Simple evaluation of Casimir invariants in finite-dimensional Poisson systems

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Abstract

In this letter we present a procedure for the calculation of the Casimir functions of finite-dimensional Poisson systems which avoids the burden of solving a set of partial differential equations, as it is usually suggested in the literature. We show how a simple algebraic manipulation of the structure matrix reduces substantially the difficulty of the problem.

Keywords: Ordinary differential equations, Poisson systems, Casimir functions.

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1. Introduction

Poisson structures [1, 2] are ubiquitous in all fields of Mathematical Physics, from dynamical systems theory [3]–[9] to fluid dynamics [10, 11], magnetohydrodynamics [11]–[14], continuous media [14, 15], superconductivity [16], superfluidity [17], chromohydrodynamics [18], etc. The association of a Poisson structure to a given physical problem (which is still an open question [3, 4, 7, 8, 19]) opens the possibility of obtaining a wide range of information about the system, which may be in the form of perturbative solutions [20]–[22], nonlinear stability analysis through the energy-Casimir method [5, 23, 24, 25], bifurcation properties and characterization of chaotic behaviour [26]–[28], or integrability results [29, 30], to cite a few.

Mathematically, a finite-dimensional dynamical system is said to have a Poisson structure if it can be written in terms of a set of ODEs of the form:

$$\dot{x}_i = \sum_{j=1}^{n} J^{ij} \partial_j H, \quad i = 1, \ldots, n,$$

(1)

where $H(x)$, which is usually taken to be a time-independent first integral, plays the role of Hamiltonian function, and $J^{ij}(x)$ are the entries of a $n \times n$ skew-symmetric structure matrix $\mathcal{J}$ verifying the Jacobi equations:

$$\sum_{l=1}^{n} (J^{li} \partial_l J^{jk} + J^{lj} \partial_l J^{ki} + J^{lk} \partial_l J^{ij}) = 0$$

(2)

Here $\partial_l$ means $\partial/\partial x^l$ and indices $i, j, k$ run from 1 to $n$. Notice that, in particular, the rank of matrix $\mathcal{J}$ may not be maximum. For example, this is the case if the dimension of the system is odd, since the rank of a skew-symmetric matrix is always even. We shall denote in what follows the rank of matrix $\mathcal{J}$ by $2m$. It can be demonstrated [2] that whenever the Poisson structure is singular (i.e., when $2m < n$) there exist $n - 2m$ independent constants of motion known as Casimir (or distinguished) functions, which are present irrespective of the form of the Hamiltonian — in other words, they are completely determined by the structure matrix. From an operational point of view, the Casimir functions $C(x)$ are the solutions of the set of partial differential equations $\mathcal{J} \cdot \nabla C = 0$, or equivalently [2, 4, 6, 31]:

$$\sum_{j=1}^{n} J^{ij} \partial_j C = 0, \quad i = 1, \ldots, 2m$$

(3)
Here we have assumed without loss of generality that the first $2m$ rows of $J$ are the linearly independent ones, a convention that we shall follow throughout.

The characterization of the Casimir functions is of central importance in the analysis of Poisson structures. They do not only provide information about the structure of the solutions of the system (since they are first integrals, whose common level sets determine the symplectic foliation of the phase space). They constitute also the basis for establishing criteria for the nonlinear stability via the aforementioned energy-Casimir method; they allow the application of reduction of order procedures [2, 32]; and they can be used in the determination of time-independent solutions of nonlinear field equations [13].

Resorting to system (3) to obtain the Casimir functions is, in general, a rather inconvenient practice. We propose here a much simpler approach, which is developed from elementary linear algebraic considerations, which leads directly to a set of $n - 2m$ ordinary differential equations. The application of our method, which is completely systematic, will be seen to be always more efficient than the resolution of (3). Moreover, the convenience of the procedure, when compared with the traditional approach, is greater for increasing dimension of the structure matrix.

2. Description of the method

Let us consider (1), and a region of the $n$-dimensional phase-space in which the rank of $J$ is constant and equal to $2m < n$. If the $2m$ first rows of $J$ are the linearly independent ones, then there exists a set of $2m \times (n - 2m)$ functions $\gamma_i^k(x)$, where $i = 2m + 1, \ldots, n$ and $k = 1, \ldots, 2m$, such that

$$J^{ij} = \sum_{k=1}^{2m} \gamma_i^k J^{kj}, \quad j = 1, \ldots, n$$

(4)

The importance of the proportionality functions $\gamma_i^k$ was already noticed by Littlejohn [31]. Let us assume for the moment that they are known (their calculation is just a technical step for which we shall give a procedure later in this section). Then, the substitution of (4) into (1) gives immediately the following relations:

$$\dot{x}^i = \sum_{k=1}^{2m} \gamma_i^k \dot{x}^k, \quad i = 2m + 1, \ldots, n$$

(5)

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These equations reveal the structure which is present in the system due to the fact that the rank of matrix $J$ is not maximum, i.e., they express all interdependencies among the system variables induced by the existence of the Casimir functions. We have therefore obtained a set of $(n - 2m)$ ordinary differential equations for the Casimirs:

$$dx^i = \sum_{k=1}^{2m} \gamma^i_k dx^k, \quad i = 2m + 1, \ldots, n$$

(6)

Note that each of these equations is to be integrated separately. It is not difficult to prove that (6) do lead to the Casimir functions: let $C^{(i)}(x)$ be a solution of the $i$-th equation, where $2m + 1 \leq i \leq n$. Then there exists a function $\eta(x)$ such that:

$$dC^{(i)} = \eta(x) \left\{ \sum_{k=1}^{2m} \gamma^i_k dx^k - dx^i \right\}$$

(7)

The $j$-th component of the vector $J \cdot \nabla C^{(i)}$ will be:

$$(J \cdot \nabla C^{(i)})^j = \sum_{k=1}^{n} J^{jk} \partial_k C^{(i)} = \eta(x) \left\{ \sum_{k=1}^{2m} J^{jk} \gamma^i_k - J^{ji} \right\} = \eta(x) \left\{ J^{ij} - \sum_{k=1}^{2m} \gamma^i_k J^{kj} \right\} = 0, \quad \forall \ j = 1, \ldots, n$$

(8)

Here we have applied the original degeneracy relations (4). This demonstrates that the result of integrating each of the $n - 2m$ equations (6) is one family of Casimir functions of matrix $J$. We know, on the other hand, that there are $n - 2m$ functionally independent Casimirs. From (7) it can be easily shown that the solutions of two different equations of the set (6) are always functionally independent. Consequently, the integration of equations (6) produces all the Casimirs of the system.

We end this section by indicating how functions $\gamma^i_k$ can be calculated. To do so we proceed to write (4) in matrix form as:

$$(\tilde{J}_{2m})^T \cdot \Gamma = (\tilde{J}_{n-2m})^T$$

(9)

where $\tilde{J}_{2m}$ is the $2m \times n$ matrix composed by the first $2m$ rows of $J$, $\tilde{J}_{n-2m}$ is the $(n - 2m) \times n$ matrix composed by the last $(n - 2m)$ rows of $J$, and

$$\Gamma = \begin{pmatrix} \gamma_{1}^{2m+1} & \cdots & \gamma_{n}^{2m+1} \\ \vdots & \ddots & \vdots \\ \gamma_{2m}^{2m+1} & \cdots & \gamma_{2m}^{n} \end{pmatrix}$$

(10)
A rank analysis of the matrix equation (9) shows immediately that there always exists a unique matrix $\Gamma$ which is the solution. In fact, since $\tilde{J}_{2m}$ is a $2m \times n$ matrix, there are $(n - 2m)^2$ redundant equations in (9). If we assume again that these redundant equations are those corresponding to the last $(n - 2m)$ rows of $(\tilde{J}_{2m})^T$, we can write (9) in the nonredundant form:

$$(J_{2m})^T \cdot \Gamma = (J_{n-2m})^T$$

where

$$J_{2m} = \begin{pmatrix} J^{1,1} & \cdots & J^{1,2m} \\ \vdots & \ddots & \vdots \\ J^{2m,1} & \cdots & J^{2m,2m} \end{pmatrix}, \quad J_{n-2m} = \begin{pmatrix} J^{2m+1,1} & \cdots & J^{2m+1,2m} \\ \vdots & \ddots & \vdots \\ J^{n,1} & \cdots & J^{n,2m} \end{pmatrix}$$

Since now $J_{2m}$ is an invertible matrix, the solution is:

$$\Gamma = (J_{n-2m} \cdot J_{2m}^{-1})^T$$

To summarize, our approach to the determination of the Casimir functions proceeds in two steps: i) calculation of $\Gamma$ through (13); and ii) the integration of (6) —each equation leading to an independent family of Casimirs. We shall now illustrate the procedure by means of some examples.

### 3. Examples

#### (I) 3D Lotka-Volterra systems

Nutku has demonstrated [33] that the 3D Lotka-Volterra equations

$$\begin{align*}
\dot{x}^1 &= x^1(\lambda + cx^2 + x^3) \\
\dot{x}^2 &= x^2(\mu + x^1 + ax^3) \\
\dot{x}^3 &= x^3(\nu + bx^1 + x^2)
\end{align*}$$

are biHamiltonian when $abc = -1$ and $\nu = \mu b - \lambda ab$. In this case, the flow can be written as a Poisson system in two different ways:

$$\dot{x} = J_1 \cdot \nabla H_1 = J_2 \cdot \nabla H_2,$$

where:

$$J_1 = \begin{pmatrix} 0 & cx^1 x^2 & b c x^1 x^3 \\ -c x^1 x^2 & 0 & -x^2 x^3 \\ -b c x^1 x^3 & x^2 x^3 & 0 \end{pmatrix}$$
\[ J_2 = \begin{pmatrix} 0 & cx^1 x^2 (ax^3 + \mu) & cx^1 x^3 (x^2 + \nu) \\ -cx^1 x^2 (ax^3 + \mu) & 0 & cx^1 x^3 (x^2 + \nu) \\ -cx^1 x^3 (x^2 + \nu) & -x^1 x^2 x^3 & 0 \end{pmatrix} \] (17)

\[ H_1 = abx^1 + x^2 - ax^3 + \nu \ln x^2 - \mu \ln x^3 \] (18)

\[ H_2 = ab \ln x^1 - b \ln x^2 + \ln x^3 \] (19)

Since the rank of both \( J_1 \) and \( J_2 \) is 2 everywhere in the positive orthant, there is always one independent Casimir. We shall apply our method to both Poisson structures.

For \( J_1 \) we have, by simple inspection:

\[ \text{(row3)} = \frac{x^3}{cx^1} \text{(row1)} + \frac{bx^3}{x^2} \text{(row2)} \] (20)

In other words, \( \gamma_1^3 = x^3/cx^1 \) and \( \gamma_2^3 = bx^3/x^2 \). The equation we must solve is then:

\[ dx^3 = \frac{x^3}{cx^1} dx^1 + \frac{bx^3}{x^2} dx^2 \] (21)

The integration of this equation is immediate and gives \( ab \ln x^1 - b \ln x^2 + \ln x^3 = \text{constant} \), which is Nutku’s result. Since any function of a Casimir is also a Casimir, the general solution would be:

\[ C = \Psi \left[ ab \ln x^1 - b \ln x^2 + \ln x^3 \right] \] (22)

with \( \Psi \) a smooth one-variable function.

Similarly, for \( J_2 \) we see that:

\[ \text{(row3)} = -\frac{x^3}{c(ax^3 + \mu)} \text{(row1)} + \frac{x^3(x^2 + \nu)}{x^2(ax^3 + \mu)} \text{(row2)} \] (23)

Consequently, \( \gamma_1^3 = -x^3/(c(ax^3 + \mu)) \) and \( \gamma_2^3 = x^3(x^2 + \nu)/(x^2(ax^3 + \mu)) \). This implies that:

\[ dx^3 = -\frac{x^3}{c(ax^3 + \mu)} dx^1 + \frac{x^3(x^2 + \nu)}{x^2(ax^3 + \mu)} dx^2 \] (24)

After integration we arrive easily at \( abx^1 + x^2 - ax^3 + \nu \ln x^2 - \mu \ln x^3 = \text{constant} \), which is the solution. In general:

\[ C = \Psi \left[ abx^1 + x^2 - ax^3 + \nu \ln x^2 - \mu \ln x^3 \right] \] (25)

It is interesting to compare this procedure with the usual method of characteristics. We shall do it for \( J_2 \). Since rank(\( J_2 \)) is two in the domain of interest,
the third equation of the system $J_2 \cdot \nabla C = 0$ is a linear combination of the first and second ones, and can therefore be suppressed. The system of PDEs we have to solve in order to determine $C$ is then:

$$cx^1 x^2 (ax^3 + \mu) \frac{\partial C}{\partial x^2} + cx^1 x^3 (x^2 + \nu) \frac{\partial C}{\partial x^3} = 0 \quad (26)$$

$$-cx^1 x^2 (ax^3 + \mu) \frac{\partial C}{\partial x^1} + x^1 x^2 x^3 \frac{\partial C}{\partial x^3} = 0 \quad (27)$$

The characteristic equations for (26) are:

$$\frac{dx^2}{cx^1 x^2 (ax^3 + \mu)} = \frac{dx^3}{cx^1 x^3 (x^2 + \nu)} , \quad dx^1 = 0 \quad (28)$$

Since $C$ is a function of three variables, we have to make two integrations from the characteristic equations. It can be found easily that $x^1 = k_1$ and $x^2 - ax^3 + \nu \ln x^2 - \mu \ln x^3 = k_2$, where $k_1$ and $k_2$ are constants of integration. Then, the general solution of equation (26) is of the form:

$$C^{(1)} = \Psi^{(1)}[x^1, x^2 - ax^3 + \nu \ln x^2 - \mu \ln x^3] \quad (29)$$

Similarly, for the second PDE (27), the system of characteristic equations is:

$$-\frac{dx^1}{cx^1 x^2 (ax^3 + \mu)} = \frac{dx^3}{x^1 x^2 x^3} , \quad dx^2 = 0 \quad (30)$$

We can obtain without difficulty that $x^2 = k_1$ and $ab x^1 - ax^3 - \mu \ln x^3 = k_2$, and then the general solution of (27) is:

$$C^{(2)} = \Psi^{(2)}[x^2, ab x^1 - ax^3 - \mu \ln x^3] \quad (31)$$

Now we must take into account that the Casimir of the system is a simultaneous solution of both (26) and (27). This means that it must be a function of the $x^i$ complying to both formats (29) and (31). After inspection, one arrives directly to the solution (25). We shall comment in Section 4 on the differences between both methods.

(II) A high-dimensional system: The light top

We shall now analyze in detail a six-dimensional example due to Weinstein [1]: The equations of motion of a rigid body anchored at one point, which moves in a gravitational field. The system variables are the entries of the angular
momentum in body coordinates, \( \mathbf{M} = (M_1, M_2, M_3) \), as well as those of the gravitational force, also in body coordinates, \( \mathbf{F} = (F_1, F_2, F_3) \). From now on, we will take the six variables in the following order: \( (M_1, M_2, M_3, F_1, F_2, F_3) \).

Then, the structure matrix and the Hamiltonian are, respectively:

\[
\mathbf{J} = \begin{pmatrix}
0 & M_3 & -M_2 & 0 & F_3 & -F_2 \\
-M_3 & 0 & M_1 & -F_3 & 0 & F_1 \\
M_2 & -M_1 & 0 & F_2 & -F_1 & 0 \\
0 & F_3 & -F_2 & 0 & 0 & 0 \\
-F_3 & 0 & F_1 & 0 & 0 & 0 \\
F_2 & -F_1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad (32)
\]

and

\[
H = \sum_{i=1}^{3} \left( \frac{M_i^2}{2I_i} + x_i F_i \right) \quad (33)
\]

In \( H \), the \( I_i \) are the principal moments of inertia, and the \( x_i \) are the coordinates of the body’s center of mass measured from the anchor point (see [1] and references therein for further details).

We shall first apply our procedure for the determination of the Casimir functions of this system. For the sake of comparison, we shall later solve the same problem through the traditional method of characteristics.

i) Solution of the problem by the present method:

Clearly, \( \text{rank}(\mathbf{J}) = 4 \), the third and the sixth rows being linear combinations of the rest. Then there are two independent Casimirs. We can find the \( \gamma_k^i \) by means of (13):

\[
\Gamma = \begin{pmatrix}
\gamma_1^3 & \gamma_2^3 & \gamma_3^3 \\
\gamma_4^3 & \gamma_5^3 & \gamma_6^3 \\
\end{pmatrix}^T = (\mathbf{J} \cdot \mathbf{J}_4^{-1})^T \quad (34)
\]

where

\[
\mathbf{J}_4 = \begin{pmatrix}
0 & M_3 & 0 & F_3 \\
-M_3 & 0 & -F_3 & 0 \\
0 & F_3 & 0 & 0 \\
-F_3 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix}
M_2 & -M_1 & F_2 & -F_1 \\
F_2 & -F_1 & 0 & 0 \\
\end{pmatrix} \quad (35)
\]

The solution is:

\[
\Gamma = \begin{pmatrix}
-F_1/F_3 & 0 & 0 \\
-F_2/F_3 & 0 & 0 \\
(F_1M_3 - M_1F_3)/F_3^2 & -F_1/F_3 & 0 \\
(F_2M_3 - M_2F_3)/F_3^2 & -F_2/F_3 & 0 \\
\end{pmatrix} \quad (36)
\]
We then have to solve independently the following two differential equations:

\[
\begin{align*}
\frac{dM_3}{F_3} &= -\frac{F_1}{F_3} dM_1 - \frac{F_2}{F_3} dM_2 + \left( \frac{F_1 M_3}{F_3^2} - \frac{M_1}{F_3} \right) dF_1 + \left( \frac{F_2 M_3}{F_3^2} - \frac{M_2}{F_3} \right) dF_2 \quad (37) \\
\frac{dF_3}{F_3} &= -\frac{F_1}{F_3} dF_1 - \frac{F_2}{F_3} dF_2 \quad (38)
\end{align*}
\]

The last one is straightforward and gives a first Casimir: 

\[C_1 = F_1^2 + F_2^2 + F_3^2 = \|F\|^2.\] Now, if we expand (37) and regroup terms we have:

\[F_1 dM_1 + F_2 dM_2 + F_3 dM_3 + M_1 dF_1 + M_2 dF_2 = M_3 \left( \frac{F_1}{F_3} dF_1 + \frac{F_2}{F_3} dF_2 \right) \quad (39)\]

Making use of equation (38) in the right-hand side of (39) leads immediately to 

\[d(M_1 F_1 + M_2 F_2 + M_3 F_3) = 0.\] Thus, the second independent Casimir is 

\[C_2 = M_1 F_1 + M_2 F_2 + M_3 F_3 = M \cdot F.\] We can write, as usual, the most general form of a Casimir as:

\[C = \Psi[F_1^2 + F_2^2 + F_3^2, M_1 F_1 + M_2 F_2 + M_3 F_3], \quad (40)\]

where \(\Psi\) is a smooth two-variable function.

\[\text{ii) Solution of the problem by the method of characteristics:}\]

We can now compare the previous procedure with the direct solution of the system of PDEs \(J \cdot \nabla C = 0\). For this, we should begin by recalling the same observation than before: Since \(\text{rank}(J) = 4\), two of the equations of the system will be redundant —which can be taken as those corresponding to the third and sixth rows of \(J\). Therefore, the system we have to solve is:

\[
\begin{align*}
M_3 \frac{\partial C}{\partial M_2} - M_2 \frac{\partial C}{\partial M_3} + F_3 \frac{\partial C}{\partial F_2} - F_2 \frac{\partial C}{\partial F_3} &= 0 \quad (41) \\
-M_3 \frac{\partial C}{\partial M_1} + M_1 \frac{\partial C}{\partial M_3} - F_3 \frac{\partial C}{\partial F_1} + F_1 \frac{\partial C}{\partial F_3} &= 0 \quad (42) \\
F_3 \frac{\partial C}{\partial M_2} - F_2 \frac{\partial C}{\partial M_3} &= 0 \quad (43) \\
-F_3 \frac{\partial C}{\partial M_1} + F_1 \frac{\partial C}{\partial M_3} &= 0 \quad (44)
\end{align*}
\]

The characteristic equations of (41) are:

\[
\frac{dM_2}{M_3} = -\frac{dM_3}{M_2} = \frac{dF_2}{F_3} = -\frac{dF_3}{F_2}, \quad dM_1 = dF_1 = 0 \quad (45)
\]
Since the unknown $C$ is a function of six variables, we have to find five constants from the characteristic equations (45) in order to construct the general solution of the PDE (41). We immediately find from (45) four of them:

$$M_1 = k_1, \quad F_1 = k_2, \quad M_2^2 + M_3^2 = k_3, \quad F_2^2 + F_3^2 = k_4$$  \quad (46)

We can derive a fifth one as follows:

$$0 \equiv M_3 dF_3 - M_3 dF_3 + F_3 dM_3 - F_3 dM_3 =$$
$$M_3 dF_3 + F_3 dM_3 + M_2 dF_2 + F_2 dM_2 =$$
$$d(M_2 F_2 + M_3 F_3)$$

Here we have made use of the characteristic equations (45). The fifth constant is thus $k_5 = M_2 F_2 + M_3 F_3$. The general solution of the PDE (41) is then:

$$C^{(1)} = \Psi^{(1)}[M_1, F_1, M_2^2 + M_3^2, F_2^2 + F_3^2, M_2 F_2 + M_3 F_3]$$  \quad (47)

The second PDE (42) can be obtained from the first one (41) if we exchange the subindexes 1 and 2. Then we can directly write:

$$C^{(2)} = \Psi^{(2)}[M_2, F_2, M_1^2 + M_3^2, F_1^2 + F_3^2, M_1 F_1 + M_3 F_3]$$  \quad (48)

For the third equation (43) we now have:

$$\frac{dM_2}{F_3} = - \frac{dM_3}{F_2}, \quad dM_1 = dF_1 = dF_2 = dF_3 = 0$$  \quad (49)

This leads to:

$$M_1 = k_1, \quad F_1 = k_2, \quad F_2 = k_3, \quad F_3 = k_4$$  \quad (50)

Since $F_2$ and $F_3$ are constants, we also arrive at $k_5 = M_2 F_2 + M_3 F_3$. Consequently, the general solution of the PDE (43) is:

$$C^{(3)} = \Psi^{(3)}[M_1, F_1, F_2, F_3, M_2 F_2 + M_3 F_3]$$  \quad (51)

And finally, we again obtain the fourth PDE (44) from the third one (43) by permutation of the subindexes 1 and 2. Therefore:

$$C^{(4)} = \Psi^{(4)}[M_2, F_1, F_2, F_3, M_1 F_1 + M_3 F_3]$$  \quad (52)

Now, the Casimir functions are simultaneous solutions of all the PDEs (41-44). Then, we now have to compare the four solutions $C^{(i)}$, for $i = 1, \ldots, 4,$
and look for those functions of $\mathbf{M}$ and $\mathbf{F}$ compatible with all of them. After inspection, it is not difficult to arrive to the two most obvious possibilities: $||\mathbf{F}||^2$ and $\mathbf{M} \cdot \mathbf{F}$, which are the two independent Casimirs already known.

4. Final remarks

We have seen how our algebraic approach allows the calculation of the Casimir functions in a quite natural and rapid way. In fact, we believe that this procedure gives some insight on how a symplectic foliation arises from the degeneracy present in a singular Poisson structure.

A comparison with the traditional method relying on the system of PDEs (3) seems to be convenient. If we wish to solve equations (3), the two simplest strategies are separation of variables and the method of characteristics.

Separation of variables, which is rather lengthy even for simple PDEs and usually requires an eigenvalue analysis of the resulting ODEs, is clearly much less efficient than our procedure.

On the other hand, we have already given in the examples a comparative solution of the problems by both our approach and the method of characteristics. Before entering in more quantitative and general arguments, two observations can be drawn from the examples: The first one is that our method is clearly less computation consuming than that of the characteristics. Notice that our technique reduces the problem to the solution of one ODE per Casimir. The number of ODEs which has been necessary to handle and the number of quadratures which must be found by the method of characteristics is certainly higher, in both examples. The second important remark is that both techniques do not lead to the same set of equations, i.e., our method is not a shortcut for the obtainment of the characteristic equations, as it can be easily checked.

Let us compare in a quantitative way the complexity of both methods. We shall give as a measure of such complexity the number of quadratures which have to be calculated in every case to determine the solution. This number is $N_a = n - 2m$ for our algebraic method, namely the corank of the structure matrix, as we already know.

In the method of the characteristics, on the other hand, we have to solve system (3), which consists of $2m$ nonredundant PDEs (the remaining $n - 2m$ equations are redundant due to the degeneracy in rank of the structure matrix, and can therefore be suppressed, as we have seen in the examples). In order to
compute the total number of quadratures in the method of characteristics, let us consider the \(i\)-th PDE of system (3). Its characteristic equations are of the form:

\[
\frac{dx^i}{f^i} = \ldots = \frac{dx^{i-1}}{f^{i-1}} = \frac{dx^{i+1}}{f^{i+1}} = \ldots = \frac{dx^n}{f^n}, \quad dx^i = 0 \tag{53}
\]

Since \(C\) is a function of \(n\) variables, we need \(n - 1\) quadratures. However, we always have a trivial one, which is \(x^i = \text{constant}\). Therefore, we only have to carry out \(n - 2\) quadratures per PDE, in general. Consequently, the total number of quadratures is \(N_c = 2m(n - 2)\) for the method of characteristics. It is then straightforward to verify that

\[
\frac{N_a}{N_c} < 1 \tag{54}
\]

in all nontrivial cases (the only situation in which (54) is not satisfied for a singular Poisson structure, is the unimportant case corresponding to a null structure matrix). When the number of Casimirs is large, for example if \(2m = 2\), we obtain \(N_a/N_c = 1/2\). When such a number is medium, i.e. for \(2m \simeq n/2\), we have that \(N_a/N_c \simeq 1/(n - 2)\), thus decreasing with increasing size of the structure matrix. Finally, when the number of Casimirs is small, say \(2m \simeq (n - 1)\), we arrive at \(N_a/N_c \simeq 1/[(n - 1)(n - 2)]\). In this case the ratio decreases as \(n^{-2}\) as \(n\) grows, and our approach is now much more economic for a large structure matrix.

Acknowledgements

This work has been supported by the DGICYT (Spain), under grant PB94-0390, and by the EU Esprit WG 24490 (CATHODE-2). B. H. acknowledges a doctoral fellowship from Comunidad Autónoma de Madrid. The authors also wish to acknowledge an anonymous referee for useful suggestions and comments.
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