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Level statistics in arithmetical and pseudo-arithmetical chaos

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Abstract

We investigate a long-standing riddle in quantum chaos, posed by certain fully chaotic billiards with constant negative curvature whose periodic orbits are highly degenerate in length. Depending on the boundary conditions for the quantum wavefunctions, the energy spectra either have uncorrelated levels usually associated with classical integrability or conform to the ‘universal’ Wigner–Dyson type although the classical dynamics in both cases is the same. The resolution turns out surprisingly simple. The Maslov indices of orbits within multiplets of degenerate length either yield equal phases for the respective Feynman amplitudes (and thus Poissonian level statistics) or give rise to amplitudes with uncorrelated phases (leading to Wigner–Dyson level correlations). The recent semiclassical explanation of spectral universality in quantum chaos is thus extended to the latter case of ‘pseudo-arithmetical’ chaos.

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1. Introduction

After more than two decades of investigations, the famous BGS conjecture [1] has recently found a semiclassical explanation [2–6]: the quantum level spectra of classically chaotic systems display fluctuations conforming to the random-matrix-theory (RMT) predictions for the Wigner–Dyson universality classes.

However, there is the notable exception of ‘arithmetical’ systems which are fully chaotic classically but display quantum spectral statistics close to Poissonian, a behavior usually associated with integrable classical motion [7–12]. Quantum mechanically, these exceptional dynamics exhibit an infinite number of the so-called Hecke operators commuting with the Hamiltonian. Therefore, the energy spectrum falls into practically independent multiplets
such that nearby levels bear no correlation. On the classical side the periodic-orbit action spectra of such systems are distinguished by a degeneracy exponentially growing with the orbit period.

On the other hand, by merely changing the boundary conditions for the quantum wavefunctions of some such exceptional arithmetical systems one can retrieve universal spectral fluctuations in the manner of Wigner and Dyson while not at all changing the classical dynamics. It is customary to speak of pseudo-arithmetical systems then [13–15].

The strikingly different quantum behavior of arithmetical and pseudo-arithmetical systems might raise doubts about the validity of the recent semiclassical explanation of universal quantum spectral fluctuations under conditions of classical chaos; after all, the classical dynamics are identical for the systems under discussion, and so appear, on first sight, the Gutzwiller-type semiclassical periodic-orbit expansions. Various suggestions were ventured for the effect of the boundary conditions, among them a distinction between the orbit classes contributing to the Selberg trace formula applicable (and exact) in the arithmetical case, and the Gutzwiller formula applicable in the pseudo-arithmetical case.

We show here that the explanation is much simpler and lies in the special properties of periodic orbits. Due to these peculiarities all equal-length orbits (save for a negligible fraction) of an arithmetical system contribute Feynman amplitudes with the same Maslov phase; their constructive interference makes for nonuniversal spectral statistics. In the pseudo-arithmetical case these phases vary randomly within a degenerate-action multiplet such that destructive interference makes the high action degeneracy ineffective. The difference between the two cases is most easily revealed for the diagonal approximation to the spectral form factor, and therefore that approximation will play a central role here; off-diagonal corrections will be discussed briefly in the end.

2. The billiard $T^*(2, 3, 8)$

We shall not deal with the alternative arithmetical/pseudo-arithmetical in full generality but prefer to work with a representative example, the so-called triangular billiard $T^*(2, 3, 8)$. That system was first considered in studies of free motion on the surface of constant negative curvature tessellated by regular octagons [16]. Desymmetrization of the regular octagon necessary to get rid of the rotational and reflection symmetry leads to a triangular fundamental domain with the angles $\pi/2, \pi/3, \pi/8$. Depicted inside the Poincaré disk $|z| \leq 1$ in the complex plane, the triangle has its hypotenuse ($N$) and the longer leg ($L$) directed along two diameters whereas the shorter leg ($M$) looks like an arc (figure 1). A classical periodic orbit folded into the triangle looks like a sequence of arcs mirror-reflected from the sides of the triangle. The classical motion is completely chaotic, all periodic orbits having the same stability index. The multiplicities in the length spectrum of periodic orbits grow exponentially with the length, like $\exp(l_\gamma/2)$, where $l_\gamma$ is made dimensionless by referral to a scale fixed by setting the curvature to $-1$; since the action is proportional to the orbit length, the action spectrum also has an exponentially growing degeneracy.

The quantum energy levels for $T^*(2, 3, 8)$ are found as the eigenvalues of the Laplace–Beltrami operator. The boundary conditions can be either Dirichlet or Neumann. There are $2^3 = 8$ quantum mechanical problems, all related to the same classical system. In problems stemming from desymmetrization of the regular octagon, the boundary conditions on the triangle sides are chosen to obtain the spectrum for a particular irreducible representation. The boundary conditions on the hypotenuse $N$ and the shorter leg $M$ must then be the same, i.e. both Dirichlet or both Neumann; four such possibilities exist all of which lead to arithmetical systems with near-Poissonian spectral statistics. The remaining four choices where $N$ and
$M$ host different boundary conditions lead to pseudo-arithmetical systems with the Wigner–Dyson statistics of the orthogonal universality class.

3. Form factor and diagonal approximation

The spectral form factor following from Gutzwiller’s trace formula is a double sum over periodic orbits:

$$K(\tau) \sim \left\langle \sum_{\gamma, \gamma^\prime} A_\gamma A_{\gamma^\prime} \exp \left[ i \left( \frac{S_\gamma - S_{\gamma^\prime}}{\hbar} - i \frac{(\mu_\gamma - \mu_{\gamma^\prime}) \pi}{2} \right) \right] \times \delta \left( \tau T_H - \frac{T_\gamma + T_{\gamma^\prime}}{2} \right) \right\rangle,$$

where $S_\gamma$, $T_\gamma$, $\mu_\gamma$, and $A_\gamma = A_{\gamma}^*$ are the action, period, Maslov index and stability coefficient of the orbit $\gamma$; the Heisenberg time $T_H = \frac{2\pi \hbar}{\Delta}$, with $\Delta$ being the mean level spacing, is used as a unit of time such that $\tau$ becomes a dimensionless time; the angular brackets $\langle \cdots \rangle$ denote averages over the energy shell and over a small $\tau$ interval.

Of special interest are the pairs of orbits obviously immune against destructive interference of their contributions, namely with the same action and Maslov index. For generic chaotic dynamics these are the trivial pairs $\gamma^\prime = \gamma$ and, if time reversal is allowed, the pairs of mutually time reversed orbits, $\gamma^\prime = \gamma^{TR}$. Discarding all other pairs one obtains Berry’s diagonal approximation \[2\]

$$K_{\text{diag}}(\tau) \sim g \sum_{\gamma} A_\gamma^2 \delta(\tau T_H - T_\gamma) \equiv g \tau ;$$

here $g$ is the average multiplicity of the action spectrum which is 1 in the absence of time reversal (unitary universality class) or 2 in the presence of time reversal (orthogonal class); the result yields the first-order term of a power series in $\tau$, in agreement with RMT \[4\].

Turning to arithmetical chaos we can carry over the foregoing reasoning, except that the multiplicity is now exponentially large, $g \propto e^{l/2}$. However, orbits with the same length bear no geometric similarity, and it is not at all obvious that their Maslov phases are the same.

As will be shown, the Maslov phases do indeed coincide for all orbits in a length multiplet, a negligible fraction apart, in the arithmetical case. Consequently, the form factor exhibits almost instant increase at $\tau \geq 0$, similar to the jump of the integrable case \[6\], $K(0) = 0$ and $K(\tau) = 1$ for $\tau > 0$.

In contrast, the Maslov index of pseudo-arithmetical systems will turn out to fluctuate randomly within each fixed-action multiplet. The usual diagonal approximation with $g = 2$
then holds in each multiplet only the pairs \( \gamma' = \gamma \) and \( \gamma' = \gamma^{\text{TR}} \) escape destructive interference. The diagonal approximation thus suggests universal spectral fluctuations.

The Maslov index of an orbit in our billiard is determined only by the number \( N_D \) of reflections from the sides with the Dirichlet boundary condition; each such reflection changes the Maslov phase by \( \pi \). Therefore, the contribution of an equal-action pair \( \gamma, \gamma' \) is \( A_{2}^{2} (-1)^{N_D-N_o} \) where \( N_D \) and \( N_o \), respectively, refer to \( \gamma \) and \( \gamma' \); in long orbits both \( N_D \) and \( N_o \) are large pseudo-random integers. We shall demonstrate that in arithmetical systems equal-length orbits have, in their overwhelming majority, \( N_o \) of the same parity (even or odd) such that contributions of all pairs of them are positive and add up. In contrast, pseudo-arithmetical systems have uncorrelated parities of \( N_D, N'_o \) and the equal-action contributions mutually cancel, apart from the standard pairs of the orthogonal universality class.

4. Numerical observations

The triangular billiard affords symbolic dynamics, and calculating its orbits involves generating all allowed sequences of symbols \( L, M, N \) each standing for the visit of the respective side. We denote by \( \lambda_{\gamma}, \mu_{\gamma}, \nu_{\gamma} \) the number of symbols \( L, M, N \) in an orbit \( \gamma \); the total number of symbols in \( \gamma \) is \( n_{\gamma} = \lambda_{\gamma} + \mu_{\gamma} + \nu_{\gamma} \). Studying up to a million orbits we find that:

(i) Orbits within a given length multiplet \( \Lambda \) almost always have \( n_{\gamma} \) with the same parity. We can therefore speak about \( \Lambda_{\text{e}} \) and \( \Lambda_{\text{o}} \)-multiplets (\( e = \text{even}, o = \text{odd} \)) depending on the parity of \( n_{\gamma}, \gamma \in \Lambda \). Exceptions are extremely rare and in fact amount to a negligible fraction: e.g. among approximately 257,000 orbits with the length \( l_{\gamma} < 16 \) grouped into more than 13,000 length multiplets, only 4 multiplets are ‘eo-degenerate’, i.e. contain orbits with both even and odd number of symbols; these offenders have lengths \( l = 10,699,996,\ 12,242,2622,\ 13,757,1382,\ 15,285,7092 \).

(ii) All orbits in a given \( \Lambda \) without eo-degeneracy have \( \lambda_{\gamma} \) of the same parity which automatically leads to a definite parity of \( \mu_{\gamma} + \nu_{\gamma} \). On the other hand, \( \mu_{\gamma} \) and \( \nu_{\gamma} \) separately have no definite parity within \( \Lambda \).

These observations, in particular the rarity of multiplets with eo-degeneracy, suffice to explain the sensitivity of the level statistics to the boundary conditions. We denote by \( \Phi_L \) the phase jump on reflection from the side \( L \), which is 0 for the Neumann and \( \pi \) for the Dirichlet condition on \( L \); similarly, \( \Phi_M \) and \( \Phi_N \) denote the phase jumps on \( M \) and \( N \). The contribution to the diagonal form factor of a pair of orbits \( (\gamma, \gamma') \) belonging to the same \( \Lambda \) is

\[
K_{\gamma \gamma'} = A_{\Lambda}^{2} e^{i(\lambda_{\gamma} - \lambda_{\gamma'}) \Phi_L + i(\mu_{\gamma} - \mu_{\gamma'}) \Phi_M + i(\nu_{\gamma} - \nu_{\gamma'}) \Phi_N} = A_{\Lambda}^{2} e^{i(\mu_{\gamma} - \mu_{\gamma'}) \Phi_M + i(\nu_{\gamma} - \nu_{\gamma'}) \Phi_N},
\]

the phase proportional to \( \Phi_L \) disappears since, in the absence of the eo-degeneracy, \( \lambda_{\gamma} = \lambda_{\gamma'} \) is even.

In arithmetical systems with the same boundary conditions on the sides \( M, N \) we have \( \Phi_M = \Phi_N \equiv \Phi_{MN} \), and the contribution of every pair within \( \Lambda \) will be positive since \( \mu_{\gamma} + \nu_{\gamma} - \mu_{\gamma'} - \nu_{\gamma'} \) is even:

\[
K_{\gamma \gamma'} = A_{\Lambda}^{2} e^{i(\mu_{\gamma} + \nu_{\gamma} - \mu_{\gamma'} - \nu_{\gamma'}) \Phi_{MN}} = A_{\Lambda}^{2}.
\]

Equation (2) then applies with the abnormally high value of \( g \) bringing about the nearly vertical rise of \( K(\tau) \) at \( \tau \approx 0 \).

On the other hand, if the boundary conditions on \( M \) and \( N \) are different (the pseudo-arithmetical case), one of \( \Phi_M, \Phi_N \) is zero and another one \( \pi \). The contribution \( K_{\gamma \gamma'} \) can then be of any sign. Summed over all pairs of a large multiplet, these contributions will interfere destructively, except the standard pairs \( \gamma' = \gamma, \gamma^{\text{TR}} \) of the orthogonal universality class.
Table 1. Average effective periodic orbit multiplicity in a pseudo-arithmetic billiard.

| Length interval | $g_{\text{eff}}$ | No of multiplets | Average size of multiplet |
|-----------------|------------------|------------------|---------------------------|
| 9–11            | 1.827            | 654              | 6.69                      |
| 11–13           | 1.947            | 1 892            | 15.51                     |
| 13–15           | 1.990            | 5 227            | 36.92                     |
| 15–17           | 2.014            | 14 272           | 88.33                     |

Numerical estimates show that the contribution of all pairs but the standard ones converges to zero as the multiplet length grows. The diagonal contribution to the form factor is therefore the Wigner–Dyson one for the orthogonal universality class with the effective periodic orbit multiplicity $g_{\text{eff}} = 2$. These observations suggest the absence of correlation of both $\mu_\gamma$ and $\nu_\gamma$ in the non-standard pairs as well as the equal probability of even and odd $\mu$ or $\nu$ within a degenerate multiplet.

As a numerical example we calculated $g_{\text{eff}}$ for the pseudo-arithmetical system with the Dirichlet boundary condition at the side $M$ and the Neumann conditions at two other sides (table 1). The effective average multiplicity was estimated as

$$g_{\text{eff}} = \left\langle \frac{(N_+ - N_-)^2}{N_+ + N_-} \right\rangle,$$

where $N_+$, $N_-$ are the numbers of orbits with even and odd number of bounces against the side $M$ within a single multiplet; averaging is done over all degenerate multiplets in the interval of lengths specified in the first column. The third and fourth columns give the number of degenerate multiplets and the average number of orbits per multiplet for the length band considered. The result appears to converge to the GOE value with growing orbit lengths.

5. Analytic reasoning

Using the group-theoretical properties [15] of $T^*(2, 3, 8)$ we have substantiated these results analytically. The symbols $L, M, N$ can be associated with transformations mapping the interior of the Poincaré disk $|z| \leq 1$ onto itself. Each elementary operation involves complex conjugation $K : z \to z^*$ followed by a Möbius transformation $z \to (az + b)/(b^*z + a^*)$; the three respective matrices

$$\rho_L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho_M = \begin{pmatrix} i \left( 1 + \frac{\sqrt{2}}{2} \right) \gamma & -i \frac{\sqrt{2}}{\gamma} \\ i \frac{\sqrt{2}}{\gamma} & -i \left( 1 + \frac{\sqrt{2}}{2} \right) \gamma \end{pmatrix},$$

$$\rho_N = \begin{pmatrix} \left( i \frac{\sqrt{2} + 1}{2} \right) \gamma & 0 \\ 0 & \left( i \frac{\sqrt{2} - 1}{2} \right) \gamma \end{pmatrix},$$

where $\alpha, \beta, \gamma$ denote the quartic irrationalities

$$\alpha = \sqrt{\sqrt{2} - 1}, \quad \beta = \sqrt{\sqrt{2}}, \quad \gamma = \sqrt{2 - \sqrt{2}}.$$

Let us write the code of the orbit $\gamma$ starting with an arbitrary symbol and multiply the associated elementary operators. The product is either a pure Möbius transformation with
a certain matrix $\rho_\gamma$, if the number of symbols $n_\gamma$ is even, or $K$ followed by $\rho_\gamma$ (odd $n_\gamma$).

The cumulative transformation leaves invariant a circle crossing the fundamental domain in the complex plane (the so-called invariant geodesic of the transformation); its part inside the domain is the piece of $\gamma$ between the two bounces against the sides given by the first and last symbol of the code. Cyclically shifting the code by one symbol one analogously gets the next orbit piece, and so forth [15]. The matrix $\rho_\gamma$ yields the orbit length as

$$2 \cosh \frac{l_\gamma}{2} = \text{Tr} \rho_\gamma, \quad n_\gamma \text{ even},$$

$$2 \sinh \frac{l_\gamma}{2} = \text{At} \rho_\gamma, \quad n_\gamma \text{ odd},$$

where $\text{At} \rho$ is the sum of the off-diagonal elements of $\rho$.

It can be shown by induction that the matrices $\rho_\gamma$ can be of two arithmetical types ([15], appendix B),

$$\rho^{(1)} = \left( \begin{array}{c}
\frac{1}{2} (u_{1,R} + u_{1,I}) \\
\frac{1}{2} (v_{1,R} + iv_{1,I}) \alpha
\end{array} \right),$$

$$\rho^{(2)} = \left( \begin{array}{c}
\frac{1}{2} (u_{2,R} + u_{2,I}) \\
\frac{1}{2} (v_{2,R} + iv_{2,I}) \gamma
\end{array} \right),$$

Here, $u_{k,R}, v_{k,R}, u_{k,I}, v_{k,I}$ are algebraic integers:

$$u_{k,R} = m_{k,R} + n_{k,R} \sqrt{2},$$

$$v_{k,R} = p_{k,R} + q_{k,R} \sqrt{2}, \quad k = 1, 2,$$

with integers $m_{k,R}, n_{k,R}, p_{k,R}, q_{k,R}$; the imaginary parts have the same appearance.Appending a symbol $L$ (pure complex conjugation) to the code of the orbit does not change the type of $\rho_\gamma$ whereas appending $M$ or $N$ toggles the type, $\rho^{(1)} \leftrightarrow \rho^{(2)}$. The matrices $\rho_M$ and $\rho_N$ belong to the type $\rho^{(2)}$; therefore, $\rho_\gamma$ belongs to the type $\rho^{(1)}$ if the sum $\mu_\gamma + \nu_\gamma$ of the number of symbols $M, N$ in the code of $\gamma$ is even, and $\rho_2$ if $\mu_\gamma + \nu_\gamma$ is odd.

Equations (6) and (8) entail four types of orbit lengths: orbits with even numbers of symbols $n_\gamma$ have the lengths

$$2 \cosh \frac{l_\gamma}{2} = u_{1,R}, \quad 2 \cosh \frac{l_\gamma}{2} = \gamma u_{2,R},$$

while orbits with odd $n_\gamma$ have the lengths

$$2 \sinh \frac{l_\gamma}{2} = \alpha v_{1,R}, \quad 2 \sinh \frac{l_\gamma}{2} = \beta v_{2,R}.$$
hand, an orbit with an even number of symbols $n_γ$ can have the same length as one with odd $n_γ$, and such equality happens for some rare combinations of $u_{k,R}, v_{k,R}$; these are the cases of eo-degeneracy mentioned above. Equating one of $l_a, l_b$ to one of $l_c, l_d$ we obtain an equation for $u, v$ equivalent to two diophantine equations for the integers $m, n, p, q$. (Not all solutions of these equations correspond to really existing length multiplets since (11), (12) are only necessary conditions.) A careful analysis of the latter equations (see the appendix for details) for the cases $l_a = l_c$ or $l_b = l_d$ reveals that the solutions form an equidistant sequence $l^{(1)}_k = sI_k, \quad k = 1, 2, \ldots$, with $sI = 2 \text{arsinh} \ 2^{-1/4}$. Similarly solutions for $l_a = l_d$ or $l_b = l_c$ are described by $l^{(2)}_k = s^{II} k, \quad k = 1, 2, \ldots$, with $s^{II} = 2 \text{arsinh} \sqrt{1+\sqrt{2}}$. For example, the exceptional length $l = 10.6999964 = 7sI$ pertains to two multiplets, (b) with $m = 138, n = 97$ and (d) with $p = 88, q = 63$.

Since the exceptional lengths appear in equidistant sequences, the multiplets with eo-degeneracy are exponentially outnumbered, as the length $l$ grows, by the multiplets of definite type and hence the parity of $λ$ and $μ + ν$. The eo-degenerate multiplets can thus be discarded for lengths corresponding to a finite fraction of $TH$.

6. Conclusion

We have shown that ‘all’ (all save for a negligible fraction of multiplets) orbits with the same length of the $T^∗(2, 3, 8)$ billiard have the same parity of the number of bounces against the longer leg $L$ of the triangle; this is also true for the parity of the sum of the number of bounces against the sides $M, N$. As a result ‘all’ orbits with the same length in the arithmetical case have the same Maslov index such that all diagonal terms in the form factor are positive. A similar mechanism must exist in all other arithmetical systems with non-vanishing Maslov phases. In pseudo-arithmetical systems this mechanism is absent, and uncorrelated Maslov phases of degenerate orbits can be expected.

We based our reasoning on the diagonal approximation. But the essence of our argument carries over to the higher order terms of the $τ$-expansion valid for $τ < 1$ [4] as well as for the behavior at times exceeding the Heisenberg time [5], as far as the pseudo-arithmetical case is concerned: the high multiplicity of length multiplets of orbits is rendered irrelevant by destructive interference of random Maslov phases. Of course, the present arguments do not amount to a proof of the multiplicity $g$ taking the value 2 in the limit of long orbits for pseudo-arithmetical dynamics, on average over some length intervals. Such a proof remains a major challenge.

For arithmetical systems the interplay of high length degeneracy and bunches of orbits that very nearly coincide in configuration space, apart from reconnections in close self-encounters, may be more difficult to capture. On the other hand, the Hecke symmetries intuitively suggest (near) Poissonian level statistics.
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Appendix. Exceptional multiplets

We briefly indicate how the existence and uniqueness of the exceptional lengths can be ascertained, for the four possible cases $l_a = l_c$, $l_a = l_d$, $l_b = l_c$, $l_b = l_d$ which we shall refer to as $ac, ad, bc, bd$. Starting with the case $ac$ we infer from equations (11) and (12) that the exceptional lengths obey

$$e^\pm = \frac{u \pm \alpha v}{2}, \quad e^{-\pm} = \frac{u - \alpha v}{2}. \quad (A.1)$$

Now take two solutions $u_0 + \alpha v_0$ and $u + \alpha v$ corresponding to the lengths $l_0, l$. Then since $\exp[\pm(\frac{u_0}{2} \pm \frac{l}{2})] = \exp(\pm \frac{l_0}{2}) \exp(\pm \frac{l}{2})$, the product

$$u' + \alpha v' = \frac{u_0 + \alpha v_0 + u + \alpha v}{2} \quad (A.2)$$

is a solution with length $l' = l + l_0$, provided the integers on the rhs are divisible by 2.

The numerically found solution of smallest length $l_I^0 = 2s = 3.057 141 838 962 00$ was $u_0 = 2 + 2\sqrt{2}$, $v_0 = 4 + 2\sqrt{2}$, or $m_0 = 2, n_0 = 2, p_0 = 4, q_0 = 2$. Substituting the latter into (A.2) we obtain

$$m' = m + 2n + 2q, \quad n' = m + n + p, \quad p' = 2m + 2n + p + 2q, \quad q' = m + 2n + p + q; \quad (A.3)$$

obviously if $m, n, p, q$ are even, then so are the primed numbers. We thus face the sequence of solutions $(\frac{u_0 + \alpha v_0 + u + \alpha v}{2})^k$ with the equidistant lengths $kl_I^0$.

The transformation (A.3) can be inverted; the doubly primed numbers

$$m'' = m + 2n - 2q, \quad n'' = m + n - p, \quad p'' = -2m - 2n + p + 2q, \quad q'' = -m - 2n + p + q; \quad (A.4)$$

yield an orbit of length $l'' = l - l_I^0$ such that a ladder of decreasing lengths is obtained.

In order to make sure that the transformations (A.3) and (A.4) really yield orbits we must check that inequalities (a) and (c) in (13) are not violated. To that end we note that the variables $x \equiv (m - n\sqrt{2})/2$, $y \equiv (p - q\sqrt{2})/2\alpha$ span an invariant subspace of the transformations (A.3) and (A.4); the ensuing transformation $(x, y) \rightarrow (x', y')$ is a two-dimensional rotation preserving $x^2 + y^2$. Since the aforementioned minimal-length solution $m_0 = 2, n_0 = 2, p_0 = 4, q_0 = 2$ has the property $x^2 + y^2 = 1$ and since that ‘normalization’ is preserved, we indeed conclude the preservation of the inequalities under study.
Finally, let us demonstrate that there are no lengths of the type \( ac \) outside the equidistant-length ladder just established. Momentarily assuming the existence of such a freak length we can apply the length reducing transformation (A.4) until arriving at a reduced length \( l'' \in [0, l'_0] \) which must obey

\[
1 < \cosh \frac{l''}{2} = \frac{m'' + n'' \sqrt{2}}{2} < 1 + \sqrt{2} = \cosh \frac{l'_0}{2},
\]

(A.5)

\[
0 < \sinh \frac{l''}{2} = \left( p'' + \sqrt{2} q'' \right) \frac{\alpha}{2} < \left( 2 + \sqrt{2} \right) \alpha = \sinh \frac{l'_0}{2}.
\]

On the other hand, the conservation of length \( l_I \) only differences are the replacements of (i) the quartic irrationality \( \alpha \) by \( \beta \) and (ii) the minimal length \( l'_0 \) by \( l''_0 = 2v_m = 4.8969 \), the latter corresponding to \( u_0 = 6 + 4 \sqrt{2} \), \( v_0 = 4 + 4 \sqrt{2} \). All other lengths are given by \( kl''_0 \), \( k = 1, 2, \ldots \). The proof repeats the previous case word for word.

To treat the case \( bd \) we employ \( \gamma = \alpha \beta \) to write the equations for the lengths as

\[
e^\frac{1}{2} = \frac{\beta (\alpha u + v)}{2}, \quad e^{-\frac{1}{2}} = \frac{\beta (\alpha u - v)}{2}.
\]

(A.6)

The smallest length corresponds to \( u_m = 2 + \sqrt{2}, v_m = \sqrt{2} \) and is exactly \( s'_1 \), i.e. half of the smallest length of the case \( ac \). Solutions of (A.6) do not have in general the group property of the case \( ac \); in particular, if \( l \) is an exceptional length of the type \( bd \), then \( 2l \) is not. Indeed, squaring the rhs of (A.6) we get \( \sqrt{2} (\alpha u \pm v)^2 / 4 \) which cannot be an algebraic number of the type \( \beta (\alpha u \pm v) / 2 \). On the other hand, all odd multiples of \( l \) do belong to the admissible type. The same reasoning as in the cases \( ac, ad \) shows that all solutions can be represented as \( (k + 1/2) l''_0 \), \( k = 0, 1, \ldots \).

The remaining case \( bc \) is related to \( bd \) by the replacements \( \alpha \leftrightarrow \beta \) and \( l'_0 \rightarrow l''_0 \). All solutions can be written as \( (k + 1/2) l''_0 \), \( k = 0, 1, \ldots \).

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