TREATING 'tHOOFT-POLYAKOV MONOPOLE AS CONSTRAINED SYSTEM

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Abstract

The 'tHooft-Polyakov monopole is treated as constrained system using the Hamilton-Jacobi method. The set of the Hamilton-Jacobi partial differential equations and the equations of motion are obtained. The quantization of the system is also discussed.

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1 Introduction

A physical system whose space consists of $2N$ degrees of freedom: $q = (q_1, \cdots, q_N)$, $p = (p_1, \cdots, p_N)$ is called a constrained system if there are relations between coordinates and momenta. In other words, in constrained systems some of the velocities $\dot{q}_i (i = 1, \cdots, N)$ cannot be expressed in terms of the coordinates and momenta. When this happens, the Lagrangian of the system is called singular. The constraints place restrictions on the possible choices of boundary conditions for the canonical coordinates. Moreover, the standard quantization methods cannot be applied directly to constrained theories. However, the basic ideas of the classical treatment and the quantization of such systems were presented first by Dirac [1]. Following Dirac, it was shown (Faddeev [2], Hanson et al [3], Dayi [4], Evans [5]) that gauge fixing conditions should be imposed for first class constraints.

The construction of the Hamilton-Jacobi partial differential equations (HJPDs) for constrained systems is of prime importance, the Hamilton-Jacobi theory provides a bridge between classical and quantum mechanics. The first study of the Hamilton-Jacobi equations for arbitrary first-order actions was initiated by Santilli [6]. Later on the canonical method has been developed [7]. In this method the equations of motion are written as total differential equations, and the set of HJPDs are obtained.

In an earlier work, quantum electrodynamics theory and pure Yang-Mills are treated as constrained systems using the canonical approach [8]. Later on this work has been extended to quantum chromodynamics theory, which is a Yang-Mills theory in the presence of quark fields [9]. In this paper we treat one
of the extended objects as a constrained system which is the ’tHooft-Polyakov monopole. In section 2, the canonical formulation is briefly reviewed and in section 3 a brief discussion of ’tHooft-Polyakov monopole is given. While in section 4 the ’tHooft-Polyakov monopole is treated as a constrained system and the equations of motion for the system are obtained. Also the quantization of the system is discussed.

2 Canonical formulation

In this formulation, if we start with a singular Lagrangian with Hessian matrix of rank \((N - r), r < N\), then the set of HJPDEs is expressed as

\[
H'_\alpha \left( x_\beta, q_\alpha, \frac{\partial S}{\partial q_\alpha}, \frac{\partial S}{\partial x_\beta} \right) = 0, \quad \alpha, \beta = 0, 1, \ldots, r
\]  

(1)

where

\[
H'_0 = p_0 + H_0,
\]

\[
H'_\mu = p_\mu + H_\mu, \quad \mu = 1, \ldots, r
\]  

(2)

and \(S\) is the total action of the theory under consideration.

According to ref. [7, 10], the equations of motion are written as total differential equations as follows,

\[
dq_a = \frac{\partial H'_\alpha}{\partial p_\alpha} dx_\alpha, \quad a = 1, 2, \ldots, N - r
\]  

(3)

\[
dp_i = -\frac{\partial H'_\alpha}{\partial q_i} dx_\alpha, \quad i = 1, 2, \ldots, N.
\]  

(4)

Following Rabei [10], these equations are integrable if and only if

\[
dH'_\mu = 0.
\]  

(5)
The canonical action $S$ can easily be obtained from the following equation \[7, 10\]

$$dS[q_i; t] = \left(-\mathcal{H}_\alpha + p_a \frac{\partial \mathcal{H}'_\alpha}{\partial p_a}\right) dx_\alpha. \quad (6)$$

Note that the canonical action integral is obtained in terms of the canonical coordinates. Also, if condition (5) is not satisfied, one consider it as new constraints and again tests the consistency condition. Hence, the canonical formulation leads to obtain the set of canonical phase space coordinates $q_a$ and $p_a$ as functions of $x_\alpha$. The Hamiltonian $H'_\alpha$ are considered as the infinitesimal generators of canonical transformations given by parameters $x_\alpha$ respectively.

In this case, the path integral representation may be written as \[11\]

$$< \text{Out}|S|\text{In} > = \int \prod_{a=1}^{N-r} dq_a dp_a \exp \left[ i \int_{x_{\alpha}}^{x'_{\alpha}} \left(-\mathcal{H}_\alpha + p_a \frac{\partial \mathcal{H}'_\alpha}{\partial p_a}\right) dx_\alpha \right]. \quad (7)$$

### 3 ’tHooft-Polyakov monopole

Perturbation theory is based upon making power expansions of the path integral around the trivial vacua \[12\]. However, there are solutions of the classical, nonlinear equations of motion that exhibit particle-like behavior that gives powerful insight into the nonperturbative behavior of these theories. A new quantum expansion can be developed around each exact solution, allowing us to explore regions that are not accessible by standard perturbation theory. In particular, these solutions give us nonperturbative information about important physical phenomena such as tunnelling and bound states.

In this paper we consider one type of classical solutions which is the monopole, or particles with magnetic charge, where first investigated by Dirac \[13\], then lately by ’tHooft \[14\] and Polyakov \[15\]. They have been found in
gauge theory with spontaneous symmetry breaking and may have cosmological significance [16].

The model consists of Higgs scalar fields $\phi_a(\vec{x}, t)$ and vector fields $W^\mu_a(\vec{x}, t)$ in $(3 + 1)$ dimensions. The index $a = 1, 2, 3$, is an internal space index, which will transform according to local (space-time dependent) $SO(3)$ transformation given below. For any given $a$, $\phi_a$ is a scalar and $W^\mu_a(\mu = 0, 1, 2, 3)$ is a vector under Lorentz transformation. The Lagrangian density is

$$
\mathcal{L} = -\frac{1}{4}G^{\mu\nu}_aG_{a\mu\nu} + \frac{1}{2}D^\mu \vec{\phi}D_\mu \vec{\phi} - V(\vec{\phi}),
$$

(8)

where $\vec{\phi}$ is the Higgs field and is given by $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, and the potential is given by

$$
V(\vec{\phi}) = \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2 + \phi_3^2 - \sigma^2)^2,
$$

(9)

where $\phi^2 = \sigma^2$ is the Higgs vacuum. The gauge field strength is $G^{\mu\nu}_a$ and given by

$$
G^{\mu\nu}_a = \partial^\mu W^\nu_a - \partial^\nu W^\mu_a - e\epsilon^{abc}W^\mu_bW^\nu_c.
$$

(10)

The Lagrangian density is invariant under the group $SO(3)$ which is generated by $T^a$ such that

$$
U = e^{i\theta^aT^a/\sigma},
$$

(11)

where $\theta^a$ is the group parameters vary with space-time.

Let the monopole configuration be centered at the origin. Energy finiteness implies that there is some radius $r_0$ such that for $r \geq r_0$

$$
D^\mu \vec{\phi} = \partial^\mu \vec{\phi} - e\vec{W}^\mu \times \vec{\phi} = 0,
$$

(12)

and

$$
\phi_1^2 + \phi_2^2 + \phi_3^2 - \sigma^2 = 0, (\Rightarrow V(\vec{\phi}) = 0).
$$

(13)
Regions of space-time, where the eqns. (12) and (13) are satisfied constitute the Higgs vacuum.

The general form of $\vec{W}_\mu$ satisfying eqn. (12), provided $\vec{\phi}$ satisfies eqn. (13) is

$$\vec{W}_\mu = \frac{1}{\sigma^2 e} \vec{\phi} \times \partial^\mu \vec{\phi} + \frac{1}{\sigma} \vec{\phi} A^\mu,$$

(14)

where $A^\mu$ is an arbitrary gauge field. So, we can write the gauge field strength as

$$G^{\mu\nu} = \frac{1}{\sigma^3 e} \vec{\phi} \cdot (\partial^\mu \vec{\phi} \times \partial^\nu \vec{\phi}) + \partial^\mu A^\nu - \partial^\nu A^\mu.$$

(15)

In the region outside the localized region, where eqns. (12) and (13) are satisfied, i.e. in the Higgs vacuum, $\mathcal{L}$ is given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{4} \left[ \frac{1}{\sigma^2 e^2} \epsilon_{ijkl} \epsilon_{rst} \phi_i \phi_r \partial^\mu \phi_j \partial^\nu \phi_k \partial^\mu \phi_s \partial^\nu \phi_t + 2(\partial^\mu A^\nu - \partial^\nu A^\mu) \partial_\mu A_\nu \right.$$

$$+ \left. \frac{4}{\sigma^3 e} \epsilon_{ijkl} \phi_i \partial^\mu \phi_j \partial^\nu \phi_k \partial^\mu A_\nu \right].$$

(17)

4 The monopole as a constrained system

The coordinates are $\phi_i(x)$ and $A_\mu(x)$ and their corresponding conjugate momenta are $\pi_i(x)$ and $\Pi_\mu(x)$ respectively. They are given by the following
expressions:

\[ \pi_l(x) = \frac{\partial L}{\partial (\partial_0 \phi_l(x))} \]

\[ = \frac{\epsilon_{ijl}}{\sigma^3 e} \dot{\phi}_i \partial^k \phi_j \left( \frac{\epsilon_{rst}}{\sigma^3 e} \dot{\phi}_r \partial_0 \phi_s \partial_k \phi_t + \partial_0 A_k - \partial_k A_0 \right). \tag{18} \]

\[ \Pi_\mu(x) = \frac{\partial L}{\partial (\partial_0 A_\mu(x))} \]

\[ = \frac{\epsilon_{rst}}{\sigma^3 e} \dot{\phi}_r \partial_\mu \phi_s \partial_0 \phi_t + \partial_\mu A_0 - \partial_0 A_\mu. \tag{19} \]

The nonvanishing Poisson brackets are

\[ \{ \phi_i(x), \pi_j(x') \} = \delta_{ij} \delta^{(3)}(x - x'). \tag{20} \]

\[ \{ A^\mu(x), \Pi_\nu(x') \} = \delta^\mu_\nu \delta^{(3)}(x - x'). \tag{21} \]

Upon quantization, these brackets have to be converted into proper commutators. The spatial components of the conjugate momentum of \( A_\mu \) field read as

\[ \Pi_i = \frac{\epsilon_{rst}}{\sigma^3 e} \dot{\phi}_r \partial_i \phi_s \partial_0 \phi_t + \partial_i A_0 - \partial_0 A_i \]

\[ = F_{i0}. \tag{22} \]

where \( i = 1, 2, 3 \) and the time component is

\[ \Pi_0 = 0 = \mathcal{H}_1. \tag{23} \]

which is a constraint.

Note that using eqn. (22), one can write the velocity \( \partial_0 A \) in terms of the momenta \( \Pi_i \) as

\[ \partial_0 A_i = \frac{\epsilon_{rst}}{\sigma^3 e} \dot{\phi}_r \partial_i \phi_s \partial_0 \phi_t + \partial_i A_0 - \Pi_i. \tag{24} \]
Substituting eqn. (24) into eqn. (18), we get

\[ \pi_l = -\frac{\epsilon_{lrt}}{e^{3\phi}} \phi_r \phi_t \phi_k \Pi_k \]

\[ = -\mathcal{H}_{2t}, \quad (25) \]

which is also a constraint. Note that eqns. (23) and (25) are called primary constraints according to Dirac.

The Hamiltonian density can be obtained as

\[ \mathcal{H}_0 = \partial_t A_i \Pi^i - \partial_0 \phi_l \mathcal{H}_{2t} - \mathcal{L} \]

\[ = -\frac{1}{2} \Pi_i \Pi^i - \partial_i A_0 \Pi^i + \frac{1}{4} F_{ij} F^{ij}. \quad (26) \]

Thus the total Hamiltonian is given by

\[ H_0 = \int \left( -\frac{1}{2} \Pi_i \Pi^i - \partial_i A_0 \Pi^i + \frac{1}{4} F_{ij} F^{ij} \right) d^3x. \quad (27) \]

Using eqn. (2), the set of HJPDEs reads as

\[ \mathcal{H}'_0 = P_0 + \mathcal{H}_0, \quad (28) \]

\[ \mathcal{H}'_1 = \Pi_0 = 0, \quad (29) \]

\[ \mathcal{H}'_{2t} = \pi_t + \mathcal{H}_{2t} = \pi_t + \frac{\epsilon_{lrt}}{e^{3\phi}} \phi_r \phi_t \phi_k, \quad (30) \]

where \( P_0 = \frac{\partial S}{\partial t}, \pi_t = \frac{\partial S}{\partial \phi_t}, \Pi_\mu = \frac{\partial S}{\partial A_\mu}, \) where \( S = S[A_\mu, \phi_l; t] \) represents the action.

The above equations can be written in a more compact form as

\[ \frac{\partial S}{\partial t} + \mathcal{H}_0 = 0, \]

\[ \frac{\partial S}{\partial A_0} = 0, \quad (31) \]

\[ \frac{\partial S}{\partial \phi_t} + \mathcal{H}_{2t} = 0, \quad (32) \]
and their simultaneous solutions determine the action $S$.

Using eqns. (3) and (4), the total differential equations are

$$
dA_\mu = \frac{\partial H'_0}{\partial \Pi_\mu} dt + \frac{\partial H'_1}{\partial \Pi_\mu} dA_0 + \frac{\partial H'_{2l}}{\partial \Pi_\mu} d\phi_l
$$

$$
dA_i = (-\Pi_i + \partial_i A_0) dt,
$$

$$
d\Pi_\mu = -\frac{\partial H'_0}{\partial A_\mu} dt - \frac{\partial H'_1}{\partial A_\mu} dA_0 - \frac{\partial H'_{2l}}{\partial A_\mu} d\phi_l
$$

$$
d\Pi_i = -\partial_i F^{\mu i} dt,
$$

$$
d\Pi_0 = \partial^i \Pi_i dt,
$$

$$
d\pi_l = -\frac{\partial H'_0}{\partial \phi_l} dt - \frac{\partial H'_1}{\partial \phi_l} dA_0 - \frac{\partial H'_{2l}}{\partial \phi_l} d\phi_l
$$

$$
= -\frac{1}{2} \epsilon_{ijkl} \partial_i \phi_j \partial_j \phi_l F^{\mu l} dt + \frac{\epsilon_{ijkl}}{\sigma^3 e} \partial_i \phi_j F^{\mu l} d\phi_l.
$$

These equations are integrable iff the total differential of eqns. (29) and (30) are equal to zero.

The vanishing of the total differential of $H'_1$ leads to a new constraint

$$
H''_1 = dH'_1 = d\Pi_0 = \partial^i \Pi_i dt = 0.
$$

Taking the total derivative of the above equation gives

$$
dH''_1 = \partial^i d\Pi_i dt
$$

$$
= -\partial_i \partial_j F^{\mu i} dt = 0,
$$

which can be shown equivalent to

$$
dH''_1 = -\partial_\mu \partial^\nu F^{\mu \nu} dt = 0,
$$

$$
= \partial_\mu K^\mu dt = 0,
$$

8
where

\[ K^\mu = - \partial_\nu F^{\mu \nu}, \]

\[ = -\frac{1}{2\sigma^3} \epsilon^{\mu \rho \eta} \epsilon_{\nu \tau} \partial_\nu \phi_l \partial_\rho \phi_r \partial_\eta \phi_t \quad (40) \]

is the conserved magnetic current provided \( A_\mu = 0 \), for details see [17].

The first set of the Euler-Lagrange equations of motion can be obtained using eqn. (34) and the constraint eqn. (35),

\[ \partial_\mu F^{\mu \nu} = 0, \quad (41) \]

which represents the equations of motion for the gauge field \( A_\mu \) and can be obtained if we take the variation of the Lagrange density, eqn. (17), with respect to the gauge field \( A_\mu \).

The second set of Euler-Lagrange equations of motion can be obtained using the integrability condition eqn. (30),

\[ d\mathcal{H}'_{2l} = 0. \quad (42) \]

It gives

\[ \frac{1}{2} \epsilon_{\nu \tau} \partial_\mu \phi_r \partial_\rho \phi_t F^{\mu \nu} = 0, \quad (43) \]

which represents the equation of motion for the scalar field \( \phi_l \).

The action can be calculated using eqn. (6) as,

\[ dS[A_\mu, \phi_l; t] = (-\mathcal{H}'_0 + \Pi_i \frac{\partial \mathcal{H}'_0}{\partial \Pi_i})dt \]
\[ + (\Pi_0 + \Pi_i \frac{\partial \mathcal{H}'_1}{\partial \Pi_i})dA_0 \]
\[ + (-\mathcal{H}'_{2l} + \Pi_i \frac{\partial \mathcal{H}'_{2l}}{\partial \Pi_i})d\phi_l. \quad (44) \]
Hence,
\[ S[A_\mu, \phi_t; t] = \int d^4x \left( -\frac{1}{2}\Pi_i\Pi^i - \frac{1}{4}F_{ij}F^{ij} \right). \] (45)

We see that the original Lagrangian eqn. (17) can be recovered using the definition of the canonical momenta.

Note that although \( \phi_t \) is introduced as coordinates in the Lagrangian, the presence of the constraints and the integrability conditions force us to treat it as a parameter like \( t \). Since the set of total differential equations is integrable, the canonical phase space coordinates \( A_i \) and \( \Pi_i \) are obtained in terms of independent parameters \( t \) and \( \phi_t \). Hence, using eqn. (7), the path integral representation for the system is calculated as
\[ <\text{Out}|S|\text{In}> = \int \prod_i dA_i d\Pi_i \exp \left[ i \int \left( -\frac{1}{2}\Pi_i\Pi^i - \frac{1}{4}F_{ij}F^{ij} \right) d^4x \right]. \] (46)

One should notice that the path integral eqn. (46) has no singular nature and it is an integration over the canonical phase space coordinates \( A^i \) and \( \Pi^i \). Moreover, it is no need to choose a gauge fixing and ghost fields, while in the usual path quantization one must choose a gauge fixing and introduce ghost fields [18].

5 Conclusion

The 'tHooft-Polyakov monopole is treated as a constrained system using the canonical method. It is observed that the Hamilton equations of motion are obtained to be in exact agreement with the Euler-Lagrange equations. In addition, the conserved current has been obtained from the integrability conditions. In our approach, we recover action of the 'tHooft-Polyakov monopole
from the equations of motion as well as from the integrability conditions without redefining the Lagrange multipliers which are necessary in Dirac’s approach [19]. Finally, the path integral quantization is obtained directly as an integration over the canonical phase space coordinates without choosing an appropriate gauge fixing.

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