Wire deconstructionism and classification of topological phases

Titus Neupert,1 Claudio Chamon,2 Christopher Mudry,3 and Ronny Thomale4

1Princeton Center for Theoretical Science, Princeton University, Princeton, New Jersey 08544, USA
2Physics Department, Boston University, Boston, Massachusetts 02215, USA
3Condensed Matter Theory Group, Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland
4Institute for Theoretical Physics, University of Würzburg, Am Hubland, D-97074 Würzburg, Germany
(Dated: March 6, 2014)

A scheme is proposed to classify integer and fractional topological quantum states of fermions in two spatial dimensions. We devise models for such states by coupling wires of non-chiral Luttinger liquids of electrons, that are arranged in a periodic array. Which inter-wire couplings are allowed is dictated by symmetry and the compatibility criterion that they can simultaneously acquire a finite expectation value, opening a spectral gap between the ground state(s) and all excited states in the bulk. First, with these criteria at hand, we reproduce the tenfold classification table of integer topological insulators, where their stability against interactions becomes immediately transparent in the Luttinger liquid description. Second, we expand the table to long-range entangled topological phases with intrinsic topological order and fractional excitations.

I. INTRODUCTION

The study of topological phases of matter is one of the most vibrant directions of research in contemporary condensed matter physics. One core accomplishment has been the theoretical modeling and experimental discovery of two-dimensional topological insulators.1–4 The integer quantum Hall effect (IQHE) was an early example of how states could be classified into distinct topological classes using an integer, the Chern number, to express the quantized Hall conductivity.5–7 In the IQHE, the number of delocalized edge channels is directly tied to the quantized Hall conductivity through the Chern number. More recently, it has been found that the symmetry under reversal of time acts as a protective symmetry for edge modes in (bulk) insulators with strong spin-orbit interactions in two and three dimensions1,8 and that these systems are characterized by a Z2 topological invariant.

The discovery of Z2 topological insulators has triggered a search for a classification of phases of fermionic matter that are distinct by some topological attribute. For non-interacting electrons, a complete classification, the tenfold way, has been accomplished in arbitrary dimensions.9,10 In this scheme, three discrete symmetries that act locally in position space – time-reversal symmetry (TRS), particle-hole symmetry (PHS), and chiral or sublattice symmetry (SLS) – play a central role when defining the quantum numbers that identify the topological insulating fermionic phases of matter within one of the ten symmetry classes (see columns 1-3 from Table I).

The tenfold way is believed to be robust to a perturbative treatment of short-ranged electron-electron interactions for the following reasons. First, the unperturbed ground state in the clean limit and in a closed geometry is given by non-degenerate filled bands of a band insulator. The band gap provides a small expansion parameter, namely the ratio of the characteristic interacting energy scale to the band gap. Second, the quantized topological invariant that characterizes the filled bands cannot change in a perturbative treatment of a short-range interaction.

On the other hand, the fate of the tenfold way when electron-electron interactions are strong is rather subtle.11–14 For example, short-range interactions can drive the system through a topological phase transition at which the energy gap closes,15,16 or they may spontaneously break a defining symmetry of the topological phase. Even when short-range interactions neither spontaneously break the symmetries nor close the gap, it may be that two phases from the non-interacting ten-fold way cease to be distinguishable in the presence of interactions. In fact, it was shown for the symmetry class BDI in one dimension by Fidkowski and Kitaev that the non-interacting Z classification was too fine in that it must be replaced by a Z8 classification when generic short-range interactions are allowed. The task of how to construct a counterpart to the tenfold way for interacting fermion (and boson) systems has thus attracted a lot of interest.17–27

The fractional quantum Hall effect (FQHE) is the paradigm for a situation by which interactions select topologically ordered ground states of a very different kind than the non-degenerate ground states from the tenfold way. On a closed two-dimensional manifold of genus g, interactions can stabilize incompressible many-body ground states with a g-dependent degeneracy. Excited states in the bulk must then carry fractional quantum numbers (see Ref. 28 and references therein). Such phases of matter, that follow the FQHE paradigm, appear in the literature under different names: fractional topological insulators, long-range entangled phases, topologically ordered phases, or symmetry enriched topological phases. In this paper we use the terminology long-range entangled (LRE) phase for all phases with nontrivial g-dependent ground state degeneracy. All other phases, i.e., those that follow the IQHE paradigm, are called short-range entangled (SRE) phases. (In doing so, we follow the terminology of Ref. 25, that differs slightly from the one used in Ref. 20. The latter counts all chiral phases irrespective of their ground state degen-
While there are nontrivial SRE and LRE phases in the absence of any symmetry constraint, many SRE and LRE phases are defined by some protecting symmetry they obey. If this protecting symmetry is broken, the topological attribute of the phase is not well defined any more. However, there is a sense in which LRE phases are more robust than SRE phases against a weak breaking of the defining symmetry. The topological attributes of LRE phases are not confined to the boundary in space between two distinct topological realizations of these phases, as they are for SRE phases. They also characterize intrinsic bulk properties. Hence, whereas gapless edge states are gaped by any breaking of the defining symmetry, topological bulk properties are robust to a weak breaking of the defining symmetry as long as the characteristic energy scale for this symmetry breaking is small compared to the bulk gap in the LRE phase.

The purpose of this paper is to implement a classification scheme for interacting electronic systems in two spatial dimensions that treats SRE and LRE phases on equal footing. To this end, we use a coupled wire construction for each of the symmetry classes from the tenfold way. This approach has been pioneered in Refs. 29 and 30 for the IQHE and in Refs. 31 and 32 for the FQHE.

To begin with, non-chiral Luttinger liquids are placed in a periodic array. In doing so, forward-scattering two-body interactions are naturally accounted for. We then assume that the back-scattering between neighboring wires is dominant over the intra-wire back-scattering. Imposing symmetries constrains the allowed tunnelings between consecutive wires. Whether a given arrangement of tunnelings truly gaps out all bulk modes, except for some ungaped edge states on the first and last wire, is verified with the help of a condition that applies to the limit of strong inter-wire tunneling. We name this condition the Haldane criterion, as introduced by Haldane in his study of the stability of non-maximally chiral edge states in the quantum Hall effect. We show that, for a proper choice of the tunnelings between neighboring wires, all bulk modes are gaped. Moreover, in five out of the ten symmetry classes of the tenfold way, there remain gapless edge states in agreement with the tenfold way. It is the many-body character of the tunnelings between neighboring wires that determines if this wire construction select a SRE or a LRE phase. Hence, this construction, predicated as it is on the strong tunneling limit, generalizes the tenfold way for SRE phases to LRE phases. It thereby delivers LRE phases that have not yet appeared in the literature before. Evidently, this edge-centered classification scheme does not distinguish between LRE phases of matter that do not carry protected gapless edge modes at their interfaces. For example, some fractional, time-reversal-symmetric, incompressible and topological phases of matter can have fractionalized excitations in the bulk, while not supporting protected gapless modes at their boundaries.

Stated in a slightly more constructive way, we can think of our approach as (1) fixing, in a first step, a given desired edge theory at the boundary, and (2) continue, in a second step, by asking whether such an edge can be consistently defined with a set of symmetry-allowed periodic tunneling terms between wires which manage to gap out all other modes. Alluding to a related strategy in philosophy, this is what we call wire deconstructionism of topological phases.

The purpose of this paper is to implement a classification scheme for interacting electronic systems in two spatial dimensions that treats SRE and LRE phases on equal footing. To this end, we use a coupled wire construction for each of the symmetry classes from the tenfold way. This approach has been pioneered in Refs. 29 and 30 for the IQHE and in Refs. 31 and 32 for the FQHE.

To begin with, non-chiral Luttinger liquids are placed in a periodic array. In doing so, forward-scattering two-body interactions are naturally accounted for. We then assume that the back-scattering between neighboring wires is dominant over the intra-wire back-scattering. Imposing symmetries constrains the allowed tunnelings between consecutive wires. Whether a given arrangement of tunnelings truly gaps out all bulk modes, except for some ungaped edge states on the first and last wire, is verified with the help of a condition that applies to the limit of strong inter-wire tunneling. We name this condition the Haldane criterion, as introduced by Haldane in his study of the stability of non-maximally chiral edge states in the quantum Hall effect. We show that, for a proper choice of the tunnelings between neighboring wires, all bulk modes are gaped. Moreover, in five out of the ten symmetry classes of the tenfold way, there remain gapless edge states in agreement with the tenfold way. It is the many-body character of the tunnelings between neighboring wires that determines if this wire construction selects a SRE or a LRE phase. Hence, this construction, predicated as it is on the strong tunneling limit, generalizes the tenfold way for SRE phases to LRE phases. It thereby delivers LRE phases that have not yet appeared in the literature before. Evidently, this edge-centered classification scheme does not distinguish between LRE phases of matter that do not carry protected gapless edge modes at their interfaces. For example, some fractional, time-reversal-symmetric, incompressible and topological phases of matter can have fractionalized excitations in the bulk, while not supporting protected gapless modes at their boundaries.

Stated in a slightly more constructive way, we can think of our approach as (1) fixing, in a first step, a given desired edge theory at the boundary, and (2) continue, in a second step, by asking whether such an edge can be consistently defined with a set of symmetry-allowed periodic tunneling terms between wires which manage to gap out all other modes. Alluding to a related strategy in philosophy, this is what we call wire deconstructionism of topological phases.
TABLE I. (Color online) Realization of a two-dimensional array of quantum wires in each symmetry class of the tenfold way, such that the ground state supports propagating gapless edge modes localized on the first and last wire that are immune to local and symmetry-preserving perturbations for any one of the symmetry classes A, AII, D, DIII, and C. The first column labels the symmetry classes according to the Cartan classification of symmetric spaces. The second column dictates if the operations for reversal of time ($\hat{\Theta}$), exchange of particles and holes ($\hat{\Pi}$), and reversal of chirality ($\hat{C}$) (see the footnote 37 for a definition of $\hat{C}$) are the generators of symmetries that square to $+1$, $-1$, or are not present in which case the entry 0 is used. The fourth column is a pictorial representation of the Hamiltonian for the two-dimensional array of quantum wires that delivers short-range entangled (SRE) gapless edge states. A wire is represented by a colored box with the minimum number of channels compatible with the symmetry class. Each channel in a wire is either a right mover ($\otimes$) or a left mover ($\odot$) that may or may not carry a spin quantum number ($\uparrow$, $\downarrow$) or a particle (yellow color) or hole (black color) attribute. The lines describe tunneling processes between consecutive wires in the array that are either of one-body type for the SRE column or of strictly many-body type for the LRE column.

| Symmetry Class | $\hat{\Theta}^2$ | $\hat{\Pi}^2$ | $\hat{C}^2$ | Short-range entangled (SRE) topological phase | Long-range entangled (LRE) topological phase |
|----------------|-----------------|----------------|------------|---------------------------------------------|---------------------------------------------|
| A              | 0               | 0              | 0          | $\bullet$ $\odot$ $\bullet$ $\odot$ ... | $\bullet$ $\odot$ $\bullet$ $\odot$ ... |
| AIII           | 0               | 0              | +          | NONE                                        | NONE                                        |
| AII            | -               | 0              | 0          | $\bullet$ $\odot$ $\bullet$ $\odot$ ... | $\bullet$ $\odot$ $\bullet$ $\odot$ ... |
| DIII           | -               | +              | +          | $\bullet$ $\odot$ $\bullet$ $\odot$ ... | $\bullet$ $\odot$ $\bullet$ $\odot$ ... |
| D              | 0               | +              | 0          | $\bullet$ $\odot$ $\bullet$ $\odot$ ... | $\bullet$ $\odot$ $\bullet$ $\odot$ ... |
| BDI            | +               | +              | +          | NONE                                        | NONE                                        |
| AI             | +               | 0              | 0          | NONE                                        | NONE                                        |
| CI             | +               | -              | +          | NONE                                        | NONE                                        |
| C              | 0               | -              | 0          | $\bullet$ $\odot$ $\bullet$ $\odot$ ... | $\bullet$ $\odot$ $\bullet$ $\odot$ ... |
| CII            | -               | -              | -          | NONE                                        | NONE                                        |

with all other anticommutators vanishing and the collective labels $a, a' = 1, \ldots, M \times N$. The notation

$$\hat{\Psi}^\dagger(x) \equiv (\hat{\psi}_1^\dagger(x) \cdots \hat{\psi}_{MN}^\dagger(x)), \quad \hat{\Psi}(x) \equiv \begin{pmatrix} \hat{\psi}_1(x) \\ \vdots \\ \hat{\psi}_{MN}(x) \end{pmatrix},$$

is used for the operator-valued row ($\hat{\Psi}^\dagger$) and column ($\hat{\Psi}$) vector fields. We assume that the many-body quantum dynamics of the fermions supported by this array of wires is governed by the Hamiltonian $\hat{H}$, whereby interactions within each wire are dominant over interactions between wires so that we may represent $\hat{H}$ as $N$ coupled Luttinger liquids, each one of which is composed of $M$ interacting fermionic channels.

By assumption, we may thus bosonize the $M \times N$ fermionic channels making up the array. To this end, we follow Ref. 38. Within Abelian bosonization, this is done by postulating first the $MN \times MN$ matrix

$$\mathcal{K} \equiv (K_{a\alpha'})$$

(2.2a)

to be symmetric with integer-valued entries. Because we are after an array of identical wires, each of which having its quantum dynamics governed by that of a Luttinger liquid, it is natural to assume that $\mathcal{K}$ is reducible,

$$K_{a\alpha'} = \delta_{ii'} K_{\gamma\gamma'}, \quad i, i' = 1, \cdots, N, \quad \gamma, \gamma' = 1, \cdots, M.$$

(2.2b)
A second $MN \times MN$ matrix is then defined by
\[ \mathcal{L} \equiv (\mathcal{L}_{aa'}) \] (2.3a)
where
\[ \mathcal{L}_{aa'} := \text{sgn}(a - a') (K^{-1}_{aa'} + 1). \] (2.3b)

Third, one verifies that, for any pair $a, a' = 1, \cdots, MN$, the Hermitian fields $\hat{\phi}_a$ and $\hat{\phi}_{a'}$, defined by the Mandelstam formula
\[ \hat{\psi}_a(x) \equiv \exp\left(\pm iK^{-1}_{aa'} \hat{\phi}_{a'}(x)\right): \] (2.4a)
as they are, obey the bosonic equal-time algebra
\[ \left[\hat{\phi}_a(x), \hat{\phi}_{a'}(x')\right] = -i\pi (K^{-1}_{aa'} \text{sgn}(x - x') + K^{-1}_{ab} L_{bc} K^{-1}_{ca'}). \] (2.4b)

Here, the notation $(\cdots)$ stands for normal ordering of the argument $(\cdots)$ and the summation convention over repeated indices is implied. In line with Eq. (2.1b), we use the notation
\[ \hat{\Phi}^T(x) \equiv \left(\hat{\phi}_1(x) \cdots \hat{\phi}_{MN}(x)\right), \quad \hat{\Phi}(x) \equiv \begin{pmatrix} \hat{\phi}_1(x) \\ \vdots \\ \hat{\phi}_{MN}(x) \end{pmatrix}, \] (2.4c)
for the operator-valued row $(\hat{\Phi}^T)$ and column $(\hat{\Phi})$ vector fields. Periodic boundary conditions along the $x$ direction parallel to the wires are imposed by demanding that
\[ K \hat{\Phi}(x + L) = K \hat{\Phi}(x) + 2\pi \mathcal{N}, \quad \mathcal{N} \in \mathbb{Z}^{MN}. \] (2.4d)

Equipped with Eqs. (2.2)–(2.4), we decompose additively the many-body Hamiltonian $\hat{H}$ for the $MN$ interacting fermions propagating on the array of wires into
\[ \hat{H} = \hat{H}_V + \hat{H}_{\{\mathcal{T}\}}, \] (2.5a)

Hamiltonian
\[ \hat{H}_V := \int dx \left( \partial_x \hat{\Phi}^T \right)(x) \mathbb{V} \left( \partial_x \hat{\Phi} \right)(x), \] (2.5b)
even though quadratic in the bosonic field, encodes both local one-body terms as well as contact many-body interactions between the $M$ fermionic channels in any given wire from the array through the block-diagonal, real-valued, and symmetric $MN \times MN$ matrix
\[ \mathbb{V} := (V_{aa'}) \equiv \left(V_{(i,\gamma)(i',\gamma')}\right) = \mathbb{I}_N \otimes (V_{\gamma\gamma'}). \] (2.5c)

Hamiltonian
\[ \hat{H}_{\{\mathcal{T}\}} := \int dx \sum_{\mathcal{T}} h_\mathcal{T}(x) \left( e^{\pm i\alpha_\mathcal{T}(x)} \prod_{a=1}^{MN} \hat{\psi}^\dagger_{\mathcal{T}}(x) \right) + \text{H.c.} \]
\[ = \int dx \sum_{\mathcal{T}} h_\mathcal{T}(x) \cos \left( \mathcal{T}^T K \hat{\Phi}(x) + \alpha_\mathcal{T}(x) \right) \] (2.5d)
is not quadratic in the bosonic fields. With the understanding that the operator-multiplication of identical fermion fields at the same point $x$ along the wire requires point splitting, and with the short-hand notation $\hat{\psi}^{-1}_a(x) = \hat{\psi}^\dagger_a(x)$, we interpret $\hat{H}_{\{\mathcal{T}\}}$ as (possibly many-body) tunnelings between the fermionic channels. Here, we introduced the set $\{\mathcal{T}\}$ comprised of all integer-valued tunneling vectors
\[ T \equiv (T_a) \] (2.5e)
obeying the condition
\[ \sum_{a=1}^{MN} T_a = 0, \] (2.5f)
and we assigned to each $T$ from the set $\{\mathcal{T}\}$ the real-valued functions
\[ h_\mathcal{T}(x) = h_\gamma^\mathcal{T}(x) \] (2.5g)
and
\[ \alpha_\mathcal{T}(x) = \alpha_\gamma^\mathcal{T}(x). \] (2.5h)

The condition (2.5f) ensures that these tunneling events preserve the total fermion number. We emphasize that the integer
\[ q := \sum_{a=1}^{MN} |T_a|/2 \] (2.6)
dictates that $\mathcal{T}$ encodes a $q$-body interaction in the fermion representation.

### III. CONDITIONS FOR A SPECTRAL GAP

Hamiltonian $\hat{H}_V$ in the decomposition (2.5) has $MN$ gapless modes. However, $\hat{H}_V$ does not commute with $\hat{H}_{\{\mathcal{T}\}}$ and the competition between $\hat{H}_V$ and $\hat{H}_{\{\mathcal{T}\}}$ can gap some, if not all, the gapless modes of $\hat{H}_V$. For example, a tunneling amplitude that scatters the right mover into the left mover of each flavor in each wire will gap out the spectrum of $\hat{H}_V$.

A term in $\hat{H}_{\{\mathcal{T}\}}$ has the potential to gap out a gapless mode of $\hat{H}_V$ if the condition (in the Heisenberg representation)$^{38,39}$
\[ \partial_x \left[ \mathcal{T}^T K \hat{\Phi}(t, x) + \alpha_\mathcal{T}(x) \right] = C_\mathcal{T}(x) \] (3.1)
holds, when applied to the ground state, for some time-independent real-valued functions $C_\mathcal{T}(x)$. The locking condition (3.1) removes the linear combination $\mathcal{T}^T K \hat{\Phi}(t, x)$ from the gapless degrees of freedom of the theory. However, not all scattering vectors $\mathcal{T}$ can simultaneously lead to such a locking due to quantum fluctuations. The set of linear combinations $\{\mathcal{T}^T K \hat{\Phi}(t, x)\}$ that
can satisfy the locking condition (3.1) simultaneously is labeled by the subset \( \{ T \}_\text{locking} \) of all tunneling matrices \( \{ T \} \) defined by Eqs. (2.5e) and (2.5f) obeying the Haldane criterion (3.2)\(^{38,39}\)

\[
T^T K T = 0 \tag{3.2a}
\]

for any \( T \in \{ T \}_\text{locking} \) and

\[
T^T K T' = 0 \tag{3.2b}
\]

pairwise for any \( T \neq T' \in \{ T \}_\text{locking} \).

**IV. REPRODUCING THE TENFOLD WAY**

Our first goal is to apply the wire construction in order to reproduce the classification of non-interacting topological insulators and superconductors within the wire construction in \((2 + 1)\) dimensions (see Table I).\(^9,10\) In this section, we will carry out the classification scheme within the bosonized description of quantum wires. Here, we will restrict the classification to one-body tunnelings within the bosonized description of quantum wires. Here, we will carry out the classification scheme within the bosonized language.

Within the classification of non-interacting Hamiltonians, superconductors are nothing but fermionic bilinears with a particle-hole symmetry. The physical interpretation of the degrees of freedom as Bogoliubov quasiparticles is of no consequence to the analysis. In particular, they still carry an effective conserved \( U(1) \) charge in the non-interacting description.

We start by showing how to implement the symmetries of the tenfold way in the bosonized language.

**A. Symmetries**

The classification is based on the presence or the absence of the TRS and the spectral PHS that are represented by antiunitary operators \( \hat{\Theta} \) and \( \hat{\Pi} \), respectively. Each of \( \hat{\Theta} \) and \( \hat{\Pi} \) can exist in two varieties in which each square to the identity operator up to the multiplicative factor \( \pm 1 \),

\[
\hat{\Theta}^2 = \pm 1, \quad \hat{\Phi}^2 = \pm 1, \tag{4.1}
\]

respectively. By assumption, the set of all degrees of freedom in each given wire is invariant under the actions of \( \hat{\Theta} \) and \( \hat{\Pi} \). If so, we can represent the actions of \( \hat{\Theta} \) and \( \hat{\Pi} \) on the fermionic fields in two steps. First, we introduce two \( M \times M \)-dimensional matrix representations \( P_\Theta \) and \( P_\Pi \) of the permutation group of \( M \) elements, which we combine into the block-diagonal \( MN \times MN \) matrices

\[
P_\Theta := 1_N \otimes P_\Theta, \quad P_\Pi := 1_N \otimes P_\Pi, \tag{4.2a}
\]

where \( 1_N \) is the \( N \times N \) unit matrix and we make sure that \( P_\Theta \) and \( P_\Pi \) represent products of transpositions so that

\[
P_\Theta = P_\Theta^{-1}, \quad P_\Pi = P_\Pi^{-1}. \tag{4.2b}
\]

Second, we introduce two column vectors \( I_\Theta \in \mathbb{Z}^M \) and \( I_\Pi \in \mathbb{Z}^M \), which we combine into the two column vectors

\[
I_\Theta := \begin{pmatrix} I_\Theta \\ I_\Theta \end{pmatrix}, \quad I_\Pi := \begin{pmatrix} I_\Pi \\ I_\Pi \end{pmatrix}, \tag{4.2c}
\]

and the \( MN \times MN \) diagonal matrices

\[
D_\Theta := \text{diag} ( I_\Theta ), \quad D_\Pi := \text{diag} ( I_\Pi ), \tag{4.2d}
\]

with the components of the vectors \( I_\Theta \) and \( I_\Pi \) as diagonal matrix elements. The vectors \( I_\Theta \) and \( I_\Pi \) are not chosen arbitrarily. We demand that the vectors \( (1 + P_\Theta) I_\Theta \) and \( (1 + P_\Pi) I_\Pi \) are made of even \( [\text{for the} +1 \text{in Eq. (4.1)}] \) and odd \( [\text{for the} -1 \text{in Eq. (4.1)}] \) integer entries only, while

\[
e^{+i\pi D_\Theta} P_\Theta = \pm P_\Theta e^{+i\pi D_\Theta}, \tag{4.2e}
\]

and

\[
e^{+i\pi D_\Pi} P_\Pi = \pm P_\Pi e^{+i\pi D_\Pi}, \tag{4.2f}
\]

in order to meet \( \hat{\Theta}^2 = \pm 1 \) and \( \hat{\Pi}^2 = \pm 1 \), respectively. The operations of reversal of time and interchanges of particles and holes are then represented by

\[
\hat{\Theta} \hat{\Psi} \hat{\Theta}^{-1} = e^{+i\pi D_\Theta} P_\Theta \hat{\Psi}, \tag{4.2g}
\]

\[
\hat{\Pi} \hat{\Psi} \hat{\Pi}^{-1} = e^{+i\pi D_\Pi} P_\Pi \hat{\Psi}, \tag{4.2h}
\]

for the fermions and

\[
\hat{\Theta} \hat{\Phi} \hat{\Theta}^{-1} = P_\Theta \hat{\Phi} + \pi K^{-1} I_\Theta, \tag{4.2i}
\]

\[
\hat{\Pi} \hat{\Phi} \hat{\Pi}^{-1} = P_\Pi \hat{\Phi} + \pi K^{-1} I_\Pi, \tag{4.2j}
\]

for the bosons. One verifies that Eq. (4.1) is fulfilled. Hamiltonian (2.5) is PHS if

\[
\hat{\Theta} \hat{H} \hat{\Theta}^{-1} = + \hat{H}. \tag{4.3a}
\]

This condition is met if

\[
P_\Theta V P_\Theta^{-1} = + V, \tag{4.3b}
\]

\[
P_\Theta K P_\Theta^{-1} = - K, \tag{4.3c}
\]

\[
h_\tau(x) = h_{-P_\Theta} \tau(x), \tag{4.3d}
\]

\[
a_\tau(x) = a_{-P_\Theta} \tau(x) - \pi T^T P_\Theta^{-1} I_\Theta. \tag{4.3e}
\]

The spectrum of Hamiltonian (2.5) is PHS if

\[
\hat{\Pi} \hat{H} \hat{\Pi}^{-1} = - \hat{H}. \tag{4.4a}
\]
This condition is met if

$$P_{\Pi} V P_{\Pi}^{-1} = -V, \quad (4.4b)$$

$$P_{\Pi} K P_{\Pi}^{-1} = +K, \quad (4.4c)$$

$$h_{\tau}(x) = h_p + P_{\Pi} \tau(x), \quad (4.4d)$$

$$\alpha_{\tau}(x) = \alpha_{+} + P_{\Pi} \tau(x) + \pi T^T P_{\Pi}^{-1} \tau(x) + \pi. \quad (4.4e)$$

Without loss of generality, we may derive conditions (4.3) and (4.4) by reproducing the derivation of Eqs. (4.3d) and (4.3e).

We shall now apply the wire construction to all topologically nontrivial symmetry classes in (2 + 1) dimensions. This will be done by specifying a set of tunneling vectors \{\tau\} for which \(h_{\tau}\) is nonvanishing for any \(\tau\) from \{\tau\}.

### B. Strategy

Our strategy consists in choosing the many-body Hamiltonian \(\hat{H} = \hat{H}_{\Pi} + \hat{H}_{\{\tau\}}\) defined in Eq. (2.5) so that (i) it belongs to any one of the ten symmetry classes from the tenfold way, (ii) all excitations in the bulk are gaped by a specific choice of the tunneling vectors \{\tau\} entering \(\hat{H}_{\{\tau\}}\) (the energy scales in \(\hat{H}_{\{\tau\}}\) are assumed sufficiently large compared to those in \(\hat{H}_{\Pi}\) so that it is \(\hat{H}_{\Pi}\) that may be thought of as a perturbation of \(\hat{H}_{\{\tau\}}\) and not the converse), and (iii) for five of the ten symmetry classes, there can be protected gapless edge states because of locality and symmetry. Steps (ii) and (iii) for each of the five symmetry classes supporting gapless edge states is represented pictorially as is shown in the fourth column of Table I. In each symmetry class, topologically trivial states that do not support protected gapless edge states in the tenfold classification can be constructed by gapping all states in each individual wire from the array as shown in the fifth column of Table I.

### C. Symmetry class A

Topological insulators in symmetry class A can be realized without any symmetry aside from the \(U(1)\) charge conservation. The wire construction starts from wires supporting spinless fermions, so that the minimal choice

\[ M = 2 \]

only counts left- and right-moving degrees of freedom. The \(K\)-matrix reads

\[ K := \text{diag}(+1, -1). \]  

The entry \(+1\) of the \(K\)-matrix corresponds to a right mover. It is depicted by the symbol \(\circ\) in the first line of Table I. The entry \(-1\) of the \(K\)-matrix corresponds to a left mover. It is depicted by the symbol \(\bigcirc\) in the first line of Table I. The operation for reversal of time in any one of the \(N\) wires is represented by [one verifies that Eq. (4.2e) holds]

\[ P_{\Theta} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_{\Theta} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]  

We define \(\hat{H}_{\{\tau\}}\) by choosing \((N - 1)\) scattering vectors, whereby, for any \(j = 1, \cdots, (N - 1),\)

\[ \tau(\gamma) := \delta_{i,j} \delta_{\gamma,2} - \delta_{i-1,j} \delta_{\gamma,1} \]  

with \(i = 1, \cdots, N\) and \(\gamma = 1, 2.\) In other words,

\[ \tau(j) := (0, 0) \cdots (0, 0, 0) \quad \text{and} \quad \tau^T(j) := (0, 0, 0, 0, 0, 0, 0, 0, 0) \]  

for \(j = 1, \cdots, N - 1.\) Intent on helping with the interpretation of the tunneling vectors, we use the \(|\gamma\rangle\)’s in Eq. (4.7b) to compartmentalize the elements within a given wire. Henceforth, there are \(M = 2\) vector components within each pair of \(|\gamma\rangle\)’s that encode the \(M = 2\) degrees of freedom within a given wire. The \(j\)th scattering vector (4.7b) labels a one-body interaction in the fermion representation that fulfills Eq. (2.5f) and breaks TRS, since the scattering vector \((0, +1)^T\) is mapped into the scattering vector \((+1, 0)^T\) by the permutation \(P_{\Theta}\) that represents reversal of time in a wire by exchanging right- with left-movers. For any \(j = 1, \cdots, (N - 1),\) we also introduce the amplitude

\[ h_{\tau(j)}(x) \geq 0 \]
and the phase
\[ \alpha_{\tau^{(j)}}(x) \in \mathbb{R} \quad (4.7d) \]
according to Eqs. (4.3d) and (4.3e), respectively. The choices for the amplitude (4.7c) and the phase (4.7d) are arbitrary.

One verifies that all \((N - 1)\) scattering vectors (4.7a) satisfy the Haldane criterion (3.2), i.e.,
\[ \mathcal{T}^{(i)T} \mathcal{K} \mathcal{T}^{(j)} = 0, \quad i, j = 1, \ldots, N - 1. \quad (4.8) \]
Correspondingly, the term \(\hat{H}_{\{\tau\}}\) gaps out \(2(N - 1)\) of the \(2N\) gapless modes of \(\hat{H}_V\). Two modes of opposite chirality that propagate along the first and last wire, respectively, remain in the low energy sector of the theory. These edge states are localized on wire \(i = 1\) and \(i = N\), respectively, for their overlaps with the gapped states from the bulk decay exponentially fast as a function of the distance away from the first and end wires. The energy splitting between the edge state localized on wire \(i = 1\) and the one localized on wire \(i = N\) that is brought about by the bulk states vanishes exponentially fast with increasing \(N\). Two gapless edge states with opposite chiralities emerge in the two-dimensional limit \(N \to \infty\).

At energies much lower than the bulk gap, the effective \(\mathcal{K}\)-matrix for the edge modes is
\[ \mathcal{K}_{\text{eff}} := \text{diag}(+1, 0|0, 0| \cdots |0, 0|0, -1). \quad (4.9) \]
Here, \(\mathcal{K}_{\text{eff}}\) follows from replacing the entries in the \(2N \times 2N\) \(\mathcal{K}\) matrix for all gapped modes by 0. The pictorial representation of the topological phase in class A with one chiral edge state per end wire through the wire construction is shown on the first row and fourth column of Table I. The generalization to an arbitrary number \(n\) of gapless edge states sharing a given chirality on the first wire that is opposite to that of the last wire is the following. We enlarge \(M = 2\) to \(M = 2n\) by making \(n\) identical copies of the model depicted in the first row and fourth column of Table I. The stability of the \(n\) chiral gapless edge states in wire 1 and wire \(N\) is guaranteed because back-scattering among these gapless edges state is not allowed kinematically within wire 1 or within wire \(N\), while back-scattering across the bulk is exponentially suppressed for \(N\) large by locality and the gap in the bulk. The number of robust gapless edge states of a given chirality is thus integer. This is the reason why \(Z\) is found in the third column on the first line of Table I.

D. Symmetry class AII

Topological insulators in symmetry class AII can be realized by demanding that \(U(1)\) charge conservation holds and that TRS with \(\hat{\Theta}^2 = -1\) holds. The wire construction starts from wires supporting spin-1/2 fermions because \(\hat{\Theta}^2 = -1\), so that the minimal choice \(M = 4\) counts two pairs of Kramers degenerate left- and right-moving degrees of freedom carrying opposite spin projections on the spin quantization axis, i.e., two pairs of Kramers degenerate helical modes. The \(K\)-matrix reads
\[ K := \text{diag}(+1, -1, -1, +1). \quad (4.10a) \]
The entries in the \(K\)-matrix represent, from left to right, a right-moving particle with spin up, a left-moving particle with spin down, a left-moving particle with spin up, and a right-moving particle with spin down. The operation for reversal of time in any one of the \(N\) wires is represented by \(\{\text{one verifies that Eq. (4.2e) holds}\}\)
\[ P_\Theta := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad I_\Theta := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \quad (4.10b) \]
We define \(\hat{H}_V\) by choosing any symmetric \(4 \times 4\) matrix \(V\) that obeys
\[ V = P_\Theta V P_\Theta^{-1}. \quad (4.11a) \]
We define \(\hat{H}_{\{\tau\}}\) by choosing \(2(N - 1)\) scattering vectors as follows. For any \(j = 1, \ldots, (N - 1)\), we introduce the pair of scattering vectors
\[ \mathcal{T}^{(j)}_{\text{so}} := (0, 0, 0, 0|\cdots|0, 0, +1, 0| -1, 0, 0, 0| \cdots |0, 0, 0, 0)^T \quad (4.11b) \]
and
\[ \mathcal{T}^{(j)}_{\text{so}} := -P_\Theta \mathcal{T}^{(j)}_{\text{so}}. \quad (4.11c) \]
The scattering vector (4.11b) labels a one-body interaction in the fermion representation that fulfills Eq. (2.5f). It scatters a left mover with spin up from wire \(j\) into a right mover with spin up in wire \(j + 1\). For any \(j = 1, \ldots, (N - 1)\), we also introduce the pair of amplitudes
\[ h_{\tau^{(j)}}(x) = h_{\tau^{(j)}}(x) \geq 0 \quad (4.11d) \]
and the pair of phases
\[ \alpha_{\tau^{(j)}}(x) = \alpha_{\tau^{(j)}}(x) \in \mathbb{R} \quad (4.11e) \]
according to Eqs. (4.3d) and (4.3e), respectively. The choices for the amplitude (4.11d) and the phase (4.11e) are arbitrary. In particular the amplitude (4.11d) can be chosen to be sufficiently large so that it is \(\hat{H}_V\) that may be thought of as a perturbation of \(\hat{H}_{\{\tau\}}\), and not the converse. The subscript \(\text{SO}\) refers to the intrinsic spin-orbit coupling. The rational for using it shall be shortly explained.

One verifies that all \(2(N - 1)\) scattering vectors (4.11a) and (4.11b) satisfy the Haldane criterion (3.2), i.e.,
\[ \mathcal{T}^{(i)T}_{\text{so}} \mathcal{K} \mathcal{T}^{(j)}_{\text{so}} = \mathcal{T}^{(i)T}_{\text{so}} \mathcal{K} \mathcal{T}^{(j)}_{\text{so}} = \mathcal{T}^{(i)T}_{\text{so}} \mathcal{K} \mathcal{T}^{(j)}_{\text{so}} = 0, \quad (4.12) \]
for \( i, j = 1, \cdots, N - 1 \). Correspondingly, the term \( \hat{H}_{(TSO)} \) gaps out \( 4(N - 1) \) of the \( 4N \) gapless modes of \( \hat{H}_V \). Two pairs of Kramers degenerate helical edge states that propagate along the first and last wire, respectively, remain in the low energy sector of the theory. These edge states are localized on wire \( i = 1 \) and \( i = N \), respectively, for their overlaps with the gaped states from the bulk decay exponentially fast as a function of the distance away from the first and end wires. The energy splitting between the edge state localized on wire \( i = 1 \) and wire \( i = N \) brought about by the bulk states vanishes exponentially fast with increasing \( N \). Two pairs of gapless Kramers degenerate helical edge states emerge in the two-dimensional limit \( N \to \infty \).

At energies much lower than the bulk gap, the effective \( K \)-matrix for the two pairs of helical edge modes is

\[
\mathcal{K} = \text{diag}(+1, -1, 0, 0)|0, 0, 0, 0| \cdots |0, 0, 0, 0|0, 0, -1, +1). \tag{4.13}
\]

Here, \( \mathcal{K} \) follows from replacing the entries in the \( 4N \times 4N \) \( K \) matrix for all gaped modes by 0. We are going to show that the effective scattering vector

\[
\mathcal{T}_{\text{eff}} := (+1, -1, 0, 0|0, 0, 0, 0| \cdots |0, 0, 0, 0|0, 0, -1, +1) \tag{4.14}
\]

with the potential to gap out the pair of Kramers degenerate helical edge modes on wire \( i = 1 \) since it fulfills the Haldane criterion (3.2), is not allowed by TRS.\(^{40}\) On the one hand, \( \mathcal{T}_{\text{eff}} \) maps to itself under reversal of time,

\[
\mathcal{T}_{\text{eff}} = -\mathcal{P}_\Theta \mathcal{T}_{\text{eff}}. \tag{4.15}
\]

On the other hand,

\[
\mathcal{T}_{\text{eff}}^\dagger \mathcal{P}_\Theta \mathcal{T}_{\text{eff}} = -1. \tag{4.16}
\]

Therefore, the condition (4.3c) for \( \mathcal{T}_{\text{eff}} \) to be a TRS perturbation is not met, for the phase \( \alpha_{\mathcal{T}_{\text{eff}}}(x) \) associated to \( \mathcal{T}_{\text{eff}} \) then obeys

\[
\alpha_{\mathcal{T}_{\text{eff}}}(x) = \alpha_{\mathcal{T}_{\text{eff}}}(x) - \pi, \tag{4.17}
\]

a condition that cannot be satisfied.

Had we imposed a TRS with \( \Theta = +1 \) instead of \( \Theta = -1 \) as is suited for the symmetry class AI that describes spinless fermions with TRS, we would only need to replace \( I_{\Theta} \) in Eq. (4.10b) by the null vector. If so, the scattering vector (4.14) is compatible with TRS since the condition (4.3e) for TRS then becomes

\[
\alpha_{\mathcal{T}_{\text{eff}}}(x) = \alpha_{\mathcal{T}_{\text{eff}}}(x), \tag{4.18}
\]

instead of Eq. (4.17). This is the reason why symmetry class AI is always topologically trivial in two-dimensional space from the point of view of the wire construction.

To address the question of what happens if we change \( M = 4 \) to \( M = 4n \) with \( n \) any strictly positive integer in each wire from the array, we consider, without loss of generality as we shall see, the case of \( n = 2 \). To this end, it suffices to repeat all the steps that lead to Eq. (4.14), except for the change

\[
\mathcal{K}_{\text{eff}} := \text{diag}(+1, -1, 0, 0; +1, -1, 0, 0|0, 0, 0, 0| \cdots |0, 0, 0, 0|0, 0, 0, 0, 0, 0, -1, +1; 0, 0, -1, +1). \tag{4.19}
\]

One verifies that the scattering vectors

\[
\mathcal{T}'_{\text{eff}} := (+1, 0, 0, 0|0, 0, 0, 0|0, 0, 0, 0|0, 0, -1, 0, 0, 0| \cdots |0, 0, 0, 0|0, 0, 0, 0, 0) \tag{4.20}
\]

and

\[
\mathcal{T}''_{\text{eff}} := (0, -1, 0, 0; +1, 0, 0|0, 0, 0, 0|0, 0, 0, 0|0, 0, 0, 0|0, 0, 0, 0| \cdots |0, 0, 0, 0) \tag{4.21}
\]

are compatible with the condition that TRS holds in that the pair is a closed set under reversal of time,

\[
\mathcal{T}'_{\text{eff}} = -\mathcal{P}_\Theta \mathcal{T}''_{\text{eff}}. \tag{4.22}
\]

One verifies that these scattering vectors fulfill the Haldane criterion (3.2). Consequently, inclusion in \( \hat{H}_{(TSO)} \) of the two cosine potentials with \( \mathcal{T}'_{\text{eff}} \) and \( \mathcal{T}''_{\text{eff}} \) entering in their arguments, respectively, gaps out the pair of Kramers degenerate helical modes on wire \( i = 1 \). The same treatment of the wire \( i = N \) leads to the conclusion that TRS does not protect the gapless pairs of Kramers degenerate edge states from perturbations when \( n = 2 \). The generalization to \( M = 4n \) channels is that it is only when \( n \) is odd that a pair of Kramers degenerate helical edge modes is robust to the most generic \( \hat{H}_{(TSO)} \) of the form depicted in the fourth column on line 3 of Table I. Since it is the parity of \( n \) in the number \( M = 4n \) of channels per wire that matters for the stability of the Kramers degenerate helical edge states, we use the group of two integers \( \mathbb{Z}_2 \) under addition modulo 2 in the third column on line 3 of Table I.

If we were to impose conservation of the projection of the spin-1/2 quantum number on the quantization axis, we
must then preclude from all scattering processes by which a spin is flipped. In particular, the scattering vectors (4.20) and (4.21) are not admissible anymore. By imposing the $U(1)$ residual symmetry of the full $SU(2)$ symmetry group for a spin-1/2 degree of freedom, we recover the group of integers $\mathbb{Z}$ under the addition that encodes the topological stability in the quantum spin Hall effect (QSHE).

We close the discussion of the symmetry class AII by justifying the interpretation of the index SO as an abbreviation for the intrinsic spin-orbit coupling. To this end, we introduce a set of $(N - 1)$ pairs of scattering vectors

$$\mathcal{T}_R^{(j)} := (0, 0, 0, 0 | \cdots | 0, +1, 0, 0) - 1, 0, 0, 0 | \cdots | 0, 0, 0, 0)^T$$

(4.23a)

and

$$\overline{\mathcal{T}}_R^{(j)} := -\mathcal{P}_R \mathcal{T}_R^{(j)}$$

(4.23b)

for $j = 1, \cdots, N - 1$. The scattering vector (4.23a) labels a one-body interaction in the fermion representation that fulfills Eq. (2.5f). The index $R$ is an acronym for Rashba as it describes a backward scattering process by which a left mover with spin down from wire $j$ is scattered into a right mover with spin up on wire $j + 1$ and conversely. For any $j = 1, \cdots, (N - 1)$, we also introduce the pair of amplitudes

$$h_{\mathcal{T}_R^{(j)}}(x) = h_{\overline{\mathcal{T}}_R^{(j)}}(x) \geq 0$$

(4.23c)

and the pair of phases

$$\alpha_{\mathcal{T}_R^{(j)}}(x) = \alpha_{\overline{\mathcal{T}}_R^{(j)}}(x) + \pi \in \mathbb{R}$$

(4.23d)

according to Eqs. (4.3d) and (4.3e), respectively. In contrast to the intrinsic spin-orbit scattering vectors, the Rashba scattering vectors (4.23a) fail to meet the Hal-dane criterion (3.2) as

$$\mathcal{T}_R^{(j)} K \mathcal{T}_R^{(j+1)} = -1, \quad j = 1, \cdots, N - 1.$$  

(4.24)

Hence, the Rashba scattering processes fail to open a gap in the bulk, as is expected of a Rashba coupling in a two-dimensional electron gas. On the other hand, the intrinsic spin-orbit coupling can lead to a phase with a gap in the bulk that supports the spin quantum Hall effect in a two-dimensional electron gas.

### E. Symmetry class D

The simplest example among the topological superconductors can be found in the symmetry class D that is defined by the presence of a spectral PHS with $\tilde{\Pi}^2 = +1$ and the absence of TRS. We note that some physical systems with a non-interacting mean-field description that falls in this symmetry class, such as a two-dimensional chiral $p$-wave superconductor, do not obey a particle-hole symmetry when generic interactions are added on top of a mean-field Hamiltonian. While symmetry class D imposes a $U(1) \times \Pi$ symmetry, a generic interacting chiral $p$-wave superconductor is protected by the $\mathbb{Z}_2$ fermion parity symmetry alone. In promoting the symmetry class D (and all other systems with PHS) from the tenfold way to interacting LRE phases, we thus obtain LRE phases which should not be thought of as generic, but rather as very special, interacting superconductors.

The wire construction starts from identical wires supporting Bogoliubov-de-Gennes quasiparticles that, for our purposes, we may think of as right- and left-moving spinless fermions each of which carry a particle or a hole label, i.e., $M = 4$. The $K$-matrix reads

$$K := \text{diag}(+1, -1, -1, +1).$$

(4.25a)

The entries in the $K$-matrix represent, from left to right, a right-moving particle, a left-moving particle, a left-moving hole, and a right-moving hole. The operation for the exchange of particles and holes in any one of the $N$ wires is represented by [one verifies that Eq. (4.2) holds]

$$P_\Pi := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad I_\Pi := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$  

(4.25b)

We define $\hat{H}_V$ by choosing any symmetric $4 \times 4$ matrix $V$ that obeys

$$V = -P_\Pi V P_\Pi^{-1}.$$  

(4.25c)

We define $\hat{H}_\mathcal{T}$ by choosing $2(N - 1)$ scattering vectors as follows. For any $j = 1, \cdots, (N - 1)$, we introduce the pair of scattering vectors

$$\mathcal{T}^{(j)} := (0, 0, 0, 0 | \cdots | 0, +1, 0, 0) - 1, 0, 0, 0 | \cdots | 0, 0, 0, 0)^T$$

(4.26a)

and

$$\overline{\mathcal{T}}^{(j)} := \mathcal{P}_\Pi \mathcal{T}^{(j)}.$$  

(4.26b)

The scattering vector (4.26a) labels a one-body interaction in the fermion representation that fulfills Eq. (2.5f). It promotes a hole-like left mover from wire $j$ to a particle-like right mover to wire $j + 1$ and conversely. For any $j = 1, \cdots, (N - 1)$, we also introduce the pair of amplitudes

$$h_{\mathcal{T}^{(j)}}(x) = h_{\overline{\mathcal{T}}^{(j)}}(x) \geq 0$$

(4.26c)

and the pair of phases

$$\alpha_{\mathcal{T}^{(j)}}(x) = \alpha_{\overline{\mathcal{T}}^{(j)}}(x) + \pi \in \mathbb{R}$$

(4.26d)

according to Eqs. (4.4d) and (4.4e), respectively. The choices for the amplitude (4.26c) and the phase (4.26d) are arbitrary. In particular the amplitude (4.26c) can be chosen to be sufficiently large so that it is $\hat{H}_V$ that may
be thought of as a perturbation of $\hat{H}_{\{T_{0}\}}$ and not the converse.

One verifies that all $2(N-1)$ scattering vectors (4.25c) and (4.26a) satisfy the Haldane criterion (3.2), i.e.,

$$T^{(i)^T}K^{(j)} = T^{(i)^T}K^{(j)} = T^{(i)^T}K^{(j)} = 0,$$  

(4.27)

for $i, j = 1, \cdots, N - 1$. Correspondingly, the term $\hat{H}_{\{T\}}$ gaps out $4(N - 1)$ of the $4N$ gapless modes of $\hat{H}_V$. Two pairs of particle-like and hole-like edge states that propagate along wire $i = 1$ with a given chirality and wire $i = N$ with the opposite chirality, respectively, remain in the low energy sector of the theory. These two pairs of particle-like and hole-like chiral edge states are localized on wire $i = 1$ and $i = N$, respectively, for their overlaps with the gaped states from the bulk exponentially fast as a function of the distance away from the first and end wires. The energy splitting between these pairs of particle-like and hole-like chiral edge states localized on wire $i = 1$ and wire $i = N$ brought about by the bulk states vanishes exponentially fast with increasing $N$. Two pairs of gapless chiral particle-like and hole-like edge states emerge in the two-dimensional limit $N \to \infty$.

At energies much lower than the bulk gap, the effective $K$-matrix for the two pairs of particle-like and hole-like chiral edge modes is

$$K_{\text{eff}} := \text{diag} (+1, 0, 0, +1|0, 0, 0, 0|0, 0, 0, -1, 0).$$  

(4.28)

Here, $K_{\text{eff}}$ follows from replacing the entries in the $4N \times 4N K$ matrix for all gaped modes by 0. The effective scattering vector

$$T_{\text{eff}} := (+1, 0, 0, 0, -1|0, 0, 0, 0| \cdots)^T,$$  

(4.29)

with the potential to gap out the pair of particle-like and hole-like chiral edge modes on wire $i = 1$ is not allowed, for it does not fulfill the Haldane criterion (3.2). Hence, the pair of emergent gapless particle-like and hole-like chiral edge modes is robust to perturbations that preserve the spectral PHS.

The pictorial representation of the topological phase in class D with one pair of particle-like and hole-like chiral edge modes per end wire through the wire construction is shown on the fifth row of Table I. The generalization to an arbitrary number $n$ of gapless pairs of particle-like and hole-like chiral edge modes sharing a given chirality on the first wire that is opposite to that of the last wire is the following. We enlarge $M = 4$ to $M = 4n$ by making $n$ identical copies of the model depicted on the fifth row of Table I. The stability of the $n$ pairs of particle-like and hole-like chiral edge modes in wire 1 and wire $N$ is guaranteed because scattering among these gapless edge states is not allowed by the Haldane criterion (3.2) within wire 1 or within wire $N$, while back-scattering across the bulk is exponentially suppressed for $N$ large by locality and the gap in the bulk. The number of robust gapless pairs of particle-like and hole-like chiral edge states of a given chirality is thus integer. This is the reason why the group of integers $\mathbb{Z}$ is found in the third column on the fifth line of Table I.

**F. Symmetry classes DIII and C**

The remaining two topological nontrivial superconducting classes DIII (TRS with $\tilde{\Theta}^2 = -1$ and spectral PHS with $\tilde{\Pi}^2 = +1$) and C (spectral PHS with $\tilde{\Pi}^2 = -1$) involve spin-1/2 fermions. Each wire thus features no less than $M = 8$ internal degrees of freedom corresponding to the spin-1/2, chirality, and particle/hole indices. The construction is very similar to the cases we already presented. Below we summarize the main issues, and relegate details to the appendices A and B.

For class DIII we need to supply (1) the $K$-matrix, (2) the pair of commuting permutation matrices $P_\Theta$ and $P_\Pi$ and their associated vectors $I_\Theta$ and $I_\Pi$, respectively, (3) the set of $4(N - 1)$ tunneling vectors

$$\left\{ T^{(j)}, P_\Theta T^{(j)}, P_\Pi T^{(j)}, P_\Pi P_\Theta T^{(j)} \right\}, \quad j = 1, \cdots, N,$$  

(4.30)

that satisfy Haldane criterion (3.2) and are needed to gap the bulk, as well as (4) the effective representations of (1) and (2) on the remaining low-energy edge modes.

For class C we need to supply (1) the $K$-matrix, (2) the permutation matrices $P_\Pi$ and its associated vectors $I_\Pi$, (3) the set of $4(N - 1)$ tunneling matrices

$$\left\{ T^{(j)}, T_2^{(j)}, T_3^{(j)}, T_4^{(j)} \right\}, \quad j = 1, \cdots, N,$$  

(4.31)

that satisfy the Haldane criterion (3.2) and are needed to gap the bulk, as well as (4) the effective representations of (1) and (2) on the remaining low-energy edge modes.

These data are represented pictorially in the fourth row of Table I.

**G. Summary**

We have provided an explicit construction by way of an array of wires supporting fermions that realizes all five insulating and superconducting topological phases of matter with a nondegenerate ground state in two-dimensional space according to the tenfold classification of band insulators and superconductors. The topological protection of edge modes in the bosonic formulation follows from imposing the Haldane criterion (3.2) along with the appropriate symmetry constraints. This construction accommodates non-interacting electrons, as well as a class of interactions within each given wire. Thus far we only considered tunneling processes of the one-body type. In the next section we shall extend the wire construction to allow many-body tunneling processes that delivers fractionalized phases with degenerate ground states.
V. FRACTIONALIZED PHASES

The power of the wire construction goes much beyond what we have used in Sec. IV to reproduce the classification of the SRE phases. In this section we describe how to construct models for interacting phases of matter with intrinsic topological order and fractionalized excitations by relaxing the condition on the tunnelings between wires that they be of the one-body type. While these phases are more complex, the principles for constructing the models and proving the stability of edge modes remain the same: All allowed tunneling vectors have to obey the Haldane criterion (3.2) and the respective symmetries.

A. Symmetry class A: Fractional quantum Hall states

First, we review the models of quantum wires that are topologically equivalent to the Laughlin state in the FQHE, following the construction in Ref. 31 for Abelian fractional quantum Hall states. Here, we want to emphasize that the choice of scattering vectors is determined by the Haldane criterion (3.2) and at the same time prepare the grounds for the construction of fractional topological insulators with TRS in Sec. V B.

We want to construct the fermionic Laughlin series of states indexed by the positive odd integer \( m \). By the same method, other fractional quantum Hall phases from the Abelian hierarchy could be constructed. The elementary degrees of freedom in each wire are spinless right- and left-moving fermions with the \( K \)-matrix

\[
K = \text{diag}(+1, -1), \quad (5.1a)
\]

as is done in Eq. (4.6a). Reversal of time is defined through \( P_\Theta \) and \( I_\Theta \) given in Eq (4.6b). Instead of Eq (4.7), the scattering vectors that describe the interactions between the wires are now defined by

\[
T^{(j)} := \left( \begin{array}{c} 0, 0 \mid \cdots \mid 0, 0 \mid \frac{m + 1}{2}, \frac{-m - 1}{2} \mid \frac{m - 1}{2}, \frac{-m + 1}{2} \mid 0, 0 \mid \cdots \mid 0, 0 \end{array} \right)^T, \quad (5.1b)
\]

for any \( j = 1, \cdots, N - 1 \) [see Table I for an illustration of the scattering process].

For any \( j = 1, \cdots, N - 1 \), the scattering (tunneling) vectors (5.1b) preserve the conservation of the total fermion number in that

\[
\sum_{a=1}^{MN} T^{(j)}_a = 0, \quad (5.2)
\]

and they encode a tunneling interaction of order \( m \) in that

\[
\frac{1}{2} \sum_{a=1}^{MN} |T^{(j)}_a| = m. \quad (5.3)
\]

As a set, all tunneling interactions satisfy the Haldane criterion (3.2), for

\[
T^{(i)T} K T^{(j)} = 0, \quad i, j = 1, \cdots, N - 1. \quad (5.4)
\]

We note that the choice of tunneling vector in Eq. (5.1b) is unique if one insists on charge conservation, compliance with the Haldane criterion (3.2), and only includes scattering between neighboring wires.

The bare counting of tunneling vectors shows that the wire model gaps out all but two modes. However, we still have to convince ourselves that the remaining two modes (i) live on the edge, (ii) cannot be gaped out by other (local) scattering vectors and (iii) are made out of fractionalized quasiparticles.

To address (i) and (ii), we note that the remaining two modes can be gaped out by a unique charge-conserving scattering vector that satisfies the Haldane criterion (3.2) with all existing scatterings, namely

\[
T^{(0)} := \left( \begin{array}{c} \frac{m + 1}{2}, \frac{-m - 1}{2} \mid 0, 0 \mid \cdots \mid 0, 0 \mid \frac{m - 1}{2}, \frac{-m + 1}{2} \end{array} \right)^T. \quad (5.5)
\]

Connecting the opposite ends of the array of wires through the tunneling \( T^{(0)} \) is not an admissible perturbation, for it violates locality in the two-dimensional thermodynamic limit \( N \to \infty \). Had we chosen periodic boundary conditions corresponding to a cylinder geometry (i.e., a tube as in Fig. 1) by which the first and last wire are nearest neighbors, \( T^{(0)} \) would be admissible. Hence, the gapless nature of the remaining modes
when open boundary conditions are chosen depends on the boundary conditions. These gapless modes have support near the boundary only and are topologically protected.

To understand point (iii) just above Eq. (5.5), we employ the following linear transformation on the bosonic field variables

$$\tilde{\Phi}(x) = W \Phi(x), \quad (5.6a)$$

where $W$ is a $MN \times MN$ block-diagonal matrix with integer entries. Its $M \times M$ blocks are given by

$$W := \frac{1}{2} \left( \begin{array}{cc} 1 - m & 1 + m \\ 1 + m & 1 - m \end{array} \right). \quad (5.6b)$$

The matrix $W/\sqrt{m}$ belongs to the group $O(1,1)$ of linear transformations with determinant $-1$ that leaves the bilinear form represented by $K$ invariant, for

$$\det W = -m. \quad (5.7)$$

Correspondingly, the commutation relations for the fields $\tilde{\Phi}(x)$ are given by Eq. (2.4b), provided we replace the matrix $K$ in Eq. (2.4b) with the matrix

$$\tilde{K} := W^T K W = \left( \begin{array}{cc} -m & 0 \\ 0 & +m \end{array} \right). \quad (5.8)$$

Moreover, the linear transformation (5.6b) changes the compactification radius of the new field $\tilde{\Phi}(x)$ relative to the compactification radius of the old field $\Phi(x)$ accordingly. Finally, for the arguments $\mathcal{T}^T \tilde{K} \tilde{\Phi}$ of the cosine terms in $\tilde{H}_{K}$ to remain form invariant, i.e.,

$$\mathcal{T}^T \tilde{K} \tilde{\Phi} \rightarrow \mathcal{T}^T \tilde{K} \tilde{\Phi}, \quad (5.9)$$

the scattering vectors (5.4) must obey the transformation law

$$\tilde{\mathcal{T}}^{(j)} := W^{-1} \mathcal{T}^{(j)} = (0, 0 | \cdots | 0, 0 | 0, 0, +1 | 1, 0, 0, 0 | \cdots | 0, 0, 0, 0)^T, \quad (5.10)$$

where $W := I_N \otimes W$ and $j = 1, \ldots, N - 1$.

In view of Eqs. (5.8) and (5.10), the remaining effective edge theory is described by

$$\tilde{\mathcal{K}}_{\text{eff}} = \text{diag}(-m, 0 | 0, 0 | \cdots | 0, 0, 0, +m). \quad (5.11)$$

This is a chiral theory at each edge that cannot be gaped by local perturbations. Equation (5.11) is precisely the edge theory for anyons with statistical angle $1/m$ and charge $e/m$,\textsuperscript{28} where $e$ is the charge of the original fermions.

**B. Symmetry Class AII: Fractional topological insulators**

Having understood how fractionalized quasiparticles emerge out of a wire construction, it is imperative to ask what other phases can be obtained when symmetries are imposed on the topologically ordered phase. Such symmetry enriched topological phases have been classified by methods of group cohomology.\textsuperscript{29} Here, we shall exemplify for the case of TRS with $\tilde{\Theta}^2 = -1$ how the wire construction can be used to build up an intuition for these phases and to study the stability of their edge theory.

The elementary degrees of freedom in each wire are spin-$1/2$ right- and left-moving fermions with the $K$-matrix

$$K := \text{diag}(+1, -1, -1, +1), \quad (5.12a)$$

as is done in Eq. (4.10a). Reversal of time is defined through $P_{\Theta}$ and $I_{\Theta}$ given in Eq (4.10b). Instead of Eq (4.11a), the scattering vectors that describe the interactions between the wires are now defined by

$$\mathcal{T}^{(j)} := \left( \begin{array}{c} 0, 0, 0 | \cdots | 0, 0, 0 | -\frac{m - 1}{2}, 0, +\frac{m + 1}{2}, 0 \\ -\frac{m - 1}{2}, 0, +\frac{m + 1}{2}, 0 | 0, 0, 0, 0 | \cdots | 0, 0, 0, 0 \end{array} \right)^T \quad (5.12b)$$

and

$$\tilde{\mathcal{T}}^{(j)} := -\mathcal{P}_{\Theta} \mathcal{T}^{(j)}, \quad (5.12c)$$

for any $j = 1, \cdots, N - 1$ and $m$ a positive odd integer.

For any $j = 1, \cdots, N - 1$, the scattering (tunneling) vectors (5.12b) preserve conservation of the total fermion number in that

$$\sum_{a=1}^{MN} \mathcal{T}^{(j)} = 0, \quad (5.13)$$
and they encode a tunneling interaction of order \( m \) in that
\[
\frac{1}{2} \sum_{a=1}^{MN} |\mathcal{T}^{(j)}| = m. \quad (5.14)
\]

They also satisfy the Haldane criterion (3.2) as a set [see Table I for an illustration of the scattering process].

Applying the transformation (5.6) with
\[
W := \frac{1}{2} \left( \begin{array}{ccc}
1-m & 0 & m+1 \\
0 & 1-m & 0 \\
m+1 & 0 & 1-m
\end{array} \right), \quad (5.15)
\]
to the bosonic fields, leaves the representation of time-reversal invariant
\[
W^{-1} P_{\Theta} W = P_{\Theta}, \quad (5.16)
\]
while casting the theory in a new form with the transformed \( \tilde{K} \)-matrix given by
\[
\tilde{K} = \text{diag}(-m, +m, +m, -m), \quad (5.17)
\]
and, for any \( j = 1, \ldots, N-1 \) of the first \( N-1 \) wires, with the transformed pair of scattering vector \( (\tilde{T}^{(j)}, \tilde{\mathcal{T}}^{(j)}) \) given by
\[
\tilde{T}^{(j)} = (0, 0, 0, 0| \cdots | 0, 0, 0, 0) + 1, 0, 0, 0| 0, 0, -1, 0| 0, 0, 0, 0 \cdots | 0, 0, 0, 0) \quad (5.18)
\]
and
\[
\tilde{\mathcal{T}}^{(j)} = (0, 0, 0, 0| \cdots | 0, 0, 0, 0| -1, 0| 0, 0, 0, 0 \cdots | 0, 0, 0, 0) \quad (5.19)
\]
When these scattering vectors have gaped out all modes in the bulk, the effective edge theory is described by
\[
K_{\text{eff}} = \text{diag}(0, 0, +m, -m| 0, 0, 0, 0 \cdots | 0, 0, 0, 0) - m, +m, 0, 0). \quad (5.20)
\]

This effective \( K \)-matrix describes a single Kramers degenerate pair of \( 1/m \) anyons propagating along the first wire and another single Kramers degenerate pair of \( 1/m \) anyons propagating along the last wire. Their robustness to local perturbations is guaranteed by TRS.

We will now show that Eq. (5.20) exhausts the edge theory of all possible abelian topological phases which are protected by TRS with \( \Theta^2 = -1 \) alone.

It was shown in Refs. 38 and 42 that the edge theory of an Abelian and TRS fractional topological insulator, consisting of Kramers degenerate pairs of edge modes, supports at most one Kramers degenerate pair of delocalized quasiparticles that are stable against disorder. This Kramers degenerate pair is characterized by the linear combination of bosonic fields
\[
\hat{\varphi}(x) := \mathcal{T}^{T} K' \hat{\Phi}(x) \quad (5.21)
\]
and its time-reversed partner
\[
\hat{\phi}(x) := \mathcal{T}^{T} K' \hat{\Phi}(x), \quad (5.22)
\]
where the tunneling vector \( \mathcal{T} \) was constructed from the microscopic information from the theory in Ref. 38 and \( K' \) is the \( K \)-matrix of a TRS bulk Chern-Simons theory from the theory in Ref. 38. [In other words, the theory encoded by \( K' \) has nothing to do a priori with the array of quantum wires defined by Eq. (5.12).] The Kramers degenerate pair of modes \( (\hat{\varphi}, \hat{\phi}) \) is stable against TRS perturbations supported on a single edge if and only if
\[
\frac{1}{2} |\mathcal{T}^{T} Q| \quad (5.23)
\]
is an odd number. Here, \( Q \) is the charge vector that determines the coupling of the different modes to the electromagnetic field. Provided \( (\hat{\varphi}, \hat{\phi}) \) is stable, its equal-time commutation relations follow from Eq. (2.4b) as
\[
[\hat{\varphi}(x), \hat{\varphi}(x')] = -i \pi \left( \mathcal{T}^{T} K' \mathcal{T} \text{sgn}(x - x') + \mathcal{T}^{T} L \mathcal{T} \right), \quad (5.24a)
\]
\[
[\hat{\phi}(x), \hat{\phi}(x')] = -i \pi \left( -\mathcal{T}^{T} K' \mathcal{T} \text{sgn}(x - x') + \mathcal{T}^{T} L \mathcal{T} \right), \quad (5.24b)
\]
where we used that \( K' \) anticommutes with \( P_{\Theta} \) according to Eq. (4.3c). By the same token, we can show that the fields \( \hat{\varphi} \) and \( \hat{\phi} \) commute, for
\[
\mathcal{T}^{T} K' \mathcal{T} = \mathcal{T}^{T} P_{\Theta} K' \mathcal{T} = -\mathcal{T}^{T} K' \mathcal{T} = 0. \quad (5.25)
\]
We conclude that the effective edge theory for any Abelian TRS fractional topological insulator build from fermions has the effective form of one Kramers degenerate pairs
\[
K_{\text{eff}} = \left( \begin{array}{cc}
\mathcal{T}^{T} K' \mathcal{T} & 0 \\
0 & -\mathcal{T}^{T} K' \mathcal{T}
\end{array} \right). \quad (5.26)
\]
and is thus entirely defined by the single integer
\[ m := T^T K' T. \]  \hfill (5.27)

With the scattering vectors (5.12c) we have given an explicit wire construction for each of these cases, thus exhausting all possible stable edge theories for Abelian fractional topological insulators.

For each positive odd integer \( m \), we can thus say that the fractionalized mode has a \( \mathbb{Z}_2 \) character: It can have either one or none stable Kramers degenerate pair of \( m \) quasiparticles.

### C. Symmetry Class D: Parafermion superconductors

In Sec. V B we have imposed TRS on the wire construction of fractional quantum Hall states and obtained the fractional topological insulator in symmetry class AII. In complete analogy, we can impose PHS with \( \tilde{K}^2 = +1 \) on the wire construction of a fractional quantum Hall state, thereby promoting it to symmetry class D. Physically, there follows a model for a superconductor with parafermion excitations, which are “fractionalized” Majorana fermions or Bogoliubov quasiparticles. As in the classification of non-interacting insulators, we treat the Bogoliubov quasiparticles under bosonization as if they were Dirac fermions. The parafermion phase is driven by interactions among the Bogoliubov quasiparticles. While we demand that these interactions fully respect the PHS for the purpose of this wire construction, we emphasize that generic interactions are not compatible with PHS. With this important caveat in mind, we forge ahead with the definition of the model.

The elementary degrees of freedom in each wire are spinless right- and left-moving fermions and holes as was defined for symmetry class D in Eqs. (4.25a)-(4.25c). We construct the fractional topological insulator using the set of PHS pairs of scattering vectors \( T^{(j)} \) and \( \tilde{T}^{(j)} := \hat{P}_\Pi T^{(j)} \), for \( j = 1, \ldots, N - 1 \) with \( T^{(j)} \) as defined in Eq. (5.12b) and the PHS as defined in Eq. (4.25b).

Performing the transformation (5.15), yields the effective edge theory

\[ \tilde{K}_{\text{eff}} = \text{diag} (0, +m, +m, 0|0, 0, 0, 0| \cdots |0, 0, 0, 0| - m, 0, 0, -m). \]  \hfill (5.28)

We note that the PHS is invariant under the transformation (5.15), for

\[ W^{-1} P_\Pi W = P_\Pi. \]  \hfill (5.29)

Thus, Eq. (5.28) describes a particle-hole pair that corresponds to a chiral parafermionic edge mode on either side of the two-dimensional array of quantum wires. The definite chirality is an important difference to the case of the fractional \( \mathbb{Z}_2 \) topological insulator discussed in Sec. V B. It guarantees that any integer number \( n \in \mathbb{Z} \) of copies of the edge theory (5.28) is stable, for no tunneling vector that acts locally on one edge can satisfy the Haldane criterion (3.2). For each \( m \), we can thus say that the parafermion mode has a \( \mathbb{Z} \) character, as does the SRE phase in symmetry class D.

### D. Symmetry classes DIII and C: More parafermions

Needed are the many-body tunneling matrices for class DIII and C. We refer the reader to the appendices A and B for their definitions. For class DIII, the edge excitations (and bulk quasiparticles) of the phase are TRS parafermions that have been discussed in one-dimensional systems.\(^{43}\)

\[ \tilde{K}_{\text{eff}} = \text{diag} (0, +m, +m, 0|0, 0, 0, 0| \cdots |0, 0, 0, 0| - m, 0, 0, -m). \]  \hfill (5.28)

### VI. DISCUSSION

In this work, we have developed a wire construction to build models of short-range entangled and long-range entangled topological phases in two spatial dimensions, so as to yield immediate information about the topological stability of their edge modes. As such, we have promoted the periodic table of integer topological phases to its fractional counterpart. The following paradigms were applied.

1. Each Luttinger liquid wire describes (spinfull or spinless) electrons. We rely on a bosonized description.
2. Back-scattering and short-range interactions within and in-between wires are added. Modes are gaped out if these terms acquire a finite expectation value.
3. A mutual compatibility condition, the Haldane criterion, is imposed among the terms that acquire an expectation value. It is an incarnation of the statement that the operators have to commute if they are to be replaced simultaneously by their expectation value.
4. A set of discrete and local symmetries are imposed on all terms in the Hamiltonian. When modes become massive, they may not break these symmetries.
5. We do not study the renormalization group flow of the interaction and back-scattering terms, but analyze the model in a strong-coupling limit.

Using this strategy, the following directions present themselves for future work. First, all examples that we...
have given include a global $U(1)$ symmetry that encodes the conservation of the total quasiparticle number, besides the TRS, PHS, and SLS of the respective symmetry classes. Topological phases in which this symmetry is broken down to a fermion parity ($\mathbb{Z}_2$) or more generally a $\mathbb{Z}_n$ symmetry, $n = 1, 2, \cdots$, could also be studied using the wire construction. In this context, the inclusion of LRE phases with non-Abelian quasiparticles in our formalism, with an appropriate extension of Haldane’s criterion, is a crucial challenge.

Second, we can impose on our wire construction additional symmetries such as a non-local inversion symmetry or such as a residual $U(1)$ spin symmetry.

Third, our construction can be extended to topological phases of systems that have bosons as their elementary degrees of freedom. For bosons, no analogue of the tenfold way exists to provide guidance. However, several works are dedicated to the classification of SRE and LRE phases of bosons, which might provide a helpful starting point.\(^{25}\)

Fourth, extensions to higher dimensions could be considered. This would, however, entail leaving the comfort zone of one-dimensional bosonization, with a necessary generalization of the Haldane criterion in a layer construction.

**ACKNOWLEDGMENTS**

This work was supported by the European Research Council through the grant TOPOLECTRICS, ERC-StG-Thomale-336012, and by DOE Grant DEF-06ER46316.

---

**Appendix A: Symmetry class C**

Class C is defined on line 9 of Table I by the operator $\hat{\Pi}$ for the PHS obeying $\hat{\Pi}^2 = -1$ with neither TRS nor chiral symmetry present (as is implied by $\hat{\Theta}^2 = 0$ and $\hat{C}^2 = 0$). In physical terms, class C describes a generic superconductor for which full spin $SU(2)$ symmetry is retained but TRS is broken. The only difference to the case of class D considered in the main text is that the number of degrees of freedom is doubled. We postulate that under PHS the following transformation rules hold

$$
\begin{align*}
    b^\dag_{\uparrow,R} &\rightarrow -b^\dag_{\downarrow,R}, & b^\dag_{\downarrow,R} &\rightarrow +b^\dag_{\uparrow,R}, \\
\end{align*}
$$

(A1)

for the creation operators of Bogoliubov-deGennes quasiparticles that are right (R) movers at the Fermi energy and carry the spin quantum numbers $\uparrow, \downarrow$. We apply the same transformation law to the creation operators of Bogoliubov-deGennes quasiparticles that are left (L) movers at the Fermi energy and carry the spin quantum numbers $\uparrow, \downarrow$.

We consider identical wires with quasiparticles of type 1 and 2, “particles” and “holes”, as well as left- and right-moving degrees of freedom. For any given wire with the basis $( b^\dag_{\uparrow,L}, b^\dag_{\downarrow,L}, b^\dag_{\uparrow,R}, b^\dag_{\downarrow,R}, b_{\uparrow,L}, b_{\downarrow,L}, b_{\uparrow,R}, b_{\downarrow,R} )$, the $K$-matrix reads

$$
K := \text{diag}(+, +, -, -, -, +, +),
$$

(A2a)

where PHS has the representation

$$
P_{\Pi} := 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
I_{\Pi} := 
\begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
1 \\
0 \\
1 \\
0
\end{pmatrix}.
$$

(A2b)
1. SRE phase

To complete the definition of an array of quantum wires realizing a SRE phase in the symmetry class C, we specify the $4(N - 1)$ tunneling vectors

$$\mathcal{T}^{(j)}_{1,\text{SRE}} := (0, 0, 0, 0, 0, 0, 0, 0, 0, m + 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

$$\mathcal{T}^{(j)}_{2,\text{SRE}} := (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, m + 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

$$\mathcal{T}^{(j)}_{3,\text{SRE}} := (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, m + 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

$$\mathcal{T}^{(j)}_{4,\text{SRE}} := (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, m + 1, 0, -1, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

for $j = 1, \cdots, N - 1$. These tunneling vectors gap out all modes in the bulk which comply both with PHS and with the Haldane criterion (3.2). Extending this construction to any integer number of layers yields the $\mathbb{Z}$ classification of class C.

2. LRE phase

To complete the definition of an array of quantum wires realizing a LRE phase in the symmetry class C, we specify the $4(N - 1)$ tunneling vectors

$$\mathcal{T}^{(j)}_{1,\text{LRE}} := (0, 0, 0, 0, 0, 0, 0, 0, 0, m + 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

$$\mathcal{T}^{(j)}_{2,\text{LRE}} := (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, m + 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

$$\mathcal{T}^{(j)}_{3,\text{LRE}} := (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, m + 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

$$\mathcal{T}^{(j)}_{4,\text{LRE}} := (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, m + 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

for $j = 1, \cdots, N - 1$. These tunneling vectors gap out all modes in the bulk which comply both with PHS and with the Haldane criterion (3.2). One verifies that there exists a linear transformation with integer entries $W$ and $|\det W| = m^2$ such that

$$\mathcal{T}^{(j)}_{l,\text{SRE}} = W^{-1} \mathcal{T}^{(j)}_{l,\text{LRE}}, \quad j = 1, \cdots, N - 1, \quad l = 1, \cdots, 4. \quad (A6)$$

The $K$-matrix transforms according to Eq. (5.8), leaving the effective edge theory

$$\tilde{K}_{\text{eff}} = \text{diag}(-m, -m, 0, 0, 0, 0, -m, -m|0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \cdots (2)$$

As with the SRE phase of symmetry class C, this is a completely chiral theory and no back-scattering mechanism could gap out modes by the Haldane criterion (3.2). Extending this construction to any integer number of layers yields a $\mathbb{Z}$ classification of the LRE phase in symmetry class C for every positive odd integer $m$.

Appendix B: Symmetry class DIII

Class DIII is defined on line 4 of Table I by the operator $\tilde{\Pi}$ for the PHS obeying $\tilde{\Pi}^2 = +1$ and with the TRS $\tilde{\Theta}^2 = -1$. In physical terms, class DIII describes a generic superconductor for which full spin $SU(2)$ symmetry is
broken but TRS is retained. We use the same basis and $K$-matrix as specified for class C in Eq. (A2a). The PHS now has the representation

$$P_\Pi := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad I_\Pi := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{(B1a)}$$

while TRS is defined by

$$P_\Theta := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_\Theta := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad \text{(B1b)}$$

### 1. SRE phase

To complete the definition of an array of quantum wires realizing a SRE phase in the symmetry class DIII, we specify the $4(N-1)$ tunneling vectors

$$\mathcal{T}_{1,\text{SRE}}^{(j)} := (0, 0, 0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0, 0, 0), \quad \text{(B2a)}$$

$$\mathcal{T}_{2,\text{SRE}}^{(j)} := (0, 0, 0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0, 0, 0), \quad \text{(B2b)}$$

$$\mathcal{T}_{3,\text{SRE}}^{(j)} := (0, 0, 0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0, 0, 0), \quad \text{(B2c)}$$

$$\mathcal{T}_{4,\text{SRE}}^{(j)} := (0, 0, 0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0, 0, 0), \quad \text{(B2d)}$$

for $j = 1, \cdots, N-1$, which gap out all the bulk modes. Here, $\mathcal{T}_{1,\text{SRE}}^{(j)}$ and $\mathcal{T}_{2,\text{SRE}}^{(j)}$ as well as $\mathcal{T}_{3,\text{SRE}}^{(j)}$ and $\mathcal{T}_{4,\text{SRE}}^{(j)}$ are pairwise related by TRS, while $\mathcal{T}_{1,\text{SRE}}^{(j)}$ and $\mathcal{T}_{3,\text{SRE}}^{(j)}$ as well as $\mathcal{T}_{2,\text{SRE}}^{(j)}$ and $\mathcal{T}_{4,\text{SRE}}^{(j)}$ are pairwise related by PHS. The phases of the corresponding cosine terms in Hamiltonian (2.5d) can be defined in consistency with both TRS and PHS as

$$\alpha_{\mathcal{T}_{1}}^{(j)} = \alpha_{\mathcal{T}_{2}}^{(j)} = \alpha_{\mathcal{T}_{3}}^{(j)} = \alpha_{\mathcal{T}_{4}}^{(j)}. \quad \text{(B3)}$$

Let us now discuss the stability of the remaining edge theory in more detail. The effective edge theory that survives is described by

$$\mathcal{K}_{\text{edge}} = \text{diag}(+, 0, 0, 0, 0, 0, 0, 0) \cdots (0, 0, 0, 0, 0, 0, 0, 0), \quad \text{(B4)}$$

Equivalently, and more compactly we can consider a model with

$$\mathcal{K}_{\text{edge}} := \text{diag}(+, -, -, +). \quad \text{(B5)}$$

in which case the PHS reduces to

$$\mathcal{P}^{\text{edge}} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{I}^{\text{edge}} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad \text{(B6)}$$

while TRS reduces to

$$\mathcal{P}^{\text{edge}} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{I}^{\text{edge}} := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \quad \text{(B7)}$$
We have two options for a back-scattering process. For one, the scattering vector
\[ \mathcal{T} = (+, -, 0, 0) \] (B8)
is not allowed, for TRS would require
\[ \alpha_T = \alpha_{-P_\alpha T} + \pi = \alpha_T + \pi. \] (B9)
On the other hand, the scattering vector
\[ \mathcal{T} = (+, 0, -, 0) \] (B10)
together with its time-reversed and particle-hole reversed partner
\[ \mathcal{T}' = -P_\alpha T = P_{\Pi} T = (0, -, 0, +) \] (B11)
would require the phase locking
\[ \alpha_T = \alpha_{T'}, + \pi \] (B12)
to comply with TRS, while PHS requires
\[ \alpha_T = \alpha_{T'} + 2\pi. \] (B13)
We conclude that there is no possibility to localize the remaining edge modes with perturbations that comply with both TRS and PHS. If we considered two layers of this wire model, edge modes in both layers can be gaped out pairwise, similar to the case of class AII that we discussed in the main text. We conclude that the SRE phase of symmetry class DIII features a \( \mathbb{Z}_2 \) topological classification.

2. LRE phase

To complete the definition of an array of quantum wires realizing a LRE phase in the symmetry class DIII, we specify the \( 4(N - 1) \) tunneling vectors
\[ \mathcal{T}^{(j)}_{1, \text{LRE}} := (0, 0, 0, 0, 0, 0, 0, 0) \cdots |0, \frac{1 - m}{2}, 0, m + 1, 0, 0, 0, 0|0, -m + 1, 0, m - 1, 0, 0, 0, 0| \cdots |0, 0, 0, 0, 0, 0, 0, 0, \] (B14a)
\[ \mathcal{T}^{(j)}_{2, \text{LRE}} := (0, 0, 0, 0, 0, 0, 0, 0) \cdots | -m + 1, 0, m - 1, 0, 0, 0, 0|0, 1 - m, 0, -m + 1, 0, m + 1, 0, 0, 0, 0| \cdots |0, 0, 0, 0, 0, 0, 0, 0, \] (B14b)
\[ \mathcal{T}^{(j)}_{3, \text{LRE}} := (0, 0, 0, 0, 0, 0, 0, 0) \cdots |0, 0, 0, 0, 0, 0, 0, 0, |m + 1, 0, 0, 0, 0, m - 1, 0, -m + 1| \cdots |0, 0, 0, 0, 0, 0, 0, 0, \] (B14c)
\[ \mathcal{T}^{(j)}_{4, \text{LRE}} := (0, 0, 0, 0, 0, 0, 0, 0) \cdots |0, 0, 0, 0, 0, 0, 0, 0, |m - 1, 0, -m + 1, 0, m + 1, 0, 1 - m| \cdots |0, 0, 0, 0, 0, 0, 0, 0, \] (B14d)
for \( j = 1, \cdots, N - 1 \). These tunneling vectors gap out all modes in the bulk which comply both with PHS and with the Haldane criterion (3.2). One verifies that there exists a linear transformation with integer entires \( W \) and \( |\text{det } W| = m^2 \) such that
\[ \mathcal{T}^{(j)}_{1, \text{SRE}} = W^{-1} \mathcal{T}^{(j)}_{1, \text{LRE}}, \quad j = 1, \cdots, N - 1, \quad l = 1, \cdots, 4. \] (B15)
The \( K \)-matrix transforms according to Eq. (5.8), leaving the effective effective edge theory
\[ \mathcal{K}_{\text{eff}} = \text{diag} \left( -m, 0, 0, +m, 0, +m, -m, 0|0, 0, 0, 0, 0, 0, 0, 0| \cdots |0, 0, 0, 0, 0, 0, 0, 0, -m, +m, 0, +m, 0, 0, -m. \right) \] (B16)
As with the SRE phase of symmetry class DIII, this edge theory is protected by PHS and TRS. Two copies of (B16), however, can be fully gaped out while preserving PHS and TRS. This yields a \( \mathbb{Z}_2 \) classification of the LRE phase in symmetry class DIII for every positive odd integer \( m \).

---

1 C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (Sep 2005), http://link.aps.org/doi/10.1103/PhysRevLett.95.146802
2. L. Fidkowski and A. Kitaev, Phys. Rev. B 83, 075103 (Feb 2011), http://link.aps.org/doi/10.1103/PhysRevB.83.075103
3. A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B 83, 075102 (Feb 2011), http://link.aps.org/doi/10.1103/PhysRevB.83.075102
4. Z.-C. Gu and X.-G. Wen, ArXiv e-prints (Jan. 2012), arXiv:1201.2648 [cond-mat.str-el]
5. Y.-M. Lu and A. Vishwanath, Phys. Rev. B 86, 125119 (Sep 2012), http://link.aps.org/doi/10.1103/PhysRevB.86.125119
6. X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (Apr 2013), http://link.aps.org/doi/10.1103/PhysRevB.87.155114
7. Y.-M. Lu and A. Vishwanath, ArXiv e-prints (Feb. 2013), arXiv:1302.2634 [cond-mat.str-el]
8. X.-G. Wen, Int. J. Mod. Phys. B 55, 1641 (1991)
9. V. M. Yakovenko, Phys. Rev. B 43, 11353 (May 1991), http://link.aps.org/doi/10.1103/PhysRevB.43.11353
10. D.-H. Lee, Phys. Rev. B 50, 10788 (Oct 1994), http://link.aps.org/doi/10.1103/PhysRevB.50.10788
11. C. L. Kane, R. Mukhopadhyay, and T. C. Lubensky, Phys. Rev. Lett. 88, 036401 (Jan 2002), http://link.aps.org/doi/10.1103/PhysRevLett.88.036401
12. J. C. Y. Teo and C. L. Kane, Phys. Rev. B 89, 085101 (Feb 2014), http://link.aps.org/doi/10.1103/PhysRevB.89.085101
13. F. D. M. Haldane, Phys. Rev. Lett. 74, 2090 (Mar 1995), http://link.aps.org/doi/10.1103/PhysRevLett.74.2090
14. X. G. Wen, Phys. Rev. B 44, 2664 (Aug 1991), http://link.aps.org/doi/10.1103/PhysRevB.44.2664
15. S. Sachdev and N. Read, Int. J. Mod. Phys. B 05, 219 (January 1991), http://www.worldscientific.com/doi/10.1142/S0217979291000158
16. C. Mudry and E. Fradkin, Phys. Rev. B 49, 5200 (Feb 1994), http://link.aps.org/doi/10.1103/PhysRevB.49.5200
17. A spectral chiral symmetry is present if there exists a chiral operator $\tilde{C}$ that is unitary and anticommutes with the Hamiltonian. In a basis in which $\tilde{C}$ is strictly block off diagonal, $\tilde{C}$ reverses the chirality. This chirality is unrelated to the direction of propagation of left and right movers which is also called chirality in this paper.
18. T. Neupert, L. Santos, S. Ryu, C. Chamon, and C. Mudry, Phys. Rev. B 84, 165107 (Oct 2011), http://link.aps.org/doi/10.1103/PhysRevB.84.165107
19. F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (Oct 1988), http://link.aps.org/doi/10.1103/PhysRevLett.61.2015
20. Even integer multiples of $T_m$ would gap the edge states, but they must also be discarded as explained in Ref. 38.
21. R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (May 1983), http://link.aps.org/doi/10.1103/PhysRevLett.50.1395
22. M. Levin and A. Stern, Phys. Rev. Lett. 103, 196803 (Nov 2009), http://link.aps.org/doi/10.1103/PhysRevLett.103.196803
23. J. Klinovaja and D. Loss, ArXiv e-prints (Dec. 2013), arXiv:1312.1998 [cond-mat.mes-hall]