A NOTE ON A FUNDAMENTAL DOMAIN FOR SIEGEL-JACOBI SPACE

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Abstract. In this paper, we study a fundamental domain for the Siegel-Jacobi space \( \text{Sp}(g, \mathbb{Z}) \ltimes \mathbb{H}_g \times \mathbb{C}^{(h,a)} \).

1. Introduction

For a given fixed positive integer \( g \), we let

\[ \mathbb{H}_g = \{ \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = ^t\Omega, \quad \text{Im} \Omega > 0 \} \]

be the Siegel upper half plane of degree \( g \) and let

\[ \text{Sp}(g, \mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid ^tMJ_gM = J_g \} \]

be the symplectic group of degree \( g \), where \( F^{(k,l)} \) denotes the set of all \( k \times l \) matrices with entries in a commutative ring \( F \) for two positive integers \( k \) and \( l \), \(^tM\) denotes the transpose matrix of a matrix \( M \) and

\[ J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \].

\( \text{Sp}(g, \mathbb{R}) \) acts on \( \mathbb{H}_g \) transitively by

\[ M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \]

where \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(g, \mathbb{R}) \) and \( \Omega \in \mathbb{H}_g \). Let \( \Gamma_g \) be the Siegel modular group of degree \( g \). C. L. Siegel [8] found a fundamental domain \( \mathcal{F}_g \) for \( \Gamma_g \backslash \mathbb{H}_g \) and calculated the volume of \( \mathcal{F}_g \). We also refer to [2], [4], [10] for some details on \( \mathcal{F}_g \).

For two positive integers \( g \) and \( h \), we consider the Heisenberg group

\[ H^{(g,h)}_\mathbb{R} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)}, \kappa + \mu^t\lambda \text{ symmetric} \} \]

endowed with the following multiplication law

\[ (\lambda, \mu; \kappa) \cdot (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t\mu' - \mu^t\lambda') \].

We define the semidirect product of \( \text{Sp}(g, \mathbb{R}) \) and \( H^{(g,h)}_\mathbb{R} \)

\[ G^{(g,h)} = \text{Sp}(g, \mathbb{R}) \ltimes H^{(g,h)}_\mathbb{R} \]

endowed with the following multiplication law

\[ (M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = (MM', (\bar{\lambda} + \lambda', \bar{\mu} + \mu'; \kappa + \kappa' + \bar{\lambda}^t\mu' - \bar{\mu}^t\lambda')) \]

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with \( M, M' \in Sp(g, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(g,h)} \) and \((\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'\). Then \( G^J \) acts on \( \mathbb{H}_g \times \mathbb{C}^{(h,g)} \) transitively by
\[
(1.2) \quad (M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = (M \cdot \Omega, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1}),
\]
where \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R}), (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)} \) and \((\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}\). We note that the Jacobi group \( G^J \) is not a reductive Lie group and also that the space \( \mathbb{H}_g \times \mathbb{C}^{(h,g)} \) is not a symmetric space. We refer to [11]-[14] and [16] about automorphic forms on \( G^J \) and topics related to the content of this paper. From now on, we write \( \mathbb{H}_{g,h} := \mathbb{H}_g \times \mathbb{C}^{(h,g)} \).

We let
\[
\Gamma_{g,h} := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}
\]
be the discrete subgroup of \( G^J \), where
\[
H_{\mathbb{Z}}^{(g,h)} = \{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)} | \lambda, \mu \in \mathbb{Z}^{(h,g)}, \kappa \in \mathbb{Z}^{(h,h)} \}.
\]

The aim of this paper is to find a fundamental domain for \( \Gamma_{g,h} \setminus \mathbb{H}_{g,h} \). This article is organized as follows. In Section 2, we review the Minkowski domain and the Siegel’s fundamental domain \( F_g \) roughly. In Section 3, we find a fundamental domain for \( \Gamma_{g,h} \setminus \mathbb{H}_{g,h} \) and present Riemannian metrics on the fundamental domain invariant under the action \((1.2)\) of the Jacobi group \( G^J \). In Section 4, we investigate the spectral theory of the Laplacian on the abelian variety \( A_{\Omega} \) associated to \( \Omega \in F_g \).

2. Review on a Fundamental Domain \( F_g \) for \( \Gamma_g \setminus \mathbb{H}_g \)

We let
\[
\mathcal{P}_g = \left\{ Y \in \mathbb{R}^{(g,g)} | Y = t' Y > 0 \right\}
\]
be an open cone in \( \mathbb{R}^N \) with \( N = g(g + 1)/2 \). The general linear group \( GL(g, \mathbb{R}) \) acts on \( \mathcal{P}_g \) transitively by
\[
(2.1) \quad g \circ Y := gY t' g, \quad g \in GL(g, \mathbb{R}), \ Y \in \mathcal{P}_g.
\]

Thus \( \mathcal{P}_g \) is a symmetric space diffeomorphic to \( GL(g, \mathbb{R})/O(g) \). For a matrix \( A \in F^{(k,l)} \) and \( B \in F^{(k,l)} \), we write \( A[B] = t'BAB \) and for a square matrix \( A, \sigma(A) \) denotes the trace of \( A \).

The fundamental domain \( \mathcal{R}_g \) for \( GL(g, \mathbb{Z}) \setminus \mathcal{P}_g \) which was found by H. Minkowski [5] is defined as a subset of \( \mathcal{P}_g \) consisting of \( Y = (y_{ij}) \in \mathcal{P}_g \) satisfying the following conditions (M.1)-(M.2) (cf. [2] p. 191 or [4] p. 123):

(M.1) \( aY^t a \geq y_{kk} \) for every \( a = (a_i) \in \mathbb{Z}^g \) in which \( a_k, \cdots, a_g \) are relatively prime for \( k = 1, 2, \cdots, g \).

(M.2) \( y_{kk+1} \geq 0 \) for \( k = 1, \cdots , g - 1 \).

We say that a point of \( \mathcal{R}_g \) is Minkowski reduced or simply M-reduced. \( \mathcal{R}_g \) has the following properties (R1)-(R6):

(R1) For any \( Y \in \mathcal{P}_g \), there exist a matrix \( A \in GL(g, \mathbb{Z}) \) and \( R \in \mathcal{R}_g \) such that \( Y = R[A] \) (cf. [2] p. 191 or [4] p. 139). That is,
\[
GL(g, \mathbb{Z}) \circ \mathcal{R}_g = \mathcal{P}_g.
\]
(R2) \( \mathcal{R}_g \) is a convex cone through the origin bounded by a finite number of hyperplanes. \( \mathcal{R}_g \) is closed in \( \mathcal{P}_g \) (cf. [4] p. 139).

(R3) If \( Y \) and \( Y[A] \) lie in \( \mathcal{R}_g \) for \( A \in GL(g, \mathbb{Z}) \) with \( A \neq \pm I_g \), then \( Y \) lies on the boundary \( \partial \mathcal{R}_g \) of \( \mathcal{R}_g \). Moreover \( \mathcal{R}_g \cap (\mathcal{R}_g[A]) \neq \emptyset \) for only finitely many \( A \in GL(g, \mathbb{Z}) \) (cf. [4] p. 139).

(R4) If \( Y = (y_{ij}) \) is an element of \( \mathcal{R}_g \), then 
\[
y_{11} \leq y_{22} \leq \cdots \leq y_{gg} \quad \text{and} \quad |y_{ij}| < \frac{1}{2} y_{ii} \quad \text{for } 1 \leq i < j \leq g.
\]

We refer to [2] p. 192 or [4] pp. 123-124.

Remark. Grenier [1] found another fundamental domain for \( GL(g, \mathbb{Z}) / \mathcal{P}_g \).

For \( Y = (y_{ij}) \in \mathcal{P}_g \), we put 
\[
dY = (dy_{ij}) \quad \text{and} \quad \partial \frac{\partial}{\partial Y} = \left( \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right).
\]

Then we can see easily that
\[
ds^2 = \sigma((Y^{-1}dY)^2)
\]
is a \( GL(g, \mathbb{R}) \)-invariant Riemannian metric on \( \mathcal{P}_g \) and its Laplacian is given by
\[
\Delta = \sigma \left( Y \frac{\partial}{\partial Y} \right)^2.
\]

We also can see that
\[
d\mu_g(Y) = (\det Y)^{-\frac{g+1}{2}} \prod_{i \leq j} dy_{ij}
\]
is a \( GL(g, \mathbb{R}) \)-invariant volume element on \( \mathcal{P}_g \). The metric \( ds^2 \) on \( \mathcal{P}_g \) induces the metric \( ds^2_{\mathbb{R}} \) on \( \mathcal{R}_g \). Minkowski [5] calculated the volume of \( \mathcal{R}_g \) for the volume element \([dY] := \prod_{i \leq j} dy_{ij}\) explicitly. Later Siegel [7], [9] computed the volume of \( \mathcal{R}_g \) for the volume element \([dY]\) by a simple analytic method and generalized this case to the case of any algebraic number field.

Siegel [8] determined a fundamental domain \( \mathcal{F}_g \) for \( \Gamma_g \backslash \mathbb{H}_g \). We say that \( \Omega = X + iY \in \mathbb{H}_g \) with \( X, Y \) real is \textit{Siegel reduced} or \textit{S-reduced} if it has the following three properties:

(S.1) \( \det(\text{Im}(\gamma \cdot \Omega)) \leq \det(\text{Im}(\Omega)) \) for all \( \gamma \in \Gamma_g \);
(S.2) \( Y = \text{Im} \Omega \) is M-reduced, that is, \( Y \in \mathcal{R}_g \);
(S.3) \( |x_{ij}| \leq \frac{1}{2} \) for \( 1 \leq i, j \leq g \), where \( X = (x_{ij}) \).

\( \mathcal{F}_g \) is defined as the set of all Siegel reduced points in \( \mathbb{H}_g \). Using the highest point method, Siegel proved the following (F1)-(F3) (cf. [2] pp. 194-197 or [4] p. 169):

(F1) \( \Gamma_g \cdot \mathcal{F}_g = \mathbb{H}_g \), i.e., \( \mathbb{H}_g = \cup_{\gamma \in \Gamma_g} \gamma \cdot \mathcal{F}_g \).

(F2) \( \mathcal{F}_g \) is closed in \( \mathbb{H}_g \).

(F3) \( \mathcal{F}_g \) is connected and the boundary of \( \mathcal{F}_g \) consists of a finite number of hyperplanes.

For \( \Omega = (\omega_{ij}) \in \mathbb{H}_g \), we write \( \Omega = X + iY \) with \( X = (x_{ij}) \), \( Y = (y_{ij}) \) real and \( d\Omega = (d\omega_{ij}) \).

We also put 
\[
\frac{\partial}{\partial \Omega} = \left( \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \quad \text{and} \quad \frac{\partial}{\partial \Omega} = \left( \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right).
\]
Then
\begin{equation}
   ds^2 = \sigma (Y^{-1}d\Omega Y^{-1}d\overline{\Omega})
\end{equation}
is a $Sp(g, \mathbb{R})$-invariant Kähler metric on $\mathbb{H}_g$ (cf. [8]) and H. Maass [3] proved that its Laplacian is given by
\begin{equation}
   \Delta_* = 4 \sigma \left( Y^t \left( \frac{\partial}{\partial \Omega} \right) \frac{\partial}{\partial \overline{\Omega}} \right).
\end{equation}
And
\begin{equation}
   dv_g(\Omega) = (\det Y)^{-(g+1)} \prod_{1 \leq i \leq j \leq g} dx_{ij} \prod_{1 \leq i \leq j \leq g} dy_{ij}
\end{equation}
is a $Sp(g, \mathbb{R})$-invariant volume element on $H_g$ (cf. [10], p. 130). The metric $ds^2_*$ given by (2.3) induces a metric $ds^2_\mathcal{F}$ on $F_g$.

Siegel [8] computed the volume of $F_g$
\begin{equation}
   \text{vol}(F_g) = 2 \prod_{k=1}^{g} \pi^{-k} \Gamma(k) \zeta(2k),
\end{equation}
where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann zeta function. For instance,
\begin{align*}
   \text{vol}(F_1) &= \frac{\pi}{3}, \\
   \text{vol}(F_2) &= \frac{\pi^3}{270}, \\
   \text{vol}(F_3) &= \frac{\pi^6}{127575}, \\
   \text{vol}(F_4) &= \frac{\pi^{10}}{200930625}.
\end{align*}

3. A Fundamental Domain for $\Gamma_{g,h} \backslash \mathbb{H}_{g,h}$

Let $E_{kj}$ be the $h \times g$ matrix with entry 1 where the $k$-th row and the $j$-th column meet, and all other entries 0. For an element $\Omega \in \mathbb{H}_g$, we set for brevity
\begin{equation}
   F_{kj}(\Omega) := E_{kj}\Omega, \quad 1 \leq k \leq h, \ 1 \leq j \leq n.
\end{equation}

For each $\Omega \in F_g$, we define a subset $P_\Omega$ of $\mathbb{C}^{(h,g)}$ by
\begin{align*}
   P_\Omega &= \left\{ \sum_{k=1}^h \sum_{j=1}^g \lambda_{kj} E_{kj} + \sum_{k=1}^h \sum_{j=1}^g \mu_{kj} F_{kj}(\Omega) \middle| 0 \leq \lambda_{kj}, \mu_{kj} \leq 1 \right\}.
\end{align*}

For each $\Omega \in F_g$, we define the subset $D_\Omega$ of $\mathbb{H}_{g,h}$ by
\begin{align*}
   D_\Omega := \{ (\Omega, Z) \in \mathbb{H}_{g,h} | Z \in P_\Omega \}.
\end{align*}

We define
\begin{align*}
   F_{g,h} := \bigcup_{\Omega \in F_g} D_\Omega.
\end{align*}

**Theorem 3.1.** $F_{g,h}$ is a fundamental domain for $\Gamma_{g,h} \backslash \mathbb{H}_{g,h}$.

**Proof.** Let $(\Omega, Z)$ be an arbitrary element of $\mathbb{H}_{g,h}$. We must find an element $(\Omega, Z)$ of $F_{g,h}$ and an element $\gamma^J = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{g,h}$ with $\gamma \in \Gamma_g$ such that $\gamma^J \cdot (\Omega, Z) = (\Omega, Z)$. Since $F_g$ is a fundamental domain for $\Gamma_g \backslash \mathbb{H}_g$, there exists an element $\gamma$ of $\Gamma_g$ and an element $\Omega$ of $F_g$ such that $\gamma \cdot \Omega = \Omega$. Here $\Omega$ is unique up to the boundary of $F_g$. 
From the definition of $P$ and $\gamma \Omega$ and $\pm$.

In the case that both $\Omega$ and $\gamma$.

If we take $\gamma = (\gamma_1, \ldots, \gamma_h)$, $Z \in P_{\Omega}$ and $\pm$.

Let $(\Omega, Z)$ and $\gamma = (\gamma_1, \ldots, \gamma_h)$ lie in the boundary of $F_g$. Therefore both of them either lie in the boundary of $F_g$ or $\gamma = \pm I_{2g}$.

In the case that both $\Omega$ and $\gamma \cdot \Omega$ lie in the boundary of $F_g$, both $(\Omega, Z)$ and $\gamma = \pm I_{2g}$, we have

$$Z \in P_{\Omega} \quad \text{and} \quad \pm (Z + \lambda \Omega + \mu) \in P_{\Omega}, \quad \lambda, \mu \in \mathbb{Z}^{(h, g)}.$$

From the definition of $P_{\Omega}$ and (3.2), we see that either $\lambda = \mu = 0$, $\gamma \neq -I_{2g}$ or both $Z$ and $\pm (Z + \lambda \Omega + \mu)$ lie on the boundary of the parallelepiped $P_{\Omega}$. Hence either both $(\Omega, Z)$ and $\gamma = (I_{2g}, (0, 0; \kappa)) \in \Gamma_{g, h}$. Consequently $F_{g,h}$ is a fundamental domain for $\Gamma_{g,h} \setminus P_{g,h}$. \hfill $\Box$

For a coordinate $(\Omega, Z) \in P_{g,h}$ with $\Omega = (\omega_{\mu\nu}) \in \mathbb{H}_g$ and $Z = (z_{kl}) \in \mathbb{C}^{(h, g)}$, we put

$$\Omega = X + iY, \quad X = (x_{1\mu}), \quad Y = (y_{1\mu}) \quad \text{real},$$
$$Z = U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \quad \text{real},$$
$$d\Omega = (d\omega_{\mu\nu}), \quad dX = (dx_{1\mu}), \quad dY = (dy_{1\mu}),$$
$$dZ = (dz_{kl}), \quad dU = (du_{kl}), \quad dV = (dv_{kl}),$$
$$d\bar{\Omega} = (d\bar{\omega}_{\mu\nu}), \quad d\bar{Z} = (d\bar{z}_{kl}).$$

$$\frac{\partial}{\partial \Omega} = \left( \frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial \omega_{\mu\nu}} \right), \quad \frac{\partial}{\partial \bar{\Omega}} = \left( \frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial \bar{\omega}_{\mu\nu}} \right),$$
$$\frac{\partial}{\partial Z} = \left( \frac{\partial}{\partial z_{11}}, \ldots, \frac{\partial}{\partial z_{hh}} \right), \quad \frac{\partial}{\partial \bar{Z}} = \left( \frac{\partial}{\partial \bar{z}_{11}}, \ldots, \frac{\partial}{\partial \bar{z}_{hh}} \right).$$

Remark. The following metric

$$ds_{g,h}^2 = \sigma (Y^{-1} d\Omega Y^{-1} d\bar{\Omega}) + \sigma (Y^{-1} V Y^{-1} d\Omega Y^{-1} d\bar{\Omega})$$
$$+ \sigma (Y^{-1} (dZ) d\bar{Z})$$
$$- \sigma (V Y^{-1} d\Omega Y^{-1} (d\bar{\Omega}) + V Y^{-1} d\bar{\Omega} Y^{-1} (dZ))$$
is a Kähler metric on $\mathbb{H}_{g,h}$ which is invariant under the action (1.2) of the Jacobi group $G_J$. Its Laplacian is given by

$$
\Delta_{g,h} = 4\sigma \left( Y^t \left( Y \frac{\partial}{\partial \Omega} \right) \frac{\partial}{\partial \Omega} \right) + 4\sigma \left( VY^{-1}tV^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial \Omega} \right) + 4\sigma \left( V^t \left( Y \frac{\partial}{\partial \Omega} \right) \frac{\partial}{\partial \Omega} \right).
$$

The following differential form

$$
dv_{g,h} = (\det Y)^{-(g+h+1)} [dX] \wedge [dY] \wedge [dU] \wedge [dV]
$$

is a $G_J$-invariant volume element on $\mathbb{H}_{g,h}$, where

$$
[dX] = \wedge_{\mu \leq \nu} dx_{\mu\nu}, \quad [dY] = \wedge_{\mu \leq \nu} dy_{\mu\nu}, \quad [dU] = \wedge_{k,l} du_{kl} \quad \text{and} \quad [dV] = \wedge_{k,l} dv_{kl}.
$$

The point is that the invariant metric $ds_{g,h}^2$ and its Laplacian are beautifully expressed in terms of the trace form. The proof of the above facts can be found in [15].

4. Spectral Decomposition of $L^2(A_\Omega)$

We fix two positive integers $g$ and $h$ throughout this section.

For an element $\Omega \in \mathbb{H}_g$, we set

$$
L_\Omega := \mathbb{Z}^{(h,g)} + \mathbb{Z}^{(h,g)}\Omega
$$

We use the notation (3.1). It follows from the positivity of $\text{Im} \Omega$ that the elements $E_{kj}, F_{kj}(\Omega)$ ($1 \leq k \leq h, 1 \leq j \leq g$) of $L_\Omega$ are linearly independent over $\mathbb{R}$. Therefore $L_\Omega$ is a lattice in $\mathbb{C}^{(h,g)}$ and the set \{ $E_{kj}, F_{kj}(\Omega) \mid 1 \leq k \leq h, 1 \leq j \leq g$ \} forms an integral basis of $L_\Omega$.

We see easily that if $\Omega$ is an element of $\mathbb{H}_g$, the period matrix $\Omega_* := (I_g, \Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2):

\begin{align*}
(\text{RC.1}) & \quad \Omega_*J_g^t\Omega_* = 0; \\
(\text{RC.2}) & \quad -\frac{1}{i}\Omega_*J_g^r\Omega_* > 0.
\end{align*}

Thus the complex torus $A_\Omega := \mathbb{C}^{(h,g)}/L_\Omega$ is an abelian variety. For more details on $A_\Omega$, we refer to [2] and [6].

It might be interesting to investigate the spectral theory of the Laplacian $\Delta_{g,h}$ on a fundamental domain $F_{g,h}$. But this work is very complicated and difficult at this moment. It may be that the first step is to develop the spectral theory of the Laplacian $\Delta_\Omega$ on the abelian variety $A_\Omega$. The second step will be to study the spectral theory of the Laplacian $\Delta_*$ (see (2.4)) on the moduli space $\Gamma_g \backslash \mathbb{H}_g$ of principally polarized abelian varieties of dimension $g$. The final step would be to combine the above steps and more works to develop the spectral theory of the Laplacian $\Delta_{g,h}$ on $F_{g,h}$. In this section, we deal only with the spectral theory $\Delta_\Omega$ on $L^2(A_\Omega)$.

We fix an element $\Omega = X + iY$ of $\mathbb{H}_g$ with $X = \text{Re} \Omega$ and $Y = \text{Im} \Omega$. For a pair $(A, B)$ with $A, B \in \mathbb{Z}^{(h,g)}$, we define the function $E_{A:B} : \mathbb{C}^{(h,g)} \to \mathbb{C}$ by

$$
E_{A:B}(Z) = e^{2\pi i (\langle tA^tU \rangle + \sigma ((B-AX)Y^{-1}tY))},
$$
Lemma 4.1. For any $A, B \in \mathbb{Z}^{(h, g)}$, the function $E_{\Omega; A, B}$ satisfies the following functional equation

$$E_{\Omega; A, B}(Z + \lambda \Omega + \mu) = E_{\Omega; A, B}(Z), \quad Z \in \mathbb{C}^{(h, g)}$$

for all $\lambda, \mu \in \mathbb{Z}^{(h, g)}$. Thus $E_{\Omega; A, B}$ can be regarded as a function on $A_{\Omega}$.

Proof. We write $\Omega = X + iY$ with real $X, Y$. For any $\lambda, \mu \in \mathbb{Z}^{(h, g)}$, we have

$$E_{\Omega; A, B}(Z + \lambda \Omega + \mu) = E_{\Omega; A, B}((U + \lambda X + \mu) + i(V + \lambda Y))$$

$$= e^{2\pi i \{ \sigma (\lambda^t A (U + \lambda X + \mu)) + \sigma ((B - AX)(Y^{-1} (V + \lambda Y))) \}}$$

$$= e^{2\pi i \{ \sigma (\lambda^t A U + \lambda^t A X) + \sigma ((B - AX)Y^{-1} Y + B^t \lambda - AX^t \lambda) \}}$$

$$= e^{2\pi i \{ \sigma (\lambda^t A U) + \sigma ((B - AX)Y^{-1} V) \}}$$

$$= E_{\Omega; A, B}(Z).$$

Here we used the fact that $\lambda^t A \mu$ and $B^t \lambda$ are integral. \hfill \Box

We use the notations in Section 3.

Lemma 4.2. The metric

$$ds_{\Omega}^2 = \sigma \left( (\text{Im} \Omega)^{-1} t(dZ) d\overline{Z} \right)$$

is a Kähler metric on $A_{\Omega}$ invariant under the action (1.2) of $\Gamma^J = Sp(g, \mathbb{Z}) \ltimes H_{Z}^{(h, g)}$ on $(\Omega, Z)$ with $\Omega$ fixed. Its Laplacian $\Delta_{\Omega}$ of $ds_{\Omega}^2$ is given by

$$\Delta_{\Omega} = \sigma \left( (\text{Im} \Omega) \frac{\partial}{\partial Z} t \left( \frac{\partial}{\partial Z} \right) \right).$$

Proof. Let $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{Z})$ and $(\tilde{\Omega}, \tilde{Z}) = \tilde{\gamma} \cdot (\Omega, Z)$ with $\Omega \in \mathbb{H}_g$ fixed. Then according to [4], p. 33,

$$\text{Im} \gamma \cdot \Omega = t(C \overline{\Omega} + D)^{-1} \text{Im} \Omega (C \Omega + D)^{-1}$$

and by (1.2),

$$d\tilde{Z} = dZ (C \Omega + D)^{-1}.$$

Therefore

$$\left( \text{Im} \tilde{\Omega} \right)^{-1} t(d\tilde{Z}) d\overline{Z}$$

$$= (C \overline{\Omega} + D) \left( \text{Im} \Omega \right)^{-1} t(C \Omega + D) \left( C \Omega + D \right)^{-1} t(dZ) d\overline{Z} (C \overline{\Omega} + D)^{-1}$$

$$= (\overline{C \Omega} + D) \left( \text{Im} \Omega \right)^{-1} t(dZ) d\overline{Z} (C \overline{\Omega} + D)^{-1}.$$
Hence if \( f \) is a differentiable function on \( A_\Omega \), then
\[
\text{Im} \hat{\Omega} \frac{\partial}{\partial Z} t \left( \frac{\partial f}{\partial Z} \right) = t (C\bar{\Omega} + D)^{-1} (\text{Im} \Omega) (C\Omega + D)^{-1} (C\bar{\Omega} + D) \frac{\partial}{\partial Z} t \left( (C\bar{\Omega} + D) \frac{\partial f}{\partial Z} \right) = t (C\bar{\Omega} + D)^{-1} \text{Im} \Omega \frac{\partial}{\partial Z} t \left( \frac{\partial f}{\partial Z} \right) (C\bar{\Omega} + D).
\]

Therefore
\[
\sigma \left( \text{Im} \hat{\Omega} \frac{\partial}{\partial Z} t \left( \frac{\partial f}{\partial Z} \right) \right) = \sigma \left( \text{Im} \Omega \frac{\partial}{\partial Z} t \left( \frac{\partial f}{\partial Z} \right) \right).
\]

By the induction on \( h \), we can compute the Laplacian \( \Delta_{\Omega} \).

\( \square \)

We let \( L^2(A_\Omega) \) be the space of all functions \( f : A_\Omega \rightarrow \mathbb{C} \) such that
\[
||f||_{\Omega} := \int_{A_\Omega} |f(Z)|^2 dv_\Omega,
\]
where \( dv_\Omega \) is the volume element on \( A_\Omega \) normalized so that \( \int_{A_\Omega} dv_\Omega = 1 \). The inner product \((\ , \)_\Omega\) on the Hilbert space \( L^2(A_\Omega) \) is given by
\[
(f, g)_\Omega := \int_{A_\Omega} f(Z) g(Z) dv_\Omega, \quad f, g \in L^2(A_\Omega).
\]

**Theorem 4.1.** The set \( \{ E_{\Omega,A,B} \mid A, B \in \mathbb{Z}^{(h,g)} \} \) is a complete orthonormal basis for \( L^2(A_\Omega) \).

Moreover we have the following spectral decomposition of \( \Delta_{\Omega} \):
\[
L^2(A_\Omega) = \oplus_{A,B \in \mathbb{Z}^{(h,g)}} \mathbb{C} \cdot E_{\Omega,A,B}.
\]

**Proof.** Let
\[
T = \mathbb{C}^{(h,g)} / (\mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}) = (\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}) / (\mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)})
\]
be the torus of real dimension \( 2h \). The Hilbert space \( L^2(T) \) is isomorphic to the \( 2hg \) tensor product of \( L^2(\mathbb{R}/\mathbb{Z}) \), where \( \mathbb{R}/\mathbb{Z} \) is the one-dimensional real torus. Since \( L^2(\mathbb{R}/\mathbb{Z}) = \oplus_{n \in \mathbb{Z}} \mathbb{C} \cdot e^{2\pi i n x} \), the Hilbert space \( L^2(T) \) is
\[
L^2(T) = \oplus_{A,B \in \mathbb{Z}^{(h,g)}} \mathbb{C} \cdot E_{A,B}(W),
\]
where \( W = P + iQ, \ P, Q \in \mathbb{R}^{(h,g)} \) and
\[
E_{A,B}(W) := e^{2\pi i \sigma (tAP + tBQ)}, \quad A, B \in \mathbb{Z}^{(h,g)}.
\]

The inner product on \( L^2(T) \) is defined by
\[
(f, g) := \int_0^1 \cdots \int_0^1 f(W) \overline{g(W)} dp_{11} \cdots dp_{hg} dq_{11} \cdots dq_{hg}, \quad f, g \in L^2(T),
\]
where \( W = P + iQ \in T, \ P = (p_{kl}) \) and \( Q = (q_{kl}) \). Then we see that the set \( \{ E_{A,B}(W) \mid A, B \in \mathbb{Z}^{(h,g)} \} \) is a complete orthonormal basis for \( L^2(T) \), and each \( E_{A,B}(W) \) is an eigenfunction of the standard Laplacian
\[
\Delta_T = \sum_{k=1}^h \sum_{l=1}^g \left( \frac{\partial^2}{\partial p_{kl}^2} + \frac{\partial^2}{\partial q_{kl}^2} \right).
\]
We define the mapping $\Phi_\Omega : T \rightarrow A_\Omega$ by

\begin{equation}
\Phi_\Omega(P + iQ) = (P + QX) + iQY, \quad P, Q \in \mathbb{R}^{(h,g)}.
\end{equation}

This is well defined. We can see that $\Phi_\Omega$ is a diffeomorphism and that the inverse $\Phi_\Omega^{-1}$ of $\Phi_\Omega$ is given by

\begin{equation}
\Phi_\Omega^{-1}(U + iV) = (U - VY^{-1}X) + iVY^{-1}, \quad U, V \in \mathbb{R}^{(h,g)}.
\end{equation}

Using (4.4), we can show that for $A, B \in \mathbb{Z}^{(h,g)}$, the function $E_{A,B}(W)$ on $T$ is transformed to the function $E_{A_\Omega,A,B}$ on $A_\Omega$ via the diffeomorphism $\Phi_\Omega$. Using (4.2) and the diffeomorphism $\Phi_\Omega$, we can choose a normalized volume element $dv_\Omega$ on $A_\Omega$ and then we get the inner product on $L^2(A_\Omega)$ defined by (4.1). This completes the proof. □

\section*{References}

[1] Grenier, D.: An analogue of Siegel's $\phi$-operator for automorphic forms for $GL(n, \mathbb{Z})$. Trans. Amer. Math. Soc. \textbf{331}, No. 1 (1992), 463-477.

[2] Igusa, J.: \textit{Theta Functions}. Springer-Verlag, Berlin-Heidelberg-New York (1971).

[3] Maass, H.: \textit{Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen}. Math. Ann. \textbf{126} (1953), 44-68.

[4] Maass, H.: \textit{Siegel modular forms and Dirichlet series}. Lecture Notes in Math., \textbf{216}, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[5] Minkowski, H.: \textit{Gesammelte Abhandlungen}. Chelsea, New York (1967).

[6] Mumford, D.: \textit{Tata Lectures on Theta I}. Progress in Math. \textbf{28}, Boston-Basel-Stuttgart (1983).

[7] Siegel, C. L.: \textit{The volume of the fundamental domain for some infinite groups}. Transactions of AMS. \textbf{39} (1936), 209-218.

[8] Siegel, C. L.: \textit{Symplectic geometry}. Amer. J. Math. \textbf{65} (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.

[9] Siegel, C. L.: \textit{Zur Bestimmung des Volumens des Fundamentalbereichs der unimodularen Gruppe}. Math. Ann. \textbf{137} (1959), 427-432.

[10] Siegel, C. L.: \textit{Topics in Complex Function Theory}. Wiley-Intersciences, New York, vol. III (1973).

[11] Yang, J.-H.: \textit{Remarks on Jacobi forms of higher degree}. Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33-58.

[12] Yang, J.-H.: \textit{Singular Jacobi forms}. Trans. of American Math. Soc. \textbf{347}, No. 6 (1995), 2041-2049.

[13] Yang, J.-H.: \textit{Construction of vector valued modular forms from Jacobi forms}. Canadian J. of Math. \textbf{47} (6) (1995), 1329-1339.

[14] Yang, J.-H.: \textit{A geometrical theory of Jacobi forms of higher degree}. Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147 or Kyungpook Math. J. \textbf{40}, no. 2 (2000), 209-237.

[15] Yang, J.-H.: \textit{Invariant metrics and Laplacians on the Siegel-Jacobi spaces}. \texttt{arXiv:math.NT/0507215} v1.

[16] Ziegler, C.: \textit{Jacobi forms of higher degree}. Abh. Math. Sem. Univ. Hamburg \textbf{59} (1989), 191-224.

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