In the present paper, we investigate blow-up and lifespan estimates for a class of semilinear hyperbolic coupled system in $\mathbb{R}^n$ with $n \geq 1$, which is a part of the so-called Nakao's type problem weakly coupled a semilinear damped wave equation with a semilinear wave equation with nonlinearities of derivative type. By constructing two time-dependent functionals and employing an iteration method for unbounded multiplier with slicing procedure, the results of blow-up and upper bound estimates for the lifespan of energy solutions are derived. The model seems to be hyperbolic-like instead of parabolic-like. Particularly, the blow-up result for one-dimensional case is optimal.

**KEYWORDS**

blow-up, damped wave equation, lifespan estimate, semilinear hyperbolic system, wave equation

**MSC CLASSIFICATION**

Primary 35L52; Secondary 35B44

1 | INTRODUCTION

The problem of critical curve, which describes the threshold condition between global (in time) existence of small data weak solutions and blow-up of small data weak solutions, of the power exponents for the weakly coupled system of wave equations and damped wave equations was proposed by Professor Mitsuhiro Nakao, Emeritus of Kyushu University (see, for instance, Nishihara and Wakasugi1,2), namely,

\[
\begin{align*}
    u_{tt} - \Delta u + u_t &= f_1(v, v_t), & x \in \mathbb{R}^n, t > 0, \\
    v_{tt} - \Delta v &= f_2(u, u_t), & x \in \mathbb{R}^n, t > 0, \\
    (u, u_t, v, v_t)(0, x) &= (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{align*}
\]

(1)

where the nonlinearities on the right-hand sides are given by the mixture of the power-type and derivative-type nonlinearities

\[
\begin{align*}
    f_1(v, v_t) &:= d_1|v|^{p_1} + d_2|v_t|^{p_2}, \\
    f_2(u, u_t) &:= d_3|u|^{q_1} + d_4|u_t|^{q_2},
\end{align*}
\]

carrying some nonnegative constants $d_1, \ldots, d_4$ and $p_1, p_2, q_1, q_2 > 1$. Here, to guarantee the hyperbolic coupled system (1) being a nonlinear problem, we have to restrict ourselves that the coefficients satisfy $d_1 + d_2 \neq 0$ and $d_3 + d_4 \neq 0$. Roughly speaking, the main difficulty to treat Nakao’s type problem is to understand varying degrees of influence from damped...
wave equations and wave equations. It is well known that $L^2$ decay properties and diffusion phenomenon hold in linear damped wave equations due to the frictional damping $u_t$. However, these effects disappear in linear wave equations, and Huygens' principle and energy conservation are valid in linear wave equations, which make huge differences of the treatments between semilinear wave equations and semilinear damped wave equations. In other words, Nakao's type problem (1) bridges a connection between semilinear damped wave equations and semilinear wave equations through a coupled system with nonlinearities of power type. We emphasize that the critical condition for the Cauchy problem (1) is still an open problem for any $d_1 + d_2 \neq 0$ and $d_3 + d_4 \neq 0$, although some partial results (viz., blow-up results) have been proved in the last years.

Indeed, some blow-up results for Nakao's type problem with power nonlinearities, namely,

$$\begin{align*}
    u_t - \Delta u + u_t &= |v|^p, \quad x \in \mathbb{R}^n, \; t > 0, \\
    v_t - \Delta v &= |u|^q, \quad x \in \mathbb{R}^n, \; t > 0, \\
    (u, v, v_t)(0, x) &= (u_0, v_0)(x), \quad x \in \mathbb{R}^n,
\end{align*}$$

the special case of the hyperbolic coupled system (1) with $d_1 = d_3 = 1, d_2 = d_4 = 0$ and $p_1 = p, q_1 = q$, have been derived in Wakasugi and Reissig.\textsuperscript{2,3} With the aim of guaranteeing existence of local (in time) solutions, let us generally assume $p, q \leq n/(n - 2)$ if $n \geq 3$ in the discussion of this paragraph. Firstly, by using a test function method, Wakasugi\textsuperscript{2} demonstrated the blow-up of local (in time) weak solutions with suitable assumption on initial data providing that

$$\max \left\{ \frac{q/2 + 1}{pq - 1} + \frac{1}{2}, \frac{q + 1}{pq - 1}, \frac{p + 1}{pq - 1} \right\} \geq \frac{n}{2}. \tag{3}$$

Condition (3) is optimal in $n = 1$ since it is equivalent to $1 < p, q < \infty$. Later, Wakasugi and Reissig\textsuperscript{3} employed an iteration method associated with slicing procedure to improve the blow-up condition (3) for $n \geq 2$. Precisely, the result in Wakasugi and Reissig\textsuperscript{3} partially improved for $n = 2, 3$ and completely improved for $n \geq 4$ such that if

$$a_N(p, q) := \max \left\{ \frac{q/2 + 1}{pq - 1}, \frac{2 + p^{-1}}{pq - 1}, \frac{1/2 + p}{pq - 1}, \frac{1}{2} \right\} > \frac{n - 1}{2}, \tag{4}$$

then every nonzero local (in time) energy solution blows up in finite time. In other words, Wakasugi and Reissig\textsuperscript{3} observed that Nakao's type problem with power nonlinearities is hyperbolic-like model rather than parabolic-like model due to fact that the component

$$\frac{2 + p^{-1}}{pq - 1} > \frac{n - 1}{2}$$

plays a dominant role when $n \geq 3$. This effect comes from the semilinear wave equations. Actually, the corresponding linearized system to (2) is hyperbolic model. Here, we stress that the nomenclatures of hyperbolic-like and parabolic-like refer to the conditions on the powers in the nonlinear term. Precisely, the parabolic-like model means the derived results are related or closed to those for the semilinear heat model, and the hyperbolic-like model means the derived results are related or closed to those for the semilinear wave model. Therefore, an interesting and viable problem is to ask the situation of nonlinearities of derivative type, that is, the hyperbolic coupled system (1) with $d_1 = d_3 = 0, d_2 = d_4 = 1$ and $p_2 = p, q_2 = q$. At this time, one may notice that the time derivative of solution not only exists in the linear part (the damped wave equation) but also appears in the nonlinear parts of both equations. We would like to understand:

Do the nonlinear terms including the time derivative of solutions change the model from hyperbolic-like (i.e., as in the nonlinear problem (2)) to parabolic-like?

We will give a possible answer from the blow-up point of view that Nakao's type problem with nonlinearities of derivative type still could be hyperbolic-like model.
In this paper, we study blow-up of solutions and lifespan estimates from the above for Nakao's type problem with derivative-type nonlinearities, namely,

\[
\begin{align*}
    u_{tt} - \Delta u + u_t &= |v_t|^p, & x \in \mathbb{R}^n, t > 0, \\
    v_{tt} - \Delta v &= |v|^q, & x \in \mathbb{R}^n, t > 0, \\
    (u, u_t, v, v_t)(0, x) &= \epsilon (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n.
\end{align*}
\]

(5)

where \( p, q > 1 \) and \( \epsilon \) is a positive parameter describing the size of initial data. As we will show in Theorem 1, the blow-up condition of Nakao's type problem (5) is strongly related to the Glassey exponent, which is the critical exponent for the semilinear wave equations with derivative-type nonlinearity (see Hidano et al. for a detailed introduction to this topic). The approach to derive our result consists in setting suitable time-dependent functionals related to a local solution and in applying an iteration method to establish lower bound estimates for these functionals. In particular, the integral iteration frame involves an unbounded exponential multiplier, whose handling will be possible through a slicing procedure of the domain of integration.

Let us recall some results for semilinear wave equations and semilinear damped wave equations, which are strongly related to Nakao's type problem (5). Concerning the weakly coupled system of semilinear wave equations

\[
\begin{align*}
    u_{tt} - \Delta u &= |v_t|^p, & x \in \mathbb{R}^n, t > 0, \\
    v_{tt} - \Delta v &= |u|^q, & x \in \mathbb{R}^n, t > 0, \\
    (u, u_t, v, v_t)(0, x) &= (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n.
\end{align*}
\]

(6)

the critical curve is given by

\[
\alpha_w(p, q) := \frac{\max\{p, q\} + 1}{pq - 1} = \frac{n - 1}{2}.
\]

(7)

Particularly, under certain integral sign assumptions for initial data, if \( \alpha_w(p, q) \geq (n - 1)/2 \), then every nontrivial local (in time) solution \((u, v)\) blows up in finite time. Considering the critical curve (7), we refer to the related works. Taking our consideration of the special case \( p = q \), the critical exponent, the so-called Glassey exponent, is given by

\[
p_{Glas}(n) := \begin{cases} 
    \infty & \text{if } n = 1, \\
    \frac{n+1}{n-1} & \text{if } n \geq 2,
\end{cases}
\]

which is also the critical exponent for the single semilinear wave equation with nonlinearity \( |u_t|^p \). One may see the criticality of the Glassey exponent for the single semilinear wave equation with nonlinearity of derivative type in several studies and references therein. Next, we turn to the weakly coupled system of semilinear classical damped wave equations with nonlinearities of derivative type as follows:

\[
\begin{align*}
    u_{tt} - \Delta u + u_t &= |v_t|^p, & x \in \mathbb{R}^n, t > 0, \\
    v_{tt} - \Delta v + v_t &= |u|^q, & x \in \mathbb{R}^n, t > 0, \\
    (u, u_t, v, v_t)(0, x) &= (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n.
\end{align*}
\]

(8)

To the knowledge of the author, results either for the global (in time) existence or for the blow-up of solutions to (8) are still unknown. However, motivated by the previous study, we conjecture that the solution of the weakly coupled system (8) does not blow up for any \( p, q > 1 \). This effect also appears in the wave equations with scale-invariant damping in the effective case (see among other things in Theorem 2.2 of Palmieri and Tu by letting parameters \( \mu_1, \mu_2 \rightarrow \infty \)). Thus, the consideration of Nakao's type problem (5) is reasonable.

**Notation:** We give some notations used in this paper. We write \( f \preceq g \) when there exists a positive constant \( C \) such that \( f \leq Cg \). We denote by \( \lceil r \rceil := \min\{C \in \mathbb{Z} : r \leq C\} \) the ceiling function. Moreover, \( B_R \) denotes the ball around the origin with radius \( R \) in \( \mathbb{R}^n \).
2 | MAIN RESULT

Let us first introduce a suitable definition of energy solutions of Nakao’s type problem (5).

**Definition 1.** Let \((u_0, u_1, v_0, v_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \times (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))\). One may say that \((u, v)\) is an energy solution of Nakao’s type problem (5) on \([0, T]\) if

\[
\begin{align*}
&u \in C([0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n)) \text{ and } u_t \in L^q_{\text{loc}}([0, T) \times \mathbb{R}^n), \\
v \in C([0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n)) \text{ and } v_t \in L^p_{\text{loc}}([0, T) \times \mathbb{R}^n),
\end{align*}
\]

satisfies \((u, v)(0, \cdot) = \epsilon(u_0, v_0)\) in \(H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)\) and the following integral relations:

\[
\begin{align*}
\int_0^t \int_{\mathbb{R}^n} (-u_t(s, x)\phi(s, x) + u_t(s, x)\phi(s, x) + \nabla u(s, x) \cdot \nabla \phi(s, x))dxds &+ \int_{\mathbb{R}^n} u_1(t, x)\phi(t, x)dx - \epsilon \int_{\mathbb{R}^n} u_1(x)\phi(0, x)dx = \int_0^t \int_{\mathbb{R}^n} |v_t(s, x)|^p\phi(s, x)dxds \\
&= T(0, t)
\end{align*}
\]

and

\[
\begin{align*}
\int_0^t \int_{\mathbb{R}^n} (-v_t(s, x)\psi(s, x) + \nabla v(s, x) \cdot \nabla \psi(s, x))dxds &+ \int_{\mathbb{R}^n} v_1(t, x)\psi(t, x)dx - \epsilon \int_{\mathbb{R}^n} v_1(x)\psi(0, x)dx = \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^q\psi(s, x)dxds \\
&= T(0, t)
\end{align*}
\]

hold for any test functions \(\phi, \psi \in C_0^\infty([0, T) \times \mathbb{R}^n)\) and any \(t \in (0, T)\).

**Remark 1.** Actually, similarly to treatments in Wakasugi and Reissig,\(^1\) by applying further steps of integration by parts in (9) as well as (10), respectively, and taking \(t \to T\), we may claim that \((u, v)\) fulfills the definition of weak solutions of Nakao’s type problem (5).

If we assume compactly supported data in a ball with radius \(R\), from Banach’s fixed-point theorem and Duhamel’s principle associated with some \(L^2\) estimates for solutions of the corresponding linear Cauchy problem to (5), one may derive local (in time) existence of weak solutions with compact support localized in a ball with radius \(R + r\) of Nakao’s type problem (5) if \(p, q > 1\) for \(n = 1, 2\), and \(1 < p, q \leq n/(n - 2)\) for \(n \geq 3\).

Let us state the blow-up result for Nakao’s type problem (5).

**Theorem 1.** Let us consider the exponents \(p, q > 1\) such that

\[
pq < \begin{cases} \infty & \text{if } n = 1, \\ \frac{2}{p_G(n)} & \text{if } n \geq 2. \end{cases}
\]

Assume that \((u_0, u_1, v_0, v_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \times (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))\) are nonnegative and compactly supported functions with supports contained in \(B_R\) for some \(R > 0\) such that \(u_t, v_t\) are not identically zero. Let \((u, v)\) be the local (in time) energy solution of Nakao’s type problem (5) according to Definition 1 with the lifespan \(T = T(\epsilon)\). Then, these solutions satisfy

\[
\supp u, \supp v \subset \{(t, x) \in [0, T) \times \mathbb{R}^n : |x| \leq R + t\}.
\]

Moreover, there exists a positive constant \(\epsilon_0 = \epsilon_0(u_0, u_1, v_0, v_1, p, q, n, R)\) such that for any \(\epsilon \in (0, \epsilon_0]\), the energy solution \((u, v)\) blows up in finite time. In addition, the upper bound estimate for the lifespan

\[
T(\epsilon) \leq C\epsilon^{-\frac{2}{(n+1)(n+1-pq)}}
\]

holds, where \(C > 0\) is a constant independent of \(\epsilon\).
Remark 2. Due to the blow-up condition \( pq < (n+1)/(n-1) \) for \( n \geq 2 \), it is trivial that \( p, q < n/(n-2) \) for \( n \geq 3 \). In other words, under the condition of exponents \( p \) and \( q \) in Theorem 1, the weak solution having compact support in \( B_{R+t} \) of Nakao’s type problem (5) locally (in time) exists.

Remark 3. In the one-dimensional case, the energy solution of Nakao’s type problem (5) blows up for all \( 1 < p, q < \infty \), which implies the optimality for \( n = 1 \). However, for the high-dimensional cases when \( n \geq 2 \), the critical curve in the \( p-q \) plane for Nakao’s type problem (5) is open due to the lack of global existence result for small data solutions.

Remark 4. Due to the fact that

\[
\left\{ (p, q) : pq < \frac{n+1}{n-1} \right\} \subset \left\{ (p, q) : \frac{1}{pq-1} > \frac{n-1}{2} \right\} \subset \left\{ (p, q) : \alpha_n(p, q) > \frac{n-1}{2} \right\}
\]

for any \( n \geq 2 \), we may claim that the nonlinear terms including time derivative of solutions, that is, \((|v|^{p}, |u|^{q})^{T}\), will weaken the blow-up range of exponents for Nakao’s type model in the \( p-q \) plane in comparison with the usual nonlinear terms \((|v|^{p}, |u|^{q})^{T}\). However, since the blow-up region for (5) that we determine in Theorem 1 is smaller in comparison to the corresponding one for (2), Nakao’s type problem with nonlinearities of derivative type still behaves as a hyperbolic-like model whose reason will be shown later. For this reason, combining the explanations of Theorem 1 and those in the recent paper,\(^3\) we may conjecture that for the general Nakao’s type problem (1) that is not necessarily with both nonlinear terms either of power nonlinearity type or of derivative type, the model still has a behavior of hyperbolic-type.

2.1 Hyperbolic-like versus parabolic-like

In this part, let us give some remarks and explanations on the blow-up conditions for energy solutions of Nakao’s type problem (5) with respect to the exponents \( p, q \) for \( n = 2 \) and \( n \geq 3 \), respectively. In the forthcoming discussion, we assume \( p, q > 1 \).

According to Figure 1, we may observe that

\[
\left\{ (p, q) : pq < p_{\text{Gla}}(n) = \frac{n+1}{n-1} \right\} \subset \left\{ (p, q) : \alpha_{\text{W}}(p, q) = \frac{\max\{p, q\} + 1}{pq - 1} > \frac{n-1}{2} \right\}
\]

(13)

for any \( n \geq 2 \). Again, \( \alpha_{\text{W}}(p, q) = (n-1)/2 \) is the critical curve in the \( p-q \) plane for the weak coupled system (6). The effect (13) is caused by the influence of friction \( u_{t} \) on the first equation of the Cauchy problem (5). For the reason of the blow-up condition \( pq < p_{\text{Gla}}(n) \), we feel that Nakao’s type problem (5) is of hyperbolic-like (wave behavior) rather than of parabolic-like.

2.2 The shifted magnitude of the curve for blow-up condition

We observe an interesting effect in the blow-up conditions for Nakao’s type problem, namely, the shifted curve with a certain magnitude from the weakly coupled system of wave equations with corresponding nonlinearities. Later, we assume \( p, q \leq n/(n-2) \) if \( n \geq 3 \) and \( p, q > 1 \) for any \( n \geq 1 \) to guarantee local (in time) existence of solutions.

Let us first consider the blow-up result for Nakao’s type problem (2) with power nonlinearities. Wakasugi and Reissig\(^3\) proved blow-up of energy solutions for \( n \geq 3 \) if the exponents \( p, q \) satisfy

\[
\frac{2 + p^{-1}}{pq - 1} > \frac{n-1}{2},
\]

or it can be rewritten by

\[
\frac{q + 2 + p^{-1}}{pq - 1} - \frac{n-1}{2} > \frac{q}{pq - 1} = \alpha_{\text{shift}}(p, q).
\]
The blow-up condition in the subcritical case for the weakly coupled system of wave equations with power nonlinearities is

\[
\frac{q + 2 + p^{-1}}{pq - 1} - \frac{n - 1}{2} > 0.
\]

In other words, we may explain the blow-up condition for Nakao's type problem (2) by a shifted curve with the magnitude \( \alpha_{\text{shift}}(p, q) \) of the weakly coupled system of wave equations with power nonlinearities.

Remarkably, this effect with same magnitude still holds for Nakao's type problem with nonlinearities of derivative type. Let us turn to the blow-up result for the coupled system (5). Our main result in Theorem 1 for \( n \geq 2 \) claims that the blow-up condition is

\[
\frac{1}{pq - 1} > \frac{n - 1}{2},
\]

or, equivalently,

\[
\frac{q + 1}{pq - 1} - \frac{n - 1}{2} > \frac{q}{pq - 1} = \alpha_{\text{shift}}(p, q).
\]

Therefore, we still may explain the blow-up condition for Nakao's type problem (2) by a shifted magnitude \( \alpha_{\text{shift}}(p, q) \) of the weakly coupled system of wave equations with nonlinearities of derivative type.

3 | PROOF OF MAIN RESULT VIA AN ITERATION ARGUMENT

3.1 | Iteration frame

In order to apply an iteration argument in the proof, we need to derive integral inequalities for some suitable time-dependent functions. To begin with, let us introduce the eigenfunction \( \Phi = \Phi(x) \) of the Laplace operator in \( n \)-dimensions Euclidean space such that

\[
\Phi(x) := \begin{cases} e^x + e^{-x} & \text{if } n = 1, \\ \int_{\mathbb{S}^{n-1}} e^{x \omega} d\sigma_{\omega} & \text{if } n \geq 2, \end{cases}
\]
where $S^{n-1}$ is the $(n-1)$-dimensional sphere. This test function $\Phi$ has been introduced in the pioneering paper.\textsuperscript{22} It fulfills
the property $\Delta \Phi = \Phi$ and the asymptotic behavior
\[ \Phi(x) \sim |x|^{-\frac{n+1}{2}} e^{|x|} \text{ as } |x| \to \infty. \] \hfill \text{(14)}

Moreover, we define the test function $\Psi = \Psi(t,x)$ with separate variables such that $\Psi(t,x) := e^{-t} \Phi(x)$. Clearly, the function $\Psi$ is a special solution of the homogeneous wave equation $\Psi_t - \Delta \Psi = 0$. By using the asymptotic behavior in (14), it immediately yields the estimate
\[ \int_{R^+} \Psi(t,x)dx \leq C_1 (R + t)^{-\frac{n+1}{2}} \] \hfill \text{(15)}
for any $t \geq 0$, where $C_1$ is a positive constant. The previous estimate (15) was shown in Lai and Takamura.\textsuperscript{13}

To construct the iteration frame, it is necessary for us to introduce some functionals with respect to $u_t$ and $v_t$ due to the derivative-type nonlinearities of the hyperbolic coupled system (5). With the aid of the above test function $\Psi$, we may define new time-dependent functionals $F_1 = F_1(t)$ and $F_2 = F_2(t)$ such that
\[ F_1(t) := \int_0^t \int_{R^+} u_t(s,x)\Psi(s,x)dxds \text{ and } F_2(t) := \int_{R^+} v_t(t,x)\Psi(t,x)dx. \] \hfill \text{(16)}

Here, we should emphasize that $F_1'(t)$ has an analogous form as $F_2(t)$, which is beneficial to process the iteration procedure later.

Due to the fact that $u, v$ are supported in a forward cone $\{(s,x) \in [0, t] \times R^n : |x| \leq R + s\}$, we can apply the definition of energy solution $(u, v)$ in the sense of Definition 1 with the test function $\Psi$ in (9) and (10). For one thing, by using integration by parts in (9) with $\phi(t,x) = \Psi(t,x)$, we have
\[ \int_0^t \int_{R^+} u_t(s,x)\Psi(s,x)dxds + \int_{R^+} (u_t(t,x)\Psi(t,x) + u(t,x)\Psi(t,x))dx \]
\[ = \varepsilon \int_{R^+} (u_0(x) + u_1(x))\Phi(x)dx + \int_0^t \int_{R^+} |v_t(s,x)|^\varepsilon \Psi(s,x)dxds, \]
which can also be rewritten by
\[ F_1'(t) + F_1(t) + \int_{R^+} u(t,x)\Psi(t,x)dx = \varepsilon \int_{R^+} (u_0(x) + u_1(x))\Phi(x)dx + \int_0^t \int_{R^+} |v_t(s,x)|^\varepsilon \Psi(s,x)dxds. \] \hfill \text{(17)}

Taking time derivative in the above equality and using $\Psi_t(t,x) = -\Psi(t,x)$ bring
\[ F_1''(t) + 2F_1'(t) - \int_{R^+} u(t,x)\Psi(t,x)dx = \int_{R^+} |v_t(t,x)|^\varepsilon \Psi(t,x)dx. \] \hfill \text{(18)}

Adding up (17) and (18), one may derive
\[ F_1''(t) + 3F_1'(t) + F_1(t) = \varepsilon \int_{R^+} (u_0(x) + u_1(x))\Phi(x)dx + \int_0^t \int_{R^+} |v_t(s,x)|^\varepsilon \Psi(s,x)dxds + \int_{R^+} |v_t(t,x)|^\varepsilon \Psi(t,x)dx. \] \hfill \text{(19)}

For another thing, we employ once integration by parts in (10) with $\psi(t,x) = \Psi(t,x)$ to get
\[ F_2(t) + \int_{R^+} v(t,x)\Psi(t,x)dx = \varepsilon \int_{R^+} (v_0(x) + v_1(x))\Phi(x)dx + \int_0^t \int_{R^+} |u_t(s,x)|^\varepsilon \Psi(s,x)dxds. \]
Similarly to the treatment of $F_1(t)$, differentiating the last equality with respect to $t$ implies
\[ F_2'(t) + F_2(t) - \int_{R^+} v(t,x)\Psi(t,x)dx = \int_{R^+} |u_t(t,x)|^\varepsilon \Psi(t,x)dx. \]
Summarizing the derived equations, one has

\[
F'_2(t) + 2F_2(t) = \epsilon \int_{\mathbb{R}^n} (v_0(x) + v_1(x))\Phi(x)dx + \int_0^t \int_{\mathbb{R}^n} |u_t(s,x)|^q\Psi(s,x)dxds + \int_{\mathbb{R}^n} |u_t(t,x)|^q\Psi(t,x)dx.
\] (20)

With the aim of constructing the iteration frame, we need to transfer (19) and (20) to suitable integral inequalities, respectively. Let us consider (19) initially. We now define a time-dependent functional:

\[
G_1(t) := F'_1(t) + \frac{3 - \sqrt{5}}{2} F_1(t) - \epsilon \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx - \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p\Psi(s,x)dxds.
\]

Remark 5. Let us now explain the choice of suitable constant coefficient \((3 + \sqrt{5})/2\) for \(F_1(t)\) in the definition of \(G_1(t)\). In the view of the left-hand side of (19), motivated by ODE studies, we would like to reduce it as

\[
e^{-\tilde{a}t}(e^{\tilde{a}t}\tilde{G}_1(t))' \quad \text{with} \quad \tilde{G}_1(t) := F'_1(t) + \tilde{b}F_1(t),
\]

where \(\tilde{a}\) and \(\tilde{b}\) are real constants to be determined later. By direct computations

\[
F''_1(t) + 3F'_1(t) + F_1(t) = F''_1(t) + (\tilde{a} + \tilde{b})F'_1(t) + \tilde{a}\tilde{b}F_1(t),
\]

we have \(\tilde{a} + \tilde{b} = 3\) and \(\tilde{a}\tilde{b} = 1\). Obviously, one of the solutions is \(\tilde{a} = (3 - \sqrt{5})/2\) and \(\tilde{b} = (3 + \sqrt{5})/2\). So, the constants in the above expected form have been deduced.

Then, it is obvious from (19) that

\[
G'_1(t) + \frac{3 - \sqrt{5}}{2} G_1(t) = F''_1(t) + 3F'_1(t) + F_1(t) - \frac{(3 - \sqrt{5})\epsilon}{2} \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx
\]

\[
- \int_{\mathbb{R}^n} |v_t(x)|^p\Psi(x)dx - \frac{3 - \sqrt{5}}{2} \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p\Psi(s,x)dxds
\]

\[
= \epsilon \int_{\mathbb{R}^n} u_0(x)\Phi(x)dx + \frac{(\sqrt{5} - 1)\epsilon}{2} \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx + \frac{\sqrt{5} - 1}{2} \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p\Psi(s,x)dxds.
\]

According to the nonnegative hypothesis on initial data \(u_0\) and \(u_1\), we are able to conclude

\[
e^{-\frac{1+\sqrt{5}}{2}t} \left( e^{-\frac{\sqrt{5}}{2}t} G_1(t) \right)' = G'_1(t) + \frac{3 - \sqrt{5}}{2} G_1(t) \geq 0,
\]

which results

\[
G_1(t) \geq e^{-\frac{1+\sqrt{5}}{2}t} G_1(0) = e^{-\frac{\sqrt{5}}{2}t} \left( F'_1(0) + \frac{3 + \sqrt{5}}{2} F_1(0) - \epsilon \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx \right) = 0.
\]

For this reason, we obtain

\[
e^{-\frac{3+\sqrt{5}}{2}t} \left( e^{-\frac{\sqrt{5}}{2}t} F_1(t) \right)' \geq \epsilon \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p\Psi(s,x)dxds.
\] (21)

By ignoring the nonnegative nonlinear integral term on the right-hand side, multiplying the previous equality by \(e^{-\frac{3+\sqrt{5}}{2}t}\), and integrating the resultant over \([0,t]\), we arrive at

\[
F_1(t) \geq e^{-\frac{3+\sqrt{5}}{2}t} F_1(0) + \frac{2\epsilon}{3 + \sqrt{5}} \left( 1 - e^{-\frac{3+\sqrt{5}}{2}t} \right) \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx \geq C_2\epsilon > 0
\] (22)
for any \( t \geq 1 \), where \( C_2 \) is a suitably positive constant depending on \( u_1 \). Here, we used nontrivial assumption on \( u_1 \) and \( F_1(0) = 0 \). Moreover, by applying inverse Hölder’s inequality and omitting the term containing initial data \( u_1 \) in (21), we find that

\[
F_1(t) \geq \int_0^t e^{\frac{\alpha_1^2}{2}(t-r)} \int_0^{\tau} |v_1(s,s)|^p \Psi(s,x) dx ds dr
\]

\[
\geq \int_0^t e^{\frac{\alpha_1^2}{2}(t-r)} \int_0^{\tau} |F_2(s)|^p \left( \int_{|x|<R+t} \Psi(s,x) dx \right)^{-\frac{p-1}{p}} dr
\]

\[
\geq C_1^{-p} \int_0^t e^{\frac{\alpha_1^2}{2}(t-r)} \int_0^{\tau} (R+s)^{-(\frac{n+1(p-1)}{2})} |F_2(s)|^p ds dr,
\]

where we used the support condition for the wave model and estimate (15).

Next, we treat (20) by constructing another time-dependent functional such that

\[
G_2(t) := F_2(t) - \frac{\varepsilon}{2} \int_{\mathbb{R}^n} v_1(x) \Phi(x) dx - \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} |u_1(s,x)|^q \Psi(s,x) dx ds.
\]  

In other words, in the light of (20), we find

\[
e^{-2t} (e^{2t} G_2(t))' = F_2'(t) + 2F_2(t) - \varepsilon \int_{\mathbb{R}^n} v_1(x) \Phi(x) dx - \frac{1}{2} \int_{\mathbb{R}^n} |u_1(t,x)|^q \Psi(t,x) dx - \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} |v_1(s,x)|^q \Psi(s,x) dx ds
\]

\[
= \varepsilon \int_{\mathbb{R}^n} v_0(x) \Phi(x) dx + \frac{1}{2} \int_{\mathbb{R}^n} |u_1(t,x)|^q \Psi(t,x) dx \geq 0,
\]

where the nonnegativity of \( v_0 \) was applied. It immediately conduces to

\[
G_2(t) \geq e^{-2t} G_2(0) = \frac{e^{-2t} \varepsilon}{2} \int_{\mathbb{R}^n} v_1(x) \Phi(x) dx \geq 0
\]

from the nonnegativity of \( v_1 \). Consequently, the nontrivial assumption on \( v_1 \) associated with relation (24) shows

\[
F_2(t) \geq \frac{\varepsilon}{2} \int_{\mathbb{R}^n} v_1(x) \Phi(x) dx = C_3 \varepsilon > 0,
\]  

with a positive constant \( C_3 \) depending on \( v_1 \), and

\[
F_2(t) \geq \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} |u_1(s,x)|^q \Psi(s,x) dx ds
\]

\[
\geq \frac{C_1^{-q}}{2} \int_0^t (R+s)^{-(\frac{n+1(p-1)}{2})} |F_1'(s)|^q ds
\]

\[
\geq \frac{C_1^{-q}}{2} \left( \int_0^t (R+s)^{\frac{n+1}{2}} ds \right)^{-(q-1)} \left( \int_0^t |F_1'(s)| ds \right)^q
\]

\[
\geq C_4(R+\cdot)^{-\frac{(n+1(p-1)}{2}} |F_1(t)|^q,
\]

with a positive constant \( C_4 \) and \( t > 1 \), where we utilized Hölder’s inequality twice associated with (15) again and

\[
|F_1(t)| = |F_1(t) - F_1(0)| = \left| \int_0^t F_1'(s) ds \right| < \int_0^t |F_1'(s)| ds.
\]
The further step is to investigate first lower bound estimates for the functionals $F_1(t)$ as well as $F_2(t)$, individually. On the one hand, we combine (25) with (23) to deduce

\[
F_1(t) \geq C_1^1 - p \epsilon_3 \epsilon^p \int_0^t e^{\frac{\alpha \epsilon^3}{2} (t-r)} \int_0^r (R + s)^{-\gamma (1-p)} \frac{1}{\epsilon} ds \, dr \\
\geq C_1^1 - p \epsilon_3 \epsilon^p (R + t)^{-\gamma (1-p)} \int_{t/2}^t e^{\frac{\alpha \epsilon^3}{2} (t-r)} \epsilon \, dr \\
\geq C_1^1 - p \epsilon_3 \epsilon^p (R + t)^{-\gamma (1-p)} \frac{3 + \sqrt{5}}{3 + \sqrt{5}} (1 - e^{-\frac{\alpha \epsilon^3}{4}}) \epsilon^p (R + t)^{-\gamma (1-p)} (t - L_1)
\]

for any $t \geq L_1 := 1 + 4/(3 + \sqrt{5})$, which provides first lower bound estimates for $F_1(t)$. On the other hand, we summarize (22) and (26). It results

\[
F_2(t) \geq C_2^q C_4^q \epsilon^q (R + t)^{-\gamma (1-p)}
\]

for any $t \geq L_1 > 1$. This choice of $L_1$ is concerned with the slicing procedure dealing with the unbounded multiplier in the next subsection.

All in all, we derived first lower bound estimates as follows:

\[
F_1(t) \geq D_1 (R + t)^{-a_1} (t - L_1)^{b_1}, \quad (27)
\]

\[
F_2(t) \geq Q_1 (R + t)^{-a_1} (t - L_1)^{b_1}, \quad (28)
\]

for any $t \geq L_1$, where the multiplicative constants are given by

\[
D_1 := \frac{C_1^1 - p \epsilon_3 \epsilon^p}{3 + \sqrt{5}} (1 - e^{-\frac{\alpha \epsilon^3}{4}}) \epsilon^p, \quad Q_1 := C_2^q C_4^q \epsilon^q,
\]

and the exponents are represented by

\[
a_1 := \frac{(n - 1)(p - 1)}{2}, \quad a_1 := \frac{(n + 1)(q - 1)}{2}, \quad \beta_1 := 1, \quad b_1 := 0.
\]

We remark that the above constants are nonnegative.

### 3.2 | Iteration argument

In this part, we will derive sequences of lower bound estimates for the functionals $F_1(t)$ and $F_2(t)$ by using some derived inequalities in the last subsection. To be specific, the following lower bounds will be proved:

\[
F_1(t) \geq D_j (R + t)^{-a_j} (t - L_j)^{b_j}, \quad (29)
\]

\[
F_2(t) \geq Q_j (R + t)^{-a_j} (t - L_j)^{b_j}, \quad (30)
\]

for any $t \geq L_j$, where $\{D_j\}_{j \geq 1}$, $\{Q_j\}_{j \geq 1}$, $\{a_j\}_{j \geq 1}$, $\{a_j\}_{j \geq 1}$, $\{\beta_j\}_{j \geq 1}$, and $\{b_j\}_{j \geq 1}$ are sequences of nonnegative real numbers that will be determined later in the iteration procedure. Motivated by the recent paper, we may define a crucial sequence $\{L_j\}_{j \geq 1}$ of the partial products of the convergent infinite product

\[
\prod_{k=1}^\infty \ell_k \quad \text{with} \quad \ell_k := 1 + \frac{4}{3 + \sqrt{5}} \frac{1}{(pq)^{1/2}} \quad \text{for any} \quad k \geq 1,
\]

(31)
that is,
\[ L_j := \prod_{k=1}^{j} \ell_k \text{ for any } j \geq 1. \] (32)

Here, we recall that \( L_1 = \ell_1 = 1 + 4/(3 + \sqrt{5}) \). Essentially, thanks to the ratio test and
\[
\lim_{k \to \infty} \frac{\ln \ell_{k+1}}{\ln \ell_k} = \lim_{k \to \infty} \frac{\ell_k}{\ell_{k+1}(pq)^{1/2}} = (pq)^{-1/2} < 1,
\]
we claim that the infinite product
\[
\prod_{k=1}^{\infty} \ell_k = \exp \left( \sum_{k=1}^{\infty} \ln \ell_k \right)
\]
is convergent. Furthermore, the desired estimates (29) and (30) for \( j = 1 \) are given in (27) and (28), respectively.

As a consequence, with the aim of demonstrating (29) and (30), we just need to prove the inductive step. In other words, by assuming that (29) and (30) hold for \( j \), one ought to prove them being valid for \( j + 1 \). Let us first substitute (30) into (23) to lead to
\[
F_1(t) \geq C_1^{1-p} Q_j \int_0^t e^{\frac{\gamma_1}{2}(t-s)} \frac{1}{(R + s)^{\frac{1}{2}(1-p)-a_s (s - L_j)^b_s s}} ds dr
\]
\[
\geq C_1^{1-p} Q_j (R + t) \frac{1}{(b_j p + 1)^{b_j p + 1}} \int_0^t e^{\frac{\gamma_1}{2}(t-s)} (s - L_j)^b_s s ds dr
\]
\[
\geq C_1^{1-p} Q_j (R + t) (b_j p + 1)^{b_j p + 1} \int_0^t e^{\frac{\gamma_1}{2}(t-s)} (s - L_j)^b_s s ds dr.
\]

In view of \( t \geq L_{j+1} = L_j \ell_{j+1} \), that is, \( L_j \leq t/\ell_{j+1} \), we may instantly shrink the interval \([L_j, t]\) into \([t/\ell_{j+1}, t]\) so that
\[
F_1(t) \geq C_1^{1-p} Q_j \int_{t/\ell_{j+1}}^t e^{\frac{\gamma_1}{2}(t-s)} (s - L_j)^b_s s ds dr
\]
\[
\geq \frac{2C_1^{1-p} Q_j}{(3 + \sqrt{5})(b_j p + 1)^{b_j p + 1}} \left( 1 - e^{\frac{\gamma_1}{2}(\ell_{j+1} - 1)} \right) (R + t) \frac{1}{(b_j p + 1)^{b_j p + 1}} \left( 1 - e^{\frac{\gamma_1}{2}(\ell_{j+1} - 1)} \right)\]
\[
\geq 2(pq)^{-\frac{1}{2}} \left( 1 - (pq)^{-\frac{1}{2}} \right) \geq 2 \left( (pq)^{\frac{1}{2}} - 1 \right) (pq)^{-j} > 0
\]
for any \( j \geq 1 \). In conclusion, it yields
\[
F_1(t) \geq \frac{4C_1^{1-p} (pq)^{\frac{j}{2} - 1} (pq)^{-j} Q_j}{(3 + \sqrt{5})(b_j p + 1)^{b_j p + 1}} (R + t) \frac{1}{(b_j p + 1)^{b_j p + 1}} \left( 1 - e^{\frac{\gamma_1}{2}(\ell_{j+1} - 1)} \right)\]
for any \( t \geq L_{j+1} \). Then, the combination of (26) as well as (29) shows
\[
F_2(t) \geq C_4 D_j (R + t) \frac{1}{(b_j p + 1)^{b_j p + 1}} \left( 1 - e^{\frac{\gamma_1}{2}(\ell_{j+1} - 1)} \right) \]
for any \( t \geq L_{j+1} \), where we used the fact that \( L_j \leq L_j \ell_{j+1} = L_{j+1} \) with \( \ell_{j+1} > 1 \).
In other words, (29) and (30) are valid for \( j + 1 \) with
\[
D_{j+1} := \frac{4C_1^{1-p} (pq)^{2j} - 1}{(3 + \sqrt{5})(b_jp + 1)\epsilon_{j+1}^{p+1}} R_j^{\alpha_j p}, \quad Q_{j+1} := C_4 D_{j+1}^p,
\]
\[
a_{j+1} := \frac{(n-1)(p-1)}{2} + a_j p, \quad a_{j+1} := \frac{(n+1)(q-1)}{2} + a_j q, \quad \beta_{j+1} = 1 + b_j p, \quad b_{j+1} := \beta_j q.
\]

### 3.3 Upper bound estimates for the lifespan

In the last subsection, we derive a sequence of lower bound estimates for \( F_1(t) \) and \( F_2(t) \), respectively. In the forthcoming part, we will demonstrate that the \( j \)-dependent lower bounds for the functionals \( F_1(t) \) and \( F_2(t) \) blow up as \( j \to \infty \) for \( t \) above a certain \( \epsilon \)-dependent threshold. At the same time, the blow-up result and upper bound estimates for the lifespan stated in Theorem 1 will be concluded.

We will begin with the explicit formulas for the sequences \( a_j, \beta_j, a_j, \) and \( b_j \), which devote to estimates for the multiplicative constants \( D_j \) and \( Q_j \).

Particularly, concerning the formulas of \( a_j \) and \( a_j \), we need to discuss the case when \( j \) is an odd integer only, which is sufficient for our proof. Taking account of the relation between \( a_j \) and \( a_j \), we may get for odd number \( j \) that
\[
a_j = \left( a_1 + \frac{2}{(n+1)(pq - 2 - (n-1))} \right) \frac{(pq)^j}{j!} - \frac{1}{2} \frac{(n+1)(pq - 2 - (n-1))}{2(pq - 1)},
\]
and similarly,
\[
a_j = \left( a_1 + \frac{2}{(n-1)(pq + 2 - (n+1))} \right) \frac{(pq)^j}{j!} - \frac{1}{2} \frac{(n-1)(pq + 2 - (n+1))}{2(pq - 1)}.
\]

Furthermore, by the definition of \( \beta_j \) and \( b_j \), one derives for odd number \( j \) that
\[
\beta_j = 1 + b_{j-1} p = 1 + \beta_{j-2} p = \sum_{k=0}^{(j-3)/2} (pq)^k + \beta_1 (pq)^{j-1} = \left( \beta_1 + \frac{1}{pq - 1} \right) (pq)^{j-1} - \frac{1}{pq - 1},
\]
\[
b_j = \beta_{j-1} q = q + b_{j-2} p = q \sum_{k=0}^{(j-3)/2} (pq)^k + b_1 (pq)^{j-1} = \left( b_1 + \frac{q}{pq - 1} \right) (pq)^{j-1} - \frac{q}{pq - 1}.
\]

For an even number \( j \), which means that \( j - 1 \) is an odd number, we make use of the previous two equalities to arrive at
\[
\beta_j = 1 + b_{j-1} p = q^{-1} \left( b_1 + \frac{q}{pq - 1} \right) (pq)^{j-1} - \frac{1}{pq - 1},
\]
\[
b_j = \beta_{j-1} q = p^{-1} \left( \beta_1 + \frac{1}{pq - 1} \right) (pq)^{j-1} - \frac{q}{pq - 1}.
\]

For this reason, it holds
\[
\beta_j \leq B_0 (pq)^{1/2} \quad \text{and} \quad b_j \leq B_1 (pq)^{1/2}
\]
for any \( j \geq 1 \), where \( B_0 = B_0(p, q) \) and \( B_1 = B_1(p, q) \) are positive constants independent of \( j \).
Before estimating the constants $D_j$ and $Q_j$ from below, we remark that
\[
\lim_{j \to \infty} \epsilon_j^{b_j - p + 1} = \lim_{j \to \infty} \epsilon_j^\rho_j \leq \lim_{j \to \infty} \exp \left( B_0 (pq)^{\frac{j}{2}} \ln \left( 1 + \frac{4}{3 + \sqrt{5}} (pq)^{-\frac{j}{2}} \right) \right) = \exp \left( \frac{4B_0}{3 + \sqrt{5}} (pq)^{\frac{j}{2}} \right) > 0
\]
so that there exists a suitable constant satisfying $1/\epsilon_j^{b_j - p + 1} \geq M > 0$ for any $j \geq 1$. As a result, the next iterated relations for the lower bounds come:
\[
D_j = \frac{4C_1^{1-p} (pq)^{\frac{j}{2}} - 1}{(3 + \sqrt{5})(b_j^{-1} + 1)} Q_j^{p/b_{j-1}} \geq \frac{4C_1^{1-p} (pq)^{\frac{j}{2}} - 1}{(3 + \sqrt{5})B_0} M (pq)^{-\frac{j}{2}} Q_j^{p/b_{j-1}}
\]
\[
\geq \frac{4C_1^{1-p} C_4 (pq)^{\frac{j}{2}} - 1}{(3 + \sqrt{5})B_0} M (pq)^{-\frac{j}{2}} Q_j^{p/b_{j-1}} = : E_0 (pq)^{-\frac{j}{2}} Q_j^{p/b_{j-1}}.
\]
and simultaneously,
\[
Q_j = C_4 D_j^{p/b_{j-1}} \geq \frac{4^q C_1^{1-p} q C_4 (pq)^{\frac{j}{2}} - 1}{(3 + \sqrt{5})q B_0} M (pq)^{-\frac{j}{2} q} Q_j^{p/b_{j-1}} = : E_1 (pq)^{-\frac{j}{2} q} Q_j^{p/b_{j-1}},
\]
with two constants $E_0 = E_0(p, q) > 0$ and $E_1 = E_1(p, q) > 0$ independent of $j$, but they depend on $M$, $C_1$, and $C_4$. Considering (33) with an odd number $j$, we take the logarithmic function on both sides to deduce
\[
\log D_j \geq pq \log D_{j-2} - \left( \frac{3}{2} j - 1 \right) \log(pq) + \log E_0
\]
\[
\geq (pq)^{\frac{j-1}{2}} \log D_1 - \frac{3}{2} \log(pq) \sum_{k=1}^{(j-1)/2} (j + 2 - 2k)(pq)^{k-1} + (\log(pq) + \log E_0) \sum_{k=1}^{(j-1)/2} (pq)^{k-1}
\]
\[
= (pq)^{\frac{j-1}{2}} \left( \log D_1 + \frac{\log(pq)}{2(pq - 1)^2} (1 - 7pq) + \frac{\log E_0}{pq - 1} \right) - \frac{\log(pq)}{2(pq - 1) + j} - \frac{\log E_0}{pq - 1},
\]
where we used the next formula in the last line of the chain inequalities:
\[
\sum_{k=1}^{(j-1)/2} (j + 2 - 2k)(pq)^{k-1} = \frac{1}{pq - 1} \left( \frac{2pq}{pq - 1} \left( \frac{3}{2} (pq)^{\frac{j-1}{2}} - 1 \right) - j \right).
\]
Thus, for all nonnegative odd numbers $j$ satisfying
\[
j \geq j_0 := \left[ \frac{2}{3} + \frac{2 \log E_0}{3 \log(pq)} - \frac{2pq}{pq - 1} \right],
\]
we conclude
\[
\log D_j \geq (pq)^{\frac{j-1}{2}} \left( \log D_1 + \frac{\log(pq)}{2(pq - 1)^2} (1 - 7pq) + \frac{\log E_0}{pq - 1} \right)
\]
\[
= (pq)^{\frac{j-1}{2}} \log \left( D_1 (pq)^{\frac{1 - 3q}{2(pq - 1)^2}} E_0^{\frac{j-1}{2}} \right) = : (pq)^{\frac{j-1}{2}} \log(E_2 \epsilon^p).
\]
for a positive constant $E_2 = E_2(p, q)$ and depending on $M, C_1, C_3,$ and $C_4$. By the same way of calculation, we may illustrate

$$
\log Q_j \geq pq \log Q_{j-2} - \left( \frac{3}{2} \right) q \log(pq) + \log E_1
\geq (pq)^{\frac{j-1}{2}} \log Q_1 - \frac{3}{2} q \log(pq) \sum_{k=1}^{(j-1)/2} (j - 2k)(pq)^{k-1} + (q \log(pq) + \log E_1) \sum_{k=1}^{(j-1)/2} (pq)^{k-1}
= (pq)^{\frac{j-1}{2}} (\log Q_1 + \frac{q \log(pq)}{2pq-1} (1 - 7pq) + \frac{\log E_1}{pq-1}) + \frac{\log(pq)}{pq-1} \left( \frac{3q}{2} \left( \frac{2pq}{pq-1} - j \right) - q \right) - \frac{\log E_1}{pq-1}.
$$

Consequently, for all nonnegative odd numbers $j$ fulfilling

$$
j \geq j_1 := \left[ \frac{2}{3} + \frac{2 \log E_1}{3q \log(pq) - 2pq} \right],
$$

we conclude

$$\log Q_j \geq (pq)^{\frac{j-1}{2}} (\log Q_1 + \frac{q \log(pq)}{2pq-1} (1 - 7pq) + \frac{\log E_1}{pq-1}) =: (pq)^{\frac{j-1}{2}} \log(\varepsilon^q)
$$

for a positive constant $E_3 = E_3(p, q)$ but depending on $M, C_1, C_2,$ and $C_4$.

Let us now denote

$$L := \lim_{j \to \infty} L_j = \prod_{j=1}^\infty \ell_j > 1.$$

Due to $\ell_j > 1$, the sequence $\{L_j\}_{j \geq 1}$ is monotonically increasing. Namely, relations (29) and (30) hold for any odd number $j \geq 1$ and any $t \geq L$.

Let us now consider an odd number $j$ such that $j \geq j_{\max} := \max\{j_0, j_1\}$. Estimate (29) can be shown by

$$F_1(t) \geq \exp \left( (pq)^{\frac{j-1}{2}} \log(\varepsilon^q) \right) (R + t)^{-\alpha} (t - L)^{\beta}
\geq \exp \left( (pq)^{\frac{j-1}{2}} \left( \log(\varepsilon^q) - \left( q_1 + \frac{n + 1}{pq - 1} \right) \log(R + t) + \left( \beta_1 + \frac{1}{pq - 1} \right) \log(t - L) \right) \right)
\times (R + t)^{(n+1)(pq-2p+1)} (t - L)^{-\frac{1}{pq-1}}
$$

for any odd number $j \geq j_{\max}$ and any $t \geq L$. Choosing $t \geq \max\{R, 2L\}$, since $R + t \leq 2t$ and $t - L \geq t/2$, the functional $F_1(t)$ can be estimated by the following way:

$$F_1(t) \geq \exp \left( (pq)^{\frac{j-1}{2}} \log \left( E_2 \varepsilon^q 2^{-\frac{(n+1)(pq-2p+1)}{2pq-1} + \frac{pq}{pq-1}} \right) \right) (R + t)^{(n+1)(pq-2p+1)} (t - L)^{-\frac{1}{pq-1}} \quad (35)
$$

for any odd number $j \geq j_{\max}$. The exponent of $t$ in (35) can be represented as follows:

$$\frac{(n+1)(pq-2p+1)}{2pq-1} - \frac{pq}{pq-1} = \frac{p(-(n+1)pq+n+1)}{2pq-1} =: pt_1(p, q, n).
$$

By our assumption that $pq < (n+1)/(n-1)$ for any $n \geq 2$ and $p, q > 1$ for any $n \geq 1$, the power of $t$ in the exponential term of (35) is positive.

In a similar way to the above, we may deduce the lower bound estimate for any odd number $j$ fulfilling $j \geq j_{\max}$

$$F_2(t) \geq \exp \left( (pq)^{\frac{j-1}{2}} \log \left( E_3 \varepsilon^q 2^{-\frac{(n+1)(pq-2p+1)}{2pq-1} + \frac{pq}{pq-1}} \right) \right) (R + t)^{(n+1)(pq-2p+1)} (t - L)^{-\frac{1}{pq-1}} \quad (36)$$
Thus, the power of $t$ in the exponential term can be represented by
\[
- \frac{(n+1)(q-1)}{2} - \frac{(n-1)pq + 2q -(n+1)}{2(pq-1)} + \frac{q}{pq-1} = \frac{q(-(n+1)pq + 2p + n + 1)}{2(pq-1)} = qT_2(p, q, n).
\]

By assuming $(n+1)pq - 2p - (n+1) < 0$, the power for $t$ in the exponential term of (36) is positive. We should emphasize that
\[
\{ (p, q) : (n+1)pq - 2p - (n+1) < 0 \} \subseteq \{ (p, q) : (n+1)pq < (n+1) \}
\]
for all $p, q > 1$ and $n \geq 1$. To put it differently, the condition $(n-1)pq < n + 1$ is sufficient to guarantee the positivity of the power for $t$ in the exponential term of (36).

Eventually, for studying upper bound estimates for the lifespan, we now can introduce $\epsilon > 0$, $\epsilon_0 = \epsilon_0(u_0, u_1, v_0, v_1, p, q, n, R) > 0$ such that
\[
\left( E^{-2} \frac{1}{2} \left( \frac{n-1}{2pq} + \frac{(n-1)pq - 2p}{2pq - 1} + \frac{q}{pq-1} \right) \right) = \frac{1}{E^{\frac{1}{2}}}, \quad \epsilon \in \{ \epsilon_0 \},
\]
for all $p, q > 1$ and $n \geq 1$. Consequently, they lead to blow-up of $(u, v)$. Thus, the result of the functional $F_1(t)$ being not defined globally in time immediately shows the solution $(u, v)$ from Definition 1 being not defined globally in time.

**Remark 6.** Indeed, since the space average of the solution is the quantity that blows up in finite time and we do consider a solution with support contained in a curved light cone (so with compact support with respect to the space variables as long as the solution exists), by means of Hölder’s inequality and the Poincaré inequality, it follows the blow-up of solution according to Definition 1. To be specific, let us recall
\[ F_1(t) = \int_0^t \int_{R^+} u(s, x) \Psi(s, x) \, dx \, ds \]
with $\Psi(t, x) = e^{-t} \Phi(x)$. Asymptotic behaviors for $\Phi(x)$ in (14) claim that
\[ \| \Phi \|^2_{L^2(B_{R+1})} \leq \int_{B_{R+1}} |x|^{-2(n-1)} e^{2|x|} \, dx \leq e^{2(R+t)} - 1. \]
Therefore, concerning $t \in [0, T]$, by using Hölder’s inequality, we get
\[ F_1(t) \leq \int_0^t e^{-t} \| \Phi \|_{L^2(B_{R+1})} \| u_t(s, \cdot) \|_{L^2(B_{R+1})} \, ds \leq C(T, R) \sup_{s \in [0, T]} \| u_t(s, \cdot) \|_{L^2} \]
with $C(T, R) > 0$, as well as
\[
F_1(t) = - \int_0^t \int_{B_{R+1}} u(s, x) \Psi_t(s, x) \, dx \, ds + \int_{B_{R+1}} u(t, x) \Psi(x) \, dx - \int_{B_{R+1}} u_0(x) \Phi(x) \, dx
\leq \int_0^t e^{-\epsilon t} \| \Phi \|_{L^2(B_{R+1})} \| u_t(s, \cdot) \|_{L^2(B_{R+1})} \, ds + e^{-\epsilon t} \| \Phi \|_{L^2(B_{R+1})} \| u(t, \cdot) \|_{L^2(B_{R+1})} + \| \Phi \|_{L^2(B_R)} \| u_0 \|_{L^2(B_R)}
\leq C(T, R) \left( \sup_{s \in [0, T]} \| u(s, \cdot) \|_{L^2(B_{R+1})} + \| u_0 \|_{L^2(B_R)} \right)
\]
with $C(T, R) > 0$, where we employed the support condition for $u$. The Poincaré inequality shows
\[ \| u(s, \cdot) \|_{L^2(B_{R+1})} \leq C(T, R) \| \nabla u(s, \cdot) \|_{L^2(B_{R+1})} \]
for $s \in [0, T)$. Consequently, they lead to blow-up of $(u, v)$ according to Definition 1.
In conclusion, these statements proved that the energy solution \((u, v)\) is not defined globally in time and, simultaneously, the lifespan of this local (in time) solution \((u, v)\) can be estimated by
\[
T(\varepsilon) \leq C\varepsilon^{-\frac{1}{p+q+1}}.
\]

The proof of the theorem is complete.

4 CONCLUDING REMARKS

Throughout this paper, we focus on blow-up phenomena for Nakao’s type problem (1) with derivative-type nonlinearities \((|v|^p, |u|^q)^T\), which is motivated by the weakly coupled system for semilinear wave equations and damped wave equations with the same nonlinearities \((|v|^p, |u|^q)^T\). Therefore, the model considered in the present paper is naturally based on two systems whose blow-up results have been derived or can be conjectured. It seems also interesting to study Nakao’s type problem (1) with the mixed nonlinear terms \((|v|^p, |u|^q)^T\) or \((|v|^p, |u|^q)^T\), namely, (1) with \(d_1 = d_4 = 1\) and \(d_2 = d_3 = 0\), or \(d_2 = d_3 = 1\) and \(d_1 = d_4 = 0\), respectively. Finally, we conjecture that some blow-up results can also be obtained for mixed nonlinearities by combining the approaches of Wakasugi and Reissig and Palmieri and Takamura and the one in this paper.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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