SECOND MAIN THEOREMS FOR MEROMORPHIC MAPPINGS AND MOVING HYPERPLANES WITH TRUNCATED COUNTING FUNCTIONS

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ABSTRACT. In this article, we establish some new second main theorems for meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \) and moving hyperplanes with truncated counting functions. Our results are improvements of the previous second main theorems for moving hyperplanes with truncated (to level \( n \)) counting functions.

1. Introduction

The second main theorem for meromorphic mappings into projective spaces with moving hyperplanes was first given by W. Stoll, M. Ru [7] and M. Shirosaki in 1990’s [9, 10], where the counting functions are not truncated. In 2000, M. Ru [6] proved a second main theorem with truncated counting functions for nondegenerate mappings of \( \mathbb{C} \) into \( \mathbb{P}^n(\mathbb{C}) \) and moving hyperplanes. After that, this result was reproved for the case of several complex variables by Thai-Quang [12]. For the case of degenerate meromorphic mappings, in [8], Ru and Wang gave a second main theorem for moving hyperplanes with counting function truncated to level \( n \). And then, the result of Ru-Wang was improved by Thai-Quang [13] and Quang-An [5]. In 2016, the author have improved and extended all those results to a better second main theorem. To state their results, we recall the following.

Let \( a_1, \ldots, a_q (q \geq n+1) \) be \( q \) meromorphic mappings of \( \mathbb{C}^m \) into the dual space \( \mathbb{P}^n(\mathbb{C})^* \) with reduced representations \( a_i = (a_{i0} : \cdots : a_{in}) \) (\( 1 \leq i \leq q \)). We say that \( a_1, \ldots, a_q \) are located in general position if \( \det(a_{i_0 i_1} \cdots a_{i_n}) \neq 0 \) for any \( 1 \leq i_0 < i_1 < \cdots < i_n \leq q \). Let \( \mathcal{M}_m \) be the field of all meromorphic functions on \( \mathbb{C}^m \). Denote by \( \mathcal{R}_{\{a_i\}_{i=1}^q} \subset \mathcal{M}_m \) the smallest subfield which contains \( \mathbb{C} \) and all \( \frac{a_{ik}}{a_{il}} \) with \( a_{il} \neq 0 \). Throughout this paper, if without any notification, the notation \( \mathcal{R} \) is always stands for \( \mathcal{R}_{\{a_i\}_{i=1}^q} \).

In 2004, M. Ru and J. Wang proved the following.

**Theorem A** [8 Theorem 1.3] Let \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) be a holomorphic map. Let \( \{a_j\}_{j=1}^q \) be moving hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \) in general position such that \( (f, a_j) \neq 0 \) (\( 1 \leq j \leq q \)). If \( q \geq 2n+1 \) then

\[
\left| \frac{q}{n(2n+1)} T_f(r) \right| \leq \sum_{i=1}^q N_f^{[n]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).
\]

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Here, by the notation “|| $P$” we mean the assertion $P$ holds for all $r \in [0, \infty)$ outside a Borel subset $E$ of the interval $[0, \infty)$ with $\int_E dr < \infty$.

In 2008, D. D. Thai and S. D. Quang improved the above result by increasing the coefficient $\frac{q}{n(2n+1)}$ in front of the characteristic function to $\frac{q}{2n+1}$. In 2016, S. D. Quang [3] improved this result to the following.

**Theorem B** [3] Theorem 1.1] Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let \{$a_j$\}$_{j=1}^q$ ($q \geq 2n-k+2$) be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_j) \neq 0$ ($1 \leq j \leq q$), where $\text{rank}_{\mathcal{R}(a_j)}(f) = k + 1$. Then the following assertion holds:

(a) $|| \frac{q}{2n-k+2} T_f(r) \leq \sum_{i=1}^q N_{(f,a_i)}^k(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a_i(r)),$
(b) $|| \frac{q-(n+2k-1)}{n+k+1} T_f(r) \leq \sum_{i=1}^q N_{(f,a_i)}^k(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a_i(r)).$

The main purpose of the present paper is to establish a stronger second main theorem for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and moving hyperplanes. Namely, we will prove the following theorem.

**Theorem 1.1.** Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let \{$a_i$\}$_{i=1}^q$ ($q \geq 2n-k+2$) be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \neq 0$ ($1 \leq i \leq q$), where $k + 1 = \text{rank}_{\mathcal{R}(a_i)}(f)$. Then the following assertions hold:

(a) $|| \frac{q-(n-k)}{n+2} T_f(r) \leq \sum_{i=1}^q N_{(f,a_i)}^1(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a_i(r)),$
(b) $|| \frac{q-2(n-k)}{k(k+2)} T_f(r) \leq \sum_{i=1}^q N_{(f,a_i)}^1(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a_i(r)).$

Here by $\text{rank}_{\mathcal{R}(a_i)}(f)$ we denote the rank of the set \{$f_0, f_1, \ldots, f_n$\} over the field $\mathcal{R}(a_i)$, where $(f_0 : f_1 : \cdots : f_n)$ is a representation of the mapping $f$.

**Remark:** 1) The assertion (a) is an improvement of Theorem B.

2) It is easy to see that $\frac{q-2(n-k)}{k(k+2)} \geq \frac{q}{n(n+2)}$. Therefore, the assertion (b) immediately implies the following corollary.

**Corollary 1.2.** With the assumptions of Theorem A, we have

$|| \frac{q}{n(n+2)} T_f(r) \leq \sum_{i=1}^q N_{(f,a_i)}^1(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a_i(r)).$

In order to prove the above result, beside developing the method used in [3, 8, 12], we also propose some new techniques. Firstly, we will rearrange the family hyperplanes in the increasing order of the values of the counting functions (of their inverse images). After that, we find the smallest number of the first hyperplanes in this order such that the sum of their counting functions exceed the characteristic functions. And then, we have to
compare the characteristic functions with this sum of counting functions with explicitly estimating the truncation level. From that, we deduce the second main theorem.

For the case where the number of moving hyperplanes is large enough, we will prove a better second main theorem as follows.

**Theorem 1.3.** With the assumptions of Theorem 1.1, we assume further more that $q \geq (n - k)(k + 1) + n + 2$. Then we have

$$
|| \frac{q}{k+2}T_f(r) \leq \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).
$$

In this case, we may see that the coefficient in front of the characteristic functions are exactly the same as the case where the mappings are assumed to be non-degenerate.

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2. **Basic notions and auxiliary results from Nevanlinna theory**

Throughout this paper, we use the standard notation on Nevanllina theory from [3,4] and [13]. For a meromorphic mapping $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$, we denote by $T_f(r)$ its characteristic function. Let $\varphi$ be a meromorphic function on $\mathbb{C}^m$. We denote by $\nu_\varphi$ its divisor, $N^k_\varphi(r)$ the counting function with the truncation level $k$ of its zeros divisor and $m(r, \varphi)$ its proximity function. The lemma on logarithmic derivative in Nevanlinna theory is stated as follows.

**Lemma 2.1** ([11, Lemma 3.11]). Let $f$ be a nonzero meromorphic function on $\mathbb{C}^m$. Then

$$
|| m\left(r, \frac{\mathcal{D}^\alpha(f)}{f}\right) = O(\log^+ T_f(r)) \ (\alpha \in \mathbb{Z}_+^m).
$$

The first main theorem states that

$$
T_\varphi(r) = m(r, \varphi) + N_\varphi^1(r).
$$

We assume that throughout this paper, the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$ is chosen so that for each given meromorphic mapping $a = (a_0 : \cdots : a_n)$ of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ then $a_0 \neq 0$. We set

$$
\tilde{a}_i = \frac{a_i}{a_0} \text{ and } \tilde{a} = (\tilde{a}_0 : \tilde{a}_1 : \cdots : \tilde{a}_n).
$$

Supposing that $f$ has a reduced representation $f = (f_0 : \cdots : f_n)$, we put $(f, a) := \sum_{i=0}^{n} f_i a_i$ and $(f, \tilde{a}) := \sum_{i=0}^{n} f_i \tilde{a}_i$.

Let $\{a_i\}_{i=1}^{q}$ be $q$ meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ with reduced representations $a_i = (a_{i0} : \cdots : a_{in}) \ (1 \leq i \leq q)$.

**Definition 2.2.** The family $\{a_i\}_{i=1}^{q}$ is said to be in general position if

$$
det(a_{ij}; 0 \leq j \leq n, 0 \leq l \leq n) \neq 0
$$

for any $1 \leq i_0 \leq \cdots \leq i_n \leq q$. 

Definition 2.3. A subset \( \mathcal{L} \) of \( \mathcal{M} \) (or \( \mathcal{M}^{n+1} \)) is said to be minimal over the field \( \mathcal{R} \) if it is linearly dependent over \( \mathcal{R} \) and each proper subset of \( \mathcal{L} \) is linearly independent over \( \mathcal{R} \).

Repeating the argument in [1] Proposition 4.5, we have the following proposition.

Proposition 2.4 (see [1] Proposition 4.5]). Let \( \Phi_0, \ldots, \Phi_k \) be meromorphic functions on \( \mathbb{C}^m \) such that \( \{\Phi_0, \ldots, \Phi_k\} \) are linearly independent over \( \mathcal{C} \). Then there exists an admissible set \( \{\alpha_i = (\alpha_i^1, \ldots, \alpha_i^m)\}_{i=0}^k \subset \mathbb{Z}_+^m \) with \( |\alpha_i| = \sum_{j=1}^m |\alpha_{ij}| \leq k \) \( (0 \leq i \leq k) \) satisfying the following two properties:

(i) \( \{D^{\alpha_i}\Phi_0, \ldots, D^{\alpha_i}\Phi_k\}_{i=0}^k \) is linearly independent over \( \mathcal{M} \), i.e., \( \det (D^{\alpha_i}\Phi_j) \neq 0 \),

(ii) \( \det (D^{\alpha_i}(h\Phi_j)) = h^{k+1} \det (D^{\alpha_i}\Phi_j) \) for any nonzero meromorphic function \( h \) on \( \mathbb{C}^m \).

3. Proof of Theorem 1.1 and Theorem 1.3

In order to prove Theorem 1.1 we need the following lemma, which is an improvement of Lemma 3.1 in [3].

Lemma 3.1. Let \( f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping. Let \( \{a_i\}_{i=1}^p \) be \( p \) meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C})^* \) in general position with \( \{f, \tilde{a}_i\}; 1 \leq i \leq q \} = \text{rank}_\mathcal{R}(f) \), where \( \mathcal{R} = \mathcal{R}\{a_i\}_{i=1}^p \). Assume that there exists a partition \( \{1, \ldots, q\} = I_1 \cup I_2 \ldots \cup I_l \) satisfying:

(i) \( \{(f, \tilde{a}_i)\}_{i \in I_i} \) is minimal over \( \mathcal{R} \), \( \{(f, \tilde{a}_i)\}_{i \in I_i} \) is linearly independent over \( \mathcal{R} \) \( (2 \leq t \leq l) \),

(ii) For any \( 2 \leq t \leq l, i \in I_i \), there exist meromorphic functions \( c_i \in \mathcal{R} \setminus \{0\} \) such that

\[
\sum_{i \in I_i} c_i (f, \tilde{a}_i) \in \left( \bigcup_{j=1}^{t-1} \bigcup_{i \in I_j} (f, \tilde{a}_i) \right) \mathcal{R}.
\]

Then we have

\[
T_f(r) \leq \sum_{i=1}^{t} \sum_{j \in I_i} N_{f(a_j)}^{|n_1|}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq p} T_{a_i}(r)),
\]

where \( n_1 = \sharp I_1 - 2 \) and \( n_t = \sharp I_t - 1 \) for \( t = 2, \ldots, l \).

Proof. Let \( f = (f_0 : \cdots : f_n) \) be a reduced representation of \( f \). By changing the homogeneous coordinate system of \( \mathbb{P}^n(\mathbb{C}) \) if necessary, we may assume that \( f_0 \neq 0 \). Without loss of generality, we may assume that \( I_1 = \{1, \ldots, k_1\} \) and

\[
I_t = \{k_{t-1} + 1, \ldots, k_t\} \ (2 \leq t \leq l), \quad \text{where} \quad 1 = k_0 < \cdots < k_l = q.
\]

Since \( \{(f, \tilde{a}_i)\}_{i \in I_1} \) is minimal over \( \mathcal{R} \), there exist \( c_{1i} \in \mathcal{R} \setminus \{0\} \) such that

\[
\sum_{i=1}^{k_1} c_{1i} \cdot (f, \tilde{a}_i) = 0.
\]

Define \( c_{1i} = 0 \) for all \( i > k_1 \). Then

\[
\sum_{i=1}^{k_t} c_{1i} \cdot (f, \tilde{a}_i) = 0.
\]
Because \( \{c_{ti}(f, \tilde{a}_i)\}_{i=k_0+1}^{k_t} \) is linearly independent over \( \mathcal{R} \), Proposition 2.4 yields that there exists an admissible set \( \{\alpha_{1(k_0+1)}, \ldots, \alpha_{k_t}\} \subset \mathbb{Z}_+^m \) (\( |\alpha_{1i}| \leq k_t - k_0 - 1 = n_1 \)) such that the matrix
\[
A_1 = (\mathcal{D}^{\alpha_{1i}}(c_{ij}(f, \tilde{a}_j)); k_0 + 1 \leq i, j \leq k_t)
\]
has nonzero determinant.

Now consider \( t \geq 2 \). By the construction of the set \( I_t \), there exist meromorphic mappings \( c_{ti} \neq 0 \) \( (k_{t-1} + 1 \leq i \leq k_t) \) such that
\[
\sum_{i=k_{t-1}+1}^{k_t} c_{ti} \cdot (f, \tilde{a}_i) \in \left( \bigcup_{j=1}^{t-1} \bigcup_{i \in I_t} (f, \tilde{a}_i) \right) \mathcal{R}.
\]
Therefore, there exist meromorphic mappings \( c_{ti} \in \mathcal{R} \) \( (1 \leq i \leq k_{t-1}) \) such that
\[
\sum_{i=1}^{k_t} c_{ti} \cdot (f, \tilde{a}_i) = 0.
\]
Define \( c_{ti} = 0 \) for all \( i > k_t \). Then
\[
\sum_{i=1}^{k_t} c_{ti} \cdot (f, \tilde{a}_i) = 0.
\]
Since \( \{c_{ti}(f, \tilde{a}_i)\}_{i=k_{t-1}+1}^{k_t} \) is \( \mathcal{R} \)-linearly independent, by again Proposition 2.4 there exists an admissible set \( \{\alpha_{t(k_{t-1}+1)}, \ldots, \alpha_{tk_t}\} \subset \mathbb{Z}_+^m \) (\( |\alpha_{ti}| \leq k_t - k_{t-1} - 1 = n_t \)) such that the matrix
\[
A_t = (\mathcal{D}^{\alpha_{ti}}(c_{ij}(f, \tilde{a}_j)); k_{t-1} + 1 \leq i, j \leq k_t)
\]
has nonzero determinant.

Consider the following \( (k_t - 1) \times k_t \) matrix
\[
T = \begin{bmatrix}
(\mathcal{D}^{\alpha_{ti}}(c_{ij}(f, \tilde{a}_j)); 1 \leq t \leq l, k_{t-1} + 1 \leq i \leq k_t; 1 \leq j \leq k_t)
\end{bmatrix}
\]
Denote by \( D_i \) the subsquare matrix obtained by deleting the \( i \)-th column of the minor matrix \( T \). Since the sum of each row of \( T \) is zero, we have
\[
\det D_i = (-1)^{i-1} \det D_1 = (-1)^{i-1} \prod_{j=1}^{l} \det A_j.
\]
Since \( \{a_i\}_{i=1}^{q} \) is in general position, we have
\[
\det(a_{ij}, 1 \leq i \leq n+1, 0 \leq j \leq n) \neq 0.
\]
By solving the linear equation system \( (f, \tilde{a}_i) = \tilde{a}_{i0} \cdot f_0 + \ldots + \tilde{a}_{in} \cdot f_n \) \( (1 \leq i \leq n+1) \), we obtain
\[
f_v = \sum_{i=1}^{n+1} A_{vi}(f, \tilde{a}_i) \quad (A_{vi} \in \mathcal{R}) \quad \text{for each } 0 \leq v \leq n.
\]
Put \( \Psi(z) = \sum_{i=1}^{n+1} \sum_{v=0}^{n} |A_{vi}(z)| \) \( (z \in \mathbb{C}^m) \). Then
\[
||f(z)|| \leq \Psi(z) \cdot \max_{1 \leq i \leq n+1} (||f, \tilde{a}_i)(z)|| \leq \Psi(z) \cdot \max_{1 \leq i \leq q} (||f, \tilde{a}_i)(z)|| \quad (z \in \mathbb{C}^m),
\]
and
\[
\int_{S(r)} \log^+ \Psi(z) \eta \leq \sum_{i=1}^{n+1} \sum_{v=0}^{n} T(r, A_{iv}) + O(1) = O(\max_{1 \leq i \leq q} T_{a_i}(r)) + O(1).
\]

Fix \( z_0 \in \mathbb{C}^m \setminus \bigcup_{j=1}^{q} (\text{Supp}(\nu_{(f, \tilde{a}_j)}^0) \cup \text{Supp}(\nu_{(f, \tilde{a}_j)}^\infty)) \). Take \( i \) (\( 1 \leq i \leq q \)) such that
\[
|(f, \tilde{a}_i)(z_0)| = \max_{1 \leq j \leq q} |(f, \tilde{a}_j)(z_0)|.
\]

Then
\[
\log \left| \frac{\det D_i(z_0)}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z_0)|} \right| \leq \log^+ \left( \frac{\Psi(z_0) \cdot \left| \frac{\det D_i(z_0)}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z_0)|} \right|}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z_0)|} \right) \leq \log^+ \left( \frac{\left| \frac{\det D_i(z_0)}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z_0)|} \right|}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z_0)|} \right) + \log^+ \Psi(z_0).
\]

Thus, for each \( z \in \mathbb{C}^m \setminus \bigcup_{j=1}^{q} (\text{Supp}(\nu_{(f, \tilde{a}_j)}^0) \cup \text{Supp}(\nu_{(f, \tilde{a}_j)}^\infty)) \), we have
\[
\log \left| \frac{\det D_i(z)}{\prod_{i=1}^q |(f, \tilde{a}_i)(z)|} \right| \leq \sum_{i=1}^{q} \log^+ \left( \frac{\left| \frac{\det D_i(z)}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z)|} \right|}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z)|} \right) + \log^+ \Psi(z).
\]

Hence
\[
(3.3) \quad \log |f(z)| \leq \log \left| \frac{\prod_{i=1}^q |(f, \tilde{a}_i)(z)|}{\det D_i(z)} \right| + \sum_{i=1}^{q} \log^+ \left( \frac{\left| \frac{\det D_i(z)}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z)|} \right|}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)(z)|} \right) + \log^+ \Psi(z).
\]

Integrating both sides of the above inequality and using Jensen’s formula and the lemma on logarithmic derivative, we have
\[
\| T_f(r) \leq N_{\Pi_{i=1}^q (f, \tilde{a}_i)}(r) - N(r, \nu_{\det D_1}) + \sum_{i=1}^{q} m \left( r, \frac{\det D_i}{\prod_{j=1,j \neq i}^q |(f, \tilde{a}_j)|} \right) + O\left( \max_{0 \leq i \leq q} T_{a_i}(r) \right).
\]

(3.4)
\[
= N_{\Pi_{i=1}^q (f, \tilde{a}_i)}(r) - N(r, \nu_{\det D_1}) + O(\log^+ T_f(r)) + O(\max_{0 \leq i \leq q} T_{a_i}(r)).
\]

Claim 3.5. \( \| N_{\Pi_{i=1}^q (f, \tilde{a}_i)}(r) - N(r, \nu_{\det D_1}) \leq \sum_{s=1}^{d} \sum_{i \in I_s} N_{(f, a_i)}^{[n_s]}(r) + O(\max_{0 \leq i \leq q} T_{a_i}(r)). \)

Indeed, fix \( z \in \mathbb{C}^m \setminus I(f) \), where \( I(f) = \{ f_0 = \cdots = f_n = 0 \} \). We call \( i_0 \) the index satisfying
\[
\nu_{(f, \tilde{a}_{i_0})}^0(z) = \min_{1 \leq i \leq n+1} \nu_{(f, \tilde{a}_i)}^0(z).
\]

For each \( i \neq i_0, i \in I_s \), we easily have
\[
\nu_{D_{sk_{n-1}}(c_{si}(f, \tilde{a}_i))}^0(z) \geq \max\{0, \nu_{(f, \tilde{a}_i)}^0(z) - n_s\} - C(2\nu_{c_{si}}^\infty(z) + \nu_{a_{i_0}}^0(z)),
\]
where \( C \) is a fixed constant.
Since each element of the matrix $D_{i_0}$ is of the form $D^{\alpha_{i_0} + 1}(c_{a_i}(f, \bar{a}_i))$ ($i \neq i_0$), one estimates

$$\nu_{D_1}(z) = \nu_{D_{i_0}}(z) \geq \sum_{s=1}^{l} \sum_{i \in I_s} \left( \max \{0, \nu_0(f, \bar{a}_i)(z) - n_s\} - (k + 1)(2\nu_{c_{a_i}}(z) + \nu_{a_{i_0}}(z)) \right).$$

We see that there exists $v_0 \in \{0, \ldots, n\}$ with $f_{a_{v_0}}(z) \neq 0$. Then by (3.2), there exists $i_1 \in \{1, \ldots, n+1\}$ such that $A_{a_{v_1}}(z) \cdot (f, \bar{a}_{i_1})(z) \neq 0$. Thus

$$\nu^0_{(f, \bar{a}_{i_0})}(z) \leq \nu^0_{(f, \bar{a}_{i_1})}(z) \leq \nu^\infty_{A_{a_{v_1}}}(z) \leq \sum_{A_{vi} \neq 0} \nu^\infty_{A_{vi}}(z).$$

Combining the inequalities (3.6) and (3.7), we have

$$\nu^0_{\prod_{i=1}^{s}(f, \bar{a}_i)}(z) - \nu_{det D_1}(z)$$

$$\leq \sum_{s=1}^{l} \sum_{i \in I_s} \left( \min \{\nu_0(f, \bar{a}_i)(z), n_s\} + (k + 1)(2\nu_{c_{a_i}}(z) + \nu_{a_{i_0}}(z)) \right) \leq \sum_{A_{vi} \neq 0} \nu^\infty_{A_{vi}}(z).$$

Integrating both sides of this inequality, we easily obtain

$$\|N^q_{\prod_{i=1}^{s}(f, \bar{a}_i)}(r) - N(r, \nu_{det D_1}) \leq \sum_{s=1}^{l} \sum_{i \in I_s} N^{|a_1|}_{(f, a_i)}(r) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).$$

The claim is proved.

From the inequalities (3.4) and the claim, we get

$$\|T_f(r) \leq \sum_{s=1}^{l} \sum_{i \in I_s} N^{|a_1|}_{(f, a_i)}(r) + O(\log^+ T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).$$

The lemma is proved.

**Proof of Theorem 1.1**

We denote by $I$ the set of all permutations of $q$-tuple $(1, \ldots, q)$. For each element $I = (i_1, \ldots, i_q) \in I$, we set

$$N_I = \{r \in \mathbb{R}^+; N^{|k|}_{(f, a_{i_1})}(r) \leq \cdots \leq N^{|k|}_{(f, a_{i_q})}(r)\};$$

$$M_I = \{r \in \mathbb{R}^+; N^{|l|}_{(f, a_{i_1})}(r) \leq \cdots \leq N^{|l|}_{(f, a_{i_q})}(r)\}.$$
• Case 1. Suppose that \( \sharp A_2 \geq n + 1 \). Since \( \{\tilde{a}_{ij}\}_{j \in A_2} \) is in general position, we have
\[
((f, \tilde{a}_{ij}); j \in A_2)_R = (f_0, \ldots, f_n)_R \supset ((f, \tilde{a}_{ij}); j \in I_1)_R \neq 0.
\]

• Case 2. Suppose that \( \sharp A_2 < n + 1 \). Then we have the following:
\[
\dim_R \left( (f, \tilde{a}_{ij}); j \in I_1 \right)_R \geq k + 1 - (n + 1 - \sharp I_1 \cup I_1') = k - n + \sharp I_1 \cup I_1',
\]
\[
\dim_R \left( (f, \tilde{a}_{ij}); j \in A_2 \right)_R \geq k + 1 - (n + 1 - \sharp A_2) = k - n + \sharp A_2.
\]

We note that \( \sharp I_1 \cup I_1' + \sharp A_2 = 2n - k + 2 \). Hence the above inequalities imply that
\[
\dim_R \left( (f, \tilde{a}_{ij}); j \in I_1 \right)_R \cap ((f, \tilde{a}_{ij}); j \in A_2)_R \neq 0.
\]

Therefore, from the above two cases, we see that \( (f, \tilde{a}_{ij}); j \in I_2 \cap (f, \tilde{a}_{ij}); j \in A_2)_R \neq 0 \).

Therefore, we may chose a subset \( I_2 \subset A_2 \) which is the minimal subset of \( A_2 \) satisfying that there exist nonzero meromorphic functions \( c_i \in R \ (i \in I_2) \),
\[
\sum_{i \in I_2} c_i(f, \tilde{a}_{ij}) \in \left( \bigcup_{i \in I_1} (f, \tilde{a}_{ij}) \right)_R.
\]

We see that \( \sharp I_2 \geq 2 \). By the minimality of the set \( I_2 \), the family \( \{(f, \tilde{a}_{ij})\}_{j \in I_2} \) is linearly independent over \( R \), and hence \( \sharp I_2 \leq k + 1 \) and
\[
\sharp (I_2 \cup I_2) \leq \min\{2n - k + 2, n + k + 1\}.
\]

Moreover, we will show that
\[
\dim \left( (f, \tilde{a}_{ij}); j \in I_1 \right)_R \cap ((f, \tilde{a}_{ij}); j \in A_2)_R = 1.
\]

Indeed, suppose contrarily there exist two linearly independent vectors \( x, y \in (f, \tilde{a}_{ij}); j \in I_1 \right)_R \cap ((f, \tilde{a}_{ij}); j \in I_2)_R \), with
\[
x = \sum_{i \in I_2} x_i(f, \tilde{a}_{ij}) \in \left( (f, \tilde{a}_{ij}); j \in I_1 \right)_R,
\]
\[
y = \sum_{i \in I_2} y_i(f, \tilde{a}_{ij}) \in \left( (f, \tilde{a}_{ij}); j \in I_1 \right)_R,
\]

where \( x_i, y_i \in R \). By the minimality of the set \( I_2 \), all functions \( x_i, y_i \) are not zero. Therefore, fixing \( i_0 \in I_2 \), we have
\[
\sum_{i \neq i_0} (x_{i_0}x_i - x_0y_i)(f, \tilde{a}_{ij}) \in \left( (f, \tilde{a}_{ij}); j \in I_1 \right)_R.
\]

Since \( x, y \) are linearly independent, the left hand side is not zero. This contradicts the minimality of the set \( I_2 \). Hence
\[
\dim \left( (f, \tilde{a}_{ij}); j \in I_1 \right)_R \cap ((f, \tilde{a}_{ij}); j \in I_2)_R = 1.
\]
On the other hand, we will see that \( \sharp I_1 \cup I_2 \leq n + 2 \). If \( \text{rank}_R \{ (f, \tilde{a}_i) \}_{j \in I_1 \cup I_2} = k + 1 \) then we stop the process.

Otherwise, by repeating the above argument, we have a subset \( I'_2 = \{ i; (f, \tilde{a}_i) \in ((f, \tilde{a}_i) \}_{j \in I_1 \cup I_2} \} \), a subset \( I_3 \) of \( A_3 = A_1 \setminus (I_1 \cup I_2 \cup I'_2) \), which satisfy the following:

- there exist nonzero meromorphic functions \( c_i \in R \ (i \in I_3) \) so that
  \[
  \sum_{i \in I_3} c_i (f, \tilde{a}_i) \in \left( \bigcup_{i \in I_1 \cup I_2} (f, \tilde{a}_i) \right)_R,
  \]
- \( \{ (f, \tilde{a}_i) \}_{j \in I_3} \) is linearly independent over \( R \),
- \( 2 \leq \sharp I_3 \leq k + 1 \) and \( \sharp (I_1 \cup \cdots \cup I_3) \leq \min \{ 2n - k + 2, n + k + 1 \} \),
- \( \dim \left( \left( (f, \tilde{a}_i); j \in I_1 \cup I_2 \right)_R \cap \left( (f, \tilde{a}_i); j \in I_3 \right)_R \right) = 1. \)

Continuing this process, we get a sequence of subsets \( I_1, \ldots, I_l \), which satisfy:

1. \( \{ (f, \tilde{a}_i) \}_{j \in I_1} \) is minimal over \( R \), \( \sharp I_1 \geq 2 \) and \( \{ (f, \tilde{a}_i) \}_{j \in I_l} \) is linearly independent over \( R \) (2 \( \leq t \leq l \)),
2. for any \( 2 \leq t \leq l, j \in I_t \), there exist meromorphic functions \( c_j \in R \setminus \{ 0 \} \) such that
   \[
   \sum_{j \in I_t} c_j (f, \tilde{a}_i) \in \left( \bigcup_{s=1}^{t-1} \bigcup_{j \in I_s} (f, \tilde{a}_i) \right)_R,
   \]
   and \( \dim \left( \left( (f, \tilde{a}_i); j \in I_1 \cup \cdots \cup I_{t-1} \right)_R \cap \left( (f, \tilde{a}_i); j \in I_t \right)_R \right) = 1. \)
3. \( \text{rank}_R \{ (f, \tilde{a}_i) \}_{j \in I_1 \cup \cdots \cup I_l} = k + 1. \)

If \( \sharp I_1 = 2 \) we will remove one element from \( I_1 \) and combine the remaining element with \( I_2 \) to become a new set \( I_1 \). Therefore, we will get a sequence \( I_1, \ldots, I_l \) which satisfy the above three properties and \( \sharp I_1 \geq 3, \sharp I_t \geq 2 \) (2 \( \leq t \leq l \)). We set \( n_1 = \sharp I_1 - 2, n_s = \sharp I_s - 1 \) (2 \( \leq s \leq l \)), \( n_0 = \max_{1 \leq s \leq l} n_s \), \( J = I_1 \cup \cdots \cup I_l \) and \( d + 2 = \sharp J \). One estimates:

\[
(n_1 + 2) + (n_2 + 1) + \cdots + (n_l + 1) = d + 2,
\]
\[
(n_1 + 1) + n_2 + \cdots + n_l = k + 1.
\]

Since the rank of the set of any \( n + 1 \) functions \( (f, \tilde{a}_i) \)'s is equal to \( k + 1 \), we have
\[
(n + 1) - \sharp(I_1 \cup \cdots \cup I_{t-1}) \geq (k + 1) - \text{rank}\{(f, \tilde{a}_i); i \in I_1 \cup \cdots \cup I_{t-1}\},
\]
i.e., \( (n + 1) - (n_1 + 2) - (n_2 + 1) - \cdots - (n_{t-1} + 1) \geq (k + 1) - (n_1 + 1) - n_2 - \cdots - n_{t-1}. \)
This implies that
\[
d + 2 \leq n + 2.
\]
On the other hand, we see that \( k + 1 + l = d + 2 \), and hence
\[
n_s = k - \sum_{i=1 \atop i \neq s}^{l} n_i \leq k - (l - 1) \leq \frac{k(k + 2)}{k + l + 1} = \frac{k(k + 2)}{d + 2}.
\]
Thus \( n_0 \leq \frac{k(k + 2)}{d + 2}. \)
Now the family of subsets $I_1, \ldots, I_l$ satisfies the assumptions of the Lemma 3.1. Therefore, we have

$$|| T_f(r) \leq \sum_{s=1}^{l} \sum_{j \in I_s} N^{[n_s]}_{(f,a_j)} + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).$$ \hspace{1cm} (3.9)$$

(a) For all $r \in N_I$ (may be outside a finite Borel measure subset of $\mathbb{R}^+$), from (3.9) we have

$$|| T_f(r) \leq \sum_{j \in J} N^{[k]}_{(f,a_j)} + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))$$

$$\leq \frac{\#J}{q - (2n - k + 2)} + \frac{\#J}{q - (2n - k + 2)} \left( \sum_{j \in J} N^{[k]}_{(f,a_j)}(r) + \sum_{j=2n-k+3}^{q} N^{[k]}_{(f,a_j)}(r) \right) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).$$

Since $\#J = d + 2 \leq n + 2$, the above inequality implies that

$$|| T_f(r) \leq \frac{n + 2}{q - (n - k)} \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)), \quad r \in N_I.$$ \hspace{1cm} (3.10)

We see that $\bigcup_{I \in \mathcal{I}} N_I = \mathbb{R}^+$ and the inequality (3.10) holds for every $r \in N_I, I \in \mathcal{I}$. This yields that

$$T_f(r) \leq \frac{n + 2}{q - (n - k)} \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))$$

for all $r$ outside a finite Borel measure subset of $\mathbb{R}^+$. Thus

$$|| \frac{q - (n - k)}{n + 2} T_f(r) \leq \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).$$

The assertion (a) is proved.
(b) We repeat the same argument as in the proof of the assertion (a). For all \( r \in M_I \) (may be outside a finite Borel measure subset of \( \mathbb{R}^+ \)) we have

\[
\| T_f(r) \leq \sum_{s=1}^{l} \sum_{j \in I_s} N_{(f,a,j)}^{[n_s]} + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
\]

\[
\leq \sum_{j \in J} n_0 N_{(f,a,j)}^{[1]} + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
\]

\[
\leq n_0 \cdot \frac{d + 2}{q - (2n - k + 2)} + d + 2 \left( \sum_{j \in J} N_{(f,a,j)}^{[1]}(r) + \sum_{j=2n-k+3}^{q} N_{(f,a,j)}^{[1]}(r) \right)
\]

\[
+ o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
\]

\[
\leq \frac{k(k+2)}{q - (2n - k + 2)} + d + 2 \sum_{i=1}^{q} N_{(f,a,i)}^{[1]}(r) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
\]

\[
\leq \frac{k(k+2)}{q - 2(n - k)} \sum_{i=1}^{q} N_{(f,a,i)}^{[1]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).
\]

Repeating again the argument in the proof of assertion (a), we see that the above inequality holds for all \( r \in \mathbb{R}^+ \) outside a finite Borel measure set. Then the assertion (b) is proved. \( \square \)

**Proof of Theorem 1.3** We denote by \( \mathcal{I} \) the set of all permutations of \( q \)-tuple \((1, \ldots, q)\). For each element \( I = (i_1, \ldots, i_q) \in \mathcal{I} \), we set

\[
N_I = \{ r \in \mathbb{R}^+ ; N_{(f,a_j)}^{[k]}(r) \leq \cdots \leq N_{(f,a_j)}^{[k]}(r) \}.
\]

We now consider an element \( I \) of \( \mathcal{I} \), for instance it is \( I = (1, \ldots, q) \). Then there is a maximal linearly independent subfamily of the set \( \{ (f, \tilde{a}_i); 1 \leq i \leq n + 1 \} \) which is of exactly \( k + 1 \) elements and contains \((f, \tilde{a}_1)\). We assume that they are \( \{(f, \tilde{a}_i); 1 = i_1 < \cdots < i_{k+1} \leq n + 1 \} \). For each \( 1 \leq j \leq k + 1 \), we set \( J = \{i_1, \ldots, i_{k+1}\} \)

\[
V_j = \left\{ i \in \{1, \ldots, q\} ; (f, \tilde{a}_j) \in \left( (f\tilde{a}_{i_1}); 1 \leq s \leq k + 1, s \neq j \right)_{\mathcal{R}} \right\}.
\]

Since the space \( (f\tilde{a}_{i_1}); 1 \leq s \leq k + 1, s \neq j)_{\mathcal{R}} \) is of dimension \( k \), the set \( V_j \) has at most \( n \) elements. Hence

\[
\# \bigcup_{j=1}^{k+1} V_j = \# \bigcup_{j=1}^{k+1} (V_j \setminus J) + (k + 1) \leq (n - k)(k + 1) + (k + 1) = (n - k + 1)(k + 1).
\]

Therefore, there exists an index \( i_0 \leq (n - k + 1)(k + 1) + 1 \) such that \( i_0 \not\in \bigcup_{j=1}^{k+1} V_j \). This yields that the set \( \{(f, \tilde{a}_j); 0 \leq j \leq k + 1\} \) is minimal over \( \mathcal{R} \). Then by Lemma 3.1 for
all $r \in N_I$ we have

$$
\|T(r, f)\| \leq \sum_{j=0}^{k+1} N_{(f, a_j)}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
$$

$$
\leq N_{(f, a_1)}^{[k]}(r) + \sum_{i=n-k+2}^{n+1} N_{(f, a_i)}^{[k]}(r) + N_{(f, a_{(n-k+1)(k+1)+1})}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
$$

$$
\leq \frac{1}{n-k+1} \left( \sum_{i=1}^{(n-k+1)(k+2)} N_{(f, a_i)}^{[k]}(r) \right) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
$$

$$
= \frac{1}{n-k+1} \left( \sum_{i=1}^{(n-k+1)(k+2)} N_{(f, a_i)}^{[k]}(r) \right) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
$$

$$
\leq \frac{k+2}{q} \sum_{i=1}^{q} N_{(f, a_i)}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).
$$

Repeating again the argument in the proof of Theorem [ ] we see that the above inequality holds for all $r \in \mathbb{R}^+$ outside a finite Borel measure set. Hence, the theorem is proved.

\[\Box\]

**References**

[1] H. Fujimoto, *Non-integrated defect relation for meromorphic maps of complete Kähler manifolds into $\mathbb{P}^{N_1}(\mathbb{C}) \times \ldots \times \mathbb{P}^{N_q}(\mathbb{C})$*, Japanese J. Math. **11** (1985), 233–264.

[2] J. Noguchi and T. Ochiai, *Introduction to Geometric Function Theory in Several Complex Variables*, Trans. Math. Monogr. 80, Amer. Math. Soc., Providence, Rhode Island, 1990.

[3] S. D. Quang, *Second main theorems for meromorphic mappings intersecting moving hyperplanes with truncated counting functions and unicity problem*, Abh. Math. Semin. Univ. Hambg. **86** (2016) 1–18.

[4] S. D. Quang, *Second main theorems with weighted counting functions and algebraic dependence of meromorphic mappings*, Proc. Amer. Soc. Math. **144** (2016), 4329–4340.

[5] S. D. Quang and D. P. An, *Unicity of meromorphic mappings sharing few moving hyperplanes*, Vietnam Math. J. **41** (2013), 383–398.

[6] M. Ru, *A uniqueness theorem with moving targets without counting multiplicity*, Proc. Amer. Math. Soc. **129** (2001), 2701–2707.

[7] M. Ru and W. Stoll, *The second main theorem for moving targets*, Journal of Geom. Anal. **1**, No. 2 (1991) 99–138.

[8] M. Ru and J. T-Y. Wang, *Truncated second main theorem with moving targets*, Trans. Amer. Math. Soc. **356** (2004), 557–571.

[9] M. Shirosaki, *Another proof of the defect relation for moving target*, Tohoku Math. J., **43** (1991), 355–360.

[10] M. Shirosaki, *On defect relations of moving hyperplanes*, Nogoya Math. J. **120** (1990), 103–112.

[11] B. Shifman, *Introduction to the Carlson - Griffiths equidistribution theory*, Lecture Notes in Math. **981** (1983), 44–89.
[12] D. D. Thai and S. D. Quang, *Uniqueness problem with truncated multiplicities of meromorphic mappings in several complex variables for moving targets*, Internat. J. Math., 16 (2005), 903-939.

[13] D. D. Thai and S. D. Quang, *Second main theorem with truncated counting function in several complex variables for moving targets*, Forum Mathematicum 20 (2008), 145-179.

[14] K. Yamanoi, *The second main theorem for small functions and related problems*, Acta Math. 192 (2004), 225-294.

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