Quasi Two-dimensional Transfer of Elastic Waves

Nicolas P. Trégourès and Bart A. van Tiggelen
Laboratoire de Physique et Modélisation des Milieux Condensés
CNRS/Université Joseph Fourier, B.P. 166, 38042 Grenoble Cedex 09, France
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Abstract

A theory for multiple scattering of elastic waves is presented in a random medium bounded by two ideal free surfaces, whose horizontal size is infinite and whose transverse size is smaller than the mean free path of the waves. This geometry is relevant for seismic wave propagation in the Earth crust. We derive a time-dependent, quasi-2D radiative transfer equation, that describes the coupling of the eigenmodes of the layer (surface Rayleigh waves, SH waves, and Lamb waves). Expressions are found that relate the small-scale fluctuations to the life time of the modes and to their coupling rates. We discuss a diffusion approximation that simplifies the mathematics of this model significantly, and which should apply at large lapse times. Finally, coherent backscattering is studied within the quasi-2D radiative transfer equation for different source and detection configurations.

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Multiple scattering studies of elastic waves have become relevant to get to a deeper understanding of the seismic Coda $Q$ \cite{1,2} and the principle of equipartition \cite{3,4}. Multiple scattering is believed to occur in the spectral regime of 1-15 Hz. Waves with larger frequencies suffer from absorption. Waves with smaller frequencies don’t see the disorder any more.

The wavelength of a transverse elastic wave at a frequency of 2 Hz is 1.7 km. This brings us to a basic problem in seismic studies of multiple scattering of elastic waves: all measurements take place at the free surface. As a result, they suffer from coherent reflections and mode-conversions. To be of any relevance to seismology, a multiple scattering theory should be capable of describing the boundaries in a very precise way, preferentially on the level of the wave equation. The standard equation of radiative transfer \cite{5} does not have this property. All phase information has been lost, and the accuracy of its spatial resolution is estimated to be some small fraction of the mean free path, estimated to be equal to 20 km at least for seismic waves which is usually still much bigger than the wavelength. The equation of radiative transfer has been studied in ultrasonics for elastic waves at much higher frequencies \cite{6–9}, for which the complication of near-field detection is much less a problem.

In this paper we present a multiple-scattering model that has been adapted to the needs of seismology. It incorporates the complex polarization properties of elastic waves as well as the mode conversions at the free surface. At the same time, we can formulate an almost conventional radiative transfer equation describing mode conversions induced by scattering. Many other contemporary phenomena, such as equipartition and coherent backscattering, can be studied for measurements taking place near the free surface.

The set-up of this paper is as follows. In section 2 we look at the wave equation for elastic waves, and we will define the Green’s function for elastic wave propagation. In section 3 we introduce small-scale fluctuations and define the ensemble-averaged Green’s function. This provides us with the extinction times of all elastic modes. They will serve to define our quasi-2D approximation. In section 4, the transport equation is derived, which describes the time-evolution of the ensemble-averaged energy-contents of all individual modes, and whose stationary solution exhibits equipartition of energy between all modes. In section 5, we discuss the application of the well-known diffusion approximation to this quasi-2D model, introducing a $N \times N$ diffusion tensor for $N$ modes. Finally, in section 6 we investigate coherent backscattering using our Quasi-2D approximation for different source and detection configurations. Section 7 is devoted to conclusions and perspectives.
II. TIME-EVOLUTION OF ELASTIC WAVES

In this section we will formulate the mathematics of elastic wave propagation in a way that is suited to apply conventional methods in wave transport. Many elements have already been discussed very thoroughly by Papanicolaou et al. [3], and some will be recalled here for convenience. We start out with Newton’s second law for the elastic displacement $u$ at time $t$ and position $r$,

$$\rho(r)\partial_t^2 u_i = \partial_j \sigma_{ij}(r) + f_i(r, t). \quad (1)$$

Here, $\rho(r)$ is the local mass density, $f(r, t)$ is an external force per unit volume, and $\sigma_{ij}(r)$ is the local traction which, by Hooke’s law, is given by [10–12]

$$\sigma_{ij}(r) = C_{ijkl}(r) \varepsilon_{kl}(r) = \lambda(r)\varepsilon_{kk}\delta_{ij} + 2\mu(r)\varepsilon_{ij}(r), \quad (2)$$

with $\varepsilon_{kl} = \frac{1}{2}(\partial_k u_l + \partial_l u_k)$ the stress tensor. As always, summation over repeated indices is assumed implicitly. The second equality applies to an isotropic elastic medium, in which case the four-rank tensor $C_{ijkl}$ can only have two independent contributions, proportional to the Lamé moduli $\lambda$ and $\mu$. Following Papanicolaou et al. [3] we shall introduce the vector field,

$$\Psi(r, t) = \left( \begin{array}{c} \sqrt{\frac{\lambda}{2}} \vec{p} \cdot \vec{u} \\ \sqrt{\frac{\mu}{2}} i\partial_t u_i \\ -i\sqrt{\mu} \varepsilon_{ij} \end{array} \right). \quad (3)$$

This vector has 9 independent components since $\varepsilon_{ij}$ is a symmetric tensor, whose trace is given by the first component. We have introduced the operator $\vec{p} = -i\nabla$. It can readily be checked that Eqs. (1) and (2) combine to the following time-evolution problem,

$$i\partial_t |\Psi(r, t)\rangle = K(r, p) \cdot |\Psi(t)\rangle + |\Psi_f(t)\rangle, \quad (4)$$

with the time-evolution operator,

$$K = \left( \begin{array}{ccc} 0 & \frac{\sqrt{\lambda}}{\sqrt{\rho}} - \frac{\varepsilon}{\sqrt{\rho}} & 0 \\ \frac{\sqrt{\mu}}{\sqrt{\rho}} \vec{p} \downarrow \sqrt{\lambda} & 0 & \frac{\varepsilon}{\sqrt{\rho}} \\ 0 & \sqrt{2\mu} \vec{L}(p) & 0 \end{array} \right) \quad (5)$$

and the external force-term $\Psi_f(r, t) \equiv (0, -\vec{f}(r, t)/\sqrt{\rho(r, 0)}$. We have introduced the third rank tensor $L_{ijk}(p) \equiv \frac{1}{2}(p_i\delta_{jk} + p_j\delta_{ik})$ and used the formal Dirac notation for vector fields
to facilitate later the more convenient mode base. The number of arrows determines the
order of the tensor. For clarity, we have put horizontal arrows when they contract in a right-
hand side product with a vector. If $\lambda$ and $\mu$ are real-valued, the matrix $K$ is manifestly
symmetrical with respect to the ordinary Cartesian scalar product. As a result,

$$\langle \Psi(t) | \Psi(t) \rangle \equiv \int dr \Psi^*(r, t) \cdot \Psi(r, t) = \int dr \frac{1}{2} \lambda(r)(\nabla \cdot u)^2 + \frac{1}{2} \rho(r)(\partial_t u)^2 + \mu(r) \text{Tr} \, \varepsilon^* \cdot \varepsilon,$$

recognized as the total elastic energy [12], is conserved in time if no external forces are
present. It is customary to split off a term $\mu (\text{curl} \, u)^2$ (describing pure shear wave energy)
and $2\mu (\text{div} \, u)^2$ (contributing to compressional energy) from the last term, leaving a rest term
$I$. This identifies four terms as “kinetic energy”, “compressional energy”, “shear energy”
and an interference term [12]. The latter vanishes for plane waves with either pure transverse
or pure longitudinal polarization.

Equation (6) can easily be Laplace-transformed ($\text{Im} \, z > 0$). This yields the solution

$$|\Psi(z)\rangle = [z - K]^{-1} [i |\Psi(t = 0)\rangle + |\Psi_f(z)\rangle].$$

(7)

The operator $[z - K]^{-1} \equiv G(z)$ will be called the Green’s function, and is introduced here
for future need. It is convenient to define $t = 0$ just before the source sets in so that
$\Psi(t = 0) = 0$ and the force field becomes the source for wave propagation.

III. EXTINCTION OF ELASTIC WAVES IN A LAYER

We consider a homogeneous elastic plate with mass density $\rho$ of infinite horizontal di-

dmension and of thickness $H$. Both sides of the plate will here be assumed to be free surfaces,
with traction-free boundary conditions. The Lamé coefficients are $\lambda$ and $\mu$, in terms of which
the $P$-wave speed is $\alpha \equiv \sqrt{(2\mu + \lambda)/\rho}$ and the $S$-wave speed is $\beta \equiv \sqrt{\mu/\rho}$. The eigenmodes
have been discussed in great detail by Weaver [13,14]. Their representation (3) can be ob-
tained straightforwardly and we shall denote them by $\{\Psi_n\}$. The index $n$ is a discrete index
that labels, at constant frequency, the symmetric and anti-symmetric Lamb and SH waves
in the plate, including the two Rayleigh surface waves (Fig. 1).

The structure of the symmetric and antisymmetric Lamb modes is extremely rich whereas
the $SH$ modes are “simple” guided waves. For the sake of clarity let us focus on the
antisymmetric branches, indicated with normal lines in Figure 1. The first antisymmetric
mode (first black dot on the right in Figure 1) is an antisymmetric Rayleigh surface mode. Its
displacement is evanescent for both the compressional and the shear component. Rayleigh
modes propagate somewhat slower than bulk $S$ or $P$ waves. As a result they lie on the right of the two dashed lines indicating the pure shear and pure compressional excitations. The second antisymmetric Lamb mode (third black dot) lies between the two dashed lines indicating the pure shear and pure compressional excitations. This mode is evanescent for its compressional component but has a propagating shear displacement. It behaves like a pure shear mode as we go away from either one of the free surfaces. As a matter of fact its potential energy is mostly shear since its compressional potential energy is negligible. Finally the antisymmetric mode most on the left in Figure 1 lies on the left of the lines that indicates pure shear and pure compressional excitation. Even deep in the plate this mode is a mixture of $P$ and $S$ displacements. As we increase the frequency, the organization of Lamb modes stays intact. One finds two surface Rayleigh modes (symmetric and antisymmetric), “evanescent $P$ but bulk $S$” modes and modes that are both bulk $S$ and bulk $P$.

By translational symmetry, the eigenmodes can be chosen proportional to transverse plane waves with wave number $k$. We will treat him initially as $\exp(i k \cdot x)/\sqrt{A}$, with a discrete contribution of $k$ to the label $n$ as a result of the periodic boundary conditions on both sides of a square plate with surface $A$, and eventually take the limit $A \to \infty$.

We will now assume the presence of disorder in the plate, to be specified more precisely later on, and calculate the Green’s function, averaged over this random disorder. The exact meaning of this averaging in seismic observations will be addressed elsewhere. This procedure is the first part to formulate a transport theory [15]. Let the disorder be represented by a perturbation $\delta K$ in the time-evolution operator: $K = K_0 + \delta K$. The ensemble-averaged “retarded” (outgoing) Green’s function at frequency $\omega$ is given by,

$$\langle G(z = \omega + i 0) \rangle = \frac{1}{\omega + i 0 - K} \equiv \frac{1}{\omega + i 0 - K_0 - \Sigma(\omega)}.$$  \hspace{1cm} (8)

This “Dyson” equation defines the mass-operator $\Sigma(\omega)$. The lowest order contribution is given by [16],

$$\Sigma(\omega) = \left\langle \frac{\delta K \cdot 1}{\omega + i 0 - K_0} \cdot \delta K \right\rangle + O(\delta K)^3. \hspace{1cm} (9)$$

Next, we can insert the complete and orthonormal set $\{\Psi_n\}$ of the homogeneous plate, defined above. Standard first-order perturbation theory yields,

$$G(\omega) = \sum_n \frac{\langle \Psi_n | \langle \Psi_n |}{\omega - \omega_n - \Sigma_n(\omega)}.$$ \hspace{1cm} (10)

with

$$\Sigma_n(\omega) = \sum_m \left\langle \frac{|\langle \Psi_n | \delta K | \Psi_m \rangle|^2}{\omega - \omega_m + i 0} \right\rangle \frac{1}{\omega - \omega_m + i 0}.$$ \hspace{1cm} (11)
The imaginary part of this parameter is negative, and is identified with $-1/2\tau_n$, where $\tau_n$ is the extinction time of mode $n$.

In general, both $\rho(\mathbf{r})$, $\lambda(\mathbf{r})$ and $\mu(\mathbf{r})$ are random variables. We will simplify the problem by assuming that $\rho(\mathbf{r})$ is constant, and that velocity fluctuations are due to fluctuations in the two Lamé coefficients: $\lambda(\mathbf{r}) = \lambda_0 + \delta\lambda(\mathbf{r})$ and $\mu(\mathbf{r}) = \mu_0 + \delta\mu(\mathbf{r})$, with $\lambda_0$ and $\mu_0$ the coefficients of the homogeneous layer. In that case,

$$
\delta K = \frac{1}{\sqrt{\rho}}\begin{pmatrix}
0 & (\delta\lambda(\mathbf{r})/2\sqrt{\lambda_0}) \mathbf{p} & \mathbf{0} \\
\mathbf{p} \downarrow (\delta\lambda(\mathbf{r})/2\sqrt{\lambda_0}) & \mathbf{0} \downarrow & \mathbf{0} \\
\mathbf{0} \downarrow \downarrow & (\delta\mu(\mathbf{r})/2\sqrt{2\mu_0}) \mathbf{L}(\mathbf{p}) & \mathbf{0} \downarrow \downarrow
\end{pmatrix}
$$

A straightforward calculation, employing integration by parts, finally leads to,

$$
\langle |\langle \Psi_n | \delta K | \Psi_m \rangle |^2 \rangle = \omega^2 \int d\mathbf{r} \int d\mathbf{r}' \{ \langle \delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}') \rangle (\nabla \cdot \mathbf{u}_n)^*(\nabla \cdot \mathbf{u}_m)(\nabla' \cdot \mathbf{u}_n')(\nabla' \cdot \mathbf{u}_m') \\
+ \langle \delta\mu(\mathbf{r})\delta\mu(\mathbf{r}') \rangle \text{Tr} \varepsilon_n^* \cdot \varepsilon_m \text{Tr} (\varepsilon'_n)^* \cdot \varepsilon'_m \\
+ \langle \delta\lambda(\mathbf{r})\delta\mu(\mathbf{r}') \rangle (\nabla \cdot \mathbf{u}_n)^*(\nabla \cdot \mathbf{u}_m)\text{Tr} (\varepsilon'_n)^* \cdot \varepsilon'_m + \text{c.c.} \}
$$

To evaluate $\Sigma_n(\omega)$ we must specify the spatial correlations between the Lamé coefficients. The simplest choice is to assume correlations that are short range with respect to the wavelength,

$$
\langle \delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}') \rangle = \sigma_{\lambda}^2(z) \delta(\mathbf{r} - \mathbf{r}') \\
\langle \delta\mu(\mathbf{r})\delta\mu(\mathbf{r}') \rangle = \sigma_{\mu}^2(z) \delta(\mathbf{r} - \mathbf{r}') \\
\langle \delta\mu(\mathbf{r})\delta\lambda(\mathbf{r}') \rangle = \sigma_{\mu\lambda}^2(z) \delta(\mathbf{r} - \mathbf{r}')
$$

Without extra difficulty, we can still allow for a depth dependence of the correlation functions. $\Sigma_n$ can now be evaluated for a big plate for which $\sum_m \to \sum_i A \int d^2k/(2\pi)^2$, including a sum over the different branches. All factors $A$ cancel if a transverse plane wave normalization $\exp(i\mathbf{k} \cdot \mathbf{x})$ is adopted. For the extinction time of mode branch $j$ at frequency $\omega$, we find

$$
\frac{1}{\tau_j(\omega)} = \omega^2 \sum_i n_i \int \frac{d^2k_i}{2\pi} W(i\mathbf{k}_i, j\mathbf{k}_j).
$$

with $n_i(\omega) \equiv k_i(\omega)/v_i(\omega)$ in terms of the group velocity $v_i = d\omega_i/dk_i$. The “mode scattering cross-section” is defined as,

$$
W(i\mathbf{k}_i, j\mathbf{k}_j) = \int_0^H dz \left\{ \sigma_{\lambda}^2(z) |\nabla \cdot \mathbf{u}_{ik}||\nabla \cdot \mathbf{u}_{jk}|^2 + \sigma_{\mu}^2(z) |\text{Tr} \varepsilon_{ik}^* \cdot \varepsilon_{jk}|^2 \\
+ 2\sigma_{\mu\lambda}^2(z) \text{Re} (\nabla \cdot \mathbf{u}_{ik}^* \nabla \cdot \mathbf{u}_{jk} \text{Tr} \varepsilon_{ik}^* \cdot \varepsilon_{jk}) \right\}
$$
We have chosen to split the factor $n_i$, so that this matrix is symmetric. According to our model the extinction time $\tau_j$ does not depend on the direction of the horizontal wave number $k_j$.

The *imaginary* part of the ensemble-averaged Green’s function is directly related to the excitations of the waves [17]. The spectral density $\mathcal{N}(\omega)$ per unit surface can be expressed as,

\[
\mathcal{N}(\omega) = -\frac{1}{\pi A} \text{Tr} \text{Im} G(\omega) = \frac{1}{\pi} \sum_i \int \frac{d^2k}{(2\pi)^2} \frac{1/2\tau_{ik}}{[\omega - \omega_{ik}]^2 + 1/4\tau_{ik}^2}.
\]

Due to scattering, all modes are spectrally broadened. The separation in wavenumber of two adjacent modes with the same frequency (see Figure 1) is typically of order $1/H$. The uncertainty in $k$ is typically $1/v_{ik}\tau_{ik}$, with $v_{ik}$ the group velocity of the mode. If

\[
\tau_{ik} > H/v_{ik},
\]

one can assume that different modes at fixed $k$ do not overlap, except at a few degeneration points where the dispersion curves for modes with different symmetry (i.e. SH and Lamb) cross. This assumption is the *Quasi Two-Dimensional Approximation* (Q2DA). Criterion (18) is typically satisfied in the Earth crust, which has $H \approx 30$ km, a typical wave speed $\beta \approx 3.5$ km/s and a mean free time $\tau > 15$ sec. In the Q2DA we find for the spectral density per unit surface $\mathcal{N}(\omega) = (2\pi)^{-1} \sum_i n_i$, showing that $n_i$, defined in Eq. (15), represents the spectral weight per unit surface of mode $i$ at frequency $\omega$ in phase space.

In the following, all time scales will be normalized by the mean free time of $S$ waves in an infinite medium with the same amount of disorder. This time depends only on $\sigma^2_\mu$, which can be related to the correlation length and the shear velocity fluctuations. For a velocity fluctuation of 2% and a correlation length of 700 m, both being typical seismic values, we get a shear mean free path $\tau_\infty^s \approx 15$s. Note that the correlation length is much smaller than the wavelength $\lambda_s = 1700$m, which justifies the short range approximation of Eqs (14c).

Figure 2 shows extinction times for different modes index, calculated from Eqs. (15) and (16), normalized by the mean free time of $S$-waves in an infinite medium. The plate thickness is $H = 20.2\lambda_s$, which has $N = 106$ modes. The disorder is chosen to be uniform in the whole plate, and the spatial correlations among the Lamé coefficients is taken equal: $\sigma^2_\lambda = \sigma^2_\mu = \sigma^2_\lambda \mu$. $SH$ modes show an extinction time very similar to the extinction time of $S$-waves in an infinite medium $\tau_\infty^s$. On the other hand the Lamb modes present a more complex pattern: Rayleigh modes clearly show a shorter extinction time, Lamb modes with an evanescent compressional component behave very much like a bulk $S$ wave. Finally, Lamb
modes with both bulk compressional and bulk shear components behave in a complicated fashion but tend to have an extinction time larger than $S$ waves in an infinite medium.

In the case of dominant $\mu$ correlation, $\sigma_\lambda^2 \ll \sigma_\mu^2$ (dominant shear velocity fluctuations) the Lamb modes with both “bulk” compressional and shear components will have a somewhat relatively larger extinction time. On the other hand, if the $\lambda$ correlation dominates, $\sigma_\lambda^2 \gg \sigma_\mu^2$, (strong compressional velocity fluctuations), the same Lamb modes with ‘bulk’ compressional and shear displacements will have the shortest extinction time.

We would like to point out that the life time of Rayleigh waves is not well described by our model since they suffer most from surface disorder (fluctuations in height), which is not included in Eqs. (14).

IV. TRANSPORT EQUATION IN A LAYER

The next task is the formulation of an elastic transport equation in the Quasi 2D Approximation. Basic observable is the ensemble-averaged intensity Green’s function $\langle G(\omega^-) \times G(\omega^+)^* \rangle$, with $\omega^\pm = \omega \pm \frac{1}{2} \Omega$. It can be expressed in the complete base $\{\Psi_n\}$ of the homogeneous plate, giving rise to the matrix element $L(\omega, \Omega)_{nn'\lambda\lambda'}$ (Figure 2). The Bethe-Salpeter equation [15,17] for this object reads,

$$L_{nn'\lambda\lambda'}(\omega, \Omega) = G_n(\omega^+)G_{n'}(\omega^-)^* \left[ \delta_{nn'}\delta_{\lambda\lambda'} + \sum_{ll'} U_{nn'\lambda\lambda'}(\omega, \Omega) L_{ll'\lambda\lambda'}(\omega, \Omega) \right].$$  (19)

with $G_n$ the Dyson Green’s function defined in Eq. (8), and a new object $U$ called the Irreducible Vertex. Upon introducing $\Delta G_{nn'}(\omega, \Omega) \equiv G_n(\omega^+) - G_{n'}(\omega^-)$ (idem for $\Delta \Sigma$) this equation can be re-arranged into,

$$[\Omega + (\omega_{n'} - \omega_{n}) - \Delta \Sigma_{nn'}] L_{nn'\lambda\lambda'}(\omega, \Omega) = \Delta G_{nn'}(\omega, \Omega) \left[ \delta_{nn'}\delta_{\lambda\lambda'} + \sum_{ll'} U_{nn'\lambda\lambda'}(\omega, \Omega) L_{ll'\lambda\lambda'}(\omega, \Omega) \right].$$  (20)

This equation is still exact. We will now carry through a number of approximations relevant to our problem. For small disorder, the vertex $U$ is given by [18],

$$U_{nn'\lambda\lambda'}(\omega, \Omega) = \langle \langle \Psi_n | \delta K | \Psi_{n'} \rangle \langle \Psi_{n'} | \delta K | \Psi_{\lambda} \rangle \rangle.$$  (21)

For short-range correlations, as specified in Eqs. (14), the vertex $U$ can be straightforwardly related to the cross-section $W(i\mathbf{k}_i, j\mathbf{k}_j)$ defined in Eq. (15). For typical wave packets is $\Omega \ll \omega$ (i.e. a wave packet contains many cycles) so that we neglect $\Omega$ in any functional dependence on frequency (“slowly varying envelope approximation”). The index $n$ consists of
one discrete branch index \( j \), and one index \( k \) that becomes continuous as \( A \to \infty \). The Q2DA neglects all overlaps between different branches, so that \( \Delta G(\omega, \Omega)_{nn'} \to 2\pi i \delta_{jj'} \delta[\omega - \omega_j(k)] \).

If we let \( k - k' = q \), and \( S_m(\omega) \) the source in mode representation, a new observable quantity \( L_{jk} \) can be defined as

\[
\sum_{mm'} \mathcal{L}_{mm'}(\omega, \Omega) S_m S_{m'}^* \equiv 2\pi \delta[\omega - \omega_{jk}] \delta_{jj'} \times L_{jk}(q, \Omega). \tag{22}
\]

In space-time the Q2D transport equation reads,

\[
\left[ \partial_t + v_j \cdot \nabla + \frac{1}{\tau_{jkj}} \right] L_{jk}(x, t) = |S_{jk}(\omega)|^2 \delta(t) \delta(x)
+ \omega^2 \sum_{j'} \int \frac{d^2k_{j'}}{2\pi} n_{j'} W(jk_{j'}, j'k_{j'}) L_{jj'}(x, t). \tag{23}
\]

We will use this equation as a starting point for our calculations. The equation is essentially two-dimensional, with a finite number of modes (of order \( 2H\omega/\beta \)) to take care of the third, vertical dimension. The great advantage of this equation is that the boundary conditions of the elastic waves have been dealt with exactly, i.e. on the level of the wave equation, contrary to conventional transport equations \[3,8,9\]. We see that \( L_{jk}(x, t) \) can be interpreted as the specific intensity of the mode \((jk)\) at frequency \( \omega \), at horizontal position \( x \), at a time \( t \) after the release of energy by the source.

The source term \( S_{jk}(\omega) \) is given by,

\[
S_{jk}(\omega) = \langle \Psi_{jk} | \Psi_f \rangle = \int d^3r f^*(r, \omega) \cdot u_{jk}(r). \tag{24}
\]

Since \( u_{jk} \) is an eigenfunction for which the energy \([1]\) has been normalized, we see that \( |S_{jk}|^2 \) has the dimension of energy. Since \((\Omega, q)\)-dependence has been neglected in the source, it emerges in our transport equation as a \( \delta(t)\delta(r) \) in space-time.

**A. Dynamics of the Equipartition Process**

Equation (23) has one very important property that has been discussed in great detail in the literature. By recalling the expression (15) for the extinction time, it follows immediately that the specific intensity with the property that its total mode energy

\[
\int d^2x L_{jk}(x, t) = \text{constant}, \tag{25}
\]

is independent of the mode-index \( j \) and independent of the horizontal direction of propagation \( k \), is a stationary solution for \( t > 0 \) of the transport equation. All solutions converges
to this solution regardless the nature and position of the source. This implies that finally all modes have an equal share in the total energy contents of the plate. This phenomenon is called *equipartition*[^1][^13][^21], and is believed to be a fundamental feature of the solution of most transport equations at large lapse times, provided absorption is absent, or at least small[^22]. According to our definition[^22], the total spectral energy per unit surface in the regime of equipartition is given by,

$$E_{\omega}(t) = \sum_j \int \frac{d^2k}{(2\pi)^2} \int d^2x \Delta_{jk}(x,t) 2\pi \delta(\omega - \omega_{jk}).$$

$$\rightarrow \text{constant} \times \sum_j n_j.$$  \hfill (26)

We will introduce the spectral energy density $E_i(x,t)$ of mode $i$ per unit surface, and its current density $J_i(x,t)$ according to,

$$E_i(x,t) \equiv \int \frac{d^2k}{(2\pi)^2} 2\pi \delta(\omega - \omega_{ik}) \Delta_{ik}(x,t) = n_i \int \frac{d^2k}{2\pi} \Delta_{ik}(x,t),$$

$$J_i(x,t) \equiv \int \frac{d^2k}{(2\pi)^2} 2\pi \delta(\omega - \omega_{ik}) \mathbf{v_i} \Delta_{ik}(x,t) = n_i \int \frac{d^2k}{2\pi} \mathbf{v_i} \Delta_{ik}(x,t).$$ \hfill (27a)

$$J_i(x,t) \equiv \int \frac{d^2k}{(2\pi)^2} 2\pi \delta(\omega - \omega_{ik}) \mathbf{v_i} \Delta_{ik}(x,t) = n_i \int \frac{d^2k}{2\pi} \mathbf{v_i} \Delta_{ik}(x,t).$$ \hfill (27b)

An exact equation of continuity can be found from Eq. (23) by integrating over $k_i$,

$$\partial_t E_i(x,t) + \nabla \cdot J_i(x,t) = \left[n_i \int \frac{d^2k}{2\pi} |S_{ik}(\omega)|^2 \right] \delta(x) \delta(t) - \sum_j C_{ij} E_j(x,t),$$ \hfill (28)

with the “mode-conversion matrix”,

$$C_{ij} = \frac{\delta_{ij}}{\tau_i} - \omega^2 n_i \int \frac{d^2k}{2\pi} W(i\mathbf{k}_i,j\mathbf{k}_j).$$ \hfill (29)

The mode-conversion matrix $C$ has an eigenvalue 0 with (left-hand) eigenvector $\{n_i\}$, associated with the equipartition. The $N-1$ nonzero eigenvalues, which can be called “Stokes parameters”, of the mode conversion matrix $C$ determine the dynamics of the equipartition process. It depends on the initial conditions, i.e. how the initial release of energy is distributed among the different modes, as described by $S_{ik}(\omega)$.

Figure 3 shows all eigenvalues of the matrix $C$ in the case of a plate of thickness $H = 20.2\lambda_S$, for which the number of modes is $N = 106$. The disorder is uniform in the whole plate and the spatial correlation among all Lamé coefficients is chosen equal: $\sigma^2_\lambda = \sigma^2_\mu = \sigma^2_\mu\lambda$. The time scale has been normalized to the mean free time of $S$-waves in an the infinite medium, which has the same amount of disorder, i.e. as described by Eqs. (14).

The largest eigenvalue (associated with the shortest life-time) has an eigenvector made of the symmetric and antisymmetric Rayleigh modes. This configuration is very sensitive
to the location of the disorder in the plate. If the plate does not have any disorder close to the two free surfaces (at the length scale of a wavelength) the Rayleigh modes, which have a penetration length of the order a wavelength, do not suffer from the disorder. As a consequence, their life-time would become very large compared to the mean free time of $S$-waves in an infinite medium. On the other hand, if the disorder is localized close to the free surface the Rayleigh modes end up with a very large eigenvalue. The eigenvectors associated with the flat plateau in Figure 3 consist of modes whose shear component strongly dominates over the compressional part. As a result their eigenvalues are very similar to the inverse shear mean free time of a $S$-wave in an infinite medium. The eigenvectors associated with the eigenvalues smaller than unity exhibit a strong compressional component. They are associated with longer life times as shown in Figure 2. Quite logically they show up with a smaller eigenvalue (associated to a longer life-time) in the mode-conversion matrix $C$.

In the case of dominant $\mu$ correlation, $\sigma_\lambda^2 \ll \sigma_\mu^2$, the picture does not change drastically since Lamb modes are always dominated by shear. For dominant $\lambda$ correlation, $\sigma_\lambda^2 \gg \sigma_\mu^2$, the structure of eigenvalues of the mode-conversion matrix $C$ is modified considerably. Eigenvalues that were previously associated with “bulk” $P$ and $S$ vectors now see their life-time becoming much shorter. An eigenvector with a small eigenvalue in Figure 3, achieves a large eigenvalue.

Figures 4 show, for different kind of sources, how the initial release of energy is distributed among the different modes. Figure 4a shows an isotropic explosion at a depth $\lambda_s/3$ from the free surface. An explosion is a purely compressional source, and does not excite any $SH$ modes. Among the Lamb modes it excites preferentially the modes that are “bulk” for both compressional and shear components as well as Rayleigh modes. A source at a larger depth will not excite the Rayleigh modes since they have a penetration length of the order of the wavelength.

Figure 4b applies for a double couple in the $xy$ plane at a depth $\lambda_s/3$ from the free surface. Contrary to the isotropic explosion, the double couple in the $xy$ plane strongly excites the $SH$ modes. Since the source is close to the free surface Rayleigh modes are excited as well. The Lamb modes which are “bulk” for the shear component but only evanescent for the compressional component are also excited.

Figures 4c, d show the mode distribution for a double couple in the plane $xz$ for two different depths of the source, $\lambda_s/3$ and $5\lambda_s$. When the source is located close to the free surface the majority of the energy is distributed among the Rayleigh modes. Two Rayleigh modes are out of scale in Figure 4c and carry half of the released energy. Alone they carry half of the total energy released. On the other hand, when the source is situated deep in the
plate the pattern becomes very rich. One can see that the Rayleigh modes are not excited
anymore.

V. DIFFUSION APPROXIMATION

Despite the many simplifications that have been carried out, the final transport equation
(23) is still difficult to solve numerically. In future work, we intend to adapt our Monte-Carlo
simulations, developed to solve the 3D radiative transfer equation [23,24], to this modified
equation. In this section we shall carry out a final and rather familiar simplification, that
facilitates a numerical solution.

The diffusion approximation is typically valid at large lapse times, when currents start
to become small. In that case, the specific intensity of mode $i$ can be written as,

$$L_{ik}(q, \Omega) = \frac{1}{n_i} \left[ E_i(q, \Omega) + \frac{2}{v_i^2} v_i \cdot J_i(q, \Omega) + \cdots \right], \quad (30)$$

with $n_i = k_i/v_i$ the density of mode $i$ in phase space introduced earlier. In real space $q$
transforms into the 2D gradient $\nabla$. Inserting the series (30) into Eq. (23) leads to the
relation

$$J_i(r, t) = -\sum_j D_{ij} \nabla E_j(r, t). \quad (31)$$

This relation is recognized as a generalized Fick’s Law [25], generalized, because it involves
different individual modes at the cost of one dimension. The diffusion matrix is given by,

$$\left(D^{-1}\right)_{ij} = 2 \left( \frac{\delta_{ij}}{v_i^2 r_i} - \frac{\omega^2}{n_j} \int \frac{d^2 \hat{k}}{2\pi} W(i\hat{k}, i\hat{k}) \frac{v_i \cdot v_j}{v_i^2 v_j^2} \right), \quad (32)$$

It is easy to check the following relation,

$$\frac{D_{ij}}{D_{ji}} = \frac{n_i}{n_j}. \quad (33)$$

Combining Eqs. (31) and (28) and transforming back to space-time yields the generalized
2D diffusion equation,

$$\partial_t E_i(r, t) - \sum_j D_{ij}(\omega) \nabla^2 E_j(r, t) = S_i(\omega) \delta(t) - \sum_j C_{ij}(\omega) E_j(r, t). \quad (34)$$

This diffusion equation is an ordinary partial differential equation that can be solved by
conventional means. For an infinite plate no boundary conditions have to be specified: the
boundary conditions at the two free surfaces have been taken care of exactly. For this reason,
the Q2D diffusion approximation is not expected to break down near the boundaries, as was noticed by Turner and Weaver for the conventional diffusion approximation [9].

Equation (34) still captures the time-evolution of the different elastic modes of the plate, and can thus be used to study polarization properties. Integrating equation (34) over the horizontal coordinate $r$ gives for the time evolution of the total modal energy

$$\partial_t E_i(t) = S_i(\omega)\delta(t) - \sum_j C_{ij}(\omega)E_j(t).$$  (35)

In fact this equation follows directly from Eq. (28) without the need to apply the diffusion approximation. Its formal solution is

$$E_i(t) = \sum_j \left[ \exp \left( -Ct \right) \right]_{ij} S_j(\omega)\theta(t > 0).$$

This can easily be evaluated using the complete set of eigenmodes of $C$.

Figure 5a shows the time evolution of the energy among the different modes for an isotropic explosion at a depth $\lambda_s/3$ from the free surface. The initial modal energy distribution was shown in Figure 4a. For the sake of clarity we only display the evolution of three sub-classes of modes (Rayleigh, Lamb, SH) and not the whole distribution. Rayleigh modes are excited but not SH modes since the source is purely compressional. As time goes on, the mode occupation changes as a result of the dynamics of the equipartition process and finally tend to the equipartitioned distribution which does not depend on nature and location of the source.

Figure 5b shows the time evolution of two “observable” energy ratios measured at the free surface: the ratio of shear to compressional potential energy, $E_s/E_p$, and the one of the horizontal to vertical kinetic energy $H^2/V^2$. After a few shear wave mean free times, the energy ratios stabilize to their predicted equipartition value $E_s/E_p = 7.19$, $H^2/V^2 = 1.77$. The ratios $E_s/E_p$ and $H^2/V^2$ increase monotonically which is due to the compressional nature of the source.

Figures 5c, d present the equipartition process for a double couple source deep in the plate ($5\lambda_s$ from the free surface). For such a source the Rayleigh modes are not excited while the other Lamb modes and SH modes are strongly excited (see Figure 4d). The initial ratio of shear to compressional energy at the free surface is higher than the one for the explosion source due to the shear nature of the source. However, in both cases the energy distributions converge towards an equipartitioned distribution which is independent of the nature of the source and its location. Note that, for an exploding source, the equipartition process takes much longer a time, typically $6\tau_s^\infty$. For the double-couple source in Figures 5c, d it is typically $\tau_s^\infty$.

It is not very difficult to show that in the equipartition regime, the generalized diffusion equation (34) further simplifies to a genuine 2D diffusion equation for the total energy
density,
\[ \partial_t E(\mathbf{r}, t) - D(\omega) \nabla^2 E(\mathbf{r}, t) = S(\omega) \delta(\mathbf{r}) \delta(t) , \]  
with diffusion constant,
\[ D(\omega) = \frac{\sum_i D_{ij}(\omega) n_j(\omega)}{\sum_j n_j(\omega)} , \]  
and source,
\[ S(\omega) = \sum_i n_i \int \frac{d^2 \hat{k}}{2\pi} |S_{ik}(\omega)|^2 . \]
Equation (37) is recognized as an equipartitioned sum of all diffusion matrix elements. A similar result was obtained for the diffusion constant in an infinite elastic medium, in terms of the individual matrix elements for \( P \) and \( S \) waves \[7,21,26\]. Equation (36) has the simple solution,
\[ E(\omega, \mathbf{r}, t) = \frac{S(\omega)}{4\pi D(\omega) t} \exp \left( -\frac{\mathbf{r}^2}{4D(\omega) t} \right) , \]
i.e. the local energy basically varies as \( E(\omega) \sim t^{-1} \times S(\omega)/D(\omega) \) at large times.

Table I shows the evolution of the ratio \( D(\omega)/D^\infty(\omega) \) as a function of the number of modes in the plate. \( D^\infty(\omega) \) is the infinite medium elastic diffusion constant, obtained by Weaver \[7\] and Ryzhik \[27\], with the same amount of disorder, \( i.e. \), as described by Eqs. (14). The ratio changes form 0.72 for \( N = 3 \) modes to 0.85 for a thick plate, \( i.e. \) \( H \approx l^* \). Our quasi-2D approximation starts to break down when the thickness of the plate exceeds the mean free path.

VI. COHERENT BACKSCATTERING NEAR THE FREE SURFACE

Coherent backscattering of waves is an interference effect that survives multiple scattering. It refers to a coherent enhancement of intensity near the source \[26\]. The effect has recently been observed with acoustic \[28\] and elastic waves \[29,30\].

We recently investigated coherent backscattering of waves in a more seismic context \[32,33\]. Specific aspects such as symmetry of the source, near field, leakage and the exact measurement process have to be understood before any seismic experiment can be considered. Our analyses so far have been done either with scalar (acoustic) waves in a disordered plate with leakage \[32\] or with fully elastic waves in an infinite medium \[33\]. The last study
established that the enhancement factor of coherent backscattering is highly dependent on both the nature of the source and on the precise parameter that is being measured. More specifically, a measurement of simply $\langle u_i(\omega)^2 \rangle$ of waves released by a “double-couple” source will hardly give rise to a coherent enhancement, so that observation is unlikely. On the other hand, the measurement of $\langle \text{div}\ u(\omega)^2 \rangle$ of waves released by an explosion source maps exactly onto the acoustic problem, which has the maximal enhancement factor of 2.

Both approaches are unable to model the coherent backscattering effect of wave propagation in the crust, whose elastic eigenmodes are not plane waves. In addition, a measurement necessarily takes place at the Earth surface, whereas the source (an earthquake or explosion) can be located at depth. In this section we will investigate coherent backscattering using our Quasi 2D transport model. Recently, De Rosny et al. [29] and Weaver et al. [30] reported the studies of coherent backscattering of elastic waves at frequencies around 1 MHz.

Our analysis will closely follow the one given in Ref. [33]. Starting point is the calculation of the vertex $L_{nn'pp'}(k, k', q)$ defined in Eq. (22) and describing the ensemble-averaged, incoherent scattering of the modes $(i, k + \frac{1}{2}q)$ and $(i', k - \frac{1}{2}q)$ into $(j, k' + \frac{1}{2}q)$ and $(j', k' - \frac{1}{2}q)$. By the reciprocity principle this object must be symmetrical with respect to left and right-hand indices. The diffusion approximation, applied to our Q2DA model yields for large lapse times,

$$L_{ii'jj'}(k, k', q) = \frac{\delta_{ii'}\delta(\omega - \omega_i k)\delta_{jj'}\delta(\omega - \omega_j k')}{-i\Omega + Dq^2 + \omega/Q}.$$  

An inverse Fourier transform with respect to $\Omega$ provides the time-dependence of the envelope of a wave packet with central frequency $\omega$. Similarly, the spatial dependence is obtained by an inverse Fourier transform over $q$, $k$ and $k'$. The result is,

$$L_{ii'jj'}(\omega, t, x_1, x_2 \to x_3, x_4) = \exp(-\omega t/Q) \frac{D}{Dt}\delta_{ii'}\delta_{jj'} n_i n_j J_0(k_i x_{12}) J_0(k_j x_{34}).$$  

The depth (i.e. $z$) dependence can be obtained by summing over the $N$ eigenfunctions $\Psi_i(z)$ at frequency $\omega$. The coherent backscattering is due to constructive interference of time-reversed waves. It can be constructed straightforwardly by interchanging the indices $(i'x_2)$ and $(j'x_4)$.

1Seismic measurements usually have access to time-correlations $\langle \psi_j(t - \frac{1}{2}\tau)\psi^*_j(t + \frac{1}{2}\tau) \rangle$ of different components of the wave function [33]. The smooth time-dependence of the envelope of the pulse at frequency $\omega$ can be obtained by Fourier transforming this object with respect to $\tau$. Signal-to-noise can usually be increased by averaging over a time window $\Delta t$ and using a finite bandwidth $\Delta \omega$ over which the signal is not expected to vary too much.
\[ C_{ij'}(\omega, t, x_1, x_2 \rightarrow x_3, x_4) = \frac{\exp\left(-\frac{\omega t}{Q}\right)}{Dt} \delta_{ij'} n_i n_j J_0(k_i x_1) J_0(k_j x_2). \] (42)

Both \( L \) and \( C \) contribute to \( \langle G(i, x_1 \rightarrow j, x_2) G^*(i', x_3 \rightarrow j', x_4) \rangle \), but \( C \) survives only close to the source, as we shall see. To calculate actual enhancement factors, we must specify source and detector. In Eq. (24) the source was already expressed in the eigenmodes \((j, k)\). Different sources will now be considered.

**A. Monopolar source at depth**

We consider the source \( f(r) \sim f_0(\omega) \delta^{(3)}(r - r_0) \), which represents a highly directional force field at position \( r_0 \), small compared to the wavelength. Equation (24) gives \( S_{jk}(\omega) \sim \omega f_0(\omega) \cdot u_{jk}(z_0) \) with \( z_0 \) the depth of the source. To simplify the analysis we will assume that the force is directed into the \( z \)-direction. This configuration was also studied by De Rosny et al. [29,31] using a thin chaotic 2D silicon cavity, with only 3 excited Lamb waves. In addition, their detection method of heterodyne laser interferometry is only sensitive to the normal displacement \( u_z(z = 0) \). In seismology, the force field above may be a simple model for a volcano eruption.

Let \( x \) be the horizontal distance between source and detector. The measured “incoherent” background is found from Eq. (41),

\[ L(x, t) \sim \frac{\exp\left(-\frac{\omega t}{Q}\right)}{Dt} f_0(\omega)^2 \sum_i n_i |u_{i,z}(0)|^2 \sum_j n_j |u_{j,z}(z_0)|^2, \] (43)

which is independent of \( x \), but still depends on the depth \( z_0 \) of the source. The ”coherent” contribution follows from Eq. (42)

\[ C(x, t) \sim \frac{\exp\left(-\frac{\omega t}{Q}\right)}{Dt} f_0(\omega)^2 \left| \sum_i n_i u_{i,z}(0) u_{i,z}(z_0)^* J_0(k_i x) \right|^2. \] (44)

As was already mentioned in previous work, the ratio \((L + C)/L\), the so-called “enhancement factor”, is independent of time at large lapse times [32]. An application of Cauchy’s inequality shows that \((L + C)/L \leq 2\), with equality only if \( x = 0 \) and if \( u_{i,z}(0) = u_{i,z}(z_0) \) for all modes \( i \). This can only be true if \( z_0 = 0 \) i.e. the source must be near the surface. In practice, to produce any measurable enhancement factor, the source must be at a depth less than the typical wavelength, as shown in Figures 6a and 6b. A source with a force direction different from normal would have a lower enhancement as well. Note that the enhancement is symmetric in azimuth around the source.
B. Isotropic Explosion

An isotropic explosion at depth $z_0$ is described by the force field $f(r, \omega) = B(\omega) \nabla \delta(r - r_0)$ \[11\]. It can easily be shown that $S_{ik}(\omega) = -B(\omega) \omega \text{div} u_{ik}(\omega)$. For a fixed frequency this depends on the mode label $i$ but, very conveniently, not on the direction $\hat{k}$ of horizontal propagation.

Let us first suppose that we measure the normal component of the displacement vector at the surface. Incoherent background and coherent enhancement are given by,

$$L(x, t) \sim \exp(-\omega t/Q) B(\omega)^2 \sum_i n_i |u_{i,z}(0)|^2 \sum_j n_j |\text{div} u_j(z_0)|^2,$$

$$C(x, t) \sim \exp(-\omega t/Q) B(\omega)^2 \left| \sum_i n_i u_{i,z}(0) \text{div} u_i(z_0) J_0(k_i x) \right|^2.$$

The resulting enhancement factor $(L + C)/L$ is plotted in dashed lines in Figure 7a as a function of the horizontal distance, and in Figure 7b for a measurement on top of an explosion source as a function of the depth $z_0$. Note that the enhancement never reaches its maximum value 2, not even when $z_0 = 0$. In an infinite medium, a measurement of any component of the displacement vector of waves released by an explosion source would have had no enhancement at all near the source \[33\]. Here, the finite enhancement is due to the nearness of a free surface.

The enhancement factor can be restored by a measurement of the dilatation ($\text{div} u$) in which case,

$$L(x, t) \sim \exp(-\omega t/Q) B(\omega)^2 \sum_i n_i |\text{div} u_i(0)|^2 \sum_j n_j |\text{div} u_j(z_0)|^2,$$

$$C(x, t) \sim B(\omega)^2 \exp(-\omega t/Q) \left| \sum_i n_i \text{div} u_i(0) \text{div} u_i(z_0)^* J_0(k_i x) \right|^2.$$

A measurement of the dilatation restores the symmetry between detector and source, and reveals the maximum enhancement factor 2 when the detector is located close to the source as shown in solid lines in Figures 7a, b.

C. Dipolar Source

We next consider a single couple at the surface with normal displacement vector, and axis along the $x$-axis. This source can be represented by the dipole $f(r, t) \sim \delta(\omega) \hat{z} \partial_x \delta^{(3)}(r - r_0)$. Such a source can be generated with laser interferometry on an elastic plate, and the resulting coherent backscattering effect was recently studied experimentally by De Rosny et al. \[31\].
The spatial derivative in the source finds its way in the Bessel functions, in the same way as was done in earlier work for the infinite system \[33\]. We derive, again for a measurement of the displacement vector in the direction normal to the surface,

\[
L(x, t) \sim \exp(-\omega t/Q) \frac{1}{D} d(\omega)^2 \sum_i n_i |u_{i,z}(0)|^2 \sum_j n_j |u_{j,z}(z_0)|^2 k_j^2 ,
\]

\[
C(x, t) \sim \exp(-\omega t/Q) \frac{\cos^2 \phi d(\omega)^2}{D} \sum_i n_i k_i u_{i,z}(0) u_{i,z}(z_0)^* J_1(k_i x)^2 . \tag{47}
\]

Two things can be noted. First, \(C\) vanishes anywhere above the source, \((x = 0)\). The enhancement is destroyed because the dipolar nature of the source is in some sense “orthogonal” to the detection of the displacement vector. Maximum enhancement actually occurs a fraction of a wavelength away from the source as shown in dashed line in Figure 8a.

Secondly, the coherent enhancement around the source has a \(\cos^2 \phi\) structure, with \(\phi\) the azimuthal angle between the dipole-axis of the source and the direction of detection. This “double-well” structure was observed by De Rosny, Tourin and Fink \[31\].

The coherent enhancement factor can be restored by a modification of the measurement. Suppose we measure the parameter \(\partial_x u_z(r, t)\). This measurement has the same symmetry as the dipolar source. We find for background and coherent enhancement,

\[
L(x, t) \sim \exp(-\omega t/Q) \frac{1}{D} d(\omega)^2 \sum_i n_i k_i^2 |u_{i,z}(0)|^2 \sum_j n_j k_j^2 |u_{j,z}(0)|^2 ,
\]

\[
C(x, t) \sim \exp(-\omega t/Q) \frac{\cos^2 \phi d(\omega)^2}{D} \times \\
\sum_i n_i k_i^2 u_{i,z}(0) u_{i,z}(z_0)^* \left[ \frac{J_1(k_i x)}{k_i x} - \frac{J_2(k_i x)}{J_1(k_i x)} \cos^2 \phi \right] \right]^2 . \tag{48}
\]

For \(x = 0\) and \(z_0 = 0\) we infer that \(L = C\), i.e the maximal enhancement can now be reached. The plot of the restored enhancement factor as the function of the horizontal distance and as the function of the source depth are shown in solid line in the Figures 8a, b. Note that the line profile is still not cylindrically symmetric, but depends on \(\phi\).

### D. Double-couple source at depth

Seismic sources have successfully been modeled as two compensating couples (dipoles) \[10\]. To facilitate observation of coherent backscattering with seismic waves we will here obtain the enhancement expected for such a source close to a free surface. In view of the complexity of the problem, we will restrict ourselves to a seismic plane that is oriented
parallel to the free surface where detection takes place. The depth of this plane is located at $z_0$.

The force field of a double-couple source is described by a symmetric, off-diagonal seismic tensor. We assume that the two dipoles are orthogonal and along the axes $x$ and $y$. The force field is then given by,

$$f(r, \omega) = M(\omega) \left( \hat{x} \partial_y + \hat{y} \partial_x \right) \delta(r - r_0).$$

(49)

with $r_0 = (0, 0, z_0)$. We can easily check that the mode representation of the source (24) is $S_{i,k} = \omega M(\omega) \left[ k_x u_{i,y}(z_0) + k_y u_{i,x}(z_0) \right]$ We will assume that the measured parameter is $\partial_y u_x + \partial_x u_y$, i.e. a certain horizontal component of the stress tensor; $(x', y')$ are the coordinates in a frame that has been rotated over an angle $\beta$ with $(x, y)$ (see Figure 9). The displacement vector of a mode $(i \cdot k)$ can be expressed as,

$$u_{ik}(z) = \left\{ u_{i,z}(z) \hat{z} + u_{i,\parallel}(z) \left[ \cos \alpha_i \hat{k} + \sin \alpha_i \hat{z} \times \hat{k} \right] \right\} \exp(i k \cdot x),$$

(50)

which introduces a new angle $\alpha_i$ independent on the direction $k$ of propagation and on depth. Lamb waves have $\alpha_i = 0$ whereas SH-waves have $\alpha_i = \frac{\pi}{2}$. We define $\phi$ as the angle between $k$ and the $x$-axis, i.e. $\hat{k} = \cos \phi \hat{x} + \sin \phi \hat{y}$. Finally, the angle $\mu$ orientates the direction of measurement $x$ in the horizontal plane with respect to the source. (see Figure 9).

The incoherent background is calculated from,

$$L(x, t) \sim \frac{\exp(-\omega t/Q)}{Dt} M(\omega)^2 \sum_i \int d^3k \left[ \partial_{x'} u_{ik,y'}(0) + \partial_{y'} u_{ik,x'}(0) \right]^2 \ Im G_{ik}(\omega)$$

$$\times \sum_j \int d^3k' \left[ \partial_{x} u_{jk',y}(r_0) + \partial_{y} u_{jk',x}(r_0) \right]^2 \ Im G_{jk'}(\omega)$$

(51)

whereas the coherent enhancement follows from,

$$C(x, t) = \frac{\exp(-\omega t/Q)}{Dt} M(\omega)^2 \times$$

$$\left| \sum_i \int d^3k \left[ \partial_{x'} u_{ik,y'}(0) + \partial_{y'} u_{ik,x'}(0) \right] \left[ \partial_{x} u_{ik,y}(r_0) + \partial_{y} u_{ik,x}(r_0) \right] \ Im G_{ik}(\omega) \right|^2.$$

(52)

These $k$-integrals can be evaluated straightforwardly and we simply quote the final result,

$$L(x, t) \sim \frac{\exp(-\omega t/Q)}{Dt} \frac{1}{4} M(\omega)^2 \sum_i n_i k_i^2 |u_{i,\parallel}(0)|^2 \sum_j n_j k_j^2 |u_{j,\parallel}(z_0)|^2.$$

(53)

Here, $u_{\parallel}$ denotes the complex amplitude of the horizontal component of the displacement vector. The coherent part is,
\[ C(x, t) \sim \frac{\exp(-\omega t/Q) \frac{1}{4} M(\omega)^2 \times}{Dt} \left| \sum \limits_i n_i k_i^2 u_{ij}(0) u_{ij}(z_0)^* \left[ \cos \beta J_0(k_i x) - \cos q_i J_4(k_i x) \right] \right|^2. \tag{54} \]

with \( q_i = 4\mu + 3\beta \) for Lamb waves and \( q_i = 4\mu + 3\beta + \pi \) for \( SH \) waves. Since \( J_4 \) term is very small, the line profile is almost isotropic around \( x = 0 \), independent on \( \mu \), and maximal for \( \beta = 0 \). The enhancement factor \( (L + C)/L \) is plotted in Figure 10a as a function of the horizontal distance for different source depth and in Figure 10b as a function of the source depth \( z_0 \) for a measurement on the top of the source. It is interesting to notice the relatively large second maximum of the coherent backscattering for a source at \( x = 0.7\lambda_s \) away from the detector (\( x \approx 1.2km \)).

VII. CONCLUSIONS AND OUTLOOK

In this paper we have investigated multiple scattering of elastic waves for a model that is adapted to the needs of seismology of the Earth’s crust: a two-dimensional solid plate with a thickness that is less than the mean free path of the waves. Contrary to other approaches, this model facilitates an exact treatment of the boundary condition at both sides of the plate, \( i.e. \) on the level of the elastic waves equation, and allows for fluctuations in the elastic constants that are depth-dependent. At the same time, we can describe the horizontal transport of waves, as well as the inter-mode mixing, by a generalized radiative transfer equation, that can be solved with conventional methods. Using this equation, we have investigated different aspects, such as surface detection, mode extinction times, equipartition, polarized sources at different depths in the plate and coherent backscattering. We believe that this study is an important step in the modelisation of seismic waves in the Earth’s crust, but may also find applications in laboratory experiments. In future studies we will try to solve our equation numerically using Monte-Carlo methods.

One final limitation of the present model has to be looked at in more detail. In this paper we have assumed a solid plate bounded by two ideal free surfaces without any leaks. The Earth’s crust is much better described by one top free surface and a solid-solid interface at 30km in depth (the so-called Moho) overlying a high-velocity mantle. In previous studies we already suggested that the resulting energy leak into the mantle may be the origin of seismic coda \[23,24\]. A future study should establish the relation between seismic coda and the individual quality factors of the modes.
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TABLE I. Ratio $D(\omega)/D^{\infty}(\omega)$ as a function of the number of modes $N(\omega)$ in the plate. $D^{\infty}(\omega)$ is the frequency-dependent diffusion constant for a 3D infinite medium, $D(\omega)$ is the frequency-dependent diffusion coefficient for our quasi-2D model with $N(\omega)$ modes, with the same kind of disorder in $\lambda$ and $\mu$. 

| $N(\omega)$ | 3   | 5   | 13  | 23  | 43  | 65  | 85  | 106 |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|
| $\frac{D(\omega)}{D^{\infty}(\omega)}$ | 0.72 | 0.56 | 0.72 | 0.77 | 0.82 | 0.84 | 0.85 | 0.85 |
FIGURES

FIG. 1. Schematic plot of the dispersion law of the elastic Rayleigh-Lamb eigenmodes in a layer bounded by two free surfaces. Bold lines indicate symmetric branches, straight lines indicate anti-symmetric modes. Only modes of different symmetry are allowed to cross. The two dashed lines indicate the pure shear or pure compressional excitations. The surface Rayleigh waves propagate somewhat slower than pure S waves.

FIG. 2. Extinction times for the different modes, calculated from Eq. (15), normalized by the mean free time of S waves in an infinite medium. The disorder is chosen to be uniform in the whole plate and the spatial correlation between the Lamé coefficients is chosen equal. The plate thickness is \( H = 20.2\lambda_s \), which has a number of \( N = 106 \) modes.

FIG. 3. Eigenvalues of the matrix \( C \) defined in Eq. (23) in the case of a plate thickness \( H = 20.2\lambda_s \) having a number of \( N = 106 \) modes normalized to the mean free time of S-waves in an infinite medium. The disorder is uniform in the whole plate and the spatial correlation between all Lamé coefficients is equal.

FIG. 4. Energy distribution among the different modes for different sources. The total energy release is normalized to 1 and the plate thickness is \( H = 20.2\lambda_s \) with \( N = 106 \) modes. A) Isotropic explosion source at a depth \( \lambda_s/3 \) from the free surface. B) Double couple source in the \( xy \) plan at a depth \( \lambda_s/3 \) from the free surface. C) Double couple source in the \( xz \) plan at a depth \( \lambda_s/3 \) from the free surface. The two Rayleigh modes are out of scale and carry half of the released energy. D) Double couple source in the \( xz \) plan at a depth \( 5\lambda_s \) from the free surface.

FIG. 5. Prediction of the diffusion equation (35) for the time evolution of the energy for different sources. The time scale has been normalized to the mean free time of the S waves in an infinite medium. The plate thickness is \( H = 20.2\lambda_s \), with \( N = 106 \) modes. A) and C) are predictions for the evolution of the energy for different modes, \( SH \) modes \( (E_{SH}) \), Lamb modes without the Rayleigh modes \( (E_{(L-R)}) \), and Rayleigh modes alone \( (E_R) \) for respectively an isotropic explosion at a depth \( \lambda_s/3 \) from the free surface and a double couple source in the \( xz \) plan at depth \( 5\lambda_s \) from the free surface. B) and D) are predictions for the potential energy ratio \( E_s/E_p \) and the ratio \( H^2/V^2 \) of the kinetic energies for the elastic displacement in different directions.
FIG. 6. Plot of the backscattering cone and enhancement factor for a monopolar source along the z axis. The normal component of the displacement field $u_z(0)$ is measured at the free surface and the plate thickness is $H = 20.2\lambda_s$ which has $N = 106$ modes. A) Plot of the backscattering cone for different depths $z_0$. B) Plot of the enhancement factor at $x = 0$ as a function of the source depth $z_0$.

FIG. 7. Plot of the backscattering cone and enhancement factor for a isotropic explosion source. Both the divergence (solid line) and normal component of the field (dashed line) are measured. The plate thickness is $H = 20.2\lambda_s$ which has $N = 106$ modes. A) Plot of the backscattering cone. B) Plot of the enhancement factor at $x = 0$ as a function of the source depth $z_0$.

FIG. 8. Plot of the backscattering cone and enhancement factor for a dipolar source. Both $\partial_r u_z(r,t)$ (solid line) and normal component $u_z(r,t)$ of the field (dashed line) are measured. The plate thickness is $H = 20.2\lambda_s$ which has $N = 106$ modes. A) Plot of the backscattering cone. B) Plot of the enhancement factor at $x = 0$ as a function of the source depth $z_0$.

FIG. 9. All angles involved in the measurement of the backscattering cone of a dislocation source at depth $z_0$. See text for discussion, $\alpha_i = 0$ for Lamb waves and $\alpha_i = \pi/2$ for $SH$ waves.

FIG. 10. Plot of the backscattering cone and enhancement factor for a double-couple source with both its axes along the free surface. The orientation of the detection is such that $\beta = 0$ and $\mu = 0$. The plate thickness is $H = 20.2\lambda_s$ which has $N = 106$ modes. A) Plot of the backscattering cone for different source depths $z_0 = 0$, $z_0 = \lambda_s/3$ and $z_0 = \lambda_s/2$. B) Plot of the enhancement factor at $x = 0$ as a function of the source depth $z_0$.
A) Isotropic Explosion Source

B) Isotropic Explosion Source

C) Double Couple Source

D) Double Couple Source
A) Monopolar Source

B) Monopolar Source
A) Isotropic Explosion Source

![Graph A: Horizontal distance vs. Enhancement factor](image1)

B) Isotropic Explosion Source

![Graph B: Depth z0 vs. Enhancement factor](image2)
A) Dipolar Source

B) Dipolar Source
A) Double-Couple Source

B) Double-Couple Source