Classification of local conservation laws of Maxwell’s equations

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Abstract

A complete and explicit classification of all independent local conservation laws of Maxwell’s equations in four dimensional Minkowski space is given. Besides the elementary linear conservation laws, and the well-known quadratic conservation laws associated to the conserved stress-energy and zilch tensors, there are also chiral quadratic conservation laws which are associated to a new conserved tensor. The chiral conservation laws possess odd parity under the electric-magnetic duality transformation of Maxwell’s equations, in contrast to the even parity of the stress-energy and zilch conservation laws. The main result of the classification establishes that every local conservation law of Maxwell’s equations is equivalent to a linear combination of the elementary conservation laws, the stress-energy and zilch conservation laws, the chiral conservation laws, and their higher order extensions obtained by replacing the electromagnetic field tensor by its repeated Lie derivatives with respect to the conformal Killing vectors on Minkowski space. The classification is based on spinorial methods and provides a direct, unified characterization of the conservation laws in terms of Killing spinors.
I. INTRODUCTION

Conservation laws play an important role in physical field theories by determining conserved quantities for the time evolution of the fields. For free electromagnetic fields, Maxwell’s equations exhibit a rich structure of conservation laws. The well-known Maxwell stress-energy tensor yields local conservation laws for energy, momentum, angular and boost momentum \([\text{I}]\) which arise from Killing vectors associated to the Poincaré symmetries of flat spacetime. In addition, because of the conformal invariance of Maxwell’s equations, the stress-energy tensor also yields local conservation laws arising from conformal Killing vectors associated to conformal symmetries of the spacetime. Interestingly, Maxwell’s equations possess another physically significant conserved tensor, given in its original form by Lipkin’s “zilch” tensor \([\text{II}]\), and subsequently generalized in Refs. \([3–6]\). This tensor yields additional local conservation laws and corresponding conserved quantities arising from conformal Killing vectors. The physical meaning of these conserved quantities is discussed in Ref. \([7]\).

More recently, new conserved quantities have been found by Fushchich and Nikitin \([8]\) through an analysis of quadratic expressions in the electromagnetic field variables whose integrals are constant in time on the solutions of Maxwell’s equations. These quantities correspond to local conservation laws associated with a new conserved tensor which is independent of the stress-energy and zilch tensors. The new conserved tensor is physically interesting since, as we point out here, it possesses odd parity, i.e. chirality, under the duality transformation interchanging the electric and magnetic fields. Hence, in a remarkable contrast to the stress-energy and zilch conservation laws, which are invariant under the duality transformation, the new chiral conservation laws distinguish between pure electric and pure magnetic fields.

All these conservation laws and underlying conserved tensors have extensions obtained by replacing the electromagnetic fields by their repeated Lie derivatives with respect to conformal Killing vectors. (See, e.g. Ref. \([8]\)). The resulting set of all such higher order
local conservation laws yields an infinite number of conserved quantities. This proliferation of conservation laws raises the immediate questions: Do Maxwell’s equations admit any other independent local conservation laws? Can a unified account be given of all the different local conservation laws and associated conserved tensors?

In this paper we answer these questions by presenting a direct, unified classification of all local conservation laws of Maxwell’s equations in flat spacetime. As a result of our classification, we are able to show that every local conservation law which is quadratic in the electromagnetic fields can be expressed as a linear combination of the stress-energy and zilch conservation laws, the chiral conservation laws, and their extensions. Moreover, we show that Maxwell’s equations have no other local conservation laws apart from elementary ones which are linear in the electromagnetic fields.

Our method is based on the general approach described in Refs. [9–11] for constructing local conservation laws for any field equations. In this approach, all local conservation laws can be obtained from adjoint symmetries which are solutions of the formal adjoint equations of the determining equations for symmetries. Ordinarily, there are additional constraint conditions which an adjoint symmetry must satisfy in order to yield a conservation law; however, we show that as a consequence of linearity of Maxwell’s equations these conditions can be by-passed in the present case.

The determining equations for adjoint symmetries of Maxwell’s equations can be elegantly solved by spinorial methods. The solutions are characterized in terms of symmetric spinorial tensors, called Killing spinors, which were first used by Penrose [12] to construct first integrals for null geodesics in black-hole spacetimes. Killing spinors also play a central role in twistor theory as the principal parts of trace-free symmetric twistors [13]. In flat spacetime, Killing spinors have an important factorization property in terms of twistors. This allows for a simple classification of all adjoint symmetries, which is pivotal in our analysis of local conservation laws of Maxwell’s equations. In particular, Killing spinors lead to a unified derivation of the stress-energy and zilch conservation laws together with the chiral conservation laws.
In Sec. II we establish some notation and summarize our main classification results. In Sec. III we describe our method. We present the details of the classification analysis in Secs. IV and V. In Sec. VI we translate between the tensor form and spinor form of our classification results. We make some concluding remarks in Sec. VII. Throughout we use the index notation and conventions of Ref. [13].

II. MAIN RESULTS

The free Maxwell’s equations for the electromagnetic field tensor $F_{\mu\nu}(x) = -F_{\nu\mu}(x)$ in four dimensional Minkowski space $M^4 = (R^4, \eta)$ are given by

$$\partial^\mu F_{\mu\nu}(x) = 0, \quad \partial^\mu * F_{\mu\nu}(x) = 0,$$

where, in the standard Minkowski coordinates $x^\mu$, $\partial_\mu = \partial/\partial x^\mu$ is the coordinate derivative, $*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} F^{\sigma\tau}$ is the dual of $F_{\mu\nu}$, $\epsilon_{\alpha\beta\sigma\tau}$ is the spacetime volume form, and indices are raised and lowered using the spacetime metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. The structure of (2.1) explicitly displays the symmetry of the field equations under the duality transformation

$$F_{\mu\nu} \rightarrow *F_{\mu\nu}, \quad *F_{\mu\nu} \rightarrow -F_{\mu\nu}.$$  \hspace{1cm} (2.2)

Let $J^q(F)$, $0 \leq q \leq \infty$, denote the coordinate space

$$J^q(F) = \{(x^\mu, F_{\mu\nu}, F_{\mu\nu,\sigma_1}, \ldots, F_{\mu\nu,\sigma_1,\ldots,\sigma_q})\},$$  \hspace{1cm} (2.3)

where each $q$-jet $(x^\mu_o, F_{o\mu\nu}, F_{o\mu\nu,\sigma_1}, \ldots, F_{o\mu\nu,\sigma_1,\ldots,\sigma_q}) \in J^q(F)$ is to be identified with a spacetime point $x^\mu = x^\mu_o$ and values of the field tensor and its derivatives at $x^\mu = x^\mu_o$,

$$F_{o\mu\nu,\sigma_1,\ldots,\sigma_p} = \partial_{\sigma_1} \cdots \partial_{\sigma_p} F_{\mu\nu}(x_o), \quad 0 \leq p \leq q,$$

where the notation (2.4) with $p = 0$ stands for $F_{o\mu\nu} = F_{\mu\nu}(x_o)$. Let $R(F)$ denote the solution space of Maxwell’s equations, which is the subspace of $J^1(F)$ defined by imposing the field equations (2.1) on $F_{\mu\nu}(x)$. The derivatives of the field equations (2.1) up to order $q$ define
the $q$-fold prolonged solution space $R^q(F) \subset J^{q+1}(F)$ associated with Maxwell’s equations. Let $D_\mu$ be the total derivative operator

$$D_\mu = \partial_\mu + \sum_{q \geq 0} F_{\alpha_\mu \nu_1 \cdots \nu_q} \partial_{F}^{\alpha_\mu \nu_1 \cdots \nu_q},$$

(2.5)

where $\partial_{F}^{\alpha_\mu \nu_1 \cdots \nu_q}$ are the partial differential operators satisfying

$$\partial_{F}^{\alpha_\mu \nu_1 \cdots \nu_q} F_{\mu_1 \nu_1 \cdots \nu_q} = \begin{cases} \delta_\alpha^{(\mu} \delta_\beta^{\nu_1} \cdots \delta_\nu_q^{(\sigma_1 \cdots \delta_\nu_q)}), & \text{if } r = q, \\ 0, & \text{if } r \neq q. \end{cases}$$

(2.6)

A local conserved current of Maxwell’s equations is a vector function $\Psi^\mu$ defined on some $J^q(F)$ satisfying

$$D_\mu \Psi^\mu = 0 \quad \text{on } R^q(F).$$

(2.7)

We refer to the integer $q$ as the order of $\Psi^\mu$. The conserved current (2.7) is trivial if

$$\Psi^\mu = D_\nu \Theta^{\mu \nu} \quad \text{on some } R^p(F),$$

(2.8)

where $\Theta^{\mu \nu} = - \Theta^{\nu \mu}$ are some functions on $J^p(F)$. Two conserved currents are considered equivalent if their difference is a trivial conserved current. We refer to the class of conserved currents equivalent to a current $\Psi^\mu$ as the conservation law associated with $\Psi^\mu$. The smallest among the orders of these equivalent currents is called the order of the conservation law.

We now write down the stress-energy, zilch, and chiral conservation laws of Maxwell’s equations. On $M^4$ let $\xi^\mu$ be a conformal Killing vector [14] and $Y^{\mu \nu} = -Y^{\nu \mu}$ be a conformal Killing-Yano tensor [15]. These are characterized, respectively, by the equations

$$\partial^{(\mu \nu)} Y = \frac{1}{4} \eta^{\mu \nu} \partial_\sigma \xi^\sigma, \quad \partial^{(\mu \nu) \alpha} = \frac{1}{3} \eta^{\mu \nu} \partial_\sigma Y^{\sigma \alpha} + \frac{1}{3} \eta^{\alpha (\mu} \partial_\sigma Y^{\nu) \sigma}.$$

(2.9)

The solutions are polynomials in the spacetime coordinates $x^\mu$,

$$\xi^\mu = \alpha_1^\mu + \alpha_2^\mu x^\nu + \alpha_3^\mu x^\sigma x^\mu - \frac{1}{2} \alpha_4^\mu x^\sigma x^\sigma,$$

(2.10)

$$Y^{\mu \nu} = \alpha_5^{\mu \nu} + \alpha_6^{[\mu \nu]} + \epsilon_{\kappa \tau} \alpha_7^{\mu \tau} x^\nu + \alpha_8^{[\mu \nu]} x^\kappa + \frac{1}{4} \alpha_8^{\mu \nu} x^\kappa x^\sigma,$$

(2.11)

with constant coefficients
\[
\begin{align*}
\alpha_1^\mu, \alpha_2^{\mu\nu} = -\alpha_2^{\mu\nu}, \alpha_3, \alpha_4^\sigma, \alpha_5^{\mu\nu} = -\alpha_5^{\mu\nu}, \alpha_6^\mu, \alpha_7^\sigma, \alpha_8^{\mu\nu} = -\alpha_8^{\mu\nu}.
\end{align*}
\] (2.12)

There are 15 linearly independent conformal Killing vectors (2.10) and 20 linearly independent conformal Killing-Yano tensors (2.11) on \(M^4\).

The stress-energy and zilch conservation laws are, respectively, given by

\[
\begin{align*}
\Psi_1^\mu(F; \xi) &= F^{\mu\sigma} F_{\nu\sigma} \xi^\nu - \frac{1}{4} F^{\nu\sigma} F_{\nu\sigma} \xi^\mu, \\
\Psi_2^\mu(F; \xi) &= *F^{\mu\sigma} (\mathcal{L}_\xi F_{\nu\sigma}) \xi^\nu - F^{\mu\sigma} (\mathcal{L}_\xi *F_{\nu\sigma}) \xi^\nu,
\end{align*}
\] (2.13) (2.14)

where

\[
\mathcal{L}_\xi F_{\alpha\beta} = \xi^\sigma F_{\alpha\beta,\sigma} - 2(\partial_\alpha \xi^\sigma) F_{\beta|\sigma}, \quad \mathcal{L}_\xi *F_{\alpha\beta} = \xi^\sigma *F_{\alpha\beta,\sigma} - 2(\partial_\alpha \xi^\sigma) *F_{\beta|\sigma}
\] (2.15)

are the standard Lie derivatives of the electromagnetic field tensor and its dual, with respect to the vector field \(\xi^\sigma\) on \(M^4\).

The chiral conservation laws are given by

\[
\begin{align*}
\Psi_3^\mu(F; \xi, Y) &= F_{\nu\sigma} (D^\mu \mathcal{L}_\xi F_{\alpha\beta}) Y^{\nu\sigma\alpha\beta} + 4 F_{\nu\sigma} (D^\mu \mathcal{L}_\xi F_{\alpha\beta}) Y^{\nu|\sigma\alpha\beta} \\
&\quad + \frac{3}{5} F_{\nu\sigma} (\mathcal{L}_\xi F_{\alpha\beta}) \partial_\mu Y^{\nu|\sigma\alpha\beta} + \frac{12}{5} F_{\nu\sigma} (\mathcal{L}_\xi F_{\alpha\beta}) \partial_{\nu} Y^{\nu|\sigma\alpha\beta},
\end{align*}
\] (2.16)

where

\[
Y^{\nu\sigma\alpha\beta} = Y^{\nu\sigma} Y^{\alpha\beta} - Y^{\nu|\alpha} Y^{\beta|\sigma} - 3\eta^{[\nu|\alpha} Y^{\beta]} Y^{\tau|\sigma]} + \frac{1}{2} \eta^{\nu|\alpha} \eta^{\beta|\sigma} Y^{\tau\lambda} Y^{\tau\lambda}.
\] (2.17)

The current (2.16) is equivalent to the first order current

\[
\begin{align*}
\Psi_4^\mu(F; \xi, Y) &= -F_{\nu\sigma} \xi^\mu (\mathcal{L}_\xi F_{\alpha\beta}) Y^{\nu\sigma\alpha\beta} - 4 F_{\nu\sigma} (\mathcal{L}_\xi F_{\alpha\beta}) Y^{\nu\sigma[\mu \eta]} Y^{\tau]\beta} \\
&\quad - 4 F_{\nu\sigma} (\mathcal{L}_\xi F_{\alpha\beta}) \partial_\gamma Y^{\nu\sigma[\mu \eta]} Y^{\tau]\beta} + \frac{3}{5} F_{\nu\sigma} (\mathcal{L}_\xi F_{\alpha\beta}) \partial_\nu Y^{\nu|\sigma\alpha\beta} \\
&\quad - \frac{8}{5} F_{\nu\sigma} (\mathcal{L}_\xi F_{\alpha\beta}) \partial_\eta Y^{\nu|\sigma\alpha\beta}.
\end{align*}
\] (2.18)

In addition to the quadratic conserved currents (2.13), (2.14), (2.16) and (2.18), Maxwell’s equations also possess linear conserved currents given by

\[
\begin{align*}
\Psi_5^\mu(F; W, \tilde{W}) &= W_{\nu} F^{\mu\nu} + \tilde{W}_{\nu} F^{\mu\nu}.
\end{align*}
\] (2.19)
where the vector functions $W_\nu(x), \tilde{W}_\nu(x)$ satisfy the adjoint Maxwell’s equations

$$\partial_{[\mu} W_{\nu]} + *\partial_{[\mu} \tilde{W}_{\nu]} = 0. \tag{2.20}$$

The previous linear and quadratic conserved currents each have higher order extensions given in terms of repeated Lie derivatives on the electromagnetic field tensor. Let

$$F^{(n)}_{\xi \mu \nu}(x) = (\mathcal{L}_\xi)^n F_{\mu \nu}(x), \tag{2.21}$$

where $F_{\mu \nu}(x)$ satisfies (2.1). It follows from the linearity and conformal invariance of Maxwell’s equations (2.1) that if $\zeta^\sigma$ is a conformal Killing vector then $F^{(n)}_{\xi \mu \nu}(x)$ satisfies Maxwell’s equations. Consequently, the replacement of $F_{\mu \nu}(x)$ by $F^{(n)}_{\xi \mu \nu}(x)$ in any conserved current of order $q$ produces a conserved current of order $q + n$, for all $n \geq 1$.

**Theorem 2.1.** Let $\xi$ be a conformal Killing vector (2.10) and $Y$ be a conformal Killing-Yano tensor (2.11), let $W_\nu(x), \tilde{W}_\nu(x)$ be solutions of (2.20), and define

$$\Psi_T^{(n)\mu}(F; \xi) = \Psi_T^{(n)\mu}(F^{(n)}; \xi), \tag{2.22}$$

$$\Psi_Z^{(n)\mu}(F; \xi) = \Psi_Z^{(n)\mu}(F^{(n)}; \xi), \tag{2.23}$$

$$\Psi_Y^{(n)\mu}(F; \xi, Y) = \Psi_Y^{(n)\mu}(F^{(n)}; \xi, Y), \tag{2.24}$$

$$\Psi_W^{(n)\mu}(F; \xi, W, \tilde{W}) = \Psi_W^{(n)\mu}(F^{(n)}; W, \tilde{W}). \tag{2.25}$$

These are non-trivial conserved currents of Maxwell’s equations (2.1) of order $n, n+1, n+1, n$, respectively.

**Remark:** The current $\Psi_W^{(n)\mu}(F; \xi, W, \tilde{W})$ is equivalent to $\Psi_W^{\mu}(F; (-\mathcal{L}_\xi)^n W, (-\mathcal{L}_\xi)^n \tilde{W})$, which is of order 0.

The extended currents in Theorem 2.1 can be obviously generalized using Lie derivatives with respect to distinct conformal Killing vectors $\zeta_1, \ldots, \zeta_n$ in place of a single repeated conformal Killing vector in (2.21). However, this generalization does not lead to any additional independent currents.
Theorem 2.2. Every local conservation law (2.7) of order \( q \geq 0 \) of Maxwell’s equations (2.1) is equivalent to a linear combination of the currents

\[
\Psi_{T}^{(n_1)\mu}(F; \xi), \quad \Psi_{Z}^{(n_2)\mu}(F; \xi), \quad \Psi_{V}^{(n_3)\mu}(F; \xi, Y), \quad \Psi_{W}^{\mu}(F; W, \tilde{W}),
\]

involving a sum over conformal Killing vectors \( \xi \) and conformal Killing-Yano tensors \( Y \) for each \( n_1, n_2, n_3 \), with \( 0 \leq n_1 \leq q, \ 0 \leq n_2 \leq q - 1, \ 0 \leq n_3 \leq q - 1 \), and solutions \( W, \tilde{W} \) of the adjoint Maxwell’s equations (2.20).

Through substitution of the expressions (2.10) for conformal Killing vectors \( \xi \) and (2.11) for conformal Killing-Yano tensors \( Y \), the currents (2.22), (2.23), (2.24) become homogeneous polynomials of degree \( 2n + 1, 2n + 2, 2n + 3 \) in the arbitrary constants (2.12). The coefficient of each monomial of these constants yields a conserved current, some of which, however, are not independent. A complete, explicit basis of independent currents of order \( q \geq 0 \) is given later in Theorem 6.5 by using the null tetrad formalism for conformal Killing vectors and conformal Killing-Yano tensors as described in Sec. VI.

Proposition 2.3. The set of stress-energy conservation laws (2.13) is a 15 dimensional vector space that admits a basis in which 4 have no explicit \( x^\mu \) dependence, 7 are linear, and 4 are quadratic, in the highest degree terms in \( x^\mu \);

The set of zilch conservation laws (2.14) is a 84 dimensional vector space that admits a basis in which 9 have no explicit \( x^\mu \) dependence, 20 are linear, 26 are quadratic, 20 are cubic, and 9 are quartic, in the highest degree terms in \( x^\mu \);

The set of chiral conservation laws (2.16) is a 378 dimensional vector space that admits a basis in which 24 have no explicit \( x^\mu \) dependence, 54 are linear, 72 are quadratic, 78 are cubic, 72 are quartic, 54 are quintic, and 24 are sextic, in the highest degree terms in \( x^\mu \).

In general, for \( n \geq 0 \), there are \( \frac{1}{3}(n+1)^2(2n+3)^2(4n+5) \) linearly independent conservation laws arising from (2.22), \( \frac{1}{3}(n+2)^2(2n+3)^2(4n+7) \) linearly independent conservation laws arising from (2.23), and \( \frac{2}{3}(n+1)(n+3)(2n+3)(2n+7)(4n+9) \) linearly independent conservation laws arising from (2.24).
Theorems 2.1 and 2.2 combined with Proposition 2.3 and Theorem 6.5 give a complete and explicit classification of all non-trivial local conservation laws of Maxwell’s equations.

We obtain conserved tensors from the stress-energy, zilch, and chiral conservation laws by first setting $\xi^\mu$ to be a constant vector, $Y^\sigma_{\tau\alpha\beta}$ to be a constant skew-tensor, and then factoring out $\xi^\mu, Y^\sigma_{\tau\alpha\beta}$. In (2.13) and (2.14), this directly leads to

$$T^\mu_\nu(F) = F^\mu\sigma F^\nu_\sigma - \frac{1}{4} \delta^\mu_\nu F^\tau\sigma F^\tau_\sigma, \quad (2.27)$$

$$Z^\mu_\nu(F) = F^\mu\sigma * F^\sigma_{(\nu,\rho)} - * F^\mu\sigma F^\sigma_{(\nu,\rho)}, \quad (2.28)$$

which are, respectively, the well-known stress-energy tensor and Lipkin’s zilch tensor. In (2.18), after some lengthy manipulations, we extract the expression

$$V^\mu_{\nu \alpha \beta \sigma \tau}(F) = F^\alpha\beta,\mu F^\nu_{\sigma,\nu} + F^\sigma_{\tau,\nu} F^\alpha\beta,\nu - 2F^\nu_{[\alpha}[\sigma,\mu F^\tau]_{\beta],\nu} + 3\eta^\sigma_{[\alpha,\nu} F^\tau_{\beta],\nu} + \frac{1}{2} \eta^\nu_{\alpha,\nu} F^\tau_{\beta],\nu} - \frac{1}{2} \eta^\nu_{\alpha,\nu} F^\tau_{\beta],\nu} - \frac{1}{2} \eta \delta^\nu_{\alpha,\nu} F^\tau_{\beta],\nu}, \quad (2.29)$$

which we call the chiral tensor.

**Theorem 2.4.** On the solutions of Maxwell’s equations (2.3), the tensors $T^\mu_\nu(F\xi^{(n)})$, $Z^\mu_\nu(F\xi^{(n)})$, $V^\mu_{\nu \alpha \beta \sigma \tau}(F\xi^{(n)})$, for $n \geq 0$, have the properties

$$D^\sigma_\mu T^\mu_\nu = 0, \quad T^\mu_\nu = T^{(\mu\nu)}, \quad T^\mu_\mu = 0, \quad (2.30)$$

$$D^\sigma_\mu Z^\mu_\rho = 0, \quad Z^\mu_\rho = Z^{(\mu\rho)}, \quad Z^\mu_\rho = 0, \quad (2.31)$$

$$D^\sigma_\mu V^{\mu\nu\alpha\beta\sigma\tau} = 0, \quad V^{\mu\nu\alpha\beta\sigma\tau} = V^{(\mu\nu)[\sigma\tau][\alpha\beta],\sigma\tau} = V^{\mu\nu[\alpha\beta\sigma\tau]} = 0, \quad V^{\mu\nu\alpha\beta\sigma}_\beta = 0, \quad (2.32)$$

$$V^{\tau[\alpha\beta]} = 0, \quad V^{\tau_{(\nu\sigma)}} = 0, \quad (2.33)$$

Moreover, under the duality transformation (2.3), $T^\mu_\nu$ and $Z^\mu_\rho$ have even parity, i.e., are invariant, while $V^{\mu\nu\alpha\beta\sigma\tau}$ has odd parity, i.e., is chiral. The same invariance properties extend to the currents $\Psi_T^{(n)\mu}(F;\xi)$, $\Psi_Z^{(n)\mu}(F;\xi)$, $\Psi_V^{(n)\mu}(F;\xi,Y)$ for all $n \geq 0$. 

9
III. PRELIMINARIES

Given a conserved current (2.7) of order $q$, one can show that there are functions $R_{\mu \sigma_1 \cdots \sigma_p}$, $\tilde{R}_{\mu \sigma_1 \cdots \sigma_p}$ on $J^{q+1}(F)$ so that $\Psi^\mu$ identically satisfies

$$D_\mu \Psi^\mu = \sum_{0 \leq p \leq q} (R_{\mu \sigma_1 \cdots \sigma_p} F^{\mu \nu, \sigma_1 \cdots \sigma_p} + \tilde{R}_{\mu \sigma_1 \cdots \sigma_p} \ast F^{\mu \nu, \sigma_1 \cdots \sigma_p}).$$  (3.1)

An application of the standard integration by parts procedure \[9\] then leads to an equivalent conserved current, which for convenience we again denote by $\Psi^\mu$ and which has order at most $2q - 1$, identically satisfying

$$D_\mu \Psi^\mu = Q_\nu F^{\mu \nu, \mu} + \tilde{Q}_\nu F^{\mu \nu, \mu}$$  (3.2)

for some functions $Q_\nu, \tilde{Q}_\nu$ on $J^r(F)$ for some $r \leq 2q$. We refer to the pair $Q = (Q_\nu, \tilde{Q}_\nu)$ as the characteristic of the conserved current $\Psi^\mu$ and call the integer $r$ the order of $Q$. If $\Phi^\mu$ is a conserved current equivalent to $\Psi^\mu$ then $Q$ is called a characteristic admitted by $\Phi^\mu$.

A function $H$ defined on some $J^p(F)$ is a total divergence if and only if it is annihilated by the Euler operators $E^{\mu \nu}_F(H) = \sum_{0 \leq k \leq p} (-1)^k D_{\sigma_1} \cdots D_{\sigma_k} \partial F^{\mu \nu, \sigma_1 \cdots \sigma_k}(H)$. Hence, all characteristics $Q$ of order $r$ are determined by the equation

$$E^{\mu \nu}_F(Q_\beta F^{\alpha \beta, \alpha, \mu} + \tilde{Q}_\beta \ast F^{\alpha \beta, \alpha, \mu}) = 0$$  on some $J^p(F)$.  (3.3)

After some manipulations, this equation yields

$$D_{[\mu} Q_{\nu]} + \ast D_{[\mu} \tilde{Q}_{\nu]} = \sum_{0 \leq k \leq r} (-1)^k D_{\sigma_1} \cdots D_{\sigma_k} (q_{\beta \mu \nu, \sigma_1 \cdots \sigma_k} F^{\alpha \beta, \alpha, \mu} + \tilde{q}_{\beta \mu \nu, \sigma_1 \cdots \sigma_k} \ast F^{\alpha \beta, \alpha, \mu})$$  (3.4)

on $J^p(F)$, where

$$q_{\beta \mu \nu, \sigma_1 \cdots \sigma_k} = \partial F^{\mu \nu, \sigma_1 \cdots \sigma_k} Q_\beta, \quad \tilde{q}_{\beta \mu \nu, \sigma_1 \cdots \sigma_k} = \partial F^{\mu \nu, \sigma_1 \cdots \sigma_k} \tilde{Q}_\beta, \quad 0 \leq k \leq r,$$  (3.5)

are the coefficients of the Fréchet derivative of $Q_\beta, \tilde{Q}_\beta$. The solutions of the determining equations (3.4) for $r \geq 0$ are the characteristics for all conserved currents of Maxwell’s equations. Furthermore, given a solution $Q = (Q_\nu, \tilde{Q}_\nu)$, one can invert the Euler operator
equations (3.3) by applying a standard homotopy operator (see [9,10]) to obtain an explicit integral formula for a current $\Psi^\mu$ in the characteristic form (3.2).

From the determining equations (3.4), we see that, on solutions of Maxwell’s equations, all characteristics $Q$ of order $r$ satisfy

$$D_{[\mu}Q_{\nu]} + \ast D_{[\mu}\tilde{Q}_{\nu]} = 0 \quad \text{on } R^r(F).$$

These equations are the adjoint of the determining equations for symmetries of Maxwell’s equations [6,16]. We refer to them as the adjoint symmetry equations and we call functions $P = (P_\nu, \tilde{P}_\nu)$ defined on $J^r(F)$ satisfying

$$D_{[\mu}P_{\nu]} + \ast D_{[\mu}\tilde{P}_{\nu]} = 0 \quad \text{on } R^r(F)$$

adjoint symmetries of order $r$ of Maxwell’s equations. Note that the gradients

$$P_\nu = D_\nu \chi, \quad \tilde{P}_\nu = D_\nu \tilde{\chi},$$

for any functions $\chi, \tilde{\chi}$ on some $J^p(F)$ are trivially a solution of (3.7). We call $P$ an adjoint gauge symmetry if it agrees with (3.8) on $R^p(F)$, and we consider two adjoint symmetries to be equivalent if their difference is an adjoint gauge symmetry. The order $r$ of an adjoint symmetry $P$ is called minimal if it is the smallest among the orders of all adjoint symmetries equivalent to $P$. If $P$ is not equivalent to an adjoint gauge symmetry then we call $P$ non-trivial.

One can easily show that if a characteristic $Q$ agrees with gradient expressions (3.8) when restricted to solutions of Maxwell’s equations, then it determines a trivial conserved current (2.8) with $\Theta^{\mu\nu} = \chi F^{\mu\nu} + \tilde{\chi} \ast F^{\mu\nu}$. The resulting conservation law reflects the well-known divergence identities

$$D_\mu(F^{\mu\nu}) = 0, \quad D_\mu(\ast F^{\mu\nu}) = 0, \quad \text{on } J^2(F),$$

which express the conservation of electric and magnetic charges in the free Maxwell’s equations. Consequently, we call a characteristic $Q$ trivial if
\[ Q_\nu = D_\nu \chi, \quad \tilde{Q}_\nu = D_\nu \tilde{\chi}, \quad \text{on } R^p(F) \quad (3.10) \]

for some functions \( \chi, \tilde{\chi} \) on \( J^p(F) \), and we consider two characteristics to be equivalent if their difference is a trivial characteristic.

Typically, as advocated e.g. in Refs. [9,10], the classification of conserved currents is based on first solving the adjoint symmetry equations and then verifying which of the solutions satisfy the determining equations for characteristics. However, a serious complication arises for Maxwell’s equations. As we will see in Sec. V, in contrast to the evolutionary PDEs studied in Refs. [9,10], Maxwell’s equations possess equivalence classes of non-trivial adjoint symmetries all of which fail to satisfy the determining equations (3.4) and, hence, are not characteristics of conserved currents. More importantly, adjoint symmetries that are equivalent to the characteristic of a conserved current typically also fail to satisfy (3.4). Thus, for a complete classification of characteristics, one needs not only to find the equivalence classes of adjoint symmetries, but also to determine whether each class admits a representative that satisfies the determining equations (3.4).

Here we circumvent these difficulties by employing a variant of the standard integral formula for constructing a conserved current from its characteristic [17,18]. Let

\[ \Phi^\mu(P) = \int_0^1 d\lambda \left( P_\nu(x, \lambda F, \lambda \partial F, \ldots, \lambda \partial^q F) F^{\mu\nu} + \tilde{P}_\nu(x, \lambda F, \lambda \partial F, \ldots, \lambda \partial^q F) \ast F^{\mu\nu} \right), \quad (3.11) \]

where \( P = (P_\nu(x, F, \partial F, \ldots, \partial^p F), \tilde{P}_\nu(x, F, \partial F, \ldots, \partial^p F)) \) is a pair of functions defined on some \( J^q(F) \), and \( \partial^p F \) stands collectively for the variables \( F_{\mu\nu,\sigma_1\ldots\sigma_p}, \ p \geq 0 \).

**Proposition 3.1.** Let \( P = (P_\nu, \tilde{P}_\nu) \) be an adjoint symmetry of order \( q \) of Maxwell’s equations. Then \( \Phi^\mu(P) \) is a conserved current of order \( q \) of Maxwell’s equations. If \( P \) is equivalent to the characteristic \( Q \) of a conserved current \( \Psi^\mu \), then the current \( \Phi^\mu(P) \) is equivalent to \( \Psi^\mu \). In particular, if \( P \) is equivalent to a trivial characteristic, then \( \Phi^\mu(P) \) is a trivial current.

**Proof**
The proof of the first and third claims amounts to a straightforward computation and will be omitted. As to the second claim, suppose that \( Q = (Q_\nu, \tilde{Q}_\nu) \) is the characteristic of a conserved current \( \Psi^\mu \). Using (3.4), we see that

\[
D_\mu \Phi^\mu(Q) = \int_0^1 d\lambda \left( \left( Q_\nu(x, \lambda F, \lambda \partial F, \ldots, \lambda \partial^q F) \right) F^{\mu\nu}_{\ldots \mu} + \tilde{Q}_\nu(x, \lambda F, \lambda \partial F, \ldots, \lambda \partial^q F) * F^{\mu\nu}_{\ldots \mu}
\right)
\]

\[
+ \lambda \sum_{0 \leq k \leq q} (-1)^k D_{\sigma_1} \cdots D_{\sigma_k} \left( q_{\nu\alpha\beta}^{\sigma_1\cdots\sigma_k}(x, \lambda F, \lambda \partial F, \ldots, \lambda \partial^q F) F^{\alpha\beta}_{\sigma_1\cdots\sigma_k} \right) + \tilde{q}_{\nu\alpha\beta}^{\sigma_1\cdots\sigma_k}(x, \lambda F, \lambda \partial F, \ldots, \lambda \partial^q F) * F^{\alpha\beta}_{\sigma_1\cdots\sigma_k} \right).
\]

A repeated integration by parts yields the expression

\[
D_\mu \Phi^\mu(Q) = \int_0^1 d\lambda \left( \left( Q_\nu(x, \lambda F, \lambda \partial F, \ldots, \lambda \partial^q F) \right) F^{\mu\nu}_{\ldots \mu}
\right)
\]

\[
+ D_\nu \Upsilon^\nu,
\]

where \( \Upsilon^\nu \) vanishes on solutions of Maxwell’s equations. Now, an application of the fundamental theorem of calculus to the above integral gives

\[
D_\mu \Phi^\mu(Q) = Q_\nu F^{\mu\nu}_{\ldots \mu} + \tilde{Q}_\nu * F^{\mu\nu}_{\ldots \mu} + D_\nu \Upsilon^\nu \quad \text{on some } J^{p+1}(F).
\]

(3.12)

Thus by (3.2), the equation \( D_\mu \Phi^\mu(Q) = D_\mu (\Psi^\mu + \Upsilon^\mu) \) holds identically on \( J^{p+1}(F) \). Consequently, we have (see, e.g. [19,20])

\[
\Phi^\mu(Q) = \Psi^\mu + \Upsilon^\mu + D_\nu \Theta^{\mu\nu} \quad \text{on } J^p(F),
\]

(3.13)

for some functions \( \Theta^{\mu\nu} = -\Theta^{\nu\mu} \). Thus, since \( \Upsilon^\mu = 0 \) on \( R^{p-1}(F) \), we see that \( \Phi^\mu(Q) \) and \( \Psi^\mu \) are equivalent conserved currents. \( \square \)

We emphasize that, as a consequence of Proposition 3.1, one can completely classify all non-trivial local conservation laws of Maxwell’s equations by the following steps:

(i) classify up to equivalence all adjoint symmetries of Maxwell’s equations;

(ii) use formula (3.11) to construct the conserved currents arising from the equivalence classes of adjoint symmetries found in step (i);
(iii) determine all equivalence classes of the conserved currents found in step (ii).

We carry out step (i) in Sec. IV and steps (ii), (iii) in Sec. V. In step (iii) we first calculate a characteristic admitted by each conserved current in step (ii) and then we determine the equivalence classes of these characteristics; finally, we find all equivalence classes of conserved currents by employing the following result.

**Theorem 3.2.** There is a one-to-one correspondence between equivalence classes of conserved currents and equivalence classes of characteristics for Maxwell’s equations (2.1).

**Corollary 3.3.** Let \( P = (P_\mu, \tilde{P}_\nu) \) be an adjoint symmetry of Maxwell’s equations. If \( P \) is not equivalent to the characteristic \( Q \) of the conserved current \( \Phi^\mu(P) \), then \( P \) is not equivalent to the characteristic of any non-trivial conserved current of Maxwell’s equations.

To prove Theorem 3.2 we start with a preliminary result.

**Lemma 3.4.** Suppose \( H \) is a function defined on \( J^q(F) \) with the form

\[
H = \sum_{0 \leq p \leq q-1} (M_\mu^{\sigma_1\cdots\sigma_p} F_{\mu\nu_1\cdots\nu_p} + \tilde{M}_\mu^{\sigma_1\cdots\sigma_p} F_{\mu\nu_1\cdots\nu_p}) ,
\]

where \( M_\mu^{\sigma_1\cdots\sigma_p} = M_\mu^{(\sigma_1\cdots\sigma_p)} \), \( \tilde{M}_\mu^{\sigma_1\cdots\sigma_p} = \tilde{M}_\mu^{(\sigma_1\cdots\sigma_p)} \), \( p \geq 0 \), are some differential functions on \( J^q(F) \). If \( H \) vanishes identically on \( J^q(F) \), then, for \( p = 0 \),

\[
M_\mu = 0, \quad \tilde{M}_\mu = 0 \quad \text{on } R^{q-1}(F), \tag{3.15}
\]

and there are functions \( N^{\sigma_1\cdots\sigma_p} = N^{(\sigma_1\cdots\sigma_p)} \), \( \tilde{N}^{\sigma_1\cdots\sigma_p} = \tilde{N}^{(\sigma_1\cdots\sigma_p)} \) on \( J^q(F) \) so that for all \( 1 \leq p \leq q-1 \),

\[
M_\mu^{\sigma_1\cdots\sigma_p} = \delta_\mu^{(\sigma_1} N^{\sigma_2\cdots\sigma_p)} , \quad \tilde{M}_\mu^{\sigma_1\cdots\sigma_p} = \delta_\mu^{(\sigma_1} \tilde{N}^{\sigma_2\cdots\sigma_p)} \quad \text{on } R^{q-1}(F). \tag{3.16}
\]

**Proof of Lemma 3.4.**

First note that we only need to prove the first equation in both (3.15) and (3.16) since the second equation follows from the first one by duality.

Apply the partial derivative operator \( \partial_F^{\alpha_1\beta_1\cdots\gamma_p} \) to (3.14) and restrict the resulting expression to \( R^{q-1}(F) \) to obtain
\[ M^{\alpha(\gamma_1 \cdots \gamma_{p-1})\eta_\nu\beta} - M^{\beta(\gamma_1 \cdots \gamma_{p-1})\eta_\nu\alpha} + \tilde{M}^{\nu(\gamma_1 \cdots \gamma_{p-1})\epsilon_\sigma\gamma_\nu \alpha \beta} = 0 \quad \text{on } \mathcal{R}^{q-1}(F). \] (3.17)

When \( p = 1 \), equation (3.17) immediately shows that \( M_{\mu} = 0 \) on \( \mathcal{R}^{q-1}(F) \). Now suppose that \( p \geq 2 \). In equation (3.17), on one hand, symmetrize over the indices \( \alpha, \gamma_1, \ldots, \gamma_p \) and, on the other hand, contract over the indices \( \beta, \gamma_p \) and then symmetrize over \( \alpha, \gamma_1, \ldots, \gamma_{p-1} \) to see that

\[ M^{(\alpha \gamma_1 \cdots \gamma_{p-1})\eta_\nu\beta} = M^{\beta(\gamma_1 \cdots \gamma_{p-1})\eta_\nu\alpha}, \quad M^{(\alpha \gamma_1 \cdots \gamma_{p-1})} = \frac{p-1}{p+2} M_\tau^{\tau(\gamma_1 \cdots \gamma_{p-1})\eta_\nu\alpha}, \] (3.18)
on \( \mathcal{R}^{q-1}(F) \). By combining these equations, we have that

\[ M^{\beta(\gamma_1 \cdots \gamma_{p-1})\eta_\nu\alpha} - \frac{p-1}{p+2} M_\tau^{\tau(\gamma_1 \cdots \gamma_{p-1})\eta_\nu\alpha} = 0 \quad \text{on } \mathcal{R}^{q-1}(F). \] (3.19)

Next choose a covector \( X_\gamma \in T^*(M^4) \). Then equation (3.19) yields

\[ (M^{\beta \gamma_1 \cdots \gamma_{p-1}} - \frac{p-1}{p+2} M_\tau^{\tau \gamma_1 \cdots \gamma_{p-1}}) X_\gamma_1 \cdots X_\gamma_{p-1} = 0 \quad \text{on } \mathcal{R}^{q-1}(F), \] (3.20)

whenever \( X_\gamma X^\gamma \neq 0 \). By continuity, equation (3.20) holds for all \( X_\gamma \in T^*(M^4) \) and thus

\[ M^{\beta \gamma_1 \cdots \gamma_{p-1}} = \frac{p-1}{p+2} M_\tau^{\tau \gamma_1 \cdots \gamma_{p-1}} \quad \text{on } \mathcal{R}^{q-1}(F). \] (3.21)

This proves the first equation in (3.16) with \( N^{\sigma_1 \cdots \sigma_{p-2}} = \frac{p-1}{p+2} M_\tau^{\tau \sigma_1 \cdots \sigma_{p-2}} \). \( \square \)

**Proof of Theorem 3.2.**

By Proposition 3.1, a conserved current with a trivial characteristic is itself trivial. Conversely, suppose that \( \Psi^\mu \) is a trivial conserved current in the characteristic form (3.2). Since \( \Psi^\mu = D_\nu \Theta^{\mu \nu} \) on some \( \mathcal{R}^q(F) \), there are functions \( R_\nu^{\tau_1 \cdots \tau_p}, \tilde{R}_\nu^{\tau_1 \cdots \tau_p} \) on \( J^{q+1}(F) \) so that

\[ \Psi^\mu = D_\nu \Theta^{\mu \nu} + \sum_{0 \leq p \leq q} (R_\nu^{\mu \sigma_1 \cdots \sigma_p} F^{\tau \nu, \tau \sigma_1 \cdots \sigma_p} + \tilde{R}_\nu^{\mu \sigma_1 \cdots \sigma_p} F^{\tau \nu, \tau \sigma_1 \cdots \sigma_p}) \quad \text{on } J^{q+1}(F). \] (3.22)

Note that we can manipulate the term \( R_\nu^{\mu \sigma_1 \cdots \sigma_p} F^{\tau \nu, \tau \sigma_1 \cdots \sigma_p} \) by using the identities

\[ (R_\nu^{\mu \sigma_1 \sigma_2 \cdots \sigma_p} - R_\nu^{(\mu \sigma_1 \sigma_2 \cdots \sigma_p)}) F^{\tau \nu, \tau \sigma_1 \sigma_2 \cdots \sigma_p} = \frac{2p}{p+1} R_\nu^{[\mu \sigma_1] \sigma_2 \cdots \sigma_p} F^{\tau \nu, \tau \sigma_1 \sigma_2 \cdots \sigma_p}, \]

\[ = D_{\sigma_1} \left( \frac{2p}{p+1} R_\nu^{[\mu \sigma_1] \sigma_2 \cdots \sigma_p} F^{\tau \nu, \tau \sigma_2 \cdots \sigma_p} \right) - \left( \frac{2p}{p+1} D_{\sigma_1} R_\nu^{[\mu \sigma_1] \sigma_2 \cdots \sigma_p} \right) F^{\tau \nu, \tau \sigma_2 \cdots \sigma_p}. \] (3.23)
The two terms in the final equality in (3.23) can be incorporated, respectively, into the terms $D_{\mu} \Theta^{\mu\nu}$ and $R_{\nu}^{\mu_1 \cdots \mu_{p-1}} F_{\tau \sigma_1 \cdots \sigma_p}$ in (3.22). Hence it is clear that by proceeding inductively we can arrange the coefficient functions $R_{\nu}^{\mu_1 \cdots \mu_{p-1}}$, $\tilde{R}_{\nu}^{\mu_1 \cdots \mu_{p-1}}$ in (3.22) to be symmetric in their upper indices:

$$R_{\nu}^{\mu_1 \cdots \mu_{p-1}} = R_{\nu}^{(\mu_1 \cdots \mu_{p-1})}, \quad \tilde{R}_{\nu}^{\mu_1 \cdots \mu_{p-1}} = \tilde{R}_{\nu}^{(\mu_1 \cdots \mu_{p-1})}, \quad 1 \leq p \leq q. \quad (3.24)$$

Now, since $\Psi^{\mu}$ satisfies (3.2), we have by (3.22) that

$$Q_{\nu} F_{\mu \nu}^{\mu_1} + \tilde{Q}_{\nu} * F_{\mu \nu}^{\mu_1} = \sum_{0 \leq p \leq q} \left( (D_{\mu} R_{\nu}^{\mu_1 \cdots \mu_{p-1}} + R_{\nu}^{\sigma_1 \cdots \sigma_{p-1}}) F_{\tau \sigma_1 \cdots \sigma_{p-1}} + (D_{\mu} \tilde{R}_{\nu}^{\mu_1 \cdots \mu_{p-1}} + \tilde{R}_{\nu}^{\sigma_1 \cdots \sigma_{p-1}}) * F_{\tau \sigma_1 \cdots \sigma_{p-1}} \right) \quad (3.25)$$
on $J^{q+2}(F)$, where

$$R_{\nu}^{\tau_1 \cdots \tau_p} = 0, \quad \tilde{R}_{\nu}^{\tau_1 \cdots \tau_p} = 0, \quad \text{if } p \geq q + 1 \text{ or } p = 0.$$

Then Lemma 3.4 together with equations (3.24) and (3.25) implies that there are functions $N^{\sigma_1 \cdots \sigma_{p-1}}$, $\tilde{N}^{\sigma_1 \cdots \sigma_{p-1}}$ so that on $R^{q+1}(F)$,

$$D_{\mu} R_{\nu}^{\mu} = Q_{\nu}, \quad D_{\mu} R_{\nu}^{\mu_1 \cdots \mu_{p-1}} + R_{\nu}^{\sigma_1 \cdots \sigma_{p-1}} = \delta^{(\sigma_1 \cdots \sigma_{p-1})} N^{\sigma_1 \cdots \sigma_{p-1}}, \quad 1 \leq p \leq q, \quad (3.26)$$

and similarly,

$$D_{\mu} \tilde{R}_{\nu}^{\mu} = \tilde{Q}_{\nu}, \quad D_{\mu} \tilde{R}_{\nu}^{\mu_1 \cdots \mu_{p-1}} + \tilde{R}_{\nu}^{\sigma_1 \cdots \sigma_{p-1}} = \delta^{(\sigma_1 \cdots \sigma_{p-1})} \tilde{N}^{\sigma_1 \cdots \sigma_{p-1}}, \quad 1 \leq p \leq q. \quad (3.27)$$

It is easy to see that these equations lead to

$$Q_{\mu} = D_{\mu} \left( \sum_{0 \leq p \leq q-1} (-1)^p D_{\sigma_1} \cdots D_{\sigma_p} N^{\sigma_1 \cdots \sigma_p} \right),$$

$$\tilde{Q}_{\mu} = D_{\mu} \left( \sum_{0 \leq p \leq q-1} (-1)^p D_{\sigma_1} \cdots D_{\sigma_p} \tilde{N}^{\sigma_1 \cdots \sigma_p} \right), \quad \text{on } R^{q+1}(F).$$

Thus $Q = (Q_{\mu}, \tilde{Q}_{\mu})$ is a trivial characteristic. \qed
IV. CLASSIFICATION OF ADJOINT SYMMETRIES

We solve the adjoint symmetry equations (3.7) by spinorial methods. Fix a complex null tetrad basis $e_{\mu}^{AA'}$ for $\eta_{\mu\nu}$, satisfying $\eta^{\mu\nu} e_\mu^{AA'} e_\nu^{BB'} = \epsilon^{AB} e_{AA'}$, with the inverse $e_\mu^{AA'}$, where $\epsilon^{AB}$ is the spin metric. In spinor form Maxwell’s equations (2.1) become

$$\partial^B A \phi_{AA'}(x) = 0, \quad \partial^B A \bar{\phi}_{A'B'}(x) = 0,$$

where $\phi_{AB}$ is the electromagnetic spinor defined by $F_{\mu\nu,\sigma} e_\mu^{AA'} e_\nu^{BB'} = \phi_{AB} e_{A'B'} + \bar{\phi}_{A'B'} e_{AB}$ and $\partial_{AA'} = e_\mu^{AA'} \partial_\mu$ is the spinorial derivative operator. The duality symmetry (2.2) of Maxwell’s equations corresponds to the transformation

$$\phi_{AB} \rightarrow -i \bar{\phi}_{AB}, \quad \bar{\phi}_{A'B'} \rightarrow i \phi_{A'B'}.$$

We let

$$\phi_{AB, C_1 \cdots C_p} = \frac{1}{2} F_{\mu\nu, \sigma_1 \cdots \sigma_p} e_\mu^{AA'} e_\nu^{BB'} e^{\sigma_1 C_1} \cdots e^{\sigma_p C_p},$$

$$\bar{\phi}_{A'B', C_1 \cdots C_p} = \frac{1}{2} F_{\mu\nu, \sigma_1 \cdots \sigma_p} e_\mu^{AA'} e_\nu^{BB'} e^{\sigma_1 C_1} \cdots e^{\sigma_p C_p}$$

denote the spinor components of the jet variables $F_{\mu\nu, \sigma_1 \cdots \sigma_p}$, $p \geq 0$. We write

$$\phi_{ABC, C_1 \cdots C_p} = \phi_{(AB, C_1 \cdots C_p)}, \quad \bar{\phi}_{A'B'C, C_1 \cdots C_p} = \bar{\phi}_{(A'B', C_1 \cdots C_p)}$$

for the symmetric derivative variables, using the convention that (4.3) to (4.5) with $p = 0$ stand for $\phi_{AB}, \bar{\phi}_{A'B'}$. Recall that the symmetric spinor variables $\phi_{ABC, C_1 \cdots C_p}$ together with the independent variables $x_{CC'} = e_\mu^{CC'} x^\mu$ form a coordinate system on the space of solutions of Maxwell’s equations (see [13]). Thus, $J^q(F)$ and $R^{q-1}(F) \subset J^q(F)$ admit the spinor coordinates

$$J^q(F) = \{(x_{CC'}, \phi_{AB, C_1} \cdots \phi_{ABC, C_1 \cdots C_p})\},$$

and

$$R^{q-1}(F) = \{(x_{CC'}, \phi_{AB, C_1} \cdots \phi_{ABC, C_1 \cdots C_p})\}.$$
One can easily verify that Maxwell’s equations are locally solvable, that is, for every $q$-jet $(x^{CC'}, \phi_0^{AB}, \ldots, \phi_{oABC\cdots C_q}^{C_1\cdots C_p}) \in R^{q-1}(F)$, $q \geq 1$, there is a solution $\varphi_{AB} = \varphi_{AB}(x)$ of (4.1) such that $\partial^{C_1\cdots C_p}_{C_1\cdots C_p} \varphi_{AB}(x_o) = \phi_{oABC\cdots C_q}$, $0 \leq p \leq q$. This is a consequence of the fact that $\phi_{AB}, \bar{\phi}_{A'B'}$ and their symmetrized derivatives form an “exact set of fields” for Maxwell’s equations, as discussed by Penrose [13].

Let $\partial^{AB C_1\cdots C_p}_{C_1\cdots C_p}$ be the partial differential operators defined by

$$
\partial^{AB C_1\cdots C_p}_{C_1\cdots C_p}(\phi_{DE,F_1\cdots F_q}) = \begin{cases} 
\epsilon (D (A_e E) B) & \epsilon (F_1 (C_1 \cdots C_p) F_q) \epsilon (C_1 (F_1' \cdots C_p' F_q')), \quad \text{if } p = q; \\
0, & \text{if } p \neq q;
\end{cases} \tag{4.8}
$$

and let $\bar{\phi}^{A'B' C_1\cdots C_p}_{C_1\cdots C_p}$ be the complex conjugate operator

$$
\bar{\phi}^{A'B' C_1\cdots C_p}_{C_1\cdots C_p} = \bar{\partial}^{AB C_1\cdots C_p}_{C_1\cdots C_p}. \tag{4.10}
$$

Let $D^A_A$ be the spinorial total derivative operator on $J^\infty(F)$ given by

$$
D^A_A = \epsilon^{\mu A}_A D_\mu = \partial^A_A + \sum_{q \geq 0} (\bar{\phi}_{DE,F_1\cdots F_q} \partial^{DE,F_1\cdots F_q}_{F_1'\cdots F_q'} + \phi^{D'E',A'B' F_1'\cdots F_q'}_{AF_1\cdots F_q} \partial^{D'E',A'B' F_1'\cdots F_q'}_{F_1\cdots F_q}). \tag{4.11}
$$

Note, due to the commutativity of partial derivatives, $D^A_A$ satisfies the identities

$$
D^{A'}_{(A} D_{A')} = 0, \quad D^{A}_{(A} D_{A')} = 0. \tag{4.12}
$$

In spinor form the adjoint symmetry equations (3.7) simply reduce to

$$
D^B_{(A} P_{B'A')} = 0 \quad \text{on } R^r(F), \tag{4.13}
$$

where we have written

$$
P^{A'A'} = P^\mu \epsilon^\mu_{A'A'} + i \tilde{P}^\mu \epsilon^\mu_{A'A'} \quad \text{on } J^r(F). \tag{4.14}
$$

We refer to the solutions $P_{A'A'}$ of equation (4.13) as spinorial adjoint symmetries of order $r$. Note that adjoint gauge symmetries (3.10) correspond here to the gradient solutions

$$
P_{A'A'} = D^A_A \chi, \tag{4.15}
$$
which satisfy the curl equation $D^A_{(C}P_{A')A'} = 0$.

Our classification of adjoint symmetries $P_{AA'}$ makes use of Killing spinors. Recall that Killing spinors $\kappa^A_{A_1\cdots A_k} = \kappa^{(A_1\cdots A_k)}(x)$ of type $(k, l)$ are the solutions of the conformally invariant equations \[13\]
\[\partial^{(B}_{(B'}\kappa^{A_1\cdots A_k)}_{A_1'\cdots A_k')} = 0. \quad (4.16)\]

For $k = l = 1$, this equation is the conformal Killing vector equation \[13\], and for $k = 0, l = 2$, the self-dual conformal Killing-Yano equation \[14\]. Hence a type $(1, 1)$ Killing spinor $\kappa^A_A$ corresponds to a complex conformal Killing vector $\xi^\mu = e^{A'}_A\kappa^A_A$, while a type $(0, 2)$ Killing spinor $\kappa_{A'B'}$ corresponds to a complex conformal Killing-Yano tensor $Y^{\mu\nu} = e^{A'}_Ae^{B'}_{A'}\kappa_{A'B'}$, satisfying the self-duality condition $*Y^{\mu\nu} = iY^{\mu\nu}$. These are polynomial expressions in the spacetime coordinates $x^{CC'}$, specifically,

$$
\xi^A_{A'} = \alpha_1^A_A + \alpha_2^A_A + \alpha_3^{A'B'}x^{AB'} + \alpha_4^{AB}x_{A'B'} + \alpha_5^{B'B}x^{AB}x_{A'B'},
$$

$$
Y_{A'B'} = \alpha_6^{A'B'} + \alpha_7^{A'A}x^{A}_{A'B'} + \alpha_8^{AB}x^{A}_{A'}x^{B'},
$$

where $\alpha_1^{AA'}, \alpha_2, \alpha_3^{A'B'}, \alpha_4^{AB}, \alpha_5^{BB'}, \alpha_6^{AA'}, \alpha_7^{AA'}, \alpha_8^{AB}$ are constant symmetric spinors. The following Lemma, which will be pivotal in our classification analysis, is a special case of the well-known factorization property \[13\] of Killing spinors in Minkowski space $M^4$.

**Lemma 4.1.** A symmetric spinor field $\xi^A_{A_1\cdots A_k}$ is a Killing spinor of type $(k, k)$ if and only if it can be expressed as a sum of symmetrized products of $k$ Killing spinors of type $(1, 1)$. A symmetric spinor field $Y^{A_1\cdots A_k}_{A_1'\cdots A_{k+4}}$ is a Killing spinor of type $(k, k + 4)$ if and only if it can be expressed as a sum of symmetrized products of two Killing spinors of type $(0, 2)$ and $k$ Killing spinors of type $(1, 1)$. There are $\frac{1}{12}(k + 1)^2(k + 2)^2(2k + 3)$ linearly independent Killing spinors of type $(k, k)$, and $\frac{1}{12}(k + 1)(k + 2)(k + 5)(k + 6)(2k + 7)$ linearly independent Killing spinors of type $(k, k + 4)$, over the complex numbers.

Let $\zeta^{CC'}$ be a Killing spinor and define

$$
\pi_{\zeta AB} = \frac{1}{2}e^{\mu}_{AA'}e^{\nu}_{B}(\zeta^\tau F_{\mu\nu} + 2(\partial_{[\mu}{\zeta^\tau}F_{\nu]})) = \zeta^{CC'}\phi_{AB,CC'} + \partial_{CC'}(A\zeta^{CC'}\phi_{B)C'}, \quad (4.18)
$$
which stands for the spinor components of the Lie derivative (2.15) of $F_{\mu\nu}$ with respect to 
$\zeta^\tau = e^{\tau}_{CC'}\zeta^{CC'}$. As a consequence of the linearity and conformal invariance of Maxwell’s 
equations (4.1), one easily sees that, for any solution $\phi_{AB}(x)$ of (4.1), $\pi_{\zeta AB}(x)$ is also a 
solution. We remark that, geometrically, $\pi_{\zeta AB}$ represents a conformally weighted Lie derivative 
of $\phi_{AB}$ (see [13]).

Given any adjoint symmetry of Maxwell’s equations, we can obtain a family of higher 
order adjoint symmetries by the action of conformal symmetries on $\phi_{AB}$. Let 

$$X_\zeta = (\pi_{\zeta AB})\partial_\phi^{AB} + (\bar{\pi}_{\zeta A'B'})\partial_{\bar{\phi}}^{A'B'},$$

(4.19)

which is the conformal symmetry in evolutionary form [1], defined on $J^1(F)$, and let $pr X_\zeta$ 
denote the prolongation of $X_\zeta$ to $J^\infty(F)$. Then, if $P_{AA'}$ is any adjoint symmetry of order $r$, 
the linearity of the adjoint symmetry equation (4.13) implies that $pr X_\zeta P_{AA'}$ is an adjoint 
symmetry of order $r+1$. This can be iterated any number of times using conformal symmetry 
generators $pr X_{\zeta_1}$, $pr X_{\zeta_2}$, etc. Note that $[pr X_{\zeta_1}, pr X_{\zeta_2}] = pr X_{[\zeta_1, \zeta_2]}$, where 
$[\zeta_1, \zeta_2]$ denotes the commutator of conformal Killing vectors $\zeta_1, \zeta_2$, which is again a conformal Killing vector.

**Proposition 4.2** Let $\xi^{AA'}, \zeta_1^{AA'}, \ldots, \zeta_p^{AA'}$ and $\kappa_{A'B'C'D'}$ be Killing spinors. Then the spinor 
functions

$$U_{AA'}(\phi; \xi) = \xi^{B}_{A'}\phi_{AB},$$

(4.20)

$$V_{AA'}(\bar{\phi}; \kappa) = \kappa_{A'B'C'D'}\bar{\phi}^{A'B'C'D'},$$

(4.21)

are adjoint symmetries of order $q = 0$ and $q = 1$, respectively. Hence their extensions

$$U_{AA'}[\xi, \zeta_1, \ldots, \zeta_p] = \frac{1}{p!} \sum_{s \in S_p} pr X_{\zeta_s(1)} \cdots pr X_{\zeta_s(p)} U_{AA'}(\phi; \xi),$$

(4.22)

$$V_{AA'}[\kappa, \zeta_1, \ldots, \zeta_p] = \frac{1}{p!} \sum_{s \in S_p} pr X_{\zeta_s(1)} \cdots pr X_{\zeta_s(p)} V_{AA'}(\bar{\phi}; \kappa),$$

(4.23)

are adjoint symmetries of order $q = p$ and $q = p + 1$, respectively, where $S_p$ denotes the 
symmetric group on the index set $\{1, \ldots, p\}$. Furthermore, (4.22) and (4.23) are equivalent 
to non-trivial adjoint symmetries with highest order terms given by, respectively,
\[ (-1)^p \xi^B \xi^C_1 \cdots \xi^C_p \phi^{C_1 \cdots C_p}_B (E_1 \cdots E_p) = 0 \]

whenever \( \xi, \zeta \) are non-zero.

The proof of Proposition 4.2 is based on straightforward computations which will be omitted.

Hereafter we use the notation in (4.22) and (4.23) with \( p = 0 \) to refer to (4.20) and (4.21).

We now let \( \omega_{AA'}(x) \) be a spinor field satisfying

\[ \partial^B (A' \omega_{B'}) = 0, \]

and we call \( W_{AA'}[\omega] = \omega_{AA'} \) an elementary adjoint symmetry of Maxwell’s equations (4.11).

**Theorem 4.3** Every adjoint symmetry \( P_{AA'} \) of order \( r \) of Maxwell’s equations (4.11) is equivalent to a sum of an elementary adjoint symmetry \( W_{AA'}[\omega] \) and a linear adjoint symmetry given by a sum of \( U_{AA}[\xi, \zeta_1, \ldots, \zeta_p] \) and \( V_{AA'}[\kappa, \varrho_1, \ldots, \varrho_q], 0 \leq p \leq r, 0 \leq q \leq r - 1, \)

involving type (1, 1) Killing spinors \( \xi, \zeta_i, \varrho_j \) and type (0, 4) Killing spinors \( \kappa \) for each \( p, q \).

**Proof.**

By replacing \( P_{AA'} \) with an equivalent adjoint symmetry we can assume that \( P_{AA'} \) depends only on \( x_{CC'} \) and \( \phi^{C_1 \cdots C_p}_{ABC_1 \cdots C_p} \), \( 0 \leq p \leq r \), i.e., \( P_{AA'} \) is a function in the coordinates of \( R^{r-1}(F) \subset J^r(F) \).

Let \( \varphi_{AB} = \varphi_{AB}(x) \) be a solution of Maxwell’s equations. Define the linearization operator

\[ L_\varphi = \sum_{p \geq 0} (\varphi_{ABK_1 \cdots K_p} \partial_{\phi}^{K_1 \cdots K_p} + \varphi_{A'B'K_1 \cdots K_p} \partial_{\phi}^{A'B'K_1 \cdots K_p}), \]

where \( \varphi_{ABK_1 \cdots K_p} = \partial_{K_1} \cdots \partial_{K_p} \varphi_{AB}, \varphi_{A'B'K_1 \cdots K_p} = \partial_{K_1} \cdots \partial_{K_p} \varphi_{A'B'K_1 \cdots K_p} \), which are each symmetric spinor fields. Note that, for any adjoint symmetry \( P_{AA'} \) of order \( r \), the linearization \( P_{\varphi_{AA'}} = L_\varphi P_{AA'} \) again satisfies the adjoint symmetry equations

\[ D^B_{(A'} P_{\varphi_{B')}B} = 0 \quad \text{on } R^r(F). \]  

By the local solvability of Maxwell’s equations, the coefficients of \( \varphi_{IJK_1 \cdots K_p}, \varphi_{IJK_1 \cdots K_p} \), \( 0 \leq p \leq r+1 \), in equation (4.27) must vanish. Thus, we find that for \( p = r+1 \) the coefficients yield the equations
A straightforward analysis of the highest order terms in (4.35) shows that the functions 

\[ p_{B'}^{(AJK_1\ldots K_{\nu})} = 0, \quad \bar{p}_{B'}^{(A'I'J'K'_1\ldots K'_{\nu})} = 0, \]  

(4.28)

and for \( 1 \leq p \leq r \), the coefficients yield the equations

\[ D_{(A'p_{B'})B}^{B} \frac{IJK_1\ldots K_p}{K_1'\ldots K_{p-1}'} - \epsilon_{(K_p')}(A'P_k') = 0 \quad \text{on} \ R^p(F), \]  

(4.29)

\[ D_{(A'\bar{p}_{B'})B}^{B} \frac{I'J'K'_1\ldots K'_p}{K_1'\ldots K_{p-1}'} + \epsilon_{(A')} \epsilon_{(K_p')} = 0 \quad \text{on} \ R^p(F), \]  

(4.30)

while for \( p = 0 \) the coefficients yield the equations

\[ D_{(A'p_{B'})B}^{B} \frac{IJ}{J'K'_1\ldots K'_p} = 0, \quad D_{(A'\bar{p}_{B'})B}^{B} \frac{I'J'}{J'K'_1\ldots K'_p} = 0 \quad \text{on} \ R^p(F), \]  

(4.31)

where we have written

\[ p_{AA'} \frac{IJK_1\ldots K_p}{K_1'\ldots K_{p-1}'} = \partial_{\phi} \frac{IJK_1\ldots K_p}{K_1'\ldots K_{p-1}'} P_{AA'}, \quad \bar{p}_{AA'} \frac{I'J'K'_1\ldots K'_p}{K_1'\ldots K_{p-1}'} = \partial_{\phi} \frac{I'J'K'_1\ldots K'_p}{K_1'\ldots K_{p-1}'} P_{AA'}, \quad 0 \leq p \leq r. \]  

(4.32)

Note that \( p_{AA'} \frac{IJK_1\ldots K_p}{K_1'\ldots K_{p-1}'} \), \( \bar{p}_{AA'} \frac{I'J'K'_1\ldots K'_p}{K_1'\ldots K_{p-1}'} \) are spinor functions of order \( r \) in the coordinates of \( R^{r-1}(F) \subset J^r(F) \) and are symmetric separately in their primed and unprimed indices excluding \( A, A' \).

By reduction of \( p_{AA'} \frac{IJK_1\ldots K_p}{K_1'\ldots K_{p-1}'} \), \( \bar{p}_{AA'} \frac{I'J'K'_1\ldots K'_p}{K_1'\ldots K_{p-1}'} \) into symmetric components, we find that the solution to the equations in (4.28) is

\[ p_{AA'} \frac{IJK_1\ldots K_r}{K_1'\ldots K_{r-1}'} = \epsilon_A \frac{IJK_1\ldots K_r}{K_1'\ldots K_{r-1}'} + \epsilon_A \frac{IJK_1\ldots K_r}{K_1'\ldots K_{r-1}'} P_{AA'}, \]  

(4.33)

\[ \bar{p}_{AA'} \frac{I'J'K'_1\ldots K'_r}{K_1'\ldots K_{r-1}'} = \epsilon_A \frac{I'J'K'_1\ldots K'_r}{K_1'\ldots K_{r-1}'} + \epsilon_A \frac{I'J'K'_1\ldots K'_r}{K_1'\ldots K_{r-1}'} \bar{P}_{AA'}, \]  

(4.34)

where \( p_{AA'} \frac{JK_1\ldots K_r}{K_1'\ldots K_{r-1}'} \), \( \bar{p}_{AA'} \frac{JK_1\ldots K_r}{K_1'\ldots K_{r-1}'} \) are symmetric spinor functions of order \( r \) depending only on the coordinates of \( R^{r-1}(F) \subset J^r(F) \). Next we substitute the expressions (4.33) and (4.34) into equations (4.29) and (4.30) with \( p = r \) and symmetrize over primed indices in the resulting expressions. This yields the equations

\[ D_{(A)BJ'}^{(I)} \frac{JK_1\ldots K_r}{K_1'\ldots K_{r-1}'} = 0, \quad D_{(K)B_1}^{(A')} \frac{B'J'K'_1\ldots K'_r}{K_1'\ldots K_{r-1}'} = 0 \quad \text{on} \ R^r(F). \]  

(4.35)

A straightforward analysis of the highest order terms in (4.35) shows that the functions

\[ p_{BJ'} \frac{JK_1\ldots K_r}{K_1'\ldots K_{r-1}'} \], \( \bar{p}_{BJ'} \frac{B'J'K'_1\ldots K'_r}{K_1'\ldots K_{r-1}'} \) must only depend on \( x^{CC'} \) and, consequently, are Killing spinors of type \( (r + 1, r + 1) \) and \( (r - 1, r + 3) \), respectively.
From equation (4.33) we have that

\[
P_{\varphi AA'} = p_{A'K_1\cdots K_r}^{J K_1 \cdots K_r} K_1' \cdots K_r' + \epsilon_{AK_1} \tilde{p}_{1 A'}^{J K_1 \cdots K_r} K_1' \cdots K_r' + \epsilon_{AK_1} \tilde{p}_{2 K_2' \cdots K_r'}^{J K_1' \cdots K_r'} K_1' \cdots K_r' + H_{AA'},
\]

(4.36)

where \(H_{AA'}\) involves the derivatives of \(\varphi_{IJ}, \tilde{\varphi}_{I'J'}\) of order at most \(r - 1\). Now by Lemma 4.1, \(p_{1 A' K_1' \cdots K_r'}\) and \(\tilde{p}_{1 A' K_1' \cdots K_r'}\) are respectively sums of symmetrized products of \(r + 1\) type \((1, 1)\) Killing spinors, \(\xi^{(A_1 \xi_{1 K_1'} \cdots \xi_{r K_r'})}\), and of a type \((0, 4)\) Killing spinor and \(r - 1\) type \((1, 1)\) Killing spinors, \(\kappa^{(A'J'K_1' K_2' \cdots K_r')_{1}}\). Consequently, by (4.24) in Proposition 4.2, there is an adjoint symmetry \(\tilde{P}_{AA'}\) equivalent to a sum of adjoint symmetries \(U_{AA'}[\xi; \xi_1, \ldots, \xi_r] \), \(V_{AA'}[\kappa; \varphi_1, \ldots, \varphi_{r-1}]\) such that the terms involving derivatives of order \(r\) of the spinor functions \(\varphi_{IJ}, \tilde{\varphi}_{I'J'}\) in the linearization \(L_{\varphi} \tilde{P}_{AA'}\) agree with the first two terms on the right-hand side of equation (4.33). Thus, after subtracting \(\tilde{P}_{AA'}\) from \(P_{AA'}\) we get an adjoint symmetry, which we again denote by \(P_{AA'}\), with the property that

\[
P_{\varphi AA'} = \epsilon_{A'K_1'} p_{2 K_2' \cdots K_r'}^{J K_1' \cdots K_r'} \varphi_{A'K_1' \cdots K_r'} + \epsilon_{AK_1} \tilde{p}_{2 K_2' \cdots K_r'}^{J K_1' \cdots K_r'} \varphi_{A'K_1' \cdots K_r'} + \tilde{H}_{AA'},
\]

(4.37)

where \(\tilde{H}_{AA'}\) involves the derivatives of \(\varphi_{IJ}, \tilde{\varphi}_{I'J'}\) of order at most \(r - 1\).

Now, in equation (4.37) observe that \(\varphi_{A'K_1' \cdots K_r'} = \partial_{A'} \varphi_{JK_1K_2 \cdots K_r}, \tilde{\varphi}_{A'K_1' \cdots K_r'} = \partial_{A'} \tilde{\varphi}_{J'K_1'K_2' \cdots K_r'}\). Consequently, our next goal is to show that there is a differential function \(\chi\) of \(x_{CC'}^p\) and \(\Phi_{ABC_1 \cdots C_p}^{C_1' \cdots C_p'}\), \(0 \leq p \leq r - 1\), so that the terms involving derivatives of order \(r\) of \(\varphi_{IJ}, \tilde{\varphi}_{I'J'}\) in \(L_{\varphi} D_{AA'} \chi\) and in (4.37) agree. For this to hold, it suffices to show that the functions \(p_2 \stackrel{IJK_1 \cdots K_{r-1}}{K_1' \cdots K_{r-1}}, \tilde{p}_2 \stackrel{IJ'K_1' \cdots K_{r-1}'}{K_1' \cdots K_{r-1}}\) are of order \(r - 1\) and satisfy the integrability conditions

\[
\partial_{\Phi}^{PQ} R_{1 \cdots R_{r-1}}^{IJK_1 \cdots K_{r-1}} = \partial_{\Phi}^{IJK_1 \cdots K_{r-1}} R_{1 \cdots R_{r-1}}^{PQ}
\]

(4.38)

\[
\partial_{\Phi}^{PQ} R_{1 \cdots R_{r-1}}^{IJK_1 \cdots K_{r-1}} = \partial_{\Phi}^{IJK_1 \cdots K_{r-1}} R_{1 \cdots R_{r-1}}^{PQ}
\]

(4.39)

\[
\partial_{\Phi}^{PQ} R_{1 \cdots R_{r-1}}^{IJK_1 \cdots K_{r-1}} = \partial_{\Phi}^{IJK_1 \cdots K_{r-1}} R_{1 \cdots R_{r-1}}^{PQ}
\]

(4.40)

In fact, if equations (4.38), (4.39), (4.40) hold, then one can verify that a function \(\chi\) with the desired properties is given by
\[
\chi = \int_0^1 \left( p_2 J K_1 \cdots K_r \frac{I}{K_1' \cdots K_r'}(x, \phi, \partial \phi, \ldots, \partial r^{-2} \phi, \lambda \partial r^{-1} \phi)\phi_{J K_1 \cdots K_r}, K_1' \cdots K_r' + p_2 K_1' \cdots K_r' \frac{I}{K_1' \cdots K_r'}(x, \phi, \partial \phi, \ldots, \partial r^{-2} \phi, \lambda \partial r^{-1} \phi)\phi_{J K_1 \cdots K_r}, K_1' \cdots K_r' \right) d\lambda,
\]

where $\partial^p \phi$ stands collectively for the variables $\phi_{A B C_1 \cdots C_p}, P \geq 0$.

The proofs of conditions (4.38), (4.39), (4.40) are based on like computations. We will therefore prove the first one and omit the proofs of the others.

From equation (4.37) we have
\[
p_{A A'} \frac{I}{K_1' \cdots K_r'} = \epsilon_A^I \epsilon_A^A \frac{I}{K_1' \cdots K_r'}.
\]
Now substitute (4.41) into equation (4.29), contract on the indices $B', K_1'$, and symmetrize over the indices $A', K_2', \ldots, K_r'$ to obtain
\[
D \frac{I}{(A P_2 J K_1 K_2 \cdots K_r)} = -p_{(A'} \frac{I}{A'' \frac{I}{K_1' K_2' \cdots K_r'}} \text{ on } R'(F).
\]

The terms of highest order in (4.42) arise from $p_2 J K_1 K_2 \cdots K_r \frac{I}{K_1' \cdots K_r'}$, which is of order at most $r$. Thus an application of $\partial^p \phi \frac{P Q R_1 \cdots R_{r+1}}{R_1' \cdots R_{r+1}'}$ and its complex conjugate to (4.42) yields
\[
\partial^p \phi \frac{P Q R_1 \cdots R_r}{R_1' \cdots R_r'} \frac{I}{R_{r+1}' \cdots R_{r+1}'} = 0, \tag{4.43}
\]
\[
\partial^p \phi \frac{P' Q' R_1' \cdots R_{r}' \cdots R_1'}{R_{r+1}' \cdots R_{r+1}'} \frac{I}{(K_1' \cdots K_r') A'} = 0. \tag{4.44}
\]

Now choose a pair of spinors $o_A, t_A$ such that $o_A \neq 0$. Then equations (4.43) and (4.44) yield
\[
\partial^p \phi \frac{P Q R_1 \cdots R_r}{R_1' \cdots R_r'} \frac{I}{R_{r+1}' \cdots R_{r+1}'} = 0, \tag{4.45}
\]
\[
\partial^p \phi \frac{P' Q' R_1' \cdots R_{r}' \cdots R_1'}{R_{r+1}' \cdots R_{r+1}'} \frac{I}{(K_1' \cdots K_r') A'} = 0. \tag{4.46}
\]

By continuity, these equations hold for all spinors $o_A, t_A$, and hence yield
\[
\partial^p \phi \frac{P Q R_1 \cdots R_r}{R_1' \cdots R_r'} \frac{I}{K_1' \cdots K_r'} = 0, \partial^p \phi \frac{P' Q' R_1' \cdots R_{r}' \cdots R_1'}{R_{r+1}' \cdots R_{r+1}'} \frac{I}{(K_1' \cdots K_r') A'} = 0. \tag{4.47}
\]
Thus $p_2 J K_1 \cdots K_r \frac{I}{K_1' \cdots K_r'}$ is of order $r - 1$.

We now return to (4.42) and apply $\partial^p \phi \frac{P Q R_1 \cdots R_r}{R_1' \cdots R_r'}$ to obtain
\[
\epsilon_{(A')(R_1')}^j \frac{1}{(P)} \frac{\partial}{\partial \phi} \frac{Q R_1 R_2 \ldots R_{p_r}}{R_{p_r}' \ldots R_{p_1}'} P_2 |J K_1 K_2 \ldots K_r \rangle = \frac{\partial}{\partial \phi} \frac{P Q R_1 \ldots R_{p}}{R_{p_r}' \ldots R_{p_1}'} \langle J K_1 K_2 \ldots K_r \rangle.
\] (4.48)

On the right-hand side of this equation we first use (4.32) and the commutativity of partial derivatives followed by substitution of (4.41) to conclude that
\[
\frac{\partial}{\partial \phi} \frac{P Q R_1 \ldots R_{p}}{R_{p_r}' \ldots R_{p_1}'} \langle J K_1 K_2 \ldots K_r \rangle = \epsilon_{(A)'}^j \frac{1}{(P)} \frac{\partial}{\partial \phi} \frac{Q R_1 R_2 \ldots R_{p_r}}{R_{p_r}' \ldots R_{p_1}'} \langle J K_1 K_2 \ldots K_r \rangle.
\] (4.49)

Substitution of (4.49) into (4.48) gives
\[
\epsilon_{(A')(R_1')}^j \frac{1}{(P)} \frac{\partial}{\partial \phi} \frac{Q R_1 R_2 \ldots R_{p_r}}{R_{p_r}' \ldots R_{p_1}'} P_2 |J K_1 K_2 \ldots K_r \rangle = \epsilon_{(R_1')}^j \frac{1}{(A') \frac{1}{(P)}} \frac{\partial}{\partial \phi} \frac{J K_1 K_2 \ldots K_r}{K_2' \ldots K_r'} P_2 |Q R_1 R_2 \ldots R_{p_r} \rangle. \] (4.50)

Finally, an analysis similar to that for (4.43) using spinor pairs \( o_A, \iota_A \) directly leads from (4.50) to equation (4.38), which completes the proof of the integrability condition for existence of \( \chi \).

In summary, we have shown that given an adjoint symmetry \( P_{A'A'} \) of order \( r \), there is an adjoint symmetry \( \hat{P}_{A'A'} \) equivalent to a sum of \( U_{A'A'}[\xi; \zeta_1, \ldots, \zeta_r], V_{A'A'}[\kappa; \vartheta_1, \ldots, \vartheta_{r-1}] \), and a differential function \( \chi \) with the property that the linearization \( L_{\phi} H_{A'A'} \) of the difference \( H_{A'A'} = P_{A'A'} - \hat{P}_{A'A'} - D_{A'A'} \chi \) involves derivatives of \( \phi_{IJ}, \tilde{\phi}_{I',J} \) only up to order \( r-1 \). Thus, due to the local solvability of Maxwell’s equations, it follows that \( H_{A'A'} \) when restricted to \( R^{r-1}(F) \subset J^r(F) \) is of order at most \( r-1 \). By the linearity of the adjoint symmetry equation (4.13), we then conclude that the adjoint symmetry \( P_{A'A'} \) is equivalent to the sum of \( \hat{P}_{A'A'} \) and an adjoint symmetry \( \hat{H}_{A'A'} \) of order at most \( r-1 \), where \( \hat{H}_{A'A'} \) is a function in the coordinates of \( R^{r-1}(F) \subset J^r(F) \) given by replacing all the variables \( \phi_{AB,K_1 \ldots K_p} \) in \( H_{A'A'} \) by the symmetric variables \( \phi_{ABK_1 \ldots K_p}, p \geq 0 \).

Now we proceed inductively by descent in the order of \( P_{A'A'} \). In the last step we see that \( P_{A'A'} \) is equivalent to a sum of the linear adjoint symmetries (4.22), \( 0 \leq p \leq r \), (4.23), \( 0 \leq p \leq r-1 \), and an elementary adjoint symmetry (4.24). This completes the proof of the Theorem. \( \Box \)
V. CLASSIFICATION OF CURRENTS

In this section we give a complete classification of the local conservation laws of Maxwell’s equations and explicitly exhibit a basis for them in terms of conformal Killing vectors and conformal Killing-Yano tensors.

We start with some preliminaries. Let \( \zeta^{CC'} \), \( \xi^{CC'} \) be type (1,1) Killing spinors, let \( \kappa^{A'B'C'D'} \) be a type (4,0) Killing spinor. We define derivatives of \( \xi^{CC'} \) and \( \kappa^{A'B'C'D'} \) with respect to \( \zeta^{CC'} \) by

\[
L_\zeta \xi^{CC'} = \zeta^{EE'} \partial_{EE'} \xi^{CC'} - \xi^{EE'} \partial_{EE'} \zeta^{CC'},
\]

(5.1)

\[
L_\zeta \kappa^{A'B'C'D'} = \zeta^{EE'} \partial_{EE'} \kappa^{A'B'C'D'} + 2(\partial_{F(A'} \zeta^{E')}^{F}) \kappa^{B'C'D')}_{E'} - \frac{3}{2}(\partial_{CC'} \zeta^{CC'}) \kappa^{A'B'C'D'}. \]

(5.2)

It is straightforward to verify that, as a consequence of the conformal invariance of the Killing spinor equation (4.16), \( L_\zeta \xi^{CC'} \) and \( L_\zeta \kappa^{A'B'C'D'} \) are Killing spinors.

Write

\[
\Delta_{AA'} = \phi_{AB,A}^{B'}, \quad \bar{\Delta}_{AA'} = \bar{\phi}_{A'B',A}^{B'},
\]

(5.3)

so that the solution space \( R(F) \) is given by \( \Delta_{AA'} = 0 \). One can then easily verify using the conformal symmetry (4.19) of Maxwell’s equations (4.1) that

\[
\text{pr} X_\zeta \Delta_{AA'} = \zeta^{CC'} D_{CC'} \Delta_{AA'} + (\partial_{AA'} \zeta^{CC'}) \Delta_{CC'} + \frac{1}{2} \partial_{CC'} \zeta^{CC'} \Delta_{AA'}
\]

\[
= \left( L^F_\zeta + \frac{1}{2} \text{div} \zeta \right) \Delta_{AA'},
\]

(5.4)

and similarly that

\[
\text{pr} X_\zeta \Delta^{AA'} = \zeta^{CC'} D_{CC'} \Delta^{AA'} - (\partial_{CC'} \zeta^{AA'}) \Delta^{CC'} + (\partial_{CC'} \zeta^{CC'}) \Delta^{AA'}
\]

\[
= \left( L^F_\zeta + \text{div} \zeta \right) \Delta^{AA'},
\]

(5.5)

where \( \text{div} \zeta = \partial_{CC'} \zeta^{CC'} \), and where \( L^F_\zeta \) denotes the natural lift of the standard spinorial Lie derivative operator \( [13] \) to spinor functions on \( J^\infty(F) \).
A. Equivalence classes of conserved currents

We now proceed to determine the equivalence classes of conserved currents arising from the classification of adjoint symmetries in Theorem 4.3.

In spinor form a conserved current $\Psi^{AA'} = \Psi^\mu e^{AA'}_\mu$ with the characteristic form (3.2) satisfies

$$D_{AA'} \Psi^{AA'} = Q_{AA'} \bar{\Delta}^{AA'} + \bar{Q}_{AA'} \Delta^{AA'},$$

(5.6)

where $Q_{AA'} = (Q_* + i\tilde{Q}_*) e^{AA'}_\mu$ stands for the spinorial characteristic of $\Psi^{AA'}$. If $\Phi^{AA'}_{AA'}$ is a conserved current equivalent to $\Psi^{AA'}$, then $Q_{AA'}$ is a spinorial characteristic admitted by $\Phi^{AA'}$. For any linear adjoint symmetry $P_{AA'}$, the conserved current arising from formula (3.11) is given by

$$\Phi^{AA'}_{AA'} = \frac{1}{2} P_{A'B'} \bar{\phi}^{A'B'} + \bar{\Phi}_{AA'}^{A'B'} \phi^{AB}.$$  

(5.7)

Let $\xi^{CC'}, \xi_1^{CC'}, \ldots, \xi_p^{CC'}$ be real Killing spinors, let $\kappa_{A'B'C'D'}$ be a Killing spinor, and let $\omega_{CC'}$ be a spinor field satisfying (4.25). The conserved currents obtained from the adjoint symmetries $U_{AA'}[\xi, \xi_1, \ldots, \xi_p]$, $iU_{AA'}[\xi, \xi_1, \ldots, \xi_p]$, $V_{AA'}[\kappa, \xi_1, \ldots, \xi_p]$, $W_{AA'}[\omega]$ through formula (5.7) are given by, respectively,

$$\Phi^{AA'}_T[\xi, \xi_1, \ldots, \xi_p] = \frac{1}{2} (U_{A'B'}[\xi, \xi_1, \ldots, \xi_p] \bar{\phi}^{A'B'} + \bar{U}_{A'B'}[\xi, \xi_1, \ldots, \xi_p] \phi^{AB}),$$

(5.8)

$$\Phi^{AA'}_Z[\xi, \xi_1, \ldots, \xi_p] = \frac{i}{2} (U_{A'B'}[\xi, \xi_1, \ldots, \xi_p] \bar{\phi}^{A'B'} - \bar{U}_{A'B'}[\xi, \xi_1, \ldots, \xi_p] \phi^{AB}),$$

(5.9)

$$\Phi^{AA'}_W[\kappa, \xi_1, \ldots, \xi_p] = \frac{i}{2} (V^{A'B'}[\kappa, \xi_1, \ldots, \xi_p] \bar{\phi}^{A'B'} + \bar{V}_{B'A'}[\kappa, \xi_1, \ldots, \xi_p] \phi^{AB}),$$

(5.10)

$$\Phi^{AA'}_W[\omega] = \omega_{B'A'} \bar{\phi}^{A'B'} + \bar{\omega}_{B'A'} \phi^{AB}.$$  

(5.11)

**Lemma 5.1.** The conserved currents (5.8), (5.9), (5.10), (5.11) admit, respectively, the spinorial characteristics

$$Q_{AA'}^T[\xi, \xi_1, \ldots, \xi_p] = \frac{1}{2} \left( U_{AA'}[\xi, \xi_1, \ldots, \xi_p] + (-1)^p \sum_{a=0}^p \sum_{s \in S_p} \frac{1}{a!(p-a)!} U_{AA'}[\mathcal{L}_{\xi_{s(a)}} \cdots \mathcal{L}_{\xi_{s(a+1)}} \xi, \xi_{s(a+1)}, \ldots, \xi_{s(p)}] \right),$$

(5.12)
\[ Q_{A'A'}^Z[\xi, \zeta_1, \ldots, \zeta_p] = \frac{1}{2} \left( U_{A'A'}[\xi, \zeta_1, \ldots, \zeta_p] + (-1)^{p+1} \sum_{a=0}^{p} \sum_{s \in S_p} \frac{1}{a!(p-a)!} U_{A'A'}[L_{\zeta_{s(1)}} \cdots L_{\zeta_{s(a)}} \xi, \zeta_{s(a+1)}, \ldots, \zeta_{s(p)}] \right), \tag{5.13} \]

\[ Q_{A'A'}^V[\kappa, \zeta_1, \ldots, \zeta_p] = \frac{1}{2} \left( V_{A'A'}[\kappa, \zeta_1, \ldots, \zeta_p] + (-1)^{p+1} \sum_{a=0}^{p} \sum_{s \in S_p} \frac{1}{a!(p-a)!} V_{A'A'}[L_{\zeta_{s(1)}} \cdots L_{\zeta_{s(a)}} \kappa, \zeta_{s(a+1)}, \ldots, \zeta_{s(p)}] \right), \tag{5.14} \]

\[ Q_{A'A'}^W[\omega] = \omega_{A'A'}, \tag{5.15} \]

where \( S_p \) denotes the symmetric group on the index set \( \{1, \ldots, p\} \). These characteristics are adjoint symmetries of order \( q_T, q_Z, q_V, q_W \), where

\[ q_T = p, \quad \text{if } p \text{ is even}, \quad q_T < p, \quad \text{if } p \text{ is odd}, \tag{5.16} \]
\[ q_Z = p, \quad \text{if } p \text{ is odd}, \quad q_Z < p, \quad \text{if } p \text{ is even}, \tag{5.17} \]
\[ q_V = p + 1, \quad \text{if } p \text{ is odd}, \quad q_V < p + 1, \quad \text{if } p \text{ is even}, \tag{5.18} \]

and \( q_W = 0 \). In particular, for \( p = 0 \), \( Q_{A'A'}^T[\xi] = U_{A'A'}[\xi] \) is of order \( q_T = 0 \), and \( Q_{A'A'}^Z[\xi] = Q_{A'A'}^V[\kappa] = 0 \).

**Remark:** It follows from Corollary 3.3, Proposition 4.2, and Lemma 5.1 that

\[ U_{A'A'}[\xi, \zeta_1, \ldots, \zeta_{2r+1}], \quad iU_{A'A'}[\xi, \zeta_1, \ldots, \zeta_{2r}], \quad V_{A'A'}[\kappa, \zeta_1, \ldots, \zeta_{2r}], \quad r \geq 0 \tag{5.19} \]

are non-trivial linear adjoint symmetries, respectively of order \( 2r + 2, 2r + 1, 2r + 1 \), none of which is equivalent to the characteristic of any non-trivial conservation law of Maxwell’s equations.

To prove Lemma 5.1 we begin with a preliminary Proposition whose proof is based on straightforward albeit somewhat lengthy computations and will be omitted.

**Proposition 5.2.** The adjoint symmetries (4.22) and (4.23) satisfy the equations

\[ D_{(A'B')B}[\xi] = \xi_{(A'B')B}, \tag{5.20} \]
\[ D_{(A'B'C'D')D'}[\kappa] = -\kappa_{A'B'C'D'} D_D' \Delta_{D'D'} - \frac{2}{5} \Delta_{D'D'} \partial_{D'} \kappa_{A'B'C'D'}, \tag{5.21} \]
\[
\frac{1}{p!} \sum_{s \in S_p} \mathcal{L}_F^F \xi_{s(p)} U_{AA'} \xi_{s(1)}, \ldots, \xi_{s(p-1)} = \frac{1}{p!} \sum_{s \in S_p} U_{AA'} \xi_{s(p)} U_{AA'} \xi_{s(1)}, \ldots, \xi_{s(p-1)} + U_{AA'} \xi_1, \ldots, \xi_p, \quad (5.22)
\]

\[
\frac{1}{p!} \sum_{s \in S_p} \mathcal{L}_F^F \xi_{s(p)} V_{AA'} \kappa_{s(1)}, \ldots, \kappa_{s(p-1)} = \frac{1}{p!} \sum_{s \in S_p} V_{AA'} \xi_{s(p)} V_{AA'} \kappa_{s(1)}, \ldots, \kappa_{s(p-1)} + V_{AA'} \kappa_1, \ldots, \kappa_p, \quad (5.23)
\]

**Proof of Lemma 5.1.**

The proof is based on similar computations for each spinorial characteristic. We therefore will prove the claim only for the characteristic \(Q_{AA'}^V \kappa_1, \ldots, \kappa_p\) and omit the rest.

First by (5.10) we have

\[
D_{AA'} \Phi_V^{AA'} \kappa_1, \ldots, \kappa_p = \frac{1}{2} (\Delta_{AB'} V_{AB'} \kappa_1, \ldots, \kappa_p) + \bar{\Delta}^{BA'} V_{BA'} \bar{\kappa}_1, \ldots, \bar{\kappa}_p
\]

\[= \frac{1}{2} \frac{1}{2} \Delta_{AB'} V_{AB'} \kappa_1, \ldots, \kappa_p + \bar{\Delta}^{BA'} V_{BA'} \bar{\kappa}_1, \ldots, \bar{\kappa}_p - \bar{\phi}^{AB'} D^A_{A'} V_{AB'} \kappa_1, \ldots, \kappa_p - \phi^{AB} D_A V_{BA'} \bar{\kappa}_1, \ldots, \bar{\kappa}_p, \quad (5.24)\]

We recall (5.23) and use equations (5.22) and (5.24) together with the fact that \(D^A_{A'}\) commutes with \(pr X_\xi\) to obtain

\[
(D^A_{A'} V_{B'})_A \kappa_1, \ldots, \kappa_p \bar{\phi}^{AB'}
\]

\[= \frac{1}{p!} \sum_{s \in S_p} (pr X_{s(1)} \cdots pr X_{s(p)} D_{A'}(A')_A \kappa) \bar{\phi}^{AB'}
\]

\[= \frac{1}{p!} \sum_{s \in S_p} (pr X_{s(1)} \cdots pr X_{s(p)} (-\kappa_{A'B'C'D'} D^C_D \Delta_{DD'} - \frac{2}{5} \sigma_{D'}^C \kappa_{A'B'C'D'} \Delta_{DD'})) \bar{\phi}^{AB'}
\]

\[= \frac{1}{p!} \sum_{s \in S_p} (-D^C_D (\kappa_{A'B'C'D'} (pr X_{s(1)} \cdots pr X_{s(p)} \Delta_{DD'})) \bar{\phi}^{AB'})
\]

\[= \frac{1}{p!} \sum_{s \in S_p} (pr X_{s(1)} \cdots pr X_{s(p)} \Delta_{DD'}) V_{DD'} [\kappa] + D_{AA'} Y_{AA'}^{AA'}
\]

\[= \frac{1}{p!} \sum_{s \in S_p} ((\mathcal{L}_F^F + \text{div } \zeta) \cdots (\mathcal{L}_F^F + \text{div } \zeta) \Delta_{DD'}) V_{DD'} [\kappa] + D_{AA'} Y_{AA'}^{AA'},
\]

where \(Y_{AA'}^{AA'}\) is a trivial conserved current. Next we integrate by parts and use the identity
\((L_{\zeta}^F + \text{div} \, \zeta)(\Lambda^{DD'}V_{DD'}[\kappa]) = D_{AA'}(\zeta^{AA'}\Lambda^{DD'}V_{DD'}[\kappa])\) \tag{5.26}

to see that

\[
(D^A_{(A'V_{B'})A}[\kappa, \zeta_1, \ldots, \zeta_p])_{\Phi}^{A'B'} = \\
\frac{1}{p!} \sum_{s \in S_p} (-1)^p \Lambda^{DD'} L_{\zeta_{s(1)}}^F \cdots L_{\zeta_{s(p)}}^F V_{DD'}[\kappa] + D_{AA'} \Upsilon_{2}^{AA'},
\]

where \(\Upsilon_{2}^{AA'}\) is a trivial conserved current. Now a repeated application of (5.23) yields

\[
(D^A_{(A'V_{B'})A}[\kappa, \zeta_1, \ldots, \zeta_p])_{\Phi}^{A'B'} = \\
(-1)^p \frac{1}{p!} \sum_{a=0}^{p} \sum_{s \in S_p} \frac{1}{(p-a)!} \Lambda^{DD'} V_{DD'}[\zeta_{s(1)}^A \cdots \zeta_{s(a)}^A \kappa, \zeta_{s(a+1)}, \ldots, \zeta_{s(p)}] + D_{AA'} \Upsilon_{2}^{AA'}. \tag{5.27}
\]

We substitute equation (5.27) and its complex conjugate into (5.24) to conclude that \(\Phi_{AA'}^{V}[\kappa, \zeta_1, \ldots, \zeta_p]\) admits the characteristic (5.14).

Finally, by equation (5.14) and Proposition 4.2, we see that \(Q_{AA'}^{V}[\kappa, \zeta_1, \ldots, \zeta_p]\) is equivalent to an adjoint symmetry with the highest order term

\[
\frac{1}{2}((-1)^p - 1) \zeta_{C_1}^{C_1} \cdots \zeta_{C_p}^{C_p} K_{A'D'E'} \Phi_{AC_1 \cdots C_p}. \tag{5.28}
\]

Hence we immediately obtain (5.18). \(\square\)

A conserved current \(\Psi^{AA'}\) of order \(q\) is linear/quadratic if it can be expressed as a homogeneous linear/quadratic polynomial in the variables \(\phi_{I,J,K_1 \ldots K_p}, 0 \leq p \leq q\), and complex conjugate variables. Let the weight of a monomial be the sum of the orders of these variables, and let the weight of a linear/quadratic current \(\Psi^{AA'}\) be the maximum of the weights of the monomials in \(\Psi^{AA'}\). This weight is called minimal if it is the smallest among the weights of all quadratic currents equivalent to \(\Psi^{AA'}\).

**Proposition 5.3.** A conserved current \(\Psi^{AA'}\) is equivalent to a linear/quadratic current of minimal weight \(w\) if and only if \(\Psi^{AA'}\) admits a characteristic \(Q_{AA'}\) equivalent to an elementary/linear adjoint symmetry \(P_{AA'}\) of minimal order \(w\).

**Proof.**

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First, by Proposition 3.1 and Theorem 3.2, the equivalence classes of linear/quadratic currents correspond to the equivalence classes of elementary/linear characteristics. Let $\Psi^{AA'}$ be a linear/quadratic current of minimal weight $w$, and let $P_{AA'}$ be an elementary/linear adjoint symmetry of minimal order $p$ equivalent to a spinorial characteristic admitted by $\Psi^{AA'}$. We now show that $w = p$.

By the standard integration by parts procedure [9], one can show that $\Psi^{AA'}$ admits the spinorial characteristic

$$Q_{AA'} = \sum_{r \geq 0} (-1)^r \epsilon_{CA} D_{E_1} \cdots D_{E_r} (T + 2) \frac{\partial}{\partial \phi} (BC E_1 \cdots E_r \Psi^{AA'}_{A'B} - \partial \phi (BC E_1 \cdots E_r \Psi^{AA'}_{A'})),$$

which is of order $q$ where $q + 1$ equals the weight of $D_{AA'} \Psi^{AA'}$. Since this weight is at most $w + 1$, we have $q \leq w$. Now since $Q_{AA'}$ is equivalent to $P_{AA'}$, which has minimal order $p$, it immediately follows that $p \leq q$ and hence $p \leq w$. On the other hand, by Proposition 3.1 we have that $\Psi^{AA'}$ is equivalent to a current of weight $p$, and hence $w \leq p$ since $\Psi^{AA'}$ has minimal weight. Thus we conclude $w = p$. ☐

**Theorem 5.4.** Every conserved current of Maxwell’s equations (4.1) is equivalent to the sum of a linear current and a quadratic current. The equivalence classes of linear currents are represented by the currents

$$\Phi^{AA'}_W[\omega],$$

where $\omega$ satisfies equation (4.23). The equivalence classes of quadratic currents of weight $w$ are represented by sums of the currents

$$\Phi^{AA'}_T[\xi, \zeta_1, \ldots, \zeta_{2r}], \quad 0 \leq r \leq \lfloor w/2 \rfloor,$$

$$\Phi^{AA'}_Z[\xi, \zeta_1, \ldots, \zeta_{2r+1}], \quad 0 \leq r \leq \lfloor (w - 1)/2 \rfloor,$$

$$\Phi^{AA'}_V[\kappa, \zeta_1, \ldots, \zeta_{2r+1}], \quad 0 \leq r \leq \lfloor w/2 \rfloor - 1,$$

involving type $(1,1)$ real Killing spinors $\xi$, $\zeta_i$, and type $(0,4)$ Killing spinors $\kappa$ for each $r$. In particular, up to quadratic currents of lower weight, a quadratic current of minimal weight $w$ is equivalent to a sum of currents given by (5.31) for $r = w/2$ and (5.33) for $r = w/2 - 1$, if $w$ is even, or (5.32) for $r = (w - 1)/2$, if $w$ is odd.
Proof.

Let $\Psi^{AA'}$ be a conserved current satisfying (5.6). Recall from (3.6) that the spinorial characteristic $Q_{AA'}$ of $\Psi^{AA'}$ satisfies the adjoint symmetry equation (4.13) so that, by Theorem 4.3, $Q_{AA'}$ is equivalent to a sum of the linear adjoint symmetries (4.22), (4.23) and the elementary adjoint symmetry (4.25). Thus by Proposition 3.1, $\Psi^{AA'}$ is equivalent to a sum of linear and quadratic conserved currents.

Clearly, by Theorem 3.2 and Lemma 5.1, any linear current is equivalent to a current $\Phi^{AA'}_W[\omega]$.

Next, by Lemma 5.1, the quadratic currents $\Phi^{AA'}_T[\xi, \xi_1, \ldots, \xi_p]$, $\Phi^{AA'}_Z[\xi, \xi_1, \ldots, \xi_p]$, $\Phi^{AA'}_V[\kappa, \xi_1, \xi_2, \ldots, \xi_{p-1}]$ of order $p \geq 0$ have the respective weights:

- $w_T = p$, $w_Z < p$, $w_V = p$, if $p$ is even,
- $w_T < p$, $w_Z = p$, $w_V < p$, if $p$ is odd. (5.34)

Consequently, by Proposition 5.3 together with Theorem 3.2 and Lemma 5.1, we see by using induction on $p$ that the currents (5.31), (5.32), (5.33) span the equivalence classes of quadratic currents $\Psi^{AA'}$ of weight at most $w$. □

Remark: The currents $\Phi^{AA'}_T[\xi]$ yield the stress-energy conservation laws (2.13), the currents $\Phi^{AA'}_Z[\xi, \xi]$ yield the zilch conservation laws (2.14), and the currents $\Phi^{AA'}_V[\kappa, \xi]$ yield the chiral conservation laws (2.16), in spinorial form.

Write $\mathcal{V}_0^W$ for the real vector space of equivalence classes of linear currents (5.30), write $\mathcal{V}_w$ for the real vector space of equivalence classes of quadratic currents of weight at most $w$, and write $\mathcal{V}_{2r}^T$, $\mathcal{V}_{2r+1}^Z$, $\mathcal{V}_{2r+2}^V$ for the respective real vector spaces of equivalence classes of currents spanned by (5.31), (5.32), (5.33), for a fixed $r \geq 0$. We remark that these spans contain non-trivial lower weight currents and consequently it is convenient to define the following quotient spaces of quadratic currents. Let

$\mathcal{N}_0^T = \mathcal{V}_0^T$, $\mathcal{N}_1^Z = \mathcal{V}_1^Z$, $\mathcal{N}_2^V = \mathcal{V}_2^V$, (5.36)

and define, for $r \geq 1$,
Finally, write $\mathcal{K}_r^R$ for the real vector space of real Killing spinors of type $(r, r)$, and write $\mathcal{K}_{r,s}$ for the complex vector space of Killing spinors of type $(r, s)$, regarded as a real vector space.

Let $\Psi^{AA'}$ be a quadratic conserved current in $\mathcal{V}^T_{2r} \oplus \mathcal{V}^V_{2r}$. By Theorem 3.2, Lemma 5.1, Proposition 4.2 and the Killing spinor factorization result of Lemma 4.1, we see that any current equivalent to $\Psi^{AA'}$ admits a spinorial characteristic $Q_{AA'}$ which is equivalent to a linear adjoint symmetry of order $2r$ of the form

$$P_{AA'} = \xi^{BC_1 \ldots C_{2r}} \phi_{ABC_1 \ldots C_{2r}} + \kappa^{C_1 \ldots C_{2r-1}} A_{BC_1 \ldots C_{2r-1}} - B_{C_1 \ldots C_{2r-1}} A_{C_1 \ldots C_{2r-1}} + H_{AA'},$$

(5.38)

where $\xi^{BC_1 \ldots C_{2r}}$ is a real Killing spinor of type $(2r + 1, 2r + 1)$, $\kappa^{C_1 \ldots C_{2r-1}} A_{BC_1 \ldots C_{2r-1}}$ is a Killing spinor of type $(2r - 1, 2r + 3)$, and where $H_{AA'}$ is of order less than $2r$. It follows from (5.38) and the identities (4.12) that the highest order terms in the spinorial curl $D_A^V Q_{AA'}$ of $Q_{AA'}$ are given by

$$\xi^{BC_1 \ldots C_{2r}} \phi_{ABC_1 \ldots C_{2r}} + \kappa^{C_1 \ldots C_{2r-1}} A_{BC_1 \ldots C_{2r-1}} - B_{C_1 \ldots C_{2r-1}} A_{C_1 \ldots C_{2r-1}}$$

(5.39)

on $R^{2r+1}(F)$, and hence the Killing spinors in (5.38) are unique for the class of conserved currents equivalent to $\Psi^{AA'}$. This yields a linear map

$$I_{2r} : \mathcal{V}^T_{2r} \oplus \mathcal{V}^V_{2r} \to \mathcal{K}_{2r+1} \bigoplus \mathcal{K}_{2r-1,2r+3}, \quad I_{2r}(\Psi) = (\xi^{BC_1 \ldots C_{2r}} \phi_{ABC_1 \ldots C_{2r}}, \kappa^{C_1 \ldots C_{2r-1}} A_{BC_1 \ldots C_{2r-1}} - B_{C_1 \ldots C_{2r-1}} A_{C_1 \ldots C_{2r-1}}),$$

(5.40)

which is well defined on the equivalence classes of currents in $\mathcal{V}^T_{2r} \oplus \mathcal{V}^V_{2r}$.

Next let $\Psi^{AA'}$ be a quadratic conserved current in $\mathcal{V}^Z_{2r+1}$. Similarly as above, we can show that any current equivalent to $\Psi^{AA'}$ admits a characteristic $Q_{AA'}$ which is equivalent to a linear adjoint symmetry of order $2r + 1$ of the form

$$P_{AA'} = i\xi^{BC_1 \ldots C_{2r+1}} \phi_{ABC_1 \ldots C_{2r+1}} + H_{AA'},$$

(5.41)

where $\xi^{BC_1 \ldots C_{2r+1}}$ is a real Killing spinor of type $(2r + 2, 2r + 2)$, and where $H_{AA'}$ is of order less than $2r + 1$. From (5.41) and (4.12) the highest order terms in the spinorial curl of $Q_{AA'}$ are given by
on $R^{2r+2}(F)$, and hence the Killing spinor in (5.41) is unique for the class of conserved currents equivalent to $\Psi^{AA'}$. This again yields a linear map

$$I_{2r+1} : \mathcal{V}_Z^{2r+1} \rightarrow \mathcal{K}_R^{2r+2}, \quad I_{2r+1}(\Psi) = \varepsilon^{BC_1\cdots C_{2r+1}} A'BCC_1\cdots C_{2r+1},$$

which is well defined on the equivalence classes of currents in $\mathcal{V}_Z^{2r+1}$.

**Theorem 5.5.** (i) The vector space $\mathcal{V}_w$ is isomorphic to the direct sum

$$\mathcal{V}_w \simeq \bigoplus_{r=0}^{[w/2]} \mathcal{N}_T^{2r} \oplus \bigoplus_{r=0}^{[w-1]/2} \mathcal{N}_Z^{2r+1} \oplus \bigoplus_{r=0}^{[w/2]-1} \mathcal{N}_V^{2r+2}. \quad (5.44)$$

Moreover, for $r \geq 0$, the vector spaces $\mathcal{N}_T^{2r}$, $\mathcal{N}_Z^{2r+1}$, $\mathcal{N}_V^{2r+2}$ are respectively isomorphic to $\mathcal{K}_R^{2r}$, $\mathcal{K}_R^{2r+2}$, $\mathcal{K}_{2r+1,2r+5}$ under the mappings (5.40) and (5.43). Consequently, these vector spaces of conserved currents have the following dimensions over the real numbers:

$$\dim \mathcal{N}_T^{2r} = \frac{1}{3}(r + 1)^2(2r + 3)^2(4r + 5), \quad (5.45a)$$

$$\dim \mathcal{N}_Z^{2r+1} = \frac{1}{3}(r + 2)^2(2r + 3)^2(4r + 7), \quad (5.45b)$$

$$\dim \mathcal{N}_V^{2r+2} = \frac{2}{3}(r + 1)(r + 3)(2r + 3)(2r + 7)(4r + 9). \quad (5.45c)$$

(ii) The vector space $\mathcal{V}_0^W$ is isomorphic to the space of solutions to Maxwell’s equations.

**Proof.**

To prove part (ii) of the Theorem, note that by Lemma 5.1 and Theorem 5.4, the equivalence classes of linear conserved currents are in one-to-one correspondence with spinor fields $\omega_{CC'}$ satisfying (4.25) up to gradient terms $\partial_{CC'} \chi$. But such spinors $\omega_{CC'}$ correspond to potentials for the electromagnetic fields via $\phi_{AB} = \partial_{(A'} \omega_{B')}$, while gradient terms $\omega_{CC'} = \partial_{CC'} \chi$ represent pure gauge potentials giving rise to the zero electromagnetic field, $\phi_{AB} = \partial_{(A} \partial_{B')} \chi = 0$. This establishes an isomorphism between solutions of Maxwell’s equations and equivalence classes of linear conserved currents.
To prove part (i) of the Theorem, we note that the isomorphism (5.44) follows from Theorem 5.4. Our goal now is to show that $\mathcal{N}_{2r}^T \simeq \mathcal{K}_{2r+1}^R$, $\mathcal{N}_{2r+1}^Z \simeq \mathcal{K}_{2r+2}^R$, $\mathcal{N}_{2r+2}^V \simeq \mathcal{K}_{2r+1,2r+5}^R$.

We first note that by construction $I_{2r}$ and $I_{2r+1}$ are linear maps which descend to $\mathcal{N}_{2r}^T \oplus \mathcal{N}_{2r}^V$ and $\mathcal{N}_{2r+1}^Z$, respectively. Hence, in order to prove that they are one-to-one, it is sufficient to show that their respective kernels are contained in $V_{2r-2}$ and $V_{2r-1}$. Let $\Psi^{AA'}$ be a quadratic current in $V_{2r}^T \oplus V_{2r}^V$ admitting a spinorial characteristic $Q_{AA'}$ such that $I_{2r}(\Psi) = 0$. Hence we have $\xi_{AA'}^{B_1 \cdots B_r} = 0$ and $\kappa_{AA'}^{C_1 \cdots C_{2r-1}} = 0$ in (5.39). It then follows from (5.38) that $Q_{AA'}$ is equivalent to a linear adjoint symmetry $P_{AA'}$ of order less than $2r$. By Proposition 5.3 we see that $\Psi^{AA'}$ is equivalent to a quadratic current of weight less than $2r$ and thus is contained in $V_{2r-2}^T \oplus V_{2r-2}^V$ by Theorem 5.4. Hence we conclude ker $I_{2r} \subset V_{2r-2}$.

By a similar argument we can show that ker $I_{2r+1} \subset V_{2r-1}$.

Therefore the mappings $I_{2r}$ and $I_{2r+1}$ are one-to-one. Next, by the factorization property of Killing spinors in Lemma 4.1, it immediately follows that the mappings are onto. This establishes the required isomorphisms.

Finally, the dimension counts of the vector spaces follow from Lemma 4.1. □

B. A Basis

We now present an explicit basis for the vector space $V_w$ of currents spanned by (5.31), (5.32), (5.33). We start by defining a basis of complex conformal Killing vectors and self-dual conformal Killing-Yano tensors as follows.

Fix a spinor basis $\{o^A, \iota^A\}$, namely, $o^A$ and $\iota^A$ are linearly independent constant spinors satisfying $o_A \iota^A = 1$. Now let

$$\begin{align*}
\xi_{0,1}^{AA'} &= o^A \overline{o}^A, & \xi_{0,2}^{AA'} &= o^A \iota^A, & \xi_{0,2}^{AA'} &= \iota^A \overline{o}^A, & \xi_{0,3}^{AA'} &= \iota^A \iota^A, \\
\xi_{1,1}^{AA'} &= x_{B_1}^A o_{B_1}^A \overline{o}, & \xi_{1,2}^{AA'} &= x_{B_1}^A \iota^A \overline{\iota}^B, & \xi_{1,3}^{AA'} &= x_{B_1}^A \overline{\iota}^A \iota^B, \\
\xi_{1,1}^{AA'} &= x_{B_1}^A o_{B_1}^A \overline{o}, & \xi_{1,2}^{AA'} &= x_{B_1}^A \iota^A \overline{\iota}^B, & \xi_{1,3}^{AA'} &= x_{B_1}^A \overline{\iota}^A \iota^B.
\end{align*}$$

(5.46)
\[ \xi_{AA'} = x^{AA'}, \quad (5.49) \]

\[ \xi_{2A'} = x^A x_{B'} o_B o, \quad \xi_{2A'} = x^A x_{B'} A' B' o, \quad \xi_{2A'} = x^A x_{B'} A' B' o, \quad (5.50) \]

and

\[ Y_{01}^{AB} = o^B o, \quad Y_{0,2}^{AB} = o^{(A} t^{B)}, \quad Y_{0,3}^{AB} = t^A t^B, \quad (5.51) \]

\[ Y_{1,1}^{AB} = x_C^{(A} o_C^{B)} o, \quad Y_{1,2}^{AB} = x_C^{(A} o_{C'}^{B)} o, \quad Y_{1,3}^{AB} = x_C^{(A} t^{B)} o, \quad Y_{1,4}^{AB} = x_C^{(A} t^{B)} o, \quad (5.52) \]

\[ Y_{2,1}^{AB} = x_{D'}^{(A} x_{D'}^{B)} o, \quad Y_{2,2}^{AB} = x_C^{(A} x_{D'}^{B)} o, \quad Y_{2,3}^{AB} = x_C^{(A} x_{D'}^{B)} o, \quad (5.53) \]

From equations (2.10) and (2.11) we have the following result.

**Proposition 5.6.** The set of 15 spinorial conformal Killing vectors (5.46) to (5.50) is a basis for the complex vector space \( K_{1,1} \) of type \((1,1)\) Killing spinors. The set of 10 spinorial conformal Killing-Yano tensors (5.51) to (5.53) is a basis for the complex vector space \( K_{2,0} \) of type \((2,0)\) Killing spinors.

To proceed, we state a preliminary result which follows immediately from Theorem 5.5, Proposition 5.6 and Lemma 4.1, and equation (4.2).

**Proposition 5.7.** Let \( \Psi^{AA'} \) be a current of weight \( q \). Then \( \Psi^{AA'} \) is equivalent to a quadratic current which is a polynomial of degree at most \( 2q + 2 \) in \( x^{CC'} \) and admits a unique decomposition into a sum of monomials each of which possess either even parity or odd parity with respect to the duality transformation (4.2).

We now define a basis for \( \mathcal{V}_w \) inductively in terms of the weight \( 0 \leq q \leq w \) and the degree \( 0 \leq p \leq 2q + 2 \). Let \( \mathcal{V}_w = \mathcal{V}_w^+ \oplus \mathcal{V}_w^- \), where \( \mathcal{V}_w^+ \) and \( \mathcal{V}_w^- \) denote the subspaces of currents with even parity (+) and odd parity (−) respectively. We consider \( \mathcal{V}_w^+ \) and \( \mathcal{V}_w^- \) separately.

To proceed, we first state the main results and then present the proofs afterwards.

The basis for \( \mathcal{V}_w^+ \) is indexed by \( s = q + 1 \) and \( p \) in addition to two pairs of integers \( (i, j) \) and \( (n, n') \) satisfying

\[ \max(0, p - s) \leq i \leq j \leq p - i, \quad 0 \leq p \leq 2s, \quad \text{and} \quad (5.54) \]
\[
\begin{aligned}
0 \leq n \leq s - p + 2i, & \quad 0 \leq n' \leq s - p + 2j, \quad \text{if } i < j; \\
0 \leq n \leq n' \leq s - p + 2i, & \quad \text{if } i = j.
\end{aligned}
\] (5.55)

Write
\[
l = s - p + i, \quad l' = s - p + j.
\] (5.56)

For each value of \(i, j, n, n', p, q\), we define the stress-energy type currents when \(q\) is even,
\[
\Psi_{T(i,j,n,n',p,q)}^{AA'}[\xi_1, \ldots, \xi_{q+1}] = \frac{1}{2} (U_B^A[A_1, \ldots, \xi_{q+1}] + U_B^A[\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}]) \phi^{A'B'} \\
+ \frac{1}{2} (\bar{U}_B^A[A_1, \ldots, \xi_{q+1}] + \bar{U}_B^A[\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}]) \phi^{AB},
\] (5.57)

and we also define the zilch type currents when \(q\) is odd,
\[
\Psi_{Z(i,j,n,n',p,q)}^{AA'}[\xi_1, \ldots, \xi_{q+1}] = \frac{i}{2} (U_B^A[A_1, \ldots, \xi_{q+1}] + U_B^A[\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}]) \phi^{A'B'} \\
- \frac{i}{2} (\bar{U}_B^A[A_1, \ldots, \xi_{q+1}] + \bar{U}_B^A[\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}]) \phi^{AB},
\] (5.59)

and
\[
\Psi_{Z'(i,j,n,n',p,q)}^{AA'}[\xi_1, \ldots, \xi_{q+1}] = \frac{1}{2} (U_B^A[A_1, \ldots, \xi_{q+1}] - U_B^A[\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}]) \phi^{A'B'} \\
- \frac{1}{2} (\bar{U}_B^A[A_1, \ldots, \xi_{q+1}] - \bar{U}_B^A[\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}]) \phi^{AB},
\] (5.60)

where \(U_A^A[A_1, \ldots, \xi_{q+1}] = \overline{U_A^A[A_1, \ldots, \bar{\xi}_{q+1}]}\). In (5.57) to (5.60) the set \(\{\xi_1, \ldots, \xi_{q+1}\}\) consists of the conformal Killing vectors (\ref{5.46}), (\ref{5.47}), (\ref{5.49}), (\ref{5.50}) with each \(\xi_{a,b}\) appearing \(\#[\xi_{a,b}]\) times according to the following count formulas:
\[
\#[\xi_{0,1}] = \min(n, l, n'),
\] (5.61)
\[
\#[\xi_{0,2}] = \max(0, \min(n - n', l - n')),
\] (5.62)
\[
\#[\bar{\xi}_{0,2}] = \max(0, \min(n' - n, l - n)),
\] (5.63)
\[ \#[\xi_{0,3}] = \max(0, \min(l - n', l - n)), \quad (5.64) \]
\[ \#[\xi_{1,1}] = \max(0, \min(n' - l', l' - l)), \quad (5.65) \]
\[ \#[\xi_{1,2}] = \max(0, l' - l - |l' - n'|), \quad (5.66) \]
\[ \#[\xi_{1,3}] = \max(0, \min(l' - n', l' - l)), \quad (5.67) \]
\[ \#[\xi_{1,4}] = s - (i + j + l + l')/2, \quad (5.68) \]
\[ \#[\xi_{1,1}] = \#[\xi_{1,2}] = \#[\xi_{1,3}] = 0, \quad (5.69) \]
\[ \#[\xi_{2,1}] = \max(0, \min(n' - 2l' + l, n - l)), \quad (5.70) \]
\[ \#[\xi_{2,2}] = \max(0, \min(n - n' + 2(l' - l), n - l)), \quad (5.71) \]
\[ \#[\xi_{2,3}] = i - \max(0, n' - 2l' + l, n - l). \quad (5.73) \]

Note that \( \#[\xi_{0,1}] + \#[\xi_{0,2}] + \#[\xi_{0,3}] = l, \#[\xi_{1,1}] + \#[\xi_{1,2}] + \#[\xi_{1,3}] = j - i, \#[\xi_{1,4}] = p - i - j, \#[\xi_{2,1}] + \#[\xi_{2,2}] + \#[\xi_{2,3}] = i, \) and so \( \sum_{a,b} \#[\xi_{a,b}] = s. \)

**Theorem 5.8** The set of all currents \((5.57)\) and \((5.58)\) for \(0 \leq p \leq 2q + 2, \quad q = 2r, \quad 0 \leq r \leq \lceil w/2 \rceil, \) together with \((5.59)\) and \((5.60)\) for \(0 \leq p \leq 2q + 2, \quad q = 2r + 1, \quad 0 \leq r \leq \lfloor (w - 1)/2 \rfloor, \) constitutes a basis for \( V_w^+. \)

The basis for \( V_w^- \) is indexed by \( s = q - 1 \) and \( p \) in addition to two pairs and a triplet of integers \((i, j, k), (n, n')\) and \((m, m')\), satisfying

\[ p' = p - k, \quad 0 \leq k \leq 4, \quad 0 \leq p' \leq 2s, \quad (5.74) \]
\[ \max(0, p' - s) \leq i \leq j \leq p' - i, \quad (5.75) \]

\[ \begin{cases} 0 \leq n \leq s - p' + 2i, & 0 \leq n' \leq s - p' + 2j, \quad \text{if } i < j; \\ 0 \leq n \leq n' \leq s - p' + 2i, & \text{if } i = j; \end{cases} \quad (5.76) \]

\[ \begin{cases} 0 \leq m \leq 4 - k, & 0 \leq m' \leq k, \quad \text{if } n = 0; \\ m = 4 - k, & m' = k, \quad \text{if } n > 0. \end{cases} \quad (5.77) \]

Write
\[ l = s - p' + i, \quad l' = s - p' + j, \quad h = \lfloor k/2 \rfloor. \] (5.78)

For each value of \( i, j, k, n, n', m, m', p, q, \) if \( q \) is even, we define the chiral type currents

\[
\Psi_{+V}^{AA'}(i,j,k,n',m,m';p,q)[Y_1, Y_2, \xi_1, \ldots, \xi_{q-1}] = \frac{1}{2}(V_A^A[k, \xi_1, \ldots, \xi_{q-1}] + V_B^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}])\phi^{AB} \\
+ \frac{1}{2}(V_B^A[k, \xi_1, \ldots, \xi_{q-1}] + V_B^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}])\phi^{AB},
\] (5.79)

\[
\Psi_{-V}^{AA'}(i,j,k,n',m,m';p,q)[Y_1, Y_2, \xi_1, \ldots, \xi_{q-1}] = \frac{i}{2}(V_A^A[k, \xi_1, \ldots, \xi_{q-1}] + V_B^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}])\phi^{AB} \\
- \frac{i}{2}(V_B^A[k, \xi_1, \ldots, \xi_{q-1}] + V_B^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}])\phi^{AB},
\] (5.80)

and

\[
\Psi_{+V'}^{AA'}(i,j,k,n',m,m';p,q)[Y_1, Y_2, \xi_1, \ldots, \xi_{q-1}] = \frac{1}{2}(V_B^A[k, \xi_1, \ldots, \xi_{q-1}] - V_B^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}])\phi^{AB} \\
+ \frac{1}{2}(V_B^A[k, \xi_1, \ldots, \xi_{q-1}] - V_B^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}])\phi^{AB},
\] (5.81)

\[
\Psi_{-V'}^{AA'}(i,j,k,n',m,m';p,q)[Y_1, Y_2, \xi_1, \ldots, \xi_{q-1}] = \frac{i}{2}(V_B^A[k, \xi_1, \ldots, \xi_{q-1}] - V_B^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}])\phi^{AB} \\
- \frac{i}{2}(V_B^A[k, \xi_1, \ldots, \xi_{q-1}] - V_B^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}])\phi^{AB},
\] (5.82)

if \( i \neq j \) or \( n \neq n' \),

where \( \kappa^{A'B'C'D'} = Y_1^{(AB}Y_2^{CD)} \), and \( \bar{V}_A^A[k, \xi_1, \ldots, \xi_{q-1}] = V_A^A[k, \bar{\xi}_1, \ldots, \bar{\xi}_{q-1}] \).

In (5.79) to (5.82) the set \( \{\xi_1, \ldots, \xi_{q-1}\} \) consists of the conformal Killing vectors (5.46), (5.47), (5.49), (5.50) with each \( \xi_{a,b} \) appearing \( \#[\xi_{a,b}] \) times according to the previous count formulas (5.61) to (5.73), and, in addition, the set \( \{Y_1, Y_2\} \) consists of the conformal Killing-Yano tensors (5.51), (5.52), (5.53) with each \( Y_{a,b} \) appearing \( \#[Y_{a,b}] \) times according to the following count formulas:

\[
\#[Y_{0,1}] = \min(2 + h - k, \max(0, m + k - h - 2)),
\] (5.83)

\[
\#[Y_{0,2}] = \max(0, \min(m, 4 + 2h - 2k - m)),
\] (5.84)

\[
\#[Y_{0,3}] = \max(0, 2 + h - k - m),
\] (5.85)

\[
\#[Y_{1,1}] = \max(0, m + k - 4 + \min(k - 2h, m')),
\] (5.86)

\[
\#[Y_{1,2}] = \max(0, m + k - 4 + \max(0, k - 2h - m')),
\] (5.87)
Note that \( \#[Y_{1,3}] = \min(k - 2h, m') - \max(0, m + k - 4 + \min(k - 2h, m')) \), \( \#[Y_{1,4}] = \max(0, k - 2h - m') - \max(0, m + k - 4 + \max(0, k - 2h - m')) \), \( \#[Y_{2,1}] = \max(0, m' - k + h) \), \( \#[Y_{2,2}] = \max(0, k - m' + \min(0, 2m' - 2k + 2h)) \), \( \#[Y_{2,3}] = \max(0, h - \max(0, m' - k + 2h)) \).

Note that \( \#[Y_{0,1}] + \#[Y_{0,2}] + \#[Y_{0,3}] = 2 - k + h \), \( \#[Y_{1,1}] + \#[Y_{1,2}] + \#[Y_{1,3}] + \#[Y_{1,4}] = k - 2h \), \( \#[Y_{2,1}] + \#[Y_{2,2}] + \#[Y_{2,3}] + \#[Y_{2,4}] = h \), and so \( \sum_{a,b} \#[Y_{a,b}] = 2 \).

**Theorem 5.9** The set of all currents (5.72) to (5.83) for \( 0 \leq p \leq 2q + 2 \), \( q = 2r + 2 \), \( 0 \leq r \leq [w/2] - 1 \), constitutes a basis for \( V_w^- \).

The proof of Theorems 5.8 and 5.9 relies on the construction of an explicit basis for Killing spinors based on the factorization Lemma 4.1.

**Lemma 5.10** (i) For \( s \geq 0 \) a basis for \( \mathcal{K}^R_{s,s} \) is given by the set of symmetrized products of conformal Killing vectors

\[
\xi_{1(A_1' \cdots A_s')}(\xi_{s,A_s}) + \bar{\xi}^{(A_1' \cdots A_s')}(\xi_{s,A_s}),
\]

\[
i\xi_{1(A_1' \cdots A_s')}(\xi_{s,A_s}) - i\bar{\xi}^{(A_1' \cdots A_s')}(\xi_{s,A_s}), \quad \text{if } i \neq j \text{ or } n \neq n', \tag{5.94}
\]

where \( \xi_1, \ldots, \xi_s \) are chosen by the count formulas (5.61) to (5.73) with \( p, i, j, n, n' \) satisfying (5.54) and (5.55).

(ii) For \( s \geq 0 \) a basis for \( \mathcal{K}_{1+s,s} \) is given by the set of symmetrized products of conformal Killing vectors and conformal Killing-Yano tensors

\[
Y_{1}^{(AB \{CD \xi_{1(A_1' \cdots A_s')} + Y_{1}^{(AB \{CD \xi_{1(A_1' \cdots A_s')},
\]

\[
iY_{1}^{(AB \{CD \xi_{1(A_1' \cdots A_s')} - iy_{1}^{(AB \{CD \xi_{1(A_1' \cdots A_s')}, \quad \text{if } i \neq j \text{ or } n \neq n', \tag{5.96}
\]

where \( \xi_1, \ldots, \xi_s, Y_1, Y_2 \) are chosen by the respective count formulas (5.61) to (5.73) and (5.83) to (5.92) with \( p, k, i, j, n, n', m, m' \) satisfying (5.74) to (5.77).

**Proof.**
Let $\xi_{A_1^{\cdots}A_s}^{A_1^{\cdots}A_s}$ be a Killing spinor of type $(s, s)$. By the factorization Lemma 4.1, one can show that $\xi_{A_1^{\cdots}A_s}^{A_1^{\cdots}A_s}$ is a sum of linearly independent monomials in $x^{CC'}$ given by

$$\xi_{A_1^{\cdots}A_s}^{A_1^{\cdots}A_s} = \sum_{i,j,p} \Gamma_{(i,j,p)}^{A_1^{\cdots}A_s} A_i \cdots A_s (x, \gamma),$$

with $i, j, p$ satisfying $i \geq \max(0, p - s)$, $j \geq \max(0, p - s)$, $i + j \leq p$, and

$$\Gamma_{(i,j,p)}^{A_1^{\cdots}A_s} (x, \gamma) = \frac{1}{2} x^{E_i^{\cdots} E_j} \left( \right) \Gamma_{A_{p-i}^{\cdots} A_s} \left( \right),$$

where $x_{A_{p-i}^{\cdots} A_s} = x_{A_{p-i}^{\cdots} A_s}$ and where $\gamma_{A_{p-i}^{\cdots} A_s}$ is a constant symmetric spinor.

Hence, by setting

$$\gamma(n,n) = 0^{C_1} \cdots 0^{C_n} \cdots 0^{C_{n+1}} \cdots 0^{C_{s+j}},$$

we obtain a basis for the real vector space $\mathcal{K}_{s,s}^R$ given by the Killing spinors

$$\xi_{A_1^{\cdots}A_s}^{A_1^{\cdots}A_s} = \sum_{i,j,p} \left( \Gamma_{(i,j,p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n')) + \Gamma_{(j,i,p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n')) \right),$$

and

$$\xi_{A_1^{\cdots}A_s}^{A_1^{\cdots}A_s} = \sum_{i,j,p} \left( i \Gamma_{(i,j,p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n')) - i \Gamma_{(j,i,p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n')) \right),$$

when $i \neq j$ or $n \neq n'$,

where $i, j, p$ satisfy (5.54) and $n, n'$ satisfy (5.55).

Now, by straightforward calculations one can verify that each conformal Killing vector product (5.93) for fixed $i, j, p, n, n'$ in the count formulas (5.61) to (5.73) is equal to the monomial $\Gamma_{(i,j,p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n')) + \Gamma_{(j,i,p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n'))$ plus a certain linear combination of monomials $\Gamma_{(i',j',p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n')) + \Gamma_{(j',i',p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n'))$ over smaller index values $0 \leq i' < i, j' < j$. Similar calculations hold for each conformal Killing vector product (5.94) in terms of monomials $i \Gamma_{(i,j,p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n')) - i \Gamma_{(j,i,p)}^{A_1^{\cdots}A_s} (x, \gamma(n,n')) \neq 0$ for $i \neq j$ or $n \neq n'$. Therefore, by induction on $i, j, p, n, n'$, it follows that the set of conformal Killing vector products (5.93) and (5.94) comprise a basis for $\mathcal{K}_{s,s}^R$.

This completes the proof of part (i). The proof of part (ii) is similar and will be omitted.

\[\square\]
Proof of Theorems 5.8 and 5.9.

Let $\Psi^{AA'}$ be any of the currents (5.57), (5.58), (5.59), (5.60). If $q$ is even, we see by Lemma 5.1 that the highest order terms in the spinorial characteristic (5.12) admitted by $\Psi^{AA'}$ are given by (5.38) with $r = (s - 1)/2$, where $\xi^{BC_1 \cdots C_{2r}}_{A'C_{1}' \cdots C_{2r}'}$ is a Killing spinor as given in Lemma 5.10(i), and where $\kappa^{C_1 \cdots C_{2r-1}}_{A'B'C'D'C_{1}' \cdots C_{2r-1}'} = 0$. Similarly, if $q$ is odd, the highest order terms in the spinorial characteristic (5.13) admitted by $\Psi^{AA'}$ are given by (5.41) with $r = s/2 - 1$, where $\xi^{BC_1 \cdots C_{2r+1}}_{A'C_{1}' \cdots C_{2r+1}'}$ is a Killing spinor as above.

Hence, the mappings (5.40) and (5.43) provide an isomorphism $V^+_w \cong \bigoplus_{r=0}^{w} K_{2r+1}$. Thus by Theorem 5.5 and Lemma 5.10 the currents in Theorem 5.8 comprise a basis for $V^+_w$.

Next let $\Psi^{AA'}$ be any of the currents (5.79), (5.80), (5.81), (5.82). We see by Lemma 5.1 that the highest order terms in the spinorial characteristic (5.14) admitted by $\Psi^{AA'}$ are given by (5.38) with $r = (s - 1)/2$, where $\kappa^{C_1 \cdots C_{2r-1}}_{A'B'C'D'C_{1}' \cdots C_{2r-1}'}$ is a Killing spinor as given in Lemma 5.10(ii) and $\xi^{BC_1 \cdots C_{2r}}_{A'C_{1}' \cdots C_{2r}'} = 0$.

Hence, the mapping (5.43) provides an isomorphism $V^-_w \cong \bigoplus_{r=0}^{[w/2]} K_{2r-1,3+2r}$. Thus by Theorem 5.5 and Lemma 5.10 the currents in Theorem 5.9 comprise a basis for $V^-_w$.  

\[ \square \]

C. Classification algorithm

To conclude, we remark that by Theorems 5.4 and 5.5, and Lemma 5.10, there is a simple algorithm for writing any given quadratic conserved current of Maxwell’s equations explicitly as a sum of currents (5.31), (5.32), (5.33) up to a trivial current. Let $\Psi^{AA'}$ be a quadratic current of weight $w \geq 0$ and proceed by the following steps:

(i) Calculate a spinorial characteristic $Q_{AA'}$ for $\Psi^{AA'}$ by (5.29).

(ii) Calculate the spinorial curl $D^A_{(C} Q_{A')}$ and equate its highest order terms to expressions (5.39) or (5.42). There are two cases to consider: Let $q$ denote the order of the curl expression. If $q$ is odd, the highest order terms must match (5.39) for some Killing spinors $\xi^{BC_1 \cdots C_{2r}}_{A'C_{1}' \cdots C_{2r}'}$ and $\kappa^{C_1 \cdots C_{2r-1}}_{A'B'C'D'C_{1}' \cdots C_{2r-1}'}$ with $r = (q - 1)/2$. Then $\Psi^{AA'}$ is equivalent to a sum of quadratic currents (5.31) and (5.33) of weight $q - 1$ plus lower order currents, with
the Killing spinors ξ, ζ1, . . . , ζ2r in (5.31) and the Killing spinors κ, ζ1, . . . , ζ2r−1 in (5.33) given by a respective Killing spinor factorization of ξBC1...C2r and κA′B′C′D′C′1...C′2r−1 as in the proof of Lemma 5.10. Similarly, if q is even, the highest order terms must match (5.42) for some Killing spinor ξBC1...C2r+1A′C′1...C′2r+1 and κC1...C2r−1A′B′C′D′C′1...C′2r−1 as in the proof of Lemma 5.10.

(iii) Subtract from ΨAA′ the quadratic current of minimal weight wQ = q − 1 determined in step (ii) and repeat these steps until q = 1.

By steps (i), (ii), (iii) we have expressed ΨAA′ as a sum of quadratic currents (5.31), (5.32), (5.33) of minimal weights at most wQ plus a trivial current.

VI. CORRESPONDENCE BETWEEN TENSORIAL AND SPINORIAL CURRENTS

A. Currents

In this section we show that the tensorial currents in Theorem 2.1 span the vector space of equivalence classes of quadratic currents in Theorem 5.4.

Let ξμ = eA′AA′ξAA′ be a real conformal Killing vector, and let Yμν = eμAA′eνBB′ (YA′B′εAB + ỸABεA′B′) be a real conformal Killing-Yano tensor. Let Fμν = eμAA′eνBB′ (φABεA′B′ + φA′B′εAB) be a solution of Maxwell’s equations. Write

\[ \phi^{(n)}_ξ AB = (pr X_ξ)^n φ_{AB}, \quad \bar{φ}^{(n)}_ξ A′B′ = (pr X^*_ξ)^n \bar{φ}_{A′B′}, \]

(6.1)

and let ξ = {ξ1, . . . , ξn}, ξ1 = . . . = ξn = ξ, denote the set consisting of n copies of a conformal Killing vector ξ. Then we have the identities

\[ U_{AA'}[ξ^{(n+1)}] = U_{AA'}(φ^{(n)}_ξ; ξ), \quad V_{AA'}[κ, ξ^{(n)}] = V_{AA'}(\bar{φ}^{(n)}_ξ; κ). \]

(6.2)
One can verify by a direct computation that the spinorial forms of the currents (2.22), (2.23), (2.24) are given by, respectively,

\[ \Psi_T^{(n)AA'}(F; \xi) = U_{BB'}^{A'}[\xi^{(n+1)}] \Phi_{\xi}^{(n)AB'} + \bar{U}_{BB'}^{A'}[\xi^{(n+1)}] \phi_{\xi}^{(n)AB}, \]  
(6.3)

\[ \Psi_Z^{(n)AA'}(F; \xi) = 2iU_{BB'}^{A'}[\xi^{(n+2)}] \Phi_{\xi}^{(n)AB'} - 2i\bar{U}_{BB'}^{A'}[\xi^{(n+2)}] \phi_{\xi}^{(n)AB}, \]  
(6.4)

\[ \Psi_V^{(n)AA'}(F; \xi, Y) = 8V_{BB'}^{A'}[\kappa, \xi^{(n+1)}] \Phi_{\xi}^{(n)AB'} + 8\bar{V}_{BB'}^{A'}[\kappa, \xi^{(n+1)}] \phi_{\xi}^{(n)AB}, \]  
(6.5)

where \( \kappa_{A'B'C'D'} = 3Y_{A'B'}Y_{C'D'}. \)

**Proposition 6.1.** The currents \( \Psi_T^{(n)AA'}(F; \xi), \Psi_Z^{(n)AA'}(F; \xi), \Psi_V^{(n)AA'}(F; \xi, Y) \) admit spinorial characteristics (5.4) given by, respectively,

\[ Q_T^{(n)AA'}[\xi] = (-1)^n2U_{AA'}[\xi^{(2n+1)}], \]  
(6.6)

\[ Q_Z^{(n)AA'}[\xi] = (-1)^n4iU_{AA'}[\xi^{(2n+2)}], \]  
(6.7)

\[ Q_V^{(n)AA'}[\xi, Y] = (-1)^n16\sum_{a=0}^{n} \frac{n!}{(n-a)!a!} V_{AA'}[(\mathcal{L}_{\xi})^{a}\kappa, \xi^{(2n+1-a)}]. \]  
(6.8)

These characteristics are adjoint symmetries of order \( q_T = 2n, q_Z = 2n + 1, q_V = 2n + 2, \) respectively.

**Proof.**

The calculation of each characteristic is similar. We will prove (6.6) and omit the proofs of the other two.

By Lemma 5.1 and equation (5.6), the spinorial characteristic of current (6.3) for \( n = 0 \) is \( Q_T^{(0)AA'}[\xi] = 2U_{AA'}[\xi] \) and thus we have

\[ D_{AA'}\Psi_T^{(0)AA'}(F; \xi) = 2U_{AA'}[\xi] \Delta_{AA'} + 2\bar{U}_{AA'}[\xi] \Delta_{AA'} + D_{AA'}Y_1^{AA'}, \]  
(6.9)

where \( Y_1^{AA'} \) is a trivial current. Now, by replacing \( \phi_{AB}^{(n)AB} \) by \( \phi_{\xi}^{(n)AB} \) in (6.9) and then using equations (6.1) and (6.2) together with equation (5.3), we obtain

\[ D_{AA'}\Psi_T^{(n)AA'}(F; \xi) = 2U_{AA'}[\xi^{(n+1)}](\text{pr X}_{\xi})^{n} \Delta_{AA'} + 2\bar{U}_{AA'}[\xi^{(n+1)}](\text{pr X}_{\xi})^{n} \Delta_{AA'} + D_{AA'}Y_2^{AA'}, \]  
(6.10)
where $\Upsilon_{AA'}^2$, $\Upsilon_{AA'}^3$ are trivial currents. By a repeated application of (5.22) and a comparison with (5.6) we conclude that $\Psi_{AA'}^T (F; \xi)$ admits the spinorial characteristic (6.6).

Lemma 6.2. The currents $\Phi_{AA'}^T [\xi, \zeta_1, \ldots, \zeta_{2r}], \Phi_{Z}^{AA'} [\xi, \zeta_1, \ldots, \zeta_{2r+1}], \Phi_{V}^{AA'} [\kappa, \zeta_1, \ldots, \zeta_{2r+1}]$ for $r \geq 0$ are, respectively, equivalent to a linear combination of the currents

$$
\Psi_{AA'}^T (F; \xi), \quad \Psi_{Z}^{AA'}(F; \xi), \quad \Psi_{V}^{AA'}(F; \xi, Y), \quad 0 \leq n \leq r,
$$

(6.11)

involving a sum over conformal Killing vectors $\zeta$ and conformal Killing-Yano tensors $Y$ for each $n$.

Proof.

The computations to prove the equivalence are similar for each pair of currents. We will prove the claim for the pair $\Phi_{AA'}^T [\xi, \zeta_1, \ldots, \zeta_{2r}], \Psi_{AA'}^T (F; \xi)$ and omit the other two.

The image of the current $\Phi_{AA'}^T [\xi, \zeta_1, \ldots, \zeta_{2r}]$ under the map (5.40) is the Killing spinor $\xi_{(C' \zeta_1 C \cdot \cdot \cdot \zeta_{2r} C_{2r})}$ which is a symmetric multilinear expression in $\xi, \zeta_1, \ldots, \zeta_{2r}$ and hence can be written as a linear combination of powers of Killing spinors of the form $\varphi_{(C' \varphi_1 \varphi \cdot \cdot \cdot \varphi_{2r} \varphi_{2r})}$. Thus by Theorem 5.5, there is a linear combination of currents $\Phi_{AA'}^T [\varphi_{(2r+1)}]$ that is equivalent to the current $\Phi_{AA'}^T [\xi, \zeta_1, \ldots, \zeta_{2r}]$ up to a current of lower weight. Hence, by induction on $r$, we have that for all $r \geq 0$ the current $\Phi_{AA'}^T [\xi, \zeta_1, \ldots, \zeta_{2r}]$ is equivalent to a linear combination of currents $\Phi_{AA'}^T [\varphi_{(2n+1)}], 0 \leq n \leq r$, each of which is equivalent to a multiple of a current (5.3) by the relation $Q_{AA'}^T [\varphi] = 2(-1)^n Q_{AA'}^T [\varphi_{(2n+1)}]$ and Proposition 3.1. Thus, the current $\Phi_{AA'}^T [\xi, \zeta_1, \ldots, \zeta_{2r}]$ lies in the equivalence class of a linear combination of the currents $\Psi_{AA'}^T (F; \xi), 0 \leq n \leq r$. □

We conclude with the following Proposition whose proof is immediate.

Proposition 6.3. Under the duality transformation (4.2), the currents $\Psi_{AA'}^T (F; \xi)$, $\Psi_{Z}^{AA'}(F; \xi), \Psi_{V}^{AA'}(F; \xi, Y)$ transform as

$$
\Psi_{AA'}^T \rightarrow \Psi_{AA'}^T, \quad \Psi_{Z}^{AA'} \rightarrow \Psi_{Z}^{AA'}, \quad \Psi_{V}^{AA'} \rightarrow -\Psi_{V}^{AA'},
$$

(6.12)
B. A basis

Here we present a basis for tensorial currents given in Theorem 2.1 by using the basis for the spinorial currents given in Theorems 5.8 and 5.9.

First, by fixing a spinor basis \( \{ o^A, \ell^A \} \) we obtain a null tetrad basis for vectors in Minkowski space,

\[
\ell^\mu = e^\mu_{\ A\ B} o^A o^{A'}, \quad n^\mu = e^\mu_{\ A\ B} \ell^A \ell^{A'},
\]

\[
m^\mu = e^\mu_{\ A\ B} o^A \ell^{A'}, \quad \bar{m}^\mu = e^\mu_{\ A\ B} \bar{o}^{A'} \ell^A.
\]  

(6.13)

Products of the basis spinors \( o^A, \ell^A \) yield a basis for complex skew-tensors in Minkowski space,

\[
e^\mu_{\ A\ B} e^\nu_{\ B' B} o^A o^{A'} \ell^{A'} = \ell^{[\mu \ n]} + i * \ell^{[\mu \ m]},
\]

(6.14)

\[
e^\mu_{\ A\ B} e^\nu_{\ B' B} o^A (\ell^{A'}) \ell^{A'} = \ell^{[\mu \ n]} + i * \ell^{[\mu \ n]},
\]

(6.15)

\[
e^\mu_{\ A\ B} e^\nu_{\ B' B} (\ell^{A'}) \ell^{A'} = n^{[\mu \ m]} + i * n^{[\mu \ m]}.
\]

(6.16)

In terms of expressions (6.13) to (6.16), the basis (5.46) to (5.50) for complex conformal Killing vectors and the basis (5.51) to (5.53) for self-dual conformal Killing-Yano tensors takes the form

\[
\xi^\mu_{0,1} = \ell^\mu, \quad \xi^\mu_{0,2} = m^\mu, \quad \xi^\mu_{0,3} = \bar{m}^\mu,
\]

(6.17)

\[
\xi^\mu_{1,1} = x_\nu (1 - i \ast) \ell^{[\mu \ n]}, \quad \xi^\mu_{1,2} = x_\nu (1 - i \ast) \ell^{[\mu \ m]}, \quad \xi^\mu_{1,3} = x_\nu (1 + i \ast) m^{[\mu \ n]},
\]

(6.18)

\[
\xi^\mu_{2,1} = x_\nu \ell^\mu, \quad \xi^\mu_{2,2} = x_\nu m^\nu - \frac{1}{2} x_\nu x_\rho \ell^\mu, \quad \xi^\mu_{2,3} = x_\nu \bar{m}^\nu - \frac{1}{2} x_\nu x_\rho \bar{m}^\mu,
\]

(6.20)

\[
\xi^\mu_{2,2} = x^\mu \ell^\nu - \frac{1}{2} x^\nu x_\rho \ell^\mu, \quad \xi^\mu_{2,3} = x^\mu m^\nu - \frac{1}{2} x^\nu x_\rho m^\mu, \quad \xi^\mu_{2,3} = x^\mu \bar{m}^\nu - \frac{1}{2} x^\nu x_\rho \bar{m}^\mu,
\]

(6.21)

\[
Y^{\mu \nu}_{0,1} = (1 + i \ast) \ell^{[\mu \ n]}, \quad Y^{\mu \nu}_{0,2} = (1 + i \ast) \ell^{[\mu \ m]}, \quad Y^{\mu \nu}_{0,3} = (1 + i \ast) n^{[\mu \ m]},
\]

(6.23)

\[
Y^{\mu \nu}_{1,1} = (1 + i \ast) x^{[\mu \ \ell]}, \quad Y^{\mu \nu}_{1,2} = (1 + i \ast) x^{[\mu \ \bar{m}]}.
\]

(6.24)
\[ Y_{1,3}^{\mu
u} = (1 + i\star)x[^{\mu}\overline{m}^{\nu}], \quad Y_{1,4}^{\mu
u} = (1 + i\star)x[^{\mu}n^{\nu}], \quad (6.25) \]

\[ Y_{2,1}^{\mu
u} = x^{\mu}x_{\sigma}(1 - i\star)\ell[^{\nu}m^{\sigma}] - x^{\nu}x_{\sigma}(1 - i\star)\ell[^{\mu}m^{\nu}] + \frac{1}{2}x^{2}(1 - i\star)\ell[^{\mu}\overline{m}^{\nu}], \quad (6.26) \]

\[ Y_{2,2}^{\mu
u} = x^{\mu}x_{\sigma}(1 - i\star)\ell[^{\nu}n^{\sigma}] - x^{\nu}x_{\sigma}(1 - i\star)\ell[^{\mu}n^{\nu}] + \frac{1}{2}x^{2}(1 - i\star)\ell[^{\mu}\overline{n}^{\nu}], \quad (6.27) \]

\[ Y_{2,3}^{\mu
u} = x^{\mu}x_{\sigma}(1 - i\star)\overline{m}[^{\nu}\overline{n}^{\sigma}] - x^{\nu}x_{\sigma}(1 - i\star)\overline{m}[^{\mu}\overline{n}^{\nu}] + \frac{1}{2}x^{2}(1 - i\star)\overline{m}[^{\mu}\overline{n}^{\nu}]. \quad (6.28) \]

Note that, geometrically, the Killing vectors (6.17) to (6.22) describe 4 null translations, 6 null boosts, 1 dilation, 4 null conformal transformations.

**Proposition 6.4.** Let \( \xi_{i} \) denote the conformal Killing vectors in (5.54) to (5.60) and (5.73) to (5.82) written in the tensorial form (6.17) to (6.22), and let \( Y_{i} \) denote the conformal Killing-Yano tensors in (5.73) to (5.82) written in the tensorial form (6.23) to (6.28). Then the tensorial forms of the currents (5.54) to (5.60), (5.73) to (5.82) are, respectively, given by

\[ \Phi_{+}^{\mu} = F^{\mu\sigma}F_{\nu\sigma}[\xi_{1}, \ldots, \xi_{q}]\xi_{q+1}^{\nu} - \frac{1}{4}F^{\mu\sigma}F_{\nu\sigma}[\xi_{1}, \ldots, \xi_{q}]\xi_{q+1}^{\nu} + c.c., \quad (6.29) \]

\[ \Phi_{T}^{\mu} = iF^{\mu\sigma}F_{\nu\sigma}[\xi_{1}, \ldots, \xi_{q}]\xi_{q+1}^{\nu} - \frac{1}{4}F^{\mu\sigma}F_{\nu\sigma}[\xi_{1}, \ldots, \xi_{q}]\xi_{q+1}^{\nu} + c.c., \quad (6.30) \]

\[ \Phi_{Z}^{\mu} = F^{\mu\sigma}F_{\nu\sigma}[\xi_{1}, \ldots, \xi_{q}]\xi_{q+1}^{\nu} - F^{\mu\sigma}F_{\nu\sigma}[\xi_{1}, \ldots, \xi_{q}]\xi_{q+1}^{\nu} + c.c., \quad (6.31) \]

\[ \Phi_{Z}^{\mu} = iF^{\mu\sigma}F_{\nu\sigma}[\xi_{1}, \ldots, \xi_{q}]\xi_{q+1}^{\nu} - i*F^{\mu\sigma}F_{\nu\sigma}[\xi_{1}, \ldots, \xi_{q}]\xi_{q+1}^{\nu} + c.c., \quad (6.32) \]

\[ \Phi_{+V}^{\mu} = F_{\nu\sigma}(D_{\nu}F_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}])Y^{[\nu\lambda\sigma\alpha\beta]}Y^{\nu\lambda\sigma\alpha\beta} + 4F_{\nu\sigma}(D_{\nu}F_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}])Y^{\nu\lambda\sigma\alpha\beta} \]

\[ + \frac{3}{5}F_{\nu\sigma}F_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}]\partial^{\mu}Y^{\nu\lambda\sigma\alpha\beta} + \frac{12}{5}F_{\nu\sigma}F_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}]\partial_{\nu}Y^{\nu\lambda\sigma\alpha\beta} + c.c., \quad (6.33) \]

\[ \Phi_{-V}^{\mu} = F_{\nu\sigma}(D^{\mu}F_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}])Y^{\nu\lambda\sigma\alpha\beta} + 4F_{\nu\sigma}(D^{\mu}F_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}])Y^{\nu\lambda\sigma\alpha\beta} \]

\[ + \frac{3}{5}F_{\nu\sigma}F_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}]\partial^{\mu}Y^{\nu\lambda\sigma\alpha\beta} + \frac{12}{5}F_{\nu\sigma}F_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}]\partial_{\nu}Y^{\nu\lambda\sigma\alpha\beta} + c.c., \quad (6.34) \]

\[ \Phi_{-V'}^{\mu} = iF_{\nu\sigma}(D^{\mu}F^{q}_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}])Y^{\nu\lambda\sigma\alpha\beta} + 4iF_{\nu\sigma}(D^{\mu}F^{q}_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}])Y^{\nu\lambda\sigma\alpha\beta} \]

\[ + \frac{3}{5}F_{\nu\sigma}F^{q}_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}]\partial^{\mu}Y^{\nu\lambda\sigma\alpha\beta} + \frac{12}{5}F_{\nu\sigma}F^{q}_{\alpha\beta}[\xi_{1}, \ldots, \xi_{q-1}]\partial_{\nu}Y^{\nu\lambda\sigma\alpha\beta} + c.c., \quad (6.35) \]

where
\[
F_{\mu \nu}^{(n)}[\xi_1, \ldots, \xi_n] = \frac{1}{n!} \sum_{s \in S_n} \mathcal{L}_{\xi_{s(1)}} \cdots \mathcal{L}_{\xi_{s(n)}} F_{\mu \nu},
\]

\[
Y^{\nu \sigma \alpha \beta} = \frac{1}{2} \sum_{s \in S_2} \left( Y^{\nu \sigma}_{s(1)} Y^{\alpha \beta}_{s(2)} - Y^{\nu [\alpha}_{s(1)} Y^{\beta \sigma]}_{s(2)} - 3\eta^{[\nu [\alpha} Y^{\beta \sigma]}_{s(1)\tau} \eta^{\nu [\tau]}_{s(2)} \right) + \frac{1}{2} \eta^{[\nu [\alpha} \eta^{\beta \sigma]}_{s(1)\tau} Y^{\nu [\tau]}_{s(2)} Y^{\nu [\tau]}_{s(2)} .
\]

Here c.c. stands for the complex conjugate of all preceding terms in an expression.

The proof of Proposition 6.4 is a straightforward computation and will be omitted.

Now, by converting the basis of currents in Theorems 5.8 and 5.9 into tensorial form using Proposition 6.4, we obtain the following tensorial basis for the vector space of quadratic currents \( V_w = V_w^+ \oplus V_w^- \) of weight at most \( w \).

**Theorem 6.5** A tensorial basis for \( V_w^+ \) is given by the set of all currents (6.29) and (6.30) for \( 0 \leq p \leq 2q + 2, \quad q = 2r, \quad 0 \leq r \leq [w/2] \), and (6.31) and (6.32) for \( 0 \leq p \leq 2q + 2, \quad q = 2r + 1, \quad 0 \leq r \leq [(w-1)/2] \), in which the conformal Killing vectors are given by the count formulas (5.61) to (5.73) indexed by \( i, j, n, n' \) satisfying (5.54) and (5.55), with currents (6.30) and (6.32) restricted to \( i \neq j \) or \( n \neq n' \). A tensorial basis for \( V_w^- \) is given by the set of all currents (6.33) to (6.36) for \( 0 \leq p \leq 2q + 2, \quad q = 2r + 2, \quad 0 \leq r \leq [w/2] - 1 \), in which the conformal Killing vectors and conformal Killing-Yano tensors are given by the respective count formulas (5.61) to (5.73) and (5.83) to (5.92) indexed by \( i, j, k, n, n', m, m' \) satisfying (5.74) to (5.77), with currents (6.34) and (6.36) restricted to \( i \neq j \) or \( n \neq n' \).

We remark that the tensorial basis currents in Theorem 6.5 can be expressed as linear combinations of the currents in Theorem 2.2 by the procedure used in the proof of Lemma 6.2.

**VII. CONCLUDING REMARKS**

In this paper we classify all local conservation laws of Maxwell’s equations in Minkowski space in a systematic fashion by classifying their characteristics. Even though Maxwell’s equations are a degenerate system of PDEs, we are able to establish a one-to-one correspondence...
dence between classes of equivalent conserved currents and classes of equivalent characteristics. We find the characteristics by solving the adjoint symmetry equations of Maxwell’s equations by means of spinorial methods, leading, essentially, to a one-to-one correspondence between classes of adjoint symmetries and Killing spinors of certain type. Interestingly, we find classes of adjoint symmetries that are not equivalent to characteristics and hence do not correspond to conserved currents. We also identify a recursion structure within the spaces of adjoint symmetries and conserved currents which is induced by Lie derivatives with respect to conformal Killing vectors. The use of spinorial methods allows us to obtain all conserved currents explicitly, in a unified manner in coordinate invariant form in terms of Killing spinors, and this leads to the identification of new chiral conserved currents along with an associated conserved tensor. In addition, by means of a factorization of Killing spinors in Minkowski space, we exhibit a basis for conserved currents of any order or weight.

Moreover, our classification extends to conservation laws of Maxwell’s equations \( \nabla^\mu F_{\mu \nu}(x) = 0 \) and \( \nabla^\mu \ast F_{\mu \nu}(x) = 0 \) in a curved background metric \( g_{\mu \nu} \). Here \( \nabla_\mu \) and \( \ast \) stand for the torsion-free covariant derivative and Hodge star operator associated to \( g_{\mu \nu} \). All local conservation laws continue to arise from adjoint symmetries of Maxwell’s equations through the integral formula (3.11). The adjoint symmetries can be obtained in a straightforward manner in spinor form and involve Killing spinors of the curved metric. Interestingly, the Killing spinor equations now possess integrability conditions [13] which lead to restrictions on the curvature tensor of \( g_{\mu \nu} \). Furthermore, additional curvature conditions arise from the determining equations for the adjoint symmetries.

Consequently, non-trivial conservation laws of Maxwell’s equations exist only for certain classes of metrics \( g_{\mu \nu} \). A complete analysis of the curvature conditions will be explored elsewhere. Of particular interest is the family of black-hole spacetime metrics, since Maxwell’s equations admit non-trivial symmetries [21] in addition to the symmetries due to spacetime isometries in the Kerr spacetime metric. The methods of Secs. [11] to [14] can be expected to resolve the issue of whether Maxwell’s equations possess corresponding conservation laws.

The relation between the local conservation laws and local symmetries of Maxwell’s
equations, in flat and curved spacetime, will be explored in a subsequent paper [16].

Finally, our methods can be extended to the analysis of local conservation laws of other physical field equations, in particular, the linearized gravity wave equation on flat and curved spacetimes.
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