Uncountable dichromatic number without short directed cycles

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Abstract

A. Hajnal and P. Erdős proved that a graph with uncountable chromatic number cannot avoid short cycles, it must contain for example $C_4$ (among other obligatory subgraphs). It was shown recently by D. T. Soukup that, in contrast of the undirected case, it is consistent that for any $n < \omega$ there exists an uncountably dichromatic digraph without directed cycles shorter than $n$. He asked if it is provable already in ZFC. We answer his question positively by constructing for every infinite cardinal $\kappa$ and $n < \omega$ a digraph of size $2^\kappa$ with dichromatic number at least $\kappa^+$ without directed cycles of length less than $n$.

1 Introduction

1.1 Background

Graphs with uncountable chromatic number and their obligatory subgraphs is a well investigated branch of infinite graph theory. It is known that every finite bipartite graph and hence every even cycle must be a subgraph of any uncountably chromatic graph. For a survey of the related results we refer to [2]. The dichromatic number $\chi(D)$ of a digraph $D$ (introduced by V. Neumann-Lara in [3]) is the smallest cardinal $\kappa$ for which $V(D)$ can be coloured with $\kappa$ many colours avoiding monochromatic directed cycles. Some of the results in connection with the dichromatic number are analogues with the corresponding theorems about the chromatic number. For example it was shown in [4] by probabilistic methods that for every $k, n < \omega$ there is a (finite) digraph $D$ with $\chi(D) \geq k$ which does not contain a directed cycle of length less than $n$ (an elegant explicit construction for such digraphs was given later by M. Severino in [5]). Because of the undirected analogue, it was natural to expect that uncountable dichromatic number makes some small directed cycles unavoidable. Galvin and Shelah showed in [6] that there is a tournament on $\omega_1$ without uncountable transitive subtournaments. It exemplifies that one can avoid directed cycles of length two (i.e., back and forth edges between a vertex pair). D. T. Soukup investigated in [1] the relation between the chromatic number of a graph and the dichromatic number of its possible orientations (which research direction was initiated by P. Erdős and V. Neumann-Lara) and the obligatory subdigraphs of uncountably dichromatic digraphs. One of his conclusions (see Theorem 3.5 of [1]) is the following:

Theorem 1.1 (D. T. Soukup). It is consistent that for every $n < \omega$ there is a digraph $D$ with dichromatic number $\kappa_1$ which does not contain directed cycles of length less than $n$.

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He asked (Question 6.7 in [1]) if the statement in Theorem 1.1 is true already in ZFC. We answer this question positively even with arbitrary large \( \kappa \) instead of \( \aleph_1 \).

### 1.2 Notation

We use standard set theoretic notation. Ordinals (in particular natural numbers) are identified with the set of the smaller ordinals. The set of functions with domain \( A \) and range \( \subseteq B \) is denoted by \( A^B \). For the concatenation of sequences \( s \) and \( z \) we write \( s \circ z \). Sequences of length 1 are not distinguished in notation from their only element. For a sequence \( s \) with length at least \( \alpha \), \( s \upharpoonright \alpha \) is its restriction to \( \alpha \). For an ordered pair \( \{ \{ u \}, \{ u, v \} \} \), we write simply \( uv \). A digraph \( D \) is a set of ordered pairs without loops (i.e., without elements of the form \( vv \)). The vertex set of a digraph \( D \) is \( V(D) := \bigcup \bigcup D \). For \( U \subseteq V(D) \), the subdigraph spanned by \( U \) is denoted by \( D[U] \). A directed cycle of length \( n \) \( (2 \leq n < \omega) \) is a digraph of the form \( \{ v_k v_{k+1} : k < n \} \) where the addition meant to be mod \( n \) and \( v_0, \ldots, v_{n-1} \) are pairwise distinct.

### 2 Main result

**Theorem 2.1.** For every infinite cardinal \( \kappa \) and \( n < \omega \), there is a digraph of size \( 2^\kappa \) with dichromatic number at least \( \kappa^+ \) which does not contain directed cycles of length less than \( n \).

**Proof.** Let \( V := \kappa^n \). For \( u \neq v \in V \), let \( uv \in D \) if for the smallest \( \xi \) with \( u(\xi) \neq v(\xi) \) we have \( v(\xi) = u(\xi) + 1 \mod n \). For a sequence \( s \) with length \( \alpha < \kappa \) and with range \( \subseteq n \), let \( V_s := \{ v \in V : v(0) = s \} \). From the definition of the edges of \( D \) the following is clear.

**Observation 2.2.** For every \( \alpha < \kappa \) and \( s \in \kappa^n \), the function \( \varphi_s \in V[V_s] \) with \( \varphi_s(v) = s \circ v \) is an isomorphism between \( D \) and \( D[V_s] \).

Theorem 2.1 follows from the following two lemmas.

**Lemma 2.3.** \( D \) does not contain directed cycles of length less than \( n \).

**Proof.** Let \( C \subseteq D \) be a directed cycle. Without loss of generality we can assume that the first coordinates of the elements of \( V(C) \) are not all the same. Indeed, otherwise let us denote the longest common initial segment of the elements of \( V(C) \) by \( s \) and we consider the directed cycle of same length as \( C \) which is the inverse image of \( C \) with respect to \( \varphi_s \) (see Observation 2.2) instead of \( C \).

By symmetry, we may assume that \( V(C) \cap V_0 \neq \varnothing \). We know that \( V(C) \subseteq V_0 \) because not all \( v \in V(C) \) start with 0 by assumption. It follows that \( C \) uses an edge that leaves \( V_0 \). It is easy to prove by induction that \( C \) visits \( V_k \) for every \( k < n \) (see Figure 1) thus \( |V(C)| \geq n \).

**Lemma 2.4.** For every colouring \( c \in V \kappa \), there is a monochromatic directed cycle in \( D \).

**Proof.** Suppose for a contradiction that colouring \( c \in V \kappa \) avoids monochromatic directed cycles. We define a continuous increasing sequence \( \langle s_\alpha : \alpha \leq \kappa \rangle \) where \( s_\alpha \in \kappa^n \) and \( c \) does not use the colours below \( \alpha \) in \( V_{s_\alpha} \). For \( \alpha = 0 \), our only choice \( s_0 := \varnothing \) satisfies the conditions. For limit ordinal \( \alpha \), we preserve the condition automatically since \( V_{s_\alpha} \supseteq \bigcap_{\beta < \alpha} V_{s_\beta} \). Suppose that \( \alpha = \beta + 1 \). Colour \( \beta \) cannot appear in every set \( V_{s_\alpha} \smallsetminus k \) \( (k < n) \) otherwise we can pick a \( v_k \in V_{s_\alpha} \smallsetminus k \) with \( c(v_k) = \beta \) for each \( k < n \) and then \( v_0, \ldots, v_{n-1} \) form a directed cycle of colour \( \beta \). Pick \( k_\beta < n \) such that colour \( \beta \) does not appear in \( V_{s_\alpha} \smallsetminus k_\beta \) and let \( s_{\beta + 1} := s_\beta \smallsetminus k_\beta \). Since \( V_{s_\alpha} \smallsetminus k_\beta \subseteq V_{s_\beta} \), by induction and by the choice of \( k_\beta \) it follows that colours below \( \beta + 1 \) do not appear in \( V_{s_{\beta + 1}} \smallsetminus k_\beta \). The recursion is done. Consider \( V_{s_\kappa} = \{ s_\kappa \} \), we have \( c(s_\kappa) \neq \xi \) for every \( \xi < \kappa \) which contradicts \( c \in V \kappa \).

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3 Open questions

**Question 3.1.** Does there exist for every infinite cardinal $\kappa$ and $n < \omega$ a digraph $D$ of size and dichromatic number $\kappa$ which does not contain directed cycles of length less than $n$.

**Question 3.2.** Is it true that for every uncountable not weakly compact cardinal $\kappa$ and $n < \omega$ there is a digraph $D$ on $\kappa$ such that for every $X \subseteq \kappa$ of size $\kappa$ $D[X]$ contains a directed cycle?

Question 3.1 is consistently true (combine Theorem 2.1 with the generalized continuum hypothesis). For a fixed $\kappa$, Question 3.2 can be forced to be true by a ccc poset, it is done (for $\kappa = \aleph_1$) in Theorem 3.5 of [1]).

References

1. D. T. Soukup, Orientations of graphs with uncountable chromatic number, J. Graph Theory 88 (2018), no. 4, 606–630, DOI 10.1002/jgt.22233. MR3818601 ✱1.1, 1.1, 3
2. P. Komjáth, The chromatic number of infinite graphs—a survey, Discrete Math. 311 (2011), no. 15, 1448–1450, DOI 10.1016/j.disc.2010.11.004. MR2800970 ✱1.1
3. V. Neumann Lara, The dichromatic number of a digraph, J. Combin. Theory Ser. B 33 (1982), no. 3, 265–270, DOI 10.1016/0095-8956(82)90046-6. MR693366 ✱1.1
4. D. Bokal, G. Fijavž, M. Juvan, P. M. Kayll, and B. Mohar, The circular chromatic number of a digraph, J. Graph Theory 46 (2004), no. 3, 227–240, DOI 10.1002/jgt.20003. MR2063373 ✱1.1
5. M. Severino, A short construction of highly chromatic digraphs without short cycles, Contrib. Discrete Math. 9 (2014), no. 2, 91–94. MR3320450 ✱1.1
6. F. Galvin and S. Shelah, Some counterexamples in the partition calculus, J. Combinatorial Theory Ser. A 15 (1973), 167–174, DOI 10.1016/s0097-3165(73)80004-4. MR0329900 ✱1.1