Local phase invariance of the free-particle Schrödinger equation in momentum space

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The local phase-invariance of the momentum-space Schrödinger equation for free-particle has been used to construct quantum kinematics that describes a motion of the particle in external $U(1)$ background gauge field. The gauge structure over the momentum space of the particle is interpreted in terms of geometrical phase accumulated by nuclear quadrupole resonance spectra [5].

**II. QUANTUM KINEMATICS OF A FREE PARTICLE**

In momentum representation, the free-particle Schrödinger equation is

$$\left(\frac{p^2}{2} - E\right)\psi(p) = 0, \quad (1)$$

where $E$ is the kinetic energy of the particle, and $p$ is the momentum, which is a multiplication operator. Eq. (1) is invariant under local change of the phase of the wave-function

$$\psi(p) \to \psi(p)e^{-i\Lambda(p)}, \quad (2)$$

which is because the momentum $p$ does not change

$$e^{-i\Lambda(p)}\hat{p}e^{i\Lambda(p)} = p. \quad (3)$$

Therefore the phase of the wave-function is not fixed by Eq. (1). Under an infinitesimal local phase transformation of Eq. (2), the wave-function changes as

$$\delta\psi(p) = -i\Lambda(p)\psi(p), \quad (4)$$

however the derivative

$$\delta[\nabla_p \psi] = -i\Lambda(p)\nabla_p \psi(p) - i(\nabla_p \Lambda(p))\psi(p) \quad (5)$$

do not change in the same way. Since the description of the free-particle motion is independent on the choice of phase, we introduce a derivative that transforms covariantly under $U(1)$ phase transformations

$$D = \nabla_p + ie\Lambda(p). \quad (6)$$
where \( e \) is a coupling constant (not necessarily \( e = 1 \)), and \( A(p) \) is a compensating gauge field, that transforms according to

\[
A(p) \rightarrow A(p) + \frac{1}{e} \nabla_p A(p),
\]

and ensures that

\[
D \delta \psi = \delta [\nabla_p \psi] + ie(\delta A) \psi + ieA \delta \psi = -i \Lambda(p) D \psi(p) \tag{8}
\]

nothing depends on the arbitrary local phase factor. A gauge-invariant one-particle displacement operator \( R = iD \) can be introduced, which satisfies the commutation relations

\[
[R_i, p_j] = i \delta_{ij}, \quad [R_i, R_j] = -ieF_{ij}(p) \tag{9}
\]

where \( F_{ij} \) is anti-symmetric second-rank tensor of the displacement field strength

\[
F_{ij} = \frac{\partial}{\partial p_i} A_j - \frac{\partial}{\partial p_j} A_i = \varepsilon_{ijk} B_k, \tag{10}
\]

\( \varepsilon_{ijk} \) is the Levi-Civita symbol and \( B_k(p) \) labels the components of the background magnetic-like field. Gauge-invariant angular momentum operator can be introduced \( L = R \times p \), however its components do not satisfy canonical commutation relations

\[
[L_i, L_j] = i \varepsilon_{ijk} L_k - i e \varepsilon_{ikl} \varepsilon_{jmn} p_l p_m F_{km} \tag{11}
\]

and \( \{L_i\} \) are not generators of spatial rotations. The canonical angular momentum algebra \( [L_i, L_j] = i \varepsilon_{ijk} L_k \) can be restored when the background magnetic-like field is rotation-symmetric \( B(p) = B(p) \hat{p} \) by the transformation

\[
L = R \times p + ep^2 B(p) \hat{p}, \tag{12}
\]

The form-factor \( B(p) \) can be determined from the requirement that the displacement operator transforms as a vector \( [L_i, R_j] = i \varepsilon_{ijk} R_k \), which is only satisfied when \( B(p) = g/p^2 \), where \( g \) is a field-strength constant. The conserved gauge-invariant rotation operator is given by

\[
L = R \times p + eg \hat{p}. \tag{13}
\]

Eq. (13) is momentum-space analogue to the angular momentum operator \( r \times (p - eA(r)) - eg \hat{r} \) of a charged particle in an external magnetic field of point monopole charge of strength \( g \). The rotation symmetry restoration term \( eg \hat{p} \) is related to the generator of gauge transformations of the wave-function \( W = eg \hat{p} \cdot \mathbf{n} \) which compensates for the non-symmetric response of the gauge-field \( A(p) \) to rotations about the unit-vector \( \mathbf{n} \).

Analogously, a gauge-invariant extension of the generator of Galilei boost transformations can be based on the operators

\[
K = p t - R, \tag{14}
\]

where \( t \) is the time evolution parameter. The boost operator components do not commute

\[
[K_i, K_j] = [R_i, R_j] = -ieF_{ij}(p) \tag{15}
\]

but are simply related to the conventional generators by change of coordinates

\[
K \rightarrow K + eA = pt - R \tag{16}
\]

Under an infinitesimal boost transformation generated by the operators \( \{K_i\} \), the coordinates \( R \) change as

\[
\delta R = \delta r - e\delta A = i[\delta \mathbf{v}, K] = \delta \mathbf{v} t + \frac{eg}{p^2} \delta \mathbf{v} \times \hat{p}, \tag{17}
\]

The first term \( \delta \mathbf{v} t \) is the infinitesimal Galilei transformation, which is supplemented by a term, which describes an apparent rotation of the momentum \( \mathbf{p} \) about the direction of the boost \( \mathbf{v} \). The accompanying rotation effect is normally suppressed at high kinetic energies \( p^2 \gg eg \). The canonical coordinates \( r \) change in conventional way as \( \delta r = \delta \mathbf{v} t \), since the variation of the gauge-field \( \delta \mathbf{A} \) can be compensated by re-definition of the phase of the wave-function.

The representation of the modified boost operators in the Hilbert space of states is based on exponentials

\[
U(v) = \exp(-iv \cdot R), \tag{18}
\]

depending on a velocity vector \( v \), with the following action onto the wave-function

\[
U(v) \psi(p) = \exp(-iv \cdot R) \exp(i(v \cdot r) \psi(p + v). \tag{19}
\]

The product of the two exponentials can be expressed by a straight-line integral

\[
\exp(-iv \cdot R) \exp(iv \cdot r) = \exp \left( ie \int_p^{p + v} dk \cdot A(k) \right) \tag{20}
\]

that connects the points \( p \) with \( p + v \). The composition law for the generalized boost transformations takes the form

\[
U(v_1) U(v_2) = \exp[i \omega_2(p, v_1, v_2)] U(v_1 + v_2), \tag{21}
\]

where

\[
\omega_2(p, v_1, v_2) = e \int_{\Delta_2} dk \cdot A(k) \tag{22}
\]

is the Berry’s phase, which is the flux of the background magnetic-like field \( B(p) \) through the triangle \( \Delta_2 \) formed by the vertices of the momenta \( p, p + v_1 \), and \( p + v_1 + v_2 \). The phase factor violates associativity of the gauge-invariant boost transformations (cf. [10]), since

\[
[U(v_1) U(v_2)] U(v_3) = e^{i\omega_3(p, v_1, v_2, v_3)} U(v_1) U(v_2) U(v_3) \tag{23}
\]
where a three co-cycle phase $\omega_3$

$$\omega_3(p, v_1, v_2, v_3) = e \oint \oint \Delta_3 \mathbf{B} \cdot d\mathbf{S}$$  \hspace{1cm} (24)$$

is the flux of the background magnetic-like field through a tetrahedron $\Delta_3$ formed by the vertices of the momenta $p, p + v_1, p + v_1 + v_2$ and $p + v_1 + v_2 + v_3$. Associativity can be restored when a Dirac-type quantization condition is satisfied

$$\omega_3(p, v_1, v_2, v_3) = 2\pi n,$$  \hspace{1cm} (25)$$

Applying the Stokes theorem to Eq. (24) gives

$$e \int \int \int d^3k \nabla_k \cdot \mathbf{B}(k) = 4\pi eg = 2\pi n$$  \hspace{1cm} (26)$$

which implies quantization of the product of the two coupling constants $eg = n/2$.

Since the background magnetic-like field is rotation symmetric, it can not be written as $\mathbf{B} = \nabla_p \times \mathbf{A}(p)$ over the entire momentum space. Locally, we can look for a gauge field $\mathbf{A}(p)$ in the form

$$\mathbf{A}(p) = A(\theta)\nabla_p \varphi,$$  \hspace{1cm} (27)$$

where $(\theta, \varphi)$ are the spherical coordinates of the momentum $p = (p, \theta, \varphi)$, the equation $\mathbf{B} = \nabla_p \times \mathbf{A}$ is solved by

$$A(\theta) = -g(1 + \cos \theta)$$  \hspace{1cm} (28)$$

The gauge field $\mathbf{A}$ exhibits unremovable coordinate-type Dirac string singularity along the line $\theta = 0$. Singularity-free gauge fields can be defined on two overlapping momentum space patches

$$\mathbf{A}_N = \frac{g}{p} \frac{1 - \cos \theta}{\sin \theta} \hat{\varphi}, \quad R_N : 0 \leq \theta < \frac{\pi}{2} + \epsilon,$$

$$\mathbf{A}_S = -\frac{g}{p} \frac{1 + \cos \theta}{\sin \theta} \hat{\varphi}, \quad R_S : \frac{\pi}{2} - \epsilon < \theta \leq \pi$$  \hspace{1cm} (29)$$

where $\mathbf{A}_N$ is regular on the northern momentum-space hemi-sphere $R_N$, while $\mathbf{A}_S$ has support on the southern hemi-sphere $R_S$. Near the equator $R_N \cap R_S$, where the gauge-field has a discontinuity, the pair of potentials can be related by a gauge transformation

$$\mathbf{A}_S \rightarrow \mathbf{A}_S - ie^{-im\varphi} \nabla_p e^{im\varphi} = \mathbf{A}_N.$$  \hspace{1cm} (30)$$

Since the gauge-field is not globally defined, the rotation operator $\mathbf{L}$ is not globally defined either. The component of the rotation operator onto the space-fixed $z$-axis is two-valued, since

$$L_z = -i\partial_\varphi + eg, \quad (\theta, \varphi) \in R_N$$  \hspace{1cm} (31)$$

or

$$L_z = -i\partial_\varphi - eg, \quad (\theta, \varphi) \in R_S$$  \hspace{1cm} (32)$$

depending on the orientation of the wave-vector $p$. However, the rotation-symmetric term that restores conventional angular momentum algebra

$$s = eg\hat{p}$$  \hspace{1cm} (33)$$

is conserved and does not depend on the momentum space patching. It is related to the helicity of the particle by

$$\mathbf{L} \cdot \hat{p} = s \cdot \hat{p} = eg = n/2,$$  \hspace{1cm} (34)$$

which is quantized topologically with integer or half integer numbers. The free-particle wave-functions of definite helicity $\mu = eg$ are eigen-functions of the operators $\mathbf{L}^2$ and $L_z$. The square of the angular momentum operator in Eq. (23) for the northern patch is then given by

$$L^2|lm\mu\rangle = l(l + 1)|lm\mu\rangle, \quad L_z|lm\mu\rangle = m|lm\mu\rangle,$$  \hspace{1cm} (36)$$

for $l = |\mu|, |\mu| + 1, \ldots$ and $-l \leq m \leq l$. Wave-functions are given by sectional (spin-weighted) Wu-Yang monopole harmonics

$$Y_{lm\mu}(\theta, \varphi) = \langle \theta, \varphi|lm\mu\rangle$$  \hspace{1cm} (37)$$

or equivalently expressed by the Jacobi polynomials $P_n^{(\alpha, \beta)}(z)$

$$Y_{lm\mu}(\theta, \varphi) = N_{lm\mu}e^{i(\mu + m)\varphi}(1 - z)^{-(\mu + m)/2} \times (1 + z)^{-(\mu - m)/2} P_{l+m}^{\mu-m, -\mu-m}(z),$$  \hspace{1cm} (38)$$

where $z = \cos \theta$ and $N_{lm\mu}$ are normalization constants. The wave-functions $Y_{lm\mu}$ of half-integer angular momentum $\mu = n/2$ correspond to spinor representations of the rotation group, since they are related to the Wigner’s rotation functions by $Y_{lm\mu}(\theta, \varphi) = D_{lm\mu}^0(\varphi, \theta, \varphi)$. The total one-particle wave-function, that is an eigen-function of $H, \mathbf{L}^2, L_z$ is characterized by four quantum numbers and given by

$$\psi_{klm\mu}(p) = \frac{\delta(p - k)}{2K} Y_{lm\mu}(\hat{p}),$$  \hspace{1cm} (39)$$

where $k = \sqrt{2E}$ is a characteristic wave-number. When $\mu = 0$, these wave-functions reduce to the conventional spherical harmonics $Y_{lm}(\theta, \varphi)$. 
III. HELCITITY AND SPIN OF A FREE PARTICLE

The effects of the gauge-field can be expressed by the non-integrable phase factor

$$\exp \left(i \int_p^q \mathbf{k} \cdot \mathbf{A}(k) \right)$$

(40)

which accompanies the translation motion of the free-particle. The gauge-potential one-form $A = \mathbf{k} \cdot \mathbf{A}(k)$ is singular and can not be defined globally over the two-dimensional unit sphere $S^2$. A description that avoids gauge patching of the momentum space can be based on constructing a Hopf bundle [11] [12] over the two-dimensional unit sphere $S^2$. We further restrict our analysis to the minimal non-trivial helicity quantum numbers $\mu = +1/2$ and $\mu = -1/2$. A regular gauge potential one-form can be defined on the three-dimensional unit sphere $S^3$ in four-dimensional Euclidean space $\mathbb{R}^4$. The 3-sphere can be parametrized by four coordinates

$$
p_1 = \cos \frac{\theta}{2} \cos \alpha
\quad p_2 = \cos \frac{\theta}{2} \sin \alpha
\quad p_3 = \sin \frac{\theta}{2} \cos(\varphi + \alpha)
\quad p_4 = \sin \frac{\theta}{2} \sin(\varphi + \alpha)
$$

(41)

such that $p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1$. These four coordinates can be grouped into a pair of complex numbers $(z_1, z_2)$ as

$$z_1 = p_1 + ip_2 = \cos \frac{\theta}{2} e^{i\alpha}, \quad z_2 = p_3 + ip_4 = \sin \frac{\theta}{2} e^{i(\varphi + \alpha)}$$

(42)

These complex coordinates are related to the spherical coordinates $\mathbf{p}(\theta, \varphi)$ on $S^2$ by the Hopf projection map $\pi : S^3 \to S^2$

$$n_1 = z_1^* z_2 + z_2^* z_1 = \sin \theta \cos \varphi$$
$$n_2 = i(z_2^* z_1 - z_1^* z_2) = \sin \theta \sin \varphi$$
$$n_3 = |z_1|^2 - |z_2|^2 = \cos \theta$$

(43)

where $(n_1, n_2, n_3)$ are the Cartesian coordinates of the unit wave-vector $\mathbf{p}$. Since locally the 3-sphere has a product form $S^2 \times S^1$, the Hopf projection has the property to eliminate the dependence on the third angle $\alpha$ by mapping the unit circle $S^1$ parameterized by $\alpha$ to a single point on $S^2(\theta, \varphi)$. The pair of complex coordinates can be grouped into a two-component spinor to label the points on $S^3$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) e^{i\alpha} \\ \sin(\theta/2) e^{i(\varphi + \alpha)} \end{pmatrix}$$

(44)

In these coordinates the Hopf projection map in Eq. (43) is written more simply

$$\mathbf{p}(\theta, \varphi) = z^\dagger \sigma z,$$

(45)

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the three Pauli matrices. A globally-defined connection one-form in these coordinates is given by

$$\omega = -iz^\dagger dz = p_1 dp_2 - p_2 dp_1 + p_3 dp_4 - p_4 dp_3 = d\alpha + \frac{1}{2} (1 - \cos \theta) d\varphi$$

(46)

which looks like the Wu-Yang potential one-form on the northern patch supplemented by an exact one-form $d\alpha$. The $S^3$ one-form is invariant under global $U(2)$ transformations of the spinor coordinates, i.e.

$$z \to U z, \quad U^\dagger U = 1, \quad \omega \to \omega$$

(47)

The $SU(2)$ subgroup acts by the matrices

$$U(\Omega, n) = \exp(i\Omega \mathbf{n} \cdot \sigma),$$

(48)

and the result projected onto the two-sphere through the Hopf map as

$$z^\dagger(U^\dagger \sigma U)z = \mathbf{p} \cos \Omega + (n \times \mathbf{p}) \sin \Omega$$

(49)

generates rotation of the unit-vector $\mathbf{p}$ on an angle $\Omega$ about the vector $\mathbf{n}$. Though the group $\{ A \in SL(2, \mathbb{C}) | \det A = 1 \}$ of linear transformations of the spinor coordinates also acts on these states, the $S^3$ connection one-form $\omega$ is not invariant under such transformations. In orthogonal coordinates $(\theta, \xi = \varphi + \alpha)$ that diagonalize the metric on the unit 3-sphere

$$ds^2 = \frac{1}{4} d\theta^2 + \cos^2 \frac{\theta}{2} d\alpha^2 + \sin^2 \frac{\theta}{2} d\xi^2$$

(50)

the gauge-potential one-form in Eq. (46) is

$$\omega = \frac{1}{2} \omega_\theta d\theta + \cos \frac{\theta}{2} \omega_\alpha d\alpha + \sin \frac{\theta}{2} \omega_\xi d\xi,$$

(51)

non-singular with components

$$\omega_\theta = 0, \quad \omega_\alpha = \cos \frac{\theta}{2}, \quad \omega_\xi = \sin \frac{\theta}{2}$$

(52)

The corresponding gauge-field strength two-form

$$F = d\omega = -iz^\dagger \wedge dz = \frac{1}{2} \sin \theta d\theta \wedge d\varphi$$

(53)

is exact and closed two-form $(dF = 0)$ on $S^3$. It gives half of the volume of the two-dimensional unit sphere $S^2$ and therefore the topology of the 3-sphere can be characterized by the first Chern number

$$c_1 = \frac{1}{2\pi} \int_{S^2} F = 1.$$

(54)

which corresponds to a conserved helicity quantum number $\mu = +1/2$. The reduction to the Wu-Yang monopole gauge potentials over the 2-sphere can be obtained by taking sections of the 3-sphere. Local sections of $S^3$ can be taken if a particular value for the phase angle $\alpha$ is
fixed, such that the metric on the 3-sphere reduces to the metric on the 2-sphere $ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$ in the north $R_N: (\theta/2 \to \theta)$ and south $R_S: (\pi - \theta)/2 \to \theta$ hemispheres, respectively. These two choices correspond to fixing $\alpha = 0$ and $\alpha = -\varphi$ in Eq. (51) and leading to the Wu-Yang monopole potentials on the two local patches

$$A_N = \frac{1}{2}(1 - \cos \theta) d\varphi, \quad A_S = \frac{1}{2}(1 + \cos \theta) d\varphi,$$

respectively. Using the property of the Hopf projection map, the regular $S^3$ connection one-form can be written in more compact way in terms of projection operators

$$\omega = d\alpha z^\dagger \frac{1}{2}(1 + \sigma_3) z + d\xi z^\dagger \frac{1}{2}(1 - \sigma_3) z$$

(56)

where $\sigma_3 = \text{diag}(+1, -1)$ is the third Pauli matrix. The eigen-states of $\sigma_3$

$$\frac{1}{2}\sigma_3 |\sigma\rangle = \sigma |\sigma\rangle$$

(57)

for $\sigma = \pm 1/2$ are two-component spinors

$$\begin{align*}
(1,0)^T &= |+\rangle, \\
(0,1)^T &= |–\rangle
\end{align*}$$

(58)

which can represent the North $N = (0, 0, 1)$ and south poles $S = (0, 0, -1)$ on $S^2$, as follows from the property of the Hopf map (cf. Eq. (43)). In terms of these two states, the $S^3$ gauge-potential one-form

$$\omega = d\alpha |(+z)^2| + d\xi |(–z)^2|$$

(59)

is written as a sum of separate probabilities for an $S^3$ point $z$ to have two different signatures ±. For fixed $\theta$, the two circle variables $\xi, \alpha$ parameterize a one-dimensional complex torus $T^2 = S^1 \times S^1(e^{i\alpha}, e^{i\xi})$ in $S^3$, and hence locally $\omega$ is viewed as one-form over the torus. The changes of $\theta$ generate a family of tori $T^2(\theta)$. A linear equi-variation of the phase angles on a curve $\gamma(s) = (\alpha(s) = s, \xi(s) = s)$ gives

$$\omega = ds,$$

(60)

i.e. $\omega$ reduces locally to an exact one-form. The unit Euclidean four-momentum changes as

$$n(s) = (\cos \frac{\theta}{2} \cos s, \cos \frac{\theta}{2} \sin s, \sin \frac{\theta}{2} \cos s, \sin \frac{\theta}{2} \sin s)$$

(61)

and describes a helix which lies in the flat torus $p_1^2 + p_2^2 = \cos^2 \theta/2, p_3^2 + p_4^2 = \sin^2 \theta/2$. When the phase angle changes are not equi-variant $\alpha \neq \xi$, the helical paths the particle follows deform continuously. The two-component spinors $z(\theta, \xi, \alpha)$ can be used to represent a quantum state of additional spin-projection variable $\sigma$ that takes only two values $\sigma = \pm 1/2$. This is because, when the particle propagates north $\hat{p} = (0, 0, 1)$, and a local section onto the north hemisphere is taken

$$A_N = d\varphi |(–z)^2| = 0,$$

(62)

then $\sigma$ ”points” north, since $|(+z)^2| = 1$. Analogously, setting $\hat{p} = (0, 0, -1)$ and taking southern local section

$$-A_S = d\varphi |(+z)^2| = 0,$$

(63)

shows that the variable $\sigma$ is ”pointing” south $|(-z)^2| = 1$. In all cases, this can be written as

$$A_\sigma = \sigma d\varphi (1 - 2 \sigma \cos \theta)$$

(64)

where the discrete variable $\sigma = \pm 1/2$ labels the two-coordinate patches. Therefore four coordinates $(p, \theta, \varphi, \sigma)$ can be used to parameterize the space, where the wave-function takes values. The invariant property of the gauge-field that correlates the spin and momentum as given by Eqs. (52) and (63) can be made explicit, by noting that the sectional spin-states $\{z(\theta, \xi, \alpha), \alpha = 0, -\varphi\}$ defined locally over the two-sphere, are eigen-states of the operator of the helicity, i.e.

$$R_N : \frac{1}{2} \hat{p} \cdot z(\theta, \varphi, 0) = \mp \frac{1}{2} z(\theta, \varphi, 0),$$

(65)

and

$$R_S : \frac{1}{2} \hat{p} \cdot z(\theta, 0, -\varphi) = \mp \frac{1}{2} z(\theta, 0, -\varphi),$$

(66)

i.e. the helicity is conserved and can be identified with the first Chern number of the Hopf bundle $c_1/2 = +1/2$. That is because the flux of the “background magnetic-like field” $B$ through the 2-sphere is the flux of the local spin vector field $s(\theta, \varphi) = z^\dagger(\sigma/2) z = +\hat{p}(\theta, \varphi)/2$ ((cf. also Eq. (53))

$$\frac{1}{2\pi} \oint_{S^2} B \cdot dS = \int \int \frac{d\Omega}{2\pi} \hat{p} \cdot s(\theta, \varphi) = 1$$

(67)

where $d\Omega = \sin \theta d\theta d\varphi$ is the area element on the two-sphere and $\hat{p}$ is the outward surface normal. In terms of the sectional spin states $\{z(\theta, \varphi), \}$, the spin-gauge fields on $S^2$ can be written as

$$A(\sigma) = -\langle \sigma(\theta, \varphi) | i \nabla_p | \sigma(\theta, \varphi) \rangle$$

(68)

The angular momentum operator in Eq. (13) takes the simpler form

$$L = r \times p + \langle \sigma(\theta, \varphi) | \left( r \times p + \frac{1}{2} \sigma \right) | \sigma(\theta, \varphi) \rangle,$$

(69)

and makes explicit the underlying total angular momentum operator folded between the states $| \pm (\theta, \varphi) \rangle$

$$J = r \times p + \frac{1}{2} \sigma$$

(70)

to be the conventional kinematic angular momentum $l = r \times p$ supplemented by a non-kinematic angular momentum operator $\frac{1}{2} \sigma$ acting on the sectional spin states $\pm (\theta, \varphi)$. Therefore, the total gauge-invariant one-particle
wave-function has an adiabatic form and representable by a product of orbital and spin-dependent factor
\[ \psi(\pm)(\theta, \varphi) = Y^{(\pm)}_{lm}(\theta, \varphi) | \pm(\theta, \varphi) \]  
(71)
where \( Y_{lm}(\theta, \varphi) \) are the Wu-Yang wave-functions. The total wave-function is gauge-invariant, since when the \( U(1) \) phase of the sectional spinor \(| \pm(\theta, \varphi) \) is locally changed, the phase of the angular wave-function rotates oppositely, and the total wave-function remains gauge-invariant.

A dual Hopf bundle \( H_{-1}(= S^3) \) corresponding to definite helicity quantum number \( \mu = \epsilon g = -1/2 \), or equivalently first Chern number \( c_1 = -1 \) can be defined in terms of the conjugate left-handed spinors \( \tilde{z} = (-z_2^*, z_1^*) \), which satisfy \( \sigma \cdot \tilde{p}\tilde{z}(\theta, \varphi) = -\tilde{z}(\theta, \varphi) \). Then the flux of the local spin vector field is
\[ \oint \oint \mathbf{B} \cdot d\mathbf{s} = \int \int d\Omega \hat{p} \cdot \mathbf{s}(\theta, \varphi) = -2\pi, \]
(72)
and that is why the flux of the "background magnetic-like field" has a negative sign, since the spin-vector \( \mathbf{s} \) points oppositely to the particle momentum \( \mathbf{p} \). Though the displacement operator \( \nabla_p \) can couple points of different helicity \( \mu = \pm 1/2 \), these states exhibit opposite Chern character \( c_1 = +1 \) and \( c_1 = -1 \) and are topologically distinct. For free-particle states, the helicity \( \mu = \pm 1/2 \) is conserved.

It can be pointed out, that the one-particle formalism can be generalized to a system of \( N \) non-interacting particles, when the momentum space is \( 3N \) dimensional. The description involves a gauge-potential one-form
\[ A = dp^\mu A_\mu(p) \]
(73)
over \( \mathbb{R}^{3N} \), where \( (\mu = 1, 2, \ldots, 3N) \) and \( p = (p_1, p_2, \ldots, p_{3N}) \) labels the points in \( \mathbb{R}^{3N} \). However, different and more complex cooperative effects occur, since \( A_\mu \) correlates the actions of single-particle angular momenta operators in non-trivial way. In particular, this cooperative effect, could describe the effect of particle inter-change and statistics.

IV. SCREENING

When there is an external potential field \( U(\mathbf{r}) \) present, the single-particle Hamiltonian is
\[ H = \frac{1}{2} \mathbf{p}^2 + U(\mathbf{r}) \]
(74)
and the corresponding Schrödinger equation for the eigen-states is
\[ H|\Psi\rangle = E|\Psi\rangle, \]
(75)
By assuming that helicity is conserved adiabatically, i.e. assume that a selection-rule \( \Delta \mu = 0 \) is satisfied. That is because the external field \( U(\mathbf{r}) \) is assumed to be topologically trivial, such that it can not change the helicity of the particle \( \mu \). On each patch, the total spin wave-function exhibits an adiabatic product form
\[ \Psi(\mathbf{p}) = \psi(\mathbf{p}) |z(\theta, \varphi)\rangle \]
(76)
and that is why the patch label is suppressed. The orbital part of the wave-function can be expanded over complete set of monopole wave-functions
\[ \psi(\mathbf{p}) = \sum_{lm} F_{lm}(\mu)p Y_{lm}(\theta, \varphi) \]
(77)
where \( \mu = \epsilon g \) is the helicity, \( l = |\mu|, |\mu| + 1, \ldots \) is the orbital angular momentum quantum number and \( m = -l, \ldots, l \) is the azimuthal quantum number. Expanding the external potential over Fourier components
\[ U(\mathbf{r}) = \sum_{\mathbf{q}} U(\mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}} \]
(78)
substituting Eq. (76) into Eq. (75), and projecting the result onto a helicity eigenstate, we obtain the equation for the orbital part of the wave-function
\[ (p^2/2 - E)\psi_E(\mathbf{p}) + \sum_{\mathbf{q}} U(\mathbf{q}) F_p(\mathbf{q}) \psi_E(\mathbf{p} - \mathbf{q}) = 0, \]
(79)
and a form-factor has been introduced
\[ F_p(\mathbf{q}) = \langle z(\mathbf{p})|e^{i\mathbf{q} \cdot \mathbf{r}}|z(\mathbf{p})\rangle = \langle z(\mathbf{p})|z(\mathbf{p} - \mathbf{q})\rangle, \]
(80)
which has the effect to screen the Fourier components of the external potential \( U(\mathbf{q}) \rightarrow U_{eff}(\mathbf{q}, \mathbf{p}) = U(\mathbf{q}) F_p(\mathbf{q}) \). The spin-dependent form-factor can be computed from the function
\[ F(t) = \langle z(\mathbf{p})|z(\mathbf{p}(t))\rangle \]
(81)
on a straight line path \( \mathbf{p}(t) = \mathbf{p}(1-t) + (\mathbf{p} - \mathbf{q})t, (0 \leq t \leq 1) \) inter-connecting the wave-vectors \( \mathbf{p} \) and \( \mathbf{q} \). The function \( F(t) \) satisfies the initial condition \( F(t = 0) = 1 \) and \( F(t = 1) = F_p(\mathbf{q}) \) gives the form-factor. Differentiating Eq. (81) with respect to the parameter \( t \) gives
\[ \dot{F}(t) = \dot{\mathbf{p}} \cdot (z(\mathbf{p})|\nabla_\mathbf{p}|z(\mathbf{p}(t))) \]
(82)
and using that \( |z(\mathbf{p}(t))| = F(t)|z(\mathbf{p}(0))| \), an equation for the Berry’s phase is obtained
\[ \dot{F}(t) = \frac{1}{i} \dot{\mathbf{p}} \cdot A(\mathbf{p}(t)) F(t), \]
(83)
where \( A(\mathbf{p}) = (z(\mathbf{p})|\nabla_\mathbf{p}|z(\mathbf{p})) \) is the gauge field. Eq. (83) can be integrated along the straight line to give at the end point \( t = 1 \) the form-factor
\[ F(t = 1) = F_{q}(\mathbf{p}) = \exp \left( i \int_{\mathbf{p} - \mathbf{q}}^{\mathbf{p}} d\mathbf{k} \cdot A(\mathbf{k}) \right). \]
(84)
The Schrödinger equation reduces to a pair of coupled equations for the wave-functions on the patches

$$(p^2/2 - E)\psi_N(p) + \sum_{q \in R_N} U(p - q) F_N(p, q) \psi_N(q) + \sum_{q \in R_S} U(p - q) F_{NS}(p, q) \psi_S(q) = 0$$

and

$$(p^2/2 - E)\psi_S(p) + \sum_{q \in R_S} U(p - q) F_S(p, q) \psi_S(q) + \sum_{q \in R_N} U(p - q) F_{SN}(p, q) \psi_N(q) = 0$$

(85)

where the patch label $N(S)$, instead of $\sigma = \pm 1/2$, is used. The form-factor for the northern patch is

$$F_N(p, q) = \exp \left( i \int_q^p dk \cdot A_N(k) \right),$$

and similarly on the southern patch it is

$$F_S(p, q) = \exp \left( i \int_q^p dk \cdot A_S(k) \right).$$

These form-factors can be evaluated from the overlap of the sectional spinors $\langle \pm(q) | \pm(p) \rangle$, for instance

$$F_N(q, p) = \cos \frac{\theta}{2} \cos \frac{\theta'}{2} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2} e^{i(\varphi - \varphi')}.$$  

(88)

where $q = (q, \theta', \varphi')$ and $p = (p, \theta, \varphi)$. When the integration path crosses the equator, the spin projection $\sigma$ onto the $z$-axis flips, i.e. the coordinate patches interchange, and the phase-factor is evaluated from (cf. also Ref. [13])

$$F_{NS}(p, q) = F_N(p, k_E) e^{i\varepsilon_{NS}(p,q)} F_S(k_E, q)$$

(89)

Here $k_E$ is an equatorial vector in the overlap region of the two patches. It is determined from the crossing of the straight line interconnecting the pair $(q, p)$ with the equatorial plane ($\theta = \pi/2$). Explicitly $k_E$ is given by

$$k_E = -q \frac{p_z - q_z}{p_z - q_z} + p \frac{q_z}{p_z - q_z} = (p \times q) \times n \frac{p \cdot n - q \cdot n}{p \cdot n - q \cdot n}$$

(90)

where $n = (0, 0, 1)$ is the unit-vector pointing along the $z$-axis. The azimuthal angle $\varphi_{NS}$ is given by

$$\tan \varphi_{NS}(p, q) = \frac{k_y}{k_x} = \frac{p_y q_z - q_y p_z}{p_z q_x - q_z p_x}$$

(91)

The spin-transition form-factor $F_{SN}$ can be obtained from symmetry relation $F_{SN}(p, q) = F_{NS}^*(-p, q)$.

The matrix elements $U_{lm\pm l'm'}(p, q)$ of the screened potential are evaluated in basis of monopole harmonics $Y_{lm\pm l}^{\pm}(\theta, \varphi)$ and coupled integral equations for the partial-wave amplitudes $F_{lm}(p)$ must be integrated numerically. This mathematical formalism can be applied to the case of simple harmonic oscillator potential $U(r) = r^2/2$, when numerical computation is not needed. The effective Hamiltonian for this particular case is

$$H_{\text{eff}} = \frac{1}{2} (i \nabla_p - A(p))^2 + \frac{1}{2} p^2$$

(92)

and can be diagonalized in the basis of the spin-weighted Wu-Yang monopole harmonics, i.e.

$$\psi(p) = Y_{lm\pm l}(\theta, \varphi) F_l(p).$$

(93)

where $\mu = \pm 1/2$ is the helicity, $l = 1/2, 3/2, \ldots$ is the orbital angular momentum quantum number and $m = -l, -l + 1, \ldots, l$. The effective Hamiltonian for radial motion is

$$H_l(p) = \frac{1}{2} \frac{d^2}{dp^2} e^{ilp} - \frac{l}{2} + \frac{l + 1}{2} - 1/4 + \frac{1}{2} p^2,$$

(94)

i.e. the gauge-field only changes the effective centrifugal barrier for radial motion. The wave-functions of Eq.(84) are analytic and given by means of generalized Laguerre polynomials

$$F_{ul}(p) = p^{l^*} e^{-p^2/2} L_{l^*+1/2}^{l^*}(p^2),$$

(95)

where $l^* = \sqrt{l(l+1)} - 1/2$ is an effective fractional angular momentum and $u = 0, 1, 2, \ldots$ is a vibrational quantum number, which counts the nodes of the momentum-space wave-functions. The energy levels of the helicity-carrying oscillator eigen-states are $l$-dependent

$$E_{ul} = 2v + \sqrt{l(l+1) + 1}$$

(96)

and for a given $v$, levels are $(2l+1)$-fold degenerate, corresponding to their independence on the magnetic quantum number $m$. In the simplest case, when the particle is spin-less $\mu = eg = 0$, the harmonic oscillator energy levels are given by $2v + l_0 + 3/2$ for $l_0 = 0, 1, 2, \ldots$. These states can be labeled by a single quantum number $N = 2v + l_0$. Each level of principal quantum number $N$ is $(N+1)(N+2)/2$-fold degenerate. In opposite, when the particle is carrying a helicity $\mu = \pm 1/2$, this degeneracy is lifted, and states can not be classified by a single quantum number $N$. This is because the effective orbital angular momentum $l^* = \sqrt{l(l+1)} - 1/2$ is fractional. Therefore the principal effect of the gauge-field is to lift degeneracy of conventional harmonic oscillator energy levels, which split depending on both the vibration quantum number $v = 0, 1, 2, \ldots$ and the orbital angular momentum quantum number $l = 1/2, 3/2, \ldots$

In the case of hydrogen atom, represented by a Coulomb potential $Ze^{-1}$, similar effect of splitting of the energy levels can occur due to the screening of the Coulomb field by the helicity-carrying particle. We expect that the effect of energy-level splitting is small, as the observed fine-structure of energy levels of hydrogen shows, and numerical computation must be made in order to verify if such an effect of spin-dependent screening is small or negligible.
V. CONCLUSION

The local phase invariance of the momentum-space Schrödinger equation has been used to describe the motion of a non-relativistic particle with spin and helicity. As a byproduct, effective one-particle Schrödinger equation of motion in external field is derived, which predicts an effect of spin-dependent screening of the external potential. The approach is applied for simple harmonic oscillator potential and shown that the effect of screening affects rotation energy level splittings.

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