LOGICS OF VARIABLE INCLUSION AND THE LATTICE OF CONSEQUENCE RELATIONS

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Abstract. In this paper, firstly, we determine the number of sublogics of variable inclusion of an arbitrary finitary logic $\vdash$ with partition function. Then, we investigate their position into the lattice of consequence relations over the language of $\vdash$.

1. INTRODUCTION

The family of logics of variable inclusion splits into two subfamilies, namely logics of left variable inclusion and logics of right variable inclusion. More precisely, given a logic $\vdash$, the two sublogics that can be defined by means of a different variable inclusion principle are

$$
\Gamma \vdash^{l} \varphi \iff \text{there is } \Delta \subseteq \Gamma \text{ s.t. } \text{Var}(\Delta) \subseteq \text{Var}(\varphi) \text{ and } \Delta \vdash \varphi,
$$

and

$$
\Gamma \vdash^{r} \varphi \iff \left\{ \begin{array}{l}
\Gamma \vdash \varphi \text{ and } \text{Var}(\varphi) \subseteq \text{Var}(\Gamma) \text{ or } \\
\Sigma \subseteq \Gamma,
\end{array} \right.
$$

where $\Sigma$ is an antithorem of $\vdash$ (see Definition 1).

Here, the logic $\vdash^{l}$ denotes the left variable inclusion companion of $\vdash$, while $\vdash^{r}$ is its right variable inclusion counterpart. The best known examples of variable inclusion logics arise when $\vdash$ is considered to be classical logic. In this case, $\vdash^{l}$ is known as paraco{}nsistent weak Kleene logic (PWK for short) [17, 16] and $\vdash^{r}$ as Bochvar logic ($B_3$)[5, 17, 16]. These two logics are semantically defined on the base of the so-called weak Kleene tables

| $\wedge$ | 0 | n | 1 |
| $\vee$ | 0 | 0 | n |
| $\neg$ | 0 | n | 1 |

as follows:

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\begin{itemize}
  \item \(\langle \text{WK}, \{1\} \rangle = B_3\)
  \item \(\langle \text{WK}\{1, n\} \rangle = \text{PWK}\), where \text{WK} is the three elements algebra induced by the above tables.
\end{itemize}

Logics of variable inclusion have recently been influential in several research areas, including the philosophy of language [1], theories of truth [27] and, of course, logic [12, 26, 7, 8, 6]. On the logical side, the fact that PWK actually corresponds to the left variable inclusion companion of classical logic is shown in [11], while [6] contains an algebraic study of PWK with the tools of modern abstract algebraic logic (AAL). The work in [7], which also adopts the AAL framework, identifies a general method to turn a complete matrix semantics for an arbitrary logic \(\vdash\) into a complete matrix semantics for its left variable inclusion companion. A similar task is accomplished in [8] for finitary right variable inclusion logics.

Of course, nothing prevents from iterating the definitions of left and right variable inclusion logics. For instance, one can define the logic \(\vdash_{\text{lr}}\), that is the right variable inclusion companion of the left variable inclusion companion of \(\vdash\). The only known example of this kind is the logic \(\text{K}_n^w\), investigated in the very recent papers [28, 20]. In general, by looking at the above definitions, it is immediate to verify that each logic of variable inclusion of \(\vdash\) is a sublogic of \(\vdash\).

The general theory of closure operators states that, given a set \(A\), the set of all the structural closure operators on \(A\) can be equipped with a (complete) lattice structure. One of the outcomes of the pioneering work of [4] and of the more recent developments in abstract algebraic logic contained in [14, 3, 15] states that there is a bijective correspondence between logics in the language \(\mathcal{L}\) and structural closure operators over the set of formulas \(\text{Fm}_\mathcal{L}\) equipped with a monoid action (whose elements represents substitutions). This perspective highlights that the investigation of the lattice of logics over a fixed language \(\mathcal{L}\) is worth pursuing.

In [26], a first attempt to determine how \(B_3\) and PWK relates with other sublogics of CL is offered. However, a general and systematic method that determines how the logics of variable inclusion of \(\vdash\) fit into the lattice of logics over \(\mathcal{L}\) (with \(\vdash\) being a finitary logic over a fixed language \(\mathcal{L}\)) is still missing.

The main aim of this paper is to fill this gap, by solving the above mentioned problem in full generality. It will turn out that the number of sublogics of variable inclusion of a logic \(\vdash\) is no greater that 8 if \(\vdash\) possesses an antitheorem, and no greater that 5 otherwise. In the final section, we consider the example of classical logic, and we
describe in a transparent way the relations among its sublogics of variable inclusion. Remarkably, it turns out that only four of these eight logics have been considered in the literature until now.

2. Preliminaries

For standard background on closure operators and abstract algebraic logic we refer the reader respectively to [9, 2] and [4, 13, 14]. Unless stated otherwise, we work within a fixed but arbitrary algebraic language. We denote algebras by $A, B, C \ldots$ respectively with universes $A, B, C \ldots$

2.1. Abstract algebraic logic. Let $Fm$ be the algebra of formulas built up over a countably infinite set $\text{Var}$ of variables. Given a formula $\varphi \in Fm$, we denote by $\text{Var}(\varphi)$ the set of variables really occurring in $\varphi$. Similarly, given $\Gamma \subseteq Fm$, we set $\text{Var}(\Gamma) = \bigcup \{\text{Var}(\gamma) : \gamma \in \Gamma\}$.

A logic is a substitution invariant consequence relation $\vdash \subseteq P(Fm) \times Fm$ in the sense that for every substitution $\sigma: Fm \rightarrow Fm$,

$\if \Gamma \vdash \varphi, \then \sigma[\Gamma] \vdash \sigma(\varphi)$.

A logic $\vdash$ is finitary when the following holds for all $\Gamma \cup \varphi \subseteq Fm$:

$\Gamma \vdash \varphi \iff \exists \Delta \subseteq \Gamma \text{ s.t. } \Delta \text{ is finite and } \Delta \vdash \varphi$.

A matrix is a pair $\langle A, F \rangle$ where $A$ is an algebra and $F \subseteq A$. In this case, $A$ is called the algebraic reduct of the matrix $\langle A, F \rangle$. Every class of matrices $M$ induces a logic as follows:

$\Gamma \vdash_M \varphi \iff \text{for every } \langle A, F \rangle \in M \text{ and hom. } h: Fm \rightarrow A$, if $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$.

A logic $\vdash$ is complete w.r.t. a class of matrices $M$ when it coincides with $\vdash_M$.

A matrix $\langle A, F \rangle$ is a model of a logic $\vdash$ when

$\if \Gamma \vdash \varphi, \then \text{for every hom. } h: Fm \rightarrow A$, if $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$.

A set $F \subseteq A$ is a (deductive) filter of $\vdash$ on $A$, or simply a $\vdash$-filter, when the matrix $\langle A, F \rangle$ is a model of $\vdash$. We denote the class of matrix models of $\vdash$ as $\text{Mod}(\vdash)$.

The following definition originates in [18], but see also [10, 24]

Definition 1. A set of formulas $\Sigma$ in an antitheorem of a logic $\vdash$ if $\sigma[\Sigma] \vdash \varphi$ for every substitution $\sigma$ and formula $\varphi$. 
Example 2. For any formula \( \varphi \), the set \( \{ \lnot (\varphi \rightarrow \varphi) \} \) is an antitheorem of all superintuitionistic logics. Similarly, the sets \( \{ \varphi, \lnot \varphi \} \), \( \{ \varphi \land \lnot \varphi \} \) are antitheorems of all the expansions of Classical logic and Strong Kleene logic.

Remark 3. Observe that if \( \vdash \) has an antitheorem, then \( \vdash \) has an antitheorem only in variable \( x \). If, moreover, \( \vdash \) is finitary, then it has a finite antitheorem only in variable \( x \).

Given two logics \( \vdash, \vdash' \) in the same language, we say that \( \vdash' \) is a sublogic of \( \vdash \) (in symbols \( \vdash' \leq \vdash \)) if for every \( \Gamma \cup \{ \varphi \} \subseteq Fm \),

\[ \Gamma \vdash \varphi \quad \text{entails} \quad \Gamma \vdash \varphi. \]

Let \( \mathcal{L} \) be an algebraic language. The set \( \mathcal{L}^\vdash \) of all logics in the language \( \mathcal{L} \) forms a complete lattice (see [29] for details), where, given \( \vdash_i, i \in I \) logics over \( \mathcal{L} \), the operations are defined as follows

\[ \bigwedge_{i \in I} \vdash_i := \bigcap_{i \in I} \vdash_i \]
\[ \bigvee_{i \in I} \vdash_i := \bigcap \{ \vdash : \vdash \geq \vdash_i \text{ for every } i \in I \}. \]

An immediate consequence is that, given a logic \( \vdash \in \mathcal{L}^\vdash \), the set of sublogics of \( \vdash \) is a sublattice of \( \mathcal{L}^\vdash \). Given a logic \( \vdash \), we denote the set of its logics of variable inclusion by \( SV(\vdash) \).

2.2. Plonka sums. The main mathematical tool that allows for a systematic study of logics of variable inclusion is an algebraic construction coming from universal algebra, and more specifically from the study of regular varieties, i.e. varieties of algebras satisfying only equations \( \sigma \approx \delta \) in which \( \text{Var}(\sigma) = \text{Var}(\delta) \). Such construction, known as Plonka sums, originates in the late 1960’s from a series of papers published by the polish mathematician J.Plonka, who first provided a general representation theorem for regular varieties.

For standard information on Plonka sums we refer the reader to [22, 21, 23, 25]. A semilattice is an algebra \( A = \langle A, \lor \rangle \), where \( \lor \) is a binary commutative, associative and idempotent operation. Given a semilattice \( A \) and \( a, b \in A \), we set

\[ a \leq b \iff a \lor b = b. \]

It is easy to see that \( \leq \) is a partial order on \( A \).

Definition 4. A direct system of algebras consists in

(i) a semilattice \( I = \langle I, \lor \rangle \);
Moreover, $f_{ii}$ is the identity map for every $i \in I$, and if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$.

Let $X$ be a direct system of algebras as above. The Plonka sum of $X$, in symbols $P(X)$ or $P(A_i)_{i \in I}$, is the algebra defined as follows. The universe of $P(A_i)_{i \in I}$ is the union $\bigcup_{i \in I} A_i$. Moreover, for every $n$-ary basic operation $f$ and $a_1, \ldots, a_n \in \bigcup_{i \in I} A_i$, we set $f_{P(A_i)_{i \in I}}(a_1, \ldots, a_n) := f_{A_i}(f_{i_j}(a_1), \ldots, f_{i_j}(a_n))$ where $a_1 \in A_{i_1}, \ldots, a_j \in A_{i_j}$ and $j = i_1 \lor \cdots \lor i_n$.

Observe that if in the above display we replace $f$ by any complex formula $\varphi$ in $n$-variables, we still have that $\varphi_{P(A_i)_{i \in I}}(a_1, \ldots, a_n) = \varphi_{A_i}(f_{i_j}(a_1), \ldots, f_{i_j}(a_n))$.

The theory of Plonka sums is strictly related with a special kind of operation:

**Definition 5.** Let $A$ be an algebra of type $\nu$. A function $\cdot : A^2 \to A$ is a partition function in $A$ if the following conditions are satisfied for all $a, b, c \in A$, $a_1, \ldots, a_n \in A^n$ and for any operation $g \in \nu$ of arity $n \geq 1$.

- **P1.** $a \cdot a = a$
- **P2.** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- **P3.** $a \cdot (b \cdot c) = a \cdot (c \cdot b)$
- **P4.** $g(a_1, \ldots, a_n) \cdot b = g(a_1 \cdot b, \ldots, a_n \cdot b)$
- **P5.** $b \cdot g(a_1, \ldots, a_n) = b \cdot a_1 \cdots \cdot a_n$

The next result makes explicit the relation between Plonka sums and partition functions:

**Theorem 6.** [21, Thm. II] Let $A$ be an algebra of type $\nu$ with a partition function $\cdot$. The following conditions hold:

1. $A$ can be partitioned into $\{A_i : i \in I\}$ where any two elements $a, b \in A$ belong to the same component $A_i$ exactly when $a = a \cdot b$ and $b = b \cdot a$.

Moreover, every $A_i$ is the universe of a subalgebra $A_i$ of $A$.

2. The relation $\leq$ on $I$ given by the rule

\[ i \leq j \iff \text{there exist } a \in A_i, b \in A_j \text{ s.t. } b \cdot a = b \]

is a partial order and $\langle I, \leq \rangle$ is a semilattice.
(3) For all \(i, j \in I\) such that \(i \leq j\) and \(b \in A_j\), the map \(f_{ij}: A_i \to A_j\), defined by the rule \(f_{ij}(x) = x \cdot b\) is a homomorphism. The definition of \(f_{ij}\) is independent from the choice of \(b\), since \(a \cdot b = a \cdot c\), for all \(a \in A_i\) and \(c \in A_j\).

(4) \(Y = \langle\langle I, \leq\rangle, \{A_i\}_{i \in I}, \{f_{ij} : i \leq j\}\rangle\) is a direct system of algebras such that \(\mathcal{P}_1(Y) = A\).

It is worth remarking that the construction of Plonka sums preserves the validity of the so-called regular identities, i.e. identities of the form \(\varphi \approx \psi\) such that \(\text{Var}(\varphi) = \text{Var}(\psi)\).

3. Matrix models for logics of variable inclusion

In this section we review how to generalize the machinery of Plonka sums up to logical matrices, in order to provide a complete matrix semantics for an arbitrary, finitary logic of variable inclusion.

3.1. Left variable inclusion logics. The definition of direct system of algebras can be extended, as follows, to logical matrices:

**Definition 7.** (Essentially [7, Definition 8])

A \(l\)-direct system of matrices consists in

(i) a semilattice \(I = \langle I, \lor\rangle\);

(ii) a family of matrices \(\{\langle A_i, F_i \rangle\}_{i \in I}\) with disjoint universes;

(iii) a homomorphism \(f_{ij}: A_i \to A_j\) such that \(f_{ij}[F_i] \subseteq F_j\), for every \(i, j \in I\) such that \(i \leq j\)

such that \(f_{ii}\) is the identity map for every \(i \in I\), and if \(i \leq j \leq k\), then \(f_{ik} = f_{jk} \circ f_{ij}\).

Given a \(l\)-direct system of matrices \(X\) as above, we set

\[\mathcal{P}_1(X) := \langle\mathcal{P}_1(A_i)_{i \in I}, \bigcup_{i \in I} F_i\rangle.\]

The matrix \(\mathcal{P}_1(X)\) is the Plonka sum of the \(l\)-direct system of matrices \(X\). Given a class \(M\) of matrices, we denote by \(\mathcal{P}_1^l(M)\) the class of all Plonka sums of \(l\)-direct systems of matrices in \(M\).

The following Theorem establishes a completeness results for left variable inclusion logics.

**Theorem 8.** ([7, Theorem 14]) Let \(\vdash\) be a logic and \(M\) be a class of matrices containing \(\langle n, \{n\}\rangle\). If \(\vdash\) is complete w.r.t. \(M\), then \(\vdash^l\) is complete w.r.t. \(\mathcal{P}_1^l(M)\).
Example 9. As paradigmatic application of the above theorem, consider the case in which \( \vdash = \vdash_{\text{CL}} \). Consider the class of matrices \( \{ \langle B_2, 1 \rangle, \langle n, n \rangle \} \), where \( B_2 \) is the two-element Boolean algebra, and \( n \) is the trivial algebra. Theorem 8 states that the following matrix is complete for \( \vdash_{\text{PWK}} \):

\[
\begin{array}{c}
 n \\
 \downarrow \\
 1 \\
 \downarrow \\
 0
\end{array}
\]

The following definition plays a central role in the algebraic study of logics of left variable inclusion.

Definition 10. A logic \( \vdash \) has a \textit{l-partition function} if there is a formula \( x \cdot y \), in which the variables \( x \) and \( y \) really occur, such that \( x \vdash x \cdot y \) and the equations \( P1, \ldots, P5 \) in Definition 5 hold in \( \text{Alg}(\vdash) \) for every \( n \)-ary connective \( f \). In this case, \( x \cdot y \) is a \textit{l-partition function} for \( \vdash \).

Remark 11. Observe that logics with a \( l \)-partition function abound in the literature (see \([7]\)). For instance, the term \( x \wedge (x \vee y) \) is a \( l \)-partition function for the above mentioned logic PWK.

3.2. Right variable inclusion logics. \textit{Right variable inclusion logics}, also called \textit{containment logics} \([19]\), are defined as follows:

Definition 12. Let \( \vdash \) be a logic, \( \vdash^r \) is the logic defined as

\[
\Gamma \vdash^r \varphi \iff \left\{ \begin{array}{l}
\Gamma \vdash \varphi \text{ and } \text{Var}(\varphi) \subseteq \text{Var}(\Gamma) \text{ or } \\
\Sigma(x) \subseteq \Gamma
\end{array} \right.
\]

where \( \Sigma(x) \) is an antitheorem of \( \vdash \).

Another possible way of extending the notion of direct system of algebras (see Definition 4) to logical matrices is the following:

Definition 13. (Essentially \([8, \text{Definition 13}]\)

A \textit{r-direct system} of matrices consists in

(i) A semilattice \( I = \langle I, \vee \rangle \).

(ii) A family of matrices \( \{ \langle A_i, F_i \rangle : i \in I \} \) such that

\( I^+ := \{ i \in I : \langle A_i, F_i \rangle : F_i \neq \emptyset \} \) is a sub-semilattice of \( I \).
(iii) a homomorphism \( f_{ij} : A_i \rightarrow A_j \), for every \( i, j \in I \) such that \( i \leq j \), satisfying also that:
- \( f_{ii} \) is the identity map for every \( i \in I \);
- if \( i \leq j \leq k \), then \( f_{ik} = f_{jk} \circ f_{ij} \);
- if \( F_j \neq \emptyset \) then \( f_{ij}^{-1}[F_j] = F_i \).

Observe that the just defined notion of \( r \)-direct system differs from the definition of \( l \)-direct system above.

Given a \( r \)-direct system of matrices \( X \), a new matrix is defined as

\[
\mathcal{P}_1(X) := \langle \mathcal{P}_1(A_i)_{i \in I}, \bigcup_{i \in I} F_i \rangle.
\]

Given a class \( M \) of matrices, \( \mathcal{P}_1^r(M) \) will denote the class of all Plonka sums of \( r \)-directed systems of matrices in \( M \).

Given a logic \( \vdash \) which is complete with respect to a class \( M \) of matrices, we set \( M^\oplus := M \cup \langle A, \emptyset \rangle \), for any arbitrary \( A \in \text{Alg}(\vdash) \). The result which provides a complete matrix semantics for an arbitrary finitary right variable inclusion logic is the following

**Theorem 14.** ([8, Theorem 19]) Let \( \vdash \) be a logic which is complete w.r.t. a class of non trivial matrices \( M \). Then \( \vdash^r \) is complete w.r.t. \( \mathcal{P}_1^r(M^\oplus) \).

**Example 15.** Recall the situation of Example 9, and consider the case in which \( \vdash = \vdash_{\text{CL}} \). Consider the class of matrices \( \{ \langle B_2, 1 \rangle, \langle n, n \rangle \} \), where \( B_2 \) is the two-element Boolean algebra, and \( n \) is the trivial algebra. Theorem 14 states that the following matrix is complete for \( \vdash_{B_3} \)

\[
\begin{array}{c|c}
& n \\
\hline
1 & 0
\end{array}
\]

**Definition 16.** A logic \( \vdash \) has a \( r \)-partition function if there is a formula \( x \ast y \), in which the variables \( x \) and \( y \) really occur, such that

(i) \( x, y \vdash x \ast y \),
(ii) \( x \ast y \vdash x \),

and the term operation \( \ast \) is a partition function in every \( A \in \text{Alg}(\vdash) \).
Remark 17. Observe that, according with Theorem 8 and Theorem 14, given $M^\vdash$ a complete class of matrices for $\vdash$ containing $\langle n, n \rangle$ as only trivial matrix, it is always possible to obtain a complete class of non trivial matrices $M$ for $\vdash^l$, and a complete class of matrices $M^*$ for $\vdash^r$ containing $\langle n, n \rangle$ as only trivial matrix. Moreover, by applying again the mentioned theorems to $M$ and $M^*$ we have that $\mathcal{P}_l'(M \cup \langle n, \emptyset \rangle)$ is complete for $\vdash^lr$ while $\mathcal{P}_l(M^*)$ is complete for $\vdash^rl$.

In what follows, we write $\bullet$ to denote any (possibly empty) sequence of elements among $\{l, r\}$. So, $\vdash \bullet$ will denote an arbitrary logic obtained by replacing $\bullet$ with a sequence of elements among $\{l, r\}$. We denote the length of a sequence $\bullet$ as $L(\bullet)$. The reading of a sequence $\bullet$ is from left to right. So, if $\bullet = u_1 \ldots u_n$ with $(u_i \in \{l, r\}$ for $1 \leq i \leq n$) the logic $\vdash \bullet$ is the logic obtained by applying the definition of $u_m$ to the logic $\vdash u_1 \ldots u_{m-1}$ for every $1 \leq m \leq n$.

An immediate consequence of Remark 17 is that $\vdash \bullet^l \geq \vdash \bullet^lr$ and $\vdash \bullet^r \geq \vdash \bullet^rl$. This fact will be useful for the next sections. From now on, unless stated otherwise, we assume that $\vdash$ is a finitary logic, and that it possesses a binary term $\pi(x, y)$ that behaves as a $r$-partition function for $\vdash^r$ and as a $l$-partition function for $\vdash^l$. Observe that a great amount of logics share this feature. For instance, the term $\pi(x, y) = x \wedge (x \vee y)$, is a partition function for classical and intuitionistic logic, as well as for every substructural and modal logic.

4. Logics without antitheorems

In this section, given an antitheorem-free logic $\vdash$, we determine the number of the sublogics of variable inclusion of $\vdash$. Then, we investigate their position within the lattice of sublogics of $\vdash$.

Lemma 18. Let $\vdash$ be a logic without antitheorems. If $\Gamma \vdash^r \bullet^l \varphi$, then there exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$ and $\text{Var}(\Delta) = \text{Var}(\varphi)$.

Proof. By induction on the length of $\bullet^l$.

(B). If $L(\bullet^l) = 0$ the proof is immediate, so it remains to consider $L(\bullet^l) = 1$. There are cases: (a) $\bullet^l = l$ or (b): $\bullet^l = r$. if (a) then $\Gamma \vdash^l \varphi$ implies $\Gamma \vdash^r \varphi$, so there exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash^r \varphi$ and $\text{Var}(\Delta) \subseteq \text{Var}(\varphi)$. This implies $\Delta \vdash^l \varphi$ and $\text{Var}(\varphi) \subseteq \text{Var}(\Delta)$. Now, suppose $\Delta \not\vdash \varphi$. This implies $\Delta \not\vdash^r \varphi$ and $\Delta \not\vdash^l \varphi$, which is in contradiction with the fact that $\Delta \vdash^r \varphi$. So $\Delta \vdash \varphi$. The case of (b) is analogous.
(IND). Suppose the statement holds for \( L(\bullet') = n \) and consider \( L(\bullet') = n + 1 \). That is, \( \bullet' \) can be of the following forms: (a) \( \bullet' = s \cup \{ l \} \) with \( L(s) = n \), or (b) \( \bullet' = s \cup \{ r \} \) with \( L(s) = n \). In the case of (a), as \( \Gamma \vdash \bullet' \varphi \) we have that there exists \( \Delta \subseteq \Gamma \) such that \( \Delta \vdash \bullet' \varphi \) and \( \text{Var}(\Delta) \subseteq \text{Var}(\varphi) \). As \( L(s) = n \), by inductive hypothesis there exists \( \Sigma \subseteq \Delta \) such that \( \Sigma \vdash \varphi \) and \( \text{Var}(\Sigma) = \text{Var}(\varphi) \). Observing that \( \Sigma \subseteq \Delta \subseteq \Gamma \) we obtain our conclusion. The case for (b) can be proved with the same strategy.

The previous Lemma 18 has the following immediate consequences:

**Corollary 19.** Let \( \vdash \) be a logic without antitheorems. Then

(i) If \( \Gamma \vdash \bullet' \varphi \) then there exists \( \Delta \subseteq \Gamma \) such that \( \Delta \vdash \varphi \) and \( \text{Var}(\Delta) = \text{Var}(\varphi) \)

(ii) \( \vdash \text{rl} \leq \vdash \text{rl} \)

**Remark 20.** Observe that every logic \( \vdash \text{rl} \) such that \( l \in \bullet \) does not have antitheorems. Indeed, let \( \vdash \) be a logic and suppose \( \Sigma(x) \) is an antitheorem for \( \vdash \text{rl} \). Let \( X \) be a \( l \)-direct system of matrices such that

(i) \( l = \{ i, j \} \) with \( i \leq j \)

(ii) \( \langle A_i, F_i \rangle \in \text{Mod}(\vdash) \) be non trivial

(iii) \( \langle A_j, F_j \rangle \) such that \( A_j = n, F_j = n \)

(iv) \( f_{ij} : A_i \rightarrow A_j \) be the unique homomorphism

Then by Theorem 8 \( \mathcal{P}_l(X) = \langle A, F \rangle \) is a model of \( \vdash \text{rl} \). The fact that \( \Sigma(x) \) is an antitheorem for \( \vdash \text{rl} \) implies \( \Sigma(x) \vdash \text{rl} y \) for \( y \in \text{Var} \). Let now \( h : \text{Fm} \rightarrow \mathcal{P}_l(A_i)_{i \in l} \) be such that \( h(x) = n, h(y) = c \) with \( c \in A_i \setminus F_i \) (note that such \( c \) exists as \( A_i \neq F_i \)). Then clearly \( h(\Sigma(x)) \subseteq F \), while \( h(y) \notin F \), a contradiction.

The following theorem characterizes the relation among the sublogics of variable inclusion of an antitheorem-free logic \( \vdash \).

**Theorem 21.** Let \( \vdash \neq \vdash \text{rl} \), \( \vdash \text{rl} \) be a logic without antitheorems. The following relations hold:

(i) \( \vdash \text{rl} \nleq \vdash \text{rl} \) and \( \vdash \text{rl} \nleq \vdash \text{rl} \)

(ii) \( \vdash \text{rl} \cap \vdash \text{rl} = \vdash \text{rl} \leq \vdash \text{rl}, \vdash \text{rl} \)

(iii) \( \vdash \text{rl} = \vdash \text{rl} \leq \vdash \text{rl} \)

*Proof.* (i) it immediately follows by noticing that \( \pi(x, y) \vdash \text{rl} x \) while \( \pi(x, y) \not\vdash \text{rl} x \) and \( x \vdash \pi(x, y) \) while \( x \not\vdash \pi(x, y) \).

(ii) As a direct consequence of Remark 17 we have \( \vdash \text{rl} \leq \vdash \text{rl} \). We now prove using contraposition that \( \vdash \text{rl} \leq \vdash \text{rl} \). So assume \( \Gamma \vdash \text{rl} \varphi \).
There are cases, namely (1) $\Gamma \not\vdash \varphi$ or (2) $\text{Var}(\varphi) \not\subseteq \text{Var}(\Gamma)$. (1) immediately implies $\Gamma \not\vdash \varphi$, so $\Gamma \not\vdash_{lr} \varphi$. If it is case of (2), assume towards a contradiction that $\Gamma \vdash_{lr} \varphi$. This entails that $\Gamma \vdash \varphi$ and that $\text{Var}(\varphi) \subseteq \text{Var}(\Gamma)$, which is a contradiction. So $\Gamma \not\vdash_{lr} \varphi$.

Now, $\vdash_{l} \cap \vdash_{r} \leq \vdash_{lr}$ follows by noticing that in the lattice of sublogics of $\vdash$ it holds $\vdash_{l} \land \vdash_{r} \vdash_{l} \cap \vdash_{r}$, and so, as $\vdash_{lr} \leq \vdash_{r}, \vdash_{l}$ it follows $\vdash_{l} \cap \vdash_{r} \leq \vdash_{lr}$. For the other direction, assume $\Gamma \vdash_{lr} \varphi$. This entails $\Gamma \vdash \varphi$ with $\text{Var}(\varphi) \subseteq \text{Var}(\Gamma)$. Furthermore, as $\vdash_{l} \leq \vdash$, we have $\Gamma \vdash \varphi$ which finally entails $\Gamma \vdash \varphi$.

Moreover, the fact that $\pi(x,y) \vdash_{r} x$ while $\pi(x,y) \not\vdash_{lr} x$ proves the desired proper inequality.

(iii) That $\vdash_{rl} \leq \vdash_{r}$ follows again by remark 17. That $\vdash_{rl} \leq \vdash_{rl}^{\bullet}$ follows immediately from Corollary 19. Now we prove $\vdash_{rl}^{\bullet} \leq \vdash_{rl}$. To this end, assume $\Gamma \vdash_{rl}^{\bullet} \varphi$. By Lemma 18 we have that there exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$ and $\text{Var}(\Delta) = \text{Var}(\varphi)$. So, it follows $\Delta \vdash_{rl} \varphi$ and, by monotonicity, we obtain $\Gamma \vdash_{rl} \varphi$. The fact that $\vdash_{rl} \leq \vdash_{rl}^{\bullet}$ is a consequence of Corollary 19.

\[\text{Remark 22.} \text{ Observe that if a logic } \vdash \text{ has a theorem } \varphi, \text{ then } \vdash_{rl} \leq \vdash_{lr}. \text{ Indeed it is immediate to verify that } \pi(x,y) \vdash_{lr} \varphi(x) \text{ while } \pi(x,y) \not\vdash_{rl} \varphi(x).\]

The following corollary summarizes the results of this section.

\[\text{Corollary 23. Let } \vdash \text{ be a logic with a partition function and without antitheorems. Then the following holds:}\]

(i) There are at most four proper sublogics of variable inclusion of $\vdash$.

(ii) The sublattice of $\mathcal{L}$ generated by $\text{SV}(\vdash)$ has (at most) six elements, and it is represented by the following Figure 4.
5. Logics with antitheorems

We now turn to the case in which the logic $\vdash$ does possess an antithorem $\Sigma(x)$. In the next Theorem 24 we assume w.l.o.g. $\Sigma(x) = \{\epsilon_1(x), \ldots, \epsilon_n(x)\}$.

**Theorem 24.** Let $\vdash$ be a logic with antitheorems. Then the following relations hold

(i) $\vdash rl \not\subseteq \vdash lr$ and $\vdash lr \not\subseteq \vdash rl$

(ii) $\vdash l \cap \vdash r \not\subseteq \vdash lr$, $\vdash rl$

(iii) $\vdash rlr \leq \vdash rl$ and $\vdash lrl \leq \vdash lr$

(iv) $\vdash rlr \leq \vdash lr \cap \vdash rl$

(v) $\vdash lr \leq \vdash lrl = \vdash lrlr \leq \vdash lr$, $\vdash lr$

(vi) $\vdash rlr \bullet = \vdash lrl \bullet$

where $\bullet$ denotes any (possibly empty) sequence of elements among $\{l, r\}$.

**Proof.** (i). Firstly we show $\vdash rl \not\subseteq \vdash lr$. To this end it is immediate to verify that $\Sigma(x) \vdash rl \pi(x, y)$ while $\Sigma(x) \nvdash lr \pi(x, y)$.

For the other inequality, first observe that

$$\text{Var}(\pi(y, z)) \subseteq \text{Var}(y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z))$$

and, moreover

$$y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z) \vdash l \pi(y, z),$$
as \( y \vdash l \, \pi(y,z) \) and \( \{y\} \subseteq \{y, \pi(e_1(x),z), \ldots, \pi(e_n(x),z)\} \). So, this proves
\[
y, \pi(e_1(x),z), \ldots, \pi(e_n(x),z) \vdash^{lr} \pi(y,z).
\]
This, together with the fact that for no \( \Delta \subseteq \{y, \pi(e_1(x),z), \ldots, \pi(e_n(x),z)\} \) it holds \( \Delta \vdash r \, \pi(y,z) \) and \( \text{Var}(\Delta) \subseteq \{y,z\} \) shows
\[
y, \pi(e_1(x),z), \ldots, \pi(e_n(x),z) \not\vdash^{rl} \pi(y,z),
\]
as desired.

(ii). We first prove \( \vdash^{l} \cap \vdash^{r} \not\geq^{lr}, \vdash^{rl} \). Let \( \Gamma \vdash^{lr} \varphi \), then, as \( \vdash^{l} \) does not have antitheorems, it must be that \( \Gamma \vdash^{l} \varphi \) and \( \text{Var}(\varphi) \subseteq \text{Var}(\Gamma) \).
This, together with \( \vdash^{l} \vdash^{r} \) entails \( \Gamma \vdash \varphi \), so \( \Gamma \vdash^{r} \varphi \). So, \( \Gamma \vdash^{l} \cap \vdash^{r} \varphi \).
That \( \vdash^{l} \cap \vdash^{r} \not\geq^{lr} \vdash^{rl} \) is proved in the same way.

As the inferences described in point (i) hold both in \( \vdash^{l} \) and \( \vdash^{r} \), we obtain \( \vdash^{lr}, \vdash^{rl} \not\leq^{l} \cap \vdash^{r} \).

(iii). The fact that \( \vdash^{lr} \leq^{rl} \) and \( \vdash^{rl} \leq^{lr} \) is a direct consequence of Remark 17.

This, together with the fact that
\[
y, \pi(e_1(x),z), \ldots, \pi(e_n(x),z) \not\vdash^{rl} \pi(y,z)
\]
and \( \Sigma(x) \not\vdash^{rlr} \pi(x,y) \) proves the desired proper inequalities.

(iv). We first prove \( \vdash^{rl} \leq^{lr} \vdash^{l} \cap \vdash^{r} \). That \( \vdash^{rl} \leq^{lr} \vdash^{rl} \) follows, again by Remark 17. Consider \( \Gamma \vdash^{rlr} \varphi \), so, as \( \vdash^{rl} \) does not have antitheorems, \( \Gamma \vdash^{rl} \varphi \) with \( \text{Var}(\varphi) \subseteq \text{Var}(\Gamma) \). This entail that there exists \( \Delta \subseteq \Gamma \), \( \Delta \vdash^{r} \varphi \) and \( \text{Var}(\Delta) \subseteq \text{Var}(\varphi) \). As, \( \vdash^{l} \leq^{l} \) we obtain \( \Delta \vdash \varphi \), so \( \Delta \vdash^{l} \varphi \) which, by monotonicity entails \( \Gamma \vdash^{l} \varphi \). Recalling that \( \text{Var}(\varphi) \subseteq \text{Var}(\Gamma) \) we conclude \( \Gamma \vdash^{lr} \varphi \).

The proper inclusion is proved by noticing that \( \Sigma(x) \vdash^{lr} \pi(x,y) \), \( \Sigma(x) \vdash^{rl} \pi(x,y) \) while \( \Sigma(x) \not\vdash^{rlr} \pi(x,y) \).

(v). As by remark 20 \( \vdash^{l}, \vdash^{lr}, \vdash^{rl} \) are logics without antitheorems, then by Lemma 18 we know that \( \Gamma \vdash^{rlr} \varphi \) entails that there exists \( \Delta \subseteq \Gamma \), \( \Delta \vdash^{r} \varphi \) and \( \text{Var}(\varphi) = \text{Var}(\Delta) \) (the same holds for \( \vdash^{lr} \) and \( \vdash^{rlr} \)). As this immediately implies \( \Delta \vdash^{lr} \varphi \) and \( \Delta \vdash^{rlr} \varphi \), by monotonicity we conclude \( \Gamma \vdash^{lr} \varphi \) and \( \Gamma \vdash^{rlr} \varphi \), so \( \vdash^{lr} = \vdash^{rlr} = \vdash^{rl} \).

It only remains to prove that \( \vdash^{rlr} = \vdash^{rl} \). To this end, it suffices to note that \( \pi(y,z), \Sigma(x) \vdash^{lr} \pi(y,x) \) while \( \pi(y,z), \Sigma(x) \not\vdash^{rlr} \pi(y,x) \).

(vi). The equality \( \vdash^{rlr} = \vdash^{rl} \) is a straightforward application of Lemma 18, using the same strategy of point (v).

The following corollary summarizes the results of the section:
Corollary 25. Let $\vdash$ be a logic with a partition function and antitheorems, then

(i) there are at most 6 proper sublogics of variable inclusion of $\vdash$.
(ii) the sublattice of $\mathcal{L}^\vdash$ generated by $\mathcal{S}V(\vdash)$ has (at most) 11 elements, and it is represented by the following Figure 5.

Figure 5

6. The Sublogics of Variable Inclusion of Classical Logic

In this final section we briefly investigate the lattice of logics of variable inclusion of Classical logic. As we already noticed, only three proper sublogics of variable inclusion of classical logics have already been investigated, namely PWK, $B_3$ and $\vdash_{K_n^\omega}$.

Let now consider $\vdash_{CL}$, the logic defined by the matrix $\langle B_2, 1 \rangle$ where $B_2$ is the two-elements Boolean algebra. The matrices that defines the sublogics of variable inclusion of classical logic are as follows:
Letting $\pi(x, y) = x \land (x \lor y)$ it is not difficult to apply Theorem 24, and to observe that there are 6 proper sublogics of variable inclusion of CL. Moreover, the lattice generated by $SV(\vdash_{CL})$ is the following:

Figure 6
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