RIGIDIFICATION OF HOMOTOPY ALGEBRAS OVER FINITE
PRODUCT SKETCHES

BRUCE R. CORRIGAN-SALTER

1. Introduction

For $n \geq 1$ let $C_n$ denote the category whose objects are natural numbers $0, 1, \ldots, n$, and that has one non-identity morphism $p^0_k: 0 \to k$ for each $k = 1, \ldots, n$. Also, by $C_0$ we will denote the category with only one object $0$ and the identity morphism only. Given a small category $B$, an $n$-fold cone in $B$ is a functor $\alpha: C_n \to B$. It will be convenient, given an $n$-fold cone $\alpha$, to denote $n$ by $|\alpha|$, $\alpha(k)$ by $\alpha_k$ and $\alpha(p^0_k)$ by $p^\alpha_k$. We will call the morphism $p^\alpha_k$ the $k$-th projection of $\alpha$.

Definition 1.1. A finite product sketch is a pair $(B, \kappa)$ where $B$ is a small category and $\kappa$ is a set of cones in $B$.

Sometimes, when this will not lead to a confusion, we will write $B$ to refer to the sketch $(B, \kappa)$.

Definition 1.2. Let $D$ be a category closed under finite products and let $(B, \kappa)$ be a finite product sketch. A strict $(B, \kappa)$-algebra with values in $D$ is a functor $A: B \to D$ such that for any cone $\alpha \in \kappa$ the morphism

$$\prod_{k=1}^{|\alpha|} A(p^\alpha_k): A(\alpha_0) \to \prod_{k=1}^{|\alpha|} A(\alpha_k)$$

is an isomorphism. For a cone $\alpha: C_0 \to B$ this condition means that $\alpha_0$ is a terminal object in $D$. A morphism of strict algebras is a natural transformation of functors.

Finite product sketches and their strict algebras have long been present in categorical algebra as a formalism for describing algebraic structures. Giving a strict $(B, \kappa)$-algebra amounts to describing some algebraic object in $D$ of the type determined by the sketch $(B, \kappa)$. We illustrate this by a few examples.

Example 1.3. Let $B$ be the category consisting of two objects $b_1, b_2$, and three non-identity morphisms $\varphi_1, \varphi_2, \mu: b_2 \to b_1$. Let $\alpha: C_2 \to B$ be given by $p^\alpha_k = \varphi_k$. 
A strict algebra over the sketch \((B, \{a\})\) with values in the category of sets, \(\text{Sets} \) is a functor

\[ A : B \to \text{Sets} \]

such that \(A(b_2) \equiv A(b_1) \times A(b_1)\) and \(A(\varphi_1), A(\varphi_2) : A(b_2) \to A(b_1)\) are the projection maps. In effect the category of strict \((B, \{a\})\)-algebras is equivalent to the category whose objects are sets \(Y\) equipped with a binary operation \(\mu_Y : Y \times Y \to Y\), and whose morphisms are maps \(f : Y \to Y'\) satisfying

\[ \mu_Y(f(y_1), f(y_2)) = f(\mu_Y(y_1, y_2)) \]

It is not hard to modify this example to obtain sketches whose algebras are sets with a binary operation that satisfies some further conditions (is unital, associative, commutative, has inverse etc.).

**Example 1.4.** Let \(B\) be a small category and let \(W \subseteq B\) be a subcategory of \(B\). For each morphism \(\varphi \in W\) let \(\alpha_\varphi : C_1 \to B\) be the functor given by \(p_1^{\alpha_\varphi} = \varphi\). Giving a strict algebra over the sketch \((B, \{\alpha_\varphi\}_{\varphi \in W})\) with values in a category \(D\) amounts to giving a functor \(A : B \to D\) such that \(A(\varphi)\) is an isomorphism for each \(\varphi \in W\).

The next few examples are somewhat more complex. Their origin will be explained later in this section.

**Example 1.5.** Let \(\Gamma^{op}\) denote the category whose objects are finite sets \([n] = \{0, 1, \ldots, n\}, n \geq 0\) and whose morphisms are maps of sets \(\varphi : [n] \to [m]\) satisfying \(\varphi(0) = 0\). For \(n \geq 1\) let \(\alpha^n : C_n \to \Gamma^{op}\) be the functor such that \(\alpha_0 = [n]\), \(\alpha_k = [1]\) for \(k > 0\) and \(p_k^{\alpha^n} : [n] \to [1]\) is the map given by

\[ p_k^{\alpha^n}(i) = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{otherwise} \end{cases} \]

Also, let \(\alpha^0 : C_0 \to \Gamma^{op}\) be given by \(\alpha_0^0 = [0]\). Giving a strict \((\Gamma^{op}, \{\alpha^n\}_{n \geq 0})\)-algebra \(A : \Gamma^{op} \to \text{Sets}\) is equivalent to defining a structure of an abelian monoid on the set \(A([1])\).

**Example 1.6.** Let \(\Delta\) be the category whose objects are finite ordered sets \([n] = \{0, 1, \ldots, n\}, n \geq 0\), and whose morphisms are non-decreasing functions. Let \(\Delta^{op}\) denote the opposite category. For \(m \geq 1\) define a sketch \((\Delta^{op}, \kappa^m)\) as follows. We have

\[ \kappa^m = \{\alpha^k\}_{k \geq 0} \]

where for \(k < m\) the functor \(\alpha^k : C_0 \to \Delta^{op}\) is given by \(\alpha_0^k = [k]\). For \(k \geq m\) let \(N_k = \binom{k}{m}\) and let \(f_k\) be a bijection between the set of natural numbers \(\{1, \ldots, N_k\}\) and the set of strictly increasing functions \(\varphi : [m] \to [k]\) satisfying \(\varphi(0) = 0\). Define \(\alpha^k : C_{N_k} \to \Delta^{op}\) by setting \(p_i^k = f_k(i)\). For \(m = 1\) giving a strict \((\Delta^{op}, \kappa^m)\)-algebra \(A : \Delta^{op} \to \text{Sets}\) amounts to defining a group structure on the set \(X([1])\). For \(m > 1\) giving a strict \((\Delta^{op}, \kappa^m)\)-algebra \(A : \Delta^{op} \to \text{Sets}\) is equivalent to specifying an abelian group structure on \(A([m])\) (see [5]).
Example 1.7. Let $\Delta$ be defined as in Example 1.6. For $n \geq 1$ and $1 \leq k \leq n$ let $i_k^n : [1] \to [n]$ be the morphism in $\Delta$ given by $i_k^n(0) = k - 1$ and $i_k^n(1) = k$. Also, let $j_1^n, j_2^n : [0] \to [n]$ be given by $j_1^n(0) = 0$ and $j_2^n(0) = n$.

Let $I$ be the category with two objects $i_0, i_1$ and one non-identity morphism $\pi : i_0 \to i_1$. Consider the product category $\Delta^{op} \times I$. For $n \geq 1$ let

$$\alpha^n : C_n \to \Delta^{op} \times I$$

be the cone such that $p_k^n$ is the morphism $i_k^n \times id_i : ([n], i_1) \to ([1], i_1)$, and let $\alpha^0$ be the 0-fold cone given by $\alpha^0_0 = ([0], i_1)$. Also, for $n \geq 0$ let

$$\beta^n, \gamma^n : C_2 \to \Delta^{op} \times I$$

be defined by $p_1^n = j_1^n \times id_{i_0}$, $p_2^n = j_2^n \times id_{i_0}$, and $p_2^n = p_2^n = id_n \times \pi$. Define

$$\kappa := \{\alpha^n\}_{n \geq 0} \cup \{\beta^n\}_{n \geq 0} \cup \{\gamma^n\}_{n \geq 0}$$

Giving a strict $(\Delta^{op} \times I, \kappa)$-algebra

$$X : \Delta^{op} \times I \to \text{Sets}$$

is equivalent to describing an associative monoid structure on the set $X([1], i_1)$ and an action of this monoid on the set $X([0], i_0)$. Giving a morphism of strict algebras $\varphi : X \to X'$ amounts to giving maps $\varphi_1 : X([1], i_1) \to X'([1], i_1)$ and $\varphi_2 : X([0], i_0) \to X([0], i_0)$ where $\varphi_1$ is a homomorphism of monoids and $\varphi_2$ preserves the action:

$$\varphi_2(mx) = \varphi_1(m)\varphi_2(x)$$

for all $x \in X([0], i_0)$ and $m \in X([1], i_1)$ (see [14]).

Finite product sketches have been shown to be very useful for describing various structures that appear in homotopy theory. In this context we need to replace the notion of a strict algebra over a sketch by a more flexible notion of a homotopy algebra. Let $\text{Spaces}$ denote the category of simplicial sets.

Definition 1.8. Let $(B, \kappa)$ be a finite product sketch. A homotopy $(B, \kappa)$-algebra is a functor

$$X : B \to \text{Spaces}$$

such that for any cone $\alpha \in \kappa$ the map

$$\prod_{|\alpha|} X(p^n_k) : X(\alpha_0) \to \prod_{|\alpha|} X(\alpha_k)$$

is a weak equivalence. For a 0-fold cone $\alpha$ this condition means that $X(\alpha_0) = *$.

The sketches described in Examples 1.4–1.7 were all constructed because of the interest in the homotopy algebras they define. Homotopy algebras over the sketch defined in Example 1.4 are functors $B \to \text{Spaces}$ that map morphisms of $W$ to weak equivalences. Categories of such functors were studied e.g. by Dwyer and
Kan in [8] who also gave several applications of such functors in homotopy theory. Homotopy algebras over the sketch defined in Example 1.5 are the special \( \Gamma \)-spaces that were introduced by Segal in [16] to describe the structure of infinite loop spaces. In [5] Bousfield showed that the homotopy category of homotopy algebras over the sketch \((\Delta^{op}, \kappa^m)\) defined in Example 1.6 is equivalent to the homotopy category of \(m\)-fold loop spaces. Finally, homotopy algebras over the sketch defined in Example 1.7 appear in the work of Prezma [14] as a formalism that describes “homotopy actions”, i.e. actions of loop spaces on topological spaces.

Obviously any strict algebra over a sketch \((B, \kappa)\) with values in \(\text{Spaces}\) is also a homotopy \((B, \kappa)\)-algebra. A natural question is if the opposite statement also holds. More precisely, we will say that a morphism \(\varphi: X \to X'\) of homotopy algebras is a weak equivalence if the map \(\varphi_b: X(b) \to X'(b)\) is a weak equivalence for all \(b \in B\). We can ask if it is true that for any homotopy \((B, \kappa)\)-algebra \(X\) there exists a strict \((B, \kappa)\)-algebra \(X'\) such that \(X' \simeq X\). We will call this a rigidification problem since it asks whether a lax, homotopy structure can be replaced by an equivalent algebraic structure.

The examples of finite product sketches listed above show that rigidification of homotopy algebras is not always possible. Take e.g. the sketch \((\Delta^{op}, \kappa^2)\) described in Example 1.6. Strict algebras over this sketch with values in \(\text{Spaces}\) are simplicial abelian groups, while homotopy algebras correspond to double loop spaces. Since it is not true that every double loop space is weakly equivalent to an abelian group, it is in general not possible to find a strict \((\Delta^{op}, \kappa)\)-algebra weakly equivalent to a given homotopy algebra.

Badzioch [1] and Bergner [4] showed that homotopy algebras can be rigidified if the sketch \((B, \kappa)\) is of a special form. In order to explain their results we will need a couple of definitions.

**Definition 1.9.** A product cone in a category \(B\) is a cone \(\alpha: C_n \to B\) such that the categorical product \(\prod_{k=1}^{n} \alpha(k)\) exists in \(B\) and that the map

\[
\prod_{k=1}^{n} \alpha(p^k_k): \alpha(0) \to \prod_{k=1}^{n} \alpha(k)
\]

is an isomorphism.

**Definition 1.10** (cf. [4, 3.1]). Let \(S\) be a set. An \(S\)-sorted algebraic theory is a finite product sketch \((T, \kappa)\) satisfying the following properties:

i) objects \(t_s \in T\) are indexed by all \(n\)-tuples, \(s = (s_1, \ldots, s_n)\) (with possible repetitions) where and \(s_i \in S\) and \(n \geq 0\);

ii) for an \(n\)-tuple \(s = (s_1, \ldots, s_n)\) the object \(t_s\) is a categorical product of \(t_{s_1}, \ldots, t_{s_n}\) (by abuse of notation we denote will denote by \(t_{s_1}\) the object of \(T\) indexed by the 1-tuple \((s_1)\);
iii) the set $\kappa$ consists of all product cones $\alpha \sigma$ indexed by $n$-tuples $s = (s_1, \ldots, s_n)$ with $n \geq 0$, where $\alpha_0 = t_\bot$ and $\alpha_k = t_{s_k}$ for $k = 1, \ldots, n$.

We will call $S$ the set of sorts for $T$. We will also call a sketch $T$ a multi-sorted algebraic theory if $T$ is an $S$-sorted algebraic theory for some set $S$. Notice that a strict algebra over a multi-sorted algebraic theory $T$ with values in $\text{Spaces}$ is just a product preserving functor $T \to \text{Spaces}$, and a homotopy $T$-algebra is a functor that preserves products up to a weak equivalence.

The result of Bergner and Badzioch can be now stated as follows:

**Theorem 1.11 ([1], [4]).** If $T$ is a multi-sorted algebraic theory then any homotopy $T$-algebra is weakly equivalent to a strict $T$-algebra.

As we have already noticed this theorem cannot be directly extended to arbitrary finite product sketches. Our main goal, however, is to show that a variant of this rigification result still holds. Given a finite product sketch $B$ we can consider the homotopy category of homotopy $B$-algebras, that is the category obtained by taking the category of all homotopy $B$-algebras and inverting weak equivalences. We will show that the following holds:

**Theorem 1.12.** For any finite product sketch $B$ there exists a simplicial multi-sorted algebraic theory $F, B'$ such that the homotopy category of homotopy $B$-algebras is equivalent to the homotopy category of homotopy $F, B'$-algebras. Moreover, the construction of $F, B'$ is functorial in $B$.

Combining this fact with Theorem 1.11 we will obtain

**Corollary 1.13.** For any finite product sketch $B$ there exists a simplicial multi-sorted algebraic theory $F, B'$ such that the homotopy category of homotopy $B$-algebras is equivalent to the homotopy category of strict $F, B'$-algebras.

Thus, the homotopy structure described by any finite product sketch is equivalent to some algebraic structure, but in general the sketches describing these two structures will be different.

As an application of Corollary 1.13 we partially resolve another natural question related to homotopy algebras over finite product sketches. Namely, assume that $G: (B_1, \kappa_1) \to (B_2, \kappa_2)$ is a morphism of sketches. That is, $G$ is a functor such that for any $\alpha \in \kappa_1$ the cone $G \alpha$ is in $\kappa_2$. Obviously if $X: (B_2, \kappa_2) \to \text{Spaces}$ is a homotopy $(B_2, \kappa_2)$-algebra then $G^* X := XG$ is a homotopy $(B_1, \kappa_1)$-algebra. One can ask what conditions on $G$ guarantee that the functor $G^*$ is an equivalence of the homotopy categories of homotopy algebras. This can be answered as follows. Since the passage from a
finite product sketch \((B, \kappa)\) to its associated multisorted algebraic theory \(\mathcal{F}_* B'\) is natural in \((B, \kappa)\) a morphism of sketches \(G: (B_1, \kappa_1) \to (B_2, \kappa_2)\) yields a functor of the simplicial algebraic theories \(G: \mathcal{F}_* B'_1 \to \mathcal{F}_* B'_2\). We will show:

**Theorem 1.14.** Consider a morphism of finite product sketches

\[ G : (B_1, \kappa_1) \to (B_2, \kappa_2) \]

Assume that \(G\) is a bijection on the sets of objects of \(B_1\) and \(B_2\) and that the induced map on the sets of cones \(\kappa_1 \to \kappa_2\) is also a bijection. The morphism \(G\) induces an equivalence of the homotopy categories of homotopy algebras if and only if the induced functor

\[ G : \mathcal{F}_* B'_1 \to \mathcal{F}_* B'_2 \]

between the associated multi-sorted algebraic theories is a weak equivalence of simplicial categories.

This fact parallels the result of Dwyer and Kan [8, 2.5] who proved an analogous statement for functors between sketches of the form described in Example 1.4, and a theorem of Badzioch [2, 1.6], that gives a similar criterion for functors between single-sorted semi-theories, i.e. finite product sketches of a specific type (cf. Definition 3.1).

### 2. Organization of Paper

We start in section 3 by considering a specific type of finite product sketches which we refer to as multi-sorted semi-theories, and in sections 4 through 7 to show that we can rigidify homotopy algebras over multi-sorted semi-theories by constructing an associated multi-sorted algebraic theory. In particular in section 4 we show that the setup, paralleling [2], can be used to define model category structures for the categories of homotopy algebras as well as strict algebras over a multi-sorted semi-theory. In section 5 we show that for any multi-sorted semi-theory \(C\) we can construct to a multi-sorted algebraic theory \(\bar{C}\) without changing the category of strict algebras. We also give an explicit, combinatorial constriction of \(\bar{C}\) in the case when \(C\) is a free semi-theory. In section 6 the initial semi-theory \(P\) is introduced and it is shown that it can be used to detect weak equivalences in the category of homotopy algebras over a multi-sorted semi-theory. In section 7 we complete the argument showing that homotopy algebras over a multi-sorted semi-theory can be rigidified as strict algebras over a certain multi-sorted algebraic theory. In section 8 we prove the variant of Theorem 1.14 for multi-sorted semi-theories. Given an arbitrary finite product sketch we show in section 9 that we can construct an associated multi-sorted semi-theory for which the homotopy category of homotopy algebras is equivalent to the one defined by the original sketch. Using this result in section 10 we prove Theorems 1.12 and 1.14.
Acknowledgment. This paper is a version of my Ph.D. thesis completed at the State University of New York at Buffalo. I would like to thank my advisor Bernard Badzioch for his incredible patience and for his valuable suggestions. I would also like to thank my loving wife for her encouragement and support.

3. Multi-Sorted Semi-Theories

While our ultimate goal in this paper is to show that we can rigidify algebras over arbitrary finite product sketches, we will first consider the rigidification problem for a specific type of finite product sketches, which we will call multi-sorted semi-theories:

Definition 3.1. Let $S$ be a set. An $S$-sorted semi-theory $(C, \kappa)$ satisfying the following properties:

i) Objects $c_s \in C$ are indexed by $n$-tuples $s = (s_1, \ldots, s_n)$ (with possible repetitions) where $s_i \in S$ and $n \geq 0$. By abuse of notation for $s_1 \in S$ we will write $c_s$ to denote the object of $C$ indexed by the 1-tuple $(s_1)$.

ii) For any $n$-tuple $s = (s_1, \ldots, s_n)$ there is a unique $n$-fold cone $\alpha_s \in \kappa$ such that $\alpha_{s_0} = c_s$, $\alpha_{s_k} = c_{s_k}$ for $k = 1, \ldots, n$. Moreover, every cone in $\kappa$ is of such form.

Given an $S$-sorted semi-theory $C$ for simplicity we will denote the $k$-th projection in the cone $\alpha$ by $p_k^\alpha$ instead of $p_k^\alpha$. We will say that $C$ is a multi-sorted semi-theory if $C$ is $S$-sorted for some set $S$.

Notice that the definition of an $S$-sorted semi-theory parallels that of an $S$-sorted algebraic theory (1.10). The only difference is that we do not assume that the cones in $\kappa$ are product cones. Our first goal will be to show that variants of theorems 1.12 and 1.14 hold for multi-sorted semi-theories:

Theorem 3.2. For any $S$-sorted semi-theory $C$ there exists an $S$-sorted algebraic theory $\overline{F}_C$ such that the homotopy category of homotopy $C$-algebras is equivalent to the homotopy category of strict $\overline{F}_C$-algebras. Moreover, the construction of $\overline{F}_C$ is functorial in $C$.

Theorem 3.3. Let $C, C'$ be $S$-sorted semi-theories, and let $G: C \to C'$ be a morphism of finite product sketches that preserves sorts, i.e. $G(c_s) = c'_s$ for any $n$-tuple $s$ in $S$. The functor $G$ induces an equivalence of the homotopy categories of homotopy algebras if and only if the induced functor

$$G: \overline{F}_C \to \overline{F}_{C'}$$

between the associated multi-sorted theories is a weak equivalence of simplicial categories.
After proving these facts we will show how they can be used to give Theorems 1.12 and 1.14 in their whole generality.

From now until Section 9 it will be convenient fix the set $S$ and assume that all multi-sorted semi-theories are $S$-sorted. Consequently, all morphisms of semi-theories will be assumed to preserve sorts and projections. The resulting category of $S$-sorted semi-theories will be denoted by $\text{SemiTh}_S$. Notice that the category of $S$-sorted algebraic theories $\text{AlgTh}_S$ is a full subcategory of $\text{SemiTh}_S$.

4. Model Categories of Strict and Homotopy Algebras

Our basic strategy for proving Theorem 3.2 will be to rephrase it in terms model categories and prove that the relevant model categories are Quillen equivalent. With this in mind our first task will be to introduce model category structures reflecting the homotopy theories of strict and homotopy algebras. Much of what is discussed here parallels the setup of [1], [2], and [4] so the presentation will be brief.

The model category of strict algebras.

Let $C$ be a multi-sorted semi-theory and let $\text{Alg}^C$ denote the full subcategory of $\text{Spaces}^C$, whose objects are strict algebras over $C$. We would like to get a model category structure on $\text{Alg}^C$ where $C$ is a multi-sorted semi-theory. To do this let us first consider the category $\text{Alg}^T$ when $T$ is an $S$-sorted algebraic theory.

We will use the following fact due to Kan:

**Theorem 4.1** ([12, 11.3.2]). Let $M$ be a cofibrantly generated model category with a set of generating cofibrations $I$ and generating acyclic cofibrations $J$. Let $N$ be a category which has all small limits and small colimits and for which there exists a pair of adjoint functors:

$F: M \rightleftarrows N: U$

with $FI = \{Fu | u \in I\}$ and $FJ = \{Fv | v \in J\}$ and

i) $FI$ and $FJ$ permit the small object argument  
ii) $U$ takes colimits of pushouts along maps in $FJ$ to weak equivalences

Then there is a cofibrantly generated model category structure on $N$ for which $FI$ is a set of generating cofibrations, $FJ$ is a set of generating acyclic cofibrations and the set of weak equivalences is the set of maps which $U$ sends to weak equivalences in $M$. Furthermore, with respect to this model category structure, $(F, U)$ is a Quillen pair.
Proposition 4.2. Let $T$ be an $S$-sorted algebraic theory. The category of strict $T$-algebras $\text{Alg}^T$ admits a model category structure defined by the following classes of morphisms:

i) weak equivalences are objectwise weak equivalences;

ii) fibrations are objectwise fibrations;

iii) cofibrations are morphisms which have the left lifting property with respect to acyclic fibrations.

Proof. In [4, §4] Bergner showed that for each $s \in S$ the evaluation functor $U_s : \text{Alg}^T \to \text{Spaces}, \quad U_s(X) = X(t_s)$ has a left adjoint $F_s : \text{Spaces} \to \text{Alg}^T$.

Consider the set $S$ as a category with identity morphisms only and let $\text{Spaces}^S$ be the category of functors $S \to \text{Spaces}$. Notice that objects of $\text{Spaces}^S$ are just assignments that associate to each element $s \in S$ a space $Y_s$. The category $\text{Spaces}^S$ has a model structure with fibrations, cofibrations, and weak equivalences defined objectwise. The forgetful functor $U : \text{Alg}^T \to \text{Spaces}^S, \quad U(X)(s) = X(t_s)$ has a left adjoint $F$ defined for $Y \in \text{Spaces}^S$ by

$F(Y) = \bigsqcup_{s \in S} F_s(Y(s))$

where the coproduct is taken in $\text{Alg}^T$.

Note that the model category $\text{Spaces}^S$ is cofibrantly generated with the set of generating cofibrations

$I = \{ \Delta[n]_{s_i} \to \Delta[n]_{s_j} | n \geq 0, s_i \in S \}$

and a set of generating acyclic cofibrations

$J = \{ V[n, k]_{s_i} \to \Delta[n]_{s_i} | n \geq 1, 0 \leq k \leq n, s_i \in S \}$. 

Here $\Delta[n]_{s_i} \in \text{Spaces}^S$ is defined by $\Delta[n]_{s_i}(s_j) = \Delta[n]$ for $s_j = s_i$ and $\Delta[n]_{s_i}(s_j)$ is the empty set if $s_j \neq s_i$. The objects $\Delta[n]_{s_i}$ and $V[n, k]_{s_i}$ in $\text{Spaces}^S$ are defined in a similar matter. Applying theorem 4.1 to the adjoint pair $(F, U)$ we get a model category structure on $\text{Alg}^T$ as described in the statement.

□

Proposition 4.2 can be easily generalized to the case of arbitrary multi-sorted semi-theories:
**Corollary 4.3.** Let C be an S-sorted semi-theory. The category of strict C-algebras, \( \text{Alg}^C \) admits a model category structure defined by the following classes of morphisms:

i) weak equivalences are objectwise weak equivalences;
ii) fibrations are objectwise fibrations;
iii) cofibrations are morphisms which have the left lifting property with respect to acyclic fibrations.

**Proof.** By [3, Chapter 4, Theorem 3.6] we get that any S-sorted semi-theory C has an associated S-sorted algebraic theory \( \hat{C} \) with the property that \( \text{Alg}^C \) and \( \text{Alg}^{\hat{C}} \) are equivalent categories, so the statement follows directly from Proposition 4.2. □

**The model category of homotopy algebras**

Let C be a multi-sorted semi-theory. Our next goal is to describe a model structure that reflects the homotopy theory of homotopy C-algebras. We can’t do this arguing along the same lines as in the case of strict algebras since the full subcategory of \( \text{Spaces}^C \) that consists of homotopy algebras is not closed under colimits. Instead, we will obtain the desired model category by localizing the functor category \( \text{Spaces}^C \). Also, since we will eventually need a model category structure of homotopy algebras over an arbitrary finite product sketch (and not just for homotopy algebras over a multi-sorted semi-theory) we will work here in this more general setting.

For a small category C the functor category \( \text{Spaces}^C \) can be equipped with two different model category structures which we will denote by \( \text{Spaces}^C_{fib} \) and \( \text{Spaces}^C_{cof} \). In both of these model categories weak equivalences are objectwise weak equivalences. In \( \text{Spaces}^C_{fib} \) fibrations are the objectwise fibrations, and in \( \text{Spaces}^C_{cof} \), cofibrations are the objectwise cofibrations. In each case the third class of morphisms is determined by the lifting properties of model categories. By [11, VIII.1.4] and [11, IX.5.1] both \( \text{Spaces}^C_{fib} \) and \( \text{Spaces}^C_{cof} \) are simplicial model categories with the following simplicial structure: for \( X \in \text{Spaces}^C \) and \( K \in \text{Spaces} \) the functor \( X \otimes K \in \text{Spaces}^C \) is given by

\[
(X \otimes K)(c) = X(c) \times K
\]

for all \( c \in C \). For \( X, Y \in \text{Spaces}^C \) by \( \text{Map}_C(X, Y) \) we will denote the associated simplicial mapping complex.

Let \((C, \kappa)\) be a finite product sketch. We will consider an additional model category structure on \( \text{Spaces}^C \) denoted \( L\text{Spaces}^C \). This category is obtained as follows. For
c ∈ C let \( C_c \in \text{Spaces}^C \) denote the functor corepresented by \( c \):
\[
C_c(d) := \text{Hom}_C(c, d)
\]
Given a \( n \)-fold cone \( \alpha \in \kappa \) consider the morphism
\[
p^{0*} := \prod_{k=1}^{n} (p_k^0)^* : \prod_{k=1}^{n} C_{\alpha_k} \rightarrow C_{\alpha_0}
\]

The category \( \text{LSpaces}^C \) is the left Bousfield localization of \( \text{Spaces}^C_{\text{fib}} \) with respect to the set \( P = \{ p^{0*} \}_{\alpha \in \kappa} \). This localization exists by [12, 4.1.1] and by the fact that \( \text{Spaces}^C_{\text{fib}} \) is a left proper cellular model category. The model category structure on \( \text{LSpaces}^C \), can be described explicitly as follows:

i) If \( X \) and \( Y \) are cofibrant objects in \( \text{Spaces}^C_{\text{fib}} \) then a map \( f : X \rightarrow Y \) is a weak equivalence in \( \text{LSpaces}^C \) if for every homotopy algebra \( Z \), fibrant in \( \text{Spaces}^C_{\text{fib}} \) the induced map of homotopy function complexes
\[
f^* : \text{Map}_C(Y, Z) \rightarrow \text{Map}_C(X, Z)
\]
is a weak equivalence of simplicial sets. If \( X \) and \( Y \) are not cofibrant then the map \( f \) is a weak equivalence in \( \text{LSpaces}^C \) if the induced map \( f' : X' \rightarrow Y' \) between cofibrant replacements of \( X \) and \( Y \) is one. We will call such map \( f \) a local equivalence to distinguish it from and objectwise weak equivalence.

ii) Cofibrations in \( \text{LSpaces}^C \) are the same as cofibrations in \( \text{Spaces}^C_{\text{fib}} \), and fibrations in \( \text{LSpaces}^C \) are morphisms with the right lifting property with respect to local equivalences which are also cofibrations.

iii) An object \( X \in \text{LSpaces}^C \) is fibrant iff \( X \) is a homotopy \((C, \kappa)\)-algebra and \( X \) is fibrant in \( \text{Spaces}^C_{\text{fib}} \).

iv) If \( f : X \rightarrow Y \) is a map of homotopy \((C, \kappa)\)-algebras then \( f \) is a local equivalence iff \( f \) is an objectwise weak equivalence.

Note 4.4. Later on we will frequently use the following, equivalent description of local equivalences that can be obtained using arguments paralleling these given in [1, Section 5]: a map \( f : X \rightarrow Y \) is a local equivalence if for any homotopy \((C, \kappa)\)-algebra \( Z \) which is a fibrant object in \( \text{Spaces}^C_{\text{cof}} \) the induced map of simplicial function complexes
\[
f^* : \text{Map}_C(Y, Z) \rightarrow \text{Map}_C(X, Z)
\]
is a weak equivalence.

Combining properties iii) and and iv) we obtain:

Proposition 4.5. Let \((C, \kappa)\) be a finite product sketch. The homotopy category of \( \text{LSpaces}^C \) is equivalent to the category obtained by taking the full subcategory of \( \text{Spaces}^C \) spanned by homotopy \((C, \kappa)\)-algebras and inverting all objectwise weak equivalences.
In other words, $\text{LSpaces}^C$ is a model category that describes the homotopy theory of homotopy $(\mathcal{C}, \kappa)$-algebras.

**The Quillen Pair between $\text{Alg}^C$ and $\text{LSpaces}^C$**

Assume now that $\mathcal{C}$ is a multi-sorted semi-theory and consider the inclusion functor

$$J_C : \text{Alg}^C \rightarrow \text{Spaces}^C$$

Our next goal will be to show that $J_C$ is the right adjoint in a Quillen pair of functors between the model categories $\text{Alg}^C$ and $\text{LSpaces}^C$. If for some semi-theory $\mathcal{C}$ we can show that this Quillen pair is a Quillen equivalence we will obtain that the rigidification problem can be solved for homotopy algebras over $\mathcal{C}$: any homotopy $\mathcal{C}$-algebra is weakly equivalent to a strict $\mathcal{C}$-algebra.

The existence of a left adjoint of the functor $J_C$ can be demonstrated using the approach used by Bergner in [4].

**Definition 4.6.** [4, 5.5] Let $\mathcal{D}$ be a small category and let $P$ be a set of morphisms in $\text{Spaces}^\mathcal{D}$. An object $Y$ in $\text{Spaces}^\mathcal{D}$ is strictly local if for every $(f : A \rightarrow B) \in P$, the induced map of the simplicial function complexes

$$f^* : \text{Map}(B, Y) \rightarrow \text{Map}(A, Y)$$

is an isomorphism of simplicial sets.

**Lemma 4.7.** [4, 5.6] For a small category $\mathcal{D}$ and a set of morphisms $P$ in $\text{Spaces}^\mathcal{D}$ let $\text{Alg}^{(\mathcal{D}, P)}$ denote the full subcategory of $\text{Spaces}^\mathcal{D}$ whose objects are strictly local diagrams. The inclusion functor

$$\text{Alg}^{(\mathcal{D}, P)} \rightarrow \text{Spaces}^\mathcal{D}$$

has a left adjoint.

Using this lemma we obtain:

**Proposition 4.8.** Let $\mathcal{C}$ be a multi-sorted semi-theory. There exists a functor

$$K_C : \text{Spaces}^C \rightarrow \text{Alg}^C$$

left adjoint to $J_C$. Furthermore, the pair $(K_C, J_C)$ is a Quillen pair between the model categories $\text{Spaces}^C_{\text{fib}}$ and $\text{Alg}^C$.

**Proof.** It is enough to notice that a strict $\mathcal{C}$-algebra is a diagram $X : \mathcal{C} \rightarrow \text{Spaces}$ which is strictly local with respect to the set $P = \{p^{\alpha x} \}_{\alpha \in \kappa}$ where $p^{\alpha x}$ is the map defined as in (1). To see that we have a Quillen pair, notice that fibrations and weak equivalences in $\text{Alg}^C$ are computed objectwise, thus $J_C$ preserves both. □
Next, we want show that \((K_C, J_C)\) is still a Quillen pair after we localize \(\text{Spaces}_{fib}^C\).

This is a consequence of the following fact:

**Lemma 4.9.** Let \(M\) and \(N\) be model categories and \(LM\) be a left Bousfield localization of \(M\). If

\[
F : M \rightleftarrows N : G
\]

is a Quillen pair such that for any fibrant object \(X \in N\) the object \(G(X)\) is fibrant in \(LM\), then

\[
F : LM \rightleftarrows N : G
\]

is also a Quillen pair.

**Proof.** By [7, A.2] we only need show that \(G : N \to LM\) preserves all fibrations between fibrant objects and preserves all acyclic fibrations. Assume then that \(f : X \to Y\) is a fibration between fibrant objects in \(N\). By assumption \(G(X)\) and \(G(Y)\) are fibrant in \(LM\). Also, since \(G : N \to M\) is a right Quillen functor the morphism \(G(f)\) is a fibration in \(M\). Using the model category structure of \(LM\) we can decompose \(G(f)\) so that

\[
G(f) = G(X) \xrightarrow{\varphi} Z \xrightarrow{\psi} G(Y)
\]

where \(\psi\) is a fibration in \(LM\) and \(\varphi\) is an acyclic cofibration in \(LM\). By [12, 3.3.14] we get that \(Z\) must be a local object, which by [12, 3.2.13] gives that \(\varphi\) is a weak equivalence in \(M\). Therefore by [12, 3.3.15] \(G(f)\) must be a fibration in \(LM\).

If \(f : X \to Y\) is an acyclic fibration in \(N\) then \(G(f)\) is an acyclic fibration in \(M\) since \(G\) is a right Quillen functor. It remains to notice that acyclic fibrations in \(LM\) are the same as acyclic fibrations in \(M\) [12, 3.3.3]. \(\square\)

**Proposition 4.10.** For any multi-sorted semi-theory the adjoint pair of functors \((K_C, J_C)\) is a Quillen pair between the model categories \(L\text{Spaces}^C\) and \(\text{Alg}^C\).

**Proof.** By Lemma 4.9 we only need to show that for any fibrant object \(X \in \text{Alg}^C\) the object \(J_C(X)\) is fibrant in \(L\text{Spaces}^C\). This is obvious since fibrant objects in \(L\text{Spaces}^C\) are homotopy algebras, fibrant in \(\text{Spaces}^C_{fib}\). \(\square\)

The next lemma gives a way of verifying that \((K_C, J_C)\) is a Quillen equivalence for a given multi-sorted semi-theory \(C\). For \(X \in \text{Spaces}^C\) let

\[
\eta_X : X \to J_CK_CX
\]

be the unit of adjunction of \((K_C, J_C)\). We have:

**Lemma 4.11.** Let \(C\) be a multi-sorted semi-theory. If the map \(\eta_c\) is a local equivalence for all \(c \in C\) then the Quillen pair \((K_C, J_C)\) is a Quillen equivalence of the model categories \(L\text{Spaces}^C\) and \(\text{Alg}^C\).
The proof of lemma 4.11 will use a couple of auxiliary facts.

First, for a simplicial model category $\mathbf{M}$ let $\mathbf{sM}$ be the category of simplicial objects in $\mathbf{M}$, i.e. the category of functors $\Delta^{op} \to \mathbf{M}$. We have the geometric realization functor

$$|−|: \mathbf{sM} \to \mathbf{M}$$

such that for $X_\bullet \in \mathbf{sM}$ the object $|X_\bullet| \in \mathbf{M}$ is the coequalizer of the diagram:

$$\bigcup_{\varphi: [n] \to [m]} X_m \otimes \Delta[n] \rightrightarrows \bigcup_{[n]} X_n \otimes \Delta[n]$$

**Lemma 4.12.** Let $\mathbf{C}$ be a small category and $\mathbf{LSpaces}^\mathbf{C}$ be the left Bousfield localization of $\mathbf{Spaces}^\mathbf{C}_{\text{fib}}$ with respect to a set of maps $P$. Assume that we have a Quillen pair

$$K: \mathbf{LSpaces}^\mathbf{C} \leftrightarrow \mathbf{M}: J$$

such that the following hold

i) for $X_\bullet \in \mathbf{sSpaces}^\mathbf{C}$ we have $JKX_\bullet \cong JK|X_\bullet|$

ii) $J$ commutes with filtered colimits

iii) $J(f)$ is a local equivalence if and only if $f$ is a weak equivalence

iv) the unit of adjunction $\eta_Y$ is a local equivalence for $Y = \bigsqcup_{i=1}^m \mathbf{C}_{c_i}$ where $\{c_i\}_{i=1}^m$ is any finite set of objects in $\mathbf{C}$

Then $(K, J)$ is a Quillen equivalence.

**Proof.** We need to show that for any cofibrant object $X \in \mathbf{LSpaces}^\mathbf{C}$ and any fibrant object $Y \in \mathbf{M}$ a morphism $f: K(X) \to Y$ is a weak equivalence in $\mathbf{M}$ if and only if its adjoint $f^\# : X \to J(Y)$ is a weak equivalence in $\mathbf{LSpaces}^\mathbf{C}$. Recall that $f^\#$ is given by the composition $f^\# = J(f)\eta_X$ where $\eta_X$ is the unit of adjunction. Since by assumption $J(f)$ is a weak equivalence if and only if $f$ is one, it will suffice to show that $\eta_X$ is a weak equivalence in $\mathbf{LSpaces}^\mathbf{C}$ for all cofibrant objects $X \in \mathbf{LSpaces}^\mathbf{C}$.

Assume now that $X, \hat{X} \in \mathbf{LSpaces}^\mathbf{C}$ are cofibrant objects, that $\eta_{\hat{X}}$ is a weak equivalence and that we also have a weak equivalence $g: \hat{X} \to X$. We claim that in such a case $\eta_X$ is also a weak equivalence. Indeed, we have a commutative diagram

$$\begin{array}{ccc}
\hat{X} & \xrightarrow{\eta_{\hat{X}}} & JK(\hat{X}) \\
g & \cong & JK(g) \\
X & \xrightarrow{\eta_X} & JK(X)
\end{array}$$

It suffices to show that $JK(g)$ is a weak equivalence in $\mathbf{LSpaces}^\mathbf{C}$. Since $g$ is weak equivalence of cofibrant objects and $K$ is a left Quillen functor by [10, Lemma 9.9]
we obtain that the morphism $K(g)$ is a weak equivalence. Therefore by assumption $JK(g)$ is a weak equivalence.

The above argument shows that in order to complete the proof it suffices to show that for any $X \in \text{LSpaces}_C$ there exists a cofibrant object $\hat{X}$ weakly equivalent to $X$ and such that $\eta_{\hat{X}}$ is a weak equivalence. In order to get an appropriate choice of $\hat{X}$ we can use a construction given by Badzioch. In [1, §3] he showed that to any $X \in \text{LSpaces}_C$ we can associate a cofibrant replacement $\hat{X}$ ($\hat{X} \simeq |FU_\bullet X|$ in the notation of [1]) such that:

1) $\hat{X}$ is obtained as the geometrical realization of a certain bisimplicial object $\hat{X}_{\bullet \bullet}$ in $\text{Spaces}_C$;
2) in each bisimplicial grading the functor $\hat{X}_{m,n} \in \text{Spaces}_C$ is given by a (possibly infinite) coproduct of corepresented functors:

$$\hat{X}_{m,n} = \bigsqcup_{i \in I} C_{c_i}$$

Since by assumption the functor $J K$ commutes with geometric realization we obtain that in order to see that $\eta_{\hat{X}}$ is a weak equivalence for all $\hat{X}$ it will suffice to show that $\eta_Z$ is a weak equivalence for any $Z = \bigsqcup_{i \in I} C_{c_i}$. Notice that if $I$ is a finite set then this holds by assumption. Assume then that $I$ is infinite, and let $P_I$ denote the category of all finite subsets of $I$ with morphisms given by inclusions of sets. We have a functor:

$$\tilde{Z}: P_I \rightarrow \text{LSpaces}_C, \quad \tilde{Z}(A) := \bigsqcup_{i \in A} C_{c_i}$$

Notice that $\text{colim}_{P_I} \tilde{Z} = Z$. Also, $\text{colim}_P J K \tilde{Z} = J K(Z)$ since by assumption $J$ commutes with filtered colimits and $K$, as a left adjoint, preserves all colimits. The map $\eta_Z$ is then a colimit of maps:

$$\eta_{\tilde{Z}(A)}: \tilde{Z}(A) \rightarrow J K(\tilde{Z}(A))$$

and since $\tilde{Z}(A)$ is a finite disjoint union of corepresented functors for all $A \in P_I$ thus by assumption the maps $\eta_{\tilde{Z}(A)}$ are local equivalences. This gives that the map

$$\text{hocolim}_{P_I} \eta_{\tilde{Z}(A)}: \text{hocolim}_{P_I} \tilde{Z}(A) \rightarrow \text{hocolim}_{P_I} J K(\tilde{Z}(A))$$

is a local equivalence. It remains to notice that since $P_I$ is a filtered category we have $\text{colim}_{P_I} \tilde{Z} \simeq \text{hocolim}_{P_I} \tilde{Z}$ and $\text{colim}_{P_I} J K(\tilde{Z}) \simeq \text{hocolim}_{P_I} J K(\tilde{Z})$.

**Lemma 4.13.** Let $C$ be a multi-sorted semi-theory. If $\eta_{C_c}$ is a local equivalence for all $c \in C$ then for a collection $\{c_i\}_{i=1}^m$ of objects in $C$, $\eta_Y$ is a local equivalence for all $Y = \bigsqcup_{i=1}^m C_{c_i}$

**Proof:** Let $C$ be an $S$-sorted semi-theory. Recall that objects $c_\xi \in C$ are indexed by $n$-tuples $\xi = (s_1, \ldots, s_n)$ where $s_k \in S$ and $n \geq 0$. In order to simplify notation we will write $C_\xi$ to denote the functor corepresented by $c_\xi$. For $i = 1, \ldots, m$ let
Let \( s = (s^1, \ldots, s^n) \) be an \( n_i \)-tuple and let \( Y = \coprod_{i=1}^m C_{s^i} \). We need to show that \( \eta_Y \) is a local equivalence. Let \( s \) denote the \( \sum_{i=1}^m n_i \)-tuple obtained by concatenating \( s^i \)’s:

\[
\underline{s} = (s^1, \ldots, s^1, \ldots, s^n, \ldots, s^n)
\]

Notice that the projections maps in the cones of \( C_s \) define functors \( p^\omega: \coprod_{i=1}^m \coprod_{k=1}^{n_i} C_{s^i_k} \rightarrow C_{\underline{s}} \) and for \( i = 1, \ldots, m \)

\[
p^\omega_i: \coprod_{k=1}^{n_i} C_{s^i_k} \rightarrow C_{\underline{s}}
\]

given by the equation (1). Consider the commutative diagram

\[
\begin{array}{ccc}
\coprod_{i=1}^m C_{\underline{s}} & \xrightarrow{\eta} & J_C K_C(\coprod_{i=1}^m C_{\underline{s}}) \\
\downarrow{p^\omega} & & \downarrow{J_C K_C(\coprod_{i=1}^m p^\omega_i)} \\
\coprod_{i=1}^m \coprod_{k=1}^{n_i} C_{s^i_k} & \xrightarrow{\eta} & J_C K_C(\coprod_{i=1}^m \coprod_{k=1}^{n_i} C_{s^i_k}) \\
\downarrow{p^\omega} & & \downarrow{J_C K_C(p^\omega)} \\
C_{\underline{s}} & \xrightarrow{\eta} & J_C K_C(C_{\underline{s}})
\end{array}
\]

Our goal is to show that the top horizontal map is a local equivalence. Notice that the bottom horizontal map is a local equivalence by assumption. The vertical arrows on the left come from localizing maps in \( \text{LSpaces}^C \), thus they are local equivalences as well. As a consequence it will be enough to show that the vertical maps on the right are local equivalences. We will show that actually more is true, namely that the maps \( K_C(\coprod_{i=1}^m p^\omega_i) \) and \( K_C(p^\omega) \) are isomorphisms in \( \text{Alg}^C \), and so that \( J_C K_C(\coprod_{i=1}^m p^\omega_i) \) and \( J_C K_C(p^\omega) \) are isomorphisms \( \text{Spaces}^C \). To see this it will be enough to check that for any strict \( C \)-algebra \( A \) the maps induced by \( K_C(\coprod_{i=1}^m p^\omega_i) \) and \( K_C(p^\omega) \) on the simplicial mapping complexes \( \text{Map}(\cdot, A) \) are isomorphisms. Notice that we have

\[
\text{Map}_C(K_C(C_{\underline{s}}), A) \cong \text{Map}_C(C_{\underline{s}}, J_C A) \cong A(c_{\underline{s}})
\]

where the first isomorphism is given by the adjunction and the second comes from the Yoneda lemma. Similarly we obtain:

\[
\text{Map}_C(K_C(\coprod_{i=1}^m C_{\underline{s}}), A) \cong \text{Map}_C(\coprod_{i=1}^m C_{\underline{s}}, J_C(A)) \cong \prod_{i=1}^m A(c_{\underline{s}})
\]
and:

\[ \text{Map}_C(KC(\coprod_{i=1}^m \coprod_{k=1}^n C_{s_i}), A) \cong \text{Map}_C(\coprod_{i=1}^m \coprod_{k=1}^n C_{s_i}, JC(A)) \cong \prod_{i=1}^m \prod_{k=1}^n A(c_{s_i}) \]

Since \( A \) is a strict algebra we also have isomorphisms

\[ A(c_{s_i}) \cong \prod_{i=1}^m \prod_{k=1}^n A(c_{s_i}) \cong \prod_{i=1}^m A(c_{s_i}) \]

These isomorphisms are given by projections in \( A \), so it follows that the isomorphisms of the mapping complexes are induced by \( KC(\coprod_{i=1}^m p_{s_i}) \) and \( KC(p_{s_i}) \).

\[ \square \]

**Proof of Lemma 4.11.** Assume that \( \eta_C \) is a local equivalence for all \( c \in C \). It will suffice to show that all assumptions of Lemma 4.12 are satisfied. By Lemma 4.13, \( \eta_Y \) is a local equivalence for all \( Y = \coprod_{i=1}^m C_{c_i} \), so assumption (iv) is satisfied. Assumption (iii) is satisfied since any weak equivalence in \( \text{Alg}^C \) is an objectwise weak equivalence and assumption (ii) is satisfied since filtered colimits in \( \text{Alg}^C \) are computed objectwise. Lastly we see that (i) is satisfied since as in [1, 6.2] we get that if \( X_\bullet \) is in \( \text{Spaces}^C_{\text{fib}} \) then \( KC|X_\bullet| \cong |KCX_\bullet| \), but this gives \( JC(KC|X_\bullet|) \cong |JC KCX_\bullet| \).

\[ \square \]

5. **Simplicial Resolution of a Multi-Sorted Semi-Theory**

Recall that the statement of Theorem 3.2 says that homotopy algebras over an \( S \)-sorted semi-theory \( C \) can be rigidified to strict algebras over a certain \( S \)-sorted algebraic theory \( \overline{F}_C \). In this section we will describe the construction of \( \overline{F}_C \).

**Completion of a semi-theory**

Recall that for a set \( S \) we denote by \( \text{AlgTh}_S \) the category of \( S \)-sorted algebraic theories and by \( \text{SemiTh}_S \) be the category of \( S \)-sorted semi-theories. In both categories morphisms are functors that preserve sorts and cones. We have an inclusion functor:

\[ R: \text{AlgTh}_S \to \text{SemiTh}_S \]

By [3, ch.4 3.6] every \( S \)-sorted semi-theory \( C \) has a functorially associated \( S \)-sorted algebraic theory \( \overline{C} \) and a morphism \( \Phi_C: C \to \overline{C} \) in \( \text{SemiTh}_S \) with the property that any functor

\[ C \to T \]

to an \( S \)-sorted algebraic theory \( T \) uniquely factors through \( \Phi_C \). Equivalently, this says that the functor \( R \) has a left adjoint

\[ L: \text{SemiTh}_S \to \text{AlgTh}_S \]
with $\Phi_C$ as the unit of adjunction. Furthermore, Barr and Wells showed that the following holds:

**Proposition 5.1.** [3, ch.4 3.6] For any semi-theory $C$ the functor $\Phi_C: C \rightarrow \bar{C}$ induces an equivalence of the categories of strict algebras

$$\Phi_C: \text{Alg}^C \rightarrow \text{Alg}^\bar{C}$$

**Definition 5.2.** Given a multi-sorted semi-theory $C$ we will call the functor $\Phi_C: C \rightarrow \bar{C}$ the completion of $C$.

**Simplicial Resolution**

Let $C$ be a small category. Following [9, 2.5] by the simplicial resolution of $C$ we will understand the simplicial category $F^\ast C$ given as follows. $F^0 C$ is the free category whose objects are the objects of $C$ and whose generators are the morphisms of $C$. For $k > 0$ we define $F^k C$ to be the free category generated by $F^{k-1} C$. Notice that for $c, d \in C$ we have a canonical map

$$\varphi_{c,d}: \text{Hom}_C(c, d) \rightarrow \text{Hom}_{F^\ast C}(c, d).$$

If $C$ is an $S$-sorted semi theory then we define an $S$-sorted semi-theory structure on $F^\ast C$ in such way that projections morphisms of cones in $F^\ast C$ are the images of projections in $C$ under the maps $\varphi_{c,d}$. Notice that in this way we have a naturally defined functor $\psi: F^\ast C \rightarrow C$, which defines a morphism of $S$-sorted semi-theories. Let $\psi^*: \text{Spaces}^C \rightarrow \text{Spaces}^{F^\ast C}$ denote the functor induced by $\psi$, and let $\psi_*$ be the left adjoint of $\psi^*$. We have

**Proposition 5.3.** The adjoint pair of functors

$$\psi_*: \text{LSpaces}^{F^\ast C} \rightleftarrows \text{LSpaces}^C: \psi^*$$

is a Quillen equivalence.

**Proof.** We use an argument analogous to the proof of [2, 4.1]. By [9, 2.6] the functor $\psi$ is a weak equivalence of categories, and so the adjunction $(\psi_*, \psi^*)$ is a Quillen equivalence of the model categories $\text{Spaces}_{fib}^C$ and $\text{Spaces}_{fib}^{F^\ast C}$. Also, the morphisms with respect to which we localize $\text{Spaces}_{fib}^{F^\ast C}$ to obtain $\text{LSpaces}^{F^\ast C}$ are precisely the images under the functor $\psi_*$ of the localizing morphisms in $\text{Spaces}_{fib}^C$. Therefore we can apply [12, 3.3.20] which says that in such situation localizations preserve Quillen equivalences. \qed

For every $k \geq 0$ consider the completion

$$\Phi_k: F_k C \rightarrow \bar{F}_k C$$

The functors $\Phi_k$ taken together define a functor of simplicial categories

$$\Phi: F^\ast C \rightarrow \bar{F}^\ast C$$
where $F^*C$ is the simplicial $S$-sorted algebraic theory which has $F^kC$ in its $k$-th simplicial dimension. Using Proposition 5.1 we get:

**Lemma 5.4.** The functor $\Phi: F^*C \to F^*\overline{C}$ induces an equivalence of categories of strict algebras

$$\Phi^*: \text{Alg}^{F^*\overline{C}} \xrightarrow{\simeq} \text{Alg}^{F^*C}$$

Let $C$ be a multi-sorted semi-theory and let $F^*C$ be the simplicial resolution of $C$. Recall that by Proposition 4.10 we have a Quillen pair of functors

$$K_{F^*,C}: \text{LSpaces}^{F^*C} \xrightarrow{\leftarrow} \text{Alg}^{F^*C}: J_{F^*,C}$$

In view of Proposition 5.3 and Lemma 5.4 in order to prove Theorem 3.2 it is enough to show that the following holds:

**Proposition 5.5.** The Quillen pair $(K_{F^*,C}, J_{F^*,C})$ is a Quillen equivalence.

By Lemma 4.11 this fact in turn reduces to the following

**Lemma 5.6.** Let $C$ be an $S$-sorted semi-theory. For an $n$-tuple $s$ let $F^*C_s \in \text{LSpaces}^{F^*C}$ denote the functor corepresented by the object $c_s \in F^*C$. The unit of adjunction of the pair $(K_{F^*,C}, J_{F^*,C})$

$$\eta_{F^*,C_s}: F^*C_s \xrightarrow{\eta} J_{F^*,C}K_{F^*,C}F^*C_s$$

is a local equivalence.

The proof of Lemma 5.6 will be given in §7 after we develop a better understanding of the algebraic completion as well as a way of detecting local equivalences. Meanwhile our last goal in this section will be to obtain an explicit description of the algebraic completion for free multi-sorted semi-theories. Our approach will parallel that of [2, §3].

Let $C$ be a free $S$-sorted semi-theory; that is $C$ is a free category such that all projections in the structure cones of $C$ are among the free generators of $C$. We will construct in a combinatorial manner a category $C'$ and later show that $C'$ is the algebraic completion of $C$. The construction of $C'$ proceeds as follows. Objects $c_s \in C'$ are the same as the objects in $C$ i.e. they are indexed by all $n$-tuples of objects of $S$ for $n \geq 0$. As before for $s_1 \in S$ we will denote by $c_{s_1}$ the object indexed by the 1-tuple $(s_1)$.

In order to describe morphisms in $C'$ assume first that $s = (s_1, ... , s_n)$ is an arbitrary $n$-tuple and let $s' \in S$. A morphism in $\text{Hom}_C(c_s, c_{s'})$ is a directed tree $T$: 

...
satisfying the following conditions:

1) the lowest vertex of $T$ has only one incoming edge;
2) all edges of $T$ are labeled with $\theta_i$, where $\theta$ is a free generator of $C$ whose codomain is an $n^i$-tuple in $S$, and where $1 \leq i \leq n^0$. If $\theta = p^s_k$, we will write $p^s_k$ instead of $(p^s_k)_k$;
3) if a vertex of $T$ has $m$ incoming edges with labels $\theta_1^{i_1}, \ldots, \theta_m^{i_m}$, and for $k = 1, \ldots, m$ the codomain of $\theta_k$ in $C$ is labeled by the $n^{i_k}$-tuple $(s^{j_k}_1, \ldots, s^{j_k}_{n^{i_k}})$ then the outgoing edge is labeled $\psi_j$ where the domain of $\psi$ in $C$ is labeled by the $m$-tuple $(s_{i_1}^{j_k}, s_{i_2}^{j_k}, \ldots, s_{i_m}^{j_k})$
4) all the initial edges of $T$ (that is, the edges starting at vertices with no incoming edges) are labeled with projections $p^s_k$, where $c_s$ is the domain of the tree
5) no non-initial edges of $T$ are labeled with projection morphisms;
6) the lowest edge is labeled with $\varphi_i$ where the codomain of $\varphi$ in $C$ is given by an $m$-tuple in $S$ whose $i$-th element is $s'$.

For the remainder of the morphisms in $C'$ let $\underline{s} = (s_1, \ldots, s_m)$ and $\underline{s'} = (s'_1, \ldots, s'_m)$, then

$$\text{Hom}_{C'}(c_{\underline{s}}, c_{\underline{s'}}) = \prod_{1 \leq i \leq m} \text{Hom}_{C'}(c_{s_i}, c_{s'_i})$$

Composition of morphisms in $C'$ composition defined the same as in [2, §3]: if $(T_1, \ldots, T_m) \in \text{Hom}_{C'}(c_{\underline{s}}, c_{\underline{s'}})$ and $W \in \text{Hom}_{C'}(c_{\underline{s}'}, c_{\underline{s}''})$ then $W \circ (T_1, \ldots, T_m) \in \text{Hom}_{C'}(c_{\underline{s}}, c_{\underline{s}''})$ is the tree obtained by grafting the tree $T_i$ in place of each initial edge of $W$ labeled $p^s_{k_i}$. In general if $(T_1, \ldots, T_m) \in \text{Hom}_{C'}(c_{\underline{s}}, c_{\underline{s}'})$ and $(W_1, \ldots, W_r) \in \text{Hom}_{C'}(c_{\underline{s}'}, c_{\underline{s}''})$ then

$$(W_1, \ldots, W_r) \circ (T_1, \ldots, T_m) = (W_1 \circ (T_{s^1_1}, \ldots, T_{s^1_m}), \ldots, W_r \circ (T_1, \ldots, T_m))$$
Let for an \( n \)-tuple \( \underline{s} = (s_1, \ldots, s_n) \) in \( S \) let \( p^\underline{s}_k : c_{\underline{s}} \to c_{s_k} \) denote the morphism in \( C' \) represented by the tree:

\[
\begin{array}{c}
\bullet \\
p^\underline{s}_k \\
\end{array}
\]

We give \( C' \) an \( S \)-sorted semi-theory structure by choosing the morphisms \( p^\underline{s}_k \) to be projections in \( C' \). It can be checked that \( \bar{C} \) is, in fact, a multi-sorted theory.

Next we define the functor

\[
\Theta_C : C \to \bar{C}
\]

which is the identity on objects, and such that \( \Theta_C(p^\underline{s}_k) = p^\underline{s}_k \). If \( \varphi : c_{\underline{s}} \to c_{\underline{s}'} \) is a generator of \( C \) which is not a projection, \( \underline{s} \) is an \( n \)-tuple, and \( \underline{s}' \) is an \( m \)-tuple then \( \Theta_C(\varphi) = (T_1, \ldots, T_m) \) where \( T_j \) is the tree:

\[
\begin{array}{c}
\bullet \\
\bullet \\
\downarrow \\
p^\underline{s}_1 \\
\vdots \\
p^\underline{s}_m \\
\varphi_j \\
\end{array}
\]

**Proposition 5.7.** The functor \( \Theta_C \) is the completion of the semi-theory \( C \) to an algebraic theory.

**Proof.** By [3, ch.4 §3] the algebraic completion of an \( S \)-sorted semi-theory \( C \) is the closure of the category \( C \) under taking finite products and this is what we constructed \( C' \) to be. \( \square \)

6. **The Initial \( S \)-Sorted Semi-Theory**

Let \( P \) be the \( S \)-sorted semi-theory whose only non-identity morphisms are projections \( p^\underline{s}_k : c_{\underline{s}} \to c_{s_k} \) for all \( n \)-tuples \( \underline{s} = (s_1, \ldots, s_n) \), \( n \geq 0 \) and \( 1 \leq k \leq n \). Clearly \( P \) is an initial object in the category of \( S \)-sorted semi-theories, i.e. for any \( S \)-sorted semi-theory \( C \) there is a unique morphism \( P \to C \) in \( \text{SemiTh}_S \).

A nice feature of \( P \) is that local equivalences in \( \text{LSpaces}^P \) are easy to detect:

**Proposition 6.1.** A map \( f : X \to Y \) in \( \text{LSpaces}^P \) is a local equivalence if the restrictions \( f_{c_s} : X(c_s) \to Y(c_s) \) is a weak equivalence of spaces for all \( s \in S \).

**Proof.** Let \( f : X \to Y \) be a map such that \( f_{c_s} : X(c_s) \to Y(c_s) \) is a weak equivalence for all \( s \in S \). Notice for any homotopy algebra \( Z \in \text{Spaces}^P \) we can find a strict
\( \mathbf{P}-\text{algebra} \ Z' \) such that \( Z' \) is fibrant in \( \mathbf{Spaces}_\text{fib}^\mathbf{P} \) and there is an objectwise weak equivalence \( Z \to Z' \). It follows that in order to show that \( f \) is a local equivalence we only need to check that the induced map of simplicial function complexes

\[
f^* : \text{Map}_\mathbf{P}(X, Z') \longrightarrow \text{Map}_\mathbf{P}(Y, Z')
\]

is a weak equivalence for all strict \( \mathbf{P}-\text{algebras} \ Z' \) that are fibrant in \( \mathbf{Spaces}_\text{fib}^\mathbf{P} \). This fact follows however from the observation that since \( Z' \) is a strict algebra we have

\[
\text{Map}_\mathbf{P}(X, Z') \cong \prod_{s \in S} \text{Map}(X(c_s), Z'(c_s))
\]

and that under this isomorphism \( f_* \) is given by the product \( \prod_{s \in S} f_*^s \) where

\[
f_*^s : \text{Map}(Y(c_s), Z(c_s)) \to \text{Map}(X(c_s), Z(c_s))
\]

is the map induced by \( f_s \). \( \square \)

**Proposition 6.2.** Let \( J_\mathbf{P} : \mathbf{P} \to \mathbf{C} \) denote the inclusion of \( \mathbf{P} \) into a free semi-theory \( \mathbf{C} \). Then the adjoint pair of functors

\[
J_\mathbf{C}^* : \mathbf{Spaces}_\text{cof}^\mathbf{P} \rightleftarrows \mathbf{Spaces}_\text{cof}^\mathbf{C} : J_\mathbf{C}^*
\]

is a Quillen pair.

*Proof.* It can be checked that if we define the functor \( J_{\mathbf{C}*} : \mathbf{Spaces}_\text{cof}^\mathbf{P} \to \mathbf{Spaces}_\text{cof}^\mathbf{C} \) as follows:

\[
J_{\mathbf{C}*}Y(e) = Y(c) \sqcup \bigsqcup_{(\varphi : c_i \to c) \in G_c} Y(c_i)
\]

Where \( Y \in \mathbf{Spaces}_\text{cof}^\mathbf{P}, \ c \in \mathbf{C}, \ G_c \) is the set of all morphisms \( \varphi : c_i \to c, \ c_i \in \mathbf{C} \) such that \( \varphi = \zeta_k \circ \zeta_{k-1} \circ \cdots \circ \zeta_1 \) with \( \zeta_1, \ldots, \zeta_k \) generators of \( \mathbf{C} \) and \( \zeta_1 \) is not a projection, then \((J_{\mathbf{C}*}, J_\mathbf{C})\) satisfies the axioms for adjoint functors. Furthermore, this description it is clear that \( J_{\mathbf{C}*} \) preserves objectwise cofibrations and weak equivalences. Therefore \((J_{\mathbf{C}*}, J_\mathbf{C}^*)\) is a Quillen pair. \( \square \)

**Corollary 6.3.** The map of semi-theories \( J : \mathbf{P} \to \mathbf{F}_\mathbf{C} \) induces Quillen pair of functors

\[
J_* : \mathbf{Spaces}_\text{cof}^\mathbf{P} \rightleftarrows \mathbf{Spaces}_\text{cof}^{\mathbf{F}_\mathbf{C}} : J_*
\]

*Proof.* This follows by a argument paralleling the proof of [2, 5.2]. The key idea is that the functors \((J_k)_*\) (coming from Proposition 6.2 with \( \mathbf{C} := \mathbf{F}_h\mathbf{C} \)) can be assembled into a functor \( J_* \) for which have \( J_* = | - | \circ J_* \) where \( | - | \) is the diagonal functor \( | - | : s\mathbf{Spaces} \to \mathbf{Spaces} \). \( \square \)
7. **Proof of Lemma 5.6.**

Since the combinatorial description of the algebraic completion of a free multi-sorted semi-theory uses a similar setup as the algebraic completion of free one-sorted semi-theory described in [2], the remainder of the proofs will be also similar. For this reason we will outline the proof of lemma 5.6, but refer to [2, §7] for details.

Let $D$ be a free $S$-sorted semi-theory and $\Phi_D : D \to \bar{D}$ be the completion of $D$ to an algebraic theory. As usual we will denote objects of $D$ and $\bar{D}$ by $d_s$ where $s$ is an $n$-tuple in $S$ for $n \geq 0$, and by $D_s$ (respectively $\bar{D}_s$) we will denote the $D$-diagram (resp. the $\bar{D}$-diagram) corepresented by $d_s$. Using the functor $\Phi_D$ we can think of $\bar{D}_s$ as a $D$-diagram. Since $\Phi_D$ is an embedding of categories $D_s$ is a subdiagram of $\bar{D}_s$.

Define a filtration of the diagram $\bar{D}_s$ by $D$-diagrams

$$\bar{D}_s^0 \subseteq \bar{D}_s^1 \subseteq \cdots \subseteq \bar{D}_s^k$$

as follows. Set $\bar{D}_s^0 := D_s$. For $k \geq 0$ we define $\bar{D}_s^{k+1}$ as the smallest $D$–subdiagram of $\bar{D}_s$ such that if $T_1, T_2, \ldots, T_m$ are trees that are elements of $\bar{D}_s^k$ then $(T_1, T_2, \ldots, T_m)$ belongs to $\bar{D}_s^{k+1}$. From the combinatorial construction of $\bar{D}$ (§5) we obtain that $\text{colim}_k \bar{D}_s^k = \bar{D}_s$.

As before let $P$ denote the initial $S$-sorted semi-theory. The unique map $P \to D$ induces a $P$-diagram structure on $D_s$, $\bar{D}_s$ and $\bar{D}_s^k$. We define a filtration of $\bar{D}_s$ by $P$-diagrams

$$s\bar{D}_s^0 \subseteq s\bar{D}_s^1 \subseteq \cdots \subseteq \bar{D}_s^k$$

where $s\bar{D}_s^0 = D_s$ and $s\bar{D}_s^{k+1}$ is the smallest $P$–subdiagram of $\bar{D}_s$ such that if $T_1, T_2, \ldots, T_m$ are elements of $\bar{D}_s^k$ then $(T_1, T_2, \ldots, T_m)$ belongs to $s\bar{D}_s^{k+1}$. We have inclusions of $P$–diagrams

$$\bar{D}_s^k \subseteq s\bar{D}_s^{k+1} \subseteq \bar{D}_s^{k+1}$$

and $\text{colim}_k s\bar{D}_s^k = \bar{D}_s$.

Using the same tree-length arguments as in [2] we can check that the filtrations $\{\bar{D}_s^k\}$ and $\{s\bar{D}_s^k\}$ have the following property:

**Lemma 7.1.** For any $D$–diagram of spaces $X : D \to \text{Spaces}$ and for $k \geq 0$ the square of simplicial mapping complexes

$$\begin{array}{ccc}
\text{Map}_D(\bar{D}_s^k, X) & \longrightarrow & \text{Map}_D(\bar{D}_s^{k+1}, X) \\
\downarrow & & \downarrow \\
\text{Map}_P(\bar{D}_s^k, X) & \longrightarrow & \text{Map}_P(s\bar{D}_s^{k+1}, X)
\end{array}$$
is a pull-back diagram.

Let \(C\) be an \(S\)-sorted semi-theory, and let \(F_*C\) be the simplicial resolution of \(C\). Consider \(F_mC\), the free multi-sorted semi-theory in the \(m\)-th simplicial dimension of \(F_*C\), and let \(\overline{F_mC}\) be the completion of \(F_mC\) to an \(S\)-sorted algebraic theory. Setting \(D := F_mC\) above we see that the \(F_mC\)-diagram \(\overline{F_mC}_\alpha\) (where \(\overline{F_mC}_\alpha(s') = \text{Hom}_{\overline{F_mC}}(c_s, c_{s'})\)) admits two filtrations

by \(F_mC\)-diagrams:

\[
F_mC_\alpha = F_mC_\alpha^{0} \subseteq F_mC_\alpha^{1} \subseteq \cdots \subseteq F_mC_\alpha
\]

and by \(P\)-diagrams:

\[
F_mC_\alpha = sF_mC_\alpha^{0} \subseteq sF_mC_\alpha^{1} \subseteq \cdots \subseteq F_mC_\alpha
\]

The first of these filtrations, gives a filtration of the diagram \(\overline{F_*C}_\alpha\) by \(F_*C\)-diagrams

\[
\overline{F_*C}_\alpha = \overline{F_*C}_\alpha^{0} \subseteq \overline{F_*C}_\alpha^{1} \subseteq \cdots \subseteq \overline{F_*C}_\alpha
\]

Similarly, the filtrations of \(F_mC_\alpha\) by \(P\)-diagrams \(sF_mC_\alpha^k\) for \(m \geq 0\) give a filtration of \(\overline{F_*C}_\alpha\) by \(P\)-diagrams

\[
\overline{F_*C}_\alpha = s\overline{F_*C}_\alpha^{0} \subseteq s\overline{F_*C}_\alpha^{1} \subseteq \cdots \subseteq \overline{F_*C}_\alpha
\]

We have

**Lemma 7.2.** For \(X \in \text{Spaces}^{F_*C}\) consider the following diagrams of simplicial function complexes:

\[
\begin{array}{ccc}
\text{Map}_{F_*C}(\overline{F_*C}_\alpha^k, X) & \xleftarrow{f} & \text{Map}_{F_*C}(\overline{F_*C}_\alpha^{k+1}, X) \\
\downarrow & & \downarrow \\
\text{Map}_P(\overline{F_*C}_\alpha^k, X) & \xleftarrow{g} & \text{Map}_P(s\overline{F_*C}_\alpha^{k+1}, X)
\end{array}
\]

This is a pullback diagram for all \(X\), and \(k \geq 0\) and \(\alpha \in \tau\).

**Proof.** This follows directly from 7.1 and [2, 6.1].

Next we wish to show that the map \(g\) from lemma 7.2 satisfies the following:

**Lemma 7.3.** Let \(X\) be a homotopy algebra fibrant in \(\text{Spaces}^{F_*C}_{\alpha \in \tau}\). For every \(k \geq 0\) the map

\[
g: \text{Map}_P(s\overline{F_*C}_\alpha^{k+1}, X) \to \text{Map}_P(\overline{F_*C}_\alpha^{k}, X)
\]

induced by an inclusion \(\iota: \overline{F_*C}_\alpha^{k} \hookrightarrow s\overline{F_*C}_\alpha^{k+1}\) is an acyclic fibration of simplicial sets.
Proof. Since $\iota_k$ is a cofibration in $\text{Spaces}_{\text{cof}}^{F,C}$ we get that $g$ is a fibration. It remains to show that $g$ is also a weak equivalence of simplicial sets.

By Corollary 6.3 if $X$ is a homotopy $F_*$-algebra fibrant in $\text{Spaces}_{\text{cof}}^{F,C}$ then it is also a homotopy $P$-algebra which is fibrant in $\text{Spaces}_{\text{cof}}^P$. Therefore we need only show that the map $\iota_k$ is a local equivalence in $\text{Spaces}_P$, but this is a result of Theorem 6.1 and the fact that $\iota_k$ restricts to an isomorphism of simplicial sets

$$F_*C^i_s(s) \xrightarrow{\bar{\eta}} sF_*C^{i+1}_s(s)$$

for all $s \in S$

Next, consider the upper map $f$ in the diagram in lemma 7.2. From lemma 7.3 we have that $g$ is an acyclic fibration and from lemma 7.2 $f$ is the base change of $g$ along

$$\text{Map}_{F,C}(\overline{F_*C^i_s}, X) \rightarrow \text{Map}_{P}(\overline{F_*C^i_s}, X).$$

so by [10, 3.14] this we obtain:

**Corollary 7.4.** Let $X$ be a homotopy algebra fibrant in $\text{Spaces}_{\text{cof}}^{F,C}$. For all $k \geq 0$ the map

$$f: \text{Map}_{F,C}(\overline{F_*C^{k+1}_s}, X) \rightarrow \text{Map}_{F,C}(\overline{F_*C^k_s}, X)$$

is an acyclic fibration of simplicial sets.

We can now give the proof of Lemma 5.6.

**Proof of Lemma 5.6.** The map $\eta_{F_*C_\Delta}: F_*C_\Delta \rightarrow J_C K_C F_*C_\Delta$ is given by the inclusion of $F_*C$-diagrams

$$F_*C_\Delta = F_*C^0_\Delta \leftarrow F_*C_\Delta = J_C K_C F_*C_\Delta$$

Moreover, $\overline{F_*C_\Delta} = \text{colim}_k \overline{F_*C^k_s}$. We have a commutative diagram:

Using Corollary 7.4 we obtain that both the top map and the map on the right are local equivalences, and so the map $\eta_{F_*C_\Delta}$ is also a local equivalence.
Recall that Lemma 5.6 was the last element we needed to complete the proof of Theorem 3.2. Therefore Theorem 3.2 is now established.

8. Proof of Theorem 3.3

We will proceed with a proof of theorem 3.3. To start we will take note of the following lemma (see also [15, 8.6]).

**Lemma 8.1.** Let $G : T \to T'$ be a functor of multi-sorted algebraic theories. Then $G$ induces an adjoint pair of functors between the categories of strict algebras

$$G_* : \text{Alg}^T \longrightarrow \text{Alg}^{T'} : G^*$$

which is a Quillen pair. Moreover, $G^*$ gives an equivalence of the homotopy theories of strict algebras iff the functor $G$ is a weak equivalence of categories.

**Proof.** The adjoint pair $(G_*, G^*)$ exists by [3, ch.4 3.5], we also see that $G^*$ preserves fibrations and weak equivalences since both are computed objectwise and $G$ preserves products so we in fact have a Quillen pair.

Now, if we have that $G^*$ gives an equivalence of homotopy categories, then in particular it’s left adjoint, $G_*$ gives the inverse. For $T_s \in T$ we have the corepresented diagram $\text{Hom}_T(T_s, -)$ which is cofibrant in $\text{Alg}^T$. By using these equivalences we get that the unit of adjunction

$$\eta : \text{Hom}_T(T_s, -) \longrightarrow G^*G_* \text{Hom}_T(T_s, -)$$

is an objectwise weak equivalence. To see that $G$ is a weak equivalence of categories, notice by adjunction and Yoneda’s lemma we get:

$$\text{Map}_T(G_*, \text{Hom}_T(T_s, -), X)$$

$$\cong \text{Map}_T(\text{Hom}_T(T_s, -), G^*X)$$

$$\cong G^*X(T_s) \cong X(G(T_s))$$

$$\cong \text{Map}_{T'}(\text{Hom}_{T'}(G(T_s), -), X)$$

for all $X \in \text{Alg}^{T'}$, but this gives

$$G_* \text{Hom}_T(T_s, -) \cong \text{Hom}_{T'}(G(T_s), -)$$

and

$$G^*G_* \text{Hom}_T(T_s, -) \cong \text{Hom}_{T'}(G(T_s), G(-))$$

so in particular the unit of adjunction for a corepresented diagram given by

$$\eta : \text{Hom}_T(T_s, -) \longrightarrow \text{Hom}_{T'}(G(T_s), G(-))$$

is an objectwise weak equivalence if and only if $G$ is a weak equivalence of categories.
To finish the proof we need only show that if $G$ is a weak equivalence of categories, then $G^*$ gives an equivalence of homotopy algebras. The proof of this direction follows the proof of [17, 3.4] and the fact that if $G$ is a weak equivalence of categories, the unit of adjunction for any corepresented diagram is an objectwise weak equivalence. Recall from the proof of proposition 4.2 that we have a pair of adjoint functors:

$$F: \text{Spaces}^S \to \text{Alg}^T: U$$

It can be seen that this is in fact a Quillen pair since in the model category of $\text{Spaces}^S$, weak equivalences, fibrations and cofibrations are all computed object-wise. By [13, 9.6] for any strict $T$-algebra $X$, we can define a simplicial object $B(X)$ in $\text{Alg}^T$ by letting $B(X)_n = (FU)^{n+1}(X)$. Furthermore, by [13, 9.8] we see that the geometric realization of $B(X)$, denoted $|B(X)|$ has the property that there exists a map $\varphi: |B(X)| \to X$ which is a weak equivalence of strict $T$-algebras. By following an analogous argument to the one given in the proof of [1, 3.6], we also see that $|B(X)|$ is a cofibrant object of $\text{Alg}^T$, thus $|B(X)|$ is a cofibrant replacement of $X$.

Now, we see that any cofibrant diagram $X$ can be replaced by $|B(X)|$, for which it can be seen that the unit of adjunction $\eta_{|B(X)|}$ is an objectwise weak equivalence if the unit of adjunction for any corepresented diagram is an objectwise weak equivalence. This follows from the argument given in the proof of [17, 3.4], which in turn gives us that the unit of adjunction $\eta_X$ is an objectwise weak equivalence, but this gives us that $(G_*, G^*)$ is a Quillen equivalence.

With that we can give the proof of theorem 3.3:

**Proof of theorem 3.3.** Let $G: C \to C'$ be a functor of multi-sorted semi-theories. Consider the diagram

$$
\begin{array}{ccc}
\text{Alg}^{F,C} & \longrightarrow & \text{Alg}^{F,C'} \\
\downarrow & & \downarrow \\
\text{LSpaces}^{F,C} & \longrightarrow & \text{LSpaces}^{F,C'} \\
\downarrow & & \downarrow \\
\text{LSpaces}^C & \longrightarrow & \text{LSpaces}^{C'}
\end{array}
$$

in which every pair of arrows represents a Quillen pairs of functors. The horizontal pairs are induced by the functor $G$ while the vertical ones come from the adjunctions of (5.3), (4.10) and (5.4). Propositions 5.3 and 5.5 imply that the vertical pairs are Quillen equivalences. By this we have that $G$ induces an equivalence of the homotopy categories of $\text{LSpaces}^C$ and $\text{LSpaces}^{C'}$ if and only if it induces
an equivalence of the homotopy categories of strict algebras $\text{Alg}^{F,C}$ and $\text{Alg}^{F,C'}$. Thus lemma 8.1 completes the proof.

\[\square\]

9. The Associated Multi-Sorted Semi-Theory

Our next goal will be to prove Theorems 1.12 and 1.14. Our strategy will be to use a reductive process: we will show that an arbitrary finite product sketch can be replaced by a multi-sorted semi-theory which has the same homotopy category of homotopy algebras. As a result Theorems 1.12 and 1.14 will follow directly from their already established analogs for multi-sorted semi-theories, i.e. Theorems 3.2 and 3.3.

The reduction of finite product sketches to multi-sorted semi-theories will be performed in two stages. First, we will show that for any finite product sketch we can construct a multi-sorted finite product sketch in a way that preserves the the homotopy theory of homotopy algebras.

**Definition 9.1.** Let $S$ be a set. An $S$-sorted finite product sketch is a finite product sketch $(C,\kappa)$ with a distinguished set of objects $\{c_s\}_{s \in S}$ indexed by $S$ with the following properties:

- for all $\alpha \in \kappa$ and $i > 0$ we have $\alpha_i \in \{c_s\}_{s \in S}$;
- for all $\alpha \in \kappa$ $\alpha_0 \notin \{c_s\}_{s \in S}$;
- if $\alpha, \beta \in \kappa$, $|\alpha| = n = |\beta|$, and $(\alpha_i)_{i=1}^n = (\beta_i)_{i=1}^n$ then $\alpha = \beta$;
- if $\alpha, \beta \in \kappa$ and $\alpha_0 = \beta_0$ then $\alpha = \beta$.

A multi-sorted finite product sketch is a finite product sketch that is $S$-sorted for some set $S$.

**Lemma 9.2.** For any finite product sketch $B$ there exists a multi-sorted finite product sketch $B'$ such that the homotopy categories of homotopy algebras over $B$ and $B'$ are equivalent. Moreover, this construction is functorial.

The proof of Lemma 9.2 is a consequence of the following fact. Given two $n$-fold cones $\alpha$ and $\beta$ we will say that these cones are isomorphic if there exists a natural transformation between them given by the set of maps $(f_i)_{i=0}^n$ where $f_i: \alpha_i \rightarrow \beta_i$ is an isomorphism for all $i$. We have:

**Lemma 9.3.** Suppose $(B, \kappa)$ and $(B', \kappa')$ are two finite product sketches. Assume also that we have a functor

$$F: B \rightarrow B'$$

such that $F$ is an equivalence of categories. Assume also that the following conditions hold:
• for each $\alpha \in \kappa$, there is $\alpha' \in \kappa'$ so that $F(\alpha) \cong \alpha'$
• for each $\alpha' \in \kappa'$, there is $\alpha \in \kappa$ so that $\alpha' \cong F(\alpha)$

then the Quillen pair of functors

$$F_* : \text{LSpaces}^B \leftrightarrows \text{LSpaces}^{B'} : F^*$$

is a Quillen equivalence.

**Proof.** First, since $F$ is an equivalence of categories it induces a Quillen equivalence

$$F_* : \text{Spaces}^B_{\text{fib}} \leftrightarrows \text{Spaces}^{B'}_{\text{fib}} : F^*$$

Since $F$ preserves cones (up to an isomorphism) we get that maps which we localize $\text{Spaces}^{B'}$ by are sent by $F^*$ to maps which we localize $\text{Spaces}^B_{\text{fib}}$ by (up to isomorphism). By [12, 3.3.20] it follows that

$$F_* : \text{LSpaces}^B \leftrightarrows \text{LSpaces}^{B'} : F^*$$

is a Quillen equivalence.

□

We can now proceed to the proof of Lemma 9.2.

**Proof.** of lemma 9.2 Let $(B, \kappa)$ be a finite product sketch. We define the category $K$ which has as its objects, the set of cones $\{\alpha | \alpha \in \kappa\}$ and for each pair of objects $\alpha, \beta \in K$ there is a unique isomorphism

$$\psi_{\alpha, \beta} : \alpha \to \beta$$

Next, let $J$ be the category with two objects 0 and 1 and the nonidentity morphisms given two inverse isomorphisms:

$$\varphi : 0 \leftrightarrow 1 : \varphi^{-1}$$

Take $B' = B \times K \times J$. For every $n$-fold cone $\alpha \in \kappa$ we have the associated $n$-fold cone in $B'$ given as follows:

$$\begin{array}{c}
(a_0, \alpha, 0) \\
\downarrow \text{id} \times \text{id} \times \varphi \\
(a_0, \alpha, 1) \\
\downarrow p_1^0 \times \text{id} \times \text{id} \\
(a_1, \alpha, 1) \\
\downarrow p_2^0 \times \text{id} \times \text{id} \\
(a_2, \alpha, 1) \\
\downarrow p_2^1 \times \text{id} \times \text{id} \\
(a_n, \alpha, 1)
\end{array}$$
Consider the sketch \((B', \kappa')\) where \(\kappa'\) is the set of all cones of the above form. Notice that \((B', \kappa')\) is a multi-sorted finite product sketch with the distinguished set of objects \(\{(\alpha_i, \alpha, 1) \mid \alpha \in \kappa, 0 < \alpha \leq |\alpha|\}\). We define the functor:

\[
F : B \rightarrow B',
\]

by \(F(b) = (b, \alpha, 0)\) and \(F(\theta : b_1 \to b_2) = \theta \times \text{id} \times \text{id}\), where \(\alpha\) is some fixed cone from \(\kappa\) (we can assume that \(\kappa\) is non-empty since \(B\) would satisfy lemma 9.2 trivially otherwise). It can be checked that \(F\) satisfies the conditions of Lemma 9.3, and so it gives a Quillen equivalence

\[
F_* : \text{LSpaces}^B \leftrightarrow \text{LSpaces}^{B'} : F^*
\]

Thus \(F^*\) induces an equivalence between homotopy categories of homotopy algebras over \(B\) and \(B'\).

Next, we will show that any multi-sorted finite product sketch can be in replaced by a multi-sorted semi-theory in a way that does not change the homotopy theory of homotopy algebras.

**Lemma 9.4.** For any multi-sorted finite product sketch \((B, \kappa)\) there exists a multi-sorted semi-theory \((B', \kappa')\) so that the homotopy category of homotopy algebras over \(B\) and \(B'\) are equivalent. Moreover, this construction is functorial in \(B\).

**Proof.** Notice that a multi-sorted finite product sketch \((B, \kappa)\) can be equivalently described as follows. There exists a set \(S\) such that objects \(b_\varepsilon \in B\) can be indexed by some of the \(n\)-tuples of \(S\) \((n \geq 0)\) and for any \(s \in S\) we have \(b_s \in S\) (as before we identify here elements of \(S\) with 1-tuples defined by these elements). Any cone \(\alpha \in \kappa\) satisfies the property that if \(\alpha_0 = b_\varepsilon\) where \(\varepsilon = (s_1, \ldots, s_n)\) then \(\alpha_k = b_{s_k}\) for \(k = 1, \ldots, n\). Moreover, for any \(b_\varepsilon \in B\) there exists a unique cone \(\alpha \in \kappa\) such that \(\alpha_0 = b_\varepsilon\). In other words the difference between \(B\) and an \(S\)-sorted semi-theory is that for some \(n\)-tuples \(\varepsilon\) in \(S\) there may be no object of \(B\) indexed by \(\varepsilon\), and thus the cone corresponding to \(\varepsilon\) will be also missing. To fix it we enlarge that category \(B\) as follows. Let \(\mathcal{S}\) denote the set of all \(n\)-tuples in \(S\):

\[
\mathcal{S} = \{\varepsilon = (s_1, \ldots, s_n) \mid s_i \in S, n \geq 0\}
\]

also, let \(\mathcal{S}_B\) denote the set of all \(n\)-tuples that index elements of \(B\):

\[
\mathcal{S}_B = \{\varepsilon \in \mathcal{S} \mid b_\varepsilon \in B\}
\]

Let \(B'\) be the category whose objects \(b_\varepsilon\) are indexed by all elements \(\varepsilon \in \mathcal{S}\), and such that \(B\) is a full subcategory of \(B'\). For each \(\varepsilon = (s_1, \ldots, s_n) \notin \mathcal{S}_B\) the category \(B'\) has morphisms \(p_\varepsilon^k : b_\varepsilon \rightarrow b_{s_k}\), and these morphisms compose freely with morphisms in \(B\). We give \(B'\) a finite product sketch structure by defining the set of cones \(\kappa'\) that consists of all cones in \(\kappa\) and for each \(\varepsilon = (s_1, \ldots, s_n) \in \mathcal{S}_B\) a cone \(\alpha_\varepsilon^k\) with \(\alpha_\varepsilon^k = b_\varepsilon^k\) for \(k = 1, \ldots, n\), and \(\alpha_\varepsilon = b_\varepsilon\) and with projections given by the morphisms \(p_\varepsilon^k\). Clearly \((B', \kappa')\) is an \(S\)-sorted semi-theory.
It remains to show that the homotopy category of homotopy algebras over \((B, \kappa)\) is equivalent to the homotopy category of homotopy algebras over \((B', \kappa')\). Let \(F: B \to B'\) be the inclusion functor, and let \(F^*: \text{Spaces}^{B'} \to \text{Spaces}^B\) be the functor induced by \(F\). The functor \(F\) has the right adjoint \(G\) which can be described as follows. For \(X \in \text{Spaces}^B\) the functor \(G(X): B' \to \text{Spaces}\) coincides with \(B'\) when restricted to \(B \subseteq B'\). For an object \(b_\underline{s} \in B'\) such that \(b_\underline{s} \notin B\) and where \(\underline{s} = (s_1, \ldots, s_n)\) we set

\[
G(X)(b_\underline{s}) = \prod_{k=1}^n X(b_{s_k})
\]

Denote by \(\text{Alg}_h^B\) and \(\text{Alg}_h^{B'}\) the full subcategories of \(\text{Spaces}^B\) and \(\text{Spaces}^{B'}\) respectively whose objects are homotopy \(B\)- (resp. \(B'\)-) algebras. Notice that both \(F^*\) and \(G\) restrict to a functors

\[
F^*: \text{Alg}_h^{B'} \leftrightarrow \text{Alg}_h^B: G
\]

Notice that composition \(F^*G\) is naturally isomorphic to the identity functor. Also, for any \(X \in \text{Alg}_h^B\) we have a natural objectwise weak equivalence \(X \xrightarrow{\sim} GF^*(X)\). Since the homotopy categories of homotopy algebras are obtained from \(\text{Alg}_h^B\) and \(\text{Alg}_h^{B'}\) by inverting all objectwise weak equivalences we obtain that \(F^*\) and \(G\) give inverse equivalences on the level of the homotopy categories. \(\square\)

10. **Proof of theorem 1.12 and theorem 1.14**

We can now use lemma 9.2 and lemma 9.4 to see that any finite product sketch \(B\) can be replaced by a multi-sorted semi-theory \(B'\) which has an equivalent homotopy category of homotopy algebras.

**Proof** of theorem 1.12 Let \(B\) be a finite product sketch and \(B'\) be the associated multi-sorted semi-theory. By theorem 3.2 we have that there is a Quillen equivalence:

\[
\text{LSpaces}^{B'} \leftrightarrow \text{Alg}^{F^*,B'}
\]

and we can compose with the Quillen equivalences from lemma 9.2 and lemma 9.4 to get a Quillen equivalence:

\[
\text{LSpaces}^B \leftrightarrow \text{Alg}^{F^*,B'}
\]

which gives us that the homotopy category of homotopy \(B\)-algebras is equivalent to the homotopy category of strict \(F^*,B'\)-algebras. \(\square\)
With this we can proceed with a proof of theorem 1.14.

**Proof.** of theorem 1.14 For two finite product sketches $B_1$ and $B_2$, let $B_1'$, $B_2'$, $B_1''$, and $B_2''$ be the associated multi-sorted finite product sketches and multi-sorted semi-theories from lemma 9.2 and lemma 9.4 respectively. Notice that for a functor of finite product sketches which is a bijection on objects and the set of cones:

$$F : B_1 \longrightarrow B_2$$

we can induce a cone preserving functor:

$$F' : B_1' \longrightarrow B_2'$$

$$(B, \alpha, i) \longrightarrow (F(B), F(\alpha), i)$$

where $F(\alpha)$ makes sense because $F$ preserves cones. This induces a functor:

$$F'' : B_1'' \longrightarrow B_2''$$

which is defined naturally on objects of the form $(B, \alpha, i)$ and which sends any added cones in $B_1''$ to the obvious added cone in $B_2''$. We see that this is a functor of multi-sorted semi-theories which is a weak equivalence if and only if it induces an equivalence of the homotopy algebras by theorem 3.3, but this only occurs if $F : B_1 \longrightarrow B_2$ induces an equivalence of the homotopy algebras. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY AT BUFFALO, SUNY, 244 MATHEMATICS BUILDING, BUFFALO, NY 14260, USA

E-mail address: brucecor@buffalo.edu