Deformation of pairs and Noether–Lefschetz loci in toric varieties

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Abstract
We continue our study of the Noether–Lefschetz loci in toric varieties and investigate deformation of pairs \((V, X)\) where \(V\) is a complete intersection subvariety and \(X\) a quasi-smooth hypersurface in a simplicial projective toric variety \(\mathbb{P}^k_{\Sigma_1} + 1/\Sigma_1\), with \(V \subset X\). The hypersurface \(X\) is supposed to be of Macaulay type, which means that its toric Jacobian ideal is Cox–Gorenstein, a property that generalizes the notion of Gorenstein ideal in the standard polynomial ring. Under some assumptions, we prove that the class \(\lambda_V \in H^{k,k}(X)\) deforms to an algebraic class if and only if it remains of type \((k, k)\). Actually we prove that locally the Noether–Lefschetz locus is an irreducible component of a suitable Hilbert scheme. This generalizes Theorem 4.2 in our previous work (Bruzzo and Montoya 15(2):682–694, 2021) and the main theorem proved by Dan (in: Analytic and Algebraic Geometry. Hindustan Book Agency, New Delhi, pp 107–115, 2017).

Keywords Noether–Lefschetz locus · Hodge locus · Toric varieties

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1 Introduction

In this short note we continue our study of the Noether–Lefschetz loci in toric varieties and investigate the deformation of pairs \((V, X)\) where \(V\) is a \(k\)-dimensional complete intersection subvariety and \(X\) a quasi-smooth ample hypersurface in a simplicial projective toric variety \(\mathbb{P}^{2k+1}_\Sigma\) of odd dimension \(2k + 1 \geq 3\), with \(V \subset X\). We make two assumptions:

- The hypersurface \(X\) is supposed to be of Macaulay type, which means that its toric Jacobian ideal is Cox–Gorenstein, a property that generalizes the notion of Gorenstein ideal in a standard polynomial ring. This will be discussed in Sect. 3. Cox–Gorenstein ideals are studied in some detail in [2].
- The local Noether–Lefschetz locus \(\text{NL}_{k, \lambda, V, U}\), also called “Hodge locus” in the literature when \(\mathbb{P}^{2k+1}_\Sigma\) is a projective space, as defined in Sect. 5, is not empty (a condition for this to happen is for instance given in [3, Lemma 3.7]). Here \(\lambda\) is the cohomology class of \(V\), and \(\beta\) is the class of \(X\) in \(\text{Pic}(\mathbb{P}^{2k+1}_\Sigma)\). Then the full Noether–Lefschetz locus \(\text{NL}_\beta\), defined as the locus in the linear system \(|\beta|\) of the points corresponding to quasi-smooth hypersurfaces whose \((k, k)\)-cohomology does not come entirely from the ambient variety \(\mathbb{P}^{2k+1}_\Sigma\), is locally analytically a finite union of Hodge loci [5].

Moreover, under the further assumption that \(\beta\) satisfies \(\beta = q \eta + \beta', n \in \mathbb{N}\), where \(q \in \mathbb{Q}_{>0}\), \(\eta\) is a primitive ample class in \(\mathbb{P}^{2k+1}_\Sigma\), and \(\beta'\) is a nef Cartier class, if \(X\) contains a \(k\)-dimensional complete intersection subvariety with \(\deg \eta V < qm_{k+1}\), where \(m_{k+1}\) is a rational number only depending on \(\mathbb{P}^{2k+1}_\Sigma\) and the choice of a polarization, we will show that its associated cohomology class \(\lambda\) deforms to an algebraic class if and only it remains of type \((k, k)\).

This extends the work of Dan in [10] and the last result of [4, Theorem 4.2] for toric varieties with higher Picard rank (there the Picard number was assumed to be one, and moreover, the result is asymptotic).

2 Infinitesimal variation of the Hodge structure

According to Batyrev and Cox in [1], the cohomology of hypersurfaces in projective simplicial toric varieties has a pure Hodge structure. In this section, we introduce its infinitesimal variation following the notions due to Carlson, Green, Griffiths and Harris in [7].

Definition 2.1 A polarized Hodge structure of weight \(n\), denoted by \(\{H_Z, H^{p,q}, Q\}\), is a Hodge structure together with a bilinear form \(Q: H_Z \times H_Z \to \mathbb{Z}\) satisfying
\[ Q(\psi, \phi) = (-1)^n Q(\phi, \psi), \]
\[ Q(\psi, \phi) = 0, \quad \psi \in H^{p,q}, \phi \in H^{p',q'} \text{ and } p \neq q', \]
\[ i^{p-q} Q(\psi, \bar{\psi}) > 0, \quad 0 \neq \psi \in H^{p,q}. \]

**Definition 2.2** An *infinitesimal variation of Hodge structure* \( \{H_Z, H^{p,q}, Q, T, \delta\} \) is given by a polarized Hodge structure together with a vector space \( T \) and linear map
\[ \delta : T \to \bigoplus_{1 \leq p \leq n} \text{Hom}(H^{p,q}, H^{p-1,q+1}) \]
that satisfies the following two conditions:
\[ \delta(\xi_1) \delta(\xi_2) = \delta(\xi_2) \delta(\xi_1), \quad \xi_1, \xi_2 \in T, \]
\[ Q(\delta(\xi) \phi, \psi) + Q(\phi, \delta(\xi) \psi) = 0 \quad \text{for } \xi \in T \text{ and } \phi \in F^p, \psi \in F^{n-p+1}. \]

Here \( F^* \) is the filtration of \( H^n \) given by
\[ F^p = \bigoplus_{i=0}^p H^{n-i,i}. \]

If \( X \to \mathbb{P}_\Sigma^d \) is a quasi-smooth hypersurface in a simplicial projective toric variety \( \mathbb{P}_\Sigma^d \) of dimension \( d \), its primitive cohomology of degree \( d-1 \) is defined by the exact sequence [1]
\[ 0 \to i^* H^{d-1}(\mathbb{P}_\Sigma^d, \mathbb{C}) \to H^{d-1}(X, \mathbb{C}) \to H^{d-1}_{\text{prim}}(X, \mathbb{C}) \to 0. \]

The pullback \( i^* \) is compatible with the Hodge structures so that the primitive cohomology has a pure Hodge structure as well.

For a quasi-smooth hypersurface \( X \) in a simplicial projective toric variety, \( \delta \) is the morphism associated via tensor-hom adjuction to \( \gamma = \sum_p \gamma_p \), where
\[ \gamma_p : T_X M_{\beta} \otimes H^{p,d-1-p}_{\text{prim}}(X) \to H^{p,d-1-p}_{\text{prim}}(X) \]
is the natural multiplication map; for more details see [3, Section 3.3]. Given an infinitesimal variation of Hodge structure of weight \( 2k \), there is an invariant associated to \( \gamma \in H_Z^{k,k} \).

**Definition 2.3** The *third invariant* associated to \( \gamma \in H_Z^{k,k} \) is
\[ H^{k,k}(-\gamma) := \{ \psi \in H^{k,k} | \langle \delta^0(\xi) \psi, \gamma \rangle = 0 \text{ for all } \xi \in T \}. \]
Let us assume $\gamma$ is the primitive part of the class of $k$-codimensional algebraic cycle $V = \sum_i n_i V_i$ in $X$ with support $\sigma(V)$. Let $I_{\sigma(V)}$ be the ideal associated to $\sigma(V)$ and denote by $H^k(\Omega^k_X(-V))$ the image of the composed map

$$H^k(X, \Omega^k_X \otimes I_{\sigma(V)}) \to H^k(X, \Omega^k_X) \to H^k_{\text{prim}}(\Omega^k_X).$$

One has the following fact [11, Observation 4.a.4].

**Lemma 2.4** $H^k(\Omega^k_X(-V)) \subseteq H^{k,k}(-\gamma)$.

This is the result we shall need later on.

### 3 Macaulay-type hypersurfaces

In this section we characterize a class of hypersurfaces in toric varieties that satisfy a generalization of the Macaulay theorem which holds for projective spaces. As we shall see, these are hypersurfaces whose toric Jacobian ideal (whose definition will be recalled later in this section) has a property which generalizes the notion of Gorenstein ideal in a polynomial ring.

The Cox ring $S$ of a complete simplicial toric variety $\mathbb{P}^d_{/\Sigma}$ is graded over the effective classes in the class group $\text{Cl}(\mathbb{P}^d_{/\Sigma})$

$$S = \sum_{\alpha \in \text{Cl}(\mathbb{P}^d_{/\Sigma})} S^\alpha, \quad S^\alpha = H^0(\mathbb{P}^d_{/\Sigma}, \mathcal{O}_{\mathbb{P}^d_{/\Sigma}}(\alpha))$$

(see e.g. [8]). Following [2], we give a definition of *Cox–Gorenstein ideal* of the Cox rings which generalizes to toric varieties the definition given by Otwinowska in [12] for projective spaces.

**Definition 3.1** A graded ideal $I$ of $S$ is said to be a *Cox–Gorenstein ideal of socle degree* $N \in \text{Cl}(\mathbb{P}^d_{/\Sigma})$ if

- the quotient $R = S/I$ is Artinian;
- $\dim_{\mathbb{C}} R^N = 1$;
- for every homogeneous class $\alpha \in \text{Cl}(\mathbb{P}^d_{/\Sigma})$, either the natural bilinear morphism (called “Poincaré duality”)

$$R^\alpha \times R^{N-\alpha} \to R^N \simeq \mathbb{C}$$

is nondegenerate, or $R^\alpha = R^{N-\alpha} = 0$.

**Example 3.2** We give here some examples of Cox–Gorenstein ideals. In all cases the proof that the relevant ideal is Cox–Gorenstein is done by direct computation.

1. $\mathbb{P}^1 \times \mathbb{P}^1$ with homogeneous coordinates $(x, y, u, v)$, and

$$I = (x^2 u - y^2 v, x v, y u, x^3, y^3, u^2, v^2, x y).$$
I is Cox–Gorenstein of socle degree \((2, 1)\).

2. \(\mathbb{P}^1 \times \mathbb{P}^2\) with homogeneous coordinates \((x, y, u, v, w)\); the annihilator of \(f = xu^2 + uvw\) in the ring of polynomial operators \(\mathbb{C}[\partial_x, \partial_y, \partial_u, \partial_v, \partial_w]\) is a Cox–Gorenstein ideal of socle degree \((2, 1)\).

3. A singular example is provided by the fake weighted projective space associated with the fan generated by \(v_1 = (-3, -2)\), \(v_2 = (1, 2)\), \(v_3 = (1, 0)\) in \(\mathbb{R}^3\). The resulting variety has class group \(\mathbb{Z} \oplus \mathbb{Z}/2\) and is a quotient \(\mathbb{P}^{5,0}\) of degree \((5, 0)\). Indeed \(R^{5,0}\) is generated by the class of the monomial \(yz^2\).

Examples of Cox–Gorenstein ideals may be given in terms of toric Jacobian ideals. For every ray \(\rho \in \Sigma(1)\) denote by \(v_\rho\) its rational generator, and by \(x_\rho\) the corresponding variable in the Cox ring. Recall that \(d\) is the dimension of the toric variety \(\mathbb{P}^d_{\Sigma}\), while we denote by \(r = \#(\Sigma(1)\) the number of rays. Given \(f \in S^\beta\), one defines its toric Jacobian ideal as

\[
J_0(f) = \left( x_{\rho_1} \frac{\partial f}{\partial x_{\rho_1}}, \ldots, x_{\rho_r} \frac{\partial f}{\partial x_{\rho_r}} \right).
\]

We recall from [1] the definition of nondegenerate hypersurface and some properties (Definition 4.13 and Proposition 4.15).

**Definition 3.3** Let \(f \in S^\beta\), with \(\beta\) an ample Cartier class. The associated hypersurface \(X_f \subset \mathbb{P}^d_{\Sigma}\) is nondegenerate if for all \(\sigma \in \Sigma\) the affine hypersurface \(X_f \cap O(\sigma)\) is a smooth codimension one subvariety of the orbit \(O(\sigma)\) of the action of the torus \(\mathbb{T}^d\).

**Proposition 3.4** (1) Every nondegenerate hypersurface is quasi-smooth.
(2) If \(f\) is generic then \(X_f\) is nondegenerate.

We collect here, with some changes in the terminology, some results that are already contained in [9, Proposition 5.3].

**Proposition 3.5** Let \(f \in S^\beta\), and let \(\{\rho_1, \ldots, \rho_d\} \subset \Sigma(1)\) be such that \(v_{\rho_1}, \ldots, v_{\rho_d}\) are linearly independent.

(1) The toric Jacobian ideal of \(f\) coincides with the ideal

\[
\left( f, x_{\rho_1} \frac{\partial f}{\partial x_{\rho_1}}, \ldots, x_{\rho_d} \frac{\partial f}{\partial x_{\rho_d}} \right).
\]

(2) The following conditions are equivalent:
(a) \( f \) is nondegenerate;
(b) the polynomials \( x_{\rho_i} \frac{\partial f}{\partial x_{\rho_i}}, i = 1, \ldots, r, \) do not vanish simultaneously on \( X_f; \)
(c) the polynomials \( f \) and \( x_{\rho_i} \frac{\partial f}{\partial x_{\rho_i}}, i = 1, \ldots, d, \) do not vanish simultaneously on \( X_f. \)

Now we define the notion of hypersurface of Macaulay type.

**Definition 3.6** Let \( f \in S^\beta \) be nondegenerate, with \( \beta \) an ample Cartier class. \( f \) is said to be of the Macaulay type if its toric Jacobian ideal \( J_0(f) \) is a Cox–Gorenstein ideal of socle degree \( N = (d + 1)\beta - \beta_0, \) where \( \beta_0 \) is the anticanonical class of \( \mathbb{P}^d. \)

**Example 3.7** 1. According to this definition, any generic smooth hypersurface in \( \mathbb{P}^d \) is of Macaulay type.
2. Macaulay-type hypersurfaces in singular toric varieties do exist; a simple example is the curve \( x + y^2 + z^2 = 0 \) in \( \mathbb{P}[1, 1, 1], \) where \( \deg x = 2 \) and \( \deg y = \deg z = 1. \)
3. Another singular example, this time with class group different from \( \mathbb{Z}, \) is provided by the fake weighted projective space of Example 3.2.3 by letting \( f = x^4 + y^2 + z^2. \) The toric Jacobian ideal is \( I = (x^4, y^4, z^2) \) and the socle degree is \( N = (8, 0). \)

Actually a result in [2] shows that every nondegenerate ample Cartier hypersurface in a simplicial projective toric variety with Picard number 1 is of Macaulay type.

**4 The tangent space to the Noether–Lefschetz locus**

From now on we assume \( d = 2k + 1. \) Let \( f \in S^\beta \) define a nondegenerate quasi-smooth hypersurface \( X \) in \( \mathbb{P}^{2k+1}_\Sigma \) and suppose \( \beta \) is ample. Moreover, we assume that the hypersurface \( X \) is of Macaulay type. Let \( N = (k + 1)\beta - \beta_0 \) and let \( J_0(f) \) be the toric Jacobian ideal associated to \( f, \) which is Cox–Gorenstein of socle degree \( 2N + \beta_0. \) Then there is a perfect pairing \( R_0^\alpha \times R_0^{2N+\beta_0-\alpha} \rightarrow R_0^{2N+\beta_0} \) for \( \alpha \leq 2N + \beta_0. \) Let us denote by \( T_0' \) the subspace of \( R_0^N \) which is the kernel of the multiplication map \( \cdot x_1, \ldots, x_r P : R_0^N \rightarrow R_0^{2N+\beta_0} \) and by \( T_0 \) its inverse image in \( S^N, \) where \( P \) is a preimage of \( \gamma \) under the natural map

\[
\begin{align*}
S^N &\longrightarrow S^N / J^N \\ P &\longmapsto \bar{P} \\ \gamma &\longmapsto H_{\text{prim}}^{k,k}(X).
\end{align*}
\]

**Definition 4.1** Let \( T \subset S \) be the \( \text{Cl}(\Sigma) \)-graded module such that \( T^\alpha \) is the largest subspace where \( T^\alpha \otimes S^{N-\alpha} \) is contained in \( T_0 \) for \( \alpha \leq N, T^N = T_0 \) and \( E^{N+\alpha} = T_0 \otimes S^\alpha \) for \( \alpha \geq 0. \)

**Remark 4.2** Note that \( T \) is a Cox–Gorenstein ideal with socle degree \( N. \)

Actually \( T^\beta \) is the tangent space of the local Noether–Lefschetz locus at \( f. \)

\footnote{For ease of notation we write \( f \) but we mean its class modulo a nonzero constant factor.}
Lemma 4.3 \( T_f \text{NL}_{\lambda, \beta} \cong T_\beta \), where \( \lambda \) is a primitive class in \( H^{k,k}(X_f, \mathbb{Q}) \).

**Proof** An overbar will denote the class in \( R = S/J \) of an element in \( S \). Now, \( H \in T_\beta \) if and only if \( \overline{PH} \otimes R^{N-\beta} \) is contained in \( T'_0 \), which is equivalent to

\[
\overline{x_0 \ldots x_\beta PH} \otimes R^{N-\beta} = 0 \quad \text{in} \quad R^{N+\beta_0},
\]

using Poincaré duality that means \( \overline{x_0 \ldots x_\beta PH} = 0 \) in \( R^{N+\beta_0} \) and equivalently \( \overline{PH} = 0 \) in \( R^{N+\beta_0} \) if and only if \( H \in T_f \text{NL}_{\lambda, \beta} \) (see [6, Theorem 6.2]). \( \square \)

Let us suppose that \( V \) is the zero locus of \( \langle A_1, \ldots, A_{k+1} \rangle \) and since \( V \subset X_f \) there exist polynomials \( K_1, \ldots, K_{k+1} \) of degree \( \beta - \deg(A_i) \) such that \( f = A_1K_1 + \cdots + A_{k+1}K_{k+1} \). Let \( I = \langle A_1, \ldots, A_{k+1}, K_1, \ldots, K_{k+1} \rangle \).

**Proposition 4.4** \( T^\alpha = I^\alpha \) for \( \alpha \leq N \).

**Proof** Let \( W_1 \) be the zero locus of \( \langle K_1, A_2, \ldots, A_{k+1} \rangle \). Since \( V \cup W_1 \) is equal to \( X_f \cap \{ A_2 = \cdots = A_{k+1} = 0 \} \), \( \lambda_V \) is equal to \( -\lambda_{W_1} \) in the primitive cohomology. Now, let us denote by \( W_2 \) the zero locus of \( K_1, \ldots, K_{k+1} \) then, as before, \( [\lambda_V]_{\text{prim}} = [\alpha W_2]_{\text{prim}}, \alpha \in \mathbb{Z} \). By Lemma 2.4 we have \( \langle A_1, \ldots, A_{k+1}, K_1, \ldots, K_{k+1} \rangle \subset T \). Since \( X \) is quasi-smooth, the ideal \( \langle A_1, \ldots, A_{k+1}, K_1, \ldots, K_{k+1} \rangle \) is Cox–Gorenstein with socle degree \( N \), the socle degree of \( T \), so that \( I \) and \( T \) coincide in degree \( \alpha \leq N \). \( \square \)

**5 Main theorem**

In this section we prove our main result. We start by recalling the construction of the local Noether–Lefschetz locus [6]. Given an ample class \( \beta \) in \( \text{Pic}(\mathbb{P}^{2k+1}) \), let

\[
U_\beta \subset \mathbb{P}(H^0(\mathbb{P}^{2k+1}), \mathcal{O}_{\mathbb{P}^{2k+1}}(\beta))
\]

be the open subset parameterizing quasi-smooth hypersurfaces and let \( \pi: \chi_\beta \to U_\beta \) be the tautological family. One considers the local system \( \mathcal{H}^{2k} = R^{2k}\pi_*\mathcal{O} \otimes U_\beta \) over \( U_\beta \).

If \( f \in U_\beta \), let \( \lambda_f \in H^{k,k}(X_f, \mathbb{Q})/i^*(H^{k,k}(\mathbb{P}^{2k+1}, \mathbb{Q})) \) be a nonzero class, and let \( U \subset U_\beta \) be a contractible open subset around \( f \). Finally, let \( \lambda \in \mathcal{H}^{2k}(U) \) be the section defined by \( \lambda_f \) and let \( \bar{\lambda} \) be its image in \( (\mathcal{H}^{2k}/F^k\mathcal{H}^{2k})(U) \), where

\[
F^k\mathcal{H}^{2k} = \mathcal{H}^{2k,0} \oplus \mathcal{H}^{2k-1,1} \oplus \cdots \oplus \mathcal{H}^{k,k}
\]

**Definition 5.1** (Local Noether–Lefschetz Locus) \( \text{NL}^{k,\beta}_{\lambda, U} = \{ G \in U \mid \bar{\lambda}_G = 0 \} \).

Let \( \eta \) be a polarization for \( \mathbb{P}^{2k+1}_\Sigma \), that we assume to be primitive in the Picard group. Given the Hilbert polynomial \( P \) of a subscheme \( V \), computed with respect to \( \eta \), we denote by \( \text{Hilb}_P \) the Hilbert scheme of closed subschemes of \( \mathbb{P}^{2k+1}_\Sigma \) with Hilbert polynomial \( P \). We denote by \( Q \) the Hilbert polynomial of quasi-smooth hypersurface in \( \mathbb{P}^{2k+1}_\Sigma \) whose class in the Picard group is \( \beta \). The flag Hilbert scheme \( \text{Hilb}_{P, Q} \)
parametrizes all pairs \((V, X)\) where \(V \in \text{Hilb}_P\) and \(X\) is a quasi-smooth hypersurface in \(\mathbb{P}^{2k+1}_\Sigma\) of class \(\beta\) containing \(V\). Let \(\text{pr}_1\) be the projection to the first component and \(\text{pr}_2\) : \(\text{Hilb}_P \times Q \to U_\beta\) the natural projection to the open set which parametrizes quasi-smooth hypersurfaces in \(\mathbb{P}^{2k+1}_\Sigma\). Note that \(\text{pr}_1(\text{Hilb}_P, Q)\) is irreducible, so that there exists a unique component in \(\text{Hilb}_P, Q\) such that \(\text{pr}_1(\text{Hilb}_P, Q)\) coincides with the parameter space for complete intersection subschemes in \(\mathbb{P}^{2k+1}_\Sigma\).

For \(Z\) a \(d\)-dimensional closed subvariety of \(\mathbb{P}^{2k+1}_\Sigma\) we define its degree as \(\deg_Z Z = [Z] \cdot \eta^d\).

**Lemma 5.2** There is a positive rational number \(m_{k+1}\) such that \(\deg_Z W \geq m_{k+1}\) for all \((k + 1)\)-dimensional closed subvarieties \(W\) of \(\mathbb{P}^{2k+1}_\Sigma\).

**Proof** Let \(a\) be the smallest integer such that \(a\eta\) is very ample. Then \(a\eta\) defines a closed embedding \(j : \mathbb{P}^{k+1}_\Sigma \to \mathbb{P}^N\) for some \(N\). Denoting by \(H\) the hyperplane class in \(\mathbb{P}^N\), one has

\[
\deg Z W = \frac{1}{a^k} j^* H^k \cdot [W] = \frac{1}{a^k} H^k \cdot j_* [W] \geq \frac{1}{a^k}
\]

and one sets \(m_{k+1} = 1/a^k\). \(\square\)

The next lemma is a version of the Bézout theorem in the present context.

**Lemma 5.3** If \(X\) is an ample Cartier hypersurface in \(\mathbb{P}^{2k+1}_\Sigma\) whose class in \(\text{Pic}(\mathbb{P}^{2k+1}_\Sigma)\) satisfies \(\beta = q\eta + \beta',\) where \(q \in \mathbb{N}_{>0}\) and \(\beta'\) is a nef Cartier class, and \(V = X \cap W\) is a \(k\)-dimensional subvariety contained in \(X\), where \(W\) is a \((k + 1)\)-dimensional closed subvariety \(W \subset \mathbb{P}^{2k+1}_\Sigma\), then \(\deg Z V \geq qm_{k+1}\).

**Proof** We shall denote by \((Z)\) the class in \(A_d(\mathbb{P}^{2k+1}_\Sigma)\) of a \(d\)-dimensional closed subvariety \(Z\) of \(\text{Pic}(\mathbb{P}^{2k+1}_\Sigma)\), and by \([Z]\) its class in \(A^{2k+1-d}(\mathbb{P}^{2k+1}_\Sigma)\). Thus we have

\[
\deg Z V = \langle \eta^k, (W) \cap [X] \rangle = \langle \eta^k \cup [X], (W) \rangle = \langle \eta^k \cup (q\eta + \beta'), (W) \rangle
\]

\[
= q \deg Z W + \langle \eta^k \cup \beta', [W] \rangle \geq qm_{k+1} + \langle \eta^k \cup \beta', [W] \rangle.
\]

Since \(\beta'\) is nef we have \(\langle \eta^k \cup \beta', [W] \rangle \geq 0\), hence the claim follows. \(\square\)

Now we state and prove the main result of this paper.

**Theorem 5.4** Assume that \(\beta\) is as in Lemma 5.3. Let \(V\) be a quasi-smooth complete intersection in \(\mathbb{P}^{2k+1}_\Sigma\) of codimension \(k + 1\) and let \(X\) be a quasi-smooth hypersurface of class \(\beta\) containing \(V\) such that \(\deg Z V < qm_{k+1}\). Assume also that \(X\) is of the Macaulay type. Then,

\(\lambda_V\) deforms to a \((k, k)\) class if and only if \(\lambda_{[V]}\) deforms to an algebraic cycle.

In particular, for a suitable open subset \(U, NL_{x, \lambda_V, U}^{k, \beta}\) is isomorphic to an irreducible component of \(U \cap \text{pr}_2(\text{Hilb}_P, Q)\), where \(P\) and \(Q\) are the Hilbert polynomials of \(V\) and \(X\), respectively.
**Proof** By the assumption on the degree of $V$, one has $\text{pr}_2(\text{Hilb}_{P,Q}) \subset \text{NL}^{k,\beta}_{\lambda V,U}$. Then,
\[
\text{codim}_U \text{pr}_2(\text{Hilb}_{P,Q}) \geq \text{codim}_U \text{NL}^{k,\beta}_{\lambda V,U} \geq \text{codim}_{T_X U} T_X \text{NL}^{k,\beta}_{\lambda V,U}.
\]
On the other hand, keeping in mind that $T^\beta = I^\beta \subset I_V^\beta$, we have a natural map $\phi$ from $T^\beta$ to $\text{Hilb}_{P,Q}$, which sends a homogeneous polynomial of degree $\beta$ to its zero locus. One has $\text{Im}(\phi) \subset \text{pr}_2(\text{Hilb}_{P,Q})$ and since the zero locus is invariant under the torus action, $\dim T^\beta > \dim \text{Im}(\phi)$. Hence,
\[
\text{codim} \text{pr}_2(\text{Hilb}_{P,Q}) \leq \text{codim} \text{Im}(\phi) \leq \text{codim} T^\beta = \text{codim} T_X \text{NL}^{k,\beta}_{\lambda V,U}.
\]
So $\text{pr}_2(\text{Hilb}_{P,Q})$ and $\text{NL}^{k,\beta}_{\lambda V,U}$ have the same dimension, which implies the claim. \(\square\)

Note that the Noether-Lefschetz locus $\text{NL}^{k,\beta}_{\lambda V}$ is nonempty as $V$ is primitive due to Lemma 5.3.

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