Elasticity and melting of skyrmion flux lattices in $p$-wave superconductors

Qi Li$^{1,2}$, John Toner$^1$, and D. Belitz$^{1,2}$

$^1$Department of Physics and Institute of Theoretical Science, University of Oregon, Eugene, OR 97403
$^2$Materials Science Institute, University of Oregon, Eugene, OR 97403

(Dated: May 5, 2019)

We analytically calculate the energy, magnetization curves ($B(H)$), and elasticity of skyrmion flux lattices in $p$-wave superconductors near the lower critical field $H_{c1}$, and use these results with the Lindemann criterion to predict their melting curve. In striking contrast to vortex flux lattices, which always melt at an external field $H > H_{c1}$, skyrmion flux lattices never melt near $H_{c1}$. This provides a simple and unambiguous test for the presence of skyrmions.

PACS numbers: 74.20.Rp; 74.25.Dw; 74.25.Qt; 74.70.Pq

The topological excitations known as skyrmions have been proposed to have many applications. After they were introduced in a nuclear physics context by Skyrme [1], variations of this concept have been shown or proposed to be important in superfluid $^3$He [2], in quantum Hall systems [3], in $p$-wave superconductors [4], and in metallic magnets [5]. Given this widespread predicted occurrence, it is desirable to find a simple experimental signature that can serve as a smoking gun indicating the presence of skyrmions. In this Letter we show that, in $p$-wave superconductors subject to an external magnetic field, the structure of the phase diagram provides an unambiguous test for the presence of skyrmions [6].

In superconductors, skyrmions compete with another, and more well-known, species of topological excitations, viz., vortices. In type-II superconductors, the latter are induced by an external magnetic field $H$ and form the famous Abrikosov flux lattice in a field range $H_{c1} < H < H_{c2}$. It is known both theoretically [7, 8, 9, 10] and experimentally [11, 12] that these flux lattices can melt. They do so near both $H_{c1}$ and $H_{c2}$, where the elastic constants of the flux lattice vanish, which in clean systems makes the root-mean-square positional thermal fluctuations $\sqrt{\langle |u(x)|^2 \rangle}$ grow without bound as these fields are approached.

Vortices involve a singular field configuration at their cores and can occur for any symmetry of the superconducting order parameter. In $p$-wave superconductors the vector character of the superconducting order parameter also allows for skyrmions, which, in contrast to vortices, are non-singular topological defects. Like vortices, each skyrmion carries a quantized magnetic flux, and for strongly type-II $p$-wave superconductors in a field $H_{c1} < H < H_{c2}$, a skyrmion lattice is predicted to have a lower energy than a vortex lattice [4]. The currently most convincing case for a $p$-wave superconductor is Sr$_2$RuO$_4$ [13], another candidate is UGe$_2$ [14].

Our central result is the prediction that one can distinguish a skyrmion lattice from a vortex lattice by considering the melting curve of the lattice. Our results for the latter are summarized in the phase diagram for clean skyrmion flux lattices shown in Fig. 1(a). This is strikingly different from that for the vortex case, Fig. 1(b): The skyrmion lattice never melts near $H_{c1}$, while vortex lattices always melt near $H_{c1}$.

![Fig. 1](image.png)

FIG. 1: (a) External field ($H$) vs. temperature ($T$) phase diagram for skyrmion flux lattices. There is a direct transition from the skyrmion flux lattice to the Meissner phase. The theory predicts the shape of the melting curve only close to $T_c$; the rest of the curve is an educated guess. (b) Same as (a) for vortex flux lattices. The vortex flux lattice always melts before $H$ reaches $H_{c1}$.

The physics behind this result is simple. The interaction between vortices falls off exponentially at large distances [15]. As a result, when $H \rightarrow H_{c1}$ from above, which causes the vortex lattice constant $R$ to grow, the elastic constants vanish exponentially with $R$ as $R \rightarrow \infty$, which, in turn, causes the root-mean-square displacement fluctuation $\sqrt{\langle |u(x)|^2 \rangle}$ to diverge exponentially. Consequently, $\sqrt{\langle |u(x)|^2 \rangle} \propto R$ as $R \rightarrow \infty$ holds. The Lindemann criterion then implies that the vortex lattice must melt with decreasing $H$ before one reaches $H_{c1}$.

For skyrmions, as first shown numerically in Ref. [4], and confirmed analytically below, the interaction potential falls off only as $1/R$. As a result, the Lamé coefficients $\mu$ and $\lambda_1$ of the skyrmion lattice vanish only as $1/R^3$ as $R \rightarrow \infty$. This leads to $\sqrt{\langle |u(x)|^2 \rangle} \propto R^{3/4} \ll R$. Hence, the skyrmion lattice never melts as $H_{c1}$ is approached. The shape of the phase diagram alone thus distinguishes skyrmion lattices from vortex lattices.
We expect this approximation to preserve the correlation work of Ref. 4, we replace the hexagonal unit cell one superconductor is always in this limit. As in the numerical solutions will be given elsewhere [17]. We begin by reviewing the numerical solution reported in Ref. [4], see Fig. 2. Here \( \lambda \) is the London penetration depth. Near \( H_{21} \) the superconductor is always in this limit. As in the numerical work of Ref. [4], we replace the hexagonal unit cell one expects in the skyrmion lattice by a circle of the same area. We expect this approximation to preserve the correct scaling of the energy.

For the first three terms of the asymptotic expansion of the energy \( E \) per skyrmion per unit length we obtain

\[
E/E_0 = 2 + (8\sqrt{6}/3) (\lambda/R) - (16/3) (\lambda/R)^2 \ln(R/\lambda). \tag{1}
\]

Here \( E_0 = (\Phi_0/4\pi\lambda)^2 \), with \( \Phi_0 = \pi\hbar c/e \) the flux quantum. This analytic result is in excellent agreement with the numerical solution reported in Ref. [4], see Fig. 2. Notice that there are no free parameters in Eq. (1).

In the remainder of this Letter we sketch the derivation of the above results. A complete account of the calculations will be given elsewhere [17]. We begin by reviewing the formulation of the skyrmion lattice problem [4].

The spin part of the order parameter for a \( p \)-wave superconductor is a complex 3-vector \( \psi(x) \). In a large regime of Landau theory parameter space (the so-called \( \beta \)-phase), all low-energy configurations of \( \psi(x) \) can be written in the form \( \psi(x) = \psi_0(\hat{n}(x) + i\hat{m}(x)) \), where \( \hat{n}(x) \) and \( \hat{m}(x) \) are real, mutually orthogonal unit vectors that vary slowly in space, and \( \psi_0 \) is a constant [4].

In the ground state, \( \hat{n} \) and \( \hat{m} \) are constants. Slow spatial variations \( \hat{n}(x) \) and \( \hat{m}(x) \) cost an energy

\[
H_L = \int d^3x \left[ \frac{1}{2} (\partial \hat{a})^2 + (\hat{n} \cdot \partial \hat{m} - a)^2 + (\nabla \times a)^2 
- 2h \cdot (\nabla \times a) \right]. \tag{2}
\]

Here we use dimensionless units where distance, vector potential \( a \), magnetic field \( h \), and energy are measured in units of \( \lambda, \Phi_0/2\pi\lambda, \Phi_0/2\pi\lambda^2 \), and \( \Phi_0^2/32\pi^3\lambda \), respectively [4]. \( \hat{l} \equiv \hat{n} \times \hat{m} \) and the last term in the London energy, Eq. (2), represents the coupling of the external magnetic field \( h \) to the magnetic induction \( b = \nabla \times a \).

A vortex is a low-energy configuration of \( \hat{n}(x) \) and \( \hat{m}(x) \) in which \( \hat{l} \) is a constant (i.e., in which \( \hat{n}(x) \) and \( \hat{m}(x) \) span the same plane for all \( x \)), but \( \hat{n}(x) \) and \( \hat{m}(x) \) rotate by \( 2\pi n \) (an integer) as one follows their evolution around any closed spatial path that encircles the path of the vortex core. Such a configuration necessarily has an undetermined, angle \( \theta(x) \) is no longer constant. In the simplest case the skyrmion is cylindrically symmetric, and \( \hat{l}(x) \) forms an angle \( \theta(x) \) with some central axis (which we will take to be the \( z \)-axis). While \( \theta(x) \) changes from 0 to \( \pi \) as one moves from infinity to the skyrmion axis, \( \hat{n}(x) \) and \( \hat{m}(x) \) rotate around \( \hat{l} \) by \( 4\pi \) on any loop enclosing the skyrmion axis.

This picture leads to the characterization of a cylindrically symmetric skyrmion in terms of the single, as yet undetermined, angle \( \theta(x) \). In polar coordinates \((r, \varphi)\) the vector fields \( \hat{n}(x) \), \( \hat{m}(x) \) and \( \hat{l}(x) \) are given by

\[
\hat{l} = \hat{e}_z \cos \theta(r) + \hat{e}_r \sin \theta(r),
\hat{n} = (\hat{e}_z \sin \theta(r) - \hat{e}_r \cos \theta(r)) \sin \varphi + \hat{e}_\varphi \cos \varphi,
\hat{m} = (\hat{e}_z \sin \theta(r) - \hat{e}_r \cos \theta(r)) \cos \varphi - \hat{e}_\varphi \sin \varphi. \tag{3}
\]

If the field \( \theta(x) \) minimizes the energy of this configuration, then that configuration is a local minimum of the London energy, Eq. (2) [4].

Inserting Eq. (3) into Eq. (2) yields the energy per unit length, in units of \( E_0 \), of a cylindrically symmetric...
skyrmion in a region of radius $R$,

$$E = \frac{1}{2} \int_0^R dr \left[ (\theta'(r))^2 + \frac{1}{r^2} \sin^2 \theta(r) \right] + \int_0^R dr \left[ \frac{r}{2} (1 + \cos \theta(r)) + a(r) \right]^2 + \int_0^R dr \left[ a(r)/r + a'(r) \right]^2. \quad (4)$$

The three terms represent the energy of a nonmagnetic skyrmion, the supercurrent energy, and the magnetic energy, respectively. Skyrmions in a lattice are not cylindrically symmetric, since the lattice is not. However, since a hexagon is well approximated by a circle of the same radius, we obtain a linear, inhomogeneous ODE of second order, This set of coupled, nonlinear ODEs must be solved numerically.

We now calculate the elastic properties of the skyrmion. We write

$$\Theta''(r) + \frac{1}{r} \Theta'(r) = \frac{-\sin \theta(r)}{r} \left[ \frac{2 + \cos \theta(r)}{r} + 2a(r) \right]. \quad (5a)$$

$$a''(r) + \frac{1}{r} a'(r) - \frac{1}{r^2} a(r) = a(r) + \frac{1}{r} \left[ 1 + \cos \theta(r) \right]. \quad (5b)$$

This set of coupled, nonlinear ODEs must be solved subject to the boundary condition $\theta(r=0) = \pi$, and $\theta(r=R) = 0$. In Ref.[4] this was done numerically using finite elements methods. Here we show that for large $R$ (i.e., near the lower critical field $H_{c1}$) an analytic solution can be obtained in the form of an asymptotic expansion in powers of $1/R$. To zeroth order, i.e., for $R \to \infty$, the l.h.s. of Eq. (5b) vanishes, so $a$ is given by

$$a_{\infty}(r) = - \left[ 1 + \cos \theta(r) \right]/r. \quad (6a)$$

Inserting this in Eq. (5a), the resulting ODE is solved by

$$\Theta_{\infty}(r) = 2 \arctan(\ell/r), \quad (6b)$$

for any $\ell$. All of these solutions fulfill the boundary condition, and $\ell$ is undetermined at this point.

We write $\theta(r) = \Theta_{\infty}(r) + \delta \Theta(r)$ and $a(r) = a_{\infty}(r) + \delta a(r)$, and require $|\delta \Theta(r)| \ll 1$ and $|\delta a(r)| \ll |1 + \cos \theta_{\infty}(r)|/r$. We do not require $|\delta \theta| < \Theta_{\infty}$, which is crucial for the success of our perturbative method. Inserting this into the ODEs, one sees that $\delta \theta$ can be written

$$\delta \Theta(r) = (1/\ell^2) g(r/\ell) + (1/\ell^4) h(r/\ell) + O(1/\ell^6), \quad (7)$$

while $\delta a$ can be expressed in terms of $\Theta_{\infty}$, $\delta \Theta$, and its derivatives. It will turn out that $\ell \propto \sqrt{R}$, so this is the desired expansion in powers of $1/R$. For the function $g$ we obtain a linear, inhomogeneous ODE of second order,

$$g''(u) + \frac{1}{u} g'(u) - \frac{u^4 - 6u^2 + 1}{u^2(1 + u^2)^2} g(u) = -64u \frac{1}{(1 + u^2)^4}. \quad (8)$$

The physical solution is the one that diverges linearly for large arguments. Standard methods give

$$g(u) = -\frac{4}{3} \frac{u^2(4 + u^2) + 2(1 + u^2) \ln(1 + u^2)}{(1 + u^2)^2}. \quad (9)$$

The parameter $\ell$ is now determined by the requirement $\theta(r = R) = 0$, which yields, to this order, $\ell = (2/3)^{1/4} R^{1/2}$. To the same order, $\delta a$ is just given by the l.h.s. of Eq. (5b) with $a$ replaced by $a_{\infty}$,

$$\delta a(r) = (16r/\ell^4)(1 + r^2/\ell^2)^{-3} + O(1/\ell^4). \quad (10)$$

Inserting these results in Eq. (4) and performing the integrals yields the first two terms on the r.h.s. of Eq. (4). This method can be continued order by order. At the next order it yields the final term in Eq. (4).

From the energy per skyrmion, Eq. (4), we can calculate the external field dependence of the equilibrium lattice constant, $R(H)$. This is done by minimizing the energy per unit volume, which is the energy per unit length per skyrmion, Eq. (4), divided by the area per skyrmion, $\pi R^2$, minus an energy density gain of $2\Phi_0 \pi R^2$ due to the external field. The latter is obtained from the $\cdot (\nabla \times \hat{a})$ term in Eq. (2) by noting that the magnetic flux $\int dx dy \cdot (\nabla \times \hat{a}) = 2\Phi_0$ for each skyrmion in the lattice. This yields the Gibbs free energy density

$$g(R) = (K/4\pi^2) \left[-\delta/R^2 + 4\sqrt{3}\lambda/(3R^3)\right], \quad (11)$$

where $K \equiv \Phi_0^2/2\pi \lambda^2$ and $\delta \equiv H/H_{c1} - 1$ with $H_{c1} \equiv K/2\Phi_0$.

For $H < H_{c1}$, $g(R)$ is minimized by $R \to \infty$; i.e., there are no skyrmions. For $H > H_{c1}$, on the other hand, the energy density is minimized at a finite $R$ given by

$$R = R_0 \equiv 2\sqrt{6}\lambda/\delta. \quad (12)$$

Thus, $H_{c1}$ is the lower critical field at which the skyrmion lattice first forms. This implies for the spatially averaged magnetic induction, which is the flux per skyrmion $2\Phi_0$ divided by the unit cell area,

$$B(H) = 2\Phi_0/(\pi R_0^2) = \delta^2 H_{c1}/3. \quad (13)$$

Hence, $B(H)$ is horizontal near $H_{c1}$, while it is vertical for vortex lattices. This result, with a slightly different numerical prefactor, was first obtained numerically in [4].

We now calculate the elastic properties of the skyrmion lattice. By symmetry, the elastic Hamiltonian of a hexagonal lattice of lines parallel to the $z$-axis in 3-d is

$$H_{c1} = \frac{1}{2} \int dx \left( 2\mu u_{\alpha \beta} u_{\alpha \beta} + \lambda_L u_{\alpha \alpha} u_{\beta \beta} + K_{\text{tilt}} |\partial_z |u|^2 \right), \quad (14)$$

where $u_{\alpha \beta} \equiv (\partial_\beta u_\alpha + \partial_\alpha u_\beta)/2$ is the strain tensor, $\alpha, \beta \in \{x, y\}$, and $u$ only has $x$ and $y$ components. $\mu$, $\lambda_L$, and $K_{\text{tilt}}$ are the shear, bulk, and tilt moduli.

Consider the energy change due to a dilation of the lattice, $R_0 \rightarrow R_0' \equiv R_0(1 + \epsilon)$ with $\epsilon \ll 1$. This corresponds to a displacement $u(x) = \epsilon x$, and a strain tensor $u_{\alpha \beta} \equiv \epsilon \delta_{\alpha \beta}$, with $x$ the projection of $x$ perpendicular to $\hat{z}$. Inserting this in $H_{c1}$ gives $E_{\text{div}}/V = 2(\mu + \lambda_L) \epsilon^2,$
and comparing with the dilation energy implied by Eq. (14),

\[
\frac{E_{\text{dil}}}{V} = g(R_0) - g(R_0) = \frac{1}{2} g''(R_0) (\epsilon R_0)^2 = \frac{K \delta^3}{96 \pi^2 \lambda^2} \epsilon^2.
\]
gives

\[
\mu + \lambda_L = K \delta^3/192 \pi^2 \lambda^2.
\]

Obtaining \( \mu \) is more difficult since we have approximated the unit cell by a circle. We use a heuristic approach. Equation (14) is of the form that would result if the skyrmions interacted via a nearest-neighbor pair potential \( U_p(r) \propto K \lambda/r \). Pretending that \( U_p(r) \) is the origin of the skyrmion energy (which should give the correct scaling of \( \mu \) with \( h \)) makes it straightforward to calculate \( \mu \), and comparing the result with Eq. (15) yields \( \lambda_L \). We obtain

\[
\mu \sim \lambda_L = K \delta^3/3 \lambda^2 \times O(1).
\]

\( K_{\text{tilt}} \) we obtain by considering a uniform tilt of the skyrmion axes away from the \( z \)-axis by a small angle \( \theta = |\partial_\theta \mathbf{u}| \). The tilt energy in Eq. (14) is the change in the \( \mathbf{b} \cdot \mathbf{h} \) energy in Eq. (2). Since, per skyrmion per unit length, this energy is (in ordinary units) \( -\Phi_0 H \cos \theta/2\pi \), tilting changes this energy by \( \Phi_0 H(1 - \cos \theta)/2\pi \approx \Phi_0 H \theta^2/4\pi = \Phi_0 H \partial_\theta \mathbf{u}^2/4\pi \). Dividing this result by the unit cell area, using Eq. (14) for \( R_0 \), and identifying the result with the \( K_{\text{tilt}} \) term in Eq. (14) gives \( K_{\text{tilt}} \) near \( H_{c1} \):

\[
K_{\text{tilt}} = (H_{c1} \delta)^2/12\pi.
\]

We can now calculate the mean-square positional fluctuations \( \langle |\mathbf{u}(x)|^2 \rangle \) by Fourier transforming Eq. (14) and using the equipartition theorem. This gives

\[
\langle |\mathbf{u}(x)|^2 \rangle_T = \frac{k_B T}{V} \sum_{\mathbf{q} \in BZ} 1/\langle \mu q_\perp^2 + K_{\text{tilt}} q_\perp^2 \rangle
\]

for the transverse fluctuations, and the same expression with \( \mu \rightarrow (2 \mu + \lambda_L) \) for the longitudinal ones. \( q_\perp \) and \( q_\parallel \) are the projections of \( \mathbf{q} \) orthogonal to and along \( z \), respectively. The Brillouin zone (BZ) of the skyrmion lattice is a hexagon of edge length \( \times 1/R_0 \) in the plane perpendicular to \( z \), and infinitely extended in the \( z \)-direction.

Since \( \mu \) and \( \lambda_L \) are comparable in magnitude, the same is true for the longitudinal and the transverse contributions to \( \langle |\mathbf{u}(x)|^2 \rangle \). Performing the integral over \( q_\perp \) yields

\[
\langle |\mathbf{u}(x)|^2 \rangle \sim \langle |\mathbf{u}(x)|^2 \rangle_T = \int_{BZ} \frac{d^2 q_\perp}{8 \pi^2} \frac{k_B T}{\mu K_{\text{tilt}} q_\perp}.
\]

A change of variables, \( q_\perp \equiv w/R_0 \), and using Eqs. (12), (16), and (17) gives, as claimed in the Introduction,

\[
\langle |\mathbf{u}(x)|^2 \rangle = O(1) \times k_B T/\sqrt{\mu K_{\text{tilt}}} R_0
\]

\[
= O(1) \times k_B T/\lambda H_{c1}^2 4^{3/2} \propto R_0^{-3/2}.
\]

The Lindemann criterion for melting is \( \Gamma_L \equiv \langle |\mathbf{u}(x)|^2 \rangle / R_0^2 > \Gamma_c = O(1) \). In our case,

\[
\Gamma_L = k_B T 4^{3/2}/H^2 c_1 4^{5/2} \times O(1).
\]

We see that, as claimed in the Introduction, the skyrmion never melts as \( H \rightarrow H_{c1} \) from above (i.e., as \( \delta \rightarrow 0 \)), since the Lindemann ratio vanishes in that limit.

We finally determine the shape of the melting curve \( H_m(T) \) near the superconducting \( T_c \). Since, in mean field theory, \( H_{c1} \propto (T_c - T) \), and \( \lambda \propto 1/(T_c - T) \), we find from Eq. (21) by putting \( \Gamma_L = O(1) \),

\[
H_m - H_{c1} \propto (T_c - T)^{5/2}.
\]

The melting curve thus quickly rises above \( H_{c1} \) with decreasing temperature, as shown qualitatively in Fig. 1(a).

This work was supported by the NSF under grant No. DMR-05-29966.

[1] T. Skyrme, Proc. Roy. Soc. A 260, 127 (1961).
[2] P. W. Anderson and G. Toulouse, Phys. Rev. Lett. 38, 508 (1977).
[3] C. Timm, S. M. Girvin, and H. A. Fertig, Phys. Rev. B 58, 10634 (1998), and references therein.
[4] A. Knigavko, B. Rosenstein, and Y. F. Chen, Phys. Rev. B 60, 550 (1999). This paper focused on UPt3, which was a leading candidate for \( p \)-wave superconductivity at the time, but is now believed to have \( f \)-wave symmetry.
[5] U. Roessler, A. Bogdanov, and C. Pfleiderer, Nature 442, 797 (2006).
[6] Our results apply to the \( p \)-wave case only. Other spin triplet states, in particular \( f \)-wave superconductors, await further investigation.
[7] B. A. Huberman and S. Doniach, Phys. Rev. Lett. 43, 950 (1979).
[8] D. R. Nelson and H. S. Seung, Phys. Rev. B 39, 9153 (1989).
[9] E. H. Brandt, Phys. Rev. Lett. 63, 1106 (1989).
[10] D. S. Fisher, Phys. Rev. B 22, 1190 (1980).
[11] P. L. Gammel, L. F. Schneemeyer, J. V. Waszczak, and D. J. Bishop, Phys. Rev. Lett. 61, 1666 (1988).
[12] H. Safar, P. L. Gammel, D. H. Huse, D. J. Bishop, J. P. Rice, and D. M. Ginsberg, Phys. Rev. Lett. 69, 824 (1992).
[13] K. D. Nelson, Z. Q. Mao, Y. M. Maeno, and Y. Liu, Science 306, 1151 (2004). The nature of the order parameter in Sr2RuO4 is, however, not quite settled yet, see Ref. 13 and references therein.
[14] K. Machida and T. Ohmi, Phys. Rev. Lett. 86, 850 (2001).
[15] M. Tinkham, Introduction to Superconductivity (McGraw-Hill, New York, 1975).
[16] P. Chaikin and T. C. Lubensky, Principles of Condensed Matter Physics (Cambridge University, Cambridge, 1995).
[17] Qi Li, John Toner, and D. Belitz, unpublished results.
[18] In principle, crystal-field effects invalidate this isotropic model, provide a mass to Goldstone modes, and cause
the skyrmion interaction to fall off exponentially. This is a weak effect that is manifest only at very long distances, as is the case, e.g., in magnets, and we ignore it.

[19] P. G. Björnsson, Y. Maeno, M. E. Huber, and K. A. Moler, Phys. Rev. B 72, 012504 (2005).