Lie–Schwinger Block-Diagonalization and Gapped Quantum Chains

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Abstract: We study quantum chains whose Hamiltonians are perturbations by bounded interactions of short range of a Hamiltonian that does not couple the degrees of freedom located at different sites of the chain and has a strictly positive energy gap above its ground-state energy. We prove that, for small values of a coupling constant, the spectral gap of the perturbed Hamiltonian above its ground-state energy is bounded from below by a positive constant \textit{uniformly} in the length of the chain. In our proof we use a novel method based on \textit{local} Lie–Schwinger conjugations of the Hamiltonians associated with connected subsets of the chain.

1. Introduction: Models and Results

In this paper, we study spectral properties of Hamiltonians of some family of quantum chains with bounded interactions of short range, including the Kitaev chain, [GST, KST]. We are primarily interested in determining the multiplicity of the ground-state energy and in estimating the size of the spectral gap above the ground-state energy of Hamiltonians of such chains, as the length of the chains tends to infinity. We will consider a family of Hamiltonians for which we will prove that their ground-state energy is finitely degenerate and the spectral gap above the ground-state energy is bounded from below by a positive constant \textit{uniformly} in the length of the chain. Connected sets of Hamiltonians with these properties represent what people tend to call (somewhat misleadingly) a “topological phase”. Our analysis is motivated by recent wide-spread interest in characterising topological phases of matter; see, e.g., [MN,NSY,BN].

Results similar to the ones established in this paper have been proven before, often using so-called “cluster expansions”; see [DFF,FFU,KT,Y,KU,DS,H,BHM,MZ] and refs. given there. The purpose of this paper is to introduce a novel method to analyse spectral properties of Hamiltonians of quantum chains near their ground-state energies. This method is based on \textit{iterative unitary conjugations} of the Hamiltonians, which serve to block-diagonalize them with respect to a fixed orthogonal projection and its orthogonal
complement; (see [DFFR] for a similar technique in a simpler context). Ideas somewhat similar to those presented in this paper have been used in work of Imbrie [I1,12], and of De Roeck et al. [DRS].

1.1. A concrete family of quantum chains. The Hilbert space of pure state vectors of the quantum chains studied in this paper has the form

$$\mathcal{H}^{(N)} := \bigotimes_{j=1}^{N} \mathcal{H}_j,$$

where $\mathcal{H}_j \simeq \mathbb{C}^M$, $\forall j = 1, 2, \ldots$, and where $M$ is an arbitrary, but $N$-independent finite integer. Let $H$ be a non-negative $M \times M$ matrix with the properties that $0$ is an eigenvalue of $H$ corresponding to an eigenvector $\Omega \in \mathbb{C}^M$, and

$$H |_{\{\Omega\}} \geq 1.$$

We define

$$H_i := \mathbb{1}_1 \otimes \cdots \otimes H \uparrow_{\text{ith slot}} \otimes \cdots \mathbb{1}_N. \quad (1.2)$$

By $P_{\Omega_i}$ we denote the orthogonal projection onto the subspace

$$\mathcal{H}_1 \otimes \cdots \otimes \{\mathbb{C} \Omega\} \otimes \cdots \otimes \mathcal{H}_N \subset \mathcal{H}^{(N)}, \quad \text{and} \quad P_{\Omega_i}^\perp := \mathbb{1} - P_{\Omega_i}. \quad (1.3)$$

Then

$$H_i = P_{\Omega_i} H_i P_{\Omega_i} + P_{\Omega_i}^\perp H_i P_{\Omega_i}^\perp,$$

with

$$P_{\Omega_i} H_i P_{\Omega_i} = 0, \quad P_{\Omega_i}^\perp H_i P_{\Omega_i}^\perp \geq P_{\Omega_i}^\perp. \quad (1.4)$$

We study quantum chains on the graph $I_{N-1,1} := \{1, \ldots, N\}$, $N < \infty$ arbitrary, with a Hamiltonian of the form

$$K_N \equiv K_N(t) := \sum_{i=1}^{N} H_i + t \sum_{i \in I_{N-1,1}, \ k \leq \bar{k}} V_{I_{k,i}}, \quad (1.5)$$

where $\bar{k} < \infty$ is an arbitrary, but fixed integer, $I_{k,i}$ is the “interval” given by $\{i, \ldots, i + k\}$, $i = 1, \ldots, N - k$, and $V_{I_{k,i}}$ is a symmetric matrix acting on $\mathcal{H}^{(N)}$ with the property that

$$V_{I_{k,i}} \text{ acts as the identity on } \bigotimes_{j \in I_{N-1,1}, \ j \notin I_{k,i}} \mathcal{H}_j, \quad (1.6)$$

and $t \in \mathbb{R}$ is a coupling constant. (We call $I_{k,i}$ the “support” of $V_{I_{k,i}}$.) Without loss of generality, we may assume that

$$\|V_{I_{k,i}}\| \leq 1. \quad (1.7)$$
A concrete example of a quantum chain we are able to analyse is the (generalised) “Kitaev chain”, which has a Hamiltonian that is a small perturbation of the following quadratic Hamiltonian:

\[ H_N := -\mu \sum_{j=1}^{N} c_j^* c_j - \sum_{j=1}^{N-1} \left( \tau c_j^* c_{j+1} + h.c. + \Delta c_j c_{j+1} + h.c. \right), \]

(1.8)

where \( c_j^*, c_j, \ j = 1, \ldots, N, \) are fermionic creation- and annihilation operators satisfying canonical anti-commutation relations, \( \mu \) is a chemical potential, \( \tau \) is a hopping amplitude, and \( \Delta \) is a pairing amplitude. Using appropriate linear combinations of the operators \( c_j^*, c_j, \) the Hamiltonian \( H_N, \) as well as certain small perturbations thereof, can be cast in the form given in (1.5). See Sect. 4 for details.

Another example of a quantum chain that can be treated with the methods of this paper is an anisotropic Heisenberg chain corresponding to a small quantum perturbation of the ferromagnetic Ising chain, with domain walls interpreted as the elementary finite-energy excitations of the ferromagnetically ordered ground-state of the chain. Examples where the dimension, \( M, \) of the Hilbert spaces \( \mathcal{H}_j \) is infinite are studied in [DFPR].

1.2. Main result. The main result in this paper is the following theorem proven in Sect. 3, (see Theorem 3.5).

**Theorem.** Under the assumption that (1.4), (1.6) and (1.7) hold, the Hamiltonian \( K_N \) defined in (1.5) has the following properties: There exists some \( t_0 > 0 \) such that, for any \( t \in \mathbb{R} \) with \( |t| < t_0, \) and for all \( N < \infty, \)

(i) \( K_N \) has a unique ground-state; and

(ii) the energy spectrum of \( K_N \) has a strictly positive gap, \( \Delta_N(t) \geq \frac{1}{2}, \) above the ground-state energy.

**Remark 1.1.** In general, the ground-state of \( K_N \) may depend on “boundary conditions” at the two ends of the chain, in which case several different ground-states may exist. A simple example of this phenomenon is furnished by the anisotropic Heisenberg chain described above, with + or − boundary conditions imposed at the ends of the chain.

Results similar to the theorem stated above have appeared in the literature; see, e.g., [DS]. The main novelty introduced in this paper is our method of proof.

We define

\[ P_{\text{vac}} := \bigotimes_{i=1}^{N} P_{\Omega_i}. \]

(1.9)

Note that \( P_{\text{vac}} \) is the orthogonal projection onto the ground-state of the operator \( K_N(t = 0) = \sum_{i=1}^{N} H_i. \) Our aim is to find an anti-symmetric matrix \( S_N(t) = -S_N(t)^* \) acting on \( \mathcal{H}_j^{(N)} \) (so that \( \exp(\pm S_N(t)) \) is unitary) with the property that, after conjugation, the operator

\[ e^{S_N(t)} K_N(t) e^{-S_N(t)} =: \tilde{K}_N(t) \]

(1.10)

is “block-diagonal” with respect to \( P_{\text{vac}}, \) \( P_{\text{vac}} := 1 - P_{\text{vac}}, \) in the sense that \( P_{\text{vac}} \) projects onto the ground-state of \( \tilde{K}_N(t), \)

\[ \tilde{K}_N(t) = P_{\text{vac}} \tilde{K}_N(t) P_{\text{vac}} + P_{\text{vac}} \tilde{K}_N(t) P_{\text{vac}}^\perp, \]

(1.11)
and

\[
\text{infspec} \left( P_{\text{vac}} \tilde{K}_N(t) P_{\text{vac}} \upharpoonright \mathcal{H}^{(N)} \right) \geq \text{infspec} \left( P_{\text{vac}} \tilde{K}_N(t) P_{\text{vac}} \upharpoonright \mathcal{H}^{(N)} \right) + \Delta_N(t),
\]

\[\text{(1.12)}\]

with \( \Delta_N(t) \geq \frac{1}{2} \), for \(|t| < t_0\), uniformly in \( N \). The iterative construction of the operator \( S_N(t) \), yielding (1.11), and the proof of (1.12) are the main tasks to be carried out. Formal aspects of our construction are described in Sect. 2. In Sect. 3, the proof of convergence of our construction of the operator \( S_N(t) \) and the proof of a lower bound on the spectral gap \( \Delta_N(t) \), for sufficiently small values of \(|t|\), are presented, with a few technicalities deferred to “Appendix A”. In Sect. 4, the example of the (generalized) Kitaev chain is studied.

**Notation.** (1) Notice that \( I_{k,q} \) can also be seen as a connected one-dimensional graph with \( k \) edges connecting the \( k + 1 \) vertices \( q, 1 + q, \ldots, k + q \), or as an “interval” of length \( k \) whose left end-point coincides with \( q \).

(2) We use the same symbol for the operator \( O_j \) acting on \( \mathcal{H}_j \) and the corresponding operator

\[
1_i \otimes \cdots \otimes 1_{j-1} \otimes O_j \otimes 1_{j+1} \cdots \otimes 1_l
\]

acting on \( \otimes_{k=i}^l \mathcal{H}_k \), for any \( i \leq j \leq l \).

(3) With the symbol “\( \subset \)” we denote strict inclusion, otherwise we use the symbol “\( \subseteq \)”.

### 2. Local Conjugations Based on Lie–Schwinger Series

In this section we describe some of the key ideas underlying our proof of the theorem announced in the previous section. We study quantum chains with Hamiltonians \( K_N(t) \) of the form described in (1.5) acting on the Hilbert space \( \mathcal{H}^{(N)} \) defined in (1.1). As announced in Sect. 1, our aim is to block-diagonalize \( K_N(t) \), for \(|t|\) small enough, by conjugating it by a sequence of unitary operators chosen according to the “Lie-Schwinger procedure” (supported on subsets of \( \{1, \ldots, N\} \) of successive sites). The block-diagonalization will concern operators acting on tensor-product spaces of the sort \( \mathcal{H}_q \otimes \cdots \otimes \mathcal{H}_{k+q} \) (and acting trivially on the remaining tensor factors), and it will be with respect to the ground-state (“vacuum”) subspace, \( \mathcal{C}(\Omega_q \otimes \cdots \otimes \Omega_{k+q}) \), contained in \( \mathcal{H}_q \otimes \cdots \otimes \mathcal{H}_{k+q} \) and its orthogonal complement. Along the way, new interaction terms are being created whose support corresponds to ever longer intervals (connected subsets) of the chain.

#### 2.1. Block-diagonalization: definitions and formal aspects. For each \( k \), we consider \( (N - k) \) block-diagonalization steps, each of them associated with a subset \( I_{k,q} \), \( q = 1, \ldots, N - k \). The block-diagonalization of the Hamiltonian will be with respect to the subspaces associated with the projectors in (2.4), (2.5), introduced below. By \((k, q)\) we label the block-diagonalization step associated with \( I_{k,q} \). We introduce an ordering amongst these steps:

\[
(k', q') \succ (k, q)
\]

\[\text{(2.1)}\]

if \( k' > k \) or if \( k' = k \) and \( q' > q \).
Our original Hamiltonian is denoted by $K_{N}^{(0,N)} := K_{N}(t)$. We proceed to the first block-diagonalization step yielding $K_{N}^{(1,1)}$. The index $(0, N)$ is our initial choice of the index $(k, q)$: all the on-site terms in the Hamiltonian, i.e., the terms $H_{i}$, are block-diagonal with respect to the subspaces associated with the projectors in (2.4), (2.5), for $l = 0$. Our goal is to arrive at a Hamiltonian of the form

\begin{align}
K_{N}^{(k,q)} &= \sum_{i=1}^{N} H_{i} + t \sum_{i=1}^{N-1} V_{l_{1},i}^{(k,q)} + t \sum_{i=1}^{N-2} V_{l_{2},i}^{(k,q)} + \cdots + t \sum_{i=1}^{N-k} V_{l_{k},i}^{(k,q)} \\
&+ t \sum_{i=1}^{N-k-1} V_{l_{k+1},i}^{(k,q)} + \cdots + t \sum_{i=1}^{2} V_{l_{N-2},i}^{(k,q)} + t V_{l_{N-1,1}}^{(k,q)}
\end{align}

(2.2)

after the block-diagonalization step $(k, q)$, with the following properties:

1. For a fixed $l_{i}, i$, the corresponding potential term may change, at each step of the block-diagonalization procedure, up to the step $(k, q) \equiv (l, i)$; hence $V_{l_{i},i}^{(k,q)}$ is the potential term associated with the interval $l_{i}, i$ in step $(k, q)$ of the block-diagonalization, and the superscript $(k, q)$ keeps track of the changes in the potential term in step $(k, q)$. (In general $V_{l_{i},i}^{(k,q)}$ is $t$-dependent though this is not reflected in our notation.) The operator $V_{l_{i},i}^{(k,q)}$ acts as the identity on the spaces $\mathcal{H}_{j}$ for $j \neq i, i+1, \ldots, i+l$; the description of how these terms are created and estimates on their norms are deferred to Sect. 3;

2. for all sets $l_{i}, i$ with $(l, i) < (k, q)$ and for the set $l_{i}, i \equiv l_{k}, q$, the associated potential $V_{l_{i},i}^{(k,q)}$ is block-diagonal with respect to the decomposition of the identity into the sum of projectors

\begin{align}
P_{l_{i},i}^{(-)} &:= P_{\Omega_{i}} \otimes P_{\Omega_{i+1}} \otimes \cdots \otimes P_{\Omega_{i+l}}, \\
P_{l_{i},i}^{(+)} &:= (P_{\Omega_{i}} \otimes P_{\Omega_{i+1}} \otimes \cdots \otimes P_{\Omega_{i+l}})^{\perp}.
\end{align}

(2.5)

**Remark 2.1.** It is important to notice that if $V_{l_{i},i}^{(k,q)}$ is block-diagonal with respect to the decomposition of the identity into

$$P_{l_{i},i}^{(+)} + P_{l_{i},i}^{(-)},$$

i.e.,

$$V_{l_{i},i}^{(k,q)} = P_{l_{i},i}^{(+)} V_{l_{i},i}^{(k,q)} P_{l_{i},i}^{(+)} + P_{l_{i},i}^{(-)} V_{l_{i},i}^{(k,q)} P_{l_{i},i}^{(-)},$$

then, for $l_{i}, i \subset l_{r}, j$, we have that

$$P_{l_{r},j}^{(+)} \left[ P_{l_{i},i}^{(+)} V_{l_{i},i}^{(k,q)} P_{l_{i},i}^{(+)} + P_{l_{i},i}^{(-)} V_{l_{i},i}^{(k,q)} P_{l_{i},i}^{(-)} \right] P_{l_{r},j}^{(-)} = 0.$$

To see that the first term vanishes, we use that

$$P_{l_{i},i}^{(+)} P_{l_{r},j}^{(-)} = 0,$$

(2.6)

while, in the second term, we use that

$$P_{l_{i},i}^{(-)} V_{l_{i},i}^{(k,q)} P_{l_{r},j}^{(-)} = P_{l_{r},j}^{(-)} V_{l_{i},i}^{(k,q)} P_{l_{i},i}^{(-)} P_{l_{r},j}^{(-)}$$

(2.7)
\[ P_{l_{r,j}}^{(+)} P_{l_{r,j}}^{(-)} = 0. \]  

(2.8)

Hence \( V_{l_{r,i}}^{(k,q)} \) is also block-diagonal with respect to the decomposition of the identity into

\[ P_{l_{r,j}}^{(+)} + P_{l_{r,j}}^{(-)}. \]

However, notice that

\[
P_{l_{r,j}}^{(-)} \left[ P_{l_{r,i}}^{(+)} V_{l_{r,i}}^{(k,q)} P_{l_{r,i}}^{(+)} + P_{l_{r,i}}^{(-)} V_{l_{r,i}}^{(k,q)} P_{l_{r,i}}^{(-)} \right] P_{l_{r,j}}^{(-)} = P_{l_{r,j}}^{(-)} V_{l_{r,i}}^{(k,q)} P_{l_{r,j}}^{(-)} \]

(2.9)

but

\[
P_{l_{r,j}}^{(+)} \left[ P_{l_{r,i}}^{(+)} V_{l_{r,i}}^{(k,q)} P_{l_{r,i}}^{(+)} + P_{l_{r,i}}^{(-)} V_{l_{r,i}}^{(k,q)} P_{l_{r,i}}^{(-)} \right] P_{l_{r,j}}^{(+)}
\]

remains as it is.

Remark 2.2. The block-diagonalization procedure that we will implement enjoys the property that the terms block-diagonalized along the process do not change, anymore, in subsequent steps.

2.2. Lie–Schwinger conjugation associated with \( I_{k,q} \). Here we explain the block-diagonalization procedure from \((k, q-1)\) to \((k, q)\) by which the term \( V_{l_{k,q}}^{(k,q-1)} \) is transformed to a new operator, \( V_{l_{k,q}}^{(k,q)} \), which is block-diagonal w.r.t. the decomposition of the identity into

\[ P_{l_{k,q}}^{(+)} + P_{l_{k,q}}^{(-)}. \]

We note that the steps of the type\(^1\) \((k, N-k) \rightarrow (k+1, 1)\) are somewhat different, because the first index (i.e., the number of edges of the interval) is changing from \(k\) to \(k+1\). Hence we start by showing how our procedure works for them. Later we deal with general steps \((k, q-1) \rightarrow (k, q)\), with \(N-k \geq q \geq 2\).

We recall that the Hamiltonian \( K_N^{(k,N-k)} \) is given by

\[
K_N^{(k,N-k)} = \sum_{i=1}^{N} H_i + t \sum_{i=1}^{N-1} V_{l_{1,i}}^{(k,N-k)} + t \sum_{i=1}^{N-2} V_{l_{2,i}}^{(k,N-k)} + \cdots + t \sum_{i=1}^{N-k} V_{l_{k,i}}^{(k,N-k)}
\]

(2.10)

\[
+ t \sum_{i=1}^{N-k-1} V_{l_{k+1,i}}^{(k,N-k)} + \cdots + t \sum_{i=1}^{2} V_{l_{N-2,i}}^{(k,N-k)} + t V_{l_{N-1,1}}^{(k,N-k)}
\]

(2.11)

and has the following properties

\(^1\) The initial step, \((0, N) \rightarrow (1, 1)\), is of this type; see the definitions in (3.10) corresponding to a Hamiltonian \( K_N \) with nearest-neighbor interactions.
1. each operator $V^{(k,N-k)}_{l,i}$ acts as the identity on the spaces $H_j$ for $j \neq i, i+1, \ldots, i+l$.

In Sect. 3 we explain how these terms are created and their norms estimated;

2. each operator $V^{(k,N-k)}_{l,i}$, with $l \leq k$, is block-diagonal w.r.t. the decomposition of the identity into the sum of projectors in (2.4), (2.5).

**Remark 2.3.** The term step is used throughout the paper with two slightly different meanings:

(i) as level in the block-diagonalization iteration, e.g., $K_N^{(k,q)}$ is the Hamiltonian in step $(k, q)$;

(ii) for the block-diagonalization procedure to switch from level $(k, q - 1)$ to level $(k, q)$, e.g., the step $(k, q - 1) \rightarrow (k, q)$.

With the next block-diagonalization step, labeled by $(k+1, 1)$, we want to block-diagonalize the interaction term $V^{(k,N-k)}_{l+1,1}$, considering the operator

$$G_{k+1,1} := \sum_{i \in l_{k+1,1}} H_i + t \sum_{l_i, i \subseteq l_{k+1,1}} V^{(k,N-k)}_{l_i,i} + \cdots + t \sum_{l_k, i \subseteq l_{k+1,1}} V^{(k,N-k)}_{l_k,i}$$

(2.12)

as the “unperturbed” Hamiltonian. This operator is block-diagonal w.r.t. the decomposition in (2.20), i.e.,

$$G_{k+1,1} = P_{k+1,1}^{(+)} G_{k+1,1} P_{k+1,1}^{(+)} + P_{k+1,1}^{(-)} G_{k+1,1} P_{k+1,1}^{(-)};$$

(2.13)

see Remarks 2.1 and 2.2. We also define

$$E_{l_{k+1,1}} := \inf \text{spec } G_{k+1,1}$$

(2.14)

and we temporarily assume that

$$G_{k+1,1} P_{k+1,1}^{(-)} = E_{l_{k+1,1}} P_{k+1,1}^{(-)}.$$

Next, we sketch a convenient formalism used to construct our block-diagonalization operations, below; (for further details the reader is referred to Sects. 2 and 3 of [DFFR]).

We define

$$\text{ad } A (B) := [A, B],$$

where $A$ and $B$ are bounded operators, and, for $n \geq 2$,

$$\text{ad}^n A (B) := [A, \text{ad}^{n-1} A (B)].$$

(2.15)

(2.16)

In the block-diagonalization step $(k+1, 1)$, we use the operator

$$U_{l_{k+1,1}} := e^{-S_{l_{k+1,1}}},$$

(2.17)

with

$$S_{l_{k+1,1}} := \sum_{j=1}^{\infty} t^j (S_{l_{k+1,1}}^j),$$

(2.18)

where
\[(S_{k+1,1})_j := ad^{-1} G_{k+1,1} \left((V_{k+1,1}^{(k,N-k)})_j\right) = \frac{1}{G_{k+1,1} - E_{k+1,1}} P_{k+1,1}^{(+)} (V_{k+1,1}^{(k,N-k)})_j P_{k+1,1}^{(-)} - h.c., \tag{2.19}\]

where \(od\) means “off-diagonal” w.r.t. the decomposition of the identity into

\[P_{k+1,1}^{(+)} + P_{k+1,1}^{(-)} \tag{2.20}\]

\[(V_{k+1,1}^{(k,N-k)})_1 := V_{k+1,1}^{(k,N-k)}, \quad \text{and, for } j \geq 2,
\]

\[(V_{k+1,1}^{(k,N-k)})_j := \sum_{p \geq 2, r_1 \geq 1 \ldots r_p \geq 1; r_1 + \cdots + r_p = j} \frac{1}{p!} \text{ad} (S_{k+1,1})_{r_1} \left( \text{ad} (S_{k+1,1})_{r_2} \cdots (\text{ad} (S_{k+1,1})_{r_p} (G_{k+1,1})) \cdots \right) + \sum_{p \geq 1, r_1 \geq 1 \ldots r_p \geq 1; r_1 + \cdots + r_p = j-1} \frac{1}{p!} \text{ad} (S_{k+1,1})_{r_1} \left( \text{ad} (S_{k+1,1})_{r_2} \cdots (\text{ad} (S_{k+1,1})_{r_p} (V_{k+1,1}^{(k,N-k)})) \cdots \right). \tag{2.21}\]

We define

\[K_{N}^{(k+1,1)} := U_{k+1,1}^* K_{N}^{(k,N-k)} U_{k+1,1}, \quad \text{with } U_{k+1,1} \text{ as in } (2.17). \tag{2.22}\]

After the block-diagonalization step labeled by \((k, q-1)\), with \(q \leq N-k\), we obtain

\[K_{N}^{(k,q-1)} = \sum_{i=1}^{N} H_{i} + t \sum_{i=1}^{N-1} V_{1,i}^{(k,q-1)} + t \sum_{i=1}^{N-2} V_{2,i}^{(k,q-1)} + \cdots + t \sum_{i=1}^{N-k} V_{k,i}^{(k,q-1)} , \tag{2.23}\]

\[+ t \sum_{i=1}^{N-k-1} V_{k+1,i}^{(k,q-1)} + \cdots + t \sum_{i=1}^{N-k} V_{k+1,i}^{(k,q-1)} + t V_{k+1,1}^{(k,q-1)} + t V_{k+1,1}^{(k,q-1)} \tag{2.24}\]

where, for all sets \(I_{k'},q'\), with \((k'q') < (k,q-1)\), and for the set \(I_{k,q-1}\), the associated \(V_{I_{k',q'}}^{(k,q-1)}\) is block-diagonal.

Next, in order to block-diagonalize the interaction term \(V_{I_{k,q}}^{(k,q-1)}\), we conjugate the Hamiltonian with the operator

\[U_{k,q} := e^{-S_{I_{k,q}}} , \tag{2.25}\]

where

\[S_{I_{k,q}} := \sum_{j=1}^{\infty} t^j (S_{I_{k,q}})_j , \tag{2.26}\]
with
\[
(S_{k,q})_j := a^{-1} G_{k,q} \left( (V_{k,q})_j \right)
\]  
(2.27)

and
\[
G_{k,q} := \sum_{i} H_i + t \sum_{I_{1,1} \subseteq I_{k,q}} V^{(k,q-1)}_{I_{1,1}} + \cdots + t \sum_{I_{k-1,1} \subseteq I_{k,q}} V^{(k,q-1)}_{I_{k-1,1}};
\]
\[
(V^{(k,q-1)}_{k,q})_1 := V^{(k,q-1)}_{k,q}
\]  
(2.28)

and, for \( j \geq 2 \),
\[
(V^{(k,q-1)}_{k,q})_j := \sum_{p \geq 2, r_1 \geq 1, \cdots, r_p \geq 1: r_1 + \cdots + r_p = j} \frac{1}{p!} \text{ad} (S_{k,q})_{r_1} \left( \text{ad} (S_{k,q})_{r_2} \cdots \text{ad} (S_{k,q})_{r_p} (G_{k,q}) \right) \cdots
\]
\[+ \sum_{p \geq 1, r_1 \geq 1, \cdots, r_p \geq 1: r_1 + \cdots + r_p = j-1} \frac{1}{p!} \text{ad} (S_{k,q})_{r_1} \left( \text{ad} (S_{k,q})_{r_2} \cdots \text{ad} (S_{k,q})_{r_p} (V^{(k,q-1)}_{k,q}) \right) \cdots \].
\]
(2.29)

We define
\[
K^{(k,q)}_N := e^{S_{k,q}} K^{(k,q-1)}_N e^{-S_{k,q}}.
\]
(2.30)

### 2.3. Gap of the local Hamiltonians \( G_{k,q} \): main argument

In the rest of this section we outline the main arguments and estimates underlying our strategy. To simplify our presentation, we consider a nearest-neighbor interaction with

\[
\|V_{I_{1,i}}\| \leq 1
\]

and \( t > 0 \) small enough. However, with obvious modifications, our proof can be adapted to general Hamiltonians of the type in (1.5).

We assume that

\[
\|V^{(k,q-1)}_{I_{1,i}}\| \leq \frac{8 \cdot t^{l-1}}{(l+1)^2}.
\]
(2.31)

(The number “8” does not have particular significance, but comes up in the inductive part of the proof of Theorem 3.4.)

We exhibit the key mechanism underlying our method, starting from the potential terms \( V^{(k,q-1)}_{I_{1,i}} \). We already know that, for any \( k > 1 \), the operator \( V^{(k,q-1)}_{I_{1,i}} \) is block-diagonalized, i.e.,

\[
V^{(k,q-1)}_{I_{1,i}} = P_{I_{1,i}}^{(+)} V^{(k,q-1)}_{I_{1,i}} P_{I_{1,i}}^{(+)} + P_{I_{1,i}}^{(-)} V^{(k,q-1)}_{I_{1,i}} P_{I_{1,i}}^{(-)}.
\]
(2.32)
Hence we can write

\[
P_{l_{k,q}}^{(+)} \left[ \sum_{i \subseteq l_{k,q}} H_i + t \sum_{l_{1,i} \subseteq l_{k,q}} V_{l_{1,i}}^{(k,q-1)} \right] P_{l_{k,q}}^{(+)} = P_{l_{k,q}}^{(+)} \left[ \sum_{i \subseteq l_{k,q}} H_i + t \sum_{l_{1,i} \subseteq l_{k,q}} P_{l_{1,i}}^{(+)} V_{l_{1,i}}^{(k,q-1)} P_{l_{1,i}}^{(+)} \right] + t \sum_{l_{1,i} \subseteq l_{k,q}} P_{l_{1,i}}^{(-)} V_{l_{1,i}}^{(k,q-1)} P_{l_{1,i}}^{(-)} P_{l_{k,q}}^{(+)}
\]

and observe that, by assumption (1.4),

\[
\sum_{i \subseteq l_{k,q}} H_i \geq \sum_{i = q}^{k+q} P_{\Omega_i}^{(+)}. \tag{2.35}
\]

We will make use of a simple, but crucial inequality proven in Corollary A.2: For \(1 \leq l \leq L \leq N - r\),

\[
\sum_{i = l}^{L} P_{l_{r,i}}^{(+)} \leq (r + 1) \sum_{i = l}^{L+q} P_{\Omega_i}^{(+)}. \tag{2.36}
\]

Due to assumption (2.31) and inequality (2.36), with \(r = 1, l = q, L = k + q - r\), we have that

\[
\pm \sum_{l_{1,i} \subseteq l_{k,q}} P_{l_{1,i}}^{(+)} V_{l_{1,i}}^{(k,q-1)} P_{l_{1,i}}^{(+)} \leq 4 \sum_{i = q}^{k+q} P_{\Omega_i}^{(+)} \tag{2.37}
\]

Hence, recalling that \(t > 0\) and combining (2.35) with (2.37), we conclude that

\[
\tag{2.34} \geq P_{l_{k,q}}^{(+)} \left[ (1 - 4t) \sum_{i = q}^{k+q} P_{\Omega_i}^{(+)} \right] P_{l_{k,q}}^{(+)} + P_{l_{k,q}}^{(+)} \left[ t \sum_{l_{1,i} \subseteq l_{k,q}} P_{l_{1,i}}^{(-)} V_{l_{1,i}}^{(k,q-1)} P_{l_{1,i}}^{(-)} \right] P_{l_{k,q}}^{(+)} = P_{l_{k,q}}^{(+)} \left[ (1 - 4t) \sum_{i = q}^{k+q} P_{\Omega_i}^{(+)} \right] P_{l_{k,q}}^{(+)} + P_{l_{k,q}}^{(+)} \left[ t \sum_{l_{1,i} \subseteq l_{k,q}} \left( V_{l_{1,i}}^{(k,q-1)} \right) P_{l_{1,i}}^{(-)} \right] P_{l_{k,q}}^{(+)} \tag{2.38}
\]

where

\[
\langle V_{l_{j,i}}^{(k,q-1)} \rangle := \langle \Omega_i \otimes \cdots \otimes \Omega_{i+j} , V_{l_{j,i}}^{(k,q-1)} \Omega_i \otimes \cdots \otimes \Omega_{i+j} \rangle. \tag{2.40}
\]
Next, substituting $P_{l_{1,i}}^{(-)} = \mathbb{1} - P_{l_{1,i}}^{(+)}$ into (2.39), we find that

$$
(2.34) \geq P_{l_{1,q}}^{(+)} \left[ (1 - 4t) \sum_{i=q}^{k+q} P_{l_{1,i}}^{\perp} - t \sum_{l_{1,i} \subset l_{k,q}} \langle V_{l_{1,i}}^{(k,q-1)} \rangle P_{l_{1,i}}^{(+)} \right] P_{l_{k,q}}^{(+)}
$$

$$
+ P_{l_{k,q}}^{(+)} \left[ t \sum_{l_{1,i} \subset l_{k,q}} \langle V_{l_{1,i}}^{(k,q-1)} \rangle P_{l_{1,i}}^{(+)} \right] P_{l_{k,q}}^{(+)}
$$

$$
\geq P_{l_{k,q}}^{(+)} \left[ (1 - 8t) \sum_{i=q}^{k+q} P_{l_{1,i}}^{\perp} \right] P_{l_{k,q}}^{(+)}
$$

$$
+ P_{l_{k,q}}^{(+)} \left[ t \sum_{l_{1,i} \subset l_{k,q}} \langle V_{l_{1,i}}^{(k,q-1)} \rangle P_{l_{1,i}}^{(+)} \right] P_{l_{k,q}}^{(+)}
$$

where, in the step from (2.41) to (2.43), we have used (2.37) and (2.31). Iterating this argument yields the following lemma.

**Lemma 2.4.** Assuming the bound in (2.31) and choosing $t (> 0)$ so small that

$$
1 - 8t - 16t \sum_{l=3}^{\infty} \frac{l^{i-2}}{l^2} > 0
$$

the following inequality holds:

$$
P_{l_{k,q}}^{(+)} G_{l_{k,q}} P_{l_{k,q}}^{(+)} \geq \left( 1 - 8t - 16t \sum_{l=3}^{k} \frac{l^{i-2}}{l^2} \right) P_{l_{k,q}}^{(+)}
$$

$$
+ P_{l_{k,q}}^{(+)} \left[ t \sum_{l_{1,i} \subset l_{k,q}} \langle V_{l_{1,i}}^{(k,q-1)} \rangle P_{l_{1,i}}^{(+)} + \cdots + t \sum_{l_{k-1,i} \subset l_{k,q}} \langle V_{l_{k-1,i}}^{(k,q-1)} \rangle \right] P_{l_{k,q}}^{(+)}
$$

$$
(2.46)
$$

**Proof.** This lemma serves to establish a bound on the spectral gap above the ground-state energy of the operator $G_{l_{k,q}}$. Proceeding as in (2.32)–(2.44) and using the definition in (2.40), we get that

$$
P_{l_{k,q}}^{(+)} G_{l_{k,q}} P_{l_{k,q}}^{(+)} \geq P_{l_{k,q}}^{(+)} \left[ (1 - 4t - 8t \sum_{l=3}^{k} \frac{l^{i-2}}{l^2} \sum_{i=q}^{k+q} P_{l_{1,i}}^{\perp} \right] P_{l_{k,q}}^{(+)}
$$

$$
+ P_{l_{k,q}}^{(+)} \left[ t \sum_{l_{1,i} \subset l_{k,q}} \langle V_{l_{1,i}}^{(k,q-1)} \rangle P_{l_{1,i}}^{(+)} + \cdots \right]
$$

$$
+ t \sum_{l_{k-1,i} \subset l_{k,q}} \langle V_{l_{k-1,i}}^{(k,q-1)} \rangle P_{l_{k-1,i}}^{(+)} P_{l_{k,q}}^{(+)}
$$

$$
\geq P_{l_{k,q}}^{(+)} \left[ (1 - 4t - 8t \sum_{l=3}^{k} \frac{l^{i-2}}{l^2} \sum_{i=q}^{k+q} P_{l_{1,i}}^{\perp} \right] P_{l_{k,q}}^{(+)}
$$

$$
+ P_{l_{k,q}}^{(+)} \left[ - t \sum_{l_{1,i} \subset l_{k,q}} \langle V_{l_{1,i}}^{(k,q-1)} \rangle P_{l_{1,i}}^{(+)} + \cdots \right]
$$

$$
(2.47)
$$
We propose to define and control an algorithm, \( \alpha_{k,q} \), determining

\[-t \sum_{l_{k-1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{k-1,i}} \rangle P^{(+)}_{l_{k-1,i}} P^{(+)}_{l_{k,q}} + P^{(+)}_{l_{k,q}} \left[ t \sum_{l_{1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{1,i}} \rangle + \cdots + t \sum_{l_{k-1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{k-1,i}} \rangle \right] P^{(+)}_{l_{k,q}} \geq P^{(+)}_{l_{k,q}} \left[ (1 - 8t - 16t \sum_{l=3}^{k} \frac{l!-2}{l^2} \sum_{i=q}^{k+q} P_{\Omega_i}^{(1)} \right] P^{(+)}_{l_{k,q}} + P^{(+)}_{l_{k,q}} \left[ t \sum_{l_{1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{1,i}} \rangle + \cdots + t \sum_{l_{k-1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{k-1,i}} \rangle \right] P^{(+)}_{l_{k,q}} \geq (1 - 8t - 16t \sum_{l=3}^{k} \frac{l!-2}{l^2} \sum_{i=q}^{k+q} P_{\Omega_i}^{(1)} + P^{(+)}_{l_{k,q}} \left[ t \sum_{l_{1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{1,i}} \rangle + \cdots + t \sum_{l_{k-1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{k-1,i}} \rangle \right] P^{(+)}_{l_{k,q}} \right. \]

where Lemma A.1 is used in the last inequality. \( \square \)

Lemma 2.4 implies that, under assumption (2.31), the Hamiltonian \( G_{I_{k,q}} \) has a spectral gap above its ground-state energy that can be estimated from below by \( \frac{1}{2} \), for \( t > 0 \) sufficiently small but independent of \( N, k, \) and \( q \), as stated in the Corollary below.

Corollary 2.5. For \( t > 0 \) sufficiently small, but independent of \( N, k, \) and \( q \), the Hamiltonian \( G_{I_{k,q}} \) has a spectral gap \( \Delta_{I_{k,q}} \geq \frac{1}{2} \) above the ground-state energy

\[ E_{I_{k,q}} = t \sum_{l_{1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{1,i}} \rangle + \cdots + t \sum_{l_{k-1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{k-1,i}} \rangle. \]

The ground-state of \( G_{I_{k,q}} \) coincides with the “vacuum”, \( \otimes_{j \in I_{k,q}} \Omega_j \), in \( \mathcal{H}_{I_{k,q}} \). We have the identity

\[ P^{(-)}_{l_{k,q}} G_{I_{k,q}} P^{(-)}_{l_{k,q}} = P^{(-)}_{l_{k,q}} \left[ t \sum_{l_{1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{1,i}} \rangle P^{(-)}_{l_{1,i}} + \cdots + t \sum_{l_{k-1,i} \in I_{k,q}} \langle V^{(k,q-1)}_{l_{k-1,i}} \rangle P^{(-)}_{l_{k-1,i}} \right] P^{(-)}_{l_{k,q}} \]

\[ = E_{I_{k,q}} P^{(-)}_{l_{k,q}}. \]

3. An Algorithm Defining the Operators \( V^{(k,q)}_{l_{i,i}} \), and Inductive Control of Block-Diagonalization

Here we address the question of how the interaction terms evolve under our block-diagonalization steps. We propose to define and control an algorithm, \( \alpha_{k,q} \), determining
a map that sends each operator $V^{(k,q-1)}_{l,i}$ to a corresponding potential term supported on the same interval, but at the next block-diagonalization step, i.e.,

$$\alpha_{l,k,q} (V^{(k,q-1)}_{l,i}) = V^{(k,q)}_{l,i}. \quad (3.1)$$

For this purpose, it is helpful to study what happens to the interaction term $V^{(k,q-1)}_{l,i}$ after conjugation with $\exp(S_{l,k,q})$, i.e., to consider the operator

$$e^{S_{l,k,q}} V^{(k,q-1)}_{l,i} e^{-S_{l,k,q}}, \quad (3.2)$$

assuming that $S_{l,k,q}$ is well defined. We start from $V^{(0,N)}_{l_0,i} := H_i$ and follow the fate of these operators and the one of the potential terms. As will follow from definition (3.9), $V^{(k,q)}_{l_0,i}$ coincides with $H_i$, for all $k$ and $q$.

We distinguish four cases; (see Fig. 1 for a graphical representation of the different cases):

1. If $I_{l,i} \cap I_{k,q} = \emptyset$ then
   $$e^{S_{l,k,q}} V^{(k,q-1)}_{l,i} e^{-S_{l,k,q}} = V^{(k,q-1)}_{l,i}, \quad (3.3)$$
   since $S_{l,k,q}$ acts as the identity on $\mathcal{H}_{l,i} := \mathcal{H}_i \otimes \cdots \otimes \mathcal{H}_{i+l}$.

2. If $I_{k,q} \subset I_{l,i}$ then
   $$e^{S_{l,k,q}} V^{(k,q-1)}_{l,i} e^{-S_{l,k,q}} = (V^{(k,q-1)}_{l,i})', \quad (3.4)$$
   where the right side is an operator acting as the identity outside $\mathcal{H}_{l,i}$.

3. If $I_{l,i} \subset I_{k,q}$ then
   $$e^{S_{l,k,q}} V^{(k,q-1)}_{l,i} e^{-S_{l,k,q}} = (V^{(k,q-1)}_{l,i})'', \quad (3.5)$$
   where the right side is an operator acting as the identity outside $\mathcal{H}_{k,q}$.

4. If $I_{l,i} \cap I_{k,q} \neq \emptyset$, with $I_{l,i} \not\subset I_{k,q}$ and $I_{k,q} \not\subset I_{l,i}$, then we use that
   $$e^{S_{l,k,q}} V^{(k,q-1)}_{l,i} e^{-S_{l,k,q}} = V^{(k,q-1)}_{l,i} + \sum_{n=1}^{\infty} \frac{1}{n!} ad^n S_{l,k,q} (V^{(k,q-1)}_{l,i}). \quad (3.6)$$

We will provide a precise definition of $V^{(k,q)}_{l,i}$ below; see Definition 3.1. To prepare the grounds, some heuristic explanations may be helpful. Each operator $V^{(k,q)}_{l,i}$ can be thought of as resulting from the following operations:

(I) A “growth process”, involving operators corresponding to shorter intervals, as described in point (4), above, by the terms on the very right side of (3.6).

(II) Operations as in points (1) and (2), or as given by the first term on the right side of (3.6), which do not change the support of the operator (i.e., they do not change the length of the interval) and leave the norm of the operator invariant.
(III) Operations as described in case (3), above, and made more explicit in the following
remarks: By including all potentials\(^2\) \(V_{I_i,i}^{(k,q-1)}\), with \(I_i,i \subset I_{k,q}\), we obtain the
operator denoted by \(G_{I_{k,q}}\). Moreover, by construction of \(S_{I_i,q}\),
\[
e^{-S_{I_{k,q}}} (G_{I_{k,q}} + t V_{I_i,i}^{(k,q-1)}) e^{-S_{I_{k,q}}} = G_{I_{k,q}} + t \sum_{j=1}^{\infty} t^{-j-1} (V_{I_i,i}^{(k,q-1)})^{diag}
\]
where “\(diag\)” indicates that the corresponding operator is block-diagonal w.r.t. to
the decomposition of the identity into \(P_{I_{k,q}}^{(-)} + P_{I_{k,q}}^{(+)}\). Hence:
(i) If \(I_i,i \equiv I_{k,q}\) we set
\[
V_{I_i,i \equiv I_{k,q}}^{(k,q)} := \sum_{j=1}^{\infty} t^{-j-1} (V_{I_i,i}^{(k,q-1)})^{diag}
\]
\[
= e^{-S_{I_{k,q}}} (\frac{G_{I_{k,q}}}{t} + V_{I_i,i}^{(k,q-1)}) e^{-S_{I_{k,q}}} = \frac{G_{I_{k,q}}}{t}.
\]  
Clearly the operator \(V_{I_i,i}^{(k,q)}\) acts as the identity outside \(\mathcal{H}_{I_{k,q}}\) but in general
\(\|V_{I_i,i}^{(k,q)}\| \neq \|V_{I_i,i}^{(k,q-1)}\|\).
(ii) If \(I_i,i \subset I_{k,q}\) we set
\[
V_{I_i,i}^{(k,q)} := V_{I_i,i}^{(k,q-1)},
\]  
which is block-diagonal w.r.t. the decomposition of the identity into \(P_{I_{k,q}}^{(+)} + P_{I_{k,q}}^{(-)}\),
too, as explained in Remark 2.1. Clearly the operator \(V_{I_i,i}^{(k,q)}\) acts as the identity
outside \(\mathcal{H}_{I_{k,q}}\) and \(\|V_{I_i,i}^{(k,q)}\| = \|V_{I_i,i}^{(k,q-1)}\|\).
Thus the net result of the conjugation of the sum of the operators \(V_{I_i,i}^{(k,q-1)}\) appearing
on the left side of (3.7) can be re-interpreted as follows:
(a) The operators \(V_{I_i,i}^{(k,q-1)}\), with \(I_i,i \subset I_{k,q}\), are kept fixed in the step \((k,q - 1) \rightarrow
(k,q)\), i.e., we define \(V_{I_i,i}^{(k,q)} := V_{I_i,i}^{(k,q-1)}\), hence
\[
G_{I_{k,q}} = \sum_{I_i,i \subset I_{k,q}} H_{I_i,i} + t \sum_{I_i,i \subset I_{k,q}} V_{I_i,i}^{(k,q-1)} + \cdots + t \sum_{I_{k-1,i} \subset I_{k,q}} V_{I_{k-1,i}}^{(k,q-1)}
\]
\[
= \sum_{I_i,i \subset I_{k,q}} H_{I_i,i} + t \sum_{I_i,i \subset I_{k,q}} V_{I_i,i}^{(k,q)} + \cdots + t \sum_{I_{k-1,i} \subset I_{k,q}} V_{I_{k-1,i}}^{(k,q)}
\]
(b) the operator \(V_{I_{k,q}}^{(k,q-1)}\) is transformed to the operator
\[
V_{I_{k,q}}^{(k,q)} := \sum_{j=1}^{\infty} t^{-j-1} (V_{I_{k,q}}^{(k,q-1)})^{diag}
\]
\(^2\) Recall that \(V_{I_{0,i}}^{(0,N)} := H_{I_i}\) and \(V_{I_{0,i}}^{(k,q)}\) will coincide with \(V_{I_{0,i}}^{(0,N)}\) for all \((k,q)\).
Fig. 1. Relative positions of intervals $I_{k,q}$ and $I_{l,i}$

which is block-diagonal, and

$$\| V_{I_{k,q}}^{(k,q)} \| \leq 2 \| V_{I_{k,q}}^{(k,q-1)} \| ,$$

as will be shown, assuming that $t > 0$ is sufficiently small.

3.1. The algorithm $\alpha_{I_{k,q}}$. In this subsection, we finally present a precise iterative definition of the operators

$$V_{I_{l,i}}^{(k,q)} := \alpha_{I_{k,q}} (V_{I_{l,i}}^{(k,q-1)})$$
in terms of the operators $V_{I_{l,i}}^{(k,q-1)}$, at the previous step $(k, q-1)$, starting from

$$V_{I_{l,i}}^{(0,N)} = H_i, \quad V_{I_{l,i}}^{(0,N)} = V_{I_{1,i}}, \quad V_{I_{l,i}}^{(0,N)} = 0 \text{ for } l \geq 2. \quad (3.10)$$

**Definition 3.1.** We assume that, for fixed $(k, q-1)$, with $(k, q-1) > (0, N)$, the operators $V_{I_{l,i}}^{(k,q-1)}$ and $S_{I_{k,q}}$ are well defined and bounded, for any $l, i$; or we assume that $(k, q) = (1, 1)$ and that the operator $S_{I_{1,1}}$ is well defined and bounded. We then define the operators $V_{I_{l,i}}^{(k,q)}$ as follows, with the warning that if $q = 1$ the couple $(k, q-1)$ is replaced by $(k-1, N-k+1)$ in (3.11)–(3.15) – see Fig. 2 for a graphical representation of the different cases (b), (c) (d-1) and (d-2), below:

(a) in all the following cases

(a-1) $l \leq k - 1;$
Notice that, according to Definition 3.1:

Remark 3.2.

For the iteration step, if $q = 1$, $V_{l,i}^{(k,q)} := V_{l,i}^{(k,q-1)}$;

(b) if $I_{l,i} \equiv I_{k,q}$, we define

\begin{equation}
V_{l,i}^{(k,q)} := \sum_{j=1}^{\infty} t^{j-1} (V_{l,i}^{(k,q-1)})_{j}^{\text{diag}};
\end{equation}

(c) if $I_{k,q} \subset I_{l,i}$ and $i, i + l \notin I_{k,q}$, we define

\begin{equation}
V_{l,i}^{(k,q)} := e^{S_{l,q}} V_{l,i}^{(k,q-1)} e^{-S_{l,q}};
\end{equation}

(d) if $I_{k,q} \subset I_{l,i}$ and either $i$ or $i + l$ belongs to $I_{k,q}$, we define

(d-1) if $i$ belongs to $I_{k,q}$, i.e., $q \equiv i$, then

\begin{equation}
V_{l,i}^{(k,q)} := e^{S_{l,q}} V_{l,i}^{(k,q-1)} e^{-S_{l,q}} + \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n!} ad^n S_{k,i} (V_{l,i}^{(k,q-1)})_{j}^{\text{diag}};
\end{equation}

(d-2) if $i + l$ belongs to $I_{k,q}$, i.e., $q + k \equiv i + l$ that means $q \equiv i + l - k$, then

\begin{equation}
V_{l,i}^{(k,q)} := e^{S_{l,q}} V_{l,i}^{(k,q-1)} e^{-S_{l,q}} + \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n!} ad^n S_{k,i} (V_{l,i}^{(k,q-1)})_{j}^{\text{diag}}.
\end{equation}

Notice that in both cases, (d-1) and (d-2), the elements of the sets $\{I_{l-i,j+i}\}_{j=1}^{k}$ and $\{I_{l-j,i}\}_{j=1}^{k}$, respectively, are all the intervals, $\mathcal{I}$, such that $\mathcal{I} \cap I_{k,q} \neq \emptyset$, $\mathcal{I} \nsubseteq I_{k,q}$, $I_{k,q} \nsubseteq \mathcal{I}$, and $\mathcal{I} \cup I_{k,q} \equiv I_{l,i}$.

Remark 3.2. Notice that, according to Definition 3.1:

- if $(k', q') > (l, i)$ then

\begin{equation}
V_{l,i}^{(k',q')} = V_{l,i}^{(l,i)};
\end{equation}

since the occurrences in cases (b), (c), (d-1), and (d-2) are excluded;

- for $k \geq 1$ and all allowed choices of $q$,

\begin{equation}
V_{l,0,i}^{(k,q)} = H_{i}
\end{equation}

due to (a-1).

In the next theorem we prove that Definition 3.1 yields operators $V_{l,i}^{(k,q)}$ consistent with the expression of the Hamiltonian $K_{N}^{(k,q)}$ given in (2.2), (2.3).

Theorem 3.3. We assume that the potentials satisfy Definition 3.1. Then the Hamiltonian $K_{N}^{(k,q)} := e^{S_{l,q}} K_{N}^{(k,q-1)} e^{-S_{l,q}}$, with $k \geq 1$ and $q \geq 2$, has the form given in (2.2), (2.3), where the operators $\{V_{l,i}^{(k,q)}\}$ are determined by the operators $\{V_{l,i}^{(k,q-1)}\}$ of the previous iteration step. If $q = 1$ the statement holds with $(k, q - 1)$ replaced by $(k - 1, N - k + 1)$. 

Proof. We study the case $q \geq 2$ explicitly, the case $q = 1$ is proven in the same way. In the expression

\[
e^{S_{I_k,q}} K_N^{(k,q-1)} e^{-S_{I_k,q}}
\]

we observe that:

- For all intervals $I_{l,i}$ with the property that $I_{l,i} \cap I_{k,q} = \emptyset$,

\[
e^{S_{I_k,q}} V_{I_{l,i}}^{(k,q-1)} e^{-S_{I_k,q}} = V_{I_{l,i}}^{(k,q-1)} =: V_{I_{l,i}}^{(k,q)}
\]

which follows from (a-2), Definition 3.1.

- With regard to the terms constituting $G_{I_{k,q}}$ (see Definition (2.28)), we get, after adding $t V_{I_{k,q}}^{(k,q-1)}$,

\[
e^{S_{I_k,q}} (G_{I_{k,q}} + t V_{I_{k,q}}^{(k,q-1)}) e^{-S_{I_k,q}}
\]

\[
= \sum_{i \subset I_{k,q}} H_i + t \sum_{I_{l,i} \subset I_{k,q}} V_{I_{l,i}}^{(k,q-1)} + \ldots
\]

\[
+ t \sum_{I_{k-1,i} \subset I_{k,q}} V_{I_{k-1,i}}^{(k,q-1)} + t \sum_{j=1}^{\infty} t^{j-1} (V_{I_{k,q}}^{(k,q-1)})^{diag}
\]
= \sum_{i \in I_{k,q}} H_i + t \sum_{I_{l,i} \subset I_{k,q}} V_{l,i}^{(k,q)} + \cdots + t \sum_{I_{k-1,i} \subset I_{k,q}} V_{k-1,i}^{(k,q)} + t V_{k,i}^{(k,q)}, \quad (3.20)

where the first identity is the result of the Lie-Schwinger conjugation, and the last identity follows from Definition 3.1, cases (a-1) and (b).

- With regard to the terms \( V_{l,i}^{(k,q-1)} \), with \( I_{k,q} \subset I_{l,i} \) and \( i, i + l \notin I_{k,q} \), the expression

\[
e^{S_{l,i}} V_{l,i}^{(k,q-1)} e^{-S_{l,i}}
\]

(3.21)

corresponds to \( V_{l,i}^{(k,q)} \), by Definition 3.1, case (c).

- With regard to the terms \( V_{l,i}^{(k,q-1)} \), with \( I_{l,i} \cap I_{k,q} \neq \emptyset \), but \( I_{l,i} \not\subset I_{k,q} \) and \( I_{k,q} \not\subset I_{l,i} \), it follows that

\[
e^{S_{l,i}} V_{l,i}^{(k,q-1)} e^{-S_{l,i}} = V_{l,i}^{(k,q-1)} + \sum_{n=1}^{\infty} \frac{1}{n!} a d^n S_{l,i} (V_{l,i}^{(k,q-1)}). \quad (3.22)
\]

The first term on the right side is \( V_{l,i}^{(k,q)} \) (see cases (a-1) and (a-3) in Definition 3.1), the second term contributes to \( V_{l,r}^{(k,q)} \), where \( I_{r,j} \equiv I_{l,i} \cup I_{k,q} \), together with further similar terms and with

\[
e^{S_{l,q}} V_{l,r}^{(k,q-1)} e^{-S_{l,q}} , \quad (3.23)
\]

where the set \( I_{r,j} \) has the property that \( I_{k,q} \subset I_{r,j} \), and either \( j \) or \( j + r \) belong to \( I_{k,q} \). Notice that the term in (3.23) has not been considered in the previous cases and corresponds to the first term on the r.h.s of (3.14) or to the analogous quantity in (3.15), where \( l \) is replaced by \( r \) and \( i \) by \( j \). \( \square \)

3.2. Block-diagonalization of \( K_N \)–control of \( \| V_{l,r}^{(k,q)} \| \). In the next theorem, we estimate the norm of \( V_{l,r}^{(k,q)} \) in terms of the norm of the potentials in the previous step \((k, q - 1)\).

For a fixed interval \( I_{r,i} \), the norm of the potential does not change, i.e., \( \| V_{l,r}^{(k,q-1)} \| = \| V_{l,r}^{(k,q)} \| \), in the step \((k, q - 1) \rightarrow (k, q)\), unless some conditions are fulfilled. To gain some intuition of this fact, the reader is advised to take a look at Fig. 1, (replacing \( l \) by \( r \)). Notice that shifting the interval \( I_{k,q} \) to the left by one site makes it coincide with \( I_{k,q-1} \). If \( I_{k,q} \) is not contained in \( I_{r,i} \) or if it is contained therein, but none of the endpoints of the interval \( I_{k,q} \) coincides with an endpoint of \( I_{r,i} \), then \( \| V_{l,r}^{(k,q)} \| = \| V_{l,r}^{(k,q-1)} \| \). Therefore, in the step \((k, q - 1) \rightarrow (k, q)\), a change of norm, i.e., \( \| V_{l,r}^{(k,q)} \| \neq \| V_{l,r}^{(k,q-1)} \| \), only happens in at most two cases, provided \( r > k \), and only in one case if \( k \) coincides with the length \( r \); and it never happens if \( r < k \).

In the theorem below we estimate the change of the norm of the potentials in the block-diagonalization steps, for each \( k \), starting from \( k = 0 \). It is crucial to control the block-diagonalization of \( V_{l,r}^{(k,q)} \) that takes place when \((k, q) \equiv (r, i)\). We have to make use of a lower bound on the gap above the ground-state energy in the energy spectrum of the Hamiltonian \( G_{l,k} \). This lower bound follows from estimate (2.31), as explained in Lemma 2.4 and Corollary 2.5. We will proceed inductively by showing that, for \( t(> 0) \)
sufficiently small but independent of $N$, $k$, and $q$, the operator-norm bound in (2.31), at step $(k, q - 1)$, $q \geq 2$ (for $q = 1$ see the footnote), yields control over the spectral gap of the Hamiltonians $G_{I_kq}$, (see Corollary 2.5), and the latter provides an essential ingredient for the proof of a bound on the operator norms of the potentials, according to (2.31), in the next step\(^3\) $(k, q)$.

**Theorem 3.4.** Assume that the coupling constant $t(> 0)$ is sufficiently small. Then the Hamiltonians $G_{I_kq}$ and $K_N^{(k,q)}$ are well defined, and

1. for any interval $I_{r,i}$, with $r \geq 1$, the operator $V_{I_{r,i}}^{(k,q)}$ has a norm bounded by
   \[
   \frac{8}{(r+1)\gamma} t^{r-1},
   \]
2. $G_{I_{k,q+1}}$ has a spectral gap $\Delta_{I_{k,q+1}} \geq \frac{1}{2}$ above the ground state energy, where $G_{I_kq}$ is defined in (2.28) for $k \geq 2$, and $G_{I_{k,q}} := H_q + H_{q+1}$.

**Proof.** The proof is by induction in the diagonalization step $(k, q)$, starting at $(k, q) = (0, N)$, and ending at $(k, q) = (N - 1, 1)$; (notice that (S2) is not defined for $(k, q) = (N - 1, 1)$).

For $(k, q) = (0, N)$, we observe that $K_N^{(k,q)} \equiv K_N$ and $G_{I_{k,q}}$ is not defined, indeed it is not needed since (S1) is verified by direct computation, because by definition

\[
\|V_{I_{l,i}}^{(0,N)}\| = \|V_{I_{l,i}}\| \leq 1,
\]

and $V_{I_{l,i}}^{(0,N)} = 0$, for $r \geq 2$. (S2) holds trivially since, by definition, the successor of $(0, N)$ is $(1, 1)$ and $G_{I_{1,1}} = H_1 + H_2$.

Assume that (S1) and (S2) hold for all steps $(k', q')$ with $(k', q') < (k, q)$. We prove that they then hold in step $(k, q)$. By Lemma A.3, (S1) and (S2) for $(k, q - 1)$ imply that $S_{k,q}$ and, consequently, that $K_N^{(k,q)}$ are well defined operators, see (2.30). In the steps described below it is understood that if $q = 1$ the pair $(k, q - 1)$ is replaced by $(k - 1, N - k + 1)$.

**Induction step in the proof of (S1)**

Starting from Definition 3.1 we consider the following cases:

**Case $r = 1$.**

Let $k > 1(= r)$ or $k = 1 = r$ but $I_{1,i}$ such that $i \neq q$. Then the possible cases are described in (a-1), (a-2), and (a-3), see Definition 3.1, and we have that

\[
\|V_{I_{l,i}}^{(k,q)}\| = \|V_{I_{l,i}}^{(k,q-1)}\|.
\]

(3.24)

Hence, we use the inductive hypothesis.

Let $k = 1$ and assume the set $I_{l,i}$ is equal to $I_{1,q}$. Then we refer to case b) and we find that

\[
\|V_{I_{l,q}}^{(1,q)}\| \leq 2\|V_{I_{l,q}}^{(1,q-1)}\| \leq 2,
\]

(3.25)

where:

- (a) the inequality $\|V_{I_{l,q}}^{(1,q)}\| \leq 2\|V_{I_{l,q}}^{(1,q-1)}\|$ holds for $t$ sufficiently small uniformly in $q$ and $N$, thanks to Lemma A.3 which can be applied, since we assume (S1) and (S2) in step $(1, q - 1)$;

\(^3\) Recall the special steps of type $(k, 1)$ with preceding step $(k - 1, N - k + 1)$. 
(b) we use \( \| V_{I_{1,q}}^{(1,q-1)} \| = \| V_{I_{1,q}}^{(1,q-2)} \| = \cdots = \| V_{I_{1,q}}^{(0,N)} \| \leq 1 \).

**Case** \( r \geq 2 \).

1) Let \( r > k \), then in cases (a-2), (a-3), and (c), see Definition 3.1, we have that

\[
\| V_{I_{r,i}}^{(k,q)} \| = \| V_{I_{r,i}}^{(k,q-1)} \| ,
\]

(3.26)

otherwise we are in case (d-1), see (3.14), that means \( q \equiv i \), and estimate

\[
\| V_{I_{r,i}}^{(k,q)} \| \leq \| V_{I_{r,i}}^{(k,q-1)} \| + \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^n S_{k,j} (V_{I_{r-j,i}}^{(k,q-1)}) \| ,
\]

(3.27)

or in case (d-2), see (3.15), that means \( q \equiv i + r - k \), and we estimate

\[
\| V_{I_{r,i}}^{(k,q)} \| \leq \| V_{I_{r,i}}^{(k,q-1)} \| + \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^n S_{k,i+j} (V_{I_{r-j,i}}^{(k,q-1)}) \| .
\]

(3.28)

Hence, for \( r > k \) and \( q > i + r - k \), we can write (assuming \( i \geq 2 \), but an analogous procedure holds if \( i = 1 \))

\[
\| V_{I_{r,i}}^{(k,q)} \| \leq \| V_{I_{r,i}}^{(k,i+r-k)} \| (3.29)
\]

\[
= \| V_{I_{r,i}}^{(k,i+r-k)} \| (3.30)
\]

\[
\leq \| V_{I_{r,i}}^{(k,i+r-k-1)} \| + \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^n S_{k,i+r-k} (V_{I_{r-j,i}}^{(k,i+r-k-1)}) \| (3.31)
\]

\[
= \| V_{I_{r,i}}^{(k,i)} \| + \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^n S_{k,i+r-k} (V_{I_{r-j,i}}^{(k,i)}) \| (3.32)
\]

\[
\leq \| V_{I_{r,i}}^{(k,i-1)} \| + \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^n S_{k,i} (V_{I_{r-j,i}}^{(k,i-1)}) \| (3.33)
\]

\[
+ \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^n S_{k,i+r-k} (V_{I_{r-j,i}}^{(k,i-1)}) \| (3.34)
\]

where

- in the step from (3.29) to (3.30) we have used that \( \| V_{I_{r,i}}^{(k,q')} \| = \| V_{I_{r,i}}^{(k,q'-1)} \| \), for all \( q \geq q' \geq i + r - k + 1 \), by the argument yielding (3.26);
● in the step from (3.30) to (3.31) we have used (3.28);
● in the step from (3.31) to (3.32) we have used that \( \| V_{lr,i}^{(k,q')} \| = \| V_{lr,i}^{(k,q'-1)} \| \), for all \( i + r - k - 1 \geq q' \geq i + 1 \), by the argument yielding (3.26), (notice that if \( r = k + 1 \) this step is trivial);
● in the step from (3.32) to (3.33) we have used (3.27);
● in the step from (3.33) to (3.34), invoking the argument yielding (3.26), we have used the following facts:
\begin{itemize}
  \item (A) \( \| V_{lr,i}^{(k,q')} \| = \| V_{lr,i}^{(k,q'-1)} \| \), for all \( i - 1 \geq q' \geq 2 \),
  \item (B) \( \| V_{lr,i}^{(k,1)} \| = \| V_{lr,i}^{(k-1,N-k+1)} \| \),
  \item (C) \( \| V_{lr,i}^{(k-1,q')} \| = \| V_{lr,i}^{(k-1,q'-1)} \| \), for all \( N - k + 1 \geq q' \geq i + r - k + 2 \).
\end{itemize}
Notice that if \( i = 2 \) then (A) is an empty statement and must be ignored, likewise if \( N = i + r \) statement (C) is empty and must be ignored.

Iterating the arguments in (3.26) and (3.27), (3.28) for the first term on the right side of (3.34) – i.e., \( \| V_{lr,i}^{(k-1,i+r-k+1)} \| \) – and observing that, by assumption, \( V_{lr,i}^{(0,N)} = 0 \) if \( r \geq 2 \), we end up finding that
\[
\| V_{lr,i}^{(k,q)} \| \leq \sum_{m=1}^{k} \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^{n} S_{m,i+r-m} (V_{lr,j,i}^{(m,i+r-m-1)}) \| + \sum_{m=1}^{k} \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^{n} S_{m,i} (V_{lr,j,i+j}^{(m,i-1)}) \| ,
\]
where the interval \( I_{m,i} \), with \( m \leq k - 1 \), has first vertex coinciding with \( i \) and last vertex equal to \( i + m \), while the set \( I_{m,i+r-m} \) has first vertex equal to \( i + r - m \) and the last vertex coinciding with \( i + r \). The intervals associated to the summands in (3.36) are displayed in Fig. 3. Notice that the last two terms on the right side of (3.34) correspond to the summands associated with \( m \equiv k \) in (3.35) and (3.36).

II
Let \( r = k \), then the interval is \( I_{r=k,i} \).
If \( i > q \) then the procedure is identical to the previous case I), except that the sum over \( m \) in (3.35), (3.36) is up to \( k - 1 \).
If \( i \leq q \) we have two possibilities:
\begin{itemize}
  \item (A) if \( i < q \) then \( \| V_{lr_{=k,i}}^{(k,q)} \| = \| V_{lr_{=k,i}}^{(k,q-1)} \| \), and we use the inductive hypothesis;
  \item (B) if \( q = i \) we refer to case (b) (of Definition 3.1) and, thanks to (S1) and (S2) of the previous step \((k, q - 1)\), we can apply Lemma A.3 and estimate
\[
\| V_{lr_{=k,i}}^{(k,q_{=i})} \| \leq 2 \| V_{lr_{=k,i}}^{(k,i-1)} \|. \tag{3.37}
\]
\end{itemize}
Then we proceed as in the previous case I). Eventually we can estimate
\[
\| V_{lr_{=k,i}}^{(k,q)} \| \leq 2 \sum_{m=1}^{k-1} \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^{n} S_{m,i+r-m} (V_{lr_{=k,i}}^{(m,i+r-m-1)}) \| + 2 \sum_{m=1}^{k-1} \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{1}{n!} \| a d^{n} S_{m,i} (V_{lr_{=k,i}}^{(m,i-1)}) \| . \tag{3.38}
\]
III) Let $r < k$. This corresponds to case (a-1) in Definition (3.1). Thus

$$\| V_{(k,q)}^{(k,q-1)} \| = \| V_{(k,q-1)}^{(k,q-1)} \|,$$

(3.40)

and we use the inductive hypothesis.

Estimate of (3.38), (3.39)

We recall formula (2.18), and we invoke properties (S1) and (S2) at the previous steps, so that we can apply Lemma A.3 and get the estimate

$$\| S_{m,i+r-m} \| \leq C \cdot t \cdot \frac{8}{(m+1)^2} t^{m-1}$$

(3.41)

and

$$\sum_{n=1}^{\infty} \frac{1}{n!} \| a d^n S_{m,i+r-m} (V_{(m,i+r-m-1)}) \| \leq C \cdot t \cdot \frac{64}{(m+1)^2} t^{m-1} t \cdot \frac{r-j-1}{(r-j+1)^2},$$

(3.42)

where the value of the constant $C$ changes from line to line.

Using the observations on the construction of the intervals (see (3.36)), the estimate in (3.42), and property (S1) at the previous steps, we can write

$$\| V_{(k,q)}^{(k,q)} \| \leq 128 \cdot C \cdot t \sum_{m=1}^{r-1} \frac{t^{m-1}}{(m+1)^2} \sum_{j=1}^{m} \frac{t^{r-j-1}}{(r-j+1)^2}$$

$$\leq C' \cdot t \sum_{m=1}^{r-1} \frac{t^{m-1}}{m^2} \frac{t^{r-m-1}}{(r-m)^2}.$$

Fig. 3. Intervals $I_{k,q}$ and $I_{r-j,i+j}$, $j = 1, 2, 3, 4$, associated with the summands in formula (3.36) for $m = k$. 


\[ \leq C' \cdot t \cdot t^{r-2} \sum_{m=1}^{r-1} \frac{1}{m^2 (r-m)^2} \]
\[ \leq \frac{8 \cdot t^{r-1}}{(r+1)^2}, \]  
(3.43)

for \( t > 0 \) sufficiently small, but independent of \( r \geq 2, k, \) and \( q, \) where \( C' \) is a universal constant.

The estimate of (3.35), (3.36) is similar.

**Induction step to prove (S2)**

Having proven (S1), we can use Lemma 2.4 and Corollary 2.5 in subsequent arguments. Hence, (S2) holds for \( t > 0 \) sufficiently small, but independent of \( N, k, \) and \( q, \) where \( C' \) is a universal constant. \( \square \)

**Theorem 3.5.** Under the assumption that (1.4), (1.6), and (1.7) hold, the Hamiltonian \( K_N \) defined in (1.5) has the following properties: There exists some \( t_0 > 0 \) such that, for any \( t \in \mathbb{R} \) with \( |t| < t_0, \) and for all \( N < \infty, \)

(i) \( K_N \equiv K_N(t) \) has a unique ground-state; and

(ii) the energy spectrum of \( K_N \) has a strictly positive gap, \( \Delta_N(t) \geq \frac{1}{2}, \) above the ground-state energy.

**Proof.** Notice that \( K_N^{(N-1,1)} \equiv G_{I_{N-1,1}} + t V_{I_{N-1,1}}^{(N-1,1)}. \) Hence we have constructed the unitary conjugation \( \exp(S_N(t)), \) see (1.10), such that the operator

\[ e^{S_N(t)} K_N(t) e^{-S_N(t)} = G_{I_{N-1,1}} + t V_{I_{N-1,1}}^{(N-1,1)} =: \tilde{K}_N(t), \]
has the properties in (1.11) and (1.12), which follow from Theorem 3.4 and from (2.46) and (2.51), for \( (k, q) = (N - 1, 1), \) where we also include the block-diagonalized potential \( V_{I_{N-1,1}}^{(N-1,1)} . \) \( \square \)

### 4. The Kitaev Chain

In this last section, we show how our method can be used to study small perturbations of the Hamiltonian of a Kitaev chain in the nontrivial phase; (see [GST]). (Similar results have been proven in [KST] for a special class of perturbations.)

Consider a chain with \( N \) sites, where, at each site \( j, \) there are fermion creation- and annihilation operators, \( c_j^*, \ c_j, \) with

\[ \{c_j, c_l\} = \{c_j^*, c_l^*\} = 0, \quad \{c_j, c_l^*\} = \delta_{j,l} , \]  
(4.1)

where \( \{A, B\} \) is the anti-commutator of \( A \) and \( B. \) The Hilbert space \( \mathcal{H} \) is spanned by the vectors obtained by applying products of creation operators, \( c_j^*, j = 1, \ldots, N, \) to the vacuum vector, which is annihilated by all the operators \( c_j. \) The Hamiltonian of the system is given by

\[ H := -\mu \sum_{j=1}^{N} c_j^* c_j - \sum_{j=1}^{N-1} \left( \tau c_j^* c_{j+1} + \tau c_{j+1}^* c_j + \Delta c_j c_{j+1} + \Delta c_{j+1}^* c_j^* \right) \]  
(4.2)
where $\mu$ is the chemical potential, $\tau \geq 0$ is the nearest-neighbor hopping amplitude, and $\Delta \geq 0$ is the p-wave pairing amplitude. By re-writing the fermion operators $c_j, c_j^*$ in terms of the Clifford generators (“Dirac matrices”)

$$\gamma_{A,j} := ic_j - ic_j^*, \quad \gamma_{B,j} := c_j^* + c_j,$$

the Hamiltonian becomes

$$H = -\frac{\mu}{2} \sum_{j=1}^{N} (1 + i\gamma_{B,j}\gamma_{A,j}) - i\sum_{j=1}^{N-1} \{ (\Delta + \tau)\gamma_{B,j}\gamma_{A,j+1} + (\Delta - \tau)\gamma_{A,j}\gamma_{B,j+1} \}.\quad (4.4)$$

If $\mu = 0$ and $\tau = \Delta = 1$, and for open boundary conditions, the system is in a “nontrivial phase”, and the corresponding Hamiltonian is denoted by $H_{\text{Kitaev}}$:

$$H_{\text{Kitaev}} := -i \sum_{j=1}^{N-1} \gamma_{B,j}\gamma_{A,j+1} = \sum_{j=1}^{N-1} (2d_j^*d_j - 1), \quad (4.5)$$

where

$$2d_j^* := \gamma_{B,j} + i\gamma_{A,j+1} = c_{j+1} - c_{j+1}^* + c_j^* + c_j, \quad 2d_0^* := -c_1^* + c_1 + c_N^* + c_N. \quad (4.6)$$

As a consequence of (4.1), the operators $d_j, d_j^*$, obey the relations

$$\{d_j, d_l\} = \{d_j^*, d_l^*\} = 0, \quad \{d_j, d_l^*\} = \delta_{j,l} \quad \text{for} \quad j, l = 0, \ldots, N - 1. \quad (4.7)$$

Notice that, for $1 \leq j \leq N - 1$,

$$c_j = \frac{d_j + d_j^* + d_{j+1}^* - d_{j-1}}{2}, \quad (4.8)$$

and

$$c_N = \frac{d_0 + d_0^* + d_{N-1}^* - d_{N-1}}{2}. \quad (4.9)$$

Consider the following local perturbations of the Hamiltonian $H$:

$$\beta \sum_{i=1}^{N-1} V_{\text{l}_{1,i}} + \beta \sum_{i=1}^{N-2} V_{\text{l}_{2,i}} + \cdots + \beta \sum_{i=1}^{N-\tilde{k}} V_{\text{l}_{k,i}} \quad (4.10)$$

where $\tilde{k}$ is $N$-independent, $\beta$ is a coupling constant, and each term $V_{\text{l}_{j,i}}$ is a hermitian operator consisting of an $N$-independent, finite sum of products of an even number of operators $\{c_l, c_l^*\}_{l=i}^{i+j}$. In (4.10) we split the sum into

$$\beta V_{\text{l}_{1,1}} + \beta V_{\text{l}_{2,1}} + \cdots + \beta V_{\text{l}_{k,1}} \quad (4.12)$$

$$+ \beta \sum_{i=2}^{N-2} V_{\text{l}_{1,i}} + \beta \sum_{i=2}^{N-3} V_{\text{l}_{2,i}} + \cdots + \beta \sum_{i=2}^{N-\tilde{k}-1} V_{\text{l}_{k,i}} \quad (4.13)$$

$$+ \beta V_{\text{l}_{1,N-1}} + \beta V_{\text{l}_{2,N-2}} + \cdots + \beta V_{\text{l}_{k,N-\tilde{k}}}. \quad (4.14)$$
and we then use the identities (4.8), (4.9) to re-write these operators in terms of the $d$- and $d^*$-operators. We then get

\begin{align}
(4.12) + (4.13) + (4.14) &
= \beta \tilde{V}_{I_{2,0}} + \beta \tilde{V}_{I_{3,0}} + \cdots + \beta \tilde{V}_{I_{k+1,0}} \\
&+ \beta \sum_{i=2}^{N-2} \tilde{V}_{I_{2,i-1}} + \beta \sum_{i=2}^{N-3} \tilde{V}_{I_{3,i-1}} + \cdots + \beta \sum_{i=2}^{N-k-1} \tilde{V}_{I_{k+1,i-1}} \\
&+ \beta \tilde{V}_{I_{2,N-2}} + \beta \tilde{V}_{I_{3,N-3}} + \cdots + \beta \tilde{V}_{I_{k+1,N-k-1}},
\end{align}

(4.15)

where, in (4.18), $N$ is identified with 0, i.e, $I_{2,N-2} := (N-2, N-1, 0)$, and the symbol

\begin{equation}
\tilde{V}_{I_{j,i}}
\end{equation}

stands for a finite sum of operators consisting of products of an even number of operators $\{d_l, d^*_l\}_{l=i+j}$.

Let $\Omega^{(d)}$ be the vector annihilated by the operators $d_j$, $j = 0, \ldots, N-1$, and define the Fock space $F_{1, \ldots, N-1}$ as the span of vectors obtained by applying products of the operators $d^*_j$, $j = 1, \ldots, N-1$, to $\Omega_1(d)$. This space can be identified with the space

\begin{equation}
\bigotimes_{j=1}^{N-1} F_j,
\end{equation}

(4.20)

where $F_j \simeq \mathbb{C}^2$ is the fermionic Fock space obtained by applying the identity and the creation operator $d^*_j$ to the vacuum vector $\Omega_1(d)$, which is annihilated by $d_j$. Likewise, we have that

\begin{equation}
\mathcal{H} \simeq F_0 \otimes \left( \bigotimes_{j=1}^{N-1} F_j \right).
\end{equation}

(4.21)

Notice that the potentials in (4.17) do not depend on the zero-mode operators. Hence we can apply the method developed in previous sections to analyse the Hamiltonian

\begin{equation}
H'_{\beta} \upharpoonright \bigotimes_{j=1}^{N-1} F_j := \left( H_{Kitaev} + \beta \sum_{i=2}^{N-2} \tilde{V}_{I_{2,i-1}} + \beta \sum_{i=2}^{N-3} \tilde{V}_{I_{3,i-1}} + \cdots + \beta \sum_{i=2}^{N-k-1} \tilde{V}_{I_{k+1,i-1}} \right) \upharpoonright \bigotimes_{j=1}^{N-1} F_j
\end{equation}

(4.22)

and show that, for $|\beta|$ sufficiently small, there is a unique ground-state and the energy spectrum is gapped, with a gap larger than 1 above the ground-state energy, uniformly in $N$.

It is straightforward to check that the Hamiltonian $H'_{\beta} : \mathcal{H} \to \mathcal{H}$ has the same spectrum as $H'_{\beta} \upharpoonright \bigotimes_{j=1}^{N-1} F_j$; but, for each eigenvalue $E$, the corresponding eigenspace is doubled, since if $\Psi_E \in \bigotimes_{j=1}^{N-1} F_j$ is an eigenvector of (4.22) corresponding to the
eigenvalue $E$ then both vectors, $\Omega_0 \otimes \Psi_E$ and $d_0^\dagger \Omega_0 \otimes \Psi_E$, are eigenvectors corresponding to the same eigenvalue $E$ of the operator $H'_\beta : \mathcal{H} \to \mathcal{H}$. We denote by $\mathcal{H}_{\beta,gs}$ the doubly-degenerate ground-state subspace of $H'_\beta : \mathcal{H} \to \mathcal{H}$.

We can now apply a Lie-Schwinger block-diagonalization procedure to the operator

$$H_\beta := H'_\beta$$

by considering $H'_\beta$ as the unperturbed Hamiltonian: By constructing a unitary operator $U$ (see [DFFR]), we can block-diagonalize $H_\beta$, so that the transformed Hamiltonian

$$U^* H_\beta U : \mathcal{H}_{\beta,gs} \to \mathcal{H}_{\beta,gs}, \quad U^* H_\beta U : (\mathcal{H} \ominus \mathcal{H}_{\beta,gs}) \to (\mathcal{H} \ominus \mathcal{H}_{\beta,gs}).$$

The distance between the spectrum of $U^* H_\beta U \upharpoonright \mathcal{H}_{\beta,gs}$ and the one of $U^* H_\beta U \upharpoonright (\mathcal{H} \ominus \mathcal{H}_{\beta,gs})$ is of order 1 provided $|\beta|$ is sufficiently small. Moreover, the operator $U^* H_\beta U \upharpoonright \mathcal{H}_{\beta,gs}$ is a $2 \times 2$ matrix that can be diagonalized. 

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A. Appendix

Lemma A.1. For any $1 \leq n \leq N$

$$\sum_{i=1}^{n} P_{\Omega_i}^\perp \geq \mathbb{1} - \bigotimes_{i=1}^{n} P_{\Omega_i} =: \left( \bigotimes_{i=1}^{n} P_{\Omega_i} \right)^\perp \quad (A.1)$$

where $P_{\Omega_i}^\perp = \mathbb{1} - P_{\Omega_i}$.

Proof. We call $P_{vac} := \bigotimes_{i=1}^{n} P_{\Omega_i}$ acting on $\mathcal{H}^{(n)} := \bigotimes_{i=1}^{n} \mathcal{H}_i$. We define

$$A_n := \sum_{j=1}^{n} P_{\Omega_j}^\perp + P_{vac} \quad (A.2)$$

Notice that all operators $P_{\Omega_j}^\perp$ and $P_{vac}$ commute with each other and are orthogonal projectors. Therefore we deduce that

$$\text{spec}(A_n) \subseteq \{0, 1, 2, \ldots, n + 1\} \quad (A.3)$$

We will show that

$$\text{Range } A_n = \mathcal{H}^{(n)} \quad (A.4)$$
If (A.4) holds then $0 \notin \text{spec}(A_n)$. By (A.3) it then follows that

$$A_n \geq I. \quad (A.5)$$

Thus, we are left with proving (A.4).

(i) Assume that $\psi$ is perpendicular to the range of $A_n$, and let $P_{\Omega_j}^\perp \psi =: \phi_j$. Then, since $\psi \perp \text{Range } A_n$, we have that

$$0 = \langle \psi, A_n \psi \rangle = \sum_{i=1; i \neq j}^n \langle \psi, P_{\Omega_i}^\perp \psi \rangle + \langle \psi, P_{\text{vac}} \psi \rangle + \langle \psi, P_{\Omega_j}^\perp \psi \rangle = \langle \psi, P_{\Omega_j}^\perp \psi \rangle \quad (A.6)$$

but

$$\langle \psi, P_{\Omega_j}^\perp \psi \rangle = \langle P_{\Omega_j}^\perp \psi, P_{\Omega_j}^\perp \psi \rangle = \langle \phi_j, \phi_j \rangle \quad (A.7)$$

where we have used that $P_{\Omega_j}^\perp$ is an orthogonal projector. We conclude that $\phi_j = 0$ for all $j$.

(ii) Let $\psi \perp \text{Range } A_n$. Then, by (i),

$$\psi = \left( \bigotimes_{j=1}^n (P_{\Omega_j}^\perp + P_{\Omega_j}) \right) \psi = \left( \bigotimes_{j=1}^n P_{\Omega_j} \right) \psi = P_{\text{vac}} \psi \quad (A.8)$$

and

$$0 = \langle \psi, A_n \psi \rangle = \langle \psi, P_{\text{vac}} \psi \rangle = \langle \psi, \psi \rangle \Rightarrow \psi = 0. \quad (A.9)$$

Thus, $\text{Range } A_n = \mathcal{H}^{(n)}$, and (A.4) is proven. \hfill \Box

From Lemma A.1 we derive the following bound.

**Corollary A.2.** For $i + r \leq N$, we define

$$P_{i,r}^{(+) := \left( \bigotimes_{k=i}^{i+r} P_{\Omega_k} \right)^\perp}. \quad (A.10)$$

Then, for $1 \leq l \leq L \leq N - r$,

$$\sum_{i=l}^L P_{i,r}^{(+) \leq (r + 1) \sum_{i=l}^{L+r} P_{\Omega_i}^\perp. \quad (A.11)$$

**Proof.** Lemma A.1 says that

$$\sum_{j=i}^{i+r} P_{\Omega_j}^\perp \geq \left( \bigotimes_{k=i}^{i+r} P_{\Omega_k} \right)^\perp. \quad (A.12)$$

By summing over $i$, from $i = l$ up to $L$, the l-h-s of (A.12), for each $j$ we get no more than $r + 1$ terms of the type $P_{\Omega_j}^\perp$ and the inequality in (A.11) follows. \hfill \Box
Lemma A.3. Assume that $t > 0$ is sufficiently small, $\|V_{l,i}^{(k,q-1)}\| \leq \frac{8}{(r+1)^2} t^{r-1}$, and $\Delta_{l,k,q} \geq \frac{1}{2}$. Then, for arbitrary $N$, $k \geq 1$, and $q \geq 2$, the inequalities

$$\|V_{l,k,q}^{(k,q)}\| \leq 2\|V_{l,k,q}^{(k,q-1)}\|$$ \quad \text{(A.13)}

$$\|S_{l,k,q}\| \leq C \cdot t \cdot \|V_{l,k,q}^{(k,q-1)}\|$$ \quad \text{(A.14)}

hold true for a universal constant $C$. For $q = 1$, $V_{l,k,q}^{(k,q-1)}$ is replaced by $V_{l,k,1}^{(k-1,N-k+1)}$ on the right side of (A.13) and (A.14).

Proof. In the following we assume $q \geq 2$; if $q = 1$ an analogous proof holds. We recall that

$$V_{l,k,q}^{(k,q)} := \sum_{j=1}^{\infty} t^{j-1} (V_{l,k,q}^{(k,q-1)})^j_{\text{diag}}$$ \quad \text{(A.15)}

and

$$S_{l,k,q} := \sum_{j=1}^{\infty} t^j (S_{l,k,q})_j$$ \quad \text{(A.16)}

with

$$(V_{l,k,q}^{(k,q-1)})_1 := V_{l,k,q}^{(k,q-1)}$$

and, for $j \geq 2$,

$$(V_{l,k,q}^{(k,q-1)})_j := \sum_{p\geq 2, r_1 \geq 1, \ldots, r_p \geq 1 \mid r_1 + \cdots + r_p = j} \frac{1}{p!} \text{ad} (S_{l,k,q})_{r_1} \left( \text{ad} (S_{l,k,q})_{r_2} \ldots (\text{ad} (S_{l,k,q})_{r_p} (G_{l,k,q})) \ldots \right) \quad \text{(A.17)}$$

$$+ \sum_{p\geq 1, r_1 \geq 1, \ldots, r_p \geq 1 \mid r_1 + \cdots + r_p = j-1} \frac{1}{p!} \text{ad} (S_{l,k,q})_{r_1} \left( \text{ad} (S_{l,k,q})_{r_2} \ldots (\text{ad} (S_{l,k,q})_{r_p} (V_{l,k,q}^{(k,q-1)})) \ldots \right). \quad \text{(A.18)}$$

and, for $j \geq 1$,

$$(S_{l,k,q})_j := \text{ad}^{-1} G_{l,k,q} ((V_{l,k,q}^{(k,q-1)})_{j\text{ad}}) = \frac{1}{G_{l,k,q} - E_{l,k,q}} P_{l,k,q}^{(+) \cdot} (V_{l,k,q}^{(k,q-1)})_j \cdot P_{l,k,q}^{(-)} - h.c. \quad \text{(A.19)}$$

and

$$(S_{l,k,q})_j := \text{ad}^{-1} G_{l,k,q} ((V_{l,k,q}^{(k,q-1)})_{j\text{ad}}) = \frac{1}{G_{l,k,q} - E_{l,k,q}} P_{l,k,q}^{(+) \cdot} (V_{l,k,q}^{(k,q-1)})_j \cdot P_{l,k,q}^{(-)} - h.c. \quad \text{(A.20)}$$

\[\text{ad}^{-1} \]
From the lines above we derive
\[
\text{ad} (S_{k,q})_{r_p} (G_{k,q}) = \text{ad} (S_{k,q})_{r_p} (G_{k,q} - E_{k,q}) = \left[ \frac{1}{G_{k,q} - E_{k,q}} P_{k,q}^{(+)} (V_{k,q}^{(k,q-1)})_{r_p} P_{k,q}^{(-)} (G_{k,q} - E_{k,q}) \right] + \text{h.c.} \tag{A.21}
\]
\[
= -P_{k,q}^{(+)} (V_{k,q}^{(k,q-1)})_{r_p} P_{k,q}^{(-)} (V_{k,q}^{(k,q-1)})_{r_p} P_{k,q}^{(+)} . \tag{A.22}
\]

We recall definition (A.20) and we observe that
\[
\| (S_{k,q})_j \| \leq 2 \| (V_{k,q}^{(k,q-1)})_j \| \leq 4 \| (V_{k,q}^{(k,q-1)})_j \| , \tag{A.23}
\]
where we use the inductive hypothesis \( \Delta_{k,q} \geq \frac{1}{2} \). (Recall that \( \Delta_{k,q} \) is the gap of \( G_{k,q} \).) Then formula (A.17) yields
\[
\| (V_{k,q}^{(k,q-1)})_j \| \leq \sum_{p=2}^{j} \frac{8p}{p!} \sum_{r_1 \geq 1, \ldots, r_p \geq 1: r_1 + \cdots + r_p = j} \| (V_{k,q}^{(k,q-1)})_{r_1} \| \| (V_{k,q}^{(k,q-1)})_{r_2} \| \cdots \| (V_{k,q}^{(k,q-1)})_{r_p} \|. \tag{A.24}
\]

From now on, we closely follow the proof of Theorem 3.2 in [DFFR]; that is, assuming \( \| V_{k,q}^{(k,q-1)} \| \neq 0 \), we recursively define numbers \( B_j, j \geq 1 \), by the equations
\[
B_1 := \| V_{k,q}^{(k,q-1)} \| = \| (V_{k,q}^{(k,q-1)})_1 \| , \tag{A.25}
\]
\[
B_j := \frac{1}{a} \sum_{k=1}^{j-1} B_{j-k} B_k , \quad j \geq 2 , \tag{A.26}
\]

with \( a > 0 \) satisfying the relation
\[
\frac{e^{8a} - 8a - 1}{a} + e^{8a} - 1 = 1 . \tag{A.27}
\]

Using (A.25), (A.26), (A.24), and an induction, it is not difficult to prove that (see Theorem 3.2 in [DFFR]) for \( j \geq 2 \)
\[
\| (V_{k,q}^{(k,q-1)})_j \| \leq B_j \left( \frac{e^{8a} - 8a - 1}{a} \right) + 2 \| V_{k,q}^{(k,q-1)} \| B_{j-1} \left( \frac{e^{8a} - 1}{a} \right) . \tag{A.28}
\]

From (A.25) and (A.26) it also follows that
\[
B_j \geq \frac{2 B_{j-1} \| V_{k,q}^{(k,q-1)} \|}{a} \quad \Rightarrow \quad B_{j-1} \leq a \frac{B_j}{2 \| V_{k,q}^{(k,q-1)} \|} , \tag{A.29}
\]
which, when combined with (A.28) and (A.27), yields

$$B_j \geq \| (V_{k,q}^{(k,q-1)})_j \|. \quad (A.30)$$

The numbers $B_j$ are the Taylor’s coefficients of the function

$$f(x) := \frac{a}{2} \left( 1 - \sqrt{1 - \left( \frac{4}{a} \cdot \| V_{k,q}^{(k,q-1)} \| \right) x} \right), \quad (A.31)$$

(see [DFFR]). Therefore the radius of analyticity, $t_0$, of

$$\sum_{j=1}^{\infty} t^{j-1} \| (V_{k,q}^{(k,q-1)})_j \| \text{diag} = \frac{d}{dt} \left( \sum_{j=1}^{\infty} \frac{t^j}{j} \| (V_{k,q}^{(k,q-1)})_j \| \text{diag} \right) \quad (A.32)$$

is bounded below by the radius of analyticity of $\sum_{j=1}^{\infty} x^j B_j$, i.e.,

$$t_0 \geq \frac{a}{4 \| V_{k,q}^{(k,q-1)} \|} \geq \frac{a}{8} \quad (A.33)$$

where we have used the assumption that $\| V_{k,q}^{(k,q-1)} \| \leq \frac{8}{(r+1)^2} t^{r-1}$ and $(0 <) t < 1$.

Thanks to the inequality in (A.23) the same bound holds true for the radius of convergence of the series $S_{k,q} := \sum_{j=1}^{\infty} t^j (S_{k,q})_j$. For $0 < t < 1$ and in the interval $(0, \frac{a}{16})$, by using (A.25) and (A.30) we can estimate

$$\sum_{j=1}^{\infty} t^{j-1} \| (V_{k,q}^{(k,q-1)})_j \| \text{diag} \leq \frac{1}{t} \sum_{j=1}^{\infty} t^j B_j \quad (A.34)$$

$$= \frac{1}{t} \cdot \frac{a}{2} \left( 1 - \sqrt{1 - \left( \frac{4}{a} \cdot \| V_{k,q}^{(k,q-1)} \| \right) t} \right) \quad (A.35)$$

$$\leq (1 + C_a \cdot t) \| V_{k,q}^{(k,q-1)} \| \quad (A.36)$$

for some $a$-dependent constant $C_a > 0$. Hence the inequality in (A.13) holds true, provided that $t > 0$ is sufficiently small but independent of $N, k, q$. In a similar way we derive (A.14). \qed

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