How group theory and school mathematics are connected: an identification of mathematics in-service teachers

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Abstract. The purpose of the article is to describe the IMTs’ identified mathematical connection between group theory and school mathematics curriculums in Indonesia. The data was gathered from the IMTs who involved in the Teachers Professional Training (TPT) program. The study used the concept-by-concept framework through a descriptive qualitative approach, to analyze how IMT’s connect the group theory and school mathematics contents. The results reveal that the IMTs’ identified connection classified as a concept by concept which align to the comparison through common features categorize. Even though the IMTs have the teaching experience both junior and senior high school, however; the identified connection tends to be too general based on the curriculum on school mathematics. The teaching experience that the subjects have, seems to affect the way the subject created the connections. Nevertheless, it shows that the IMTs could recall and create the mathematical connection between group theory and school mathematics with adjusted to the school curriculum. For mathematics teachers’ education programs, the results could be a consideration for designing a mathematics school textbook which is strongly supported by group theory concept as university mathematics.

1. Introduction
Most of the Teachers Training Institutions (TTI) in Indonesia which prepare prospective secondary mathematics teachers, offer the students with advanced mathematical knowledge such as group theory as part of abstract algebra. Abstract algebra which is included as advanced algebra contributes to provide an understanding of how numbers system are structured [1]. It also supports the mathematics pre-service teachers (PMTs) to explain the concepts in school mathematics such as the terminology of the inverse and identity element which means it could develop concept images and concept definitions [2]. The inverse and identity element concepts from group theory are also taught in the school mathematics curriculum; therefore, it is important to strengthen and broaden the understanding of mathematics [3]. Furthermore, abstract algebra also important to connect the content, approaches, and principles to communicate the relevance of mathematics to the students [4], for example, the knowledge from group theory particularly can help secondary mathematics teachers explain the closeness property in Integers.

However, IMTs do not grasp how to use the knowledge from abstract algebra in teaching school mathematics [5]. Moreover, the advanced mathematical knowledge which acquired from university is difficult to articulate in the using of school teaching [6]. Furthermore, teachers consider to this knowledge is not relevant and useless because there is no connection between the university and school mathematics curriculum [7].
Whereas, several studies has been conducted to show how the knowledge from abstract algebra support the school mathematics content and also how this knowledge integrated into practice [8][9][10][11][12][13][14]. However, most of them show the general connection without a direct relation from university mathematics to school mathematics or vice versa. For example, Wasserman [13] described that abstract algebra support the mathematical knowledge for teaching of the structure of sets, inverses, arithmetic properties, and solving equations. These findings attempt to address the second part of Klein’s [15] double discontinuity that abstract algebra have a connection to school mathematics teaching, but did not demonstrate the precise connection of concepts from university mathematics and school mathematics. Therefore, the current study intends to emphasize the mathematical connections from university and school level as a concept-by-concept connection [16]. In this article, concept-by-concept connections are defined as an explicit connection from the group theory’s topics or concepts to build another topics or concepts in school mathematics or vice versa.

Since the mathematical connection is important to increase the mathematical understanding [17]; therefore, it is crucial to address Klein’s [15] double discontinuity. Klein [15] conceptualized the concept of the double discontinuity, in this context the TTI curriculum offers the group theory course, but it does not connected from the knowledge from school mathematics. The second discontinuity is that when PMTs teach in school setting, group theory does not related to school mathematics content.

Regarding to this fact, the study aims to describe the IMTs’ identified mathematical connection between group theory and school mathematics. This can be utilized to design a school mathematics textbook that bridges the gap between the curriculum in school mathematics and university mathematics.

2. Methods

The current study employed a descriptive qualitative approach [18]. Data was collected from two IMTs, the junior and senior high school mathematics teacher who enrolled in the TPT program in one TTI located in central Java. The research procedures were conducted through six sequential stages. In the initial stage, the IMT’s were interviewed to obtain an overview of how they perceive the connection between group theory and school mathematics content as the initial description for conducting a teacher professional development (TPD). In the next step, the IMTs did the activity provided in the TPD which aimed to remind them of the group theory content in the university. The activity is that analyzing the material in the group theory from the provided textbook by identifying the material which is possibly related to school mathematics. In the third stage, the IMTs analyzed the school mathematics topics from the syllabus document by identifying the possible material which is related to algebra. In the fourth stage, the IMTs’ created the connections between group theory and school mathematics contents based on their experience in teaching. In the fifth stage, the research subjects and the researchers did a discussion into Forum Group Discussion (FGD) to confirm the identified connection that has been found by the research subjects. In the last stage, the researchers analyzed the identified connections from the research subject based on the created connected regarding the syllabus and Suominen’s [11] framework and the categorization of mathematical connection from table 1. The concept-by-concept connection of each group is presented in table 2 and table 3. Data were analyzed in three steps: data condensation, data display, and concluding [19].

| Category                       | Description                                                                 |
|--------------------------------|----------------------------------------------------------------------------|
| Alternative representation     | One concept is represented in different ways.                               |
| Comparison through common      | Two concepts share some features in common, which allow a                  |
| features                       | comparison through the concepts being similar, the same or not             |
|                                | the same.                                                                  |
| Equivalent representation      | One concept could be represented in different ways symbolically.           |
| Generalization                 | One concept is an example of a specific instance of another concept         |
| Hierarchical relationship      | One concept is a component of or included in another concept.              |
3. Results and discussion

This part firstly presents the IMTs’ work which is followed by the interpretations to draw the findings. Then, the findings are discussed to elaborate the implication of it for the mathematics teacher education program. Table 2 and table 3 show the IMTs’ work.

3.1. Results

Table 2. The identified connection between group theory and school mathematics by Subject 1.

| No | Concepts/Procedures/Facts in Group Theory | School Algebra Concepts | Material | Senior High School Level |
|----|----------------------------------------|-------------------------|----------|-------------------------|
| 1  | Definition of Function                  | Definition of Function  | Functions| 10th Grade              |
| 2  | Definition of Surjective, Injective and Bijective | Definition of Surjective, Injective and Bijective | Functions | 10th Grade |
| 3  | Concept of range, domain, and codomain | Concept of range, domain, and codomain | Functions | 10th Grade |
| 4  | Definition of Group and properties      | Associative properties of matrices in multiplication: $(A + B) + C = A + (B + C)$ | Matrices | 10th Grade |
|    | (1) closeness                           |                         |          |                         |
|    | (2) associative                         |                         |          |                         |
|    | (3) identity element                    |                         |          |                         |
|    | (4) an inverse element                  |                         |          |                         |
| 5  | Concepts of Commutative Group           | Associative properties of matrices in multiplication: $(AB)C = A(BC)$ | Matrices | 10th Grade |
|    |                                        |                         |          |                         |
|    |                                        | Associative properties of vector addition | Vectors | 10th Grade |
|    |                                        | Identity matrices      | Matrices | 10th Grade |
|    |                                        | $AI = IA = A$           |          | Grade                  |
|    |                                        | Inverse matrices       | Matrices | 10th Grade |
|    |                                        | $AA^{-1} = A^{-1}A = I$ |          | Grade                  |
|    |                                        | The inverse of function composition | Function composition | 11th Grade |
|    |                                        |                         |          |                         |
|    |                                        |                         |          |                         |
|    |                                        |                         |          |                         |
| No | Concepts/Procedures/Facts in Group Theory | School Algebra Concepts | Material | Senior High School Level |
|----|----------------------------------------|------------------------|----------|-------------------------|
| 6  | Theorem: the uniqueness of the identity element in a group. | Commutative properties in matrices addition \((A + B) = (B + A)\) | Matrices Addition | 10th Grade |
| 7  | Theorem: the uniqueness of inverse of a group element | Identity Matrix \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] | Matrices | 10th Grade |
| 8  | Properties of the group, for every \(a, b \in G\), \((a \circ b)^{-1} = b^{-1} o a^{-1}\) | If \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) and \(ad - bc \neq 0\) then the matrices inverse of \(A\) (written as \(A^{-1}\)). \[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\] | Matrices | 10th Grade |
| 9  | Suppose \((G, o)\) is a group and \(a \in G\). For any natural numbers \(n\), then \(a^n = a \cdot a \cdot a \cdot \cdots \cdot a\) \(n\) times \(\) if \(G\) group under addition then \(a \cdot n = a + a + \cdots + a\) \(n\) term = \(na\) | The inverse of composition function \((f \circ g)^{-1} = g^{-1} o f^{-1}\) \(\) | Matrices | 10th Grade |
| 10 | Composition of permutation | The inverse of the square matrix \((AB)^{-1} = B^{-1} A^{-1}\) \(\) | Matrices | 10th Grade |
| 11 | Cyclic Group | The inverse of the composition function \((BA)^{-1} = A^{-1} B^{-1}\) \(\) | Exponent | 10th Grade |
| 12 | Permutation Group | Suppose \((G, o)\) is a group and \(a \in G\). For any natural numbers \(n\), then \(a^n = a \cdot a \cdot a \cdot \cdots \cdot a\) \(n\) times \(\) if \(G\) group under addition then \(a \cdot n = a + a + \cdots + a\) \(n\) term = \(na\) | Composition of functions | 10th Grade |
| 13 | Definition of the cyclic group | Composition function \(|(a^m)^{-1} = (a^{-1})^m|\) \(\) | Exponent | 10th Grade |

Exponent properties 
\[
\begin{align*}
(a^m)^{-1} &= (a^{-1})^m \\
 a^m \cdot a^n &= a^{m+n} \\
 a^m \cdot a^n &= a^{m-n} \\
 (a^m)^n &= (a)^{mxn} \\
 (a \cdot b)^n &= a^n \cdot b^n
\end{align*}
\]
Table 2 shows that subject 1 draw the connections based on concept-by-concept from group theory to the functions, matrices, vectors, matrix addition, matrix determinant, matrices inverse, inverse function, composition functions, the inverse of composition functions, and enumeration rules. These identified connections are recognized as the concept by concept based on the classification from the previous studies [11][20][21]. One of the specific connections from table 2, for instance, \( an = a \cdot a \cdot a \cdot \ldots \cdot a \) \( n \) times is the concept of cyclic group which support the basic concept of exponents properties which is learnt at school mathematics. Subject 1 are able to identify into detail and systematically based on the school mathematics curriculum particularly in the senior high school.

**Table 3.** The identified connection between group theory and school mathematics by subject 2.
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| Concept/Procedures/Facts in Group Theory | School Algebra Concepts | Junior High School Level |
|----------------------------------------|-------------------------|-------------------------|
| $y \circ a = b$ has a unique solution in $G$. | The identity element under addition is 0, meanwhile, the identity element under multiplication is 1. | $7^{th}$ Grade |
| Theorem: the uniqueness of identity element in a group | For every integer under addition, it only has an inverse ($x \text{ the inverse of } -x$) | $7^{th}$ Grade |
| Theorem: the uniqueness of inverse of each group element. | Subsets |
| Cyclic Group | Determining numbers’ pattern and repetitive pattern | $8^{th}$ Grade |

Table 3 describes that subject 2 could create the connection from group theory which is connected to the group properties such as closeness, associativity, inverse, and identity element property in integers, commutative property under addition and multiplication, the cancellation law and the uniqueness solution in algebraic operations, subset, determining numbers’ pattern and repetitive pattern.

The identified connection are observed as the concept by concept connection based on the categorization of the previous studies [11][20][21]. One of the specific connections from table 3, the group properties which embed on the integers in addition to showing the closeness, associativity properties, the inverse and identity element which is taught in school mathematics particularly in junior high school. Subject 2 can identify and elaborate the school algebra concept without present the general content in school mathematics.

3.2. Findings
From subject 1 which is shown in table 2, the general content from senior high school was created besides elaborate the specific concept that closely related. It describes that the subject could show and map the concepts precisely into details from university mathematics to the school mathematics. From table 3, the subject presented the elaborated concept without showing the general content from the school material. Overall, both subjects could identify the knowledge from group theory into general curriculum in school mathematics level.

Nevertheless, the subjects have limitation to reveal all the concept-by-concept connections from both curriculums. They only present the limited number of common concepts from group theory. This finding allows with Klein’s double discontinuity [15] which is the group theory that was taught in the university mathematics did not related to school mathematics topics and vice versa. Therefore, the IMTs have a difficulty to match concept by concept from both levels.

Table 4 presents the analysis of both syllabus group theory and school mathematics in the Indonesian context and connected to Suominen’s work of mathematical connections [11] which align to the concept by concept connection. The comparison of the connection that has been identified by IMTs employs the table 4.

| Table 4. Concept-by-concept connection. |
|----------------------------------------|--------------------------------|
| Group Theory Concept | Secondary School Mathematics Concept |
| Structure Algebra and its Properties | Identity |
| | Inverse |
| | Function and domain |
| | Solving linear equations |
| | Number systems and operators |
| Binary Operation | Domain |
| | Function |
| | Composition of function |
3.3. Discussion

Regarding the table 2 and table 3, identified mathematical connections between group theory and school mathematics in junior and high school, the research subjects can create the connections into detail and systematically after involving in TPD by reviewing the material from group theory textbook. The identifications accommodate several identified connections which are compiled by the researcher. Even though the IMTs have the teaching experience both junior and senior high school, however; the identified connection tends to be too general based on the curriculum on school mathematics. Nevertheless, it shows that the IMTs could recall and create the connection between group theory and school mathematics with adjusted to the school curriculum.

The IMTs’ identified connections of group theory and school mathematics in this study do not completely describes all the connections as stated in the prior studies [11][20][21] since it is presented based on general curriculum both in junior and senior high school. The most possible reasons that enable the IMTs only to identify the connections in a general school mathematics curriculum is that the time-lapse when the subjects had taken the group theory subject in the university and the practicing teaching at school at present. Therefore, they can only relate the big picture that has been adjusted to the school mathematics curriculum. Even though the IMTs also involve in the TPD program, however; it is only conducted in the short period time. Further, the longer period time of conducting the program are expected could help the TTI and school mathematics teachers to tackle Klein’s double discontinuity. The second reason is that no connection established in the learning process from both school mathematics and group theory which is aligned to the Klein’s double discontinuity [15]. Therefore, it is important for lecturers who teach group theory in the TTI to consider the connections from both levels. This is in line to the findings from the previous study [22] that the lecturers should provide the teaching resource which facilitates the PMTs to recognize the connections of the university mathematics and school mathematics particularly for group theory course. Therefore, when the PMTs have an opportunity to teach in a school setting they could easily relate the fundamental concept in group theory to support advanced concepts in school mathematics contents.

Overall, the results show that IMTs could create identified connection between group theory and school mathematics based on general mathematics curriculums of junior and high school. It is expected that the IMTs will recognize the important the course that they had taken to build a fundamental concept for school mathematics content. As the findings reveal the limitation of identified the mathematical by the IMTs; therefore, it is important to consider the collaboration between TTI and school mathematics teachers to connect the group theory, particularly, and school mathematics for addressing the Klein’s double discontinuity. The finding could be a reference for developing a school mathematics textbook that accommodates the connection between advanced algebra as a foundation for school mathematics contents. However, this study only exposes the IMTs’ identified connection without intending to
investigate how they use this connection to support school mathematics teaching. Therefore, further studies are suggested to analyze how IMTs use these connections into teaching practice.

4. Conclusion
This study describes the IMTs’ identified mathematical connections between group theory and school mathematics’ content both junior and senior high school. Even though the IMTs have the teaching experience both junior and senior high school, however; the identified connection tends to be too general based on the curriculum on school mathematics. The teaching experience that the subjects have, seems to affect the way the subject created the connections. Nevertheless, it shows that the IMTs could recall and create the connection between group theory and school mathematics with adjusted to the school curriculum. The constraints of IMTs in classifying the mathematical connections might be the time gap when the subjects have taken the group theory subject in the university and the practicing teaching at school at present. Even though, the TPD has been provided to enable the subjects in reviewing the material from the group theory textbook. However, they might find it difficult to specifically write the comprehensive connections which accommodate all the possibilities connected material. Therefore, to cope this condition it is crucial for the lecturers who teach group theory, particularly, to pay more attention to collaborate with the school mathematics teachers. The collaboration in designing the school mathematics textbook which provides as many as the connection between school mathematics and university algebra is expected to anticipate the second Klein’s discontinuity.

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