Lelong numbers of currents of full mass intersection

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In memory of Nessim Sibony

Abstract

We study Lelong numbers of currents of full mass intersection on a compact Kähler manifold in a mixed setting. Our main theorems cover some recent results due to Darvas-Di Nezza-Lu. The key ingredient in our approach is a new notion of products of pseudoeffective $(1,1)$-classes which captures some “pluripolar part” of the “total intersection” of given pseudoeffective $(1,1)$-classes.

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1 Introduction

Let $X$ be a compact Kähler manifold of dimension $n$. For every closed positive current $S$ on $X$, we denote by $\{S\}$ its cohomology class. For cohomology $(q,q)$-classes $\alpha$ and $\beta$ on $X$, we write $\alpha \leq \beta$ if $\beta - \alpha$ can be represented by a closed positive $(q,q)$-current.

Let $\alpha_1, \ldots, \alpha_m$ be pseudoeffective $(1,1)$-classes, where $1 \leq m \leq n$. Let $T_j$ and $T'_j$ be closed positive $(1,1)$-currents in $\alpha_j$ for $1 \leq j \leq m$ such that $T_j$ is more singular than $T'_j$, i.e, potentials of $T_j$ is smaller than those of $T'_j$ modulo an additive constant. By monotonicity of non-pluripolar products (see [27, Theorem 1.1] and also [7, 12, 30]), there holds

\[ \{\langle T_1 \wedge \cdots \wedge T_m \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \rangle\}. \]  

(1.1)

We refer to the beginning of Section 2 for a brief recap of non-pluripolar products.

We are interested in comparing the singularity types of $T_j$ and $T'_j$ when the equality in (1.1) occurs. Given the generality of the problem, it is desirable to formulate it in a more concrete way. In what follows, we focus on the important setting where $T_1, \ldots, T_m$ are of full mass intersection (i.e, $T'_j$'s have minimal singularities in their cohomology classes).

Let us recall that $T_1, \ldots, T_m$ are said to be of full mass intersection if the equality in (1.1) occurs for $T'_j$ to be a current with minimal singularities $T_{j,\text{min}}$ in $\alpha_j$ for $1 \leq j \leq m$. 
This is independent of the choice of \( T_{j,\min} \). The last notion has played an important role in complex geometry, for example, see [2, 7, 11, 14, 19, 23, 28, 29]. We also notice that a connection of the notion of full mass intersection with the theory of density currents (see [20]) was established in [26], see also [22].

One of the most basic objects to measure the singularity of a current is the notion of Lelong numbers. We refer to [15] for its basic properties. Hence, the purpose of this paper is to compare the Lelong numbers of \( T \) and \( T_{j,\min} \) when \( T_1, \ldots, T_m \) are of full mass intersection. To go into details, we need some notions.

Let \( S \) be a closed positive current on \( X \) and \( x \) be a point in \( X \). Denote by \( \nu(S, x) \) the Lelong number of \( S \) at \( x \). One can compute \( \nu(S, x) \) as follows. We write \( S = dd^c \psi \) for some psh function \( \psi \) defined on an open neighborhood \( U \) of \( x \) such that \( U \) is a local chart of \( X \) which we identify with an open subset in \( \mathbb{C}^n \) and the point \( x \) corresponds to the origin in \( \mathbb{C}^n \). Then we have

\[
\nu(S, x) = \max \{ \gamma \in \mathbb{R}_{\geq 0} : \psi(z) \leq \gamma \log |z| + O(1) \text{ near } 0 \},
\]

see [15] Chapter III]. Let \( V \) be an irreducible analytic subset of \( X \). By Siu’s analytic semi-continuity of Lelong numbers ([15, 24]), for every \( x \in V \) outside some proper analytic subset of \( V \), we have

\[
\nu(S, x) = \min_{x' \in V} \nu(S, x').
\]

The last number is called the generic Lelong number of \( S \) along \( V \) and is denoted by \( \nu(S, V) \).

Let \( \alpha \) be a pseudoeffective \((1, 1)\)-class on \( X \). Following [16], we recall that \( \alpha \) is said to be big if there is a Kähler current in \( \alpha \), i.e., there is a closed positive current \( T \) in \( \alpha \) such that \( T \geq \omega \) for some Kähler form \( \omega \) on \( X \). Let \( T_{\alpha,\min} \) be a current with minimal singularities in \( \alpha \) (see [16], page 41-42 for definition). We denote by \( \nu(\alpha, V) \) the generic Lelong number of \( T_{\alpha,\min} \) along \( V \). This number is independent of the choice of \( T_{\alpha,\min} \). It is clear that for every current \( S \in \alpha \), we have \( \nu(S, V) \geq \nu(\alpha, V) \). Here is our first main result.

**Theorem 1.1.** Let \( 1 \leq m \leq n \) be an integer. Let \( \alpha_1, \ldots, \alpha_m \) be big cohomology classes in \( X \) and let \( T_j \) be a closed positive \((1, 1)\)-currents in \( \alpha_j \) for \( 1 \leq j \leq m \). Let \( V \) be a proper irreducible analytic subset of \( X \) of dimension \( \geq n - m \). Assume that \( T_1, \ldots, T_m \) are of full mass intersection. Then there exists an index \( 1 \leq j \leq m \) such that

\[
\nu(T_j, V) = \nu(\alpha_j, V). \tag{1.2}
\]

We note that when \( \alpha_1, \ldots, \alpha_m \) are Kähler, Theorem 1.1 was proved in [27 Theorem 1.2]; see also the discussion after Corollary 1.4 below. The proof presented there is not applicable in the setting of Theorem 1.1.

When \( \dim V = n - m \), the above result is optimal because in general, it might happen that there is only one index \( j \) satisfying (1.2); see Example 3.5. However, motivated from the Kähler case, we wonder whether it is true that the number of \( 1 \leq j \leq m \) such that \( \nu(T_j, V) = \nu(\alpha_j, V) \) is at least \( \dim V - (n - m) + 1 \) (recall \( V \not\subseteq X \)).

In the case where \( m = n \), our above result can be improved quantitatively as follows.
Theorem 1.2. Let $\mathcal{B}_0$ be a closed cone in $H^{1,1}(X, \mathbb{R})$ which is contained in the cone of big $(1,1)$-classes of $X$. Then, there exists a constant $C > 0$ such that for every $x_0 \in X$, every $\alpha_j \in \mathcal{B}_0$ and every closed positive $(1,1)$-current $T_j \in \alpha_j$ for $1 \leq j \leq n$, we have

$$\int_X \left( \langle \Lambda_{j=1}^n \alpha_j \rangle - \{ \langle \Lambda_{j=1}^n T_j \rangle \} \right) \geq C \left( \nu(T_1, x_0) - \nu(\alpha_1, x_0) \right) \cdots \left( \nu(T_n, x_0) - \nu(\alpha_n, x_0) \right).$$

(1.3)

The dependence of $C$ on $\mathcal{B}_0$ is necessary, see Example 3.4. We have the following direct consequences of Theorem 1.1. 

Corollary 1.3. Let $1 \leq m \leq n$ be an integer. Let $\alpha$ be a big class and let $T \in \alpha$ be a closed positive $(1,1)$-current so that $\{ \langle T^m \rangle \} = \langle \alpha^m \rangle$. Let $V$ be an irreducible analytic subset of $X$ of dimension at least $n - m$. Then there holds

$$\nu(T, V) = \nu(\alpha_j, V).$$

In particular, if $\alpha$ is big and nef, then $T$ has zero Lelong number at a generic point in $V$.

Recall that $\langle \alpha^m \rangle$ is defined to be the cohomology class of $\langle T_{\alpha, \min}^m \rangle$, where $T_{\alpha, \min}$ is a current with minimal singularities in $\alpha$, see Section 2 below for details. Combining Corollary 1.3 with results in [4, 8], we recover the following known result.

Corollary 1.4. Let $\theta$ be a smooth closed $(1,1)$-form in a big cohomology class $\alpha$. Let $\varphi$ be a $\theta$-psh function of full Monge-Ampère mass, i.e,

$$\{ \langle (dd^c \varphi + \theta)^n \rangle \} = \langle \alpha^n \rangle.$$

Let $\varphi_{\alpha, \min}$ be a $\theta$-psh function with minimal singularities. Then, we have

$$\mathcal{I}(t \varphi) = \mathcal{I}(t \varphi_{\alpha, \min})$$

(1.4)

for every $t > 0$, where for every quasi-psh function $\psi$ on $X$, we denote by $\mathcal{I}(\psi)$ the multiplier ideal sheaf associated to $\psi$.

Corollary 1.4 was proved in [10, 12, 13] (hence answering a question posed in [18]); see also [21] for the case where $\theta$ is Kähler. In fact, [12] gives a stronger fact which we describe below. For every closed positive $(1,1)$-current $T'$ with $\int_X \langle T'^m \rangle > 0$, Theorem 1.3 in [12] gives a characterization (in terms of certain plurisubharmonic rooftop envelopes) of potentials of every closed positive $(1,1)$-current $T$ cohomologous to $T'$ such that $T$ is less singular than $T'$ and

$$\int_X \langle T^m \rangle = \int_X \langle T'^m \rangle.$$

Consequently, the multiplier ideal sheafs associated to the potentials of $T$ and $T'$ are the same by arguments from the proof of [13, Theorem 1.1]. Nevertheless, in the present setting of our main results, it is unclear how to formulate such a characterization because either $T_1, \ldots, T_m$ can be different or $m \leq n$ (even if one takes $T_1 = \cdots = T_m$). In fact,
a direct analogue of the envelope characterization given [12] is not true in our setting when \( m \leq n \); see the comment after Theorem 1.1 and [12], Remark 3.3.

Let us now have a few comments on our approach. Due to the above discussions, we present here a completely new strategy to the study of singularity of currents of full mass intersection. We stress that although our main results only involve the usual non-pluripolar products, the notion of relative non-pluripolar products introduced in [27] will play an essential role in our proof. The reason, which will be more clear later, is that relative non-pluripolar products allow us to better control the loss of masses.

The key ingredient in our proof of Theorem 1.1 is a new notion of products of pseudoeffective classes which was briefly mentioned in [27], Remark 4.5. This new product of pseudoeffective classes is bounded from below by the positive product introduced in [5, 7]. The feature is that this new product also captures some pluripolar part of “total intersection” of classes. This explains why we have a better control on masses.

Theorem 1.2 is a direct consequence of the proof of Theorem 1.1. We underline that our arguments in the proof of Theorem 1.1 are not quantifiable as soon as \( \dim V \geq 2 \). This is due to the fact that we need to use the blowup along \( V \) and the desingularization of \( V \) (in case \( V \) is singular). Despite of this, we think that it is still reasonable to expect an estimate similar to Theorem 1.2 in the case where \( V \) is of higher dimension.

Finally, in view of the above discussion of results in [12], one can wonder what should be expected for the equality case of (1.1) when \( T_j^\prime \)'s are not necessarily of minimal singularities. It is not unrealistic to hope that our approach can be extended to this setting. But there are non-trivial obstructions. To single out one: the condition that \( T_j^\prime \)'s have minimal singularities are needed in our proof of Theorem 1.1 because we will use the fact that there are Kähler currents with analytic singularities which are more singular than \( T_j \) for every \( j \).

The paper is organized as follows. In Section 2, we present basic properties of relative non-pluripolar products and introduce the above-mentioned notion of products of pseudoeffective classes. Theorems 1.1 and 1.2 are proved in Section 3.

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## 2 Relative non-pluripolar products

We first recall some basic facts about relative non-pluripolar products. This notion was introduced in [27] as a generalization of the usual non-pluripolar products given in [3, 7, 21]. To simplify the presentation, we only consider the compact setting.

Let \( X \) be a compact Kähler manifold of dimension \( n \). Let \( T_1, \ldots, T_m \) be closed positive \((1, 1)\)-currents on \( X \). Let \( T \) be a closed positive current of bi-degree \((p, p)\) on \( X \). By [27], we can define the \( T\)-relative non-pluripolar product \( \langle \wedge_{j=1}^m T_j \wedge T \rangle \) in a way similar to that of the usual non-pluripolar product. For readers' convenience, we recall how to do it.
Write $T_j = dd^c u_j + \theta_j$, where $\theta_j$ is a smooth form and $u_j$ is a $\theta_j$-psh function. Put

$$R_k := 1_{c_{p_j} > -k} \wedge_{j=1}^n (dd^c \max \{u_j, -k\} + \theta_j) \wedge T$$

for $k \in \mathbb{N}$. By the strong quasi-continuity of bounded psh functions ([27, Theorems 2.4 and 2.9]), we have

$$R_k = 1_{c_{p_j} > -k} \wedge_{j=1}^n (dd^c \max \{u_j, -l\} + \theta_j) \wedge T$$

for every $l \geq k \geq 1$. A similar equality also holds if we use local potentials of relative non-pluripolar products. We refer to [27, Proposition 3.5] for more properties of relative non-pluripolar products.

Let $\alpha_1, \ldots, \alpha_m$ be pseudoeffective $(1, 1)$-classes on $X$. Recall that by using a monotonicity of relative non-pluripolar products ([27, Theorem 1.1]), we can define the cohomology class $\{\langle \alpha_1 \wedge \cdots \wedge \alpha_m \wedge T \rangle\}$ which is the one of the current $\langle \wedge_{j=1}^m T_{j, \min} \wedge T \rangle$, where $T_{j,\min}$ is a current with minimal singularities in $\alpha_j$ for $1 \leq j \leq m$. When $T$ is the current of integration along $X$, we write $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ for $\{\langle \alpha_1 \wedge \cdots \wedge \alpha_m \wedge T \rangle\}$. By [27, Proposition 4.6], the class $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ is equal to the positive product of $\alpha_1, \ldots, \alpha_m$ defined in [7, Definition 1.17] provided that $\alpha_1, \ldots, \alpha_m$ are big.

In the next paragraph, we are going to introduce a related notion of products of $(1, 1)$-classes. This idea was already suggested in [27]. This new notion will play a crucial role in our proof of Theorem 1.1. We are interested in the case where $T$ is of bi-degree $(1, 1)$. We recall the following key monotonicity property.

**Theorem 2.1. ([27, Remark 4.5])** Let $X$ be a compact Kähler manifold and let $T_1, \ldots, T_m, T$ be closed positive $(1, 1)$-currents on $X$. Let $T'_j$ and $T'$ be closed positive $(1, 1)$-currents in the cohomology class of $T_j$ and $T$ respectively such that $T'_j$ is less singular than $T_j$ for $1 \leq j \leq m$ and $T'$ is less singular than $T$. Then we have

$$\{\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \wedge T' \rangle\}.$$ 

Recall that for closed positive $(1, 1)$-currents $P$ and $P'$ on $X$, we say that $P'$ is less singular than $P$ if for every global potential $u$ of $P$ and $u'$ of $P'$, then $u \leq u' + O(1)$.

**Proof.** Since this result is crucial for us, we will present its proof below. Write $T_j = dd^c u_j + \theta_j$, $T'_j = dd^c u'_j + \theta_j$, where $\theta_j$ is a smooth form and $u'_j, u_j$ are negative $\theta_j$-psh functions, for every $1 \leq j \leq m$. Similarly, we have $T = dd^c \varphi + \eta$, $T' = dd^c \varphi' + \eta'$.

**Step 1.** Assume for the moment that $T_j, T'_j$ are of the same singularity type for every $1 \leq j \leq m$ and $T, T'$ are also of the same singularity type. We will check that

$$\{\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle\} = \{\langle T'_1 \wedge \cdots \wedge T'_m \wedge T' \rangle\}. \quad (2.1)$$

5
Since \( T_j, T'_j \) are of the same singularity type, we have \( \{ u_j = -\infty \} = \{ u'_j = -\infty \} \) and \( w_j := u_j - u'_j \) is bounded. We have similar properties for \( \varphi, \varphi' \). Let \( A := \bigcup_{j=1}^m \{ u_j = -\infty \} \) which is a complete pluripolar set. Put \( u_{jk} := \max \{ u_j, -k \} \), \( u'_{jk} := \max \{ u'_j, -k \} \) and

\[
\psi_k := k^{-1} \max \{ \sum_{j=1}^n (u_j + u'_j), -k \} + 1 \tag{2.2}
\]

which is quasi-psh and \( 0 \leq \psi_k \leq 1 \), \( \psi_k(x) \) increases to 1 for \( x \not\in A \). We have \( \psi_k(x) = 0 \) if \( u_j(x) \leq -k \) or \( u'_j(x) \leq -k \) for some \( j \). Put \( w_{jk} := u_{jk} - u'_{jk} \). Since \( w_j \) is bounded, we have

\[
|w_{jk}| \lesssim 1 \tag{2.3}
\]

on \( X \). Let \( J, J' \subset \{ 1, \ldots, m \} \) with \( J \cap J' = \emptyset \). Put

\[
R_{J,J,k} := \wedge_{j \in J} (dd^c u_{jk} + \theta_j) \wedge \wedge_{j' \in J'} (dd^c u'_{jk} + \theta_{j'}) \wedge T
\]

and

\[
R_{J,J'} := \langle \wedge_{j \in J} (dd^c u_j + \theta_j) \wedge \wedge_{j' \in J'} (dd^c u'_{j'} + \theta_{j'}) \rangle T.
\]

Let

\[
B_k := \cap_{j \in J} \{ u_j > -k \} \cap \cap_{j' \in J'} \{ u'_{j'} > -k \}.
\]

Observe

\[
0 \leq 1_{B_k} R_{J,J,k} = 1_{B_k} R_{J,J'}
\]

for every \( J, J', k \). Put \( \hat{R}_{J,J'} := 1_{X \setminus A} R_{J,J'} \). The last current is closed positive. Using the fact that \( \{ \psi_k \neq 0 \} \subset B_k \setminus A \), we get

\[
\psi_k \hat{R}_{J,J'} = \psi_k R_{J,J'} = \psi_k R_{J,J'}.
\tag{2.4}
\]

Put \( p' := n - \lvert J \rvert - \lvert J' \rvert - p - 1 \). By Claim in the proof of [27, Proposition 4.2], for every \( j'' \in \{ 1, \ldots, m \} \setminus (J \cup J') \) and every closed smooth form \( \Phi \) of bi-degree \( (p', p') \) on \( X \), we have

\[
\lim_{k \to \infty} \int_X \psi_k dd^c w'_{j''} \wedge \hat{R}_{J,J'} \wedge \Phi = 0. \tag{2.5}
\]

Let

\[
S_0 := \langle T_1 \wedge \cdots \wedge T_n \rangle T - \langle T'_1 \wedge \cdots \wedge T'_n \rangle T
\]

and

\[
S_1 := \langle T_1 \wedge \cdots \wedge T_n \rangle T_1 - \langle T'_1 \wedge \cdots \wedge T'_n \rangle T, \quad S_2 := \langle T'_1 \wedge \cdots \wedge T'_n \rangle (T - T').
\]

We have \( S_0 = S_1 + S_2 \). Using \( T_{jk} = T'_{jk} + dd^c w_{jk} \), one can check that

\[
\int_X \psi_k S_1 \wedge \Phi = \sum_{s=1}^m \int_X \psi_k \wedge_{j=1}^{s-1} T'_{jk} \wedge dd^c w_{sk} \wedge \wedge_{j=s+1}^{m} T_{jk} \wedge T \wedge \Phi
\]
for every closed smooth $\Phi$. This together with (2.5) yields
\[
\langle S_1, \Phi \rangle = \lim_{k \to \infty} \langle \psi_k S_1, \Phi \rangle = 0. \tag{2.6}
\]

Let $\varphi_l := \max\{\varphi, -l\}$ and $\varphi'_l := \max\{\varphi', -l\}$ for $l \in \mathbb{N}$. By [27, Theorem 2.2], observe
\[
\int_X \psi_k S_2 \wedge \Phi = \lim_{l \to \infty} \int_X \psi_k dd^c(\varphi_l - \varphi'_l) \wedge T'_{1k} \wedge \cdots T'_{mk} \wedge \Phi. \tag{2.7}
\]

Since $\varphi_l - \varphi'_l$ is bounded uniformly in $l \in \mathbb{N}$, reasoning as in the proof of (2.5), we see that the term under limit in the right-hand side of (2.7) converges to 0 as $k \to \infty$ uniformly in $l$. Hence
\[
\int_X \psi_k S_2 \wedge \Phi \to 0
\]
as $k \to \infty$. Consequently, we get $\int_X \psi_k S \wedge \Phi \to 0$ as $k \to \infty$. In other words, (2.1) follows. This finishes Step 1.

**Step 2.** Consider now the general case, i.e, $T'_j$ and $T'$ are less singular than $T_j$ and $T$ respectively. Without loss of generality, we can assume that $u'_j \geq u_j$ and $\varphi' \geq \varphi$. For $l \in \mathbb{N}$, put $u'_j := \max\{u_j, u'_j - l\}$ which is of the same singularity type as $u'_j$. Notice that $dd^c u'_j + \theta_j \geq 0$. Similarly, put $\varphi' := \max\{\varphi, \varphi' - l\}$ and $T^l := dd^c\varphi' + \eta \geq 0$.

Since $X$ is Kähler, the family of currents $\langle \wedge_{j=1}^m (dd^c u'_j + \theta_j) \wedge T^l \rangle$ parameterized by $l$ is of uniformly bounded mass. Let $S$ be a limit current of the last family as $l \to \infty$. Since $u'_j, u'_j$ are of the same singularity type for every $j$ and $\varphi', \varphi'$ are so, using Step 1, we see that
\[
\{S\} = \{\langle \wedge_{j=1}^m T'_j \wedge T' \rangle \}. \tag{2.8}
\]
On the other hand, since $u'_j, \varphi'$ decrease to $u_j, \varphi$ as $l \to \infty$ respectively, we can apply [27, Lemma 4.1] (and [27, Theorem 2.2]) to get
\[
S \geq \langle \wedge_{j=1}^m T_j \wedge T \rangle.
\]
This combined with (2.8) gives the desired assertion. The proof is finished. 

We note here the following remark which could be useful for other works.

**Remark 2.2.** Let $P$ and $P'$ be closed positive $(1, 1)$-currents and $Q$ a closed positive currents such that $P'$ is less singular than $P$ and potentials of $P$ and potentials of $Q$ are integrable with respect to the trace measure of $Q$. Put $T := P \wedge Q$ and $T' := P' \wedge Q$. Then Theorem 2.1 still holds for these $T', T$ with the same proof. The only minor modification is that the potentials $\varphi, \varphi'$ of $T, T'$ in the last proof are replaced by those of $P, P'$.

For a $(1, 1)$-current $P$, recall that the polar locus $I_P$ of $P$ is the set of $x \in X$ so that the potentials of $P$ are equal to $-\infty$ at $x$. By abuse of language, we say that a closed positive current $T$ has no mass on a Borel set $A \subset X$, if the trace measure of $T$ has no mass on $A$.

For every pseudoeffective $(1, 1)$-class $\beta$ in $X$, we define its polar locus $I_{\beta}$ to be that of a current with minimal singularities in $\beta$. This is independent of the choice of a current with minimal singularities. We have the following.
Lemma 2.3. Assume that $T$ is of bi-degree $(1, 1)$. Then we have

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \langle T_1 \wedge \cdots \wedge T_m \wedge (1_{X \setminus I_T} T) \rangle,$$

(2.9)

In particular, $T$ has no mass on $I_T$, then

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle.$$

Proof. By [27, Proposition 3.6], we get

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = 1_{X \setminus I_T} \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle.$$

(2.10)

Now using (2.10) and [27, Proposition 3.5] (vii) gives (2.9). This finishes the proof. □

Let $1 \leq l \leq m$. Let $\alpha_1, \ldots, \alpha_m, \beta$ be pseudoeffective $(1, 1)$-classes of $X$. Let $T_{j, \min}, T_{\max}$ be currents with minimal singularities in the classes $\alpha_j, \beta$ respectively, where $l \leq j \leq m$. By Theorem [2.1] the class

$$\{\langle T_1 \wedge \cdots \wedge T_{l-1} \wedge T_{l, \min} \wedge \cdots \wedge T_{m, \min} \wedge T_{\max} \rangle\}$$

is a well-defined pseudoeffective class which is independent of the choice of $T_{\min}$ and $T_{j, \sqrt{2}}$ for $l \leq j \leq m$. We denote the last class by

$$\{\langle T_1 \wedge \cdots \wedge T_{l-1} \wedge \alpha_l \wedge \cdots \wedge \alpha_m \wedge T_{\max} \rangle\}.$$

For simplicity, when $l = 1$, we remove the bracket $\{\}$ from the last notation.

The following result holds for the class $\{\langle T_1 \wedge \cdots \wedge T_{l-1} \wedge \alpha_l \wedge \cdots \wedge \alpha_m \wedge T_{\max} \rangle\}$ but to avoid cumbersome notations (while keeping the essence of the statements), we only write it for $l = 1$.

Proposition 2.4. (i) The product $\langle \wedge_{j=1}^m \alpha_j \wedge \beta \rangle$ is symmetric and homogeneous in $\alpha_1, \ldots, \alpha_m$.

(ii) If $\beta'$ is a pseudo-effective $(1, 1)$-class, then

$$\langle \wedge_{j=1}^m \alpha_j \wedge \beta \rangle + \langle \wedge_{j=1}^m \alpha_j \wedge \beta' \rangle \leq \langle \wedge_{j=1}^m \alpha_j \wedge (\beta + \beta') \rangle.$$

(iii) Let $1 \leq l \leq m$ be an integer. Let $\alpha_l', \ldots, \alpha_l''$ be a pseudoeffective $(1, 1)$-class such that $\alpha_l' \geq \alpha_j$ for $1 \leq j \leq l$. Assume that there is a current with minimal singularities in $\beta$ having no mass on $I_{\alpha_l'' - \alpha_j}$ for every $1 \leq j \leq l$. Then, we have

$$\langle \wedge_{j=1}^l \alpha_j' \wedge \wedge_{j=l+1}^m \alpha_j \wedge (\beta + \beta') \rangle \geq \langle \wedge_{j=1}^m \alpha_j \wedge (\beta + \beta') \rangle.$$

(iv) If there is a current with minimal singularities in $\beta$ having no mass on proper analytic subsets on $X$, then the product $\{\langle \wedge_{j=1}^m \alpha_j \wedge \beta \rangle\}$ is continuous on the set of $(\alpha_1, \ldots, \alpha_m)$ such that $\alpha_1, \ldots, \alpha_m$ are big.

(v) We have

$$\langle \wedge_{j=1}^m \alpha_j \wedge \beta \rangle \leq \langle \wedge_{j=1}^m \alpha_j \wedge (\beta + \beta') \rangle$$

and the equality occurs if there is a current with minimal singularities $P$ in $\beta$ such that $P = 0$ on $I_P$. 
Proof. We see that \((v)\) is a direct consequence of Lemma 2.3 and the definition of the product \((\wedge_{j=1}^{m} \alpha \wedge \beta)\). The other desired statements can be proved by using arguments similar to those in the proof of [27, Proposition 4.6]; see also [9] for related materials. This finishes the proof.

The following result will be useful later.

**Lemma 2.5.** Let \(\alpha\) be a big class and let \(T_{\alpha,\text{min}}\) be a current with minimal singularities in \(\alpha\). Let \(T\) be a current in \(\alpha\). Then, the current \(T_{\alpha} := 1_{I_{T_{\alpha,\text{min}}}} T_{\alpha,\text{min}}\) is a linear combination of currents of integration along irreducible hypersurfaces of \(X\), and we have

\[
T_{\alpha} \leq 1_{I_{\alpha}} T. \tag{2.11}
\]

In particular, for every pluripolar set \(A\), if \(T\) has no mass on \(A\), then neither does \(T_{\alpha,\text{min}}\).

**Proof.** Recall that \(I_{\alpha} = I_{T_{\alpha,\text{min}}}\). By Demailly’s analytic approximation of \((1,1)\)-currents ([16]), there exists a Kähler current with analytic singularities \(P\) in \(\alpha\). It follows that \(I_{\alpha}\) is contained in a proper analytic subset \(V\) of \(X\). This together with the fact that \(\text{Supp} T_{\alpha}\) is contained in the closure of \(I_{\alpha}\) implies that \(T_{\alpha}\) is supported on \(V\).

Since \(T_{\alpha}\) is of bi-dimension \((n-1,n-1)\), using the first support theorem [15, Page 141], we see that \(T_{\alpha}\) is supported on the union of hypersurfaces of \(X\) contained in \(V\). Now the second support theorem [15, Page 142-143] implies that \(T_{\alpha}\) must be a linear combination of currents of integration along hypersurfaces. Hence the first desired assertion follows.

We prove (2.11). It is enough to consider the case where \(1_{I_{T_{\alpha,\text{min}}}} T_{\alpha,\text{min}}\) is nonzero. Let \(W\) be the support of the last current. By the above observation, \(W\) is a hypersurface. Since \(T\) is less singular than \(T_{\alpha,\text{min}}\), we get

\[
\nu(T, x) \geq \nu(T_{\alpha,\text{min}}, x)
\]

for every \(x\). In particular, the generic Lelong number of \(T\) along every irreducible component \(W'\) of \(W\) is greater than or equal to that of \(T_{\alpha,\text{min}}\) along \(W'\). We deduce that \(T \geq 1_{I_{T_{\alpha,\text{min}}}} T_{\alpha,\text{min}}\). Hence, (2.11) follows.

Let \(A\) be a pluripolar set in \(X\). Let \(\varphi_{\text{min}}\) be a potential of \(T_{\alpha,\text{min}}\). We have

\[
T_{\alpha,\text{min}} = 1_{\{\varphi_{\text{min}} > -\infty\}} T_{\alpha,\text{min}} + 1_{\{\varphi_{\text{min}} = -\infty\}} T_{\alpha,\text{min}}.
\]

Denote by \(I_1, I_2\) the first and second term in the right-hand side of the last equality respectively. By (2.11) and the hypothesis, we see that \(I_2\) has no mass on \(A\). We now show that \(I_1\) satisfies the same property.

If \(\{\varphi_{\text{min}} > -\infty\}\) is open, then it is clear that \(I_1\) has no mass on \(A\) because \(\varphi_{\text{min}}\) is locally bounded on the open set \(\{\varphi_{\text{min}} > -\infty\}\). However in general, when \(\{\varphi_{\text{min}} > -\infty\}\) is not necessarily open, some more arguments are needed. Recall that \(I_1\) is actually equal to the non-pluripolar product \(\langle T_{\alpha,\text{min}} \rangle\) of \(T_{\alpha,\text{min}}\) itself (e.g, by applying [27, Proposition 3.6 (i)] to \(T \equiv 1\) and \(m = 1\)). Since the current \(\langle T_{\alpha,\text{min}} \rangle\) has no mass on pluripolar sets, we see that \(I_1\) has no mass on \(A\). Hence, \(T_{\alpha,\text{min}}\) has no mass on \(A\). This finishes the proof.

We note that (2.11) actually holds in a much more general setting; see [1, Lemma 4.1].
3 Proof of Theorems 1.1 and 1.2

We will sometimes use the notations $\gtrsim, \lesssim$ to denote the inequalities $\geq, \leq$ modulo some strictly positive multiplicative constant independent of parameters in consideration. For every analytic set $W$ in a complex manifold $Y$, we denote by $[W]$ the current of integration along $W$.

Let $X$ be a compact Kähler manifold. Let $\alpha_1, \ldots, \alpha_m$ be big classes in $X$. Let $T_{j, \text{min}}$ be a current with minimal singularities in $\alpha_j$ and

\[ T_{\alpha_j} := 1_{I_{\alpha_j}} T_{j, \text{min}} \]

(recall here that $I_{\alpha_j}$ is the set of $x \in X$ such that potentials of $T_{j, \text{min}}$ are equal to $-\infty$ at $x$). By Lemma 2.5 the current $T_{\alpha_j}$ is a linear combination of currents of integration along irreducible hypersurfaces of $X$. In view of proving Theorem 1.1, we first explain how to reduce the problem to the case where $T_{\alpha_j}$’s are zero.

**Lemma 3.1.** For every $j$, the class $\alpha_j - \{T_{\alpha_j}\}$ is big and there holds

\[ \langle \wedge_{j=1}^m \alpha_j \rangle = \langle \wedge_{j=1}^m (\alpha_j - \{T_{\alpha_j}\}) \rangle. \]  

(3.1)

**Proof.** Let $\omega$ be a Kähler form on $X$. Fix an index $1 \leq j \leq m$. Let $W_j$ be the support of $T_{\alpha_j}$. Consider a Kähler current $P_j \in \alpha_j$. By Lemma 2.5 the set $W_j$ is a hypersurface (or empty), and $P_j - T_{\alpha_j}$ is a closed positive current. Note that

\[ P_j - T_{\alpha_j} = P_j \gtrsim \omega \]

on $X \setminus W_j$. Since $\omega$ is smooth, we get $P_j - T_{\alpha_j} \gtrsim \omega$ on $X$. In other words, $P_j - T_{\alpha_j}$ is a Kähler current. Hence, $\alpha_j - \{T_{\alpha_j}\}$ is big.

It remains to prove (3.1). The inequality direction “$\geq$” is clear because $\alpha_j \geq \alpha_j - \{T_{\alpha_j}\}$. To get the converse inequality, one only needs to notice that

\[ \langle \wedge_{j=1}^m T_{j, \text{min}} \rangle = \langle \wedge_{j=1}^m (T_{j, \text{min}} - T_{\alpha_j}) \rangle \]

which is true because both sides are currents which have no mass on

\[ W := \cup_{j=1}^m W_j \]

(which is a closed pluripolar set) and are equal on $X \setminus W$ (which is an open subset of $X$). The proof is finished.

Let $T_j \in \alpha_j$ be a closed positive current as in Theorem 1.1. By Lemma 2.5, we have $1_{T_j} T_j \geq T_{\alpha_j}$. It follows that $T_j - T_{\alpha_j}$ is positive. Using the fact that $T_{\alpha_j}$ is supported on proper analytic subsets on $X$ gives

\[ \langle \wedge_{j=1}^m T_j \rangle = \langle \wedge_{j=1}^m (T_j - T_{\alpha_j}) \rangle. \]

This combined with Lemma 3.1 yields that $\langle T_1 - T_{\alpha_1} \rangle, \ldots, \langle T_m - T_{\alpha_m} \rangle$ are of full mass intersection. Hence, by considering $T_j - T_{\alpha_j}, \alpha_j - \{T_{\alpha_j}\}$ instead of $T_j, \alpha_j$, we can assume, from now on, that $T_{\alpha_j}$ is zero as desired.
Assume for the moment that $V$ is a smooth submanifold of $X$ of dimension $\leq n-1$. Let $\sigma : \hat{X} \to X$ be the blowup of $X$ along $V$. Denote by $\hat{V}$ the exceptional hypersurface. Let $\omega$ be a Kähler form on $X$. Let $\omega_\lambda$ be a closed smooth form cohomologous to $-\{\hat{V}\}$ so that the restriction of $\omega_\lambda$ to each fiber of the natural projection from $\hat{V}$ to $V$ is strictly positive (the existence of such a form is classical, see [25, Lemma 3.25]). Thus, there exists a strictly positive constants $c_V$ satisfying that

\[ \hat{\omega} := c_V \sigma^* \omega + \omega_\lambda > 0 \quad (3.2) \]

We note that when $\dim V = n-1$, by convention, we put $\hat{X} := X$, $\sigma := \text{id}$, $\hat{V} := V$, $c_V := 1$ and $\omega_\lambda := 0$.

For every closed positive current $S$ on $X$, let $\lambda_S$ be the generic Lelong number of $S$ along $V$. By a well-known result on Lelong numbers under blowups (see [5, Corollary 1.1.8]), the generic Lelong number of $\sigma^* S$ along $\hat{V}$ is equal to $\lambda_S$. Hence, we can decompose

\[ \sigma^* T_j = \lambda_T j [\hat{V}] + \eta_j, \quad \sigma^* T_{j,\min} = \lambda_{T,j,\min} [\hat{V}] + \eta_{j,\min}, \]

where $\eta_j$ and $\eta_{j,\min}$ are currents whose generic Lelong numbers along $\hat{V}$ are zero. Since $T_{j,\min}$ is less singular than $T_j$, we have $\lambda_T j \geq \lambda_{T,j,\min}$.

Let

\[ \gamma_j := \{\eta_j\}, \quad \gamma_{j,\min} := \{\eta_{j,\min}\}, \quad \beta := \{[\hat{V}]\}. \]

These classes are important in the sequel. By [6, 17], the class $\gamma_{j,\min}$ is big. For every closed smooth $(n-m, n-m)$-form $\Phi$, using the fact that $T_{j,\min}$ has minimal singularities and the monotonicity of non-pluripolar products gives

\[ \int_X \langle \wedge_{j=1}^m T_{j,\min} \rangle \wedge \Phi = \int_{\hat{X}} \langle \wedge_{j=1}^m \eta_{j,\min} \rangle \wedge \sigma^* \Phi = \int_{\hat{X}} \langle \wedge_{j=1}^m \gamma_{j,\min} \rangle \wedge \sigma^* \Phi. \quad (3.3) \]

**Lemma 3.2.** We have

\[ \langle \wedge_{j=1}^m \eta_j \rangle \leq \langle \wedge_{j=1}^m \eta_{j,\min} \hat{\eta}_m \rangle, \quad \langle \wedge_{j=1}^m \eta_{j,\min} \rangle = \langle \wedge_{j=1}^m \eta_{j,\min} \gamma_{j,\min} \rangle, \quad (3.4) \]

and

\[ \langle \wedge_{j=1}^m \gamma_{j,\min} \rangle = \langle \wedge_{j=1}^m \gamma_{j,\min} \hat{\gamma}_{m,\min} \rangle \quad (3.5) \]

**Proof.** The first desired inequality in (3.4) is clear by Proposition 2.4. Observe that $1_{I_{m,\min}} \eta_{m,\min}$ has no mass on $\hat{V}$ because the generic Lelong number of $\eta_{m,\min}$ along $\hat{V}$ is equal to zero. We deduce that

\[ 1_{I_{m,\min}} \eta_{m,\min} = 1_{I_{m,\min}} \setminus \hat{V} \eta_{m,\min} \leq \sigma^*(1_{\sigma(I_{m,\min})} T_{m,\min}) \leq \sigma^*(1_{I_{m,\min}} T_{m,\min}) = 0. \]

Hence, $\eta_{m,\min}$ has no mass on $I_{m,\min}$.

We now prove (3.5). Let $Q_m$ be a current with minimal singularities in $\gamma_{m,\min}$. By Lemma 2.5 and the fact that $\gamma_{m,\min}$ is big, we see that

\[ 1_{I_{Q_m}} Q_m \leq 1_{\eta_{m,\min}} \eta_{m,\min} = 0. \]

Hence, $Q_m$ has no mass on $I_{Q_m}$. Using this and Lemma 2.3 gives the desired equality and finishes the proof. 

[11]
Fix a norm \( \| \cdot \| \) on \( H^{1,1}(X, \mathbb{R}) \). For \( 1 \leq j \leq m \), let \( P_j \) be a Kähler current with analytic singularities in \( \alpha_j \). Let \( \epsilon > 0 \) be a constant small enough so that \( P_j \geq \epsilon \omega \) for every \( 1 \leq j \leq m \).

**Lemma 3.3.** For every constant \( \delta \in (0, 1) \), there exist a constant \( c_\delta > 0 \) and a Kähler current with analytic singularities \( Q_j \in \gamma_{j, \text{min}} - c_\delta \beta \) for \( 1 \leq j \leq m \) such that \( I_{Q_j} \) does not contain \( \hat{V} \), and \( Q_j \geq \frac{\delta \epsilon}{2c_V} \hat{\omega} \), and

\[
\frac{\delta \epsilon}{2c_V} \leq c_\delta \leq (c\|\alpha_j\| + \frac{\epsilon}{2c_V}) \delta, \tag{3.6}
\]

for some constant \( c \) > 0 independent of \( \delta, \beta \) and \( \alpha_j \). In particular, the currents with minimal singularities in \( \gamma_{j, \text{min}} - c_\delta \beta \) has no mass on \( \hat{V} \), and the current \([\hat{V}]\) has no mass on the polar locus of the class \( \gamma_{j, \text{min}} - c_\delta \beta - \frac{\delta \epsilon}{2c_V} \{ \hat{\omega} \} \).

**Proof.** Using Demailly’s analytic approximation of currents (16) applied to the Kähler current \( (1 - \delta)T_{j, \text{min}} + \delta P_j \) for \( \delta \in (0, 1) \), we obtain that for every \( \delta \in (0, 1) \), there exits a Kähler current \( P_{j, \delta} \) with analytic singularities in the class \( \alpha_j \) such that \( P_{j, \delta} \) is less singular than \( (1 - \delta)T_{j, \text{min}} + \delta P_j \) and

\[
P_{j, \delta} \geq \delta \epsilon \omega / 2. \tag{3.7}
\]

We deduce that

\[
\lambda_{T_{j, \text{min}}} \leq \lambda_{P_{j, \delta}} \leq \lambda_{T_{j, \text{min}}} + a_j \delta, \tag{3.8}
\]

where \( a_j := \lambda_{P_j} - \lambda_{T_{j, \text{min}}} \geq 0 \). Write

\[
\sigma^* P_{j, \delta} = \lambda_{P_{j, \delta}} [\hat{V}] + \eta_{j, \delta}.
\]

Since \( P_{j, \delta} \) has analytic singularities, so does \( \eta_{j, \delta} \) and the polar locus of \( \eta_{j, \delta} \) is an analytic subset of \( X \) which doesn’t contain \( \hat{V} \). Hence, \([\hat{V}]\) has no mass on the polar locus of \( \eta_{j, \delta} \).

Recall that by the choice of \( \omega_h \), we have \( \omega_h \in -\beta \). By (3.7) and (3.2), we also get

\[
Q_j := \eta_{j, \delta} + \frac{\delta \epsilon}{2c_V} \omega_h \geq \frac{\delta \epsilon}{2c_V} \hat{\omega}.
\]

The last current is in the class

\[
\gamma_{j, \text{min}} - c_\delta \beta,
\]

where

\[
c_\delta := \lambda_{P_{j, \delta}} - \lambda_{T_{j, \text{min}}} + (\delta \epsilon)/(2c_V) \leq (\lambda_{P_j} - \lambda_{T_{j, \text{min}}} + \epsilon/(2c_V)) \delta
\]

by (3.8). Since \( P_j \) is a current in \( \alpha_j \), we get \( \lambda_{P_j} \leq c \|\alpha_j\| \) for some positive constant \( c \) independent of \( \alpha_j \). Hence, (3.6) follows.

We have proved that there is a Kähler current with analytic singularities \( Q_j \) in \( \gamma_{j, \text{min}} - c_\delta \beta \) such that \( \hat{V} \not\subset I_{Q_j} \). It follows that \( Q_j \) has no mass on \( \hat{V} \). Using this and Lemma 2.5 yields that the currents with minimal singularities in \( \gamma_{j, \text{min}} - c_\delta \beta \) has no mass on \( \hat{V} \). The last desired assertion is also immediate because the polar locus of \( Q_j - \frac{\delta \epsilon}{2c_V} \hat{\omega} \) does not contain \( \hat{V} \). This finishes the proof. \( \square \)
End of the proof of Theorem 1.1 Let
\[ b_j := \lambda T_j - \lambda T_{j,\min} \geq 0. \]

Note that \( \gamma_j = \gamma_{j,\min} - b_j \beta \). Suppose on contrary that \( b_j > 0 \) for every \( j \). Recall that we are assuming that \( V \) is smooth. The case where \( V \) is singular is dealt with later.

Let \( c_\delta \) be the constant associated to a number \( \delta \in (0, 1) \) as in Lemma 3.3. Let \( \epsilon \) be the constant appearing in (3.6). Put
\[ \delta_j := (\epsilon \| \alpha_j \| + \frac{\epsilon}{2c\nu})^{-1}b_j \]
for \( 1 \leq j \leq m \). Note that since \( b_j \leq \| \alpha_j \| \), we can increase \( \epsilon \) in order to have \( \delta_j \in (0, 1) \).

By (3.6), we get \( c_\delta \leq b_j \) for every \( j \). Let \( \gamma'_{j,\min} := \gamma_{j,\min} - c_\delta \beta \). By Lemma 3.3 and the fact that
\[ I_{\gamma'_{j,\min} - \gamma_j} = I_{(b_j - c_\delta \beta)\beta} \subset \hat{V}, \]
we obtain that the currents with minimal singularities in \( \gamma'_{m,\min} \) has no mass on \( I_{\gamma'_{j,\min} - \gamma_j} \).

This combined with Proposition 2.4 (iii) gives
\[ \{ \langle \wedge_{j=1}^m \gamma_j \rangle \} = \langle \wedge_{j=1}^{m-1} \gamma_j \wedge \gamma_m \rangle \leq \langle \wedge_{j=1}^{m-1} \gamma_j \wedge \gamma'_{m,\min} \rangle \leq \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \gamma'_{m,\min} \rangle. \]

Using the supper-additivity of products of classes (Proposition 2.4 (ii)), we get
\[ \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \gamma'_{m,\min} \rangle \leq \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \gamma_{m,\min} \rangle - c_\delta \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \beta \rangle. \]

Let \( I \) be the first term in the right-hand side in the last inequality. Recall that the currents with minimal singularities in \( \gamma_{m,\min} \) has no mass on \( \hat{V} \). The last set contains \( I_\beta \). Hence, using Proposition 2.4 (iii) implies
\[ I \leq \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \gamma_{m,\min} \rangle. \]

Consequently,
\[ \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \gamma'_{m,\min} \rangle \leq \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \gamma_{m,\min} \rangle - c_\delta \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \beta \rangle \]
\[ \leq \langle \wedge_{j=1}^m \gamma_{j,\min} \rangle - c_\delta \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \wedge \hat{V} \rangle \]
by Lemma 3.2. Now let \( \Phi \) be a closed smooth positive \((n - m, n - m)\)-form on \( X \). Put \( M_j := \frac{\delta_j \epsilon}{2c\nu} \). Note that by (3.6), we get \( M_j \leq c_\delta \) for every \( j \). Taking into account Lemma 3.3 and Proposition 2.4 (iii), we see that
\[ \int_X \langle \wedge_{j=1}^m \gamma_{j,\min} \wedge \hat{V} \rangle \wedge \sigma^* \Phi \geq M_1 \cdots M_{m-1} \int_X \hat{\omega}^{m-1} \wedge \sigma^* \Phi. \]

Consequently, we obtain
\[ \int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \Phi = \int_X \langle \wedge_{j=1}^m \eta_j \rangle \wedge \sigma^* \Phi \]
\[ \leq \int_X \langle \wedge_{j=1}^m \gamma_{j,\min} \rangle \wedge \sigma^* \Phi - M_1 \cdots M_m \langle \hat{V} \rangle \wedge \sigma^* \Phi, \hat{\omega}^{m-1} \rangle \]
which is, by (3.3), equal to

$$\int_X \langle \wedge_{j=1}^m T_{j,\min} \rangle \wedge \Phi - M_1 \cdots M_m \langle [\hat{V}] \wedge \sigma^* \Phi, \hat{\omega}^{m-1} \rangle.$$  

Using this and the hypothesis that

$$\int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \Phi = \int_X \langle \wedge_{j=1}^m T_{j,\min} \rangle \wedge \Phi,$$  

we infer that $[\hat{V}] \wedge \sigma^* \Phi = 0$ for every closed smooth $(n - m, n - m)$-form $\Phi$. The last property means that $[V] \wedge \Phi = 0$ for every closed smooth $(n - m, n - m)$-form $\Phi$. By choosing $\Phi := \omega^{n-m}$, we obtain a contradiction because $\dim V \geq n - m$. This finishes Step 1 of the proof. We observe that we didn’t fully use the assumption that $\{ \langle \wedge_{j=1}^m T_j \rangle \} = \{ \langle \wedge_{j=1}^m T_{j,\min} \rangle \}$. We only needed that there is a closed positive smooth $(n - m, n - m)$-form $\Phi$ on $X$ such that (3.10) holds and $[V] \wedge \Phi \neq 0$. We will use this remark in the next paragraph.

We now explain how to treat the case where $V$ is not necessarily smooth. By Hironaka’s desingularization, there is $\sigma' : X' \to X$ which is a composition of consecutive blowups along smooth centers starting from $X$ so that the centers don’t intersect the regular part of $V$ and the strict transform $V'$ of $V$ by $\sigma'$ is smooth. Note that $V'$ is of the same dimension as $V$.

Let $T'_j := \sigma'^* T_j$ and $\alpha'_j := \sigma'^* \alpha_j$. One should note that $T'_1, \ldots, T'_m$ might not be of full mass intersection, however, we still have

$$\int_X \langle \wedge_{j=1}^m T'_j \rangle \wedge \sigma'^* \Phi = \int_X \langle \wedge_{j=1}^m \alpha_j \rangle \wedge \Phi = \int_{X'} \langle \wedge_{j=1}^m \alpha'_j \rangle \wedge \sigma'^* \Phi,$$  

for every closed smooth $(n - m, n - m)$-form $\Phi$ on $X$. We will use $\Phi := \omega^{n-m}$. Observe that

$$[V'] \wedge \sigma'^* \Phi \neq 0$$

because $\sigma'$ is a biholomorphism on an open Zariski set containing the regular part of $V$ and $[V] \wedge \Phi \neq 0$ (here we use $\dim V \geq n - m$). This together with (3.11) and the observation at the end of Step 1 allows us to apply Step 1 to $X', \alpha'_j$ and $T'_j$ to obtain that there exist an index $j_0$ such that

$$\nu(T'_{j_0}', V') = \nu(\alpha'_{j_0}, V').$$

On the other hand, by construction of $\sigma'$, we get $\nu(T'_{j_0}, V') = \nu(T_j, V)$ for every $j$, a similar property also holds for $T_{j,\min}$. It follows that

$$\nu(T_{j_0}, V) = \nu(\alpha'_{j_0}, V') \leq \nu(T'_{j_0,\min}, V') = \nu(T_{j_0,\min}, V) \leq \nu(T_{j_0}, V).$$

Hence, we get $\nu(T_{j_0,\min}, V) = \nu(T_{j_0}, V)$. This finishes the proof. \hfill \square

We now present the proof of Theorem 1.2.
Proof of Theorem [1.2] Let $\omega$ be a fixed Kähler form on $X$. Observe that by homogeneity, in order to prove the desired inequality, it suffices to consider $\alpha_j/\|\alpha_j\|$ in place of $\alpha_j$. Hence, from now on, without loss of generality, we can assume that $\alpha_j \in \mathcal{B}_0 \cap \mathcal{J}$, where $\mathcal{J}$ is the unit sphere in $H^{1,1}(X, \mathbb{R})$. Since $\mathcal{B}_0$ is closed and contained in the big cone, we deduce that $\mathcal{B}_0 \cap \mathcal{J}$ is compact in the big cone. It follows that there exist a constant $\epsilon > 0$ such that for every $\alpha \in \mathcal{B}_0 \cap \mathcal{J}$, there exists a current with analytic singularities $P \in \alpha$ such that $P \geq \epsilon \omega$. In particular, we obtain currents with analytic singularities $P_j \in \alpha_j$ such that $P_j \geq \epsilon \omega$ for $1 \leq j \leq m$.

Now, we follow the arguments in the proof of Theorem [1.1]. One only needs to review carefully the constants involving in estimates used there. Our submanifold $V$ is now the point set $\{x_0\}$. Let the notations be as in the proof of Theorem [1.1]. By the construction of $\tilde{X}$, the constant $c_V > 0$ in (3.2) can be chosen to be independent of $x_0$. As in the proof of Theorem [1.1] put

$$b_j := \nu(T_j, x_0) - \nu(\alpha_j, x_0), \quad \delta_j := \left(\epsilon \|\alpha_j\| + \frac{\epsilon}{2c_V}\right)^{-1} b_j, \quad M_j := \frac{\delta_j \epsilon}{2c_V}$$

for $1 \leq j \leq n$, where $c$ is a constant big enough depending only on $X$ (and a fixed Kähler form $\omega$ on $X$ and a fixed norm on $H^{1,1}(X, \mathbb{R})$). Since $\alpha_1, \ldots, \alpha_n \in \mathcal{B}_0 \cap \mathcal{J}$, we get

$$\delta_j \gtrsim b_j,$$

and the constant $\epsilon$ can be chosen independent of $\alpha_1, \ldots, \alpha_n$. Using (3.9) for $\Phi$ to be the constant function equal to 1 gives

$$\int_X (\langle \wedge_{j=1}^n \alpha_j \rangle - \langle \wedge_{j=1}^n T_j \rangle) \geq M_1 \cdots M_n = \frac{\delta_1 \epsilon}{2c_V} \cdots \frac{\delta_n \epsilon}{2c_V} \gtrsim b_1 \cdots b_n.$$

The proof is finished.

Example 3.4. Let $Y$ be a compact Kähler manifold and $\theta$ be a semi-positive $(1, 1)$-form in $Y$ such that there is a current $P$ in $\{\theta\}$ with $\nu(P, x_0) > 0$ for some $x_0 \in Y$ (one can take, for example, $Y$ to be the complex projective space and $\theta$ to be its Fubini-Study form). Let $X := Y^2$ and $\alpha := \pi_1^* \{\theta\}$ which is a semi-positive class, where $\pi_1 : Y^2 \to Y$ is the projection to the first component. We have $\int_X \alpha^{2 \dim Y} = 0$. Hence, $\alpha$ is not big. Let $\omega$ be a Kähler form on $X$. Let $\alpha_\epsilon := \alpha + \epsilon \{\omega\}$. We have

$$\int_X \alpha_\epsilon^{2 \dim Y} \rightarrow \int_X \alpha^{2 \dim Y} = 0.$$

Hence, if the constant $C$ in Theorem [1.2] were independent of $\mathcal{B}_0$, then (1.3) for $x_0$ would hold for $\alpha_j := \alpha_\epsilon$ and $T_j = \pi_1^* P$ for every $j$ for some constant $C$ independent of $\epsilon$. Letting $\epsilon \to 0$ gives a contradiction because the left-hand side converges to 0, whereas the right-hand side converges to a positive constant.

Proof of Corollary [1.4] We explain how to obtain Corollary [1.4] from Corollary [1.3]. Let $\rho : X' \to X$ be a smooth modification of $X$ and $E$ an irreducible hypersurface in $X'$. Let
\( \varphi' := \varphi \circ \rho, \varphi_{a, \min}' := \varphi_{a, \min} \circ \rho, \theta' := \rho \ast \theta \) and \( \alpha' := \rho \ast \alpha \). Since non-pluripolar products have no mass on pluripolar sets, we have

\[
\langle (dd^c \varphi' + \theta')^n \rangle = \langle \alpha'^n \rangle = \langle \alpha^n \rangle > 0,
\]

and a similar equality also holds if \( \varphi' \) is replaced by \( \varphi_{a, \min}' \) (note that we don’t know if the latter is a quasi-psh function with minimal singularities in \( \alpha' \); anyway we will only need that \( \varphi_{a, \min}' \) is of full Monge-Ampère mass in \( \alpha' \)). By a well-known result in [6], the class \( \alpha' \) is big.

Applying Corollary 1.3 to \( dd^c \varphi' + \theta' \) and \( V := E \), we obtain that the generic Lelong number of \( \varphi' \) along \( E \) is equal to \( \nu(\alpha', E) \). We also get an analogous property for \( \varphi_{a, \min}' \) by applying Corollary 1.3 to \( dd^c \varphi_{a, \min}' + \theta' \). It follows that the generic Lelong numbers of \( \varphi' \) and \( \varphi_{a, \min}' \) along \( E \) are equal. Now using this property and [4, Corollary 10.18] (or [8] Theorem A) gives the desired assertion. The proof is finished.

We end the paper with the example mentioned in Introduction.

**Example 3.5.** Let \( X := \mathbb{P}^n \) and \( [x_0 : x_1 : \cdots : x_n] \) the homogeneous coordinates. Let \( \omega \) be the Fubini-Study form on \( \mathbb{P}^n \). Let \( 2 \leq m \leq n \) be an integer. Consider

\[
V := \{ [x_0 : \cdots : x_n] \in \mathbb{P}^n : x_j = 0, \quad 0 \leq j \leq m - 1 \},
\]

and

\[
T_j := dd^c (|x_0|^2 + \cdots + |x_m|^2) = dd^c |x_0|^2 + \cdots + |x_m|^2 + \omega
\]

for \( 1 \leq j \leq m - 1 \). We have \( \dim V = n - m \). Put \( T_m := \omega \). Observe that the currents \( T := T_1 \wedge \cdots \wedge T_m \) and \( T' := T_1 \wedge \cdots \wedge T_{m-1} \) are well-defined (classically) by [13] Corollary 4.11, Page 156. Moreover since \( V \) is of dimension \( n - m \), and \( T' \) is of bi-dimension \( (n - m + 1, n - m + 1) \), we see that the trace measure of \( T' \) has no mass on \( V \) by [15] Page 141. This combined with the fact that \( T_m \) is smooth yields that the trace measure of \( T \) also has no mass on \( V \). Using this and the fact that \( T_j \) is smooth outside \( V \), we obtain

\[
T = \langle T_1 \wedge \cdots \wedge T_m \rangle
\]

(both sides have no mass on \( V \)). It follows that \( T_1, \ldots, T_m \) are of full mass intersection, but \( \nu(T_j, V) > 0 \) for \( 1 \leq j \leq m - 1 \), and \( \nu(T_m, V) = 0 \).

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