BOUNDS ON KRONECKER AND $q$-BINOMIAL COEFFICIENTS

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Abstract. We present a lower bound on the Kronecker coefficients of the symmetric group via the characters of $S_n$, which we apply to obtain various explicit estimates. Notably, we extend Sylvester’s unimodality of $q$-binomial coefficients $\binom{n}{k}_q$ as polynomials in $q$ to derive sharp bounds on the differences of their consecutive coefficients.

1. Introduction

The Kronecker coefficients are perhaps the most challenging, deep and mysterious objects in Algebraic Combinatorics. Universally admired, they are beautiful, unapproachable and barely understood. For decades since they were introduced by Murnaghan in 1938, the field lacked tools to study them, so they remained largely out of reach. However, in recent years a flurry of activity led to significant advances, spurred in part by the increased interest and applications to other fields. We refer to [PP5] for a detailed survey of these advances and further references.

In this paper, we focus on lower bounds for the Kronecker coefficients. We are motivated by applications to the $q$-binomial (Gaussian) coefficients, and by connections to the Geometric Complexity Theory (see §6.2). The tools are based on technical advances in combinatorial representation theory obtained in recent years, see [BOR, CDW, CHM, Man, Val], and our own series of papers [PP1, PP2, PP3, PPV]. In fact, here we give several extensions of our earlier work.

The Kronecker coefficients $g(\lambda, \mu, \nu)$ are defined by:

\[(1.1) \quad \chi^{\lambda} \otimes \chi^{\mu} = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) \chi^{\nu}, \quad \text{where } \lambda, \mu \vdash n,\]

where $\chi^\alpha$ denotes the character of the irreducible representation $S^\alpha$ of $S_n$ indexed by partition $\alpha \vdash n$. They are integer and nonnegative by definition, have full $S_3$ symmetry, and satisfy a number of further properties (see [2.2]). In contrast with their “cousins” Littlewood–Richardson (LR) coefficients, they lack a combinatorial interpretation or any meaningful positive formula, and thus harder to compute and to estimate.

Our first main result is a lower bound of the Kronecker coefficients $g(\lambda, \mu, \mu)$ for multiplicities in tensor squares of self-conjugate partitions:

**Theorem 1.1.** Let $\mu = \mu'$ be a self-conjugate partition and let $\hat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n$ be the partition of its principal hooks. Then:

$$g(\lambda, \mu, \mu) \geq |\chi^{\lambda}[\hat{\mu}]|, \quad \text{for every } \lambda \vdash n.$$
While it is relatively easy to obtain various trivial upper bounds on the Kronecker coefficients (see e.g. (2.1)), this is the only general lower bound that we know. The theorem strengthens a qualititative result \( g(\lambda, \mu, \mu) \geq 1 \) given in [PPV, Lemma 1.3], used there to prove a special case of the Saxl conjecture (see §6.4). We use the bound to give a new proof of Stanley’s Theorem 4.1, from [Sta1].

Our second main result is motivated by an application of bounds for Kronecker coefficients to the \( q \)-binomial coefficients, defined as:

\[
\left( \begin{array}{c} m + \ell \\ m \end{array} \right)_q = \frac{(q^{m+1} - 1) \cdots (q^{m+\ell} - 1)}{(q - 1) \cdots (q^\ell - 1)} = \sum_{n=0}^{\ell m} p_n(\ell, m) q^n,
\]

where \( p_n(\ell, m) \) is also the number of partitions of \( n \) which fit inside an \( \ell \times m \) rectangle. In 1878, Sylvester proved unimodality of the coefficients:

\[
p_0(\ell, m) \leq p_1(\ell, m) \leq \ldots \leq p_{\ell m/2}(\ell, m) \geq \ldots \geq p_{\ell m}(\ell, m),
\]

see [Syl]. In [PP1], we used the Kronecker coefficients to prove strict unimodality:

\[
(1.2) \quad p_k(\ell, m) - p_{k-1}(\ell, m) \geq 1, \quad \text{for} \quad 2 \leq k \leq \ell m/2, \quad \ell, m \geq 8.
\]

Here we further strengthen this result as follows.

**Theorem 1.2.** There is a universal constant \( A > 0 \), such that for all \( m \geq \ell \geq 8 \) and \( 2 \leq k \leq \ell m/2 \), we have:

\[
p_k(\ell, m) - p_{k-1}(\ell, m) > A \frac{2\sqrt{s}}{s^{1/4}}, \quad \text{where} \quad s = \min\{2k, \ell^2\}.
\]

This lower bound is almost sharp, in a sense that it gives a correct exponential behavior of the difference, but perhaps not the base of the exponent. We discuss the upper bounds in §5.3 (see also §6.6).

The proof of the theorem has several ingredients. We use the above mentioned Stanley’s theorem, an extension of analytic estimates in the proof of Almkvist’s Theorem (Theorem 2.2), and the monotonicity property of the Kronecker coefficients (Theorem 2.1). Most crucially, we use the following connection between the Kronecker and \( q \)-binomial coefficients:

**Lemma 1.3 (Two Coefficients Lemma).** Let \( n = \ell m \), \( \tau_k = (n-k, k) \), where \( 1 \leq k \leq n/2 \). Then:

\[
g(m^\ell, m^\ell, \tau_k) = p_k(\ell, m) - p_{k-1}(\ell, m).
\]

This simple but very useful lemma was first proved in [Val, §4] and later in [PP1], but is implicit in [MY, PP2]. Note that it immediately implies Sylvester’s unimodality theorem.

The rest of the paper is structured as follows. We begin with a quick recap of definitions, notations and some basic results we are using (Section 2). We prove Theorem 1.1 in Section 3 and give applications of the theorem in Section 4. We then prove Theorem 1.2 in Section 5. We conclude with final remarks and open problems (Section 6).

## 2. Definitions and basic results

### 2.1. Partitions and Young diagrams.

We adopt the standard notation in combinatorics of partitions and representation theory of \( S_n \), as well as the theory of symmetric functions (see e.g. [Mac, Sta2]).

Let \( \mathcal{P} \) denote the set of integer partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \). We write \( |\lambda| = n \) and \( \lambda \vdash n \), for \( \lambda_1 + \lambda_2 + \ldots = n \). Let \( \mathcal{P}_n \) the set of all \( \lambda \vdash n \), and let \( P(n) = |\mathcal{P}_n| \) the number of
partitions of \( n \). We use \( \ell(\lambda) \) to denote the number of parts of \( \lambda \), and \( \lambda' \) to denote the conjugate partition. Define addition of partitions \( \alpha, \beta \in \mathcal{P} \) to be their addition as vectors:

\[
\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots).
\]

We denote by \( \chi^\lambda \) the character of the irreducible representation \( S^\lambda \) of \( S_n \) corresponding to \( \lambda \). Denote by \( f^\lambda = \chi^\lambda[1^n] \) the dimension of \( S^\lambda \). Finally, hooks of partition \( \mu \) are defined by \( h_{ij} = \mu_i + \mu_j' - i - j + 1 \), and the integers \( (h_{11}, h_{22}, \ldots) \) are called principal hooks. When \( \mu = \mu' \), the sequence of principal hooks is exactly partition \( \hat{\mu} \) defined in Theorem 1.1.

### 2.2. Kronecker coefficients

It is well known that

\[
g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\omega \in S_n} \chi^\lambda(\omega) \chi^\mu(\omega) \chi^\nu(\omega).
\]

This implies that Kronecker coefficients have full \( S_3 \) group of symmetry:

\[
g(\lambda, \mu, \nu) = g(\mu, \lambda, \nu) = g(\lambda, \nu, \mu) = \ldots
\]

We will use the following monotonicity property:

**Theorem 2.1 (Man).** Suppose \( \alpha, \beta, \gamma \) are partitions of \( n \), such that the Kronecker coefficients \( g(\alpha, \beta, \gamma) > 0 \). Then for any partitions \( \lambda, \mu, \nu \) with \( |\lambda| = |\mu| = |\nu| \) we have

\[
g(\lambda + \alpha, \mu + \beta, \nu + \gamma) \geq g(\lambda, \mu, \nu).
\]

We also have the following trivial upper bound (see e.g. [Sta2, Exc. 7.83]) :

\[
(2.1) \quad g(\lambda, \mu, \nu) \leq \min\{f^\lambda, f^\mu, f^\nu\} \quad \text{for all} \quad \lambda, \mu, \nu \vdash n.
\]

### 2.3. Partition asymptotics

Denote by \( P'(n) = P(n) - P(n - 1) \) the number of partitions into parts \( \geq 2 \). Recall that \( P'(n) \geq 1 \) for all \( n \geq 2 \), and the following Hardy–Ramanujan and Roth–Szekeres formulas, respectively:

\[
(2.2) \quad P(n) \sim \frac{1}{4\sqrt{3n}} e^{\pi \sqrt{\frac{2}{3}n}}, \quad P'(n) \sim \frac{\pi}{\sqrt{6n}} P(n) \quad \text{as} \quad n \to \infty,
\]

see [RS] (see also [ER, p. 59]).

Denote by \( b_k(n) \) the number of partitions of \( k \) into distinct odd parts \( \leq 2n - 1 \). We have:

\[
\prod_{i=1}^{n} (1 + q^{2i-1}) = \sum_{k=0}^{n^2} b_k(n) q^k.
\]

**Theorem 2.2 (Almkvist).** The following sequence is symmetric and unimodal:

\[
(\Diamond) \quad b_2(n), b_3(n), \ldots, b_{n^2-2}(n).
\]
Denote by $\chi_\downarrow$ the restriction of the $S_n$-representation $\chi$ to $A_n$, and by $\psi_\uparrow$ the induced $S_n$-representation of the $A_n$-representation $\psi$. We refer to [4K, §2.5] for basic results in representation theory of $A_n$. Recall that if $\nu \neq \nu'$, then $\chi_\downarrow^{\nu} = \chi_\downarrow^{\nu'}$ is irreducible in $A_n$. Similarly, if $\nu = \nu'$, then $\chi_\downarrow^{\nu} = \psi_\downarrow^{\nu} \oplus \psi_\downarrow^{\nu'}$, where $\psi_\downarrow^{\nu}$ are irreducible in $A_n$, and are related via $\psi_\downarrow^{\nu}([12]) = \psi_\downarrow^{\nu'}([\pi])$.

Consider now the conjugacy classes of $A_n$ and the corresponding character values. Denote by $C^\alpha$ the conjugacy class of $S_n$ of permutations of cycle type $\alpha$, and by $D \subset P$ the set of partitions into distinct odd parts. We have two cases:

(1) For $\alpha \notin D$, we have $C^\alpha$ is also a conjugacy class of $A_n$. Then
$$\chi^{\nu}([C^\alpha]) = \chi^{\nu'}([C^\alpha]) \quad \text{if} \quad \nu \neq \nu', $$
$$\psi^{\nu}([C^\alpha]) = \frac{1}{2} \chi^{\nu'}([C^\alpha]) \quad \text{if} \quad \nu = \nu'. $$

(2) For $\alpha \in D$, we have $C^\alpha = C^\alpha_+ \cup C^\alpha_-$, where $C^\alpha_+$ are conjugacy classes of $A_n$. Then
$$\chi^{\nu}([C^\alpha_+]) = \chi^{\nu'}([C^\alpha_+]) \quad \text{if} \quad \nu \neq \nu', $$
$$\psi^{\nu}([C^\alpha_+]) = \frac{1}{2} \chi^{\nu'}([C^\alpha_+]) \quad \text{if} \quad \nu = \nu' \quad \text{and} \quad \alpha \neq \bar{\alpha}, $$
$$\psi^{\nu}([C^\alpha_-]) - \psi^{\nu}([C^\alpha_+]) = \pm e_\nu \quad \text{if} \quad \nu = \nu' \quad \text{and} \quad e_\nu = (\nu_1 \nu_2 \cdots)^{1/2} > 0.$$ 

Now, by the Frobenius reciprocity, for every $\mu = \mu'$ we have:
$$\langle \psi^{\mu}_{\downarrow}, \chi^{\alpha} \rangle = \langle \psi^{\mu}_{\downarrow}, \chi^{\alpha}_{\downarrow} \rangle,$$
which is nonzero exactly when $\alpha = \mu$ and so $\psi^{\mu}_{\downarrow} = \chi^\mu$. This implies
$$g(\lambda, \mu, \mu) = \langle \chi^{\mu} \otimes \chi^\lambda, \chi^{\mu} \rangle = \langle \chi^\mu, \chi^\lambda \rangle_{\downarrow}$$
$$\langle \chi^\lambda \rangle_{\uparrow} = \langle \chi^{\mu} \otimes \chi^{\lambda}, \psi^{\mu}_{\uparrow} \rangle = \langle \chi^{\mu} \otimes \chi^\lambda \rangle_{\downarrow} \psi^{\mu}_{\uparrow}$$

(3.1)

We can now estimate the Kronecker coefficient in the theorem. First, decompose the following tensor product of the $A_n$ representations:
$$\psi^{\mu}_{\uparrow} \otimes \chi^\lambda_{\downarrow} = \oplus \tau m_\tau \psi^\tau,$$
where $\psi^\tau$ are all the irreducible representations of $A_n$, the coefficients $m_\tau$ are their multiplicities in the above tensor product, and $\tau$ goes over the appropriate indexing.

Note that for any character $\chi$ of $S_n$ and $\pi \in A_n$ we trivially have $\chi_\downarrow[\pi] = \chi[\pi]$. Evaluating that tensor product on the classes $C^\alpha_{\pm}$ gives
$$(\psi^{\mu}_{\uparrow} \otimes \chi^\lambda_{\downarrow})[C^\alpha_{\pm}] = \chi^{\downarrow}_{\pm} - \psi^{\mu}_{\downarrow}[C^\alpha_{\pm}] = \chi^{\lambda}_{\downarrow} \chi^{\mu}_{\pm}.$$

On the other hand, evaluating the right-hand side of equation (3.2) gives
$$(\psi^{\mu}_{\uparrow} \otimes \chi^\lambda_{\downarrow})[C^\alpha_{\pm}] = \chi^{\downarrow}_{\pm} - \psi^{\mu}_{\downarrow}[C^\alpha_{\pm}] = \chi^{\lambda}_{\downarrow}[C^\alpha_{\pm}] e_\nu.$$ 

Here we used the fact that all characters are equal at the two classes $C^\alpha_{\pm}$ except for the ones corresponding to $\mu$. Equating the evaluations and using $e_\nu > 0$, we obtain
$$m^{\mu}_{\uparrow} - m^{\mu}_{\downarrow} = \chi^\lambda[C^\alpha].$$
This immediately implies
\[ \max\{m_{\mu+}, m_{\mu-}\} \geq |\chi^\lambda[C^{\mu}]| \]
\[ \text{(3.3)} \]

On the other hand, since all inner products are nonnegative, the equation \( (3.1) \) gives
\[ g(\lambda, \mu, \mu) \geq \max \left\{ \langle \psi_+^\mu \otimes \chi^{\lambda^\downarrow}, \psi_+^\mu \rangle, \langle \psi_+^\mu \otimes \chi^{\lambda^\downarrow}, \psi_-^\mu \rangle \right\} = \max\{m_{\mu+}, m_{\mu-}\}, \]
and now equation \( (3.3) \) implies the result. \( \square \)

4. Bounds on Kronecker coefficients via characters

4.1. Stanley’s theorem. We give a new proof of the following technical result by Stanley [Sta1, Prop. 11]. Our proof uses Theorem 1.1 and Almkvist’s Theorem 2.2. Both results are crucially used in the next section.

\textbf{Theorem 4.1} (Stanley). The following polynomial in \( q \) is symmetric and unimodal
\[ \binom{2n}{n}_q - \prod_{i=1}^{n} (1 + q^{2i-1}) . \]
\[ \text{Proof.} \]
Let \( \mu = (n^n) \) and \( \tau_k = (n^2 - k, k) \), where \( k \leq n^2/2 \). By the two coefficients lemma (Lemma 1.3), we have
\[ g(\lambda, \mu, \mu) = p_k(n, n) - p_{k-1}(n, n). \]
By the Jacobi-Trudi identity and the Murnaghan–Nakayama rule, we have:
\[ \chi^{\tau_k}[\mu] = \chi^{(n^2-k)\circ(k)}[\mu] - \chi^{(n^2-k+1)\circ(k-1)}[\mu] = b_k(n) - b_{k-1}(n). \]
(cf. [PP2, PPV]). Applying Theorem 1.1 with \( \lambda = \tau_k \) and \( \mu \) as above, we have:
\[ p_k(n, n) - p_{k-1}(n, n) = g(\lambda, \mu, \mu) \geq \chi^\lambda[\mu] = b_k(n) - b_{k-1}(n). \]
The last equality follows from Almkvist’s Theorem 2.2. Reordering the terms, we conclude
\[ p_k(n, n) - b_k(n) \geq p_{k-1}(n, n) - b_{k-1}(n) , \]
which implies unimodality. The symmetry is straightforward. \( \square \)

4.2. Asymptotic applications. Let \( \rho_m = (m, m \smile 1, \ldots, 2, 1) \) be the staircase shape, \( n = |\tau_m| = \binom{m+1}{2} \). The coefficient \( g(\rho_m, \rho_m, \nu) \) first appeared in connection with the Saxl conjecture [PPV], and was further studied in [Val] §8.

For simplicity, let \( m = 1 \mod 4 \), so \( n \) is even and \( \widehat{\rho}_m = (1, 5, \ldots, 2m-1) \). Let \( \tau_k = (n-k, k) \). Applying Theorem 1.1 and the Murnaghan–Nakayama rule as above, we have
\[ g(\rho_m, \rho_m, \tau_k) \geq |\chi^{\tau_k}[\mu]| = P_R(k) - P_R(k-1), \]
where \( P_R(k) \) is the number of partitions of \( k \) into \( R = \{1, 5, \ldots, 2m-1\} \).

In the “small case” \( k \leq 2m \), by the Roth–Szekeres theorem [RS], we have:
\[ g(\rho_m, \rho_m, \tau_k) \geq P_R(k) - P_R(k-1) \sim \frac{\pi \sqrt{2}}{3k^{3/2}} e^{\pi \sqrt{k/6}}, \]
i.e. independent of \( n \). On the other hand, by equation \( (2.1) \), we have
\[ g(\rho_m, \rho_m, \tau_k) \leq f^{\tau_k} < \frac{n^k}{k!}, \]
leaving a substantial gap between the upper and lower bounds. For \( k = O(1) \) bounded, Theorem 8.10 in \cite{Val}, gives
\[
g(\rho_m, \rho_m, \tau_k) \sim m^k \sim (2n)^{k/2} \quad \text{as} \quad n \to \infty,
\]
suggesting that the upper bound is closer to the truth. In fact, the proof in \cite{Val} seems to hold for all \( k = o(m) \).

In the “large case” \( k = n/2 \sim m^2/4 \), the Odlyzko–Richmond result (\cite[Thm. 3]{OR}) gives
\[
g(\rho_m, \rho_m, \tau_k) \geq P_R(k) - P_R(k - 1) \sim \frac{3^{3/2}}{2^{15/4} \sqrt{\pi m^3}} 2^{m/4} \sim \frac{3^{3/2}}{2^{47/4} \sqrt{\pi k^{3/2}}} 2^{\sqrt{k}/2}.
\]

For the upper bound, equation (3.1) gives
\[
g(\rho_m, \rho_m, \tau_k) \leq f_{\tau_k} \lesssim \frac{1}{\sqrt{\pi k^{3/2}}} 4^{k}.
\]

4.3. Lower bounds for border equal partitions. Two partitions \( \lambda, \mu \vdash n \) are called \( s \)-border equal if they have equal the first \( s \) principal hooks. By \( \lambda^{(s)} \) denote the partition with the first \( s \) rows and \( s \) columns removed.

**Corollary 4.2.** Let \( \lambda, \mu \vdash n \) be \( s \)-border equal partitions such that \( \mu = \mu' \) is self-conjugate. Denote by \( \alpha = \lambda^{(s)}, \beta = \mu^{(s)} \), and let \( \hat{\beta} = (2\beta_1 - 1, 2\beta_2 - 3, \ldots) \vdash n \). Then:
\[
g(\lambda, \mu, \mu) \geq |\chi^{(s)}[\hat{\beta}]|.
\]

**Proof.** By the Murnaghan–Nakayama rule, for the \( s \)-border equal partitions \( \lambda \) and \( \mu \), there is a unique way to fit the first \( s \) rim hooks of length \( \langle \tilde{\mu}_1, \ldots, \tilde{\mu}_s \rangle \) into the shape \( \lambda \). Therefore, we have
\[
|\chi^{\lambda}[\tilde{\mu}]| = |\chi^{\alpha}[\hat{\beta}]|.
\]
Now Theorem 1.1 implies the result. \( \square \)

**Example 4.3.** Fix \( r \geq 4 \) and \( s \geq 0 \). Consider \( \lambda = (2r^2 + 2r + s + 1)^{s+1}(s + 1)2r^2+2r \), \( \mu = (2r^2 + 2r + s + 1)^s(2r + s + 1)^{2r+1}s2r^2 \), and observe that \( |\lambda| = |\mu|, \mu = \mu' \). Furthermore, \( \lambda \) and \( \mu \) are \( s \)-border equal. We therefore have:
\[
g(\lambda, \mu, \mu) \geq 0.37 \left\{ \frac{2\sqrt{2k}}{(2k)^{9/4}} \right\} \geq \frac{4r}{3(2r)^{9/2}}, \quad \text{where} \quad k = 2r(r + 1).
\]

Indeed, in notation of the corollary, we have \( \alpha = (2r^2 + 2r + 1, 12r^2+2r) \), \( \beta = (2r + 1)^{2r+1} \), and \( \hat{\beta} = (4r + 1, 4r - 1, \ldots; 3, 1) \). Using Corollary 4.2, the Murnaghan–Nakayama rule and the Giambelli formula as in \cite{PPV}, we conclude that
\[
g(\lambda, \mu, \mu) \geq |\chi^{(s)}[\hat{\beta}]| \geq b_k(2r + 1) - b_{k-1}(2r + 1).
\]

Now the inequality (4.1) follows from Theorem 5.3, given in the next section (note that \( r \geq 4 \) implies \( k \geq 26 \), assumed in the theorem). We omit the details.

4.4. **Bounds via the rim hook bijection.** Fix an integer \( r \geq 1 \). Following \cite{PPV}, let \( \mu = \gamma_{6r} \) be a unique self conjugate partition with principal hooks \( \hat{\mu} = (36r - 3, 36r - 9, \ldots, 9, 3) \), which we called the caret shape. Let \( \lambda = (108r^2)^{3} \vdash n \), where \( n = 324r^2 \).

**Corollary 4.4.** For \( n = 324r^2 \) and partitions \( \lambda = (n/3)^3, \mu = \gamma_{6r} \) as above, we have:
\[
c\sqrt{n} \leq g(\lambda, \mu, \mu) \leq 3^n,
\]
for some universal constant \( c > 1 \).
Proof. To compute $|\chi^\lambda[\hat{\mu}]|$, recall the abacus construction (see [JK §2.7]). By Nakayama’s theorem, all rim hook tableaux have the same sign in this case, and thus

$$|\chi^\lambda[\hat{\mu}]| = \#\text{RH}(\lambda, \hat{\mu}),$$

where $\text{RH}(\lambda, \hat{\mu})$ is the set of rim hook tableaux of shape $\lambda$ and type $\hat{\mu}$. From the abacus construction, or equivalently the rim hook bijection (see [FS]), the number of such tableaux is equal to the number of ways to partition $\{1, 3, \ldots, 12r - 1\}$ into three groups of total size $36r^2$. Therefore,

$$\#\text{RH}(\lambda, \hat{\mu}) = e^{\theta(r)}.$$

Here the upper bound is trivial, but for the corollary we need the lower bound. Merge the $i$-th largest and $i$-th smallest part for all $i$, to obtain $3r$ groups of size $12r$. This gives

$$\#\text{RH}(\lambda, \hat{\mu}) \geq 3^r r^r r^r \sim \sqrt[3]{3 \pi 2} 3^{3r}.$$

Now Theorem 1.1 implies the lower bound in the corollary.

For the upper bound, use (2.1). By the hook-length formula applied to $\lambda$ (see e.g. [AR §13]), or a standard Young tableaux combinatorial interpretation, we easily have

$$g(\lambda, \mu, \mu) \leq f^\lambda \leq 3^n,$$

as desired. □

Remark 4.5. The corollary can be readily generalized by replacing 3 with any other odd number $d$ in the definition of the caret shape and then obtain such exponential bounds the Kronecker coefficient with $\lambda = (n/d)^d$, where $n = dk^2$ for some $k$. More generally, we can take any $\mu$ with any set of distinct principal hooks divisible by odd $d > 1$, and any $\lambda$ with empty $d$-core, but computing or even estimating $\#\text{RH}(\lambda, \hat{\mu})$ becomes difficult.

5. Bounds on the $q$-binomial coefficients

5.1. Analytic estimates. The proof of Almkvist’s Theorem 2.2 is based on the following technical results.

Lemma 5.1 ([Alm]). For $3 \leq k \leq 2n + 1$, we have:

$$b_k(n) - b_{k-1}(n) = b_k(n - 1) - b_{k-1}(n - 1).$$

Similarly, for $2n + 2 \leq k \leq (n - 1)^2/2$, we have:

$$b_k(n) - b_{k-1}(n) = b_k(n - 1) - b_{k-1}(n - 1) + b_{k-2n+1}(n - 1) - b_{k-2n}(n - 1).$$

Lemma 5.2 ([Alm]). For $n \geq 83$ and $(n - 1)^2/2 \leq k \leq n^2/2$, we have:

$$b_k(n) - b_{k-1}(n) \geq C \frac{2^n}{n^{9/2}}, \quad \text{where} \quad C = \frac{3\sqrt{3}}{\sqrt{2} \pi^2} \approx 0.37.$$

Based on this setup, we refine Almkvist’s Theorem 2.2 as follows

Theorem 5.3. For any $n \geq 31$, and $26 \leq k \leq n^2/2$ we have:

$$b_k(n) - b_{k-1}(n) \geq C \frac{2\sqrt{2k}}{(2k)^{9/4}}, \quad \text{where} \quad C \text{ is as above.}$$
\textbf{Proof.} Denote  
\[ \vartheta_k(n) = b_k(n) - b_{k-1}(n). \]
First, let \( n \geq 83 \) and \( (n - 1)^2/2 \leq k \leq n^2/2 \). By Lemma \ref{lem:recurrence_relations}, we have
\[ (5.1) \vartheta_k(n) \geq C \frac{2^n}{n^{9/2}} \geq C \frac{2^{\sqrt{2k}}}{(2j)^{9/4}}, \]
where the last inequality follows since the function \( f(x) = \log 2^{\sqrt{x}} - 9/4 \log(x) \) is increasing.

The recurrence relations in Lemma \ref{lem:recurrence_relations} and Almkvist’s theorem \( \vartheta_k(n) \geq 0 \) give
\[ \vartheta_k(n) \geq \vartheta_k(n-1) \quad \text{for all} \quad 3 \leq k \leq n^2/2, \quad k \neq 2n+1 \]
and \( \vartheta_{2n+1}(n) = \vartheta_{2n+1}(n-1) - 1 = \vartheta_{2n+1}(n-4) \). Now, let \( r \) be such that \( (n-r-1)^2/2 \leq k \leq (n-r)^2/2 \), and \( n-r \geq 83 \). Applying (5.1) to \( (n-r) \), we conclude:
\[ \vartheta_k(n) \geq \vartheta_k(n-r) \geq C 2^{\sqrt{2k}} \frac{1}{(2k)^{9/4}}. \]

Next, we check by computer that the inequality in the Theorem holds for all \( n \in \{31, \ldots, 83\} \) and \( 26 \leq k \leq n^2/2 \). Finally, for \( k \leq 83^2/2 \) and \( n > 83 \), we apply the inequalities of Lemma \ref{lem:recurrence_relations} repeatedly to obtain
\[ \vartheta_k(n) \geq \vartheta_k(83) \geq C 2^{\sqrt{2k}} \frac{1}{(2k)^{9/4}}. \quad \square \]

\textbf{Corollary 5.4.} Let \( n \geq 8, 1 \leq k \leq n^2/2, \mu = (n^n) \) and \( \tau_k = (n-k, k) \). Then
\[ g(\mu, \mu, \tau_k) \geq C \frac{2^{\sqrt{2k}}}{(2k)^{9/4}}, \quad \text{where} \quad C = \frac{\sqrt{27/8} \pi}{2}. \]

\textbf{Proof.} Following the proof of Stanley’s Theorem \ref{thm:algebra} for all \( 26 \leq k \leq n^2/2 \) and \( n \geq 31 \), Theorem \ref{thm:algebra} gives:
\[ g(\mu, \mu, \tau_k) = p_k(n, n) - p_{k-1}(n, n) \geq b_k(n) - b_{k-1}(n) \geq C \frac{2^{\sqrt{2k}}}{(2k)^{9/4}}. \]
For the remaining values of \( n \) and \( k \) we check the inequality by computer, noticing that \( p_k(n, n) = p_k(26, 26) \) when \( k \leq 26 \). \quad \square

\textbf{5.2. Partitions in rectangles.} By Lemma \ref{lem:recurrence_relations} we have
\[ \delta_k(\ell, \ell) := p_k(\ell, \ell) - p_{k-1}(\ell, \ell) = g(m^{\ell, \ell}, \tau_k). \]

\textbf{Theorem 5.5.} Let \( 8 \leq \ell \leq m \) and \( 1 \leq k \leq m\ell/2 \). Define \( n \) as
\[ n = \begin{cases} \lfloor \ell^2/8 \rfloor, & \text{when} \; \ell m \; \text{is even}, \\ \lfloor (\ell^2/2) - 1 \rfloor, & \text{when} \; \ell m \; \text{is odd}, \end{cases} \]
and let \( v = \min(k, n^2/2) \). Then:
\[ \delta_k(\ell, \ell) \geq C \frac{2^{\sqrt{\pi}}}{v^{9/4}}, \quad \text{where} \quad C = \frac{3\sqrt{3}}{\sqrt{2} \pi^2}. \]

\textbf{Proof.} We apply Theorem \ref{thm:algebra} to bound the Kronecker coefficient for rectangles with an appropriate Kronecker coefficient for a square and then apply Corollary \ref{cor:algebra}.

By strict unimodality \ref{lem:unimodality}, we have that \( g(m^{\ell, \ell}, (m\ell - k, k)) > 0 \) for all \( \ell, m \geq 8 \). By Corollary \ref{cor:algebra} we can assume \( \ell < m \). Assume first that \( \ell > 16 \).
First, suppose that $\ell m$ is even and let $n = 2\lfloor \frac{\ell - 8}{2} \rfloor$. Then for any $1 < k \leq \frac{\ell m}{2}$ we can find $1 \neq r \leq \frac{(m-n)\ell}{2}$ and $1 \neq s \leq \frac{m\ell}{2}$, such that $k = r + s$. Take $s = \min(k, n\ell/2)$ . Let $\tau_k = (m\ell - k, k)$ and $\tau_r = ((m - n)\ell - r, r)$, $\tau_s = (n\ell - s, s)$. Apply Theorem 2.1 to the triples $((m - n)\ell, (m - n)\ell, \tau_r)$ and $(n\ell, n\ell, \tau_s)$ to obtain

$$\delta_k(\ell, m) = g(m\ell, m\ell, \tau_k) \geq \max_g(m\ell, m\ell, \tau_r, g(n\ell, n\ell, \tau_s)) \geq \delta_s(\ell, n).$$

Similarly, dividing the $n \times \ell$ rectangle into $n \times n$ square and $n \times (n - \ell)$ rectangle, where $n\ell$ is again even, we have

$$\delta_s(\ell, n) \geq \delta_v(n, n),$$

where $s' = \min(s, n^2/2) = \min(k, n^2/2)$.

In the case that both $\ell$ and $m$ are odd, the only case where the above reasoning fails is when $k = \lfloor m\ell/2 \rfloor$ and $r, s$ don’t exist. In this case we take $n = 2\lfloor \frac{\ell - 8}{2} \rfloor - 1$ and we can always find $r, s$. In summary, we have that

$$\delta_k(\ell, m) \geq \delta_v(n, n),$$

where

$$n = \begin{cases} 2\lfloor \frac{\ell - 8}{2} \rfloor, & \text{when } \ell m \text{ is even} \\ 2\lfloor \frac{\ell - 8}{2} \rfloor - 1, & \text{when } \ell m \text{ is odd} \end{cases}$$

and $v = \min(k, n^2/2)$. Now apply Corollary 5.4 to bound $\delta_v(n, n)$ and obtain the result for $\ell > 16$.

When $\ell \leq 16$, and $m \geq 24$, we can apply the same reasoning as above to show $\delta_k(\ell, m) \geq \delta_v(\ell, 16)$. Then for $\ell, m \leq 16$ the statement is easily verified by direct calculation. \hfill \Box

**Proof of Theorem 1.2** For $n \geq \ell - 9$, the desired inequality then follows from Theorem 5.5 and the observation that

$$\frac{2^{n/\sqrt{2}}}{n^{9/2}} \geq 2^{-9/\sqrt{2}} \frac{2^{\ell/\sqrt{2}}}{\ell^{9/2}}.$$  

Taking $A = 2^{-9/\sqrt{2}} C \approx 0.00449$ gives the desired inequality for all values. \hfill \Box

5.3. Upper bounds. Let $k \leq \ell \leq m$ and $n = \ell m$. We have:

$$\delta_k(\ell, m) = p_k(\ell, m) - p_{k-1}(\ell, m) = P(k) - P(k - 1) = P'(k) \approx \frac{\pi}{12 \sqrt{2} k^{3/2}} e^{\pi \sqrt{k/2}}$$

Compare this with the lower bound in Theorem 1.2

$$\delta_k(\ell, m) > A \frac{2^{\sqrt{2}k}}{(2k)^{9/4}}.$$  

There is only room to improve the base of exponent here:

$$2^{\sqrt{2}} \approx 2.26 \text{ to } e^{\pi \sqrt{4}} \approx 13.00.$$  

In fact, using our methods, the best lower bound we can hope to obtain is

$$e^{\pi \sqrt{n/2}} \approx 3.61,$$

which is the base of exponent in the Roth–Szekeres formula for the number $b_k(n)$ of unrestricted partitions into distinct odd parts, where $n \geq k$.  

For a different extreme, let $m = \ell$ be even, and $k = m^2/2$. We have the following sharp upper bound:

$$\delta_k(m, m) \leq p_k(m, m) \sim \sqrt{\frac{3}{\pi m}} \binom{2m}{m} \sim \frac{\sqrt{3}^m}{\pi m}.$$ 

On the other hand, the lower bound in Theorem 1.2 gives:

$$\delta_k(m, m) > A \frac{2^m}{m^{9/2}}.$$ 

Again, we cannot improve the base of the exponent 2 with our method, simply because the total number of partitions into distinct odd parts $\leq 2m - 1$, is equal to $2^m$.

6. Final remarks

6.1. The results of this paper originally appeared in preprint [PP4], which also contained other results. We then split [PP4] into two parts, moving the lower bounds into this paper, and various (known and new) upper bounds are shifted to a survey [PP5]. We also added new applications in Section 4.

6.2. It is rather easy to justify the importance of the Kronecker coefficients in Combinatorics and Representation Theory. Stanley writes: “One of the main problems in the combinatorial representation theory of the symmetric group is to obtain a combinatorial interpretation for the Kronecker coefficients” [Sta2].

The Geometric Complexity Theory (GCT) is a more recent interdisciplinary area, where computing the Kronecker coefficients is crucial (see [MS]). Bürgisser voices a common complaint of the experts: “frustratingly little is known about them” [Bür]. We refer to [PP3, PP5] for details and further references.

Part of this work is motivated by questions in GCT. Specifically, the experts seem to be interested in estimating the coefficients

$$g(m^\ell, m^\ell, \lambda), \quad \text{where} \quad \lambda \vdash \ell m.$$ 

Both theorems 1.1 and 1.2 are directly applicable to this case, when $m = \ell$ and $\lambda = \lambda'$, and when $\ell(\lambda) = 2$, respectively. We plan to return to this problem in the future.

6.3. The notion of s-border equal partitions given in §4.3 is perhaps new but rather natural. It would be interesting to see if a stronger bound

$$g(\lambda, \mu, \mu) \geq g(\lambda^{(s)}, \mu^{(s)}, \mu^{(s)})$$

holds under assumptions of Corollary 4.2. Note also that one cannot drop $\mu = \mu'$ assumption here. For example, when $\lambda = \mu = (2, 2, 1)$ and $s = 1$, we have $\lambda^{(1)} = \mu^{(1)} = (1)$, and $g(\lambda, \mu, \mu) = 0$ while $g(\lambda^{(1)}, \mu^{(1)}, \mu^{(1)}) = 1$.

6.4. In a special case $\lambda = \mu = \mu'$, Theorem 1.1 and the Murnaghan–Nakayama rule gives a weak bound $g(\mu, \mu, \mu) \geq 1$ proved earlier in [BB]. In [PPV], we apply a qualitative version of Theorem 1.1 to a variety of partitions generalizing hooks and two-row partitions. Unfortunately, computing the characters of $S_n$ in #P-hard [PP3]. It is thus unlikely that this approach can give good bounds for general tensor squares $g(\lambda, \mu, \mu)$ (cf. [Val]).
6.5. As we showed in §5.3, there is a gap in the base of the exponent between lower and upper bounds even for $\delta_k(m, m)$. Finding sharper lower bounds in this case would be very interesting.

On the other hand, there is perhaps also room for improvement for the rectangular case $\delta_k(\ell, m)$, where $\ell = o(m)$ and $k$ is in the “middle range” $\ell^2/2 \leq k \leq \ell m/2$. Since our proof of a lower bound in this case relies crucially on Theorem 2.1 and no other tools in this case are available, extending the inequality in the theorem would be very useful.

6.6. As we mentioned in the introduction, the first lower bound $\delta_k(\ell, m) \geq 1$ was obtained by the authors in [PP1]. This was quickly extended in followup papers [Dha] and [Zan], both of which employing O’Hara’s combinatorial proof [O’H]. First, Zanello’s proof gives:

$$\delta_k(\ell, m) > d, \quad \text{for } \ell \geq d^2 + 5d + 12, \quad m \geq 2d + 4, \quad 4d^2 + 10d + 7 \leq k \leq \ell m/2,$$

see the proof of Prop. 4 in [Zan]. Similarly, Dhand’s proves somewhat stronger bounds

$$\delta_k(\ell, m) > d, \quad \text{for } \ell, m \geq 8d, \quad 2d \leq k \leq \ell m/2,$$

see Theorem 1.1 in [Dha]. Ignoring constraints of $\ell$ and $m$, the bounds give $\delta_k = \Omega(\sqrt{k})$ and $\delta_k = \Omega(k)$, which are substantially inferior to our $\delta_k = \exp \Omega(\sqrt{k})$ bound in Theorem 1.2.

In conclusion, let us mention that the sequence $p_k(\ell, m)$ has remarkably sharp asymptotics bounds, including the central limit theorem (CLT) with the error bound [Tak] and the Hardy–Ramanujan type formula [AA]. When $\ell$ is fixed, sharp asymptotic bounds are given in [SZ].

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