Mathematical and geometry model of a spherical gear with three degree of freedom

Demin Wang¹, Zhenquan Xu¹, Jun Wang², Xianting Lu¹ and Baichao Wang¹*, Junsheng Wang², Xuzhong Zhang²

¹Mechanical and Electrical Engineering, Changchun University Of Science And Technology, Changchun, Jilin, 130022, China
²Inner Mongolia First Machinery Group Co.LTD

*Corresponding author’s e-mail: 442036581@qq.com

Abstract. This paper proposes a spherical gear pair with three degrees of freedom based on meshing transmission principle. The kinematic meshing relationship between the spherical gear pair is illustrated with its pitch spheres and pitch cones. This determines that the spherical gear surface is completed with two tooth surfaces of ring-involute spherical gear and a semi-spherical bevel gear. Mathematical models and numerical results of the two tooth surfaces are presented for obtaining geometry model of the spherical gear pair with the CAD three-dimensional molding technique. This study provides the basis for future studies to design and analyze a novel spherical gear.

1. Introduction
The traditional gears such as spur gear have a single degree of freedom, which only transmits one-dimensional motion with the fixed relative position. However, spherical gear which can transmit multidimensional rotational motion has been applied in modern transmission fields such as bio-simulation machine and robot flexible joint, rather than traditional gear[1].

Most of the developed spherical gears have two degrees of freedom (2-DOF), and it is mainly divided into Trallfa gear and ring-involute spherical gear (RISG). The Trallfa gear has a transmission error in theory due to the tooth profile design[2]. In order to solve the problem of weak bearing capacity, easy wear and difficult manufacturing process, the tooth profile of Trallfa gear was modified into circular tooth and ring involute tooth in succession[3][4]. However, the transmission error still existed. The RISG overcame the transmission error and had stable continuous meshing transmission and easy machinability[5]. Unfortunately, few studies have focused on spherical gear with 3-DOF.

2. Transmission principle analysis of spherical gear
The designed spherical gear engagement with 3-DOF is considered as pure-rolling motions between its pitch spheres. As depicted in figure 1, the pure-rolling motions along x, y and z-axis correspond to motion transmissions with 3-DOF. The latitude and longitude lines are planar and spherical involute pitch circle, respectively, and the latitude is also pitch cone edge. Meshing transmissions along p axis combined with x and y axis and z-axis are considered as pure-rolling motion between any latitude lines and pitch cones, respectively.

The tooth surface of RISG is generated by planar involute revolving around z-axis, whose engagement is considered as pure-rolling motion between any planar involute pitch circle[6]. The
designed SSBG is used to connect shafts with variable intersecting axis angle. The spherical involute generated by unwrapping the surface of the pitch cone can participate in meshing while the two pitch spheres roll on each other. Therefore, teeth surfaces of the designed spherical gear are made up of tooth surfaces of RISG and SSBG, which means the gear is generated by Boolean operation implementations between the teeth of RISG and SSBG.

Figure 1. The meshing transmission of spherical gear pair.

Figure 2. The convex involute profile of RISG

3. Tooth surface of RISG

The RISG is produced by a convex or concave planar involute profile revolving around z-axis. The left-hand and right-hand involutes in convex teeth profile are symmetric about z-axis, as shown in figure 2, and the left-hand or right-hand involutes in concave teeth profile have the symmetry. In figure 2, the origin \( O_1 \) of frame \( S_1(\alpha_1x_1y_1z_1) \) is the center of pitch sphere, and \( \beta(k) \) is the azimuth angle of the start point \( Q_l \) in left-hand planar involute. The tooth surface of RISG in the frame \( S_1 \) is represented by[7]

\[
\begin{bmatrix}
  r_{11}^{(1)} \\
  r_{12}^{(1)} \\
  r_{13}^{(1)}
\end{bmatrix} =
\begin{bmatrix}
  r_i [\sin(u_i + \varphi) - u_i \cos(u_i + \varphi)] \cos v \\
  r_i [\sin(u_i + \varphi) - u_i \cos(u_i + \varphi)] \cos v \\
  r_i [\cos(u_i + \varphi) + u_i \sin(u_i + \varphi)]
\end{bmatrix}
\]

(1)

where \( r_b \) and \( u_1 \) are base circle radius of RISG and the sum of evolving angle and pressure angle, respectively. \( v \) is revolving angle of planar involute around z-axis. For convex planar involute profile, the left-hand and right-hand involutes are expressed by substituting \( \beta(k) \) and \( \beta(k) + \theta_v \) for \( \varphi \) in equation (1), respectively. Using the expressions \( \beta(k) + \pi/z_1 \) and \( \beta(k) + \theta_v + \pi/z_1 \) to substitute for \( \varphi \), the left-hand and right-hand involutes in concave teeth profile are illustrated, respectively; \( z_1 \) is the teeth number of RISG. The equations of angle \( \theta_v \) and azimuth angle \( \beta(k) \) are given as[8]

\[
\beta(k) = -\pi / 2z_1 - \text{inv}(\alpha_p) + 2k \pi / z_1, k = 1, \cdots, |k| \leq z_1 / 4 - 1/2
\]

(2)

\[
\theta_v = \pi / z_1 - 2[\text{inv}(\alpha) - \text{inv}(\alpha_p)]
\]

(3)

where \( \alpha \) is pressure angle at any point of planar involute with the case of \( \alpha_p = 20^\circ \) at pitch circle. For convex planar involute profile, the left-hand planar involute lies on the negative side of the \( x_1 \)-axis for the case of \( k = 0 \), which lies on the positive side for concave planar involute profile.

The design parameters of RISG were chosen as follows: \( z_1 = 21 \), the module \( m = 2 \), and transmission ratio along x-axis \( i_v = 1 \). The tooth surface of RISG represented by equations. (1)-(3) has been implemented in MATLAB, as shown in figure 3.
4. Mathematical models of SSBG surface

In this section, based on the transmission principle and the above proposed formulation of tooth surface of RISG, a tooth surface equation of SSBG has been derived. In figure 4, the $i_v$ equates pure-rolling ratio between two planar involute pitch circles; $\gamma_{p1}$ and $\gamma_{p2}$ are pitch-cone angle (see also figure 1) depicted with dashed lines before the pure-rolling motion. By applying the laws of sines and Stewart's theorem, the relations of triangle $po_1o_2$ are written bellow

\[
\begin{align*}
\beta_i v \cdot \beta_r 1 & = \gamma_{p1} \\
\beta_i v \cdot \beta_r 2 & = \gamma_{p2} \\
y_1 & = \sin \gamma_{p1} \\
y_2 & = \sin \gamma_{p2} \\
z_1 & = \cos \beta_i \\
z_2 & = \cos \beta_r \\
\end{align*}
\]

where $\rho$ is the length of generatrix of pitch cone, and $\beta$ is rotation angle of x-axis. Referring to equations (4)-(5), the pitch cone and transmission ratio along z-axis, $i_s$, can be determined by the design parameters of the tooth surface RISG, according to the law of sine:

\[
i_s = \frac{\sin \gamma_{p2}}{\sin \gamma_{p1}} = \frac{i \cos \beta}{\cos \beta}
\]
\( \beta \). The angle of rotation is defined as \( \Delta \theta_s \), and the tooth surface RISG in frame \( S_1(o_1x_1y_1z_1) \) can be represented as

\[
\begin{bmatrix}
\cos \Delta \theta_s & -\sin \Delta \theta_s & 0 & 0 \\
\sin \Delta \theta_s & \cos \Delta \theta_s & 0 & 0 \\
0 & 0 & -1 & l_1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\rho \sin \gamma \sin \psi \sin \theta_s \\
\rho \sin \gamma \cos \theta_s \sin \psi \\
\rho \cos \gamma \\
l_1 - \rho \cos \gamma
\end{bmatrix}
= 
\begin{bmatrix}
-\rho \sin \gamma \sin (\Delta \theta_s - \psi) \\
-\rho \sin \gamma \cos (\Delta \theta_s - \psi) \\
l_1 - \rho \cos \gamma \\
1
\end{bmatrix}
\]

(7)

where base cone angle is given by \( \gamma_b = \sin^{-1}(\sin \gamma \cos \phi_p) \), and \( \phi_p \) is pressure angle of spherical involute. The angle subtended tooth thickness of SSBG and the polar angle[10] are written as

\[
\psi = \frac{1}{\sin \gamma_b} \cos^{-1} \left( \frac{\cos \gamma}{\cos \gamma_b} \right) - \cos^{-1} \left( \frac{\tan \gamma_b}{\tan \gamma} \right)
\]

(8)

\[
\Delta \theta_s = \frac{\theta_{sb}(\beta) - \theta_{sb}(0)}{2}
\]

(9)

that provides \( \theta_{sb} = \theta_{sb}(\gamma_b) = \pi/N + 2\psi_p \) for base circle tooth thickness, and the involute angle \( \psi_p \) is obtained from equation (8) by substituting the pressure angle \( \phi_p \) for angle \( \phi \). The equation of \( \Delta \theta_s \) is derived as

\[
\Delta \theta_s = \Delta \psi_p
\]

(10)

According to equations (7)-(10), the parameters of tooth surface of SSBG are \( \gamma \) and \( \beta \). The range of values for \( \gamma \) depends on teeth number of SSBG, and \( \beta \) is defined particularly in the interval \( 0 \leq \beta \leq \pi/2 \).

For given the design parameters of RISG, the value of transmission ratio along z-axis, \( i_s = 1 \), is obtained while \( i_v = 1 \) in equation (6). On the basis of the geometry model of RISG founded in MATLAB and above formulations, the tooth surface of SSBG is simulated with \( N = 12 \), as shown in figure 6.

![Figure 6. The tooth surface of SSBG](image)

5. The geometry model of spherical gear pairs

The two tooth surfaces of RISG and SSBG have been implemented in MATLAB, then three-dimensional coordinate data of the two tooth surfaces are generated to import in Pro/E. The coordinate data is written as IBL extension format file to form a series of spherical involutes and planar involute profile. The spherical involutes are proposed to form the tooth surface of SSBG by boundary-blend command in Pro/E. The tooth surface of RISG is generated by the planar involute directly rotating around z-axis.

In order to complete the tooth of spherical gear with the two surfaces of RISG and SSBG, three sets of SSBG’s teeth are matched with the teeth of RISG to generate spherical pinion and spherical driven gear, as shown in figure 7. The three sets of SSBG’s teeth number are chosen as follows: 6, 10 and 16.
The teeth of spherical pinion are constructed with the planar involute profile cutting the three sets of SSBG’s teeth by revolving features in Pro/E. The spherical driven gear is obtained by AND operation of Boolean algorithm between RISG and the three sets of SSBG’s teeth.

![Figure 7. The solid geometry model of SSBG](image)

**6. Conclusion**

Based on the above work, we have analysed meshing transmission and conjugate motion of a gear set with 3-DOF to design the spherical gear completed with the tooth surface of RISG and RISG. This represents a new approach to design spherical gear with 3-DOF. The parametric formulation of the tooth surface of RISG was proposed and performed in MATLAB code determined by design parameters. Based on the meshing transmission principle, a mathematical model of the tooth surface of SSBG was derived by the design parameters of RISG. The coordinate data of the tooth surface of RISG and SSBG in MATLAB was imported into PROE, then the solid model of spherical gear was obtained with the help of feature-based modeling approach. This is the first study to our knowledge to present a spherical gear whose tooth surfaces profile is combined with two different gear tooth surfaces.

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