Dynamic versus thermodynamic approach to non-canonical equilibrium

Mauro Bologna\textsuperscript{1}, Michele Campisi\textsuperscript{2}, Paolo Grigolini\textsuperscript{1,2,3}

\textsuperscript{1}Center for Nonlinear Science, University of North Texas, P.O. Box 305370, Denton, Texas 76203-5370

\textsuperscript{2}Dipartimento di Fisica dell’Università di Pisa and INFM

Piazza Torricelli 2, 56127 Pisa, Italy

\textsuperscript{3}Istituto di Biofisica del Consiglio Nazionale delle Ricerche, Area della Ricerca di Pisa, Via Alfieri 1, San Cataldo, 56010, Ghezzano-Pisa, Italy

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Abstract

We study the dynamic and thermodynamic origin of non-canonical equilibria, and we discuss their connection with the generalized central limit theorem and the micro-canonical Boltzmann principle. We reach the conclusion that the zeroth law of thermodynamics and the Boltzmann principle are fulfilled thanks to an apparent fault turned into a benefit: the dynamic approach can only produce a truncated form of inverse power law equilibrium.

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I. INTRODUCTION

This paper aims at establishing the most convenient theoretical framework accounting for the emergence of non-canonical form of equilibrium. As pointed out by the authors of Ref. [1], one of the big merits of the non-extensive thermodynamics of Tsallis [2] is that of making popular the discussion of non-canonical equilibria that a few years earlier might have caused the severe criticism of the referees [1]. We address the same problem from within the dynamic perspective of Ref. [3] and we reach conclusions that support theoretically the existence a form of non-canonical equilibrium, in apparent accordance with the reasons of the success of non-extensive thermodynamics [4]. Actually, as we shall see, our conclusions significantly depart from the tenets of non-extensive thermodynamics à la Tsallis, and restate the importance of the ordinary Boltzmann principle [5–7], in accordance with the point of view of Gross, illustrated in one of the papers of these Proceedings [8].

II. JAYNES APPROACHES TO LÉVY PROCESSES

It has been recently pointed out [9] that the adoption of the method of entropy maximization à la Jaynes [10,11], with the Shannon entropy replaced by the Tsallis entropy, does not yield directly the Lévy distribution, but a probability density function $\Pi(x)$ that for reiterated application of the convolution yields the stable Lévy distribution. Here we show that, in principle, the method of entropy maximization yields exactly the Lévy distribution, if a deliberate use is made of the information nature of the Jaynes method [10,11].

Let us discuss this way of proceeding in the Gaussian case, first. Let us imagine that the problem to solve has to do with determining the most probable form of a square summable function $f(x)$ belonging to the real axis $[-\infty, \infty]$. We know that this function is symmetric around $x = 0$ and we know the first two non-vanishing terms of its Taylor series expansion about $x = 0$,

$$f(x) = \frac{c_0}{2\pi} - \frac{c_2}{2\pi}x^2 + \ldots \quad (1)$$
It is convenient to stress that in this case the information available to us is expressed by

\[ f(0) = \frac{c_0}{2\pi} \]  

and

\[ \frac{d^2}{dx^2} f(x)|_{x=0} = -\frac{c_2}{2\pi}. \]  

It is apparently difficult to proceed with the method of entropy maximization to guess
the unknown form of this function. However, this is made easy, if we move from the \( \text{x}\) -
representation to the \( \text{k}\) - or Fourier-representation of the function \( f \). We notice, in other
words, that the information available to us can be expressed under the form of moment
constraints if we quit the representation of \( f \) as a function of \( x \), and we focus on the Fourier
transform of \( f(x) \), denoted by \( \hat{f}(k) \). In fact, the knowledge of \( c_0 \) and \( c_2 \) can be used to set
the constraints

\[ \int dk \hat{f}(k) dk = c_0 \]  

and

\[ \int dk \hat{f}(k) k^2 dk = c_2. \]  

According to the principle of entropy maximization \cite{[12]}, we have to look for the maximum
of the Shannon entropy

\[ S[f] = - \int dk \hat{f}(k) ln \hat{f}(k), \]  

while taking into account the constraints of Eq. (4) and Eq.(5) by means of the Lagrange
multiplier method. This yields

\[ \hat{f}(k) = Ae^{-\tau k^2}, \]  

this resulting form being the Fourier transform of an ordinary Gaussian distribution.
The derivation of the Lévy distribution from the proper extension of these arguments is easy. We assume that the second piece of information available to us, rather than being expressed in terms of the second-order derivative of Eq. (3), is given by
\[ \frac{d^\alpha}{d|x|^\alpha} f(x)|_{x=0} = \frac{c_\alpha}{2\pi}, \]
while the form of the first piece of information of Eq. (2) is kept unchanged. Note that the symbol \( \frac{d^\alpha}{d|x|^\alpha} \) denotes the symmetric fractional derivative [13], defined by its action on the Fourier space, which reads:
\[ \mathcal{F}\left\{ \frac{d^\alpha}{d|x|^\alpha} f(x) \right\} = -|k|^\alpha \hat{f}(k). \]
This means that in the Fourier space the constraint on the second moment of Eq. (5) is now replaced by
\[ \int dk \hat{f}(k)|k|^\alpha dk = c_\alpha. \]
In this case the method of entropy maximization yields
\[ \hat{f}(k) \propto e^{-\tau|k|^\alpha}, \]
which is well known to be the Fourier transform of an \( \alpha \)-stable Lévy process [14].

In conclusion, we have solved the problem raised by Shlesinger and Montroll [14] about the derivation of Lévy processes from a maximum entropy formalism, without departing from the extensive form of Shannon entropy and without using strange logarithmic constraints, as proposed by these authors [14]. However, this interesting conclusion does not establish the thermodynamic character of Lévy processes, since the information approach in this case might depart from the Boltzmann principle [5–8].

**III. ON THE WEST-SESHADRI STATISTICS**

To set the dynamic approach in the proper perspective, we remind the reader about the deep connection between the Boltzmann principle and dynamics established by the authors
of Ref. [3]. An oscillator, with coordinate $x$ and velocity $v$ is coupled to a dynamic system, called *booster* to emphasize the fact that no thermodynamic property is already used, as it happens with the ordinary thermal baths. The Hamiltonian coupling is given by $\kappa x \xi$, $\xi$ being the coordinate of one of the particles of the booster, referred to as *doorway* variable. The booster is a dynamic system in a condition of strong chaos. We do not assign to it any thermodynamic property, but a given energy $E$ and we assume that the condition of strong chaos makes it reach the micro-canonical state. We write the Liouville equation of the whole system, oscillator plus bath, and we derive from this equation, with a projection method, the equation of motion of the oscillator. We prove that, under the condition of time scale separation, this reduced Liouville equation becomes equivalent to the Fokker-Planck equation, thereby leading to a canonical form of equilibrium. No use of thermodynamics has been done to reach this important conclusion, but only dynamic properties have been invoked [3]. At this stage we assume that the width of the resulting oscillator equilibrium distribution can be identified with the temperature $T$ of the booster, and we obtain [3]

$$k_B T = \left[ \frac{\partial}{\partial E} \ln W(E) \frac{\partial}{\partial E} \ln <\xi^2> + Re \Phi_\xi(\omega) \right]^{-1}, \quad (12)$$

where $W(E)$ denotes the volume of booster multidimensional surface with energy $E$ and $\Phi_\xi(\omega)$ is the Laplace transform of the correlation function of the doorway variable $\xi$, evaluated at $\omega$, the oscillator frequency. We know that $W(E)$ is proportional to $E^N$, $N$ being the number of degrees of freedom of the booster. Thus, in the thermodynamic limit only the first term within the square bracket of Eq.(12) survives, thereby recovering the ordinary form of Boltzmann principle. It is evident that this nice connection between dynamics and thermodynamics is made possible by the fact that, using the language of the advocates of non-extensive statistical mechanics [2,4], extensive conditions apply. This is so because of the time scale separation between system of interest and booster, resting on the fact that the function $\Phi_\xi(t)$ is integrable. Furthermore, the oscillator of interest is coupled to only one particle of the system. The coupling with all the particles of the booster would prevent us from recovering the ordinary form of Boltzmann principle. In this paper, we focus our attention
on the case where the former extensive condition is broken, and \( \lim_{t \to \infty} \Phi_t(t) = \text{const}/t^\beta \), with \( \beta < 1 \). We shall see that this dynamic condition generates a special kind of non-canonical equilibrium. To prepare the ground for this interesting form of non-canonical equilibrium, illustrated in Section II B, we have to discuss first the case of free diffusion.

A. Free Diffusion

We have now to address a crucial issue, that concerning the subtle difference between Lévy flight and Lévy walk. Let us imagine a random walker that walks according to the following prescription. At regular intervals of time, 0, \( T, 2T, \ldots \), we draw the random numbers \( \eta_1, \eta_2, \eta_3, \ldots \), playing the role of random velocities. These random numbers are characterized by the probability density \( P(\eta) \) given by

\[
P(\eta) = \frac{1}{2}(\mu - 1) \frac{W^{\mu-1}}{(W + |\eta|)^\mu},
\]

the factor of 1/2 taking into account the fact that the probability for the walker to make jumps in the positive direction is the same as that of making jumps in the negative direction. Note that our discussion rests on the special case where the first moment of this distribution is finite, while the second is infinite, namely,

\[
2 < \mu < 3.
\]

The reason for this choice is transparent. The fact that \( \mu < 3 \) ensures that the second moment of the distribution is infinite, thereby making it possible for us to depart from the Gaussian basin of attraction, which would correspond to the case of canonical distribution. On the other hand, we have to ensure that the first moment is finite, so as to establish, as we shall see, a connection with a satisfactory dynamic approach.

Let us consider the case where the random drawing is carried out a given number of times, \( n \). This means that the random walker, moving along an one-dimensional path, the \( x \)-axis, at “time” \( n \) is found in the position
\[ x(n) = (\eta_1 + \eta_2 + \ldots \eta_n)T. \]  

(15)

Let us imagine now that this process is repeated an infinitely large number of times, so as to build up a distribution density yielding the probability for us to find the random walker at the position \( x \) at time \( n \), \( p(x,n) \). According to the generalized central limit theorem [13], the Fourier transform, \( \hat{p}(k,n) \), in the limiting condition of very large \( n \)'s, gets the analytical form

\[ \hat{p}(k,n) = \exp(-b|k|^\alpha n), \]  

(16)

where

\[ \alpha \equiv \mu - 1. \]  

(17)

In the physical condition corresponding to Eq. (14), the diffusion coefficient \( b \) reads

\[ b = -W^{\mu-1} \sin \left( \frac{\pi \mu}{2} \right) \frac{\Gamma(3-\mu)}{(\mu-2)}. \]  

(18)

It is important to point out that the asymptotic form of Eq. (16) is reached after applying the convolution procedure for a finite number of times, \( n_{\text{crit}} \), which is fixed to be of the order of 10 [13]. Thus, we make the assumption of considering a coarse grained time scale \( \bar{n} \), where also the infinitesimal change \( d\bar{n} \) fits the important property \( d\bar{n} \gg n_{\text{crit}} \).

Let us now consider another random walk prescription. This has to do with drawing the random numbers \( \tau_i \)'s, with the probability density \( \psi(\tau) \) given by

\[ \psi(\tau) = (\mu - 1) \frac{T^{\mu-1}}{(T + \tau)^\mu}. \]  

(19)

This means that we can build up an infinite sequence of numbers, which are then used to make a random walker walk with the following rules. At time \( t = 0 \), when the first random number, \( \tau_1 \), is selected, we also toss a coin to decide whether the random walker has to move in the positive or in the negative direction. The random walker walks with a velocity of constant intensity \( W \). Thus, tossing a coin serves the purpose of establishing whether the velocity of the random walker is \( W \) (head) or \(-W\) (tail). This condition of uniform motion
lasts for a time interval of duration $\tau_1$. At the end of this condition of uniform motion, a
new number, $\tau_2$, is randomly drawn, and a new velocity direction is established by another
coin tossing. It is important to stress that the physical condition of Eq. (14) corresponds to
the non-vanishing mean time $<\tau>$, whose explicit expression is:

$$<\tau> = \frac{T}{\mu - 2}. \quad (20)$$

We denote as *event* the joint process of random drawing of a number and of coin tossing.
We consider a time scale characterized by the property $t >> <\tau>$. It is evident that the
number of events that occurred prior to a given time $t$ is given by

$$n = \frac{t}{<\tau>}. \quad (21)$$

Note that we set also the condition that $n > n_{crit}$, thereby implying that rather than the
absolute time $t$ we are considering the coarse time $\bar{t}$ defined by

$$\bar{t} = \bar{n} <\tau>. \quad (22)$$

If we adopt this coarse-grained time scale the position occupied by the particle
according to Eq. (15) becomes indistinguishable from the random walker position prescribed
by

$$x(\bar{t}) = \xi_1 \tau_1 + \xi_2 \tau_2 + \ldots \xi_n \tau_n, \quad (23)$$

where $\xi_i$ denotes a stochastic variable with only two possible values, either $W$ or $-W$, a
variable that keeps the same sign for the whole time duration of a time interval between two
nearest-neighbor events. We can also connect $P(\eta)$ to $\psi(\tau)$ as follows [9]

$$P(\eta) = \frac{T}{2W} \psi\left(\frac{|\eta|T}{W}\right). \quad (24)$$

As pointed out in Ref. [3], this has the effect of making Lévy diffusion compatible with a
dynamic and Hamiltonian derivation.
B. Fluctuation-dissipation without a finite time scale

In 1982 West and Seshadri [16] made an interesting proposal to derive a form of non-canonical equilibrium. This is described by the Langevin equation

\[ \frac{dx}{dt} = -\gamma x(t) + \eta_L(t). \]  

(25)

Note that West and Seshadri [16] assumed the variable \( \eta \) to be a Lévy stochastic process. This means that the continuous time \( t \) of their treatment must be identified with the coarse-grained time of Section IIA. This is an important aspect that has fundamental consequences on the dynamic realization of Lévy processes, and, consequently, on our dynamic approach to non-canonical equilibrium. The equilibrium distribution emerging from Eq. (25) can be easily evaluated by noticing that the probability distribution \( p(x, t) \) is driven by the following equation of motion

\[ \frac{d}{dt}p(x, t) = [\gamma \frac{d}{dx} + \frac{d^\alpha}{d|x|^\alpha}]p(x, t). \]  

(26)

This is the density picture equivalent to Eq. (25). Both equations afford an attractive picture of dynamics driven by both dissipation and fluctuation. In the ordinary case the fluctuation process is described by a second-order differential operator. The anomalous case here under discussion forces us to replace the second-order derivative with a fractional differential operator. The Fourier transform of Eq. (26) obeys the time evolution equation

\[ \frac{\partial}{\partial t} \hat{p}(k, t) = -b|k|^{\alpha} \hat{p}(k, t) - \gamma k \frac{\partial}{\partial k} \hat{p}(k, t). \]  

(27)

This equation yields the equilibrium distribution

\[ \hat{p}(k, \infty) = \exp \left( -\frac{b}{\alpha \gamma} |k|^{\alpha} \right), \]  

(28)

which we refer to as West-Seshadri (WS) non-canonical equilibrium. It is important to point out that this form of equilibrium distribution has slow tails inversely proportional to \( |x|^\mu \). In other words, in the case \( 2 < \mu < 3 \) the equilibrium distribution keeps unchanged the power law nature of the original fluctuation.
At this stage we have to fit the request of making our treatment compatible with a Hamiltonian derivation \[3\]. As pointed out in Ref. \[17\], in accordance with the Hamiltonian formulation advocated by Zaslavsky \[18\], this condition is fulfilled by replacing Eq.(25) with
\[
\frac{dx}{dt} = -\gamma x(t) + \xi(t),
\]  
where \(\xi(t)\) is the stochastic process described in Section IIA. This has apparently the effect of making the WS statistics compatible with a Hamiltonian derivation. However, Eq.(29) yields an equilibrium that is not exactly equivalent to the predictions of the WS statistics. In fact, it is straightforward to prove that the trajectories moving always in the same direction, namely, those exploring the largest distances from the origin, cannot overcome, in the positive direction, the distance \(x = x_{\text{max}} = \frac{W}{\gamma}\) and, in the negative direction, the distance \(x = x_{\text{min}} = -\frac{W}{\gamma}\). In other words, in the long-time limit, \(\frac{1}{\gamma} \gg T\), we realize a truncated Lévy equilibrium. In Section V we shall see that this property becomes the key ingredient to ensure thermalization between a Gauss and a Lévy system.

IV. THE MICRO-CANONICAL BOLTZMANN PRINCIPLE AND THE GENERALIZED CENTRAL LIMIT THEOREM

The central idea of this section is borrowed from Rajagopal and Abe \[19\]. In fact, these authors made the remarkable observation that according to Khinchin \[20\] ordinary statistical mechanics rests on the central limit theorem thereby making it reasonable to expect that the non-extensive statistical mechanics is based on the Generalized Central Limit Theorem (GCLT) \[15\]. We find this observation very appealing and we want to discuss in this section its consequences from within our perspective that, as mentioned in Section I, is also based on the micro-canonical Boltzmann principle, in accordance with other authors \[5-7\].

Let us assume that we know the energy distribution of a small subsystem of a macroscopic system that is assumed to obey the micro-canonical Boltzmann principle. The small subsystem is denoted, for the sake of simplicity as particle. Let us denote by \(p(e)de\) the
probability that the particle energy \( e \) is found in the small interval \([e, e + de]\). We do not take position on the explicit form of this energy distribution. We only make the assumption that

\[
\lim_{e \to \infty} p(e) = \frac{\text{const}}{e^{\nu+1}},
\]

with \( 0 < \nu < 2 \). This is compatible with both Lévy and Tsallis statistics. Let us assume that the macroscopic system consists of \( N \) independent particles and let us define the energy per particle, \( \epsilon \):

\[
\epsilon = \frac{\sum_{j=1}^{N} e_j}{N}.
\]

Since the \( N \) particles are independent the ones from the others, we obtain for \( \epsilon \) the following expression

\[
P_N(\epsilon) = \int \prod [de_j p(e_j)] \delta \left( \epsilon - \frac{\sum_{j=1}^{N} e_j}{N} \right).
\]

The characteristic function of the distribution \( P_N(\epsilon) \), \( \hat{P}_N(k) \), is related to the characteristic functions of the single particle probability distribution \( p(e_j), \hat{p}(e_k) \), by

\[
\hat{P}_N(k) = \hat{p}^N \left( \frac{k}{N} \right).
\]

The key aspect of the search we are doing is the following one. We study the limiting case of \( N \to \infty \) for the purpose of assessing if the characteristic function becomes equivalent to the Fourier transform of a delta of Dirac. If this happens, then the non-canonical equilibrium under study is compatible with the micro-canonical principle. If it does not, the non-canonical equilibrium distribution is found to be incompatible with the micro-canonical principle.

We skip the discussion of the case \( \nu < 1 \), which is proved to be incompatible with the micro-canonical condition, and we focus our attention on the condition:

\[
1 < \nu < 2.
\]

In this case the first moment is finite and is denoted by the symbol \( a \), namely
\[ a \equiv < e > = \int e p(e)de, \quad (35) \]

and the Lévy-Gnedenko theorem affords an analytical expression for the asymptotic distribution of the variable

\[ u \equiv \frac{\sum_{k=1}^{N} e_k - Na}{N^{\frac{1}{\nu}}}. \quad (36) \]

We denote by \( \hat{L}_N(k) \) the characteristic function of the variable \( u \), and we obtain

\[
\lim_{N \to \infty} \hat{L}_N(k) = \exp \left[ ik\gamma - b|k|^{\nu} \left( 1 + \beta \frac{k}{|k|} \tan \left( \frac{\pi \nu}{2} \right) \right) \right],
\]

and

\[
\hat{L}_N(k) = e^{-ikaN^{\frac{1}{\nu}}} \hat{p}^N \left( \frac{k}{N^{\frac{1}{\nu}}} \right),
\]

thereby yielding

\[
\lim_{N \to \infty} \hat{p}^N \left( \frac{k}{N^{\frac{1}{\nu}}} \right) = \exp \left[ ik\gamma + ikaN^{1-\frac{1}{\nu}} - b|k|^{\nu} \left( 1 + i\beta \frac{k}{|k|} \tan \frac{\pi \nu}{2} \right) \right],
\]

and consequently:

\[
\lim_{N \to \infty} \hat{p}^N \left( \frac{k}{N} \right) = \exp \left[ ik\gamma N^{\frac{1}{\nu} - 1} + ika - b|k|^{\nu} N^{1-\nu} \left( 1 + i\beta \frac{k}{|k|} \tan \frac{\pi \nu}{2} \right) \right]. \quad (39)
\]

In this case we see that increasing \( N \) rather than a broader and broader distribution makes the right hand side of Eq.(39) identical to \( \exp(ika) \), namely the Fourier transform of a delta Dirac. Thus, this case is compatible with the micro-canonical equilibrium.

At this stage it would be straightforward to prove that the Tsallis non-canonical equilibrium is compatible with the Boltzmann principle provided that the entropic index \( q \) fulfills the condition \( 1 < q < 7/5 \). The WS statistics, on the contrary, would be incompatible with the Boltzmann principle. However, as we shall see in Section V, the dynamic approach to WS statistics, thanks to the fact that the long tails are truncated, fits both the Boltzmann principle and the zeroth law of thermodynamics at the same time.
V. THERMAL EQUILIBRIUM BETWEEN A NON-CANONICAL AND A CANONICAL SYSTEM AND CONCLUSIONS

All the moments of a truncated Lévy process are finite, thereby fitting the Khinchin prescriptions for an equilibrium distribution to be compatible with the Boltzmann principle. The same property makes this form of non-canonical equilibrium compatible with the zeroth principle of thermodynamics. In Ref. [1] it has been shown that the finite second moment of the non-canonical distribution can be evaluated analytically and its explicit expression is

\[
\langle x^2(\infty) \rangle = \xi^2 > T^{\mu-2} \exp[\gamma(\mu - 2)T] \frac{\gamma(3 - \mu, \gamma T)}{\Gamma(3 - \mu)},
\]

where \( \Gamma(\alpha, z) \) is the incomplete Gamma function. This conclusion can be turned into a benefit. We think in fact that it makes it possible to establish in a natural way the thermal equilibrium between a system with Lévy statistics and one with ordinary Gauss statistics. This naturally emerges from the theoretical approach established years ago in Ref. [21] to study the process of heat transfer from a hotter to a warmer system. Let us imagine that a Gaussian oscillator, with temperature \( T \), is weakly coupled to a truncated Lévy process, whose temperature has to be assessed using the Gauss system as a thermometer. The thermometer equilibrium is shown [21] to depend only on the second moment of the Lévy system, thereby making it possible to establish naturally the thermalization between the two systems, and also to measure the Lévy temperature by means of the Gauss thermometer. In conclusion, the adoption of the dynamic perspective of Ref. [3], extended to the case of a booster producing fluctuations with infinite correlation time, yields a non-canonical form of equilibrium, which is compatible with the Boltzmann principle and with the zeroth principle of thermodynamics.
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