FLAT TRACE ESTIMATES FOR ANOSOV FLOWS

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ABSTRACT. We prove a high energy flat trace estimate for the modified resolvent of the generator of an Anosov flow. This fills a gap in the proof of the local trace formula in [JiZw17] and is a by-product of the authors’ ongoing project of its generalization to Axiom A flows.

1. Introduction

This note is a by-product of the authors’ ongoing project on the local trace formula for Axiom A flows, which leads to the discovery of some issues in [JiZw17]. Since the situation for Anosov flows is simpler than the one for Axiom A flows, we give here a separate presentation to fix the issues in [JiZw17].

Let \( X \) be a smooth compact manifold, \( \varphi_t : X \to X \) be an Anosov flow generated by a smooth vector field \( V \), and \( P = -iV \), Jin–Zworski [JiZw17] proved the following local trace formula relating the Pollicott–Ruelle resonances \( \text{Res}(P) \) to the lengths of closed geodesics.

**Theorem 1.** For any \( A > 0 \) there exists a distribution \( F_A \in \mathcal{S}'(\mathbb{R}) \) supported in \([0, \infty)\) such that

\[
\sum_{\mu \in \text{Res}(P), \text{Im}(\mu) > -A} e^{-i\mu t} + F_A(t) = \sum_{\gamma} \frac{T^\#_\gamma \delta(t - T^\gamma)}{|\det(I - P^\gamma)|}, \quad t > 0
\]

in \( \mathcal{D}'((0, \infty)) \), where the sum on the right hand side is taken over all closed geodesics, \( P^\gamma \) is the Poincaré map, and

\[
|\hat{F}_A(\lambda)| = O_{A, \varepsilon}(|\lambda|^{2n+1+\varepsilon}), \quad \text{Im } \lambda < A - \varepsilon
\]

for any \( \varepsilon > 0 \).

The last estimate (1.1) has been modified comparing to [JiZw17, (1.5)]. The additional loss of \( \varepsilon \) in the exponent in (1.1) comes from the following mistake in [JiZw17]: rescaling from [JiZw17, (4.20)] back to [JiZw17, (4.1)], we should gain an additional \( h \) from the derivative changing from \( \frac{d}{dz} \) to \( \frac{d}{d\lambda} \), but also have \( |z| = h|\lambda| \sim h^{1/2} \) and thus the result should be \( h^{-2n} \sim \lambda^{4n} \). However, we can go back to the setting of [DyZw16, Proposition 3.4] and replace \( h^{1/2} \) by any \( h^\varepsilon \) with \( \varepsilon \in (0, 1) \) arbitrarily small. This way we also replace
the bound in [JiZw17, (4.19)] and [JiZw17, (4.20)] by \( h^{-(2-\varepsilon)n-2} \) and thus we obtain the bound in (1.1). In section 3, we will give a simpler proof for a weaker high energy flat trace estimate, comparing to [JiZw17, Proposition 3.1], see Theorem 2. From this, the bound in [JiZw17, (4.20)] becomes \( h^{-2n-2} \), but still gives the same bound in (1.1). The advantage is that we can avoid the complicated construction for complex absorbing potential \( Q \) as in [JiZw17, §2.5].

In [JiZw17], the proof for the high energy flat trace estimate [JiZw17, Proposition 3.1] was incomplete as it relied on the following flawed statement ([JiZw17, (2.14)]) about the semiclassical wavefront set for the resolvent \( R_h(z) = (hP - z)^{-1} \):

\[
WF_h'(R_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}),
\]

which was used to deduce the same statement [JiZw17, (2.19)] for the modified resolvent \( \tilde{R}_h(z) = (hP - iQ - z)^{-1} \). However, \( \tilde{R}_h(z) \) has poles which are exactly the Pollicott–Ruelle resonances. Even in the set where it is well-defined, it is not clear that the kernel is \( h \)-tempered uniformly in \( z \), and thus \( WF_h'(R_h(z)) \) may not be defined. To remedy this issue, we analyze the modified resolvent \( \tilde{R}_h(z) \) directly to give the statement [JiZw17, (2.19)], which is the correct statement eventually used in the proof of Theorem 1 in [JiZw17]. This will be done in Proposition 2.1 in Section 2.

For more details on the notations we refer to [JiZw17]. For preliminaries on semiclassical analysis we refer to Zworski [Zw12] and Dyatlov–Zworski [DyZw19, Appendix E]. For other recent developments concerning trace formulas for Pollicott-Ruelle resonances, see [Je20], [Je21].

2. Wavefront set estimates

In this section, we fix the issue in [JiZw17] by proving the following semiclassical wavefront set estimate for the modified resolvent \( \tilde{R}_h(z) \). We briefly recall the notations from [JiZw17]: Let \( Q \) be the absorbing potential as in [JiZw17], to be more precise, we require

- \( WF_h(Q) \subset \{ |\xi| < 1 \} \);
- \( \sigma_h(Q) > 0 \) on \( \{ |\xi| \leq 1/2 \} \);
- and \( \sigma_h(Q) \geq 0 \) everywhere.

The additional requirement in [JiZw17, §2.5] is used to improve the power in the flat trace estimate (3.1) and we will give a simpler argument in Secion 3 to avoid the complications. In [DyZw16, Proposition 3.4], it is shown that for fixed \( C_1, C_2, \varepsilon > 0 \), \( \tilde{P}_h(z) = hP - iQ - z \) is invertible for \( z \in [-C_1 h^\varepsilon, C_1 h^\varepsilon] + i[-C_2 h, 1] \) and its inverse satisfies the following estimate

\[
\| \tilde{R}_h(z) \|_{\mathcal{H}_h^* \to \mathcal{H}_h^*} \leq C h^{-1}.
\]
Here $\mathcal{H}_h^s = H_{sG(h)}$ is the semiclassical anisotropic Sobolev space defined in [DyZw16, §3.3] and $s > 0$ is a parameter chosen large enough depending on $C_1$ and $C_2$. The weight function $G(h)$ is constructed in a way that $\tilde{P}_h(z) : D_h^s := \{u \in \mathcal{H}_h^s : \tilde{P}_h(z)u \in \mathcal{H}_h^s\} \to \mathcal{H}_h^s$ is invertible. In the following we will only use the fact that

$$H_h^s \subset \mathcal{H}_h^s \subset H_h^{-s},$$

where $H_h^s$ is the usual semiclassical Sobolev spaces on $X$.

**Proposition 2.1.** We have

$$\text{WF}'_h(\bar{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^s \times E_u^s) \setminus \{0\}) \quad (2.1)$$

where $\Omega_+$ is the flowout

$$\Omega_+ = \{ (e^{ithp}(y, \eta), y, \eta) : p(y, \eta) = 0 \} \subset T^*(X \times X) \simeq T^*X \times T^*X,$$

and $\kappa : T^*(X \times X) \setminus \{0\} \to S^*(X \times X)$ is the natural projection map.

**Remark 2.2.** Note that $S^*(X \times X) \neq S^*X \times S^*X$, hence there are difficulties to deal with the fiber infinity directly. In fact, unlike the finite part of the wavefront set $T^*(X \times X) \simeq T^*X \times T^*X$, there is no natural way to identify the element in $S^*X \times S^*X$ where $S^*X = \kappa(T^*X \setminus \{0\})$ with the element in $S^*(X \times X) = \kappa(T^*(X \times X) \setminus \{0\})$. However, we do have the natural identification of the diagonal elements $\Delta(S^*X) = \kappa(\Delta(T^*X) \setminus \{0\})$.

The rest of this section will be devoted to the proof of Proposition 2.1. We will follow the strategy of [DyZw16, Proposition 3.4], where the authors prove the estimate for the finite part of $\text{WF}'_h(\bar{R}_h(z))$. To deal with the wavefront set at fiber infinity we introduce another small parameter $h > 0$ (which will play the role of $|\langle \xi, \eta \rangle|^{-1}$).

**Step 1:** Let $p^{-1}(0) = \{(x, \xi) \in T^*X : p(x, \xi) = 0\} \supset E_u^s \cup E_u^s$, we first show a weaker statement:

$$\text{WF}'_h(\bar{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^s \times p^{-1}(0)) \setminus \{0\}). \quad (2.2)$$

Suppose $(x_0, \xi_0, y_0, \eta_0) \in \{ |\langle \xi, \eta \rangle| = 1 \} \setminus (\Delta(T^*X) \cup \Omega_+ \cup (E_u^s \times p^{-1}(0)))$, then as in [DyZw16, Proposition 3.4], using the propagation estimate ([DyZw16, Proposition 2.5]) and the radial source estimate ([DyZw16, Proposition 2.6]), we can find a sufficiently large $\rho > 0$, neighbourhoods $U$ of $(x_0, \rho\xi_0)$ and $W$ of $(y_0, \rho\eta_0)$, and $A, B \in \Psi^0_h(X)$ such that

- $U \subset \ell_{h}(A)$ and $A$ is microlocally supported near $(x_0, \rho\xi_0)$;
- $B$ is microlocally supported in a neighbourhood of $\{e^{-iH_p}(x_0, \rho\xi_0) : t \geq 0\}$ and

$$\{ |\xi| \leq 1 \} \cup W) \cap \text{WF}_h(B) = \emptyset. \quad (2.3)$$
there exists a constant $C > 0$, for any $h$-tempered $u \in \mathcal{D}'(X)$,
\[
\|A u\|_{\mathcal{H}^s} \leq C h^{-1} \|B \tilde{P}_h(z) u\|_{\mathcal{H}^s} + \mathcal{O}(h^{\infty}) \|u\|_{H^{-N}}.
\]
Here we use the condition $(x_0, \xi_0, y_0, \eta_0) \notin E^s \times p^{-1}(0)$ to guarantee $\text{WF}_h(B) \cap \{|\xi| \leq 1\} = \emptyset$ in (2.3) when $\rho > 0$ is large enough. We can also assume that
\[
A = \text{Op}_h(a), \quad B = \text{Op}_h(b), \quad Q = \text{Op}_h(q)
\]
with symbols $b \in S^0$ and $a, q \in C^\infty$ independent of $h$, and $\text{supp} q \subset \{|\xi| \leq 1\}$ so that $\text{supp} q \cap \text{supp} b = \emptyset$. Here $\text{Op}_h$ denotes a semiclassical quantization on a compact manifold, see [DyZw19, Appendix E].

Now we introduce another small parameter $\tilde{h} \to 0^+$ independent of $h$ to describe the behaviour of the semiclassical Fourier transform as $(\xi, \eta) \to \infty$ in a conic neighborhood of $(\xi_0, \eta_0)$. Replacing $h$ by $\tilde{h} h$ in the estimate (2.4), we get
\[
A_{\tilde{h}} = \text{Op}_{\tilde{h} h}(a), \quad B_{\tilde{h}} = \text{Op}_{\tilde{h} h}(b), \quad Q_{\tilde{h}} = \text{Op}_{\tilde{h} h}(q) \in \Psi^0_{\tilde{h} h}(X)
\]
such that
\[
\|A_{\tilde{h}} u\|_{\mathcal{H}^s_{\tilde{h} h}} \leq C(\tilde{h} h)^{-1} \|B_{\tilde{h}}(\tilde{h} h) P - \tilde{h} z - i Q_{\tilde{h}} u\|_{\mathcal{H}^s_{\tilde{h} h}} + \mathcal{O}((\tilde{h} h)^{\infty}) \|u\|_{H^{-N}_{\tilde{h} h}},
\]
\[
U \subset \ell h_{\tilde{h}}(A_{\tilde{h}}), \quad (\{|\xi| \leq 1\} \cup W) \cap \text{WF}_{\tilde{h} h}(B_{\tilde{h}}) = \emptyset.
\]
Note $z \in [-C_1 h^\varepsilon, C_1 h^\varepsilon] + i[-C_2 h, 1]$ implies $\tilde{h} z \in [-C_1(\tilde{h} h)^\varepsilon, C_1(\tilde{h} h)^\varepsilon] + i[-C_2 \tilde{h} h, 1]$. However we wish to recover $\tilde{P}_h$ in estimate (2.5), and this require us to replace $Q_{\tilde{h}}$ by $\tilde{h} Q$ and to deal with the $Q$ term. We need the following lemma:

Lemma 2.3. For every $N \in \mathbb{N}$,
\[
\|B_{\tilde{h}} Q u\|_{H^N_{\tilde{h} h}} = \mathcal{O}(h^{\infty} \tilde{h}^\infty) \|u\|_{H^{-N}_{\tilde{h} h}}.
\]

Proof. Using a partition of unity argument we can reduce to the case $M = \mathbb{R}^n$ and assume that all the symbols are compactly supported in $\mathbb{R}^{2n}$. Recall that (e.g. [Zw12, Theorem 4.23]) for a sufficiently large constant $M > 0$ only depending on $n = \dim M$,
\[
\|\text{Op}_h(a)\|_{L^2 \to L^2} \lesssim \|a\|_{S_{0, M}}, \quad \|a\|_{S_{k, M}} := \sum_{|\alpha|+|\beta| \leq M} \|\langle \xi \rangle^{|\alpha|-k} \partial_x^{\alpha} \partial_\xi^{\beta} a(x, \xi)\|_{L^\infty}.
\]
Therefore for any $N \geq 0$, we can estimate
\[
\|B_{\tilde{h}} Q\|_{H^{-N}_{\tilde{h} h} \to H^N_{\tilde{h} h}} = \|\langle \tilde{h} D \rangle^N B_{\tilde{h}} Q \langle \tilde{h} D \rangle^N\|_{L^2 \to L^2} \lesssim \|\langle \tilde{h} \xi \rangle^N \#b(x, \tilde{h} \xi) \#q(x, \xi) \#\langle \tilde{h} \xi \rangle^N\|_{S_{0, M}}.
\]
Recall that
\[
(a, b) \mapsto a \# b \text{ is a continuous bilinear map from } S^k \times S^\ell \text{ to } S^{k+\ell} \text{ for any } k, \ell \in \mathbb{R},
\]
Therefore we have
\[ q \in \{ |\xi| \leq 1 \} \]
and when \( \operatorname{supp} a \cap \operatorname{supp} b = \emptyset \), \( a \# b = \mathcal{O}_{S^{k+\varepsilon}}(h^\infty) \), or more precisely, for any \( m \geq 0 \), any \( \hbar \) \( \mathcal{O}_{S^{k+\varepsilon}}(h^\infty) \), or more precisely, for any \( m \geq 0 \), any seminorm of \( a \# b \) in \( S^{k+\varepsilon} \) is bounded by \( h^m \) times the product of a seminorm of \( a \) in \( S^k \) and a seminorm of \( b \) in \( S^\varepsilon \).

Now \( \operatorname{supp} q \subset \{ |\xi| \leq 1 \} \) and the wavefront set condition (2.3) shows that for any \( \hbar \in (0, 1) \), we have \( \operatorname{supp} b(x, \hbar \xi) \cap \operatorname{supp} q(x, \xi) = \emptyset \), therefore for any \( m \geq 1 \), there exists \( M \) depending only on \( N, M \) and \( m \), such that uniformly for sufficiently small \( h, \hbar > 0 \)
\[
\| B_{\hbar} Q \|_{H^{-N}_{\hbar} \rightarrow H^{-N}_{\hbar}} \lesssim h^m \| (\hbar \xi)^N \|_{S^{N, M'}}^{2} \| b(x, \hbar \xi) \|_{S^{m, M'}} \| q(x, \xi) \|_{S^{-m-2N, M'}}.
\]

Finally, we note that \( \| (\hbar \xi)^N \|_{S^{N, M'}}^{2} \) is uniformly bounded in \( \hbar > 0 \) and since \( \{ \xi = 0 \} \cap \operatorname{supp} b = \emptyset \),
\[
\| b(x, \hbar \xi) \|_{S^{m, M'}} = \mathcal{O}(h^m).
\]
We conclude that uniformly for sufficiently small \( h, \hbar > 0 \),
\[
\| B_{\hbar} Q \|_{H^{-N}_{\hbar} \rightarrow H^{-N}_{\hbar}} = \mathcal{O}(h^m \hbar^m),
\]
and since \( m \) can be chosen arbitrarily large, this concludes the proof. \( \square \)

Now we go back to (2.5) and taking \( u(x) = \tilde{R}_h(z)(\psi(x)e^{ix \rho \eta_0/h}) \) (here we choose a local coordinates and identify a neighborhood of \( x_0 \) to subset of \( \mathbb{R}^n \)) where \( \operatorname{supp} \psi \times \{ \rho \eta_0 \} \subset W \), the wavefront set condition (2.3) for \( B \) gives
\[
\| B_{\hbar} Q \|_{H^{-N}_{\hbar} \rightarrow H^{-N}_{\hbar}} = \mathcal{O}(h^\infty \hbar^\infty), \quad \| B_{\hbar} (\psi(x)e^{ix \rho \eta_0/h}) \|_{H^{-N}_{\hbar}} = \mathcal{O}(h^\infty \hbar^\infty).
\]
Therefore we have
\[
\| A_{\hbar} u \|_{H^s_{\hbar}} \leq C h^{-1} \| B_{\hbar} \tilde{P} u \|_{H^s_{\hbar}} + C(h\hbar)^{-1} \| B_{\hbar} Q \tilde{u} \|_{H^s_{\hbar}} + C h^{-1} \| B_{\hbar} Qu \|_{H^s_{\hbar}} + \mathcal{O}((h\hbar)^\infty) \| u \|_{H^{-N}_{\hbar}}
\]
\[
= \mathcal{O}(h^{-1}) \| B_{\hbar} (\psi(x)e^{ix \rho \eta_0/h}) \|_{H^s_{\hbar}} + \mathcal{O}(h^\infty \hbar^\infty) \| u \|_{H^{-N}_{\hbar}}
\]
\[
= \mathcal{O}(h^\infty \hbar^\infty).
\]
This means \( \operatorname{WF}_{\hbar}(u) \cap U = \emptyset \), and thus if \( \chi \in C^\infty(X) \) and \( \operatorname{supp} \chi \times \{ \rho \xi_0 \} \subset U \), then
\[
\int \chi(x)e^{-ix \rho \xi_0/h \hbar} \tilde{R}_h(z)(\psi(x)e^{ix \rho \eta_0/h}) dx = \mathcal{O}(h^\infty \hbar^\infty).
\]
Moreover, by construction it is easy to see the estimate is locally uniform in \((x_0, \xi_0, y_0, \eta_0)\). Therefore by the equivalent definition of semiclassical wavefront sets using the semiclassical Fourier transform (see [Al08, Definition 3.2]), \( \kappa(x, \xi_0, y_0, \eta_0) = \kappa(x, \rho \xi_0, y_0, \rho \eta_0) \notin \operatorname{WF}_{\hbar}(\tilde{R}_h(z)) \cap S^*(X \times X) \) and we have (2.2).
Step 2: The previous method does not work for \((x_0, \xi_0, y_0, \eta_0) \in E_u^* \times p^{-1}(0)\) since \(WF_h(B)\) has to intersect the zero section \(\{\xi = 0\}\). Here we argue by duality. Suppose \((x_0, \xi_0, y_0, \eta_0) \in \{\|\langle \xi, \eta \rangle \| = 1\} \setminus (\Delta \cup \Omega_+ \cup (p^{-1}(0) \times E_u^*))\), we consider the following operator
\[
-\tilde{P}_h(z)^* := -hP^* - iQ - (-\tilde{z}),
\]
acting on \(H_h^{-s}\). We see that this corresponds to the reversed Anosov flow \(\varphi_{-t}\) generated by \(-V\) and \(z \in [-C_h \varepsilon, C_1 h^\varepsilon] + i[-C_2 h, 1]\) also gives \(-\tilde{z}\) in the same region. We can repeat the same argument with the opposite propagation direction we get \(\tilde{P}_h(z)^*\) is invertible, with inverse \(\tilde{R}_h(z)^*: H_h^{-s} \to H_h^{-s}\) satisfying
\[
\|\tilde{R}_h(z)^*\|_{H_h^{-s} \to H_h^{-s}} \leq C h^{-1}.
\]
Moreover, there exist \(\rho > 0, U = \text{nbhd}(x_0, \rho \xi_0)\) and \(W = \text{nbhd}(y_0, \rho \eta_0)\) such that for \(\text{supp } \psi \times \{\rho \xi_0\} \subset W\) and \(\text{supp } \chi \times \{\rho \eta_0\} \subset U\) we have
\[
\int \psi(x) e^{-ix \cdot \rho \xi_0/h} \tilde{R}_h(z)^*(\chi(x) e^{ix \cdot \rho \eta_0/h}) dx = O(h^\infty),
\]
and the estimate is locally uniform in \((x_0, \xi_0, y_0, \eta_0)\). Therefore
\[
\kappa(y_0, \eta_0, x_0, \xi_0) \notin WF_h(\tilde{R}_h(z)^*) \cap S^*(X \times X).
\]
Since the Schwartz kernel of \(\tilde{R}_h(z)^*\) is \(\overline{K(y, x)}\) if the \(K(x, y)\) is the Schwartz kernel of \(\tilde{R}_h(z)\), we have \(\kappa(x_0, \xi_0, y_0, \eta_0) \notin WF_h(\tilde{R}_h(z)) \cap S^*(X \times X)\) and thus
\[
WF'_h(\tilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (p^{-1}(0) \times E_u^*)) \setminus \{0\}.
\]
Combining this with (2.2) we get the desired estimate (2.1) and finish the proof of Proposition 2.1.

3. Flat trace estimates

In this section, we present a simpler argument than the one in [JiZw17] to give the following flat trace estimate (see [JiZw17, Proposition 3.1]). The result is slightly weaker than the original one in [JiZw17], but avoid using [NoZw15, Proposition 10.3] and thus the assumption [JiZw17, (2.7)] for the complex absorbing potential \(Q\).

**Theorem 2.** Fix any \(\varepsilon > 0\), the flat trace
\[
T(z) = \text{tr}^b(e^{-it_0 h^{-1} \tilde{P}_h(z)} \tilde{R}_h(z))
\]
is well-defined and holomorphic for \(z\) in \([-C_1 h^\varepsilon, C_1 h^\varepsilon] + i[-C_2 h, 1]\). Moreover, we have
\[
T(z) = O(h^{-2n-2}).
\]

To prove it we need a wavefront set estimate for the Schwartz kernel of \(e^{-it_0 h^{-1} \tilde{P}_h(z)} \tilde{R}_h(z)\):
Lemma 3.1.

\[
WF_h'(e^{-it_0h^{-1}R_h(z)}\tilde{R}_h(z)) \cap S^*(X \times X) \subset \\
\kappa(\{(x, \xi, y, \eta) : (e^{-t_0H_p(x, \xi, y, \eta)} ) \in \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\} \text{ or } \xi = 0, \eta \neq 0\}).
\]

Proof. Proposition 2.1 gives

\[
WF_h'(\tilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}).
\]

Thus

\[
WF_h'(e^{-it_0V}\tilde{R}_h(z)) \cap S^*(X \times X) \subset \\
\kappa(\{(x, \xi, y, \eta) : (e^{-t_0H_p(x, \xi, y, \eta)} ) \in \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}).
\]

We have

\[
e^{-i0P} - e^{-it_0h^{-1}(hP-iQ)} = h^{-1}\int_0^{t_0} e^{-i(t_0-t)P}Qe^{-ith^{-1}(hP-iQ)}dt,
\]

and using \(WF_h'(Q) \cap S^*(X \times X) = \emptyset\) and [Al08, Lemma 3.7(iii)], we can compute

\[
WF_h'(e^{-i(t_0-t)P}Q e^{-ith^{-1}(hP-iQ)} \tilde{R}_h(z)) \cap S^*(X \times X) \subset (X \times \{0\}) \times S^*X.
\]

Therefore

\[
WF_h'(e^{-it_0h^{-1}R_h(z)}\tilde{R}_h(z)) \cap S^*(X \times X) \\
\subset \left(WF_h'(e^{-i0P} \tilde{R}_h(z)) \cup \bigcup_{t=0}^{t_0} WF_h'(e^{-i(t_0-t)P}Q e^{-ith^{-1}(hP-iQ)} \tilde{R}_h(z)) \right) \cap S^*(X \times X) \\
\subset \kappa(\{(x, \xi, y, \eta) : (e^{-t_0H_p(x, \xi, y, \eta)} ) \in \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\} \text{ or } \xi = 0, \eta \neq 0\}).
\]

Theorem 2 then follows from the following general lemma.

Lemma 3.2. Let \(X\) be an \(n\)-dimensional smooth manifold and \(m \in \mathbb{R}\). If \(P(h) : C^\infty(X) \rightarrow \mathcal{D}'(X)\) is \(h\)-tempered and satisfies

- \(WF_h'(P(h)) \cap \Delta(S^*X) = \emptyset\);
- \(\|AP(h)B\|_{L^2 \rightarrow L^2} = \mathcal{O}(h^{-m})\) for any \(A, B \in \Psi^\mathrm{comp}_h(X)\);

then \(\text{tr}^h(P(h))\) is well-defined with

\[
\text{tr}^h(P(h)) = \mathcal{O}(h^{-2n-m}).
\]

Proof. Since \(WF_h'(P(h)) \cap \Delta(S^*X) = \emptyset\), we have \(WF'((P(h)) \cap \Delta(T^*X) = \emptyset\), it is then a classical theorem (see e.g. [Hö83, Theorem 8.2.4]) that the flat trace is well-defined as long as the wavefront set does not intersect the diagonal.
Let $u = K_h$ be the Schwartz kernel of $P(h)$, $\iota : X \rightarrow X \times X$ be the diagonal embedding, then for $\chi \in C^\infty(X)$, $\varphi(x, y) = \psi(x)\psi(y) \in C^\infty(X \times X)$ supported near the diagonal,

$$\langle \iota^* (\varphi u), \chi \rangle = \langle \varphi u, \iota^* \chi \rangle = \frac{1}{(2\pi h)^{2n}} \int F_h(\varphi u) I_{X,h}(\xi, \eta) d\xi d\eta \quad (3.2)$$

where

$$I_{X,h}(\xi, \eta) = \int \chi(x)e^{ix(\xi+\eta)/h}dx.$$

If $|\xi + \eta| > |\xi|/C$, then

$$I_{X,h}(\xi, \eta) = O(h^{\infty}(|\xi| + |\eta|)^{-\infty}).$$

Thus we only need to consider the case when $(\xi, \eta)$ lies in a small conical neighbourhood of $\{\xi + \eta = 0\}$ or in a neighbourhood of $\{\xi = \eta = 0\}$.

(i) When $|\xi| + |\eta| \leq C$ is bounded, we have for some $A, B \in \Psi_h^{\text{comp}}(X)$

$$|F_h(\varphi u)| = |\langle P(h)B(\psi(y)e^{-i\eta/y/h}), A(\psi(x)e^{-ix/\xi/h}) \rangle| + O(h^{\infty}) \\
\lesssim \|AP(h)B\|_{L^2 \rightarrow L^2} + O(h^{\infty}) \\
= O(h^{-m}).$$

(ii) When $(\xi, \eta)$ is near fiber infinity and in a small conic neighbourhood of $\{\xi + \eta = 0\}$ which does not intersect $WF'_h(P(h))$, we have

$$F_h(\varphi u) = O(h^{\infty}(|\xi| + |\eta|)^{-\infty})$$

thanks to the wavefront condition $WF'_h(P(h)) \cap \Delta(S^*X) = \emptyset$.

Now (3.2) gives us

$$|\langle \iota^* (\varphi u), \chi \rangle| = h^{-2n}O(h^{-m}) = O(h^{-2n-m})$$

and a partition of unity argument finishes the proof. \qed

Proof of Theorem 2. The operator $\tilde{R}_h(z) : H^s_h \rightarrow H^s_h$ is bounded and thus $h$-tempered. Lemma 3.1 gives

$$WF'_h(e^{-it_0h^{-1}\tilde{R}_h(z)}\tilde{R}_h(z)) \cap \Delta(S^*X) = \emptyset$$

if we choose $t_0 > 0$ smaller than the least length of the closed orbits. For any $A, B \in \Psi_h^{\text{comp}}(X)$ recall

$$e^{-it_0P} - e^{-it_0h^{-1}(hP-iQ)} = h^{-1} \int_0^{t_0} e^{-ith^{-1}(hP-iQ)}Qe^{-i(t_0-t)P}dt,$$
we have
\[
\|Ae^{-it_0^2} \tilde{R}_h(z) B\|_{L^2 \to L^2} \\
\lesssim \|Ae^{-it_0^2} \tilde{R}_h(z) B\|_{L^2 \to L^2} + \frac{1}{h} \int_0^{t_0} \|Ae^{-ith^{-1}(hP-iQ)} Q e^{-i(t_0-t)^2} \tilde{R}_h(z) B\|_{L^2 \to L^2} dt \\
\lesssim \|Ae^{-it_0^2} \tilde{R}_h(z) B\|_{\mathcal{H}_h^s \to \mathcal{H}_h^s} + \frac{1}{h} \int_0^{t_0} \|Q e^{-i(t_0-t)^2} \tilde{R}_h(z) B\|_{L^2 \to L^2} dt \\
= O(h^{-1}) + h^{-1} \int_0^{t_0} \|Q e^{-i(t_0-t)^2} \tilde{R}_h(z) B\|_{\mathcal{H}_h^s \to \mathcal{H}_h^s} dt \\
= O(h^{-2}).
\]

Here we use the fact that on compact sets in the phase space $L^2$ norm is equivalent to any $\mathcal{H}^s$ norm. Now the claim follows from Lemma 3.2.

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