Nonexponential motional damping of impurity atoms in Bose-Einstein condensates

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We demonstrate that the damping of the motion of an impurity atom injected at a supercritical velocity into a Bose-Einstein condensate can exhibit appreciable deviations from the exponential law on time scales of $10^{-5}$ s.

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The decay of an unstable quantum system has long been known to deviate from exponential law for both very short and very long times [1] – [4]. The short-time deviation from the exponential decay gives rise to either slowdown or speedup of the decay by frequently repeated measurements, known, respectively, as the quantum Zeno effect (QZE) [1] and the anti-Zeno effect (AZE) [2,3]. A general unified theory of the QZE and AZE has been given in Ref. [4]. Among the proposed realizations of the QZE/AZE are photodetachment of negative ions [5] and radiative decay of excited atoms in cavities [3], photonic band gap structures [6] or in the presence of a magnetic or leptonic bath, which is the time-scale of non-Markovian (memory) time of the relevant (electro-) quanta (phonons) by atoms into a BEC as a candidate bath, as in radiative decay [11] or in beta-decay [12]. Here we propose the emission of sound waves from a trapping potential resembles nuclear alpha-decay [10] rather than decay via quanta emission into a bath, as in radiative decay [11] or in beta-decay [12].

The difficulty impeding the demonstration of nonexponential decay via quanta emission has been the short non-Markovian (memory) time of the relevant (electromagnetic or leptonic) bath, which is the time-scale of the effect [4]. We propose the emission of sound quanta (phonons) by atoms into a BEC as a candidate process for the observation of nonexponential relaxation, taking advantage of the rather long memory time of the condensate. The detailed understanding of such decoherence processes is important for the envisaged use of atomic BECs in metrology and interferometry [13].

Let us consider an “impurity” atom moving in an atomic BEC. Atoms of another isotope or the same isotope but in a different internal (hyperfine) state can be viewed as impurities as long as their density is small enough not to modify considerably the BEC excitation spectrum. At “supercritical” velocities, namely, above the speed of sound in the BEC, the impurity atom is decelerated due to phonon creation in the BEC. The rate of such a process according to the standard Fermi golden rule (i.e., assuming exponential decay of the amplitude of the initial state) has been calculated for both a uniform BEC [14] and a harmonically trapped BEC [15] and has been determined experimentally [16]. We shall step beyond this approach and calculate more generally the time evolution of the impurity-atom motion in such a system.

The state vector of the system can be written as (we set $\hbar = 1$)

$$|\psi(t)\rangle = \alpha_{in}(t)|\text{in}\rangle + \int \frac{d^3q}{(2\pi)^3} \beta_q(t)|q\rangle,$$

where the initial state $|\text{in}\rangle$ corresponds to an impurity atom of mass $m_1$ moving at the velocity $\mathbf{V}_1$ in a BEC with no elementary excitation, and $|q\rangle$ denotes the state where one elementary excitation of the BEC with the momentum $q$ is present (correspondingly, the impurity momentum is changed to $m_1\mathbf{V}_1 - q$). The initial conditions are, naturally,

$$\alpha_{in}(0) = 1, \quad \beta_q(0) = 0.$$  

The set of equations describing motional damping of the impurity atom is

$$i\dot{\alpha}_{in} = \left(\frac{m_1\mathbf{V}_1^2}{2} + \tilde{g}_{12}n\right)\alpha_{in} + \tilde{g}_{12}\sqrt{n}\int \frac{d^3q}{(2\pi)^3} (u_q - v_q)\beta_q,$$

$$i\dot{\beta}_q = \left[\left(\frac{m_1\mathbf{V}_1 - q}{2m_1}\right)^2 + \epsilon(q) + \tilde{g}_{12}n\right] \beta_q + \tilde{g}_{12}\sqrt{n}(u_q - v_q)\alpha_{in}.$$  

Here $n$ is the uniform BEC density, $\tilde{g}_{12}$ is the effective interspecies coupling constant, $u_q = \{|E_{HF}(q)/\epsilon(q) + 1/2\}^{1/2}$ and $v_q = (u_q^2 - 1)^{1/2}$ are the Bogoliubov transformation coefficients, and $\epsilon(q) = |E_{HF}^2(q) - \mu|^{1/2}$ is the energy of the elementary excitation with momentum $q$ [17]. Here we have introduced the Hartree-Fock excitation energy $E_{HF} = q^2/(2m_2) + \mu$ and the chemical potential of the BEC $\mu = 4\pi a_{22}n/m_2$, $a_{22}$ being the intraspecies s-wave scattering length for the condensed atoms of mass $m_2$. Correspondingly, the speed of sound in the BEC is $c_s = \sqrt{\mu/m_2}$. 


To proceed, we have to reckon with the renormalization of the interspecies coupling constant [18]. The renormalized constant \( \tilde{g}_{12} \) is expressed in terms of the bare coupling constant \( g_{12} \) as

\[
\tilde{g}_{12} = g_{12} \left[ 1 + 2mg_{12}(2\pi)^{-3} \int d^3q q^{-2} \right],
\]

with \( g_{12} = 2\pi a_{12}/m, a_{12} \) being the interspecies s-wave scattering length and \( m = m_1m_2/(m_1 + m_2) \) being the reduced mass. In the approximate (perturbative) solution of Eqs. (3, 4) we shall keep the terms up to the second order in the bare constant \( g_{12} \).

The easiest way to solve Eqs. (3, 4) is by the Laplace transformation (cf. Ref. [6]). We adopt the interaction representation, wherein the probability amplitude of the initial state is

\[
\alpha(t) = \alpha_{in}(t) \exp(i(m_1V_1^2/2 + g_{12}n)t).
\]

The algebraic solution for the Laplace transform of \( \alpha(t) \), \( \overline{\alpha}(s) = \int_0^\infty dt \exp(-st)\alpha(t) \) has the form

\[
\overline{\alpha}(s) = \left[ s + \overline{\Omega}(s) \right]^{-1},
\]

\[
\overline{\Omega}(s) = g_{12}^2n \int d^3q \frac{\left( (u_q - v_q)^2 + 2im \right)}{(2\pi)^3} \left[ \frac{1}{s + i\Delta(q)} \right].
\]

\( \Delta(q) = \epsilon(q) + q^2/(2m_1) - qV_1 \) being the energy mismatch between the the states \( |in \rangle \) and \( |q \rangle \). The second term in the square brackets in Eq. (8) arises from the coupling-constant renormalization in Eq.(5) and compensates for the ultraviolet divergence of the first term. This compensation is completely analogous to that of the electron mass renormalization in calculations of the radiative shift of an atomic optical transition [19].

Equations (3, 4) yield over a broad time interval, excluding very short times, exponential decay of \( \alpha(t) \propto \exp\{-\gamma/2 + i\omega_s\}t \), with the rate \( \gamma \) and the frequency shift \( \omega_s \)

\[
\gamma = \lim_{s \to 0} 2 \text{Re} \left( I(s) \right), \quad \omega_s = \lim_{s \to 0} \text{Im} \left( \overline{\Omega}(s) \right).
\]

The relaxation rate \( \gamma \) can be calculated within the exact Bogoliubov theory using Fermi’s golden rule [14,16].

It is possible to obtain a nonexponential analytical solution for \( \alpha(t) \) in the Hartree-Fock (HF) approximation [20] (hereafter we label all the quantities in this approximation by HF): \( \epsilon_{HF}(q) = E_{HF}(q), u_{HF} = 1, v_{HF} = 0 \). Then the integral in Eq.(8) can be evaluated analytically, yielding in Eq. (9) the exponential decay rate

\[
\gamma_{HF} = \gamma_0 [1 - e_{HF}^2/V_1^2]^{1/2} \quad \text{if} \quad V_1 > e_{HF} \equiv \sqrt{2\mu/m_2} \quad \text{and zero otherwise.}
\]

Here \( \gamma_0 = 4\pi a_{12}^2 V_1 \) is the collision rate calculated for the impurity atom using the Fermi golden rule in the limit \( a_{22} \to 0 \) (an ideal BEC). Finally, in the case \( V_1 > e_{HF} \) we obtain

\[
\overline{\alpha}_{HF}(s) = \left[ \gamma_{HF} \left( \frac{s}{\gamma_{HF}} + \frac{1}{2} \sqrt{1 + \frac{is}{\gamma_{HF}K}} \right) \right]^{-1},
\]

where \( K = (mV_1^2/2 - \mu)/\gamma_{HF} \) is the energy of collision of the impurity atom with a condensate atom in their center-of-mass frame (including the correction due to the BEC chemical potential \( \mu \) scaled to \( \gamma_{HF} \)). By inverting such a Laplace transform [21] we get

\[
\alpha_{HF}(t) = \frac{e^{it\Delta}}{\xi_1 - \xi_2} \left[ \xi_1 e^{i\sqrt{\xi_1}t} - \xi_2 e^{i\sqrt{\xi_2}t} \right],
\]

where \( \varphi(z) = \exp(z^2)erfc(-z), \text{erfc}(z) \) being the complementary error function, and \( \xi_{1,2} = -\Xi/2 \pm \sqrt{\Xi^2/4 - i\Lambda} \) are the roots of the quadratic equation \( \xi^2 + \Xi \xi + i\Lambda = 0 \). The coefficients of the latter equation are \( \Xi = \gamma_0\sqrt{t/(2mV_1^2)} \) and \( \Lambda = mV_1^2/2 - \mu \).

![Fig. 1. Numerically calculated logarithm of survival probability \( P(t) \) of the initial state of impurity atoms in a ^87Rb BEC plotted versus dimensionless time \( \gamma t \) for \( n = 10^{14} \text{cm}^{-3} \) (c_s = 0.2 cm/s); \( m_1 = m_2, a_{12} = 3a_{22} \). Filled circles: \( V_1 = 3c_s, \gamma = 1.1 \cdot 10^3 \text{s}^{-1} \). Open circles: \( V_1 = 7c_s, \gamma = 4.4 \cdot 10^3 \text{s}^{-1} \). Solid line: exponential law \( \exp(-\gamma t) \). Inset: typical behavior of logarithm of \( P(t) \) for the case of subcritical \( V_1 \) (practically any \( V_1 \leq 0.7c_s \)). Note the different scaling \( \text{(by} \mu) \) of the horizontal axis of the inset plot.](image)

Although our numerical results (Fig. 1) show that the HF approximated solution Eq. (11) is rather crude, it nonetheless provides a qualitative guidance to the physical behavior. Simple scaling considerations lead us to the conclusion that \( \alpha_{HF} \) depends on two parameters: the dimensionless time variable \( \gamma_{HF}t \) and \( K \) [Eq. (10)]. Equation (11) predicts that at \( t \to 0 \) the decay is more rapid than exponential, so the survival probability \( P(t) = |\alpha(t)|^2 \) behaves in the HF approximation as \( P_{HF}(t) \approx 1 - 4Re(\xi_1 + \xi_2)/\sqrt{t/\pi} \), becoming exponential at larger times, \( P_{HF}(t) \approx \exp(-\gamma_{HF}t) \). The deviation from exponential decay is appreciable only for \( K \lesssim 1 \).
There are two ways to attain $K < 1$. One is to take a small difference between the impurity atom velocity $V_1$ and the critical velocity, but this would reduce the damping rate, which may be experimentally inconvenient. A much better way is to strive for a large interspecies scattering length $a_{12}$, as discussed below.

The results of our numerical calculations based on Eqs. (7, 8), which use the exact expressions for $\epsilon(q)$, $u_\eta$ and $v_\eta$ instead of the HF approximation, clearly reveal a deviation from exponential decay for small times. Under such conditions, frequent measurements would accelerate the decay, causing the anti-Zeno effect (AZE). Alternatively, one may accelerate the decay by periodically modulating the coupling of the initial state to the continuum [4], instead of repeated projective measurements. This can be done by changing the impurity velocity using a sequence of Bragg or Raman laser pulses [22].

The inset to Fig. 1 shows that if $V_1 < c_s$, the survival probability first decreases and then approaches the constant value of about 0.85, almost independent on $V_1$. This behavior reveals the physical reason for the short-time nonexponential decay: the initial conditions Eq.(2) imply that, initially, the impurity atom is surrounded by no virtual phonons, while in the steady state, the impurity atom must be surrounded by a cloud of virtual phonons (cf. the polaronic effect for electrons in a crystal [23]). Thus the nonexponential stage of the decay is associated with the formation of such a phonon cloud.

The faster the impurity atom moves, the weaker its coupling to the phonon cloud is. Therefore the decrease of $\omega_s$ corresponds to the vanishing of nonexponential decay effects as $V_1$ increases. This behavior is displayed in Fig. 2 by the numerically calculated [from Eq. (9)] decay rate $\gamma$ and frequency shift $\omega_s$.

Our numerical studies of Eqs. (3, 4) always yield decay acceleration at short times, typically on the scale of $10^{-5}$ s. But should one not expect, from general considerations [1], $P(t) = 1 - \text{const} \cdot t^2$ at $t \to 0$, in accordance with the QZE? To answer this question, we should apply the general theory developed in Ref. [4], whereby the short-time behavior is determined by the spectrum (i.e., the dependence on the emitted quantum energy $\epsilon$) of the reservoir response $G(\epsilon)$. This spectrum is given by the interaction matrix element squared multiplied by the density of the reservoir states. In our case, we find that

$$G(\epsilon) = \left[ \frac{2 \pi a_{12}(u_\eta - v_\eta)}{m} \right]^2 \frac{\epsilon g^2 dq}{2 \pi^2 \delta \sigma}$$  \hspace{1cm} (12)

monotonously increases with the emitted phonon energy $\epsilon$. According to Ref. [4], if the energy uncertainty $\sim \frac{1}{t^2}$ associated with the finite observation time $t$ covers the energy range where $G(\epsilon)$ increases, then decay acceleration (AZE) takes place. However, our approach [Eqs.(3, 4)] leading to Eq.12 is valid only for small transferred momenta. If $q \lesssim r_0^{-1}$, where $r_0 \sim 10^{-7}$ cm is the characteristic radius of the interatomic potential, we cannot consider $a_{12}$ as constant any more. Instead, the interaction matrix element decreases with $q$ in this range. In the inset of Fig. 2 we schematically display the spectrum of $G(\epsilon)$, including its decreasing part, whose detailed calculation is beyond the scope of this Letter. This spectrum implies that at very short times ($\lesssim 10^{-9}$ s), the energy uncertainty broadening covers the whole profile of $G(\epsilon)$, thereby giving rise to the QZE.

![FIG. 2. Numerically calculated frequency shift $\omega_s$ and the exponential decay rate $\gamma$ (scaled to $\gamma_0$) versus the impurity dimensionless velocity $V_1/c_s$. Both $\omega_s/\gamma_0$ and $\gamma/\gamma_0$ display a universal behaviour, independent of the BEC density and the atomic species. Inset: schematic representation of the spectrum $G(\epsilon)$; solid line: the increasing part given by Eq.(12); dashed line: the remaining part, which decreases at $\epsilon \sim 1/(mr_0^2)$. The arrows indicate ranges of the energy uncertainty corresponding to AZE and QZE.

The parameters used in Fig. 1 for a BEC of $^{87}$Rb may correspond to impurity atoms of the same isotope but in a different hyperfine state, obtained by a short Raman pulse. This method of impurity atom admixing conforms to the initial conditions of Eq.(2). However, to reach appreciable nonexponentiality by this method, one has to enhance interspecies scattering either by means of interspecies Feshbach resonance or via laser-induced dipole-dipole interactions [24]. Another possibility is the use of two-isotope mixture, for example, a BEC of $^{87}$Rb atoms with admixed fermionic $^{40}$K atoms. Such a choice is of particular interest, since the large interspecies scattering length in this mixture [25], as well as the mass ratio between these two elements seem very promising for experimental search of nonexponential decay effects. We note that the condition on the impurity velocity in this case is opposite to that of the aforementioned case of impurities generated from a BEC by a Raman pulse. Indeed, the $^{40}$K atoms at rest co-exist with the $^{87}$Rb BEC for a time long enough to form virtual phonon clouds around them. If then the $^{40}$K atoms suddenly acquire, by the action
of a Bragg or Raman pulse, the velocity $V_1 \gg c_s$, the initial conditions, instead of Eq.(2), should assume the pre-existence of the phonon cloud (prior to the pulse), which vanishes when the impurities attain high (super-critical) velocities. Thus we expect that in the case of a two-element ultracold mixture, nonexponential decay features are most pronounced for $V_1 \gg c_s$.

To conclude, we have outlined the possibility of observing deviations from exponential decay for unstable momentum states of impurity atoms moving in a BEC with a supercritical velocity on time scales of $10^{-5}$ s. The effects of finite temperature and bosonic enhancement in this process will be a subject of a separate work.

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[1] R.G. Winter, Phys. Rev. 123, 1503 (1961); L.A. Khalifin, Zh. Eksp. Teor. Fiz. 33, 1371 (1957) [Sov. Phys. JETP 6, 1053 (1958)]; L. Fonda, G.C. Ghirardi, and A. Rimini, Rep. Prog. Phys. 41, 587 (1978); B. Misra and E.C.G. Sudarshan, J. Math. Phys. 18, 756 (1977); W.M. Itano, D.J. Heinzen, J.J. Bollinger, and D.J. Wineland, Phys. Rev. A 41, 2295 (1990).

[2] A.M. Lane, Phys. Lett. A 99, 359 (1983); W.C. Schieve, L.P. Horwitz, and J. Levitan, Phys. Lett. A 136, 264 (1989); P. Facchi, H. Nakazato, and S. Pascazio, Phys. Rev. Lett. 86, 2699 (2001).

[3] A.G. Kofman and G. Kurizki, Phys. Rev. A 54, R3750 (1996).

[4] A.G. Kofman and G. Kurizki, Nature (London) 405, 546 (2000); A.G. Kofman and G. Kurizki, Phys. Rev. Lett. 87, 270405 (2001).

[5] K. Rzązewski, M. Lewenstein, and J.H. Eberly, J. Phys. B 15, L661 (1982).

[6] A.G. Kofman, G. Kurizki, and B. Sherman, J. Mod. Opt. 41, 353 (1994).

[7] I.E. Mazets and L.B. Shifrin, Phys. Lett. A 229, 73 (1997).

[8] S.R. Wilkinson, C.F. Bharucha, M.C. Fischer, K.W. Madison, P.R. Morrow, Q. Niu, B. Sundaram, and M.G. Raizen, Nature (London) 387, 575 (1997).

[9] M.C. Fischer, B. Gutiérrez-Medina, and M.G. Raizen, Phys. Rev. Lett. 87, 040402 (2001).

[10] G. Gamow, Z. Phys. 51, 204 (1928); E.U. Condon and R.W. Gurney, Nature 122, 439 (1928).

[11] W. Heitler, The Quantum Theory of Radiation (Clarendon Press, Oxford, 1954).

[12] E. Fermi, Nuclear Physics (Univ. of Chicago Press, Chicago, 1950).

[13] S. Choi et al., Phys. Rev. A 63, 065601 (2001); S. Gupta, K. Dieckmann, Z. Hadzibabic, and D.E. Pritchard, Phys. Rev. Lett. 89, 140401 (2002); J.A. Dunningham, K. Burnett, and S.M. Barnett, Phys. Rev. Lett. 89, 150401 (2002); A. Görlitz et al., Phys. Rev. Lett. 90, 090401 (2003).

[14] E. Timmermans and R. Côté, Phys. Rev. Lett. 80, 3419 (1998).

[15] Z. Idziaszek, K. Rzążeński, and M. Wilkens, J. Phys. B 32, L205 (1999).

[16] A.P. Chikkatur, A. Görlitz, D.M. Stamper-Kurn, S. Inouye, S. Gupta, and W. Ketterle, Phys. Rev. Lett. 85, 483 (2000).

[17] N.N. Bogoliubov, J. Phys. (U.S.S.R.) 11, 23 (1947).

[18] E.M. Lifshitz and L.P. Pitaevskii, Statistical Physics, Pt.2 (Butterworth-Heinemann, Oxford, 1980); S.T. Belyaev, Zh. Eksp. Teor. Fiz. 34, 433 (1958).

[19] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, Atom-Photon Interactions (Wiley, NY, 1992); H.A. Bethe, Phys. Rev. 72, 339 (1947).

[20] B.D. Esry, Phys. Rev. A 55, 1147 (1997).

[21] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions (National Bureau of Standards, Washington, DC, 1964).

[22] J. Stenger, S. Inouye, A.P. Chikkatur, D.M. Stamper-Kurn, D.E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 82, 4569 (1999); J. Steinmacher, R. Ozeri, N. Katz, and N. Davidson, Phys. Rev. Lett. 88, 120407 (2002).

[23] A. Ishihara, Statistical Physics (Academic Press, NY, 1971).

[24] G. Kurizki, S.Giovanazzi, D. O’Dell, and A.I. Artemiev, Springer Lecture Notes in Physics, 602, 382 (2002).

[25] G. Roati, F. Riboli, G. Modugno, and M. Inguscio, Phys. Rev. Lett. 89, 150403 (2002).