Introduction

For a non-singular scheme $Y$ over $\mathbb{C}$ the hyper-cohomology of the algebraic de Rham complex calculates the analytic cohomology $H^i(Y^{an}, \mathbb{C})$. For singular $Y$ there is no straightforward generalization of this calculation: indeed, it is the algebraic side that causes problems. Grothendieck has introduced the algebraic infinitesimal site $Y_{inf}$. Moreover, as explained in [13], when $Y$ admits an embedding as a closed subscheme of a smooth scheme $X$, one can also consider the completion of the de Rham complex $\Omega^*_X$ along $Y$: $\Omega^*_{X|Y}$. At this point one has three different cohomologies

$$ H^i(Y, \Omega^*_{X|Y}), $$

$$ H^i(Y_{inf}, \mathcal{O}_{Y_{inf}}), $$

$$ H^i(Y^{an}, \mathbb{C}). $$

The isomorphism between (2) and (3) was proved by Grothendieck ([11]) only in the case of a smooth scheme over $\mathbb{C}$. The isomorphism between (1) and (3) was proved by Herrera-Liebermann ([14]), in the case of $Y$ proper over $\mathbb{C}$, while Deligne (unpublished), and Hartshorne ([13, Chapter IV, Theorem (I.1)]) proved it for a general (not necessarily proper) scheme over $\mathbb{C}$. A direct statement asserting the isomorphism of these cohomology groups for arbitrary $\mathbb{C}$-schemes $Y$ cannot be found in literature, although all the necessary ingredients are given. The proof presented in this paper, if applied to classical schemes, can be used to fill this gap (see §1). Of course the generalization of this problem to the case of mixed or finite characteristic has been carefully studied by Berthelot and Ogus.

On the other hand, in more recent years the notion of scheme and the properties of schemes have been generalized by the introduction of log schemes. Among the expected features of log schemes, there is the fact that log smooth schemes (which are in general singular as schemes) should behave like classical smooth schemes and moreover should also be related to analytic schemes. The goal of the present work is to introduce the log scheme analogues of (1), (2), (3) over $\mathbb{C}$, and prove the isomorphisms between them.

With these ideas in mind we first consider the analogue of Grothendieck’s Infinitesimal Site ([11]) in the logarithmic context (see also [18] for positive characteristic). We work with pro-crystals and

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we link them to the logarithmic stratification on pro-objects. If we suppose that there exists an exact closed immersion of fs log schemes $Y \hookrightarrow X$, with $X$ log smooth over $\mathbb{C}$, then we can define the Log De Rham Cohomology of $Y$ over $\mathbb{C}$, as the hyper-cohomology of the complex $\omega_{X/Y}^*$, where $X|Y$ represents the formal completion of the log scheme $X$ along its closed log subscheme $Y$. We give a direct proof of the existence of an isomorphism between the Log Infinitesimal Cohomology of $Y$ over $\mathbb{C}$, and its Log De Rham Cohomology, namely we prove the following isomorphism

$$H^i(Y^\log_{\inf}, \mathcal{O}_{Y^\log_{\inf}}) \cong H_{DR,\log}(Y/\mathbb{C}) =: \mathbb{H}(Y, \omega_{X/Y}^*)$$

This result was proved by Shiho ([21]) for a log smooth log scheme over $\mathbb{C}$.

For the remaining isomorphisms, we were inspired by a work of K. Kato and C. Nakayama ([21, Theorem (0.2), (2)]). Given an fs (ideally) log smooth log scheme $X$ over $\mathbb{C}$, Kato and Nakayama associate a topological space $X_{an}^{\log}$ and show that the algebraic Log De Rham cohomology of $X$ (which is defined as the hyper-cohomology of the log De Rham complex $\omega_X^*$) is isomorphic to the cohomology of the constant sheaf $\mathbb{C}$ on $X_{an}^{\log}$, i.e.

$$H_{DR,\log}(X/\mathbb{C}) =: \mathbb{H}(X, \omega_X^*) \cong H(X_{an}^{\log}, \mathbb{C})$$

(4)

We will prove an analogue of (4), for a general fs log scheme $Y$ over $\mathbb{C}$. In this case, the Log De Rham Cohomology of $Y$ is defined as before (for $Y$ admitting an exact closed immersion). In §2 we extend the theory of Kato-Nakayama ([21]) to the log formal setting. To this end, we first introduce a ringed topological space $(X|Y)^{\log}$, associated to the log formal analytic space $(X|Y)^{an}$, with sheaf of rings $\mathcal{O}_{(X|Y)^{an}}$ (Definition 2.2). The underlying topological space $(X|Y)^{\log}$ of this ringed space coincides with $Y_{an}^{\log}$. We construct the complex $\omega_{(X|Y)^{an}}^*$ (Definition 2.5 and (33)), which is a sort of “formal analogue” of the complex $\omega_{X^{an}}^*$, introduced by Kato-Nakayama for a log smooth log scheme $X$ ([21, (3.5)]).

Later, in [3] we give a “formal version” of the Deligne Poincaré Residue map ([6, (3.6.7.1)]), in the particular case of a smooth scheme $X$ over $\mathbb{C}$, endowed with log structure given by a normal crossing divisor $D$, and $Y \hookrightarrow X$ a closed subscheme, with the induced log structure ([3,2]). We show that this map is an isomorphism. It is useful for describing the cohomology of the complex $\omega_{(X|Y)^{an}}^*$.

Using that description, we can prove the Log Formal Poincaré Lemma (Theorem 4.1): given a general fs log scheme $Y$ over $\mathbb{C}$, the Betti Cohomology of its associated topological space $Y_{an}^{\log}$ is isomorphic to the hyper-cohomology of the complex $\omega_{(X|Y)^{an}}^*$.

In [5] we show that $\omega_{(X|Y)^{an}}^*$ is quasi-isomorphic to $\mathbb{R}\tau_* \omega_{(X|Y)^{an}}^*$ (Proposition 5.2), where $\tau: Y_{an}^{\log} \rightarrow Y^{an}$ is the canonical (continuous, proper and surjective) Kato-Nakayama map of topological spaces. Finally, we prove that there exists an isomorphism in cohomology $\mathbb{H}(Y^{an}, \omega_{(X|Y)^{an}}^*) \cong \mathbb{H}(Y, \omega_{X/Y}^*)$ between the Analytic and the Algebraic Log De Rham Cohomology (Theorem 5.3). We conclude with the main theorem of this article (Theorem 5.1): the cohomology of the constant sheaf $\mathbb{C}$ on the topological space $Y_{an}^{\log}$, associated to an fs log scheme $Y$, is isomorphic to the Log De Rham Cohomology of $Y$,

$$H^i(Y_{an}^{\log}, \mathcal{O}_{Y_{an}^{\log}}) \cong H_{DR,\log}(Y/\mathbb{C}) \cong H(Y_{an}^{\log}, \mathbb{C})$$

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Preliminaries

Notations: by $S$ we denote the logarithmic scheme $\text{Spec} \mathbb{C}$ endowed with the trivial log structure, and, by a log scheme, we mean a logarithmic scheme over $S$, whose underlying scheme is a separated
C-scheme of finite type. Moreover, if \( A \) is a complex of sheaves and \( k \in \mathbb{N} \), then \( A \left[ -k \right] \) is the complex defined in degree \( j \) as \( A^{j+k} \).

### 0.1 The Logarithmic Infinitesimal Site

Given a log scheme \( X \), endowed with a fine log structure \( M \), we denote by \( \text{InfLog}(X/S) \) the Logarithmic Infinitesimal Site of \( X \) over \( S \). It is given by 4-uples \((U,T,M_T,i)\), where \( U \) is an étale scheme over \( X \), \((T,M_T)\) is a scheme with a fine log structure over \( S \), \( i \) is an exact closed immersion \((U,M) \hookrightarrow (T,M_T)\) over \( S \), defined by a nilpotent ideal on \( T \), i.e. \( i \) is a nilpotent exact closed immersion. Morphisms, coverings (for the usual étale topology), and sheaves on \( \text{InfLog}(X/S) \) are defined in the usual way. The category of all sheaves on \( \text{InfLog}(X/S) \) is a ringed topos, called the Logarithmic Infinitesimal Site of \( X \) over \( S \), and denoted by \( (X/S)_{\text{log}}^{\text{inf}} \).

### 0.2 Pro-Crystals and Logarithmic Stratification

Let \( X \) be a log smooth log scheme. For the definition of pro objects we refer to \([1, 8, 11, 23, 6.2]\).

**Definition 0.1** Let \( \{F_k\}_{k \in K} \) be a pro object. It is a pro-crystal if, given \( g: (U', T', M_{T'}, i') \rightarrow (U, T, M, i) \) a morphism in \( \text{InfLog}^{\text{inf}} \), there exists an isomorphism of pro objects

\[
\{ g^* F_k(U, T, M, i) \}_{k \in K} \cong \{ F_k(U', T', M_{T'}, i') \}_{k \in K}
\]
i.e. \( \{ g^* F_k \}_{k \in K} \cong \{ F_k T' \}_{k \in K} \). In a similar way we can define Artin-Rees pro-crystals (see \([23, \text{Proposition 0.5.1}]\)).

For each integer \( i \geq 0 \), let \( \Delta^1_{\log}(i) \) be the \( i \)-th log infinitesimal neighbourhood of the diagonal \((X, M) \hookrightarrow (X, M) \times_S (X, M)\), and let \( \Delta^2_{\log}(i) \) be the \( i \)-th log infinitesimal neighbourhood of \((X, M) \times_S (X, M) \hookrightarrow (X, M) \times_S (X, M) \times_S (X, M)\) (where the fiber product is taken in the category of fine log schemes). We have the canonical projections \( p_1(i), p_2(i): \Delta^1_{\log}(i) \rightarrow (X, M) \), and \( p_{31}(i), p_{32}(i), p_{21}(i): \Delta^2_{\log}(i) \rightarrow \Delta^1_{\log}(i) \). We denote by \( \mathcal{P}^{\nu,i}_{X,\log} \) the structural sheaf of rings \( \mathcal{O}\Delta^\nu_{\log}(i) \), for each \( \nu = 1, 2 \), \( i \geq 0 \). In particular, we can regard \( \mathcal{P}^{1,1}_{X,\log} \) as an \( \mathcal{O}_X \)-module in two ways, via the canonical projections \( p_1(i), p_2(i) \). So, we call the left \( \mathcal{O}_X \)-module structure (resp. right \( \mathcal{O}_X \)-module structure) on \( \mathcal{P}^{1,1}_{X,\log} \) the structure given by \( p_1(i) \) (resp. \( p_2(i) \)).

We introduce a logarithmic stratification on the category of pro-coherent \( \mathcal{O}_X \)-modules. We could define a logarithmic stratification “at any level” of the pro object, and consider the pro-category of log stratified \( \mathcal{O}_X \)-modules. But this stratification would be too restrictive for our purpose. We need to work with a larger category and, to this aim, we introduce the logarithmic stratification as a pro-morphism.

**Definition 0.2** [Definition 1.3] Let \( \{F_k\}_{k \in K} \) be a pro-coherent \( \mathcal{O}_X \) module. A logarithmic stratification on \( \{F_k\}_{k \in K} \) is a pro-morphism

\[
\{F_k\}_{k} \xrightarrow{s(F_k)} \{F_k\}_{k} \otimes \{\mathcal{P}^{1,i}_{X,\log}\}_{i}
\]
such that the coidentity diagram

\[
\begin{array}{ccc}
\{F_k\}_{k} & \xrightarrow{s(F_k)} & \{F_k\}_{k} \otimes \{\mathcal{P}^{1,i}_{X,\log}\}_{i} \\
\downarrow{id_{\{F_k\}_{k}}} & & \downarrow{id_{\{F_k\}_{k}} \otimes (id_{\mathcal{O}_X})_{i}} \\
\{F_k\}_{k} & \xrightarrow{id_{\{F_k\}_{k}}} & \{F_k\}_{k}
\end{array}
\]

and the coassociativity diagram

\[
\begin{array}{ccc}
\{F_k\}_{k} & \xrightarrow{s(F_k)} & \{F_k\}_{k} \otimes \{\mathcal{P}^{1,i}_{X,\log}\}_{i} \\
\downarrow{s(F_k)} & & \downarrow{s(F_k) \otimes (id_{\mathcal{O}_X})_{i}} \\
\{F_k\}_{k} \otimes \{\mathcal{P}^{1,i}_{X,\log}\}_{i} & \xrightarrow{id_{\{F_k\}_{k}} \otimes s(F_k)} & \{F_k\}_{k} \otimes \{\mathcal{P}^{1,i}_{X,\log}\}_{i} \otimes \{\mathcal{P}^{1,i}_{X,\log}\}_{i}
\end{array}
\]
are commutative, where \( q_{i,j} : \mathcal{P}^{1,i}_{X,\log} \rightarrow \mathcal{P}^{1,j}_{X,\log} \) are the natural compatible maps, and \( s_{\mathcal{P}^{1,i}_{X,\log}} = \{ \delta_{X}^{j,i} \}_{(i,j)} \) (see \cite{24} Lemma 3.2.3) for the definition of \( \delta_{X}^{j,i} : \Delta_{\log}^{1}(i) \times_{(X,M)} \Delta_{\log}^{1}(j) \rightarrow \Delta_{\log}^{1}(i + j) \).

**Theorem 0.3** [\cite{3} Theorem 3.2]. The following two categories are equivalent:
(a) the category of pro-crystals on InfLog\((X/S)\);
(b) the category of pro-sheaves \( \{ \mathcal{M}_{n} \}_{n} \) on \( X \), endowed with a logarithmic stratification.

**Remark 0.4** In fact, our pro-crystals are actually Artin-Rees pro-crystals and one could refine the previous result on these objects.

Now let \( i : Y \hookrightarrow X \) be an exact closed immersion of fs log schemes, with \( X \) log smooth over \( S \). We consider the direct image functor \( i^{\log}_{inf,*} : Y^{\log}_{inf} \rightarrow X^{\log}_{inf} \). For a crystal \( \mathcal{F} \) of \( Y^{\log}_{inf} \), we briefly describe the construction of the direct image \( i^{\log}_{inf,*} \mathcal{F} \), in characteristic zero. Let \( (U, T, M_{T}, j) \in \text{InfLog}(\{(X,M)\}/S) \), then we consider the fiber product (in the category of fs log schemes) \( U_{Y} = (Y, N) \times_{(X,M)} (U, M) \). By base change, the map \( U_{Y} \hookrightarrow (U,M) \) is an exact closed immersion. Since \( j : (U,M) \hookrightarrow (T,M_{T}) \) is a nilpotent exact closed immersion, \( U_{Y} \hookrightarrow (T,M_{T}) \) is also an exact closed immersion, and we can take the \( n \)-th infinitesimal neighborhood of \( U_{Y} \) inside \( (T,M_{T}) \), and denote it by \( (T_{n}, M_{n}) \).

Let \( \lambda_{n} : (T_{n}, M_{n}) \hookrightarrow (T, M_{T}) \). Then, \( (i^{\log}_{inf,*} \mathcal{F})(U,T,M_{T},j) = : \lim_{n} \lambda_{n}^{*} \mathcal{F}(U_{Y},T_{n},M_{n},j_{n}) \). So, the Artin-Rees pro-crystal \( \{ \mathcal{F}_{n} \}_{n} \) is a basis of \( \text{InfLog}(\{(X,M)\}/S) \), associated to \( i^{\log}_{inf,*} \mathcal{F} \), is defined on \( (U, T, M_{T}, j) \in \text{InfLog}(\{(X,M)\}/S) \) as \( \mathcal{F}_{n}(U,T,M_{T},j) := \lambda_{n}^{*} \mathcal{F}(U_{Y},T_{n},M_{n},j_{n}) \) \cite{24} Proposition 0.5.1]. In particular, the Artin-Rees pro-crystal \( \{ \mathcal{O}_{n} \}_{n} \) associated to \( i^{\log}_{inf,*} \mathcal{O}_{Y,log} \) is in fact defined, on each \( (U, T, M_{T}, j) \), as \( \{ \{ \mathcal{O}_{n} \}_{n} \}_{(U,T,M_{T},j)} := \{ \lambda_{n}^{*} \mathcal{O}_{T_{n}} \}_{n} \).

**Remark 0.5** From Theorem 0.3, the log stratified \( \mathcal{O}_{X} \)-pro-module associated to the (Artin-Rees) pro-crystal \( \{ \mathcal{O}_{n} \}_{n} \) is equal to \( \{ \{ \mathcal{O}_{n} \}_{(X,X,M,\text{id})} \}_{n} = \{ \mathcal{O}_{n} \mathcal{O}_{Y} \}_{n} = \{ \mathcal{O}_{X}/\mathcal{F} \}_{n} \), where \( \mathcal{F} \subset \mathcal{O}_{X} \) is the ideal of definition of \( Y \) into \( X \), and \( Y_{n} \) is the \( n \)-th log infinitesimal neighborhood of \( Y \hookrightarrow X \). Moreover, \( \lim_{n} \mathcal{O}_{X}/\mathcal{F}^{n} \mathcal{O}_{X} \mathcal{O}_{Y} \).

### 0.3 Linearization of the Log De Rham complex

Let \( \omega_{X}^{\log} \) be the log De Rham complex of the log smooth log scheme \( X \) over \( S \). As in the classical case \cite{3} p. 2.17], we denote the complex of Artin-Rees pro-coherent \( \mathcal{O}_{X} \) modules which is the linearization of \( \omega_{X}^{\log} \) by \( \{ L_{X}(\omega_{X}^{\log})_{i} \}_{i} \), i.e.

\[
\{ L_{X}(\omega_{X}^{\log})_{i} \}_{i} = : \{ \mathcal{P}^{1,i}_{1} \mathcal{O}_{X} \}_{i} \otimes \mathcal{O}_{X} \omega_{X}^{\log} \tag{5}
\]

Since, for all \( i, j \in \mathbb{N} \), there exist maps \( \mathcal{P}^{1,i+j}_{X,\log} \otimes \omega_{X}^{\log} \rightarrow \mathcal{P}^{1,i}_{X,\log} \otimes \omega_{X}^{\log} \otimes \mathcal{P}^{1,j}_{X,\log} \), each term of (5) has a canonical logarithmic stratification, in the sense of Definition \cite{12} (3 Construction 2.14]).

We have a local description of the differential maps \( \{ L_{X}(d_{X})_{i} \}_{i} \) of this complex. Indeed, let \( M^{i} \) be the log structure on \( \Delta_{\log}^{1}(i) \). Let \( U \rightarrow X \) be an \( \text{étale} \) morphism of schemes, and let \( m \in \Gamma(U,M) \).

Then, there exists uniquely an element \( u_{m,i} \in \Gamma(U,(\mathcal{P}^{1,i}_{X,\log})) \subset \Gamma(U,M^{i}) \), such that \( p_{2}(i)^{*}(m) = p_{1}(i)^{*}(m)u_{m,i} \) \cite{21} pp. 43, 44]. In particular, we have that \( u_{m,i} - 1 \in \ker \Gamma(U,(\mathcal{P}^{1,i}_{X,\log})) \rightarrow \Gamma(U,\mathcal{O}_{X})) \) \cite{24} Lemma 3.2.7].

Let now \( x \in X \), and \( t_{1},...,t_{r} \in M_{x} \) be such that \( \{ d\log t_{j} \}_{1 \leq j \leq r} \) is a basis of \( \omega_{X,x}^{\log} \). We can restrict to an \( \text{étale} \) neighborhood \( U \) of \( x \), and suppose that \( \{ d\log t_{j} \} \) is a local basis of \( \omega_{X}^{\log} \) on \( U \). Let \( u_{j,i} \), \( 1 \leq j \leq r, \ i \geq 0 \), be the elements in \( \Gamma(U,(\mathcal{P}^{1,i}_{X,\log}))^{*} \) such that \( p_{2}(i)^{*}(t_{j}) = p_{1}(i)^{*}(t_{j})u_{j,i} \), as above. We put \( \xi_{j,i} := u_{j,i} - 1 \in \Gamma(\Delta_{\log}^{1}(i),\mathcal{P}^{1,i}_{X,\log}) \) (note that the \( \xi_{j,i} \)'s are compatible with respect to \( i \)).
Proposition 0.6 In the above notations, 
(1) [24 Lemma 3.2.7], for each \( i \geq 0 \), the \( \mathcal{O}_X \)-module \( \mathcal{P}^{1,i}_{X,\log} \) is locally free with basis 

\[
\{ \xi^a_i := \prod_{j=1}^r \xi^a_{j,i} | 0 \leq \sum_{j=1}^r a_j \leq i \}
\]

where \( a = (a_1, \ldots, a_r) \) is a multi-index of length \( r \). In particular, \( \{\xi_{1,i}, \ldots, \xi_{r,i}\} \) is a basis for the locally free \( \mathcal{O}_X \)-module \( \mathcal{P}^{1,1}_{X,\log} \), étale locally at \( x \);

(2) [24 Proposition 3.2.5], there exists a canonical isomorphism of \( \mathcal{O}_X \)-modules \( \mathcal{H}/\mathcal{H}^2 \xrightarrow{\cong} \omega_X \), where \( \mathcal{H} := \text{Ker} \{\Delta^*: \mathcal{P}^{1,1}_{X,\log} \to \mathcal{O}_X \} \) (with \( \Delta: X \to \Delta^1_{\log}(1) \) the exact closed immersion). Under this identification, the local basis \( \{d\log t_j\}_{1 \leq j \leq r} \) of \( \omega_X \) is identified with \( \{\xi_{j,1}\}_{1 \leq j \leq r} \).

Therefore, for each \( i \geq 0 \), the map \( L_X(d_X^i) \) is the \( \mathcal{O}_X \)-linear map 

\[
L_X(d_X^i): \mathcal{P}^{1,i}_{X,\log} \otimes_{\mathcal{O}_X} \omega_X^k \to \mathcal{P}^{1,i-1}_{X,\log} \otimes_{\mathcal{O}_X} \omega_X^{k+1}
\]

defined, for \( a \in \mathcal{O}_X, \omega \in \omega_X^k \), and \( n_j \in \mathbb{N} \) such that \( n_1 + \ldots + n_r \leq i \), by setting

\[
L_X(d_X^i)(a_{\xi_{1,i}}^n \cdots \xi_{r,i}^n \otimes \omega) = a \cdot \sum_{j=1}^{n_r} n_j \xi_{1,i-1}^n \cdots \xi_{r,i-1}^n \otimes d\log t_j \wedge \omega + a \xi_{1,i-1}^n \cdots \xi_{r,i-1}^n \otimes d\omega (6)
\]

0.4 Log Formal Completion and Kato-Nakayama Topological Space

Let \( X \) be a log scheme, endowed with fs log structure \( M \), and let \( i: Y \hookrightarrow X \) be an exact closed immersion. We refer to [18 (5.8)] and [24 Proposition-Definition 3.2.1], for the definition of logarithmic \( n \)-infinitesimal neighborhoods (in \( \mathbb{N} \)).

Definition 0.7 The log formal completion \( \hat{X}/Y \) of \( X \) along \( Y \) is the classical formal completion of the scheme \( X \) along its closed subscheme \( Y \), endowed with log structure given by the inverse image of \( M \) via the canonical map \( \hat{X}/Y \to X \).

We note that, if the closed immersion \( i: Y \hookrightarrow X \) is not exact, it is also possible to define the log formal completion of \( X \) along \( Y \), but in this article we will always work with exact closed immersions.

Let now \( X^\text{an} \) be the (fs) log analytic space associated to \( X \). Kato-Nakayama define the topological space \( X^\text{an}_{\log} \) associated to \( X^\text{an} \) as the set \( \{(x,h) | x \in X^\text{an}, h \in \text{Hom}(M^{gp}, \mathbb{S}^1), h(f) = f(x)/|f(x)|, \text{for any } f \in \mathcal{O}_{X^\text{an}} \} \) (where \( \mathbb{S}^1 = \{x \in \mathbb{C}; |x| = 1\} \)). Let now \( \beta: P \to M \) be a fixed local chart for \( X^\text{an} \), with \( P \) an fs monoid. The topology on \( X^\text{an}_{\log} \) is locally defined as follows:

Definition 0.8 In the local chart \( \beta, X^\text{an}_{\log} \) is identified with a closed subset of \( X^\text{an} \times \text{Hom}(P^{gp}, \mathbb{S}^1) \), via the map \( X^\text{an}_{\log} \to X^\text{an} \times \text{Hom}(P^{gp}, \mathbb{S}^1); (x,h) \mapsto (x,h_P), \) where \( h_P \) is the composite \( P^{gp} \to M^{gp}_x \stackrel{h}{\to} \mathbb{S}^1 \). So, \( X^\text{an}_{\log} \) is locally endowed with the topology induced from the natural topology on \( X^\text{an} \times \text{Hom}(P^{gp}, \mathbb{S}^1) \).

This local topology does not depend on the choice of the chart, so it induces a well defined global topology on \( X^\text{an}_{\log} \). There exists a surjective map of topological spaces \( \tau: X^\text{an}_{\log} \to X^\text{an}; (x,h) \mapsto x \), which is continuous and proper ([21 Lemma (1.3)]). Though \( X^\text{an}_{\log} \) in general is not an analytic space, it is still endowed with a nice sheaf of rings \( \mathcal{O}_{X^\text{an}} \). Indeed, \( \mathcal{L}_X \) be the sheaf of abelian groups on \( X^\text{an}_{\log} \) which represents the “logarithms of local sections of \( \tau^{-1}(M^{gp}) \)” ([21 (1.4)]). There exists an exact sequence of sheaves of abelian groups

\[
0 \to \tau^{-1}(\mathcal{O}_{X^\text{an}}) \xrightarrow{k} \mathcal{L}_X \xrightarrow{\exp} \tau^{-1}(M^{gp}/\mathcal{O}_{X^\text{an}}) \to 0
\]
If we consider commutative $\tau^{-1}(\mathcal{O}_{X^{\text{an}}})$-algebras $\mathcal{A}$ on $X^{\text{an}}_{\log}$, endowed with a homomorphism $\mathcal{L}_X \rightarrow \mathcal{A}$ of sheaves of abelian groups which commutes with $k$, then $\mathcal{O}^{\log}_{X^{\text{an}}}$ is the universal object among such $\mathcal{A}$ ([21 (3.2)]).

We suppose that $X$ satisfies the following hypothesis ([21 Theorem (0.2), (2)]).

(*) Locally for the étale topology, there exists an fs monoid $P$, an ideal $\Phi$ of $P$, and a morphism $f : X \rightarrow \text{Spec}(\mathbb{C}[P]/(\Phi))$ of log schemes over $S$, such that the underlying morphism of schemes is smooth, and the log structure on $X$ is associated to $P \rightarrow \mathcal{O}_X$.

**Remark 0.9** We note that, if $X$ is (ideally) log smooth over $S$, then it satisfies hypothesis (*), because $X$ is a filtered semi-toroidal variety ([17 Definition 5.2], [16 Definition (1.5)], [9 Proposition II.1.0.11]).

**Theorem 0.10** [21 Theorem (0.2), (2)]. Let $X$ be an fs (ideally) log smooth log scheme over $S$ (see Remark 0.9). Then, there exists a canonical isomorphism

$$\mathbb{H}^q(X, \omega_X) \cong H^q(X^{\text{an}}_{\log}, \mathbb{C}), \text{ for all } q \in \mathbb{Z}.$$  

## 1 Log Infinitesimal and Log De Rham Cohomologies

From now on, let $Y$ be a generic fs log scheme over $S$, endowed with log structure $M_Y$. We suppose there exists an exact closed immersion $i : Y \hookrightarrow X$ (i.e. $i^*M_X \cong M_Y$), where $X$ is an fs log smooth log scheme over $S$, with log structure $M_X$. The Log De Rham complex of the log scheme $Y$ is the complex $(\omega_{X,Y} \cong \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X,Y} \cdot \hat{d})$, where $\hat{d}$ is the integrable connection induced by the differential $d_X$ of $\omega_X$. Then, we define the Log De Rham Cohomology of $Y$ as the hyper-cohomology of the log De Rham complex $\omega_{X,Y}$, i.e. $H^{DR,log}_{Y/S} = \mathbb{H}(Y, \omega_{X,Y})$.

Let $\mathscr{S}$ be the ideal of definition of $Y$ in $X$. For each fixed $\nu \geq 0$, we consider the diagonal immersion of fine log schemes $X \hookrightarrow X^\nu$, where $X^\nu$ is the fiber product over $S$ of $\nu + 1$ copies of $(X, M_X)$ over $S$ (the fiber product being that of the category of fine log schemes over $S$). We denote by $\Delta_{X, log}^{\nu,i}(i)$ the $i$-th log infinitesimal neighbourhood of the diagonal of $X^\nu$, and by $\mathcal{P}^{\nu,i}_{X, log}$ its structural sheaf of rings $\mathcal{O}_{\Delta_{X, log}^{\nu,i}(i)} = \mathcal{O}_{X^\nu}/\mathcal{K}_{\nu,i+1}$, where $\mathcal{K}_{\nu}$ is the ideal of definition of the log scheme $X$ inside $X^\nu$.

Now, if we fix $\nu$ and vary $i \in \mathbb{N}$, we get the Artin-Rees pro-object of sheaves $\{\mathcal{P}^{\nu,i}_{X, log}\}_i$ on $X$. On the other hand, if we fix $i$ and vary $\nu \in \mathbb{N}$, we get a sheaf on the simplicial log smooth log scheme

$$\ldots \rightarrow X^\nu \rightarrow \ldots \rightarrow X^1 = X \times_S X \rightarrow X$$

which is the following cosimplicial sheaf of rings on $X$

$$0 \rightarrow \mathcal{O}_X \xrightarrow{d_1 - d_0} \mathcal{P}^{1,i}_{X, log} \xrightarrow{d_2 - d_1 + d_0} \mathcal{P}^{2,i}_{X, log} \rightarrow \ldots \rightarrow \mathcal{P}^{\nu,i}_{X, log} \rightarrow \ldots$$  

(7)

where the maps are given by the alternating sum of the faces of the simplicial log scheme $\{X^\nu\}_\nu$. If we vary $\nu$ and $i$, we get a cosimplicial sheaf of Artin-Rees $\mathcal{O}_X-$pro modules $\{\mathcal{P}^{\nu,i}_{X, log}\}_{\nu,i}$.

We define the cosimplicial Artin-Rees pro-object $\{Q^{\nu,i}_{\log}\}_{\nu,i}$, by setting

$$Q^{\nu,i}_{\log} = : \mathcal{P}^{\nu+1,i}_{X, log}$$  

(8)

for every $\nu, i \geq 0$. Then, for each $\nu \geq 0$, there is a canonical homomorphism of pro-rings $\alpha^{\nu,i}_{\log} : \mathcal{P}^{\nu,i}_{X, log} \rightarrow Q^{\nu,i}_{\log}$, defined by the canonical injection $\{0, 1, \ldots, \nu\} \hookrightarrow \{0, 1, \ldots, \nu, \nu + 1\}$. So, we have a homomorphism of cosimplicial pro-rings

$$\{\alpha^{\nu,i}_{\log}\} : \{\mathcal{P}^{\nu,i}_{X, log}\} \rightarrow \{Q^{\nu,i}_{\log}\}$$  

(9)

Let $\mathcal{N}$ be an $\mathcal{O}_X-$module. As in the classical case, we define the cosimplicial pro-ring

$$\{Q^{\nu,i}_{\log}(\mathcal{N})\} = \{Q^{\nu,i}_{\log}\}_i \otimes_{\mathcal{O}_X} \mathcal{N}$$  

(10)
We note that, for fixed $\nu \geq 0$, $\{Q_{\nu,i}^{\nu,i}(\mathcal{N})\}_i$ is clearly a $Q_{\nu,i}^{\nu,i}$-module, and so an $O_X$-bimodule with the obvious left and right structures. Moreover, if we regard $\{Q_{\nu,i}^{\nu,i}(\mathcal{N})\}_i$ as a cosimplicial pro-module on $\{\mathcal{P}_{X,i}^{\nu,i}(\mathcal{N})\}_i$ (by restriction of scalars, via $\{Q_{\nu,i}^{\nu,i}(\mathcal{N})\}_i$), we see that it is the cosimplicial pro-module associated with the $O_X$-pro-module with canonical stratification $\{Q_{\nu,i}^{\nu,i}(\mathcal{N})\}_i = \{\mathcal{P}_{X,i}^{\nu,i}(\mathcal{N})\}_i \otimes O_X \mathcal{N}$. Indeed, for each $\nu \leq \mu$, $\{Q_{\nu,i}^{\nu,i}(\mathcal{N})\}_i$ is obtained from $\{Q_{\nu,i}^{\nu,i}(\mathcal{N})\}_i$, by base change with respect to any of the canonical morphisms $\{\mathcal{P}_{X,i}^{\nu,i}(\mathcal{N})\}_i \rightarrow \{\mathcal{P}_{X,i}^{\mu,i}(\mathcal{N})\}_i$. Now, for each integer $k \geq 2$, we consider the differential operator $d^k$ of the log De Rham complex,

$$d^k : \omega_X^k \rightarrow \omega_X^{k+1}$$

Following [11, p. 347], for each $\nu \geq 0$, $d^k$ induces a homomorphism of Artin-Rees pro-objects

$$Q_{\nu,i}^{\nu,i}(d^k) : Q_{\nu,i}^{\nu,i}(\omega_X^k) \rightarrow Q_{\nu,i}^{\nu,i}(\omega_X^{k+1})$$

and we get the following cosimplicial complex of Artin-Rees pro-objects,

$$\{Q_{\nu,i}^{\nu,i}(\omega_X)\}_i \rightarrow \{Q_{\nu,i}^{\nu,i}(\omega_X^2)\}_i \rightarrow \ldots \rightarrow \{Q_{\nu,i}^{\nu,i}(\omega_X^k)\}_i \rightarrow \ldots$$

(11)

The cosimplicial complex of Artin-Rees pro-objects $\{Q_{\nu,i}^{\nu,i}(\omega_X)\}_i$ is a resolution of $\omega_X$ (Čech resolution). Indeed, we consider the double complex of $O_X$-pro-modules

$$\omega_X \xrightarrow{d_0} \{Q_{\nu,i}^{\nu,i}(\omega_X)\}_i \xrightarrow{d_1,d_0} \{Q_{\nu,i}^{\nu,i}(\omega_X)\}_i \xrightarrow{d_2,d_1+d_0} \ldots \rightarrow \{Q_{\nu,i}^{\nu,i}(\omega_X)\}_i \rightarrow \ldots$$

(12)

where the maps are obtained from the cosimplicial maps [7] (with respect to the cosimplicial index $\nu$), by “forgetting one face” ([5, p. 12]). Then, one can show that [11, p. 347] is locally homotopic to zero, by using the degenerations maps of the cosimplicial complex [7] ([2, §V, Lemma 2.2.1]). Now, we apply to [13] the additive functor $\{O_X/\mathcal{I}^n\}_{n \in \mathbb{N}} \otimes O_X (-)$, in the category of pro-coherent $O_X$-modules. Since it respects the local homotopies, we find that the complex

$$\{O_X/\mathcal{I}^n\}_n \otimes \omega_X \xrightarrow{d_0} \{O_X/\mathcal{I}^n\}_n \otimes \{Q_{\nu,i}^{\nu,i}(\omega_X)\}_i \xrightarrow{d_1,d_0} \{O_X/\mathcal{I}^n\}_n \otimes \{Q_{\nu,i}^{\nu,i}(\omega_X)\}_i \xrightarrow{d_2,d_1+d_0} \ldots$$

(13)

is also locally homotopic to zero.

We give now a sort of “Log Poincaré Lemma” in characteristic zero.

**Theorem 1.1** The complex of Artin-Rees $O_X$-pro modules

$$[O_X \xrightarrow{d_0} \{L_X(\omega_X)\}_i] = [O_X \xrightarrow{d_0} \{\mathcal{P}_{X,i}^{1,i}(\mathcal{N})\}_i \rightarrow \{\mathcal{P}_{X,i}^{1,i}(\mathcal{N})\}_i \otimes O_X \omega_X^1 \rightarrow \{\mathcal{P}_{X,i}^{1,i}(\mathcal{N})\}_i \otimes O_X \omega_X^2 \rightarrow \cdots]$$

(14)

is locally homotopic to zero.

**Proof.** From (13), it follows that the composition $O_X \rightarrow \{L_X(\omega_X)\}_i$ is zero, so [13] is in fact a complex of $O_X$-pro modules. It is easy to show that the complexes

$$[O_X \xrightarrow{d_0} \mathcal{P}_{X,i}^{1,i}(\mathcal{N}) \xrightarrow{L_X(d_0)} \mathcal{P}_{X,i}^{1,i}(\mathcal{N}) \otimes O_X \omega_X^1 \xrightarrow{L_X(d_1)} \mathcal{P}_{X,i}^{1,i-1}(\mathcal{N}) \otimes O_X \omega_X^2 \xrightarrow{L_X(d_2)} \ldots]$$

are locally homotopic to zero, for each $i \in \mathbb{N}$. Indeed, from Proposition 0.6, we can define the following maps on the local basis,

$$\mathcal{P}_{X,i}^{1,i-1} \otimes O_X \omega_X^1 \xrightarrow{s_p} \mathcal{P}_{X,i}^{1,i} \otimes O_X \omega_X^p$$

$$\begin{array}{c}
\xi_{i-1} \otimes \ldots \otimes \xi_{i-1} \otimes \xi_i \otimes \ldots \otimes \xi_p \\
\oplus \xi_{i-1} \otimes \ldots \otimes \xi_{i-1} \otimes \xi_i \otimes \ldots \otimes \xi_p
\end{array}$$

where $0 \neq \alpha_1 + \ldots + \alpha_r = k \leq i - p$, and extend them by linearity. It is easy to compute that $s_0 \circ d_0 = id$, and $L_X(d_0)_{i-1} \circ s_p + s_{p+1} \otimes L_X(d_0)_{i-1} = id$, for each $p \geq 0$ (see also [8, Theorem 2.2] for an alternative proof, in the classical case).
Now, since the additive functor \( \{ \mathcal{O}_X/\mathcal{I} \}_n \otimes_{\mathcal{O}_X} \mathcal{O}_X(-) \) respects the local homotopies, by Theorem\(^\text{[11]}\) the following complex of Artin-Rees \( \mathcal{O}_X\)-pro modules

\[
\{ \mathcal{O}_X/\mathcal{I} \}_n \xrightarrow{d_0} \{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \{ \mathcal{P}_{\mathcal{I}, \log}^{1,i} \}_i \longrightarrow \{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \{ \mathcal{P}_{\mathcal{I}, \log}^{1,i} \}_i \otimes_{\mathcal{O}_X} \mathcal{O}_X \longrightarrow \ldots
\]  

is also locally homotopic to zero, in the category of pro-coherent \( \mathcal{O}_X \)-modules.
We consider now the following double complex (**)

\[
\begin{array}{c}
\{ \mathcal{O}_X/\mathcal{I} \}_n \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \{ \mathcal{P}_{\mathcal{I}, \log}^{1,i} \}_i \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \{ \mathcal{P}_{\mathcal{I}, \log}^{1,i} \}_i \otimes \mathcal{O}_X \otimes \mathcal{O}_X \longrightarrow \ldots
\end{array}
\]

Now, from (14), all the columns of (**) except the first, are locally homotopic to zero. Moreover, from (17), the second row of (**) is also locally homotopic to zero. The \((\nu + 1)\)-th row of this double complex \((\nu \geq 2)\) is obtained from the second row by tensorizing \((\mathcal{O}_X)\) with the \((\mathcal{I}\) stratified Artin-Rees pro-object \( \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \). Indeed, for each \( \nu \geq 0 \), \( \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \cong \mathcal{P}_{\mathcal{I}, \log}^{\nu-1,i} \otimes_{\mathcal{O}_X} \mathcal{O}_X \). So, since the second row is locally homotopic to zero and the additive functor \( \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \otimes_{\mathcal{O}_X} \mathcal{O}_X \) respects the local homotopies, we see that each row of (**), except the first, is also locally homotopic to zero. Therefore, we can conclude that the double complex \( \{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \mathcal{O}_X \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \) is a resolution of both the first column \( \{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \), and the first row \( \{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \mathcal{O}_X \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \) of (**) Then, since all pro-systems satisfy the Mittag-Leffler condition, we get the two following isomorphisms in cohomology,

\[
H(Y, \lim \lim \lim \mathcal{O}_X/\mathcal{I} \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \otimes \mathcal{P}^i \otimes \mathcal{O}_X \otimes \mathcal{O}_X) \cong H(Y, \lim \lim \lim \mathcal{O}_X/\mathcal{I} \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \otimes \mathcal{O}_X \otimes \mathcal{O}_X) \]

\[
H_{D, \log}((Y, N)/C) := H(Y, \lim \lim \lim \mathcal{O}_X/\mathcal{I} \otimes \mathcal{O}_X \otimes \mathcal{O}_X) \cong H(Y, \lim \lim \lim \mathcal{O}_X/\mathcal{I} \otimes \mathcal{O}_X \otimes \mathcal{O}_X) \]

\[
(18)
\]

\[
(19)
\]

Remark 1.2 Since the Artin-Rees \( \mathcal{O}_X \)-pro module \( \{ \mathcal{O}_X/\mathcal{I} \}_n \) is endowed with a log stratification (see Remark\(^\text{[05]}\)), we have isomorphisms, for any \( \nu, k \geq 0 \),

\[
\{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \cong \{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \mathcal{O}_X \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \]

and so there is an identification

\[
\{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \mathcal{O}_X \otimes \mathcal{O}_X \cong \{ \mathcal{O}_X/\mathcal{I} \}_n \otimes \mathcal{O}_X \otimes \mathcal{O}_X \]

In order to calculate \( H(Y^\log_{inf}, \mathcal{O}^\log_{inf}) \), from the exact closed immersion \( Y \rightarrow X \) one can define the sheaf \( \hat{Y} := \lim \hat{Y}_n \) on \( \text{InfLog} \) \((Y_n \) being the \( n \)-th log infinitesimal neighborhood of \( Y \) in \( X \)), which covers the final object of \( Y^\log_{inf} \) (III \( \S 5.2 \)). If \( Y^\log_{inf} \) is the log formal completion of \( X^\nu \) along its closed log subscheme \( Y \rightarrow X \rightarrow X^\nu \), we can introduce the (Čech) cosimplicial sheaf on \( Y \) equal to \( \mathcal{O}_X(\mathcal{X}^\nu) \) (this is denoted by \( \mathcal{F}^\nu \) in III \( \S 338 \), when \( F = \mathcal{O}^\log_{inf} \)). Then, by means of \( \hat{Y} \) and this Čech cosimplicial sheaf, we have that \( H(Y^\log_{inf}, \mathcal{O}^\log_{inf}) \cong H(Y^\log_{inf}, \mathcal{O}^\log_{inf}) \) (III \( \S 5.1 \) and p. 339). Finally, since \( \lim \lim \mathcal{O}_X/\mathcal{I} \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \), as cosimplicial sheaf on \( Y \), is equal to \( \mathcal{O}_X(\mathcal{X}^\nu) \),

\[
H(Y, \lim \lim \mathcal{O}_X/\mathcal{I} \otimes \mathcal{P}_{\mathcal{I}, \log}^{\nu,i} \cong H(Y^\log_{inf}, \mathcal{O}^\log_{inf}) \]

\[
(20)
\]
From (18), (19) and (20), we conclude that there exists an isomorphism between the log De Rham cohomology of $Y$ and its log Infinitesimal cohomology, namely

$$H_{DR, log}((Y, N)/\mathbb{C}) \cong H^i_{\log \inf}((Y, N)/\mathbb{C}, \mathcal{O}_{Y, \log}^{\inf})$$ (21)

**Remark 1.3** Forgetting log structure, one could use analogous techniques in the classical setting (which are nothing but a miscellanea of those ones indicated in (11)) and obtain the isomorphism (21) for any scheme $Y$ over $S$ (without log). This fact, together with the result proved by Hartshorne in [13 Chapter IV, Theorem I.1], gives the isomorphisms between (1), (2) and (3) in the Introduction.

## 2 The Complex $\omega^{\cdot, \log}_{X|Y}$

With $X$ and $Y$ as in §1, let $Y_{an}$, $X_{an}$ be the associated fs log analytic spaces, and let $i_{an}: Y_{an} \hookrightarrow X_{an}$ be the corresponding analytic exact closed immersion. When the context obviates any confusion, we will omit the superscript $(-)_{an}$ in denoting the associated analytic spaces.

We consider the closed analytic subspaces $Y_k$ of $X$, defined by the ideals $\mathcal{I}^k$, with $k \in \mathbb{N}$. On each such $Y_k$ we consider the log structure induced by $M_X$, i.e., if $i_k: Y_k \hookrightarrow X$ is the closed immersion, then we take $M_{Y_k} = i_k^* M_X$. We have a sequence of exact closed immersions, which we denote by $\varphi_k$,

$$Y = Y_1 \hookrightarrow Y_2 \hookrightarrow Y_3 \hookrightarrow \cdots \hookrightarrow Y_k \hookrightarrow \cdots \hookrightarrow X$$

Therefore, we have a projective system of rings $\{\mathcal{O}_{Y_k} \cong i_k^{-1}(\mathcal{O}_X/\mathcal{I}^k); \varphi_k: \mathcal{O}_{Y_{k+1}} \to \mathcal{O}_{Y_k}\}_{k \geq 1}$, where the transition maps $\varphi_k$ are surjective. Moreover, the diagram

$$\varphi_k^{-1}(\mathcal{O}_{Y_{k+1}}) \to M_{Y_{k+1}} \quad \varphi_k^{-1}(\mathcal{O}_{Y_{k+1}}) \to \mathcal{O}_{Y_k}$$

is commutative, for each $k \geq 1$. Since $M_X$ is a fine log structure on $X$, each $Y_k$ is endowed with a fine log structure $i_k^* M_X$, which we denote more by $M_k$.

The closed immersion $i_k: Y_k \hookrightarrow X$ is exact, for each $k$, so ([13 (1.4.1)])

$$M_k/\mathcal{O}_{Y_k} = i_k^* M_X/\mathcal{O}_{Y_k} \cong i_k^{-1}(M_X/\mathcal{O}_X) = (M_X/\mathcal{O}_X)|_{Y_k}$$ (23)

**Remark 2.1** Since the underlying topological space of each $Y_k$ is equal to $Y$, it follows from (23) that $M_k/\mathcal{O}_{Y_k} \cong (M_X/\mathcal{O}_X)|_{Y_k}$, for each $k \geq 1$. Therefore, if we consider the associated sheaf of groups $(M_k/\mathcal{O}_{Y_k})^{gp} \cong M_k^{gp}/\mathcal{O}_{Y_k}^{*}$, we have that, for each $k \geq 1$,

$$M_k^{gp}/\mathcal{O}_{Y_k}^{*} \cong (M_X/\mathcal{O}_X)^{gp}|_{Y_k}$$ (24)

Let $Y^{\log}$ (resp. $X^{\log}$) be the Kato-Nakayama topological space associated to $Y_k$ (resp. to $X$), and let $\tau_k: Y^{\log} \to Y_k$ (resp. $\tau_X: X^{\log} \to X$) be the corresponding surjective, continuous and proper map of topological spaces ([0.3.2]). We now consider the “formal” analytic space $X|_{Y}$, which is $Y_{an}$ as topological space, and whose structural sheaf is

$$\mathcal{O}_{X|Y} =: \lim_{\leftarrow k} \mathcal{O}_{Y_k} \cong \lim_{\leftarrow k} i_k^{-1}(\mathcal{O}_X/\mathcal{I}^k)$$

Now, since the closed immersion $i: Y \hookrightarrow X$ is exact, the formal completion of the fs log analytic space $X$ along the closed log subspace $Y$ is equal to the classical completion $X|_{Y}$, endowed with the log structure induced by $M_X$ (Definition [0.7]). So, if $i_{X|Y}: X|_{Y} \hookrightarrow X$, then the log structure on $X|_{Y}$ is $i_{X|Y}^* M_X$. We denote it by $M_{X|Y}$. We now define a ringed topological space $((X|_{Y})^{log}, \mathcal{O}_{X|Y}^{log})$, associated to the formal fine log analytic space $X|_{Y}$.
Lemma 2.3 [21, Lemma (3.3)] With the previous notation, let \( i \) for \( x \).

By [21, Lemma (3.3)], applied to \( X \),

Proof. By [21, Lemma (3.3)], applied to \( X \), the isomorphism \( O_{X,y}^{\log} \cong \tau_{X}(O_{X})_{y}[T_{1}, \ldots, T_{n}] \) implies that \( O_{(X|Y),y}^{\log} \cong \tau_{y}(O_{X|Y})_{y} \otimes \tau_{X}(O_{X})_{y} \tau_{X}(O_{X})_{y}[T_{1}, \ldots, T_{n}] \cong \tau_{X}(O_{X|Y})_{y}[T_{1}, \ldots, T_{n}] \cong O_{(X|Y),x}[T_{1}, \ldots, T_{n}] \).

Lemma 2.4 [21, Lemma (3.4)] Let \( r \in \mathbb{Z} \). We define a filtration \( \hat{\text{fil}}_{r}(O_{X|Y}^{\log}) \) on \( O_{X|Y}^{\log} \) by

\[
\hat{\text{fil}}_{r}(O_{X|Y}^{\log}) = \tau_{X}(O_{X|Y}) \otimes \tau_{X}(O_{X}) \text{fil}_{r}(O_{X}^{\log})
\] (27)

(\text{where } \hat{\text{fil}}_{r}(O_{X}^{\log}) \text{ is defined by Kato-Nakayama as } \text{Im}(\tau_{X}(O_{X}) \otimes \mathbb{Z} (\bigoplus_{j=1}^{\infty} \text{Sym}^{j}_{X|Y}) \rightarrow O_{X|Y}^{\log})). Then, the canonical map

\[
\tau_{X}(O_{X|Y}) \otimes \tau_{X}(O_{X}) \subset \text{fil}_{1}(O_{X}^{\log})/\text{fil}_{0}(O_{X}^{\log})
\]

induces the following isomorphism

\[
\tau_{X}(O_{X|Y}) \otimes \mathbb{Z} \tau_{X}(\text{Sym}^{r}_{X}(O_{X}^{\log})) \cong \hat{\text{fil}}_{r}(O_{X|Y}^{\log})/\hat{\text{fil}}_{r-1}(O_{X|Y}^{\log})
\] (28)

Proof. By [21, Lemma (3.4)], for any \( r \geq 0 \), we have an isomorphism

\[
\tau_{X}(O_{X|Y}) \otimes \mathbb{Z} \tau_{X}(\text{Sym}^{r}_{X}(O_{X}^{\log})) \cong \hat{\text{fil}}_{r}(O_{X|Y}^{\log})/\hat{\text{fil}}_{r-1}(O_{X|Y}^{\log})
\]

(29)

So,

\[
\tau_{X}(O_{X|Y}) \otimes \tau_{X}(O_{X}) \tau_{X}(\text{Sym}^{r}_{X}(O_{X}^{\log})) \cong \hat{\text{fil}}_{r}(O_{X|Y}^{\log})/\hat{\text{fil}}_{r-1}(O_{X|Y}^{\log})
\]

and, by (29), this is isomorphic to

\[
\tau_{X}(O_{X|Y}) \otimes \tau_{X}(O_{X}) \text{fil}_{r}(O_{X}^{\log})/\text{fil}_{r-1}(O_{X}^{\log})
\]

(30)

Now, since the functor \( \tau_{X}(O_{X|Y}) \otimes \tau_{X}(O_{X}) \text{fil}_{r}(O_{X}^{\log})/\text{fil}_{r-1}(O_{X}^{\log}) \) is right exact, it follows that (30) is isomorphic to \( \hat{\text{fil}}_{r}(O_{X|Y}^{\log})/\hat{\text{fil}}_{r-1}(O_{X|Y}^{\log}) \).

Using the Kato-Nakayama complex \( \omega_{X}^{\log} \) associated to the fs log smooth log analytic space \( X \) over \( S \) ([21 (3.5)]), we can now give the following.
Definition 2.5 In the previous notation, for any $q \in \mathbb{N}$, $0 \leq q \leq \text{rk}_X \omega_X^1$, we define the following sheaf on $Y^{\log}$

$$\omega_{X|Y}^{q, \log} : = \mathcal{O}_{X|Y}^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega_X^q)$$

(31)

Since $X$ is log smooth over $S$, it follows that $\omega_X^1$ is a locally free $\mathcal{O}_X$-module of finite type, and so $\omega_{X|Y}^{q, \log}$ is a locally free $\mathcal{O}_{X|Y}$-module of finite type. Moreover, since $\omega_X^{q, \log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega_X^q)$ (21 (3.5)), using the definition of $\mathcal{O}_{X|Y}^{\log}$, we can also write

$$\omega_{X|Y}^{q, \log} = \tau^{-1}(\mathcal{O}_{X|Y}^{\log}) \otimes_{\tau^{-1}(\mathcal{O}_X)} \omega_X^{q, \log}$$

(32)

Now, since the differential $d^q : \omega_X^{q, \log} \to \omega_X^{q+1, \log}$ is induced by that of $\omega_X$, we have that $d^1(\mathcal{G}^r) \subseteq \mathcal{G}^{r-1}\omega_X^1$, for any $r \geq 0$. Thus, $\hat{d}^i$ can be extended to get a differential

$$\hat{d}^i : \omega_{X|Y}^{q, \log} \to \omega_{X|Y}^{q+1, \log}$$

(33)

Therefore, we obtain a complex $\omega_{X|Y}^{\log}$, whose differentials $\hat{d}^i$ satisfy $\hat{d}^1(x) = d\log(\exp(x))$, for each element $x \in \mathcal{L}_X$, and $\hat{d}^i(y) \in \mathcal{G}^{r-1}\omega_X^1$, for each element $y \in \mathcal{G}^r$, $r \geq 0$.

3 Formal Poincaré Residue Map

In this section, we want to give a “formal version” of the Poincaré Residue map given by Deligne (3.1.5.2)). We consider an fs log scheme $Y$, with log structure $M_Y$, and an exact closed immersion $i : Y \to X$, where $X$ is an fs log smooth log scheme, with log structure $M_X$. We also suppose that the underlying scheme of $X$ is smooth over $S$, and its log structure $M_X$ is given by a normal crossing divisor $D \to X$, i.e. $M_X = j_*\mathcal{O}_U \cap \mathcal{O}_X \to \mathcal{O}_X$, where $j : U = X - D \to X$ is the open immersion. Let $X_{an}, Y_{an}$ be the log analytic spaces associated to $X$ and $Y$, which we will simply denote by $X, Y$, when no confusion can arise.

We take the log De Rham complex $\omega_X = \Omega_X(\log M_X) = \Omega_X(\log D)$. Its completion $\omega_{X|Y}$, along the closed subscheme $Y$ of $X$, satisfies

$$\omega_{X|Y}^i \cong \omega_X^i \otimes_{\mathcal{O}_X} \mathcal{O}_{X|Y}$$

for each $i$, $0 \leq i \leq n = \dim X$, because the $\mathcal{O}_X$-modules $\omega_X^i$ are locally free, and so coherent.

We denote by $d_i^\hat{}$ the differential $\omega_{X|Y}^i \to \omega_{X|Y}^{i+1}$ of the complex $\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X|Y}$. We consider the weight filtration $W$ on $\omega_X$ (3.1.5.1)): since each term $W_k(\omega_X^i) = \Omega_X^{i-k} \wedge \omega_X^k$ is a locally free $\mathcal{O}_X$-module, this filtration induces an increasing filtration $\hat{W}$ on $\omega_{X|Y}$, defined by

$$\hat{W}_k(\omega_{X|Y}^i) = : W_k(\omega_X^i) \otimes_{\mathcal{O}_X} \mathcal{O}_{X|Y}$$

for each $0 \leq k \leq i \leq n$.

Since $\hat{W}_k(\omega_{X|Y}^i) = \text{Im}\{\omega_X^i \otimes \mathcal{O}_X \Omega_X^{i-k} \otimes \mathcal{O}_X \mathcal{O}_{X|Y} \wedge \otimes d_i^\hat{} \wedge \omega_X^k \otimes \mathcal{O}_{X|Y}\}$, we can write the term $\hat{W}_k$ of the filtration as $\omega_X^i \wedge \Omega_X^{i-k}$, where $\omega_{X|Y}^i$ is the completion of the classical De Rham complex $\Omega_X$ along $Y$.

Moreover, we note that $\hat{W}_k(\omega_{X|Y}^i)$ is a locally free $\mathcal{O}_{X|Y}$-submodule of $\omega_{X|Y}^i$, for each $i$.

We suppose now that, locally at a point $x \in Y \to X$, the normal crossing divisor $D$ is the union of smooth irreducible components $D = D_1 \cup ... \cup D_r$, where each component $D_i$ is locally defined by the equation $z_i = 0$ (for a local coordinate system $\{z_1, ..., z_n\}$ of $X$ at $x$). Let $S^k$ be the set of strictly increasing sequences of indices $\sigma = (\sigma_1, ..., \sigma_k)$, where $\sigma_i \in \{1, ..., r\}$, and let $D_\sigma = D_{\sigma_1} \cap ... \cap D_{\sigma_k}$. Let $D_k = \bigcup_{\sigma \in S^k} D_\sigma$ and $D^k$ be the disjoint union $\coprod_{\sigma \in S^k} D_\sigma$. Moreover, let $\pi^k : D^k \to X$ be the canonical map. Then, locally at $x$, the $\mathcal{O}_{X|Y}$-submodule $\hat{W}_k(\omega_{X|Y}^i)$ of $\omega_{X|Y}^i$ can be written as

$$\hat{W}_k(\omega_{X|Y}^i) = \sum_{\sigma \in S^k} \Omega_{X|Y}^{i-k} \wedge \text{dlog}z_{\sigma_1} \wedge ... \wedge \text{dlog}z_{\sigma_k}$$
for each $0 \leq i \leq n$. Therefore, the elements of $\hat{W}_k(\omega^i_{X|Y})$ are locally linear combinations of terms $\eta \wedge \text{dlog} z_{\sigma_1} \wedge ... \wedge \text{dlog} z_{\sigma_k}$, with $\eta \in \Omega^{i-k}_{X|Y}$.

Let $Y_k = D_k \cap Y$, and $Y^k = \coprod_{\sigma \in S_k} (Y_\sigma)$, with $Y_\sigma = D_\sigma \cap Y$. We have the following cartesian diagram

$$
\begin{array}{c}
Y^k \\ \pi_Y \downarrow \\
Y \\ \downarrow \\
X
\end{array}
$$

(34)

Since each intersection $D_\sigma$ is smooth over $S$, we can take the sheaf of classical differential $i$–forms $\Omega^i_{D_k}$ over $D^k$; then, $\pi^k_*(\Omega^i_{D_k}) \cong \bigoplus_{\sigma \in S_k} (i_\sigma^* \Omega^i_{D_\sigma})$, where $i_\sigma : D_\sigma \hookrightarrow X$.

So, $(\pi^k_*(\Omega^i_{D_k}))|_Y \cong (\bigoplus_{\sigma \in S_k} (i_\sigma^* \Omega^i_{D_\sigma}))|_Y \cong \bigoplus_{\sigma \in S_k} (i_\sigma^* \Omega^i_{D_\sigma}|_Y)$.

From the cartesian diagram

$$
\begin{array}{c}
Y_\sigma \\ \downarrow \\
Y \\ \downarrow \\
X
\end{array}
$$

(35)

we deduce the map $i_\sigma : D_\sigma|_Y \hookrightarrow X|_Y$. From [12] (Corollaire (10.14.7)), it follows that

$$(i_\sigma^* \Omega^i_{D_\sigma}|_Y) \cong \hat{i}_\sigma^* (\Omega^i_{D_\sigma|_Y})$$

and then

$$(\pi^k_*(\Omega^i_{D_k}))|_Y \cong \bigoplus_{\sigma \in S_k} (\hat{i}_\sigma^* (\Omega^i_{D_\sigma|_Y}))$$

(36)

### 3.1 The Formal Poincaré Residue

In [6] (3.1.5.2), Deligne defines a map of complexes

$$\text{Res} : \text{Gr}_k^W (\Omega^*_X(\log D)) \rightarrow \pi^k_*(\Omega^i_{D_k}(\epsilon^k)[-k])$$

(37)

for each $k \leq n$, called the Poincaré Residue map, where $\epsilon^k$ is defined as in [6] (3.1.4), and represents the orientations of the intersections $D_\sigma$ of $k$ components of $D$. Given a local section $\eta \wedge \text{dlog} z_{\sigma_1} \wedge ... \wedge \text{dlog} z_{\sigma_k} \in \text{Gr}_k^W (\Omega^p_X(\log D))$, with $\eta \in \Omega^p_{X|Y}$, the map Res sends it to $\eta|_{D_\sigma} \otimes (\text{orientation } \sigma_1...\sigma_k)$.

Deligne proved that Res is an isomorphism of complexes ([6]). Moreover, from [6] (3.1.8.2), the following sequence of isomorphisms

$$\mathbb{R}^k j_* C \cong \mathcal{H}^k (j_* \Omega^p_{X}) \cong \mathcal{H}^k (\Omega^*_X(\log D)) \cong \epsilon^k_X$$

implies that there exists an identification

$$\epsilon^k_X \cong C \otimes \bigwedge^k M_X^{\text{op}} / \mathcal{O}_X^*$$

(38)

($M_X$ is the log structure on $X$ associated to the normal crossing divisor $D$, and $\epsilon^k_X$ is the direct image of $\epsilon^k$ via the map $\pi^k : D^k \rightarrow X$ [6] (3.1.4.1)). Using diagram (34), and (36), we can extend the Deligne Poincaré Residue map to the formal case.

**Definition 3.1** In the previous notation, we define the map

$$\text{Res}^p : \text{Gr}_k^W (\omega^p_{X|Y}) \cong \text{Gr}_k^W (\omega^p_X \otimes \mathcal{O}_X |_Y) \rightarrow \pi^k_*(\Omega^{p-k}_{D|Y}(\epsilon^k))$$

(39)

as

$$\eta \wedge \text{dlog} z_{\sigma_1} \wedge ... \wedge \text{dlog} z_{\sigma_k} \longmapsto \eta|_{(D_\sigma|_Y)} \otimes (\text{orientation } \sigma_1...\sigma_k)$$

where $\eta \in \Omega^{p-k}_{X|Y}$. 

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We consider the completion $\hat{\text{Res}}^p$ of the Deligne Poincaré Residue map \([37]\), in degree $p$, along the closed subscheme $Y$. We will prove that the maps $\hat{\text{Res}}^p$ induce the following $\mathcal{O}_X$–linear isomorphism of complexes

$$
\text{Res} : \text{Gr}_k^W(\omega^p_{X\mid Y}) \xrightarrow{\sim} \hat{\pi}_*^k\Omega_{D^k\mid Y^k}(\varepsilon^k)[-k]
$$

(40)
for each $k \leq n$.

To this end, we briefly recall the classical construction of the Deligne Poincaré Residue map. So, given $\sigma \in S^k$, we consider the application

$$
\rho_\sigma : \Omega^{p-k}_X \longrightarrow \text{Gr}_k^W(\omega^p_X)
$$

which is locally defined by

$$
\rho_\sigma(\eta) =: \eta \wedge \text{dlog} z_{\sigma_1} \wedge \ldots \wedge \text{dlog} z_{\sigma_k}
$$

(42)

This map does not depend on the choice of the local coordinates $z_i$ \([7, 3.6.6]\). Moreover we have that

$$
\rho_\sigma(z_{\sigma_i} \cdot \beta) = 0 \quad \text{and} \quad \rho_\sigma(dz_{\sigma_i} \wedge \gamma) = 0
$$

(43)

for all sections $\beta \in \Omega^{p-k}_X$, and $\gamma \in \Omega^{p-1-k}_X$. Therefore $\rho_\sigma$ factorizes into

$$
\begin{align*}
\Omega^{p-k}_X & \xrightarrow{\rho_\sigma} i_{\sigma*}\Omega^{p-k}_{D^k} \otimes (\text{orientation } \sigma_1 \ldots \sigma_k) \\
\text{Gr}_k^W(\omega^p_X) & \xrightarrow{\sqrt{\varphi_\sigma}} 
\end{align*}
$$

Thus, all these maps being locally compatible with the differentials, the maps $\overline{\rho}_\sigma$ define a morphism of complexes

$$
\varphi : \pi_*^{k}\Omega_{D^k}(\varepsilon^k)[-k] \longrightarrow \text{Gr}_k^W(\omega^p_X)
$$

(44)

This morphism is locally defined by \([42]\), and it is a global morphism on $X$: it is an isomorphism of complexes. Its inverse isomorphism $\text{Gr}_k^W(\omega^p_X) \longrightarrow \pi_*^{k}\Omega_{D^k}(\varepsilon^k)[-k]$ is the Deligne Poincaré Residue map $\text{Res}^*$ \([7, (3.6.7.1)]\). In view of this construction, we can see that $\text{Res}^*$ is an $\mathcal{O}_X$–linear morphism of complexes: so we deduce that the maps $\hat{\text{Res}}^p$ in \([39]\), are compatible with the differentials induced from $\omega^p_{X\mid Y}$, and $\Omega^{p-k}_{D^k\mid Y^k}[-k]$, because $\hat{\text{Res}}^p$ comes from the $\mathcal{O}_X$–linear map $\text{Res}^p$ by completion along $Y$.

Indeed, we note that

$$
\text{Gr}_k^W(\omega^p_{X\mid Y}) \cong \text{Gr}_k^W(\omega^p_X) \otimes_{\mathcal{O}_X} \mathcal{O}_{X\mid Y}
$$

and, from \([46]\), we have that

$$
(\pi_*^{k}\Omega_{D^k}(\varepsilon^k)[-k]_{\mid Y}) \cong \bigoplus_{\sigma \in S^k} i_{\sigma*}(\Omega_{D^k\mid Y^k})(\varepsilon^k)[-k] \cong \hat{\pi}_*^{k}\Omega_{D^k\mid Y^k}(\varepsilon^k)[-k]
$$

So we conclude that the morphism of complexes $\text{Res}^*$ \([10]\) is an isomorphism, for each $k \leq n$.

**Remark 3.2.** We can also construct the morphism $\text{Res}^*$ using a formal version of the classical construction of $\text{Res}^*$, described in \([11], [42], [13], [44]\). Indeed, we can define the map

$$
\rho_{\sigma\mid Y} : \Omega^{p-k}_{X\mid Y} \longrightarrow \text{Gr}_k^W(\omega^p_{X\mid Y})
$$

(45)

which is the completion along $Y$ of \([11]\), and is locally defined as in \([42]\), but with $\eta \in \Omega^{p-k}_{X\mid Y}$. Then, we can see that this map $\rho_{\sigma\mid Y}$ factorizes into

$$
\begin{align*}
\Omega^{p-k}_{X\mid Y} & \xrightarrow{\rho_{\sigma\mid Y}} i_{\sigma*}\Omega^{p-k}_{D^k\mid Y} \otimes (\text{orientation } \sigma_1 \ldots \sigma_k) \\
\text{Gr}_k^W(\omega^p_{X\mid Y}) & \xrightarrow{\sqrt{\varphi_{\sigma\mid Y}}} 
\end{align*}
$$

(46)

which is the formal analogue of \([13]\). We conclude that there exists an isomorphism of complexes on $X$

$$
\overline{\varphi}_{\mid Y} : \pi_*^{k}\Omega_{D^k\mid Y^k}(\varepsilon^k)[-k] \longrightarrow \text{Gr}_k^W(\omega^p_{X\mid Y})
$$

(47)

whose inverse isomorphism is exactly $\text{Res}^*$.
3.2 Cohomology of $\omega_{X|Y}$

From the isomorphism $\hat{\text{Res}}^*$, and from (38), we deduce that, when $X$ is smooth over $S$, with log structure given by a normal crossing divisor $D$ on $X$, and $Y$ is a closed subscheme of $X$, then

\[ \mathcal{H}^q(\text{Gr}^W_k(\omega_{X|Y})) \cong \mathbb{C}_{Y^k} \otimes_{\mathbb{C}} \mathcal{E}_X^k \cong \mathbb{C}_{Y^k} \otimes_{\mathbb{Z}} \bigwedge^k M_X^p / \mathcal{O}_X^* \quad \text{if} \quad q = k \]  

(48)

and

\[ \mathcal{H}^q(\text{Gr}^W_k(\omega_{X|Y})) = 0 \quad \text{if} \quad q \neq k \]  

(49)

Therefore, we deduce that, for each point $x \in Y \cap D$, there exists an isomorphism

\[ \mathcal{H}^q(\omega_{X|Y})_x \cong \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^q (M_X^p / \mathcal{O}_X)_x \]  

(50)

4 Formal Log Poincaré Lemma

In this section, we generalize the logarithmic version of the Poincaré Lemma, proved by Kato-Nakayama ([21, Theorem (3.8)]) in the case of a (ideally) log smooth log analytic space (i.e. a log analytic space satisfying the assumption (0.4) in [21]). We extend this result to the case of a generic fs log analytic space over $S$, and prove the following

**Theorem 4.1** Let $i : Y \hookrightarrow X$ be like in (2) and let $\omega^{\text{log}}_{X|Y}$ be the complex introduced in Definition 2.5 with differential maps given by (33). Then, there exists a quasi-isomorphism

\[ \mathbb{C}_{Y^{\text{log}}} \xrightarrow{\cong} \omega^{\text{log}}_{X|Y} \]  

(51)

To prove this theorem we first need some preliminary results. The methods of the proof are similar to those in [21]. Let $Y$ and $X$ be as in (2). Let $P \rightarrow M_X$ be a chart, with $P$ a toric (or simply fs) monoid. Let $p$ be a prime ideal of $P$ which is sent to $0 \in \mathcal{O}_X$ under $P \rightarrow M_X \rightarrow \mathcal{O}_X$. Let $T$ be the fs log analytic space whose underlying space is the same as that of $X$ but whose log structure $M_T$ is associated to $P \smallsetminus p \rightarrow \mathcal{O}_T$. Similarly, let $Z$ be the closed log subspace of $T$ whose underlying space is the same as that of $Y$ and whose log structure is the inverse image of $M_T$. We have the following commutative diagram of fine log analytic spaces

\[ \begin{array}{ccc}
(Y, i^* M_X) & \xrightarrow{i} & (X, M_X) \\
\downarrow & & \downarrow \\
(Z, i_T^* M_T) & \xleftarrow{i_T} & (T, M_T)
\end{array} \]  

(52)

where the vertical maps are the identity over the underlying analytic spaces. We also note that, since the closed immersions $i$ and $i_T$ are both exact, the log formal analytic space $T|Z$ coincides with the classical formal analytic space $X|Y$ and so

\[ \omega^{T|Z} \cong \omega_T \otimes_{\mathcal{O}_T} \mathcal{O}_{X|Y} \]  

(53)

We introduce now a filtration on the complex $\omega_{X|Y}$. So, for $q, r \in \mathbb{Z}$, let $F_r^p \omega_X^q$ be the $\mathcal{O}_X$—subsheaf of $\omega_X^q$ defined by $F_r^p \omega_X^q = 0$, if $r < 0$; $F_r^p \omega_X^q = \text{Im} \{ \omega_X^q \otimes \omega_T^{q-r} \rightarrow \omega_X^q \}$, if $0 \leq r \leq q$; $F_r^p \omega_X^q = \omega_X^q$, if $q \leq r$ ([21, Fil I in Lemma (4.4)])

On the complex $\omega_{X|Y}$ we consider the induced filtration

\[ \bar{F}_r^p \omega_{X|Y} = F_r^p \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X|Y} \]
Lemma 4.2  [21 Lemma (4.4)] In the previous context,
(1) $\hat{\mathcal{F}}^p_0\omega_{X|Y}$ is a subcomplex of $\omega_{X|Y}$.
(2) For any $r \in \mathbb{Z}$, there is an isomorphism of complexes
\begin{equation}
\bigwedge^r (P^{op}/(P \setminus \mathfrak{p})^{op}) \otimes_{\mathbb{Z}} \omega_{T|Z}[-r] \overset{\cong}{\longrightarrow} \hat{\mathcal{F}}^p_0\omega_{X|Y}/\hat{\mathcal{F}}^p_{r-1}\omega_{X|Y}
\end{equation}
whose degree $q$ part is given by
\begin{equation}
(p_1 \wedge \ldots \wedge p_r) \otimes_{\mathbb{Z}} (\eta \otimes \mathcal{O}_X f) \longrightarrow \text{dlog}(p_1) \wedge \ldots \wedge \text{dlog}(p_r) \wedge (\eta \otimes \mathcal{O}_X f)
\end{equation}
where $p_1, ..., p_r \in P^{op}$, $\eta \otimes \mathcal{O}_X f \in \omega_{T|Z}^{\omega} \cong \omega_{T} \otimes \mathcal{O}_X \mathcal{O}_{X|Y}$. The differential of the left side is equal to
\begin{equation}
(id \otimes \hat{d}), \text{ where } \hat{d} \text{ is the differential of } \omega_{T|Z}.
\end{equation}

**Proof.** (2). By applying the functor $(-) \otimes \mathcal{O}_{X|Y}$ to the exact sequence of coherent sheaves 4.4.1 in [21 Lemma (4.4)], and using (53), we have
\begin{equation}
0 \longrightarrow \omega_{T|Z}^1 \longrightarrow \omega_{X|Y}^1 \longrightarrow \mathcal{O}_{X|Y} \otimes_{\mathbb{Z}} P^{op}/(P \setminus \mathfrak{p})^{op} \longrightarrow 0
\end{equation}

\[\blacksquare\]

Let $P$ be an fs (or toric) monoid and let $X$ be the log analytic space $\text{Spec } \mathbb{C}[P]$, endowed with log structure $P \longrightarrow \mathcal{O}_X$. Let $i: Y \hookrightarrow X$ be an exact closed immersion, where $Y$ is a fine log analytic space endowed with the induced log structure. We fix a point $x \in Y$. Since $i: Y \hookrightarrow X$ is exact, via the canonical isomorphism
\begin{equation}
\omega_{X|Y}^1 \cong \mathcal{C}[P]_{|Y} \otimes_{\mathbb{Z}} P^{op}
\end{equation}
the map $P^{op} \longrightarrow \omega_{X|Y}^1$, sending $p \in P^{op}$ to $\text{dlog } p$, corresponds to the map sending $p$ to $1 \otimes p$ ([22 §3]). The image of this map is contained in the closed $1$–forms. Therefore, we get a map
\begin{equation}
M^{\omega^{op}}_{X,x}/\mathcal{O}_{X,x}^* \cong P^{op} \longrightarrow \mathcal{H}^1(\omega_{X|Y}^1)
\end{equation}
and, by cup product, we deduce a map
\begin{equation}
\bigwedge^q (M^{\omega^{op}}_{X,x}/\mathcal{O}_{X,x}^*) \cong \bigwedge^q P^{op} \longrightarrow \mathcal{H}^q(\omega_{X|Y}^1)
\end{equation}

Let $\mathfrak{b}$ be the prime ideal of $P$ which is the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ under $P \longrightarrow \mathcal{O}_{X,x}$. We denote by $X(\mathfrak{b})$ the closed analytic subspace $\text{Spec } \mathbb{C}[P]/(\mathfrak{b})$ of $X$. The underlying analytic space of $X(\mathfrak{b})$ is equal to $\text{Spec } \mathbb{C}[P \setminus \mathfrak{b}]$, and $x$ belongs to its smooth open analytic subspace $\text{Spec } \mathbb{C}[P \setminus \mathfrak{b}]^{\omega}$, where the log structure is trivial. Let $Y(\mathfrak{b})$ be the fiber product
\begin{equation}
\begin{array}{c}
Y(\mathfrak{b}) \hookrightarrow X(\mathfrak{b}) \\
\downarrow \quad \downarrow \\
Y \hookrightarrow X
\end{array}
\end{equation}
which is a closed subspace of $X(\mathfrak{b})$. Moreover, let $(X|Y)(\mathfrak{b})$ be the completion $X(\mathfrak{b})|Y(\mathfrak{b})$ of $X(\mathfrak{b})$ along its closed subspace $Y(\mathfrak{b})$, and let $\omega^{\mathfrak{b}}_{X|Y}(\mathfrak{b})$ be the formal log complex of $(X|Y)(\mathfrak{b})$, with differential maps $\hat{d}_{\mathfrak{b}}$.

We denote by $(\mathcal{O}_{X|Y}(\mathfrak{b}))_{\hat{d}_{\mathfrak{b}}=0}$ the kernel of $\hat{d}_{\mathfrak{b}}: \mathcal{O}_{X|Y}(\mathfrak{b}) \longrightarrow \omega_{X|Y}(\mathfrak{b})$.

**Lemma 4.3  [21 Lemma (4.5)]** In the previous context,
(1) if we restrict to some open neighbourhood of $x$ in $X(\mathfrak{b})$ ($x \in Y$), there exists an isomorphism
\begin{equation}
\mathbb{C} \cong (\mathcal{O}_{X|Y}(\mathfrak{b}))_{\hat{d}_{\mathfrak{b}}=0}
\end{equation}
and a quasi-isomorphism
\[ \mathbb{C} \xrightarrow{\cong} (\omega_{(X|Y)(b)})_x \]  
\[ (58) \]

(2) Let \( b \omega_{X|Y} \) be the subcomplex of \( \omega_{X|Y} \) whose degree \( q \) part is defined to be the \( \mathcal{O}_X \)-subsheaf of \( \omega_{X|Y} \) generated by \( b \omega_{X|Y} \), with \( b \in \mathfrak{b} \). For any \( q \), the map
\[ \bigwedge^q (M_{X|Y}^p / \mathcal{O}_X^p)_x \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \mathcal{H}^q(\omega_{X|Y} / b \omega_{X|Y})_x \]  
\[ (59) \]
which is induced by the map \([55]\), is bijective.

(3) The stalk at \( x \) of the canonical map of complexes
\[ \omega_{X|Y} \rightarrow \omega_{X|Y} / b \omega_{X|Y} \]  
\[ (60) \]
is a quasi-isomorphism.

**Proof.** We first note that the complex \( \omega_{X|Y} / b \omega_{X|Y} \) is isomorphic to \( \omega_{(X|Y)(b)} \). Indeed, since \( \mathfrak{b} \) is an ideal of the monoid \( P \), by \([21]\) Lemma (3.6), (2), \( \omega_X^I / b \omega_X^I \cong \omega_{X(b)}^I \), and it follows that
\[ \omega_{X|Y} / b \omega_{X|Y} \cong \omega_X / b \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X|Y \cong \omega_{X|Y}(b) \otimes_{\mathcal{O}_X} \mathcal{O}_X|Y \cong \omega_{X|Y}(b) \]

We start to prove (1) and (2). We may restrict ourselves to the open neighbourhood \( \text{Spec} \mathbb{C}[(P \setminus \mathfrak{b})^0] \) of \( x \) in \( X(b) \), and consider the restriction of \( Y(b) \) to this open neighborhood. So, \( x \) belongs to the closed subspace \( Y(b) \cap \text{Spec} \mathbb{C}[(P \setminus \mathfrak{b})^0] \) of the non-singular analytic space \( \text{Spec} \mathbb{C}[(P \setminus \mathfrak{b})^0] \), where the log structure is trivial.

In this local situation, \( \omega_{(X|Y)(b)} \cong \Omega_{X(b)|Y(b)} \), and, from \([13]\) Theorem 2.1, we know that this complex is a resolution of the constant sheaf \( \mathbb{C}_Y(b) \) over \( Y(b) \). Therefore, the stalk at \( x \) of \( (\mathcal{O}_X|Y(b))^d = 0 \) is isomorphic to \( \mathbb{C} \), and there is a quasi-isomorphism
\[ \mathbb{C} \xrightarrow{\cong} (\Omega_{X(b)|Y(b)})_x \]
so (1) is proved. Now, we apply Lemma \([12]\) by taking \( X(b) \), \( P \), \( \mathfrak{b} \) as \( X \), \( P \) and \( \mathfrak{p} \). We consider \( \mathcal{H}^q \) of both sides of Lemma \([12]\) (2), and take the stalk at \( x \). Then, \( \mathcal{H}^q(\tilde{F}_{\mathfrak{b}}(\omega_{(X|Y)(b)}) / \tilde{F}_{\mathfrak{b}}^{-1}(\omega_{(X|Y)(b)}))_x \) is isomorphic to \( \bigwedge^r P_{\mathfrak{b}^0} / (P \setminus \mathfrak{b})^0 \otimes_{\mathbb{Z}} \mathcal{H}^{q-r}(\Omega_{X(b)|Y(b)})_x \), which is isomorphic to \( \bigwedge^r P_{\mathfrak{b}^0} / (P \setminus \mathfrak{b})^0 \otimes_{\mathbb{Z}} \mathbb{C} \) if \( q = r \) and is zero if \( q \neq r \). Therefore, since \( \omega_{X|Y} / b \omega_{X|Y} \cong \omega_{(X|Y)(b)} \), the stalk at \( x \) of \( \mathcal{H}^q(\omega_{X|Y} / b \omega_{X|Y}) \) is isomorphic to \( \bigwedge^r (M_{X|X(x)} / \mathcal{O}_{X|Y})_x \otimes_{\mathbb{Z}} \mathbb{C} \), so (2) is proved.

To prove (3), we consider the particular case where \( P = \mathbb{N}^r \), for some \( r \geq 0 \). In this situation, \( X \cong \mathbb{C}^r \) as an analytic space, with canonical log structure given by a normal crossing divisor \( D \) of \( X \), and \( Y \) is a closed analytic subspace of \( X \), with induced log structure. Then, the complex \( \omega_{X|Y} \cong \Omega_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X|Y \). Therefore, we are reduced to the case analyzed in \([32]\) and we can use the isomorphism \([50]\) to describe the stalk at \( x \) of \( \mathcal{H}^{q}(\omega_{X|Y}) \). So, by applying Lemma \([13]\) (2), \( \mathcal{H}^{q}(\omega_{X|Y} / b \omega_{X|Y})_x \cong \bigwedge^q (M_{X|X}^p / \mathcal{O}_{X|Y}^p)_x \otimes_{\mathbb{Z}} \mathbb{C} \), which is isomorphic to \( \mathcal{H}^{q}(\omega_{X|Y})_x \), as \([50]\).

Now, we prove (3) in the general situation. For a non-empty ideal \( I \) of the monoid \( P \), we can consider the toric variety \( \bar{B}_I(\text{Spec} \mathbb{C}[P]) \), which we get from \( X \) by "blowing-up" along \( I \) as in \([20]\) §I, Theorem 10. It is endowed with a canonical log structure \([13]\) (3.7(1))).

**Note.** From \([19]\) Proposition (9.8)], and \([20]\) §I, Theorem 11, it is possible to choose an ideal \( \bar{I} \) of \( P \), such that, if \( \tilde{X} = \bar{B}_I(\text{Spec} \mathbb{C}[P]) \), with log structure \( \tilde{M} \), then, for any \( y \in \tilde{X} \), \( \tilde{M}/\mathcal{O}_{X|Y}^y \) is isomorphic to \( \mathbb{N}^r(y) \), for some \( r(y) \geq 0 \). Let \( f: \tilde{X} \rightarrow X \) be the proper map, corresponding to the
“blowing-up” of $X$ along $\tilde{I}$. We can suppose $\tilde{X} \cong \mathbb{C}^r$, for some $r \in \mathbb{N}$, i.e. $X$ to be a non-singular log analytic space, with canonical log structure $\mathbb{N}^r \to \mathbb{C}[\mathbb{N}^r]$, namely, the log structure given by a normal crossing divisor $\tilde{D}$.

Then, we consider the following cartesian diagram

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{i} & \tilde{X} \\
\downarrow{f_{\tilde{Y}}} & & \downarrow{f} \\
Y & \xrightarrow{i} & X
\end{array}
$$

(61)

where $\tilde{Y} = f^{-1}(Y)$ is a closed subspace of $\tilde{X}$, and we suppose it to be endowed with the inverse image of the log structure $M$. We denote by $\tilde{f}$ the morphism $\tilde{f}: \tilde{X} \to \tilde{Y}$ (deduced from the cartesian diagram (61)). We also note that the vertical maps in (61) are log-étale, so

$$
\omega_{\tilde{X}} \cong f^* \omega_X \tag{62}
$$

Then, from (62), we get

$$
\omega_{\tilde{X}|\tilde{Y}} \cong \tilde{f}^* \omega_{X|Y} \tag{63}
$$

Moreover, by [20, §I, Corollary 1. c)], there exists a quasi-isomorphism

$$
\mathcal{O}_X \xrightarrow{\cong} \mathbb{R}f_* \mathcal{O}_{\tilde{X}} \tag{64}
$$

Since $f$ is proper, and $X, \tilde{X}$ are schemes of finite type over $S$, applying [13, §II, Proposition (6.2)] to the structural sheaf $\mathcal{O}_{\tilde{X}}$, we get

$$
(\mathbb{R}f_* \mathcal{O}_{\tilde{X}})|_Y \cong \mathbb{R}\tilde{f}_* (\mathcal{O}_{\tilde{X}|\tilde{Y}}) \tag{65}
$$

and, from the isomorphism (64), we get

$$
\mathcal{O}_{X|Y} \overset{\cong}{\xrightarrow{} \mathbb{R}\tilde{f}_* (\mathcal{O}_{\tilde{X}|\tilde{Y}}) \tag{66}
$$

Therefore, since the $\mathcal{O}_{\tilde{X}}$-module (resp. $\mathcal{O}_X$-module) $\omega_{\tilde{X}}^q$ (resp. $\omega_X^q$) is free of finite rank, for any $q$, from (65) and (66), we finally get an isomorphism in the derived category

$$
\omega_{X|Y} \cong \mathbb{R}\tilde{f}_* \omega_{\tilde{X}|\tilde{Y}} \tag{67}
$$

Let now $\tilde{x} \in \tilde{Y}$ be such that $f(\tilde{x}) = x$. Let $\tilde{b}$ be the prime ideal of $\mathbb{N}^r$ equal to the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ under $\mathbb{N}^r \to \mathcal{O}_{X,x}$. We consider the log analytic closed subspace $\tilde{X}(\tilde{b})$ of $\tilde{X}$, and its closed log subspace $\tilde{Y}(\tilde{b}) \subseteq \tilde{X}(\tilde{b})$, defined as in (60).

We consider the following commutative diagram

$$
\begin{array}{ccc}
\omega_{X|Y} & \xrightarrow{\cong} & \omega_{X|Y}/b\omega_{X|Y} \cong \omega_{(X|Y)(b)} \\
\downarrow & & \downarrow \\
\mathbb{R}\tilde{f}_* \omega_{\tilde{X}|\tilde{Y}} & \xrightarrow{\cong} & \mathbb{R}\tilde{f}_* (\omega_{\tilde{X}|\tilde{Y}}/b\omega_{\tilde{X}|\tilde{Y}}) \cong \mathbb{R}\tilde{f}_* (\omega_{(\tilde{X}|\tilde{Y})(b)} \tag{68}
\end{array}
$$

Since we have just proved (3) in the case $P = \mathbb{N}^r$, $r \geq 0$, the lower horizontal arrow is an isomorphism at $x$. Therefore, the map

$$
\mathcal{H}^q (\omega_{X|Y})_x \to \mathcal{H}^q (\omega_{X|Y}/b\omega_{X|Y})_x \tag{69}
$$

is injective. Moreover, by Lemma [43] (1), $\mathcal{H}^0 (\omega_{X|Y}/b\omega_{X|Y})_x \cong \mathcal{H}^0 (\omega_{(X|Y)(b)})_x \overset{\cong}{\to} \mathbb{C}$, and also $\mathcal{H}^0 (\omega_{X|Y}/b\omega_{X|Y}) \cong \mathcal{H}^0 (\omega_{(X|Y)(b)}) \cong \mathbb{C}$. Thus, we find that the composition map

$$
\mathbb{C} \cong \mathcal{H}^0 (\omega_{X|Y}/b\omega_{X|Y})_x \to \mathcal{H}^0 (\mathbb{R}\tilde{f}_* (\omega_{X|Y}/b\omega_{X|Y}))_x \to \mathcal{H}^0 (\omega_{X|Y}/b\omega_{X|Y})_x \cong \mathbb{C}
$$
is the identity map, and so

\[ \mathcal{H}^0(\omega_{X|y}/b\omega_{X|y})_x \rightarrow \mathcal{H}^0(\mathcal{L}_x(\omega_{X|y}/b\omega_{X|y}))_x \]  

(70)

is injective. Now, from diagram (68), since the composed map \( \omega_{X|y} \rightarrow \mathcal{L}_x(\omega_{X|y}/b\omega_{X|y}) \) is an isomorphism at \( x \), it follows that the map (70) is also surjective, and so it is an isomorphism. Therefore, from diagram (68), the map

\[ \mathcal{H}^0(\omega_{X|y})_x \rightarrow \mathcal{H}^0(\omega_{X|y}/b\omega_{X|y})_x \]  

(71)

is also an isomorphism, and we can conclude that \( \mathcal{H}^0(\omega_{X|y})_x \cong \mathbb{C} \).

Now, the isomorphism (59), factorizes through

\[ \bigwedge^q(M^{gp}_{X,x}/\mathcal{O}_{X,x}) \otimes_{\mathbb{Z}} \mathcal{H}^0(\omega_{X|y})_x \rightarrow \mathcal{H}^q(\omega_{X|y})_x \rightarrow \mathcal{H}^q(\omega_{X|y}/b\omega_{X|y})_x \]

and we can conclude that the map (69) is also surjective, and so it is an isomorphism.

From Lemma 4.3 we can deduce the following

**Proposition 4.4** [21, Proposition (4.6)] Let \( Y \) be an fs log analytic space over \( S \), and let \( i: Y \hookrightarrow X \) be an exact closed immersion of \( Y \) into an fs log smooth log analytic space \( X \). Then, for all \( q \in \mathbb{Z} \), there is an isomorphism

\[ \bigwedge^q(M^{gp}_{X}/\mathcal{O}_X)^Y \otimes_{\mathbb{Z}} \mathcal{H}^q(\omega_{X|y})_x \rightarrow \mathcal{H}^q(\omega_{X|y}/b\omega_{X|y})_x \]  

(72)

induced by the map \( \hat{d}\log: M^{gp}_{X,Y} \rightarrow \omega^1_{X|Y} \).

**Proof.** Since the question is local on \( X \), we may assume that \( X = \text{Spec} \mathbb{C}[P] \), where \( P \) is an fs monoid. Let \( x \in Y \), and let \( b \subset P \) be the inverse image of the maximal ideal of \( \mathcal{O}_{X,x} \). Now, by Lemma 4.3 (3),

\[ \mathcal{H}^q(\omega_{X|y})_x \cong \mathcal{H}^q(\omega_{X|y}/b\omega_{X|y})_x \]

and, by Lemma 4.3 (2),

\[ \mathcal{H}^q(\omega_{X|y}/b\omega_{X|y})_x \cong \bigwedge^q(M^{gp}_{X,x}/\mathcal{O}_{X,x})_x \otimes_{\mathbb{Z}} \mathbb{C} \]

for each point \( x \in Y \).

Now, we use Lemma 4.3 and Proposition 4.4 to prove a “formal version” of the logarithmic Poincaré Lemma.

**Proof.** (of Theorem 4.1) In the previous notation, let \( x \in Y \), \( y \in Y^\log \) be such that \( \tau(y) = x \). Let \( \{t_1, ..., t_n\} \) be a family of elements of \( \mathcal{L}_{X,x} \) whose image via the map \( \exp_x: \mathcal{L}_{X,x} \rightarrow M^{gp}_{X,x}/\mathcal{O}_{X,x} \) is a \( \mathbb{Z} \)-basis of \( M^{gp}_{X,x}/\mathcal{O}_{X,x} \).

Let \( R \) be the polynomial ring \( \mathbb{C}[T_1, ..., T_n] \). From Lemma 2.2, the stalk in \( y \) of \( \mathcal{O}_{Y|y}^\log \) is isomorphic to \( \mathcal{O}_{X|y,x}^\log [T_1, ..., T_n] \), where each \( t_i \) corresponds to \( T_i \). Therefore, we consider the \( \mathbb{C} \)-linear homomorphism

\[ R \rightarrow \mathcal{O}_{X|y,y}^\log \]  

(73)

which sends \( T_i \mapsto t_i \), for \( i = 1, ..., n \). Since

\[ \mathbb{C} \rightarrow \Omega_{R/\mathbb{C}} \]
is a quasi-isomorphism, it is sufficient to prove that there exists a quasi-isomorphism
\[ \Omega_{R/C} \xrightarrow{\cong} \omega_{X|Y,y}^{\log} \] (74)

To this end, we introduce a filtration on \( \Omega_{R/C} \) as follows: for any \( r \in \mathbb{Z} \), let \( \text{Fil}_r(\Omega_{R/C}) \) be the subcomplex of \( \Omega_{R/C} \) whose degree \( q \) part is the \( \mathbb{C} \)-submodule of \( \Omega_{R/C}^q \) generated by elements of the type \( f \cdot \gamma \), with \( f \in R \) an element of degree \( \leq r \), and \( \gamma \in \Omega_{R/C}^r \).

We also introduce a filtration on \( \omega_{X|Y,y}^{\log} \); for any \( r \in \mathbb{Z} \), let \( \text{Fil}_r(\omega_{X|Y,y}^{\log}) \) be the subcomplex of \( \omega_{X|Y,y}^{\log} \) whose degree \( q \) part \( \text{Fil}_r(\omega_{X|Y,y}^{q,\log}) \) is defined as

\[
\text{Fil}_r(\omega_{X|Y,y}^{q,\log}) =: \text{fil}_r(\mathcal{O}_{X|Y,y}^{\log}) \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega_X^q)
\]

where \( \text{fil}_r(\mathcal{O}_{X|Y,y}^{\log}) \) is the filtration defined in Lemma 2.4.

Then, by Lemma 2.4,
\[
\text{Fil}_r(\omega_{X|Y,y}^{q,\log})/\text{Fil}_{r-1}(\omega_{X|Y,y}^{q,\log}) \cong \tau^{-1} \left( \omega_{X|Y,y}^q \otimes_{\mathbb{Z}} \text{Sym}_\mathbb{Z}^n \left( \mathcal{M}_X^{\mathbb{R}} / \mathcal{O}_X^* \right) \right)
\]

and, by Proposition 4.4, for any \( q \),
\[
\mathcal{H}^q \left( \tau^{-1} \left( \omega_{X|Y,y}^q \otimes_{\mathbb{Z}} \text{Sym}_\mathbb{Z}^n \left( \mathcal{M}_X^{\mathbb{R}} / \mathcal{O}_X^* \right) \right) \right) \cong \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge_{i=1}^n \mathbb{Z} T_i \otimes_{\mathbb{Z}} \text{Sym}_\mathbb{Z}^n (\bigoplus_{i=1}^n \mathbb{Z} T_i)
\]

On the other hand, \( \text{Fil}_r(\Omega_{R/C})/\text{Fil}_{r-1}(\Omega_{R/C}) \) is the complex
\[
q \mapsto \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge_{i=1}^n \mathbb{Z} T_i \otimes_{\mathbb{Z}} \text{Sym}_\mathbb{Z}^n (\bigoplus_{i=1}^n \mathbb{Z} T_i)
\]

which is isomorphic to \( \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge_{i=1}^n \mathbb{Z} T_i \otimes_{\mathbb{Z}} \text{Sym}_\mathbb{Z}^n (\mathcal{M}_X^{\mathbb{R}} / \mathcal{O}_X^*) \). The differentials of this complex are zero.

Therefore, for any \( r \in \mathbb{Z} \), there exists a quasi-isomorphism
\[
\text{Fil}_r(\Omega_{R/C})/\text{Fil}_{r-1}(\Omega_{R/C}) \xrightarrow{\cong} \text{Fil}_r(\omega_{X|Y,y}^{q,\log})/\text{Fil}_{r-1}(\omega_{X|Y,y}^{q,\log})
\]

and this implies that there is a quasi-isomorphism \( \Omega_{R/C} \cong \omega_{X|Y,y}^{\log} \) for each point \( y \in \text{Ylog} \). ■

5 Log De Rham and Log Betti Cohomologies

The goal of this section is to compare the Log Betti Cohomology \( H(Y^{\log}, \mathbb{C}) \) of an fs log scheme \( Y \), with its Algebraic Log De Rham Cohomology \( \mathbb{H}(Y, \omega_{X|Y}) \). Therefore, we begin with

**Theorem 5.1** Let \( Y \) be an fs log scheme over \( S \), and let \( i: Y \hookrightarrow X \) be an exact closed immersion of \( Y \) into an fs log smooth log scheme \( X \). Then, for any \( q \in \mathbb{Z} \), there exists an isomorphism
\[ H^q(Y^{\log}, \mathbb{C}) \cong \mathbb{H}^q(Y^{\log}, \omega_{X|Y}^{\log}) \] (76)

**Proof.** In the previous section, we have checked that \( H^q(Y^{\log}, \mathbb{C}) \cong \mathbb{H}^q(Y^{\log}, \omega_{X|Y}^{\log}) \), for any \( q \in \mathbb{Z} \). So, we will first show that the Log Betti Cohomology of \( Y \) is isomorphic to the Analytic Log De Rham Cohomology \( \mathbb{H}(Y^{an}, \omega_{X|Y}^{\log}) \) (Proposition 5.2). Finally, we will check that the algebraic log De Rham complex \( \omega_{X|Y}^{\log} \) is quasi-isomorphic to its associated analytic log complex \( \omega_{X|Y}^{\log} \) (Theorem 5.3). ■
 Proposition 5.2 \cite{21} (4.8), 4.8.5. There exists a quasi-isomorphism

\[ R \tau^*_s(\omega^\log_{X|Y}) \cong \omega^\log_{(X|Y)_{an}} \]  

\[ (77) \]

**Proof.** We consider the composed map

\[ \mathbb{C} \otimes \mathbb{Z} \left( M^\text{op}_X / \mathcal{O}_X^* \right)_Y \overset{\cong}{\to} \mathcal{H}^q(\omega^\log_{(X|Y)_{an}}) \to R^q \tau^*_s \mathcal{C}_Y^\log \]  

\[ (78) \]

(where the second map is given by the natural arrow \( \omega^\log_{(X|Y)_{an}} \to R \tau^*_s(\omega^\log_{X|Y}) \)). Since the first map is an isomorphism by Proposition 4.4, it is sufficient to prove that there is a quasi-isomorphism

\[ R^q \tau^*_s \mathcal{C}_Y^\log \cong \mathbb{C} \otimes \mathbb{Z} \left( M^\text{op}_X / \mathcal{O}_X^* \right)_Y \]

To prove this, we apply \cite{21} Lemma (1.5), taking the constant sheaf \( \mathcal{E} \) on \( Y^\text{an} \). We have a canonical isomorphism

\[ (R^q \tau^*_s \mathcal{C}_Y^\log \cong) R^q \tau^*_s \tau^* \mathcal{C}_{Y^\text{an}} \cong \mathbb{C} \otimes \mathbb{Z} \left( M^\text{op}_Y / \mathcal{O}_Y^* \right)_Y \]

where the sheaf \( M^\text{op}_Y / \mathcal{O}_Y^* \) is isomorphic to \( (M^\text{op}_X / \mathcal{O}_X^*)_Y \), by \cite{23}. \( \blacksquare \)

We will now compare the algebraic log De Rham complex \( \omega^\log_{X|Y} \) with its associated analytic log De Rham complex \( \omega^\log_{(X|Y)_{an}} \), and we will show that they are quasi-isomorphic.

**Theorem 5.3** In the previous notation, let \( g: X^\text{an} \to X \) be the canonical morphism. If we consider the cartesian diagram

\[ \begin{array}{ccc}
Y^\text{an} & \overset{i}{\hookrightarrow} & X^\text{an} \\
g^Y \downarrow & & g \downarrow \\
Y & \overset{i}{\hookrightarrow} & X
\end{array} \]  

\[ (79) \]

then the morphism

\[ \omega^\log_{X|Y} \to R \hat{g}^* \omega^\log_{(X|Y)_{an}} \]  

\[ (80) \]

induces an isomorphism in cohomology

\[ \mathbb{H} (Y, \omega^\log_{X|Y}) \cong \mathbb{H} (Y^\text{an}, \omega^\log_{(X|Y)_{an}}) \]  

\[ (81) \]

**Proof.** We may assume that \( X = \text{Spec} \mathbb{C}[P] \), endowed with the canonical log structure (with \( P \) a toric monoid), and \( Y \hookrightarrow X \) is a closed log subscheme of \( X \), with the induced log structure. We divide the proof into two steps:

1) We begin by proving the assertion in the case where \( P = \mathbb{N}^r \), for some \( r \in \mathbb{N} \), i.e. in the case of a smooth scheme \( X \) over \( S \), with log structure given by a normal crossing divisor \( D \hookrightarrow X \). Then, by the formal Poincaré residue isomorphism \( \text{[40]} \), for each \( k \leq n \), we have the following identifications

\[ \mathbb{H}^q(Y, \text{Gr}_k^W(\omega^\log_{X|Y})) \cong \mathbb{H}^{q-k}(Y, \mathcal{Z}_k^\omega_{D^k|Y^k} \mathcal{E}^k) \]  

\[ (82) \]

Moreover, by \cite{33} \text{[IV]},

\[ \mathbb{H}^{q-k}(Y, \mathcal{Z}_k^\omega_{D^k|Y^k} \mathcal{E}^k) \cong \mathbb{H}^{q-k}(Y^\text{an}, \mathcal{Z}_k^\omega_{D^k,\text{an}} \mathcal{E}^k) \cong H^{q-k}(Y^{k,\text{an}}, \mathbb{C}) \]

and so

\[ \mathbb{H}^q(Y, \text{Gr}_k^W(\omega^\log_{X|Y})) \cong \mathbb{H}^q(Y^\text{an}, \text{Gr}_k^W(\omega^\log_{(X|Y)_{an}})) \]

for each \( k \), \( 0 \leq k \leq n \). Therefore, we can conclude that the morphism \( \text{[80]} \) induces the isomorphism

\[ \mathbb{H} (Y, \omega^\log_{X|Y}) \cong \mathbb{H} (Y^\text{an}, \omega^\log_{(X|Y)_{an}}) \]
2) We now prove the assertion for a generic toric monoid \( P \). We can take \( \tilde{I}, \tilde{X}, \) and \( f: \tilde{X} \to X \), as in the Note interpolated in the proof of Lemma 4.3. We consider the cartesian diagram (61), for the algebraic and analytic cases. Then, by arguments similar to those in the proof of Lemma 4.3, (63), (66), (67), we can conclude that, in the algebraic setting,
\[
\omega_{\tilde{X}|\tilde{Y}} \cong \mathbb{R} \tilde{f}_* \omega_{\tilde{X}|\tilde{Y}}
\]
and similarly, in the analytic setting,
\[
\omega_{(X|Y)^{an}} \cong \mathbb{R} \tilde{f}^{an}_* \omega_{(X|Y)^{an}}
\]
Therefore, to prove the assertion it is sufficient to check that there exists an isomorphism \( \mathbb{H} (\tilde{Y}, \omega_{\tilde{X}|\tilde{Y}}) \cong \mathbb{H} (\tilde{Y}^{an}, \omega_{(X|Y)^{an}}) \). But this follows from step 1), because \( \tilde{X} = \text{Spec} \mathbb{C}[\mathbb{N}^r] \), for some \( r \in \mathbb{N} \), endowed with canonical log structure \( \mathbb{N}^r \to \mathbb{C}[\mathbb{N}^r] \).}

References

[1] F. Baldassarri, M. Cailotto, L. Fiorot, Poincaré Duality for Algebraic De Rham Cohomology, Manuscripta Math., \textbf{114}, 61-116, (2004).

[2] P. Berthelot, Cohomologie Cristalline des Schémas de Caractéristique \( p > 0 \), Lecture Notes Math., \textbf{407}, Springer-Verlag, New York, Berlin, (1974).

[3] P. Berthelot, A. Ogus, Notes on Crystalline Cohomology, Princeton University Press and University of Tokyo Press, Princeton, New Jersey, (1978).

[4] M. Cailotto, A note on Logarithmic Differential Operators, Preprint, Università di Padova, (2003).

[5] A. Chambert-Loir, Cohomologie Cristalline: un survol, Exposition Math., \textbf{16}, 333-382, (1998).

[6] P. Deligne, Théorie de Hodge II, Publ. Math. I.H.E.S., \textbf{40}, 5-58, (1971).

[7] F. El Zein, Introduction à la théorie de Hodge mixte, Trans. of the Amer. Math. Soc., \textbf{275}, n. 1, 71-106, (January 1983).

[8] L. Fiorot, Stratified Pro-Modules, Preprint, Università di Padova, (2004).

[9] M. Fornasiero, De Rham Cohomology for Log Schemes, Ph.D. Thesis, Università di Padova, (2004).

[10] A. Grothendieck, On the De Rham cohomology of algebraic varieties, Publ. Math. I.H.E.S., \textbf{29}, 95-103, (1966).

[11] A. Grothendieck, Crystals and the De Rham Cohomology of Schemes (Notes by I. Coates and O. Jussila), Dix exposés sur la Cohomologie des Schémas, North Holland, 306-358, (1968).

[12] A. Grothendieck, J. A. Dieudonné, EGA I, Springer-Verlag, Berlin, (1971).

[13] R. Hartshorne, On the De Rham Cohomology of Algebraic Varieties, Publ. Math., \textbf{45}, 5-99, (1975).

[14] M. Herrera, D. Liebermann, Duality and De Rham Cohomology of Infinitesimal Neighbour-hoods, Inv. Math., \textbf{13}, 97-124, (1971).
[15] L. Illusie, *Report on Crystalline Cohomology*, Proceedings of Symposia in Pure Mathematics, **29**, Amer. Math. Soc., 459-478, (1975).

[16] L. Illusie, K. Kato, and C. Nakayama, *Quasi-unipotent Logarithmic Riemann-Hilbert Correspondence*, Preprint, (2003).

[17] M. N. Ishida, *Torus embeddings and de Rham complexes*, Advanced Studies in Pure Math., **11**, Commutative Algebra and Combinatorics, 111-145, (1987).

[18] K. Kato, *Logarithmic Structures of Fontaine-Illusie*, Algebraic Analysis, Geometry, and Number Theory, The Johns Hopkins Univ. Press, Baltimore, 191-224.

[19] K. Kato, *Toric Singularities*, Amer. J. Math., **116**, 1073-1099, (1994).

[20] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal Embeddings, I*, Lecture Notes Math., **339**, (1973).

[21] K. Kato, C. Nakayama, *Log Betti Cohomology, Log étale Cohomology, and Log De Rham Cohomology of Log Schemes over $\mathbb{C}$*, Kodai Math. J., **22**, 161-186, (1999).

[22] A. Ogus, *Logarithmic De Rham Cohomology*, Preprint, (1997).

[23] A. Ogus, *The Convergent Topos in Characteristic $p$*, The Grothendieck Festschrift, Vol. III, 133-162, Progr. Math., **88**, Birkhuser Boston, (1990).

[24] A. Shiho, *Crystalline Fundamental Groups I- Isocrystals on Log Crystalline Site and Log Convergent Site*, J. Math. Sci. Univ. Tokyo, **7**, n. 4, 509-656, (2000).