Do quantum fluctuations stabilize an inverted pendulum? A dynamical system study

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We explore analytically the quantum dynamics of a point mass pendulum using the Heisenberg equation of motion. Choosing as variables the mean position of the pendulum, a suitably defined generalised variance and a generalised skewness, we set up a dynamical system which reproduces the correct limits of simple harmonic oscillator like and free rotor like behaviour. We then find the unexpected result that the quantum pendulum released from and near the inverted position executes oscillatory motion around the classically unstable position. The behaviour of the dynamical system for the quantum pendulum follows closely the behaviour of the Kapitza pendulum where the point of support is vibrated vertically with a frequency higher than the critical value needed to stabilize the inverted position. A somewhat similar phenomenon has recently been observed in the non equilibrium dynamics of a spin - 1 Bose-Einstein Condensate.

The quantum dynamics of a point mass pendulum has been most exhaustively treated by Leibscher and Schmidt [1]. Prior to that certain specific features of the dynamics was discussed in Ref. [2], [3]. Cook and Zaidins [2] estimated the time and fall of a pendulum due to the quantum fluctuations. Doncheski and Robinett [3] discussed in detail the limiting cases of a simple harmonic oscillator and the free rotor and focussed on the issues of the wave packet revival.

In a recent communication, Gerving et al. [4] have studied the non equilibrium dynamics of a spin-1 Bose-Einstein Condensate initialized in an unstable state which is analogous in the mean field limit to the exactly inverted pendulum. They have measured the evolution of this state along a separation caused by quantum fluctuations. Subsequently in a thesis submitted to the physics department of Georgia Institute of Technology, Gerving [5] presents in chapter 7 a numerical calculation of the motion of a wave packet which is initially centred around the unstable equilibrium point of the pendulum. Surprisingly, the mean angular displacement of the quantum pendulum shows oscillation around the unstable equilibrium point. The variance has oscillatory behaviour as well and there is a

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marked skewness which also oscillates in time. Clearly the quantum fluctuations are stabilizing the unstable fixed point of the classical pendulum. This is reminiscent of the Kapitza pendulum \[6\] where a high frequency vertical vibration of the point of support stabilizes the unstable position. In this work, we use the average angular displacement, a suitably defined generalised variance and a generalised skewness as the dynamical variables to set up a dynamical system approach which clearly establishes the stabilization of the classical unstable point by angular fluctuations.

The pendulum is described by the angular variable \( \theta \) and the conjugate momentum \( p_\theta \). We start with Heisenberg equations of motion for any operator \( O \) and take an expectation value in any quantum state \( \psi(\theta, t) \) to write

\[
\dot{\hat{O}} = \frac{i}{\hbar} [\hat{O}, H] + \frac{i}{\hbar} \frac{\partial \hat{O}}{\partial t} \tag{1}
\]

where,

\[
H = \frac{p_\theta^2}{2m} + m\omega^2 t^2 (1 - \cos \theta) \tag{2}
\]

and

\[
\langle \hat{O} \rangle = \int_{-\pi}^{\pi} \psi^*(\theta, t) \hat{O} \psi(\theta, t) d\theta \tag{3}
\]

We have set the mass \( m \) and the length of the pendulum to unity and will restore them at the end of the calculation. We will denote the average angular displacement of the pendulum for a quantum state to be \( \phi \) (i.e \( \phi = \langle \theta \rangle = \int_{-\pi}^{\pi} \psi^*(\theta, t) \hat{\theta} \psi(\theta, t) d\theta \) and the two applications of Eq. (1) leads to

\[
\frac{d^2 \phi}{dt^2} + \omega^2 \langle \sin \theta \rangle = 0 \tag{4}
\]

As is obvious the mean position does not follow the classical trajectory. Writing

\[
\langle \sin \theta \rangle = \langle \sin(\theta - \phi + \phi) \rangle \\
= \langle \cos(\theta - \phi) \rangle \sin \phi + \langle \sin(\theta - \phi) \rangle \cos \phi \tag{5}
\]

we find,

\[
\frac{d^2 \phi}{dt^2} = -\omega^2 \sin \phi + \omega^2 \langle 1 - \cos(\theta - \phi) \rangle \sin \phi \\
- \omega^2 \langle \sin(\theta - \phi) \rangle \cos \phi \tag{6}
\]

The last two terms on the left hand side represent the effect of quantum fluctuations on the dynamics of the mean angular displacement. We define a generalised variance

\[
V = \langle 1 - \cos(\theta - \phi) \rangle \tag{7}
\]

and a generalised skewness

\[
S = \langle \sin(\theta - \phi) \rangle \tag{8}
\]

For small fluctuations about the mean, \( V \simeq \frac{\langle (\theta - \phi)^2 \rangle}{2} \) and \( S \simeq -\frac{1}{6} \langle (\theta - \phi)^3 \rangle \), which are the usual definitions of variance and skewness apart from the factors of 1/2 and 1/6. For discussion
of quantum dynamics in non periodic situations using the usual variance \((\langle \theta^2 \rangle - \langle \theta \rangle^2)\) and subsequent Gaussian approximations, one should consult Ref. \[9\], \[10\], \[11\] and \[12\]. We need to find the dynamics of \(V\) and \(S\) without making any small angle approximation. This calls for repeated applications of Eq. \((6)\). The issue to be settled is that of the appearance of higher moments. We have used a factorizing scheme, where any correlations of the form \(\langle [1 - \cos(\theta - \phi)]^2 \rangle\), \(\langle \sin(\theta - \phi)[1 - \cos(\theta - \phi)] \rangle\), \(\langle p^2(1 - \cos(\theta - \phi)) \rangle\) has been replaced by \(V^2\), \(SV, p^2 V\) etc. and care has been taken to ensure that the simple harmonic oscillator limit is correct. Since the Hamiltonian is a constant of motion, we use,}

\[
e = \frac{\langle p^2 \rangle}{2} + \omega^2 \langle 1 - \cos \theta \rangle
\]

\[
= \frac{\langle p^2 \rangle}{2} + \omega^2 (1 - (1 - V) \cos \phi + S \sin \phi)
\]

as a parameter of the problem. Long but straightforward algebra now leads to

\[
\frac{d^2 V}{dt^2} = (2e - 2\omega^2 - \frac{\hbar^2}{4m^2 l^4})(1 - V)
\]

\[
+ \omega^2 (2 - 6V + 3V^2) \cos \phi
\]

\[
- 2\omega^2 S(1 - V) \sin \phi - 3\omega^2 S \sin \phi
\]

\[
+ \omega^2 S^2 \cos \phi - 2\dot{\phi} \dot{S} - (1 - V) \dot{\phi}^2
\]

\[
(10)
\]

Our dynamical system comprises of Eqs. \((6)\), \((10)\), and \((11)\). We should point out that in the limit of very low energies i.e. \(e/\omega^2 << 1\), the system behaves like a simple pendulum and for \(\omega \to 0\), it is like a free rotor.

We now look at the fixed points of our dynamical system. There are three of them.

\(A\) \(\phi = S = 0, V = V_o\).

The value of \(V_o\) is found to be,

\[
6\omega^2 V_o = 2e + 4\omega^2 - \frac{\hbar^2}{4m^2 l^4}
\]

\[
\pm \left[ (2e + 4\omega^2 - \frac{\hbar^2}{4m^2 l^4})^2
\right.
\]

\[
- 12\omega^2 \left( 2e - \frac{\hbar^2}{4m^2 l^4} \right)
\]

\[
1/2
\]

\[
(12)
\]

\(B\) \(\phi = \pi, S = 0, V = V_o\).

The value of \(V_o\) is found to be,

\[
6\omega^2 V_o = 8\omega^2 - 2e + \frac{\hbar^2}{4m^2 l^4}
\]

\[
\pm \left[ (2e - 8\omega^2 - \frac{\hbar^2}{4m^2 l^4})^2
\right.
\]

\[
+ 12\omega^2 \left( 2e - 4\omega^2 - \frac{\hbar^2}{4m^2 l^4} \right)
\]

\[
1/2
\]

\[
(13)
\]

\(C\) The third fixed point is found to be,
$\phi = \phi^*, V = V_o^*, S = S^*$  \hspace{1cm} (14)

$\phi^* \simeq \frac{\pi}{2}, \overline{V_o}^* \simeq 1, S^* < 0$

We see from Eqs. (12) and (13) that there are two possible values of $V_o$ and $\overline{V_o}$. The choice of the relevant value is made by analysing the stability properties of the fixed points. Linearising about the fixed point $A$, we get,

\begin{align*}
\frac{d^2 \delta \phi}{dt^2} &= -\omega^2 (1 - V_o) \delta \phi - \omega^2 \delta S \hspace{1cm} (15a) \\
\frac{d^2 \delta V}{dt^2} &= -\left(\frac{2e}{m} + 4\omega^2 - 6\omega^2 V_o \right) \delta V - \frac{h^2}{4m^2l^4} \delta V \hspace{1cm} (15b) \\
\frac{d^2 \delta S}{dt^2} &= -\left(\frac{2e}{m} - 2\omega^2 + 5\omega^2 (1 - V_o) \right) \delta S - \frac{h^2}{4m^2l^4} \delta S \hspace{1cm} (15c)
\end{align*}

For $\delta V$ to execute small oscillations about $V_o$ as a mark of stability, it is essential that we choose the negative sign in Eq. (12). An identical argument for the fixed point $(B)$ reveals that the positive sign is relevant for Eq. (13).

We want to check that our dynamical system does reproduce the known quantum dynamics in the limits of $e \gg \omega^2$ and $e \ll \omega^2$. For $e \ll \omega^2$, we have a simple harmonic oscillator. The fixed point variance $V_o$ has the value $\frac{e}{2\omega^2}$. In terms of $\Delta^2 = \langle \theta^2 \rangle - \langle \theta \rangle^2$, we have the variable $V \simeq \Delta^2$. In the $\omega \gg e$ limit, the dynamics of $V$ is given by $V = V_o + \delta V$, where $\delta V$ satisfies

$$\delta \ddot{V} + 4\omega^2 \delta V = 0 \hspace{1cm} (16)$$

and hence in terms of the quantity $\Delta^2$, we have,

$$\Delta^2 = \frac{e}{\omega^2} + A \cos 2\omega t + B \sin 2\omega t \hspace{1cm} (17)$$

If we take an initial Gaussian wave packet having width $\Delta_o$, i.e. $\psi(x, t = 0) = \frac{1}{\sqrt{\pi \Delta_o^2}} \exp \frac{- (x-a)^2}{2\Delta_o^2}$, then $\frac{\Delta}{\Delta_o} = \frac{h^2}{4\Delta_o^2 \omega^2} + \frac{\Delta^2}{4}$. With such a packet $\frac{d\Delta^2}{dt} = 0$ at $t = 0$ which makes $B = 0$. Hence $A + \frac{e}{\omega^2} = \frac{\Delta^2}{2}$ and Eq. (16) becomes

$$\Delta^2 = \frac{e}{\omega^2} + \left(\frac{\Delta^2}{2} - \frac{e}{\omega^2}\right) \cos 2\omega t \hspace{1cm} (18)$$

If we choose $\Delta^4_o = \frac{h^2}{\omega^2}$ (in $m = 1$ units), we find that $\Delta^2$ remains fixed in time with the initial width. This is exactly as it should be since $\Delta^4_o = \frac{h}{\omega}$ corresponds to the coherent state. Hence the dynamics of the large $\omega$ limit correctly reproduces the essential feature in that limit. This is shown in Fig. 1. For the numerics shown, the term $\frac{h^2}{4m^2l^4}$ is a small quantity in comparison with energy, $e$, hence defining it as $\epsilon$ and choosing it to be $1/16$, we proceed with the numerics in the harmonic oscillator limit and all subsequent plots.

We now turn to the free rotor limit, where $\omega \to 0$. From Eq. (11) this leads to, as expected,
$B = 0$, and if $V = V_o$ at $t = 0$, then,

$$V = V_o + (1 - V_o)(1 - \cos(2e' t^{0.5})) \quad (21)$$

If the initial width $V_o$ is small (large energy), then clearly for times $t << \sqrt{\frac{1}{2\omega}}$, $V = V_o + \frac{(1-V_o)(2e')}{2} t \simeq V_o + \frac{h^2}{4 V_o}$, since $e' \simeq e = \frac{\langle p \rangle}{2}$ which is the standard free particle (no periodicity) limit. Fig. (2) shows the corresponding plots of Eqs. (6), (10) and (11) in the free particle limit. From Fig. (2a), one sees that $\phi(t)$, with an initial momentum, increases linearly with time while $V(t)$ oscillates about 1 and $S(t)$ shows oscillation about zero. The numerical solution of Eqs. (6), (10) and (11) show the expected behaviour in Fig. 2.

Confident that the system works in the limiting situations, we now show the numerical results of Eqs. (6), (10), and (11) about the fixed points $(A)$ and $(B)$ for an intermediate value of $\frac{\omega}{e'}$. Fig. (3) shows the results for mean initial angular displacement of $\pi/6$ radians. The initial variance is given near the fixed point Eq. (12), the negative value of $V_o$ being the relevant one here, and a small initial skewness. The derivatives of $\phi$, $V$ and $S$ are specified as zero at $t = 0$.

\[
V = 1 + A \cos(2e')^{0.5} t + B \sin(2e')^{0.5} t \quad (20)
\]

Once again for an initially real wave packet,
(a) Plot of \( \phi(t) \) with \( \phi(0) = 0.1 \) and initial momentum \( \dot{\phi}(0) = 0.1 \)

(b) \( V(t) \) with \( V(0) = 1 \) and \( \dot{V}(0) = 0 \). \( V(t) \) shows small fluctuations about its fixed point value of 1.

(c) \( S(t) \) with \( S(0) = 0.01 \) and \( \dot{S}(0) = 0 \). \( S(t) \) oscillates about zero with amplitude 0.01

FIG. 2: Numerics for \( \phi(t) \), \( V(t) \), \( S(t) \) with \( \epsilon = 1.5 \), \( \omega = 0 \) and initial momentum \( \dot{\phi}(0) = 0.1 \)

(a) Plot of \( \phi(t) \) with \( \phi(0) = \pi/6 \) and \( \dot{\phi}(0) = 0 \)

(b) \( V(t) \) with \( V(0) = 0.55 \) and \( \dot{V}(0) = 0 \). The dashed line is the fixed point \( V_o = 0.55813 \).

(c) \( S(t) \) with \( S(0) = 0.1 \) and \( \dot{S}(0) = 0 \)

FIG. 3: Numerics for \( \phi(t) \), \( V(t) \) and \( S(t) \) about the fixed point \( A \) with \( \epsilon = 1.5 \) and \( \omega = 1 \).
We now turn to the unstable fixed point of the classical pendulum. In our case this is the fixed point $\phi = \pi$, $V = V_o$ and $S = 0$ with the positive sign in Eq. (13) being the relevant one. The dynamics is stable (i.e. periodic around $V_o$) for this choice of sign in Eq. (10) as can be found by performing stability analysis with Eqs. (6) and (11). For $\phi = \pi + \delta \phi$ in Eq. (6) and $S = \delta S$ in eq. (11), we have,

$$\frac{d^2 \delta \phi}{dt^2} = \omega^2 (1 - V_o) \delta \phi + \omega^2 \delta S$$

(22)

$$\frac{d^2 \delta S}{dt^2} = - \left( \frac{2e}{m} - 2\omega^2 - \frac{\hbar^2}{4m^2 l^4} + 5\omega^2 (V_o - 1) \right)$$

(23)

The stability matrix has a negative trace and a positive determinant which ensures that the fixed point is a centre and the dynamics about the vertical i.e. upside down position is oscillatory. In Figs. (4) and (5) we show the existence of oscillation about the vertical position. This is the unexpected feature of the quantum pendulum. Fig. (4) describes the nature of $\phi(t), V(t)$ and $S(t)$ with $\phi(0) = 2.8$. Fig. (5) shows the nature of $\phi(t)$ when its released on either sides of $\pi$ and one very close to $\pi$.

A linear stability around the fixed position $C$ of Eq. (14) shows the fixed point to be unstable. Thus the fixed point structure representing the quantum pendulum has an identical structure to that of the Kapitza pendulum which we now recall.

The Kapitza pendulum has its point of sup-
port vibrated at a high frequency \( \Omega \), so that the equation of motion is given by,

\[
\ddot{\theta} + \omega^2 (1 + \epsilon \cos \Omega t) \sin \theta = 0
\]  

(24)

where \( \omega = \sqrt{\frac{g}{l}} \), is the natural frequency of the pendulum. The dynamics can be split into a slow (frequency of \( O(\omega) \)) and a fast (frequency of \( O(\Omega) \)) as \( \theta = \theta_s + \theta_f \), with the high frequency oscillation taken to be a small perturbation around the primary solution \( \theta_s \). For \( \theta_f \ll \theta_s \), \( \ddot{\theta}_s + \omega^2 \theta_s = 0 \) is the primary dynamics. The fast variation satisfies, to the lowest order, the dynamics

\[
\ddot{\theta}_f + \omega^2 \cos \theta_s \theta_f = -\epsilon \omega^2 \cos \Omega t \sin \theta_s,
\]  

(25)

leading to the approximate solution \( \theta_f \approx \frac{\epsilon \omega^2 \Omega^2}{4 \Omega^2} \sin \theta_s \). Inserting this \( \theta_f \) back into Eq. (25), leads to the \( O(\epsilon^2) \) dynamics of \( \theta_s \) as,

\[
\ddot{\theta}_s + \omega^2 \sin \theta_s + \frac{\epsilon^2 \omega^4}{4 \Omega^2} \sin 2 \theta_s = 0.
\]  

(26)

The effective potential for this dynamics is,

\[
V_{\text{eff}} = -\cos \theta_s - \frac{\epsilon^2 \omega^4}{4 \Omega^2} \cos 2 \theta_s.
\]  

(27)

The fixed point of Eq. (27) for \( \frac{\epsilon^2 \omega^2}{4 \Omega^2} > 1 \) are clearly \( \theta_s = 0, \theta_s = \pi \) and \( \cos \theta_s = -\frac{\Omega^2}{\omega^4} \). The stability can be understood from the extrema of \( V_{\text{eff}} \). The extremum \( \theta_s = 0 \) is clearly a minimum. The extremum \( \theta_s = \pi \) is a minimum for \( \frac{\epsilon^2 \omega^2}{4 \Omega^2} > 1 \) which corresponds to the stabilization of the inverted position. In this situation,
the third fixed point $\theta_s = \cos^{-1} \left( -\frac{\Omega^2}{\omega^2} \right)$ exists and is easily seen to be unstable. This is a completely analogous situation to the quantum pendulum discussed before.

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