Research Article

Some new facts about group $G$ generated by the family of convergent permutations

Abstract: The aim of this paper is to present some new and essential facts about group $G$ generated by the family of convergent permutations, i.e. the permutations on $\mathbb{N}$ preserving the convergence of series of real terms. We prove that there exist permutations preserving the sum of series which do not belong to $G$. Additionally, we show that there exists a family $G$ (possessing the cardinality equal to continuum) of groups of permutations on $\mathbb{N}$ such that each one of these groups is different than $G$ and is composed only from the permutations preserving the sum of series. This result substantially strengthens some old Pleasants’ result.

Keywords: Convergent permutations, Divergent permutations, $b$-connected permutations

MSC: 40A05, 05A99

1 Introduction

The family of all permutations on $\mathbb{N}$ will be denoted by $\mathcal{P}$. A permutation $p \in \mathcal{P}$ is called a convergent permutation if for every convergent series $\sum a_n$ of real terms the $p$-rearranged series $\sum a_{p(n)}$ is convergent as well. A family of all convergent permutations will be denoted by $\mathcal{C}$. If we replace in the definition of convergent permutation $p$ series $\sum a_n$ of real terms by series $\sum v_n$ of vector terms, and even by series of normed abelian semigroup terms, the respective set of all convergent permutations will be the same as in the real series case. As Stoller [1] observed for some normed abelian groups (e.g. the p-adics) the set of permutations preserving convergence is larger than $\mathcal{C}$.

Permutations belonging to the family $\mathcal{D} := \mathcal{P} \setminus \mathcal{C}$ will be called the divergent permutations. A group of permutations generated by $\mathcal{C}$ will be denoted by $\mathcal{G}$. We know that $\mathcal{G} \neq \mathcal{P}$ (Pleasants [2, 3], see also Corollary 4.2 in Section 4).

Let $A, B \in \mathcal{P}$. Then the following family of permutations of $\mathbb{N}$

$$\{p \in \mathcal{P} : p \in A \quad \text{and} \quad p^{-1} \in B\}$$

will be denoted by $AB$. After Kronrod [4] and Wituła we call

a) elements of $\mathcal{C}\mathcal{C}$ – the two-sided convergent permutations,

b) elements of $\mathcal{C}\mathcal{D}$ – the one-sided convergent permutations,

c) elements of $\mathcal{D}\mathcal{C}$ – the one-sided divergent permutations,

d) elements of $\mathcal{D}\mathcal{D}$ – the two-sided divergent permutations.
The symbol $\circ$ denotes here the composition of subsets of $\mathcal{P}$, i.e.
$$B \circ A = \{q \circ p(\cdot) : q(p(\cdot)) : q \in B \text{ and } p \in A\}$$
for any nonempty subsets $A, B$ of $\mathcal{P}$. We say that a set $A \subseteq \mathcal{P}$ is algebraically big if $A \circ A = \mathcal{P}$. For example, the family $\mathcal{DD}$ is only the double-sided family of permutations defined in a)-d), which is algebraically big. Moreover, we note that (see [5, 6]):
$$\mathcal{G} \cap \mathcal{DD} \neq \emptyset,$$
$$\mathcal{G} = \mathcal{CD} \circ \mathcal{DE} \cup \mathcal{DE} \circ \mathcal{CD} \circ \mathcal{DE} \cup \mathcal{CD} \circ \mathcal{DE} \circ \mathcal{CD} \circ \mathcal{DE} \cup \ldots$$
$$= \mathcal{DE} \circ \mathcal{CD} \circ \mathcal{DE} \circ \mathcal{CD} \cup \mathcal{DE} \circ \mathcal{CD} \circ \mathcal{DE} \circ \mathcal{CD} \cup \ldots$$
and
$$\mathcal{CD} \cup \mathcal{DE} \subseteq \mathcal{CD} \circ \mathcal{DE} \subseteq \mathcal{DE} \circ \mathcal{CD} \subseteq \mathcal{CD} \circ \mathcal{DE} \circ \mathcal{CD} \circ \mathcal{DE} \subseteq \ldots$$
$$\mathcal{CD} \cup \mathcal{DE} \subset \mathcal{CD} \circ \mathcal{DE} \subseteq \mathcal{DE} \circ \mathcal{CD} \subseteq \mathcal{CD} \circ \mathcal{DE} \circ \mathcal{CD} \circ \mathcal{DE} \subseteq \ldots$$
We think that all these inclusions are really strict. We are working now on some details of the proof of this conjecture.

For the shortness of notation we will write $q(p(n))$, respectively) instead of $q \circ p(q \circ p(n))$, respectively).

By $\mathcal{S}$ we denote the family of all permutations on $\mathbb{N}$ preserving the sum of series, it means permutations $\mathcal{S}$ such that if $\sum a_n$ is the convergent series with real terms and the $p$–rearranged series $\sum a_{p(n)}$ is also convergent then $\sum a_n = \sum a_{p(n)}$, i.e. the sums of both series are the same\(^1\). The set $\mathcal{S}$ is also algebraically big (Stoller [1]). More precisely Stoller proved that the set
$$\mathcal{D}(1) := \{p \in \mathcal{P} : p[\{1, 2, \ldots, n\}] = \{1, 2, \ldots, n\} \text{ for infinitely many } n \in \mathbb{N}\}$$
is algebraically big and we have $\mathcal{D}(1) \subset \mathcal{S}$ [6]. It will be proven in Section 4 that there exists the family $\mathcal{G}$ (possessing the cardinality equal to continuum) of groups of permutations on $\mathbb{N}$ such that for every $G \in \mathcal{G}$ we have $G \subset \mathcal{S}$ and $G \setminus \mathcal{S} \neq \emptyset$.

All subsets of family $\mathcal{P}$, introduced here, are nonempty. Some basic algebraic type properties of all these and some others subsets of $\mathcal{P}$ are discussed in papers [5–10].

## 2 Technical notations and auxiliary results

Let $A$ be a nonempty subset of $\mathbb{N}$, let $\{a_n\}$ denote the increasing sequence of all elements of $A$. Subset $I$ of $A$ is said to be an interval of $A$, providing that it has the form
$$\{a_n, a_n+1, \ldots, a_n+m\}$$
for some $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. So, only the nonempty intervals are discussed here.

We say that two intervals $I$ and $J$ of the given nonempty subset $A$ of $\mathbb{N}$ are separated if the set $I \cup J$ is not an interval of $A$.

Let $A$ be a nonempty subset of $\mathbb{N}$ and let $B$ be a nonempty subset of $A$. Then the notation set $B$ is a union of $n$ (or of at most $n$, or of at least $n$, respectively) MSI(A) means that there exists a family $\mathcal{I}$ of $n$ (or of at most $n$, or of at least $n$, respectively) intervals of set $A$ such that each two intervals are separated. In short we say that the family $\mathcal{I}$ form $n$ (or of at most $n$, or of at least $n$, respectively) mutually separated intervals of $A$.

For the given nonempty set $A \subseteq \mathbb{N}$ we will denote by $S(A)$ the set of all permutations of set $A$. Certainly we have $\mathcal{P} = S(\mathbb{N})$. In particular, $S_n := S(\{1, 2, \ldots, n\})$ denotes the symmetric group of degree $n$ for every $n \in \mathbb{N}$.

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\(^1\) Symbol $\sum a_n$, depending on the context of considerations, denotes either the respective series, i.e. the sequence of partial sums $\left\{ \sum_{k=1}^{n} a_k \right\}_{n=1}^{\infty}$, or the sum of this series if it is convergent.
Next, for every $p \in \mathcal{P}$ and an infinite set $A \subseteq \mathbb{N}$ by $c(p|_A)$ we denote

$$\sup\{n \in \mathbb{N} : \text{ there exists an interval } B \subseteq A \text{ such that the set } p(B) \text{ is a union of } n \text{ MSI } (p(A))\}.$$  \hfill (1)

Instead of $c(p|_B)$ we will write $c(p)$ for every $p \in \mathcal{P}$. We know (see [11, 12]) that permutation $p \in \mathcal{P}$ is a convergent permutation if and only if $c(p) \in \mathbb{N}$ (for the other characterizations of convergent permutations on $\mathbb{N}$ see also [13]).

Furthermore, if $A \subseteq \mathbb{N}$ and both sets $A$ and its complement are infinite then for every $p \in \mathcal{P}$ such that $p(A) = A$ we get that $p$ is a convergent permutation on $A$ (which by definition means that if the given real series $\sum_{n \in A} a_n$ is convergent then also the respective $p$-rearranged series $\sum_{n \in A} a_{p(n)}$ is convergent) if and only if $c(p|_A) \in \mathbb{N}$ (the value of $c(p|_A)$ is defined by (1)). The family of all convergent permutations $p \in \mathcal{P}$ on $A$ such that $p(A) = A$ will be denoted by $\mathcal{C}(A)$.

The symbol $\sum_{n} (p)$ denotes the convergence class of permutation $p$, i.e. the family of all convergent real series $\sum a_n$ for which the $p$–rearranged series $\sum a_{p(n)}$ is also convergent. We note that the following fundamental results hold true.

**Theorem 2.1 (Fundamental Theorem for Inclusion Relation Between the Convergence Classes).** Let $p, q \in \mathcal{P}$. If $\sum(p) \subseteq \sum(q)$ then for every infinite set $A \subseteq \mathbb{N}$ we have

$$\text{if } c(p|_{p^{-1}(A)}) < \infty \text{ then } c(q|_{q^{-1}(A)}) < \infty.$$  

If equality $\sum(q) = \sum(q)$ holds then for every infinite set $A \subseteq \mathbb{N}$ the following relations are satisfied:

- either $c(p|_{p^{-1}(A)}) < \infty$ and $c(q|_{q^{-1}(A)}) < \infty$,
- or both values $c(p|_{p^{-1}(A)})$ and $c(q|_{q^{-1}(A)})$ are simultaneously infinite.

If $A \subseteq \mathbb{N}$ and both sets $A$ and its complement are infinite, and $p(A) = q(A) = A$, $p \in \mathcal{D}(A)$, $q \in \mathcal{C}(A)$, then either $\sum(p) = \sum(q)$ or $\sum(p) \setminus \sum(q) \neq \emptyset$ and $\sum(q) \setminus \sum(p) \neq \emptyset$ (we say in this case that $p$ and $q$ are incomparable). Moreover, the respective permutations $p, q$ could be always chosen to be incomparable.

At last, if there exist two disjoint infinite sets $A, B \subseteq \mathbb{N}$ such that

$$c(p|_{p^{-1}(A)}) < \infty, \quad c(q|_{q^{-1}(A)}) = \infty,$$

$$c(p|_{p^{-1}(B)}) = \infty \quad \text{and} \quad c(q|_{q^{-1}(B)}) < \infty$$

then the permutations $p$ and $q$ are incomparable.

If additionally we suppose that there exist a partition $\{I_n\}_{n=1}^{\infty}$ of $\mathbb{N}$, created from the successive intervals of $\mathbb{N}$, such that

$$p(I_n) = q(I_n) = I_n$$

for every $n \in \mathbb{N}$ (which implies that $p$ and $q$ belong to $\mathcal{D}(1)$) and if there exists an infinite set $A \subseteq \mathbb{N}$ such that

$$p \in \mathcal{D}\left( \bigcup_{n \in A} I_n \right), \quad q \in \mathcal{C}\left( \bigcup_{n \in A} I_n \right),$$

and

$$\sup\{k : \text{ there exists an interval } I \subseteq \bigcup_{n \in A} I_n \text{ such that each of the sets } \text{ } p(I) \setminus q(I) \text{ and } q(I) \setminus p(I) \text{ is a union of at most } k \text{ MSII } \} < \infty,$$

then $\sum(p) \subseteq \sum(q)$ (the strict inclusion holds here).

Detailed proof of Theorem 2.1 will be omitted here since it can be easily deduced from the definitions of concepts used in here.

We would like to make one more remark. The Reader can feel a great deficiency about the sufficient conditions given in the last part of Theorem 2.1, which implies that inclusion $\sum(p) \subseteq \sum(q)$ can be satisfied (however these conditions are very useful in analysis of permutations presented in other places of the paper). We will additionally respond to these doubts in such a way that it is impossible to obtain this inclusion under very general assumptions,
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because in paper [15] we will present the example of set $A \subset \mathbb{N}$ such that this set and its complement are both infinite and of the respective permutations $p, q \in \mathcal{D}$ such that

$$c(p|_{p^{-1}(A)}) < \infty, \quad c(q|_{q^{-1}(A)}) = \infty,$$

$$c(p|_{p^{-1}(\mathbb{N}\setminus A)}) < \infty \text{ and } c(q|_{q^{-1}(\mathbb{N}\setminus A)}) < \infty.$$

Nevertheless, permutations $p$ and $q$ are incomparable.

**Definition 2.2.** The family $S$, having the power of continuum, of increasing sequences of positive integers will be called the Sierpinski family if for any $a, b \in S$, $a = (a_n)_{n=1}^{\infty}$, $b = (b_n)_{n=1}^{\infty}$, the sets of values of $a$ and $b$ are almost disjoint, i.e. there exists $k \in \mathbb{N}$ such that $a_m \neq b_n$ for all $m, n \in \mathbb{N}, m, n \geq k$, whenever $a \neq b$.

**Definition 2.3.** Let $b, n \in \mathbb{N}$. We say that a permutation $\rho$, belonging to $S_n$, is $b$–connected if for each interval $I \subseteq [1, n]$ the image $\rho(I)$ is the union of at most $b$ MSI.

**Definition 2.4.** By the symbol $S_n(k, b)$, for $b, n, k \in \mathbb{N}$, we will denote the family of all permutations $\rho \in S_n$ which can be presented in the form of composition of $k$ permutations belonging to $S_n$, it means in the form $\rho = q_k q_{k-1} \ldots q_1$, where $q_i \in S_n$, $i = 1, 2, \ldots, k$, and, simultaneously,

$$q_i \text{ and } q_j^{-1} \text{ are the } b\text{–connected permutations}$$

either for any indices $i \in 2\mathbb{N}$ and $j \in (2\mathbb{N} - 1)$ or for any indices $i \in (2\mathbb{N} - 1)$ and $j \in 2\mathbb{N}$, such that $i, j \leq k$.

The following lemma, coming from paper [2], gives an estimation of the number of $b$–connected permutations belonging to group $S_n$.

**Lemma 2.5.** The number of $b$–connected permutations belonging to group $S_n$ is majorized by $8b \ln n^{2b n}$. In the sequel we obtain

$$\text{card } S_n(k, b) \leq 2 (8b \ln n)^{2b n k}.$$

### 3 Some algebraic facts on the family $\mathfrak{P} \setminus G$

Let us start with two results of technical character.

**Lemma 3.1.** Let $G \subset \mathfrak{P}$ be a group of permutations. If $q, p, p^n \in \mathfrak{P} \setminus G$ for some $n \in \mathbb{N}, n \geq 2$, then for every $k \in \mathbb{N}, k < n$, we have either $p^{-k} q \in \mathfrak{P} \setminus G$ or $p^{n-k} q \in \mathfrak{P} \setminus G$.

**Proof.** Suppose that for some $k < n$ we have

$$p^{-k} q \in G \quad \text{and} \quad p^{n-k} q \in G.$$

Hence, also $p^n = (p^{n-k} q)(q^{-1} p^k) = (p^{n-k} q)(p^{-k} q)^{-1} \in G$ which is impossible. $\square$

**Lemma 3.2.** Let $G \subset \mathfrak{P}$ be a group of permutations. Then the following relations hold true

i) $$(\mathfrak{P} \setminus G)^{-1} = \mathfrak{P} \setminus G,$$

ii) $$\{g\} \circ (\mathfrak{P} \setminus G) = (\mathfrak{P} \setminus G) \circ \{g\} = \mathfrak{P} \setminus G$$

for every $g \in G$, 

iii) $$G \subset (\mathfrak{P} \setminus G) \circ (\mathfrak{P} \setminus G).$$
iv) If there exists \( g \in \mathcal{P} \setminus G \) such that also \( g^2 \in \mathcal{P} \setminus G \), then \( \mathcal{P} \setminus G \) is the algebraically big subset of \( \mathcal{P} \). Moreover, for any \( \varphi, \psi \in \mathcal{P} \setminus G \) at least one of the compositions \( g\varphi, \psi g \) or \( \varphi \psi \) also belongs to \( \mathcal{P} \setminus G \). Furthermore, either \( g\varphi \in \mathcal{P} \setminus G \) or \( g^{-1}\varphi \in \mathcal{P} \setminus G \), and either \( \varphi g \in \mathcal{P} \setminus G \) or \( \varphi g^{-1} \in \mathcal{P} \setminus G \).

**Proof.** Since \( G \) is a group, therefore relations i) and ii) are obvious. Let \( \gamma \in G, g \in \mathcal{P} \setminus G \). Then also \( g^{-1} \in \mathcal{P} \setminus G \) which implies that \( \gamma = (\gamma g)^{-1} \in (\mathcal{P} \setminus G) \circ (\mathcal{P} \setminus G) \), i.e. relation iii) holds. If \( \varphi \in \mathcal{P} \setminus G \) and additionally \( g^2 \in \mathcal{P} \setminus G \), then by Lemma 3.1 either \( \varphi g \in \mathcal{P} \setminus G \) or \( \varphi g^{-1} \in \mathcal{P} \setminus G \) which implies \( \varphi = (\varphi g)^{-1} = (\varphi g^{-1})g \in (\mathcal{P} \setminus G) \circ (\mathcal{P} \setminus G) \), i.e. \( \mathcal{P} \setminus G \subset (\mathcal{P} \setminus G) \circ (\mathcal{P} \setminus G) \) and by i) relation \( \mathcal{P} = (\mathcal{P} \setminus G) \circ (\mathcal{P} \setminus G) \) holds.

Suppose that \( \varphi \varphi, \psi g \) and \( \varphi \psi \) all belong to \( G \). Then also

\[
g^2 = g\varphi(\psi \psi)^{-1}\varphi g \in G,
\]

which gives the contradiction. \( \square \)

**Corollary 3.3.** All three relations i) - iii) hold for the group \( \mathcal{G} \).

We prove that by item iv) of Lemma 3.2 the set \( \mathcal{P} \setminus G \) is the algebraically big subset of \( \mathcal{P} \). To this aim we need a special case \((k = 3)\) of the following intriguing result.

**Theorem 3.4.** For every \( k \in \mathbb{N}, k > 1 \), equation \( p^k = i \delta_3 \) possesses the solution \( p \in \mathcal{P} \setminus G \) such that \( p^i \in \mathcal{P} \setminus G \) for every \( i = 1, 2, ..., k - 1 \).

**Proof.** From Lemma 2.5 and the Stirling formula [14] we get that there exists an increasing sequence of natural numbers \( \{n(u) : u \in \mathbb{N}\} \) such that \( S_{n(u)} \setminus S_{n(u)}(u, u) \neq \emptyset \) for every \( u \in \mathbb{N} \). Let \( \overline{p}_u \in (S_{n(u)} \setminus S_{n(u)}(u, u)) \) for every \( u \in \mathbb{N} \).

Let us fix \( k \in \mathbb{N}, k > 1 \), and let \( \{I_u\} \) be the increasing sequence of intervals (which means that \( i < j \) for any \( i \in I_u, j \in I_{u+1}, u \in \mathbb{N} \)) forming a partition of \( \mathbb{N} \) and satisfying condition card \( I_u = k \cdot n(u) \), for every \( u \in \mathbb{N} \). Let us put

\[
J_u = [(k - 2) n(u) + \min I_u, (k - 1) n(u) + \min I_u]
\]

for \( u \in \mathbb{N} \).

The expected permutation \( p \) is defined in the following way

\[
\begin{align*}
p(i n(u) + j + \min I_u) &= (i + 1) n(u) + j + \min I_u, \\
p((k - 2) n(u) + j + \min I_u) &= (k - 1) n(u) + \overline{p}_u(j + 1) - 1 + \min I_u, \\
p((k - 1) n(u) + j + \min I_u) &= \overline{p}_u^{-1}(j + 1) - 1 + \min I_u,
\end{align*}
\]

for \( i = 0, 1, ..., k - 3 \) and \( j = 0, 1, ..., n(u) - 1 \).

The verification of condition \( p^k = i \delta_3 \) is trivial, whereas the fact that \( p^i \in (\mathcal{P} \setminus G) \) for every \( i = 1, 2, ..., k - 1 \) is a consequence of the fact that composition of the following three mappings: the increasing mapping of interval \([1, n(u)]\) onto interval \( J_u \), the restriction of \( p \) to interval \( J_u \) and the increasing mapping of interval \([(k - 1) n(u) + \min I_u, \max I_u]\) onto interval \([1, n(u)]\), is equal to \( \overline{p}_u \) for every \( u \in \mathbb{N} \).

Now, let us suppose that there exist permutations \( p_i \in \mathcal{P}, i = 1, 2, ..., n \), such that

\[
p = p_n p_{n-1} ... p_1
\]

and

\[
either \quad p_i \in \mathcal{C} \quad or \quad p_i^{-1} \in \mathcal{C},
\]

for every \( i = 1, 2, ..., n \).

Let \( J \) be an interval of natural numbers. We denote by \( p_J \) and \( p_{i, J} \) the restrictions of permutations \( p \) and \( p_i \), respectively, to sets \( J \) and

\[
p_{i-1} p_{i-2} ... p_1 p_0(J).
\]
respectively, for each \( i = 1, 2, \ldots, n \), where \( p_0 := \text{id}_S \). Then the following decomposition holds

\[
q_{n,J} pq^{-1}_{n,J} = (q_{n,J} p_{n,J} q_{n,J}^{-1})(q_{n-1,J} p_{n-1,J} q_{n-1,J}^{-1})\cdots(q_{2,J} p_{2,J} q^{-1}_{2,J})(q_{1,J} p_{1,J} q_{0,J}^{-1}),
\]

where \( q_i, J, 1 \leq i \leq n \), is the increasing mapping of set \( p_ip_{i-1}\ldots p_1(J) \) onto interval \([1, \text{card}(J)]\) and \( q_{0,J} := \text{id}_S \).

Let us notice that for each \( i = 1, 2, \ldots, n \) we have \( q_i J p_i J q_{i-1,J}^{-1} \in S_{\text{card}(J)} \) and this is the \( b \)-connected permutation for \( b = c(p_i) \) or \( b = c(p_i^{-1}) \), depending on the fact whether \( p_i \in \mathcal{C} \) or \( p_i^{-1} \in \mathcal{C} \). Justification is needed only for the last property.

So let us suppose that there exists \( 1 \leq i \leq n \) such that \( p_i \in \mathcal{C} \) and simultaneously \( \xi := q_i J p_i J q_{i-1,J}^{-1} \) is not the \( b \)-connected permutation. Then there exists subinterval \( \Lambda \) of interval \([1, \text{card}(J)]\) such that the set \( \xi(\Lambda) \) is a union of at least \( (b+1) \) \( \text{MSI} \). Let \( \mathcal{B} \) be the family of mutually separated intervals of natural numbers forming the decomposition of set \( \xi(\Lambda) \). Auxiliarly we take

\[
\alpha = \min q_{i-1,J}^{-1}(\Lambda) \quad \text{and} \quad \beta = \max q_{i-1,J}^{-1}(\Lambda).
\]

Then from the fact that \( q_{i-1,J}^{-1} \) is the increasing mapping we get the following relations

\[
\alpha = q_{i-1,J}^{-1}(\min \Lambda) \quad \text{and} \quad \beta = q_{i-1,J}^{-1}(\max \Lambda)
\]

and

\[
[\alpha, \beta] \cap q_{i-1,J}^{-1}([1, \text{card}(J)]) = q_{i-1,J}^{-1}(\Lambda).
\]

Let us notice one more property. For any \( \Theta, \Xi \in \mathcal{B} \), \( \Theta \neq \Xi \),

\[
\text{if} \quad q_{i-1,J}^{-1}(\Theta) < q_{i-1,J}^{-1}(\Xi) \quad \text{then the following relation holds}
\]

\[
\text{(max} q_{i-1,J}^{-1}(\Theta), \min q_{i-1,J}^{-1}(\Xi)) \cap p_i q_{i-1,J}^{-1}([1, \text{card}(J)]) \neq \emptyset.
\]

From the relations (4), (5) and (6) we are able to deduce that the set \( p_i([\alpha, \beta]) \), as well as the set \( \xi(\Lambda) \), is the union of at least \( (b+1) \) \( \text{MSI} \), since the numbers from the set \( p_i([\alpha, \beta]) \cup q_{i-1,J}^{-1}(\Lambda) \) “do not fill” completely all the “holes” between sets \( q_{i-1,J}^{-1}(\Theta) \) and \( q_{i-1,J}^{-1}(\Xi) \) for any \( \Theta, \Xi \in \mathcal{B} \), \( \Theta \neq \Xi \). Thereby we get the contradiction with definition of number \( b \) and, in consequence, with the assumption that \( \xi \) is not the \( b \)-connected permutation.

Thus, if we assume that

\[
v > n + \max_{1 \leq i \leq n} \{c(q_i) : q_i = p_i \text{ if } p_i \in \mathcal{C}, \quad q_i = p_i^{-1} \text{ otherwise}\}
\]

then \( p_u = q_n J p_J q_{J-1,J}^{-1} \notin S_n(u, u) \) for \( J = J_u \) and for every \( u \in \mathbb{N}, u > v \), which contradicts the assumption.

\[
\square
\]

**Corollary 3.5.** The set \( \mathcal{P} \setminus \mathcal{G} \) is algebraically big.

Referring to Theorem 3.4, as well as to item iv) of Lemma 3.2, we present one more important result.

**Theorem 3.6.** Let \( p \in \mathcal{P} \setminus \mathcal{G} \). If also \( p^2 \in \mathcal{P} \setminus \mathcal{G} \), then there exists a family \( B_p \subset \mathcal{P} \setminus \mathcal{G} \) such that \( \text{card} B_p = \epsilon \) and for every \( q \in B_p \) we have

\[
pq \in \mathcal{C} \quad \text{and} \quad qp \in \mathcal{C}.
\]

The proof of this result is based on the following unexpected result, a proof of which will be presented in paper [16], since we have received this result exactly in the course of preparing this paper. Thus, the proof of Theorem 3.6 is the next point of its application.

**Theorem 3.7.** For every \( p \in \mathcal{D} \) there exist a family \( A_p \subset \mathcal{C} \times \mathcal{C} \) satisfying the following conditions:

i) \( \text{card} A_p = \epsilon \),

ii) for every \( (\alpha, \delta) \in A_p \) we have \( \alpha p = p\delta \in \mathcal{D} \),

iii) if \( (\sigma, \delta), (\varphi, \psi) \in A_p \) and \( (\sigma, \delta) \neq (\varphi, \psi) \), then \( \alpha \neq \varphi \) and \( \delta \neq \psi \).
Proof of Theorem 3.6. By item i) of Lemma 3.2 we have \( p^{-1} \in P \setminus G \). Thus, from Theorem 3.7 there exists a family \( A_{p^{-1}} \subset C \times C \) possessing the respective properties i) – iii) given above. In the sequel, if \( (\sigma, \delta) \in A_{p^{-1}} \) then \( \sigma p^{-1} = p^{-1} \delta \in P \setminus G \), because of item i) of Lemma 3.2 and
\[
p(p^{-1} \delta) = \delta \in C \quad \text{and} \quad (\sigma p^{-1}) p = \sigma \in C.
\]
It is sufficient to set
\[
B_p := \{ p^{-1} \delta : (\sigma, \delta) \in A_{p^{-1}} \}.
\]
We get \( \text{card } B_p = c \) because of the condition iii) of Theorem 3.7.

Let us finish this section with one more result concerning the semigroups with one generator.

Proposition 3.8. Let \( G \subseteq S \subseteq P \). Let us suppose that \( G \) is a group of permutations and \( S \) is a semigroup of permutations such that \( (S \setminus G)^{-1} \cap S = \emptyset \). Then for any two \( p, q \in S \setminus G \) the relation
\[
pq \in S \setminus G
\]
is true as well.

In the sequel, we obtain \( \{ p^n : n \in \mathbb{N} \} \subseteq S \setminus G \) and \( \{ p^{-n} : n \in \mathbb{N} \} \subseteq P \setminus S \).

Proof. Let \( p, q \in S \setminus G \). If \( pq \in G \) then \( p^{-1} = q(pq)^{-1} \in S \) which is impossible. Similarly, if \( p^{-n} \in S \) for some \( n \in \mathbb{N} \) then also \( p^{-1} = p^{n-1}(p^{-n}) \in S \) which again contradicts the assumptions.

Corollary 3.9. Proposition 3.8 holds for the following examples of pairs \( (G, S) \):
\[
G = \mathcal{C} \mathcal{C} \quad \text{and} \quad S = \mathcal{C},
\]
\[
G = \mathcal{C} \cap \mathcal{C} \mathcal{C} \quad \text{and} \quad S = \mathcal{C} \cap \mathcal{C},
\]
\[
G = \mathcal{C} \cap G \quad \text{and} \quad S = G,
\]
\[
G = \text{Fin} \quad \text{and} \quad S = \mathcal{C} \mathcal{C} \cup S = \mathcal{C},
\]
where symbol Fin denotes the family of all almost everywhere identical permutations \( p \in P \), whereas \( \mathcal{C} \) (see [7]) denotes the family of all permutations \( p \in P \) for which there exists a finite partition \( N_1, N_2, \ldots, N_k \) of set \( \mathbb{N} \) such that all restrictions \( p|_{N_i}, i = 1, 2, \ldots, k \) are the increasing mappings. We note that \( \mathcal{C} \cap (P \setminus \mathcal{C}) \neq \emptyset \), i.e., \( \mathcal{C} \cap (P \setminus \mathcal{C}) \neq \emptyset \) (the respective example can be easily created).

4 Decomposition of the family \( \mathcal{O}(1) \) and generalization of Pleasants’ result

Let \( \{ n_k \}_{k=1}^{\infty} \) be an increasing sequence of positive integers such that
\[
\lim_{k \to \infty} \sup(n_{k+1} - n_k) = \infty
\]
and \( n_1 = 1 \). Then we denote by \( G(\{ n_k \}_{k=1}^{\infty}) \) the family of all permutations \( p \in P \) defined in the following way
\[
p|_{[n_k, n_{k+1})} \in S_{n_{k+1} - n_k},
\]
for every \( k \in \mathbb{N} \), which by definition means that
\[
p([n_k, n_{k+1})) = [n_k, n_{k+1})
\]
and
\[
q_k \in S_{n_{k+1} - n_k},
\]
where \( q_k(i) := p(n_k + i - 1) \) for every \( i = 1, 2, \ldots, n_{k+1} - n_k \) and \( k \in \mathbb{N} \).

We note the following facts.
Theorem 4.1. The family $G := G\{n_k\}_{k=1}^{\infty}$ is the group of permutations on $\mathbb{N}$ satisfying the following conditions:

i) $G \cap \mathcal{D} = G \cap \mathcal{D}(1)$ and $G \setminus \mathcal{D}(1) \subseteq \mathcal{C}$,

ii) $G \subseteq \mathcal{G}$,

iii) $G \cap \mathcal{C} \neq \emptyset$, $G \cap \mathcal{C} \neq \emptyset$, $G \cap \mathcal{C} \neq \emptyset$, and $G \cap \mathcal{D} \neq \emptyset$.

iv) $G \setminus G \neq \emptyset$.

Corollary 4.2 (P.A.B. PLEASANTS [2]). We have $G \neq \mathcal{P}$.

Corollary 4.3. We have $\mathcal{G} \neq G$.

Remark 4.4. Corollary 4.3 results also from two following facts: Stoller’s result saying that $\mathcal{D}(1)$ is algebraically big which implies that $\mathcal{G}$ is algebraically big and from Corollary 4.2 ($G$ is a group, so $G \circ G = G$).

Proof of Theorem 4.1. i) The first equality follows from the fact that

$$p([1,n_k]) = [1,n_k],$$

for every $k \in \mathbb{N}$.

ii) This inclusion is obvious (see [6]).

iii) First we note that $id_{\mathbb{N}} \in G \cap \mathcal{C} \mathcal{C}$. Next let us choose the infinite set $K \subseteq \mathbb{N}$ of indices such that

$$\lim_{K \ni k \to \infty} (n_{k+1} - n_k) = \infty$$ (7)

and

$$n_{k+1} - n_k \geq 3,$$

for every $k \in K$.

Let us set $p|_{\mathcal{I}} = id_{\mathcal{I}}$, where $\mathcal{I} := \bigcup_{k \in \mathbb{N} \setminus K} [n_k, n_{k+1})$ and

$$p(n_k - 1 + i) = n_k - 1 + 2i,$$

for every $i = 1, 2, \ldots, [(n_{k+1} - n_k + 1)/2]$, and $p$ is the increasing mapping of interval $([(n_{k+1} - n_k + 1)/2], n_{k+1})$ onto set

$$\{n_{k+1}, n_k\} \setminus p([n_k, n_k - 1 + [(n_{k+1} - n_k + 1)/2]])$$

for every $k \in K$. From (7) it follows that $p \in \mathcal{C} \mathcal{D} \cap G$. Certainly we have $p^{-1} \in \mathcal{C} \mathcal{D} \cap G$.

Let $K_1$ and $K_2$ form a partition of the set $K$ such that both of them are infinite. Now if we put

$$q(i) = \begin{cases} p(i) & \text{for } i \in [n_k, n_{k+1}) \text{ and } k \in K_1, \\ p^{-1}(i) & \text{for } i \in [n_k, n_{k+1}) \text{ and } k \in K_2, \end{cases}$$

where $p$ is defined as above and $q(j) = j$ for $j \in \mathcal{I}$, then we get $q \in \mathcal{C} \mathcal{D} \cap G$.

iv) Let us fix the increasing sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers such that

$$S_n \setminus S_n(u, u) \neq \emptyset,$$ (8)

for every $u = 1, 2, \ldots$, where $n := n_{k+1} - n_u$. Let us also fix

$$p_u \in S_n \setminus S_n(u, u),$$

for every $u = 1, 2, \ldots$, and define

$$p(i) = p_u(i)$$

for every $i \in [n_{k_u}, n_{k_{u+1}})$ and $u = 1, 2, \ldots$. Further, we set $p = id_{\mathbb{N}}$ where $N := \mathbb{N} \setminus \bigcup_{u \in \mathbb{N}} [n_{k_u}, n_{k_{u+1}})$. Certainly

$p \in G$. We intend to prove that $p \notin G$. Condition (8) implies that $p$ is not a composition of the form $q_1^{-1}q_2\ldots q_u^{-1}u$, either $q_1^{-1}q_2\ldots q_u^{-1}u$, $\mathcal{C}$ such that $c(q_i) \leq u$ for $i = 1, 2, \ldots, u$ for each $u \in \mathbb{N}$, which in view of definition of the group generated by the given set of generators means that $p \notin G$. $\square$
The following last result of this paper could be deduced from the proof of Theorem 4.1 on the basis of its proof.

**Theorem 4.5.** There exists a family \( \{G_x : x \in \mathbb{R}\} \) of subgroups of \( \mathcal{P} \) satisfying conditions i) – iv) of Theorem 4.1. Moreover, for every pair \( x, y \in \mathbb{R}, x \neq y, \) there exist the subsets \( G_x' \subset G_x \) and \( G_y' \subset G_y, \) both having the power of continuum and such that any two permutations \( p \in G_x' \) and \( q \in G_y' \) are incomparable which means that the following relations hold
\[
\sum (p) \setminus \sum (q) \neq \emptyset \quad \text{and} \quad \sum (q) \setminus \sum (p) \neq \emptyset.
\]

**Proof.** First we fix a Sierpiński’s family \( S, D \) of increasing sequences of positive integers. We can also suppose that if \( \{n_k\}_{k=1}^{\infty} \in S \) then \( \lim_{k \to \infty} (n_{k+1} - n_k) = \infty. \)

Next, with each \( \{n_k\}_{k=1}^{\infty} \in S \) we connect a family
\[
G'(\{n_k\}_{k=1}^{\infty}) \subset G(\{n_k\}_{k=1}^{\infty})
\]
defined in the following way. Firstly, for every divergent permutation \( p \) on \( \mathbb{N} \) we set
\[
q(n_k) = n_{p(k)}, \quad k \in \mathbb{N},
\]
and
\[
q|_I = \text{id}_I, \quad \text{where} \quad I = \mathbb{N} \setminus \{n_k\}_{k=1}^{\infty}.
\]

Secondly, let us define \( G' \) to be the family of all permutations \( q \) defined in this way. Certainly we have
\[
q|_{\{n_k\}_{k=1}^{\infty}} \in \mathcal{D}(\{n_k\}_{k=1}^{\infty})
\]
for every \( \{n_k\}_{k=1}^{\infty} \in S \) and permutation \( p \in \mathcal{D}, \) and since any two different sequences \( \{n_k\}_{k=1}^{\infty} \) and \( \{m_l\}_{l=1}^{\infty} \) from \( S \) are almost disjoint we get the relations
\[
\sum (\varphi) \setminus \sum (q) \neq \emptyset \quad \text{and} \quad \sum (q) \setminus \sum (\varphi) \neq \emptyset
\]
which hold for any two \( q \in G'(\{n_k\}_{k=1}^{\infty}) \) and \( \varphi \in G'(\{m_l\}_{l=1}^{\infty}). \) Both these families \( G' \) possess the power of continuum since card \( \mathcal{D} = \mathfrak{c}. \)

The above relations from the Theorem 2.1 can be deduced. \( \Box \)

**Reflection about Theorem 4.5**

Why this result is so important to us? Because in our considerations we do not want to get away from the roots of our research. We succeeded in this theorem to give the analytical weight to the distinguished subgroups of group \( \mathcal{P} \) differentiating these subgroups. It is certainly the incomparability relation of permutations related to their convergence classes. We think that this phenomenon is valuable because the excessive algebraization of this research may deprive it of the natural power and inquisitiveness resulting from its connection with the theory of series.

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