Scalar fields in (2+1) dimensions coupled to Einstein-Cartan gravity

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We couple a conformal scalar field in (2+1) dimensions to Einstein-Cartan gravity. The field equations are obtained by a variational principle. Einstein-Cartan equations are not solved analytically. These equations are solved numerically with 4th order Runge-Kutta method. From the numerical solution, we make an ansatz for the rotation parameter in the proposed metric, which gives an analytical solution for the scalar field for asymptotic regions.

I. INTRODUCTION

In a series of three recent papers [1, 2] dark energy is explained in terms of metric-scalar couplings with torsion. In these papers, Sur and Bhatia discuss ”the replacement of the cosmological constant $\Lambda$ with a scalar field non-minimally coupled to curvature and torsion to overcome the problems of the cosmological constant due to the ”fine tuning” problem [4–10] to explain the driving of the late time cosmic acceleration [3, 11, 12].” Sur and Bhatia state that [3] ”although observations greatly favor the $\Lambda$CDM model (cold dark matter with $\Lambda$) [13–16] there is still some room for models which replace the cosmological constant with scalar field coupled to torsion”. This fact motivates researchers to investigate models which couple fields with torsion as well as curvature.

The Einstein-Cartan theory is the simplest generalization of Einstein’s general relativity theory, allowing the possibility of relating space-time with torsion. It reduces to Einstein’s original theory when the torsion vanishes. The Einstein’s general relativity is in agreement with all experimental facts in the domain of macrophysics. It has been argued, however, in the microscopic level space-time must have a non-vanishing torsion, and so, microscopic gravitational interactions should be described by the Einstein-Cartan theory [17]. It has been also shown that torsion is required for a complete theory of gravitation [18]. The spin of matter, as well as its mass plays a dynamical role in Einstein-Cartan theory. All the available theoretical evidence that argues for admitting spin and torsion into a gravitational theory is summarized in Refs. [19–21].

The spin-gravity coupling has been paid much attention and appeared in the work of several authors, who have been mainly interested in the study of the matter fields, namely, scalar, gauge, and spinor fields [22, 23].

Among the other recent papers on the arXiv for Einstein-Cartan theory, one can cite the paper by Ivanov and Wellenzohn [24] where the torsion field acts as the origin of the cosmological constant or dark energy density. Still another paper treats helicity effects of solar neutrinos using a dynamic torsion field [25].

The non-minimally coupled scalar field is of interest for general relativistic gravitational theories, and plays an important role in inflationary cosmology [26]. Exact general solutions of the Einstein-Cartan equations for open Friedmann models containing a non-minimally coupled scalar field with an arbitrary coupling constant have been obtained [27]. Galiakhmetov continued working on this field and wrote several papers where the scalar field coupled to torsion and curvature gave rise to interesting results [28, 32].

Studying models in lesser dimensions to disclose some properties of similar models in (3+1) dimensions has been a common method in quantum field theory. General relativity in (2+1) dimensions has become an increasingly popular endeavor to understand the basic features of the gravitational dynamics [33].

The study of (2+1) dimensional gravity led to a number of outstanding results, among which the discovery of the Bañados, Teitelboim and Zanelli (BTZ) black hole was of particular importance [34, 35]. Einstein gravity in (2+1) dimensions coupled to a scalar field is studied in the literature [36–40]. The interest in Einstein-Cartan theory of gravity in (2+1) dimensions has also grown in recent years [41, 46].

In this work we study the Einstein-Cartan gravity in (2+1) dimensions conformally coupled to a scalar field. Conformally coupling refers to the fact the matter term in the action is invariant under conformal transformations. We should note that in the framework of Einstein-Cartan theory, a scalar field non-minimally coupled to gravity gives rise to torsion, even though the scalar field has zero spin. In this paper, we present the geometrical apparatus necessary for the formulation of the Einstein-Cartan theory. By variation of the action function with respect to metric and torsion we obtain the field equations in a general form. We attempt to solve these equations numerically. Studying the plots of these equations, we make an ansatz for the rotation parameter in the proposed metric ending up in an analytical solution for the scalar field.

The work organized as follows: In Section 2 we introduce notation and definitions used throughout this work. We set up the total action function of the scalar field with gravitation. We give explicit form of the Klein-Gordon equation and Einstein-Cartan field equations. In Section 3 we work out the Einstein-Cartan field equations which after simplification reduce to the system of first and second order differential equations. These equations can not
be solved analytically. By using the methods of 4th order Runge-Kutta we give the numerical solutions. From our
graphs, we can conjecture the form of the angular momentum parameter $J$. For this case we can find the asymptotic
form of the scalar field analytically. In Section 4 we conclude with some final remarks and perspectives.

II. EINSTEIN-CARTAN GRAVITY WITH CONFORMAL SCALAR FIELDS IN (2+1) DIMENSIONS

The Einstein-Cartan theory of gravity is the closest theory with torsion to general relativity. We used the massless
scalar field as the source of torsion. The torsion arises from the scalar field, as just as mass and charge give rise to a
field, so does torsion.

We take a homogeneous and isotropic universe, i.e. we assume that our solutions will be functions of only the radial
coordinate $r$, a circularly symmetric solution. The line element (2+1) dimensional space-time is given by

$$ds^2 = -(v(r) + \frac{J^2(r)}{r^2})dt^2 + w^2(r)dr^2 + (rd\phi + \frac{J(r)}{r}dt)^2$$

in plane polar coordinates $(t, r, \phi)$.

We consider a massless scalar field non-minimally coupled to Einstein-Cartan gravity in (2+1) space-time dimensions
in the presence of a cosmological constant $\Lambda$.

The action for the gravitational field with torsion is given by

$$S = \int \sqrt{-g}L d^3x.$$  \hspace{1cm} (2)

The Lagrangian for Einstein-Cartan gravity with a massless scalar field can be written as:

$$L = L_G + L_M + L_I.$$ \hspace{1cm} (3)

Here $L_G$ represents the Lagrangian of the gravitational field

$$L_G = \frac{1}{2\kappa} (R - 2\Lambda),$$ \hspace{1cm} (4)

$L_M$ represents the Lagrangian of the matter field

$$L_M = -\frac{1}{2} \nabla^\mu \varphi \nabla_\mu \varphi,$$ \hspace{1cm} (5)

and $L_I$ represents the interaction between the gravitational field and the matter field

$$L_I = -\frac{1}{2} \xi R \varphi^2$$ \hspace{1cm} (6)

with (2+1) space-time metric tensor $g_{\mu\nu}$, the determinant of the metric tensor $g$, the scalar field $\varphi$, the Einstein
gravitational constant $\kappa$, the non-minimally coupling constant $\xi$ and the Ricci scalar of the Riemann-Cartan space-
time $R$.

The Greek indices $(\mu, \nu, \sigma)$ refer to the space-time, and they run over 1, 2, 3. A covariant derivative of the covariant
metric tensor vanishes $\nabla_\rho g_{\mu\nu} = 0$. This condition is referred to as metricity or metric compatibility of the affine
connection. From physical arguments Riemann-Cartan space-time is assumed to possess a connection

$$\Gamma^\rho_{\mu\nu} = \{^\rho_{\mu\nu}\} - K_{\mu\nu}^\rho.$$ \hspace{1cm} (7)

\{^\rho_{\mu\nu}\} are Christoffel symbols of the second kind which are symmetric in their covariant indices built up from the metric
tensor $g_{\mu\nu}$

$$\{^\rho_{\mu\nu}\} = g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}),$$ \hspace{1cm} (8)

and already appearing in general relativity.

As in Einstein-Cartan theory, the anti-symmetric part of the connection $\Gamma^\rho_{\mu\nu}$ defines Cartan’s torsion tensor,

$$T_{\mu
u}^\rho = \frac{1}{2}(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}).$$ \hspace{1cm} (9)
\( K_{\mu\nu}^{\rho} \) is the contortion tensor, which is given in terms of the torsion tensor by

\[
K_{\mu\nu}^{\rho} = -T_{\mu\nu}^{\rho} + T_{\mu}^{\rho\nu} - T_{\rho\mu
u}.
\] (10)

Cartan’s torsion tensor \( T_{\mu\nu}^{\rho} \) and the contortion tensor \( K_{\mu\nu}^{\rho} \), in contrast to the Einstein-Cartan theory, both vanish identically in conventional general relativity \[47\].

The contracted torsion tensor

\[
T_{\mu} = T_{\mu\nu}^{\nu}
\] (11)

is the torsion trace vector \[48\]. The torsion can interact with a scalar field only through its trace \[49\].

The connection (7) is used to define the covariant derivative of a contravariant vector,

\[
\nabla_{\nu} A^\mu = \partial_{\nu} A^\mu + \Gamma_{\nu\rho}^{\mu} A^\rho.
\] (12)

The Riemann-Cartan curvature tensor is defined by using the connection (7), and is given by

\[
R_{\sigma\mu\nu}^{\rho} = \partial_{\mu} \Gamma_{\nu\sigma}^{\rho} - \partial_{\nu} \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\lambda}^{\rho} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\rho} \Gamma_{\mu\sigma}^{\lambda}.
\] (13)

The Ricci tensor of the Riemann-Cartan connection is defined as \( R_{\mu\nu} = R_{\rho\mu\nu}^{\rho} \). The scalar curvature of the Riemann-Cartan space-time is given as follows

\[
R = g^{\mu\nu} R_{\mu\nu}.
\] (14)

The curvature scalar \( R \) can be presented in the form \( R = \tilde{R} + R(T) \), where \( \tilde{R} \) is the Riemannian part of the curvature built from the Christoffel symbols; \( R(T) \) is the part of which is obtained from the covariant derivative of the torsion tensor \[49\].

A. Explicit form of the Klein-Gordon equation and Cartan equations

The Klein-Gordon equation in an external gravitational field with torsion is considered. By Hamilton’s principle, the variation of the total action \( S \) for the gravitational field, matter field and interaction of torsion with matter field vanishes \( \delta S = 0 \).

By varying the total action with respect to the scalar field \( \varphi \)

\[
\frac{\partial (\sqrt{-g} L)}{\partial \varphi} - \partial_{\nu} \frac{\partial (\sqrt{-g} L)}{\partial (\partial_{\nu} \varphi)} = 0,
\] (15)

the Klein-Gordon equation obtained as follows:

\[
-16\kappa^2 w \varphi \varphi''(J^2 + r^2 v)^2 + (J^2 + r^2 v)(\kappa \xi (8 \xi + 1) \varphi^2 - 1)(\kappa \xi \varphi^2 - 1)
(2w^2(J^2 + r^2 v) + \varphi'(-2w'(J^2 + r^2 v) + 2JwJ' + rw(rv' + 2v))
+ \xi \varphi(\kappa \xi \varphi^2 - 1)^2(2(J^2 + r^2 v)(2JwJ'' + rvw' - 2rvw')
- 2rv'(rvw'(J^2 + r^2 v) - w(3J^2 + r^2 v)) + 4J^2 uv - 4J'w'(w'(J^2 + r^2 v)
+rw(rv' + 2v)) - wJ'(J^2 + 3r^2 v - r^2 w^2) = 0.
\] (16)

Here ‘ denotes the derivative with respect to \( r \).

From the line element (1), the affine connection (7), the torsion tensor (9) and the contortion tensor (10), the Cartan field equations are obtained in the PhD thesis prepared by Hasan Tuncay Özcélik (YTU 2016) \[50\].

Varying the total action with respect to the contortion \( K_{\mu\nu}^{\rho} \)

\[
\frac{\partial (\sqrt{-g} L)}{\partial K_{\mu\nu}^{\rho}} - \partial_{\nu} \frac{\partial (\sqrt{-g} L)}{\partial (\partial_{\nu} K_{\mu\nu}^{\rho})} = 0,
\] (17)

gives the Cartan field equations \[50\]. Solving these equations, we can arrive non-zero components of the contortion tensor

\[
K_{12}^{1} = K_{32}^{3} = -U, \quad K_{11}^{2} = -\frac{v}{w^2}U, \quad K_{13}^{2} = K_{31}^{2} = \frac{J}{w^2}U, \quad K_{33}^{2} = \frac{r^2}{w^2}U.
\] (18)
Here contortion coefficients are obtained in terms of an arbitrary function $U$

$$U = \frac{2\kappa\phi'}{\kappa\xi\phi^2 - 1}$$

(19)

The non-zero components of the torsion tensor (9) are

$$T_{12}^1 = T_{23}^3 = -\frac{U}{2}$$

(20)

By means of the relation (11) the trace of the torsion can be obtained as $T_2 = U$.

**B. Explicit form of the Einstein field equations**

The variation of the total action with respect to the metric tensor $g_{\mu\nu}$

$$\frac{\partial(\sqrt{-g}L)}{\partial g_{\mu\nu}} - \partial_\mu \left(\frac{\partial(\sqrt{-g}L)}{\partial g_{\nu\rho}}\right) + \partial_\rho \left(\frac{\partial(\sqrt{-g}L)}{\partial g_{\mu\rho}}\right) = 0,$$

(21)

yields the Einstein field equations

$$4r(J^2 + r^2v)(\kappa\xi\phi^2 - 1)(\Lambda w^3 - 2\kappa\xi\phi'(w - rw') + rw\phi'')$$

$$+ 2\kappa\phi^2(J^2 + r^2v)(\kappa\xi(4\xi + 1)\phi^2 + 4\xi - 1)$$

$$+ (\kappa\xi\phi^2 - 1)^2(4rw'(J^2 + r^2v) - wJ' - 2J^2) = 0,$$

(22)

$$2\kappa(\phi')^2(J^2 + r^2v)(\kappa\xi(8\xi + 1)\phi^2 - 1) + (\kappa\xi\phi^2 - 1)^2(J^2 + 2rv')$$

$$+ (\kappa\xi\phi^2 - 1)(4\kappa\phi\phi'(2J^2 + r(2v' + 2v)) - 4\Lambda w^2(J^2 + r^2v)) = 0,$$

(23)

In this section solutions of Einstein-Cartan equations in (2+1) dimensional space-time is given by considering scalar field as external source for torsion of space-time.

**III. SOLUTIONS**

**A. Case $J(r) \neq 0$**

From the Einstein field equation (22) we find

$$\phi''(r) = \left(\frac{4w(J^2(-\kappa\xi\phi^2 + \Lambda r^2w^2 + 1) + \Lambda r^4vw^2) + r(\kappa\xi\phi^2 - 1)(4w'(J^2 + r^2v)}{wJ'(AJ - rJ'))/(8\kappa\xi r^2\phi w(J^2 + r^2v)) + (\phi'^2(\kappa\xi(4\xi + 1))\phi^2$$

$$+ 4\xi - 1))/((4\phi(\kappa\xi\phi^2 - 1)) + \phi'(w'/w - 1/r).$$

(26)

Using the above equation, equation (23) can be solved as

$$\phi'(r) = \left(-2\kappa(\phi')^2(J^2 + r^2v)(\kappa\xi(8\xi + 1)\phi^2 - 1) - 8\kappa\xi\phi'(\kappa\xi\phi^2 - 1)(J^2 + rv)$$

$$+ (\kappa\xi\phi^2 - 1)(4\Lambda w^2(J^2 + r^2v) + (J')^2(1 - \kappa\xi\phi^2))]/(2r(\kappa\xi\phi^2 - 1)(\kappa\xi\phi^2 + 2\kappa\xi\phi\phi' - 1))$$

(27)
and

\[ \varphi'(r) = (-2\kappa^2 \xi^2 \varphi^3 (2J' + r(r' + 2v)) + 2\kappa \xi \varphi (2J' + r(r' + 2v)) + p\sqrt{2} \]
\[
\sqrt{(\kappa^2 \varphi^2 - 1)(2\kappa^2 \varphi^2 (\kappa \xi \varphi^2 - 1)(2J' + r(r' + 2v)))^2 + (J^2 + r^2 v)}
\]
\[
(\kappa \xi (8\xi - 1) \varphi^2 - 1)(4J^2 \Lambda w^2 - (\kappa \xi \varphi^2 - 1)(J^2 + r^2 v) + 4\Lambda r^2 w^2)))/(2(\kappa \xi (J^2 + r^2) (\kappa \xi (8\xi + 1) \varphi^2 - 1))
\]

(28)

where \( p \) is \pm 1.

Substituting \( \varphi''(r) \) (26) and \( \varphi'(r) \) (27) into Einstein field equation (24) we obtain

\[ w'(r) = (w(8J(\kappa \xi \varphi^2 - 1)) (J^2 - \kappa \xi \varphi^2 + \Lambda r^2 w^2 + 1) + \Lambda r^2 w^2) + r(4J^2 + r^2 v)
\]
\[
(-2\kappa \xi \varphi^3 (4J - r)^2 \varphi^2 (\kappa \xi \varphi^2 - 1) + rJ'' \varphi^2 (\kappa \xi \varphi^2 - 1)^2 - Jkr^2 
\]
\[
(\kappa \xi (16\xi + 1) \varphi^2 - 1) + r^2 J''^2 (\kappa \xi \varphi^2 - 1)^2 - 6JrJ^2 (\kappa \xi \varphi^2 - 1)^2 + 2J'(k \varphi^3 (J^2 + r^2 v)(4\xi \varphi (\kappa \xi \varphi^2 - 1) + r^2 \varphi'(\kappa \xi (16\xi + 1) \varphi^2 - 1)) - 2(\kappa \xi \varphi^2 - 1)(J^2 - (2\kappa \xi \varphi^2 + \Lambda r^2 w^2 + 2) + r^2 v(\kappa \xi \varphi^2 + \Lambda r^2 w^2 - 1))))/(4J^2 + r^2 v)(rJ' - 2J)(\kappa \xi \varphi^2 - 1)(\kappa \xi \varphi^2 + 2\kappa \xi r \varphi \varphi' - 1)).
\]

(29)

Substituting \( \varphi''(r) \), \( \varphi'(r) \) and \( w'(r) \) into the Klein-Gordon equation (16) and the Einstein equation (25) we can obtain

\[ J''(r) = (2Jk(12\Lambda \xi^2 \varphi^2 w^2 (J^2 + r^2 v)(\kappa \xi \varphi^2 - 1) + \varphi'(\kappa \xi \varphi^2 - 1)) + \varphi'(\kappa \xi \varphi^2 - 1))
\]
\[
(\kappa \xi (\xi (8\xi + 1) \varphi^2 - 1) + 4\kappa \xi r \varphi \varphi') + 4\kappa \xi r \varphi (J^2 \Lambda w^2 (\kappa \xi (6\xi + 1) \varphi^2 - 1) + v((\kappa \xi \varphi^2 - 1)^2 + \Lambda r^2 w^2 (\kappa \xi (6\xi + 1) \varphi^2 - 1))) + \kappa \xi r^3 \varphi (J^2)^2 \varphi'(\kappa \xi \varphi^2 - 1)^2
\]
\[
-6Jk^2 \varphi' (\kappa \xi \varphi^2 - 1)^2 - J'(k \varphi^3 (J^2 + r^2 v)((\kappa \xi \varphi^2 - 1)
\]
\[
(\kappa \xi (\xi (8\xi + 1) \varphi^2 - 1) + 4\kappa \xi r \varphi \varphi') + 2\kappa \xi r \varphi (2Jk(2\Lambda r^2 w^2 (\kappa \xi (6\xi + 1) \varphi^2 - 1)
\]
\[
-5(\kappa \xi \varphi^2 - 1)^2) + r^2 v((\kappa \xi \varphi^2 - 1)^2 + 2\Lambda r^2 w^2 (\kappa \xi (6\xi + 1) \varphi^2 - 1))) + (J^2 - r^2 v)(r \varphi'(\kappa \xi \varphi^2 - 1)(\kappa \xi \varphi^2 - 12\Lambda r^2 w^2 + 2 - 1))/r(J^2 + r^2 v)(\kappa \xi \varphi^2 - 1)(\kappa \xi \varphi^2 + 2\kappa \xi r \varphi \varphi' - 1)).
\]

(30)

From the equation (14), the curvature scalar of space-time with torsion can be obtained as

\[ R = \frac{6\Lambda w^2 + k \varphi^2}{w^2(1 - \kappa \xi \varphi^2)}
\]

(31)

The Ricci tensor components are given in Ref. 50. In the absence of torsion, the Ricci tensor reduces to \( R = 6\Lambda \).

The equations \( \varphi''(r) \) (26), \( \varphi'(r) \) (27), \( w'(r) \) (29) and \( J''(r) \) (30) cannot be solved in analytical forms. We will give only the numerical solutions using the methods of 4th order Runge-Kutta.

We set \( \kappa = 1, \xi = \frac{1}{3}, \Lambda = 10^{-8}, \varphi(1) = 10, \varphi'(1) = -2.065, v(1) = 10^2, v'(1) = \Lambda, w(1) = 10^{-1}, J(1) = 10^{-2} \) and \( J'(1) = 10^{-4} \).

The plots of the scalar field \( \varphi(r) \) and \( \varphi'(r) \) with the angular momentum \( J(r) \neq 0 \) are given in Fig. 1. From the Fig. 1 we can see that the scalar field goes to zero \( r \) goes to infinity.

The plots of the metric components \( v(r) \) and \( w(r) \) with the angular momentum \( J(r) \neq 0 \) are given in Fig. 2.

The plots of the angular momentum \( J(r) \) and \( J'(r) \) are given in Fig. 3.

The plot of the Ricci scalar with the angular momentum \( J(r) \neq 0 \) is given in Fig. 4.

**B. Case \( J(r) = ar^2 + b \)**

From the \( J(r) \) and \( J'(r) \) in Fig. 3, we can propose an approximate functional form of the angular momentum \( J(r) \) as \( J(r) = ar^2 + b \). If we substitute this form of \( J(r) \) into \( \varphi''(r) \) (26), we can obtain

\[ \varphi''(r) - \frac{\varphi'(r)}{r} - \frac{3(3\varphi^2 - 8)\varphi'^2}{(\varphi^2 - 8)\varphi} = 0.
\]

(32)
FIG. 1: The plots of the scalar field $\varphi(r)$ and $\varphi'(r)$ with the angular momentum $J(r) \neq 0$ are plotted by Runge-Kutta method with respect to $r$. We set $\kappa = 1$, $\xi = \frac{1}{8}$, $\Lambda = 10^{-8}$, $\varphi(1) = 10$, $\varphi'(1) = -2.065$, $v(1) = 10^2$, $v'(1) = \Lambda$, $w(1) = 10^{-1}$, $J(1) = 10^{-2}$ and $J'(1) = 10^{-4}$.

FIG. 2: The plots of the metrics components $v(r)$ and $w(r)$ with the angular momentum $J(r) \neq 0$ by Runge-Kutta method with respect to $r$. We set $\kappa = 1$, $\xi = \frac{1}{8}$, $\Lambda = 10^{-8}$, $\varphi(1) = 10$, $\varphi'(1) = -2.065$, $v(1) = 10^2$, $v'(1) = \Lambda$, $w(1) = 10^{-1}$, $J(1) = 10^{-2}$ and $J'(1) = 10^{-4}$.

FIG. 3: The plots of the angular momentum $J(r)$ and $J'(r)$ are plotted by Runge-Kutta method with respect to $r$. We set $\kappa = 1$, $\xi = \frac{1}{8}$, $\Lambda = 10^{-8}$, $\varphi(1) = 10$, $\varphi'(1) = -2.065$, $v(1) = 10^2$, $v'(1) = \Lambda$, $w(1) = 10^{-1}$, $J(1) = 10^{-2}$ and $J'(1) = 10^{-4}$.

1. Case $\kappa \xi \varphi^2 >> 1$

If we solve the above equation (32), the solution can be written as follows

$$\varphi(r) = \frac{C_1}{(r^2 + C_2)^{1/8}}$$

where $C_1$ and $C_2$ are constants.
FIG. 4: The Ricci scalar $R$ with the angular momentum $J(r) \neq 0$ is plotted by Runge-Kutta method with respect to $r$. We set $\kappa = 1$, $\xi = \frac{1}{8}$, $\Lambda = 10^{-8}$, $\varphi(1) = 10$, $\varphi'(1) = -2.065$, $v(1) = 10^2$, $v'(1) = \Lambda$, $w(1) = 10^{-1}$, $J(1) = 10^{-2}$ and $J'(1) = 10^{-4}$.

2. Case $0 < \kappa \xi \varphi^2 << 1$

From the equation (32), it is possible to express the scalar field as

$$\varphi(r) = \frac{C_3}{\sqrt{r^2 + C_4}}$$

(34)

where $C_3$ and $C_4$ are constants.

C. Case $J(r) = 0$

Noting the agreement between the case with and without torsion, we investigate if torsion has a significant effect on the scalar field $\varphi(r)$.

The non-rotating black hole we take the metric as

$$ds^2 = -(v(r) + \frac{J^2(r)}{r^2})dt^2 + w^2(r)dr^2 + (rd\phi + \frac{J(r)}{r}dt)^2.$$  

(35)

In order to express the zero-torsion and torsion parts of the Klein-Gordon equation and the Einstein field equations, we can write the connection (7) as follows.

$$\Gamma^\rho_{\mu\nu} = \{\rho_{\mu\nu}\} - \alpha K_{\mu\nu} \rho.$$  

(36)

The first term in equation (7) gives zero-torsion part and the second term gives the contribution from the torsion. When $\alpha = 0$ it means zero-torsion. When $\alpha = 1$ it means that there is a torsion.

From equation (15) we obtain the Klein-Gordon equation

$$k^2 \xi^2 (\alpha 8 \xi + 1) \varphi^4 ((rv' + v)\varphi' + rv\varphi'') - 2k\xi(\alpha 4 \xi + 1) \varphi^2 ((rv' + v)\varphi' + rv\varphi'')$$

$$+ 2k\xi^2 \varphi^3 (4(\alpha - 1)\alpha k\xi rv\varphi'^2 - rv'' - 2v') + \xi \varphi(r(v'' - 8\alpha k\xi v\varphi'^2) + 2v')$$

$$+ k^2 \xi^3 \varphi^5 (rv'' + 2v') + (rv'(r) + v)\varphi' + rv\varphi''' = 0.$$  

(37)

From equation (21) we find the Einstein field equations

$$kv(\alpha 8 k\xi^2 r \varphi^2 \varphi^2 / (k\xi \varphi^2 - 1) + (1 - 4\xi)r\varphi^2 - 4\xi \varphi(r\varphi'' + \varphi'))$$

$$- v'(2k\xi r\varphi' + k\xi \varphi^2) - 2A r = 0,$$

$$kv\varphi'((\alpha 2 8 k\xi^2 r \varphi^2 \varphi') / (k\xi \varphi^2 - 1) + 4\xi \varphi) + r\varphi'$$

$$+ v'(2k\xi r\varphi' + k\xi \varphi^2) - 2A r = 0,$$

$$- k\xi \varphi^2 (k(\alpha 8 \xi - 4 \xi + 1) \varphi'^2 + 2(\Lambda + v'')) + k^2 \xi^2 \varphi^4 v'' + 4k^2 \xi^2 \varphi^3 (v' \varphi' + v\varphi'')$$

$$- 4k\xi \varphi(v' \varphi' + v\varphi'') + k(1 - 4\xi)v\varphi'^2 + 2A + v'' = 0.$$  

(40)
From the Einstein field equation (39) we obtain
\[ v'(r) = (-k\xi r^2(k(\alpha^2 8\xi + 1)v\varphi'^2 - 2\Lambda) - 4k^2\xi^2 v\varphi^3\varphi' + 4k\xi v\varphi' + kr\varphi'^2 \\
- 2\Lambda r)/( (k\xi v^2 - 1)(2k\xi v\varphi' + k\xi v^2 - 1)). \] (41)

From the above equation we can obtain \( v''(r) \). Substituting \( v'(r) \) and \( v''(r) \) into the Einstein field equation (38) we find
\[ \varphi''(r) - \frac{\alpha 4k\xi \varphi(r)\varphi'(r)^2}{k\xi \varphi(r)^2 - 1} + \frac{2(\xi - 1)\varphi'(r)^2}{2\xi \varphi(r)} = 0. \] (42)

1. Case \( \alpha = 0 \)

We set \( k = 1 \) and \( \xi = \frac{1}{8} \) in the above equation, we obtain the scalar field
\[ \varphi(r) = \frac{C_5}{\sqrt{r + C_6}} \] (43)
where \( C_5 \) and \( C_6 \) are constants. This solution is in agreement with those made in Hortaçsu, Özçelik and Yapıskan [38], where the coupling to torsion was absent. In both cases the scalar field goes to zero as \( r \) goes to infinity.

2. Case \( \alpha = 1 \) and \( \kappa\xi\varphi^2 \gg 1 \)

If we solve the equation (42), the solution can be written as follows
\[ \varphi(r) = \frac{C_7}{(r + C_8)^{1/6}} \] (44)
where \( C_7 \) and \( C_8 \) are constants.

3. Case \( \alpha = 1 \) and \( 0 < \kappa\xi\varphi^2 << 1 \)

From the equation (42), it is possible to express the scalar field as
\[ \varphi(r) = \frac{C_9}{\sqrt{r + C_{10}}} \] (45)
where \( C_9 \) and \( C_{10} \) are constants.

4. Case \( \alpha = 1 \)

Setting \( \xi \) is equal to \( \frac{1}{8} \) in equation (41), the scalar field \( \varphi(r) \) can be found as the inverse of the function \( f(x) \)
\[ \varphi(r) = f^{-1}(r + A). \] (46)

Here the function \( f(x) \) is
\[ f(x) = \frac{4}{2x} + \frac{4k}{8 + kxx} + k\log[k - \frac{8}{2x}] \] (47)
where \( A \) and \( B \) are constants.
IV. CONCLUSIONS

We studied in (2+1) dimensions Einstein-Cartan gravity conformally coupled to a massless scalar field $\varphi$. By considering variations with respect to the scalar field, the metric tensor and contortion tensor, the Klein-Gordon equation and Einstein-Cartan field equations are obtained. We give numerically solutions of Einstein-Cartan equations with $J(r) \neq 0$ in (2+1) dimensional space-time. We derived black hole solutions in (2+1) dimensions with $J(r) = ar^2 + b$ and $J(r) = 0$ interacting with a scalar field.

If the angular momentum has $J(r) = ar^2 + b$ form and the scalar field with torsion is too small $(0 < \kappa \xi \varphi^2 \ll 1)$, the scalar field $\varphi(r)$ behaves like $r^{-1}$. If the angular momentum is $J(r) = 0$ and the scalar field with torsion is too small $(0 < \kappa \xi \varphi^2 \ll 1)$, the scalar field $\varphi(r)$ behaves like $r^{-1/2}$. From these it can be deduced that the angular momentum is an effect on the scalar field.

If the angular momentum $J(r) = ar^2 + b$ and the scalar field with torsion is too large $(\kappa \xi \varphi^2 \gg 1)$, the scalar field $\varphi(r)$ behaves like $r^{-1/4}$. If the angular momentum is $J(r) = 0$ and the scalar field with torsion is too large $(\kappa \xi \varphi^2 \gg 1)$, the scalar field $\varphi$ behaves like $r^{-1/6}$. If these two conditions are compared, it can be concluded that the angular momentum is an effect on the scalar field.

If the angular momentum is $J(r) = 0$ and the scalar field without torsion, the scalar field $\varphi(r)$ behaves like $r^{-1/2}$.

If the angular momentum is $J(r) = 0$ and the scalar field with torsion, the scalar field is the inverse of the function $f(x)$.

We note that the torsion has a significant effect on the scalar field in (2+1) dimensions.

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