ADDITION FORMULAS FOR \( q \)-SPECIAL FUNCTIONS

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Abstract. A general addition formula for a two-parameter family of Askey-Wilson polynomials is derived from the quantum \( SU(2) \) group theoretic interpretation. This formula contains most of the previously known addition formulas for \( q \)-Legendre polynomials as special or limiting cases. A survey of the literature on addition formulas for \( q \)-special functions using quantum groups and quantum algebras is given.

1. Survey and introduction

Many of the well-known special functions, such as the Jacobi polynomials and Bessel functions, satisfy addition formulas, which can be found in e.g. [1], [12], [63]. Often there exists a group theoretic interpretation of such an addition formula. This means that there exists a group \( G \) and a representation \( t \) of \( G \) in a Hilbert space \( V \) such that for a suitable basis \( \{e_n\} \) of \( V \) the matrix elements \( t_{n,m}: G \to \mathbb{C} \) defined by \( t_{n,m}(g) = \langle t(g)e_m, e_n \rangle \) are known in terms of special functions. Then the homomorphism property

\[
t_{n,m}(gh) = \sum_p t_{n,p}(g)t_{p,m}(h), \quad g, h \in G
\]

(1.1)

gives an addition formula for a suitably chosen basis of \( V \) and for certain elements \( g, h \in G \). Usually we number the basis such that \( t_{0,0} \) is left and right invariant with respect to a certain subgroup \( K \) (a spherical function) and we use (1.1) for \( n = m = 0 \). See Vilenkin [61] and Vilenkin and Klimyk [62] for more information.

Before the advent of quantum groups and quantum algebras (or \( q \)-algebras) only a few addition formulas were known for \( q \)-special functions. Addition formulas for \( q \)-Krawtchouk polynomials and \( q \)-Hahn polynomials were proved by Dunkl [11] and Stanton [56] in the late 70’s using the interpretation of these polynomials on Chevalley groups over a finite field. See also [55] for a partial result (a product formula). This method is limited to \( q \)-Krawtchouk and \( q \)-Hahn polynomials, cf. the list in [57]. Also, by a purely analytic method Rahman and Verma [54] have shown that also the continuous \( q \)-ultraspherical polynomials satisfy an addition formula similar to Gegenbauer’s addition formula for the ultraspherical polynomials.

1991 Mathematics Subject Classification. 33D80, 33D45, 33D55, 17B37.

Research supported by a Fellowship of the Research Council of the Katholieke Universiteit Leuven and the Netherlands Organization for Scientific Research (NWO) under project number 610.06.100.

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With the introduction of quantum groups and quantum algebras (or quantised universal enveloping algebras) a new setting for various $q$-special functions emerged, see e.g. the survey papers by Koornwinder [40], Noumi [46] and the author [35] for the quantum group approach and Floreanini and Vinet [16], [18] and references given later in this section for the quantum algebra approach. For a good introduction to quantum groups, quantised universal enveloping algebras and their applications we refer to Chari and Pressley [8], which also contains a wealth of references on the subject.

From this setting two approaches for deriving addition formulas for $q$-special functions can be obtained. To understand the quantum group approach we recall that a quantum group $A_q(G)$ is a deformation (with deformation parameter $q$) of a function algebra $A(G)$ on the group $G$. For a compact group $G$ this would mean the polynomials on $G$ or, in Woronowicz’s [64] approach, the continuous functions on $G$. We assume that the comultiplication
\[
\Delta: A(G) \to A(G) \otimes A(G) \cong A(G \times G), \quad (\Delta f)(g, h) = f(gh),
\]
is well-defined and we demand that $\Delta: A_q(G) \to A_q(G) \otimes A_q(G)$ survives the deformation unchanged (in a suitable sense). In the deformation the representations of $G$ correspond to corepresentations of $A_q(G)$ and for a suitable basis of the representation space we get matrix elements $t_{n,m} \in A_q(G)$ satisfying the cohomomorphism property
\[
(1.2) \quad \Delta(t_{n,m}) = \sum_p t_{n,p} \otimes t_{p,n},
\]
which must be considered as the analogue of (1.1). If the matrix elements can be expressed in terms of $q$-special functions we consider (1.2) as an (implicit) addition formula. Usually $A_q(G)$ is a non-commutative algebra and thus we need to investigate the representations of $A_q(G)$ in order to convert (1.2) into an addition formula involving $q$-special functions in commuting variables. Usually, $n = m = 0$ in (1.2) and $t_{0,0}$ can be considered as a spherical function. In this paper we show that in the case of the quantum $SU(2)$ group we can obtain a general addition formula for Askey-Wilson polynomials from (1.2) from which various known examples can be obtained as special or limiting cases.

Let us now briefly consider the quantum algebra approach. A quantum algebra is a deformation $U_q\mathfrak{g}$ of the universal enveloping algebra $U\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$. The representation theory of $U_q\mathfrak{g}$ is usually similar to the representation theory of $\mathfrak{g}$. Classically we can obtain elements of the corresponding group $G$ by exponentiating Lie algebra elements. In the quantum algebra approach the action of $\exp_q(\alpha_1 X_1) \ldots \exp_q(\alpha_n X_n)$ is calculated in a representation of $U_q\mathfrak{g}$. Here $\exp_q$ can be one of the $q$-analogues of the exponential function, $\alpha_i$ are scalars and $X_i$ are generators of $U_q\mathfrak{g}$. For a suitable basis $\{f_m\}$ of the representation space we get
\[
(1.3) \quad \exp_q(\alpha_1 X_1) \ldots \exp_q(\alpha_n X_n) f_m = \sum_k U_{m,k}(\alpha_1, \ldots, \alpha_n) f_k.
\]
The matrix coefficients $U_{m,k}(\alpha_1, \ldots, \alpha_n)$ can be calculated in terms of special functions in $\alpha_1, \ldots, \alpha_n$. By working with explicit realisations, in which the $f_m$ correspond to certain
special functions, addition formulas can be derived from (1.3). This method is motivated by the classical relation between Lie algebras and special functions as described in Miller’s book [44] and references for the $q$-algebra approach are given later on.

It should be observed that in the quantum group case the coalgebra structure, and in particular the comultiplication $\Delta$, is needed to find the addition formula (1.2) in non-commuting variables. The algebra structure of $A_q(G)$ is needed to transform (1.2) into an addition formula for $q$-special functions in commuting variables, although identities for $q$-special functions in non-commuting variables are of interest in their own right. In contrast, in the quantum algebra approach only the algebra structure is needed, and there are examples of quantum algebras used in relation with $q$-special functions which do not carry a bialgebra structure.

Let us now give some references to the literature for several addition formulas using one of these methods. For addition formulas for $q$-Legendre polynomials from the quantum group theoretic point of view see Koornwinder [39], Masuda et. al [45], Noumi and Mimachi [47, 48, 49, 50], Vaksman and Soibelman [59] for (implicit) addition formulas in non-commuting variables and the author [31, 33, 37], Koornwinder [41] (analytically proved by Rahman [53]) and Noumi and Mimachi [47] for addition formulas involving only commuting variables. For addition formulas for $q$-Bessel functions from the quantum group theoretic point of view see the author [32, 34]. For the $q$-algebra point of view see Floreanini and Vinet [14, 19, 21], Kalnins and Miller [27], Kalnins, Miller and Mukherjee [29]. For analytic proofs of related addition formulas for $q$-Bessel functions see the author [30, 36], Koelink and Swarttouw [38], Koornwinder and Swarttouw [43], Rahman [52], Swarttouw [58]. Of these references [30], [43] are closely related to the $q$-algebra approach and [36] is related to the quantum group approach. Using the quantum group approach it is possible to derive an addition formula for $q$-disk polynomial involving little $q$-Jacobi polynomials, cf. Floris [22], Floris and Koelink [23]. The announcement [9] by Dijkhuizen and Noumi suggests that generalisations to Askey-Wilson polynomials might be possible in this context. Using other quantum algebras it is possible to obtain addition formulas for general $r\varphi_s$ (Floreanini and Vinet [17]), for basic Lauricella $\varphi_D$ (Floreanini, Lapointe and Vinet [13]), for continuous analogues of addition formulas (Floreanini and Vinet [20], Kalnins and Miller [27]), for $q$-Laguerre polynomials (Kalnins, Manocha and Miller [26], Kalnins and Miller [27], Kalnins, Miller and Mukherjee [28]). See Groza and Kachurik [25] for addition and product formulas from the quantum $SU(2)$ group interpretation of $q$-Krawtchouk, $q$-Hahn and $q$-Racah polynomials.

In this paper we show how to derive an explicit addition formula for Askey-Wilson polynomials involving 3 parameters from the implicit addition formula, i.e. involving non-commuting variables. So we start off with an identity of the type as in (1.2) and we convert it into an identity for Askey-Wilson polynomials. In §2 we recall results on the quantum $SU(2)$ group and its relation with Askey-Wilson polynomials. In §3 we consider suitable vectors in the representation space of an irreducible representation of $A_q(SU(2))$ in which all the elements of $A_q(SU(2))$ under consideration act as a multiplication operator or as a shift operator. From this we obtain in §4 an addition formula for Askey-Wilson polynomials from which various known addition formulas can be obtained as special and limit cases.
The notation for q-shifted factorials and q-hypergeometric series follows the excellent book [24] by Gasper and Rahman.

Acknowledgement. I thank Mizan Rahman and Sergeî Suslov for sending their preprint [4] and for answering a number of questions. Most of the work for this paper was done at the Katholieke Universiteit Leuven, and I thank Walter Van Assche and Alfons Van Daele for their hospitality.

2. Generalised matrix elements on the quantum SU(2) group

In this section we recall the relation between the quantum SU(2) group and the Askey-Wilson polynomials. We also give the appropriate version of (1.2) in this case, which is the starting point for the derivation of the addition formula. References for this section are [47, 51] and [35], from which the notation has been taken and where further references can be found.

\( \mathcal{A}_q(SU(2)) \) is the complex unital associative algebra generated by \( \alpha, \beta, \gamma, \delta \) subject to the relations

\[
\begin{align*}
\alpha \beta &= q \beta \alpha, & \alpha \gamma &= q \gamma \alpha, & \beta \delta &= q \delta \beta, & \gamma \delta &= q \delta \gamma, \\
\beta \gamma &= \gamma \beta, & \alpha \delta - q \beta \gamma &= \delta \alpha - q^{-1} \beta \gamma &= 1
\end{align*}
\]

for some constant \( q \in \mathbb{C} \). With a \( * \)-operator given by

\[
\begin{align*}
\alpha^* &= \delta, & \beta^* &= -q \gamma, & \gamma^* &= -q^{-1} \beta, & \delta^* &= \alpha
\end{align*}
\]

the algebra \( \mathcal{A}_q(SU(2)) \) becomes a \( * \)-algebra for real \( q \), and from now on we fix \( 0 < q < 1 \).

We use the following realisation of \( \mathcal{A}_q(SU(2)) \) by a non-faithful representation. An irreducible infinite dimensional \( * \)-representation \( \pi \) of \( \mathcal{A}_q(SU(2)) \) in the Hilbert space \( \ell^2(\mathbb{Z}_+) \) with orthonormal basis \( \{ e_n \mid n \in \mathbb{Z}_+ \} \) is given by

\[
\pi(\alpha)e_n = \sqrt{1 - q^{2n}}e_{n-1}, \quad \pi(\gamma)e_n = q^n e_n,
\]

where we use the convention \( e_{-p} = 0 \) for \( p \in \mathbb{N} \). Note that \(-q\pi(\gamma) = \pi(\beta)\). For all \( \xi \in \mathcal{A}_q(SU(2)) \), \( \pi(\xi) \) is a bounded operator on \( \ell^2(\mathbb{Z}_+) \). The one-dimensional \( * \)-representation \( \tau_\phi : \mathcal{A}_q(SU(2)) \to \mathbb{C} \) is defined by

\[
(2.1) \quad \tau_\phi(\alpha) = e^{i\phi}, \quad \tau_\phi(\gamma) = 0.
\]

The algebra \( \mathcal{A}_q(SU(2)) \) is an example of a Hopf \( * \)-algebra. The comultiplication \( \Delta \), which is a \( * \)-homomorphism of \( \mathcal{A}_q(SU(2)) \to \mathcal{A}_q(SU(2)) \otimes \mathcal{A}_q(SU(2)) \), is given on the generators by

\[
(2.2) \quad \Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma.
\]

There exist elements \( b^l_{i,j}(\tau, \sigma) \in \mathcal{A}_q(SU(2)), \ l \in \frac{1}{2}\mathbb{Z}_+, \ i, j \in \{-l, -l+1, \ldots, l-1, l\}, \ \sigma, \tau \in \mathbb{R} \cup \{ \infty \} \) such that

\[
(2.3) \quad \Delta(b^l_{i,j}(\tau, \sigma)) = \sum_{n=-l}^{l} (D.b^l_{i,n}(\tau, \mu)) \otimes b^l_{n,j}(\mu, \sigma), \quad \forall \ \mu \in \mathbb{R} \cup \{ \infty \}.
\]
Here $D : \mathcal{A}_q(SU(2)) \to \mathcal{A}_q(SU(2))$ is the bijective algebra homomorphism given by $D.\alpha = q^{-\frac{i}{2}}\alpha$, $D.\beta = q^{\frac{i}{2}}\beta$, $D.\gamma = q^{-\frac{\tau}{2}}\gamma$ and $D.\delta = q^{\frac{\tau}{2}}\delta$. The case of most interest to us is $l \in \mathbb{Z}_+$, $i = j = 0$. So (2.3) is an example of the cohomorphism property (1.2).

In order to describe the generalised matrix elements $b_{i,j}^{l}(\tau, \sigma)$ in more detail we introduce certain simple elements of $\mathcal{A}_q(SU(2))$. In case $l = \frac{1}{2}$ the $2 \times 2$-matrix $b^{\frac{1}{2}}(\tau, \sigma)$ is given by

\[
\frac{1}{\sqrt{(1 + q^{2\sigma})(1 + q^{2\tau})}} \begin{pmatrix}
\alpha_{\tau, \sigma}, & \beta_{\tau, \sigma} \\
\gamma_{\tau, \sigma}, & \delta_{\tau, \sigma}
\end{pmatrix} = b^{1/2}(\tau, \sigma) = \begin{pmatrix}
b^{1/2}_{-1/2, -1/2}, & b^{1/2}_{1/2, 1/2} \\
b^{1/2}_{1/2, -1/2}, & b^{1/2}_{1/2, 1/2}
\end{pmatrix}
\]

with

\[
\begin{align*}
\alpha_{\tau, \sigma} &= q^{1/2}\alpha - iq^{\sigma-1/2}\beta + iq^{\tau+1/2}\gamma + q^{\sigma+\tau-1/2}\delta, \\
\beta_{\tau, \sigma} &= -q^{\sigma+1/2}\alpha - iq^{-1/2}\beta - iq^{\sigma+\tau+1/2}\gamma + q^{\tau-1/2}\delta, \\
\gamma_{\tau, \sigma} &= -q^{\tau+1/2}\alpha + iq^{\tau+\sigma-1/2}\beta + iq^{1/2}\gamma + q^{\sigma-1/2}\delta, \\
\delta_{\tau, \sigma} &= q^{\sigma+1/2}\alpha + iq^{-1/2}\beta - iq^{\sigma+1/2}\gamma + q^{-1/2}\delta.
\end{align*}
\]

We let $\sigma, \tau \in \mathbb{R} \cup \{\infty\}$, e.g. $\alpha_{\infty, \sigma} = q^{1/2}\alpha - iq^{\sigma-1/2}\beta$ and $\alpha_{\infty, \infty} = q^{1/2}\alpha$. Then it follows that

\[
\begin{align*}
\alpha_{\tau, \sigma} &= \alpha_{\tau, \infty} + q^{\sigma}\beta_{\tau, \infty}, & \beta_{\tau, \sigma} &= \beta_{\tau, \infty} - q^{\sigma}\alpha_{\tau, \infty}, \\
\gamma_{\tau, \sigma} &= \gamma_{\tau, \infty} + q^{\sigma}\delta_{\tau, \infty}, & \delta_{\tau, \sigma} &= \delta_{\tau, \infty} - q^{\sigma}\gamma_{\tau, \infty}.
\end{align*}
\]

Next we define

\[
\rho_{\tau, \sigma} = \frac{1}{2}(q^{-\tau-\sigma-1}\alpha_{\tau+1, \sigma+1}\delta_{\tau, \sigma} - q^{-\tau-\sigma-1} - q^{\tau+\sigma+1})
\]

\[
= \frac{1}{2}(q^{-\tau-\sigma}\beta_{\tau+1, \sigma-1}\gamma_{\tau, \sigma} + q^{\sigma-\tau-1} + q^{\tau-\sigma+1}),
\]

cf. e.g. [37, prop. 3.3] for the last equality. Note that $\rho_{\tau, \sigma}^* = \rho_{\tau, \sigma}$. The limit case $\sigma \to \infty$ of $\rho_{\tau, \sigma}$ is defined by $\rho_{\tau, \infty} = \lim_{\sigma \to \infty} 2q^{\sigma+\tau-1}\rho_{\tau, \sigma} = q^{-1}\beta_{\tau, \infty}\gamma_{\tau, \infty} + q^{2\tau}$.

In order to be able to express $b_{i,j}^{l}(\tau, \sigma)$ in these simple terms we need Askey-Wilson polynomials $p_n(x; a, b, c, d|q)$, where we follow the normalisation as in Askey and Wilson [5, (1.15)], see also [24, (7.5.2)]. We use the following notation for the Askey-Wilson polynomials as $q$-analogues of the Jacobi polynomials;

\[
p_{n}^{(\alpha, \beta)}(x; s, t|q) = p_{n}(x; q^{1/2}t/s, q^{1/2+\alpha}s/t, -q^{1/2}/(st), -stq^{1/2+\beta}|q).
\]

Observe that $\alpha, \beta \to \infty$ in (2.7) gives Askey-Wilson polynomials with two parameters set to zero, which are the Al-Salam–Chihara polynomials. So the Al-Salam–Chihara polynomials can be considered as the Hermite case of the $q$-Jacobi polynomials (2.7), and in this form they play a role in the sequel.
The generalised matrix elements $b^l_{i,j}(\tau, \sigma)$ can be expressed using these simple elements and the Askey-Wilson polynomials. Here we use the following four cases. For $n \in \mathbb{Z}_+$ we have

$$
\begin{align*}
  b^l_{n,0}(\tau, \sigma) &= d^l_{\tau,\sigma} c^n_{n,0}(\tau, \sigma) p_{l-n}^{(n,n)}(\rho_{\tau,\sigma}; q^\tau, q^\sigma | q^2), \\
  b^l_{0,n}(\tau, \sigma) &= d^l_{\sigma,\tau} c^n_{0,n}(\tau, \sigma) p_{l-n}^{(n,n)}(\rho_{\tau,\sigma}; q^\sigma, q^\tau | q^2), \\
  b^l_{-n,0}(\tau, \sigma) &= d^l_{-\sigma,-\tau} c^n_{-n,0}(\tau, \sigma) p_{l-n}^{(n,n)}(\rho_{\tau,\sigma}; q^{-\tau}, q^{-\sigma} | q^2), \\
  b^l_{0,-n}(\tau, \sigma) &= d^l_{-\tau,-\sigma} c^n_{0,-n}(\tau, \sigma) p_{l-n}^{(n,n)}(\rho_{\tau,\sigma}; q^{-\sigma}, q^{-\tau} | q^2)
\end{align*}
$$

with the so-called minimal elements in $A_q(SU(2))$ defined by

$$
\begin{align*}
  c^n_{n,0}(\tau, \sigma) &= q^{n(1-\sigma)} \prod_{j=0}^{n-1} \delta_{\tau+2n-1-2j,\sigma-1} \gamma_{\tau+2n-2-2j,\sigma}, \\
  c^n_{0,n}(\tau, \sigma) &= q^{n(1-\tau)} \prod_{j=0}^{n-1} \delta_{\tau-1,\sigma+2n-1-2j} \beta_{\tau,\sigma+2n-2-2j}, \\
  c^n_{-n,0}(\tau, \sigma) &= q^{-n(\sigma+2\tau-2n)} \prod_{j=0}^{n-1} \beta_{\tau-2n+1+2j,\sigma-1} \alpha_{\tau-2n+2+2j,\sigma}, \\
  c^n_{0,-n}(\tau, \sigma) &= q^{-n(2\sigma+\tau-2n)} \prod_{j=0}^{n-1} \gamma_{\tau-1,\sigma+1+n+2j} \alpha_{\tau,\sigma+2+n+2j},
\end{align*}
$$

(2.8)

and with the constant given by

$$
\begin{align*}
  d^l_{\tau,\sigma} &= \frac{C^l,0(\sigma)C^l,n(\tau)q^{-l}}{(q^{2l+2n+2}; q^2)_{l-n}} = d^l_{\sigma,-\tau}, \\
  C^{l,j}(\sigma) &= q^{l+j} \left[ \frac{2l}{l-j} \right]^{1/2} \left( \frac{1 + q^{-4j-2\sigma}}{(1 + q^{-2\sigma})(-q^{2-2\sigma}; q^2)_{l-j}(-q^{2+2\sigma}; q^2)_{l+j}} \right)^{1/2}.
\end{align*}
$$

(2.9)

Note that $C^{l,j}(\sigma) = C^{l,-j}(-\sigma)$. Since we work in a non-commutative algebra, we have to be careful about the ordering in the product. We use the convention that $\prod_{i=0}^{k} \psi_i = \psi_0 \psi_1 \ldots \psi_k$.

The following identity in the $A_q(SU(2))$ is the starting point for the ‘addition formula’. The identity is obtained from (2.3) with $i = j = 0$ after applying $id \otimes \tau_{\phi/2}$. We assume...
from now on that $\tau, \sigma, \mu \in \mathbb{R}$. Explicitly,

$$d_{\tau,\rho}^{(0,0)}(\rho_{\tau,\sigma}^\phi; q^\tau, q^\sigma|q^2) = d_{\mu,\rho}^{(0,0)}(\cos \phi; q^\mu, q^\sigma|q^2)p_{l}^{(0,0)}(D, \rho_{\tau,\mu}; q^\mu, q^\sigma|q^2)$$

$$+ \sum_{n=1}^{l} d_{\mu,\tau}^{l,n}d_{\rho,\sigma}^{l,n}e^{-in\phi}(-q^{1+\sigma+\mu}e^{i\phi}, q^{1-\sigma+\mu}e^{i\phi}; q^2)_{n}p_{l-n}^{(n,n)}(\cos \phi; q^\mu, q^\sigma|q^2)$$

$$\times (D, \rho_{\tau,\mu})p_{l-n}^{(n,n)}(D, \rho_{\tau,\mu}; q^\mu, q^\sigma|q^2)$$

(2.10)

$$\sum_{n=1}^{l} d_{\mu,-\gamma}^{l,n}d_{\rho,\sigma}^{l,n}e^{-in\phi}(-q^{-1-\sigma-\mu}e^{-i\phi}, q^{1+\sigma-\mu}e^{-i\phi}; q^2)_{n}p_{l-n}^{(n,n)}(\cos \phi; q^{-\mu}, q^{-\sigma}|q^2)$$

$$\times (D, \rho_{\tau,\mu})p_{l-n}^{(n,n)}(D, \rho_{\tau,\mu}; q^{-\mu}, q^{-\sigma}|q^2),$$

where $\rho_{\tau,\sigma}^\phi = (id \otimes \tau_{\phi/2})\Delta(\rho_{\tau,\sigma})$. To this identity in $A_q(SU(2))$ we apply the $*$-representation $\pi$ in order to obtain an identity in the space $B(\ell^2(\mathbb{Z}_+))$ of bounded linear operators.

Note that if we apply the one-dimensional $*$-representation $\tau_{\theta/2}$ to (2.10) instead of $\pi$, we obtain the (degenerate) addition formula for the Askey-Wilson polynomials, cf. Noumi and Mimachi [47, thm. 4], see also [31, (3.15)], [35, (8.1)].

3. Basis of the representation space

To turn (2.10) into an identity for $q$-special functions we have to study the operators $\pi(\rho_{\tau,\sigma}^\phi)$ and $\pi(D, \rho_{\tau,\sigma})$. These operators are given as a five term recurrence operator, but it turns out that $\ell^2(\mathbb{Z}_+)$ has an orthogonal basis of eigenvectors of $\pi(\rho_{\tau,\infty})$ in which these operators are tridiagonal. This gives the opportunity to determine the eigenvectors in terms of orthogonal polynomials, and in this case the Al-Salam–Chihara polynomials are involved. The action of the minimal elements on these eigenvectors is given by certain shift operators, and thus the action of each of the generalised matrix elements can be calculated.

So we first recall the following basis of $\ell^2(\mathbb{Z}_+)$ and the action of certain operators in this basis, cf. [33, §4], [37, §3].

**Proposition 3.1.** (i) $\ell^2(\mathbb{Z}_+)$ has an orthogonal basis of the form $v_\lambda = v_\lambda(q^\tau)$, where $\lambda = -q^{2n}$, $n \in \mathbb{Z}_+$, $\lambda = q^{2\tau+2n}$, $n \in \mathbb{Z}_+$. The vector $v_\lambda$ is an eigenvector of the self-adjoint operator $\pi(\rho_{\tau,\infty})$ for the eigenvalue $\lambda$. Moreover, with the normalisation $\langle v_\lambda, e_0 \rangle = 1$ we have

$$\langle v_\lambda, v_\lambda \rangle = q^{-2n}(q^2; q^2)_n(-q^{-2\tau}; q^2)_n(-q^{2\tau}; q^2)_\infty, \quad \lambda = -q^{2n},$$

$$\langle v_\lambda, v_\lambda \rangle = q^{-2n}(q^2; q^2)_n(-q^{2\tau+2}; q^2)_n(-q^{-2\tau}; q^2)_\infty, \quad \lambda = q^{2\tau+2n}.$$  

(ii) For $\lambda = -q^{2n}$, $\lambda = q^{2\tau+2n}$, $n \in \mathbb{Z}_+$, we have

$$\pi(\alpha_{\tau,\infty})v_\lambda(q^\tau) = iq^{\frac{1}{2}-\tau}(1 + \lambda)v_{\lambda/q^2}(q^{\tau-1}), \quad \pi(\beta_{\tau,\infty})v_\lambda(q^\tau) = iq^{\frac{1}{2}}v_{\lambda}(q^{\tau-1}),$$

$$\pi(\gamma_{\tau,\infty})v_\lambda(q^\tau) = iq^{\frac{1}{2}}(q^{2\tau} - \lambda)v_{\lambda}(q^{\tau+1}), \quad \pi(\delta_{\tau,\infty})v_\lambda(q^\tau) = -iq^{\frac{1}{2}+\tau}v_{\lambda q^2}(q^{\tau+1}),$$
with the convention \( v_{-q^{-2}}(q^\tau) = 0 = v_{q^{2r-2}}(q^\tau) \).

**Remark 3.2.** The basis of proposition 3.1 induces the orthogonal decomposition \( \ell^2(\mathbb{Z}_+) = V_1 \oplus V_2 \), where \( V_1 \), respectively \( V_2 \), is spanned by \( v_{-q^{2n}} \), \( n \in \mathbb{Z}_+ \), respectively \( v_{q^{2r+2n}} \), \( n \in \mathbb{Z}_+ \). When needed we use \( V_1^\tau = V_1 \) to stress the dependence on \( \tau \).

From (2.2) and (2.1) we see that \((id \otimes \tau_{\phi/2}) \circ \Delta\) multiplies \( \alpha \) by \( e^{i\phi/2} \) and \( \beta \) by \( e^{-i\phi/2} \). From this we see that \((id \otimes \tau_{\phi/2}) \circ \Delta\) multiplies \( \alpha_{\tau,\infty} \) and \( \gamma_{\tau,\infty} \) by \( e^{i\phi/2} \) and \( \beta_{\tau,\infty} \) and \( \delta_{\tau,\infty} \) by \( e^{-i\phi/2} \) and from this we find the action of \((id \otimes \tau_{\phi/2}) \circ \Delta\) on \( \alpha_{\tau,\sigma} \), etcetera, cf. (2.5). Next recall (2.6) to find

\[
2q^{r+\sigma}(\pi \otimes \tau_{\phi/2})\Delta(\rho_{\tau,\sigma}) - q^{2\sigma-1} - q^{2r+1} \mid v_\lambda = (\pi \otimes \tau_{\phi/2})\Delta(\beta_{\tau+1,\sigma-1}\gamma_{\tau,\sigma})v_\lambda = \pi(e^{-i\phi/2}\beta_{\tau+1,\infty} - q^{\sigma-1}e^{i\phi/2}\alpha_{\tau+1,\infty})(e^{i\phi/2}\gamma_{\tau,\infty} + q^\sigma e^{-i\phi/2}\delta_{\tau,\infty})v_\lambda,
\]

and this can be calculated explicitly by proposition 3.1(ii). We obtain

\[
(3.1) \quad 2(\pi \otimes \tau_{\phi/2})\Delta(\rho_{\tau,\sigma})v_\lambda = e^{-i\phi}q^{\lambda q^2} + e^{i\phi}q^{-1}(1 - q^{-2\tau}\lambda)(1 + \lambda)v_{\lambda/q^2} + \lambda q^{1-\tau}(q^{-\sigma} - q^\sigma)v_\lambda.
\]

For convenience we restrict our attention from now on to the subspace \( V_1 \) of \( \ell^2(\mathbb{Z}_+) \). We use the notation \( v_n = v_n(q^\tau) \) for the orthogonal basis \( v_{-q^{2n}} = v_{-q^{2n}}(q^\tau) \), \( n \in \mathbb{Z}_+ \), of \( V_1 \). Put \( \lambda = -q^{2n} \) in (3.1) and compare the recurrence relation with the three-term recurrence relation for the Al-Salam–Chihara polynomials \( h_n(x) = h_n(x; s, t|q) = \lim_{\alpha, \beta \to \infty} p^{(\alpha, \beta)}_n(x; s, t|q) \), cf. [5, (1.24)], [24, (7.5.3)], [10].

\[
(3.2) \quad 2xh_n(x) = h_{n+1}(x) + (t - t^{-1})q^{n+\frac{1}{2}}s^{-1}h_n(x) + (1 - q^n)(1 + q^ns^{-2})h_{n-1}(x),
\]

to see that the operator \((\pi \otimes \tau_{\phi/2})\Delta(\rho_{\tau,\sigma})\) can be realised as the multiplication operator on a suitable weighted \( L^2 \)-space. This is the content of the following proposition, which follows from the spectral theory of Jacobi matrices, cf. Berezanskií [6, Ch. VII, §1], Dombrowski [8].

**Proposition 3.3.** Denote by \( dm \) the orthogonality measure for the Al-Salam–Chihara polynomials \( h_n(\cdot; q^\tau, q^\sigma|q^2) \) normalised by \( \int_{\mathbb{R}} dm(x) = (-q^{2\tau}; q^2)_\infty \) and define the mapping \( \Lambda: V_1 \to L^2(dm) \) by

\[
\Lambda: v_n(q^\tau) \mapsto q^{-n}e^{in\phi}h_n(\cdot; q^\tau, q^\sigma|q^2).
\]

Then \( \Lambda \) extends to a unitary operator and \( \Lambda((\pi \otimes \tau_{\phi/2})\Delta(\rho_{\tau,\sigma})) = M\Lambda \), where the multiplication operator \( M: L^2(dm) \to L^2(dm) \) is defined by \( Mf: x \mapsto xf(x) \).

**Remark 3.4.** The fact that \( \Lambda \) extends to a unitary operator follows from the determinacy of the moment problem for the Al-Salam–Chihara polynomials. The multiplication operator \( M \) is a bounded operator on \( L^2(dm) \), since the support of the orthogonality measure \( dm \) is compact.
Next we study $\pi(D,\rho_{\tau,\sigma})$. In a completely analogous way we prove

$$2\pi(D,\rho_{\tau,\sigma})v_\lambda = q^2 v_{\lambda q^2} + q^{-2}(1 - q^{-2\tau})\lambda(1 + \lambda)v_{\lambda/q^2} + \lambda q^{1-\tau}(q^{-\sigma} - q^{\sigma})v_\lambda.$$ 

Restricting to $V_1$ we see that we can find eigenvectors of $\pi(D,\rho_{\tau,\sigma})$ for the eigenvalue $y$ of the form $\sum_{n=0}^{\infty} p_n(y)v_n$ if and only if the $p_n$’s satisfy

$$2yp_0(y) = q^{-2}(1 + q^{2n+2-2\tau})(1 - q^{2n+2})p_{n+1}(y) + q^{1-\tau+2n}(q^{\sigma} - q^{-\sigma})p_n(y) + q^2 p_{n-1}(y),$$

$$2yp_0(y) = q^{-2}(1 + q^{2-2\tau})(1 - q^2)p_{1}(y) + q^{1-\tau}(q^{\sigma} - q^{-\sigma})p_0(y),$$

so that the eigenvector is completely determined by $p_0(y)$. Here we use the convention that $v_{-1} = 0$. So we find the eigenvector

$$(3.3) \quad u_y = u_y(q^\tau, q^{\sigma}) = \sum_{n=0}^{\infty} \frac{q^{2n}h_n(y; q^{\tau}, q^{\sigma}|q^2)}{(q^2, -q^{2-2\tau}; q^2)_n} v_n(q^\tau), \quad y \in \text{supp}(dm),$$

by using the three-term recurrence relation (3.2). To see that $u_y \in \ell^2(\mathbb{Z}_+)$ for $y$ in the support of $dm$ we can use the fact that the similarly defined vector with $q^{2n}$ replaced by $q^n$ is a generalised eigenvector of the self-adjoint operator $\pi(\rho_{\tau,\sigma})$ and then use [6, Ch. VII, §1.1, (1.24)]. Another way to see this is to use the asymptotic properties of the Al-Salam–Chihara polynomials as $n \to \infty$, cf. e.g. Askey and Ismail [3, §3.1].

**Proposition 3.5.** The operators $\pi(D,\alpha_{\tau,\sigma})$ and $\pi(D,\beta_{\tau,\sigma})$ map $V_1^\tau$ into $V_1^{\tau-1}$ and the operators $\pi(D,\gamma_{\tau,\sigma})$ and $\pi(D,\delta_{\tau,\sigma})$ map $V_1^\tau$ into $V_1^{\tau+1}$. Moreover,

$$\pi(D,\alpha_{\tau,\sigma}) u_y(q^\tau, q^{\sigma}) = \frac{iq^{1+\sigma}}{1 + q^{2-2\tau}}(1 + 2yq^{1-\tau-\sigma} + q^{2-2\tau-2\sigma}) u_y(q^{-1}, q^{\sigma-1}),$$

$$\pi(D,\beta_{\tau,\sigma}) u_y(q^\tau, q^{\sigma}) = \frac{iq}{1 + q^{2-2\tau}}(1 - 2yq^{1+\sigma-\tau} + q^{2+2\sigma-2\tau}) u_y(q^{-1}, q^{\sigma+1}),$$

$$\pi(D,\gamma_{\tau,\sigma}) u_y(q^\tau, q^{\sigma}) = i(1 + q^{2\tau}) u_y(q^{\tau+1}, q^{\sigma-1}),$$

$$\pi(D,\delta_{\tau,\sigma}) u_y(q^\tau, q^{\sigma}) = -iq^{\sigma}(1 + q^{2\tau}) u_y(q^{\tau+1}, q^{\sigma+1}).$$

**Proof.** First observe that $D,\gamma_{\tau,\infty} = q^{-\frac{1}{2}}\gamma_{\tau,\infty}$ and $D,\delta_{\tau,\infty} = q^{\frac{1}{2}}\delta_{\tau,\infty}$. Now (2.4) and proposition 3.1(ii) show that $\pi(D,\gamma_{\tau,\sigma})$ maps $V_1^\tau$ into $V_1^{\tau+1}$. And similarly for the other operators.

To prove the second statement we note that the results for $D,\gamma_{\tau,\sigma}$ and $D,\delta_{\tau,\sigma}$ imply the results for $D,\beta_{\tau,\sigma}$ and $D,\alpha_{\tau,\sigma}$ by (2.6), since $u_y$ is an eigenvector of $\pi(D,\rho_{\tau,\sigma})$. Recall, cf. [37, cor. 3.3],

$$(D,\gamma_{\tau,\sigma})(D,\rho_{\tau,\sigma}) = (D,\rho_{\tau+1,\sigma-1})(D,\gamma_{\tau,\sigma}).$$

Since the eigenspaces of $\pi(D,\rho_{\tau,\sigma})$ in $V_1^\tau$ are one-dimensional, this implies

$$\pi(D,\gamma_{\tau,\sigma})u_y(q^\tau, q^{\sigma}) = C u_y(q^{\tau+1}, q^{\sigma-1})$$
for some constant $C$. To calculate $C$ we take the inner product with $v_0(q^{\tau+1})$ and use that $\pi$ is a $\ast$-representation to get
\[
C(-q^{2\tau}; q^2)_\infty = \langle u_y(q^\tau, q^\sigma), \pi((D, \gamma_{\tau, \sigma})^*)v_0(q^{\tau+1}) \rangle
= \langle u_y(q^\tau, q^\sigma), \pi(-q^{-\frac{1}{2}}\beta_{\tau+1, \infty} + q^{\tau-\frac{1}{2}}\alpha_{\tau+1, \infty})v_0(q^{\tau+1}) \rangle = i(-q^{2\tau}; q^2)_\infty
\]
by (2.5), $\gamma_{\tau, \infty} = -\beta_{\tau+1, \infty}$, $\delta_{\tau, \infty} = q^{-1}\alpha_{\tau+1, \infty}$ and proposition 3.1(ii). This implies the value for $C$. The statement for $\pi(D, \delta_{\tau, \sigma})$ is proved analogously. □

**Remark 3.6.** We can rewrite the result of proposition 3.5 as identities for Al-Salam–Chihara polynomials by use of (3.3), (2.5) and proposition 3.1(ii) to find special cases of
\[
(1 - q^{2n}abcd)(1 - 2ay + a^2)p_n(y; aq,b,c,d|q) = -ap_{n+1}(y; a,b,c,d|q) + (1 - abq^n)(1 - acq^n)(1 - adq^n)p_n(y; a,b,c,d|q),
\]
which can be proved from the orthogonality relations for the Askey-Wilson polynomials, and of
\[
(1 - q^{2n-2}abcd)p_n(y; a/q,b,c,d|q) = (1 - q^{n-2}abcd)p_n(y; a,b,c,d|q)
- \frac{a}{q}(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})p_{n-1}(y; a,b,c,d|q)
\]
which is a special case of the connection coefficients in [5, §6].

From proposition 3.5 and (2.8) we obtain the following corollary by iteration.

**Corollary 3.7.** With $y = (z + z^{-1})/2$ we have
\[
\pi(D, c_{0,n}^n(\tau, \sigma)) u_y(q^{\tau}, q^{\sigma}) = q^{n(\tau+\sigma+n)}(q^{1+\sigma-\tau}z, q^{1+\sigma-\tau}/z; q^2)_n u_y(q^{\tau}, q^{\sigma+2n}),
\]
\[
\pi(D, c_{0,-n}^n(\tau, \sigma)) u_y(q^{\tau}, q^{\sigma}) = (-1)^n q^{n(\tau-\sigma+n)}
\times (-q^{1-\sigma-\tau}z, -q^{1-\sigma-\tau}/z; q^2)_n u_y(q^{\tau}, q^{\sigma-2n}).
\]

4. **Addition formula for Askey-Wilson polynomials**

The results of section 3 allow us to calculate the action of the elements in (2.10) under the representation $\pi$ on suitable vectors. In this section we show how we can obtain a very general type of addition formula for the Legendre case, i.e. $\alpha = \beta = 0$, of the $q$-Jacobi polynomials defined in (2.7). We also show how this formula covers known addition formulas for $q$-Legendre polynomials and we end with some open problems.

Apply $\pi$ to (2.10) and let the resulting identity in $B(\ell^2(\mathbb{Z}_+))$ act on the eigenvector $u_y(q^{\tau}, q^{\mu})$ of $\pi(D, \rho_{\tau, \mu})$. The action of the right hand side of (2.10) follows from corollary 3.7. To the resulting identity we apply the unitary operator $\Lambda$ of proposition 3.3, which also shows how the left hand side of (2.10) looks under $\Lambda$. In order to find the result we
need the action of $\Lambda$ on the right hand side of (2.10). For this we have to calculate the following function of $x$ in the Hilbert space $L^2(dm)$;

$$
(4.1) \quad (A \eta_y(q^\tau, q^\mu))(x) = \sum_{m=0}^{\infty} \frac{q^m e^{im\phi}}{(q^2 - q^{-2} - q^2)^m} h_m(y; q^\tau, \tau, q^2) h_m(x; q^\tau, \tau | q^2).
$$

Note that in case $\mu = \sigma$ this is the Poisson kernel for the Al-Salam–Chihara polynomials at $t = qe^{i\phi}$. This absolutely convergent expression, which can be considered as a non-symmetric Poisson kernel for the Al-Salam–Chihara polynomials, has been evaluated by Askey, Rahman and Suslov [4, §14, Case II] in terms of a very-well-poised $8\varphi_7$-series. Explicitly, if we define for $|t| < 1$, $x = \cos \psi$, $y = \cos \theta$ the function $P(t; x, y; \tau; \sigma, \mu) =$

$$
(4.2) \quad \left(\frac{-q^{-\sigma-\tau}t, q^{-1+\sigma+\tau}te^{i\theta}, q^{-1+\sigma-\tau}te^{-i\theta}, q^{-1+\mu-\tau}te^{i\phi}, q^{-1+\mu+\tau}te^{-i\phi}; q^2)^{\infty}}{(q^{2+\sigma-\tau}t, te^{i\theta}, te^{-i\theta}, te^{i\phi}, te^{-i\phi}; q^2)^{\infty}}\right)
\times sW_7(q^{\mu+\sigma-2\tau}t, -q^{\sigma+\tau}t, q^{1+\mu-\tau}e^{i\theta}, q^{1+\mu+\tau}e^{i\phi}, q^{1+\sigma-\tau}e^{i\phi}, q^{1+\sigma+\tau}e^{-i\phi}; q^2, -q^{-\sigma-\tau}t),
$$

then $(A \eta_y(q^\tau, q^\mu))(x) = P(qe^{i\phi}; x, y; \tau; \sigma, \mu)$. Here we use the standard notation for very-well-poised basic hypergeometric series as in [24, §2.1];

$$
sW_7(a; b, c, d, e, f; q, z) = s\varphi_7\left(\sqrt{a}, -\sqrt{a}, a \sqrt{a}, b, c, d, e, f; q, z\right).
$$

These remarks prove the following addition theorem for the Askey-Wilson polynomials.

**Theorem 4.1.** We have the following ‘addition formula’ for Askey-Wilson polynomials;

$$
\begin{align*}
&d_{\tau, \sigma}^{l, 0} P_{l}(0, 0)(x; q^\tau, q^\sigma | q^2) P(qe^{i\phi}; x, y; \tau; \sigma, \mu) = \\
&\quad A_0 p_{l}(0, 0)(z; q^\mu, q^\tau | q^2) P(qe^{i\phi}; x, y; \tau; \sigma, \mu) \\
&\quad + \sum_{n=1}^{l} A_n e^{-im\phi}(-q^{1+\sigma+\mu}e^{i\phi}, q^{1-\sigma+\mu}e^{i\phi}; q^2)_n (q^{1+\mu-\tau}e^{i\theta}, q^{1+\mu+\tau}e^{-i\theta}; q^2)_n \\
&\quad \times p_{l-n}^{(n, 0)}(z; q^\mu, q^\tau | q^2) p_{l-n}^{(n, 0)}(y; q^\mu, q^\tau | q^2) P(qe^{i\phi}; x, y; \tau; \sigma, \mu + 2n) \\
&\quad + \sum_{n=1}^{l} B_n e^{-im\phi}(-q^{1-\sigma-\mu}e^{i\phi}, q^{1+\sigma-\mu}e^{i\phi}; q^2)_n (-q^{1-\mu-\tau}e^{i\theta}, -q^{1-\mu+\tau}e^{-i\theta}; q^2)_n \\
&\quad \times p_{l-n}^{(n, 0)}(z; q^\mu, q^\sigma | q^2) p_{l-n}^{(n, 0)}(y; q^\mu, q^\sigma | q^2) P(qe^{i\phi}; x, y; \tau; \sigma, \mu - 2n)
\end{align*}
$$

with $A_n = d_{\mu, \tau}^{l, n} d_{\mu, \sigma}^{l, n} d_{\mu, \tau + n}^{n, \tau + n}$, $B_n = (-1)^n d_{\mu, -\tau}^{l, n} d_{\mu, -\sigma}^{l, n} d_{\mu, -\tau + n}^{n, -\tau - n}$, $x = \cos \psi$, $y = \cos \theta$, $z = \cos \phi$ and $d_{\mu, \sigma}$ and the Poisson kernel defined by (2.9) and (4.2).

Initially theorem 4.1 only holds for $x = \cos \psi$ almost everywhere as an identity in $L^2(dm)$, but by continuity it holds everywhere.
Remark 4.2. Theorem 4.1 has been derived from the identity (2.10) after applying π to it. We have only considered the resulting operator identity restricted to the subspace $V_1^\tau$ of $L^2(\mathbb{Z}_+^\tau)$. If we study the restriction to the subspace $V_2^\tau$, we obtain the same formula after replacing $\tau$, $\sigma$ and $\mu$ by $-\tau$, $-\sigma$ and $-\mu$. Replacing π by any other irreducible infinite-dimensional representation of $A_q(SU(2))$ does not lead to greater generality.

Remark 4.3. (i) To recover the (degenerate) addition formula for the 2-parameter family of Askey-Wilson polynomials, i.e. the result of applying $\tau_{\theta/2}$ to (2.10), we formally replace $e^{i\theta}$ by $q^{-1}e^{i\theta}$, $e^{i\phi}$ by $q^{-1}e^{i\phi}$, and $e^{i\psi}$ by $q^{-1}e^{i\theta+i\phi}$. For this choice of parameters the very-well-poised $s_3\varphi_7$-series in the non-symmetric Poisson kernel reduces to a very-well-poised $6\varphi_5$-series, which can be summed by [24, (II.20)].

(ii) In [37] the case $i = j = 0$, $\mu \to \infty$ of (2.3) is turned into an addition formula for the 2-parameter family of Askey-Wilson polynomials involving big $q$-Jacobi polynomials in the sum and the role of the very-well-poised $s_3\varphi_7$-series is taken over by $q$-Laguerre polynomials, i.e. the polynomials obtained from (2.7) by letting $\beta \to \infty$. In [37] $\pi \otimes \pi$ is used as the representation of $A_q(SU(2)) \otimes A_q(SU(2))$ instead of $\pi \otimes \tau_{\phi/2}$ as here. It is possible to obtain this result formally as a limit case of theorem 4.1 as follows. Replace $e^{i\phi}$ by $q^{\mu+\sigma-1}z^{-1}$ and let $\mu \to \infty$ such that $2q^{\mu+\tau-1}\cos \theta$ tends to $y$. Observe that then $2q^{\mu+\sigma-1}\cos \phi$ tends to $z$. Then we can use the limit transition of the Askey-Wilson polynomials to the big $q$-Jacobi polynomials as described by Koornwinder [42, prop. 6.1]. It remains to consider the limit case $\mu \to \infty$ of the non-symmetric Poisson kernel (4.2), which can be done directly in the very-well-poised $s_3\varphi_7$-series. The result is a $3\varphi_2$-series which can be written as a $q$-Laguerre polynomial when $y$ and $z$ are mass points of the orthogonality measure for the corresponding big $q$-Jacobi polynomials by simple transformations for $3\varphi_2$-series.

The limit case can also be evaluated directly from the infinite sum (4.1). Using the limit as described in [42, prop. 6.1] we get

$$q^{m\mu}h_m(\cos \theta; q^\tau, q^{\mu+2n}|q^2) \to y^m q^{(1-\tau)m}(-y^{-1}q^{-2n}; q^2)_m$$

as $\mu \to \infty$. The resulting sum is then a generating function for the Al-Salam–Chihara polynomials, which has been evaluated by Sergei Suslov (private communication). Explicitly, for the Al-Salam–Chihara polynomial defined by

$$s_m(\cos \psi; a, b, |q) = a^{-m}(ab|q)_m 3\varphi_2\left(\frac{q^{-m}, ae^{i\psi}, ae^{-i\psi}}{ab, 0}; q, q\right)$$

we have the generating function

$$\sum_{m=0}^\infty \frac{(u; q)_m t^m}{(q, ab; q)_m} s_m(\cos \psi; a, b |q) = \frac{(ute^{-i\psi}; q)_\infty}{(te^{-i\psi}; q)_\infty} 3\varphi_2\left(\frac{u, ae^{i\psi}, ae^{-i\psi}}{ab, ute^{-i\psi}}; q, te^{i\psi}\right).$$

Again for the choices of $y$ and $z$ as above this can be rewritten as a $q$-Laguerre polynomial.

In general it seems hard to reduce theorem 4.1 to a polynomial identity, but for the special case $\sigma = \tau = \mu = 0$ we can obtain the following addition formula for the continuous $q$-Legendre polynomials, cf. Rahman and Verma [54, (1.24)], see also [31, thm. 4.1].
**Corollary 4.4.** For the continuous $q$-ultraspherical polynomials defined by

$$C_n(\cos \psi; \beta | q) = \sum_{k=0}^{n} \frac{(\beta; q)_k(\beta; q)_{n-k}}{(q; q)_k(q; q)_{n-k}} e^{i(n-2k)\psi}$$

we have

$$C_1(\cos \psi; q^2 | q^4) = q^l C_1(\cos \phi; q^2 | q^4)C_1(\cos \theta; q^2 | q^4) +$$

$$\sum_{n=1}^{\infty} \frac{q^{l-2n}(1 + q^{4n})(q^2; q^4)_n}{(-q^2; q^2)_{2n}(q^4; q^4)_n} \left[ \frac{l + n}{2n} \right]^{-1} C_{l-n}(\cos \theta; q^{2+4n} | q^4)C_{l-n}(\cos \phi; q^{2+4n} | q^4)$$

$$\times e^{-in(\theta + \phi)}(q^2 e^{2i\theta}, q^2 e^{2i\phi}; q^4)_n 4\varphi_3 \left( \begin{array}{c} q^{-4n}, q^4n, q e^{i(\theta + \phi)} e^{i\psi}, q e^{i(\phi + \theta)} e^{-i\psi} \\ q^2, q^2 e^{2i\phi}, q^2 e^{2i\theta} \end{array} ; q^4, q^4 \right).$$

**Proof.** We can apply the same reductions as in [31, §4] to theorem 4.1. First observe

$$P(t; \cos \theta, \cos \psi; 0; 0, 0) = \frac{(t^2; q^4)_{\infty}}{(te^{i\psi + i\theta}, te^{i\psi - i\theta}, te^{i\theta - i\psi}, te^{-i\psi - i\theta}; q^4)_{\infty}}, \quad |t| < 1.$$  

The last identity can be observed from $h_n(x; 1, 1 | q^2) = H_n(x | q^4)$, (use e.g. the three-term recurrence relation) and using Rogers’s expression for the Poisson kernel of the continuous $q$-Hermite polynomials [2], [7] or by applying the summation formula [24, (II.18)]. Now the corollary follows from the following lemma, which has been proved by Mizan Rahman (private communication) using transformation and summation theorems for basic hypergeometric series.

**Lemma 4.5.** For $n \in \mathbb{Z}_+$ and the Poisson kernel $P$ defined by (4.2) we have

$$q^{n^2}(qe^{i\theta}, q^{-i\theta}; q^2)_n P(qe^{i\phi}; \cos \psi, \cos \theta; 0, 0, 2n)$$

$$+ (-1)^n q^{n^2}(-qe^{i\theta}, -q^{-i\theta}; q^2)_n P(qe^{i\phi}; \cos \psi, \cos \theta; 0, 0, -2n) =$$

$$\frac{2q^{-in\theta}(q^{2(i\theta + i\phi)}, q^{2i\phi}; q^4)_{\infty}}{(qe^{i\phi + i\psi + i\theta}, q e^{i\phi + i\psi - i\theta}, q e^{i\phi + i\theta - i\psi}, q e^{i\phi - i\psi - i\theta}; q^4)_{\infty}}$$

$$\times 4\varphi_3 \left( \begin{array}{c} q^{-4n}, q^4n, q e^{i(\theta + \phi)} e^{i\psi}, q e^{i(\phi + \theta)} e^{-i\psi} \\ q^2, q^2 e^{2i\phi}, q^2 e^{2i\theta} \end{array} ; q^4, q^4 \right).$$

Of course, it also works the other way round; assuming theorem 4.1 and corollary 4.4 gives lemma 4.5. Since Rahman’s proof is limited to the case $\sigma = \tau = \mu = 0$ we do not give it here. □

**Remark 4.6.** (i) Because of the similarity between (2.3) and the group theoretic proof of the addition formula for the Legendre polynomials, cf. [61, §III.4.2], [62, §6.6.1], we want to have a limit transition of theorem 4.1 to the addition formula for the Legendre polynomials as $q \uparrow 1$. A straightforward limit does not seem possible in general, but it does work for the Rahman-Verma addition formula in corollary 4.4, cf. [54]. Also the more sophisticated
technique of Van Assche and Koornwinder [60], which can be used to handle the result for the case \( \mu = \infty \), is not applicable, since we do not have a three-term recurrence relation for the non-symmetric Poisson kernel in \( \mu \), i.e. a simple relation between the non-symmetric Poisson kernels with \( \mu - 2 \), \( \mu \) and \( \mu + 2 \).

(ii) Usually an addition formula leads to a product formula. In this case this can be done if \( P(qe^{i\phi}; x, y; \tau; \sigma, \mu + 2n) \) is part of a set of e.g. biorthogonal rational functions in \( x \) with respect to \( n \). This seems not known.

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