Closure and Spanning $k$-Trees

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Abstract In this paper, we propose a new closure concept for spanning $k$-trees. A $k$-tree is a tree with maximum degree at most $k$. We prove that: Let $G$ be a connected graph and let $u$ and $v$ be nonadjacent vertices of $G$. Suppose that $\sum_{w \in S} d_G(w) \geq |V(G)| - 1$ for every independent set $S$ in $G$ of order $k$ with $u, v \in S$. Then $G$ has a spanning $k$-tree if and only if $G + uv$ has a spanning $k$-tree. This result implies Win’s result (Abh Math Sem Univ Hamburg, 43:263–267, 1975) and Kano and Kishimoto’s result (Graph Comb, 2013) as corollaries.

Keywords Spanning tree · $k$-tree · Closure

1 Introduction

All graphs considered in this paper are only simple and finite. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [1].

Bondy and Chvátal [2] introduced the closure concept, and showed that it plays an important role for the existence of cycles, paths, and other subgraphs in graphs. In this
paper, we consider a closure concept for spanning $k$-trees, and refer the reader to the survey [3] on closure concept. A $k$-tree is a tree with maximum degree at most $k$. Win [6] obtained a degree sum condition for the existence of spanning $k$-trees.

**Theorem 1** (Win [6]) Let $k \geq 2$ be an integer, and let $G$ be a connected graph. If $\sum_{v \in S} d_G(v) \geq |V(G)| - 1$ for every independent set $S$ in $G$ of order $k$, then $G$ has a spanning $k$-tree.

Recently, Kano and Kishimoto [4] considered a closure concept for spanning $k$-trees, and proved the following theorem.

**Theorem 2** (Kano and Kishimoto [4]) Let $k \geq 2$ be an integer, and let $G$ be an $m$-connected graph. Let $u$ and $v$ be two nonadjacent vertices of $G$. Suppose that $d_G(u) + d_G(v) \geq |V(G)| - m(k - 2) - 1$. Then $G$ has a spanning $k$-tree if and only if $G + uv$ has a spanning $k$-tree.

In this paper, we give a closure result which implies the above theorems as corollaries.

**Theorem 3** Let $k \geq 2$ be an integer, and let $G$ be a connected graph. Let $u$ and $v$ be two nonadjacent vertices of $G$. Suppose that $\sum_{w \in S} d_G(w) \geq |V(G)| - 1$ for every independent set $S$ in $G$ of order $k$ such that $u, v \in S$. Then $G$ has a spanning $k$-tree if and only if $G + uv$ has a spanning $k$-tree.

We now show that a graph satisfying the condition of Theorem 2 also satisfies that of Theorem 3.

**Proof of Theorem 2** Assume that $G$ is an $m$-connected graph and satisfies $d_G(u) + d_G(v) \geq |V(G)| - m(k - 2) - 1$ for some $u, v \in V(G)$ with $uv \notin E(G)$. Since $|V(G)| - m(k - 2) - 1 \geq |V(G)| - \delta(G)(k - 2) - 1 \geq |V(G)| - \sum_{w \in T} d_G(w) - 1$ for every independent set $T \subseteq V(G) \setminus \{u, v\}$ of order $k - 2$, we have $\sum_{w \in S} d_G(w) \geq |V(G)| - 1$ for every independent set $S \subseteq V(G)$ of order $k$ such that $u, v \in S$. Hence $G$ satisfies the condition of Theorem 3.

\[ \square \]

2 Proof of Theorem 3

We prove a slightly stronger theorem than Theorem 3. For a graph $G$ and $S \subseteq V(G)$ with $|S| \geq k$, let $\Delta_k(S; G) := \max \{ \sum_{x \in X} d_G(x) : X \text{ is a subset of } S \text{ of order } k \}$. If there is no confusion, then we abbreviate $\Delta_k(S; G)$ to $\Delta_k(S)$.

**Theorem 4** Let $k \geq 2$ be an integer, and let $G$ be a connected graph. Let $u$ and $v$ be two nonadjacent vertices of $G$. Suppose that there exists no independent set of order $k + 1$ containing both $u$ and $v$, or $\Delta_k(S) \geq |V(G)| - 1$ for every independent set $S$ in $G$ of order $k + 1$ such that $u, v \in S$. Then $G$ has a spanning $k$-tree if and only if $G + uv$ has a spanning $k$-tree.

1. The degree condition of Theorem 4 is best possible in the following sense.

Let $G$ be a complete bipartite graph $K_{n,n(k-1)+2}$ with partite sets $X$ and $Y$ such that $|X| = n$ and $|Y| = n(k - 1) + 2$, where $n \geq 1$ and $k \geq 2$. Let $u$ and $v$ be
two vertices of $Y$. Then $\Delta_k(S) = nk = |V(G)| - 2$ for every independent set $S$ of order $k + 1$ such that $u, v \in S$, and $G + uv$ has a spanning $k$-tree. But $G$ has no spanning $k$-tree, because if $G$ has a spanning $k$-tree $T$, then $|V(G)| - 1 = |V(T)| - 1 = |E(T)| \leq k|X| = kn = |V(G)| - 2$, a contradiction.

2. The closure $cl^{\Delta}(G)$ obtained from Theorem 4 is well-defined. Let $G_1$ and $G_2$ be graphs obtained from $G$ by recursively joining pairs of nonadjacent vertices which satisfy the condition of Theorem 4 until there exists no such a pair. Let $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ be the sequences of edges added to $G$ in obtaining $G_1$ and $G_2$, respectively. Suppose that $e_1, e_2, \ldots, e_l \in E(G_2)$ and $e_{l+1} \notin E(G_2)$. Let $e_{l+1} := uv$ and $H := G + e_1 + \cdots + e_l$. Then, by the definition of $G_2$, there exists an independent set $S$ in $G_2$ of order $k + 1$ such that $u, v \in S$ and $\Delta_k(S; G_2) \leq |V(G_2)| - 2 = |V(G)| - 2$. Since $H$ is a subgraph of $G_2$, $S$ is an independent set in $H$ and $\Delta_k(S; G_2) \geq \Delta_k(S; H)$. By the choice of $e_{l+1}$, we have $\Delta_k(S; H) \geq |V(H)| - 1 = |V(G)| - 1$. Hence $|V(G)| - 2 \geq \Delta_k(S; G_2) \geq \Delta_k(S; H) \geq |V(G)| - 1$, a contradiction. Hence $e_1, e_2, \ldots, e_m \in E(G_2)$. Similarly, we can obtain $f_1, f_2, \ldots, f_n \in E(G_1)$. This implies that $G_1 = G_2$, and so $cl^{\Delta}(G)$ is well-defined.

3. Theorem 4 implies a result due to Neumann-Lara and Rivera-Campo. Neumann-Lara and Rivera-Campo [5] obtained an independence number condition for the existence of spanning $k$-trees. (In fact, they proved a stronger result as we mention in Sect. 3.)

**Theorem 5** (Neumann-Lara and Rivera-Campo [5]) Let $k \geq 2$ be an integer, and let $G$ be a connected graph. If there exists no independent set of order $k + 1$, then $G$ has a spanning $k$-tree.

If a graph $G$ satisfies the hypothesis of Theorem 5, then $cl^{\Delta}(G)$ is complete, and hence Theorem 4 implies Theorem 5.

**Proof of Theorem 4** For a subgraph $H$ of a graph $G$ and a vertex $v \in V(H)$, we denote the set of neighbors of $v$ in $H$ by $N_H(v)$, and let $d_H(v) := |N_H(v)|$.

If $G$ has a spanning $k$-tree, then trivially also $G + uv$ has a spanning $k$-tree. Hence we prove the converse.

Suppose that $G + uv$ has a spanning $k$-tree $T$ and $G$ does not have a spanning $k$-tree. Then $T - uv$ consists of two trees $T_1$ and $T_2$ such that $u \in V(T_1)$ and $v \in V(T_2)$. Note that for $i = 1, 2$, $T_i$ is a $k$-tree in $G$, and $d_{T_1}(w) = d_{T}(w)$ for $w \in V(T_i) \setminus \{u, v\}$, $d_{T_1}(u) \leq k - 1$ and $d_{T_2}(v) \leq k - 1$. Since $G$ is a connected graph, there exist $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$ with $w_1w_2 \in E(G)$. Choose $w_1$ and $w_2$ such that $d_{T_1}(w_1) + d_{T_2}(w_2)$ is as small as possible. Since $G$ does not have a spanning $k$-tree, it follows that for some $i = 1, 2$, there exists no $k$-tree $S_i$ such that $V(S_i) = V(T_i)$ and $d_{S_i}(w_i) \leq k - 1$. Without loss of generality, we may assume that there exists no $k$-tree $S_1$ such that $V(S_1) = V(T_1)$ and $d_{S_1}(w_1) \leq k - 1$. (1)

Hence we have $d_{T_1}(w_1) = k$. Then $w_1 \neq u$ because $d_{T_1}(u) \leq k - 1$.

Let $T_3 := T_1 \cup T_2 + w_1w_2$ and let $F_0, \ldots, F_k$ be $k + 1$ components of $T_3 - w_1$. Since $F_i$ is a tree, there exists a vertex $x_i$ of $F_i$ with $d_{T_1 \cup T_2}(x_i) \leq k - 1$ for $0 \leq i \leq k$. Let $X := \{x_0, x_1, \ldots, x_k\}$. We can choose $X$ so that $u, v \in X$, because $d_{T_1}(u) \leq k - 1$ and

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Suppose that \( d_T(v) \leq k - 1 \). Without loss of generality, we may assume that \( d_G(x_0) = \min\{d_G(x_i) : 0 \leq i \leq k\} \). Let \( \{z_i\} := N_{T_i}(w_1) \cap V(F_i) \) for each \( 0 \leq i \leq k \). We regard \( F_0 \) as a rooted tree with root \( z_0 \) and \( F_i \) as a rooted tree with root \( x_i \) for \( 1 \leq i \leq k \).

**Claim 1** Let \( i, j \) be integers with \( 0 \leq i \neq j \leq k \). Then \( d_{T_1 \cup T_2}(y) = k \) for all \( y \in N_G(x_i) \cap V(F_j) \).

**Proof** Suppose that \( d_{T_1 \cup T_2}(y) \leq k - 1 \) for some \( y \in N_G(x_p) \cap V(F_q) \), where \( p, q \) are integers with \( 0 \leq p \neq q \leq k \). If \( v \in \{x_p, x_q\} \), then \( T' := T_1 \cup T_2 + x_p y \) is a spanning \( k \)-tree in \( G \), a contradiction. Hence \( v \not\in \{x_p, x_q\} \). Then \( S_1 := T_1 - w_1 z_q + x_p y \) is a \( k \)-tree with \( V(S_1) = V(T_1) \) and \( d_{S_1}(w_1) = k - 1 \). This contradicts (1). \( \square \)

By Claim 1 and the choice of \( x_0 \), we obtain the following.

**Claim 2** \( X \) is an independent set in \( G \), and \( \Delta_k(X) = \sum_{i=1}^k d_G(x_i) \).

We define

\[
Y_j := \bigcup_{1 \leq i \neq j \leq k} \left( N_G(x_i) \cap V(F_j) \right) \quad \text{for } 1 \leq j \leq k
\]

and

\[
Y_0 := \bigcup_{1 \leq i \leq k-1} \left( N_G(x_i) \cap V(F_0) \right).
\]

For \( 0 \leq i \leq k \) and \( z \in V(F_i) \), we denote the parent and the children of \( z \) in \( F_i \) by \( z^- \) and \( ch(z) \), respectively and we let \( Y_i^+ := \bigcup_{y \in Y_i} ch(y) \).

**Claim 3** \( Y_i^+ \cap N_G(x_i) = \emptyset \) for each \( 1 \leq i \leq k \), and \( Y_0^+ \cap N_G(x_k) = \emptyset \).

**Proof** First, suppose that there exists \( y \in Y_p^+ \cap N_G(x_p) \) for some \( 1 \leq p \leq k \). Then \( y^- \in N_G(x_q) \) for some \( 1 \leq q \neq p \leq k \). If \( v \in \{x_p, x_q\} \), then \( T_1 \cup T_2 - y^- + x_p y + x_q y^- \) is a spanning \( k \)-tree in \( G \), a contradiction. Otherwise, \( S_1 := T_1 - y^- - w_1 z_p + x_p y + x_q y^- \) is a \( k \)-tree and \( d_{S_1}(w_1) = k - 1 \). This contradicts (1). Next, suppose that there exists \( y \in Y_0^+ \cap N_G(x_k) \). Then \( y^- \in N_G(x_r) \) for some \( 1 \leq r \leq k - 1 \). If \( v \in \{x_0, x_r\} \), then \( T_1 \cup T_2 - y^- + x_k y + x_r y^- \) is a spanning \( k \)-tree in \( G \), a contradiction. Assume that \( x_k = v \). Then \( x_k \in V(T_2) \) and \( y \in V(T_1) \), and the minimality of \( d_{T_1}(w_1) + d_{T_2}(w_2) \) and \( d_{T_1}(y) + d_{T_2}(x_k) \leq k + k - 1 \) yields that \( d_{T_2}(w_2) \leq k - 1 \). Therefore \( T_3 = w_1 z_0 - y^- + x_k y + x_r y^- \) is a spanning \( k \)-tree in \( G \), a contradiction. If \( v \not\in \{x_0, x_r, x_k\} \), then \( S'_1 := T_1 - w_1 z_0 - y^- + x_k y + x_r y^- \) is a \( k \)-tree with \( V(S'_1) = V(T_1) \) and \( d_{S'_1}(w_1) = k - 1 \). This contradicts (1).

**Claim 4** \( z_i \notin N_G(x_j) \) for each \( 0 \leq i \neq j \leq k \).

**Proof** Suppose that \( z_p \in N_G(x_q) \) for some \( 0 \leq p \neq q \leq k \). Assume that \( x_p = v \). Then \( z_p = w_2 \) and the minimality of \( d_{T_1}(w_1) + d_{T_2}(w_2) \) yields that \( k + d_{T_2}(w_2) = d_{T_1}(w_1) + d_{T_2}(w_2) \leq d_{T_1}(x_q) + d_{T_2}(z_p) \leq k - 1 + d_{T_2}(w_2) \), a contradiction. Assume
that \( x_q = v \). Then note that \( d_G(w_2) \leq k - 1 \) by the choice of \( w_1 \) and \( w_2 \). Thus, \( T_3 - z_p w_1 + x_q z_p \) is a spanning \( k \)-tree in \( G \), a contradiction. If \( v \notin \{ x_p, x_q \} \), then \( S_1 := T_1 - z_p w_1 + x_q z_p \) is a \( k \)-tree with \( V(S_1) = V(T_1) \) and \( d_{S_1}(w_1) = k - 1 \), which contradicts (1).

**Claim 5** \(| Y_i^+ | = (k - 1)| Y_i | \) for each \( 0 \leq i \leq k \).

**Proof** By Claim 4, \( z_i \notin Y_i \) for all \( 0 \leq i \leq k \), and hence \( d_{F_i}(y) = d_{T_1 \cup T_2}(y) \) for all \( y \in Y_i \). It follows from Claim 1 that \( |ch(y)| = d_{F_i}(y) - 1 = k - 1 \) for all \( y \in Y_i \).

Since \( F_i \) is a tree, \( ch(y_1) \cap ch(y_2) = \emptyset \) for every \( y_1, y_2 \in Y_i \) with \( y_1 \neq y_2 \). Therefore we obtain
\[
|Y_i^+| = \sum_{y \in Y_i} |ch(y)| = (k - 1)|Y_i| \text{ for each } 0 \leq i \leq k.
\]

By Claims 3–5, for \( 1 \leq h \leq k \), we obtain
\[
|N_G(x_h) \cap V(F_h)| \leq |V(F_h)| - |\{x_h\}| - |Y_h^+| \\
= |V(F_h)| - 1 - (k - 1)|Y_h| \\
\leq |V(F_h)| - 1 - \sum_{1 \leq i \leq k, i \neq h} |N_G(x_i) \cap V(F_h)|
\]
and
\[
|N_G(x_k) \cap V(F_0)| \leq |V(F_0)| - |\{z_0\}| - |Y_0^+| \\
= |V(F_0)| - 1 - (k - 1)|Y_0| \\
\leq |V(F_0)| - 1 - \sum_{1 \leq i \leq k - 1} |N_G(x_i) \cap V(F_0)|.
\]

Therefore we deduce that
\[
\sum_{i=1}^{k} |N_G(x_i) \cap V(F_j)| \leq |V(F_j)| - 1 \text{ for each } 0 \leq j \leq k. \tag{2}
\]

Since \( d_G(x_i) \leq |\{w_1\}| + \sum_{j=0}^{k} |N_G(x_i) \cap V(F_j)| \) for each \( 1 \leq i \leq k \), it follows from the inequality (2) that
\[
\Delta_k(X) = \sum_{i=1}^{k} d_G(x_i) \\
\leq \sum_{i=1}^{k} \left( |\{w_1\}| + \sum_{j=0}^{k} |N_G(x_i) \cap V(F_j)| \right) \\
\leq k + \sum_{j=0}^{k} (|V(F_j)| - 1) \\
\leq |V(G)| - 2,
\]
a contradiction.
3 Problem

In this section, we propose a problem concerning a closure involving the independence number and the connectivity. Let $\alpha(G)$ and $\kappa(G)$ be the independence number and the connectivity of $G$, respectively. Neumann-Lara and Rivera-Campo [5] obtained the following result.

**Theorem 6** (Neumann-Lara and Rivera-Campo [5]) Let $k \geq 2$ be an integer, and let $G$ be a graph. If $\alpha(G) \leq (k - 1)\kappa(G) + 1$, then $G$ has a spanning $k$-tree.

We can consider the following problem as a closure result for Theorem 6. For a graph $G$ and $u, v \in V(G)$ with $uv \not\in E(G)$, let $\alpha(u, v; G)$ be the cardinality of a maximum independent set containing $u$ and $v$. For a graph $G$ and $u, v \in V(G)$, the local connectivity $\kappa(u, v; G)$ is defined to be the maximum number of internally-disjoint paths connecting $u$ and $v$ in $G$.

**Problem 7** Let $k \geq 2$ be an integer, and let $G$ be a graph. Let $u$ and $v$ be two nonadjacent vertices of $G$. Assume that $\alpha(u, v; G) \leq (k - 1)\kappa(u, v; G) + 1$. Then $G$ has a spanning $k$-tree if and only if $G + uv$ has a spanning $k$-tree.

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