Generalization of the CHSH inequality for detecting entanglement between two-mode light states

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Detection of entanglement between two modes of light can be performed via a Bell-type experiment involving Mach-Zehnder interferometers. Such an experiment considers two distant laboratories, each possessing two settings of a Mach-Zehnder interferometer terminated by a photodetector. We generalize the CHSH inequality for such a system to n settings of the interferometer and check for the maximum possible violations. We also comment on the states achieving such violations and theoretically verify that the n MZI settings + photodetector arrangement can detect entanglement for two important classes of two-mode light states, namely, entangled coherent states and two-mode squeezed vacuum.

I. INTRODUCTION

Entangled quantum systems have grown in importance for technological as well as fundamental scientific applications. The advantage of quantum non-locality has been proved in various fields such as quantum communication [1, 2], metrology [3, 4] and computation [5, 6].

Entangled modes of light typically, are useful in photonic quantum metrology schemes [7], where the purpose is to achieve the quantum limit of measurement [8]. The states are multiphotonic states and are represented by combinations of Fock states and not a single Fock state. Therefore, performing measurements, which scales exponentially with number of constituent photons (m), on such states for the purpose of quantum state tomography [9] becomes difficult with growing m. As an alternative for performing state tomography, Bell-CHSH inequality [10, 11] is one of the most popular methods of verifying entanglement in a system. The violation of this inequality suggests that deterministic and local hidden variable models are not sufficient for explaining quantum correlations. Therefore, a recent work describes a Bell-CHSH inequality which can detect entanglement for two-mode light states [12]. The proposed experimental arrangement for this CHSH inequality uses a Mach-Zehnder Interferometer (MZI) fed with a strong coherent state paired with a photodetector having binary outcomes, possessed by each of two parties. Also, it was considered that each party has two measurement settings, as required for the conventional CHSH formulation.

However, it remains to be verified that if such a CHSH inequality for detecting entanglement in two-mode states of light can be improved by generalizing the inequality. For this purpose, we consider n measurement settings possessed by each of two parties. Each of these n measurements produce outcomes of ±1. A particular generalization of the CHSH inequality for such n measurement settings has been reported in [13]. To implement such a generalization, n-MZI measurement settings are required by each party. The number of measurement settings can be easily changed by changing the intensity of the input coherent state of the MZI; each intensity corresponds to one measurement setting. Therefore, by changing the intensity n times, the n measurement settings can be obtained per party. Thus, one can proceed with testing for the generalized CHSH inequality for this system without any added cost.

In this paper, we intend to check if the maximal violation obtained by the generalization of the CHSH inequality to n-MZI settings can be saturated to that given in [13]. Further, we also look into the structure of the states achieving maximal violation. Certain states, under the class of two-mode entangled states of light, such as entangled coherent states (ECS) [14] and two-mode squeezed vacuum (TMSV) [15] are important for various applications in quantum metrology [16, 17]. These states have also been experimentally generated [18, 19]. Thus, we also check if the CHSH inequality proposed using n-MZI settings can detect entanglement for ECS and TMSV.

The paper is organized as follows: Sec. II describes the proposed experimental arrangement and corresponding formulation of the CHSH inequality for n measurement settings per party. In Sec. A, we choose an appropriate basis that spans the subspace of each party, and thus to represent observables in the generalized CHSH inequality. In Sec. III, we report and comment the results for maximal violation obtained by the n-MZI settings and the states achieving them. Finally, in Sec. IV, we discuss the violations obtained by n-MZI settings when detecting entanglement in two important classes of two-mode states of light: entangled coherent states and two-mode squeezed vacuum.
II. GENERALIZED CHSH INEQUALITY

In general, for $n$ dichotomic observables (of output values $\pm 1$) per party, the generalized CHSH inequality is given by [13]:

$$|\mathbb{E}(\sum_{i=1}^{n} X_i \otimes Y_i + \sum_{i=1}^{n-1} X_{i+1} \otimes Y_i - X_1 \otimes Y_1)| \leq 2n\cos\left(\frac{\pi}{2n}\right)$$

(1)

where $\{X_1, ..., X_n\}$ and $\{Y_1, ..., Y_n\}$ are dichotomic observables employed by Lab X and Y, respectively, corresponding to their $n$ independent measurement settings. The quantum bound (RHS of Eq 1) is tight for appropriate choice of state and measurement settings [13] and the classical bound is $(2n - 2)$.

Let us assume, that we have a two mode state of light. The modes are separated and sent to two laboratories (Lab X and Lab Y) that perform measurements on their modes using Mach-Zehnder interferometers (MZI), each fed with a strong coherent state of light ($\langle \alpha \rangle = \hat{D}(\alpha) |0\rangle$) at one input port [20]. MZI in such setting implements the displacement operator $\hat{D}(T\alpha) = e^{T\alpha a^\dagger - T^* a^\dagger} \hat{a}^\dagger$ [21] on the state of the second input port [20]. Here $\hat{a}$ ($\hat{a}^\dagger$) is the photon annihilation (creation) operator [21] and $T$ is effective transmittivity of the MZI.

Further we have assumed that the desired output port of the MZI is terminated by a photodetector having binary outcomes (i.e., measures only zero or non-zero photons). Hence the corresponding measurement observable is of the form:

$$A(\beta) = I - 2 |\beta\rangle\langle\beta|$$

where $\beta$ is the total displacement produced by the MZI on a state on its input. Let the displacements implemented by MZIs in Lab X and Y be $\{\beta_1, ..., \beta_n\}$ and $\{\gamma_1, ..., \gamma_n\}$, respectively. The corresponding observables are $\{A(\beta_1), ..., A(\beta_n)\}$ and $\{A(\gamma_1), ..., A(\gamma_n)\}$.

Therefore, the observables in [Eq. 1] are:

$$X_i = A(\beta_i) = I - 2 |\beta_i\rangle\langle\beta_i|$$

$$Y_i = A(\gamma_i) = I - 2 |\gamma_i\rangle\langle\gamma_i|$$

(3)

Thus, we can write the LHS of CHSH inequality ($S$) in [Eq. 1] as:

$$S = \sum_{i=1}^{n} A(\beta_i) \otimes A(\gamma_i) + \sum_{i=1}^{n-1} A(\beta_{i+1}) \otimes A(\gamma_i) - A(\beta_1) \otimes A(\gamma_1)$$

(4)

In [12], it has been proven, that for $n = 2$, the maximal violation of the CHSH inequality can be achieved by an appropriate choice of displacements in both (MZI+photodetector) settings possessed by labs Lab X and Lab Y. Now, to detect entanglement by a larger ($n > 2$) number of settings, the classical bound $(2n - 2)$ must be violated, i.e., $(2n - 2) < \mathbb{E}(S)_{\text{max}} \leq 2n \cos\left(\frac{\pi}{2n}\right)$. Further, we check the MZI settings in both labs maximising the violation and how close to the maximal violation $2n \cos\left(\frac{\pi}{2n}\right) - (2n - 2)$ can it be.

III. RESULTS

In this section, we discuss a number of optimisation results we have obtained analysing the generalised CHSH inequality with observables originating from MZI setups and for various families of experimentally accessible states. We have obtained the results numerically and the Appendices B1 and B2a describe the details of our codes.

A. Maximal Eigenvalues of CHSH matrix

As discussed earlier, the generalised CHSH inequality is maximally violated in a pure state represented by an eigenvector of $S$ (4) related to its maximal eigenvalue. The maximal possible violation is equal to $2n \cos\left(\frac{\pi}{2n}\right) - 2(n - 2)$, in particular for $n = 2$ we obtain $2\sqrt{2} - 2$ - a maximal violation of the standard CHSH inequality.

First, we maximise the violations for $n \in \{3, 4, 5\}$ where the parameters are $\{\beta_i\}, \{\gamma_i\}$, for $i \in \{1, ..., n\}$. We minimise the probability of getting stuck in a local maximum by repeating procedure with a number of randomly chosen starting points.

First we perform optimisation for $n = 3, 4, 5$. The optimal sets of $\{\beta_i\}$ and $\{\gamma_i\}$ are shown in the figure 1. We observe that the sets $\{\beta_i\}$ and $\{\gamma_i\}$ each behave co-linearly in the complex plane, with almost equal spacing between subsequent values.

Using this non-trivial observation, we therefore conjecture that the maximal violation for all $n$ is achieved by the sets of $\{\beta_i\}$ and $\{\gamma_i\}$ being arithmetical sequences.

Further, the common phase $\varphi_\beta$ of $\{\beta_i\}$ and the common phase $\varphi_\gamma$ of $\{\gamma_i\}$ can be made zero by appropriate rotations in the complex plane, realised by a local unitary operators $\exp(-i\varphi_\beta a^\dagger a) \otimes \exp(-i\varphi_\gamma a^\dagger a)$. Hence we can assume $\{\beta_i\}$ and $\{\gamma_i\}$ to be arithmetic sequences of displacements.

Furthermore, we can translate both sequences to begin in zero by a local unitary operator being a tensor product of appropriate displacement operators: $D(-\beta_0) \otimes D(-\gamma_0)$ in the Hilbert space of each mode.

Finally, we observe that the module of the differences between subsequent $\beta$s and subsequent $\gamma$s are almost equal.

Thus, we bring the following assumption into our numerical model:

Assumption 1. 1A For a given $n$, $\{\beta_1, \beta_2, ..., \beta_n\}$ and $\{\gamma_1, \gamma_2, ..., \gamma_n\}$ are equal and form a real arithmetic sequences starting from $\beta_1 = 0$ and $\gamma_1 = 0$. They are defined by one real parameter $\Delta = \beta_{i+1} - \beta_i = \gamma_{i+1} - \gamma_i$. 

Initially, we performed numerical optimization over $4n$ real parameters as $n$ complex parameters are required for each party. As a consequence of assumption 1, we reduced the number of parameters to 1. This will allow us to proceed with optimisation to higher dimensions.

Under the assumption 1, we observe a violation for $n \in [2,40]$ as The Fig. 2 shows the optimisation results for $n \in \{2, \ldots, 19\}$ (blue triangles) in comparison to the theoretical bound $D(n) = E(S) - 2n + 2$ (red points). We observe a significant shift between these values for $n > 2$. The maximal violation achievable by MZI setups is obtained for $n = 7$.

At the end of this subsection, we plot $\Delta$ as a function of $n$, denoted in the following as $\Delta_n$. We verify that for $n = 2$, $|\Delta_2| = \sqrt{10}2$, which is in accordance to what we had found for maximal violation when $n = 2$ [12].

In the Fig. 3(a), we plot $|\Delta_n|$ vs. $n$. We observe a decreasing behaviour. Its inverse, $1/|\Delta_n|$ vs $n$ behaves almost linearly with growing $n$ [see Fig. 3(b)].

### B. States for Maximal Violation

In the Sec. III A, we have maximised the maximal eigenvalue of the LHS of the generalised CHSH inequality (1) for $n = 2, \ldots, 39$. Now we want to discuss the properties of the corresponding eigenvectors. In the following we will used the arithmetic progression $\Delta$, resulting from the optimization protocol, to re-calculate the sets of displacements $\{\beta_i\}$ and $\{\gamma_i\}$, the corresponding two-mode observable $S$ and the eigenvector corresponding to its maximal eigenvalue.

To verify our results, we check for the case $n = 2$. The numerical model yields the eigenvector corresponding to maximal violation as: $\text{numpy.array}([-1.30656296, 1.30656296, 0.54119609, -1.30656296])$ which can be re-written as:

$$|\psi_2\rangle = \frac{1}{\sqrt{2} - \sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ \sqrt{2} - 1 \\ -1 \end{bmatrix}.$$  \hfill (5)

$|\psi_2\rangle$, unitary equivalent to the maximally entangled state, is exactly what we have found analytically for in our previous work [12]. Therefore, we proceed to comment on the states for $n > 2$-settings.

The entries of vectors $|\psi_n\rangle$ are real (see the Remark 3 in the Appendix B). In the Fig. 4(a)-(d) we plot the values of entries of $|\psi_n\rangle$ in the computation (orthonormal) basis for $n = 3, 6, 9, 12$ after reshaping ($n^2 \times 1$) column vectors to $(n \times n)$ matrices.

Next we calculate the Schmidt coefficients [22, 23] of vectors $\Psi_n$ performing the singular value decomposition of the corresponding $(n \times n)$ matrices [see Figs. 4(e)-(h)]. We observe that even for high $n$ the Schmidt rank of $|\psi_n\rangle$ is 4 and the first two Schmidt coefficients dominate.
FIG. 3. Variation of $|\Delta_n|$ with number of settings ($n$): (a) Calculated absolute values of $\beta_{i+1} - \beta_i = \gamma_{i+1} - \gamma_i = \Delta_n$ (for a given $n$) plotted against number of settings ($n$). We observe a decreasing function which seems to be falling $\propto 1/n^2$. (b) Calculated values of $1/|\Delta_n|$ (black dots) are plotted with $n$ (number of settings).

FIG. 4. Eigenvectors for maximum violation and their Schmidt coefficients: The $n \times n$ eigenvector-matrices (reshaped from $(n^2 \times 1)$ column vectors) are plotted for (a) $n = 3$, (b) $n = 6$, (c) $n = 9$ and (d) $n = 12$. The color bar on the right-end indicates the scalar values associated with the entries of the matrices plotted. (e)-(h) Bar graphs showing the magnitude of the Schmidt coefficients ($\lambda_j, j = 1$ to $n$) for the corresponding eigenvectors plotted above are shown.

Hence the maximal violation achievable by MZI settings is realised on states far from maximally entangled ones. We illustrate it calculating their Von Neumann entropies [23, 24] out of their Schmidt coefficients:

$$S_n(\lambda) = - \sum_{i=1}^{n} |\lambda_i|^2 \log_2(|\lambda_i|^2)$$

and comparing the result to Von Neumann entropy of maximally entangled state $= \log_2(n)$ for $n = 2, \ldots, 19$ [see Fig. 5]

For $n = 2$, we obtain $S_2(\lambda) = 0.9999 \simeq 1$, hence the state is maximally entangled. For higher values of $n$, $S_n(|\psi_n\rangle)$ grows slowly, staying close to 1 (in agreement with two dominating Schmidt coefficients). The Von Neumann entropies of $|\psi_n\rangle$, $n = 3, 6, 9, 12$, whose entries and Schmidt coefficients have been illustrated in Figs. 4(e)-(h), are: $S_3(\lambda) = 1.0008, S_6(\lambda) = 1.0676, S_9(\lambda) = 1.1314$ and $S_{12}(\lambda) = 1.1760$, respectively.
FIG. 5. Von Neumann Entropies $S_n(\lambda)$ with number of settings ($n$): The Von Neumann entropies ($S_n(\lambda)$) calculated from the Schmidt coefficients ($\{\lambda_j\}$, $j = 1$ to $n$) [in red] and the normalization factor ($\log_2(n)$) [in blue] are plotted with $n$ (number of measurement settings). We observe a sub-logarithmic behaviour for $S_n(\lambda)$, which becomes almost constant near 1 for growing $n$.

IV. OPTIMAL VIOLATION FOR EXPERIMENTALLY ACHIEVABLE STATES

In the previous sections, we have observed maximum violation of the CHSH inequality for generalized ($n$) number of settings. Further, we calculated the state vectors for which such violation can be achieved. We have found that the state vector causing maximal violation for $n = 2$ measurement settings can be written as:

$$|\Psi\rangle = \frac{1}{\sqrt{2 - \sqrt{2}}} \left\{ \left[ |\beta_1\rangle - |\beta_2\rangle \right] \otimes \left[ |\gamma_1\rangle - |\gamma_2\rangle \right] - (2 - \sqrt{2}) |\beta_1\rangle \otimes |\gamma_3\rangle \right\}$$  

(7)

up to normalization and under certain conditions [12].

However, experimental realisation of such states and their applications have not been reported yet. Therefore, in this section we will discuss how the generalised CHSH inequality (1) is useful to detect entanglement of certain experimentally viable states: entangled coherent states [18] and two-mode squeezed vacuum [19], known for their importance for applications in quantum metrology. We describe the numerics for this section in Appendix B 2 b.

A. Entangled Coherent States

In this section, we focus on a new algorithm for obtaining a violation of the CHSH inequality for an experimentally achievable, two-mode light state: entangled coherent states, $|\Psi_{ECS}\rangle$, which are of the form:

$$|\Psi_{ECS}\rangle = N_\alpha (a |\alpha\rangle \otimes |0\rangle + |0\rangle \otimes |\alpha\rangle)$$  

(8)

where $N_\alpha = 1/\sqrt{1 + |a|^2 + 2e^{-|\alpha|^2} \text{Re}(a)}$ is the normalization factor [18].

Remark 1. Observe, that acting with displacement operators in both subsystems $D(\eta) \otimes D(\epsilon)$ one can obtain a more general state $N_\alpha ((\alpha + \eta) \otimes |\epsilon\rangle + |\eta\rangle \otimes (|\alpha + \epsilon\rangle)$. Local unitary operators $\hat{D}(\epsilon)$ and $\hat{D}(\eta)$ can be performed by using a Mach-Zehnder interferometer fed by a strong coherent state [12, 20]. We will not consider these more general states, as they are related to (8) by a local unitary operation and have the same amount of entanglement.

The expectation value of the observable $S$ (4) w.r.t. a particular state vector $|\Psi_{ECS}\rangle$ is:

FIG. 6. Violations achieved by various protocols: Comparison of numerically generated plots for the maximum violation obtained by the $n$-MZI settings [in blue], MZI+ECS setting [in black], MZI+TMSV [in yellow] and theoretically calculated [in red] for generalized Bell-CHSH inequality as given by [13]. The plots have been generated for $n \in [2, 20)$. 

$|\Psi_{ECS}\rangle = N_\alpha (a |\alpha\rangle \otimes |0\rangle + |0\rangle \otimes |\alpha\rangle)$
where we have used the fact than $A(\beta) = D(\beta)A(0)D^\dagger(\beta)$ and the properties of displacement operators [21].

We perform a numerical optimisation of $\mathbb{E}(S)_{ECS}$ for each $n = 2, \ldots, 19$ to find the maximal violation in this class of states. We discuss our results for this in Sec. IV C.

**B. Two-Mode Squeezed Vacuum States**

Another interesting class of experimentally accessible states is two-mode squeezed vacuum [19] states. Such states (denoted by $|\Psi_{TMSV}\rangle$) can be achieved when the squeezing operator $(S(\xi))$ acts on two-mode vacuum $(|0\rangle \otimes |0\rangle)$ as:

$$|\Psi_{TMSV}\rangle = S(\xi) |0\rangle \otimes |0\rangle = \exp\{\xi^* a\hat{b} - \xi a\hat{b}^\dagger\} |0\rangle \otimes |0\rangle$$

(11)

$$\mathbb{E}(S)_{TMSV} = \langle \Psi_{TMSV}(r)|S|\Psi_{TMSV}(r)\rangle$$

$$= \langle \Psi_{TMSV}(r)|\left(\sum_{i=1}^{n} A(\beta_i) \otimes A(\gamma_i) + \sum_{i=1}^{n-1} A(\beta_{i+1}) \otimes A(\gamma_i) - A(\beta_1) \otimes A(\gamma_n)\right)|\Psi_{TMSV}(r)\rangle$$

(13)

Introducing $g(r, \beta_i, \gamma_j) = \langle \Psi_{TMSV}(r)|A(\beta_i) \otimes A(\gamma_j)|\Psi_{TMSV}(r)\rangle$:

$$g(r, \beta_i, \gamma_j) = 1 - 2 e^{-|\beta_i|^2/\cosh^2 r} + e^{-|\gamma_j|^2/\cosh^2 r} - 2 \exp\{ -|\beta_i|^2 - |\gamma_j|^2 - 2Re(\beta_i\gamma_j) \tanh r \}/\cosh^2 r$$

(14)

We obtain the following:

$$\mathbb{E}(S)_{TMSV} = \sum_{i=1}^{n} g(r, \beta_i, \gamma_i) + \sum_{i=0}^{n-1} g(r, \beta_{i+1}, \gamma_i)$$

$$- g(r, \beta_1, \gamma_n)$$

(15)

Performing the numerical optimization of the above, we obtain the violation ($D(n)_{TMSV} = \mathbb{E}(S)_{TMSV} - 2n + 2$) that our MZI settings can achieve for TMSV states.

We compare and discuss the results for ECS and TMSV families in the following section.
C. Observations

In the Fig. 6 we illustrate the comparison of the maximal violation of the generalised CHSH inequality with MZI+photodetector arrangement. We see that the maximal violation is obtainable for TMSV as input states when each of two labs uses MZI+photodetector in each lab. For ECS, the maximal violation can be achieved for positive diagonal entries. It can be easily seen that for $n > 10$, the violation becomes negligible. However, as we intend to calculate eigenvalues of the global observable $S$, we need an orthonormal basis, because the spectrum of a matrix is invariant on unitary transformations.

An obvious path would be to obtain an orthonormal basis of span{[β_i]} via the Gram-Schmidt orthonormalization [25]. For the $n = 2$ case, we easily perform it to get:

$$
|e_1⟩ = |β_1⟩,
|e_2⟩ = \frac{|β_2⟩ - |β_1⟩⟨β_1|β_2⟩}{\sqrt{1 - exp(-|β_1 - β_2|^2)}}.
$$

Appendix A: Orthonormal Basis

In this appendix, we focus on obtaining appropriate bases to represent $A(β_i)$ and $A(γ_i)$. Let us first consider one subsystem, i.e., say Lab X whose detection observables are $\{A(β_i)\}$, $i = 1$ to $n$. Now, due to the definition of $A(β)$ given in [Eq. 2], we can write these observables in the basis formed by coherent states $|β_i⟩$. However, as we intend to calculate eigenvalues of the global observable $S$, we need an orthonormal basis, because the spectrum of a matrix is invariant on unitary transformations.

Now, the Cholesky decomposition of $G = L^†L$, where $L$ is a lower triangular $n \times n$ matrix with real and positive diagonal entries. It can be easily seen that $(BL^{−11})^†(BL^{−11}) = I_n$. Thus, the columns of $\infty \times n$ matrix $BL^{−11}$ are orthonormal to each other. Let us call the vectors corresponding to these columns (combinations of $β$s) by: $|e_1⟩, ..., |e_n⟩$. The one has:

$$
BL^{−11} = [|e_1⟩ | e_2⟩ ... | e_n⟩] \text{ and,}
B = [|β_1⟩ | |β_2⟩ ... | |β_n⟩⟩ = [|e_1⟩ | |e_2⟩ ... | |e_n⟩]L^† (A2)
$$

Thus the $i$th column-elements of $L^†$ are the coefficients of $|β_i⟩$ in the orthonormal basis $\{|e_i⟩\}$. Considering $|β_i⟩ = L^†|e_i⟩$, we have:

$$A(β_i) = 1 - 2L^†|e_i⟩⟨e_i|L \quad (A3)$$

Similarly, for Lab Y, we can repeat the same procedure to find the orthonormal basis from $\{γ_i⟩\}$. On doing so, we obtain $γ_i⟨γ_i| = K^†|f_i⟩⟨f_i|K$ where $H = K^†K$ is the Cholesky-decomposed Gram matrix ($H$) corresponding to $γ_i⟩$.
Appendix B: Numerical Analysis

In this Appendix we describe the details of implementation and refer to the python codes accessible in a GitHub repository [https://github.com/maduragd/generalization-of-CHSH](https://github.com/maduragd/generalization-of-CHSH). We use the python programming language with numerical packages numpy, scipy and matplotlib.

1. Numerical Orthonormalisation

As described in the Appendix A, first we have to express a set coherent state vectors \( \{\beta_1, \ldots, \beta_n\} \) living in the Hilbert space of one subsystem in an orthonormal basis of the subspace spanned by them.

First, for the given set of displacements, we generate the Gram matrix \( G \) of the corresponding coherent state vectors with entries \( G_{ij} = (\beta_i | \beta_j) = \exp(-((\beta_i - \beta_j)^2 + \beta_i^* \beta_j - \beta_i \beta_j^*)/2) \). Next we calculate the matrix \( L \) of \( G = LL^\dagger \) using numpy.linalg.cholesky.

During the procedure, computation errors such as “Matrix is not positive definite” sometimes occur. This happens because during the calculations some diagonal elements becomes negative due to numerical inaccuracy. This situation takes place, if during the minimisation procedure, two displacements become close to each other. To rectify this error, we add a correction factor \((3 \times \text{modulus of smallest eigenvalue of } G)\) to the diagonal terms in the matrix. Thus, we obtain columns of \( L^\dagger \) as the coefficients of \( \{|\beta_i\rangle\} \) in the orthonormal basis \( \{|e_i\rangle\} \).

Remark 2. In case all \( \{\beta_i\} \) are real, \( L = L^\dagger \).

Remark 3. In case all \( \{\beta_i\} \) and \( \{\gamma_i\} \) are real, the matrix \( S \) given by (4) is a real symmetric matrix, and its eigenvectors have symmetric entries.

2. Maximal Violation of CHSH inequality

a. General states

Once we are able to express \( |\beta\rangle_S \) in an orthonormal basis, we can define a function prescribing the maximum eigenvalue of the matrix \( S \) (4) to given sets of displacements \( \{\beta_i\} \) and \( \{\gamma_i\} \) \((4n \text{ random numbers, } n \text{ real and } n \text{ imaginary numbers for each party})\). Now we are able to maximise this function (strictly speaking: minimise its negative) using scipy.optimize.minimise. We start from random sequences of displacements and repeat the procedure a number of times to minimise he probability of getting stuck in a local minimum.

In this way we obtain the maximal violation for the CHSH inequality using \( n \text{-MZI} \) settings. Now, for \( n \in [3,5] \), we obtain the sequences \( \{\beta_i\} \) and \( \{\gamma_i\} \) for which the violation is obtained under such a protocol, and plot them in the complex plane. The whole code is under n3to6_general.betas.gammas.py.

Next, under the Assumption 1, we consider real arithmetic sequences of \( \{\beta_i\} \) and \( \{\gamma_i\} \) with \( \beta_i = \gamma_i, \beta_1 = \gamma_1 = 0 \). This reduces the randomly generated numbers to 1 for \( \forall n \) which is \( \Delta_n = \beta_{n+1} - \beta_i = \gamma_{n+1} - \gamma_i \) for a given \( n \). The code n_MZI.py performs numerical maximisation of the maximal eigenvalue of \( S \) (4) w.r.t. the above assumptions for \( n \in \{2, \ldots, 19\} \). Next the results are serialized to a file max_chsh_eig.pi using _pickle. The code in each run updates the stored values if yields a better value than a previous one.

The code Plots_Violation.py produces a plot of the maximal violation vs. \( n \). It also plots the behaviour of \( |\Delta_n| \) and \( 1/|\Delta_n| \) vs. \( n \) using the results stored in max_chsh_eig.pi.

For the analysis of the states realising maximal violation, we use the the results pickled in max_chsh_eig.pi. We recover the eigenvectors corresponding to the maximal eigenvalues of the CHSH matrix \( S \), reconstructed from \( \Delta_n \), using numpy.linalg.eig. This is realised by the code Eigenvectors_maxViolation.py. We store the entries of the maximal eigenvectors in max_chsh_states.pi. Further, loading the results stored in this file, we calculate the Schmidt coefficients using Singular Value Decomposition of \( n \times n \) matrix composed from reshaping \( n^2 \times 1 \) column vector of the entries of the corresponding eigenvector using numpy.linalg.svd. Then the results \((n \times n)\) eigenvector matrices and their corresponding Schmidt coefficients are plotted by the code Plots_Eigenvectors_SchmidtCoeffs.py. Further, using the serialised data we calculate Von-Neumann entropy of these state vectors and plots them in the code VonNeumann_Enropies.py.

b. Experimentally achievable states

For the cases when the laboratories using the \( n \)-MZI settings share a ECS or TMSV state, we consider the expected value of \( S \) in \(|\Psi_E\rangle \) or \(|\Psi_{TMSV}\rangle \), i.e., \( \langle \Psi_E | S | \Psi_E \rangle \) or \( \langle \Psi_{TMSV} | S | \Psi_{TMSV} \rangle \), respectively. Thus, we deal with a function dependent on \( \beta_i, \gamma_i \) and relevant parameters describing \( |\Psi_E\rangle \) or \(|\Psi_{TMSV}\rangle \) given by an algebraic expression instead of the maximal eigenvalue of a matrix. Again, we maximise this function (given by (9) or (15) for ECS and TMSV states respectively) by running scipy.optimize.minimise on its negative, and serialize the results using pickle.

The calculation are made in the codes ECS.py and TMSV.py respectively. The results are stored in max_chsh_ecs.pi and max_chsh_tmsv.pi, respectively. Finally, the maximal violations are plotted for all four cases: \( n \)-MZI settings, MZI+ECS, MZI+TMSV and theoretically expected in Plots_Violation.py.
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