Weak Solution of a Doubly Degenerate Parabolic Equation

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Abstract

This paper studies a fourth-order, nonlinear, doubly-degenerate parabolic equation derived from the thin film equation in spherical geometry. A regularization method is used to study the equation and several useful estimates are obtained. The main result of this paper is a proof of the existence of a weak solution of the equation in a weighted Sobolev space.

1 Introduction

The classical thin film equation of the form

\[ u_t + \left( |u|^n u_{xxx} \right)_x = 0, \quad (1) \]

describes the behavior of a thin viscous film on a flat surface under the effect of surface tension. In 1990, Bernis and Friedman \[2\] published their pioneering analytical result on equation (1). This equation is a degenerate parabolic equation since \(|u|^n = 0\) when \(u = 0\). Therefore, Bernis and Friedman \[2\] used a regularization method and modified the equation to

\[ u_t + \left( (|u|^n + \epsilon) u_{xxx} \right)_x = 0, \]

with a small positive parameter \(\epsilon\), on which the Schauder estimates in \[10\] can be applied. Bernis and Friedman \[2\] proved the existence of non-negative weak solutions of equation (1) when \(n \geq 1\). In addition, they showed that for \(n \geq 4\), with a positive initial condition, there exists a unique positive classical solution. In 1994, Bertozzi et al. \[3\] generalized this result to \(n \geq \frac{7}{2}\). In 1995, Beretta et al. \[4\] showed that this positivity-preserving property holds for almost every time \(t\) in the case \(n \geq 2\). Regarding the long time behavior, Carrillo and Toscani \[5\] proved the convergence to a self-similar solution for equation (1) with \(n = 1\) and Carlen and Ulusoy \[7\] gave an upper bound on the distance from the self-similar solution. A similar result on a cylindrical surface was obtained in \[6\].

In this paper, we consider the related equation

\[ u_t + \left( |u|^n (1 - x^2) \left( (1 - x^2) u_x \right)_{xx} \right)_x = 0, \quad \text{in } \Omega \times (0, T_0) \quad (2) \]

where \(n \geq 1\) and \(\Omega = (-1, 1)\). This equation arises from the thin film equation in spherical geometry. In \[12\], the authors examined the dynamics of a thin liquid film coating the outer surface of a sphere rotating around its vertical axis in the presence of gravity. The evolution equation describing the thickness of the thin film was shown to be given by

\[ h_t + \frac{1}{\sin \theta} \cos \theta \left[ \sin \theta h^3 \left( a \sin \theta + b \sin \theta \cos \theta + c \left( 2h + \frac{1}{\sin \theta} (\sin \theta h_t) \right) \right) \right]_\theta = 0, \]

where \(\theta\) is the polar coordinate \((0 < \theta < \pi)\), \(h(\theta, t)\) is the thickness of the thin film and \(a\), \(b\) and \(c\) are parameters describing the effects of gravity, rotation and surface tension, respectively. Using the change of variable \(x = -\cos \theta\), the equation becomes:

\[ u_t + \left[ u^3 (1 - x^2) \left( a - bx + c \left( 2h + ((1 - x^2)u_x) \right) \right) \right]_x = 0. \quad (3) \]
Lemma 2. 

Equation (2) results from collecting the highest order terms of equation (3) and replacing $u^3$ with $|u|^n$ to be able study the role of exponent $n$ as is standard practice for the thin-film equation.

Equation (2) is somewhat similar to equation (1). However, (2) loses its parabolicity at $x = \pm 1$ or when $u = 0$, which makes it a doubly-degenerate parabolic equation. In this paper, we will use a similar regularization method to prove the existence of non-negative weak solutions of the equation. Since the equation is doubly degenerate, the existence can be proven only in a weighted Sobolev space.

2 Weighted Sobolev Space

In this section, we will give the definition of weighted Sobolev spaces and provide two related lemmas which will be used in section three.

Definition 1. [Weighted Sobolev space] Let $\Omega$ be a bounded interval and $w(x)$, $v_0(x)$ and $v_1(x)$ be nonnegative continuous functions defined on $\Omega$. The weighted $L^2$ space with weight $w$ is defined as

$$L^2_w(\Omega) = \left\{ u \mid \int_{\Omega} u(x)^2 w(x) \, dx < \infty \right\}$$

where the weighted $L^2$ norm is given by

$$\|u\|_{L^2_w} = \left( \int_{\Omega} u(x)^2 w(x) \, dx \right)^{1/2}.$$ 

The weighted Sobolev space $W^{1,p}(\Omega; v_0, v_1)$ is the set of functions $u$ such that the norm

$$\|u\|_{1,p,v_0,v_1} := \left( \int_{\Omega} |u|^p v_0 \, dx + \int_{\Omega} |
abla u|^p v_1 \, dx \right)^{1/p}$$

is finite. Specifically, the weighted $H^1$ space with weight $w$ is defined as

$$H^1_w(\Omega) = \left\{ u \mid \|u\|_{H^1_w} = \|u\|_{L^2_w} + \|u_x\|_{L^2_w} < \infty \right\}.$$ 

For the purposes of this paper, we take $\Omega = (-1,1)$ and $w(x) = 1 - x^2$. In order to prove the existence of a weak solution in $H^1_w(\Omega)$, we will use two lemmas. The first is the generalized Nirenberg inequality [4] given below.

Lemma 2. Let $p, q, r, \alpha, \beta, \gamma, \sigma$ and $a$ be real numbers satisfying $p, q \geq 1$, $r > 0$, $0 \leq a \leq 1$, $\gamma = a\sigma + (1-a)\beta$, $\frac{1}{p} + \frac{a}{n} > 0$, $\frac{1}{q} + \frac{\beta}{n} > 0$ and $\frac{1}{r} + \frac{\gamma}{n} > 0$. There exists a positive constant $C$ such that the following inequality holds for all $u \in C^{\infty}_0(\mathbb{R}^n)$ ($n \geq 1$)

$$\|x|\gamma u\|_{L^r} \leq C \|x|\sigma |Du|\|_{L^p} a \|x|\beta u\|_{L^q}^{1-a}$$

if and only if

$$\frac{1}{r} + \frac{\gamma}{n} = a \left( \frac{1}{p} + \frac{\alpha - 1}{n} \right) + (1-a) \left( \frac{1}{q} + \frac{\beta}{n} \right)$$

and

$$\begin{cases} 1 \leq \alpha - \sigma & \text{if} & a > 0 \\ \alpha - \sigma \leq 1 & \text{if} & a > 0 \quad \text{and} \quad \frac{1}{p} + \frac{a - 1}{n} = \frac{1}{r} + \frac{\gamma}{n} \end{cases}$$

The next lemma is an embedding theorem from weighted Sobolev spaces to weighted continuous function spaces [4]. Before introducing the lemma, we give several more definitions.
Definition 3. [Weighted Hölder space] Let $\Omega$ be a bounded interval and $w_0(x)$ and $w_1(x)$ be nonnegative continuous functions defined on $\Omega$. The weighted Hölder space $C^\lambda(\Omega; w_0, w_1)$ is the set of functions $u$ such that the norm
\[ ||u||_{C(\Omega; w_0, w_1)} := \sup_{t \in \Omega} |u(t)w_0(t)| + \sup_{t \in \Omega} H_\lambda(t, u) \]
is finite, where
\[ H_\lambda(t, u) = \sup_{s \neq t} \left( w_1(s) \left| \frac{|u(s) - u(t)|}{|s - t|^\lambda} \right| \right). \]
With the definitions above, we have the following lemma.

Lemma 4. [Embedding Theorem] Let $p \in (n, \infty)$, $\lambda \in (0, 1 - \frac{n}{p})$, and let $w_1(x)$, $v_1(x)$ and $r(t)$ be nonnegative continuous functions for $i = 1, 2$. Define
\[ D_w(t) := \sup_{s \in B(t, r(t))} \frac{w(s)}{w(t)}, \quad C_w(t) := \inf_{s \in B(t, r(t))} \frac{w(s)}{w(t)}, \quad S_j^\lambda \{ r, w_i, v_j \}(t) := r(t)^{\lambda j - n - \lambda p_i} w_i(t)^{\lambda j} v_j(t). \]
If
\[ \sup_t \left( \frac{D_w(t)}{C_v(t)} S_j \{ r, w_0, v_1 \}(t) \right) < \infty, \quad i = 0, 1 \]
and
\[ \sup_t \left( \frac{D_w(t)}{C_v(t)} S_j \{ r, w_1, v_1 \}(t) \right) < \infty, \quad i = 0, 1 \]
then
\[ W^{1,p}(\Omega; v_0, v_1) \hookrightarrow C^\lambda(\Omega; w_0, w_1). \]

3 Regularization and Existence of Weak Solution

In this section, we study the equation
\[ u_t + (|u|^n (1 - x^2) (1 - x^2) u_x)_x = 0, \quad x \in \Omega = (-1, 1), \quad t \in (0, T_0) \]
where $n \geq 1$ and $T_0 > 0$, with the initial condition
\[ u(x, 0) = u_0(x) \in H^1(-1, 1) \]  \hspace{1cm} (4)
and boundary conditions
\[ (1 - x^2)u_x = (1 - x^2) (1 - x^2) u_x)_x = 0 \text{ at } x = \pm 1. \]  \hspace{1cm} (5)
Since this equation is doubly degenerate, we use $\epsilon$ and $\delta$ to regularize the equation as
\[ u_{\epsilon, \delta} + ((u_{\epsilon, \delta})^n + \epsilon) (1 - x^2 + \delta) (1 - x^2 + \delta) (u_{\epsilon, \delta})_x)_x = 0, \quad x \in \Omega = (-1, 1), \quad t \in (0, T_0) \]  \hspace{1cm} (6)
where $\epsilon, \delta > 0$. The corresponding regularized initial and boundary conditions become
\[ u_{\epsilon, \delta}(x, 0) = u_{\epsilon, \delta 0}(x) \in H^1(-1, 1) \]
and
\[ (u_{\epsilon, \delta})_x = ((1 - x^2 + \delta) (u_{\epsilon, \delta})_x)_x = 0 \text{ at } x = \pm 1. \]

The existence of a solution of (6) in a small time interval is guaranteed by the Schauder estimates in (10). Now suppose that $u_{\epsilon, \delta}$ is a solution of equation (6) and that it is continuously differentiable
with respect to the time variable and fourth order continuously differentiable with respect to the spatial variable. In order to get an \textit{a priori} estimation of $u_{\epsilon, \delta}$, we multiply both sides of equation (6) by $(1 - x^2 + \delta)(u_{\epsilon, \delta})_x$ and integrate over $\Omega \times (0, T)$. This gives us

\[
\int_0^T \int_{-1}^1 \left[ ((u_{\epsilon, \delta})^n + \epsilon) (1 - x^2 + \delta) \left( (1 - x^2 + \delta)(u_{\epsilon, \delta})_x \right)_x \right] ((1 - x^2 + \delta)(u_{\epsilon, \delta})_x) \, dx dt
\]

\[
+ \int_0^T \int_{-1}^1 (u_{\epsilon, \delta})_t (1 - x^2 + \delta)(u_{\epsilon, \delta})_x \, dx dt = 0.
\]

Integrating by parts, we get

\[
\frac{1}{2} \int_0^T \int_{-1}^1 (1 - x^2 + \delta) \frac{d}{dt} \left| (u_{\epsilon, \delta})_x \right|^2 \, dx dt
\]

\[
+ \int_0^T \int_{-1}^1 (u_{\epsilon, \delta})^n + \epsilon) (1 - x^2 + \delta) \left( (1 - x^2 + \delta)(u_{\epsilon, \delta})_x \right)_x \, dx dt = 0.
\]

The second term of equation (7) is non-negative and the first term is non-positive. It follows that

\[
\int_{-1}^1 (1 - x^2 + \delta)(u_{\epsilon, \delta})_x^2 \, dx \leq \int_{-1}^1 (1 - x^2 + \delta)(u_{\epsilon, \delta, 0})_x^2 \, dx \leq C_0
\]

where $C_0 > 0$ is some constant. This shows that the family of functions $\{u_{\epsilon, \delta}\}$ is uniformly bounded in the $L^2_w$ norm.

To obtain a uniform bound for $\{u_{\epsilon, \delta}\}$, we apply lemma 2 and set the parameters as follows:

\[
n = 1, \gamma = \frac{1}{2}, r = p = 2, a = 1, \sigma = \frac{1}{2}, \alpha = \frac{3}{2}
\]

Then we obtain the inequality

\[
\| x^2 u \|_{L^2} \leq C \| x^2 u_x \|_{L^2}.
\]

Combining (8) and (9), we have

\[
\int_{-1}^1 (1 - x^2)(u_{\epsilon, \delta})^2 \, dx \leq \frac{1}{2} \int_{-1}^0 (1 + x)(u_{\epsilon, \delta})^2 + \frac{1}{2} \int_0^1 (1 - x)(u_{\epsilon, \delta})^2 \, dx
\]

\[
\leq \frac{1}{2} C_1 \left( \int_{-1}^0 (1 + x)^3(u_{\epsilon, \delta})^2 \, dx + \int_0^1 (1 - x)^3(u_{\epsilon, \delta})^2 \, dx \right)
\]

\[
\leq C_1 \int_{-1}^1 (1 - x^2)(u_{\epsilon, \delta})^2 \, dx \leq C_2.
\]

where $C, C_1, C_2 > 0$ are constants. Hence $\{u_{\epsilon, \delta}\}$ is uniformly bounded.

Next we take $n = 1, p = 2, \lambda = \frac{1}{2}, r = 1, w_0 = v_0 = 1$ and $w_1 = v_1 = w = 1 - x^2$ in Lemma 4. It is easy to check that with these parameters and weights, the conditions of Lemma 4 hold. Thus we have

\[
H^1_w(\Omega) \hookrightarrow C^{1/2}_w(\Omega),
\]

which means that

\[
|(1 - x^2)(u_{\epsilon, \delta}(x_1, t) - u_{\epsilon, \delta}(x_2, t))| \leq C_3 |x_1 - x_2|^{1/2}, \forall x_1, x_2 \in \Omega.
\]

Using the same method as Lemma 2.1, we can prove similarly that

\[
|(1 - x^2)(u_{\epsilon, \delta}(x, t_1) - u_{\epsilon, \delta}(x, t_2))| \leq C_3 |t_1 - t_2|^{1/8}, \forall t_1, t_2 \in (0, T).
\]
The inequalities (12) and (13) show the existence of an weighted upper bound on the $C^{1/2,1/8}_{x,t,w}$-norm of $u_{\epsilon,\delta}$ that is independent of $\epsilon$ and $\delta$. By the Arzelà-Ascoli theorem, this equicontinuous property, together with the uniformly boundedness of the weighted $H^1$ norm given by (8) and (10), shows that as $\epsilon \to 0$ and $\delta \to 0$, every sequence $\{u_{\epsilon,\delta}\}$ has a subsequence $\{\tilde{u}_{\epsilon,\delta}\}$ such that

$$\tilde{u}_{\epsilon,\delta} \to u$$

uniformly in $\Omega \times (0,T)$. We establish in the following theorem that this $u$ is a weak solution of (2).

**Theorem 5.** Any function $u$ obtained by (14) has the following properties:

1. $u$ is continuous in weighted Hölder space in $x$ with order $\frac{1}{2}$ and in $t$ with order $\frac{1}{8}$.
2. $u$ satisfies the boundary conditions (6) and initial condition (4).
3. $u$ is a weak solution of equation (2) in the following sense:

$$\int_0^T \int_{-1}^1 u_{t}\phi_t \, dx dt + \int_0^T \int_{|x|}^1 (|u|^n (1-x^2) (1-x^2) u_x) \phi_x \, dx dt = 0$$

for all $\phi \in \text{Lip}(\Omega \times (0,T))$, $\phi = 0$ at $t = 0$ and $t = T$. Here $P = \Omega \times (0,T) \setminus \{(x,t) | u = 0\}$.

4. If the initial value $u_0$ is non-negative, then

$$u \geq 0.$$ 

**Proof.** Parts (1) and (2) of the theorem can be derived directly from (14). To prove (3), notice that

$$\int_0^T \int_{-1}^1 u_{\epsilon,\delta} \phi_t \, dx dt + \int_0^T \int_{-1}^1 (|u_{\epsilon,\delta}|^n + \epsilon) (1-x^2 + \delta) ((1-x^2) (u_{\epsilon,\delta})_x)_x \phi_x \, dx dt = 0,$$

so we have

$$\int_0^T \int_{-1}^1 u_{\epsilon,\delta} \phi_t \, dx dt + \int_0^T \int_{-1}^1 (|u_{\epsilon,\delta}|^n (1-x^2) ((1-x^2) (u_{\epsilon,\delta})_x)_x) \phi_x \, dx dt + \epsilon \int_0^T \int_{-1}^1 (1-x^2 + \delta) ((1-x^2) (u_{\epsilon,\delta})_x)_x \phi_x \, dx dt + \epsilon \int_0^T \int_{-1}^1 (1-x^2 + \delta) ((1-x^2) (u_{\epsilon,\delta})_x)_x \phi_x \, dx dt = 0.$$ 

Using equation (7), we can show that the last two terms go to 0 as $\epsilon \to 0$ and $\delta \to 0$. We also have

$$\lim_{\xi \to 0} \int_{\{u < \xi\}} (u_{\epsilon,\delta})^n (1-x^2) (1-x^2) (u_{\epsilon,\delta})_x_2 \, dx dt = 0.$$ 

From (15) and (16), we can get (3) of the theorem.

In order to prove (4) of the theorem, we first define function $g_\epsilon(s)$ and $G_\epsilon(s)$ as following:

$$g_\epsilon(s) = - \int_{s}^{A} \frac{dr}{|r|^n + \epsilon},$$

$$G_\epsilon(s) = - \int_{s}^{A} g_\epsilon(r) \, dr,$$

where A is an uniform upper bound for $u_{\epsilon,\delta}$ for all $\epsilon$ and $\delta$. Now we multiply equation (6) by $g_\epsilon(u_{\epsilon,\delta})$ and integrate over $\Omega \times (0,T)$ to get

$$\int_0^T \int_{\Omega} g_\epsilon(u_{\epsilon,\delta}) (u_{\epsilon,\delta} t + (|u_{\epsilon,\delta}|^n + \epsilon) (1-x^2 + \delta) ((1-x^2 + \delta) (u_{\epsilon,\delta})_x)_x) \, dx dt = 0.$$ 

(17)
Note that \[ G'_{\epsilon}(s) = g'_{\epsilon}(s) = \frac{1}{|s|^n + \epsilon}, \]
so we have \[ g_{\epsilon}(u_{\epsilon, \delta})(u_{\epsilon, \delta})_t = (G_{\epsilon}(u_{\epsilon, \delta}(x, t)))_t. \]
After using this and integration by parts, equation (17) becomes
\[
\int_{\Omega} G_{\epsilon}(u_{\epsilon, \delta}(x, T)) \, dx - \int_{\Omega} G_{\epsilon}(u_{\epsilon, \delta}(x, 0)) \, dx + \int_{0}^{T} \int_{\Omega} \left((1 - x^2 + \delta)(u_{\epsilon, \delta})_x \right)^2 \, dx \, dt = 0.
\]
As \( u_{\epsilon, \delta}(x, 0) \) is bounded, we have
\[
\int_{\Omega} G_{\epsilon}(u_{\epsilon, \delta}(x, T)) \, dx < C. \tag{18}
\]
Assume there exists a point \((x_0, t_0)\) such that \( u(x_0, t_0) < 0 \). Because \( u_{\epsilon, \delta}(x, t) \) uniformly converges to \( u(x, t) \), we can choose \( \epsilon_0 > 0 \) and \( \xi > 0 \) such that
\[
u_{\epsilon, \delta}(x, t_0) < -\xi, \text{ if } |x - x_0| < \xi, \epsilon < \epsilon_0.
\]
Then
\[
\lim_{\epsilon \to 0} G_{\epsilon}(u_{\epsilon, \delta}(x, t_0)) = -\lim_{\epsilon \to 0} \int_{u_{\epsilon, \delta}(x, t_0)}^{A} g_{\epsilon}(s) \, ds \geq -\lim_{\epsilon \to 0} \int_{-\delta}^{0} g_{\epsilon}(s) \, ds = \infty. \tag{19}
\]
The last step is because \( \lim_{\epsilon \to 0} g_{\epsilon}(s) = -\infty \) when \( s < 0 \). Since (19) conflicts with (18), we have proved (4) of the theorem. \( \square \)

References

[1] Beretta, Elena, Michiel Bertsch, and Roberta Passo. "Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation." Archive for rational mechanics and analysis 129.2 (1995): 175-200.

[2] Bernis, Francisco, and Avner Friedman. "Higher order nonlinear degenerate parabolic equations." Journal of Differential Equations 83.1 (1990): 179-206.

[3] Bertozzi, Andrea L., et al. "Singularities and similarities in interface flows." Trends and perspectives in applied mathematics. Springer New York, 1994. 155-208.

[4] Brown, R. C., and B. Opic. "Embeddings of weighted Sobolev spaces into spaces of continuous functions." Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences. Vol. 439. No. 1906. The Royal Society, 1992.

[5] Burchard, Almut, Marina Chugunova, and Benjamin K. Stephens. "Convergence to equilibrium for a thin-film equation on a cylindrical surface." Communications in Partial Differential Equations 37.4 (2012): 585-609.

[6] Caffarelli, Luis, Robert Kohn, and Louis Nirenberg. "First order interpolation inequalities with weights." Compositio Mathematica 53.3 (1984): 259-275.

[7] Carlen, Eric A., and Ulusoy, Süleyman. "Asymptotic equipartition and long time behavior of solutions of a thin-film equation." Journal of Differential Equations 241.2 (2007): 279-292.

[8] Carrillo, José A., and Giuseppe Toscani. "Long-Time Asymptotics for Strong Solutions of the Thin Film Equation." Communications in mathematical physics 225.3 (2002): 551-571.
[9] Chugunova, Marina, John R. King, and Roman M. Taranets. "Uniqueness of the regular waiting-time type solution of the thin film equation." European Journal of Applied Mathematics 23.04 (2012): 537-554.

[10] Friedman, Avner. "Interior estimates for parabolic systems of partial differential equations." J. Math. Mech 7.3 (1958): 393-417.

[11] Giacomelli, Lorenzo, Hans Knüpfer, and Felix Otto. "Smooth zero-contact-angle solutions to a thin-film equation around the steady state." Journal of Differential Equations 245.6 (2008): 1454-1506.

[12] Kang, D., A. Nadim, and M. Chugunova. "Dynamics and equilibria of thin viscous coating films on a rotating sphere," Journal of Fluid Mechanics 791 (2016): 495-518.