Optimal tuning of a Linear Quadratic Regulator for Position Control using Particle Swarm Optimisation

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Abstract. This paper discusses the stabilising problems of a nonlinear and multivariate underactuated systems, also known as Inverted Pendulum (IP) systems. The attitude control of a nonlinear IP system is thus considered. One of the most important techniques, a Linear Quadratic Regulator (LQR), is used in this paper to design a controller for an IP system. The structure of LQR parameters is based on weight matrices, and the adjustment of the weight matrices parameters is the most challenging part of this, being normally computed by repeating trials and adjusting based on successful or unsuccessful outcomes. The difficulty in guessing the optimum matrices and achieving automatic adjustment of the weighting matrices is mitigated by the introduction of a metaheuristic algorithm to investigate the optimal solution; this is based on a swarm of particles moving in a virtual space. The proposed approach successfully stabilised the IP system in the upright position and removed perturbations; the simulated results thus showed satisfactory performance of the suggested control method.

Keywords: metaheuristic algorithm, optimal control, particle swarm optimization, the linear quadratic regulator

1. Introduction

Many studies have addressed the problem of motion control in an Inverted Pendulum (IP) system, which is generally considered an underactuated system [1]–[3]. A system is considered underactuated when the number of degrees of freedom is greater than the number of actuators [4]–[7]. One of the greatest challenges of this type of system is how to control dynamic movement, and IP systems thus provide an important nonlinear prototype for the analysis, validation, and evaluation of different control techniques[8]–[10]. In [3], a sliding-mode control (SMC) was proposed to control the motions of inverted pendulums on a cart; the designed controller featured a linear sliding surface. Vinodh Kumar and Jerome in [8] utilised various control techniques on a double IP system, and Proportional-Integral-Derivative (PID) controllers, two loop PID controllers, and linear quadratic regulators (LQR) have also been implemented and studied in real-time applications. LQR is one of the most widely used types of optimal control algorithm, and it has been extensively used for controller design. The theory of LQR thus provides a useful outline for how to balance input consumption and output states.
The design of LQR is related to the selection of weight matrices $Q$ and $R$, which must be determined correctly to develop the desired satisfactory performances for the closed-loop control system [11], [12]. Lingyan et al. [13] used an LQR controller to investigate stabilizing an IP system with both simulation and experimental work. The simulation results showcased the ability of the proposed controller to stabilize the IP system. Another study focused on combining different control methods to achieve better performance of trajectory tracking and balancing. Rahmani et al. [14] used two techniques to stabilise the IP system by utilizing a modern combination of fuzzy off-line systems and a nonlinear backstepping planner. In order to realise improvements in the performance of unsteady IP systems and to design a fuzzy controller [15], a new multi-local linear model based on the Takagi–Sugeno method was developed.

A large number of research studies utilising evolutionary optimisation algorithms in control system design [6], [16]–[18] have also been undertaken. Sen and Kalyoncu [16] discussed the problem of optimising the weighing matrices’ parameters in the LQR controller to achieve the desired dynamic response, with the authors proposing a new swarm optimisation method called the Bee Algorithm (BA) to determine effective control gains and to maintain the stability of control of the IP with minimum effort. In the same manner, Singh et al. [19] used a PSO method with a PID sliding mode control to balance the system in the upper position with minimum errors.

Control theory, in general, deals with many challenges affecting the stability and controllability of systems. These challenges can thus be categorised into several parts. The first one is external challenges associated with the unexpected modification of parameters in the system, which is also known as uncertainty. The second is external challenges related to disturbance associated with the measuring of the data; implementation of any controller thus depends on the reliability of the mathematical model used in the system.

This paper describes the design and implementation of an optimal controller for balancing an inverted pendulum on a cart in the inverted equilibrium position. The LQR controller is used to determine the controller gains. However, it is normally hard to estimate values for the $Q$ and $R$ matrices where these are calculated manually through trial-and-error procedures; this procedure is time-consuming, very tedious, and inaccurate. Therefore, in this study, the selection of the best combination of weighting matrices ($Q$ and $R$) was based on the PSO [20] algorithm. A better understanding the relationship between weight matrices ($Q$ and $R$) of the LQR controller should thus lead to obtaining satisfying results with minimum error margins and reasonable settling times.

The remainder of this paper is arranged as follows. Section 2 describes an inverted pendulum system and derives its mathematical model. In section 3, the PSO is demonstrated. Section 4 introduces how the PSO was used for tuning the LQR parameters in this work, and Section 5 discusses the results. Conclusions are thus given in section 6.

### 2. Modelling of an Inverted Pendulum

The IP system is represented in figure 1. It consists of a rotationally movable pendulum fixed to the top midpoint of a movable cart that can move freely to the right and to the left across a horizontal path by means of a servo motor. The motion of the cart is controlled by a control signal sent to the servo motor, which is applied to dominate the motion of the cart. When the IP is exposed to force, the motor will perform a mechanical action that horizontally drives the cart in a straight path and thus moves the pendulum rotationally. The applied voltage to the servo motor is considered to be the input signal for the IP. The outputs are related for both the pendulum and the cart, and the state outputs of the pendulum are demonstrated by the angle and angular velocity while the state outputs of the cart are demonstrated by position and the velocity.
Table 1. Inverted Pendulum parameters

| Symbol | Parameter                                           | Value                  |
|--------|-----------------------------------------------------|------------------------|
| F      | Input force to the IP system.                       | $kg \cdot m/s^2$       |
| g      | Gravity                                             | $9.81 \ m/s^2$         |
| l      | Length of the pendulum.                             | 0.3                    |
| m      | Mass of the pendulum                                | 0.2 kg                 |
| M      | Cart mass                                           | 0.5 kg                 |
| I      | Moment of inertia of the pendulum.                  | 0.006 kg.m$^2$         |
| B      | Coefficient of viscosity friction between the ground | 0.1 Ns/m               |
|        | surface and the cart.                               |                        |
| x      | Horizontal displacement for the cart                | m                      |
| $\theta$ | The angle of the Pendulum.                        | Degree                 |
| b      | Coefficient of viscosity friction between pendulum on| 0.0004 N/m/s           |
|        | and the cart.                                       |                        |

Analysing the diagram of the IP system illustrated in figure 1 allows the dynamics of the system to be derived[10], [21]. These dynamics are described by two nonlinear equations: cart location is represented by equation (1), and rotational movement for the pendulum is represented by equation (2).

\[
\begin{align*}
(m + M)\dddot{x} + b_1\dot{x} + ml\dddot{\theta} \cos \theta - ml\dot{\theta}^2 \sin \theta &= F \quad (1) \\
\dot{\theta}(I + ml^2) + ml\dddot{x} \cos \theta - mglsin \theta + b_2\dot{\theta} &= 0 \quad (2)
\end{align*}
\]

Considering $\cos(\theta) \approx 1$, $\sin(\theta) \approx 0$ and $\dot{\theta}^2 = 0$, and substituting these assumptions into equation (1) and equation (2), new equations are obtained that represent the linearized model at the equilibrium point.

\[
\begin{align*}
(m + M)\dddot{x} + b_1\dot{x} + ml\dddot{\theta} &= F \quad (3) \\
\dot{\theta}(I + ml^2) + ml\dddot{x} + b_2\dot{\theta} &= 0 \quad (4)
\end{align*}
\]

The state space model for a continuous system is thus as in equations (5):
\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \]

The state space model of the IP can therefore be arranged as follows:

\[
\begin{bmatrix}
\dot{x} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & -m^2l^2g & -b_1(I + ml^2) & 1 & 0 \\
0 & 0 & (m + M)I + Mml^2 & (m + M)I + Mml^2 & (m + M)I + Mml^2 & b_2ml \\
0 & mlg(m + M) & mlb_1 & (m + M)I + Mml^2 & (m + M)I + Mml^2 & -b_2(m + M)
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x} \\
\theta \\
\dot{\theta}
\end{bmatrix} + \ldots
\]

\[ y = [x_\theta] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \]

In equation 9, the state vector \([x \quad \dot{x} \quad \theta \quad \dot{\theta}]^T\) represents the state vector for the IP system where \(x\) represents the cart position, \(\theta\) represents the direction angle of the pendulum from the upper equilibrium position, \(\dot{x}\) represents the speed of the cart, and \(\dot{\theta}\) represents the angular velocity of the pendulum.

3. Particle swarm optimization (PSO)

PSO is a metaheuristic algorithm inspired by nature [21] that models the swarming behaviours of fishes and birds. The population is directed by a specific cognitive consciousness and follows a social consciousness that is applied to each particle in the swarm. Particles in the swarm at the beginning are placed randomly in the search space with arbitrary speeds, and each particle is defined as a possible solution for the optimisation problem. Figure 2 illustrates the mechanisms of PSO.

Every particle is capable of evaluating its own position, which is then saved in memory as the best performance of that particle; these values represent the cognitive consciousness. For each iteration, particles refer to other particles to determine their positions and best performance, representing social consciousness. All particles then change their speed and position according to the data thus obtained. During this process, performance is evaluated using a weight function that characterises the controller’s overall performance. As the particles gather around one local or global optimal point, this represents the optimal solution. The speed and position of each particle in the swarm is modelled in the following equations:

\[ v_i(t + 1) = wv_i(t) + r_1C_1(p_i(t) + x_i(t)) + r_2C_2(g(t) + x_i(t)) \]

\[ x_i(t + 1) = x_i(t) + v_i(t + 1) \]

\[ w = \frac{2 - \varphi - \sqrt{\varphi^2 - 4\varphi}}{2} \]

where \(\varphi_1\) and \(\varphi_2\): Constriction coefficients.

\(C_1\) and \(C_2\): Acceleration coefficients.
$r_1$ and $r_2$: Uniform distribution numbers.
$r_1C_1(p_i(t) + x_i(t))$: Cognitive component.
$r_2C_2(g(t) + x_i(t))$: Social component.

![PSO flowchart](image)

**Figure 2.** PSO flowchart

4. **Tuning LQR using PSO algorithm**

The locations of the eigenvalues are the most important properties in such closed-loop systems, though it is hard to find appropriate positions for them. For this reason, many strategies have been implemented to find a solution to this problem. The benefit of the closed-loop system is that it allows manipulation of the state dynamics for the system based on feedback from output. By adding a state such that feedback $u = -Kx$, the ability to control the dynamics of the system and to place the roots in the required position to maintain stability and robustness is developed. The schematic diagram of LQR is shown in figure 3.

![Schematic diagram of LQR](image)

**Figure 3.** Schematic diagram of LQR

Performance for the controllers can be achieved through the addition of numerous fitness functions; the general form is represented as

$$J = \int_0^\infty (x^TQx + u^TRu)dt \quad (9)$$

In this study, an Integral Absolute Error (IAE) performance function was used to calculate the error at each iteration of the PSO algorithm. The particles thus tracked the lowest error that appeared in the
life cycle of the algorithm in order to find the optimum values for the diagonal, symmetric and positive matrices (R and Q). Figure 4 illustrates the approach to applying PSO with an LQR controller.

![Figure 4: PSO with LQR controller.](image)

5. Results and Discussion

MATLAB software was used to implement the PSO algorithm and also to design the full-state feedback controller for IP, as shown in figure 5.

![Figure 5. Full-state feedback controller for IP in Simulink](image)

Usually, algorithms without constraints such as PSO take additional time and processing efforts. In order to reduce these, constraints were applied to this algorithm within suitable ranges to operate the controlled system. These ranges were based on the experience of the designer. The particles thus held random values only within these constraints, and at each iteration, the particles were not permitted to move outside these boundaries. Particles that would have moved outside these boundaries were thus held in the same position until the next iteration. Table 2 shows the ranges selected.
Table 2. Suitable ranges (constraints)

| Parameter | Value |
|-----------|-------|
| Q1        | 0     |
|           | 7000  |
| Q2        | 0     |
|           | 7000  |
| Q3        | 0     |
|           | 7000  |
| Q4        | 0     |
|           | 7000  |
| R         | 0     |
|           | 7000  |

The resulting 5-dimensional shape can thus be considered as a convex optimisation problem with the solution is somewhere inside it. The boundaries for the dimensions represent the suitable ranges for the parameters of the weight matrices Q and R that form the LQR controller. Particles moving inside the 5-dimensional space search for the optimal solution to minimise the cost function. The simulation of the IP dynamics thus represents the angle and the position of the pendulum and the cart respectively, as illustrated in the following results. These results are arranged in three cases: the first case is for the initial values; the second case shows the performance after several successive iterations; and the final case is for the final values, with the fitness function being minimised. These latter are considered to be the optimal values.

Case 1: Initial Values

![Figure 6. Performance of x.](image-url)
Figures 6 and 7 show high levels of variation between the responses where the particles are separated randomly in the search space, leading to random values (positions) for the particles inside the suitable ranges as suggested in table 2.

**Case 2: After several successive iterations**

After several successive iterations, the particles start to gather near each other, gradually covering a smaller area depending on the cognitive and social behaviours developed. This gathering affects the responses for the pendulum angle and cart position, and it is usually near or around the optimal values. In figures 8 and 9, a reduction in variation compared to the previous simulations as seen in figures 3 and 4 can be noted as the simulations move towards the best solution.
Case 3: Final iteration
In the final iteration, the performance as demonstrated in figure 10 and 11 shows no variation; most of the particles have accumulated at the optimal value.
From the previous figures, in each iteration, the particles move inside the suggested constraints to record 5-dimensional data \((Q_1, Q_2, Q_3, Q_4, R)\) in a high-dimension matrix. This matrix includes many “best” solutions generated during the particles’ quest in the search space. The designed algorithm generates matrices sequentially in order to isolate the optimal values among the best values. Based on the LQR algorithm, the optimum values of the weight matrices \(Q\) and \(R\) are thus used to calculate the feedback states. The optimal values for the weighting matrices’ parameters are shown in Table 3, and from these optimal values, the state feedback gain is computed as seen in Table 4.

**Table 3. Weighting matrices parameter during PSO lifecycle**

| 1\(^{st}\) iteration | 10\(^{th}\) iteration | 20\(^{th}\) iteration | 30\(^{th}\) iteration | 40\(^{th}\) iteration | 50\(^{th}\) iteration (Optimal) |
|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------------|
| \(Q_1=1892\)          | \(Q_1=315.4825\)      | \(Q_1=139.0442\)      | \(Q_1=138.542\)       | \(Q_1=137.8268\)      | \(Q_1=137.8095\)            |
| \(Q_2=4.4411\)        | \(Q_2=5710.2\)        | \(Q_2=5707.5\)        | \(Q_2=5706.9\)        | \(Q_2=5706.9\)          | \(Q_2=5706.9\)              |
| \(Q_3=3750.9\)        | \(Q_3=3784.8\)        | \(Q_3=3.774.6\)       | \(Q_3=3774.7\)        | \(Q_3=3775\)            | \(Q_3=3775\)               |
| \(Q_4=629\)           | \(Q_4=1236.4\)        | \(Q_4=1229.7\)        | \(Q_4=1223.3\)        | \(Q_4=1223.5\)          | \(Q_4=1223.5\)             |
| \(R=3539\)            | \(R=633.8437\)        | \(R=441.2994\)        | \(R=436.1078\)        | \(R=436.1259\)          | \(R=436.1262\)             |
| Error=0.1640          | Error=0.0933          | Error=0.0857          | Error=0.0856          | Error=0.0855            | Error=0.0855               |

**Table 4. Optimal state feedback gains**

| State feedback gain | Value  |
|---------------------|--------|
| K1                  | -0.5621|
| K2                  | -4.0543|
| K3                  | 29.0135|
| K4                  | 5.7731 |
6. Conclusion
The paper focused on utilising the PSO algorithm to find the optimum parameters of an LQR controller. Those parameters were applied to determine state feedback gains, and the optimal values of the gain control were utilised to balance the IP in the upright equilibrium position. From the simulated results, mixing the LQR algorithm and the PSO algorithm successfully controls both the angle of the pendulum and the position of the cart. Thus, the designed controller can maintain the IP in an upper balancing point with only reasonable and minor oscillations, allowing the cart to reach the required posture. The response characteristics also meet the demands of the design specification. Future work should include studying the robustness of the controller by simulating the system response both without and with disturbance.

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