Long Run Feedback in the Broker Call Money Market

ALEX GARIVALTIS*
Northern Illinois University

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Abstract

I unravel the basic long run dynamics of the broker call money market, which is the pile of cash that funds margin loans to retail clients (read: continuous time Kelly gamblers). Call money is assumed to supply itself perfectly inelastically, and to continuously reinvest all principal and interest. I show that the relative size of the money market (that is, relative to the Kelly bankroll) is a martingale that nonetheless converges in probability to zero. The margin loan interest rate is a submartingale that converges in mean square to the choke price $r_\infty := \nu - \sigma^2/2$, where $\nu$ is the asymptotic compound growth rate of the stock market and $\sigma$ is its annual volatility. In this environment, the gambler no longer beats the market asymptotically a.s. by an exponential factor (as he would under perfectly elastic supply). Rather, he beats the market asymptotically with very high probability (think 98%) by a factor (say 1.87, or 87% more final wealth) whose mean cannot exceed what the leverage ratio was at the start of the model (say, 2 : 1). Although the ratio of the gambler’s wealth to that of an equivalent buy-and-hold investor is a submartingale (always expected to increase), his realized compound growth rate converges in mean square to $\nu$. This happens because the equilibrium leverage ratio converges to 1 : 1 in lockstep with the gradual rise of margin loan interest rates.

Keywords: Interest Rates; Margin Loans; Kelly Criterion; Risk Sharing

JEL Classification Codes: D14; D53; E41; E43; G11; G17; G24

Asymptotic Margin Rate $= \text{Stock Market CAGR} - 0.5 \times (\text{Volatility})^2$

(1)

*Assistant Professor of Economics, School of Public and Global Affairs, 514 Zulauf Hall, DeKalb IL 60115. E-mail: agarivaltis1@niu.edu. ORCID iD: 0000-0003-0944-8517.
“There are two sorts of wealth-getting, as I have said; one is a part of household management, the other is retail trade: the former necessary and honorable, while that which consists in exchange is justly censured; for it is unnatural, and a mode by which men gain from one another. The most hated sort, and with the greatest reason, is usury, which makes a gain out of money itself, and not from the natural object of it. For money was intended to be used in exchange, but not to increase at interest. And this term interest, which means the birth of money from money, is applied to the breeding of money because the offspring resembles the parent.”

—Aristotle, Politics

“According to Laplace, the state of the world at a given instant is defined by an infinite number of parameters, subject to an infinite number of differential equations. If some universal mind could write down all these equations and integrate them, it could then predict with complete exactness, according to Laplace, the entire evolution of the world in the infinite future.”

—Andrey N. Kolmogorov
1 Introduction

This paper studies a system of stochastic differential equations that purports to express the iron laws of dynamical equilibrium behavior in the broker call money market, a market that exists for the sake of funding stock brokers’ margin loans to retail clients.

We assume that the (aggregate) demand side of the market is comprised of continuous time Kelly (1956) gamblers. Kelly’s seminal (1956) article takes up the problem of repeated bets on independent horse races for which the gambler has a better-quality estimate of the win probabilities than does the bookie, whose beliefs are implicit in the posted odds. The common-sense insight is that the (stationary) nature of the problem dictates that the gambler should always bet a fixed fraction of his wealth on this (favorable) opportunity; for obvious reasons, the Kelly fraction is chosen so as to optimize the asymptotic continuously-compounded per-bet capital growth rate. Leo Breiman (1961) demonstrated the competitively superior properties of the Kelly Criterion: namely, that it asymptotically almost surely beats any “essentially different” strategy by an exponential factor; and it has the shortest mean waiting time for hitting a distant wealth goal.

In our problem, each horse race has been replaced by a differential tick $dt$ of the market clock, whereby the stock market index $S_t$ undergoes a fluctuation $dS_t$ that determines the gambler’s profit-and-loss. Unlike betting on a horse race (where you should not bet 100% of your wealth because you will eventually lose it all), it is highly advisable, given a low enough margin loan interest rate, to bet more than 100% of your wealth on every little movement of the stock market. To fix ideas, let us assume that the S&P 500 index multiplies itself at an expected (logarithmic) rate of $\nu := 9\%$ a year, with $\sigma := 15\%$ annual (log-) volatility. If we were to imagine, in passing, that
σ tends to zero, then it becomes clear by continuity considerations that we should be falling all over ourselves in order to borrow money at a margin rate of, say, 5%.

In the context of leveraged investment, Kelly’s fixed fraction betting scheme implies an ostensibly counter-intuitive trading mechanic. To illustrate, let us assume that we have $100, and that we resolve to act so as to maintain a constant $2 \times$ level of exposure to the S&P 500 index. This means that we must try to always maintain a margin loan (debit) balance equal to the level of our account equity; the loan-to-value ratio must always be 50%. Now, assume that we wake up tomorrow and the stock market has gapped up 10%; our new account equity is $120 = \frac{\text{Assets}}{\text{Liabilities}}$ = $220 - $100. Whatever were our good reasons to lever the initial $100 twice over, it seems natural that they should continue to apply to our newer, wealthier self. Note well that the market has effectively chosen for us the new leverage ratio of $1.83 : 1$, with a corresponding loan-to-value ratio of 45.5%. Thus, our fixed-fraction betting scheme dictates that we must borrow and invest an additional $20 (which is equal to the profits just earned). Although this behavior gives off the optics of a trend-following, performance-chasing, or market-timing scheme, the simple fact is that we are just going back to the well so as to carry on exploiting the opportunity to borrow at a low price. To be clear, the “trend” in question is the exponential growth of corporate earnings and dividends that is manifest in the high drift rate of the log-price of the market index.

The flip side of the coin is that the gambler must unwind this very process when the market goes down; for, suppose that on the next day, the index price gets divided by a factor of 1.1 (for a loss of 9.1%), e.g. it returns full circle back to its original level. Our portfolio assets have thereby dwindled in value to $240 \div 1.1 = $218.18 (against liabilities of $120), for a net equity of $98.18. Thus, the “sideways” motion of the asset price has caused us to underperform a buy-and-hold investor; we have
been “whipsawed” by the cold arithmetic of buying high and selling low. On that score, our leverage ratio has just ballooned to 2.22×, which makes for a loan-to-value ratio of 55%. In order to remedy this (overlevered) situation, our scheme dictates that we must liquidate $21.82 of assets, e.g. the amount of money that we just lost in the fire.

This simple example makes it abundantly clear just what is the fundamental trade-off that is faced by the continuous time Kelly gambler. On the one hand, we expect to earn the spread between the margin loan interest rate and the compound growth rate of the market index; on the other, we must deduct the ongoing costs of the whipsaw effect, which become more pronounced with higher levels of volatility in the underlying. The sweet spot that perfectly balances these two considerations (check with David Luenberger 1998) amounts to the magic leverage ratio $b^* := 0.5 + (\nu - r_L) \div \sigma^2$, where $r_L$ is the margin loan interest rate.

If we assume that margin loans are supplied perfectly elastically (e.g. a horizontal supply curve), then the continuous time Kelly gambler has access to a permanent source of funding that allows him to beat the market asymptotically almost surely by an exponential factor (cf. with Garivaltis 2019b). This “bucket shop”\(^1\) environment, with its unlimited supply of Saps willing to provide cash to their betters for a song, has obvious practical defects from a meta-perspective. Namely, on a long enough time horizon, the Kelly gamblers must inevitably own every single dollar of stock market capitalization.

Thus, in order to get a realistic equilibrium outcome, we have decided in this paper to stand the supply curve on its head. The Saps, who are in possession of a giant pool of call money, are now assumed to supply it perfectly inelastically at the going rate (e.g. as determined by a vertical supply curve). Hapless though they are,\(^1\)

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\(^1\)A vivid expression that can be found in Merton (1992).
they nonetheless manage to multiply their capital at an exponential rate; this paper assumes that all principal and interest payments are continuously reinvested in the money market.

How is this natural and straightforward market structure going to shake itself out in the long run, given all the reverberatory effects of so many random vibrations in the asset price? These pages contain the answer.

2 The Model

We assume that the stock market index or ETF has \( N \) shares outstanding, and its price per share \( S_t \) evolves according to the geometric Brownian motion

\[
dS_t := S_t \times (\mu \, dt + \sigma \, dW_t),
\]

where \( S_0 \) is the (given) initial price at time 0. Here, \( \mu \) denotes the annual drift rate, \( \sigma \) is the annual volatility, and \( W_t \) is a standard Brownian motion. The log-price evolves according to (cf. Paul Wilmott 2001)

\[
d(\log S_t) = \nu \, dt + \sigma \, dW_t,
\]

where \( \nu := \mu - \sigma^2/2 \) is the almost-sure asymptotic continuously-compounded capital growth rate

\[
\nu = \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{S_t}{S_0} \right) = \mathbb{E}\left[ d(\log S_t) \right]/dt.
\]

We assume that, at every instant \( t \), there is a quantity \( q_t \) of loan money ("broker call money") that is supplied inelastically to the retail brokerage market. The money market charges a continuously-compounded interest rate of \( r_L(t) \) per year for the
duration of the differential time step \([t, t + dt]\), where \(r_L(t)\) will be determined below in equilibrium. Naturally, we assume that the money market continuously reinvests all proceeds (both principal and interest), and so the size of the money market evolves according to

\[
dq_t := q_t \times r_L(t) \times dt, \tag{5}
\]

or

\[
d(\log q_t) = r_L(t) \times dt, \tag{6}
\]

where \(q_0\) is exogenously given. Thus, we have the relation

\[
q_t = q_0 \times \exp \left\{ \int_0^t r_L(s) ds \right\}. \tag{7}
\]

The demand side of the broker call money market is supposed to be constituted by a (representative) continuous time Kelly (1956) gambler, that “bets” the fraction \(b_t \in [1, \infty)\) of his wealth on the stock market for the differential time step \([t, t + dt]\). In so doing, since \(b_t \geq 1\), he has borrowed the quantity \(q_t := (b_t - 1) \times V_t\) from the money market; the loan must be repaid (both principal and interest) “on call” at time \(t + dt\). Starting from a given initial value of \(V_0\), the gambler’s fortune \(V_t\) evolves according to the stochastic differential equation

\[
dV_t := \frac{bV_t}{S_t} \times \frac{dS_t}{\text{number of shares}} \quad \text{profit/loss per share} - \frac{q_t \times r_L(t) \times dt}{\text{interest paid (} = dq_t)} = V_t \times \left\{ (b_t \mu + (1 - b_t)r_L(t)) dt + b_t \sigma dW_t \right\}. \tag{8}
\]

Applying Itô’s Lemma (cf. with Thomas Mikosch 1998) to the transformed process
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\[ V_t \mapsto \log V_t, \text{ we see that the gambler’s log-fortune evolves according to} \]

\[ d(\log V_t) = \left\{ b_t \mu + (1 - b_t) r_L(t) - \frac{\sigma^2 b_t^2}{2} \right\} dt + b_t \sigma dW_t. \] (9)

The gambler’s expected continuously-compounded growth rate over \([t, t + dt]\) is equal to \((\text{cf. with David Luenberger 1998})\)

\[ \Gamma(b, r_L) := \text{Growth Rate}(b, r_L) = \frac{\mathbb{E}[d(\log V_t)]}{dt} = r_L + (\mu - r_L)b - \frac{\sigma^2 b_t^2}{2}. \] (10)

The Kelly bet (cf. with Edward O. Thorp\(^2\) 2006, 2017) for the next tick of the market clock \((dt)\) is, by definition, the fraction of wealth that maximizes the growth rate:

\[ b_t = b^*(r_L) := \arg \max_{b \geq 1} \Gamma(b, r_L) = \frac{\mu - r_L}{\sigma^2} = \frac{1}{2} + \frac{\nu - r_L}{\sigma^2}. \] (11)

Note well that the maximized (instantaneous expected continuously-compounded) growth rate is a stochastic process\(^3\) \((\Gamma_t)_{t \geq 0}\) that fluctuates according to the prevailing margin loan interest rate \(r_L(t)\); substituting the Kelly bet (11) into the objective function (10), we get the expressions

\[ \Gamma_t := \max_{b \geq 1} \Gamma(b, r_L(t)) = \Gamma(b_t, r_L(t)) \]

\[ = r_L(t) + \frac{1}{2} \left[ \frac{\mu - r_L(t)}{\sigma} \right]^2 = r_L(t) + \frac{\sigma^2 b_t^2}{2}. \] (12)

\(^2\)Who made a few such bets himself.

\(^3\)The author is aware that this is an abuse of notation, albeit a very natural one that should cause no confusion.
We will require the fact that the process \((\Gamma_t)_{t \geq 0}\) is bounded:

\[
\mu - \sigma^2/2 \leq \Gamma_t \leq \frac{\mu^2}{2\sigma^2}. \tag{13}
\]

The minorant \(\mu - \sigma^2/2 \equiv \Gamma(1, r_L(t))\) is the growth rate of an unlevered investor \((b := 1)\) who just buys the market index, and holds. The majorant \(\mu^2 / (2\sigma^2)\) is the Kelly growth rate that obtains when the margin loan interest rate is zero. To put it differently, we have

\[
\frac{\partial}{\partial r_L} \left\{ \max_{b \geq 1} \Gamma(b, r_L(t)) \right\} = 1 - \frac{\mu - r_L(t)}{\sigma^2} = 1 - b_t < 0. \tag{14}
\]

The instantaneous demand curve for margin loans is

\[
q_t = \left( \frac{\mu}{\sigma^2} - 1 \right) V_t - \frac{V_t}{\sigma^2} \times r_L(t). \tag{15}
\]

The corresponding instantaneous inverse demand curve is

\[
r_L = \mu - \sigma^2 \left( 1 + \frac{q_t}{V_t} \right) = \mu - \sigma^2 \times b_t, \tag{16}
\]

and the (price) elasticity of instantaneous demand for margin loans is

\[
\epsilon^d(q_t) := - \frac{r_L(t)}{q_t} \times \frac{dq_t}{dr_L(t)} = \left( \frac{\mu}{\sigma^2} - 1 \right) \times \frac{V_t}{q_t} - 1. \tag{17}
\]

Since all \(q_t\) dollars of call money are supplied inelastically by the money market, we
have the vertical supply curve

\[
\text{Quantity Supplied} := \begin{cases} 
q_t & \text{if } r_L(t) > 0 \\
[0, q_t] & \text{if } r_L(t) = 0.
\end{cases}
\]  

(18)

Intersecting supply and demand, we get the equilibrium interest rate

\[
r_L(t) = \max \left( \mu - \sigma^2 - \sigma^2 \times \frac{q_t}{V_t}, 0 \right) = (\mu - \sigma^2[1 + q_t/V_t])^+.
\]  

(19)

where \(\mu - \sigma^2 = \nu - \sigma^2/2\) is the choke price of margin debt, and \(x^+ := \max(x, 0)\) denotes the positive part of the number \(x\). On account of the equilibrium price (19), we get the formula

\[
b_t = \min \left( 1 + q_t/V_t, \frac{\mu}{\sigma^2} \right).
\]  

(20)

Thus, our dynamical model amounts to the following three assumptions:

(I.) All \(q_t\) dollars of call money are supplied inelastically.

(II.) The broker call money market continuously reinvests all its interest \(dq_t\) and principal \(q_t\).

(III.) All margin loans are issued to continuous time Kelly gamblers; the loans pass through costlessly from the money market, with no additional markup from stock brokers.

To help visualize this environment, Figure 1 plots the supply and demand curves for both \(t := 0\) and \(t := 10\) years later, along with the corresponding sample path \((q_t, r_L(t))_{0 \leq t \leq 10}\) in the price-quantity plane. The simulation (50,000 steps, \(\Delta t := 1.75\) hours) used the parameter values \((q_0, V_0, \nu, \sigma) := (1, 1, 0.09, 0.15)\).

**Lemma 1.** The size of the money market relative to Kelly gamblers’ total equity has
Figure 1: Random vibrations of the supply and demand for margin loans over the course of a decade, as generated by the sample path \((q_t, r_L(t))_{0 \leq t < 10}\) in the price-quantity plane. The (50,000-step) simulation used the parameter values \((q_0, V_0, \nu, \sigma) := (1, 1, 0.09, 0.15)\).

The following upper bound:

\[
\frac{q_t}{V_t} \leq \frac{q_0}{V_0} \times \exp\left(-\frac{\sigma^2 t}{2} - \sigma \int_0^t b_s dW_s\right).
\] (21)

Thus, the margin loan interest rate is bounded from below by the expression:

\[
r_L(t) \geq \mu - \sigma^2 \left(1 + \frac{q_0}{V_0} \exp\left(-\frac{\sigma^2 t}{2} - \sigma \int_0^t b_s dW_s\right)\right).
\] (22)

**Proof.** First, we note the fact that \(q_t \leq q_0 \times e^{(\mu - \sigma^2)t}\), e.g. the call money market can never compound its money any faster than the choke price \(\mu - \sigma^2\). That is, looking at the interest rate expression \((19)\), we have the upper bound \(r_L(t) \leq \mu - \sigma^2\); juxtaposing this inequality with the integral \((7)\) yields \(q_t \leq q_0 \times \exp\{(\mu - \sigma^2)t\}\), as promised. On
the other hand, the gambler’s fortune is bounded below by the quantity

\[ V_t \geq \exp \left\{ (\mu - \sigma^2/2)t + \sigma \int_0^t b_s dW_s \right\}. \]  

(23)

For, looking at the differential equation (9), and bearing in mind that \( \Gamma_t \geq \mu - \sigma^2/2 \), we have

\[ d(\log V_t) = \Gamma_t dt + b_t \sigma dW_t \geq (\mu - \sigma^2/2) dt + b_t \sigma dW_t \]  

(24)

\[ \Rightarrow \log(V_t/V_0) \geq (\mu - \sigma^2/2)t + \sigma \int_0^t b_s dW_s. \]  

(25)

Combining the majorant \( q_t \leq q_0 \times e^{(\mu-\sigma^2)t} \) with the minorant (23) for \( V_t \), we get the stated result (21).

In plain language, Lemma 1 says that the locally expected growth rate of the gambler’s fortune always exceeds the expected growth rate of the market index; on the other hand, the compound growth rate of the broker call money market (at most \( \mu - \sigma^2 \)) is expected to be lower than that of the market index. To be sure, the actual dynamics of the relative market size \( (q_t/V_t)_{t \geq 0} \) is ultimately determined by the realized path \( (W_t)_{t \geq 0} \) of the Brownian motion that drives all uncertainty in the economy. But based on the expected difference in the exponential growth rates of \( V_t \) and \( q_t \), it is clear that after the elapse of many years (read: decades or centuries), the chances are high that the aggregate quantity of call money will be small in relation to the total bankrolls of continuous time Kelly gamblers.

**Theorem 1.** The size of the broker call money market relative to Kelly gamblers’ total equity converges in probability\(^4\) to zero:

\[ \lim_{t \to \infty} \frac{q_t}{V_t} = 0. \]  

(26)

\(^4\)It emphatically does not converge in mean square, as we will show below.
Thus, the leverage ratio of Kelly gamblers converges in probability to 1 ($b_\infty := \lim_{t \to \infty} b_t = 1$) and the margin loan interest rate converges in probability\(^5\) to the choke price:

$$r_\infty := \lim_{t \to \infty} r_L(t) = \mu - \sigma^2 = \nu - \frac{\sigma^2}{2}. \quad (27)$$

The growth rate process $\Gamma_t$ converges in probability to the buy-and-hold growth rate:

$$\lim_{t \to \infty} \Gamma_t = \nu = \mu - \sigma^2/2. \quad (28)$$

**Proof.** Since the Kelly bet $b_t = \min(1 + q_t/V_t, \mu/\sigma^2)$ is a continuous function of the ratio $q_t/V_t$, and the interest rate $r_L(t) = \mu - \sigma^2 b_t$ is in turn a continuous function of $b_t$, it suffices to show that $q_t/V_t$ converges in probability to 0, since probability limits are preserved by continuous transformations. There follows $\lim_{t \to \infty} \Gamma_t = \Gamma(1, \mu - \sigma^2) = \mu - \sigma^2/2$.

Thus, let $\epsilon$ be any positive real number. Applying Lemma 1, we get the relations

$$1 \geq \text{Prob}\left\{ \frac{q_t}{V_t} \leq \epsilon \right\} \geq \text{Prob}\left\{ -\frac{1}{t} \int_0^t b_s dW_s - \frac{\log(\epsilon V_0/q_0)}{\sigma t} \leq \frac{\sigma}{2} \right\} = 1 \quad \text{as} \quad t \to \infty. \quad (29)$$

That is, note that the process $X_t := -t^{-1}\left\{ \int_0^t b_s dW_s + \log(\epsilon V_0/q_0)/\sigma \right\}$ converges to zero in mean square: we have

$$\lim_{t \to \infty} \mathbb{E}[X_t] = \lim_{t \to \infty} -\frac{\log(\epsilon V_0/q_0)}{\sigma t} = 0 \quad (30)$$

and, combining the Itô isometry (cf. Tomas Björk 1998) with the bound $1 \leq b_s \leq \mu \leq \mu - \sigma^2$, they do indeed converge in mean square to 1 and $\mu - \sigma^2$, respectively. The ratio $q_t/V_t$ has no such bounds; it may take any value in $(0, +\infty)$.

\(^5\)Since the processes $b_t$ and $r_L(t)$ are bounded ($1 \leq b_t \leq \mu/\sigma^2$ and $0 \leq r_L(t) \leq \mu - \sigma^2$), they do indeed converge in mean square to 1 and $\mu - \sigma^2$, respectively. The ratio $q_t/V_t$ has no such bounds; it may take any value in $(0, +\infty)$. 

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\( \mu/\sigma^2 \), we get
\[
\text{Var}[X_t] = \frac{1}{t^2} \int_0^t \mathbb{E}[b_s^2] ds \leq \frac{\mu^2}{\sigma^2 t} \to 0, \tag{31}
\]
so that \( \lim_{t \to \infty} \text{Var}[X_t] = 0 \). Since the process \( (X_t)_{t \geq 0} \) converges to zero in mean square, it certainly converges to zero in probability; in particular, this means that
\[
\lim_{t \to \infty} \mathbb{P}\left\{ X_t \leq \frac{\sigma}{2} \right\} = 1. \tag{32}
\]
By the squeeze theorem, then, we have obtained the desired result: for all \( \epsilon > 0 \),
\[
\lim_{t \to \infty} \mathbb{P}\left\{ \frac{q_t}{V_t} \leq \epsilon \right\} = 1. \tag{33}
\]

**Corollary 1.** The margin loan interest rate \( r_L(t) \) converges in mean square to the choke price \( r_\infty := \mu - \sigma^2 = \nu - \sigma^2/2 \) and the Kelly bet \( b_t \) converges in mean square to 1. The instantaneous Kelly growth rate \( \Gamma_t \) converges in mean square to the buy-and-hold growth rate \( \nu = \mu - \sigma^2/2 \).

**Proof.** It suffices to show that \( b_t \) converges in mean square to 1; then, on account of the linear relationship \( r_L(t) = \mu - \sigma^2 b_t \), we will have \( \lim_{t \to \infty} \mathbb{E}[r_L(t)] = \mu - \sigma^2 \) and \( \lim_{t \to \infty} \text{Var}[r_L(t)] = 0 \).

To this end, let \( \epsilon \) be any positive number, and let \( R_t := q_t/V_t \) denote the relative size of the call money market. We have
\[
\mathbb{E}[(b_t - 1)^2] = \mathbb{P}\{R_t < \epsilon\} \mathbb{E}[(b_t - 1)^2|R_t < \epsilon] + \mathbb{P}\{R_t \geq \epsilon\} \mathbb{E}[(b_t - 1)^2|R_t \geq \epsilon]
\leq 1 \times \epsilon^2 + \mathbb{P}\{R_t \geq \epsilon\} \times (\mu/\sigma^2 - 1)^2. \tag{34}
\]
Using the fact that \( \lim_{t \to \infty} \Pr\{R_t \geq \epsilon\} = 0 \), we see that the following relation must obtain for every \( \epsilon > 0 \):

\[
\limsup_{t \to \infty} \mathbb{E}[(b_t - 1)^2] \leq \epsilon^2.
\]  

(35)

Since the \( \limsup_{t \to \infty} \) of the mean-squared error is smaller than every positive number, we get the inequalities

\[
\liminf_{t \to \infty} \mathbb{E}[(b_t - 1)^2] \leq \limsup_{t \to \infty} \mathbb{E}[(b_t - 1)^2] = 0 \leq \liminf_{t \to \infty} \mathbb{E}[(b_t - 1)^2],
\]

(36)

which implies that \( \lim_{t \to \infty} \mathbb{E}[(b_t - 1)^2] = 0 \).

Finally, turning our attention to the instantaneous Kelly growth rate \( \Gamma_t = r_L(t) + \sigma^2 b_t^2 / 2 \), it now suffices to show that \( b_t^2 \) converges to 1 in mean square; then, since \( r_L(t) \overset{m.s.}{\to} \mu - \sigma^2 \), we will have \( \Gamma_t \overset{m.s.}{\to} \mu - \sigma^2 + \sigma^2 / 2 \times 1 = \mu - \sigma^2 / 2 \), as promised. Accordingly, we bound the mean-squared error

\[
\mathbb{E}[(b_t^2 - 1)^2] = \mathbb{E}[(b_t - 1)^2(b_t + 1)^2] \leq \mathbb{E}[(b_t - 1)^2] \times \left( \frac{\mu}{\sigma^2} + 1 \right)^2 \to 0,
\]

(37)

which proves that \( b_t^2 \overset{m.s.}{\to} 1 \).

In plain English: as time goes on, there are some (exceedingly rare) sample paths of the experiment whereby the Kelly gambler performs very poorly in relation to the money market; the ratio \( q_t / V_t \) therefore spikes and the Kelly bet hits the upper bound \( b_t = \mu / \sigma^2 \) under an interest rate of zero. However, these rare events make a negligible contribution to the mean-squared error \( \mathbb{E}[(b_t - 1)^2] \), precisely because the Kelly gambler’s mantra prevents him from betting more than \( \bar{b} := \mu / \sigma^2 \), even when he is offered an interest rate of zero. After many years \( t \) have elapsed, the density of \( b_t \) becomes concentrated around 1, albeit with a long tail that spans the interval \([1, \mu / \sigma^2]\).
Figure 2: 100-year sample path of the Kelly leverage ratios \( b_t \) and corresponding margin loan interest rates \( r_L(t) \), for the parameters \((q_0, V_0, \nu, \sigma, \mu, r_\infty) := (1, 1, 0.09, 0.15, 0.1012, 0.0787)\). The means and standard deviations were estimated from 50,000 Monte Carlo simulations of 50,000 steps each (\( \Delta t := 17.5 \) hours).

Figure 2 illustrates the corollary by plotting a 100-year, 50,000-step sample path of the Kelly leverage ratios and margin loan interest rates in an economy generated by the parameters \((q_0, V_0, \nu, \sigma, \mu, r_\infty) := (1, 1, 0.09, 0.15, 0.1012, 0.0787)\). For context, the Figure provides Monte Carlo estimates of the expected values \( \{E[b_t], E[r_L(t)]\}_{0 \leq t \leq 100} \) and standard deviations \( \{\text{Std}(b_t), \text{Std}(r_L(t))\}_{0 \leq t \leq 100} \) that were generated from 50,000 simulations of 50,000 steps each (\( \Delta t := 17.5 \) hours). In the same vein, Figure 3 gives a 30-year sample path of the instantaneous growth rate process \( \Gamma_t \) for the same parameters, along with Monte Carlo estimates of the functions \( t \mapsto E[\Gamma_t] \) and \( t \mapsto \text{Std}(\Gamma_t) \) that were computed from 40,000 simulations (\( \Delta t := 6.6 \) hours).

**Theorem 2.** The relative size \( q_t/V_t \) of the money market is a martingale (in spite of the fact that it converges in probability to zero); the Kelly bet \( b_t \) is a supermartingale (e.g. it is always expected to decrease) and the margin loan interest rate \( r_L(t) \) is a submartingale (e.g. it is always expected to increase).
Figure 3: 30-YEAR SAMPLE PATH OF THE OPTIMUM GROWTH RATE PROCESS \((\Gamma_t)\) FOR THE PARAMETERS \((q_0, V_0, \nu, \sigma) := (1, 1, 0.09, 0.15)\), ALONG WITH MONTE CARLO ESTIMATES (40,000 SIMULATIONS, \(\Delta t := 6.6\) HOURS) OF THE FUNCTIONS \(t \mapsto \mathbb{E}[\Gamma_t]\) AND \(t \mapsto \text{Std}(\Gamma_t)\).

**Proof.** We apply the quotient rule of the Itô calculus (cf. with Ovidiu Calin 2015) to the ratio \(q_t/V_t\):

\[
d\left(\frac{q_t}{V_t}\right) = dq_t V_t - q_t dV_t - \frac{dq_t dV_t}{V_t^2} + \frac{q_t}{V_t^3} \times (dV_t)^2. \tag{38}
\]

According to the Itô multiplication table (e.g. Paul Wilmott 1998), we have \(dq_t \times dV_t = 0\) and \((dV_t/V_t)^2 = b_t^2 \sigma^2 \times dt\). Thus, one calculates that

\[
d\left(\frac{q_t}{V_t}\right) = \frac{q_t}{V_t} \times \left[ (r_L + b_t^2 \sigma^2)dt - \frac{dV_t}{V_t} \right] = \frac{q_t}{V_t} \times \left[ b_t (\sigma^2 b_t - \mu + r_L(t)) dt - b_t \sigma dW_t \right] = -\sigma \frac{q_t b_t}{V_t} dW_t. \tag{39}
\]
Thus, the stochastic process \((q_t/V_t)_{t \geq 0}\) is a martingale, since it has zero drift and admits the (Itô) integral representation

\[
\frac{q_t}{V_t} = \frac{q_0}{V_0} - \sigma \int_0^t \frac{q_s b_s}{V_s} dW_s.
\]  

(40)

On account of the fact that \(b_t = \min(1 + q_t/V_t, \mu/\sigma^2)\) is a concave function of the martingale \(q_t/V_t\), we conclude that \((b_t)_{t \geq 0}\) is a supermartingale, e.g. it is always expected to decrease (cf. with Lawrence Evans 2010). Likewise, the interest rate \(r_L(t) = \max(\mu - \sigma^2 - \sigma^2 q_t/V_t, 0)\) is a submartingale, since it is a convex function of a martingale.

Thus, although the chances are high that the ratio \(R_t := q_t/V_t\) is very low in the long run, it nevertheless has a constant mean \(\mathbb{E}[R_t] \equiv q_0/V_0\); this happens on account of a few sample paths for which the stock market dramatically underperforms the broker call money market. The (unconditionally) expected interest rate \(\mathbb{E}[r_L(t)]\) is an increasing function of time that converges to \(\mu - \sigma^2\); conditional on the current state of things at time \(t\), the expected margin rate \(\mathbb{E}[r_L(t + \Delta t)|r_L(t)]\) at any time in the future is greater than or equal to the current observation \(r_L(t)\). However, the expected increases in the interest rate (and attendant decreases in the aggregate leverage ratio) are disturbed by so many random vibrations of the stock market. The margin loan interest rate responds pro-cyclically to random noise in the financial markets; the leverage ratios of continuous time Kelly gamblers respond counter-cyclically. But the underlying signal (that is, the exponential growth of asset prices) suffices to generate a permanent uptrend in margin loan interest rates.

Corollary 2. The probability of the margin loan interest rate ever hitting zero (be-
between now and kingdom come) has the following majorant:

$$\text{Prob}\{r_L(t) \text{ is ever } 0\} \leq 1 - \frac{r_L(0)}{r_\infty} = 1 - \frac{\text{Current Interest Rate}}{\text{Choke Price}}$$

(41)

**Proof.** The condition that the margin loan interest rate \( r_L(t) \) hits zero at least once over a given horizon \([0, T]\) is equivalent to the condition that the ratio \( q_t/V_t \) breaches \( \mu/\sigma^2 - 1 \) at least once. Since \( (q_t/V_t)_{t \geq 0} \) is a positive martingale, Doob’s martingale inequality obtains (cf. Lawrence Evans 2010); in our context, this inequality amounts to

$$\text{Prob}\left\{ \max_{0 \leq t \leq T} \frac{q_t}{V_t} \geq \frac{\mu}{\sigma^2} - 1 \right\} \leq \frac{\mathbb{E}[q_T/V_T]}{\mu/\sigma^2 - 1} = \frac{q_0}{V_0} \times \frac{\sigma^2}{\mu - \sigma^2} = 1 - \frac{r_L(0)}{r_\infty},$$

(42)

where we have used the fact that \( \mathbb{E}[q_T/V_T] \equiv q_0/V_0 \). Taking the limit of the inequality (42) as \( T \to \infty \), we obtain the desired result, that

$$\text{Prob}\left\{ \sup_{t \geq 0} \frac{q_t}{V_t} \geq \frac{\mu}{\sigma^2} - 1 \right\} \leq 1 - \frac{r_L(0)}{r_\infty}. \quad (43)$$

Thus, if the current margin loan interest rate amounts to 70% of the choke price, then the chance of it ever hitting zero is at most 30%. If the current rate is 20% of the asymptotic interest rate, then the chance of it ever reaching zero is at most 80%, etc. Table 1 illustrates the majorant for different stock market volatilities and compound-annual (logarithmic) growth rates, assuming that the money market begins on par with the gambler’s fortune \( (q_0/V_0 := 1) \). Naturally, the bound becomes tighter as the stock market parameters become more favorable (higher \( \nu \), lower \( \sigma \)); it also tightens with the relative scarcity of loanable funds (lower \( q_0/V_0 \)).
Example 1. As of this writing, the broker call money rate (as reported by Bankrate.com) is 4.25%. Assuming the stylized parameters \((\nu, \sigma) := (0.09, 0.15)\) for the S&P 500 index, we get a choke price of 7.9%. Thus, we reckon that the chance of the margin loan interest rate ever hitting zero is at most \(4.25 \div 7.9 = 54\%\).

Proposition 1. We have the following bounds on the (unconditional) standard deviation of the relative market size \(q_t/V_t\):

\[
\frac{q_0}{V_0} \times \sqrt{\exp[\sigma^2 t]} - 1 \leq \text{Std} \left( \frac{q_t}{V_t} \right) \leq \frac{q_0}{V_0} \times \sqrt{\exp[(\mu/\sigma)^2 t]} - 1. 
\]

In particular, \(\lim_{t \to \infty} \text{Std}(q_t/V_t) = +\infty\).

Proof. For notational convenience, we let \(F(t) := \mathbb{E}[(q_t/V_t)^2]\) denote the second moment of the relative size process. Recalling the (Itô) integral representation

\[
\frac{q_t}{V_t} = \frac{q_0}{V_0} - \sigma \int_0^t \frac{q_s b_s}{V_s} dW_s, 
\]

the Itô isometry implies that

\[
\text{Var} \left[ \frac{q_t}{V_t} \right] = F(t) - (q_0/V_0)^2 = \sigma^2 \int_0^t \mathbb{E} \left[ \left( \frac{q_s b_s}{V_s} \right)^2 \right] ds, 
\]

which, upon differentiating, gives us

\[
\frac{dF}{dt} = \sigma^2 \mathbb{E} \left[ \left( \frac{q_t b_t}{V_t} \right)^2 \right]. 
\]

Now, bearing in mind that \(1 \leq b_t^2 \leq \mu^2/\sigma^4\), we have the inequalities

\[
\sigma^2 F(t) \leq \frac{dF}{dt} \leq \left( \frac{\mu}{\sigma} \right)^2 F(t),
\]

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or equivalently,
\[ \sigma^2 \leq \frac{d}{dt} \log F(t) \leq \left( \frac{\mu}{\sigma} \right)^2. \]  \hspace{1cm} (49)

Integrating the inequalities (49) and simplifying, we obtain the theoretical bounds

\[ F(0) \times \{ \exp[\sigma^2t] - 1 \} \leq F(t) - F(0) \leq F(0) \times \{ \exp[(\mu/\sigma)^2t] - 1 \}. \]  \hspace{1cm} (50)

Remembering that \( F(0) = (q_0/V_0)^2 \) and \( \text{Var}[q_t/V_t] = F(t) - F(0) \), taking the square root of (50) yields the stated result.

Thus, although the martingale \((q_t/V_t)_{t \geq 0}\) converges in probability to zero, its standard deviation grows to infinity at a geometric rate. Figure 4 plots these theoretical bounds, along with Monte Carlo estimates of the true standard deviation, for \( t \in [0, 2] \) assuming the parameters \( q_0 := 1, V_0 := 1, \nu := 0.09, \sigma := 0.15, \) and \( \mu := \nu + \sigma^2/2 = 0.1012. \) The (deterministic) function \( t \mapsto \text{Std}(q_t/V_t) \) was estimated from 100,000 experiments of 100,000 steps each; the corresponding step size was \( \Delta t := 10.5 \) minutes.

Note that the population standard deviation \( \sqrt{\text{E}[(q_t/V_t)^2] - (q_0/V_0)^2} \) is required to be increasing on account of the fact that the process \((q_t/V_t)^2\) is a submartingale (e.g. it is a convex function of \( q_t/V_t \)). For the sake of visualization, Figure 5 plots a 100-year sample path of \( q_t/V_t \) for the same deep parameters \((q_0, V_0, \nu, \sigma) := (1, 1, 0.09, 0.15)\); the experiment consisted of 100,000 steps, for a step size of 8.8 hours.

**Theorem 3.** The Kelly gambler’s realized continuously-compounded capital growth rate over \([0, T]\) (namely, \( \log(V_T/V_0)/T \)) converges in mean square to the stock market growth rate \( \nu = \mu - \sigma^2/2; \) the realized continuously-compounded growth rate of the money market over \([0, T]\) (namely, \( \log(q_T/q_0)/T \)) converges in mean square to the choke price \( r_\infty = \nu - \sigma^2/2 = \mu - \sigma^2. \)
### Table 1: Upper bounds on the probability of the margin loan interest rate ever hitting zero over $t \in [0, +\infty)$, for different stock market volatilities and growth rates, assuming that the money market starts on par with the gambler’s fortune ($q_0/V_0 := 1$).

| Ratio $(q_0/V_0)$ | Vol. $(\sigma)$ | CAGR $(\nu)$ | Current/Choke $(r_L(0)/r_\infty)$ | Majorant of $P\{r_L$ is ever 0$\}$ | Actual Prob. $(Monte Carlo$ est.$^*$) |
|------------------|----------------|-------------|-------------------------------|---------------------------------|----------------------------------|
| 1                | 10% 9%         | 7.5%/8.5% | $\leq 11.8\%$                  | 9.3% (0.18%)                     |
| 1                | 15% 9%         | 5.6%/7.9% | $\leq 28.6\%$                  | 26.3% (0.28%)                    |
| 1                | 20% 9%         | 3%/7%     | $\leq 57.1\%$                  | 55.1% (0.31%)                    |
| 1                | 10% 8%         | 6.5%/7.5% | $\leq 13.3\%$                  | 10.8% (0.2%)                     |
| 1                | 15% 8%         | 4.6%/6.9% | $\leq 32.8\%$                  | 30.1% (0.29%)                    |
| 1                | 20% 8%         | 2%/6%     | $\leq 66.7\%$                  | 63.8% (0.3%)                     |
| 1                | 10% 7%         | 5.5%/6.5% | $\leq 15.4\%$                  | 12.7% (0.21%)                    |
| 1                | 15% 7%         | 3.6%/5.9% | $\leq 38.3\%$                  | 36.1% (0.3%)                     |
| 1                | 20% 7%         | 1%/5%     | $\leq 80\%$                    | 78% (0.26%)                      |
| 1                | 10% 6%         | 4.5%/5.5% | $\leq 18.2\%$                  | 15.2% (0.23%)                    |
| 1                | 15% 6%         | 2.6%/4.9% | $\leq 46.2\%$                  | 43.6% (0.31%)                    |
| 1                | 20% 6%         | 0%/4%     | $\leq 100\%$                   | 100%                             |
| 1                | 10% 5%         | 3.5%/4.5% | $\leq 22.2\%$                  | 18.7% (0.25%)                    |
| 1                | 15% 5%         | 1.6%/3.9% | $\leq 58.1\%$                  | 56% (0.31%)                      |
| 1                | 20% 5%         | 0%/3%     | $\leq 100\%$                   | 100%                             |

*Percentage of all simulations for which the margin loan interest rate hit the zero bound (standard errors in parentheses). 25,000 simulations per estimate, spanning 200 years each, 25,000 steps per simulation, $\Delta t := 2.92$ days.
Monte Carlo estimates of $\text{Std}(q_t/V_t)$ over $t \in [0, 2]$, assuming the parameters $q_0 := 1, V_0 := 1, \nu := 0.09, \sigma := 0.15$, and $\mu := \nu + \sigma^2/2 = 0.1012$. Estimates computed from the simulation of 100,000 sample paths of 100,000 steps each ($\Delta t := 10.5$ minutes).

Proof. On account of the expression

$$\frac{\log(q_T/q_0)}{T} = \frac{1}{T} \int_0^T r_L(t)dt,$$

it follows that

$$\mathbb{E} \left[ \frac{\log(q_T/q_0)}{T} \right] = \frac{1}{T} \int_0^T \mathbb{E}[r_L(t)]dt$$

is the average value of the (deterministic) function $t \mapsto \mathbb{E}[r_L(t)]$ over the interval $[0, T]$. Since $r_L(t)$ converges in mean square to $r_\infty$, we have the relation $\lim_{t \to \infty} \mathbb{E}[r_L(t)] = r_\infty$; thus, the average value of the function $t \mapsto \mathbb{E}[r_L(t)]$ must also converge to $r_\infty$. It remains to show that

$$\lim_{T \to \infty} \text{Var} \left[ \frac{\log(q_T/q_0)}{T} \right] = 0.$$
Figure 5: 100-YEAR SAMPLE PATH OF $q_t/V_t$, GENERATED BY THE PARAMETERS $q_0 := 1, V_0 := 1, \nu := 0.09, \sigma := 0.15$, AND $\mu := \nu + \sigma^2/2 = 0.1012$. 100,000 STEPS, STEP SIZE = 8.8 HOURS. CROSSING THE BLUE (DASHED) BARRIER RESULTS IN A MARGIN LOAN INTEREST RATE OF ZERO.

To this end, we invoke the formula (cf. with Hoel, Port, and Stone 1972)

$$\text{Var}\left[\frac{\log(q_T/q_0)}{T}\right] = \frac{1}{T^2} \int_0^T \int_0^T \text{Cov}(r_L(s), r_L(t)) \, ds \, dt. \quad (54)$$

The Cauchy-Schwarz inequality (e.g. check with T.T. Soong 1973) says that

$$\text{Cov}(r_L(s), r_L(t)) \leq \text{Std}(r_L(s)) \times \text{Std}(r_L(t)). \quad (55)$$

Hence, since the right-hand-side of (55) is multiplicatively separable in the variables $s$ and $t$, the double integral (54) is majorized by the square of the unidimensional integral $\int_0^T \text{Std}(r_L(t)) \, dt$; this gives us the variance bound

$$\text{Var}\left[\frac{\log(q_T/q_0)}{T}\right] \leq \left[ \frac{1}{T} \int_0^T \text{Std}(r_L(t)) \, dt \right]^2 \to 0 \text{ as } T \to \infty. \quad (56)$$
The right-hand-side of (56) converges to zero as $T \to \infty$ because it is the average value of the (deterministic) function $t \mapsto \text{Std}(r_L(t))$, which itself converges to zero on account of the fact that $r_L(t)$ converges in mean square to $r_{\infty}$. This proves that the realized money market growth rate $\log(q_T/q_0)/T$ converges in mean square to the choke price $r_{\infty} = \mu - \sigma^2$.

Turning our attention to the realized compound-growth rate of the Kelly bankroll over $0 \leq t \leq T$, we integrate the left-hand-side of (24) and obtain the expression

$$\frac{\log(V_T/V_0)}{T} = \frac{1}{T} \int_0^T \Gamma_t \, dt + \frac{\sigma}{T} \int_0^T b_t dW_t =: x_T + y_T. \quad (57)$$

Bearing in mind that $\mathbb{E}[y_T] \equiv 0$, we get

$$\mathbb{E}\left[ \frac{\log(V_T/V_0)}{T} \right] = \mathbb{E}[x_T] = \frac{1}{T} \int_0^T \mathbb{E}[\Gamma_t] \, dt, \quad (58)$$

which is the average value of the deterministic function $t \mapsto \mathbb{E}[\Gamma_t]$ over the interval $[0, T]$. Since the stochastic process $(\Gamma_t)_{t \geq 0}$ converges in mean square to $\mu - \sigma^2/2$, we of course have $\lim_{t \to \infty} \mathbb{E}[\Gamma_t] = \mu - \sigma^2/2$; accordingly, the average value (58) of the function $t \mapsto \mathbb{E}[\Gamma_t]$ must also converge to $\mu - \sigma^2/2$ as $T \to \infty$.

To complete the proof, we proceed to demonstrate that that $\lim_{T \to \infty} \text{Var}[x_T + y_T] = 0$. On account of the triangle inequality $\text{Std}(x_T + y_T) \leq \text{Std}(x_T) + \text{Std}(y_T)$, it suffices to show that $\lim_{T \to \infty} \text{Var}[x_T] = 0$ and $\lim_{T \to \infty} \text{Var}[y_T] = 0$. We are already familiar with the fact that $y_T$ converges to zero in mean square; mutatis mutandis, analogous
to what we just did with the interest rate $r_L(t)$, we write

$$\text{Var}\left[\frac{1}{T} \int_0^T \Gamma_t \, dt\right] = \frac{1}{T^2} \int_0^T \int_0^T \text{Cov}(\Gamma_s, \Gamma_t) \, ds \, dt$$

$$\leq \frac{1}{T^2} \left[ \int_0^T \text{Std}(\Gamma_s) \, ds \right] \left[ \int_0^T \text{Std}(\Gamma_t) \, dt \right]$$

$$= \left[ \frac{1}{T} \int_0^T \text{Std}(\Gamma_t) \, dt \right]^2 \to 0. \tag{59}$$

The last bracketed expression in (59) converges to 0 as $T \to \infty$ because it is the average value of the function $t \mapsto \text{Std}(\Gamma_t)$ over the interval $[0, T]$, a function whose value itself converges to 0 as $t \to \infty$.

To illustrate the Theorem, Figure 6 plots the realized growth rate series

$$\frac{1}{t} (\log(q_t/q_0), \log(S_t/S_0), \log(V_t/V_0)) \tag{60}$$

that obtained from a 200-year, 200,000-step simulation of the model economy generated by the parameters $(q_0, V_0, \nu, \sigma) := (1, 1, 0.09, 0.15)$.

**Theorem 4** (Change of Numéraire). The ratio $V_t/S_t$ of the gambler’s fortune to the price of one unit of the market index (e.g. the value of the bankroll as measured in shares of the ETF) is a submartingale (always expected to increase). The total size $q_t/S_t$ of the money market, as expressed in units of this numéraire, is a supermartingale that converges in probability to zero. The aggregate wealth in the model $(q_t+V_t)/S_t$ (money market plus gambler’s equity) is a supermartingale when expressed
Figure 6: The realized continuously-compounded capital growth rates in a 200-year, 200,000-step simulation of the model economy, under the parameters \((q_0, V_0, \nu, \sigma) := (1, 1, 0.09, 0.15)\).

in shares of the market index. Consequently, the ratio

\[
\text{Kelly Gambler's Relative Growth Factor} := \frac{V_t/V_0}{S_t/S_0}
\]  

has the property that

\[
\mathbb{E} \left[ \frac{V_t/V_0}{S_t/S_0} \right] \leq 1 + \frac{q_0}{V_0} \text{ for all } t.
\]  

The relative growth factor \((V_t/V_0)/\div (S_t/S_0)\) amounts to the ratio of the Kelly gambler’s bankroll to the wealth of a buy-and-hold investor \((b \equiv 1)\) who started with the same initial capital. Note that, although the cumulative outperformance realized by the Kelly gambler over \([0,t]\) is always expected to increase, it fails to grow to infinity at an exponential rate (as it would under perfectly elastic supply of margin loans). Rather, the asymptotic relative growth factor is a finite, random quantity that may even turn out to be less than 1 (albeit with low probability). At the start of
the model, the Kelly gambler cannot expect to ever achieve more than \(1 + q_0/V_0\) times the wealth of a buy-and-hold investor who started with the same amount of money. Say, if the initial interest rate is positive and the initial leverage ratio is \(b_0 := 2\) (the maximum allowed by U.S. Regulation-T), then we cannot expect to achieve more than double the final wealth of an equivalent buy-and-hold investor.

**Proof.** Applying the Itô quotient rule to the process \((V_t/S_t)_{t \geq 0}\), one calculates that

\[
d \left( \frac{V_t}{S_t} \right) = \frac{V_t}{S_t} \times \left\{ (b_t - 1)(\mu - \sigma^2 - r_L(t))dt + \sigma(b_t - 1)dW_t \right\}. \tag{63}
\]

Thus, \((V_t/S_t)_{t \geq 0}\) is a submartingale because of its positive drift, which obtains on account of the fact that \(b_t > 1\) and \(r_L < \mu - \sigma^2\). A similar calculation shows that

\[
d \left( \frac{q_t}{S_t} \right) = \frac{q_t}{S_t} \times \left\{ -(\mu - \sigma^2 - r_L(t))dt - \sigma dW_t \right\}, \tag{64}
\]

whence \((q_t/S_t)_{t \geq 0}\) is a supermartingale because of its negative drift rate. Combining equations (63) and (64), and simplifying, we obtain

\[
d \left( \frac{q_t + V_t}{S_t} \right) = d \left( \frac{q_t}{S_t} \right) + d \left( \frac{V_t}{S_t} \right) = \frac{(b_t - 1)V_t - q_t}{S_t} \left\{ (\mu - \sigma^2 - r_L(t))dt + \sigma dW_t \right\}. \tag{65}
\]

Recalling that \(b_t = \min(1 + q_t/V_t, \mu/\sigma^2) \leq 1 + q_t/V_t\), we see that \((q_t + V_t)/S_t\) is a supermartingale, since its drift is \(\leq 0\). With these facts in hand, we have the inequalities

\[
\mathbb{E} \left[ \frac{V_t}{S_t} \right] \leq \mathbb{E} \left[ \frac{q_t + V_t}{S_t} \right] \leq \frac{q_0 + V_0}{S_0}, \tag{66}
\]

where we have used the fact that \(\mathbb{E}[(q_t + V_t)/S_t]\) is a decreasing function of time.
Multiplying (66) through by \( S_0/V_0 \), we get the promised result:

\[
\mathbb{E} \left( \frac{V_t/V_0}{S_t/S_0} \right) \leq 1 + \frac{q_0}{V_0}.
\]

Finally, for the sake of demonstrating that \( \text{plim}_{t \to \infty} q_t/S_t = 0 \), we start with the upper bound

\[
\frac{q_t}{S_t} \leq \frac{q_0}{S_0} \times \exp \left( -\frac{\sigma^2 t}{2} + \sigma W_t \right);
\]

If \( \epsilon \) is any positive number, then

\[
\text{Prob} \left\{ \frac{q_t}{S_t} \leq \epsilon \right\} \geq \text{Prob} \left\{ \frac{q_0}{S_0} \exp \left( -\frac{\sigma^2 t}{2} + \sigma W_t \right) \leq \epsilon \right\} = \text{Prob} \left\{ \frac{W_t}{\sqrt{t}} \leq \frac{\log(\epsilon S_0/q_0)}{\sigma \sqrt{t}} + \frac{\sigma \sqrt{t}}{2} \right\} = N \left( \frac{\log(\epsilon S_0/q_0)}{\sigma \sqrt{t}} + \frac{\sigma \sqrt{t}}{2} \right),
\]

where \( N(\bullet) \) denotes the cumulative normal distribution function. Thus, we have

\[
1 \geq \lim_{t \to \infty} \text{Prob} \left\{ \frac{q_t}{S_t} \leq \epsilon \right\} \geq N(\infty) = 1,
\]

which is the desired result.

Figure 7 supplements Theorem 4 by plotting a 100-year (100,000-step) sample path of the time series \( V_t/S_t, q_t/S_t, (q_t + V_t)/S_t, \) and \( r_L(t) \) for the parameters \((q_0, V_0, \nu, \sigma) := (1, 1, 0.08, 0.2)\). For this particular simulation (\( \Delta t := 8.8 \) hours), we made the stock market index less favorable than it was in our previous experiments (lower \( \nu \), higher \( \sigma \)) so as to highlight the model’s behavior when the margin loan interest rate hits zero very frequently on its way up to \( r_\infty \). To help visualize the population statistics under this change of numéraire, Figure 8 provides a 300-year
Figure 7: 100-year sample path of the stochastic processes $V_t/S_t$, $(q_t + V_t)/S_t$, $q_t/S_t$, and $r_L(t)$ for the parameters $(q_0, V_0, \nu, \sigma) := (1, 1, 0.08, 0.2)$. 100,000 steps, $\Delta t := 8.8$ hours.

plot of the time functions $E[V_t/S_t] \pm \text{Std}(V_t/S_t)$, $E[q_t/S_t]$, and $E[(q_t + V_t)/S_t]$ for these same parameters.

Figure 9 gives a density estimate (Epanechnikov kernel, bandwidth := 0.0193) for the random variable $\text{ms-lim}_{t \to \infty} [(V_t/V_0) \div (S_t/S_0)]$, based on 100,000 simulations generated by the parameters $(q_0, V_0, \nu, \sigma) := (1, 1, 0.09, 0.15)$. For these sample paths, on average, the Kelly gambler achieved a relative growth factor of 1.87, or 87% more final wealth than the equivalent buy-and-hold investor. Note that the Kelly Criterion underperformed buy-and-hold on 2.1% of all sample paths (cf. the asymptotic CDF, which is supplied in the right half of the Figure).

3 Summary and Conclusions

This paper established the core dynamical behavior of the broker call money market, which supplies cash to stock brokers for the sake of funding margin loans to retail clients. We assumed (naturally) that the demand side of the market is comprised of continuous time Kelly gamblers, who size their bets over each tick $dt$ of the clock
so as to maximize their expected continuously-compounded capital growth rate over 
\([t, t+dt]\). Ordinarily, under perfectly elastic supply of margin loans (cf. with Garivaltis
2019a), the Kelly gambler is able to beat the market asymptotically almost surely,
and by an exponential factor to boot.

To model the powerful long run feedback effects that these sophisticated investors
must have on the equilibrium price of margin debt, we assumed that the production
side of market amounts to a giant pool of cash that supplies itself inelastically and
continuously reinvests all principal and interest. Thus, although the total size of the
money market \((q_t)\) grows to infinity at a geometric rate, this rate of supply expansion
is lower than the asymptotic growth rate \(\nu\) of the market index \((S_t)\) and the expected
compound growth rate of the Kelly bankroll \((V_t)\). Proceeding with this intuition,
we found that the relative market size \((q_t/V_t)_{t\geq 0}\) is a martingale (whose variance
Figure 9: **Empirical distribution of the random variable** $\text{ms-lim}_{t \to \infty} \left[ (V_t/V_0) \div (S_t/S_0) \right]$, **based on 100,000 simulations generated by the parameters** $(q_0, V_0, \nu, \sigma) := (1, 1, 0.09, 0.15)$. **Epanechnikov kernel bandwidth** $= 0.0193$; the Kelly Criterion underperformed buy-and-hold on 2.1% of all sample paths.

tends to infinity) that nonetheless converges to zero in probability\(^6\); $(q_t/S_t)_{t \geq 0}$ is a supermartingale that converges to zero in probability. Consequently, the margin loan interest rate is a submartingale (always expected to increase) that converges in mean square to the choke price $r_\infty = \nu - \sigma^2/2$, where $\sigma$ is the annual log-volatility of the stock market. If the relative size of the money market becomes unexpectedly large (e.g. due to bad stock market performance), then the margin loan interest rate may happen to hit zero on its way up to $r_\infty$; we found a nice rule of thumb for bounding the chances of this ever happening (from here to eternity): the probability is at most $1 - (\text{Current Interest Rate} \div \text{Choke Price})$. Based on numerical solutions of the differential equations, we observed that this majorant is typically within 3% of the actual value.

In the same vein, we concluded that the Kelly leverage ratio $(b_t)_{t \geq 0}$ is a supermartingale that converges in mean square to 1 : 1; thus, the very success of the

\(^6\)But *not* in mean square!
leveraged investor causes a gradual degradation of the quality of his opportunity set. This is manifest in the asymptotic distribution of the Kelly gambler’s performance relative to the a buy-and-hold investor with the same starting capital, e.g. the random variable \( \text{ms-lim}_{t \to \infty} [(V_t/V_0) \div (S_t/S_0)] \). In fact, when all feedback effects are considered, the Kelly gambler no longer beats the market by an exponential factor; his asymptotic compound growth rate, namely \( \text{ms-lim}_{t \to \infty} [\log(V_t/V_0)/t] \), is equal to the stock market growth rate, \( \nu \). The realized money market growth rate \( \log(q_t/q_0)/t \) converges in mean square to \( r_\infty \).

We demonstrated that the leveraged investor’s relative growth factor \((V_t/V_0) \div (S_t/S_0)\) is a submartingale (always expected to increase); however, its limiting expected value is at most \( 1 + q_0/V_0 \). Thus, if the money market starts out on par with the Kelly bankroll \((q_0/V_0 := 1)\), then the Kelly gambler cannot expect to achieve any more than double the final wealth of the equivalent buy-and-hold investor. Simulation studies (using the stylized parameter values \((\nu, \sigma) := (0.09, 0.15)\) for the S&P 500 index) indicate that the asymptotic relative growth factor (which is negatively-skewed) has a mean of 1.87 and a standard deviation of 0.24; the Kelly Criterion eventually beat the market in 97.9% of all (100,000) simulations. The greatest final relative growth factor ever achieved in simulation was 2.13; in a select few of the experiments, the gambler blew himself up spectacularly: the empirical minimum final growth relative was 0.012. Pray that that never happens to you.

\textit{Northern Illinois University}

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