Suppression of decoherence with entanglement in the XXX central spin model

Qing-Kun Wan,1,2,∗ Hai-Long Shi,3,4,∗ Xu Zhou,2,† Xiao-Hui Wang,2,5,† and Wen-Li Yang1,5

1Institute of Modern Physics, Northwest University, Xi’an 720127, China
2School of Physics, Northwest University, Xi’an 720127, China
3State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China
4University of Chinese Academy of Sciences, Beijing 100049, China
5Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xi’an 720127, China

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Maintaining coherence of a qubit is of vital importance for realizing a large-scale quantum computer in practice. In this work, we study the central spin decoherence problem in the XXX central spin model (CSM) and focus on the quantum states with different initial entanglement, namely, intra-bath entanglement or system-bath entanglement. We analytically obtain the closed-form evolutions of fidelity, concurrence, global entanglement, and the relative entropy of coherence to describe the quantum dynamics in the CSM. For initial states with only intra-bath entanglement (the Greenberger-Horne-Zeilinger-bath or the W-bath), their leading amplitudes of fidelity evolution both scale as \( O(1/N) \) with \( N \) being the number of bath spins. However, when the central spin and the first bath spin constitute a maximal entangled pair in the initial state, the amplitude scaling of fidelity evolution declines from \( O(1/N) \) to \( O(1/N^2) \). That shows appropriate initial system-bath entanglement is contributive to suppress central spin decoherence. The role of such system-bath entanglement is further clarified by observing the trade-off relation between quantum coherence and entanglement dynamics. In addition, with the help of system-bath entanglement, we eventually realize quantum coherence-enhanced dynamics for the central spin where the consumption of bath entanglement is shown to play a central role.

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I. INTRODUCTION

A spin of a localized electron in semiconductor quantum dots (QDs) is a promising candidate for a physical matter qubit—the elementary unit of a quantum computer [1–6]. A key challenge of realizing solid-state quantum computation is suppressing electron spin relaxation, namely, decoherence, so that quantum information can be stored and manipulated without loss in a sufficiently long coherence time. Decoherence of an electron spin is induced by the inevitable hyperfine interaction with the surrounding nuclei [7–10]. This hyperfine interaction is captured by the Hamiltonian of the central spin model (CSM) as follows

\[
H = \sum_{j=1}^{N} A_j S_0 \cdot S_j
\]

(1)

where a central spin \( S_0 \) is coupled to a spin bath of \( N \) nuclei \( S_j \) via an inhomogeneous hyperfine interaction \( A_j \). Increasing attention has been payed to this model for seeking theoretical guidance of suppressing central spin decoherence [11–16].

A great deal of investigations to the decoherence problem were mainly focused on some special initial product states. For the initial state with a fully polarized bath, the central spin polarization function \( \langle S_0^z(t) \rangle \) quantifying the degree of amplitude decoherence oscillates with a frequency \( \sim \sum_{j=1}^{N} A_j \) and an amplitude of order \( O(1/N) \) about a mean value [17, 18] while for an unpolarized bath as the initial bath the corresponding oscillation with a frequency \( O(\sqrt{N}) \) and an amplitude \( O(1) \) [8, 19]. This observation indicates that the decay of the central spin can be suppressed through polarized nuclear spins [20, 21]. Recently, Floquet resonances have been suggested to realize such a polarization-based decoupling of the central spin from its environment in the CSM [11].

Entanglement, as a fundamental quantum resource, takes responsibility for most quantum information processes [22, 23]. A paradigmatic example is the quantum teleportation where the use of maximally entangled states ensures the success of deterministic remote quantum state transfer [24]. The implementation of this quantum technology in quantum dot chains or spin chains has been verified to be possible beyond the classical teleportation scheme [25, 26]. This hints that quantum information can be protected with the entanglement generated by the spin chains. A natural question arises whether entanglement contained in initial states can protect the central spin from decoherence. It was shown that decoherence of the central spin can be suppressed by using persistent entanglement in the bath [27]. And Ref. [28] identified a coherence-preserving phase by the evolution of the bath concurrence. There is, however, still a lack of a comprehensive understanding of the role of initial

∗These authors contributed equally to this work.
†Electronic address: xhwang@nwu.edu.cn
bath entanglement played in the decoherence problem, especially for initial system-bath entanglement. Moreover, in multipartite systems entanglement can transfer during the dynamical process [29, 30] as well as can be transformed into quantum coherence via local incoherent operations and classical communications provided by quantum resource theories [31]. Thus we expect to uncover the role of initial entanglement in the decoherence problem from the perspective of quantum resource.

The inhomogeneous CSM (1) is exactly solvable by the Bethe ansatz [32–34]. However, the difficulty of solving the Bethe ansatz equations prohibits full analytical access to the evolutions of quantum dynamics. We bypass this obstacle by considering the homogeneous CSM in this paper. The dynamics in the homogeneous model can be analytically calculated while providing some valid insights for the inhomogeneous CSM [18, 35]. In Sec. II, we briefly review the homogeneous CSM and its exact solutions. Some coherence and entanglement measures, such as fidelity, the relative entropy of coherence, global entanglement provide a possible understanding of CSM in Sec. IV. Eventually, we realize coherence-enhanced dynamics for some initial states with special system-bath entanglement where the consumption of bath entanglement is emphasized to explain the increase in central spin coherence. A conclusion is made in Sec. V.

II. THE CENTRAL SPIN MODEL AND QUANTUM CORRELATIONS

We consider a single electron confined in a quantum dot in which decoherence of the electron is induced by the homogeneous hyperfine interaction with surrounding nuclei. Set \( A_j = 2 \) in the Hamiltonian (1), then

\[
H = 2 \sum_{j=1}^{N} S_0 \cdot S_j, \tag{2}
\]

where \( S_0^\alpha \) and \( S_j^\alpha \) (\( \alpha = x, y, z \)) denote spin-1/2 operators of the central spin and the \( j \)-th bath spin respectively. This model is a simplified Gaudin model, yet it still exhibits rich phenomena and is an ideal model for analytically investigating the decoherence problem. By introducing a bath spin operator \( S_b = \sum_{j=1}^{N} S_j \) and a total spin operator \( S_{tot} = S_b + S_0 \), the Hamiltonian can be rewritten as

\[
H = S_{tot}^2 - S_b^2 - S_0^2, \tag{3}
\]

For a given initial state \(| \Psi(0) \rangle \), our goal is to obtain the wave function under a unitary time evolution of the Hamiltonian (2), i.e., \(| \Psi(t) \rangle = \exp(-iHt) | \Psi(0) \rangle \), then reduce the density matrix \( \rho(t) = | \Psi(t) \rangle \langle \Psi(t) | \) to some specified lattice sites, and eventually use these reduced density matrices to calculate fidelity, global entanglement, concurrence, and the relative entropy of coherence. For convenience, we use the following notation:

\[
| s_{[L]}, s_{[L-1]}, \ldots, s_{[M]}, s_{[L]} \rangle_{[L]}, \quad 1 \leq M \leq L, \tag{4}
\]

to denote a \( L \)-qubit state which is the eigenstate of \( S_{[K]}^z = (\sum_{j=L-K+1}^{L} S_j^z)^2 \) and \( S_{[K]}^z = \sum_{j=L-K+1}^{L} S_j^z \) with eigenvalues \( s_{[K]}^x(s_{[K]} + 1) \) and \( s_{[K]}^y \), respectively. For instance, the \( N \)-qubit GHZ state [48] \(| \text{GHZ} \rangle_{[N]} = (|↑⟩^N + |↓⟩^N)}/\sqrt{2} \) can be rewritten as \(| N/2, N/2, \ldots, N/2 \rangle_{[N]} \rangle \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle} \rangle_{[N]} \rangle}

where \( s_b = N/2, N/2 - 1, \ldots, 0 \) for \( N \) being even and \( s_b = N/2, N/2 - 1, \ldots, 1/2 \) for \( N \) being odd.

The calculation of \(| \Psi(t) \rangle \) is easy to carry out once we decompose \(| \Psi(0) \rangle \) into the eigenstates of the Hamiltonian (2). This decomposition process can be implemented by repeatedly using the following relations:
reduced density matrices of

More details can be found in Ref. [18]. Note that the expression of eigenstates (5) is inconvenient to evaluate the trace preserving quantum operations mapping incoher-
fined and has an operational property in which the

deftly, phase decoherence, which can be characterized via fi-
cence where the relative entropy of coherence is an excel-
larly, when \( \rho \) reaches to the minimal value zero [50]. As a
is necessary to quantitatively characterize the degree of
decoherence. A widely-used one is fidelity

where \( \rho_{0}(t) \) denotes the initial reduced density
matrix of the central spin and \( \rho_{0}(t) = T_{R}(\exp(-iHt)\rho_{0}(0)\exp(iHt)) \). Fidelity is bounded
between 0 and 1. If \( \rho_{0}(t) \) is the same as \( \rho_{0}(t) \) then the
fidelity equals to one, whereas if \( \rho_{0}(t) \) is different from
\( \rho_{0}(t) \) then the fidelity is strictly less than one. Particularly,
when \( \rho_{0}(0) \) and \( \rho_{0}(t) \) are perfectly distinguishable,
\( \rho_{0}(t) \) are supported on orthogonal subspaces, the
fidelity reaches to the minimal value zero [50]. As a
consequence, the smaller fidelity indicates the central
spin is more easily decohered.

In a broad context, decoherence refers to the changes
of quantum states, including amplitude decoherence and
phase decoherence, which can be characterized via fi-
delity, \( \langle S_{z}^{2}(t) \rangle \) or \( \langle S_{+}^{2}(t) \rangle \). In the field of quantum
information, quantum coherence is unambiguously defined
and has an operational property in which the states
without off-diagonal elements in their density ma-
trices are incoherent states and the incoherent opera-
tions \( A \rho_{\text{CPTP}} \) are defined to be completely positive and
trace preserving quantum operations mapping incoherent
states into incoherent states. By attaching other
reasonable requirements to coherence measures \( C \), e.g.,
\( C(\rho) \geq C(A\rho_{\text{CPTP}}(\rho)) \), Ref. [36] estab-
lished a rigorous mathematical framework for quantifying quantum
cohere where the relative entropy of coherence is an excel-

Quantum coherence quantified by the relative entropy
has been argue to be a quantum resource. Thus, we
will apply fidelity to characterize the decoherence pro-
blem in the next section while use the relative entropy of
coherence to investigate the possibility of dynamically
enhancing central spin coherence in Sec. IV.

The effects of entanglement on CSM dynamics can be
elicited via two entanglement measures: concurrence
and global entanglement. For bipartite entangled states,
the reduced density matrices of each subsystem are not
pure. This leads to the definition of concurrence for a
pure state \( |\psi\rangle_{01} \) as [51]

where the minimization is over all possible ensemble
decompositions \( \rho_{01} = \sum_{j} p_{j} |\psi_{j}\rangle \langle \psi_{j}| \). We will use concurrence
to calculate the evolution of bipartite entanglement
between the central spin and the first bath spin \( \rho_{01}(t) \).
The closed form of the concurrence for two-qubit state
\( |\psi_{01}\rangle \) is [51]

where \( \lambda_{1}(t), \lambda_{2}(t), \lambda_{3}(t), \lambda_{4}(t) \) are the square roots of
the eigenvalues of \( \rho_{01}(t)\rho_{01}(t) \) satisfying \( \lambda_{1}(t) \geq \lambda_{2}(t) \geq \lambda_{3}(t) \geq \lambda_{4}(t) \). \( \rho_{01}(t) \) is the reduced density matrix of
the spin 0 and 1 by tracing out other spins and \( \rho_{01}(t) =
\sigma_{0}^{y} \otimes \sigma_{0}^{y} \rho_{01}(t) \sigma_{0}^{y} \otimes \sigma_{0}^{y} \). For a \( N \)-spin bath, the global
entanglement measure will be adopted to characterize its
multipartite entanglement which is given explicitly via
[52]

Here \( S(\sigma) = -\text{Tr}[\sigma \ln(\sigma)] \) is the von Neumann entropy
and \( \rho_{0}^{D}(t) \) is obtained by deleting all off-diagonal ele-
ments in the reduced density matrix of the central spin.
where \( \rho_j(t) \) is the reduced density matrix of the \( j \)-th bath spin. This quantity not only reflects the global nature but also is an easy one to analytical calculate.

### III. DECOHERENCE PROBLEM

#### A. The Product Bath

Before discussing entangled baths, we first consider an initial state with a product bath given by

\[
|\Psi_P(0)\rangle = |\downarrow\rangle|\uparrow\uparrow\cdots\uparrow\rangle_{[N]},
\]

where the central spin is spin-down denoted by \( |\downarrow\rangle \) and \( N \) bath spins constitute a fully polarized bath denoted by \( |\uparrow\uparrow\cdots\uparrow\rangle_{[N]} \). This quantum state can be rewritten as \( |\downarrow\rangle |N/2, (N - 1)/2, N/2\rangle_{[N]} \) in our notation (4) and then by Eq. (6) we obtain the state after a unitary time evolution as follows

\[
|\Psi_P(t)\rangle = \frac{1}{\sqrt{N + 1}} e^{-\frac{i}{N + 1} \frac{2}{N} t} |N + 1, N, N - 1, N - 1\rangle_{[N + 1]}
- \frac{N}{N + 1} e^{\frac{i}{N + 1} \frac{2}{N} t} |N - 1, N, N - 1, N - 1\rangle_{[N + 1]}.
\]

Applying Eq. (7) to the above state, it becomes

\[
|\Psi_P(t)\rangle = \frac{\sqrt{N} (e^{-\frac{i}{N + 1} \frac{2}{N} t} - e^{\frac{i}{N + 1} \frac{2}{N} t})}{N + 1} |\uparrow\rangle |N, N - 1, N - 2\rangle_{[N]}
+ \frac{e^{-\frac{i}{N + 1} \frac{2}{N} t} + Ne^{\frac{i}{N + 1} \frac{2}{N} t}}{N + 1} |\downarrow\rangle |N, N - 1, N - 2\rangle_{[N]},
\]

which allows us to directly calculate the reduced density matrix for the central spin, i.e.,

\[
\rho_0^P(t) = a_{11}(t) |\uparrow\rangle \langle \uparrow| + (1 - a_{11}(t)) |\downarrow\rangle \langle \downarrow|,
\]

with \( a_{11}(t) = 2N[1 - \cos((N + 1)t)]/(N + 1)^2 \). By definition (8) the evolution of fidelity is obtained from (17) as

\[
F_0^P(t) = 1 - a_{11}(t) = 1 - \frac{2N}{(N + 1)^2} [1 - \cos((N + 1)t)].
\]

This simple form (18) provides rich insights into the central spin decoherence problem. The dynamic term in Eq. (18) describes an oscillation with no long-time decay where the frequency scales as \( O(N) \) and the amplitude scales as \( O(1/N) \) in the thermodynamic limit \( N \to \infty \), which has been pointed out in Ref. [18] by calculating \( \langle S_0^z(t) \rangle \). The absence of a long-time decay can be understood from the energy differences of eigenstates that determine transition frequencies. In homogeneous CSM, the distribution of gaps \( g_i = E_{i+1} - E_i \) of adjacent energies is almost uniform, see Eq. (5), and thus the fidelity evolution (18) displays a periodic behavior even for more complex initial state settings (30, 35). When the couplings between the central spin and the bath spins become inhomogeneous, the distribution of \( g_i \) is no longer uniform leading to a long-time decay of \( \langle S_0^z(t) \rangle \) [8, 19] but \( \langle S_0^z(t) \rangle \) will not tend to a stable value. Eventually, \( \langle S_0^z(t) \rangle \) for such a fully polarized bath reaches to a persistent oscillation with an amplitude of \( O(1/N) \) [18]. On the other hand, if the CSM with a disorder magnetic field for bath spins instead of inhomogeneous couplings, a phase transition will occur from the eigenstate thermalization hypothesis (ETH) phase to the many-body localization (MBL) phase when disorder overs interaction. Such phenomenon has been also witnessed by level statistics \( \min(g_i, g_{i+1})/\max(g_i, g_{i+1}) \) [53] and a long-time decay will occur [54, 55]. The difference from the inhomogeneous case is that the oscillation of \( \langle S_0^z(t) \rangle \) decays completely to a constant in the CSM with a disorder field. Based on the above observations, non-uniformity of level statistics caused by disorder magnetic fields or inhomogeneous couplings takes main responsibility for the emergence of long-time decays. Considering that there is no long-time decay in our model (2), we will use the scaling of leading oscillation amplitude of fidelity evolution to characterize central spin decoherence and call the scaling of leading oscillation amplitude of \( X \) the amplitude scaling of \( X \) for convenience.

The global entanglement is another important quantity to describe the evolution of bath entanglement. In order to evaluate it, we need in principle all single-qubit reduced density matrices of (15). Due to the permutation symmetry for the bath, it is enough to calculate the reduced density matrix for the first bath spin \( \rho_j^P(t) \) and the reduced density matrices of other bath spins are all the same as it, i.e., \( \rho_j^P(t) = \rho_i^P(t), j = 2, 3, \ldots, N \). Using Eq. (7) again, the quantum state of (16) becomes

\[
|\Psi_P(t)\rangle = \frac{\sqrt{N - 1} (e^{-\frac{i}{N + 1} \frac{2}{N} t} - e^{\frac{i}{N + 1} \frac{2}{N} t})}{N + 1} |\uparrow\rangle |N - 1, N - 3\rangle_{[N - 1]}
+ \frac{e^{-\frac{i}{N + 1} \frac{2}{N} t} + Ne^{\frac{i}{N + 1} \frac{2}{N} t}}{N + 1} |\downarrow\rangle |N - 1, N - 3\rangle_{[N - 1]},
\]

where the subscript 1 refers to the first bath spin. Form the above expression, it is easy to check that \( \rho_j^P(t) \) has the form of

\[
\rho_j^P(t) = b_{11}(t) |\uparrow\rangle \langle \uparrow| + (1 - b_{11}(t)) |\downarrow\rangle \langle \downarrow|,
\]

with \( b_{11}(t) = 1 - 2[1 - \cos((N + 1)t)]/(N + 1)^2 \). Substituting Eq. (20) into the definition of global entanglement (13) we get

\[
E_0^P(t) = \frac{8}{(N + 1)^2} [1 - \cos((N + 1)t)] - \frac{16}{(N + 1)^2} [1 - \cos((N + 1)t)]^2.
\]

The first term of Eq. (21) determines the amplitude scaling of bath entanglement, \( O(1/N^2) \), that is smaller than the amplitude scaling of fidelity (18) \( O(1/N) \). This shows that the central spin does not receive an intense impact
from the fully polarized bath and the $O(1/N)$-oscillation of fidelity indicates that a strong magnetic field can suppress decoherence by polarizing a large number of bath spins [20, 21].

B. Entangled Baths

Now we try to seek a decoherence suppression beyond $O(1/N)$ by considering entangled baths, namely, the GHZ-bath: $|GHZ\rangle_{[N]} = (|\uparrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle)/\sqrt{2}$ and the $W$-bath [56]: $|W\rangle_{[N]} = (|\uparrow\rangle \otimes \cdots \otimes |\uparrow\rangle + |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle)/\sqrt{N}$. The initial states are given by

$$
|\psi_{GHZ}(0)\rangle = |\uparrow\rangle |GHZ\rangle_{[N]}
$$

$$
|\psi_{W}(0)\rangle = |\uparrow\rangle |W\rangle_{[N]}
$$

Similarly we use Eq. (6) to determine the states after a unitary time evolution,

$$
|\psi_{GHZ}(t)\rangle = \frac{1}{\sqrt{N(N+1)}}
\begin{pmatrix}
N+1 & N & N-1 \\
2 & 2 & 2
\end{pmatrix}_{[N+1]} |\uparrow\rangle
+ \frac{1}{\sqrt{N(N+1)}}
\begin{pmatrix}
N & N-1 & N-1 \\
2 & 2 & 2
\end{pmatrix}_{[N+1]} |\downarrow\rangle
$$

$$
|\psi_{W}(t)\rangle = \frac{1}{\sqrt{N}}
\begin{pmatrix}
N+1 & N & N-1 \\
2 & 2 & 2
\end{pmatrix}_{[N]} |\uparrow\rangle
+ \frac{1}{\sqrt{N}}
\begin{pmatrix}
N & N-1 & N-3 \\
2 & 2 & 2
\end{pmatrix}_{[N]} |\downarrow\rangle
$$

and then in virtue of Eq. (7) to obtain the reduced density matrices with respect to the central spin and the first bath spin, i.e.,

$$
\rho_{01}^{GHZ}(t) =
\begin{pmatrix}
c_{11}(t) & 0 & 0 & 0 \\
0 & c_{22}(t) & c_{23}(t) & 0 \\
0 & c_{23}(t)^* & c_{33}(t) & 0 \\
0 & 0 & 0 & c_{44}(t)
\end{pmatrix},
$$

$$
\rho_{01}^{W}(t) =
\begin{pmatrix}
d_{11}(t) & 0 & 0 & 0 \\
0 & d_{22}(t) & d_{23}(t) & 0 \\
0 & d_{23}(t)^* & d_{33}(t) & 0 \\
0 & 0 & 0 & d_{44}(t)
\end{pmatrix},
$$

where the matrix elements read:

$$
c_{22}(t) = -\frac{1}{(N+1)^2} \cos((N+1)t) - 1,
$$

$$
c_{23}(t) = \frac{1}{2(N+1)^2} - \frac{N}{2(N+1)^2} + \frac{N}{2(N+1)^2} e^{-it(N+1)}
- \frac{1}{2(N+1)^2} e^{it(N+1)},
$$

$$
c_{33}(t) = \frac{1}{2(N+1)^2} + \frac{N^2}{2(N+1)^2} + \frac{N}{(N+1)^2} \cos((N+1)t),
$$

$$
c_{44}(t) = \frac{1}{2(N+1)^2} - \frac{N}{2(N+1)^2} + \frac{N}{2(N+1)^2} \cos((N+1)t),
$$

$$
d_{11}(t) = \frac{4(N-1)(N-2)}{N(N+1)^2} [1 - \cos((N+1)t)],
$$

$$
d_{22}(t) = \frac{8(N-1)}{N(N+1)^2} [1 - \cos((N+1)t)],
$$

$$
d_{33}(t) = 1 - \frac{4}{N} + \frac{4(N-1)^2}{N(N+1)^2} \cos((N+1)t) - 1,
$$

$$
d_{44}(t) = 1 - d_{11}(t) - d_{22}(t) - d_{33}(t),
$$

$$
d_{23}(t) = \frac{2(N-1)^2}{N(N+1)^2} [e^{-it(N+1)} - 1]
- \frac{4(N-1)}{N(N+1)^2} [e^{-it(N+1)} - 1].
$$

With these explicit expressions of reduced density matrices, it is effortless to calculate fidelity for the central spin and global entanglement for the bath. The corresponding results are as follows

$$
F_0^{GHZ}(t) = 1 - \frac{N}{(N+1)^2} [1 - \cos((N+1)t)],
$$

$$
E_b^{GHZ}(t) = 1 - \frac{4}{(N+1)^2} [1 - \cos((N+1)t)],
$$

$$
F_0^{W}(t) = 1 - \frac{4(N-1)}{(N+1)^2} [1 - \cos((N+1)t)],
$$

$$
E_b^{W}(t) = \frac{16(N-1)(N-2)}{N^2(N+1)^2} [\cos((N+1)t) - 1]^2
- \frac{4(N-1)}{N^2(N+1)^2} [\cos((N+1)t) - 1].
$$

It is observed from Eqs. (26) and (28) that the amplitude scalings of fidelity for the GHZ-bath and for the $W$-bath are both $O(1/N)$ although the $W$ state and the GHZ state belong to distinct entanglement classes in the entanglement classification problem [57]. The same amplitude scaling of fidelity for the entangled baths and the product bath (18) indicates that such entangled baths (22) can not provide a more effective decoherence suppression. One possible reason for this phenomenon is that the initial states we have considered so far are all in a product form i.e. $|\uparrow\rangle \otimes |\downarrow\rangle$, lacking of system-bath entanglement, which causes the failure of bath entanglement to affect the dynamics of the central spin intensely. To demonstrate this point, we extract the amplitude scalings of global entanglement from Eqs. (27, 29): $O(1/N^2)$ for the GHZ-bath case and $O(1/N^2)$ [59] for the $W$-bath case. They are less than or equal to the product bath case $O(1/N^2)$ (21). Therefore one expects the states with system-bath entanglement will establish more effective suppressions of decoherence than product states.
C. System-Bath Entangled Pairs

The initial state with a maximal entangled pair between the central spin and the first bath spin, 
\[
\psi_{EP}(0) = \frac{1}{\sqrt{2}} (|\uparrow\rangle |\uparrow\rangle + |\downarrow\rangle |\downarrow\rangle |\uparrow\uparrow\cdots\uparrow\rangle)_{[N-1]}.
\] (30)
is considered in this subsection. As before we use Eqs. (6) and (7) to derive the reduced density matrices after a unitary time evolution of the Hamiltonian (2),
\[
\rho_{EP}^2(t) = \begin{bmatrix}
1 - e_{22}(t) & 0 & 0 \\
0 & e_{22}(t) & 0 \\
0 & 0 & 0
\end{bmatrix},
\]
\[
\rho_{01}^{EP}(t) = \begin{bmatrix}
f_{11}(t) & 0 & 0 & 0 \\
0 & f_{22}(t) & f_{23}(t) & 0 \\
0 & f_{23}(t) & f_{33}(t) & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\] (31)
where \(\rho_{EP}^2(t)\) denotes the reduced density matrix of the second bath spin and \(\rho_{01}^{EP}(t)\) denotes the reduced density matrix of the central spin and the first bath spin. The matrix elements in Eq. (31) are given by
\[
e_{22}(t) = -\frac{2}{N(N+1)} \cos t + (N-1) - N^2/(N+1) \cos(Nt) - 1 - \frac{2(N-1)}{N(N+1)^2} \cos((N+1)t) - 1,
\]
\[
f_{22}(t) = \frac{1}{2} + \frac{2(N-1)}{N(N+1)} \cos t - (N-1)^2/(N+1) \cos(Nt) - 1 - \frac{2(N-1)}{N(N+1)^2} \cos((N+1)t) - 1,
\]
\[
f_{23}(t) = \frac{1}{2} + \frac{N-1}{(N+1)^2} e^{-i(N+1)t} - 1 + \frac{N-1}{N(N+1)} e^{-it} - 1 - \frac{N-1}{2N(N+1)^2} e^{-i(N+1)t} - 1,
\]
\[
f_{33}(t) = \frac{1}{2} + \frac{2(N-1)}{(N+1)^2} \cos((N+1)t) - 1,
\]
\[
f_{11}(t) = 1 - f_{22}(t) - f_{33}(t).
\] (32)
The fidelity of the central spin is founded by substituting \(\rho_{01}^{EP}(t)\) into the definition (8),
\[
F_0^{EP} = \frac{1}{2} \left(1 + \sqrt{1 - \left(\frac{2(N-1)\cos((N+1)t) - 1}{N^2} \right)^2}\right).
\] (33)
Note that when considering the initial state (30) we need two reduced density matrices \(\rho_{1}^{EP}(t)\) and \(\rho_{2}^{EP}(t)\) \([= \rho_{EP}^j(t), j = 3, 4, \ldots, N]\) to determine the evolution of global entanglement, which is as follows
\[
E_0^{EP} = 2 - \frac{2}{N} \Tr(\rho_1^{EP}(t)^2) - \frac{2(N-1)}{N} \Tr(\rho_2^{EP}(t)^2)
\]
\[
= \frac{4f_{22}(t)}{N} (1 - f_{22}(t)) + \frac{4(N-1)}{N} e_{22}(t) (1 - e_{22}(t))
\]
\[
\approx \frac{1}{N} - \frac{8(N-1)}{(N+1)N^2} \cos t - 1
\]
where the higher-order oscillation terms are omitted in the thermodynamic limit.

A direct simplification of Eq. (33) gives the amplitude scaling of fidelity being \(O(1/N^2)\). That is to say, the state with a system-bath entangled pair (30) outperforms than the product state (14) in suppressing of central spin decoherence. To emphasize the role of the entangled pair, we use the reduced density matrix \(\rho_{01}^{EP}(t)\) to calculate its bipartite entanglement \(E_{01}^{EP}(t)\) measured by the concurrence (12). Fig. 1 shows an obvious trade-off relation between \(E_0^{EP}(t)\) and \(E_{01}^{EP}(t)\), i.e., the bipartite entanglement between the entangled pair decreases as the bath entanglement increases. Therefore, the central spin can build a strong dynamics correlation with the bath spins where the entangled pair acts as a bridge and finally an \(O(1/N^2)\) suppression of decoherence occurs.

Figure 1: Time evolutions of fidelity \(F_0(t)/F_{0\text{max}}\) for the central spin, global entanglement \(E_0(t)/E_{0\text{max}}\) for the bath, and concurrence \(E_{01}(t)/E_{01\text{max}}\) for the entangled pair. The initial state is selected as (30) and the number of bath spins is set to \(N = 50\). The lines from top to bottom represent \(F_0(t)/F_{0\text{max}}, E_0(t)/E_{0\text{max}},\) and \(E_{01}(t)/E_{01\text{max}}\), respectively.

IV. ENTANGLEMENT ENHANCES COHERENCE

In the previous section, fidelity is utilized to quantify decoherence of the central spin. The reduced density matrices of the central spin \(\rho_0(t)\) discussed before do not contain off-diagonal elements \((|\uparrow\rangle \langle\downarrow|, |\downarrow\rangle \langle\uparrow|)\) during time evolutions and thus only the effect of amplitude decoherence is involved. In this section, we fix our attention on the phase decoherence problem and aim to improve the quantum coherence of the central spin encoded in the off-diagonal elements of its density matrix.
The relative entropy of coherence $C_0(t)$ (9) is applied in here to characterize such quantum coherence since $C_0(t)$ is zero if and only if the reduced density matrix of the central spin has only diagonal elements.

The ability of entangled pairs to prevent central spin decoherence is shown in Sec. (III). Similarly, one expects to improve the coherence of the central spin during the dynamics with the help of some system-bath entangled pairs. Then, we consider a set of initial states as follows

$$\rho(\theta) = \rho^{EP}_{[2]}(\theta) \otimes \rho^{bath}_{[N-1]},$$

where the central spin and the first bath spin constitute an entangled pair $\rho^{EP}_{[2]}$ while all bath spins except the first bath spin constitute a $(N-1)$-qubit product state $\rho^{bath}_{[N-1]} = |\uparrow \uparrow \cdots \uparrow\rangle \langle \uparrow \uparrow \cdots \uparrow|_{[N-1]}$. The parameter $\theta$ is required to adjust the bipartite entanglement of the entangled pair but, at the same time, to keep the coherence of the central spin unchanged. It was proved in Ref. [58] that an arbitrary non-maximal entangled two-qubit pure state $|\phi_{01}\rangle = a|\uparrow \uparrow\rangle + b|\uparrow \downarrow\rangle + c|\downarrow \uparrow\rangle + d|\downarrow \downarrow\rangle$ can be parametrized in terms of six angles $(\chi, \theta_0, \theta_1, \phi_0, \phi_1, \gamma)$:

$$a = \left[ \cos \chi \cos \theta_0 \cos \theta_1 \cos \frac{\phi_0 + \phi_1}{2} + \sin \chi \sin \theta_0 \sin \theta_1 \cos \frac{\phi_1 - \phi_0}{2} \right] e^{-i\theta_0 + \phi_1},$$

$$b = \left[ \cos \chi \cos \theta_0 \sin \theta_1 \cos \frac{\phi_0 + \phi_1}{2} - \sin \chi \sin \theta_0 \cos \theta_1 \sin \frac{\phi_1 - \phi_0}{2} \right] e^{i\theta_0 - \phi_1},$$

$$c = \left[ \cos \chi \sin \theta_0 \cos \theta_1 \cos \frac{\phi_0 - \phi_1}{2} + \sin \chi \sin \theta_0 \sin \theta_1 \sin \frac{\phi_1 + \phi_0}{2} \right] e^{i\theta_0 + \phi_1},$$

$$d = \left[ \cos \chi \sin \theta_0 \sin \theta_1 \cos \frac{\phi_0 - \phi_1}{2} - \sin \chi \sin \theta_0 \cos \theta_1 \sin \frac{\phi_1 + \phi_0}{2} \right] e^{-i\theta_0 - \phi_1}.$$

This parameterization has a geometric intuition. For instance, the reduced density matrix of the central spin can be expressed as

$$\rho_0 = \text{Tr}_1(|\phi\rangle \langle \phi|) = \frac{I + r_0 \cdot \sigma}{2},$$

with $r_0 = (\cos \chi, \theta_0, \phi_0)$ in spherical coordinates and the reduced density matrix of the first bath spin is in the same form (36) with $r_1 = (\cos \chi, \theta_1, \phi_1)$. The parameter $\chi$ is not only related to the norms of the Bloch vectors $r_0$ and $r_1$, but also determines the value of the concurrence by $E_{01}(|\phi\rangle) = \sin \chi$. Note that this parameterization (36) excludes the maximal entangled two-qubit pure state, i.e., $\chi \neq \pi/2$, since for $\chi = \pi/2$ the norms of the Bloch vectors $r_0$ and $r_1$ both are zero, which is absurd. Then we take $\theta_1$ as the parameter $\theta$ in Eq. (35) and fix the other angles $(\chi, \theta_0, \phi_0, \phi_1, \gamma)$ to $(\pi/3, \pi/2, 0, 0, 0)$ as an example. In this setting, the reduced density matrix of the central spin is a constant matrix and thus the coherence of the central spin no longer depends on the parameter $\theta$. However, at this time, the concurrence is also constant due to $\chi$ being fixed. According to the definition (11), concurrence depends on the optimal ensemble decomposition of a given density matrix. The unique ensemble decomposition of a quantum state $\rho$ up to a phase factor exists only when $\rho$ is a pure state. Thus, we mix the state $|\phi\rangle$ and $(|\uparrow \uparrow\rangle \langle \uparrow \uparrow| + |\downarrow \downarrow\rangle \langle \downarrow \downarrow|)/2$ with equal possibility 1/2 to construct a set of entangled pairs,

$$\rho^{EP}_{[2]}(\theta) = \left( \frac{1}{2} |\phi\rangle \langle \phi| + \frac{1}{4} |\uparrow \uparrow\rangle \langle \uparrow \uparrow| + \frac{1}{4} |\downarrow \downarrow\rangle \langle \downarrow \downarrow| \right),$$

and expect their optimal decompositions to be different for different $\theta$. Here,

$$|\phi\rangle = a|\uparrow \uparrow\rangle + b|\uparrow \downarrow\rangle + c|\downarrow \uparrow\rangle + d|\downarrow \downarrow\rangle,$$

$$a = \frac{\sqrt{6}}{4} \cos \theta + \frac{\sqrt{2}}{4} \sin \theta, \quad b = \frac{\sqrt{6}}{4} \sin \theta - \frac{\sqrt{2}}{4} \cos \theta,$$

$$c = \frac{\sqrt{6}}{4} \cos \theta - \frac{\sqrt{2}}{4} \sin \theta, \quad d = \frac{\sqrt{6}}{4} \sin \theta + \frac{\sqrt{2}}{4} \cos \theta.$$

For the entangled pair (37), the reduced density matrix of the central spin reads

$$\rho_0(\theta) = \frac{2I + r_0 \cdot \sigma}{4},$$

with $r_0 = (1/2, \pi/2, 0)$ in spherical coordinates and the value of coherence is a constant of $|5(\log_2 5)/8 + 3(\log_3 3)/8 - 2| \simeq 0.0456$ according to Eq. (9). In Fig. 2(b) we plot the concurrence of the entangled pair (37) versus $\theta$ where the concurrence first increases, then remains constant, and finally decreases to the initial value.

Having a set of initial states (35) with the same initial coherence value for the central spin but different initial system-bath entanglement values, we are going to investigate their dynamics. The explicit expression of $\rho_0(t; \theta) = \text{Tr}_{2,3,...,N}(e^{-iHt}\rho(\theta)e^{iHt})$ is found in the Appendix. Omitting $O(1/N)$ terms in $\rho_0(t; \theta)$, the reduced density matrix of the central spin is simplified to

$$\rho_0(t; \theta) \simeq \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{12}^*(t) & 1 - A_{11}(t) \end{bmatrix},$$

with $A_{11}(t) \simeq 1/2$ and $A_{12}(t) \simeq 1/8 + ac(e^{-it(N+1)} - 1)/2 + bd(e^{-it(N-1)} - 1)/2$. It follows that the evolution of central spin coherence is

$$C_0(t; \theta) \simeq 1 - H_b(\lambda(t; \theta)),$$

where $H_b(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ is the binary entropy function and $\lambda(t; \theta)$ is one of the eigenvalues of $\rho_0(t; \theta)$,

$$\lambda(t; \theta) = \frac{1}{2} + \frac{1}{8} \sqrt{1 + \left( \frac{1}{2} + \cos(2\theta) \right) (1 - \cos(2t))}.$$

Notice that $H_b(x) = H_b(1-x)$ and $H_b(x)$ is a monotonically decreasing function of $x$ when $1/2 \leq x \leq 1$.

According to Eqs. (41) and (42), we observe that when $0 \leq \theta < \pi/3$, i.e., $[1/2 + \cos(2\theta)] > 0$, the coherence of the central spin $C_0(t; \theta)$ first increases and then declines to its initial value in a time period, (see Fig. 2(c)), while $C_0(t)$ behaves just opposite as $\pi/3 < \theta \leq \pi$, (see Fig. 2(d)). Therefore, for the system-bath entangled pairs $\rho^{EP}_{[2]}(\theta)$ with the condition of $0 \leq \theta < \pi/3$, the central spin coherence will increase over time.
To shed light on the origin of such dynamics behaviors, the bath entanglement (13) is plotted in Figs. 2(c)-2(f) by means of the reduced density matrices (46) given in the Appendix. As seen from Figs. 2(c)-2(d) and 2(e)-2(f), the trade-off relation holds for the evolutions of central spin coherence $C_0(t)$ and bath entanglement $E_b(t)$. Thus, in the case of $0 \leq \theta < \pi/3$, the increase in central spin coherence is derived from the loss of bath entanglement. The initial coherence is, however, nonzero only for the central spin and the first bath spin. It is necessary to calculate the evolution of coherence for the first bath spin $C_1(t)$ to exclude the possibility of coherence transfer between the central spin and the first bath spin leading to the gain of coherence in the central spin.

Similarly, we omit the $\mathcal{O}(1/N)$ terms in $\rho_{01}(t; \theta)$ (46) and trace out its central spin degrees of freedom to obtain the evolution of the reduced density matrix for the first bath spin,

$$\rho_1(t; \theta) \simeq \begin{pmatrix} 1 - B_{22}(t) & B_{12}(t) \\ B_{12}^*(t) & B_{22}^*(t) \end{pmatrix},$$

with $B_{22}(t) \simeq 1/4 + d^2/2 + b^2/2$ and $B_{12}(t) = ab\exp(-it)/2 + cd\exp(it)/2$. By definition (9), the coherence of the first bath spin is given by

$$C_1(t; \theta) \simeq H_b\left(\frac{1}{2} + \frac{\cos \theta}{8}\right) - H_b(\lambda'(t; \theta)), \quad (44)$$

where

$$\lambda'(t; \theta) = \frac{1}{2} + \frac{1}{8} \sqrt{\frac{3}{2} + \cos(2\theta) - \left(\frac{1}{2} + \cos(2\theta)\right)\cos(2t)}.$$

It is obvious from Eqs. (41-42) and (44-45) that the monotonicity of $C_1(t)$ and $C_0(t)$ is the same, which confirms that the bath entanglement is the main source of coherence gains in the central spin as $0 \leq \theta < \pi/3$.

### V. CONCLUSION

We investigated the role of bath entanglement in the central spin decoherence problem by obtaining exact evolutions of fidelity, global entanglement, concurrence, and the relative entropy of coherence. The closed-form expressions of them have been obtained in this paper and in the thermodynamic limit we extracted their amplitude scalings summarized in Tab. I. Here the degree of decoherence for the central spin is characterized by the amplitude scaling of fidelity $F_0(t)$ while the bath entanglement is described by the global entanglement $E_b(t)$. For the initial states with only bath entanglement, the entangled bath ($W$-bath or GHZ-bath) do not outperform than the product bath in suppression of central spin decoherence. In contrast to the above situation, however, the

| Scalings Product bath | $W$-Bath | GHZ-Bath | Entangled pair |
|-----------------------|---------|----------|---------------|
| $F_0(t)$              | $O(1/N)$| $O(1/N)$| $O(1/N)$ |
| $E_b(t)$              | $O(1/N^2)$| $O(1/N^2)$| $O(1/N^2)$ |

| Table I: Amplitude scalings of four different initial states |
state with a maximal entangled pair between the central spin and the first bath spin establishes an $O(1/N^2)$ decoherence suppression, as well as, the amplitude scaling of bath entanglement is the same as $O(1/N^2)$, which highlights the potential ability of initial system-bath entanglement to provide effective suppressions of decoherence. Additionally, such a system-bath entangled pair leads to the trade-off relation between $E_0(t)$ and $E_{01}(t)$ implying a dynamic correlation between the central spin and the bath spins. Since quantum coherence is embedded in the off-diagonal elements of the density matrix, the relative entropy of coherence has been used to describe coherence dynamics for the central spin. We provided a method to construct some initial states with suitable entangled pairs where the coherence of the central spin can be improved over its initial value in dynamics. By calculating the evolutions of coherence and bath entanglement, we confirmed that this part of increased coherence comes from the consumption of the bath entanglement. Our research reveals the significance of system-bath entangled pairs in the use of bath entanglement to suppress decoherence of the central spin.

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**APPENDIX**

The reduced density matrices of the entangled pairs (37) and of the second bath spin at arbitrary times are given by

$$\rho_{01}(t; \theta) = \begin{bmatrix} D_{11}(t) & D_{12}(t) & D_{13}(t) & D_{14}(t) \\ D_{21}(t) & D_{22}(t) & D_{23}(t) & D_{24}(t) \\ D_{31}(t) & D_{32}(t) & D_{33}(t) & D_{34}(t) \\ D_{41}(t) & D_{42}(t) & D_{43}(t) & D_{44}(t) \end{bmatrix}, \quad \rho_2(t; \theta) = \begin{bmatrix} 1 - E_{22}(t) & E_{12}(t) \\ E_{12}(t) & E_{22}(t) \end{bmatrix}, \quad (46)$$

where the matrix elements are

$$D_{11}(t) = \frac{1}{4} + \frac{\alpha^2}{2} - \frac{(1 + 2d^2)(N - 2)(\cos((N - 1)t) - 1)}{N^2(N - 1)} - \frac{2(1 + 2d^2)(N - 2)}{(N + 1)^2N^2}(\cos t - \cos(Nt))$$

$$+ \frac{N - 1}{(N + 1)^2N} \left\{ \frac{-N - 2}{N} + (b + c)(b - cN) \right\} \left\{ \cos((N + 1)t) - 1 \right\}$$

$$+ \frac{-b(b + c)(N - 1)}{(N + 1)^2N}(\cos t - 1) + \frac{b(N - 1)(b - cN)}{(N + 1)^2N} \left\{ \cos(Nt) - 1 \right\},$$

$$D_{12}(t) = \frac{b^2}{2} - \frac{2 + d^2(N - 2)^2}{N^2(N - 1)} \left\{ \cos((N - 1)t) - 1 \right\} + \frac{\cos t - 1}{N(N + 1)} \left\{ \frac{2(N - 2)}{N} + \frac{4d^2(N - 2)}{N} + b(b + c)(N - 1) \right\}$$

$$+ \frac{\cos(Nt) - 1}{(N + 1)^2N} \left\{ \frac{-2(N - 2) - 4d^2(N - 2) + b(b - cN)(N - 1)}{N} \right\},$$

$$D_{44}(t) = \frac{2d^2 + 1}{4(N + 1)^2N^2} \left\{ 4(N - 1) \cos((N - 1)t) + [16(N + 1) - 10(N + 1)^2 + 2(N + 1)^3] \cos t \right\}$$

$$+ \frac{[6(N + 1)^2 - 16(N + 1)] \cos(Nt) + [2(N + 1)^3 - 6(N + 1)^2] \cos((N - 1)t)}{N}$$

$$+ \frac{8 - 4(N + 1) + 11(N + 1)^2 - 6(N + 1)^3 + (N + 1)^4}{N},$$

$$D_{12}(t) = \frac{ab}{2N} + \frac{ab(N - 1)}{2N} \left( e^{-it(N + 1)} - 1 \right)$$

$$+ \frac{1}{2N(N + 1)(N - 1)} \left\{ \frac{2bd(N - 1)^2}{N} \left( e^{-itN} - 1 \right) + \frac{bd(N + 1)(N - 1)(N - 2)}{N} \left( e^{-it(N - 1)} - 1 \right) \right\}$$

$$- \frac{(b + c)(N - 1)(N - 2)}{N(1 + 2d^2)(N - 2)} \left( e^{-itN} - 1 \right) + \frac{2d(b + c)(N - 1)^2}{N(1 + 2d^2)(N - 2)} \left( e^{-it(N + 1)} - 1 \right) - \frac{2d(N - 1)^2}{N} \left( e^{it} - 1 \right).$$
\[ D_{13}(t) = \frac{ac}{2} - \frac{bd(N-1)^2 + d(b+c)(N-2)N}{2(N+1)^2N^2} (e^{-itN} - 1) + \frac{bd(N-2)}{2N^2} (e^{-it(N-1)} - 1) - \frac{bd(N-1)}{(N+1)^2N^2} (e^{it} - 1) \]

\[ D_{14}(t) = \frac{ad}{2} + \frac{ad(N-2)}{2N} (e^{-it} - 1) + \frac{ad(N-1)}{2N(N+1)} (e^{-it(N-1)} - 1) , \]

\[ D_{23}(t) = \frac{bc}{2} - \frac{d^2}{2} \left\{ \frac{2 + (N-1)(N-2)}{(N+1)^2N^2} (e^{-it} - 1) + \frac{2(N-3)}{(N+1)^2N^2} (e^{-itN} - 1) + \frac{4(N-1)}{(N+1)^2N^2} (e^{-it(N+1)} - 1) \right\} \]

\[ D_{24}(t) = \frac{bd}{2} + \frac{d(b+c)}{2N(N+1)} (e^{-it} - 1) + \frac{d(b+c)(N-1)}{2N(N+1)^2N^2} (e^{-it(N+1)} - 1) + \frac{d(b-cN)}{2N(N+1)} (e^{-it} - 1) , \]

\[ D_{34}(t) = \frac{cd}{2} + \frac{d(b+c)}{2N(N+1)} (e^{-it} - 1) - \frac{d(b-cN)}{(N+1)^2N^2} (e^{-it(N+1)} - 1) + \frac{bd(N-1)}{2(N+1)^2N^2} (e^{-it} - 1) + \frac{bd(N-1)(N-2)}{2N(N+1)} (e^{-it(N-1)} - 1) , \]

\[ E_{22}(t) = \frac{1}{(N+1)^2N^2(N-1)^3} \left( - \frac{1+2d^2}{2} (N^2 - 3N + 4) - 2(1 + 2d^2)(N-2) - b(b+c)N(N-1) \right) (\cos t - 1) \]

\[ + \frac{N-2}{N^2(N-1)^2} \left( \frac{1}{2} - d^2(N+1) - \frac{2(N-1)}{N-2} \right) (\cos(N-1)t) - 1 \]

\[ + \frac{1}{(N+1)^2N(N-1)} \left( (N-3)d^2 + \frac{1}{2} - b(b-cN)(N-1) \right) (\cos(Nt) - 1) \]

\[ + \frac{1}{(N+1)^2N^2(N-1)} \left( (N-1)d^2 + \frac{1}{2} + (b+c)N(N-1) \right) (\cos((N+1)t) - 1) \]

\[ E_{12}(t) = \frac{-bd}{(N+1)^2N^2} (e^{-itN} - 1) + \frac{1}{2N(N+1)} (b-bN)(N-1) \left( a - \frac{2d}{N+1} \right) \frac{d(b+c)(N-1)}{N+1} (e^{-it(N+1)} - 1) \]

\[ + 2(N+1)^2N(N-1) \left[ \frac{d(b-cN)(N-2)(N+1) + 2bd(N-1)}{(N+1)^2N} (e^{it} - 1) - \frac{ab}{2N} + \frac{d(b+c)}{2(N+1)^2N^2} \right] (e^{-it} - 1) \]

\[ + \frac{d(b-cN)(N-1)}{(N+1)^2N^2} (e^{it(N+1)} - 1) - \frac{bd(N+1)(N-2)}{2N^2(N-1)} (e^{-it(N-1)} - 1) + \frac{d(b-cN)}{(N+1)^2N^2(N-1)} \left( e^{itN} - 1 \right) . \]
