Derivation of anisotropic dissipative fluid dynamics from the Boltzmann equation

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Fluid-dynamical equations of motion can be derived from the Boltzmann equation in terms of an expansion around a single-particle distribution function which is in local thermodynamical equilibrium, i.e., isotropic in momentum space in the rest frame of a fluid element. However, in situations where the single-particle distribution function is highly anisotropic in momentum space, such as the initial stage of heavy-ion collisions at relativistic energies, such an expansion is bound to break down. Nevertheless, one can still derive a fluid-dynamical theory, called anisotropic dissipative fluid dynamics, in terms of an expansion around a single-particle distribution function, \( f_0 \), which incorporates (at least parts of) the momentum anisotropy via a suitable parametrization. We construct such an expansion in terms of polynomials in energy and momentum in the direction of the anisotropy and of irreducible tensors in the two-dimensional momentum subspace orthogonal to both the fluid velocity and the direction of the anisotropy. From the Boltzmann equation we then derive the set of equations of motion for the irreducible moments of the deviation of the single-particle distribution function from \( f_0 \). Truncating this set via the 14-moment approximation, we obtain the equations of motion of anisotropic dissipative fluid dynamics.

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I. INTRODUCTION

Fluid dynamics is the effective theory for the long-wavelength, small-frequency dynamics of a given system. As such, it finds widespread application in many different areas of physics. Its basic equations represent nothing but the conservation of particle number and energy-momentum, i.e., in special-relativistic notation:

\[
\partial_\alpha N^\alpha = 0, \quad \partial_\mu T^{\mu\nu} = 0,
\]

where \( N^\mu \) is the particle four-current and \( T^{\mu\nu} \) is the energy-momentum tensor. These five equations are not closed, since they involve in general 14 unknowns, the four components of the particle four-current and the ten components of the (symmetric) energy-momentum tensor. Closure, and thus a unique solution, is possible under certain simplifying assumptions.

The most drastic assumption is to reduce \( N^\mu \) and \( T^{\mu\nu} \) to the so-called ideal-fluid form,

\[
N_0^\mu = n_0 u^\mu, \quad T_0^{\mu\nu} = c_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu},
\]

where \( n_0, c_0, \) and \( P_0 \) are the particle number density, the energy density, and the thermodynamic pressure, respectively, \( u^\mu = \gamma (1, v) \) is the fluid four-velocity \( [v \text{ is the fluid three-velocity and } \gamma = (1 - v^2)^{-1/2}, \text{ such that } u^\mu u_\mu = 1, \text{ in units where } c = 1 \) and where the metric tensor is \( g^{\mu\nu} = \text{diag}(+,-,-,-) \), and \( \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \) is the projector onto the three-dimensional subspace orthogonal to \( u^\mu \), \( \Delta^{\mu\nu} u_\nu = u_\mu \Delta^{\mu\nu} = 0 \). Now the equations of motion (1) contain only six unknowns, and providing a thermodynamic equation of state (EoS) of the form \( P_0(c_0, n_0) \) closes them.

The microscopic picture underlying ideal fluid dynamics is that the fluid is in local thermodynamical equilibrium, e.g. for dilute gases at each space-time point \( x^\mu \) the single-particle distribution function assumes the form

\[
f_{0k}(\alpha_0, \beta_0 E_{ku}) = [\exp(-\alpha_0 + \beta_0 E_{ku}) + a]^{-1}.
\]

Here, \( \alpha_0 = \mu \beta_0 \), where \( \mu \) is the chemical potential associated with the particle density \( n_0 \), \( \beta_0 = 1/T \) is the inverse temperature (in units where \( k_B = 1 \)) and \( E_{ku} \equiv k^\mu u_\mu \), where \( k^\mu = (k_0, k) \) is the particle four-momentum. In the local rest (LR) frame of the fluid, \( u^\mu_{LR} = (1, 0, 0, 0) \), \( E_{ku} = \sqrt{k^2 + m_0^2} \) is the (relativistic on-shell) energy of a particle with mass \( m_0 \). In local thermodynamical equilibrium the quantities \( \alpha_0, \beta_0, \) and \( u^\mu \) are functions of \( x^\mu \). In global thermodynamical equilibrium, they assume constant values at all \( x^\mu \). The quantity \( a = 1 \) for Fermi–Dirac and \( a = -1 \) for Bose–Einstein statistics, respectively. Boltzmann statistics is obtained in the limit \( a \to 0 \). Equations (3) represents

1 At relativistic energy scales, pair-creation processes require that particle-number conservation is replaced by net-charge conservation. Bearing this in mind, for the sake of notational simplicity, we nevertheless keep the former in the present work.
the well-known Jüttner distribution function \[1, 2\], which for \( a = 0 \) is identical to the relativistic Maxwell-Boltzmann distribution function.

The particle four-current and the energy-momentum tensor are simply the first and second moments of \( f_{0k} \),

\[
N_0^\mu = \langle k^\mu \rangle_0 \ , \ T_0^{\mu\nu} = \langle k^\mu k^\nu \rangle_0 \ ,
\]

where (in units where \( h = 1 \))

\[
\langle \cdots \rangle_0 = \int dK \langle \cdots \rangle f_{0k} .
\]

Here, \( dK = g d^3k / [(2\pi)^3 k_0] \), where \( g \) counts the number of internal degrees of freedom. Inserting Eq. (3) into Eq. (4) it can be shown that \( |\delta N^\mu| \ll |N_0^\mu|, |\delta T^{\mu\nu}| \ll |T_0^{\mu\nu}| \). As explained above, \( N_0^\mu \) and \( T_0^{\mu\nu} \) are completely determined by five independent quantities (for a given equation of state \( P_0(c_0, n_0) \)). Therefore, \( \delta N^\mu \) and \( \delta T^{\mu\nu} \) involve in total nine independent quantities. Thus, one has to specify nine additional equations in order to close the equations of motion (11). These 14 equations then define a theory of dissipative (or viscous) fluid dynamics.

Unlike ideal fluid dynamics, in dissipative fluid dynamics the fluid four-velocity, and thus the choice of the LR frame of the fluid, is not uniquely defined. The two most popular choices are the Eckart frame \[2\], where

\[
u^\mu = \frac{N^\mu}{\sqrt{N^\alpha N_\alpha}}
\]

is proportional to the flow of particles, and the Landau frame \[3\], where

\[
u^\mu = \frac{T^{\mu\nu} u_\nu}{\sqrt{u_\alpha T^{\alpha\beta} T_{\beta\gamma} u^\gamma}}
\]

is proportional to the flow of energy. In the Eckart frame, the nine independent variables, or dissipative currents, which enter besides \( n_0, c_0 \), and \( u^\mu \), are the bulk viscous pressure \( \Pi \) (which is the difference between the isotropic pressure \( P = -\frac{1}{2} \Delta_{\mu\nu} T^{\mu\nu} \) and the thermodynamic pressure \( P_0 = -\frac{1}{2} \Delta_{\mu\nu} T_0^{\mu\nu} \), \( \Pi \equiv P - P_0 \) or, equivalently, up to a factor \( -1/3 \) equal to the trace of \( \Delta T^{\mu\nu} \), \( \Pi = -\frac{1}{3} \Delta_{\mu\nu} \delta T^{\mu\nu} \)), the shear-stress tensor \( \pi^{\mu\nu} \) [which is the trace-free part of \( \delta T^{\mu\nu} \), \( \pi^{\mu\nu} = \Delta_{\alpha\beta}^{\mu\nu} \delta T^{\alpha\beta} \), where \( \Delta_{\alpha\beta}^{\mu\nu} = \frac{2}{3} \Delta_{\alpha\beta}^{\mu\nu} + \Delta^{\mu\nu} \Delta_{\alpha\beta} - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \)], and the energy diffusion current \( W^\mu \) (which is the flow of energy relative to the flow of particles, \( W^\mu = \Delta_{\alpha\beta}^{\mu\nu} \delta T^{\alpha\beta} \)). In the Landau frame, the latter is replaced by the particle diffusion current \( V^\mu \) (which is the flow of particles relative to the flow of energy, \( V^\mu = \Delta^{\alpha\mu} \delta N^{\alpha} \)).

Providing equations for the dissipative currents closes the equations of motion (11) and defines a theory of dissipative fluid dynamics. However, there is considerable freedom in choosing these additional equations. Using the second law of thermodynamics one can show \[3\] that, to leading order, the dissipative currents must be proportional to gradients of \( \alpha_0, \beta_0 \), and \( u^\mu \), e.g. in the Landau frame,

\[
\Pi = -\zeta \partial_\alpha u^\alpha , \quad V^\mu = \kappa_n \Delta^{\mu\nu} \partial_\nu \alpha_0 , \quad \pi^{\mu\nu} = 2\eta \Delta_{\alpha\beta}^{\mu\nu} \delta^{\alpha\beta} u^\nu ,
\]

where \( \zeta \) is the bulk-viscosity, \( \eta \) the shear-viscosity, and \( \kappa_n \) the particle-diffusion coefficient, respectively. Equations (9) are the relativistic analogues of the Navier-Stokes equations \[3, 4\]. Since the dissipative currents are of first order in gradients, one also speaks of a first-order theory of dissipative fluid dynamics. Note that the dissipative currents are...
expressed solely in terms of the quantities $\alpha_0, \beta_0,$ and $u^\mu$, which are already present in ideal fluid dynamics. Inserting Eqs. (9) into the equations of motion (11) thus closes the latter.

Unfortunately, directly using Eqs. (9) in the equations of motion (11) renders them parabolic, leading to an unstable and acausal theory [3], at least in the relativistic case. The reason is that, in Eqs. (11), the dissipative currents (the left-hand side of these equations) act instantaneously to the dissipative forces (the right-hand sides). This unphysical behavior can be cured [3] by assuming a non-vanishing relaxation time for the dissipative currents, as suggested in Refs. [3 8 11]. In its simplest form, these equations read

\[
\begin{align*}
\tau_\Pi D\Pi + \Pi &= -\zeta \partial_\mu u^\mu , \\
\tau_\mu \Delta^\mu_\nu DV_\nu + V^\mu &= \kappa_0 u^\nu \partial_\nu \alpha_0 , \\
\tau_\Pi \Delta^\mu_\alpha D\pi_{\alpha\beta} + \pi^\mu = 2\eta \Delta^\mu_\alpha \partial^\alpha u^\beta ,
\end{align*}
\]

(10)

where $D \equiv u^\mu \partial_\mu$ is the comoving derivative and $\tau_\Pi, \tau_\mu,$ and $\tau_\Pi$ are the relaxation times associated with the individual dissipative currents. Note that these equations promote the dissipative currents to dynamical variables, which relax towards their Navier-Stokes values [3] on timescales given by $\tau_\Pi, \tau_\mu,$ and $\tau_\Pi$, respectively. On account of the Navier-Stokes equations [3], the dissipative currents themselves are already of first order in gradients. Since the equations (10) contain derivatives of the dissipative currents, these equations are formally of second order in gradients. One refers to formulations of dissipative fluid dynamics which contain terms of second order in gradients as second-order theories of dissipative fluid dynamics.

The microscopic picture behind dissipative fluid dynamics is that the single-particle distribution function $f_k$ is, albeit not identical to the local-equilibrium form [3], still sufficiently close to it, i.e.,

\[
f_k = f_{0k} + \delta f_k ,
\]

(11)

where the deviation $\delta f_k$ from local thermodynamical equilibrium being small, $|\delta f_k| \ll 1$. The equations of motion of dissipative fluid dynamics can be derived from this microscopic picture using the Boltzmann equation [12 12],

\[
k^\mu \partial_\mu f_k = C[f] ,
\]

(12)

where $C[f]$ is the collision integral. Employing the microscopic definitions of the particle four-current and the energy-momentum tensor,

\[
N^\mu = \langle k^\mu \rangle , \quad T^{\mu\nu} = \langle k^\mu k^\nu \rangle ,
\]

(13)

where, analogous to Eq. (3),

\[
\langle \cdots \rangle = \int dK \langle \cdots \rangle f_k ,
\]

(14)

the equations of motion (11) are nothing but the zeroth and first moments of the Boltzmann equation (assuming that the collision integral respects particle-number and energy-momentum conservation). Closure of the equations of motion (11) can be achieved by considering particle-number and energy-momentum conservation. This strategy has been pioneered by the authors of Refs. [4 11]. Honoring their work, the resulting equations (the most simple form of which reads as given in Eqs. (11)) are commonly referred to as Israel-Stewart (IS) equations. Similar approaches have been pursued in Refs. [12 16].

Recently, following the original idea of Grad [17], the deviation $\delta f_k$ has been expanded in a set of orthogonal polynomials in energy, $E_{k\mu} = k^\nu a_\mu$, and irreducible tensors in momentum, $1, k^{(\mu)}, k^{(\mu)k^{(\nu)}}, \ldots$, where the angular brackets denote the symmetrized and, for more than one Lorentz index, tracefree projection orthogonal to $u^\mu$, for details see Refs. [12 13]. Subsequently, the equations of motion of dissipative fluid dynamics have been derived by a systematic truncation (based on power counting in Knudsen and inverse Reynolds numbers) of the set of moments of the Boltzmann equation [12 20]. The lowest-order truncation gives the IS equations (10), including all other terms that are of second order in gradients (equivalent to terms of second order in either Knudsen or inverse Reynolds number, or of first order in the product of these). The advantage of this approach is that it can be systematically improved and applied to situations where the Knudsen or inverse Reynolds numbers are not small [21].

As a power series in gradients (or in Knudsen and inverse Reynolds number), the range of applicability of dissipative fluid dynamics is (at least formally) restricted to situations where the deviation $\delta f_k$ of the single-particle distribution function $f_k$ from the one in local thermodynamical equilibrium, $f_{0k}$, is small. There are, however, situations where gradients, or $\delta f_k$, respectively, become so large that a power-series expansion is expected to break down. In this case, one should modify the above described approach to (dissipative) fluid dynamics, explicitly taking into account deviations from local thermodynamical equilibrium to all orders. One of these situations is the initial stage of
ultrarelativistic heavy-ion collisions. In this case, the gradient of the fluid velocity in beam \((z-)\) direction is of the order of the inverse lifetime of the system, \(\partial_z v_z \sim 1/t\), which can in principle become arbitrarily large as one approaches the moment of impact of the colliding nuclei (at \(t = 0\)). This large gradient is reflected in a single-particle distribution function which is highly anisotropic in the \(z\)-direction in momentum space.

Fluid dynamics for anisotropic single-particle distribution functions have been studied a long time ago. Incidentally, the physics motivation was very similar to the case described above, namely to account for momentum-space anisotropies in the initial stage of heavy-ion collisions, although at that time the available beam energies were orders of magnitude smaller. Recently, Florkowski, Martinez, Ryblewski, and Strickland proposed a formulation of anisotropic fluid dynamics based on a single-particle distribution function, which is anisotropic in momentum space and whose specific form is motivated by the gluon fields created in the initial stages of a heavy-ion collision. In this formulation, \(f_k\) is assumed to have a spheroidal shape in the LR frame, which is deformed in the \(z\)-direction with respect to a spherically symmetric \(f_{0k}\). The degree of anisotropy is quantified by a single parameter. On the other hand, having in mind applying their formalism to the evolution of the fluid in the mid-rapidity region of a heavy-ion collision, where there is no conserved net-charge, the authors of Refs. [24–31] assumed that there is no parameter like \(\alpha_0\) which controls the particle-number (or more precisely, net-charge) density. Thus, besides energy-momentum conservation [the second equation (4)], a single additional equation is necessary to close the system of equations of motion. The simplest possibility is the zeroth moment of the Boltzmann equation, with the collision integral taken in relaxation-time approximation. Another possibility is the entropy equation (with a non-vanishing source term, as entropy must grow in non-equilibrium situations). But in principle, also higher moments of the Boltzmann equation could serve to determine the anisotropy parameter.

In this formulation of anisotropic fluid dynamics, while the anisotropy parameter is a function of space-time, the form of the single-particle distribution function as a function of the anisotropy parameter always remains the same. Even though one may generalize this idea by introducing additional parameters to capture the anisotropy to an even better degree, this does not change the fact that one is always restricted by the chosen form of the anisotropic distribution function. In this sense, this approach is rather a generalization of ideal fluid dynamics, where it is assumed that the single-particle distribution always has the form of dissipative fluid dynamics. Even so, when compared to ideal fluid dynamics, this approach includes dissipative effects due to the anisotropic single-particle distribution function.

Generalizing dissipative fluid dynamics to a theory of anisotropic dissipative fluid dynamics, one should take an anisotropic single-particle distribution function, called \(f_{0k}\) in the following, as the starting point for an expansion of the general single-particle distribution function, i.e.,

\[
f_k = f_{0k} + \delta f_k.
\]

While this looks similar to Eq. (11), the rationale behind an expansion around \(f_{0k}\) instead of around \(f_{0k}\) as in Eq. (11) is the following: in the case of a pronounced anisotropy, \(\delta f_k\) in Eq. (11) may be of similar magnitude (or even larger) than \(f_{0k}\), i.e., an expansion around the local equilibrium distribution converges badly. However, taking a suitably chosen \(f_{0k}\), we ensure that \(|\delta f_k| \ll |\delta f_k|\), so that the convergence properties of the series expansion are vastly improved.

This strategy has been applied in Ref. [37] to derive equations of motion for anisotropic dissipative fluid dynamics (or, as called there, “viscous anisotropic hydrodynamics”), based on the method presented in Refs. [17, 38]. However, the form of \(\delta f_k\) was a simple linear function of the tensors \(k^\mu, k^\mu k^\nu\) with 14 unknown coefficients. These were then expressed in terms of the 14 fluid-dynamical variables \(N^\mu, T^{\mu\nu}\) (or an equivalent set of variables) via a linear mapping procedure (which employs the so-called Landau-matching conditions and the choice of LR frame). This strategy is analogous to that of Ref. [11] for the derivation of “ordinary” dissipative fluid dynamics.

The disadvantages of this approach were explained in the introduction of Ref. [18], the main one being that it is not systematically improvable. In order to provide an improvable framework in the case of “ordinary” dissipative fluid dynamics, it was essential to use a set of orthogonal polynomials in energy, \(E_{ku} = k^\mu u_\mu\), and irreducible tensors in momentum, \(1, k^{(\mu)}, k^{(\mu)k^\nu}, \ldots\), in the expansion of \(\delta f_k\). This was used for an expansion of \(\delta f_k\) in deriving equations of motion for anisotropic dissipative fluid dynamics in Ref. [39]. However, in the case of anisotropic dissipative fluid dynamics, besides \(u^\mu\) there is an additional space-like four-vector, \(l^\mu\), which defines the direction of the anisotropy (as explained above, usually taken to be the \(z\)-direction) and can be chosen to be orthogonal to \(u^\mu\), \(l^\mu u_\mu = 0\). In place of \(\delta f_k\) in Eq. (11), one now needs to expand \(\delta f_k\) in Eq. (13). This expansion involves orthogonal polynomials in the two variables \(E_{ku}\) and the particle momentum in the direction of the anisotropy, \(E_{kl} = -k^\mu l_\mu\), as well as irreducible tensors which are orthogonal to both \(u^\mu\) and \(l^\mu\). A derivation of anisotropic dissipative fluid dynamics along these lines is the main goal of the present paper. In this way, we provide a systematically improvable framework for anisotropic dissipative fluid dynamics.

This paper is organized as follows. In Sec. [11] we introduce the tensor decomposition of fluid-dynamical variables
with respect to the time-like fluid four-velocity \( u^\mu \) and the space-like four-vector \( l^\mu \) \((l^\mu l_\mu = -1)\), which is usually chosen to point into the direction of the spatial anisotropy. In Secs. III and IV we study two limiting cases of this tensor decomposition. The first is the well-known ideal-fluid limit where only tensor structures proportional to \( u^\mu \) and \( \Delta^{\mu\nu} \) appear in the moments of the single-particle distribution function \( f_{0k} \). The second is the anisotropic case, where the single-particle distribution function also depends on \( l^\mu \) besides \( u^\mu \) and where tensor structures proportional to \( u^\mu \), \( l^\mu \), their direct product, and the two-space projector orthogonal to both \( u^\mu \) and \( l^\mu \) \([40, 43]\),

\[
\Xi^{\mu\nu} \equiv \Delta^{\mu\nu} + l^{\mu}l^{\nu} = g^{\mu\nu} - u^{\mu}u^{\nu} + l^{\mu}l^{\nu} ,
\]

appear in the moments of the single-particle distribution function. In Sec. V we present the expansion of the single-particle distribution function \( f_{0k} \) around the anisotropic state \( \tilde{f}_{0k} \). In analogy to Refs. [12, 18], this is done in terms of an orthogonal basis of irreducible tensors in momentum space. However, in contrast to previous work, these tensors are not only orthogonal to \( u^\mu \) but also to \( l^\mu \). Then, in Sec. VI taking moments of the Boltzmann equation we derive the equations of motion for the irreducible moments of the single-particle distribution function up to tensor-rank two. These equations are not yet closed and need to be truncated in order to derive the fluid-dynamical equations of motion in terms of conserved quantities, i.e., the particle four-current \( N^\mu \) and the energy-momentum tensor \( T^{\mu\nu} \). In Sec. VII we study the explicit form of the collision integral. Finally, in Sec. VIII we give the derivation of the fluid-dynamical equations of motion in the 14–moment approximation. Section IX concludes this work with a summary and an outlook. Details of our calculations are delegated to various appendices.

We adopt natural units, \( \hbar = c = k_B = 1 \), throughout this work. The symmetrization of tensor indices is denoted by \( \{} \) around Greek indices, e.g., \( A^{(\mu\nu)} = (A^{\mu\nu} + A^{\nu\mu})/2 \) while \( [\{} \) means the antisymmetrization of indices, i.e., \( A^{[\mu\nu]} = (A^{\mu\nu} - A^{\nu\mu})/2 \). The projection of an arbitrary four-vector \( A^\mu \) onto the directions orthogonal to \( u^\mu \) will be denoted by \( \langle \rangle \) around Greek indices, \( A^{\langle\mu\rangle} = \Delta^{\mu\rho} A_\rho \). The projection of an arbitrary four-vector \( A^\mu \) onto the directions orthogonal to both \( u^\mu \) and \( l^\mu \) will be denoted by \( \{\} \) around indices, \( A^{(\mu\nu)} = \Xi^{\mu\nu} A_\nu \). Projections of higher-rank tensors will be denoted in a similar manner. In case of an arbitrary anisotropy the four-momentum of particles is \( k^\mu = E_{k_0} u^\mu + E_{k_1} l^\mu + k^{(\mu)} \), where \( k^{(\mu)} = \Xi^{\mu\nu} k_\nu \) are the components of the momentum orthogonal to \( u^\mu \) and \( l^\mu \).

II. FLUID-DYNAMICAL VARIABLES

In this section we introduce the tensor decomposition of the fluid-dynamical variables with respect to the time-like fluid four-velocity \( u^\mu \), as well as the decomposition with respect to both \( u^\mu \) and the space-like four-vector \( l^\mu \). Details of the calculation can be found in Apps. A and B.

Equation (13) gives the microscopic definition of the particle 4-current and the energy-momentum tensor in terms of the first and second moment of the single-particle distribution function \( f_{0k} \). Using Eqs. (17), (18), and the fact that the energy-momentum tensor is symmetric, the tensor decomposition of \( N^\mu \) and \( T^{\mu\nu} \) with respect to \( u^\mu \) and \( \Delta^{\mu\nu} \) reads

\[
N^\mu \equiv \langle k^\mu \rangle = n u^\mu + V^\mu , \tag{17}
\]

\[
T^{\mu\nu} \equiv \langle k^{(\mu)} k^{(\nu)} \rangle = e u^{\mu} u^{\nu} - P \Delta^{\mu\nu} + 2 W^{(\mu} u^{\nu)} + \pi^{\mu\nu} , \tag{18}
\]

where the angular brackets denote the momentum-space average defined in Eq. (13). On the one hand, the various quantities appearing in the tensor decomposition on the right-hand side can be expressed in terms of different projections of the particle four-current and the energy-momentum tensor. On the other hand, employing Eqs. (17), (18), these quantities can be identified as moments of the single-particle distribution function \( f_{0k} \). Following this strategy, the scalar coefficients in Eqs. (17), (18), i.e., the particle density \( n \), the energy density \( e \), and the isotropic pressure \( P \), read

\[
n \equiv \langle E_{k_0} \rangle = N^\mu u_\mu , \tag{19}
\]

\[
e \equiv \langle E_{k_1}^2 \rangle = T^{\mu\nu} u_\mu u_\nu , \tag{20}
\]

\[
P \equiv -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle = -\frac{1}{3} T^{\mu\nu} \Delta_{\mu\nu} , \tag{21}
\]

while the particle and energy-momentum diffusion currents \( V^\mu \) and \( W^\mu \), respectively, are

\[
V^\mu \equiv \langle k^{(\mu)} \rangle = \Delta^\mu_N N^\nu , \tag{22}
\]

\[
W^\mu \equiv \langle E_{k_0} k^{(\mu)} \rangle = \Delta^\mu_W T^{\alpha\beta} u_\beta . \tag{23}
\]
Both are orthogonal to the flow velocity, $V^\mu u_\mu = W^\mu u_\mu = 0$. Finally, the shear-stress tensor

$$\pi^{\mu\nu} \equiv \langle k^{(\mu} k^{\nu)} \rangle = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta},$$  \hspace{1cm} (24)

is the part of the energy-momentum tensor that is symmetric, $\pi^{\mu\nu} = \pi^{\nu\mu}$, traceless, $\pi^{\mu\nu} g_{\mu\nu} = 0$, and orthogonal to the flow velocity $\pi^{\mu\nu} u_\mu = 0$. Here $\Delta^{\mu\nu}_{\alpha\beta}$ is the projection tensor defined in Eq. (14), such that $k^{(\mu} k^{\nu)} = \Delta^{\mu\nu}_{\alpha\beta} k^\alpha k^\beta$.

The particle four-current and the energy-momentum tensor can also be decomposed with respect to $u^\mu$, the space-like four-vector $l^\mu$, and the projection tensor $\Xi^{\mu\nu}$. Using Eqs. (11), (12) and the fact that the energy-momentum tensor is symmetric, we obtain

$$N^\mu = n u^\mu + n l^\mu + V^\mu_l,$$  \hspace{1cm} (25)

$$T^{\mu\nu} = e u^\mu u^\nu + 2 M u^\mu l^\nu + P_l l^\mu l^\nu - P_{\perp} \Xi^{\mu\nu} + 2 W_{\perp}^{\mu\nu} + 2 W_{\perp}^{\mu l \nu} + \pi^{\mu\nu}.$$  \hspace{1cm} (26)

Here, in addition to the previously defined quantities, we have denoted the part of the particle diffusion current in the $l^\mu$ direction by $n_l$, while the particle diffusion current orthogonal to both four-vectors is denoted by $V^\mu_l$. The pressure in the transverse direction is denoted by $P_{\perp}$, while the pressure in the longitudinal direction is $P_l$. The projection of the energy-momentum tensor in both $w^\mu$ and $l^\mu$ directions is denoted by $M$. The projection in either the $u^\mu$ or $l^\mu$ direction and orthogonal to both directions is denoted by $W_{\perp}^{\mu\nu}$ or $W_{\perp}^{\mu l \nu}$, respectively. The only rank-two tensor in the subspace orthogonal to both distinguished four-vectors is given by the "transverse" shear-stress tensor $\pi^{\mu\nu}$. Note that the various subscripts $u$ (for projection onto the direction of $u^\mu$), $l$ (for projection onto the direction of $l^\mu$) and $\perp$ (for projection onto the direction “perpendicular” to both $w^\mu$ and $l^\mu$) serve as reminders of the directions that the various quantities are projected onto.

These newly defined quantities can either be expressed in terms of different projections of the conserved particle four-current and energy-momentum tensor or, with the help of Eqs. (23), (25), identified as moments of $f_k$, i.e.,

$$n_l \equiv \langle E_{kl} \rangle = -N^\mu l_\mu,$$  \hspace{1cm} (27)

$$M \equiv \langle E_{kl} E_{kl} \rangle = -T^{\mu\nu} u_\mu l_\nu,$$  \hspace{1cm} (28)

$$P_l \equiv \langle E_{kl}^2 \rangle = T^{\mu\nu} l_\mu l_\nu,$$  \hspace{1cm} (29)

$$P_{\perp} \equiv -\frac{1}{2} \langle \Xi^{\mu\nu} k_\mu k_\nu \rangle = -\frac{1}{2} T^{\mu\nu} \Xi_{\mu\nu},$$  \hspace{1cm} (30)

and

$$V^\mu_{\perp} \equiv \langle k^{(\mu} \rangle = \Xi^{\mu}_{\nu} N^{\nu},$$  \hspace{1cm} (31)

$$W_{\perp}^{\mu\nu} \equiv \langle E_{kl} k^{(\mu} \rangle = \Xi^{\mu}_{\alpha} T^{\alpha\beta} u_\beta,$$  \hspace{1cm} (32)

$$W_{\perp}^{\mu l \nu} \equiv \langle E_{kl} k^{(\mu} \rangle = -\Xi^{\mu}_{\nu} T^{\alpha\beta} l_\beta,$$  \hspace{1cm} (33)

$$\pi_{\perp}^{\mu\nu} \equiv \langle k^{(\mu} k^{\nu)} \rangle = \Xi^{\mu\nu}_{\alpha\beta},$$  \hspace{1cm} (34)

From these definitions it is evident that $V^\mu_{\perp} u_\mu = V_{\perp}^{\mu l}_\mu = 0$ as well as $W_{\perp}^{\mu\nu} u_\mu = W_{\perp}^{\mu l \nu} l_\mu = 0$ and $W_{\perp}^{\mu l \nu} u_\mu = W_{\perp}^{\mu l l \nu} = 0$. The transverse shear-stress tensor $\pi_{\perp}^{\mu\nu}$ is the part of the energy-momentum tensor that is symmetric, $\pi_{\perp}^{\mu\nu} = \pi_{\perp}^{\nu\mu}$, traceless, $\pi_{\perp}^{\mu\nu} g_{\mu\nu} = 0$, and orthogonal to both preferred four-vectors, $\pi_{\perp}^{\mu\nu} u_\mu = \pi_{\perp}^{\mu\nu} l_\mu = 0$. Here $\Xi^{\mu\nu}_{\alpha\beta}$ is the projection tensor defined in Eq. (15), such that $k^{(\mu} k^{\nu)} = \Xi^{\mu\nu}_{\alpha\beta} k^\alpha k^\beta$.

Note that the latter decomposition of the conserved quantities with respect to $u^\mu$, $l^\mu$, and $\Xi^{\mu\nu}$ was, to our knowledge, already given by Barz, Kämpfer, Lukács, Martinác, and Wolf as early as the late 1980’s [23], and was later on used in Refs. [24, 25]. Recently, Florkowski, Martinez, Ryblewski, and Strickland [24, 31] proposed the so-called anisotropic hydrodynamics formalism based on a specific distribution function that has an anisotropic spheroidal shape in momentum space in the LR frame [32]. This particular ansatz leads to a less general form of the energy-momentum tensor, which only features a pressure anisotropy, $P_l \neq P_{\perp}$, while other terms listed in Eqs. (27)–(34) vanish equivalently. This was later improved in Ref. [33] based on Eq. (16) but the decomposition still differs from Eqs. (23), (25). For example $\tilde{\pi}^{\mu\nu}$ from Eq. (25g) of Ref. [37] is not orthogonal to the four-vector specifying the direction of the anisotropy (i.e., in our case $l^\mu$).

Comparing Eq. (21) to Eqs. (29)–(34) the longitudinal and transverse pressure components are related to the isotropic pressure as

$$P = \frac{1}{3} (P_l + 2 P_{\perp}).$$  \hspace{1cm} (35)
This result is independent on how far off the system is from local thermodynamical equilibrium. In case that $P = P_\perp = \Pi_0$, the pressure is isotropic, but the system may not be in local thermodynamical equilibrium, because the bulk viscous pressure
\[ \Pi_{iso} \equiv P - P_0, \]
may be non-zero. Here, $P_0$ is the pressure in local thermodynamical equilibrium.

Furthermore, not only the isotropic pressure separates into two parts but also the diffusion currents $V^\mu$ and $W^\mu$ are split according to the direction defined by $l^\mu$ and the direction perpendicular to it,
\[
V^\mu = n_l l^\mu + V_\perp^\mu, \quad W^\mu = M l^\mu + W_\perp^\mu.
\]
Finally, using Eq. (A16), we can show that
\[
\pi^{\mu\nu} = \pi_\perp^{\mu\nu} + 2 W^{(\mu l^\nu)} + \frac{1}{3} (P_l - P_\perp) (2 l^\mu l^\nu + \Xi^{\mu\nu}).
\]
Equations (35) – (39) relate the fluid-dynamical quantities decomposed with respect to $l^\mu$ and $W^\mu$ to those decomposed with respect to $\mu^\mu$, $\Delta^{\mu\nu}$, and $\Xi^{\mu\nu}$. Note that, in terms of independent degrees of freedom, these two decompositions are completely equivalent. In general, $N^{\mu\nu}$ has four while $T^{\mu\nu}$ has ten independent components. The decomposition with respect to (a given four-vector) $u^\mu$ and the projector $\Delta^{\mu\nu}$ also contains 14 independent dynamical variables. These are the three scalars $n$, $e$, and $P$, the two vectors $V^\mu$ and $W^\mu$, each with three independent components, while the shear-stress tensor $\pi^{\mu\nu}$ has five independent components. On the other hand, the decomposition with respect to (given) $u^\mu$, $l^\mu$, and $\Xi^{\mu\nu}$ has the six scalars $n$, $e$, $n_l$, $M$, $P_l$, and $P_\perp$, the three vectors $V_\perp^\mu$, $W_\perp^\mu$, and $W_\perp^\mu$, with two independent components each, whereas the shear-stress tensor in the transverse direction, $\pi_\perp^{\mu\nu}$, possesses only two independent components.

So far, the four-vectors $u^\mu$ and $l^\mu$ were arbitrary quantities. However, commonly the (time-like) four-vector $u^\mu$ is supposed to have a physical meaning, namely the fluid four-velocity. In this case, it becomes a dynamical quantity with three independent degrees of freedom. The choice of the fluid four-velocity is not unique. The two most popular choices to fix the LR frame of the fluid have already been discussed in the Introduction, the Eckart frame (2), Eq. (7), which follows the flow of particles, and the Landau frame (3), Eq. (8), which follows the flow of energy. Consequently, in the Eckart frame, there is no diffusion of particles (or charges) relative to $u^\mu$.

In this section, we discuss the moments of the single-particle distribution function in local thermodynamical equilibrium. The equilibrium moments of tensor-rank $n$ are defined as
\[
\mathcal{T}^{B_1 \cdots \mu_n}_{1} = \langle E_{k_n}^1 k^{\mu_1} \cdots k^{\mu_n} \rangle_0,
\]
where the angular brackets denote the average over momentum space defined in Eq. (5). The subscript $i$ on this quantity reflects the power of $E_{k\nu}$ in the definition of the moment. Due to the fact that the equilibrium distribution function depends only on the quantities $\alpha_0$, $\beta_0$, and the flow velocity $u^\mu$, the equilibrium moments can be expanded in terms of $u^\mu$ and the projector $\Delta^{\mu\nu}$ as

$$I^{\mu_1 \cdots \mu_n}_{i \cdots n} = \sum_{q=0}^{[n/2]} (-1)^q b_{nq} I_{i+n,q} \Delta^{\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} u^{\mu_{2q+1} \cdots \mu_n}},$$  \hspace{1cm} (44)$$

where $n$, $q$ are natural numbers while the sum runs over $0 \leq q \leq [n/2]$. Here, $[n/2]$ denotes the largest integer which is less than or equal to $n/2$. The symmetrized tensors $\Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} u^{\mu_{2q+1} \cdots \mu_n})}$ are discussed in App. C.

The symmetrization yields

$$b_{nq} = \frac{n!}{2^q q! (n-2q)!}$$

(45)

distinct terms, see App. C. Finally, $I_{nq}$ are the so-called relativistic thermodynamic integrals that only depend on the equilibrium variables $\alpha_0$ and $\beta_0$,

$$I_{nq} (\alpha_0, \beta_0) = \frac{(-1)^q}{(2q + 1)!} \left( E_{k\nu}^{n-2q} (\Delta^{\alpha \beta k_\alpha k_\beta})^q \right)_0,$$  \hspace{1cm} (46)$$

where the double factorial is defined as $(2q + 1)! = (2q + 1)! / (2q)!$. Note that the thermodynamic integrals with index $q = 0$ are directly given by the moments (43) of rank zero,

$$I_{00} \equiv I_i.$$  \hspace{1cm} (47)$$

Analogously, the thermodynamic integrals with index $q = 1$ can be obtained from a projection of the moments (43) of rank two,

$$I_{1+2,1} \equiv -\frac{1}{3} I_i^{\mu\nu} \Delta_{\mu\nu} = -\frac{1}{3} (m_0^2 I_i - I_{i+2}),$$  \hspace{1cm} (48)$$

where we made use of the explicit form of $\Delta_{\mu\nu}$, of the on-shell condition $k^\mu k_\mu = m_0^2$, and again employed Eq. (43).

The so-called auxiliary thermodynamic integrals are defined as

$$J_{nq} \equiv \left( \frac{\partial I_{nq}}{\partial \alpha_0} \right)_{\beta_0} = \frac{(-1)^q}{(2q + 1)!} \int dK E_{k\nu}^{n-2q} (\Delta^{\alpha \beta k_\alpha k_\beta})^q f_{0k} (1 - a f_{0k}) .$$  \hspace{1cm} (49)$$

Note that

$$\left( \frac{\partial I_{nq}}{\partial \beta_0} \right)_{\alpha_0} = \frac{(-1)^q}{(2q + 1)!} \int dK E_{k\nu}^{n+1-2q} (\Delta^{\alpha \beta k_\alpha k_\beta})^q f_{0k} (1 - a f_{0k}) \equiv -J_{n+1,q} .$$  \hspace{1cm} (50)$$

Using the definition (4) of the conserved quantities and Eqs. (43) – (45),

$$N^\mu_0 \equiv T^\mu_0 = I_{10} u^\mu,$$  \hspace{1cm} (51)$$

$$T^\mu_0 \equiv T_0^\mu = I_{20} u^\mu u^\nu - I_{21} \Delta^{\mu\nu} .$$  \hspace{1cm} (52)$$

Tensor-projecting these quantities, we obtain

$$I_{10} \equiv N^\mu_0 u_\mu = n_0 = \langle E_{k\nu} \rangle_{00} \equiv I_1 ,$$  \hspace{1cm} (53)$$

$$I_{20} \equiv T_0^\mu u_\mu u_\nu = e_0 = \langle E^2 \rangle_0 \equiv I_2 ,$$  \hspace{1cm} (54)$$

$$I_{21} \equiv -\frac{1}{3} T_0^\mu \Delta_{\mu\nu} = P_0 = -\frac{1}{3} \langle \Delta_{\mu\nu} k_\mu k_\nu \rangle_0 \equiv -\frac{1}{3} (m_0^2 I_0 - I_2) .$$  \hspace{1cm} (55)$$

Here we used Eqs. (10) – (21), the explicit form of $\Delta^{\mu\nu}$, as well as Eq. (43). We note that the left- and right-hand sides of these equations are consistent with the relations (47) and (48). Using Eqs. (53) – (55), we see that the tensor decompositions (51) and (52) correspond to the usual ideal-fluid form (2) of the conserved quantities.
IV. ANISOTROPIC STATE

In this section, we discuss the case where the single-particle distribution function has an anisotropic shape in momentum space. Let us assume that this function differs from the local equilibrium distribution function \( f_0 \) such that it is a function of the scalars \( \tilde{\alpha}, \tilde{\beta}_u, \) and \( \tilde{\beta}_l \) as well as of two distinct four-vectors, the flow velocity \( u^\mu \) and the vector \( l^\mu \) parametrizing the direction of the anisotropy. All these quantities are functions of \( x^\mu \). Such distribution functions are common in plasma physics where the presence of magnetic fields introduces a momentum anisotropy and so the particle momenta parallel and perpendicular to the magnetic field are different; as in the case of the bi-Maxwellian, the drifting Maxwellian, or the loss-cone distribution functions \([44]\). Analogously, \( \tilde{\beta}_l \) can be thought of as an additional parameter characterizing the temperature difference between the directions parallel and perpendicular to the \( z \)-axis.

As discussed in the Introduction, we will denote the anisotropic single-particle distribution function as \( \tilde{f}_{0k} = \tilde{f}_{0k} \left( \tilde{\alpha}, \tilde{\beta}_u E_{ku}, \tilde{\beta}_l E_{kl} \right) \). At this point, the functional dependence on \( \tilde{\beta}_l E_{kl} \) does not need to be specified. All we need to know is that this combination of variables parametrizes the momentum anisotropy.

We will also assume that

\[
\lim_{\beta_l \to 0} \tilde{f}_{0k} \left( \tilde{\alpha}, \tilde{\beta}_u E_{ku}, \tilde{\beta}_l E_{kl} \right) = f_{0k} \left( \tilde{\alpha}, \tilde{\beta}_u E_{ku} \right),
\]

i.e., that in the limit of vanishing anisotropy the single-particle distribution function assumes the local-equilibrium form \( f_0 \). The assumption \( \text{(56)} \) has no impact on our formulation of anisotropic dissipative fluid dynamics, it is merely physically natural. We furthermore demand that

\[
\left( \frac{\partial \tilde{f}_{0k}}{\partial \tilde{\alpha}} \right)_{\tilde{\beta}_u, \tilde{\beta}_l} = \tilde{f}_{0k} \left( 1 - a \tilde{f}_{0k} \right).
\]

This further constraint on the form of \( \tilde{f}_{0k} \) is naturally respected in the limit of vanishing anisotropy, cf. Eq. \( \text{(3)} \). There is no real physical reason that we should require it also for \( \tilde{\beta}_l \neq 0 \), but it simplifies the following calculations. For instance, the spheroidal distribution function proposed in Ref. \([32]\) and used in Refs. [24–31] satisfies the above constraints and can be used for explicit calculations.

In analogy to Eq. \( \text{(36)} \) we now introduce a set of generalized moments of \( \tilde{f}_{0k} \) of tensor-rank \( n \),

\[
\tilde{I}_{ij}^{\mu_1 \cdots \mu_n} = \left\langle E_{ku}^i E_{kl}^j k^{\mu_1} \cdots k^{\mu_n} \right\rangle_{\tilde{0}},
\]

where, similarly to Eqs. \( \text{(55)} \) and \( \text{(44)} \),

\[
\left\langle \cdots \right\rangle_{\tilde{0}} = \int dK \left\langle \cdots \right\rangle_{\tilde{f}_{0k}},
\]

and the subscripts \( i \) and \( j \) denote the powers of \( E_{ku} \) and \( E_{kl} \), respectively. These generalized moments can be expanded in terms of the two four-vectors \( u^\mu \), \( l^\mu \), and the tensor \( \Xi^{\mu \nu} \),

\[
\tilde{I}_{ij}^{\mu_1 \cdots \mu_n} = \sum_{q=0}^{[n/2]} \sum_{r=0}^{n-2q} (-1)^q b_{nqr} \tilde{I}_{i+n+j+r,q} \Xi^{\mu_1 \mu_2 \cdots \mu_{2q+1}, \mu_{2q+1} \cdots \mu_{n+r+1}} E_{ku}^i E_{kl}^j \Xi^{\mu \nu} k^{\mu} l^{\nu},
\]

where \( n, r, \) and \( q \) are natural numbers; \( r \) counts the number of four-vectors \( l^\mu \) and \( q \) the number of \( \Xi \) projectors in the expansion. The symmetrized tensor products \( \Xi^{\mu_1 \mu_2 \cdots \mu_{2q+1}, \mu_{2q+1} \cdots \mu_{n+r+1}} \) are discussed in App. \([C]\) The symmetrization yields

\[
b_{nqr} \equiv \frac{n!}{2^q q! r! (n - r - 2q)!}
\]

distinct terms, see App. \([C]\) Finally, the generalized thermodynamic integrals \( \tilde{I}_{nqr} \) are defined as

\[
\tilde{I}_{nqr} \left( \tilde{\alpha}, \tilde{\beta}_u, \tilde{\beta}_l \right) = \frac{(-1)^q}{(2q)!!!} \left\langle E_{ku}^{n-r-2q} E_{kl}^{r} \left( \Xi^{\mu \nu} k^{\mu} l^{\nu} \right)^q \right\rangle_{\tilde{0}},
\]
where the double factorial of an even number is \((2q)!! \equiv 2^q q!\). The corresponding generalized auxiliary thermodynamic integrals are defined with the help of Eq. (57) and similarly to Eq. (19),

\[
\hat{J}_{n\nu} = \left( \frac{\partial \hat{J}_{n\nu}}{\partial \alpha} \right)_{\hat{n},\hat{\beta}_a,\hat{\beta}_l} = \frac{(-1)^q}{(2q)!!} \int dK E_{ku}^{n\nu-2q} E_{kl}^{n\nu} \left( \Xi_{\mu\nu} k_{\mu} k_{\nu} \right)^q \hat{f}_{0k} \left( 1 - \alpha \hat{f}_{0k} \right). \tag{63}
\]

As in Eqs. (17) and (18) one can easily show that, in the case \(q = 0\), comparison of Eqs. (58) and (62) leads to

\[
\hat{I}_{i+j,0} = \hat{I}_{ij}, \tag{64}
\]

while for \(q = 1\), using the explicit form of \(\Xi_{\mu\nu}\) and the on-shell condition \(k^\mu k_\mu = m^2\), comparison of Eqs. (58) and (62) yields

\[
\hat{I}_{i+j+2,1} = -\frac{1}{2} \hat{T}_{ij} \Xi_{\mu\nu} = -\frac{1}{2} \left( m_0^2 \hat{I}_{ij} - \hat{I}_{i+j+2} + \hat{I}_{i+j+2} \right). \tag{65}
\]

Note that these quantities can also be used in (local) thermodynamic equilibrium, see for example Eq. (122). The conventional relativistic thermodynamic integrals \((10), (19)\) are then recovered as linear combinations of those defined in Eqs. (62), (63), as shown in App. \((9)\). Furthermore, also note that the moment defined in the first line of Eq. (66) of Ref. [37] is (because of the \(\Delta\) instead of the \(\Xi\) projector in the integrand) not equivalent to the one defined in Eq. (55).

It is instructive to explicitly write down the tensor decomposition of the generalized moments (60) of \(\hat{f}_{0k}\). The conserved quantities read

\[
\hat{N}_\mu \equiv \hat{T}_{0\mu} = \hat{I}_{100} u_\mu + \hat{I}_{110} l_\mu, \tag{66}
\]

\[
\hat{T}_{\mu\nu} \equiv \hat{T}_{0\mu\nu} = \hat{I}_{200} u^\mu u_\nu + 2 \hat{I}_{120} u^\mu l_\nu + \hat{I}_{220} l^\mu l_\nu - \hat{I}_{201} \Xi_{\mu\nu}. \tag{67}
\]

The coefficients can be obtained by appropriate tensor projections of these quantities. According to Eqs. (19), (20) and Eqs. (24) – (30) we obtain

\[
\hat{I}_{100} \equiv \hat{N}_\mu u_\mu = \hat{n} = \langle E_{ku} \rangle_0 \equiv \hat{\bar{I}}_{10}, \tag{68}
\]

\[
\hat{I}_{110} \equiv -\hat{N}_\mu l_\mu = \hat{\eta} = \langle E_{kl} \rangle_0 \equiv \hat{\bar{I}}_{01}, \tag{69}
\]

\[
\hat{I}_{200} \equiv \hat{T}_{\mu\nu} u_\mu u_\nu = \hat{e} = \langle E_{ku}^2 \rangle_0 \equiv \hat{\bar{I}}_{20}, \tag{70}
\]

\[
\hat{I}_{210} \equiv -\hat{T}_{\mu\nu} u_\mu l_\nu = \hat{M} = \langle E_{ku} E_{kl} \rangle_0 \equiv \hat{\bar{I}}_{11}, \tag{71}
\]

\[
\hat{I}_{220} \equiv \hat{T}_{\mu\nu} l_\mu l_\nu = \hat{P}_t = \langle E_{kl}^2 \rangle_0 \equiv \hat{\bar{I}}_{02}, \tag{72}
\]

\[
\hat{I}_{201} \equiv -\frac{1}{2} \hat{T}_{\mu\nu} \Xi_{\mu\nu} = \hat{P}_\perp = -\frac{1}{2} \langle \Xi_{\alpha\beta} k_{\alpha} k_{\beta} \rangle_0 \equiv -\frac{1}{2} \left( m_0^2 \hat{\bar{I}}_{00} - \hat{\bar{I}}_{20} + \hat{\bar{I}}_{02} \right). \tag{73}
\]

Note that \(\hat{e}, \hat{P}_t\), and \(\hat{P}_\perp\) are related to each other, see Eq. (123), hence they are not independent variables.

In general, the conserved quantities \((59), (57)\) contain eleven unknowns: the scalar quantities \(\hat{n}, \hat{\eta}, \hat{\bar{I}}_{10}, \hat{\bar{I}}_{01}, \hat{M}, \hat{P}_t, \) and \(\hat{P}_\perp\), and the vectors \(u^\mu\) (three independent components) and \(l^\mu\) (two independent components). At the end of Sec. \(11\) we had already discussed our choice of \(l^\mu\) which is completely determined by \(u^\mu\), so there remain nine unknowns. The choice of a LR frame (Eckart or Landau) eliminates either \(\hat{n}\) or \(\hat{\eta}\), hence leaving eight unknowns. However, once \(\hat{f}_{0k} \left( \hat{\alpha}, \hat{\beta}_a E_{ku}, \hat{\beta}_l E_{kl} \right)\) is specified, the remaining five scalar unknowns \((\hat{n}, \hat{\eta}, \hat{\bar{I}}_{10}, \hat{\bar{I}}_{01}, \hat{M})\), and – depending on the choice of the LR frame – either \(\hat{\bar{I}}_{10}\) or \(\hat{\bar{I}}_{01}\) are not independent variables anymore: they are functions of the three independent variables \(\hat{e}, \hat{\beta}_a, \hat{\beta}_l\). This reduces the number of independent variables to six. Five constraints are provided by the five equations of motion \(\partial_\mu \hat{N}_\mu = 0\) and \(\partial_\mu \hat{T}_{\mu\nu} = 0\). In the ideal-fluid limit, \(\hat{\beta}_l \rightarrow 0\), and the system of equations of motion is closed. For arbitrary \(\hat{\beta}_l\), however, we need an additional equation of motion to close the system of equations. This will effectively describe the decay of the momentum anisotropy of the distribution function and the approach of the system to local thermal equilibrium. This auxiliary equation can be provided, for example, from the higher moments of the Boltzmann equation as is usually done in kinetic theory.

It is instructive to repeat this discussion from a slightly different perspective. For very large times, any closed system described by the Boltzmann equation will reach global thermodynamical equilibrium. If, in this process, the system first reaches local thermodynamical equilibrium, the evolution towards global equilibrium is governed by ideal fluid dynamics. In this case, it is advantageous to explicitly exhibit the equilibrium EoS \(P_0(e_0, n_0)\) in the fluid-dynamical equations of motion. Usually, this is done via the Landau matching conditions. These conditions
require that particle density $n$ and energy density $\epsilon$ in a general non-equilibrium state are equal to those of a fictitious (local) thermodynamical equilibrium state, $n = n_0(\alpha_0, \beta_0)$, $\epsilon = \epsilon_0(\alpha_0, \beta_0)$. These equations implicitly determine the intensive parameters $\alpha_0$, $\beta_0$ in the distribution function \[3\] pertaining to the fictitious (local) equilibrium state.

Analogously, for the anisotropic state the Landau matching conditions read $\hat{n}(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l) = n_0(\alpha_0, \beta_0)$ and $\hat{e}(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l) = e_0(\alpha_0, \beta_0)$, which is equivalent to

\[
\begin{align*}
\left(\hat{N}^\mu - N^\mu_0\right) u_\mu &= 0 , \\
\left(T^{\mu\nu} - T_0^{\mu\nu}\right) u_\mu u_\nu &= 0 ,
\end{align*}
\]

where $N_0^\mu$ and $T_0^{\mu\nu}$ were defined in Eq. \[2\]. These dynamical matching conditions will determine the two parameters of a fictitious equilibrium state, $\alpha_0 = \alpha_0(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l)$ and $\beta_0 = \beta_0(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l)$, as function of the three scalar parameters $\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l$ pertaining to the anisotropic state.

In principle, there are infinitely many possibilities to extend an equilibrium distribution function by an additional free parameter, $\hat{\beta}_l$, each one resulting in a different set of thermodynamical relations, i.e., the latter are not universal. For example, choosing $\hat{\beta}_l$ as a free intensive variable that is related to some new conjugate extensive quantity, as in Ref. \[23\], we recover the conventional laws of thermodynamics only in the equilibrium limit, $\hat{\beta}_l \to 0$. Therefore, without specifying the exact form of the anisotropic distribution function we cannot derive any thermodynamic relation, and in particular the EoS, from kinetic theory.

Using the Landau matching conditions \[73\], \[75\] the tensor decomposition of the conserved quantities reads

\[
\begin{align*}
\hat{N}^\mu &= n_0 w^\mu + \hat{n}_l l^\mu , \\
\hat{T}^{\mu\nu} &= e_0 w^\mu w^\nu + 2 \hat{P}_l l^\mu l^\nu - \hat{P}_\perp \Xi^{\mu\nu} ,
\end{align*}
\]

The equilibrium EoS is now introduced by writing the isotropic pressure \[63\] in the form

\[
\hat{P} \equiv \frac{1}{3} \left(\hat{P}_l + 2 \hat{P}_\perp\right) \equiv P_0(\alpha_0, \beta_0) + \hat{\Pi}(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l) ,
\]

which at the same time defines the bulk viscous pressure $\hat{\Pi}$ with respect to the pressure in local equilibrium and hence can be used to eliminate either $\hat{P}_l$ or $\hat{P}_\perp$ in Eq. \[77\].

Making the connection to the equilibrium EoS is advantageous when the system is close to the ideal-fluid limit. However, for the anisotropic state it does not solve the problem that the five conservation equations do not determine all independent variables. We are thus left with six independent variables, say $n_0$, $e_0$, and $\hat{P}_l$ (or $\hat{\Pi}$) and the three components of $w^\mu$, which means that we must supply one additional equation of motion.

As mentioned above, the tensor decompositions \[73\], \[77\] were obtained previously in Refs. \[24\]–\[31\] based on the distribution function given in Ref. \[32\]. This spheroidal single-particle distribution function leads to $\hat{n}_l = \hat{M} = 0$, hence exclusively to a pressure anisotropy. Furthermore, in these refs. the case of vanishing $\hat{\alpha}$ was considered, which for a massless ideal gas EoS also leads to a vanishing bulk viscous pressure. This also leaves the freedom to use the zeroth moment of the Boltzmann equation as additional input to close the equations of motion.

V. EXPANSION AROUND THE ANISOTROPIC DISTRIBUTION FUNCTION

In principle, $f_k$ is a solution of the Boltzmann equation. However, if we are only interested in the low-frequency, large-wavenumber limit of the latter, we may consider the (much simpler) fluid-dynamical equations of motion. In order to derive them from the Boltzmann equation, the method of moments is particularly well suited \[11\], \[12\], \[13\], where $f_k$ is expanded around the distribution function $f_{0k}$ of a fictitious (local) equilibrium state. The corrections are written in terms of the irreducible moments of $\delta f_k \equiv f_k - f_{0k}$. Then, the infinite set of moments of the Boltzmann equation provide an infinite set of equations of motion for these irreducible moments. Conventional dissipative fluid dynamics then emerges by truncating this set and expressing the irreducible moments in terms of the fluid-dynamical variables, for more details see Ref. \[18\].

Here we will follow the same strategy, except expanding $f_k$ around $\hat{f}_{0k}$ instead of $f_{0k}$, see also Refs. \[57\], \[58\]. Nevertheless, at each step it is instructive to also recall the conventional expansion of $f_k$ around $f_{0k}$. Hence,

\[
f_k = f_{0k} + \delta f_k \equiv \hat{f}_{0k} + \hat{\delta} f_k ,
\]
where it is implicitly assumed that the corrections $\delta f_k$, $\delta \hat{f}_k$ fulfill $|\delta f_k| \ll f_0k$ and $|\delta \hat{f}_k| \ll \hat{f}_0k$. The rationale behind the expansion around $f_0k$ instead of around $\hat{f}_0k$ is that, in the case of a pronounced anisotropy, $|\delta \hat{f}_k| \ll |\delta f_k|$, so that the convergence properties of the former series expansion are vastly improved over those of the latter. Without loss of generality we write these corrections as follows,

\begin{align}
\delta f_k &= f_0k \left( 1 - a_{f0k} \right) \phi_k, \\
\delta \hat{f}_k &= f_0k \left( 1 - a_{f0k} \right) \hat{\phi}_k,
\end{align}

where $\phi_k$ and $\hat{\phi}_k$ are measures of the deviation of $f_k$ from $f_0k$ or $\hat{f}_0k$, respectively.

We recall [18, 19] that $\phi_k$ can be expanded in terms of a complete and orthogonal set of irreducible tensors $1$, $k(\mu)$, $k(\mu k \nu)$, $k(\mu k \nu k \lambda)$, …, where

$$k^{(\mu_1 \cdots k \mu_\ell)} = \Delta^{\mu_1 \cdots \mu_\ell} k^{\mu_1 \cdots k \mu_\ell}.$$  

The symmetric and traceless projection tensor $\Delta^{\mu_1 \cdots \mu_\ell}$ is defined in Eq. (81). By definition, this tensor and thus $k^{(\mu_1 \cdots k \mu_\ell)}$ are orthogonal to $u^\mu$. For an arbitrary function $F(E_{ku})$ the irreducible tensors satisfy the following orthogonality condition

$$\int dK \ F(E_{ku}) \ k^{(\mu_1 \cdots k \mu_\ell)} k^{(\nu_1 \cdots k \nu_n)} = \frac{\ell^!}{(2\ell + 1)!} \Delta^{\mu_1 \cdots \mu_\ell \nu_1 \cdots \nu_n} \int dK \ F(E_{ku}) \ \left( \Delta^{\alpha \beta} k_\alpha k_\beta \right)^\ell,$$

for the proof see App. [2]. The expansion of $\phi_k$ in terms of a complete and orthogonal set of irreducible tensors [52] now reads

$$\phi_k = \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} c^{(\mu_1 \cdots \mu_\ell)}_n k^{(\mu_1 \cdots k \mu_\ell)} \mathcal{P}^{(\ell)}_{kn},$$

where $c^{(\mu_1 \cdots \mu_\ell)}_n$ are coefficients given in Eq. (26) of Ref. [18], and the polynomials in $E_{ku}$ are defined as

$$\mathcal{P}^{(\ell)}_{kn} = \sum_{i=0}^{n} a^{(\ell)}_{ni} E_{ku}^i.$$

Finally, we can write the distribution function as

$$f_k = f_0k + f_0k \left( 1 - a_{f0k} \right) \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho^{(\mu_1 \cdots k \mu_\ell)}_n \mathcal{H}^{(\ell)}_{kn},$$

see Eq. (30) of Ref. [18]. Here,

$$\mathcal{H}^{(\ell)}_{kn} = \frac{W^{(\ell)}}{\ell!} \sum_{i=0}^{n} a^{(\ell)}_{ni} \mathcal{P}^{(\ell)}_{ki},$$

with $W^{(\ell)} = (-1)^\ell / J_{2\ell,\ell}$, while the irreducible moments of $\delta f_k$ are defined as

$$\rho^{(\mu_1 \cdots k \mu_\ell)}_i \equiv \left( E_{ku} k^{(\mu_1 \cdots k \mu_\ell)} \right)_i,$$

where

$$\langle \cdots \rangle_0 \equiv \langle \cdots \rangle_0 = \int dK \ \langle \cdots \rangle \ \delta f_k.$$  

One can easily show substituting Eq. (80) into Eq. (88) and using the orthogonality condition (83) and the definition of the auxiliary thermodynamic integrals (19) that all irreducible moments are linearly related to each other [for more details see, Eq. (72) of Ref. [20]],

$$\rho^{(\mu_1 \cdots k \mu_\ell)}_i \equiv (-1)^\ell \ell! \sum_{n=0}^{N_\ell} \rho^{(\mu_1 \cdots k \mu_\ell)}_{n} \mathcal{P}^{(\ell)}_{ni}.$$
where

\[ \gamma_{ln}^{(\ell)} = \frac{W^{(\ell)}}{\ell!} \sum_{n=\ell}^{N_\ell} \sum_{i=0}^{n'} a_{ln}^{(\ell)} F_{n'i}^{(\ell)} I_{i+\ell}^{2\ell,\ell}. \]  

(91)

Note that these relations are also valid for moments with negative powers of \( E_{ku} \) in terms of the ones with positive \( \ell \), for details see, Eq. (65) of Ref. [18].

Similarly, the correction \( \delta \hat{f}_k \) to the anisotropic state is expanded in terms of another complete, orthogonal set of reducible tensors, 1, \( k^{(\mu)} \), \( k^{(\mu \nu \lambda)} \), \( k^{(\mu \nu \rho)} \), \( \cdots \), where

\[ k^{(\mu_1 \cdots \mu_p)} = \Xi^{\mu_1 \cdots \mu_p} k^{\mu_1} \cdots k^{\mu_p}. \]  

(92)

Here, the symmetric and traceless projection tensor \( \Xi^{\mu_1 \cdots \mu_p} \) is defined in Eq. (E7). By definition, this tensor and thus \( k^{(\mu_1 \cdots \mu_p)} \) are orthogonal to both \( u^\mu \) and \( l^\mu \). For an arbitrary function of both \( E_{ku} \) and \( E_{kl} \), say \( \hat{F}(E_{ku}, E_{kl}) \), the irreducible tensors satisfy the following orthogonality condition,

\[ \int dK \hat{F}(E_{ku}, E_{kl}) k^{(\mu_1 \cdots \mu_p)} k^{(\nu_1 \cdots \nu_q)} = \frac{\delta_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_q}}{2!} \int dK \hat{F}(E_{ku}, E_{kl})(\Xi^{\alpha \beta} k_\alpha k_\beta)^p, \]  

(93)

for the proof see App. [8]. The expansion of \( \delta \hat{f}_k \) now reads

\[ \delta \hat{f}_k = \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \sum_{m=0}^{N_\ell-n} c_{n}^{(\mu_1 \cdots \mu_p)} k^{(\mu_1 \cdots \mu_p)} P_{knm}^{(\ell)}. \]  

(94)

Here, similarly to Eq. (S3), \( N_\ell \) truncates the (in principle) infinite sums over \( n \) and \( m \) at some natural number. Furthermore, \( c_{n}^{(\mu_1 \cdots \mu_p)} \) are some coefficients that will be determined later, while the \( P_{knm}^{(\ell)} \) form an orthogonal set of polynomials in both \( E_{ku} \) and \( E_{kl} \),

\[ P_{knm}^{(\ell)} = \sum_{\ell=0}^{n} \sum_{m=0}^{m} a_{knm}^{(\ell)} E_{ku}^\ell E_{kl}^m, \]  

(95)

where \( a_{knm}^{(\ell)} \) are coefficients that are independent of \( E_{ku} \) and \( E_{kl} \). These multivariate polynomials in \( E_{ku} \) and \( E_{kl} \) are constructed to satisfy the following orthonormality relation,

\[ \int dK \hat{\omega}^{(\ell)} P_{knm}^{(\ell)} P_{knm'}^{(\ell')} = \delta_{nn'} \delta_{mm'}, \]  

(96)

where the weight is defined as

\[ \hat{\omega}^{(\ell)} = \frac{\hat{W}^{(\ell)}}{(2\ell)!} (\Xi^{\alpha \beta} k_\alpha k_\beta)^\ell \int_0^1 \left( 1 - a \hat{f}_k \right). \]  

(97)

The normalization constant \( \hat{W}^{(\ell)} \) and the coefficients \( a_{knm}^{(\ell)} \) can be found via the Gram-Schmidt orthogonalization procedure, see App. [G]. Note that for \( m = 0 \), the multivariate polynomials \( P_{knm}^{(\ell)} \) defined in Eq. (95) naturally reduce to the polynomials \( P_{kn}^{(\ell)} \), for more details see App. [G].

Finally, in complete analogy to the expansion (S3), the distribution function can be written as

\[ f_k = \hat{f}_k + \hat{f}_k \left( 1 - a \hat{f}_k \right) \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \sum_{m=0}^{N_\ell-n} \hat{p}_{knm}^{(\ell)} k^{(\mu_1 \cdots \mu_p)} P_{knm}^{(\ell)}. \]  

(98)

Here we introduced the irreducible moments of \( \delta \hat{f}_k \),

\[ \hat{p}_{ij}^{\mu_1 \cdots \mu_p} \equiv \langle E_{ku} E_{kl} k^{(\mu_1 \cdots \mu_p)} \rangle_\delta, \]  

(99)

where

\[ \langle \cdots \rangle_\delta \equiv \langle \cdots \rangle_0 - \int dK \langle \cdots \rangle_0 = \int dK \langle \cdots \rangle \delta \hat{f}_k. \]  

(100)
Note the differences to the irreducible moments \[ \delta_f \]: apart from the weight factor $\delta f_k$ instead of $\delta f_k$ they carry two indices $i$ and $j$, to indicate the different powers of $E_{ku}$ and $E_{kl}$ and finally, the irreducible tensors $k^{(\mu_1 \ldots k_{\mu \ell})}$ appear instead of $k^{(\mu_1 \ldots k_{\mu \ell})}$. Furthermore, the coefficients $\hat{H}^{(l)}_{kmn}$ are defined as

$$
\hat{H}^{(l)}_{kmn} = \frac{\hat{W}^{(l)}(t)}{\ell!} \sum_{i=n}^{N_{\ell}} \sum_{j=m}^{N_{\ell}} a_{injm}^{(l)} p_{kj}^{(l)},
$$

(101)

For the proof of Eq. (101) we first determine the coefficients $c^{(\mu_1 \ldots k_{\mu \ell})}_{nm}$ in Eq. (102). With the help of Eqs. (103), (104), and (105) one can prove that

$$
c^{(\mu_1 \ldots k_{\mu \ell})}_{nm} = \frac{\hat{W}^{(l)}(t)}{\ell!} \left\langle p_{knm}^{(l)} k^{(\mu_1 \ldots k_{\mu \ell})} \right\rangle \delta_{\ell}. 
$$

(102)

Inserting this into Eq. (104) and employing Eq. (105), we obtain

$$
\hat{\delta}_k = \sum_{\ell=0}^{\infty} \frac{\hat{W}^{(l)}(t)}{\ell!} \sum_{n=0}^{N_{\ell}} \sum_{m=0}^{N_{\ell}} \sum_{i=0}^{n} \sum_{j=0}^{m} a_{injm}^{(l)} \hat{\rho}_{\mu_1 \ldots k_{\mu \ell}}^{(\mu_1 \ldots k_{\mu \ell})}, \nonumber 
$$

(103)

where we have used the definition of $\rho_{\mu_1 \ldots k_{\mu \ell}}^{(\mu_1 \ldots k_{\mu \ell})}$. Now, by renaming $n \leftrightarrow i$ and $m \leftrightarrow j$ and cleverly reordering the sums, we obtain Eq. (101) with $\hat{H}^{(l)}_{kmn}$ as defined in Eq. (101).

Note that, similarly to Eq. (101), once we truncate the expansion at some finite $N_{\ell}$, any irreducible moment of $\delta f_k$ with corresponding tensor rank is linearly related to the moments that appear in the truncated expansion,

$$
\hat{\rho}_{\mu_1 \ldots k_{\mu \ell}}^{(\mu_1 \ldots k_{\mu \ell})} \equiv (-1)^{\ell} \frac{N_{\ell}}{\ell!} \sum_{n=0}^{N_{\ell}} \sum_{m=0}^{N_{\ell}} \sum_{i=0}^{n} \sum_{j=0}^{m} \hat{\rho}_{\mu_1 \ldots k_{\mu \ell}}^{(\mu_1 \ldots k_{\mu \ell})},
$$

(104)

where

$$
\gamma_{injm}^{(l)} = \frac{\hat{W}^{(l)}(t)}{\ell!} \sum_{n'=n}^{N_{\ell}} \sum_{m'=m}^{N_{\ell}} \sum_{i'=0}^{n'} \sum_{j'=0}^{m'} a_{i'n'm'i'j'}^{(l)} a_{n'm'i'j'}^{(l)},
$$

(105)

We also remark that the irreducible moments with negative powers $i$ and $j$ of $E_{ku}$ and $E_{kl}$, respectively, can be expressed in terms of the ones with positive indices.

**VI. EQUATIONS OF MOTION FOR THE IRREDUCIBLE MOMENTS**

The space-time evolution of the single-particle distribution function $f_k$ of a single-component, weakly interacting, dilute gas is given by the relativistic Boltzmann equation (12). Considering only binary collisions, the collision term on the right-hand side is

$$
C[f] = \frac{1}{2} \int dK'dP'dP' \left[ W_{pp' \rightarrow kk'} f_{p'} f_{p'} (1 - a_{f_k}) (1 - a_{f_{k'}}) - W_{kk' \rightarrow pp'} f_{k} f_{k'} (1 - a_{f_p}) (1 - a_{f_{p'}}) \right].
$$

(106)

Here the factors $1 - a_{f_k}$ represent the corrections from quantum statistics. The factor $1/2$ appears if the colliding particles are indistinguishable. The invariant transition rate $W_{kk' \rightarrow pp'}$ satisfies detailed balance, $W_{kk' \rightarrow pp'} = W_{pp' \rightarrow kk'}^*$, and is symmetric with respect to exchange of momenta, $W_{kk' \rightarrow pp'} = W_{kk' \rightarrow pp'}^{*}$.

In order to derive equations of motion for the irreducible moments of $\delta f_k$, we proceed along the lines of Ref. 18, except that we express the derivatives using Eq. (122) instead of Eq. (119) and thus we rewrite the Boltzmann equation (12) as an evolution equation for the correction $\delta f_k$ instead of $f_k$,

$$
D \delta f_k = -D \delta f_0 + E_{ku}^{-1} \left( E_{kl} D_l \delta f_k + E_{kl} D_l \delta f_0 - k^\mu \hat{\nabla}_\mu \delta f_0 - k^\mu \hat{\nabla}_\mu \delta f_k \right) + E_{ku}^{-1} C \left[ \delta f_0 + \delta f_k \right].
$$

(107)

Here $D = u^\mu \partial_\mu$, $D_l = -u^\mu \partial_\mu$, and $\hat{\nabla}_\mu = \Xi_{\mu \nu} \partial^\nu$. 

Now we form moments of Eq. (107), which leads to an infinite set of equations of motion for the irreducible moments \( \rho_{ij}^{(\mu_1 \cdots \mu_r)} \). Defining

\[
D \rho_{ij}^{(\mu_1 \cdots \mu_r)} \equiv \Xi_{\nu_1 \cdots \nu_r}^{\mu_1 \cdots \mu_r} \Delta \rho_{ij}^{\nu_1 \cdots \nu_r},
\]

as well as

\[
C_{ij}^{(\mu_1 \cdots \mu_r)} \equiv \Xi_{\nu_1 \cdots \nu_r}^{\mu_1 \cdots \mu_r} \int dK \ E_{\nu_1}^i E_{\nu_2}^j k^{\nu_1} \cdots k^{\nu_r} C[f],
\]

we obtain, after a long but straightforward calculation, the equation of motion for the irreducible moments of tensor-rank zero

\[
D \rho_{ij} = C_{i-1,j} - D \tilde{\xi}_{i,j} + D \tilde{\xi}_{i-1,j} + (i \tilde{\xi}_{i-1,j} + j \tilde{\xi}_{i,j-1}) l_{\alpha} D u^\alpha - \left[ \left( i - 1 \right) \tilde{\xi}_{i-2,j} + (j + 1) \tilde{\xi}_{i,j} \right] l_{\alpha} D u^\alpha + \frac{1}{2} \left[ m_0^2 (i - 1) \tilde{\xi}_{i-2,j} - (i + 1) \tilde{\xi}_{i,j} + (i - 1) \tilde{\xi}_{i-2,j} + (j + 1) \tilde{\xi}_{i,j} \right] \tilde{\theta},
\]

where \( \tilde{\theta} = \tilde{\nabla}_\mu u^\mu, \tilde{\theta}_l = \tilde{\nabla}_\mu \theta^\mu, \tilde{\theta}^{\mu\nu} = \partial_i (\mu \nu), \) and \( \tilde{\theta}_l^{\mu\nu} = \partial_i (\mu \nu) \).

Similarly, the time-evolution equation for the tensor-rank one is

\[
D \tilde{\rho}_{ij}^{(\mu)} = C_{i-1,j} - \frac{1}{2} \tilde{\nabla}_\mu \left( m_0^2 \tilde{\xi}_{i-1,j} - \tilde{\xi}_{i-1,j} + \tilde{\xi}_{i-1,j} \right) + \frac{1}{2} \left[ m_0^2 i \tilde{\xi}_{i-1,j} - (i + 2) \tilde{\xi}_{i,j} + (i + 2) \tilde{\xi}_{i,j} \right] l_{\alpha} D u^\alpha - \frac{1}{2} \left[ m_0^2 j \tilde{\xi}_{i,j} - j \tilde{\xi}_{i+2,j} + (j + 2) \tilde{\xi}_{i,j} \right] l_{\alpha} D u^\alpha + \frac{1}{2} \left[ m_0^2 (i - 1) \tilde{\xi}_{i-2,j} - (i + 1) \tilde{\xi}_{i,j} + (i - 1) \tilde{\xi}_{i-2,j} + (j + 1) \tilde{\xi}_{i,j} \right] \tilde{\theta},
\]

where \( \tilde{\theta} = \tilde{\nabla}_\mu u^\mu, \tilde{\theta}_l = \tilde{\nabla}_\mu \theta^\mu, \tilde{\theta}^{\mu\nu} = \partial_i (\mu \nu), \) and \( \tilde{\theta}_l^{\mu\nu} = \partial_i (\mu \nu) \).
where \( \tilde{\omega}^{\mu\nu} = \Xi^{\mu\nu} \Xi^{\rho\lambda} \partial_{[\alpha u \beta]} \) and \( \tilde{\omega}^{\mu}_{\nu} = \Xi^{\mu\alpha} \Xi^{\rho\alpha} \partial_{[\alpha l \beta]} \).

Finally, the equation of motion for the irreducible moments of tensor-rank two is

\[
D \hat{\rho}_{[\mu \nu]} = -\xi^{[\mu \nu]}
\]

\[
+ \frac{1}{4} \left\{ m_0^2 [j + (i + 1) \hat{\rho}_{i, j + 1} - (i + 1) \hat{\rho}_{i, j + 2}] - 2 (i + 1) \hat{\rho}_{i, j + 2} + (i + 3) \hat{\rho}_{i, j + 3} + (i + 1) \hat{\rho}_{i, j + 4} \right\} \hat{\sigma}^{\mu\nu}
\]

\[
+ \frac{1}{4} \left\{ m_0^2 \hat{D}_{i, j - 1} - 2 m_0^2 \left[j \hat{D}_{i, j - 1} + (i + 2) \hat{D}_{i, j + 1} + j \hat{D}_{i, j + 1} - (i + 2) \hat{D}_{i, j + 2} + (i + 3) \hat{D}_{i, j + 3} + (i + 1) \hat{D}_{i, j + 4} \right] \hat{\sigma}^{\mu\nu}
\]

\[
+ \frac{1}{4} \left\{ m_0^2 [j \hat{\rho}_{i, j - 1} - (i + 1) \hat{\rho}_{i, j + 1} - (j + 2) \hat{\rho}_{i, j + 1} + (j + 4) \hat{\rho}_{i, j + 3} + (j + 1) \hat{\rho}_{i, j + 3} \right\} \hat{\sigma}^{\mu\nu}
\]

\[
+ \frac{1}{2} \left[m_0^2 \left[j \hat{\rho}_{i, j - 1} - (i + 4) \hat{\rho}_{i, j + 1} i \hat{\rho}_{i, j + 2} + (i + 3) \hat{\rho}_{i, j + 3} + (i + 1) \hat{\rho}_{i, j + 4} \right] \hat{D}^{\mu\nu}
\]

\[
+ \frac{1}{2} \left[m_0^2 \left[i \hat{\rho}_{i, j - 1} - (i + 1) \hat{\rho}_{i, j + 1} - (i + 3) \hat{\rho}_{i, j + 3} + (i + 1) \hat{\rho}_{i, j + 3} \right] \hat{D}^{\mu\nu}
\]

\[
+ \frac{1}{2} \left[m_0^2 \left[(i + 1) \hat{\rho}_{i, j - 1} - (i + 3) \hat{\rho}_{i, j + 1} + (j + 5) \hat{\rho}_{i, j + 1} \right] \hat{D}^{\mu\nu}
\]

\[
- \frac{1}{2} \left[m_0^2 \left[(i + 1) \hat{\rho}_{i, j - 1} - (i + 3) \hat{\rho}_{i, j + 1} + (j + 5) \hat{\rho}_{i, j + 1} \right] \hat{D}^{\mu\nu}
\]

\[
- \frac{1}{2} \left[m_0^2 \left[(i + 1) \hat{\rho}_{i, j - 1} - (i + 3) \hat{\rho}_{i, j + 1} + (j + 5) \hat{\rho}_{i, j + 1} \right] \hat{D}^{\mu\nu}
\]

Since the fluid-dynamical equations of motion do not contain quantities of tensor rank higher than two, we do not explicitly quote the equations of motion for the irreducible moments \( \hat{\rho}^{[\mu_1 \ldots \mu_\ell]} \) with \( \ell \geq 3 \). The equations of motion of relativistic dissipative fluid dynamics for an anisotropic reference state can now be obtained from these general equations for different values of \( i \) and \( j \). This will be further elaborated in Sec. [VII].

VII. COLLISION INTEGRALS

In order to derive the fluid-dynamical equations of motion of the fluid, we need to consider the collision terms \( \Xi^{\mu\nu} \), which appear in Eqs. (111) - (122). Exchanging integration variables \( (p, p') \leftrightarrow (k, k') \), we can rewrite Eq. (109) as

\[
C_{ij}^{[\mu_1 \ldots \mu_\ell]} = \frac{1}{2} \int dK dP dP' f_k f_{k'} (1 - a f_p) (1 - a f_{p'}) W_{kk' \rightarrow pp'} \left( E_{p_k} E_{p_{k'}}^{\mu_1} \ldots p_{\mu_\ell} \right) = \int dK dP dP' f_k f_{k'} (1 - a f_p) (1 - a f_{p'}) W_{kk' \rightarrow pp'} \left( E_{p_k} E_{p_{k'}}^{\mu_1} \ldots p_{\mu_\ell} \right)
\]

(113)

As a consequence of the conservation of particle number as well as energy and momentum in binary collisions, we have

\[
C_{00} = C_{10} = C_{01} = C_{11}^{(\mu)} = 0
\]

(114)

for any distribution function \( f_k \).

Now inserting the distribution function from Eq. (125) into Eq. (113) and neglecting terms proportional to \( \delta f_k \delta f_{k'} \), we obtain

\[
C_{ij}^{[\mu_1 \ldots \mu_\ell]} = \hat{C}_{ij}^{[\mu_1 \ldots \mu_\ell]} + \tilde{C}_{ij}^{[\mu_1 \ldots \mu_\ell]}.
\]

(115)
Here,
\[
\hat{c}_{ij}^{(\mu_1 \cdots \mu_L)} = \frac{1}{2} \int dKdK'dPdP'W_{kk'\rightarrow pp'}\hat{f}_{0k}\hat{f}_{0k'}(1-a\hat{f}_{0p})(1-a\hat{f}_{0p'}) \left( E_{pu} E_{p'u'}^{(\mu_1 \cdots \mu_L)} - E_{ku} E_{k'u'}^{(\mu_1 \cdots \mu_L)} \right).
\]

In local thermodynamical equilibrium, i.e., replacing \( \hat{f}_{0k} \) by \( f_{0k} \), such a collision integral vanishes due to the symmetry of the collision rate, \( W_{kk'\rightarrow pp'} = W_{pp'\rightarrow kk'} \), and energy conservation in binary elastic collisions, \( E_{pu} + E_{p'u'} = E_{ku} + E_{k'u'} \). However, for the anisotropic state characterized by \( \hat{f}_{0k} \) this is a priori not the case. Thus, if we consider the fluid dynamics of such a system, without additional corrections from the irreducible moments of \( \delta \hat{f}_{0k} \), as was done in Refs. \[24\]–\[31\], the microscopic collision dynamics contained in the term \( \hat{c}_{ij}^{(116)} \) is solely responsible for the approach towards local thermodynamic equilibrium.

Let us now turn to the second term in Eq. \[115\] and, in order to simplify the discussion, consider the case of Boltzmann statistics, \( a = 0 \). Then,
\[
\hat{L}_{ij}^{(\mu_1 \cdots \mu_L)} = \frac{1}{2} \sum_{r=0}^{\infty} \sum_{n=0}^{N_r} \sum_{m=0}^{N_r-n} \hat{\rho}_{rm}^{(\mu_1 \cdots \mu_L)} \int dKdK'dPdP'W_{kk'\rightarrow pp'}\hat{f}_{0k}\hat{f}_{0k'} \times \left( k_{(\mu_1} \cdots k_{\nu_1)}^r \hat{H}_{km}^{(r)} + k_{(\nu_1}^r k_{\nu_2)}^r \hat{H}_{km}^{(r)} \right) \left( E_{pu} E_{p'u'}^{(\mu_1 \cdots \mu_L)} - E_{ku} E_{k'u'}^{(\mu_1 \cdots \mu_L)} \right).
\]

In analogy to Eqs. \[50\] and \[51\] of Ref. \[13\], this expression can be rewritten as, see App. \[4\] for details,
\[
\hat{L}_{ij}^{(\mu_1 \cdots \mu_L)} = \sum_{n=0}^{N_r} \sum_{m=0}^{N_r-n} \hat{\rho}_{nm}^{(\mu_1 \cdots \mu_L)} \hat{A}_{injm}^{(f)}, \tag{118}
\]

where
\[
\hat{A}_{injm}^{(f)} = \frac{1}{2} \int dKdK'dPdP'W_{kk'\rightarrow pp'}\hat{f}_{0k}\hat{f}_{0k'} \times \left( k_{(\mu_1}^r k_{\nu_1)}^{r'} \hat{H}_{knm}^{(r')} + k_{(\nu_1}^{r'} k_{\nu_2)}^{r'} \hat{H}_{knm}^{(r')} \right) \left( E_{pu} E_{p'u'}^{(\mu_1 \cdots \mu_L)} - E_{ku} E_{k'u'}^{(\mu_1 \cdots \mu_L)} \right). \tag{119}
\]

Similar to the tensor \( \hat{c}_{ij}^{(\mu_1 \cdots \mu_L)} \) in Eq. \[116\], the coefficients \( \hat{A}_{injm}^{(f)} \) contain information about the microscopic interactions. The difference is that the tensor \( \hat{L}_{ij}^{(\mu_1 \cdots \mu_L)} \) is linearly proportional to the irreducible moments \( \hat{\rho}_{nm}^{(\mu_1 \cdots \mu_L)} \). This means on the one hand that \( \hat{L}_{ij}^{(\mu_1 \cdots \mu_L)} = 0 \) if we consider a state characterized by \( \hat{f}_{0k} \) without corrections \( \sim \delta \hat{f}_{0k} \), such as in Refs. \[24\]–\[31\]. On the other hand, the linear proportionality of \( \hat{L}_{ij}^{(\mu_1 \cdots \mu_L)} \) to \( \hat{\rho}_{nm}^{(\mu_1 \cdots \mu_L)} \) means that the coefficients \( \hat{A}_{injm}^{(f)} \) are inversely proportional to the relaxation time scales for the irreducible moments.

In the remainder of this section, we discuss a widely used simplification for the collision integral, the so-called relaxation-time approximation (RTA) \[14\]. In this case, the full collision integral is replaced by the following relativistically invariant expression \[17\],
\[
C[f] \equiv \frac{E_{ku}}{\tau_{rel}} (f_k - f_{0k}) = -\frac{E_{ku}}{\tau_{rel}} \left( \hat{f}_{0k} + \delta \hat{f}_{0k} - f_k \right), \tag{120}
\]
where we used Eq. \[22\] and \( \tau_{rel} \) is the so-called relaxation time, which is assumed to be independent of the particle four-momenta and determines the time scale over which \( f_k \) relaxes towards \( f_{0k} \). In our case this approximation translates with the help of Eq. \[118\] and the definitions \[48\], \[91\] into
\[
C_{ij}^{(\mu_1 \cdots \mu_L)} = -\frac{1}{\tau_{rel}} \left( \hat{T}_{ij}^{(\mu_1 \cdots \mu_L)} + \hat{\rho}_{ij}^{(\mu_1 \cdots \mu_L)} - \hat{T}_{ij}^{(\mu_1 \cdots \mu_L)} \right), \tag{121}
\]
where we defined the generalized moments of the local equilibrium distribution function \( f_{0k} \) of tensor-rank \( \ell \),
\[
\lim_{\beta \rightarrow 0} \hat{T}_{ij}^{\mu_1 \cdots \mu_L} \equiv \hat{T}_{ij}^{\mu_1 \cdots \mu_L} = \left( E_{ku} E_{k'u'}^{(\mu_1 \cdots \mu_L)} \right), \tag{122}
\]
for more details see App. \[3\]. Note that for symmetry reasons \( \hat{T}_{ij}^{\mu_1 \cdots \mu_L} = 0 \) for all odd \( j \). For the scalar collision integrals, this simplifies to
\[
C_{i-1,j} = -\frac{1}{\tau_{rel}} \left( \hat{T}_{ij} + \hat{\rho}_{ij} - \hat{T}_{ij} \right). \tag{123}
\]
However, for any \( \ell \geq 1 \), \( \hat{\mathcal{I}}_{ij}^{(\mu_1 \cdots \mu_\ell)} = I_{ij}^{(\mu_1 \cdots \mu_\ell)} = 0 \), and we have
\[
\hat{c}_{i=1,j}^{(\mu_1 \cdots \mu_\ell)} = -\frac{1}{\tau_{rel}} \hat{\rho}_{ij}^{(\mu_1 \cdots \mu_\ell)} , \quad \ell \geq 1 .
\] (124)
This means that, in RTA, for any \( \ell \geq 1 \) we have \( \hat{c}_{ij}^{(\mu_1 \cdots \mu_\ell)} \equiv 0 \) and \( A_{\ell njm}^{(\ell)} = -\delta_{n0} \delta_{jm}/\tau_{rel} \).

\[\text{VIII. FLUID-DYNAMICAL EQUATIONS OF MOTION}\]

In this section, we derive the fluid-dynamical equations of motion from the equations of motion for the irreducible moments, Eqs. (111) – (112). We split the discussion into two subsections. In the first, we derive the conservation equations for particle number and energy-momentum. These equations are special, since for them the collision integrals vanish identically. In the second subsection, we discuss the remaining equations which are necessary to close the system of equations of motion. As expected, these equations correspond to relaxation equations for the dissipative quantities.

A. Conservation equations

For the conservation equations, the collision integrals vanish on account of particle-number and energy-momentum conservation in binary collisions, cf. Eq. (114). We shall show below that, from this equation, we can identify

(i) the conservation equation for particle number as the equation of motion (110) for \((i, j) = (1, 0)\), because \( C_{00} = 0 \),

(ii) the conservation of energy as the equation of motion (110) for \((i, j) = (2, 0)\), because \( C_{10} = 0 \),

(iii) the conservation equation for the momentum in \(l^\mu\) direction as the equation of motion (110) for \((i, j) = (1, 1)\), because \( C_{01} = 0 \), and

(iv) the conservation equation for the momentum in the direction transverse to \(l^\mu\) as the equation of motion (111) for \((i, j) = (1, 0)\), because \( \sigma_{\ell 00}^{(\ell)} = 0 \).

In order to prove this, we first have to write the respective equations for the irreducible moments of \( \delta f_k \) in terms of the usual fluid-dynamical variables. Using Eq. (99), the definition of \( \delta f_k \), and Eqs. (19), (20), (27) – (29) and (31)– (33), as well as Eqs. (68) – (72) we obtain
\[
\begin{align*}
\hat{\rho}_{10} &= n - \hat{n} ,
\hat{\rho}_{20} &= e - \hat{e} ,
\hat{\rho}_{01} &= n - \hat{n} ,
\hat{\rho}_{11} &= M - \hat{M} ,
\hat{\rho}_{02} &= \hat{P}_t - \hat{P}_e ,
\hat{\rho}_{\mu0}^\mu &= V_{\perp}^\mu ,
\hat{\rho}_{1\mu}^\mu &= W_{\perp u}^\mu ,
\hat{\rho}_{\mu\nu}^\mu &= \pi_{\perp \mu\nu} .
\end{align*}
\] (125) (126) (127) (128) (129) (130) (131) (132) (133)
Inserting these identities into Eq. (110) for \((i, j) = (1, 0)\) leads to the conservation equation for particle number,
\[
\partial_t n + Dn = D_t n_t + n \hat{\theta} + n_t \hat{\theta}_t + n_i l_{\mu} D_{\mu} w_i - n_l l_\mu D_{\mu} w_l - V_{\perp}^\mu D_{\mu} + \hat{\nabla}_u V_{\perp}^\mu = 0 .
\] (134)
Analogously, inserting them into Eq. (110) for \((i, j) = (2, 0)\) we obtain the energy conservation equation,
\[
\begin{align*}
\begin{align*}
\hat{u}_\nu \partial_{\mu} T_{\nu\mu} &\equiv D e - D_t M + (e + P_{\perp}) \hat{\theta} + M \hat{\theta}_t + (e + P_t) l_{\mu} D_{\mu} w - 2M l_{\mu} D_{\mu} w
\end{align*}
\end{align*}
\] (135)
Repeating this for Eq. (111) in the case \((i, j) = (1, 1)\) yields the conservation of momentum in \(l^\mu\) direction,

\[
l_\mu \partial_\mu T^{\mu \nu} \equiv -DM + D_\mu P_1 - M \tilde{\theta} + (P_\perp - P_\parallel) \tilde{\theta}_1 - 2M l_\mu D_\mu u^\nu + (e + P_1) l_\mu Du^\mu
\]

\[
+ W_{\perp \perp}^\mu u_\mu + 2W_{\parallel \perp}^\mu D_\mu l_\mu - W_{\perp \parallel}^\mu l_\nu \tilde{\nabla}_\mu u^\nu - \tilde{\nabla}_\mu W_{\perp \parallel}^\mu - \pi_{\perp \parallel}^{\mu \nu} \tilde{\nabla}_{(\mu,l_{\nu})} = 0 .
\] (136)

Finally, using Eq. (111) for \((i, j) = (1, 0)\), we obtain the conservation of momentum in the direction transverse to \(l^\mu\),

\[
\Xi_\nu \partial_\mu T^{\mu \nu} \equiv -\tilde{\nabla}^\alpha P_\alpha + DW^\alpha_\perp \equiv \Xi_\nu \partial_\mu T^{\mu \nu} = -\tilde{\nabla}^\alpha P_\alpha + DW^\alpha_\perp \equiv \Xi_\nu \partial_\mu T^{\mu \nu}
\]

\[
\equiv -\tilde{\nabla}^\alpha P_\alpha + DW^\alpha_\perp + W_{\perp \perp}^\mu \partial_\mu - \tilde{\nabla}_\mu W_{\perp \mu}^\nu - \pi_{\perp \parallel}^{\mu \nu} \tilde{\nabla}_{(\mu,l_{\nu})} = 0 .
\] (137)

Of course, the conservation equations (134) – (137) also follow from taking the four-derivative of the tensor decompositions (26) and (20) and projecting onto the directions given by \(u^\mu, l^\mu\), and \(\Xi^{\mu \nu}\). Note that irreducible moments of higher tensor rank, like \(\hat{\rho}_{ij}^{\hat{\mu} \hat{\nu} \lambda}\) do not appear in Eq. (137), since the coefficients of these terms in Eq. (111) vanish for \(i = 1\) and \(j = 0\).

Now we need to find the connection between the actual state of the fluid to a fictitious anisotropic state where the single-particle distribution function is given by \(\bar{f}_{0,k}\), i.e., we need to determine the three intensive quantities \(\hat{a}, \hat{\beta}_u\), and \(\hat{\beta}_l\) of this state in terms of the fluid-dynamical variables of the actual state. As usual, we impose the Landau matching conditions for particle-number and energy density \(n = \bar{n}(\hat{a}, \hat{\beta}_u, \hat{\beta}_l)\) and \(e = \bar{e}(\hat{a}, \hat{\beta}_u, \hat{\beta}_l)\), which are equivalent to

\[
\left( N^\mu - \hat{N}^\mu \right) u_\mu \equiv \hat{\rho}_{10} = 0 ,
\] (138)

\[
\left( T^{\mu \nu} - \hat{T}^{\mu \nu} \right) u_\mu u_\nu \equiv \hat{\rho}_{20} = 0 .
\] (139)

Since these are only two constraints, but we have three intensive quantities, we require an additional matching condition. In analogy to the Landau matching conditions (138), (139), we may assume that

\[
\left( T^{\mu \nu} - \hat{T}^{\mu \nu} \right) l_\mu l_\nu \equiv \hat{\rho}_{02} = 0 ,
\] (140)

meaning that the pressure in the direction of the anisotropy is equal to its value in the fictitious anisotropic reference state, i.e., \(P_1 = \hat{P}((\hat{a}, \hat{\beta}_u, \hat{\beta}_l))\), and hence any correction from \(\hat{\delta}_{l_{\nu}}\) to this pressure component vanishes. However, the choice for the third matching condition is not unique [neither are the usual Landau matching conditions (138), (139), see e.g. Refs. [48, 49], because we could have demanded for example also the equivalence of the transverse pressures, \(P_\perp = \hat{P}_\perp(\hat{a}, \hat{\beta}_u, \hat{\beta}_l)\), together with Eqs. (138), (139) to infer the parameters \(\hat{a}, \hat{\beta}_u\), and \(\hat{\beta}_l\). Another alternative could be to enforce \(\hat{n}_l = n_l\), or \(\hat{M} = M\), or even \(P_\perp - P_1 = \hat{P}_\perp - \hat{P}_1\) as advocated in Ref. [57].

Furthermore, there is the choice of the LR frame of the fluid. Eckart’s definition (10) leads to

\[
\hat{\rho}^{\mu}_{00} = 0 , \quad \hat{\rho}_{01} \equiv -\hat{n}_l ,
\] (141)

which is equivalent to \(V^\mu_{\perp} = 0\) and \(n_l = 0\). On the other hand, Landau’s definition (11) implies

\[
\hat{\rho}_{10} = 0 , \quad \hat{\rho}_{11} \equiv -\hat{M} ,
\] (142)

which leads to \(W^\mu_{\perp \perp} = 0\) and \(M = 0\). Thus, for both choices of the LR frame, we obtain a simplification of the conservation equations (134) – (137), where several terms vanish identically. Note that the functional form of the anisotropic distribution function might explicitly lead to \(\hat{n}_l = 0\) and/or \(\hat{M} = 0\), as is the case in Refs. [24, 26, 51]. In this case, using \(n_l = n_l\) or \(\hat{M} = M\) as an additional dynamical matching condition would be meaningless.

Finally, note that we may also introduce the bulk viscous pressure, but now with respect to the anisotropic state instead of the local equilibrium state, i.e.,

\[
\Pi \equiv P - \hat{P}((\hat{a}, \hat{\beta}_u, \hat{\beta}_l)) = -\frac{1}{3} (\Delta^{\mu \nu} k_\mu k_\nu)_{\parallel} = -\frac{m_\perp^2}{3} \hat{\rho}_{00} .
\] (143)

Using Eqs. (35), (36), and (78) we obtain

\[
\Pi \equiv \frac{1}{3} \left[ \left( P_1 - \hat{P}_1 \right) + 2 \left( P_\perp - \hat{P}_\perp \right) \right] = \frac{1}{3} (P_1 + 2P_\perp) - P_0 - \hat{\Pi} \equiv \Pi_{100} - \hat{\Pi} ,
\] (144)
which can be used to express either $P_l$ or $P_{\perp}$ by $\Pi$ in the conservation equations (134) – (137). Also note that this equation is equivalent to Eq. (28) of Ref. [37] with the following identification of quantities: $\Pi \equiv \Pi_{\text{AHYDRO}}$, and $\Pi_{\text{iso}} \equiv \Pi$.

With the matching conditions (138) – (140), Landau’s choice (142) for the LR frame, and the bulk viscous pressure (143) replacing the transverse pressure $P_{\perp}$, the conservation equations (134) – (137) simplify,

\begin{align}
0 & = \dot{D}n + \tilde{n} (l_{\mu} D_{\mu} u^\mu + \tilde{\theta}) - D_l n_l + n_l \left( \tilde{\theta}_l - l_{\mu} D_{\mu} u^\mu \right) - V^\mu_l \left( D u_{\mu} + D_l i_{\mu} \right) + \nabla_\mu V^\mu_l , \tag{145} \\
0 & = D \dot{e} + \left( \dot{\tilde{\theta}} + \tilde{P}_l \right) l_{\mu} D_{\mu} u^\mu + \left( \dot{\tilde{\theta}} + \tilde{P}_l + \frac{3}{2} \Pi \right) \tilde{\theta} + W^\mu_{\perp l} \left( D u_{\mu} - l_{\nu} \nabla_\nu u^\nu \right) - \pi^\mu_{\perp l} \sigma_{\mu l} , \tag{146} \\
0 & = \left( \dot{\tilde{\theta}} + \tilde{P}_l + \frac{3}{2} \Pi \right) \Xi_\nu D_{\nu} u^\nu - \nabla_\nu \left( \tilde{P}_l + \frac{3}{2} \Pi \right) \tilde{\theta}_l + W^\mu_{\perp l} \left( D u_{\mu} + 2 D_l i_{\mu} + l_{\nu} \nabla_\nu u^\nu \right) - \nabla_\nu W^\mu_{\perp l} - \pi^\mu_{\perp l} \sigma_{\nu l} \nu , \tag{147} \\
0 & = \left( \dot{\tilde{\theta}} + \tilde{P}_l + \frac{3}{2} \Pi \right) \Xi_\nu D_{\nu} u^\nu - \tilde{\omega}_l + W^\mu_{\perp l} \left( D u_{\mu} + D_l i_{\mu} \right) + \nabla_\nu \nabla_\mu \pi^\mu_{\perp l} . \tag{148}
\end{align}

These five equations contain 14 unknowns: the scalar quantities $\dot{\alpha}$, $\dot{\beta}, \dot{\dot{\beta}}$ (which determine the quantities $\tilde{n}$, $n_l$, $\dot{\tilde{\theta}}$, $\dot{\tilde{\theta}}$, $P_l$, $P_{\perp}$, $n_l$, and $\Pi$, the three independent components of the fluid four-velocity $u^\mu$, the two independent components of $V^\mu_l$, the two independent components of $W^\mu_{\perp l}$, and the two independent components of $\pi^\mu_{\perp l}$). Thus, we need to specify nine additional equations of motion. This will be done in the next subsection, resorting again to the system (110) – (112) of equations of motion for the irreducible moments.

B. Relaxation equations in the 14-moment approximation

In this subsection, we outline the derivation of the nine additional equations of motion which close the system of conservation equations (134) – (137). As advertised above, to this end we again use the equations of motion (110) – (112) for the irreducible moments.

We will also make use of the fact that

\begin{align}
\dot{D}n & = \frac{\partial \tilde{n}}{\partial \alpha} \dot{\alpha} + \frac{\partial \tilde{n}}{\partial \beta_{\alpha}} \dot{\beta}_{\alpha} + \frac{\partial \tilde{n}}{\partial \beta_{\dot{\alpha}}} \dot{\beta}_{\dot{\alpha}} , \\
\dot{D}e & = \frac{\partial \tilde{\theta}}{\partial \alpha} \dot{\alpha} + \frac{\partial \tilde{\theta}}{\partial \beta_{\alpha}} \dot{\beta}_{\alpha} + \frac{\partial \tilde{\theta}}{\partial \beta_{\dot{\alpha}}} \dot{\beta}_{\dot{\alpha}} . 
\end{align}

Replacing the total derivatives with comoving derivatives, making use of the conservation equations (145) – (148), and neglecting terms of third order in dissipative quantities such as $\Pi, n_l, V^\mu_l, W^\mu_{\perp l}, \pi^\mu_{\perp l}$, or in gradients
of \( \dot{n}, \dot{n}_t, \dot{\rho}_t, \dot{P}_t, \dot{P}_\perp, \mu, l^\mu \), we derive the following equations for the comoving derivatives of \( \dot{\alpha} \) and \( \dot{\beta}_\perp \):

\[
D \dot{\alpha} = \left( \frac{\partial (\dot{\epsilon}, \dot{n})}{\partial (\beta_\perp, \dot{\alpha})} \right)^{-1} \left\{ \frac{\partial \dot{n}}{\partial \beta_\perp} \left( \dot{\epsilon} + \dot{P}_t \right) + \frac{\partial \dot{n}}{\partial \beta_\perp} \left( \dot{\epsilon} + \dot{P}_\perp \right) - \frac{\partial \dot{n}}{\partial \beta_\perp} \left( \dot{\epsilon} + \dot{P}_t \right) \right\},
\]

\[
D \dot{\beta}_\perp = \left( \frac{\partial (\dot{\epsilon}, \dot{n})}{\partial (\beta_\perp, \dot{\alpha})} \right)^{-1} \left\{ \frac{\partial \dot{n}}{\partial \beta_\perp} \left( \dot{\epsilon} + \dot{P}_t \right) + \frac{\partial \dot{n}}{\partial \beta_\perp} \left( \dot{\epsilon} + \dot{P}_\perp \right) - \frac{\partial \dot{n}}{\partial \beta_\perp} \left( \dot{\epsilon} + \dot{P}_t \right) \right\},
\]

where we have employed the notation

\[
\frac{\partial (X, Y)}{\partial (x, y)} = \frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} = \frac{\partial X}{\partial y} \frac{\partial Y}{\partial x}
\]

for the Jacobi determinant of partial derivatives. Note that the comoving derivatives \( D \dot{\alpha}, D \dot{\beta}_\perp \) depend also on the comoving derivative of the non-equilibrium parameter \( D \dot{\beta}_\parallel \), which is determined by a relaxation equation, see below.

In this work, the form of \( f_{\text{eq}} \) will not be explicitly specified. Thus, we cannot explicitly compute the transport coefficients of anisotropic dissipative fluid dynamics. For this reason, we leave the establishment of a systematically improvable framework for this theory following the approach of Ref. [18] to future work. Here, we restrict ourselves to the derivation of the equations in the simplest possible case: the 14-moment approximation, i.e., \( N_0 = 2, N_1 = 1, N_2 = 0 \). This assumption simplifies the discussion tremendously, since we no longer have to find the eigenmodes of the linearized Boltzmann equation as done in Ref. [18].

All we need to do is to close the system of equations of motion \([110] - [112]\) by employing Eq. \([105]\) in order to express the irreducible moments entering these equations in terms of the variables entering the conservation equations \([154] - [157]\): the Landau matching conditions \([133], [134]\) together with the third matching condition \([140]\), the choice \([142]\) of LR frame, and the definition \([143]\) of the bulk viscosity.

In addition, since the conservation equations \([133], [134]\) only involve quantities up to tensor rank two, we neglect irreducible tensor moments of rank higher than two, i.e., \( \hat{\rho}^{(i)}_{\mu\nu} = 0 \) for \( \ell \geq 3 \) in Eqs. \([111], [112]\), cf. Ref. [15]. Finally, since we want to derive relaxation equations for the dissipative quantities, we need to choose pairs of indices \((i, j)\) in Eqs. \([125] - [132] \), for which the collision integrals \( C^{(i, j)}_{\mu\nu} \) do not vanish. This choice is not unique, as any values for the indices \((i, j)\) are possible (except those that correspond to the conservation equations) and lead to a closed system of equations of motion. In the following, for the sake of simplicity the case where the indices \((i, j)\) take the lowest possible values will be considered.

Following this procedure, we obtain:
(i) From Eq. (110) for \((i, j) = (0, 0)\) the relaxation equation for the bulk viscous pressure \(P\),
\[
D_{\Pi} = \frac{m_0^2}{3} C_{-1,0} - \bar{\zeta}_\perp l_\mu D_{\Pi u^\mu} - \bar{\zeta}_\parallel \bar{\theta} - \bar{\zeta}_{\perp\perp} \bar{\theta}_I - \bar{\kappa}_\Pi D_I \bar{\alpha} - \bar{\kappa}_n D_I \bar{\beta}_n - \bar{\kappa}_I D_I \bar{\beta}_I + \bar{\tau}_{\Pi} D_{\Pi \bar{\beta}_I}.
\]
\[
- \left( \beta_{\Pi}^0 \Pi + \bar{\beta}_n^0 n_1 \right) l_\mu D_{\Pi u^\mu} - \left( \beta_{\Pi}^0 \Pi - \bar{\beta}_n^0 n_1 \right) \bar{\theta} - \left( \bar{\epsilon}_\Pi^0 \Pi + \bar{\epsilon}_n^0 n_1 \right) \bar{\theta}_I \bar{\zeta}_\perp + \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \bar{\theta}_I + \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \bar{\theta}_I
\]
\[
+ \left( \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \right) \left( D_\Pi u_{\mu} - l_\mu \bar{\nabla}_\mu u^\nu \right) + \left( \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \right) D_\Pi I + \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \bar{\theta}_I - V^\mu_{\perp} D_\Pi u^\mu,
\]
where we have made use of Eqs. (151), (152). Due to the anisotropy, there are now two (instead of one) bulk
viscosity coefficients \([42, 43]\): one coefficient multiplying \(l_\mu D_{\Pi u^\mu}\) for the fluid expansion in the direction of the
anisotropy, and one multiplying \(\bar{\theta}\) for the expansion in the direction transverse to \(l^\mu\). A third bulk viscosity
coefficient, \(\zeta_{\perp\perp}\), is related to the gradient of \(l^\mu\) in the direction transverse to both \(u^\mu\) and \(l^\mu\). Note that there are
first-order terms in the equation for the bulk viscosity which are proportional to gradients in \(l^\mu\)-direction of the
variables \(\bar{\alpha}, \bar{\beta}_n, \) and \(\bar{\beta}_I\) which do not appear in the isotropic case. There is also a term \(D_{\Pi \bar{\beta}_I}\) which couples the relaxation equation for the bulk viscous pressure to that for the anisotropy parameter \(\bar{\beta}_I\), see item (iii) below. The transport coefficients appearing in Eq. (157) are listed in Appendix II.

Finally, let us comment on the collision term \(-m_0^2 C_{-1,0}/3\). Although it is complicated in general, we expect that it is dominated by a term similar in structure to the RTA \([42]\),
\[
- \frac{m_0^2}{3} C_{-1,0} \sim \frac{1}{\tau_\Pi} \left( \frac{m_0^2}{3} \left( \bar{I}_{00} - \bar{I}_{00} \right) - I \right),
\]
where we have indicated the characteristic relaxation time for the bulk viscous pressure by \(\tau_\Pi\). The last term \(-I/\tau_\Pi\) reflects the nature of Eq. (157) as relaxation equation for the bulk viscous pressure: the bulk viscous pressure relaxes towards its Navier-Stokes value (given by the first-order terms in Eq. (157)) on a time scale \(\tau_\Pi\). This term already appears in the isotropic case \([13]\). However, the first term \(\bar{I}_{00} - \bar{I}_{00}\) is new: it represents an additional force that drives the system towards the isotropic limit.

(ii) From Eq. (110) for \((i, j) = (0, 1)\) the relaxation equation for the diffusion current \(n_1\) in the direction of the
anisotropy,
\[
D_{\Pi I} = C_{-1,1} + \bar{\kappa}_n D_I \bar{\alpha} + \bar{\kappa}_n D_I \bar{\beta}_n + \bar{\kappa}_I D_I \bar{\beta}_I + \bar{\zeta}_I l_\mu D_{\Pi u^\mu} - \bar{\zeta}_{\perp} \bar{\theta} - \bar{\zeta}_{\perp\perp} \bar{\theta}_I - V^\mu_{\perp} D_\Pi u^\mu,
\]
\[
- \left( \beta_{\Pi}^0 \Pi + \bar{\beta}_n^0 n_1 \right) l_\mu D_{\Pi u^\mu} - \left( \beta_{\Pi}^0 \Pi - \bar{\beta}_n^0 n_1 \right) \bar{\theta} - \left( \bar{\epsilon}_\Pi^0 \Pi + \bar{\epsilon}_n^0 n_1 \right) \bar{\theta}_I \bar{\zeta}_\perp + \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \bar{\theta}_I + \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \bar{\theta}_I
\]
\[
+ \left( \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \right) \left( D_\Pi u_{\mu} - l_\mu \bar{\nabla}_\mu u^\nu \right) + \left( \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \right) D_\Pi I + \bar{\lambda}_{\Pi}^0 \Pi + \bar{\lambda}_{\Pi}^0 W_{\Pi} W^\mu_{\perp} \bar{\theta}_I - V^\mu_{\perp} D_\Pi u^\mu,
\]
where we have employed Eqs. (147) and (148). The transport coefficients in Eq. (159) are listed in App. II.2.
Note that the last term in the first line couples the relaxation equation for \(n_1\) to the time evolution of \(l^\mu\).

We also comment on the collision term \(C_{-1,1}\). Again, we make the assumption that it is dominated by a term similar in structure to the RTA \([42]\),
\[
C_{-1,1} \sim -\frac{1}{\tau_n} \left( \bar{I}_{01} + \bar{\rho}_{01} - \bar{I}_{01} \right) \equiv -\frac{1}{\tau_n} \left( n_1 - I_{01} \right),
\]
where we have indicated the characteristic relaxation time for the diffusion current \(n_1\) by \(\tau_n\). For symmetry reasons \(I_{01} \equiv 0\), so that we are left with a term \(-n_1/\tau_n\), which is needed such that Eq. (159) is a relaxation equation for \(n_1\).
(iii) from Eq. (110) for \((i, j) = (0, 2)\) an evolution equation for the anisotropic pressure in the longitudinal direction \(\hat{P}_l\):

\[
D\hat{P}_l = C_{-1.2} + \kappa_\alpha^{\mu} D_l \hat{\alpha} + \kappa_\mu\nu D_l \hat{\beta}_\mu + \kappa_{\Delta}^{\mu} D_l \hat{\beta}_\Delta + \zeta_l^{\mu} l_\mu D_l u^\mu - \zeta_{1,\perp}^{\mu} \tilde{\theta} - \zeta_{1,\perp}^{\mu} \tilde{\theta}_l - 2 W_1^{\mu} D_l \mu
\]

from Eq. (111) for \((i, j) = (0, 2)\) an evolution equation for \(\hat{V}_{\perp}^\mu\) and \(\hat{V}_{\perp}^{\mu\nu}\) on the left-hand side as well as Eqs. (151), (152), it is straightforward to rewrite Eq. (161) into a relaxation equation for the parameter \(\hat{\beta}_l\).

Finally, let us comment on the collision integral \(C_{-1.2}\). Let us again assume that this is dominated by an RTA-like term,

\[
C_{-1.2} \sim -\frac{1}{\tau_l} \left( \hat{P}_l - \mathcal{Z}_{02} \right) .
\]

(iv) from Eq. (111) for \((i, j) = (0, 0)\) the relaxation equation for \(V_{\perp}^\mu = \beta_{\perp 0}^\mu\),

\[
DV_{\perp}^{(\mu)} = C_{1.0}^{(\mu)} + \kappa_\alpha^{\mu} \tilde{\nabla}_\alpha \hat{\alpha} + \kappa_\mu\nu \tilde{\nabla}^{\mu} \hat{\beta}_\mu + \kappa_{\Delta}^{\mu} \tilde{\nabla}^{\mu} \hat{\beta}_\Delta + \zeta_l^{\mu} (\Xi_l^{\mu} D_l u^\nu - l_\nu \tilde{\nabla}_\nu u^\nu) + \tilde{\zeta}^{\mu} S_{\perp}^{\nu} D_l u^\nu
\]

\[
- \beta_{1,\perp}^{\nu} \left( \Xi_l^{\nu} D_l u^\nu - l_\nu \tilde{\nabla}_\nu u^\nu \right) - \beta_{1,\perp}^{\nu} \left( \Xi_l^{\nu} D_l u^\nu - l_\nu \tilde{\nabla}_\nu u^\nu \right) + \tilde{\zeta}^{\mu} \Xi_l^{\nu} D_l u^\nu - 2 \tilde{\zeta}^{\mu} \tilde{\nabla}_\nu u^\nu
\]

where we have employed Eqs. (167) and (168). The last term in the first line couples the relaxation equation for \(V_{\perp}^\mu\) to the time evolution of \(l^\mu\). The transport coefficients appearing in Eq. (164) are listed in App. (114). In RTA, the collision term \(C_{1.0}^{(\mu)} \sim - V_{\perp}^\mu / \tau_V\), such that Eq. (163) is a relaxation-type equation for \(V_{\perp}^\mu\).

(v) from Eq. (111) for \((i, j) = (0, 1)\) the relaxation equation for the energy-momentum flow in the direction of the
anisotropy $W_{\perp}^{\mu
u}$.  

$$
D W_{\perp}^{\mu
u} = C_{\perp,1}^{(\mu
u)} - k_W \nabla^\mu \tilde{\alpha} - k_W \nabla^\nu \tilde{\beta}_u - k_W \nabla^\nu \tilde{\beta}_l + 2 \eta_W \Xi_\mu D u^\nu - 2 \eta_W l_u \nabla^\mu u^\nu + 2 \eta_W \Xi_\nu D l^\mu
$$

$$
= \left[ \left( \frac{\tau_W}{2} \right) \Xi_\mu + \pi_\mu^{\alpha\nu} \right] D l^\nu
$$

$$
- \beta_W^{\mu\nu} \Xi_\mu D u^\nu + \beta_W^{\mu\nu} l_u \nabla^\mu u^\nu - \left( \beta_W^{\mu\nu} - \beta_W^{\mu\nu} \eta_W l_u \nabla^\mu u^\nu \right)
$$

$$
+ \left( \delta_W^{\mu\nu} + \delta_W^{\mu\nu} W_{\perp,1}^{\mu\nu} \right) \nabla^\mu \theta_l + \left( \nabla^\mu \theta_l + \delta_W^{\mu\nu} W_{\perp,1}^{\mu\nu} \right) \theta_l
$$

$$
+ \frac{\eta_W}{\eta_W} \nabla^\mu \nu l_u \nabla^\mu u^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu
$$

$$
+ \left( \delta_W^{\mu\nu} + \delta_W^{\mu\nu} W_{\perp,1}^{\mu\nu} \right) D l^\nu
$$

The transport coefficients in this equation are listed in App. H.5. The term in the second line couples the relaxation equation for $W_{\perp}^{\mu\nu}$ to the time evolution of $l^\mu$. In RTA, the collision term $C_{\perp,1}^{(\mu\nu)} \sim -W_{\perp,1}^{\mu\nu}/\tau_W$, such that Eq. (165) is a relaxation-type equation for $W_{\perp}^{\mu\nu}$.

(vii) From Eq. (122) for $(i, j) = (0, 0)$ the relaxation equation for the transverse stress tensor $\pi_{\mu\nu}^{\perp \perp}$,

$$
D \pi_{\perp}^{\mu\nu} = C_{\perp,0}^{(\mu\nu)} + 2 \eta_W \nabla^\mu u^\nu + 2 \eta_W \delta^\mu_{\parallel} \delta^\nu_{\parallel} - 2 W_{\perp,1}^{\mu\nu} D l^\nu
$$

$$
- \delta_W^{\mu\nu} \tilde{\alpha} - 2 \nabla^\mu \nu l_u \nabla^\mu u^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu
$$

$$
+ \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu - \delta_W^{\mu\nu} \Xi_\mu D l^\nu
$$

$$
+ \left( \delta_W^{\mu\nu} + \delta_W^{\mu\nu} W_{\perp,1}^{\mu\nu} \right) \nabla^\mu \theta_l + \left( \nabla^\mu \theta_l + \delta_W^{\mu\nu} W_{\perp,1}^{\mu\nu} \right) \theta_l
$$

$$
+ \left( \delta_W^{\mu\nu} + \delta_W^{\mu\nu} W_{\perp,1}^{\mu\nu} \right) D l^\nu
$$

$$
- \lambda_W^{\mu\nu} W_{\perp,1}^{\mu\nu} D l_{\alpha} + \lambda_W^{\mu\nu} W_{\perp,1}^{\mu\nu} W_{\perp,1}^{\nu\alpha} D l_{\alpha} + \lambda_W^{\mu\nu} W_{\perp,1}^{\mu\nu} D l_{\alpha} + \lambda_W^{\mu\nu} W_{\perp,1}^{\mu\nu} W_{\perp,1}^{\nu\alpha} D l_{\alpha}
$$

$$
- \lambda_W^{\mu\nu} W_{\perp,1}^{\mu\nu} D l_{\alpha} + \lambda_W^{\mu\nu} W_{\perp,1}^{\mu\nu} W_{\perp,1}^{\nu\alpha} D l_{\alpha}
$$

The transport coefficients in this equation are listed in App. H.6. The last term in the first line couples the relaxation equation for $\pi_{\mu\nu}^{\perp \perp}$ to the time evolution of $l^\mu$. In RTA, the collision term $C_{\perp,0}^{(\mu\nu)} \sim -\pi_{\mu\nu}^{\perp \perp}/\tau_W$, such that Eq. (166) is a relaxation-type equation for $\pi_{\mu\nu}^{\perp \perp}$.

We conclude this section by a couple of remarks:

(a) In the relaxation equations (167), (159), (161), (162), (163), and (164), at a given order (in gradients and dissipative quantities) all terms appear which are allowed by Lorentz symmetry. In principle, the corresponding transport coefficients are all independent quantities. However, close inspection reveals that several of them are proportional (or identical) to other coefficients. At this stage, we perceive this to be an artifact of the 14-moment approximation (but we cannot exclude that this will persist also at higher order in the moment expansion).

(b) The power-counting scheme in the relaxation equations (157), (159), (161), (162), (163), and (164) is similar to that of Ref. [18], i.e., in terms of powers of Knudsen and inverse Reynolds numbers. Note, however, that the latter are now proportional to the bulk viscous pressure [141] and the dissipative quantities defined in Eqs. (27) and (28). At this point, let us remark that the collision integral (116) drives the relaxation of $f_{\perp 0}$ towards the local equilibrium distribution $f_{\perp 0}$. Thus, for a strong anisotropy it can in principle become arbitrarily large and is thus not part of the above power-counting scheme.

(c) In the particular case of an anisotropic fluid specified by $f_{\perp 0}$ alone, i.e., when the dissipative quantities with respect to the anisotropic reference state vanish $\tilde{\rho}_{\parallel 0}^{\mu\nu} = 0$, there is only one additional equation needed for closure. This is so, since in general $\bar{u}_l$, $\bar{\rho}_l$, and $\Pi$ are not independent variables. Furthermore, in the case of $f_{\perp 0}$ this leads to $\bar{u}_l = M = 0$ we are left with an equation for $\bar{\rho}_l$. 


(d) Our approach is analogous to the method given in Ref. [18], with the exception that now the reference state is different from the local thermodynamic equilibrium state. Therefore, due to this specific functional difference, anisotropic fluid dynamics embodies an arbitrary local momentum-space anisotropy and hence extends the ideal fluid-dynamical approach of a local equilibrium distribution function. This in turn also means that, when compared to ideal fluids, such anisotropic fluids correspond to a specific class of dissipative fluids with an additional independent variable. More specifically, anisotropic fluids require the knowledge of the equation of state specified by \( \hat{f}_k \) and only after that can one proceed with the solution of the more general equations of anisotropic dissipative fluid dynamics. Furthermore, compared to the usual dissipative fluid-dynamical approach, in anisotropic dissipative fluid dynamics there are only eight additional dissipative quantities which "relax" to the anisotropic state characterized by six independent variables.

IX. CONCLUSIONS AND OUTLOOK

Anisotropic fluid dynamics allows for a macroscopic description of systems when the microscopic single-particle distribution exhibits a strong anisotropy in momentum space in the local rest frame of the fluid. In this paper, starting from the relativistic Boltzmann equation, we have derived the equations of motion for the irreducible moments of the deviation of the single-particle distribution function from a given anisotropic reference state. These equations of motion are given in Eqs. (110) – (112) up to tensor-rank two. For the derivation, we constructed a new orthogonal basis in terms of multivariate polynomials in both energy \( E_k \) and momentum \( E_k \) in the direction of the anisotropy, specified by the space-like four-vector \( \vec{l} \), as well as irreducible tensors in momentum space orthogonal to both the fluid four-velocity \( u^\mu \) and \( l^\mu \).

We then derived the equations of anisotropic dissipative fluid dynamics in the Landau frame, Eqs. (145) – (148), (157), (159), (161), (164), (165), and (166), from the equations of motion for the irreducible moments in the 14-moment approximation, substituting the irreducible moments by the fluid-dynamical variables, Eqs. (156).

Our treatment is general in the sense that we did not specify the form of the anisotropic single-particle distribution function. This, however, is mandatory before one can explicitly compute the transport coefficients and solve the equations of motion. We will leave this study to future work.

Besides applications to heavy-ion collisions, we envisage that the framework introduced here will also be useful in formulating a theory of relativistic dissipative magneto-hydrodynamics [42, 43]. In this case, the magnetic-field vector \( B^\mu = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} u_\nu \) assumes the role of \( l^\mu \).

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Appendix A: Tensor decompositions with respect to fluid flow and the direction of anisotropy

The velocity of fluid-dynamical flow is specified in terms of the time-like four-vector

\[
u^\mu \left( t, x \right) = \gamma \left( 1, v^x, v^y, v^z \right),
\]

which is taken to be normalized,

\[
u^\mu \nu_\mu = 1.
\]

Hence, the Lorentz-gamma factor is \( \gamma = \left( 1 - v^2 \right)^{-1/2} \), and the four-flow velocity contains only three independent components: the three components of the fluid three-velocity \( \vec{v} = (v^x, v^y, v^z) \).

In order to specify the direction of a possible anisotropy in a given system, we define a space-like four-vector \( l^\mu \),

\[
l^\mu \nu_\mu = -l^2,
\]

which is taken to be normalized,

\[
l^\mu l_\mu = 1.
\]

Hence, the direction \( l^\mu \) can be expressed in terms of the components of the fluid three-velocity \( \vec{v} \).
where \( l^2 > 0 \) characterizes the strength of the anisotropy. Note that \( l^\mu \equiv z^\mu \) in the notation of Refs. [31, 57]. Furthermore, \( l^\mu \) is taken to be orthogonal to the four-flow velocity,

\[
u^\mu l_\mu = 0 .
\]

An anisotropy could be caused by an external magnetic field. In this case, \( l^\mu \) would be conveniently chosen to point into the direction of this field. In the context of heavy-ion collisions, this four-vector would be chosen to point into the direction of the beam axis (usually the \( z \)-axis).

In general, a space-like four-vector can be written in the form

\[
l^\mu (t, \mathbf{x}) = l_\gamma (1, \ell^\gamma, \ell^\eta, \ell^\zeta) ,
\]

where \( \gamma (1, 1, 1) \) follows from the normalization condition \([A3]\). If one is only interested in the direction of the anisotropy, and not its magnitude, \( l^\mu \) can be normalized to one, i.e., \( l \equiv 1 \). The orthogonality of the normalized \( l^\mu \) to the flow velocity \([A3]\) gives the constraint

\[
u^\mu l_\mu \equiv \gamma \gamma (1 - \nu \cdot \ell) = 0 ,
\]

which may serve to express one component of \( l^\mu \) by the others, provided the corresponding component of \( \nu^\mu \) does not vanish. Thus, in general a normalized \( l^\mu \) has two independent components. For instance, for purely longitudinal flow

\[
u^\mu = \gamma^\mu (1, 0, 0, \nu_z^2) ,
\]

with \( \gamma \equiv (\ell^2 - 1)^{-1/2} \), and one can determine \( \ell^\mu \) from Eq. \([A6]\) as \( \ell^\mu = 1/\nu_z^2 \). Without loss of generality it is possible to set \( \ell^\mu = \ell^0 = 0 \), such that \( l^\mu \) is completely specified,

\[
l^\mu = \gamma^\mu (1, 0, 0, 1/\nu_z^2) \equiv \gamma^\mu (0^+, 0, 0, 1) .
\]

One may use this form even if the flow is three-dimensional, since it still fulfills the requirements \([A3]\) (with \( l = 1 \)) and \([A4], [A9], [A2], [A34]\). In the co-moving frame or LR frame of matter, \( u^\mu_{LR} = (1, 0, 0, 0) \), (independent of the physical meaning of the four-velocity), hence the direction of the anisotropy corresponds to the longitudinal or \( z \)-direction of the coordinate system, \( l^\mu_{LR} = (0, 0, 0, 1) \).

In fluid dynamics, one usually introduces a tensor

\[
\Delta^{\mu\nu} \equiv \gamma^{\mu\nu} - u^\mu u^\nu ,
\]

which is symmetric \( \Delta^{\mu\nu} = \Delta^{\nu\mu} \) and projects onto the three-dimensional space orthogonal to the four-flow velocity of matter, \( \Delta^{\mu\nu} u_\mu = 0 \), \( \Delta^{\mu\mu} = 3 \). Since \( l^\mu \) is already orthogonal to \( u^\mu \), cf. Eq. \([A3]\), we have \( \Delta^{\mu\nu} l_\nu \equiv l^\mu \).

An anisotropy in a system singles out another direction besides the direction of fluid flow. In our case, this is the direction characterized by \( l^\mu \). Thus, it is natural to generalize the projection operator \([A3]\) to a new symmetric projection operator onto the two-dimensional subspace orthogonal to both \( u^\mu \) and \( l^\mu \) \([A2], [A3]\),

\[
\Xi^{\mu\nu} \equiv \gamma^{\mu\nu} - u^\mu u^\nu + l^\mu l^\nu = \Delta^{\mu\nu} + l^\mu l^\nu ,
\]

where \( \Xi^{\mu\nu} = \Xi^{\nu\mu} \), \( \Xi^{\mu\mu} u_\mu = \Xi^{\mu\nu} l_\nu = 0 \), and \( \Xi = 2 \). Note also that \( \Delta^{\mu\nu} \Xi_{\mu\nu} = 2 \).

Let us briefly remind the reader of our notational conventions (see Sec. I). The projection of an arbitrary four-vector \( A^\mu \) orthogonal to \( u^\mu \) will be denoted by

\[
A^{(\mu)} = \Delta^{\mu\nu} A_\nu ,
\]

while the projection orthogonal to both \( u^\mu \) and \( l^\mu \) will be denoted by

\[
A^{(\mu)} = \Xi^{\mu\nu} A_\nu .
\]

The corresponding projections of arbitrary rank-two tensors are defined as

\[
A^{(\mu\nu)} = \Delta^{\mu\nu}_\alpha \delta^{\alpha\beta} A^{\beta} ,
\]

\[
A^{(\mu\nu)} = \Xi^{\mu\nu}_\alpha \delta^{\alpha\beta} A^{\beta} ,
\]

where the corresponding symmetric, orthogonal, and traceless projection operators are

\[
\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2} \left( \Delta^{\mu\nu}_\alpha \delta^{\beta\gamma} + \Delta^{\nu\mu}_\beta \delta^{\gamma\alpha} \right) - \frac{1}{3} \Delta^{\mu\nu} \delta^{\alpha\beta} ,
\]

\[
\Xi^{\mu\nu}_{\alpha\beta} = \frac{1}{2} \left( \Xi^{\mu\nu}_\alpha \delta^{\beta\gamma} + \Xi^{\nu\mu}_\beta \delta^{\gamma\alpha} \right) - \frac{1}{2} \Xi^{\mu\nu} \delta^{\alpha\beta} ,
\]

where \( \alpha, \beta = 1, 2 \).
while the projection operators for higher-rank tensors following are specified in App. [E].

We remark that one can trade the orthogonal projection [A14] for the projection [A15],

\[
\Delta_{\mu\nu}^{\alpha\beta} = \Xi_{\alpha\beta} - 2l_{\alpha} (\Xi_{\beta})^{l_{\beta}} \mid l_{\nu} + \frac{1}{6} (\Xi_{\alpha\beta} + 2l_{\alpha} l_{\beta}) (\Xi_{\mu\nu} + 2l_{\mu} l_{\nu}) , \tag{A16}
\]

which can be obtained by inserting \(\Delta^{\mu\nu} = \Xi^{\mu\nu} - l^\mu l^\nu\), cf. Eq. [A9], into Eq. [A14].

The tensor decomposition of a four-vector with respect to \(u^\mu\) and \(\Delta^{\mu\nu}\) reads

\[
A^\mu = A^\nu u_\nu u^\mu + A^{(\mu)} , \tag{A17}
\]

while that with respect to \(u^\mu\), \(l^\mu\), and \(\Xi^{\mu\nu}\) reads

\[
A^\mu = A^\nu u_\nu u^\mu - A^\nu l_\nu l^\mu + A^{(\mu)} . \tag{A18}
\]

Analogously, the tensor decomposition of the four-gradient with respect to \(u^\mu\) and \(\Delta^{\mu\nu}\) reads

\[
\partial_\mu = u_\mu D + \nabla_\mu , \tag{A19}
\]

where the comoving time-derivative and the four-gradient are defined as,

\[
D = u^\mu \partial_\mu , \tag{A20}
\]

\[
\nabla_\mu \equiv \Delta^{\mu\nu} \partial_\nu = \partial_\mu , \tag{A21}
\]

In the LR frame \(D\) is the time derivative, \(D_{LR} = (\partial/\partial t, 0, 0, 0)\), while \(\nabla_{LR}^\mu = (0, \partial/\partial x, \partial/\partial y, \partial/\partial z)\) is the three-gradient.

Similarly, we decompose \(\partial_\mu\) with respect to \(u^\mu\), \(l^\mu\), and \(\Xi^{\mu\nu}\) as

\[
\partial_\mu = u_\mu D + l_\mu D_l + \nabla_\mu , \tag{A22}
\]

where

\[
D_l = -l^\mu \partial_\mu , \tag{A23}
\]

\[
\nabla_\mu = \Xi^{\mu\nu} \partial_\nu = \partial_\mu . \tag{A24}
\]

Comparing Eq. [A19] to Eq. [A22] it is immediately apparent that in the anisotropic case the usual gradient operator is split into two parts,

\[
\nabla_\mu = l_\mu D_l + \nabla_\mu , \tag{A25}
\]

where according to the specific choice of Eq. [A1] in the LR frame \(D_{LR} = (0, 0, 0, \partial/\partial z)\) corresponds to the derivative in the direction of the anisotropy, while \(\nabla_{LR}^\mu = (0, \partial/\partial x, \partial/\partial y, \partial/\partial z)\) is the spatial derivative in the remaining transverse directions orthogonal to both \(u^\mu\) and \(l^\mu\).

Note that on account of the normalization and orthogonality conditions [A2], [A3], and [A4], we have the identities

\[
u^\mu \partial_\mu u_\nu = u^\nu D u_\nu = u^\nu l_\nu D_l = l^\nu \nabla_\nu l_\nu = 0 , \tag{A26}
\]

\[
\nu^\mu \partial_\mu l_\nu = \nu^\nu D l_\nu = \nu^\nu l_\nu D_l = \nu^\nu \nabla_\nu l_\nu = 0 , \tag{A27}
\]

as well as

\[
\nu^\mu \partial_\mu u_\nu = -\nu^\mu \partial_\mu l_\nu , \quad \nu^\nu D u_\nu = -\nu^\nu D l_\nu , \quad \nu^\nu l_\nu D_l = -\nu^\nu D_l u_\nu , \quad \nu^\nu \nabla_\nu l_\nu = -\nu^\nu \nabla_\nu u_\nu . \tag{A28}
\]

The decomposition of a rank-two tensor with respect to \(u^\mu\) and \(\Delta^{\mu\nu}\) reads

\[
A^{\mu\nu} = A_{\alpha\beta} u^\alpha u^\beta \nu^\mu \nu^\nu + \Delta^{\alpha\beta} A_{\alpha\beta} u^\beta \nu^\mu + \Delta^{\nu\beta} A_{\nu\beta} \nu^\alpha \nu^\mu + \frac{1}{3} A^\nu_{\alpha\beta} \Delta_{\alpha\beta} \nu^\mu + \Delta^{\alpha\beta} A_{\alpha\beta} + \Delta^{\nu\beta} A_{\nu\beta} A^{\alpha\beta} . \tag{A29}
\]

Applying this to the gradient of the four-flow velocity, we obtain with Eq. [A20] the well-known relativistic Cauchy-Stokes formula,

\[
\partial_\mu u_\nu = u_\mu D u_\nu + \frac{1}{3} \nu \Delta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} . \tag{A30}
\]
where

\[ \theta \equiv \nabla_\mu u^\mu \]  

is the expansion scalar,

\[ \sigma^{\mu\nu} \equiv \partial^{(\mu} u^{\nu)} = \Delta_{\alpha\beta}^{\mu\nu} \partial^\alpha u^\beta = \frac{1}{2} (\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3} \theta \Delta_{\mu\nu} \]  

is the shear tensor and

\[ \omega^{\mu\nu} \equiv \Delta_\alpha^{\mu\nu} \partial^{\alpha} u^\beta = \frac{1}{2} (\nabla^\mu u^\nu - \nabla^\nu u^\mu) \]  

is the vorticity.

Finally, the vorticities (A33) and (A38) are antisymmetric and relate via

\[ \omega^{\mu\nu} = \Delta_\alpha^{\mu\nu} \partial^{\alpha} u^\beta = \frac{1}{2} (\nabla^\mu u^\nu - \nabla^\nu u^\mu) \]  

is the vorticity.

The tensor decomposition of a rank-two tensor with respect to \( u^\mu, l^\mu \), and \( \Xi^{\mu\nu} \) reads

\[ A^{\mu\nu} = A^{\alpha\beta} u_\alpha u_\beta u^\mu u^\nu + A^{\alpha\beta} l_\alpha l_\beta l^\mu l^\nu - A^{\alpha\beta} u_\alpha l_\beta u^\mu u^\nu - A^{\alpha\beta} u_\alpha u_\beta l^\mu l^\nu + \Xi^{\mu\alpha} A^{\alpha\beta} u^\beta u^\nu - \Xi^{\mu\alpha} A^{\alpha\beta} l^\beta l^\nu + \Xi^{\mu\alpha} \Xi^{\alpha\beta} + \Xi^{\mu\alpha} A^{\alpha\beta} + \Xi^{\mu\alpha} \Xi^{\alpha\beta} \]  

(A34)

With this decomposition we obtain the counterpart of the relativistic Cauchy-Stokes formula (A30),

\[ \partial_\mu u_\nu = u_\mu D u_\nu + l_\mu D l u_\nu + \frac{1}{2} \theta \Xi_{\mu\nu} - l_\beta l_\nu \nabla_\mu u^\beta + \hat{\sigma}_{\mu\nu} + \hat{\omega}_{\mu\nu} , \]  

(A35)

where we have again made use of Eq. (A26). Note that \( -l_\beta l_\nu \nabla_\mu u^\beta = -l_\beta l_\nu (\nabla_\mu \nabla^\nu v^\beta) \) can in principle also be further separated into a symmetric and an antisymmetric part. In Eq. (A35) we defined the following quantities in the subspace orthogonal to both \( u^\mu \) and \( l^\mu \): the transverse expansion scalar

\[ \hat{\theta} \equiv \nabla_\mu u^\mu , \]  

(A36)

the transverse shear tensor

\[ \hat{\sigma}_{\mu\nu} \equiv \delta^{(\mu} \nabla^{\nu)} = \nabla^{(\mu} \nabla^{\nu)} - \frac{1}{2} \hat{\theta} \Xi_{\mu\nu} + l_\beta l_\nu \nabla_\mu u^\beta , \]  

(A37)

and the transverse vorticity

\[ \hat{\omega}_{\mu\nu} \equiv \Xi^{\mu\alpha} \Xi^{\nu\beta} \partial_{[\alpha} u_{\beta]} = \nabla^{(\mu} \nabla^{\nu)} - l_\beta l_\nu \nabla_\mu u^\beta . \]  

(A38)

Note that the shear tensor (A32) and vorticity (A33) are orthogonal to the flow velocity, i.e., \( \sigma^{\mu\nu} u_\nu = \omega^{\mu\nu} u_\nu = 0 \), but \( \sigma^{\mu\nu} l_\nu \neq 0 \), \( \omega^{\mu\nu} l_\nu \neq 0 \), while the transverse shear tensor (A37) and the transverse vorticity (A38) are orthogonal to both four-vectors, \( \hat{\sigma}_{\mu\nu} u_\nu = \hat{\sigma}_{\mu\nu} l_\nu = 0 \) and \( \hat{\omega}_{\mu\nu} u_\nu = \hat{\omega}_{\mu\nu} l_\nu = 0 \).

The expansion scalars (A31) and (A36) are related to each other through Eq. (A25),

\[ \theta = l_\mu D l u^\mu + \hat{\theta} . \]  

(A39)

The shear tensors (A32) and (A37) are symmetric by definition and related to each other through Eq. (A16),

\[ \sigma^{\mu\nu} = \hat{\sigma}^{\mu\nu} + (l^{[\mu} \Xi^{\nu]}) D l u^\beta - l_\beta l^{[\mu} \nabla^{\nu]} u^\beta + \frac{1}{6} \left( \hat{\theta} - 2 l_\beta D l u^\beta \right) \left( \Xi^{\mu\nu} + 2 l^{[\mu} l^{\nu]} \right) . \]  

(A40)

Finally, the vorticities (A33) and (A38) are antisymmetric and related via

\[ \omega^{\mu\nu} = \hat{\omega}^{\mu\nu} + [l^{[\mu} D l u^{\nu]} + l^{[\mu} (l^{[\nu]} \nabla)^{\nu]} u^\beta . \]  

(A41)

Similarly to the Cauchy-Stokes formulae (A30) and (A35) for \( \partial_\mu l_\nu \), we also need the decompositions of \( \partial_\mu l_\nu \),

\[ \partial_\mu l_\nu \equiv u_\mu D l_\nu + \frac{1}{3} \hat{\theta} \Delta_{\mu\nu} + u_\beta u_\nu \nabla_\mu l^\beta + \sigma_{\mu\nu} + \omega_{\mu\nu} \]

\[ = u_\mu D l_\nu + l_\mu D l_\nu + \frac{1}{2} \hat{\theta} \Xi_{\mu\nu} + u_\beta u_\nu \nabla_\mu l^\beta + \hat{\sigma}_{\mu\nu} + \hat{\omega}_{\mu\nu} . \]  

(A42)
where \( u_\alpha u_\nu \nabla_\mu l^\beta = u_\beta u_(\mu \nabla_\nu )l^\beta \) and we defined the quantities
\[
\theta_1 \equiv \nabla_\mu l^\mu ,
\]  
where we used \( \Delta = \frac{1}{2} \theta_1 \Delta^{\mu \nu} - u_\beta u_(\mu \nabla_\nu )l^\beta \) ,
\]  
\[
\sigma_1^{\mu \nu} \equiv \Delta^{(\mu \nu)} \equiv \nabla_\mu l^\nu - \frac{1}{3} \theta_1 \Delta^{\mu \nu} - u_\beta u_(\mu \nabla_\nu )l^\beta ,
\]  
\[
\tilde{\sigma}_1^{\mu \nu} \equiv \Delta^{(\mu \nu)} \equiv \nabla_\mu l^\nu - \frac{1}{2} \theta_1 \Xi^{\mu \nu} - u_\beta u_(\mu \nabla_\nu )l^\beta ,
\]  
\[
\omega_1^{\mu \nu} \equiv \Delta^{\mu \nu} \Delta^{(\nu \beta)} \partial_\alpha l_\beta = \nabla_\nu l^{\mu} + u_\beta u_(\mu \nabla_\nu )l^\beta ,
\]  
\[
\tilde{\omega}_1^{\mu \nu} \equiv \Xi^{\mu \nu} \Xi^{\alpha \beta} \partial_\alpha l_\beta = \nabla_\nu l^{\mu} + u_\beta u_(\mu \nabla_\nu )l^\beta .
\]  
Note that \( \sigma_1^{\mu \nu} u_\nu \neq 0 \) and \( \omega_1^{\mu \nu} u_\nu \neq 0 \), but \( \tilde{\sigma}_1^{\mu \nu} u_\nu = \tilde{\sigma}_1^{\mu \nu} l_\nu = 0 \) and \( \tilde{\omega}_1^{\mu \nu} u_\nu = \tilde{\omega}_1^{\mu \nu} l_\nu = 0 \). The relationships between the various quantities are
\[
\theta_1 = \tilde{\theta}_1 ,
\]  
\[
\sigma_1^{\mu \nu} = \tilde{\sigma}_1^{\mu \nu} + l^{(\mu} \Xi^{\nu)} D_\nu l^\beta + \frac{1}{6} \tilde{\theta}_1 (\Xi^{\mu \nu} + 2 l^{\mu} l^\nu ) ,
\]  
\[
\omega_1^{\mu \nu} = \tilde{\omega}_1^{\mu \nu} + l^{(\mu} \Xi^{\nu)} D_\nu l^\beta .
\]  

Appendix B: Tensor decomposition of four-momentum

In this appendix, we apply the result of App. A to derive the tensor decomposition of four-momentum \( k_\mu = (k^0, k^x, k^y, k^z) \). According to Eq. (A17),
\[
k_\mu = E_{ku} u_\mu + k^{(\mu)} ,
\]  
where
\[
E_{ku} = k_\mu u_\mu , \quad k^{(\mu)} = \Delta^{\mu \nu} k_\nu .
\]  

The physical meaning of these quantities becomes apparent in the LR frame: \( E_{k_L} = k_0 \) is the energy while \( k^{(x)} = (0, k^x, k^y, k^z) \) is the three-momentum.

Analogously, one can tensor-decompose \( k_\mu \) using Eq. (B1),
\[
k_\mu = E_{ku} u_\mu + E_{kl} l^\mu + k^{(\mu)} ,
\]  
where
\[
E_{kl} = -k^\mu l_\mu , \quad k^{(\mu)} = \Xi^{\mu \nu} k_\nu .
\]  

Comparing Eqs. (B2) and (B3), it is obvious that \( k^{(\mu)} = E_{k_L} l^\mu + k^{(\mu)} \). In the LR frame and with the choice (A7) for \( l^\mu \), the quantities defined in Eq. (B3) are \( E_{k_L} = (0, 0, 0, k^2) \), i.e., the component of three-momentum in \( l^\mu \)-direction, and \( k^{(\mu)} = (0, k^x, k^y, 0) \), i.e., the components of three-momentum orthogonal to \( l^\mu \).

For on-shell particles,
\[
k_\mu k_\mu \equiv E_{ku}^2 + k^{(\mu)} k^{(\mu)} = \Delta^{\alpha \beta} k_\alpha k_\beta ,
\]  
\[
k^{(\mu)} k^{(\mu)} = \Xi^{\alpha \beta} k_\alpha k_\beta .
\]  

From Eq. (A29),
\[
k_\mu k_\nu = E_{ku}^2 u^\mu u^\nu + \frac{1}{3} k^{(\mu)} k^{(\mu)} \Delta^{\mu \nu} + 2 E_{ku} k^{(\mu)} u_\mu u^\nu + k^{(\mu)} k^{(\mu)} ,
\]  
while from Eq. (A34)
\[
k_\mu k_\nu = E_{ku}^2 u^\mu u^\nu + E_{kl}^2 l^\mu l^\nu + 2 E_{ku} E_{kl} l^\mu u_\mu l^\nu + 2 E_{ku} k^{(\mu)} u_\mu u^\nu + 2 E_{kl} k^{(\mu)} l^\mu l^\nu + \frac{1}{2} k^{(\mu)} k^{(\mu)} \Xi^{\alpha \beta} k_\alpha k_\beta = k^{(\mu)} k^{(\mu)} .
\]  

Higher-rank tensors formed from dyadic products of \( k_\mu \) can be decomposed in a similar manner.
Appendix C: Thermodynamic integrals and properties

In this appendix, we compute the thermodynamic integrals \( I_{i+n,q} \) and \( I_{i+j+n,j+r,q} \) in Eqs. (11) and (20). They are obtained by suitable projections of the tensors \( T_{ij}^{\mu_1 \cdots \mu_n} \) and \( T_{ij}^{\mu_1 \cdots \mu_n} \).

Let us first focus on the integrals \( I_{i+n,q} \). The coefficient \( b_{nq} \) in Eq. (11) is defined as the number of distinct terms in the symmetrized tensor product

\[
\Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})} = \frac{1}{b_{nq}} \sum_{p=2}^{nq} \Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})},
\]

where the sum runs over all distinct permutations of the \( n \) indices \( \mu_1, \ldots, \mu_n \). The total number of permutations of \( n \) indices is \( n! \). There are \( q \) projection operators \( \Delta^{(\mu_1 \mu_2)} \) and \( n-2q \) factors of \( \mu_{ij} \). Permutations of the order of the \( \Delta^{(\mu_1 \mu_2)} \) and of the \( \mu_{ij} \) among themselves do not lead to distinct terms, so we need to divide the total number \( n! \) by \( q!(n-2q)! \). Finally, since \( \Delta^{(\mu_1 \mu_2)} \) is a symmetric projection operator, a permutation of its indices does not lead to a distinct term. Therefore, there are \( q \) such projection operators, and there are \( 2^q \) permutations that do not lead to distinct terms. Hence, the total number of distinct terms in the symmetrized tensor product is

\[
b_{nq} = \frac{n!}{2^q q!(n-2q)!} = \frac{n!(q-1)!}{(2q)! (n-2q)!},
\]

which is identical to Eq. (A2) of Ref. [11].

In order to obtain the thermodynamic integrals \( I_{i+n,q} \) by projection of the tensors \( T_i^{\mu_1 \cdots \mu_n} \), it is advantageous to use the orthogonality relation

\[
\Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})} \Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})} = \frac{(2q+1)!}{b_{nq}} \delta_{qq'},
\]

cf. Eq. (A3) of chapter VI.1 of Ref. [12]. Since later on we will generalize this result to the case involving \( l \)'s and \( \Xi \)'s, we give the proof in some detail. First, it is clear that if \( q \neq q' \) there are terms where a \( \mu_{ij} \) gets contracted with a \( \Delta^{(\mu_1 \mu_2)} \), which gives zero. The existence of the Kronecker delta is thus easily explained and we only need to prove Eq. (C3) for \( q = q' \). Second, as the same set of indices is symmetrized on both tensor products, it actually suffices to keep the set of indices fixed on one tensor, say in the order \( \mu_1, \ldots, \mu_{2q}, \mu_{2q+1}, \ldots, \mu_n \), and symmetrize only the one on the other,

\[
\Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})} \Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})} = \frac{1}{b_{nq}} \sum_{p=2}^{nq} \Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})},
\]

where we used Eq. (C1). Among the terms in the sum over all distinct permutations, only those survive where the indices on the \( u \)'s are \( \mu_{2q+1}, \ldots, \mu_n \), just as in the term in front of the sum. (Otherwise, a \( \mu_{ij} \) will be contracted with a \( \Delta^{(\mu_1 \mu_2)} \), which gives zero.) Permutations among these indices do not lead to distinct terms. Using \( \mu_{ij} \mu_{ij} = 1 \), we thus obtain

\[
\Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})} = \frac{1}{b_{nq}} \sum_{p=2}^{nq} \Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})},
\]

where the sum now runs only over the distinct permutations of \( 2q \) indices \( \mu_1, \ldots, \mu_{2q} \) on the \( \Delta \) projectors. There are in total \( (2q)!/(2^q q!) \equiv (2q-1)! \) distinct terms, so that we obtain

\[
\Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})} = \frac{(2q-1)!}{b_{nq}} \Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})}.
\]

The proof of Eq. (C3) is completed by proving that

\[
\Delta^{(\mu_1 \mu_2 \cdots \mu_{2q-1} \mu_{2q} \mu_{2q+1} \cdots \mu_{nq})} = 2q + 1.
\]
Using the definition of the symmetrized tensor, 
\[ \Delta_{\mu_1 \mu_2} \ldots \Delta_{\mu_{2q+1} \mu_{2q+2}} \Delta_{(\mu_1 \mu_2) \ldots \Delta_{\mu_{2q+1} \mu_{2q+2}}} = \frac{2^{q+1}(q+1)!}{(2q+2)!} \Delta_{\mu_1 \mu_2} \ldots \Delta_{\mu_{2q+1} \mu_{2q+2}} \sum_{\tau} \Delta_{\mu_1 \mu_2} \ldots \Delta_{\mu_{2q+1} \mu_{2q+2}}. \]  
(C8)

Consider the contraction of \( \Delta_{\mu_{2q+1} \mu_{2q+2}} \) with the sum over distinct permutations of \( 2q+2 \) indices \( \mu_1, \ldots, \mu_{2q+2} \). There is one term in the sum where both indices are on the same \( \Delta \) projector. This term is \( \sim \Delta_{\mu_{2q+1} \mu_{2q+2}} \Delta_{\mu_{2q+1} \mu_{2q+2}} \equiv 3 \). Then, there are \( 2q \) terms where the indices \( \mu_{2q+1} \) and \( \mu_{2q+2} \) are on different projectors, say \( \Delta_{\mu_{2q+1} \mu_{2q+2}} \). Contracting with \( \Delta_{\mu_{2q+1} \mu_{2q+2}} \) gives a term \( \sim \Delta_{\mu_1 \mu_2} \), where both indices are from the set \( \mu_1, \ldots, \mu_{2q} \). Putting this together and using Eq. (C7) gives
\[ \Delta_{\mu_1 \mu_2} \ldots \Delta_{\mu_{2q+1} \mu_{2q+2}} \Delta_{(\mu_1 \mu_2) \ldots \Delta_{\mu_{2q+1} \mu_{2q+2}}} = \frac{2^{q+1}(q+1)!}{(2q+2)!} \Delta_{\mu_1 \mu_2} \ldots \Delta_{\mu_{2q-1} \mu_{2q}} (2q + 3) \sum_{\tau} \Delta_{\mu_1 \mu_2} \ldots \Delta_{\mu_{2q-1} \mu_{2q}} \equiv 2q + 3, \text{ q.e.d.} \]  
(C9)

With the orthogonality relation (C3), we now easily find by projecting Eq. (14) that
\[ I_{i+n,q} = \frac{(-1)^q}{(2q+1)!} \int dK^+ dK^- d\kappa \delta_{K^+ \kappa} \frac{1}{(2q+1)!} \Delta_{\mu_1 \mu_2} \ldots \Delta_{\mu_{2q-1} \mu_{2q}} (2q + 3) \sum_{\tau} \Delta_{\mu_1 \mu_2} \ldots \Delta_{\mu_{2q-1} \mu_{2q}} \equiv 2q + 3, \]  
(C10)

where we used the definition (43) of the thermodynamic average \( \langle \ldots \rangle_0 \), the second line yields Eq. (18).

Now we compute the thermodynamical integrals \( \tilde{I}_{i+j+n,j+r,q} \) in Eq. (26). In that equation, we introduced the symmetrized tensor products
\[ \Xi(\mu_1 \mu_2) \ldots \Xi(\mu_{2q-1} \mu_{2q}) \ldots \Xi(\mu_{2q+r} \mu_{2q+r+1}) \ldots \Xi(\mu_n) \equiv 1 \sum_{\tau} \Xi(\mu_1 \mu_2) \ldots \Xi(\mu_{2q-1} \mu_{2q}) \ldots \Xi(\mu_{2q+r} \mu_{2q+r+1}) \ldots \Xi(\mu_n), \]  
(C11)

where \( b_{n \mu} \) is the number of terms in the sum over distinct permutations of the \( n \) indices \( \mu_1, \ldots, \mu_n \). Again there are in total \( n! \) different permutations of the indices. There are \( q \) projection operators \( \Xi(\mu_1 \mu_2) \), \( r \) factors of \( \mu_{2q+r} \), \( n-r-2q \) factors of \( \mu_{2q+r+1} \), and of the \( \mu_{2q+r} \) among themselves to lead to distinct terms. Likewise, permutations of the two indices of the symmetric projection operator \( \Xi(\mu_1 \mu_2) \) do not lead to distinct terms. Thus the total number of distinct terms in the symmetrized tensor product is
\[ b_{n \mu} = \frac{n!}{2q! r! (n-r-2q)!} = \frac{n!}{2q! r! (n-r-2q)!}. \]  
(C12)

A suitable projection of the tensor \( \tilde{I}_{i+j+n,j+r,q} \) is now found by employing the orthogonality relation
\[ \Xi(\mu_1 \mu_2) \ldots \Xi(\mu_{2q-1} \mu_{2q}) \ldots \Xi(\mu_{2q+r} \mu_{2q+r+1}) \ldots \Xi(\mu_n) \Xi(\mu_1 \mu_2) \ldots \Xi(\mu_{2q-1} \mu_{2q}) \ldots \Xi(\mu_{2q+r} \mu_{2q+r+1}) \ldots \Xi(\mu_n) = (-1)^{2q} \delta_{q q'} \delta_{r r'} \].  
(C13)

In order to prove this relation, we first note that the Kronecker deltas are easily explained by the fact that if \( q \neq q' \) or \( r \neq r' \), there are terms where a \( \mu_{2q+r} \) or an \( \mu_{2q+r+1} \) are either contracted with each other or with projection operators \( \Xi(\mu_1 \mu_2) \), \( \Xi(\mu_1 \mu_2) \), which gives zero. We thus need to prove Eq. (C13) only for \( q = q' \), \( r = r' \). Again, since both sets of
indices are symmetrized, we may keep one set fixed, i.e.,
\[
\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_n}(\mu_1\mu_2\ldots\mu_n)
\]
\[
= \frac{1}{b_{nrq}}\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2n+1}}(\mu_1\mu_2\ldots\mu_{2n+1})
\]
\[
= \frac{1}{b_{nrq}} \left(-1\right)^r (2q)! \sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2n+2}}(\mu_1\mu_2\ldots\mu_{2n+2}),
\]
where in the next-to-last step we used the fact that only those permutations in the sum are non-vanishing where the indices \(\mu_{2q+1},\ldots,\mu_{2q+r}\) are on \(l^s\) and \(\mu_{2q+r+1},\ldots,\mu_{2n}\) are on \(u^s\). Then, we exploited \(u^su_\mu = 1\) and \(l^sl_\mu = -1\). We now prove by complete induction that
\[
\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2n-1}}(\mu_1\mu_2\ldots\mu_{2n-1}) = \left(\frac{2^q q!}{(2q)!}\right)^2.
\]
This holds obviously for \(q = 1\), since \(\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2} = 1 = 2^2 / 2\). We now assume that Eq. (C15) holds for \(q\) and prove it for \(q + 1\):
\[
\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+2}}(\mu_1\mu_2\ldots\mu_{2q+2})
\]
\[
= \frac{2q+1}{(2q+2)!} \sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+2}}(\mu_1\mu_2\ldots\mu_{2q+2}),
\]
Consider the contraction of \(\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+2}}(\mu_1\mu_2\ldots\mu_{2q+2})\) with the sum over distinct permutations of \(2q+1\) indices \(\mu_1,\ldots,\mu_{2q+2}\). There is one term in the sum where both indices are on the same \(\Xi\) projector. This term is \(\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+2}}(\mu_1\mu_2\ldots\mu_{2q+2}) \equiv 2\). Then, there are \(2q\) terms where the indices \(\mu_{2q+1}\) and \(\mu_{2q+2}\) are on different projectors, say \(\Xi_{\mu_{2q+1}}(\mu_{2q+2})\). Contracting with \(\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+1}}(\mu_1\mu_2\ldots\mu_{2q+1})\) gives a term \(\Xi_{\mu_1\mu_2} \cdots \Xi_{\mu_{2q+1}}\), where both indices are from the set \(\mu_1,\ldots,\mu_{2q}\). Putting this together and using Eq. (C15) gives
\[
\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+1}}(\mu_1\mu_2\ldots\mu_{2q+1})
\]
\[
= \frac{2q+1}{(2q+2)!} \sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+2}}(\mu_1\mu_2\ldots\mu_{2q+2}),
\]
\[
\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+1}}(\mu_1\mu_2\ldots\mu_{2q+1})
\]
\[
= \frac{2q+1}{(2q+2)!} \sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+2}}(\mu_1\mu_2\ldots\mu_{2q+2}),
\]
\[
\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+1}}(\mu_1\mu_2\ldots\mu_{2q+1})
\]
\[
= \frac{2q+1}{(2q+2)!} \sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+2}}(\mu_1\mu_2\ldots\mu_{2q+2}),
\]
which is Eq. (C15) for \(q + 1\). Now we insert Eq. (C15) into Eq. (C14) and obtain
\[
\sum_{(\mu)\in\mathbb{Z}_+^n} (\xi)_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+1}}(\mu_1\mu_2\ldots\mu_{2q+1})
\]
\[
= \frac{1}{b_{nrq}} \left(-1\right)^r (2q)! \left(\frac{2^q q!}{(2q)!}\right)^2
\]
\[
= \frac{1}{b_{nrq}} \left(-1\right)^r 2^q q!.
\]
With \((2q)! = 2^q q!\), we obtain Eq. (C13), q.e.d.

The thermodynamic integrals \(I_{i+j+n+j+r,q}\) are now obtained by a projection of Eq. (69). The orthogonality relation (C13) leads to the result
\[
I_{i+j+n+j+r,q} = \frac{(-1)^{q+r}}{(2q)!} \int dK E_{k_l}^{i+j+n} \langle \xi_{\mu_1\mu_2}\ldots\xi_{\mu_{2q+1}}(\mu_1\mu_2\ldots\mu_{2q+1}) \rangle_f \langle\langle\cdots\rangle\rangle.
\]
where we used the definition (58) of the tensor \(\hat{L}_{ij}^{\mu_1\cdots\mu_n}\). With the definition of the average \(\langle\langle\cdots\rangle\rangle_0\), the second line yields Eq. (62).
Note that since $E_{kn}^i = (k^i u_\mu)^i$ and $E_{kl}^j = (-k^i l_\mu)^j$ the tensors \((13), (58)\) immediately follow from the projection of higher-rank tensors on tensors built from $u$'s and $l$'s,

\[
\begin{align*}
J_{i}^{\mu_1 \cdots \mu_n} &= u_{(\mu_1} \cdots u_{\mu_i)} J_0^{(\mu_1 \cdots \mu_{n+i})}, \\
\bar{J}_{ij}^{\mu_1 \cdots \mu_n} &= (-1)^j u_{(\alpha_1} \cdots u_{\alpha_j} l_{\beta_1} \cdots l_{\beta_j)} \bar{J}_{40}^{(\alpha_1 \cdots \alpha_j \beta_1 \cdots \beta_j \mu_1 \cdots \mu_n)}.
\end{align*}
\tag{C20}
\]

Other useful relations are obtained from contracting two indices of the tensors \((13), (58)\) with a $\Delta$ resp. a $\Xi$ projector,

\[
\begin{align*}
J_{i}^{\mu_1 \cdots \mu_n} \Delta_{\mu_{n-1} \mu_n} &= m_0^2 J_{i}^{\mu_1 \cdots \mu_{n-2}} - \bar{J}_{i}^{\mu_1 \cdots \mu_{n-2}}, \\
\bar{J}_{ij}^{\mu_1 \cdots \mu_n} \Xi_{\mu_{n-1} \mu_n} &= m_0^2 \bar{J}_{ij}^{\mu_1 \cdots \mu_{n-2}} - \bar{\bar{J}}_{ij}^{\mu_1 \cdots \mu_{n-2}} + \bar{J}_{ij}^{\mu_1 \cdots \mu_{n-2}}.
\end{align*}
\tag{C22}
\]

The conventional and the generalized thermodynamic integrals \((16), (19), (22)\), and \((53)\) obey useful recursion relations, which are given here. Replacing \((\Delta^{\alpha \beta} k_\alpha k_\beta)^{q+1} = (\Delta^{\alpha \beta} k_\alpha k_\beta)^q (m_0^2 - E_{kn}^2)\) in Eq. \(19\) we obtain for $0 \leq q \leq n/2$,

\[
\begin{align*}
I_{n+2,q} &= m_0^2 I_{nq} + (2q + 3) I_{n+2,q+1}, \\
J_{n+2,q} &= m_0^2 J_{nq} + (2q + 3) J_{n+2,q+1}.
\end{align*}
\tag{C24}
\]

Correspondingly, using $\Xi^{\mu \nu} k_\mu k_\nu = m_0^2 - E_{kn}^2 + E_{kl}^2$ in Eq. \(52\) we get

\[
\begin{align*}
\hat{I}_{n+2,r,q} - \hat{I}_{n+2,r+2,q} &= m_0^2 \hat{I}_{nq} + (2q + 2) \hat{I}_{n+2,r,q+1}, \\
\hat{J}_{n+2,r,q} - \hat{J}_{n+2,r+2,q} &= m_0^2 \hat{J}_{nq} + (2q + 2) \hat{J}_{n+2,r,q+1}.
\end{align*}
\tag{C26}
\]

For $n = r = q = 0$ Eqs. \(C24), \(C26)\) read

\[
\begin{align*}
I_{20} &= m_0^2 I_{00} + 3I_{21}, \\
\hat{I}_{200} &= m_0^2 \hat{I}_{000} + \hat{I}_{220} + 2\hat{I}_{201}.
\end{align*}
\tag{C28}
\]

In the massless limit this leads to the familiar relations $c_0 = 3P_0$ and $\hat{c} = \hat{P}_1 + 2\hat{P}_2$.

**Appendix D: Thermodynamic integrals in the equilibrium limit**

In this appendix we derive some properties of the generalized moments \((58)\) and the corresponding generalized thermodynamic integrals \((32)\) in the limit of local thermodynamic equilibrium, see Eq. \((122)\).

The generalized moments \((122)\) can also be expanded in terms of the four-vectors $u^\mu$, $l^\mu$, and $\Xi^{\mu \nu}$, just as in Eq. \((60)\), where the corresponding thermodynamic integrals are defined similarly to Eq. \((32)\),

\[
I_{nq} = \frac{(-1)^q}{(2q)!} \int dK E_{kn}^{n-r-2q} E_{kl}^{2q} (\Xi^{\mu \nu} k_\mu k_\nu)^q f_{0k}.
\tag{D1}
\]

Making use of Eq. \((A9)\), of the binomial theorem, and of the definition of the double factorial for even arguments it is straightforward to obtain a relation between the thermodynamical integrals $I_{nq}$ and $I_{nrq}$,

\[
\begin{align*}
I_{nq} &= \frac{(-1)^q}{(2q + 1)!} \int dK E_{kn}^{n-r-2q} (\Xi^{\mu \nu} k_\mu k_\nu - E_{kn}^2)^q f_{0k} \\
&= \frac{1}{(2q + 1)!} \sum_{r=0}^{q} \frac{2q-r)!}{r!} I_{n,2r,q-r}.
\end{align*}
\tag{D2}
\]

E.g. for $q = 0, 1, 2$ we have

\[
\begin{align*}
I_{n0} &= I_{n00}, \\
I_{n1} &= \frac{1}{3} (2I_{n01} + I_{n20}), \\
I_{n2} &= \frac{1}{15} (8I_{n02} + 4I_{n21} + I_{n40}).
\end{align*}
\tag{D3}
\]

Note that the corresponding auxiliary thermodynamical integrals $J_{nqr}$ may be defined similarly to Eq. \((53)\), and obviously will lead to analogous relations.
Furthermore, using \( df_{0k}/dE_{kn} = -\beta_0 f_{0k} (1 - \alpha f_{0k}) \), after an integration by parts Eq. (49) can be rewritten as a relation between the conventional thermodynamic and auxiliary integrals,

\[
\beta_0 J_{nq} = I_{n-1,q-1} + (n-2q) I_{n-1,q} .
\]

Similarly, for the auxiliary thermodynamical integrals \( J_{n \nu q} \) we obtain (as long as \( r \geq 2, q \geq 1 \))

\[
\beta_0 J_{n \nu q} \equiv I_{n-1,r,q-1} + (n-r-2q) I_{n-1,r,q}
= (r-1) I_{n-1,r-2,q} + (n-r-2q) I_{n-1,r,q} .
\]

Comparing the right-hand sides, we obtain the identity

\[
I_{n-1,r,q-1} = (r-1) I_{n-1,r-2,q} .
\]

E.g., for \( n = 3, r = 2, q = 1 \) we obtain the equivalence of the longitudinal and transverse pressures in thermodynamical equilibrium,

\[
I_{220} = I_{201} .
\]

The main thermodynamic relations are also obtained from integration by parts, namely

\[
dI_{nq} (\alpha_0, \beta_0) \equiv \left( \frac{\partial I_{nq}}{\partial \alpha_0} \right)_{\beta_0} d\alpha_0 + \left( \frac{\partial I_{nq}}{\partial \beta_0} \right)_{\alpha_0} d\beta_0
= J_{nq} d\alpha_0 - J_{n+1,q} d\beta_0 ,
\]

and similarly

\[
dI_{n \nu q} (\alpha_0, \beta_0) \equiv \left( \frac{\partial I_{n \nu q}}{\partial \alpha_0} \right)_{\beta_0} d\alpha_0 + \left( \frac{\partial I_{n \nu q}}{\partial \beta_0} \right)_{\alpha_0} d\beta_0
= J_{n \nu q} d\alpha_0 - J_{n+1,r,q} d\beta_0 .
\]

**Appendix E: Irreducible projection operators**

In this appendix, we present the irreducible projection operators necessary to derive the irreducible moments of \( \delta f_k \) or \( \delta \tilde{f}_k \). We start by recalling the definition of the irreducible projection operators in the first case \([12, 18, 20]\),

\[
\Delta_{\mu_1, \ldots, \mu_n}^{\nu_1, \ldots, \nu_n} = \sum_{q=0}^{[n/2]} C(n, q) \frac{1}{\mathcal{N}_{nq}} \sum_{\mathcal{P}_\mu \mathcal{P}_\nu} \Delta_{\mu_1, \mu_2}^{\nu_1, \nu_2} \cdots \Delta_{\mu_{2q-1}, \mu_{2q}}^{\nu_{2q-1}, \nu_{2q}} \Delta_{\nu_{2q+1}, \nu_{2q+2}}^{\mu_{2q+1}, \mu_{2q+2}} \cdots \Delta_{\mu_n}^{\nu_n} .
\]

Here, \([n/2]\) denotes the largest integer less than or equal to \( n/2 \), the coefficients \( C(n, q) \) are defined as

\[
C(n, q) = (-1)^q \frac{(n!)^2}{(2n)!} \frac{(2n - 2q)!}{q!(n - q)!(n - 2q)!} .
\]

and the second sum in Eq. (E1) runs over all distinct permutations \( \mathcal{P}_\mu \mathcal{P}_\nu \) of \( \mu \)- and \( \nu \)-type indices. The coefficient in front of this sum is just the inverse of the total number of these distinct permutations,

\[
\mathcal{N}_{nq} \equiv \frac{1}{(n-2q)!} \left( \frac{n!}{(2q)!} \right)^2 .
\]

This number can be explained as follows: \((n!)^2\) is the number of all permutations of \( \mu \)- and \( \nu \)-type indices. In order to obtain the number of distinct permutations, one has to divide this by the number \((2n)!\) of permutations of \( \mu \)- and \( \nu \)-type indices on the same \( \Delta \) projectors (where only projectors with only \( \mu \)- and only \( \nu \)-type indices are considered), and by the number \((q!)^2\) of trivial reorderings of the sequence of these projectors. Finally, one also has to divide by the number \((n-2q)!\) of trivial reorderings of the sequence of projectors with mixed indices.

The projectors (E1) are symmetric under exchange of \( \mu \)- and \( \nu \)-type indices,

\[
\Delta_{\mu_1, \ldots, \mu_n}^{\nu_1, \ldots, \nu_n} = \Delta_{\nu_1, \ldots, \nu_n}^{\mu_1, \ldots, \mu_n} ,
\]

(E4)
and traceless with respect to contraction of either \( \mu \)- or \( \nu \)-type indices,
\[
\Delta_{\mu_1\cdots\mu_n} g_{\mu_i\mu_j} = \Delta_{\mu_1\cdots\mu_n} g^{\nu_i\nu_j} = 0 \quad \text{for any } i,j.
\]  
(E5)
Moreover, upon complete contraction,
\[
\Delta_{\mu_1\cdots\mu_n} \equiv \Delta_{\mu_1\cdots\mu_n}^{\nu_1\nu_2\cdots\nu_n} g_{\mu_1\nu_1} \cdots g_{\mu_n\nu_n} = 2n + 1,
\]  
(E6)

cf. Eq. (23) in chapter VI.2 of Ref. [12].

Analogously, in the case where we decompose tensors with respect to both \( u^\mu \) and \( l^\mu \) the irreducible projection operators read
\[
\Xi_{\mu_1\cdots\mu_n} = \sum_{q=0}^{[n/2]} \frac{1}{N_{nq}} \sum_{p_\mu p_\nu} \Xi_{\mu_1\mu_2} \cdots \Xi_{q-1\mu_2} \Xi_{q+1\nu_2} \cdots \Xi_{n\nu_2} \Xi_{q\nu_2} \Xi_{n-q\nu_2+1} \cdots \Xi_{n\nu_n}. 
\]  
(E7)
The coefficient in front of the sum over distinct permutations is the same as in Eq. (E1). The coefficient \( \hat{C}(n, q) \) will be determined below. Just as the projectors \( [E1] \), the projectors \( [E7] \) are also symmetric under the interchange of \( \mu \)- and \( \nu \)-type indices,
\[
\Xi_{\mu_1\cdots\mu_n} = \Xi_{(\mu_1\cdots\mu_n)},
\]  
(E8)
and traceless with respect to contraction of either \( \mu \)- or \( \nu \)-type indices,
\[
\Xi_{\mu_1\cdots\mu_n} g_{\mu_i\mu_j} = \Xi_{\mu_1\cdots\mu_n} g^{\nu_i\nu_j} = 0 \quad \text{for any } i,j.
\]  
(E9)

This relation forms the basis for the determination of the coefficients \( \hat{C}(n, q) \), as we shall show now. [The calculation closely follows that given in Ref. [12] for the coefficients \( C(n, q) \), cf. Eq. (E9)].

Without loss of generality, let us consider the contraction \( [E9] \) of the projector \( [E7] \) with respect to the two indices \( \mu_1, \mu_2 \). For the following arguments, it is advantageous to replace the sum over distinct permutations \( P_\mu P_\nu \) of the \( n \) \( \mu \)- and \( \nu \)-type indices by the sum over all permutations \( \mathcal{P}_\mu \mathcal{P}_\nu \) (with in total \( (n!)^2 \) different terms). In the various terms of this sum, the indices \( \mu_1, \mu_2 \) can appear in four different ways (for the sake of notational simplicity, we omit the other \( \Xi \) projectors in these terms):

1. \( \Xi_{\mu_1\mu_2} \Xi_{\mu_3\mu_2} \times g_{\mu_1\mu_2} \to \Xi_{\mu_1\mu_2} \cdot \Xi^{\mu_3\mu_2} \cdot \Xi^{2q(2q-2) \text{ terms}}, \)
2. \( \Xi_{\mu_1\mu_2} \Xi_{\mu_3\mu_2} \to \Xi_{\mu_1\mu_2} \cdot \Xi_{\mu_3\mu_2} \cdot 2 \cdot 2q(n-2q) \text{ terms}, \)
3. \( \Xi_{\mu_1\mu_2} \Xi_{\mu_3\mu_2} \to \Xi_{\mu_1\mu_2} \cdot \Xi_{\mu_3\mu_2} \cdot (n-2q) \cdot (n-2q-1) \text{ terms}, \)

The arrow symbolizes contraction with \( g_{\mu_1\mu_2} \), with the corresponding result shown to the right of the arrow. On the far right we also denoted the number of times that such terms occur in the sum over all permutations. Computing this number is a simple combinatorial exercise: for case (i), the index \( \mu_3 \) can appear in \( 2q \) different positions and the index \( \mu_2 \) in the remaining \( 2q-2 \) positions. For case (ii), there are \( 2q \) different positions for the index \( \mu_1 \), but then the position of \( \mu_2 \) is fixed. For case (iii), there are \( 2q \) positions for \( \mu_1 \) and \( n-2q \) positions for \( \mu_2 \). Interchanging \( \mu_1 \leftrightarrow \mu_2 \) gives another factor of 2. Finally, for case (iv) there are \( n-2q \) positions for \( \mu_1 \) and \( n-2q-1 \) remaining positions for \( \mu_2 \).

The cases (i) – (iii) generate terms of the form
\[
\Xi_{\mu_3\mu_4} \cdots \Xi_{q+1\mu_2} \Xi_{\mu_3\mu_2} \cdots \Xi_{n\mu_n},
\]  
while case (iv) generates terms of the form
\[
\Xi_{\mu_3\mu_4} \cdots \Xi_{2q+3\mu_2} \Xi_{\mu_3\mu_2} \cdots \Xi_{n\mu_n}.
\]  
(E10)

Note that, because of the contraction of \( \mu_1, \mu_2 \), only \( n-2 \) indices of the \( \mu \)-type are to be permuted. Collecting all terms (i) – (iv) with the correct prefactors, Eq. (E10) reads
\[
0 = \sum_{q=0}^{[n/2]} \hat{C}(n, q) \frac{1}{(n!)^2} \sum_{\mathcal{P}_\mu \mathcal{P}_\nu} \left\{ 2q [2q - 2 + 2 + 2(n - 2q)] \Xi_{\mu_1\mu_2} \cdots \Xi_{2q-3\mu_2} \Xi_{\mu_3\mu_2} \cdots \Xi_{n\mu_n} + (n - 2q)(n-2q-1) \Xi_{\mu_1\mu_2} \cdots \Xi_{2q+1\mu_2} \Xi_{\mu_3\mu_2} \cdots \Xi_{n\mu_n} \right\},
\]  
(E12)
where we relabelled the \( \mu \)-type indices \( \mu_i \to \mu_{i-2} \). We now observe that the \( q = 0 \)-term does not contribute to the first term in curly brackets, while the \( q = [n/2] \)-term does not contribute to the last term (no matter whether \( n \) is even or odd). Thus, we may write

\[
0 = \frac{1}{(n!)^2} \sum_{p_\mu \cdots p_{\nu}} \left\{ \sum_{q=1}^{[n/2]} \left[ \frac{[n/2]!}{(n-2q)!} \cdot \frac{2q(2n-2q)!}{(2n-2q-2)!} \cdot \frac{(n-q)!}{q!(n-2q)!} \cdot \sum_{\nu_1} \cdots \sum_{\nu_{n-2}} \sum_{\mu_1} \cdots \sum_{\mu_{n-2}} g_{\mu_1} \cdots g_{\mu_{n-2}} \right] \right\} .
\]

(E13)

Substituting \( q \to q - 1 \) in the second sum, we observe that the product of the \( \Xi \)-projectors is actually identical in both sums, so that we obtain

\[
0 = \sum_{q=1}^{[n/2]} \left[ \frac{[n/2]!}{(n-2q)!} \cdot \frac{2q(2n-2q)!}{(2n-2q-1)!} \cdot \frac{(n-q)!}{q!(n-2q)!} \cdot \sum_{\nu_1} \cdots \sum_{\nu_{n-2}} \sum_{\mu_1} \cdots \sum_{\mu_{n-2}} g_{\mu_1} \cdots g_{\mu_{n-2}} \right] .
\]

(E14)

In order to fulfill this relation, we have to demand that the term in brackets vanishes, which leads to the recursion relation

\[
\hat{C}(n, q) = - \frac{(n-2q+2)(n-2q+1)}{2q(2n-2q)} \hat{C}(n, q-1) .
\]

(E15)

If we set

\[
\hat{C}(n, 0) \equiv 1 \quad \text{for all } n ,
\]

(E16)

the solution is

\[
\hat{C}(n, q) = (-1)^q \frac{1}{2q!} \frac{n!}{(n-2q)!} \frac{(2n-2q-2)!}{(2n-2q)!} = (-1)^q \frac{1}{4^q} \frac{(n-q)!}{q!(n-2q)!} \frac{n}{n-q} ,
\]

(E17)

where we used the definition of the double factorial for even numbers.

Upon complete contraction,

\[
\Xi_{\mu_1 \cdots \mu_n} = \Xi_{\nu_1 \cdots \nu_n} g_{\mu_1} \cdots g_{\mu_n} = 2 .
\]

(E18)

In order to prove this relation, it is advantageous to first prove the recursion relation

\[
\Xi_{\nu_1 \cdots \nu_n} g_{\mu_1} = \Xi_{\nu_1 \cdots \nu_{n-1}} ,
\]

(E19)

valid for any \( i, j \) and \( n > 1 \). Equation \( \text{(E19)} \) then immediately follows on account of \( \Xi_{\mu_1 \cdots \mu_n} = \Xi_{\mu_1} = 2 \).

Equation \( \text{(E19)} \) is proved as follows. First note that, since \( \Xi_{\mu_1 \cdots \mu_n} \) is symmetric in all indices, we may without loss of generality choose \( i = j = n \) in Eq. \( \text{(E19)} \). Then, as already done in the derivation of Eq. \( \text{(E17)} \), it is advantageous to replace in Eq. \( \text{(E17)} \) the sum over distinct permutations \( p_\mu p_\nu \) by the sum over all permutations \( p_\mu p_\nu \) (with in total \( (n!)^2 \) different terms). Inserting this into the left-hand side of Eq. \( \text{(E19)} \), we see that contraction of the indices \( \mu_n, \nu_n \) generates five different types of terms:

(i) \( \Xi_{\nu_1 \cdots \nu_n} g_{\mu_1} \to \Xi_{\nu_1}^{\mu_1} \), \( 2q \cdot 2q \) terms,

(ii) \( \Xi_{\nu_1 \cdots \nu_n} g_{\mu_1} \to \Xi_{\nu_1}^{\mu_1} \), \( 2q(n-2q) \) terms,

(iii) \( \Xi_{\nu_1 \cdots \nu_n} g_{\mu_1} \to \Xi_{\nu_1}^{\nu_1} \), \( 2q(n-2q) \) terms,

(iv) \( \Xi_{\nu_1 \cdots \nu_n} g_{\mu_1} \to \Xi_{\nu_1}^{\nu_1} \), \( (n-2q)(n-2q-1) \) terms,

(v) \( \Xi_{\nu_1} \to 2 \), \( (n-2q) \) terms.
The number of terms is easily explained as follows: in case (i), there are \(2q\) possible positions for the index \(\mu\) and \(2q\) possible positions for the index \(\nu\). In cases (ii) and (iii), there are \(2q\) positions for the index \(\mu\) and \(n - 2q\) resp. \(2q\) positions for the index \(\nu\). In case (iv), there are \(n - 2q\) positions for the index \(\mu\) and a remaining \(n - 2q - 1\) positions for the index \(\nu\). Finally, in case (v) there are \(n - 2q\) possibilities (equal to the number of projectors with mixed indices) to have the indices \(\mu\) and \(\nu\) occurring at the same projector. One observes that upon contraction, in case (i) (and after suitably relabelling indices) one generates terms of the form

\[
\Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-3} \mu_{2q-2}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-3} \nu_{2q-2}} \Xi^{\mu_{2q-1}} \cdots \Xi^{\mu_{n-1}},
\]

i.e., terms with \(q - 1\) projectors \(\Xi^{\mu_i \mu_i}, q - 1\) projectors \(\Xi_{\nu_i \nu_i}\), and \(n - 2q + 1\) projectors \(\Xi^\mu\). On the other hand, in all other cases (ii) – (v) one generates terms of the form

\[
\Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-1} \mu_{2q}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-2} \nu_{2q}} \Xi^{\mu_{2q+1}} \cdots \Xi^{\mu_{n-1}},
\]

i.e., terms with \(q\) projectors \(\Xi^{\mu_i \mu_i}, q\) projectors \(\Xi_{\nu_i \nu_i}\), and \(n - 2q - 1\) projectors \(\Xi^\mu\).

To proceed, it is advantageous to consider the case of \(n\) even and \(n\) odd separately. Let us first focus on the (somewhat simpler) case of \(n\) even, where \([n/2] = n/2\). Collecting the results obtained so far, the left-hand-side of Eq. (E19) can be written as

\[
\Xi_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_n} g_{\mu_n} = \sum_{q=0}^{n/2} \hat{C}(n, q) \frac{1}{(n!)^2} \sum_{\rho_r^{-1} \rho_c^{-1}} \left[ (n - 2q)(n + 2q + 1) \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-1} \mu_{2q}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-2} \nu_{2q}} \Xi^{\mu_{2q+1}} \cdots \Xi^{\mu_{n-1}} + 4q^2 \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-3} \mu_{2q-2}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-3} \nu_{2q-2}} \Xi^{\mu_{2q-1}} \cdots \Xi^{\mu_{n-1}} \right].
\]

We now note that the first term in brackets does not contribute for \(q = n/2\), while the second term does not contribute for \(q = 0\),

\[
\Xi_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_n} g_{\mu_n} = \sum_{q=0}^{n/2-1} \hat{C}(n, q) (n - 2q)(n + 2q + 1) \frac{1}{(n!)^2} \sum_{\rho_r^{-1} \rho_c^{-1}} \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-1} \mu_{2q}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-2} \nu_{2q}} \Xi^{\mu_{2q+1}} \cdots \Xi^{\mu_{n-1}} + \sum_{q=1}^{n/2} \hat{C}(n, q) 4q^2 \frac{1}{(n!)^2} \sum_{\rho_r^{-1} \rho_c^{-1}} \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-3} \mu_{2q-2}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-3} \nu_{2q-2}} \Xi^{\mu_{2q-1}} \cdots \Xi^{\mu_{n-1}}.
\]

Substituting the summation index \(q \to q + 1\) in the second sum, we observe that the product of projectors becomes identical to the one in the first sum, so that we can write

\[
\Xi_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_n} g_{\mu_n} = \sum_{q=0}^{n/2-1} \hat{C}(n, q) (n - 2q)(n + 2q + 1) + \hat{C}(n, q + 1) 4(q + 1)^2 \frac{1}{(n!)^2} \sum_{\rho_r^{-1} \rho_c^{-1}} \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-1} \mu_{2q}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-2} \nu_{2q}} \Xi^{\mu_{2q+1}} \cdots \Xi^{\mu_{n-1}}.
\]

With the definition (E17) one now convinces oneself that

\[
\frac{1}{n!} \left[ \hat{C}(n, q) (n - 2q)(n + 2q + 1) + \hat{C}(n, q + 1) 4(q + 1)^2 \right] = \hat{C}(n - 1, q).
\]

Reverting the sum over all permutations of \(n - 1\) indices of \(\mu\)– and of \(\nu\)-type to the one over distinct permutations and using the definition (E17) and the fact that for even \(n\) one has \(n/2 - 1 = [(n - 1)/2]\), one then arrives at Eq. (E19).

Let us now consider the case of \(n\) odd, where \([n/2] = [(n - 1)/2] = (n - 1)/2\). For \(q < (n - 1)/2\), the arguments of the previous case of \(n\) even can be taken over unchanged. However, when \(q = (n - 1)/2\) things become more subtle: in the cases (ii), (iii), and (iv), one observes that after contraction of the indices, no further projectors of the type \(\Xi_{\nu_1}^\nu\) occur. We thus treat the case \(q = (n - 1)/2\) separately. Collecting the results obtained so far, the left-hand-side
of Eq. (E19) can be written as
\[ \Xi_{\mu_1 \cdots \mu_n} \Xi_{\nu_1 \cdots \nu_n} = \sum_{q=0}^{(n-1)/2-1} \frac{1}{(n!)^2} \sum_{p_{\mu} \cdot p_{\nu}^{-1}} \left[ (n-2q)(n+2q+1) \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-1} \mu_{2q}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-1} \nu_{2q}} \Xi^{\mu_{2q+1} \cdots \mu_n} + 4q^2 \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{2q-2} \mu_{2q}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{2q-2} \nu_{2q}} \Xi^{\mu_{2q+1} \cdots \mu_n} - \right. 
\left. + (n-1)^2 \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_n-2 \mu_n-1} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{n-2} \nu_{n-1}} \right]. \tag{E24} \]

Noting that the term in the second line does not contribute for \( q = 0 \), we may combine the terms in the second and fourth line to a sum that runs from \( q = 1 \) to \( (n-1)/2 \). Substituting \( q \to q+1 \) in that sum, we observe that this sum can be combined with the one in the first line, similar to Eq. (E22). With Eq. (E23) this yields
\[ \Xi_{\mu_1 \cdots \mu_n} \Xi_{\nu_1 \cdots \nu_n} = \sum_{q=0}^{(n-1)/2-1} \frac{1}{(n!)^2} \sum_{p_{\mu} \cdot p_{\nu}^{-1}} \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{n-2} \mu_{n-1}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{n-2} \nu_{n-1}} + \frac{2}{n} \Xi^{\mu_1 \mu_2} \cdots \Xi^{\mu_{n-2} \mu_{n-1}} \Xi_{\nu_1 \nu_2} \cdots \Xi_{\nu_{n-2} \nu_{n-1}} \tag{E25} \]

With Eq. (E17) one proves that
\[ \frac{2}{n} \hat{C} \left( n, \frac{n-1}{2} \right) = \hat{C} \left( n-1, \frac{n-1}{2} \right). \tag{E26} \]

Then, the last term in Eq. (E25) just represents the missing \( q = (n-1)/2 \)-term of the sum and we again obtain Eq. (E21), q.e.d.

Note that relations (E20) and (E18) mean that the projection of an arbitrary tensor of rank \( n \) with respect to either (E11) or (E7), i.e., \( A^\rho_{\mu_1 \cdots \mu_n} \Delta_{\nu_1 \cdots \nu_n} \) or \( A^\rho_{\mu_1 \cdots \mu_n} \Xi_{\nu_1 \cdots \nu_n} \), has \( 2n+1 \) or \( 2 \) independent tensor components, respectively.

In order to prove this, we note that an arbitrary tensor \( A^\rho_{\mu_1 \cdots \mu_n} \) of rank \( n \) in \( d \)-dimensional space-time has \( d^n \) independent components, because each of the \( n \) indices can assume \( d \) distinct values. Now consider a rank-\( n \) tensor which is completely symmetric with respect to the interchange of indices. This tensor can be constructed from the arbitrary tensor \( A^\rho_{\mu_1 \cdots \mu_n} \) via symmetrization,
\[ A^{\mu_1 \cdots \mu_n}_{\rho} = \frac{1}{n!} \sum_{\rho_{\mu}} A^\rho_{\mu_1 \cdots \mu_n}, \tag{E27} \]
where the sum over \( \rho_{\mu} \) runs over all \( n! \) permutations of the \( \mu \)-type indices. The number of independent tensor components of such a symmetric tensor is given by the number of combinations with repetition to draw \( n \) elements from a set of \( d \) elements,
\[ N_{dn} \left( A^{\mu_1 \cdots \mu_n}_{\rho} \right) = \frac{(n+d-1)!}{n! (d-1)!}. \tag{E28} \]

Let us now demand in addition that this tensor is traceless,
\[ 0 = A^{(\mu_1 \cdots \mu_n)}_{\rho} g_{\mu_1-\mu_n} = A^{(\mu_1 \cdots \mu_{n-2})}_{\rho}, \tag{E29} \]
where the right-hand side defines a new symmetric tensor of rank \( n-2 \). According to Eq. (E28), this tensor has
\[ N_{dn} \left( A^{(\mu_1 \cdots \mu_{n-2})}_{\rho} \right) = \frac{(n+d-3)!}{(n-2)! (d-1)!}, \tag{E30} \]
independent components. This is also the number of constraints by which the number of independent components of the original symmetric tensor \( A^\rho_{(\mu_1 \cdots \mu_n)} \) is reduced, if we demand that it is traceless in addition to being symmetric.
Thus, the number of independent components of a symmetric traceless tensor is

\[
N_{dn} \left( A_d^{(\mu_1 \cdots \mu_n)} \right) = N_{dn} \left( A_d^{(\mu_1 \cdots \mu_n)} \right) - N_{dn} \left( A_d^{(\mu_1 \cdots \mu_n-2)} \right)
\]

\[
= \frac{(n + d - 1)!}{n! (d-1)!} \left[ \frac{(n + d - 3)!}{(n-2)! (d-1)!} - \frac{(n + d - 3)!}{n! (d-2)!} \right] \left( 2n + d - 2 \right).
\]  

(E31)

Let us now require in addition that such a symmetric traceless tensor is orthogonal to a given four-vector \( u^\mu \),

\[
0 = A_d^{(\mu_1 \cdots \mu_n)} u_{\mu_n} = A_{d, tr}^{(\mu_1 \cdots \mu_n-1)}.
\]

(E32)

The right-hand side defines a new symmetric traceless tensor of rank \( n - 1 \) which, according to Eq. (E31), has

\[
N_{dn} \left( A_{d, tr}^{(\mu_1 \cdots \mu_n-1)} \right) = \frac{(n + d - 4)!}{(n-1)! (d-2)!} \left( 2n + d - 4 \right)
\]

(E33)

independent components. This number reduces the number of independent components of the original symmetric traceless tensor, if we demand in addition that it is orthogonal to \( u^\mu \); thus the latter has

\[
N_{dn} \left( A_{d, tr, \text{ortho}}^{(\mu_1 \cdots \mu_n)} \right) = N_{dn} \left( A_{d, tr}^{(\mu_1 \cdots \mu_n)} \right) - N_{dn} \left( A_{d, tr}^{(\mu_1 \cdots \mu_n-1)} \right)
\]

\[
= \frac{(n + d - 3)!}{n! (d-3)!} \left(2n + d - 2\right) - \frac{(n + d - 4)!}{(n-1)! (d-2)!} \left(2n + d - 4\right)
\]

\[
= \frac{(n + d - 4)!}{n! (d-3)!} \left(2n + d - 3\right)
\]

(E34)

independent components. Comparing this equation to Eq. (E31) we realize that the orthogonality constraint has effectively reduced the number of dimensions by one unit, \( d \rightarrow d - 1 \). Subsequently demanding orthogonality to another four-vector \( l^\mu \) would reduce the number of dimensions by another unit, etc.

Now taking \( d = 4 \), Eq. (E34) tells us that any symmetric traceless tensor of rank \( N \), which is orthogonal to \( u^\mu \), has

\[
N_{4n} \left( A_{d=4, tr, \text{ortho}}^{(\mu_1 \cdots \mu_n)} \right) = 2n + 1 \text{ independent components.}
\]

If this tensor is in addition orthogonal to another four-vector \( l^\mu \), then Eq. (E34) applies replacing \( d = 4 \) by \( d = 3 \), and we obtain

\[
N_{3n} \left( A_{d=3, tr, \text{ortho}}^{(\mu_1 \cdots \mu_n)} \right) = 2 \text{ independent components.}
\]

This result is independent of the tensor rank \( n \).

In the following, we list the irreducible projection operators which are necessary for the derivation of Eqs. (110) – (112). In the case of rank-one tensors, the irreducible projection operator (E1) is trivially given by

\[
\Delta^{\mu_1 \nu_1} = C(1,0) \frac{1}{N_{10}} \Delta^{\mu_1 \nu_1},
\]

(E35)

with \( C(1,0) = N_{10} = 1 \). Analogously, the irreducible projection operator (E7) is

\[
\Xi^{\mu_1 \nu_1} = \hat{C}(1,0) \frac{1}{N_{10}} \Xi^{\mu_1 \nu_1},
\]

(E36)

with \( \hat{C}(1,0) = N_{10} = 1 \).

For rank-two tensors, the irreducible projection operator (E9) is

\[
\Delta^{\mu_1 \nu_2 \nu_1 \nu_2} = C(2,0) \frac{1}{N_{20}} \left( \Delta^{\mu_1 \nu_1} \Delta^{\nu_2 \nu_2} + \Delta^{\mu_1 \nu_2} \Delta^{\nu_1 \nu_1} \right) + C(2,1) \frac{1}{N_{21}} \Delta^{\mu_1 \nu_2} \Delta^{\nu_1 \nu_2}
\]

\[
= \Delta^{\mu_1 \nu_1} \Delta^{\nu_2 \nu_2} - \frac{1}{3} \Delta^{\mu_1 \nu_2} \Delta^{\nu_1 \nu_2}.
\]

(E37)

Analogously, the irreducible projection operator (E7) reads

\[
\Xi^{\mu_1 \nu_2 \nu_1 \nu_2} = \hat{C}(2,0) \frac{1}{N_{20}} \left( \Xi^{\mu_1 \nu_1} \Xi^{\nu_2 \nu_2} + \Xi^{\mu_1 \nu_2} \Xi^{\nu_1 \nu_1} \right) + \hat{C}(2,1) \frac{1}{N_{21}} \Xi^{\mu_1 \nu_2} \Xi^{\nu_1 \nu_2}
\]

\[
= \Xi^{\mu_1 \nu_1} \Xi^{\nu_2 \nu_2} - \frac{1}{2} \Xi^{\mu_1 \nu_2} \Xi^{\nu_1 \nu_2}.
\]

(E38)

We also give the irreducible projection operators for rank-three tensors:

\[
\Delta^{\mu_1 \nu_2 \nu_3 \nu_1 \nu_2 \nu_3} = \frac{1}{3} \left( \Delta^{\mu_1 \nu_1} \Delta^{\nu_2 \nu_3} \Delta^{\nu_3 \nu_1} + \Delta^{\mu_1 \nu_2} \Delta^{\nu_1 \nu_3} \Delta^{\nu_3 \nu_2} + \Delta^{\mu_1 \nu_3} \Delta^{\nu_1 \nu_2} \Delta^{\nu_2 \nu_3} \right)
\]

\[
- \frac{3}{5} \Delta^{\mu_1 \nu_2} \Delta^{\mu_3 \nu_3} \Delta^{\nu_1 \nu_2}.
\]

(E39)
and
\[
\Xi_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} = \frac{1}{3} \left( \Xi_{\mu_1 \mu_2 \mu_3 (\nu_2 \equiv \nu_3) \nu_3} + \Xi_{\mu_1 \mu_2 \mu_2 (\nu_1 \equiv \nu_3) \nu_3} + \Xi_{\mu_1 \mu_3 \mu_2 (\nu_2 \equiv \nu_1) \nu_3} \right) \\
- \frac{3}{4} \Xi_{(\mu_1 \mu_2 \equiv \mu_3) (\nu_3 \equiv \nu_1 \nu_2)} .
\]  
(E40)

Finally, for rank-four tensors, we obtain for the irreducible projection operators:
\[
\Delta_{\mu_1 \mu_2 \mu_3 \mu_4 \nu_1 \nu_2 \nu_3 \nu_4} = \frac{1}{4!} \sum_{P \neq P_0} \Delta_{\mu_1 \nu_1} \Delta_{\mu_2 \nu_2} \Delta_{\mu_3 \nu_3} \Delta_{\mu_4 \nu_4} \\
- \frac{3}{4} \Delta_{(\mu_1 \mu_2 \equiv \mu_3) (\nu_3 \equiv \nu_1 \nu_2 \nu_4) \nu_4} \\
- \frac{3}{4} \Delta_{(\mu_1 \mu_2 \equiv \mu_4) (\nu_1 \nu_2 \equiv \nu_3 \nu_4) \nu_3} \\
- \frac{3}{4} \Delta_{(\mu_1 \mu_3 \equiv \mu_4) (\nu_1 \nu_3 \equiv \nu_2 \nu_4) \nu_2} \\
+ \frac{3}{35} \Delta_{\nu_1 (\mu_2 \equiv \mu_3 \equiv \mu_4) (\nu_2 \equiv \nu_3 \nu_4) \nu_4} ,
\]  
(E41)

and
\[
\Xi_{\mu_1 \mu_2 \mu_3 \mu_4 \nu_1 \nu_2 \nu_3 \nu_4} = \frac{1}{4!} \sum_{P \neq P_0} \Xi_{\mu_1 \mu_2 \mu_3 \mu_4 \nu_1 \nu_2 \nu_3 \nu_4} \\
- \frac{1}{4} \Xi_{(\mu_1 \mu_2 \equiv \mu_4) (\nu_1 \equiv \nu_2 \nu_3 \nu_4) \nu_3} \\
- \frac{1}{4} \Xi_{(\mu_1 \mu_3 \equiv \mu_4) (\nu_1 \equiv \nu_2 \nu_3 \nu_4) \nu_2} \\
+ \frac{1}{8} \Xi_{(\mu_1 \mu_2 \equiv \mu_3 \equiv \mu_4) (\nu_1 \equiv \nu_2 \nu_3 \nu_4) \nu_4} .
\]  
(E42)

In both expressions, the first sum runs over all distinct permutations of the four \( \mu \)- and the four \( \nu \)-type indices.

The irreducible projection operators \([21]\) acting on tensors formed by the \( n \)-adic product of a four-vector, i.e., \( \mathbf{A}^{\mu_1 \cdots \mu_n} = A^{\mu_1} \cdots A^{\mu_n} \), lead to the following expressions for \( n = 1, \ldots, 4 \):
\[
A^{(\mu_1)} = \Delta_{\mu_1 \nu_1} A_{\nu_1} ,
\]  
(E43)
\[
A^{(\mu_1, \mu_2)} = \Delta_{\mu_1 \mu_2 \nu_1 \nu_2} A_{\nu_1} A_{\nu_2} = A^{(\mu_1)} A^{(\mu_2)} - \frac{1}{3} \Delta_{\mu_1 \mu_2} (\Delta_{\alpha \beta} A_{\alpha} A_{\beta}) ,
\]  
(E44)
\[
A^{(\mu_1, \mu_2, \mu_3)} = \Delta_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} A_{\nu_1} A_{\nu_2} A_{\nu_3} = A^{(\mu_1)} A^{(\mu_2)} A^{(\mu_3)} - \frac{3}{5} \Delta_{(\mu_1 \mu_2 \equiv \mu_3) (\Delta_{\alpha \beta} A_{\alpha} A_{\beta})} ,
\]  
(E45)
\[
A^{(\mu_1, \mu_2, \mu_3, \mu_4)} = \Delta_{\mu_1 \mu_2 \mu_3 \mu_4 \nu_1 \nu_2 \nu_3 \nu_4} A_{\nu_1} A_{\nu_2} A_{\nu_3} A_{\nu_4} \\
= A^{(\mu_1, \mu_2)} A^{(\mu_3)} A^{(\mu_4)} - \frac{3}{14} \Delta_{(\mu_1 \mu_2 \equiv \mu_3) A_{\alpha} A_{\beta}} (\Delta_{\alpha \beta} A_{\alpha} A_{\beta}) \\
- \frac{3}{4} \Delta_{(\mu_1 \mu_2 \equiv \mu_4) A_{\alpha \beta}} (\Delta_{\alpha \beta} A_{\alpha} A_{\beta}) - \frac{3}{14} \Delta_{(\mu_1 \mu_2 \equiv \mu_3) A_{\alpha} A_{\beta}} (\Delta_{\alpha \beta} A_{\alpha} A_{\beta}) \\
- \frac{3}{35} \Delta_{(\mu_1 \mu_2 \equiv \mu_3 \equiv \mu_4) (\Delta_{\alpha \beta} A_{\alpha} A_{\beta})^2} .
\]  
(E46)

Similarly, for the irreducible projection operators \([27]\) we obtain
\[
A^{(\mu_1)} = \Xi_{\mu_1 \nu_1} A_{\nu_1} ,
\]  
(E47)
\[
A^{(\mu_1, \mu_2)} = \Xi_{\mu_1 \mu_2 \nu_1 \nu_2} A_{\nu_1} A_{\nu_2} = A^{(\mu_1)} A^{(\mu_2)} - \frac{1}{2} \Xi_{\mu_1 \mu_2} (\Xi_{\alpha \beta} A_{\alpha} A_{\beta}) ,
\]  
(E48)
\[
A^{(\mu_1, \mu_2, \mu_3)} = \Xi_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} A_{\nu_1} A_{\nu_2} A_{\nu_3} = A^{(\mu_1)} A^{(\mu_2)} A^{(\mu_3)} - \frac{3}{4} \Xi_{(\mu_1 \mu_2 \equiv \mu_3) (\Xi_{\alpha \beta} A_{\alpha} A_{\beta})} ,
\]  
(E49)
\[
A^{(\mu_1, \mu_2, \mu_3, \mu_4)} = \Xi_{\mu_1 \mu_2 \mu_3 \mu_4 \nu_1 \nu_2 \nu_3 \nu_4} A_{\nu_1} A_{\nu_2} A_{\nu_3} A_{\nu_4} \\
= A^{(\mu_1)} A^{(\mu_2, \mu_3, \mu_4)} A^{(\mu_4)} - \frac{1}{4} \Xi_{(\mu_1 \mu_2 \equiv \mu_3 \equiv \mu_4) A_{\alpha} A_{\beta}} (\Xi_{\alpha \beta} A_{\alpha} A_{\beta}) \\
- \frac{1}{4} \Xi_{(\mu_1 \mu_2 \equiv \mu_4) A_{\alpha \beta}} (\Xi_{\alpha \beta} A_{\alpha} A_{\beta}) - \frac{1}{4} \Xi_{(\mu_1 \mu_2 \equiv \mu_3 \equiv \mu_4) A_{\alpha} A_{\beta}} (\Xi_{\alpha \beta} A_{\alpha} A_{\beta}) \\
- \frac{1}{8} \Xi_{(\mu_1 \mu_2 \equiv \mu_3 \equiv \mu_4) (\Xi_{\alpha \beta} A_{\alpha} A_{\beta})^2} .
\]  
(E50)
Appendix F: Orthogonality properties and conditions

In this appendix, we derive the orthogonality conditions \[^{[83]}\] and \[^{[93]}\]. The derivation utilizes the relations

\[ k^{(\mu_1} k^{\mu_2} k_{(\mu_1} \cdots k_{\mu_\ell)} = \frac{\ell!}{(2\ell - 1)!!} (\Delta^{\alpha\beta} k_{(\alpha} k_{\beta)})^\ell , \]  
\[ (F1) \]

and

\[ k^{(\mu_1} k^{\mu_2} k_{(\mu_1} \cdots k_{\mu_\ell)} = \frac{1}{2\ell - 1} (\Xi^{\alpha\beta} k_{\alpha} k_{\beta})^\ell . \]  
\[ (F2) \]

The first relation is proved as follows. We first note that

\[ k^{(\mu_1} k^{\mu_2} k_{(\mu_1} \cdots k_{\mu_\ell)} = \Delta^{\mu_1 \cdots \mu_\ell}_{\beta_1 \cdots \beta_\ell} \Delta^{\alpha_1 \cdots \alpha_\ell}_{\mu_1 \cdots \mu_\ell} k_{\alpha_1} \cdots k_{\alpha_\ell} k_{\beta_1} \cdots k_{\beta_\ell} = \Delta^{\alpha_1 \cdots \alpha_\ell}_{\beta_1 \cdots \beta_\ell} k_{\alpha_1} \cdots k_{\alpha_\ell} k_{\beta_1} \cdots k_{\beta_\ell} . \]  
\[ (F3) \]

Now we insert the explicit form \[^{[51]}\] of the projection operator and note that the contraction of all indices with the momenta reduces the second sum (including the prefactor \(1/N_{\nu q}\)) to just a factor of \((\Delta^{\alpha\beta} k_{\alpha} k_{\beta})^\ell\),

\[ k^{(\mu_1} k^{\mu_2} k_{(\mu_1} \cdots k_{\mu_\ell)} = \sum_{q_0}^{|\ell/2|} C(\ell, q) (\Delta^{\alpha\beta} k_{(\alpha} k_{\beta)})^\ell . \]  
\[ (F4) \]

The Legendre polynomials \(P_\ell(z)\) have the representation \[^{[50]}\]

\[ P_\ell(z) = \frac{1}{2^\ell} \sum_{q=0}^{|\ell/2|} (-1)^q \frac{(2\ell - 2q)!}{q!(\ell - q)! (\ell - 2q)!} z^{\ell - 2q} = \frac{1}{2^\ell \ell!(\ell)!^2} \sum_{q=0}^{|\ell/2|} C(\ell, q) z^{\ell - 2q} . \]  
\[ (F5) \]

Since for all \(\ell\)

\[ 1 \equiv P_\ell(1) = \frac{1}{2^\ell \ell!(\ell)!^2} \sum_{q=0}^{|\ell/2|} C(\ell, q) , \]  
\[ (F6) \]

we derive the identity

\[ \sum_{q=0}^{|\ell/2|} C(\ell, q) = \frac{2^\ell \ell!(\ell)!^2}{\ell!(\ell - 1)!^2} = \frac{\ell!}{2\ell - 1} \frac{\ell!}{(\ell! - 2)!} = \frac{\ell!}{(\ell! - 1)!} , \]  
\[ (F7) \]

where we have used the definition of the double factorial for odd numbers. Inserting this into Eq. \[^{[F4]}\] proves Eq. \[^{[F1]}\].

We now prove Eq. \[^{[F2]}\]. Analogously to Eq. \[^{[F4]}\] we obtain

\[ k^{(\mu_1} k^{\mu_2} k_{(\mu_1} \cdots k_{\mu_\ell)} = \sum_{q_0}^{|\ell/2|} \tilde{C}(\ell, q) (\Xi^{\alpha\beta} k_{\alpha} k_{\beta})^\ell . \]  
\[ (F8) \]

The Chebyshev polynomial of the first kind \(T_\ell(z)\) has the representation \[^{[51]}\]

\[ T_\ell(z) = \frac{\ell}{2} \sum_{q=0}^{|\ell/2|} (-1)^q \frac{(\ell - q - 1)!}{q!(\ell - 2q)!} (2z)^{\ell - 2q} = 2^{\ell - 1} \sum_{q=0}^{|\ell/2|} (-1)^q \frac{1}{4^q} \frac{(\ell - q)!}{q!(\ell - 2q)!} \frac{\ell}{\ell - q} z^{\ell - 2q} . \]  
\[ (F9) \]

Since \(T_\ell(1) = 1\) for all \(\ell\) \[^{[51]}\], we obtain with Eq. \[^{[E14]}\]

\[ \sum_{q=0}^{|\ell/2|} \tilde{C}(\ell, q) = \frac{1}{2\ell - 1} , \quad \text{q.e.d.} \]  
\[ (F10) \]

The orthogonality condition \[^{[83]}\] is obtained from an integral of the type

\[ M^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_\ell)} = \int dK F(E_{ku}) k^{(\mu_1} k^{\mu_2} k_{(\nu_1} \cdots k_{\nu_\ell)} , \]  
\[ (F11) \]
which is a tensor of rank $(\ell + n)$ that is (separately) symmetric under the permutation of $\mu$-type and $\nu$-type indices. In Appendix A of Ref. [18] it is proven that tensors of this type must obey the relation

$$M^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_n)} = \delta_{\ell n} \mathcal{M} \Delta_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_\ell}, \quad \text{(F12)}$$

where $\mathcal{M}$ is an invariant scalar that can be computed by completely contracting the indices of $M^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_n)}$.

Substituting this into Eq. (F17) leads to Eq. (118).

In Appendix A of Ref. [18] it is proven that tensors of this type must obey the relation

$$M^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_n)} = \delta_{\ell n} \mathcal{M} \Delta_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_\ell}, \quad \text{(F12)}$$

where $\mathcal{M}$ is an invariant scalar that can be computed by completely contracting the indices of $M^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_n)}$.

Similarly one derives Eq. (F33). We define a rank-$(\ell + n)$ tensor

$$\hat{M}^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_n)} = \int dK F(E_{ku}, E_{kl}) k^{(\mu_1 \cdots \mu_\ell)} k_{(\nu_1 \cdots \nu_n)}, \quad \text{(F14)}$$

which is (separately) symmetric under permutations of the $\mu$- and $\nu$-type indices and depends solely on the fluid four-velocity $u^\mu$ and the four-vector $l^\mu$. Therefore, $\hat{M}^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_n)}$ must be constructed from tensor structures made of $u^\mu$, $l^\mu$, and $\Xi^\mu\nu$. Furthermore, $\hat{M}^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_n)}$ must be orthogonal to $u^\mu$ as well as to $l^\mu$, which implies that it can only be constructed from combinations of the projection operators $\Xi^\mu\nu$, and henceforth the rank of the tensor, $\ell + n$, must be an even number. Now, following the arguments presented in Appendix A of Ref. [18] one can prove that

$$\hat{M}^{(\mu_1 \cdots \mu_\ell)}_{(\nu_1 \cdots \nu_n)} = \delta_{\ell n} \hat{\mathcal{M}} \Xi_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_\ell}, \quad \text{(F15)}$$

where $\hat{\mathcal{M}}$ is a scalar. This is the analogue to Eq. (F12). Using Eqs. (E18) and (F2), we finally obtain

$$\hat{\mathcal{M}} \equiv \frac{1}{\Xi_{\mu_1 \cdots \mu_\ell}^{\nu_1 \cdots \nu_n}} \int dK \hat{F}(E_{ku}, E_{kl}) k^{(\mu_1 \cdots \mu_\ell)} k_{(\nu_1 \cdots \nu_n)}, \quad \text{(F16)}$$

Finally, we also prove Eq. (118). We first rewrite the collision integral (17) as

$$F^{(\mu_1 \cdots \mu_\ell)}_{ij} = \sum_{r=0}^{\infty} \sum_{n=1}^{N_r} \sum_{m=0}^{N_m-n} \rho^{\mu_1 \cdots \mu_\ell} (A_{injm})^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_r}, \quad \text{(F17)}$$

where, following similar arguments as above, the tensor $(A_{injm})^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_r}$ can be shown to possess the property

$$(A_{injm})^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_r} = \delta_{\ell r} A^{(\ell)}_{injm} \Xi^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_r}. \quad \text{(F18)}$$

Substituting this into Eq. (F17) leads to Eq. (118).

Appendix G: The polynomial coefficients in the 14-moment approximation

Here we construct the complete set of orthonormal polynomials in both $E_{ku}$ and $E_{kl}$ using the Gram-Schmidt orthogonalization procedure in the 14-moment approximation, i.e., where $N_0 = 2$, $N_1 = 1$, and $N_2 = 0$, cf. Refs. [18] [19]. With Eq. (44), one observes that we only need to determine the polynomials $P^{(0)}_{k00}$, $P^{(0)}_{k01}$, $P^{(0)}_{k10}$, $P^{(0)}_{k11}$, $P^{(0)}_{k20}$, $P^{(1)}_{k01}$, $P^{(1)}_{k10}$, and $P^{(2)}_{k00}$.

Using the orthonormality condition (46) for $n = m = n' = m' = 0$ and Eq. (63) we first obtain the value of the normalization constant in Eq. (47):

$$\hat{W}^{(\ell)} = \frac{(-1)^{\ell}}{\sqrt{2\pi r, 0, \ell}}. \quad \text{(G1)}$$
Then, the orthonormality condition (96) can be written with the help of Eqs. (63) and (95) as

\[ J_{2\ell,0,\ell} \delta_{nn'} \delta_{mm'} = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{r=0}^{n'} \sum_{s=0}^{m'} a_{nmjr}^{(e)} a_{n'rm's}^{(e)} J_{i+r+s+2\ell,j+s,\ell}. \]  

(G2)

From this equation, we will successively construct the polynomials.

(i) \( P_{k00}^{(0)}, P_{k00}^{(1)}, P_{k00}^{(2)} \): For any \( \ell \) and \( n = n' = m = m' = 0 \), we obtain from Eq. (G2)

\[ 1 = a_{0000}^{(e)} \equiv P_{k00}^{(0)}. \]  

(G3)

This determines the polynomials \( P_{k00}^{(0)}, P_{k00}^{(1)}, \text{and} P_{k00}^{(2)}. \)

(ii) \( P_{k10}^{(0)} \): Consider Eq. (G2) for \( \ell = 0, n = n' = 1, m = m' = 0 \), as well as for \( \ell = 0, n = 1, n' = 0, m = m' = 0 \).

Solving these two equations for the coefficients \( a_{1000}^{(0)} \) and \( a_{1100}^{(0)} \) leads to the result

\[ \begin{align*}
\frac{a_{1000}^{(0)}}{a_{1100}^{(0)}} &= -\frac{J_{100}}{J_{000}}, \\
\left( a_{1100}^{(0)} \right)^2 &= \frac{J_{200}^2}{\hat{D}_{10}},
\end{align*} \]  

(G4)

where

\[ \hat{D}_{aq} = \hat{J}_{n-1,0,q} \hat{J}_{n+1,0,q} - J_{aq}^2. \]  

(G5)

This uniquely determines the polynomial \( P_{k10}^{(0)} = a_{1000}^{(0)} + a_{1100}^{(0)} E_{ku}. \)

(iii) \( P_{k20}^{(0)} \): Consider Eq. (G2) for \( \ell = 0, n = n' = 2, m = m' = 0 \), for \( \ell = 0, n = 2, n' = 1, m = m' = 0 \), and for \( \ell = 0, n = 2, n' = 0, m = m' = 0 \). From these three equations one obtains

\[ \begin{align*}
\frac{a_{2000}^{(0)}}{a_{2200}^{(0)}} &= \frac{\hat{D}_{20}}{\hat{D}_{10}}, \\
\frac{a_{2100}^{(0)}}{a_{2200}^{(0)}} &= \frac{\hat{G}_{12}}{\hat{D}_{10}}, \\
\left( a_{2100}^{(0)} \right)^2 &= \frac{J_{000} \hat{D}_{10}}{J_{200} \hat{D}_{20} + J_{300} \hat{G}_{12} + J_{400} \hat{D}_{10}},
\end{align*} \]  

(G6)

where

\[ \hat{G}_{nm} = \hat{J}_{n00} \hat{J}_{m00} - \hat{J}_{n-1,0,q} \hat{J}_{m+1,0,q}. \]  

(G7)

This uniquely determines the polynomial \( P_{k20}^{(0)} = a_{2000}^{(0)} + a_{2100}^{(0)} E_{ku} + a_{2200}^{(0)} E_{ku}^2. \)

(iv) \( P_{k10}^{(1)} \): Consider Eq. (G2) for \( \ell = 1, n = n' = 1, m = m' = 0 \), as well as for \( \ell = 1, n = 1, n' = 0, m = m' = 0 \).

From these two equations one obtains

\[ \begin{align*}
\frac{a_{1000}^{(1)}}{a_{1100}^{(1)}} &= -\frac{\hat{J}_{001}}{\hat{J}_{201}}, \\
\left( a_{1100}^{(1)} \right)^2 &= \frac{\hat{J}_{201}^2}{\hat{D}_{31}},
\end{align*} \]  

(G8)

which uniquely determines the polynomial \( P_{k10}^{(1)} = a_{1000}^{(1)} + a_{1100}^{(1)} E_{ku}. \)

(v) \( P_{k01}^{(0)} \): Consider Eq. (G2) for \( \ell = 0, n = n' = 0, m = m' = 1 \), as well as for \( \ell = 0, n = n' = 0, m = 1, m' = 0 \).

From these two equations one obtains

\[ \begin{align*}
\frac{a_{0010}^{(0)}}{a_{0011}^{(0)}} &= -\frac{\hat{J}_{110}}{\hat{J}_{000}}, \\
\left( a_{0011}^{(0)} \right)^2 &= \frac{\hat{J}_{000}^2}{\hat{D}_{110}},
\end{align*} \]  

(G9)

where we defined

\[ \hat{D}_{nq} = \hat{J}_{n-1,0,q} \hat{J}_{n+1,0,q} - J_{nq}^2. \]  

(G10)

This uniquely determines the polynomial \( P_{k01}^{(0)} = a_{0010}^{(0)} + a_{0011}^{(0)} E_{kl}. \)
(vi) $P_{k02}^{(0)}$: Consider Eq. (G2) for $\ell = 0$, $n = n' = 0$, $m = m' = 2$, for $\ell = 0$, $n = n' = 0$, $m = 2$, $m' = 1$, and for $\ell = 0$, $n = n' = 0$, $m = 2$, $m' = 0$. From these three equations one obtains

$$\frac{a_{0020}^{(0)}}{a_{0022}^{(0)}} = \frac{\hat{D}_{220}}{\hat{D}_{110}}, \quad \frac{a_{0021}^{(0)}}{a_{0022}^{(0)}} = \frac{\hat{G}_{1122}}{\hat{D}_{110}} \quad \left(\frac{a_{0022}^{(0)}}{a_{0022}^{(0)}}\right)^2 = \frac{\hat{J}_{000} \hat{D}_{110}}{\hat{J}_{220} \hat{D}_{220} + \hat{J}_{330} \hat{G}_{1122} + \hat{J}_{440} \hat{D}_{110}},$$

where we defined

$$\hat{G}_{nmmp} = \hat{J}_{n00} \hat{J}_{mp0} - \hat{J}_{n-1,r-1,0} \hat{J}_{m+1,p+1,0}.$$

This uniquely determines the polynomial $P_{k02}^{(0)} = a_{0020}^{(0)} + a_{0021}^{(0)} E_{k\ell} + a_{0022}^{(0)} E_{k\ell}^2$.

(vii) $P_{k01}^{(1)}$: Consider Eq. (G2) for $\ell = 1$, $n = n' = 0$, $m = m' = 1$ and for $\ell = 1$, $n = n' = 0$, $m = 1$, $m' = 0$. From these two equations one obtains

$$\frac{a_{0010}^{(1)}}{a_{0011}^{(1)}} = -\frac{\hat{J}_{311}}{\hat{J}_{201}}, \quad \left(\frac{a_{0011}^{(1)}}{a_{0011}^{(1)}}\right)^2 = \frac{\hat{J}_{210}^2}{\hat{D}_{311}}.$$ 

which uniquely determines the polynomial $P_{k01}^{(1)} = a_{0010}^{(1)} + a_{0011}^{(1)} E_{k\ell}$.

(viii) $P_{k11}^{(0)}$: Consider Eq. (G2) for $\ell = 0$, $n = n' = 1$, $m = m' = 1$, for $\ell = 0$, $n = n' = 1$, $m = 1$, $m' = 0$, for $\ell = 0$, $n = 1$, $n' = 0$, $m = m' = 1$, and for $\ell = 0$, $n = 1$, $n' = 0$, $m = 1$, $m' = 0$. From these four equations one obtains

$$\frac{a_{0101}^{(0)}}{a_{1111}^{(0)}} = -\frac{\hat{J}_{310} \hat{G}_{2210} - \hat{J}_{210} \hat{D}_{210} - \hat{J}_{200} \hat{G}_{2221}}{\hat{J}_{210} \hat{G}_{2100} - \hat{J}_{200} \hat{D}_{110} - \hat{J}_{100} \hat{G}_{2111}}.$$ 

(G14)

$$\frac{a_{0101}^{(0)}}{a_{1111}^{(0)}} = -\frac{\hat{J}_{310} \hat{G}_{2100} + \hat{J}_{210} \hat{G}_{2112} + \hat{J}_{100} \hat{D}_{210}}{\hat{J}_{210} \hat{G}_{2100} - \hat{J}_{200} \hat{D}_{110} - \hat{J}_{100} \hat{G}_{2111}}.$$ 

(G15)

$$\frac{a_{1110}^{(0)}}{a_{1111}^{(0)}} = \frac{\hat{J}_{310} \hat{G}_{2100} - \hat{J}_{110} \hat{D}_{110} - \hat{J}_{210} \hat{G}_{2111}}{\hat{J}_{210} \hat{G}_{2100} - \hat{J}_{200} \hat{D}_{110} - \hat{J}_{100} \hat{G}_{2111}}.$$ 

(G16)

and

$$\left(\frac{1}{a_{1111}^{(0)}}\right)^2 = \frac{\hat{J}_{210}}{\hat{J}_{000}} + \left(\frac{a_{1010}^{(0)}}{a_{1111}^{(0)}}\right)^2 + \frac{\hat{J}_{220}}{\hat{J}_{000}} \left(\frac{a_{1011}^{(0)}}{a_{1111}^{(0)}}\right)^2 + \frac{\hat{J}_{200}}{\hat{J}_{000}} \left(\frac{a_{310}^{(0)}}{a_{1111}^{(0)}}\right)^2 + 2 \frac{\hat{J}_{210}}{\hat{J}_{000}} \frac{a_{1010}^{(0)}}{a_{1111}^{(0)}} \frac{a_{1011}^{(0)}}{a_{1111}^{(0)}} + 2 \frac{\hat{J}_{310}}{\hat{J}_{000}} \frac{a_{1010}^{(0)}}{a_{1111}^{(0)}} \frac{a_{1011}^{(0)}}{a_{1111}^{(0)}} + 2 \frac{\hat{J}_{110}}{\hat{J}_{000}} \frac{a_{1010}^{(0)}}{a_{1111}^{(0)}} \frac{a_{1011}^{(0)}}{a_{1111}^{(0)}} + 2 \frac{\hat{J}_{310}}{\hat{J}_{000}} \frac{a_{1010}^{(0)}}{a_{1111}^{(0)}} \frac{a_{1011}^{(0)}}{a_{1111}^{(0)}}.$$ 

(G17)

In this equation, we refrained from explicitly inserting the coefficients (G14) – (G16), because the resulting expression becomes too unwieldy. With Eqs. (G14) – (G17), the polynomial $P_{k11}^{(0)} = a_{1010}^{(0)} + a_{1011}^{(0)} E_{k\ell} + a_{1110}^{(0)} E_{k\ell} + a_{1111}^{(0)} E_{k\ell} E_{k\ell}$ is uniquely determined.

We remark that the results for $m = 0$ are formally similar to the polynomials $P_{k\ell}^{(l)}$ with coefficients $a_{n\ell}^{(l)}$ from Eq. (S5), see for example Eqs. (91) – (99) of Ref. [19]. The reason is that the polynomials $P_{k\ell}^{(l)}$ are the $m = 0$ case of the more general multivariate polynomials $P_{k\ell m}$.

Finally, with these results we can explicitly compute the $\tilde{P}_{km}^{(l)}$ coefficients from Eq. (104) for the cases relevant for the 14-moment approximation.

Appendix H: Transport coefficients

In this appendix, we list the expressions for the transport coefficients in Eqs. (157), (159), (161), (163), (165), and (166), as obtained in the 14-moment approximation.
1. Bulk viscous pressure, Eq. (157)

Using Eq. (153) the three bulk viscosity coefficients in Eq. (157) are

\[
\zeta_l = \frac{m_0^2}{3} \left\{ \dot{\mathcal{L}}_{-2,2} - \dot{\mathcal{L}}_{00} - \dot{n} \gamma^{(0)}_{-2,0,2,1} - \dot{M} \gamma^{(0)}_{-2,1,2,1} + \left[ \frac{\partial (\dot{e}, \dot{n})}{\partial (\beta_u, \alpha)} \right]^{-1} \left[ \frac{\partial (\dot{\mathcal{L}}_{00}, \dot{n})}{\partial (\beta_u, \alpha)} (\dot{e} + \dot{P}_l) - \frac{\partial (\dot{\mathcal{L}}_{00}, \dot{e})}{\partial (\beta_u, \alpha)} \dot{n} \right] \right\},
\]
\( \text{H1} \)

\[
\zeta_\perp = \frac{m_0^2}{6} \left\{ -m_0^2 \dot{\mathcal{L}}_{-2,0} - \dot{\mathcal{L}}_{00} - \dot{n} \left( m_0^2 \gamma^{(0)}_{-2,0,0,1} + \gamma^{(0)}_{-2,0,2,1} \right) + \dot{M} \left( m_0^2 \gamma^{(0)}_{-2,1,0,1} + \gamma^{(0)}_{-2,1,2,1} \right) 
+ 2 \left[ \frac{\partial (\dot{e}, \dot{n})}{\partial (\beta_u, \alpha)} \right]^{-1} \left[ \frac{\partial (\dot{\mathcal{L}}_{00}, \dot{n})}{\partial (\beta_u, \alpha)} (\dot{e} + \dot{P}_\perp) - \frac{\partial (\dot{\mathcal{L}}_{00}, \dot{e})}{\partial (\beta_u, \alpha)} \dot{n} \right] \right\},
\]
\( \text{H2} \)

\[
\zeta_{\perp l} = \frac{m_0^2}{3} \left( -\dot{\mathcal{L}}_{-1,1} + \dot{n} \gamma^{(0)}_{-1,0,1,1} + \dot{M} \gamma^{(0)}_{-1,1,1,1} \right).
\]
\( \text{H3} \)

The diffusion coefficients are

\[
\bar{m}^\Pi = \frac{\partial \bar{\zeta}_{\perp l}}{\partial \alpha}, \quad \bar{m}^\Pi_u = \frac{\partial \bar{\zeta}_{\perp l}}{\partial \beta_u}, \quad \bar{m}^\Pi_l = \frac{\partial \bar{\zeta}_{\perp l}}{\partial \beta_l}.
\]
\( \text{H4} \)

The coefficient which couples the relaxation equation for the bulk viscous pressure to the one for the variable \( \beta_l \) is

\[
\bar{z}^\Pi_l = \frac{m_0^2}{3} \left[ \frac{\partial (\dot{e}, \dot{n})}{\partial (\beta_u, \alpha)} \right]^{-1} \frac{\partial (\dot{\mathcal{L}}_{00}, \dot{e}, \dot{n})}{\partial (\beta_l, \beta_u, \alpha)},
\]
\( \text{H5} \)

where the last factor is a generalization of Eq. (153) to (3 × 3) matrices,
Finally, the coefficients in front of the second-order terms are

\[ \begin{align*}
\beta_{ll}^2 &= 1 - \gamma_{2,0,2,0}^0, \\
\beta_{nn}^2 &= \frac{m_0^2}{3} \gamma_{2,0,2,1}^0, \\
\eta_{ll}^2 &= \frac{1}{2} \left( 1 + m_0^2 \gamma_{2,0,0,0}^0 + \gamma_{2,0,2,0}^0 + m_0^2 \left[ \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right] - \frac{1}{\partial (\beta_u, \dot{\alpha})} \right), \\
\eta_{nn}^2 &= \frac{m_0^2}{6} \left( m_0^2 \gamma_{2,0,0,1}^0 + \gamma_{2,0,2,1}^0 \right), \\
\xi_{ll}^2 &= \gamma_{-1,0,1,0}^0, \\
\xi_{nn}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,1,1}^0 + \frac{\dot{\xi} + \dot{P}_l}{\dot{\xi} + \dot{P}_l} \left[ \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right] - \frac{1}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{ll}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{nn}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{ll}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{nn}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{ll}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{nn}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{ll}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{nn}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{ll}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{nn}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{ll}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}, \\
\xi_{nn}^2 &= \frac{m_0^2}{3} \left\{ \gamma_{-1,0,0,0}^1 - \frac{\partial (\dot{\xi}, \ddot{n})}{\partial (\beta_u, \dot{\alpha})} \right\}.
\end{align*} \]
The transport coefficients that couple the evolution of $n_l$ to the expansion scalars are

$$\bar{\varepsilon}_l^{\text{n}} = \gamma_{-1,0}^{(0)} - n_l \gamma_{-2,0}^{(0)} - \tilde{M} \gamma_{-1,0}^{(0)},$$

$$\bar{\varepsilon}_l^{\text{u}} = \gamma_{-1,1}^{(0)} - n_l \gamma_{-2,1}^{(0)} + \tilde{M} \gamma_{-1,2}^{(0)},$$

$$\bar{\varepsilon}_l^{\ell} = \gamma_{-1,1}^{(0)} - n_l \gamma_{-2,1}^{(0)} + \tilde{M} \gamma_{-1,2}^{(0)}.$$

The second-order transport coefficients are

$$\bar{\varepsilon}_l^{\text{n}} = \frac{3}{2} m_0^2 \gamma_{-2,0}^{(0)} - \bar{\varepsilon}_l^{\text{n}} = 2 - \gamma_{-2,0}^{(0)},$$

$$\bar{\varepsilon}_l^{\text{u}} = \frac{3}{2} m_0^2 \gamma_{-2,1}^{(0)} + \tilde{M} \gamma_{-2,1}^{(0)},$$

$$\bar{\varepsilon}_l^{\ell} = \frac{3}{2} m_0^2 \gamma_{-2,1}^{(0)} + \tilde{M} \gamma_{-2,2}^{(0)}.$$

The first-order diffusion coefficients in Eq. (161) are

$$\kappa_a = \frac{\partial \bar{\varepsilon}_l^{\text{n}}}{\partial \alpha} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \alpha} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \beta_a} (\tilde{M} \gamma_{-1,0}^{(0)} + \tilde{M} \gamma_{-1,1}^{(0)}),$$

$$\kappa_u = \frac{\partial \bar{\varepsilon}_l^{\text{u}}}{\partial \alpha} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \alpha} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \beta_u} (\tilde{M} \gamma_{-1,0}^{(0)} + \tilde{M} \gamma_{-1,1}^{(0)}),$$

$$\kappa_l = \frac{\partial \bar{\varepsilon}_l^{\text{u}}}{\partial \beta_l} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \beta_l} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \beta_l} (\tilde{M} \gamma_{-1,0}^{(0)} + \tilde{M} \gamma_{-1,1}^{(0)}).$$

3. **Longitudinal pressure, Eq. (161)**

The first-order diffusion coefficients in Eq. (161) are

$$\kappa_a = \frac{\partial \bar{\varepsilon}_l^{\text{n}}}{\partial \alpha} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \alpha} (\tilde{M} \gamma_{-1,0}^{(0)} + \tilde{M} \gamma_{-1,1}^{(0)}),$$

$$\kappa_u = \frac{\partial \bar{\varepsilon}_l^{\text{u}}}{\partial \alpha} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \alpha} (\tilde{M} \gamma_{-1,0}^{(0)} + \tilde{M} \gamma_{-1,1}^{(0)}),$$

$$\kappa_l = \frac{\partial \bar{\varepsilon}_l^{\text{u}}}{\partial \beta_l} - \frac{\partial \bar{\varepsilon}_l^{\ell}}{\partial \beta_l} (\tilde{M} \gamma_{-1,0}^{(0)} + \tilde{M} \gamma_{-1,1}^{(0)}).$$
while the viscosity coefficients are

\[
\xi_l = \tilde{\xi}_{-2,4} - 3 \tilde{P}_l - \tilde{n}_l \gamma_{-2,0,4,1} - \tilde{M} \delta_{-2,1,4,1}, \quad (H42)
\]

\[
\xi^V_l = \frac{3}{2} \left[ m_0^2 \tilde{\xi}_{-2,2} + \tilde{P}_l + \tilde{\xi}_{-2,4} - \tilde{n}_l \left( m_0^2 \gamma_{-2,0,2,1} + \gamma_{-2,0,4,1} \right) - \tilde{M} \left( m_0^2 \gamma_{-2,1,2,1} + \gamma_{-2,1,4,1} \right) \right], \quad (H43)
\]

\[
\xi^V_{1,l} = m_0^2 \tilde{\xi}_{-1,1} + 3 \tilde{\xi}_{-1,3} - \tilde{n}_l \left( m_0^2 \gamma_{-1,0,1,1} + 2 \gamma_{-1,0,3,1} \right) - \tilde{M} \left( m_0^2 \gamma_{-1,1,1,1} + 2 \gamma_{-1,1,3,1} \right). \quad (H44)
\]

The second-order transport coefficients are

\[
\beta_n^l = \frac{3}{m_0} \gamma_{-2,0,4,0}, \quad \beta_{\perp}^l = \gamma_{-2,0,4,1}, \quad (H45)
\]

\[
\delta_l^l = \frac{3}{2m_0} \left( m_0^2 \gamma_{-2,0,2,0} + \gamma_{-2,0,4,0} \right), \quad \delta_{\perp}^l = \frac{1}{2} \left( m_0^2 \gamma_{-2,0,2,1} + \gamma_{-2,0,4,1} \right), \quad (H46)
\]

\[
\epsilon_l^l = \frac{3}{m_0} \left( m_0^2 \gamma_{-1,0,1,0} + 2 \gamma_{-1,0,3,0} \right), \quad \epsilon_{\perp}^l = m_0^2 \gamma_{-1,0,1,1} + 2 \gamma_{-1,0,3,1}, \quad (H47)
\]

\[
\Pi_{\perp}^l = \frac{3}{m_0} \left( \gamma_{-2,0,2,0} + \gamma_{-2,0,4,0} \right), \quad \Pi_{\perp}^l = \gamma_{-2,0,2,1} + \gamma_{-2,0,4,1}, \quad (H48)
\]

\[
\Sigma_{\perp}^{I_{10}} = \frac{\partial \Pi}{\partial \alpha}, \quad \Sigma_{\perp}^{I_{1u}} = \frac{\partial \Pi}{\partial \beta_u}, \quad \Sigma_{\perp}^{I_{11}} = \frac{\partial \Pi}{\partial \beta_1}, \quad (H49)
\]

\[
\Sigma_{\perp}^{I_{1v}} = \frac{\partial \Pi}{\partial \alpha}, \quad \Sigma_{\perp}^{I_{1u}} = \frac{\partial \Pi}{\partial \beta_u}, \quad \Sigma_{\perp}^{I_{11}} = \frac{\partial \Pi}{\partial \beta_1}, \quad \Sigma_{\perp}^{I_{1W}} = \frac{\partial \Pi}{\partial \beta_1}, \quad \Sigma_{\perp}^{I_{1l}} = \frac{\partial \Pi}{\partial \beta_1}, \quad (H50)
\]

\[
\lambda_{\perp}^{I_{1u}} = \gamma_{-2,0,3,0}, \quad \lambda_{\perp}^{I_{1W}} = \gamma_{-2,0,3,1}, \quad \lambda_{\perp}^{I_{1l}} = 2 \lambda_{\perp}^{I_{1u}}, \quad \lambda_{\perp}^{I_{1W}} = 2 \lambda_{\perp}^{I_{1l}}, \quad \lambda_{\perp}^{I_{1l}} = 4 \gamma_{-1,0,1,0}. \quad (H51)
\]

4. Diffusion current in the direction perpendicular to \( u^\mu \) and \( l^\nu \), Eq. \( (164) \)

The diffusion coefficients in Eq. \( (164) \) are

\[
\kappa_{\alpha} = \frac{1}{2} \frac{\partial}{\partial \alpha} \left[ -m_0^2 \tilde{\xi}_{-1,0} + \tilde{n} \tilde{\xi}_{-1,2} + \tilde{n}_l \left( m_0^2 \gamma_{-1,0,0,1} + \gamma_{-1,0,2,1} \right) + \tilde{M} \left( m_0^2 \gamma_{-1,1,0,0} + \gamma_{-1,1,2,1} \right) \right], \quad (H52)
\]

\[
\kappa_{\alpha} = \frac{1}{2} \frac{\partial}{\partial \beta_u} \left[ -m_0^2 \tilde{\xi}_{-1,0} + \tilde{n} \tilde{\xi}_{-1,2} + \tilde{n}_l \left( m_0^2 \gamma_{-1,0,0,1} + \gamma_{-1,0,2,1} \right) + \tilde{M} \left( m_0^2 \gamma_{-1,1,0,1} + \gamma_{-1,1,2,1} \right) \right], \quad (H53)
\]

\[
\kappa_{\alpha} = \frac{1}{2} \frac{\partial}{\partial \beta_l} \left[ -m_0^2 \tilde{\xi}_{-1,0} + \tilde{n} \tilde{\xi}_{-1,2} + \tilde{n}_l \left( m_0^2 \gamma_{-1,0,0,1} + \gamma_{-1,0,2,1} \right) + \tilde{M} \left( m_0^2 \gamma_{-1,1,0,1} + \gamma_{-1,1,2,1} \right) \right], \quad (H54)
\]

while the viscosity coefficients are

\[
\zeta^{V}_u = \frac{1}{2} \left[ m_0^2 \tilde{\xi}_{-2,1} + \tilde{\xi}_{-2,3} - \tilde{n}_l \left( m_0^2 \gamma_{-2,0,1,1} + \gamma_{-2,0,3,1} \right) - \tilde{M} \left( m_0^2 \gamma_{-2,1,1,1} + \gamma_{-2,1,3,1} \right) \right], \quad (H55)
\]

\[
\zeta^{V}_l = \frac{1}{2} \left[ m_0^2 \tilde{\xi}_{-2,1} + 3 \tilde{\xi}_{-2,3} - \tilde{n}_l \left( m_0^2 \gamma_{-1,0,0,1} + 3 \gamma_{-1,0,2,1} \right) - \tilde{M} \left( m_0^2 \gamma_{-1,1,0,1} + 3 \gamma_{-1,1,2,1} \right) \right] + \frac{\tilde{n}}{\tilde{e} + \tilde{P}_l} (\tilde{P}_l - \tilde{P}_l). \quad (H56)
\]
The second-order transport coefficients are

\[ \beta^V_\Pi = \frac{3}{2m_0} \left( m_0^2 \gamma_{2,0,1,0} + \gamma_{2,0,3,0} \right), \quad \beta^V_{\Pi} = \frac{3}{2m_0} \left( m_0^2 \gamma_{1,0,0,0} + 3 \gamma_{1,0,2,0} \right) - \frac{3}{2} \frac{\hat{n}}{\hat{e} + \hat{P}_\perp} \frac{\hat{e} + \hat{P}_l}{\hat{e} + \hat{P}_\perp}, \]  
(575)

\[ \beta^W_\Pi = \frac{1}{2} \left( 1 + m_0^2 \gamma_{2,0,1,1} + \gamma_{2,0,3,1} \right), \quad \beta^W_{\Pi} = 1 - \beta^V_\Pi, \quad \beta^W_{\Pi} = \frac{1}{2} \left( m_0^2 \gamma_{1,0,0,1} + 3 \gamma_{1,0,2,1} \right), \]  
(576)

\[ \delta^V_W = \frac{1}{2} \left( m_0^2 \gamma_{2,0,0,0} + \gamma_{2,0,2,0} \right) - 1, \quad \delta^W_W = \frac{1}{2} \left( m_0^2 \gamma_{2,0,0,1} + \gamma_{2,0,2,1} \right), \]  
(577)

\[ \tilde{e}^V_W = \frac{1}{2} \left( 1 + \gamma_{2,0,1,1} + \gamma_{2,0,3,1} \right), \quad \tilde{e}^W_W = \frac{3}{2} \gamma_{1,0,1,1} + 1, \quad \tilde{e}^V_W = \frac{1}{2} \left( m_0^2 \gamma_{1,0,0,1} + \gamma_{1,0,2,1} \right), \]  
(578)

\[ \tilde{e}^V_\Pi = \frac{1}{2} \left( m_0^2 \gamma_{2,0,0,0} + \gamma_{2,0,2,0} \right) - \frac{3}{2} \frac{\hat{n}}{\hat{e} + \hat{P}_\perp}, \quad \tilde{e}^W_\Pi = \frac{1}{2} \left( m_0^2 \gamma_{1,0,0,1} + \gamma_{1,0,2,1} \right), \]  
(579)

\[ \tilde{e}^V_W = \frac{3}{2} \gamma_{1,0,1,1} + \frac{\hat{n}}{\hat{e} + \hat{P}_\perp}, \quad \tilde{e}^W_\Pi = \frac{3}{2} \gamma_{1,0,1,1} + 1 + \frac{\hat{n}}{\hat{e} + \hat{P}_\perp} - \frac{\hat{n}}{\hat{e} + \hat{P}_\perp}, \]  
(580)

\[ \tilde{e}^V_W = \frac{3}{2} \gamma_{1,0,1,1} + \frac{\hat{n}}{\hat{e} + \hat{P}_\perp}. \]  
(581)

5. Shear-tensor component in \( l^\nu \)-direction, Eq. (165)
The coefficient coupling Eq. (165) to the time evolution of $l^\mu$ is

\[
\tilde{\tau}_l^W = \frac{1}{2} \left( m_0^2 \tilde{\mathcal{F}}_{00} - \tilde{\mathcal{C}} + 3 \tilde{\mathcal{H}}_l \right).
\]  

(H77)

The second-order transport coefficients are

\[
\begin{align*}
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-2,0,2,0} + \gamma^{(0)}_{-2,0,4,0} \right), \\
\delta_{\Pi}^W &= \delta_{\Pi}^W, \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,1,0} + 2 \gamma^{(0)}_{-1,0,3,0} \right), \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,1,1} + 2 \gamma^{(0)}_{-1,0,3,1} \right), \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,2,0} + \gamma^{(0)}_{-2,0,3,0} \right), \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,1,0} + \gamma^{(0)}_{-1,0,3,0} \right), \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,1,1} + \gamma^{(0)}_{-1,0,3,1} \right), \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,2,0} + \gamma^{(0)}_{-2,0,3,0} \right), \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,1,0} + \gamma^{(0)}_{-1,0,3,0} \right), \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,1,1} + \gamma^{(0)}_{-1,0,3,1} \right), \\
\delta_{\Pi}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,2,0} + \gamma^{(0)}_{-2,0,3,0} \right).
\end{align*}
\]  

(H78)

The shear tensor in the direction perpendicular to $u^\alpha$ and $l^\nu$, Eq. (166)

\[
\begin{align*}
\tilde{\tau}_{\alpha}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,0,0} + \gamma^{(0)}_{-2,0,2,0} \right), \\
\tilde{\tau}_{\alpha}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,0,1} + \gamma^{(0)}_{-2,0,2,1} \right), \\
\tilde{\tau}_{\alpha}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,1,0} + \gamma^{(0)}_{-2,0,3,0} \right), \\
\tilde{\tau}_{\alpha}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,1,1} + \gamma^{(0)}_{-2,0,3,1} \right), \\
\tilde{\tau}_{\alpha}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,2,0} + \gamma^{(0)}_{-2,0,4,0} \right), \\
\tilde{\tau}_{\alpha}^W &= \frac{3}{2m_0^2} \left( m_0^2 \gamma^{(0)}_{-1,0,2,1} + \gamma^{(0)}_{-2,0,4,1} \right).
\end{align*}
\]  

(H86)

The shear viscosity coefficients in Eq. (165) are

\[
\begin{align*}
\tilde{\eta} &= \frac{1}{8} \left[ 3 \tilde{\mathcal{C}} - 2 l_l - m_0^4 \tilde{\mathcal{F}}_{-2,0} - 2 m_0^2 \left( \tilde{\mathcal{F}}_{00} + \tilde{\mathcal{F}}_{-2,2} \right) - \tilde{\mathcal{F}}_{-2,4} + \tilde{n}_l \left( m_0^4 \gamma^{(0)}_{-2,0,0,1} + 2 m_0^2 \gamma^{(0)}_{-2,0,2,1} + \gamma^{(0)}_{-2,0,4,1} \right) + \tilde{M} \left( m_0^4 \gamma^{(0)}_{-2,1,0,1} + 2 m_0^2 \gamma^{(0)}_{-2,1,2,1} + \gamma^{(0)}_{-2,1,4,1} \right) \right] \, , \\
\tilde{\eta} &= \frac{1}{2} \left[ -m_0^4 \tilde{\mathcal{F}}_{-1,1} - \tilde{\mathcal{F}}_{-1,3} + \tilde{n}_l \left( m_0^4 \gamma^{(0)}_{-1,0,1,1} + \gamma^{(0)}_{-1,0,3,1} \right) + \tilde{M} \left( m_0^4 \gamma^{(0)}_{-1,1,1,1} + \gamma^{(0)}_{-1,1,3,1} \right) \right] \, .
\end{align*}
\]  

(H89)

(H90)
The second-order transport coefficients are

\[
\delta^\pi_\tau = \frac{3}{2} + m^2_0 \gamma^{(2)}_{\tau,0,0,0} + \gamma^{(2)}_{\tau,0,2,0,0}, \quad \tilde{\delta}^\pi_\tau = 2\gamma^{(2)}_{\tau,0,1,0,0}, \quad \tau^\pi_\tau = \frac{4}{3} \left( \delta^\pi_\tau - \frac{1}{2} \right),
\]

\[
\lambda^\pi_\eta = \frac{3}{4m^2_0} \left[ m^4_0 \gamma^{(0)}_{\eta,0,0,0} + 2m^2_0 \gamma^{(0)}_{\eta,0,2,0} + 1 \right] + \gamma^{(0)}_{\eta,0,4,0} + \gamma^{(0)}_{\eta,0,2,2},
\]

\[
\lambda^\pi_n = \frac{1}{4} \left( m^2_0 \gamma^{(0)}_{n,0,0,1} + 2m^2_0 \gamma^{(0)}_{n,2,0,1} + \gamma^{(0)}_{n,0,4,1} \right),
\]

\[
\lambda^\pi_{n\ell} = \frac{3}{4m^2_0} \left( m^2_0 \gamma^{(0)}_{n,1,0,1} + \gamma^{(0)}_{n,1,0,3} \right), \quad \lambda^\pi_{nl} = m^2_0 \gamma^{(0)}_{n,1,1,1} + \gamma^{(0)}_{n,1,0,3,1},
\]

\[
\bar{\lambda}^\pi_V = \frac{1}{2} \left( m^2_0 \gamma^{(1)}_{1,0,0,0} + \gamma^{(1)}_{1,0,2,0} \right), \quad \bar{\lambda}^\pi_W = \frac{1}{2} \left( m^2_0 \gamma^{(1)}_{1,0,0,1} + \gamma^{(1)}_{1,0,2,1} \right),
\]

\[
\bar{\lambda}^\pi_{\eta\alpha} = \frac{2}{\pi} \gamma^{(2)}_{\eta,2,0,2,0} + 1,
\]

\[
\lambda^\pi_{V\ell} = \frac{1}{2} \left( m^2_0 \gamma^{(1)}_{1,0,0,0} + 5\gamma^{(1)}_{2,0,3,0} \right), \quad \lambda^\pi_{W\ell} = \frac{3}{2} \left( m^2_0 \gamma^{(1)}_{2,0,0,1} + \gamma^{(1)}_{2,0,2,1} \right), \quad \lambda^\pi_{W\perp} = \lambda^\pi_{Wu} + 2, \quad \lambda^\pi_{V\perp} = \lambda^\pi_{Vl} + \frac{3}{2} \left( m^2_0 \gamma^{(1)}_{1,0,0,0} + 5\gamma^{(1)}_{2,0,3,0} \right).
\]

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