Invariants in Quantum Geometry

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Abstract

In our earlier work, we used Einstein-Hilbert path integrals to quantize gravity in loop quantum gravity. The observables are actually area and curvature of a surface and volume of a three dimensional manifold, which are quantized into operators acting on quantum states. The quantum states are defined using a set of non-intersecting loops in \( \mathbb{R} \times \mathbb{R}^3 \).

A successful quantum theory of gravity in \( \mathbb{R} \times \mathbb{R}^3 \) should be invariant under the diffeomorphism group. This means that the eigenvalues of the quantized operators should yield topological invariants for the surfaces and three dimensional manifold. In this article, we would like to discuss some of these invariants which appear in loop quantum gravity. We will also define and discuss an equivalence class of loops to be considered in loop quantum gravity.

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General Relativity is a theory invariant under diffeomorphism of the underlying manifold. Furthermore, the idea of quantum gravity being invariant of diffeomorphism is also mentioned in [17] and also evident in [23]. Therefore, in any attempt to quantize gravity, one has to preserve the diffeomorphism constraint. See [2].

As stated in [20], according to a theorem by Geroch, any globally hyperbolic 4-manifold has to be of the form \( \mathbb{R} \times M \), whereby \( M \) is a 3-manifold. See [4]. For a start, we consider our 4-manifold to be \( \mathbb{R} \times \mathbb{R}^3 \), which will be our ambient space.

In the study of Loop Quantum Gravity, one of the key players would be loops in \( \mathbb{R}^4 \), which one can think of as a 1-dimensional manifold. As seen in [17], any computations done in the quantum general relativity should be invariant under equivalence class of these loops. This implies that the theory should give us loop invariants. So it is important that we discuss some topological invariants of loops.

Another geometrical object that appears in Loop Quantum Gravity would be surface, with or without boundary. As seen in [11], [14] and [18], the surface plays an important role in the quantization of area and curvature.

Finally, a 3-dimensional compact region, viewed as a 3-dimensional manifold, also appears in the quantization process of volume, see [13] and also [15]. Put all these together, one can see that Loop Quantum Gravity is a study of submanifolds in a 4-dimensional manifold.

In all the calculations discussed in the articles cited above, one finds that topological invariants appear. This is necessary, for if a successful theory of Loop Quantum Gravity is achieved, then the calculations should be invariant under the diffeomorphism group, as implied in [17].

One such invariant is the linking number between submanifolds. Linking is a topological concept; two loops are linked if it is impossible to translate one loop by an arbitrary distance from
the other loop without the two objects actually crossing one another. In the case of a loop and a surface, we can also link them together in a 4-manifold. Finally, the idea extends to that of a 3-dimensional spatial region and a set of finite points, viewed as a 0-dimensional manifold.

But a set of loops in $\mathbb{R}^4$ do not have a non-trivial linking-number, as one can topologically deformed the loops without crossing, to yield trivial linking-number. In order to a non-trivial theory of Loop Quantum Gravity, we need to define an equivalence relation on the set of loops.

1 Hyperlinks in $\mathbb{R}^4$

In physics, the speed of light $c$ and the Planck’s constant $\hbar$ play an important role. In this article, we set $c = \hbar = 1$.

Consider our ambient space $\mathbb{R}^4 \equiv \mathbb{R} \times \mathbb{R}^3$, whereby $\mathbb{R}$ will be referred to as the time-axis and $\mathbb{R}^3$ is the spatial 3-dimensional Euclidean space. Fix the standard coordinates on $\mathbb{R}^4 \equiv \mathbb{R} \times \mathbb{R}^3$, with time coordinate $x_0$ and spatial coordinates $(x_1, x_2, x_3)$. Let $\pi_0 : \mathbb{R}^4 \to \mathbb{R}^3$ denote this projection.

Let $\{e_i\}_{i=1}^3$ be the standard basis in $\mathbb{R}^3$. And $\Sigma_i$ is the plane in $\mathbb{R}^3$, containing the origin, whose normal is given by $e_i$. So, $\Sigma_1$ is the $x_2 - x_3$ plane, $\Sigma_2$ is the $x_3 - x_1$ plane and finally $\Sigma_3$ is the $x_1 - x_2$ plane.

Note that $\mathbb{R} \times \Sigma_i \cong \mathbb{R}^3$ is a 3-dimensional subspace in $\mathbb{R}^4$. Here, we replace one of the axis in the spatial 3-dimensional Euclidean space with the time-axis. Let $\pi_i : \mathbb{R}^4 \to \mathbb{R} \times \Sigma_i$ denote this projection.

For a finite set of non-intersecting simple closed curves in $\mathbb{R}^3$ or in $\mathbb{R} \times \Sigma_i$, we will refer to it as a link. If it has only one component, then this link will be referred to as a knot. A simple closed curve in $\mathbb{R}^4$ will be referred to as a loop. A finite set of non-intersecting loops in $\mathbb{R}^4$ will be referred to as a hyperlink in this article. We say a link or hyperlink is oriented if we assign an orientation to its components.

We need to consider the space of hyperlinks in $\mathbb{R} \times \mathbb{R}^3$, which is too big for our consideration. Given any hyperlink in $\mathbb{R} \times \mathbb{R}^3$, it is ambient isotopic to the trivial hyperlink, a disjoint union of trivial loops. So the equivalence class of ambient isotopic hyperlinks will give us only the trivial hyperlink, which is too trivial. Hence we will instead consider a special equivalence class of hyperlinks.

**Definition 1.1** (Time-like separation)
Let $p, q$ be 2 points in $\mathbb{R} \times \mathbb{R}^3$, with coordinates $(x_0, x_1, x_2, x_3)$ and $(y_0, y_1, y_2, y_3)$ respectively. We say $p$ and $q$ are time-like separated if the Minkowski distance between $p, q$,

$$
\sum_{i=1}^3 (x_i - y_i)^2 - (x_0 - y_0)^2 < 0.
$$

We also say $p$ and $q$ are space-like separated if the Minkowski distance between $p, q$,

$$
\sum_{i=1}^3 (x_i - y_i)^2 - (x_0 - y_0)^2 > 0.
$$

When 2 points are time-like separated, we see that their time components must be different. Notice that we use the Minkowski metric to define the time-like separation. In Quantum Field
Theory, one uses the Minkowski metric. But in General Relativity, this is no longer correct as the metric is actually a variable and the stress-energy tensor would determine the correct metric by solving the Einstein’s equations.

This means that there should be no background metric and the quantized theory of gravity should be background independent. Therefore, one should not use Minkowski metric and since there is no notion of a preferred metric, there is no such thing as time-like separation. See [15] and [20]. Nevertheless, we will still borrow the term ‘time-like’ and define the following special class of hyperlinks we would like to consider.

**Definition 1.2 (Time-like hyperlink)**

Let \( L \) be a hyperlink. We say it is a time-like hyperlink if given any 2 distinct points \( p \equiv (x_0, x_1, x_2, x_3), q \equiv (y_0, y_1, y_2, y_3) \in L, \ p \neq q, \)

1. \( (T1) \sum_{i=1}^{3}(x_i - y_i)^2 > 0; \)
2. \( (T2) \) if there exists \( i, j, i \neq j \) such that \( x_i = y_i \) and \( x_j = y_j, \) then \( x_0 - y_0 \neq 0. \)

We make the following remarks, which is immediate from the definition.

**Remark 1.3**

1. In Condition T1, we insist that any 2 distinct points in a hyperlink are separated in 3-dimensional spatial space \( \mathbb{R}^3. \) This is to ensure that when we project the hyperlink in \( \mathbb{R}^3, \) we obtain a link.
2. Conditions T1 and T2 imply that given a hyperlink \( L, \) for each \( i=1, 2, 3, \) \( \pi_i(L) \in \mathbb{R} \times \Sigma_i \) is a link. Furthermore, they guarantee that for each crossing as defined in Subsection [7], its algebraic crossing number and its time-lag are well-defined.

**Definition 1.4**

Two oriented hyperlinks \( L \) and \( L' \) in \( \mathbb{R} \times \mathbb{R}^3 \) are time-like isotopic to each other if there is an orientation preserving continuous map \( F : \mathbb{R} \times \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R} \times \mathbb{R}^3, \) such that

1. \( F_0 \) is the identity map;
2. \( F_1 \) is a homeomorphism from \( \mathbb{R} \times \mathbb{R}^3 \) to \( \mathbb{R} \times \mathbb{R}^3; \)
3. \( F_1(L) = L' \);
4. each \( F_t(L) \) is a time-like hyperlink.

In other words, two time-like hyperlinks \( L_1 \) and \( L_2 \) in \( \mathbb{R} \times \mathbb{R}^3 \) are time-like isotopic if \( L_1 \) can be continuously deformed to \( L_2 \) while remaining time-like.

**Remark 1.5**

By definition, \( L \) and \( L' \) in \( \mathbb{R} \times \mathbb{R}^3 \) are time-like isotopic to each other if

1. \( \pi_0(L) \) and \( \pi_0(L') \) are ambient isotopic to each other in \( \mathbb{R}^3; \)
2. \( \pi_i(L) \) and \( \pi_i(L') \) are ambient isotopic to each other in \( \mathbb{R} \times \Sigma_i, i = 1, 2, 3. \)

**Definition 1.6**

Let \( L_1 \) and \( L_2 \) be two hyperlinks. Two hyperlinks \( L_1 \) and \( L_2 \) in \( \mathbb{R} \times \mathbb{R}^3 \) are time-like equivalent, written \( L_1 \sim_h L_2, \) if
• $\pi_0(L_1)$ and $\pi_0(L_2)$ are links in $\mathbb{R}^3$, and when projected onto $\Sigma_i$ as defined in Definition 4.3 in [9] to form a link diagram, are equivalent to each other via Reidemeister Moves;

• for each $i = 1, 2, 3$, $\pi_1(L_1)$ and $\pi_1(L_2)$ are links in $\mathbb{R} \times \Sigma_i$, and when projected onto $\Sigma_i$ to form a link diagram, are equivalent to each other via Reidemeister Moves.

Recall two links in $\mathbb{R}^3$ are ambient isotopic to each other if and only if the projection of the two links on a plane, say $\Sigma_3$, form two link diagrams, which are equivalent to each other via Reidemeister Moves. We have a similar result for time-like hyperlinks.

**Proposition 1.7** Two hyperlinks $L_1$ and $L_2$ in $\mathbb{R} \times \mathbb{R}^3$ are time-like equivalent if and only if $L_1$ and $L_2$ are time-like isotopic.

Henceforth, throughout this article, all our hyperlinks will be time-like and we consider equivalence classes of such hyperlinks using the equivalent relation $\sim_h$ in Definition 1.6.

### 1.1 Link Diagrams

The next thing that we want to define is the hyperlinking number of a hyperlink. This should be thought of as a generalization of the linking number of a link. Indeed, one can calculate the hyperlinking number from a link diagram, as described in Definition 4.3 in [9].

Any link in $\mathbb{R}^3$ can be represented by a link diagram, up to isotopy. This allows us to study links using link diagrams. Two link diagrams $D$ and $D'$ are (planar) isotopic if there exists an isotopy $h$ of $\mathbb{R}^2$ such that $h(1, D) = D'$. To check if $D$ and $D'$ are isotopic, it suffices to show that $D$ can be obtained from $D'$ by a sequence of Reidemeister Moves.

A crossing $p$ on a link diagram is represented (up to isotopy) either by

- $\times \quad$ or $\times \quad$.

We assign the value $\varepsilon(p) := +1$ for the diagram on the left; $\varepsilon(p) := -1$ for the diagram on the right. Note that $\varepsilon(p)$ is also known as the algebraic crossing number of the crossing $p$.

Given 2 oriented simple closed curves $\mathcal{I}$ and $\mathcal{L}$ which are non-intersecting in $\mathbb{R}^3$, project it on $\Sigma_i$ and write $\text{DP}(\Sigma_i; \mathcal{I}, \mathcal{L})$ to denote the set of all crossings in a link diagram of curves $\mathcal{I}$ and $\mathcal{L}$. Define the linking number between $\mathcal{I}$ and $\mathcal{L}$,

$$\text{lk}(\mathcal{I}, \mathcal{L}) := \sum_{p \in \text{DP}(\Sigma_i; \mathcal{I}, \mathcal{L})} \varepsilon(p).$$

The linking number between 2 oriented curves is an invariant up to ambient isotopy, so it does not matter which plane we project it onto.

Recall a hyperlink is a finite set of non-interesting simple closed curves in $\mathbb{R} \times \mathbb{R}^3$ and considered as time-like, as defined in Definition 1.2. We can project a hyperlink on $\Sigma_i$ to form a link diagram as before. Suppose each crossing $p$ on a link diagram is formed from projecting 2 arcs $\mathcal{C}$ and $\mathcal{C}$ from respective loops $\mathcal{I}$ and $\mathcal{L}$.

Let $\bar{x} = (x_0, x) \in \mathcal{C}$ and $\bar{y} = (y_0, y) \in \mathcal{C}$ respectively, with $x, y \in \mathbb{R}^3$, such that $p$ is the projection of $x$ and $y$ onto the plane $\Sigma_i$. Note that $x_0$ and $y_0$ are the time components of $\bar{x}$ and $\bar{y}$ respectively.
Define the time-lag of \( p \) by
\[
\text{sgn}(p; l_0 : l_0) = \begin{cases} 
1, & x_0 < y_0; \\
-1, & x_0 > y_0,
\end{cases}
\]
and the hyperlinking number between \( \tilde{l} \) and \( l \) as
\[
\text{sk}(\tilde{l}, l) := \sum_{k=1}^{3} \sum_{p \in \text{DP}(\Sigma_k; \hat{y}_k, \hat{\nu}_k)} \varepsilon(p) \cdot \text{sgn}(p; l_0 : l_0).
\]

**Theorem 1.8** Consider two oriented time-like hyperlinks, \( \mathcal{T} = \{ l_1^u \}_{u=1}^{n} \), \( L = \{ l_1^v \}_{v=1}^{n} \) in \( \mathbb{R} \times \mathbb{R}^3 \) with non-intersecting (closed) loops. From these 2 oriented hyperlinks, form a new oriented time-like hyperlink, denoted by \( \chi(\mathcal{T}, L) \).

Define for each \( u = 1, \ldots, n \),
\[
\text{sk}(\mathcal{T}, L) \equiv \text{sk}(L, L) := \sum_{u=1}^{n} \sum_{v=1}^{n} \text{sk}(l_1^u, l_1^v),
\]
calculated from \( \chi(\mathcal{T}, L) \). Then, it is invariant under the equivalence relation defined in Definition 1.6.

**Proof.** To show that \( \text{sk}(\mathcal{T}, L) \) is invariant under the equivalence relation, it suffices to show that it is invariant under Reidemeister Moves II and III.

Given any set of non-intersecting loops \( \tilde{L} \subset \mathbb{R} \times \mathbb{R}^3 \), the linking number of the projected link \( \pi_0(\tilde{L}) \) is a link invariant, independent on which plane we choose to project \( \pi_0(\tilde{L}) \) on. Thus, using \( \Sigma_i \) to compute the linking number will yield the same value for each \( i = 1, 2, 3 \).

Label the arcs by 1, 2, 3, according to which arc appears first in terms of time, the arc that appears first will be assigned 1, the one that appears last will be assigned 3. Note that under Reidemeister Move III as shown above, the algebraic crossing number \( \varepsilon \) does not change. Likewise, the assignment of 1, 2, 3 to the arcs will remain unchanged. Hence algebraic crossing number times the time-lag of the crossing remains unchanged under Reidemeister Move III.
Redeimeister II is given by the above diagram. Notice that 2 crossings are removed. Suppose the arc labelled $K$ occurs before the arc on the right. Then the time-lag for both crossings are the same and thus the algebraic crossing number times the time-lag for each crossing is of opposite sign, so it gives us zero, which corresponds to the diagram on the right.

The only problem now is when the time-lag for the top crossing is of opposite sign with the time-lag for the bottom crossing. Under the equivalence relation, the hyperlink must also be projected onto $\mathbb{R} \times \Sigma_i$.

The diagram on the left in the above diagram shows the arcs from $\pi_0(\tilde{L})$ projected onto $\Sigma_i$; the diagram on the right in the diagram shows the corresponding arcs from $\pi_i(\tilde{L})$ which are projected onto $\Sigma_i$. Hence, we cannot apply Reidemeister Move II to remove the crossings shown on the right in the above diagram.

Thus $\text{sk}(\tilde{L}, \tilde{L})$ remains unchanged under Reidemeister Moves II and III, and so it is an invariant under the equivalence relation defined in Definition 1.6.

2 Surfaces

Choose an orientable, closed and bounded surface $S \subset \mathbb{R}^4$, with or without boundary. If it has a boundary $\partial S$, then $\partial S$ is assumed to be a time-like hyperlink. Do note that we allow $S$ to be disconnected. Pick a time-like loop $l \subset \mathbb{R}^4$, disjoint from $S$, and project them onto $\mathbb{R}^3$, denoted by $\pi_0(l)$ and $\pi_0(S)$ respectively, such that $\pi_0(l)$ intersect $\pi_0(S)$ at finitely many points. We also assume that $\partial S$, together with $l$, form a time-like hyperlink and any ambient isotopy of $\partial S \cup l$ should be time-like. From page 43 in [5], one can define a linking number between a surface $S$ and a loop as follows.

Note that when we project $S$ inside $\mathbb{R}^3$, $\pi_0(S)$ may not be a surface. But we only consider ambient isotopic equivalence classes of surface. Thus, for each point $q \in S$, we can choose a smaller surface $q \in S' \subset S$, such that $\pi_0(S')$ is indeed a surface inside $\mathbb{R}^3$. So we shall assume that $\pi_0(S)$
is a surface. Furthermore, we assume that \( \pi_0(l) \) intersects \( \pi_0(S) \) at finitely many points and we do not allow \( \pi_0(l) \) to be tangent to \( \pi_0(S) \) at any such intersection points.

Let \( \text{DP}(\pi_0; l, S) \) denote the set of finitely many intersection points between \( \pi_0(l) \) and \( \pi_0(S) \), henceforth termed as piercings. Choose a non-zero normal \( n_S \) on \( \pi_0(S) \) and orientate \( \pi_0(l) \) and let \( \nu_l \) be the non-zero tangent vector along the oriented curve \( \pi_0(l) \) in \( \mathbb{R}^3 \). For a piercing \( p \), define the orientation of \( p \), \( \text{sgn}(p; l, S) \), which takes the value +1 if \( n_S(p) \cdot \nu_l(p) > 0 \); −1 if \( n_S(p) \cdot \nu_l(p) < 0 \).

For each such piercing \( p \), let \( (x_0, x) \in S \), \( (y_0, y) \in l \) such that \( p = \pi_0(x_0, x) = x \) and \( p = \pi_0(y_0, y) \), so \( p = x = y \in \mathbb{R}^3 \). Define the height of \( p \),

\[
\text{ht}(p; l, S) = \begin{cases} 
1, & x_0 < y_0; \\
-1, & x_0 > y_0.
\end{cases}
\]

Define the algebraic piercing number of \( p \) as

\[
\varepsilon(p) := \text{sgn}(p; l, S) \cdot \text{ht}(p; l, S).
\]

And we define the linking number between \( l \) and \( S \) as

\[
\text{lk}(l, S) := \sum_{p \in \text{DP}(\pi_0; l, S)} \varepsilon(p).
\]

If \( S \) has no boundary, then the linking number is invariant under ambient isotopy of \( S \) and \( l \). If \( S \) has a time-like hyperlink \( \partial S \) as boundary, then the linking number is invariant under ambient isotopy, provided \( \partial S \) and \( l \) remains time-like during the isotopy process. And for a time-like hyperlink \( L = \{l^u\}_{u=1}^n \), which is entangled with \( \partial S \) to form a time-like hyperlink, define the linking number between \( L \) and \( S \), as

\[
\text{lk}(L, S) := \sum_{u=1}^n \text{lk}(l^u, S).
\]

In fact, we can project \( S \) inside \( \mathbb{R} \times \Sigma \) using the projection \( \pi_i \). Because the linking number between \( l \) and \( S \) is a topological invariant in \( \mathbb{R}^4 \), it is independent on how we project it inside a hypersurface inside \( \mathbb{R}^4 \). Thus, we could have chosen to project it inside \( \mathbb{R} \times \Sigma \) and obtain the linking number between \( l \) and \( S \).

We will call a projection of a hyperlink \( L \) and a surface \( S \) in \( \mathbb{R}^3 \) as a link-surface diagram. Just as in a link diagram, there might be many link-surface diagrams that represent a hyperlink \( L \) and a surface \( S \). Let \( \text{DP}(\pi_0; L, S) \) denote the set of piercings between \( \pi_0(L) \) and \( S \) and \( |\text{DP}(\pi_0; L, S)| \) is the total number of piercings in the set. Now, \( |\text{DP}(\pi_0; L, S)| \) is not invariant under ambient isotopy of a hyperlink \( L \) and a surface \( S \).

**Definition 2.1** (Piercing number)
We define the piercing number as \( \nu_S(L) \), which is the minimum of piercings of any link-surface diagram for \( L \) and \( S \). This is a topological invariant between \( L \) and \( S \).

**Remark 2.2** This is analogous to the minimum crossing number of a knot.
3 Framed hyperlinks

Let $v$ be a non-tangential vector field on a knot $\gamma$. Now shift the knot along $\epsilon v$, whereby $\epsilon > 0$ and is small. Call this shifted curve $\gamma':=\gamma+\epsilon v$. A framing of a knot is a homotopy class of normal vector fields on $\gamma$ where two normal vector fields are said to be homotopic if they can be deformed into one another within the class of normal vector fields. Thus, the vector field $v$ defines a frame for the knot $\gamma$ and $\{\gamma, v\}$ is called a framed knot or ribbon.

Now consider $\gamma$ and $\gamma'$ as separate knots. Project it down onto $\Sigma_3$ plane to form a link diagram as above. A half-twist is formed when a displaced copy of an arc inside $\gamma'$ twirls around the original arc in $\gamma$, which projects onto a plane to form a crossing $q$. Thus, we define the algebraic crossing of the half-twist as $\varepsilon(q)$. For a more detailed description of a half-twist, we refer the reader to [10].

A framed link $L \subset \mathbb{R}^3$ will be a finite set of non-intersecting simple closed curves, whereby each component knot is equipped with a frame. We can project it onto a plane as described above. Two link diagrams represent the same framed link, if one diagram can be obtained from the other by a sequence of Reidemeister moves I, II and III. Reidemeister move I says that twisting in one direction, followed by twisting in the opposite direction, undo both twists. See page 217 in [7], Figure 8.2. Two framed links $L$ and $L'$ are ambient isotopic to each other if when projected onto the same plane to form link diagrams $D$ and $D'$ respectively, we can obtain $D'$ from $D$ by a finite sequence of Reidemeister moves.

Since twisting a ribbon in the positive direction, followed by twisting it in the negative direction, undo all the twists, hence we can and will assume that all the half-twists on a knot have the same algebraic number. Therefore, we can and will always assume that the total number of half-twists is a minimum on a framed knot. Obviously, the set of half-twists in a link-diagram will depend on which plane we project the framed knot on. However, the total number of half-twists will be the same and is an even number, independent of the plane we choose.

A framed hyperlink will be a hyperlink whereby each component loop is disjoint from $\mathbb{R}^3$, when projected in $\mathbb{R}^3$ to form a knot $\pi_0(l^n)$, is equipped with a frame. When we say a framed hyperlink $L$ is time-like isotopic to framed hyperlink $L'$, we mean that $L$ and $L'$ are time-like isotopic as in Definition 1.4 and furthermore, $\pi_0(L)$ and $\pi_0(L')$ are ambient isotopic as framed links in $\mathbb{R}^3$. We will also say that they are time-like equivalent, as framed hyperlinks.

Let $R$ be a bounded and possibly disconnected 3-dimensional manifold inside spatial space $\mathbb{R}^3$, containing all of its boundary. We will term $R$ as a compact region and view $R \subset \{0\} \times \mathbb{R}^3$. Let $l$ be a framed loop $l^n$, i.e. the projected knot $\pi_0(l)$ is equipped with a framing and so we add half-twists to a link diagram of $\pi_0(l)$. We also assume that $l$ is disjoint from $R$. In [10], we project $\pi_0(l)$ onto $\Sigma_3$ using the projection $\tilde{\pi}_3: \mathbb{R}^3 \rightarrow \Sigma_3$ to obtain a graph, each vertex has valency 2 or 4. Those vertices with valency 2 are actually the half-twists we described earlier on.

It is impossible to define a linking number $l$ with a bounded 3-dimensional region $R \subset \mathbb{R}^3$ inside $\mathbb{R}^4$, so there is no notion of a linking number between $l$ and $R$. But recall we always assume that all the half-twists on the link diagram of $\pi_0(l)$ has the same sign, so indeed we will have the minimum even number of half-twists on any link diagram of $\pi_0(l)$. Therefore, we can define a topological invariant, the confinement number between $l$ and $R$, by counting how many of the half-twists in a link diagram of $\pi_0(l)$ lie inside the planar set $\tilde{\pi}_3(R)$ by projecting $R$ onto the same plane.

Notation 3.1 Let $l$ be a framed loop and $\tilde{\pi}_3 : \mathbb{R}^3 \rightarrow \Sigma_3$. Given a compact region $R$, project $R$ to be a planar set $\tilde{\pi}_3(R) \subset \Sigma_3$ and $\pi_0(l)$ to be a graph as defined in [10], such that each vertex has valency 2 or 4. We define $\text{TDP}(l; R)$ to be the set of all half-twists from the knot $l$, i.e. vertices
with valency 2 which are in the interior of $\tilde{\pi}_3(R)$. We further assume that in the projection, all the half-twists in the graph from $\pi_0(l)$ have the same sign.

**Remark 3.2** We could have chosen a different plane $\Sigma_i$, but the invariant we are going to define later will be independent of the choice of plane. See [11] and [12].

Note that half-twists lie in a link diagram. However, we can ‘lift’ these half-twists and represent them as nodes on $\pi_0(l)$. Therefore, in future, we will now view a framed knot as a knot in $\mathbb{R}^3$, but with nodes attached to it. We can define the algebraic number of a node to be equal to the algebraic crossing number of its corresponding half-twist.

For a framed loop $l$, let $\text{Nd}(\pi_0(l))$ be the set of nodes on a projected loop $\pi_0(l)$, all of them have the same sign for its algebraic number. This means that we have the minimum number of nodes on the framed knot. Equivalence class of framed loop $l$ allows us to move the nodes in the framed knot $\pi_0(l)$, so we should view $\text{Nd}(\pi_0(l))$ as an equivalence class.

If we view $\text{Nd}(\pi_0(l)) \subset \mathbb{R}^3$ as being 0-dimensional manifold and it does not cross the boundary surface $\partial R$ of a compact region $R$, then we can define the confinement number, a topological invariant.

**Definition 3.3** (Confinement number)
Let $l$ be a framed loop and define $\text{Nd}(\pi_0(l)) \subset \mathbb{R}^3$, the finite set of all nodes on the projected loop $\pi_0(l) \subset \mathbb{R}^3$. We define the confinement number between $l$ and $R$ as

$$\nu_R(l) := |\text{Nd}(\pi_0(l)) \cap R|.$$ 

If $L = \{l^1, \ldots, l^n\}$ is a hyperlink, then we define

$$\text{Nd}(\pi_0(L)) := \bigcup_{u=1}^n \text{Nd}(\pi_0(l^u))$$ and $$\nu_R(L) := \sum_{u=1}^n \nu_R(l^u).$$

**Remark 3.4** If there is no frame assigned, hence no nodes, then the set of nodes will be the empty set.

Notice that we count the number of nodes from the framed knot, which lie inside the interior of $R$, under the assumption that the set $\text{Nd}(\pi_0(l))$ is disjoint from the boundary of $\partial R$ of $R$. We allow the knot $\pi_0(l)$ and the region $R$ to be deformed up to ambient isotopy, as long as the set $\text{Nd}(\pi_0(l))$ remains disjoint from the boundary $\partial R$. By definition, we only consider the minimum number of nodes on the knot, so we are not allowed to add more half-twists.

### 4 Summary

We would like to end off by summarizing all the facts we have have discussed. Our ambient space is $\mathbb{R} \times \mathbb{R}^3$, a 4-manifold. We consider the following sub-manifolds as follows.

Let $\mathcal{T}$ and $\mathcal{L}$ be two distinct hyperlinks, the former will be termed matter hyperlink, and latter geometric hyperlink. The hyperlinks are expected to be time-like and together, they form a time-like hyperlink $\chi(\mathcal{T}, \mathcal{L})$. 
Equip the matter hyperlink with a frame, so that $\pi_0(\mathcal{T})$ is a framed link. That is, add in nodes to $\pi_0(\mathcal{T})$. Further assume that we have the minimum number of nodes, which is equivalent to all the nodes from the same component knot in $\pi_0(\mathcal{T})$ have the same sign. The set of nodes should be thought of as an equivalence class, denoted by $\text{Nd}(\pi_0(\mathcal{T}))$.

Consider a (possibly disconnected) bounded surface $S$ in $\mathbb{R}^4$, with or without boundary. If it has boundary $\partial S$, then $\partial S \cup \chi(\mathcal{T}, \mathcal{L})$ must be a time-like hyperlink. Finally we have a (possibly disconnected) bounded 3-dimensional manifold in $\{0\} \times \mathbb{R}^3 \subset \mathbb{R}^4$, assumed to be closed, hence termed as a compact region. We assume that $\chi(\mathcal{T}, \mathcal{L})$ do not intersect $S$ and $R$.

**Theorem 4.1** Consider the triple $\{S, R, \chi(\mathcal{T}, \mathcal{L})\}$ as described above. Note that $\chi(\mathcal{T}, \mathcal{L})$ do not intersect $S$ and $R$. We define a time-like equivalence relation and say $\{S, R, \chi(\mathcal{T}, \mathcal{L})\}$ is time-like equivalent to $\{S, R, \chi(\mathcal{T}, \mathcal{L})\}$ if

1. $[\partial S \cup \chi(\mathcal{T}, \mathcal{L})] \sim_h [\partial S \cup \chi(\mathcal{T}, \mathcal{L})]$;
2. $S$ is ambient isotopic to $S$;
3. $R$ is ambient isotopic to $R$;
4. throughout the ambient isotopy, the hyperlink do not intersect the surface and the compact region and the set of nodes do not intersect the boundary of the compact region.

Henceforth, we will call its equivalence class as a time-like triple. Then,

1. the hyperlinking number $\text{sk}(\mathcal{T}, \mathcal{L})$ of $\chi(\mathcal{T}, \mathcal{L})$;
2. the piercing number $\nu_S(\mathcal{T})$ between $S$ and $\mathcal{T}$;
3. the confinement number $\nu_R(\mathcal{T})$ between $R$ and the framed hyperlink $\mathcal{T}$;
4. the linking number $\text{lk}(\mathcal{L}, S)$ between $\mathcal{L}$ and $S$,

are all invariant under the time-like isotopy as described above.

**Remark 4.2** Note that linking numbers for knots do not appear in quantum geometry. Instead, they showed up in quantized Chern-Simons theory, as shown in [9]. The idea of using Chern-Simons theory to obtain knot invariants was first described in [23].

Quantum geometry was developed in the mid-nineties by several researchers and a detailed reference can be found in [3], [16], and [21]. From these articles, one can see that quantum geometry is in a way, describing the discretization of space-time.

But make no mistake. It is not discrete geometry. In [1], the authors describe quantum geometry as a theory of interaction between quantum matter and quantum geometry. Evidently in this article, quantum matter and geometry are represented by matter and geometric hyperlinks respectively. More generally, quantum geometry is a topological theory, which focuses on how sub-manifolds are ‘linked’ in $\mathbb{R}^4$ or any globally hyperbolic 4-manifold, rather than the ambient manifold itself. But what sets it apart from other topological theories?

In topology, area, volume and curvature have no meaning. One needs to define a metric or a connection to define these quantities. Unlike topological theory, in quantum geometry, we do have notions of area and volume in quantum geometry, without using a metric or connection.
In Loop Quantum Gravity, one uses the Einstein-Hilbert action to define a path integral. See [12]. By averaging area of the surface or volume of a region over all (degenerate) metric, one can quantize the area and volume into its corresponding operators. This was done in [11] and [13]. The eigenvalues are computed from the piercing and confinement numbers respectively, which are topological invariants of the time-like triple discussed above. The discrete eigenvalues hence show that space-time is discretized.

In a similar manner, we can quantize curvature of a surface by averaging over all connections on ambient space, into an operator. Quantized curvature now becomes a topological invariant, computed using the linking number between a surface and a hyperlink. See [14].

In quantum geometry, there is no preferred choice of metric or connection, hence no classical background geometry is introduced. So, area and curvature of a surface $S$ and the volume of a compact region $R$, represented up to ambient isotopy, do not make any sense. The time-like triple $(S, R, \chi(L, \mathcal{L}))$ considered in quantum geometry gives meaning to area, volume and curvature and turn these physical quantities into topological invariants.

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