Conceptual analysis of quantum history theory

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ABSTRACT. We give formal content to some concepts, naturally stemming from consistent
history approach (CHA), which are not formalized in the standard formulation of the theory.
The outcoming (extended) conceptual basis is used to perform a formal, conceptually trans-
parent analysis of some debated questions in CHA. As results, the problems raised by contrary
inferences of Kent are ruled out, whereas some prescriptions of the theory cannot be mantained.

RESUMÉ. Nous fournissons contenu formel à quelques concepts, sortant naturellement de
la théorie des histoires consistantes (CHA), qui ne sont pas formalisés dans la formulation
usuel. La base conceptuelle (extendue) qui en sort est usée pour effectuer une analyse formel
conceptuellement transparente de quelques problèmes débattus dans CHA. Nous avons comme
resultat que le problèmes soulevés par les inferences contraires de Kent sont résolus; toutefois,
des prescriptions de la théorie ne peuvent pas être maintennes.

Key words: Foundations of quantum theory, Consistent history approach, contrary inferences
1. Introduction

In recent works [1][2] Bassi and Ghirardi have argued that the consistent history approach (CHA) to quantum physics, first proposed by Griffiths [3], leads to logical contradiction with some assumptions "necessary for a sound interpretation of the theory" [1]. These criticisms to CHA generalize those moved by Kent [4], who proved that CHA "allows contrary inferences from the same empirical data". In his replies Griffiths argues that the criticisms are the result of a misunderstanding of his theory [5]. However, Griffiths’ arguments did not convince Bassi and Ghirardi [6]. On the other hand, also Kent [7] considered the answer of Griffiths and Hartle [8] to his own criticism unsatisfactory [4]. Hence, the scientific debate seems to have ended without a definitive clarification of the subject.

We extend the formal apparatus of the theory, by introducing the notion of support of a family of histories. This theoretical extension allows us to perform a conceptually transparent analysis of the above mentioned criticisms. The conclusions of our analysis partly agree with Griffiths; for instance the conceptual difficulties raised by contrary inferences are ruled out. But they also entail that some prescriptions of the theory cannot be maintained, in partial agreement with Bassi and Ghirardi.

In section 3 we present a brief synthesis of the debate. To do this,
we have to recall, in section 2, some standard concepts of CHA. In section 4 we introduce the formal tools we need for the conceptual analysis performed in sections 5 and 6. In particular, in section 5 we deal with the criticism of Bassi and Ghirardi; while our analysis shares Griffiths’ conclusion that there is no universal truth functional in quantum history theory, it entails that we cannot maintain the assumption that every family of histories can be chosen, in agreement with Bassi and Ghirardi. In section 6 we show that our analysis of contrary inferences removes conceptual difficulties, also without the necessity of adopting a weaker interpretation of perpendicular (i.e. mutually exclusive) events, as done by Griffiths.

2. Consistent History Quantum Theory

Let $\mathcal{H}$ be the Hilbert space of the standard quantum description, in Heisenberg’s picture, of the physical system. Throughout this paper we assume that $\mathcal{H}$ has a finite dimension $N$. A decomposition of the identity is a finite set $\mathbf{E} = \{E^{(1)}, E^{(2)}, ..., E^{(k)}\}$ of projection operators such that $E^{(i)} \perp E^{(j)}$ if $i \neq j$ and $\sum_{i=1}^{k} E^{(i)} = 1$. Given a finite, ordered sequence of times $(t_1, t_2, ..., t_n)$, for each time $t_k$ we consider a decomposition of the identity $\mathbf{E}_k$ whose projections represent (in Heisenberg picture) events
which may occur at time $t_k$. A sequence $h = (E_1, E_2, ..., E_n)$ such that $E_k \in E_k$, i.e. in Cartesian product $\mathcal{E} = E_1 \times E_2 \times \cdots \times E_n$, is called elementary history. The family $\mathcal{C}$ of histories, generated by $E_1, E_2, ..., E_n$, is the set of all sequences $h = (E_1, E_2, ..., E_n)$ such that $E_k = \sum_{i=1}^{\text{some}} E_{k}^{(i)}$. Given every history $h$, we define the bounded operator $C_h = E_n \cdot E_{n-1} \cdot \cdots \cdot E_1$.

Each history $h = (E_1, E_2, ..., E_n)$ can be identified with subset $h \subseteq \mathcal{E}$, where $h = \{ \hat{h} = (\hat{E}_1, \hat{E}_2, ..., \hat{E}_n) \in \mathcal{E} \mid \hat{E}_k \leq E_k, \ \forall k \}$.

The physical interpretation of CHA, which makes it a physical theory, is based on the principle which establishes that if a given family $\mathcal{C}$ satisfies a criterion of consistency, then

(I) The set of all elementary histories of $\mathcal{C}$ is a “sample space of mutually exclusive elementary events, one and only one of which occurs” [8].

The occurrence of history $h$ means that an elementary history $\hat{h} \in h$ occurs.

The basic physical notion of the theory is that of occurrence of a history, whose meaning is the following.

(O) A given history $h = (E_1, E_2, ..., E_n)$ occurs if all events $E_1, E_2, ..., E_n$ objectively occur at respective times $t_1, t_2, ..., t_n$. The occurrence of a history is an objective physical fact, independent of the performance of a measurement which reveals this occurrence.
Hence, in this theory measurements reveal properties already objectively possessed by the physical system. This is remarkably different from standard quantum theory, which, on the contrary, can make statistical predictions only about outcomes of measurements, and cannot describe the occurrence of a history \( h = (E_1, E_2, ..., E_n) \) (consider the case \([E_k, E_j] \neq 0\)).

The criterion for establishing whether \( \mathcal{C} \) is consistent is given by the following rule of CHA.

**Rule 1.** A family \( \mathcal{C} \) is consistent if and only if it is weakly decohering, i.e., if \( \text{Re}(\text{Tr}(C_{h_1}C_{h_2}^*)) = 0 \) for all \( h_1, h_2 \in \mathcal{E}, h_1 \neq h_2 \).

The occurrences of histories are the empirical facts the theory is concerned with; CHA postulates that their statistics obeys the following rule.

**Rule 2.** If \( \mathcal{C} \) is consistent, then number \( p(h) = \frac{1}{N} \text{Tr}(C_hC_h^*) \) is the probability of occurrence of history \( h \) for every \( h \in \mathcal{C} \).

It can be proved that if \( \mathcal{C} \) is weakly decohering then, coherently with statement (I), for every \( h \in \mathcal{C} \) we have \( p(h) = \sum_{\hat{h} \in h} p(\hat{h}) \).

Following [9], to make inferences about occurrences of histories, the initial data, i.e. the available information about the physical system, must be expressed as sentences involving histories of a consistent family.
$\mathcal{C}$, and assumed as true, i.e. objectively realized sentences. At this point it is possible to derive conclusions by making logical reasonings in which histories of $\mathcal{C}$ are regarded, according to (I), as events of a classical sample space. According to CHA these conclusions have to be regarded as empirically valid sentences. However, another set of conclusions could be derived by using another consistent family $\mathcal{C}'$ in which the same initial data can be expressed. A conclusion of this new set could conflict with the conclusions drawn from $\mathcal{C}$. In Griffiths’ theory these kinds of conflicts are avoided by means of the introduction of the following.

**Rule 3.** All valid physical inferences are those obtained by using rule 2 within a single consistent family $\mathcal{C}$. In general, different conclusions drawn by using distinct consistent families do not hold together.

### 3. Debated questions

*Single family* rule 3 is at the root of the criticisms mentioned in section 1. To describe the criticisms we make use of the contrary inferences discovered by Kent [4].

#### 3.1. Contrary inferences

Kent was able to find two histories $h_1 = (E_0, E_1, E_2)$ and $h_2 = (E_0, F_1, E_2)$,
with $E_1 \perp F_1$, belonging to two different consistent families $C_1$ and $C_2$ respectively, such that according to the rules of CHA

$$\begin{cases} p(E_1 \mid E_0, E_2) = 1 & \text{in family } C_1 \\ p(F_1 \mid E_0, E_2) = 1 & \text{in family } C_2 \\ p(E_0, E_2) \neq 0 & \text{(in both families).} \end{cases}$$

If we take as initial data the occurrence of $E_0$ and $E_2$, this becomes a very striking clear example of the above mentioned conflict. Indeed, these initial data are compatible with both families $C_1$ and $C_2$ (third equation in (1)). Therefore, when history $h_0 = (E_0, E_2)$ occurs we conclude that

\begin{enumerate}
\item[(α)] event $E_1$ occurs at time $t_1$, by reasoning with $C_1$,
\item[(β)] event $F_1$ occurs at time $t_1$, by reasoning with $C_2$.
\end{enumerate}

Since $E_1 \perp F_1$ means that $E_1$ and $F_1$ are mutually exclusive events, according to Kent the simultaneous validity of (α) and (β) makes it “hard to take it [CHA] seriously as a fundamental theory in its present form” [4]. Even though single family rule 3 formally prevents contrary inferences from yielding theoretical contradiction [8], “Any formalism [...] can be made free from contradiction by such a restriction” [7].

The answer of Griffiths ([10], Appendix A) was that the problem arises because Kent assigns to ‘contrary’ histories $h_1$ and $h_2$ the classical-logic meaning of word contrary, i.e. that the occurrence of $h_1$ always implies
the non-occurrence of $h_2$, as stated by axiom 3 in section 6 of present paper. But, according to Griffiths, this cannot be done because $h_1$ and $h_2$ cannot be compared without violating single family rule 3.

### 3.2. Ordered consistency

A proposal to solve the problem is due to Kent himself. He proposed to replace the original criterion of consistency, i.e. by a more restrictive one he called ordered consistency [11]. Kent defined the ordering $h_1 \leq h_2$ iff $E_k \leq F_k$ for all $k$, where $h_1 = (E_1, E_2, ..., E_k, ...)$ and $h_2 = (F_1, F_2, ..., F_k, ...)$. History $h_1$ is said to be ordered consistent if $h_1$ belongs to a consistent family and if $h_1 \leq h_2$ implies $\text{Tr}(C_{h_1}\rho C_{h_1}^*) \leq \text{Tr}(C_{h_2}\rho C_{h_2}^*)$, for every $h_2$ belonging to a consistent family. When all histories of a consistent family $\mathcal{C}$ are ordered consistent, then $\mathcal{C}$ is said to be ordered consistent. Then Kent proved that contrary inferences are forbidden if the new criterion of consistency is adopted. However, it must be noticed that the sense of the proposal of Kent was not to suggest that the ordered consistent formalism is the “right” interpretation of quantum theory: “The aim here is thus not to propose the ordered consistent approach as a plausible fundamental interpretation of quantum theory, but to suggest that the range of natural and useful mathematical definitions of types of quantum history is wider than previously understood.” [11]. In the present work a different
approach to the problem of contrary inferences is followed; in section 6 we show that contrary inferences do not yield contradiction if the conceptual basis introduced in section 4 is adopted.

3.3. More general criticism

The more general criticism raised by Bassi and Ghirardi also gives a formal content to that of Kent. They take into account four assumptions, labelled as (a), (b), (c) and (d) in [1]. The first two, (a) and (b), essentially reflect the content of our rule 1 and rule 2. The third assumption, (c), reflects the meaning of the notion of occurrence of history as expressed by (O):

(c) The occurrence of a given history “cannot depend from the decoherent [i.e. consistent] family one is considering.”

Bassi and Ghirardi proved that these assumptions lead to logical contradiction with the following fourth assumption.

(d) “Any decoherent family must be taken into account”, because “some supporters [of CHA] insist in claiming that there are no privileged families.” [1].

In his reply [5][12] Griffiths uses essentially two arguments.

1. The derivation of the contradiction violates the single family rule.

2. “The conceptual difficulty goes away if one supposes that the two in-
compatible frameworks are being used to describe two distinct physical systems that are described by the same initial data” [13].

All replies have not convinced [7][6] the critics of CHA. We shall attempt to explain synthetically the reason for such a disagreement by using example 1. Suppose that the known data about a single physical system $s$ are that $E_0$ and $E_2$ occur. In order to establish whether $E_1$ occurs or not, a physicist can use family $C_1$ and, in accordance with (1), he finds that $E_1$ occurs. But another physicist, for the same individual system $s$, could choose $C_2$, and he must conclude that $F_1$ occurs. The fact that $E_1$ occurs or $F_1$ occurs seems to depend on the physicist’s choice of family $C_1$ or $C_2$. But, according to CHA itself, the occurrence of a history, once established by means of the theory, is an objective fact. As a consequence, both $E_1$ and $F_1$ should occur. But this final conclusion is rejected by everybody because $E_1 \perp F_1$.

We see that replies 1 and 2 above do not provide a satisfactory answer to the problem. Therefore, the question

what is the event which occurs for this $s$, $E_1$ or $F_1$?

remains open.
4. Conceptual basis

To begin our analysis, we take into account basic principles (I) and (O). Principle (I) entails that, given a consistent family \( \mathcal{C} \), each time the fact that ‘an elementary history of \( \mathcal{C} \) occurs and all other elementary histories do not occur’ objectively takes place, this holds for one individual concrete sample of the physical system. Then, for every family \( \mathcal{C} \) we can postulate the existence of a set \( c(\mathcal{C}) \), whose elements represent all concrete physical systems, such that for each individual \( s \in c(\mathcal{C}) \) every history of \( \mathcal{C} \) either occurs or does not occur. In CHA language, an individual concrete system \( s \) belongs to \( c(\mathcal{C}) \) if and only if every history \( h \in \mathcal{C} \) either occurs or does not occur (makes sense) for this \( s \). The introduction of set \( c(\mathcal{C}) \), we shall call support of \( \mathcal{C} \), allows us to formally express the consistency of a family by means of a simple definition.

**Definition 1.** A family \( \mathcal{C} \) is consistent if and only if \( c(\mathcal{C}) \neq \emptyset \).

Now we shall establish, in a coherent fashion, basic principle (P), and general Axioms 1 and 2. Let \( s \) be an individual physical system. If \( s \notin c(\mathcal{C}) \), then a history of \( \mathcal{C} \) does not necessarily make sense for \( s \), even if \( \mathcal{C} \) is consistent. In this case to ask for the occurrence of a history \( h \in \mathcal{C} \) is generally speaking as meaningless, as, for instance, to ask for the political
tendency of an electron. Then, the following principle must hold:

(P) the (sufficient) condition which makes the conclusions of logical reasonings based on a family $C$ valid is

$$s \in c(C).$$ (2)

Now we introduce the first formal axiom.

**Axiom 1.** Let $C_1, C_2$ be two families of histories. Then

$$C_1 \subseteq C_2 \implies c(C_2) \subseteq c(C_1).$$

Let us explain why Axiom 1 should hold. Since the validity of Axiom 1 is obvious when $c(C_2) = \emptyset$, we consider the case in which $c(C_2) \neq \emptyset$. If $s \in c(C_2)$, then there is an elementary history $h_2$ of $C_2$ which occurs, and all other elementary histories of $C_2$ do not occur for this $s$. From $C_1 \subseteq C_2$ it follows that all elementary histories of $C_1$ form a set, denoted by $E_1$, of albeit non-elementary histories of $C_2$ ($C_1$ is a coarser graining than $C_2$). Only one history $h_1$ among those of $E_1$ occurs in correspondence with this $s$, because there is a unique $h_1 \in E_1$ such that $h_2 \in h_1$, and all other $h \in E_1$ do not occur (see (I)). Therefore, it is possible to state that only one elementary history of $C_1$ occurs and all the others do not occur for this individual system $s$, thus $s \in c(C_1)$. 
Now we proceed to state axiom 2. If \( h \in \mathcal{C} \), by \( c_1(h; \mathcal{C}) \) (resp., \( c_0(h; \mathcal{C}) \)) we denote the subset of \( c(\mathcal{C}) \) whose elements are the systems for which \( h \) occurs (resp., does not occur). It is obvious that

\[
c_0(h; \mathcal{C}) = c(\mathcal{C}) \setminus c_1(h; \mathcal{C}) \quad \text{and} \quad c(\mathcal{C}) = \bigcup_{h \in \mathcal{E}} c_1(h; \mathcal{C}). \tag{3}
\]

We adopt the notion of occurrence of history expressed by \((O)\), according to which the occurrence of \( h = (E_1, E_2, ..., E_n) \) is equivalent to the occurrences of all events \( E_1, E_2, ...E_n \), without making reference to the family which \( h \) belongs to. Coherently, we state the further following axiom.

**Axiom 2.** If \( h \in \mathcal{C} \cap \mathcal{C}' \), then \( c_1(h; \mathcal{C}) \cap c_0(h; \mathcal{C}') = c_1(h; \mathcal{C}') \cap c_0(h; \mathcal{C}) = \emptyset \).

In other words, \( h \) cannot both occur as history of \( \mathcal{C} \) and do not occur as history of \( \mathcal{C}' \), for the same system \( s \).

Now we use axioms 1 and 2 to introduce the notion of *truth functional* stemming from our approach. Let \( X \) be a set of histories. The *family generated by \( X \)* is \( \mathcal{C}(X) = \cap_{X \subseteq \mathcal{C}} \mathcal{C} \). For instance, family \( \mathcal{C}(\{h\}) \) generated by a single history \( h = (E_1, E_2, ..., E_n) \) is the family having \( \mathcal{E}(h) = \{(F_1, F_2, ..., F_n) \mid F_k \in \{E_k, E'_k\}\} \) as set of elementary histories; indeed, \( h \in \mathcal{C} \) implies \( \mathcal{C}(\{h\}) \subseteq \mathcal{C} \).

Through Axiom 1 we find that \( h \in \mathcal{C} \) implies \( c(\mathcal{C}) \subseteq c(\mathcal{C}(\{h\})) = \bigcup_{h \in \mathcal{C}} c(\mathcal{C}) \). Therefore, \( c(h) \equiv c(\mathcal{C}(\{h\})) \) is the set of all concrete physical
systems for which single history $h$ occurs or does not occur ($h$ makes sense). By $c_1(h)$ (resp., $c_0(h)$) we denote the subset of those systems for which $h$ occurs (resp., does not occur). Of course

$$c_0(h) = c(h) \setminus c_1(h), \quad c_1(h) = c(h) \setminus c_0(h).$$

(4)

Then for each physical system $s$, we can define the mapping

$$t_s : \cup_{s \in c(C)} \mathcal{C} \to \{0, 1\}, \quad t_s(h) = \begin{cases} 1 & \text{if } s \in c_1(h) \\ 0 & \text{if } s \in c_0(h) \end{cases},$$

(5)

called truth functional relative to $s$. If $t_s(h) = 1$, then $h$ occurs as history of $\mathcal{C} \{h\}$, and hence it occurs as history of whatever family $\mathcal{C}$ such that $h \in \mathcal{C}$ and $s \in c(\mathcal{C})$.

In order to perform our conceptual analysis of CHA, we have to consider the formal tools just established, together with standard axioms of CHA, which we here formulate as axioms CHA.1 and CHA.2.

**Axiom CHA.1.** A family $\mathcal{C}$ is consistent if and only if it is weakly decohering.

**Axiom CHA.2.** Let $\mathcal{C}$ be a consistent family. If $h \in \mathcal{C}$, then $p(h) = \frac{1}{N} Tr(C_h C^*_h)$ is the probability of occurrence of history $h$.

The existence of (non-empty) set $c(\mathcal{C})$ for every (consistent) family $\mathcal{C}$ is a logical consequence of the notion (O) of occurrence of history. Whether a given concrete sample $s$ of the physical system belongs to $c(\mathcal{C})$ or not is a
question the physicist can try to answer by using the tools provided by the theoretical apparatus, as the axioms, and the initial data at his disposal; for instance, see our discussion of the problem of contrary inferences in section 6. However, due to the intrinsically stochastic character of CHA [13], a general characterization of $c(C)$ cannot be given, though its existence cannot be denied without affecting the objectivity of the occurrences of histories, which is one of the peculiar basic ideas of CHA.

On the other hand, we encounter a similar situation in other valuable physical theories. For instance, classical statistical mechanics assumes that many micro-states correspond to one macro-state, and that the micro-state of an individual system (e.g. a gas) in a known macro-state is unique at a particular time $t$. While such an assumption is very useful in developing the theory, in general the theory itself is not able to establish the particular micro-state the system occupies when its macro-state is known.

5. Conceptual analysis

Now we shall demonstrate explicitly that our analysis leads to the conclusion that assumption (d) in section 3 does not hold. Assumption (c) still holds, and we get a more precise understanding of it. In so doing, we
can also point out how the standard interpretation should be modified.

First, we consider assumption (c). Because of Axiom 2 and (5) it is clear that we have to agree with the following idea expressed by Bassi and Ghirardi: “Does the truth value of the considered history depend on the decoherent family to which it may belong? We think that the answer must be “no”. ” [6]. However, now we have a deeper understanding. The fact that a given history $h_0$ occurs (or does not occur) for a physical system $s_0$ entails that a family $C_0$ exists such that

$$h_0 \in C_0 \quad \text{and} \quad s_0 \in c(C_0).$$  \hspace{1cm} (6)

For instance, in virtue of Axiom 1, family $C_0 = C(\{h_0\})$ fulfills these requirements. All histories of $C_0$ make sense for $s_0$. However, the eventuality that for a given system $s$ history $h \in C$ occurs, where $C$ is a consistent family, but $s \notin c(C)$ is logically possible; remark 3 in section 6 provides an explicit example. In general, therefore, the fact that a family $C$ is consistent, and $h \in C$ does not imply that the inferences obtained by means of reasonings based on $C$ hold for an arbitrary physical system $s$ for which $h$ occurs.

Furthermore, assumption (c) cannot be generally interpreted in the sense that truth functional $t_s$ must be defined on all consistent families. Indeed, set $D_s$ which $t_s$ is defined on, is just $D_s = \cup_{s \in c(C)} C$. Therefore, we
also agree with Griffiths’ conclusion [12] stating that in quantum physics 
there is no universal truth functional.

Now we come to assumption (d). Once again, principle (P) denies 
that all consistent families are valid bases for quantum reasonings. Only 
if a family $\mathcal{C}$ satisfies (2), can the inferences based on $\mathcal{C}$ be considered 
valid. However, the failure of assumption (d) does not yield conceptual 
difficulties, because whether a family can be used or not depends on con-
dition (2), rather than on the physicist’s (subjective) choice. Even though 
a general criterion for establishing whether (2) holds is not available, un-
der certain circumstances we can state that (2) certainly holds. For in-
stance, suppose that the following statement holds for $s$: “history $h_0$
occurs” (initial data). Then $s \in c_1(h_0)$, and hence $s \in c(\mathcal{C}(\{h_0\}))$, are 
true statements. Thus, all sentences obtained by logical deductions based 
on $\mathcal{C}(\{h_0\})$ hold for $s$.

Family $\mathcal{C}(\{h_0\})$ admits several refinements, i.e. various families $\mathcal{C}$ may 
exist such that $\mathcal{C}(\{h_0\}) \subseteq \mathcal{C}$. Deductions based on a refinement $\mathcal{C}$ in gen-
eral do not hold for $s \in c(\mathcal{C}(\{h_0\}))$; indeed, as already argued, $\mathcal{C}(\{h_0\}) \subseteq \mathcal{C}$ 
implies, by Axiom 1, $c(\mathcal{C}) \subseteq c(\mathcal{C}(\{h_0\}))$, and therefore $s \in c(\mathcal{C})$ does not 
necessarily follow. Of course, all inferences obtained by reasoning with 
$\mathcal{C}(\{h_0\})$ can be obtained with $\mathcal{C}$, since $\mathcal{C}(\{h_0\}) \subseteq \mathcal{C}$, therefore the two set
of sentences cannot contradict each other, but inferences involving histories in \( \mathcal{C} \setminus \mathcal{C}(\{h_0\}) \) in general do not make sense if \( s \in c(\mathcal{C}(\{h_0\})) \setminus c(\mathcal{C}) \).

The whole argument immediately extends to the more general case in which the data consist in the occurrence of a given set \( X \) of histories, by replacing \( \mathcal{C}(\{h\}) \) with \( \mathcal{C}(X) \).

Thus, there is a profound difference between the “coarsest” family \( \mathcal{C}(X) \) compatible with the available initial data relative to a physical system \( s \) and any refinement \( \mathcal{C} \) of \( \mathcal{C}(X) \): conclusions drawn by using \( \mathcal{C}(X) \) are true, i.e. objective facts, whereas the truth of conclusions obtained from a refinement is not ensured by the truth of the data \( (X) \) alone. For instance, if a reasoning made in \( \mathcal{C}(X) \) leads us to infer that a certain history \( h_1 \in \mathcal{C}(X) \) occurs, such an occurrence must be considered an objective fact. On the contrary, when the occurrence of a history \( h_2 \in \mathcal{C} \setminus \mathcal{C}(X) \) is inferred by a reasoning made in \( \mathcal{C} \), such an occurrence cannot be considered certain.

Remark 1. Griffiths puts forward a precise strategy for choosing the family to be used: “Use the smallest, or coarsest framework which contains both the initial data and the additional properties of interest in order to analyse the problem.” [13]. In the present approach Griffiths’ strategy is not always valid. Indeed, coarsest family \( \mathcal{C} \) containing, besides the initial
data, the properties of interest as well, does not satisfy (2) in general.

Thus, a conclusion of our analysis is that assumption (d) cannot be maintained, while assumption (c) holds.

REMARK 2. Kent considered single family rule 3 an unnatural expedient to avoid contradiction [7] and this was at root of the criticisms; we see that that it is sufficient our quite natural principle (P) to avoid conflicts between conclusions drawn from different families. However, since in Griffiths’ theory the conclusions drawn in different families hold together in the case that these families are ‘compatible’, we recall the definition of compatibility.

DEFINITION 2. Two consistent families $C_1$ and $C_2$ are compatible if a third consistent family $C$ exists such that $C_1 \cup C_2 \subseteq C$.

Compatibility implies $c(C) \subseteq c(C_1) \cap c(C_2)$; therefore, according to our approach, the conclusions drawn from $C_1$ and $C_2$ hold together only for those systems $s$ such that $s \in c(C_1) \cap c(C_2)$, in particular if $s \in c(C)$; on the contrary, if $s \in c(C_1)$ and $s \notin c(C_2)$, then a conclusion drawn from $C_2$ does not necessarily hold for this $s$ (see remark 3 in the next section for a concrete example).
6. Re-interpreting contrary inferences

It is worthwhile seeing whether the particular situation of contrary inferences, described in example 1, can be interpreted without encountering conceptual difficulties.

Let $E$ and $F$ be two mutually orthogonal projections. The family generated by $E$ and $F$, i.e. $\mathcal{C}(\{E, F\})$, has 3 elementary (one-event) histories: $\mathcal{E} = \{E, F, G = 1 - (E + F)\}$; it is the smallest family containing $E$ and $F$. Then, whenever both $E$ and $F$ make sense, all histories in $\mathcal{C}(\{E, F\})$ must make sense too. Therefore, we can state the following proposition.

**Proposition 1.** If $E \perp F$, then $s \in c(E) \cap c(F)$ implies $s \in c(\mathcal{C}(\{E, F\}))$ and, therefore, $c_1(E) \cap c_1(F) = \emptyset$.

Proposition 1 says that two perpendicular projections represent mutually exclusive events.

Kent’s example 1 would be a proof of a contradiction if $\mathcal{C}_1$ and $\mathcal{C}_2$ were compatible families. Indeed, if $\mathcal{C}_1$ and $\mathcal{C}_2$ were compatible, there would be a consistent family $\mathcal{C}$ such that $\mathcal{C}_1 \cup \mathcal{C}_2 \subseteq \mathcal{C}$. Then, from axiom 1

$$c(\mathcal{C}) \subseteq c(\mathcal{C}_1) \cap c(\mathcal{C}_2)$$

would follow; furthermore $h_0 = (E_0, E_2) \in \mathcal{C}$. By $p(E_0, E_2) \neq 0$ in (1)
we find that there exists $\hat{s} \in c_1(h_0) \cap c(C)$. By (7), $\hat{s} \in c(C_1) \cap c(C_2)$. Therefore, by (1) we should conclude $\hat{s} \in c_1(E_1)$ and $\hat{s} \in c_1(F_1)$. Thus we have a contradiction with $c_1(E_1) \cap c_1(F_1) = \emptyset$ following from proposition 1, since $E_1 \perp F_1$.

However, $C_1$ and $C_2$ in example 1 are not compatible families, therefore this argument does not apply. On the contrary, the occurrence of history $h_0 = (E_0, E_2)$ does not give rise to conceptual difficulties. The condition to be satisfied to avoid the contradiction is

$$c_1(h_0) \cap c(C_1) \cap c(C_2) = \emptyset.$$  \hfill (8)

Indeed, if $s \in c_1(h_0) \cap c(C_1) \cap c(C_2) \neq \emptyset$, then $s \in c_1(E_1) \cap c_1(F_1) \neq \emptyset$ would follow from (1), in contradiction with Prop.1.

If $C_1$ and $C_2$ are not compatible, (8) is logically consistent with the occurrence of $h_0$. Indeed, from

$$h_0 \in C_1 \cap C_2,$$  \hfill (9)

by axiom 1 we get

$$c(C_1) \cup c(C_2) \subseteq c(h_0),$$

which is consistent with (8). In particular, when $h_0$ occurs, i.e. $s \in c_1(h_0) \subseteq c(h_0)$, we have 3 distinct possibilities, all consistent with (8); namely
p) \( s \in c(C_1) \), and then \( E_1 \) occurs. In such a case consistency with (8) requires that \( s \notin c(C_2) \), therefore the second equation in (1), stating that ‘\( h_0 \) occurs implies \( F_1 \) occurs’, does not hold for this \( s \). There are two possibilities regarding the occurrence of \( F_1 \):

\[ p_1) \ s \in c(F_1) \setminus c(C_2). \]  
This possibility is consistent because \( F_1 \in C_2 \) implies \( c(C_2) \subseteq c(F_1) \). By axioms 2 and proposition 1 we must conclude that \( F_1 \) does not occur, i.e. \( s \in c_0(F_1) \);

\[ p_2) \ s \notin c(F_1), \]  
therefore \( F_1 \) does not make sense, i.e. it neither occurs nor does not occur.

q) \( s \in c(C_2) \), and then \( F_1 \) occurs. In such a case we have for \( E_1 \) the same conclusions of item (a) for \( F_1 \).

r) \( s \notin c(C_1) \cup c(C_2) \). In this case no inference about \( E_1 \) or \( F_1 \) can be drawn from the data and (1).

Which of the alternative, and mutually exclusive, possibilities (p), (q) and (r) above is actually realized with our initial data \( (s \in c_1(h_0)) \) is a question which cannot be answered without further data.

Remark 3. The foregoing argument provides a concrete example in which a statement drawn from a consistent family \( (C(\{h_0\})) \) holds, whereas another statement drawn from another family compatible with \( C(\{h_0\}) \)
does not hold. Indeed, statement “$h_0$ occurs” must hold for some $s \in c(C(\{h_0\}))$, because $p(h_0) \neq 0$, whereas at least one of the statements “$E_1$ occurs” or “$F_1$ occurs”, drawn in correspondence with initial datum “$h_0$ occurs” from $C_1$ or $C_2$, must not hold, where both $C_1$ and $C_2$ are compatible with $c(C(\{h_0\}))$.

Remark 4. In [13] Griffiths presents another argument for showing the absence of any contradiction in CHA. In discussing the 3-boxes paradox, quite equivalent to contrary inferences, he shows that the contradiction arises when Axiom 3 below is assumed to hold.

**Axiom 3.** If $E \perp F$, then

i) $s \in c(E) \cap c(F)$ implies $s \in c(C(\{E, F\}))$,

ii) $c_1(E) \subseteq c_0(F)$.

This axiom relies more on implications drawn from perpendicularity, than proposition 1. Griffiths argues that the validity of this axiom is misleading in CHA, and therefore the contradiction disappears once Axiom 3 is ruled out. Our consistent interpretation of contrary inferences actually does not make use of Axiom 3. This may therefore give rise to the suspicion that our solution of contrary inferences works only because we have not assumed Axiom 3. On the contrary, our argument above can be repeated
along the same lines with Axiom 3 instead of proposition 1. The results turn out to be the same, with the only difference that possibility \((p_2)\) in item \((p)\) can no longer occur.
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