Influence of the shape of a conducting chamber on the stability of rigid ballooning modes in a mirror trap

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Abstract
MHD stabilization of flute and ballooning modes in an axisymmetric mirror trap is studied under the assumption of strong finite Larmor radius effect that suppresses all perturbations with azimuthal numbers \( m \geq 2 \) and makes the \( m = 1 \) mode ‘rigid’. The rigid mode can be effectively suppressed using perfectly conducting lateral wall without any additional means of stabilization or in combination with end MHD anchors. Numerical calculations were carried out for an anisotropic plasma produced in the course of neutral beam injection into the minimum of the magnetic field at the right angle to the trap axis. The stabilizing effect of the conducting shell made of a straightened cylinder is compared with a proportional chamber, which, on an enlarged scale, repeats the shape of the plasma column. It is confirmed that for convincing wall stabilization of the rigid modes, the plasma beta (\( \beta \), the ratio of the plasma pressure to the magnetic field pressure) must exceed some critical value \( \beta_{c1} \). When conducting lateral wall is combined with conducting end plates imitating MHD end anchors, there are two critical betas and, respectively, two stability zones \( \beta < \beta_{c1} \) and \( \beta > \beta_{c2} \) that can merge, making the entire range \( 0 < \beta < 1 \) of betas allowable for stable plasma confinement. The dependence of the critical betas on the plasma anisotropy, mirror ratio, width of the vacuum gap between the plasma column and the lateral wall, radial pressure profile and the axial magnetic field profile is examined.

Keywords: MHD stability, ballooning modes, LoDestro equation, gas-dynamic trap, compact axisymmetric toroid, wisconsin HTS axisymmetric mirror

1. Introduction
Continuing the study of large-scale magnetohydrodynamic (MHD) instabilities, begun in our papers [1, 2], below we present new results of studying the so-called wall stabilization of rigid flute and ballooning perturbations with azimuthal number \( m = 1 \) in an axially-symmetric mirror trap (also called linear or open trap) with anisotropic plasma. According to the physics of the case, wall stabilization of a plasma with a sufficiently high pressure is achieved by inducing image (Foucault’s) currents in the conducting walls of the vacuum chamber surrounding the plasma column. These currents are directed oppositely to the diamagnetic currents in the plasma edge, and opposite currents are known to repel each other. Such repulsion returns the floating ‘tongues’ of plasma back to the axis of the trap.

Under the conditions of a real experiment, small-scale flute and ballooning oscillations are readily suppressed by finite Larmor radius effects (FLR effects) [3]. The FLR effects cannot stabilize oscillations with an azimuthal number \( m = 1 \), but make them ‘rigid’ in a certain sense. In two papers [1, 2] we successively analyzed the wall stabilization of isotropic and anisotropic plasmas in a model where the shape of a perfectly conducting lateral (side) wall on an enlarged scale repeated the shape of the plasma column, i.e. the ratio of the radius of...
the conducting wall $r_w$ to the plasma radius $a$ is the same in all sections, $r_w/a = \text{const}$, as shown in figure 1(a). It is difficult to fabricate a conducting chamber of this shape, if only because the shape of the plasma stub changes with a change in the plasma pressure. Nevertheless, it is this model that has been used in most of the previous theoretical studies, since it simplifies the calculation. In the following, a conducting lateral wall of this shape will be called proportional chamber and will be denoted by the abbreviation ‘Pr’.

In this paper, we examine for the first time the stability of the flute-ballooning mode $m = 1$ in a straightened chamber (denoted by the abbreviation ‘St’) with a constant radius $r_w = \text{const}$ as shown in figure 1(b). We present the results of calculating the stability zones both in the absence of additional end MHD anchors, which are traditionally used to suppress flute oscillations in open axially symmetric traps, and in their presence.

In the first case, hereinafter referred to as lateral wall stabilization and denoted by ‘Lw’ for brevity, there is one stability zone at a sufficiently large beta that exceeds the critical value, $\beta > \beta_{a2}$. In the second case, there are two such zones: the first one is for small beta, $\beta < \beta_{a1}$, the second one is for large beta, $\beta > \beta_{a2}$. These two zones can merge when the vacuum gap between the plasma column and the inner surface of the shell is sufficiently narrow. We call this case combined wall stabilization and denote it by the abbreviation ‘Cw’.

The existence of the lower stability zone at small betas is well known, since $\beta_{a1}$ is just the threshold of ballooning instability in axially symmetric mirror traps, where the flute mode is suppressed by the end MHD anchors. As for the upper zone, its appearance due to the effect of the surrounding conducting wall gradually became recognized after the publications of many authors, the most attention of which was attracted by the conference report of the recently deceased Berk [4], but this was neither the first publication nor the most clear. A complete history of research on the ballooning mode instability can be found in [1, 5].

Our goal is to calculate the critical values of $\beta_{a1}, \beta_{a2}$ and identify the dependence of these values on the degree of plasma anisotropy, radial profile of plasma pressure, axial profile of the magnetic field, and the width of the vacuum gap between the plasma and the lateral wall. We also simulate MHD anchors by conducting end plates placed at different locations behind the throat of magnetic field plugs.

The same goals we had when examined earlier the case of proportional chamber. To simplify the comparison of stability zones for the proportional and straightened chambers, we further use the same four radial pressure profiles and the same three magnetic field profiles as in [1, 2].

In an effort to reduce the inevitable duplication of part of the content of [1, 2], we skip the review of publications by other authors and restrict ourselves to a couple of remarks. First, we point out that the key equation that describes flute and ballooning oscillations $m = 1$ in an open axially symmetric trap belongs to LoDestro [6]. Secondly, we note that the effect of the shape of a conducting wall on its stabilizing properties was previously discussed by Li et al [7], and also by Li et al [8]. The first work assumed that stabilizing wall extends axially only over a part of the distance between the mirror trap midplane and the throat. In a model of this arrangement, a wall was used which was near the plasma surface in the bad curvature region and distant from the plasma surface in the good curvature region. A variational method was used to solve the equations of a pre-LoDestro type for both regions assuming sharp-boundary radial profile of the plasma pressure. The authors of the second paper assumed that a certain length of magnetic field has a series of ripples in it. They concluded that with isotropic pressure wall stabilization is possible if the plasma beta value is higher than 50%, provided that the conducting wall is very close to the plasma surface.

In section 2, we reproduce the LoDestro equation with all the notation, but without detailed explanations, and from the very beginning we use dimensionless variables. Also, in dimensionless notation, in section 3 we repeat (with some changes) the formulas that model the anisotropic pressure distribution in a mirror trap under normal injection of neutral beams (normal NBI); the changes are due to the transition to a more economical parameterizations in numerical calculations of the coefficients of the LoDestro equation. In sections 4–6, the results of calculations are given first for the case when there are no other means of MHD stabilization besides the lateral conducting wall, and then for the case when conducting plates are installed in magnetic mirrors or behind them, which simulate the effect of end MHD anchors of different strength. Finally, section 7 summarizes our observations and conclusions.

2. LoDestro equation

The LoDestro equation, named after LoDestro [6], is a second-order ordinary differential equation for the function

$$\phi(z) = a(z) B_r(z) \zeta_n(z),$$

(1)
of one coordinate $z$ along the axis of axial symmetry of an open trap. This function is just a product of the local radius of the plasma column boundary $a = a(z)$, the vacuum magnetic field $B_r = B_r(z)$ and a small displacement $\xi_\alpha = \xi_\alpha(z)$ of the magnetic column from the trap axis. In its final form, the LoDestro equation reads

$$0 = \frac{d}{dz} \left[ \Lambda + 1 - \frac{2 \langle \bar{\rho} \rangle}{B_r^2} \frac{d\phi}{dz} \right] + \phi \left[ - \frac{d}{dz} \left( \frac{B_r'}{B_r} + \frac{2a'}{a} \right) \left( 1 - \frac{\langle \bar{\rho} \rangle}{B_r^2} \right) + \frac{\omega^2}{B_r^2} \frac{\langle \rho \rangle}{B_r^2} - \frac{2 \langle \bar{\rho} \rangle a''}{B_r^2 a'} - \frac{1}{2} \left( \frac{B_r'}{B_r} + \frac{2a'}{a} \right)^2 \left( 1 - \frac{\langle \bar{\rho} \rangle}{B_r^2} \right) \right], \tag{2}$$

where the derivative $d/dz$ in the first two lines acts on all factors to the right of it, the prime (‘) is a shortcut for $d/dz$, and $\omega$ is the oscillation frequency. Other notations are defined as follows

$$\frac{a^2}{2} = \int_0^1 \frac{d\psi}{B}, \tag{3}$$

$$\frac{r^2}{2} = \int_0^\psi \frac{d\psi}{B}, \tag{4}$$

$$B_r^2 = B_0^2 - 2p_\perp, \tag{5}$$

$$a_r(z) = \frac{2}{B_r(z)}, \tag{6}$$

$$\bar{p} = \frac{p_\perp + p_\parallel}{2}, \tag{7}$$

$$\langle \bar{\rho} \rangle = \frac{2}{a^2} \int_0^1 \frac{d\psi}{B} \bar{p}, \tag{8}$$

$$\Lambda = \frac{r_w^2 + a^2}{r_w^2 - a^2}. \tag{9}$$

The equation (4) expresses radial coordinate $r$ in terms of magnetic flux $\psi$ through a disk of radius $r$ in a current plane $z$. The true magnetic field $B = B(\psi, z)$, reduced by diamagnetic currents at the plasma boundary, in the paraxial (long-thin) approximation, is related to the vacuum magnetic field $B_r = B_r(z)$ by equation (5) of the transverse equilibrium. Kinetic theory proves that the transverse and longitudinal plasma pressures can be considered as functions of the true magnetic field $B$ and the magnetic flux $\psi$, i.e. $p_\perp = p_\perp(B, \psi)$, $p_\parallel = p_\parallel(B, \psi)$ (see, for example, [9]). In equation (2) one must assume that $B$ is already expressed in terms of $\psi$ and $z$, so $p_\perp = p_\perp(\psi, z)$, $p_\parallel = p_\parallel(\psi, z)$. The angle brackets in equation (2) denote the mean value of an arbitrary function $\psi$ and $z$ over the plasma cross section. In particular, the average value $\langle \rho \rangle$ of the density $\rho = \rho(\psi, z)$ is calculated using a formula similar to (8).

Parameter $\Lambda$ is, generally speaking, a function of the $z$ coordinate. It implicitly depends on the plasma parameters and the magnetic field through the dependence of the plasma column radius $a = a(z)$ on them. In the special case of a proportional chamber, when $r_w(z)/a(z) = \text{const}$, the function $\Lambda(z)$ becomes a constant, which simplifies the equation somewhat. Namely, for the sake of such simplification, in our previous studies we assumed that $\Lambda = \text{const}$. Now we drop this simplification and consider a completely realistic case of a conducting chamber in the form of a straightened cylinder, that is, we assume that $r_w = \text{const}$. Comparison of the sizes of the stability zones for the proportional and straightened conducting chambers, which in a sense are antipodes, allow us to estimate how strong the influence of the shape of the conducting chamber on the stability of ballooning oscillations is.

Traditionally, two types of boundary conditions are considered. In the presence of conducting end plates straightly in the magnetic mirrors at $z \pm 1$, the boundary condition

$$\phi(\pm 1) = 0 \tag{10}$$

should be chosen. In a more general case, when the conducting end plate is installed somewhere in the behind-the-mirror region, namely, in the plane with coordinates $z = \pm z_{\text{end}}$, the zero boundary condition must obviously be assigned to this plane:

$$\phi(\pm z_{\text{end}}) = 0. \tag{11}$$

By solving the LoDestro equation with the boundary condition (11), it is possible to simulate the effect of end MHD anchors with different stability margins.

If the plasma ends are electrically isolated, the boundary condition

$$\phi'(\pm z_{\text{end}}) = 0 \tag{12}$$

is applied at $z = \pm z_{\text{end}}$. As a rule, it implies that other methods of MHD stabilization in addition to stabilization by a conducting lateral wall are not used.

An obvious fit to the LoDestro equation with boundary conditions (10) and (11) or (12) is the trivial solution $\phi \equiv 0$. To eliminate the trivial solution, we impose a normalization in the form of one more condition

$$\phi(0) = 1. \tag{13}$$

Taking into account the symmetry of the magnetic field in actually existing open traps with respect to the median plane $z = 0$, it suffices to find a solution to the LoDestro equation at a half the distance between the magnetic mirrors, for example, in the interval $0 < z < 1$. Due to the same symmetry, the desired function $\phi(z)$ must be even, therefore

$$\phi'(0) = 0. \tag{14}$$

It is convenient to search for a solution to the LoDestro equation by choosing the boundary conditions (13) and (14).
In theory, a second-order linear ordinary differential condition with two boundary conditions must always have a solution. However, the third boundary condition (10), (11) or (12) can only be satisfied for a certain combination of the problem parameters. If the parameters of the plasma, magnetic field, and geometry of the lateral conducting wall are given, the third boundary condition should be considered as a non-linear equation for the frequency squared $\omega^2$. If the lowest root of such an equation is greater than zero, MHD oscillations with azimuthal number $m = 1$ are stable; if $\omega^2 < 0$, then instability takes place. On the margin of the stability zone $\omega^2 = 0$. In this case, the solution of the boundary-value problem (2), (13) and (14) with the additional boundary condition (12) or (10) gives the critical value of beta, $\beta_{cr}$ and/or $\beta_{cr2}$.

As it was mentioned in [2], the LoDestro equation (2) with boundary conditions (10), (11) or (12) constitutes the standard Sturm–Liouville problem. At first glance, it may seem that the solution of such a problem is rather standard. However, the equation has the peculiarity that its coefficients can be singular. In the anisotropic pressure model, formulated in section 3, the singularity appears near the minimum of the magnetic field in the limit $\beta \to 1$. By some indications, our predecessors were aware of the singularity problem. Unfortunately, they did not leave us recipes for dealing with this singularity. The current version of the our numerical code uses the shooting method to solve the Sturm–Liouville problem, as described in section 5 of [1].

### 3. Plasma pressure and magnetic field

In this article, we study wall stabilization using the same model of an anisotropic plasma that was introduced in our previous article [2]. It approximately simulates the pressure distribution in an open trap with normal NBI. Known examples of such traps provide compact axisymmetric toroid (CAT), a middle-scale device operating at the Budker Institute of Nuclear Physics [10–12], and a larger Wisconsin HTS axisymmetric mirror (WHAM), which is under construction at the Wisconsin University in Madison [13–15]. To avoid possible misunderstanding it is worth to remind (see [2]) that it would be severe mistake to choose completely arbitrary functions of longitudinal and transverse plasma pressure. Firstly, because the functions of longitudinal and transverse pressure are related by the equation of longitudinal equilibrium. Therefore, transverse pressure is expressed through longitudinal pressure, and longitudinal pressure through transverse pressure. But even more important is the fact that the functions of longitudinal and transverse pressure must be expressed through integrals of the distribution function of plasma particles, which should satisfy the kinetic equation. It is not obvious in advance that arbitrarily chosen pressure functions will satisfy the specified requirements. The pressure function used by us satisfies all known requirements. The proper particle distribution functions can be restored under some additional assumptions as can be shown in the next paper [16].

![Figure 2.](image)

**Figure 2.** Axial profile of the vacuum magnetic field (21) for a fixed ratio $M/R = 8$. Index $q$ governs axial profiles of the field and varies from $q = 2$, denoting a ‘quasi-parabolic’ model, to $q = 8$, which approaches a ‘flat hole’ model.

In addition, an undeniable advantage of this pressure model is that it allows a significant part of the calculations to be performed in an analytical form. In addition, such a pressure distribution is not subject to mirror and firehose instabilities. Below we repeat the necessary formulas, writing them in dimensionless variables, such that $B = B_0 = 1$ at the ‘turning point’, where the pressure vanishes:

\[
p_{\perp}(B, \psi) = p_0 f_k(\psi) \left(1 - B^2\right), \tag{15}\n\]

\[
p_{\parallel}(B, \psi) = p_0 f_k(\psi) (1-B)^2, \tag{16}\n\]

\[
\bar{p}(B, \psi) = p_0 f_k(\psi) (1-B). \tag{17}\n\]

It is assumed here and below that the entire region between the stop point and magnetic mirror throat is occupied by relatively cold plasma, so that $p_{\perp} = p_{\parallel} = \bar{p} = 0$ for $B > 1$. Dimensionless function $f_k(\psi)$ describes radial profile of the plasma pressure. It is defined as

\[
f_k(\psi) = \begin{cases} 1 - \psi^k, & \text{if } 0 \leq \psi \leq 1 \\ 0, & \text{otherwise} \end{cases}\tag{18}\n\]

for integer values of index $k$, and for $k = \infty$ is expressed in terms of a $\theta$-function such that $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$:

\[
\theta(\psi) = \theta(1-\psi). \tag{19}\n\]

For the model (15), equation (5) can be solved to explicitly express the magnetic field $B$ in terms of the vacuum field $B_v$:

\[
B(\psi, z) = \sqrt{\frac{B_v^2(z) - 2p_0f_k(\psi)}{1 - 2p_0f_k(\psi)}}. \tag{20}\n\]

Choosing the function $B_v(z)$ for the calculations presented below, we use the second of the two models that were use in our first work [1]:

\[
B_v(z) = |1 + (M - 1) \sin^6(\pi z/2)|/R. \tag{21}\n\]

Figure 2 visualizes this model for fixed ratio of $M$ to $R$. 
Parameter beta, $\beta$, is defined as the maximum of the ratio $2p_\perp/B_0^2$:

$$\beta = \max \left( \frac{2p_\perp}{B_0^2} \right). \quad (22)$$

The maximum is reached on the trap axis (where $\psi = 0$) at the vacuum field minimum (where $\min(B_\perp) = 1/R$), so that

$$\beta = \frac{2(R^2 - 1)p_0}{1 - 2p_0}. \quad (23)$$

Parameter $p_0$ can vary within $0 < p_0 < 1/2R^2$, and $\beta \to 1$ for $p_0 \to 1/2R^2$. For $\beta > 1$, plasma equilibrium is impossible, since equation (5) does not have a continuous solution [17].

In our internal classification, isotropic plasma variant is designated ‘A0’, the anisotropic pressure (15)–(17), which is formed during normal NBI, is designated ‘A1’. Particular device configurations exploiting conducting lateral wall stabilization is be labeled by the shortcut ‘Lw’ (which stands for Lateral Wall), and the configurations exploiting combination of the lateral wall stabilization with end conducting plates installed at the throat of the magnetic plug is marked by the shortcut ‘Cw’ (Combined Wall). The design of a proportional chamber is labeled ‘Pr’ (Proportional). Thus, the label ‘A1-LwPr’ in a figure caption means that plasma with anisotropic pressure of the ‘A1’ type is confined in a device equipped with a proportional conducting shell without any end MHD anchors. The abbreviation ‘St’ (Straightened) is reserved for a straightened shell.

4. Stabilization by lateral wall

Equation (2), supplemented by the boundary conditions (12)–(14) constitutes the Sturm–Liouville problem on half the interval $0 \leq z \leq 1$ between the magnetic plugs. Its solution at $\omega^2 = 0$ gives the critical value of beta, $\beta_{c2}$, in the case when end MHD anchors are not used for MHD stabilization. The particularities of solving this problem for the models of anisotropic pressure adopted by us are described in detail in [2], so we immediately proceed to the analysis of the results obtained.

Our calculations were carried out for the magnetic field (21) with mirror ratios chosen from the set $M \in \{24, 16, 8, 4, 2\}$ for most of possible commutations of the parameters $k \in \{1, 2, 4, \infty\}$, $q \in \{2, 4, 8\}$ and discrete values $\lambda_0 \equiv \Lambda(0) = \{1, 001, 002, \ldots, 450, 500\}$ of the function $\Lambda(z)$ at $z = 0$. Parameter $R$ varied from $R = 1.1$ to $R = M$ taking discrete values, as a rule, from the set $R \in \{1.1, 1.2, 1.5, 2, 4, 8, 16, 24\}$.

Numbers of figures in other articles below will be superscripted with the reference number to the corresponding article in the bibliography.

First of all, we computed the second critical beta $\beta_{c2}$ in St chamber at the closest location of the conducting wall to the surface of the plasma column to compare it with similar computation in [2] for Pr chamber. Figure 4 shows a series of graphs for the case $\Lambda_0 = 500$, when the conducting wall almost touches the plasma column surface in its widest section $z = 0$. They illustrate the dependence of $\beta_{c2}$ on parameter $R$, which...
Figure 4. Second critical beta for magnetic field \((2)\) vs. the stop point mirror ratio \(R\) for a set of mirror ratios \(M \in \{16, 2\}\) and indices \(q \in \{2, 4, 8\}\) in the limit \(\Lambda(0) \rightarrow \infty\). The stability zone for the radial profile with index \(k\) is located above and to the left of the margin curve of the color, which is related to \(k\) in the legend under the bottom row. The common zone, where all radial profiles are stable, is shaded in blue. The odd and even rows contain graphs for the A1-LwPr and A1-LwSt configurations, respectively. The left column shows graphs for a ‘quasi-parabolic’ magnetic field with the index \(q = 2\), and the right column shows the results for the ‘flat hole’ magnetic field with \(q = 8\).

characterizes the spatial width of the pressure peak (the peak is wider for larger \(R\)).

Within each graph, it is not difficult to detect a trend towards a decrease in \(\beta_{cr}^2\) with an increase in the steepness (sharpness) of the radial pressure profile as the index \(k\) increases from \(k = 1\) to \(k = \infty\) for a fixed pair of parameters \(q\) and \(M\). Comparison of graphs in even rows (drawn for A1-LwPr configuration) and odd rows (A1-LwSt) shows that \(\beta_{cr}^2\) is certainly lower in the case of a proportional chamber over the entire range of \(R\). This fact is especially noticeable for a moderate value of \(R\) and has a completely obvious explanation. Indeed, even at equal ratios \(r_w/a_0 = \text{const}\) (in odd rows) and \(r_w/a_0\) (in even rows) average vacuum gap in much wider in the second case as can be seen in figure 1. However, the difference between the Pr and St configurations is apparently not catastrophic, since the plasma pressure peak in the case of normal NBI is located directly in the middle plane of the trap. It is evident that similar study for oblique NBI can lead to an opposite conclusion.

The second observation is that the straightened chamber stabilizes the smoothest radial pressure profile with index \(k = 1\) noticeably worse, in the sense that the range of values of the \(R\) parameter at which this profile can be stabilized is noticeably narrower. In the next section 5, we will see that in the presence of end MHD anchors, on the contrary, the smoothest radial profile is the easiest to stabilize.

Figure 5 contains graphs of \(\beta_{cr}^2\) versus ratio \(r_w/a_0 = \sqrt{(\Lambda_0 + 1)/(\Lambda_0 - 1)}\) for mirror ratios \(M \in \{4, 16\}\) at \(q = 4\)
Figure 5. Second critical beta versus ratio $r_w/a_0$ for model magnetic field (21) and anisotropic plasma pressure model (15), simulating normal NBI, $q = 4$, $M = 4$, $R \in \{1.1, 1.2, 1.5, 2, 4\}$, and $M = 16$, $R = 4$. The stability zone for the radial profile with index $k$ is located above and to the left of the curve of the corresponding color, which is indicated in the legend under the bottom row of graphs. The zone, where all studied radial pressure profiles are stable, is covered with blue shading. If only orange zone is shaded, the most smooth radial profile $k = 1$ is always unstable. The odd and even rows contain graphs for the A1-LwPr and A1-LwSt configurations, respectively.

and a set of values of $R$. Comparison of consecutive graphs from (a) to (e) once more demonstrate strong dependence of critical betas on parameter $R$ in case of proportional wall shape. Graphs from (g) to (k) demonstrate the same tendency in case of straightened lateral wall. Passing from graph (e) to (f) and from (k) to (l) illustrates absence of strong dependency on the mirror ratio. The effect of reshaping magnetic field axial profile governed by parameter $q$ as even more weak and therefore graphs for other values of $q$ are not shown in figure 5.

It can be assumed that under the conditions of a real experiment, the shape of the conducting chamber will be something in between the proportional and straightened chambers. In this sense, these two chambers are two antipodes, and the results of calculations for these chambers determine the boundaries of the interval of the expected value $\beta_{cr2}$ for a real experiment. However, it is currently not entirely clear whether abrupt changes in the radius $r_w(z)$ of the conducting chamber can have a strong stabilizing effect. Such changes can imitate the placement of a massive limiter near the plasma column in the region of unfavorable curvature of magnetic field lines.

Summing up this section, we can conclude that although MHD plasma stabilization by means of only perfectly conducting lateral wall without the use of accompanying stabilization methods is possible in principle, it requires either a sufficiently strong plasma anisotropy or sufficiently large values of beta. In this case, a natural question arises, how to transfer the plasma to such a state through the intermediate stages of a startup.
5. Combined wall stabilization

To adapt our calculation to the case of wall stabilization in combination with MHD end anchors, we should replace the boundary condition (12) with the boundary condition (10), which means that the plasma is frozen into the conducting end plates. And again, as mentioned in section 4, the features of solving this problem for the model of anisotropic pressure and vacuum magnetic field adopted in this paper are described in detail in [2]. There, formulas were obtained that refine the boundary conditions in a form suitable for a conducting chamber of any shape, so we will immediately proceed to the analysis of the results obtained.

Following same scheme as in [1, 2], we performed a series of calculations for the vacuum magnetic field (21) with three values of the index $q \in \{2, 4, 8\}$ and a mirror ratio from the limited set $M \in \{24, 16, 8, 4\}$. Parameter of anisotropy $R \leq M$ was taken from the list $R \in \{1, 1.5, 2, 4, 8, 16, 24\}$.

A series of graphs in figure 6 illustrates the results of calculations for the case $R = M$, when the plasma anisotropy is minimal and the instability zone has its maximum dimensions. Such a zone in this and subsequent figures lies between the lower and upper branches of each curve, the color of which is individual for each pressure profile with a given index $k \in \{1, 2, 4, \infty\}$. The lower and upper branches of such a curve are $\beta_{\text{cr}1}$ and $\beta_{\text{cr}2}$, respectively. If only one branch is shown, as in the case of the blue curve for the $k = 1$ profile in figure 6(a), we interpret it as $\beta_{\text{cr}1}$. If there is no curve of a certain color on a graph at all, then we consider that the corresponding pressure profile is stable in the entire range $0 < \beta < 1$ of beta and
Figure 7. Stability map for model magnetic field (21) and anisotropic plasma pressure (15) at combined MHD stabilization by lateral wall and MHD anchors at \( q = 4, M = 16 \) and \( R \in \{1.1, 1.2, 1.5, 2, 4, 8\} \). The unstable zone is located between the lower \( \beta_{cr1}(r_w/a_0) \) and upper branches \( \beta_{cr2}(r_w/a_0) \) of a margin curve of a fixed color. Only margins for \( k \in \{\infty, 4\} \) are found for indicated parameters; it means that the radial profiles \( k \in \{2, 1\} \) are everywhere stable within the range shown in the graphs. Correspondence of the index \( k \) to the color of margin curves is shown at the bottom of the figure. Common zone of stability is shaded. It is determined by the plasma with a sharp boundary \( (k = \infty) \), for which it has the minimum size. The odd and even rows contain graphs for the A1-CwPr and A1-CwSt configurations, respectively.

As expected, in the straightened chamber (see even rows), the zone of instability is definitely smaller. In other words, in a straightened chamber, the vacuum gap (measured in the median plane) must be smaller in order to ensure the stability of a plasma with any radial profile for any value of \( \beta \). Note also that for comparable values of \( r_w/a_0 \) the lower stability zone \( 0 < \beta < \beta_{cr1} \) in the straightened chamber is definitely narrower.

The graphs in figure 7 illustrate the evolution of the margins between the stability and instability zones as the degree of plasma anisotropy changes, but axial profile of magnetic field is fixed (by \( q = 4 \)). The differences between CwPr and CwSt configurations are minimal at maximum anisotropy (which corresponds to plots (a), \( R = 1.1 \)) and maximum at minimal anisotropy (plots (f), \( R = 8 \)).
Figure 8. Stability map A1-BwSt4 for model magnetic field (21), straightened conducting wall and anisotropic plasma pressure model (15) at combined stabilization with weak MHD anchors simulated by conducting end plates installed at mirror ratio $S = 4$ behind the magnetic mirror throat; $q \in \{2, 4, 8\}$, set of mirror ratios $M \in \{24, 16, 8, 4\}$ and minimum anisotropy $R = M$. Instability zone is located between $\beta_{cr1}(r_w/a_0)$ (lower branch) and $\beta_{cr2}(r_w/a_0)$ (upper branch of the same curve). Common zone of stability is shaded. It is determined by the plasma with a sharp boundary ($k = \infty$), for which it has the minimum size. Correspondence of the colors and markers of the curves to the index $k$ is shown under the bottom row of the graphs. Compare with figure 6.

6. Blind wall stabilization

End perfectly conducting plates simulating the effect of the end MHD stabilizers can be located not only in the throat of magnetic mirror. For example, the GDT facility has ring limiters (see, e.g. [18]). They are located between the magnetic mirror and the stop point of fast ions. Such a limiter in the context of rigid mode study can be imitated by a conducting end plate installed in a plane with local mirror ratio $S$ such that $R < S < M$. Generally speaking, end plate can be placed both in front of the magnetic plug and behind it. In our internal classification, a configuration with a ring limiter in the region $0 < z < 1$ is designated ‘Rw’ (from the words ‘Ring Wall’) with the addition of a numerical value $S$, while configurations with a conducting end plate behind the magnetic mirror, that is, in the region $z > 1$, has the shortcut ‘Bw’ (from the words ‘Blind Wall’), with the addition of the number $S$ as well.

Below we assume that the end conducting plates are moved from the magnetic mirror to the behind-mirror region, namely, to the plane where the current mirror ratio is $S = 4$. It means that the boundary condition $\phi = 0$ should be set not in the plane $z = \pm 1$, as equation (10) prescribes, but in the plane $z = z_{\text{cond}}$, where $R(z_{\text{cond}}) = S/R$, as prescribed by equation (11). A similar transfer of the boundary condition was previously used in [19] to simulate an MHD stabilizer with a moderate stability margin. The results of such a calculation are drawn in figures 8 and 9. Comparing figures 6 and 8 or
Figure 9. Stability map A1-BwSt4 for model magnetic field (21) and anisotropic plasma pressure model (15), simulating normal NBI at combined stabilization by straightened conducting lateral shell and the end MHD anchors located beyond mirror throat at mirror ratio $S = 4$; $M = 16$, $R \in \{1.1, 1.2, 1.5, 2, 4, 8\}$, $q = 4$. Instability zone is located between the lower $\beta_{\text{cr}}(r_w/a_0)$ and upper $\beta_{\text{cr}}(r_w/a_0)$ branches of the margin curve of the same color. Correspondence of the colors and markers of the margin curves to the index $k$ is shown under the bottom row. Shaded common zone of stability is located to the left of margin curve for a plasma with a sharp boundary ($k = \infty$). Compare with figure 7. Both the odd and even rows contain graphs for the A1-BwSt4 configurations.

Figure 10. Boundary of lower stability zone $\beta_{\text{cr}}$ for the magnetic field (21) and anisotropic plasma model (15) simulating normal NBI, with stabilization by end conducting plates installed outside the central cell at local mirror ratio $S = 4$; no lateral conducting wall is assumed. Stability zone for the radial profile with index $k$ is located below the corresponding curve. The absence of a curve with particular color means that the corresponding profile is stable over the entire range $0 < \beta < 1$. Correspondence of the colors and markers of the curves to index $k$ is shown under the bottom row of the graphs. Compare with figure 12 [2] in [2].
figures 7 and 9 it is easy to see that the second figure in each pair has larger instability zones. In other words, the weakening of the MHD end stabilizer expands the zone of instability, with smoother radial pressure profiles becoming unstable first.

In contrast to the wall stabilization maps shown in figures 4 and 5 in section 4, figures 6–9 demonstrate a strong dependence of the instability zone size on the magnetic field profile characterized by the index \( q \). As can be seen from these figures, the width of the instability zone \( \beta_{\text{cr}1} < \beta < \beta_{\text{cr}2} \) are maximum for the most smooth magnetic field profile with index \( q = 2 \). It is minimal at \( q = 8 \).

The shape of the conducting shell obviously does not matter in the limit \( r_a / a_0 \to \infty \), since this limit means that the lateral wall is actually removed. We would reproduce figure 12 [2] if we repeat the calculation in this limit for the straightened chamber. That figure shows graphs of \( \beta_{\text{cr}1} \) versus \( R \) for different values of the mirror ratio \( M \) and index \( q \) in the limit \( r_a / a = \infty \).

When discussing figure 12 [2] in that article, it was said that at a large distance of the lateral conducting wall from the plasma, the upper stability zone becomes extremely thin (or even disappears for radially smooth profiles), and it is located in the immediate vicinity of the boundary \( \beta = 1 \), where the paraxial approximation fails. In this case, the lower zone remains sufficiently wide, which is in full agreement with the results of calculating the threshold of small-scale ballooning instability [5, 20, 21]. According to those calculations, an analogue of \( \beta_{\text{cr}1} \) for such instability even without taking into account the effects of FLR, is estimated to be of the order of 0.6 \( \div \) 0.7.

The analysis of that figure in [2] unexpectedly showed that a large-scale ballooning perturbation with azimuth number \( m = 1 \) in a plasma with a fairly smooth radial profile can be stable over the entire allowable range \( 0 < \beta < 1 \) of the beta parameter even in the absence of a lateral conducting wall. This requires that the degree of anisotropy be sufficiently high, and the mirror ratio, on the contrary, not too large. It was found that the magnetic field profiles with narrow and steep mirror plugs are more stable.

In order not to tritely repeat figure 12 [2], we once again assumed that the end conducting plates were relocated from the throat of magnetic mirror to a location with mirror ratio \( S = 4 \) behind the plug. The results are shown in figure 10. Their comparison with figure 12 [2] for the A1-CwPr configuration shows that the decrease in \( \beta_{\text{cr}1} \) can reach 30%. In addition, the range of values of the mirror ratio for which the entire range \( 0 < \beta < 1 \) is stable for smooth pressure profiles has been significantly reduced. Such a trend was quite expected for a weaker end-face MHD stabilizer. We did not make computation for the A1-BwPr4 configuration since we do not consider it realistic.

It may be surprising that the instability zone disappears at \( r_a / a \to \infty \) as the radial pressure profile is smoothed out. On the one hand, such a statement may seem doubtful. On the other hand, as noted above, even for small-scale perturbations, the value of \( \beta_{\text{cr}1} \) is quite large. Intuitively, it seems that with respect to the rigid ballooning mode, the value of \( \beta_{\text{cr}1} \) should be even larger, since it is more difficult to deform the entire plasma column than a thin magnetic tube.

7. Conclusions

In this paper, within the framework of an anisotropic pressure model simulating the normal injection of neutral beams into a relatively cold target plasma, we studied the feasibility of stabilization of the \( m = 1 \) rigid flute and ballooning modes in an axially symmetric mirror trap using a perfectly conducting cylindrical wall in the form of a straightened cylindrical chamber. Unlike most previous works, which assumed a not quite realistic stepwise plasma pressure radial profile and the shape of the conducting chamber repeated the shape of the plasma column on an enlarged scale, we considered a set of diffuse pressure profiles with different degrees of sharpness of the plasma edge, as well as several variants of the axial profile of the vacuum magnetic field with different mirror ratios and different width of the magnetic mirrors.

As expected, the conducting wall in the form of a straightened cylinder has a weaker stabilizing effect on ballooning perturbations compared to a proportional chamber, but the attenuation does not seem to be critically large. It can be assumed that under the conditions of a real experiment, the shape of the conducting shell will be something in between the proportional and straightened chambers. In this sense, these two shapes are two antipodes, and the results of calculations for these shapes determine the range of the expected values of \( \beta_{\text{cr}2} \) for a real experiment.

Admitting as more than encouraging the conclusion about the weak influence of the shape of the conducting chamber on its stabilizing properties, it is necessary to make a reservation that this conclusion is based on the analysis of only one model of anisotropic pressure, which imitates the plasma that is formed by normal NBI to the minimum of the magnetic field at the right angle to the axis of the trap. It should be expected that with oblique injection, the influence of the shape of the lateral conducting wall will increase markedly.

Our calculations also confirmed some of the earlier conclusions. In particular, several graphs have shown that a perfectly conducting side wall is the best at stabilizing balloontype perturbations in a plasma with a steep radial pressure profile. In contrast, MHD end stabilizers are best at stabilizing the smoothest radial pressure profiles. The influence of the axial profile of the magnetic field is minimal for MHD stabilization by means of lateral wall only. On the contrary, stabilization with the help of end MHD stabilizers is very sensitive to the value of the mirror ratio and the features of the magnetic field profile. The degree of anisotropy is also decisive. The larger it is, the larger the stability zone in the space of other parameters of the problem. Thus, these two methods of stabilization perfectly complement each other.

Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this
field of study. The data that support the findings of this study are available upon reasonable request from the authors.

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References

[1] Kotelnikov I, Zeng Q, Prikhodko V, Yakovlev D, Zhang K, Chen Z and Yu J 2022 Wall stabilization of the rigid ballooning m = 1 mode in a long-thin mirror trap Nucl. Fusion 62 096025
[2] Kotelnikov I, Prikhodko V and Yakovlev D 2023 Wall stabilization of high-beta anisotropic plasmas in an axisymmetric mirror trap Nucl. Fusion 63 066027
[3] Rosenbluth M, Krall N and Rostoker N 1962 Finite larmor radius stabilization of “weakly” unstable confined plasmas Nucl. Fusion Part I 143
[4] Berk H et al 1985 Stabilization of an axisymmetric mirror cell and trapped particle modes Plasma Physics and Controlled Nuclear Fusion Research 1984 (Nuclear Fusion, Suppl.) vol 2 (IAEA) pp 321–35
[5] Kotelnikov I, Lizunov A and Zeng Q 2022 On the stability of small-scale ballooning modes in axisymmetric mirror traps Plasma Sci. Technol. 24 015102
[6] LoDestro L L 1986 The rigid ballooning mode in finite-beta axisymmetric plasmas with diffuse pressure profiles Phys. Fluids 29 2329
[7] Li X Z, Kesner J and Lane B 1987 MHD stabilization of a high beta mirror plasma partially enclosed by a conducting wall Nucl. Fusion 27 101
[8] Li Z-Z, Kesner J and LoDestro L 1987 Wall stabilized high beta mirror plasma in a rippled magnetic field Nucl. Fusion 27 1259
[9] Newcomb W A 1981 Equilibrium and stability of collisionless systems in the paraxial limit J. Plasma Phys. 26 529
[10] Bagryansky P A et al 2016 Status of the experiment on magnetic field reversal at binp AIP Conf. Proc. 1771 030015
[11] Akhmetov T, Davydenko V, Ivanov A and Murakhitin S 2018 Sustainment of high-beta mirror plasma by neutral beams Plasma Phys. Technol. 5 125
[12] Sudnikov A and Soldatkin E 2019 Review of recent advances and new ideas in development of the open magnetic traps AIP Conf. Proc. 2179 020026
[13] Anderson J et al 2020 Introducing the Wisconsin HTS Axisymmetric Mirror (APS Division of Plasma Physics Meeting 2020) (abstract id.CP20.003)
[14] Egedal J, Endrizzi D, Forest C and Fowler T 2022 Fusion by beam ions in a low collisionality, high mirror ratio magnetic mirror Nucl. Fusion 62 126053
[15] Endrizzi D et al 2023 Physics basis for the wisconsin hts axisymmetric mirror (wham) J. Plasma Phys. 89 975890501
[16] Kotelnikov I 2024 On the stability of the m = 1 rigid ballooning mode in a mirror trap with high-beta sloshing ions (in preparation)
[17] Lansky J M 1993 On the paraxial equilibrium of the finite β plasma in open magnetic configuration Technical Report BudkerINP 93-96 (Budker Institute of Nuclear Physics)
[18] Bagryansky P A et al 2011 Confinement of hot ion plasma with β = 0.6 in the gas dynamic trap Fusion Sci. Technol. 59 31
[19] Kesner J 1985 Axisymmetric, wall-stabilized tandem mirrors Nucl. Fusion 25 275
[20] Bushkova O A and Mirnov V V 1986 Influence of the configuration of the magnetic field on the MHD stability of the gas-dynamic trap VANT (Questions of Atomic Sci. and Tech., ser. Nucl. Fus., p 19 (in Russian)
[21] Ryutov D D and Stupakov G V 1980 New results in the theory of MHD-stability and transport processes in ambipolar traps Plasma Physics and Controlled Nuclear Fusion Research, 8th Conf. Proc. (Brussel, 1–10 July 1980) vol 1 (International Atomic Energy Agency) pp 119–32
[22] Bagryansky P A, Chen Z, Kotelnikov I A, Yakovlev D V, Prikhodko V V, Zeng Q, Bai Y, Yu J, Ivanov A and Wu Y 2020 Development strategy for steady-state fusion volumetric neutron source based on the gas-dynamic trap Nucl. Fusion 60 036005
[23] Kotelnikov I, Chen Z, Bagryansky P, Sudnikov A, Zeng Q, Yakovlev D, Wang F, Ivanov A and Wu Y 2020 Summary of the 2nd international workshop on gas-dynamic trap based fusion neutron source (GDT-FNS) Nucl. Fusion 60 067001
[24] Prikhodko V V, Chen Z, Kotelnikov I A, Yakovlev D V, Yu J and Zeng Q 2021 Simulation of plasma parameters for ALIANCE project, Issues of atomic science and technology Thermonucl. Fusion 44 166
[25] Yang W, Zeng Q, Chen C, Chen Z, Song J, Wang Z, Yu J, Yakovlev D and Prikhodko V 2021 Shielding design and neutronics calculation of the GDT based fusion neutron source aliance Fusion Eng. Des. 164 112221
[26] Chen Z et al 2022 Summary of the 3rd international workshop on gas-dynamic trap based fusion neutron source (GDT-FNS) Nucl. Fusion 62 067001
[27] Yakovlev D et al 2022 Conceptual design of the ALIANCE-T mirror experiment Nucl. Fusion 62 076017