Quantization of electromagnetic fields
in a circular cylindrical cavity

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We present a quantization procedure for the electromagnetic field in a circular cylindrical cavity with perfectly conducting walls, which is based on the decomposition of the field. A new decomposition procedure is proposed; all vector mode functions satisfying the boundary conditions are obtained with the help of this decomposition. After expanding the quantized field in terms of the vector mode functions, it is possible to derive the Hamiltonian for this quantized system.

PACS:42.50.-p, 03.70.+k
1 Introduction

The behavior of an atom depends on the mode structure of the electromagnetic field surrounding it [1]. The mode structure in a cavity is determined by its boundary conditions which are quite different from the periodic boundary conditions used in Lorentz-covariant electrodynamics. There are many interesting effects on the interaction of the atom with the field with various boundary conditions. It is possible to calculate them and compare them with experimental data [2]. Thanks to many new “high-tech” experimental instruments, cavity electrodynamics has become one of the most active branches of physics in recent years.

In quantum electrodynamics, we use the plane waves satisfying the periodic boundary condition whose quantization volume eventually becomes infinite. As a result, the orientation of the field is not essential. On the other hand, the field in a cavity must satisfy the boundary conditions and the orientation of the field with respect to cavity walls becomes an important factor. For this reason, we have to use vector mode functions to deal with this problem. These functions satisfy the boundary conditions and the boundary effect in turn is contained in the functions. Thus, for a given cavity, the most important step in the quantization process is to construct the vector mode functions.

Indeed, several authors have investigated the quantization procedure for the field in cavities [3]. They have studied the field quantization, the emission rate, and the atomic level shifts in the cavities with one or two infinite plane mirrors. Cavity electrodynamics with rectangular coordinate system does not present further significant mathematical problems. However, the case is quite different for cavities with curved boundary conditions. Even for circular cylindrical or spherical cavities, the quantization procedure has not yet been firmly established. The purpose of the present paper is to discuss in detail the mathematical problems in constructing the vector mode functions with a circular cylindrical boundary condition. We hope to discuss the spherical cavity in a later paper.

Recently, we have carried out the field quantization for several different rectangular boundary conditions including a rectangular tube using the vector mode functions [4]. These functions have been derived with the help of an orthogonal matrix. However, the procedure developed there is not applicable to other cavities in a straightforward manner.

In this paper, we present a more general method of obtaining the vector mode functions and apply it to the field quantization in a circular cylindrical cavity with perfectly conducting walls. This will be summarized in two theorems.

With this in mind, we first decompose the field into three parts in the circular cylindrical coordinates in the following section. Then, in Sec. 3, we apply it to the circular cylindrical cavity. After obtaining all vector mode functions by using the decomposition, we arrive at the quantized field and the quantized electromagnetic Hamiltonian in Sec. 4.

2 Decomposition of Electromagnetic fields

Let us now derive the decomposition formula for the electromagnetic field from Maxwell’s equations. Using the cylindrical coordinate system, we shall decompose the solutions of Maxwell’s equations into three terms, in preparation for the field quantization within the
circular cylindrical cavity in Sec. 3. We emphasize here again that this decomposition process is the key to the quantization.

Maxwell’s equations for the electric field \( E \) and the magnetic field \( B \) in free space are given by

\[
\nabla \cdot E = 0, \quad (2.1) \\
\nabla \cdot B = 0, \quad (2.2) \\
\nabla \times E + \partial_t B = 0, \quad (2.3) \\
\nabla \times B - \frac{1}{c^2} \partial_t E = 0, \quad (2.4)
\]

where \( c \) is the velocity of light in free space and \( \partial_t = \partial / \partial t \). As is well known, it follows from Maxwell’s equations that the fields \( E \) and \( B \) satisfy the wave equations:

\[
\left( \Delta - \frac{1}{c^2} \partial_t^2 \right) E = 0, \quad \left( \Delta - \frac{1}{c^2} \partial_t^2 \right) B = 0, \quad (2.5)
\]

where \( \Delta \) is the Laplacian operator.

The electromagnetic field \( E \) and \( B \) can be written in the cylindrical coordinates \((r, \varphi, z)\) as

\[
E = E_T + E_z, \quad B = B_T + B_z, \quad (2.6)
\]

where \( E_T = e_r E_r + e_\varphi E_\varphi \) is the transverse component of the field and \( E_z = e_z E_z \). Here \( e_r, e_\varphi, \) and \( e_z \) are the unit vectors in the \( r, \varphi, \) and \( z \) directions, respectively. Similarly, \( \nabla \times E, \nabla \cdot E, \) and \( \nabla \phi \) (\( \phi \) a scaler function) can also be divided into two parts. For example, \( \nabla \times E = \nabla_T \times E + \nabla_z \times E \), where the first term contains the derivatives with respect to \( r \) and \( \varphi \), while the second term contains the derivative with respect to \( z \). For simplicity, the derivatives with respect to \( r, \varphi, \) and \( z \) are described as \( \partial / \partial r = \partial_r, \partial / \partial \varphi = \partial_\varphi \) and \( \partial / \partial z = \partial_z \), respectively.

Since \( \nabla_z \times E_z = 0 \), Eq. (2.3) gives

\[
\partial_t B_z = -\nabla_T \times E_T, \quad \partial_t B_T = -\nabla_T \times E_z - \nabla_z \times E_T. \quad (2.7)
\]

Similarly, from Eq. (2.4), we have

\[
\frac{1}{c^2} \partial_t E_z = \nabla_T \times B_T, \quad \frac{1}{c^2} \partial_t E_T = \nabla_T \times B_z + \nabla_z \times B_T. \quad (2.8)
\]

Equations (2.7) and (2.8) give

\[
\frac{1}{c^2} \partial_t^2 E_T = \nabla_T \times \partial_t B_z - \nabla_z \times (\nabla_T \times E_z + \nabla_z \times E_T), \\
\frac{1}{c^2} \partial_t^2 B_T = -\frac{1}{c^2} \nabla_T \times \partial_t E_z - \nabla_z \times (\nabla_T \times B_z + \nabla_z \times B_T), \quad (2.9)
\]

which leads to

\[
\frac{1}{c^2} \partial_t^2 E_T + \nabla_z \times \nabla_z \times E_T - \nabla_T \times \nabla_T \times E_z = \nabla \times \partial_t B_z - \nabla \times \nabla \times E_z, \\
\frac{1}{c^2} \partial_t^2 B_T + \nabla_z \times \nabla_z \times B_T - \nabla_T \times \nabla_T \times B_z = -\frac{1}{c^2} \nabla \times \partial_t E_z - \nabla \times \nabla \times B_z, \quad (2.10)
\]

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where we have used $\nabla_T \times E_z = \nabla \times E_z$. Let us note that
\[
\nabla_z \times \nabla_z \times E_T = -\triangle E_T,
\n\nabla_T \times \nabla_T \times E_z = -\triangle_T E_z,
\n\frac{1}{c^2} \partial_t^2 E_T = (\triangle E)_T = \triangle E_T,
\]
(2.11)
where $(\triangle E)_T$ is the transverse part of $\triangle E$; the last equation holds in the present coordinates, although it is not correct in general. (For example, it is not correct in the spherical coordinates.) Then, from Eq. (2.10), we arrive at
\[
\nabla_T E = -\nabla \times \nabla E_z + \nabla \times \partial_t B_z,
\n\nabla_T B = -\frac{1}{c^2} \nabla \times \partial_t E_z - \nabla \times \nabla \times B_z.
\]
(2.12)

To rewrite Eq. (2.12), we must obtain an expression for the components $E_z$ and $B_z$. Suppose that the field is in a finite region and expand $E_z$ and $B_z$ in terms of a certain complete system of functions with mode $s$:
\[
E_z(r, t) = \sum_s (E_{zs}(r, t) + c.c.),
\nB_z(r, t) = \sum_s (B_{zs}(r, t) + c.c.),
\]
(2.13)
where
\[
E_{zs}(r, t) = \tilde{E}_{zs}(r)e^{-i\omega_{s1}t}, \quad B_{zs}(r, t) = \tilde{B}_{zs}(r)e^{-i\omega_{s2}t}.
\]
(2.14)
Here $\omega_{s\sigma}$ ($\omega_{s\sigma} \geq 0$, $\sigma = 1, 2$) is determined by using given boundary conditions. Since $E_z$ and $B_z$ satisfy the wave equation, their components satisfy the Helmholtz equations:
\[
\triangle E_{zs} = -k_{s1}^2 E_{zs}, \quad \triangle B_{zs} = -k_{s2}^2 B_{zs},
\]
(2.15)
where
\[
k_{s\sigma}^2 = \frac{\omega_{s\sigma}^2}{c^2}.
\]
(2.16)
We assume that the components satisfy
\[
\partial_z^2 E_{zs} = -h_{s1}^2 E_{zs}, \quad \partial_z^2 B_{zs} = -h_{s2}^2 B_{zs},
\]
(2.17)
where $h_{s\sigma}^2$ is also determined by the boundary conditions. Then we have two dimensional Helmholtz equations:
\[
\triangle_T E_{zs} = -g_{s1}^2 E_{zs}, \quad \triangle_T B_{zs} = -g_{s2}^2 B_{zs},
\]
(2.18)
where
\[
g_{s\sigma}^2 = k_{s\sigma}^2 - h_{s\sigma}^2.
\]
(2.19)
Now we define two functions $F_{\sigma}$ from $E_{zs}$ and $B_{zs}$ with $g_{s\sigma}^2 \neq 0$ as
\[
F_{\sigma}(r, t) = \sum_{g_{s\sigma}^2 \neq 0} [F_{s\sigma}(r, t) + c.c.] = \sum_{g_{s\sigma}^2 \neq 0} [\tilde{F}_{s\sigma}(r) e^{-i\omega_{s\sigma}t} + c.c.],
\]
(2.20)
where
\[ F_{s1} = \frac{E_{zs}}{g_{s1}^2}, \quad F_{s2} = \frac{B_{zs}}{g_{s2}^2}. \] (2.21)

The functions \( F_\sigma \) and their components \( F_{s\sigma} \) satisfy the same equations as the \( z \) components of the field:
\[ \triangle F_\sigma = -\frac{1}{c^2} \frac{\partial}{\partial t} F_\sigma, \quad \triangle_T F_{s\sigma} = -g_{s\sigma}^2 F_{s\sigma} \] (2.22)

From Eqs. (2.21) and (2.22) the component \( F_{s\sigma} \) is a solution of the Poisson equation
\[ \triangle_T F_{s1} = -E_{zs}, \quad \triangle_T F_{s2} = -B_{zs} \] (2.23)
and then the functions \( \triangle_T F_\sigma \) satisfy
\[ \triangle_T F_1 = -\sum \left( g_{s1}^2 F_{s1} + \text{c.c.} \right) = -\sum \left( E_{zs} + \text{c.c.} \right), \]
\[ \triangle_T F_2 = -\sum \left( g_{s2}^2 F_{s2} + \text{c.c.} \right) = -\sum \left( B_{zs} + \text{c.c.} \right). \] (2.24)

On the other hand, if there is a component \( E_{zs} \) or \( B_{zs} \) with \( g_{s\sigma}^2 = 0 \), Eq. (2.18) reduces to the two dimensional Laplace equation: \( \triangle_T E_{zs} = 0 \) or \( \triangle_T B_{zs} = 0 \). Now define \( E_{0z} \) and \( B_{0z} \) as
\[ E_{0z} = \sum \left( E_{zs} + \text{c.c.} \right), \quad B_{0z} = \sum \left( B_{zs} + \text{c.c.} \right), \] (2.25)
which satisfy
\[ \triangle_T E_{0z} = 0, \quad \triangle_T B_{0z} = 0. \] (2.26)

Then we have a useful expression for \( E_z \) and \( B_z \); for any \( E_z \) and \( B_z \) there exist functions \( F_\sigma \) given by Eq. (2.20) and functions \( E_{0z} \) and \( B_{0z} \) are given by Eq. (2.25) such that
\[ E_z = -\triangle_T F_1 + E_{0z}, \quad B_z = -\triangle_T F_2 + B_{0z}. \] (2.27)

It is worth emphasizing again that the functions \( F_\sigma \) are constructed with the components \( E_{zs} \) and \( B_{zs} \) with \( g_{s\sigma}^2 \neq 0 \), while \( E_{0z} \) and \( B_{0z} \) consist of the components with \( g_{s\sigma}^2 = 0 \).

Using Eq. (2.27) and defining \( F_\sigma \) by
\[ F_\sigma = e_z F_\sigma, \] (2.28)
we can rewrite Eq. (2.12) as
\[ \triangle_T \left( E - \nabla \times \nabla \times F_1 + \nabla \times \partial_t F_2 \right) = -\nabla \times \nabla \times E_{0z} + \nabla \times \partial_t B_{0z}, \] (2.29)
\[ \triangle_T \left( B - \frac{1}{c^2} \nabla \times \partial_t F_1 - \nabla \times \nabla \times F_2 \right) = -\frac{1}{c^2} \nabla \times \partial_t E_{0z} - \nabla \times \nabla \times B_{0z}, \] (2.30)
where \( E_{0z} = e_z E_{0z} \) and \( B_{0z} = e_z B_{0z} \). Let us define \( E_0 \) and \( B_0 \) as the quantities in the parentheses at the left hand side in Eqs. (2.29) and (2.30), respectively. The results of this section are then summarized in the following theorem.
Theorem 1: The field can be decomposed into three components as follows:

\[ \mathbf{E} = \nabla \times \nabla \times \mathbf{F}_1 - \nabla \times \partial_t \mathbf{F}_2 + \mathbf{E}_0, \]
\[ \mathbf{B} = \frac{1}{c^2} \nabla \times \partial_t \mathbf{F}_1 + \nabla \times \nabla \times \mathbf{F}_2 + \mathbf{B}_0, \]  
(2.31)

where \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \) satisfy

\[ \Delta_T \mathbf{E}_0 = -\nabla \times \nabla \times \mathbf{E}_{0z} + \nabla \times \partial_t \mathbf{B}_{0z}, \]
\[ \Delta_T \mathbf{B}_0 = -\frac{1}{c^2} \nabla \times \partial_t \mathbf{E}_{0z} - \nabla \times \nabla \times \mathbf{B}_{0z}. \]  
(2.32)

Theorem 1 plays a central role in performing field quantization in this paper. Take the \( z \) component of Eq. (2.31), we find that the \( z \) components of \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \) are given by \( \mathbf{E}_{0z} \) and \( \mathbf{B}_{0z} \) in Eq. (2.27), respectively. Furthermore, Eq. (2.32) is consistent with Eq. (2.12).

Let us consider the physical meaning of the above decomposition formula. First it is worth emphasizing that each term in Eq. (2.31) is a solution to Maxwell’s equations. That is, each term containing \( \mathbf{F}_\sigma \) (\( \mathbf{F}_\sigma \) term) satisfies Eqs. (2.1) - (2.4), which is easily shown by using the fact that \( \mathbf{F}_\sigma \) satisfies the wave equation (2.22). As a result, the third term \( \mathbf{E}_0, \mathbf{B}_0 \) is also a solution to Maxwell’s equations, so that it satisfies Eq. (2.12), as mentioned above.

Next take a particular example of the boundary condition satisfying \( \mathbf{E}_{0z}, \mathbf{B}_{0z} = 0 \). Then the field \( \mathbf{E}_0, \mathbf{B}_0 \) describes so-called the TEM (transverse electromagnetic) mode \( [3] \). Since the \( F_1 \) term has no \( z \) component of the magnetic field, it becomes TM (transverse magnetic), while the \( F_2 \) term becomes TE (transverse electric), because it does not contain the \( z \) component of the electric field.

In the above particular case where \( E_{0z}, B_{0z} = 0 \), Maxwell’s equations for the field \( \mathbf{E}_0, \mathbf{B}_0 \) reduce to

\[ \nabla_T \cdot \mathbf{E}_0 = 0, \quad \nabla_T \times \mathbf{E}_0 = 0, \]
\[ \nabla_T \cdot \mathbf{B}_0 = 0, \quad \nabla_T \times \mathbf{B}_0 = 0, \]
\[ \nabla_z \times \mathbf{E}_0 + \partial_t \mathbf{B}_0 = 0, \]
\[ \nabla_z \times \mathbf{B}_0 - \frac{1}{c^2} \partial_t \mathbf{E}_0 = 0. \]  
(2.33)

3 Determination of Functions \( F_\sigma \)

We are considering here the cavity enclosed by a circular cylindrical wall with radius \( a \) and height \( L \): \( r < a, \ 0 < z < L \). We assume that the cavity has perfectly conducting walls at \( z = 0, L \) and at \( r = a \). The tangential component of the electric field \( \mathbf{E}|_{\text{tan}} \) and the normal component of the magnetic field \( \mathbf{B}|_{\text{norm}} \) must accordingly vanish at the boundary of the cavity.

The above boundary condition reduces to that for the \( z \) components

\[ E_z = 0, \quad \partial_r B_z = 0, \quad (r = a), \]  
(3.1)
\[ B_z = 0, \quad \partial_z E_z = 0, \quad (z = 0, \ L). \]  
(3.2)
It is easy to get the second condition in Eq. (3.1) if we take the \( \varphi \) component of Eq. (2.4). Equation (2.1) leads to the second condition in Eq. (3.2).

Before obtaining the components \( E_{zs} \) and \( B_{zs} \), we show that the following lemma.

**Lemma 1**: \( E_0 = 0, B_0 = 0 \).

**Proof**: First we show that \( E_{0z}, B_{0z} = 0 \). Consider the component \( E_{0z} \), which satisfies the Laplace equation \( \Delta_T E_{0z} = 0 \) [see Eq. (2.20)] with the boundary condition \( E_{0z} = 0 \) at the boundary \( (r = a) \). Then we have \( E_{0z} = 0 \); this is a well known property of the Laplace equation.

Next consider the component \( B_{0z} \) satisfying \( \Delta_T B_{0z} = 0 \) with the boundary condition \( \partial_r B_{0z} = 0 (r = a) \). Then \( B_{0z} \) must be a constant, i.e., independent of the variables \( r \) and \( \varphi \): \( B_{0z} = f(z, t) \). Applying the two-dimensional Gauss’ theorem to \( \nabla \cdot B_0 = \nabla \cdot B_{0T} + \partial_z B_{0z} = 0 \), we get \( \partial_z B_{0z} = 0 \), because \( B_{0r} = 0 (r = a) \) and, as a result, \( B_{0z} \) is independent of the variables \( r, \varphi, \) and \( z \). On the other hand, \( B_{0z} \) must be zero at \( z = 0, L \), which leads to \( B_{0z} = 0 \).

As mentioned in the preceding section, the field \( E_0, B_0 \) becomes TEM satisfying Eq. (2.33), because \( E_{0z}, B_{0z} = 0 \). There exists no TEM in the present cavity. That is, from Eq. (2.33) we have \( \nabla_T \times B_0 = 0 \) and \( \nabla_T \cdot B_0 = 0 \). As a result, there exists a function \( \varphi \) such that \( B_0 = -\nabla_T \varphi \), which gives \( B_0 = 0 \) by the boundary condition at \( r = a \). Similarly, we get a vector function \( A \) such that \( E_0 = \nabla \times A \). The boundary condition leads us to \( E_0 = 0 \). Q.E.D.

Since \( E_0, B_0 = 0 \), Theorem 1 shows that the field is constructed from the functions \( F_\sigma \). Consequently, from Eq. (2.20), the field only contains the components with \( g_{s\sigma}^2 \neq 0 \). Moreover, it is easy to prove that \( g_{s\sigma}^2 > 0 \) by using Gauss’ theorem and the boundary condition.

Next we solve the Helmholtz equation (2.13) for the components \( E_{zs} \) and \( B_{zs} \) under the boundary conditions (3.1) and (2.2). Since \( g_{s\sigma}^2 > 0 \), the solution is given by

\[
E_{zs}(r, t) = C_{s1}(t)J_m(\chi_{s1}r/a) e^{i\eta z/L} \cos(n\pi z/L),
B_{zs}(r, t) = C_{s2}(t)J_m(\chi_{s2}r/a) e^{i\eta z/L} \sin(n\pi z/L),
\]

where the mode index is \( s = (m, \mu, n) (m = 0, \pm 1, \pm 2, \cdots; \mu = 1, 2, 3, \cdots; n = 0, 1, 2, \cdots) \), \( J_m \) is the Bessel function of the first kind, \( \chi_{s1} \equiv \chi_{m\mu 1} \) is the \( \mu \)th zero point of \( J_m \), and \( \chi_{s2} \equiv \chi_{m\mu 2} \) is the \( \mu \)th zero point of \( J'_m \), the derivative of \( J_m \). It follows from Eq. (2.14) that \( C_{s\sigma}(t) \propto \exp(-i\omega_{s\sigma} t) \).

From the solution (3.3) we have

\[
\Delta_T E_{zs} = -(k_{s1}^2 - (n\pi/L)^2)E_{zs} = -(\chi_{s1}^2/a^2)E_{zs},
\Delta_T B_{zs} = -(k_{s2}^2 - (n\pi/L)^2)B_{zs} = -(\chi_{s2}^2/a^2)B_{zs},
\]

which gives

\[
g_{s\sigma}^2 = (\chi_{s\sigma}/a)^2, \quad k_{s\sigma}^2 = (\chi_{s\sigma}/a)^2 + (n\pi/L)^2.
\]

Let us next obtain the functions \( F_\sigma \), which are defined in Eqs. (2.20) and (2.21); they are given by

\[
F_{s1}(r, t) = \frac{a^2 E_{zs}(r, t)}{\chi_{s1}} \equiv i \sqrt{\frac{h \omega_{s1}}{2\varepsilon_0}} a_{s1}(t) \psi_{s1}(r),
\]
\[ F_{s2}(r, t) = \frac{a^2 B_{s2}(r, t)}{\lambda_{s2}^2} = i \sqrt{\frac{\hbar \omega}{2\varepsilon_0}} a_{s2}(t) \psi_{s2}(r), \] (3.6)

where we have introduced \( a_{s\sigma} \) and \( \psi_{s\sigma} \):

\[ a_{s\sigma}(t) = a_{s\sigma}(0)e^{-i\omega_{s\sigma}t}, \] (3.7)

\[ \psi_{s1}(r, t) = c_{s1} J_m(\chi_{s1} r/a) e^{im\varphi} \cos(n\pi z/L), \] (3.8)

\[ \psi_{s2}(r, t) = c_{s2} J_m(\chi_{s2} r/a) e^{im\varphi} \sin(n\pi z/L), \] (3.9)

where \( c_{s\sigma} \) are normalization constants. The functions \( \psi_{s\sigma} \) have the orthonormality property, which is used in quantization in the next section.

**Lemma 2:**

\[ \int d\mathbf{r} \psi_{s\sigma}^* (\mathbf{r}) \psi_{s'\sigma} (\mathbf{r}) = \frac{1}{2} |c_{s\sigma}|^2 V \alpha_{s\sigma} \delta_{s's'}, \] (3.10)

where \( \int d\mathbf{r} = \int_\text{cavity} r dr d\varphi dz, V \) is the cavity volume, and

\[ \alpha_{s1} = \alpha_{m\mu_1} = J_{m+1}(\chi_{m\mu_1}), \]

\[ \alpha_{s2} = \alpha_{m\mu_2} = J_m(\chi_{m\mu_2}) - J_{m+1}(\chi_{m\mu_2}). \] (3.11)

Here the quantity \( \cos(n\pi z/L) \) in Eq. (3.8) must be changed to \( 1/\sqrt{2} \) when \( n = 0 \).

**Proof:** The orthonormality property (3.10) is easily derived from the following equality for the Bessel functions:

\[ \int_0^a dr r J_m(\chi_{m\sigma} r/a) J_m(\chi_{m'\sigma} r/a) = \frac{1}{2} a^2 \alpha_{s\sigma} \delta_{m'm'}. \] (3.12)

To prove Eq. (3.12), let us first observe the following two integrals [6]

\[ \int_0^1 dx x J_m(\chi_i x) J_m(\chi_j x) = \frac{1}{\chi_i^2 - \chi_j^2} \left[ \chi_i J_{m+1}(\chi_i) J_m(\chi_j) - \chi_j J_{m+1}(\chi_j) J_m(\chi_i) \right] \]

\[ = \frac{1}{\chi_i^2 - \chi_j^2} \left[ -\chi_i J_m(\chi_i) J_m(\chi_j) + \chi_j J_m(\chi_i) J_m(\chi_j) \right], \] (3.13)

\[ \int_0^1 dx x J_m^2(\chi_x x) = \frac{1}{2} \left[ J_m^2(\chi_i) - J_{m-1}(\chi_i) J_{m+1}(\chi_i) \right] \]

\[ = \frac{1}{2} \left[ J_m^2(\chi_i) - J_{m+1}(\chi_i) - 2J_m(\chi_i) J_{m+1}(\chi_i) \right], \] (3.14)

where \( \chi_i \) \((i = 1, 2, \cdots)\) is a positive real number and \( \chi_i \neq \chi_j \).

Considering that \( \chi_{m\mu_1} \) is the \( \mu \)th zero point of \( J_m(x) \) and that \( J_{m-1}(\chi_{m\mu_1}) + J_{m+1}(\chi_{m\mu_1}) = 0 \), from Eqs. (3.13) and (3.14), we have

\[ \int_0^1 dx x J_m(\chi_{m\mu_1} x) J_m(\chi_{m'\mu'1} x) = \frac{1}{2} \alpha_{m\mu_1} \delta_{\mu\mu'}, \] (3.15)

where \( \alpha_{m\mu_1} \) is given by Eq. (3.11). If we change the variable \( x \) to \( r/a \), Eq. (3.15) gives Eq. (3.12) for \( \sigma = 1 \).

Similarly, Eqs. (3.13) and (3.14) lead to

\[ \int_0^1 dx x J_m(\chi_{m\mu_2} x) J_m(\chi_{m'\mu'2} x) = \frac{1}{2} \alpha_{m\mu_2} \delta_{\mu\mu'}, \] (3.16)

where \( \alpha_{m\mu_2} \) is given by Eq. (3.11). Equation (3.16) gives us Eq. (3.12) for \( \sigma = 2 \). Q.E.D.
4 Vector Mode Functions and Field Quantization

After defining the vector mode functions, we quantize the field and obtain the quantized Hamiltonian. The mode functions are constructed according to the decomposition formula given in Eq. (2.31)

In the cavity with the perfectly conducting walls, the decomposition (2.31) in Theorem 1 reduces to

\[
E = \nabla \times \nabla \times F_1 - \nabla \times \partial_t F_2, \\
B = \frac{1}{c^2} \nabla \times \partial_t F_1 + \nabla \times \nabla \times F_2, 
\]  

(4.1)
because of \( E_0, B_0 = 0 \) [see Lemma 1].

Substituting the functions \( F_{\sigma} \) in Eq. (3.6) into \( E \) in Eq. (4.1), we find

\[
E(r, t) = i \sum_{s\sigma} \sqrt{\frac{\hbar \omega_{s\sigma}}{2\varepsilon_0}} \left[ a_{s\sigma}(t) u_{s\sigma}(r) - a^*_{s\sigma}(t) u^*_{s\sigma}(r) \right], 
\]

(4.2)

where the vector mode functions \( u_{s\sigma} \) are given by

\[
u_{s1} = \nabla \times \nabla \times e_z \psi_{s1}, \quad u_{s2} = i\omega_{s2} \nabla \times e_z \psi_{s2}. \]

(4.3)

Similarly, substituting \( F_{\sigma} \) into \( B \) and considering the equality

\[
\nabla \times u_{s1} = k_{s1}^2 \nabla \times e_z \psi_{s1}, \]

(4.4)

we obtain

\[
B(r, t) = \sum_{s\sigma} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{s\sigma}}} \left[ a_{s\sigma}(t) \nabla \times u_{s\sigma}(r) + a^*_{s\sigma}(t) \nabla \times u^*_{s\sigma}(r) \right]. 
\]

(4.5)

Before obtaining the quantized field and Hamiltonian, let us present some properties of the mode functions.

**Lemma 3:**

\[
\nabla \cdot u_{s\sigma} = 0, \\
(\Delta + k_{s\sigma}^2) u_{s\sigma} = 0, \\
|u_{s\sigma}|_{\tan} = 0, \quad |u_{s\sigma}|_{\text{norm}} = 0, \quad \text{(on walls)}, \\
\int c \, du^*_{s\sigma}(r) \cdot u_{s'\sigma'}(r) = \delta_{ss'} \delta_{\sigma\sigma'}, \]

(4.6) (4.7) (4.8) (4.9)

**Proof:** We give a proof of the orthonormality property (4.9), because it is obvious to prove the other equations. From Eq. (1.3), the mode function \( u_{s1} \) is rewritten as

\[
u_{s1} = \nabla (\nabla \cdot e_z \psi_{s1}) - \Delta e_z \psi_{s1} = k_{s1}^2 e_z \psi_{s1} + \nabla (\partial_z \psi_{s1}), \]

(4.10)

which gives

\[
u^*_{s1} \cdot u_{s'1} = \left[ k_{s1}^2 k_{s'1}^2 - k_{s1}^2 h_{s'1}^2 \right] \psi^*_{s1} \psi_{s'1} \\
+ k_{s'1}^2 \partial_z (\partial_z \psi^*_{s1}) \psi_{s'1} + \nabla (\partial_z \psi^*_{s1}) \nabla (\partial_z \psi_{s'1}).
\]

(4.11)
Taking into account Gauss’ theorem and the boundary conditions, we find that the last two terms in Eq. (4.11) have no effect on the volume integration of \( u_{s1} \cdot u_{s'1} \). Equation (4.11) gives the orthonormality relation for \( u_{s1} \):

\[
\int_c d\mathbf{r} u_{s1}^*(\mathbf{r}) \cdot u_{s'1}(\mathbf{r}) = \delta_{ss'},
\]

where we have used Eq. (3.10) and set

\[
c_{s1} = \sqrt{\frac{2c^2a^2}{V\alpha_{s1}\chi_{s1}^2\omega_{s1}^2}}.
\]

In the case of \( u_{s2} \), we make use of the equality

\[
(\nabla \times \mathbf{e}_z \psi_{s2}^*) \cdot (\nabla \times \mathbf{e}_z \psi_{s'2}) = \mathbf{e}_z \psi_{s2}^* \cdot (\nabla \times \nabla \times \mathbf{e}_z \psi_{s'2}) + \nabla \cdot [\mathbf{e}_z \psi_{s2}^* \times (\nabla \times \mathbf{e}_z \psi_{s'2})]
\]

\[
= \frac{\chi_{s2}^2}{d^2} \psi_{s2}^* \psi_{s'2} \nabla \cdot (\psi_{s2}^* \nabla T \psi_{s'2}).
\]

The orthonormality property of \( u_{s2} \) is then obtained from Eq. (3.10) if we set the normalization constant:

\[
c_{s2} = \sqrt{\frac{2d^2}{V\alpha_{s2}\chi_{s2}^2\omega_{s2}^2}}.
\]

Finally we show that the mode functions \( u_{s1} \) and \( u_{s'2} \) are orthogonal to each other. Since \( u_{s2} \) has no \( z \) component, we see that

\[
u_{s1}^* \cdot u_{s'2} = \int_c d\mathbf{r} \nabla \cdot (\nabla \times \mathbf{e}_z \psi_{s1}^*) \cdot (\nabla \times \mathbf{e}_z \psi_{s'2})
\]

\[
= i\omega_{s'2} \nabla \cdot \left[ (\partial_z \psi_{s1}^*) (\nabla \times \mathbf{e}_z \psi_{s'2}) \right],
\]

which has no effect on the volume integration of \( u_{s1}^* \cdot u_{s'2} \). Q.E.D.

We are now ready to carry out the field quantization. To get the quantized field the functions \( a_{s\sigma}(t) \) are regarded as annihilation operators, which annihilate photons with indices \( s\sigma \) and satisfy the commutation relation:

\[
[a_{s\sigma}(t), a_{s'\sigma'}^\dagger(t)] = \delta_{ss'}\delta_{\sigma\sigma'},
\]

where \( a_{s'\sigma'}^\dagger(t) \) are creation operators. Then the field given by Eqs. (4.2) and (4.5) becomes

\[
\int_c d\mathbf{r} (\nabla \times \mathbf{u}_{s\sigma}^*(\mathbf{r})) \cdot (\nabla \times \mathbf{u}_{s'\sigma'}(\mathbf{r})) = k_{s'\sigma}^2 \int_c d\mathbf{r} \mathbf{u}_{s\sigma}^*(\mathbf{r}) \cdot \mathbf{u}_{s'\sigma'}(\mathbf{r}).
\]

These are summarized in the following theorem.

**Theorem 2:** The quantized field and the Hamiltonian are given by

\[
\mathbf{E}(\mathbf{r}, t) = i \sum_{s\sigma} \sqrt{\frac{\hbar \omega_{s\sigma}}{2c^2}} \left[ a_{s\sigma}(t) \mathbf{u}_{s\sigma}(\mathbf{r}) - a_{s\sigma}^\dagger(t) \mathbf{u}_{s\sigma}^*(\mathbf{r}) \right],
\]

\[
\mathbf{B}(\mathbf{r}, t) = \sum_{s\sigma} \sqrt{\frac{\hbar}{2c^2 \omega_{s\sigma}}} \left[ a_{s\sigma}(t) \nabla \times \mathbf{u}_{s\sigma}(\mathbf{r}) + a_{s\sigma}^\dagger(t) \nabla \times \mathbf{u}_{s\sigma}^*(\mathbf{r}) \right],
\]

\[
\mathbf{H}_R = \sum_{s\sigma} \frac{1}{2} \hbar \omega_{s\sigma} \left( a_{s\sigma}^\dagger a_{s\sigma} + a_{s\sigma} a_{s\sigma}^\dagger \right) = \sum_{s\sigma} \hbar \omega_{s\sigma} \left( a_{s\sigma}^\dagger a_{s\sigma} + \frac{1}{2} \right).
\]
Theorem 2 will play a fundamental role when we investigate the electromagnetic property of a physical system in the circular cylindrical cavity. There are also important issues in the present cavity including, for example, spontaneous (stimulated) emission, absorption, the atomic energy shifts, and the Casimir effect; they will be investigated with the help of Theorem 2.

It is necessary to expand the field in terms of the vector mode functions as in Theorem 2. According to Fourier analysis, the field can be expanded in terms of, for instance, the periodic functions as in ordinary quantum electrodynamics. However, in this case, the operators $a_{s\sigma}$ and $a_{s\sigma}^\dagger$ cannot describe the physical photons. Consider the situation where the field consists of only one mode corresponding to the photon with $a_{s\sigma}u_{s\sigma}$, where $u_{s\sigma}$ satisfies the periodic boundary condition. This must be impossible, because this field does not satisfy the boundary conditions; the photon is then fictitious.

5 Conclusions

The quantization procedure of the field in the cavity without $E_0, B_0$ is as follows: (a) obtain the decomposition formula (2.31) in the circular cylindrical coordinates; (b) solve the Helmholtz equations (2.15) for the components $E_{zs}$ and $B_{zs}$ under the boundary conditions (3.4) and (3.2); (c) determine the functions $F_\sigma$, which are the solutions of the Poisson equation (2.23), from $E_{zs}, B_{zs}$; (d) substitute $F_\sigma$ into the decomposition formula (2.31) and obtain the vector mode functions satisfying the orthonormality property (4.9); (e) then we arrive at the quantized field and Hamiltonian in Theorem 2.

As has been shown explicitly in Theorem 2 in Sec. 4, the energy $\hbar\omega_{s\sigma}$ of the photon depends on the polarization index as well as the wave number index. This is one of the characteristics of the circular cylindrical cavity. The photon energy is independent of the polarization index in a rectangular cavity and in free space.

In the entire process of quantization, the decomposition formula in Theorem 1 plays the essential role. We must find such a formula in an appropriate coordinate system if we study other types of cavities. Also, if a cavity allows the third term $E_0, B_0$ in (2.31), we have to construct a methodology to deal with the problem. We hope to discuss the quantization of electromagnetic fields inside a spherical cavity in a future publication.

Acknowledgments

We would like to thank Prof. Y. Ezawa, Dr. R. Ray, Dr. G. Gat, Dr. S. Kobayashi, Dr. Kurennoy, and Mr. K. Ohshiro for numerous valuable discussions. We are also grateful to Prof. M. S. Kim for bringing some of the references to our attention.

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