Numerical Double Sumudu Transform for Nonlinear Mixed Fractional Partial Differential Equations

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Abstract: In this paper, we are going to study a nonlinear fractional partial differential equations by double Sumudu transformation coupled with Adomian decomposition method or with variational iteration method. For this case, we choose an important fractional partial differential equations, such as parabolic-hyperbolic with mixed fractional derivatives (homogeneous and nonhomogeneous) types. Some examples are given here to illustrate efficiency of this method.

1 Introduction
There are no general methods for solving fractional partial differential equations, however, integral transform method is one of the famous methods in order to get the solution of some linear fractional partial differential equations [1, 2]. The non-linear equations cannot be generally be solved by linear integral transform due to the non-linearity terms, so we must couple these transforms with a known numerical method Double Sumudu transform used to solve, wave and Poisson equations [3] and nonhomogeneous wave equation with non-constant coefficients [4]. In this work, we focus on double Sumudu transform for solving some of linear / nonlinear mixed fractional partial differential equations, coupled with Adomian decomposition method [5] to solve such equations.

First of all, we start with definition of double Sumudu transform and some of it’s properties.

2 Preliminaries
In this section, we give some important definitions and notation which are needed in our work.

Definition 1 [6] The Riemann–Liouville fractional integral of order \( \alpha \) for a function \( f \) is defined as
\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx, \quad \alpha > 0
\]
The R-L fractional integral operator has the following properties:
- \( I^\alpha \) is linear
- \( I^0 = I \)
- \( I^{\alpha}\alpha \)

Definition 2 [6] The Caputo fractional derivative of positive order \( a \) for a function \( f \) is defined as
Some properties of fractional Caputo derivative and fractional R-L integral are:

Definition 3 [6] A two parameters Mittag-Leffler function is defined as:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \quad \alpha, \beta \in \mathbb{C}, \quad R(\alpha), \quad R(\beta) > 0$$

Lemma 1 A Mittag-Leffler function has an interesting properties [6]:

- $$E_{1,1}(x) = e^x$$
- $$E_{1,1}(x^2) = \cosh x$$
- $$xE_{1,1}(x^3) = \sinh x$$
- $$x^2E_{1,1}(x^2) = \sin x$$

Lemma 2 The Caputo derivatives of Mittag-Leffler are given as [7]:

- $$E_{\alpha,\beta}^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(x+\alpha+k)^n}{k! \Gamma(\alpha k + \beta)} \quad n \in \mathbb{N}$$
- $$\frac{\partial}{\partial x} (E_{\alpha,\beta}(ax^\alpha)) = a E_{\alpha,\beta}(ax^\alpha) \quad a > 0, \quad \alpha \in \mathbb{R}$$
- $$\frac{d}{dt} \left( t^{\alpha-1} E_{\alpha,\beta}(at^\alpha) \right) = t^{\alpha-1} E_{\alpha,\beta-1}(at^\alpha) \quad \gamma > 0$$

3 Fundamental Facts of the Single and Double Sumudu transform

The following definitions and properties of single and double Sumudu transform are necessary for our work. For all, we consider Sumudu transform of a function and it’s inverse is exist.

The following definition is introduced by (Nyimyua, 2007).

Definition 4 The Sumudu transform for the exponent order function $$f(x)$$ is given by

$$S[f(x)] = T(u) = \frac{1}{u} \int_{0}^{\infty} e^{-ux} f(x) \, dx \quad x > 0$$

And the inverse Sumudu transform of $$T(u)$$ is defined by:

$$S^{-1}[T(u)] = f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{ux} T(u) \frac{du}{u}$$

$$u$$ 

$$Re(u) < a$$
The Sumudu transform of the important function "Mittag-Leffler function" is given by

\[ S\{e^{\beta t}E_{\alpha,\beta}(\lambda t)\} = \frac{\mu^{\alpha-1}}{1-\lambda \mu} \]

In Table 1, Sumudu transform for some famous functions are given.

**Definition 5** The double Sumudu transform for the function \( f(t, x) \) is given by [8]

\[ S_{2}\{f(x, t)\} = T(u, v) = \frac{1}{uv} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{(x+1)}{uv}} f(x, t) dt \, dx \]

We state here some important properties of double Sumudu transform which are need

- \( S_{2}\{f(ax)g(bt)\} = T(au)G(bv) \)
- \( S_{2}\{f(x, t)\} = u^n T(u, v) \)

**Theorem 1** [9] If the double Sumudu transform of the function \( f(x, t) \) given by \( S_{2}\{f(x, t)\} = T(u, v) \), then:

\[ S_{2}\{x^n f(x, t)\} = u^n \sum_{k=0}^{n} a_k^n \, u^k \frac{\partial^k}{\partial u^k} T(u, v) \]

\[ S_{2}\{x^n t^m f(x, t)\} = u^n v^m \sum_{k=0}^{m} a_k^m \, b_k^n \, u^k \frac{\partial^k}{\partial u^k} \frac{\partial^{m-k}}{\partial v^{m-k}} T(u, v) \]

where \( a_0^n = 1, \, a_1^n = 1, \, a_k^n = \frac{\partial^k}{\partial u^k} T(u, v) \), \( k = 1, 2, \ldots, n \), \( a_k^n = \frac{\partial^k}{\partial u^k} T(u, v) \), similarly for \( b_k^m \)

The double Sumudu transform of the partial Caputo fractional derivatives are given in the following theorem

**Theorem 2** [10] Let the exponent order function \( f(x, t) \) has a continuous partial derivatives on \( R^+ \times R^+ \) and \( n-1 < a < n, \, m-1 < b < m \) then:

\[ S_{2}\{D_{x}^{a} f(x, t)\} = u^{-a} [T(u, v) - \sum_{j=0}^{a-1} u^j T_j(u, 0)] \]

\[ S_{2}\{D_{t}^{b} f(x, t)\} = v^{-b} [T(u, v) - \sum_{j=0}^{b-1} v^j T_j(u, 0)] \]

\[ S_{2}\{D_{x}^{a} D_{t}^{b} f(x, t)\} = u^{-a} v^{-b} [T(u, v) - \sum_{j=0}^{a+b-1} u^j v^j T_j(u, 0)] \]

\[ \sum_{j=0}^{a-1} \sum_{j=0}^{b-1} u^j v^j \frac{\partial^{a+b-j}}{\partial u^{a-j} \partial v^{b-j}} f(0, 0) \]

Where \( T_j(u, 0) = S_{2}\{\frac{\partial^j}{\partial x^j} f(0, t)\} \) and \( T_j(u, 0) = S_{2}\{\frac{\partial^j}{\partial x^j} f(x, 0)\} \)

**Lemma 3**

\[ S\{D_{x}^{a} D_{t}^{b} f(x, t)\} = v^{-a} \left( T(u, v) - \sum_{j=0}^{a-1} u^j T_j(u, 0) \right) - \sum_{j=0}^{b-1} v^j S\{D_{t}^{j} f(x, 0)\} \]

**Proof:**

According to Definition (5), and properties of fractional integral and Caputo derivatives, we get:

\[ S\{D_{x}^{a} D_{t}^{b} f(x, t)\} = S\{v^{m-a} D_{x}^{m-a} D_{t}^{b} f(x, t)\} = v^{m-a} S\{D_{x}^{m-a} D_{t}^{b} f(x, t)\} \]

\[ = v^{-a} \left( T(u, v) - \sum_{j=0}^{a-1} u^j T_j(u, 0) \right) - \sum_{j=0}^{b-1} v^j S\{D_{t}^{j} D_{x}^{a} f(x, 0)\} \]
3 A Decomposition Double Sumudu Transform Method (ADSTM)

To give an overview of the method, we consider the following nonlinear partial differential equation

\[ Lf(x,t) + Rf(x,t) = h(x,t) \]  

(1)

With auxiliary conditions, were \( L \) is linear operator, \( N \) is nonlinear operator and \( h \) is inhomogeneous term.

If we take the double Sumudu transform of both sides of equation (1), then we have

\[ S[Lf] = S[h - Rf] \Rightarrow T(u,v) = g(u,v) - K(u,v)S[Rf] \]

(2)

Where \( \Theta \) is obtained from definition of double Sumudu transform of the function \( f \) and the source term \( h \). Through operating the double inverse Sumudu transform for equation (2), we conclude that

\[ f(x,t) = S^{-1}[g(u,v)] - S^{-1}[K(u,v)S[Rf]] \]

(3)

Now, by consider the unknown function \( \hat{f}(x,t) \) as

\[ f = \sum_{i=0}^{\infty} f_i \]

(4)

And the nonlinear terms \( Rf \) as infinite series of the Adomian polynomials \( A_i \) [10]

\[ Nf = \sum_{i=3}^{\infty} A_i(f_0,f_1,f_2,...), \quad i = 0,1,2,... \]

(5)

Where

\[ A_i = \frac{1}{i!} \frac{d^i}{dt^i} \left[ N \left( \sum_{n=0}^{\infty} \lambda^n f_n \right) \right] \]

So, if we choose \( f_0 = S^{-1}[g(u,v)] \) and substitute (4) and (5) in equation (3), we obtain the solution recursively as

\[ f_1 = -S^{-1}[K(u,v)S[A_0]] \]

\[ f_2 = -S^{-1}[K(u,v)S[A_1]] \]

\[ f_3 = -S^{-1}[K(u,v)S[A_2]] \]

\[ f_4 = -S^{-1}[K(u,v)S[A_3]] \]

\[ \vdots \]

\[ f_{n+1} = -S^{-1}[K(u,v)S[A_n]] \]

4 Applications of Mixed Partial Differential Equations

Example 1 We consider the non-homogeneous parabolic- hyperbolic fractional mixed partial differential equation

\[ \left( \frac{\partial^{\alpha}}{\partial t^{\alpha}} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^\beta}{\partial x^\beta} \right) f(x,t) = \Gamma(\alpha + \beta + 1) \left( \frac{2^{\alpha} - \Gamma(1 + \beta)}{\Gamma(\alpha + 1)} t^\alpha \right) \]

(6)

With conditions

\[ f(0,0) = 0, D_x^0 f(0,t) = \Gamma(1 + \beta) t^{\alpha+\beta}, \quad D_t^\alpha f(x,0) = 0 = D_t^\beta f(x,0) \]

\[ D_x D_t^\alpha f(0,t) = 0, \quad f_x(0,t) = 0, \quad f_x(0,0) = 0, \quad f(0,t) = f(x,0) = 0 \]

Solution

By using Lemma (3) and Lemma(4), we obtain

\[ S[D_t^\alpha D_x^\beta f] = u^{-\alpha(\alpha+\beta)} T(u,v), \quad S[D_x^\alpha D_t^\beta f] = u^{-\alpha\beta} v^{-\beta} T(u,v), \quad S[D_x^\beta D_t^\alpha f] = u^{-\alpha\beta} v^{-\beta} T(u,v), \]

\[ S[D_x^\alpha D_t^\beta f] = u^{-\alpha\beta} (T(u,v) - \Gamma(1 + \beta) \Gamma(1 + \alpha + \beta) u^\alpha v^{\alpha+\beta}) \]

Hence, operating double Sumudu transform for Eq. (6), we have:
And by applying inverse double Sumudu transformation of the last equation, we have the exact solution of Eq.(6) as

\[ f(x, t) = x^\alpha t^\beta \]

When \( \alpha = 1, \beta = 2 \), the standard equation (6) has \( f(x, t) = x^2 t^2 \) as an exact solution.

**Example 2** Consider the following nonlinear parabolic-hyperbolic differential equation

\[ (D_x^2 - D_x^4)(D_x^2 - D_x^4)f(x,t) = (D_xf)^2 - (D_xf)^2 \]

(7)

With the conditions

\[ f(x, 0) = f_t(x, 0) = f_{xx}(x, 0) = e^x, f(0, t) = f_x(0, t) = f_{xx}(0, t) = f_{xxx}(0, t) = e^t \]

**Solution**

Single Sumudu transformation of the conditions yields:

\[ T(u, 0) = T_1(u, 0) = T_2(u, 0) = \frac{1}{1 - u}, \quad T(0, v) = T_1(0, v) = T_2(0, v) = T_3(0, v) = \frac{1}{1 - v} \]

And double Sumudu transformation of the integer order partial derivatives are:

\[ S(D_x^2 f) = v^{-3} \left( T(u, v) - \frac{1 + v + u^2}{1 - u} \right), \quad S(D_x^2 D_x^2 f) = v^{-2} u^{-1} \left( T(u, v) - \frac{1 + u}{1 - v} \right), \]

\[ S(D_x^2 D_x^2 D_x^2 f) = v^{-3} u^2 \left( T(u, v) - \frac{1 + u}{1 - v} \right) \]

Then, by taking double Sumudu transformation of (8) we obtain:

\[ \left( u^4 - u^2 v^2 - u^2 v + v^2 \right) T(u, v) = \frac{u^4(1 + v + u^2)}{1 - u} - u^2 v^2 \left( \frac{1 + u}{1 - v} + \frac{u^2}{1 - u} \right) \]

By simplify, we have:

\[ T(u, v) = \frac{1}{1 - u} + \frac{u^2 v^4}{u^4 - u^2 v^2 - u^2 v + v^2} \]

Taking inverse double Sumudu transformation, we get:

\[ f(x, t) = e^{x+t} + S^{-1} \left[ \frac{u^2 v^4}{u^4 - u^2 v^2 - u^2 v + v^2} S(f_x)^2 - (f_x)^2 \right] \]

Now, for the nonlinear terms, we applied the ADM method [9, 5], with \( f_0 = e^{x+t} \)

\[ A_0 = f_0^2 \quad B_0 = f_0^2 \]
\[ A_1 = 2f_0 f_1 \quad B_1 = 2f_0 f_1 \]
\[ A_2 = 2f_0 f_2 + f_1^2 \quad B_2 = 2f_0 f_2 + f_1^2 \]
\[ A_3 = 2f_0 f_3 + 2f_1 f_2 \quad B_3 = 2f_0 f_3 + 2f_1 f_2 \]
\[ \vdots \]

Since \( A_i = B_i, \forall i \in \mathbb{N} \), then \( S(f_x)^2 - (f_x)^2 = 0 \)

So, the approximate solution of equation (7) is
\begin{align*}
f(x, t) &= \sum_{i=0}^{n} f_i(x, t) = e^{x + t}
\end{align*}

Which is an exact solution.

Example 3 Consider the following non-linear parabolic - hyperbolic fractional differential equation

\begin{equation}
\left( D^\alpha_t - D^\beta_x \right) \left( D^\alpha_t - D^\beta_x \right) f(x, t) = \left( D^\beta_x f \right)^2 \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2
\tag{8}
\end{equation}

With conditions

\begin{align*}
f(x, 0) &= f_t(x, 0) = f_{tt}(x, 0) = E_\beta(x^\beta) + x E_{\beta,2}(x^\beta), \\
f(0, t) &= f_x(0, t) = E_{\beta - \alpha}(x^{\beta - \alpha})
\end{align*}

Solution:

Single Sumudu transformation of the fractional conditions are:

\begin{align*}
T_0(u, 0) &= T_1(u, 0) = T_2(u, 0) = \frac{1 + u}{1 - u^\beta}, \quad T_3(0, v) = T_4(0, v) = \frac{1}{1 - v^{\beta - \alpha}}
\end{align*}

By taking double Sumudu transformation of equation (8), as above example, then we obtain:

\begin{equation}
S\left[ \left( D^\alpha_t - D^\beta_x \right) \left( D^\alpha_t - D^\beta_x \right) f(x, t) \right] = S\left[ \left(D^\beta_x f \right)^2 - \left( D^\beta_x f \right)^2 \right]
\tag{9}
\end{equation}

\begin{align*}
S\left(D^\alpha_t D^\beta_x f \right) &= v^{-(\alpha + \beta)} \left[T(u, v) - \frac{(1 + u)(1 + v + u^\beta)}{1 - u^\beta} \right] \\
S\left(D^\alpha_t^2 D^\beta_x f \right) &= v^{-\alpha} u^{-\beta} \left[T(u, v) - \frac{(1 + u)(1 + u + u^\beta)}{1 - u^\beta} \right] \\
S\left(D^\alpha_x D^\beta_x f \right) &= u^{-\beta} \left[T(u, v) - \frac{(1 + u)(1 + v^\beta)}{1 - v^{\beta - \alpha}} \right]
\end{align*}

Substitute in equation (9), we have

\begin{align*}
&\left[ v^{-(\alpha + \beta)} - v^{-\alpha} u^{-\beta} - v^{-\alpha} u^{-\beta} + u^{-\beta} \right] T(u, v) = \left[ v^{-(\alpha + \beta)} \frac{(1 + u)(1 + v + u^\beta)}{1 - u^\beta} \right] - v^{-\alpha} u^{-\beta} \left[ \frac{(1 + u)(1 + v + u^\beta)}{1 - u^\beta} \right] + v^{-\alpha} u^{-\beta} \left[ \frac{(1 + u)(1 + v^\beta)}{1 - v^{\beta - \alpha}} \right] +
\end{align*}

\begin{align*}
S\left[ \left(D^\beta_x f \right)^2 - \left( D^\beta_x f \right)^2 \right] &
\end{align*}

So,

\begin{align*}
(u^\beta - v^\beta)(u^\beta - v^\alpha)(1 - v^{\beta - \alpha})(1 - u^\beta) T(u, v) &= u^{2\beta}(1 + u)(1 + v + v^2)(1 - v^{\beta - \alpha}) - u^\beta v^\alpha (1 + u)(1 - u^\beta - v^{\beta - \alpha} - u^\beta v^{\beta - \alpha}(1 + u) \left[ 1 + u^\beta \left( (1 + v^\beta)(1 - v^{\beta - \alpha}) - 1 \right) \right] + v^{\beta + \alpha}(1 + u)(1 - u^\beta) + u^{2\beta} v^{\beta + \alpha} S\left[ \left(D^\beta_x f \right)^2 - \left( D^\beta_x f \right)^2 \right]
\end{align*}

By simplified, we have:

\begin{align*}
T(u, v) &= \frac{(1 + u)(u^\beta - v^\beta)(u^\beta - v^\alpha)(1 - v^{\beta - \alpha})(1 - u^\beta)}{(u^\beta - v^\beta)(u^\beta - v^\alpha)(1 - v^{\beta - \alpha})(1 - u^\beta)} + \frac{u^{2\beta} (1 + u)(1 + v)(1 - v^{\beta - \alpha})(v - v^\alpha)(v^\beta - 1)(v^{\beta - \alpha} - v^\alpha)}{(u^\beta - v^\beta)(u^\beta - v^\alpha)(1 - v^{\beta - \alpha})(1 - u^\beta)} + \frac{u^{2\beta} v^{\beta + \alpha} S\left[ \left(D^\beta_x f \right)^2 - \left( D^\beta_x f \right)^2 \right]}{(u^\beta - v^\beta)(u^\beta - v^\alpha)(1 - v^{\beta - \alpha})(1 - u^\beta)}
\end{align*}

Note that, the second fraction of linear part is goes to zero when \( \alpha \to 1, \quad \beta \to 2 \)

So, by applying the inverse double Sumudu transformation for the first term, we can find an approximation solution of the linear part of equation (9) as

\begin{equation}
f(x, t) = \left[ E_\beta(x^\beta) + x E_{\beta,2}(x^\beta) \right] E_{\beta - \alpha}(x^{\beta - \alpha})
\end{equation}
Now, it is time to apply ADM to derive the approximate solution of equation (9) as following:

\[ A_7 = \left( D_\beta x^2 f_0 \right)^2, \quad B_7 = \left( D_\beta x^2 f_0 \right)^2\]
\[ A_7 = 2(D_\beta x^2 f_0)(D_\beta x^2 f_0), \quad B_7 = 2(D_\beta x^2 f_0)(D_\beta x^2 f_0)^2\]
\[ A_7 = 2(D_\beta x^2 f_0)(D_\beta x^2 f_0) + (D_\beta x^2 f_0)^2, \quad B_7 = 2(D_\beta x^2 f_0)(D_\beta x^2 f_0)^2 + (D_\beta x^2 f_0)^2\]
\[ A_7 = 2(D_\beta x^2 f_0)(D_\beta x^2 f_0) + 2(D_\beta x^2 f_0)(D_\beta x^2 f_0)^2\]

By considering

\[ f_0(x,t) = \left[ E_\beta\left(x^\beta\right) + x E_\beta.z(x^\beta)\right] E_{\alpha-\beta}(t^{\alpha-\beta}), \quad \text{and the fact} \quad D_\alpha^\beta E_\beta(x^\beta) = E_\beta(x^\beta)\]

Then for all \( t \in N \) we can conclude that \( D_\alpha^\beta f_i = D_\alpha^\beta f_i \)

Thus \( A_i = B_i \quad \forall \ i \in N \) and the nonlinear term of Eq. (9) is vanish.

Hence the approximate solution of Eq. (8) is:

\[ f(x,t) = \sum_{i=0}^n f_i(x,t) = \left[ E_\beta\left(x^\beta\right) + x E_\beta.z(x^\beta)\right] E_{\alpha-\beta}(t^{\alpha-\beta})\]

Also, it is an exact solution for equation (8) in this case.

Exact and approximate solutions for equation (8), are plotted in figure 1.

Example 4 Consider the following inhomogeneous fractional nonlinear partial differential equation with variable constant

\[D_\alpha^{\beta}f(x,t) - x^2 = \frac{-X}{36}(D_\alpha^{\beta}f(x,t))^2, \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2\]  \hspace{1cm} (10)

With condition \( f(x,0) = 0 \)

Solution:

Operated the Double Sumudu transform for both sides of equation(10)

\[ S[D_\alpha^{\beta}f(x,t) - x^2] = S\left[\frac{-X}{36}(D_\alpha^{\beta}f(x,t))^2\right]\]
\[ v^{-\alpha}\mathcal{T}(u,v) - 6u^2 = S\left[\frac{-X}{36}(D_\alpha^{\beta}f(x,t))^2\right]\]
\[ \mathcal{T}(u,v) = 6u^2v^{-\alpha} + v^{-\alpha}S\left[\frac{-X}{36}(D_\alpha^{\beta}f(x,t))^2\right]\]
Operating the inverse double Sumudu transform, we get
\[ f(x, t) = \frac{x^{3\alpha}}{\Gamma(1 + \alpha)} + s^{-1} \left[ \frac{x}{\frac{36}{\Gamma(5\alpha)}} \left( D_x^\beta s f(x, t) \right)^2 \right] \]

Applying ADDSTM, with \( f_0 = \frac{x^{3\alpha}}{\Gamma(1 + \alpha)} \), to get
\[ A_0 = (D_x^\beta f_0)^2 \]
\[ A_1 = 2(D_x^\beta f_0)(D_x^\beta f_1) \]
\[ A_2 = 2(D_x^\beta f_0)(D_x^\beta f_2) + (D_x^\beta f_2)^2 \]
\[ A_3 = 2(D_x^\beta f_0)(D_x^\beta f_3) + 2(D_x^\beta f_1)(D_x^\beta f_2) \]
\[ \vdots \]
\[ f_1 = s^{-1} \left[ \frac{x}{\frac{36}{\Gamma(5\alpha)}} \left( \frac{1}{\Gamma(1 + 2\alpha)} x^{3 - 2\alpha} \right)^2 \right] \]
\[ f_2 = s^{-1} \left[ \frac{x}{\frac{36}{\Gamma(5\alpha)}} \left( \frac{12\Gamma(1 + 2\alpha)\Gamma(8 - 2\beta)}{\Gamma^2(4 - \beta)\Gamma(8 - 3\beta)\Gamma(1 + 3\alpha)} x^{10 - 4\beta - 4\alpha} \right)^2 \right] \]
\[ f_3 = s^{-1} \left[ \frac{x}{\frac{36}{\Gamma(5\alpha)}} \left( \frac{12\Gamma(12 - 4\beta)}{\Gamma(4 - \beta)\Gamma(12 - 5\beta)} + \frac{\Gamma(8 - 2\beta)}{\Gamma(8 - 3\beta)} x^{15 - 6\beta - 7\alpha} \right)^2 \right] \]

Consequently,
\[ f_1 = s^{-1} \left[ \frac{x}{\frac{36}{\Gamma(5\alpha)}} \left( \frac{1}{\Gamma(1 + 2\alpha)} x^{3 - 2\alpha} \right)^2 \right] = s^{-1} \left[ \frac{x}{\frac{36}{\Gamma(5\alpha)}} \left( \frac{1}{\Gamma(1 + 2\alpha)} x^{3 - 2\alpha} \right)^2 \right] = ax^{7 - 2\beta - 2\alpha} \]
\[ f_2 = s^{-1} \left[ \frac{x}{\frac{36}{\Gamma(5\alpha)}} \left( \frac{12\Gamma(1 + 2\alpha)\Gamma(8 - 2\beta)}{\Gamma^2(4 - \beta)\Gamma(8 - 3\beta)\Gamma(1 + 3\alpha)} x^{10 - 4\beta - 4\alpha} \right)^2 \right] = bx^{11 - 4\beta - 2\alpha} \]

By the same manipulate

Where
\[ \alpha = \frac{-\Gamma(1 + 2\alpha)}{\Gamma(1 + 3\alpha)\Gamma^2(4 - \beta)}, \quad b = \frac{\Gamma(1 + 4\alpha)\Gamma(1 + 2\alpha)\Gamma(8 - 2\beta)}{3\Gamma(1 + 5\alpha)\Gamma^3(4 - \beta)\Gamma(8 - 3\beta)\Gamma(1 + 3\alpha)}, \quad c = \frac{\Gamma(1 + 6\alpha)\Gamma^2(12 - 4\beta) + \Gamma(8 - 2\beta)^2}{\Gamma^2(4 - \beta)\Gamma(12 - 5\beta) + \Gamma(8 - 3\beta)^2} \]

Hence, the approximate solution of equation (10) according to the ADDSTM is
\[ f(x, t) = \sum_{n=0}^{\infty} f_n(x, t) = f_0 + f_1 + f_2 + f_3 + \cdots \]
\[ = \frac{x^{3\alpha}}{\Gamma(1 + \alpha)} + ax^{7 - 2\beta - 2\alpha} + bx^{11 - 4\beta - 2\alpha} + cx^{15 - 6\beta - 7\alpha} + \cdots \]
\[ = x^{3\alpha} \left( \frac{1}{\Gamma(1 + \alpha)} + ax^{7 - 2\beta - 2\alpha} + bx^{11 - 4\beta - 2\alpha} + cx^{15 - 6\beta - 7\alpha} + \cdots \right) \]
For standard case $\alpha = 1; \ \beta = 2$, the approximate solution after three iteration is
\[ f(x,t) = x^3 \left( t - \frac{1}{3} t^3 + \frac{12}{15} t^5 - \frac{17}{315} t^7 + \cdots \right) = x^3 \tanh \tanh t \]
in closed form.

Which is an exact solution for standard equation (10).

Table 1 Sumudu transform for some famous functions.

| $f(x)$       | $S[f(x)] = T(u)$       | $f(x)$       | $S[f(x)] = T(u)$       |
|--------------|------------------------|--------------|------------------------|
| 1            | 1                      | $x^\alpha$   | $\Gamma(1 + \alpha)u^\alpha$ |
| $e^x$        | $\frac{1}{1 - au}$    | $\sin \sin ax$ | $\frac{au}{1 + a^2u^2}$ |
| $\cos \cos ax$ | $\frac{1}{1 + a^2u^2}$ | $\sinh \sinh ax$ | $\frac{au}{1 - a^2u^2}$ |
| $\frac{x^\alpha e^{\alpha x}}{\Gamma(1 + \alpha)}$ | $\frac{u^\alpha}{(1 - au)^{1+\alpha}}$ | $\cosh \cosh ax$ | $\frac{1}{1 - a^2u^2}$ |

5 Discussion

In this paper, double Sumudu transform method is proved to be an efficient tool for solving linear mixed fractional differential equations, in special case, with constant coefficients. However, for the non-linear case, we proved double Sumudu transform coupled with numerical method 'ADMM' is a powerful role to solve such equations. All result which obtained by help of Matlab and Mathcad programs.

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