Abstract

Biological synaptic plasticity exhibits nonlinearities that are not accounted for by classic Hebbian learning rules. Here, we introduce a simple family of generalized nonlinear Hebbian learning rules. We study the computations implemented by their dynamics in the simple setting of a neuron receiving feedforward inputs. These nonlinear Hebbian rules allow a neuron to learn tensor decompositions of its higher-order input correlations. The particular input correlation decomposed and the form of the decomposition depend on the location of nonlinearities in the plasticity rule. For simple, biologically motivated parameters, the neuron learns eigenvectors of higher-order input correlation tensors. We prove that tensor eigenvectors are attractors and determine their basins of attraction. We calculate the volume of those basins, showing that the dominant eigenvector has the largest basin of attraction. We then study arbitrary learning rules and find that any learning rule that admits a finite Taylor expansion into the neural input and output also has stable equilibria at generalized eigenvectors of higher-order input correlation tensors. Nonlinearities in synaptic plasticity thus allow a neuron to encode higher-order input correlations in a simple fashion.

1 Introduction

In Hebbian learning, potentiation of the net synaptic weight between two neurons is driven by the correlation between pre- and postsynaptic activity [1]. That postulate is a cornerstone of the theory of synaptic plasticity and learning [2][3]. In its basic form, the Hebbian model leads to runaway potentiation or depression of synapses, since the pre-post correlation increases with increasing synaptic weight [4]. That runaway potentiation can be stabilized by supplemental homeostatic plasticity dynamics [5], by weight dependence in the learning rule [6][7], or by synaptic scaling regulating a neuron’s total synaptic weight [8][9]. In 1982, Erkki Oja observed that a linear neuron with Hebbian plasticity and synaptic scaling learns the first principal component of its inputs [10]. In 1985, Oja and Karhunen proved that this is a global attractor of the Hebbian dynamics [11]. This led to a fountain of research on unsupervised feature learning in neural networks [12][13].

Principal component analysis (PCA) describes second-order features of a random variable. Both naturalistic stimuli and neural activity can, however, exhibit higher-order correlations [14][15]. Canonical models of retinal and thalamic processing whiten inputs, removing pairwise features [16][22]. Beyond-pairwise features, encoded in tensors, can provide a powerful substrate for learning from data [23][27].

The basic Hebbian postulate does not take into account fundamental nonlinear aspects of biological synaptic plasticity in cortical pyramidal neurons. First, synaptic plasticity depends on beyond-pairwise activity correlations [28][33]. Second, spatially clustered and temporally coactive synapses exhibit
We assume that there is a rich literature on computationally motivated forms of nonlinear Hebbian learning (section 3.2). Here, we will prove that these biologically motivated nonlinearities allow a neuron to learn higher-order features of its inputs.

We study the dynamics of a simple family of generalized Hebbian learning rules, combined with synaptic scaling (eq. 1). Equilibria of these learning rules are invariants of higher-order input correlation tensors. The order of input correlation (pair, triplet, etc.) depends on the pre- and postsynaptic nonlinearities of the learning rule. When the only nonlinearity in the plasticity rule is postsynaptic, the steady states are eigenvectors of higher-order input correlation tensors \cite{42,43}. We prove that these eigenvectors are attractors of the generalized Hebbian plasticity dynamics and characterize their basins of attraction.

Then, we study further generalizations of these learning rules. We show that any plasticity model (with a finite Taylor expansion in the synaptic input, neural output, and synaptic weight) has steady states that generalize these tensor decompositions to multiple input correlations, including generalized tensor eigenvectors. We show that these generalized tensor eigenvectors are stable equilibria of the learning dynamics. Due to the complexity of the arbitrary learning rules, we are unable to fully determine their basins of attraction. We do find that these generalized tensor eigenvectors are in an attracting set for the dynamics, and characterize its basin of attraction. Finally, we conclude by discussing extensions of these results to spiking models and weight-dependent plasticity.

2 Results

Take a neuron receiving $K$ inputs $x_i(t)$, $i \in [K]$, each filtered through a connection with synaptic weight $J_i(t)$ to produce activity $n(t)$. We consider synaptic plasticity where the evolution of $J_i$ can depend nonlinearly on the postsynaptic activity $n(t)$, the local input $x_i(t)$, and the current synaptic weight $J_i(t)$. We model these dependencies in a learning rule $f$:

$$J_i(t + dt) = \frac{J_i(t) + dt}{\tau} f_i(n(t), x_i(t), J_i(t)) \left| \left| \frac{J(t) + dt}{\tau} f(n(t), x(t), J(t)) \right| \right|$$

(1)

The parameter $a$ sets the output-dependent nonlinearity of the learning rule, $b$ sets the input-dependent nonlinearity, and $c$ sets its dependence on the current synaptic weight. Eq. 1 assumes a simple form for these nonlinearities; we discuss arbitrary nonlinear learning rules in section 3.2. We assume that $a$ and $b$ are positive integers, as in higher-order voltage or spike timing–dependent plasticity (STDP) models \cite{44,45}. The scaling by the norm of the synaptic weight vector, $\left| \left| J \right| \right|$, models homeostatic synaptic scaling \cite{8,10}. Bold type indicates a vector, matrix, or tensor (depending on the variable) and regular font with lower indices indicates elements thereof. Roman type denotes a random variable ($x$).

We assume that $x(t)$ is drawn from a stationary distribution with finite moments of order $a + b$. Combined with a linear neuron, $n = J^T x$, and a slow learning rate, $\tau \gg dt$, this implies the following dynamics for $J$ (appendix A.1):

$$\tau \dot{J}_i = J_i^c \sum_{\alpha} \mu_{i,\alpha} (J^\otimes \alpha)_\alpha - J_i \sum_{j,\alpha} J_j^{c+1} \mu_{j,\alpha} (J^\otimes \alpha)_\alpha.$$  

(2)

In eq. 2, $J_i = dJ_i/dt, \alpha = (j_1, \ldots, j_a)$ is a multi-index, and $\otimes$ is the vector outer product; $J^\otimes \alpha$ is the $a$-fold outer product of the synaptic weight vector $J$. $\mu$ is a higher-order moment (correlation) tensor of the inputs:

$$\mu_{i,\alpha} = \langle x_i^\otimes (x^\otimes \alpha)_\alpha \rangle_x$$

(3)

where $\langle \rangle_x$ denotes the expectation with respect to the distribution of the inputs. $\mu$ is an $(a + 1)$-order tensor containing an $(a + b)$-order joint moment of $x$. The order of the tensor refers to its number of indices, so a vector is a first-order tensor and a matrix a second-order tensor. $\mu$ is cubical; each mode of $\mu$ has the same dimension $K$. In the first term of eq. 2 for example, $\sum_{\alpha} \mu_{i,\alpha} (J^\otimes \alpha)_\alpha$ takes the dot product of $J$ along modes 2 through $a + 1$ of $\mu$.

2.1 Steady states of nonlinear Hebbian learning

If we take $a = b = 1$, then $\mu$ is the second-order correlation of $x$ and $\alpha$ is just the index $j$. With $c = 0$ also, eq. 2 reduces to Oja’s rule and $J$ is guaranteed to converge to the dominant eigenvector
While the top eigenvector of a matrix can be computed efficiently, computing the top eigenvector weight vector and each of the first 10 eigenvectors (components of the Tucker decomposition) was not symmetric (fig. 1b). The mean squared error of the canonical polyadic (CP) approximation $c$ we will usually focus on parameter sets with (e.g., fig. 1e, f), around which the weights fluctuated due to the finite learning timescale $\tau$. Any individual realizations of the weight dynamics, we computed the overlap between the final synaptic equilibrium is guaranteed to be some eigenvector of the input correlation tensor $\mu$. At steady states of eq. 2 where $J = 0$, the first eigenvector of $J$ is a trivial steady state. At steady states of eq. 2 where $J_i \neq 0$,

$$\sum \alpha \mu_{i,\alpha}(J^{\otimes a})_{\alpha} = \lambda J_i^{1-c}$$

where $\lambda(\mu, J) = \sum_{j,\alpha} J_j^{c+1} \mu_{j,\alpha}(J^{\otimes a})_{\alpha}$, (4)

so that $J$ is invariant under the multilinear map of $\mu$ except for a scaling by $\lambda$ and element-wise exponentiation by $1-c$. For two parameter families $(a, b, c)$, eq. 4 reduces to different types of tensor eigenvalue/vector equation $c$ for $\mu$ [10, 11]. We next briefly describe these and some of their properties.

First, if $a+c = 1$, we have the tensor eigenvalue/vector equation

$$\sum \alpha \mu_{i,\alpha}(J^{\otimes a})_{\alpha} = \lambda J_i$$

Qi called these $\lambda, J$ an E-eigenpair [42] and Lim called them the $\ell^2$-eigenpair [43]. In general, a tensor may have infinitely many such eigenpairs. If the spectrum of a $K$-dimensional tensor of order $a + 1$ is finite, however, there are $(a^{K-1})/(a-1)$ eigenvalues counted with multiplicity, and the spectrum of a symmetric tensor is finite [50, 51]. (If $b = 1$, $\mu$ is symmetric.) Unlike the steady states when $a + c = 1$, these eigenpairs are invariant under orthogonal transformations [42]. For non-negative $\mu$, there exists a positive eigenpair [52]. It may not be unique, however, unlike the largest eigenpair for $a + c = 1$ (an anti-Perron-Frobenius result) [51]. In the remainder of this paper we will usually focus on parameter sets with $c = 0$ and use “tensor eigenvector” to refer to those of eq. 5.

2.2 Dynamics of nonlinear Hebbian learning

For the linear Hebbian rule, $(a, b, c) = (1, 1, 0)$, Oja and Karhunen proved that the first principal component of the inputs is a global attractor of eq. 2 [11]. We thus asked whether the first tensor eigenvector is a global attractor of eq. 2 when $c = 0$ but $(a, b) \neq (1, 1)$. We first simulated the nonlinear Hebbian dynamics. For the inputs $x$, we whitened $35 \times 35$ pixel image patches sampled from the Berkeley segmentation dataset (fig. 1a; [53]). For $b \neq 1$, the correlation of these image patches was not symmetric (fig. 1b). The mean squared error of the canonical polyadic (CP) approximation of these tensors was higher for $b = 1$ than $b = 2$ (fig. 1d). It decreased slowly past rank $\sim 10$, and the rank of the input correlation tensors was at least $30$ (fig. 1d).

The nonlinear Hebbian learning dynamics converged to an equilibrium from random initial conditions (e.g., fig. 1e, f), around which the weights fluctuated due to the finite learning timescale $\tau$. Any equilibrium is guaranteed to be some eigenvector of the input correlation tensor $\mu$ (section 2.1). For individual realizations of the weight dynamics, we computed the overlap between the final synaptic weight vector and each of the first 10 eigenvectors (components of the Tucker decomposition) of the corresponding input correlation $\mu$ [47, 54]. The dynamics mostly converged to the first eigenvector. For a non-negligible fraction of initial conditions, however, the nonlinear Hebbian rule converged to subdominant eigenvectors (fig. 1f, h). The input correlations $\mu$ did have a unique dominant eigenvector (fig. 1, blue), but the dynamics of eq. 2 did not always converge to it. This finding stands in contrast to the standard Hebbian rule, which must converge to the first eigenvector if it is unique [11].

While the top eigenvector of a matrix can be computed efficiently, computing the top eigenvector of a tensor is, in general, NP-hard [55]. To understand the learning dynamics further, we examined them analytically. Our main finding is that with $(b, c) = (1, 0)$ in the generalized Hebbian rule, eigenvectors of $\mu$ are attractors of eq. 2. Contrary to the case when $a = 1$ (Oja’s rule), the dynamics are thus not guaranteed to converge to the first eigenvector of the input correlation tensor when $a > 1$. The first eigenvector of $\mu$ does, however, have the largest basin of attraction.
Figure 1: Convergence of higher-order Hebbian plasticity to tensor eigenvectors of natural image correlations. a) Example \( P \times P \) image patches (\( P = 35 \) pixels). b) Third-order input correlation \( \mu \) that drives plasticity for \( a = 1, b = 2 \). c) Third-order input correlation \( \mu \) that drives plasticity for \( a = 2, b = 1 \). d) Mean squared error of the CP approximation of the input correlations. (Curves: mean across 4 initializations for the alternating least-squares computation of the CP, with standard deviation within the line thickness.) e) Learning dynamics on natural inputs: five randomly selected synaptic weights with \( (a,b,c) = (2,1,0) \). f) Example final synaptic weight configuration with \( (a,b,c) = (2,1,0) \). g) Overlap of the synaptic weights with their final closest singular vector, \( U_1^T J \). Solid line: mean over 10 samples of natural image patches and 10 realizations of the weight dynamics for each input patch. Shaded region: standard deviation. Initial conditions for \( J \) are chosen uniformly on the unit sphere. h) Histogram of the closest singular vector to the final synaptic weights for \( (a,b,c) = (2,1,0) \).

**Theorem 1.** In eq. 2 take \( (b,c) = (1,0) \). Let \( \mu \) be a cubical, symmetric tensor of order \( a + 1 \) and odeco with \( R \) components:

\[
\mu = \sum_{r=1}^{R} \lambda_r (U_r)^{\otimes a+1}
\]

where \( U \) is a matrix of unit-norm orthogonal eigenvectors: \( U^T U = I \). Let \( \lambda_i > 0 \) for each \( i \in [R] \) and \( \lambda_i \neq \lambda_j \) \( \forall (i,j) \in [R] \times [R] \) with \( i \neq j \). Then for each \( k \in [R] \):

1. With any odd \( a > 1 \), \( J = \pm U_k \) are attracting fixed points of eq. 2 and their basin of attraction is \( \bigcap_{i \in [R] \backslash k} \{ J : |U_i^T J/|U_k^T J| < (\lambda_k/\lambda_i)^{1/(a-1)} \} \). Within that region, the separatrix of \( +U_k \) and \( -U_k \) is the hyperplane orthogonal to \( U_k^T \) : \( \{ J : U_k^T J = 0 \} \).
2. With any even positive \( a \), \( J = U_k \) is an attracting fixed point of eq. 2 and its basin of attraction is \( \{ J : U_k^T J > 0 \} \cap_{i \in [R] \backslash k} \{ J : U_i^T J/|U_k^T J| < (\lambda_k/\lambda_i)^{1/(a-1)} \} \).
3. With any even positive \( a \), \( J = 0 \) is a neutrally stable fixed point of eq. 2 with basin of attraction \( \{ J : \sum_{j=1}^{R} (U_j^T J)^2 < 1 \land U_k^T J < 0 \ \forall \ k \in [R] \} \).

**Remark.** Each component of the orthogonal decomposition of an odeco tensor \( \mu \) (eq. 5) is an eigenvector of \( \mu \). If \( R < (a^{k-1})/(a-1) \), there are additional eigenvectors. The components of the orthogonal decomposition are the robust eigenvectors of \( \mu \): the attractors of its multilinear map [26, 56]. The non-robust eigenvectors of an odeco tensor are fixed by its robust eigenvectors and their eigenvalues [37].
The proof of theorem [1] is given in appendix A.2. To prove theorem [1] we project \( J \) onto the eigenvectors of \( \mu \), and study the dynamics of the loadings \( v = U^T J \). This leads to the discovery of a collection of unstable manifolds: each pair of axes \((i, k)\) has an associated unstable hyperplane \( v_i = v_k (\lambda_i/\lambda_k)^{1/(a-1)} \) (and if \( a \) is odd, also the corresponding hyperplane with negative slope). These partition the phase space into the basins of attraction of the eigenvectors of \( \mu \).

For example, consider a fourth-order input correlation (corresponding to \( a = 3 \) in eq. [1]) with two eigenvectors. The phase portrait of the loadings is in fig. 2a, with the nullclines in black and attracting and unstable manifolds in blue. The attracting manifold is the unit sphere. There are two unstable hyperplanes that partition the phase space into the basins of attraction of \((0, \pm 1)\) and \((\pm 1, 0)\), where the synaptic weights \( J \) are an eigenvector of \( \mu \).

For even \( a \), only the unstable hyperplanes with positive slope survive (fig. 2b, blue line). The unit sphere (fig. 2b, blue) is attracting from any region where at least one loading is positive. Its vertices \([1] \times [1]\) are equilibria; \((v_1, v_2) = (1, 0)\) and \((0, 1)\) are attractors and the unstable hyperplane separates their basins of attraction. For the region with all loadings negative that is the basin of attraction of the origin, noise will drive the system away from zero towards one of the eigenvector equilibria.

By partitioning the phase space of \( J \) into basins of attraction for eigenvectors of \( \mu \), theorem [1] also allows us to determine the volumes of those basins of attraction. The basins of attraction are open sections of \( \mathbb{R}^R \) so we measure their volume relative to that of a large hypercube.

**Corollary 1.1.** Let \( V_k \) be the relative volume of the basin of attraction for \( J = U_k \). For odd \( a > 1 \),

\[
V_k = R^{-1} \prod_{i=1}^{R} \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a-1)}
\]  \( (7) \)

**Corollary 1.2.** Let \( V_k \) be the relative volume of the basin of attraction for \( J = U_k \). For even positive \( a \),

\[
V_k = 2^{1-R} \left( R^{-1} \prod_{i} \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a-1)} + (R-1)^{-1} \sum_{j \neq k \neq i} \prod_{l \neq i, l \neq j} \left( \frac{\lambda_k}{\lambda_l} \right)^{1/(a-1)} \right)
\]  \( + (R-2)^{-1} \sum_{j,i,k \neq l} \prod_{l} \left( \frac{\lambda_k}{\lambda_l} \right)^{1/(a-1)} + \ldots + 1 \)  \( (8) \)

The calculations for corollaries [1.1] and [1.2] are given in appendix A.2. We see that the volumes of the basins of attraction depend on the spectrum of \( \mu \), its rank \( R \), and its order \( a \). The result for odd \( a \) also

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Figure 2: Example dynamics of the loadings, \( v = U^T J \), for rank-two input correlations. Black curves: nullclines of \( v_1, v_2 \). Blue curves: stable sets and separatrices of the basins of attraction of the sparse attractors for \( v \).  

**a)** Odd \( a \) (\( a = 3 \)). The stable set is the unit sphere, and the separatrices are \( v_2 = \pm v_1 \left( \frac{\lambda_1}{\lambda_2} \right)^{1/(a-1)} \).  

**b)** Even \( a \) (\( a = 2 \)). The unstable set is \( v_2 = v_1 \left( \frac{\lambda_1}{\lambda_2} \right)^{1/(a-1)} \) (solid blue line). In (a,b), \( \lambda_1, \lambda_2 \) = (3, 1).  

**c)** Phase portrait for a two-term learning rule (eq. 9). All parameters of the input correlation tensors \( \lambda_{mr}, A_i \) are equal to one. Dashed blue curve: \( \mathcal{L} = \{ v : \sum_{m,j} \lambda_{mj} v^m_j v^{m+1} = 0 \} \).
We then approximate \( f \), truncating its expansion after those

While eigenvectors of \( \mu \) with small eigenvalues contribute little to values of the input correlation \( \mu \), they can have a large impact on the basins of attraction. The volume of the basin of attraction of eigenvector \( k \) is proportional to \( \lambda_k^{R/(a-1)} \). An eigenvector with eigenvalue \( \epsilon \) scales the basins of attraction for the other eigenvectors by \( \epsilon^{-1/(a-1)} \). The relative volume of two eigenvectors’ basins of attraction is, however, unaffected by the other eigenvalues whatever their amplitude. With \( a \) odd, the ratio of the volumes of the basins of attraction of eigenvectors \( k \) and \( j \) is \( V_k/V_j = (\lambda_k/\lambda_j)^{R/(a-1)} \).

We see in theorem 1 that attractors of eq. 2 are points on the unit hypersphere, \( S \). While eigenvectors of \( \mu \) are robust eigenvectors is an attracting equilibrium manifold and its basin of attraction is defined by each eigenvector’s basins of attraction boundaries with the other \( R - d \) robust eigenvectors.

2.3 Arbitrary learning rules

So far we have studied phenomenological plasticity rules in the particular form of eq. 1. The neural output \( n \), input \( x_i \), and synaptic weight \( J_i \) were each raised to a power and then multiplied together. Changes in the strength of actual synapses are governed by complex biochemical, transcriptional, and regulatory pathways [58]. We view these as specifying some unknown function of the neural output, input, and synaptic weight, \( f(n, x, J) \). That function might not have the form of eq. 1. So, we next investigate the dynamics induced by arbitrary learning rules \( f \). We see here that under a mild condition, any equilibrium of the plasticity dynamics will have a similar form as the steady states of eq. 2. If \( f \) does not depend on \( J \) except through \( n \), equilibria will be generalized eigenvectors of higher-order input correlations.

The Taylor expansion of \( f \) around zero is:

\[
\dot{f}_i(n(t), x_i(t), J_i(t)) = \sum_{m=1}^{\infty} A_m n^a_m(t) x_i^b_m(t) J_i^c_m(t) \tag{9}
\]

where the coefficients \( A_m \) are partial derivates of \( f \). We assume that there exists a finite integer \( N \) such that derivatives of order \( N + 1 \) and higher are negligible compared to the lower-order derivatives. We then approximate \( f \), truncating its expansion after those \( N \) terms. With a linear neuron, synaptic scaling, and slow learning, this implies the plasticity dynamics

\[
\tau \dot{J}_i = \sum_m J_i^c_m \sum_{\alpha_m} m \mu_{i,\alpha_m} (J^{\otimes \alpha_m})_{\alpha_m} - J_i \sum_{\alpha_m} J_i^m \sum_{\alpha_m} m \mu_{i,\alpha_m} (J^{\otimes \alpha_m})_{\alpha_m} \tag{10}
\]

where \( m \mu_{i,\alpha_m} = A_m (x_i^b_m (x^{\otimes \alpha_m})_{\alpha_m})_I \). At steady states where \( J_i \neq 0 \),

\[
\sum_m J_i^c_m \sum_{\alpha_m} m \mu_{i,\alpha_m} (J^{\otimes \alpha_m})_{\alpha_m} = \lambda J_i, \text{ where } \lambda(J, \{m \mu\}) = \sum_m \sum_{\alpha_m} J_i^m \sum_{\alpha_m} m \mu_{j,\alpha} (J^{\otimes \alpha_m})_{\alpha} \tag{11}
\]

If each \( c_m = 0 \), this is a kind of generalized tensor eigenfunction:

\[
\sum_{m,\alpha_m} m \mu_{i,\alpha} (J^{\otimes \alpha_m})_{\alpha} = \lambda J_i \tag{12}
\]
so that $J$ is invariant under the combined action of the multilinear maps $m_\mu$ (which are potentially of different orders). If $a_1 = a_2 = \cdots = a_m$, then this can be simplified to a tensor eigenvector equation by summing the input correlations $m_\mu$. If different terms of the expansion of $f$ generate different-order input correlations, however, the steady states are no longer necessarily equivalent to tensor eigenvectors. If there exists a synaptic weight vector $J$ that is an eigenvector of each of those input correlation tensors, $\sum_\alpha m_\mu i_\alpha J^{(\alpha)} = \lambda J$, for each $m$, then that configuration $J$ is a steady state of the plasticity dynamics with each $c_m = 0$.

We next investigated whether these steady states were attractors in simulations of a learning rule with a contribution from two-point and three-point correlations ($a = (1, 2), b = 1, c = 0, A = (1, 1/2)$ in eq. 9). As before, we used whitened natural image patches for the inputs $x$ (fig. 1a). The two- and three-point correlations of those image patches had similar first eigenvalues, but the spectrum of the two-point correlation decreased more quickly than the three-point correlation (fig. 3a). The first three eigenvectors of the different correlations overlapped strongly (fig. 3b). This was not due to a trivial constant offset since the inputs were whitened. With this parameter set, the synaptic weights usually converged to the (shared) first eigenvector of the input correlations (fig. 3c, d, e (blue)).

![Figure 3](image)

**Figure 3:** Multi-term Hebbian plasticity rule converges to input correlation tensors’ shared eigenvectors. We study a learning rule with two terms, driven by the second- and third-order input correlations ($a = (1, 2), b = 1, c = 0$). **a**) Tucker spectra of the natural image patches’ correlation tensors. Solid lines: mean over 10 samples of 200 image patches. Shaded regions: standard deviation. **b**) Overlap of the eigenvectors of the image patches’ two- and three-point correlations (mean over 10 samples of 200 image patches). **c, d**) Example dynamics for the two-term learning rule. Transparency increases with the eigenvector number in **d**. **e**) Distribution of the final alignment to the first eigenvector over 10 random initial conditions for each of 10 samples of image patches. **f**) Final alignment to the two- and three-point correlations’ eigenvectors as a function of the weight on the two-point correlation, $A_1$. $A_2 = 1$.

We next asked how the weights of the different input correlations in the learning rule (the parameters $A_1, A_2$) affected the plasticity dynamics. When the learning rule weighted the inputs’ three-point correlation more strongly than the two-point correlation ($A = (1/2, 1)$), the dynamics converged almost always to the first eigenvector of the three-point correlation (fig. 3f, blue vs orange). Without loss of generality, we then fixed $A_2 = 1$ and varied the amplitude of $A_1$. As $A_1$ increased, the learning dynamics converged to equilibrium increasingly aligned with the top eigenvectors of the input correlations (fig. 3f). For sufficiently negative $A_1$, the dynamics converged to steady states that were neither eigenvectors of the two-point input correlation nor any of the top 20 eigenvectors of the three-point input correlation (fig. 3f).

Earlier, we saw that in single-term learning rules, the only attractors were robust eigenvectors of the input correlation (theorem). The dynamics of eq. usually converged to the first eigenvector.
because it had the largest basin of attraction (corollaries 1.1, 1.2). Here, we saw that at least for some parameter sets, the dynamics of a multi-term generalized Hebbian rule may not converge to an input eigenvector. This suggested the existence of other attractors for the dynamics of eq. 10.

We next investigated the steady states of multi-term nonlinear Hebbian rules analytically. We focused on the case when the different input correlations generated by the learning rule all have a shared set of eigenvectors. In this case, those shared eigenvectors are all stable equilibria of eq. 10. They are not, however, the only stable equilibria. We see in a simple example that there can be equilibria that are linear combinations of those eigenvectors with all negative weights (in fig. 2a, the fixed point in the lower left quadrant on the unit circle). In fact, any stable equilibrium that is not a shared eigenvector must be such a negative combination. Our results are summarized in the following theorem:

**Theorem 2.** In eq. 10 take \( b = 1, c = 0, a \in \mathbb{Z}^N \), and consider \( N \) cubical, symmetric tensors, \( m\mu \), each of order \( a_m + 1 \), for \( m \in [N] \), that are mutually odec into \( R \) components:

\[
m\mu = \sum_{r=1}^{R} \lambda_{mr} U_{r} \otimes U_{r} \otimes \cdots \otimes U_{r}
\]

with \( U^T U = I \). Let \( \lambda_{mr} \geq 0 \) and \( \sum_{m} \lambda_{mr} > 0 \) for each \( m, r \in [N] \times [R] \). Let

\[
S(J) = \sum_{i=1}^{R} (U_i^T J)^2 \text{ and } L(J) = \sum_{m=1}^{N} \sum_{i=1}^{R_m} \lambda_{mi} (U_i^T J)^{a_m + 1}.
\]

Then:

1. \( S^* = \{ J : S(J) = 1 \wedge L(J) > 0 \} \) is an attracting set for eq. 10 and its basin of attraction includes \( \{ J : L(J) > 0 \} \).

2. For each \( k \in [R] \), \( J = U_k \) is a stable equilibrium of eq. 10.

3. For each \( k \in [R] \), \( J = -U_k \) is a stable equilibrium of eq. 10 if \( \sum_{m} \lambda_{mk} (-1)^{a_m} < 0 \) (and unstable if \( \sum_{m} \lambda_{mk} (-1)^{a_m} > 0 \)).

4. Any other stable equilibrium must have \( U_k^T J \leq 0 \) for each \( k \in [R] \).

The claims of theorem 2 are proven in appendix A.2. Similar to theorem 1, we see that the robust eigenvectors of each input correlation generated by the learning rule are stable equilibria of the learning dynamics. The complexity of eq. 10 has kept us from determining their basins of attraction. We can, however, make several guarantees. First, in a large region, the unit sphere is an attracting set for the dynamics of eq. 10. Second, the only stable fixed points are either the eigenvectors of \( \pm \mu \) or combinations of the eigenvectors of \( \mu \) with only nonpositive weights. This is in contrast to the situation where the learning rule has only one term; then theorem 1 guarantees that the only attractors are eigenvectors.

### 3 Discussion

We have analyzed biologically motivated plasticity dynamics that generalize the Oja rule. One class of these compute tensor eigenvectors. We proved that without a multiplicative weight-dependence in the plasticity, those eigenvectors are attractors of the dynamics (theorem 1, figs. 1a, 2a, b). Contrary to Oja’s rule, the first eigenvector of higher-order input correlations is not a unique attractor. Rather, each eigenvector \( k \) has a finite basin of attraction, the size of which is proportional to \( \lambda_k^{b/a−1} \). If there are \( d \) codominant eigenvectors (\( \lambda_1 = \lambda_2 = \ldots = \lambda_d \)), the \( d \)-sphere they span is an attracting equilibrium manifold (corollary 1.4 in appendix A.2). Furthermore, steady states of any plasticity model with a finite Taylor polynomial in the neural output and inputs are generalized eigenvectors of multiple input correlations. These steady states are stable and attracting (theorem 2, figs. 2a, b).

#### 3.1 Spiking neurons and weight-dependence

While biological synaptic plasticity is certainly more complex than the simple generalized Hebbian rule of eq. 11, neural activity is also more complex than the linear model \( n = J^T x \). We examined the
simple linear-nonlinear-Poisson spiking model and a generalized spike timing–dependent plasticity (STDP) rule ([45]; appendix A.3). Similar to eqs. 2 and 10, we can write the dynamical equation for $J$ as a function of joint cumulant tensors of the input (eq. 56 in appendix A.3). These dynamics have a different structure than eqs. 2 and 10.

We focused here mainly on learning rules with no direct dependence on the synaptic weight ($c = 0$ in eq. 1, $c \neq 0$ in eq. 9). When $c \neq 0$, the learning dynamics cannot be simply analyzed in terms of the loading onto the input correlations’ eigenvectors. We studied the learning dynamics with weight-dependence for two simple families of input correlations: diagonal $\mu$ and piecewise-constant rank one $\mu$ (appendix A.4). In both cases, we found that with eigenvectors of those simple input correlations were also attractors of the plasticity rule. With diagonal input correlations, sparse steady states with one nonzero synapse are always stable and attracting when $a + c > 0$, but if $a + c \leq 0$ synaptic weights converge to solutions where all weights have the same magnitude (fig. A.4.1). With rank one input correlations, multiplicative weight-dependence can interfere with synaptic scaling and lead to an instability for the neurons’ total synaptic amplitude (fig. A.4.2).

3.2 Related work and applications

There is a rich literature on generalized or nonlinear forms of Hebbian learning. We briefly discuss the most closely related results, to our knowledge. The family of Bienenstock, Cooper, and Munro (BCM) learning rules supplement the classic Hebbian model with a stabilizing sliding threshold for potentiation rather than synaptic scaling [59]. BCM rules balance terms driven by third- and fourth-order joint moments of the pre- and postsynaptic activity [60]. A triplet STDP model with rate-dependent depression and uncorrelated Poisson spiking has BCM dynamics [45] and can develop selective (sparse) connectivity in response to rate- or correlation-based input patterns [61]. If the input is drawn from a mixture model then under a BCM rule, the synaptic weights are guaranteed to converge to the class means of the mixture [62].

Learning rules with suitable postsynaptic nonlinearities can allow a neuron to perform independent component analysis (ICA) [63, 64]. These learning rules optimize the kurtosis of the neural response. In contrast, we show that a simple nonlinear Hebbian model learns tensor eigenvectors of higher-order input correlations. Those higher-order input correlations can determine which features are learned by gradient-based ICA algorithms [65]. Taylor and Coombes showed that a generalization of the Oja rule to higher-order neurons can also learn higher-order correlations [66], which can allow learning independent components [67]. In that model, however, the synaptic weights $J$ are a higher-order tensor.

Computing the robust eigenvectors of an odec tensor $\mu$ by power iteration has $O(K^{a+1})$ space complexity: it requires first computing $\mu$. The discrete-time dynamics of eq. 1 correspond to streaming power iteration, with $O(K)$ space complexity [68–70]. Eq. 2 defines limiting continuous-time dynamics for tensor power iteration, exposing the basins of attraction.

Oja’s rule inspired a generation of neural algorithms for PCA and subspace learning [12, 13]. Local learning rules for approximating higher-order correlation tensors may also prove useful, for example in neuromorphic devices [71, 73].

Code availability

The code associated with figures one and three is available at https://github.com/gocker/TensorHebb.

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A Appendix

A.1 A nonlinear Hebbian learning rule: model and notation

We take a neuron receiving $K$ time-varying inputs $x_i(t)$, each filtered through a connection with synaptic weight $J_i$ to produce activity $n(t)$. We consider learning rules where the update of $J_i$ can depend on the postsynaptic activity $n(t)$, the local input $x_i(t)$, and the current synaptic weight $J_i(t)$. We encode these dependencies in a learning rule $f$:

$$ f(n(t), x_i(t), J_i(t)) = n^a x_i^b J_i^c $$

where $a, b \in \mathbb{Z}^+$ and $c \in \mathbb{R}$. Finally, we assume that the synaptic weights are homeostatically regulated [9]. Together,

$$ J_i(t + dt) = J_i(t) + (dt/\tau) \frac{f(n(t), x_i(t), J_i(t))}{\| J + (dt/\tau) f \|_p} $$

where $dJ_i = J_i(t + dt) - J_i(t)$, $\tau$ is a learning timescale, and $\| x \|_p$ is the $\ell^p$ norm of $x$.

We will model the neuron as a simple linear unit, $n(t) = (Jx)(t)$. Note that taking $n(t) = (Jx)^a(t)$ and $f(n(t), x_i(t), J_i(t)) = n(t)x_i^a(t)J_i^c(t)$ yields the same weight update as (eq. 1). As observed by [74], it is the composition of the neural nonlinearity and the output-dependent nonlinearity in the learning rule that determines the effective nonlinearity of the output-dependence in learning.

A power-law neural transfer function has been shown to approximate biological models near their spiking threshold [25][6].

We follow Oja [10] and expand in powers of $dt$ (equivalently, in $1/\tau$ or $dt/\tau$) which yields to linear order,

$$ \tau \frac{dJ_i}{dt} = f(n(t), x_i(t), J_i(t)) - J_i \sum_j J_j |J_j|^{p-2} f(n(t), x_j(t), J_j(t)) $$

$$ = n^a x_i^b J_i^c - J_i n^a \sum_j J_j |J_j|^{p-2} x_i^b J_j^c $$

We suppressed the time-dependence of $x$ and $J$ here and onwards. We will assume, as is standard, that learning is slow ($dt/\tau \ll 1$). In this case, individual changes in synaptic weights are small. If the inputs $x$ are stationary (at least within a timescale $T$, $dt \ll T \ll \tau$) and have finite joint moments up to order $a + 1$, the dynamics average over the statistics of $x$ [77] so that

$$ \tau \dot{J}_i = J_i^c \sum_\alpha \mu_{i,\alpha}(J^{\otimes \alpha})_\alpha - J_i \sum_j J_j^{c+1} |J_j|^{p-2} \mu_{j,\alpha}(J^{\otimes \alpha})_\alpha $$

where $\dot{J}_i(t) = (1/T) \int_{t-T}^t dt' (dJ_i/dt)(t')$ and $\mu_{i,\alpha} = \langle x_i^b(x^{\otimes \alpha})_\alpha \rangle_x \approx (1/T) \int_{t-T}^t dt' x_i^a(t)(x^{\otimes \alpha})_\alpha(t)$ is a $(a + b)$-order joint moment of $x$ and an $(a + 1)$-order tensor. The order of the tensor refers to its number of indices, so a vector is a first-order tensor and a matrix a second-order tensor. Since $\mu$ is a correlation tensor of $x$ each of its modes has the same range, $1, \ldots, K$. We also use multi-index notation: $\alpha = k_1, k_2, \ldots, k_a$. Sums over any index run from 1 to $K$ unless otherwise specified.

A.2 Proofs

**Theorem 1.** In eq. 2 take $(b, c) = (1, 0)$. Let $\mu$ be a cubical, symmetric tensor of order $a + 1$ and orthogonally decomposable (odeco) into $R$ components:

$$ \mu = \sum_{r=1}^R \lambda_r (U_r)^{\otimes a+1} $$

where $U$ is a matrix of unit-norm orthogonal $E$-eigenvectors: $U^T U = I$. Let $\lambda_i > 0$ for each $i \in [R]$ and $\lambda_i \neq \lambda_j \forall (i, j) \in [R] \times [R]$ with $i \neq j$. Then for each $k \in [R]$:

1. With any odd $a > 1$, $J = \pm U_k$ are attracting fixed points of eq. 2 and their basin of attraction is $\bigcap_{i \in [R]} k \{ J : (U_r^T J U_r^T J) < (\lambda_i/\lambda_j)^{a-1} \}$. Within that region, the separatrix of $+U_k$ and $-U_k$ is the hyperplane orthogonal to $U_k^T$: $\{ J : U_k^T J = 0 \}$. 


2. With any even positive $a$, $J = U_k$ is an attracting fixed point of eq. 2 and its basin of attraction is $\{ J : U^T_k J \geq 0 \} \cap \{ J : U^T_k J < (\frac{\lambda_k}{\lambda_1})^{\frac{1}{(a-1)}} \}$. 

3. With any even positive $a$, $J = 0$ is a neutrally stable fixed point of eq. 2 with basin of attraction $\{ J : \sum_{j=1}^R (U^T_j J)^2 < 1 \land U^T_k J < 0 \forall k \in [R] \}$.

Proof: Note that because the eigenvalues of $\mu$ are distinct, $U$ is unique [47]. Reshaping $\mu$ from an order $a + 1$ tensor with each fiber of length $K$ to $\mu(n)$, a $K \times K^{a}$ matrix with the rows equal to mode $n$ of $\mu$, yields the matricized form [23]

$$\mu(n) = U \Lambda (U^{\otimes a})^T \tag{6}$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_R)$ and $\otimes$ is the columnwise Kronecker-Rao product; $U^{\otimes a}$ is the $a$-fold Kronecker-Rao product of $U$ with itself, a $K^a \times R$ matrix. Let $J(t) \equiv Uv(t)$. From the mixed-product property of the Kronecker and dot products,

$$J^{\otimes a} = U^{\otimes a} v^{\otimes a} \tag{7}$$

where $\otimes$ is the Kronecker product; $J^{\otimes a}$ is the $a$-fold Kronecker product of $J$ with itself, a $K^a \times R^a$ matrix. (For vectors, the Kronecker product is the vector outer product.) We insert the decomposition of $\mu(n)$ and this projection of $J$ into the plasticity dynamics, eq. 7 with $c = 0$:

$$\tau J = \mu(n) J^{\otimes a} - J \circ J^T \mu(n) J^{\otimes a}$$

$$\tau U \dot{v} = U \Lambda (U^{\otimes a})^T \Sigma v^{\otimes a} - U \circ U^T U \Lambda (U^{\otimes a})^T U v^{\otimes a} \tag{8}$$

where $\circ$ is the elementwise product. Since $U$ is orthogonal so are $U^{\otimes a}$ and $U^{\otimes a}$ and $(U^{\otimes a})^T U^{\otimes a} = I$, where $I$ is a $R \times R^a$ identity matrix that picks out diagonal elements of $v^{\otimes a}$. Let $\Sigma = \Lambda I$, so

$$\tau U \dot{v} = U \Lambda I v^{\otimes a} - U \circ v^T \Lambda I v^{\otimes a}$$

$$\tau \dot{v} = \Sigma v^{\otimes a} - v \circ v^T \Sigma v^{\otimes a}, \text{ or}$$

$$\tau \dot{v}_i = a^a \lambda_i - \dot{v}_i \sum_{j=1}^R \lambda_j a_j^{a+1} \tag{9}$$

Note that for any $i$, $v_i = 0$ is an equilibrium. In the standard Oja rule ($a = 1$), all other equilibria are on the unit sphere, $S = \{ v : \sum_i v_i^2 = 1 \}$, which is a globally attracting manifold [11]. Is this still the case? Let

$$S(v) = \sum_i v_i^2, L(v) = \sum_j \lambda_j v_j^{a+1} \tag{10}$$

so

$$\frac{\tau}{2} S = L(1 - S) \tag{11}$$

If $a$ is odd, $L(v) > 0$ for any $v$, so $S = 1$ is a global attractor for $S$ and $S$ is a globally attracting manifold for $v$. If $a$ is even, $S$ is attracting from regions where $L(v) > 0$ but repelling from regions where $L(v) < 0$.

What points in $S$ are fixed points? From eq. 9 we have for each $i$ either $v_i^* = 0$ or it obeys the fixed point equation

$$v_i^* = (L^*/\lambda_i)^{1/((a-1)}} \tag{12}$$

where $L^* = L(v^*)$. This implies that if $v_i^* \neq 0$,

$$v_i^* = \lambda_i^{1/((a-1))} \left( \sum_{j, v_j \neq 0} \lambda_j^{2/((a-1))} \right)^{-1/2} \tag{13}$$

In what follows, we will see that the sparse points on $S$, with one $v_k = 1$ ($\pm 1$ if $a$ odd) and the rest at 0, are the only attractors and determine their basins of attraction.
Are the sparse points stable? The Jacobian of eq. (14) is
\[ \frac{\partial \dot{y}_i}{\partial v_k} = \delta_{ik} \left( a \lambda_i v_i^{a-1} - (a + 2) \lambda_i v_i^{a+1} - \sum_{j \neq i} \lambda_j v_j^{a+1} \right) + (1 - \delta_{ik}) v_i (a + 1) \lambda_k v_k^a \] (14)

At a sparse fixed point, the Jacobian is diagonal. At a sparse \( v^* \) with 1 at element \( j \), the Jacobian eigenvalues are \(-2 \lambda_j\) and, with multiplicity \( R - 1\), \(-\lambda_j\) so those positive sparse points are stable. At a sparse vector with -1 at element \( j \), the Jacobian eigenvalues are \( 2 \lambda_j (-1)^a \), and with multiplicity \( R - 1, \lambda_j (-1)^a \). So the negative sparse points are stable if \( a \) is odd and unstable if \( a \) is even.

What are the basins of attraction of the stable sparse points? Can any non-sparse equilibria be stable? Following [11], we consider the change of variables
\[ y_i = \frac{v_i}{v_k}, \quad i \neq k \] (15)

for some \( v_k \neq 0 \). We examine the joint dynamics of the loading onto the normalizing eigenvector, \( v_k \), and the relative loadings onto the other eigenvectors, \( y_i \). These are
\[ \dot{v}_k = v_k^a \left( \lambda_k - v_k^2 \sum_{j \neq k} \lambda_j v_j^{a+1} \right) \]
(16)
The nullclines for \( v_k \) are \( v_k^* \in \{0, \pm \sqrt{\lambda_k/(\lambda_k + \sum_{j \neq k} \lambda_j y_j^{a+1})}\} \) (\( v_k = 0 \) is a hyperplane equilibrium for the whole system, but the coordinate transform is singular at it). Checking \( \text{sign}(\dot{v}_k) \) on either side of \( v_k^* = \pm \sqrt{\lambda_k/(\lambda_k + \sum_{j \neq k} \lambda_j y_j^{a+1})} \) reveals that those nullclines for \( v_k \) are both attracting with respect to \( v_k \) for any finite \( y_i \).

The stability of \( v_k^* = 0 \) depends on the parity of \( a \). The \( y \)-nullclines are \( y_i^* \in \{0, \pm (\lambda_i/\lambda_k)^{1/(a-1)}\} \). (If \( a \) is odd we have both roots, while if \( a \) is even we have only the + root.) These nullclines also depend on the parity of \( a \).

Case 1: odd \( a \). First consider \( v_k^* = 0 \). Note that with \( a \) odd, \( \lambda_k + \sum_{j=2}^R \lambda_j y_j^{a+1} \geq \lambda_k > 0 \). From eq. (16), \( \text{sign}(\dot{v}_k) = \text{sign}(v_k) \) when \( a \) is odd. So, \( v_k \) is repelled from 0.

Now consider the \( y \)-nullclines. For any \( v_k \neq 0 \), checking the sign of \( \dot{y}_i \) reveals that \( y_i = 0 \) is an attractor and \( y_i = \pm (\lambda_i/\lambda_k)^{1/(a-1)} \) are both repellers. Recall that \( v_k^* = \pm \sqrt{\lambda_k/(\lambda_k + \sum_{j \neq k} \lambda_j y_j^{a+1})} \) is attracting along \( v_k \) for any finite \( y_i \). Together, the only attracting equilibria in \( v_k, y \) can be at \( v_k = \pm 1 \) and each \( y_i = 0 \). Since the columns of \( U \) are orthonormal, \( v_k = U_k^T J = \pm 1 \) implies that \( J = \pm U_k \). The basin of attraction of these equilibria are defined by the other, unstable nullclines.

Back in the space of \( v \), those are the unstable hyperplanes \( v_i/v_k = \pm (\lambda_i/\lambda_k)^{1/(a-1)} \) and the repelling nullcline \( v_k^* = 0 \). Each unique pair \( v_i, v_k \) generates one such pair of unstable hyperplanes, all passing through the origin. All of the equilibria points \( v^* \in S \) identified earlier, with at least two nonzero elements, lie on at least one of those unstable hyperplanes and are thus unstable. Since \( S \) is a global attractor, these \( R(R - 1) + 1 \) hyperplanes partition \( \mathbb{R}^R \) into the basins of attraction of each sparse point with one \( v_k = \pm 1 \) and the others 0, which are the only attractors.

Case 2: even \( a \). Let \( v_k > 0 \) in the definition of \( y \). (If \( L(v) > 0 \) and \( a \) even, at least one \( v_k \) must be positive.) Checking the sign of \( \dot{y}_i \) then reveals that for each \( i \neq k, y_i^* = 0 \) is an attractor while \( y_i^* = (\lambda_i/\lambda_k)^{1/(a-1)} \) is repelling. So the hyperplane \( v_i = v_k (\lambda_i/\lambda_k)^{1/(a-1)} \) is unstable. Each unique pair of axes \( v_i, v_k \) has such an unstable hyperplane where \( v_k > 0 \).

If \( v_k(0) > 0 \), \( v_k \) cannot cross zero since \( v_k = 0 \) is an equilibrium for the whole system. Furthermore, if \( v_k = \epsilon w_k \),
\[ \tau \dot{w}_k = \epsilon^{a-1} \lambda_k w_k^a + \mathcal{O}(\epsilon^{a+1}) \] (17)
so \( v_k \) is repelled by 0 from above. So if \( v_k > 0 \) and each \( y_i < (\lambda_i/\lambda_k)^{1/(a-1)} \), the relative loadings \( y_i \) will all approach zero. Let each \( y_i = 0 \), so
\[ \tau \dot{v}_k = \lambda_k v_k^a (1 - v_k^2) \] (18)
with a stable equilibrium at \( v_k = 1 \). (The equilibrium \( v_k = -1 \) is excluded by construction.) Each sparse point with \( v_k = 1 \), \( y = 0 \) corresponds to a sparse equilibrium for the loadings with \( v_i = 0 \), \( i \neq k \). Together, the only attracting equilibria in \( v_k, y \) can be at \( v_k = \pm 1 \) and each \( y_i = 0 \). Since the columns of \( U \) are orthonormal, \( v_k = U_k^T J = \pm 1 \) implies that \( J = \pm U_k \). The basin of attraction of these equilibria are defined by the other, unstable nullclines.

Each positive sparse point, with \( v_k = 1 \), lies inside a section of \( \mathbb{R}^R \) bounded by the \( R - 1 \) repelling hyperplanes \( v_i = v_k (\lambda_k / \lambda_i)^{1/(a - 1)} \) and/or the repelling axis \( v_k = 0 \), and each such region contains one such sparse point. So, those hyperplanes divide \( \mathbb{R}^R \) into the basins of attraction for the columns of \( U \).

Finally, \( v = 0 \) is an equilibrium of eq. 9. At \( v = 0 \), the Jacobian of eq. 9 is identically 0. Let \( v < 0 \) elementwise. Then \( L(v) < 0 \) so if \( S(v) < 1 \) then \( \dot{S} < 0 \) (eq. 11). As \( S(v) \to 0 \), no \( v_k \) can become positive because \( v_k = 0 \) is an equilibrium. In the section under \( S \) with \( v < 0 \), \( v \) thus approaches the origin. \( \square \)

Corollary 1.1. Let \( V_k \) be the relative volume of the basin of attraction for \( J^* = U_k^T \). For odd \( a > 1 \),

\[
V_k = R^{-1} \prod_{i=1}^{R} \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a-1)} 
\]

Proof. We will compute the volume \( V_k \) of the basin of attraction for \( U_k^T \) directly. Take \( a \) odd. Then from theorem 1

\[
V_k = \int D J \prod_{i=1}^{K} \theta \left( C_{ik} - \frac{U_k^T J}{U_k^T J} \right) 
\]

where \( D J = \prod_{i=1}^{K} d J_i \) and \( C_{ik} = \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a-1)} \). We change variables to the relative loadings, \( v = U_k^T J \); the Jacobian factor is \( \text{vol}(U) \), the product of the singular values of \( U \). Since \( U \) is orthogonal, its singular values are all 1. The integrals over \( v_i, i \neq k \), all factorize:

\[
\int_{-\infty}^{\infty} dv_i \theta (C_{ik} |v_k| - |v_i|) = 2C_{ik} |v_k|
\]

which leaves

\[
V_k = 2R^{-1} \prod_{i \neq k} C_{ik} \int_{-\infty}^{\infty} dv_k |v_k|^{R-1}
\]

We choose bounds \(-x, x\) for \( \int dv_k \) and compute \( V(x) \):

\[
V_k(x) = R^{-1} (2x)^R \prod_{i \neq k} C_{ik}
\]

Note that \( C_{kk} = 1 \), so \( \prod_{i \neq k} C_{ik} = \prod_{i} C_{ik} \). \( V_k(x) \) is the volume of the set of weight vectors with projection at least \( x \) onto \( U_k^T \) that will converge to \( U_k \). We recognize the factor of \( (2x)^R \) as the volume of an \( R \)-dimensional hypercube with edge lengths \( 2x \). Normalizing by the volume of the hypercube concludes the calculation.

For even \( a \), integrating each \( v_j \) over \((0, \infty)\) and normalizing by the hypercube volume \( x^R \), rather than \( (2x)^R \), yields the same result. The region with all \( v_i > 0 \) is certainly part of the basin of attraction of \( v_k = 1 \), but not the whole basin which requires only \( v_k > 0 \). This result thus provides a lower bound of the volume of the basins of attraction for even \( a \). \( \square \)

Corollary 1.2. Let \( V_k \) be the relative volume of the basin of attraction for \( J^* = U_k^T \). For even positive \( a \),

\[
V_k = 2^{-R} \left( R^{-1} \prod_{i} \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a-1)} + (R - 1)^{-1} \sum_{j \neq k} \prod_{i \neq j} \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a-1)} \right) 
\]

\[
+ (R - 2)^{-1} \sum_{j, l \neq k, i \neq j, l} \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a-1)} + \ldots + 1 \]

(24)
Proof. For even $a$, we compute

$$V_k = \int DJ \theta (U_k^T J) \prod_{i=1}^{R} \theta \left( C_{ik} - \frac{U_i^T J}{U_k^T J} \right)$$

(25)

as in corollary 1.1. The integrals over $v_i, i \neq k$ all factorize:

$$\int_{-\infty}^{\infty} dv_i \theta (C_{ik} v_k - v_i)$$

(26)

and we choose a bound of $-x$ for them, leaving

$$V_k(x) = \int_{0}^{x} dv_k \prod_{i \neq k} (C_{ik} v_k + x)$$

(27)

The proper normalization here is by $2^{R-1} x^R$, with a factor of $x$ from $\int dv_k$ and $(2x)^{R-1}$ from the integrals $\int dv_i$. \qed

**Corollary 1.3.** Let $S$ be the unit $R$-sphere in $\mathbb{R}^R$ and $S_k$ be the section of $S$ in the basin of attraction of $J^* = U_k^T$. If $a$ is odd, the surface area of $S_k$ is

$$A_k = 2 \tan^{-1} C_{jk} \prod_{i=1}^{R-1} \Gamma \left( \frac{R - i}{2} \right) \left( \frac{\sqrt{\pi}}{\Gamma \left( \frac{R + 1}{2} \right)} - \frac{1 + C_{ik}^2}{\Gamma \left( \frac{R - j + 2}{2} \right)} \right) F_1 \left( \frac{1}{2}, \frac{R - i}{2}, \frac{R - j + 2}{2}, \frac{1}{1 + C_{ik}^2} \right).$$

If $a$ is even, the surface area of $S_k$ is

$$S_k = \left( \tan^{-1} C_{jk} + \frac{\pi}{2} \right) \prod_{i=1}^{R-1} 2 \Gamma \left( \frac{R - i}{2} \right) \left( \frac{2\sqrt{\pi}}{\Gamma \left( \frac{R + 1}{2} \right)} - \frac{1 + C_{ik}^2}{\Gamma \left( \frac{R - j + 2}{2} \right)} \right) F_1 \left( \frac{1}{2}, \frac{R - i}{2}, \frac{R - j + 2}{2}, \frac{1}{1 + C_{ik}^2} \right).$$

(28)

**Proof.** We calculate the surface area by transforming to spherical coordinates with an azimuthal angle $\theta_j \in [0, 2\pi)$ and $R - 2$ polar angles $\theta_i \in [0, \pi]$. Without loss of generality, we place $U_k^T$ at $\theta_i = 0, \theta_j = \pi/2$ for each $j$.

Take $a$ odd. The unstable hyperplanes bounding the basin of attraction for $U_k^T$ are defined by the azimuthal angle $\theta_i = \pm \tan^{-1} (\lambda_k/\lambda_i)^{1/a-1}$ and polar angles $\theta_j = \pi/2 \pm \tan^{-1} (\lambda_k/\lambda_i)^{1/a-1}$. Let $C_{ik} = (\lambda_k/\lambda_i)^{1/a-1}$. The surface area of $S_k$ with $a$ odd is:

$$S_k = \int_{-\tan^{-1} C_{jk}}^{\tan^{-1} C_{jk}} d\theta_j \prod_{i=1, i \neq j}^{R-1} \int_{-\tan^{-1} C_{ik}}^{\tan^{-1} C_{ik}} \sin^{R-i-1} \theta_i$$

(30)

Take $a$ even. The unstable hyperplanes bounding the basin of attraction for $U_k^T$ are defined by the azimuthal angle $\theta_i = \tan^{-1} (\lambda_k/\lambda_i)^{1/a-1}$ and the polar angles $\theta_j = \pi/2 + \tan^{-1} (\lambda_k/\lambda_i)^{1/a-1}$. The other bounds for the basin of attraction are that they have positive loadings, $U_k^T J > 0$. Those correspond to the azimuthal angle $-\pi/2$ and the polar angles $\pi$. The surface area of $S_k$ with $a$ even is:

$$S_k = \int_{-\pi}^{\tan^{-1} C_{jk}} d\theta_j \prod_{i=1, i \neq j}^{R-1} \int_{-\tan^{-1} C_{ik}}^{\tan^{-1} C_{ik}} \sin^{R-i-1} \theta_i$$

(31)
As in the proof of theorem 1, let \( J \) where each of order 1. For even \( a \), the upper limit is always \( \pi \). For small \( \lambda_k \), those polar factors thus approach 1. This raises the hope that those products might be truncated. The number of eigenvalues is, however, exponentially large in \( K := a^{n−1}/a−1 \) and standard algorithms for computing the singular value decomposition of a tensor have space complexity \( O(K^{n+1}) \). We computed the first 20 singular vectors (eigenvectors) of \( \mu \) and they did not decay to negligible values within those.

**Remark.** For small eigenvalues \( \lambda_k \), the limits of integration for the polar factors (e.g., \( \int_0^{2\pi} \tan^{−1} C_{ik} d\theta_i \sin^{R−i−1} \theta_i \)) approach 0 and \( \pi \). For even \( a \), the upper limit is always \( \pi \). For small \( \lambda_k \), those polar factors thus approach 1. This raises the hope that those products might be truncated. The number of eigenvalues is, however, exponentially large in \( K := a^{n−1}/a−1 \) and standard algorithms for computing the singular value decomposition of a tensor have space complexity \( O(K^{n+1}) \). We computed the first 20 singular vectors (eigenvectors) of \( \mu \) and they did not decay to negligible values within those.

**Corollary 1.4.** In eq. 2 take \((b, c) = (1, 0)\). Let \( \mu \) be a cubical, symmetric tensor of order \( a + 1 \) and rank \( R \), as in theorem 4 but with \( d \) equal eigenvalues. Let \( D \) be the index set of those equal eigenvalues. Let \( S_D \in \mathbb{R}^D \) be the unit \( d \)-sphere spanned by \( \{ U_j^T : j \in D \} \) and let \( −S_D \) be the unit \( d \)-sphere spanned by \( \{ −U_j^T : j \in D \} \). Then:

1. With any odd \( a > 1 \), \( S_D \) and \( −S_D \) are attracting equilibrium manifolds of eq. 2. The basin of attraction for \( S_D \) is \( \bigcup_{k \in D} \bigcap_{i \notin D} \left\{ J : \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a−1)} < \frac{U_j^T J}{u_j^T J} < \left( \frac{\lambda_i}{\lambda_k} \right)^{1/(a−1)} \right\} \). The basin of attraction for \( −S_D \) is \( \bigcup_{k \in D} \bigcap_{i \notin D} \left\{ J : \left( \frac{\lambda_k}{\lambda_i} \right)^{1/(a−1)} > \frac{U_j^T J}{u_j^T J} > \left( \frac{\lambda_i}{\lambda_k} \right)^{1/(a−1)} \right\} \).

2. With any even positive \( a \), \( S_D \) is an attracting equilibrium manifold of eq. 2 and its basin of attraction is \( \bigcup_{k \in D} \left\{ J : U_k^T J > 0 \right\} \bigcap_{i \notin D} \left\{ J : \frac{U_j^T J}{u_j^T J} < \left( \frac{\lambda_i}{\lambda_j} \right)^{1/(a−1)} \right\} \).

**Proof.** Since \( d \) eigenvalues of \( \mu \) are equal, the eigendecomposition of \( \mu \) is not unique. Call \( U_D^T \) be the set of the \( d \) eigenvectors with equal eigenvalues. Let

\[
U' = UT
\]

where \( T \) is an orthogonal transformation within the subspace spanned by \( U_D^T \). For any such \( T \), the columns of \( U' \) are also eigenvectors of \( \mu \). Note that for any \( i \notin D \), \( U_i^T = U_i^T \).

As in the proof of theorem 1, let \( J = U'v \). Pick one \( k \in D \) and choose \( T \) such that \( v_j = 0 \) for each \( j \in D \), \( j \neq k \). This is a fixed point for the \( d \) loadings \( v_j \). For the remaining \( R - d + 1 \) loadings, the proof of theorem 1 follows.

In particular, for odd \( a \), \( v_k = 1 \) is an attractor with basin of attraction \( \bigcap_{i \in [R] \setminus D} \left\{ J : −(\lambda_k/\lambda_i)^{1/(a−1)} < v_i/v_k < (\lambda_i/\lambda_k)^{1/(a−1)} \right\} \). This holds for each \( k \in D \). Together, the basin of attraction for \( U_D^T \) is the union of those basins of attraction. Similarly, the basin of attraction for \( v_k = −1 \) is \( \bigcap_{i \in [R] \setminus D} \left\{ J : −(\lambda_k/\lambda_i)^{1/(a−1)} > v_i/v_k > (\lambda_i/\lambda_k)^{1/(a−1)} \right\} \), and the basin of attraction for \( −U_D^T \) is the union of those.

Recall that the eigendecomposition of \( \mu \) is invariant under orthogonal transformations. That is, prior to choosing \( T \) above, the eigenvectors \( U_D^T \) can be replaced by any unit-norm linear combination thereof. Any point on \( S_D \) or \( −S_D \) is thus an attractor with the same basin of attraction defined above.

For even \( a \), the same argument applies; the boundaries of the basins of attraction are as specified in theorem 1.

**Theorem 2.** In eq. 10 take \( b = 1, c = 0, a \in \mathbb{Z}^+_m \), and consider \( N \) cubical, symmetric tensors, \( m \mu \), each of order \( a \) for \( m \in [N] \), that are mutually orthogonally decomposable into \( R \) components:

\[
m \mu = \sum_{r=1}^R \lambda_{mr} U_r^T \otimes U_r^T \otimes \cdots \otimes U_r^T
\]
Then:

1. \( S^* = \{ J : S(J) = 1 \land L(J) > 0 \} \) is an attracting set for eq. (10) and its basin of attraction includes \( \{ J : L(J) > 0 \} \).

2. For each \( k \in [R] \), \( J = U_k \) is a stable equilibrium of eq. (10).

3. For each \( k \in [R] \), \( J = -U_k \) is a stable equilibrium of eq. (10) if \( \sum_m \lambda_k (-1)^{a_m} < 0 \) (and unstable if \( \sum_m \lambda_k (-1)^{a_m} > 0 \)).

4. Any other stable equilibrium must have \( U_k^T J \leq 0 \) for each \( k \in [R] \).

**Proof.** We will prove the claims in the order of their statement in theorem 2. Let \( J(t) = U(v(t)) \); we again study the dynamics for the loadings:

\[
\tau \dot{v}_i = \sum_m \lambda_{mi} v_i^{a_m} - v_i \sum_m \lambda_{mj} v_j^{a_m+1}
\]

(35)

Let

\[
S(v) = \sum_i v_i^2, \quad L(v) = \sum_m \lambda_{mj} v_j^{a_m+1}
\]

(36)

At a fixed point for \( v \), \( S \) and \( L \) must also be at a fixed point. The dynamics of \( S \) are

\[
\frac{T}{2} S = L(1 - S)
\]

(37)

with fixed points at \( S = 1, L = 0 \). Let \( S = \{ v : S(v) = 1 \} \), the unit sphere, and \( L = \{ v : L(v) = 0 \} \). All fixed points \( v^* \) must be in \( S \) or \( L \). A fixed point has each \( v_i \) at a root of \( \sum_m \lambda_{mi} v_i^{a_m} = v_i(v) \). Furthermore, from eq. (35) we have that at a fixed point for any \( i \), either \( v_i = 0 \) or it obeys the fixed point equation

\[
L(v) = \sum_m \lambda_{mi} v_i^{a_m-1}
\]

(38)

\( S \) is attracting from above the boundary set \( L = \{ v : L(v) = 0 \} \). If \( v \) starts above \( L \), will it remain so? Is \( L \) attracting or repelling? Let \( L(v^*) = \epsilon \). Then

\[
\tau \dot{v}_i = \sum_m \lambda_{mi} v_i^{a_m} + O(\epsilon)
\]

(39)

Let \( v = v^* + \epsilon w \), where \( v^* \in L \) and \( w_i = \sum_m \lambda_{mi} (v^*)^{a_m} \), so

\[
L(v^* + \epsilon w) = \epsilon \sum_m \lambda_{mj} (v^*_j)^{a_m} \lambda_{mj} (v^*_j)^{a_n} + O(\epsilon^2)
\]

\[
= \epsilon \sum_j \left( \sum_m \lambda_{mj} (v^*_j)^{a_m} \right)^2 + O(\epsilon^2) \geq 0
\]

(40)

Points \( v^* \in L \), if perturbed, will either 1) move above \( L \) or 2) if \( \sum_j \left( \sum_m \lambda_{mj} (v^*_j)^{a_m} \right)^2 = 0 \), stay on \( L \). So if \( L(v) > 0 \) at some time \( t \), \( L(v) \geq 0 \) for all subsequent times and \( S^* = \{ v \in S | L(v) \geq 0 \} \) is an attracting set for \( v \).

The sparse vectors \( v^* \) with one element at \( \pm 1 \) and the others at 0 are in \( S \). They correspond to equilibria for \( J \) at the columns of \( \pm U \). Are those equilibria stable? The Jacobian of eq. (35) is

\[
\frac{\partial \dot{v}_i}{\partial v_k} = \delta_{ik} \sum_m \left( a_m \lambda_{mi} v_i^{a_m-1} - (a_m + 2) \lambda_{mi} v_i^{a_m+1} - \sum_{j \neq i} \lambda_{mj} v_j^{a_m+1} \right) + (1 - \delta_{ik}) v_i \sum_m (a_m + 1) \lambda_{mk} v_k^{a_m}
\]

\[
\frac{\partial \dot{v}_i}{\partial v_k} = \delta_{ik} \sum_m a_m \lambda_{mi} v_i^{a_m-1} (1 - v_i^2) + (1 - \delta_{ik}) v_i \left( L v_k + \sum_m a_m \lambda_{mk} v_k^{a_m} \right)
\]

(41)
where we used the fixed point condition eq. 38. At a sparse fixed point, the Jacobian is diagonal. At a sparse vector \( v^* \) with 1 at element \( j \), the Jacobian eigenvalues are \(-2 \sum m \lambda_{mj} \) and, with multiplicity \( R - 1 \), \(-\sum m \lambda_{mj}(-1)^{am}\) so those are stable. At a sparse vector with -1 at element \( j \), the Jacobian eigenvalues are \( 2 \sum m \lambda_{mj}(-1)^{am}\) and, with multiplicity \( R - 1 \), \( \sum m \lambda_{mj}(-1)^{am} \). So the sparse points with -1 at element \( j \) are stable if \( \sum m \lambda_{mj}(-1)^{am} < 0 \) and unstable if \( \sum m \lambda_{mj}(-1)^{am} > 0 \).

Next we will study non-sparse equilibria. We again study the dynamics of the relative loadings \( y_i/v_k, i \neq k \), for some \( v_k \neq 0 \):

\[
\tau \dot{y}_i = \frac{y_i}{v_k} \sum_{m=1}^{N} v_k^{a_m-1} \left( \lambda_{mi} y_i^{a_m-1} - \lambda_{mk} \right)
\]

\[
\tau \dot{v}_k = \frac{v_k}{v_k} \sum_{m=1}^{N} v_k^{a_m} \left( \lambda_{mk} - v_k^2 \left( \lambda_{mk} + \sum_{j \neq k} \lambda_{mj} y_j^{a_m+1} \right) \right)
\]

with nullclines for each \( y_i \) at 0 and the other roots of \( \sum_m v_k^{a_m-1} \left( \lambda_{mi} y_i^{a_m-1} - \lambda_{mk} \right) \), and nullclines for \( v_k \) at 0 and the other roots of \( \sum_m v_k^{a_m} \left( \lambda_{mk} - v_k^2 \left( \lambda_{mk} + \sum_{j \neq k} \lambda_{mj} y_j^{a_m+1} \right) \right) \). A fixed point for \( v \) must also be a fixed point for \( (v_k, y) \) for any \( k \) with \( v_k \neq 0 \). If such a fixed point \( \bar{v} \) has at least two nonzero elements \( \bar{v}_i \), they must correspond to each \( y_i \) on a nonzero nullcline.

Consider the \( y_i \)-nullclines at the roots of \( \sum_m v_k^{a_m-1} \left( \lambda_{mi} y_i^{a_m-1} - \lambda_{mk} \right) \). Let \( \bar{y}_i(t) = y_i + \epsilon w_i(t) \), where \( \sum_m v_k^{a_m-1} \left( \lambda_{mi} y_i^{a_m-1} - \lambda_{mk} \right) = 0 \). The dynamics of \( w_i \) are

\[
\tau \dot{w}_i = \sum_m (a_m - 1)\lambda_{mi} v_k^{a_m-1} y_i^{a_m-1} + O(\epsilon)
\]

These nullclines \( \bar{y}_i \) are stable if \( \sum_m (a_m - 1)\lambda_{mi} v_k^{a_m-1} y_i^{a_m-1} < 0 \), or equivalently \( \sum_m (a_m - 1)\lambda_{mj} v_k^{a_m-1} y_i^{a_m-1} < 0 \). This is only possible if \( v_i < 0 \): a condition directly on \( v \). A stable fixed point must thus have \( v_i \leq 0 \) for each \( i \neq k \). This is true for any \( k \) with \( v_k \neq 0 \). So, a stable fixed point must have \( v_i \leq 0 \) for each \( i \in [R] \).

\( \square \)

### A.3 Spiking models

So far, we have discussed learning in a neuron model with two major simplifying assumptions. First, the neural output \( n \) depended only on the current input \( x(t) \). Synaptic kinetics, however, exhibit nonzero time constants so that neural activity depends also on the recent history of its inputs. Second, the neural output was a continuous, linear function of the inputs. Cortical neurons, however, spike. We next relax these two assumptions. We introduce a generalized spike timing–dependent plasticity (STDP) rule:

\[
f (n(t), x_i(t), J_i(t)) = A^T (n^a x_i^b J_i^c)
\]

where \( A = A(s) \) is the STDP kernel, a scalar function of each of the \( a \) post-post lags, \( b \) pre-post lags and \( c \) synaptic weight lags. Here, the notation \( A^T X \) denotes a functional inner product, integrating over the time lags of the STDP kernel \( A \) and the tensor \( X \) (eq. 1). We use this functional notation for simplicity and to emphasize the similarity with the simpler model of eq. 1. The case \( a = 1, b = 1, c = 0 \) corresponds to classic pair-based STDP [28, 38] while \( a = 2, b = 1, c = 0 \) corresponds to triplet STDP [45]. The commonly used triplet STDP model has two terms: a pair-based depression and triplet-based potentiation. Here we first discuss STDP rules with one term and then consider an arbitrary expansion of a plasticity model in STDP kernels [44]. Similarly to for eq. 1 combining eq. 14 with a homeostatic normalization of the synaptic weights and a separation of timescales between the neural and plasticity dynamics leads to

\[
\tau J_i = A^T (\langle n^a x_i^b \rangle_{n,x} J_i^c) - J_i \sum_j J_j A^T (\langle n^a x_j^b \rangle_{n,x} J_j^c)
\]

where \( \langle n^a x_i^b \rangle_{n,x}(t,s) \) is an order \( a + b \) joint moment density (correlation function) of the output spike train and the inputs (which might be spike trains or any process admitting a finite joint moment of this order). \( \langle \rangle_{n,x} \) is the expectation over the joint density of the inputs \( x \) and the activity \( n \).
In contrast to the original case of eq. [2] these dynamics depend on a joint moment of the inputs and output, rather than on just the input correlation. To calculate this joint moment, we will model the postsynaptic activity as conditionally Poisson. With two additional assumptions, we can recast eq. [45] in a form that depends only on $J$ and statistics of $x$. First we take the neural transfer function to be a power-law nonlinearity, which matches the effective nonlinearity of mechanistic spiking models in fluctuation-driven regimes \[75, 81\] and experimental observations \[82, 83\]. Second, we will assume that the input to the nonlinearity is non-negative, restricting the average over $p(x)$ to one over the samples of $x$ that can drive spiking.

With the STDP rule of eq. [44] homeostatic regulation of the $p$-norm of the synaptic weights, and a separation of timescales between activity and plasticity, the plasticity dynamics are

$$\tau \dot{J}_i = A^T \left( \langle n^a x_i^b \rangle_{n,x} J_i^c \right) - J_i \sum_j J_j |J_j|^{p-2} A^T \left( \langle n^a x_j^b \rangle_{n,x} J_j^c \right)$$  \hspace{1cm} (46)$$

where for fixed $i$ and $t$ we introduce the inner product over functions:

$$A^T \left( \langle n^a x_i^b \rangle_{n,x} J_i^c \right) (t) = \int_{-\infty}^{\infty} Ds \ A(s) \ n(t) \prod_{i=1}^{a-1} n(t + s_i) \prod_{j=a}^{a+b} x_i(t + s_j) \prod_{k=a+b+1}^{a+b+c} J_i(t + s_k)$$  \hspace{1cm} (47)$$

with integration measure $Ds = \prod_{i=1}^{a+b+c} ds_i$. Now we must determine the input-output joint moment $\langle n^a x_i^b \rangle_{n,x}$. This will depend on the input distribution, $p(x)$, and the model for the neural activity $n(t)$. We take $n(t)$ to be a Poisson process with stochastic intensity

$$r_x(t) = \phi \left( G^T x(t) + \lambda(t) \right)$$  \hspace{1cm} (48)$$

where $G(t, s) = J(t) \circ W(s)$ and $G^T x(t) = \sum_j \int_0^\infty ds \ G_j(t-s)x_j(s)$. That is, $J$ is a vector of synaptic weights and $W$ is a vector of coupling kernels for each synapse. We fix the integral of each elements of $W$ at 1, so $J$ sets the amplitude of synaptic interactions. $\lambda(t)$ models a deterministic drive. We assume that $W$ is fixed and plasticity only affects the weights, $J$. We will also assume that $G^T x(t) + \lambda(t) \geq 0$.

Our strategy to compute the joint moment $\langle n^a x_i^b \rangle_{n,x}$ has two parts. First, we decompose the joint moment into cumulants. Second, we write each of those cumulants as a tensor product of $J$ and a cumulant of $x$. Only the second step depends on the neuron model.

The joint moment $\langle n^a x_i^b \rangle_{n,x}$ can be decomposed into a Bell polynomial in its cumulants:

$$\left\langle n(t) \prod_{i=1}^{b} x_i(t + s_i) \prod_{m=1}^{a-1} n(t + s_{b+m}) \right\rangle_{n,x} = \sum_{\pi \in \Pi} \prod_{(P,Q) \in \pi} \left\langle \prod_{j \in P} n(t + s_j)x_i(t + s_k) \right\rangle_{n,x}$$  \hspace{1cm} (49)$$

where $\Pi$ is the set of all partitions of the time lags $(0, s_1, \ldots, s_{a+b-1})$. ($\Pi$ also corresponds to the set of all partitions of the $a$ factors of $n$ and $b$ factors of $x_i$ appearing in the joint moment. The first lag, 0, corresponds to $n(t)$.) For one such partition $\pi \in \Pi$, each of its blocks $(P,Q)$ contains indices $j,k$ for the time lags corresponding to factors of $n$ or $x$. In one block $(P,Q)$ of the partition $\pi$, $P$ is the set of indices $j$ correspond to factors of $n$ while $Q$ is the set of indices $k$ corresponding to factors of $x$.

We will compute the joint expectation by factorizing $p(n,x) = p(n|x)p(x)$. This will allow us to write a each joint cumulant of $n, x$ as a tensor product of $J$ and a cumulant of $x$. Given $x$, a cumulant of $n$ is

$$\left\langle n(t) \prod_{m=1}^{M} n(t + s_m) \right\rangle_{n,x} = r_x(t) \prod_{m=1}^{M} \delta(s_m)$$  \hspace{1cm} (50)$$

We will take $\phi(x) = |x|^d_x$ so a joint cumulant of $n, x$ is

$$\left\langle n(t) \prod_{m=1}^{M} n(t + s_{N+m}) \prod_{n=1}^{N} x_n(t + s_n) \right\rangle_{n,x} = \left\langle \n(t) \prod_{m=1}^{M} \n(t + s_{N+m}) \right\rangle_{n,x} \prod_{n=1}^{M} \delta(s_{N+m})$$  \hspace{1cm} (51)$$

$$\left\langle \left[ G^T x \right]_d^d (t) \prod_{n=1}^{M} x_n(t + s_n) \right\rangle_{x} \prod_{m=1}^{M} \delta(s_{N+m})$$  \hspace{1cm} (51)$$
and if $G^T x \geq 0$ for all $x$ then $[G^T x]^d_+ = \sum_\alpha (G^d)^T(x^d)_\alpha$ so

$$
\left\langle \left( \sum_\alpha \alpha_1 \right) \cdots \left( \sum_\alpha \alpha_n \right) \right\rangle_{n,x} = \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right)_\alpha \left( t, s_1, \ldots, s_b \right) \prod_{m=1}^{b-1} \delta(s_{b+m})
$$

These expansions of the input-output joint moment have a similar structure to the expansion of arbitrary learning rules (section 2.3) with one main difference: the exponent of the neural transfer function, $d$, also determines the relevant input moments because of the Poisson cumulants of $n$.

For example, take $a = b = 1$. The relevant joint moment $\langle n^d x^d \rangle_{n,x}$ is

$$\langle n(t)x_i(t+s) \rangle_{n,x} = \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t) \langle x^d \rangle (t+s) + \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t) \langle x^d \rangle (t+s) \rangle_x
$$

$$= \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t) \langle x^d \rangle (t+s) \rangle_x
$$

where $\langle x^d \rangle_{n,x} = \langle (x^d)_{n,x} \rangle - \langle x_{n,x} \rangle$ denotes the second cumulant of $(x^d)_\alpha$ and $x_i$, not a $d + 1$-order cumulant of $x$, since the factor of $x^d$ arises from the intensity of $n$. For $a = b = 1$, the decomposition of $\langle n^d x^d \rangle_{n,x}$ reduces to just the inner product of $G^d$ with a $d + 1$-order moment of the inputs, evaluated at one set of time lags. The decomposition of $\langle n^d x^d \rangle_{n,x}$ does not always reduce to just one term like that. As a second example, take a simple triplet STDP rule ($a = b = 1$). The relevant joint moment $\langle n^d x^d \rangle_{n,x}$ is

$$\langle n(t)n(t+s)x_i(t+s) \rangle_{n,x} = \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t) \langle x^d \rangle (t+s) \rangle_x \delta(s_2)
$$

$$+ \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t) \delta(s_2) \langle x^d \rangle (t+s) \rangle_x
$$

$$+ \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t) \langle x^d \rangle (t+s) \rangle_x \delta(s_2)
$$

$$+ \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t+s) \langle x^d \rangle (t+s) \rangle_x
$$

We can recognize two moments of the input here, combining the first and second lines and either the third or fourth with the fifth:

$$\langle n(t)n(t+s)x_i(t+s) \rangle_{n,x} = \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t) \langle x^d \rangle (t+s) \rangle_x \delta(s_2)
$$

$$+ \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t) \sum_\beta \left( G^d \right)^T_\beta \left( x^d \right) (t) \langle x^d \rangle (t+s) \rangle_x
$$

$$+ \sum_\alpha \left( G^d \right)^T_\alpha \left( x^d \right) (t+s) \langle x^d \rangle (t+s) \rangle_x
$$

where all expectations on the right-hand side are with respect to the input distribution, $p(x)$.

As discussed above, any joint moment $\langle n^d x^d \rangle_{n,x}$ can be decomposed into joint cumulants $\langle n^d x^d \rangle_{n,x}$. Each of those joint cumulants can be expressed as a tensor product of $G$ with a cumulant of $x$. To isolate the synaptic weights $J$, let $y = W^T x$ so $(G^d)^T_\alpha (x^d)_\alpha = (J^d)_\alpha ((W^T x)^d)_\alpha = (J^d)_\alpha (y^d)_\alpha$.

Since $y = W^T x$, joint cumulants of $x$, $y$ are cumulants of $x$. So we can write any joint cumulant of $n$, $x$ as a tensor product of $J$ with a cumulant of $x$. Using this and the cumulant decomposition of
We will analyze fixed points of eq. 58 and their stability. If we begin by analyzing inputs with constant-diagonal correlations, $\mu$ assuming $c \neq 0$ in eq. 1. In biological plasticity, this might not be the case. Within dendritic branches, spatially clustered and temporally coactive synapses [40] exhibit cooperative plasticity [36, 41]. Multiplicative weight-dependence also stabilizes Hebbian spike timing–dependent plasticity distributions [6, 7, 84]. As a first step towards incorporating these effects, we consider the dynamics of eq. 2 with weight-dependence also stabilizing Hebbian spike timing–dependent plasticity distributions [6, 7, 84].

A.4 Weight-dependent plasticity

Above, we examined the dynamics of the generalized Hebbian rule with no direct weight-dependence ($c = 0$ in eq. 1). In biological plasticity, this might not be the case. Within dendritic branches, spatially clustered and temporally coactive synapses [40] exhibit cooperative plasticity [36, 41]. Multiplicative weight-dependence also stabilizes Hebbian spike timing–dependent plasticity distributions [6, 7, 84]. As a first step towards incorporating these effects, we consider the dynamics of eq. 2 with $c \neq 0$.

In this case, steady states of the plasticity dynamics (eq. 2) are new kind of tensor decomposition: $J$ is invariant under $\mu$ up to a scaling and elementwise exponentiation. Are these steady states attractors of eq. 2? Unfortunately, the approach we used to prove theorem 1 does not allow us to answer this question. We next outline the impediment.

Assuming $\mu$ is symmetric and odeo, inserting the orthogonal decomposition (eq. 5) and projecting $J$ onto its factors (as in the proof of theorem 1) yields the dynamics for the eigenvector loadings $x$:

$$\tau \dot{x} = (Ux)^{\sigma} \circ \Sigma x^{\otimes a} - x \circ (x^T U)^{\sigma + 1} U \Sigma x^{\otimes a + 1}$$  \hspace{1cm} (57)

where $x^{\sigma}$ is the elementwise power of $x$ and $U$ is the matrix with columns composed of the orthogonal components of $\mu$ (eq. 5). (Compare this to eq. 9 in the proof of theorem 1.) If $c \neq 0$, the dynamics of the loadings $x$ are not closed but depend on the structure of the factors in $U$. A general analysis of how $U$ impacts the evolution of $x$ for $c \neq 0$ is beyond the scope of this study. We will instead consider input distributions that impart simple structure to $\mu$ and analyze the fixed points of eq. 2 for them.

In this section we also generalize the learning dynamics to incorporate a constraint on any $p$-norm of the synaptic weight vector, rather than only its Euclidean norm. This introduces a factor of $|J_j|^{p-2}$ into the second right-hand-side term of eq. 2 (appendix A.1).

A.4.1 Diagonal input correlations

We begin by analyzing inputs with constant-diagonal correlations, $\mu_d = \sigma \delta_{d}$ with $\sigma > 0$. These could arise if at each time $t$, only one synapse can be activated and the remaining inputs are 0. In that case the only nonzero contribution to $\mu$ would be $(x_i^{k+\sigma})$. It is possible in this case $\mu = 0$, for example if $x_i \sim \mathcal{N}(0, 1)$ and $a + b$ is odd. Then the leading-order contribution to $J_i$ would be supralinear in $dt$. In this case eq. 2 reduces to

$$\frac{\tau}{\sigma} \dot{J}_i = J_i^{a+c} - J_i \sum_j J_j^{a+c-1} |J_j|^p$$  \hspace{1cm} (58)

We will analyze fixed points of eq. 58 and their stability. If $J$ is a steady state of eq. 58 its Jacobian matrix is

$$\frac{\tau}{\sigma} \frac{dJ_i}{dJ_k} = \delta_{ik} \left((a+c)J_i^{c+a-1} - \sum_j J_j^{a+c+1} |J_j|^{p-2}\right) - (a + c + p - 1) J_i J_j^{a+c} |J_k|^{p-2}$$  \hspace{1cm} (59)
We will see that sparse connectivity, with one synaptic weight at 1 and the rest at zero, is always a stable equilibrium. In addition, sparse connectivity with one weight at −1 is stable if \( a + c \) is odd. In addition to these fully sparse steady states, we identify partially sparse and uniform-magnitude equilibria and conditions for their stability. We first state and prove these results. Then, we present simulation results showing that even when other stable equilibria exist, the learning dynamics tend to converge to the fully sparse equilibria.

**Theorem 3.** Let \( \mu \in \mathbb{R}^{K \times K \times \ldots \times K} = \sigma \delta \) be a diagonal tensor of order \( a + 1 \) with all diagonal elements equal to \( \sigma \), \( \sigma > 0 \). Let \( a + c = 1 \) with \( a \geq 1 \). Then the \( \ell^p \)-sphere in \( \mathbb{R}^K \) with unit radius is an attracting slow manifold of eq. [2]

**Proof.** Setting \( \dot{J} = 0 \) in eq. [58] yields the steady-state requirement

\[
J^* = J^* \left( \sum_j |J_j|^p \right)
\]  

(60)

Whenever \( ||J^*||_p = 1 \), \( \left( \sum_j |J_j|^p \right) = 1 \) and vice versa. Any \( J^* \) with \( ||J^*||_p = 1 \) is thus a steady state of eq. [2] if \( ||J^*||_p \neq 1 \), the only steady state is \( J^* = 0 \).

We next consider the linear stability to perturbations around an element \( J^* \) of the \( \ell^p \)-sphere. The Jacobian at \( J^* \) is of rank one:

\[
\frac{d\dot{J}_i}{dJ_j} = -\frac{\sigma p}{\tau} J_j^* |J_j^*|^p / J_k^*
\]  

(61)

with eigenvalues \( -(\sigma p/\tau) \hat{\lambda} \), where \( \hat{\lambda} \) is an eigenvalue of \( A \), \( A_{ik} = \dot{J}_i^* |J_k^*|^p / J_k^* \). The characteristic equation for \( A \) is

\[
J^* \sum_k |J_k^*|^p \hat{v}_k = \hat{\lambda} \hat{v}_i
\]  

(62)

where \( \hat{v} \) is an eigenvector of \( A \). Matching indices, the eigenvector \( \hat{v} \) with unit \( \ell^p \)-norm is identical to \( J^* \) and it has eigenvalue \( \hat{\lambda} = 1 \). For \( a + c = 1 \), any \( J^* \) on the \( p \)-sphere is thus a steady state with one Jacobian eigenvalue \( -(\sigma p/\tau) \), corresponding to the eigenvector \( J^* \). The remaining \( K - 1 \) eigenvalues are zero, so the orthogonal complement of \( J^* \) is a slow subspace for the linearized dynamics. Each point on the \( \ell^p \) \( K \)-sphere has such a slow subspace. Together, the \( \ell^p \) \( K \)-sphere is a linearly stable slow manifold. Is it globally attracting? Let

\[
L = \sum_{i=1}^K |J_i|^p
\]  

(63)

The total derivative of \( L \) with respect to time is

\[
\frac{dL}{dt} = p \sum_i J_i |J_i|^p - 2 J_i \frac{\sigma}{\tau} L (1 - L)
\]  

(64)

which has a stable fixed point at \( L = 1 \) and an unstable point at \( L = 0 \). The full synaptic weight dynamics thus must admit a globally attracting subspace on the \( \ell^p \) \( K \)-sphere. Those dynamics are symmetric with respect to rotations of the axes, so that subspace must be the full sphere.

**Remark.** Theorem 3 generalizes the corresponding result for Oja’s rule that, when the inputs are zero-mean and uncorrelated \( (\mu_{ij} = \sigma \delta_{ij}) \), the \( \ell^2 \)-sphere is a slow manifold of its dynamics. On it, however, the mean-field dynamics of eq. [58] vanish - so a full accounting of the weight dynamics must examine fluctuations.

To illustrate these results, we simulated the learning dynamics with individually presented, identically distributed (standard normal) inputs. Since at each time point only one input is presented, the input correlation tensors are diagonal. We first examined the classic Oja rule, taking \( (a, b, c) = (1, 1, 0) \). As expected, the synaptic weights exhibited random motion (fig. [A.4.1a]). Their \( p \)-norm was fixed and synaptic weights initialized off the unit \( p \)-sphere quickly converged onto it as predicted by Theorem 3 (fig. [A.4.1b]).

Next we examined a different parameter set with \( a + c = 1 \): \( (a, c) = (2, -1) \). We kept \( b = 1 \). In this case, we observed the synaptic weights converge to a sparse solution with one nonzero synapse with
magnitude 1 (fig. 4.11). This convergence occurred over a longer timescale than the convergence to the unit sphere for \((a, c) = (1, 0)\). For that previous parameter set, we did not observe synaptic weights converge to these sparse solutions even over this longer timescale (simulation not shown). With \((a, c) = (2, -1)\), the dynamics converged to sparse equilibria for different values of \(p\) and for synaptic weights initialized with different variances (fig. 4.11). This solution is on the unit \(p\)-sphere so does not contradict Theorem 3. It is, however, more particular. Next we examine sparse and partially-sparse equilibria, and their stability, for integer-valued \(a + c\). We begin by examining even \(a + c\), then odd.

**Theorem 4.** Take \(a + c\) even and \(\mu \in \mathbb{R}^{K \times K \times \ldots \times K} = \sigma \delta\) be a diagonal tensor of order \(a + 1\) with all diagonal elements equal. Let \(\{J^*_i \in \mathbb{R}^K\}\) be the set of \(n\)-sparse vectors with \(n\) nonzero elements \(|J^*_i| = n^{-1/p}\). Any such vector where all elements share a sign, \(J^*_i = \xi n^{-1/p}\) where \(\xi \in \{-1, 1\}\), is a steady state of eq. 58.

**Proof.** Let \(J^*\) be a \(n\)-sparse vector with nonzero elements \(J^*_i = \xi_i n^{-1/p}\), where \(\xi_i \in \{-1, 1\}\). Note that with \(\xi = 1\), \(J^*\) is a steady state solution of eq. 58.

Without loss of generality, permute \(J^*\) so that its first \(n\) elements are nonzero and last \(K - n\) elements are zero. Now, set one element \(\xi_i = -1\) and insert this solution for \(J\) into the steady-state condition. Since \(a + c\) is even, this yields

\[
1 - n = \sum_{j=1, j \neq i}^n \xi_j
\]

\(\xi_j \in \{-1, 1\}\), so this requires that \(\xi = -1\). If one element of \(\xi\) is negative, all must be. The \(n\)-sparse vector with nonzero elements \(J^*_i = -n^{-1/p}\) is also a steady state of eq. 58.

**Corollary 4.1.** If \(c = 0\) and \(a = 2\), and \(\mu\) has finitely many \(E\)-eigenvectors, then \(\{J^*_i\}\) contains all the steady states of eq. 2.
Proof. Since $c = 0$, steady state solutions of eq. 5 are also E-eigenvectors of $\mathbf{\mu}$. If $\mathbf{\mu}$ is a tensor of order 3 with finitely many E-eigenvectors, then it has $2^K - 1$ E-eigenvectors, counted with multiplicity $[50][51]$.

The set of $n$-sparse vectors with elements $J^*_i = \xi_i n^{-1/p}$, where $\xi \in \{-1, 1\}$, contains steady states of eq. 58. There are

$$\sum_{n=1}^{K} \binom{K}{n} = 2^K - 1$$

such steady states with $\xi = 1$. The corresponding E-eigenvalues are $\lambda = \sum_{j,\alpha} J_j |J_j|^p - 2 \mathbf{\mu}_{j,\alpha} (J_j^2)_\alpha = \sigma \xi n^{-1/p}$. The factor of $\xi$ cancels out in the E-eigenvector/E-eigenvalue equation. So with $c = 0, a = 2$, each of the $n$-sparse steady states with nonzero elements $J^*_i = \xi_i n^{-1/p}/p$ is proportional to a steady state of eq. 58, since the constant of proportionality would obtain a power of 2 in one term of eq. 58 and a power of $2 + p$ in the other term.

Theorem 5. Take $a + c$ odd and $\mathbf{\mu} \in \mathbb{R}^{K \times K \times \cdots \times K} = \sigma \mathbf{\delta}$ be a diagonal tensor of order $a + 1$ with all diagonal elements equal. Let $\{J^*_i \in \mathbb{R}^K\}$ be the set of $n$-sparse vectors with $n$ nonzero elements $J^*_i = \xi_i n^{-1/p}$ where each $\xi_i \in \{-1, 1\}$. Any such $J^*$ is a steady state of eq. 58.

Proof. Let $J^*$ be an $n$-sparse vector with nonzero elements $J^*_i = \xi_i n^{-1/p}$, where $\xi_i \in \{-1, 1\}$. Since $a + c$ is odd, $a + c = 1$ is even and the steady-state condition for $J^*$ is invariant to $\xi$. Each such $J^*$ is a steady state of eq. 58.

Corollary 5.1. If $c = 0$, $a = 3$, and $\mathbf{\mu}$ has finitely many E-eigenvectors, then $\{J^*_i\}$ contains all the steady states of eq. 2.

Proof. The proof follows the same construction as for Theorem 4.1. Each steady state of eq. 58 corresponds to an E-eigenvector of $\mathbf{\mu}$. For $a = 3$, there are (if finitely many) $(3^K - 1)/2$ E-eigenvectors of $\mathbf{\mu}$. The set of $n$-sparse vectors with elements $J^*_i = \xi_i n^{-1/p}$, where $\xi_i \in \{-1, 1\}$, contains steady states of eq. 58. There are $\sum_{n=1}^{K} 2^n (K)^n = 3^K - 1$ such steady states. For each $n$, two of them are equal up to a global sign change which will cancel out with the E-eigenvalue in the E-eigenvalue / E-eigenvector equation. Any other vector $J^*$ proportional to an E-eigenvector of $\mathbf{\mu}$ would not be a steady state of eq. 58, since the constant of proportionality would obtain a power of 3 in one term of eq. 58 and a power of $3 + p$ in the other term and $p \geq 1$. So these steady states are all of the weight vectors corresponding to the E-eigenvectors of $\mathbf{\mu}$, and they correspond to all of the E-eigenvectors.

Theorem 6. Let $\mathbf{\mu} \in \mathbb{R}^{K \times K \times \cdots \times K} = \sigma \mathbf{\delta}$ be a diagonal tensor of order $a + 1$ with all diagonal elements equal. Let $\{J^*_i\}$ be the set of $n$-sparse vectors with $n$ nonzero elements and $K - n$ zero elements, with nonzero elements $J^*_i = \xi_i n^{-1/p}$ where $\xi_i \in \{-1, 1\}$, $a + c \neq 1$. Then the vectors in $\{J^*_i\}$ that are linearly stable steady states of eq. 2 are:

1. Fully sparse solutions with one synaptic weight at 1, unless $a + c < 1$
2. Fully sparse solutions with one synaptic weight at -1, unless either a) $a + c$ is even and $a + c > 1$ or b) $a + c$ is odd and $a + c < 1$.
3. All $n$-sparse vectors with each $\xi_i = 1$, if $a + c = 0$.
4. Flat solutions at $J = K^{-1/p}1$, if $a + c \leq 0$ and even (if $p = 1$ it is marginally stable).
5. $n$-sparse solutions with $m \geq 1$ weights at $-n^{-1/p}$ and $n - m$ weights at $n^{-1/p}$, if $a + c < 1$ and odd.

Remark. If $c = 0$ and $a \in \{2, 3\}$, then $\{J^*_i\}$ contains all steady states of eq. 58, so the only stable steady states of eq. 58 are those described. Otherwise there might be others.
Proof. We separate the proof into sections describing the different equilibria. We begin with the fully sparse equilibria with one nonzero weight $J_j = \xi$, where $\xi \in \{ -1, 1 \}$.

Fully sparse equilibria. The Jacobian, eq. (59) reduces to

$$\frac{\tau}{\sigma} \frac{d J_I}{d J_K} = -\delta_{ik} \xi^{a+c-1} (a + c - 1 - \delta_{ij} (a + c + p - 1))$$

(66)

where $j$ is fixed. The Jacobian is diagonal and its eigenvalues are $\lambda_1 = -\xi^{a+c-1} (a + c - 1)$, with algebraic multiplicity $K - 1$, and $\lambda_2 = -\xi^{a+c-1} (a + c + p - 1)$. The fully sparse equilibrium with $\xi = 1$ is thus stable unless $a + c < 1$. The fully sparse equilibrium with $\xi = -1$ is unstable if either 1) $a + c$ is odd $a + c < 1$ or 2) $a + c$ is even and $a + c > 1$. The opposite conditions guarantee stability. If $a + c = 1$ the sparse solution is neutrally stable.

Now let the first $1 < n \leq K$ weights be nonzero and $J_j = \xi_j n^{-1/p}$, $j = 1, \ldots, n$. The $n$-sparse solution has Jacobian

$$\frac{\tau}{\sigma} \frac{d J_I}{d J_K} = -\delta_{ik} n^{-(a+c+p-1)/p} \left( \sum_{j=1}^{n} \xi_j^{a+c-1} \right) + \theta(n-i)\theta(n-k) \left( \delta_{ik} (a + c) \xi_i^{a+c-1} n^{-1} - (a + c + p - 1) n^{-1} \right)$$

(67)

We will first consider the case when $a + c$ is even and then when $a + c$ is odd.

Partially sparse and flat equilibria: $a + c$ even. In this case, all $n$ nonzero weights have the same sign, $\xi$, and

$$\frac{\tau}{\sigma} \frac{d J_I}{d J_K} = -\xi \delta_{ik} n^{-(a+c+1)/p} + \theta(n-i)\theta(n-k) \left( \delta_{ik} (a + c) \xi_i n^{-(a+c+1)/p} - (a + c + p - 1) n^{-(a+c+1)/p} \right)$$

(68)

where $\theta(x)$ is the Heaviside step function. The Jacobian is the sum of a diagonal matrix and a block-constant matrix. It is similar to a block-diagonal matrix of the form

$$\begin{pmatrix} x e_n e_n^T & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y I_{K-n} \end{pmatrix}$$

(69)

where $I_q$ is the $q \times q$ identity matrix and $e_n = (1, 0, \ldots, 0)$, and the Jacobian eigenvalues are

$$\frac{\tau}{\sigma} \lambda_1 = x + z = (a + c) n^{-(a+c+1)/p} (\xi - n^{-(p-1)/p}) - (p - 1) n^{-(a+c+p-1)/p},$$

$$\frac{\tau}{\sigma} \lambda_2 = x = \xi n^{-(a+c+1)/p} (a + c),$$

(70)

with algebraic multiplicity $n - 1$

$$\frac{\tau}{\sigma} \lambda_3 = y = -\xi n^{-(a+c+1)/p},$$

with algebraic multiplicity $K - n$

If $1 < n < K$, the latter two eigenvalues guarantee instability whether $a + c > 0$ or $a + c < 0$, since they share $\xi = \pm 1$. Let $n = K$, so $\lambda_2$ doesn’t exist. In this case,

$$\lambda_1 = K^{-(a+c+p)/p} \left( (a + c) \left( \xi K^{(p+1)/p} - K^2 \right) + K^{1/p} (1 - p) \right)$$

(71)

and $\lambda_1$ is negative if

$$(a + c) \left( K^{2-1/p} - \xi K \right) < 1 - p$$

(72)

We can determine the behavior of $\lambda_1$ by recalling that $p \geq 1$ so $K^{2-1/p} \geq K$ with equality at $p = 1$. If $p = \xi = 1$, then $\lambda_1 = 0$ and the flat equilibria has an associated slow direction. The equilibrium, $J = K^{-1/p} 1$, is then marginally stable if $\lambda_2 \leq 0$, which occurs when $a + c < 0$.

If $p > 1$ and $a + c > 0$ then $\lambda_1 > 0$ for any $K$ whether $\xi = 1$ or $\xi = -1$. If $p > 1$ and $a + c < 0$, then $\lambda_1 < 0$ for either sign of $\xi$. In that case, $\lambda_2 < 0$ only if $\xi = 1$. So for $p > 1$ and even $a + c$, the uniform steady states with $\xi = 1$ is stable if $a + c < 0$ and unstable if $a + c > 0$.

If $a + c = 0$, $\lambda_2 = 0$ and there are $n - 1$ slow directions associated with each $n$-sparse equilibrium (since, in the basis of eq. (59) these eigenvalues are associated with the unit basis eigenvectors).
Inspection of $\lambda_1, \lambda_3$ reveals that $n$-sparse equilibria are linearly stable with $\xi = 1$ and unstable with $\xi = -1$.

Partially sparse and flat equilibria: $a + c$ odd. Let $n \geq 2$. Without loss of generality, let the first $0 \leq m \leq n$ nonzero weights be negative, the next $n - m$ weights be positive, and the remaining $K - n$ weights be 0. The Jacobian is

$$
\frac{\tau}{\sigma} \frac{dJ}{dJ_k} = -\delta_{ik} n^{-(a+c-1)/p} \theta(n-i) \theta(n-k) \left( \delta_{ik} (a + c) n^{-(a+c-1)/p} - (a + c + p - 1) n^{-(a+c+p-1)/p} \xi_i \xi_k \right)
$$

which is a sum of block-diagonal and block-constant matrices,

$$
\begin{pmatrix}
xI_n & 0 \\
0 & yI_{K-n}
\end{pmatrix} + \begin{pmatrix}
C & 0 \\
0 & 0
\end{pmatrix}
$$

where $C$ is a $n \times n$ block matrix, with entries $C_{ik} \propto \xi_i \xi_k$. We can calculate the eigenvalues of $C$ by noticing that it is the sum of constant and diagonal matrices. The final Jacobian eigenvalues are

$$
\begin{align*}
\frac{\tau}{\sigma} \lambda_1 &= n^{-(a+c-1)/p} (a + c - 1), \\
\frac{\tau}{\sigma} \lambda_{2+} &= n^{-(a+c-1)/p} (a + c - 1) \left( 1 + (p/n) \left( m - (n-m) + 2 + \sqrt{(n-2)^2 + 4m(n-m)} \right) / 2 \right), \\
\frac{\tau}{\sigma} \lambda_{2-} &= n^{-(a+c-1)/p} (a + c - 1) \left( 1 + (p/n) \left( m - (n-m) + 2 - \sqrt{(n-2)^2 + 4m(n-m)} \right) / 2 \right), \\
\frac{\tau}{\sigma} \lambda_3 &= n^{-(a+c-1)/p} (a + c - 1) (1 + 2p/n), \\
\frac{\tau}{\sigma} \lambda_4 &= -n^{-(a+c-1)/p}, \text{ exists if } n < K
\end{align*}
$$

(75)

If $a + c = 1$, these are all zero except $\lambda_4$ which is negative. Take $a + c \neq 1$ and odd. $a + c$ might be positive or negative. If $a + c > 1, \lambda_1$ and $\lambda_3$ guarantee instability. If $a + c < 1$ then $\lambda_1, \lambda_3, \lambda_4$ are all negative and the only possible instability is in $\lambda_{2\pm}$. The discriminant appearing inside the square root in $\lambda_{2\pm}$, $D$, is strictly increasing with respect to $n$. Take $\lambda_{2+}$. If $a + c < 1$, then for fixed $n$ it is maximized at $m = 0$:

$$
\lambda_{2+} \leq \frac{\sigma}{\tau} n^{-(a+c-1)/p} (a + c - 1) \left( 1 + \frac{p}{2n} \right) < 0
$$

(76)

so $\lambda_{2+} < 0$ and $\lambda_{2-}$ determines the stability. If $a + c < 1$ then for fixed $n$, $\lambda_{2-}$ is also maximized at $m = 0$:

$$
\lambda_{2-} \leq \frac{\sigma}{\tau} n^{-(a+c-1)/p} (a + c - 1) \left( 1 + \frac{2p}{n} - p \right)
$$

(77)

If $a + c < 1$ and $p = 1$, that upper bound is always negative. If instead $p > 1$ and $n < 2p/(p-1)$, then the upper bound for $\lambda_{2-}(m)$ is positive: as long as $m$ is sufficiently small, $\lambda_{2-}$ can be positive. $\lambda_{2-}$ is negative if

$$
m > \frac{n(1-p) + \sqrt{n^2(p^2 - 1) + 2p^2(1-n)}}{2np}
$$

(78)

and $\lambda_{2-}$ is positive if the inequality is reversed. That bound is less than or equal to

$$
0 < \frac{1 - p + \sqrt{p^2 - 1}}{2p} < 1
$$

(79)

and approaches it from below as $n \to \infty$. So for $a + c < 1$ and odd (i.e., negative) at least one negative synaptic weight is required to stabilize a $n$-sparse steady state.

We have constructed a number of steady states for the nonlinear Hebbian dynamics with weight dependence and examined conditions for their stability. If $c \neq 0$ and $a + c \neq 1$, there are always $K$ stable sparse equilibria. In several cases, there are also other stable equilibria also (theorem 6). eq. [2] is a limiting deterministic description (large $\tau$) of an underlying stochastic dynamics, eq. [3]. Here we asked whether the fixed points we described above accurately describe the stochastic system. To examine the learning dynamics with diagonal input correlations, we presented i.i.d inputs to one synapse at a time. Since at each time point only one input is presented, the input correlation tensors
are diagonal. We examined parameter sets in each of the cases of theorem 6. For odd \( a + c > 0 \), the only stable \( n \)-sparse equilibria are fully sparse with one weight at 1 or -1 (fig. A.4.1b). These were also the only equilibrium we observed over 50 randomly chosen initial conditions (fig. A.4.1a). For even \( a + c > 0 \), the only stable equilibrium described in theorem 6 is fully sparse with one weight at 1. For such parameters, that was the only equilibrium we observed (fig. A.4.1d).

![Figure 5: Dynamics of nonlinear Hebbian plasticity rules with weight-dependence and diagonal input correlations. For all panels, we used \( K = 10 \) inputs. The learning rate was \( \eta = 10^{-2} \) for all panels except e-h, which had \( \eta = 10^{-3} \). a) Convergence to the sparse solution with one \( J_i = 1 \) for \( a + c > 0 \) and odd. b) Histogram of final synaptic weight values after \( T = 10^3 \) time steps, across 50 random initial conditions. Synaptic weights were averaged over the final 500 time points to smooth out fluctuations for visualization. c) Convergence to the sparse solution with one \( J_i = 1 \) for \( a + c > 0 \) and even. d) Histogram of final synaptic weight values (as in panel a). e) Convergence to the flat solution at \( J = K^{-1/p} \) for \( a + c = 0 \). f) Histogram of final synaptic weight values (as in panel a). g) Convergence to a bimodal distribution with 2 synaptic weights at \(-K^{-1/p}\) and the remaining 8 at \( K^{-1/p} \). h) Histogram of final synaptic weight values (as in panel a).

For \( a + c = 0 \), theorem 6 describes a combinatorial explosion of equilibria: each of the \( n \)-sparse steady states is stable. There are \( \sum_{n=1}^{K} \binom{K}{n} = 2^K - 1 \) such points, each with \( n - 1 \) neutrally stable directions. In simulations, we only observed convergence to the flat solution with \( n = K \) and all weights at \( K^{-1/p} \) (fig. A.4.1c, f). The stochastic dynamics we simulated contain terms proportional to \( J_i^2 \); this is the origin of the powers of \( c \) in eq. [58]. Since \( c < 0 \) these factors explode for \( J_i \to 0 \). So the only partially sparse solution consistent with the stochastic dynamics is the one with \( n = K \) nonzero weights.

Finally, for \( a + c < 0 \), theorem 6 describes an even greater combinatorial explosion of equilibria. Each \( n \)-sparse steady state with \( 1 < m < n \) negative weights and \( n - m \) positive weights is linearly stable. There are \( \sum_{n=1}^{K} \binom{K}{n} \sum_{m=1}^{n} \binom{n}{m} = 3^K - 2^K \) such equilibria. As before, however, if any \( J_i \to 0 \) the stochastic dynamics would explode because of the factors \( J_i^3 \). (\( a + c < 0 \) requires \( c < 0 \) since \( a > 0 \) by assumption.) So again, we see that the only possible steady states for the stochastic dynamics have \( K \) nonzero weights (fig. A.4.1g, h). In this case there are \( \binom{K}{m} \) equilibria with \( m \) negative synaptic weights and \( \sum_{m=1}^{K} \binom{K}{m} = 2^K - 1 \) such equilibria in total. With odd \( a + c < 0 \), any of these are stable and we observed convergence to various of them (fig. A.4.1g, h). For even \( a + c < 0 \), only the flat solution with all weights at \( K^{-1/p} \) are linearly stable. In simulations, we did...
not observe convergence to this solution. Instead we observed large fluctuations characterized by prolonged excursions of individual synaptic weights (fig. A.4.1, b). When \( a + c \neq 1 \), the dynamics of the synaptic weight norm are not closed. With \( a + c < 0 \) and even, the unit-norm \( \ell^p \)-sphere appeared unstable in simulations (fig. A.4.1c, d).

![Figure 6](image_url)

Figure 6: Large fluctuations in synaptic weights for \( a + c < 0 \) and even. a) Example dynamics for two different parameter sets. b) Evolution of the synaptic weight norm over 20 realizations. c) Impact of decreasing the learning rate.

### A.4.2 Rank one input correlations

Let \( \mu = r^{a+1} \), the \( (a+1) \)-fold outer product of the vector \( r \). This corresponds to the case of constant inputs. The dynamics reduce to

\[
\tau \dot{J}_i = r_i J^c_i (r^T J)^a - J_i (r^T J)^a \sum_j r_j J_j^{-1} |J_j|^p
\]

and at steady states,

\[
r_i J^c_i (r^T J)^a = J_i (r^T J)^a \sum_j r_j J_j^{-1} |J_j|^p
\]

Weights orthogonal to the input direction, \( r^T J = 0 \), are a steady state. Otherwise, we see that \( J_i = 0 \) is always a steady state for \( c > 0 \). If \( r_i = 0 \), then either \( J_i = 0 \) or \( \sum_j r_j J_j^{-1} |J_j|^p = 0 \). If \( J \) is a steady state, the Jacobian simplifies to exactly 0 so that direction defines a slow subspace of the linearized dynamics.

By definition, \( r \) is an E-eigenvector of \( \mu \) with eigenvalue \( ||r||^2_2 \) and \( \mu \) has a rank one CP decomposition in \( r \). So if \( (b,c) = (1,0) \), \( J = r \) is an attracting steady state of eq. 80 (theorem 1). Here we focus on the dynamics with weight-dependence. We study the simple case of \( c = 1 \) and a piecewise constant \( r \) with \( n \) elements equal to \( r \), and the remaining zero. We see that in this case, the unit-norm \( n \)-sphere is an equilibrium set for the dynamics and determine when it is stable.
Theorem 7. Let \( \mu \in \mathbb{R}^{K \times K \times \ldots \times K} = r^{a+1} \) be a rank one tensor of order \( a + 1 \), the \((a + 1)\)-fold outer product of \( r \), where \( r \in \mathbb{R}^K \). Let \( r \) be \( n \)-sparse and piecewise constant, with \( n \) nonzero elements equal to \( r \) and the remaining \( K - n \) elements zero. Let

\[
M(J) = \sum_{i=1}^{K} J_i
\]

and name \( S \) the unit \( \ell^p \)-sphere in \( \mathbb{R}^K \), with nonzero elements on the dimensions corresponding to the nonzero elements of \( r \). If \( c = 1 \) then

1. The \( K - n \) elements of \( J \) corresponding to the zero elements of \( r \) have a fixed point at zero. It is stable if \( M(J) > 0 \) and unstable if \( M(J) < 0 \).

2. \( S \) is a slow manifold for the dynamics of the remaining \( n \) synaptic weights. It is stable if \( a \) is odd or \( r > 0 \) and unstable if both \( a \) is even and \( r < 0 \).

Proof. Let \( r \) be \( n \)-sparse and piecewise constant, with its first \( n \) elements equal to \( r \) and the remaining \( K - n \) elements zero. Assume that \( J \neq 0 \). We proceed in order of the claims. First consider the \( K - n \) inputs where \( r_i = 0 \). For \( c = 1 \), their dynamics are

\[
\tau J_i = -r^{a+1}M^a(J)L(J)J_i
\]

where

\[
L(J) = \sum_{i=1}^{K} |J_i|^p
\]

\( L \geq 0 \) by definition with equality only at \( J = 0 \). So if \( r^{a+1}M^a > 0 \), these weights will converge to a steady state at zero. If \( r^{a+1}M^a < 0 \), these weights will diverge exponentially. If \( M \) is fixed at 0 these weights are stable.

Second consider the dynamics of the \( n \) weights with nonzero \( r_i \), which reduce to

\[
\tau \dot{J}_i = r^{a+1}M^a(J)(1 - L(J))J_i
\]

and the steady state condition for \( J_i \) is that either \( J_i = 0 \), \( M = 0 \) or \( L = 1 \). So we have steady states for the first \( n \) elements of \( J \) on either the \( \ell^p \)-sphere or on the hyperplane orthogonal to \( 1 \) (and the trivial steady state \( J_i = 0 \)). Next we examine stability for those \( n \) weights at one such point \( J^* \).

From eq. 82, the Jacobian matrix at \( J^* \) has rank one

\[
\tau \frac{dJ_i}{dJ_k} = -pr^{a+1}M^a(J^*)J_i^*J_j^*|J_i|^p
\]

It has one eigenvalue \(-|p/\tau| r^{a+1}M^a(J^*) \sum_i J_i J_i^* |J_i|^p\), with associated eigenvector \( J^* \). The remaining \( n - 1 \) eigenvalues are zero, and the orthogonal complement of \( J^* \) is their slow eigenspace. Each point \( J^* \) on the \( \ell^p \)-sphere has such a slow eigenspace so the full sphere is a slow manifold. To determine the stability of the unit-norm \( \ell^p \)-sphere we will examine the dynamics of the synaptic weight norm. The dynamics of \( L \) and \( M \) form a closed system:

\[
\tau \dot{L} = pr^{a+1}M^aL(1 - L)
\]

\[
\tau \dot{M} = r^{a+1}M^{a+1}(1 - L)
\]

There are two line equilibria on \( M = 0 \) and \( L = 1 \) and the Jacobian determinant is \( pr^{a+3}M^{a+1}(1 - L)^2 \), which is zero on either of those line equilibria so a linear stability analysis is uninformative. Recall that \( L \geq 0 \) by definition. There are three relevant cases for the dynamics. When \( a \) is odd, all factors of \( r \) are positive and so is \( M^{a+1} \). When \( a \) is even, the sign of \( r \) impacts the sign of \( M \). We next examine the three cases: 1) \( a \) odd, 2) \( a \) even and \( r > 0 \) and 3) \( a \) even and \( r < 0 \).

First take \( a \) odd (fig. A.4.2b). Then \( L = 1 \) is attracting when \( M > 0 \) but repelling when \( M < 0 \). \( M \) is always increasing for \( L < 1 \) and decreasing for \( L > 1 \). With \( a \) even and \( r > 0 \), \( L = 1 \) is always attracting (fig. A.4.2b). \( M = 0 \) is attracting for \( L > 1 \) and vice versa. If \( a \) is even and \( r < 0 \), \( L = 1 \) is always repelling. In this case, if \( L(0) > 1 \) the synaptic weights will explode while if \( L(0) < 1 \) the synaptic weights will evolve towards the stable equilibrium \( L = 0, M = 0 \) (fig. A.4.2c). This corresponds to \( J = 0 \). In sum, the unit-norm solution \( L = 1 \) can be attracting or repelling. It is attracting if \( a \) is odd, or \( a \) even with \( r > 0 \). It is repelling if \( a \) is even and \( r < 0 \).
Figure 7: Dynamics of the synaptic weight norm (phase portraits). The vectors $\dot{L}, \dot{M}$ are plotted with unit norm. For each case, we show two corresponding parameter sets. 

a) Case 1: $a$ odd. 

b) Case 2: $a$ even and $r > 0$. 

c) Case 3: $a$ even and $r < 0$. 
