Deciding Asynchronous Hyperproperties for Recursive Programs

JENS OLIVER GUTSFELD, University of Münster, Germany
MARKUS MÜLLER-OLM, University of Münster, Germany
CHRISTOPH OHREM, University of Münster, Germany

We introduce a novel logic for asynchronous hyperproperties with a new mechanism to identify relevant positions on traces. While the new logic is more expressive than a related logic presented recently by Bozzelli et al., we obtain the same complexity of the model checking problem for finite state models. Beyond this, we study the model checking problem of our logic for pushdown models. We argue that the combination of asynchronicity and a non-regular model class studied in this paper constitutes the first suitable approach for hyperproperty model checking against recursive programs.

CCS Concepts: • Theory of computation → Modal and temporal logics; Semantics and reasoning: Verification by model checking; Logic and verification.

Additional Key Words and Phrases: Temporal Logic, Hyperproperties, Automata Theory, Model Checking, Pushdown Systems, Asynchronicity

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1 INTRODUCTION

In recent years, hyperproperties have received increased interest in verification, static analysis and other areas of computer science. While traditional trace properties provide a unifying concept for phenomena that can be captured by considering traces of a system individually, hyperproperties provide such a concept for phenomena that require us to look at multiple traces of a system simultaneously. For example, A state annotated with the proposition p must eventually be reached is a trace property while The number of occurrences of p is the same on all traces is a hyperproperty. Many important requirements in information security like observational determinism or non-interference can be described by hyperproperties [Clarkson and Schneider 2010]. They also provide a natural framework for the analysis of concurrent systems [Bonakdarpour et al. 2018].

As traditional specification logics like LTL are suitable for trace properties only, new hyperlogics were developed to specify hyperproperties. A prominent example is HyperLTL [Clarkson et al. 2014] which adds quantification over named traces to LTL and thus enables the simultaneous analysis of multiple traces. These hyperlogics first only followed traces synchronously, but software is inherently asynchronous [Baumeister et al. 2021], especially concurrent software [Finkbeiner 2017], and therefore new hyperlogics that can relate traces at different time points are required. For

Authors’ addresses: Jens Oliver Gutsfeld, University of Münster, Germany, jens.gutsfeld@uni-muenster.de; Markus Müller-Olm, University of Münster, Germany, markus.mueller-olm@uni-muenster.de; Christoph Ohrem, University of Münster, Germany, christoph.ohrem@uni-muenster.de.

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example, when checking information-flow policies on concurrent programs, traces might only be required to be equivalent up to stuttering [Zdancewic and Myers 2003] and thus matching observation points on different traces are not perfectly aligned. Another example for an asynchronous hyperproperty is the hyperproperty The number of occurrences of \( p \) is the same on all traces from above since matching \( p \)-positions on different traces may be arbitrarily far apart. In [Gutsfeld et al. 2021], a systematic study of asynchronous hyperproperties was conducted, including the introduction of the temporal fixpoint calculus \( H_\mu \). While \( H_\mu \) is able to capture the class of asynchronous hyperproperties nicely, its model checking problem is highly undecidable and even the decidable fragments presented in [Gutsfeld et al. 2021] have a high complexity. Later, the asynchronous hyperlogic HyperLTL\( S \) was introduced in [Bozzelli et al. 2021] that has an interesting decidable fragment for the model checking problem with lower complexity, simple HyperLTL\( S \). It extends HyperLTL by modalities that jump from the current position on each trace to the next position where some formula from a set of LTL formulae defining an indistinguishability criterion takes a different value. While accounting for asynchronicity is a necessary feature of hyperlogics for software systems, current model checking procedures for logics such as HyperLTL\( S \) are still insufficient as they only handle finite models which cannot capture many programs suitably due to the lack of a representation for the call stack. Moreover, expressivity of asynchronous hyperlogics can be increased largely beyond simple HyperLTL\( S \) without increasing the complexity of the model checking problem for finite models. For example, HyperLTL\( S \) lacks the ability to express arbitrary \( \omega \)-regular properties and cannot express properties like The number of occurrences of \( p \) is the same on all traces.

In this paper, we address these shortcomings by introducing a new asynchronous hyperlogic based on the linear-time \( \mu \)-calculus extending HyperLTL\( S \) in the following respects: 1) it provides a simpler jump mechanism that directly characterises positions of interest instead of an indistinguishability criterion. 2) it supports different jump criteria for different traces; 3) for the specification of the jump criterion and basic properties of single traces, it allows linear-time \( \mu \)-calculi formulae with CaRet-like modalities [Alur et al. 2004], i.e. modalities inspecting the call/return behaviour of recursive programs; and 4) it offers fixpoint operators in multitrace formulae. Moreover, we provide variants of the modalities with a well-aligned semantics to enable decidability of model checking for pushdown systems (PDS), a well established model of recursive programs. This novel concept requires the traces under consideration to have a similar call-return behaviour.

We call the new logic mumbling \( H_\mu \) where the notion of mumbling is a counterpart to the notion of stuttering from HyperLTL\( S \) similar to how stuttering and mumbling are used as counterparts in a classic paper by Brookes [Brookes 1996]: Stuttering describes the repetition of equal states while mumbling describes suppression of intermediate states. It turns out that mumbling is a more powerful jump mechanism than stuttering if only LTL modalities are used in jump criteria. Surprisingly, the difference vanishes if arbitrary fixpoint operators are allowed for the definition of jump criteria.

The use of fixpoints on the trace and multitrace level gives mumbling \( H_\mu \) the power to specify arbitrary \( \omega \)-regular properties on both levels. Despite this and the other additions, the model checking problem against finite state models has the same complexity as for the less expressive logic simple HyperLTL\( S \) under analogous restrictions necessary for decidability. In addition, it turns out that the model checking problem for PDS is decidable for mumbling \( H_\mu \) with well-aligned modalities and the above-mentioned restriction even though already synchronous HyperLTL model checking is undecidable for such models [Pommellet and Touili 2018]. Thus, our approach provides the first model checking algorithm for an asynchronous hyperlogic on PDS. Moreover, it is the first application of CaRet-like non-regular operators to the hyperproperty setting. In summary:
• We introduce mumbling $H_\mu$, an asynchronous hyperlogic with several extensions compared to HyperLTL_S and present examples highlighting the merits of the new logic (Section 3).
• We show that the finite state model checking complexity for mumbling $H_\mu$ coincides with that of simple HyperLTL_S under an analogous restriction despite the extensions (Section 4).
• We introduce well-aligned modalities and present a technique able to handle these modalities for decidable PDS model checking (Section 5). The technique is also of independent interest as it can be transferred to other hyperlogics for decidable PDS model checking.
• We compare mumbling to stuttering with respect to expressivity and show that it is more expressive for criteria defined by LTL formulae and equally expressive in the presence of fixpoints (Section 6). These results require some heavy technical work due to the intricacies of the definition of stuttering and mumbling modalities.

Due to lack of space, some technical proofs and constructions can be found in an extended version made available via arxiv [Gutsfeld et al. 2023].

2 PRELIMINARIES

Without further ado, we introduce notation, models and results used throughout the paper. This section may be skipped on first reading and can be consulted for reference later.

Pushdown Systems and Kripke Structures. We start by introducing models for recursive systems and systems with a finite state space. For this, let $AP$ be a finite set of atomic propositions, $\Theta$ be a finite set of stack symbols and $\bot \notin \Theta$ be a special bottom of stack symbol. We model recursive systems by structures $\mathcal{PD} = (S, S_0, R, L)$ called pushdown systems (PDS) where $S$ is a finite set of control locations, $S_0 \subseteq S$ is a set of initial control locations and $L: S \rightarrow 2^{AP}$ is a labeling function. The transition relation $R \subseteq (S \times S) \cup (S \times S \times \Theta) \cup (S \times \Theta \times S)$ consists of three kinds of transitions: Internal transitions from $S \times S$, push transitions from $S \times S \times \Theta$ and return transitions from $S \times \Theta \times S$. The semantics of a PDS $\mathcal{PD} = (S, S_0, R, L)$ is based on configurations, i.e. pairs $c = (s, u)$ where $s \in S$ is a control location and $u \in \Theta^* \bot$ is a stack content ending in $\bot$. The set of all configurations of $\mathcal{PD}$ is denoted by $C(\mathcal{PD})$. For the definition of the semantics, let $c = (s, u)$ and $c' = (s', u')$ be configurations. We call $c'$ an internal successor of $c$, denoted by $c \rightarrow_{\text{int}} c'$, if there is a transition $(s, s') \in R$ and $u = u'$. We call $c'$ a call successor of $c$, denoted by $c \rightarrow_{\text{call}} c'$, if there is a transition $(s, s', \theta) \in R$ and $u' = \theta u$. We call $c'$ a return successor of $c$, denoted by $c \rightarrow_{\text{return}} c'$, if there is a transition $(s, \theta, s') \in R$ and $u = u'$. A path of $\mathcal{PD}$ is an infinite alternating sequence $p = c_0c_1c_2\cdots \in (C(\mathcal{PD}) \cdot \{\text{int, call, ret}\})^\omega$ such that $c_0 = (s_0, \bot)$ for some $s_0 \in S_0$ and $c_i \rightarrow_{\text{int}} c_{i+1}$ holds for all $i \geq 0$. Paths of a system induce sequences of visible system behaviour called traces; an infinite trace is an infinite sequence from $\text{Traces} := (2^{AP} \cdot \{\text{int, call, ret}\})^\omega$ and a finite trace is a finite sequence from $(2^{AP} \cdot \{\text{int, call, ret}\})^+ \cdot 2^{AP}$. The trace induced by the path $p$ is $L(c_0)m_0L(c_1)m_1\cdots \in \text{Traces}$ where $L((s, u))$ is given by $L(s)$. We write $\text{Paths}(\mathcal{PD})$ for the set of paths of $\mathcal{PD}$ and $\text{Traces}(\mathcal{PD})$ for the set of traces induced by paths in $\text{Paths}(\mathcal{PD})$. Our model for finite state systems, Kripke structures, is defined as a special case of a PDS $\mathcal{PD} = (S, S_0, R, L)$ where $R \subseteq S \times S$, i.e. a PDS with only internal transitions. In order to highlight this case, we use $\mathcal{K}$ instead of $\mathcal{PD}$ to denote Kripke structures. As all transition labels are int in traces generated by Kripke structures, we omit these labels and write traces as sequences from $(2^{AP})^\omega$. Also, we introduce fair variants of these two system models. A fair pushdown system is a pair $(\mathcal{PD}, F)$ where $\mathcal{PD} = (S, S_0, R, L)$ is a PDS and $F \subseteq S$ is a set of target states. $\text{Paths}(\mathcal{PD}, F)$ is the set of paths of $\mathcal{PD}$ that visit states in $F$ infinitely often (fair paths). Then, $\text{Traces}(\mathcal{PD}, F)$ is the set of traces induced by $\text{Paths}(\mathcal{PD}, F)$. A fair Kripke structure is defined analogously.

Words and traces. We introduce our notation for common operations on words. For infinite words $w = w_0w_1\cdots \in \Sigma^\omega$ over an alphabet $\Sigma$, we use $w(i) = w_i$ to denote the letter at position
i of \( w \) and \( w[i] = w_i w_{i+1} \ldots \) for the suffix of \( w \) starting at position \( i \). Furthermore, for \( i \leq j \) we write \( w[i, j] = w_i w_{i+1} \ldots w_j \) for the subword from position \( i \) to position \( j \) of \( w \). For traces \( tr = P_0 m_0 P_1 m_1 \ldots \), we slightly alter these notations in order to improve readability and write \( tr(i) \) for the symbol \( P_i \), \( tr[i] \) for the infinite trace \( P_i m_i P_{i+1} m_{i+1} \ldots \) and \( tr[i, j] \) for the finite trace \( P_i m_i \ldots P_{j-1} m_{j-1} P_j \). The same applies to paths. For finite and infinite traces \( tr \), we use \( tr[t] \) to denote their restriction to their transition symbols, i.e. if \( tr = P_0 m_0 P_1 m_1 \ldots m_{n-1} P_n \) then \( tr[i] = m_0 m_1 \ldots m_{n-1} \). Also, we introduce some successor and predecessor functions as in [Alur et al. 2004].

Intuitively, the global successor always moves to the next index and the backwards predecessor moves to the previous index while the abstract successor skips over procedure calls and the caller predecessor moves back to the point where the current procedure was called. Formally, we define several functions \( f \): \( \text{Traces} \times \mathbb{N}_0 \to \mathbb{N}_0 \) (or partial functions \( f \): \( \text{Traces} \times \mathbb{N}_0 \rightsquigarrow \mathbb{N}_0 \)) where \( \mathbb{N}_0 \) is the set of natural numbers including zero that are interpreted as follows: If \( f(tr, i) = j \), then \( f \) moves from \( tr(i) \) to \( tr(j) \). We define the global successor function \( \text{succ}_g \): \( \text{Traces} \times \mathbb{N}_0 \to \mathbb{N}_0 \) by \( \text{succ}_g(tr, i) = i + 1 \). The backwards predecessor function \( \text{succ}_b \): \( \text{Traces} \times \mathbb{N}_0 \rightsquigarrow \mathbb{N}_0 \) is partial where \( \text{succ}_b(tr, i) = i - 1 \) if \( i > 0 \) and is undefined otherwise. For the definition of the remaining two functions, let \( \text{calls}(tr, i, j) = |\{k \mid i < k < j \text{ and } tr_{[i]}(k) = \text{call}\}| \) be the number of calls between positions \( i \) and \( j \) on \( tr \) and \( \text{rets}(tr, i, j) = |\{k \mid i < k < j \text{ and } tr_{[i]}(k) = \text{ret}\}| \) be the number of returns between positions \( i \) and \( j \) on trace \( tr \). Then, the abstract successor function \( \text{succ}_a \): \( \text{Traces} \times \mathbb{N}_0 \rightsquigarrow \mathbb{N}_0 \) is the partial function such that \( \text{succ}_a(tr, i) = i + 1 \), if \( tr_{[i]}(i) = \text{int} \), \( \text{succ}_a(tr, i) = \min S \), where \( S = \{ j \mid j > i, \text{calls}(tr, i, j) = \text{rets}(tr, i, j) \} \), if \( tr_{[i]}(i) = \text{call} \) and the set \( S \) is non-empty, and is undefined otherwise. Our definition of the abstract successor differs slightly from that in [Alur et al. 2004], where it is defined on words over an extended alphabet \( \Sigma \times \{ \text{int}, \text{call}, \text{ret} \} \) and moves from a call to the matching return. Instead, we move from the propositional position before a call to the propositional position after the matching return, which is more natural in our scenario since it ensures that both positions have the same stack level. Finally, the caller function \( \text{succ}_c \): \( \text{Traces} \times \mathbb{N}_0 \rightsquigarrow \mathbb{N}_0 \) is the partial function such that \( \text{succ}_c(tr, i) = \max S \), where \( S = \{ j \mid j < i, \text{calls}(tr, j, i) = \text{rets}(tr, j + 1, i) \} \), if the set \( S \) is non-empty, and is undefined otherwise.

**Multi-Automata.** In one of our constructions, we use multi-automata [Bouajjani et al. 1997] to represent certain sets of configurations of a PDS. Formally, let \( \mathcal{PD} = (S, S_0, R, L) \) be a PDS with \( S = \{s_1, \ldots, s_m\} \) and stack alphabet \( \Theta \). A \( \mathcal{PD}\)-multi-automaton is a tuple \( \mathcal{A} = (Q, Q_0, \rho, F) \) where \( Q \) is a finite set of states, \( Q_0 = \{q_1, \ldots, q_m\} \subseteq Q \) is a set of initial states, \( \rho: Q \times \Theta \to 2^Q \) is a transition function and \( F \) is a set of final states. The transition relation \( \rho \subseteq Q \times \Theta^* \times Q \) is the smallest relation such that (i) \( q \rightarrow^\epsilon q \) for all \( q \in Q \) and (ii) \( q \rightarrow_u q' \) and \( q' \in \rho(q', \theta) \) implies \( q \rightarrow_{u, \theta} q' \). A configuration \( c = (s, u) \) is recognised by \( \mathcal{A} \) iff \( q_i \rightarrow_u q \) for some \( q \in F \). By slight abuse of notation, we sometimes identify \( q_i \) with \( s_i \) and write \( q_i = s_i \) for \( q_i \in Q \) and \( s_i \in S \). The set of configurations recognised by \( \mathcal{A} \) is denoted by \( C(\mathcal{A}) \). The following result is a corollary of Proposition 3.1 in [Bouajjani et al. 1997].

**Proposition 2.1.** For any fair pushdown system \((\mathcal{PD}, F)\), there is a \( \mathcal{PD}\)-multi-automaton \( \mathcal{A} \) with size linear in \(|\mathcal{PD}|\) such that \( C(\mathcal{A}) = \{c \in C(\mathcal{PD}) \mid \text{there is a fair path in } (\mathcal{PD}, F) \text{ starting in } c \} \).

**Visibly Pushdown Automata.** Next, we use visibly pushdown automata [Alur and Madhusudan 2004] in some of our constructions. These automata are a variant of conventional pushdown automata, i.e. automata with access to a stack, but have better closure and decidability properties. Their input alphabet is called a finite visibly pushdown alphabet, i.e. an alphabet \( \Sigma = \Sigma_i \cup \Sigma_e \cup \Sigma_x \) partitioned into alphabets \( \Sigma_i \) of internal symbols, \( \Sigma_e \) of call symbols and \( \Sigma_x \) of return symbols. Like PDS, they are defined over a finite stack alphabet \( \Theta \) and a special bottom of stack symbol \( \bot \notin \Theta \). Formally, a (nondeterministic) visibly pushdown automaton (VPA) over \( \Sigma \) and \( \Theta \) is a tuple
\( \mathcal{A} = (Q, Q_0, \rho, F) \) where \( Q \) is a finite set of states, \( Q_0 \subseteq Q \) is a set of initial states and \( F \subseteq Q \) is a set of final states. The transition function \( \rho : Q \times \Sigma \rightarrow 2^Q \cup 2^{Q \times (\Theta^* \perp)} \) allows transitions of three different types: (i) if \( \sigma \in \Sigma_t \), then \( \rho(q, \sigma) \in 2^Q \) holds and \( \rho(q, \sigma) \) is a set of internal transitions, (ii) if \( \sigma \in \Sigma_c \), then \( \rho(q, \sigma) \in 2^{Q \times \Theta} \) holds and \( \rho(q, \sigma) \) is a set of call transitions, and (iii) if \( \sigma \in \Sigma_r \), then we have \( \rho(q, \sigma) \in 2^{Q \times (\Theta^* \perp)} \) and \( \rho(q, \sigma) \) is a set of return transitions. Intuitively, seeing a symbol from \( \Sigma_t, \Sigma_c \) and \( \Sigma_r \) forces a VPA to make an internal, a call and a return transition, respectively.

Formally, a run of a VPA \( \mathcal{A} \) over an infinite word \( w_0 w_1 \cdots \in \Sigma^\omega \) is an infinite sequence \( (q_0, u_0) (q_1, u_1) \cdots \in (Q \times \Theta^* \perp)^\omega \) such that \( q_0 \in Q_0, u_0 = \bot \) and for all \( i \geq 0 \) (i) if \( w_i \in \Sigma_t \), then \( q_{i+1} \in \rho(q_i, w_i) \) and \( u_i = u_{i+1} \), (ii) if \( w_i \in \Sigma_c \), then \( q_{i+1}, \theta \in \rho(q_i, w_i) \) and \( u_{i+1} = \theta u_i \) for some \( \theta \in \Theta \), and (iii) if \( w_i \in \Sigma_r \), then \( (q_{i+1}, \theta) \in \rho(q_i, w_i) \) and either \( u_i = \theta u_{i+1} \) for \( \theta \in \Theta \) or \( u_i = u_{i+1} = \bot \). A run \((q_0, u_0)(q_1, u_1) \cdots \in (Q \times \Theta^* \perp)^\omega\) is accepting iff \( q_i \in F \) for infinitely many \( i \). A VPA \( \mathcal{A} \) accepts a word \( w \) iff there is an accepting run of \( \mathcal{A} \) over \( w \). We use \( \mathcal{L}(\mathcal{A}) \) to denote the set of words accepted by \( \mathcal{A} \). For VPA, the following proposition holds:

**Proposition 2.2 ([Alur and Madhusudan 2004])**. For any VPA, there is a VPA with an exponentially larger number of states for the complement language. The VPA emptiness problem is in \( \text{PTIME} \).

**2-way Alternating Jump Automata and their subclasses.** We now define 2-way Alternating Jump Automaton [Bozzelli 2007], a model that provides a direct way to navigate over input words using the global and abstract successor as well as backwards and caller predecessor types previously defined in this section. The corresponding functions defined on traces previously are straightforwardly extended to words over a visibly pushdown alphabet \( \Sigma \). Also, we use \( \Theta = \{g, a, b, c\} \) for the set of corresponding directions. A 2-way Alternating Jump Automaton (2-AJA) is a tuple \( \mathcal{A} = (Q, Q_0, \rho, \Omega) \) where \( Q \) is a finite set of states, \( Q_0 \subseteq Q \) is a set of initial states, \( \Omega : Q \rightarrow \{0, 1, \ldots, k\} \) is a priority assignment and \( \rho : Q \times \Sigma \rightarrow \mathcal{B}^+(\Theta \times Q \times Q) \) is a transition function where \( \mathcal{B}^+(\Theta \times Q \times Q) \) denotes positive boolean formulae over \( (\Theta \times Q \times Q) \). In the transition function, a triple \((\text{dir}, q, q')\) denotes that if the \( \text{dir}\)-successor or predecessor exists in the current position \( i \), the automaton starts a copy in state \( q \) at this successor or predecessor and else starts a copy in state \( q' \) at position \( i + 1 \). We assume that every 2-AJA has two distinct states \( \text{true} \) and \( \text{false} \) with priority 0 and 1, respectively, such that \( \rho(\text{true}, \sigma) = (g, \text{true}, \text{true}) \) and \( \rho(\text{false}, \sigma) = (g, \text{false}, \text{false}) \) for all \( \sigma \in \Sigma \). We define several commonly used automata models as special cases of 2-AJA. In particular, an Alternating Parity Automaton (APA) is a 2-AJA with a transition function that maps to \( \mathcal{B}^+(\{g\} \times Q \times Q) \). An APA with a priority assignment \( \Omega \) where \( \Omega(q) \in \{0, 1\} \) for every \( q \) and a transition function \( \rho \) mapping to disjunctions only is called a Non-deterministic Büchi Automaton (NBA). As usual for NBA, we define the acceptance condition by the set \( F \) of states with priority 0 and write \( \rho(q, \sigma) \) as a set of states.

We now define the semantics of 2-AJA. A tree \( T \) is a subset of \( \mathbb{N} \) such that for every node \( t \in \mathbb{N} \) and every positive integer \( n \in \mathbb{N} : t \cdot n \in T \) implies (i) \( t \in T \) (we then call \( t \cdot n \) a child of \( t \)), and (ii) for every \( 0 < m < n, t \cdot m \in T \). We assume every node has at least one child. A path in a tree \( T \) is a sequence of nodes \( t_0 t_1 \cdots \) such that \( t_0 = \epsilon \) and \( t_{i+1} \) is a child of \( t_i \) for all \( i \in \mathbb{N} \). A \((q, j)\)-run of a 2-AJA over an infinite word \( w = w_0 w_1 \cdots \in \Sigma^\omega \) is a \( \mathbb{N} \times Q \)-labelled tree \((T, r)\) where \( r : T \rightarrow \mathbb{N} \times Q \) is a labelling function that satisfies \( r(\epsilon) = (j, q) \) and for all \( t \in T \) labelled \( r(t) = (i, q') \) we have a set \((\{\text{dir}_1, q_1', q_2'\}, \ldots, (\text{dir}_l, q_l', q_l')\})\) satisfying \( \rho(q', w_i) \) and children \( t_1, \ldots, t_l \) that are labelled as follows: for all \( 1 \leq h \leq l, \text{if} \text{succ}_{\text{dir}_h}(w, i) \) is undefined, then \( r(t_h) = (i + 1, q_h') \), else \( r(t_h) = (\text{succ}_{\text{dir}_h}(w, i), q_h') \). A \((q, j)\)-run of an AJA is accepting iff for every path in the run the lowest priority occuring infinitely often on that path is even. \( \mathcal{A} \) accepts a word \( w \) iff there is an accepting \((q_0, 0)\)-run of \( \mathcal{A} \) over \( w \) for some \( q_0 \in Q_0 \). We write \( \mathcal{L}(\mathcal{A}) \) for the set of words accepted by \( \mathcal{A} \). For 2-AJA and their subclasses, the following propositions hold:
Proposition 2.3 ([Bozzelli 2007]). For every 2-AJA with n states, there is a VPA with a number of states exponential in n accepting the same language.\footnote{The definition of abstract successors in [Bozzelli 2007] differs slightly from the one we use here. However, a 2-AJA using our definition can straightforwardly be translated to an equivalent 2-AJA using the definition from [Bozzelli 2007] so that Proposition 2.3 also applies to the definition presented here.}

Proposition 2.4 ([Dax and Klaedtke 2008]). For any APA with n states and k priorities, there is an NBA with \(2^{O(n \cdot k \cdot \log(n))}\) states accepting the same language.

Proposition 2.5. The emptiness problem is in PSPACE for APA and in NLOGSPACE for NBA.

Proposition 2.5 can be found e.g. in [Demri et al. 2016].

**Functions.** We introduce two notations for functions that are used throughout the paper. For a function \(f\), we use \(f[a \mapsto b]\) for the function defined by \(f[a \mapsto b](a) = b\) and \(f[a \mapsto b](a') = f(a')\) for all \(a' \neq a\). We also need a function for nested exponentials, which we define as \(g_{c,p}(0, n) := p(n)\) and \(g_{c,p}(d + 1, n) := e_{g_{c,p}}^{c,p}(d, n)\) for a constant \(c > 1\) and a polynomial \(p\). We say that a function \(f\) is in \(O(g(d, n))\) if \(f\) is in \(O(g_{c,p}(d, n))\) for some constant \(c > 1\) and polynomial \(p\).

## 3 A MUMBLING HYPERLOGIC

In this section, we introduce mumbling \(H_\mu\). In Section 3.1, we define the syntax, explain it on a conceptual level and also introduce relevant notations and conventions. Then, in Section 3.2, we present some example applications of mumbling \(H_\mu\) suitable for the model checking of recursive programs. Finally, we define the semantics of the logic formally in Section 3.3.

### 3.1 Syntax of Mumbling \(H_\mu\)

Mumbling \(H_\mu\) is inspired by the hyperlogic HyperLTL\(_S\) [Bozzelli et al. 2021]. Like HyperLTL\(_S\), mumbling \(H_\mu\) is a hyperlogic with trace quantification and asynchronous progression on traces. Unlike HyperLTL\(_S\) however, it is a fixpoint calculus, has more expressive atomic properties, and has a simpler jump criterion.

**Definition 3.1 (Syntax of mumbling \(H_\mu\)).** Let \(N\) be a set of trace variables and \(\chi_\omega, \chi_1\) be disjoint sets of fixpoint variables. We define three types of mumbling \(H_\mu\) formulae by the following grammar:

- **hyperproperty formulae**
  \[
  \varphi := \exists \pi . \varphi | \forall \pi . \varphi | \psi
  \]

- **multitrace formulae**
  \[
  \psi := [\delta]_{\pi} | X | \psi \lor \psi | \neg \psi | \bigodot \Delta \psi | \mu X . \psi
  \]

- **trace formulae**
  \[
  \delta := ap | Y | \delta \lor \delta | \neg \delta | \bigodot \delta | \mu Y . \delta
  \]

where \(\pi \in N\) is a trace variable, \(X \in \chi_\omega\) and \(Y \in \chi_1\) are fixpoint variables, \(\Delta : N \rightarrow \delta\) is a successor assignment, \(ap \in AP\) is an atomic proposition and \(f \in \{g, a, c\}\) is a successor or predecessor type.

We introduce some additional syntactical notions. A multitrace formula \(\psi\) is *closed* if every fixpoint variable used in it is bound, i.e. if in \(\psi\) as well as all its maximal trace subformulae \(\delta\), fixpoint variables \(X\) and \(Y\) only occur inside fixpoint formulae \(\mu X . \psi'\) and \(\mu Y . \delta'\), respectively. We call a hyperproperty formula \(\varphi\) closed if its maximal multitrace subformula \(\psi\) is closed and additionally, every trace variable \(\pi\) used in \(\psi\) is bound by a quantifier. As usual, we assume that fixpoint variables occur *positively* in closed formulae, i.e. in scope of an even number of negations inside the corresponding fixpoint formula. We write \(\text{Sub}(\varphi)\) for the set of subformulae of \(\varphi\) and \(\text{base}(\varphi)\) for the set of base formulae of \(\varphi\), i.e. the set of trace formulae occurring in a test \([\delta]_{\varphi}\) or a successor assignment \(\Delta(\pi)\) of \(\varphi\). The size \(|\varphi|\) of a hyperproperty formula \(\varphi\) is defined as the number of its distinct subformulae. The same definitions apply to trace and multitrace formulae \(\delta\) and \(\psi\). Before introducing further definitions and examples, we informally describe the intuition behind each type of
formula. Trace formulae (denoted $\delta$) specify properties of single traces. Here, atomic propositions $ap$ express that $ap$ holds on the current position of the trace. Progress is made via next operators $\circ^f \delta$, which expresses that the $f$-successor or predecessor exists in the current position and satisfies $\delta$. Here, $f$ can be one of three kinds of successors or predecessors: $g$ for a global successor, $a$ for an abstract successor, and $c$ for the caller. The latter two successor and predecessor types allow to express richer properties on traces generated by pushdown systems rather than Kripke structures.

In addition, we have disjunction $\delta \lor \delta$, negation $\neg \delta$ and fixpoints $\mu Y. \delta$ to express more involved properties. Formulae of this kind essentially correspond to the logic VP-$\mu$TL from [Bozzelli 2007], a variant of the linear time $\mu$-calculus $\mu$TL [Vardi 1988] with various non-regular next operators as introduced by the logic CaRet [Alur et al. 2004]. Multitrace formulae (denoted $\psi$) express hyperproperties on a set of named traces $\pi_1, \ldots, \pi_n$. Basic properties $[\delta]_\pi$ express that the trace formula $\delta$ holds in the current position on trace $\pi$. So-called successor assignments $\Delta$ assigning a formula $\delta$ to each trace $\pi$ describe points of interest on the traces. The next operator $\circ^\Delta \psi$ advances each trace $\pi$ to the next position where $\Delta(\pi)$ holds and checks for $\psi$ on the resulting suffixes. This next operator is inspired by, but different from, the one of HyperLTL, which advances every trace to the next point where the valuation of some formula $\psi$ from a set of trace formulae $\Gamma$ differs from the current valuation. Also, note that $\circ^f \delta$ and $\circ^\Delta \psi$, being formulae on different levels, operate quite differently: The former advances a single trace to the $f$-successor or predecessor while the latter advances all traces according to a successor assignment $\Delta$ simultaneously. Again, we have disjunction $\psi \lor \psi$, negation $\neg \psi$ and fixpoints $\mu X. \psi$ for more complex properties. Finally, hyperproperty formulae (denoted $\phi$) express hyperproperties. Here, we extend specifications $\phi$ by trace quantifiers $\exists \pi. \phi$ and $\forall \pi. \phi$ expressing that for some or each trace of a system, respectively, $\phi$ holds if $\pi$ is bound to that trace.

We use common syntactic sugar: In trace formulae $\delta$, we use $true \equiv ap \lor \neg ap$, $false \equiv \neg true$, $\delta \land \delta' \equiv \neg (\neg \delta \lor \neg \delta')$, $\delta \rightarrow \delta' \equiv \neg \delta \lor \delta'$, $\delta \leftrightarrow \delta' \equiv (\delta \rightarrow \delta') \land (\delta' \rightarrow \delta)$ and $\nu Y. \delta \equiv \neg \mu Y. \neg \delta[Y/Y]$. We use the same abbreviations for multitrace formulae $\psi$. Additionally, we borrow some LTL-modalities as derived operators in order to improve readability: $\cal F^f \delta \equiv \mu Y. \delta \lor \circ^f Y$, $\cal G^d \delta \equiv \neg \cal F^d \neg \delta$ and $\delta_1 \cal U^f \delta_2 \equiv \mu Y. \delta_1 \lor (\delta_1 \land \circ^f Y)$. Again, we use the same abbreviations for formulae $\psi$, this time using $\circ^\Delta$ operators instead of $\circ^f$ operators. Using some of these connectives and commonly known equivalences, we can impose additional restrictions on the syntax of mumbling $H_\mu$. In particular, we assume a positive form where negation only occurs directly in front of atomic propositions $ap$ in trace formula and only occur in front of tests $[\delta]_\pi$ in multitrace formulae. The operator $\circ^f$ in trace formulae is not self-dual for $f \in \{a, c\}$, i.e. the equivalence $\circ^f \delta \equiv \neg \circ^f \neg \delta$ does not hold. We thus use a dual version $\circ^d \delta \equiv \neg \circ^f \neg \delta$ for these two operators to obtain a positive form. Intuitively, while the normal next operator is equivalent to $false$ when the associated successor or predecessor type is undefined, the dual operator is equivalent to $true$ in this case. Next, we assume a strictly guarded form where every fixpoint variable has to be preceded directly by a next operator. Finally, we assume that every fixpoint variable $X$ is bound by exactly one fixpoint construction $\mu X. \psi$ or $\nu X. \psi$. The same applies to fixpoint variables $Y$ in trace formulae. As any formula can be transformed into an equivalent formula meeting these requirements, they do not form proper restrictions. They do, however, help us make the automata constructions in Sections 4 and 5 clearer.

We now define fragments and variants of the logic. For trace formulae, $\mu$TL [Vardi 1988] is the syntactic fragment where only the next operator $\circ^f \delta$ is used. If additionally, fixpoints are only used in $\delta_1 \cal U^f \delta_2$ formulae, we obtain the logic LTL. Next, we introduce a name for the logic that uses only a subset of trace formulae. We use mumbling $H_\mu$ with basis $B$ to denote the subset of mumbling $H_\mu$ where $base(\phi) \subseteq B$ for all formulae $\phi$. We sometimes write mumbling $H_\mu$ with full basis instead of...
mumbling $H_\mu$ to denote the full logic. Finally, we denote the subset of mumbling $H_\mu$ where all $\bigcirc^\Delta$ operators use the same successor assignment $\Delta$ as *mumbling $H_\mu$ with unique mumbling*. In order to compare mumbling with the jump mechanism from HyperLTL$_S$ [Bozzelli et al. 2021], we define *stuttering $H_\mu$* as a variant of mumbling $H_\mu$ where $\bigcirc^\Gamma$ operators are used instead of $\bigcirc^\Delta$. Given a stuttering assignment $\Gamma : N \rightarrow 2^\delta$, the operator $\bigcirc^\Gamma$ advances each trace $\pi$ to the next position with a different valuation of some $\delta \in \Gamma(\pi)$. We call a jump criterion $\Gamma$ a *stuttering assignment* in order to highlight the difference to *successor assignments* $\Delta$: An assignment $\Gamma$ specifies positions that are similar and can thus be skipped, while an assignment $\Delta$ specifies positions that are of special interest and thus should be advanced to. For this variant, the notions of basis and unique stuttering are defined analogously to the main logic.

### 3.2 Example Properties

Let us discuss the utility of mumbling $H_\mu$ for the verification of recursive programs using some example hyperproperties and verification scenarios. We focus on properties with unique mumbling, since they are of particular practical interest due to their decidable model checking problem.

As a first example, consider an asynchronous variant of the information flow policy observational determinism [Clarkson and Schneider 2010]. Intuitively, it states that a system looks deterministic to a low security user who cannot inspect the secret variables of the system. More precisely, it requires that if two executions of a system initially match on inputs $I$ visible to a low security user, then they match on outputs $O$ visible to that user all the time. An earlier formulation of this property in HyperLTL from [Clarkson et al. 2014] required the progress in between observation points to be synchronous, i.e. the same number of steps has to be made on all traces. However, this is an unrealistic assumption for many systems. A different formulation of the property in HyperLTL$_S$ from [Bozzelli et al. 2021] approached the problem by allowing consecutive steps with the same observable outputs on one trace to be matched by a (possibly different) number of steps with the same outputs on the other. However, this formulation can only model a user that is unable to identify that outputs have been performed unless they differ from previous outputs. We suggest a new formulation using the jump mechanism of mumbling $H_\mu$. Explicitly labelling observation points by an atomic proposition $\text{obs}$ allows us to model many different kinds of low security observers. Our variant of observational determinism is expressed by the formula

$$
\forall \pi_1, \forall \pi_2, (\bigwedge_{a \in I} [ap]_{\pi_1} \leftrightarrow [ap]_{\pi_2}) \rightarrow G^{\{\pi_1 \mapsto \text{obs}, \pi_2 \mapsto \text{obs}\}} \left(\bigwedge_{a \in O} [ap]_{\pi_1} \leftrightarrow [ap]_{\pi_2}\right).
$$

We can formulate a stronger variant of this property with different successor criteria for different traces. When given a labelling with $\text{obs}_1$ and $\text{obs}_2$ modelling two different observers, we can use the successor assignment $\{\pi_1 \mapsto \text{obs}_1, \pi_2 \mapsto \text{obs}_2\}$ instead of the previous one. Then, the property requires the system to have indistinguishable behaviour even for two observers who can inspect different sets of states. Note that the use of different successor criteria enables a trace to fulfil both the role of being observed by the first and being observed by the second observer. This variant still implies the previous requirement of indistinguishability of two traces $tr_1, tr_2$ inspected by the same observer as the variant asserts that $tr_1$ observed by observer one is equivalent to $tr_2$ observed by observer two which in turn is equivalent to $tr_2$ observed by observer one.

Similarly, one can formulate asynchronous variants of other information flow policies. Clarkson and Schneider, for instance, model a version of non-interference as a hyperproperty with quantifier alternation [Clarkson and Schneider 2010]. It requires that for all traces, there exists a trace without high security inputs such that the two traces are indistinguishable to a low security user who can only inspect atomic propositions from a set $L$. An asynchronous variant of this requirement can be expressed by a modification of a HyperLTL formula from [Clarkson et al. 2014] in which a trace
without high security inputs is modelled by a trace in which all these inputs have been replaced by a dummy symbol \( dum \):

\[
\forall \pi_1. \exists \pi_2. ([G^d dum]_{\pi_1} \wedge G^{\{\pi_1 \to obs, \pi_2 \to obs\}} (\bigwedge_{ap \in L} [ap]_{\pi_1} \leftrightarrow [ap]_{\pi_2}).
\]

Here, we use a non-atomic test to state that high security inputs on \( \pi_2 \) are replaced by \( dum \) in all positions including those not inspected by the successor criterion \( obs \). As the test is performed on the first position of the trace, this is an example of filtering traces bound by a quantifier. Indeed, trace filtering motivated Bozzelli et al. [Bozzelli et al. 2021] to specifically include single trace formulae checked on the initial position in their decidable fragment of HyperLTL by a specific condition in the fragment’s definition. In contrast, these tests are integrated in mumbling \( H_\mu \) naturally and can be used on later positions as well. For example, assuming call positions for a procedure labelled with \( pr \) in the fragment’s definition. In contrast, these tests are integrated in mumbling \( H_\mu \) naturally and can be used on later positions as well. For example, assuming call positions for a procedure \( pr \) labelled with \( pr \) in the call stack on trace \( \pi \). This can prove useful since sometimes in information flow, the requirement of indistinguishability for low security users need not be as strict, e.g., if information is declassified when it is sent via an encrypted message. In such a case, we wouldn’t want to require indistinguishability inside a procedure \( pr \) that is used to send encrypted messages. By replacing the requirement \( \bigwedge_{ap \in L} [ap]_{\pi_1} \leftrightarrow [ap]_{\pi_2} \) in the non-interference property with

\[
(\neg[F^c pr]_{\pi_1} \wedge \neg[F^c pr]_{\pi_2}) \rightarrow (\bigwedge_{ap \in L} [ap]_{\pi_1} \leftrightarrow [ap]_{\pi_2}),
\]

we require indistinguishability only when neither \( \pi_1 \) nor \( \pi_2 \) is currently inside the procedure \( pr \).

So far, we have focussed on what hyperproperties can be expressed in mumbling \( H_\mu \) and only implicitly considered the system model. Besides Kripke structures, for which model checking specifications with unique mumbling is decidable, we consider pushdown systems for which hyperproperty verification is inherently difficult: As we will see in Section 5, the model checking problem for pushdown systems is undecidable already for a fixed hyperproperty from the literature that is expressible in synchronous hyperlogics like HyperLTL. While this implies that further restrictions are needed for decidability, we want these restrictions to be as lax as possible in order to be able to analyse as many systems as possible precisely. In this paper, we propose well-alignedness, a condition introduced and discussed later. Intuitively, while traces satisfying this condition must reach the same stack level on all observation points, they may differ e.g., by executing procedures in between. As motivation for this restriction, consider the following two lines of thought. First, one of the main motivations for studying the verification of hyperproperties are security hyperproperties like the ones presented in this section. These hyperproperties express in different ways that certain traces of a system are very similar. We argue in Section 5.1 that it is reasonable to expect that in systems constructed with the aim to have very similar traces, stack actions along these traces are alike as well. Since pairs of traces from such systems satisfy well-alignedness by construction, they can be analysed precisely with the methods developed in this paper. Secondly, a precise analysis under well-alignedness is also possible for many systems in which stack actions are not perfectly aligned. For example, a scenario where one execution uses a recursive procedure call in between observation points while another one only performs iterative calculations constitutes a strong deviation from a perfect alignment of stack actions. However, differences like this are still allowed under well-alignedness. Thus, a precise analysis is possible in this scenario as well.

### 3.3 Semantics of Mumbling \( H_\mu \)

We now formally define the semantics of mumbling \( H_\mu \). We do this incrementally, starting with trace formulae, then moving on to multitrace and hyperproperty formulae and introducing required notation on the way. The semantics of a trace formula \( \delta \) is defined with respect to a trace
A trace assignment \( \chi \) assigns sets of positions to fixpoint variables. Intuitively, \( [\delta]^r \subseteq \mathbb{N}_0 \) is the set of indices \( i \) such that if each fixpoint variable \( Y \) is interpreted to hold in the positions given by the set \( \mathcal{V}(Y) \), \( \delta \) holds on the suffix \( tr[i] \) of \( tr \).

**Definition 3.2 (Trace semantics).** The semantics of trace formulae is given by:

\[
\begin{align*}
[ap]^r & := \{ i \in \mathbb{N}_0 \mid ap \in tr(i) \} \\
[\bigcirc \delta]^r & := \{ i \in \mathbb{N}_0 \mid \text{succ}(tr, i) \in [\delta]^r \} \\
[\mu Y. \delta]^r & := \{ I \subseteq \mathbb{N}_0 \mid [\delta]^r_{tr[I-Y-I]} \subseteq I \} \\
[\delta_1 \lor \delta_2]^r & := [\delta_1]^r \cup [\delta_2]^r \\
[-\delta]^r & := \mathbb{N}_0 \setminus [\delta]^r \\
[\mathcal{V}(Y)]^r & := \mathcal{V}(Y)
\end{align*}
\]

We use \( \mathcal{V}_0 := \lambda Y. \emptyset \) for the empty fixpoint variable assignment over \( \chi \) and write \( [\delta]^r \) for \( [\delta]^r_{\mathcal{V}_0} \). For the semantics of multitrace formulae, we introduce the notion of trace assignments. A trace assignment is a partial function \( \Pi : N \to Traces \). If \( \Pi \) maps to traces from \( T \subseteq Traces \) only, we say that is is a trace assignment over \( T \). Inumming \( H_\mu \), progress is made via successor assignments \( \Delta \) that assign a trace formula \( \delta \) to every trace. For single traces, we define \( \text{succ}_\delta : Traces \times \mathbb{N}_0 \to \mathbb{N}_0 \) such that \( \text{succ}_\delta(tr, i) = \min S \), where \( S = \{ j \mid j > i, j \in [\delta]^r \} \), if the set \( S \) is non-empty, and \( \text{succ}_\delta(tr, i) = i + 1 \) otherwise. Thus, \( \text{succ}_\delta \) advances a trace to the next position where \( \delta \) holds, if one exists, and the immediate successor otherwise. For trace assignments and a successor assignment \( \Delta \), progress is described by the function \( \text{succ}_\Delta \) that is defined as \( \text{succ}_\Delta(\Pi, (v_1, \ldots, v_n)) = (\text{succ}_{\Delta(\pi_1)}(\Pi(\pi_1), v_1), \ldots, \text{succ}_{\Delta(\pi_n)}(\Pi(\pi_n), v_n)) \). We also define the \( i \)-fold application of both of these successor functions: \( \text{succ}_{\Delta}^i \) is the \( i \)-fold application of the \( \delta \)-successor function defined by \( \text{succ}_{\Delta}^0(tr, j) = j \) and \( \text{succ}_{\Delta}^{i+1}(tr, j) = \text{succ}_{\Delta}(\text{succ}_{\Delta}^i(tr, j)) \). \( \text{succ}_{\Delta}^i \) is defined analogously. For stuttering assignments \( \Gamma \), we introduce similar notations: For a set \( y \) of trace formulae, we define \( \text{succ}_\gamma : Traces \times \mathbb{N}_0 \to \mathbb{N}_0 \) such that \( \text{succ}_\gamma(tr, i) = \min \{ j \mid j > i, i \in [\gamma]^r \} \), if the set is non-empty, and \( \text{succ}_\gamma(tr, i) = i + 1 \) otherwise. \( \text{succ}_\gamma \) and its \( i \)-fold application is then defined analogous to the same notion for \( \Delta \).

The semantics of a multitrace formula \( \psi \) is defined with respect to a trace assignment \( \Pi \) and fixpoint variable assignment \( \mathcal{W} : \chi \to 2^\mathbb{N}_0 \) where \( n = |\text{dom}(\Pi)| \). In the definition, \( [\psi]_\mathcal{W}^\Pi \subseteq \mathbb{N}_0^n \) is the set of vectors \( (v_1, \ldots, v_n) \) such that in the context of fixpoint variable assignment \( \mathcal{W} \), the combination of suffixes \( \Pi(\pi_1)[v_1], \ldots, \Pi(\pi_n)[v_n] \) satisfies \( \psi \).

**Definition 3.3 (Multitrace semantics).** The semantics of multitrace formulae is given by:

\[
\begin{align*}
[[[\delta]]_\mathcal{W}]^\Pi & := \{ (v_1, \ldots, v_n) \in \mathbb{N}_0^n \mid v_i \in [\delta]]^\Pi \} \\
[[\bigcirc \Delta \psi]]_\mathcal{W}^\Pi & := \{ v \in \mathbb{N}_0^n \mid \text{succ}_\Delta(\Pi, v) \in [\psi]_\mathcal{W}^\Pi \} \\
[[\mu X. \psi]]_\mathcal{W}^\Pi & := \{ V \subseteq \mathbb{N}_0^n \mid [\psi]_\mathcal{W}^{\Pi[X \to V]} \subseteq V \} \\
[[\psi_1 \lor \psi_2]]_\mathcal{W}^\Pi & := [[\psi_1]]_\mathcal{W}^\Pi \cup [[\psi_2]]_\mathcal{W}^\Pi \\
[[-\psi]]_\mathcal{W}^\Pi & := \mathbb{N}_0^n \setminus [[\psi]]_\mathcal{W}^\Pi \\
[[\mathcal{W}(X)]]_\mathcal{W}^\Pi & := \mathcal{W}(X)
\end{align*}
\]

As formalised in the extended version [Gutsfeld et al. 2023], \( \mu Y. \delta \) and \( \mu X. \psi \) characterise fixpoints. We again use \( \mathcal{W}_0 := \lambda X. \emptyset \) for the empty fixpoint variable assignment over \( \chi_0 \) and write \( [\psi]^\Pi \) for \( [\psi]_{\mathcal{W}_0}^\Pi \). Now, we define the semantics of hyperproperty formulae. In this definition, \( \Pi \vdash_T \varphi \) denotes that the trace assignment \( \Pi \) over \( T \) satisfies \( \varphi \).

**Definition 3.4 (Hyperproperty semantics).** The semantics of hyperproperty formulae is given by:

\[
\begin{align*}
\Pi \vdash_T \exists \pi. \varphi & \iff \Pi(\pi \mapsto tr) \vdash_T \varphi \text{ for some } tr \in T \\
\Pi \vdash_T \forall \pi. \varphi & \iff \Pi(\pi \mapsto tr) \vdash_T \varphi \text{ for all } tr \in T \\
\Pi \vdash_T \varphi & \iff (0, \ldots, 0) \in [\varphi]^\Pi
\end{align*}
\]

For closed hyperproperty formulae \( \varphi \), we write \( T \models \varphi \) if \( \{ \} \models T \varphi \) where \( \{ \} \) is the trace assignment with empty domain and \( PD \models \varphi \) if \( \text{Traces}(PD) \models \varphi \). For fair pushdown systems \( (PD, F) \) this is straightforwardly extended: \( (PD, F) \models \varphi \) if \( \text{Traces}(PD, F) \models \varphi \).

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Remark 3.5. On traces generated by a Kripke structure $\mathcal{K}$, $\Diamond^3\delta$ is equivalent to $\Diamond^5\delta$ and $\Diamond^5\delta$ is equivalent to $false$. Thus, any hyperproperty formula $\varphi$ can be translated to a hyperproperty formula $\varphi'$ without these two operators such that $\mathcal{K} \models \varphi$ iff $\mathcal{K} \models \varphi'$.

We investigate the following decision problems:

- **Fair Finite State Model Checking**: given a closed mumbling $H_\mu$ hyperproperty formula $\varphi$ and a fair Kripke structure $(\mathcal{K}, F)$, decide whether $(\mathcal{K}, F) \models \varphi$ holds.
- **Fair Pushdown Model Checking**: given a closed mumbling $H_\mu$ hyperproperty formula $\varphi$ and a fair PDS $(\mathcal{P}D, F)$, decide whether $(\mathcal{P}D, F) \models \varphi$ holds.

Note that the fair model checking problem is stronger than the traditional model checking problem since an instance of the latter can trivially be transformed into an instance of the former by declaring all states of the input structure target states. It is convenient to consider this stronger variant for the reduction in Section 4.1.

4 FAIR FINITE STATE MODEL CHECKING

In this section, we solve the fair finite state model checking problem. We show that the complexity is the same as for HyperLTL$_S$ model checking despite the addition of fixpoints, non-atomic tests and a new jump criterion. We consider two restrictions. The first one is a restriction to unique mumbling. This is necessary as the problem is already undecidable for HyperLTL$_S$ without the corresponding restriction [Bozzei et al. 2021] which transfers to mumbling $H_\mu$ using the reduction from Theorem 6.1 (presented in Section 6).

**Theorem 4.1.** The finite state model checking problem for mumbling $H_\mu$ is undecidable.

The second restriction is to consider only the basis $AP$. As we show in Section 4.1, this is not a proper restriction since the model checking problem for the full basis can be reduced to this fragment. Afterwards, we present an algorithm for model checking with the two restrictions in Section 4.2. Under these restrictions, both subsections also prepare us for the procedure for pushdown model checking in Section 5: The reduction is suitable for both model checking variants and the pushdown model checking procedure will have the same general structure as the one for finite state systems.

4.1 Restriction of the Basis

We start this section by showing how the fair model checking problem for mumbling $H_\mu$ with full basis can be reduced to the fair model checking problem for mumbling $H_\mu$ with basis $AP'$ for an extended set of atomic propositions $AP' \supseteq AP$. The reduction has the nice property that it keeps the number of successor assignments the same, which is crucial for decidability. It thus allows us to focus our efforts on developing a model checking procedure for mumbling $H_\mu$ with an atomic basis since such a procedure can be combined with the reduction to obtain a procedure for the full logic. Even though we want to solve the finite state model checking problem first, we present a more general construction that works for both Kripke structures and PDS. Our construction is inspired by a similar construction from [Bozzei et al. 2021] and uses their idea to track the satisfaction of formulae by newly introduced atomic propositions. However, we cannot directly apply their results since (i) we need to track the satisfaction of formulae from a more expressive logic requiring a more powerful type of automaton and (ii) the reduction must also work for PDS.

Conceptually, we proceed as follows. Given a mumbling $H_\mu$ hyperproperty formula $\varphi$ and a fair PDS $(\mathcal{P}D, F)$, we transform $\varphi$ into a formula $\varphi'$ over basis $AP'$ for an extended set of atomic propositions $AP' \supseteq AP$ and $(\mathcal{P}D, F)$ into a fair PDS $(\mathcal{P}D', F')$ such that $(\mathcal{P}D, F) \models \varphi$ iff $(\mathcal{P}D', F') \models \varphi'$. The main idea is to track satisfaction of the formulae $\delta$ in $base(\varphi)$ by atomic propositions $at(\delta)$ in the translation. This is done by first constructing a VPA $\mathcal{A}_{base(\varphi)}$ that ensures for every formula $\delta$
in base(φ) that at(δ) is encountered iff δ indeed holds in this position of the input word of Abase(φ).
We intersect this automaton with (PD, F) to obtain the system (PD′, F′) that is properly labelled with at(δ) labels. Then, we replace tests [δ]π or jump criteria Δ(π) in φ by [at(δ)]π or at(Δ(π)), respectively, to obtain formula φ′ with basis AP′.

We describe the construction of a VPA A_B for arbitrary finite sets B of closed trace formulae over AP. For this, we first introduce some notation. We expand the set of atomic propositions AP by APδ := {at(δ) | δ ∈ B} to obtain AP_B := AP ∪ APδ and expand traces from (2AP · {int, call, ret})ω to (2AP · {int, call, ret})ω. For a word w ∈ (2AP · {int, call, ret})ω, we use (w)_AP to denote the restriction of w to (2AP · {int, call, ret})ω. Additionally, let cl(B) be the least set C of trace formulae such that (i) B ⊆ C, (ii) C is closed under semantic negation, that is if δ ∈ C then δ′ ∈ C, where δ′ is the positive form of ¬δ, and (iii) if δ ∈ Sub(δ′) and δ′ ∈ C then δ ∈ C.

We now sketch the construction. The goal is to construct a VPA A_B that recognizes all traces tr with the property that for all δ ∈ B, at(δ) holds in a position on tr iff δ holds on this position on the trace’s restriction, (tr)_AP. Depending on whether we have a PDS or Kripke structure, we construct a 2-AJA or APA first. This automaton loops on an initial state and conjunctively moves to a module checking δ for every atomic proposition at(δ) encountered and to a module checking ¬δ′ for every atomic proposition at(δ′) not encountered. These modules are constructed using established techniques for transforming fixpoint formulae into automata: We introduce a state qδ for each δ ∈ cl(B). Its transition function can either check δ directly, if it is an atomic formula, or move to states for the subformulae of δ using suitable transitions, if it is not. Fixpoints introduce loops in the automaton. The priorities are assigned to reflect the nature and nesting of the fixpoints. The details of this construction are given in [Gutsfeld et al. 2023]. Note that due to Remark 3.5, we can assume base(φ) to not contain formulae using Oω or Oω operators when considering the fair finite state model checking problem. Our construction introduces non global moves only for these operators, so an APA suffices in this case. Applying Proposition 2.3 or Proposition 2.4 to the automaton constructed so far, we obtain a VPA or NBA with the following properties:

**Lemma 4.2.** Given a set of closed trace formulae B over AP, one can construct a VPA A_B over 2AP · {int, call, ret} with a number of states exponential in |AP_B| satisfying:

1. for all w ∈ L(A_B), i ≥ 0 and δ ∈ cl(B), we have: at(δ) ∈ w(i) iff i ∈ [δ](w)_AP.
2. for each trace tr ∈ Traces, there exists w ∈ L(A_B) such that tr = (w)_AP.

If B is a set of μTL formulae, then A_B is an NBA.

The intersection of (PD, F) and Abase(φ) is described in [Gutsfeld et al. 2023]. We obtain:

**Lemma 4.3.** Let φ be a mumbling H_μ hyperproperty formula with full basis and (PD, F) be a fair PDS. There is an extended set of atomic propositions AP′ ⊇ AP such that one can construct a mumbling H_μ formula φ′ of size O(|φ|) with basis AP′ and a fair PDS (PD′, F′) of size O(|PD| · 2^p(|φ|)) for a polynomial p such that (PD, F) |= φ iff (PD′, F′) |= φ′. Moreover, φ and φ′ have the same number of successor assignments. If PD is a Kripke structure, then PD′ is also a Kripke structure.

### 4.2 Fair Finite State Model Checking

Now, we show how to decide the fair model checking problem for mumbling H_μ with unique mumbling and basis AP. We borrow the idea from [Bozelli et al. 2021] to build a Kripke structure whose traces represent summarised variants of the original Kripke structure’s traces and then analyse these traces synchronously. In contrast to [Bozelli et al. 2021], where decidability for HyperLTL̂ model checking is obtained by reduction to the model checking problem for synchronous HyperLTL, we present a direct model checking procedure here. This also introduces ideas for the model checking procedure in Section 5.
We show how to check \((\mathcal{K}, F) \models \varphi\) for a fair Kripke structure \((\mathcal{K}, F)\) and a closed hyperproperty formula \(\varphi := Q_0 \pi_n \ldots Q_i \pi_1 \psi\) with basis \(AP\) and unique successor assignment \(\Delta\). We use \(\varphi_i\) to denote the subformula \(Q_i \pi_1 \ldots Q_i \pi_1 \psi\) with the \(i\) innermost quantifiers. As special cases, we have \(\varphi_0 = \psi\) and \(\varphi_n = \psi\). In a nutshell, we inductively construct automata \(\mathcal{A}_{\varphi_i}\) that are equivalent to the formulae \(\varphi_i\) in a certain sense. If the modes of progression of formulae and automata match, the notion of \(\mathcal{K}\)-equivalence from [Finkbeiner et al. 2015] is suitable. We adapt this notion first. In this definition, trace assignments \(\Pi\) over \(\mathcal{T}\) with \(\Pi(\pi_i) = P_0^i P_1^i \ldots \in (2^{AP})^{\omega}\) are encoded by words \(w_\Pi \in ((2^{AP})^n)^\omega\) with \(w_\Pi(j) = (P_j^0, \ldots, P_j^n)\):

**Definition 4.4 \((\mathcal{T}\text{-equivalence})\).** Given a set of traces \(\mathcal{T}\), a closed hyperproperty formula \(\varphi\) and automaton \(\mathcal{A}\), we call \(\mathcal{A}\) \(\mathcal{T}\)-equivalent to \(\varphi\), iff for all trace assignments \(\Pi\) over \(\mathcal{T}\) binding the free trace variables in \(\varphi\), we have \(\Pi \models \varphi\) iff \(w_\Pi \in L(\mathcal{A})\).

In our current setup, however, we deal with formulae that advance trace assignments asynchronously in accordance with a successor assignment \(\Delta\) such that the modes of progression of formulae differ from that of the automata to be used. We thus define a new notion of equivalence that also respects successor assignments. In this definition, we need the notation \(\Pi^\Delta\) for a trace assignment \(\Pi\) that is summarised with respect to a successor assignment \(\Delta\), i.e. where all positions that are skipped by \(\Delta\) are left out. For a trace formula \(\delta\) and a trace \(tr\), the trace summary \(\text{sum}_\delta(tr)\) is given by \(\text{sum}_\delta(tr)(i) = tr(\text{succ}_\delta^{\neg}(tr, 0))\). Then, \(\Pi^\Delta\) is given by \(\Pi^\Delta(\pi) = \text{sum}_{\Delta(\pi)}(\Pi(\pi))\).

**Definition 4.5 \((\Delta, \mathcal{T}\text{-equivalence})\).** Given a set of traces \(\mathcal{T}\), a hyperproperty formula \(\varphi\) with unique successor assignment \(\Delta\) and automaton \(\mathcal{A}\), we call \(\mathcal{A}\) \((\Delta, \mathcal{T}\text{-equivalence})\)-equivalent to \(\varphi\), iff for all trace assignments \(\Pi\) over \(\mathcal{T}\) binding the free trace variables in \(\varphi\), we have \(\Pi \models \varphi\) iff \(w_{\Pi^\Delta} \in L(\mathcal{A})\).

In the case where \(\varphi\) is closed, the equivalence in this definition reduces to \(\mathcal{T} \models \varphi\) iff \(w \in L(\mathcal{A})\) for the unique word \(w\) over the single letter alphabet of empty tuples. Thus, model checking a fair Kripke structure \((\mathcal{K}, F)\) against a formula \(\varphi\) with unique successor assignment \(\Delta\) can be reduced to an emptiness test on an automaton \(\mathcal{A}\) that is \((\Delta, \text{Traces(} \mathcal{K}, F)\text{-equivalent})\) to \(\varphi\).

Now that this notion is established, we present the inductive construction of the automata \(\mathcal{A}_{\varphi_i}\).

In the base case, where \(\varphi_0 = \psi\), we reuse an automaton construction for synchronous \(H_\mu\) from [Gutsfeld et al. 2021] as the automaton \(\mathcal{A}_\psi\). For this purpose, we need a connection between \(\mathcal{T}\)-equivalence and \((\Delta, \mathcal{T}\text{-equivalence})\) that we establish next. Let \(\psi^\Delta\) be the variant of \(\psi\) where \(\Delta\) is replaced with the synchronous successor assignment \(\Delta^\lambda = \lambda \pi. \text{true}\). Since we only have atomic tests, \(\psi^\Delta\) belongs to the synchronous fragment of \(H_\mu\) from [Gutsfeld et al. 2021].

**Lemma 4.6.** Let \(\psi\) be a closed multitrace formula with unique successor assignment \(\Delta\) and basis \(AP\) and let \(\mathcal{A}_\psi\) be an automaton that is \(\mathcal{T}\text{-equivalent})\) to \(\psi^\Delta\) for all sets of traces \(\mathcal{T}\). Then, \(\mathcal{A}_\psi\) is \((\Delta, \mathcal{T}\text{-equivalent})\) to \(\psi\) for all sets of traces \(\mathcal{T}\).

The following theorem is a combination of Theorem 5.2 and 6.1 from [Gutsfeld et al. 2021]:

**Theorem 4.7 ([Gutsfeld et al. 2021]).** Let \(\psi\) be a quantifier-free closed synchronous \(H_\mu\) formula. There is an APA \(\mathcal{A}_\psi\) of size linear in \(|\psi|\) that is \(\mathcal{T}\text{-equivalent})\) to \(\psi^\Delta\) for all sets of traces \(\mathcal{T}\). \(^2\)

Together, Lemma 4.6 and Theorem 4.7 give us:

**Theorem 4.8.** For any closed multitrace formula \(\psi\) with unique successor assignment \(\Delta\) and basis \(AP\), there is an APA \(\mathcal{A}_\psi\) with size linear in \(|\psi|\) that is \((\Delta, \mathcal{T}\text{-equivalent})\) to \(\psi^\Delta\) for all sets of traces \(\mathcal{T}\).

\(^2\)Note that in [Gutsfeld et al. 2021], the definition of \(\mathcal{K}\)-equivalence considers free predicates and offset indices. Since we are only concerned with closed formulae, we can use a simpler definition here. Another minor difference is that the definition in [Gutsfeld et al. 2021] considers paths of a Kripke structures \(\mathcal{K}\) rather than general trace sets \(\mathcal{T}\).
Starting with the automaton $\mathcal{A}_{\varphi_0}$ from Theorem 4.8, we inductively construct automata $\mathcal{A}_{\varphi_i}$ that are $(\Delta, \text{Traces}(\mathcal{K}, F))$-equivalent to $\varphi_i$. For $i \geq 1$, we have $\varphi_i = Q_i \pi_i \varphi_{i-1}$ and construct an NBA $\mathcal{A}_{\varphi_i}$ with input alphabet $(2^{AP})^{n-i}$ from the NBA $\mathcal{A}_{\varphi_{i-1}}$ with input alphabet $(2^{AP})^{n-i+1}$ and the structure $(\mathcal{K}, F)$. Note that $\mathcal{A}_{\varphi_0}$ can indeed be assumed to be given as an NBA by Proposition 2.4. Since $\varphi$ has basis $AP$, we know that $\Delta(\pi_i) = \text{ap}$ for some $\text{ap} \in AP$. We transform $(\mathcal{K}, F)$ into a fair Kripke structure $(\mathcal{K}_{\text{ap}}, F_{\text{ap}})$ such that $\text{Traces}(\mathcal{K}_{\text{ap}}, F_{\text{ap}}) = \text{sum}_{\text{ap}}(\text{Traces}(\mathcal{K}, F))$ where $\text{sum}_{\text{ap}}(T) = \{ \text{sum}_{\text{ap}}(tr) | tr \in T \}$ is the straightforward extension of $\text{sum}_{\text{ap}}$ to sets. Then, we can use a standard construction for handling quantifiers as used e.g. for HyperLTL [Finkbeiner et al. 2015] with the difference that we use $(\mathcal{K}_{\text{ap}}, F_{\text{ap}})$ instead of $(\mathcal{K}, F)$. In short, when $Q_i$ is an existential quantifier, we build the product of $\mathcal{A}_{\varphi_{i-1}}$ and $(\mathcal{K}_{\text{ap}}, F_{\text{ap}})$ and perform a projection on the components of the input alphabet other than the one representing $\pi_i$. Universal quantifiers are handled using complementation. For this, an NBA can be interpreted as an APA, complemented without size increase, and then turned into an NBA again using Proposition 2.4. In order to avoid further exponential costs in the model checking procedure, we restrict the following theorem to formulæ where the outermost quantifier is an existential one. Outermost universal quantifiers can be handled by constructing the automaton for the negation of the formula instead. The details of this construction and the proof of the following theorem can be found in [Gutsfeld et al. 2023].

**Theorem 4.9.** Let $(\mathcal{K}, F)$ be a fair Kripke structure and let $\varphi$ be a hyperproperty formula with unique successor assignment $\Delta$, an outermost existential quantifier, basis $AP$ and quantifier alternation depth $k$. There is an NBA $\mathcal{A}_\varphi$ of size $O(g(k + 1, |\varphi| + \log(|\mathcal{K}|)))$ that is $(\Delta, \text{Traces}(\mathcal{K}, F))$-equivalent to $\varphi$.

By combining the model checking procedure from this subsection with the reduction from Lemma 4.3, we obtain a model checking procedure for $H_\mu$ with full basis. From corresponding bounds for HyperLTL [Rabe 2016], we can derive matching lower bounds for the complexity of the model checking problem for fixed structure and formula, respectively. Overall, we obtain:

**Theorem 4.10.** The fair finite state model checking problem for mumbling $H_\mu$ with unique mumbling and alternation depth $k$ is complete for $k\text{EXPSPACE}$. For fixed formulæ, the problem is complete for $(k - 1)\text{EXPSPACE}$ if $k \geq 1$ and complete for $\text{NLOGSPACE}$ if $k = 0$.

## 5 FAIR PUSHDOWN MODEL CHECKING

Now, we tackle the fair model checking problem for pushdown systems. By the next theorem, the restriction to unique mumbling is not enough to obtain a decidable model checking problem on its own. The theorem follows from a straightforward reduction from HyperLTL model checking against PDS which is known to be undecidable [Pommellet and Touili 2018].

**Theorem 5.1.** Pushdown model checking for mumbling $H_\mu$ with unique mumbling is undecidable.

Undecidability of pushdown model checking does not only apply to specially crafted formulæ; it also applies to relevant information flow policies. An example is generalised non-interference, one of the information flow properties that motivated the introduction of HyperLTL [Clarkson et al. 2014]. It is described by the HyperLTL formulæ $\varphi_{\text{GNI}} := \forall \pi_1 \forall \pi_2 \exists \pi_3. (G \land_{l \leq L} l_{\pi_1} \leftrightarrow l_{\pi_3}) \land (G \land_{h \in H} h_{\pi_2} \leftrightarrow h_{\pi_1})$. A proof by reduction from the equivalence problem for pushdown automata can be found in the extended version [Gutsfeld et al. 2023].

**Theorem 5.2.** Checking Generalised Non-Interference is undecidable for pushdown systems.

In order to regain decidability, we propose to replace the standard successor operator by well-aligned successor operators. After introducing these operators in Section 5.1, we present a corresponding model checking procedure for pushdown systems in Section 5.2.
5.1 Well-Alignedness

In many applications, hyperproperties are used to specify that different executions of a system satisfying certain conditions are sufficiently similar. This is particularly the case for applications from the realm of security where hyperproperties such as Observational Determinism require that executions of a system are so similar that they are indistinguishable from the perspective of a low security user. In such situations, we expect that systems specifically crafted to satisfy these properties can be constructed such that outputs visible to the attacker are generated in the same procedures or at least at the same stack level in many cases despite the deviations of the executions induced by differences in secret data.

We develop well-aligned next operators for a precise analysis in such situations. Informally, these operators $\bigcirc_w^\Delta$ coincide with the normal next operators $\bigcirc^\Delta$ but additionally require that the subtraces that are skipped by them start on a common stack level, end on a common stack level, and the lowest stack level they encounter is the same. Nevertheless, the $\text{call}$ and $\text{ret}$ behaviour on different traces may differ widely, e.g., by executing procedures unmatched by the other traces between observed positions. Thus, well-alignedness still covers a wide range of interesting behaviour. In particular, for systems constructed as described above, the aligned next operator $\bigcirc_w^\Delta$ coincides with the standard next operator $\bigcirc^\Delta$ and opens the way to analyse hyperproperties for recursive systems by automatic methods. Note also that the formula $\psi_{wa} = G^w_\Delta \text{true}$ (where $G^w_\Delta$ is the well-aligned analogue to $G^\Delta$) expresses explicitly that the traces under consideration are well-aligned with respect to $\Delta$ indefinitely. This formula can be used either to require certain properties captured by a subformula $\psi_{pr}$ for well-aligned evolutions only by using $\psi_{wa}$ as a pre-condition as in $\psi_{wa} \rightarrow \psi_{pr}$ or to require well-alignedness in addition to the property as in $\psi_{wa} \land \psi_{pr}$. The addition of the formula $\psi_{wa}$ as a pre-condition or a conjunct of subformulae preserves unique mumbling such that the resulting formulae still belong to the fragment for which model checking for pushdown systems is decidable. Given these considerations and given the undecidability results for the logic with respect to pushdown systems, we believe that the approximation by well-aligned successors is a useful approach to address recursive systems in an automated verification method for hyperproperties.

In order to formalise the notion of well-aligned traces, we define the $\text{ret-call}$ profile of traces via a notion of abstract summarisation. Intuitively, an abstract summarisation is a sequence of transition symbols progressing a trace while taking an abstract successor whenever possible and the $\text{ret-call}$ profile is the number of $\text{ret}$ and $\text{call}$ symbols left that cannot be summarised in an abstract step. Then, well-aligned traces are those that share the same $\text{ret-call}$ profile. Formally, the abstract summarisation $abssum(tr) \in \{\text{abs, call, ret}\}^*$ of a finite trace $tr$ is constructed from $tr$ as described next. Let $tr_{abs}$ be the version of $tr$ where every $\text{int}$ symbol is replaced with $\text{abs}$. We construct a maximal sequence $tr_0, tr_1, \ldots, tr_l$ with $tr_0 = tr_{abs}$ and $abssum(tr) = tr_{l|ts}$ such that for all $i < l$, $tr_{i+1}$ is obtained from $tr_i$ in the following way: if $tr_i = P_0, m_0, \ldots, P_n$, let $j_1 < n$ be the minimal index such that there is $j_2 > j_1$ with $m_{j_1} = \text{call}$ and $\text{succ}_a(tr_i, j_1) = j_2$. Then $tr_{i+1} = P_0 m_0 \ldots m_1 m_{j_1} \text{abs} P_{j_1} m_{j_2} \ldots P_n$. It is easy to see that the sequence is unique and can be constructed for every finite trace. Thus, $abssum(tr)$ is well-defined. From the definition of abstract successors, it is also easy to see that $abssum(tr)$ is contained in the regular language $(\text{abs}' \text{ret})^c(\text{abs}' \text{call})^c\text{abs}'$ for some $r, c \in \mathbb{N}_0$. We then call $(r, c)$ the $\text{ret-call}$ profile of $tr$. We define:

**Definition 5.3.** We call finite traces $tr_1, \ldots, tr_n$ well-aligned iff they have the same $\text{ret-call}$ profile.

As an example, consider three traces $tr_1, tr_2$ and $tr_3$ with $tr_{1|ts} = \text{call} \cdot \text{ret} \cdot \text{ret} \cdot \text{int} \cdot \text{call} \cdot \text{int} \cdot \text{call}$, $tr_{2|ts} = \text{ret} \cdot \text{call} \cdot \text{ret} \cdot \text{int} \cdot \text{call}$ and $tr_{3|ts} = \text{call} \cdot \text{ret} \cdot \text{call} \cdot \text{int} \cdot \text{ret} \cdot \text{int} \cdot \text{call}$. Then $abssum(tr_1) = \text{abs} \cdot \text{ret} \cdot \text{abs} \cdot \text{call} \cdot \text{abs} \cdot \text{call}$, $abssum(tr_2) = \text{ret} \cdot \text{abs} \cdot \text{abs} \cdot \text{call}$ and $abssum(tr_3) = \text{abs} \cdot \text{abs} \cdot \text{abs} \cdot \text{call}$.
therefore \(tr_1, tr_2\) and \(tr_3\) have the \(\text{ret-call}\) profiles \((1, 2), (1, 2)\) and \((0, 1)\), respectively. This means that \(tr_1\) and \(tr_2\) are well-aligned while \(tr_1\) and \(tr_3\) are not.

Intuitively, the main insight underlying our analysis is that well-aligned traces can be progressed in tandem using a single stack, even though they have different \(\text{call}\) and \(\text{ret}\) behaviour. For this, sequences of \(\text{abs}\) moves can be turned into internal steps and the different traces can synchronise their stack actions on the \(r\) common \(\text{ret}\) and \(c\) common \(\text{call}\) moves.

We now define a well-aligned variant, \(\text{succ}_\Lambda^w\), of the successor function \(\text{succ}_\Lambda\). Let \(\Pi\) be a trace assignment with \(\Pi(\pi_i) = tr_i\) and \(v = (v_1, \ldots, v_n), v' = (v'_1, \ldots, v'_n)\) be vectors such that \(v'_i = \text{succ}_\Lambda(\pi_i)(tr_i, v_i)\). We define \(\text{succ}_\Lambda^w\) as the partial function such that \(\text{succ}_\Lambda^w(\Pi, v) = \text{succ}_\Lambda(\Pi, v)\), if \(tr_1[v_1, v'_1], \ldots, tr_n[v_n, v'_n]\) are well-aligned, and is undefined otherwise. From now on, we use a version of \(\Lambda\) that uses this successor operator in its semantics: \([\Lambda^w w]\)_W \(\subseteq \{v \in \mathbb{N}_0^n \mid \text{succ}_\Lambda^w(\Pi, v)\} \in [\psi]_W\). Notice that this operator is not self-dual. However, we can easily introduce its dual version \(\Lambda^D\) with the following semantics: \([\Lambda^D w]\)_W \(\subseteq \{v \in \mathbb{N}_0^n \mid \text{succ}_\Lambda^w(\Pi, v)\} \in [\psi]_W\). On traces generated from Kripke structures, the semantics of both these next operators coincides with that of \(\Lambda\). Moreover, for formulae in positive form, replacing the standard next operator with \(\Lambda\) or \(\Lambda^D\) leads to formulae that under- or overapproximate the semantics of the original formula, respectively.

### 5.2 Fair Pushdown Model Checking

We now proceed with the model checking procedure. For this purpose, let \((\mathcal{P}, D, F)\) be a fair Pushdown System over the stack alphabet \(\Theta\) and \(\varphi := Q_n \pi_n \ldots Q_1 \pi_1 \psi\) be a hyperproperty formula with basis \(\mathcal{A}\) that uses a single successor assignment \(\Lambda\) and well-aligned next operators. We again write \(\varphi_i\) for the subformula \(Q_i \pi_i \ldots Q_1 \pi_1 \psi\) and have \(\varphi_0 = \psi\) and \(\varphi_n = \varphi\). Also, we again build an automaton \(\mathcal{A}_\varphi\) that is in a certain sense equivalent to \(\varphi\) in order to reduce the fair model checking problem to an emptiness test of an automaton. Here, we define a slightly different notion of equivalence compared to Definition 4.5 that also respects well-alignedness.

For this purpose, we introduce the well-aligned encoding \(w_\Lambda^\mathcal{A}\) of a trace assignment \(\Pi\). Intuitively, in addition to the propositional symbols \(\mathcal{P}\) already occurring in the previous encoding \(w_\mathcal{A}\), the well-aligned encoding contains \(\text{ret}\) and \(\text{call}\) symbols according to the \(\text{ret-call}\) profile of the well-aligned subtraces that are skipped by \(\Delta\) as well as \(\top\)-symbols where these subtraces are not well-aligned. Before we can formally define this encoding, we need notation for the number of steps for which the well-aligned next operator is defined on a trace assignment \(\Pi\). For this, let \(\text{prog}_w(\Pi, \Lambda, i) = (\text{succ}_\Lambda^w)^i(\Pi, (0, \ldots, 0))\) be the progress made by \(i\) steps of the well-aligned \(\Lambda\) successor operator on the trace assignment \(\Pi\). Note that \(\text{prog}_w(\Pi, \Lambda, i)\) may be undefined for certain indices \(i\). We call the supremum of the set \(\{i \in \mathbb{N}_0 \mid \text{prog}_w(\Pi, \Lambda, i)\}\) the length of the \(\Lambda\)-well-aligned prefix of \(\Pi\) and denote it by \(\text{wapref}(\Pi, \Lambda)\). For a formal definition of \(w_\Lambda^\mathcal{A}\), let \(\Pi\) be a trace assignment over \(\mathcal{T}\) with \(\Pi(\pi_i) = tr_i \in \text{Traces}\), let \(\Lambda(\pi_i) = \delta_i\) and let \(P_j = \text{tr}_i(\text{succ}_\delta_j(\text{tr}_i, 0))\). For \(j < \text{wapref}(\Pi, \Lambda)\), let \((r_j, c_j)\) be the \(\text{ret-call}\) profile of the finite trace that is skipped by step \(j\) on \(\text{tr}_i\), i.e. the \(\text{ret-call}\) profile of \(\text{tr}_i[\text{succ}_j^-(\text{tr}_i, 0), \text{succ}_j^+(\text{tr}_i, 0)]\). Since step \(j\) is well-aligned, \((r_j, c_j)\) is the \(\text{ret-call}\) profile of the finite traces corresponding to step \(j\) on all other traces \(\text{tr}_i\) as well. Moreover, let \(\mathcal{P}_j = (P_j^1, \ldots, P_j^n)\).

We define

\[
w_\Lambda^\mathcal{A}_i := \mathcal{P}_0 \cdot \{\text{ret}\}^{\varphi_0} \cdot \{\text{call}\}^{\varphi_0} \cdot \mathcal{P}_1 \cdot \{\text{ret}\}^{\varphi_1} \cdot \{\text{call}\}^{\varphi_1} \cdot \ldots \in (2^{\mathcal{AP}})^n \cdot \{\text{ret}\}^* \cdot \{\text{call}\}^*
\]

if \(\text{wapref}(\Pi, \Lambda) = \infty\) and

\[
w_\Lambda^\mathcal{A}_i := \mathcal{P}_0 \cdot \{\text{ret}\}^{\varphi_0} \cdot \{\text{call}\}^{\varphi_0} \cdot \ldots \cdot \mathcal{P}_{\text{wapref}(\Pi, \Lambda)} \cdot \{\top\}^\infty \in (2^{\mathcal{AP}})^n \cdot \{\text{ret}\}^* \cdot \{\text{call}\}^* \cdot \{\top\}^*
\]
if \( \text{wapref}(\Pi, \Delta) \in \mathbb{N} \). For single traces \( tr \), we also define \( w^\delta_{tr} = w^\Lambda_{\{\} \{\} \{\} \{\} \{\}} \) where \( \text{dom}(\Pi) = \{\pi\} \), \( \Pi(\pi) = tr \) and \( \Delta(\pi) = \delta \). For the empty trace assignment \( \{\} \), we say that \( w^\Lambda_{\{\}} \) is a well-aligned encoding of \( \{\} \) if it is contained in the language \((\in \cdot \{\text{ret}\}^* \cdot \{\text{call}\}^* \}^* \). Thus, unlike trace assignments assigning at least one trace, \( \{\} \) has multiple encodings. Based on this encoding, we adapt our notion of equivalence between formulae and automata:

**Definition 5.4 (Aligned \((\Delta, T)\)-equivalence).** Given a set of traces \( T \), a hyperproperty formula \( \varphi \) with well-aligned next operators and unique successor assignment \( \Delta \) as well as an automaton \( A \), we call \( A \) aligned \((\Delta, T)\)-equivalent to \( \varphi \), iff for all trace assignments \( \Pi \) over \( T \) binding the free trace variables in \( \varphi \), we have

- \( \Pi \models_T \varphi \) iff \( w^\Lambda_{\Pi} \in L(A) \), if \( \Pi \neq \{\} \) and
- \( \Pi \models_T \varphi \) iff \( w^\Lambda_{\Pi} \in L(A) \) for some encoding \( w^\Lambda_{\Pi} \) of \( \{\} \), otherwise.

From the second requirement, we can see that model checking a fair PDS \((\mathcal{P}, \mathcal{D}, F)\) against a formula \( \varphi \) can be solved by intersecting an automaton that is \((\Delta, \text{Traces}(\mathcal{P}, \mathcal{D}, F))\)-equivalent to \( \varphi \) with an automaton for the encodings of \( \{\} \) and testing the resulting automaton for emptiness.

We now have the necessary tools and notation for our construction. The process is similar to that in Section 4. We first construct an APA that is aligned \((\Delta, \text{Traces}(\mathcal{P}, \mathcal{D}, F))\)-equivalent to the inner formula \( \psi \) and then inductively handle the quantifiers of formulae \( \varphi_i \) for \( i \geq 1 \). Unlike in Section 4, where we relied on a connection to synchronous formulae, we construct the automaton \( A_\psi \) explicitly here in order to cope with the distinction between well-aligned and non-well-aligned parts of the trace assignment encoded by the input word. In this construction, we do not care about the behaviour on words that do not represent well-aligned encodings as such words do not matter for aligned \((\Delta, T)\)-equivalence.

As in the construction of \( A_B \) from Section 4.1, we use established techniques to transform fixpoint formulae into automata and introduce a state \( q^\psi \) for every subformula \( \psi' \) of \( \psi \) in the construction of \( A_\psi \). The transition function of \( q^\psi \) moves to states for the subformulae of \( \psi' \) in a suitable manner when encountering \( \mathcal{P} \) - symbols and skips \( \text{ret} \) - and \( \text{call} \) - symbols. In order to handle well-aligned encodings and the two variants of the next operator, we have two copies \((q^\psi, t)\) and \((q^\psi, f)\) of each state. Intuitively, the bit \( b \) in a state \((q^\psi, b)\) indicates whether we accept or reject if we encounter a \( T \) - symbol indicating that the next step is not well-aligned. Thus, for \( \psi' = \bigcirc^\Lambda_w \psi'' \), we transition to \((q^\psi, f)\) to indicate that for \( \psi' \) to hold, the next step has to be well-aligned. Likewise, for \( \psi' = \bigcirc^\Lambda_w \psi'' \), we transition to \((q^\psi, t)\) to indicate that if the next step is not well-aligned, \( \psi' \) holds. The priorities are again assigned to reflect the nature and nesting of fixpoints. The details of this construction can be found in the extended version [Gutsfeld et al. 2023].

**Theorem 5.5.** For any closed multitrace formula \( \psi \) with well-aligned next operators, unique successor assignment \( \Delta \) and basis \( \mathcal{A}_B \), there is an APA \( A_\psi \) of size linear in \( |\psi| \) that is aligned \((\Delta, T)\)-equivalent to \( \psi \) for all sets of traces \( T \).

Similar to Section 4.2, we now handle the quantifiers and inductively construct an automaton \( A_{\varphi_i} \) that is aligned \((\Delta, \text{Traces}(\mathcal{P}, \mathcal{D}, F))\)-equivalent to \( \varphi_i \). The general idea of the construction for an existential quantifier is the same as in that section: On input of an encoding \( w^\varphi_{\Pi'} \) of a trace assignment \( \Pi' \) binding \( n - i \) trace variables, we simulate a trace \( tr \) of \((\mathcal{P}, \mathcal{D}, F)\) in the state space of the automaton and feed the encoding \( w^\varphi_{\Pi'} \) of the trace assignment \( \Pi' = \Pi[\pi_i \mapsto tr] \) binding \( n - i + 1 \) trace variables into the inductively given automaton \( A_{\varphi_{i-1}} \). However, there are a number of difficulties compared to the construction in the finite state case. First of all, our construction has to handle the \( \text{call} \) and \( \text{ret} \) behaviour of the system and the well-aligned encoding. We thus construct a VPA instead of an APA here. Moreover, we have to handle the fact that \( \Pi \) and \( \Pi' \) can
be non-well-aligned from some point onward. The easier case is where the lengths of the well-aligned prefixes of \( \Pi \) and \( \Pi' \) coincide. In this case, we can just feed the \( \top \)-symbols from the input into \( A_{\rho_{i-1}} \). The more difficult case is where the length of the well-aligned prefix of \( \Pi \) is strictly greater than that of \( \Pi' \). We handle this case by nondeterministically guessing a point where the next step is not well-aligned, checking that this is indeed the case by finding a call on \( tr \) matched by a ret on \( w^w_\Pi \) (or any other combination of non matching behaviour) and feeding \( \top \)-symbols into \( A_{\rho_{i-1}} \). In both cases, we cannot continue simulating the stack behaviour of both \( w^w_\Pi \) and \( tr \) since the behaviour is not well-aligned. Thus, we stop simulating \( tr \) in these cases and just check that the prefix up to that point can be extended to a fair trace using Proposition 2.1. Before we perform the main construction, we need two auxiliary constructions which we present first.

First, for \( \Delta(\pi_r) = ap \), we transform \((PD,F)\) into a pushdown system \((PD_{ap},F_{ap})\) with \( PD_{ap} = (S_{ap},S_{0,ap},R_{ap},L_{ap}) \), a structure that progresses the well-aligned encodings \( w^ap_\Pi \) of traces \( \Pi \) by simulating finite traces in between inspected states based on their abstract summarisations. More precisely, a finite subtrace with ret-call profile \((r,c)\) is simulated by first making \( r \) ret-steps (each corresponding to a part abs\(^*\) ret in the abstract summarisation), followed by \( c \) call-steps (each corresponding to a part abs\(^*\) call in the abstract summarisation) and finally one int-step (corresponding to the final abs\(^*\) part in the abstract summarisation) in \((PD_{ap},F_{ap})\). The final int-step comes in handy when reading the propositional symbols of an inspected state in the construction of \( A_{\rho_i} \). This transformed structure is used later to obtain the encoding of \( \Pi[\pi_i \mapsto tr] \) for a trace \( tr \) of \((PD,F)\) by composing \( w^wa_\Pi \) with \( w^ap_\Pi \) generated from \((PD_{ap},F_{ap})\). The transformation to \((PD_{ap},F_{ap})\) is done in two steps. We first construct an intermediate structure \((PD',F')\) with two copies of each state reachable by the jump criterion ap. This structure has an int-step between the two copies in order to ensure that one step corresponding to the final abs\(^*\) part of a trace’s abstract summarisation is made whenever such a state is visited. In that structure, we calculate abstract successors and build ret, call and int transitions corresponding to abs\(^*\) ret, abs\(^*\) call and abs\(^*\) parts of the abstract summarisation, respectively. A formal description of this construction can be found in the extended version [Gutsfeld et al. 2023].

Secondly, in order to check whether prefixes of paths of \((PD_{ap},F_{ap})\) can be extended into fair paths, we use the multi-automaton \( A_{\psi_i} = (Q_{\psi_i},Q_{0,\psi_i},\rho_{\psi_i},F_{\psi_i}) \) from Proposition 2.1. Since multi-automata read stacks top-down while we build stacks bottom-up in our main construction, we will use this automaton in reverse, i.e. we will start in final states and aim to reach initial states by following its transitions backwards. For this, we assume that the automaton is reverse-total, i.e. we assume that for all \( q' \in Q_{PD} \) and \( \theta \in \Theta \), there is a state \( q \in Q_{PD} \) such that \( q' \in \rho_{PD}(q,\theta) \). Intuitively, this means that every state has a predecessor. This can be achieved easily by introducing an additional non-initial state.

We now describe the construction of \( A_{\psi_i} \) for \( Q_i = \exists_i \). We assume that the VPA \( A_{\rho_{i-1}} \) is inductively given by \((Q_{\rho_{i-1}},Q_{0,\rho_{i-1}},\rho_{\rho_{i-1}},F_{\rho_{i-1}})\) over the visibly pushdown alphabet \( \Sigma = \Sigma_1 \cup \Sigma_c \cup \Sigma_\tau \) with \( \Sigma_1 = (2^{AP})^{n-1+i} \cup \{\top\} \), \( \Sigma_c = \{\text{call}\} \) and \( \Sigma_\tau = \{\text{ret}\} \) and stack alphabet \( \Theta'^{-1} \times Q_{PD}^{n-1} \). For the inner formula \( \phi_0 = \psi \), we have an APA from Theorem 5.5 that is transformed into an NBA with Proposition 2.4 and then as a VPA that pushes and pops empty tuples () when reading call and ret symbols. The automaton \( A_{\psi_i} = (Q_{\psi_i},Q_{0,\psi_i},\rho_{\psi_i},F_{\psi_i}) \) has the input alphabet \( \Sigma' = \Sigma_1' \cup \Sigma_c' \cup \Sigma_\tau' \) with \( \Sigma_1' = (2^{AP})^{n-i} \cup \{\top\} \) and stack alphabet \( \Theta'^{-1} \times Q_{PD}^{n-1} \). Its state sets are given by:

\[
Q_{\psi_i} = Q_{\rho_{i-1}} \times S_{ap} \times \{0,1\} \times Q_{PD} \times \{wa,ua\} \cup \{q_\top \mid q \in Q_{\rho_{i-1}}\}
\]
\[
Q_{0,\psi_i} = Q_{0,\rho_{i-1}} \times S_{0,ap} \times \{0\} \times F_{PD} \times \{wa\}
\]
\[
F_{\psi_i} = F_{\rho_{i-1}} \times S_{ap} \times \{1\} \times Q_{PD} \times \{wa\} \cup \{q_\top \mid q \in F_{\rho_{i-1}}\}
\]

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\[ \rho_{\phi_1}((q, s, b, q_{PD}, wa), \mathcal{P}) = \{(q', s', b', q_{PD}, al) \mid (s, s') \in R_{ap}, q' \in \rho_{\phi_{1-1}}(q, L(s) + \mathcal{P}), \xi(b, b', s, q), al \in \{wa, ua\} \} \]

\[ \rho_{\phi_1}((q, s, b, q_{PD}, wa), \text{ret}) = \{(q', s', b', q_{PD}, wa), (\theta + \theta_v, q_{PD} + q_o) \mid (s, \theta, s') \in R_{ap}, (q', \theta_v, q_o) \in \rho_{\phi_{1-1}}(q, \text{ret}), \xi(b, b', s, q) \} \]

\[ \rho_{\phi_1}((q, s, b, q_{PD}, wa), \text{call}) = \{(q', s', b', q_{PD}, wa), (\theta + \theta_v, q_{PD} + q_o) \mid (s, s', \theta) \in R_{ap}, (q', \theta_v, q_o) \in \rho_{\phi_{1-1}}(q, \text{call}), \xi(b, b', s, q), q_{PD} \in \rho_{PD}(q_{PD}, \theta) \} \]

\[ \rho_{\phi_1}((q, s, b, q_{PD}, wa), \top) = \begin{cases} \{q'_\top \mid q' \in \rho_{\phi_{1-1}}(q, \top) \} & \text{if } s = q_{PD} \\ \emptyset & \text{otherwise} \end{cases} \]

\[ \rho_{\phi_1}(q_{\top}, \sigma) = \begin{cases} \{q'_\top \mid q' \in \rho_{\phi_{1-1}}(q_{\top}) \} & \text{if } \sigma \in \{\top, \mathcal{P}\} \\ \{\{q'_\top, (\theta_w, q_w)\} \mid q' \in \rho_{\phi_{1-1}}(q_{\top}) \} & \text{otherwise} \end{cases} \]

\[ \rho_{\phi_1}((q, s, b, q_{PD}, ua), \mathcal{P}) = \begin{cases} \{q'_\top \mid q' \in \rho_{\phi_{1-1}}(q, \top) \} & \text{if } \xi_{\text{call}}(s, q_{PD}) \text{ or } \xi_{\text{ret}}(q, s, b, q_{PD}, ua) \\ \emptyset & \text{otherwise} \end{cases} \]

\[ \rho_{\phi_1}((q, s, b, q_{PD}, ua), \text{ret}) = \begin{cases} \{\{q'_\top, (\theta_w, q_w)\} \mid q' \in \rho_{\phi_{1-1}}(q_{\top}) \} & \text{if } \xi_{\text{int}}(s, q_{PD}) \text{ or } \xi_{\text{call}}(s, q_{PD}) \\ \{\{q'_\top, (\theta_w, q_w)\} \mid q' \in \rho_{\phi_{1-1}}(q_{\top}) \} & \text{otherwise} \end{cases} \]

\[ \rho_{\phi_1}((q, s, b, q_{PD}, ua), \text{call}) = \begin{cases} \{\{q'_\top, (\theta_w, q_w)\} \mid q' \in \rho_{\phi_{1-1}}(q_{\top}) \} & \text{if } \xi_{\text{int}}(s, q_{PD}) \text{ or } \xi_{\text{ret}}(q, s, b, q_{PD}, ua) \\ \{\{q'_\top, (\theta_w, q_w)\} \mid q' \in \rho_{\phi_{1-1}}(q_{\top}) \} & \text{otherwise} \end{cases} \]

\[ \rho_{\phi_1}((q, s, b, q_{PD}, ua), \top) = \emptyset \]

![Fig. 1. Definition of \(\rho_{\phi_1}\)](image)

The transition rules are given in Figure 1 where \(\xi(b, b', s, q)\) is the condition \(b \neq b'\) iff \(b = 0\) and \(s \in F_{ap}\) or \(b = 1\) and \(q \in F_{\phi_{1-1}}\) in the first three cases. Additionally, we use the conditions \(\xi_{\text{int}}(s, q_{PD})\) for \((s, s') \in R_{ap}\) and \(s' = q_{PD}\) for some \(s', \xi_{\text{call}}(s, q_{PD})\) for \((s, s', \theta) \in R_{ap}\) and \(q_{PD} \in \rho_{PD}(s', \theta)\) for some \(s'\) and \(\theta\) and \(\xi_{\text{ret}}(q, s, b, q_{PD}, ua)\) for \((s, \theta, s') \in R_{ap}\) and \(\text{tos}(q, s, b, q_{PD}, ua)\)\(^3\) = \(\theta + \theta_o, s' + q_o\) for some \(s', \theta, \theta_o, \theta_o\) and \(q_o\). Intuitively, \(\xi_m\) applied to \(s\) means that an \(m\)-transition which leads to an extension into a fair path is possible in \(s\). Furthermore, we write \((P_1, \ldots, P_{n-1}) \in (\mathcal{P}\mathcal{P})^{n-1}\) as \(\mathcal{P}\), \((P, P_1, \ldots, P_{n-1}) \in (\mathcal{P}\mathcal{P})^{n-1}\) as \(\mathcal{P} + \mathcal{P}\), \((\theta_1, \ldots, \theta_{i-1})\) as \(\theta_v, (\theta, \theta_1, \ldots, \theta_{i-1})\) as \(\theta_v, (\theta, \theta_1, \ldots, \theta_{i-1})\) as \(\theta_v, (\theta, \theta_1, \ldots, \theta_{i-1})\) as \(\theta_v, (\theta, \theta_1, \ldots, \theta_{i-1})\) as \(\theta_v\). Analogously, we use \(q_o, q_{PD} + q_o\) and \(q_w\).

Intuitively, the automaton reads an encoding \(w_{II}^\lambda\) as follows: it starts in its copy \(wa\) reading the prefix containing only \(\mathcal{P}\), \(\text{ret}\) and \(\text{call}\) symbols (lines 1-6 in Figure 1). Here, it simulates both \((\mathcal{P}\mathcal{D}_{ap}, F_{ap})\) to check for an encoding of a trace \(tr\) and \(\mathcal{A}_{\phi_{1-1}}\) to check whether \(w_{II}^\lambda\) for the trace assignment \(\Pi' = \Pi \pi_i \mapsto tr\) is accepted. We use a standard construction to combine the Büchi conditions of \((\mathcal{P}\mathcal{D}_{ap}, F_{ap})\) and \(\mathcal{A}_{\phi_{1-1}}\) into one: A bit \(b\) indicates whether we have seen a state of \(F_{ap}\) and it is reset to 0 when a state from \(F_{\phi_{1-1}}\) is seen. This is expressed in the formula \(\xi(b, b', s, q)\) and makes sure that only the runs satisfying both Büchi conditions are accepting. Additionally, we

---

3By \(\text{tos}(q, s, b, q_{PD}, ua)\) we mean the current top of stack symbol in state \((q, s, b, q_{PD}, ua)\). Since the top of stack symbol can be stored in the state, we can assume w.l.o.g. that this information is available.
track a reverse-run of $A_{PD}$ in the forth component of a state. This is done by starting in a final state of $A_{PD}$ and updating the state to match a predecessor of the previous state whenever making a call transition. Additionally, we store the old state in the stack in order to enable backtracking of the reverse-run when making a return transition. When this reverse-run ends in state $s$ (which is checked in the conditions $\xi_n$), this indicates that there is a continuation into a fair path of ($PD_{qp}, F_{qp}$) starting in $s$ with the current stack content. At any point in the prefix, the automaton can nondeterministically move to its copy $ua$ (line 1-2) to check whether there is a mismatch in the encodings of $tr$ and $w^\Lambda_I$ (lines 11 ff.). Here, it accepts iff $A_{\phi_{i-1}}$ accepts when reading only $\top$ symbols from this point onwards. This is checked in states $q_T$ (line 9-10). Since we do not follow the existentially quantified path in this part of the automaton anymore, a transition into this part of the automaton can only be made if there is a continuation of the path into a fair path. Finally, it can also enter states $q_T$ when encountering a $\top$ symbol (line 7-8) since that means that both $w^\Lambda_I$ and $w^\Lambda_F$ are not well-aligned from this point onward. For universal quantifiers, we use complementation as in Section 4.2. For this, we use Proposition 2.2 since $A_{\phi_i}$ is given as a VPA instead of an NBA.

**Theorem 5.6.** Let ($PD, F$) be a fair pushdown system and $\phi$ a closed hyperproperty formula with well-aligned next operators, unique successor assignment $\Delta$, an outermost existential quantifier, basis $AP$ and quantifier alternation depth $k$. There is a VPA $A_{\phi}$, of size $O(g(k + 1, |\phi| + \log(|PD|)))$ that is aligned ($\Delta, \text{Traces}(PD, F)$)-equivalent to $\phi$.

**Proof.** (Sketch) The part of the claim about the size of $A_{\phi}$ can be seen by inspecting the construction. For the inner formula $\psi$, we know that $|A_{\psi}|$ is linear in $|\psi|$ for the APA $A_{\psi}$ from Theorem 5.5. An alternation removal construction to transform it into an NBA increases the size to exponential in $|\psi|$. Complementation constructions are performed using Proposition 2.2 for each quantifier alternation, each further increasing the size exponentially. Finally, the size measured in $|PD|$ is one exponent smaller since the structure is first introduced into the automaton after the first alternation removal construction.

Using the notation $\phi_i = Q_i \pi_i \ldots Q_0 \pi_0, \psi$ with special cases $\phi_0 = \psi$ and $\phi_n = \phi$, we show that $A_{\phi_i}$ is ($\Delta, \text{Traces}(PD, F)$)-equivalent to $\phi_i$ by induction on $i$. The base case immediately follows from Theorem 5.5. In the inductive step, the more interesting case is that where $Q_i$ is an existential quantifier since the case for a universal quantifier is a corollary from the proof for an existential quantifier. For this case, we show both directions of the required claim separately. In the first direction, we can directly use the induction hypothesis and then have to discriminate cases based on the length of the well-aligned prefixes of $\Pi$ and $\Pi[\pi_i \mapsto tr]$ since each of these cases induces a different form for the accepting run we construct. In the other direction, we discriminate cases based on the length of the well-aligned prefix of $\Pi$ and additionally on the form of the accepting run of the automaton to construct a trace $tr$ and trace assignment $\Pi[\pi_i \mapsto tr]$ on which we can use the induction hypothesis. In both directions, the most interesting case is the one where the length of the well-aligned prefix of $\Pi$ is strictly greater than that of $\Pi[\pi_i \mapsto tr]$.

A detailed proof can be found in the extended version [Gutsfeld et al. 2023].

Again combining the procedure from this section with the reduction from Lemma 4.3, we obtain a fair model checking procedure for PDS. Additionally, we can derive lower bounds for the complexity from finite state HyperLTL model checking [Rabe 2016] and LTL pushdown model checking [Bouajjani et al. 1997]. We obtain:

**Theorem 5.7.** The fair pushdown model checking problem for alternation depth $k$ mumbling $H_k$ with unique mumbling and well-aligned successor operators is in $(k+1)\text{EXPTIME}$ and in $k\text{EXPTIME}$ for fixed formulae. For $k \geq 1$, it is $k\text{EXPSPACE}$-hard and $(k-1)\text{EXPSPACE}$-hard for fixed formulae. For $k = 0$, it is $\text{EXPTIME}$-complete.
6 EXPRESSIVENESS OF STUTTERING AND MUMBLING

In this section, we compare the two jump mechanisms stuttering and mumbling with respect to expressiveness. It is easy to write a formula expressing that some formula from a set changes its valuation from this point on a trace to the next. This can be used to mimic the behavior of a stuttering next operator by a mumbling next operator. There is a slight mismatch in positions visited by the operators but this can be accounted for by shifting tests with a next operator. This translation can be used to obtain the following results:

**Theorem 6.1.** Fair pushdown and finite state model checking for stuttering \( H_\mu \) can be reduced in linear time to fair pushdown and finite state model checking for mumbling \( H_\nu \) respectively.

**Lemma 6.2.** Mumbling \( H_\mu \) with unique mumbling and basis LTL (resp. full basis) is at least as expressive as stuttering \( H_\mu \) with unique stuttering and basis LTL (resp. full basis).

Detailed proofs can be found in [Gutsfeld et al. 2023]. On the other hand, there are cases where stuttering cannot mimic the behaviour of mumbling. For example, consider a formula \( (\{p\} \cdot \emptyset)^\omega \) and mumbling criterion \( p \). While mumbling visits every other position, it is easy to see that stuttering must necessarily visit every position on this trace independently of the stuttering criterion since the postfixes of this trace coincide in every other position. When considering the basis LTL, this mismatch in expressivity between the jump criteria cannot be compensated on the level of formulae. For this, consider the hyperproperty \( H = \{T \subseteq (2^{AP})^\omega \mid \forall tr, tr' \in T. |\{i \mid p \in tr(i)\}| = |\{i \mid p \in tr'(i)\}| \} \) expressing that all traces of a set have the same number of \( p \)-positions. We show:

**Lemma 6.3.** The hyperproperty \( H \) is expressible in mumbling \( H_\mu \) with unique mumbling and basis \( AP \) while not expressible in stuttering \( H_\mu \) with unique stuttering and basis LTL.

**Proof.** The first part of this claim, namely expressing \( H \) in mumbling \( H_\mu \) with unique mumbling, is straightforward and can be found in the extended version [Gutsfeld et al. 2023].

For the second part, we first adapt some of the theorems from Section 4 to stuttering \( H_\mu \). In particular, we define \((\Gamma, T)\)-equivalence in the obvious way. It is easy to see that the results of Theorem 4.8 carry over to this notion of equivalence. Additionally, we use a claim about LTL in which we write \( nd(\delta) \) for the nesting depth of next operators in \( \delta \).

**Claim 1.** For all trace formulae \( \delta \in \text{LTL} \) with \( nd(\delta) = n \) and traces \( tr \), we have \( i \in [\delta]^n \) iff \( i + 1 \in [\delta]^{n+1} \) if there is a set \( P \subseteq AP \) such that \( tr(j) = P \) for all \( i \leq j \leq i + n + 1 \).

This claim can easily be established by induction (see the extended version [Gutsfeld et al. 2023]). It generalises Theorem 4.1 from the classic paper [Wolper 1981] about the expressivity of LTL.

Assume towards contradiction that there is a hyperproperty formula \( \varphi = Q_0 \pi_0 \ldots Q_1 \pi_1 \psi \) from stuttering \( H_\mu \) with unique stuttering expressing the property \( H \). Let \( \varphi_i = Q_i \pi_1 \ldots Q_1 \pi_1 \psi \) for \( i \leq n \), with \( \varphi_0 = \psi \) and \( \varphi_n = \varphi \). Let \( \Gamma \) be the stuttering assignment used in \( \varphi \) and \( \Gamma(\pi_i) = \gamma_i \). We say that a jump criterion \( \gamma \in 2^\delta \) makes a type one step on a trace \( tr \) at position \( i \) if \( \text{succ}_\gamma(tr, i) \) is given by the first case in the definition of \( \text{succ}_\gamma \). Similarly, we say that \( \gamma \) makes a type two step on \( tr \) at position \( i \) if \( \text{succ}_\gamma(tr, i) \) is given by the second case. Finally, we say that \( \gamma \) makes a type one/two step (without specifying a position) if it makes a type one/two step at position 0.

We choose a trace \( tr \) from \((2^{\{p\}})^\omega\) with finitely many \( p \)-positions maximising

\[
|\{i \in \{1, \ldots, n\} \mid \gamma_i \text{ makes a type one step on } tr\}|
\]

(1)

Since \( tr \) has only finitely many \( p \)-positions, we can write it as \( tr = tr_s \cdot \emptyset^\omega \). Let

\[
\text{TypeOnePos} = \{i \in \{1, \ldots, n\} \mid \gamma_i \text{ makes a type one step on } tr\}
\]

and

\[
\text{TypeTwoPos} = \{i \in \{1, \ldots, n\} \mid \gamma_i \text{ makes a type two step on } tr\}.
\]
For all $\hat{t}r \in (2^{|\phi|})^\ast$, we have

\[ \forall i \in \text{TypeOnePos} : \gamma_i \text{ makes a type one step on } \hat{t}r \cdot tr \]  
\[ \forall i \in \text{TypeTwoPos} : \gamma_i \text{ makes a type two step on } \hat{t}r \cdot tr. \]  

Here, Property (2) follows from the fact that when $\gamma$ makes a type one step at position $i$, then it also makes a type one step for all earlier positions $j \leq i$. Property (3) follows from the fact that $tr$ maximises the quantity in (1): if Property (3) would not hold for some $\hat{t}r$, then $\hat{t}r \cdot tr$ would have more type one positions than $tr$ given that (2) holds.

We transform $\psi$ in the same way as in the reduction presented in Section 4.1. That is, for every $\gamma \in \text{base}(\varphi)$, we introduce a fresh atomic proposition $at(\gamma)$ and replace tests and stuttering criteria in $\psi$ with the respective atomic propositions. This yields a formula $\psi_{at}$. For such formulae, we properly label traces with these atomic propositions, i.e. we extend each position in $\varphi$ to move over $\text{succ}_\gamma$ with the respective atomic propositions. This yields a formula $\psi_{at}$. For such formulae, we properly label traces with these atomic propositions, i.e. we extend each position in $\varphi$ to move over $\text{succ}_\gamma$ with the set $\{at(\gamma) \mid i \in [\psi]^{\Pi(\pi)}\}$ to obtain a trace $tr_{at}$. Analogously, we define variants $\Pi_{at}$ of trace assignments $\Pi$ and $T_{at}$ of sets of traces $T$. It is straightforward to see that

\[ (0, \ldots, 0) \in [\psi]_{at} \text{ iff } (0, \ldots, 0) \in [\psi_{at}]_{at} \]  

for all trace assignments $\Pi$. It is also clear that the $\Gamma$-variant of Theorem 4.8 is applicable to $\psi_{at}$ since this formula has an atomic basis.

Let $A_{\psi}$ thus be the automaton for $\psi_{at}$ according to Theorem 4.8 and let $|A_{\psi}|$ be the number of states of $A_{\psi}$. Let $l = |tr_{at}| + 3 \cdot nd(\psi) + |A_{\psi}| + 3$. Consider the following two sets of traces $T = \{tr_{0}, tr_{1}\}$ with $tr_{0} = \{p\}^l \cdot tr$ and $tr_{1} = \{p\}^l \cdot \emptyset^l \cdot tr$ as well as $T' = \{tr'_{0}, tr'_{1}\}$ with $tr'_{0} = \{p\}^{l+|\hat{A}_{\psi}|} \cdot \emptyset^l \cdot tr$ and $tr'_{1} = \emptyset^l \cdot \hat{A}_{\psi} \cdot \{p\}^l \cdot tr$. It is easy to see that $T \in H$ while $T' \notin H$.

For any trace assignment $\Pi$ over $T$, let $\Pi'$ be the trace assignment defined by $\Pi'(\pi_j) = tr'_{0}$ if $\Pi(\pi_j) = tr_{0}$ and $\Pi'(\pi_j) = tr'_{1}$ if $\Pi(\pi_j) = tr_{1}$. Below, we show by induction over $j$ that for all trace assignments $\Pi$ over $T$ and $j \in \{0, \ldots, n\}$, $\Pi \models T \varphi_j$ implies $\Pi' \models T' \varphi_j$. For $j = n$, this would mean that $T \in H$, i.e. $T \models \varphi_n$, implies $T' \models \varphi_n$, i.e. $T' \notin H$, a contradiction.

In the base case, assume that $\Pi \models T \varphi$. By Property (4), we have $(0, \ldots, 0) \in [\psi_{at}]_{at}$. Since $A_{\psi}$ is $(\Gamma, T_{at})$-equivalent to $\psi_{at}$ by the $\Gamma$-variant of Theorem 4.8, we have an accepting run of $A_{\psi}$ on $w_{at}$. Consider this accepting run of $A_{\psi}$ after $l - |A_{\psi}| - nd(\psi) = 2 = |tr_{at}| + 2 \cdot nd(\psi) + 1$ steps. This situation is depicted in Figure 2. On the one hand, for all $i \in \text{TypeOnePos}$, the suffix left to read in component $i$ of $w_{at}$ is $0^{\omega}$. For $tr_{at}$, this is due to the fact that by Claim 1 and Property (2), it takes at most $nd(\psi)$ applications of succ$_\gamma$ to move over the prefix $\{p\}^l$, by the same argument it takes at most $nd(\psi)$ applications of succ$_\gamma$ to move over $\emptyset^l$ and finally, it takes at most $|tr_{at}|$ applications of succ$_\gamma$ to move over $tr$. The argumentation for $tr_{at}$ is analogous. This case is represented in lines one and three in Figure 2. On the other hand, for all $i \in \text{TypeTwoPos}$, the suffix left to read in component $i$ of $w_{at}$ is either $\{p\}^{l+|A_{\psi}|+nd(\psi)+2} \cdot \emptyset^l \cdot tr$ if $\Pi(\pi_i) = tr_{0}$ or $\emptyset^l \cdot \hat{A}_{\psi} \cdot \{p\}^l \cdot tr$ if $\Pi(\pi_i) = tr_{1}$.
\[ \Pi(\pi_i) = tr_1. \] This is due to the fact that by Property (3), \( \gamma_i \) makes only type two steps on \( tr_0 \) and \( tr_1 \). This case is represented in lines two and four in Figure 2. Thus, during the next \( |A_\psi| + 1 \) steps, the automaton reads the same symbols on all traces: For all \( i \in TypeOnePos \), there are only \( \emptyset \)-symbols in these positions on both \( tr_0 \) and \( tr_1 \), which by Claim 1 and the fact that the suffix is \( \emptyset^o \) all have the same extended labelling in \( tr_0^o \) and \( tr_1^o \). For all \( i \in TypeTwoPos \), there are only \( \{p\} \)-symbols on \( tr_0 \) and only \( \emptyset \)-symbols on \( tr_1 \) in these positions. By Claim 1 and the fact that the next nd(\( \psi \)) + 1 steps after these steps are also \( \{p\} \)- or \( \emptyset \)-symbols on \( tr_0 \) and \( tr_1 \), respectively, the extended labelling in \( tr_0^o \) and \( tr_1^o \) is the same for these steps as well. During these \( |A_\psi| + 1 \) steps where the same symbol is seen on each trace, at least one state \( q \) of \( A_\psi \) is visited twice. Let \( k \) be the number of steps between the two visits of \( q \). We add \( |A_\psi| \cdot \{p\} \)-positions to the \( \{p\} \)-prefix of \( tr_0 \) to obtain \( tr_0' \) and \( |A_\psi| \cdot \emptyset \)-positions to the \( \emptyset \)-prefix of \( tr_1 \) to obtain \( tr_1' \). This situation is depicted in Figure 3. Since \( |A_\psi| \) is a multiple of \( k \), we do not change the acceptance of \( A_\psi \) as the run can repeat the loop from \( q \) to \( q \) \( \frac{|A_\psi|}{k} \) times and then proceed as before: By the same argument as before, the extended labelling on the added positions is the same as on the position directly after. Thus, (i) for \( i \in TypeOnePos \), the same number of applications of \( succ_\gamma_i \) as before are needed to skip over \( \{p\} |_{tr_0} |A_\psi| | \emptyset_0^{+} |A_\psi| \) due to Claim 1, thus the run is again in the \( \emptyset \)-suffix after \( |tr_a| + 2 \cdot nd(\psi) + 1 \) steps where \( A_\psi \) can loop from \( q \) to \( q \) without changing the suffix of the trace to be processed and (ii) for \( i \in TypeTwoPos \), the loops from \( q \) to \( q \) read exactly the additional symbols. The parts of the traces where these loops are taken are marked in red in Figure 3 (areas with solid border). Consequently, we have an accepting run of \( A_\psi \) over \( w_{\Pi_\psi^{\omega}} \) and conclude \( \Pi' = T_? \emptyset \) by again using Property (4) and the fact that \( A_\psi \) is also \( (\Gamma, T'_a)\)-equivalent to \( \psi_\ast \).

The inductive step considers the quantifiers \( Q_1, \ldots, Q_n \) and follows straightforwardly from the semantics of quantifiers and the induction hypothesis.

Combining Lemma 6.2 and Lemma 6.3, we obtain:

**Theorem 6.4.** Mumbling \( H_\mu \) with basis LTL and unique mumbling is strictly more expressive than stuttering \( H_\mu \) with basis LTL and unique stuttering.

As simple HyperLTL\(_S\), the decidable fragment of HyperLTL\(_S\) from [Bozzelli et al. 2021], can straightforwardly be embedded into stuttering \( H_\mu \) with unique stuttering, these results also directly imply that the hyperproperty \( H \) is not expressible in simple HyperLTL\(_S\) and that mumbling \( H_\mu \) with unique mumbling is strictly more expressive than simple HyperLTL\(_S\). Surprisingly, the lower expressivity of stuttering can be compensated exploiting the power of fixpoints:
Lemma 6.5. Stuttering $H_\mu$ with unique stuttering and full basis is at least as expressive as mumbling $H_\mu$ with unique mumbling and full basis.

Proof. We show this lemma by presenting a translation from a mumbling $H_\mu$ formula $\varphi$ with unique mumbling to an equivalent stuttering $H_\mu$ formula $\phi$ with unique stuttering.

Let $\varphi = Q_0\pi_0 \ldots Q_n\pi_n.\psi$ be a mumbling $H_\mu$ formula with unique mumbling using the successor assignment $\Delta$. We assume $\varphi$ is in positive form, i.e. negation occurs only in front of tests in $\psi$. As in other proofs, we define $\varphi_1 = Q_1\pi_1 \ldots Q_n\pi_n.\psi$ with $\varphi_0 = \psi$ and $\varphi_n = \varphi$. We assume w.l.o.g. that every test in $\psi$ is either only applied on the first position of a trace or only on later positions. This can be achieved by unrolling fixpoints so that all tests are either unguarded (and thus only apply to the first position) or in scope of at least one $\bigcirc^\Delta$ operator (and thus only apply to later positions).

We first define a stuttering criterion $\Gamma$ by specifying $\Gamma(\pi)$ for each trace variable $\pi$. For this, let $\pi \in \{\pi_1, \ldots, \pi_n\}$ be a trace variable with $\delta = \Delta(\pi)$ as well as $[\delta_1]_{\pi_1}, \ldots, [\delta_m]_{\pi_m}$ be the tests applied on $\pi$. We introduce $\bigcirc^\delta \delta'$ as an abbreviation for the trace formula $\bigcirc(\neg \bigcirc^\delta (\pi \land \delta'))$. Intuitively, $\bigcirc^\delta \delta'$ asserts that (i) there is a future position where $\delta$ holds and (ii) $\delta'$ holds at the next $\delta$ position. For $j \in \{1, \ldots, m\}$, we define formulae $\gamma_0, \gamma_j$ and $\tilde{y}_j$ which we explain later:

$$\gamma_0 = (\mathcal{F} \mathcal{G} \neg \delta) \land \mu Y.((\bigcirc^\delta \bigcirc^\delta \mathcal{G} \neg \delta) \lor (\bigcirc^\delta \bigcirc^\delta Y))$$

$$\gamma_j = (\mathcal{G} \mathcal{F} \delta) \land \mu Y.((\bigcirc^\delta \neg \delta_j \lor \bigcirc^\delta (\delta_j \land \bigcirc^\delta (\delta_j \land Y)))$$

$$\tilde{y}_j = (\mathcal{G} \mathcal{F} \mathcal{F}) \land \mu Y.((\bigcirc^\delta \delta_j \lor \bigcirc^\delta (\neg \delta_j \land \bigcirc^\delta (\neg \delta_j \land Y))).$$

We set $\Gamma(\pi) = \{\gamma_0, \gamma_1, \ldots, \gamma_m, \tilde{y}_1, \ldots, \tilde{y}_m\}$ and replace every test $[\delta_j]_{\pi}$ in scope of a $\bigcirc^\Delta$ operator by $[(\neg \mathcal{U}(\delta \land \delta_j)) \lor (\delta_j \land \neg \mathcal{F} \delta)]_{\pi}$. After doing so for all trace variables $\pi$, we replace every next operator $\bigcirc^\Delta$ with $\bigcirc^\Gamma$, obtaining a multitrace formula $\hat{\varphi}$. $\hat{\varphi}$ is then given as $Q_0\pi_0 \ldots Q_n\pi_n.\hat{\varphi}$. The equivalence of $\varphi$ and $\hat{\varphi}$ follows from the following claim in which $\hat{\varphi}_i$ is defined analogous to $\varphi_i$:

Claim 2. For all sets of traces $T$ and trace assignments $\Pi$ over $T$, $\Pi \models^T \varphi_i$ iff $\Pi \models^T \hat{\varphi}_i$.

A formal proof of this claim by induction on $i$ can be found in the extended version [Gutsfeld et al. 2023]. Here, we explain the intuition of the translation. First, consider a trace where $\delta$ is true on a finite number of positions. This case is illustrated in Figure 4. On such traces, the formulae $\gamma_j$ or $\tilde{y}_j$ do not change their valuation since their first conjunct is never fulfilled. We thus use $\gamma_0$ to progress on such traces. Intuitively, the formula expresses that (i) there are only finitely many $\delta$-positions on the trace (expressed by $\mathcal{F} \mathcal{G} \neg \delta$) and (ii) there is an odd number of $\delta$-positions after the current position (expressed by the fixpoint formula $\mu Y.((\bigcirc^\delta \bigcirc^\delta \mathcal{G} \neg \delta) \lor (\bigcirc^\delta \bigcirc^\delta Y))$). In this formula, $\bigcirc^\delta \bigcirc^\delta \mathcal{G} \neg \delta$ identifies the positions with exactly one $\delta$-position after them. Additionally, the use of $\bigcirc^\delta \bigcirc^\delta Y$ in each fixpoint iteration advances by two $\delta$-positions and thus expresses that a position satisfying the fixpoint is an even number of $\delta$-positions away from the base case. Since the number of $\delta$-positions left on the trace is decreased by one whenever a $\delta$-position is encountered, $\gamma_0$ changes its valuation exactly at the positions where $\delta$ holds.

Fig. 4. Valuation of $\gamma_0$ on trace with finitely many $\delta$-positions. The numbers in the line labeled # indicate the number of $\delta$-positions after the current position.
We use formulae based on the linear time \( \mathcal{L} \) of hyperlogics was developed based on variants of established temporal logics like LTL and CTL*.

Asynchronous hyperproperties were first systematically studied in Clarkson et al. 2014]. A plethora of hyperlogics was developed based on variants of established temporal logics like LTL and CTL* [Clarkson et al. 2014], QPTL [Rabe 2016] or PDL-\( \Delta \) [Gutsfeld et al. 2020]. All these approaches only concern synchronous hyperproperties.

Next, consider a trace where \( \delta \) holds infinitely often. This case is illustrated in Figure 5. On such traces, the formula \( \gamma_0 \) does not change its valuation since the first conjunct is never fulfilled. Here, we use formulae \( \gamma_j \) and \( \tilde{\gamma}_j \) to progress on the trace. For these formulae, the first conjunct \( GF \delta \) is used to identify the case that there is an infinite number of \( \delta \)-positions. Additionally, we have:

1. \( \gamma_j \) is satisfied on positions with an even number of positions that satisfy \( \delta_j \land \delta \) after the current position and before the next position satisfying \( \neg \delta_j \land \delta \). The base case of the fixpoint formula \( (\bigcirc \delta \neg \delta_j) \) identifies positions with no further \( \delta \)-positions between them and the next position where \( \neg \delta_j \land \delta \) is true. Each fixpoint iteration (by \( \bigcirc \delta (\delta_j \land \bigcirc \delta (\delta_j \land Y)) \)) advances by two \( \delta \)-positions where \( \delta_j \) is true as well.

2. Analogously, \( \tilde{\gamma}_j \) is satisfied on positions with an even number of positions that satisfy \( \neg \delta_j \land \delta \) after the current position and before the next position satisfying \( \delta_j \land \delta \).

As a consequence of (1) and (2), either \( \gamma_j \) or \( \tilde{\gamma}_j \) changes its value on all \( \delta \)-positions if there are infinitely many \( \delta \)-positions where \( \delta_j \) holds and infinitely many \( \delta \)-positions where \( \neg \delta_j \) holds, i.e., the valuation of \( \delta_j \) on \( \delta \)-positions changes infinitely often. If this is not the case, i.e., if the valuation of \( \delta_j \) is constant for all \( j \in \{1, \ldots, m\} \) on \( \delta \)-positions from some point onward, \( \gamma_j \) or \( \tilde{\gamma}_j \) change value on all \( \delta \)-positions up to that point. Thus, \( \Gamma \) advances a trace exactly like \( \Delta \) except in situations where there are infinitely many \( \delta \)-positions and the valuation of all tests \( \delta_j \) is constant on \( \delta \)-positions on the suffix of the trace. However, in this case we can use the fact that the valuation for all tests is constant and perform future tests on arbitrary \( \delta \)-positions. This is done by replacing tests \( [\delta_j]_\pi \) by \( [\neg \delta U(\delta \land \delta_j)]_\pi \). The disjunct \( [\neg \delta U(\delta \land \delta_j)]_\pi \) is equivalent to \( \gamma_j \), if the stuttering assignment has correctly advanced to a \( \delta \)-position and tests at the next \( \delta \)-position if the stuttering assignment has not correctly advanced. Additionally, \( (\delta_j \land \neg F \delta) \) accounts for the tests that are performed on the suffix where \( \delta \) does not hold in the case with finitely many \( \delta \)-positions.

From Lemma 6.2 and Lemma 6.5, we conclude:

**Theorem 6.6.** Mumbling \( H_\mu \) with unique mumbling and stuttering \( H_\mu \) with unique stuttering are expressively equivalent.

7 RELATED WORK

Hyperproperties were first systematically studied in [Clarkson and Schneider 2010]. A plethora of hyperlogics was developed based on variants of established temporal logics like LTL and CTL* [Clarkson et al. 2014], QPTL [Rabe 2016] or PDL-\( \Delta \) [Gutsfeld et al. 2020]. All these approaches only concern synchronous hyperproperties.

In [Gutsfeld et al. 2021] the logic \( H_\mu \) for asynchronous hyperproperties was introduced. It is based on the linear time \( \mu \)-calculus \( \mu TL \) with an asynchronous notion of progress on different
paths and is one inspiration for the logic presented in this paper. However, $H_j$ does not include abstract modalities or a jump mechanism and has only been considered on finite models. The same holds true for the logics presented in [Baumeister et al. 2021; Bonakdarpour et al. 2020] that make use of trajectories to model asynchronous progress. Bozzelli et al. [Bozzelli et al. 2021] recently introduced an asynchronous variant of HyperLTL based on a mechanism to specify an indistinguishability criterion for positions on traces, another inspiration for the logic in the current paper. Another logic for asynchronous hyperproperties is observation-based HyperLTL [Beutner and Finkbeiner 2022]. The concept of observation points in that logic is similar to our notion of mumbling. However, due to a different choice of infinite state system model and verification technique, precise decidability or complexity results cannot be provided in [Beutner and Finkbeiner 2022] whereas our work does. Additionally, they do not consider the expressiveness of different jump criteria. In [Bozzelli et al. 2022], different asynchronous hyperlogics are compared with respect to expressivity. As opposed to the study of expressiveness in this paper, [Bozzelli et al. 2022] compares the unrestricted versions of the logics rather than focussing on decidable fragments.

There are only two other approaches for model checking hyperlogics against pushdown models that we are aware of. The approach of [Pommellet and Touili 2018] consists of model checking HyperLTL against a regular over- or underapproximation of the pushdown model. This approach, however, considers neither asynchronicity nor non-regular modalities and its restrictions are unrelated to the notion of well-alignedness we introduce. The other approach, [Bajwa et al. 2023], uses quantification over stack access patterns to align the stack actions of different traces. A preprint version of the current paper is discussed in the related work section of [Bajwa et al. 2023] which suggests that their approach might be inspired by the notion of well-aligned modalities. They only cover synchronous hyperproperties where a common stack access pattern corresponds to a special case of our notion of well-alignedness.

Finally, there are two approaches to hyperlogics that are orthogonal to the one using named quantifiers and thus only indirectly related to the current work. In logics with team semantics [Gutsfeld et al. 2022; Krebs et al. 2018; Virtema et al. 2021], a formula is evaluated over multiple traces (teams) at once instead of only a single one. The adoption of team semantics seems to lead to logics expressively incomparable to our approach. Other logics add an equal-level predicate to first- and second-order logics [Coenen et al. 2019; Finkbeiner 2017; Spelten et al. 2011]. The work [Coenen et al. 2019] discovered that these logics can be placed in an expressiveness hierarchy with synchronous hyperlogics with trace quantification while the work [Bozzelli et al. 2022] suggests that this may not be the case for asynchronous hyperlogics with trace quantification. Finally, we note these two approaches also have not yet been considered for the verification of recursive programs.

8 CONCLUSION

We proposed a novel logic for the specification and verification of asynchronous hyperproperties. In addition to other extensions, the logic provides a new jump mechanism on traces that is simpler yet more expressive for LTL jump criteria than a related mechanism used by the logic HyperLTL$_S$. Under an assumption necessary for decidability, we provided a model checking algorithm for both finite and pushdown models, the first model checking algorithm for asynchronous hyperproperties on pushdown models. For the finite state case, the complexity of the model checking procedure coincides with that of simple HyperLTL$_S$ despite the increased expressiveness. For the pushdown case, we introduced a concept called well-alignedness as an enabler for decidability. The ability to model check pushdown systems in conjunction with the ability to handle asynchronicity and the abstract, non-regular modalities renders our algorithm a promising approach for automatic verification of hyperproperties on recursive programs.
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REFERENCES

Rajeev Alur, Kousha Etessami, and P. Madhusudan. 2004. A Temporal Logic of Nested Calls and Returns. In Tools and Algorithms for the Construction and Analysis of Systems, 10th International Conference, TACAS 2004, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2004, Barcelona, Spain, March 29 - April 2, 2004, Proceedings (Lecture Notes in Computer Science, Vol. 2988), Kurt Jensen and Andreas Podelski (Eds.). Springer, 467–481. https://doi.org/10.1007/978-3-540-24730-2_35

Rajeev Alur and P. Madhusudan. 2004. Visibly Pushdown Languages. In Proceedings of the 36th Annual ACM Symposium on Theory of Computing, Chicago, IL, USA, June 13-16, 2004, László Babai (Ed.). ACM, 202–211. https://doi.org/10.1145/1007352.1007390

Ali Bajwa, Minjjan Zhang, Rohit Chadha, and Mahesh Viswanathan. 2023. Stack-Aware Hyperproperties. In Tools and Algorithms for the Construction and Analysis of Systems - 29th International Conference, TACAS 2023, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2022, Paris, France, April 22-27, 2023, Proceedings, Part I (Lecture Notes in Computer Science, Vol. 13993), Sriram Sankaranarayanan and Natasha Sharygina (Eds.). Springer, 308–325. https://doi.org/10.1007/978-3-031-30823-9_16

Jan Baumeister, Norine Coenen, Borzoo Bonakdarpour, Bernd Finkbeiner, and César Sánchez. 2021. A Temporal Logic for Asynchronous Hyperproperties. In Computer Aided Verification - 33rd International Conference, CAV 2021, Virtual Event, July 20-23, 2021, Proceedings, Part I (Lecture Notes in Computer Science, Vol. 12759), Alexandra Silva and K. Rustan M. Leino (Eds.). Springer, 694–717. https://doi.org/10.1007/978-3-030-81685-8_33

Raven Beutner and Bernd Finkbeiner. 2022. Software Verification of Hyperproperties Beyond k-Safety. In Computer Aided Verification - 34th International Conference, CAV 2022, Haifa, Israel, August 7-10, 2022, Proceedings, Part I (Lecture Notes in Computer Science, Vol. 13371), Sharon Shoham and Yakir Vizel (Eds.). Springer, 341–362. https://doi.org/10.1007/978-3-031-13185-1_17

Borzoo Bonakdarpour, Pavithra Prabhakar, and César Sánchez. 2020. Model Checking Timed Hyperproperties in Discrete-Time Systems. In NASA Formal Methods - 12th International Symposium, NFM 2020, Moffett Field, CA, USA, May 11-15, 2020, Proceedings (Lecture Notes in Computer Science, Vol. 12229), Ritchie Lee, Sumit Jha, and Anastasia Madvidou (Eds.). Springer, 311–328. https://doi.org/10.1007/978-3-030-55754-6_18

Borzoo Bonakdarpour, César Sánchez, and Gerardo Schneider. 2018. Monitoring Hyperproperties by Combining Static Analysis and Runtime Verification. In Leveraging Applications of Formal Methods, Verification and Validation. Verification - 8th International Symposium, ISoLA 2018, Limassol, Cyprus, November 5-9, 2018, Proceedings, Part II (Lecture Notes in Computer Science, Vol. 11245), Tiziana Margaria and Bernhard Steffen (Eds.). Springer, 8–27. https://doi.org/10.1007/978-3-030-03421-4_2

Ahmed Bouajjani, Javier Esparza, and Oded Maler. 1997. Reachability Analysis of Pushdown Automata: Application to Model-Checking. In CONCUR ’97: Concurrency Theory, 8th International Conference, Warsaw, Poland, July 1-4, 1997, Proceedings (Lecture Notes in Computer Science, Vol. 1243), Antoni W. Mazurkiewicz and Józef Winkowski (Eds.). Springer, 135–150. https://doi.org/10.1007/3-540-63141-0_10

Laura Bozelli. 2007. Alternating Automata and a Temporal Fixpoint Calculus for Visibly Pushdown Languages. In CONCUR 2007 – Concurrency Theory, Luis Caires and Vasco T. Vasconcelos (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 476–491. https://doi.org/10.1007/978-3-540-74407-8_32

Laura Bozelli, Adriano Peron, and César Sánchez. 2021. Asynchronous Extensions of HyperLTL. In 36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021. IEEE, 1–13. https://doi.org/10.1109/LICS52264.2021.9470583

Laura Bozelli, Adriano Peron, and César Sánchez. 2022. Expressiveness and Decidability of Temporal Logics for Asynchronous Hyperproperties. In 33rd International Conference on Concurrency Theory, CONCUR 2022, September 12-16, 2022, Warsaw, Poland (LIPIcs, Vol. 243), Bartek Klin, Slawomir Lasota, and Anca Muscholl (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 27:1–27:16. https://doi.org/10.4230/LIPIcs.CONCUR.2022.27

Stephen D. Brookes. 1996. Full Abstraction for a Shared-Variable Parallel Language. Inf. Comput. 127, 2 (1996), 145–163. https://doi.org/10.1006/inco.1996.0056

Michael R. Clarkson, Bernd Finkbeiner, Masoud Koleini, Kristopher K. Micinski, Markus N. Rabe, and César Sánchez. 2014. Temporal Logics for Hyperproperties. In Principles of Security and Trust, Martin Abadi and Steve Kremer (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 265–284. https://doi.org/10.1007/978-3-642-54792-8_15

Michael R. Clarkson and Fred B. Schneider. 2010. Hyperproperties. J. Comput. Secur. 18, 6 (Sept. 2010), 1157–1210. https://doi.org/10.5233/jcs-2009-0393
