Riemannian and Non-commutative Geometry in Physics

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Abstract

We feel that non-commutative geometry is to particle physics what Riemannian geometry is to gravity. We try to explain this feeling.

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We feel that non-commutative geometry is as fundamental to physics as Minkowskian and Riemannian geometry. Let us try to explain this by comparing the standard model of particle physics and general relativity. From a chronological point of view, this comparison is difficult, because Riemannian geometry existed well before general relativity. However, the field theoretic approach allows to introduce general relativity in close analogy to classical electrodynamics without use of Riemannian geometry. Therefore this approach is well suited for our comparison.

So let us imagine a world ignoring Riemannian geometry where physicists try to describe gravity. They are inspired by Maxwell who takes a field $A$ of spin 1, a second order differential operator $D_{\text{Max}}$ and writes down his field equation

$$D_{\text{Max}} A = \frac{1}{c^2 \epsilon_0} j,$$

where $j$ is the source, charge density and currents, and $\epsilon_0$ is the proportionality constant from Coulomb’s law. After many ingenious and expensive experiments and theoretical trials and errors, the physicists agree on the standard model of gravity. It starts from a particular spin 2 field $g$, and a second order differential operator $D_{\text{Ein}}$. The field equation is

$$D_{\text{Ein}} g = -\frac{8\pi G}{c^4} T,$$

where the source $T$ is energy-momentum density and currents, and $G$ is the proportionality constant from Newton’s universal law of gravity. Although in perfect agreement with experiment, this standard model has drawbacks: who ordered spin 2? Maxwell’s differential operator $D_{\text{Max}}$ contains 8 summands, the gravitational one $D_{\text{Ein}}$ results from brute force and contains roughly 80 000 summands. Some of these summands still are unaccessable to experiment. At this stage, Riemannian geometry is discovered, the spin 2 field is recognized as the metric and the differential operator $D_{\text{Ein}}$ is recognized as the curvature if the unknown summands are chosen properly. Most physicists say: so what, just fancy mathematics. Some dream of a geometric unification of all forces. Later, even more expensive experiments will test the predictions of Riemannian geometry coming from the unknown summands.

If, in the real world, we qualify general relativity as revolution, we have several criteria.

- Postdiction: the theory correctly reproduces experimental data, that remain unexplained in the old theories, e.g. the precession of perihelia of Mercury.
- Prediction: the theory can be in contradiction with future experimental data, e.g. deflection of light.
- New concepts, e.g. curved spacetimes, absence of universal time.
- Reticence of the majority.

Our purpose is to explain that for non-commutative geometry the analogue of $g$ in the imaginary world is the Higgs field, the analogue of $D_{\text{Ein}}$ is the Lagrangian of the standard
model of electro-weak and strong interactions. Postdictions of the theory are that fermions sit in fundamental representations, that weak interactions violate parity, that strong interactions are vector like and

\[ \rho := \frac{g_1^2 + g_2^2}{g_2^2} \frac{m_W^2}{m_Z^2} = 1, \]

\[ m_e < m_W < m_t/\sqrt{3}, \]

\[ 2g_1^{-2} > g_2^{-2} + g_3^{-2}/3. \]

There is also a prediction, the mass of the Higgs, accessible to experiment in about ten years. New concepts are fuzzy spacetimes — that is spacetimes with an uncertainty relation — and discrete spacetimes.

1 The Establishment

Let us briefly summarize today’s established theory of particles and interactions. It is a particular Yang-Mills-Higgs theory. To get started, we view this class of theories as black box or slot machine. The input comes in two parts, bills and coins. The output is a particle phenomenology, that is cross sections, branching ratios, life times ... To decide whether a particular input is a winner, its corresponding output is confronted with millions of experimental numbers that cost billions of Swiss Francs.

1.1 Bills and coins

The Yang-Mills-Higgs machine has four slots for one bill each. In the first of these slots you are supposed to put a finite dimensional, real, compact Lie group \( G \). For the remaining slots choose three unitary representations \( \rho_L, \rho_R, \rho_S \) defined on Hilbert spaces \( \mathcal{H}_L, \mathcal{H}_R, \mathcal{H}_S \). These Hilbert spaces will accommodate the left- and right-handed fermions and the Higgs scalars.

After having eaten these four bills, the machine will ask you for coins, real or complex numbers. The number of coins depends on the chosen bills.

- An invariant scalar product on the Lie algebra \( \mathfrak{g} \) of \( G \). This choice is parameterized by one positive number \( g \), the ‘gauge coupling’, for every simple factor in \( G \), e.g.

\[ (b, b') := \frac{1}{g_1^2} b \bar{b}', \quad b, b' \in u(1), \]

\[ (a, a') := \frac{2}{g_2^2} \text{tr}(a^* a'), \quad a, a' \in su(n). \]

- An invariant, positive polynomial \( V(\varphi), \varphi \in \mathcal{H}_S \) of order 4, the ‘Higgs potential’. We want this potential to break \( G \) spontaneously. This means that no invariant vector
in $\mathcal{H}_S$ minimizes $V$. For example if $G = SU(2)$ with the fundamental representation $\mathcal{H}_S = \mathbb{C}^2$, the most general Higgs potential is

$$V(\varphi) = \lambda(\varphi^*\varphi)^2 - \frac{\mu^2}{2}\varphi^*\varphi, \quad \varphi \in \mathcal{H}_S, \quad \lambda, \mu > 0.$$ 

- One complex number or ‘Yukawa coupling’ $g_Y$ for every trilinear invariant — i.e. for every one dimensional invariant subspace, ‘singlet’ — in the decomposition of the representation associated to $(\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S) \oplus (\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S^*)$. For example if $G = SU(2)$, $\mathcal{H}_L = \mathbb{C}^2$, $\mathcal{H}_R = \mathbb{C}$, $\mathcal{H}_S = \mathbb{C}^2$ there is one singlet:

$$\sum_{j=1}^{2} \bar{\psi}_{Lj} \psi_{Rj} \varphi_j, \quad \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix} \in \mathcal{H}_L, \quad \psi_R \in \mathcal{H}_R, \quad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathcal{H}_S.$$

Physicist have been playing on this slot machine for the last thirty years. One winner clearly emerged, the so called standard model. Its bills are

$$G = SU(3) \times SU(2) \times U(1)$$

$$\mathcal{H}_L = \bigoplus_1^3 \left[ (1, 2, -\frac{1}{2}) \oplus (3, 2, \frac{1}{2}) \right], \quad (1)$$

$$\mathcal{H}_R = \bigoplus_1^3 \left[ (1, 1, -1) \oplus (3, 1, \frac{2}{3}) \oplus (3, 1, -\frac{1}{3}) \right], \quad (2)$$

$$\mathcal{H}_S = (1, 2, -\frac{1}{2}), \quad (3)$$

where $(n_3, n_2, y)$ denotes the tensor product of an $n_3$ dimensional representation of $SU(3)$, an $n_2$ dimensional representation of $SU(2)$ and the one dimensional representation of $U(1)$ with hypercharge $y$:

$$\rho(e^{i\theta}) = e^{iy\theta}, \quad y \in \mathbb{Q}, \ \theta \in [0, 2\pi).$$

Some vocabulary: particles are basis elements. The spin 1 particles, the gauge bosons, span the Lie algebra $\mathfrak{g}$ of the group $G$. The eight basis elements of $su(3)$ are called gluons. They are massless and mediate the strong interactions, e.g. nuclear fusion, fission, $\alpha$-decay. The remaining $su(2) \oplus u(1)$ is spanned by the photon — Maxwell’s old friend and later found responsible for $\gamma$-decay — and three massive bosons, the $W^+$, $W^-$ and $Z$. They mediate the weak interactions, e.g. $\beta$-decay. The spin $\frac{1}{2}$ particles or fermions come in three identical copies, ‘generations’. The first generation of $\mathcal{H}_L$ is spanned by the left-handed parts (Weyl spinors) of the electronic neutrino, the electron and the up and down quarks. The first two are called leptons, from the greek word for mild, because sitting in $SU(3)$ singlets they are not subject to strong interactions. The other two left-handed generations are spanned by the muonic neutrino, the muon, the charm and strange quarks, and the tau neutrino, the tau, the top and bottom quarks. $\mathcal{H}$ is spanned by the right-handed parts of the same particles, except that there are
no right-handed neutrinos. Consequently the neutrinos are massless. The particle count for
the spin 0 particles, scalars, is a little bit more complicated. Not all basis elements of \( \mathcal{H}_S \)
correspond to physical scalars. There is only one in the standard model. It is called Higgs
scalar and is still being searched for.

Because of the high degree of reducibility in the bills, there are many coins, among them
27 Yukawa couplings. Not all of them have a physical meaning. They can be converted into 18
physically significant, positive numbers \[1\], three gauge couplings,

\[
g_1 = 0.3575 \pm 0.0001, \quad g_2 = 0.6507 \pm 0.0007, \quad g_3 = 1.207 \pm 0.026,
\]
eleven particle masses,

\[
m_W = 80.22 \pm 0.26 \text{ GeV}, \quad m_H > 58.4 \text{ GeV},
\]
\[
m_e = 0.51099906 \pm 0.00000015 \text{ MeV}, \quad m_u = 5 \pm 3 \text{ MeV}, \quad m_d = 10 \pm 5 \text{ MeV},
\]
\[
m_\mu = 0.105658389 \pm 0.000000034 \text{ GeV}, \quad m_c = 1.3 \pm 0.3 \text{ GeV}, \quad m_s = 0.2 \pm 0.1 \text{ GeV},
\]
\[
m_\tau = 1.7771 \pm 0.0005 \text{ GeV}, \quad m_t = 176 \pm 18 \text{ GeV}, \quad m_b = 4.3 \pm 0.2 \text{ GeV},
\]

and quark mixings. These mixings are given in form of a unitary matrix, the Cabbibo-
Kobayashi-Maskawa matrix

\[
C_{KM} := \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}.
\]

For physical purposes it can be parameterized by three angles \( \theta_{12}, \theta_{23}, \theta_{13} \) and one \( CP \)
violating phase \( \delta \):

\[
C_{KM} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & s_{23}c_{13}
\end{pmatrix},
\]

with \( c_{kl} := \cos \theta_{kl}, \ s_{kl} := \sin \theta_{kl} \). The absolute values of the matrix elements are:

\[
\begin{pmatrix}
0.9753 \pm 0.0006 & 0.221 \pm 0.003 & 0.004 \pm 0.002 \\
0.221 \pm 0.003 & 0.9745 \pm 0.0007 & 0.040 \pm 0.008 \\
0.010 \pm 0.006 & 0.039 \pm 0.009 & 0.9991 \pm 0.0004
\end{pmatrix}.
\]

Every body agrees that the standard model is ugly, too ugly to be a fundamental theory.

1.2 The general rules

Let us now have a closer look at the inside of the Yang-Mills-Higgs machine. It produces
a Lagrangian that consists of five separate pieces. The first is the Yang-Mills Lagrangian,
well motivated physically as non-abelian generalization of the famous Maxwell Lagrangian,
\( G = U(1) \). Also on the mathematical side, this Lagrangian needs no further introduction. Its
fundamental field are the gauge bosons or connection, \( A \in \Omega^1(M, g) \), a 1-form on the spacetime manifold \( M \) with values in the Lie algebra \( g \):

\[
\mathcal{L}_{YM}[A] = \frac{1}{4}(F, *F),
\]

where \( F := dA + \frac{1}{2}[A, A] \) denotes the field strength or curvature of \( A \), \(*\) is the Hodge star, and \((\cdot, \cdot)\) is the chosen invariant scalar product on \( g \).

The second piece is the Dirac Lagrangian. It is geometrically as noble as the Yang-Mills Lagrangian.

\[
\mathcal{L}_D[A, \psi_L, \psi_R] = \psi_L^* \bar{\mathcal{D}} \psi_L + \psi_R^* \bar{\mathcal{D}} \psi_R,
\]

where \( \psi_L \) is a left-handed spinor with values in \( \mathcal{H}_L \), \( \psi_L^* \) is its dual with respect to the scalar product in \( \mathcal{H}_L \), \( \bar{\mathcal{D}} \) is the covariant Dirac operator, \( \mathcal{D} \psi_L := d\psi_L + \tilde{\rho}_L(A)\psi_L \). We denote by \( \tilde{\rho} \) the Lie algebra representation belonging to the group representation \( \rho \). For \( G = U(1) \) these two Lagrangians yield the very successful quantum electrodynamics and for \( G = SU(3) \), \( \mathcal{H}_L = \mathcal{H}_R = \mathbb{C}^3 \) we get the present day theory for strong interactions, quantum chromodynamics.

In order to incorporate weak interactions and to give masses to gauge bosons and fermions, one is forced to break the symmetry \( G \) spontaneously. This is where the patchwork starts. One has to postulate the existence of scalars, till now unobserved. They are 0-forms with values in \( \mathcal{H}_S \),

\[
\varphi \in \Omega^0(M, \mathcal{H}_S).
\]

One also has to add three more Lagrangian pieces involving the scalars, the Klein-Gordon Lagrangian

\[
\mathcal{L}_{KG}[A, \varphi] = \frac{1}{2} \mathcal{D} \varphi^* \mathcal{D} \varphi,
\]

the Higgs potential and the Yukawa terms

\[
\mathcal{L}_{Yu}[\psi_L, \psi_R, \varphi] = \sum_{j=1}^{n} g_{Yj} (\psi_L^*, \psi_R, \varphi)_j + \sum_{j=n+1}^{m} g_{Yj} (\psi_L^*, \psi_R, \varphi^*)_j + \text{complex conjugate}.
\]

To summarize, the standard model has two major shortcomings, the general rules of Yang-Mills-Higgs look artificial, as well as the input, bills and coins, singled out by nature. Nevertheless the standard model has resisted to an extremely detailed experimental analysis where all concurrent models have failed.

## 2 The Revolution

The non-commutative formulation improves the situation on all three levels, general rules, bills and coins.
2.1 General rules

The Yang-Mills and Dirac Lagrangians have a geometric origin and Alain Connes found a natural generalization of some of them to non-commutative geometry [2]. Connes and Lott have considered these Lagrangians in the particular case of a product geometry of an ordinary four dimensional spacetime geometry by a zero dimensional non-commutative geometry. There a miracle happens [2, 3]. When decomposing the non-commutative versions of the Yang-Mills and Dirac Lagrangians in terms of ordinary fields they retrieve of course the ordinary Yang-Mills and Dirac Lagrangians. Simultaneously and free of charge, they also get the other three pieces, the Klein-Gordon Lagrangian, the symmetry breaking Higgs potential, and some Yukawa terms. Every such Connes-Lott model yields a particular Yang-Mills-Higgs model. The contrary is far from being true, how far will be discussed in terms of bills and coins in the following subsections.

2.2 Bills

Since the introduction of quantum mechanics, we are used to the description of non-commutative spaces in terms of involution algebras. A zero dimensional non-commutative space is given by a finite dimensional, real involution algebra $\mathcal{A}$. The group $G$ of the ensuing gauge model will be the group of unitaries of $\mathcal{A}$

$$\{a \in \mathcal{A} \mid a^* a = a a^* = 1\}$$

or possibly a subgroup thereof. In order to construct a differential calculus on the non-commutative space, Connes introduces two algebra representations $\rho_L$ and $\rho_R$ on Hilbert spaces $\mathcal{H}_L$ and $\mathcal{H}_R$ such that $\rho_L \oplus \rho_R$ is faithful. In the finite dimensional case, this implies that $\mathcal{A} = M_n(\mathbb{R}), M_n(\mathbb{C})$ or $M_n(\mathbb{H})$, $\mathbb{H}$ denoting the quaternions, and that the algebra representations are copies of the defining representation. For $M_n(\mathbb{C})$, there is—in addition to the defining representation—its conjugate. In terms of bills of the resulting Yang-Mills-Higgs model we have the following irreducible possibilities:

$$G = O(n, \mathbb{R}), \quad \mathcal{H}_{L,R} = \mathbb{C}^n,$$

$$G = U(n) \text{ or } SU(n), \quad \mathcal{H}_{L,R} = \mathbb{C}^n,$$

$$G = USp(n), \quad \mathcal{H}_{L,R} = \mathbb{C}^{2n}.$$

The restriction on the group bill is mild, only the exceptional groups are excluded. The restrictions on the two fermionic bills is appreciable, e.g. $U(1)$ only admits hypercharge -1, or 1, $SU(2)$ only has one irreducible representation, the two dimensional one, while in the general setting there is an infinite number to choose from.

The restriction on the scalar bill is spectacular. It comes out to be a group representation, a unitary representation of the group of unitaries, and is restricted by the fermionic bills: its Hilbert space is an invariant subspace,

$$\mathcal{H}_S \subset (\mathcal{H}_L^* \otimes \mathcal{H}_R) \oplus (\mathcal{H}_R^* \otimes \mathcal{H}_L). \quad (4)$$
This invariant subspace is entirely determined by the coins.

One is of course tempted to build models with a simple algebra and/or irreducible fermion representations. Besides phenomenological shortcomings, all such models have a degenerate vacuum, an invariant vector in $\mathcal{H}$, that minimizes the Higgs potential. All popular Grand Unifies Theories are excluded in Connes and Lott’s approach. Similarly, all left-right symmetric models are excluded, because the constraint $\mathfrak{H}$ forbids spontaneous parity violation. The minimal non-commutative model without degeneracy turns out to be the $SU(2) \times U(1)$ model of weak interactions with two generations of leptons:

$$\begin{align*}
\mathcal{H}_L &= \bigoplus_1^2 (2, 0), \\
\mathcal{H}_R &= \bigoplus_1^2 (1, -1).
\end{align*}$$

Comparing with (1-2), we see that the hypercharges are wrong. They are corrected by the inclusion of strong interactions.

This inclusion requires a new ingredient $\mathfrak{H}$, a real structure or — in physical terms — a generalization of charge conjugation to non-commutative geometry. The existence of a real structure implies additional constraints on the fermion representations. The representations (1-3) of the standard model have four features.

- The weak interaction $SU(2)$ violates parity maximally, it acts only on left-handed fermions.
- The strong interaction $SU(3)$ is vectorial, it acts in the same way on left- and right-handed fermions.
- The scalars transform under $SU(2)$, implying spontaneous breaking of $SU(2)$ that renders its gauge bosons, $W^+, W^-$ and $Z$, massive.
- The scalars do not transform under $SU(3)$. It remains unbroken and its gauge bosons, the gluons, massless.

In a Yang-Mills-Higgs theory these four features are independent, not so in the non-commutative approach. We already stated that the scalar representation is not chosen and the two last features follow from the two first. On top, the existence of a real structure implies that the first feature implies the second $\mathfrak{H}$.

The existence of a real structure is intimately related to another mathematical property, a non-commutative version of Poincaré duality which puts still another constraint on the fermion representations. It turns out that this constraint is fulfilled in the standard model (1-2). However, slightly modifying $\mathcal{H}_R$ by adding right-handed neutrinos — a modification compatible with all constraints so far $\mathfrak{H}$ — violates this additional constraint $\mathfrak{H}$.
2.3 Coins

In a Yang-Mills-Higgs model, that comes from a Connes-Lott model, the coins cannot be chosen independently. In an arbitrary Yang-Mills-Higgs model the choice of coins is a point in the space of direct products of intervals. In a Connes-Lott model this point must lie in a subspace. This subspace is a submanifold with interesting structure. Depending on the choice of bills, this submanifold may be of the same dimension as its surrounding hypercube or not. Due to the high degree of reducibility of its fermionic Hilbert space, the standard model is in the first case. Its Connes-Lott submanifold is an open subset of its Yang-Mills-Higgs hypercube given by the inequalities

\[ m_e < m_W < m_t / \sqrt{3}, \]
\[ 2g_1^{-2} > g_2^{-2} + g_3^{-2} / 3 \]
\[ m_{H \text{min}} < m_H < m_{H \text{max}}. \]

The bounds on the Higgs mass are complicated functions of the other coins. The fact that the non-commutative constraints on the parameters of the standard model are inequalities rather than equations may be important to insure their stability under renormalization flow. On the other hand, for the experimental values of the parameters

\[ \frac{m_{H \text{max}} - m_{H \text{min}}}{(m_{H \text{max}} + m_{H \text{min}}) / 2} \simeq \frac{m_\tau^2 - m_e^2}{m_t^2} \simeq 10^{-4} \]

and for all practical purpose the Higgs mass is fixed. To our knowledge, this is the first mass relation, that comes with a (small) conceptual uncertainty and we call it a fuzzy relation. We stress that the fuzziness of the Higgs mass requires the existence of at least two generations.

3 Conclusion

The first miracle of non-commutative geometry applied to particle physics concerns the general rules. Here, this geometry answers the question: Who ordered the Higgs.

The two subspaces of bills and coins accessible to a Connes-Lott model have interesting structure and are tiny compared to the two Yang-Mills-Higgs hypercubes, fig. 1 and 2. To us, it is a second miracle that the two points defining the standard model fall into these tiny subspaces, at least as long as the Higgs mass is unknown.

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**Figure captions**

Fig.1: An artist’s partial view of the space of bills of all Yang-Mills-Higgs models and some of its subspaces. GUT stands for ‘Grand Unified Theories’, i.e. $G$ simple. $L - R$ stands for left-right symmetric models, i.e. $\mathcal{H}_L = \mathcal{H}_R$. SM stands for standard model and $CL$ for Connes-Lott models.

Fig.2: Partial view of the space of coins of the standard model, lower and upper bounds of the Higgs mass as a function of the top and $\tau$ masses, all other coins are set to their experimental values. For the experimental value, $m_\tau = 1.8$ GeV, the two bounds differ by $10^{-2}$ GeV in the indicated range of $m_t$. 
