Singularity of dynamical maps

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For a dynamical map $\Lambda(t,0)$, which sends a state $\rho(0)$ of quantum open system to a state $\rho(t) = \Lambda(t,0)\rho(0)$, the decomposition law $\Lambda(t,0) = \Lambda(t,t_c)\Lambda(t_c,0)$ may break down at a specific time $t_c$. In this paper, we present a method to find the singular points $t_c$ and propose a measure for the singularity of the dynamical map. Two examples are portrayed to illustrate the method, the measure of singularity for these singular points is calculated and discussed. An extension to high-dimensional system is presented.

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I. INTRODUCTION

The actual dynamics of any real open quantum system is expected to deviate to some extent from the Markovian evolution. This deviation can be measured by non-Markovianity and it has attracted much attention in recent years, leading to a deeper understanding of quite a few issues in the theory of open quantum system\textsuperscript{1,8}. Non-Markovian systems can be found in many branches of physics, including quantum optics\textsuperscript{3,10}, solid state physics\textsuperscript{11}, quantum chemistry\textsuperscript{12}, and quantum information processing\textsuperscript{13}. Since non-Markovian dynamics modifies monotonic decay of quantum coherence, it may protect quantum entanglement in composite systems for longer time than standard Markovian evolution\textsuperscript{14}. In particular it may protect the system against the sudden death of entanglement\textsuperscript{15}. Therefore, it is interesting to quantify the non-Markovianity within the description of quantum open system.

There are two approaches to quantify the measure of the degree of non-Markovianity. One approach is based on the idea of the composition law which is essentially equivalent to the idea of divisibility\textsuperscript{16}. This approach was used recently in Ref.\textsuperscript{17,18} to construct the measure of non-Markovianity, quantifying actually the deviation of the dynamical map from divisibility. Another approach is as in Ref.\textsuperscript{19}, where the authors define non-Markovianity dynamics as the information flow from the environment back into the system, the measure manifests itself as an increase in the distinguishability of pairs of evolving quantum states, and the information is identified to be the Fisher information\textsuperscript{20}.

The measure for non-Markovianity proposed in Ref.\textsuperscript{17} is based on the completely positive divisibility of a dynamical map: a trace preserving completely positive map $\Lambda(t,0)$ is completely positive divisible (CP-divisibility) if it can be written as,

$$\Lambda(t_2,0) = \Lambda(t_2,t_1)\Lambda(t_1,0),$$

and $\Lambda(t_2,t_1)$ is completely positive for any $t_2$ and $t_1$ ($t_2 > t_1 > 0$). By contrast, we say that the map $\Lambda(t_2,0)$ is positively divisible (P-divisibility) if $\Lambda(t_2,t_1)$ sends states into states but it is only positive, and that $\Lambda(t_2,0)$ is indivisible if neither P-divisibility nor CP-divisibility holds.

In this paper, we shall consider the other situation where,

$$\Lambda(t_2,0) \neq \Lambda(t_2,t_c)\Lambda(t_c,0),$$

at a special time $t_c$, $t_2 < t_c < 0$. We will refer to this instance of time $t_c$ as the singular point of the dynamical map $\Lambda(t_2,0), t_2 \in (0,\infty)$. Taking a qubit (two-level system) as an example, a method to find the singular point is presented, a measure to quantify this singularity is proposed and discussed.

This paper is organized as follows. In Sec.II, we present a general formalism for a qubit dynamics, exhibiting the method to find the singular point $t_c$. A measure to quantify the singularity is constructed. Two examples, one describes a qubit coupled to a harmonic oscillator bath and the other includes a qubit coupled to a finite spin bath, are given to illustrate the critical point in Sec.III, the measure of singularity is also calculated and discussed in this section. A generalization of the representation to $d$-dimensional open systems is presented in Sec. IV. Finally, we conclude our results in Sec. V.

II. GENERAL FORMALISM FOR A QUBIT DYNAMICAL MAP

Consider a dynamical map $\Lambda(t,0)$ for a qubit (or two-level system), which sends an arbitrary initial state $\rho(0) = (1 + \vec{n}(0) \cdot \vec{\sigma})/2$ with Bloch vector $\vec{n}(0) = (n_x(0), n_y(0), n_z(0))$ into a state $\rho(t)$,

$$\rho(t) = \Lambda(t,0)\rho(0) = \frac{1}{2}(1 + \vec{n}(t) \cdot \vec{\sigma}).$$
Without loss of generality, the Bloch vector \( \vec{n}(t) = (n_x(t), n_y(t), n_z(t)) \) can be written as,

\[
\vec{n}(t) = \vec{n}(0) \cdot D(t) + \vec{f}(t),
\]

where \( D(t) \) is a 3 x 3 matrix and \( \vec{f}(t) \) is a time-dependent vector.

Now we elicit the condition for \( \Lambda(t_2, 0) \neq \Lambda(t_2, t_c) \Lambda(t_c, 0) \). To this aim we introduce an ancilla \( A \) and define,

\[
M_{SA} \equiv \Lambda(t_2, t_c) \otimes I_A(\Phi_{SA} | \Phi_{SA}),
\]

where \( |\Phi_{SA}\rangle = |0\rangle_S \otimes |0\rangle_A + |1\rangle_S \otimes |1\rangle_A \) is an unnormalized maximally entangled state of the qubit and ancilla, and \( I_A \) denotes the identity operator of the ancilla. Since the ancilla is also a qubit, \( M_{SA} \) can be written as,

\[
M_{SA} = \frac{1}{2}(x + \vec{r} \cdot \vec{\sigma}_A + \vec{s} \cdot \vec{\sigma}_S + \vec{\sigma}_S \cdot V \cdot \vec{\sigma}_A)
\]

\[
= \frac{1}{2}(I, \vec{\sigma}_S) F \left( \frac{1}{\vec{\sigma}_A} \right).
\]

Here \( x \) is a constant, \( \vec{r} \) and \( \vec{s} \) are vectors, \( I \) is an identity matrix, \( F = \left( \begin{array}{cc} x & \vec{r} \\ \vec{s} & V \end{array} \right) \) and \( V \) is a 3 x 3 matrix, which is determined by the map \( \Lambda(t_2, t_c) \) and will be derived in the following. If \( \Lambda(t_2, 0) = \Lambda(t_2, t_c) \Lambda(t_c, 0) \), the map \( \Lambda(t_2, t_c) \) would send the state \( \rho(t_c) = \Lambda(t_c, 0) \rho(0) \) to state \( \rho(t_2) \). In terms of \( M_{SA} \), this can be expressed as,

\[
\rho(t_2) = \Lambda(t_2, t_c) \rho(t_c) = \text{Tr}_A[M_{SA} I_S \otimes \rho_A(t_c)]
\]

\[
= \frac{1}{2}(I, \vec{\sigma}_S) F \left( \frac{1}{\vec{n}(t_c)} \right).
\]

Writing \( \rho(t_2) = \frac{1}{2}(I, \vec{\sigma}_S) \left( \frac{1}{\vec{n}(t_2)} \right) \), we obtain from Eq. (7),

\[
\left( \begin{array}{c} 1 \\ \vec{n}(t_2) \end{array} \right) = F \left( \begin{array}{c} 1 \\ \vec{n}(t_c) \end{array} \right) = \left( \begin{array}{c} x + \vec{r} \cdot \vec{n}(t_c) \\ \vec{s} + \vec{n}(t_c) \cdot V \end{array} \right).
\]

It is easy to find that,

\[
x = 1, \quad \vec{r} = 0, \quad \vec{n}(t_2) = \vec{n}(t_c) \cdot V + \vec{s}.
\]

Considering that \( \rho(0) \) is an arbitrary initial state, namely \( \vec{n}(0) \) is arbitrary, Eqs. (4-5) together yield,

\[
D(t_2) = D(t_c) \cdot V,
\]

\[
\vec{f}(t_2) = \vec{f}(t_c) \cdot V + \vec{s}.
\]

The condition for \( \Lambda(t_2, 0) \neq \Lambda(t_2, t_c) \Lambda(t_c, 0) \) now is equivalent to that there does not exist a matrix \( V \) to satisfy Eq. (10). If the determinant of \( D(t_c) \) is non-zero, we have

\[
V = D^{-1}(t_c) \cdot D(t_2),
\]

and

\[
\vec{s} = \vec{s}(t_2, t_c) = \vec{f}(t_2) - \vec{f}(t_c) \cdot V.
\]

From the above derivations, we find that once \( \vec{r}, \vec{s}, V \) are (uniquely or non-uniquely) established for any \( t_c \) \( t_2 > t_c > 0 \), the decomposition \( \Lambda(t_2, 0) = \Lambda(t_2, t_c) \Lambda(t_c, 0) \) holds true, namely there are no singular points in the time interval \([0, t_2] \). We notice that the null-determinant of \( D(t_c) \) plays a key role in finding \( V \), it can thus be taken as a condition to find the singular point \( t_c \) if the matrix \( D(t_2) \) is of full rank. Mathematically, the necessary and sufficient condition for Eq. (10) to have no solution is that the rank of \( D(t_c) \) must be smaller than the rank of \([D(t_c)]D(t_2)], where [D(t_c)]D(t_2) \) is an augmented matrix obtained by attaching the columns of \( D(t_2) \) to the columns of \( D(t_c) \).

To quantify the singularity of the singular point, we introduce the trace distance, \( D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} |\rho_1 - \rho_2| \), which is an appropriate measure for the distinguishability between two quantum states \( \rho_1 \) and \( \rho_2 \). Here |\( A \rangle = \sqrt{AA^\dagger} \). We define the singularity measure of a dynamical map \( \Lambda(t_2, 0) \) at time \( t_c \) by

\[
S_\Lambda(t_c) = \max_{\rho(0)} D(\rho(T), \rho_{t_c}(T)),
\]

where \( \rho(T) \equiv \Lambda(T, 0) \rho(0) \) and \( \rho_{t_c}(T) \equiv \Lambda(t_c, 0) \rho(0) \) is the solution of the dynamical process taking \( \rho(t_c) = \Lambda(t_c, 0) \rho(0) \) as initial state. The maximum is taken over all initial states and the final time \( T \).

### III. EXAMPLES

In this section, we will present two examples to illustrate the singular point and the measure of singularity.

The first example is a dephasing model that consists of a spin-1/2 particle coupling to a spin-bath. The coupling Hamiltonian commutes with the free Hamiltonian of the central spin, thus the central spin conserves its energy. In the second example, we consider a dissipative system, the energy of the system is no longer conserved.

#### A. A two-level system coupling to a finite spin bath

Consider a central spin-1/2 coupling to a bath of \( N \) spin-1/2 particles. The interaction Hamiltonian is,

\[
H = \sum_{k=1}^{N} A_k \sigma_z \sigma_z^k,
\]

where \( A_k = A/\sqrt{N} \) represents the coupling constants. Assume the initial state of the whole system is \( \rho_0(t_0) \otimes (\frac{1}{N+1} I) \), i.e., all spins in the reservoir are in a maximal mixed state. The density matrix of the central spin at
time $t$ takes,
\[ \rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix} = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix} e^{-\frac{1}{2} [\sigma_+ \sigma_-]} \cos \left( \frac{2 \Lambda t}{\sqrt{N}} \right) \rho. \]  
(15)

In terms of dynamical map, the dynamics can be represented as, $\Lambda(t, 0) \rho = \frac{1}{2} (1 - \cos N \left( \frac{2 \Lambda t}{\sqrt{N}} \right)) \sigma_+ \rho \sigma_- + \frac{1}{2} (1 + \cos N \left( \frac{2 \Lambda t}{\sqrt{N}} \right)) \rho$. This is equivalent to the following master equation,
\[ \dot{\rho} = \gamma(t) \mathcal{L}(\rho), \]  
(16)

where $\mathcal{L}(\rho) = \sigma_+ \rho \sigma_- - \rho$, and the time-dependent decay rate is $\gamma(t) = A \sqrt{N} \tan \left( \frac{2 \Lambda t}{\sqrt{N}} \right)$. This model is discussed in several papers as a typical example to quantify non-Markovianity.

Writing $\rho(t)$ in Eq. (15) in the form of Eq. (4), we find
\[ D(t) = \begin{pmatrix} C(t) & 0 & 0 \\ 0 & C(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
\[ \tilde{f}(t) = 0, \]  
(17)

where $C(t) = \cos N \left( \frac{2 \Lambda t}{\sqrt{N}} \right)$. The singular point $t_c$ can be found by solving $C(t_c) = 0$, it yields,
\[ t_c = \frac{\sqrt{N}}{4A} (2n + 1) \pi, \quad n = 0, 1, 2, \ldots \]  
(18)

By the definition of the measure of singularity, we obtain,
\[ S_\Lambda(t_c) = \max_{|n| \leq T} |C(T)||\rho_{12}(0)|; \]  
(19)

where $T$ is a time, $T > t_c$, and $\rho_{12}(0)$ denotes the element of the initial density matrix $\rho(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix}$.

After a simple algebra, we have $S_\Lambda(t_c) = \frac{1}{\Lambda}$ for any singular point given in Eq. (18). It is interesting that the singularity measure of these singular points are equal. Indeed, examining the dynamical map $\Lambda(t, 0)$, we find that the features of $\Lambda(t, 0)$ around any $t_c$ are the same.

\section{The damping J-C model}

This example consists of a two-level system coupling to a reservoir at zero temperature. The reservoir consists of infinite number of harmonic oscillators that is also referred in the literature as the spin-boson model. The Hamiltonian for such a system reads,

\[ H = H_0 + H_I, \]  
(20)

where $H_0 = \sum_{k} \hbar \omega_k b_k^\dagger b_k$, $H_I = \sigma_+ B + \sigma_- B^\dagger$, and $B = \sum_{k} g_k b_k$. The Rabi frequency of the two-level system and the frequency for the $k$-th harmonic oscillator are denoted by $\omega_0$ and $\omega_k$, respectively. $b_k^\dagger$ and $b_k$ are the creation and annihilation operators of $k$-th oscillator, which couples to the system with coupling constant $g_k$.

This model is exactly solvable. Assuming the system and the reservoir initially uncorrelated, we can obtain a time-dependent master equation in the interaction picture,
\[ \dot{\rho} = -i \frac{\epsilon(t)}{2} [\sigma_+ \sigma_-, \rho] + \gamma(t) (\sigma^- \rho \sigma^+ - \frac{1}{2} \sigma^+ \sigma^- \rho - \frac{1}{2} \rho \sigma^+ \sigma^-), \]  
(21)

where $\epsilon(t) = -2 \Im \left( \frac{\epsilon(t)}{2} \right)$ and $\gamma(t) = -2 \Re \left( \frac{\epsilon(t)}{2} \right)$, $\epsilon(t)$ plays the role of Lamb shift and $\gamma(t)$ is the decay rate. Both $\epsilon(t)$ and $\gamma(t)$ are time-dependent. $\epsilon(t)$ is determined by $\dot{\epsilon}(t) = -\int_0^t f(t - \tau) e c(\tau) d\tau$, where $f(t - \tau) = \int d\omega J(\omega) e^{i(\omega_0 - \omega)(t - \tau)}$ is the environmental correlation function. In the derivation of the master equation, the reservoir is assumed in its vacuum at $t = 0$.

Consider the following spectral density, $J(\omega) = \frac{1}{\pi} \frac{\gamma_0^2 \lambda^2}{(\omega - \omega_0)^2 + \lambda^2}$, where $\gamma_0$ represents the coupling constant between the system and reservoir, $\lambda$ defines the spectral width of the coupling at the resonance point $\omega_0$. For the spectral density $J(\omega)$, we have $\epsilon(t) = 0$, $\epsilon(t) = e^{\omega_i^2 / 2} = \frac{1}{\sqrt{2\pi}} e^{\omega_0^2 / 4}$. Moreover, we have
\[ \gamma(t) = \frac{2 \gamma_0^2 \lambda \sinh(dt/2)}{d \cosh(dt/2) + \lambda \sinh(dt/2)}. \]  
(22)

with $d = \sqrt{\lambda^2 - 2 \gamma_0^2 \lambda}$ in Eq. (21). Assume the system initially in $\rho(0) = \begin{pmatrix} \rho_{ec}(0) & \rho_{eg}(0) \\ \rho_{ge}(0) & \rho_{gg}(0) \end{pmatrix}$, by the effective Hamiltonian approach [21], we have the density matrix at time $t$, $\rho(t) = \begin{pmatrix} \rho_{ec}(t) & \rho_{eg}(t) \\ \rho_{ge}(t) & \rho_{gg}(t) \end{pmatrix}$, where
\[ \rho_{ec}(t) = \rho_{ec}(0) e^{-i \int_0^t \gamma(t') dt'}, \]
\[ \rho_{gg}(t) = 1 - \rho_{ec}(t), \]
\[ \rho_{eg}(t) = \rho_{ge}^*(t) = \rho_{ge}(t) = e^{-\frac{i}{2} \int_0^t \gamma(t') dt'} \rho_{gg}(0). \]  
(23)

It is easy to show that the matrix $D(t)$ and $\tilde{f}(t)$ in this example are
\[ D(t) = \begin{pmatrix} D_{11}(t) & 0 & 0 \\ 0 & D_{22}(t) & 0 \\ 0 & 0 & D_{33}(t) \end{pmatrix} \]  
(24)

and
\[ \tilde{f}(t) = (f_x, f_y, f_z) = (0, 0, (e^{-i \int_0^t \gamma(t') dt'} - 1)), \]  
(25)

where
\[ D_{11} = D_{22} = e^{-\frac{i}{2} \int_0^t \gamma(t') dt'}, \]
\[ D_{33} = D_{11}^*, \]  
(26)
We find from Eq. (24) that $D_{ij}(t_c) = 0$, $j = 1, 2, 3$, gives the singular points. $D_{ij}(t_c) = 0$ can happen only when $\gamma_0/\lambda > 1/2$. Noticing that $D_{11}(t) = e^{-\lambda t/2}[\cos(d_0 t/2) + \frac{1}{d_0} \sin(d_0 t/2)]$ for $\gamma_0/\lambda > 1/2$, where $d_0 = \sqrt{|\lambda^2 - 2\gamma_0 \lambda|}$, we obtain the $n$th singular point $t_c^{(n)} = \frac{1}{d_0} \left( \cos^{-1} \left( \frac{1}{\sqrt{1 - 2\gamma_0 \lambda / \lambda^n}} \right) + n\pi \right)$, $n = 0, 1, 2, \ldots$. At these singular points, the singularity measure can be given by maximizing the distance $D(\rho(T), \rho_c(T)) = \frac{1}{2} \sqrt{D_{33}(T) (n_1^2(0) + n_2^2(0)) + D_{33}(T) (1 + n_3(0))^2}$ over $T$ and the Bloch vector $n(t) = (n_1(0), n_2(0), n_3(0))$ with constraint $0 \leq n_1^2(0) + n_2^2(0) + n_3^2(0) \leq 1$. Simple algebra shows that the maximum of the $n$th singular point arrives at $T = T_c^{(n)} = \frac{2(n+1)^2}{\lambda}, n = 0, 1, 2, \ldots$, and $n_3(0) = \frac{D_{33}(T_c^{(n)})}{1 - D_{33}(T_c^{(n)})}, n_1^2(0) + n_2^2(0) = 1 - n_3^2(0)$. The measure of singularity for the $n$th singular point $t_c^{(n)}$ is then $S_\Lambda(t_c^{(n)}) = \frac{1}{2} \left[ \frac{e^{-\lambda t_c^{(n)}}}{1 - e^{-\lambda t_c^{(n)}}} \right]$, for $0 \leq D_{33}(T_c^{(n)}) < 0.5$.

IV. EXTENSION TO $d$-DIMENSIONAL SYSTEMS

We consider now an arbitrary dynamical map $\Lambda(t, 0)$ with $t > 0$ for a quantum $d$-dimensional system (qudit). Let $\{\lambda_{\mu}\}_{\mu=1}^d$ with $d = 2^d - 1$ be a set of traceless qudit observable satisfying $\text{Tr}(\lambda_{\mu}) = \delta_{\mu \nu}$. Together with the identity operator they form an orthonormal basis for all the qudit operators. Thus we have expansions $\varrho = \langle I + n \cdot \vec{\lambda} \rangle/\lambda$ for the initial state and $\Lambda(t, 0) \varrho = \langle I + n \cdot \vec{\lambda} \rangle/\lambda$ with $n(t) = D(t) \cdot n + \epsilon(t)$ for the final state $\varrho(t) = \Lambda(t, 0) \varrho$. Here $n(t) = \text{Tr}(\rho \vec{\lambda})$, $i(t) = \text{Tr}(\rho \vec{\lambda})$, and $\epsilon(t) = \text{Tr}(\langle I(t) \rangle \vec{\lambda})$ are real vectors and $D(t)$ is an $n \times n$ real matrix with matrix elements given by $\langle [D(t)]_{\mu \nu} \rangle = \text{Tr}(\langle I(t) \rangle \lambda_{\mu} \lambda_{\nu})$. The linear trace-preserving map $\Lambda(t, 0)$ is determined uniquely by $D(t)$ and $\epsilon(t)$ and vice versa. For later use we denote by $V(t) = \{\tilde{a} \in R^d | D(t) \cdot \tilde{a} = 0\}$ the null space of $D(t)$.

Let $t > t_c > 0$ and consider the possible decomposition of a dynamical map $\Lambda(t, 0) = \Lambda(t, t_c) \Lambda(t_c, 0)$ for some linear trace-preserving map $\Lambda(t_c, 0)$. Any linear qudit map, e.g., $\Lambda(t, t_c)$, is in a one-to-one correspondence with a 2-qudit operator, e.g., $\Lambda(t_c, t_c) \otimes \mathbb{I}(\Phi)$ with $\Phi$ being the projector of the subnormalized 2-qudit state $|\Phi\rangle = \sum_n |n, n\rangle$. We denote by $S$ the $n \times n$ matrix with elements $\langle [S]_{\mu \nu} \rangle = \text{Tr}(R \lambda_{\mu} \lambda_{\nu} \vec{\lambda}^T )/d$ for $\mu, \nu = 1, 2, \ldots, n = d^2 - 1$ and $\vec{\lambda} = \text{Tr}(R \lambda \otimes I)/d$. For a trace-preserving map it holds $\text{Tr}(R \vec{\lambda}^T ) = 0$ and therefore $R$ and $\vec{\lambda}$ determine uniquely $R$ and consequently the linear trace-preserving map $\Lambda(t, t_c)$. By definition the linear map $\Lambda(t, t_c)$ is a possible decomposition if and only if for an arbitrary initial state $\varrho$ it holds $\varrho(t) = \Lambda(t, 0) \varrho = \Lambda(t, t_c) \Lambda(t_c, 0) \varrho = \Lambda(t, t_c) \varrho(t_c)$. 

Lemma. If a qudit operator $O$ satisfies $\langle \psi | O | \psi \rangle = 0$ for an arbitrary pure qudit state $|\psi\rangle$ then $O = 0$.

Proof. Let $V_{12} = \sum_{ij} |i, j\rangle \langle j, i|$ be the swapping operator of 2 qudits and $I_{12}$ be the identity operator. From
the following identity
\[ W_{12} := \int d\psi |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| = \frac{I_{12} + V_{12}}{d(d+1)} \]
(30)
it follows that \( O = d(d+1) Tr_1\left( (O_1 \otimes I_2) W_{12} \right) = 0 \).

**Theorem.** Given a qudit channel \( \Lambda(t,0) \) with \( t > t_c > 0 \) there exists a linear trace-preserving map \( \Lambda(t,t_c) \) such that \( \Lambda(t,0) = \Lambda(t,t_c) \Lambda(t_c,0) \) if and only if \( V(t_c) \subseteq V(t) \). Moreover the decomposition \( \Lambda(t,t_c) \) is unique if and only if det \( D(t_c) \neq 0 \).

**Proof.** Necessity (only if part), i.e., \( \Lambda(t,0) = \Lambda(t,t_c) \Lambda(t_c,0) \) infers \( V(t_c) \subseteq V(t) \). From Eq.\(^{29} \) it follows that for arbitrary \( \tilde{g} \) it holds \( Tr(\tilde{X}\tilde{g}(t)) = Tr(\tilde{X}\tilde{g}(t_c)) \). Taking into account \( \tilde{n}(t) = Tr(\tilde{X}\tilde{g}(t)) = (D(t)\cdot \tilde{n} + \tilde{c}(t)) \) for \( t \) and \( t_c \) we see that \( (D(t) - S \cdot D(t_c)) \cdot \tilde{n} = S \cdot \tilde{c}(t_c) + \tilde{r} - \tilde{c}(t) \) must hold for arbitrary \( \tilde{n} = Tr(\tilde{g} \tilde{\lambda}) \) with \( \tilde{g} \) being a density matrix. If we let \( \tilde{n} = 0 \) with corresponding state being \( g = I/d \) then we obtain \( \tilde{r} = \tilde{c}(t) - S \cdot \tilde{c}(t_c) \). As a result \( \tilde{r} = 0 \), \( \tilde{c}(t) = 0 \), \( S \cdot \tilde{c}(t_c) = 0 \), for arbitrary \( \tilde{n} \) and density state \( \tilde{g} \) where \( \tilde{g} = \sum_{\mu} \tilde{\lambda}_\mu \tilde{\rho}_\mu \). From \( S \cdot \tilde{c}(t_c) = 0 \) we have \( V(t_c) \subseteq V(t) \) since \( D(t) \cdot \tilde{n} = 0 \) infers \( D(t) \cdot \tilde{n} = S \cdot D(t_c) \cdot \tilde{n} = 0 \).

Sufficiency (if part), i.e., \( V(t_c) \subseteq V(t) \) infers \( \Lambda(t,0) = \Lambda(t,t_c) \Lambda(t_c,0) \). Let \( \{ \tilde{c}_i \}_{i=1}^K \) span \( V(t_c) \) where \( K = \dim V(t_c) \) and \( \{ \tilde{e}_i \}_{i=1}^K \) span the orthogonal complement \( V(t_c) \). As a result \( D(t_c) \cdot \tilde{c}_i = 0 \) and thus \( D(t) \cdot \tilde{c}_i = 0 \) for \( i = 1, \ldots, K \). Since we have assumed \( V(t_c) \subseteq V(t) \). The equation \( D(t) = S \cdot D(t_c) \) is equivalent to \( D(t_c) \cdot \tilde{c}_i = S \cdot D(t_c) \cdot \tilde{c}_i \) for \( i = 1, \ldots, n \). Since \( D(t_c) \cdot \tilde{c}_i = D(t) \cdot \tilde{c}_i = 0 \) for \( i = 1, \ldots, K \) the equation becomes \( D'(t) = S \cdot D'(t_c) \) with \( D'(t_c) = [D(t_c)\tilde{c}_K, D(t_c)\tilde{c}_{K+1}, \ldots, D(t_c)\tilde{c}_n] \) and \( D'(t) = [D(t)\tilde{c}_K, D(t)\tilde{c}_{K+1}, \ldots, D(t)\tilde{c}_n] \) being of dimension \( n \times (n-K) \). Since the rank of \( D(t_c) \) is \( n-K \) there are exactly \( n-K \) linearly independent row vectors of \( D'(t_c) \). Therefore it is always possible to expand each row vector of \( D'(t) \), an \((n-K)\)-dimensional vector, by those \( n \) row vectors of \( D'(t_c) \), i.e., for any given \( i \) there exist real numbers \( S_{ik} \) such that
\[
[D'(t)]_{i1}, [D'(t)]_{i2}, \ldots, [D'(t)]_{i(n-K)} = \sum_{k=1}^n S_{ik} ([D'(t)]_{k1}, [D'(t)]_{k2}, \ldots, [D'(t)]_{k,n-K}).
\]
The \( n \times n \) matrix \( S \) formed by the coefficients \( S_{ik} \) with \( i,k = 1,2,\ldots,n \) satisfies \( D(t) = S \cdot D(t_c) \) and together with \( \tilde{r} = \tilde{c}(t) - S \cdot \tilde{c}(t_c) \) determines \( R \) and thus \( \Lambda(t,t_c) \), i.e., \( \Lambda(t,t_c) \Lambda(t_c,0) \). Moreover if and only if \( K = 0 \), i.e., \( V(t_c) \) is empty, i.e., \( det(D(t_c)) \neq 0 \), and \( S \) as well as \( \tilde{r} = \tilde{c}(t) - S \cdot \tilde{c}(t_c) \) is unique and therefore \( \Lambda(t,t_c) \) is unique.

We note that the condition \( V(t_c) \subseteq V(t) \) is equivalent to \( \text{Rank}[D(t_c)|D(t)] = \text{Rank}[D(t_c)] \), where \( [D(t_c)|D(t)] \) is an augmented matrix. The decomposition \( \Lambda(t,0) = \Lambda(t,t_c) \Lambda(t_c,0) \) does not exist if and only if \( \text{Rank}[D(t_c)|D(t)] > \text{Rank}[D(t_c)] \).

**V. CONCLUSION**

In summary, we have explored the singular point \( t_c \) where the dynamical map \( \Lambda(t,0) \neq \Lambda(t,t_c) \Lambda(t_c,0) \), i.e., the dynamical map is indistinguishable at the instance of time \( t_c \). We quantify the singularity of the singular point \( t_c \) and present examples to show the singularity. Until now these points were not aware in the divisibility-based measure of non-Markovianity, hence it would contribute to the understanding of quantum non-Markovian process.

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