MODEL THEORETIC PROPERTIES OF THE URYSOHN SPHERE

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Abstract. We characterize model theoretic properties of the Urysohn sphere in continuous logic. In particular, we show that the theory of the Urysohn sphere is SOP\(_n\) for all \(n \geq 3\), but does not have the (finitary) strong order property. In the process, we give necessary and sufficient conditions for when a partially defined symmetric function on a finite set can be extended to a metric on that set. Our second main result is a geometric characterization of dividing independence in the theory of the Urysohn sphere. We further show that this characterization satisfies the extension axiom, and so forking and dividing are the same for complete types.

1. Introduction

The Urysohn sphere is the unique complete separable metric space of diameter 1, which isometrically embeds every separable metric space of diameter \(\leq 1\). As a structure in model theory, the Urysohn sphere is most naturally studied by way of continuous logic. The goal of this paper is to answer two model theoretic questions about the theory of the Urysohn sphere in this setting. These questions are:

(1) How complicated is the theory with respect to commonly considered model theoretic dividing lines?

(2) What is the nature of forking independence in this theory?

Concerning both questions, much of what is known about the Urysohn sphere can be found in [8], where the authors characterize thorn forking and show that the continuous theory is rosy (with respect to finitary imaginaries). They also include an argument, due to Pillay, that the Urysohn sphere is not simple, and remark that unpublished work of Berenstein and Usvyatsov has demonstrated that the Urysohn sphere is SOP\(_3\), but without the strict order property (SOP).

In the classical setting, theories with SOP\(_3\) and NSOP\(_\infty\) can be further stratified by Shelah’s hierarchy of \(n\)-strong order properties (SOP\(_n\)). Therefore, our answer to the first question will be a placement of the Urysohn sphere in this hierarchy. In particular, we show that the Urysohn sphere is SOP\(_n\) for all \(n \geq 3\), but does not have the (finitary) strong order property, which we denote SOP\(_\infty\). It is standard result in classical logic that the strict order property implies the strong order property. This has been verified for continuous logic in unpublished work of Usvyatso. We also show that the Urysohn sphere has TP\(_2\). Altogether this suggests that the Urysohn sphere is, in some sense, as complicated as an NSOP\(_\infty\) theory can be.

On the other hand we will show that, despite its complexity, the Urysohn sphere has a rather nice characterization of forking independence. Forking independence was originally used to study stable theories, and was later shown to be meaningful in simple theories (see [9]) and, more recently, in NTP\(_2\) theories (see [3]). We will
show that the Urysohn sphere gives an example of a theory with TP₂ in which nonforking still has a meaningful geometric interpretation. Along the way we show that forking is the same as dividing for complete types.

The outline of the paper is as follows. In Section 2 we introduce the Urysohn sphere as a continuous structure and record the basic facts we need. In Section 3 we define and reformulate SOPₙ and SOP∞ in the setting of continuous logic. This raises the question of satisfying partial types containing only statements about the metric, i.e., partially defined metric spaces. So in Section 4 we prove a general result characterizing when a partial symmetric function on a finite set can be extended to a metric on that set. Section 5 contains the main classification of the Urysohn sphere with respect to SOPₙ. At this point, we turn our attention towards forking independence, beginning in Section 6 with definitions and facts about forking and dividing in continuous logic. In Section 7, we first give a geometric characterization of dividing independence. We then show that this characterization satisfies the extension axiom, and so forking and dividing are the same for complete types. Finally, we discuss some corollaries of these results. For example, we note a relationship between forking independence and the stationary independence relation given in [12], which is used by the authors there to prove that the isometry group of the Urysohn sphere is simple. In Section 8 we characterize stationary types in the Urysohn sphere. In particular, we show all stationary types are algebraic. Finally, in Section 9 we discuss how the situation changes in first order logic, and how our results can be adapted to that setting.

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2. Continuous logic and the Urysohn sphere

In this section we fix some notation and terminology, define the Urysohn sphere, and discuss the logic of bounded metric structures we will use to study it. We take the following definitions from [10].

Definition 2.1. Given a metric space (X, d), a Katětov map is a function f : X → [0, 1] such that for all x, y ∈ X,

\|f(x) - f(y)\| \leq d(x, y) \leq f(x) + f(y).

Let E(X) be the set of all Katětov maps on X.

Note that Katětov maps correspond to one-point metric space extensions of X in the following way. If f ∈ E(X) then the extension X ∪ \{a\} where d(x, a) = f(x) is a one-point metric space extension of X. Conversely if X ∪ \{a\} is a metric space extension of X, then x ↦ d(x, a) is a Katětov map.

Definition 2.2. A space X has the extension property if for all finite A ⊆ X and f ∈ E(A) there is some x ∈ X such that d(x, a) = f(a) for all a ∈ A.

Recall that a separable metric space X is universal if every separable metric space is isometric to a subspace of X, and ultrahomogeneous if every isometry between finite subsets of X extends to an isometry of X.
Theorem 2.3. [10] A complete separable metric space has the extension property if and only if it is universal and ultrahomogeneous. Moreover, up to isometry there is a unique complete separable metric space with these properties.

Definition 2.4. The Urysohn sphere, $U$, is the unique complete separable metric space, which is universal and ultrahomogeneous for separable metric spaces of diameter $\leq 1$.

We will consider the Urysohn sphere as a metric structure in continuous logic. An in-depth introduction can be found in [3]. See [8] and [15] for important results about the Urysohn sphere as a continuous structure. For us, the salient points are the following.

1. We consider $U$ in the “empty language” containing only the metric $d$.
2. The theory of $U$ is $\aleph_0$-categorical and has quantifier elimination in this language (see [15]). Therefore complete types are entirely determined by distances. In particular, if $M \models \text{Th}(U), C \subseteq M$ and $\bar{a} = (a_1, \ldots, a_n) \in M$ then $\text{tp}(\bar{a}/C)$ is completely determined by
   \[
   \{d(x_i, x_j) = d(a_i, a_j) : 1 \leq i, j \leq n\} \cup \{d(x_i, c) : 1 \leq i \leq n, c \in C\}.
   \]

Convention. We use $M$ to denote a sufficiently saturated “monster” model of a complete continuous theory $T$. When $T = \text{Th}(U)$, we use $U \models \text{Th}(U)$. The letters $a, b, c, \ldots$ denote singleton elements and $A, B, C, \ldots$ denote sets. Most sets are “small”, written $A \subseteq M$, which means $A \subseteq M$ and $M$ is $|A|^{+}$-saturated. We will use $\bar{a}, \bar{b}, \bar{c}, \ldots$ to denote tuples of elements; unless otherwise stated, these tuples are finite.

We will only be considering metric spaces of diameter $\leq 1$. Therefore, throughout this paper, when carrying out calculations with distances we use addition truncated at 1. We also adopt the convention that $\sup \emptyset = 0$ and $\inf \emptyset = 1$.

3. The strong order property

The strong order property is a dividing line in first-order logic, first defined in [11]. In this section, we translate the definition to continuous logic.

Definition 3.1.

1. For $n \geq 3$, $T$ has the $n$-strong order property, $\text{SOP}_n$, if there is a formula $\varphi(\bar{x}, \bar{y})$ such that
   \[
   (i) \text{ there is a sequence } (\bar{a}^i)_{i < \omega} \text{ in } M, \text{ with } \varphi(\bar{a}^i, \bar{a}^j) = 0 \text{ for all } i < j,
   \]
   \[
   (ii) \{\varphi(\bar{x}^i, \bar{x}^{i+1}) = 0 : 1 \leq i < n\} \cup \{\varphi(\bar{x}^n, \bar{x}^1) = 0\} \text{ is unsatisfiable, i.e.}
   \]
   \[
   \inf_{\bar{x}^1, \ldots, \bar{x}^n} \max\{\varphi(\bar{x}^1, \bar{x}^2), \ldots, \varphi(\bar{x}^{n-1}, \bar{x}^n), \varphi(\bar{x}^n, \bar{x}^1)\} > 0.
   \]
2. $T$ has the strong order property, $\text{SOP}_\infty$, if there is a formula $\varphi(\bar{x}, \bar{y})$ such that
   \[
   (i) \text{ there is a sequence } (\bar{x}^i)_{i < \omega} \text{ in } M, \text{ with } \varphi(\bar{a}^i, \bar{a}^j) = 0 \text{ for all } i < j,
   \]
   \[
   (ii) \text{ for } n \geq 1, \{\varphi(\bar{x}^i, \bar{x}^{i+1}) = 0 : 1 \leq i < n\} \cup \{\varphi(\bar{x}^n, \bar{x}^1) = 0\} \text{ is unsatisfiable.}
   \]

We leave it to the reader to verify that for all $n \geq 3$, if $T$ has $\text{SOP}_{n+1}$ then $T$ has $\text{SOP}_n$. Moreover, if $T$ has $\text{SOP}_\infty$ then $T$ has $\text{SOP}_n$ for all $n \geq 3$. See [11] for more on these properties.
Next, we give reformulations of these definitions. They will be easier to work with because they use complete types rather than single formulas.

**Definition 3.2.** Suppose \((\vec{a}_i)_{i<\omega}\) is an indiscernible sequence and \(p(\vec{x}, \vec{y}) = \text{tp}(\vec{a}_0, \vec{a}_1)\).

For \(n \geq 1\), \((\vec{a}_i)_{i<\omega}\) is \(n\)-cyclic if

\[
p(\vec{x}^1, \vec{x}^2) \cup p(\vec{x}^2, \vec{x}^3) \cup \ldots \cup p(\vec{x}^{n-1}, \vec{x}^n) \cup p(\vec{x}^n, \vec{x}^1)
\]

is satisfiable.

The following facts are straightforward exercises, which were pointed out to us by Lynn Scow.

**Proposition 3.3.** Suppose \(T\) is a complete theory.

(a) Given \(n \geq 3\), \(T\) has SOP\(_n\) if and only if there is an indiscernible sequence that is not \(n\)-cyclic.

(b) If \(T\) has SOP\(_\infty\) then there is an indiscernible sequence that is not \(n\)-cyclic for any \(n \geq 1\).

With these facts, we can translate questions about the strong order property to questions about satisfiability of unions of complete types. By quantifier elimination, these are really questions about the existence of certain metric spaces. When we take the union of complete types, as in the definition of SOP\(_n\) and SOP\(_\infty\), we are considering metric spaces where some distances are specified, but others are possibly left open. Therefore we will need a way to know when we can consistently fill in the open distances. The next section will give one such strategy.

4. Partial semimetric spaces and a condition for their satisfiability

**Definition 4.1.** Let \(X\) be a set and \(f : \text{dom}(f) \subseteq X \times X \rightarrow [0,1]\) be a partial function.

1. \(f\) is reflexive if for all \((x, y) \in \text{dom}(f)\), \(f(x, y) = 0\) if and only if \(x = y\).

2. \(f\) is symmetric if for all \(x, y \in X\), if \((x, y) \in \text{dom}(f)\) then \((y, x) \in \text{dom}(f)\) and \(f(x, y) = f(y, x)\).

3. \(f\) is a semimetric if it is reflexive and symmetric.

If \(f\) is a partial semimetric on \(X\), then \((X, f)\) is a partial semimetric space. A partial semimetric \(f\) is consistent if there is a metric on \(X\), which extends \(f\).

We think of a partial semimetric \(f\) on a set \(X\) as assigning distances between some pairs of points in \(X\), while leaving other distances open. Our goal is to find a criterion for when such an \(f\) can be extended to a metric on \(X\). For the rest of the section, we will only work with semimetrics \(f\) such that \(\text{dom}(f)\) contains the diagonal \(\Delta(X)\) of \(X\).

Fix a partial semimetric space \((X, f)\), with \(X\) finite. We construct a canonical total semimetric \(d_f : X \times X \rightarrow [0,1]\) extending \(f\). First, we set some notation.

Given \(x, y \in X\), define

\[
f_{\max}(x, y) = \min_{(x, z), (z, y) \in \text{dom}(f)} f(x, z) + f(y, z).
\]

The value \(f_{\max}(x, y)\) can be interpreted as follows: if \(d\) is a metric extending \(f\) then, by the triangle inequality, \(f_{\max}(x, y)\) is an upper bound for \(d(x, y)\).

The total semimetric \(d_f\) is constructed via an increasing chain \(f = f^0 \subset f^1 \subset \ldots\) of partial semimetrics on \(X\). At each stage we compute a canonical upper bound
$\gamma(x, y)$ for each open distance $(x, y)$. Then we set $\gamma$ to be the minimum of the $\gamma(x, y)$, and assign the distance $\gamma$ to all open pairs $(x, y)$ such that $\gamma(x, y) = \gamma$. We now describe the construction in full detail.

**Construction of $d_f$.** Let $f^0 = f$ and suppose we have defined $f^0 \subseteq \ldots \subseteq f^k$, with $\text{dom}(f^0) \subseteq \ldots \subseteq \text{dom}(f^k)$. If $\text{dom}(f^k) = X \times X$, we let $d_f = f^k$. Otherwise, define

$$\gamma_{k+1} = \min_{(x,y) \notin \text{dom}(f_k)} f^k_{\max}(x,y).$$

We construct $f^{k+1}$ extending $f^k$ by setting

$$\text{dom}(f^{k+1}) = \text{dom}(f^k) \cup \{(x, y) \notin \text{dom}(f_k) : f^k_{\max}(x, y) = \gamma_{k+1}\},$$

and, for $(x, y) \in \text{dom}(f^{k+1}) \setminus \text{dom}(f^k)$, setting $f^{k+1}(x, y) = \gamma_{k+1}$. Note that $f^{k+1}$ is a partial semimetric and $\text{dom}(f^k) \subseteq \text{dom}(f^{k+1})$.

Since $X$ is finite, there must be some $k$ such that $\text{dom}(f_k) = X \times X$. Let $k_f$ be the minimal such $k$, and define $d_f = f^{k_f}$. This gives us a total semimetric $d_f : X \times X \to [0, 1]$.

The total semimetric $d_f$ will play a key role in determining when $f$ can be extended to a metric. In fact, we will show that if there is any metric extending $f$, then $d_f$ is such a metric. In order to prove this, we need a few more definitions.

**Definition 4.2.** Let $(X, f)$ be a partial semimetric space.

1. A sequence $\vec{x} = (x_0, \ldots, x_n)$ of elements of $X$ is an $f$-sequence if $(x_0, x_1)$, $(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_0, x_n) \in \text{dom}(f)$.
2. Given an $f$-sequence $\vec{x} = (x_0, \ldots, x_n)$, define

   $$f[\vec{x}] = f(x_0, x_1) + f(x_1, x_2) + \ldots + f(x_{n-1}, x_n).$$

3. For $n \geq 2$, $f$ is $n$-transitive if $f(x_0, x_n) \leq f[\vec{x}]$ for all $f$-sequences $\vec{x}$ of length $n + 1$.

Note that $2$-transitive is equivalent to the statement that the triangle inequality holds on $\text{dom}(f)$. Also note that since we are assuming $\Delta(X) \subseteq \text{dom}(f)$, we have that $n$-transitivity implies $m$-transitivity for all $m \leq n$.

The main result of this section rests on the following key lemma.

**Lemma 4.3.** Let $(X, f)$ be a finite partial semimetric space. Given $0 \leq k < k_f$ and $n \geq 2$, if $f^k$ is $2^n$-transitive then $f^{k+1}$ is $2^{n-1}$-transitive.

**Proof.** Assume $f^k$ is $2^n$-transitive. Let $m = 2^{n-1}$ and suppose $\vec{x} = (x_0, \ldots, x_m)$ is an $f^{k+1}$-sequence. We want to show that $f^{k+1}(x_0, x_m) \leq f^{k+1}[\vec{x}]$.

**Case 1:** $(x_0, x_m) \notin \text{dom}(f^k)$. Then $f^{k+1}(x_0, x_m) = \gamma_{k+1}$.

**Subcase 1.1:** $(x_i, x_{i+1}) \notin \text{dom}(f^k)$ for some $0 \leq i < m$. Then

$$f^{k+1}(x_0, x_m) = \gamma_{k+1} = f^{k+1}(x_i, x_{i+1}) \leq f^{k+1}[\vec{x}].$$

**Subcase 1.2:** $(x_i, x_{i+1}) \in \text{dom}(f^k)$ for all $0 \leq i < m$. Then, in particular, we have $(x_0, x_1) \in \text{dom}(f^k)$ and $(x_0, x_m) \notin \text{dom}(f^k)$ so we may choose $1 \leq j < m$ maximal such that $(x_0, x_j) \in \text{dom}(f^k)$. These assumptions imply $(x_0, x_j), (x_j, x_{j+1}) \in$
We have \( \gamma_k \) and \( (x_0, x_{j+1}) \notin \text{dom}(f^k) \). By definition of \( \gamma_{k+1} \) and \( j \)-transitivity of \( f^k \), we have

\[
\begin{align*}
f^{k+1}(x_0, x_m) &= \gamma_{k+1} \\
&\leq f^k_{\text{max}}(x_0, x_{j+1}) \\
&\leq f^k(x_0, x_j) + f^k(x_j, x_{j+1}) \\
&\leq f^k(x_0, x_1) + \ldots + f^k(x_{j-1}, x_j) + f^k(x_j, x_{j+1}) \\
&= f^{k+1}(x_0, x_1) + \ldots + f^{k+1}(x_{j-1}, x_j) + f^{k+1}(x_j, x_{j+1}) \\
&\leq f^{k+1}[\bar{x}].
\end{align*}
\]

Case 2: \( (x_0, x_m) \in \text{dom}(f^k) \). Then \( f^{k+1}(x_0, x_m) = f^k(x_0, x_m) \). Note that if \( 0 \leq i < m \) is such that \( (x_i, x_{i+1}) \notin \text{dom}(f^k) \), then, by construction, we can fix some \( y_i \in X \) such that \( (x_i, y_i), (y_i, x_{i+1}) \in \text{dom}(f^k) \) and \( f^{k+1}(x_i, x_{i+1}) = f^k(x_i, y_i) + f^k(y_i, x_{i+1}) \). Therefore

\[
\begin{align*}
f^{k+1}(x_i, x_{i+1}) &= \begin{cases} 
  f^k(x_i, x_{i+1}) & \text{if } (x_i, x_{i+1}) \in \text{dom}(f^k) \\
  f^k(x_i, y_i) + f^k(y_i, x_{i+1}) & \text{if } (x_i, x_{i+1}) \notin \text{dom}(f^k).
\end{cases}
\end{align*}
\]

Let \( \bar{z} \) be the \( f^k \)-sequence obtained from \( \bar{x} \) where, for each \( 0 \leq i < m \) such that \( (x_i, x_{i+1}) \notin \text{dom}(f^k) \), we insert \( y_i \) between \( x_i \) and \( x_{i+1} \). Note that \( f^{k+1}[\bar{z}] = f^k[\bar{z}] \), and the length of \( \bar{z} \) is at most \( 2m = 2^n \). By \( 2^n \)-transitivity of \( f^k \), we have

\[
f^{k+1}(x_0, x_m) = f^k(x_0, x_m) \leq f^k[\bar{z}] = f^{k+1}[\bar{x}]. \qedhere
\]

Now we can prove the conditions for when a partial semimetric can be extended to a total metric.

**Theorem 4.4.** Suppose \((X, f)\) is a finite partial semimetric space. The following are equivalent:

(i) \( f \) is consistent;
(ii) \( f \) is \( n \)-transitive for all \( n \geq 2 \);
(iii) \( d_f \) is a metric on \( X \).

**Proof.**

(i) \( \Rightarrow \) (ii): Suppose \( f \) is consistent. Let \( d \) be a metric on \( X \) extending \( f \). Given \( n \geq 2 \) and an \( f \)-sequence \( (x_0, \ldots, x_n) \), we have

\[
f(x_0, x_n) = d(x_0, x_n) \leq d(x_0, x_1) + \ldots + d(x_{n-1}, x_n) = f(x_0, x_1) + \ldots + f(x_{n-1}, x_n).
\]

(ii) \( \Rightarrow \) (iii): We already have that \( d_f \) is a semimetric on \( X \). So we only need to show that \( d_f \) satisfies the triangle inequality, i.e. is \( 2 \)-transitive. If \( k_f = 0 \) then \( d_f = f \) is \( 2 \)-transitive by (ii). So we may assume \( k_f \geq 1 \). By (ii), \( f \) is \( 2^{k_f+1} \)-transitive. By repeated application of Lemma 4.3 we obtain that \( d_f = f^{k_f} \) is \( 2 \)-transitive.

(iii) \( \Rightarrow \) (i): Trivial. \( \Box \)

From the proof we see that to show \( f \) is \( n \)-transitive for all \( n \geq 2 \), it suffices show \( 2^{k_f+1} \)-transitivity. The reader may verify that \((|X| - 1)\)-transitivity would also suffice.

If \((X, f)\) is a finite partial semimetric space and \( f \) is consistent, then \( d_f \) can be regarded as the “maximal” metric on \( X \) extending \( f \). This is made precise by the following proposition, whose proof we leave as an exercise.
Proposition 4.5. Suppose \((X, f)\) is a finite partial semimetric space and \(d\) is a metric on \(X\) extending \(f\). Then \(d(x, y) \leq d_f(x, y)\) for all \(x, y \in X\).

Remark 4.6. The construction of \(d_f\) generalizes the familiar amalgamation of metric spaces (see [10]). In particular, given finite metric spaces \(X\) and \(Y\), define \(f : (X \cup Y)^2 \to [0, 1]\) such that \(f|_{X \times X} = d_X\) and \(f|_{Y \times Y} = d_Y\). Then \(f\) is a consistent partial semimetric, and a straightforward verification shows that \(d_f\) is precisely the metric on the usual metric space amalgamation of \(X\) and \(Y\).

5. Classification of \(\text{Th}(\mathcal{U})\)

The goal of this section is to place \(\text{Th}(\mathcal{U})\) in the SOP\(_n\) hierarchy. Fix \(\mathcal{U}\), a sufficiently saturated “monster” model of \(\text{Th}(\mathcal{U})\). By saturation, we have the following fact.

Proposition 5.1. If \(A \subset \mathcal{U}\) and \(X\) is a metric such that \(A \subseteq X\) and \(\mathcal{U}\) is \(|X|\)-saturated, then \(X\) isometrically embeds into \(\mathcal{U}\) over \(A\).

Lemma 5.2. Suppose \((a^i)_{i<\omega}\) is an indiscernible sequence in \(\mathcal{U}\), with \(|a^0| = k\) for some \(k \geq 1\). Given \(1 \leq i, j \leq k\), define \(\varepsilon_{i, j} = d(a^i, a^j)\).

(a) For any \(m \geq 1\) and \(i_0, \ldots, i_m \in \{1, \ldots, k\}\),
\[
\varepsilon_{i_0, i_m} \leq \varepsilon_{i_0, i_1} + \varepsilon_{i_1, i_2} + \ldots + \varepsilon_{i_{m-1}, i_m}.
\]

(b) For any \(m \geq 1\) and \(i_0, \ldots, i_m \in \{1, \ldots, k\}\), if there are \(0 \leq s < t \leq m\) such that \(i_s = i_t\) then
\[
\varepsilon_{i_s, i_0} + \varepsilon_{i_1, i_2} + \ldots + \varepsilon_{i_{m-1}, i_m}.
\]

Proof. Part (a): By indiscernibility,
\[
\varepsilon_{i_0, i_m} = d(a^0_{i_0}, a^1_{i_m})
\leq d(a^0_{i_0}, a^1_{i_1}) + d(a^1_{i_1}, a^2_{i_m}) + \ldots + d(a^{m-1}_{i_{m-1}}, a^m_{i_m})
= \varepsilon_{i_0, i_1} + \varepsilon_{i_1, i_2} + \ldots + \varepsilon_{i_{m-1}, i_m}.
\]

Part (b): We assume \(0 < s < t < m\) (the cases \(s = 0\) and \(t = m\) are similar and left to the reader). By part (a),
\[
\varepsilon_{i_s, i_m} = \varepsilon_{i_s, i_0} + \varepsilon_{i_0, i_1} + \varepsilon_{i_1, i_2} + \varepsilon_{i_{m-1}, i_m},
\]
\[
\varepsilon_{i_s, i_t} = \varepsilon_{i_s, i_s} + \varepsilon_{i_s, i_{s+1}} + \varepsilon_{i_{s+1}, i_{s+2}} + \varepsilon_{i_{m-1}, i_t}.
\]

By indiscernibility and the triangle inequality we have
\[
\varepsilon_{i_s, i_0} = d(a^1_{i_s}, a^2_{i_0})
\leq d(a^1_{i_s}, a^0_{i_0}) + d(a^0_{i_0}, a^3_{i_s}) + d(a^3_{i_s}, a^2_{i_0})
= \varepsilon_{i_s, i_0} + \varepsilon_{i_s, i_s} + \varepsilon_{i_0, i_0}
\leq \varepsilon_{i_0, i_0} + \varepsilon_{i_1, i_2} + \ldots + \varepsilon_{i_{m-1}, i_m}.
\]

These transitivity properties will be useful when trying to show that certain indiscernible sequences are \(n\)-cyclic for some \(n\). Specifically, we now show that the task of proving an indiscernible sequence is \(n\)-cyclic reduces to just checking transitivity for sequences like those in the previous lemma.
Definition 5.3. Let $(X, f)$ be a partial semимetric space.

(1) If $\bar{x} = (x_0, \ldots, x_n)$ is a sequence of elements of $X$, and $0 \leq i < j \leq n$, let $\bar{x}[i, j] := (x_i, x_{i+1}, \ldots, x_j)$.

(2) If $\bar{x} = (x_0, \ldots, x_n)$ is a sequence of elements of $X$, then a subsequence of $\bar{x}$ is a sequence of the form $(x_0, x_{i_1}, \ldots, x_{i_k}, x_n)$, for some $0 < i_1 < \ldots < i_k < n$. If $1 \leq k \leq n - 2$ then the subsequence is proper.

Lemma 5.4. Suppose $(\bar{a}^i)_{i<\omega}$ is an indiscernible sequence in $\mathcal{U}$, with $|\bar{a}^0| = k$ for some $k \geq 1$. Given $1 \leq i, j \leq k$, define $\epsilon_{i,j} = d(a^0_i, a^1_j)$. Given $n \geq 2$, the following are equivalent:

(i) $(\bar{a}^i)_{i<\omega}$ is n-cyclic;

(ii) for all $i_1, \ldots, i_n \in \{1, \ldots, k\}$, $\epsilon_{i_1, i_1} \leq \epsilon_{i_1, i_2} + \epsilon_{i_2, i_3} + \ldots + \epsilon_{i_{n-1}, i_n}$.

Proof. Fix $n \geq 2$ and an indiscernible sequence $(\bar{a}^i)_{i<\omega}$, with $|\bar{a}^0| = k$, for some $k \geq 1$. We define the partial semimetric space $(X, f)$ as follows.

$$X = \{x^l_i : 1 \leq l \leq n, 1 \leq i \leq k\}$$

$$\text{dom}(f) = \{(x^l_i, x^m_j) : 1 \leq i, j \leq k, 1 \leq l \leq n\} \cup \{(x^l_i, x^m_j) : 1 \leq i, j \leq k, 1 \leq l < n\}$$

$$f(x^l_i, x^m_j) = d(a^0_i, a^0_j) \text{ for all } 1 \leq i \leq j \leq k, 1 \leq l \leq n$$

$$f(x^l_i, x^{l+1}_j) = d(a^0_i, a^1_j) \text{ for all } 1 \leq i, j \leq k, 1 \leq l < n$$

$$f(x^{l}_i, x^m_j) = d(a^1_i, a^0_j) \text{ for all } 1 \leq i, j \leq k$$

By Proposition 5.1 $(\bar{a}^i)_{i<\omega}$ is n-cyclic if and only if $(X, f)$ is consistent. Together with Theorem 4.4 we obtain

$(\dagger)$ $(\bar{a}^i)_{i<\omega}$ is n-cyclic if and only if $f$ is $m$-transitive for all $m \geq 2$.

$(i) \Rightarrow (ii)$: If $(\bar{a}^i)_{i<\omega}$ is n-cyclic then for all $i_1, \ldots, i_n \in \{1, \ldots, k\}$, it follows from $(\dagger)$ that

$$\epsilon_{i_1, i_1} = f(x^1_{i_1}, x^n_{i_1}) \leq f(x^1_{i_1}, x^2_{i_2}) + \ldots + f(x^{n-1}_{i_{n-1}}, x^n_{i_n}) = \epsilon_{i_1, i_2} + \ldots + \epsilon_{i_{n-1}, i_n}.$$ 

$(ii) \Rightarrow (i)$: Assume $(ii)$ holds. By $(\dagger)$, it suffices to prove by induction that $f$ is $m$-transitive for all $m \geq 2$. For the base case, we are just claiming that the triangle inequality holds for $f$, wherever it is defined. We only need to consider triangles in $\text{dom}(f)$ that do not already appear in $(\bar{a}^i)_{i<\omega}$.

First, we have triangles in $\text{dom}(f)$ of the form $\{x^l_r, x^m_s, x^{l+1}_t\}$, for some $r, s, t \in \{1, \ldots, k\}$. But these can still be found in $(\bar{a}^i)_{i<\omega}$ as $\{a^1_r, a^0_s, a^0_t\}$ and $\{a^1_r, a^1_s, a^0_t\}$, respectively.

If $n \neq 3$ then we have dealt with all possible triangles in $\text{dom}(f)$. When $n = 3$, we have additional triangles of the form $\{x^l_r, x^2_s, x^l_t\}$, for some $r, s, t \in \{1, \ldots, k\}$. In this case, the distances in the triangle are $\epsilon_{r,s}$, $\epsilon_{r,s}$, and $\epsilon_{s,t}$. By $(ii)$, the triangle inequality holds for these distances.

For the induction step, fix $m > 2$ and assume that $f$ is $j$-transitive for all $j < m$. Suppose we have an $f$-sequence $\bar{v} = (v_0, \ldots, v_m)$ from $X$. Case 1: some proper subsequence of $\bar{v}$ is an $f$-sequence. Let $\tilde{v} = (v_0, \ldots, v_j)$ be this subsequence, where $j < m$, $v_0 = u_0$, and $v_j = u_m$. For $0 \leq t \leq j - 1$, set $\tilde{u}_t = \bar{v}[v_t, v_{t+1}]$. By induction, we have

$$f(u_0, u_m) = f(v_0, v_j) \leq f(v_0, v_t) + \ldots + f(v_{j-1}, v_j) \leq f[\tilde{u}_0] + \ldots + f[\tilde{u}_{j-1}] = f[\tilde{u}]$$.
Case 2: no proper subsequence of $\bar{u}$ is an $f$-sequence. Note that $(u_0, \ldots, u_m) = (x_{i_0}^e, \ldots, x_{i_m}^e)$ for some $1 \leq e_t \leq n$ and $1 \leq i_t \leq k$.

Claim: $e_s \neq e_t$ for all $s \neq t$.

Proof: For a contradiction, suppose $s < t$ and $e_s = e_t$. If $s + 1 < t$ then $(u_0, \ldots, u_s, u_t, \ldots, u_m)$ is a proper $f$-subsequence of $\bar{u}$, since $(x_{i_s}^e, x_{i_t}^e) = (x_{i_s}^e, x_{i_t}^e)$ is in $\text{dom}(f)$. Therefore we may assume $t = s + 1$. If $t < m$ then $(x_{i_s}^e, x_{i_{t+1}}^e) = (x_{i_s}^e, x_{i_{t+1}}^e)$ is in $\text{dom}(f)$ and so $(u_0, \ldots, u_s, u_{t+1}, \ldots, u_m)$ is a proper $f$-subsequence of $\bar{u}$. So we may assume $t = m$. We have $(x_{i_{s-1}}^e, x_{i_t}^e) = (x_{i_{s-1}}^e, x_{i_t}^e) \in \text{dom}(f)$, and so $(u_0, \ldots, u_{s-1}, u_t, \ldots, u_m)$ is a proper $f$-subsequence of $\bar{u}$.

By the definition of $\text{dom}(f)$, and since $\bar{u}$ is an $f$-sequence, it follows from the claim that there must be $\sigma \in S_n$, some power the permutation $(1 \ 2 \ \ldots \ n)$, such that either

$$(\sigma(e_0), \ldots, \sigma(e_m)) = (1, \ldots, n), \text{ or } (\sigma(e_0), \ldots, \sigma(e_m)) = (n, \ldots, 1).$$

Note that if $\varphi : X \rightarrow X$ is such that $\varphi(x_i^e) = x_i^{\varphi(e)}$, then for all $x, y \in X$, we have $f(x, y) = f(\varphi(x), \varphi(y))$. Therefore we may assume $(e_0, \ldots, e_m)$ is either $(1, \ldots, n)$ or $(n, \ldots, 1)$.

Next, note that $f(u_0, u_m) \leq f(\bar{u})$ if and only if $f(u_0, u_0) \leq f([u_0, u_{m-1}, \ldots, u_0])$. Therefore we may assume $(e_0, \ldots, e_m) = (1, \ldots, n)$. In particular, $m = n - 1$ and

$$\bar{u} = (x_{i_0}, \ldots, x_{i_{n-1}}).$$

By (ii), we have

$$f(x_{i_0}, x_{i_{n-1}}^n) = \epsilon_{i_{n-1}, i_0} \leq \epsilon_{i_0, i_1} + \ldots + \epsilon_{i_{n-2}, i_{n-1}} = f(\bar{u}),$$

as desired. \hfill \Box

We are now ready prove the main result of this section, a corollary of which will be the desired classification of $\text{Th}(U)$ in the SOF$_n$ hiearchy.

**Theorem 5.5.** Fix $n \geq 1$.

(a) Any indiscernible sequence in $U$ of tuples of length $n$ is $(n + 1)$-cyclic.

(b) There is an indiscernible sequence in $U$ of tuples of length $n$ that is not $n$-cyclic.

**Proof.** Part (a): Suppose $(\bar{a}^l)_{l<\omega}$ is an indiscernible sequence in $U$, with $|\bar{a}^l| = n$. To show that $(\bar{a}^l)_{l<\omega}$ is $(n + 1)$-cyclic, we use the characterization of Lemma 5.4.

If $i_0, \ldots, i_n \in \{1, \ldots, n\}$ then there are $0 \leq s < t \leq n$ such that $i_s = i_t$. By Lemma 5.4, we have $\epsilon_{i_{n-1}, i_0} \leq \epsilon_{i_0, i_1} + \ldots + \epsilon_{i_{n-1}, i_n}$.

Part (b): We construct $(\bar{a}^l)_{l<\omega}$, where $|\bar{a}^l| = n$, and define distances as follows. Given $k \leq l < \omega$,

$$d(a^k, a^l) = \begin{cases} \frac{i-1}{n} & \text{if } k < l \text{ and } i \leq j, \text{ or } k = l \text{ and } i < j \\ \frac{j-1}{n} & \text{if } k < l \text{ and } i > j. \end{cases}$$

We leave it to the reader to check that this definition satisfies the triangle inequality and so we have actually defined an indiscernible sequence in $U$. It remains to show that this sequence is not $n$-cyclic. Let $p(\bar{x}, \bar{y}) = \text{tp}(\bar{a}^l, \bar{a}^1)$ and suppose, towards a contradiction, that

$$p(\bar{x}^1, \bar{x}^2) \cup \ldots \cup p(\bar{x}^{n-1}, \bar{x}^n) \cup p(\bar{x}^n, \bar{x}^1)$$
Definition 6.1. Fix $C \subseteq M$.

1. A partial type $\pi(\bar{x}, \bar{b})$ divides over $C$ if there is a $C$-indiscernible sequence $(\bar{a}_i)_{i < \omega}$ such that $\bar{b}^i = \bar{b}$ and $\bigcup_{i < \omega} \pi(\bar{x}, \bar{b}^i)$ is unsatisfiable.

2. A partial type $\pi(\bar{x}, \bar{b})$ forks over $C$ if there some $D \supseteq \bar{b}C$ such that any extension of $\pi(\bar{x}, \bar{b})$ to a complete type over $D$ divides over $C$. 

6. Forking and Dividing in Continuous Logic

We now turn our attention toward the second main question of this paper: characterizing forking independence in the Urysohn sphere. The first step is to establish definitions and basic results about forking and dividing in the setting of continuous logic.

Let $M$ be a sufficiently saturated model of a complete continuous theory $T$. In this section, $\bar{x}, \bar{y}, \bar{a}, \bar{b}, \ldots$ denote tuples of possibly infinite length.
(3) Define ternary relations on \( A, B \subset M \),
\[
A \downarrow^d_C B \text{ if and only if } \text{tp}(A/BC) \text{ does not divide over } C,
\]
\[
A \downarrow^f_C B \text{ if and only if } \text{tp}(A/BC) \text{ does not fork over } C.
\]

The following facts about these notions are straightforward arguments in classical logic (see e.g. [13]), and their proofs can be translated directly to continuous logic.

**Fact 6.2.** Suppose \( A, B, C \subset M \).

1. \( A \downarrow^d_C B \) if and only if \( \bar{a} \downarrow^d_C \bar{b} \) for all finite \( \bar{a} \in A \) and \( \bar{b} \in B \).
2. \( A \downarrow^d_C B \) if and only if \( A \downarrow^d_{dcl(C)} B \).

Concerning the Urysohn sphere, we will show that \( \downarrow^d \) and \( \downarrow^f \) coincide, i.e. forking and dividing are the same for complete types. Our strategy will be to use the following characterization of when this happens. This is a standard result in classical logic (see e.g. [11]), and we detail the translation to continuous logic.

**Theorem 6.3.** The following are equivalent.

1. For all \( A, B, C \subset M \), \( A \downarrow_C^d B \) if and only if \( A \downarrow_C^f B \).
2. Nondividing satisfies extension, i.e. for all \( A, B, C \subset M \), if \( A \downarrow_C^d B \) and \( D \supseteq BC \) then there is \( A' \equiv_{BC} A \) such that \( A' \downarrow_C^d D \).
3. For all finite tuples \( \bar{a}, \bar{b} \in M \) and \( C = dcl(C) \subset M \), if \( \bar{a} \downarrow^d_C \bar{b} \) and \( b_\ast \in M \) is a singleton then there is \( \bar{a}' \equiv_{BC} \bar{a} \) such that \( \bar{a}' \downarrow^d_C b\ast \).

**Proof.** The equivalence of (i) and (ii) can be argued exactly as in classical logic (see [11]). The implication (ii) \( \Rightarrow \) (iii) is trivial; so we prove (iii) \( \Rightarrow \) (ii).

Assume (iii) and suppose we have \( A, B, C \subset M \) such that \( A \downarrow^d_C B \). Fix an enumeration \( A = (a_i)_{i < \lambda} \) and let \( \bar{x} = (x_i)_{i < \lambda} \) be a tuple of variables. Suppose \( D \supseteq BC \). Define
\[
\Sigma = \{ \varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \in L(dcl(C)), \ \bar{b} \in D, \ \varphi(\bar{x}, \bar{b}) = 0 \text{ divides over } dcl(C) \},
\]
where each formula uses only finitely many variables and parameters.

By compactness, for any \( \varphi \in \Sigma \) we can find \( \epsilon_\varphi > 0 \) such that \( \varphi(\bar{x}, \bar{b}) \leq \epsilon_\varphi \) divides over \( dcl(C) \). Define
\[
p(\bar{x}) = \text{tp}_x(A/B dcl(C)) \cup \{ \varphi(\bar{x}, \bar{b}) \geq \epsilon_\varphi : \varphi(\bar{x}, \bar{b}) \in \Sigma \}.
\]

**Claim:** \( p(\bar{x}) \) is satisfiable.

**Proof:** By compactness, we may reduce to a finite subset \( p_0(\bar{x}) \subseteq p(\bar{x}) \). Then \( p_0(\bar{x}) \) is implied by a type of the form
\[
\pi(\bar{x}) = \text{tp}_x(\bar{a}/\bar{b} dcl(C)) \cup \{ \varphi(\bar{x}, \bar{b}, \bar{d}) \geq \epsilon_\varphi : \varphi(\bar{x}, \bar{b}, \bar{d}) = 0 \text{ divides over } dcl(C) \},
\]
where \( \bar{a} \in A, \bar{b} \in B, \bar{d} \in D \) are finite tuples. By Fact 6.2
\[
A \downarrow^d_C B \Rightarrow A \downarrow^d_{dcl(C)} B \Rightarrow \bar{a} \downarrow^d_{dcl(C)} \bar{b}.
\]
Let \( \bar{d} = (d_1, \ldots, d_n) \). Given \( 0 \leq k \leq n \), suppose we have \( \bar{a}^k \equiv_{b dcl(C)} \bar{a} \) such that \( \bar{a}^k \downarrow^d_{dcl(C)} \bar{b}(d_{i \leq k}) \) (where \( k = 0 \) is satisfied with \( \bar{a}^0 = \bar{a} \)). By (ii), there is \( \bar{a}^{k+1} \equiv_{b(d_{i \leq k} dcl(C))} \bar{a}^k \) such that \( \bar{a}^{k+1} \downarrow^d_{dcl(C)} \bar{b}(d_{i \leq k+1}) \). This constructs \( \bar{a}^n \equiv_{b dcl(C)} \bar{a} \) such that \( \bar{a}^n \downarrow^d_{dcl(C)} \bar{b} \).
To finish the proof of the claim, we show $\bar{a}^n \models \pi(x)$. We have $\bar{a}^n \equiv_{\text{dcl}(C)} \bar{a}$ so if $\bar{a}^n \not\models p(\bar{x})$ then it follows that there is some $\varphi(\bar{x}, \bar{b}, \bar{d})$, which divides over $\text{dcl}(C)$, such that $\varphi(\bar{a}^n, \bar{b}, \bar{d}) < \epsilon_\varphi$. By definition of $\epsilon_\varphi$, this means $\text{tp}(\bar{a}^n/\bar{b}\text{dcl}(C))$ divides over $\text{dcl}(C)$, which contradicts the choice of $\bar{a}^n, \bar{f}$.

By the claim, let $A'$ be a realization of $p(\bar{x})$. We clearly have $A' \equiv_{\text{dcl}(C)} A$, and we want to show $A' \downarrow_d \text{dcl}(C) D$. But if $A' \downarrow_d \text{dcl}(C) D$ then there is some formula $\varphi(\bar{x}, \bar{b}) \in \Sigma$, such that "$\varphi(\bar{x}, \bar{b}) = 0$" $\in \text{tp}(A'/D)$. But this contradicts "$\varphi(\bar{x}, \bar{b}) \geq \epsilon_\varphi$" $\in p(\bar{x})$. By Fact 6.2 we have $A' \equiv_{BC} A$ and $A' \downarrow_C D$, as desired. \hfill \Box

7. Forking and dividing in $\text{Th}(U)$

In this section, we first characterize nondividing in the Urysohn sphere. We then show that this characterization satisfies condition (ii) of Theorem 6.3. As a result, forking and dividing are the same for complete types, and we will have given a purely geometric characterization of both notions of independence.

We continue to work in a monster model $\mathbb{U} \models \text{Th}(U)$, and use the conventions specified at the end of Section 2.

7.1. Characterization of dividing. Given $C \subset \mathbb{U}$, with $|C| < \kappa$, and $b_1, b_2 \in \mathbb{U}$ define

$$d_{\text{min}}(b_1, b_2/C) = \sup_{c \in C} |d(b_1, c) - d(b_2, c)|,$$

and

$$d_{\text{max}}(b_1, b_2/C) = \inf_{c \in C} (d(b_1, c) + d(b_2, c)).$$

Note that $d_{\text{max}}$ is reminiscent of the upper bounds used in Section 4. The significance of these values can be seen via the following result, the proof of which is left to the reader.

**Proposition 7.1.** Fix $C \subset \mathbb{U}$ and $b_1, b_2 \in \mathbb{U}$. Given $\gamma \in [0, 1]$, the following are equivalent:

(i) there are $(a_1, a_2) \models \text{tp}_{\text{all}}(b_1/C) \cup \text{tp}_{\text{all}}(b_2/C)$ such that $d(a_1, a_2) = \gamma$,

(ii) $d_{\text{min}}(b_1, b_2/C) \leq \gamma \leq d_{\text{max}}(b_1, b_2/C)$.

Note also that if $C \subseteq D$ then it follows that $d_{\text{max}}(b_1, b_2/D) \leq d_{\text{max}}(b_1, b_2/C)$ and $d_{\text{min}}(b_1, b_2/D) \geq d_{\text{min}}(b_1, b_2/C)$.

The values given by $d_{\text{max}}$ and $d_{\text{min}}$ will play a significant role in our characterization of forking independence in the Urysohn sphere. The first step is to prove a lemma which relates $d_{\text{max}}$ and $d_{\text{min}}$ to the possible behavior of indiscernible sequences in $\mathbb{U}$.

**Lemma 7.2.** Let $\bar{b} = (b_1 \ldots b_n)$ and $C \subset \mathbb{U}$.

(a) Suppose $(\bar{b})_{i \in \omega}$ is a $C$-indiscernible sequence with $\bar{b}^0 \equiv_C \bar{b}$. Then for all $l \neq k$, $1 \leq i, j \leq n$,

$$d_{\text{min}}(b_i, b_j/C) \leq d(b_i, b_j) \leq d_{\text{max}}(b_i, b_j/C).$$

(b) There is a $C$-indiscernible sequence $(\bar{b})_{i \in \omega}$, with $\bar{b}^0 \equiv_C \bar{b}$, such that $d(b_k, b_j) = d_{\text{max}}(b_k, b_j/C)$ for all $k \neq l$ and $1 \leq i, j \leq n$.

(c) There is a $C$-indiscernible sequence $(\bar{b})_{i \in \omega}$, with $\bar{b}^0 \equiv_C \bar{b}$, such that $d(b_k, b_j) = d_{\text{min}}(b_k, b_j/C)$ for all $k \neq l$ and $1 \leq i, j \leq n$. 
Proof. Part (a) is clear by the triangle inequality and the fact that for all \( l \neq k, \ 0 \leq i, j \leq n \), and for all \( c \in C \),
\[
|d(b_i, c) - d(b_j, c)| = |d(b_i, c) - d(b_j, c)| \quad \text{and} \quad d(b_i, c) + d(b_j, c) = d(b_i, c) + d(b_j, c).
\]

We prove part (b). The proof of part (c), which is similar, is left to the reader. Suppose (b) is false. Then the following is unsatisfiable:
\[
\Gamma := \{ d(y_i^k, c) = d(b_i, c) : 1 \leq i \leq n, \ k \in \omega, \ c \in C \}
\]
\[
\cup \{ \langle y_i^k, y_j^k \rangle = d_{\max}(b_i, b_j/C) : 1 \leq i, j \leq n, \ k \in \omega \}
\]
\[
\cup \{ d(y_i^k, y_j^k) = d_{\max}(b_i, b_j/C) : 1 \leq i, j \leq n, \ k < l < \omega \}.
\]
This means we must have a failure of the triangle inequality in these distances. The possible violations are:
\begin{enumerate}
\item \( d_{\max}(b_i, b_j/C) > d(b_i, c) + d(c, b_j) \) for some \( 1 \leq i, j \leq n \) and \( c \in C \);
\item \( d_{\max}(b_i, b_j/C) > d_{\max}(b_i, b_k/C) + d_{\max}(b_j, b_k/C) \) for some \( 1 \leq i, j, k \leq n \);
\item \( d(b_i, c) > d_{\max}(b_i, b_j/C) + d(b_j, c) \) for some \( 1 \leq i, j \leq n \) and \( c \in C \).
\end{enumerate}
By definition, for all \( 1 \leq i, j \leq n \) and \( c \in C \),
\[
d_{\max}(b_i, b_j/C) \leq d(b_i, c) + d(b_j, c).
\]
So (1) is impossible. If (2) holds then there are \( c, c' \) in \( C \) such that,
\[
d_{\max}(b_i, b_j/C) > d(b_i, c) + d(b_k, c) + d(b_k, c') + d(b_j, c')
\]
\[
\geq d(b_i, c') + d(b_j, c') \geq d_{\max}(b_i, b_j/C),
\]
which is a contradiction. Finally, if (3) holds then for some \( 1 \leq i, j \leq n \) and \( c, c' \in C \) we have
\[
d(b_i, c) > d(b_i, c') + d(b_k, c') + d(b_k, c) \geq d(b_i, c),
\]
which is a contradiction. Thus \( \Gamma \) is satisfiable and the desired \( C \)-indiscernible sequence exists. \( \square \)

We are now ready to characterize dividing independence in \( U \) for 1-types.

**Lemma 7.3.** Suppose \( B, C \subset U \) and \( a \in U \). Then \( \perp^a_C B \) if and only if for all \( b_1, b_2 \in B \),
\[
d(a, b_1) + d(a, b_2) \geq d_{\max}(b_1, b_2/C) \quad \text{and} \quad |d(a, b_1) - d(a, b_2)| \leq d_{\min}(b_1, b_2/C).
\]

**Proof.** (\( \Leftarrow \)): Suppose \( \perp^a_C B \). By Fact 6.2 we may assume \( B \) is finite, say \( B = \bar{b} = (b_1, \ldots, b_n) \). Let \( p(x, \bar{y}) \) denote \( \text{tp}(a, \bar{b}/C) \). Then there is an indiscernible sequence \( I = (\bar{b}) \in \omega \) such that \( \bar{b}^I \equiv_C \bar{b} \) and \( \bigcup_{i \in \omega} p(x, \bar{b}) \) is unsatisfiable, and therefore contains some failure of the triangle inequality. The possible failures are:
\begin{enumerate}
\item \( d(b_l^I, b_m^I) > d(a, b_i) + d(a, b_j) \) for some \( b_i, b_j \in \bar{b} \) and \( 1 \leq l, m \leq n \);
\item \( d(a, b_j) > d(a, b_i) + d(b_l^I, b_m^I) \) for some \( b_i, b_j \in \bar{b} \) and \( 1 \leq l, m \leq n \).
\end{enumerate}
Suppose (1) holds. By Lemma 7.2, \( d(b_l^I, b_m^I) \leq d_{\max}(b_l, b_m/C) \), thus
\[
d(a, b_i) + d(a, b_j) < d_{\max}(b_i, b_j/C),
\]
as desired.

Suppose (2) holds. Then \( |d(a, b_j) - d(a, b_i)| > d(b_l^I, b_m^I) \). By Lemma 7.2, \( d(b_l^I, b_m^I) \geq d_{\min}(b_l, b_m/C) \), thus
\[
|d(a, b_j) - d(a, b_i)| > d_{\min}(b_l, b_m/C),
\]
as desired.

$(\Rightarrow)$: First suppose there are $b_1, b_2 \in B$ such that

$$d(a, b_1) + d(a, b_2) < d_{\max}(b_1, b_2/C).$$

Let $\bar{b} = (b_1, b_2)$ and let $p(x, \bar{y})$ denote $tp(a, \bar{b}/C)$. By Lemma 7.2 there is a $C$-indiscernible sequence $\langle \bar{b}^l \rangle_{l \in \omega}$ such that for all $l, m \in \omega$ and $i, j \in \{1, 2\}$,

$$d(b^i_l, b^i_j) = d(b^j_l, b^j_m) = d_{\max}(b_1, b_2/C).$$

Then

$$\pi(x, \bar{b}) := \{d(x, b^0_1) = d(a, b_1), d(x, b^1_2) = d(a, b_2), d(b^0_1, b^0_2) = d_{\max}(b_1, b_2/C)\}$$

is contained in $p(x, \bar{b}^0) \cup p(x, \bar{b}^1)$. But $\pi(x, \bar{b})$ is unsatisfiable by our assumption, and so $a \not\equiv_{C} B$.

Finally, suppose there are $b_1, b_2 \in B$ such that

$$|d(a, b_1) - d(a, b_2)| > d_{\min}(b_1, b_2/C).$$

By a similar argument as above, we use Lemma 7.2 to obtain $a \not\equiv_{C} B$. □

The following result shows that characterizing dividing for 1-types is sufficient to characterize dividing for all types.

**Lemma 7.4.** $A \equiv_{C} B$ if and only if for every $a \in A$, $a \equiv_{C} B$.

**Proof.** $(\Rightarrow)$: Trivial.

$(\Leftarrow)$: By Fact 6.2 we may assume $A$ and $B$ are finite. Let $A = \{a_1, \ldots, a_N\}$ and $B = \bar{b} = (b_1, \ldots, b_M)$. Suppose $tp(A/BC)$ divides over $C$. We show that for some $a \in A$, $tp(a/BC)$ divides over $C$. By assumption, there is a $C$-indiscernible sequence $\langle \bar{b}^l \rangle_{l \in \omega}$, with $\bar{b}^0 = \bar{b}$, such that if $p(x, \bar{y}) = tp(\bar{a}, \bar{b}/C)$, then $\bigcup_{l \in \omega} p(x, \bar{b}^l)$ is unsatisfiable. In other words, the following is unsatisfiable:

$$\{d(x, x_j) = d(a_i, a_j) : 1 \leq i, j \leq N\} \cup \{\{d(x, b^{l}_{i}) = d(a_i, b_{j}) : 1 \leq i \leq N, 1 \leq j \leq M, 1 \leq l \leq \omega\} \cup tp(I/C).$$

So there is a failure of the triangle inequality in these distances. The only possible failures are of the form

$$d(a_i, b_j) > d(a_i, b_k) + d(b^l_i, b^m_j) \text{ or } d(b^l_i, b^m_j) > d(a_i, b_j) + d(a_i, b_k).$$

Thus, by Lemma 7.3 there is some $1 \leq i \leq N$ such that $tp(a_i/bC)$ divides over $C$, as desired.

Combining Lemma 7.3 and Lemma 7.4 we now have a full characterization of dividing for complete types.

**Theorem 7.5.** If $A, B, C \subseteq U$ then $A \equiv_{C} B$ if and only if for all $b_1, b_2 \in B$

$$d_{\max}(b_1, b_2/AC) = d_{\max}(b_1, b_2/C) \text{ and } d_{\min}(b_1, b_2/AC) = d_{\min}(b_1, b_2/C).$$

**7.2. Extension for dividing independence.** The next goal is to show our characterization of $\equiv_{C}$ satisfies condition (iii) of Theorem 6.3. We will need the following fact, which can be found in [8].

**Fact 7.6.** If $C \subseteq U$ then acl($C$) = dcl($C$) = $\overline{C}$, the usual metric space closure.

**Theorem 7.7.** Suppose $\bar{a}, \bar{b} \in U$ are finite tuples and $C = \overline{C} \subseteq U$. If $\bar{a} \equiv_{C} \bar{b}$ and $b_a \in U$ is a singleton, then there is $\bar{a}^\prime \equiv_{C} \bar{a}$ such that $\bar{a}^\prime \equiv_{C} \bar{b}_{b_a}$. 

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[2] Hrushovski, E. (1994). Stable groups. In Proofs from the Book (pp. 318-322). Springer.

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The proof of this theorem requires several lemmas. The strategy is to prove Theorem 7.7 first when \( \bar{a} \) is a singleton, and then when \( \bar{a} \) is a 2-tuple, which together will imply the general case.

First, fix \( C = \bar{C} \subseteq \mathbb{U} \). Since \( C \) is closed, it follows that for any \( b_1, b_2 \in \mathbb{U} \) there are \( c, c' \in C \) such that

\[
d_{\max}(b_1, b_2/C) = d(b_1, c) + d(b_2, c) \quad \text{and} \quad d_{\min}(b_1, b_2/C) = |d(b_1, c') - d(b_2, c')|.
\]

We will use this fact repeatedly without further mention.

Fix \( b_\ast \in \mathbb{U} \) and \( \bar{b} = (b_1, \ldots, b_n) \in \mathbb{U} \). For \( 1 \leq i \leq n \), let \( \delta_i = d_{\min}(b_\ast, b_i/C) \) and \( \epsilon_i = d_{\max}(b_\ast, b_i/C) \).

**Definition 7.8.** Fix \( a \in \mathbb{U} \).

1. Define

\[
L^* = \{ |d(b_\ast, x) - d(a, x)| : x \in \bar{b}C \}
\]
\[
U^* = \{ d(b_\ast, x) + d(a, x) : x \in \bar{b}C \}
\]
\[
L = \{ \epsilon_i - d(a, b_i) : 1 \leq i \leq n \} \cup \{ d(a, b_i) - \delta_i : 0 \leq i \leq n \} \cup \{ d(b_\ast, C) \}
\]
\[
U = \{ d(a, b_i) + \delta_i : 0 \leq i \leq n \}
\]

Let \( L(a) = \sup(L^* \cup U) \) and \( U(a) = \inf(U^* \cup U) \).

2. Given \( \gamma \in [0, 1] \), we define \( p_\ast^\gamma(x) := \text{tp}_\ast(a/bC) \cup \{ d(x, b_\ast) = \gamma \} \). Note that \( p_\ast^\gamma(x) \) is a possibly inconsistent 1-type over \( \bar{b}C b_\ast \).

**Lemma 7.9.** Fix \( a \in \mathbb{U} \) such that \( a \downarrow^d \bar{b} \).

(a) Suppose \( \gamma \in [0, 1] \) is such that \( L(a) \leq \gamma \leq U(a) \). Then \( p_\ast^\gamma \) is satisfiable.

Moreover if \( a' \models p_\ast^\gamma \) then \( a' \equiv_C a \) and \( a' \downarrow^d \bar{b} b_\ast \).

(b) \( L(a) \leq U(a) \), and so Theorem 7.7 holds when \( \bar{a} = a \) is a singleton.

**Proof.** Part (a): The fact that \( L^* \leq \gamma \leq U^* \) implies \( p_\ast^\gamma \) is satisfiable by Proposition 7.1. Suppose \( a' \models p_\ast^\gamma \). By construction, \( a' \equiv \bar{b} b_\ast \). Moreover, \( L \leq \gamma \leq U \) implies \( d(a', b_\ast) \geq d(b_\ast, C) \) and for all \( 1 \leq i \leq n \),

\[
|d(a', b_i) - d(a, b_i)| \leq \delta_i \quad \text{and} \quad d(a', b_\ast) + d(a, b_i) \geq \epsilon_i.
\]

Therefore \( a' \downarrow^{d} \bar{b} b_\ast \) by Theorem 7.3.

Part (b): Fix \( a \in L \cup L^* \) and \( \beta \in U \cup U^* \). We want to show \( a \leq \beta \).

Case 1: \( a \in L^* \) and \( \beta \in U^* \). By the triangle inequality, \( a \leq d(b_\ast, a) \leq \beta \).

Case 2: \( a \in L^* \) and \( \beta \in U \), or \( a \in L \) and \( \beta \in U^* \). We do one typical example and leave the rest to the reader. Given \( 1 \leq i, j \leq n \), we show \( d(a, b_i) - d(b_\ast, b_i) \leq d(a, b_i) + d(b_\ast, C) \) and for all \( 1 \leq i \leq n \),

\[
|d(a', b_i) - d(a, b_i)| \leq \delta_i \quad \text{and} \quad d(a', b_\ast) + d(a, b_i) \geq \epsilon_i.
\]

Therefore \( a' \downarrow^d \bar{b} b_\ast \) by Theorem 7.3.

Note that \( a \downarrow^d \bar{b} \) implies there is \( c \in C \) such that \( d(a, b_i) - d(a, b_j) \leq d(b_\ast, b_i) + |d(b_\ast, c) - d(b_j, c)| \).

There are again several cases to consider depending on the relative order of \( d(b_\ast, c) \), \( d(b_j, c) \), and \( d(b_i, c) \). We show the case when \( d(b_\ast, c) \leq d(b_j, c) \leq d(b_i, c) \). In this
Case 3: $\alpha \in L$ and $\beta \in U$. Again, we do one example. Given $1 \leq i, j \leq n$, we show

$$
\epsilon_i - d(a, b_i) \leq \delta_j + d(a, b_j),
$$
i.e. there is $c \in C$ such that

$$
d(b_i, c) + d(b_j, c) - d(b_i, a) \leq d_m(b_i, b_j/c) + d(b_j, a).
$$

Note that $a \downarrow^{d} b$ implies there is $c \in C$ such that $d(b_i, c) + d(b_j, c) \leq d(a, b_i) + d(a, b_j)$, so

$$
d(b_i, c) - d(a, b_i) \leq d(a, b_j) - d(b_j, c).
$$

Adding $d(b_i, c)$ to both sides yields:

$$
d(b_i, c) - d(a, b_i) + d(b_i, c) \leq (-d(b_j, c) + d(b_i, c)) + d(a, b_j)
\leq d_m(b_i, b_j/c) + d(b_j, a),
$$
as desired. The other cases are similar.

**Lemma 7.10.** Suppose $a_1, a_2 \in U$ and $a_1a_2 \downarrow^{d} b$. Then $|U(a_1) - U(a_2)| \leq d(a_1, a_2) \leq U(a_1) + U(a_2)$.

**Proof.** Let $L_i = L(a_i)$ and $U_i = U(a_i)$. By Lemma 7.9 we have $L_i \leq U_i$. We show $d(a_1, a_2) \leq U_1 + U_2$. The proof that $|U_1 - U_2| \leq d(a_1, a_2)$ is similar and left to the reader.

**Case 1:** $U_1 = d(a_1, b_i) + \delta_i$. Note that $U_2 \geq L_2 \geq d(a_2, b_i) - \delta_i$, so we have

$$
U_1 + U_2 \geq U_1 + d(a_2, b_i) - \delta_i = d(a_1, b_i) + d(a_2, b_i) \geq d(a_1, a_2).
$$

**Case 2:** $U_1 = d(a_1, b_i) + d(b_i, b_i)$. Note that $U_2 \geq L_2 \geq d(a_1, b_i) - d(b_i, b_i)$, so we have

$$
U_1 + U_2 \geq U_1 + d(a_1, b_i) - d(b_i, b_i) = d(a_1, b_i) + d(a_2, b_i) \geq d(a_1, a_2).
$$

**Case 3:** $U_1 = d(a_1, c_j) + d(b_i, c_j)$. Note that $U_2 \geq L_2 \geq d(a_1, c_j) - d(b_i, c_j)$, so we have

$$
U_1 + U_2 \geq U_1 + d(a_1, c_j) - d(b_i, c_j) = d(a_1, c_j) + d(a_2, c_j) \geq d(a_1, a_2).$$

**Lemma 7.11.** Suppose $a_1, a_2 \in U$ and $a_1a_2 \downarrow^{d} B$. Then there is $a_1'a_1 \equiv_{BC} a_1a_2$ such that $a_1'a_2 \downarrow^{d} \tilde{c}b$, and so Theorem 7.7 holds when $\tilde{a}$ is a 2-tuple.

**Proof.** Define the type

$$
p(x_1, x_2) = tp_{x_1, x_2}(a_1, a_2/BC) \cup \{d(x_1, b) = U(a_1)\} \cup \{d(x_2, b) = U(a_2)\}.
$$

By quantifier elimination, $p(x_1, x_2)$ is a complete 2-type over $\tilde{b}CB$. Suppose $(a_1', a_2') \models p(x_1, x_2)$. Then $a_1'a_2' \equiv_{BC} a_1a_2$ and, using Theorem 7.6 and Lemma 7.10 we have $a_1'a_2' \downarrow^{d} \tilde{b}$. So it suffices to show that $p(x_1, x_2)$ is consistent, which, as usual, means checking that there is no violation of the triangle inequality. The only non-trivial inequality to check is

$$
|U(a_1) - U(a_2)| \leq d(a_1, a_2) \leq U(a_1) + U(a_2),
$$
which is given to us by Lemma 7.10. \qed
Proof of Theorem 7.7. Suppose \( \bar{a} \downarrow_C b \), with \( \bar{a} = (a_1, \ldots, a_m) \). Define the type
\[
p(\bar{x}) := \text{tp}_k(\bar{a}/\bar{b}C) \cup \{d(x_l, b_*) = U(a_l) : 1 \leq l \leq n\}.
\]
By Theorem 7.5 and Lemma 7.10, if \( \bar{a}' \equiv_{C \bar{a}} \bar{a} \) and \( \bar{a}' \downarrow_C \bar{b}b_* \).
So we just need to show \( p(\bar{x}) \) is consistent. Suppose \( p(\bar{x}) \) implies a violation of the triangle inequality among three points \( v_1, v_2, v_3 \). If \( \{v_1, v_2, v_3\} \subseteq \bar{a}\bar{b}C \), or if \( \{v_1, v_2, v_3\} \subseteq \bar{b}Cb_* \), then we get a violation of the triangle equality among points already existing in \( U \). Thus we may assume one of the following holds:
\[
(1) \quad v_1 = a_l \in \bar{a}, \quad v_2 \in \bar{c}, \quad v_3 = b_*.
(2) \quad v_1 = a_k \in \bar{a}, \quad v_2 = a_l \in \bar{a}, \quad v_3 = b_1.
\]
If (1) holds then \( p|_{x_1} \) is inconsistent, contradicting Lemma 7.9. If (2) holds then \( p|_{x_k, x_l} \) is inconsistent, contradicting the proof of Lemma 7.11.

7.3. Characterization of forking independence. Combining Theorem 6.3, Theorem 7.5, and Theorem 7.7, we have completed the full characterization of forking and dividing for complete types in the Urysohn sphere.

Theorem 7.12. Let \( A, B, C \subset U \). Then the following are equivalent:

(i) \( A \not\downarrow_C B \),
(ii) \( A \not\downarrow_C B \),
(iii) for all \( b_1, b_2 \in B \),
\[
d_{\text{max}}(b_1, b_2/AC) = d_{\text{max}}(b_1, b_2/C) \quad \text{and} \quad d_{\text{min}}(b_1, b_2/AC) = d_{\text{min}}(b_1, b_2/C).
\]

The following properties of nonforking are straightforward from this characterization.

Corollary 7.13. Let \( A, B, C \subset U \).

(a) \( A \not\downarrow_C B \) if and only if \( a \not\downarrow_C b_1b_2 \) for all \( a \in A \) and \( b_1, b_2 \in B \).
(b) If \( A \not\downarrow_C B \) and \( D \supseteq C \) then \( A \not\downarrow_D B \).

In [14], Tent and Ziegler define the following ternary relation on finite subsets of the unbounded Urysohn space:
\[
A \downarrow_C B \iff \text{for all } a \in A, b \in B, \quad d(a, b) = \min\{d(a, c) + d(c, b) : c \in C\}.
\]
This relation (which is defined using unbounded addition) is shown to satisfy the axioms for what is called a stationary independence relation, and is used in [14] to show the isometry group of the Urysohn space is boundedly simple. The same authors use \( \downarrow \) (defined with bounded addition) in [12] to show that the isometry group of \( U \) is simple.

We can generalize the definition to arbitrary small subsets of \( U \) by
\[
A \downarrow_C B \iff \text{for all } a \in A, b \in B, \quad d(a, b) = d_{\text{max}}(a, b/C).
\]
This ternary relation is related to nonforking in the following way.

Theorem 7.14. For all \( A, B, C \subset U \), \( A \not\downarrow_C B \) implies \( A \not\downarrow_C B \) and \( B \not\downarrow_C A \).
The reader may verify this directly using our characterization of nonforking. However, in an unpublished result, the second author has shown that in any theory $T$, if $\perp$ is a stationary independence relation (as defined in [14]) on subsets of a monster model $M \models T$, then $A \perp C B$ implies $A \downarrow^d C B$ and $B \downarrow^d C A$.

The fact that forking and dividing are the same for complete types means that these notions of independence satisfy several nice properties.

**Fact 7.15.** Suppose $\perp^d$ and $\perp^f$ are the same for some theory $T$.

(a) Nondividing satisfies **full existence**, i.e. for all sets $A, B, C$ there is $A' \equiv_C A$ such that $A' \perp^d C B$.

(b) All sets are **extension bases for nonforking**, i.e. if $p$ is a partial type over a set $C$ then $p$ does not fork over $C$.

Details can be found in [1] or [13]. The translation to continuous logic is left to the reader.

The questions of whether forking equals dividing and what sets are extension bases are closely related. In particular, if the theory is simple then forking and dividing are the same for formulas and all sets are extension bases. This result has been generalized to classical NTP$_2$ theories: if $T$ is a classical NTP$_2$ theory and $C$ is an extension base for nonforking then a formula forks over $C$ if and only if it divides over $C$ (see [5]). It is interesting to ask how these results can be extended to theories with TP$_2$ (e.g. the Urysohn sphere). In [6], the first author has shown that in the theory of the generic $K_n$-free graph, forking and dividing are the same for complete types, all sets are extension bases for nonforking, but forking and dividing are not the same for all formulas. We have shown here that in the Urysohn sphere all sets are extension bases for nonforking, and forking and dividing are the same for complete types. This raises the following question.

**Question 7.16.** Are forking and dividing the same for formulas in the Urysohn sphere?

### 8. Stationary types

In this section, we use the characterization of nonforking to show that a type is stationary if and only if it is algebraic.

**Definition 8.1.** Let $C \subset M \models T$ and $p \in S_n(C)$. Then $p$ is **stationary** if for all $B \supseteq C$, there is a unique nonforking extension of $p$ to $S_n(B)$.

As with many other notions around nonforking, the study of stationary types began in stable theories. One important fact is that if $T$ is stable and $M \models T$ then any type over $M$ is stationary. Therefore, when extending nonforking types over a model, there is a unique choice of extension.

**Example 8.2.** Suppose $T$ is a theory in which nonforking satisfies extension, that is no type forks over its parameter set. Then the most trivial kind of stationary type is one of the form $\text{tp}(\bar{a}/C)$ where $\bar{a} \in \text{dcl}(C)$. In this case, the type is stationary because there is a unique extension to any larger set (which is a nonforking extension by the assumption on $T$). Note that conversely, if a type has a unique extension to any larger set then it must be of the form $\text{tp}(\bar{a}/C)$, with $\bar{a} \in \text{dcl}(C)$. 


The main result of this section will be to show that in $\text{Th}(\mathcal{U})$, the only stationary types are the trivial kind in the previous example. In other words, if a type has more than one extension to any larger parameter set then it has more than one nonforking extension to any larger parameter set.

Towards characterizing stationary types, we begin by characterizing when a 1-type has a unique nonforking extension to a parameter set obtained by adding a single element.

**Lemma 8.3.** Let $C \subset \mathcal{U}$ and $a \in \mathcal{U}$. The following are equivalent.

1. $a \in \overline{C}$.
2. For all $D \supseteq C$ and any $b \in \mathcal{U}$, $d_{\min}(a, b/D) = d(a, b) = d_{\max}(a, b/D)$.

**Proof.** (i) $\Rightarrow$ (ii): Suppose $a \in \overline{C}$. Let $(c_n)_{n<\omega}$ be a sequence in $C$ converging to $a$. Let $b \in \mathcal{U}$ and $D \supseteq C$. Then for all $n < \omega$,

$$|d(a, c_n) - d(b, c_n)| \leq d_{\min}(a, b/D) \leq d(a, b) \leq d_{\max}(a, b/D) \leq d(a, c_n) + d(b, c_n).$$

But

$$|d(a, c_n) - d(b, c_n)| \rightarrow d(a, b) \text{ and } d(a, c_n) + d(b, c_n) \rightarrow d(a, b),$$

so the result follows.

(ii) $\Rightarrow$ (i): If (ii) holds then it follows that $d_{\max}(a, a/C) = d_{\min}(a, a/C) = 0$. Therefore there is a sequence $(c_n)_{n=0}^{\infty}$ in $C$ such that $(d(a, c_n) + d(a, c_n)) \rightarrow 0$. It follows that $a \in \overline{C}$. □

**Theorem 8.4.** Let $C \subset \mathcal{U}$ and $a, b \in \mathcal{U}$. The following are equivalent:

1. $\text{tp}(a/C)$ has a unique nonforking extension to $S_1(Cb)$;
2. $\text{tp}(a/C)$ has a unique extension to $S_1(Cb)$;
3. $d_{\min}(a, b/C) = d_{\max}(a, b/C)$.

**Proof.** (i) $\Rightarrow$ (iii): Suppose $\text{tp}(a/C)$ has a unique nonforking extension to $S_1(Cb)$. Claim: $\max\{d_{\min}(a, b/C), d(b, C)\} = d_{\max}(a, b/C)$.

Proof: Note that $\max\{d_{\min}(a, b/C), d(b, C)\} \leq d_{\max}(a, b/C)$. Therefore, by (i), it suffices to show that if $\gamma \in [0, 1]$ is such that

$$\max\{d_{\min}(a, b/C), d(b, C)\} \leq \gamma \leq d_{\max}(a, b/C),$$

then there is $a' \vdash d_C b$ such that $d(a', b) = \gamma$. Fix such a $\gamma$. By Proposition [#A.4.1], there is $a' \equiv_C a$ such that $d(a', b) = \gamma$. To show $a' \vdash d_C b$, note that $d_{\min}(b, b/C) = 0 = d_{\min}(b, b/AC)$ is always true. Moreover,

$$d_{\max}(b, b/C) = d(b, C) + d(b, C) \leq \gamma + \gamma = d(a', b) + d(a', b),$$

and so $d_{\max}(b, b/C) \leq d_{\max}(b, a'/C)$. If (iii) fails, then by the claim, $d(b, C) = d_{\max}(a, b/C)$. Then for all $c \in C$, there is $c' \in C$ such that $d(a, c') + d(b, c') \leq d(b, c)$. Let $(c_n)_{n<\omega}$ be a sequence in $C$ such that $d(b, c_n) \rightarrow d(b, C)$. Then for all $n < \omega$,

$$d(b, C) \leq d(b, c'_n) \leq d(a, c'_n) + d(b, c'_n) \leq d(b, c_n).$$

By the Squeeze Theorem, $d(a, c'_n) \rightarrow 0$, so $a \in \overline{C}$. By Lemma [#8.3], $d_{\min}(a, b/C) = d_{\max}(a, b/C)$.

(iii) $\Rightarrow$ (ii): Suppose $a'$ realizes an extension of $\text{tp}(a/C)$ to $Cb$. Then $\text{tp}(a'/Cb)$ is completely determined by choice of $d(a', b)$. By Proposition [#7.1] and (ii), there is a unique choice.
(ii) ⇒ (i): We have shown that nonforking satisfies existence, so this is immediate. □

Combining previous results, we obtain the following corollary.

**Corollary 8.5.** Let $C \subseteq U$ and $a \in U$. The following are equivalent:

(i) $\text{tp}(a/C)$ is stationary;

(ii) $d_{\text{min}}(a, b/C) = d_{\text{max}}(a, b/C)$ for all $b \in U$;

(iii) $a \in C$.

The next step is to show that stationary of $n$-types is determined by stationarity of $1$-types.

**Proposition 8.6.** Let $C \subseteq B \subseteq U$. The following are equivalent.

(i) $p \in S_n(C)$ has a unique nonforking extension to $S_n(B)$;

(ii) for all $1 \leq i \leq n$ and $b \in B$, $p|_{x_i}$ has a unique nonforking extension to $S_1(Cb)$.

*Proof.* (i) ⇒ (ii): Suppose for some $1 \leq i \leq n$ and $b \in B$, $p|_{x_i}$ has two distinct nonforking extensions $q$ and $q'$ to $S_1(Cb)$. Without loss of generality, assume $i = 1$. Let $r$ and $r'$ be nonforking extensions to $S_1(B)$ of $q$ and $q'$, respectively. So $r$ and $r'$ are distinct nonforking extensions of $p|_{x_1}$ to $S_1(B)$.

**Claim:** If $a$ realizes a nonforking extension of $p|_{x_1}$ to $S_1(B)$ then there some $\bar{a}$, realizing a nonforking extension of $p$ to $S_n(B)$, such that $a_1 = a$.

*Proof:* We have $a_1 \downarrow_C B$. By extension, there is $(a_2, \ldots, a_n) \models p(a_1, x_2, \ldots, x_n)$ such that $a_2, \ldots, a_n \downarrow_{Ca_1} Ba_1$. By a transitivity property of dividing (see [13, Proposition 7.1.6]), we have $\bar{a} \downarrow_C B$, where $\bar{a} = (a_1, \ldots, a_n)$. So $\bar{a}$ is the desired realization of a nonforking extension of $p$ to $S_n(B)$.

By the claim, $r$ and $r'$ yield distinct nonforking extensions of $p$ to $S_n(B)$.

(ii) ⇒ (i): Fix $p \in S_n(C)$. Suppose $\bar{a}$ and $\bar{a}'$ realize distinct (nonforking) extensions of $p$ to $S_n(B)$. Note that $\bar{a} \equiv_C \bar{a}'$ so by quantifier elimination there must be some $1 \leq i \leq n$ and $b \in B$ such that $d(a_i, b) \neq d(a_i', b)$. Therefore $a_i$ and $a_i'$ realize distinct (nonforking) extensions of $p|_{x_i}$ to $S_1(Cb)$.

Note that the proof of (i) ⇒ (ii) holds in any theory where $\downarrow^d = \downarrow^f$.

**Corollary 8.7.** Let $C \subseteq B$ and $p \in S_n(C)$. Then $p$ has a unique nonforking extension to $S_n(B)$ if and only if $p$ has a unique extension to $S_n(B)$.

*Proof.* The forward direction follows by existence of nonforking extensions. Conversely, combining Theorem 8.4 and Proposition 8.6, we have that if $p$ has a unique nonforking extension to $S_n(B)$ then for all $1 \leq i \leq n$ and $b \in B$, $p|_{x_i}$ has a unique extension to $S_1(Cb)$. As in the proof of Proposition 8.6(ii) ⇒ (i), it follows that $p$ has a unique extension to $S_n(B)$.

We are now ready to characterize stationary types in $\text{Th}(U)$.

**Corollary 8.8.** Let $C$ be a set and $p \in S_n(C)$. Then $p$ is stationary if and only if $p$ is algebraic, i.e., $p = \text{tp}(\bar{a}/C)$ for some $\bar{a} \in C$.

*Proof.* Let $\bar{a} \models p$. By Proposition 8.8 $p$ is stationary if and only if $\text{tp}(a_i/C)$ is stationary for all $1 \leq i \leq n$. By Corollary 8.3 this is equivalent to $a_i \in C$ for all $1 \leq i \leq n$. □
9. Final remarks on first order logic

The Urysohn sphere is a natural mathematical object that can also be considered as a first order structure. A common approach is to use a relational language for distance inequalities \( d(x, y) \leq r \). In this situation, we have two unavoidable complications.

1. The theory is not \( \aleph_0 \)-categorical.
2. Saturated models contain infinitely close elements realizing \( \{d(x, y) > 0\} \cup \{d(x, y) \leq r : r \in (0, 1]\} \). Other nonstandard distances are also possible.

These facts can be seen as supporting arguments for considering the Urysohn sphere as a structure in continuous logic, where neither problem arises. However, the first order approach has led to interesting results. For example:

1. In [4], the authors construct the theory of the rational Urysohn sphere in first order logic and show it is non-simple and NSOP. They then use infinitely close elements to demonstrate that the theory does not eliminate hyperimaginaries.
2. The notion of a Urysohn space can be expanded to include distance sets other than \( \mathbb{R}^\geq 0 \) or the interval \([0, 1]\). These structures are not always complete metric spaces, and cannot all be studied in continuous logic.

In [7], the authors give necessary and sufficient conditions for when a countable subset \( D \subseteq \mathbb{R}^\geq 0 \) admits a countable homogeneous metric space with distance set \( D \), which embeds every finite metric space with distances in \( D \). Natural examples are \( D = \mathbb{Q} \cap [0, 1] \), which gives the rational Urysohn sphere, and \( D = \mathbb{Q}^\geq 0 \), which gives the rational Urysohn space. Motivated by the above observations, we can ask how our results would apply to other Urysohn spaces. The following example is one where our results can be directly applied.

**Example 9.1.** The construction of the Urysohn sphere in first order logic, given in [4], is done by way of first defining, for \( n \geq 1 \), the free \( n^{th} \) root of the complete graph. This structure is precisely the Urysohn space with distance set \( \{0, 1, \ldots, n\} \), and we denote its first order theory by \( T_n \). In this case, \( T_n \) is \( \aleph_0 \)-categorical and, more importantly, nonstandard distances do not arise in saturated models. Therefore our methods in continuous logic can be translated directly to first order logic to obtain the following results.

1. For \( n \geq 3 \), \( T_n \) is TP\(_2\), SOP\(_n\) and NSOP\(_{n+1}\).
2. For complete types in \( T_n \), forking and dividing are the same and have the same geometric characterization.

For general Urysohn spaces, the first order theory becomes more complicated in the sense that nonstandard distances are introduced and saturated models cannot necessarily be formulated as usual metric spaces. However, in the presence of certain nice conditions (e.g. quantifier elimination), our methods here lay the groundwork for obtaining similar results for the first order theories of these Urysohn spaces. For example, after accounting for nonstandard distances, we can adapt our proofs to show NSOP\(_\infty\) and SOP\(_n\), for all \( n \geq 3 \), for both the theory of the rational Urysohn sphere and the theory of the rational Urysohn space. A work in preparation will consider these cases, along with the full generality of arbitrary distance sets \( D \).
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