Entanglement in Three Coupled Harmonic Oscillators

Abdeldjalil Merdaci\textsuperscript{a} and Ahmed Jellal\textsuperscript{b}

\textsuperscript{a}Physics Department, College of Science, King Faisal University, PO Box 380, Alahsa 31982, Saudi Arabia
\textsuperscript{b}Laboratory of Theoretical Physics, Faculty of Sciences, Chouaib Doukkali University, PO Box 20, 24000 El Jadida, Morocco

Abstract

We develop an approach in solving exactly the problem of three-body oscillators including general quadratic interactions in the coordinates for arbitrary masses and couplings. We introduce a unitary transformation of three independent angles to end up with a diagonalized Hamiltonian. Using the representation theory of the group SU(3), we explicitly determine the solutions of the energy spectrum. Considering the ground state together with reduced density matrix, we derive the corresponding purity function that is giving rise to minimal and maximal entanglement under suitable conditions. The cases of realizing one variable among three is discussed and know results in literature are recovered.

PACS numbers: 03.67.Bg, 03.65.-w, 02.20.Sv

Keywords: Three coupled harmonic oscillators, group SU(3), representation theory, reduced density matrix, purity function.
1 Introduction

Historically, the notion of entanglement has been related to the most quantum mechanical exotic concepts such as Schrödinger cat [1], Einstein-Podolsky-Rosen paradox [2] and violation of Bell’s inequalities [3]. Despite its conventional significance, entanglement has gained, in the last decades, a renewed interest mainly because of the development of the quantum information science [4]. It has been revealed that it lies at the heart of various communication and computational tasks (called measurement-based quantum computation) that cannot be implemented classically. It is believed that entanglement is the main ingredient of the quantum speed-up in quantum computation [4]. Moreover, several quantum protocols such as teleportation, quantum dense coding, and so on [5–11] are exclusively realized with the help of entangled states.

Entanglements of many particles (three or more) are fascinating quantum systems, especially when the entanglement is maximal [12]. In fact, from a large entangled state of many parts of the system and performing some measurements on certain parts of such state it turns out that one can get some information about the state of the rest of the system. In general, it is not easy to analyze the entanglement of three or more particles because of the complicity of the problem. Indeed, for a system of two qubits, it is easy to decide whether the system is entangled or not and here the positivity of the partial trace is a necessary and sufficient condition for separability. However for a system of three qubits described by the states \((\psi_a, \psi_b, \psi_c)\) things start to be little bit complicated, because one has to consider three bipartitions of the whole system and look at their separabilities. Now, it may happen that the state \(\psi_a\) will be separable with respect to the first partition but not to the last partition of the state \(\psi_c\). One might even find that all three partial traces are positive but the separability of all three qubits In principle, one can have states completely inseparable, separable with respect to one or two bipartitions, states separable with respect to all three bipartitions but not completely separable, and fully separable states.

We are interested to the entanglement of three-body system and before doing so, let us mention some relevant works. Indeed, a general scheme and realizable procedures for generating three particle entanglements out of just two pairs of entangled particles from independent emissions was studied [12]. The dynamics of mixedness and entanglement was examined by solving the time-dependent Schrödinger equation for three coupled harmonic oscillator system with arbitrary time-dependent frequency and coupling constants parameters, assuming that part of oscillators is inaccessible and remaining oscillators accessible [13]. On the other hand, an analysis of three coupled oscillators composed of three-mode interaction (Stokes, anti-Stokes and phonon) was developed using the representation theory of a Lie algebra [14].

Motivated by [12–14], we consider a system of three coupled harmonic oscillators and study the entanglement. After some transformations and using the \(SU(3)\) representation, we solve the eigenvalue equation to obtain the solutions of the energy spectrum. By extracting the ground state wavefunction, we determine the corresponding reduced density matrix and then the purity function in terms of the physical parameters. We show that our system can be minimally and maximally entangled under suitable conditions. To show the validity of our findings, we recover already published significant results concerning two coupled harmonic oscillators [15]. This will be done by distinguishing three different cases according to choices of the frequencies limits and adequate variables.
The present paper is organized as follows. In section 2, we consider the Hamiltonian of three coupled harmonic oscillators and introduce relevant transformations. To get the eigenvalues and eigenstates, we use the representation theory of the group $SU(3)$ in section 3. Section 4 deals with the entanglement in ground state, which is done by calculating the purity function and discuss its minimal and maximal values. In section 5, we study interesting limits to show the relevance of our results. We conclude our results in the final section and finish with two Appendices (A: for representation of $SU(3)$, B: for the two remaining limits).

2 Hamiltonian model

We consider a system of three coupled harmonic oscillators of different masses ($m_1, m_2, m_3$), frequencies ($\omega_1, \omega_2, \omega_3$) and couplings ($D_{12}, D_{13}, D_{23}$). It is described by the Hamiltonian

$$H_1 = \frac{1}{2} \left( \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_3} + m_1\omega_1^2 x_1^2 + m_2\omega_2^2 x_2^2 + m_3\omega_3^2 x_3^2 + D_{12}x_1x_2 + D_{13}x_1x_3 + D_{23}x_2x_3 \right)$$

which can be written as

$$H_2 = \frac{1}{2m} \left( p_1^2 + p_2^2 + p_3^2 \right) + \frac{m}{2} \left( \omega_1^2 X_1^2 + \omega_2^2 X_2^2 + \omega_3^2 X_3^2 \right) + m (J_{12} X_1 X_2 + J_{13} X_1 X_3 + J_{23} X_2 X_3)$$

after rescaling the phase space variables

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \mu_1^{-1} X_1 \\ \mu_2^{-1} X_2 \\ \mu_3^{-1} X_3 \end{pmatrix}, \quad \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \mu_1 P_1 \\ \mu_2 P_2 \\ \mu_3 P_3 \end{pmatrix}$$

where the involved parameters are given by

$$m = (m_1 m_2 m_3)^{\frac{1}{3}}, \quad \mu_i = \left( \frac{m_i}{m} \right)^{\frac{1}{3}}, \quad \mu_1 \mu_2 \mu_3 = 1, \quad J_{ij} = \frac{D_{ij}}{2\sqrt{m_i m_j}}$$

and the indices $i, j = 1, 2, 3$.

Since $H_2$ involves interacting terms, then a straightforward investigation of the basic features of the system is not an easy task. Nevertheless, one can overcome such situation by writing the Hamiltonian in matrix form, such as

$$H_2 = \frac{1}{2m} \sum_{i,j=1}^{3} P_i \delta_{ij} P_j + \frac{1}{2} m \sum_{i,j=1}^{3} X_i \mathcal{R}_{ij} X_j$$

where the two matrices take the forms

$$\mathcal{R}_{ij} = \begin{pmatrix} \omega_1^2 & J_{12} & J_{13} \\ J_{12} & \omega_2^2 & J_{23} \\ J_{13} & J_{23} & \omega_3^2 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$
3 Group SU(3) and diagonalization

We will now see how to use a faithful matrix representation to diagonalize the Hamiltonian \( H_2 \) by introducing the generators of Lie group SU(3), see Appendix A. To start let us write the matrix \( R_{ij} \) in terms of the generators \( \lambda_i \)

\[
R_{ij} = J_{12} \lambda_1 + J_{13} \lambda_4 + J_{23} \lambda_6 + \text{diag} \left( \omega_1^2, \omega_2^2, \omega_3^2 \right) \tag{7}
\]

and make a rotation with three angles (\( \varphi, \phi, \theta \))

\[
\begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix} = M
\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix} \tag{8}
\]

such that the matrix is of the form

\[
M = e^{i\varphi \lambda_7} e^{i\phi \lambda_2} e^{i\theta \lambda_5} \tag{9}
\]

and explicitly reads as

\[
M = \begin{pmatrix}
\cos \theta \cos \phi & \sin \phi & \cos \phi \sin \theta \\
-\sin \theta \sin \varphi - \cos \theta \cos \varphi \sin \phi & \cos \varphi \cos \phi & \cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi \\
-\sin \theta \cos \varphi + \cos \theta \sin \varphi \sin \phi & -\cos \varphi \sin \phi & \cos \theta \cos \varphi + \sin \theta \sin \varphi \sin \phi
\end{pmatrix} \tag{10}
\]

Using (7) together with (10), we show the relation

\[
R_{ij} = M \text{diag} \left( \Sigma_1^2, \Sigma_2^2, \Sigma_3^2 \right) M^{-1} \tag{11}
\]

with the quantities

\[
\omega_1^2 = \left( \Sigma_1^2 \cos^2 \theta + \Sigma_3^2 \sin^2 \theta \right) \cos^2 \phi + \Sigma_2^2 \sin^2 \phi \tag{12}
\]

\[
\omega_2^2 = 2 \left( \Sigma_1^2 \cos^2 \theta + \Sigma_3^2 \sin^2 \theta \right) \cos^2 \varphi + \frac{1}{2} \Sigma_1^2 \Sigma_3^2 \sin^2 \omega + \frac{1}{2} \Sigma_2^2 \Sigma_3^2 \sin^2 \omega \tag{13}
\]

\[
\omega_3^2 = 2 \left( \Sigma_1^2 \cos^2 \theta + \Sigma_3^2 \sin^2 \theta \right) \cos^2 \varphi + \frac{1}{2} \Sigma_1^2 \Sigma_3^2 \sin^2 \omega + \frac{1}{2} \Sigma_2^2 \Sigma_3^2 \sin^2 \omega \tag{14}
\]

\[
J_{12} = -\frac{\left( \Sigma_1^2 - \Sigma_2^2 \right) \cos^2 \theta + \left( \Sigma_2^2 - \Sigma_3^2 \right) \sin^2 \theta}{2} \sin 2\phi \cos \varphi - \frac{\Sigma_1^2 - \Sigma_3^2}{2} \sin 2\theta \cos \phi \sin \varphi \tag{15}
\]

\[
J_{13} = \frac{\left( \Sigma_1^2 - \Sigma_2^2 \right) \cos^2 \theta + \left( \Sigma_1^2 - \Sigma_3^2 \right) \sin^2 \theta}{2} \sin 2\phi \sin \varphi - \frac{\Sigma_1^2 - \Sigma_2^2}{2} \sin 2\theta \cos \phi \cos \varphi \tag{16}
\]

\[
J_{23} = \frac{\Sigma_1^2 - \Sigma_2^2}{2} \sin 2\theta \sin \phi \cos 2\varphi - \frac{\Sigma_2^2 \cos^2 \phi + \left( \Sigma_1^2 \cos^2 \theta + \Sigma_3^2 \sin^2 \theta \right) \sin^2 \phi - \Sigma_1^2 \cos^2 \theta - \Sigma_3^2 \sin^2 \theta}{2} \sin 2\phi \tag{17}
\]

From these, we map the Hamiltonian \( H_2 \) as follows

\[
H_3 = \frac{1}{2m} \left( P_1^2 + P_2^2 + P_3^2 \right) + \frac{m}{2} \left( \Sigma_1 q_1^2 + \Sigma_2 q_2^2 + \Sigma_3 q_3^2 \right) \tag{18}
\]

meaning that our system becomes decoupled (three decoupled harmonic oscillators) and therefore the energy spectrum can be easily obtained. One more thing, the result obtained by studying quantum propagator for some classes of three-dimensional three-body systems [16] can be recovered from our results just by taking the special case \( \Sigma_3 \rightarrow 0 \). In fact, one of the three oscillators becomes a free particle and then the Hamiltonian can be diagonalized using tow rather than three free parameters.
By introducing the new set of parameters $\rho, \zeta$ and $\kappa$

$$\varpi = (\Sigma_1 \Sigma_2 \Sigma_3)^{1/2}, \quad e^{\zeta - \rho} = \frac{\Sigma_1}{\varpi}, \quad e^{\kappa - \zeta} = \frac{\Sigma_2}{\varpi}, \quad e^{\rho - \kappa} = \frac{\Sigma_3}{\varpi}$$

we write the Hamiltonian (18) as

$$H_3 = \frac{1}{2m} \left( P_1^2 + P_2^2 + P_3^2 \right) + \frac{m}{2} \varpi^2 \left( e^{2(\zeta - \rho)} q_1^2 + e^{2(\kappa - \zeta)} q_2^2 + e^{2(\rho - \kappa)} q_3^2 \right)$$

which is now describing three decoupled harmonic oscillators with the same mass $m$ and three different frequencies. Then solving the eigenvalue equation

$$H_4 \mid n_1, n_2, n_3 \rangle = E_{n_1, n_2, n_3} \mid n_1, n_2, n_3 \rangle$$

we end up with the eigenvalues

$$E_{n_1, n_2, n_3} = \hbar \varpi \left( e^{\zeta - \rho} n_1 + e^{\kappa - \zeta} n_2 + e^{\rho - \kappa} n_3 + \frac{e^{\zeta - \rho} + e^{\kappa - \zeta} + e^{\rho - \kappa}}{2} \right)$$

as well as the normalized wavefunctions

$$\psi_{n_1, n_2, n_3} (q_1, q_2, q_3) = \left( \frac{m \varpi}{\pi \hbar} \right)^{3/4} \frac{1}{\sqrt{n_1 + n_2 + n_3 + 1}} e^{\frac{m \varpi}{2\hbar} \left( e^{\zeta - \rho} q_1^2 + e^{\kappa - \zeta} q_2^2 + e^{\rho - \kappa} q_3^2 \right)}$$

$$\times H_{n_1} \left( \sqrt{\frac{m \varpi e^{\zeta - \rho}}{\hbar}} q_1 \right) H_{n_2} \left( \sqrt{\frac{m \varpi e^{\kappa - \zeta}}{\hbar}} q_2 \right) H_{n_3} \left( \sqrt{\frac{m \varpi e^{\rho - \kappa}}{\hbar}} q_3 \right)$$

where $n_1, n_2, n_3$ are three integer numbers and $H_n$ are Hermite polynomials, with $i = 1, 2, 3$. It is clearly seen that (23) is tensor product of three independent solutions such that each one is corresponding to a harmonic oscillator in one dimension.

To express (23) in terms of old variables $(x_1, x_2, x_3)$ we use the reciprocal transformations of (8) to write the new variables $(q_1, q_2, q_3)$ as

$$q_1 = \mu_1 \cos \theta \cos \phi x_1 - \mu_2 \left( \sin \theta \sin \phi + \cos \theta \cos \phi \sin \phi \right) x_2 - \mu_3 \left( \sin \theta \cos \varphi - \cos \theta \sin \phi \sin \varphi \right) x_3$$

$$q_2 = \mu_1 \sin \phi x_1 + \mu_2 \cos \phi \cos \varphi x_2 - \mu_3 \cos \phi \sin \varphi x_3$$

$$q_3 = \mu_1 \cos \phi \sin \theta x_1 + \mu_2 \left( \cos \theta \sin \varphi - \sin \theta \cos \phi \sin \varphi \right) x_2 + \mu_3 \left( \cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi \right) x_3$$

and then replace in (23) to get the exact wavefunctions $\psi_{n_1, n_2, n_3} (x_1, x_2, x_3)$ solutions of the problem of three coupled harmonic oscillators with quadratic interaction described by the Hamiltonian (1). We emphasis that these solutions are general and derived without any assumption or approximation. On the other hand, $\psi_{n_1, n_2, n_3} (x_1, x_2, x_3)$ can be used to deal with other issues related to our system.

4 Entanglement in ground state

To analyze the entanglement of the three-body system, we first determine the ground state. Indeed, from general solutions (23) together with variable changes (24–26), we extract the ground state wavefunction

$$\psi_{0,0,0} (x_1, x_2, x_3) \sim e^{-A \mu_1^2 x_1^2 - B \mu_2^2 x_2^2 - C \mu_3^2 x_3^2 + 2\mu_1 \mu_2 \Gamma_1 x_1 x_2 + 2\mu_1 \mu_3 x_1 x_3 + 2\Gamma_2 x_2 x_3}$$
such that all parameters read as

\[
A = \alpha \cos^2 \theta \cos^2 \phi + \beta \sin^2 \phi + \gamma \cos^2 \phi \sin^2 \theta
\]  

(28)

\[
B = \alpha (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2 + \beta \cos^2 \phi \cos^2 \varphi + \gamma (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^2
\]  

(29)

\[
C = \alpha (\sin \theta \cos \varphi - \cos \theta \sin \phi \sin \phi)^2 + \beta \cos^2 \phi \sin^2 \varphi + \gamma (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \phi)^2
\]  

(30)

\[
\Gamma_{12} = \alpha \cos \theta \cos \phi (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi) - \beta \sin \phi \cos \phi \cos \varphi
\]

\[-\gamma \cos \phi \sin \theta \cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)
\]  

(31)

\[
\Gamma_{13} = -\alpha \cos \theta \cos \phi (-\sin \theta \cos \varphi + \cos \theta \sin \phi \sin \varphi) + \beta \sin \phi \cos \phi \sin \varphi
\]

\[-\gamma \cos \phi \sin \theta (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)
\]  

(32)

\[
\Gamma_{23} = \alpha (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi) (-\sin \theta \cos \varphi + \cos \theta \sin \phi \sin \varphi)
\]

\[+ \beta \cos \phi \cos \phi \cos \phi \sin \varphi - \gamma (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi) (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)
\]  

(33)

\[
\alpha = \frac{m\rho}{2} e^{r-\rho}, \quad \beta = \frac{m\varphi}{2} e^{K-\varphi}, \quad \gamma = \frac{m\kappa}{2} e^{\theta-\kappa}.
\]  

(34)

Once the ground state wavefunction corresponding to our system is obtained, we now return to explicitly determine the reduced density matrix. Then based on the standard definition

\[
\rho^A_{\text{red}}(x_1, x'_1) = \frac{\int \psi_{0,0,0}(x_1, x_2, x_3) \psi_{0,0,0}^{*}(x'_1, x_2, x_3) \, dx_2 dx_3}{\int \psi_{0,0,0}(x_1, x_2, x_3) \psi_{0,0,0}^{*}(x_1, x_2, x_3) \, dx_1 dx_2 dx_3}
\]

(35)

we find the result

\[
\rho^A_{\text{red}}(x_1, x'_1) = \sqrt{\frac{2L - \omega}{\pi}} \, e^{-L(x_1^2 + x'^2_1) + \omega x_1 x'_1}
\]

(36)

and the two parameters are given by

\[
L = A \mu_1^2 - \frac{\mu_2^2 \mu_3^2}{2C \mu_3^2} \left( \frac{\mu_1 \mu_2 \Gamma_{12} + \mu_1 \mu_3 \Gamma_{13} + \mu_2 \mu_3 \Gamma_{23}}{C \mu_3^2} \right)^2
\]

(37)

\[
w = \frac{\mu_2^2 \mu_3^2}{C \mu_3^2} + \left( \frac{\mu_1 \mu_2 \Gamma_{12} + \mu_1 \mu_3 \Gamma_{13} + \mu_2 \mu_3 \Gamma_{23}}{B \mu_3^2 - \frac{\mu_2^2 \mu_3^2}{C \mu_3^2}} \right)^2.
\]

(38)

Now we are in the final stage to talk about entanglement of our system. Indeed, the corresponding purity function can be obtained as

\[
P = \int \rho^A_{\text{red}}(x_1, x'_1) \rho^A_{\text{red}}(x'_1, x_1) \, dx_1 dx'_1 = \sqrt{\frac{2L - \omega}{2L + \omega}}
\]

(39)

Replacing different quantities and after straightforward calculation we end up with the final form of such function

\[
P = \frac{1}{\sqrt{e^{r-\rho} \cos^2 \theta \cos^2 \phi + e^{r-\rho} \sin^2 \phi + e^{r-\rho} \cos^2 \phi \sin^2 \theta}}
\]

\times \frac{1}{\sqrt{e^{r-\rho} (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2 + e^{r-\rho} \cos^2 \phi \cos^2 \varphi + e^{r-\rho} (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^2}}
\]

\times \frac{1}{\sqrt{e^{r-\rho} (-\sin \theta \cos \varphi + \cos \theta \sin \phi \sin \varphi)^2 + e^{r-\rho} \cos^2 \phi \sin^2 \varphi + e^{r-\rho} (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2}}
\]

(40)

At this level, we have some comments in order. Indeed (40) is actually depending on a set of six parameters \((\rho, \varsigma, \kappa, \theta, \varphi, \phi)\) and therefore one may consider different configurations to numerically
analyze the behavior of our system. This will be not done here because our concern is to give an exact solution of the problem under consideration. Nevertheless, we can still talk about minimal and maximal values of the purity function to give some ideas about the entanglement of our system. More precisely, two situations will be analyzed with respect to the strength of the coupling parameters \((\varsigma, \rho, \kappa)\), which will allow us to see how much the present system is entangled.

We start with the weak coupling that is characterized by taking the limit \(J_{12}, J_{13}, J_{23} \to (0, 0, 0)\) where the angles \((\theta, \phi, \varphi) \to (\theta_w, \phi_w, \varphi_w)\) and the coupling \((\varsigma, \rho, \kappa) \to (\varsigma_w, \rho_w, \kappa_w)\). In this case, (15-17) and (19) reduce to the following

\[
(\theta_w, \phi_w, \varphi_w) = (0, 0, 0), \quad e^{\varsigma_w - \rho_w} = \frac{\omega_1}{(\omega_1 \omega_2 \omega_3)^{\frac{1}{2}}}, \quad e^{\kappa_w - \omega_w} = \frac{\omega_2}{(\omega_1 \omega_2 \omega_3)^{\frac{1}{2}}}, \quad e^{\rho_w - \kappa_w} = \frac{\omega_3}{(\omega_1 \omega_2 \omega_3)^{\frac{1}{2}}}
\] (41)

which can be implemented into (40) to get the maximal value of the purity function

\[
P(\varsigma_w, \rho_w, \kappa_w, \theta_w, \phi_w, \varphi_w) = 1
\] (42)

showing that the system is completely separable and therefore there is no entangled states because of the entropy \(S = 1 - P\).

Now we consider the strong coupling limit corresponding to the limit \((\varsigma, \rho, \kappa) \to (\varsigma_s, \rho_s, \kappa_s)\) and \((\theta, \phi, \varphi) \to (\theta_s, \phi_s, \varphi_s)\). Doing this process to obtain the limit

\[
(\varsigma_s - \rho_s, \kappa_s - \varsigma_s, \rho_s - \kappa_s) \to (\pm \infty, \pm \infty, \pm \infty)
\] (43)

and therefore the purity function (40) reduces to the following quantity

\[
P(\varsigma_s, \rho_s, \kappa_s, \theta_s, \phi_s, \varphi_s) \to 0
\] (44)

telling us that our system is maximally entangled because of \(S = 1\). This summarizes that there are two extremely values of the purity function those could be reached as long as the coupling parameter takes small or large values.

5 Limiting cases

Now we will see how to derive some results already know in literature, which concern three limiting cases to distinguish in terms of the coupling parameters where the first one will be treated below and tow remaining will be summarized in Appendix B. We emphasis that all such cases will give the same results but main differences are how to fix the physical parameters and choose coordinate variables. To get the solutions of two coupled harmonic oscillators in variables \((x_1, x_2)\) we simply require the limits \(D_{13}, D_{23} \to 0\), which correspond to \(J_{13}, J_{23} \to 0\). These operations restrict the Hamiltonian \(H_1\) to the following

\[
H_1 \to H_0 + \frac{p_1^2}{2m_1} + \frac{1}{2} m_3 \omega_3^2 x_3^2
\] (45)

where \(H_0\) is the Hamiltonian of the two coupled harmonic oscillators in \((x_1, x_2)\) variables

\[
H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2} m_1 \omega_1^2 x_1^2 + \frac{1}{2} m_2 \omega_2^2 x_2^2 + \frac{1}{2} D_{12} x_1 x_2.
\] (46)
By taking $J_{13}, J_{23} \to 0$ in (16-17) we obtain $\theta \to 0, \varphi \to 0$ and

$$
\begin{align*}
\omega_1^2 &\to \Sigma_1^2 \cos^2 \phi + \Sigma_2^2 \sin^2 \phi = \frac{\Sigma_1^2 + \Sigma_2^2}{2} + \frac{\Sigma_1^2 - \Sigma_2^2}{2} \cos 2\phi \\
\omega_2^2 &\to \Sigma_2^2 \cos^2 \phi + \Sigma_1^2 \sin^2 \phi = \frac{\Sigma_1^2 + \Sigma_2^2}{2} - \frac{\Sigma_1^2 - \Sigma_2^2}{2} \cos 2\phi, \\
\omega_3^2 &\to \Sigma_3^2 \\
J_{12} &\to -\frac{\Sigma_1^2 - \Sigma_2^2}{2} \sin 2\phi
\end{align*}
$$

showing that the reciprocal expressions take the forms

$$
\begin{align*}
\Sigma_1^2 &= \frac{\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_1^2 \cos 2\phi + \omega_2^2 \cos 2\phi}{2} = k_{12} e^{-2\eta_{12}} \\
\Sigma_2^2 &= \frac{\omega_1^2 + \omega_2^2 - \omega_3^2 - \omega_1^2 \cos 2\phi + \omega_2^2 \cos 2\phi}{2} = k_{12} e^{-2\eta_{12}} \\
\Sigma_3^2 &= \omega_3^2, \quad \xi = (\Sigma_1 \Sigma_2 \Sigma_3)^{\frac{1}{3}} \to (k_{12} \omega_3)^{\frac{1}{3}} \\
e^{-\rho} &= \frac{\sqrt{k_{12} e^{-\eta_{12}}}}{(k_{12} \omega_3)^{\frac{1}{3}}}, \quad e^{-\zeta} = \frac{\sqrt{k_{12} e^{-\eta_{12}}}}{(k_{12} \omega_3)^{\frac{1}{3}}}, \quad e^{\rho - \kappa} = \frac{\omega_3}{(k_{12} \omega_3)^{\frac{1}{3}}}
\end{align*}
$$

where we have set

$$
e^{\pm 2\eta_{12}} = \frac{\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + J_{12}^2}}{2k_{12}}, \quad k_{12} = \sqrt{\frac{2}{\omega_1^2 \omega_2^2 - J_{12}^2}}. \tag{55}
$$

It is clearly seen that the above set are those used in our previous work [15] to decouple the problem of two harmonic oscillators in $(x_1, x_2)$ variables

$$
H = \left( \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m}{2} k_{12} e^{2\eta_{12}} q_1^2 + \frac{m}{2} k_{12} e^{-2\eta_{12}} q_2^2 \right) + \frac{p_3^2}{2n_3} + \frac{1}{2} n_3 \omega_3^2 q_3^2 \tag{56}
$$

whose eigenvalues and the eigenstates are given by

$$
E_{n_1, n_2, n_3} = \hbar \sqrt{k_{12}} \left( e^{\eta_{12}} n_1 + e^{-\eta_{12}} n_2 + \cosh \eta_{12} \right) + \hbar \omega_3 \left( n_3 + \frac{1}{2} \right) \tag{57}
$$

$$
\psi_{n_1, n_2, n_3}(x_1, x_2, x_3) = \frac{\left( \frac{m \omega_3}{n_3} \right)^{\frac{1}{2}} \left( \frac{m \sqrt{k_{12}}}{n_3} \right)^{\frac{1}{2}}}{\sqrt{2^{n_1+1} + 3^{n_2+1} + 5^{n_3+1}}} e^{-\frac{m}{2} \left( \sqrt{k_{12}} e^{\eta_{12}} q_1^2 + \sqrt{k_{12}} e^{-\eta_{12}} q_2^2 + \omega_3 q_3^2 \right)} \\
\times H_{n_1} \sqrt{\frac{m \sqrt{k_{12}} e^{\eta_{12}}}{n_1}} q_1 H_{n_2} \sqrt{\frac{m \sqrt{k_{12}} e^{-\eta_{12}}}{n_2}} q_2 H_{n_3} \sqrt{\frac{m \omega_3}{n_3}} q_3 \tag{58}
$$

with the variables

$$
q_1 = \mu_1 \cos \phi x_1 - \mu_2 \sin \phi x_2, \quad q_2 = \mu_1 \sin \phi x_1 + \mu_2 \cos \phi x_2, \quad q_3 = \mu_3 x_3 \tag{59}
$$

The corresponding purity function from can be derived from (40) as limiting case

$$
P(\rho, \zeta, \kappa, \theta = 0, \varphi = 0, \phi) = \frac{1}{\sqrt{e^{-\eta_{12}} \cos^2 \phi + e^{\eta_{12}} \sin^2 \phi} (e^{\eta_{12}} \sin^2 \phi + e^{-\eta_{12}} \cos^2 \phi) e^{\rho - \kappa}} = \frac{1}{\sqrt{e^{-\eta_{12}} \cos^2 \phi + e^{\eta_{12}} \sin^2 \phi} (e^{\eta_{12}} \sin^2 \phi + e^{-\eta_{12}} \cos^2 \phi)} = P_{0,0}(\eta_{12}, \phi) \tag{60}
$$

which coincides exactly with that obtained in our previous work [15]. The two other limiting cases are discussed in Appendix B and the obtained results are similar except that the configurations of physical parameters together with variable coordinates are not the same.
6 Conclusion

We have studied the problem of three coupled harmonic oscillators involving general coupling between coordinates. In doing so, different transformations have been introduced to finally end up with the solutions of the energy spectrum. More precisely, the representation theory of the group $SU(3)$ was employed to get a diagonalizable Hamiltonian describing three decoupled harmonic oscillators. Later on, the reciprocal transformations were used to express the general solutions of the interacting system in terms of the initial coordinates. The obtained results are general and derived without making use of any assumption or approximation.

Subsequently, focused on the ground state wavefunction we have calculated the corresponding reduced density matrix. This was used to explicitly determine the purity function in terms of different physical parameters of of three coupled harmonic oscillators as well as obtain its minimal and maximal values. To check the validity of results, we have inspected three limiting cases, which have been done by realizing one among three oscillators. In each case we have established the corresponding conditions as well as the convenient variable changes. In all cases, we have obtained the same entanglement as has been reported in [15].

The present work will not remain at this stage, in fact we plane to investigate other issues using the obtained results so far. First question arises is what about the corresponding thermodynamic properties of our system and second one can we talk about its dynamics by considering frequencies time dependent? All these questions and related matters are actually under consideration.

Acknowledgments

AM acknowledges the Deanship of Scientific Research at King Faisal University for the financial support under Nasher Track (Grant No. 186262). The generous support provided by the Saudi Center for Theoretical Physics (SCTP) is highly appreciated by AJ.

Appendix A: $SU(3)$ algebra

We recall some mathematical tools related to a Lie group, which have been used in our work. Indeed, the generators of the group $SU(3)$ are given by [17]

$$
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{align*}
$$

(61)
where the Gell-Mann matrices $\lambda_i$, that are analog of the Pauli matrices for the group $SU(2)$, satisfy the $SU(3)$ commutation relations

$$[\lambda_j, \lambda_k] = 2i\sum_l f^{jkl} \lambda_l$$

and the structure constants $f^{ijk}$ of the Lie algebra are given by

$$f^{123} = 1,$$

$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}$$

We can check the useful identities

$$e^{-i\phi_\lambda_2} \lambda_6 e^{+i\phi_\lambda_2} = \cos \phi \lambda_6 - \sin \phi \lambda_4$$

$$e^{-i\phi_\lambda_2} \lambda_4 e^{+i\phi_\lambda_2} = \cos \phi \lambda_4 + \sin \phi \lambda_6$$

$$e^{-i\phi_\lambda_2} \lambda_1 e^{+i\phi_\lambda_2} = \mathrm{diag} (-\sin 2\phi, \sin 2\phi, 0) + \cos (2\phi) \lambda_1$$

$$e^{-i\phi_\lambda_2} \mathrm{diag} (a, b, c) e^{+i\phi_\lambda_2} = \mathrm{diag} \left( a \cos^2 \phi + b \sin^2 \phi, b \cos^2 \phi + a \sin^2 \phi, c \right) + \frac{a-b}{2} \sin 2\phi \lambda_1$$

$$e^{-i\theta_\lambda_5} \lambda_6 e^{i\theta_\lambda_5} = \cos \theta \lambda_6 - \sin \theta \lambda_1$$

$$e^{-i\theta_\lambda_5} \lambda_4 e^{i\theta_\lambda_5} = \cos \theta \lambda_4 + \sin \theta \lambda_6$$

$$e^{-i\theta_\lambda_5} \lambda_1 e^{i\theta_\lambda_5} = \mathrm{diag} (-\sin 2\theta, 0, \sin 2\theta) + \cos 2\theta \lambda_4$$

$$e^{-i\theta_\lambda_5} \mathrm{diag} (a, b, c) e^{i\theta_\lambda_5} = \mathrm{diag} \left( a \cos^2 \theta + c \sin^2 \theta, b, c \cos^2 \theta + a \sin^2 \theta \right) + \frac{a-c}{2} \sin 2\theta \lambda_4$$

$$e^{-i\varphi_\lambda_7} \lambda_4 e^{+i\varphi_\lambda_7} = \cos \varphi \lambda_4 + \sin \varphi \lambda_1$$

$$e^{-i\varphi_\lambda_7} \lambda_6 e^{+i\varphi_\lambda_7} = \cos \varphi \lambda_6 - \sin \varphi \lambda_1$$

$$e^{-i\varphi_\lambda_7} \lambda_1 e^{+i\varphi_\lambda_7} = \mathrm{diag} (0, -\sin 2\varphi, \sin 2\varphi) + \cos 2\varphi \lambda_6$$

$$e^{-i\varphi_\lambda_7} \mathrm{diag} (a, b, c) e^{+i\varphi_\lambda_7} = \mathrm{diag} \left( a \cos^2 \varphi + c \sin^2 \varphi, b \cos^2 \varphi + c \sin^2 \varphi, b \cos^2 \varphi \right) + \frac{b-c}{2} \sin 2\varphi \lambda_6$$

**Appendix B: More couplings**

For the two coupled harmonic oscillator in Variables $(x_1, x_3)$, we have the limits $D_{12}, D_{23} \rightarrow 0$ implying that $J_{12}, J_{23} \rightarrow 0$ and then from (15) and (17) we deduce that $\varphi \rightarrow 0, \phi \rightarrow 0$. From previous equations (12-14) and (16), we can deduce

$$\omega_1^2 \rightarrow \Sigma_1^2 \cos^2 \theta + \Sigma_3^2 \sin^2 \theta = \frac{\Sigma_1^2 + \Sigma_2^2}{2} + \frac{\Sigma_1^2 - \Sigma_2^2}{2} \cos 2\theta$$

$$\omega_2^2 \rightarrow \Sigma_2^2$$

$$\omega_3^2 \rightarrow \Sigma_3^2 \cos^2 \theta + \Sigma_1^2 \sin^2 \theta = \frac{\Sigma_1^2 + \Sigma_2^2}{2} - \frac{\Sigma_1^2 - \Sigma_2^2}{2} \cos 2\theta$$

$$J_{13} \rightarrow -\frac{\Sigma_1^2 - \Sigma_2^2}{2} \sin 2\theta$$

and variables take the form

$$q_1 = \mu_1 \cos \theta x_1 - \mu_3 \sin \theta x_3, \quad q_2 = \mu_2 x_2, \quad q_3 = \mu_1 \sin \theta x_1 + \mu_3 \cos \theta x_3$$
The corresponding purity from (40) can be derived as limiting case

\[ P(\rho, \varsigma, \kappa, \theta, \phi = 0, \phi = 0) = \frac{1}{\sqrt{(e^{\rho - \varsigma} \cos^2 \theta + e^{\rho - \kappa} \sin^2 \theta)}(e^{\rho - \kappa} \sin^2 \theta + e^{\rho - \rho} \cos^2 \theta)} = P_{0,0}(\eta_{13}, \theta). \]  

(81)

As concerning the two coupled harmonic oscillator in variables \((x_2, x_3)\), We put \(D_{12}, D_{13} \to 0\) giving \(J_{12}, J_{13} \to 0\) and from (15-16) we deduce that \(\theta \to 0, \phi \to 0\). Also from (12-14) and (17) we have

\[ \begin{align*}
\omega_1^2 &\to \Sigma_1^2 \\
\omega_2^2 &\to \Sigma_2^2 \cos^2 \varphi + \Sigma_3^2 \sin^2 \varphi = \frac{\Sigma_2^2 + \Sigma_3^2}{2} + \frac{\Sigma_2^2 - \Sigma_3^2}{2} \cos 2\varphi \\
\omega_3^2 &\to \Sigma_3^2 \cos^2 \varphi + \Sigma_2^2 \sin^2 \varphi = \frac{\Sigma_2^2 + \Sigma_3^2}{2} - \frac{\Sigma_2^2 - \Sigma_3^2}{2} \cos 2\varphi \\
J_{23} &\to -\frac{\Sigma_2^2 - \Sigma_3^2}{2} \sin 2\varphi. 
\end{align*} \]

(82-85)

The corresponding variables are given by

\[ q_1 = \mu_1 x_1, \quad q_2 = \mu_2 \cos \varphi x_2 - \mu_3 \sin \varphi x_3, \quad q_3 = \mu_2 \sin \varphi x_2 + \mu_3 \cos \varphi x_3 \]

(86)

as well as the purity

\[ P(\rho, \varsigma, \kappa, \theta = 0, \varphi = 0) = \frac{1}{\sqrt{(e^{-\rho - \kappa} \cos^2 x + e^{-\rho - \rho} \sin^2 x)}(e^{-\rho - \kappa} \sin^2 x + e^{-\rho - \rho} \cos^2 x)} = P_{0,0}(\eta_{23}, \varphi) \].

(87)

References

[1] E. Schrödinger, Naturwissenschaften 23 (1935) 807.

[2] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47 (1935) 777.

[3] J.S. Bell, ”Speakable and Unspeakable in Quantum Mechanics” (Cambridge University Press, 1987).

[4] Charles. H. Benett and Peter W. Shor, IEEE Transactions on Information Theory 44 (1998) 2724.

[5] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W.K. Wootters, Phys. Rev. Lett. 70 (1993) 1895.

[6] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. 69 (1992) 2881.

[7] A.K. Ekert, Phys. Rev. Lett. 67 (1991) 661.

[8] M. Murao, D. Jonathan, M.B. Plenio and V. Vedral, Phys. Rev. A 59 (1999) 156.

[9] C.A. Fuchs, Phys. Rev. Lett. 79 (1997) 1162.

[10] R. Rausschendorf and H. Briegel, Quantum computing via
[11] D. Gottesman and I. Chuang, Nature 402 (1999) 390.

[12] Anton Zeilinger, Michael A. Horne, Harald Weinfurter, and Marek Åžukowski, Phys. Rev. Lett. 78 (1997) 3031.

[13] D. Park, Quantum Inf Process 18 (2019) 282.

[14] M. M Sebawe Abdalla and M A Bashir, Quantum Semiclass. Opt. 10 (1998) 415.

[15] A. Jellal, F. Madouri and A. Merdaci, J. Stat. Mech. (2011) P09015.

[16] A. de Souza Dutra, Ann. Phys 321 (2006) 1092.

[17] M Gell-Mann and Y Ne’eman, The Eightfold Way, W A Benjamin (1964). Also see https://en.wikipedia.org/wiki/Gell-Mann_matrices.