Some higher moments of deep inelastic structure functions at next-to-next-to leading order of perturbative QCD

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Abstract
We present the analytic next-to-next-to-leading QCD calculation of some higher moments of deep inelastic structure functions in the leading twist approximation. We give results for the moments $N=1,3,5,7,9,11,13$ of the structure function $F_3$. Similarly we present the moments $N=10,12$ for the flavour singlet and $N=12,14$ for the non-singlet structure functions $F_2$ and $F_L$. We have calculated both the three-loop anomalous dimensions of the corresponding operators and the three-loop coefficient functions of the moments of these structure functions.
1 Introduction

The determination of the next-to-next-to-leading (NNL) order QCD approximation for the structure functions of deep inelastic scattering has become important for the understanding of perturbative QCD and necessary for an accurate comparison of perturbative QCD with the increasing precision of experiments. Such calculations however are rather complicated and hence a complete NNL result does not exist as of yet. The one-loop anomalous dimensions were calculated in [1, 2]. In [3] (see also the references therein) the complete one-loop coefficient functions were obtained. Anomalous dimensions at 2-loop order were obtained in [4, 5, 6, 7, 8, 9, 10, 11, 12]. The 2-loop coefficient functions were calculated in [13, 14, 15, 16, 17, 18, 19, 20], [21].

Analytical results of the 3-loop anomalous dimensions and coefficient functions of the moments of $F_2$ and $F_L$ are only known for the moments $N=2, 4, 6, 8$ in both the singlet and non-singlet case and additionally for $N=10$ in the non-singlet case from [22] and [23]. In addition the Gross-Llewellyn Smith sum rule, which corresponds to the first moment of $F_3^{p+p+\bar{p}}$ has been calculated at this order [24].

For a complete reconstruction of the $x$-dependence of the structure functions via an inverse Mellin-transformation one would need the moments for all $N$ (that is either all even or all odd integer values). Additionally one needs them both for $F_2$ and $F_3$ in order to untangle the various quark and gluon contributions. The determination of the NNL approximation for generic $N$ is work in progress [25], but probably will not be finished in the near future.

The available moments of $F_2$ have been used by a number of authors to make a reconstruction of the complete structure functions at NNL by a variety of means [23, 26, 27, 28]. Additionally they can be used to obtain a better value of $\alpha_S$ [29] It should be clear that it is important to have as large a number of moments as possible. First, these results can be immediately used to increase the precision of phenomenological investigations of deep inelastic scattering [30]. Second, the moments will be a very important check for the new methods and programs needed for the determination of the 3-loop results for arbitrary $N$. Unfortunately it is not very easy to increase the number of moments, because each new moment requires roughly five times the computer resources that its predecessor needs. With the advent of better computers this means that by now it has been possible to obtain two more moments for the singlet case and one additional moment for the non singlet $F_2$ case. This should allow for instance a somewhat better determination of $\alpha_S$. More important however is the determination of the first seven odd moments of $F_3^{p+p+\bar{p}}$ to three loops. To this end we used the the same programs as in [23], state of the art computers and a new version of the symbolic manipulation program FORM [31] that supports now 64-bit architectures and to some extent parallel computers (see also [32]). We could push the limit in these calculations to include two new moments ($N=10, 12$) in the calculation of the flavour singlet structure functions $F_L$ and $F_2$, and two new moments ($N = 12, 14$) for the flavour nonsinglet structure functions $F_L$ and $F_2$. Additionally we have computed the moments $N=3, 5, 7, 9, 11, 13$ of the structure function $F_3$. We do not expect more moments to become available before the complete results for all $N$ will be presented.

2 The formalism

This calculation follows the one presented in [23] (see also [21]) in every detail, so we only will give a very short review on the methods used.

We need to calculate the hadronic part of the amplitude for unpolarized deep inelastic scattering
which is given by the hadronic tensor

\[ W_{\nu \mu}(x, Q^2) = \frac{1}{4\pi} \int d^4z e^{i q \cdot z} \langle p, \text{nucl}| J_\mu(z) J_\nu(0)|\text{nucl}, p \rangle \]

\[ = \left( g_{\mu\nu} - q_\mu q_\nu \right) \frac{1}{2 \varepsilon} F_L(x, Q^2) \]

\[ + \left( -g_{\mu\nu} - p_\mu p_\nu \right) \frac{4x^2}{q^2} - \left( p_\mu q_\nu + p_\nu q_\mu \right) \frac{2x}{q^2} \frac{1}{2\varepsilon} F_2(x, Q^2) \]

\[ + i \varepsilon_{\mu\nu\rho\sigma} \frac{p^\rho q^\sigma}{p \cdot q} F_3(x, Q^2) \]  

(1)

where the \( J^\mu \) are either electromagnetic or weak hadronic currents and \( x = Q^2/(2p \cdot q) \) is the Bjorken scaling variable with \( 0 < x \leq 1 \). \( Q^2 = -q^2 \) is the transferred momentum and \( |p, \text{nucl}| \) is the nucleon state with momentum \( p \). In these equations spin averaging is assumed. The longitudinal structure function \( F_L \) is related to the structure function \( F_1 \) by \( F_L = F_2 - 2xF_1 \). For electron-nucleon scattering \( J^\mu \) is the electromagnetic quark current and \( F_3 \) vanishes. For neutrino-nucleon scattering \( J^\mu \) is an electroweak quark current which has an axial vector contribution and \( F_3 \), which describes parity violating effects that arise from vector and axial-vector interference, will not vanish.

Using the dispersion relation technique one can relate the hadronic tensor to the following 4-point Green functions:

\[ W^{\mu\nu}(p, q) = \frac{1}{2\pi} \text{Im} T^{\mu\nu}(p, q), \quad T_{\mu\nu}(p, q) = i \int d^4z e^{i q \cdot z} \langle p, \text{nucl}| T[J_\mu(z)J_\nu(0)]|\text{nucl}, p \rangle. \]

Applying a formal operator product expansion in terms of local operators to the time-ordered product of two quark currents leads to:

\[ i \int d^4z e^{i q \cdot z} T[J_{\nu_1}(z)J_{\nu_2}(0)] = \sum_{N,J} \left( \frac{1}{Q^2} \right)^N \left[ \left( g_{\nu_1\nu_2} - \frac{q_{\nu_1} q_{\nu_2}}{q^2} \right) \right] q_{\nu_1} q_{\nu_2} C_{J}^{\nu_1,\nu_2}( \frac{Q^2}{\mu^2}, a_s ) \]

\[ - \left( g_{\nu_1\nu_1} g_{\nu_2\nu_2} q^2 - g_{\nu_1\nu_2} g_{\nu_2\nu_1} q^2 - g_{\nu_2\nu_1} g_{\nu_1\nu_2} q^2 + g_{\nu_1\nu_1} g_{\nu_2\nu_2} q^2 \right) C_{J}^{\nu_1,\nu_2}( \frac{Q^2}{\mu^2}, a_s ) \]

\[ + i \varepsilon_{\nu_1\nu_2\nu_3\nu_4} g_{\nu_3\nu_4} q_{\nu_3} q_{\nu_4} C_{J}^{\nu_1,\nu_2}( \frac{Q^2}{\mu^2}, a_s ) \int q_{\mu_3} \cdots q_{\mu_n} O^J(\nu_1,\ldots,\nu_n)(0) + \text{higher twists,} \]

\[ j = \alpha, \psi, G \]

Here we have introduced the notation \( a_s = \alpha_s/(4\pi) = g^2/(4\pi)^2 \) and everything is assumed to be renormalized. The sum over \( N \) runs over the standard set of the spin-\( N \) twist-2 irreducible flavour non-singlet quark operators and the singlet quark and gluon operators:

\[ O^{\alpha}(\nu_1,\ldots,\nu_n) = \bar{\psi} \lambda^\alpha \gamma^\mu_1 D^\mu_2 \cdots D^\mu_n \psi, \quad \alpha = 1, 2, \ldots, (n^2 - 1) \]

\[ G^\alpha(\nu_1,\ldots,\nu_n) = \bar{\psi} \gamma^\nu_1 D^\nu_2 \cdots D^\nu_n \psi \]

Application of this OPE to Eq. (1) leads to an expansion for the unphysical values \( x \to \infty \). From the proper analytical continuation to the physical region \( 0 < x \leq 1 \) one finds for the moments of the structure functions \( F_2, F_L \) and \( F_3 \):

\[ M_{k,N-2} = \frac{1}{0} \int dx x^{N-2} F_k(x, Q^2) = \sum_{i=\alpha, \psi, G} C_{i,N}^k \frac{Q^2}{\mu^2}, a_s \right) A_i^{\nu_1,\ldots,\nu_n} \]

\[ (2) \]

\[ M_{2,N-1} = \frac{1}{0} \int dx x^{N-1} F_2(x, Q^2) = \sum_{i=\alpha} C_{i,N} \frac{Q^2}{\mu^2}, a_s \right) A_i^{\nu_1,\ldots,\nu_n} \]

\[ (3) \]
with the spin averaged matrix elements

\[ \langle p, \text{nucl}|O^{(\mu_1 \cdots \mu_N)}|p, \text{nucl} \rangle = p^{(\mu_1} \cdots p^{\mu_N)} A_{\text{nucl},N} \left( \frac{p^2}{\mu^2} \right) \]  

(5)

In the derivation of (3) one needs the symmetry properties of \( T_{\mu \nu} \) under \( x \to -x \). This is why one can only find either even or odd moments from these equations, dependent on the process under consideration. For \( F_3 \) we will only consider the flavor non-singlet contributions, due to the properties of the operators \( O^\psi \) and \( O^G \) under charge conjugation there should not be a singlet contribution (see e.g. [33]).

The scale-dependence of the coefficient functions is then covered by the renormalization group equations:

\[ \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) \frac{\partial}{\partial a_s} - \gamma^k_N \right] C^k_{i,N} \left( \frac{Q^2}{\mu^2}, a_s \right) = 0, \quad i = 2, 3, L \]  

(6)

\[ \sum_{k=\psi,G} \left[ \left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) \frac{\partial}{\partial a_s} \right) \delta^{jk} - \gamma^k_N \right] C^k_{i,N} \left( \frac{Q^2}{\mu^2}, a_s \right) = 0, \quad i = 2, L; \ j = \psi, G \]  

(7)

The non-singlet coefficient functions and anomalous dimensions don’t depend on the index \( \alpha \), and we have adopted the conventional collective denotation “ns” for them.

3 The even moments of \( F_2 \) and \( F_L \)

Equation (2) is a relation between operators and does not depend on the hadronic states to which the OPE is applied.

Using the method of projectors [34] one can find both the coefficient functions and the anomalous dimensions for the even moments of \( F_2 \) and \( F_L \) as defined in Eq. (6) from the following 4-point Green functions:

\[ T_{\mu \nu}^{\gamma \gamma} = i \int d^4 z e^{i q \cdot z} \langle p, \text{quark} | T [J^\gamma(z) J^\gamma(0)] | \text{quark} p \rangle \]  

(8)

\[ T_{\mu \nu}^{\gamma \gamma} = i \int d^4 z e^{i q \cdot z} \langle p, \text{gluon} | T [J^\gamma(z) J^\gamma(0)] | \text{gluon} p \rangle \]  

(9)

Applying to Eqs. (8) the projectors

\[ P_N \equiv \left[ \frac{q^{(\mu_1} \cdots q^{\mu_N)}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} \right] \bigg|_{p=0} \]  

(10)

and, to project out the different Lorentz projections (these projectors are valid in \( D = 4 - 2\epsilon \) and for the leading twist approximation):

\[ P_L = -\frac{q^2}{(p \cdot q)^2} p^\mu p^\nu \]  

\[ P_2 = -\frac{3 - 2\epsilon}{2 - 2\epsilon} \frac{q^2}{(p \cdot q)^2} p^\mu p^\nu + \frac{1}{2 - 2\epsilon} g^{\mu \nu} \]

as well as the corresponding flavour projections results in the equations:

\[ T_{k,N}^{\gamma \gamma,\text{quark}} \left( \frac{Q^2}{\mu^2}, a_s, \frac{1}{\epsilon} \right) = \left( C_{k,N}^\psi \left( \frac{Q^2}{\mu^2}, a_s, \frac{1}{\epsilon} \right) Z_N^\psi \left( a_s, \frac{1}{\epsilon} \right) + C_{k,N}^G \left( \frac{Q^2}{\mu^2}, a_s, \frac{1}{\epsilon} \right) Z_N^G \left( a_s, \frac{1}{\epsilon} \right) \right) A_{\text{quark},N} \left( \frac{q^2}{\mu^2} \right) ] \]  

(11)
From these equations the coefficient functions and from the $Z^N$ the anomalous dimensions can be calculated in the usual way.

It should be mentioned that in the Eqs. (11) on the left hand side after applying the projectors (10) we are left with only diagrams of the massless propagator type, a problem solved at 3-loop order long ago [35] and implemented in an efficient way in the FORM package MINCER [36]. On the right hand side only the tree level diagrams contributing to the Matrix elements survive.

It turns out that much computing time can be saved when calculating additionally Green functions with external ghosts to get rid of the unphysical polarization states of the external gluons instead of using the very complicated projection onto physical states. Also, from Eqs. (11) one can determine the $Z^{GG}_N$ and $Z^{G\psi}_N$ only to order $\alpha^2_s$. To obtain the $\alpha^3_s$-contributions one can calculate additionally Green functions with external scalar fields $\phi$ that couple to gluons only at tree level. Altogether, to obtain the coefficient functions and anomalous dimensions for the even moments with $N=10,12$ of $F_2$ and $F_L$ the following diagrams had to be calculated (q=quark, g=gluon, $\gamma=$photon, h=ghost, $\phi=$scalar field):

| Diagram | tree | 1-loop | 2-loops | 3-loops | Lorentz projections |
|---------|------|--------|---------|---------|---------------------|
| $q\gamma q\gamma$ | 1    | 3      | 27      | 413     | 2                   |
| $q\phi q\phi$ | 1    | 24     | 697     | 1       |
| $g\gamma g\gamma$ | 2    | 20     | 366     | 2       |
| $h\gamma h\gamma$ | 2    | 53     | 2       |
| $g\phi g\phi$ | 1    | 11     | 241     | 1266    | 1                   |
| $h\phi h\phi$ | 11   | 241    | 1266    | 1       |
| Total | 3    | 23     | 399     | 10846   |

The $q^{(\mu_1,\ldots,\mu_N)}$ in Eq. (10) are the harmonic (i.e. symmetrical and traceless) part of the tensor $q^{\mu_1,\ldots,\mu_N}$. The number of terms in these harmonic tensors explodes as $N$ increases and this is the real limitation in these calculations considering the computing time as well as disk-space usage. In spite of a very efficient implementation of these tensors (see [37]) for $N=12$, singlet and $N=14$, nonsinglet, individual diagrams had a disk space usage up to and over 100 GB. Altogether the calculation of all the above diagrams for $N=10,12$ took approximately 5 weeks on a Compaq Server with 8 Alpha 21264 processors running at 700 MHz, 4 GB of RAM and 12×17 GB of disk-space. The $N=14$ nonsinglet calculation took comparable resources.

4 The odd moments of $F_3$

The coefficient functions and anomalous dimensions of the odd moments of the structure function $F_3$ can be obtained in the same way as the non-singlet part of $F_2$ and $F_L$ but now considering the time-ordered product of one vector current $V^\mu$ and one axial vector current $A^\nu$. The axial current introduces the appearance of a $\gamma_5$ and some care has to be taken to treat it correctly within the framework of dimensional regularization. We adopt the definition used in [24]:

$$\gamma_\mu \gamma_5 = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} \gamma^\nu \gamma^\rho \gamma^\sigma$$

Projecting out the flavour non-singlet part and the corresponding Lorentz structure with:

$$P_3 = -\frac{1}{(1-2\epsilon)(2-2\epsilon)} \epsilon^{\mu\nu\alpha\beta} \frac{P_a q_\beta}{p \cdot q}$$

one finds products of metric tensors which have to be considered as $D$-dimensional objects. Since this definition of $\gamma_5$ in $D$ dimensions violates the axial Ward identity one needs to renormalize $A^\mu$ with a renormalization constant $Z_A$ and additionally apply a finite renormalization with $Z_5$, both of these constants are given to 3-loop order in [24]. Combining all this finally leads to

$$Z_A(a_s, \frac{1}{\epsilon}) Z_5(a_s, \epsilon) T^{wa}_{C3,N}(\frac{Q^2}{\mu^2}, a_s, \epsilon) = C_{3,N}(\frac{Q^2}{\mu^2}, a_s, \epsilon) Z^{-1}_N(a_s, \frac{1}{\epsilon}) A^{wa}_{N,\text{tree}}(\epsilon)$$
Due to the $\gamma_5$ insertion at one of the vertices, some of the symmetries that were used to minimize the number of diagrams could not be applied in this case and to determine the $T_{3,N}^{\text{NNL}}$ 1076 ($= 1 + 4 + 55 + 1016$) diagrams had to be evaluated, which took about 6 weeks for the moments $N=1,3,5,7,9,11,13$.

5 Results

Using the strategies sketched in the previous section we find the following results for the coefficient functions and anomalous dimensions. Again following Ref. [23], we present the combined singlet and non-singlet results for $F_2$ and $F_L$ in terms of flavour factors which are defined in the following table for $n_f$ number of flavours:

| $f_{l2}$ | $f_{l11}$ | $f_{l02}$ | $f_{l2}^{\eta}$ | $f_{l11}^{\eta}$ |
|----------|----------|----------|-----------------|-----------------|
| non-singlet | $1$ | $\frac{3}{n_f} \sum_{j=1}^{n_f} e_f$ | $0$ | $-1$ |
| singlet | $1$ | $\frac{1}{n_f} \left( \sum_{j=1}^{n_f} e_f \right)^2$ | $1$ | $1$ |

The numerical values of the anomalous dimensions for the spin-even operators contributing to $F_2$ and $F_L$ that now are known to NNL order are:

$$\gamma_{2}^{\psi\psi} = 3.55555556 a_s + a_s^2 (48.32921811 - 3.160493827 n_f - 1.975308642 f_{l02} n_f)$$
$$+ a_s^3 (859.4478732 - 133.4381617 n_f - 1.229080933 n_f^2)$$
$$+ f_{l02} (-42.21182429 n_f - 3.445816187 n_f^2))$$

$$\gamma_{4}^{\psi\psi} = 6.77777778 a_s + a_s^2 (86.28665021 - 6.553580247 n_f - 0.1060246914 f_{l02} n_f)$$
$$+ a_s^3 (1515.562363 - 244.728592 n_f - 2.108515775 n_f^2)$$
$$+ f_{l02} (-5.17013312 n_f - 0.6789278464 n_f^2))$$

$$\gamma_{6}^{\psi\psi} = 9.003174603 a_s + a_s^2 (108.0184697 - 8.62925674 n_f - 0.0208048883 f_{l02} n_f)$$
$$+ a_s^3 (1891.827779 - 307.4236889 n_f - 2.570638992 n_f^2)$$
$$+ f_{l02} (-2.526091192 n_f - 0.2884556035 n_f^2))$$

$$\gamma_{8}^{\psi\psi} = 10.45820106 a_s + a_s^2 (123.7764525 - 10.14583662 n_f - 0.006586485074 f_{l02} n_f)$$
$$+ a_s^3 (2164.091836 - 352.3116596 n_f - 2.882493484 n_f^2)$$
$$+ f_{l02} (-1.682156519 n_f - 0.1620816452 n_f^2))$$

$$\gamma_{10}^{\psi\psi} = 11.5969216 a_s + a_s^2 (136.2741775 - 11.34594534 n_f - 0.0026944007 f_{l02} n_f)$$
$$+ a_s^3 (2379.919952 - 387.6422968 n_f - 3.115523145 n_f^2)$$
$$+ f_{l02} (-1.256562245 n_f - 0.1054295071 n_f^2))$$

$$\gamma_{12}^{\psi\psi} = 12.53336293 a_s + a_s^2 (146.6771964 - 12.34063169 n_f - 0.001302012 f_{l02} n_f)$$
$$+ a_s^3 (2559.641948 - 416.9091301 n_f - 3.300349343 n_f^2)$$
$$+ f_{l02} (-0.9957297081 n_f - 0.07498848634 n_f^2))$$

$$\gamma_{14}^{\psi,NS} = 13.32896733 a_s + a_s^2 (155.6071962 - 13.19066078 n_f)$$
$$+ a_s^3 (2714.031720 - 441.9494048 n_f - 3.452881831 n_f^2)$$
\[ \gamma_2^{G} = -0.666666667 \, a_s \, n_f - 7.543209877 \, a_s^2 \, n_f \\
+ a_s^3 \left( -37.62337275 \, n_f + 12.11248285 \, n_f^2 \right) \]
\[ \gamma_4^{G} = -0.366666667 \, a_s \, n_f + 1.290703704 \, a_s^2 \, n_f \\
+ a_s^3 \left( 33.58149273 \, n_f + 6.06027262 \, n_f^2 \right) \]
\[ \gamma_6^{G} = -0.2619047619 \, a_s \, n_f + 2.761104812 \, a_s^2 \, n_f \\
+ a_s^3 \left( 33.4160135 \, n_f + 3.537682102 \, n_f^2 \right) \]
\[ \gamma_8^{G} = -0.205555556 \, a_s \, n_f + 3.243957223 \, a_s^2 \, n_f \\
+ a_s^3 \left( 28.7612615 \, n_f + 2.225433112 \, n_f^2 \right) \]
\[ \gamma_{10}^{G} = -0.1696969697 \, a_s \, n_f + 3.407168695 \, a_s^2 \, n_f \\
+ a_s^3 \left( 23.93704198 \, n_f + 1.449828678 \, n_f^2 \right) \]
\[ \gamma_{12}^{G} = -0.1446886447 \, a_s \, n_f + 3.438705999 \, a_s^2 \, n_f \\
+ a_s^3 \left( 19.63230379 \, n_f + 0.9524545464 \, n_f^2 \right) \]
\[ \gamma_2^{G\psi} = -3.55555556 \, a_s + a_s^2 \left( -48.32921811 + 5.135802469 \, n_f \right) \\
+ a_s^3 \left( -859.4478372 + 175.649986 \, n_f + 4.674897119 \, n_f^2 \right) \]
\[ \gamma_4^{G\psi} = -0.9777777778 \, a_s + a_s^2 \left( -16.1752428 + 0.6182716049 \, n_f \right) \\
+ a_s^3 \left( -315.276255 + 39.82571027 \, n_f + 1.801843621 \, n_f^2 \right) \]
\[ \gamma_6^{G\psi} = -0.5587301587 \, a_s + a_s^2 \left( -9.496317796 + 0.08884857647 \, n_f \right) \\
+ a_s^3 \left( -188.9088124 + 19.67944546 \, n_f + 1.087843741 \, n_f^2 \right) \]
\[ \gamma_8^{G\psi} = -0.391543915 \, a_s + a_s^2 \left( -6.757603506 - 0.07061952353 \, n_f \right) \\
+ a_s^3 \left( -134.7055042 + 12.37544544 \, n_f + 0.7536013741 \, n_f^2 \right) \]
\[ \gamma_{10}^{G\psi} = -0.3016835017 \, a_s + a_s^2 \left( -5.297576945 - 0.1348941718 \, n_f \right) \\
+ a_s^3 \left( -104.911278 + 8.796702078 \, n_f + 0.5579674847 \, n_f^2 \right) \]
\[ \gamma_{12}^{G\psi} = -0.2455322455 \, a_s + a_s^2 \left( -4.398625917 - 0.1639529655 \, n_f \right) \\
+ a_s^3 \left( -86.18107998 + 6.735609285 \, n_f + 0.4293379075 \, n_f^2 \right) \]
\[ \gamma_2^{GG} = 0.666666667 \, a_s \, n_f + 7.543209877 \, a_s^2 \, n_f \\
+ a_s^3 \left( 37.62337275 \, n_f - 12.11248285 \, n_f^2 \right) \]
\[ \gamma_4^{GG} = a_s^2 \left( 128.178 - 13.64948148 \, n_f \right) + a_s \left( 12.6 + 0.666666667 \, n_f \right) \\
+ a_s^3 \left( 2066.19278 - 401.3127939 \, n_f - 10.43150645 \, n_f^2 \right) \]
\[ \gamma_6^{GG} = a_s^2 \left( 183.0538144 - 20.46684666 \, n_f \right) + a_s \left( 17.78571429 + 0.666666667 \, n_f \right) \\
+ a_s^3 \left( 2987.042058 - 566.6373298 \, n_f - 10.78060861 \, n_f^2 \right) \]
\[ \gamma_8^{GG} = a_s^2 \left( 219.6240988 - 24.69926432 \, n_f \right) + a_s \left( 21.26666667 + 0.666666667 \, n_f \right) \\
+ a_s^3 \left( 3609.35419 - 673.9430658 \, n_f - 11.20133837 \, n_f^2 \right) \]
\[ \gamma_{10}^{GG} = a_s^2 \left( 247.6655484 - 27.82178573 \, n_f \right) + a_s \left( 23.92337662 + 0.666666667 \, n_f \right) \\
+ a_s^3 \left( 4089.236943 - 755.1340541 \, n_f - 11.57068198 \, n_f^2 \right) \]
\[ \gamma_{12}^{GG} = a_s^2 \left( 270.6428892 - 30.31377688 \, n_f \right) + a_s \left( 26.08168498 + 0.666666667 \, n_f \right) \\
+ a_s^3 \left( 4483.563048 - 821.1236576 \, n_f - 11.88665683 \, n_f^2 \right) \]

The corresponding coefficient functions read:
\[
\begin{align*}
C_{2,2}^\psi &= 1 + 0.444444444 a_s + a_s^2 (17.69376589 - 5.33333333 n_f - 2.189300412 \, \text{fl}_{02} \, n_f) \\
&\quad + a_s^3 (442.7409693 - 165.1971095 \, n_f - 24.09201335 \, \text{fl}_{11} \, n_f + 6.030272415 \, n_f^2) \\
&\quad + \text{fl}_{02} (-79.04486142 \, n_f + 3.25504478 \, n_f^2)) \\
C_{2,4}^\psi &= 1 + 6.066666667 a_s + a_s^2 (142.3434719 - 16.98791358 \, n_f + 0.4858308642 \, \text{fl}_{02} \, n_f) \\
&\quad + a_s^3 (416.267888 - 901.2351626 \, n_f - 18.21884618 \, \text{fl}_{11} \, n_f + 23.35503924 \, n_f^2) \\
&\quad + \text{fl}_{02} (16.64834849 \, n_f - 2.298630689 \, n_f^2)) \\
C_{2,6}^\psi &= 1 + 11.17671958 a_s + a_s^2 (302.398735 - 28.0130504 \, n_f + 0.486878285 \, \text{fl}_{02} \, n_f) \\
&\quad + a_s^3 (10069.63085 - 1816.322929 \, n_f - 16.14271761 \, \text{fl}_{11} \, n_f + 42.66273116 \, n_f^2) \\
&\quad + \text{fl}_{02} (24.11778813 \, n_f - 1.525489143 \, n_f^2)) \\
C_{2,8}^\psi &= 1 + 15.52989418 a_s + a_s^2 (470.807419 - 37.9248228 \, n_f + 0.3859585393 \, \text{fl}_{02} \, n_f) \\
&\quad + a_s^3 (17162.37245 - 2787.297692 \, n_f - 15.09203827 \, \text{fl}_{11} \, n_f + 61.91177997 \, n_f^2) \\
&\quad + \text{fl}_{02} (22.33201938 \, n_f - 1.036308122 \, n_f^2)) \\
C_{2,10}^\psi &= 1 + 19.30061568 a_s + a_s^2 (639.210663 - 46.86131842 \, n_f + 0.3045901308 \, \text{fl}_{02} \, n_f) \\
&\quad + a_s^3 (24953.13497 - 3770.10212 \, n_f - 14.85874451 \, \text{fl}_{11} \, n_f + 80.52097973 \, n_f^2) \\
&\quad + \text{fl}_{02} (19.53359559 \, n_f - 0.7372434464 \, n_f^2)) \\
C_{2,12}^\psi &= 1 + 22.62841097 a_s + a_s^2 (804.5854321 - 54.99446579 \, n_f + 0.2451231747 \, \text{fl}_{02} \, n_f) \\
&\quad + a_s^3 (33171.45501 - 4746.440949 \, n_f - 14.30541028 \, \text{fl}_{11} \, n_f + 98.3483124 \, n_f^2) \\
&\quad + \text{fl}_{02} (16.98652635 \, n_f - 0.5471547625 \, n_f^2)) \\
C_{2,14}^\psi &= 1 + 2.561093284 a_s + a_s^2 (965.8132564 - 62.46549093 \, n_f) \\
&\quad + a_s^3 (41657.11568 - 5708.215623 \, n_f - 13.73240102 \, \text{fl}_{11} \, n_f + 115.3919490 \, n_f^2)) \\
C_{2,2}^G &= -0.5 a_s \, n_f - 8.918338961 a_s^2 n_f \\
&\quad + a_s^3 (-130.7340963 \, n_f + 29.37933515 \, n_f^2 - 0.9007972776 \, \text{fl}_{11} \, n_f^2) \\
C_{2,4}^G &= -0.738888889 a_s \, n_f - 14.27158692 a_s^2 n_f \\
&\quad + a_s^3 (-346.4612756 \, n_f + 46.52017564 \, n_f^2 - 1.611816512 \, \text{fl}_{11} \, n_f^2) \\
C_{2,6}^G &= -0.7051587302 a_s \, n_f - 20.06849828 a_s^2 n_f \\
&\quad + a_s^3 (-715.0372438 \, n_f + 61.28545096 \, n_f^2 - 1.496036938 \, \text{fl}_{11} \, n_f^2) \\
C_{2,8}^G &= -0.6440873016 a_s \, n_f - 23.17873524 a_s^2 n_f \\
&\quad + a_s^3 (-996.5038709 \, n_f + 68.66467304 \, n_f^2 - 1.286400915 \, \text{fl}_{11} \, n_f^2) \\
C_{2,10}^G &= -0.5861279461 a_s \, n_f - 24.76678064 a_s^2 n_f \\
&\quad + a_s^3 (-1201.206903 \, n_f + 72.23614791 \, n_f^2 - 1.094394334 \, \text{fl}_{11} \, n_f^2) \\
C_{2,12}^G &= -0.5358430591 a_s \, n_f - 25.51669345 a_s^2 n_f \\
&\quad + a_s^3 (-1351.047836 \, n_f + 73.7936445 \, n_f^2 - 0.9344248731 \, \text{fl}_{11} \, n_f^2) 
\end{align*}
\]
The numerical values of the anomalous dimensions for the odd moments of $F_3$ read:
\[ \gamma_{1}^{\text{NS}} = 0 \]
\[ \gamma_{3}^{\text{NS}} = 5.55555556 a_{s} + a_{s}^{2} \left( 70.88477366 - 5.12345679 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 1244.913602 - 196.4738081 n_{f} - 1.762002743 n_{f}^{2} \right) \]
\[ \gamma_{5}^{\text{NS}} = 8.088888889 a_{s} + a_{s}^{2} \left( 98.19940741 - 7.68691358 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 1720.942172 - 278.1581739 n_{f} - 2.366211248 n_{f}^{2} \right) \]
\[ \gamma_{7}^{\text{NS}} = 9.780952381 a_{s} + a_{s}^{2} \left( 116.4158903 - 9.437457798 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 2036.492478 - 330.8816595 n_{f} - 2.739358023 n_{f}^{2} \right) \]
\[ \gamma_{9}^{\text{NS}} = 11.05820106 a_{s} + a_{s}^{2} \left( 130.3414045 - 10.77682428 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 2277.19805 - 370.5905277 n_{f} - 3.006446616 n_{f}^{2} \right) \]
\[ \gamma_{11}^{\text{NS}} = 12.08581049 a_{s} + a_{s}^{2} \left( 141.6907901 - 11.86411897 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 2473.311857 - 402.691565 n_{f} - 3.21273599 n_{f}^{2} \right) \]
\[ \gamma_{13}^{\text{NS}} = 12.94606135 a_{s} + a_{s}^{2} \left( 151.2989044 - 12.78102552 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 2639.409887 - 429.7314605 n_{f} - 3.37996738 n_{f}^{2} \right) \]

and the coefficient functions are:

\[ C_{3,1}^{\text{NS}} = 1 - 4 a_{s} + a_{s}^{2} \left( -73.33333333 + 5.33333333 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( -2652.154437 + 513.3100408 n_{f} - 11.35802469 n_{f}^{2} \right) \]
\[ C_{3,3}^{\text{NS}} = 1 + 1.666666667 a_{s} + a_{s}^{2} \left( 14.25404015 - 6.742283951 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( -839.7638717 - 45.09953407 n_{f} + 1.747689309 n_{f}^{2} \right) \]
\[ C_{3,5}^{\text{NS}} = 1 + 7.748148148 a_{s} + a_{s}^{2} \left( 173.000629 - 19.39801646 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 4341.081057 - 96.2756356 n_{f} + 22.24125078 n_{f}^{2} \right) \]
\[ C_{3,7}^{\text{NS}} = 1 + 12.72248677 a_{s} + a_{s}^{2} \left( 345.9910777 - 30.52332666 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 11119.00053 - 1960.237096 n_{f} + 43.10377964 n_{f}^{2} \right) \]
\[ C_{3,9}^{\text{NS}} = 1 + 16.9152381 a_{s} + a_{s}^{2} \left( 520.0059615 - 40.35464229 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 18771.99642 - 2975.924131 n_{f} + 63.17127673 n_{f}^{2} \right) \]
\[ C_{3,11}^{\text{NS}} = 1 + 20.54831329 a_{s} + a_{s}^{2} \left( 690.8719666 - 49.17096968 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 26941.47987 - 3984.411605 n_{f} + 82.24581704 n_{f}^{2} \right) \]
\[ C_{3,13}^{\text{NS}} = 1 + 23.762374745 a_{s} + a_{s}^{2} \left( 857.1778817 - 57.18099124 n_{f} \right) \]
\[ \quad + a_{s}^{3} \left( 35426.82868 - 4976.080869 n_{f} + 100.3509187 n_{f}^{2} \right) \]

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References

[1] D. J. Gross and F. Wilczek, Phys. Rev. D8, 3633 (1973).
[2] D. J. Gross and F. Wilczek, Phys. Rev. D9, 980 (1974).
[3] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. Rev. D18, 3998 (1978).
[4] E. G. Floratos, D. A. Ross, and C. T. Sachrajda, Nucl. Phys. B129, 66 (1977).
[5] Nucl. Phys. B139, 545 (1978).
[6] E. G. Floratos, D. A. Ross, and C. T. Sachrajda, Nucl. Phys. B152, 493 (1979).
[7] A. Gonzalez-Arroyo, C. Lopez, and F. J. Yndurain, Nucl. Phys. B153, 161 (1979).
[8] A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B166, 429 (1980).
[9] C. Lopez and F. J. Yndurain, Nucl. Phys. B183, 157 (1981).
[10] W. Furmanski and R. Petronzio, Phys. Lett. B97, 437 (1980).
[11] G. Curci, W. Furmanski, and R. Petronzio, Nucl. Phys. B175, 27 (1980).
[12] R. Hamberg and W. L. van Neerven, Nucl. Phys. B379, 143 (1992).
[13] D. W. Duke, J. D. Kimel, and G. A. Sowell, Phys. Rev. D25, 71 (1982).
[14] A. Devoto, D. W. Duke, J. D. Kimel, and G. A. Sowell, Phys. Rev. D30, 541 (1984).
[15] D. I. Kazakov and A. V. Kotikov, Nucl. Phys. B307, 721 (1988).
[16] D. I. Kazakov and A. V. Kotikov, Phys. Lett. B291, 171 (1992).
[17] D. I. Kazakov, A. V. Kotikov, G. Parente, O. A. Sampayo, and J. S. Guillen, Santiago de Compostela Univ. - US-FT-90-06 (90, rec. Jul.) 6 p.
[18] W. L. van Neerven and E. B. Zijlstra, Phys. Lett. B272, 127 (1991).
[19] E. B. Zijlstra and W. L. van Neerven, Phys. Lett. B273, 476 (1991).
[20] E. B. Zijlstra and W. L. van Neerven, Nucl. Phys. B383, 525 (1992).
[21] S. Moch and J. A. M. Vermaseren, Nucl. Phys. B573, 853 (2000), hep-ph/9912355.
[22] S. A. Larin, T. van Ritbergen, and J. A. M. Vermaseren, Nucl. Phys. B427, 41 (1994).
[23] S. A. Larin, P. Nogueira, T. van Ritbergen, and J. A. M. Vermaseren, Nucl. Phys. B492, 338 (1997), hep-ph/9605317.
[24] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B259, 345 (1991).
[25] J. A. M. Vermaseren and S. Moch, (2000), hep-ph/0004235.
[26] G. Parente, A. V. Kotikov, and V. G. Krivokhizhin, Phys. Lett. B333, 190 (1994), hep-ph/9405290.
[27] A. Kataev, A. V. Kotikov, G. Parente and A. V. Sidorov, Phys. Lett. B388, 179 (1996), hep-ph/9605367.
[28] W. L. van Neerven and A. Vogt, Nucl. Phys. B568, 263 (2000), hep-ph/9907472.
[29] J. Santiago and F. J. Yndurain, Nucl. Phys. B563, 45 (1999).
A Conventions

Here we give the complete expressions for the newly computed moments and coefficient functions.

The notation of the color factors is as usual: The Casimir operators of the fundamental and adjoint representation are denoted by $C_F$ and $C_A$ and their values for the color group SU(3) are $\frac{4}{3}$ and $3$, respectively. Additionally we are using the symmetric structure constants of SU(3) for which $d^{abc}d^{abc} = \frac{40}{3}$. For the trace normalization of the fundamental representation we have inserted $T_F = \frac{1}{2}$. For a generic SU($n$) group the number of generators equals $N_A = n^2 - 1$.

The values of the Riemann $\zeta$ function are written as $\zeta_n = \zeta(n)$. The only check for the newly computed 3-loop results is from a large $n_f$-expansion of [38] where the $n_f^2$ terms are calculated. These terms are in agreement with ours. To test the new parallel version of FORM we have also recalculated the lower moments for $F_2$ and $F_L$ and found complete agreement with [23] except for one missing term. In $C_{G2}^G$ there should be the extra term $a_s^3n_fC^2_{A}(-\frac{300}{3}ζ_5)$. The numerical values given in this reference are correct.

It should be noted that the terms in $d^{abc}d^{abc}$ enter for the first time at the three loop level and help in the determination of $P_{qq}^S - P_{q\bar{q}}^S$.

B Results for the moments of $F_2$ and $F_L$

$$\gamma^{GG}_{10} = \begin{cases} 
\frac{18421}{2310} + a_s n_f \left( \frac{2}{3} \right) \\
+ a_s^2 C_A \left[ \frac{339202487377}{12326391000} \right] + a_s^2 C_F n_f \left[ \frac{9284182}{4492125} \right] + a_s^2 C_A n_f \left[ \frac{17481908}{1715175} \right] \\
+ a_s^3 C_A \left[ \frac{2390835263828750523}{157859652036240000} \right] + a_s^3 C_F C_A n_f \left[ \frac{3374081335517123191}{3624328745730000} + \frac{17831164}{190575} \zeta_3 \right] \\
+ a_s^3 C_F C_A n_f \left[ \frac{3009386129483453}{3883209370425000} - \frac{1344}{3025} \zeta_3 \right] \right] \\
+ a_s^3 C_F n_f \left[ \frac{43228502023851731}{2196562876200000} - \frac{17746492}{190575} \zeta_3 \right] \right] \\
+ a_s^3 C_A n_f \left[ \frac{453912946493}{420260754375} + a_s^3 C_A n_f \left[ \frac{2752314359}{815051160} \right] \right] \\
\end{cases}$$
\[ \gamma_{12}^{GG} = a_s C_A \left[ \frac{71203}{8190} + a_s n_f \left[ \frac{2}{3} + a_s^2 C_A^2 \left[ \frac{16519839244157}{549353259000} \right] \right] \right] + a_s^2 C_A n_f \left[ \frac{23220103}{2108106} + a_s^2 C_F n_f \left[ \frac{74823503}{36540504} \right] \right] + a_s^3 C_A \left[ \frac{699311987380651614841}{42112541869725600000} \right] + a_s^3 C_F C_A n_f \left[ \frac{49959541388602890695713}{498238162032042432000} + \frac{58085396}{585585} \right] + a_s^3 C_F n_f \left[ \frac{158}{507} \right] + a_s^3 C_A n_f \left[ \frac{57902906}{585585} \right] + a_s^3 C_F n_f^2 \left[ \frac{50627726543561953}{45626205314289600} \right] + a_s^3 C_A n_f^2 \left[ \frac{2635361358193}{759677078160} \right] \]

\[ \gamma_{10}^{G\psi} = a_s C_F \left[ \frac{112}{495} \right] + a_s^2 C_F C_A \left[ \frac{133349533}{80858250} \right] + a_s^2 C_F^2 \left[ \frac{88631998}{121287375} \right] + a_s^2 C_F n_f \left[ \frac{74368}{735075} \right] + a_s^3 C_F C_A \left[ \frac{165894431274725803}{48324383276400000} - \frac{645584}{81675} \right] + a_s^3 C_F^2 C_A \left[ \frac{4224421791031474951}{21745972474380000} + \frac{645584}{27225} \right] + a_s^3 C_F \left[ \frac{6457897459084371893}{326189587115700000} - \frac{1291168}{81675} \right] + a_s^3 C_F C_A n_f \left[ \frac{18846629176433}{47069204490000} + \frac{1792}{495} \right] + a_s^3 C_F n_f \left[ \frac{529979902254031}{1294403123475000} - \frac{1792}{495} \right] + a_s^3 C_F n_f^2 \left[ \frac{152267426}{363862125} \right] \]
$$\gamma_{12}^{G\psi} = a_s C_F \left[ -\frac{79}{429} + a_s^2 C_F C_A \left[ -\frac{70863259553}{50645138544} \right] + a_s^2 C_F^2 \left[ \frac{9387059226553}{13927413099600} \right] \right]$$

$$+ a_s^2 C_F n_f \left[ \frac{14257247}{115945830} \right]$$

$$+ a_s^3 C_F C_A \left[ -\frac{2475196999767240119153}{149471448609612729600000} - \frac{46179137}{6441435} \zeta_3 \right]$$

$$+ a_s^2 C_F C_A \left[ \frac{115896599183122809974023}{5872092623949071520000} + \frac{46179137}{2147145} \zeta_3 \right]$$

$$+ a_s^3 C_F^3 \left[ -\frac{2165927305724992215894703}{1130377830110196276670000} + \frac{92358274}{6441435} \zeta_3 \right]$$

$$+ a_s^3 C_F C_A n_f \left[ -\frac{64190493078139789}{158540879918384000} + \frac{1264}{429} \zeta_3 \right]$$

$$+ a_s^3 C_F^2 n_f \left[ -\frac{140144001326440151}{13981172914164456000} + \frac{1264}{429} \zeta_3 \right]$$

$$+ a_s^3 C_F n_f^2 \left[ -\frac{134540243934417}{417822392988800} \right]$$

$$\gamma_{10}^{\psi G} = a_s n_f \left[ -\frac{28}{165} + a_s^2 C_F n_f \left[ -\frac{379479917}{125779500} + a_s^2 C_A n_f \left[ \frac{373810079}{150935400} \right] \right] \right]$$

$$+ a_s^2 C_F C_A n_f \left[ \frac{926990216580622991}{24162191638200000} + \frac{643396}{27225} \zeta_3 \right]$$

$$+ a_s^3 C_F n_f \left[ -\frac{10919800485363213833}{54364931185950000} + \frac{17712}{3025} \zeta_3 \right]$$

$$+ a_s^3 C_F^2 n_f \left[ -\frac{21025430857658971}{1022738270400000} + \frac{483988}{27225} \zeta_3 \right]$$

$$+ a_s^3 C_F n_f^2 \left[ -\frac{1584713325754369}{2588806246950000} + a_s^3 C_A n_f^2 \left[ \frac{1669885489}{7906140000} \right] \right]$$

$$\gamma_{12}^{\psi G} = a_s n_f \left[ -\frac{79}{546} \right]$$

$$+ a_s^2 C_F n_f \left[ \frac{9256843807}{3197294100} + a_s^2 C_A n_f \left[ \frac{653436358741}{268572704400} \right] \right]$$

$$+ a_s^3 C_F C_A n_f \left[ \frac{404603250021008148641959}{104630014026728910720000} + \frac{171207527}{8198190} \zeta_3 \right]$$

$$+ a_s^3 C_F C_A n_f \left[ \frac{2960118366121154148615047}{1438662692867522522400000} + \frac{2563}{507} \zeta_3 \right]$$

$$+ a_s^3 C_F n_f \left[ -\frac{10876559569107463644949}{543532540395917440000} + \frac{129763817}{8198190} \zeta_3 \right]$$

$$+ a_s^3 C_F n_f^2 \left[ -\frac{149081947693135635881}{249119081016021216000} \right]$$

$$+ a_s^3 C_A n_f^2 \left[ \frac{226617401255197}{4399220898072000} \right]$$
\[ \gamma_{10}^{\psi \psi} = a_s C_F \left[ + \frac{12055}{1386} \right] \]
\[ + a_s^2 C_F C_A \left[ + \frac{19524247733}{523908000} \right] + a_s^2 C_F^2 \left[ \frac{9579051036701}{1331250228000} \right] \]
\[ + a_s^2 C_F n_f \left[ - \frac{2451995507}{288149400} \right] + a_s^2 n_f n_f \left[ \frac{27284}{13476375} \right] \]
\[ + a_s^3 C_F C_A^2 \left[ + \frac{9409156857976453}{435681892800000} + \frac{151796299}{8004150} \zeta_3 \right] \]
\[ + a_s^3 C_F^2 C_A \left[ - \frac{1638998205648833}{465937579800000} - \frac{151796299}{2668050} \zeta_3 \right] \]
\[ + a_s^3 C_F^3 \left[ \frac{220771130080736405687}{127866318149354400000} + \frac{151796299}{4002075} \zeta_3 \right] \]
\[ + a_s^3 C_F C_A n_f \left[ - \frac{9007773127403}{3890016900000} - \frac{48220}{693} \zeta_3 \right] \]
\[ + a_s^3 C_F^2 n_f \left[ - \frac{7552207321041127}{1230075210672000} + \frac{48220}{693} \zeta_3 \right] \]
\[ + a_s^3 C_F n_f^2 \left[ \frac{2799590156887}{11981252052000} \right] \]
\[ + a_s^3 n_f n_f \left[ - \frac{1028766412107043}{5177612493900000} - \frac{12544}{27225} \zeta_3 \right] \]
\[ + a_s^3 n_f n_f \left[ + \frac{209966063746798}{485401171303125} + \frac{12544}{27225} \zeta_3 \right] \]
\[ + a_s^3 n_f n_f \left[ - \frac{33230913134}{420260754375} \right] \]
\[ \gamma_{12}^{\psi \psi} = a_s C_F \left[ -\frac{423424}{45045} \right] + a_s^2 C_F C_A \left[ \frac{19487270392267}{486972486000} \right] + a_s^2 C_F^2 \left[ \frac{5507868301548461}{731189187729000} \right] + a_s^2 C_F n_f \left[ \frac{90143221429}{9739449720} \right] + a_s^2 \bar{n}_{02} C_F n_f \left[ \frac{19487270392267}{486972486000} \right] + a_s^2 C_F C_A \left[ -\frac{60166424590272000000}{26349133569002244000000} \right] + a_s^2 C_F^2 C_A \left[ \frac{13527011350}{450909450} \right] + a_s^2 C_F^2 C_A \left[ \frac{98204412073910020058227}{26349133569002244000000} \right] + a_s^2 C_F^3 \left[ \frac{25648239313}{676350675} \right] + a_s^2 C_F C_A n_f \left[ \frac{25478252190337435009}{10529124303297600000} \right] + a_s^2 C_F n_f \left[ \frac{35346062280941906036867}{526982671380044880000} \right] + a_s^2 C_F n_f \left[ \frac{6515583387858701}{26322810758244000} \right] + a_s^2 \bar{n}_{02} C_F C_A n_f \left[ \frac{609748954384494691}{32215877488028288000} \right] + a_s^2 \bar{n}_{02} C_F^2 n_f \left[ \frac{86033255402443256197}{2192247912941098670080} \right] + a_s^2 \bar{n}_{02} C_F n_f \left[ \frac{2566080055386457}{45626205314289600} \right] \]

\[ \gamma_{14}^{n s} = a_s C_F \left[ +\frac{180121}{18018} \right] + a_s^2 C_F C_A \left[ \frac{288858136265399}{6817614804000} \right] + a_s^2 C_F n_f \left[ \frac{481761665447}{48697248600} \right] + a_s^2 C_F \left[ -\frac{22819142381313407}{2924756750916000} \right] + a_s^3 C_F C_A n_f \left[ \frac{92553116363319241549}{36851935061541600000} \right] + a_s^3 C_F^2 C_A \left[ \frac{126653245164236390142889}{515927090861582400000} \right] + a_s^3 C_F n_f \left[ \frac{68176166257767019}{26322810758244000} \right] + a_s^3 C_F C_A \left[ -\frac{960013337169036214858387}{2459252466440209440000} \right] + a_s^3 C_F n_f \left[ \frac{37908544797975614512733}{526982671380044880000} \right] + a_s^3 C_F \left[ -\frac{40552395746064871242211709}{2373793443321412161960000} \right] + a_s^4 C_F \left[ \frac{3663695353}{96621525} \right] \]
\[
C_{2,10}^\psi = 1 + a_2 C_F \left[ \frac{206299}{138600} \right] + a_2^2 C_F C_A \left[ \frac{6124093193824187}{29045459520000} + \frac{104674}{1155} \zeta_3 \right] \\
+ a_2^2 C_F n_f \left[ \frac{561457267429757}{1597500273600} \right] + a_2^2 C_F^2 \left[ \frac{558708799987324013}{147690252806400} + \frac{88798}{1155} \zeta_3 \right] \\
+ a_2^2 n_2 C_F n_f \left[ \frac{3584203788491}{1568973483000} \right] + a_2^2 C_F C_A n_f \left[ \frac{21664424926039357214987}{23550349033411200000} + \frac{10519793104}{42567525} \zeta_3 - \frac{24110}{693} \zeta_4 \right] \\
+ a_2^2 C_F C_A^2 \left[ \frac{70922111996545793995237}{23550349033411200000} - \frac{14713925739913}{6243237000} \zeta_3 \right] \\
+ \frac{151796299}{16008300} \frac{190858}{231} \frac{\zeta_4}{\zeta_5} + \frac{57084428048515515111}{996360920644320000} + \frac{48220}{1871} \zeta_3 \\
+ a_3^2 C_F n_f^2 \left[ \frac{1635000930492693389608829}{836943173341288000000} + \frac{1430215936081}{6163195500} \zeta_3 \right] \\
- \frac{151796299}{5336100} \frac{22658}{99} \frac{\zeta_4}{\zeta_5} \\
+ a_3^2 C_F^2 n_f \left[ \frac{1521387460036994061010049}{272000653133589936000000} - \frac{3997754476}{42567525} \zeta_3 + \frac{24110}{693} \zeta_4 \right] \\
+ a_3^2 C_F^3 \left[ \frac{32477795323709206237760610131}{921558128166270316800000000} + \frac{2182208825245282}{1622461215375} \zeta_3 \right] \\
+ \frac{151796299}{8004150} \frac{75212}{99} \frac{\zeta_4}{\zeta_5} \\
+ a_3^2 n_2 C_F C_A n_f \left[ \frac{3566946294536415188593}{8611405998544800000000} - \frac{79527463}{56600775} \zeta_3 - \frac{6272}{27225} \zeta_4 \right] \\
+ a_3^2 n_2 C_F n_f^2 \left[ \frac{148475806971656561}{246412190336775000} + \frac{33008}{735075} \zeta_3 \right] \\
+ a_3^2 n_2 C_F^2 n_f \left[ \frac{422577250875954453771617}{5877283908650682600000000} - \frac{12949105012}{11037151125} \zeta_3 + \frac{6272}{27225} \zeta_4 \right] \\
+ a_3^2 n_1 n_f \left[ \frac{d_{abc} d_{abc}}{n} \right] \left[ \frac{3753913187503}{3520661760000} + \frac{81388}{606375} \frac{\zeta_3}{\zeta_5} - \frac{448}{33} \zeta_5 \right] 
\]
\[
C_{2,12}^\psi = 1 + a_s C_F \left[ + \frac{183473419}{10810800} \right] \\
+ a_s^2 C_F C_A \left[ + \frac{2138849987332252399}{8774270257480000} - \frac{1477711}{15015} \right] \zeta_3 \\
+ a_s^2 C_F^2 \left[ + \frac{1217110915702407798941}{131745667845011220000} + \frac{1261726}{15015} \right] \zeta_3 \\
+ a_s^2 C_F n_f \left[ - \frac{57904356630607013}{14038832404396800} + 183473419 + 10810800 \right] \\
+ a_s^2 C_F C_A n_f \left[ - \frac{8371262622937204073275967}{75885504678724627020000} + \frac{36842282041}{133783650} \right] \zeta_3 \\
+ a_s^2 C_F n_f^2 \left[ + \frac{169396}{45045} \zeta_4 \right] + a_s^2 C_F C_A^2 \left[ + \frac{163181620367687907864404054279}{45531302807235877632000000} \right] \\
+ a_s^2 C_F n_f \left[ + \frac{641004330821357}{24348643000} \zeta_3 + \frac{25648239313}{2705402700} \zeta_4 + \frac{2642336}{3003} \zeta_5 \right] \zeta_3 \\
+ a_s^3 C_F n_f \left[ - \frac{2003755100996957438423887}{2845796425524223520000} + \frac{3387392}{1216215} \right] \zeta_3 \\
+ a_s^3 C_F^2 C_A \left[ - \frac{4843191896526849157300333804109}{1709131279126617566112000000} + \frac{498562279669051}{97491891697200} \right] \zeta_3 \\
+ a_s^3 C_F n_f \left[ - \frac{525648239313}{901800900} \zeta_4 - \frac{241094}{1287} \zeta_5 \right] \zeta_3 \\
+ a_s^3 C_F n_f^2 \left[ + \frac{848389810670975600831798557}{11300041514518867223577600} - \frac{1326399435709}{12174312150} \right] \zeta_3 \\
+ a_s^3 C_F^3 \left[ + \frac{169396}{45045} \zeta_4 \right] + a_s^3 C_F C_A n_f \left[ + \frac{16317036739377553536363889790633}{3421680820811486746735622400000} \right] \\
+ a_s^3 C_F n_f \left[ + \frac{42634681331415644}{24920904127125} \zeta_3 + \frac{25648239313}{1352701350} \zeta_4 - \frac{88004}{99} \zeta_5 \right] \zeta_3 \\
+ a_s^3 C_F n_f \left[ + \frac{865659031583141388806418902631}{26931765610480021619328000000} - \frac{310947622803}{3165321159000 \zeta_3} \right] \\
+ a_s^3 C_F^2 C_A n_f \left[ - \frac{6241}{39039} \zeta_4 \right] + a_s^3 C_F n_f^2 \left[ - \frac{1960867603733060624851}{440469467961803640000} + \frac{2780}{9583} \right] \zeta_3 \\
+ a_s^3 C_F^3 n_f \left[ + \frac{3528586559068805780843042423}{5290168244915718532368000000} - \frac{751723347541}{791330289750} \right] \zeta_3 \\
+ a_s^3 C_F n_f \left[ + \frac{6241}{39039} \zeta_4 \right] + a_s^3 C_F n_f \left[ + \frac{809917806143013559}{67963572576800000} + \frac{622064791}{851350500 \zeta_3} \right] \\
+ a_s^3 C_F n_f \left[ + \frac{809917806143013559}{67963572576800000} + \frac{622064791}{851350500 \zeta_3} \right]
\[ C_{2,14}^{\text{ns}} = 1 + a_s C_F \left[ \frac{90849502}{4729725} \right] + a_s^2 C_F C_A \left[ \frac{1345455725874078802801 - 315626}{491359134153888000} \right] + a_s^2 C_F n_f \left[ \frac{1644267296654871017}{350970810109920000} \right] + a_s^2 C_F^2 \left[ \frac{301129555074311360000 + 271010}{3003} \right] \]

\[ + a_s^3 C_F C_A n_f \left[ \frac{28812973254576289068812626927}{2257593764192112266592000000} + \frac{31112773830559}{103481653275} \right] \]

\[ - \frac{360242}{9009} \zeta_4 \]

\[ + a_s^3 C_F C_A^2 \left[ \frac{78236211156350047175125815627621}{1896378761921374303372800000} - \frac{83168919211026563}{28974862917000} \right] \]

\[ + \frac{3663695353}{386486100} \zeta_4 + \frac{2738146}{3003} \zeta_5 \]

\[ + a_s^3 C_F n_f^2 \left[ \frac{2361466163828853440218087}{28457064254522423520000} + \frac{720484}{243243} \right] \]

\[ + a_s^3 C_F^2 C_A \left[ \frac{64116556842577537005311631547547}{17091312791266167566112000000} - \frac{136829927960626061}{81243243081000} \right] \]

\[ - \frac{3663695353}{128828700} \zeta_4 - \frac{865562}{9009} \zeta_5 \]

\[ + a_s^3 C_F n_f \left[ \frac{35845184538120717240043774137}{3773406720149673358752000000} - \frac{12775152582499}{103481653275} \right] \]

\[ + a_s^3 C_F^2 \left[ \frac{1292440777998503126842828006600921}{727107174422449033681319760000000} + \frac{3556746663996971701}{1695029480644500} \right] \]

\[ + \frac{3663695353}{193243050} \zeta_4 - \frac{9512108}{9009} \zeta_5 \]

\[ + a_s^3 f l_{11} n f \left[ \frac{d^{abc} d^{abc}}{n} \right] \]

\[ = \frac{637395762233410021}{49587712574400000} + \frac{78376866703}{65553988500} \zeta_3 - \frac{352}{21} \zeta_5 \]
\[ C_{\ell,10}^\psi = a_s C_F \left[ \frac{4}{11} \right] + a_s^2 C_F C_A \left[ \frac{89670761}{8731800} - \frac{48}{11} \zeta_3 \right] + a_s^2 C_F n_f \left[ -\frac{163679}{114345} \right] + a_s^2 C_F^2 \left[ -\frac{1999510607}{528273900} + \frac{96}{11} \zeta_3 \right] + a_s^2 C_F n_f^2 \left[ -\frac{415796}{8085825} \right] + a_s^3 C_F C_A n_f \left[ -\frac{176183576988227323}{1699159381920000} + \frac{55485434}{1216215} \zeta_3 \right] + a_s^3 C_F n_f^2 \left[ -\frac{6796637527680000}{11320155} \right] + a_s^3 C_F^2 \left[ \frac{2366034921481985137}{6796637527680000} - \frac{95022195887}{187297110} \zeta_3 + \frac{3760}{11} \zeta_5 \right] + a_s^3 C_F n_f^2 \left[ -\frac{63272639}{11320155} \right] + a_s^3 C_F^2 C_A \left[ \frac{323139848004267269}{3354750574560000} + \frac{22904191}{17325} \zeta_3 - \frac{14240}{11} \zeta_5 \right] + a_s^3 C_F n_f \left[ -\frac{9048874326307637}{190368782604000} - \frac{1174256}{15015} \zeta_3 \right] + a_s^3 C_F^3 \left[ \frac{887562386698645967383}{3166213592269728000} - \frac{357031607224}{468242775} \zeta_3 + \frac{13440}{11} \zeta_5 \right] + a_s^3 C_F n_f^2 \left[ -\frac{68379915239899511}{434919444948760000} - \frac{36224}{147015} \zeta_3 \right] + a_s^3 C_F n_f^2 \left[ \frac{21670644503}{156897348300} \right] + a_s^3 C_F n_f^2 \left[ -\frac{319520059852805113}{282697642166940000} + \frac{1373248}{1911195} \zeta_3 \right] + a_s^3 C_F n_f^2 \left[ \frac{5073093424963}{528099264000} - \frac{1820773}{363825} \zeta_3 + \frac{160}{11} \zeta_5 \right] \]
\[ C_{L,12}^{\alpha} = a_s C_F \left[ \frac{4}{13} \right] + a_s^2 C_F C_A \left[ \frac{7126442885209}{811620810000} - \frac{48}{13} \zeta_3 \right] \\
+ a_s^2 C_F n_f \left[ \frac{2201663}{1756755} \right] + a_s^2 C_F^2 \left[ \frac{12296141077867}{5275535265000} + \frac{96}{13} \zeta_3 \right] \\
+ a_s^2 F_{102} C_F n_f \left[ \frac{16072451}{502431930} \right] + a_s^2 C_F n_f^2 \left[ \frac{5203557911}{1027701673} \right] \\
+ a_s^3 C_F C_A n_f \left[ \frac{6133505476182519657}{658070268956100000} + \frac{52026975}{52026975} \right] \\
+ a_s^3 C_F C_A^2 \left[ \frac{148978189688273389644411}{52645621516488000000} - \frac{54676524051417}{10145260125} \zeta_3 + \frac{5600}{13} \zeta_5 \right] \\
+ a_s^3 C_F^2 C_A \left[ \frac{1508964059584735294818791}{26349133569002244000000} - \frac{3065115509662}{2029052025} \zeta_3 + \frac{2160}{13} \zeta_5 \right] \\
+ a_s^3 C_F n_f^2 \left[ \frac{12143703744472185053}{316848648015900000} - \frac{47440688}{675675} \zeta_3 \right] \\
+ a_s^3 C_3 \left[ -\frac{16091344678479668687707841}{4281734209628646500000} - \frac{1018015542576}{10145260125} \zeta_3 + 1600 \zeta_5 \right] \\
+ a_s^3 F_{102} C_F C_A n_f \left[ \frac{592278098533386728681}{5766645398893800000000} - \frac{561855512}{3381753375} \zeta_3 \right] \\
+ a_s^3 F_{102} n_f^2 \left[ + \frac{435058406339681}{5280810800265000} \right] \\
+ a_s^3 F_{102} C_F n_f \left[ -\frac{65983065928499265747263}{856348409992579230000000} - \frac{1570446544}{3381753375} \zeta_3 \right] \\
+ a_s^3 F_{111} n_f \frac{a_{abc} a_{abc}}{n} \left[ -\frac{503821438649257451}{62302510670400000} - \frac{302982523}{70945875} \zeta_3 + \frac{160}{13} \zeta_5 \right]

\[ C_{L,14}^{\alpha} = a_s C_F \left[ \frac{4}{15} \right] + a_s^2 C_F C_A \left[ + \frac{3736751546509}{486972486000} - \frac{16}{5} \zeta_3 \right] \\
+ a_s^2 C_F n_f \left[ -\frac{2263109}{2027025} \right] \\
+ a_s^2 C_F^2 \left[ -\frac{6706232197}{49691070000} + \frac{32}{5} \zeta_3 \right] \\
+ a_s^3 C_F C_A n_f \left[ -\frac{14497944461225084969643}{1708589716489656000000} + \frac{16947716309}{44972525} \zeta_3 \right] \\
+ a_s^3 C_F C_A^2 \left[ + \frac{170413131627782389591041}{751779475255486400000} - \frac{47503400282843}{82785322620} \zeta_3 + \frac{1552}{3} \zeta_5 \right] \\
+ a_s^3 C_F n_f^2 \left[ + \frac{422957746}{91216125} \right] \\
+ a_s^3 C_F C_A^2 \left[ + \frac{1016513978878471248683819}{5269826713800448800000} + \frac{451799344112951}{30435780375} \zeta_3 + 2016 \zeta_5 \right] \\
+ a_s^3 C_F n_f^2 \left[ + \frac{58345189864914275683}{18645324287089500000} - \frac{438969448}{6891885} \zeta_3 \right] \\
+ a_s^3 C_F \left[ -\frac{83285060672914320168970479}{179141082692152592000000} - \frac{640509641943719}{517408266375} \zeta_3 + \frac{5888}{3} \zeta_5 \right] \\
+ a_s^3 F_{111} n_f \frac{a_{abc} a_{abc}}{n} \left[ -\frac{110339419075223314037}{15793686454946400000} - \frac{17420356529}{4682427750} \zeta_3 + \frac{32}{3} \zeta_5 \right] \]
$$C_{2,10}^G = a_s n_f \left[ \frac{4352}{7425} \right]$$

$$+ a_s^2 C_{A n f} \left[ \frac{651112454591}{185952142800} - \frac{54}{55} \zeta_3 \right]$$

$$+ a_s^2 C_{F n f} \left[ \frac{72533010722807}{6973215480000} + \frac{108}{55} \zeta_3 \right]$$

$$+ a_s^3 C_{A n f}^2 \left[ \frac{30367858943250477461}{1739677797950400000} - \frac{18965}{137214} \zeta_3 \right]$$

$$+ a_s^3 C_{A n f}^2 \left[ -\frac{11456308998205006311847}{6697759522109040000000} + \frac{427785744377}{7924108500} \zeta_3 \right]$$

$$+ \frac{241994}{27225} \zeta_4 + \frac{8}{33} \zeta_5$$

$$+ a_s^3 C_{F C A n f} \left[ \frac{110850413975318446061177}{24877392510690720000000} + \frac{7772983651}{7358100750} \zeta_3 \right]$$

$$- \frac{321698}{27225} \zeta_4 + \frac{244}{11} \zeta_5$$

$$+ a_s^3 C_{F n f}^2 \left[ \frac{25164738348656825457229}{13993533287263530000000} - \frac{2762978063}{1226350125} \zeta_3 \right]$$

$$+ a_s^3 C_{F n f}^2 \left[ \frac{26284376777719892724358177}{2350913592260273040000000} + \frac{1782724946402}{77260057875} \zeta_3 \right]$$

$$+ \frac{8856}{3025} \zeta_4 - \frac{1048}{33} \zeta_5$$

$$+ a_s^3 b_{11}^2 n_f^2 \frac{g^{abc} d_{abc}}{N_A} \left[ -\frac{11661390042871}{128024064000} - \frac{29154339}{107800} \zeta_3 + \frac{4408}{11} \zeta_5 \right]$$
\[
C_{2,12}^G = a_s n_f \left[ \frac{-8110049}{15135120} \right] + a_s^2 C_{AN_f} \left[ \frac{395407355563386331}{10558130155372800} - \frac{11}{13} \zeta_3 \right] + a_s^2 C_{F n_f} \left[ \frac{6258789011950819}{598533455520000} + \frac{22}{13} \zeta_3 \right] + a_s^3 C_{AN_f}^2 \left[ \frac{11029762895415525512630679}{62778008416037346432000} - \frac{1574723}{8049132} \zeta_3 \right] + a_s^3 C_{A n_f}^2 \left[ \frac{18295361799349570193991309242431}{1028303777854691734556160000000} + \frac{6726830671058291}{132943488678000} \zeta_3 \right] + \frac{129763817}{16396380} \left[ \frac{4}{21} \zeta_3 \right] + a_s^3 C_{F C_{AN_f}} \left[ \frac{9699486762665487150445457223019}{1885223592733601513352960000000} + \frac{789241683301607}{132943488678000} \zeta_3 \right] - \frac{171207527}{16396380} \left[ \frac{4390}{273} \zeta_3 \right] + a_s^3 C_{F n_f}^2 \left[ \frac{25016184592875203289560351501}{13465882905240010809664000000} - \frac{13373036672}{7193911725} \zeta_3 \right] + a_s^3 C_{F n_f} \left[ \frac{19438251371991420503794320835327}{1555309464005221248516192000000} + \frac{542538591728921}{33235872169500} \zeta_3 \right] + \frac{2563}{1014} \left[ \frac{2220}{91} \zeta_3 \right] + a_s^3 \Pi_{11}^{\theta_4 n_f} \left[ \frac{1656600471440498533}{102979356480000000} - \frac{82799792129}{180589500} \zeta_3 + \frac{62436}{91} \zeta_3 \right]
\]

\[
C_{L,10}^G = a_s n_f \left[ \frac{2}{33} \right] + a_s^2 C_{AN_f} \left[ \frac{2460678191}{1056547800} \right] + a_s^2 C_{F n_f} \left[ \frac{509195549}{704365200} \right] + a_s^3 C_{A n_f}^2 \left[ \frac{140853814103239}{2196562876200} - \frac{4}{33} \zeta_3 \right] + a_s^3 C_{A n_f}^2 \left[ \frac{12721909429609774861}{162369092870400000} + \frac{5906419}{24012450} \zeta_3 - \frac{80}{11} \zeta_3 \right] + a_s^3 C_{F n_f}^2 \left[ \frac{148061845621707638477}{2638511326891440000} - \frac{210786157}{31216185} \zeta_3 + \frac{320}{11} \zeta_3 \right] + a_s^3 C_{F C_{AN_f}} \left[ \frac{491491787586698683}{188465094777960000} \right] + a_s^3 C_{F n_f} \left[ \frac{9053411269935853949}{3769301895559200000} + \frac{860883476}{156080925} \zeta_3 - \frac{320}{11} \zeta_3 \right] + a_s^3 \Pi_{11}^{\theta_4 n_f} \left[ \frac{112883693141257}{4526565120000} + \frac{347273501}{4365900} \zeta_3 - \frac{1280}{11} \zeta_3 \right]
\]
\[ C_{L,12}^G = a_s n_f \left[ \frac{4}{91} + a_s^2 C_{A n_f} + \frac{4978992299}{2685727044} + a_s^2 C_{F n_f} - \frac{98593150597}{179847793125} \right] \\
+ a_s^3 C_{A n_f}^2 \left[ \frac{116919410865341069}{22683482755683750} - \frac{8}{91} \zeta_3 \right] \\
+ a_s^3 C_{A n_f}^2 \left[ \frac{96912603479263207273229}{1467873372990024000000} - \frac{16493287}{717341625} \zeta_3 - \frac{480}{91} \zeta_5 \right] \\
+ a_s^3 C_{F C A n_f} \left[ \frac{2084160567995424969918773}{46709827690503978000000} - \frac{9006563627}{2152024875} \zeta_3 + \frac{1920}{91} \zeta_5 \right] \\
+ a_s^3 C_{F n_f}^2 \left[ \frac{49342300647723867029}{24328035255470821875} - \frac{1696}{6825} \zeta_3 \right] \\
+ a_s^3 C_{F n_f}^2 \left[ \frac{28171159611508188453551}{15569942563501326000000} + \frac{2347741862}{717341625} \zeta_3 - \frac{1920}{91} \zeta_5 \right] \\
+ a_s^3 n_f^2 d_{abc} d_{abc} N_A \left[ \frac{2232852976776993919}{56638646064000000} + \frac{11589108697}{993242250} \zeta_3 - \frac{15776}{91} \zeta_5 \right] \\

C \quad \text{Results for the moments of } F_3 \\

\[ \gamma_{3 ns} = a_s C_F \left[ \frac{25}{6} + a_s^2 C_{F C A} + \frac{535}{27} + a_s^2 C_{F n_f} - \frac{415}{108} + a_s^2 C_{F}^2 \right] \left[ \frac{2035}{432} \right] \\
+ a_s^3 C_{F C A n_f} \left[ \frac{62249}{3888} - \frac{100}{3} \zeta_3 \right] + a_s^2 C_{F C A}^2 \left[ -\frac{889433}{7776} + \frac{55}{3} \zeta_3 \right] \\
+ a_s^3 C_{F n_f}^2 \left[ \frac{2569}{1944} + a_s^3 C_{F C A}^2 \right] \left[ -\frac{311213}{1552} - 55 \zeta_3 \right] \\
+ a_s^3 C_{F}^2 n_f \left[ \frac{203627}{7776} + \frac{100}{3} \zeta_3 \right] + a_s^3 C_{F}^3 \left[ -\frac{244505}{1552} + \frac{110}{3} \zeta_3 \right] \\
+ a_s^3 n_f d_{abc} d_{abc} \left[ \frac{205}{288} \right] \\

\[ \gamma_{5 ns} = a_s C_F \left[ \frac{91}{15} + a_s^2 C_{F C A} + \frac{73223}{2700} + a_s^2 C_{F n_f} - \frac{7783}{1350} + a_s^2 C_{F}^2 \right] \left[ -\frac{2891}{500} \right] \\
+ a_s^3 C_{F C A n_f} \left[ \frac{38587}{2000} - \frac{728}{15} \zeta_3 \right] + a_s^3 C_{F C A}^2 \left[ -\frac{305342801}{1944000} + \frac{144}{75} \zeta_3 \right] \\
+ a_s^3 C_{F n_f}^2 \left[ \frac{215621}{121500} + a_s^3 C_{F C A}^2 \right] \left[ -\frac{63892213}{2430000} + \frac{1414}{25} \zeta_3 \right] \\
+ a_s^3 C_{F}^2 n_f \left[ \frac{5494973}{135000} - \frac{728}{15} \zeta_3 \right] + a_s^3 C_{F}^3 \left[ -\frac{51831073}{3037500} + \frac{2828}{75} \zeta_3 \right] \\
+ a_s^3 n_f d_{abc} d_{abc} \left[ \frac{931}{4050} \right] \]
\[
\gamma_{7}^{\text{NS}} = a_{s}C_{F} \left[ + \frac{1027}{140} + a_{s}^{2}C_{F}C_{A} \left[ + \frac{67710257}{2116800} \right] + a_{s}^{2}C_{F}n_{f} \left[ + \frac{3745727}{529200} \right] \right] + a_{s}^{3}C_{F}^{2} \left[ \frac{106801937}{16000000} + a_{s}^{2}C_{F}C_{A}n_{f} \left[ + \frac{2257057261}{106686720} \right] + \frac{1369936511}{666792000} \right] + a_{s}^{3}C_{F}C_{A} \left[ + \frac{219582793861}{11854080000} + \frac{92741}{4900} \right] + a_{s}^{3}C_{F}n_{f}^{2} \left[ + \frac{3150205788689}{62239200000} \right] + a_{s}^{3}C_{F}^{3} \left[ + \frac{151689902637457}{8712748800000} + \frac{92741}{2450} \right] + a_{s}^{3}n_{f}d_{abc}d_{abc} \left[ + \frac{56527729}{5080320000} \right] \\
\gamma_{9}^{\text{NS}} = a_{s}C_{F} \left[ + \frac{1045}{126} + a_{s}^{2}C_{F}C_{A} \left[ + \frac{242855129}{6804000} \right] + a_{s}^{2}C_{F}n_{f} \left[ + \frac{19247947}{23514000} \right] \right] + a_{s}^{3}C_{F}^{2} \left[ + \frac{6993510271}{10001880000} + a_{s}^{2}C_{F}C_{A}n_{f} \left[ + \frac{63405201707}{2813028750} \right] + \frac{20297329837}{90016920000} \right] + a_{s}^{3}C_{F}C_{A} \left[ + \frac{74118331602493}{3600676800000} + \frac{1253219}{66150} \right] + a_{s}^{3}C_{F}n_{f}^{2} \left[ + \frac{630263834317}{280052640000} \right] + a_{s}^{3}C_{F}^{3} \left[ + \frac{1382928556849837}{793949234400000} + \frac{1253219}{33075} \right] + a_{s}^{3}n_{f}d_{abc}d_{abc} \left[ + \frac{13172190779}{200037600000} \right] \\
\gamma_{11}^{\text{NS}} = a_{s}C_{F} \left[ + \frac{31408}{3465} + a_{s}^{2}C_{F}C_{A} \left[ + \frac{1448235599}{37422000} \right] + a_{s}^{2}C_{F}n_{f} \left[ + \frac{512808781}{576298800} \right] \right] + a_{s}^{3}C_{F}^{2} \left[ + \frac{2454220717793}{33281557000} + a_{s}^{3}C_{F}C_{A}n_{f} \left[ + \frac{1031510572686647}{43568189280000} \right] + \frac{2869611542843}{1198125205200000} \right] + a_{s}^{3}C_{F}C_{A} \left[ + \frac{390549244457621303}{1742727571200000} + \frac{151689577}{8004150} \right] + a_{s}^{3}C_{F}n_{f}^{2} \left[ + \frac{22143086957669083141}{127866318149354400000} \right] + a_{s}^{3}C_{F}^{3} \left[ + \frac{5083969985783}{116181838080000} \right]
\]
$$\gamma_{13}^{\text{ns}} = a_s C_F \left[ + \frac{874733}{90090} + a_s^2 C_F C_A \left[ + \frac{31236494566127}{75512756000} \right] + a_s^2 C_F n_f \left[ - \frac{93360116539}{9739449720} \right] \right]$$

$$\left. + a_s^2 C_F^2 \left[ - \frac{22445960639039759}{292475675916000} \right] \right]$$

$$\left. + a_s^3 C_F C_A n_f \left[ \frac{-90849626920977361109}{3685193506154160000} - \frac{3498932}{45045} \zeta_3 \right] \right]$$

$$\left. + a_s^3 C_F C_A^2 \left[ + \frac{41070753377638233401027}{171975696953860800000} + \frac{3662719609}{193243050} \zeta_3 \right] \right]$$

$$\left. + a_s^3 C_F n_f^2 \left[ - \frac{66727681292862571}{26322810758244000} \right] \right]$$

$$\left. + a_s^3 C_F n_f \left[ \frac{1400874681707762602284653}{38888786996630141600000} - \frac{3662719609}{64414350} \zeta_3 \right] \right]$$

$$\left. + a_s^3 C_F n_f^2 \left[ - \frac{36688336888519925613757}{5269826713800448800000} + \frac{3498932}{45045} \zeta_3 \right] \right]$$

$$\left. + a_s^3 C_F n_f \left[ - \frac{40795447722180713788820819}{2373793443231412161960000} + \frac{3662719609}{96621525} \zeta_3 \right] \right]$$

$$\left. + a_s^3 C_F \left[ \frac{62160363128061559}{1984334964852240000} \right] \right]$$

$$+ a_s n_f \frac{d^{abc} d^{abc}}{n} \left[ - \frac{11}{3} + 8 \zeta_3 \right]$$

$$C_1^{\text{ns}} = 1 + a_s C_F \left[ - \frac{3}{2} \right]$$

$$\left. + a_s^2 C_F C_A \left[ - \frac{23}{27} + a_s^2 C_F n_f \left[ + \frac{4}{3} \right] + a_s^2 C_F \left[ + \frac{21}{2} \right] \right] \right]$$

$$\left. + a_s^2 C_F C_A n_f \left[ + \frac{3555}{27} + 24 \zeta_3 - \frac{80}{3} \zeta_5 \right] + a_s^3 C_F C_A^2 \left[ - \frac{10874}{27} + \frac{440}{3} \zeta_5 \right] \right]$$

$$\left. + a_s^3 C_F n_f^2 \left[ - \frac{230}{27} \right] + a_s^3 C_F C_A \left[ + \frac{1241}{9} - \frac{176}{3} \zeta_3 \right] + a_s^3 C_F n_f \left[ - \frac{133}{18} - \frac{40}{3} \zeta_3 \right] \right]$$

$$\left. + a_s^2 C_F \left[ - \frac{3}{2} \right] + a_s n_f \frac{d^{abc} d^{abc}}{n} \left[ - \frac{11}{3} + 8 \zeta_3 \right] \right]$$

$$C_3^{\text{ns}} = 1 + a_s C_F \left[ + \frac{5}{4} \right]$$

$$\left. + a_s^2 C_F C_A \left[ + \frac{5209}{144} - 33 \zeta_3 \right] + a_s^2 C_F n_f \left[ - \frac{4369}{864} \right] + a_s^2 C_F^2 \left[ - \frac{34763}{10368} + 16 \zeta_3 \right] \right]$$

$$\left. + a_s^2 C_F C_A n_f \left[ - \frac{24877649}{699840} + \frac{18539}{405} \zeta_4 - \frac{50}{3} \zeta_4 \right] \right]$$

$$\left. + a_s^3 C_F C_A^2 \left[ + \frac{92517547}{699840} - \frac{225337}{405} \zeta_4 + \frac{55}{6} \zeta_4 + 390 \zeta_5 \right] \right]$$

$$\left. + a_s^3 C_F n_f \left[ + \frac{12125}{69984} + \frac{100}{81} \zeta_3 \right] + a_s^3 C_F^2 C_A \left[ + \frac{222157399}{559872} + \frac{206}{9} \zeta_3 - \frac{55}{2} \zeta_4 - 260 \zeta_5 \right] \right]$$

$$\left. + a_s^3 C_F n_f \left[ - \frac{4292227}{34992} + \frac{527}{9} \zeta_3 + \frac{50}{3} \zeta_4 \right] \right]$$

$$\left. + a_s^3 C_F \left[ \frac{104473}{373248} + \frac{2183}{324} \zeta_3 + \frac{55}{3} \zeta_4 - 40 \zeta_5 \right] + a_s n_f \frac{d^{abc} d^{abc}}{n} \left[ - \frac{19477}{5184} + 5 \zeta_3 \right] \right]$$

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\[
C^\text{ns}_5 = 1 + a_s C_F \left[ \begin{array}{c}
+ \frac{523}{90} \\
+ \frac{30312449}{324000} - \frac{276}{5} \zeta_3 \\
+ \frac{a^2_s C_F C_A}{1620000} + \frac{188}{5} \zeta_3 \\
+ \frac{a^3_s C_F C_A}{122472000} + \frac{1413442}{14175} \zeta_3 - \frac{364}{15} \zeta_4 \\
+ \frac{a^3_s C_F C_A}{163296000} - \frac{30424087}{28350} \zeta_3 + \frac{707}{75} \zeta_4 + 548 \zeta_5 \\
+ \frac{a^3_s C_F n f}{54000} + \frac{728}{405} \zeta_3 \\
+ \frac{a^3_s C_F C_A}{255150000} - \frac{512039}{7875} \zeta_3 - \frac{707}{25} \zeta_4 - 180 \zeta_5 \\
+ \frac{a^3_s C_F n f}{127575000} + \frac{205712}{4725} \zeta_3 + \frac{364}{15} \zeta_4 \\
+ \frac{a^3_s C_F}{2187000000} + \frac{4054586}{10125} \zeta_3 + \frac{1414}{75} \zeta_4 - 376 \zeta_5 \\
+ \frac{a^3_s n f}{-24801551 + 17496000 + 86}{n} + \frac{86}{45} \zeta_3
\end{array} \right]
\]

\[
C^\text{ns}_7 = 1 + a_s C_F \left[ \begin{array}{c}
+ \frac{48091}{5040} \\
+ \frac{502042084559}{3556224000} - \frac{4917}{70} \zeta_3 \\
+ \frac{a^2_s C_F n f}{248935680000} + \frac{1836}{35} \zeta_3 \\
+ \frac{a^3_s C_F C_A}{672126336000} + \frac{4838581}{33075} \zeta_3 - \frac{1027}{35} \zeta_4 \\
+ \frac{a^3_s C_F C_A}{336063168000} - \frac{134407981}{88200} \zeta_3 + \frac{92741}{9800} \zeta_4 + \frac{14108}{21} \zeta_5 \\
+ \frac{a^3_s C_F n f}{336063168000} + \frac{2054}{945} \zeta_3 \\
+ \frac{a^3_s C_F C_A}{188195374080000} - \frac{450138923}{4630500} \zeta_3 - \frac{278223}{9800} \zeta_4 - \frac{4118}{21} \zeta_5 \\
+ \frac{a^3_s C_F n f}{9801842400000} + \frac{483863}{26460} \zeta_3 + \frac{1027}{35} \zeta_4 \\
+ \frac{a^3_s C_F}{2143741798400000} + \frac{27358836137}{3704400} \zeta_3 + \frac{92741}{4900} \zeta_4 - \frac{11224}{21} \zeta_5 \\
+ \frac{a^3_s n f}{-1521158314519}{n} + \frac{529}{504} \zeta_3
\end{array} \right]
\]
\[
C_{9}^{\text{ns}} = 1 + a_s C_F \left[ + \frac{17761}{1400} \right] \\
+ a_s^2 C_F C_A \left[ + \frac{62479500073621}{3429216000} - \frac{2858}{35} \right] + a_s^2 C_F n_f \left[ - \frac{363260000687}{12002256000} \right] \\
+ a_s^2 C_F^2 \left[ + \frac{1008189504000}{105} \right] \\
+ a_s^3 C_F C_A n_f \left[ - \frac{15324545569897512927}{213880201920000} + \frac{121711969}{654885} \right] + a_s^3 C_F n_f \left[ - \frac{49634}{63} \right] \\
+ a_s^3 C_F^2 C_A \left[ + \frac{15496996833630287207}{68052791520000} - \frac{3969000}{132300} \right] + a_s^3 C_F n_f^2 \left[ + \frac{4180}{1701} \right] \\
+ a_s^3 C_F^2 n_f \left[ + \frac{11113414791170240118909}{57164344876800000} - \frac{4449188411}{4167450} \right] + a_s^3 C_F C_A n_f \left[ - \frac{22156972}{3274425} + \frac{2090}{63} \right] \\
+ a_s^3 C_F^3 f \left[ - \frac{85781612300063814847}{400150411437600000} + \frac{95897878894}{93767625} \right] + a_s^3 C_F^3 \left[ - \frac{1253219}{66150} - \frac{5092}{9} \right] \\
+ a_s^3 n_f d_{abc} d^{abc} \left[ - \frac{390642677252603}{756142128000000} + \frac{95839}{1175} \right] \\
C_{11}^{\text{ns}} = 1 + a_s C_F \left[ + \frac{12815983}{831600} \right] \\
+ a_s^2 C_F C_A \left[ + \frac{56653434996617}{2593346400} - \frac{104947}{1155} \right] + a_s^2 C_F n_f \left[ - \frac{117825956269669}{3195000547200} \right] \\
+ a_s^2 C_F^2 \left[ + \frac{25436786950269217}{401278204000000} + \frac{84262}{1155} \right] \\
+ a_s^2 C_F C_A n_f \left[ - \frac{5339200983355026311}{188402792267289600000} + \frac{206111663893}{93648550} \right] + a_s^2 C_F n_f^2 \left[ + \frac{209072}{231} \right] \\
+ a_s^2 C_F C_A n_f \left[ + \frac{58243229827221303967}{11596309206443200000} + \frac{251264}{93555} \right] \\
+ a_s^2 C_F^2 C_A \left[ + \frac{3789468677596002057158737}{13600326566794968000000} - \frac{52724308943557}{57687598800} \right] \\
+ a_s^3 C_F C_A \left[ + \frac{46958}{99} \right] \\
+ a_s^3 C_F C_A n_f \left[ - \frac{7504061852485890170936687}{108802612534597440000000} + \frac{2560572011}{85135050} \right] + a_s^3 C_F n_f \left[ - \frac{125632}{3465} \right] \\
+ a_s^3 C_F^2 \left[ + \frac{358252596866767334349510147}{3544454339100103968000000} + \frac{156213323360401}{124804708875} \right] \\
+ a_s^3 C_F^2 \left[ + \frac{49124}{99} \right] \\
+ a_s^3 n_f d_{abc} d^{abc} \left[ - \frac{1182009270543683429}{32206605515776000000} + \frac{6758}{14175} \right] \\
\]

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\[ C_{13}^{ns} = 1 + a_s C_F \left[ \frac{84292133}{4729725} \right] + a_s^2 C_F C_A \left[ \frac{13708277501580459849}{54595493504320000} + \frac{1480186}{15015} \zeta_3 \right] \]
\[ + a_s^2 C_F n_f \left[ \frac{301032882286150933}{7019416202198400} \right] \]
\[ + a_s^2 C_F^2 \left[ \frac{523344930735326145457}{60226591014862272000} + \frac{1210906}{15015} \zeta_3 \right] \]
\[ + a_s^3 C_F C_A n_f \left[ \frac{879328426004243254681619899}{796797799126627856560000} + \frac{3046809425749}{12174312150} \zeta_3 - \frac{1749466}{45045} \zeta_4 \right] \]
\[ + a_s^3 C_F C_A^2 \left[ \frac{64502684578679520269283647601}{185919486462879833664000000} - \frac{37039408217872}{14203364175} \zeta_3 \right] \]
\[ + \frac{3662719609}{386486100} \zeta_4 + \frac{3086186}{3003} \zeta_5 \]
\[ + a_s^3 C_F n_f^2 \left[ \frac{2043359170316436833750887}{28457064254522423522000} + \frac{3498932}{1216215} \zeta_3 \right] \]
\[ + a_s^3 C_F^2 C_A \left[ \frac{9081431864860519767694223551709}{2441616113038023938016000000} - \frac{2292102879924479}{48745945848600} \zeta_3 \right] \]
\[ - \frac{3662719609}{128828700} \zeta_4 - \frac{6487178}{9009} \zeta_5 \]
\[ + a_s^3 C_F n_f^3 \left[ \frac{1506092840163232093412698913}{22196510118527490345600000} - \frac{313559848919}{6087156075} \zeta_3 + \frac{1749466}{45045} \zeta_4 \right] \]
\[ + a_s^3 C_F^3 \left[ \frac{298680730292949187763497559489677}{4277102601435843419528000000} + \frac{142217314466274683}{99707616580500} \zeta_3 \right] \]
\[ + \frac{3662719609}{193243050} \zeta_4 - \frac{235868}{693} \zeta_5 \]
\[ + a_s^3 n_f \left[ \frac{65696095155620706347863}{23835831597805106800000} + \frac{11802932}{33108075} \zeta_3 \right] \]