ON THE CLASSIFICATION OF COMPLETE AREA-STATIONARY AND
STABLE SURFACES IN THE SUB-RIEMANNIAN SOL MANIFOLD

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Abstract. We study the classification of area-stationary and stable $C^2$ regular surfaces in the space of the rigid motions of the Minkowski plane $E(1,1)$, equipped with its sub-Riemannian structure. We construct examples of area-stationary surfaces that are not foliated by sub-Riemannian geodesics. We also prove that there exist an infinite number of $C^2$ area-stationary surfaces with a singular curve. Finally we show the stability of $C^2$ area-stationary surfaces foliated by sub-Riemannian geodesics.

Contents

1. Introduction 1
2. Preliminaries 3
3. Characteristic curves in $E(1,1)$ 5
4. Complete area-stationary surfaces with non-empty singular set in $E(1,1)$ 6
5. Complete area-minimizing surfaces in $E(1,1)$ 8

References 10

1. INTRODUCTION

The study of the sub-Riemannian area functional in three-dimensional pseudo-hermitian manifolds and in other sub-Riemannian spaces has been largely investigated in the last years, see [1 2 3 4 6 8 9 10 11 12 13 14 17 18 19 20 21 22 23 26 27 28 30], among others.

One of the more interesting questions concerning the sub-Riemannian area functional is:

Problem 1. Which are the area-minimizing surfaces in a given three-dimensional contact sub-Riemannian manifold?

A surface $\Sigma$ is area-minimizing if $A(\Sigma) \leq A(\tilde{\Sigma})$, for any compact deformation $\tilde{\Sigma}$ of $\Sigma$. To answer the previous question, a natural preliminary step is the study of the area-stationary surfaces, the critical points of the area functional.

Problem 2. Which are the area-stationary surfaces in a given three-dimensional contact sub-Riemannian manifold?

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References
We will consider these questions in the class of $C^2$ regular surfaces. For a general introduction about the study of the area functional in sub-Riemannian spaces, we refer the interested reader to \cite{7} and \cite{16}, that treat the case of $\mathbb{H}^n$ and the contact sub-Riemannian manifolds respectively.

In Sasakian space forms, the classification of $C^2$ area stationary surfaces was given in \cite{21} in the case of the Heisenberg group $\mathbb{H}^n$ and in \cite{28} for the Sasakian structures of $S^3$ and $\widetilde{SL}_2(\mathbb{R})$. In the case of pseudo-hermitian three-manifolds that are not Sasakian, the only known results concerning Problem 1 and Problem 2 are given in \cite{17}, where the group of the rigid motions of the Euclidean plane $E(2)$ is studied.

Concerning the three-dimensional pseudo-hermitian manifolds, we have the following classification result, \cite[Theorem 3.1]{25}, in terms of the Webster scalar curvature $W$ and of the pseudo-hermitian torsion $\tau$.

**Proposition 1.1.** Let $M$ be a simply connected contact 3-manifold, homogeneous in the sense of Boothby and Wang, \cite{5}. Then $M$ is one of the following Lie group:

1. if $M$ is unimodular
   - the first Heisenberg group $\mathbb{H}^1$ when $W = |\tau| = 0$;
   - the three-sphere group $SU(2)$ when $W > 2|\tau|$;
   - the group $\widetilde{SL}(2,\mathbb{R})$ when $-2|\tau| < W < 2|\tau|$;
   - the group $E(2)$, universal cover of the group of rigid motions of the Euclidean plane, when $W = 2|\tau| > 0$;
   - the group $E(1,1)$ of rigid motions of Minkowski 2-space, when $W = -2|\tau| < 0$;

2. if $M$ is non-unimodular, the Lie algebra is given by
   
   \[ [X, Y] = \alpha Y + 2T, \quad [X, T] = \gamma Y, \quad [Y, T] = 0, \quad \alpha \neq 0, \]

   where $\{X, Y\}$ is an orthonormal basis of $\mathcal{H}$, $J(X) = Y$ and $T$ is the Reeb vector field.

   In this case $W < 2|\tau|$ and when $\gamma = 0$ the structure is Sasakian and $W = -\alpha^2$.

About the models of the unimodular case, Problem 1 and Problem 2 are not investigated only for the case of the Sol geometry, modeling by the space $E(1,1)$, and its study is the aim of this work.

After some preliminaries, the paper is organized as follow.

In Section 3 we compute explicitly the coordinates of the characteristic curves with given initial conditions. These curves play an important role in the study of area-stationary surfaces, since the regular part $\Sigma - \Sigma_0$ of a surface $\Sigma$ is foliated by characteristic curves, that are not in general sub-Riemannian geodesics, since $E(1,1)$ is characterized by a non-vanishing pseudo-hermitian torsion.

Section 4 is the core of the paper. We first characterize the $C^2$ complete, area-stationary surfaces immersed in $E(1,1)$ with singular points or singular curves that are sub-Riemannian geodesics. On the other hand, for the first time in the three-dimensional pseudo-hermitian setting, we also find examples of area-stationary surfaces that are not foliated by sub-Riemannian geodesics. We stress that these examples form an infinite family, i.e., given an horizontal curve $\Gamma$, we can construct an area-stationary surface having $\Gamma$ as singular set $\Sigma_0$.

Finally in Section 5 we prove that complete area-stationary surfaces with non-empty singular set, whose characteristic curves are sub-Riemannian geodesics, are stable. We also find three families of non-singular planes that are area-minimizing, using a calibration argument.
We remark that Section 5 opens two interesting questions. Is a stable complete area-stationary surface in $E(1,1)$ with a singular curve always foliated by sub-Riemannian geodesics in $\Sigma - \Sigma_0$? Do some other complete stable area-stationary surfaces in $E(1,1)$ with empty singular set exist?

2. Preliminaries

2.1. The group $E(1,1)$ of rigid motions of the Minkowski plane. We consider the group of rigid motions of the Minkowski plane $E(1,1)$, that is a unimodular Lie group with a natural sub-Riemannian structure. As a model of $E(1,1)$ we choose as underlying manifold $\mathbb{R}^3$ with the following orthonormal basis of left-invariant vector fields

$$\begin{align*}
X &= \frac{\partial}{\partial z} \\
Y &= \frac{1}{\sqrt{2}} \left(-e^z \frac{\partial}{\partial x} + e^{-z} \frac{\partial}{\partial y}\right) \\
T &= \frac{1}{\sqrt{2}} \left(e^z \frac{\partial}{\partial x} + e^{-z} \frac{\partial}{\partial y}\right)
\end{align*}$$

(2.1)

We have that $\{X,Y\}$ is an orthonormal basis of the horizontal distribution $\mathcal{H}$ and $T$ is the Reeb vector field. The scalar product of two vector fields $W$ and $V$ with respect to the metric induced by the basis $\{X,Y,T\}$ will be often denoted by $\langle W,V \rangle$. This structure of $E(1,1)$ is characterized by the following Lie brackets, [24].

$$\begin{align*}
[X,Y] &= -T \\
[X,T] &= -Y \\
[Y,T] &= 0
\end{align*}$$

(2.2)

In fact, applying [17, eq. 9.1 and eq. 9.3] we obtain that the Webster scalar curvature is $W = -1/2$ and the matrix of the pseudo-hermitian torsion $\tau$ in the $X,Y,T$ basis is

$$\begin{pmatrix}
0 & -\frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.$$ 

The following derivatives can be easily computed

$$\begin{align*}
\nabla_X X &= 0, \quad \nabla_Y X &= 0, \quad \nabla_T X = \frac{1}{2} Y, \\
\nabla_X Y &= 0, \quad \nabla_Y Y &= 0, \quad \nabla_T Y = -\frac{1}{2} X,
\end{align*}$$

(2.3)

where $\nabla$ denotes the pseudo-hermitian connection, [15]. Furthermore we have the characterization $-2|\tau|^2 = W < 0$ peculiar of $E(1,1)$, [25]. We also define the involution $J$, the so-called complex structure, on $\mathcal{H}$ by $J(X) = Y$ and $J(Y) = -X$.

2.2. The geometry of regular surfaces in $E(1,1)$. We consider a $C^1$ surface $\Sigma$ immersed in $E(1,1)$. We define the sub-Riemannian area of $\Sigma$ as

$$A(\Sigma) = \int_{\Sigma} |N_H| d\Sigma,$$

where $N_H$ denotes the projection of the Riemannian unit normal $N$ to $\mathcal{H}$ and $d\Sigma$ denotes the Riemannian area element on $\Sigma$. In the sequel we always denote by $N$ the inner unit normal.
The singular set $\Sigma_0$ is composed by the points in which $T\Sigma$ coincides with $H$. Outside $\Sigma_0$, we can define the horizontal unit normal

$$\nu_h := \frac{N_h}{|N_h|}$$

and the characteristic vector field as $Z := J(\nu_h)$. It is straightforward to verify that $\{Z, S\}$ is an orthonormal basis of $T\Sigma$ outside $\Sigma_0$, where

$$S := \langle N, T \rangle \nu_h - |N_H| T.$$ 

Finally, outside $\Sigma_0$, we define the mean curvature of $\Sigma$ by

$$H := -\langle \nabla Z \nu_h, Z \rangle.$$ 

Given a surface $\Sigma$ as zero level set of a function $u : \Omega \subset E(1,1) \to \mathbb{R}$, we can express

$$\nu_h = -\frac{u_z X + \frac{1}{\sqrt{2}} (-e^{z} u_x + e^{-z} u_y) Y}{\sqrt{u_z^2 + \frac{1}{2} (-e^{z} u_x + e^{-z} u_y)^2}}$$

and

$$Z = \frac{\frac{1}{\sqrt{2}} (-e^{z} u_x + e^{-z} u_y) X - u_z Y}{\sqrt{u_z^2 + \frac{1}{2} (-e^{z} u_x + e^{-z} u_y)^2}}.$$ 

We define a minimal surface as a surface with vanishing mean curvature $H$.

**Proposition 2.1.** Let $\Sigma$ be a minimal surface defined as the zero level set of a $C^2$ function $u : \Omega \subset E(1,1) \to \mathbb{R}$. Then $u$ satisfies the equation

$$u_{zz}(-e^{z} u_x + e^{-z} u_y)^2 + u_x^2(-e^{2z} u_{xx} - 2u_{xy} + e^{-2z} u_{yy})$$

$$-u_z(-e^{z} u_x + e^{-z} u_y)(-2e^{z} u_{xx} + e^{-z} u_x + 2e^{-z} u_{yy} - e^{-z} u_y) = 0$$

on $\Omega$.

**Proof.** From (2.4), (2.5) and (2.6) we can find that $u$ has to satisfy

$$Y(u)^2 X(X(u)) - Y(u) X(u) Y(X(u)) - Y(u) X(u) X(Y(u)) + X(u)^2 Y(Y(u)) = 0$$

on $\Omega$. Now, using (2.4), we can transform (2.8) into (2.7).

We will call (2.7) the minimal surface equation.

**Remark 2.2.** From (2.8), it is immediate to note that a surface $\Sigma$ satisfying $u_z \equiv 0$ or $-e^z u_x + e^{-z} u_y \equiv 0$ is always minimal.

In the following Lemma, we compute some important quantities related to the torsion and the geometry of a surface. It follows from [17, eq. 9.8],

**Lemma 2.3.** Let $\Sigma$ be a $C^1$ surface in $E(1,1)$, then we have

$$\langle \tau(Z), Z \rangle = -\langle Z, X \rangle \langle Z, Y \rangle = \langle \nu_h, X \rangle \langle \nu_h, Y \rangle = \langle \nu_h, \nu_h \rangle,$$

$$\langle \tau(Z), \nu_h \rangle = \frac{1}{2} (|Z|^2 - \langle Z, X \rangle^2).$$
3. Characteristic curves in $E(1,1)$

In this section we will study the equation of the integral curves of $Z$ on $\Sigma$, that are known as characteristic curves. It is well-known that a surface with constant mean curvature $H$ is foliated by characteristic curves in $\Sigma - \Sigma_0$. In general, a characteristic curve is an arc-length parametrized horizontal curve $\gamma$ in $E(1,1)$, that satisfies the equation

$$\nabla_\gamma \dot{\gamma} + HJ(\dot{\gamma}) = 0,$$

where $\dot{\gamma}$ denotes the tangent vector along $\gamma$ and $H$ is the (constant) curvature of $\gamma$. We stress that a curve $\gamma$ satisfying (3.1) is not a sub-Riemannian geodesic. In fact a characteristic curve $\gamma$ is a sub-Riemannian geodesic if and only if $H = 0$ and $\dot{\gamma}$ satisfies the additional equation

$$\langle \tau(\dot{\gamma}), \dot{\gamma} \rangle = 0,$$

see [29, Proposition 15], that forces $\gamma$ to be an integral curve of $X$ or $Y$ by Lemma 2.3.

**Proposition 3.1.** Let $\gamma$ be a characteristic curve in $E(1,1)$ with curvature $H = 0$. Then $\gamma$ belongs to the family of curves

$$\gamma(t) = (x_0 + \dot{x}_0 t, y_0 + \dot{y}_0 t, z_0)$$

or to the family

$$\gamma(t) = \left( x_0 + \frac{\dot{x}_0}{z_0} (e^{z_0 t} - 1), y_0 - \frac{\dot{y}_0}{z_0} (e^{-z_0 t} - 1), z_0 + \dot{z}_0 t \right),$$

where $\gamma(0) = (x_0, y_0, z_0)$ and $\dot{\gamma}(0) = (\dot{x}_0, \dot{y}_0, \dot{z}_0)$.

**Proof.** We consider the curve $\gamma : I \rightarrow \Sigma$, where $I$ denotes an interval. We express $\gamma(t) = (x(t), y(t), z(t))$ and we get

$$\dot{\gamma}(t) = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z},$$

(3.5)

$$= \dot{z} X + \frac{1}{\sqrt{2}} (ye^z - xe^{-z}) Y + \frac{1}{\sqrt{2}} (ye^z + xe^{-z}) T,$$

since

$$\frac{\partial}{\partial x} = \frac{1}{\sqrt{2}} e^{-z} (T - Y),$$

$$\frac{\partial}{\partial y} = \frac{1}{\sqrt{2}} e^z (Y + T).$$

From (3.5) and the fact that $\gamma$ is horizontal, we have

$$\dot{y} e^z + \dot{x} e^{-z} = 0.$$

(3.6)

Now $\nabla_\gamma \dot{\gamma} = 0$ is equivalent to the system

$$\nabla_\gamma \dot{\gamma} = 0,$$

(3.7)

$$\begin{cases}
\dot{z} = \dot{z}_0 \\
\dot{y} e^z - \dot{x} e^{-z} = c_0
\end{cases},$$

where $\dot{z}_0$ and $c_0$ are constants. We distinguish two cases. The first one corresponds to $\dot{z}_0 = 0$. This means that $z = z_0$, with $z_0 \in \mathbb{R}$, and so (3.6) and (3.7) are reduced to

$$\begin{cases}
2\dot{y} = e^{-z_0} c_0 \\
2\dot{x} = -e^{z_0} c_0
\end{cases},$$

(3.8)
that implies \( \gamma(t) = (x_0 - e^{z_0}(c_0/2)t, y_0 + e^{-z_0}(c_0/2)t, z_0) \), where \( c_0 \neq 0 \) and \( x_0, y_0 \in \mathbb{R} \).

The second possibility is \( z_0 \neq 0 \), that implies \( z(t) = z_0 + z_0t \), with \( z_0 \in \mathbb{R} \). In this case integrating (4.3) we obtain
\[
\gamma(t) = (x_0 + \frac{c_0}{2z_0} e^{z_0}, y_0 + \frac{c_0}{2z_0} e^{z_0}, z_0 + \frac{c_0}{2} e^{-z_0} t),
\]
where \( \gamma(0) = (x_0, y_0, z_0) \). Finally, to conclude the result, we note that
\[
\frac{c_0}{2} = \frac{\gamma_0 e^{z_0}}{e^{z_0}} = -\dot{x}_0 e^{-z_0}.
\]

\[\square\]

4. Complete area-stationary surfaces with non-empty singular set in \( E(1,1) \)

4.1. Complete area-stationary surfaces containing isolated singular points. The local structure of a constant mean curvature surface \( \Sigma \) of class \( C^2 \), in a neighborhood of a singular point, is well understood. In fact, applying [17, Theorem 5.3], we have

**Lemma 4.1.** Let \( \Sigma \) be a \( C^2 \) oriented immersed surface with constant mean curvature \( H \) in \( E(1,1) \). If \( p \in \Sigma \) is an isolated singular point, then, there exists \( r > 0 \) and \( \lambda \in \mathbb{R} \) such that the set described as
\[
D_r(p) = \{ \gamma^H_{p,v}(s)| v \in T_p \Sigma, |v| = 1, s \in [0,r]\},
\]
is an open neighborhood of \( p \) in \( \Sigma \), where \( \gamma^H_{p,v} \) denote the characteristic curve starting from \( p \) in the direction \( v \) with curvature \( H \).

First we construct the unique example, up to contact isometries, of a minimal surface with isolated singular points.

**Proposition 4.2.** Let \( \Sigma \) be a \( C^2 \) complete, area-stationary surface immersed in \( E(1,1) \) with \( H = 0 \) and with an isolated singular point \( p_0 = (x_0, y_0, z_0) \). Then \( \Sigma = \{ (x, y, z) \in E(1,1) : e^{z_0}y + x = 0 \} \).

**Proof.** By Lemma 4.1 the only possible way to construct a complete area-stationary surface, with a singular point \( p_0 \), is to consider the union of all characteristic curves \( \gamma \) of curvature 0 with initial conditions \( \gamma(0) = p_0 \) and \( \dot{\gamma}(0) \in T_{p_0} \Sigma = \mathcal{H}_{p_0} \), \( |\dot{\gamma}(0)| = 1 \). We can suppose \( p_0 = 0 \), since \( E(1,1) \) is homogeneous.

We consider the initial velocities
\[
\dot{\gamma}(0) = \cos(\alpha) X(0) + \sin(\alpha) Y(0)
\]
\[
= \cos(\alpha) \frac{\partial}{\partial z}(0) + \frac{\sin(\alpha)}{\sqrt{2}} \left( -\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) \right),
\]
for \( \alpha \in [0,2\pi] \). In this way we obtain as characteristic curves
\[(4.1) \quad \gamma(0) = \left( -\frac{\sin(\alpha)}{\sqrt{2}\cos(\alpha)}(e^{\cos(\alpha)t} - 1), -\frac{\sin(\alpha)}{\sqrt{2}\cos(\alpha)}(e^{-\cos(\alpha)t} - 1), \cos(\alpha)t \right),
\]
for \( \alpha \in [0,2\pi] \) and \( \gamma(t) = (0, 0, t) \) for \( \alpha = 0 \). At this point it is easy show that \( \Sigma \) is the zero level set of the function \( e^2y + x = 0 \) (or equivalently \( e^{-2}x + y = 0 \)), that satisfies (2.7).  
\[\square\]
4.2. Complete area-stationary surfaces containing singular curves. From [17 Corollary 5.4] we have

**Lemma 4.3.** Let $\Sigma$ be a $C^2$ minimal surface with non-empty singular set $\Sigma_0$ immersed in $E(1,1)$. Then $\Sigma$ is area stationary if and only if the characteristic curves meet the singular curves orthogonally with respect the metric $\langle \cdot, \cdot \rangle$, induced by the orthonormal basis \( (2.1) \).

In the following lemma, we prove that a minimal area-stationary surface cannot contain more than a singular curve.

**Lemma 4.4.** Let $\Sigma$ be a $C^2$ complete, minimal, area-stationary surface, containing a singular curve $\Gamma(\varepsilon, t)$, immersed in $E(1,1)$. Then $\Sigma$ cannot contain more singular curves.

**Proof.** We consider a singular curve $\Gamma(\varepsilon) = (x(\varepsilon), y(\varepsilon), z(\varepsilon))$ in $\Sigma$. Then, as $\Sigma$ is foliated by characteristic curves, we can parametrize it by the map

\[ F(\varepsilon, t) = \gamma(\varepsilon, t) = (x(\varepsilon, t), y(\varepsilon, t), z(\varepsilon, t)), \]

where $\gamma(\varepsilon)$ is the characteristic curves with initial data $\gamma(0) = \Gamma(\varepsilon)$ and

\[ \dot{\gamma}(0) = J(\Gamma(\varepsilon)) = \dot{z}(\varepsilon) J(X) = \frac{1}{\sqrt{2}} (\dot{y}(\varepsilon) e^{z(\varepsilon)} - \dot{x}(\varepsilon) e^{-z(\varepsilon)}) J(Y) \]

\[ = \frac{1}{\sqrt{2}} (\dot{z}(\varepsilon) e^{z(\varepsilon)}, \dot{z}(\varepsilon) e^{-z(\varepsilon)}, \dot{x}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)}). \]

We define the function $V_x(t) := (\partial F/\partial \varepsilon)(t, \varepsilon)$ that is a smooth Jacobi-like vector field along $\gamma(\varepsilon)$, \[17\] Section 4. We have that, in a singular point $(\varepsilon, t)$, the vertical component of $V_x$

\[ \langle V_x, T \rangle(\varepsilon, t) = \frac{\partial x}{\partial \varepsilon}(\varepsilon, t)e^{-z(\varepsilon, t)} + \frac{\partial y}{\partial \varepsilon}(\varepsilon, t)e^{z(\varepsilon, t)} \]

vanishes. We suppose that $\Gamma$ is not an integral curve of $X$ or $Y$. Then, from the following expression of the component of $F(\varepsilon, t)$

\[ x(\varepsilon, t) = x(\varepsilon) + \frac{\dot{z}(\varepsilon) e^{z(\varepsilon)}}{\dot{x}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)}} (e^{(\dot{z}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)})t/\sqrt{2}} - 1) \]

\[ y(\varepsilon, t) = y(\varepsilon) - \frac{\dot{z}(\varepsilon) e^{-z(\varepsilon)}}{\dot{x}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)}} (e^{-(\dot{z}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)})t/\sqrt{2}} - 1) \]

\[ z(\varepsilon, t) = z(\varepsilon) + \frac{\dot{x}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)}}{\sqrt{2}} t, \]

we have

\[ \langle V_x, T \rangle(\varepsilon, t) = \left\{ \frac{\dot{z}(\varepsilon) e^{-z(\varepsilon)} + \frac{\dot{z}(\varepsilon) e^{z(\varepsilon)}}{\dot{x}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)}} (e^{(\dot{z}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)})t/\sqrt{2}} - 1) \left( \frac{\dot{x}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)}}{\dot{x}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)}} \right)^2 \right\}, \]

that vanishes only for the values $(\varepsilon, 0)$, for positive values of $t$. On the other hand, if $\Gamma$ is an integral curve of $Y$, we get

\[ x(\varepsilon, t) = x(\varepsilon) \]

\[ y(\varepsilon, t) = y(\varepsilon) \]

\[ z(\varepsilon, t) = z(\varepsilon) + \frac{\dot{x}(\varepsilon) e^{-z(\varepsilon)} - \dot{y}(\varepsilon) e^{z(\varepsilon)}}{\sqrt{2}} t; \]
if $\Gamma$ is an integral curve of $X$, we have

$$x(\varepsilon, t) = x(\varepsilon) - \frac{\dot{x}(\varepsilon)e^{z(\varepsilon)}}{\sqrt{2}}t$$

(4.6)

$$y(\varepsilon, t) = y(\varepsilon) + \frac{\dot{y}(\varepsilon)e^{-z(\varepsilon)}}{\sqrt{2}}t$$

$$z(\varepsilon, t) = z(\varepsilon).$$

In both cases, the singular set is only the curve $\Gamma(\varepsilon)$.

The vertical component of $V_\varepsilon$ can be computed more directly using [17 Proposition 4.3], since $H = 0$. On the other hand, the explicit computation of the components of the parametrization $F(\varepsilon, t)$ allows us to characterize all the $C^2$ area-stationary complete surfaces with a singular curve that is a characteristic curve of curvature 0. We stress that, when the characteristic curves are sub-Riemannian geodesics, these examples can also be constructed from Remark 2.2.

**Proposition 4.5.** Let $\Sigma$ be an area-stationary surface with $H = 0$, with a singular curve $\Gamma$ that is a characteristic curve of curvature 0. Then, if $\Gamma$ is a sub-Riemannian geodesic, $\Sigma$ belongs to one of the families

(i) \{ax + by + c = 0 : (x, y, z) \in E(1, 1), a, b, c \in \mathbb{R}\};

(ii) \{e^{z-z_0}(y - y_0) + e^{z_0-z}(x - x_0) = 0 : (x, y, z) \in E(1, 1), x_0, y_0, z_0 \in \mathbb{R}\}.

Otherwise, we suppose that $\Gamma$ is a characteristic curve passing through $(x_0, y_0, z_0)$ with velocity $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$, $\dot{x}_0, \dot{y}_0, \dot{z}_0 \neq 0$. We can parametrize $\Sigma$ by $F : \mathbb{R}^2 \to E(1, 1)$, with $F(\varepsilon, t) = (x(\varepsilon, t), y(\varepsilon, t), z(\varepsilon, t))$ and

$$x(\varepsilon, t) = x_0 + \frac{\dot{x}_0}{\sqrt{2}}e^{\varepsilon} - 1 + \frac{\dot{z}_0e^{\varepsilon} + \dot{z}_0e^{-\varepsilon}}{\dot{x}_0e^{-z_0} - \dot{y}_0e^{z_0}}(e^{(\dot{x}_0e^{-z_0} - \dot{y}_0e^{z_0})t/\sqrt{2}} - 1)$$

(4.7)

$$y(\varepsilon, t) = y_0 + \frac{\dot{y}_0}{\sqrt{2}}e^{\varepsilon} - 1 - \frac{\dot{z}_0e^{\varepsilon} - \dot{z}_0e^{-\varepsilon}}{\dot{x}_0e^{-z_0} - \dot{y}_0e^{z_0}}(e^{-(\dot{x}_0e^{-z_0} - \dot{y}_0e^{z_0})t/\sqrt{2}} - 1)$$

$$z(\varepsilon, t) = z_0 + \dot{z}_0e^\varepsilon + \frac{\dot{x}_0e^{-z_0} - \dot{y}_0e^{z_0}}{\sqrt{2}}t.$$

**Remark 4.6.** We note that the surfaces parametrized by \([17]\) are the first examples of area-stationary surfaces that are not foliated by sub-Riemannian geodesics in three-dimensional contact sub-Riemannian manifolds, up to our knowledge. In fact this phenomena do not appear in the roto-translation group, \([17]\) Lemma 10.4], even if its pseudo-hermitian torsion is non-vanishing. In that case, the presence of two singular curves force the the surface to be foliated by sub-Riemannian geodesics or to be not area-stationary. On the other hand, it is well-known that a minimal surface is foliated by sub-Riemannian geodesics in any three-dimensional Sasakian manifold.

**Remark 4.7.** Given any horizontal curve $\Gamma = (x(\varepsilon), y(\varepsilon), z(\varepsilon))$ in $E(1, 1)$, we stress that \([4,3]\) provide a parametrization $F(\varepsilon, t) : \mathbb{R}^2 \to \Sigma \subset E(1, 1)$ of a complete area-stationary surface $\Sigma$ with $\Sigma_0 = \Gamma$.

5. Complete area-minimizing surfaces in $E(1, 1)$

5.1. Complete area-minimizing surfaces with empty singular set. In \([17]\) Proposition 9.8] is shown a general necessary condition for the stability of a non-singular surface in
pseudo-hermitian Lie groups. This condition state that the quantity
\[ W - \langle \tau(Z), v_h \rangle = \langle v_h, Y \rangle^2 - 1 = \langle Z, X \rangle^2 - 1 \]
has to be always non-positive. This condition is trivial in \( E(1, 1) \) due to the negativity of the Webster scalar curvature. On the other hand it has been used strongly in the classification of the stable, area-stationary surfaces without singular points in the manifolds \( \mathbb{H}^1 \), \( SU(2) \) and \( \tilde{E}(2) \), see [17, 21, 28]. In any way, we can prove the following

**Proposition 5.1.** The families of planes

(i) \( \{ x + c = 0 : (x, y, z) \in E(1, 1), c \in \mathbb{R} \} \);

(ii) \( \{ y + c = 0 : (x, y, z) \in E(1, 1), c \in \mathbb{R} \} \);

(iii) \( \{ z + c = 0 : (x, y, z) \in E(1, 1), c \in \mathbb{R} \} \);

are area-stationary, foliated by sub-Riemannian geodesics, and area-minimizing.

**Proof.** We prove the result for \( \Sigma = \{ x = 0 : (x, y, z) \in E(1, 1) \} \), since all the cases are similar. In this case, from (2.5) and (2.6), we have
\[ v_h = Y \quad Z = -X. \]
So the integral curves of \( Z \) are sub-Riemannian geodesics and \( \Sigma_0 = \emptyset \). Now Remark 2.2 implies that \( \Sigma \) is area-stationary. Finally we can foliate a neighborhood of \( \Sigma \) in \( E(1, 1) \) simply translating \( \Sigma \). We obtain a foliation by area-stationary surfaces and a standard calibration argument imply that \( \Sigma \) is area-minimizing, see for example [3], [26] or [27, § 5]. \( \square \)

**Remark 5.2.** The family of planes \( \{ ax + by + cz + d = 0 : (x, y, z) \in E(1, 1), a, b, c, d \in \mathbb{R} \} \) are not minimal, since they do not satisfy (2.7).

A very natural question is: are the planes in Lemma 5.1 the unique complete area-minimizing surfaces with empty singular set in \( E(1, 1) \)?

We have only been able to find the following sufficient condition

**Lemma 5.3.** Let \( \Sigma \) be a \( C^2 \) complete oriented minimal surface immersed in \( E(1, 1) \), with empty singular set \( \Sigma_0 \). If on \( \Sigma \) there holds \( \langle N, T \rangle \leq 0 \), then \( \Sigma \) is stable.

**Proof.** Taking into account the expression for stability operator for non-singular surfaces in [17, Lemma 8.3], we only need to show that
\[ 2Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) + \left( \frac{\langle N, T \rangle}{|N_h|} \right)^2 \leq 0 \]
on \( \Sigma \). Given a point \( p \) in \( \Sigma \), let \( I \) an open interval containing the origin and \( \alpha : I \to \Sigma \) a piece of the integral curve of \( S \) passing through \( p \). Consider the characteristic curve \( \gamma_{\varepsilon}(s) \) of \( \Sigma \) with \( \gamma_{\varepsilon}(0) = \alpha(\varepsilon) \). We define the map \( F : I \times \mathbb{R} \to \Sigma \) given by \( F(\varepsilon, s) = \gamma_{\varepsilon}(s) \) and denote \( V(s) := (\partial F / \partial \varepsilon)(0, s) \) which is a Jacobi-like vector field along \( \gamma_0 \), see [17, Proposition 4.3]. Denoting by \( \dot{\gamma} \) the derivatives of functions depending on \( s \), and the covariant derivative along \( \gamma_0 \) respect to \( \nabla \) and \( \gamma_0 \) by \( Z \). Using [17] Lemma 3.1, Eq. 4.4 and Eq. 4.5 we get
\begin{align}
\langle V, T \rangle(0) &= -|N_H|, \\
\langle V, T \rangle'(0) &= -\langle N, T \rangle, \\
\langle V, T \rangle''(0) &= -|N_H| \left( Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) + \left( \frac{\langle N, T \rangle}{|N_h|} \right)^2 \right)
\end{align}
It is easy to show that \( g(V,T) \) never vanishes along \( \gamma_0 \) as \( \Sigma_0 \) is empty, see [17, Proof of Lemma 9.5]. On the other hand, by [17, Proposition 4.3] and Lemma 2.3, we have that \( \langle V, T \rangle \) satisfies the ordinary differential equation
\[
\langle V, T \rangle'''(s) - \langle Z, X \rangle^2 \langle V, T \rangle'(s) = 0
\]
along \( \gamma_0 \). We suppose that \( \langle Z, X \rangle \neq 0 \). Taking into account the initial conditions \((5.1), (5.2)\) and \((5.3)\), we obtain
\[
\langle V, T \rangle(s) = a \cosh(|\langle Z, X \rangle|s) + b \sinh(|\langle Z, X \rangle|s) + c,
\]
where
\[
a = \frac{|N_H|}{\langle X, Z \rangle^2} \left( Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) + \frac{\langle N, T \rangle}{|N_h|} \right)^2,
\]
\[
b = -\frac{\langle N, T \rangle}{|\langle Z, X \rangle|}
\]
and
\[
c = -|N_H| - \frac{|N_H|}{\langle X, Z \rangle^2} \left( Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) + \frac{\langle N, T \rangle}{|N_h|} \right)^2.
\]
We have that \( \langle V, T \rangle(s) \neq 0 \) implies
\[
a + b = \frac{|N_H|}{\langle X, Z \rangle^2} \left( Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) + \frac{\langle N, T \rangle}{|N_h|} \right)^2 - \frac{\langle N, T \rangle}{|\langle Z, X \rangle|} \leq 0.
\]
Then we can conclude
\[
2Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) + \left( \frac{\langle N, T \rangle}{|N_h|} \right)^2 \leq 2 \left( Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) + \frac{\langle N, T \rangle}{|N_h|} \right)^2 \leq 2|\langle Z, X \rangle| \langle N, T \rangle \leq 0
\]
on \( \gamma_0 \). Now since the choice of \( p \) is arbitrary, we get the statement.

If \( \langle Z, X \rangle = 0 \), we conclude that \( \Sigma \) is stable if and only if \( \langle N, T \rangle = 0 \), by [17, Proposition 9.8].

Remark 5.4. We note that the surfaces described in the points (i), (ii), (iii) of Proposition 5.1 are characterized by \( \langle N, T \rangle = -e^z/\sqrt{2} \), \( \langle N, T \rangle = -e^z/\sqrt{2} \) and \( \langle N, T \rangle \equiv 0 \) respectively, where \( N \) denotes the inward unit normal on \( \Sigma \). In the third family the planes are vertical surfaces and they satisfy \( W - \langle \tau(Z), v_h \rangle \equiv 0 \).

Taking into account the geometric invariants of \( E(1,1) \), we expect the existence of other examples of complete oriented minimal surface with empty singular set.

5.2. Complete area-minimizing surfaces with non-empty singular set. We consider the stability operator constructed in [17, Theorem 8.6].

Lemma 5.5. Let \( \Sigma \) be a \( C^2 \) oriented minimal surface immersed in \( E(1,1) \), with singular set \( \Sigma_0 \) and \( \partial \Sigma = \emptyset \). If \( \Sigma \) is stable then, for any function \( u \in C^0_b(\Sigma) \) such that \( Z(u) = 0 \) in a tubular neighborhood of a singular curve and constant in a tubular neighborhood of an
isolated singular point, we have $Q(u) \geq 0$, where
\[
Q(u) := \int_{\Sigma} \left( |N_h|^{-1} Z(u)^2 + |N_H|(1 + \langle Z, Y \rangle^2) - (|N_H|(1/2 - \langle Z, Y \rangle^2) - \langle \nabla_s \nu_h, Z \rangle)^2 \right) d\Sigma \\
+ 4 \int_{(\Sigma_0)_c} \langle N, T \rangle \langle Z, Y \rangle^2 \langle Z, \nu \rangle u^2 d(\Sigma_0)_c + \int_{(\Sigma_0)_c} S(u)^2 d(\Sigma_0)_c.
\]
Here $d(\Sigma_0)_c$ is the Riemannian length measure on $(\Sigma_0)_c$ and $\nu$ is the external unit normal to $(\Sigma_0)_c$.

**Corollary 5.6.** Let $\Sigma$ be a plane in the family \{ax + by + c = 0 : (x, y, z) \in E(1, 1), a, b, c \in \mathbb{R}\}. Then $\Sigma$ is stable.

**Proof.** We know that $\Sigma$ is area-stationary with a singular line, obtained intersecting $\Sigma$ with the plane $z = \log \sqrt{b/a}$. From (2.6) we get
\[
Z = -\frac{be^{-z} + ae^{-z}}{-be^{-z} + ae^{-z}} X,
\]
which is orthogonal to the singular line. Since \[
\langle \nabla_s \nu_h, Z \rangle = \langle \nabla_s Y, X \rangle = \frac{|N_H|}{2},
\]
we have that the stability operator $Q(u)$ is always non-negative for any admisible test function $u$. \qed

**Remark 5.7.** The planes \{ax + by + c = 0 : (x, y, z) \in E(1, 1), a, b, c \in \mathbb{R}\} are also area-minimizing, by calibration arguments.

**Corollary 5.8.** We consider the surface $\Sigma = \{e^z y + e^{-z} x = 0 : (x, y, z) \in E(1, 1)\}$. Then $\Sigma$ is stable.

**Proof.** From (2.6) we get
\[
Z = -\frac{(e^z y - e^{-z} x) Y}{e^z y - e^{-z} x}
\]
and $\Sigma_0 = \{(0, 0, z) : (x, y, z) \in E(1, 1)\}$. From (4.3) we have
\[
\langle \nabla_s \nu_h, Z \rangle = \langle \nabla_s Y, X \rangle = -\frac{|N_H|}{2},
\]
which implies
\[
Q(u) = \int_{\Sigma} \left( |N_h|^{-1} Z(u)^2 + 2|N_H|^2 u^2 \right) d\Sigma + \int_{\Sigma_0} S(u)^2 d\Sigma_0 + 4 \int_{\Sigma_0} u^2 d\Sigma_0 \geq 0,
\]
for all admisible $u$. \qed

**Corollary 5.9.** The surfaces defined in Proposition 5.2 are stable.

**Proof.** For simplicity we will prove the statement in the case of $x_0 = y_0 = z_0 = 0$. We note that, since $\Sigma_0 = (0, 0, 0)$, the argument in the proof of Lemma 5.3 works and the condition $\langle N, T \rangle = -1 + e^z)/\sqrt{2} \leq 0$ is a sufficient condition for the stability in the complementary of any tubular neighborhood of $\Sigma_0$. Finally we observe that the stability operator in Lemma 5.5 does not give contributions of the singular set in the case of isolated singular points. \qed
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