Derivation of Cameron-Liebler line classes

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Abstract

We construct a new infinite family of Cameron-Liebler line classes in PG(3, q) with parameter $x = \frac{q^2 + 1}{2}$ for all odd $q$.

1 Introduction

Let PG(3, q) denote the 3-dimensional projective space over the finite field $\mathbb{F}_q$. For a set $\mathcal{L}$ of lines in PG(3, q), let $\overline{\mathcal{L}}$ denote the complementary set of lines. A spread of PG(3, q) is a set of $q^2 + 1$ lines that partition the set of points.

We say that $\mathcal{L}$ is a Cameron-Liebler line class with parameter $x$ in PG(3, q), if there exists a non-negative integer $x$ such that, for every spread $S$ of PG(3, q), one has:

$$|S \cap \mathcal{L}| = x.$$
It can be seen from the definition that \( L \) is then a Cameron-Liebler line class with parameter \( q^2 + 1 - x \), so that we may assume \( x \leq \frac{q^2+1}{2} \). An empty set of lines \( (x = 0) \), the set of all lines in a plane \( (x = 1) \) or, dually, through a point \( (x = 1) \) are trivial examples of Cameron-Liebler line classes. If the point is not in the plane, then the union of the previous two examples with \( x = 1 \) gives a slightly less trivial Cameron-Liebler line class with parameter \( x = 2 \).

Cameron-Liebler line classes first appeared in the study [3] of collineation groups of \( \text{PG}(n, q) \), \( n \geq 3 \), that have equally many orbits on lines and on points (and were given their name in [12]). Under the Klein correspondence, Cameron-Liebler line classes are translated to tight sets of the Klein quadric being, thus, a special case of a tight set of a polar space (see [5, 1]). For more comprehensive background on this topic, we refer to the recent papers [7], [9], [11], [8], [1].

It was conjectured in [3] that the only Cameron-Liebler line classes are the examples mentioned above, i.e., \( x \leq 2 \). The first counterexample was found by Drudge [6] in \( \text{PG}(3, 3) \) with \( x = 5 \), which was generalised later by Bruen and Drudge [2] to an infinite family having parameter \( x = \frac{q^2+1}{2} \) for all odd \( q \). The first counterexample in characteristic 2 was found in [10]. With the aid of computer and using some clever ideas about possible symmetries of Cameron-Liebler line classes, Rodgers [14] constructed many more new examples for certain \( x \) and prime powers \( q \). Very recently, some of them have been shown in [11], [7] to be a part of a new infinite family of Cameron-Liebler line classes with parameter \( x = \frac{q^2+1}{2} \) for \( q \equiv 5 \) or \( 9 \) (mod 12). (In fact, a line class of the family found in [1], [7] has parameter \( \frac{q^2-1}{2} \), however, it is disjoint with a plane, which is a Cameron-Liebler line class with parameter \( \frac{q^2-1}{2} + 1 \).)

In this note, we first describe a switching-like operation in Cameron-Liebler line classes that satisfy some necessary conditions (see Lemma 2.1). We then show in Lemma 2.3 that these conditions may only hold for line classes with \( x = q^2 \) or \( x = \frac{q^2+1}{2} \). Applying this switching operation to the line classes found by Bruen and Drudge, we construct another infinite family of Cameron-Liebler line classes in \( \text{PG}(3, q) \) with parameter \( x = \frac{q^2+1}{2} \) for all odd \( q \), and show that they are not equivalent to the line classes of Bruen and Drudge, unless \( q = 3 \) (see Theorem 3.3).

## 2 Switching in Cameron-Liebler line classes

For a point \( P \) and a plane \( \pi \) of \( \text{PG}(3, q) \), let \( \text{Star}(P) \) and \( \text{Line}(\pi) \) denote the set of all lines on \( P \) or in \( \pi \), respectively.

**Lemma 2.1** Let \( \mathcal{L} \) be a Cameron-Liebler line class such that there exists an incident point-plane pair \( (P, \pi) \) satisfying the following conditions:

1. \( (\text{Line}(\pi) \setminus \text{Star}(P)) \cap \mathcal{L} = \emptyset \),
2. \( \text{Star}(P) \setminus \text{Line}(\pi) \subseteq \mathcal{L} \).


Then
\[ L' := L \cup (\text{Line}(\pi) \setminus \text{Star}(P)) \setminus (\text{Star}(P) \setminus \text{Line}(\pi)) \]
is a Cameron-Liebler line class with the same parameter.

**Proof:** For any spread \( S \) of \( \text{PG}(3, q) \) we have that \( S \) contains either a line of \( \text{Star}(P) \cap \text{Line}(\pi) \), or a line \( \ell \in \text{Line}(\pi) \setminus \text{Star}(P) \) and a line \( m \in \text{Star}(P) \setminus \text{Line}(\pi) \). In the former case, \( S \cap L = S \cap L' \), while in the latter case \( S \cap L' = (S \cap L) \cup \{m\} \setminus \{\ell\} \). Thus, \( |S \cap L'| = |S \cap L| \) holds in both cases, and so \( L' \) is a Cameron-Liebler line class.

Let \( L \) be a Cameron-Liebler line class, and \( \ell \) a line of \( \text{PG}(3, q) \). Then \( \ell \) lies in \( q + 1 \) planes \( \pi_1, \ldots, \pi_{q+1} \) and contains \( q + 1 \) points \( P_1, \ldots, P_{q+1} \). Define the square matrix \( T(\ell) = (t_{ij}) \) of size \( q + 1 \) with integer entries given by
\[ t_{ij} := |((\text{Line}(\pi_i) \setminus \text{Star}(P_j)) \setminus \{\ell\}) \cap L|, \quad 1 \leq i, j \leq q + 1. \]

The set consisting of the matrix \( T \), and every matrix obtained from this one by a permutation of the rows and a permutation of the columns is called the pattern of \( \ell \) with respect to \( L \). We represent each pattern by one of its matrices. This concept was introduced in [9], where the following result has been proved.

**Proposition 2.2** Let \( L \) be a Cameron-Liebler line class with parameter \( x \), let \( \ell \) be a line of \( \text{PG}(3, q) \), and \( T = (t_{ij}) \) the pattern of \( \ell \).

(a) For any \( i \in \{1, \ldots, q + 1\} \)
\[ \sum_{j=1}^{q+1} t_{ij} = |\text{Line}(\pi_i) \cap L \setminus \{\ell\}| \quad \text{and} \quad \sum_{j=1}^{q+1} t_{ji} = |\text{Star}(P_i) \cap L \setminus \{\ell\}|. \]

(b) For all \( k, l \in \{1, \ldots, q + 1\} \)
\[ \sum_{i=1}^{q+1} t_{il} + \sum_{j=1}^{q+1} t_{kj} = \begin{cases} x + (q + 1)t_{kl} & \text{if } \ell \notin L \\ x + (q + 1)(t_{kl} + 1) - 2 & \text{if } \ell \in L. \end{cases} \]

(c) \( t_{kl} + t_{rs} = t_{ks} + t_{rl} \) for all \( k, l, r, s \in \{1, \ldots, q + 1\} \).

(d)
\[ \sum_{i,j=1}^{q+1} t_{ij}^2 = \begin{cases} q^3 + q^2 + (x - 1)^2 + q(x - 1) & \text{if } \ell \notin L \\ q^2 + q(x + x) & \text{if } \ell \in L. \end{cases} \]

**Lemma 2.3** Let \( L \) be a Cameron-Liebler line class such that there exists an incident point-plane pair \((P, \pi)\) satisfying the conditions of Lemma 2.1. Then the parameter \( x \) of \( L \) is equal to \( q^2 \) or \( \frac{q^2 + 1}{2} \).
Proof: Up to taking the complement to a line set and the point-plane duality in PG(3, q), we may assume that there exists a line $\ell$ of $\text{Star}(P) \cap \text{Line}(\pi) \setminus \mathcal{L}$. Let $T$ be the pattern of $\ell$ such that, without loss of generality, its first row corresponds to $\pi$, and its first column corresponds to $P$. Then the conditions of Lemma 2.1 imply that $t := t_{11} = |\text{Star}(P) \cap \text{Line}(\pi) \setminus \mathcal{L}|$, and $t_{1,j} = q$ and $t_{j,1} = 0$ for all $j \in \{2, \ldots, q+1\}$. By Proposition 2.2 (c), we see that $t_{ij} = q - t_{11}$ for all $i, j \in \{2, \ldots, q+1\}$.

Further, Proposition 2.2 (b) applied to the first row and column of $T$, and Proposition 2.2 (d) applied to the pattern $T$ give the following equations:

$$\begin{cases} t + q^2 + t = x + t(q + 1), \\ t^2 + q^3 + q(t - t)^2 = x(q + x), \end{cases}$$

which yield $t = 0$ and $x = q^2$ (and thus $\mathcal{L}$ is the complement to a Cameron-Liebler line class with parameter 1), or $t = \frac{q^2 + 1}{2}$ and $x = \frac{q^2 + 1}{2}$.

3 Application of switching

From Lemma 2.3 we see that the only non-trivial case, where the switching operation of Lemma 2.1 may be applied, is the case $x = \frac{q^2 + 1}{2}$. There exist at least two infinite families of Cameron-Liebler line classes with parameter $x = \frac{q^2 + 1}{2}$, namely, the first counterexamples to the Cameron-Liebler conjecture constructed by Bruen and Drudge in [2] and the line classes recently found in [1] and independently in [7]. Fortunately, the former satisfy the conditions of Lemma 2.1 (while the latter do not), and applying the switching operation indeed produces a new Cameron-Liebler line class, not equivalent to the original one, if $q > 3$. In this section we give the necessary details.

First of all, let us recall the construction by Bruen and Drudge. Let $q$ be an odd prime power, and $\mathcal{Q}$ an elliptic quadric of PG(3, q) with the corresponding quadratic form $Q$. The set of $q + 1$ tangents $\mathcal{T}_P$ to a point $P \in \mathcal{Q}$ can be divided into two subsets, say $\mathcal{T}_P^1, \mathcal{T}_P^2$, of size $(q + 1)/2$ each, depending on whether a tangent line contains a point $P' \neq P$ such that $Q(P')$ is a square in $\mathbb{F}_q$. Note if $Q(P')$ is a square in $\mathbb{F}_q$, then all the points on the tangent $PP'$ satisfy this property, as $Q(P + cP') = c^2 Q(P')$.

Denote by $\mathcal{T}_i^j$ the set $\cup_{P \in \mathcal{Q}} \mathcal{T}_P^i$, $i \in \{1, 2\}$. Let $S$ and $E$ be the sets of secant and external lines to $\mathcal{Q}$, respectively. Then any of

$$S \cup \mathcal{T}_i, \ E \cup \mathcal{T}_j, \ i, j \in \{1, 2\},$$

is a Cameron-Liebler line class of parameter $\frac{q^2 + 1}{2}$.

Since all these line classes are equivalent under the action of PTL(4, q) and the polarity induced by $Q$ (see [5]), we may choose, without loss of generality, $\mathcal{L}$ to be $S \cup \mathcal{T}$1. For a point $P_1$ of $\mathcal{Q}$ and its tangent plane $\tau_{P_1}$, one can see that

$$(\text{Line}(\tau_{P_1}) \setminus \text{Star}(P_1)) \subset E \subset \overline{\mathcal{L}}, \ \text{Star}(P_1) \setminus \text{Line}(\tau_{P_1}) \subset S \subset \mathcal{L},$$

so that $(P_1, \tau_{P_1})$ satisfies the condition of Lemma 2.1 and the line class $\mathcal{L}'$ defined by

$$\mathcal{L}' := \mathcal{L} \cup (\text{Line}(\tau_{P_1}) \setminus \text{Star}(P_1)) \setminus (\text{Star}(P_1) \setminus \text{Line}(\tau_{P_1})).$$
is a Cameron-Liebler line class with parameter $\frac{q^2+1}{2}$.

Our aim now is to show that $\mathcal{L}'$ is not equivalent to $\mathcal{L}$ unless $q = 3$. For $q = 3$, we can either apply Drudge’s classification of Cameron-Liebler line classes in $\text{PG}(3, 3)$ [6] that determined that, up to equivalence, there is a unique Cameron-Liebler line class with parameter 5, or it can be checked with the aid of computer that $\mathcal{L}'$ is projectively equivalent to $\mathcal{L}$ for this value of $q$. From now on, we assume that $q > 3$.

**Lemma 3.1** A plane $\pi$ of $\text{PG}(3, q)$ contains $\frac{q+1}{2}$, or $\frac{q(q+1)}{2}$, or $\frac{(q+1)(q+2)}{2}$ lines of $\mathcal{L}$.

**Proof:** If $\pi$ is a tangent plane to $\mathcal{Q}$, then $|\text{Line}(\pi) \cap \mathcal{L}| = \frac{q+1}{2}$ by the construction of $\mathcal{L}$. Suppose that $\pi$ is a secant plane so that $\pi \cap \mathcal{Q}$ is a conic. Under the polarity, say $\rho$, induced by $\mathcal{Q}$, every tangent line to the conic in $\pi$ is mapped to a tangent line to $\mathcal{Q}$ on $\rho(\pi)$. Therefore, all tangent lines to the conic in $\pi$ are either in $\mathcal{T}^1$ or in $\mathcal{T}^2$. In the former case, $\pi$ contains $(\frac{q+1}{2}) + q + 1$ lines from $\mathcal{L}$, in the latter case $|\text{Line}(\pi) \cap \mathcal{L}| = \left(\frac{q+1}{2}\right)$.

**Lemma 3.2** A point $P$ of $\text{PG}(3, q)$ is on $q^2 + \frac{q+1}{2}$, or $\frac{q(q+1)}{2}$, or $\frac{q(q+1)}{2} + 1$ lines of $\mathcal{L}$.

**Proof:** If $P \in \mathcal{Q}$, then $|\text{Star}(P) \cap \mathcal{L}| = \frac{q+1}{2} + q^2$ by the construction of $\mathcal{L}$. Suppose that $P \notin \mathcal{Q}$. If $P$ is on a tangent line from $\mathcal{T}^i$ for $i \in \{1, 2\}$, then all tangent lines to $\mathcal{Q}$ through $P$ are in $\mathcal{T}^i$. Let $P'$ be a point of $\mathcal{Q}$ such that $PP'$ is a tangent line to $\mathcal{Q}$, and consider all secant planes $\pi_1, \ldots, \pi_q$ containing the line $PP'$. Recall that every point not on a conic in a projective plane of odd order lies on 0 or 2 tangents, see [13, 15]. Since $\pi_i \cap \mathcal{Q}$ is a conic, and $PP'$ is a tangent line to the conic, we conclude that $P$ lies on 2 tangents and $\frac{q-1}{2}$ secants to $\pi_i \cap \mathcal{Q}$. Thus, $|\text{Star}(P) \cap \mathcal{L}| = \frac{q(q-1)}{2}$, if $PP' \in \mathcal{T}^2$, or $|\text{Star}(P) \cap \mathcal{L}| = \frac{q(q-1)}{2} + q + 1$, if $PP' \in \mathcal{T}^1$.

**Theorem 3.3** The line classes $\mathcal{L}$ and $\mathcal{L}'$ are not equivalent under the action of $\text{PGL}(4, q)$ or a duality.

**Proof:** Following the notation from the above, one can see that the plane $\tau_{P_1}$ contains $\frac{q+1}{2} + q^2$ lines of $\mathcal{L}'$. Since, for a point $P_2 \in \mathcal{Q}$, $P_2 \neq P_1$, one has $\tau_{P_1} \cap \tau_{P_2} \in \mathcal{E}$, the plane $\tau_{P_2}$ contains $\frac{q+1}{2} + 1$ lines of $\mathcal{L}'$. It now follows from Lemmas 3.1, 3.2 that the intersection numbers of $\mathcal{L}'$ with respect to planes and points of $\text{PG}(3, q)$ are different from those of $\mathcal{L}$ or $\overline{\mathcal{L}}$.

We also note that $\mathcal{L}'$ is not equivalent to a line class of the family found in [1], [7], since there is no plane (or, dually, a point with all lines on it) contained in or disjoint from $\mathcal{L}'$. In particular, in $\text{PG}(3, 5)$, there exist at least three pairwise non-equivalent Cameron-Liebler line classes with $x = \frac{q^2+1}{2} = 13$ (namely, the example by Bruen and Drudge, its switched mate by Theorem 3.3 and the example found in [7] and [11]). In fact, up to equivalence, these are the only Cameron-Liebler line classes with given $x$ in $\text{PG}(3, 5)$ (the details will be given elsewhere).
The line class $L'$ contains only the one incident point-plane pair, namely, $(P_1, \tau_{P_1})$, satisfying the conditions of Lemma 2.1 and, clearly, switching of $L'$ with respect to it gives the line class $L$. Since, for $q > 3$, there is a unique switched mate for $L'$ (namely, $L$), it follows that its stabiliser $G_{L'}$ is a subgroup of the stabiliser $G_L$. The stabiliser $G_L$ of a Bruen-Drudge line class is a subgroup of index two of $\mathrm{PGL}(4, q)$, i.e., the subgroup that fixes $T^1$ and $T^2$. Thus, $G_{L'}$ is the stabiliser of the point $P_1$ in $G_L$. Then, for $q = p^h$, where $p$ is a prime, $G_{L'}$ has order $q^2(q^2 - 1)h$, and is isomorphic to $\mathrm{AGL}(1, q^2) \rtimes C_h$.

We expect that the only non-trivial Cameron-Liebler line classes satisfying the conditions of Lemma 2.1 are the examples of Bruen and Drudge and their switched mates.

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