, a much better bound is possible. Indeed, in that case the only possible value of \( e \) is \( e = 1 \), and in that case the only possible \( \mathcal{L}' = \mathcal{K} \), since one can see that \( \mathcal{K} \otimes (\mathcal{L}')^{-1} \) must have degree 0 and \( h^0 \geq 1 \). Thus the bound for the number of \( \mathcal{L} \) is just \( N_1^{\text{irr}} \), the number of points on \( C \), which is bounded by \( q+4q^{1/2}+1 \); we can divide this by \( |J| \) to get our final bound. A further improvement can be obtained by keeping the eigenvalues \( \alpha_1, \ldots, \alpha_4 \) for Frobenius in
both $N_i^{irr}$ and $|J|$, and optimizing the ratio $N_i^{irr}/|J|$ over the possible $\alpha_i$ satisfying the Weil bounds and the Poincaré duality relations $\alpha_1\alpha_2 = \alpha_3\alpha_4 = g$.

**Remark 2.14.** If we are a bit more careful with our bound for $\binom{2g-2}{d}$ that goes into part (1) of Corollary 2.5, we can probably remove the factor of $g$, but this does not seem worth the effort here.

We conclude this section with a precise bound on how likely it is that a line bundle of degree $g - 1$ has any global sections, under the same hypotheses as Theorem 2.12.

**Proposition 2.15.** Assume $g \geq 2$ and $q \geq 16^g$. Then the probability that a uniformly randomly chosen element $\mathcal{L} \in \text{Pic}^{g-1}(C(F))$ has $h^0(\mathcal{L}) \geq 1$ is at most $1.7/q$.

**Proof.** Using Remark 2.10 and our various estimates above, we see that this probability is bounded above by

$$
\frac{N_{g-1}}{|J|} \leq \frac{(g-1)(1 + q^{-1/2})2g}{(q^{1/2} - 1)^{2g}} = \frac{1}{q - 1} \left( \frac{q}{q - 1} \right)^{2g}.
$$

The fraction $q/(q - 1)$ is at most $256/255 \approx 1.004$, while the quantity in square brackets is bounded above by $[(1 + 4^{-g})/(1 - 4^{-g})]^{2g}$, which, for $g \geq 2$, does not exceed its value at $g = 2$, by part (3) of Lemma 2.1. Now the value at $g = 2$ is approximately 1.65, so the constant in our upper bound is approximately $(1.004)(1.65)$, which is less than 1.7. \qed

3. Application to typical divisors

We apply our results from Section 2 to bound the number of typical divisors on a curve, in the sense of [KM13]. We begin by recalling the definition of a typical divisor from that article.

**Definition 3.1.** We first set up the context in which we work. From now on, $C$ comes equipped with a distinguished rational point $P_{\infty} \in C(F(q))$, and the definition of a typical divisor on $C$ depends on the choice of $P_{\infty}$. We use the point $P_{\infty}$ (viewed as a divisor $D_0 = P_{\infty}$ of degree 1) to identify $\text{Pic}^d C$ with $\text{Pic}^0 C$ whenever convenient, as we did just after Definition 2.2.

1. A divisor $D$ on $C$ is called good if $D$ is effective, $F_q$-rational, and disjoint from $P_{\infty}$. We will assume that the degree $d = \deg D$ satisfies $d \geq g$, and will also refer to the corresponding line bundle $\mathcal{L} = \mathcal{O}_C(dP_{\infty} - D) \in \text{Pic}^0 C(F)$. $\mathcal{L}.$

2. We define the following $F_q$-vector spaces that will appear frequently in this section: for $N \in \mathbb{Z}$,

$$
W^N = H^0(C, \mathcal{O}_C(NP_{\infty})),
$$
$$
W^N_D = H^0(C, \mathcal{O}_C(NP_{\infty} - D)) = H^0(C, \mathcal{L}((N - d)P_{\infty})).
$$

We will usually view $W^N$ and $W^N_D$ as Riemann-Roch spaces, hence as subsets of the function field $F_q(C)$; for example, $W^N_D \subset W^{N+1}_D$.

3. A good divisor $D$ is called semi-typical if $W^{d+g-1}_D = 0$, or equivalently if $H^0(C, \mathcal{L}((g - 1)P_{\infty})) = 0$.

4. A good divisor $D$ is called typical if there exist $s \in W^{d+g}_D = H^0(C, \mathcal{L}(gP_{\infty}))$ and $t \in W^{d+g+1}_D = H^0(C, \mathcal{L}((g + 1)P_{\infty}))$, satisfying

$$
sw^{2g} + tw^{2g-1} + w^{d+g-1} = w^{d+3g}.
$$

}\]
The above definition of semi-typicality depends only on the corresponding line bundle \( \mathcal{L} \) (via its twist \( \mathcal{L}((g-1)P_\infty) \)), and not on the particular good divisor \( D \), nor on its degree \( d \). In [KM13], we also proved that typicality of a divisor also depends only on \( \mathcal{L} \). The following is an equivalent characterization of typicality in terms of \( \mathcal{L} \).

**Proposition 3.2.** Let \( D \) be a good divisor of degree \( d \geq g \), with \( g \geq 1 \). Then \( D \) is typical if and only if the following conditions hold for the associated line bundle \( \mathcal{L} = \mathcal{O}_C(dP_\infty - D) \):

1. The line bundles \( \mathcal{L} \) and \( \mathcal{L}^{-1} \) are semi-typical, i.e., \( h^0(\mathcal{L}((g-1)P_\infty)) = h^0(\mathcal{L}^{-1}((g-1)P_\infty)) = 0 \).
2. The line bundle \( \mathcal{L}((g+1)P_\infty) \) is base point free.

**Proof.** The fact that the above conditions are equivalent to typicality is implicit in [KM13], but in the interest of making this article self-contained we will include a streamlined version of the argument.

We first show that the conditions above imply that \( \mathcal{L} \) is typical. We know from semi-typicality that \( W_{D, t}^{d+g-1} = 0 \). To find our desired \( s, t \in W^{d+g+1}_D \), we observe that \( \dim W^{d+g+1}_D \geq 2 \) by Riemann-Roch, but that in the inclusions \( 0 = W^{d+g-1}_D \subset W^{d+g+1}_D \subset W^{d+g+1}_D \), the dimensions increase by at most 1. Thus \( \dim W^{d+g}_D = 1 \) and \( \dim W^{d+g+1}_D = 2 \), and we choose accordingly any generator \( s \in W^{d+g}_D - \{0\} \), and any \( t \in W^{d+g+1}_D - W^{d+g}_D \), from which it follows that \( \{s, t\} \) is a basis for \( W^{d+g+1}_D = H^0(C, \mathcal{L}((g+1)P_\infty)) \). Viewing \( s \) and \( t \) as elements of the function field \( \mathbb{F}_q(C) \), this means moreover that \( s \) and \( t \) have poles at \( P_\infty \) of exact orders \( d + g \) and \( d + g + 1 \), respectively. Hence their divisors have the form

\[
\text{div } s = -(d + g)P_\infty + D + A, \quad \text{div } t = -(d + g + 1)P_\infty + D + B,
\]

where \( A \) and \( B \) are good divisors of degrees \( g \) and \( g + 1 \), respectively. Moreover, the fact that \( \mathcal{L}((g+1)P_\infty) = \mathcal{O}_C((d + g + 1)P_\infty - D) \) is base point free means that \( A \) and \( B \) are disjoint. Since \( A \) and \( B \) are disjoint, it follows that \( sW^{2g}_D \cap tW^{2g-1}_D = W^{d+3g}_{D+A} \cap W^{d+3g}_{D+B} = W^{d+3g}_{D+A+B} \). This last space is isomorphic, via division by \( st \), to \( H^0(C, \mathcal{O}_C((-d + g - 1)P_\infty + D)) = H^0(C, \mathcal{L}^{-1}((g-1)P_\infty)) = 0 \), by semi-typicality of \( \mathcal{L}^{-1} \). Hence \( \dim(sW^{2g}_D + tW^{2g-1}_D) = (g + 1) + g = \dim W^{d+3g}_D \). On the other hand, \( sW^{2g}_D + tW^{2g-1}_D = W^{d+3g}_{D+A} + W^{d+3g}_{D+B} \subset W^{d+3g}_D \), and we deduce that \( sW^{2g}_D + tW^{2g-1}_D = W^{d+3g}_D \). This implies that \( (sW^{2g}_D + tW^{2g-1}_D) \cap W^{d+g-1}_D = W^{d+g-1}_D = 0 \) (since \( \mathcal{L} \) is semi-typical), hence that \( \dim(sW^{2g}_D + tW^{2g-1}_D + W^{d+g-1}_D) = \dim W^{d+3g}_D + \dim W^{d+g-1}_D = (2g + 1) + d = \dim W^{d+3g}_D \), and we have proved (3.2). Hence \( \mathcal{L} \) is typical, as desired.

We now prove the converse, namely that if \( D \) is typical, then \( \mathcal{L} \) satisfies the above conditions. This is fairly close to running the above argument backwards, but needs some details to be filled in. From (3.2), we deduce by counting dimensions that the sum is direct (and, even before that, that \( s \) and \( t \) are nonzero). Hence

\[
\text{dim}(sW^{2g}_D + tW^{2g-1}_D) = 2g + 1 = \dim W^{d+3g}_D, \quad (sW^{2g}_D + tW^{2g-1}_D) \cap W^{d+g-1}_D = 0.
\]
Since we anyhow have $sW^{2g} + tW^{2g-1} \subset W^{d+3g}$, we deduce that these spaces are equal; from the last intersection above, it follows that $W^{d+9g-1} = 0$, so $D$ is semi-typical. Moreover, from $sW^{2g} + tW^{2g-1} = W^{d+3g}$, it follows that $s$ and $t$, when viewed as sections of $\mathcal{L}((g+1)P_\infty) = \mathcal{O}_C((-d + g + 1)P_\infty - D)$, cannot have any common vanishing at an irreducible effective divisor $E$ (which could be $P_\infty$), since we would then have $sW^{2g} + tW^{2g-1} \subset W^{d+3g} \subseteq W^{d+3g}$. This shows that $\mathcal{L}((g+1)P_\infty)$ is base point free, and that the divisors of $s$ and $t$ are as in (2.15); this uses the fact that $D$ is semi-typical for the precise behavior at $P_\infty$. Finally, the fact that $sW^{2g} \cap tW^{2g-1} = 0$ implies that $\mathcal{L}^{-1}$ is semi-typical, because the existence of a nonzero $u \in H^0(C, \mathcal{O}_C((-d + g - 1)P_\infty + D))$ would give rise to a nonzero $ust = s(u) = t(u) \in sW^{2g} \cap tW^{2g+1}$. 

The above result allows us to view typicality and semi-typicality as properties of a line bundle $\mathcal{L} \in \text{Pic}^0 C(\mathbb{F}_q)$. In this context, we immediately obtain a bound on how likely it is that a uniformly randomly chosen line bundle $\mathcal{L}$ is not typical or semi-typical. If $g = 1$, then all nontrivial line bundles $\mathcal{L} \not\simeq \mathcal{O}_C$ are easily seen to be typical. We therefore limit ourselves to $g \geq 2$ and moderately large $q$, and obtain our desired bound.

**Theorem 3.3.** Assume $g \geq 2$ and $q \geq 16^3$. Then the probability that a uniformly randomly chosen element $\mathcal{L} \in \text{Pic}^0 C(\mathbb{F}_q)$ is not typical is at most $(16^3 \cdot g + 3.4)/q$. The probability that $\mathcal{L}$ is not semi-typical is at most $1.7/q$.

**Proof.** Since $\mathcal{L} \in \text{Pic}^0 C(\mathbb{F}_q)$ is chosen uniformly at random, we see that also $\mathcal{L}(NP_\infty) \in \text{Pic}^N C(\mathbb{F}_q)$ is chosen uniformly at random for any $N$, as is $\mathcal{L}^{-1}(NP_\infty)$. The statement on semi-typicality is now Proposition 2.13 applied to $\mathcal{L}((g-1)P_\infty)$. The statement on typicality follows from bounding the probability that at least one of the conditions in Proposition 3.2 fails, using Proposition 2.12 applied to $\mathcal{L}((g-1)P_\infty)$ and $\mathcal{L}^{-1}((g-1)P_\infty)$, and Theorem 2.12 applied to $\mathcal{L}((g+1)P_\infty)$. 

As a small final observation, we note that the probability that at least one of $\mathcal{L}$ and $\mathcal{L}^{-1}$ in the above theorem is not typical is bounded by $(16^3 \cdot 2g + 3.4)/q$, due to the probability that $\mathcal{L}^{-1}((g+1)P_\infty)$ is not base point free.

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