Geometric probabilities for a cluster of needles and a lattice of rectangles

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Abstract
A cluster of $n$ needles ($1 \leq n < \infty$) is dropped at random onto a plane lattice of rectangles. Each needle is fixed at one end in the cluster centre and can rotate independently about this centre.

The distribution of the relative number of needles intersecting the lattice is shown to converge uniformly to the limit distribution as $n \to \infty$.

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1 Introduction

We consider a cluster $\mathcal{Z}_n$ of $n$ needles thrown at random onto a plane lattice $\mathcal{R}_{a,b}$ of rectangles (see Fig. 1). The fundamental cell of $\mathcal{R}_{a,b}$ is a rectangle with side lengths $a$ and $b$. All needles of $\mathcal{Z}_n$ are connected in the centre of $\mathcal{Z}_n$. Each needle has equal length 1.

![Fig. 1: Lattice $\mathcal{R}_{a,b}$ and cluster $\mathcal{Z}_n$](image)

We assume $\min(a,b) \geq 2$ so that the cluster $\mathcal{Z}_n$ can intersect at most one of the vertical lines of $\mathcal{R}_{a,b}$ and (at the same time) one of the horizontal lines of $\mathcal{R}_{a,b}$ (except sets with measure zero). A random throw of $\mathcal{Z}_n$ onto $\mathcal{R}_{a,b}$ is defined as follows: After throwing $\mathcal{Z}_n$ onto $\mathcal{R}_{a,b}$ the coordinates $x$ and $y$ of the centre point are random variables uniformly distributed in $[0,a]$ and $[0,b]$ resp.; the angle $\phi_i$ between the $x$-axis and the needle $i$ is for $i \in \{1, \ldots, n\}$ a random variable uniformly distributed in $[0,2\pi]$. All $n+2$ random variables are stochastically independent. In the following $\lambda := 1/a$ and $\mu := 1/b$ with $0 \leq \lambda, \mu \leq 1/2$ will be used.
2 Intersection probabilities

The intersection probabilities for this problem are derived in [1]. In this section the results are summarised, that are necessary for the following investigations.

\( p_n(i), i \in \{0, \ldots, 2n\} \), denotes the probability of exactly \( i \) intersections between \( Z_n \) and \( R_{a,b} \).

Due to existing symmetries it is sufficient to consider only the subset \( F \) of the fundamental cell (Fig. 1). For the calculations it is necessary to consider \( F \) as union of five subsets \( F_1, \ldots, F_5 \) (Fig. 2):

\[
\begin{align*}
F_1 &= \{(x,y) \in \mathbb{R}^2 | 1 \leq x \leq a/2, 1 \leq y \leq b/2\}, \\
F_2 &= \{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq 1, 1 \leq y \leq b/2\}, \\
F_3 &= \{(x,y) \in \mathbb{R}^2 | 1 \leq x \leq a/2, 0 \leq y \leq 1\}, \\
F_4 &= \{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq 1, \sqrt{1-x^2} \leq y \leq 1\}, \\
F_5 &= \{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}.
\end{align*}
\]

Fig. 2: \( F = F_1 \cup \ldots \cup F_5 \)

With \( p_n(i \mid (x,y)) \) we denote the conditional probability, that \( Z_n \) with centre point \((x,y)\) \( F \) has exactly \( i \) intersections with \( R_{a,b} \). Considering the case distinctions for the subsets \( F_m \) the probabilities are calculated with

\[
p_n(i) = \int \int_{F} p_n(i \mid (x,y)) f_1(x) f_2(y) \, dx \, dy = \sum_{m=1}^{5} \int \int_{F_m} p_n(i \mid (x,y)) f_1(x) f_2(y) \, dx \, dy,
\]

where

\[
f_1(x) = \begin{cases} 2/a & \text{for } 0 \leq x \leq a/2, \\ 0 & \text{else} \end{cases}, \quad \text{and} \quad f_2(y) = \begin{cases} 2/b & \text{for } 0 \leq y \leq b/2, \\ 0 & \text{else} \end{cases}
\]

are the density functions of \( x \) and \( y \). We get

\[
p_n(i) = \frac{4}{ab} \sum_{m=1}^{5} \int \int_{F_m} p_n(i \mid (x,y)) \, dx \, dy = 4\lambda\mu \sum_{m=1}^{5} \int \int_{F_m} p_n(i \mid (x,y)) \, dx \, dy. \tag{1}
\]

The conditional intersection probabilities for centre point \((x,y)\) \( F_1 \) are given by

\[
p_n(0 \mid (x,y)) = 1, \quad p_n(1 \mid (x,y)) = 0, \quad p_n(2 \mid (x,y)) = 0.
\]
For \((x, y) \in \mathcal{F}_m, m \in \{2, 3, 4\}\), we have
\[
p_n(i \mid (x, y)) = \binom{n}{i} q_1(x, y)^i (1 - q_1(x, y))^{n-i}, \quad i \in \{0, 1, \ldots, n\},
\]
\[
p_n(i \mid (x, y)) = 0, \quad i \in \{n + 1, \ldots, 2n\},
\]
with
\[
q_1(x, y) = \begin{cases} 
\frac{1}{\pi} \arccos x, & \text{if } (x, y) \in \mathcal{F}_2, \\
\frac{1}{\pi} \arccos y, & \text{if } (x, y) \in \mathcal{F}_3, \\
\frac{1}{\pi} (\arccos x + \arccos y), & \text{if } (x, y) \in \mathcal{F}_4.
\end{cases}
\]
For \((x, y) \in \mathcal{F}_5\), we have
\[
p_n(i \mid (x, y)) = \sum_{j=0}^{\lfloor i/2 \rfloor} \binom{n}{i-j} \binom{n-i-j}{j} q_2^n q_1^{i-j} (1 - q_1 - q_2)^{n-i-j}, \quad i \in \{0, 1, \ldots, 2n\},
\]
where \(q_1 = q_1(x, y) = 1/2, q_2 = q_2(x, y) = \frac{1}{2\pi}(\arccos y - \arcsin x)\), and \([i/2]\) denotes the integer part of \(i/2\).

3 Distribution functions

In the following let \(X_n\) denote the ratio
\[
\frac{\text{number of intersections between } Z_n \text{ and } R_{a,b}}{n}.
\]
We shall investigate the asymptotic behaviour of the distribution functions
\[
F_n(x) = P(X_n \leq x) = \begin{cases} 
0 & \text{for } -\infty < x < 0, \\
\sum_{i=0}^{\lfloor nx \rfloor} p_n(i) & \text{for } 0 \leq x < 2, \\
1 & \text{for } 2 \leq x < \infty,
\end{cases}
\]
as \(n \to \infty\), where \(p_n(i)\) is defined by (1).

**Theorem.** As \(n \to \infty\), the random variables \(X_n\) converge weakly to the random variable \(X\), whose distribution function is given by
\[
F(x) = \begin{cases} 
0 & \text{for } -\infty < x < 0, \\
1 - 2(\lambda + \mu) \cos \pi x + 2(2 \cos \pi x - \pi x \sin \pi x) \lambda \mu & \text{for } 0 \leq x < \frac{1}{2}, \\
1 + 2\pi(x - 1) \lambda \mu \sin \pi x & \text{for } \frac{1}{2} \leq x < 1, \\
1 & \text{for } 1 \leq x < \infty.
\end{cases}
\]
Moreover, it holds the uniform convergence \(\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| = 0\).

**Proof.** The proof of the weak convergence is based on the method of moments. According to the Fréchet-Shohat theorem (see e.g. [3, pp. 81/82]) we have to show that for each \(k \in \mathbb{N}\) the sequence of moments \(E(X_n^k) = \int_{-\infty}^{\infty} x^k dF_n(x)\) converges to \(E(X^k) = \int_{-\infty}^{\infty} x^k dF(x)\) as \(n \to \infty\) and the moments \(E(X^k), k \in \mathbb{N}\), uniquely determine \(F\).
Since $F$ is a distribution function that is constant outside the interval $[0,1]$, it is uniquely determined by its moments. These moments are given by

$$E(X^k) = [2\pi(\lambda + \mu) - 6\pi\lambda\mu] \int_0^{1/2} x^k \sin \pi x \, dx - 2\pi^2\lambda\mu \int_0^{1/2} x^{k+1} \cos \pi x \, dx$$

$$+ 2\pi\lambda\mu \int_{1/2}^1 x^k [\sin \pi x - \pi(1 - x) \cos \pi x] \, dx, \ k \in \mathbb{N}.$$  \hspace{1cm} (2)

For the moments $E(X_n^k), k \in \mathbb{N},$ we find

$$E(X_n^k) = \int_{-\infty}^\infty x^k \, dF_n(x) = \sum_{i=0}^{2n} \left( \frac{i}{n} \right)^k p_n(i)$$

$$= \sum_{i=0}^{2n} \left( \frac{i}{n} \right)^k 4\lambda\mu \sum_{m=1}^5 \int_{\mathcal{F}_m} p_n(i \mid (x, y)) \, dx \, dy$$

$$= 4\lambda\mu \sum_{i=0}^{2n} \int_{\mathcal{F}_m} E(X_n^k \mid (x, y)) \, dx \, dy,$$

where $E(X_n^k \mid (x, y))$ is the conditional $k$-th moment of $X_n$ given the cluster centre in $(x, y)$. Now let us consider the subsets $\mathcal{F}_1, \ldots, \mathcal{F}_5$:

For centre point $(x, y) \in \mathcal{F}_1$ and any $k \in \mathbb{N}$ we have $E(X^k \mid (x, y)) = 0$ and therefore

$$\lim_{n \to \infty} \int_{\mathcal{F}_1} E(X_n^k \mid (x, y)) \, dx \, dy = 0.$$  

For centre point $(x, y) \in \mathcal{F}_m, m \in \{2,3,4\}$, and $i \in \{n+1, \ldots, 2n\}$ all conditional probabilities $p_n(i \mid (x, y)) = 0$. Hence we have

$$E(X_n^k \mid (x, y)) = \sum_{i=0}^n \left( \frac{i}{n} \right)^k p_n(i \mid (x, y)) = \sum_{i=0}^n \left( \frac{i}{n} \right)^k \left( \frac{n}{i} \right) q_1(x, y)^i (1 - q_1(x, y))^{n-i}.$$  

$E(X_n^k \mid (x, y))$ is the Bernstein polynomial of the function $x^k$. In the interval $0 \leq q_1(x, y) \leq 1$ it converges uniformly to $q_1(x, y)^k$ as $n \to \infty$ (see e.g. [2, p. 222]). It follows that $E(X_n^k \mid (x, y))$ converges uniformly to $q_1(x, y)^k$ in $\mathcal{F}_m, m \in \{2,3,4\},$ that is

$$\lim_{n \to \infty} \sup_{(x, y) \in \mathcal{F}_m} \left| E(X_n^k \mid (x, y)) - q_1(x, y)^k \right| = 0, \ k \in \mathbb{N}.$$  

Owing to the uniform convergence we can exchange limit and integral and get

$$\lim_{n \to \infty} \int_{\mathcal{F}_m} E(X_n^k \mid (x, y)) \, dx \, dy = \int_{\mathcal{F}_m} \lim_{n \to \infty} E(X_n^k \mid (x, y)) \, dx \, dy$$

$$= \int_{\mathcal{F}_m} q_1(x, y)^k \, dx \, dy, \ m \in \{2,3,4\}.$$  

For $(x, y) \in \mathcal{F}_2$ we have $q_1(x, y) = \frac{1}{\pi} \arccos x$, hence

$$\lim_{n \to \infty} \int_{\mathcal{F}_2} E(X_n^k \mid (x, y)) \, dx \, dy = \int_{y=1}^{\arccos(x)/\pi} \int_{x=0}^{1/2} u^k \sin \pi x \, du \, dx$$

$$= (b/2 - 1)\pi \int_0^{1/2} u^k \sin \pi u \, du = \frac{1 - 2\mu}{2\mu} \pi \int_0^{1/2} u^k \sin \pi u \, du.$$  

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For \((x, y) \in \mathcal{F}_3\) we have \(q_1(x, y) = \frac{1}{\pi} \arccos y\), hence
\[
\lim_{n \to \infty} \iint_{\mathcal{F}_3} E(X_n^k | (x, y)) \, dx \, dy = \int_{x=1}^{a/2} \int_{y=0}^{1} \left( \frac{\arccos y}{\pi} \right)^k \, dy \, dx = (a/2 - 1)\pi \int_0^{1/2} u^k \sin \pi u \, du = \frac{1 - 2\lambda}{2\lambda} \pi \int_0^{1/2} u^k \sin \pi u \, du.
\]

For \((x, y) \in \mathcal{F}_4\) we have \(q_1(x, y) = \frac{1}{\pi} (\arccos x + \arccos y)\) and therefore
\[
\lim_{n \to \infty} \iint_{\mathcal{F}_4} E(X_n^k | (x, y)) \, dx \, dy = \int_{y=0}^{1} \int_{x=\sqrt{1-y^2}}^{1} \left( \frac{\arccos x + \arccos y}{\pi} \right)^k \, dx \, dy.
\]

For centre point \((x, y) \in \mathcal{F}_5\) we have
\[
E(X_n^k | (x, y)) = \sum_{i=0}^{2n} \binom{i}{n}^k p_n(i \mid (x, y))
\]
with
\[
p_n(i \mid (x, y)) = \sum_{j=0}^{[i/2]} \binom{n}{i - j} \binom{i - j}{j} q_2^{i-j} q_1^{(1-q_1-q_2)^{n-i+j}},
\]
where \(q_1 = q_1(x, y) = 1/2\) and \(q_2 = q_2(x, y) = \frac{1}{2\pi} (\arccos y - \arcsin x)\). Using the lemma in [2, p. 219] we show that \(E(X_n^k | (x, y)) \to (q_1(x,y) + 2q_2(x,y))^k\) uniformly as \(n \to \infty\). At first we may write the expectation (3) as
\[
E(X_n^k | (x, y)) = \int_{t=0}^{2} t^k \, dF_n(t \mid (x, y)),
\]
where \(F_n(t \mid (x, y))\) is the conditional distribution of the random variable \(X_n\) for fixed cluster centre \((x, y)\).

By \(Z_i, i \in \{1, \ldots, n\}\), we denote the random number of intersections between needle \(i\) and \(\mathcal{R}_{a,b}\) given the cluster centre in \((x, y)\) and by \(M_n\) the arithmetic mean \((Z_1 + \ldots + Z_n)/n\). We have \(E(Z_i) = q_1 + 2q_2\), \(E(Z_i^2) = q_1 + 4q_2\) and therefore \(\text{Var}(Z_i) = E(Z_i^2) - [E(Z_i)]^2 = q_1 + 4q_2 - (q_1 + 2q_2)^2\). Furthermore we find
\[
E(M_n) = E(Z_1/n) + \ldots + E(Z_n/n) = E(Z_1) = q_1 + 2q_2.
\]

Since the random variables \(Z_1, \ldots, Z_n\) are independent and identically distributed we have
\[
\text{Var}(M_n) = \text{Var}(Z_1/n) + \ldots + \text{Var}(Z_n/n) = \frac{1}{n} \text{Var}(Z_1) = \frac{q_1 + 4q_2 - (q_1 + 2q_2)^2}{n}.
\]
We put \(D := \{(q_1, q_2) \in \mathbb{R} \mid 0 \leq q_1 \leq 1, 0 \leq q_2 \leq 1 - q_1\}\). The function \(g : D \to \mathbb{R}, g(q_1, q_2) := q_1 + 4q_2 - (q_1 + 2q_2)^2\) has its maximum in the point \((1/2, 0)\) with \(g(1/2, 0) = 1\). Hence \(\text{Var}(M_n) \leq 1/n\) and therefore \(\text{Var}(M_n) \to 0\) as \(n \to \infty\). From [2, p. 219] it follows that (4) converges uniformly to \((q_1 + 2q_2)^k\) as \(n \to \infty\).

Now we get
\[
\lim_{n \to \infty} \iint_{\mathcal{F}_n} E(X_n^k | (x, y)) \, dx \, dy = \iint_{\mathcal{F}_n} \lim_{n \to \infty} E(X_n^k | (x, y)) \, dx \, dy
\]
\[
= \iint_{\mathcal{F}_n} [q_1(x,y) + 2q_2(x,y)]^k \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \left( \frac{\arccos x + \arccos y}{\pi} \right)^k \, dx \, dy.
\]
The calculation of the inner integrals yields
\[ \lim_{n \to \infty} \int_{F_4 \cup F_5} \mathbb{E}(X^k_n \mid (x, y)) \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{1} \left( \frac{\arccos x + \arccos y}{\pi} \right)^k \, dx \, dy. \]

We simplify this integral, that we denote by \( I_{45} \). With the substitutions \( \arccos x = \pi u \) and \( \arccos y = \pi v \) (\( dx = -\pi \sin \pi u \, du \) and \( dy = -\pi \sin \pi v \, dv \)) it follows, that
\[ I_{45} = \int_{0}^{1/2} \int_{0}^{1/2} (u + v)^k \sin \pi u \sin \pi v \, du \, dv. \]

With \( z := u + v \) and considering \( z \) as a constant we get \( dz = du \) and
\[ I_{45} = \int_{v=0}^{1/2} \int_{z=v}^{v+1/2} z^k \sin \pi (z - v) \sin \pi v \, dz \, dv. \]

Changing the order of integrations gives
\[ I_{45} = \int_{z=0}^{1/2} z^k \int_{v=0}^{z} \sin \pi (z - v) \sin \pi v \, dz \, dv + \int_{z=1/2}^{1} z^k \int_{v=z-1/2}^{1} \sin \pi (z - v) \sin \pi v \, dz \, dv. \]

The calculation of the inner integrals yields
\[ I_{45} = \frac{\pi}{2} \int_{0}^{1/2} z^k \left[ \sin \pi z - \pi \cos \pi z \right] dz + \frac{\pi}{2} \int_{1/2}^{1} z^k \left[ \sin \pi z - \pi (1 - z) \cos \pi z \right] dz. \]

As summary of the preceding results we get
\[
\lim_{n \to \infty} \mathbb{E}(X^k_n) = 4\lambda \mu \left( 1 - \frac{2\mu}{2\mu} \frac{\pi}{\pi} \int_{0}^{1/2} x^k \sin \pi x \, dx + \frac{1 - 2\lambda}{2\lambda} \int_{0}^{1/2} x^k \sin \pi x \, dx \right)
+ \frac{\pi}{2} \int_{0}^{1/2} x^k \left[ \sin \pi x - \pi x \cos \pi x \right] dx
+ \frac{\pi}{2} \int_{1/2}^{1} x^k \left[ \sin \pi x - \pi (1 - x) \cos \pi x \right] dx
= [2\pi(\lambda + \mu) - 6\pi \lambda \mu] \int_{0}^{1/2} x^k \sin \pi x \, dx - 2\pi^2 \lambda \mu \int_{0}^{1/2} x^{k+1} \cos \pi x \, dx
+ 2\pi \lambda \mu \int_{1/2}^{1} x^k \left[ \sin \pi x - \pi (1 - x) \cos \pi x \right] dx. \tag{5}
\]

The comparison of (5) with (2) shows, that \( \lim_{n \to \infty} \mathbb{E}(X^k_n) = \mathbb{E}(X^k) \) for \( k \in \mathbb{N} \). It follows that \( F_n \) converges weakly to \( F \) as \( n \to \infty \).

From the weak convergence it follows that \( F_n \) converges uniformly to \( F \) in all points of continuity of \( F \). \( F \) is a continuous function, if \( \lambda = 1/2 \) and \( \mu = 1/2 \). If \( \lambda \neq 1/2 \) or \( \mu \neq 1/2 \), \( F \) is continuous except in the point 0. For this case we consider the convergence of \( F_n(0) \) as \( n \to \infty \). The probability that \( Z_n \) does not intersect \( R_{a,b} \) is given by
\[ p_n(0) = 4\lambda \mu \int_F p_n(0 \mid (x, y)) \, dx \, dy = 4\lambda \mu \int_F q_0(x, y)^n \, dx \, dy, \]
where \( q_0(x, y) \) denotes the probability that a single needle with one end point in the cluster centre \( (x, y) \) has no intersections with \( R_{a,b} \). For almost every \( (x, y) \in F \setminus F_1 \) we have \( q_0(x, y) < 1 \) and
therefore \( q_0(x, y)^n \to 0 \) as \( n \to \infty \). For every \((x, y) \in \mathcal{F}_1\) we have \( q_0(x, y) = 1 \). With Lebesgue’s dominated convergence theorem we find

\[
\lim_{n \to \infty} p_n(0) = 4\lambda \mu \lim_{n \to \infty} \iint_{\mathcal{F}} q_0(x, y)^n \, dx \, dy = 4\lambda \mu \iint_{\mathcal{F}_1} \, dx \, dy = (1 - 2\lambda)(1 - 2\mu).
\]

It follows that \( F_n(0) \to F(0) \) as \( n \to \infty \). Hence the convergence \( F_n \to F \) is completely uniform. So the proof is complete.

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