Geometric construction of elliptic integrable systems and $\mathcal{N} = 1^*$ superpotentials

S. Prem Kumar  
*Department of Physics*  
*University of Wales Swansea*  
*E-mail: s.p.kumar@swan.ac.uk*

Jan Troost  
*Center for Theoretical Physics*  
*MIT*  
*77 Mass Ave*  
*Cambridge, MA 02139 USA*  
*E-mail: troost@mit.edu*

**Abstract:** We show how the elliptic Calogero-Moser integrable systems arise from a symplectic quotient construction, generalising the construction for $A_{N-1}$ by Gorsky and Nekrasov to other algebras. This clarifies the role of (twisted) affine Kac-Moody algebras in elliptic Calogero-Moser systems and allows for a natural geometric construction of Lax operators for these systems. We elaborate on the connection of the associated Hamiltonians to superpotentials for $\mathcal{N} = 1^*$ deformations of $\mathcal{N} = 4$ supersymmetric gauge theory, and argue how non-perturbative physics generates the elliptic superpotentials. We also discuss the relevance of these systems and the associated quotient construction to open problems in string theory. In an appendix, we use the theory of orbit algebras to show the systematics behind the folding procedures for these integrable models.
1. Introduction

Integrable systems are an interesting subclass of physical theories to study, because we can explicitly calculate a lot of the physical properties of these theories, and because integrable systems find surprisingly rich applications in many other fields, like four-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetric field theory, and string theory.

The specific integrable systems we want to study are elliptic Calogero-Moser models. These form the top of a pyramid of integrable systems (see e.g. [1, 3]), since the (affine) Toda systems, the trigonometric and the rational models can be derived from them by taking appropriate limits in parameter and phase space. Diverse

$^1$Strictly speaking, the Calogero Moser systems appear as degenerations of Ruijsenaars-Schneider systems which can be interpreted in terms of a five dimensional theory (see e.g. [8] and references therein).
Lax operators with spectral parameter have been constructed for these theories using mostly algebraic methods \[4, 5, 6, 7, 8\], but a good geometrical understanding of the full set of elliptic models is lacking.

Our primary motivation for studying these models is from the point of view of deriving superpotentials for \(\mathcal{N} = 1\) supersymmetric gauge theories obtained as mass-deformations of finite \(\mathcal{N} = 2\) and \(\mathcal{N} = 4\) theories with general gauge groups. The superpotential for the \(\mathcal{N} = 1^*\) theory (the \(\mathcal{N} = 1\) preserving mass-deformation of \(\mathcal{N} = 4\)) for \(SU(N)\) gauge group was shown to coincide in \[9\] with the elliptic Calogero-Moser Hamiltonian associated to the \(A_{N-1}\) root system.

A direct derivation of this proposal was provided in \[10\] via a hyperkahler quotient construction following the ideas of \[11\] wherein the \(SU(N), \mathcal{N} = 2^*\) theory (the \(\mathcal{N} = 2\) preserving mass-deformation of \(\mathcal{N} = 4\)) was realised on the worldvolume of a Type IIA brane setup. This configuration was mapped by a series of dualities onto the Higgs branch of a “magnetic” mirror theory. Following this procedure in string theory for other gauge groups is an extremely attractive program, but one which runs into difficulties due to the lack of knowledge about the associated brane configurations. Nevertheless, one can try to gain insight into this approach by studying the geometry of the integrable systems associated to other root systems. Therefore, following \[12\], we want to point out a geometrical derivation of a certain form of the Lax operators for all elliptic integrable systems using a symplectic quotient construction. This also yields an alternative, geometrical proof of the integrability of the system.

In \[12\] this derivation of the integrable model and its Lax operator was done for the elliptic Calogero-Moser model based on the \(A_{N-1}\) root system. In sections \[3\] and \[4\] we extend their analysis. Our construction will naturally incorporate affine algebras. Specifically, this will help in interpreting the role of twisted affine algebras in twisted elliptic Calogero-Moser models. In section \[5\] we elaborate on the relevance of these integrable systems for supersymmetric field theory, particularly in connection with the superpotentials of mass deformed \(\mathcal{N} = 4\) theories. We discuss the various non-perturbative contributions to the superpotentials. We also speculate on the relevance of these models to brane configurations in string theory, in analogy to the important role for the (spin) elliptic integrable model corresponding to the \(A^{(1)}_{N-1}\) algebra \[13, 14, 15\] in this context. Finally, in appendix \[A\] we point out how the techniques developed in the analysis of fixed point theories in conformal field theory \[13\] explain the systematics behind folding procedures in integrable systems.

2. Current algebra on the torus

In this section we describe the symplectic quotient construction relevant for the elliptic integrable systems. For the elliptic Calogero-Moser model based on the root
system of the $A_{N-1}$ algebra, this was done in [12]. We indicate how to extend that analysis to include all elliptic integrable models.

It is not surprising that we have to start with a current algebra on the torus, since the complexified elliptic integrable model has a double periodicity. Consider then a torus $\Sigma_\tau$ with modular parameter $\tau$. We choose a holomorphic differential $\omega$ such that for the $\alpha$ and $\beta$ cycle of the torus we have:

\[ \int_\alpha \omega = 1 \]  

\[ \int_\beta \omega = \tau. \]  

We will refer to the $\alpha$-direction as direction $x^1$ and the $\beta$-direction as direction $x^2$ with periodicities $\omega_1$ and $\omega_2$ respectively.

As in [12] we consider the algebra $\bar{\mathfrak{g}}_{\Sigma_\tau}$ of maps $\phi$ from an elliptic curve $\Sigma_\tau$ to a complexified simple Lie algebra $\bar{\mathfrak{g}}$ [15]. In order to be able to deal with twisted elliptic integrable models as well as ordinary elliptic integrable models, we need to distinguish two cases. In the first case, the map $\phi$ is periodic in both directions of the torus:

\[ \phi(x^1 + \omega_1, x^2) = \phi(x^1, x^2) \]

\[ \phi(x^1, x^2 + \omega_2) = \phi(x^1, x^2). \]  

(2.3)

In the second case, the map $\phi$ is twisted around the $\beta$-cycle of the torus. Suppose $T_a$ forms a basis of $\bar{\mathfrak{g}}$ and $\sigma$ is an outer automorphism of the Lie algebra $\bar{\mathfrak{g}}$ of order $l$ (namely $\sigma^l = 1$). Then the map $\phi$ satisfies the twisted boundary conditions:

\[ \phi(x^1 + \omega_1, x^2) = \phi(x^1, x^2) \]

\[ \phi^\sigma(x^1, x^2 + \omega_2)T_a = \phi^\sigma(x^1, x^2)\sigma(T_a). \]  

(2.4)

A central extension [15] of the algebra of maps is defined by our holomorphic differential $\omega \in H^{(1,0)}(\Sigma_\tau)$ and an $H^*(1,0)(\Sigma_\tau)$ valued two-cocycle

\[ c(X,Y) = \int_{\Sigma_\tau} \omega \wedge <X, dY> \]  

(2.5)

where $X, Y \in \bar{\mathfrak{g}}^{\Sigma_\tau}$ and the brackets $<,>$ indicate an invariant bilinear form on $\bar{\mathfrak{g}}$. The cotangent bundle of the extended algebra $\bar{\mathfrak{g}}^{\Sigma_\tau}$ consists of elements $(\phi, c, \bar{A}, \kappa)$ where $\phi : \Sigma_\tau \to \bar{\mathfrak{g}}$, $\bar{A} \in \Omega^{(0,1)} \otimes \bar{\mathfrak{g}}$ and $c, \kappa \in C$. We have a pairing

\[ <(\bar{A}, \kappa), (\phi, c)> = \kappa.c + \int_{\Sigma_\tau} \omega \wedge \text{tr}\phi\bar{A}. \]  

(2.6)

The current group acts naturally on the cotangent bundle as gauge transformations, and the action preserves the standard symplectic form on the cotangent bundle. The moment map for the action on the cotangent bundle is [15]:

\[ \mu_1 = \kappa \bar{\partial}\phi + [A_z, \phi], \]  

(2.7)
which naturally takes values in the dual of the simple Lie algebra \( \bar{g} \) we started out with – which we identify with \( \bar{g} \) via the Killing form.

### 3. Punctures

As in [12, 14, 16, 17], we can introduce punctures in the Riemann surface, and extend the moment map to include the action of the current group on vector spaces attached at points of the Riemann surface. We will be interested in cases where there is only one puncture in the Riemann surface.

Before we give the form of the additional piece in the moment map, we recall a few properties of (twisted) affine Kac-Moody algebras. In the case of twisted boundary conditions, the algebra \( \bar{g} \) naturally splits into subalgebras \( \bar{g}_j \) that consist of the eigenvectors of \( \sigma \) with eigenvalue \( e^{2\pi j/l} \), as is well-known from the theory of affine Kac-Moody algebras ([18, 19, 20, 21]) to which we refer for more details.²

Note that the \( \bar{g}_j \) are representation spaces for the \( \bar{g}_0 \). (Recall that \( l \) is the order of the outer automorphism \( \sigma \).)

Thus, we can split the field \( \phi \) according to the periodicity of its components, and their decomposition in terms of step and Cartan subalgebra (CSA) generators of \( \bar{g}_0 \) and weight spaces of the other \( \bar{g}_j \):

\[
\phi = \sum_{j=0}^{l-1} \left( \phi_j^r H_r^j + \phi_j^\lambda E_\lambda^j \right). \tag{3.1}
\]

Here, the index \( j \) denotes the periodicity of the component of \( \phi_j^r \) in the \( \beta \)-direction while the label \( r \) (implicitly summed over) indexes the CSA generators or, alternatively the multiplicity of the zero weight in the \( g_0 \) module spanned by \( \bar{g}_j \). Similarly, the upper label \( \lambda \) in the second term denotes roots or weights of \( \bar{g}_0 \), depending on whether \( j \) is zero or non-zero, respectively. In the following, we will concentrate on all affine algebras except \( A_{2k}^{(2)} \), for ease of notation only. In that case, the weight space for \( g_j \) is always the set of short roots \( \{\alpha_s\} \).

Next, we motivate an additional piece in the moment map. Firstly, we consider the following formal sums over non-CSA generators in the standard realisation of the current algebra. For an untwisted current algebra, we sum ³:

\[
\sum_{\alpha \in \Delta, n \in \mathbb{Z}} E^0_\alpha \otimes e^{inx} = \delta_1(x) \sum_\alpha E^0_\alpha \tag{3.2}
\]

²The affine Kac-Moody algebra is a subalgebra of the current algebra associated to the \( \beta \)-cycle of the torus. The periodic functional dependence on \( x^1 \) is not crucial in the present section and is implicit in the following. We will then often denote \( x^2 \) by \( x \).

³Notation: \( \Delta \) for the root system of \( \bar{g}_0 \), \( \alpha \) for roots and \( \alpha_{l,s} \) for the long and the short roots of \( \bar{g}_0 \).
where \( \delta_l(x) \) is a \( \delta \)-function with periodicity \( 2\pi l \). For all twisted algebras (except for \( A_{2k}^{(2)} \)) we consider the sums:

\[
\sum_{\alpha_l, n} E^0_{\alpha_l} \otimes e^{inx} = \delta_1(x) \sum_{\alpha_l} E^0_{\alpha_l} \\
\sum_{\alpha_s, n} E^j_{\alpha_s} \otimes e^{i(ln+j)x/l} = (\delta_l(x) + \ldots + e^{2\pi i(l-1)j/l} \delta_l(x + (l-1))) \sum_{\alpha_s} E^j_{\alpha_s},
\]

where \( j \) now runs from 0 to \( l - 1 \). For the algebra \( A_{2k}^{(2)} \) we will not write out the detailed formulas, but a similar computation would naturally involve the three Weyl orbits in the affine root system.

Now, we recall that to recover the trigonometric integrable model in [22], the extra piece in the moment map (\( \mu_2 \) or RHS) was chosen such that all non-CSA generators were weighted equally. We will take a similar road in this current algebra system, and our choice will turn out to be the appropriate one to recover the standard elliptic integrable models.\(^4\) Thus, when we consider a sum over roots as our RHS (and sum over the KK modes in the \( x^1 \)-direction in a similar manner) for the untwisted case, we choose to weigh all contributions equally with weights \( \nu \), and find a RHS:

\[
\mu_2 = -\nu \sum_{\alpha \in \Delta} E^0_{\alpha} \delta_1(z).
\]

For the twisted case, we weigh the contributions from each Weyl orbit of \( \bar{g}_0 \) given in (3.3) with weights \( \nu_{l,s} \). Strictly speaking, it is necessary to work out in detail how these choices for the moment map arise from a symplectic quotient of a vector space inserted at the puncture(s).\(^5\) Our educated guess will turn out to yield the right results. It would be interesting to include more punctures and extend this analysis to Gaudin and spin models (see e.g. [25] [26]).

### 4. The zero level submanifold

We want to study the zero level submanifold \( \mu_1 + \mu_2 = 0 \). We will treat the twisted case (except for \( A_{2k}^{(2)} \)) in detail – the first, untwisted case has an easier analogue. First of all, using the current group, we assume we can conjugate \( A_{2k} \) to a constant element of the CSA of \( \bar{g}_0 \).\(^6\) The equation for the level zero submanifold splits nicely

\(^4\)Note that is also equivalent to the choice of moment map made in [14] for the \( A_{N-1} \) elliptic integrable model.

\(^5\)In the light of the later remarks on realizing these models using brane configurations, a careful analysis of these vector spaces should reveal information on the open string degrees of freedom living on D-branes in the presence of \( ON \)-planes.

\(^6\)This is always possible for untwisted affine Kac-Moody algebras as proven in [23]. For twisted affine algebras there is no similar theorem known to us.
into different parts. The equations for the generators with zero weight under $\tilde{g}_0$, are:

$$\kappa \tilde{\partial} \phi_0^r = 0$$
$$\kappa \tilde{\partial} \phi_0^s = 0. \quad (j \neq 0) \quad (4.1)$$

The solutions are $\phi_0^r = p^r = $ constant for the first equation and $\phi_0^s = 0$ for the second, since that field component has to be both constant and satisfy twisted boundary conditions. Note that our choice for the total moment map was judicious, in that a single puncture on the CSA would have lead to a contradictory equation for the periodic excitations.\(^7\)

For the other weights the equations read:

$$\kappa \tilde{\partial} \phi_{\alpha_i}^r + <a, \alpha_i> \phi_{\alpha_i}^r = \nu_l \delta_l(z)$$  \quad (4.2)
$$\kappa \tilde{\partial} \phi_{\alpha_s}^s + <a, \alpha_s> \phi_{\alpha_s}^s = \nu_s(\delta_l(z) + \ldots + e^{2\pi i(l-1)/l} \delta_l(z + (l-1)i))$$

Here we chose $A_z = a \in \tilde{h}_0$, the CSA of $\tilde{g}_0$, and for simplicity we put $\omega_1 = 1$ and $\omega_2 = i$. To solve these equations, we introduce new variables $\psi$ which are related to the $\phi$ components as:

$$\phi_{\alpha_i}^r \equiv \exp\left(\pi <a, \alpha_i> (z - \bar{z})/\kappa \tau_2\right) \theta_{11}^r(z)$$
$$\phi_{\alpha_s}^s \equiv \exp\left(\pi <a, \alpha_s> (z - \bar{z})/\kappa \tau_2\right) \theta_{11}^s(z). \quad (4.3)$$

These new fields are meromorphic functions on the torus, and they satisfy boundary conditions:

$$\psi_{\alpha_i}^r(z + 1) = \psi_{\alpha_i}^r(z)$$ \quad (4.4)
$$\psi_{\alpha_s}^s(z + i) = e^{2\pi i j/l} e^{-2\pi i^2 / \kappa} \theta_{11}^s(\delta_l(z) + \ldots + e^{2\pi i(l-1)/l} \delta_l(z + (l-1)i)) \quad (\text{any } j, \lambda) \quad (4.5)$$

For the long root excitations periodic in the $x^2$ direction, the solution to the boundary conditions and pole structure are given by \(^8\):

$$\psi_{\alpha_i}^r = \nu'_{\alpha_i} \theta_{11}(z + <a, \alpha_i>/\kappa)$$ \quad (4.6)

where $\nu'_{\alpha_i}$ is related to $\nu_l$ via the residue of the $\theta_{11}$ function at 0. The solution for the short root excitations reads

$$\psi_{\alpha_s}^s = \nu'_{\alpha_s} \theta_{11}(z + \omega_2 + <a, \alpha_s>/\kappa)$$ \quad (4.7)

\(^7\)The absence of CSA components in the moment map may be reminiscent of the nature of the algebraic approach in [24], based on representations of the Coxeter group (generated by the roots only).

\(^8\)See appendix B for our standard $\theta$-function conventions.
The Lax operator of the elliptic integrable system can then be identified with $\phi$, and the quadratic Hamiltonian $\text{Tr}(\phi^2)$ of the system is then given by (up to an unimportant constant):

$$H_2 = \frac{p^2}{2} + \nu_1^2 \sum_{\alpha l} \mathcal{P}(\langle a, \alpha_l \rangle)$$

$$+ l \nu_2^2 \sum_{\alpha_s} \sum_{j=0}^{l-1} \mathcal{P}(\langle a, \alpha_s \rangle + j\omega_2)$$

(4.8)

which matches with the (twisted) elliptic Hamiltonians of [7]. Note that the derivation was made with a twist in the $\omega_2$ direction (because of standard conventions on $\theta$-functions). We might as well have taken the twist to be in any other cycle of the torus (and in particular the cycle associated to $\omega_1$). As we already remarked, it would be interesting to generalize the geometric derivation of Lax operators and Hamiltonians to other integrable systems by generalising our choice of moment map.

5. **Superpotentials for $\mathcal{N} = 1^*$ theories**

One of our primary motivations for studying the geometry of the elliptic integrable systems is based on the intimate relationship between classical integrable models in two dimensions and the Coulomb branch of supersymmetric gauge theories, noted first in [28, 29]. This relationship was made precise in the work of [14] where it was argued that the spectral curve of an integrable model, namely the $SU(N)$ Hitchin system coincides with the Seiberg-Witten curve for $SU(N)$, $\mathcal{N} = 2$ SUSY gauge theory with a massive adjoint hypermultiplet (known as the $\mathcal{N} = 2^*$ theory i.e. an $\mathcal{N} = 2$ preserving mass deformation of $\mathcal{N} = 4$ theory). In particular, the moduli of the Donagi-Witten curve which are the gauge-invariant order parameters on the Coulomb branch of the $\mathcal{N} = 2^*$ gauge theory are identified with the Hamiltonians of the integrable model. The $SU(N)$ Hitchin system was also identified with the elliptic Calogero-Moser model associated to the $A_{N-1}$ root system in [30]. This connection between elliptic Calogero-Moser systems and $\mathcal{N} = 2^*$ theories has been extended to arbitrary gauge groups $G$ in [31, 32].

5.1 **$\mathcal{N} = 1^*$ superpotential - from field theory**

An important consequence of the above connection between the Hamiltonians of the Calogero-Moser system and the $\mathcal{N} = 2^*$, $SU(N)$ gauge theory is that the vacuum value of the superpotential for the corresponding $\mathcal{N} = 1^*$ theory ($\mathcal{N} = 1$ preserving mass-deformation of $\mathcal{N} = 4$ theory) then coincides with the quadratic Hamiltonian of the integrable model [7]. Several direct checks of this superpotential were given in [3]. In particular it was shown that a class of equilibrium configurations of the
superpotential were in one to one correspondence with the massive vacua of $SU(N)$, $\mathcal{N} = 1^*$ gauge theory.

It is natural to guess that the above conclusion extends to superpotentials for $\mathcal{N} = 1^*$ theories with other gauge groups as well. Specifically, the superpotentials would correspond to the quadratic Hamiltonians of elliptic Calogero-Moser models associated to the corresponding root systems \[9, 33, 34\]. The general argument for this follows from viewing $\mathcal{N} = 1^*$ gauge theory with gauge group $G$ as softly-broken $\mathcal{N} = 2^*$ gauge theory with the same gauge group and hypermultiplet mass $\nu$. The $\mathcal{N} = 2^*$ theory has a Coulomb branch where the effective superpotential vanishes. On soft breaking to $\mathcal{N} = 1^*$ via a mass term for the adjoint chiral multiplet $\Phi$ in the $\mathcal{N} = 2$ vector multiplet, the theory acquires a superpotential, which for small $\mu$ has the form

$$W_{\text{eff}} = \mu \langle \text{Tr}\Phi^2 \rangle = \mu u_2, \quad (5.1)$$

where we define $u_k \equiv \langle \text{Tr}\Phi^k \rangle$ as the gauge-invariant order-parameters on the Coulomb branch. As in \[35\], the above superpotential can be argued to be exact, i.e. valid for large $\mu$ as well. The $\mathcal{N} = 2^*$ gauge theory has a $U(1)_J$ global symmetry which is a subgroup of the $SU(2)_R$ symmetry. The scalar components of the $\mathcal{N} = 2$ hypermultiplet carry charges $+1$ under $U(1)_J$, while the adjoint scalar $\Phi$ is neutral under this transformation. The $\mathcal{N} = 1^*$ theory inherits this $R$-symmetry provided the mass parameter $\mu$ is assigned a charge $+2$. Since the superpotential must have $R$-charge $+2$, the only term with this charge and consistent with the requirement of holomorphy and analyticity in the variables, $u_k, \mu$ and $\nu$ is $\mu u_2$ which must therefore be the exact value of the low-energy superpotential. The correspondence between elliptic Calogero-Moser models and $\mathcal{N} = 2^*$ gauge theories identifies the gauge-invariant order parameters $\{u_k\}$ with the conserved Hamiltonians (action variables) of the associated integrable models. In particular, $u_2$ is directly identified with the quadratic Hamiltonian of the integrable system.

Thus the superpotential for the $\mathcal{N} = 1^*$ theory with gauge group $G$ is

$$W_{\text{eff}} = \mu u_2 \quad (5.2)$$

where we have simply replaced $u_2$ with the Hamiltonian of the elliptic integrable model associated to the algebra of $G$, derived in Eq.\(4.8\). Note that we have used the twisted Hamiltonians where the twist has been performed in the $\omega_1$ direction. This choice was made in anticipation of the periodicity properties of the physical degrees of freedom from the gauge theory viewpoint. The complexified coupling constant of the underlying $\mathcal{N} = 4$ theory coincides with the complex structure $\tau$ of the torus $\Sigma_\tau$ on which the Weierstrass functions are defined. It is important to note that the twisted Calogero-Moser systems naturally make an appearance for non-simply laced
We further remark that the weighting factors $\nu_{l,s}$ are fixed in terms of the field theory parameters to be $\nu_l = \nu$, the hypermultiplet mass and $\nu_s = \nu/l$. These parameters can be fixed by comparing the conjectured elliptic superpotential in a certain limit (the affine Toda limit to be discussed below) to explicit field theory computations of the superpotential in that limit.

The dynamical variables $a$, of the integrable system have a natural physical, gauge theory interpretation when the $\mathcal{N} = 1^*$ theory is compactified on $\mathbb{R}^3 \times S^1$. In the latter context $a$ represent a complex combination of the Wilson lines and the dual photons of the 3D effective theory. In particular, in the classical $\mathcal{N} = 1^*$ theory on $\mathbb{R}^3 \times S^1$, the Wilson line $a_1$ around the $S^1$ is a modulus and may be chosen to lie in the Cartan subalgebra. Generic VEVs for the Wilson lines break $G \to U(1)^r$ where $r = \text{rank}(G)$. Symmetry of the theory under large gauge transformations (i.e. gauge transformations that twist around the $S^1$ by an element of the center of $G$), requires the Wilson lines to be periodic variables under $a_1 \to a_1 + 2\pi \omega^*$. Here $\omega^*$ is an element of the co-weight lattice. In addition, in the 3D effective theory we may exchange the $r$ photons for dual scalar fields $a_2$ which are also periodic due to the quantization of magnetic charge, under $a_2 \to a_2 + 2\pi \omega$ with $\omega$ an element of the weight lattice. Supersymmetry then allows us to combine these scalars into a complex scalar field $a = i(\tau a_1 + a_2)$ which forms the lowest component of a corresponding chiral superfield. The low-energy, chiral sector of the theory i.e. the superpotential is then expected to be a holomorphic function of this periodic variable.

As we explain below the elliptic superpotential in Eq.(5.2) arises from semiclassical configurations corresponding to 3D-monopoles, carrying topological charge in general, with action proportional to $\exp(<a, \alpha^*>)$, indicating that the effective superpotential is a holomorphic function of $<a, \alpha^*>$. Importantly, when $\alpha$ is a long root, a straightforward consequence of the periodicities of $a_1$ and $a_2$ is that $<a, \alpha^*_l> = <a, \alpha_l>$ must be periodic on the torus with complex structure $\tau$. In addition, on general grounds $u_2$ must have modular weight 2. Thus terms proportional to $\mathcal{P}<a, \alpha_l>$ must naturally appear in the superpotential as a consequence of ellipticity on $\Sigma_{\tau}$. On the other hand, for short roots $\alpha_s$, the periodicity of the variable $<a, \alpha_s>$ is different. In particular $<a, \alpha_s>$ is a periodic variable with periods $\omega_2$ and $\omega_1/l$ where $l = 2/\alpha_s^2$. This explains the appearance of the twisted Weierstrass functions which involve a sum over the standard Weierstrass functions with arguments shifted by multiples of $\omega_1/l$ as in Eq.(5.2) leading to the required periodicity. The twisted functions also ensure that in the semiclassical limit, they give rise to terms proportional to $\exp <a, \alpha_{s}^*>$. Finally, as $u_2$ is a dimension two operator in the $\mathcal{N} = 2^*$ theory where the only mass scale is $\nu$, $u_2 \propto \nu^2$. This explains the mass dependence of the $\mathcal{N} = 1^*$ superpotential $W_{\text{eff}} = \mu u_2$.

For generic root systems, it turns out to be a difficult task to gather direct evidence for the superpotential. In particular, the classification of the extrema of this superpotential is difficult in general. Such a classification and subsequent comparison
with semiclassical predictions (obtained by combining the classical analysis of vacua in \[36\] with the associated Witten indices), would provide a strong test of the elliptic superpotential. In the \( A_{N-1} \) case, the classification of massive \( \mathcal{N} = 1^* \) vacua is particularly simple and elegant. Each such vacuum simply corresponds to an extremum of \( W_{eff} \) wherein the \( N \ a's \) form a lattice \( \Gamma' \) on the torus \( \Sigma_{\tau} \).

\( 9 \) The total number of such configurations = \( \sum \) divisors of \( N \), coincides with the semiclassical vacuum counting.

### 5.2 Trigonometric limit

Some evidence in favor of this superpotential for generic \( G \) may be obtained by examining the trigonometric limit. This is the limit in which the adjoint hypermultiplet with mass \( \nu \) is decoupled, keeping fixed the effective dynamical scale of the 4D \( \mathcal{N} = 2^* \) theory \( i.e. \nu \to \infty \) and \( \tau \to i\infty \) with \( \Lambda^2 = \nu^2 \exp(2\pi i \tau / c_2(\mathcal{G})) \) fixed. Here \( c_2(\mathcal{G}) \) is the dual Coxeter number for gauge group \( \mathcal{G} \). In this limit the 4D theory reduces to softly broken pure \( \mathcal{N} = 2 \) SUSY Yang-Mills, and the (twisted) elliptic superpotential above reduces precisely to the superpotential found in \[37\] for \( \mathcal{N} = 1 \) SUSY Yang-Mills on \( R^3 \times S^1 \), provided we choose the parameters \( \nu_l = \nu \) and \( \nu_s = \nu/l \). As discussed in \[37\] the latter superpotential assumes the usual form of the affine Toda potential after a rescaling and redefinition of variables. As shown in detail in \[37\], this affine Toda superpotential arises from the \( r + 1 \) types of fundamental BPS monopole contributions in the 3D effective theory on \( R^3 \times S^1 \). \( r \) of these monopoles carry charges \( \alpha^* \) in the co-root lattice where \( \alpha \) is a simple root and have action \( \sim \exp(\langle \alpha^*, a \rangle) \). In addition there is one type of fundamental monopole that carries negative magnetic charge given by the lowest root \( \alpha^*_0 \) and one unit of 4D -instanton charge with action \( \sim \exp(\langle \alpha^*_0, a \rangle + 2\pi i \tau) \).

### 5.3 Semiclassical configurations

At weak-coupling, the superpotential of the \( \mathcal{N} = 1^* \) theory on \( R^3 \times S^1 \) must arise from semiclassical, non-perturbative contributions. As usual, in the 3D effective theory we expect to have BPS monopoles carrying charges in the co-root lattice with action \( \sim \exp(k \langle \alpha^*, a \rangle) \) associated to some positive root \( \alpha \). In the four supercharge, \( \mathcal{N} = 1^* \) theory, each of these BPS monopoles has only two supersymmetric fermion zero modes. All additional zero modes whose existence is predicted by the Callias index theorem must be lifted as they are not protected by supersymmetry. Such a lifting of zero modes that are not protected by SUSY was demonstrated explicitly for the sixteen supercharge theory in 3 dimensions in \[38, 39\]. Hence all these monopoles with two exact zero modes will contribute to the low-energy \( \mathcal{N} = 1^* \) superpotentials. Note that since the monopole charge can be any positive root, including non-simple roots

\[9\] Specifically, if the torus \( \Sigma_{\tau} \) is defined by the lattice \( \Gamma \), then \( \Gamma \) must be an order \( N \) sublattice of \( \Gamma' \).
they include configurations that are charged under two different magnetic $U(1)$’s. This situation is in contrast to the affine Toda superpotential of [37] where only monopoles corresponding to simple roots could contribute to the superpotential.

The theory on $R^3 \times S^1$, also has BPS semiclassical configurations carrying both magnetic charge and 4D-instanton or topological charge. These configurations are obtained by shifting the asymptotic value of the Wilson line variable $< a_1, \alpha^* >$ by a multiple of its period i.e. $2\pi n$. Such configurations contribute terms proportional to $\exp(k < \alpha^*, a >) \exp(2\pi i kn \tau)$. The presence of such topologically charged BPS states also permits the existence of well-defined, semiclassical solutions with magnetic charge given by a negative root and non-zero 4D topological charge, with action proportional to $\exp(-k < a, \alpha^* >) \exp(2\pi i kn \tau)$. It must be emphasized that these are not anti-monopoles. The appearance of such states with negative magnetic charge in the theory on $R^3 \times S^1$ has an elegant description in the D-brane picture of the sixteen supercharge theory [40]. In this picture, fundamental monopoles (associated to simple roots) correspond to D1-branes stretching between neighboring D3-branes in a stack of $N$ parallel, separated D3-branes. However, when the D3-branes are placed on a transverse circle, a new kind of fundamental monopole appears, stretching between the first and the last D3-brane, associated to the lowest root of the corresponding affine algebra. Instanton charge is associated to D1-branes that wind all around the compact direction and can be related by T-duality to D0-branes dissolved in D4’s i.e. instantons of the corresponding theory. Various combinations of such D1-brane segments can give rise to all possible monopole configurations discussed above.

Finally, there are of course contributions from ordinary 4D instantons as well. Once again, all these configurations carrying 4D instanton number have only two exact fermion zero modes which are protected by SUSY and are expected to contribute to the superpotential. Therefore, based on these general arguments, we expect the low-energy $\mathcal{N} = 1^*$ superpotential to be an elliptic function with the following semiclassical expansion,

$$ W_{eff} = \nu^2 \mu \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau} + \nu^2 \mu \sum_{\alpha} \left[ \sum_{k=1}^{\infty} b_{k,\alpha} e^{k < a, \alpha^* >} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} e^{2\pi i kn \tau} (c_{k,n,\alpha} e^{k < a, \alpha^* >} + d_{k,n,\alpha} e^{-k < a, \alpha^* >}) \right]. $$

The semicalssical expansion of the (twisted) elliptic superpotentials in Eq.(5.2) has precisely this form as can be easily seen by expanding out the Weierstrass functions.

### 5.4 The superpotential from branes and a quotient construction

Perhaps the most direct way of obtaining the elliptic superpotential for $\mathcal{N} = 1^*$ theory is via the use of mirror symmetry on the field theory realised on the world-volume of a Type IIA brane setup. This involves a four-dimensional version of mirror
symmetry introduced in [11, 41] and was used to derive $N = 1^*$ superpotentials for the $A_{k-1}$ quiver models in [10], which include the $SU(N), N = 1^*$ theory. The key ingredient was the brane configuration [12, 13] of intersecting D4-branes and NS5-branes compactified in the conventional $x^6$ and $x^3$ directions. Using Type IIA/IIB dualities, the Coulomb branch of the “electric” mass-deformed $SU(N)$ theory was mapped to the Higgs branch of a an $SU(N)$ gauge theory on $R^3 \times T^2$ with impurities (punctures with attached vector spaces) on the torus $T^2$. This torus has complex structure $\tau$ and must be identified with $\Sigma_\tau$ introduced earlier. The Higgs branch equations of the mirror theory are given in terms of the gauge field on the torus $A_\tau$ and the adjoint scalar $\phi$, by D-flatness equations which are precisely Hitchin’s equations reduced to 2 dimensions. (The Wilson lines of the gauge field $A_\tau$ around the two cycles of $\Sigma_\tau$ are precisely the dual photons and Wilson lines of the 3D effective theory on $R^3 \times T^2$). As discussed in [10, 11], these conditions can be thought of as moment map equations for a hyperkahler quotient. In particular the moduli space of the Higgs branch equations is precisely a hyperkahler quotient with respect to the group of smooth maps $\Sigma_\tau \to U(N)$ at the zero level of the moment map. It was shown that this moduli space corresponds to the zero level manifold associated to the $A_{N-1}$ elliptic Calogero-Moser system. The geometric construction of elliptic integrable models is thus very naturally realised in the context of brane setups in string theory. The symplectic quotient construction presented in this paper generalises the above to integrable models associated to general root systems.

The problems in trying to extend the $N = 2^*$ configuration involving spiraling [13] D4-branes to other simple gauge groups are well-known [44, 45]. In [14] brane configurations realizing the $N = 2^*$ theory involving O6-planes and D4- and NS5-branes were put forward, but their corresponding curves [31] proved difficult to read off from the brane geometry. In the spirit of [11] one could try to find a corresponding mirror theory, then to study the Higgs branch of the impurity theory on the torus. This program is attractive, but runs into difficulties as the dual (IIB) theory will involve $ON5$-planes (see e.g. [16] for properties of the $ON5^0$-plane), S-dual to $O5$-planes, and the properties of these planes are in general insufficiently studied to determine the exact mirror impurity theory. It would be interesting to study these theories further, from string theory, as well as using inspiration from the geometry of the integrable systems worked out in this paper. In particular, one could guess that the mirror impurity theory would have a zero level manifold corresponding to an integrable system treated in this letter, although this remains to be demonstrated.

6. Conclusions and future directions

A lot can be said on the open problem posed in Section 5.4, but we restrict ourselves to a few more remarks. One can try to move forward on the integrable system side

$^{10}$We thank Ami Hanany for interesting discussions on this topic.
of the problem, using a correspondence between folding procedures and orientifold planes [47] (and in particular, by sharpening the dictionary between the algebraic and geometrical quantities associated to orientifold planes), and the analysis of T-duality and S-duality in the theory of integrable systems (e.g. [48]). Another issue that would come into play would be the suitability of the form of the Lax operators for taking a pure $N = 2$ limit [5] [49]. As mentioned earlier, on the field theory side, one needs to check whether the semi-classical analysis of the number of vacua of the $\mathcal{N} = 1^*$ theory [36] agrees with a possible guess for the quantum superpotential as was done for the $SU(N)$ gauge group in [4]. This involves determining the classical minima of the potential of these elliptic integrable system which seems to be an unsolved and non-trivial problem. It should yield an interesting picture for the phases of $SO(N)$ and $Sp(2N)$ mass-deformed gauge theories, and their transformation properties under the duality group. As described in the previous section, in the $SU(N)$ case, classification of certain minima corresponding to massive vacua of the $\mathcal{N} = 1^*$ theory requires classifying certain lattices. This picture is also directly reflected in the Donagi-Witten curves for the $\mathcal{N} = 2^*$ theory wherein at certain points on the moduli space which directly descend to massive $\mathcal{N} = 1^*$ vacua upon perturbation, the curve degenerates into an unramified $N$-fold cover of the torus $\Sigma_\tau$. Thus the massive vacua can be located by finding all the possible genus one $N$-fold covers of $E_\tau$, a problem that is identical to the classification of lattices $\Gamma'$ such that $\Gamma$ (the lattice defining the torus $\Sigma_\tau$) is an order $N$ sublattice of $\Gamma'$. It would be extremely interesting to understand if and how this picture generalises to other gauge groups. Note also that one can read off the stable solutions for the integrable system (i.e. the minima for the potential) directly from the explicit solution for the $A_{N-1}^{(1)}$ elliptic model [4], confirming the above picture. Unfortunately, no analogous solution for the other elliptic Calogero-Moser models is known.

Furthermore, by analogy to the $SU(N)$ case [10], one can speculate on the relevance of other spin systems (e.g. [24]) associated to other root systems to field theories with product gauge groups, but in the light of the difficulties sketched above it seems to early to make this analogy stick. We point it out to show that there remains much ground to be covered.

In summary, in this paper we have analysed the geometry of elliptic integrable systems following [12]. We clarified the role of (twisted) affine subalgebras in these integrable systems. We moreover laid bare some of the systematics of the folding procedures used to obtain new integrable systems from old ones, using the systematic results derived in the context of fixed point conformal field theories. We pointed out possible extensions of our work, which include the geometric interpretation to Gaudin and spin models, and we elaborated on the relevance of our analysis to supersymmetric field theories and on the possible relevance to brane configurations in

\[11\] We thank the authors of this paper for explaining their results.
string theory. We believe we pointed out interesting directions for future research in this context. In general, we again demonstrated the cross-fertilisation of current algebra, integrable system physics, conformal field theory, supersymmetric field theory and string theory and tried to knit them tighter together.

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A. Foldings and orbit algebras

This technical appendix makes the link between the folding procedure of [7] and the framework of orbit algebras of [13]. This appendix is separate from the main line of development of the paper, but we believe it is useful because it sheds more light on the systematics behind the folding procedures which were put to good use in integrable models, and specifically in elliptic Calogero-Moser models. We hope it will also turn out to be useful to clarify the relation between the geometric properties of orientifold planes and algebraic foldings.

For the technical details, we refer to the two papers, [7] and [13], that develop their formalism separately, and we recall merely the ingredients that enable us to clearly lay down the map between the two formalisms.\textsuperscript{12}

Indeed, the folding used in [7] corresponds to modding out an affine algebra by an outer automorphism of the Dynkin diagram $(\hat{\omega})$. The action on the Dynkin diagram induces an action $\bar{\omega}^*$ on the simple roots, as in [7]. The action on the root system is:

$$\bar{\omega}^* \alpha^{(i)} = \bar{\alpha}^{(\hat{\omega}i)} \quad \text{for} \quad i \neq \hat{\omega}^{-1}(0)$$

$$\bar{\omega}^* \bar{\alpha}^{\hat{\omega}^{-1}(0)} = -\bar{\theta}$$

(A.1)

where $\bar{\theta}$ is the highest root of the untwisted affine lie algebra $g^{(1)}$. From the framework developed in [13], it is clear that the action of $\bar{\omega}^*$ is naturally extended (as an affine Weyl transformation) to the whole weight space. In particular, the action on the Dynkin components of vectors in the weight space is [13]:

$$(\bar{\omega}^* \bar{\lambda})^j = (\bar{\lambda})^{\hat{\omega}^{-1}(j)} \quad \text{for} \quad j \neq \hat{\omega}(0)$$

\textsuperscript{12}We will use the notation of [13] throughout this section and refer to that paper and [21] for our notations. The main difference with the body of the paper is the bar over the simple roots of the simple Lie algebra $g$. The index 0 refers to the zeroth, affine root.

\textsuperscript{13}Denoted $A$ in that paper.
\[(\tilde{\omega}^* \tilde{\lambda}) \tilde{\omega}(0) = \tilde{k}_\lambda - \sum_{j=1}^r \tilde{a}_j \tilde{\lambda}^j. \quad (A.2)\]

Now, it is not difficult to see\(^\text{14}\) that these transformation rules coincide precisely with the ones guessed in [7]. In particular the second transformation rule in (A.2) takes into account the shift by a fundamental weight that was introduced ad hoc in [7]. Indeed, our first gain in this analysis is a systematic derivation of the fact that in [7] this shift always turned out to correspond to the fundamental weight associated to the node \(\tilde{\omega}(0)\). We see now that this has a natural explanation when we realize that the Dynkin diagram automorphism is actually associated to an affine Weyl transformation, as explained in [13].

We gain a little more when we realize that in [13] the orbit algebras for all affine algebras were classified. Moreover, it was remarked ([13] p. 18) that the subalgebra pointwise fixed under the automorphism has a Cartan matrix which is the transpose of the Cartan matrix of the orbit algebra. Thus we can trivially extend the table in [13] (p. 13) to include a column with the invariant subalgebras. We merely dualize (transpose) the algebras in their last column. This demonstrates on the one hand that all cases in [7] are indeed present in the classification of [13], and that we can identify the Cartan torus of the pointwise fixed algebra with the reduced phase space of [7]. More importantly, by examining the classification table, we notice that by folding an affine Dynkin diagram in any other way than the foldings exhibited in [7], we will not get a new integrable system, but we will merely recover the ones we already new. This explains the systematics behind the folding procedures, and shows that the trial and error procedure of [7] covered all cases.

It is satisfying to show that the analysis of [7] is complete and to link the folding procedure with the algebra used in fixed point conformal field theories. This raises the question of whether there are more applications of this formalism in the realm of integrable systems. It would be nice if one could make a precise connection between, say, the application of orbit algebras in the theory of moduli spaces of flat connections of non-simply connected gauge groups (as in [27]), and integrable systems on tori. A further direction in which to proceed would be to include current algebras and connections \(\tilde{A}\) with a twist on a two-torus, and examine the integrable systems that arise from the geometrical setup in algebraic terms.

**B. Conventions**

**B.1 Theta-functions**

Our conventions are such that

\[\theta_{11}(z + 1) = -\theta_{11}(z) \quad (B.1)\]

\(^{14}\)Take the inner product of the transformation rule in [7] (3.3) with the simple roots of \(g\).
\[ \theta_{11}(z + i) = e^{-2\pi i z} \theta_{11}(z), \]  

(B.2)

and the \( \theta_{11} \) function has zeroes at \( n + m i \) with \( n, m \in \mathbb{Z} \). For simplicity only, we have chosen the periods to be \( \omega_1 = 1 \) and \( \omega_2 = i \), and we have \( \tau = i \).

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