Abstract.

In this paper we complete the integration of the conformally flat pure radiation spacetimes with a non-zero cosmological constant $\Lambda$, and $\tau \neq 0$, by considering the case $\Lambda + \tau \bar{\tau} \neq 0$. This is a further demonstration of the power and suitability of the generalised invariant formalism (GIF) for spacetimes where only one null direction is picked out by the Riemann tensor. For these spacetimes, the GIF picks out a second null direction, (from the second derivative of the Riemann tensor) and once this spinor has been identified the calculations are transferred to the simpler GHP formalism, where the tetrad and metric are determined. The whole class of conformally flat pure radiation spacetimes with a non-zero cosmological constant (those found in this paper, together with those found earlier for the case $\Lambda + \tau \bar{\tau} = 0$) have a rich variety of subclasses with zero, one, two, three, four or five Killing vectors.

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1 Introduction

1.1 Integration in GHP formalism

The method of integration within the Geroch-Held-Penrose (GHP) formalism using GHP operators [15] pioneered by Held [19, 20] and developed by Edgar and Ludwig [6, 8, 25, 9] has been shown to be particularly useful and efficient in spacetimes where two null directions are picked out by the geometry.
The 'optimal situation' is when the GHP formalism generates internally a set of tables involving the first derivative GHP operators for each of four real zero-weighted \emph{intrinsic} scalars ('coordinate candidates') and a table for one complex weighted \((p \neq 0 \neq q)\) \emph{intrinsic} scalar (which describes the spin \(((p - q)/2)\) and boost \(((p + q)/2)\) gauge). An important ingredient within this method is the repeated application of the GHP commutator equations; in particular, it is essential that these commutators be applied to these five scalars in order to extract all the information residing in the GHP commutators \[6\]. Held’s original hope was that these five tables, by themselves, would be complete and involutive: in general, we now know that they will not be, since the manipulations — especially the application of the commutators to the coordinate candidates — will generate additional scalars and their associated tables; however taking all these tables together ensures a complete and involutive system. The development and applications of this method can be found in \[19, 20, 6, 8, 9, 23, 3, 4\].

One of the intriguing aspects of this operator method in the GHP formalism is that the 'optimal situation' — where the calculations for constructing a metric are, in principle, simplest — is for spaces lacking Killing vectors, and lacking spin and boost isotropy freedom, \[9\]. (On the other hand, it is well known that the NP tetrad formalism \[33\] has been particularly useful in investigating spaces with isotropy freedom and/or Killing vectors.) From the theory and structure of the GHP formalism, it follows that the presence of the full quota of four \emph{intrinsic} coordinate candidates is directly linked to the absence of Killing vectors, while the presence of a complex weighted \emph{intrinsic} scalar is directly linked to the absence of spin and boost isotropy freedom; in spaces with no Killing vectors there will be four \emph{intrinsic} coordinates within the formalism, and in spaces with no spin and boost isotropy freedom there will be one complex \emph{intrinsic} weighted scalar. In the situation where less than four zero-weighted \emph{intrinsic} scalars are supplied directly by the GHP formalism, it is then necessary to introduce replacements for the absent coordinate candidate(s) indirectly — each via a table which is complementary to the complete explicit set of GHP equations, being entirely consistent with them; in particular the table(s) for the \emph{complementary} zero-weighted scalar(s) (\emph{complementary coordinate candidate(s)}) must be consistent with the GHP commutators. In an analogous manner, in the situation where a complex weighted \emph{intrinsic} scalar is not supplied directly by the GHP formalism, it is then necessary to introduce a replacement for this absent complex scalar (or absent part of the complex scalar) indirectly — via a table which is complementary to the complete explicit set of GHP equations; the table for the \emph{complementary weighted scalar} also must be consistent with all the GHP equations including the GHP commutators. In addition, all the new tables must be consistent with each other. It is emphasised that it is essential that the full quota of five scalars be obtained (by supplementing intrinsic ones with \emph{complementary} ones, where necessary), and that the GHP commutators be applied to each, so as to ensure that \emph{all} the information in the GHP commutators is extracted.

Using this integration procedure in the GHP formalism, in addition to appli-
cations to spacetimes without Killing vectors, there have been applications to spaces with Killing vectors (mostly to spaces with one Killing vector) [6], [8], [23], and a technique has emerged for introducing the tables for the complementary coordinate candidates. For example, in the case of a spacetime with only three intrinsic coordinate candidates $x_1, x_2, x_3$ and requiring a complementary coordinate candidate $\tilde{x}_4$, the idea is to make use of a related 'generic' spacetime (where it exists) with the analogous three intrinsic coordinates candidates $x_1, x_2, x_3$ plus a fourth intrinsic coordinate $x_4$; then to introduce the complementary coordinate candidate $\tilde{x}_4$ via a 'copy' of the table for the coordinate candidate $x_4$, but in addition ‘freeing’ this complementary coordinate candidate $\tilde{x}_4$ from any direct links which $x_4$ had with the remaining explicit elements of the GHP formalism.

It has been shown that spacetimes constructed by this method not only can be easily analysed for their Killing vector structure, but the explicit form of the Killing vectors (and even the homothetic vectors) can be found in a comparatively easy manner [9].

1.2 Integration in GIF

The generalised invariant formalism (GIF) of Machado Ramos and Vickers [30], [31], [32] generalises the GHP formalism by building the null rotation freedom of the second null direction into the formalism, which means that the GIF is built around only one spinor $o_A$. However, the formal set of equations for the GIF is considerably more complicated than for the GHP formalism, involving spinor differential operators; this means that calculations — especially involving the GIF commutators — are much more involved. Nevertheless, an analogous integration method [13], [10], [11] using operators of the GIF has been developed. Once again, the ‘optimal situation’ is when the formalism generates a set of tables involving first derivative operators for each of four real zero-weighted intrinsic scalars and for one complex weighted intrinsic scalar: but in addition, we need to generate a table for a second intrinsic spinor $I_A$ (which is not parallel to the first spinor $o_A$). As with the GHP procedure, the integration technique in the GIF relies heavily on repeated applications of the commutators, and it is essential that the GIF commutators be applied to the full quota of five scalars, and also to the second spinor $I_A$. Hence, if we are investigating a less than ‘optimal situation’ which fails to generate the full quota of intrinsic scalars, then complementary scalars will need to be introduced indirectly via tables as in the GHP method, while if we cannot generate a second unique intrinsic spinor, a complementary spinor will also need to be introduced indirectly via a table.

The very first investigation using the GIF integration method was for the class of conformally flat pure radiation spacetimes (with zero cosmological constant) [13]. The pure radiation component of the Ricci tensor immediately picks out one null direction $o_A$, and the generic class of these spacetimes admits no Killing vectors; so these spaces were particularly well suited for investigation by this approach in the GIF [13]. A second intrinsic spinor $I_A$ was obtained after a little
manipulation in the GIF: the optimal situation was achieved in the generic case — with no Killing vectors; and all four intrinsic zero-weighted scalars, together with an intrinsic complex weighted scalar, were generated internally within the GIF. The GIF commutators were applied to the second spinor and to the five scalars. As a consequence, a complete and involutive set of tables was obtained in the GIF. For the non-generic case, which required one complementary coordinate candidate — corresponding to the presence of one Killing vector — a parallel calculation in the GIF was carried out, with a table for the complementary coordinate 'copied' from the generic case. Finally, for both cases, by identifying the spinor $I_A$ with the second dyad spinor $\iota_A$ of the GHP formalism, the investigations were transferred into, and completed in, the GHP formalism; the final step involved deducing the tetrad vectors from the GHP tables for the four coordinate candidates, and hence the metric. In [13], the part of the investigation in the GIF which established the complete and involutive set of GIF tables was the most complicated; in particular because of the repeated use of the complicated GIF commutators.

In fact, with the benefit of hindsight, it is now clear that it was not necessary to carry out all of these GIF calculations in [13]: the crucial step in the GIF was to generate explicitly this second intrinsic spinor $I_A$ from within the GIF formalism. As soon as this spinor was found and the GIF commutators applied to it, then $I_A$ could have been identified as the second spinor $\iota_A$ in the dyad for the GHP formalism; at this stage, the investigation could have been immediately transferred to the GHP formalism. Had this earlier transfer been made, the latter part of the complicated GIF calculations in [13] could have been replaced with simpler GHP calculations.

On the other hand, it is emphasised that when we fail to obtain a unique intrinsic second spinor we cannot take this short cut to the GHP formalism. The absence of a second unique intrinsic spinor is linked to the presence of null isotropy freedom, and we have recently considered such a situation — a subclass of conformally flat pure radiation spacetimes with a cosmological constant — and demonstrated how such a problem is solved by the GIF method [11].

1.3 Outline of paper

In this paper we wish to develop the operator method further and more efficiently in the GIF formalism, by investigating the remainder of the conformally flat pure radiation spacetimes with a cosmological constant. First of all, we wish to illustrate the point emphasised above: when we are able to determine a second unique intrinsic spinor $I_A$ in the GIF formalism we will demonstrate how to transfer to the simpler GHP formalism, and thus reduce the amount of calculations. In order to do this we will need to obtain the GHP commutators, which are crucial to simplifying our analysis; we shall demonstrate how to do this in a simple manner. Secondly, by investigating a class of spacetimes with a richer Killing vector structure than the cases looked at before, we will learn more about how to introduce tables for complementary coordinate candidates in the GIF, by modifying the techniques which have been outlined above for the
GHP formalism.
The approach being adopted in this paper, (as was also adopted in [13] and [11]), is to attempt to generate the complex scalar and as many coordinates as possible intrinsically, i.e., directly from elements of the GIF/GHP formalisms; it is only when it is clear that no more intrinsic coordinate candidates are available that we introduce complementary coordinate candidates by their respective tables. An advantage of this approach is that subsequently we can easily interpret the Killing vector structure and Karlhede classification from this version of the metric, which will involve the maximum number of ‘good’ coordinates. An alternative approach would be to introduce complementary coordinates, even when we suspect that there may exist intrinsic coordinates which we have not exploited; such an approach may be advantageous when the intrinsic choices lead to very complicated calculations. This alternative approach was carried out in our investigation of Petrov type N pure radiation spacetimes [10]; to avoid misunderstandings, we emphasise the different approach in that paper to this one.

So now we will further develop the GIF operator method by generalising the earlier derivation [13] of the metric for conformally flat pure radiation spaces to include the case of a non-zero cosmological constant. In [11] we looked at the subclass of these spacetimes for which we were unable to find a second unique intrinsic spinor due to the presence of one degree of null isotropy freedom; in this paper we look at the other subclass where there is no null isotropy freedom, and a second unique intrinsic spinor $I_A$ is quickly generated within the GIF. This means that we can quickly transfer to the GHP formalism and so minimise the calculations. These spacetimes will illustrate further refinements of our method, and they will also be shown to have a richer Killing vector structure.

More details of the philosophy and techniques of the GIF operator integration procedure has been given in [13], [10], [11], so we will not repeat these discussions in this paper, but rather we will only summarise the relevant parts of the GIF which are needed in this paper. In Section 2 we describe the differential operators, and the equations for the class of spaces under consideration are given in Section 3. In Section 4 we discuss the principles of the early transfer from the GIF to the GHP formalism, and how to obtain the GHP commutator equations. In the beginning of Section 5 in order to make comparison easy, we carry through the integration procedure initially keeping close to the pattern of the calculations in [13], obtaining a table for the crucial second spinor $I_A$. As soon as we obtain this second unique intrinsic spinor $I_A$, and extract additional information by applying the GIF commutators to it, we translate all the results into the GHP formalism; in the latter part of this section we show, by a straightforward relabelling and rearranging of some of the coordinate candidates and unknown functions, that their GHP tables can be put into a much simpler form, so that we can more easily complete the application of the commutators to these tables. Finally, from these tables, we write down the tetrad and the metric explicitly.

The procedure in Section 5 is dependent on the condition that the four zero-weighted scalars, to which we assign the role of coordinate candidates, are func-
tionally independent and hence can play the role of coordinates; indeed, if we make the assumption that none of these scalars are constants, then a check of the determinant formed from their four tables shows that all four scalars are in fact functionally independent. On the other hand, it is found that although three of the four coordinate candidates cannot be constant, the other one may be; in addition, we make some other assumptions in our calculations which exclude some other special cases. Hence the tetrad and metric obtained in Section 5 are not the most general that can be obtained for this class of spacetimes. In Section 6, we extend our approach to include one of the special cases which were excluded in the analysis in Section 5.

In Section 7 we consider the remaining special case and discuss in more detail the introduction and role of complementary coordinates, and how to copy their tables; also in that section we put together all the subclasses and present the most general form for the metric. In Section 8 we summarise the methods and results of this paper together with those of [11].

2 GIF

A full explanation of the formalism is given in [31], [32]. For the purpose of this paper, the summaries given in [13] and especially [11] are sufficient.

In this subsection, we will list only those equations to which we will make direct reference. The GIF differential operators $\mathbf{p}$, $\mathbf{\partial}$, $\mathbf{V}$ and $\mathbf{\partial}'$ act on properly weighted symmetric spinors to produce symmetric spinors of different valence and weight. Although the definition of the differential operators appears quite complicated, the fact that they take symmetric spinors to symmetric spinors means that one can write down the equations in a more compact and index free notation. In this compacted notation we have the following useful identities for scalars of weight $\{p, q\}$,

$$ (\mathbf{V}'\eta) \cdot \mathbf{\sigma} = \frac{1}{2} \{ (\mathbf{\partial}'\eta) - qT\eta \} \quad (1) $$

$$ (\mathbf{V}'\eta) \cdot \mathbf{o} = \frac{1}{2} \{ (\mathbf{\partial}\eta) - pT\eta \} \quad (2) $$

$$ (\mathbf{\partial}'\eta) \cdot \mathbf{o} = \frac{1}{2} \{ (\mathbf{p}\eta) - pR\eta \} \quad (3) $$

$$ (\mathbf{\partial}\eta) \cdot \mathbf{\sigma} = \frac{1}{2} \{ (\mathbf{p}\eta) - qR\eta \} \quad (4) $$

$$ (\mathbf{V}'\eta) \cdot \mathbf{o} \cdot \mathbf{\sigma} = \frac{1}{4} \{ (\mathbf{p}\eta) - pR\eta - qR\eta \} \quad (5) $$

For a spinor $\eta$ the above contractions become more complicated. For example for a valence (1,0)-spinor $\eta_A$ of weight $\{p, q\}$ we get

$$ (\mathbf{V}'\eta) \cdot \mathbf{o} = \frac{1}{2} \{ \mathbf{V}'(\eta \cdot \mathbf{o}) + (\mathbf{\partial}\eta) - (p - 1)T\eta \} \quad (6) $$
The equations

We are concerned with the Petrov type O pure radiation spaces with non-zero Ricci scalar, and in fact we begin with identical equations to those in [11], but we shall repeat them here for easy reference. In the usual way, we choose $\mathbf{o}_A$ to be aligned with the propagation direction of the radiation, so that the Ricci spinor takes the form

$$\Phi_{AAB'} = \Phi o_A o_B \bar{\sigma}_A \bar{\sigma}_B$$

(12)

where $\Phi = \Phi_{22}$ is a real scalar field of weight {2, 2}; all the other curvature components, except the Ricci scalar $\Lambda$, vanish.

For this class of spaces the well known property of the vanishing of the spin coefficients $\kappa, \sigma, \rho$ means that in the GIF

$$K = 0$$
$$S = 0$$
$$R = 0$$

(13)

but

$$T_{AA'} = \tau o_A \bar{\sigma}_A$$

(14)
where the scalar $\tau$ has weight \{1, $-1$\}. Notice that $\tau$ and $\Phi_{22}$ are both invariant under the group of null rotations so that they can be used instead of their GIF spinor equivalents; this gives a considerable simplification in the GIF notation. The GIF equations are:

(i) GIF Ricci equations:

$$\mathcal{D}_\tau = 0$$  
$$\partial\tau = \tau^2$$  
$$\partial^\prime\tau = \tau\tau + 2\Lambda$$  

(ii) GIF Bianchi equations:

$$\mathcal{D}\Phi = 0$$  
$$\partial\Phi = \tau\Phi$$  
$$\partial^\prime\Phi = \tau\Phi$$  

$$\mathcal{D}\Lambda = 0$$  
$$\partial\Lambda = 0$$  
$$\partial^\prime\Lambda = 0$$  
$$\mathcal{D}\Lambda = 0$$  

(iii) GIF commutators (applied to a general symmetric spinor $\eta$ of weight \{p, q\} and with $N$ unprimed and $N'$ primed indices):

$$\left(\mathcal{D}\mathcal{D}' - \mathcal{D}'\mathcal{D}\right)\eta = (\tau\mathcal{D} + \tau\mathcal{D}')\eta + (p - N)\Lambda\eta + (q - N')\Lambda\eta$$  
$$\left(\mathcal{D}\mathcal{D}' - \mathcal{D}'\mathcal{D}\right)\eta = 2\Lambda(\eta \cdot \sigma)$$  
$$\left(\mathcal{D}\mathcal{D}' - \mathcal{D}'\mathcal{D}\right)\eta = -2\Lambda(\eta \cdot \sigma)$$  
$$\left(\mathcal{D}\mathcal{D}' - \mathcal{D}'\mathcal{D}\right)\eta = -\tau\mathcal{D}\eta - \Phi(\eta \cdot \sigma)$$  
$$\left(\mathcal{D}\mathcal{D}' - \mathcal{D}'\mathcal{D}\right)\eta = -\tau\mathcal{D}^\prime\eta - \Phi(\eta \cdot \sigma)$$

where $(\eta \cdot \sigma)$ is the $(N - 1, N')$-spinor $\eta_{A_1, \ldots, A_{N-1, A_{N'}}}$, and $(\eta \cdot \tilde{o})$ is the $(N, N' - 1)$-spinor $\eta_{A_1, \ldots, A_{N-1, A_{N'}}}$, and if the contraction is not possible then these terms are set to zero.

These GIF equations contain all the information for the type O pure radiation metrics with non-zero Ricci scalar. We emphasize that we assume throughout that constant $\Lambda \neq 0$ as well as $\tau \neq 0$.

We noted in [11] that the type O pure radiation metrics with non-zero Ricci scalar divided naturally into two cases

(i) $\Lambda + \tau\tau \neq 0$
(ii) $\Lambda + \tau\tau = 0$

In [11] we considered the second of these subclasses; in the present paper we consider the first.
It will be convenient to introduce
\[ k \equiv (\Lambda + \tau^2)/2\tau \]
for notational convenience\(^1\), and so throughout this paper we will assume
\[ k \neq 0. \]

4 Transfering from GIF to the GHP formalism

As emphasised in the Introduction, calculations in the GIF can be long and complicated, and a careful examination of the details of [13] reveals that there is some redundancy in the techniques introduced there. In particular, the complete and involutive set of tables for all of the scalar quantities were obtained in terms of GIF operators, whereas we really only need the simpler GHP version of the tables in order to deduce the metric. In fact, in a particular calculation such as [13] and in the present paper, once we have used the GIF to obtain the table for the second spinor \( I \), and applied the GIF commutators to this table, we can then identify \( I \) with the second dyad spinor \( \iota \iota \iota \) in the GHP formalism and immediately translate any existing tables into the GHP formalism, which means that we can then complete the calculations for the remaining tables in the GHP formalism.

For the subsequent calculations we will need the GHP commutator equations, which can be obtained from the GIF commutator equations by projecting on the appropriate number of \( \iota \iota \iota \), \( \bar{\iota} \iota \iota \) spinors and using (8), (9), (10), (11). An alternative method, which will be quicker for our purpose, is to make use of the GHP commutator equations as quoted in [15] (specialised to this class of spacetimes),

\[
(\mathcal{P} \mathcal{P}' - \mathcal{V} \mathcal{V}') \eta = \left( (\bar{\tau} - \tau') \partial + (\tau - \bar{\tau}') \partial' + p(\tau \tau' + \Lambda) + q(\bar{\tau} \bar{\tau}' + \Lambda) \right) \eta \\
(\mathcal{V} \partial - \partial \mathcal{V}) \eta = -\bar{\tau}' \mathcal{P} \eta \\
(\partial \partial' - \partial' \partial) \eta = \left( (\bar{\rho}' - \rho') \mathcal{V} - p\Lambda + q\Lambda \right) \eta \\
(\mathcal{V}' \partial - \partial \mathcal{V}') \eta = \left( \rho' \partial + \bar{\sigma}' \partial' - \tau \mathcal{V}' - \bar{\kappa}' \mathcal{V} - q\bar{\tau} \bar{\sigma}' - p\rho' \tau \right) \eta
\]

(29)

where \( \eta \) is an arbitrary scalar of weight \( \{p, q\} \).

Of course now we encounter the problem that the GHP formalism involves the spin coefficients \( \tau', \sigma', \mu', \kappa' \) which are missing from the GIF. However, assuming that we have obtained a table for \( I \) in our GIF analysis, once we have identified \( I \) with the second dyad spinor \( \iota \), we can use this table to obtain directly these additional four spin coefficients as follows

\[ \tau' = -\mathcal{L}^B D (\iota_B) = -\mathcal{L}^B \mathcal{V} (\iota_B) = -\mathcal{L}^B \mathcal{L}^{C'} \mathcal{L}^{C'} \mathcal{P}_{C'C'} (\iota_B) \]

\(^1\)This quantity \( k \) is closely related to the quantity \( \kappa \) in [33] and to \( k \) in [16]; any of these quantities can be used to classify the conformally flat pure radiation spaces (as well as more general Petrov types) into different subclasses.
\[ \rho' = -i B^C \tilde{C}' D \partial_{CC'D'}(t_B) \]
\[ \sigma' = -i B^C \tilde{C}' D \partial_{C'D'}(t_B) \]
\[ \kappa' = -i B^C \tilde{C}' D \partial_{C'D'}(t_B) \]

(30)

5 The integration procedure for \( \Lambda + \tau \neq 0 \): the generic case.

5.1 Preliminary rearrangement.

The Riemann tensor and the spin coefficients supply three real scalars which can easily be rearranged to give one real zero-weighted \((\tau \bar{\tau})\) and two real weighted scalars, \(\Phi\) and \(\text{arg}(\tau / \bar{\tau})\). However, in order to keep the presentation of subsequent calculations to a minimum and to have easy comparison with [13], it will be convenient to rearrange slightly these three scalars, and use instead the zero-weighted scalar

\[ A = \frac{1}{\sqrt{2\tau \bar{\tau}}} \]  

(31)

and the weighted scalars\(^2\)

\[ P = \sqrt{\frac{\tau}{\bar{\tau}}} \]  

(32)

\[ Q = \sqrt{\frac{\Phi}{2\tau \bar{\tau}}} \]  

(33)

where \(P\) is a complex scalar of weight \(\{1, -1\}\), with \(P \bar{P} = \frac{1}{2}\); and \(Q\) is a real scalar of weight \(\{-1, -1\}\). (As well as \(\Phi = \frac{Q^2}{\bar{\tau}} \neq 0 \neq \Lambda\), we are assuming \(\tau = P/A \neq 0\), and so each of \(A\), \(P\), \(Q\), will always be defined and different from zero.)

These particular choices enable us to replace the Ricci equations with

\[ \Phi' A = 0 \]
\[ \partial A = -2P(\Lambda A^2 + 1/2) = -2P \bar{k} \]
\[ \partial' A = -2\bar{P}(\Lambda A^2 + 1/2) = -2\bar{P} \bar{k} \]

(34)

\[ \Phi'(\bar{P} Q) = 0 \]
\[ \partial'(\bar{P} Q) = \frac{1}{2} Q \Lambda A \]
\[ \partial'(\bar{P} Q) = -3Q \bar{P}^2 \Lambda A \]

(35)

\(^2\)We have retained the notation \(P, Q\) which was used in [13] for these two weighted scalars; note the slightly different definitions compared with \(P, Q\) used in [11] when considering the case \(\Lambda + \tau \bar{\tau} = 0\). Care needs to be taken when comparing with the various quantities labelled with \(P, Q\) (sometimes \(p, q\)) in [34], [16], [1], [2], [10] and other references.
where we now have

\[ k = \Lambda A^2 + 1/2 \neq 0 \]

At various steps in the sequel it will be obvious that we are assuming \( k \neq 3/2 \); however, this is not an additional restriction since we can deduce from the partial table (34) for \( A \) that this condition must always be satisfied.

### 5.2 Constructing a table for I and applying commutators to I.

For our integration procedure we begin by completing the partial table (35) for the \( \{−2,0\} \) weighted scalar \( PQ \),

\[
\begin{align*}
\mathbf{b}(\overline{PQ}) &= 0 \\
\partial(\overline{PQ}) &= \frac{1}{2} \Lambda AQ \\
\partial'(\overline{PQ}) &= -3\Lambda AQ\overline{P}^2 \\
\mathbf{v}(\overline{PQ}) &= J
\end{align*}
\]

(36)

where we have completed the table with some spinor \( J \), which is as yet undetermined.

We know from (1) and (2) that

\[
\mathbf{b}(\overline{PQ}) \cdot \mathbf{o} = \partial(\overline{PQ}) + 2\tau PQ = \partial(\overline{PQ}) + \frac{Q}{A}
\]

(38)

Substituting (36) we can then write

\[
J = -\left(\frac{Q}{A} + \frac{1}{2} \Lambda AQ\right)I + 3\Lambda AQ\overline{P}^2\mathbf{1}
\]

(39)

where

\[
I \cdot \mathbf{o} = 0
\]

(40)

and

\[
I \cdot \mathbf{o} = -1
\]

(41)

Hence \( I \) is a \((1,0)\) valence spinor, and from

\[
\left(\mathbf{b}(\overline{PQ})\right)_{AB,A'B'} = -\left(\frac{Q}{A} + \frac{1}{2} \Lambda AQ\right)I_{(A'B')}\mathbf{o}_A\mathbf{o}_B + 3\Lambda AQ\overline{P}^2\mathbf{1}_{(A'B')}\mathbf{o}_A\mathbf{o}_B
\]

(42)

we conclude that its weight is \( \{-1,0\} \).

It is important to note two crucial properties of the new spinor \( I \). Firstly, for this whole class of spaces, \( I \) can never be zero, nor parallel to \( \mathbf{o} \). Secondly, for the subclass under consideration in this paper, the spinor \( I \) is given uniquely in terms of the elements of the GIF formalism and so is an intrinsic spinor; this can
be seen when we solve for $I$ from (22) and its complex conjugate remembering that $\kappa \neq 0$ in this paper.
(However, it is emphasised, on the contrary, for the subclass of spaces defined by $\kappa = 0$, that the spinor $I$ is not given uniquely in terms of the elements of the GIF formalism and so is not an intrinsic spinor; this is the subclass of spaces considered in [11].)

It will be useful in the sequel to have separate tables for $P$ and $Q$ which are easily determined from (26) as follows:

\[
\begin{align*}
P & = 0 \\
\partial P & = -2\Lambda AP^2 \\
\theta P & = \Lambda A \\
\psi P & = \frac{2P^2\kappa}{A} - \frac{\kappa}{A} \\
& \quad \text{(43)}
\end{align*}
\]

\[
\begin{align*}
P & = 0 \\
\partial Q & = -\Lambda AQP \\
\theta Q & = -\Lambda AQ\bar{P} \\
\psi Q & = -\frac{QP(\frac{3}{2} - \kappa)}{A} - \frac{Q\bar{P}(\frac{3}{2} - \kappa)}{A} \\
& \quad \text{(44)}
\end{align*}
\]

Our first mission is to find the table for $I$ which should follow from applying the commutators to the table for $(PQ)$; but first of all we will need to complete the partial table (24) for $A$ as an expression for $\Psi A$ will be required. We obtain

\[
\begin{align*}
P A & = 0 \\
\partial A & = -2P\kappa \\
\theta A & = -2\bar{P}\kappa \\
\psi A & = C \\
& \quad \text{(45)}
\end{align*}
\]

where we have completed the table with a spinor $C$, which is as yet undetermined. It follows from (11) and (2) that

\[
\begin{align*}
C \cdot \bar{\sigma} & = (\Psi A) \cdot \bar{\sigma} = (\theta A) = -2\bar{P}\kappa \\
C \cdot o & = (\Psi A) \cdot o = (\partial A) = -2P\kappa \\
& \quad \text{(46)}
\end{align*}
\]

Therefore

\[
C = \frac{Q}{A}C\kappa^2 + 2P\kappa I + 2\bar{P}\kappa\bar{I} \\
& \quad \text{(48)}
\]

and so $C$ is a Hermitian $(1,1)$ type spinor of weight $\{2,2\}$, with $C$ a zero-weighted real scalar, as yet undetermined (with the factor $\kappa^2/A$ introduced for brevity and convenience of subsequent presentation).
We are now able to apply the commutators to the table for \((\overline{P}Q)\) which yields a partial table for the spinor \(I\); we obtain

\[
\begin{align*}
\mathfrak{d}I &= \frac{3\Lambda A\overline{P}}{(\frac{3}{2} - k)} \\
\partial I &= -\frac{\Lambda QCk(1 - 4\Lambda A^2)}{4 (\frac{3}{2} - k)} - \frac{3\Lambda A\overline{P}}{(\frac{3}{2} - k)} \mathfrak{t} \\
\partial' I &= \frac{3\Lambda CQk}{8P^2(\frac{3}{2} - k)} - \frac{3\Lambda A\overline{P}}{(\frac{3}{2} - k)} I \\
\mathfrak{p}' I &= W
\end{align*}
\]

where we have completed the table with some spinor \(W\) as yet undetermined.

In a similar manner as for previous tables, but this time, using (6) and (7) we find that

\[
\begin{align*}
\mathfrak{d}C &= -\frac{4}{Q(\frac{3}{2} - k)} \\
\partial C &= -\frac{P\Lambda AC(5\Lambda A^2 - 2)}{(\frac{3}{2} - k)} + \frac{4}{Q(\frac{3}{2} - k)} \mathfrak{t} \\
\partial' C &= -\frac{P\Lambda AC(5\Lambda A^2 - 2)}{(\frac{3}{2} - k)} + \frac{4}{Q(\frac{3}{2} - k)} I \\
\mathfrak{p}' C &= L
\end{align*}
\]

where \(L\) is a Hermitian \((1, 1)\) type spinor of weight \(\{2, 2\}\) determined, from (1) and (2), to be:

\[
\begin{align*}
L &= \frac{Q}{A} L + \left(\frac{P\Lambda AC(5\Lambda A^2 - 2)}{(\frac{3}{2} - k)}\right) I + \left(\frac{P\Lambda AC(5\Lambda A^2 - 2)}{(\frac{3}{2} - k)}\right) \mathfrak{t} \\
&\quad - \frac{1}{Q} \left(\frac{4}{(\frac{3}{2} - k)}\right) \mathfrak{t}
\end{align*}
\]

We have completed the table, in the same manner as we did for previous tables, with a zero-weighted real scalar \(L\), as yet undetermined.
The theory requires that we also apply the commutators to the table for $I$, which yields a partial table for complex $W$,

\[ \mathcal{P}W = \frac{\Lambda C(5\Lambda A^2 + 4) k^2}{Q \left( \frac{3}{2} - k \right)^2} \]
\[ \partial W = -2P + \frac{\Lambda^2 C^2 A k^2 (-8\Lambda^2 A^4 + 28\Lambda A^2 + 7)}{8P \left( \frac{3}{2} - k \right)^2} - \frac{3\Lambda A W}{2P \left( \frac{3}{2} - k \right)} - \frac{\Lambda A W}{P} \]
\[ -\frac{\Lambda k L (1 - 4\Lambda A^2)}{4P \left( \frac{3}{2} - k \right)} - \frac{\Lambda C k^2 (5\Lambda A^2 + 4)}{Q \left( \frac{3}{2} - k \right)^2} I \]
\[ \partial' W = \frac{3\Lambda^2 C^2 A k^2 (4\Lambda A^2 + 5)}{8P \left( \frac{3}{2} - k \right)^2} - \frac{\Lambda A k W}{P \left( \frac{3}{2} - k \right)} + \frac{3\Lambda L k}{4P \left( \frac{3}{2} - k \right)} \]
\[ -\frac{\Lambda C k^2 (5\Lambda A^2 + 4)}{Q \left( \frac{3}{2} - k \right)^2} I \]  

So we have obtained a core element required in our analysis: a new spinor $I$ which is not parallel to $o$. We have also constructed its table, and then we applied the GIF commutators to $I$ in order to extract further information. Since $I$ is uniquely defined in terms of intrinsic elements of the GIF, we can now transfer these tables into the GHP formalism and carry out subsequent calculations in the GHP formalism.

5.3 Transferring to the GHP formalism

We now identify this spinor $I$ with the second dyad spinor $\iota$ of the GHP formalism. Then the two tables for the zero weighted $A, C$ can be immediately translated into the ordinary GHP scalar operators,

\[ \mathcal{P}A = 0 \]
\[ \partial A = -2Pk \]
\[ \partial' A = -2\overline{P}k \]
\[ \mathcal{P}A' = \frac{Q}{A} Ck^2 \]  

\[ \mathcal{P}C = -\frac{4}{Q(\frac{3}{2} - k)} \]
\[ \partial C = -\frac{\overline{P}\Lambda AC(5\Lambda A^2 - 2)}{(\frac{3}{2} - k)} \]
\[ \partial' C = -\frac{\overline{P}\Lambda AC(5\Lambda A^2 - 2)}{(\frac{3}{2} - k)} \]
\[ \mathcal{P}C' = \frac{Q}{A} L \]  

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This translation is carried out using (8), (9), (10), (11), and is especially simple since the operators are acting on scalars. The table for complex $W$ can also be easily rewritten in GHP operators, but it will be more convenient to write down two tables for the real and imaginary parts of $W$ by putting,

\[
M = \frac{1}{2}(W + \overline{W}) - A
\]

\[
B = \frac{i}{2}(W - \overline{W})
\]

which gives,

\[
\mathcal{P}M = \frac{\Lambda C k^2(5\Lambda A^2 + 4)}{Q \left(\frac{3}{2} - k\right)^2}
\]

\[
\partial M = -\frac{2\Lambda A^2 P k}{\left(\frac{3}{2} - k\right)} + \frac{\Lambda^2 P A C^2 k^3 (-\Lambda A^2 + \frac{1}{2})}{\left(\frac{3}{2} - k\right)^2} + \frac{\Lambda P L k^2}{\left(\frac{3}{2} - k\right)} - \frac{3\Lambda A P M}{\left(\frac{3}{2} - k\right)}
\]

\[
\partial' M = -\frac{2\Lambda A^2 \overline{P} k}{\left(\frac{3}{2} - k\right)} + \frac{\Lambda^2 \overline{P} A C^2 k^3 (-\Lambda A^2 + \frac{1}{2})}{\left(\frac{3}{2} - k\right)^2} + \frac{\Lambda \overline{P} L k^2}{\left(\frac{3}{2} - k\right)} - \frac{3\Lambda A \overline{P} M}{\left(\frac{3}{2} - k\right)}
\]

and

\[
\mathcal{P}B = 0
\]

\[
\partial B = i \left(-2P k - \Lambda^2 P A C^2 k^2 - \Lambda P L k - 2\Lambda A P M\right)
\]

\[
\partial' B = -i \left(-2\overline{P} k - \Lambda^2 \overline{P} A C^2 k^2 - \Lambda \overline{P} L k - 2\Lambda A \overline{P} M\right)
\]

In the table (49) for $\mathbf{I}$ we will now make the substitution $W = A + M - iB$: this table is needed to calculate $\tau', \rho', \sigma', \kappa'$ at the end of this subsection. However, we shall have no further need of this table in constructing the metric since it only deals with the choice of direction for the second spinor; on the other hand, we will use the table when we consider the Karlhede classification.

From (36) and (8), (9), (10), (11), the GHP tables for the weighted scalars $P, Q$, are

\[
\mathcal{P} P = 0
\]

\[
\partial P = -2P^2 \Lambda A
\]

\[
\partial' P = \Lambda A
\]

\[
\mathcal{P} P = 0
\]
\[ \begin{align*}
\mathbf{PQ} &= 0 \\
\partial Q &= -QPA \\
\partial' Q &= -Q\overline{P}A \\
\overline{P}Q &= 0
\end{align*} \tag{61} \]

We began with GIF tables (34), (36), for the zero-weighted scalar \( A \) and the weighted complex scalar \( (PQ) \), from which the GIF commutators generated GIF tables (51), (49), for the zero-weighted scalar \( C \) and the spinor \( I \) respectively; furthermore, the GIF commutators acting on the GIF table for \( I \) generated the partial GIF table (53), for the complex zero-weighted \( W \). We have now given the equivalent GHP tables (60), (61), and (54), (55), respectively for the weighted scalars \( P, Q \) and the zero-weighted scalars \( A, C \) as well as the partial GHP tables (58), (59) for the zero-weighted scalars \( M, B \).

In addition to extracting information by applying the GIF commutators to the table for \( I \), the theory requires that we obtain complete tables for four real zero-weighted scalars (coordinate candidates) and one complex weighted scalar, and apply the commutators to all five of these scalars. Already we have applied the GIF commutators to the table for the complex weighted scalar \( PQ \); the zero-weighted scalars \( A, C, M, B \) suggest themselves as the four coordinate candidates, and hence we will need to ensure that the commutators are applied to all four, to extract all the information.

Then, providing that these scalars are functionally independent, they can be adopted as coordinates. It will be easier to check for this functional independence after we have simplified the structure of the tables and after we have also completed the calculation by applying the commutators to all four candidates.

For subsequent calculations we will require the GHP commutators, which in turn require the missing four GHP spin coefficients. These four GHP spin coefficients follow immediately from (30) and the table for \( I \) (49), and are given by

\[ \begin{align*}
\tau' &= -B^B C^C D^D' \psi_{CCD'} (\iota_B) = \frac{3\Lambda PA^2}{\left( \frac{3}{2} - k \right)} \\
\rho' &= -B^B C^C D^D' \vartheta_{CCD'} (\iota_B) = \frac{\Lambda Qck(1 - 4\Lambda A^2)}{4 \left( \frac{3}{2} - k \right)} \\
\sigma' &= -B^B C^C D^D' \vartheta_{CDD'} (\iota_B) = -\frac{3\Lambda Qck}{8P^2 \left( \frac{3}{2} - k \right)} \\
\kappa' &= -B^B C^C D^D' \varphi_{CCD'} (\iota_B) = \frac{\overline{P}Q^2}{A} (A + M - iB)
\end{align*} \tag{62} \]

and of course \( \tau = P/A \); these should now be substituted into the GHP commutators (29).
5.4 Simplifying and completing the tables in the GHP operators

We have already obtained a GHP table \((54)\) for the real zero-weighted scalar \(A\), and in addition applied the commutators, which is required for a coordinate candidate. From this, we have also obtained a GHP table \((55)\) for the real zero-weighted scalar \(C\), and when we apply the GHP commutators we obtain the partial table for the real zero-weighted scalar \(L\),

\[
\partial L = \frac{-18\Lambda^2 C A^3 k}{Q (\frac{3}{2} - k)^2} - \frac{\Lambda LA P(4\Lambda A^2 - 1)}{(\frac{3}{2} - k)} + \frac{4P}{(\frac{3}{2} - k)}(A + M + iB)
\]

\[
\partial' L = \frac{-\Lambda C^2 P k^2(11\Lambda A^2 - 2)}{(\frac{3}{2} - k)^2} - \frac{\Lambda LA P(4\Lambda A^2 - 1)}{(\frac{3}{2} - k)}
\]

\[
+ \frac{4P}{(\frac{3}{2} - k)}(A + M - iB)
\]

(63)

So we can adopt \(C\) as a second coordinate candidate and add the partial table for \(L\) to our equations. We would next like to complete the partial tables for two of \(M, B, L\), and then apply the commutators to each to exploit them as two more coordinate candidates. However, it is clear from the complicated partial tables above that these calculations would be long, and it will be easier if we first do a little rearranging and relabelling. The simpler the form which we can obtain for our tables for the four coordinate candidates, the simpler the form will be for the associated metric.

A direct substitution of \(M\) by \(T\) via

\[
T = -\frac{1}{2k^\mp}M - \frac{\Lambda k^\mp(9k + 4)}{8\sqrt{2}}C^2 - \frac{\Lambda A k^\mp}{2\sqrt{2}}L
\]

(64)

enables the complicated partial table for \(M\) to be replaced with a very simple table for \(T\),

\[
\forall T = 0
\]

\[
\partial T = 0
\]

\[
\partial' T = 0
\]

\[
VT = \frac{Q k^\mp}{A}F
\]

(65)

which we have completed in the usual way for the, as yet undetermined, zero-weighted scalar function \(F\). (Once again, the particular choice of factors multiplying the unknown function \(F\) is simply to ensure that \(F\) is a zero-weighted
scalar, and to shorten the presentation of the details of the subsequent calculations.) So we decide to replace $M$ with $T$ as a third coordinate candidate. It now remains to get a simpler replacement for the rather complicated tables (59) and (63), for $B$ and $L$ respectively.

Making a direct substitution of $L$ with $S$ in (59) via

$$S = (2k + \Lambda^2 AC^2 k^2 + \Lambda Lk + 2\Lambda AM)/\Lambda k^{1/2}$$  \hspace{1cm} (66)$$
gives the simpler form

$$PB = 0$$
$$\partial B = -iP\Lambda k^{1/2} S$$
$$\partial' B = iP\Lambda k^{1/2} S$$  \hspace{1cm} (67)$$
as well as replacing the complicated partial table for $L$ with the simpler partial table for $S$

$$PS = 0$$
$$\partial S = 4iP\Lambda \frac{3}{2} B$$
$$\partial' S = -4iP\Lambda \frac{3}{2} B$$  \hspace{1cm} (68)$$

The term in $L$ in the table (55) for $C$ will now be replaced, using (66) and (64), with

$$L = \left( S + \sqrt{2\Lambda} \right) - \left( \frac{2k^{1/2}}{\Lambda} + \frac{9\Lambda^2 A^3 C^2 k^{3/2}}{4} \right) /k^{1/2} (\frac{3}{2} - k)$$  \hspace{1cm} (69)$$

These two partial tables (67) and (68) are much simpler in appearance than (59) and (63), and our next step would seem to be to choose $B$ or $S$ as the fourth coordinate candidate and complete its table in the usual manner, and then apply the commutators to it; the other scalar function would then be defined by its partial table. Unfortunately, because of the coupled nature of $B$ or $S$ in the two tables (67) and (68) the subsequent application of the commutators to such an arrangement gets very complicated; therefore it is more convenient to make one more rearrangement.

So we make a substitution of $B$ with $V$ by

$$V = 2B/S$$  \hspace{1cm} (70)$$

Clearly this substitution is not valid when $S = 0$; so we shall assume $S \neq 0$ in the remainder of this section, and we shall later have to look at the special case $S = 0$ separately.

This means that now a comparatively simple table for $V$ replaces the partial table (67) for $B$,

$$PV = 0$$
$$\partial V = -2iP\Lambda \frac{3}{2}(V^2 + \Lambda)$$
$$\partial' V = 2iP\Lambda \frac{3}{2}(V^2 + \Lambda)$$
$$PV = \frac{Q\Lambda \frac{3}{2}(V^2 + \Lambda)H}{2\Lambda}$$  \hspace{1cm} (71)$$
which we have completed in the usual way with the, as yet undetermined, zero-weighted scalar function $H$.

In order to obtain a still simpler form for this table, and a corresponding simpler form for the metric, we can now divide across the whole table by $(V^2 + \Lambda)$ and by integration define an alternative coordinate candidate to $V$ with a simpler table.

However, it is important to note that in order to integrate with respect to $V$ we have made the assumption that $V \neq \text{constant}$; this assumption also ensures that $V^2 + \Lambda \neq 0$. Hence we will need to consider separately $V = \text{constant}$ as a special case.

So we define

$$X = \int \frac{dV}{V^2 + \Lambda} = \frac{1}{\sqrt{|\Lambda|}} \tan[h]^{-1}\left(\frac{V}{\sqrt{|\Lambda|}}\right)$$  \hspace{1cm} (72)

where we have introduced this compact notation

$$\tan[h]^{-1}\left(\frac{V}{\sqrt{|\Lambda|}}\right) = \begin{cases} -\tanh\left(\sqrt{\Lambda} x\right) & \text{for } \Lambda < 0 \\ \tan^{-1}\left(\frac{V}{\sqrt{\Lambda}}\right) & \text{for } \Lambda > 0 \end{cases}$$  \hspace{1cm} (73)

and we have now the table

\begin{align*}
\partial X &= 0 \\
\partial' X &= 2iP^\frac{1}{2} \\
\partial' X &= 2iP^\frac{1}{2}
\end{align*}

$$\Psi X = \frac{Qk^\frac{1}{2}}{2A}H$$  \hspace{1cm} (74)

Since this table turns out to be more manageable, we will adopt $X$ as the fourth coordinate candidate.

The partial table for $S$ is now modified to

\begin{align*}
\partial S &= 0 \\
\partial S &= 2iP^\frac{1}{2}S\sqrt{|\Lambda|} \tan[h](\sqrt{|\Lambda|} X) \\
\partial' S &= -2iP^\frac{1}{2}S\sqrt{|\Lambda|} \tan[h](\sqrt{|\Lambda|} X).
\end{align*}

where

$$\tan[h](\sqrt{|\Lambda|} x) = \begin{cases} -\tanh(\sqrt{-\Lambda} x) & \text{for } \Lambda < 0 \\ \tan(\sqrt{\Lambda} x) & \text{for } \Lambda > 0 \end{cases}$$  \hspace{1cm} (75)

Earlier, we postponed applying the GIF commutators to the two real scalars $B, M$, so we need to apply the GHP commutators equivalently to their replacements, the two real zero-weighted scalars $T, X$. Applying the GHP commutators (29) to (65) and (74) gives the simple partial tables for $F$ and $H$ respectively,

\begin{align*}
\partial F &= 0 \\
\partial F &= 0 \\
\partial' F &= 0
\end{align*}

$$\partial' F = 0$$  \hspace{1cm} (77)
\begin{align*}
\forall H &= 0 \\
\partial H &= 0 \\
\partial^\prime H &= 0 \quad (78)
\end{align*}

The rather extensive relabelling and rearranging — to obtain the two tables (65) and (74) and the constraints (75), (77) and (78) — which we have just carried out was in order to obtain such simple and manageable forms. Clearly the gradient vector \( \nabla F \) is parallel to \( \nabla T \); this means that the scalar function \( F \) is an arbitrary function of only the one coordinate candidate \( T \) (and independent of the other coordinate candidates \( X, C, A \)). Similarly, from (78) the function \( H \) is also an arbitrary function of only the one coordinate candidate \( T \). The function \( S \) in (75) has a more complicated structure; we shall find it as the solution of a partial differential equation when we translate into explicit coordinates.

Since we have applied the commutators to \( I \) and to \( P \) and \( Q \), as well as to \( A, C, T, X \), we have obtained, in an explicit form, all the information about this class of spaces. So we have completed the formal integration procedure for these spaces; all the information has been extracted in the generic case, by which we mean the case where we have assumed that the four zero-weighted real scalar functions, \( A, C, T, X \) are functionally independent; these are our coordinate candidates which we intend to adopt as coordinates.

In summary, we note that we have complete tables (54), (55), (65), (74), for the four zero-weighted real scalar functions, \( A, C, T, X \) respectively; \( L \) in (55) is replaced by \( S \) from (69). Clearly our tables for the zero-weighted scalars \( A, C, T, X \) and for the weighted scalars \( P \) and \( Q \) are not complete and involutive by themselves, since they contain also the zero-weighted scalar functions \( S, H, F \). However, by applying the commutators to these four scalars we have obtained the constraint equations in the form of the partial tables (75), (77), (78) for these additional scalar functions, which, taken together with the tables (54), (55), (65), (74), (69), (81) supply a complete and involutive system.

In the remainder of this section we will obtain the coordinate version of the tetrad vectors, and hence the metric.

As we emphasised in the last subsection, before we can adopt the coordinate candidates as coordinates, we must confirm that they are functionally independent. First of all we check on the possibility of these four scalars being constant: since we are assuming in this section that \( k \neq 0 \), then none of \( A, C, X \) can be constant, but \( T \) may be. From the tables it follows that \( T \) is constant iff \( F = 0 \). Hence, in this section, the additional assumption that \( F \neq 0 \) is sufficient to ensure that none of the coordinate candidates are constant. Moreover, when we assume that none of the coordinate candidates are constant, a check of the determinant formed from their four tables (65), (55), (61), (74), confirms that the four coordinate candidates are indeed functionally independent — providing \( F \neq 0 \). Hence we will complete this section for the generic case with the additional assumption \( F \neq 0 \) ensuring that the coordinate candidates \( A, C, T, X \) can be adopted as explicit coordinates.
In addition, we must not forget that in order that $X$ could be a coordinate candidate, we made the additional assumptions that $V \neq \text{constant}$, and $S \neq 0$. In Section 6 we will look separately at the special case $V = \text{constant}$, and in Section 7 we will investigate the special cases with $F = 0 = S$.

5.5 Using coordinate candidates as coordinates

If we now make the obvious choice of the coordinate candidates as coordinates

$$t = T, \quad c = C, \quad a = A, \quad x = X$$ \hspace{1cm} (79)

the above four tables for the zero-weighted scalars enable us to immediately write down the tetrad vectors in the coordinates $t, c, a, x$,

$$l^i = \frac{1}{Q} \left( 0, -\frac{4}{\left(\frac{3}{2} - k\right)}, 0, 0 \right)$$

$$m^i = P \left( 0, -\frac{\Lambda(5\Lambda a^2 - 2)ac}{\left(\frac{3}{2} - k\right)}, -2k, -2i\frac{\pi}{2} \right)$$

$$n^i = \frac{Q}{a} \left( Fk^\frac{\pi}{2}, L, k^2c, \frac{Hk^\frac{\pi}{2}}{2} \right)$$ \hspace{1cm} (80)

where the function $L$ is given in terms of $S$ by (69), the functions $S, H, F$ are respectively solutions of the partial tables (75), (78), (77), and now $k = \Lambda a^2 + 1/2$ from (28).

As noted in the last section, $F$ and $H$ respectively will be arbitrary functions of only the one coordinate $t$, so we will write $-4F = \alpha_2(t)$ and $-2H = \alpha_3(t)$ — subject to the restrictions made in the calculations in this section that $F \neq 0$ which implies that $\alpha_2(t) \neq 0$ (note there is no restriction on $\alpha_3(t)$, which is a completely arbitrary function of $t$, including the zero function).

The partial table (75) for $S$ now becomes, via the tetrad, a system of partial differential equations in the chosen coordinates,

$$\frac{\partial S}{\partial c} = 0$$

$$2k \frac{\partial S}{\partial a} + 2i\frac{\pi}{2} \frac{\partial S}{\partial x} = -2i\frac{\pi}{2} S\sqrt{|\Lambda|} \tan[\sqrt{|\Lambda|} x]$$ \hspace{1cm} (81)

which shows that $S$ is independent of the coordinates $c$ and $a$, and we easily find the solution using (72)

$$S(t, x) = \alpha_1(t) \cos[\sqrt{|\Lambda|} x]$$ \hspace{1cm} (82)

where $\cos[\sqrt{|\Lambda|} x]$ is given by

$$\cos[\sqrt{|\Lambda|} x] = \begin{cases} \cosh(\sqrt{-\Lambda} x) & \text{for } \Lambda < 0 \\ \cos(\sqrt{\Lambda} x) & \text{for } \Lambda > 0 \end{cases}$$ \hspace{1cm} (83)
and $\alpha_1(t)$ is an arbitrary function of $t$, excluding the zero function, since we are assuming $S \neq 0$ in this section.

It follows immediately from the equation

$$g^{ij} = 2^{l(i,n)} - 2^{m(i,m)}$$

that the metric $g^{ij}$, in the coordinates $t, c, a, x$, is given by

$$g^{ij} = \begin{pmatrix}
0 & \frac{k^{1/4} \alpha_2(t)}{a(\frac{3}{2} - k)} & 0 & 0 \\
\frac{k^{1/4} \alpha_2(t)}{a(\frac{3}{2} - k)} & \frac{8}{a(\frac{3}{2} - k)} & \frac{k^{1/4} \alpha_3(t)}{a(\frac{3}{2} - k)} & \frac{2k(5\Lambda a^2 + 1)c}{a(\frac{3}{2} - k)} \\
0 & \frac{2k(5\Lambda a^2 + 1)c}{a(\frac{3}{2} - k)} & -4k^2 & 0 \\
0 & \frac{k^{1/4} \alpha_3(t)}{a(\frac{3}{2} - k)} & 0 & -4k
\end{pmatrix}$$

(85)

where $k = \Lambda a^2 + 1/2$, and $Z$ is given in terms of $S$ from (69) by,

$$Z = \frac{3}{2} - k k^{1/2} \left( L + \frac{\Lambda^2 (5\Lambda a^2 - 2)^2 a^3 c^2}{8(\frac{3}{2} - k)} \right)$$

$$= \frac{3}{2} - k k^{1/2} \left( L + \frac{9\Lambda^2 a^3 c^2 k^{3/2}}{4} + \frac{k^{1/2} \Lambda^2 (5\Lambda a^2 - 2)^2 a^3 c^2}{8} \right)$$

$$= \alpha_1(t) \cos[h](\sqrt{|\Lambda|} x) + 2\sqrt{2} at - \frac{2k^{1/2}}{\Lambda}$$

$$+ \frac{\Lambda^2 a^3 c^2 (25\Lambda^2 a^4 - 2\Lambda a^2 + 13)}{8}$$

(86)

where $\cos[h](\sqrt{|\Lambda|} x)$ is given by (83), and $\alpha_3(t)$ is completely arbitrary.

We must remember that, in order to justify taking $t$ as a coordinate, we have assumed that $\alpha_2(t) \neq 0$, and in order to justify taking $x$ as a coordinate, we have assumed that $\alpha_1(t) \neq 0$; furthermore we have assumed at certain stages in our calculations that $V \neq \text{constant}$. So this metric is not necessarily the most general form for this class of spacetimes.

In the following sections we will first look at the excluded cases separately, and then obtain a more general form of the metric which will include all such previously excluded cases.

6 The integration procedure for $\Lambda + \tau \neq 0$: special case $V = \text{constant}$, and combined case

6.1 The special case with $V = \text{constant}$

When we substitute the condition $V = \text{constant}$ into (71) we find that this case can only occur for a negative cosmological constant. So if we write

$$\lambda = \pm \sqrt{-\Lambda}$$
then we find $V = \lambda$. The calculations in Section 5 up to (68) are still valid. Since neither $X$, nor constant $V$, can be a coordinate as in the last section, we must find a replacement coordinate candidate which is functionally independent of the other three $A, C, T$. We shall continue to assume in this section that $F \neq 0 \neq S$. Substitution of $V = \lambda$ into (70) modifies (68) to give the table for $B$ for this special case,

$$
\begin{align*}
\mathfrak{p}B &= 0 \\
\partial B &= 2i\mathcal{P}k^{\frac{1}{2}}\lambda B \\
\partial' B &= -2i\mathcal{P}k^{\frac{1}{2}}\lambda B \\
\mathfrak{p}' B &= -\frac{Q}{2A}k^{\frac{1}{2}}\lambda BG
\end{align*}
$$

(87)

The real zero-weighted scalar $G$ — as yet undetermined — has been chosen to complete the table in the usual manner.

This comparatively simple table suggests $B$ as the replacement coordinate candidate; this of course will require that $B \neq$ constant, but from (87) we then see that the only possible constant value is $B = 0$. However, from (67) it follows that $S = 0$, and this special class has been excluded from this section.

But an even simpler table is obtained by the substitution

$$
e^{-\lambda Y} = |B|
$$

(88)

giving

$$
\begin{align*}
\mathfrak{p}Y &= 0 \\
\partial Y &= -2i\mathcal{P}k^{\frac{1}{2}} \\
\partial' Y &= 2i\mathcal{P}k^{\frac{1}{2}} \\
\mathfrak{p}' Y &= \frac{Q}{2A}k^{\frac{1}{2}}G
\end{align*}
$$

(89)

So preferring $Y$ as our fourth coordinate candidate, we apply the commutators to get

$$
\begin{align*}
\mathfrak{p}G &= 0 \\
\partial G &= 0 \\
\partial' G &= 0
\end{align*}
$$

(90)

The tables (54), (65), (55) respectively for the other three coordinate candidates $A, T, C$ and the partial table (77) for the function $F$, are unchanged. $L$ is replaced in (55) by $S$ from (69), which in return is replaced by $Y$ from

$S = 2B/\lambda = \frac{2}{\lambda}e^{-\lambda Y}$

from (88) (remembering there is a $\pm$ included in our definition of $\lambda$).

We have already noted that $A$ and $C$ cannot be constants, and although $T$ may be, we are excluding that possibility in this section (since $F \neq 0$); furthermore,
it is clear that \( Y \) cannot be constant (remembering \( k \neq 0 \neq \lambda \)). Moreover, an examination of the determinant of the four tables (54), (55), (65) and (89) shows that the four scalars \( A, C, T \) and \( Y \) are functionally independent and therefore can be chosen as coordinates. So we now make the obvious choice of the coordinate candidates as coordinates, \( t = T, \quad c = C, \quad a = A, \quad y = Y \).

We can write down the tetrad vectors in these coordinates by means of the tables (54), (55), (65) and (89),

\[
t^i = \frac{1}{Q}(0, -\frac{4}{(\frac{3}{2} - k)}, 0, 0) \\
m^i = P\left(0, -\frac{\Lambda(5\Lambda a^2 - 2)ac}{(\frac{3}{2} - k)}, -2k, -2ik^\frac{1}{2}\right) \\
\bar{m}^i = \bar{P}(0, -\frac{\Lambda(5\Lambda a^2 - 2)ac}{(\frac{3}{2} - k)}, -2k, 2ik^\frac{1}{2}) \\
n^i = Q\left(F k^\frac{1}{2}, L, k^\frac{1}{2} c, \frac{k^\frac{1}{4} G}{2}\right)
\]

Since the function \( G \) is a solution of the partial table (90) we can write \(-2G = \beta_3(t)\), and similarly \(-4F = \beta_2(t)\); both are arbitrary functions of \( t \), but the latter has the constraint that \( \beta_2(t) \neq 0 \). The metric in \( t, c, a, y \) coordinates is therefore given by

\[
g^{ij} = \begin{pmatrix}
0 & \frac{k^{1/4} \beta_3(t)}{a(\frac{3}{2} - k)} & \frac{k^{1/4} \beta_3(t)}{a(\frac{3}{2} - k)} & Z \\
-\frac{8k(5\Lambda a^2 - 2)ac}{a^{1/2}(\frac{3}{2} - k)} & 0 & -4k^2 & 0 \\
0 & -\frac{2(5\Lambda a^2 - 2)ac}{a^{1/2}(\frac{3}{2} - k)} & 0 & -4k \\
0 & \frac{k^{1/4} \beta_3(t)}{a(\frac{3}{2} - k)} & \frac{k^{1/4} \beta_3(t)}{a(\frac{3}{2} - k)} & 0
\end{pmatrix}
\]

where

\[
Z = (\frac{3}{2} - k)k^{1/2}\left(L + \frac{\Lambda^2(5\Lambda a^2 - 2)^2a^2c^2}{8(\frac{3}{2} - k)}\right)
\]

\[
= S + 2\sqrt{2}at - \frac{2k^{1/2}}{\Lambda} + \frac{9\Lambda^2 a^3 c^2 k^{3/2}}{4} + \frac{k^{1/2} \Lambda^2(5\Lambda a^2 - 2)^2a^3c^2}{8}
\]

\[
= \frac{2}{\Lambda} e^{-\lambda y} + 2\sqrt{2}at - \frac{2k^{1/2}}{\Lambda} + \frac{\Lambda^2 k^{1/2} a^3 c^2 (25\Lambda^2 a^4 - 2\Lambda a^2 + 13)}{8}
\]

and \( k = \lambda a^2 + 1/2 \) from (28).

We emphasise that this case only exists for negative \( \Lambda = -\lambda^2 \).
6.2 Generic case combined with special case, $V = \text{ constant}$

It will be useful to place this special case (with the cosmetic change $y \to x$, and $\beta_2(t) \to \alpha_2(t), \beta_3(t) \to \alpha_3(t)$) alongside the generic metric obtained in the previous section; so we combine the result in the previous subsection with the generic result in Section 5 to present the metric in the coordinates $t, \epsilon, a, x$, given by

$$g^{ij} = \begin{pmatrix}
0 & \frac{k^{1/4} \alpha_2(t)}{a(f-k)} & 0 & 0 \\
\frac{k^{1/4} \alpha_2(t)}{a(f-k)} & \frac{a}{k} \frac{\sqrt{\Lambda} |x|}{a(f-k)} Z & -2k(5\Lambda^2 a^4 + 1)c & \frac{k^{1/4} \alpha_3(t)}{a(f-k)} \\
0 & -2k(5\Lambda^2 a^4 + 1)c & -4k^2 & 0 \\
0 & \frac{k^{1/4} \alpha_3(t)}{a(f-k)} & 0 & -4k
\end{pmatrix}$$

where $\alpha_3(t)$ is an arbitrary function of $t$ including the zero function, whereas $\alpha_2(t)$ is an arbitrary function of $t$ excluding the zero function, and $k = \Lambda a^2 + 1/2$.

There are two possibilities for $Z$:

(i) $Z = \alpha_1(t)\cos[h](\sqrt{\Lambda}|x|) + 2\sqrt{2}at - \frac{2k^{1/2}}{\Lambda} + \frac{\Lambda^2 k^{1/2} a^3 c^2 (25\Lambda^2 a^4 - 2\Lambda a^2 + 13)}{8}$

from (86) where $\alpha_1(t) \neq 0$ is an arbitrary function of $t$ excluding the zero function, and $\cos[h](\sqrt{\Lambda}|x|)$ is given by (93).

(ii) $Z = \frac{2}{\lambda} e^{-\lambda x} + 2\sqrt{2}at - \frac{2k^{1/2}}{\Lambda} + \frac{\Lambda^2 k^{1/2} a^3 c^2 (25\Lambda^2 a^4 - 2\Lambda a^2 + 13)}{8}$

from (93).

Note that case (i) exists for positive and negative cosmological constant, but case (ii) only exists for negative $\Lambda$, with $\lambda = \pm \sqrt{-\Lambda}$.

7 The most general form for the metric when $\Lambda + \tau \Phi \neq 0$.

7.1 Preliminaries to generalisations

We have not yet got the most general version of the metric because in Section 5 we assumed that $T$ was not a constant in order to be able to choose it as a coordinate candidate, and we also assumed that $S \neq 0$ in order to be able to choose $X$ as a coordinate candidate.

We begin with the excluded case where $T$ is a constant. In such a situation, clearly $F = 0$ so we cannot instead use $F$ as a coordinate candidate, but we
still have the possibility of choosing \( H \) or \( S \) as a coordinate candidate. Once we make such a choice then we could continue in a similar manner as in the last section, building our tables, and hence the tetrad, around our four coordinate candidates. However, if \textbf{neither} of the other functions \( H, S \) is functionally independent of the original three coordinates, then it will \textbf{not} be possible to find a replacement candidate \textbf{directly}; we emphasise that in such circumstances no additional independent quantities can be generated by any \textit{direct} manipulations of the tables and the commutators. In such a situation we still need a replacement candidate in order to extract the remaining information from the commutators. So rather than treating the special case \( F = 0 \) separately, we will extend the generic result to include this special case as well.

We shall now show, instead, that a \textit{complementary coordinate candidate} to replace \( T \) can easily be found, and then, using this coordinate, we will obtain a generalisation of the metric (94) which includes all possible values for \( T \), including a constant.

Secondly we consider the excluded case \( S = 0 \), and for this case we find that not only can we not construct \( X \) (or \( V \)) as a coordinate candidate, but that we cannot generate \textit{directly} any replacement coordinate candidate. We shall now show, instead, that a \textit{complementary coordinate candidate} to replace \( X \) can easily be found, and then, using this coordinate, we will first obtain this excluded case \( S = 0 \) separately; we will then obtain a generalisation of the metric (94) which includes this additional special case, \( S = 0 \).

### 7.2 Finding a complementary coordinate candidate to replace \( T \)

The results in Section 5 up to the end of subsection 5.4 apply as before; the only difference here is that we \textit{interpret} them differently. When we are interpreting our tables and choosing our explicit coordinate candidates we will now consider only the three zero-weighted real scalars \( A, C, X \) as coordinate candidates while the zero-weighted scalar \( T \) is not now included as a coordinate candidate, and so there is now no hindrance to it acquiring a constant value, even zero. A related change is that since \( T \) is no longer a coordinate candidate, we no longer need its \textit{complete} table, nor the resulting partial table for \( F \); however we still need the \textit{partial} table for \( T \) since it is a result of applying the commutators to \( I \), and so is still a crucial component of the analysis.

\[
\begin{align*}
\partial T &= 0 \\
\partial^\prime T &= 0 \\
\partial'' T &= 0
\end{align*}
\]  

(97)

So, clearly we do not have our full quota of \textit{four} coordinate candidates, but we do not wish to use any of the remaining intrinsic quantities from the tables, since it would involve the additional assumption of that quantity being non-constant. It is now very important to note that all the \textit{direct} information which can be obtained from the intrinsic elements of the GHP formalism is in these tables,
and no amount of further manipulation of the equations with the commutators will generate a replacement coordinate candidate which is functionally independent of the other three $A, C, X$. On the other hand, we have not yet extracted all the information from the commutators since we have only applied them to three zero-weighted scalars. So we require a fourth zero-weighted scalar — functionally independent of the other three $A, C, X$ — which will be the fourth coordinate candidate, and also enable us to extract any remaining information implicit in the commutators. Since there is no such intrinsic zero-weighted scalar which we can generate directly in the GHP formalism, we introduce it indirectly via its table, which will have to be consistent with all the explicit equations in the GHP formalism, and in particular with the GHP commutators.

In fact, we get a strong hint from Section 5.4 by looking at the table (65) for the coordinate $T$ (which is the missing coordinate candidate in this case); so we consider the possibility of the existence of a real zero-weighted scalar $\tilde{T}$, which satisfies the table:

\[ \begin{align*} 
\nabla \tilde{T} &= 0 \\
\frac{\partial}{\partial \tilde{T}} &= 0 \\
\frac{\partial'}{\partial \tilde{T}} &= 0 \\
\nabla' \tilde{T} &= -4Q^A_k^4 
\end{align*} \]

(A direct copy of (65) would suggest the table:

\[ \begin{align*} 
\nabla \tilde{T} &= 0 \\
\frac{\partial}{\partial \tilde{T}} &= 0 \\
\frac{\partial'}{\partial \tilde{T}} &= 0 \\
\nabla' \tilde{T} &= Q^F(\tilde{T})_k^4 
\end{align*} \]

where $F(\tilde{T})$ is an arbitrary function of $\tilde{T}$, excluding the zero function. However it is easy to see that a simple coordinate transformation $\tilde{T} \to -4 \int F(\tilde{T}) d\tilde{T}$ gives the simpler version (98).

So we have chosen a zero-weighted real scalar $\tilde{T}$ defined by its table (98), whose structure we have ‘copied’ from the table structure (65) of $T$.

It is important to appreciate the different natures of $T$ and $\tilde{T}$. In Section 5 $T$ was defined directly in terms of intrinsic elements of the formalism, and so was itself an intrinsic coordinate candidate, and the table (65) was a consequence of its definition; on the other hand, the complementary coordinate candidate $\tilde{T}$ is not defined in terms of intrinsic quantities of the formalism, but rather as the integral of the table (98). Hence, the introduction of the coordinate candidate

\[\text{For easy reference, in an extended case, we will label by } \tilde{T} \text{ a complementary coordinate candidate which replaces an intrinsic coordinate candidate } T \text{ in the corresponding generic case; but we emphasise this is not to imply any direct link between the two quantities, it simply points us to the source of the hint which suggested the table for the complementary coordinate candidate.}\]
\( \tilde{T} \), via the table (98), is structurally different from the usual direct identification of coordinates with elements of the formalism: \( C, A, X \) are intrinsic coordinate candidates, while \( T \) is a complementary coordinate candidate.

It is straightforward to confirm that this choice of table (98) is consistent with the GHP commutators (29) and creates no inconsistency with the other tables. So, compared to Section 5, we have simply replaced the fourth intrinsic coordinate candidate \( T \) with the complementary coordinate candidate \( \tilde{T} \) defined via its table (98) whose structure was ‘copied’ from the table (65) for \( T \); in addition we remember that the real zero-weighted quantity \( T \) now satisfies (97). Clearly \( T \) now is a function of only the one coordinate candidate \( \tilde{T} \), i.e., \( T(\tilde{T}) \). The remaining tables are unchanged.

### 7.3 Finding a complementary coordinate candidate to replace \( X \)

The results in Section 5 up to the end of subsection 5.3 apply as before; and we shall also assume the results up to equation (69).

When we make the substitution \( S = 0 \) into (68) we find that the table collapses giving \( B = 0 \). This means that the table for \( B \), (59) also collapses. No action with the commutators is able to generate any new information directly from the existing GHP equations. At this stage we are left with only the GHP tables for the three coordinate candidates \( A, C, T \) and the GHP tables for the weighted scalars, \( P, Q \). However we need a table for a fourth coordinate candidate in order to be able to extract all the information from the GHP commutators. So we require a fourth zero-weighted scalar — functionally independent of the other three \( A, C, T \) — which will be the fourth coordinate candidate, and also enable us to extract any remaining information implicit in the commutators. So, in a similar manner to the last subsection, we introduce a complementary coordinate candidate indirectly via its table, which will have to be consistent with all the explicit equations in the GHP formalism, and in particular with the GHP commutators.

Also, as in last section, we get a strong hint from Section 5.4 by looking at the table (74) for the coordinate \( X \) (which is the missing coordinate candidate in this case); so we consider the possibility of the existence of a real zero-weighted scalar \( \tilde{X} \), which satisfies the table

\[
\begin{align*}
P\tilde{X} &= 0 \\
\partial\tilde{X} &= -2iP\kappa^{\frac{1}{2}} \\
\partial'\tilde{X} &= 2i\kappa^{\frac{1}{2}} \\
\mathcal{V}'\tilde{X} &= \frac{Q\kappa^{\frac{1}{2}}}{2A}H
\end{align*}
\]

(100)

where we also assume \( H(t) \).

Again we have adopted the convention of labelling by \( \tilde{X} \) a complementary coordinate candidate which replaces an intrinsic coordinate candidate \( X \) in the corresponding generic case.
It is straightforward to confirm that this choice of table \( \text{(100)} \) is consistent with the GHP commutators \( \text{(29)} \) and creates no inconsistency with the other tables. Furthermore, we note since \( \tilde{X} \) is a complementary coordinate candidate which does not occur except in its own table, that we could have made an even simpler choice of table, by choosing \( H = 0 \) (which can easily be confirmed by a coordinate transformation \( \tilde{X} \rightarrow \tilde{X} + \int \frac{H(t)}{4} dt \)). However, we shall not make that simplification, for presentation reasons.

We can therefore present this special case in the coordinates \( t, c, a, \tilde{x} \), as

\[
g^{ij} = \begin{pmatrix}
0 & \frac{k^{1/4} \alpha_2(t)}{a(\frac{3}{2} - k)} & Z & 0 \\
\frac{k^{1/4} \alpha_2(t)}{a(\frac{3}{2} - k)} & 0 & -2k(5\Lambda^2 a^4 + 1)c & 0 \\
Z & -2k(5\Lambda^2 a^4 + 1)c & 0 & 0 \\
0 & 0 & 0 & -4k^2
\end{pmatrix}
\]  

(101)

where \( k = \Lambda a^2 + 1/2 \), and \( Z \) is given by,

\[
Z = 2\sqrt{2}at - \frac{2k^{1/2}}{\Lambda} + \frac{\Lambda^2 k^{1/2} a^3 c^2 (25\Lambda^2 a^4 - 2\Lambda a^2 + 13)}{8}
\]  

(102)

and \( \alpha_3(t) \) is completely arbitrary, while \( \alpha_2(t) \) is arbitrary, except for the zero function.

It is clear that this special case simply fills the gap in our original case (85), (86) by now including the case \( \alpha_1(t) = 0 \) which was excluded there.

### 7.4 The most general metric

The metric (94) gives the most general form of the metric for this class of spaces — under the additional restrictions that no Killing vectors are present. This follows from the existence of four intrinsic coordinates; this is also confirmed in [12] where we consider the detailed invariant Karlhede classification of this class of metrics. In subsection 7.2 we saw how to generalise (94) to include the possibility of the coordinate \( \tilde{t} \) being a complementary coordinate, so that this more general class also permits the existence of a Killing vector. The special case (101) just deduced in subsection 7.3 can also easily be generalised in the same manner by replacing \( t \) with a complementary coordinate \( \tilde{t} \); this special case could then be listed alongside the generalisation of (94). However it is more convenient to simply incorporate (101) into the generalisation of (94) discussed in subsection 7.2, by just removing the restriction \( \alpha_1(t) \neq 0 \). It is easy to confirm that the tables for the respective complementary candidates \( \tilde{T} \) and \( \tilde{X} \) are consistent with all the other tables, and with each other.

Hence we generalise the combined metric form (94) given in the last section by replacing the intrinsic coordinate candidate \( T \) and its table with the complementary coordinate \( \tilde{T} \) and its table, and the intrinsic coordinate candidate \( X \)
γ

... in this very special case γ

coordinate — except in this very special case general form, although it is obvious that this coordinate is in fact an intrinsic (i)

Z

transformation t = T, c = C, a = A, x = X,
given by

g^{ij} = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}(\frac{3}{2} - k)} a^2 \gamma_1(\tilde{t}) & 0 \\
\frac{1}{\sqrt{2}(\frac{3}{2} - k)} a^2 \gamma_1(\tilde{t}) & \frac{1}{k} a^2 \gamma_2(\tilde{t}) & 0 \\
0 & 0 & -4k^2
\end{pmatrix} (103)

where γ3(\tilde{t}) is an arbitrary function of \tilde{t} including the zero function, and k = Λa^2 + 1/2. There are two possibilities for Z,

(i) Z = γ1(\tilde{t})\cos[h(\sqrt{|Λ|\tilde{x}})] + 2\sqrt{2}a\gamma_2(\tilde{t}) - \frac{2k^{1/2}}{Λ} + \frac{Λ^{2k^{1/2}a^3c^2(25Λ^2a^4 - 2Λa^2 + 13)}}{8} (104)

where γ1(\tilde{t}) and γ2(\tilde{t}) are arbitrary functions of \tilde{t} including the zero function.

(ii) Z = \frac{2}{Λ}e^{-\frac{λ\tilde{x}}{2}} + 2\sqrt{2}a\gamma_2(\tilde{t}) - \frac{2k^{1/2}}{Λ} + \frac{Λ^{2k^{1/2}a^3c^2(25Λ^2a^4 - 2Λa^2 + 13)}}{8} (105)

where γ2(\tilde{t}) is an arbitrary function of \tilde{t} including the zero function. The changes α1(\tilde{t}) → γ1(\tilde{t}), α2(\tilde{t}) → γ2(\tilde{t}), α3(\tilde{t}) → γ3(\tilde{t}) are simply cosmetic.

Note that case (i) exists for positive and negative cosmological constant, but case (ii) only exists for negative Λ, with λ = ±\sqrt{−Λ}.

When we compare the metric (94) with the above metric (103) where Z is given by (104) or (105), we can easily demonstrate that the latter is a special case of the former, by making the coordinate transformation t = γ2(\tilde{t})/2\sqrt{2}t, and identifying γ1(\tilde{t}) = γ1(γ2^{-1}(2\sqrt{2}t)) = α1(t) and γ3(\tilde{t}) = γ3(γ2^{-1}(2\sqrt{2}t)) = α3(t), we confirm that the former case is included in the latter. However, the latter also permits γ2(\tilde{t}) to be constant, even zero; this is a possibility missing from the former.

It is trivial to confirm that the special subclass (101) is simply the special case of (i) given by γ1(\tilde{t}) = 0. We note that we have used the notation \tilde{x} in this general form, although it is obvious that this coordinate is in fact an intrinsic coordinate — except in this very special case γ1(\tilde{t}) = 0. Finally, we note again that in this very special case γ1(\tilde{t}) = 0 a simple coordinate transformation gives γ3(\tilde{t}) = 0, but leaves everything else unchanged.

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8 Summary and Discussion

The class of conformally flat pure radiation spacetimes with a non-zero cosmological constant which have been studied in [11] and in this paper has provided a very good laboratory for developing techniques and increasing our experience in the GIF formalism. We have shown how the method in [13] which was used to investigate conformally flat pure radiation spacetimes (with \( \tau \neq 0 \)) can be developed to investigate the more complicated situation where, in addition, there is a non-zero cosmological constant; in [11] we have found the subclass of conformally flat pure radiation spacetimes with negative cosmological constant \( \Lambda = -\tau \bar{\tau}, \tau \neq 0 \), while in this paper we have found the remaining subclass with \( \Lambda + \tau \bar{\tau} \neq 0 \neq \tau \).

An important new development in this paper is the realisation that we do not need to work the whole integration procedure in the GIF, but rather we can change to the GHP formalism once the GIF has generated the second spinor and we have extracted information by applying the GIF commutators to this spinor; since calculations in the GIF can be long and complicated, it is a considerable advantage to be able to transfer to the GHP formalism for the bulk of the calculations and only use the GIF in the initial calculations associated with the determination of the second spinor. In this paper, the simplification of the tables in subsection 5.4, which was crucial in order to obtain such a manageable form for the eventual metric, would not have been so transparent and would have been much more complicated in GIF.

This integration procedure within the GIF/GHP formalism is particularly suited to spaces with four intrinsic coordinates; spaces with less than four intrinsic coordinates may appear to pose more difficulties. Another important development in this paper is a fuller understanding of how ‘generic’ results help to suggest additional special cases; in the case where it is suspected that there exists additional special cases to the generic case, the structure of tables for complementary coordinates can be ‘copied’ from the corresponding intrinsic coordinates.

In addition, in [11] we learned how to treat the one dimensional isotropy freedom of a null rotation. These various calculations and results are enabling us gradually to build up our experience and skill in the GIF/GHP formalism, with the ultimate goal of tackling even more complicated situations in the future. The actual metrics which we have obtained have been confirmed with Maple.

It is clear from the most general form of the metric, and the fact that it is — as much as possible — presented in essential coordinates, that there will be subclasses with zero, one and two Killing vectors. There is in fact a rich symmetry structure in the whole class of conformally flat pure radiation spacetimes with non-zero cosmological constant, and the full details are presented in [12].

As well as increasing our experience and expertise in the GIF operator integration method, this particular class of spaces is interesting in its own right. The analogous spacetimes with zero cosmological constant investigated in [13] revealed some complications and subtleties in the computer classification programmes [37], [18]; it will be interesting to see how the computer programmes handle these new spacetimes, and especially the existence of one degree of null...
isotropy. It will also be interesting to explore the physical interpretation of the spacetimes in this paper and in [11], along the lines investigated in [10] for the spaces with zero cosmological constant; the wide variety of individual subclasses with a range from zero to five Killing vectors give a rich area of investigation. In some classes of spacetimes the addition of a cosmological constant makes little significant difference. On the otherhand, the addition of a non-zero cosmological constant has made a significant difference to vacuum Petrov type D spaces [3], [3], [4]. Moreover, its addition to the non-expanding Kundt class of spacetimes significantly complicates the equations: some classes of Petrov type N with non-zero cosmological constant have been found and analysed [31], [1], [2], [35], as well as some of Petrov type II [35]; recently a detailed and comprehensive derivation and analysis of Petrov type III non-expanding vacuum spacetimes with non-zero cosmological constant has been carried out in [16].

It may be suspected that these various examples of Type II, III and N spaces just mentioned will specialise in the conformally flat limit to the spaces under consideration in this paper. However, that is not necessarily so, since, in at least some of those investigations, properties of a non-zero Weyl tensor were built into the analysis. Furthermore, even if the conformally flat limit does exist in some of the investigations, the form of the metric may be much more complicated than in our version where we have built the structure around the conformally flat properties from the beginning. It has therefore been of interest to see how our method supplies ‘good’ coordinates, simple differential equations, and a very manageable form for the metrics. However, a number of terms have square roots, as well as trigonometric and hyperbolic functions, and absolute value functions have been used in the calculations; these will put restrictions on the range of the coordinates, and there will be alternative, and possibly more general, coordinate systems to consider. It remains to investigate the whole class of these spacetimes found via GIF, considering in more detail the coordinate systems, and comparing with the conformally flat limits of these various other investigations.

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