DESTROYING SATURATION WHILE PRESERVING PRESATURATION AT AN INACCESSIBLE: AN ITERATED FORCING ARGUMENT

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Abstract. We prove that nonsaturated, presaturated ideals can exist at inaccessible cardinals, answering both a question of Foreman and of Cox and Eskew. We do so by iterating a generalized version of Baumgartner and Taylor’s forcing to add a club with finite conditions along an inaccessible cardinal, and invoking Foreman’s Duality Theorem.

ideals, saturation, presaturation, iterated forcing, Foreman’s Duality Theorem

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1. Introduction

It is a classical result of Solovay [16] that the nonstationary ideal $NS_\kappa$ always has a $\kappa$-sized disjoint family of nonstationary sets; that is, in modern parlance, we say that $NS_\kappa$ is not $\kappa$-saturated. One can argue Solovay’s theorem using generic ultrapowers. Suppose for sake of contradiction that $NS_\kappa$ is $\kappa$-saturated; then a $V$-generic filter $G$ for $(\mathcal{P}(\kappa)/NS_\kappa \setminus \emptyset, \supseteq_{NS_\kappa})$ is a $V$-$\kappa$-complete $V$-normal $V$-ultrafilter with wellfounded ultrapower $Ult(V,G)$. Ultrapower arguments then yield a stationary set $S \subseteq \kappa$ in $V$ that is no longer stationary in $Ult(V,G)$, hence is nonstationary in $V[G]$. But since $NS_\kappa$ was assumed to be $\kappa$-saturated, our forcing has the $\kappa$-chain condition and hence $S$ must be stationary in $V[G]$; this is a contradiction.

Solovay then asked whether $NS_\kappa$ is $\kappa^+$-saturated, and subsequent work by Gitik and Shelah in [11] showed that $NS_\kappa$ is not $\kappa^+$-saturated, except when $\kappa = \omega_1$. Here, it is consistent (e.g. in the presence of Martin’s Maximum, c.f. [7]) for $NS_{\omega_1}$ to be $\omega_2$-saturated. Likewise, the nonstationary ideal on $\mathcal{P}_\kappa(\lambda)$ (for $\lambda \geq \kappa$) is known not to be $\kappa^+$-saturated unless $\kappa = \lambda = \omega_1$. This was due to Burke, Foreman, Gitik, Magidor, Matsubara, and Shelah; a summary and the proof of the case $\kappa = \lambda = \omega_1$ can be found in [10].

However, there are still useful arguments that can be written just assuming that $\mathcal{P}(\kappa)/NS_\kappa$ is precipitous, i.e. induces a wellfounded ultrapower $Ult(V,G)$. For instance, this simplifies Silver’s original argument in [15] that if $SCH$ fails at a singular cardinal, then the first singular cardinal at which $SCH$ fails must have countable cofinality.

One can also ask whether there is any ideal on $\kappa$ that is $\kappa$-saturated, $\kappa^+$-saturated, or even just precipitous. Results here are well-established and comprehensive.

The existence of exactly $\kappa$-saturated or $\kappa^+$-saturated ideals on inaccessible $\kappa$ are equiconsistent with a measurable cardinal. This was first shown by Kunen and Paris in [12], with weakly compact being compatible with $\kappa^+$-saturation (and it was known since early work of Lévy and Silver that a $\kappa$-saturated ideal on $\kappa$...
prevents \( \kappa \) from being weakly compact). Subsequently, Boos showed that an exactly \( \kappa^+ \)-saturated ideal on \( \kappa \) can exist at a non-weakly compact \( \kappa \) in [3].

As for successor cardinals, the consistency results are more striking. Certain arguments show that if \( \kappa \) carries a \( \kappa \)-saturated ideal, then \( \kappa \) must be weakly Mahlo, and hence not a successor. Proofs can be found in [2] and [17]. However, \( \kappa^+ \)-saturated ideals can occur at successor \( \kappa \); the known ways to achieve this come from forcing over models with huge cardinals as done by Kunen in [13] and Laver in [14].

Ideals on arbitrary sets \( Z \) project downwards to subsets \( Z' \) of \( Z \), and it is natural to ask whether regularity of the inverse embedding implies nice saturation properties of the projected ideal:

**Question 1.1** ([8], Question 13 of Foreman). Let \( n \in \omega \) and let \( \mathcal{I} \) be an ideal on \( Z \subseteq \mathcal{P}(\kappa^{+(n+1)}) \). Let \( \mathcal{J} \) be the projection of \( \mathcal{I} \) from \( Z \) to some \( Z' \subseteq \mathcal{P}(\kappa^{+n}) \). Suppose that the canonical homomorphism from \( \mathcal{P}(Z')/\mathcal{J} \) to \( \mathcal{P}(Z)/\mathcal{I} \) is a regular embedding. Is \( \mathcal{J} \) \( \kappa^{+(n+1)} \)-saturated?

The answer is no; prior work by Cox and Zeman in [6] established counterexamples. Later work by Cox and Eskew provided a template for finding counterexamples as follows. We observe that \( \mathcal{J} \) a \( \kappa^{+n+1} \)-saturated ideal on \( \kappa^{+n} \) induces a wellfounded generic ultrapower and preserves \( \kappa^{+n+1} \). So we will say that an ideal \( \mathcal{J} \) on \( \kappa^{+n} \) is \( \kappa^{+n+1} \)-presaturated if \( \mathcal{J} \) induces a wellfounded generic ultrapower and preserves \( \kappa^{+n+1} \). Our template is then:

**Fact 1.1** ([4], corollary of Theorem 1.2). Any \( \kappa^{+n+1} \)-presaturated, non-\( \kappa^{+n+1} \) saturated ideal on \( \kappa^{+n} \) provides a counterexample to Question 1.1.

To construct such ideals for successor cardinals \( \kappa = \mu^+ \) (with \( \mu \) regular and mild assumptions on cardinal arithmetic), Cox and Eskew in [4] generalized a forcing of Baumgartner and Taylor in [1] to add a club subset \( C \) of \( \kappa \) with \( < \mu \)-conditions. (Baumgartner and Taylor’s original version in [1] was for \( \mu = \omega \).) This \( C \) prevented \( \kappa^+ \)-saturated ideals on \( \kappa \) from existing in the generic extension. At the same time, their forcing was strongly proper; with use of Foreman’s Duality Theorem [8], a powerful tool for computing properties of ideals in generic extensions, Cox and Eskew were then able to argue that their forcing preserved the \( \kappa^+ \)-presaturation of a large class of ideals (including \( \kappa^+ \)-saturated ideals) in the generic extension.

This produces a generic extension in which all \( \kappa^{+n+1} \)-saturated ideals on \( \kappa^{+n} \) in the ground universe have induced \( \kappa^{+n+1} \)-presaturated, non-\( \kappa^{+n+1} \)-saturated ideals in the generic extension.

It remained open as to whether the above could be done for \( n = 0 \) and \( \kappa \) an inaccessible cardinal; this was the content of Question 8.5 of [4] and further clarifications provided in [5].

This paper’s central result establishes that Question 1.1 is consistently false at \( \kappa \) inaccessible, by an argument analogous to that of Theorem 4.1 of [4]:

**Theorem 1.2.** Suppose \( V \) is a universe of \( \text{ZFC+GCH} \) with an inaccessible cardinal \( \kappa \) admitting \( \kappa \)-complete, \( \kappa^+ \)-saturated ideals on \( \kappa \). Then there is a poset \( \mathbb{Q} \) such that:

\[(i) \, V^\mathbb{Q} \models \text{“there are no } \kappa \text{-complete, } \kappa^+ \text{-saturated ideals on } \kappa \text{”}\]
(ii) If $I \in V$ is a $\kappa$-complete, normal, $\kappa^+$-saturated ideal on $\kappa$, then $V^Q \models \text{"} \bar{T} \text{ is } \kappa^+\text{-presaturated"}$

where $\bar{T} = \{ A \in \mathcal{P}^{V^Q}(\kappa) \mid \exists N \in I \ A \subseteq N \}$.

We can further generalize Theorem 1.2(ii) as follows:

**Theorem 1.3.** With the same assumptions, there is a $Q$ such that if $\delta > \kappa$ is a regular cardinal, $I \in V$ is normal, fine, $\delta$-presaturated ideal on $Z$ of uniform completeness $\kappa$ such that

- $\Vdash_{\mathcal{B}_I} |\dot{j}_I(\kappa)| = \delta < \dot{j}_I(\kappa)$ where $\dot{j}_I$ is a name for the generic elementary embedding $j_I : V \rightarrow M$ added by $\mathcal{B}_I := \mathcal{P}(Z)/I$;
- $\mathcal{B}_I$ is proper on $IA_{<\delta}$;

then in $V^Q$,

- $\bar{T}$ is not $\delta$-saturated
- but $\bar{T}$ is $\delta$-presaturated

where $\bar{T}$ is as above.

Here, $IA_{<\delta}$ is the collection of internally approachable structures of length $< \delta$; we will give a precise definition later.

**Remark 1.4.** It will turn out that the same $Q$ will work for both Theorem 1.2 and Theorem 1.3.

**Remark 1.5.** In [4], the analogous theorem (Theorem 4.1(2)) argued that there is an $S \in \bar{T}^+$ such that $\bar{T} \upharpoonright S$ is not $\delta$-saturated, but it is $\delta$-presaturated.

The use of such an $S$ was required there due to the forcing involved not being $\kappa$-cc.

This paper is structured as follows. Section 2 presents the preliminary definitions and facts pertinent to this paper. Section 3 introduces the forcing iteration $Q$ of Theorems 1.2(i), 1.2(ii), and 1.3. Section 4 shows that saturated ideals are sundered from $V^Q$. Section 5 proves that a portion of presaturated posets remain presaturated in $V^Q$. Section 6 concludes and catalogs some conjectures.

2. **Preliminaries and notations**

Here are some definitions, theorems, and notations we use.

For a cardinal $\kappa$, we will write $\text{Reg}_\kappa$ for the set of regular cardinals below $\kappa$, and $\text{cof}(\kappa)$ for the proper class of cardinals of cofinality $\kappa$.

If $\mathbb{P}$ is a notion of forcing in $V$, we will variously use $V^\mathbb{P}$ or $V[G]$ to refer to the generic extension of $V$ by $\mathbb{P}$.

We will further take for granted that the reader is familiar with forcing, iterated forcing, and ultrapowers.

**Definition 2.1** (ideals). Let $\kappa$ be a cardinal. An ideal $I$ on $\kappa$ is a subset of $\mathcal{P}(\kappa)$ such that:

- $\emptyset \in I$, $\kappa \notin I$
(2) If $A \in I$ and $B \subseteq A$ then $B \in I$

(3) If $A, B \in I$ then $A \cup B \in I$

For $\mu \in \text{Reg}_\kappa$, the ideal $I$ is said to be $\mu$-complete if whenever $\lambda \in \text{Reg}_\mu$ and $\langle A_\alpha \mid \alpha < \lambda \rangle \subseteq I$ then

\[
\bigcup_{\alpha < \lambda} A_\alpha
\]

is also in $I$.

The ideal $I$ is said to be normal if whenever $\langle A_\alpha \mid \alpha < \kappa \rangle \subseteq I$, we have that the diagonal union

\[
\bigcup_{\alpha < \kappa} A_\alpha := \{ \beta < \kappa \mid \exists \alpha < \beta \beta \in A_\alpha \}
\]

is also in $I$.

An ideal is principal if it contains a cofinite set; for our purposes, ideals are always assumed to be nonprincipal.

For an ideal $I$ on $\kappa$, we define $I^+ := \{ S \subseteq \kappa \mid S \notin I \}$.

For example, $\text{NS}_\kappa$, the collection of nonstationary sets on $\kappa$, forms a normal ideal; its dual filter is the club filter on $\kappa$, and $(\text{NS}_\kappa)^+$ is the collection of stationary sets on $\kappa$.

**Definition 2.2.** If $I$ is an ideal on $\kappa$ then we may define an equivalence relation $\simeq_I$ on $\mathcal{P}(\kappa)$ by $A \simeq B$ if and only if $(A \setminus B) \cup (B \setminus A) \in I$.

We say that $A \leq_I B$ if $A \setminus B \in I$.

We may consider the equivalence classes $\mathcal{P}(\kappa)/I := \{ [A]_{\simeq_I} \mid A \subseteq \kappa \}$ as a poset with partial order $\leq_I$.

Given $I$ an ideal on $\kappa$, we will write $\mathcal{B}_I := (\mathcal{P}(\kappa)/I) \setminus \{ [\emptyset]_{\simeq_I} \}$; when thinking of $\mathcal{B}_I$ as a poset, we will implicitly use the partial ordering $\leq_I$ and in many cases, $\mathcal{B}_I$ will be a separative notion of forcing (or even a complete Boolean algebra).

The above two definitions are Definitions 2.1, 2.17, and 2.18 of [8].

The following two definitions summarizes some forcing properties of posets that will come in handy:

**Definition 2.3 (Chain condition, presaturation, and closure).** Let $(\mathbb{P}, \leq)$ be a poset. We say that:

(i) $(\mathbb{P}$, as Theorem 4.2) $\mathbb{P}$ is $\mu$-presaturated if for every $\lambda < \mu$ and every family $\langle A_\alpha \mid \alpha < \lambda \rangle$ of antichains, there are densely many $p \in \mathbb{P}$ such that for all $\alpha$, $\{ q \in A_\alpha \mid p \parallel q \}$ has cardinality $< \mu$.

Note that $\mu$-cc implies $\mu$-presaturation.

(ii) $\mathbb{P}$ is $< \kappa$-closed if whenever $\tau < \kappa$ and $\langle p_\alpha \mid \alpha < \tau \rangle$ is a $\leq$-decreasing sequence in $\mathbb{P}$, there is a $p \in \mathbb{P}$ such that $p \leq p_\alpha$ for all $\alpha < \tau$.

(iii) $\mathbb{P}$ is $< \kappa$-directed closed ($< \kappa$-dc) if whenever $D \subseteq \mathbb{P}$ is a directed set, such that whenever $p \in D$, $q \leq p$ such that whenever $p \in D$, $q \leq p$.

1In [8], a different version of normality is taken to be definitional, and the equivalence of these two versions is Proposition 2.19 of [8].

2that is, for all $p, q \in D$, there is an $r \in D$ such that $r \leq p, q$.
(iv) $P$ is $\mu$-preserving (for $\mu$ a $V$-cardinal) if $V^P \models "\mu \text{ is a cardinal}"$

Some of these properties have analogues for ideals as well. For $I$ an ideal on $\kappa$, we will say that $I$ is $\mu$-saturated if $\mathcal{B}_I$ has the $\mu$-cc. Additionally, we say that $I$ is $\mu$-presaturated if $\mathcal{B}_I$ is $\mu$-presaturated, and $I$ is $\mu$-preserving if $V^{\mathcal{B}_I} \models "\mu \text{ is a cardinal}"$.

For ideals, these notions relate to each other and yet another notion:

**Definition 2.4** (Definition 2.4 of [8]). An ideal $I$ is said to be precipitous if whenever $U$ is a $\mathcal{B}_I$-generic object over $V$, $Ult(V, U)$ is well-founded.

These properties have the following chain of implications:

**Theorem** (Folklore). Let $I$ be a $\kappa$-complete normal ideal on $\kappa$. Then:

$I$ is $\kappa^+$-saturated $\implies I$ is $\kappa^+$-presaturated $\implies I$ is precipitous

and $I$ is $\kappa^+$-presaturated $\iff I$ is precipitous and $\kappa^+$-preserving

Presaturation can be pushed downwards through an iteration:

**Lemma 2.5** (Lemma 2.12 of [4]). If $P \ast \dot{Q}$ is $\kappa$-presaturated then $P$ is $\kappa$-presaturated and $1_p \Vdash \dot{Q}$ is $\kappa$-presaturated.

Whether the converse holds is currently an open problem; this appears as Question 8.6 of [4].

Next we go over the notion of properness and relate properness and closedness to presaturation.

Let $\delta$ be regular uncountable, and let $H \supseteq \delta$. Then we write $P_\delta(H)$ for all subsets of $H$ of size $< \delta$, and $P^*_\delta(H)$ to denote the set of all $x \in P_\delta(H)$ such that $x \cap \delta \in \delta$.

**Definition 2.6.** Let $P$ be a notion of forcing, $\theta$ sufficiently large so that $P \in H_\theta$, and $M \prec (H_\theta, \in, P)$.

We say that $p \in P$ is an $(M, P)$-master condition if for every dense $D \in M$, $D \cap M$ is predense below $p$; equivalently, $p \Vdash P[M[\dot{G}_P] \cap V = M$.

Additionally, we say that $p$ is an $(M, P)$-strong master condition if for every $p' \leq p$, there is some $p'_M \in M \cap P$ such that every extension of $p'_M$ in $M \cap P$ is compatible with $p'$.  

Further, $P$ is (strongly) proper with respect to $M$ if every $p \in M \cap P$ has a $q \leq p$ such that $q$ is an $(M, P)$-(strong) master condition.

We say that $P$ is (strongly) $\delta$-proper on a stationary set if there is a stationary subset $S$ of $P^*_\delta(H_\theta)$ such that for every $M \in S$, $M \prec (H_\theta, \in, P)$ and $P$ is (strongly) proper with respect to $M$.

Note that $\{M \in P^*_\delta(H_\theta) \mid M \prec (H_\theta, \in, P)\}$ is a club subset of $P^*_\delta(H_\theta)$; so a forcing being $\delta$-proper on a stationary set really only depends on the properness condition.

**Fact 2.7.** If $P$ is $\delta$-proper on a stationary set, then $P$ is $\delta$-presaturated.

\footnote{It is straightforward to see that strong master conditions are also master conditions.}
This fact appears as Fact 2.8 of [4], with proof; their proof, in turn, generalizes a result of Foreman and Magidor in the case of \( \delta = \omega_1 \) (namely, Proposition 3.2 of [9]).

For the posets we will be working with, we will have a specific stationary subset witnessing \( \delta \)-properness:

**Definition 2.8.** For \( \delta \) regular and \( \theta \gg \delta \), we say that \( IA_{<\delta} \subseteq \mathcal{P}_\delta^+(H_\theta) \), the “internally approachable sets of length \( < \delta \)”, is the collection of all \( M \in \mathcal{P}_\delta^+(H_\theta) \), with \( |M| = |M \cap \delta| \), that are *internally approachable*, i.e. such that there is a \( \zeta < \delta \) and a continuous \( \subseteq \)-increasing sequence \( \langle N_\alpha \mid \alpha < \zeta \rangle \) whose union is \( M \), such that \( \bar{N} \upharpoonright \alpha \in M \) for all \( \alpha < \zeta \).

In a sense, internal approachability is preserved by any generic extension:

**Fact 2.9.** Suppose \( \mathbb{P} \) is a poset, \( M < (H_\theta, \in, \mathbb{P}), \langle N_\alpha \mid \alpha < \zeta \rangle \) witnesses that \( M \in IA_{<\delta} \), and \( G \) is \( (V, \mathbb{P}) \)-generic. Then in \( V[G] \), \( \langle N_\alpha[G] \mid \alpha < \zeta \rangle \) witnesses that \( M[G] \in IA_{<\delta} \). (Without loss of generality, we may assume that \( \mathbb{P} \in N_0 \).)

It is a standard fact that \( IA_{<\delta} \) is stationary. The following lemma makes clear its utility:

**Lemma 2.10.** Let \( \delta \) be regular and uncountable. Then:

(i) If \( \mathbb{P} \) is \( \delta \)-cc and \( M < (H_\theta, \in, \mathbb{P}) \) is an element of \( \mathcal{P}_\delta^+(H_\theta) \) (i.e. if \( M \cap \delta \in \delta \)), then \( 1_{\mathbb{P}} \) is an \( (M, \mathbb{P}) \)-master condition; in particular \( \mathbb{P} \) is \( \delta \)-proper on \( \mathcal{P}_\delta^+(H_\theta) \).

(ii) If \( \mathbb{Q} \) is \( < \delta \)-closed then \( \mathbb{Q} \) is \( \delta \)-proper on \( IA_{<\delta} \).

(iii) If \( \mathbb{P} \) is \( \delta \)-proper on \( IA_{<\delta} \) and \( \Vdash_{\mathbb{P}} \mathbb{Q} \) is \( \delta \)-cc or \( \Vdash_{\mathbb{P}} \mathbb{Q} \) is \( < \delta \)-closed then \( \mathbb{P} \ast \mathbb{Q} \) is \( \delta \)-proper on \( IA_{<\delta} \).

This is roughly Fact 2.9 out of [4]. The following proof is largely reproduced from [4] as well.

**Proof.** For part (i), let \( A \in M \) be a maximal antichain in \( \mathbb{P} \). Since \( |A| < \delta \) and \( M \cap \delta \in \delta \), we have that \( A \subseteq M \). Thus \( 1_{\mathbb{P}} \Vdash M[G] \cap \bar{V} = M \), so \( 1_{\mathbb{P}} \) is a master condition for \( M \).

Part (ii) is due to Foreman and Magidor in [9].

As for part (iii), let \( G \) be \( \mathbb{P} \)-generic over \( V \). Suppose that \( M < (H_\theta, \in, \mathbb{P} \ast \mathbb{Q}) \) and \( M \in IA_{<\delta} \). By Fact 2.9 combined with (i) and (ii), \( \mathbb{P} \) forces that \( \mathbb{Q} \) is proper with respect to \( M[G] \). Hence \( \mathbb{P} \ast \mathbb{Q} \) is proper with respect to \( M \). \( \square \)

Presaturation has a useful corollary:

**Fact 2.11.** If \( \mathbb{P} \) is \( \lambda \)-presaturated for \( \lambda \) regular then

\[
\Vdash_{\mathbb{P}} cof^V(\geq \lambda) = cof^{V[G]}(\geq \lambda)
\]

The above fact has a partial converse. We will not make use of it, but it is another known way to argue that certain iterations of presaturated forcings are presaturated:

**Fact 2.12.** If \( \mathbb{P} \) is \( \lambda^+ \omega \)-cc for some regular \( \lambda \geq \omega_1 \) and

\[
\forall n \in \omega \ \Vdash_{\mathbb{P}} cof^{V[G]}((\lambda^n)^V) \geq \lambda
\]
then $\mathbb{P}$ is $\lambda$-presaturated.

This appears as Fact 2.11 in [4], which in turn is a generalization of Theorem 4.3 of [1].

**Fact 2.13.** For a $\kappa$-complete, $\kappa^+$-saturated ideal $I \in V$, if $U$ is a $\mathcal{B}_I$-generic filter over $V$ then in $V[U]$, $\kappa^\text{Ult}(V,U) \subseteq \text{Ult}(V,U)$; that is, $\text{Ult}(V,U)$ is closed under $\kappa$-sequences from $V[U]$.

This follows from Propositions 2.9 and 2.14 of [8].

We will sometimes write $\text{Ult}(V,I)$ to denote $\text{Ult}(V,U)$, and will also write $j_I$ to denote $j_U : V \to \text{Ult}(V,U)$.

If $I \in V$ is an ideal on $\kappa$ and $\mathbb{P}$ is a notion of forcing understood from context, then we will write $\mathcal{T} := \{ N \in \mathcal{P}^{V^\mathbb{P}}(\kappa) \mid \exists A \in I \ N \subseteq A \}$.

The following two simplified versions of Foreman’s Duality Theorem will be useful later:

**Lemma 2.14.** For a $\kappa$-complete, $\kappa^+$ saturated $I \in V$, $\mathcal{T}$ is $\kappa^+$-saturated in $V^\mathcal{Q}$ if and only if $\Vdash_{\mathcal{B}_I} j_I(\mathcal{Q})$ is $\kappa^+$-cc.

This appears as Corollary 7.21 in [8].

**Theorem 2.15.** Let $I$ be a $\kappa$-complete normal precipitous ideal in $V$ and $\mathcal{Q}$ be a $\kappa$-cc poset. Then there is a canonical isomorphism witnessing that

$$\mathcal{B}(\mathcal{Q} * \mathcal{T}) \cong \mathcal{B}(\mathcal{B}_I * j_I(\mathcal{Q}))$$

where $\mathcal{B}(\mathbb{P})$ refers to the Boolean completion of $\mathbb{P}$.

This statement appears in [4] as Fact 2.24, and is a corollary of Theorem 7.14 of [8].

3. **The Forcing Iteration**

Through the rest of this paper, suppose GCH and fix $\kappa$ to be an inaccessible cardinal.

Over cardinals below $\kappa$, we will define a forcing iteration that will destroy $\kappa^+$-saturation but preserve $\kappa^+$-presaturation for ideals on $\kappa$, by adding, for each $\mu < \kappa$, $\mu$ regular, a club subset $C_\mu$ of $\mu^+$ using $< \mu$-conditions. This club $C_\mu$ will fail to contain certain ground model sets, in the sense that if $X \in V$ and $|X| \geq \mu$ then $X \not\subseteq C_\mu$.

Towards this end:

**Definition 3.1.** Let $\mu < \kappa$ be a regular cardinal. Let $\mathbb{P}(\mu)$ be the collection of all conditions $(s,f)$ such that:

1. $s \in [\mu^+ \setminus \mu]^{<\mu}$
2. $f : s \to [\mu^+ \setminus \mu]^{<\mu}$ and if $\xi, \xi' \in s$ with $\xi < \xi'$ then $f(\xi) \subseteq \xi'$.

We say $(s,f) \leq (t,g)$ if $s \supseteq t$ and whenever $\xi \in t$, $f(\xi) \supseteq g(\xi)$. 
For each \((s, f) \in P(\mu)\), \(s\) can be thought of as approximating \(C_\mu\), in the sense that \((s, f) \Vdash s \subseteq C_\mu\) (in fact, we will later define \(C_\mu = \bigcup_{(s, f) \in G} s\), for \(G\) a \(\mathbb{P}(\mu)\)-generic filter over \(V\)).

Additionally, \(f\) can be thought of as “banning” certain ordinals from ever appearing in \(\dot{C}_\mu\), in the sense that if \(\alpha < s, \beta > s\), and \(f(\alpha) \nsubseteq \beta\), then:

- it must be the case that \(s \cap (\alpha, \beta] = \emptyset\). Otherwise, if \(\gamma \in s \cap (\alpha, \beta]\), we would have that \(\beta \in f(\alpha)\) and \(\beta \notin \gamma\). Hence \(f(\alpha) \nsubseteq \gamma\), contradicting conditionhood of \((s, f)\).

- Additionally, \((s, f) \Vdash \dot{C}_\mu \cap (\alpha, \beta] = \emptyset\). This is since for every \((t, g) \leq (s, f), \beta \in g(\alpha)\); hence \(t \cap (\alpha, \beta] = \emptyset\).

**Lemma 3.2.** If \(\mu\) is a regular cardinal, then \(\mathbb{P}(\mu)\) has the following properties:

1. \(|\mathbb{P}(\mu)| = \mu^+\) hence \(\mathbb{P}(\mu)\) has the \(\mu^+\)-cc.
2. \(\mathbb{P}(\mu)\) is \(<\mu\)-directed closed.
3. If \(\theta \geq \mu^+, M \prec (H_\theta, \in, \mu^+)\), and \(M \cap \mu^+ \in \mu^+ \cap \text{cof}(\mu)\), then \(\mathbb{P}(\mu)\) is strongly proper for \(M\). Hence \(\mathbb{P}(\mu)\) preserves \(\mu^+\).
4. If \(G\) is \(\mathbb{P}(\mu)\)-generic over \(V\), then in \(V[G]\), we have that

\[
C_\mu := \bigcup_{(s, f) \in G} s
\]

is a club subset of \(\mu^+\) such that if \(X \in V\) and \(|X|^V \geq \mu\), then \(X \nsubseteq C_\mu\).
5. \(\mathbb{P}(\mu)\) is not \(\mu^+\)-cc below any condition.

**Proof.** The proofs are exactly as in Lemma 4.4 in \([3]\), where here (1) follows from assuming GCH.

For the sake of clarity, we will prove (3) and (4).

To see that (3) holds, let \(\theta \geq \mu^+, M \prec (H_\theta, \in, \mu^+)\), and \(M \cap \mu^+ \in \mu^+ \cap \text{cof}(\mu)\); suppose that \((s, f) \in \mathbb{P}(\mu) \cap M\). Observe that \(\mu^\mu = \mu\) and \(M \prec (H_\theta, \in, \mu^+, \mu)\). Let \(\delta = M \cap \mu^+\); since \(\mu^\mu = \mu^+\) as witnessed in \(H_\theta\), we have that there is a bijection \(\phi : \mu^+ \rightarrow [\mu^+]^\mu\) such that \(\phi(\mu) = M\). Without loss of generality, we may assume that for each \(\beta < \mu^+\) with \(\text{cf}(\beta) = \mu\), \(\phi \upharpoonright \beta\) surjects onto \(\beta^\mu\).

We wish to show that \(<\mu(M \cap \mu^+) \subseteq M\). Let \(\delta = M \cap \mu^+\) and suppose that \(b \in [\delta]^\mu\). Since \(\text{cf}(\delta) = \mu\), we have that \(\sup b < \delta\). But then by choice of \(\phi\), there is an \(\alpha < \sup b\) such that \(\phi(\alpha) = \beta\), and since \(\sup b < \delta\), \(\alpha \in M\). Thus \(b \in M\), and so we have shown

\(<\mu(M \cap \mu^+) \subseteq M\)

Since \(|s| < \mu \leq M \cap \mu^+\), we thus have that \(s \subseteq M\) and hence \(M \cap \mu^+ \nsubseteq s = \text{dom}(f)\). Further, if \(\xi \in s\) then \(f(\xi) \in M \cap [\mu^+]^\mu\); since \(\mu \subseteq M\) and \(\theta\) is sufficiently large, \(f(\xi) \subseteq M \cap \mu^+\).

Thus the following condition \((s', f')\) extends \((s, f)\):

\[(s', f') := (s \searrow (M \cap \mu^+), f \searrow (M \cap \mu^+ \rightarrow \{M \cap \mu^+\}))\]

We now must argue that \((s', f')\) is a strong master condition for \((M, \mathbb{P}(\mu))\). Let \((t, h) \leq (s', f')\). Then \(t_M := t \cap M\) is a \(<\mu\)-sized subset of \(M \cap \mu^+\), hence \(t_M \in M\). Further, since \((t, h) \leq (s', f')\), we have that
Proposition 3.4.

We define an Easton support iteration forcing $X$ bounded below $\text{sup}(\text{otp} X)$. Hence we may assume that $\text{otp} X = \alpha$, with $\alpha < \kappa$.

To complete the proof of strong properness, let $(u, g) \in M \cap \mathbb{P}(\mu)$, $(u, g) \leq (t, h)$. Then let $F : u \cup t \rightarrow [\mu^+]^{<\mu}$, $F(\xi) = g(\xi)$ if $\xi \in u$, and $F(\xi) = h(\xi)$ otherwise. Then $(u \cup t, F) \in \mathbb{P}(\mu)$ and $(u \cup t, F) \leq (u, g), (t, h)$.

Since $(u, g)$ was arbitrary, we have shown that every extension of $(t, h)$ in $\mathbb{P}(\mu) \cap M$ is compatible with $(t, h)$. Thus $(s', f')$ is a strong master condition. This completes our proof of $\textbf{[3]}$.

To see that $\textbf{(i)}$ holds, we have three things to show:

(i) $C_\mu$ is unbounded in $\mu^+$

(ii) $C_\mu$ is closed

(iii) If $X \in V$ and $|X|^V \geq \mu$ then $X \not\subseteq C_\mu$

To see (i), let $(s, f) \in \mathbb{P}(\mu)$ and let $\alpha < \mu^+$. By definition of $\mathbb{P}(\mu)$, $|s| < \mu$ and for each $\beta \in s$, $f(\beta)$ is a $<\mu$-sized subset of $\mu^+$. Hence $\sup_{\beta \in s} \sup f(\beta) < \mu^+$, so let $\delta$ be such that $\sup_{\beta \in s} \sup f(\beta) < \delta < \mu^+$. Then $p := (s \smallsetminus \delta, f \smallsetminus (\delta \rightarrow \emptyset))$

is a condition below $(s, f)$ such that $p \VDash \delta \in \check{C}_\mu$; thus $C_\mu$ is unbounded.

To see (ii), we argue contrapositively. Let $\beta \in \mu^+ \setminus (\mu + 1)$ and suppose $(s, f) \in \mathbb{P}(\mu)$ is such that $(s, f) \Vdash \check{\beta} \notin \hat{C}_\mu$. We will argue that $(s, f) \Vdash \check{\beta} \notin \text{Lim}(\hat{C}_\mu)$. Observe that there must be an $\alpha \in s \cap \beta$ such that $f(\alpha) \not\subseteq \beta$; for otherwise, we would have that for all $\alpha \in s \cap \beta$, $f(\alpha) \subseteq \beta$, hence $(s \smallsetminus \beta, f \smallsetminus (\beta \rightarrow \emptyset))$ would be a condition below $(s, f)$ forcing $\beta \in \check{C}_\mu$. By conditionhood of $(s, f)$, there is a unique such $\alpha$ and $\alpha$ is the largest element of $s \cap \beta$. Additionally, no extension $(t, g)$ of $(s, f)$ can have that $t \cap (\alpha, \beta) \neq \emptyset$, and hence $(s, f) \Vdash "\check{\alpha}\textrm{ is the largest element of } \check{C}_\mu \cap \check{\beta}"$. Thus $(s, f) \Vdash \check{\beta} \notin \text{Lim}(\hat{C}_\mu)$.

To see (iii), let $X \in V$ with $|X|^V \geq \mu$ and let $(s, f) \in \mathbb{P}(\mu)$. Observe that without loss of generality we may assume that $X \subseteq \mu^+ \setminus (\mu + 1)$. Further, by taking an initial segment of $X$ we may assume that $\text{otp}(X) = \mu$ and hence that $\text{cf}(\text{sup}(X)) = \mu$. Since $|s| < \mu$ and $\text{sup}(X)$ has cofinality $\mu$, $s \cap \text{sup}(X)$ is bounded below $\text{sup}(X)$.

Now we have two cases. If there is a $\xi \in s \cap \text{sup}(X)$ such that $f(\xi) \not\subseteq \text{sup}(X)$, let $\rho \in f(\xi) \setminus \text{sup}(X)$. Then $(s, f) \Vdash \check{C}_\mu \cap (\xi, \rho] = \emptyset$ and hence $(s, f) \Vdash "\check{C}_\mu \cap \check{X} \text{ is bounded below } \text{sup}(\check{X})"$. Thus $X \not\subseteq C_\mu$.

Otherwise, let $\zeta = \sup \{\text{sup}(f(\xi)) \mid \xi \in s \cap \text{sup}(X)\}$. Since each $f(\xi) \subseteq \text{sup}(X)$ and $\mu$ is regular, $\zeta < \text{sup}(X)$. Let $p = (s \smallsetminus \zeta, f \smallsetminus (\zeta \rightarrow \{\text{sup}(X)\}))$. Then $p \leq (s, f)$ and $p \Vdash \text{max}(\check{C}_\mu \cap \text{sup}(X)) = \zeta$. Hence $p \Vdash X \not\subseteq \check{C}_\mu$.

Thus $X \not\subseteq C_\mu$. This completes our proof of $\textbf{(4)}$. \hfill \Box

Definition 3.3. We define an Easton support iteration forcing $Q = \langle Q_\mu \star \check{C}(\mu) \mid \mu < \kappa \rangle$ as follows:

For each $\mu < \kappa$, if $\mu$ is regular in $V^{Q_\mu}$, let $\mathbb{C}(\mu) = \mathbb{P}(\mu)$ as above, and otherwise let $\mathbb{C}(\mu)$ be the trivial forcing.

Proposition 3.4. If $\nu < \kappa$ is regular in $V$, then $\nu$ is still regular in $V^{Q_\nu}$.
Proof. This breaks into three cases:

1. \( \nu \) is inaccessible
2. \( \nu = \tau^+ \), for \( \tau \) a regular cardinal
3. \( \nu = \lambda^+ \), for \( \lambda \) a singular cardinal

If \( \nu \) is inaccessible, then by Lemma 3.2(1) for all \( \mu < \nu \), \( C(\mu) \) is \( \mu^{++}-cc \), hence is \( \nu-cc \). Thus by Easton support, \( Q_\nu \) is also \( \nu-cc \) so preserves \( \nu \).

If \( \nu = \tau^+ \) where \( \tau \) is regular, we may decompose \( Q_\nu \) as

\[
Q_\tau \ast \hat{P}(\tau)
\]

Since \( \tau \) is regular, \( |Q_\tau| = \tau \) hence is \( \nu-cc \). Thus \( Q_\tau \) preserves \( \nu \). By Lemma 3.2(3) \( \hat{P}(\tau) \) preserves \( \nu \). Thus \( \hat{Q}_{\geq \nu} \) preserves \( \nu \).

If \( \nu = \lambda^+ \) where \( \lambda \) is singular, we decompose \( Q_\nu \) as

\[
Q_\lambda \ast \hat{P}(\nu)
\]

Here, the situation is more complicated, since now \( |Q_\lambda| = \lambda^+ = \nu \). So we must verify more directly that \( \nu \) is preserved.

So observe that if \( \nu \) is collapsed, then \( V^{Q_\lambda} \models |\nu| \leq |\lambda| \) and since \( \lambda \) is singular, we would have a \( Q_\lambda \)-name \( \dot{f} : \delta \to \check{\nu} \) for a cofinal sequence in \( \check{\nu} \) for some regular cardinal \( \delta < \lambda \).

But we may decompose \( Q_\lambda \) into

\[
Q_\delta \ast \hat{P}(\delta) \ast \hat{Q}_{\geq \delta^+}
\]

Now, \( \hat{Q}_{\geq \delta^+} \) is \( < \delta^+-\text{directed closed} \), so \( \hat{Q}_{\geq \delta} \) could not have added such an \( f \). Additionally, \( \hat{P}(\delta) \) satisfies the \( \delta^{++}-cc \), hence is \( \nu-cc \). Thus \( \hat{P}(\delta) \) also could not have added \( f \). Finally, \( Q_\delta \) satisfies the \( \delta^+-cc \), hence is also \( \nu-cc \). Thus \( Q_\delta \) could not have added such an \( f \) either.

As in the successor of a regular case, \( \hat{P}(\nu) \) and \( \hat{Q}_{\geq \nu} \) preserve \( \nu \) as well. \( \square \)

Corollary 3.5. \( Q \) preserves cardinals.

Proof. Since \( \kappa \) is inaccessible, \( Q \) is, by Lemma 3.2(1) an Easton support iteration of \( \kappa-cc \) posets hence is \( \kappa-cc \). Thus \( Q \) preserves cardinals \( \geq \kappa \).

For \( \nu < \kappa \) regular, we have that \( Q = Q_\nu \ast \hat{C}(\nu) \ast \hat{Q}_{\geq \nu} \). By the preceding proposition, \( Q_\nu \) preserves \( \nu \). By Lemma 3.2(3) \( \hat{C}(\nu) \) preserves \( \nu \). And by Lemma 3.2(2) \( \hat{Q}_{\geq \nu} \) is \( < \nu^+-\text{directed closed} \) hence preserves \( \nu \). \( \square \)

Remark 3.6. Note that \( |Q| = \kappa \) so \( Q \) preserves \( GCH_{\geq \kappa} \).

By Lemma 3.2 each \( P(\mu), \mu < \kappa \) regular, preserves \( GCH \); hence \( Q \) preserves \( GCH_{\leq \kappa} \) as well.

4. Destroying Saturation

Since \( Q \) projects to each \( Q_\mu \ast \hat{P}(\mu) \), \( \mu < \kappa \) regular, we may, for each such \( \mu \), let \( G_\mu \) be the restriction of the \( Q \)-generic \( G \) to \( P(\mu) \) and define \( C_\mu = \{ \xi \mid \exists (s, f) \in G_\mu \xi \in s \} \). By Lemma 3.2(4) \( C_\mu \) is a club subset of \( \mu^+ \) in \( V^{Q_\mu \ast \hat{P}(\mu)} \) and for every \( X \in V^{Q_\mu} \) such that \( X \subseteq [\mu, \mu^+) \) and \( X \) has \( V^{Q_\mu} \)-cardinality \( \geq \mu \), \( X \not\subseteq C_\mu \).
**Proposition 4.1.** Suppose that $I \in V$ is $\kappa$-complete, normal, and $\kappa^+$-saturated. Then in $V^\mathcal{Q}$, $\mathcal{T}$ is not $\kappa^+$-saturated.

Before we prove this, it will be helpful to isolate a lemma on what $j_I(\mathcal{Q})$ looks like in $\text{Ult}(V, I)$:

**Lemma 4.2.** Let $I$ be a $\kappa$-complete, normal, fine precipitous ideal. Then in $\text{Ult}(V, I)$, $j_I(\mathcal{Q}) \cong \mathcal{Q} \ast \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is a name for an Easton support iteration $\langle \mathbb{R}_\lambda \ast \dot{\mathcal{C}}(\lambda) \mid \lambda \in [\kappa, j(\kappa)] \rangle$, such that if $\lambda$ is regular, $\mathcal{C}(\lambda) = \mathbb{P}(\lambda)$, and $\mathcal{C}(\lambda)$ is the trivial forcing otherwise.

**Proof.** This follows from the elementarity of $j_I$. \qed

**Remark 4.3.** This is unlike a $\lambda$-complete, $\lambda^+$-saturated ideal $J$ on $\lambda$ a successor cardinal; for $\lambda$ a successor cardinal, we would have that $j_J(\lambda) = \lambda^+$. The argument can be found in [8].

**Proof of Proposition 4.1.** By Lemma 4.2 in $V^{\mathcal{B}_I}$, $j_I(\mathcal{Q}) \cong \mathcal{Q} \ast \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is an Easton support iteration $\langle \mathbb{R}_\lambda \ast \dot{\mathcal{C}}(\lambda) \mid \lambda \in [\kappa, j_I(\kappa)] \rangle$ as in the lemma.

Since $\mathbb{P}(\alpha)$ is not $\kappa^+$-saturated for all $\alpha \in [\kappa, j_I(\kappa)]$ regular, $j_I(\mathcal{Q})$ is not $\kappa^+$-saturated. So by Lemma 2.13 in $V^\mathcal{Q}$, $\mathcal{T}$ is not $\kappa^+$-saturated. \qed

We now prove Theorem 1.2(i).

**Proof of Theorem 1.2(i).** Let $G$ be $\mathcal{Q}$-generic, and suppose that in $V[G]$ there is a $\kappa$-complete, $\kappa^+$-saturated ideal $\mathcal{J}$ on $\kappa$.

Let $U$ be $P(\kappa)/\mathcal{J}$-generic over $V[G]$, and let $j : V[G] \to \text{Ult}(V[G], U)$ be the generic ultrapower.

Let $N = \bigcup_{\alpha \in \text{ORD}} j(V_\alpha)$. Then $j(\mathcal{Q}) \in N$ and hence $\text{Ult}(V[G], U) = N[g']$ for some $g' \in V[G \ast U]$ which is $j(\mathcal{Q})$-generic over $N$.

Observe that $\kappa$ is still inaccessible in $N[g']$ by inaccessibility in $V[G]$ and by being the critical point of $j$.

Since $j(\kappa) > \kappa$ and $j(\kappa)$ is a cardinal in $N[g']$, $j(\kappa) > (\kappa^+)^{N[g']} \geq (\kappa^+)^{V[G]}$ (by $\kappa$-closure and $\kappa^+$-saturation of $\mathcal{J}$). Further, by the usual ultrapower argument, $[j(\kappa)] \leq 2^\kappa = \kappa^+$.

So $j(\kappa)$ is not a cardinal in $V$, but by Fact 2.13 $N[g']$ is closed under $\kappa$-sequences from $V[G]$.

Work in $N[g']$. Let $g'$ be the projection of $j(\mathcal{Q})$ to $\mathbb{P}(\kappa)$, and let

$$C_\kappa = \bigcup_{(s, f) \in g'} s$$

Then

$$N[g'] \models C_\kappa \text{ is club in } \kappa^+ \text{ and } \forall X \in N[X]^N \geq \kappa, \, X \not\in C_\kappa$$
Since $V[G * U]$ is a $\kappa^+$-cc extension of $V$, we may let $D \in V$ be such that in $V[G * U]$, $D$ is a club subset of $C_\kappa$. Let $E \subseteq D$ be in $V$, $(o.t.(E))^V = \kappa$, $\alpha = \text{sup } E$; since $cf(\alpha) = \kappa$, let $\phi : \kappa \to \alpha$ be a normal increasing sequence.

Let $E' = \text{lim}(E) \cap \text{ran}(\phi)$.

Then $E' \subseteq D$ and $|E'|^V = \kappa$ since $\kappa$ is inaccessible. Further, $j(\phi) \in N$ and $j(\phi) \restriction \kappa : \kappa \to \kappa^+$ is also in $N$.

Thus $\text{ran}(j(\phi) \restriction \kappa) \in N$ and $j''E' \subseteq \text{ran}(j(\phi) \restriction \kappa) \subseteq j''\alpha$.

But $j''E' = \text{ran}(j(\phi) \restriction \kappa) \cap j(E') \in N$; and since $E' = \{ \beta \in \text{ran}(\phi) \mid j(\beta) \in j(E') \}$, we have that $E'$ is a subset of $C_\kappa$ with $|E'|^N = \kappa$ and $E' \subseteq [\kappa, \kappa^+]$

This contradicts Statement 1.3(ii), and hence $\mathcal{J}$ cannot be $\kappa^+$-saturated.

5. Preserving Presaturation

We now prove Theorem 1.3(ii).

Proof of Theorem 1.3(ii) Let $\mathcal{J} \in V$ be a $\kappa$-complete, normal, $\kappa^+$-saturated ideal in $V$. Work in $V^\mathcal{J}$ and let $U$ be the generic ultrafilter. Then in $\text{Ult}(V, U)$, by Lemma 2.5, $\text{Ult}(Q) \cong Q * P(\kappa) * \mathbb{R}$, where $\mathbb{R}$ is an Easton support iteration $(\mathcal{R}_\lambda * \mathbb{C}(\lambda) \mid \lambda \in [\kappa^+, j(\kappa))]$, such that if $\lambda$ is regular, $\mathbb{C}(\lambda) = P(\lambda)$, and $\mathbb{C}(\lambda)$ is the trivial forcing otherwise.

We will argue that $\mathcal{B}_I * j_I(Q)$ is $\kappa^+$-proper on a stationary set, and hence is $\kappa^+$-presaturated.

Observe that $\mathcal{B}_I$ is $\kappa^+$-cc. Since $\mathcal{B}_I$ is $< \kappa$-closed, in $\text{Ult}(V, U)$, $Q$ is still $\kappa$-cc (hence $\kappa^+$-cc). Thus, in $\text{Ult}(V, U)$, $\mathcal{B}_I * Q$ is $\kappa^+$-cc and hence is $\kappa^+$-proper on $P^*_\kappa(H_\theta)$ for all sufficiently large $\theta$.

The difficulty comes in assuring $P(\kappa)$ and $\mathbb{R}$ preserve the properness on a stationary set.

Work in $\text{Ult}(V, U)^Q$. Here, $P(\kappa)$ is proper on $S := \{ M \prec (H_\theta, \in, \kappa^+) \mid |M| = |M \cap \kappa^+| = \kappa \text{ and } M \cap \kappa^+ = \text{cof}(\kappa) \}$, and by the $< \kappa^+$-directed closedness of $\mathbb{R}$ and Fact 2.9, $\text{Ult}(P(\kappa))$ is proper on $IA_{<\kappa^+}$. But not only is $S$ stationary, $S$ is a club subset of $P^*_\kappa(H_\theta)$, and hence $S \cap IA_{<\kappa^+}$ is also stationary.

Thus $\mathcal{B}_I * j_I(Q)$ is $\kappa^+$-proper on a stationary subset of $P^*_\kappa(H_\theta)^V$, hence is $\kappa^+$-presaturated. But by Theorem 2.14, $\mathcal{B}_I * j_I(Q) \cong Q * \mathcal{B}_P$, then by Lemma 2.5, $\mathcal{B}_I * j_I(Q)$ is $\kappa^+$-presaturated.

A more general argument will prove Theorem 1.3.

Proof of Theorem 1.3. This argument breaks into two cases.

Case 1: $\delta$ inaccessible. By Theorem 2.14, we once again have that

$$\mathcal{B}(Q * \mathcal{B}_P) \cong \mathcal{B}(\mathcal{B}_I * j_I(Q))$$

and by a slight modification of Lemma 4.2

$$j_I(Q) = \mathcal{Q} * (j_I(Q)) \restriction [\kappa, \delta] * P(\delta) * (j_I(Q)) \restriction [\delta^+, j_I(\kappa))$$

where

- $\mathcal{Q}$ is $\kappa$-cc, hence $\delta$-cc
Case 2: \( \delta \) is a successor cardinal with \( \rho^+ = \delta \). Theorem 2.15 and Lemma 4.2 now give that
\[
j_I(Q) = \hat{Q} \ast (j_I(Q)) \upharpoonright [\kappa, \rho] \ast \mathbb{P}(\rho) \ast j_I(Q) \upharpoonright [\delta, j_I(\kappa)]
\]
where
- \( \hat{Q} \) is \( \delta \)-cc
- \( (j_I(Q)) \upharpoonright [\kappa, \rho] \) is an Easton support iteration of \( \delta \)-cc posets
- \( \mathbb{P}(\rho) \) is proper on \( S := \{ M < (H_\theta, \in, \delta) \mid |M| = |M \cap \delta| = \rho \text{ and } M \cap \delta \in \text{cof}(\rho) \} \)
- \( j_I(Q) \upharpoonright [\delta, j_I(\kappa)] \) is an Easton support iteration of \( < \delta \)-directed closed posets

Here, we have that in \( V, \mathcal{B}_I \) is proper on \( IA_{<\delta} \) by assumption. Additionally, in \( \text{Ult}(V, U)^Q, j_I(Q) \) is proper on \( S \cap IA_{<\delta} \) which is also stationary in \( \mathcal{P}_\delta^*(H_\theta) \) for sufficiently large \( \theta \); this is by Lemma 2.10.

Thus \( \mathcal{B}_I \ast j_I(Q) \) is \( \delta \)-proper on a stationary set, hence, by Lemma 2.10, is \( \delta \)-presaturated.

Theorem 2.15 and Lemma 2.5 then tell us that \( \mathcal{B}_I \) is \( \delta \)-presaturated.

\( \Box \)

6. Conclusions and Questions

We thus have that in \( V^Q \), \( \kappa^+ \)-saturated ideals on \( \kappa \) in \( V \) are no longer \( \kappa^+ \)-saturated, but remain \( \kappa^+ \)-presaturated. Hence we have counterexamples to Question 5.1 at inaccessible cardinals.

It seems plausible that \( \mathcal{Q} \) is not the only forcing that accomplishes this:

**Question 6.1.** Observe that Proposition 4.1 and the proof of Theorem 1.2(ii) only required arguing that \( \mathcal{Q} \) is \( \kappa \)-cc and if \( I \) is a \( \kappa \)-complete, \( \kappa^+ \)-saturated ideal in \( V \), then in \( V^{\mathcal{B}_I} \), \( j_I(Q) \) is not \( \kappa \)-cc but is \( \kappa^+ \)-presaturated.

**Can we extend the proof of Theorem 1.2(ii) to any exactly \( \kappa \)-cc forcing?**

Using Fact 2.12, Cox and Eskew argued in [4] that their forcing \( \mathbb{P} \) preserved the \( \kappa^+ \)-presaturation of a much larger class of ideals on \( \kappa \); this was possible because in their context, \( j_I(\mathbb{P}) \) was \( \delta^{+\omega} \)-cc. This naturally leads to the following question

**Question 6.2.** Does \( \mathcal{Q} \) preserve the \( \delta \)-presaturation of all \( \delta \)-presaturated ideals on \( \kappa \)?

However, for us, \( j_I(Q) \) will not be \( \delta^{+\omega} \)-cc, so Fact 2.12 does not apply. This is why we only show that ideals that are \( \delta \)-proper on \( IA_{<\delta} \) remain \( \delta \)-presaturated; \( \delta \)-saturated ideals are \( \delta \)-proper on \( IA_{<\delta} \), so this was sufficient for our purposes. We would need more powerful tools to argue that all \( \kappa^+ \)-presaturated ideals in \( V \) remain \( \kappa^+ \)-presaturated in \( V^Q \).
References

[1] James E. Baumgartner and Alan D. Taylor. Saturation Properties of Ideals in Generic Extensions. II. Transactions of the American Mathematical Society, 271(2):587, jun 1982.

[2] James E. Baumgartner, Alan D. Taylor, and Stanley Wagon. On Splitting Stationary Subsets of Large Cardinals. J. Symbolic Logic, 42(2):203–214, 1977.

[3] William Boos. Boolean extensions which efface the Mahlo property. Journal of Symbolic Logic, 39(2):254–268, jun 1974.

[4] Sean Cox and Monroe Eskew. Strongly proper forcing and some problems of Foreman. Transactions of the American Mathematical Society, 371(7):5039–5068, dec 2018.

[5] Sean Cox and Noah Schoem. Reference request: destroying saturation at an inaccessible? https://mathoverflow.net/q/315754, 2018.

[6] Sean Cox and Martin Zeman. Ideal Projections and Forcing Projections. The Journal of Symbolic Logic, 79(4):1247–1285, dec 2014.

[7] M. Foreman, M. Magidor, and S. Shelah. Martin’s Maximum, Saturated Ideals, and Non-Regular Ultrafilters. Part I. The Annals of Mathematics, 127(1):1, jan 1988.

[8] Matthew Foreman. Ideals and Generic Elementary Embeddings. In Handbook of Set Theory, pages 885–1147. Springer Netherlands, Dordrecht, 2010.

[9] Matthew Foreman and Menachem Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. Annals of Pure and Applied Logic, 76(1):47–97, nov 1995.

[10] Matthew Foreman and Menachem Magidor. Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $\mathcal{P}_\kappa(\lambda)$. Acta Math., 186(2):271–300, 2001.

[11] Moti Gitik and Saharon Shelah. Less saturated ideals. Proceedings of the American Mathematical Society, 125(5):1523–1531, may 1997.

[12] K. Kunen and J. B. Paris. Boolean extensions and measurable cardinals. Annals of Mathematical Logic, 2(4):359–377, 1971.

[13] Kenneth Kunen. Saturated Ideals. Journal of Symbolic Logic, 43(1):65–76, 1978.

[14] Richard Laver. Saturated Ideals and Nonregular Ultrafilters. Studies in Logic and the Foundations of Mathematics, 109:297–305, jan 1982.

[15] Jack Silver. On the singular cardinals problem II. Israel Journal of Mathematics, 28(1-2):1–31, mar 1977.

[16] Robert M. Solovay. Real-valued measurable cardinals. pages 397–428. 1971.

[17] Stanislaw Ulam. On measure theory in general set theory (doctoral dissertation). Wisconsin. Mat., 33:155–168, 1997.