SECOND DERIVATIVE TEST FOR
ISOMETRIC EMBEDDINGS IN $L_p$

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Abstract. An old problem of P. Levy is to characterize those Banach spaces which embed isometrically in $L_p$. We show a new criterion in terms of the second derivative of the norm. As a consequence we show that, if $M$ is a twice differentiable Orlicz function with $M'(0) = M''(0) = 0$, then the $n$-dimensional Orlicz space $\ell^n_M$, $n \geq 3$ does not embed isometrically in $L_p$ with $0 < p \leq 2$. These results generalize and clear up the recent solution to the 1938 Schoenberg’s problem on positive definite functions.

1. Introduction

In 1938, Schoenberg [28] asked a question on positive definite functions an equivalent formulation of which is as follows: for which $p \in (0, 2), n \geq 2, q > 2$ is the space $\ell^n_q$ isometric to a subspace of $L_p$? The solution was completed in 1991 and since then there have appeared a few more proofs (see historical remarks below). However, all those solutions were a little artificial in the sense that the nature of the result was buried under technical details. The first proof [13] depended heavily on calculations involving the Fourier transform of distributions. The later proof of Zastavny [29, 30] was simpler and led to more general results, but was still quite technical.

The approach developed in this article not only leads to further generalizations, but also seems to clear things up. To support this ambitious statement, let us show a very simple argument which, unfortunately, is false. It was known to P.Levy [19] that an $n$-dimensional normed space $B = (\mathbb{R}^n, \| \cdot \|)$ embeds isometrically in $L_p$, $p > 0$ if and only if there exists a finite Borel measure $\mu$ on the unit sphere $\Omega$ in $\mathbb{R}^n$ so that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_\Omega |(x, \xi)|^p \, d\mu(\xi),$$

where $(x, \xi)$ stands for the scalar product. Let us formally take the second derivatives by $x_1$ in both sides of (1). We get

$$p(p - 1)\|x\|^{p-2} (\|x\|_{x_1}')^2 + p\|x\|^{p-1} \|x\|_{x_1}'' = p(p - 1) \int_\Omega |(x, \xi)|^{p-2} \xi_1^2 \, d\mu(\xi).$$

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Suppose (and this is the case for the spaces $\ell^n_q$ with $q > 2$) that $\|x\|_{x_1}'(0, x_2, x_3) = \|x\|''_{x_2}(0, x_2, x_3) = 0$ for every $(x_2, x_3) \in \mathbb{R}^2 \setminus \{0\}$. Let us show that under this condition $B$ cannot be isometric to a subspace of $L_p$. In fact, putting $x_1 = 0$ in (2) we get zero in the left-hand side. On the other hand, the function under the integral in the right-hand side is non-negative, and the integral can be equal to zero only if the measure $\mu$ is supported in the hyperplane $\xi_1 = 0$. But this contradicts to the assumption that the dimension of $B$ is $n$.

That this reasoning is wrong follows immediately from the fact that every two-dimensional normed space (including the spaces $\ell^n_q$, $q > 2$) embeds isometrically in $L_p$ for every $p \in (0, 1]$. However, if $n \geq 3$ the solution to Schoenberg’s problem is that the spaces $\ell^n_q$, $q > 2$ do not embed in $L_p$ with $0 < p < 2$, so there is a chance that the argument can be repaired.

The mistake was that for $p \in (0, 1]$ the integral in the right-hand side of (2) might diverge because $p - 2 \leq -1$. This, however, can be fixed to a certain extent using the so-called technique of embedding in $L_{-p}$ introduced in [16]. This technique employs the connection between the Radon and Fourier transforms to define and study the representation (1) in the case of negative $p$. After relatively short repairments, which clearly distinguish the two-dimensional case, we get the main result of this article:

**Theorem 1.** Let $X$ be a three-dimensional normed space with a normalized basis $e_1, e_2, e_3$ so that:

(i) For every fixed $(x_2, x_3) \in \mathbb{R}^2 \setminus \{0\}$, the function $x_1 \to \|x_1 e_1 + x_2 e_2 + x_3 e_3\|$ has continuous second derivative everywhere on $\mathbb{R}$, and

$$\|x\|_{x_1}'(0, x_2, x_3) = \|x\|''_{x_2}(0, x_2, x_3) = 0;$$

(ii) There exists a constant $K$ so that, for every $x_1 \in \mathbb{R}$ and every $(x_2, x_3) \in \mathbb{R}^2$ with $\|x_2 e_2 + x_3 e_3\| = 1$, we have $\|x\|''_{x_2}(x_1, x_2, x_3) \leq K$;

(iii) The convergence in the limit $\lim_{x_1 \to 0} \|x\|''_{x_1}(x_1, x_2, x_3) = 0$ is uniform with respect to $(x_2, x_3) \in \mathbb{R}^2$ with $\|x_2 e_2 + x_3 e_3\| = 1$.

Then, for every $0 < p \leq 2$, $X$ is not isometric to a subspace of $L_p$.

This criterion generalizes the solution to Schoenberg’s problem, and it can be applied in some situations where previously known criteria do not work. In Section 3, we use Theorem 1 to prove that Orlicz spaces $\ell^n_M$, $n \geq 3$ do not embed in $L_p$ with $0 < p \leq 2$ if $M$ is twice continuously differentiable on $[0, \infty)$ and $M'(0) = M''(0) = 0$.

Before we proceed with our repairments, let us make a few historical remarks. The problem of how to check whether a given Banach space is isometric to a subspace of $L_p$ was raised by P. Levy [19] in 1937. A well-known fact is that a Banach space embeds isometrically in a Hilbert space if and only if its norm satisfies the

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} < \infty \]
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parallelogram law \[6,12\]. However, as shown by Neyman \[26\], subspaces of $L_p$ with $p \neq 2$ cannot be characterized by a finite number of equations or inequalities.

P. Levy \[19\] pointed out that an $n$-dimensional normed space $B = (\mathbb{R}^n, \| \cdot \|)$ embeds isometrically in $L_p$, $p > 0$ if and only if the norm admits the representation (1), and, on the other hand, for $0 < p \leq 2$ the representation (1) exists if and only if the function $\exp(-\|x\|^p)$ is positive definite, and, hence, is the characteristic function of a stable measure. The equivalence of isometric embedding in $L_p$ with $0 \leq p \leq 2$ and positive definiteness of the function $\exp(-\|x\|^p)$ was established precisely by Bretagnolle, Dacunha-Castelle and Krivine \[2\] who used this fact to show that the space $L_q$ embeds isometrically in $L_p$ if $0 < p < q \leq 2$.

For a long time, the connection with stable random vectors and positive definite functions had been the main source of results on isometric embedding in $L_p$ (see \[1,11,15,17,18,23,24\]). However, it turns out to be quite difficult to check whether $\exp(-\|x\|^p)$ is positive definite for certain norms. For example, the following 1938 Schoenberg’s problem \[28\] was open for more than fifty years: for which $p \in (0, 2)$ is the function $\exp(-\|x\|^p)$ positive definite, where $\|x\|_q$ is the norm of the space $\ell^q_n$, $2 < q \leq \infty$? As it was mentioned at the beginning of the paper, an equivalent formulation asks whether the spaces $\ell^q_n$ embed in $L_p$ with $0 < p \leq 2$. After a few partial results \[4\], the final answer was given in \[22\] for $q = \infty$ and in \[13\] for $2 < q < \infty$ : if $n \geq 3$ the spaces $\ell^q_n$, $2 < q \leq \infty$ do not embed isometrically in $L_p$, $0 < p < 2$. A well-known fact \[5,10,20\] is that every two-dimensional Banach space embeds in $L_p$ for every $p \in (0, 1]$. The spaces $\ell^2_q$, $2 < q \leq \infty$ do not embed in $L_p$ with $1 < p \leq 2$ (see \[4\]).

The solution to Schoenberg’s problem in \[13\] was based on the following Fourier transform characterization of subspaces of $L_p$ (see \[14\]): for every $p > 0$ which is not an even integer, an $n$-dimensional Banach space is isometric to a subspace of $L_p$ if and only if the Fourier transform of the function $\Gamma(-p/2)\|x\|^p$ is a positive distribution on $\mathbb{R}^n \setminus \{0\}$. In \[3\], this criterion was applied to Lorentz spaces.

Not very long after the paper \[13\] appeared, Zastavny \[29,30\] proved that a three dimensional space is not isometric to a subspace of $L_p$ with $0 < p \leq 2$ if there exists a basis $e_1, e_2, e_3$ so that the function

$$ (y, z) \mapsto \|xe_1 + ye_2 + z e_3\|_x(1, y, z)/\|e_1 + ye_2 + z e_3\|, \ y, z \in \mathbb{R} $$

belongs to the space $L_1(\mathbb{R}^2)$. This criterion provides a new proof of Schoenberg’s conjecture. Zastavny also showed a stronger result that there are no non-trivial positive definite functions of the form $f(\|x\|_q)$. (For $q = \infty$ that result was established in \[22\]; the result for $2 < q < \infty$ was shown independently by Lisitsky \[21\])

We refer the reader to \[15,25\] for more on isometric embedding of Banach spaces in $L_p$ and its connections with positive definite functions and isotropic random vectors. Note that the case $p = 1$ is related to the theory of zonoids. Geometric
characterizations of zonoids and related results of convex geometry can be found in [7,9,27].

2. Proof of the second derivative test

We start with notation and simple remarks. As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable functions (test functions) in $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi \rangle = (2\pi)^n \langle f, \phi \rangle$ for every test function $\phi$. If $p > -1$ and $p$ is not an even integer, then the Fourier transform of the function $h(z) = |z|^p$, $z \in \mathbb{R}$ is equal to $(|z|^p)'(t) = c_p |t|^{-1-p}$ (see [8, p.173]), where $c_p = \frac{2^{p+1}}{\Gamma((-p)/2)}$. The well-known connection between the Radon transform and the Fourier transform is that, for every $\xi \in \Omega$, the function $t \to \hat{\phi}(t\xi)$ is the Fourier transform of the function $z \to R\phi(\xi; z) = \int_{(x,\xi)=z} \phi(x) \, dx$ ($R$ stands for the Radon transform).

Throughout this section, we remain under the conditions and notation of Theorem 1. We denote by $\|x\|_{x_1}$ and $\|x\|_{x_2}''$ the first and second partial derivatives by $x_1$ of the norm $\|x_1 e_1 + x_2 e_2 + x_3 e_3\|_3$.

Remarks. (i) It is easy to see that, for every continuous, homogeneous of degree 1, positive outside of the origin function $f$ on $\mathbb{R}^n$ and every $\alpha > -n$, the function $f^\alpha$ is locally integrable on $\mathbb{R}^n$. In particular, for any $p > 0$, the function $(x_2, x_3) \to \|x_2 e_2 + x_3 e_3\|^{p-2}$ is locally integrable on $\mathbb{R}^2$.

(ii) A simple consequence of the triangle inequality is that $-1 \leq \|x\|_{x_2}'' \leq 1$ at every point $x \in \mathbb{R}^3$ with $(x_2, x_3) \neq 0$.

(iii) For every fixed $(x_2, x_3) \in \mathbb{R}^2 \setminus \{0\}$, $\|x\|$ is a convex differentiable function of $x_1$ whose derivative at zero is equal to zero. Therefore, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we have $\|x\| \geq \|x_2 e_2 + x_3 e_3\|$.

(iv) The function $\|x\|_{x_2}''$ is non-negative, homogeneous of degree -1. Let $K$ be the constant from the condition (ii) of Theorem 1. Then, for every $x_1 \in \mathbb{R}$ and every $(x_2, x_3) \in \mathbb{R}^2 \setminus \{0\}$, the second derivative $\|x\|_{x_2}''(x_1, x_2, x_3)$ is equal to

$$\frac{1}{\|x_2 e_2 + x_3 e_3\|} \|x\|''_{x_1}(x_1, x_2, x_3) = \frac{x_1}{\|x_2 e_2 + x_3 e_3\|'} \frac{x_2}{\|x_2 e_2 + x_3 e_3\|'} \frac{x_3}{\|x_2 e_2 + x_3 e_3\|'}$$

which is less or equal to $K/\|x_2 e_2 + x_3 e_3\|$ by the condition (ii) of Theorem 1.

We are ready to prove the main result of this paper.

Proof of Theorem 1. For every $0 < p_1 < p_2 \leq 2$, the space $L_{p_2}$ embeds isometrically in $L_{p_1}$, so it suffices to prove the theorem for $p \in (0,1)$.

Suppose that $X$ is isometric to a subspace of $L_p$ with $0 < p < 1$. Then, by (1), there exists a measure $\mu$ on the unit sphere $\Omega$ in $\mathbb{R}^3$ so that, for every $x \in \mathbb{R}^3$,

$$\|x\|^p = \int |(x,\xi)|^p \, d\mu(\xi).$$
Applying functions in both sides of the latter equality to a test function $\phi$, and using the Fubini theorem and the connection between the Radon transform and the Fourier transform, we get

\[
\langle \|x\|^p, \phi \rangle = \int_{\mathbb{R}^3} \|x\|^p \phi(x) \, dx = \int_\Omega d\mu(\xi) \int_{\mathbb{R}^3} \|(x, \xi)\|^p \phi(x) \, dx = \\
\int_\Omega d\mu(\xi) \int_{\mathbb{R}} |t|^p \left( \int_{(x, \xi) = t} \phi(x) \, dx \right) \, dt = \int_\Omega \langle |t|^p, R\phi(t; \xi) \rangle \, d\mu(\xi) = \\
\frac{1}{(2\pi)^3} c_p \int_\Omega \langle |t|^{-1-p}, \hat{\phi}(t\xi) \rangle \, d\mu(\xi).
\]

By the connection between the Fourier transform and differentiation, we have

\[
(\partial^2 \phi/\partial x_i^2)^\wedge(\xi) = -\xi_i^2 \hat{\phi}(\xi),
\]

and it follows from (3) that

\[
\langle (\|x\|^p)''_{x^2_i}, \phi \rangle = \langle \|x\|^p, \frac{\partial^2 \phi}{\partial x_i^2} \rangle = -\frac{1}{(2\pi)^3} c_p \int_\Omega \xi_i^2 d\mu(\xi) \int_{\mathbb{R}} |t|^{1-p} \hat{\phi}(t\xi) \, dt.
\]

Let $\phi_n(x_1, x_2, x_3) = h_n(x_1) u(x_2, x_3)$, where

\[
h_n(x_1) = \left(n/\sqrt{2\pi}\right) \exp(-x_1^2 n^2/2), \quad u(x_2, x_3) = (1/(2\pi)) \exp(-(x_2^2 + x_3^2)/2).
\]

Then $\hat{\phi}_n(\xi_1, \xi_2, \xi_3) = \exp(-\xi_1^2/2n^2) \exp(-((\xi_2^2 + \xi_3^2)/2).$ Clearly, $\int_{\mathbb{R}} h_n(x_1) \, dx_1 = 1$, and $\lim_{n \to \infty} \int_{|x_1| > \delta} h_n(x_1) \, dx_1 = 0$ for every $\delta > 0$.

Note that the number $c_p$ is negative for every $p \in (0, 2)$ (see the expression for $c_p$ in the beginning of this section). Applying (4) to the function $\phi_n$, we get

\[
\langle (\|x\|^p)''_{x^2_i}, \phi_n \rangle = -2^{\frac{p}{2}} \Gamma(1 - \frac{p}{2}) \frac{1}{(2\pi)^3} c_p \int_\Omega \xi_i^2 (\xi_1^2/n^2 + \xi_2^2 + \xi_3^2/t^2) \, d\mu(\xi) \geq \\
-2^{\frac{p}{2}} \Gamma(1 - \frac{p}{2}) \frac{1}{(2\pi)^3} c_p \int_\Omega \xi_i^2 \, d\mu(\xi).
\]

By Lemma 1 below, we can make the left-hand side of (5) as small as we want. Therefore, $\int_\Omega \xi_i^2 \, d\mu(\xi) = 0$, and the measure $\mu$ is supported in the hyperplane $\xi_1 = 0$. This contradicts to the assumption that the space $X$ is three-dimensional, so $X$ is not isometric to a subspace of $L_p$. \qed

In the following lemma the norm has the same properties as in Theorem 1 and the functions $\phi_n$, $h_n$, $u$ are the same as in the proof of Theorem 1.

**Lemma 1.** For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that, for every $n > N$,

\[
\langle (\|x\|^p)''_{x^2_i}, \phi_n \rangle \leq \epsilon.
\]

**Proof.** First we use the Fubini theorem to show that

\[
\langle (\|x\|^p)''_{x^2_i}, \phi_n \rangle = \langle \|x\|^p, \frac{\partial^2 \phi_n}{\partial x_i^2} \rangle = \\
\int_{\mathbb{R}^3} \|x\|^p \frac{\partial^2 \phi_n}{\partial x_i^2} \, dx = \int_\Omega d\mu(\xi) \int_{\mathbb{R}^3} \|(x, \xi)\|^p \frac{\partial^2 \phi_n}{\partial x_i^2} \, dx = \\
\int_\Omega \int_{\mathbb{R}} \xi_i^2 \left( \int_{(x, \xi) = t} \frac{\partial^2 \phi_n}{\partial x_i^2} \, dx \right) \, dt = \int_\Omega \int_{\mathbb{R}} \xi_i^2 \left( \int_{(x, \xi) = t} \frac{\partial \phi_n}{\partial x_i} \right) \, dt \, d\mu(\xi) = \\
\frac{1}{(2\pi)^3} c_p \int_\Omega \int_{\mathbb{R}} \xi_i^2 \left( \int_{(x, \xi) = t} \frac{\partial \phi_n}{\partial x_i} \right) \, dt \, d\mu(\xi) = \\
\frac{1}{(2\pi)^3} c_p \int_\Omega \int_{\mathbb{R}} \xi_i^2 \left( \int_{(x, \xi) = t} \frac{\partial \phi_n}{\partial x_i} \right) \, dt \, d\mu(\xi) = \\
\frac{1}{(2\pi)^3} c_p \int_\Omega \int_{\mathbb{R}} \xi_i^2 \left( \int_{(x, \xi) = t} \frac{\partial \phi_n}{\partial x_i} \right) \, dt \, d\mu(\xi) =.\]
\[
\int_{\mathbb{R}^3} \|x\|^p \frac{\partial^2 \phi_n}{\partial x_1^2} \, dx = \int_{\mathbb{R}^3 \setminus \{0\}} u(x_2, x_3) \, dx_2 \, dx_3 \left( \int_{\mathbb{R}} \|x\|^p \frac{\partial^2 h_n}{\partial x_1^2} \, dx_1 \right) = \\
\int_{\mathbb{R}^2 \setminus \{0\}} \left( \|x\|^p \right)''_{x_1} h_n(u_2, x_3) \, dx_2 \, dx_3.
\]

(6)

We restrict the outer integral to \(\mathbb{R}^2 \setminus \{0\}\) (which does not change the value of the integral) to formally exclude the point \((x_2, x_3) = 0\), where the derivative \((\|x\|^p)'_{x_1}\) is a distribution in terms of the \(\delta\)-function of \(x_1\).

By the condition (i) of Theorem 1 and Remarks (i), (ii), (iv), for every fixed \((x_2, x_3) \in \mathbb{R}^2 \setminus \{0\}\), the derivative \((\|x\|^p)''_{x_1}\) is a locally integrable continuous function of the variable \(x_1\) on \(\mathbb{R}\), so the expression in (6) can be written as

\[
\int_{\mathbb{R}^2 \setminus \{0\}} \left( \int_{\mathbb{R}_c} (\|x\|^p)''_{x_1} h_n(x_1) \, dx_1 \right) u(x_2, x_3) \, dx_2 \, dx_3 = \\
\int_{\mathbb{R}^2 \setminus \{0\}} \left( \int_{\mathbb{R}_c} (p(p-1)\|x\|^{p-2}\|x\|_{x_1}')^2 + p\|x\|^{p-1}\|x\|_{x_1}'' h_n(x_1) \, dx_1 \right) u(x_2, x_3) \, dx_2 \, dx_3.
\]

Since \(p < 1\), the first term under the integral by \(x_1\) is negative, so, to prove Lemma 1, it suffices to show that

\[
\lim_{n \to \infty} \int_{\mathbb{R} \times (\mathbb{R}^2 \setminus \{0\})} \|x\|^{p-1}\|x\|_{x_1}' h_n(x_1) \, u(x_2, x_3) \, dx = 0.
\]

Let \(\epsilon > 0\). By Remark (i), the function \(\|x_2 e_2 + x_3 e_3\|^{p-2}\) is locally integrable in \(\mathbb{R}^2\), so

\[
L = \int_{\mathbb{R}^2} \|x_2 e_2 + x_3 e_3\|^{p-2} u(x_2, x_3) \, dx_2 \, dx_3 < \infty.
\]

Besides, there exists \(c > 0\) so that

\[
\int_{\{(x_2, x_3) : \|x_2 e_2 + x_3 e_3\| < c\}} \|x_2 e_2 + x_3 e_3\|^{p-2} u(x_2, x_3) \, dx_2 \, dx_3 < \frac{\epsilon}{3K},
\]

where \(K\) is the number defined in Remark (iv). By Remark (iv), we have

\[
\int_{\mathbb{R} \times \{(x_2, x_3) : \|x_2 e_2 + x_3 e_3\| < c\}} \|x\|^{p-1}\|x\|_{x_1}' h_n(x_1) \, u(x_2, x_3) \, dx \leq \\
K \int_{\mathbb{R} \times \{(x_2, x_3) : \|x_2 e_2 + x_3 e_3\| < c\}} \|x\|^{p-1}\|x_2 e_2 + x_3 e_3\|^{-1} h_n(x_1) u(x_2, x_3) \, dx < \frac{\epsilon}{3},
\]

where we apply Remark (iii) to show that \(\|x\|^{p-1} \leq \|x_2 e_2 + x_3 e_3\|^{-1}\) (note that \(p - 1 < 0\)), and then use the inequality (8) and the fact that \(\int_{\mathbb{R}} h_n(t) \, dt = 1\).

By the condition (iii) of Theorem 1, \(\lim_{x_1 \to 0} \|x\|_{x_1}'(x_1, x_2, x_3) = 0\) uniformly with respect to \((x_2, x_3) \in \mathbb{R}^2\) with \(\|x_2 e_2 + x_3 e_3\| = 1\), so there exists \(\delta > 0\) so that if \(|x_1| < \delta\), then \(\|x_{x_1} h_n(x_1) \| < \frac{\epsilon}{3K}\).
\[ \delta \text{ then } \|x\|''_1(x_1, x_2, x_3) < \frac{\epsilon}{3L} \text{ for every } (x_2, x_3) \in \mathbb{R}^2 \text{ with } \|x_2e_2 + x_3e_3\| = 1. \]

The second derivative of the norm is a homogeneous function of degree -1. Therefore, if \((x_2, x_3) \in \mathbb{R}^2 \setminus \{0\}\), and \(\frac{|x|}{\|x_2e_2 + x_3e_3\|} < \delta\), then

\[
\|x\|''_1(x_1, x_2, x_3) = \frac{1}{\|x_2e_2 + x_3e_3\|} \|x\|''_1(\|x_2e_2 + x_3e_3\|) < \frac{\epsilon}{3L\|x_2e_2 + x_3e_3\|}.
\]

Denote by \(A_1, A_2\) the sets \(\{x_1 : |x_1| < \delta c\} \times \{(x_2, x_3) : \|x_2e_2 + x_3e_3\| \geq c\}\) and \(\{x_1 : |x_1| > \delta c\} \times \{(x_2, x_3) : \|x_2e_2 + x_3e_3\| \geq c\}\) in \(\mathbb{R}^3\). Then, by (10), Remark (iii) and the fact that \(\int_{\mathbb{R}} h_n(t) \, dt = 1\),

\[
\int_{A_1} \|x\|^{p-1} \|x\|''_1 h_n(x_1) \, u(x_2, x_3) \, dx \leq \frac{\epsilon}{3L} \int_{A_1} \|x_2e_2 + x_3e_3\|^{p-2} h_n(x_1) \, u(x_2, x_3) \, dx \leq \frac{\epsilon}{3}.
\]

Finally, suppose that \(n\) is large enough so that \(\int_{|x_1| > \delta c} h_n(x_1) \, dx_1 < \epsilon/(3KL)\). Then we use the estimate of Remark (iv) and Remark (iii) to show that

\[
\int_{A_2} \|x\|^{p-1} \|x\|''_1 h_n(x_1) \, u(x_2, x_3) \, dx \leq K \int_{A_2} \|x_2e_2 + x_3e_3\|^{p-2} h_n(x_1) \, u(x_2, x_3) \, dx \leq \frac{\epsilon}{3}.
\]

Since \(\epsilon\) is an arbitrary positive number, the result of Lemma 1 follows from (9), (11), (12) and the fact that the second derivative of a convex function is non-negative. \(\Box\)

**Remark.** Note that the statement of Theorem 1 is not true for two-dimensional spaces all of which embed isometrically in \(L_p\) for every \(p \in (0, 1]\). The reason that the proof of Theorem 1 does not work for two-dimensional spaces is that, unlike the function \(\|x_2e_2 + x_3e_3\|^{p-2}\) on \(\mathbb{R}^2\), the function \(|x_2|^{p-2}\) is not locally integrable on \(\mathbb{R}\) if \(0 < p \leq 1\).

### 3. Application to Orlicz spaces

An Orlicz function \(M\) is a non-decreasing convex function on \([0, \infty)\) such that \(M(0) = 0\) and \(M(t) > 0\) for every \(t > 0\).

For an Orlicz function \(M\), the norm of the \(n\)-dimensional Orlicz space \(\ell_M^n\) is defined by the equality \(\sum_{k=1}^{n} M(\|x_k\|/\|x\|) = 1\) for all \(x \in \mathbb{R}^n \setminus \{0\}\).
Theorem 2. Let $M$ be an Orlicz function so that $M \in C^2([0, \infty))$, $M'(0) = M''(0) = 0$. Then, for every $0 < p \leq 2$, the three-dimensional Orlicz space $\ell^3_M$ does not embed isometrically in $L_p$.

Proof. We are going to show that the norm of the space $\ell^3_M$ satisfies the conditions of Theorem 1. Since the Orlicz norm is an even function with respect to each variable, it suffices to consider the points $x = (x_1, x_2, x_3)$ with non-negative coordinates. We denote by $e_1, e_2, e_3$ the standard normalized basis in $\ell^3_M$.

The function $M'$ is non-decreasing, continuous on $[0, \infty)$ and $M'(0) = 0$. Since $M(0) = 0$ and $M(t) > 0$ for every $t > 0$, the function $M'$ can not be equal to zero on an interval, so $M'(t) > 0$ for every $t > 0$.

Let $x = (x_1, x_2, x_3)$ with $(x_2, x_3) \neq 0$. Then one of the numbers $x_2 M'(x_2/\|x\|)$ or $x_3 M'(x_3/\|x\|)$ is positive. By implicit differentiation,

$$\|x\|_{x_1} = \frac{\|x\| M'(\frac{x_1}{\|x\|})}{x_1 M'(\frac{x_1}{\|x\|}) + x_2 M'(\frac{x_2}{\|x\|}) + x_3 M(\frac{x_3}{\|x\|}).}$$

Also,

$$\|x\|_{x_1}'' = \frac{(\|x\| - \|x\|_{x_1})^2 M''(\frac{x_1}{\|x\|}) + x_2 (\|x\|_{x_1})^2 M''(\frac{x_2}{\|x\|}) + x_3 (\|x\|_{x_1})^2 M''(\frac{x_3}{\|x\|})}{\|x\|^2 (x_1 M'(\frac{x_1}{\|x\|}) + x_2 M'(\frac{x_2}{\|x\|}) + x_3 M'(\frac{x_3}{\|x\|))}.$$

The condition (i) of Theorem 1 follows from the fact that $M'(0) = M''(0) = 0$.

Let us show that the norm satisfies the condition (ii) of Theorem 1. Denote by $c = \min\{x_2 M'(x_2/2) + x_3 M'(x_3/2) : \|x_2 e_2 + x_3 e_3\| = 1, x_2, x_3 \geq 0\}$. Since $M'$ is a continuous function and $M'(t) > 0$ for $t > 0$, we have $c > 0$. Let $d = \max_{t \in [0,1]} M''(t)$.

Clearly, $x_i \leq \|x\|$, $i = 1, 2, 3$. Therefore, using also Remark (ii) and positivity of $\|x\|_{x_1}'$, we have $0 \leq \|x\| - \|x\|_{x_1} \leq \|x\|$, and $(\|x\| - \|x\|_{x_1})^2 / \|x\|^2 \leq 1$.

Consider any $x_2, x_3 \geq 0$ with $\|x_2 e_2 + x_3 e_3\| = 1$. Then $x_2, x_3 \leq 1$. If $x_1 \in [0, 1]$ then $1 \leq \|x\| \leq 2$, hence, $x_i / \|x\| \geq (x_i/2)$, $i = 1, 2, 3$. We get from (14) that $\|x\|_{x_1}'' \leq 3d/c$.

If $x_1 > 1$ then $x_1 / \|x\| > 1/2$, and (14) implies $\|x\|_{x_1}'' \leq 3d / M'(1/2)$, because $M'$ is an increasing function.

Since in both cases $x_1 \in [0, 1]$ and $x_1 > 1$ we estimated the second derivative by constants which do not depend on the choice of $x_2, x_3$ with $\|x_2 e_2 + x_3 e_3\| = 1$, we get the condition (ii) of Theorem 1.

Finally, let us show the condition (iii) of Theorem 1. Let $c$ be as defined above. Denote by $M'$ a continuous increasing function whose value at zero is zero, for every $\epsilon > 0$ there exists $\delta > 0$ so that $M'(t) < c\epsilon/2$ if $x_1 < \delta$. Then, by (13), for every $x_2, x_3$ with $\|x_2 e_2 + x_3 e_3\| = 1$, $\|x\|_{x_1}'(x_1, x_2, x_3) \leq \epsilon$ if $x_1 < \min(1, \delta)$, which proves that the first derivative converges to zero uniformly. Similarly, we use (14), the uniform convergence of the first derivative and the fact that $M''$ is continuous and $M''(0) = 0$ to prove that the second derivative of the norm also converges to zero uniformly. □
Clearly, the class of Orlicz functions satisfying the conditions of Theorem 2 includes all the functions $M(t) = |t|^q$, $q > 2$. Also, if $M$ satisfies the conditions of Theorem 2, the spaces $\ell^n_M$ with $n \geq 3$ (as well as the infinite dimensional spaces $\ell_M, L_M([0,1]))$ do not embed isometrically in $L_p$ with $0 < p \leq 2$. Therefore, for every $0 < p \leq 2$ and the norm of each of those spaces, the function $\exp(-\|x\|^p)$ is not positive definite. The latter result generalizes the solution to the Schoenberg’s problem.

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