Newton-Cartan Geometry and the Quantum Hall Effect

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We construct an effective field theory for quantum Hall states, guided by the requirements of nonrelativistic general coordinate invariance and regularity of the zero mass limit. We propose Newton-Cartan geometry as the most natural formalism to construct such a theory. Universal predictions of the theory are discussed.

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I. INTRODUCTION

The fractional quantum Hall (FQH) states [1, 2] are the most nontrivial states in condensed matter physics. The observed quantization of the Hall conductivity originates from nontrivial topological properties of the ground state. The topological nature of FQH states are also expressed through the ground state degeneracy, nontrivial statistics of quasiparticles, and edge modes. What make the problem of the FQH effect challenging is the crucial role played by interactions.

Many theoretical approaches have been suggested for the description of the FQH states. Two closely related approaches—the composite boson [3] and composite fermion [4] approaches—have the advantages of being field theories, enabling powerful theoretical tools. However, one serious problem of these approaches is the unnaturalness of the massless limit—the limit in which the Coulomb energy scale is much smaller than the cyclotron energy. This problem exhibits itself in the tension between Kohn’s theorem and the existence of excitations at the Coulomb energy scales.

At energies much lower than the gap, it is usually believed that all universal information about the quantum Hall states is described by a pure Chern-Simons theory (“hydrodynamic theory”) [5], which encodes the quantized Hall conductivity. There are, however, universal properties related to transport beside the Hall conductivity, at least in clean systems with rotation and Galilean symmetries. The Hall viscosity (also called odd viscosity or Lorentz shear) [6–10] is found to be a robust characteristic of gapped states and is proportional to the shift [11]. It has also been shown that the $q^2$ correction to the Hall conductivity ($q$ being the wavenumber of the longitudinal electric field) has a universal coefficient which is related to the Hall viscosity [12, 13]. These universal characteristics of the quantum Hall systems go beyond the framework of the conventional hydrodynamic theory.

In this paper, we develop a field-theoretical formalism to capture these new universal features of quantum Hall systems. We will only concern ourselves with physics at distance scales much larger than the magnetic length, and energies far below the gap. Universal
results derived in this paper are valid for the gapped FQH states with $\nu < 1$ as well as the $\nu = 1$ integer quantum Hall (IQH) state. The formalism makes use of a geometrical structure called the Newton-Cartan geometry, which was originally developed for the purpose of reformulating Newton’s gravity in a coordinate invariant way [14–16]. We found that the mathematical machinery of the Newton-Cartan geometric structure is particularly useful for developing an effective field theory describing the quantum Hall states.

The structure of the paper is as follows. In Sec. II we review symmetry properties of the microscopic theory, and the requirements for an effective theory description. Section III contains a brief overview of the Newton-Cartan geometry. In Sec. IV we construct the most general effective theory consistent with the requirements put forward in Sec. II. The physical consequences are derived in Sec. V. Finally, Sec. VI contains concluding remarks.

II. SYMMETRY CONSIDERATIONS

A. Nonrelativistic diffeomorphism

A system of nonrelativistic particles (electrons) has several conservation laws. It is well known that the conservation of particle number is related to a gauge invariance of the action describing electrons in an external electromagnetic field. Similarly, conservation of momentum is related to a gauge invariance in the theory describing electrons interacting with the gauge field and the metric. Let us first consider the case of noninteracting particles. We can couple the system to the external gauge field $A_\mu$ and the metric $h_{ij}$ in the following way,

$$S = \int d^3 x \sqrt{h} \left[ \frac{i}{2} \bar{\psi} i \gamma^\mu D_\mu \psi - \frac{h^{ij}}{2m} D_i \psi \gamma^j D_j \psi + \frac{g B}{4m} \bar{\psi} \gamma^\mu \gamma^5 \psi \right],$$

where $D_\mu = \partial_\mu - i A_\mu$, $h^{ij}$ is the inverse of $h_{ij}$, $h = \det h_{ij}$, $B = (\partial_1 A_2 - \partial_2 A_1)/\sqrt{h}$, and $g$ is the g-factor of the electrons, assumed to be fully polarized. The metric is only a spatial metric; time is absolute. At the end, we will be mostly interested in physics occurring in flat space, but introducing the metric turns out to be a useful intermediate step. There are some ambiguities in the coupling of matter to the metric, but for the purpose of this paper, the simplest coupling (1) is sufficient. The magnetic moment term in (1) modifies the dynamics only when the magnetic field is inhomogeneous, but even when $B$ is constant it modifies the expression for the current $j^i = \delta S/\delta A_i$, adding to it a magnetization current.

The action is invariant under gauge transformations: $\delta \psi = i \alpha \psi$, $\delta A_\mu = \partial_\mu \alpha$. By direct computation, one can also verify that it is invariant with respect to time-dependent general
coordinate transformations, characterized by the gauge parameters $\xi^k(t, x)$,

\begin{align}
\delta \psi &= -\xi^k \partial_k \psi, \\
\delta A_0 &= -\xi^k \partial_k A_0 - A_k \xi^k + \frac{g}{4} \varepsilon^{ij} \partial_i (h_{jk} \xi^k), \\
\delta A_i &= -\xi^k \partial_k A_i - A_k \partial_i \xi^k - m h_{ik} \xi^k, \\
\delta h_{ij} &= -\xi^k \partial_k h_{ij} - h_{kj} \partial_i \xi^k - h_{ik} \partial_j \xi^k.
\end{align}

Here $\varepsilon^{ij} = \epsilon^{ij}/\sqrt{h}$, and $\epsilon^{ij}$ is the totally antisymmetric symbol with $\epsilon^{12} = -\epsilon^{21} = 1$. Equations (2) correspond to time-dependent coordinate transformations $x^k \rightarrow x^k + \xi^k(t, x)$. The $g = 0$ version of this invariance was considered previously in Refs. [12, 17]. In this case, the invariance can be thought of as a nonrelativistic limit of a relativistic coordinate invariance [17]. It has also been shown that interactions can be introduced to the system in a way which respects the general coordinate invariance [12].

**B. Requirements for the effective theory**

The problem of finding the electromagnetic and gravitational response of a quantum Hall fluid is that of finding the effective action $S[A_0, A_i, h_{ij}]$. By the “effective action” here we simply mean the generating functional that one would obtain, in the path-integral formalism, if one was able to perform the path integral over the electron field $\psi$. However such direct integration is feasible only for IQH states but not for FQH states. We hence will have to rely on general principles.

Because the quantum Hall states are gapped, $S$ can be expanded in Taylor series over powers of fields and derivatives. Our goal is only to find the lowest terms in the derivative expansion of $S$.

The first requirement is that $S$ is gauge invariant and general coordinate invariant,

\begin{align}
S[A_0 + \partial_0 \alpha, A_i + \partial_i \alpha, h_{ij}] &= S[A_0, A_i, h_{ij}], \\
S[A_0 + \delta A_0, A_i + \delta A_i, h_{ij} + \delta h_{ij}] &= S[A_0, A_i, h_{ij}] + O(\xi^2).
\end{align}

where in the second equation $\delta A_0$, $\delta A_i$, and $\delta h_{ij}$ are given in Eqs. (2). Next, we note that $A_0$ enters the action (1) only through the combination $A_0 + gB/4m$, hence the electromagnetic responses of systems with different g-factors are related. Suppose $S_g[h_{ij}, A_0, A_i]$ is the effective action determining the response of a system with g-factor $g$. Then

$$S_g[A_0, A_i, h_{ij}] = S_g' \left[ A_0 + \frac{g - g'}{4m} B, A_i, h_{ij} \right].$$

Hence if one could find $S$ for one particular value of $g$, then one would know $S$ for all $g$'s.

The special value of the g-factor is $g = 2$. At this value, the lowest Landau level is completely degenerate at zero energy, even when the magnetic field $B$ is inhomogeneous and the metric $h_{ij}$ is nontrivial [18–20]. In this case, if one sends all higher Landau levels...
to infinite energy by taking \( m \to 0 \), effective action describing states at the lowest Landau level should remain finite. Thus, another requirement is the existence of a regular limit

\[
\lim_{m \to 0} S_{g=2}[h_{ij}, A_0, A_i].
\]  

(6)

In particular, for any \( g \neq 2 \) the limit \( m \to 0 \) in \( S \) is singular (unless \( S \) does not depend on \( A_0 \), which is unphysical). We note that the transformation laws (2) are not singular in the limit \( m \to 0 \).

C. The necessity to improve the standard hydrodynamic theory

After integrating out all dynamical fields, the standard hydrodynamic theory [5] gives a Chern-Simons action involving the external gauge potential,

\[
S_{\text{CS}}[A] = \frac{\nu}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda.
\]  

(7)

This action encodes the Hall effect with Hall conductivity \( \sigma_{xy} = \nu/2\pi \) (in units of \( e^2/\hbar \)). On the other hand, we expect the effective theory to respect the symmetry (2) of the microscopic theory. Under the general coordinate transformations (2b), (2c), the Chern-Simons action changes,

\[
\delta S_{\text{CS}}[A] = \frac{\nu}{2\pi} \int d^3x \epsilon^{\hat{i}j} \left( mE_i - \frac{g}{4} \partial_i B \right) h_{jk} \dot{\xi}^k.
\]  

(8)

So the action is not invariant under general coordinate transformations unless we take the limit \( m = 0 \) and the g-factor is zero, \( g = 0 \). The root of the problem is that, except for this particular case, \( A_0 \) and \( A_i \) do not transform like the components of a one-form.

Thus we conclude, from symmetries alone, that the Chern-Simons action (7) cannot be the complete effective action for the quantum Hall states for generic \( m \) and \( g \). Can this action be the complete action in the regime \( g = 0, m \to 0 \)? It is easy to argue that it cannot be. Indeed, as noted above, the effective action must be singular in the limit \( m \to 0 \) if \( g \neq 2 \); at the same time, (7) is completely regular. Another way to say the same thing is that, if (7) was the effective action for \( g = 0 \), the action for \( g = 2 \) would be, according to Eq. (5),

\[
S_{g=2}[A] = \frac{\nu}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{\nu}{4\pi m} \int d^3x \sqrt{h} B^2,
\]  

(9)

which becomes singular when \( m \to 0 \), in contradiction with the regularity of the \( m \to 0 \) limit at \( g = 2 \). Thus, (7) cannot be the complete action for any value of \( g \) and \( m \).

This conclusion may appear trivial, because generically one expects that, after integration over \( \psi \), terms with all numbers of derivatives are generated. However, by showing that (7) does not satisfy the general requirements, we anticipate that some of these higher-derivative terms are completely fixed by symmetries and regularity in the massless limit.

There is a second deficiency of the action (7): it does not encode the shift and the Hall viscosity. Before discussing a new, improved action, we need to discuss the Newton-Cartan geometry that underlies its construction.
III. NEWTON-CARTAN GEOMETRY

The Newton-Cartan geometric structure first appeared in Cartan’s reformulation of Newton’s gravity in coordinate-invariant language [14] and was subsequently developed by others (see, e.g., Refs. [15, 16]). We give here a short, self-contained summary of aspects of Newton-Cartan’s gravity relevant for this work, with special emphasis on the case of (2+1) dimensions.

A. The geometric structure

A Newton-Cartan geometry is a structure consisting of

- a manifold, on which one can choose any system of coordinates \( x^\mu \), and where tensors are defined by transformation properties under coordinate change;
- a degenerate metric \( h^{\mu \nu} \) with one zero eigenvalue, and all other eigenvalues being positive;
- a one-form \( n = n_\mu dx^\mu \), which is a closed form in the torsionless version of Newton-Cartan geometry;
- A vector \( v^\mu \), called the velocity vector, which satisfies \( n \cdot v = 1 \).

From \((h^{\mu \nu}, n_\mu, v^\mu)\) one can define a unique metric tensor with lower indices \( h_{\mu \nu} \) by requiring

\[
h^{\mu \lambda} h_{\lambda \nu} = \delta_\nu^\mu - v^\mu n_\nu, \quad h_{\mu \nu} v^\nu = 0.
\]

A symmetric connection can be introduced,

\[
\Gamma^\lambda_{\mu \nu} = v^\lambda \partial_\mu n_\nu + \frac{1}{2} h^{\lambda \rho} (\partial_\mu h_{\rho \nu} + \partial_\nu h_{\mu \rho} - \partial_\rho h_{\mu \nu}).
\]

It is easy to check that (11) transforms as required for a connection. Covariant derivative defined with the connection (11) possesses many interesting properties:

\[
\nabla_\lambda h^{\mu \nu} = 0, \quad \nabla_\lambda n_\mu = 0, \quad h_{\alpha [\mu} \nabla_{\nu]} v^\alpha = 0,
\]

\[
v^\lambda \nabla_\lambda h_{\mu \nu} = 0, \quad v^\lambda \nabla_\lambda v^\mu = 0, \quad h^{\mu \alpha} h^{\nu \beta} \nabla_\lambda h_{\alpha \beta} = 0, \quad h^{\alpha [\mu} \nabla_{\alpha} v^{\nu]} = 0.
\]

In fact, the connection (11) is uniquely determined if one requires three conditions in Eqs. (12).

The Newton-Cartan structure arises naturally from dimensional reduction along a light-cone direction. Consider a space with one extra dimension, parameterized by the coordinates \( x^M = (x^-, x^\mu) \), and with the metric

\[
ds^2 = G_{MN} dx^M dx^N = 2n_\mu dx^- dx^\mu + h_{\mu \nu} dx^\mu dx^\nu
\]
The metric of this space is not degenerate and so can be inverted,\[ G_{MN} = \begin{pmatrix} 0 & n_\nu \\ n_\mu & h_{\mu\nu} \end{pmatrix}, \quad G^{MN} = \begin{pmatrix} 0 & v^\nu \\ v_\mu & h^{\mu\nu} \end{pmatrix}. \tag{15} \]

The Christoffel symbols $\Gamma^L_{MN}$, when indices are restricted to those different from $x^-$, coincide with (11).

The Newton-Cartan formalism allows equations to be written in any system of coordinates. However, there is a special class of coordinate systems where the time $x^0$ is chosen to be the “global time.” The global time $t$ is defined through $n = dt$ (recall that $n$ is a closed one-form). We will call any coordinate system where $t$ is chosen as the time coordinate, $x^0 = t$, a global-time coordinate system. Note that after fixing $x^0 = t$, there is still a freedom of choosing the spatial coordinates $x^i$. This gauge freedom is parameterized by the functions $\xi^i(t, x)$, corresponding to $x^i \rightarrow x'^i = x^i + \xi^i(t, x)$.

In global-time coordinate systems the components of $n_\mu$ are $n_\mu = (1, \vec{0})$. Due to $h^{\mu\nu} n_\nu = 0$, in such a coordinate system the components of $h^{\mu\nu}$ are

\[ h^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & h^{ij} \end{pmatrix}. \tag{16} \]

The velocity $v^\mu$ and $h_{\mu\nu}$ can be parameterized through the spatial components of the velocity, $v^i$,

\[ v^\mu = \begin{pmatrix} 1 \\ v^i \end{pmatrix}, \quad h_{\mu\nu} = \begin{pmatrix} v^2 & -v_j \\ -v_i & h^{ij} \end{pmatrix}, \tag{17} \]

where, $h^{ij}$ is the inverse matrix of $h_{ij}$ and, for notational convenience, we denote $v_i = h_{ij} v^j$ and $v^2 = v^i v_i$. (Note: $v_i$ are not the spatial components of a spacetime co-vector, and $v^2$ is not the square of a spacetime vector).

Under spatial reparameterization ($\xi^0 = 0$) the form (16) is preserved while $h_{ij}$ transforms as

\[ \delta h_{ij} = -\xi^k \partial_k h^{ij} - h_{ik} \partial_j \xi^k - h_{kj} \partial_i \xi^k. \tag{18} \]

Notice that this is the same as Eq. (2d). Later in our construction of the Newton-Cartan geometry of the quantum Hall states, the external metric $h_{ij}$ will play the role of the spatial metric of the geometry. The spatial components of the velocity vector $v^i$ transform as

\[ \delta v^i = -\xi^k \partial_k v^i + v^k \partial_k \xi^i + \dot{\xi}^i. \tag{19} \]

The last term in Eq. (19) justifies calling $v^i$ the “velocity.” For example, under a Galilean boost with $\xi^i = V^i t$, one has $v^i \rightarrow v^i + V^i$.

The Newton-Cartan geometric structure can be visualized as a collection of Riemannian spaces, one space at each moment of time, with a spatial metric on each time slice and with a velocity field connecting Riemannian spaces at different times. Parallel transport within a time slice can be done with the use of the metric $h_{ij}$, but parallel transport from one time slice to another requires the velocity vector $v^i$. 
B. The shear tensor

The shear tensor $\sigma_{\mu\nu}$ can be defined without using the connection as

$$\sigma_{\mu\nu} = \mathcal{L}_v h_{\mu\nu} = v^\lambda \partial_\lambda h_{\mu\nu} + h_{\lambda\nu} \partial_\mu v^\lambda + h_{\mu\lambda} \partial_\nu v^\lambda.$$  \hspace{1cm} (20)

The shear tensor is symmetric and satisfies $v^\mu \sigma_{\mu\nu} = 0$. The covariant derivatives of $v^\mu$ and $h_{\mu\nu}$ then can be expressed in terms of the shear tensor,

$$\nabla_\mu v^\nu = \frac{1}{2} \sigma_{\mu\lambda} h^{\lambda\nu}, \quad \nabla_\lambda h_{\mu\nu} = -\sigma_{\lambda(\mu} n_{\nu)}.$$  \hspace{1cm} (21)

In a global-time coordinate system, the spatial components of the shear tensor are

$$\sigma_{ij} = \nabla_i v_j + \nabla_j v_i + \dot{h}_{ij},$$  \hspace{1cm} (22)

where the covariant derivatives in the last equation are defined with respect to the spatial metric $h_{ij}$. This justifies the name “shear tensor.” The other components of $\sigma_{\mu\nu}$ are uniquely fixed by $v^\mu \sigma_{\mu\nu} = 0$. From $\sigma_{\mu\nu}$ we can construct the traced and traceless part,

$$\sigma = h^{\mu\nu} \sigma_{\mu\nu} = 2\nabla_\mu v^\mu, \quad \hat{\sigma}_{\mu\nu} = \sigma_{\mu\nu} - \frac{1}{d} h_{\mu\nu} \sigma,$$  \hspace{1cm} (23)

where $d$ is the number of spatial dimensions. In the rest of this paper we take $d = 2$.

C. The spin connection

The spin connection plays an important role in our construction of the effective action for the quantum Hall state. We assume the Newton-Cartan space is $(2 + 1)$ dimensional. Let us define at each point a pair of vectors (a vielbein) $e^a_{\mu}$, $a = 1, 2$ so that $n_\mu e^{a\mu} = 0$ and

$$h^{\mu\nu} = \sum_{a=1}^2 e^{a\mu} e^{a\nu}.$$  \hspace{1cm} (24)

By lowering the index we can define $e^a_\mu = h_{\mu\nu} e^{a\nu}$ with the properties

$$h_{\mu\nu} = \sum_{a=1}^2 e^a_\mu e^a_\nu, \quad v^\mu e^a_\mu = 0.$$  \hspace{1cm} (25)

We will also chose to orient the basis vectors $e^a$ so that $e^{\lambda\mu\nu} e^{ab} n_\lambda e^{a\mu} e^{b\nu} > 0$.

The spin connection can be defined as

$$\omega_\mu = \frac{1}{2} e^{ab} e^{a\nu} \nabla_\mu e^b_\nu.$$  \hspace{1cm} (26)

In global-time coordinate systems, the vielbein vectors have components

$$e^{a\mu} = (0, e^{ai}), \quad e^a_\mu = (-v_i e^{ai}, e^a_i),$$  \hspace{1cm} (27)
and the components of the spin connection are

\[ \omega_0 = \frac{1}{2} \left( \epsilon^{ab} e^a \partial_0 e^b + \epsilon^{ij} \partial_i v_j \right), \]  

(28)

\[ \omega_i = \frac{1}{2} \left( \epsilon^{ab} e^a \partial_i e^b - \epsilon^{jk} \partial_j h_{ki} \right). \]  

(29)

The last term in \( \omega_0 \) is the vorticity if \( v_i \) is interpreted as the velocity field of a flow. The spin connection \( \omega_\mu \) transforms like an abelian gauge field under \( O(2) \) local rotation of the vielbein \( e^a \). The field strength tensor \( \omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \) is independent of the choice of the vielbein; its spatial component is the scalar curvature:

\[ \omega_{12} = \partial_1 \omega_2 - \partial_2 \omega_1 = \frac{1}{2} \sqrt{h} R. \]  

(30)

**IV. EFFECTIVE FIELD THEORY OF THE QUANTUM HALL STATE.**

A. Improved gauge potentials

Let us recall that (7) does not respect diffeomorphism invariance, since \( A_\mu \) does not transform as a one-form. To write down a diffeomorphism invariant action, we imagine the quantum Hall state to live in a Newton-Cartan geometry. The external metric \( h_{ij} \) transforms correctly under spatial coordinate transformations and hence can be taken as the metric of the Newton-Cartan geometric structure. We lack, however, a ready-made velocity field \( v^\mu \), but for a moment let us assume that such a field has somehow emerged dynamically. With \( v^\mu \) at hand, let us consider the following object,

\[ \tilde{A}_0 = A_0 - \frac{m}{2} v^2 - \frac{g}{4} \epsilon^{ij} \partial_i v_j, \]  

(31)

\[ \tilde{A}_i = A_i + m v_i, \]  

(32)

where \( v_i = h_{ij} v^j, \) \( v^2 = v^i v_i \). This object is interesting for the following reason. First, under gauge transformations, \( \tilde{A}_\mu \) transforms like a gauge potential: \( \tilde{A}_\mu \rightarrow \tilde{A}_\mu + \partial_\mu \alpha \). Second, using Eqs. (2) and (19), we find that \( \tilde{A}_\mu \) transforms like a one-form under diffeomorphism,

\[ \delta \tilde{A}_\mu = -\xi^\lambda \partial_\lambda \tilde{A}_\mu - \tilde{A}_\lambda \partial_\mu \xi^\lambda, \]  

(33)

where \( \xi^\mu = (0, \xi^i) \). Thanks to these properties, \( \tilde{A}_\mu \) will be especially useful for our future discussion.

B. A Chern-Simons effective action

We will search for an action \( S[A_\mu, h_{ij}] \) with the required symmetry properties. First we perform a Legendre transform of the action with respect to the transverse part of \( A_\mu \) to write

\[ S[A_\mu, h_{ij}] = S_j[j^\mu, h_{ij}] - \int d^3x \sqrt{h} j^\mu (\partial_\mu \varphi - A_\mu). \]  

(34)
\[ S_j[j^\mu, h_{ij}] \] contains the same amount of information as \( S[A_\mu, h_{ij}] \). Extremizing the right hand side of Eq. (34) with respect to \( j^\mu \) and \( \varphi \) we should obtain the functional \( S[A_\mu, h_{ij}] \). We note that, since the action \( S[A_\mu, h_{ij}] \) contains a Chern-Simons term, the action \( S_j \) contains a nonlocal contribution. To separate out this contribution we introduce a gauge field \( a_\mu \) and rewrite the action as

\[
S = \frac{\nu}{4\pi} \int d^3 x \epsilon^{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda - \int d^3 x \sqrt{h} j^\mu (\partial_\mu \varphi - A_\mu + a_\mu) + S_j^{\text{loc}}[j^\mu, h_{ij}],
\]

where \( S_j^{\text{loc}} \) is now a local functional of its variables. Next denote \( j^\mu = \rho v^\mu, v^\mu = (1, v^i) \). Since \( j^\mu \) transforms like a vector, \( v^\mu \) is also a vector. The action (35) is still not written in an explicitly invariant form since \( A_\mu \) does not transform like a one form. Thus we separate out a part from \( S_j^{\text{loc}} \),

\[
S_j^{\text{loc}}[j^\mu, h_{ij}] = S'[\rho, v^i, h_{ij}] + \int d^3 x \sqrt{h} \rho \left( \frac{mv^2}{2} - \frac{g}{4} \varepsilon^{ij} \partial_i v_j \right).
\]

Then Eq. (35) becomes

\[
S = \frac{\nu}{4\pi} \int d^3 x \epsilon^{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda - \int d^3 x \sqrt{h} \rho v^\mu (\partial_\mu \varphi - A_\mu + a_\mu) + S'[\rho, v^i, h_{ij}].
\]

Since the two first terms are diffeomorphism invariant, \( S'[\rho, v^i, h_{ij}] \) should also be diffeomorphism invariant.

**C. Coupling of composite boson to spin connection**

We now show that there is a universal contribution to \( S' \) responsible for the shift and the Hall viscosity of the quantum Hall liquid. Before giving a general argument, it is instructive to go over a heuristic argument based on the flux attachment procedure.

Assume we are dealing with a Laughlin fraction, where \( \nu = 1/(2p + 1) \). The variable \( \varphi \) in Eq. (37) can be interpreted as the phase of the condensate of “composite bosons,” obtained from attaching \( 1/\nu = 2p + 1 \) flux quanta to the original fermions. We now argue that in a curved background, such composite bosons should couple to the metric through the spin connection.

Let us recall that the shift \( S \) is defined as the offset in the linear relationship between the number of particles in the ground state of a quantum Hall state and the number of magnetic flux quanta \( N_{\phi} \): \( N_{\phi} = \nu^{-1} N - S \) [11]. For definiteness let us take \( \nu = 1/3 \), where \( S = 1/\nu = 3 \), and consider the ground state on a sphere. For \( 3N \) flux quanta through the sphere, there are \( N + 1 \) electrons in the ground state. Let us perform the standard flux attachment procedure, attaching \(-3\) statistical flux quanta to each electron. The total flux through the sphere is now \( 3N - 3(N + 1) = -3 \). This seems to contradict the fact that the composite bosons form a condensate without any vortices. (Note that this problem does not arise on a torus.) To resolve this problem, one needs to assume that the composite boson
is coupled to the spin connection, so that the curvature of the sphere supplies the missing 3 flux quanta. The total curvature flux through a sphere is 2, so the composite boson should carry charge $\frac{3}{2}$ with respect to the spin connection.

Thus we have found that for a general Laughlin’s fraction $\nu = \frac{1}{2p + 1}$, the spin connection charge of the composite boson is $s = \frac{1}{2\nu}$. This means that the covariant derivative of the condensate phase $\varphi$ should be defined as

$$D_\mu \varphi = \partial_\mu \varphi - A_\mu - s\omega_\mu + a_\mu. \quad (38)$$

We now present the general argument. If $\varphi$ couples to the gauge field and the metric as in Eq. (38), then on a closed manifold, the field $\varphi$ can be free of singularities only if the total flux of the gauge field coupled to it, $A_\mu + s\omega_\mu - a_\mu$, through the whole space is zero:

$$\int d^2 x \sqrt{g} \left( B + \frac{s}{2} R - b \right) = 0, \quad (39)$$

where we have used Eq. (30). Notice, however, that variation of the action (37) with respect to $a_0$ implies $\rho = \nu b / 2\pi$. Thus, the equation above reads $N_\phi + s\chi - \nu^{-1} N = 0$, where $\chi$ is the Euler characteristics of the manifold. Since $\chi = 2$ for a sphere, the relationship between $s$ and the shift $S$ is $s = S / 2$.

### D. The action

From the previous discussion, we separate a term proportional to $\rho v^\mu \omega_\mu$ from $S'$ and write the action in the final form,

$$S = \int d^3 x \left( \frac{\nu}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \sqrt{h} \rho v^\mu D_\mu \varphi \right) + S_0[\rho, v^i, h_{ij}], \quad (40)$$

with $D_\mu \varphi$ defined in Eq. (38).

So far we have not assumed any particular value for $g$. It is useful to assume that (40) is written for $g = 2$, so that $S_0$ is regular in the limit $m \to 0$. For a general $g$, we use Eq. (5) to write

$$S_g = \int d^3 x \left( \frac{\nu}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \sqrt{h} \rho v^\mu D_\mu \varphi + \frac{g}{8m} \rho \epsilon^{\mu\nu\lambda} n_\mu \tilde{F}_{\nu\lambda} \right) + S_0[\rho, h^{\mu\nu}, n_\mu, v^\mu]. \quad (41)$$

Note that the definition of $\tilde{A}_\mu$ involves $g$, and that the new term modifies the electromagnetic current:

$$j^\mu = \frac{\delta S_g}{\delta A_\mu} = \rho v^\mu + \frac{g - 2}{4m} \epsilon^{\lambda\mu\nu} n_\lambda \partial_\nu \rho. \quad (42)$$

The action (41) satisfies all conditions outlined in Sec. (II B). From construction, we should regard all the fields that have been introduced ($\rho, v^i, a_\mu, \varphi$) as dynamical fields, with respect to which the action is extremized.
Except for one possible topological term, $\int \omega d\omega$, all contributions to $S_0$ depend on microscopic physics, and hence non-universal. The term $\int \omega d\omega$ does not affect quantities computed later in Sec. V. A full classification of all possible terms in $S_0$ is beyond the scope of this paper. Some of these terms are

$$S_0 = -\int d^3x \sqrt{h} \left[ \epsilon_i(\rho) + \alpha_1(\rho) \partial_\mu \rho \partial_\nu \rho + \alpha_2(\rho) \partial_\mu A_\nu + \alpha_3(\rho) \sigma_\alpha \right]$$

$$+ \alpha_4(\rho) \omega_{\mu\nu} h^{\nu\lambda} \partial_\lambda \rho + \cdots \right], \quad (43)$$

where $\epsilon_i, \alpha_1, \alpha_2$, etc. can be arbitrary functions of $\rho$. The function $\epsilon_i$ has the meaning of the interaction energy of the quantum Hall state with density $\rho$.

Our formalism is applicable equally for gapped FQH states with $\nu < 1$ and the IQH state with $\nu = 1$. The difference between them is only in $S_0$. For example, by dimensional counting, the coefficients $\alpha_2$ and $\alpha_3$ in Eq. (43) should be proportional to the inverse Coulomb gap in the FQH case and the inverse cyclotron energy in the IQH case.

Putting Eq. (41) to flat space we find the Lagrangian

$$L = \frac{\nu}{4\pi} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{m\rho v^2}{2} - \rho D_0 \varphi - \rho v^i D_i \varphi + \frac{s - 1}{2} \rho \nabla \times v + \frac{g - 2}{4m} \rho B + S_0[\rho, v^i], \quad (44)$$

where in this equation $D_\mu \varphi = \partial_\mu \varphi - A_\mu + a_\mu$, $\nabla \times v \equiv \varepsilon^{ij} \partial_i v_j$. The field equations can be obtained by varying the action,

$$\rho = \frac{\nu}{2\pi} b, \quad (45a)$$

$$\rho v^i = \frac{\nu}{2\pi} \varepsilon^{ij} e_j, \quad (45b)$$

$$\frac{m\rho v^2}{2} - D_0 \varphi - v^i D_i \varphi + \frac{s - 1}{2} \rho \nabla \times v + \frac{g - 2}{4m} \rho B + \frac{\delta S_0}{\delta \rho} = 0, \quad (45c)$$

$$m\rho v_i - \rho D_i \varphi + \frac{s - 1}{2} \epsilon_{ij} \partial_j \rho + \frac{\delta S_0}{\delta v_i} = 0. \quad (45d)$$

E. Comparison to the bosonic Chern-Simons theory

Let us compare the action obtained above with that of the standard bosonic Chern-Simons theory [3]. The latter is summarized by the following Lagrangian,

$$L = \frac{\nu}{4\pi} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + i \psi \psi - \frac{1}{2m_*} D_i \psi \psi - V(\psi \psi). \quad (46)$$

Changing variables to $\psi = \sqrt{\rho} e^{i\varphi}$, it becomes

$$L = \frac{\nu}{4\pi} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \rho D_0 \varphi - \frac{\rho}{2m_*} |\nabla \varphi|^2 - \frac{|\nabla \rho|^2}{2m_*} - V(\rho). \quad (47)$$

To bring the Lagrangian to the form similar to Eq. (44), we introduce an auxiliary field $v_i$,

$$L = \frac{\nu}{4\pi} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \rho D_0 \varphi - \rho v_i D_i \varphi + \frac{m_* \rho}{2} v^2 - \frac{|\nabla \rho|^2}{2m_*} - V(\rho). \quad (48)$$
One can see some similarities with Eq. (44), but there are obvious differences. One difference is the term containing $s$ in Eq. (44). As it is clear from the calculations in the next Section, this term is essential to reproduce correct next-to-leading order corrections to electromagnetic response at finite wave numbers. The field equations following from Eq. (48) can be interpreted as the Euler hydrodynamic equation of a fluid with a constraint relating the density and the vorticity [21]. Equations (45) can also be recast as hydrodynamic equations, but the form of the equations is more complicated and we will not write them down here. We only note that these equations contain the information about the Hall viscosity, absent in the Euler equation of Ref. [21].

V. PHYSICAL CONSEQUENCES

We discuss some implications of the hydrodynamic theory. We will concentrate on linear response to external field.

A. Gravitational response: Hall viscosity

Let us turn on a weak time-dependent, spatially uniform, and traceless gravitational perturbation,

$$h_{ij} = \delta_{ij} + \tilde{h}_{ij}(t), \quad \tilde{h}_{ii} = 0.$$  \hfill (49)

Due to rotational symmetry, such a perturbation cannot excite, to linear order, perturbations of $\rho$ and $v^i$. The action then reduces to

$$S = s\rho \int d^3x \omega_0 = \frac{1}{2} s\rho \int d^3x \epsilon^{ijk} \tilde{h}_{ij} \partial_0 \tilde{h}_{ik},$$ \hfill (50)

from which we find that the Hall viscosity is

$$\eta_H = \frac{\rho s^2}{2}.$$

This relationship was derived in Ref. [10].

B. Electromagnetic response: preliminaries

We now assume the space is flat, and discuss linear response to electromagnetic perturbations. Such linear response is parameterized by the polarization tensor $\Pi^{\mu\nu}$,

$$j^\mu(\omega, q) = \Pi^{\mu\nu}(\omega, q) A_\nu(\omega, q).$$ \hfill (52)

Current conservation restricts the form of $\Pi^{\mu\nu}$ to three independent functions, $\Pi_{0,1,2}(\omega, q)$,

$$\Pi^{00} = q^2 \Pi_0,$$  \hfill (53)

$$\Pi^{0i} = \omega q_i \Pi_0 - i \epsilon^{ij} q_j \Pi_1, \quad \Pi^{i0} = \omega q_i \Pi_0 + i \epsilon^{ij} q_j \Pi_1$$  \hfill (54)

$$\Pi^{ij} = \omega^2 \delta^{ij} \Pi_0 + i \epsilon^{ij} \omega \Pi_1 + (q^2 \delta^{ij} - q^i q^j) \Pi_2.$$  \hfill (55)
In order to find $\Pi^{\mu\nu}$ we consider small perturbations

$$B = B_0 + \tilde{B}, \quad E_i = 0 + E_i, \quad \rho = \rho_0 + \tilde{\rho}, \quad v^i = 0 + \tilde{v}^i, \text{etc.} \quad (56)$$

where $\rho_0 = \nu B_0 / 2\pi$. The linearized equation can be written as

$$\tilde{\rho} + \frac{(s-1)\nu}{4\pi \rho_0} \nabla^2 \tilde{\rho} = \frac{\nu}{2\pi} \left( \delta B + m \nabla \times v + \frac{1}{\rho_0} \nabla \times \frac{\delta S_0}{\delta \vec{v}} \right), \quad (57)$$

$$m \dot{v}_i - \epsilon_{ij} B_0 v_j - \frac{s-1}{2} \left[ \epsilon_{ij} \partial_j (\nabla \cdot v) + \partial_i (\nabla \times v) \right] + \frac{1}{\rho_0} \partial_i \frac{\delta S_0}{\delta v_i} - \partial_i \frac{\delta S_0}{\delta \rho} = E_i + \frac{g-2}{4m} \partial_i B. \quad (58)$$

After solving these equations for $\tilde{\rho}$ and $\tilde{v}^i$, the current can be computed from Eq. (42). Clearly, a full calculation of $\Pi^{\mu\nu}$ requires a knowledge of $S_0$. However, certain statements about the behavior of $\Pi_0, 1, 2$ at small $q$ can be made without knowing the coefficients of terms appearing in $S_0$ [Eq. (43)]. We will present the results, mostly without detailed derivations, as they follow in a quite straightforward manner from the linearized equations above.

**C. $\Pi_0$, Kohn’s theorem, and static susceptibility**

First, for $\Pi_0$

$$\Pi_0 = \frac{\nu m B_0}{2\pi B_0^2 - m^2 \omega^2} + O(q^2). \quad (59)$$

The fact that the $\Pi_0$ is completely determined at $q = 0$ is the content of Kohn’s theorem: the response of the system to homogeneous electric field is independent of interactions. It is still instructive to derive Kohn’s theorem directly from the field equations. Consider a situation when the magnetic field is uniform, and the electric field is uniform and time-dependent $E(t)$. In this case we expect $\rho$ to remains constant and $v^i = \tilde{v}^i(t)$. Eq. (58) now becomes

$$m \dot{v}_i = E_i + \epsilon_{ij} v_j B, \quad (60)$$

which is just the equation of motion of the center of mass, which is independent of interactions.

The static susceptibility is

$$\chi(q) = -\Pi^{00}(0, q) = -\frac{\nu m}{2\pi B_0} q^2 + O(q^4). \quad (61)$$

In the limit $m \to 0$ the $q^0$ part of $\Pi_0$ vanishes, as the $q^2$ term in $\chi(q)$. The first nonzero contribution ($q^2$ in $\Pi_0$ and $q^4$ in the static susceptibility) comes from the $\hat{\sigma}^2$ term in $S_0$.

**D. $\Pi_1$, density in inhomogeneous magnetic field, and Hall conductivity at finite wavenumbers**

Next, for $\Pi_1$ we expand over $q^2$,

$$\Pi_1(\omega, q) = \Pi_1^{(0)}(\omega) + (q\ell_B)^2 \Pi_1^{(2)}(\omega), \quad \ell_B = \frac{1}{\sqrt{B}}. \quad (62)$$
The $q^0$ part is determined completely

$$\Pi_1^{(0)}(\omega) = \frac{\nu}{2\pi} \frac{B^2}{B^2 - m^2\omega^2}.$$  \hspace{1cm} (63)

but $\Pi_1^{(2)}$ is universal only at zero frequency, and narrowly speaking only in the limit $m \to 0$. For a nonzero $m$ we need to know the interaction energy as a function of $\rho$, $\epsilon_i(\rho)$,

$$\Pi_1^{(2)}(0) = \frac{\nu}{2\pi} \left[ \frac{s}{2} - 1 + \frac{g}{4} - \frac{\nu m}{2\pi} \epsilon''_i(\rho_0) \right].$$  \hspace{1cm} (64)

There are two physical predictions one can draw from $\Pi_1$. For simplicity let us take the limit $m \to 0$. The first prediction is a formula giving the number density in an static inhomogeneous magnetic field,

$$\rho = \frac{\nu}{2\pi} B - \left( \frac{s}{2} - 1 + \frac{g}{4} \right) \nabla^2 \ln B + O(\nabla^4).$$  \hspace{1cm} (65)

One can show that this relationship, as written, remains valid when the variation of $B$ is not small but of order 1.

The second prediction is for $\sigma_{xy}$ at finite wavenumber $q$. We assume that the magnetic field $B$ is constant and there is a static scalar potential $A_0(x)$, inducing a static longitudinal electric field $E = \nabla A_0$. The Hall current is

$$j = \frac{\nu}{2\pi} \left[ E - \left( \frac{s}{2} - 1 + \frac{g}{4} \right) \ell_B^2 \nabla^2 E \right] \times \hat{z}.$$  \hspace{1cm} (66)

In particular, when $g = 0$ the result of Ref. [12] is reproduced. Formulas similar to Eqs. (65) and (66) were obtained in Refs. [22, 23]. In fact, the action (44) without the $S_0$ term coincides with the one proposed in Ref. [23].

E. $\Pi_2$ and current in static inhomogeneous magnetic field

$\Pi_2$ is singular if $g \neq 2$ in the limit $m \to 0$, and the leading $m^{-1}$ behaviors of the $q^0$ and $q^2$ terms, at zero frequency, are universal,

$$\Pi_2(0, q) = -\frac{(2-g)\nu}{4\pi m} \left[ 1 + \left( \frac{s}{2} - 3 + \frac{g}{8} \right) (q\ell_B)^2 \right] + O(m^0, q^4).$$  \hspace{1cm} (67)

This expression determines the current in a static inhomogeneous magnetic field,

$$j^i = -\frac{(2-g)\nu}{4\pi m} \epsilon^{ij} \left[ 1 - \left( \frac{s}{2} - 3 + \frac{g}{8} \right) \ell_B^2 \nabla^2 \right] \partial_j B.$$  \hspace{1cm} (68)

For $g = 2$, however, the first term in $\Pi_2(0, q)$ is $m^0$ and not universal.

The results that we derived is valid for any gapped quantum Hall states with Galilean invariance. Thus, they can be verified for the simplest integer quantum Hall state of non-interacting electrons with $\nu = 1$. For this case, the polarization tensor has been computed previously [24], and the results can be checked to agree with our results when one puts in the latter $g = 0$ and $s = 1/2$. 
VI. CONCLUSION

In this paper we have shown how one can construct an effective field theory of the quantum Hall state which respects nonrelativistic diffeomorphism invariance. The most convenient mathematical framework turns out to be the Newton-Cartan geometry, previously considered in the literature in a different context. One of the most attractive features of the formalism is regularity of the massless limit $m \to 0$. The action cleanly separates universal physics from non-universal physics. The latter is is parameterized by an action $S_0$, whose form is restricted by the Newton-Cartan symmetry. Even without any dynamical information, one can already make several predictions, including the $q^2$ correction to the static Hall conductivity.

In this paper we have concentrated our attention to the regime of very low frequencies. To treat the physics at the scale of the Coulomb gap, we need to know more information about the non-universal part of the action, $S_0$. Knowing that $S_0$ depends on the Coulomb energy allows us to conclude, for example, that the $q^2$ part of the Hall conductivity $\Pi_1$ has nontrivial frequency dependence at the Coulomb energy scale. This is consistent with the results of Ref. [13].

In our formalism, among the components that make up the Newton-Cartan geometry, only the velocity $v^i$ is treated as a dynamical variable. The metric $h_{ij}$ is simply the background metric. On the other hand, it has been suggested recently [25] that some “internal metric” plays the role of a dynamic degree of freedom in FQH systems. We hope that future investigations will elucidate the connection between our approach and the suggestion of Ref. [25].

Finally, the implications for edge modes, for quasiholes and quasiparticles need to be investigated. The insights that we obtain in this paper may be useful for the construction of holographic models of quantum Hall systems.

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