Representation formula for the entropy and functional inequalities

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Abstract

We prove a stochastic formula for the Gaussian relative entropy in the spirit of Borell’s formula for the Laplace transform. As an application, we give simple proofs of a number of functional inequalities.

1 Introduction: Borell’s formula

Let $\gamma_d$ be the standard Gaussian measure on $\mathbb{R}^d$:

$$
\gamma_d(dx) = \frac{e^{-|x|^2/2}}{(2\pi)^{d/2}} dx
$$

where $|x| = \sqrt{x \cdot x}$ denotes the Euclidean norm of $x$. In [5, 6] Borell proves the following representation formula. Given a standard $d$-dimensional Brownian motion $B$ and a bounded function $f : \mathbb{R}^d \to \mathbb{R}$ we have

$$
\log \left( \int_{\mathbb{R}^d} e^f \, d\gamma_d \right) = \sup_u \left[ E \left( f(B_1 + \int_0^1 u_s \, ds) - \frac{1}{2} \int_0^1 |u_s|^2 \, ds \right) \right],
$$

(1)

where the supremum is taken over all random processes $u$, say bounded and adapted to the Brownian filtration. Among other applications, he derives easily the Prékopa-Leindler inequality. The name Borell’s formula may be unfair to Boué and Dupuis who in an earlier paper [7] obtained a stronger result, allowing the function $f$ to depend on the whole path $(B_t)_{t \in [0,1]}$ (see Theorem 9 below for a precise statement). Anyway, Borell and Boué-Dupuis agree that representation formulas such as (1) arose much earlier in optimal control theory, particularly in Fleming and Soner’s work [14], and Borell should definitely be credited for bringing these techniques in the context of functional inequalities.

The present article deals with relative entropy. Let $(\Omega, \mathcal{A}, m)$ be a measured space and $\mu$ be a probability measure. The relative entropy of $\mu$ is defined by

$$
H(\mu \mid m) = \int_{\Omega} \frac{d\mu}{dm} \log \left( \frac{d\mu}{dm} \right) \, dm \quad \text{if} \quad \mu \ll m
$$

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and $H(\mu \mid m) = +\infty$ otherwise. It is well known that there is a Legendre duality between relative entropy and logarithmic Laplace transform:

$$H(\mu \mid m) = \sup_f \left( \int f \, d\mu - \log \left( \int \Omega e^f \, dm \right) \right).$$  \hspace{1cm} (2)

The purpose of this article is to prove a representation formula for the Gaussian relative entropy, both in $\mathbb{R}^d$ and in the Wiener space, providing the entropy counterparts of the results mentioned above. All these formulas have a common feature: Girsanov’s theorem. However, our approach is somewhat different from that of Borell and Boué-Dupuis: it draws a connection with the work of Föllmer [15, 16] which makes the whole argument arguably simpler. As an application, we give new, unified and simple proofs of a number of Gaussian inequalities.

2 Representation formula for the entropy

This section contains the main results of the article. Let us recall a couple of classical facts about relative entropy, see for instance [24, section 10] and the references therein. If $A$ is the Borel $\sigma$-field of a Polish topology on $\Omega$ then it is enough to take the supremum over bounded and continuous function in (2). In particular the map $\mu \mapsto H(\mu \mid m)$ is lower semicontinuous with respect to the topology of weak convergence of measures. If $T : (\Omega, A) \to (\Omega', A')$ is a measurable map then

$$H(\mu \circ T^{-1} \mid m \circ T^{-1}) \leq H(\mu \mid m)$$  \hspace{1cm} (3)

and assuming that $H(\mu \mid m) < +\infty$, equality occurs if and only if the density $d\mu/dm$ is a function of $T$.

We now describe the setting of the article. Let $W$ be the space of continuous paths

$$\{ w \in C^0([0, +\infty); \mathbb{R}^d), \ w_0 = 0 \}$$

equipped with the topology of uniform convergence on compact intervals. Let $B$ be the associated Borel $\sigma$-field and let $\gamma$ be the Wiener measure on $(W, B)$. Let $x_t : w \mapsto w_t$ be the coordinate process and $(\mathcal{G}_t)_{t \geq 0}$ be the natural filtration of $x$.

It is well known that $B$ coincides with the smallest $\sigma$-field containing $\cup_{t \geq 0} \mathcal{G}_t$.

Let $H$ be the Cameron-Martin space: a path $U$ belongs to $H$ if there exists $u \in L^2([0, +\infty); \mathbb{R}^d)$ such that

$$U_t = \int_0^t u_s \, ds, \quad t \geq 0.$$  

The norm of $U$ in $H$ is then defined by

$$\| U \| = \left( \int_0^{+\infty} |u_s|^2 \, ds \right)^{1/2}.$$
The Cauchy-Schwarz inequality shows that the Hilbert space \( \mathbb{H} \) embeds continuously in \( \mathbb{W} \). Given a probability space \((\Omega, \mathcal{A}, P)\) equipped with a filtration \((\mathcal{F}_t)_{t \geq 0}\) we call \textit{drift} any adapted process \( U \) which belongs to \( \mathbb{H} \) almost surely. Lastly, our Brownian motions are always \( d \)-dimensional, standard and always start from 0.

### 2.1 The upper bound

We shall use repeatedly Girsanov’s formula, see \([19, \text{chapter 6}]\).

**Proposition 1.** Let \( B \) be a Brownian motion defined on some filtered probability space \((\Omega, \mathcal{A}, P, \mathcal{F})\) and let \( U \) be a drift. Letting \( \mu \) be the law of \( B + U \), we have

\[
H(\mu \mid \gamma) \leq \frac{1}{2} E \|U\|^2.
\]

**Proof.** Write \( U_t = \int_0^t u_s \, ds \) and assume for the moment that \( \|U\|^2 = \int_0^\infty |u_s|^2 \, ds \) is uniformly bounded. Then by Novikov’s criterion

\[
M_t = \exp\left(-\int_0^t u_s \cdot dB_s - \frac{1}{2} \int_0^t |u_s|^2 \, ds\right), \quad t \geq 0
\]
is a uniformly integrable martingale and Girsanov’s formula applies. Under \( dQ = M_\infty \, dP \)

the process \( X := B + U \) is a Brownian motion. Therefore \( X \) has law \( \mu \) and \( \gamma \) under \( P \) and \( Q \), respectively. Then by \([9]\)

\[
H(\mu \mid \gamma) \leq H(P \mid Q) = -E \log(M_\infty) = \frac{1}{2} E \|U\|^2,
\]

which concludes the proof when \( \|U\| \) is bounded. In the general case, define the stopping time

\[
T_n = \inf\{t \geq 0, \int_0^t |u_s|^2 \, ds \geq n\},
\]

let \( U_n \) be the stopped process \((U_n)_t = U_{t \wedge T_n} \) and \( \mu_n \) be the law of \( B + U_n \). With probability 1 we have \( \|U\|^2 < +\infty \), thus \( T_n \to +\infty \) and \( U_n \to U \) in \( \mathbb{H} \), hence in \( \mathbb{W} \). Therefore \( \mu_n \to \mu \) weakly. Also \( E\|U_n\|^2 \to E\|U\|^2 \) by monotone convergence. Thus, using the lower semicontinuity of the entropy (observe that \( \mathbb{W} \) is a Polish space)

\[
H(\mu \mid \gamma) \leq \liminf_n H(\mu_n \mid \gamma) \leq \liminf_n \frac{1}{2} E\|U_n\|^2 = \frac{1}{2} E\|U\|^2.
\]

**Remark.** It follows immediately that when \( E\|U\|^2 < +\infty \), the law of \( B + U \) is absolutely continuous with respect to the Wiener measure \( \gamma \). Let us point out that this is actually true for all drifts \( U \), even if \( E\|U\|^2 = +\infty \), see \([19, \text{chapter 7}]\).

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3
2.2 Föllmer’s drift

Let us address the question whether, given a probability measure $\mu$ on $\mathcal{W}$, equality can be achieved in (4). Recall that $(x_t)_{t \geq 0}$ is the coordinate process on Wiener space $(\mathcal{W}, \mathcal{B}, \gamma)$ and that $(\mathcal{G}_t)_{t \geq 0}$ is its natural filtration. The following is due to Föllmer [15, 16].

**Theorem 2.** Let $\mu$ be a measure on $(\mathcal{W}, \mathcal{B})$ having density $F$ with respect to $\gamma$. There exists an adapted process $u$ such that under $\mu$ the following holds.

1. The process $U_t = \int_0^t u_s \, ds$ belongs to $\mathbb{H}$ almost surely.
2. The process $y = x - U$ is a Brownian motion.
3. The relative entropy of $\mu$ is
   \[ H(\mu \mid \gamma) = \frac{1}{2} \mathbb{E}^\mu \|U\|^2. \]

We sketch the proof for completeness.

**Proof.** Throughout $\mathbb{E}^\gamma$ and $\mathbb{E}^\mu$ denote expectations with respect to $\gamma$ and $\mu$ respectively. On $\mathcal{G}_t$ the measure $\mu$ has density
   \[ F_t := \mathbb{E}^\gamma (F \mid \mathcal{G}_t), \]
with respect to $\gamma$. A standard martingale argument shows that
   \[ \mu(\inf_{t \geq 0} F_t > 0) = \mu(F > 0) = 1. \] (5)

Since Brownian martingales can be represented as stochastic integrals there exists an adapted process $v$ satisfying
   \[ \gamma \left( \int_0^{+\infty} |v_s|^2 \, ds < +\infty \right) = 1 \] (6)
and
   \[ F_t = 1 + \int_0^t v_s \cdot dx_s, \quad t \geq 0. \]

Let $u$ be the process defined by
   \[ u_t = 1_{\{F_t > 0\}} (F_t)^{-1} v_t. \]

It is adapted and (5) and (6) yield
   \[ \mu \left( \int_0^{+\infty} |u_s|^2 \, ds < +\infty \right) = 1, \]
which is the first assertion of the theorem.
The assertion 2 follows from Girsanov’s formula, see [19] Theorem 6.2.
Under $\mu$, we have
\[
F_t = 1 + \int_0^t F_s u_s \cdot dx_s \\
= 1 + \int_0^t F_s u_s \cdot dy_s + \int_0^t F_s |u_s|^2 ds.
\]
Applying Itô’s formula (recall that $F$ is positive and $y$ is a Brownian motion under $\mu$) we obtain
\[
\log(F) = \int_0^{+\infty} u_s \cdot dy_s + \frac{1}{2} \int_0^{+\infty} |u_s|^2 ds.
\]
If $E^\mu \|U\|^2 < +\infty$ the local martingale part in the equation above is integrable and has mean 0 so that
\[
H(\mu | \gamma) = E^\mu \log(F) = \frac{1}{2} E^\mu \|U\|^2.
\]
Again, a localization argument shows that this equality remains valid when $E^\mu \|U\|^2 = +\infty$, see [15] Lemma (2.6)].

To finish this subsection, we give a formula for Föllmer’s drift when the underlying density has a Malliavin derivative, we refer to the first chapter of [20] for the (little amount of) Malliavin calculus we shall use. For suitable $F$: $\mathbb{W} \to \mathbb{R}$ we let $DF: \mathbb{W} \to \mathbb{H}$ be the Malliavin derivative of $F$. The domain of $D$ in the space $L^2(\mathbb{W}, \mathcal{B}, \gamma)$ is denoted by $D^2$. If $F \in D^2$ then the Clark-Ocone formula asserts
\[
E^\gamma(F \mid \mathcal{G}_t) = 1 + \int_0^t E^\gamma(D_s F \mid \mathcal{G}_s) \cdot dx_s, \quad t \geq 0.
\]
We obtain the following result.

**Lemma 3.** When $F \in D^2$ the process $u_t$ given by Theorem 2 is
\[
u_t = \frac{E^\gamma(D_t F \mid \mathcal{G}_t)}{E^\gamma(F \mid \mathcal{G}_t)} 1_{\{E^\gamma(F \mid \mathcal{G}_t) > 0\}}.
\]
This implies that $\mu$-almost surely
\[
u_t = E^\mu \left( \frac{D_t F}{F} \mid \mathcal{G}_t \right).
\]

### 2.3 Optimal drift in a strong sense

According to Theorem 2, the filtered probability space $(\mathbb{W}, \mathcal{B}, \mu, \mathcal{G})$ carries a Brownian motion $y$. The process $x = y + U$ has law $\mu$ and the drift $U$ satisfies
\[
H(\mu | \gamma) = \frac{1}{2} E^\mu \|U\|^2.
\]
Still, it remains open whether given a probability space, a filtration and a Brownian motion, there exists a drift achieving equality in (4). It this section, we show that this is indeed the case, under some restriction on the measure $\mu$. The approach is taken from the article [4] in which Baudoin treats the case of Brownian bridges (see subsection 2.5 below). We refer to [21] for the background on stochastic differential equations.

**Theorem 4.** Let $B$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{F})$. Let $\mu$ be a measure on $\mathbb{W}$, absolutely continuous with respect to $\gamma$ and let $u_t: \mathbb{W} \to \mathbb{R}^d$ be the associated Föllmer process. If the stochastic differential equation

$$X_t = B_t + \int_0^t u_s(X) \, ds, \quad t \geq 0$$

has the pathwise uniqueness property, then it has a unique strong solution. This solution $X$ satisfies the following.

1. The process $U_t = \int_0^t u_s(X) \, ds$ belongs to $\mathbb{H}$ almost surely.
2. The process $X$ has law $\mu$.
3. The relative entropy of $\mu$ is given by

$$H(\mu | \gamma) = \frac{1}{2} \mathbb{E}\|U\|^2.$$  

Proof. According to Theorem [2] on $(\mathbb{W}, \mathcal{B}, \mu)$ the coordinate process $x$ satisfies

$$x_t = y_t + \int_0^t u_s(x) \, ds$$

where $y$ is a Brownian motion. Therefore (7) has a weak solution. By Yamada and Watanabe’s theorem, if pathwise uniqueness holds then (7) has a unique strong solution. Moreover, since pathwise uniqueness implies uniqueness in law, the solution $X$ has law $\mu$. The rest of Theorem 4 concerns the law of $X$, so it is contained in Theorem [2].

We end this section by showing that for a reasonably large class of measures $\mu$, the stochastic differential equation (7) does satisfy the pathwise uniqueness property.

**Definition 5.** Let $\mathcal{S}$ be the class of probability measures on $(\mathbb{W}, \mathcal{B}, \gamma)$ having a density of the form

$$F(w) = \Phi(w_{t_1}, \ldots, w_{t_n})$$

for some integer $n$, for some sample $0 \leq t_1 < t_2 < \cdots < t_n$ and for some function $\Phi: (\mathbb{R}^d)^n \to \mathbb{R}$ satisfying

- $\Phi$ is Lipschitz.
• $\nabla \Phi$ is Lipschitz.

• There exists $\epsilon > 0$ such that $\Phi \geq \epsilon$.

**Lemma 6.** If $\mu$ belongs to $S$ then the equation (7) has the pathwise uniqueness property.

**Proof.** Let $\mu$ have density $F$ given by (8). Then $F \in D^2$ and

$$DF(w) = \sum_{i=1}^{n} \nabla_i \Phi(w_{t_1}, \ldots, w_{t_n}) 1_{[0,t_i]}$$

where $\nabla_i \Phi$ is the gradient of $\Phi$ in the $i$-th variable. By Lemma 3, the process associated to $\mu$ is

$$u_t(w) = \mathbb{E}[DF(w) | G_t]$$

$$= \sum_{i=1}^{n} \mathbb{E}[\nabla_i \Phi(w_{t_1}, \ldots, w_{t_n}) | G_t] 1_{[0,t_i]}(t).$$

It is enough to prove that there is a constant $C$ such that

$$|u_t(w) - u_t(\tilde{w})| \leq C \sup_{0 \leq s \leq t} |w_s - \tilde{w}_s|.$$  \hspace{1cm} (9)

for all $t \geq 0$ and for all $w, \tilde{w} \in W$. Fix $t \geq 0$ and assume that $t_k \leq t < t_{k+1}$ for some $k \in \{0, \ldots, n-1\}$. By the Markov property of the Brownian motion

$$\mathbb{E}[\Phi(w_{t_1}, \ldots, w_{t_n}) | G_t] = \Psi(w_{t_1}, \ldots, w_{t_k}, w_{t_{k+1}})$$

where $\Psi(x_1, \ldots, x_k, x)$ equals

$$\int_{\mathbb{W}} \Phi(x_1, \ldots, x_k, x + w_{t_{k+1}-t}, \ldots, x + w_{t_n-t}) \gamma(dw).$$

Then observe that $\|\Psi\|_{\text{lip}} \leq \|\Phi\|_{\text{lip}}$. We have a similar property when $0 \leq t < t_1$ and when $t_n \leq t$. The argument applies also to $\nabla_i \Phi$. The inequality (9) follows easily.

To sum up, we have the following representation formula.

**Theorem 7.** Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ be a filtered probability space and let $B: \Omega \to \mathbb{W}$ be a Brownian motion. For all $\mu \in S$ we have

$$H(\mu | \gamma) = \min_U \left( \frac{1}{2} \mathbb{E}[\|U\|^2] \right)$$

where the minimum is on all drifts $U$ such that $B + U$ has law $\mu$. 

7
2.4 The Boué and Dupuis formula

In this subsection the previous results are translated in terms of log-Laplace using the following lemma.

**Lemma 8.** Let $f : \mathbb{W} \to \mathbb{R}$ bounded from below. For every positive $\epsilon$ there exists $\mu \in S$ such that

$$\log\left(\int_{\mathbb{W}} e^f \, d\gamma\right) \leq \int_{\mathbb{W}} f \, d\mu - H(\mu \mid \gamma) + \epsilon.$$  \hspace{1cm} (10)

**Proof.** By monotone convergence we can assume that $f$ is also bounded from above, and that $\int e^f \, d\gamma = 1$. Set $F = e^f$ and let $\mu$ be a probability measure on $\mathbb{W}$. Using $t \log(t) \leq |t - 1| + |t - 1|^2/2$ we get

$$H(\mu \mid \gamma) - \int f \, d\mu \leq \int \left| \frac{G}{F} - 1 \right| F \, d\gamma + \frac{1}{2} \int \left| \frac{G}{F} - 1 \right|^2 F \, d\gamma$$

$$\leq \|F - G\|_{L^1(\gamma)} + C\|F - G\|_{L^2(\gamma)}^2$$

where $G$ is the density of $\mu$ and $C$ is some constant (recall that $f$ is bounded below). Therefore, it is enough to prove that there exists $\mu \in S$ whose density $G$ is arbitrarily close to $F$ in $L^2(\gamma)$. This is left to the reader. \hfill \square

Here is the Boué and Dupuis formula.

**Theorem 9.** For every function $f : \mathbb{W} \to \mathbb{R}$ measurable and bounded from below, we have

$$\log\left(\int_{\mathbb{W}} e^f \, d\gamma\right) = \sup_U \left[ E\left(f(B + U) - \frac{1}{2} \|U\|^2\right)\right],$$

where the supremum is taken over all drifts $U$.

This is actually slightly more general than the result in [7], which concerns the space $C([0, T], \mathbb{R}^d)$ for some finite time horizon $T$.

**Proof.** Let $U$ be a drift and $\mu$ be the law of $B + U$. By Proposition 11 and the entropy/log-Laplace duality

$$E\left(f(B + U) - \frac{1}{2} \|U\|^2\right) \leq \int f \, d\mu - H(\mu \mid \gamma) \leq \log\left(\int_{\mathbb{W}} e^f \, d\gamma\right).$$

On the other hand, given $\epsilon > 0$, there exists a probability measure $\mu \in S$ satisfying (10). Since $\mu \in S$, Theorem 7 asserts that there exists a drift $U$ such that $B + U$ has law $\mu$ and satisfying

$$H(\mu \mid \gamma) = \frac{1}{2} E\|U\|^2.$$ 

Then (10) becomes

$$\log\left(\int_{\mathbb{W}} e^f \, d\gamma\right) \leq E\left(f(B + U) - \frac{1}{2} \|U\|^2\right) + \epsilon,$$

which concludes the proof. \hfill \square
2.5 Brownian bridges

A measure $\mu$ on $W$ satisfying

$$
\mu(dw) = \rho(w_1) \gamma(dw)
$$

(11)

where $\rho$ is some density on $(\mathbb{R}^d, \gamma_d)$ is said to be a Brownian bridge. It can be seen as the law of a Brownian motion conditioned to have law $\rho(x)\gamma_d(dx)$ at time 1.

**Lemma 10.** Let $\nu$ have density $\rho$ with respect to $\gamma_d$, we have

$$
\mathcal{H}(\nu \mid \gamma_d) = \inf_{\mu} \left( \mathcal{H}(\mu \mid \gamma) \right)
$$

where the infimum is on all probability measures satisfying $\mu \circ (x_1)^{-1} = \nu$. The infimum is attained when $\mu$ is the bridge (11).

In other words, among all processes having law $\nu$ at time 1, the bridge minimizes the relative entropy. This is essentially a particular case of (3), see also [4] and [17, page 161].

Assume that $\rho$ is differentiable and that $\nabla \rho \in L^2(\gamma_d)$. Then $F(w) = \rho(w_1)$ belongs to $\mathbb{D}^2$ and has Malliavin derivative

$$
DF(w) = \nabla \rho(w_1) \mathbf{1}_{[0,1]}.
$$

By Lemma 3 the Föllmer process of the bridge $\mu$ is such that

$$
u_t = \mathbb{E}^\mu \left( \nabla \log(\rho)(w_1) \mid \mathcal{G}_t \right) \mathbf{1}_{[0,1]}(t), \quad \mu - a.s.
$$

We obtain the following result.

**Lemma 11.** Under $\mu$, the process $(\nu_t)_{t \in [0,1]}$ is a martingale. In particular

$$
\mathbb{E}^\mu(\nu_t) = \mathbb{E}^\mu \nabla \log(\rho)(w_1) = \mathbb{E}^\gamma \nabla \rho(w_1) = \int_{\mathbb{R}^d} x \, \nu(dx).
$$

Now assume that $\rho$ and $\nabla \rho$ are Lipschitz and that $\rho \geq \epsilon$, so that the bridge $\mu$ belongs to $\mathcal{S}$. It is easily seen that $\nu_t$ can also be written as

$$
u_t(w) = \nabla \log P_{1-t}(w_1) \mathbf{1}_{[0,1]}(t),
$$

where $P_t$ denotes the heat semigroup on $\mathbb{R}^d$:

$$
\partial_t P_t = \frac{1}{2} \Delta P_t.
$$

The stochastic differential equation (7) becomes

$$
X_t = B_t + \int_0^{t \wedge 1} \nabla \log(P_{1-s}\rho)(X_s) \, ds, \quad t \geq 0.
$$

(12)

By Lemma 6 there is a unique strong solution. Combining Lemma 10 with Theorem 4 we obtain the following dual formulation of Borell’s result (1).
**Theorem 12.** Let \( \nu \) and \( \rho \) be as above. Then

\[
H(\nu \mid \gamma_d) = \inf_{U} \left( \frac{1}{2} \mathbb{E}[\|U\|^2] \right)
\]

where the infimum is taken on all drifts \( U \) satisfying \( B_1 + U_1 = \nu \) in law. The infimum is attained by the drift

\[
U_t = \int_0^{t \wedge 1} \nabla \log(P_{1-s} \rho)(X_s) \, ds,
\]

where \( X \) is the unique solution of \((12)\).

### 3 Applications

Following Borell, we now derive functional inequalities from the representation formula. Let us point out that in all but one applications we use Proposition 1 and Theorem 2 rather than Theorem 7.

#### 3.1 Transportation cost inequality

Let \( T_2 \) be the transportation cost for the Euclidean distance squared: given two probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}^d \)

\[
T_2(\mu, \nu) = \inf\left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi(x, y) \right)^{1/2}, \tag{13}
\]

where the infimum is taken over all couplings \( \pi \) of \( \mu \) and \( \nu \), namely all probability measures on the product space \( \mathbb{R}^d \times \mathbb{R}^d \) having marginals \( \mu \) and \( \nu \). There is a huge literature about this optimization problem, usually referred to as Monge-Kantorovitch problem, see Villani’s book [25]. Talagrand’s inequality asserts that

\[
T_2(\nu, \gamma_d) \leq 2 H(\nu \mid \gamma_d)
\]

for every probability measure \( \nu \) on \( \mathbb{R}^d \). The purpose of this subsection is to prove a Wiener space version of this inequality.

On Wiener space the natural definition of \( T_2 \) involves the norm of the Cameron-Martin space \( \mathbb{H} \): given two probability measures \( \mu, \nu \) on \( (\mathbb{W}, \mathcal{B}) \)

\[
T_2(\mu, \nu) = \inf\left( \int_{\mathbb{W} \times \mathbb{W}} \|w - w'\|^2 \, d\pi(dw, dw') \right),
\]

where the infimum is taken over all couplings \( \pi \) of \( \mu \) and \( \nu \) such that \( w - w' \in \mathbb{H} \) for \( \pi \)-almost all \( (w, w') \).

**Theorem 13.** Let \( \mu \) be a probability measure on \( (\mathbb{W}, \mathcal{B}) \). Then

\[
T_2(\mu, \gamma) \leq 2 H(\mu \mid \gamma).
\]
Here is a short proof based of Theorem 2. Fair enough, Feyel and Üstünel [13] have a very similar argument.

**Proof.** Assume that $\mu$ is absolutely continuous with respect to $\gamma$ (otherwise $H(\mu \mid \gamma) = +\infty$). According to Theorem 2 there exists a Brownian motion $B$ and a drift $U$ such that $B + U$ has law $\mu$ and

$$H(\mu \mid \gamma) = \frac{1}{2} \mathbb{E} \|U\|^2.$$  

Then $(B, B + U)$ is a coupling of $(\gamma, \mu)$ and by definition of $T_2$

$$T_2(\mu, \gamma)^2 \leq \mathbb{E} \|U\|^2 = 2 H(\mu \mid \gamma).$$

Let us point out that Talagrand’s inequality can be recovered easily from this theorem, applying it to a Brownian bridge. Details are left to the reader.

### 3.2 Logarithmic Sobolev inequality

In this section we prove the logarithmic Sobolev inequality for the Wiener measure, which extends the classical log-Sobolev inequality for the Gaussian measure, due to Gross [18]. When $\mu$ is a measure on $(\mathbb{W}, \mathcal{B}, \gamma)$ with density $F$ such that $DF$ is well defined, the Fisher information of $\mu$ is

$$I(\mu \mid \gamma) = \int_{\mathbb{W}} \frac{\|DF\|^2}{F} \, d\gamma = \int_{\mathbb{W}} \left\| \frac{DF}{F} \right\|^2 \, d\mu.$$

**Theorem 14.** Let $\mu$ have density $F$ with respect to $\gamma$ and assume that $F \in D^2$. Then

$$H(\mu \mid \gamma) \leq \frac{1}{2} I(\mu \mid \gamma).$$

**Proof.** We consider the probability space $(\mathbb{W}, \mathcal{B}, \mu)$. Recall that $(\mathcal{G}_t)_{t \geq 0}$ is the filtration of the coordinate process. By Theorem 2 and Lemma 3 letting

$$u_t = \mathbb{E}^\mu \left( \frac{D_tF}{F} \mid \mathcal{G}_t \right)$$

we have

$$H(\mu \mid \gamma) = \frac{1}{2} \mathbb{E}^\mu \int_0^\infty |u_t|^2 \, dt.$$  

By Jensen’s inequality

$$\mathbb{E}^\mu |u_t|^2 \leq \mathbb{E}^\mu \left( \frac{D_tF}{F} \right)^2$$

so that

$$H(\mu \mid \gamma) \leq \frac{1}{2} \mathbb{E}^\mu \left\| \frac{DF}{F} \right\|^2$$

which is the result.
This may not be the most straightforward proof, see [9]. Let us emphasize that applying (14) to a Brownian bridge yields the usual log-Sobolev inequality. More precisely, let $\nu$ be a probability measure on $\mathbb{R}^d$ having a smooth density $\rho$ with respect to $\gamma_d$ and let $\mu$ be the measure on $\mathbb{W}$ given by

$$\mu(dw) = \rho(w_1)\gamma(dw).$$

Then $H(\nu \mid \gamma_d) = H(\mu \mid \gamma)$. On the other hand letting $F(w) = \rho(w_1)1_{[0,1]}$, we have

$$DF(w) = \nabla\rho(w_1),$$

which implies easily that $I(\nu \mid \gamma_d) = I(\mu \mid \gamma)$. Thus (14) becomes

$$H(\nu \mid \gamma_d) \leq \frac{1}{2}I(\nu \mid \gamma_d).$$

### 3.3 Shannon’s inequality

Given a random vector $\eta$ on $\mathbb{R}^d$ having density $\rho$ with respect to the Lebesgue measure, Shannon’s entropy is defined as

$$S(\eta) = -\int_{\mathbb{R}^d} \rho \log(\rho) \, dx.$$ 

In other words $S(\eta) = -H(\nu \mid \lambda_d)$ where $\nu$ is the law of $\eta$ and $\lambda_d$ is the Lebesgue measure on $\mathbb{R}^d$.

**Theorem 15.** Let $\eta, \xi$ be independent random vectors on $\mathbb{R}^d$ and $\theta \in [0, \pi/2]$

$$S(\cos(\theta)\eta + \sin(\theta)\xi) \geq \cos^2(\theta)S(\eta) + \sin^2(\theta)S(\xi).$$ (15)

This inequality plays a central role in information theory, see [12] for an overview on the topic.

**Proof.** Let $\nu_{\theta}$ be the law of $\cos(\theta)\eta + \sin(\theta)\xi$. By Theorem 2 Lemma 10 and Lemma 11 there exists a Brownian motion $X$ and a drift $U$ such that

- $X_1 + U_1$ has law $\nu_0$,
- $H(\nu_0 \mid \gamma_d) = E\|U\|^2/2$,
- $E(U) = E(\eta)\, 1_{[0,1]}$.

Similarly, there exists a Brownian motion $Y$ and a drift $V$ satisfying the corresponding properties for $\nu_{\pi/2}$. Besides, we can clearly assume that $Y$ is independent of $X$. Then $\cos(\theta)X + \sin(\theta)Y$ is a Brownian motion and

$$\cos(\theta)X_1 + \sin(\theta)Y_1 + \cos(\theta)U_1 + \sin(\theta)V_1$$

has law $\nu_{\theta}$. By Proposition 1 and Lemma 10

$$H(\nu_{\theta} \mid \gamma_d) \leq \frac{1}{2}E\|\cos(\theta)U + \sin(\theta)V\|^2.$$
Denoting the inner product in $\mathbb{H}$ by $\langle \cdot, \cdot \rangle$ we have
\[ \mathbb{E}(U, V) = \langle \mathbb{E}U, \mathbb{E}V \rangle = \langle \mathbb{E}\eta, \mathbb{E}\xi \rangle, \]
so that
\[ H(\nu_\theta | \gamma_d) \leq \cos(\theta)^2 H(\nu_0 | \gamma_d) + \sin(\theta)^2 H(\nu_{\pi/2} | \gamma_d) \]
\[ + \cos(\theta) \sin(\theta) \langle \mathbb{E}\eta, \mathbb{E}\xi \rangle. \]

This is easily seen to be equivalent to (15).

### 3.4 Brascamp-Lieb inequality

Let us focus on a family of inequalities dating back to Brascamp and Lieb’s article [8] on optimal constants in Young’s inequality. Since then a number of nice alternate proofs have been discovered, see [9,10] and the survey article [1]. This subsection is inspired by the (unpublished) proof of Maurey relying on Borell’s formula.

Let $E$ be a Euclidean space, let $E_1, \ldots, E_m$ be subspaces and for all $i$ let $P_i$ be the orthogonal projection with range $E_i$. The crucial hypothesis is the so-called frame condition: there exist $c_1, \ldots, c_m$ in $\mathbb{R}_+$ such that
\[ \sum_{i=1}^m c_i P_i = \text{id}_E. \]

(16)

Let $x \in E$, we then have $|x|^2 = (\sum c_i P_i x) \cdot x$ and since $P_i$ is an orthogonal projection
\[ |x|^2 = \sum_{i=1}^m c_i |P_i x|^2. \]

(17)

From now on $\mathbb{W}$ denotes the space of continuous paths taking values in $E$ and starting from 0 and $\gamma$ denotes the Wiener measure on $\mathbb{W}$. The spaces $\mathbb{W}_i$ and measures $\gamma_i$ are defined similarly.

**Theorem 16.** Under the frame condition, for every probability measure $\mu$ on $\mathbb{W}$ we have
\[ H(\mu | \gamma) \geq \sum_{i=1}^m c_i H(\mu_i | \gamma_i), \]
where $\mu_i = \mu \circ P_i^{-1}$ is the push-forward of $\mu$ by the projection $P_i$.

**Proof.** According to Theorem 2 there exists a standard Brownian motion $B$ on $E$ and a drift $U$ such that $B + U$ has law $\mu$ and
\[ H(\mu | \gamma) = \frac{1}{2} \mathbb{E}||U||^2. \]

Since $P_i$ is an orthogonal projection, the process $P_i B$ is a standard Brownian motion on $E_i$. Also $P_i B + P_i U$ has law $\mu \circ P_i^{-1} = \mu_i$. By Proposition 4
\[ H(\mu_i | \gamma_i) \leq \frac{1}{2} \mathbb{E}||P_i U||^2, \quad i = 1, \ldots, m. \]
On the other hand, the frame condition (17) implies easily that
\[ \|U\|^2 = \sum_{i=1}^{n} c_i \|P_i U\|^2 \]
pointwise. Taking expectation yields the result.

As observed by Carlen and Cordero [10], this super-additivity property of the relative entropy is equivalent to the following Brascamp-Lieb inequality.

**Corollary 17.** Under the frame condition, given \( m \) functions \( F_i: \mathbb{W}_i \to \mathbb{R}_+ \), we have
\[
\int_{\mathbb{W}} \prod_{i=1}^{m} (F_i \circ P_i)^{c_i} \, d\gamma \leq \prod_{i=1}^{m} \left( \int_{\mathbb{W}_i} F_i \, d\gamma_i \right)^{c_i}.
\]

When the functions \( F_i \) depend only on the point \( w_1 \) rather than on the whole path \( w \) we recover the usual Brascamp-Lieb inequality for the Gaussian measure.

### 3.5 Reversed Brascamp-Lieb inequality

Again \( E \) is a Euclidean space and \( E_1, \ldots, E_m \) are subspaces satisfying the frame condition (16). Observe that if \( x_1, \ldots, x_m \) belong to \( E_1, \ldots, E_m \) respectively, then for any \( y \in E \), the Cauchy-Schwarz inequality and (17) yield
\[
\left( \sum_{i=1}^{m} c_i x_i \right) \cdot y = \sum_{i=1}^{m} c_i (x_i \cdot P_i y) \\
\leq \left( \sum_{i=1}^{m} c_i |x_i|^2 \right)^{1/2} \left( \sum_{i=1}^{m} c_i |P_i y|^2 \right)^{1/2} \\
= \left( \sum_{i=1}^{m} c_i |x_i|^2 \right)^{1/2} |y|.
\]

Hence
\[
\sum_{i=1}^{m} c_i x_i \mid y \mid \leq \sum_{i=1}^{m} c_i |x_i|^2.
\]

Let \( S_i \) be the class of probability measures on \( E_i \) which satisfy the conditions of Definition [5] replacing \( \mathbb{R}^d \) by \( E_i \). Here is the reversed version of Theorem 16.

**Theorem 18.** Given \( m \) probability measures \( \mu_1, \ldots, \mu_m \) belonging to \( S_1, \ldots, S_m \) respectively, there exist \( m \) processes \( X_1, \ldots, X_m \) (defined on the same probability space) such that

1. \( X_i \) has law \( \mu_i \) for all \( i = 1, \ldots, m \).
2. Letting $\mu$ be the law of $\sum c_i X_i$ we have

$$H(\mu \mid \gamma) \leq \sum_{i=1}^{m} c_i H(\mu_i \mid \gamma_i).$$

Proof. Again let $B$ be a standard Brownian motion on $E$. For $i = 1, \ldots, m$, the process $P_i B$ is a standard Brownian motion on $E_i$. Since $\mu_i \in \mathcal{S}_i$ there exists a drift $U_i$ such that the process $X_i = P_i B + U_i$ has law $\mu_i$ and

$$H(\mu_i \mid \gamma_i) = \frac{1}{2} \mathbb{E}\|U_i\|^2.$$

Let $X = \sum c_i X_i$ and let $\mu$ be the law of $X$. Since $\sum c_i P_i$ is the identity of $E$

$$X = B + \sum_{i=1}^{m} c_i U_i.$$

By Proposition 11 we get

$$H(\mu \mid \gamma) \leq \frac{1}{2} \mathbb{E}\|\sum_{i=1}^{m} c_i U_i\|^2.$$

On the other hand (18) easily implies that

$$\|\sum_{i=1}^{m} c_i U_i\|^2 \leq \sum_{i=1}^{m} c_i \|U_i\|^2,$$

pointwise. Taking expectation we get the result.

This sub-additivity property of the entropy is a multi-marginal version of the displacement convexity property put forward by Sturm [22]. By duality, we obtain the following reversed Brascamp-Lieb inequality.

**Corollary 19.** Assuming the frame condition, given $m$ functions $F_i : \mathbb{W}_i \rightarrow \mathbb{R}_+$ bounded away from 0, and a function $G : \mathbb{W} \rightarrow \mathbb{R}_+$ satisfying

$$\prod_{i=1}^{m} F_i(w_i)^{c_i} \leq G\left(\sum_{i=1}^{m} c_i w_i\right)$$

for all $(w_1, \ldots, w_m) \in \mathbb{W}_1 \times \cdots \times \mathbb{W}_m$, we have

$$\prod_{i=1}^{m} \left(\int_{\mathbb{W}_i} F_i \, d\gamma_i\right)^{c_i} \leq \int_{\mathbb{W}} G \, d\gamma.$$

Proof. By Lemma 8 for every $i$, there exists a measure $\mu_i \in \mathcal{S}_i$ such that

$$\log\left(\int_{\mathbb{W}_i} F_i \, d\gamma_i\right) \leq \int_{\mathbb{W}_i} \log(F_i) \, d\mu_i - H(\mu_i \mid \gamma_i) + \epsilon.$$
Let $X_1, \ldots, X_m$ be the random processes given by the previous theorem, let

$$X = \sum c_i X_i$$

and let $\mu$ be the law of $X$. Then by duality and the hypothesis (19) we get

$$\log \left( \int_W G \, d\gamma \right) \geq \mathbb{E} \log(G)(X) - H(\mu \mid \gamma) \geq \mathbb{E} \left( \sum_{i=1}^m c_i \log(F_i)(X_i) \right) - H(\mu \mid \gamma).$$

Since $H(\mu \mid \gamma) \leq \sum c_i H(\mu_i \mid \gamma_i)$, this is at least

$$\sum c_i \left( \log \left( \int_{W_i} F_i \, d\gamma_i \right) - \epsilon \right).$$

Letting $\epsilon$ tend to 0 yields the result. 

Again when the functions depend only on the value of the path at time 1, we recover the reversed Brascamp-Lieb inequality for the Gaussian measure, which is due to Barthe [2].

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