Order properties for single machine scheduling with non-linear cost

Nikhil Bansal* Christoph Dürr†‡ Nguyen Kim Thang§ Óscar C. Vásquez¶

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Abstract

We consider the classical scheduling problem on a single machine, on which we need to schedule sequentially $n$ given jobs. Every job $j$ has a processing time $p_j$ and a priority weight $w_j$, and for a given schedule a completion time $C_j$. In this paper we consider the problem of minimizing the objective value $\sum_j w_j C_j^\beta$ for some fixed constant $\beta > 0$. This non-linearity is motivated for example by the learning effect of a machine improving its efficiency over time, or by the speed scaling model. Except for $\beta = 1$, the complexity status of this problem is open. Past research mainly focused on the quadratic case ($\beta = 2$) and proposed different techniques to speed up exact algorithms. This paper proposes new dominance properties and generalizations of existing rules to non-integral $\beta$. An experimental study evaluates the impact of the proposed rules on the exact algorithm A*.

Keywords  Scheduling; Single Machine; Nonlinear cost function; Pruning Rules; Algorithm A*

1 Introduction

In a typical scheduling problem we have to order $n$ given jobs, each with a different processing time, so to minimize some problem specific cost function. Every job $j$ has a processing time $p_j$ and a priority weight $w_j$. A schedule is defined by a ranking $\sigma$, and the completion time of job $j$ is defined as $C_j := \sum_i p_i$, where the sum ranges over all jobs $i$ such that $\sigma_i \leq \sigma_j$. The goal is to produce a schedule that minimizes some cost function involving the jobs weights and the completion times.

A popular objective function is the weighted average completion time $\sum w_j C_j$ (omitting the normalization factor $1/n$). It has been known since the 1950’s that optimal schedules are precisely the orders following an decreasing Smith-ratio $w_j/p_j$, as has been shown by a simple exchange argument (Smith 1956).

In this paper we consider the more general objective function $\sum w_j C_j^\beta$ for some constant $\beta > 0$, and denote it by $1||\sum w_j C_j^\beta$. Several motivations have been given in the literature for this objective.

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*Department of Mathematics and Computer Science, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands n.bansal@tue.nl
†Sorbonne Universités, UPMC Univ Paris 06, UMR 7606, LIP6, F-75005 Paris, France
‡CNRS, UMR 7606, LIP6, F-75005 Paris, France christoph.durr@lip6.fr
§IBISC, University Evry Val d’Essonne, 23 blv. de France, 91034 Evry, France thang@ibisc.fr
¶Universidad de Santiago, Departamento de Ingeniería Industrial, 3769 Av. Ecuador, Santiago-Chile oscar.vasquez@usach.cl
For example it can model the standard objective $\sum w_j C_j$, but on a machine changing its execution speed continuously. This could result from a learning effect, or the continuous upgrade of its resources, or from a wear and tear effect, resulting in a machine which works less effective over time. A recent motivation comes from the speed scaling scheduling model. In Dürr et al. (2014) and Megow and Verschae (2013) the problem of minimizing total weighted completion time plus total energy consumption was studied, and both paper reduced this problem to the problem considered in this paper for a constant $1/2 \leq \beta \leq 2/3$. However as we mention later in the paper, most previous research focused on the $\beta = 2$ case, as the objective function then represents a trade off between maximum and average weighted completion time.

## 2 Dominance properties

The complexity status of the problem $1\| \sum w_j C_j^\beta$ is open in the sense that neither polynomial time algorithms nor NP-hardness proofs are known. For $\beta = 1$ the problem is polynomial, as has been shown by a simple exchange argument. When $i,j$ are adjacent jobs in a schedule, then the order $ij$ is preferred over $ji$ whenever $w_i/p_i > w_j/p_j$. In this case we denote this property by $i \prec_\ell j$. Assume for simplicity that all jobs $k$ have a distinct ratio $w_k/p_k$, which is called the Smith-ratio. Under this condition $\prec_\ell$ defines a total order on the jobs, that leads to the unique optimal schedule.

For general $\beta$ values, the situation is not so simple, as the effect of exchanging two adjacent jobs depends on their position in the schedule. So for two jobs $i,j$ none of $i \prec_\ell j, j \prec_\ell i$ might hold, which is precisely the difficulty of this scheduling problem.

However it would be much more useful if for some jobs $i,j$ we knew that an optimal schedule always schedules $i$ before $j$, no matter if they are adjacent or not. This property is denoted by $i \prec_g j$. In this case the search space could be roughly divided by 2, which could help improving exhaustive search procedures. Section 8 contains an experimental study on the impact of this information to the performance of some search procedure.

Therefore in the past several attempts have been proposed to characterize the property $i \prec_g j$ as function of the job parameters $p_i, w_i, p_j, w_j$ and of $\beta$. Several sufficient conditions have been proposed, which however are also necessary only in some restricted cases.

## Related work

Embarrassingly very little is known about the computational complexity of this problem, except for the special case $\beta = 1$ which was solved in the 1950’s (Smith 1956). In that case scheduling jobs in order of decreasing Smith ratio $w_j/p_j$ leads to the optimal schedule.

Two research directions were applied to this problem, approximation algorithms and branch and bound algorithms. The former have been proposed for the even more general problem $1\| \sum f_j(C_j)$, where every job $j$ is given an increasing penalty function $f_j(t)$, that does not need to be of the form $w_j t^\beta$. A constant factor approximation algorithm has been proposed by Bansal and Pruhs (2010) based on a geometric interpretation of the problem. The approximation factor has been improved from 16 to $2 + \epsilon$ via a primal-dual approach by Cheung and Shmoys (2011). The simpler problem $1\| \sum w_j f(C_j)$ was considered in Epstein et al. (2010), who provided a $4 + \epsilon$ approximation algorithm for the setting where $f$ is an arbitrary increasing differentiable penalty function chosen by the adversary after the schedule has been produced. A polynomial time approximation scheme has been provided by Megow and Verschae (2013) for the problem $1\| \sum w_j f(C_j)$, where $f$ is an
arbitrary monotone penalty function.

Finally, Höhn and Jacobs (2012c) derived a method to compute the tight approximation factor of the Smith-ratio-schedule for any particular monotone increasing convex or concave cost function. In particular for $f(t) = t^\beta$ they obtained for example the ratio 1.07 when $\beta = 0.5$ and the ratio 1.31 when $\beta = 2$.

Concerning branch-and-bound algorithms, several papers give sufficient conditions for the global order property, and analyze experimentally the impact on branch and bound algorithms of their contributions. Previous research focused mainly on the quadratic case $\beta = 2$, see Townsend (1978), Bagga and Karlra (1980), Sen et al. (1990), Alidaee (1993), Croce et al. (1993), Szwarc (1998). Mondal and Sen (2000) conjectured that $\beta = 2 \land (w_i \geq w_j) \land (w_i/p_i > w_j/p_j)$ implies the global order property $i \prec_g j$, and provided experimental evidence that this property would significantly improve the runtime of a branch-and-prune search. Recently, Höhn and Jacobs (2012a) succeeded to prove this conjecture. In addition they provided a weaker sufficient condition for $i \prec_g j$ which hold for any integer $\beta \geq 2$. An extensive experimental study analyzed the effect of these results on the performance of the branch-and-prune search.

3 Our contribution

Some of the previously proposed sufficient conditions for $i \prec_g j$ were ad-hoc, some used more involved functional analysis, but none were also necessary in generality. This situation created the need for a unified precise characterization of $i \prec_g j$, that would hold in all cases.

In contrast the condition $i \prec_\ell j$ is fairly easy to characterize, using standard functional analysis, as has been described in the past by Höhn and Jacobs (2012a) for $\beta = 2$. This characterization holds in fact for any value of $\beta$ and for completeness we describe it in section 5.

As $i \prec_g j$ implies $i \prec_\ell j$, the strongest condition we could hope for a characterization of $i \prec_g j$ is precisely $i \prec_\ell j$. It would give to a local exchange property a broader impact on the structure of optimal schedules, and have a strong implication on the effect of non-local exchanges.

Experimental results seem to indicate that this property is the right candidate for a characterization, as well as previous results on particular cases. For example Höhn and Jacobs (2012a) showed that if $\beta = 2$ and $p_j \leq p_i$ then $i \prec_g j$ if and only if $i \prec_\ell j$. The same characterization has been shown for a related objective function, where one wants to maximize $\sum w_j C_j^{-1}$ (Vásquez 2014). This situation motivates us to state the following conjecture.

**Conjecture 1** For any $\beta > 0$ and all jobs $i, j$, $i \prec_g j$ if and only if $i \prec_\ell j$.

We succeed to show this claim in the case $\beta \geq 1$. Our proof distinguishes the cases $p_j < p_i$ and $p_j \geq p_i$, and applies different techniques to each of them. In case $0 < \beta < 1$ the situation is still incomplete. We were able to show the conjecture in case $p_j \geq p_i$, but for $p_j < p_i$, we could only provide a sufficient but not necessary condition.

While these results do not tackle the problem of the computational complexity of the problem, they nevertheless provide a deeper insight in its structure, and in addition speed up exhaustive search techniques in practice. This is due to the fact that with the conditions for $i \prec_g j$ provided in this paper it is now possible to conclude $i \prec_g j$ for a significant portion of job pairs, for which previous known conditions failed. In the final section 8 of this paper, we study experimentally the impact of our contributions on the procedure A* for this problem.
For $0 < \beta < 1$, the boundaries are defined by functions which are named from (a) to (e). The last 2

\[ j \preceq_i \preceq j \]

using a similar representation as in Höhn and Jacobs (2012a). Every point in the diagram represents

a job $j$ with respect to some fixed job $i$. The space is divided into regions where $i \preceq j$ holds or

$j \preceq i$ or none. These regions contain subregions where we know that the stronger condition $\preceq_g$

holds. The boundaries are defined by functions which are named from (a) to (e). The last 2
diagrams also indicate the related theorems.

4 Technical lemmas

This section contains several technical lemmas used in the proof of our main theorems.

Lemma 1 For $0 < \beta < 1$, $a < b$ and $p_i > p_j$,

\[ \left( \frac{p_i}{p_j} \right)^{1-\beta} \frac{f(b + p_i) - f(a + p_i)}{f(b + p_j) - f(a + p_j)} \geq 1. \]

Proof For this purpose we define the function

\[ g(x) := x^{1-\beta} (f(b + x) - f(a + x)) \]
and show that \( g \) is increasing, which implies \( g(p_i)/g(p_j) \geq 1 \) as required. So we have to show \( g'(x) > 0 \) in other words

\[
(1 - \beta)x^{-\beta}(f(b + x) - f(a + x)) + x^{1-\beta}(f'(b + x) - f'(a + x)) \geq 0 \\
\iff (1 - \beta)x^{-\beta}((b + x)^{\beta} - (a + x)^{\beta}) + x^{1-\beta}\beta((b + x)^{\beta-1} - (a + x)^{\beta-1}) \geq 0 \\
\iff (b + x)^{\beta-1}((1 - \beta)(b + x) + \beta x) - (a + x)^{\beta-1}((1 - \beta)(a + x) + \beta x) \geq 0.
\]

To establish the last inequality, we introduce another function

\[
r(z) := (z + x)^{\beta-1}((1 - \beta)(z + x) + \beta x)
\]

and show that \( r(z) \) is increasing, implying \( r(b) \geq r(a) \). By analyzing its derivative we obtain

\[
r'(z) = (\beta - 1)(z + x)^{\beta-2}((1 - \beta)(z + x) + \beta x) + (1 - \beta)(z + x)^{\beta-1} \\
= (z + x)^{\beta-2}(1 - \beta)((\beta - 1)(z + x) - x\beta + z + x) \\
= (z + x)^{\beta-2}(1 - \beta)(\beta z),
\]

which is positive as required. This concludes the proof. \( \square \)

Some of our proofs are based on particular properties which are enumerated in the following lemma.

**Lemma 2** The function \( f(t) = t^\beta \) defined for \( \beta \geq 1 \) satisfies the following properties.

1. \( f(x) \geq 0 \) for \( x \geq 0 \).
2. \( f \) is convex and non-decreasing, i.e. \( f', f'' \geq 0 \).
3. \( f' \) is log-concave (i.e. \( \log(f') \) is concave), which implies that \( f''/f' \) is non-increasing. Intuitively this means that \( f \) does not increase much faster than \( e^x \).
4. For every \( b > 0 \), the function \( g_b(x) = f(b + e^x) - f(b) \) is log-convex in \( x \). Intuitively this means that \( f(b + e^x) - f(b) \) increases faster than \( e^{cx} \) for some \( c > 0 \). Formally this means

\[ gyf'(b + y)/(f(b + y) - f(b)) \quad (1) \]

is increasing in \( y \).

The proof is based on standard functional analysis and is omitted.

**Lemma 3** For \( a < b \), the fraction

\[
\frac{f'(b) - f'(a)}{f(b) - f(a)}
\]

• is decreasing in \( a \) and decreasing in \( b \) for any \( \beta > 1 \)
• and is increasing in \( a \) and increasing in \( b \) for any \( 0 < \beta < 1 \).
Proof First we consider the case $\beta > 1$. We can write $f'(b) - f'(a) = \int_a^b f''(x)dx$ and $f(b) - f(a) = \int_a^b f'(x)dx$. Note that $f''$ and $f'$ are non-negative. By $\beta > 1$ and Lemma 1, $f'$ is log-concave, which means that $f''(x)/f'(x)$ is non-increasing in $x$. This implies

$$\int_a^b \frac{f''(b)}{f'(b)} f'(x)dx \leq \int_a^b \frac{f''(b)}{f'(b)} f'(x)dx \leq \int_a^b \frac{f''(a)}{f'(a)} f'(x)dx$$

Hence

$$\frac{\int_a^b f''(b)dx}{\int_a^b f'(b)dx} \leq \frac{\int_a^b f''(b)dx}{\int_a^b f'(b)dx} \leq \frac{\int_a^b f''(a)dx}{\int_a^b f'(a)dx}$$

For positive values $u_1, u_2, u_3, v_1, v_2, v_3$ with $u_1/v_1 \leq u_2/v_2 \leq u_3/v_3$ we have

$$\frac{u_1 + u_2}{v_1 + v_2} \leq \frac{u_2 + u_3}{v_2 + v_3}.$$ 

We use this property for $a' < a < b < b'$

$$\frac{u_1}{v_1} = \frac{\int_{a'}^{b} f''(x)dx}{\int_{a'}^{b} f'(x)dx}, \quad \frac{u_2}{v_2} = \frac{\int_{a}^{b} f''(x)dx}{\int_{a}^{b} f'(x)dx}, \quad \frac{u_3}{v_3} = \frac{\int_{a'}^{a} f''(x)dx}{\int_{a'}^{a} f'(x)dx}$$

and obtain

$$\frac{\int_{a'}^{b} f''(x)dx}{\int_{a'}^{b} f'(x)dx} \leq \frac{\int_{a}^{b} f''(x)dx}{\int_{a}^{b} f'(x)dx} \leq \frac{\int_{a'}^{a} f''(x)dx}{\int_{a'}^{a} f'(x)dx}.$$ 

For the case $0 < \beta < 1$ the argument is the same using the fact that $f''(x)/f'(x)$ is non-decreasing in $x$. □

The previous lemma permits to show the following corollary.

Corollary 1 For $t \geq 0$ let the function $q$ be defined as

$$q(t) := \frac{f(t + p_j) - f(t)}{f(t + p_i) - f(t)}.$$ 

For $p_i > p_j$, if $\beta > 1$ then $q$ is increasing and if $0 < \beta < 1$ then $q$ is decreasing.

Proof We only prove the case $\beta > 1$, the other case is similar. Showing that $q(t)$ is increasing, it suffices to show that

$$\ln q(t) = \ln(f(t + p_j) - f(t)) - \ln(f(t + p_i) - f(t))$$

is increasing. To this purpose we notice that the derivative

$$\frac{f'(t + p_j) - f'(t)}{f(t + p_i) - f(t)} - \frac{f'(t + p_i) - f'(t)}{f(t + p_j) - f(t)}$$

is positive by Lemma 1 and $p_i > p_j$. □
Lemma 4 If $\beta > 1$, $a < b$ and
\[ g(x) = x \frac{f(b + x) - f(x + a)}{f(b + x) - f(b)}, \]
then $g(x)$ is a non-decreasing function of $x$.

Proof Equivalently, we show that $\ln g(x) = \ln(x) + \ln \left( f(b + x) - f(a + x) \right) - \ln \left( f(b + x) - f(b) \right)$ is non-decreasing. Taking derivative of the right hand side, we show
\[ \frac{1}{x} + \frac{f'(b + x) - f'(a + x)}{f(b + x) - f(a + x)} - \frac{f'(b + x)}{f(b + x) - f(b)} \geq 0. \]
By log-concavity of $f'$ and Lemma 3, the second term is minimized when $a$ approaches $b$, and hence is at least $f''(b + x)/f'(b + x)$. Therefore it is enough to show that
\[ \frac{1}{x} + \frac{f''(b + x)}{f'(b + x)} - \frac{f'(b + x)}{f(b + x) - f(b)} \geq 0, \]
which is equivalent in showing that
\[ \ln(x) + \ln(f'(b + x)) - \ln \left( f(b + x) - f(b) \right) \]
is non-decreasing $x$. The later derives from the fact that $xf'(b + x)/(f(b + x) - f(b))$ is non-decreasing, which follows from assumption in [1].

5 Characterization of the local order property

To simplify notation, throughout the paper we assume that no two jobs have the same processing time, weight or Smith-ratio (weight over processing time). For convenience we extend the notation of the penalty function $f$ to the makespan of schedule $S$ as $f(S) := f(\sum_{i \in S} p_i)$. Also we denote by $F(S)$ the cost of schedule $S$.

In order to analyze the effect of exchanging adjacent jobs, we define the following function on $t \geq 0$
\[ \phi_{ij}(t) := \frac{f(t + p_i + p_j) - f(t + p_j)}{f(t + p_i + p_j) - f(t + p_i)}. \]
Note that $\phi_{ij}(t)$ is well defined since $f$ is strictly increasing by assumption and the durations $p_i, p_j$ are non-zero. This function $\phi_{ij}$ permits us to analyze algebraically the local order property, since
\[ i \prec_j \iff \forall t \geq 0 : \phi_{ij}(t) < \frac{w_i}{w_j}. \quad (2) \]

The following technical lemmas show properties of $\phi_{ij}$ and relate them to properties of $f$.

Lemma 5 If $p_i \neq p_j$ then $\phi_{ij}$ is strictly monotone, in particular:
- If $p_i > p_j$ and $\beta > 1$, then $\phi_{ij}$ is strictly increasing.
- If $p_i < p_j$ and $\beta > 1$, then $\phi_{ij}$ is strictly decreasing.
• If $p_i > p_j$ and $\beta < 1$, then $\phi_{ij}$ is strictly decreasing.

• If $p_i < p_j$ and $\beta < 1$, then $\phi_{ij}$ is strictly increasing.

**Proof** We only show this statement for the first case $p_i > p_j$ and $\beta > 1$, and the other cases are similar. In order to show that $\phi_{ij}$ is strictly increasing we prove that $\ln \phi_{ij}$ is increasing. For this we analyze its derivative which is

\[
\frac{f'(t + p_i + p_j) - f'(t + p_j)}{f(t + p_i + p_j) - f(t + p_j)} - \frac{f'(t + p_i + p_j) - f'(t + p_i)}{f(t + p_i + p_j) - f(t + p_i)}.
\]

The derivative is positive by Lemma 3.

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**Figure 2:** Examples of the function $\phi_{ij}(t)$ for $\beta = 0.5$ and $\beta = 2$, as well as for the cases $p_i > p_j$ and $p_i < p_j$.

**Lemma 6** For any jobs $i, j$, we have $\lim_{t \to \infty} \phi_{ij}(t) = p_i/p_j$.

**Proof** Proof. By the mean value theorem, for any differentiable function $f$ and $y, x$ it holds that

\[
f(y) = f(x) + (y - x)f'(\eta)
\]

for some $\eta \in [x, y]$. Thus

\[
(t + p_i + p_j)^\beta - (t + p_i)^\beta = p_j(t + p_i + \eta)^{\beta-1}
\]

for some $\eta \in [0, p_j]$ and

\[
(t + p_i + p_j)^\beta - (t + p_j)^\beta = p_i(t + p_i + \gamma)^{\beta-1}
\]

for some $\gamma \in [0, p_i]$. Moreover, for any $\beta > 0$ and $\alpha > 0$, $\lim_{t \to \infty} (t + \alpha)^{\beta-1}/t^{\beta-1} = 1$. Therefore,

\[
\lim_{t \to \infty} \phi_{ij}(t) = \lim_{t \to \infty} \frac{(t + p_i + p_j)^\beta - (t + p_j)^\beta}{(t + p_i + p_j)^\beta - (t + p_i)^\beta} = \frac{p_i}{p_j}.
\]

These two lemmas permit to characterize the local order property, see Figure 1.

**Lemma 7** For any two jobs $i, j$ we have $i \prec_{\ell} j$ if and only if

- $\beta \geq 1$ and $p_i \leq p_j$ and $w_j/w_i \leq \phi_{ji}(0)$ or
- $\beta \geq 1$ and $p_i \geq p_j$ and $w_j/w_i \leq p_j/p_i$ or
- $0 < \beta \leq 1$ and $p_i \leq p_j$ and $w_j/w_i \leq p_j/p_i$ or
- $0 < \beta \leq 1$ and $p_i \geq p_j$ and $w_j/w_i \leq \phi_{ji}(0)$.
6 The global order property

In this section we characterize the global order property of two jobs \( i, j \) in the convex case \( \beta > 1 \), and provide sufficient conditions on the concave case \( 0 \leq \beta < 1 \). Our contributions are summarized graphically in Figure 1.

6.1 Global order property for \( p_i \leq p_j \)

In this section we give the proof of the conjecture in case \( i \) has processing time not larger than \( j \). Intuitively this seems the easier case, as exchanging \( i \) with \( j \) in the schedule \( AjBi \) makes jobs from \( B \) complete earlier. However the benefit of the exchange on these jobs cannot simply be ignored in the proof. A simple example shows why this is so. Let \( i, j, k \) be 3 jobs with \( p_i = 4, w_i = 1, p_j = 8, w_j = 1.5, p_k = 1, w_k = 0 \). Then \( i \prec j \), but exchanging \( i, j \) in the schedule \( jki \) increases the objective value, as \( F(ikj) = 4^2 + 1.5 \cdot 13^2 = 269.5 \) while \( F(jki) = 1.5 \cdot 8^2 + 13^2 = 265 \). Now if we raise \( w_k \) to 0.3, then we obtain an interesting instance. It satisfies the assumption \( F(jki) < F(jik) \) mentioned in the previous section, and \( jki \) is the optimal schedule, but it cannot be shown with an exchange argument from \( ikj \) without taking into account the gain on job \( k \) during the exchange.

**Theorem 1** The implication \( i \prec j \Rightarrow i \prec g j \) holds when \( p_i \leq p_j \).

**Proof** The proof holds in fact for any increasing penalty function \( f \). Let \( A, B \) be two arbitrary job sequences. We will show that the schedule \( AjBi \) has strictly higher cost than one of the schedules \( AijB, AiBj \).

First if \( F(AjBi) \geq F(AjiB) \), then by \( i \prec j \) we have even \( F(AjBi) > F(AijB) \). So for the remaining case we assume \( F(AjBi) < F(AjiB) \) and will show \( F(AjBi) > F(AijB) \). By \( i \prec j \) it would be enough to show the even stronger inequality

\[
F(AjiB) - F(AijB) < F(AjBi) - F(AiBj),
\]

or equivalently

\[
F(AjiB) - F(AjBi) < F(AijB) - F(AiBj).
\]

The right hand side is positive by assumption, so it would be enough to show

\[
\min \phi_{ji}(t) \left( F(AjiB) - F(AjBi) \right) < F(AijB) - F(AiBj), \tag{3}
\]

since \( \phi_{ji}(t) > 1 \) by \( p_i < p_j \).

We introduce the following notation. Denote the jobs in \( B \) by \( 1, \ldots, k \) for some integer \( k \), and for some job \( 1 \leq h \leq k \) denote by \( l_h \) the total processing time of all jobs from 1 to \( h \). We show the inequality, by analyzing separately the contribution of jobs \( h \in B \), and of the jobs \( i, j \). By definition of \( \phi_{ji} \) we have

\[
\phi_{ji}(l_h) \left( f(p_i + p_j + l_h) - f(p_j + l_h) \right) = f(p_i + p_j + l_h) - f(p_i + l_h),
\]

which implies

\[
\min_{t} \phi_{ji}(t) w_h \left( f(p_i + p_j + l_h) - f(p_j + l_h) \right) \leq w_h \left( f(p_i + p_j + l_h) - f(p_i + l_h) \right). \tag{4}
\]
To analyze the contribution of jobs $i, j$ we observe that by $i \prec_\ell j$ we have
\[ w_j \leq \min \phi_{ji} w_i \]
which implies
\[ \min_t \phi_{ji}(t) w_i \left( f(a + p_i + p_j) - f(a + p_i + p_j + l_k) \right) \leq w_j \left( f(a + p_i + p_j) - f(a + p_i + p_j + l_k) \right). \]
(5)

Summing up (4) with (5) yields (3) as required, and completes the proof. \(\square\)

6.2 Global order property for $\beta > 1$

**Theorem 2** The implication $i \prec_\ell j \Rightarrow i \prec_g j$ holds when $\beta \geq 1$.

**Proof** By Theorem 1 it suffices to consider the case $p_j < p_i$. Assume $i \prec_\ell j$ and consider a schedule $S$ of the form $AjBi$ for some job sequences $A, B$.

The proof is by induction on the number of jobs in $B$. The base case $B = \emptyset$ follows from $i \prec_\ell j$.

For the induction step, we assume that $A'jB'i$ is suboptimal for all job sequences $A', B'$ where $B'$ has strictly less jobs than $B$. Formally we denote $B$ as the job sequence 1, 2, ..., $k$ for some $k \geq 1$.

If for some $1 \leq h \leq k$ we have
\[ F(AjBi) \geq F(A(12...h)j(h+1,...,k)i), \]
then by induction we immediately obtain that $AjBi$ is suboptimal. Therefore we assume
\[ \forall 1 \leq h \leq k : \ F(AjBi) < F(A(12...h)j(h+1,...,k)i). \] (6)

Then it remains to show that $F(AjBi) > F(AiBj)$ to establish sub-optimality of $AjBi$.

For the remainder of the proof, we introduce the following notations. We denote by $a$ the total processing time of $A$. In addition we use $h$ and $h'$ to index the jobs in $B$, and denote by $l_h$ the total processing time of jobs 1, 2, ..., $h$, and by $b = a + l_k$ the total processing time of $AB$. We also introduce the expressions
\[ \delta_h := f(a + p_i + l_h) - f(a + l_h) \]
and
\[ \gamma_h^i := f(a + p_i + l_h) - f(a + p_i + l_{h-1}) \]
and define $\delta_h^j, \gamma_h^j$ similarly.

Equations (6) imply that
\[ \sum_{h' = 1}^{h} w_{h'} \left( f(a + p_j + l_{h'}) - f(a + p_j + l_{h'}) \right) \leq w_j \left( f(a + p_j + l_h) - f(a + p_j) \right) \]
\[ = w_j \sum_{h' = 1}^{h} \left( f(a + p_j + l_{h'}) - f(a + p_j + l_{h'-1}) \right) \] (7)

where we use the convention that $l_0 = 0$. 10
We restate (7) as follows: For each $1 \leq h \leq k$,
\begin{equation}
   w_1 \delta^i_1 + \ldots + w_h \delta^i_h \leq w_j (\gamma^i_1 + \ldots + \gamma^i_h)
\end{equation}

For $a < b$ define
\[ g(x) = \frac{x f(b + x) - f(x + a)}{f(b + x) - f(b)}. \]

By Lemma 4, $g$ is non-decreasing in $x$.

We need to show that $F(AiBj) \leq F(AjBi)$. As $p_i \geq p_j$ by case assumption, when we move from $AjBi$ to $AiBj$, the completion times of $j$ and the jobs in $B$ increase and that of $i$ decreases. Thus the statement is equivalent to showing that
\begin{align*}
   \sum_{h=1}^{k} w_h \left( f(a + p_i + l_h) - f(a + p_j + l_h) \right) \\
   \leq w_i \left( f(a + p_j + p_i + l_k) - f(a + p_i) \right) - w_j \left( f(a + p_j + p_i + l_k) - f(a + p_j) \right)
\end{align*}

(9)

Now, by the local optimality of $i$ and $j$, it holds that
\[ w_j \left( f(a + p_j + p_i + l_k) - f(a + p_j + l_k) \right) \leq w_i \left( f(a + p_i + p_j + l_k) - f(a + p_i + l_k) \right), \]

thus to show (9) it suffices to show that
\begin{equation}
   \sum_{h=1}^{k} w_h \left( f(a+p_i+l_h)-f(a+p_j+l_h) \right) \leq w_i \left( f(a+p_i+l_k)-f(a+p_i) \right) - w_j \left( f(a+p_j+l_k)-f(a+p_j) \right). \tag{10}
\end{equation}

We reformulate (10) as
\[ \sum_{h=1}^{k} w_h (\delta^i_h - \delta^j_h) \leq w_i \sum_{h=1}^{k} \gamma^i_h - w_j \sum_{h} \gamma^j_h. \]

Since $w_i \geq w_j p_i/p_j$ by Lemma 7, it suffices to show that
\begin{equation}
   \sum_{h=1}^{k} w_h (\delta^i_h - \delta^j_h) \leq w_j \sum_{h=1}^{k} \left( \frac{p_i}{p_j} \gamma^i_h - \gamma^j_h \right). \tag{11}
\end{equation}

We define for every job $1 \leq h \leq k$
\[ q_h := \frac{\delta^i_h}{\delta^j_h} - \frac{\delta^i_{h+1}}{\delta^j_{h+1}} \]

where we use the convention $\delta^i_{k+1}/\delta^j_{k+1} = 1$. Note that by Corollary 1, all $q_h$ are non-negative.

Multiplying for a given $h$, equation (8) by $q_h$, and summing over all $1 \leq h \leq k$ we obtain
\[ \sum_{h=1}^{k} w_h \delta^i_h \left( \sum_{h' \geq h} q_{h'} \right) \leq w_j \sum_{h=1}^{k} \gamma^i_h \left( \sum_{h' \geq h} q_{h'} \right). \]
As the sum over \( q_h \) telescopes, we can rewrite the above as

\[
\sum_{h=1}^{k} w_h (\delta_h^i - \delta_h^j) \leq w_j \sum_{h=1}^{k} \gamma_h^j (\delta_h^i / \delta_h^j - 1)
\]

Thus to prove (11), it suffices to show that

\[
\gamma_h^j (\delta_h^i / \delta_h^j - 1) \leq (p_i / p_j) \gamma_h^i - \gamma_h^j
\]
or equivalently,

\[
\delta_h^i / \delta_h^j \leq (p_i / p_j) (\gamma_h^i / \gamma_h^j)
\]

But this is exactly what Lemma 4 gives us. In particular, we get

\[
\gamma_h^j / \delta_h^j p_j \leq p_i \gamma_h^i / \delta_h^i
\]

which concludes the proof.

\[\square\]

6.3 Global order property for \( 0 < \beta < 1 \) and \( p_j \leq p_i \)

**Theorem 3** The implication \( i \ll \ell \ j \Rightarrow i \ll \ell \ j \) holds when \( p_j \leq p_i \), \( w_i / w_j \geq (p_i / p_j)^{2-\beta} \) and \( 0 < \beta < 1 \).

**Proof** The proof is along the same lines as the previous one. Hence we assume (6) and need to show \( F(AjBi) > F(AiBj) \), and for this purpose show

\[
F(ABji) - F(ABij) < F(AjBi) - F(AiBj),
\]

where the left hand side is positive by \( i \ll \ell \ j \). Equivalently we have to show

\[
F(ABji) - F(AjBi) < F(ABij) - F(AiBj).
\]  

(12)

First we claim that

\[
\frac{w_i}{w_j} > \left( \frac{p_i}{p_j} \right)^{1-\beta}.
\]  

(13)

Indeed, from Lemma 7 we have \( w_j / w_i \leq \phi_{ji}(0) \). By Lemma 5, the function \( \phi_{ji}(t) \) is increasing, and by Lemma 6, it is upper bounded by \( p_j / p_i \). Hence \( w_j / w_i \leq p_j / p_i \), and for \( 0 < \beta < 1 \) and \( p_j \leq p_i \) inequality (13) follows.

Therefore by Lemma 1 we have

\[
\frac{w_i}{w_j} \frac{f(b + p_i) - f(a + p_i)}{f(b + p_j) - f(a + p_j)} \geq \left( \frac{p_i}{p_j} \right)^{1-\beta} \frac{f(b + p_i) - f(a + p_i)}{f(b + p_j) - f(a + p_j)} \geq 1.
\]  

(14)
For convenience we denote $\Delta(t) := \sum_{h \in B} w_h(f(a + l_h + t) - f(a + l_h))$. We have

$$0 < F(ABji) - F(AjBi)$$

$$= w_j(f(b + p_j) - f(a + p_j)) - \Delta(p_j)$$

$$< \frac{w_i}{w_j} (f(b + p_i) - f(a + p_i)) (w_j(f(b + p_j) - f(a + p_j)) - \Delta(p_j))$$

$$= w_i(f(b + p_i) - f(a + p_i)) - \frac{w_i}{w_j} (f(b + p_i) - f(a + p_i)) \Delta(p_j)$$

$$< w_i(f(b + p_i) - f(a + p_i)) - \frac{w_i}{w_j} (f(b + p_i) - f(a + p_i)) \min_{t \geq 0} \frac{f(t + p_j) - f(t)}{f(t + p_i) - f(t)} \Delta(p_i).$$

The first inequality follows from assumption [10] with $h = k$. The second inequality follows from [14]. The third inequality holds since $f(t + p_j) < f(t + p_i)$ for all $t \geq 0$.

In order to upper bound the later expression by

$$w_i(f(b + p_i) - f(a + p_i)) - \Delta(p_i) = F(ABij) - F(AiBj)$$

as required, it suffices to show

$$\frac{w_i}{w_j} (f(b + p_i) - f(a + p_i)) \min_{t \geq 0} \frac{f(t + p_j) - f(t)}{f(t + p_i) - f(t)} \geq 1.$$ 



By $0 < \beta < 1$ and Corollary [1] the $\frac{f(t+p_j) - f(t)}{f(t+p_i) - f(t)}$ is decreasing, and the limit when $t \to \infty$ is $p_j/p_i$, by a similar analysis as in the proof of Lemma [6]. Therefore

$$\min_{t \geq 0} \frac{f(t+p_j) - f(t)}{f(t+p_i) - f(t)} \geq \frac{p_j}{p_i}.$$ 

Hence

$$\frac{w_i}{w_j} (f(b + p_i) - f(a + p_i)) \min_{t \geq 0} \frac{f(t + p_j) - f(t)}{f(t + p_i) - f(t)}$$

$$\geq \left( \frac{p_i}{p_j} \right)^{2-\beta} f(b + p_i) - f(a + p_i) \frac{p_j}{p_i}$$

$$\geq \left( \frac{p_i}{p_j} \right)^{1-\beta} f(b + p_i) - f(a + p_i) \frac{p_j}{p_i}$$

$$\geq 1.$$ 

where the second inequality follows from the theorem hypothesis and the last inequality from Lemma [6]. This concludes the proof of the theorem. 

\[ \square \]

7 Generalization

We can provide some technical generalizations of the aforementioned theorems. For any pair of jobs $i, j$, and job sequence $T$ of total length $t$, we denote by $i \prec_{l(t)} j$ the property $F(Tij) \leq F(Tji)$. Now suppose that none of $i \prec_0 j$ or $j \prec_0 j$ holds, and say $p_i \geq p_j$ and $\beta > 1$. Then from Lemma [5] it follows that there exist a unique time $t$, such that for all $t' \leq t$ we have $i \prec_{l(t')} j$ and for all $t' \geq t$ we have $j \prec_{l(t')}$ i. These properties are denoted respectively by $i \prec_{\ell(0,t]} j$ and $j \prec_{\ell(t,\infty)} j$. In case $p_i \leq p_j$, $\beta > 1$ or $p_i \geq p_j$, $0 < \beta < 1$, we have the symmetric situation $j \prec_{\ell(0,t]} i$ and $i \prec_{\ell(t,\infty)} j$. 

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This notation can be extended also to the global order property. If for every job sequences $A, B$ with $A$ having total length at least $t$ we have $F(A\omega Bj) \leq F(AjBi)$, then we say that $i, j$ satisfy the global order property in the interval $[t, \infty)$ and denote it by $i \prec_{g[t,\infty]} j$. The property $i \prec_{g[0,t]} j$ is defined similarly for job sequences $A, B$ of total length at most $t$.

The proof of Theorem 1 actually shows the stronger statement: if $\beta > 1$ and $p_i \geq p_j$, then $j \prec_{[t,\infty]} i$ implies $j \prec_{g[0,t]} i$. The same implication does not hold for interval $[0, t]$, as shown by the following counter example. It consists of the 3-job instance for $\beta = 2$ with $p_1 = 13, w_i = 7, p_j = 8, w_j = 5, p_k = 1, w_k = 1$. For $t = 19/18$, we have $i \prec_{[0,t]} j$ and $j \prec_{[t,\infty]} i$. But the unique optimal solution is the sequence $jki$, meaning that we don’t have $i \prec_{g[0,t]} j$.

These generalizations can be summarized as follows.

| $i \prec_{[0,t]} j$ | $j \prec_{[t,\infty]} i$ |
|----------------------|----------------------|
| $j \prec_{g[0,t]} i$ | $i \prec_{g[t,\infty]} j$ |
| $p_i \leq p_j$       | $p_j \geq p_i$       |
| $0 < \beta < 1$     | $\beta > 1$         |

### 8 Experimental study

We conclude this paper with an experimental study, evaluating the impact of the proposed rules on the performance of a search procedure. Following the approach described in [Hölm and Jacobs (2012a)](Hölm and Jacobs (2012a)), we consider the Algorithm A* by [Hart et al. (1972)](Hart et al. (1972)). The search space is the directed acyclic graph consisting of all subsets $S \subseteq \{1, \ldots, n\}$. Note that the potential search space has size $2^n$ which is already less than the space of the $n!$ different schedules. In this graph for every vertex $S$ there is an arc to $S \setminus \{j\}$ for any $j \in S$. It is labeled with $j$, and has cost $w_j t^\beta$ for $t = \sum_{i \in S} p_i$. Every directed path from the root $\{1, \ldots, n\}$ to $\{}$ corresponds to a schedule of an objective value being the total arc cost. The algorithm A* finds a shortest path in the graph, and as Dijkstra’s algorithm uses a priority queue to select the next vertex to explore. But the difference of A* is that it uses as weight for vertex $u$ not only the distance from the source to $u$, but also a lower bound on the distance from $u$ to the destination.

Figure 3 illustrates the DAG explored by A* for $\beta = 2$ on the instance consisting of the following processing time, priority weight pairs: $(10,5),(10,5),(11,3),(13,6),(8,4),(12,6)$. Arcs are labeled with their weight. Pruning rules for example permitted to remove arcs that leave vertex $\{3, 4, 6\}$. The last two arcs have the same weight, as the lower bound on single job sets is tight.

Formally A* explores the graph using a priority queue containing arcs pointing to vertices that still need to be visited. An arc $(S, S')$ has a weight corresponding to the distance from the root to $S$ through this arc plus a basic lower bound of the optimum cost of scheduling $S$, which we choose to be simply $\sum_{i \in S} w_i p_i^\beta$.

Pruning is done when constructing the list of outgoing arcs at some vertex $S$. Potentially every job $i \in S$ can generate an arc, but order properties might prevent that. Let $j$ be the label of the arc leading to $S$ (assuming $S$ is not the root). Let $t_1 = \sum_{k \in S} p_k$. Now if $j \prec_{(t_1-p_i)} i$, then no arc is generated for job $i \in S$. The same thing happens when there is a job $k \in S$ with $i \prec_{g[0,t_1]} k$. In a search tree such a pruning would cut the whole subtree attached to that arc, but in a directed acyclic graph the improvement is not so dramatic, as the typical indegree of a vertex is linear in $n$.

A simple additional pruning could be done when remaining jobs to be scheduled form a trivial subinstance. By this we mean that all pairs of jobs $i, j$ from this subinstance are comparable with
the order $\prec_{\ell[0,t]}$. In that case the local order is actually a total order, which describes in a simple manner the optimal schedule for this subinstance. In that case we could simply generate a single path from the node $S$ to the target vertex $\{1, \ldots, n\}$. However experiments showed that detecting this situation is too costly compared with the benefit we could gain from this pruning.

8.1 Random instances

We adopt the model of random instances described by Höhn and Jacobs. All previous experimental results were made by generating processing times and weights uniformly from some interval, which leads to easy instances, since any job pair $i,j$ satisfies with probability $1/2$ global precedence, i.e. $i \prec_g j$ or $j \prec_g i$. As an alternative, Höhn and Jacobs (2012a) proposed a random model, where the Smith-ratio of a job is selected according to $2N(0, \sigma^2)$ with $N$ being the normal distribution centered at 0 with variance $\sigma$. Therefore for $\beta = 2$ the probability that two jobs satisfy global precedence depends on $\sigma$, since the Höhn-Jacobs-1 rule compares the Smith-ratios among the jobs.

We adopted their model for other values of $\beta$ as follows. When $\beta > 1$, the condition for $i \prec_g j$ of our rules can be approximated, when $p_j/p_i$ tends to infinity, by the relation $w_i/p_i \geq \beta w_j/p_j$. Therefore in order to obtain a similar “hardness” of the random instances for the same parameter $\sigma$ for different values of $\beta > 1$, we choose the Smith-ratio according to $2N(0, \beta^2 \sigma^2)$. This way the ratio between the Smith-ratios of two jobs is a random variable from the distribution $2\beta^2N(0, \sigma^2)$, and the probability that this value is at least $\beta$ depends only on $\sigma$.

However when $\beta$ is between 0 and 1, the condition for $i \prec_g j$ of our rule can be approximated when $p_j/p_i$ tends to infinity by the relation $w_i/p_i \geq 2w_j/p_j$, and therefore we choose the Smith-ratio of the jobs according to the $\beta$-independent distribution $2N(0, \beta^2 \sigma^2)$.

The instances of our main test sets are generated as follows. For each choice of $\sigma \in \{0.1, 0.2, \ldots, 1\}$ and $\beta \in \{0.5, 0.8, 1.1, \ldots, 3.2\}$. We generated 25 instances of 20 jobs each. The processing time of
every job is uniformly generated in \{1, 2, \ldots, 100\}. Then the weight is generated according to the above described distribution. Note that the problem is independent on scaling of processing time or weights, motivating the arbitrary choice of the constant 100.

8.2 Hardness of instances

As a measure of the hardness of instances, we consider the portion of job pairs \( i, j \) which satisfy global precedence. By this we mean that we have either \( i \prec_{g[0,t_1]} j \) or \( j \prec_{g[0,t_1]} i \) for \( t_1 \) being the total processing time over all jobs excepting jobs \( i, j \). Figure 5 shows this measure for various choices of \( \beta \).

The results depicted in Figure 5 confirm the choice of the model of random instances. Indeed the hardness of the instances seems to depend only little on \( \beta \), except for \( \beta = 2 \) where particular strong precedence rules have been established. In addition the impact of our new rules is significant, and further experiments show how this improvement influences the number of generated nodes, and therefore the running time. Moreover it is quite visible from the measures that the instances are more difficult to solve when they are generated with a small \( \sigma \) value.

8.3 Comparison between forward and backward variant

In this section, we consider two variants of the above mentioned algorithm. In the forward approach, a partial schedule describes a prefix of length \( t \) of a complete schedule and is extended to its right along an edge of the search tree, and in this variant the basic lower bound is \( \sum_{i \in S} w_i (t + p_i)^\beta \). However in the backward approach, a partial schedule \( S \) describes a suffix of a complete schedule and is extended to its left. Kaindl et al. (2001) give experimental evidence that the backward variant generates for some problems less nodes in the search tree, and this fact has also been observed by Höhn and Jacobs (2012a).

We conducted an experimental study in order to find out which variant is most likely to be more efficient. The results are shown in Figure 6. The values are most significative for small \( \sigma \) values, since for large values the instances are easy anyway and the choice of the variant is not very important. The results indicate that without our rules the forward variant should be used only when \( \beta < 1 \) or \( \beta = 2 \), while with our rules the forward variant should be used only when \( \beta > 1 \).

Later on, when we measured the impact of our rules in the subsequent experiments, we compared the behavior of the algorithm using the most favorable variant dependent on the value of \( \beta \) as described above.

8.4 Timeout

During the resolution a timeout was set, aborting executions that needed more than a million nodes. In Figure 4 we show the fraction of instances that could be solved within the limited number of nodes. From these experiments we measure the instance sizes that can be efficiently solved, and observe that this limit is of course smaller when \( \sigma \) is small, as the instances become harder. But we also observe that with the usage of our rules much larger instances can be solved.

When \( \beta \) is close to 1, and instances consist of jobs of almost equal Smith-ratio, the different schedules diverge only slightly in cost, and intuitively one has to develop a schedule prefix close to the makespan, in order to find out that it cannot lead to the optimum. However for \( \beta = 2 \), the Mondal-Sen-Höhn-Jacobs rule make the instances easier to solve than for other values of \( \beta \), even
close to 2. Note that we had to consider different instance sizes, in order to obtain comparable results, as with our rules all 20 job instances could be solved.

![Figure 4: Proportion of instances which could be solved within the imposed time limit of a million nodes, with (below) and without (above) the new rules.](image)

8.5 Improvement factor

In this section we measure the influence on the number of nodes generated during a resolution when our rules are used. For $\beta = 2$ we compare our performance with the Mondal-Sen-Höhn-Jacobs rule, while for other values of $\beta$ we compare with the Sen-Dileepan-Ruparel rule. For fairness we excluded instances where the timeout was reached without the use of our rules. Figure 7 shows the ratio between the average number of generated nodes when the algorithm is run with our rules, and when it is run without our rules. Clearly this factor is smaller for $\beta = 2$, since the Mondal-Sen-Höhn-Jacobs rules apply here.

We observe that the improvement factor is more important for hard instances, i.e. when $\sigma$ is small. From the figures it seems that this behavior is not monotone, for $\beta = 1.1$ the factor is less important with $\sigma = 0.1$ than with $\sigma = 0.3$. However this is an artifact of our pessimistic measurements, since we average only over instances which could be solved within the time limit, so in the statistics we filtered out the really hard instances.

9 Performance measurements for $\beta = 2$

For $\beta = 2$, the authors of Höhn and Jacobs (2012a) provide several test sets to measure the impact of their rules in different variants, see Höhn and Jacobs (2012b). For completeness we selected two data sets from their collection to compare our rules with theirs.

The first set called set-n contains for every number of jobs $n = 1, 2, \ldots, 35$, 10 instances generated with parameter $\sigma = 0.5$. This file permits to measure the impact of our rules as a function on the instance size.

The second test set that we considered is called set-T and contains for every parameter $\sigma = 0.100, 0.101, 0.102, \ldots, 1.000$ 3 instances of 25 jobs. Results are depicted in figure 8.
10 Performance depending on input size

In addition we show the performance of the algorithm with our rules, in dependence on the number of jobs. Figure [9] shows for different number of jobs the number of generated nodes averaged over 100 instances generated with different $\sigma$ parameters, exposing an expected running time which strongly depends on the hardness of the instances.

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Figure 5: Proportion of job pairs that satisfy a global precedence relation as function of the parameter $\sigma$ used in the random generation of the instances for 60 job instances.
Figure 6: Proportion of instances for which the forward variant generated less nodes than the backward variant. The values are plotted as function of $\sigma$, both for the resolution with our new rules and without.
Figure 7: Average improvement factor as function of $\beta$ and $\sigma^2$
Figure 8: Improvement ratio for test sets set-n (left) and set-T (right).

Figure 9: Average number of nodes in dependence on the size of the instances, generated with $\sigma = 0.1$ on the left and $\sigma = 0.5$ on the right.