A JUSTIFICATION OF THE TIMOSHENKO BEAM MODEL THROUGH $\Gamma$-CONVERGENCE

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ABSTRACT. We validate the Timoshenko beam model as an approximation of the linear-elasticity model of a three-dimensional beam-like body. Our validation is achieved within the framework of $\Gamma$-convergence theory, in two steps: firstly, we construct a suitable sequence of energy functionals; secondly, we show that this sequence $\Gamma$-converges to a functional representing the energy of a Timoshenko beam.

1. INTRODUCTION

Understanding the relation between three-dimensional elasticity and the lower-dimensional theories of elastic structures is a long-standing quest in rational continuum mechanics. All classic models for shell-, plate-, and beam-like bodies rely on some Ansätze about their kinematic and/or static behaviour, motivated by their thinness or slenderness, that is, by the smallness of one or two of their dimensions. Such Ansätze are expedient to put together mathematical models that are both simple and capable to provide good-enough predictions for plenty of the intended applications. However, as is always the case for intuition-based models, an all-important experimental confirmation does not replace for a rigorous justification, that is, validation as a convincing approximation of an accepted parent theory.

Justification of lower-dimensional structure theories has been attempted in a number of ways, some essentially analytical in nature, like the method of asymptotic expansion [5] and the functional analysis methods accounted for in [1, 6], and some essentially mechanical, like the method of formal scaling [10] and the method based on a thickness-wise expansion [15]: the predictions of lower-dimensional models can also be corroborated by error estimates [7]. In the past couple of decades, a noticeable amount of work has been devoted to provide a justification of various structure theories within the framework of $\Gamma$-convergence; in particular, Reissner-Mindlin’s theory of shearable plates has been considered in [12, 13, 14]. In this paper, for the first time, a justification via $\Gamma$-convergence is given for Timoshenko’s sheareable beam theory [17], the one-dimensional theory that parallels the two-dimensional Reissner-Mindlin’s theory.

Heuristically, justification of beam models can be achieved by calibrating the convergence rate of the three-dimensional elastic energy (briefly, the energy) with respect to the diameter of the cross section. Anzellotti et al. [1] and Bourquin et al. [3] were first to apply the theory of $\Gamma$-convergence to deduce unshearable beam and plate models within the setting of three-dimensional linear elasticity. Their results have been generalized in several ways, and different models of beams,
plates, and shells have been studied, within the linear as well as the nonlinear
elasticity framework. We refrain from citing the large literature on the subject; we
only mention a paper by Mora and Müller [11], where a justification of a bending-
torsion beam model is provided within the non-linear elasticity framework.

In the engineering community, Bernoulli-Navier’s and Timoshenko’s beam theo-
ries are well accepted and constantly used in applications. As mentioned, the former
model has been fully justified since long by means of the theory of $\Gamma$-convergence,
whereas a similar justification for the latter was lacking. In our opinion, this was due
to the fact that the “classical” procedure used in the study of dimension-reduction
problems was excessively “constrained”. The constraints in question were relaxed
in [12, 13], so as to deduce the Reissner-Mindlin’s plate model; the procedure there
used was later classified in [14] as a special case of a general scheme. Briefly, for
a given three-dimensional problem (the so-called “real problem”), the scheme con-
sists in defining a problem sequence whose variational limit approximates the “real
problem”.

Here is a summary of the contents to follow. In Section 2 we present the real
problem for a linearly elastic and transversely isotropic three-dimensional beam-like
body; in Section 3 we recall Timoshenko’s kinematical assumptions; and, in Section
4, along the lines proposed in [14] and further discussed in [16], we introduce a
sequence of variational problems, one of which is the real problem. The main
compactness results are deduced in Section 5, while the $\Gamma$-limit result is stated and
proved in Section 6; interestingly, the $\Gamma$-limit turns out to be the energy functional
of the one-dimensional Timoshenko’s beam theory. Finally, in Section 8 we show
that the $\Gamma$-limit problem approximates the real problem well.

As to notation, throughout this work Greek indices $\alpha, \beta$ take values in the set
$\{1, 2\}$, Latin indices $i, j$ in the set $\{1, 2, 3\}$. We use $L^2(A; B)$ and $H^1(A; B)$ to
denote, respectively, the Lebesgue and Sobolev spaces of functions defined over the
set $A$ and taking values in the vector space $B$; in case $B = \mathbb{R}$, we simply write
$L^2(A)$ and $H^1(A)$; the corresponding norms are denoted by $\| \cdot \|_{L^2(A; B)}$ and
$\| \cdot \|_{H^1(A; B)}$.

2. THE REAL PROBLEM

We consider a three-dimensional body occupying a domain under form of a right
cylinder $\Omega_r = \omega_r \times (0, L) \subset \mathbb{R}^3$ of length $L$, whose cross-section $\omega_r \subset \mathbb{R}^2$ is an open
bounded simply-connected set with Lipschitz boundary $\partial \omega_r$. For $d_r$ the diameter
of the cross-section, we set $\varepsilon_r := d_r / L$ and we call $\Omega_r$ beam-like, because we take
$\varepsilon_r \ll 1$.

We denote by $e_i$ the vectors of an orthonormal basis, with vectors $e_\alpha$ tangent to
$\omega_r$, where the origin of the Cartesian frame we use is located, and vector $e_3$ in the
direction of beam axis. For definiteness and simplicity, we stipulate that the body
is clamped on the part $\partial D \Omega_r = \omega_r \times \{0\}$ of its boundary. Moreover, we denote by
$\mathcal{U}_r$ the space of admissible displacements:

$$\mathcal{U}_r = \{ \mathbf{u} \in H^1(\Omega_r; \mathbb{R}^3) \mid \mathbf{u} = 0 \text{ on } \partial D \Omega_r \} ;$$

we measure the admissible strains point-wise by means of the symmetric tensor

$$E \mathbf{\varepsilon} := \frac{1}{2} (\nabla \mathbf{\varepsilon} + (\nabla \mathbf{\varepsilon})^T) ;$$

and we denote by $\mathbb{R}^{3 \times 3}_{\text{sym}}$ the collection of all symmetric linear mappings of $\mathbb{R}^3$ into it-
self. Finally, we assume the material to be linearly elastic and transversely isotropic.
with respect to the $e_3$ direction, so that the elastic-energy density per unit volume is given by

$$W(E) := \frac{1}{2} \left[ 2\mu \left( E_{11}^2 + E_{22}^2 \right) + \lambda \left( E_{11} + E_{22} \right)^2 + 2\tau_2 E_{33} \left( E_{11} + E_{22} \right) \\
+ 4\mu E_{12}^2 + 4\gamma \left( E_{31}^2 + E_{32}^2 \right) + \tau_1 E_{33}^2 \right],$$

where the material moduli $\mu, \lambda, \tau_1, \tau_2, \tau_1^2 < 0$, and $\gamma$, satisfy the following inequalities:

$$\mu > 0, \gamma > 0, \tau_1 > 0, \tau_1 (\lambda + \mu) - \tau_1^2 > 0;$$

consequently, the elastic-energy density is positively bounded below by the strain norm, in the sense that there is a positive constant $C$ such that $W(E) \geq C|E|^2$ for every $E \in \mathbb{R}^{3 \times 3}$.

In this section and henceforth, we systematically make use of a subscript or superscript 'r' as a reminder of the fact that all kernel letters carrying that modifier are used in connection with a real three-dimensional equilibrium problem for a beam-like body.

The elastic potential $W^r : \mathcal{U}_r \rightarrow \mathbb{R}$ associated with the beam-like body $\Omega_r$ is:

$$W^r(\pi) = \int_{\Omega_r} W(E\pi) \, d\mathbf{x}, \quad \text{for all } \pi \in \mathcal{U}_r.$$

Since we let $\Omega_r$ be subjected to a distance load $b^r : \Omega_r \rightarrow \mathbb{R}^3$ and a contact load $c^r : \partial N \Omega_r \rightarrow \mathbb{R}^3$, with $\partial N \Omega_r = \partial \Omega_r \setminus \partial D \Omega_r$, the load potential is:

$$F^r(\pi) := \int_{\Omega_r} b^r \cdot \pi \, d\mathbf{x} + \int_{\partial N \Omega_r} c^r \cdot \pi \, d\mathbf{x}.$$

Finally, the total potential is:

$$\Pi^r(\pi) := W^r(\pi) - F^r(\pi).$$

Hereafter we denote by $\overline{u}^r_{\min}$ the unique minimizer of $\Pi^r$:

$$\overline{u}^r_{\min} = \arg\min_{\pi \in \mathcal{U}_r} \Pi^r(\pi).$$

3. The Timoshenko Ansatz and its mechanical interpretation

Roughly speaking, according to the kinematical Ansatz on which Timoshenko’s beam model is based, the beam’s cross-section is regarded as rigid, while the beam’s axis may deform arbitrarily. In fact, a prototypical Timoshenko displacement field has the following form:

$$u^r = u_1(x_3)e_1 + u_2(x_3)e_2 + (u_3(x_3) + x_2\psi_1(x_3) - x_1\psi_2(x_3)) e_3,$$

where $u_i$ and $\psi_\alpha$ are real-valued functions defined on $(0, L)$.

It was shown in [8] that, to within rigid global displacements, the above displacement field is the general solution of the following set of PDEs:

$$2(Eu)_{\alpha \beta} = u_{\alpha, \beta} + u_{\beta, \alpha} = 0,$$

$$2((Eu)_{3\alpha})_{, \beta} = u_{3, \alpha, \beta} + u_{\alpha, 3, \beta} = 0.$$

The following precise kinematical interpretation of (3.1) was given: for each fixed axial coordinate $x_3$, the first-order PDEs in (3.2) imply that cross-section fibers neither change their length nor change their mutual angle, while the second-order PDEs imply that the change in angle between an axial fiber and a cross-section fiber does not depend on the direction of the latter. It was also remarked in [8]
that the each of the above PDEs can be interpreted as a non-dissipative internal-constraint condition, and the reaction stresses and hyperstresses maintaining those constraints were determined.

In [10], it was suggested that both shearable-structure theories, Reissner-Mindlin’s for plates and Timoshenko’s for beams, are derivable from a three-dimensional elastic-energy functional that includes a second-order strain contribution. This suggestion will be taken into account in the construction of the energy sequence given here below.

4. A SEQUENCE OF VARIATIONAL PROBLEMS

In this section we construct a sequence of variational problems, parameterized by $\varepsilon$, such that the “real problem” presented in Section 2 is the element of the sequence whose parameter $\varepsilon$ is equal to $\varepsilon_r$. Our construction of the typical problem in this sequence is achieved in a number of scaling steps, both for data and candidate solutions.

4.1. Domain. For $\alpha$ positive, let $R^\alpha := \text{diag}(\alpha, \alpha, 1)$ be a diagonal $3 \times 3$ matrix; moreover, for $x^r \equiv (x_1^r, x_2^r, x_3^r)$ a typical point of $\Omega_r$ and for any given $\varepsilon > 0$, let $\Omega$ and $\Omega_\varepsilon$ denote the sets in $\mathbb{R}^3$ whose typical points are, respectively, $x \equiv (x_1, x_2, x_3) = R^{1/\varepsilon} x^r$ and $x^\varepsilon \equiv (\varepsilon x_1, \varepsilon x_2, x_3) = R^\varepsilon x$ (note that, consequently, $x^\varepsilon = R^{\varepsilon/\varepsilon_r} x^r$). It follows from these definitions that

$$\Omega_\varepsilon = \omega_\varepsilon \times (0, L), \quad \text{where} \quad \omega_\varepsilon := \frac{\varepsilon}{\varepsilon_r} \omega_r,$$

and that

$$\Omega = \omega \times (0, L), \quad \text{where} \quad \omega = \omega_1 := \frac{1}{\varepsilon_r} \omega_r.$$

Therefore, $\omega_\varepsilon$ and $\omega$ are nothing but the domains in $\mathbb{R}^2$ obtained by homothetic rescaling of $\omega_r$ by, respectively, factors $\varepsilon/\varepsilon_r$ and $1/\varepsilon_r$. Note for later use that the following relationships hold:

- $dv^\varepsilon = \varepsilon^2 dv$, between the volume measures of $\Omega_\varepsilon$ and $\Omega$;
- $da^\varepsilon = \varepsilon^2 da$, between the area measures of $\omega_\varepsilon$ and $\omega$.

4.2. Displacement and strain fields. Given a displacement field $\mathbf{u^r} : \Omega_r \to \mathbb{R}^3$, we let the scaled displacement $u^\varepsilon : \Omega \to \mathbb{R}^3$ be defined by

$$u^\varepsilon(x) := R^\varepsilon \mathbf{u^r}(R^\varepsilon x);$$

in Cartesian components,

$$u^\varepsilon_1 = \varepsilon \mathbf{u^r}_1 \circ R^\varepsilon, \quad u^\varepsilon_2 = \varepsilon \mathbf{u^r}_2 \circ R^\varepsilon, \quad u^\varepsilon_3 = \mathbf{u^r}_3 \circ R^\varepsilon.$$

It follows from definition (4.1) that

$$\nabla u^\varepsilon = R^\varepsilon (\nabla \mathbf{u^r}) R^\varepsilon,$$

and hence that

$$Eu^\varepsilon = R^\varepsilon (E \mathbf{u^r}) R^\varepsilon;$$

we call

$$E^\varepsilon u^\varepsilon := (R^\varepsilon)^{-1} Eu^\varepsilon (R^\varepsilon)^{-1}$$
the scaled strain, and we record here its component form:

\[
E^\varepsilon u^\varepsilon = \begin{pmatrix}
\frac{(E u^\varepsilon)_{11}}{\varepsilon^2} & \frac{(E u^\varepsilon)_{12}}{\varepsilon} & \frac{(E u^\varepsilon)_{13}}{\varepsilon} \\
\frac{(E u^\varepsilon)_{21}}{\varepsilon^2} & \frac{(E u^\varepsilon)_{22}}{\varepsilon} & \frac{(E u^\varepsilon)_{23}}{\varepsilon} \\
\frac{(E u^\varepsilon)_{31}}{\varepsilon} & \frac{(E u^\varepsilon)_{32}}{\varepsilon} & \frac{(E u^\varepsilon)_{33}}{\varepsilon}
\end{pmatrix}.
\]

4.3. Elastic and load potentials, total potential.

(i) Let

\[
W^\varepsilon(E) := \frac{1}{2} \left[ 2\mu E_{\alpha\beta} E_{\alpha\beta} + \lambda (E_{\alpha\alpha})^2 + 2\tau_2 E_{33} E_{\alpha\alpha} + 4\gamma \left( \frac{\varepsilon}{\varepsilon_{r}} \right)^2 E_{33} E_{\alpha\alpha} + \tau_1 E_{33}^2 \right]
\]

(note that \(W^\varepsilon (E) = W(E)\)). The elastic potential is the functional

\[
W^\varepsilon (u^\varepsilon) := \int_\Omega W^\varepsilon \left( E^\varepsilon u^\varepsilon \right) \, dv
\]

\[
+ \frac{1}{2} \tau R \int_\Omega \left( \frac{\varepsilon - \varepsilon_{r}}{\varepsilon} \right)^2 \sum_{\alpha, \beta} (u_{3,\alpha\beta}^\varepsilon + u_{\alpha,3\beta}^\varepsilon)^2 \, dv, \quad \tau R > 0.
\]

defined over the collection of all elements \(v \in H^1 (\Omega; \mathbb{R}^3)\) such that \((v_{3,\alpha\beta} + v_{\alpha,3\beta}) \in L^2 (\Omega)\) (cf. definition (5.1)).

(ii) For \(b^r : \Omega_r \to \mathbb{R}^3\) and \(\sigma^r : \partial_N \Omega_r \to \mathbb{R}^3\), the real distance and contact loads introduced in Section 2 we let

\[
b^r (x) := (R^r)^{-1} \sigma^r (R^r x)
\]

and

\[
c^r (x) = \begin{cases}
(R^r)^{-1} \sigma^r (R^r x) & \text{on } \omega \times \{ L \}, \\
\frac{1}{3} (R^r)^{-1} \sigma^r (R^r x) & \text{on } \partial\omega \times (0, L)
\end{cases}
\]

be the scaled loads, and we assume that \(b^r \in L^2 (\Omega; \mathbb{R}^3)\) and \(c^r \in L^2 (\partial_N \Omega; \mathbb{R}^3)\).

The load potential is the functional \(F : H^1 (\Omega; \mathbb{R}^3) \to \mathbb{R}\) defined by

\[
F (u^\varepsilon) := \int_\Omega b^r \cdot u^\varepsilon \, dv + \int_{\partial\omega \Omega} c^r \cdot u^\varepsilon \, da.
\]

(iii) the total potential is

\[
\Pi^\varepsilon (u^\varepsilon) := W^\varepsilon (u^\varepsilon) - F (u^\varepsilon).
\]

**Remark 1.** We close this section by observing that energies \(\Pi^r\) and \(\Pi^\varepsilon\) coincide, to within a multiplicative constant. Precisely, with the use of the above data and solution scalings, it can be shown that

\[
\Pi^r (\Pi^\varepsilon) = \varepsilon^2 \Pi^\varepsilon (u^\varepsilon).
\]

Since the multiplicative constant \(\varepsilon^2\) does not affect the minimization process, we deduce that, up to a change of variables, the minimizers of \(\Pi^\varepsilon\) and \(\Pi^r\) coincide.
Moreover, for we let

\[ \Pi^\varepsilon (u) := \begin{cases} \Pi^\varepsilon (u) & \text{for } u \in \mathcal{A}, \\ \infty & \text{for } u \in L^2(\Omega; \mathbb{R}^3) \setminus \mathcal{A}. \end{cases} \]

Moreover, from the first line of the inequalities above we deduce that all sequences \( \{u^\varepsilon\} \) are bounded in \( L^2 (\Omega; \mathbb{R}^3) \), without renaming it, as follows:

\[ \Pi^\varepsilon (u) > 0 \quad \text{for } u \in \mathcal{A}, \quad \infty \quad \text{for } u \in L^2(\Omega; \mathbb{R}^3) \setminus \mathcal{A}. \]

We view \( \mathcal{A} \) as the set of admissible displacements of a beam-like body, and \( TD \subset \mathcal{A} \) as the subset of all displacements compatible with Timoshenko’s kinematic Ansatz.

**Lemma 1** (Compactness lemma). Let a given sequence \( \{u^\varepsilon\} \subset \mathcal{A} \) be such that

\[ \sup_{\varepsilon} \Pi^\varepsilon (u^\varepsilon) < \infty. \]

Then, there are a subsequence \( \{u^\varepsilon\} \) (not relabeled) and an element \( u \) of \( TD \) such that \( u^\varepsilon \rightharpoonup u \) in \( H^1 (\Omega; \mathbb{R}^3) \). Moreover,

\[ (Eu^\varepsilon)_{\alpha\beta} \to 0 \quad \text{in } L^2 (\Omega). \]

**Proof.** By means of Hölder’s inequality, (2.2), the trace theorem for \( H^1 \)-functions, and Korn’s inequality, which holds because \( u^\varepsilon = 0 \) on \( \omega \times \{x_3 = 0\} \), we find:

\[
\Pi^\varepsilon (u^\varepsilon) \geq C \sum_{\alpha, \beta} \int_\Omega \left( (E u^\varepsilon)_{\alpha\beta}^2 + \varepsilon^2 (E u^\varepsilon)_{\alpha\beta}^2 + (E u^\varepsilon)_{\beta\alpha}^2 + \frac{1}{\varepsilon^2} (u^\varepsilon_{3, \alpha\beta} + u^\varepsilon_{\alpha, 3\beta})^2 \right) dx \\
- \| b^\varepsilon \|_{L^2(\Omega)} \| u^\varepsilon \|_{L^2(\Omega)} - \| c^\varepsilon \|_{L^2(\partial\Omega)} \| u^\varepsilon \|_{L^2(\partial\Omega)},
\]

\[ \geq C_1 \| \mathcal{E} u^\varepsilon \|_{L^2(\Omega)}^2 - C_2 \| u^\varepsilon \|_{H^1(\Omega)}, \]

\[ \geq C_3 \| u^\varepsilon \|_{H^1(\Omega)}^2 - C_2 \| u^\varepsilon \|_{H^1(\Omega)}, \]

\[ \geq C_4 \| u^\varepsilon \|_{H^1(\Omega)}^2 - C_5. \]

It follows from assumption (5.2) that \( \{u^\varepsilon\} \) is a bounded sequence in \( H^1 (\Omega; \mathbb{R}^3) \). Moreover, from the first line of the inequalities above we deduce that all sequences

\[ \left\{ \frac{(Eu^\varepsilon)_{\alpha\beta}}{\varepsilon^2} \right\}, \left\{ (Eu^\varepsilon)_{\alpha3} \right\}, \left\{ (Eu^\varepsilon)_{33} \right\}, \left\{ \frac{u^\varepsilon_{3, \alpha\beta} + u^\varepsilon_{\alpha, 3\beta}}{\varepsilon} \right\}, \]

are bounded in \( L^2 (\Omega) \). Hence, up to a subsequence, \( u^\varepsilon \rightharpoonup u \) in \( H^1 (\Omega; \mathbb{R}^3) \) for some \( u \in H^1 (\Omega; \mathbb{R}^3) \). In addition, from the fact that \( \left\{ \frac{(Eu^\varepsilon)_{\alpha\beta}}{\varepsilon^2} \right\} \) is a bounded sequence in \( L^2 (\Omega) \), it follows that \( u^\varepsilon_{\alpha, \beta} + u^\varepsilon_{\beta, \alpha} = 2 (Eu^\varepsilon)_{\alpha\beta} \to 0 \) in \( L^2 (\Omega) \); and, \( u^\varepsilon_{\alpha, \beta} + u^\varepsilon_{\beta, \alpha} \rightharpoonup u_{\alpha, \beta} + u_{\beta, \alpha} \) in \( L^2 (\Omega) \); we conclude that \( u_{\alpha, \beta} + u_{\beta, \alpha} = 0 \). Finally, with a similar argument, we can show that \( u^\varepsilon_{3, \alpha\beta} + u^\varepsilon_{\alpha, 3\beta} \rightharpoonup u_{3, \alpha\beta} + u_{\alpha, 3\beta} = 0 \) in \( L^2 (\Omega) \). Hence, \( u \in TD \). \( \square \)
6. The $\Gamma$-limit of $\{\Pi^\varepsilon\}$

In this section we analyze the $\Gamma$-limit of the sequence of functionals $\Pi^\varepsilon$. For

$$W^\varepsilon_\tau(E_{i3}) := \min_{g_{\alpha\beta}} \left\{ \Pi^\varepsilon \left( \sum_{\alpha} E_{\alpha 3} (e_\alpha \otimes e_3 + e_3 \otimes e_\alpha) + E_{33} e_3 \otimes e_3 \right. \right.$$

$$\left. \left. + \sum_{\alpha,\beta} \frac{g_{\alpha\beta}}{2} [e_\alpha \otimes e_\beta + e_\beta \otimes e_\alpha] \right\}, \right.$$

(6.1)

a simple calculation shows that

$$W^\varepsilon_\tau(E_{i3}) = \frac{1}{2} [4 \gamma (\varepsilon^2 (E_{13}^2 + E_{23}^2) + (\tau_1 - \frac{\tau_2^2}{\lambda + \mu})E_{33}^2].$$

(6.2)

Define

$$W_\tau(E_{i3}) := W^1_\tau(E_{i3}) = \frac{1}{2} [4 \gamma \varepsilon^2 (E_{13}^2 + E_{23}^2) + (\tau_1 - \frac{\tau_2^2}{\lambda + \mu})E_{33}^2],$$

so that

$$W^\varepsilon_\tau(E_{i3}) = W_\tau(\varepsilon E_{13}, \varepsilon E_{23}, E_{33}).$$

(6.3)

Define $\mathcal{W}_\tau : L^2(\Omega; \mathbb{R}^3) \to \mathbb{R}$ by

$$\mathcal{W}_\tau(u) := \begin{cases} \int_\Omega W_\tau(\varepsilon (Eu)_{i3}) \, dx, & u \in T \mathcal{D}, \\ \infty, & \text{otherwise} \end{cases}$$

(6.5)

and $\Pi : L^2(\Omega; \mathbb{R}^3) \to \mathbb{R}$ by

$$\Pi(u) := \begin{cases} \mathcal{W}_\tau(u) - \mathcal{F}(u), & u \in T \mathcal{D}, \\ \infty, & \text{otherwise} \end{cases}$$

(6.6)

**Theorem 2.** The sequence $\{\Pi^\varepsilon\}$ $\Gamma$-converges to $\Pi$ in the $L^2(\Omega; \mathbb{R}^3)$-topology, that is to say,

(i) (liminf inequality) for every sequence $\{u^\varepsilon\} \subset L^2(\Omega; \mathbb{R}^3)$ and for every $u \in L^2(\Omega; \mathbb{R}^3)$ such that $u^\varepsilon \to u$ in $L^2(\Omega; \mathbb{R}^3)$,

$$\liminf_{\varepsilon \to 0} \Pi^\varepsilon(u^\varepsilon) \geq \Pi(u);$$

(6.7)

(ii) (recovery sequence) for every $u \in L^2(\Omega; \mathbb{R}^3)$, there is a sequence $\{u^\varepsilon\} \subset L^2(\Omega; \mathbb{R}^3)$ such that $u^\varepsilon \to u$ in $L^2(\Omega; \mathbb{R}^3)$ and

$$\limsup_{\varepsilon \to 0} \Pi^\varepsilon(u^\varepsilon) \leq \Pi(u).$$

(6.8)

**Proof.** We start by proving (i). Let $\{u^\varepsilon\} \subset L^2(\Omega; \mathbb{R}^3)$ and $u \in L^2(\Omega; \mathbb{R}^3)$ such that $u^\varepsilon \to u$ in $L^2(\Omega; \mathbb{R}^3)$. The inequality in (6.7) is nontrivial only for

$$\liminf_{\varepsilon \to 0} \Pi^\varepsilon(u^\varepsilon) < \infty.$$

Thus, (by passing, if needed, to a subsequence) we may assume that

$$\lim_{\varepsilon \to 0} \Pi^\varepsilon(u^\varepsilon) = \liminf_{\varepsilon \to 0} \Pi^\varepsilon(u^\varepsilon) < \infty.$$
By Lemma 11, the sequence \( \{ u^\varepsilon \} \) converges weakly in \( H^1 (\Omega; \mathbb{R}^3) \) to an element \( u \in \mathcal{T}D \). Thus,

\[
\lim_{\varepsilon \to 0} \Pi^\varepsilon (u^\varepsilon) \geq \liminf_{\varepsilon \to 0} \left\{ \int_{\Omega} W^\varepsilon \left( E^\varepsilon u^\varepsilon \right) \, dx - \mathcal{F} (u^\varepsilon) \right\},
\]

\[
\geq \liminf_{\varepsilon \to 0} \left\{ \int_{\Omega} W^\varepsilon \left( (E^\varepsilon u^\varepsilon)_{13} \right) \, dx - \mathcal{F} (u^\varepsilon) \right\},
\]

\[
= \liminf_{\varepsilon \to 0} \left\{ \int_{\Omega} W^\varepsilon \left( (E^\varepsilon u^\varepsilon)_{13} \right), \varepsilon (E^\varepsilon u^\varepsilon)_{23}, (E^\varepsilon u^\varepsilon)_{33} \right) \, dx - \mathcal{F} (u^\varepsilon) \right\},
\]

\[
= \liminf_{\varepsilon \to 0} \left\{ \int_{\Omega} W^\varepsilon \left( (Eu^\varepsilon)_{13}, (Eu^\varepsilon)_{23}, (Eu^\varepsilon)_{33} \right) \, dx - \mathcal{F} (u^\varepsilon) \right\},
\]

\[
\geq \int_{\Omega} W^\varepsilon \left( (Eu)_{13}, (Eu)_{23}, (Eu)_{33} \right) \, dx - \mathcal{F} (u),
\]

\[
= W^\varepsilon (u) - \mathcal{F} (u) = \Pi(u).
\]

We point out that: in the first inequality we dispensed with second-order terms; the second inequality makes use of (5.1); in the third line we applied (6.3); the last inequality follows from the convexity of \( W^\varepsilon \) and the continuity of \( \mathcal{F} \).

We now prove (ii). Let \( u \in L^2 (\Omega; \mathbb{R}^3) \). Since inequality (6.8) is trivial for \( u \notin \mathcal{T}D \), we only need consider the case when \( u \in \mathcal{T}D \), and hence has the representation specified in (6.2).

At first, we restrict attention to functions \( u^0_1, \psi_a \) belonging to \( C^\infty (0, L) \) and equal to zero in a neighborhood of \( x_3 = 0 \); note that these functions form a dense subset of \( H^1_D (0, L) \). We consider the sequence whose typical term is:

\[
u^\varepsilon := u + \varepsilon^2 \hat{u},
\]

where \( \hat{u} \) is defined by

\[
\hat{u}_1 := -\eta (x_1 u^0_{3,3} + x_1 x_2 \psi_{1,3} - \frac{x_1^2}{2} \psi_{2,3} + \frac{x_2^2}{2} \psi_{2,3}),
\]

\[
\hat{u}_2 := -\eta (x_2 u^0_{3,3} - x_2 x_1 \psi_{2,3} + \frac{x_2^2}{2} \psi_{1,3} - \frac{x_1^2}{2} \psi_{1,3}),
\]

\[
\hat{u}_3 := 0,
\]

(6.9)

with \( \eta := \tau_2 / (\mu + \lambda) \). A simple computation yields:

\[
(E^\varepsilon u^\varepsilon)_{11} = (Eu)_{11} = -\eta (Eu)_{33},
\]

\[
(E^\varepsilon u^\varepsilon)_{12} = 0,
\]

\[
(E^\varepsilon u^\varepsilon)_{22} = (Eu)_{22} = -\eta (Eu)_{33},
\]

\[
(E^\varepsilon u^\varepsilon)_{\alpha3} = (Eu)_{\alpha3} + \varepsilon^2 (E\hat{u})_{\alpha3},
\]

\[
(E^\varepsilon u^\varepsilon)_{33} = (Eu)_{33}.
\]

Thus,

\[
W^\varepsilon (u^\varepsilon) = \int_{\Omega} \frac{1}{2} [4 \mu \eta^2 (Eu)_{33}^2 + 4 \lambda \eta^2 (Eu)_{33}^2 + 2 \tau_2 (Eu)_{33} (-2 \eta (Eu)_{33})
\]

\[
+ \tau_1 (Eu)_{33}^2 + (\varepsilon_r \varepsilon) \eta (Eu)_{\alpha3} + \varepsilon^2 (E\hat{u})_{\alpha3}^2]
\]

\[
+ \sum_{\alpha, \beta} \frac{\varepsilon_r \varepsilon}{\varepsilon^2} \varepsilon^2 (\hat{u}_{\alpha, \beta} + \hat{u}_{\alpha, \beta})^2] \, dx.
\]
Upon rearranging, we obtain:
\[
W^\varepsilon(u^\varepsilon) = \int_\Omega \frac{1}{2} \left[ (\tau_1 - \frac{\tau^2}{\mu + \lambda}) (Eu)_{33}^2 + \left( \frac{1}{\varepsilon_r} \right)^2 4\gamma ((Eu)_{a3} + \varepsilon^2 (E\ddot{u})_{a3})^2 + \sum_{\alpha, \beta} (\varepsilon - \varepsilon^2) \varepsilon^2 (\dot{u}_{3,\alpha\beta} + \dot{u}_{\alpha,3\beta})^2 \right] \, dx.
\]

By passing to the limit, it follows that
\[
\lim_{\varepsilon \to 0} W^\varepsilon(u^\varepsilon) = \int_\Omega \frac{1}{2} \left[ (\tau_1 - \frac{\tau^2}{\mu + \lambda}) (Eu)_{33}^2 \right] \, dx = W_r(u);
\]

since
\[
\lim_{\varepsilon \to 0} \mathcal{F}(u^\varepsilon) = \mathcal{F}(u),
\]
the proof of (ii) is achieved, under the smoothness assumptions stated at the beginning of this paragraph.

The general \( u \in TD \) case is handled by a standard diagonalization argument.

Indeed, let \( \{u_k\} \subset TD \cap C^\infty(\Omega; \mathbb{R}^3) \) such that \( u_k \to u \) in \( H^1(\Omega; \mathbb{R}^3) \); moreover, let \( \{u^\varepsilon_k\} \) be the recovery sequence for \( u_k \) as defined by (6.9). Since
\[
\lim \lim_{k \to \infty} \lim_{\varepsilon \to 0} \|u^\varepsilon_k - u\|_{H^1(\Omega)} = 0,
\]
and
\[
\lim \lim_{k \to \infty} \Pi^\varepsilon(u^\varepsilon_k) = \lim_{k \to \infty} \Pi(u_k) = \Pi(u)
\]
we can find an increasing map \( \varepsilon \to k_\varepsilon \) such that \( u^\varepsilon_{k_\varepsilon} \to u \) in \( H^1(\Omega; \mathbb{R}^3) \) and
\[
\lim \sup_{\varepsilon \to 0} \Pi^\varepsilon(u^\varepsilon_{k_\varepsilon}) \leq \lim_{k \to \infty} \lim_{\varepsilon \to 0} \Pi^\varepsilon(u^\varepsilon_k) = \Pi(u).
\]

\[\square\]

**Theorem 3.** Let \( u_{\min}^\varepsilon \) be the minimizer of \( \Pi^\varepsilon \), and let \( u_{\min} \) be the minimizer of \( \Pi \). Then,

(i) the sequence \( \Pi^\varepsilon(u_{\min}^\varepsilon) \) converges to \( \Pi(u_{\min}) \);

(ii) the sequence \( \{u_{\min}^\varepsilon\} \) converges to \( u_{\min} \) strongly in \( H^1(\Omega; \mathbb{R}^3) \).

**Proof.** As to the first claim, given that \( \sup_{\varepsilon} \{\Pi^\varepsilon(u_{\min}^\varepsilon)\} < \infty \), it follows from Lemma 1 that \( \{u_{\min}^\varepsilon\} \) converge weakly in \( H^1(\Omega; \mathbb{R}^3) \) (up to a subsequence) to \( u_{\lim} \in TD \). For a general \( u \in L^2(\Omega; \mathbb{R}^3) \), by (ii) of Theorem 2 there exists a recovery sequence \( \{u^\varepsilon\} \) such that
\[
\Pi(u) \geq \lim \sup_{\varepsilon \to 0} \Pi^\varepsilon(u^\varepsilon) \geq \lim \sup_{\varepsilon \to 0} \Pi^\varepsilon(u_{\min}^\varepsilon).
\]

From (i) of Theorem 2 we deduce that
\[
(6.10) \quad \Pi(u) \geq \lim \sup_{\varepsilon \to 0} \Pi^\varepsilon(u_{\min}^\varepsilon) \geq \lim \inf_{\varepsilon \to 0} \Pi^\varepsilon(u_{\min}^\varepsilon) \geq \Pi(u_{\lim}).
\]

Now, \( u \) can be chosen arbitrarily; we take \( u = u_{\min} \), and obtain \( \Pi(u_{\min}) \geq \Pi(u_{\lim}) \); as \( \Pi(u_{\min}) \leq \Pi(u_{\lim}) \), part (i) of the theorem follows and \( u_{\min}^\varepsilon \rightharpoonup u_{\min} \) in \( H^1(\Omega; \mathbb{R}^3) \).
As to the second claim, by means of (1.7), (6.1), and (6.3), we deduce that

\[
W^\varepsilon (u^\varepsilon_{\min}) - W^\tau (u_{\min}) \geq \int_\Omega [W^\varepsilon (E^\varepsilon u^\varepsilon_{\min}) - W^\tau ((Eu_{\min})_{i3})] \, dx,
\]

(6.11)

\[
= \int_\Omega [W^\varepsilon (E^\varepsilon u^\varepsilon_{\min}) - W^\tau ((E^\varepsilon u_{\min})_{i3})] \, dx,
\]

\[
\geq \int_\Omega [W^\varepsilon ((E^\varepsilon u^\varepsilon_{\min})_{i3}) - W^\tau ((E^\varepsilon u_{\min})_{i3})] \, dx.
\]

On recalling the form of $W^\varepsilon$ given in (6.2), it follows after some computations, with the aid of (2.2), that there exists a constant $C > 0$ for which

\[
\int_\Omega [W^\varepsilon ((E^\varepsilon u^\varepsilon_{\min})_{i3}) - W^\tau ((E^\varepsilon u_{\min})_{i3})] \, dx,
\]

\[
\geq C \sum_i \left( \| (Eu^\varepsilon_{\min})_{i3} - (Eu_{\min})_{i3} \|^2_{L^2} + \int_\Omega [(Eu^\varepsilon_{\min})_{i3} - (Eu_{\min})_{i3}] \, (Eu_{\min})_{i3} \, dx \right).
\]

Combination of this inequality with (6.11) yields:

\[
\Pi^\varepsilon (u^\varepsilon_{\min}) - \Pi (u_{\min}) \geq C \sum_i \left( \| (Eu^\varepsilon_{\min})_{i3} - (Eu_{\min})_{i3} \|^2_{L^2}
\right.

\[\left. + \int_\Omega [(Eu^\varepsilon_{\min})_{i3} - (Eu_{\min})_{i3}] \, (Eu_{\min})_{i3} \, dx \right) + F (u^\varepsilon_{\min}) - F (u_{\min}).
\]

As $u^\varepsilon_{\min} \to u_{\min}$ in $H^1 (\Omega; \mathbb{R}^3)$, we deduce that

\[
\int_\Omega [(Eu^\varepsilon_{\min})_{i3} - (Eu_{\min})_{i3}] \, (Eu_{\min})_{i3} \, dx \to 0,
\]

thence, by part (i) of the theorem, we find

\[
0 \geq \limsup_{\varepsilon \to 0} \sum_i \| (Eu^\varepsilon_{\min})_{i3} - (Eu_{\min})_{i3} \|^2_{L^2}.
\]

Thus, $(Eu^\varepsilon_{\min})_{i3} \to (Eu_{\min})_{i3}$ in $L^2 (\Omega)$. As the sequence $\{\Pi^\varepsilon (u^\varepsilon_{\min})\}$ is bounded, it follows from Lemma 1 that $(Eu^\varepsilon_{\min})_{\alpha\beta} \to 0 = (Eu_{\min})_{\alpha\beta}$ in $L^2 (\Omega)$; hence, $Eu^\varepsilon_{\min} \to Eu_{\min}$ in $L^2 (\Omega; \mathbb{R}^{3 \times 3})$. An application of Korn’s inequality concludes the proof. □

7. The $\Gamma$-limit potential in terms of limit displacements

Once again, recall that, for every $u \in TD$, there are $u^0_x, \psi_\alpha \in H^1_D (0, L)$ such that

(7.1) \[ u = u^0_1 (x_3) e_1 + u^0_2 (x_3) e_2 + (u^0_3 (x_3) + x_2 \psi_1 (x_3) - x_1 \psi_2 (x_3)) e_3 \]

(cf. (5.2)). It follows that the non-null associated strain components are:

$(Eu)_{13} = \frac{1}{2} (u^0_{1,3} - \psi_2), \quad (Eu)_{23} = \frac{1}{2} (u^0_{2,3} + \psi_1), \quad (Eu)_{33} = u^0_{3,3} + x_2 \psi_1 - x_1 \psi_2.$
Thus, in view of (6.3), the elastic potential (6.5) reads:

\[ W_r(u) = \frac{1}{2} \int_0^L \int_\omega \left( \gamma \varepsilon_r^{-2} [(u_{1,3}^0 - \psi_2)^2 + (u_{2,3}^0 + \psi_1)^2] \\
+ (\tau_1 - \frac{\tau_2^2}{\lambda + \mu}) (u_{3,3}^0 + x_2 \psi_{1,3} - x_1 \psi_{2,3})^2 \right) da, \]

whence, on choosing for the first two Cartesian axes the principal axes of inertia of the cross-section \( \omega \), we arrive at the one-dimensional elastic-energy functional of Timoshenko’s beam theory:

\[ W_r(u) = \frac{1}{2} \int_0^L \left( \frac{A^2}{\varepsilon_r^2} [(u_{1,3}^0 - \psi_2)^2 + (u_{2,3}^0 + \psi_1)^2] \\
+ (\tau_1 - \frac{\tau_2^2}{\lambda + \mu}) (u_{3,3}^0 + J_1 \psi_{1,3}^2 + J_2 \psi_{2,3}^2) \right) dx_3, \]

where

\[ A := \int_\omega da, \quad J_1 := \int_\omega x_2^2 da, \quad J_2 := \int_\omega x_1^2 da, \]

denote, respectively, the area and the moments of inertia with respect to the \( x_1 \) and \( x_2 \) axes.

It also follows from (7.1) that the load potential (4.8) takes the form:

\[ F(u) = \int_\Omega \left( b_1^0 u_1^0 + b_2^0 u_2^0 + b_3^0 u_3^0 + x_2 b_3^0 \psi_1 - x_1 b_3^0 \psi_2 \right) dv \\
+ \int_{\partial_\omega \Omega} \left( c_1^0 u_1^0 + c_2^0 u_2^0 + c_3^0 u_3^0 + x_2 c_3^0 \psi_1 - x_1 c_3^0 \psi_2 \right) da, \]

whence we deduce the one-dimensional load functional of Timoshenko’s beam theory:

\[ F(u) = \int_0^L \left( f_1 u_1^0 + f_2 u_2^0 + f_3 u_3^0 + m_1 \psi_1 + m_2 \psi_2 \right) dx_3 \\
+ F_1 u_1^0(L) + F_2 u_2^0(L) + F_3 u_3^0(L) + M_1 \psi_1(L) + M_2 \psi_2(L), \]

where, for almost every \( x_3 \in (0, L) \), we have set

\[ f_i(x_3) := \int_\omega b_i^0(x_1, x_2, x_3) da + \int_{\partial_\omega} c_i^0(x_1, x_2, x_3) ds, \]

\[ m_1(x_3) := \int_\omega x_2 b_3^0(x_1, x_2, x_3) da + \int_{\partial_\omega} x_2 c_3^0(x_1, x_2, x_3) ds, \]

\[ m_2(x_3) := - \int_\omega x_1 b_3^0(x_1, x_2, x_3) da - \int_{\partial_\omega} x_1 c_3^0(x_1, x_2, x_3) ds. \]

and

\[ F_i := \int_\omega c_i^0(x_1, x_2, L) da, \quad M_1 := \int_\omega x_2 c_3^0(x_1, x_2, L) da, \quad M_2 := - \int_\omega x_1 c_3^0(x_1, x_2, L) da. \]

\[ \text{Under the assumptions,} \]

\[ \int_\omega x_3 da = 0 \quad \text{and} \quad \int_\omega x_1 x_2 da = 0. \]
Interestingly, for the elastic potential as in (7.2) and the load potential as in (7.3), the total potential (6.6) may be decomposed into an axial-stretching part $\Pi_a$ and a bending part $\Pi_b$:

$$\Pi(u) = \Pi_a(u_0^3) + \Pi_b(u_0^0, \psi_\alpha),$$

where

$$\Pi_a(u_0^3) := \frac{1}{2}(\tau_1 - \frac{\tau_2^2}{\lambda + \mu}) A \int_0^L (u_{3,3}^0)^2 \, dx_3 - \int_0^L f_3 u_0^3 \, dx_3 + F_3 u_0^3(L),$$

and

$$\Pi_b(u_0^0, \psi_\alpha) := \frac{1}{2}(\tau_1 - \frac{\tau_2^2}{\lambda + \mu}) \int_0^L (u_{1,3}^0 - \psi_2^2 + (u_{2,3}^0 + \psi_1)^2 \, dx_3 + \frac{1}{2}\int_0^L f_1 u_1^0 + f_2 u_2^0 + m_1 \psi_1 + m_2 \psi_2 \, dx_3 - F_1 u_1^0(L) - F_2 u_2^0(L) - M_1 \psi_1(L) - M_2 \psi_2(L).$$

Thus, the minimization problem splits into two independent problems: the one for $u_0^3$, to be determined by minimizing the axial-stretching potential $\Pi_a$, the other for $u_0^0$ and $\psi_\alpha$, to be determined by minimizing the bending potential $\Pi_b$.

**Remark 2.** In the isotropic case, the elastic-energy density defined in (2.1) only depends on the two parameters $\lambda$ and $\mu$, because we have that $\gamma = \mu$, $\tau_1 = \lambda + 2\mu$, $\tau_2 = \lambda$. Consequently, the elastic modulus appearing in both the axial-stretching and the bending potentials reduces to the Young modulus of the material:

$$\tau_1 - \frac{\tau_2^2}{\lambda + \mu} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

**Remark 3.** The parameter $\varepsilon_r$ enters both the three-dimensional limit elastic-energy density (6.3) and the elastic part of the one-dimensional bending potential (7.6) (but not the elastic part of the axial-stretching potential (6.5)). If, ceteris paribus, we were to let $\varepsilon_r \to 0$ in (6.3), we would achieve effortlessly a justification of the Bernoulli-Navier beam model, whose total potential obtains by letting $\varepsilon_r \to 0$ in (7.6). In fact, in the envisaged limit, the shear strains $E_{\alpha 3}$ would be forced to converge to zero, which is tantamount to take $\psi_2 = u_{1,3}^0$ and $\psi_1 = -u_{2,3}^0$ in (7.1). This remark supports the engineer idea that the Bernoulli-Navier model is fine for very slender beams, whereas the Timoshenko model is preferable whenever beams are not so slender.

### 8. Summary and conclusions

In Section 2 we have defined the total potential $\Pi'$ of a linearly elastic three-dimensional beam-like body; we have denoted the minimizer of this potential by

$$\overline{\Pi}_{\text{min}} = \arg\min_{\Pi \in \Pi} \Pi'.$$
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and we have denoted the ratio between the diameter of the cross-section and the length of the beam by $\varepsilon_r$, a parameter that measures the slenderness of the beam-like body at hand. In Section 3, we have recorded the form of the Timoshenko displacement field, and we have briefly discussed its mechanical significance. In Section 4 we have constructed an $\varepsilon$-sequence of functionals $\Pi^\varepsilon$, such that $\Pi^\varepsilon$ is proportional to the ‘real’ functional $\Pi^r$ (see (4.10)). After studying, in Section 5, the compactness properties of sequences with bounded energy, we have identified the $\Gamma$-limit of the sequence $\Pi^\varepsilon$ in Section 6. The $\Gamma$-limit $\Pi^r$ turns out to be the total potential of a Timoshenko beam; in Section 7, we have written it in terms of the fields that parameterize the class of Timoshenko’s displacements (see (6.6) and (7.4)-(7.6)), and we have shown that these parameter fields can be determined by solving two independent minimum problems for, respectively, axial stretching and bending.

In accordance with the notation introduced in Theorem 3, let

$$u^\varepsilon_{\text{min}} = \arg\min_{u \in A} \Pi^\varepsilon(u) \quad \text{and} \quad u_{\text{min}} = \arg\min_{u \in T\mathcal{D}} \Pi(u).$$

The minimizer $u_{\text{min}}$ is a Timoshenko-type displacement. Since $\Pi^r$ essentially coincides with $\Pi^\varepsilon$, we deduce that $\Pi^r_{\text{min}}$ essentially coincides with $u^\varepsilon_{\text{min}}$. The exact relation between them follows immediately from (4.10) and (4.1), and is

$$(8.1) \quad u^\varepsilon_{\text{min}}(x) = R^\varepsilon \Pi^r_{\text{min}}(R^\varepsilon x).$$

In Theorem 3 we have shown that

$$u^\varepsilon_{\text{min}} \to u_{\text{min}} \text{ in } H^1(\Omega; \mathbb{R}^3);$$

thus, we can loosely say that, for small $\varepsilon$, $u^\varepsilon_{\text{min}}$ is well approximated by $u_{\text{min}}$. In particular, for $\varepsilon_r$ very small, $u^\varepsilon_{\text{min}}$ is well approximated by $u_{\text{min}}$; we concisely write this as $u^\varepsilon_{\text{min}} \approx u_{\text{min}}$. By (8.1) we therefore find an approximation of the “real” displacement $\Pi^r_{\text{min}}$, namely,

$$\Pi^r_{\text{min}}(x^r) \approx (R^\varepsilon)^{-1} u_{\text{min}}(R^{1/\varepsilon_r} x^r).$$

This relation states that the “real” displacement $\Pi^r_{\text{min}}$ is well approximated by a Timoshenko-type displacement and that such an approximation can be constructed with the use of the minimizer of the $\Gamma$-limit we found.

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