A Theory of Dimension

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\textit{Dedicated to R.V. Kadison on the occasion of his seventieth birthday}

\textbf{Abstract.} In which a theory of dimension related to the Jones index and based on the notion of conjugation is developed. An elementary proof of the additivity and multiplicativity of the dimension is given and there is an associated trace. Applications are given to a class of endomorphisms of factors and to the theory of subfactors. An important role is played by a notion of amenability inspired by the work of Popa.

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1. Introduction

The starting point for this paper is the operation of conjugation on finite dimensional objects in a strict tensor $C^*$–category. The terminology derives from group theory where to every unitary representation of a group there corresponds a conjugate representation defined on the conjugate Hilbert space. It includes as a special case the notion of conjugate endomorphisms of a $C^*$–algebra which was effectively first introduced in [5], Thm. 3.3. If we look on a $C^*$–algebra as a $C^*$–category with a single object then an endomorphism becomes an endofunctor and the conjugate endomorphism is just the left (and indeed the right) adjoint endofunctor. In this sense, the concept is just a special case of the concept of adjoint functor. The notion of conjugate objects in a strict tensor $C^*$–category is also not really new either since it is implicit, for example, in the notion of dual object in [26].

When conjugates are understood in the sense of adjoint functors then the objects in question are necessarily finite dimensional in a sense which will be made precise. In the case of von Neumann algebras it corresponds to the finiteness of the index first introduced by Jones [11]. The relation between the index and the square of the dimension was first established in [14] whereas the concept of dimension goes back to [4] where it was referred to as the statistical dimension and defined using the permutation symmetry on the strict tensor $C^*$–category. This had the great merit that the existence of conjugates for finite dimensional objects could be established. However, although it was clear that the dimensions of objects with conjugates were independent of the choice of permutation symmetry it was not clear that the dimension could be defined in the absence of any permutation symmetry and hence made sense in any strict tensor $C^*$–category. Again there is a related notion of the rank of an object with a dual defined for symmetric tensor categories [26] whose definition again involves the permutation symmetry and which, in general, depends on the choice of that symmetry.

In Section 2, the properties of conjugation are developed for finite dimensional objects in a strict tensor $C^*$–category. In particular we show how there is a related notion of conjugation on arrows and associated left and right scalar products on the category. We introduce the notions of left and right inverses for objects and prove an analogue of the Pimsner–Popa inequality [21], already established in the particular context of [4].

In Section 3, the dimension is introduced in the special case that the tensor unit is irreducible and shown to be additive on direct sums and multiplicative on tensor products. It is therefore a Perron–Frobenius eigenvalue. It can also be characterized by a property of minimality related to the concept of minimal index. As a consequence a strict tensor $^*$–functor cannot increase dimensions. We introduce the Jones projections associated with conjugation and conclude that the range of the dimension has the same a priori restrictions as the square root of the Jones index. Associated with the dimension there is a canonical trace defined on the full subcategory of finite dimensional objects.

In Section 4, we briefly consider the status of conjugates and dimension in a braided tensor $C^*$–category showing that the braiding gives rise to a central element $\kappa$ and that the braiding is automatically compatible with conjugates.

In Section 5, we study model endomorphisms canonically associated with an object in a strict tensor $C^*$–category. The starting point is the graded $C^*$–algebra
\(\mathcal{O}_\rho\) associated with an object \(\rho\) in a strict tensor \(C^*\)-category introduced in [6]. When \(d(\rho) < \infty\), the canonical trace defines a trace on the grade zero subalgebra which hence extends to a unique gauge invariant state on \(\mathcal{O}_\rho\). This state is shown to be a KMS state with respect to gauge transformations. Taking a weak closure in the corresponding GNS–representation gives a von Neumann algebra \(\mathcal{M}_\rho\) associated intrinsically with \(\rho\). \(\mathcal{M}_\rho\), like \(\mathcal{O}_\rho\), comes equipped with a canonical endomorphism \(\hat{\rho}\). An important role is played here by a notion of amenability inspired by Popa’s notion of a strongly amenable inclusion of factors. Roughly speaking amenability means that if we pass from \(\rho\) to \(\hat{\rho}\) there are no new intertwining operators. In fact, the theory is developed in greater generality, replacing the standard left inverse which defines the canonical trace by an arbitrary faithful left inverse \(\varphi\) to get a von Neumann algebra \(\mathcal{M}_\varphi\) equipped with a canonical endomorphism. We illustrate our notion with examples.

In Section 6, we present applications to subfactors and the duality theory for finite dimensional Hopf algebras. We introduce the notion of abstract \(Q\)-system, a categorical version of the \(Q\)-systems introduced in [18]. We show that there is a 1–1 correspondence between isomorphism classes of amenable \(Q\)-systems and of standard AFD inclusions.

Section 7 has quite a different goal: the formalism of tensor \(C^*\)-categories is not sufficiently general to cover all interesting applications. Instead, we need to use 2–\(C^*\)-categories. This generalization presents no difficulties and has been deferred to Section 7 merely to avoid concepts that are superficially technical. Here we present examples of 2–\(C^*\)-categories to illustrate the range of the resulting theory. We then generalize the basic notions of conjugation and dimension, again providing some key examples to illustrate the utility of this approach. Finally, it is shown in an appendix how a 2–\(C^*\)-category can completed so that it has subobjects and (finite) direct sums.

This paper by no means exhausts the basic themes of conjugation and dimension so we would like to conclude this introduction with some comments on work in progress. In the first place, although, in defining dimension, we have restricted ourselves to the case that the endomorphisms of the tensor unit \(\iota\) reduce to the complex numbers, the definitions used and examples presented in [6] already establish the need for a more general notion. In the course of this work, it has on occasions proved irksome to have to add the hypothesis that a von Neumann algebra is a factor just to be able to talk about the dimension of an endomorphism. However, although a notion of dimension can in fact be developed in more generality on the basis of the ideas presented here, there may still be room for improvement so details will be deferred to a sequel. Again our notion of conjugation in terms of adjoint functors is restricted to the finite dimensional case despite the fact that the notion of conjugate representation makes sense in infinite dimensions and the fact that conjugates always exist for normal endomorphisms of von Neumann algebras [15]. We propose to give a general definition covering the infinite dimensional case elsewhere. Although the results on inclusions we have obtained in Section 6 using the canonical endomorphism are relatively complete in themselves, it would still be of interest to treat the inclusion directly as a \(\sigma\)-unit in an appropriate 2–\(C^*\)-category. It would furthermore be of interest to look at duality at the level of abstract \(Q\)-systems.
2. Conjugation in Tensor $C^*$–Categories

In this section we will work with strict tensor $C^*$–categories, called strict monoidal $C^*$–categories in [6], §1, where the formal definition may be found. The change in name is justified since the set of arrows between two objects is automatically a linear space and opportune because of the popularity and familiarity of the term tensor as opposed to monoidal. Briefly, such a category $\mathcal{T}$ with objects denoted by $\rho, \sigma, \tau,$ etc. has a space of arrows between $\rho$ and $\sigma$, $(\rho, \sigma)$, which is a complex Banach space. Composition of arrows denoted by $\circ$ is bilinear and the adjoint $*$ is an involutive contravariant functor acting as the identity on objects. The norm satisfies the $C^*$–property: $\|R^* \circ R\| = \|R\|^2$. This makes $\mathcal{T}$ into a $C^*$–category. To be a strict tensor $C^*$–category, we need an associative bilinear bifunctor $\otimes: \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ with a unit $\iota$ and commuting with $\ast$. To save space and render expressions more legible, the functor $\otimes$ applied to a pair of objects $\rho$ and $\sigma$ will be denoted simply $\rho \sigma$ rather than $\rho \otimes \sigma$ as would be more conventional.

We begin by defining the notion of conjugation for finite dimensional objects of an arbitrary strict tensor $C^*$–category $\mathcal{T}$. This will eventually lead us to a notion of dimension related to the Jones index. We define a full subcategory $\mathcal{T}_f$ of $\mathcal{T}$ as follows: an object $\rho$ of $\mathcal{T}$ lies in $\mathcal{T}_f$ if there is an object $\bar{\rho}$ of $\mathcal{T}$ and $R \in (\iota, \bar{\rho}\rho)$ and $\bar{R} \in (\iota, \rho \bar{\rho})$ such that

$$\bar{R}^* \otimes 1_\rho \circ 1_\rho \otimes R = 1_\rho; \quad R^* \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{R} = 1_{\bar{\rho}}.$$  

Here $\iota$ denotes the tensor unit in $\mathcal{T}$. The definition is symmetric in $\rho$ and $\bar{\rho}$ so that $\bar{\rho}$ is also an object of $\mathcal{T}_f$. The above equations will be referred to as the conjugate equations and $\bar{\rho}$ will be said to be a conjugate for $\rho$. In essence, the above condition means that $\rho$ is finite dimensional. However, one should bear in mind that $\rho$ may fail to be an object of $\mathcal{T}_f$ because its conjugate, although existing in a larger $C^*$–category containing $\mathcal{T}$ as a full tensor $\ast$–subcategory, fails to exist in $\mathcal{T}$. For this reason, it is advisable to choose $\mathcal{T}$ from the outset so that it contains all the necessary objects. In particular, it is wise to take $\mathcal{T}$ to have subobjects and (finite) direct sums.

Actually, as was pointed out in [5], the basic mathematical concept underlying the notion of conjugation is that of adjoint functor. If $\rho$ and $\bar{\rho}$ are objects in a tensor $C^*$–category, then the functor of tensoring on the left by $1_\rho$ has the functor of tensoring on the left by $1_{\bar{\rho}}$ as a right adjoint precisely when we can find $R$ and $\bar{R}$ as above. The condition for adjunction takes on such a simple form because a tensor $C^*$–category has a tensor unit and a $\ast$–operation and because of the particular nature of the functors involved. The symmetry between $\rho$ and $\bar{\rho}$ shows that $1_{\bar{\rho}} \otimes$ is also a left adjoint for $1_\rho \otimes$. The same conditions are also equivalent to $\otimes 1_{\bar{\rho}}$ being a left (or right) adjoint to $\otimes 1_\rho$. From the better known alternative equivalent definition of adjoint functors it follows, as can easily be verified, that $\bar{\rho}$ is a conjugate for $\rho$ if and only if there are isomorphisms $i_{\sigma, \tau}: (\rho \sigma, \tau) \to (\sigma, \bar{\rho} \tau)$ natural in $\sigma$ and $\tau$.

To make the basic construction explicit, we state it in the form of a lemma:

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1It is convenient to consider only strict tensor categories because the generality achieved by dropping this restriction is minimal [19].
Lemma 2.1. If $R$, $\tilde{R}$ satisfy the conjugate equations for $\rho$, then for any $\rho_1$ and $\rho_2$, the maps

\[
S \mapsto 1_\rho \otimes S \circ R \otimes 1_{\rho_1}, \quad S \in (\rho_1, \rho_2),
\]
\[
S' \mapsto \tilde{R}^* \otimes 1_{\rho_2} \circ 1_\rho \otimes S', \quad S' \in (\rho_1, \tilde{\rho}_2).
\]

are inverses of one another. Similarly,

\[
T \mapsto T \otimes 1_\rho \circ 1_{\rho_1} \otimes \tilde{R}, \quad T \in (\rho_1, \rho_2),
\]
\[
T' \mapsto 1_{\rho_2} \otimes R^* \circ T' \otimes 1_\rho, \quad T' \in (\rho_1, \rho_2, \tilde{\rho})
\]

are inverses of one another.

Proof.

\[1_\rho \otimes \tilde{R}^* \otimes 1_{\rho_2} \circ 1_{\tilde{\rho}_2} \otimes S' \circ R \otimes 1_{\rho_1} = 1_\rho \otimes \tilde{R}^* \otimes 1_{\rho_2} \circ R \otimes 1_{\tilde{\rho}_2} \circ S' = 1_{\tilde{\rho}_2} \circ S' = S',\]
\[\tilde{R}^* \otimes 1_{\rho_2} \circ 1_{\tilde{\rho}_2} \otimes S \circ 1_\rho \otimes R \otimes 1_{\rho_1} = S \circ \tilde{R}^* \otimes 1_{\rho_2} \circ 1_\rho \otimes R \otimes 1_{\rho_1} = S \circ 1_{\rho_1} = S.\]

In the special case of group representations, this is in essence the Frobenius Reciprocity Theorem.

From the above lemma it follows in particular that any $R_1 \in (\iota, \rho_1, \rho)$ is of the form $R_1 = X \otimes 1_\rho \circ R$ for a unique $X \in (\rho_1, \rho)$ and $R_1$ also defines a conjugate for $\rho$ if and only if $X$ is invertible. The corresponding $\tilde{R}_1 \in (\iota, \rho_1, \rho)$ is given by $\tilde{R}_1 = 1_\rho \otimes X^{*^{-1}} \circ \tilde{R}$. Similarly, any $R' \in (\iota, \rho', \rho')$ is of the form $R' = 1_\rho \otimes Y \circ R$ for a unique $Y \in (\rho, \rho')$ and $R'$ defines a conjugate for $\rho'$ if and only if $Y$ is invertible. The corresponding $\tilde{R}' \in (\iota, \rho', \rho')$ is given by $\tilde{R}' = Y^{*^{-1}} \otimes 1_\rho \circ \tilde{R}$. Another obvious consequence of the above isomorphisms is that the spaces $(\rho, \rho^\prime), (\rho, \tilde{\rho})$ and $(\tilde{\rho}, \tilde{\rho})$ are isomorphic. In particular $\rho$ is irreducible if and only if $\tilde{\rho}$ is irreducible.

We now show that slightly weaker conditions suffice to define a conjugate.

Lemma 2.2. Two objects $\rho$ and $\tilde{\rho}$ in a strict tensor $\mathcal{C}^*$-category $\mathcal{T}$ are conjugate if and only if there are $R \in (\iota, \rho, \rho')$ and $\tilde{R} \in (\iota, \rho, \tilde{\rho})$ such that $\tilde{R}^* \otimes 1_\rho \circ 1_\rho \otimes R$ is invertible and the positive linear mapping $\omega$ on $(\rho, \rho')$ → $(\iota, \iota)$ defined by

\[\omega(T) = R^* \circ T \otimes 1_\rho \circ R, \quad T \in (\rho, \rho'),\]

is faithful.

Proof. If $R$ defines a conjugate for $\rho$ then $T \in (\rho, \tilde{\rho})$ and $T \otimes 1_\rho \circ R = 0$ imply $T = 0$ so that $\omega$ is faithful. Conversely, given $R$ and $\tilde{R}$ as in the statement of the lemma, let $X = \tilde{R}^* \otimes 1_\rho \circ 1_\rho \otimes R \in (\rho, \rho)$, then replacing $\tilde{R}$ by $X^{*^{-1}} \otimes 1_\rho \circ \tilde{R}$, we see that we may assume that $X = 1_\rho$. But now

\[R = 1_\rho \otimes \tilde{R}^* \otimes 1_\rho \circ 1_{\tilde{\rho}_2} \otimes R \circ R = 1_\rho \otimes \tilde{R}^* \otimes 1_\rho \circ R \otimes 1_{\tilde{\rho}_2} \circ R\]

and since $\omega$ is faithful, we conclude that $1_\rho \otimes \tilde{R}^* \circ R \otimes 1_\rho = 1_\rho$ and $\rho$ and $\tilde{\rho}$ are conjugates, completing the proof. \qed

In the category $\mathcal{T}_f$ we can define a conjugation on arrows. We pick for each object $\rho$ with a conjugate an $R_{\rho} \in (\iota, \rho, \rho')$ defining this conjugate and set for $T \in (\rho, \rho')$

\[T^* := 1_{\rho'} \otimes \tilde{R}_{\rho}^* \circ 1_{\rho'} \otimes T^* \otimes 1_\rho \circ R_{\rho'} \otimes 1_\rho \in (\tilde{\rho}, \rho').\]
This mapping is just the composition of the mappings

\[(\rho, \rho') \rightarrow (\iota, \rho' \bar{\rho}) \xrightarrow{\phi} (\rho' \tilde{\rho}, \iota) \rightarrow (\bar{\rho}, \rho'),\]

where the first and last mappings are defined as discussed above using the properties of adjoint functors. A simple computation shows that \(T^*\) is the unique element of \((\bar{\rho}, \rho')\), satisfying either

\[T^* \otimes 1_\rho \circ R_\rho = 1_{\bar{\rho}} \otimes T^* \circ R_{\rho'} \quad \text{or} \quad 1_{\rho'} \otimes T^{**} \circ \bar{R}_{\rho'} = T \otimes 1_{\bar{\rho}} \circ \bar{R}_{\rho}.
\]

At the same time our choice \(\rho \mapsto R_\rho\) serves to define an \((\iota, \iota)\)-valued scalar product on \(\mathcal{T}_f\) setting for \(S, T \in (\rho, \rho')\)

\[(S, T)_\ell := R_{\rho'}^* \circ 1_{\bar{\rho}} \otimes (S^* \circ T) \circ R_\rho.
\]

We obviously have

a) \((S, S)_\ell \geq 0,

b) \((S, X \circ T)_\ell = (X^* \circ S, T)_\ell,

c) \text{whenever the above expressions are well defined.}

**Lemma 2.3.** The mapping \(T \mapsto T^*\) is antilinear and has the following properties

a) \(1^*_\rho = 1_{\bar{\rho}}.

b) \(S^* \circ T^* = (S \circ T)^*\)

Furthermore if \((\cdot, \cdot)_\ell\) is tracial, i.e. if \((S, T)_\ell = (T^*, S^*)_\ell\) then

c) \(T^{**} = T^*\).

**Proof.** a) is trivial and b) follows from the above formula and \((S \circ T)^* = T^* \circ S^*\). To prove c) we note that if \(S, T \in (\rho, \rho')\) then

\[(S, T)_\ell = R_{\rho'}^* \circ 1_{\bar{\rho}} \otimes S^* \circ T^{**} \otimes 1_{\rho'} \circ R_{\rho'}
\]

whereas

\[(T^*, S^*)_\ell = R_{\rho'}^* \circ T^{**} \otimes 1_\rho \circ 1_{\bar{\rho}} \otimes S^* \circ R_{\rho'} = R_{\rho'}^* \circ 1_{\bar{\rho}} \otimes S^* \circ T^{**} \otimes 1_{\rho'} \circ R_{\rho'}.
\]

Since, by Lemma 2.1, \(1_{\bar{\rho}} \otimes S \circ R_\rho\) is an arbitrary element of \((\iota, \bar{\rho} \rho')\), c) follows. \(\square\)

**Theorem 2.4.** The subcategory \(\mathcal{T}_f\) is closed under conjugates and tensor products. If \(\mathcal{T}\) has (finite) direct sums and subobjects then \(\mathcal{T}_f\) is closed under direct sums and subobjects.

**Proof.** \(\mathcal{T}_f\) is obviously closed under conjugates and to see that it is closed under tensor products let \(R_i \in (\iota, \bar{\rho}_i \rho_i)\) and \(\bar{R}_i \in (\iota, \rho_i \bar{\rho}_i)\) define a conjugate for \(\rho_i, i = 1, 2\), then

\[R := 1_{\bar{\rho}_2} \otimes R_1 \otimes 1_{\rho_2} \circ R_2; \quad \bar{R} := 1_{\rho_1} \otimes \bar{R}_2 \otimes 1_{\bar{\rho}_1} \circ \bar{R}_1
\]

solve the conjugate equations for \(\rho_1 \rho_2\). To check closure under direct sums, we suppose \(W_i \in (\rho_i, \rho)\) and \(\bar{W}_i \in (\bar{\rho}_i, \bar{\rho})\), \(i = 1, 2\) to be isometries realizing \(\rho\) and \(\bar{\rho}\) as direct sums and set

\[R = \sum_i \bar{W}_i \otimes W_i \circ R_i; \quad \bar{R} = \sum_i W_i \otimes \bar{W}_i \circ \bar{R}_i.
\]
A simple computation again shows that $R$ and $\bar{R}$ solve the conjugate equations for $\rho$. It remains to check closure under subobjects. Suppose that $\rho$ is an object of $\mathcal{T}_f$ and $W \in (\rho', \rho)$ is an isometry. Let $R, \bar{R}$ solve the conjugate equations for $\rho$, then, by Lemma 2.3,

$$1_{\bar{\rho}} \otimes (W \circ W^*) \circ R = F \otimes 1_{\rho} \circ \rho,$$

where $F \in (\bar{\rho}, \bar{\rho})$ is a (not necessarily orthogonal) projection. Since $\mathcal{T}$ is closed under subobjects there is a corresponding subobject $\bar{\rho}'$ of $\bar{\rho}$ and hence an $\bar{X} \in (\bar{\rho}', \bar{\rho})$ such that $\bar{Y} \circ \bar{X} = 1_{\bar{\rho}'}$ and $\bar{X} \circ \bar{Y} = F$. Define

$$R' = \bar{Y} \otimes W^* \circ \rho; \quad \bar{R}' = W^* \otimes \bar{X}^* \circ \bar{R}.$$

A computation now shows that $R'$ and $\bar{R}'$ solve the conjugate equations for $\rho'$, completing the proof. $\square$

We define a left inverse $\varphi$ for an object $\rho$ in a strict tensor $C^*$–category $\mathcal{T}$ to be a set

$$\varphi_{\sigma, \tau} : (\rho\sigma, \rho\tau) \rightarrow (\sigma, \tau)$$

for objects $\sigma, \tau$ of $\mathcal{T}$ of linear mappings which are natural in $\sigma$ and $\tau$, i.e. given $S \in (\sigma, \sigma')$ and $T \in (\tau, \tau')$ we have

$$\varphi_{\sigma', \tau'}(1_{\rho} \otimes T \circ X \circ 1_{\rho} \otimes S^*) = T \circ \varphi_{\sigma, \tau}(X) \circ S^*, \quad X \in (\rho\sigma, \rho\tau),$$

and furthermore satisfy

$$\varphi_{\sigma\pi, \tau\pi}(X \otimes 1_{\pi}) = \varphi_{\sigma, \tau}(X) \otimes 1_{\pi}, \quad X \in (\rho\sigma, \rho\tau)$$

for each object $\pi$ of $\mathcal{T}$. We will say that $\varphi$ is positive if $\varphi_{\sigma, \sigma}$ is positive for each $\sigma$ and normalized if

$$\varphi_{\iota, \iota}(1_{\rho}) = 1_{\iota}.$$

When $\varphi$ is positive, we say that $\varphi$ is faithful if $\varphi_{\sigma, \sigma}$ is faithful for each object $\sigma$.

By way of explanation of the name left inverse, this term has already been used in connection with endomorphisms of a $C^*$–algebra $\mathcal{A}$. When $\varphi$ is a left inverse of an endomorphism $\rho$ and we define for each pair of endomorphisms $\sigma, \tau$ of $\mathcal{A}$ and $X \in (\rho\sigma, \rho\tau)$

$$\varphi_{\sigma, \tau}(X) = \varphi(X) \in (\sigma, \tau),$$

then we get a left inverse for the object $\rho$ of $\text{End} \mathcal{A}$. We may think of a left inverse for an object in a tensor $C^*$–category as being some kind of virtual object: $\varphi_{\sigma, \tau}$ corresponds to tensoring on the left by this virtual object looked at as a map from $(\rho\sigma, \rho\tau)$ to $(\sigma, \tau)$. If $\rho$ has a conjugate then we may always construct non–zero left inverses; for suppose $R, S \in (\iota, \bar{\rho}\rho)$ and define

$$\varphi_{\sigma, \tau}(X) := S^* \otimes 1_{\tau} \circ 1_{\bar{\rho}} \otimes X \circ R \otimes 1_{\sigma}, \quad X \in (\rho\sigma, \rho\tau),$$

then $\varphi$ is obviously a left inverse which is positive if $S = R$ and is normalized if $S^* \circ R = 1_{\iota}$. Note that when $S = R$ defines a conjugate for $\rho$ then we get a faithful left inverse.
Lemma 2.5. Suppose $\rho$ has a conjugate $\bar{\rho}$ then every left inverse is of the above form. More precisely if $R \in (\iota, \bar{\rho}\rho)$ defines a conjugate for $\rho$ then left inverses for $\rho$ are in $1 - 1$ correspondence with the elements $Y$ of $(\bar{\rho}, \bar{\rho})$ by taking $S^* = R^* \circ Y \otimes 1_{\rho}$ in the above formula and a left inverse is positive if and only if $Y$ is positive.

Proof. Let $X \in (\rho\sigma, \rho\tau)$ then trivially,

$$X = 1_{\rho} \otimes R^* \otimes 1_{\tau} \circ \bar{R} \otimes 1_{\rho\tau} \circ X \circ \bar{R}^* \otimes 1_{\rho\sigma} \circ 1_{\rho} \otimes R \otimes 1_{\sigma}$$

$$= 1_{\rho} \otimes R^* \otimes 1_{\tau} \circ 1_{\bar{\rho}} \otimes X \circ (\bar{R} \circ \bar{R}^*) \otimes 1_{\rho\sigma} \circ 1_{\rho} \otimes R \otimes 1_{\sigma}.$$ 

Hence, if $\varphi$ is a left inverse for $\rho$ we have

$$\varphi_{\sigma, \tau}(X) = R^* \otimes 1_{\tau} \circ 1_{\bar{\rho}} \otimes X \circ \varphi_{\bar{\rho}, \bar{\rho}}(\bar{R} \circ \bar{R}^*) \otimes 1_{\rho\sigma} \circ R \otimes 1_{\sigma}.$$ 

Since $\varphi_{\bar{\rho}, \bar{\rho}}(\bar{R} \circ \bar{R}^*) \in (\bar{\rho}, \bar{\rho})$ we see that every left inverse is of the stated form and is positive if and only if $\varphi_{\bar{\rho}, \bar{\rho}}(\bar{R} \circ \bar{R}^*)$ is positive. 

A left inverse $\varphi$ is thus uniquely determined by $\varphi_{\bar{\rho}, \bar{\rho}}(\bar{R} \circ \bar{R}^*)$. Notice that, alternatively, every left inverse for $\rho$ is of the form

$$\varphi_{\sigma, \tau}(X) = S^* \otimes 1_{\tau} \circ 1_{\bar{\rho}} \otimes X \circ R \otimes 1_{\sigma}, \quad X \in (\rho\sigma, \rho\tau),$$

where $S$ is a uniquely determined but arbitrary element of $(\iota, \bar{\rho}\rho)$. Hence by Lemma 2.1, a left inverse $\varphi$ is also uniquely determined by $\varphi_{\iota, \iota}$.

Before giving the basic inequality on left inverses associated with solutions of the conjugate equations, we need a preliminary lemma.

Lemma 2.6. Let $S \in (\iota, \sigma)$ then

$$S \circ S^* \leq (S^* \circ S) \otimes 1_{\sigma}; \quad S \circ S^* \leq 1_{\sigma} \otimes (S^* \circ S) \quad .$$

Proof. $S^* \circ S \circ S^* = (S^* \circ S) \otimes 1_{\iota} \circ 1_{\iota} \otimes S^* = S^* \circ (S^* \circ S) \otimes 1_{\sigma}$. We now have a positive element $X := S \circ S^*$ and a positive central element $Z := (S^* \circ S) \otimes 1_{\sigma}$ in the C*-algebra $(\sigma, \sigma)$ such that $X^2 = XZ$. Hence $X \leq Z$, since $\pi(X) \leq \pi(Z)$ for each irreducible representation $\pi$. The second inequality is proved similarly.

Lemma 2.7. Let $R$ and $\bar{R}$ solve the conjugate equations for $\rho$ and let $\varphi$ be the associated left inverse, then

$$X^* \circ X \leq \bar{R}^* \circ \bar{R} \otimes 1_{\rho\sigma} \circ 1_{\rho} \otimes \varphi_{\sigma, \sigma}(X^* \circ X), \quad X \in (\rho\sigma, \rho\tau).$$

Proof. Set $Y := 1_{\bar{\rho}} \otimes X \circ R \otimes 1_{\sigma}$ then $\bar{R}^* \circ 1_{\rho\tau} \circ 1_{\rho} \otimes Y = X$. Hence $X^* \circ X = 1_{\rho} \otimes Y^* \circ (\bar{R} \circ \bar{R}^*) \otimes 1_{\rho\tau} \circ 1_{\rho} \otimes Y$. Applying Lemma 2.6 with $S = \bar{R} \in (\iota, \rho\bar{\rho})$

$$X^* \circ X \leq (\bar{R}^* \circ \bar{R}) \otimes 1_{\rho\sigma} \circ 1_{\rho} \otimes (Y^* \circ Y) = (\bar{R}^* \circ \bar{R}) \otimes 1_{\rho\sigma} \circ 1_{\rho} \otimes \varphi_{\sigma, \sigma}(X^* \circ X).$$

□
Actually this inequality is the best possible in the sense that $\bar{R}^* \circ \bar{R}$ is the smallest element of $(\iota, \iota)$ for which this inequality holds. This is seen by taking $X^* \circ X = \bar{R} \circ \bar{R}^* \in (\rho \bar{\rho}, \rho \bar{\rho})$.

As a consequence of this inequality there are various finiteness properties of the objects $\rho$ of $\mathcal{T}_f$. For example, the image of $(\rho, \rho)$ under a representation is finite dimensional whenever the image of the centre is finite dimensional.

Now just as we have introduced the notion of a left inverse of an object $\rho$ in a tensor $C^*$–category, so we can introduce the notion of a right inverse replacing tensoring on the left by $\rho$ by tensoring on the right. Thus a right inverse $\psi$ for $\rho$ is a set of mappings

$$\psi_{\sigma, \tau} : (\sigma \rho, \tau \rho) \to (\sigma, \tau),$$

natural in $\sigma$ and $\tau$, i.e. such that given $S \in (\sigma, \sigma')$ and $T \in (\tau, \tau')$ we have

$$\psi_{\sigma', \tau'}(T \otimes \iota_\rho \circ X \circ S^* \otimes \iota_\rho) = T \circ \psi_{\sigma, \tau}(X) \circ S^*, \quad X \in (\sigma \rho, \tau \rho),$$

and satisfying

$$\psi_{\pi, \pi, \pi}(1_\pi \otimes X) = 1_\pi \otimes \psi_{\sigma, \tau}(X), \quad X \in (\sigma \rho, \tau \rho).$$

Since this notion results from that of left inverse by dualizing with respect to $\otimes$, all the above results have their counterparts for right inverses. In particular, if $R, \bar{R}$ solve the conjugate equations for $\rho$ there is an associated right inverse given by

$$\psi_{\sigma, \tau}(X) = 1_\tau \otimes \bar{R}^* \circ X \otimes \iota_{\bar{\rho}} \circ 1_\sigma \otimes \bar{R}, \quad X \in (\sigma \rho, \tau \rho).$$

In the same way, we have another $(\iota, \iota)$–valued scalar product $(\cdot, \cdot)_r$ on $\mathcal{T}_f$ associated with a choice $\rho \mapsto \bar{R}_\rho$ for each object $\rho$ of $\mathcal{T}_f$, namely if $S, T \in (\rho, \rho')$,

$$(S, T)_r := \bar{R}_\rho^* \circ (S^* \circ T) \otimes \iota_{\bar{\rho}} \circ \bar{R}_\rho.$$  

Objects $\rho$ of the form $\bar{\sigma} \sigma$ will be termed canonical since they are an abstract analogue of the canonical endomorphisms introduced in [17]. They have some special properties which are well known in the theory of adjoint functors. In fact, let $R \in (\iota, \bar{\sigma} \sigma)$ and $\bar{R} \in (\iota, \sigma \bar{\sigma})$ satisfy the conjugate equations then $R \in (\iota, \rho)$ and $M := 1_{\bar{\sigma}} \otimes \bar{R}^* \otimes 1_{\sigma} \in (\rho^2, \rho)$. We have

$$M \circ M \otimes 1_\rho = M \circ 1_\rho \otimes M; \quad M \circ R \otimes 1_\rho = 1_\rho; \quad M \circ 1_\rho \otimes R = 1_\rho$$

and since these are the associative and unit laws, we can, following the standard practice in category theory, say that they make $\rho$ into an algebra in the tensor $C^*$–category $\mathcal{T}$ in question. Since $*$ is a duality for $\mathcal{T}$, this is equivalent to saying that $M^*$ and $R^*$ make $\rho$ into a cogebras in $\mathcal{T}$. We also have a further relation

$$1_\rho \otimes M \circ M^* \otimes 1_\rho = M^* \circ M$$

which is not part of the theory of adjoint functors as it involves both $M$ and $M^*$. We shall, however, see in Section 7, that, in our context, it is actually a consequence of the other equations.

Note that if $F : \mathcal{T}_1 \to \mathcal{T}_2$ is a strict tensor $^*$–functor then $F$ maps solutions of the conjugate equations into solutions for the image object and intertwines the associated left and right inverses. Hence $F$ maps conjugates onto conjugates, canonical objects onto canonical objects and induces a strict tensor $^*$–functor from $\mathcal{T}_{1f}$ to $\mathcal{T}_{2f}$.
3. Dimension and Trace

We turn now to the related notions of dimension and trace which are central to this paper. In fact, in view of their importance, we want, throughout this section, to restrict ourselves to developing this theory in the important special case that $(\iota, \iota) = \mathbb{C}$, i.e. that $\iota$ is irreducible. In the language of von Neumann algebras this corresponds to defining the index just for the case of subfactors.

We note the following simple corollary of Lemma 2.2.

**Lemma 3.1.** Two irreducible objects $\rho$ and $\bar{\rho}$ are conjugate if and only if there are $R \in (\iota, \bar{\rho}\rho)$ and $\bar{R} \in (\iota, \rho\bar{\rho})$ such that $\bar{R}^* \otimes 1_\rho \circ 1_\rho \otimes R \neq 0$.

The next result lends support to the interpretation of the objects of $T_f$ as being finite dimensional.

**Lemma 3.2.** When $\iota$ is irreducible and $\rho$ is an object of $T_f$ then $(\rho, \rho)$ is finite dimensional. In particular if $T$ has subobjects, $\rho$ is just a finite direct sum of irreducibles.

**Proof.** Let $X_1, X_2, \ldots, X_n$ be positive elements of $(\rho, \rho)$ of norm 1 with $\sum_i X_i \leq 1_\rho$, then Lemma 2.7 shows that $n \leq \bar{R}^* \circ \bar{R} \circ R^* \circ R$, so that $(\rho, \rho)$ is finite dimensional. □

We study further the set of solutions of the conjugate equations for given $\rho$ and $\bar{\rho}$ defining equivalence classes and normalizing solutions. On this basis, we can define a well behaved notion of dimension for the corresponding left inverses. We shall then show that there exists a privileged class of normalized solutions called the standard solutions. This privileged class will be characterized in a variety of ways and serves to define a dimension for the objects themselves.

The pertinent notion of equivalence is characterized in the following lemma.

**Lemma 3.3.** Two solutions $R, \bar{R}$ and $R', \bar{R}'$ of the conjugate equations for $\rho, \bar{\rho}$ define the same left inverse for $\rho$ if and only if there is a unitary $U \in (\bar{\rho}, \bar{\rho})$ such that $R' = U \otimes 1_\rho \circ R$.

**Proof.** Since a left inverse $\varphi$ is uniquely determined by $\varphi_{\iota, \iota}, R$ and $R' = U \otimes 1_\rho \circ R$ define the same left inverse if and only if

$$R^* \circ (U^* \circ U) \otimes X \circ R = R^* \circ 1_\rho \otimes X \circ R, \ X \in (\rho, \rho),$$

i.e. if and only if

$$R^* \circ 1_\rho \otimes X \circ (U^* \circ U)^{**} \circ R, \ X \in (\rho, \rho),$$

where $^{**}$ is the map from $(\bar{\rho}, \bar{\rho})$ to $(\rho, \rho)$ defined by $R$. Since the left inverse is faithful, we conclude that $(U^* \circ U)^{**} = 1_\rho$ so by Lemma 2.3, $U^* \circ U = 1_\rho$ as required. □

We can, of course, equally well define an equivalence class by requiring $\bar{R}$ and $\bar{R}'$ to define the same left inverse for $\bar{\rho}$. However, this will define the same equivalence class if and only if the map $^*$ commutes with the adjoint hence, by Lemma 2.3c if and only if $\varphi_{\iota, \iota}$ is tracial.

On the basis of Lemma 3.3, we make the following definition which gives an analogue of the square root of the index relative to a conditional expectation in the theory of subfactors [9].
Definition 3.4. Let $\varphi$ be the left inverse of $\rho$ defined by a solution $R, \bar{R}$ of the conjugate equations, then we define the dimension $d(\varphi)$ of $\varphi$ by

$$d(\varphi) = \|R\| \|\bar{R}\|.$$ 

Note that if $\mu > 0$, then $d(\mu \varphi) = d(\varphi)$, so the dimension is independent of the normalization of $\varphi$. For this reason, it is convenient to restrict oneself to normalized solutions $R, \bar{R}$, i.e. those for which $\|R\| = \|\bar{R}\|$. In this case, we have

$$d(\varphi) = R^* \circ R = \bar{R}^* \circ \bar{R} = d(\bar{\varphi}),$$

where $\bar{\varphi}$ denotes the left inverse of $\bar{\rho}$ defined by $\bar{R}$.

Lemma 3.5. Let $R, \bar{R}$ be normalized solutions of the conjugate equations for $\rho, \bar{\rho}$ and $\varphi$ the corresponding left inverse. Then $d(\varphi) = 1$ if and only if $R$ is unitary.

Proof. Clearly, if $R$ is unitary, $d(\varphi) = 1$. Conversely, if $d(\varphi) = 1$ then computing $X^* \circ X$ for $X = R \otimes 1_\rho - 1_\rho \otimes R$, we see that $\bar{R} \otimes 1_\rho = 1_\rho \otimes R$. Similarly $R \otimes 1_\bar{\rho} = 1_\bar{\rho} \otimes \bar{R}$. Thus

$$R \circ R^* = 1_\bar{\rho} \otimes R^* \circ R \otimes 1_\rho = 1_\rho \otimes \bar{R}^* \otimes 1_\rho \circ 1_\bar{\rho} \otimes \bar{R} \otimes 1_\rho = 1_\bar{\rho},$$

and $R$ is unitary. □

We now show that our notions behave well under direct sums and tensor products. Let $W_i \in (\rho_i, \rho)$ and $\bar{W}_i \in (\bar{\rho}_i, \bar{\rho})$ be isometries expressing $\rho$ and $\bar{\rho}$ as a direct sum of irreducibles and let $R_i \in (i, \bar{\rho}_i \rho_i)$ and $\bar{R}_i \in (i, \rho_i \bar{\rho}_i)$ be solutions of the conjugate equations then

$$R = \sum_i \bar{W}_i \otimes W_i \circ R_i, \quad \bar{R} = \sum_i W_i \otimes \bar{W}_i \circ \bar{R}_i$$

solve the conjugate equations for $\rho, \bar{\rho}$ and are said to be a direct sum of the original solutions $R_i, \bar{R}_i$. They are normalized if the original solutions were normalized. This notion is compatible with equivalence classes and we have:

$$R^* \circ R = \sum_i R_i^* \circ R_i,$$

and hence for the corresponding left inverses:

$$d(\varphi) = \sum_i d(\varphi_i).$$

Turning to the product, a trivial computation gives the following result.

Lemma 3.6. Let $R_i \in (i, \rho_i \bar{\rho}_i)$ and $\bar{R}_i \in (i, \bar{\rho}_i \rho_i)$ define a conjugate for $\rho_i, i = 1, 2$, then

$$R := 1_{\bar{\rho}_2} \otimes R_1 \otimes 1_{\rho_2} \circ R_2; \quad \bar{R} := 1_{\rho_1} \otimes \bar{R}_2 \otimes 1_{\bar{\rho}_1} \circ \bar{R}_1$$

define a conjugate for $\rho_1 \rho_2$. If the $R_i$ are normalized, so is $R$.

It is clear that this notion of product is compatible with equivalence classes and that for the corresponding left inverses we have, with an obvious notation,

$$d(\varphi) = d(\varphi_1) d(\varphi_2).$$
If $\rho$ is a zero object, i.e. if $(\rho, \rho) = 0$ then $d(\varphi) = 0$. When $R, \bar{R}$ define a conjugate for a non–zero object $\rho$, the defining equations show that

$$1 \leq \|\bar{R}^* \otimes 1_\rho\| \|1_\rho \otimes R\| \leq \|\bar{R}\| \|R\|,$$

so that $d(\varphi) \geq 1$.

We now establish a connection with the work of Jones [11]. If $R \in (\iota, \bar{\rho} \rho)$ and $\bar{R} \in (\iota, \rho \bar{\rho})$ satisfy the conjugate equations, we let $E$ and $\bar{E}$ be the range projections

$$E := \|R\|^{-2} R \circ R^*; \quad \bar{E} := \|\bar{R}\|^{-2} \bar{R} \circ \bar{R}^*.$$

These will be called the associated Jones projections. Note that they are unique up to a unitary equivalence within an equivalence class of solutions, where the unitary in question can be chosen from the subalgebras $(\bar{\rho}, \bar{\rho}) \otimes 1_\rho$ and $1_\rho \otimes (\bar{\rho}, \bar{\rho})$, respectively. Since

$$1_\rho \otimes (R \circ R^*) \circ (\bar{R} \circ \bar{R}^*) \otimes 1_\rho \otimes (R \circ R^*) = 1_\rho \otimes (R \circ R^*),$$

we have

$$1_\rho \otimes E \circ \bar{E} \otimes 1_\rho \otimes E = \|R\|^{-2} \cdot \|\bar{R}\|^{-2} 1_\rho \otimes E$$

and similarly

$$1_{\bar{\rho}} \otimes \bar{E} \circ E \otimes 1_{\bar{\rho}} \otimes \bar{E} = \|R\|^{-2} \cdot \|\bar{R}\|^{-2} 1_{\bar{\rho}} \otimes \bar{E}.$$

These are the Jones relations and by looking at the associated representations of the braid group, we may conclude as in [11] that

$$\|R\|^{-2} \cdot \|\bar{R}\|^{-2} \in \{4 \cos^2 \frac{\pi}{k} : k \in \mathbb{N}, k \geq 3\} \cup [4, \infty).$$

We note that if $\varphi$ is a normalized faithful left inverse, then by Lemma 2.7, $d(\varphi)$ is the smallest constant $c$ such that, for all objects $\sigma, \tau$, we have

$$X^* \circ X \leq c^2 1_\rho \otimes \varphi_{\sigma, \sigma}(X^* \circ X), \quad X \in (\rho \sigma, \rho \tau).$$

In the special case that $\mathcal{T} = \text{End} A$ for a unital $C^*$–algebra $A$, we can regard $\varphi$ as a left inverse for the endomorphism $\rho$ by setting

$$\varphi(X) := (R^* R)^{-1} R^* \bar{\rho}(X) R$$

and $\mathcal{E} := \rho \varphi$ is a conditional expectation from $A$ onto $\rho(A)$. Setting $Y = \bar{\rho}(X) R$, cf. Lemma 2.6, we have

$$X^* X = \rho(Y^*) \bar{R} R \rho(Y) \leq \rho(Y^* Y) \bar{R} \circ \bar{R} = \mathcal{E}(X^* X) R^* R \bar{R} \circ \bar{R}.$$

Hence $d(\varphi)^{-2}$ is the Pimsner–Popa bound of $\mathcal{E}$, in other words, $\text{Ind} \mathcal{E} = d(\varphi)^2$.

If $\varphi$ is a linear mapping of $A$ into $A$ such that

$$\varphi(\rho(A) B \rho(C)) = A \varphi(B) C, \quad A, B, C \in A,$$
then for each pair $\sigma, \tau$ of endomorphisms of $A$,
\[ \varphi(X)\sigma(A) = \varphi(X \rho \sigma(A)) = \varphi(\rho \tau(A)X) = \tau(A)\varphi(X), \quad X \in (\rho, \rho \tau). \]

Thus we get an associated left inverse of $\rho$ in $\text{End}A$ by taking $\varphi_{\sigma, \tau}$ to be the restriction of $\varphi$ to $(\rho, \rho \tau)$.

Thus for each object $\rho$, there are two sets of Jones projections, those of the form $E$ associated with faithful right inverses, and those of the form $\bar{E}$ associated with faithful left inverses. For each faithful left or right inverse, there is a unique unitary equivalence class of Jones projections. Each Jones projection $E$ determines a solution of the conjugate equations up to a scalar.

The vital step towards introducing a dimension for the objects of $T_f$ rather than their left inverses is to choose a special equivalence class of solutions of the conjugate equations. We first take this step for irreducible objects by picking the unique equivalence class of normalized solutions.

In the general case, we write $\rho$ as a direct sum of irreducibles and say that $R, \bar{R}$ is a standard solution if $R$ is of the form
\[ R = \sum_i \bar{W}_i \otimes W_i \circ R_i, \]
where $R_i \in (\iota, \bar{\rho}_i \rho_i)$ is a normalized solution for an irreducible object $\rho_i$ and $W_i \in (\rho_i, \rho)$ and $\bar{W}_i \in (\bar{\rho}_i, \bar{\rho})$ are isometries expressing $\rho$ and $\bar{\rho}$ as a direct sum of irreducibles. As we know we then have
\[ \bar{R} = \sum_i W_i \otimes \bar{W}_i \circ \bar{R}_i \]
and $R, \bar{R}$ is a normalized solution. With an obvious notation, the corresponding left inverses are related by
\[ \varphi(X) = \sum_i \varphi_i(W_i^* \circ X \circ W_i). \]

It is easy to see that $\varphi$ is independent of the choice of the $W_i$ so that we get in this way a single well determined equivalence class of solutions of the conjugate equations. This left inverse will be called the standard left inverse, a notion introduced in [6] in the context of a symmetric tensor $C^*$-category.

**Lemma 3.7.** If $\rho \mapsto R_\rho$ is standard then the corresponding scalar product $(\cdot, \cdot)$, defined as in Section 2, is independent of the choice of standard solutions and is tracial.

**Proof.** The first statement follows from the uniqueness of the standard left inverse; to prove the second, we use Lemma 2.3c. Now when the $R_\rho$ are standard, $R_\rho$ and $R_{\rho'}$ have the form:
\[ R_\rho = \sum_i \bar{W}_i \otimes W_i \circ R_{\rho_i}; \quad R_{\rho'} = \sum_j \bar{W}_j' \otimes W_j' \circ R_{\rho_j'}; \]
where, without loss of generality, we may suppose that, when $\rho_i$ and $\rho_j'$ are equivalent, they are actually equal. If $T \in (\rho, \rho')$ then
\[ T^* \otimes 1_\rho \circ \sum_i \bar{W}_i \otimes W_i \circ R_{\rho_i} = 1_{\rho'} \otimes T^* \circ \sum_j \bar{W}_j' \otimes W_j' \circ R_{\rho_j'} \]
Multiplying on the left by $\bar{W}' \otimes \bar{W}_i$, we conclude that

$$W_j^* \circ T \circ \bar{W}_i = (W_j^* \circ T \circ \bar{W}_i)^* = \bar{W}_i^* \circ T^* \circ W_j' = \bar{W}'_j \circ T^{**} \circ \bar{W}_i.$$ 

Thus $T^{**} = T^{***}$ as required. □

There is a notion of trace is implicit in the above scalar product. To make it explicit, given $T \in (\rho, \rho)$, we write $tr(T) := (1_\rho, T)$. It is easy to compute the trace of a projection $E$. In fact, if we write $E = W \circ W^*$, where $W \in (\sigma, \rho)$ is an isometry then

$$tr(E) = (1_\rho, E) = (W^*, W^*) = (W, W) = tr(1_\sigma).$$

Thus the trace of $E$ is just the dimension of the standard left inverse of $\sigma$. This shows that the trace on our full subcategory is entirely determined by the dimensions of the standard left inverses.

We now define $d(\rho)$ for an object $\rho$ to be the dimension of its standard left inverse $\varphi$. We then have $d(\rho) = d(\bar{\rho})$ so that $d(\rho)$ depends only on the unitary equivalence class of $\rho$ and is additive on direct sums. Since it is additive, it follows from Theorem 2.4 that an object of dimension $< 2$ is necessarily irreducible.

Let us now look at a few simple examples. If we take our strict tensor $C^*$–category to be a category of finite dimensional Hilbert spaces then every object is a finite direct sum of 1–dimensional objects and the dimension is the usual one. More generally, if we look at a category of finite-dimensional, continuous, unitary representations of a compact group then we can take

$$R\lambda = \lambda \sum_i \bar{e}_i \otimes e_i; \quad \bar{R}\lambda = \lambda \sum_i e_i \otimes \bar{e}_i; \quad \lambda \in \mathbb{C};$$

where $e_i$ is an orthonormal basis of the underlying Hilbert space and $\bar{e}_i$ is the conjugate basis of the conjugate Hilbert space. Thus we again have the usual notion of dimension.

Turning to compact quantum groups in the sense of Woronowicz, we again get a strict tensor $C^*$–category with $(\iota, \iota) = \mathbb{C}$ by taking unitary representations on objects of a strict tensor $C^*$–category of Hilbert spaces. For a unitary representation on a finite–dimensional Hilbert space, we get solutions of the conjugate equations by taking

$$R\lambda = \lambda \sum_i f_i \otimes J^{-1} f_i; \quad \bar{R}\lambda = \lambda \sum_i e_i \otimes J e_i; \quad \lambda \in \mathbb{C};$$

where $e_i$ is again an orthonormal basis of the underlying Hilbert space and $f_i$ is an orthonormal basis of the Hilbert space of the conjugate. However $J$ is now just an invertible antilinear intertwiner and is not necessarily antiunitary. Since

$$R^* \circ R = Tr J^{-1} J^{-1}; \quad \bar{R}^* \circ \bar{R} = Tr J^* J,$$

the dimension will no longer coincide with the Hilbert space dimension in general.

Let us compute a simple example, the case of $SU_q(2)$, with $-1 < q < 1$. A few remarks will clarify the general structure. It is known that the fusion rules for the irreducible representations are the same as in the classical case of the dual of $SU(2)$
The irreducible representations of $SU_q(2)$ are indexed by \{\rho_i, i = 0, 1, 2 \ldots \}, \rho_0 = id and \rho_1 the fundamental representation, and we have

$$\rho_i \otimes \rho_j = \bigoplus_{k = |i-j|}^{i+j} \rho_k.$$ 

This means the principal graph of tensoring by \rho_1 is $A_{\infty}$. Now $d(\rho_0) = 1$ and we shall see later that the dimension is multiplicative on tensor products. Hence it suffices to compute $d := d(\rho_1)$ since the other dimensions can be computed from the recurrence relation

$$d(\rho_{n+1}) = d(\rho_n)d - d(\rho_{n-1}).$$

The task of computing $d := d(\rho_1)$ is facilitated by the fact that \rho_1 is self-conjugate. It can be easily checked that with the matrix realization of Eq. (1.34) of [30], we can take the antilinear intertwiner $J$ to be given by

$$Je_1 = q^{-1}e_2; \quad Je_2 = e_1;$$

where $e_1, e_2$ are the canonical basis of $\mathbb{C}^2$. Thus we conclude that $d(\rho_1) = q + q^{-1}$.

Before being able to show that the dimension is multiplicative, we need to examine the relationship between (faithful) left and right inverses. There are in fact two natural ways of passing from a left inverse to a right inverse. One way is to take a solution $R, \bar{R}$ of the conjugate equations and to pass from the left inverse defined by $R$ to the right inverse defined by $\bar{R}$. Obviously, this procedure depends only on the equivalence class of the solutions and is compatible with direct sums and products. The alternative procedure is to start with a left inverse \varphi and look for the right inverse \psi such that

$$\varphi_{\iota,\iota} = \psi_{\iota,\iota}.$$ 

This condition is again compatible with taking direct sums. However, it is less obvious that it is compatible with taking direct products.

**Theorem 3.8.** Let \varphi and \psi be left and right inverses of \rho and \varphi’ and \psi’ left and right inverses of \rho’ such that

$$\varphi_{\iota,\iota} = \psi_{\iota,\iota} \quad \varphi’_{\iota,\iota} = \psi’_{\iota,\iota},$$

then the corresponding product left and right inverses for $\rho \rho’$, \varphi’ \varphi and \psi \psi’ satisfy

$$(\varphi’ \varphi)_{\iota,\iota} = (\psi \psi’)_{\iota,\iota}.$$ 

**Proof.** Let $X \in (\rho \rho’)$, then

$$\varphi’_{\iota,\iota} \varphi_{\rho’,\rho’}(X) = \psi_{\iota,\iota} \varphi_{\rho’,\rho’}(X),$$

since $\varphi_{\rho’,\rho’}(X) \in (\rho’,\rho’)$. Now any right inverse commutes with any left inverse, so the above equals

$$\varphi_{\iota,\iota} \psi’_{\rho,\rho}(X) = \psi_{\iota,\iota} \psi’_{\rho,\rho}(X),$$

again since $\psi’_{\rho,\rho}(X) \in (\rho,\rho)$.

It is clear that the standard left inverse agrees with the standard right inverse of \rho on $(\rho,\rho)$ because it is trivially true on irreducibles and the equality is compatible with direct sums. Alternatively, one may observe that the standard right inverse may equally well be used to define the dimension and the trace. What is important is the following characterization of the standard solutions.
Lemma 3.9. Let $R \in (\iota, \bar{\rho})$ and $\bar{R} \in (\iota, \rho \bar{\rho})$ solve the conjugate equations for $\rho, \bar{\rho}$ then the corresponding left and right inverses $\varphi$ and $\psi$ for $\rho$,

$$
\begin{align*}
\varphi_{\sigma,\tau}(X) &= R^* \otimes 1_\tau \circ 1_{\bar{\rho}} \otimes X \circ R \otimes 1_\sigma, && X \in (\rho\sigma, \rho\tau); \\
\psi_{\sigma,\tau}(X) &= 1_\tau \otimes \bar{R}^* \circ X \otimes 1_{\bar{\rho}} \circ 1_\sigma \otimes \bar{R}, && X \in (\sigma\rho, \tau\rho);
\end{align*}
$$
satisfy $\varphi_{\iota,\iota} = \psi_{\iota,\iota}$ if and only if $R$ is standard.

Proof. We know that $\varphi_{\iota,\iota} = \psi_{\iota,\iota}$ when $R$ is standard. However any other choice of $R$ and $\bar{R}$ has the form $R' = 1_{\bar{\rho}} \otimes Y \circ R$ and $\bar{R}' = Y^{*-1} \otimes 1_{\bar{\rho}} \circ \bar{R}$, where $Y \in (\rho, \rho)$ is invertible. Thus the corresponding left and right inverses $\varphi'$ and $\psi'$ satisfy $\varphi'_{\iota,\iota} = \psi'_{\iota,\iota}$ if and only if

$$
\text{tr}Y^{*-1}XY = \text{tr}Y^{-1}XY^{*-1}, \quad X \in (\rho, \rho).
$$

This is the case if and only if $\text{tr}XYY^* = \text{tr}X(YY^*)^{-1}, X \in (\rho, \rho)$ and taking $X$ to be an eigenprojection of the positive invertible operator $YY^*$, we see that $YY^* = I$ so that $Y$ is unitary and $R'$ is standard.

The multiplicativity of the dimension now follows as a corollary.

Corollary 3.10. If $\rho_1$ and $\rho_2$ have conjugates, then

a) $d(\rho_1\rho_2) = d(\rho_1)d(\rho_2)$

b) If $\varphi_1$ and $\varphi_2$ are standard left inverses for $\rho_1$ and $\rho_2$, then $\varphi$ is a standard left inverse for $\rho_1\rho_2$, where $\varphi_{\sigma,\tau} = \varphi_2{\sigma,\tau} \varphi_1{\rho_1\sigma,\rho_1\tau}$.

c) $(S_1 \otimes S_2, T_1 \otimes T_2) = (S_1, T_1)(S_2, T_2), \quad S_i, T_i \in (\rho_i, \rho_i'), \quad i = 1, 2.$

We have therefore given a simple general proof of the multiplicativity of the dimension and hence of the minimal index. Of course, there are other proofs known in different contexts, e.g. [10, 16], but we must here draw attention to the simple proof given in [13] for inclusions of $C^*$-algebras with centre $\mathbb{C}I$. To recognize the relation with our proof, we suppose that our category is $\text{End}A$, where $A$ is a $C^*$-algebra with centre $\mathbb{C}I$. If the endomorphism $\rho$ of $A$ has conjugates defined by $R, \bar{R}$, we have a corresponding conditional expectation $\mathcal{E}$ of $A$ onto $\rho(A)$ defined by

$$
\mathcal{E}(A) := a \rho(R^* \bar{\rho}(A)R), \quad A \in A,
$$

where $a = \|R\|^{-2}$. Now

$$
\begin{align*}
a^{-1} \bar{R}^* \mathcal{E}(\bar{R}A) &= \bar{R}^* \rho(R^* \rho(\bar{R}A)R) = A,
\end{align*}
$$

so Proposition 2 of [13] tells us that $\mathcal{E}$ is minimal if and only if

$$
\begin{align*}
a^{-1} \bar{R}^* X \bar{R} &= c \mathcal{E}(X), \quad X \in (\rho, \rho),
\end{align*}
$$

for some constant $c > 0$. But this just means that

$$
\psi_{\iota,\iota}(X) = ca^2 \varphi_{\iota,\iota}(X), \quad X \in (\rho, \rho).
$$

where $\varphi$ and $\psi$ are the left and right inverses of $\rho$ defined by $R, \bar{R}$. Hence by Lemma 3.7, $c^{1/4}a^{1/2}R$ is standard and $d(\rho) = c^{1/2}a\|R\|^2 = c^{1/2}$. So $\text{Index}\mathcal{E} = d(\rho)^2$. In particular, we see that the proof of multiplicativity in [13] rests on the same criterion as our proof.

We now give an alternative method of characterizing standard solutions and the dimension based on the concept of minimality and related to using minimal conditional expectations in the theory of the index.
Theorem 3.11. Let \( R, \bar{R} \) define a conjugate \( \bar{\rho} \) for \( \rho \) then
\[
R^* \circ R \circ \bar{R}^* \circ \bar{R} \geq d(\rho)^2,
\]
with equality holding if and only if some multiple of \( R \) is standard.

Proof. The inequality itself results by taking \( X = 1_\rho \) in Lemma 2.7. Suppose \( R, \bar{R} \) are standard, then if \( R', \bar{R}' \) also define \( \bar{\rho} \) as a conjugate for \( \rho \), \( R' = 1_{\bar{\rho}} \otimes X \circ R \) and \( \bar{R}' = X^{*^{-1}} \otimes 1_{\bar{\rho}} \circ \bar{R} \), where \( X \in (\rho, \rho) \) is invertible. Hence
\[
R'^* \circ R' = R^* \circ 1_{\bar{\rho}} \otimes (X^* \circ X) \circ R = \text{tr}X^* \circ X,
\]
\[
\bar{R}'^* \circ \bar{R}' = \bar{R}^* \circ (X^{-1} \circ X^{*-1}) \otimes 1_{\bar{\rho}} \circ \bar{R} = \text{tr}(X^* \circ X)^{-1}.
\]
Now writing \( X^* \circ X = \sum_i \mu_i E_i \) and \( (X^* \circ X)^{-1} = \sum_i \mu_i^{-1} E_i \), where the \( E_i \) are mutually orthogonal projections in \( (\rho, \rho) \), we get
\[
R'^* \circ R' \circ \bar{R}'^* \circ \bar{R}' = \sum_{i,j} \mu_i \mu_j^{-1} \text{tr}(E_i)\text{tr}(E_j).
\]
Since \( \mu_i \mu_j^{-1} + \mu_j \mu_i^{-1} \geq 2 \), we deduce that
\[
R'^* \circ R' \circ \bar{R}'^* \circ \bar{R}' \geq \sum_{i,j} \text{tr}E_i \text{tr}E_j = (\text{tr}1_\rho)^2 = d(\rho)^2.
\]
Thus we have rederived the required inequality and we see that equality holds only if each \( \mu_i = 1 \), i.e. only if \( X^* \circ X = I \), i.e. only if \( R' \) is standard. \( \Box \)

Let us now consider the behaviour of the dimension under a functor in the light of the above theorem. If \( F : \mathcal{T}_1 \to \mathcal{T}_2 \) is a strict tensor \(*\)-functor where the tensor unit is irreducible in \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) then \( F(R) \) defines a conjugate for \( F(\rho) \) whenever \( R \in (\iota, \bar{\rho} \rho) \) defines a conjugate for \( \rho \). It follows that
\[
dl(F(\rho)) \leq d(\rho).
\]
In special cases, we have equality here. This is trivially the case when \( d(\rho) = 1 \) but it is also true when \( F(\rho) \) is irreducible or, more generally, when \( F \) induces an isomorphism of \( (\rho, \rho) \) and \( (F(\rho), F(\rho)) \). Note that the mapping of \( (\rho \sigma, \rho \tau) \) into \( (F(\rho \sigma), F(\rho \tau)) \) is injective whenever \( \rho \) has a conjugate.

More interesting is the fact that equality holds whenever there are just a finite number of equivalence classes of irreducibles. Let \( \rho_i, i \in I \), be a maximal set of pairwise inequivalent irreducibles and set \( d_i := d(\rho_i) \) then since the dimension is additive and multiplicative,
\[
d(\rho)d_i = \sum_j m_{ij}^{\rho_i} d_j,
\]
where \( m_{ij}^{\rho_i} \) is the dimension of \( (\rho_j, \rho \rho_i) \). Thus \( d(\rho) \) is a Perron-Frobenius eigenvalue of the matrix \( m_{ij}^{\rho} \). Since a \(*\)-functor preserves direct sums, we conclude that, whenever there is a unique Perron-Frobenius eigenvalue, as is the case when \( I \) is finite, \( F(d(\rho)) = d(\rho) \). In particular, when \( I \) is finite there can only be a tensor
*-functor into the category of Hilbert spaces if the dimension takes on integral values.

If we consider the matrix \( m_{ij}^\rho \) as an operator on \( \ell^2(I) \) then

\[
\lim_{n \to \infty} \dim(\rho^n, \rho^n) \frac{1}{\sqrt{n}} = \|m^\rho\| \leq d(\rho).
\]

This may be proved as the corresponding result in the theory of subfactors [23]. We conclude from this that if \( F \) is a faithful tensor *-functor then

\[
\|m^\rho\| \leq \|m^{F(\rho)}\| \leq d(F(\rho)) \leq d(\rho).
\]

In particular, if \( \|m^\rho\| = d(\rho) \), these inequalities are all equalities. The question of when we have equality \( \|m^\rho\| = d(\rho) \) is also of interest in the discussion of amenability given in §5. Of course, by the Perron–Frobenius Theorem, this equality must hold whenever the category in question has a finite set of equivalence classes of irreducibles. However, there are a few other very elementary but relevant remarks. Obviously, \( m^{\rho \oplus \sigma} = m^\rho + m^\sigma \). Consequently the equality \( d(\rho \oplus \sigma) = \|m^{\rho \oplus \sigma}\| \) implies both \( d(\rho) = \|m^\rho\| \) and \( d(\sigma) = \|m^\sigma\| \). Similarly, \( m^{\rho \otimes \sigma} = m^\sigma m^\rho \) and we can draw the analogous conclusion provided neither \( \rho \) nor \( \sigma \) is a zero object. Finally, \( m^\bar{\rho} = m^{\rho^*} \) and we deduce that \( d(\bar{\rho} \rho) = \|m^{\bar{\rho}}\| \) if and only if \( d(\rho) = \|m^\rho\| \).

If we define a dimension function to be a function from the set of objects with conjugates to \( \mathbb{R}_+ \), depending only on the equivalence class and additive on direct sums and multiplicative on tensor products and taking equal values on conjugate objects, then \( \rho \mapsto d(F(\rho)) \) is a dimension function, whenever \( F \) is a strict tensor *-functor. Furthermore, if \( f \) is any dimension function \( f(\rho) \) is a Perron-Frobenius eigenvalue of the matrix \( m^\rho_{ij} \). Hence a necessary condition for the existence of a tensor *-functor into the category of Hilbert spaces is for there to be an integral-valued dimension function. Thus the representation categories considered by Woronowicz in [30] trivially have integral-valued dimension functions even when the intrinsic dimensions of the objects themselves are not integral. If there is to be a right regular object, i.e. an object \( \rho \) with the property that \( \rho \sigma \) is equivalent to a finite multiple of \( \sigma \) for all \( \sigma \) with conjugates then we again need an integral-valued dimension function. The condition even becomes sufficient if we are allowed to choose \( \rho \) from a suitable closure of the category under direct sums.

In the next section devoted to braided tensor \( C^* \)-categories, we shall meet further conditions which guarantee that \( d(F(\rho)) = d(\rho) \). Note, however, that in any case the set of objects \( \rho \) such that \( F(d(\rho)) = d(\rho) \) is closed under finite direct sums, conjugates and subobjects.
4. Braided Tensor $C^*$–Categories
We now consider the question of conjugates in strict symmetric, or braided tensor $C^*$–categories. We begin by proving that if $\rho$ has a conjugate then a left inverse $\phi$ for $\rho$ is uniquely determined by the value it takes on $\varepsilon(\rho, \rho)$.

**Lemma 4.1.** Let $R \in (\iota, \check{\rho} \check{\rho})$ define a conjugate for $\rho$ and let $\varepsilon$ be a braiding then
\[
\phi_{\check{\rho}, \check{\rho}}(\check{R} \circ \check{R}^*) = 1_{\check{\rho}} \otimes \check{R}^* \circ \varepsilon(\rho, \check{\rho}) \otimes 1_{\check{\rho}} \circ \phi_{\rho, \rho}(\varepsilon(\rho, \rho)) \otimes 1_{\check{\rho}} \circ \check{R} \otimes 1_{\check{\rho}}.
\]
In particular, $\phi$ is uniquely determined by the value it takes on $\varepsilon(\rho, \rho)$.

**Proof.** $\phi_{\check{\rho}, \check{\rho}}(\check{R} \circ \check{R}^*) = \phi_{\check{\rho}, \check{\rho}}(1_{\rho \check{\rho}} \otimes \check{R}^* \circ \check{R} \otimes 1_{\rho \check{\rho}}) = 1_{\check{\rho}} \otimes \check{R}^* \circ \phi_{\check{\rho}, \check{\rho}}(\check{R} \otimes 1_{\rho}) \otimes 1_{\check{\rho}},$
\[
\check{R} \otimes 1_{\rho} = \varepsilon(\rho, \rho) \circ 1_{\rho} \otimes \check{R} = 1_{\rho} \otimes \varepsilon(\rho, \rho) \circ \varepsilon(\rho, \rho) \circ 1_{\check{\rho}} \otimes 1_{\rho} \otimes \check{R}
\]
Applying $\phi$ to this equation and substituting leads to the desired result and $\phi$ is uniquely determined by the value it takes on $\varepsilon(\rho, \rho)$ since we have seen in the course of proving Lemma 2.5 that is uniquely determined by the value it takes on $\check{R} \circ \check{R}^*$.

To render the statement of results as simple as possible, we shall suppose in what follows that our category has conjugates. We now show, using the standard left inverse, that a braiding $\varepsilon$ gives rise to a central element of the category.

**Theorem 4.2.** Let $\varepsilon$ be a braiding and $\phi$ the standard left inverse for $\rho$ and set
\[
\kappa(\rho) := d(\rho) \phi_{\rho, \rho}(\varepsilon(\rho, \rho))
\]
then $\rho \mapsto \kappa(\rho) \in (\rho, \rho)$ defines an element of the centre of the category i.e. if $T \in (\rho, \sigma)$,
\[
T \circ \kappa(\rho) = \kappa(\sigma) \circ T.
\]
Furthermore, we have
\[
\kappa(\rho_1 \rho_2) = \kappa(\rho_1) \otimes \kappa(\rho_2) \circ \varepsilon(\rho_2, \rho_1) \circ \varepsilon(\rho_1, \rho_2).
\]
so that $\kappa$ is multiplicative if and only if the braiding is in fact a symmetry.

**Proof.** We have
\[
\kappa(\rho) = \sum_i R_i^* \otimes 1_{\rho} \circ 1_{\check{\rho}_i} \otimes W_i^* \otimes 1_{\rho} \circ 1_{\check{\rho}_i} \otimes \varepsilon(\rho, \rho) \circ 1_{\check{\rho}_i} \otimes W_i \otimes 1_{\rho} \circ R_i \otimes 1_{\rho},
\]
where $R = \sum_i \check{W}_i \otimes W_i \circ R_i$ is the usual expression for a standard arrow $R$ defining a conjugate for $\rho$. A simple computation now gives $W_j^* \circ \kappa(\rho) = \kappa(\rho_j) \circ W_j^*$ and $\kappa(\rho) \circ W_j = W_j \circ \kappa(\rho_j)$. But now if we pick $T \in (\rho, \sigma)$ and isometries $V_k \in (\sigma_k, \sigma)$ with $\sum_k V_k \circ V_k^* = 1_{\sigma}$ then
\[
T \circ \kappa(\rho) = \sum_{i, k} V_k \circ V_k^* \circ T \circ W_i \circ W_i^* \circ \kappa(\rho).
\]
Since we may suppose that $\rho_i$ and $\sigma_k$ are either inequivalent or equal so that $V_k^* \circ T \circ W_i$ is just a multiple of $1_{\rho_i}$, $\kappa$ is indeed central. The formula for $\kappa(\rho_1 \rho_2)$
follows from Corollary 3.9b and the following computations where, to be concise, we write $\varepsilon$ for $\varepsilon(\rho_1\rho_2, \rho_1\rho_2)$ and $\varepsilon_{ij}$ for $\varepsilon(\rho_i, \rho_j)$ and omit indicating the objects involved when acting with the left inverses $\varphi_1$ and $\varphi_2$ of $\rho_1$ and $\rho_2$

$$\varepsilon = 1_{\rho_1} \otimes \varepsilon_{12} \otimes 1_{\rho_2} \circ \varepsilon_{11} \otimes 1_{\rho_2\rho_2} \circ 1_{\rho_1\rho_1} \otimes \varepsilon_{22} \circ 1_{\rho_1} \otimes \varepsilon_{21} \otimes 1_{\rho_2}.$$ 

thus

$$\varphi_1(\varepsilon) = \varepsilon_{12} \otimes 1_{\rho_2} \circ \varphi_1(\varepsilon_{11}) \otimes 1_{\rho_2\rho_2} \circ 1_{\rho_1} \otimes \varepsilon_{22} \circ 1_{\rho_1} \otimes \varepsilon_{21} \otimes 1_{\rho_2}$$

$$= 1_{\rho_2} \otimes \varphi_1(\varepsilon_{11}) \otimes 1_{\rho_2} \circ \varepsilon_{12} \otimes 1_{\rho_2} \circ 1_{\rho_1} \otimes \varepsilon_{22} \circ 1_{\rho_1} \otimes \varepsilon_{21} \otimes 1_{\rho_2}$$

$$= 1_{\rho_2} \otimes \varphi_1(\varepsilon_{11}) \otimes 1_{\rho_2} \circ \varepsilon(\rho_1\rho_2, \rho_2) \circ \varepsilon_{21} \otimes 1_{\rho_2}$$

$$= 1_{\rho_2} \otimes \varphi_1(\varepsilon_{11}) \otimes 1_{\rho_2} \circ 1_{\rho_2} \otimes \varepsilon(\rho_1\rho_2, \rho_2)$$

$$= 1_{\rho_2} \otimes \varphi_1(\varepsilon_{11}) \otimes 1_{\rho_2} \circ 1_{\rho_2} \otimes \varepsilon_{22} \circ 1_{\rho_1} \circ 1_{\rho_2} \otimes \varepsilon_{12}$$

Applying $\varphi_2$ gives

$$\varphi_2\varphi_1(\varepsilon) = \varphi_1(\varepsilon_{11}) \otimes \varphi_2(\varepsilon_{22}) \circ \varepsilon_{21} \circ \varepsilon_{12}.$$

yielding the required result.

Note that since $\rho \mapsto \kappa(\rho)$ is central the value $\in \mathbb{C}$ it takes on irreducible objects depends only on the equivalence class of the object. Furthermore, as for any central element,

$$\kappa(\rho) = \sum_n \mu_n E_n$$

where the $E_n$ are the minimal central projections in $(\rho, \rho)$ and the eigenvalues $\mu_n$ of $\kappa(\rho)$ are simply the values of $\kappa$ on the corresponding irreducibles.

If we introduce a conjugation $\cdot$ then by Lemma 2.3 $\rho \mapsto \kappa(\overline{\rho}) \cdot := \kappa^\cdot(\rho)$ is again central and obviously when $\rho$ is irreducible the scalar $\kappa^\cdot(\rho)$ is just the complex conjugate of the scalar $\kappa(\overline{\rho})$. Thus this involution on the centre of our category is independent of the choice of conjugation.

We now prove the following result which shows that a braiding is in a sense automatically compatible with conjugates and that $\kappa(\rho)$ plays another role.

**Lemma 4.3.** Let $R \in (\iota, \bar{\iota})$ define a conjugate for $\rho$ then $\varepsilon(\overline{\rho}, \rho) \circ R$ defines a conjugate for $\overline{\rho}$. $R$ standard implies $\varepsilon(\overline{\rho}, \rho) \circ R$ standard for all $\rho$ if and only if $\kappa$ is unitary. Furthermore, when $R$ is standard

$$\varepsilon^{-1*}(\overline{\rho}, \rho) \circ R = \kappa(\rho)^* \otimes 1_{\overline{\rho}} \circ \overline{R},$$

$$\varepsilon(\overline{\rho}, \rho) \circ R = \kappa^\cdot(\rho)^{-1*} \otimes 1_{\overline{\rho}} \circ \overline{R}.$$ 

**Proof.** If $R$ defines a conjugate then $\varepsilon(\overline{\rho}, \rho) \circ R \neq 0$ and hence trivially defines a conjugate for $\overline{\rho}$ when $\rho$ is irreducible. Furthermore the question of whether $\varepsilon(\overline{\rho}, \rho) \circ R$ defines a conjugate for $\overline{\rho}$ is independent of the choice of $R$. Thus we may choose $R$ to be standard writing $R = \sum_i W_i \otimes W_i \circ R_i$ as in the definition of standard. But then

$$\varepsilon(\overline{\rho}, \rho) \circ R = \sum_i W_i \otimes W_i \circ \varepsilon(\overline{\rho_i}, \rho_i) \circ R_i$$

which again obviously defines a conjugate for $\overline{\rho}$. Since $\varepsilon^{-1*}(\overline{\rho}, \rho) \circ R$ defines a conjugate for $\overline{\rho}$, we know that

$$\varepsilon^{-1*}(\overline{\rho}, \rho) \circ R = X^* \otimes 1_{\overline{\rho}} \circ \overline{R},$$
where $X \in (\rho, \rho)$ is invertible. Now

$$X = (\tilde{R}^* \circ X \otimes 1_{\tilde{\rho}}) \otimes 1_{\rho} \circ 1_{\rho} \otimes R = (R^* \circ \varepsilon^{-1}(\rho, \tilde{\rho})) \otimes 1_{\rho} \circ 1_{\rho} \otimes R$$

$$= R^* \otimes 1_{\rho} \circ 1_{\tilde{\rho}} \otimes \varepsilon(\rho, \rho) \otimes \varepsilon^{-1}(\rho, \tilde{\rho}) \circ 1_{\rho} \otimes R = R^* \otimes 1_{\rho} \circ 1_{\tilde{\rho}} \otimes \varepsilon(\rho, \rho) \circ R \otimes 1_{\rho} = \kappa(\rho).$$

Now $\varepsilon(\tilde{\rho}, \rho)^{-1} \circ R = \kappa(\rho)^* \otimes 1_{\tilde{\rho}} \circ \tilde{R}$ so $R = \varepsilon(\tilde{\rho}, \rho)^* \otimes 1_{\tilde{\rho}} \circ \tilde{R} = 1_{\tilde{\rho}} \otimes \kappa(\rho)^* \circ \varepsilon^*(\rho, \tilde{\rho}) \circ \tilde{R}$ thus reversing the roles of $\rho$ and $\tilde{\rho}$ we conclude that

$$\varepsilon^*(\tilde{\rho}, \rho) \circ R = 1_{\tilde{\rho}} \otimes \kappa(\rho)^{-1} \circ \tilde{R} = \kappa^*(\rho)^{-1} \otimes 1_{\tilde{\rho}} \circ \tilde{R},$$

yielding the required result since passing from $\varepsilon$ to $\varepsilon^*$ means passing from $\kappa$ to $\kappa^*$.

Now $\varepsilon(\tilde{\rho}, \rho) \circ R$ will be standard whenever $R$ is standard if this is the case for all irreducibles $\rho$. But this means that the scalars $\kappa^*(\rho)^{-1}$ must be a phase so that $\kappa$ is unitary. Conversely, if $\kappa$ is unitary and $R$ is standard then so is $\varepsilon(\tilde{\rho}, \rho) \circ R$. $\square$

Note that the above computations show that if the braiding is unitary then it will be of use shortly.

Let $\phi$ be a normalized left inverse of an object $\rho$ with conjugates, then if $\kappa$ is unitary and $\varphi_{\rho, \rho}(\varepsilon(\rho, \rho))$ is a multiple of $1_{\rho}, \varphi$ is standard.

Proof. Let $W_i \in (\rho_i, \rho)$ be isometries with $\rho_i$ irreducible and $\sum_i W_i \circ W_i^* = 1_{\rho}$, then there are normalized left inverses $\varphi_i$ of $\rho_i$ such that

$$\varphi_{i, \sigma, \tau}(S) \varphi_{i, i}(W_i \circ W_i^*) = \varphi_{\sigma, \tau}(W_i \otimes 1_{\tau} \circ S \circ W_i^* \otimes 1_{\sigma}), \quad S \in (\rho_i \sigma, \rho_i \tau).$$

Hence

$$\varphi_{i, \rho_i, \rho_i}(\varepsilon(\rho_i, \rho_i)) \varphi_{i, i}(W_i \circ W_i^*) = \varphi_{\rho_i, \rho_i}(W_i \otimes 1_{\rho_i} \circ \varepsilon(\rho_i, \rho_i) \circ W_i^* \otimes 1_{\rho_i}) = W_i^* \circ \varphi_{\rho, \rho}(\varepsilon(\rho, \rho)) \circ W_i,$$

and when $\varphi_{\rho, \rho}(\varepsilon(\rho, \rho)) = \lambda 1_{\rho}$, we conclude, since any normalized left inverse of $\rho_i$ is standard, that

$$d(\rho_i) \lambda 1_{\rho_i} = \kappa(\rho_i) \varphi_{i, i}(W_i \circ W_i^*).$$
If $\kappa$ is unitary, taking absolute values and summing over $i$ gives $d(\rho)|\lambda|=1$. Hence 

$$\varphi_{\lambda\nu}(W_i \circ W_i^*) d(\rho) = d(\rho_i)$$

and finally

$$d(\rho) \lambda 1_\rho = d(\rho) \lambda \sum_i W_i \circ W_i^* = \sum_i W_i \circ \kappa(\rho_i) \circ W_i^* = \kappa(\rho),$$

showing that $\varphi$ is indeed standard. \hfill \Box

Now the trace we have introduced in §3 depends only on the notion of conjugate. But it does make essential use of the $^*$-operation and to define it we were forced to restrict ourselves to standard arrows defining conjugates. However, in a braided tensor category there is another natural trace which is not positive, in general, and depends on the braiding. On the other hand, this trace can be defined without reference to the $^*$-operation and can be defined using arrows defining conjugation without requiring $R$ to be standard. We set for $S,T \in (\rho,\rho)$

$$(S,T)_\varepsilon = \bar{R}^* \circ \varepsilon(\bar{\rho},\rho) \circ 1_{\bar{\rho}} \otimes (S^* \circ T) \circ R,$$

where $R \in (\iota,\bar{\rho},\rho)$ defines a conjugate for $\rho$. We see immediately, that this expression is independent of the choice of $R$. It is also easy to check that it is a trace using the argument of Lemma 4.1. However, this can also be seen by relating the above expression to the trace defined previously. In fact, by Lemma 4.3,

$$\varepsilon^*(\rho,\bar{\rho}) \circ \bar{R} = 1_{\bar{\rho}} \otimes \kappa(\rho)^{-1} \circ R,$$

$$(S,T)_\varepsilon = (S,T\kappa(\rho)^{-1})$$

and since $\rho \rightarrow \kappa(\rho)^{-1}$ is in the centre of the category it follows at once that

$$(S,T)_\varepsilon = (T^*,S^*)_\varepsilon$$

This trace has no reason in general to be multiplicative however if $\varepsilon$ is a symmetry

$$(S \otimes S',T \otimes T')_\varepsilon = (S,T)_\varepsilon (S',T')_\varepsilon.$$
5. Amenable Objects

As an application of our results, we construct $C^*$–algebras $O_\rho$ and von Neumann algebras $M_\rho$ equipped in a canonical way with an endomorphism $\hat{\rho}$ starting from suitable objects $\rho$ in a tensor $C^*$–category. In this way, we get examples of subfactors $\hat{\rho}(M_\rho) \subset M_\rho$ and, making contact with the work of Popa, we introduce the notion of an amenable object in a tensor $C^*$–category.

In this section $\mathcal{T}$ always denotes a strict tensor $C^*$–category, we will consider objects with conjugates in the sense of Section 2. By considering the full $C^*$–subcategory they generate, we may also assume that all objects have conjugates. Moreover it is always possible to add objects and hence assume, as we shall do, that $\mathcal{T}$ has direct sums and subobjects (see Appendix).

We now fix an object $\rho$ of the $C^*$–category $\mathcal{T}$ and review the construction of the $C^*$–algebra $O_\rho$ given in [6]. We first form the graded $*$–algebra

$$\dot{O}_\rho \equiv \bigoplus_{k \in \mathbb{Z}} \dot{O}_\rho^{(k)}$$

where

$$\dot{O}_\rho^{(k)} = \lim_{\to} (\rho^r, \rho^{r+k})$$

and the embedding $(\rho^r, \rho^s) \to (\rho^{r+1}, \rho^{s+1})$ is given by tensoring on the right with the identity of $\rho : X \mapsto X \otimes 1_\rho$, which may, for the purposes of this paper, be supposed to be injective. The $C^*$–algebra $O_\rho$ is defined as the completion of $\dot{O}_\rho$ with respect to the unique $C^*$–norm making the expectation $\epsilon : \dot{O}_\rho \to \dot{O}_\rho^{(0)}$,

$$\epsilon \equiv \int_\mathbb{T} \alpha_t dt,$$

continuous, where $\alpha : \mathbb{T} \to \text{Aut}(O_\rho)$ is the gauge action corresponding to the grading, namely

$$\alpha_t(X) = t^k X, \quad X \in \dot{O}_\rho^{(k)}.$$

We now recall that $O_\rho$ is equipped with a natural endomorphism $\hat{\rho}$ determined by

$$\hat{\rho}(X) = 1_\rho \otimes X, \quad X \in (\rho^r, \rho^s), \quad r, s \in \mathbb{N}_0,$$

where we identify arrows between powers of $\rho$ with their images in $O_\rho$. Furthermore, with this identification, we have

$$(\rho^r, \rho^s) \subset (\hat{\rho}^r, \hat{\rho}^s), \quad r, s \in \mathbb{N}_0.$$

Now let $\varphi$ be a positive left inverse for $\rho$ then if $X \in (\rho\sigma, \rho\tau)$,

$$\varphi_{\sigma,\tau}(X)^* \circ \varphi_{\sigma,\tau}(X) \leq \|\varphi_{\ell,\ell}(1_\rho)\| \varphi_{\sigma,\sigma}(X^* \circ X) \leq \|\varphi_{\ell,\ell}(1_\rho)\|^2 \|X\| \varphi_{\sigma,\ell}(1_\rho) \otimes 1_\sigma,$$

so that

$$\|\varphi_{\sigma,\tau}(X)\| \leq \|\varphi_{\ell,\ell}(1_\rho)\| \|X\|.$$
Now, by definition, a left inverse commutes with tensoring on the right by 1_\rho and it hence induces a linear mapping \( \hat{\varphi} \) of each \( O_\rho^{(k)} \) into itself and extends by continuity to each \( O_\rho^{(k)} \) and we obviously have

\[
\begin{align*}
\hat{\varphi}(S^*) &= \varphi(S)^*, \\
\hat{\varphi}(T\hat{\rho}(S)) &= \hat{\varphi}(T)S, \\
\hat{\varphi}(\hat{\rho}(S)T) &= S\hat{\varphi}(T)
\end{align*}
\]

The argument leading to Theorem 4.12 of [6] shows that \( \hat{\varphi} \) extends to a positive linear mapping of \( O_\rho \), also denoted by \( \hat{\varphi} \) and satisfying the above equations. Obviously, \( \hat{\varphi}(I) \) is the image in \( O_\rho \) of \( \varphi_{t,t}(1_\rho) \). If \( \varphi \) is associated with a solution \( R, \bar{R} \) of the conjugate equations for \( \rho \), then Lemma 2.7 shows that

\[
\|X^*X\| \leq \|\bar{R}^* \circ \bar{R}\| \|\hat{\varphi}(X^*X)\|, \quad X \in \rho^*, \rho^*.
\]

The analogous inequality therefore holds for \( X \in O_\rho^{(0)} \) so that \( \hat{\varphi} \) is faithful on \( O_\rho^{(0)} \).

From now on we suppose that \( \varphi \) is normalized, then iterating \( \hat{\varphi} \) gives an \((t, t)\)-valued state \( \omega \) on \( O_\rho^{(0)} \), i.e. a unit-preserving positive linear mapping \( \omega : O_\rho^{(0)} \to (t, t) \), with the property that

\[
\omega(S) = \hat{\varphi}^r(S), \quad S \in (\rho^r, \rho^r).
\]

By composing this state with the conditional expectation \( \epsilon \), we extend it to a gauge–invariant \((t, t)\)-valued state, also denoted by \( \omega \) on \( O_\rho \).

At this point we make the simplifying assumption that, as in Section 3,

\[
(t, t) = \mathbb{C},
\]

since the following construction would otherwise have to depend on the choice of an auxiliary faithful state on the Abelian C*-algebra \((t, t)\). Letting \((H_\omega, \xi_\omega, \pi_\omega)\) be the Gelfand-Naimark-Segal triple associated with \( \omega \), we denote by

\[
\mathcal{M}_\varphi \equiv \pi_\omega(O_\rho)^{''}
\]

the von Neumann algebra generated by \( \pi_\omega(O_\rho) \).

**Lemma 5.1.** There is a unique isometry \( V \) determined by

\[
V\pi_\omega(X)\xi_\omega = \pi_\omega(\hat{\rho}(X))\xi_\omega, \quad X \in O_\rho,
\]

and we have

\[
V\pi_\omega(X) = \pi_\omega(\hat{\rho}(X))V; \quad \hat{\varphi}(X) = V^*\pi_\omega(X)V, \quad X \in O_\rho.
\]

Hence \( \hat{\rho} \) extends to a normal endomorphism of \( \mathcal{M}_\varphi \) and \( \hat{\varphi} \) to a normal left inverse of \( \hat{\rho} \) on \( \mathcal{M}_\varphi \). Consequently, \( \hat{\rho}\hat{\varphi} \) is a normal conditional expectation of \( \mathcal{M}_\varphi \) onto \( \hat{\rho}(\mathcal{M}_\varphi) \). If \( \hat{\varphi} \) is faithful on \( O_\rho^{(0)} \), \( \pi_\omega \) is faithful and \( \hat{\rho}\hat{\varphi} \) is faithful.

**Proof.** The existence and uniqueness of the isometry \( V \) is evident. Since \( \hat{\rho} \) is an endomorphism, \( V\pi_\omega(X) = \pi_\omega(\hat{\rho}(X))V \). Thus, if \( X_1, X_2 \in O_\rho \),

\[
(\pi_\omega(X_1)\xi_\omega, V^*\pi_\omega(X)V\pi_\omega(X_2)\xi_\omega) = (\pi_\omega(X_1)\xi_\omega, \hat{\varphi}(X)\pi_\omega(X_2)\xi_\omega),
\]

so that 

\[
\pi_\omega(X_1)\xi_\omega = \pi_\omega(\hat{\rho}(X))\xi_\omega,
\]

and hence 

\[
V\pi_\omega(X) = \pi_\omega(\hat{\rho}(X))V.
\]

Then, since \( \hat{\varphi}(X) = V^*\pi_\omega(X)V \), we have

\[
\hat{\varphi}(X) = V^*\pi_\omega(X)V.
\]

Hence, \( \hat{\rho} \) extends to a normal endomorphism of \( \mathcal{M}_\varphi \) and \( \hat{\varphi} \) to a normal left inverse of \( \hat{\rho} \) on \( \mathcal{M}_\varphi \). Consequently, \( \hat{\rho}\hat{\varphi} \) is a normal conditional expectation of \( \mathcal{M}_\varphi \) onto \( \hat{\rho}(\mathcal{M}_\varphi) \). If \( \hat{\varphi} \) is faithful on \( O_\rho^{(0)} \), \( \pi_\omega \) is faithful and \( \hat{\rho}\hat{\varphi} \) is faithful.
showing that \( \hat{\varphi}(X) = V^*\pi_\omega(X)V \), as required. If \( \hat{\varphi} \) is faithful on \( \mathcal{O}_\rho^{(0)} \), \( \pi_\omega \) is faithful on \( (\rho^r, \rho^s) \), \( r \in \mathbb{N} \) and hence on \( \mathcal{O}_\rho \). Since \( \omega \) is gauge-invariant, \( \pi_\omega \) is faithful by the uniqueness of the norm on \( \hat{\mathcal{O}}_\rho \). \( \square \)

Note that since \( \omega \) is gauge-invariant, we have an action of the gauge group on \( \mathcal{M}_\varphi \) compatible with that on \( \mathcal{O}_\rho \) and that

\[ \pi_\omega(\rho^r, \rho^s) \subset \left( \hat{\rho}^r, \hat{\rho}^s \right). \]

A case of particular interest is when \( \varphi \) is the standard left inverse of \( \rho \) and we will then write \( \mathcal{M}_\rho \) in place of \( \mathcal{M}_\varphi \) since the von Neumann algebra is uniquely determined by \( \rho \). Furthermore, in this case \( \omega|_{\mathcal{O}_\rho^{(0)}} \) is tracial.

**Lemma 5.2.** \( \omega \) is a faithful state of \( \mathcal{M}_\varphi \). In particular, \( \pi_\omega \) is faithful and \( \xi_\omega \) is a separating vector for \( \mathcal{M}_\varphi \).

**Proof.** We shall show that \( \omega \) is a KMS state of \( \mathcal{O}_\rho \) for a certain 1–parameter group of automorphisms. To this end, let \( e^{-H} \in (\rho, \rho) \) be the positive invertible operator such that

\[ \varphi(X) = tr(X \circ e^{-H}), \quad X \in (\rho, \rho). \]

If \( e^{-H_n} \) denotes the \( n \)-th tensor power of \( e^{-H} \), then

\[ \varphi^n(X) = tr(X \circ e^{-H_n}), \quad X \in (\rho^n, \rho^n). \]

Now there is a unique continuous 1–parameter group \( \sigma_t \) of automorphisms of \( \mathcal{O}_\rho \) such that

\[ \sigma_t(X) = e^{itH_s} \circ X \circ e^{-itH_r}, \quad X \in (\rho^r, \rho^s). \]

To show that \( \omega \) is a KMS state for this 1–parameter group, it suffices to pick \( X, Y \in (\rho^r, \rho^s) \) and compute:

\[ \omega(Y^* \circ \sigma_t(X)) = tr(Y^* \circ e^{itH_s} \circ X \circ e^{-itH_r} e^{-H_r}) \]

and performing an analytic continuation, we get

\[ \omega(Y^* \circ \sigma_{t+i}(X)) = \omega(\sigma_t(X) \circ Y^*). \]

\( \square \)

Note that in the special case of \( \mathcal{M}_\rho \) our modular automorphisms \( \sigma_t \) are just gauge transformations, more precisely, we have \( \sigma_t = \tilde{\alpha}_s \) with \( s = d(\rho)^{-it} \), where \( \tilde{\alpha}_s \) denotes the normal extension of \( \alpha_s \) to \( \mathcal{M}_\rho \).

We remark that \( t \mapsto \sigma_t \) may also be defined directly in terms of the category \( \mathcal{T}_\rho \). In fact, a state on a \( C^* \)-category can be understood as a state \( \omega_\sigma \) on \( (\sigma, \sigma) \) for each object \( \sigma \) of the category. When we have a faithful state on a \( W^* \)-category, there is a corresponding modular group of automorphisms of the category[12]. \( \omega \) defines a faithful state on \( \mathcal{T}_\rho \) and the corresponding modular group is a group of automorphisms of the tensor category inducing the automorphisms \( \sigma_t \) of \( \mathcal{M}_\varphi \).

We define \( \mathcal{M}_\varphi^{(k)} \) as the \( k \)-spectral subspace of \( \tilde{\alpha} \), namely \( \mathcal{M}_\varphi^{(k)} \) is the weak closure of \( \mathcal{O}_\rho^{(k)} \). To avoid certain special cases which will be of no interest in what follows we assume that \( \mathcal{O}_\rho^{(k)} \) is non-zero for some \( k \neq 0 \) and that \( d(\rho) > 1 \).
**Corollary 5.3.** $\mathcal{M}_\rho$ is a factor if the homogeneous part $\mathcal{M}_\rho^{(0)}$ is a factor, i.e. if $\omega$ is an extremal trace of $O_\rho^{(0)}$. In this case $\mathcal{M}_\rho$ is a factor of type $\text{III}_\mu$ with $\mu = d(\rho)^{-n}$ where $n = \min\{k \in \mathbb{N} \mid \exists r \text{ s.t. } (\rho^r, \rho^{r+k}) \neq 0\}$.

**proof.** The center $Z(\mathcal{M}_\rho)$ of $\mathcal{M}_\rho$ is pointwise invariant under the modular group $\sigma$ hence it belongs to $\mathcal{M}_\rho^{(0)}$ showing that if $\mathcal{M}_\rho^{(0)}$ is a factor then $\mathcal{M}_\rho$ is a factor too. Since the centralizer $\mathcal{M}_\rho^{(0)}$ of $\omega$ is a factor, the Connes invariant\cite{[3]} $S(\mathcal{M}_\rho)$ is equal to the exponential of the spectrum of the modular group $\sigma$ of $\omega$, proving the result. \qed

**Lemma 5.4.** Let $X \in \mathcal{M}_\varphi$ then $\hat{\varphi}^n(X) \rightarrow \omega(X)I$ in the $s$–topology as $n \rightarrow \infty$. Consequently, the following conditions are equivalent:

a) $\varphi(X) = X$,
b) $\hat{\rho}(X) = X$,
c) $X \in (\iota, \iota)$.

Furthermore, $\hat{\rho}^n(X) \rightarrow \omega(X)I$ in the $\sigma$–topology as $n \rightarrow \infty$. Thus $\omega$ is the unique normal $\hat{\rho}$–invariant state and $\varphi$ is the unique left inverse of $\hat{\rho}$ having a normal invariant state. More generally, if $X, Y, A \in \mathcal{M}_\varphi$ and $A$ has grade $k \geq 0$, then

$$\hat{\rho}^n(X)A\hat{\rho}^n(Y) \rightarrow \omega(X\hat{\rho}^k(Y))A$$

in the $\sigma$–topology as $n \rightarrow \infty$.

**Proof.** If $X \in (\rho^r, \rho^{r+1})$ then $\hat{\varphi}(X) = \omega(X)I$. Given $X \in \mathcal{M}_\varphi^{(0)}$ and $\varepsilon > 0$, we may pick $r \in \mathbb{N}$ and $Y \in (\rho^r, \rho^{r+1})$ such that $\|Y\| \leq \|X\|$ and $\|\omega(Y)I - \hat{\varphi}(X)\|\omega\| < \varepsilon$. Hence $\|\hat{\varphi}^n(Y - X)\|\omega\| = \|\omega(Y)I - \hat{\varphi}^n(X)\|\omega\| < \varepsilon$, for $n \geq r$, and $|\omega(Y - X)| < \varepsilon$. Now $(\iota, \rho^k)$ is finite dimensional and is mapped into itself by $\varphi$. Any eigenvalue has modulus $\leq 1$ and if we know that there are no eigenvalues on the unit circle for $k \neq 0$ then $\hat{\varphi}^n(X) \rightarrow 0 = \omega(X)I$ as $n \rightarrow \infty$ for $X \in (\iota, \rho^k)$, hence for $X \in \cup_r (\rho^r, \rho^{r+k})$ and, arguing as above, even for $X \in \mathcal{M}_\varphi^{(k)}$. However, $\hat{\varphi}(X) = \lambda X$ with $|\lambda| = 1$ implies $\hat{\varphi}(XX^*) \geq \hat{\varphi}(X)\varphi(X)^* = XX^*$. Thus, evaluating in the state $\omega$, we see that $XX^* = \omega(XX^*)I$ and this implies that $X = 0$ since $(\iota, \rho^k)$ would otherwise contain a unitary operator which is impossible as we are assuming $d(\rho) > 1$. Thus $\hat{\varphi}^n(X) \rightarrow \omega(X)I$ in the $s$-topology, as required. To show that $\hat{\rho}^n(X) \rightarrow \omega(X)I$ in the $\sigma$–topology, it suffices to show that $\hat{\rho}^n(X)\xi$ tends weakly to $\omega(X)\xi$. However if $Y \in \mathcal{M}_\varphi$, $\omega(Y\hat{\rho}^n(X)) = \omega(\hat{\varphi}^n(Y)X) \rightarrow \omega(Y)\omega(X)$ as required. $\omega$ is now the unique normal $\hat{\rho}$–invariant state. Thus the associated modular group commutes with $\hat{\rho}$ and leaves $\hat{\rho}(\mathcal{M}_\varphi)$ globally stable. Hence, by a result of Takesaki \cite{[27]}, $\hat{\rho}\varphi$ is the unique normal condition expectation leaving $\omega$ invariant or, equivalently, since any left inverse of $\hat{\rho}$ is normal and any state it leaves invariant is $\hat{\rho}$–invariant, $\varphi$ is the unique left inverse with a normal invariant state.

To complete the proof, it suffices, since $\omega$ is faithful to show that for each $\ell \in \mathbb{Z}$, $r \in \mathbb{N}$ and $S \in (\rho^r, \rho^{r+\ell})$,

$$\omega(S\hat{\rho}^n(X)A\hat{\rho}^n(Y)) \rightarrow \omega(SA)\omega(X\hat{\rho}^k(Y))$$

But for $n \geq r$ and $\ell \geq 0$, we have

$$\omega(S\hat{\rho}^n(X)A\hat{\rho}^n(Y)) = \omega(\hat{\rho}^{n+\ell}(X)SA\hat{\rho}^n(Y)) = \omega(\hat{\rho}^\ell(X)\varphi^n(SA)Y)$$
whose limit as $n \to \infty$ is $\omega(\hat{\rho}^\ell(X)Y)\omega(SA)$. If $\ell \leq 0$, a similar computation gives a limit $\omega(X\hat{\rho}^{-\ell}(Y))\omega(SA)$. Noting that $\omega(SA) = 0$ unless $\ell = -k$, we have the required result.

The last result shows that our systems $(\mathcal{M}_\varphi, \hat{\rho})$ have some asymptotic Abelian-ness and a cluster property. These can be used, as usual, to derive further interesting properties.

**Lemma 5.5.** Let $r \in \mathbb{N}_0$ and $E, E'$ be two projections in $\mathcal{M}_\varphi^{(0)}$. If there is an $X \in \mathcal{M}_\varphi$ with $E\hat{\rho}^r(X)E' \neq 0$, then there is a $Y \in \mathcal{M}_\varphi^{(0)}$ such that $E\hat{\rho}^r(Y)E' \neq 0$.

**Proof.** We may suppose that $X$ has grade $k$. Then letting $F$ be the left support of $\hat{\rho}^r(X)E'$, $F \in \mathcal{M}_\varphi^{(0)}$ and $EF \neq 0$. Now $E\hat{\rho}^r+n(X^*)F \neq 0$ for some $n$ since

$$E\hat{\rho}^r+n(X^*)F\hat{\rho}^r+n(X) \rightarrow EF\omega(X^*X) \neq 0.$$ 

But then $E\hat{\rho}^r(\hat{\rho}^n(X^*)X)E' \neq 0$, completing the proof. \qed

**Corollary 5.6.** Let $\tau$ denote the restriction of $\hat{\rho}$ to $\mathcal{M}_\varphi^{(0)}$ then

$$(\tau^r, \tau^r) = (\hat{\rho}^r, \hat{\rho}^r) \cap \mathcal{M}_\varphi^{(0)}, \quad r \in \mathbb{N}_0.$$ 

**Proof.** Let $E$ be a projection in $(\tau^r, \tau^r)$ then $E\hat{\rho}^r(Y)(I - E) = 0$ for all $Y \in \mathcal{M}_\varphi^{(0)}$. Hence by Lemma 5.5, $E\hat{\rho}^r(X)(I - E) = 0$ for all $X \in \mathcal{M}_\varphi$, i.e. $E \in (\hat{\rho}^r, \hat{\rho}^r)$. \qed

Note that for $r = 0$ this corollary relates the centres of the two von Neumann algebras. We shall see later, that, with slightly strengthened hypotheses, cf. Corollary 5.12, $(\hat{\rho}^r, \hat{\rho}^r)$ is in the zero grade part so that the spaces of intertwiners will in fact coincide. We also give at this stage a result of a similar nature which we will need later on.

**Proposition 5.7.** Let $\mathcal{M}_\omega^{(0)}$ denote the fixed-point algebra of $\mathcal{M}_\varphi^{(0)}$ under the modular group of $\omega$ and $\mu$ the restriction of $\hat{\rho}$ to $\mathcal{M}_\omega^{(0)}$. then

$$(\mu^r, \mu^r) = (\hat{\rho}^r, \hat{\rho}^r) \cap \mathcal{M}_\omega^{(0)}, \quad r \in \mathbb{N}_0.$$ 

**Proof.** We must just prove the analogue of Lemma 5.5. The only point to note is that for the analogous proof to go through instead of taking $X$ to be of grade $k$ we must take it to transform like a character under the modular group. This can be done since the finite–dimensional spaces $(\rho^r, \rho^r)$ are invariant under the modular group and their union is a total set in $\mathcal{M}_\varphi$. \qed

If we consider a pair $(\mathcal{M}, \rho)$ consisting of a factor $\mathcal{M}$ and a normal endomorphism $\rho$ and if $\mathcal{M}$ admits a unique normal $\rho$–invariant state and this state is faithful then we may define the Jones index of $\rho$ to be $d(\varphi)^2$, where $\varphi$ is the unique left inverse of $\rho$ leaving the state invariant. We have just seen in Lemma 5.4 how this situation arises in our context. When the state in question is a trace, this definition coincides with the usual definition of the Jones index for the inclusion $\rho(\mathcal{M}) \subset \mathcal{M}$. This index continues to be multiplicative in the sense that if endomorphisms $\rho$, $\sigma$, and $\rho\sigma$ of
If \( X \) is as in the \( s \)-topology as supposed hence just a multiple of \( Y = \omega \) and multiplying on the left by \( k \rightarrow \infty \), we can compute \( \mu \); it is just

\[
\omega(\theta(r + k, 1)\theta(r, 1)^*) = \sum_j \lambda_j^{k+1} \neq 0,
\]
where the sum is taken over the eigenvalues of the density matrix determining the left inverse $\Phi$, cf. Lemma 5.2. Writing $\theta(r; k) := \theta(r + k, 1)\theta(r, 1)^*$ for short, $\Phi(\theta(r; k)) \rightarrow \mu I$ and this would imply that $\omega(\Phi(\theta(r; k)^*\Phi(\theta(r; k)))$ tends to $\mu^2 I$. However, choosing an orthonormal basis $\psi_i$ of $H$ consisting of eigenvectors of the density matrix determining $\Phi$, we have $\hat{\Phi}(Y) = \sum_i \lambda_i \psi_i^* Y \psi_i$. Thus

$$\hat{\Phi}(\theta(r; k)) = \sum_i \lambda_i \psi_i^* \rho^r(\psi_i).$$

A further computation shows that

$$\omega(\hat{\Phi}(\theta(r; k)^*\Phi(\theta(r; k))) = \sum_i \lambda_i^3 \neq \mu^2$$

and this contradiction shows that $X = 0$ and completes the proof. \hfill \Box

We now take our object $\rho$ to be a continuous unitary representation of a compact group $G$ on $H$. Now $O_\rho$ can be identified with the fixed points of $O_H$ under the action of $G$ induced by $\rho$ [7]. Since the standard left inverse of $\rho$ is just the restriction of the standard left inverse of $H$, $M_\rho$ is just the von Neumann subalgebra of $M_H$ generated by $O_\rho$. As the state $\omega$ on $M_H$ is invariant under the action of $G$, $M_\rho$ is the fixed-point algebra of $M_H$ under the action of $G$ induced by $\rho$. More generally, any faithful left inverse $\varphi$ of $\rho$ extends to a $G$–invariant faithful left inverse $\Phi$ of $H$. Hence, arguing as above, $M_\varphi$ is the fixed–point algebra of $M_{\hat{\varphi}}$ under the action of $G$ induced by $\rho$.

It follows from Lemma 5.8, that $M_{\varphi'} \cap M_{\hat{\varphi}} = CI$ and that $M_{\varphi^{(0)}}$ and hence $M_\varphi$ are factors but this lemma also yields a stronger result.

**Theorem 5.9.** Let $\rho$ be a continuous unitary representation of a compact group $G$ of finite dimension $d > 1$ then

$$(\rho^r, \rho^s) = (\hat{\rho}^r, \hat{\rho}^s), \quad r, s \in \mathbb{N}_0$$

on $M_\varphi$.

**Proof.** Let $X \in (\hat{\rho}^r, \hat{\rho}^s)$ and $\psi \in H^r$, $\psi' \in H^s$ then $\psi'^* X \psi \in M_{\varphi'} \cap M_{\hat{\varphi}}$ so that $\psi'^* X \psi \in CI$. Thus $X \in (H^r, H^s) \cap M_\varphi = (\rho^r, \rho^s)$ as required. \hfill \Box

On the basis of the above result, it is clear that two systems of the form $(M_{\hat{\varphi}}, \hat{\rho})$ are isomorphic if and only if the density matrices defining the left inverses $\Phi$ are unitarily equivalent.

Now S. Popa [22] has introduced various notions of amenability for inclusions of factors with finite index, at the same time providing a classification. In particular the graph $\Gamma_{N,M}$ of an inclusion $N \subset M$ of factors is said to be amenable if $\|\Gamma_{N,M}\|^2$ equals the minimal index of the inclusion. The analogue condition for an object $\rho$ of a tensor $C^*$–category is the equality $d(\rho) = \|m^\rho\|$ discussed at the end of §3. We prefer, however, to take the conclusions of Theorem 5.9 as the definition of an object $\rho$ being amenable. The analogous property for $\hat{\rho}$ considered as an endomorphism of the $C^*$–algebra and proved as Theorem 3.5 of [7] will be termed $C^*$–amenable. A special object in a symmetric tensor $C^*$–category is $C^*$–amenable, Lemma 4.14 of [6]. In the same way if $\varphi$ is a normalized left inverse for $\rho$, we say that $\varphi$ is amenable if the corresponding property holds when $\hat{\rho}$ denotes the induced
endomorphism of $M_\varphi$. This definition is closely related to one of the consequences of what Popa terms strong amenability, namely that the relative commutants in a Jones tunnel coincide with the relative commutants in the associated standard model $N_{st}^\ast \subset M_{st}^\ast$, i.e.

$$N_i^{st'} \cap N_j^{st} = N_i' \cap N_j.'$$

Associated with an object $\rho$ in a tensor $C^*$-category $T$ we have a tensor $C^*$-category, $T_\rho$, whose objects are the integers $\mathbb{N}_0$ and where $(r, s) = (\rho^r, \rho^s)$, $r, s \in \mathbb{N}_0$, with the structure inherited from $T$. In practice the objects $\rho^r$ will usually be distinct, in particular when $d(\rho) > 1$, and $T_\rho$ can be identified with the full subcategory of $T$ whose objects are the powers of $\rho$. When $\rho$ is an object of $T_f$, the construction of $M_\varphi$ provides a natural *-functor $F_\varphi$ of $T_\rho$ into the full tensor $C^*$-subcategory of $\text{End}(M_\varphi)$ whose objects are the powers of $\hat{\rho}$. $F_\varphi$ maps $\rho^r$ to the endomorphism $\hat{\rho}^r$ and an arrow $T$ into its embedding in $M_\varphi$ again denoted by $T$. Then $\varphi$ amenable just means that the functor $F_\varphi$ is full, i.e. that

$$(\hat{\rho}^r, \hat{\rho}^s) = (\rho^r, \rho^s)$$

for all integers $r, s \geq 0$.

Note that we are assuming $(i, i) = \mathbb{C}$ hence, if $\varphi$ is amenable, then $M_\varphi$ is a factor since

$$Z(M_\varphi) = (i, i) = (i, i) = \mathbb{C}.$$  

In our discussion of $O_\rho$ and $M_\varphi$ we have, up till now, made no direct use of a conjugate $\bar{\rho}$ for $\rho$; indirectly, of course, it has been used to provide a standard left inverse. The problem, of course, is that $\bar{\rho}$ cannot, in general, be regarded as an endomorphism of $O_\rho$ or $M_\varphi$ nor can solutions $R, \hat{R}$ of the conjugate equations, in general, be interpreted as elements of $O_\rho$ or $M_\varphi$. However, the following results provide a certain substitute.

**Lemma 5.10.** Let $\rho$, an object of $T_f$, have a conjugate $\bar{\rho}$ that is a direct sum of subobjects of tensor powers of $\rho$ and let $F : T_\rho \to \hat{T}$ be a tensor *-functor into a strict tensor $C^*$-category $\hat{T}$ with subobjects and (finite) direct sums. Then $\hat{\rho} := F(\rho)$ has a conjugate in $\hat{T}$ and for every left inverse $\varphi$ of $\rho$ associated with solutions of the conjugate equations there is a left inverse $\hat{\varphi}$ of $\hat{\rho}$ such that, with an obvious abuse of notation, $F\varphi = \hat{\varphi}F$. If the units of $T$ and $\hat{T}$ are irreducible then $d(\hat{\rho}) \leq d(\rho)$. If, furthermore, the map from $(\rho, \rho)$ to $(\hat{\rho}, \hat{\rho})$ induced by $F$ is surjective then $d(\hat{\rho}) = d(\rho)$.

**Proof.** $\hat{\rho}$ is a direct sum of subobjects of tensor powers of $\rho$. Take a direct sum of objects representing the images of the corresponding projections under $F$. This can be done since $\hat{T}$ has subobjects and direct sums. The resulting object should be a conjugate for $F(\rho)$. This can be checked by taking solutions of the conjugate equations for $\rho$, using them to construct solutions of the conjugate equations for $\hat{\rho}$ and performing the necessary computations. Those who wish to avoid explicit computations can take the following longer route. The obvious full and faithful canonical tensor *-functor from $T_\rho$ to $\hat{T}$ extends to a full and faithful tensor *-functor between the canonical completions $C(T_\rho)$ and $C(\hat{T})$ under subobjects and direct sums (see Appendix). Since $C(T_\rho)$ contains an object whose image is a conjugate for $\rho$, this object must be a conjugate for $\rho$ considered as an object of $C(T_\rho)$. Now $F$ extends to a tensor *-functor $C(F) : C(T_\rho) \to C(\hat{T})$ so $C(F)(\rho)$ has
a conjugate which can be taken to be in $\hat{T}$ as every object of $C(\hat{T})$ is equivalent to an object of $\hat{T}$. The remaining assertions are now obvious. □

**Proposition 5.11.** Let $\rho$ have a conjugate $\hat{\rho}$ which is a direct sum of subobjects of tensor powers of $\rho$ then $\hat{\rho}$ has a conjugate $\hat{\rho}$ in $\text{End}\mathcal{M}_\varphi$ and the index of the inclusion $\hat{\rho}(\mathcal{M}_\varphi) \subset \mathcal{M}_\varphi$ with respect to the normal conditional expectation $\mathcal{E} := \hat{\rho}\hat{\varphi}$ is

$$[\mathcal{M}_\varphi : \hat{\rho}(\mathcal{M}_\varphi)]_{\mathcal{E}} = d(\varphi)^2.$$ 

In particular, if $\mathcal{M}_\varphi$ is a factor, $d(\hat{\rho}) \leq d(\rho)$ and if $\varphi$ is amenable, $d(\hat{\rho}) = d(\rho)$.

**Proof.** We apply Lemma 5.10 to the functor $F_\varphi$ from $\mathcal{T}_\rho$ to $\text{End}\mathcal{M}_\varphi$ noting that the image category not only has subobjects but also direct sums since $\mathcal{M}_\varphi$ is properly infinite. It only remains to remark that the formula for the index follows from [15], since we have a solution of the conjugate equations for $\hat{\rho}$ with associated left inverse $d(\varphi)\hat{\varphi}$. □

We now want to give conditions implying that $\varphi$ is amenable. One possibility is to follow the path taken in [6] when proving (Lemma 4.14) that special objects in a symmetric tensor $C^*$–category are $C^*$–amenable. We will first generalize the most important step, the Grading Lemma, Lemma 4.5 of [6].

**Lemma 5.12.** Let $\rho$ be a unital endomorphism of a $C^*$–algebra $A$ and suppose that for each $m \in \mathbb{N}$ there is an isometry $S_m \in (\iota, \rho^{n(m)})$ such that

$$S_m^*\rho^m(S_m) = \lambda_m,$$

where $\lambda_m = \lambda_m^*, \rho(\lambda_m) = \lambda_m, \|\lambda_m\| < 1$ and

$$\prod_m \frac{1 + \|\lambda_m\|}{1 - \|\lambda_m\|} < +\infty.$$

Then $[\rho]^k$, the norm closure of $[\rho]^k = \bigcup_{r, r+k \geq 0}(\rho^r, \rho^{r+k}), \ k \in \mathbb{Z}$, determine a grading of the $C^*$–algebra that they generate and the canonical map of $\mathcal{O}_\rho$ into $A$ is injective.

**Proof.** Let

$$X = X_0 + \sum_{k=-n}^{-m} + \sum_{k=m}^n X_k$$

with $X_k \in [\rho]^k$ then, setting

$$X' = (I - \rho^r(\lambda_m))^{-1}(\rho^r(S_m^*)X\rho^r(S_m) - \rho^r(\lambda_m)X),$$

we have

$$X' = X_0 + \sum_{k=-n}^{-m+1} + \sum_{k=m+1}^n X'_k,$$

where

$$X'_k = (I - \rho^r(\lambda_m))^{-1}(\rho^r(S_m^*)\rho^{r+k}(S_m) - \rho^r(\lambda_m))X_k \in [\rho]^k.$$
provided \( r \) is chosen so that each \( X_k \in (\rho^r, \rho^{r+k}) \). We conclude that \( X = 0 \) implies \( X_0 = 0 \). But \( X = 0 \) implies \( X^*X = 0 \) and \( (X^*X)_0 = \sum_k X_k^*X_k \) so \( X_k = 0 \). Thus the linear span of the \([\hat{\rho}]^k\) is a \( \mathbb{Z} \)-graded \(*\)-algebra \([\hat{\rho}]\). Now \( \|X\| \leq \frac{(1 + \|\lambda_m\|)}{(1 - \|\lambda_m\|)}\|X\| \). Thus

\[
\|X_0\| \leq \prod_m \frac{1 + \|\lambda_m\|}{1 - \|\lambda_m\|}\|X\| < +\infty.
\]

This computation shows that the projection \( m_0 \) onto the zero grade part of \([\hat{\rho}]\) is continuous for any \( C^*\)-norm on \([\hat{\rho}]\) (cf. Lemma 2.10 of [7]). The argument of Lemma 2.11 of [7] now shows that \([\hat{\rho}]\) has a unique \( C^*\)-norm extending the \( C^*\)-norm on \([\hat{\rho}]^0\). But \([\rho]^0\) being an inductive limit of \( C^*\)-algebras admits a unique \( C^*\)-norm. Hence \([\hat{\rho}]\) admits a unique \( C^*\)-norm which must be the \( C^*\)-norm defined by both \( O_\rho \) and the ambient \( C^*\)-algebra \( A \). Thus the canonical map of \( O_\rho \) into \( A \) is injective and the \([\hat{\rho}]^k\) determine the grading on the image. \( \square \)

We immediately have the following corollary.

**Corollary 5.13.** Let \( \rho \) be an object in a tensor \( C^*\)-category and suppose that for each \( m \in \mathbb{N} \) there exist an isometry \( S_m \in (\iota, \rho^{n(m)}) \) such that

\[
S_m^* \otimes 1_{\rho^m} \circ 1_{\rho^m} \otimes S_m := \lambda_m,
\]

satisfies the conditions in Lemma 5.11. If \( \hat{\rho} \) denotes the canonical endomorphism on \( O_\rho \) then \([\hat{\rho}]^k = O_\rho^{(k)}\). In the same way if \( \hat{\rho} \) denotes the canonical endomorphism on \( M_\varphi \), where \( \varphi \) is a normalized left inverse for \( \rho \), then \([\hat{\rho}]^k \) is \( \sigma \)-dense in \( M^{(k)}_\varphi \).

In all applications of the above corollary to date, \( \lambda_m = \pm d^{-m} 1_{\rho^m} \). Apart from the case of special objects in a symmetric tensor \( C^*\)-category (cf. Lemma 3.7 of [6]), the result applies to real objects of dimension greater than one in an arbitrary tensor \( C^*\)-category. These are objects \( \rho \) for which there is a solution \( R \in (\iota, \rho^2) \), \( \tilde{R} = R \) of the conjugate equations. In particular \( \rho \) is self-conjugate. To see that the corollary can be applied, it suffices to note that any power \( \rho^m \) of a real object is real. The corresponding operator \( R \) can be computed inductively using the formula for products given in the proof of Theorem 2.4:

\[
R_m = 1_\rho \otimes R_{m-1} \otimes 1_\rho \circ R; \quad R_1 = R.
\]

Thus \( R_m^* \circ R_m = d^m \) where \( d := R^* \circ R \). We can now take \( S_m = \sqrt{d^{-m}} R_m \) in Corollary 5.13.

It is easily checked that the direct sum of real objects is real and that an object of the form \( \sigma \oplus \bar{\sigma} \) or \( \sigma \bar{\sigma} \) is real. If \( \rho \) is self-conjugate and irreducible then we can either choose \( \tilde{R} = R \) or \( \tilde{R} = -R \). The second case we call pseudoreal adopting the terminology used for group representations. We may now give a characterization of real objects in terms of irreducibles.

**Theorem 5.14.** An object \( \rho \) is real if and only if \((\sigma, \rho)\) and \((\bar{\sigma}, \rho)\) have the same dimension for any irreducible \( \sigma \) and \((\sigma, \rho)\) has even dimension for any pseudoreal irreducible \( \sigma \).

**Proof.** If \( \rho \) satisfies the conditions of the theorem, it is real by the above remarks. On the other hand, if \( \rho \) is real, it is self-conjugate so the first condition must be
satisfied. Now suppose $\sigma$ is a pseudoreal irreducible, then consider the antilinear map $X \mapsto X^\bullet$ on $(\sigma, \rho)$ where we agree to take $\tilde{R}_\rho = R_\rho$ and $\tilde{R}_\sigma = -R_\sigma$. In this case, it is easily checked that $X^{\bullet\bullet} = -X$. This implies that $(\sigma, \rho)$ has even dimensions.

This characterization of real objects and the above formulae make it clear that we may restrict ourselves to standard solutions of the conjugate equations when defining real objects.

If we can take $\tilde{R} = -R$ as in the case of a pseudoreal irreducible then again Corollary 5.12 is applicable; the formulae differ from the real case only by the presence of a minus sign when $m$ is odd. It is thus natural to ask whether Corollary 5.12 may be applied to any self-conjugate object of dimension greater than one. We do not know the answer.

The next lemma provides a simpler and more general way of proving $C^*$–amenability than that taken in Lemma 4.14 of [6].

**Lemma 5.15.** Suppose that $[\hat{\rho}]^k = O^k_\rho$ and that, for some $n \in \mathbb{N}$, $(\iota, \rho^n) \neq 0$ then $\rho$ is $C^*$–amenable.

**Proof.** If $S \in (\iota, \rho^n)$ is an isometry, so is $S^{\otimes n} \in (\iota, \rho^{mn})$, $m \in \mathbb{N}$. Suppose $X \in \left(\hat{\rho}^r, \hat{\rho}^s\right)$ then $X = \hat{\rho}^r(V^*)X\hat{\rho}^s(V)$ for any isometry $V$. Since $[\hat{\rho}]^k = O^k_\rho$, we can find $X_m \in (\rho^{r^k(m)n}, \rho^{s^k(m)n})$ converging in norm to $X$. Picking an isometry $S_m \in (\iota, \rho^{k(m)n})$ and setting $Y_m := \hat{\rho}^s(S_m^*)X_m\hat{\rho}^r(S_m) \in (\rho^r, \rho^s)$ we have

$$\|Y_m - X\| \leq \|\hat{\rho}^s(S_m^*)(X_m - X)\hat{\rho}^r(S_m)\| \leq \|X_m - X\|.$$  

But $(\rho^r, \rho^s)$ is closed so $X \in (\rho^r, \rho^s)$ and $\rho$ is $C^*$–amenable. □

We now give a criterion for $\varphi$ to be amenable in the form of a theorem.

**Theorem 5.16.** Let $\rho$ be real or irreducible and pseudoreal then $\rho$ is $C^*$–amenable. Let $R \in (\iota, \rho^n)$ be such that

$$R^*XR = d(\varphi)\hat{\varphi}(X), \quad X \in (\rho, \rho),$$

then $\varphi$ is amenable if and only if

$$R^*XR = d(\varphi)\hat{\varphi}(X), \quad X \in (\hat{\rho}, \hat{\rho}).$$

Hence, in particular, in this case $d(\rho) = d(\hat{\rho})$ and $M_\varphi$ is a factor.

**Proof.** The $C^*$–amenability of $\rho$ follows from Corollary 5.13 and Lemma 5.15. Now define $S_m$ as following Corollary 5.13. This corollary shows that $(\hat{\rho}^r, \hat{\rho}^s) \subset M_\rho^{(s-r)}$. Hence, given $X \in (\hat{\rho}^r, \hat{\rho}^s)$ there exists $X_m \in (\rho^{r+k(m)}, \rho^{s+k(m)})$ uniformly bounded such that $X_m \xi_\omega$ tends to $X \xi_\omega$. Set $Y_m = \hat{\rho}^s(S_{k(m)}^*)X_m\hat{\rho}^r(S_{k(m)}) \in (\rho^r, \rho^s)$ and we have

$$\|(Y_m - X)\xi_\omega\|^2 = \|\hat{\rho}^s(S_{k(m)}^*)(X_m - X)\hat{\rho}^r(S_{k(m)})\xi_\omega\|^2$$

$$\leq \|(X_m - X)\hat{\rho}^r(S_{k(m)})\xi_\omega\|^2 = \omega(\hat{\rho}^r(S_{k(m)}^*)(X_m - X)\hat{\rho}^r(S_{k(m)})).$$

Now $(X_m - X)^*(X_m - X) \in (\rho^{r+k(m)}, \rho^{r+k(m)})$ and for any such element $B$ we have

$$\hat{\rho}^r(S_{k(m)}^*)B\hat{\rho}^r(S_{k(m)}) = \hat{\psi}^{k(m)}(B),$$
where \( \hat{\psi} \) is the right inverse of \( \hat{\rho} \) such that \( \hat{\psi}_{i,t} = \hat{\varphi}_{i,t} \). Although we do not, a priori, know that \( \mathcal{M}_\varphi \) is a factor, we may still conclude that the same relation holds between the powers of \( \hat{\varphi} \) and \( \hat{\psi} \), since the computation in Theorem 3.8 is valid without assuming the tensor unit to be irreducible. Since, \( \omega \circ \hat{\varphi} = \omega \), we get

\[
\omega(\hat{\psi}^{k(m)}(B)) = \omega(\hat{\varphi}^r \hat{\psi}^{k(m)}(B)) = \omega(\hat{\psi}^{r+k(m)}(B)) = \omega(\hat{\varphi}^{r+k(m)}(B)) = \omega(B).
\]

We conclude that \( \|(Y_m - X)\xi_\omega\| \leq \|(X_m - X)\xi_\omega\| \) so that \( Y_m\xi_\omega \) converges to \( X\xi_\omega \). Since \( \xi_\omega \) is separating and \( Y_m \) is uniformly bounded, \( Y_m \) converges to \( X \) in the strong operator topology. But \( (\rho^r, \rho^s) \) is closed in this topology, so \( X \in (\rho^r, \rho^s) \) and \( \varphi \) is amenable. The converse is trivial. \( \square \)

The following result is an immediate corollary.

**Corollary 5.17.** Let \( \rho \) be real or irreducible and pseudoreal then \( \varphi \) is amenable if and only if \( (\rho, \rho) = (\hat{\rho}, \hat{\rho}) \).

In a way, the natural condition for \( \varphi \) to be amenable would be simply that \( d(\rho) = d(\hat{\rho}) \). However, we do not know a priori that \( \mathcal{M}_\varphi \) is a factor so that \( d(\hat{\rho}) \) might not be defined. Therefore, in the next result, we add a condition which guarantees that \( \mathcal{M}_\varphi \) is a factor and which is anyway automatically fulfilled whenever \( \mathcal{M}_\varphi \) is a factor.

**Theorem 5.18.** Let \( \rho \) be real or irreducible and pseudoreal and suppose there is an \( R \in (i, \rho^2) \) such that

\[
R^*XR = d(\varphi)\hat{\varphi}(X), \quad X \in (\hat{\rho}, \hat{\rho}),
\]

then \( \mathcal{M}_\varphi \) is a factor and if \( d(\rho) = d(\hat{\rho}) \) then \( \varphi \) is amenable.

**Proof.** Let \( Z \) be in the centre of \( \mathcal{M}_\varphi \) then \( R^*ZR = ZR^*R = Zd(\varphi) \). Hence \( Z = \hat{\varphi}(Z) \) so that \( \mathcal{M}_\varphi \) is a factor by Lemma 5.4. Since \( d(\rho) = d(\hat{\rho}) \), a standard \( R' \in (i, \rho^2) \) for \( \rho \) will also be standard for \( \hat{\rho} \). Hence

\[
R'^*XR' = R'^*\hat{\rho}(X)R', \quad X \in (\hat{\rho}, \hat{\rho}).
\]

Now there is an invertible \( Y \in (\rho, \rho) \) such that \( 1_\rho \otimes Y \circ R' \) induces \( d(\varphi)\hat{\varphi} \). Hence

\[
R'^*\hat{\rho}(Y^*XY)R' = d(\varphi)\hat{\varphi}(X), \quad X \in \mathcal{M}_\varphi,
\]

so for \( X \in (\hat{\rho}, \hat{\rho}) \) this is just \( R'^*Y^*XYR' \) and taking \( Y \otimes 1_\rho \circ R' \) for \( R \) in Theorem 5.16, the result follows. \( \square \)

**Corollary 5.19.** Let \( \rho \) be real or irreducible and pseudoreal then if \( \varphi \) is not amenable, either \( \mathcal{M}_\varphi \) is not a factor, or if it is a factor \( d(\rho) > d(\hat{\rho}) \).

**Corollary 5.20.** If \( \rho \) is real or irreducible and pseudoreal and \( d(\rho) = ||m^\rho|| \) then \( \varphi \) is amenable if and only if \( \mathcal{M}_\varphi \) is a factor.

**Proof.** \( d(\rho) = ||m^\rho|| \leq ||m^\rho|| \leq d(\hat{\rho}) \leq d(\rho) \) so that we have equality everywhere. \( \square \)

The next results provide us with a class of examples of amenable objects.
Proposition 5.21. Let $\rho$ be real or irreducible and pseudoreal and let $\mathcal{M}_\varphi$ be a factor then $\hat{\varphi}$ is amenable.

Proof. It is obvious from Lemma 5.12 that $\mathcal{M}_\varphi = \mathcal{M}_{\hat{\varphi}}$ and $\hat{\rho} = \hat{\varrho}$. □

For our next example, we would like to make contact with the work of Popa. To facilitate this we give a result relating an endomorphism $\hat{\rho}$ and its restriction to the grade zero part.

Proposition 5.22. Let $\rho$ be real or irreducible and pseudoreal and let $\tau$ denote the restriction of $\hat{\rho}$ to $\mathcal{O}_\rho^{(0)}$ then

$$(\hat{\rho}^r, \hat{\rho}^r) = (\tau^r, \tau^r), \quad r \in \mathbb{N}_0,$$

hence $\|m_{\hat{\rho}}\| = \|m_{\tau}\|$. Furthermore, if $\mathcal{O}_\rho$ has trivial centre, $d(\hat{\rho}) = d(\tau)$. The analogous statement holds if $\mathcal{O}_\rho$ is replaced by $\mathcal{M}_\varphi$. In the case of $\mathcal{M}_\varphi$, the Jones indices of $\hat{\rho}$ and $\tau$ coincide.

Proof. To prove the equality $(\hat{\rho}^r, \hat{\rho}^r) = (\tau^r, \tau^r)$, it suffices to consider the case of $\mathcal{M}_\varphi$ since this implies the result for $\mathcal{O}_\rho$. However, this case is a consequence of Corollary 5.6 and Corollary 5.12. Now by the characterisation of left inverses in Lemma 2.5, every left inverse $\chi$ of $\hat{\rho}$ preserves the grading and defines a left inverse of $\tau$ by restriction. Since $(\hat{\rho}, \hat{\rho}) = (\tau, \tau)$, every left inverse of $\tau$ arises in this way. In particular, the standard left inverse arises in this way. However, the dimensions of the faithful left inverses cannot change on restriction since the corresponding Jones projections have grade zero. Hence $d(\hat{\rho}) = d(\tau)$. For the same reason, the Jones indices of $\hat{\rho}$ and $\tau$ coincide. □

Proposition 5.23. Let $\mathcal{N} \subset \mathcal{M}$ be a proper inclusion of properly infinite factors with normal conditional expectation $\mathcal{E}$ and $\Gamma_{\mathcal{N}, \mathcal{M}}$ amenable and ergodic then $\varphi$ is amenable, where $\varphi$ is the left inverse of the canonical endomorphism $\gamma : \mathcal{M} \to \mathcal{N} \subset \mathcal{M}$ induced by $\mathcal{E}$ (i.e. the composition of $\mathcal{E}$ and its dual).

Proof. Saying that $\Gamma_{\mathcal{N}, \mathcal{M}}$ is ergodic in the sense of Popa just means that the completion of $\mathcal{E}$ is a factor of type $II_1$ [24]. For our purposes, we do not need to assume that $\mathcal{E}$ induces a trace on the relative commutants in the tunnel and interpret ergodicity to mean simply that $\mathcal{M}_{\varphi}^{(0)}$ is a factor. But by Proposition 5.22, $\mathcal{M}_{\varphi}$ is then a factor. Since amenability of $\Gamma_{\mathcal{N}, \mathcal{M}}$ in the sense of Popa means that $\|m_{\gamma}\| = \|\Gamma_{\mathcal{N}, \mathcal{M}}\|^2$ is the minimal index of the inclusion, i.e. $d(\gamma) = \|m_{\gamma}\|$, and $\gamma$ is real, the result now follows from Corollary 5.20. □

We therefore see that amenability in the sense of Popa is closely related to our use of the term. Indeed, by using Popa’s results we obtain a partial converse to Corollary 5.20. Our argument involves the endomorphism $\mu$, the restriction of $\hat{\rho}$ to $\mathcal{M}_{\varphi}^{(0)}$.

Lemma 5.24. The following conditions are equivalent.

a) Some Jones projection of $\hat{\rho}$ lies in $\mathcal{M}_{\varphi}^{(0)}$.

b) Some Jones projection for $\hat{\phi}$ lies in $\mathcal{M}_{\varphi}^{(0)}$.

c) Some Jones projection for the standard left inverse of $\hat{\rho}$ lies in $\mathcal{M}_{\varphi}^{(0)}$.

Under these conditions every normalized faithful left inverse of $\mu$ has a unique extension to a faithful normalized left inverse of $\hat{\rho}$ and some Jones projection for
this extension lies in $\mathcal{M}_\omega^{(0)}$. Furthermore $\|m^\mu\| \leq \|m^\rho\|$, $d(\mu) = d(\rho)$ and the Jones indices of $\mu$ and $\rho$ coincide. These equivalent conditions are implied by the following: there is a solution $R, \tilde{R} \in (i, \hat{\rho}^2)$ of the conjugate equations such that both $R$ and $\tilde{R}$ define $\hat{\phi}$. In this case, we say that $\hat{\phi}$ is selfconjugate.

Proof. We first show the equivalence of the three conditions. Suppose $R, \tilde{R} \in (i, \hat{\rho}^2)$ solve the conjugate equations and are such that $\tilde{R}R^* \in \mathcal{M}_\omega^{(0)}$. Then $\sigma_\iota(\tilde{R}) = a^t R, t \in \mathbb{R}$, for some $a \in \mathbb{R}_+$. If $\chi$ is the corresponding faithful normalized left inverse of $\hat{\rho}$, then $\chi(\mathcal{M}_\omega^{(0)}) \subseteq \mathcal{M}_\omega^{(0)}$ and $\chi$ restricts to a faithful normalized left inverse of $\mu$. By Lemma 2.5, every faithful normalized left inverse of $\mu$ is of the form $A \mapsto \chi(X^*AX)$ for some $X \in (\mu, \mu)$ with $\chi(X^2X) = 1$. Now $(\mu, \mu) \subset (\hat{\rho}, \hat{\rho})$ by Proposition 5.7, so the same formula defines a faithful normalized left inverse of $\hat{\rho}$ and, since $X^{-1} \tilde{R} R^* X^{-1} \in \mathcal{M}_\omega^{(0)}$, there is an associated Jones projection in $\mathcal{M}_\omega^{(0)}$. Since any faithful normalized left inverse of $\hat{\rho}$ is uniquely determined by its values on a Jones projection, every faithful normalized left inverse of $\mu$ has a unique extension to a faithful normalized left inverse of $\hat{\rho}$. Since both $\hat{\phi}$ and the standard left inverse leave $\mathcal{M}_\omega^{(0)}$ globally stable, $a$ implies both $b$ and $c$ so the conditions are obviously equivalent. Now any faithful normalized left inverse of $\mu$ has the same dimension as its extension to a normalized faithful left inverse of $\hat{\rho}$ since a Jones projection for the latter lies in $\mathcal{M}_\omega^{(0)}$. It is now clear that $\|m^\mu\| \leq \|m^\rho\|$, $d(\mu) = d(\rho)$ and that the Jones indices of $\mu$ and $\hat{\rho}$ coincide. Now suppose that $R$ and $\tilde{R}$ both define $\hat{\phi}$. Then if $S \in (i, \hat{\rho}^2),$

$$\omega(SA) = \omega(\hat{\rho}^2(A)S) = \omega(A\hat{\phi}^2(S)), \quad A \in \mathcal{M}_\phi.$$ 

Hence $\sigma_\iota(S) = \hat{\phi}^2(S)$. However, since $R$ defines $\hat{\phi}$, $\hat{\phi}(R) = d(\phi)^{-1}R$, and since $\tilde{R}$ also defines $\hat{\phi}$, $\hat{\phi}^2(\tilde{R}) = d(\phi)^{-2}\tilde{R}$ so $\tilde{R}$ transforms like a character under the modular group. Thus the corresponding Jones projection, which is a Jones projection for $\hat{\phi}$ lies in $\mathcal{M}_\omega^{(0)}$. The equivalent conditions are therefore fulfilled whenever $\hat{\phi}$ is selfconjugate. \qed

**Theorem 5.25.** Let $\rho$ be real or irreducible and pseudoreal. If $\phi$ is amenable and selfconjugate then $d(\rho) = \|m^\mu\|$. The same conclusion holds if $\phi$ is amenable and $\omega$ is tracial on $\mathcal{M}_\phi^{(0)}$.

Proof. When $\phi$ is amenable, Proposition 5.7 shows that $\mathcal{M}_\omega^{(0)}$ is a factor and that $\cup_r(\mu^r, \mu^r), r \in \mathbb{N}_0$ is dense in $\mathcal{M}_\omega^{(0)}$. Thus $\mu(\mathcal{M}_\omega^{(0)}) \subseteq \mathcal{M}_\omega^{(0)}$ is a strongly amenable inclusion of type $II_1$–factors by Theorem 4.1.2(iv) of [23]. Hence $\|m^\mu\| = d(\mu)$, see the remark following the proof of Theorem 5.3.2 of [23]. But if $\phi$ is selfconjugate, then by Lemma 5.24, $d(\rho) = d(\mu) = \|m^\mu\| \leq \|m^\rho\| \leq d(\rho)$, as required. If $\omega$ is tracial on $\mathcal{M}_\phi^{(0)}$, one may either note that the equivalent conditions of Lemma 5.24 are satisfied, or, for a more elementary proof use the endomorphism $\tau$ and Proposition 5.22. \qed

We do not know whether Theorem 5.25 is valid for all amenable $\phi$. The results we have obtained justify the term “amenable” for $\phi$ by analogy of Popa’s use of strongly amenable for the graph of an inclusion [24].

We now consider further examples to complement the above discussion. We begin with the fundamental representation of $SU_q(2), \rho_1$, first taking $q$ to be a root of unity. The dual of $SU_q(2)$, with $q$ a root of the unity, contains only finitely many
irreducible objects; it is a rational tensor $C^*$–category [8]. As a consequence of Popa’s classification [23] we have

**Proposition 5.26.** If $q^n = 1$ then $\hat{\rho}_1$ is an irreducible endomorphism of $\mathcal{M}_{\rho_1}$. The inclusion $\hat{\rho}_1(\mathcal{M}_{\rho_1}^{(0)}) \subset \mathcal{M}_{\rho_1}^{(0)}$ is isomorphic to Wenzl’s subfactor [29].

**Proof.** The inclusion $\hat{\rho}_1(\mathcal{M}_{\rho_1}^{(0)}) \subset \mathcal{M}_{\rho_1}^{(0)}$ and Wenzl’s subfactor have the same standard invariant (they both generate a tensor category isomorphic to the dual of $SU_q(2)$), hence they are isomorphic inclusions [23]. \qed

Now suppose that $q \in (0,1)$. Since the principal graph of tensoring by $\rho_1$ is $A_\infty$, $\mathcal{M}_{\rho_1}^{(0)}$, the grade zero part of $\mathcal{M}_{\rho_1}$, is a factor [11]: indeed we have

**Proposition 5.27.** $\mathcal{M}_{\rho_1}$ is a factor of type $III_\lambda$, $\lambda = (q + q^{-1})^2$.

**Proof.** $\mathcal{M}_{\rho_1}^{(0)}$ is a factor, hence $\mathcal{M}_{\rho_1}$ is a factor by Corollary 5.3. Since in this case $(\rho_1^{-r}, \rho_1^{r+1}) = 0, r \in \mathbb{N}_0$ but $(\iota, \rho_1^2) \neq 0$, the statement follows by Corollary 5.3 and the computation of $d(\rho_1)$ in Section 3 or compare e.g. [28]. \qed

We therefore have an inclusion of factors $\hat{\rho}_1(\mathcal{M}_{\rho_1}) \subset \mathcal{M}_{\rho_1}$ associated with the fundamental representation of $SU_q(2)$,

**Proposition 5.28.** $d(\hat{\rho}_1) = 2$.

**Proof.** $\hat{\rho}_1(\mathcal{M}_{\rho_1}^{(0)}) \subset \mathcal{M}_{\rho_1}^{(0)}$ is an inclusion of type $II_1$–factors with Jones index $d(\rho_1)^2$. The inclusion is known to be reducible, in fact it is just given by the direct sum of two automorphisms. Hence if $\tau$ denotes the restriction of $\hat{\rho}_1$ to $\mathcal{M}_{\rho_1}^{(0)}$, $d(\tau) = 2$. However, $d(\tau) = d(\rho_1)$ by Proposition 5.22. \qed

**Corollary 5.29.** The fundamental representation of $SU_q(2)$, $q \in (-1,1)$, is not amenable.

There are uncountably many inclusions of factors with graph $A_\infty$ and hence uncountably many Ocneanu connections [20] on the graph $A_\infty$.

We note that Corollary 5.29 yields an example of a non–amenable tensor $C^*$–category whose tensor product is commutative. Now there is a certain belief that Abelianness is responsible for the amenability of the dual of a locally compact group. This example shows that matters are not quite so simple and that one has to distinguish different notions of Abelianness. The notion in the following result is sufficient for amenability.

**Lemma 5.30.** Let $\rho$ satisfy the conditions of Corollary 5.13 and suppose there is a unitary $\varepsilon \in (\rho^2, \rho^2)$ such that

$$\varepsilon(s,1) \circ X \otimes 1_\rho = 1_\rho \otimes X \circ \varepsilon(r,1), \quad X \in (\rho^r, \rho^s),$$

where

$$\varepsilon(0,1) = 1_\rho, \quad \varepsilon(r,1) = \varepsilon \otimes 1_\rho^{r-1} \circ 1_\rho \otimes \varepsilon(r-1,1)$$

then on $\mathcal{O}_\rho$ we have

$$\hat{\rho}(X) = \lim_{s \to \infty} \varepsilon(s+1,1)X\varepsilon(s,1)^*, \quad X \in \mathcal{O}_\rho^{(k)}$$

and in particular

$$\varepsilon(s,1)X = \hat{\rho}(X)\varepsilon(r,1), \quad X \in (\hat{\rho}^r, \hat{\rho}^s).$$
The analogous relations hold on $\mathcal{M}_\varphi$ if the limit is understood in the $s$–topology provided the state $\omega$ defined by $\varphi$ satisfies

$$\omega(\varepsilon X \varepsilon^*) = \omega(X) \quad X \in \mathcal{M}_\varphi^{(0)},$$

thus in particular if $\omega$ is tracial on $\mathcal{M}_\varphi^{(0)}$. More generally, the condition is fulfilled if and only if $\varepsilon$ commutes with $e^{-H_2}$, where $e^{-H_2}$ is as in Lemma 5.2. The $\varphi$ that satisfy this condition are amenable. In particular, $\rho$ is amenable.

Proof. In the case of $\mathcal{O}_\rho$ the proof follows Corollary 4.7 and Lemma 4.14 of [6]. In the case of $\mathcal{M}_\varphi$, it suffices to show that $\varepsilon(s + k, 1)X\varepsilon(s, 1)^*\xi_\omega \to \hat{\rho}(X)\xi_\omega$ since $\xi_\omega$ is separating for $\mathcal{M}_\varphi$. Now if $Y \in (\rho^r, \rho^{r+k})$ and $s \geq r$,

$$\|(\varepsilon(s+k, 1)X\varepsilon(s, 1)^* - \hat{\rho}(X))\xi_\omega\| \leq \|\varepsilon(s+k, 1)(X-Y)\varepsilon(s, 1)^*\xi_\omega\| + \|\hat{\rho}(X-Y)\xi_\omega\| = 2\|X-Y)\xi_\omega\|,$$

as required. The rest of the proof follows as for $\mathcal{O}_\rho$ replacing convergence in the norm by convergence in the $s$–topology. This is possible since $\hat{\varphi}$ is continuous in this topology (cf. Lemma 5.1).

Of course, we can find an $\varepsilon \in (\rho^2, \rho^2)$ with the required properties if $\mathcal{T}_\rho$ admits a unitary braiding. However, the above conditions are weaker and are known to be fulfilled when $\rho$ is the regular representation of a Kac–system in the sense of §6 of [1]. We will refer to $\varepsilon(\cdot, 1)$ as a unitary half–braiding. Thus a unitary half–braiding suffices for amenability but a non-unitary braiding, as in the case of $SU_q(2)$, $q \in (-1, 1)$ does not.

We end this section by summing up the result of the previous lemma in the form of a theorem.

**Theorem 5.31.** Let $\rho$ be a real object or an irreducible pseudoreal object in a strict tensor $C^*$–category. Then if $\rho$ admits a unitary half–braiding it is $C^*$–amenable and amenable and in fact $\varphi$ is amenable whenever the state $\omega$ associated with $\varphi$ is tracial on $\mathcal{M}_\varphi^{(0)}$. 
6. Inclusions of Factors and Hopf Algebras

As a further application of our methods, we give a description of two classes of inclusions of factors, the standard AFD inclusions and the standard graded AFD inclusions. As compared with the last section, there is a change of objective. There it could be said that we were studying endomorphisms of von Neumann algebras starting with objects in a tensor $C^*$–category. Here instead, we will be studying inclusions of von Neumann algebras and we need to start from more particular data.

We thus define an abstract $Q$–system to be a pair $(\lambda, S)$ consisting of an object $\lambda$ in a strict tensor $C^*$–category $\mathcal{T}$ with $d(\lambda) > 1$ and an isometry $S \in (\lambda, \lambda^2)$ such that

(a) $1_\lambda \otimes S \circ S = S \otimes 1_\lambda \circ S$

(b) $\begin{cases} S^* \circ 1_\lambda \otimes T = 1_\lambda \\ T^* \otimes 1_\lambda \circ S = 1_\lambda \end{cases}$

for some (unique) $T \in (\iota, \lambda)$.

If in addition

(c) $(\iota, \lambda)$ is one–dimensional, the $Q$–system will be said to be irreducible.

Apart from occasional remarks, we shall continue to suppose, as in §5, that the tensor unit $\iota$ is irreducible. We recall from Section 2 that if $\iota$ is irreducible we automatically have such a structure whenever $\lambda$ is a canonical object. If $\iota$ is reducible, it is not clear that we can take $S$ to be an isometry since it is not clear that $R^* \circ R$ is invertible. Invertibility in fact corresponds to the notion of finite scalar index in the theory of inclusions of von Neumann algebras. An isomorphism of abstract $Q$–systems will be an isomorphism of tensor $C^*$–categories $\mathcal{T}_\lambda$ preserving the distinguished intertwiners $T, S$.

Note that the object $\lambda$ is selfconjugate and even real, indeed the conjugate equation is satisfied with $\lambda = \bar{\lambda}$, $R = \overline{R} = S \circ T$. We denote the corresponding normalized left inverse by $\varphi$ so that

$$d(\lambda) \leq d(\varphi) = ||T||^2 ||S||^2 = ||T||^2.$$ 

We shall see however that $d(\lambda) = ||T||^2$ if the $Q$-system is irreducible. It should be noted that not all faithful normalized left inverses of $\lambda$ can arise in this manner. In particular, any such $\varphi$ is obviously selfconjugate.

When $\mathcal{T}$ is the category of unital normal endomorphisms of a von Neumann algebra $\mathcal{M}$, it was shown in [18] that $\lambda$ is the canonical endomorphism corresponding to a uniquely determined von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$. The main step in the proof consisted in writing an explicit formula for a conditional expectation onto this subalgebra, namely

$$A \mapsto S^* \lambda(A) S.$$ 

This step has a categorical analogue. We shall not need a formal definition of a conditional expectation on a $C^*$–category here. We will only deal with those arising from left and right inverses. If $\varphi$ is a left inverse for $\rho$ then defining for each pair of objects $\sigma, \tau$,

$$\delta_{\sigma, \tau}(X) := 1_\rho \otimes \varphi_{\sigma, \tau}(X), \quad X \in (\rho \sigma, \rho \tau)$$

We shall see however that $d(\lambda) = ||T||^2$ if the $Q$-system is irreducible. It should be noted that not all faithful normalized left inverses of $\lambda$ can arise in this manner. In particular, any such $\varphi$ is obviously selfconjugate.

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$$A \mapsto S^* \lambda(A) S.$$ 

This step has a categorical analogue. We shall not need a formal definition of a conditional expectation on a $C^*$–category here. We will only deal with those arising from left and right inverses. If $\varphi$ is a left inverse for $\rho$ then defining for each pair of objects $\sigma, \tau$,
gives us a conditional expectation onto the image of the functor of tensoring on the left by $1_\rho$. Similarly, we get a conditional expectation onto the image of the functor of tensoring on the right by $1_\rho$ by starting with a right inverse. The idea here in dealing with a canonical object $\lambda$ is to imagine that $\lambda = \rho \bar{\lambda}$ and write down the conditional expectation onto the image of tensoring on the left by $\rho$. We work on the category $\mathcal{T}_\lambda$ and define for $r, s \in \mathbb{N}$,

$$\delta_{r,s}(X) := S^* \otimes 1_{\lambda_{r-1}} \circ 1_\lambda \otimes X \circ 1_{\lambda_{r-1}}, \quad X \in (\lambda^r, \lambda^s).$$

It is easily checked that $\delta_{r,r}$ is faithful and a completely positive projection of norm one and the same holds for a matrix with entries $\delta_{r,s}$, $r, s = 1, 2, \ldots, n$. Since $S$ is in the image of $\delta$ by $a)$, $S \circ S^*$ is also in that image. In fact $\delta_{2,2}(S \circ S^*) \geq \delta_{1,2}(S) \circ \delta_{1,2}(S)^* = S \circ S^*$ and letting $\delta_{2,2}$ act on the difference $\delta_{2,2}(S \circ S^*) - S \circ S^*$ we get 0, thus $\delta_{2,2}(S \circ S^*) = S \circ S^*$ because $\delta$ is faithful. Computing $X^* \circ X$ for $X := 1_\lambda \otimes S^* \circ S \otimes 1_\lambda - S \circ S^*$, we see that for any abstract $Q$–system we have

$$1_\lambda \otimes S^* \circ S \otimes 1_\lambda = S \circ S^*.$$  

This condition was included as part of the definition of a $Q$–system in [18].

**Lemma 6.1.** The image of $\delta_{r,s}$ is the set of $Y \in (\lambda^r, \lambda^s)$ such that

$$1_\lambda \otimes Y \circ S \otimes 1_{\lambda_{r-1}} = S \otimes 1_{\lambda_{s-1}} \circ Y.$$  

**Proof.** It is a simple computation using $a)$ and $d)$ to show that $Y = \delta_{r,s}(X)$ satisfies the required relation. On the other hand if $Y$ satisfies the relation, $Y = \delta_{r,s}(Y)$.

On the basis of this lemma, it is clear that the image of $\delta$ is a $C^*$–subcategory which is closed under tensor products. It fails to be a tensor $C^*$–category because the tensor unit is not in the subcategory. There is also another conditional expectation, obtained conceptually by thinking of a right inverse for $\bar{\rho}$ and leading to another $C^*$–subcategory closed under tensor products. The formulae are obtained by interchanging tensoring on the left by $\lambda$ with tensoring on the right by $\lambda$. This second possibility will, however, play no role here.

Now an abstract $Q$–system gives rise to an inclusion $\mathcal{N}_\varphi \subset \mathcal{M}_\varphi$ of von Neumann algebras, where $\mathcal{N}_\varphi$ denotes the von Neumann subalgebra generated by the above subcategory. Furthermore, we have an endomorphism $\hat{\lambda}$ of $\mathcal{M}_\varphi$ and intertwiners $T \in (\iota, \hat{\lambda})$, $S \in (\hat{\lambda}, \hat{\lambda}^2)$ satisfying the relations $a)$ and $b)$ above. Thus we have a $Q$–system in the sense of [18] except that $\mathcal{M}_\varphi$ might fail to be a factor. It follows by the analysis in [18] that $\hat{\lambda}$ is the canonical endomorphism of $\mathcal{M}_\varphi$ with respect to the subalgebra $\mathcal{N}_\varphi$ since this is just the image of $\mathcal{M}_\varphi$ under the conditional expectation $\mathcal{E}$, $\mathcal{E}(X) := S^* \hat{\lambda}(X)S$. Thus

$$\mathcal{N}_\varphi = \{X \in \mathcal{M}_\varphi : SX = \hat{\lambda}(X)S\}.$$  

In particular, $\mathcal{N}_\varphi$ is stable under gauge transformations. The normal conditional expectation will be understood as part of the data when talking about inclusions and we write $\mathcal{N}_\varphi^\mathcal{E} \subset \mathcal{M}_\varphi$ to emphasise this. The $Q$–system not only singles out the conditional expectation but also a particular choice of the Jones tunnel and a set of Jones projections. The Jones tunnel is

$$\cdots \hat{\lambda}(\mathcal{N}_\varphi) \subset \hat{\lambda}(\mathcal{M}_\varphi) \subset \mathcal{N}_\varphi \subset \mathcal{M}_\varphi.$$  

The Jones projections are as follows: $E_0 = d(\varphi)^{-1}TT^*$, $E_1 = SS^*$, and $E_{i+2} = \hat{\lambda}(E_i)$, $i \in \mathbb{N}_0$. The Jones index of the inclusion is $d(\varphi)$. 

Lemma 6.2. If $X \in (\hat{\lambda}^r, \hat{\lambda}^s)$ then

$$d^{r+s}E_0E_1 \cdots E_{2s}XE_{2r} \cdots E_1E_0 = \hat{\lambda}(X)E_0,$$

where $d = d(\varphi)$.

Proof.

$$TT^*SS^* \cdots \hat{\lambda}^{s-1}(SS^*)\hat{\lambda}^s(TT^*)X\hat{\lambda}^r(TT^*) \cdots TT^* = T\hat{\lambda}^s(T^*)X\hat{\lambda}^r(T)T^* = d^2\hat{\lambda}(X)E_0,$$

as required. $\square$

Passing to the grade zero part we get an inclusion $N^{(0)}_\varphi \subseteq M^{(0)}_\varphi$ of von Neumann algebras and we denote by $\tau$ the restriction of $\hat{\lambda}$ to the homogeneous part. Since the Jones projections are of grade zero,

$$\cdots \tau(N^{(0)}_\varphi) \subseteq \tau(M^{(0)}_\varphi) \subseteq N^{(0)}_\varphi \subseteq M^{(0)}_\varphi$$

is a Jones tunnel. Furthermore, by Corollary 5.6

$$(\tau^r, \tau^r) = (\hat{\lambda}^r, \hat{\lambda}^r), \quad r \in \mathbb{N}_0.$$ 

Now the union of the relative commutants in the tunnel is just $\cup_r (\tau^r, \tau^r)$ and is hence dense in $M^{(0)}_\varphi$. In fact our inclusion is just derived by the canonical procedure from the relative commutants in the tunnel. To see this we need only verify that the unique normal $\tau$–invariant state $\omega$ satisfies

$$\omega(\mathcal{E}_i(X)) = \omega(X), \quad X \in N_{i-1}, \ i \in \mathbb{N}_0,$$

where $\mathcal{E}_i$ is the conditional expectation from $N_{i-1}$ onto $N_i$ defined by

$$\mathcal{E}_i(X)E_{i-1} = E_{i-1}XE_{i-1}.$$ 

Since $\mathcal{E}_{i+2}\tau(X) = \tau\mathcal{E}_i(X)$ and $\omega$ is $\tau$–invariant, it is enough to test $i = 1, 2$ and hence alternatively to show that

$$\omega(\mathcal{E}_1\mathcal{E}_0(X)) = \omega(X), \quad X \in M^{(0)}_\varphi.$$ 

However, we have

$$E_0\mathcal{E}_0(X)E_0 = d^{-2}TT^*S^*\tau(X)STT^* = \tau\hat{\varphi}(X)E_0,$$

so that $\mathcal{E}_1\mathcal{E}_0 = \tau\hat{\varphi}$ and the result is now obvious. As far as the zero grade part goes, our construction therefore yields inclusions of a very special type. In fact, when $\omega$ is just a trace on $M^{(0)}_\varphi$ so that we have a $II_1$–factor, we have just verified that our inclusion is strongly amenable in the sense of [23].

Our next goal is to show that our inclusion $N^{(0)}_\varphi \subseteq M^{(0)}_\varphi$ can be reconstructed from its zero grade part. In fact, it is just the cross product of its grade zero part by an endomorphism. To, see this we note that $\tau$ is a canonical endomorphism in the sense of Choda[2] since we have a faithful normal $\tau$–invariant state, $E_0M^{(0)}_\varphi E_0 = \tau(M^{(0)}_\varphi)E_0$.
by Lemma 6.2 and the Jones projections fulfill the Jones relations. We can therefore define the cross product $M$ of $M_\varphi^{(0)}$ by the endomorphism $\tau_0$, $\tau_0(X) := \tau(X)E_0$. $M$ is generated as a von Neumann algebra by $M_\varphi^{(0)}$ and an isometry $W$ satisfying 

$$WXW^* = \tau(X)W, \quad X \in M_\varphi^{(0)}.$$ 

$\tau$ extends uniquely to an endomorphism $\tilde{\tau}$ of $M$ such that

$$\tilde{\tau}(W) = d(\varphi)E_2E_1W.$$ 

Furthermore, it is easy to see that $M$ carries a continuous action $\alpha$ of the circle with $\alpha_t(W) = tW$, $t \in \mathbb{T}$ and $M_\varphi^{(0)}$ as fixed-point algebra. In fact, the restriction of $\omega$ to $M_\varphi^{(0)}$ extends to an $\alpha$–invariant and hence faithful state $\tilde{\omega}$ on $M$.

We now compare $M$, $\tilde{\tau}$ and $M_\varphi$.

Theorem 6.3. Let $(S, \lambda)$ be an abstract $Q$–system and $\varphi$ the associated left inverse of $\lambda$, then $M_\varphi$ is just the cross-product of $M_\varphi^{(0)}$ by the endomorphism $X \mapsto \lambda(X)E_0$ of $M_\varphi^{(0)}$ and $\hat{\lambda}$ is the canonical extension of its restriction to $M_\varphi^{(0)}$.

We now want to characterize the two classes of inclusion discussed in this section and begin with the appropriate definitions. Suppose we have an inclusion of properly infinite factors $N \subset M$ so that there is a $Q$–system $(\lambda, S)$, where $\lambda$ is an endomorphism of $M$ and $S \in M$ satisfies $E(X) = S^*\lambda(X)S, \quad X \in M$. Since any other such $Q$–system is of the form $\lambda', S'$, where $\lambda' = \text{ad}(U)\lambda$ and $S' = US$, $U$ being a unitary of $N$, $U$ induces an isomorphism of abstract $Q$–systems. Hence different associated $Q$–systems yield isomorphic inclusions $N_\varphi \subset M_\varphi$. If this inclusion is isomorphic to the original inclusion, we say that $N_\varphi \subset M_\varphi$ is a standard graded AFD inclusion. On the other hand, even if our factors are not properly infinite, we can choose a Jones tunnel and construct the associated inclusion obtained from the relative commutants. Again, as is well known, different choices of the tunnel lead to isomorphic inclusions. If this inclusion is isomorphic to the original inclusion, we say that $N_\varphi \subset M_\varphi$ is a standard AFD inclusion.

Theorem 6.4. Every standard AFD inclusion is the grade zero part of a standard graded AFD inclusion.

Proof. If we tensorialize a Jones tunnel with an infinite factor, the inclusion derived from the relative commutants remains unchanged. The tensorialized inclusion,
however, has an associated \(Q\)–system. This \(Q\)–system defines a model graded inclusion whose zero grade part is, as we have seen above, derived from the relative commutants in the associated Jones tunnel.

Of course, the standard graded AFD inclusion defined by an abstract \(Q\)–system \((\lambda, S)\) depends only on the \(Q\)–system \((\hat{\lambda}, S)\). \((\lambda, S)\) and \((\hat{\lambda}, S)\) are isomorphic if and only if the associated left inverse \(\varphi\) of \(\lambda\) is amenable. In this case, we say that the \(Q\)–system \((\lambda, S)\) is amenable. Hence, we can sum up the above discussion in the following way.

**Corollary 6.5.** There is a natural 1-1 correspondence between isomorphism classes of amenable abstract \(Q\)–systems, standard graded AFD inclusions and standard AFD inclusions.

Of course, it is hardly surprising that the image of the irreducible amenable \(Q\)–systems under this correspondence is precisely the set of irreducible inclusions. To see this we note that in a \(Q\)–system \((\lambda, S)\), there is a 1–1 correspondence between the elements of \((\iota, \lambda)\) and those elements \(Y\) of \((\lambda, \lambda)\) that satisfy \(Y \otimes 1_{\lambda} \circ S = S \circ Y\). \(X \in (\iota, \lambda)\) corresponds to \(S^* \circ 1_{\lambda} \otimes X \in (\lambda, \lambda)\). However, the relative commutant of \(N_\varphi\) in \(M_\varphi\) is easily seen to be \(\{ Y \in (\hat{\lambda}, \hat{\lambda}) : YS = SY\}\). Now, the relative commutant of \(N^{(0)}_\varphi\) in \(M^{(0)}_\varphi\) is just \(\{ Y \in (\tau, \tau) : YE_1 = E_1 Y\}\), where \(E_1 = SS^*\). It is not so obvious that this relative commutant is just the grade zero part of the previous relative commutant but the question is closely related to that resolved in Corollary 5.6 and the same techniques work.

**Lemma 6.6.**

\[ N^{(0)}_\varphi' \cap M^{(0)}_\varphi = N_\varphi' \cap M_\varphi. \]

**Proof.** Let \(E, E'\) be two projections in \(M^{(0)}_\varphi\). If there is an \(X \in N_\varphi\) with \(EXE' \neq 0\), we see as in the proof of Lemma 5.5 that there is a \(Y \in N^{(0)}_\varphi\) such that \(EYE' \neq 0\). The result now follows by the argument of Corollary 5.6.

Now the Jones index of an irreducible inclusion coincides with the minimal index. Hence for an irreducible \(Q\)–system \(d(\lambda) = \|T\|\).

Let us call a \(Q\)–system rational if the completion of \(T_\lambda\) under subobjects contains only finitely many inequivalent irreducible objects. This is obviously corresponds to the notion of finite depth in the theory of subfactors. As one sees from the class of examples discussed following Corollary 5.6, in our framework, the Jones index does not necessarily coincide with the minimal index for an inclusion of factors of finite depth, i.e. such inclusions are not necessarily extremal in Popa’s terminology[23]. This is because we have not imposed the condition that \(\omega\) defines a trace on the grade zero part. The Jones index, therefore, has no reason to be a Perron–Frobenius eigenvalue. For the same reason, we do not know whether a rational \(Q\)–system is necessarily amenable since \(M_\varphi\) might possibly fail to be a factor. Of course, if \(\omega\) is a trace on \(M^{(0)}_\varphi\) then the \(Q\)–system is amenable since finite depth inclusions are amenable in Popa’s setting[23].

As an application, we characterize finite–dimensional Hopf algebras in terms of tensor \(C^*\)–categories.

Notice that, as a consequence of Popa’s classification [23], we also have a bijective correspondence between amenable (finite depth) irreducible inclusions of
\(II_1\)-factors and amenable (finite depth) irreducible inclusions of injective \(III_1\)-factors.

**Theorem 6.7.** There is a bijective correspondence between
(i) Irreducible abstract \(Q\)-systems \((\lambda, S)\) such that
\[
\lambda \otimes \lambda \simeq \lambda \oplus \lambda \oplus \cdots \oplus \lambda \quad (d \text{ times } \lambda)
\]
(ii) Finite-dimensional complex Hopf algebras of dimension \(d\).

**Proof.** This follows by Corollary 6.2 and the results in [18]. □

We recall from § 6 of [18] that to each \(Q\)-system defined concretely in terms of a von Neumann algebra there is a dual \(Q\)-system. We intend in a sequel to study this duality directly in terms of abstract \(Q\)-systems.

7. 2–\(C^*\)-Categories

The formalism we have presented so far based on the notion of tensor \(C^*\)-categories is not sufficiently general to cover all interesting applications and we must generalize to 2–\(C^*\)-categories. In fact, this does not present the slightest difficulty and we could have worked with the more general notion from the outset but for our wish to avoid unfamiliar and hence superficially technical concepts.

Let us begin by understanding the typical situation not yet covered by our formalism. Suppose we wish to consider the minimal index of an inclusion of factors \(\mathcal{N} \subset \mathcal{M}\) directly. To fit in with our formalism to date, we need an object in a tensor \(C^*\)-category. If \(\mathcal{M}\) and \(\mathcal{N}\) are isomorphic, we can compose the inclusion with an isomorphism to obtain an endomorphism which is an object in a tensor \(C^*\)-category. The intrinsic dimension of the endomorphism is independent of the choice of isomorphism and serves to define the index of the inclusion. However, there is a simpler way of looking at things. We should look at morphisms rather than endomorphisms of \(C^*\)-algebras. There is no difficulty in generalizing the concept of conjugate. A morphism \(\rho\) from \(\mathcal{A}\) to \(\mathcal{B}\) has a morphism \(\bar{\rho}\) from \(\mathcal{B}\) to \(\mathcal{A}\) as a conjugate if there are \(R \in (\iota_\mathcal{A}, \bar{\rho}\rho)\) and \(\bar{R} \in (\iota_\mathcal{B}, \rho\bar{\rho})\) such that
\[
\bar{R}^* \rho(R) = 1_B; \quad R^* \bar{\rho}(\bar{R}) = 1_A.
\]

As before, \(\iota\) denotes the identity automorphism of the algebra in question. All we need therefore is the correct way of formalizing this situation and this is provided for by the notion of a 2–\(C^*\)-category.

A 2-category may be conveniently defined as a set with two composition laws, \(\otimes\) and \(\circ\), each making it into a category. In addition
a) Every \(\otimes\)-unit is an \(\circ\)-unit,
b) The \(\otimes\)-composition of \(\circ\)-units is a \(\circ\)-unit.
c) \(S' \otimes T' \circ S \otimes T = (S' \circ S) \otimes (T' \circ T)\) whenever the right hand side is defined.

A 2-category is a 2–\(C^*\)-category if, given any pair of \(\otimes\)-units, the category whose elements are the subset having that pair as left and right \(\otimes\)-units respectively and \(\circ\) as composition is a \(C^*\)-category and if furthermore the \(\otimes\)-composition is bilinear.

To illustrate this with the motivating example, we get a 2–\(C^*\)-category by taking the set of intertwining operators of morphisms between elements of a given set of unital \(C^*\)-algebras. \(\circ\)-composition is just the ordinary composition of intertwining operators. If \(S\) is an intertwining operator between \(\sigma\) and \(\sigma'\) and \(T\) an intertwiner
between $\tau$ and $\tau'$ then $S \otimes T$ is defined if and only if the composition $\sigma \tau$ (and hence also $\sigma' \tau'$) of morphisms is defined and $S \otimes T := S\sigma(T)$ considered as an intertwiner from $\sigma \tau$ to $\sigma' \tau'$. The $\circ$–units are therefore the unit intertwiners for the morphisms and may be identified with the morphisms themselves and the $\otimes$–units are the unit intertwiners for the identity automorphisms and may be identified with the algebras themselves.

But this is only one example and not necessarily the most convenient one, even for studying inclusions of factors, so we give further examples and stress the point that the subject matter of this paper, conjugation, dimension and trace, is just an elementary part of abstract theory of 2–$C^*$–categories.

Of course, the classical example of a 2–category is the 2–category of natural transformations between functors between categories. Hence there is an obvious generalization of our motivating example which involves replacing unital $C^*$–algebras by $C^*$–categories and taking the set of bounded natural transformations between *–functors between $C^*$–categories. We refrain from giving details since we shall not pursue this example here. The basic definitions of the terms involved can be found in [12].

Of more relevance to the present paper is the fact that the motivating example can be generalized by considering the set of intertwiners of morphisms from elements of a given set of $C^*$–algebras with unit into matrix algebras over the given set. $T$ is an intertwiner between two such morphisms $\rho$ and $\sigma$, written $T \in (\rho, \rho')$, if

$$T\rho(A) = \sigma(A)T, \quad A \in \mathcal{A}. $$

If $\tau : \mathcal{B} \mapsto M_r(\mathcal{C})$ then $\tau \rho : \mathcal{A} \mapsto M_{rn}(\mathcal{C})$ where $\tau \rho(A)$ is defined by letting $\tau$ act on each component of $\rho(A)$. To give an explicit formula, if $\rho_{ij}(A)$ denotes the $ij$–th entry of $\rho(A)$, then

$$(\tau \rho)_{ij} = \tau_{i_1j_1} \rho_{i_2j_2} \quad \text{where} \quad i = i_1 + ri_2, \quad j = j_1 + rj_2. $$

If $R \in (\rho, \rho')$ and $T \in (\tau, \tau')$ then

$$T \otimes R = T\tau(R) = \tau'(R)T$$

is an element of $(\tau \rho, \tau' \rho')$. This 2–$C^*$–category has the advantage of being closed under finite direct sums and subobjects.

We may also modify the above examples to cater for $C^*$–algebras without unit. To do this, given two morphisms $\rho$ and $\sigma$ from $\mathcal{A}$ to matrix algebras over $\mathcal{B}$, an intertwiner $T$ must be allowed to be in the appropriate matrix algebra over the multiplier algebra $M(\mathcal{B})$ and must satisfy the condition that $T = \lim T\rho(X_\alpha)$ where $X_\alpha$ is an approximate unit in $\mathcal{A}$.

In much the same spirit one could consider Hermitian bimodules over a set of $C^*$–algebras, cf. [25], §8. Thus if $\mathcal{A}$ and $\mathcal{B}$ are $C^*$–algebras we need a right Hermitian $\mathcal{B}$–module, i.e. a right Banach $\mathcal{B}$–module $X$ with a $\mathcal{B}$–valued inner product $<,>$, conjugate–linear in the first variable and linear in the second, such that, for all $x, y \in X$ and $B \in \mathcal{B}$,

a) $<x, x> \geq 0$,
b) \( \langle x, yB \rangle = \langle x, y \rangle B \),
c) \( \langle x, y \rangle^* = \langle y, x \rangle \),
d) \( \|x\|^2 = \|\langle x, x \rangle\| \).

\( \mathcal{A} \) then acts on the left on \( X \); in other words we have a morphism of \( \mathcal{A} \) into the \( C^* \)--algebra of \( \mathcal{B} \)--module maps of \( X \) into itself with adjoint. If \( Y \) is then an Hermitian \( \mathcal{B} \)--\( \mathcal{C} \)--bimodule, we can define the tensor product \( X \otimes \mathcal{B} Y \) as an Hermitian \( \mathcal{A} \)--\( \mathcal{C} \)--bimodule spanned by elements \( x \otimes y, x \in X, y \in Y \) with a \( \mathcal{C} \)--valued inner product such that

\[
\langle x \otimes y, x' \otimes y' \rangle = \langle y, \langle x, x' \rangle y' \rangle.
\]

Since \( X \otimes \mathcal{B} Y \) is not defined uniquely by this procedure but only up to isomorphism, we will not get a \( 2--C^* \)--category whose elements are bimodule homomorphisms since the \( \otimes \)--composition will not be strictly associative. Rather than generalizing, and hence complicating, our formalism to deal with a problem which already arises at the level of tensor products of Hilbert spaces, we content ourselves with a framework where the \( \otimes \)--composition is strictly associative.

Consider a \( C^* \)--category and pick for each object \( A \) a \( C^* \)--subalgebra \( \mathcal{A} \) of \( (A,A) \).

An Hermitian bimodule associated with this data is a closed subspace \( X \) of some \( (A,B) \) such that

a) \( XA \subset X \),

b) \( BX \subset X \)

c) \( x^*x', \quad x,x' \in X \).

If \( X \subset (A,B) \) and \( Y \subset (B,C) \) are two such Hermitian bimodules their tensor product is the closed subset of \( (A,C) \) generated by the elements of the form \( yx, x \in X, y \in Y \). The resulting Hermitian bimodule will be denoted by \( YX \). If now \( X' \subset (A,B) \) and \( Y' \subset (B,C) \) are two further Hermitian bimodules and if \( S \) is a bimodule homomorphism from \( X \) to \( X' \) having an adjoint \( S^* \) in the sense that

\[
x'^*Sx = (S^*x')^*x, \quad x \in X, \quad x' \in X'.
\]

and \( T \) is a bimodule homomorphism from \( Y \) to \( Y' \) then \( T \otimes S \) is a bimodule homomorphism from \( YX \) to \( Y'X' \). Furthermore, if \( \circ \) denotes the composition of bimodule homomorphisms then we have

\[
(T' \otimes S') \circ (T \otimes S) = (T' \circ T) \otimes (S' \circ S),
\]

whenever the right hand side is defined. The set of such bimodule homomorphisms forms a \( 2--C^* \)--category.

Note that if we take a \( C^* \)--category with a single object, i.e. a unital \( C^* \)--algebra, \( \mathcal{A} \), and take \( \mathcal{C}I \) as our distinguished \( C^* \)--subalgebra then the resulting \( 2--C^* \)--category is just the tensor \( C^* \)--category of Hilbert spaces in \( \mathcal{A} \).

Another interesting special case is to take the distinguished \( C^* \)--subalgebra of \( (A,A) \) to be \( (A,A) \) itself, for each object \( A \). The resulting \( 2--C^* \)--category then depends intrinsically on the \( C^* \)--category we started from and each of the associated bimodules \( X \in (A,B) \) is bi-Hermitian in the sense that \( X \) has a second inner product \( x'x^* \) taking values in \( (B,B) \) and

\[
x''(x'^*x) = (x''x'^*)x, \quad x, x', x'' \in X,
\]
where the identity is regarded as relating the two inner products and the module actions.

After these illustrative examples we turn to the concept of conjugation for finite dimensional \(\sigma\)-units. If \(\mathcal{T}\) is a \(2-C^*\)-category then we can define a full \(2-C^*\)-subcategory \(\mathcal{T}_f\) by saying that a \(\sigma\)-unit \(\rho\) is in \(\mathcal{T}_f\) if there exists a \(\sigma\)-unit \(\tilde{\rho}\) and \(R \in (\iota_A, \tilde{\rho} \rho)\) and \(\bar{R} \in (\iota_B, \rho \tilde{\rho})\) such that

\[
\bar{R}^* \otimes 1 \rho \circ 1 \rho \otimes R = 1 \rho; \quad R^* \otimes 1 \tilde{\rho} \circ 1 \tilde{\rho} \otimes \bar{R} = 1 \tilde{\rho}.
\]

Here \(\iota_A\) and \(\iota_B\) are the right and left \(\otimes\)-units of \(\rho\). These equations will again be referred to as the conjugate equations. With this definition there are obvious analogues of the results of Section 2 whose exact formulation can be left to the reader. There are also obvious analogues of the results of Section 3 provided \(\iota_A\) and \(\iota_B\) are irreducible but the more challenging task is to provide analogues without making this assumption.

Here we shall content ourselves with showing that the formalism of \(2-C^*\)-categories that has been developed here allows one to interpret the Jones index of an inclusion of factors as the square of the dimension of a suitable \(\sigma\)-unit in a \(2-C^*\)-category. The most obvious approach, namely to consider the inclusion as a \(\sigma\)-unit in our motivating example of \(2-C^*\)-category, viz. the \(2-C^*\)-category of intertwining operators of morphisms between elements of a given set of unital \(C^*\)-algebras fails in general because the conjugate fails to exist in this \(2-C^*\)-category. Instead, we need the generalization, discussed above, where we allow intertwiners between morphisms into matrix algebras over the given set of \(C^*\)-algebras.

Let \(\mathcal{N} \subset \mathcal{M}\) be an inclusion of factors with finite index and let \(\mathcal{M}_1 = (\mathcal{M}, \epsilon_{\mathcal{N}})\) be the basic construction. Then the existence of a Pimsner–Popa basis provides one with an isomorphism of \(\mathcal{M}_1\) with a reduced subalgebra of some \(M_n(\mathcal{N})\) and hence the inclusion of \(\mathcal{M}\) into \(\mathcal{M}_1\) provides a morphism of \(\mathcal{M}\) into \(M_n(\mathcal{N})\). This is a \(\sigma\)-unit in our category and is a conjugate for the \(\sigma\)-unit coming from the original inclusion. More specifically, if \(\xi_i \in \mathcal{M}, i = 1, 2, \ldots, n\) denotes the Pimsner–Popa basis[21] and \(\rho\) denotes the inclusion of \(\mathcal{N}\) into \(\mathcal{M}\), then \(\tilde{\rho}\), a morphism of \(\mathcal{M}\) into \(M_n(\mathcal{N})\) is given by:

\[
\tilde{\rho}_{ij}(m) = E_{\mathcal{N}}(\xi_i^* m \xi_j),
\]

where \(E_{\mathcal{N}}\) is the conditional expectation of \(\mathcal{M}\) onto \(\mathcal{N}\) appearing in the definition of the Pimsner–Popa basis. We get a solution \(R, \bar{R}\) of the conjugate equations for \(\rho\) by setting

\[
\bar{R}_i = \xi_i^*; \quad R_i = E(\xi_i^*).
\]

Computing we see that \(R^* \circ R = 1_{\mathcal{N}}\) and \(\bar{R}^* \circ \bar{R} = \sum_i \xi_i \xi_i^*\). Furthermore the image of the Jones projection \(\epsilon_{\mathcal{N}}\) in \(M_n(\mathcal{N})\) is just \(R \circ R^*\). Hence bearing in mind that the inequality of Lemma 2.7 is best possible and the Pimsner–Popa characterization of the Jones index[21], we see how the index defined by Jones for an inclusion of type \(II_1\)-factors fits into the framework considered here. When the inclusion \(\rho\) is irreducible, in particular when the index is less than 4, the Jones index is just \(d(\rho)^2\).
Appendix

In this appendix we give a canonical procedure for completing a 2–$C^*$–category $\mathcal{T}$ to give a 2–$C^*$–category $C(\mathcal{T})$ having (finite) direct sums and subobjects. Before describing the construction, we remark that it may, at times, prove more convenient to think of completion in other terms. Thus in the von Neumann algebra context one might prefer to tensor with an infinite factor $\mathcal{N}$, passing for example from $\text{End}\mathcal{M}$ to $\text{End}(\mathcal{M} \otimes \mathcal{N})$.

We first discuss how to add subobjects. The resulting 2–$C^*$–category has as elements triples $(F,T,E)$ of elements of $\mathcal{T}$, where $E$ and $F$ are projections and $T \circ E = T = F \circ T$. The composition laws are defined as follows: $(F,T,E) \circ (F',T',E')$ is defined if and only if $F' = E$ when the composition is $(F,T \circ T',E')$. $(F,T,E) \otimes (F',T',E')$ is defined if and only if $E \otimes E'$, $F \otimes F'$ and hence $T \otimes T'$ are defined and the composition is given by $(F,T,E) \otimes (F',T',E') = (F \otimes F',T \otimes T',E \otimes E')$. The adjoint is of course given by $(F,T,E)^* = (E,T^*,F)$, and $\|(F,T,E)\| = \|T\|$.

Obviously, $(E,E,E)$ is a right and $(F,F,F)$ a left $\circ$–unit for $(F,T,E)$. If $A$ and $B$ are right and left $\otimes$–units for $E$, respectively, then $(A,A,A)$ and $(B,B,B)$ are right and left $\otimes$–units for $(F,T,E)$. Hence, the set of $\otimes$–units remains unchanged and the $\otimes$–composition of $\circ$–units is a $\circ$–unit. To show that we have a 2–$C^*$–category, it only remains to check the interchange law and that $\otimes$ commutes with taking adjoints and this will be left to the reader as an exercise.

To add direct sums, we take as elements of the completed 2–$C^*$–category finite matrices with entries from the original 2–$C^*$–category $\mathcal{T}$ having (finite) direct sums and subobjects. Before describing the construction, we remark that it may, at times, prove more convenient to think of completion in other terms. Thus in the von Neumann algebra context one might prefer to tensor with an infinite factor $\mathcal{N}$, passing for example from $\text{End}\mathcal{M}$ to $\text{End}(\mathcal{M} \otimes \mathcal{N})$.

We denote the resulting 2–$C^*$–category by $C(\mathcal{T})$. It is obvious that a 2–$*$–functor $F$ between 2–$C^*$–categories extends in a canonical manner to a 2–$*$–functor $C(F)$.
between their respective completions. Of course, in the special case that $F$ is a tensor $^*$–functor, $C(F)$ will also be a tensor $^*$–functor.

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