Spinfoams: summing or refining?

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Abstract. We consider two alternative prescriptions for the continuum limit in spinfoam gravity – summing over all foams vs. refining a single foam –, and argue that, under certain conditions, they are the same.

The spinfoam program is based on truncations of the quantum gravitational field on finite 2-complexes, aka foams. How does one take the continuum limit in this context? Should one sum over all foams, as in perturbative quantum field theory? Or should one refine a single foam, as in lattice gauge theory? Both alternatives have been considered in the spinfoam literature [1, 2, 3, 4], and – to this day – neither has won the consensus. Here, we argue that they may turn out to be equivalent: summing and refining could be the two faces of the same limit [5].

At the heuristic level, there are good reasons to believe that such a coincidence could arise. Indeed, recall that spinfoam quantum gravity operates without a background metric, for it is meant to be the quantum theory of the metric itself. This means that each foam is both a lattice, viz. a discretization of spacetime, and a Feynman diagram, viz. a virtual history of field quanta.

To make this intuition precise, the key mathematical concept is that of net. Let $X$ be a (Hausdorff) topological space. A net $N$ is an $X$-valued function on a directed set $S$, i.e. a partially ordered set in which any two elements have an upper bound. A net is convergent to $N_\infty \in X$, if for any neighborhood $U$ of $N_\infty$, there is $s_U \in S$ such that $N(s) \in U$ whenever $s \geq s_U$. Thanks to the directedness of $S$, the limit $\lim_S N = N_\infty$, when it exists, is unique.

Define a foam as a triple $\Gamma = (V,E,F)$ with $V$ a finite set of vertices, $E$ a set of ordered pairs of vertices $e = (s(e), t(e))$, called edges, and $F$ a finite set of faces. Here, we call face a finite sequence of edges $f = (e_1^{n_1}, ..., e_n^{n_n}, ..., e_m^{m_m})$, where $t(e_i^{n_i}) = s(e_{i+1}^{n_{i+1}})$, $n_i = \pm 1$ and $n_f + 1 := 1$. Say that two foams are isomorphic if there exists a bijective map between the vertex sets $V_1$ and $V_2$ respecting all adjacency relations, and call $u$-foam an isomorphism class $[\Gamma]$ of foams ($u$ stands for “unlabeled”).

It is not hard to see that the (discrete) set $\mathcal{F}$ of all $u$-foams with a given boundary forms a directed set with respect to the relation defined by $[\Gamma_1] \leq [\Gamma_2]$ if $\Gamma_1$ can be embedded into $\Gamma_2$. In a nutshell, this is because, given any two foams, it is always possible to attach to them as many vertices, edges and faces as necessary to embed them both into a third, finer, foam.

A spinfoam model $Z$ can then be thought of as a net over $\mathcal{F}$, with values in a Hilbert space $\mathcal{H}$ of “quantum states”. This observation suggests an obvious definition for the refinement limit of $Z$, namely the limit $Z_\infty = \lim_\mathcal{F} Z$. What is the relation between this limit and the sum $\sum_{[\Gamma] \in \mathcal{F}} Z([\Gamma])$ (assuming both exist)?

See e.g. [6] for more details on the physical interpretation of spinfoam gravity.

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As a rule, spinfoam amplitudes $Z([\Gamma])$ are defined as sums over colorings $\sigma$ of the faces of a foam $\Gamma \in [\Gamma]$ by spins, i.e. unitary irreducible representations of some compact Lie group. To each such coloring is associated an amplitude $z(\Gamma, \sigma)$, and

$$Z([\Gamma]) = \sum_{\sigma \in \text{Col}(\Gamma)} z(\Gamma, \sigma), \quad (1)$$

where Col($\Gamma$) is the set of face-colorings of $\Gamma$. We can also consider the alternative definition

$$Z^*( [\Gamma] ) = \sum_{\sigma \in \text{Col}^*(\Gamma)} z(\Gamma, \sigma), \quad (2)$$

where now Col($\Gamma$) denotes the set of face-colorings of $\Gamma$ with trivial representations excluded (non-zero spins).

The relationship between the continuum limit $Z_\infty$ and the sum of $Z([\Gamma])$ over all u-foams with a fixed boundary is then the following: under a general condition on the amplitude $Z_{\Gamma}(\sigma)$, which we detail in the next paragraph, we have

$$Z([\Gamma]) = \sum_{[\Gamma'] \subseteq [\Gamma]} Z^*([\Gamma']), \quad (3)$$

and hence, passing to the limit (assuming it exists),

$$Z_\infty = \sum_{[\Gamma] \in \mathcal{F}} Z^*([\Gamma]). \quad (4)$$

This is our central result. It states that refining foams in the model $Z$ is the same as summing over foams in the model $Z^*$. (Note that, due to the peculiar net structure, all foams appear in (4) only in the following sense: every u-foam $[\Gamma]$ appears in one finite sum (3) whose value can be chosen arbitrarily close to $Z_\infty$.)

The condition on $z(\Gamma, \sigma)$ for the relation (4) to hold is one of cylindrical consistency. First, observe that each coloring $\sigma$ of a foam $\Gamma$ comes with a multiplicity, related to the symmetries of $\Gamma$. The multiplicity $|\sigma|_\Gamma$ of a coloring $\sigma \in \text{Col}(\Gamma)$ is the number of colorings $\sigma'$ such that $\sigma = \sigma' \circ \phi$, with $\phi$ an automorphism of $\Gamma$. Then we say that the amplitude

$$a(\Gamma, \sigma) := |\sigma|_\Gamma z(\Gamma, \sigma), \quad (5)$$

is cylindrically consistent if $a(\Gamma, \sigma) = a(\Gamma', \sigma')$ when $(\Gamma', \sigma')$ is a trivial extension of $(\Gamma, \sigma)$, that is when $\Gamma$ is a subfoam of $\Gamma'$, $\sigma$ and $\sigma'$ coincide on the faces of $\Gamma$ and $\sigma'$ is trivial on the other faces of $\Gamma'$.

Under this condition, the proof of (3) is straightforward. First, observe that the subfoams of $\Gamma$ index a partition of Col($\Gamma$), in which each each class is made of the trivial extensions of a given subfoam $\Gamma' \subset \Gamma$:

$$\text{Col}(\Gamma) = \bigsqcup_{\Gamma' \subset \Gamma} \text{Col}^*(\Gamma'), \quad (6)$$

This implies that

$$Z([\Gamma]) = \sum_{\Gamma' \subset \Gamma} \left( \sum_{\sigma' \in \text{Col}^*(\Gamma')} |\sigma'|_{\Gamma'}^{-1} a(\Gamma, \sigma') \right). \quad (7)$$

Second, check that we have

$$|\sigma'|_{\Gamma} = |\sigma'|_{\Gamma'} N_{\Gamma', \Gamma}, \quad (8)$$
with $N_{\Gamma',\Gamma}$ the number of subfoams of $\Gamma$ isomorphic to $\Gamma'$. Third, use cylindrical consistency to get

$$Z([\Gamma]) = \sum_{\Gamma' \subset \Gamma} N_{\Gamma',\Gamma}^{-1} Z^*([\Gamma'])$$

and conclude to (3).

This result in principle applies equally well to the Euclidean and Lorentzian spinfoam models of quantum gravity [6] – assuming convergence issues can be suitably addressed, of course.

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