Gauss–Bonnet cosmology with induced gravity and a non-minimally coupled scalar field on the brane

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Received 10 May 2008
Accepted 2 June 2008
Published 30 June 2008

Online at stacks.iop.org/JCAP/2008/i=06/a=032
doi:10.1088/1475-7516/2008/06/032

Abstract. We construct a cosmological model with a non-minimally coupled scalar field on the brane, where Gauss–Bonnet and induced gravity effects are taken into account. This model has 5D character at both high and low energy limits but reduces to 4D gravity for intermediate scales. While induced gravity is a manifestation of the IR limit of the model, the Gauss–Bonnet term and non-minimal coupling of the scalar field and induced gravity are essentially related to the UV limit of the scenario. We study the cosmological implications of this scenario focusing on the late time behavior of the solutions. In this setup, non-minimal coupling plays the role of an additional fine-tuning parameter that controls the initial density of the predicted finite density big bang. Also, non-minimal coupling has important implications for the bouncing nature of the solutions.

Keywords: cosmology with extra dimensions, cosmological applications of theories with extra dimensions
1. Introduction

The idea that we and all standard matter live on a brane embedded in a higher dimensional bulk has attracted a lot of attention [1]–[3]. From this viewpoint, extra dimensions are accessible only for gravitons and possibly non-standard matter. The setup of Randall and Sundrum (RSII) considers the observable universe as a 3-brane with positive tension embedded in five-dimensional anti-de Sitter bulk. In the low energy limit, the 5D graviton is localized on the brane due to the warped geometry of the bulk. The notion of AdS/CFT correspondence helps us to understand this property since the 4D gravity is coupled to a conformal field in the RS model [4,5].

The effect of the bulk on the brane can be determined by the effective mass $M$ of the bulk fluid that is measured by a bulk observer at the brane. For a spherically symmetric brane, this mass can be considered as the effective gravitational mass of the bulk. This mass depends on the brane scale factor and the proper time on the brane. In the case where the bulk observer is comoving with the bulk fluid, the mass is assumed to be comoving. However, for matter components such as a bulk radiation fluid, there is no comoving observer [6,7].

On the other hand, the model proposed by Dvali, Gabadadze and Porrati (DGP) is a radiative correction; the bulk is a flat Minkowski spacetime, but a reduced gravity term appears on the brane without tension. This model is different in the respect that it predicts deviations from the standard four-dimensional gravity over large distances. In this scenario, the transition between four-dimensional and higher dimensional gravitational potentials arises due to the presence of both the brane and bulk Einstein terms in the action. The existence of a higher dimensional embedding space allows for the existence of
bulk or brane matter which can certainly influence the cosmological evolution on the brane. This model has a rich phenomenology discussed in [8]. Maeda et al have constructed a braneworld scenario which combines the Randall–Sundrum II (RSII) model and the DGP model [9]. In this combination, an induced curvature term appears on the brane in the RSII model. This model has been called the warped DGP braneworld in the literature [10]. The existence of the induced gravity term leads to a self-accelerating branch in the brane evolution [11, 12].

Braneworld models with the scalar fields minimally or non-minimally coupled to gravity have been studied extensively (see [13] and references therein). The introduction of non-minimal coupling (NMC) is not just a matter of taste; it is forced upon us in many situations of physical and cosmological interest. For instance, NMC arises at the quantum level when quantum corrections to the scalar field theory are considered. Even if for the classical, unperturbed theory this NMC vanishes, it is necessary for the renormalizability of the scalar field theory in curved space. In most theories used to describe inflationary scenarios, it turns out that a non-vanishing value of the coupling constant cannot be avoided. In general relativity, and in all other metric theories of gravity in which the scalar field is not part of the gravitational sector, the coupling constant necessarily assumes the value of $\frac{1}{6}$. The study of the asymptotically free theories in an external gravitational field with a Gauss–Bonnet term shows a scale dependent coupling parameter. Asymptotically free grand unified theories have a non-minimal coupling depending on a renormalization group parameter that converges to the value of $\frac{1}{6}$ or to any other initial conditions depending on the gauge group and on the matter content of the theory. An exact renormalization group study of the $\lambda \phi^4$ theory shows that $\text{NMC} = \frac{1}{6}$ is a stable infrared fixed point. Also in the large $N$ limit of the Nambu–Jona–Lasinio model, we have $\text{NMC} = \frac{1}{6}$. In the $O(N)$ symmetric model with $V = \lambda \phi^4$, NMC is generally non-zero and depends on the coupling constants of the individual bosonic components. Higgs fields in the standard model have $\text{NMC} = 0$ or $\frac{1}{6}$. Only a few investigations produce a zero value (for a more complete discussion of these issues we refer the reader to papers by Faraoni, especially [14] and references therein). In view of the above results, it is then natural to incorporate an explicit NMC between the scalar field and induced Ricci scalar on the brane.

On the other hand, in a braneworld scenario, the radiative corrections in the bulk lead to higher curvature terms. At high energies, the Einstein–Hilbert action will acquire quantum corrections. The Gauss–Bonnet (GB) combination arises as the leading bulk correction in the case of the heterotic string theory [15]. This term leads to second-order gravitational field equations linear in the second derivatives in the bulk metric which is ghost-free [16]–[18], a property of the curvature invariant of the Gauss–Bonnet term.

Inclusion of the Gauss–Bonnet term in the action results in a variety of novel phenomena which certainly affect the cosmological consequences of these generalized braneworld setups, although these corrections are smaller than the usual Einstein–Hilbert terms [19]–[21]. Moreover, the zero mode of the graviton has been localized in the GB model [22]. The cosmological evolution corresponding to the RS model in the presence of a bulk GB term has been considered in [17], [23]–[28]; see also [29]. Also the case of a minimally coupled scalar field with GB gravity has been discussed extensively [30]–[33].

In the presence of a GB term with induced gravity, there are different cosmological scenarios, even if there is not any matter in the bulk [34]. In this paper, we generalize
the previous studies to the case where a scalar field non-minimally coupled to induced curvature is present on the brane in the presence of radiative corrections. We first review briefly the Brown–Maartens–Papantonopoulos–Zamarias (BMPZ) setup [35]. Then we generalize this setup to the more general framework of scalar–tensor theories. We show that relating to the BMPZ scenario, there are several interesting features which certainly affect the cosmological dynamics on the brane. Since Gauss–Bonnet and induced gravity effects are related to two extremes of the scenario (UV and IR limits), inclusion of stringy effects via the Gauss–Bonnet term leads to a finite density big bang [35]. This interesting feature has been explained in a fascinating manner via \( T \)-duality of string theory [36]. On the other hand, non-minimal coupling itself accounts for a non-singular soft big bang scenario [37]. In our setup, the existence of non-minimal coupling of the scalar field and induced gravity on the brane controls the initial density of this finite big bang scenario. In other words, in this framework, non-minimal coupling with its special fine-tuning (see [38] and references therein) plays the role of a parameter that can control density of matter fields at the beginning of the universe. As we will show, incorporation of both GB and non-minimal coupling effects will enhance the special characteristics of the BMPZ scenario.

2. DGP inspired scalar–tensor theories

The action of the DGP scenario in the presence of a non-minimally coupled scalar field on the brane can be written as follows [39]:

\[
S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g^{(5)}} \left[ R^{(5)} - 2\Lambda_5 \right]
+ \left[ \frac{r}{2\kappa_5^2} \int d^4x \sqrt{-g} \left( \alpha(\phi)R - 2\kappa_4^2 g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - 4\kappa_4^2 V(\phi) - 4\kappa_4^2 \lambda \right) \right]_{y=0},
\]

(1)

where we have included a general non-minimal coupling \( \alpha(\phi) \) in the brane part of the action. \( y \) is the coordinate of the fifth dimension and we assume that the brane is located at \( y = 0 \). \( g^{(5)}_{AB} \) is a five-dimensional bulk metric with Ricci scalar \( R^{(5)} \), while \( g_{\mu\nu} \) is the induced metric on the brane with induced Ricci scalar \( \bar{R} \). \( g_{AB} \) and \( g^{\mu\nu} \) are related via \( g_{\mu\nu} = \delta^A_\mu \delta^B_\nu g_{AB} \). \( \lambda \) is the brane tension (constant energy density) and \( r \) is the crossover scale that is defined as follows:

\[
r = \frac{\kappa_5^2}{2\kappa_4^2} = \frac{M_5^2}{2M_4^3}.
\]

(2)

The generalized cosmological dynamics of this setup is given by the following Friedmann equation [34, 40]:

\[
\varepsilon \sqrt{H^2 - \frac{\Upsilon}{a^4} - \frac{\Lambda_5}{6} + \frac{K}{a^2}} = r\alpha(\phi) \left( H^2 + \frac{K}{a^2} \right) - \frac{\kappa_5^2}{6} (\rho + \rho_\phi + \lambda)
\]

(3)

where \( \varepsilon = \pm 1 \) corresponds to two possible branches of DGP cosmology and \( \Upsilon \) is the bulk black hole mass which is related to the bulk Weyl tensor. This mass, as generalized dark radiation, induces mirage effects in the evolution, and the gravitational effect of the bulk matter on the brane evolution can be described in terms of this mass as measured by a bulk observer at the location of the brane (the DGP limit has a Minkowski bulk \( \Lambda_5 = 0 \) with \( \Upsilon = 0 \)). A part of the effect of the non-minimal coupling of the scalar field \( \phi \) with
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gravity is hidden in the definition of the effective energy density. Assuming the following line element:
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + b^2(y, t) dy^2 = -n^2(y, t) dt^2 + a^2(y, t) \gamma_{ij} dx^i dx^j + b^2(y, t) dy^2, \]

where \( \gamma_{ij} \) is a maximally symmetric three-dimensional metric defined as \( \gamma_{ij} = \delta_{ij} + k x_i x_j / (1 - kr^2) \), the energy density of a non-minimally coupled scalar field on the brane is given as follows [39,41]:
\[ \rho_\phi = \left[ \frac{1}{2} \dot{\phi}^2 + n^2 V(\phi) - 6 \alpha' H \dot{\phi} \right]_{y=0}, \]  

(4)

where \( H = \dot{a}/a \) is the Hubble parameter, \( \alpha' = d\alpha/d\phi \) and \( \dot{\phi} = d\phi/dt \).

If we consider a flat brane (\( K = 0 \)) with \( \lambda = 0 \) and also a Minkowski bulk (\( \Lambda_5 = 0, \Upsilon = 0 \)), then we can write equation (3) as follows:
\[ H^2 = \pm \frac{H}{r \alpha(\phi)} + \frac{\kappa_4^2}{3 \alpha(\phi)} (\rho + \rho_\phi - 6 \alpha' H \dot{\phi}) \]  

(5)

where \( \rho_\phi = [(1/2) \dot{\phi}^2 + n^2 V(\phi)]_{y=0} \). The DGP model has two branches, i.e. \( \varepsilon = \pm 1 \), corresponding to two different embeddings of the brane in the bulk. The behaviors of two branches at high energies and low energies are summarized as follows.

In the high energy limit we find
\[ \text{DGP}(\pm) : \quad H^2 = \pm \frac{\kappa_4^2}{3 \alpha(\phi)} (\rho + \rho_\phi), \]  

(6)

while in low energy limit we have
\[ \text{DGP}(+) : \quad H \rightarrow \frac{1}{r \alpha(\phi)} - 2 \frac{\kappa_4^2 \alpha' \dot{\phi}}{\alpha(\phi)} \]  

DGP(−) : \quad H = 0.

(7)

In terms of dimensionless variables introduced in [35]
\[ h = Hr, \quad \mu = \frac{r \kappa_5^2}{6} \rho, \quad \mu' = \frac{r \kappa_5^2}{6} \rho_\phi, \quad \sigma = \frac{r \kappa_5^2}{6} \lambda, \quad \tau = \frac{t}{r}, \]  

(8)

we find
\[ h^2 = \pm \frac{h}{\alpha(\phi)} + \frac{(\mu + \mu')}{\alpha(\phi)} - 2 h \kappa_4^2 \frac{d\alpha}{\alpha(\phi) d\tau}. \]  

(9)

The solutions of this equation for \( h \) are as follows:
\[ h = \pm \frac{1}{2 \alpha(\phi)} - \frac{\kappa_4^2}{2 \alpha(\phi)} \frac{d\alpha}{d\tau} + \sqrt{1 + 4 \kappa_4^2 (d\alpha/d\tau) + 4 (\kappa_4^2 (d\alpha/d\tau))^2 + 4 \alpha(\phi) (\mu + \mu')} \]  

(10)

The negative root is not suitable since in the limit of \( \mu + \mu' \rightarrow 0 \), with this sign one cannot recover the low energy limit of the model highlighted in (7).
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Figure 1. Two possible branches of the DGP inspired non-minimal model. The non-minimal coupling of the scalar field is assumed to be positive and the brane is considered to be tensionless, $\sigma = 0$.

Table 1. The values of $\gamma$ in the different tests.

| Test         | $\gamma$          |
|--------------|--------------------|
| SNIa         | $0.0278^{+0.0033}_{-0.0278}$ |
| SNIa + LSS   | $0.000^{+0.005}_{-0.000}$   |
| SNIa + LSS + $H(z)$ | $0.000^{+0.003}_{-0.000}$ |

Figures 1 and 2 show the behavior of these solutions with some specific values of non-minimal coupling$^3$. The upper sign in relation (10) is related to DGP(+) and the lower sign to DGP(−). It is seen that there is a late time self-acceleration in the DGP(+) branch similar to that of the minimal case; however, in the minimal case when $\mu \rightarrow 0$ then $h \rightarrow 1$, whereas in the non-minimal case, in this limit, i.e. $\mu + \mu' \rightarrow 0$, we have $h \rightarrow 1/\alpha(\phi) - 2(\kappa_4^2/\alpha(\phi))(d\alpha/d\tau)$. In the DGP(+) branch, the end point is a vacuum de Sitter state and an anti-de Sitter state for positive and negative non-minimal coupling respectively, whereas in the DGP(−) branch, the end point is a Minkowski state. The effect of non-minimal coupling in this case is to shift the end point of the DGP(+) branch. Depending on the value that $\alpha(\phi)$ can attain [38], the late time acceleration of the universe can be fine-tuned properly.

Now for $\lambda \neq 0$, the solutions of dimensionless Friedmann equation are as follows:

$$ h = \pm \frac{1}{2\alpha(\phi)} - \frac{\kappa_4^2}{\alpha(\phi)} \frac{d\alpha}{d\tau} + \sqrt{1 + 4\kappa_4^2(d\alpha/d\tau) + 4(\kappa_4^2(d\alpha/d\tau))^2 + 4\alpha(\phi)(\mu + \mu' + \sigma)} 2\alpha(\phi). $$

In plotting all of the figures in this paper, equation (41) acts as a condition on the values that $\alpha$ can attain. The possible values of $\gamma$ extracted from observational data are shown in table 1. Using the SNIa + LSS $+ H(z)$ test, we obtain $\alpha \geq 0.01$ and $\alpha \leq -0.01$ approximately. For simplicity in drawing figures we have assumed that the scalar field has no dynamics, i.e. $d\phi/d\tau = 0$. The time dependent non-minimal coupling will be discussed at the end of the paper.

$^3$ In plotting all of the figures in this paper, equation (11) acts as a condition on the values that $\alpha$ can attain. The possible values of $\gamma$ extracted from observational data are shown in table 1. Using the SNIa + LSS test, we obtain $\alpha \geq 0.01$ and $\alpha \leq -0.01$ approximately. For simplicity in drawing figures we have assumed that the scalar field has no dynamics, i.e. $d\phi/d\tau = 0$. The time dependent non-minimal coupling will be discussed at the end of the paper.
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Figure 2. Two possible branches of the DGP inspired non-minimal model. The non-minimal coupling of the scalar field is assumed to be negative and the brane is considered to be tensionless, $\sigma = 0$.

Figure 3. Two possible branches of the DGP inspired non-minimal model with a negative tension brane and positive non-minimal coupling. We have set $\alpha(\phi) = 0.01, 0.02$ and $\sigma = -4$.

These solutions are shown in figures 3 and 4 with $\sigma \neq 0$. For negative tension in the DGP(+) branch, the end point is a vacuum de Sitter state and there is a self-acceleration, whereas in the DGP(−) branch the solutions terminate at finite density (figure 3). For positive tension, both of the solutions (DGP(±)) have self-acceleration and the end points are vacuum de Sitter states (figure 4).

The existence of the energy density ($\lambda$) on the brane gives rise to a shift of the solutions. Moreover, the existence of $\alpha(\phi)$ leads to further shift of these solutions.
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Figure 4. Two possible branches of the DGP inspired non-minimal model with positive brane tension and positive non-minimal coupling. We have set $\alpha(\phi) = 0.01, 0.02$ and $\sigma = 10$.

The DGP model is an IR modification of general relativity. In the UV limit, stringy effects will play an important role. From this viewpoint, for discussing both UV and IR limits of the scenario simultaneously, the DGP model is not sufficient and we should incorporate stringy effects via inclusion of the Gauss–Bonnet terms.

3. Gauss–Bonnet braneworlds

The Gauss–Bonnet term with coupling constant $\beta$ is written as follows:

$$L_{\text{GB}} = R^{(5)2} - 4R_{ab}^{(5)}R^{(5)ab} + R_{abcd}^{(5)}R^{(5)abcd}$$

where $R^{(5)}$ is the curvature scalar of the five-dimensional bulk spacetime. These corrections have their origin in stringy effects and the most general action should involve both the Gauss–Bonnet and the Einstein–Hilbert terms in 5D theory. The GB term is present only in the bulk action

$$S_{\text{bulk}} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g^{(5)}} \left[ R^{(5)} - 2\Lambda_5 + \beta \left( R^{(5)2} - 4R_{ab}^{(5)}R^{(5)ab} + R_{abcd}^{(5)}R^{(5)abcd} \right) \right], \quad (12)$$

where $\beta$ is the Gauss–Bonnet coupling which can be positive or negative in the classical GB theory. For $\beta$ negative, it has been seen in [42] that this braneworld model leads to antigravity or tachyon modes on the brane. However, in the presence of a bulk scalar field, these effects are not present even with negative $\beta$.

The Friedmann equation in the presence of Gauss–Bonnet effects is as follows [43]:

$$H^2 = \frac{C_+ + C_- - 2}{8\beta} - \frac{K}{a^2}, \quad (13)$$
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where

\[ C_{\pm} = \left[ \sqrt{\left( 1 + \frac{4}{3} \beta \Lambda_5 + 8 \beta \frac{Y}{a^4} \right)^{3/2} + \frac{\beta \kappa_4^2 (\rho + \lambda)^2}{2}} \pm \frac{\kappa_4^2 (\rho + \lambda)}{\sqrt{\beta}} \right]^{2/3}. \]  

This equation is a cubic equation with three possible roots. For \( \rho > 0 \) there is only one real root.

The behaviors of GB model at high and low energies are as follows [35]:

\[ H \gg \alpha^{-1/2} \rightarrow H^2 \propto \rho^{2/3}, \]  

high energy limit (15)

and

\[ H \ll \alpha^{-1/2} \rightarrow H^2 \propto \rho^2, \]  

low energy limit (16)

4. Gauss–Bonnet induced gravity with a non-minimally coupled scalar field on the brane

As we have explained, the Gauss–Bonnet effect is a high energy stringy effect. On the other hand, non-minimal coupling of a scalar field and induced gravity on the brane is forced upon us for several compelling reasons. Some of these reasons have their origin in pure quantum field theoretical considerations [14]. Then it is natural to incorporate both Gauss–Bonnet and non-minimal coupling effects to have a more reliable framework for treating cosmological dynamics. The action of the GBIG (Gauss–Bonnet term in the bulk and the induced gravity term on the brane) scenario in the presence of a non-minimally coupled scalar field on the brane can be written as follows:

\[ S = \frac{1}{2 \kappa_5^2} \int d^5x \sqrt{-g} \left( R^{(5)} - 2 \Lambda_5 + \beta \left( R^{(5)2} - 4 R^{(5)ab} R^{(5)ab} + R^{(5)abcd} R^{(5)abcd} \right) \right) \]

\[ + \left[ \frac{r}{2 \kappa_5^2} \int d^4x \sqrt{-g} \left( \alpha(\phi) R - 2 \kappa_4^2 g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - 4 \kappa_4^2 V(\phi) - 4 \kappa_4^2 \lambda \right) \right] \]

where \( \beta \) and \( r \) are the GB coupling constant and IG crossover scale respectively. The relation for energy conservation on the brane is as follows:

\[ \dot{\rho} + \dot{\rho}_{\phi} + 3H(1 + \omega)(\rho + \rho_{\phi}) = 6 \alpha' \phi \left( H^2 + \frac{K}{a^2} \right) \]

where \( \omega = (p + p_{\phi})/(\rho + \rho_{\phi}) \) with \( p \) and \( \rho \) the pressure and density of ordinary matter. Since for ordinary matter, \( \dot{\rho} + 3H(\rho + P) = 0 \), the non-minimal coupling of the scalar field and induced curvature on the brane leads to the non-conservation of the scalar field effective energy density [41].

The cosmological dynamics of the model is given by the following generalized Friedmann equation:

\[ \left[ 1 + \frac{8}{3} \beta \left( H^2 + \frac{\Psi}{2} + \frac{K}{a^2} \right) \right]^2 \left( H^2 - \Psi + \frac{K}{a^2} \right) = \left[ r \alpha(\phi) H^2 + r \alpha(\phi) \frac{K}{a^2} - \frac{\kappa_4^2}{6} (\rho + \rho_{\phi} + \lambda) \right]^2. \]  

(19)
This equation describes the cosmological evolution on the brane with tension and a non-minimally coupled scalar field on the brane. The bulk contains a black hole mass and a cosmological constant. \( \Psi \) is defined as follows:

\[
\Psi + 2\beta \Psi^2 = \frac{\Lambda_5}{6} + \frac{\Upsilon}{l^2}.
\]  

(20)

If \( \beta = 0 \), the model reduces to the DGP model, while for \( r = 0 \) we recover the Gauss–Bonnet model. Here we restrict our study to the case where the bulk black hole mass vanishes (\( \Upsilon = 0 \)) and therefore \( \Psi + 2\beta \Psi^2 = \frac{\Lambda_5}{6} \). The bulk cosmological constant in the presence of the GB term is given by \( \Lambda_5 = -\frac{6}{l^2 + 1} \beta / l^4 \), where \( l \) is the bulk curvature. For a spatially flat brane (\( K = 0 \)), the Friedmann equation is given by

\[
\left[ 1 + \frac{8 \beta}{3r^2} \left( H^2 + \frac{\Psi}{2} \right) \right]^2 (H^2 - \Psi) = \left[ r \alpha(\phi) H^2 - \frac{\kappa_4^2}{6} (\rho + \rho_\phi + \lambda) \right]^2.
\]  

(21)

We define the following dimensionless quantities:

\[
\gamma = \frac{8 \beta}{3r^2}, \quad \chi = \frac{r^2}{l^2}, \quad \psi = \Psi r^2,
\]  

(22)

where the dimensionless Friedmann equation takes the following form:

\[
\left[ 1 + \gamma \left( h^2 + \frac{\psi}{2} \right) \right]^2 (h^2 - \psi) = \left[ \alpha(\phi) h^2 - \left( \mu + \mu' + \sigma - 2 \frac{d\alpha(\phi)}{d\tau} h \kappa_4^2 \right) \right]^2.
\]  

(23)

To find the cosmological dynamics of our model, we should solve this equation in an appropriate parameter space. In what follows, we consider the Minkowski and AdS bulks and investigate their cosmological consequences.

4.1. Minkowski bulk (\( \psi = 0 \)) with a tensionless brane (\( \sigma = 0 \))

In this case, the non-minimal GBIG Friedmann equation takes the following form:

\[
(1 + \gamma h^2)^2 h^2 = \left[ \alpha(\phi) h^2 - \left( \mu + \mu' + \sigma - 2 \frac{d\alpha(\phi)}{d\tau} h \kappa_4^2 \right) \right]^2.
\]  

(24)

It is straightforward to show that in this case

\[
\frac{d(\mu + \mu')}{d(h^2)} = -\frac{(1 + \gamma h^2)(3\gamma h^2 + 1)}{2 \left( \alpha(\phi) h^2 - (\mu + \mu') + 2 \frac{d\alpha(\phi)}{d\tau} h \kappa_4^2 \right)}
\]

\[
+ \frac{2 \left[ \alpha(\phi)^2 h^2 + 3\alpha(\phi) \frac{d\alpha(\phi)}{d\tau} h \kappa_4^2 - (\mu + \mu') \left( \alpha(\phi) + \frac{1}{h} \frac{d\alpha(\phi)}{d\tau} \kappa_4^2 \right) + 2 \left( \frac{d\alpha(\phi)}{d\tau} \kappa_4^2 \right)^2 \right]}{2 \left( \alpha(\phi) h^2 - (\mu + \mu') + 2 \frac{d\alpha(\phi)}{d\tau} h \kappa_4^2 \right)}
\]  

(25)
The initial Hubble rate and density are given by substituting the result $d(\mu + \mu')/d(h^2) = 0$ into equation (24). This leads us to the following relations:

$$h_i = \frac{\alpha(\phi) + \sqrt{\alpha(\phi)^2 - 3\gamma + 6\gamma \frac{\alpha(\phi)}{d\tau} \kappa_4^2}}{3\gamma},$$

(26)

$$\left(\mu + \mu'\right)_i = \frac{2\alpha(\phi)^3 - 9\alpha(\phi)\gamma + 18\alpha(\phi) \frac{\alpha(\phi)}{d\tau} \kappa_4^2 \gamma + 2 \left(\alpha(\phi)^2 - 3\gamma + 6\gamma \frac{\alpha(\phi)}{d\tau} \kappa_4^2\right)^{3/2}}{27\gamma^2}. \tag{27}$$

Before proceeding further, we should stress two important points here. Firstly, the presence of the GB term removes the big bang singularity in this setup, and the universe starts with an initial finite density. The Gauss–Bonnet effect is essentially a string inspired effect in the bulk which in combination with the pure DGP scenario leads to a finite big bang proposal on the brane. A consequence of string inspired field theories is the existence of a minimal observable length of the order of the Planck length [44]–[46]. One cannot probe distances smaller than this fundamental length. In fact a string cannot live on a scale smaller than its length. This feature leads us to generalize the standard Heisenberg uncertainty relation to incorporate this Planck scale effect [47,48]. The existence of this minimal observable length essentially removes the spacetime singularity and acts as a UV cutoff of the corresponding field theory (see for instance [49] which discusses inflation with minimum length cutoff; see also [50]). So, in principle the existence of a finite density big bang is supported at least from this viewpoint [51]; see also [52]. Secondly, non-minimal coupling of a scalar field with induced gravity on the brane controls the value of the initial density. This is not the only importance of non-minimal coupling of a scalar field and induced gravity. In fact non-minimal coupling provides a mechanism for generating spontaneous symmetry breaking at the Planck scale on the brane [53]. In this respect and on the basis of the arguments presented in the introduction on the importance of the non-minimal coupling, non-minimal coupling of a scalar field and induced gravity on the brane itself is a high energy correction of the theory and it is natural to expect this effect to couple with stringy effects on the Planck scale. In fact in this setup, we encounter a smoother behavior due to the Gauss–Bonnet term (a finite density big bang) and the late time effects of the non-minimally coupled scalar field component. These effects together provide a more reliable cosmological scenario.

The behavior of $h$ with respect to $\mu + \mu'$ is shown in figure 5. This figure shows also that the GBIG1 and GBIG2 branches have self-acceleration for some positive values of the non-minimal coupling in the same manner as the DGP(+) branch, whereas the GBIG3 branch, like DGP(−), has no self-acceleration. This is similar to the case for the pure DGP or GB model alone where there is a big bang singularity. The self-accelerating GBIG2 branch is not a physical solution since it is accelerating throughout its evolution. For negative values of non-minimal coupling, the GBIG3 and GBIG2 have self-acceleration while GBIG1 has no such property. Now it is easy to show that

$$\left(\mu + \mu'\right) \rightarrow \infty : \quad \gamma \rightarrow 0$$

$$\left(\mu + \mu'\right) \rightarrow 0 : \quad \gamma \rightarrow \frac{\alpha(\phi)^2}{4(1 - 2(\frac{d\alpha}{d\tau})\kappa_4^2)}.$$
In the minimal case, the maximum value of $\gamma$ leads to a minimum value of $h_i$. Here, in the presence of non-minimal coupling of the scalar field and induced gravity with $\gamma_{\text{max}} = \alpha(\phi)^2/(4(1 - 2(d\alpha/d\tau)^2\kappa_4^2))$, we cannot conclude that this leads to a minimum value for $h$. Using equation (26), the value of $h_i$ for $\gamma_{\text{max}}$ is given as follows:

$$h_i = \frac{2(1 - 2(d\alpha/d\tau)^2\kappa_4^2)}{\alpha(\phi)}.$$  \hspace{1cm} (28)

When $\gamma = \gamma_{\text{max}}$ and $\mu + \mu' = 0$, there is a vacuum brane with de Sitter expansion. The $h$ asymptotic value is obtained when $\mu + \mu' \to 0$ in equation (24):

$$h_\infty^2 + \frac{(2\gamma - \alpha^2)}{\gamma^2}h_\infty^4 - \frac{4\alpha(d\alpha/d\tau)^2\kappa_4^2}{\gamma^2}h_\infty^3 + \frac{[1 - 4((d\alpha/d\tau)^2\kappa_4^2)^2]}{\gamma^2}h_\infty^2 = 0.$$  \hspace{1cm} (29)

This equation has four non-zero roots, but two of them are negative and unacceptable. When $\gamma \to 0$, we should have the non-minimal DGP model.

From equation (24) one can deduce

$$(\mu + \mu') = \alpha(\phi)h^2 - h(\gamma h^2 + 1) + 2\frac{d\alpha}{d\tau}\kappa_4^2 h,$$  \hspace{1cm} (30)

where $h_\infty \leq h < h_i$. Since $(\mu + \mu')_\infty = 0$, from equation (30) it follows that

$$\gamma = \frac{\alpha(\phi)h_\infty - 1 + 2(d\alpha/d\tau)^2\kappa_4^2}{h_\infty^2}.$$  \hspace{1cm} (31)

By expanding $\mu + \mu'$ to first order in $h^2 - h_\infty^2$, we find

$$h^2 = h_\infty^2 + \frac{2(\alpha(\phi)h_\infty^2 + 2(d\alpha/d\tau)^2\kappa_4^2 h_\infty)}{\alpha(\phi)h_\infty^2 (2 - 6(d\alpha/d\tau)^2\kappa_4^2 - \alpha(\phi)h_\infty) - 8((d\alpha/d\tau)^2\kappa_4^2)^2 + 4(d\alpha/d\tau)^2\kappa_4^2}(\mu + \mu').$$  \hspace{1cm} (32)
By comparison with equation (5), we find the following effective four-dimensional Newton constant:

\[
G = \left[ \frac{(\alpha(\phi) h_\infty^2 + 2(\alpha/\tau^2) R_4^2 h_\infty)}{\alpha(\phi) h_\infty (2 - 6(\alpha/\tau^2) R_4^2 - \alpha(\phi) h_\infty) - 8((\alpha/\tau^2) R_4^2)^2 + 4(\alpha/\tau^2) R_4^2} \right] \frac{\alpha(\phi) G_5}{r},
\]

where \(G_5 = \kappa_5^2 / 8\pi\) and \(G = \kappa_4^2 / 8\pi\) are five- and four-dimensional gravitational constants respectively. From equation (33) we obtain a relation between Hubble rates and density, we set \(d(\phi) h_\infty^2 / rH_0^2 = \alpha(\phi)/rH_0 - 8((\alpha/\tau^2) R_4^2)^2 + 4(\alpha/\tau^2) R_4^2\). This is an interesting result since there is no effect of non-minimal coupling in the non-minimal DGP(+) limit at late time and the relation between \(M_5^3\) and \(M_p^2\) as follows:

\[
M_5^3 \sim \left[ \frac{(\alpha(\phi) rH_0^2 + 2(\alpha/\tau^2) R_4^2 rH_0)}{\alpha(\phi) rH_0 (2 - 6(\alpha/\tau^2) R_4^2 - \alpha(\phi) rH_0) - 8((\alpha/\tau^2) R_4^2)^2 + 4(\alpha/\tau^2) R_4^2} \right] \frac{\alpha(\phi) M_p^2}{r}.
\]

Here \(H_\infty \sim H_0\), and we see the important role played by non-minimal coupling in this setup. In principle, one can fine-tune the value of the non-minimal coupling such that the fundamental scale of the bulk is reduced to values in the range accessible for the next generation of accelerators. To compare with the DGP(+) limit, when \(\mu + \mu' \rightarrow 0\), that is at late time, we have \(rH_0 \rightarrow 1/\alpha(\phi) - 2(\kappa_5^2/\alpha(\phi))(\alpha/\tau^2)\); therefore this equation in this limit reduces to \(M_5^3 \sim M_p^2/r\). This is not surprising since essentially a part of the motivation for inclusion of non-minimal coupling has its origin in the quantum field theoretical considerations (the renormalizability of quantum field theory in a curved background and quantum corrections to the scalar field theory). From this viewpoint, non-minimal coupling shows its importance mainly in the high energy UV sector of the theory while apparently DGP(+) gives the IR sector of the theory free of stringy and strong quantum field theoretical effects. We should stress that non-minimal coupling of the scalar field and induced gravity on the DGP brane modifies the crossover scale [39].

The above argument is restricted to the limit \(\mu + \mu' \rightarrow 0\) which is related to the late time stage of evolution. In equation (34), when \(\alpha(\phi)\) attains different values, \(M_5\) increases or decreases relative to its value in the DGP limit. When \(rH_0 \rightarrow 2(1 - 2(\alpha/\tau^2) R_4^2)/\alpha(\phi), M_5\) increases. We should stress that these results are sensitive to the sign of the non-minimal coupling, explicitly.

4.2. Minkowski bulk (\(\psi = 0\)) with brane tension (\(\sigma \neq 0\))

In this case the Friedmann equation is as follows:

\[
(1 + \gamma h^2)^2 h = \left[ \alpha(\phi) h^2 - (\mu + \mu' + \sigma - 2(\alpha(\phi)/\tau^2) h_\infty^2) \right]^2.
\]

The effect of brane tension is similar to considering a cosmological constant on the brane. In this case, there are three possible solutions (GBIG1–3). To find the initial and final Hubble rates and density, we set \(d(\mu + \mu')/d(h^2) = 0\). The initial and final Hubble rates, denoted by \(h_i\) and \(h_e\) respectively, are given by

\[
h_{i,e} = \frac{\alpha(\phi) \pm \sqrt{\alpha(\phi)^2 - 3\gamma + 6\gamma(\alpha(\phi)/\tau^2) R_4^2}}{3\gamma},
\]
Figure 6. Solutions of the Friedmann equation with a negative tension brane in a Minkowski bulk. We have assumed $\gamma = 0.003$, $\sigma = -4$, $\psi = 0$, $\alpha(\phi) = 0.2$ and $-0.2$ for solid and dotted curves respectively. The figures are plotted with different scales to highlight important features.

and the initial and final density are calculated as follows:

$$ (\mu + \mu')_{i,e} = \frac{2\alpha(\phi)^3 - 9\alpha(\phi)\gamma + 18\alpha(\phi)\frac{d\alpha(\phi)}{d\tau}\kappa_4^2\gamma \pm 2\left(\alpha(\phi)^2 - 3\gamma + 6\frac{d\alpha(\phi)}{d\tau}\kappa_4^2\right)^{3/2}}{27\gamma^2} - \sigma, $$

(37)

where the plus sign is for the initial state and the minus sign shows the final state. These points have $h' = 0$. The point $h = 0$ is the place where GBIG3 loiters, which is given by

$$ (\mu + \mu')_i = -\sigma $$

(38)

and $h_{\infty}$ is given by

$$ h_{\infty}^6 + \left(\frac{2\gamma - \alpha^2}{\gamma^2}\right)h_{\infty}^4 - \left(\frac{4\alpha(\phi)}{\gamma^2}\kappa_4^2 h_{\infty}^3 + \left[1 + 2\alpha\sigma - 4\left(\frac{d\alpha(\phi)}{d\tau}\kappa_4^2\right)^2\right]h_{\infty}^2ight) - \frac{27\gamma^2}{\gamma^2} - \frac{\sigma^2}{\gamma^2} = 0. $$

(39)

To obtain this relation we have set $\mu + \mu' = 0$ in equation (35).

The relation between $h$ and $\mu + \mu'$ is shown in figure 6 for a negative tension brane. In this figure for positive non-minimal coupling, the GBIG1 and GBIG2 branches start with an initial Hubble rate and density $(h_i, (\mu + \mu')_i)$, whereas the GBIG3 branch does not remove the big bang singularity. GBIG1 and GBIG3 terminate at a finite Hubble rate and density $(h_e, (\mu + \mu')_e)$, whereas GBIG2 terminates in a vacuum de Sitter state. On the other hand, for negative non-minimal coupling, the GBIG2 branch has a big bang singularity and this is a self-accelerating solution. GBIG1 and GBIG3 start without a big bang singularity $(h_i, (\mu + \mu')_i)$ and both of them have self-acceleration, but GBIG3 throughout evolution loiters and then evolves. A universe which undergoes a period of loitering is an attractive alternative to standard cosmologies. Generally a loitering
universe is an expanding Friedmann universe that undergoes a phase of slow expansion with a redshift of \( z \sim 3-5 \). It is believed that the large scale structure of the universe is formed during this semi-static phase [54].

Figure 7 is the result for a positive tension brane. Here with positive non-minimal coupling, the three solutions have self-acceleration. There are finite densities for GBIG1 and GBIG2 but GBIG3 has a big bang singularity. For negative non-minimal coupling, these solutions are not physically reliable since for all of them the density is negative.

Note also that \( \sigma_i, e \) and \( \sigma_l \) are quantities for which \( \mu_i, e = 0 \) and \( \mu_l = 0 \) respectively.

From equation (37), \( \sigma_{i,e} \) are given in terms of \( \gamma \) and \( \alpha(\phi) \) by

\[
\sigma_{i,e} = \frac{2\alpha^3(\phi) - 9\alpha(\phi)\gamma + 18\alpha(\phi)\frac{d\alpha(\phi)}{d\tau}\kappa_4^2\gamma \pm 2\left(\alpha^2(\phi) - 3\gamma + 6\gamma\frac{d\alpha(\phi)}{d\tau}\kappa_4^2\right)^{3/2}}{27\gamma^2},
\]

and from equation (38), \( \sigma_l = 0 \). In the non-minimal case, the maximum value of \( \gamma \) is \( \gamma_{\text{max}} = \alpha^2(\phi)/(3 - 6(d\alpha(\phi)/d\tau)\kappa_4^2) \). For \( \gamma = \alpha^2(\phi)/(3 - 6(d\alpha(\phi)/d\tau)\kappa_4^2) \), the GBIG1 branch disappears, since \( h_i = h_e \) at \( \gamma_{\text{max}} \). Actually the point \( h_i = h_e \) is now a point of inflection. The requirement of having a real value for the square root in equations (36) and (37) leads us to the following relation:

\[
\gamma \leq \frac{\alpha^2(\phi)}{3 - 6(d\alpha(\phi)/d\tau)\kappa_4^2}.
\]

On the basis of this relation, the range of variation of \( \gamma \) depends on the non-minimal coupling coefficient directly. The latest observational constraints on the values of \( \gamma \) are listed in table 1 [55]. In this framework, we can constraint non-minimal coupling of a scalar field and an induced Ricci scalar using observational data. Considering a conformal coupling of the scalar field and induced gravity on the brane defined as \( \alpha(\phi) = (1 - \xi \phi^2) \), we can constrain \( \xi \) on the basis of the constraints imposed on \( \gamma \) presented in table 1.
A detailed study of constraints on the non-minimal coupling of the scalar field and gravity based on various observational data and theoretical techniques is summarized in [38] (see also [56]–[64] for more details). For a time independent scalar field, $\alpha(\phi)$ will be time independent also. Using the result of SN1a + LSS + $H(z)$ (the third line of table 1) for $\gamma$ and relation (41), we obtain $\alpha \geq 0.01$ and $\alpha \leq -0.01$. Since $\alpha(\phi) = (1 - \xi \phi^2)$, we find

$$-\sqrt{\frac{0.99}{\xi}} \leq \phi \leq \sqrt{\frac{0.99}{\xi}} ,$$

for $\alpha \geq 0.01$ and

$$\phi \geq \sqrt{\frac{1.01}{\xi}} \quad \text{and} \quad \phi \leq -\sqrt{\frac{1.01}{\xi}} ,$$

for $\alpha \leq -0.01$. The values of $\xi$ are constrained to be in the ranges of $\xi \leq 0.99\phi^{-2}$ and $\xi \geq 1.01\phi^{-2}$. As we see, these constraints are dependent on the scalar field, and this is reasonable since the dynamics of the scalar field essentially affects its coupling to gravity. From this viewpoint, variation of gravitational coupling as a field in scalar–tensor gravity can be attributed to variation of non-minimal coupling.

### 4.3. AdS bulk ($\psi \neq 0$) with brane tension ($\sigma \neq 0$)

For $\psi \neq 0$, the bulk is AdS since $\Lambda_5 \neq 0$. We should solve equation (23) for this case. The condition $h = 0$ gives two solutions:

$$\mu_{b,c} = \mp \sqrt{-\psi} \left( 1 + \frac{\psi}{2} \right) - \sigma ,$$

where $\mu_c$ is the density at the point where GBIG3 collapses (corresponding to the plus sign), while $\mu_b$ is the density of the new bouncing point for GBIG4 (corresponding to the minus sign). In the previous subsection (Minkowski bulk) there is a loitering point. Here this point separates into bouncing and collapsing points. It is interesting that when $\psi = -5$, the GBIG4 branch disappears. Although the exact value of $\psi$ for disappearance of this branch is not important and depends on the choice of parameters, the fact that in principle this branch can disappear in a suitable parameter space is an important result.

To obtain turning points of the branches, we calculate $d(\mu + \mu')/d(h^2)$ as follows:

$$\frac{d(\mu + \mu')}{d(h^2)} = -\left[ 1 + \gamma(h^2 + \frac{\psi}{2}) \right] \left[ 3\gamma(h^2 - \frac{\psi}{2}) + 1 \right] 2 \left( \alpha(\phi)h^2 - (\mu + \mu' + \sigma) + 2\frac{d\alpha(\phi)}{d\tau} \kappa_4^2 \right)$$

$$+ 2 \left[ \alpha(\phi)^2h^2 + 3\alpha(\phi)\frac{d\alpha(\phi)}{d\tau} \kappa_4^2 - (\mu + \mu' + \sigma) \left( \alpha(\phi) + \frac{1}{h} \frac{d\alpha(\phi)}{d\tau} \kappa_4^2 \right) + 2 \left( \frac{d\alpha(\phi)}{d\tau} \kappa_4^2 \right)^2 \right] 2 \left( \alpha(\phi)h^2 - (\mu + \mu' + \sigma) + 2\frac{d\alpha(\phi)}{d\tau} \kappa_4^2 \right) .$$

(43)

By substituting $d(\mu + \mu')/d(h^2) = 0$, these points can be obtained by solving the equation

$$\frac{3}{2} \gamma h^3 + \left( \frac{1}{2} - \frac{3}{4} \gamma \psi \right) h - \alpha h \sqrt{h^2 - \psi} - \frac{d\alpha}{d\tau} \kappa_4^2 \sqrt{h^2 - \psi} = 0 .$$

(44)
Gauss–Bonnet cosmology with induced gravity and a non-minimally coupled scalar field on the brane

Figure 8. The solutions of the Friedmann equation with a positive brane tension in AdS bulk. We have chosen $\gamma = 0.003$, $\sigma = 10$, $\psi = -5$, $\alpha(\phi) = 0.2, 0.3$.

There are three roots (which are presented in the appendix), two of which are complex. Using equation (23), $h_\infty$ for AdS bulk satisfies the following equation:

$$
h_\infty^6 + \frac{(2\gamma - \alpha^2)}{\gamma^2} h_\infty^4 - \frac{4\alpha(d\alpha/d\tau)\kappa_4^2}{\gamma^2} h_\infty^3
+ \frac{1 + 2\alpha \sigma - \psi \gamma (1 + (3/4)\psi \gamma) - 4(\kappa_4^2)}{\gamma^2} h_\infty^2
+ \frac{4\sigma(d\alpha/d\tau)\kappa_4^2}{\gamma^2} h_\infty - \frac{[\psi(1 + \psi/2)^2 + \sigma^2]}{\gamma^2} = 0.
$$

(45)

In comparison with the minimal case [35], the behaviors of the branches are changed considerably. To see these differences, we obtain numerical solutions of the above equation for different values of $\psi$. The results of this calculations are shown in figures 8–10.

As these figures show, $\alpha(\phi)$, as the non-minimal coupling of the scalar field and induced gravity on the brane, controls the initial density and the age of the universe in the sense that these quantities are sensitive to the proposed value of the non-minimal coupling. For large values of $\alpha(\phi)$, the universe’s age and its initial density are higher than for the case with a small value of $\alpha(\phi)$. Moreover, on increasing $\alpha(\phi)$, one of the solutions, that is, GBIG4, disappears from the set of the solutions (note that these results are obtained for constant values of $\psi$ and $\sigma$ while $\alpha(\phi)$ is variable). The GBIG4 branch gives a bouncing cosmological solution. A bouncing universe goes from an era of accelerated collapse to an expanding phase without displaying any singularity. In the bouncing universe, the equation of state parameter of the matter content, $\omega$, must transit from $\omega < -1$ to $\omega > -1$ [65]. However, current observational data show that the equation of state parameter $\omega$ was larger than $-1$ in the past and is less than $-1$ today [66, 67]. In our framework, we have seen that in a suitable domain of parameter space, on increasing $\alpha(\phi)$ values the GBIG4 branch containing a bouncing cosmology will disappear. Even for a constant $\alpha(\phi)$ there is no bouncing solution for any values of $\psi$. For example with
Gauss–Bonnet cosmology with induced gravity and a non-minimally coupled scalar field on the brane

Figure 9. The solutions of the Friedmann equation with a negative brane tension in AdS bulk and on two different scales. We have set $\gamma = 0.003$, $\sigma = -4$, $\psi = -1$, $\alpha(\phi) = 0.2$, 0.3.

Figure 10. The solutions of the Friedmann equation with a negative brane tension in AdS bulk and on two different scales. $\gamma = 0.003$, $\sigma = -4$, $\psi = -1$, $\alpha(\phi) = 1$, i.e. the minimal case.

$\psi = -5$, this branch disappears completely. These arguments show that inclusion of non-minimal coupling of a scalar field and induced gravity on the brane can be used to fine-tune braneworld cosmological models on the basis of observational data.

5. Time evolution of the branches

In this section we discuss the more general case of a time dependent non-minimal coupling. All arguments presented in the preceding sections can be reconsidered in this time dependent framework. We focus on the Minkowski bulk ($\psi = 0$) with brane tension ($\sigma \neq 0$) as an example. Starting with Friedmann equation (35), we try the following ansatz:

$$\phi(t) \propto t^{-\nu}$$
Gauss–Bonnet cosmology with induced gravity and a non-minimally coupled scalar field on the brane

Figure 11. Solutions of the Friedmann equation with a negative tension brane in a Minkowski bulk. We have assumed $\gamma = 0.003$, $\sigma = -4$, $\psi = 0$, $\nu = 0.9$, $\kappa_5^2 = 1$, $\xi = 0.99\phi^2$.

in order to investigate the late time behavior of this scenario. As has been mentioned at the end of section 4.2, the values of $\xi$ are constrained to be in the ranges of $\xi \leq 0.99\phi^2$ and $\xi \geq 1.01\phi^2$. By adapting these conditions and choosing $\alpha(\phi) = 1 - \xi\phi^2$, equation (35) can be rewritten as follows:

\[
(1 + \gamma h^2)^2 h^2 = \left[(1 - \xi t^{-2\nu})h^2 - (\mu + \mu' + \sigma - 2\kappa_5^2\nu t^{-2\nu}h)\right]^2
\]

(47)

where the proportionality constant in (46) has been set equal to unity. The constraints on $\xi$ are now time dependent as $\xi \geq 0.99t^{-2\nu}$ and $\xi \leq 1.01t^{-2\nu}$. The results of numerical solution of this equation are shown in figures 11 and 12 for different values of $\xi$. Note that the three graphs of figure 11 (and also figure 12 with a different value of the non-minimal coupling coefficient $\xi$) correspond to the three branches of figure 6, but now with time variation of the non-minimal coupling. These solutions are various possibilities for the GBIG scenario with a time varying non-minimally coupled scalar field on the brane. For instance, figures 11(a) and 12(a) correspond to a GBIG2 branch of the scenario. On the other hand, figures 11(b) and 12(b) correspond to a GBIG1 branch and, finally, figures 11(c) and 12(c) correspond to a GBIG3 branch. The main point to stress here is

Journal of Cosmology and Astroparticle Physics 06 (2008) 032 (stacks.iop.org/JCAP/2008/i=06/a=032)
the fact that in the presence of explicit time evolution of the scalar field, these branches show more or less the same late time behavior as was discussed in previous sections.

Finally the issue of stability of the self-accelerated solutions should be stressed here. It has been shown that the self-accelerating branch of the DGP model contains a ghost at the linearized level [68]. The ghost carries negative energy density and it leads to the instability of the spacetime. The presence of the ghost can be related to the infinite volume of the extra dimension in the DGP setup. When there are ghost instabilities in the self-accelerating branch, it is natural to ask what the results of solution decay are. One possible answer to this question is as follows: since the normal branch solutions are ghost-free, one could think that the self-accelerating solutions may decay into the normal branch solutions. In fact for a given brane tension, the Hubble parameter in the self-accelerating universe is larger than that of the normal branch solutions. Then it is possible to have nucleation of bubbles of the normal branch in the environment of the self-accelerating branch solution. This is similar to the false vacuum decay in de Sitter space. However, there are arguments against this kind of reasoning which suggest that the
self-accelerating branch does not decay into the normal branch by forming normal branch bubbles [68]. It was also shown that the introduction of a Gauss–Bonnet term for the bulk does not help to overcome this problem [69]. In fact, it is still unclear what the end state of the ghost instability is in the self-accelerated branch of DGP inspired setups (for more details see [68]). On the other hand, non-minimal coupling of the scalar field and induced gravity provides a new degree of freedom which requires special fine-tuning, and this may provide a suitable basis on which to treat ghost instability. As we have shown, non-minimal coupling of a scalar field and induced gravity has the capability of removing bouncing solutions. It seems that this additional degree of freedom also has the capability of providing the background for a more reliable solution to ghost instability. This issue deserves a new research program.

6. Summary and conclusions

The DGP model modifies the IR sector of general relativity. In the UV limit of a reliable theory, stringy effects should play an important role. From this viewpoint, for discussing both UV and IR limits of the scenario simultaneously, the DGP model is not sufficient alone and we should incorporate stringy effects via inclusion of the Gauss–Bonnet terms. The presence of a GB term removes the big bang singularity, and the universe starts with an initial finite density. Non-minimal coupling of a scalar field and induced gravity on the brane, which is motivated for several compelling reasons, controls the value of the initial density in a finite big bang cosmology on the brane. This is not the only importance of non-minimal coupling of a scalar field and induced gravity; non-minimal coupling provides a mechanism for generating spontaneous symmetry breaking at the Planck scale on the brane. In this respect, non-minimal coupling of a scalar field and induced gravity on the brane is itself a high energy correction of the theory and it is natural to expect this effect to couple with stringy effects on the Planck scale. Investigation of the late time behavior of the DGP scenario with GB and non-minimal coupling effects provides a framework for constraining non-minimal coupling using recent observational data. In our model, the values of non-minimal coupling are constrained so that the values that $\xi$ can attain are constrained to be in the ranges of $\xi \leq 0.99\phi^{-2}$ and $\xi \geq 1.01\phi^{-2}$. In our setup, these constraints are dependent on the scalar field dynamics and this is reasonable since the dynamics of a scalar field essentially affects its coupling to gravity. One of the main outcomes of our analysis is the implication of non-minimal coupling for bouncing cosmologies. In the bouncing universe, the equation of state parameter of the matter content, $\omega$, must transit from $\omega < -1$ to $\omega > -1$. However, current observational data show that the equation of state parameter $\omega$ was larger than $-1$ in the past and is less than $-1$ today. In our framework, we have seen that with a suitable parameter space, on increasing $\alpha(\phi)$ values, the GBIG4 branch containing a bouncing cosmology will disappear. Even for a constant $\alpha(\phi)$ there is no bouncing solution for any values of $\psi$. These arguments show that inclusion of non-minimal coupling of a scalar field and induced gravity on the brane can be used to fine-tune braneworld cosmological models on the basis of observational data. Although most of the arguments in the paper are based on a time independent non-minimal coupling, as we have shown, inclusion of an explicit time dependence of non-minimal coupling will not change the physical nature of the solutions. Finally we have discussed the issue of ghost instabilities in self-accelerated solutions and possible impacts of a Gauss–Bonnet term and non-minimal coupling on this issue.
Appendix. Three roots of equation (44)

\[
h_1 = \frac{A}{3(3\gamma - \alpha)} - (6(3\gamma - \alpha)(2 - 3\gamma \psi + 2\alpha \psi) - 4A^2)/ \]
\[
[3 \times 2^{2/3}(3\gamma - \alpha)(-216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3 + [4(6(3\gamma - \alpha)(2 - 3\gamma \psi + 2\alpha \psi) - 4A^2) - 216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3)]^{1/2})^{1/3} + \frac{1}{6 \times 2^{1/3}(3\gamma - \alpha)}(-216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3)]^{1/3} / 12 \times \sqrt{2/3}(3\gamma - \alpha)
\]
\[
h_2 = \frac{A}{3(3\gamma - \alpha)} - ((1 + i\sqrt{3})(6(3\gamma - \alpha)(2 - 3\gamma \psi + 2\alpha \psi) - 4A^2))/ \]
\[
[6 \times 2^{2/3}(3\gamma - \alpha)(-216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3 + [4(6(3\gamma - \alpha)(2 - 3\gamma \psi + 2\alpha \psi) - 4A^2) - 216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3)]^{1/2})^{1/3} - \frac{(1 - i\sqrt{3})}{12 \times 2^{1/3}(3\gamma - \alpha)}(-216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3)]^{1/3} / 12 \times \sqrt{2/3}(3\gamma - \alpha)
\]
\[
h_3 = \frac{A}{3(3\gamma - \alpha)} - ((1 - i\sqrt{3})(6(3\gamma - \alpha)(2 - 3\gamma \psi + 2\alpha \psi) - 4A^2))/ \]
\[
[6 \times 2^{2/3}(3\gamma - \alpha)(-216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3 + [4(6(3\gamma - \alpha)(2 - 3\gamma \psi + 2\alpha \psi) - 4A^2) - 216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3)]^{1/2})^{1/3} - \frac{(1 + i\sqrt{3})}{12 \times 2^{1/3}(3\gamma - \alpha)}(-216\gamma A - 1620\gamma^2 \psi A + 72\alpha A + 972\gamma \psi \alpha A - 144\psi \alpha^2 A + 16A^3)]^{1/3} / 12 \times \sqrt{2/3}(3\gamma - \alpha)
\]
Gauss–Bonnet cosmology with induced gravity and a non-minimally coupled scalar field on the brane

\[ + 16A^3 + \left[ 4\left( 3\gamma - \alpha \right) - 2 - 3\gamma\psi + 2\alpha\psi \right] - 4A^2 \right)^3 \\
+ \left( - 216\gamma A - 1620\gamma^2\psi A + 72\alpha A + 972\gamma\psi\alpha A \\
- 144\psi\alpha^2 A + 16A^3 \right)^{1/3} \]

where \( A = (\alpha/d\tau)\kappa_4^2 \).

References

[1] Arkani-Hamed N, Dimopoulos S and Dvali G, 1998 Phys. Lett. B 429 263 [SPIRES] [hep-ph/9803315]
Arkani-Hamed N, Dimopoulos S and Dvali G, 1999 Phys. Rev. D 59 086004 [SPIRES] [hep-th/9807344]
Antoniadis I, Arkani-Hamed N, Dimopoulos S and Dvali G, 1998 Phys. Lett. B 436 257 [SPIRES]
[hep-ph/9804398]

[2] Randall L and Sundrum R, 1999 Phys. Rev. Lett. 83 3370 [SPIRES]
Randall L and Sundrum R, 1999 Phys. Rev. Lett. 83 4690 [SPIRES]

[3] Dvali G, Gabadadze G and Porrati M, 2000 Phys. Lett. B 485 208 [SPIRES] [hep-th/0005016]
Dvali G and Gabadadze G, 2001 Phys. Rev. D 63 056007 [SPIRES] [hep-th/0008054]
Dvali G, Gabadadze G, Kolanoči M and Nitti F, 2002 Phys. Rev. D 65 024031 [SPIRES] [hep-th/0106058]

[4] Maldeca J M, 1996 Adv. Theor. Math. Phys. 2 231 [SPIRES] [hep-th/9711200]
Witten E, 1998 Adv. Theor. Math. Phys. 2 565 [SPIRES] [hep-th/9803131]

[5] Maldecena J M, 2002 Adv. Theor. Math. Phys. 2 231 [SPIRES] [hep-th/9711200]
Witten E, 1998 Adv. Theor. Math. Phys. 2 565 [SPIRES] [hep-th/9803131]

[6] Apostolopoulos P S and Tetradis N, 2005 Phys. Rev. D 71 043506 [SPIRES] [hep-th/0412246]

[7] Apostolopoulos P S and Tetradis N, 2006 Phys. Rev. Lett. B 26 409 [SPIRES] [hep-th/0509182]

[8] Lue A, 2000 Phys. Rev. 423 1 [astro-ph/0510068]

[9] Maeda K, Mizuno S and Torii T, 2003 Phys. Rev. D 68 024033 [SPIRES] [gr-qc/0303039]

[10] Cai R-G and Zhang H, 2004 J. Cosmol. Astropart. Phys. JCAP08(2004)017 [SPIRES] [hep-th/0403234]

[11] Dvalin C, Dvali G R and Gabadadze G, 2001 Phys. Rev. D 64 104002 [SPIRES] [hep-th/0109124]

[12] Dvalin C, Dvali G R and Gabadadze G, 2002 Phys. Rev. D 65 044023 [SPIRES] [astro-ph/0105068]

[13] Farakos K and Pasipulaires P, 2007 Phys. Rev. D 75 024018 [SPIRES] [hep-th/0610010]

[14] Atazadeh K and Sepangi H R, 2006 Phys. Lett. B 643 76 [SPIRES] [gr-qc/0610017]

[15] Nozari K, 2007 Phys. Lett. B 652 159 [SPIRES] [hep-th/07070719]

[16] Heydari-Fard M and Sepangi H R, 2007 Phys. Rev. D 75 064010 [SPIRES] [gr-qc/0702061]

[17] Faraoni V, 1996 Phys. Rev. D 53 6813 [SPIRES]
Faraoni V, 2000 Phys. Rev. D 62 023504 [SPIRES] [gr-qc/0002091]

[18] Gross D J and Sloan J H, 1987 Nucl. Phys. B 291 41 [SPIRES]

[19] Charmousis C and Dufaux J F, 2002 Class. Quantum Grav. 19 4671 [SPIRES] [hep-th/0202107]

[20] Davis S C, 2003 Phys. Rev. D 67 024030 [SPIRES] [hep-th/0208205]

[21] Binefur P, Charmousis C, Davis S C and Dufaux J F, 2002 Phys. Lett. B 544 183 [SPIRES] [hep-th/0206089]

[22] Maeda K and Torii T, 2004 Phys. Rev. D 69 024002 [SPIRES] [hep-th/0309152]
See also Aliev A N, Cebeci H and Dereli T, 2006 Class. Quantum Grav. 23 591 [SPIRES] [hep-th/0507121]
Aliev A N, Cebeci H and Dereli T, 2007 Class. Quantum Grav. 24 3425 [SPIRES] [gr-qc/0703011]

[23] Lidsey J E and Nuens N J, 2003 Phys. Rev. D 67 103510 [SPIRES] [astro-ph/0303168]

[24] Apostolopoulos P S, Brouzakis N, Tetradis N and Tzavara E, 2007 Preprint 0708.0469 [hep-th]

Journal of Cosmology and Astroparticle Physics 06 (2008) 032 (stacks.iop.org/JCAP/2008/i=06/a=032)
Gauss–Bonnet cosmology with induced gravity and a non-minimally coupled scalar field on the brane

[20] Dufaux J F, Lidsey J E, Maartens R and Sami M, 2004 Phys. Rev. D 70 083525 [SPIRES] [hep-th/0404161]

[21] Tsujikawa S, Sami M and Maartens R, 2004 Phys. Rev. D 70 063525 [SPIRES] [astro-ph/0406078]

[22] Neupane I P, 2000 J. High Energy Phys. JHEP09(2000)040 [SPIRES] [hep-th/0008190]

Neupane I P, 2001 Phys. Lett. B 512 137 [SPIRES] [hep-th/0104226]

Meissner K A and Olechowski M, 2001 Phys. Rev. Lett. 86 3708 [SPIRES] [hep-th/0009122]

Cho Y M, Neupane I and Wesson P S, 2002 Nucl. Phys. B 621 388 [SPIRES] [hep-th/0104227]

[23] Charmousis C and Dufaux J F, 2002 Class. Quantum Grav. 19 4671 [SPIRES] [hep-th/0202107]

[24] Kim J E, Kyae B and Lee H M, 2000 Nucl. Phys. B 582 296 [SPIRES] [hep-th/0004005]

Cho Y M and Neupane I P, 2003 Int. J. Mod. Phys. A 18 2703 [SPIRES] [hep-th/0112227]

Nojiri S, Odintsov S D and Ogushi S, 2002 Int. J. Mod. Phys. A 17 4809 [SPIRES] [hep-th/0205187]

Gravanis E and Willison S, 2003 Phys. Lett. B 562 118 [SPIRES] [hep-th/0209076]

de Rham C and Tolley A J, 2006 J. Cosmol. Astropart. Phys. JCAP07(2006)004 [SPIRES] [hep-th/0605122]

[25] Germani C and Sopuerta C F, 2002 Phys. Rev. Lett. 88 231101 [SPIRES] [hep-th/0202060]

[26] Kofinas G, Maartens R and Papantonopoulos E, 2003 J. High Energy Phys. JHEP10(2003)066 [SPIRES] [hep-th/0307138]

Cai R G, Zhang H S and Wang A, 2005 Commun. Theor. Phys. 44 948 [SPIRES] [hep-th/0505186]

Brown R A, 2006 Preprint gr-qc/0602050

Heydari-Fard M and Sepangi H R, 2007 Phys. Rev. D 75 064010 [SPIRES] [gr-qc/0702061]

Yin S, Wang B, Abdalla E and Lin C Y, 2007 Preprint 0708.0992 [hep-th]

Wu S F, Chatrabhuti A, Yang G H and Zhang P M, 2007 Preprint 0708.1038 [astro-ph]

Maeda H, Sahni V and Shtanov Y, 2007 Preprint 0708.3237 [gr-qc]

Bouhmadi-Lopez M and Vargas Moniz P, 2008 Preprint 0804.4484

[27] Kim J E, Kyae B and Lee H M, 2000 Phys. Rev. D 62 045013 [SPIRES]

[28] Charmousis C and Dufaux J F, 2004 Phys. Rev. D 70 106002 [SPIRES]

Nojiri S and Odintsov S D, 2000 J. High Energy Phys. JHEP07(2000)049 [SPIRES]

[29] Koivisto T and Mota D F, 2007 Phys. Lett. B 644 104 [SPIRES] [astro-ph/0606078]

Koivisto T and Mota D F, 2007 Phys. Rev. D 75 023518 [SPIRES] [hep-th/0609155]

[30] Movromatos N M and Rizos J, 2000 Phys. Rev. D 62 124004 [SPIRES]

[31] Neupane I P, 2000 J. High Energy Phys. JHEP09(2000)040 [SPIRES] [hep-th/0008190]

[32] Charmousis C, Davis S C and Dufaux J F, 2003 J. High Energy Phys. JHEP12(2003)029 [SPIRES]

[33] Giovannini M, 2001 Phys. Rev. D 64 124004 [SPIRES]

Brown R A, 2007 Preprint gr-qc/0701083

Brown R A et al, 2005 J. Cosmol. Astropart. Phys. JCAP11(2005)008 [SPIRES] [gr-qc/0508116]

Alexander S et al, 2000 Phys. Rev. D 62 103509 [SPIRES]

Novello M and Elbaz E, 1994 Nuovo Cimento B 109 741

Nozari K and Sadatian S D, 2008 Mod. Phys. Lett. at press [0710.0058]

Nozari K, 2007 J. Cosmol. Astropart. Phys. JCAP09(2007)003 [SPIRES] [hep-th/07081611]

[35] Defayet C, 2001 Phys. Lett. B 502 199 [SPIRES] [hep-th/0010186]

Bouhmadi-Lopez M and Wands D, 2005 Phys. Rev. D 71 024010 [SPIRES] [hep-th/0408061]

Davis S C, 2005 Phys. Rev. D 72 024026 [SPIRES] [hep-th/0410065]

Davis S C, 2003 Phys. Rev. D 67 024030 [SPIRES] [hep-th/0208205]

Gross D J and Mendle P F, 1988 Nucl. Phys. B 303 407 [SPIRES]

Gross D J, 1988 Phys. Rev. Lett. 60 1229 [SPIRES]

Amati D et al, 1989 Phys. Lett. B 216 41 [SPIRES]

[36] Konishi K, Paffuti G and Provero P, 1990 Phys. Lett. B 234 276 [SPIRES]

[37] Garay L J, 1995 Int. J. Mod. Phys. A 10 145 [SPIRES] [gr-qc/9403008]

Witten E, 1997 Phys. Today 49 24

Amelino-Camelia G et al, 1997 Mod. Phys. Lett. A 12 2029 [SPIRES]

Kempf A et al, 1995 Phys. Rev. D 52 1108 [SPIRES]

Hossenfelder S et al, 2003 Phys. Lett. B 575 85 [SPIRES]

Hossenfelder S, 2006 Phys. Rev. D 73 105013 [SPIRES] [hep-th/0603032]

Nozari K, 2005 Phys. Lett. B 629 41 [SPIRES] [hep-th/0508078]

[38] Ashoorioon A, Kempf A and Mann R B, 2005 Phys. Rev. D 71 023503 [SPIRES] [astro-ph/0410139]

[39] Reuter M and Schwidtw J-M, 2006 J. High Energy Phys. JHEP01(2006)070 [SPIRES]

Battisti M V and Montani G, 2007 Phys. Lett. B 656 96 [SPIRES] [gr-qc/0703023]

Nozari K and Fazlpour B, 2006 Gen. Rel. Grav. 38 1661 [SPIRES]
Gauss–Bonnet cosmology with induced gravity and a non-minimally coupled scalar field on the brane

Bertolami O and Carvalho C, 2007 Preprint 0710.2743 [hep-th]
Sahni V, Feldman H A and Stebbins A, 1992 Astrophys. J. 385 1 [SPIRES]
Chiba T, 1999 Phys. Rev. D 60 083508 [SPIRES]
Koh S, Kim S P and Song D J, 2005 Phys. Rev. D 72 043523 [SPIRES]
Carroll S M, 1998 Phys. Rev. Lett. 81 3067 [SPIRES]
Futamase T, Rothman T and Matzner R, 1989 Phys. Rev. D 39 405 [SPIRES]
Hancock S et al, 1994 Nature 367 333 [SPIRES]
Bilandzic A and Prokopec T, 2007 Preprint 0704.1905