Geodesic completeness of generalized space-times

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Abstract. We define the notion of geodesic completeness for semi-Riemannian metrics of low regularity in the framework of the geometric theory of generalized functions. We then show completeness of a wide class of impulsive gravitational wave space-times.

Keywords: Semi-Riemannian geometry, low regularity, completeness, impulsive gravitational waves

MSC classes: 46F30 (Primary), 83C15, 83C35 (Secondary)

1. Introduction

The geometric theory of generalized functions (GKOS01) based on Colombeau algebras (Col85) is by now a well-established field within generalized functions. It has proved to be widely applicable in geometric situations, such as Lie-group analysis of differential equations (e.g. KO00, DKP02, KK06), wave-type equations on Lorentzian manifolds (GMS09, HKS12) and various problems in general relativity (see SV06, NS13 for an overview).

The applications to relativity in particular include the study of the geometry of impulsive gravitational waves (introduced by Penrose (Pen72), for a thorough review see Pod02) which are key-examples for exact space-times modeling a gravitational wave pulse. The simplest of these geometries arises as the impulsive limit of plane fronted gravitational waves with parallel rays (pp-waves) described by the line-element (e.g. GP09, Ch. 17)

\[ ds^2 = -2dudv + dx^2 + dy^2 + H(x,y,u)du^2, \]

on \( \mathbb{R}^4 \), where \( H \) is a smooth function. Since the field equations put no restriction on the \( u \)-dependence of \( H \) one can perform the so-called impulsive limit by basically setting (for details see e.g. GP09, Ch. 20)

\[ H(x,y,u) = f(x,y) \delta(u), \]

where \( f \) is smooth and \( \delta \) denotes the Dirac-measure. The resulting space-time is flat everywhere but on the null-hypersurface \( \{ u = 0 \} \), where a gravitational wave impulse is located. In KS99 it was shown that the geodesic equation for these geometries possess unique and globally defined solutions in nonlinear generalized functions, although the global aspect was not emphasized there. It only recently came back into focus in the context of causality theory for Lorentzian metrics of low regularity (CGT12, KSS14).

This work is in particular motivated by a recent result of the authors (SS12) which provides a completeness statement for a wider class of impulsive radiative geometries, which have been called impulsive N-fronted waves with parallel rays (INPWs). These space-times are the impulsive limits of geometries studied originally by Brinkmann in the context of conformal mappings of Einstein spaces (Br25) and are of the following form: Let \( (N,h) \) be a connected Riemannian manifold of dimension \( n \), set \( M = N \times \mathbb{R}^2 \) and equip \( M \) with the line element

\[ ds^2 = dh^2 + 2dudv + H(x,u)du^2, \]

where \( dh^2 \) denotes the line element of \( (N,h) \). Moreover \( (u,v) \) are global null-coordinates on the 2-dimensional Minkowski space \( \mathbb{R}^2 \) and \( H : N \times \mathbb{R} \to \mathbb{R} \) is a smooth function. The causality and
the geodesics of such models have been studied in a series of papers \cite{CFS03,FS03,CFS04,FS06} since they allow to shed light on some of the peculiar causal properties of plane waves (i.e., pp-waves \cite{BEE96} with \(H(x^1,x^2,u) = A_{ij}(u)x^i x^j\), see e.g. \cite[Ch. 13]{BEE96}). INPWs now arise as the impulsive limit of \(\mathcal{M}\), i.e., upon setting \(H(x,u) = f(x)\delta(u)\). (4)

Now for the above mentioned completeness result \cite{SS12} the INPW-metric \(g\) was replaced by a net of regularizing metrics \(g_\varepsilon\) where the Dirac-\(\delta\) is replaced by a strict delta-net \(\delta_\varepsilon\) (for a precise definition see below). More explicitly it deals with the net of line elements

\[
\begin{align*}
    ds_\varepsilon^2 &= dh^2 + 2dudv + f(x)\delta_\varepsilon(u)du^2,
\end{align*}
\]

on \(M\), which physically amounts to viewing the impulsive wave as a limit of extended sandwich waves with small support but increasing amplitude of the “profile function” \(\delta_\varepsilon\). The result now states that given any geodesic \(\gamma\) in (the smooth) space-time \((M, g_\varepsilon)\) (for \(\varepsilon\) fixed) there is \(\varepsilon_0\) small enough, such that \(\gamma\) can be defined for all values of an affine parameter provided \(\varepsilon \leq \varepsilon_0\). Also \(\varepsilon_0\), for which the geodesic \(\gamma\) becomes complete can be explicitly estimated in terms of (derivatives of) \(f\) and the initial data of \(\gamma\). The obvious drawback of this statement is that \(\varepsilon_0\) depends on \(\gamma\) i.e., in general there is no uniform \(\varepsilon_0\) which renders the space-times \((M, g_\varepsilon)\) for fixed \(\varepsilon\) geodesically complete for all \(\varepsilon \leq \varepsilon_0\).

In this paper we define a notion of completeness for generalized metrics that will allow to formulate the above completeness result in clear analogy to classical completeness statements. This, in particular, beautifully exhibits the virtues of a well-founded theory of generalized functions.

For convenience of the reader and to keep this presentation self-contained we start with a brief review of semi-Riemannian geometry within the geometric theory of generalized functions. At the end of section \ref{sec:gr} we define geodesic completeness and in section \ref{sec:geoc} we prove geodesic completeness of large classes of impulsive gravitational waves. Finally we discuss associated distributions of the global geodesics.

2. Generalized semi-Riemannian geometry

Colombeau algebras of generalized functions \cite{Col88} are differential algebras which contain the vector space of distributions and display maximal consistency with classical analysis. Here we review Lorentzian geometry based on the special Colombeau algebra \(\mathcal{G}(M)\), for further details see \cite{KS02, KS02b} and \cite[Sec. 3.2]{GKS04}.

Let \(M\) be a smooth, second countable Hausdorff manifold. Denote by \(\mathcal{E}(M)\) the set of all nets \((u_\varepsilon)_{\varepsilon \in (0,1]} =: (f)\) in \(C^\infty(M)^\gamma\) depending smoothly on \(\varepsilon\). Note that smooth dependence on the parameter (which was not assumed in the earlier references) renders the theory technically more pleasant, while not changing any of the basic properties, see the discussion in \cite[Section 1]{BK12}. The algebra of generalized functions on \(M\) \cite{DRD94} is defined as the quotient \(\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)\) of moderate modulo negligible nets in \(\mathcal{E}(M)\), where the respective notions are defined by the following asymptotic estimates

\[
\begin{align*}
    \mathcal{E}_M(M) := \{ (u_\varepsilon) \in \mathcal{E}(M) : \forall K \subset M \forall P \in \mathcal{P} \exists N : \sup_{p \in K} |P u_\varepsilon(p)| = O(\varepsilon^{-N}) \},
\end{align*}
\]

\[
\begin{align*}
    \mathcal{N}(M) := \{ (u_\varepsilon) \in \mathcal{E}_M(M) : \forall K \subset M \forall m : \sup_{p \in K} |u_\varepsilon(p)| = O(\varepsilon^m) \},
\end{align*}
\]

where \(\mathcal{P}\) denotes the space of all linear differential operators on \(M\). Elements of \(\mathcal{G}(M)\) are denoted by \(u = [(u_\varepsilon)]\). With componentwise operations and the Lie derivative with respect to smooth vector fields \(\xi \in \mathfrak{X}(M)\) defined by \(L_\xi u := \[(L_\xi u_\varepsilon)\]\), \(\mathcal{G}(M)\) is a fine sheaf of differential algebras. There exist embeddings \(\iota \circ \mathcal{D}'(M)\) into \(\mathcal{G}(M)\) that are sheaf homomorphisms and render \(C^\infty(M)\) a subalgebra of \(\mathcal{G}(M)\). Another, more coarse way of relating generalized functions in \(\mathcal{G}(M)\) to distributions is based on the notion of association: \(u \in \mathcal{G}(M)\) is called associated with \(v \in \mathcal{G}(M)\), \(u \approx v\), if \(u_\varepsilon - v_\varepsilon \to 0\) in \(\mathcal{D}'(M)\). A distribution \(w \in \mathcal{D}'(M)\) is called associated with \(u\) if \(u \approx \iota(w)\).
The ring of constants in $G(M)$ is the space $\mathbb{R}$ of generalized numbers, which form the natural space of point values of Colombeau generalized functions. These, in turn, are uniquely characterized by their values on so-called compactly supported generalized points.

A similar construction is in fact possible for any locally convex space $F$ in place of $C^\infty(M)$ ([Gar03]), in particular, $F = \Gamma(M, E)$, the space of smooth sections of a vector bundle $E \to M$. The resulting space $\Gamma_G(M, E)$ then is the $G(M)$-module of generalized sections of the vector bundle $E$ and can be written as

$$\Gamma_G(M, E) = G(M) \otimes_{C^\infty(M)} \Gamma(M, E) = L_{C^\infty(M)}(\Gamma(M, E^*), G(M)).$$

$\Gamma_G$ is a fine sheaf of finitely generated and projective $G$-modules. For the special case of generalized tensor fields of rank $r,s$ we use the notation $G^r_s(M)$, i.e.

$$G^r_s(M) \cong L_{G(M)}(G^0_r(M)^k, G^0_s(M)^r; G(M)).$$

Observe that this allows the insertion of generalized vector fields and one-forms into generalized tensors, which is essential when dealing with generalized metrics which we define as follows: $g \in G_0^0(M)$ is called a generalized pseudo-Riemannian metric if it is symmetric $(g(\xi, \eta) = g(\eta, \xi)$ $\forall \xi, \eta \in X(M)$), its determinant $\det g$ is invertible in $G$ (equivalently $|\det(g_{ij})| > \varepsilon^m$ for some $m$ on compact sets), and it possesses a well-defined index $\nu$ (the index of $g_\nu$ equals $\nu$ for $\varepsilon$ small). By a “globalization Lemma” in ([HKS12, Lem. 4.3]) any generalized metric $g$ possesses a representative $(g_\varepsilon)_\varepsilon$ such that each $g_\varepsilon$ is a smooth metric globally on $M$.

Based on this definition, many notions from (pseudo-)Riemannian geometry can be extended to the generalized setting. In particular, any generalized metric induces an isomorphism between generalized vector fields and one-forms, and there is a unique Levi-Civita connection $\nabla$ corresponding to $g$. This provides a convenient framework for non-smooth pseudo-Riemannian geometry and for the analysis of space-times of low regularity in general relativity which extends the “maximal distributional” setting of [GT87] ([SV09, Ste08]).

Finally we want to discuss geodesics in generalized semi-Riemannian manifolds (for details see [KS02][5]). To this end we have to introduce generalized functions taking values in the manifold $M$. More precisely, the space of generalized functions defined on a manifold $N$ taking values in $M$, $G[N, M]$ is again defined as a quotient of moderate modulo negligible nets $(f_\varepsilon)_\varepsilon$ of maps from $N$ to $M$, where we call a net moderate (negligible) if $(\psi \circ f_\varepsilon)_\varepsilon$ is moderate (negligible) and for all smooth $\psi : M \to \mathbb{R}$.

The induced covariant derivative of a generalized vector field $\xi$ on a generalized curve $\gamma = [(\gamma_\varepsilon)_\varepsilon] \in G[J, M]$ (with $J$ a real interval), can be defined componentwise and gives again a generalized vector field $\xi^\gamma$ on $\gamma$. In particular, a geodesic in a generalized pseudo-Riemannian manifold is a curve $\gamma_\varepsilon \in G[J, M]$ satisfying $\gamma'' = 0$. Equivalently the usual local formula holds, i.e.,

$$\left[ \left( \frac{d^2 \gamma^k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{d \gamma^i}{dt} \frac{d \gamma^j}{dt} \right) \right] = 0,$$

where $\Gamma^k_{ij} = [(\Gamma^k_{ij})_\varepsilon]$ denotes the Christoffel symbols of the generalized metric $g = [(g_{\varepsilon})_\varepsilon]$.

We now give the following (natural) definition.

**Definition 2.1.** (Geodesic completeness for generalized metrics) Let $g \in G_0^0(M)$ be a generalized semi-Riemannian metric. Then the generalized space-time $(M, g)$ is said to be geodesically complete if every geodesic $\gamma$ can be defined on $\mathbb{R}$, i.e., every solution of the geodesic equation

$$\gamma'' = 0,$$

is in $G[\mathbb{R}, M]$.

### 3. Geodesic completeness of impulsive gravitational wave space-times

In this section we prove geodesic completeness for a large class of impulsive gravitational waves. More precisely, we will turn the distributional metrics discussed in the introduction into generalized...
metrics and then show that their geodesics can be defined for all values of the parameter, i.e., they belong to \( G[\mathbb{R}, M] \).

We begin by defining the very general class of regularizations used to turn the distributional metrics of impulsive wave space-times into generalized metrics.

**Definition 3.1.** A generalized delta-function is an element \( D \in G(\mathbb{R}) \) that has a strict delta net \((\delta_\varepsilon)_{\varepsilon \in I} \) as a representative, that is \((\delta_\varepsilon)_{\varepsilon \in I} \) satisfies the following properties

1. \( \text{supp}(\delta_\varepsilon) \subseteq (-\varepsilon, \varepsilon) \forall \varepsilon \in I \),
2. \( \int_{\mathbb{R}} \delta_\varepsilon(x) dx \to 1 \) for \( (\varepsilon \searrow 0) \) and
3. \( \exists C > 0 : \| \delta_\varepsilon \|_{L^1} = \int_{\mathbb{R}} |\delta_\varepsilon(x)| dx \leq C \forall \varepsilon \in I \).

We may now define the generalized metrics used in the following, the impulsive \( pp \)-wave (cf. (1), (2))

\[
\text{ds}^2 = -2 du dv + dx^2 + dy^2 + f(x, y) D(u) du^2,
\]

on \( M = \mathbb{R}^4 \), and the INPW (cf. (3), (4))

\[
\text{ds}^2 = dh^2 + 2 dv du + f(x) D(u) du^2,
\]

on \( M = N \times \mathbb{R}^2 \), where as in (3), \( (N, h) \) is a connected \( n \)-dimensional Riemannian manifold, which from now on we suppose to be complete. Here \( D \) denotes an arbitrary generalized delta function. For the impulsive \( pp \)-waves completeness follows from earlier results. More precisely, by [KS99 Thm. 1] the geodesic equation has (unique) solutions in \( G[\mathbb{R}, M] \) so that we may state the following result.

**Corollary 3.2.** (to [KS99 Thm. 1] — Completeness of impulsive \( pp \)-waves) The generalized space-time \((\mathbb{R}^4, g)\) with the metric \( g \) given by (2) is geodesically complete.

This result is of course a special case of completeness of INPWs (just set \( N = \mathbb{R}^2 \) with the flat metric) which we prove next. We first have to derive an analog of [KS99, Thm. 1], which will be based on [SS12, Thm. 3.2].

As detailed in [SS12, Sec. 2] it is possible to choose the coordinate \( u \) as an affine parameter along the geodesics, thereby only excluding trivial geodesics parallel to the impulse. Hence the geodesic equations of the space-time (10) take the form

\[
\dot{v}(u) = - \sum_{j=1}^n \frac{\partial f}{\partial x^j} (x(u)) \dot{x}^j (u) D(u) - \frac{1}{2} f(x(u)) \dot{D}(u),
\]

\[
\nabla^h_{\dot{x}} \dot{x}(u) = \frac{1}{2} \text{grad}^h (f(x(u))) \ D(u),
\]

where \( \nabla^h \) denotes the covariant derivative on \((N, h)\), \( \text{grad}^h \) is the gradient with respect to \( h \) on \( N \) and \((x^1, \ldots, x^n) \) are coordinates on \( N \). We immediately see that the \( v \)-equation is linear and decouples from the rest of the system. So it can simply be integrated and we mainly have to deal with the second equation, which actually is the disturbed geodesic equation on \( N \) with a potential given by \( \frac{1}{2} \text{grad}^h (f) D \).

We now give a global existence and uniqueness result for the system (11), where we conveniently choose data in front of the impulse at \( u = -1 \).

**Theorem 3.3.** (Existence and uniqueness for geodesics in INPW) Let \( D \in G(\mathbb{R}) \) be a generalized delta function, \( f \in C^\infty(N) \), let \( v_0, \dot{v}_0 \in \mathbb{R}, \ x_0 \in N \) and \( \dot{x}_0 \in T_{x_0} N \). The initial value problem (11) with data

\[
v(-1) = v_0, \ x(-1) = x_0, \ \dot{v}(-1) = \dot{v}_0, \ \dot{x}(-1) = \dot{x}_0,
\]

has a unique solution \((v, x) \in G[\mathbb{R}, \mathbb{R} \times N] \).

This immediately gives our main result.

**Corollary 3.4.** (Completeness of INPW) The generalized space-time \((M, g)\) with the metric \( g \) given by (11) is geodesically complete.
In the (uniqueness part of the) proof of the theorem we need the following variant of [SS12, Lem. A.2].

**Lemma 3.5.** Let $F_1 \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}^n)$, $F_2 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, let $J := [-1, 1]$, $k \in C^\infty(J, \mathbb{R}^n)$ be bounded, let $x_0, \dot{x}_0 \in \mathbb{R}^n$, let $b > 0$, $c > 0$ be given and let $(\delta_\varepsilon)_\varepsilon$ be a strict delta net with $L^1$-bound $C > 0$. Define $I_1 := \{ x \in \mathbb{R}^n : |x-x_0| \leq b \}$, $I_2 := \{ x \in \mathbb{R}^n : |x-x_0| \leq c + C\|F_2\|_{L_{1,\infty}} \}$ and $I_3 := I_1 \times I_2$. Moreover set

$$\alpha := \min(1, \frac{b}{\|x_0\| + \|F_1\|_{L_{1,\infty}} + C\|F_2\|_{L_{1,\infty}} + \|k\|_{L_{\infty}} \|F_1\|_{L_{1,\infty}} + \|k\|_{L_{\infty}}}) \quad (12)$$

Then the regularized problem

$$\begin{cases}
\ddot{x} = F_1(x, \dot{x}) + F_2(x)\delta_\varepsilon + k, \\
x(-\varepsilon) = x_0, \quad \dot{x}(-\varepsilon) = \dot{x}_0,
\end{cases} \quad (13)$$

has a unique solution $x_\varepsilon$ on $J_\varepsilon := [-\varepsilon, \alpha - \varepsilon]$. Moreover $x_\varepsilon$ and $\dot{x}_\varepsilon$ are locally uniformly bounded. Finally the result remains true if we replace $k$ by a net $(k_\varepsilon)_\varepsilon$ in $C^\infty(J, \mathbb{R}^n)$ which is uniformly bounded.

The proof is obtained by adapting the proof of [SS12, Lem. A.2], so that it is not necessary to give it here. Note that by classical ODE-theory Lemma 3.5 gives global uniqueness (not only in the function space $X_\varepsilon$ used in the proof of [SS12, Lem. A.2]).

**Proof of the Theorem.** We proceed in three steps. First we need to obtain a ”solution candidate”, i.e., a net of smooth solutions $(\nu_\varepsilon, x_\varepsilon)_\varepsilon$ defined for all of $\mathbb{R}$ (at least for small $\varepsilon$) of the regularized initial value problem

$$ \begin{align*}
\tilde{v}_\varepsilon &= -\delta_\varepsilon \sum_{j=1}^n \frac{\partial f}{\partial x^j}(x_\varepsilon)\dot{x}_\varepsilon^j - \frac{1}{2} f(x_\varepsilon)\delta_\varepsilon, \\
\ddot{x}_\varepsilon^k &= -\sum_{i,j=1}^n \Gamma_{ij}^{k(N)}(x_\varepsilon)\dot{x}_\varepsilon^i \dot{x}_\varepsilon^j + \frac{1}{2} \delta_\varepsilon \sum_{m=1}^n h^{km}(x_\varepsilon)\frac{\partial f}{\partial x^m}(x_\varepsilon), \\
v_\varepsilon(-1) &= v_0, \quad x_\varepsilon(-1) = x_0, \quad \dot{v}_\varepsilon(-1) = \dot{v}_0, \quad \dot{x}_\varepsilon(-1) = \dot{x}_0,
\end{align*} \quad (14)$$

where $\Gamma_{ij}^{k(N)}$ denotes the Christoffel symbols of the (smooth) metric $h$ on $N$. By [SS12, Thm. 3.2] we obtain such a solution for all $\varepsilon$ smaller than a certain $\varepsilon_0$ (which depends on $\dot{x}_0$ as well as on $\Gamma_{ij}^{k(N)}$ and $\text{grad}^k(f)$ on a neighborhood of the point where the geodesic crosses the impulse). Since we are interested in generalized solutions we may choose $(\nu_\varepsilon, u_\varepsilon)$ arbitrarily (yet smoothly depending on $\varepsilon$) for all $\varepsilon_0 \leq \varepsilon \leq 1$.

In the second step we prove existence of solutions in $G[\mathbb{R}, \mathbb{R} \times N]$, that is we establish that the net obtained in the first step is moderate. To this end we have to show that $v_\varepsilon$ and $\psi \circ x_\varepsilon$ are moderate for arbitrary smooth $\psi : N \to \mathbb{R}$. But by Lemma 3.5 (with with $b > 0$, $c > 0$, $k = 0$, $F_1(y, z)^k := -\Gamma_{ij}^{k(N)(y)}(y)z^i z^j$, $F_2(y)^k := \frac{1}{2} h^{km}(y)\frac{\partial f}{\partial x^m}(y)$) the solution $x_\varepsilon$ and its derivative $\dot{x}_\varepsilon$ are even uniformly bounded on compact subsets of $\mathbb{R}$. Using the differential equation inductively, we see that in fact even all higher order derivatives of $x_\varepsilon$ are also locally uniformly bounded. Now from the $v$-equation it follows that $\tilde{v}_\varepsilon$ obeys an $O(\varepsilon^{-2})$-estimate and inductively all higher order derivatives obey $O(\varepsilon^{-k})$-estimates. The estimates for $v_\varepsilon$ and $\ddot{v}_\varepsilon$ simply follow by integration. Smooth dependence on $\varepsilon$ is immediate.

In the third step it remains to show uniqueness. To this end suppose that $(\tilde{v}, \tilde{x}) \in G[\mathbb{R}, \mathbb{R} \times N]$ is a solution of (11) as well. Writing $\tilde{v} = [(\tilde{v}_\varepsilon)_\varepsilon]$ and $\tilde{x} = [(\tilde{x}_\varepsilon)_\varepsilon]$, there exist $[(a_\varepsilon)_\varepsilon] \in N(\mathbb{R})$,
\[(b_\varepsilon)_\varepsilon \in \mathcal{N}(N)\text{ and negligible generalized numbers } [(c_\varepsilon)_\varepsilon], [(\dot{c}_\varepsilon)_\varepsilon], [(d_\varepsilon)_\varepsilon], [(\dot{d}_\varepsilon)_\varepsilon], \text{ such that}\]
\[
\begin{align*}
\ddot{v}_\varepsilon &= -\delta_\varepsilon \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(\tilde{x}_\varepsilon)\dot{x}^j_\varepsilon - \frac{1}{2}f(\tilde{x}_\varepsilon)\delta_\varepsilon + a_\varepsilon, \\
\ddot{\tilde{x}}_\varepsilon^k &= -\sum_{i,j} \Gamma_{ij}^{k(N)}(\tilde{x}_\varepsilon)\dot{x}^i_\varepsilon\dot{x}^j_\varepsilon + \frac{1}{2}\delta_\varepsilon \sum_{m=1}^{n} h_{km}(\tilde{x}_\varepsilon)\frac{\partial f}{\partial x^m}(\tilde{x}_\varepsilon) + b_\varepsilon^k,
\end{align*}
\]
\[
\tilde{v}_\varepsilon(-1) = v_0 + c_\varepsilon, \quad \tilde{x}_\varepsilon(-1) = x_0 + d_\varepsilon.
\]

We have to show that \((x_\varepsilon - \tilde{x}_\varepsilon)_\varepsilon\) is negligible. To this end we will, however, also estimate \((\dot{x}_\varepsilon - \dot{\tilde{x}}_\varepsilon)_\varepsilon\).

Now by Lemma [5.5] we know that \((x_\varepsilon)_\varepsilon, (\tilde{x}_\varepsilon)_\varepsilon\) are locally uniformly bounded. Moreover using the integral formulas for \(x_\varepsilon\) respectively for \(\tilde{x}_\varepsilon\) and that \((a_\varepsilon)_\varepsilon, (b_\varepsilon)_\varepsilon\) are negligible, we obtain that

\[
\forall T > 0 \forall \eta \in \mathbb{N} \exists K_1, K_2 > 0 \exists \eta > 0 \text{ such that } \forall \varepsilon \in (0, \eta) \forall u \in [-T, T]:
\]
\[
\begin{align*}
|x_\varepsilon(u) - \tilde{x}_\varepsilon(u)| &\leq K_1\varepsilon^2 + \int_{-\varepsilon}^{u} |F_1(x_\varepsilon(r), \dot{x}_\varepsilon(r)) - F_1(\tilde{x}_\varepsilon(r), \dot{\tilde{x}}_\varepsilon(r))|drds + \\
&\int_{-\varepsilon}^{u} \int_{-\varepsilon}^{s} |F_2(x_\varepsilon(s), \dot{x}_\varepsilon(s)) - F_2(\tilde{x}_\varepsilon(s), \dot{\tilde{x}}_\varepsilon(s))|\delta_\varepsilon(r)drds \\
&\leq K_1\varepsilon^2 + C_3 \int_{-\varepsilon}^{u} \int_{-\varepsilon}^{s} |(x_\varepsilon(s) - \tilde{x}_\varepsilon(s)) + |\dot{x}_\varepsilon(s) - \dot{\tilde{x}}_\varepsilon(s)||drds + \\
&C_4 \int_{-\varepsilon}^{u} \int_{-\varepsilon}^{s} |(x_\varepsilon(s) - \tilde{x}_\varepsilon(s))|drds.
\end{align*}
\]

Similarly for the derivatives:
\[
\begin{align*}
|x_\varepsilon(u) - \tilde{x}_\varepsilon(u)| &\leq K_2\varepsilon^2 + \int_{-\varepsilon}^{u} |F_1(x_\varepsilon(s), \dot{x}_\varepsilon(s)) - F_1(\tilde{x}_\varepsilon(s), \dot{\tilde{x}}_\varepsilon(s))|ds + \\
&\int_{-\varepsilon}^{u} |F_2(x_\varepsilon(s)) - F_2(\tilde{x}_\varepsilon(s))|\delta_\varepsilon(s)ds \\
&\leq K_2\varepsilon^2 + C_3 \int_{-\varepsilon}^{u} |(x_\varepsilon(s) - \tilde{x}_\varepsilon(s)) + |\dot{x}_\varepsilon(s) - \dot{\tilde{x}}_\varepsilon(s)||ds + \\
&C_4 \int_{-\varepsilon}^{u} |(x_\varepsilon(s) - \tilde{x}_\varepsilon(s))|\delta_\varepsilon(s)ds.
\end{align*}
\]

Here we have used the mean value theorem to obtain the constants \(C_3, C_4\). Adding these two inequalities and setting \(\psi(u) := |x_\varepsilon(u) - \tilde{x}_\varepsilon(u)| + \|\dot{x}_\varepsilon(u) - \dot{\tilde{x}}_\varepsilon(u)\| \text{ (for } u \in [-T, T]) \text{ yields}\)
\[
\psi(u) \leq (K_1 + K_2)\varepsilon^2 + \int_{-\varepsilon}^{u} (C_3 + C_4|\delta_\varepsilon(s)|)\psi(s)ds + \int_{-\varepsilon}^{u} \int_{-\varepsilon}^{s} (C_3 + C_4|\delta_\varepsilon(r)|)\psi(r)drds.
\]

Then by a generalization of Gronwall’s inequality (due to Bykov [BS92 Thm.11.1]) we get that
\[
\psi(u) \leq (K_1 + K_2)\varepsilon^2 \exp \left( \int_{-\varepsilon}^{u} (C_3 + C_4|\delta_\varepsilon(s)|)ds + \int_{-\varepsilon}^{u} \int_{-\varepsilon}^{s} (C_3 + C_4|\delta_\varepsilon(r)|)drds \right) \leq K'\varepsilon^g,
\]
where we used the fact that \(u \in [-T, T]\) and the uniform \(L^1\)-bound on \(\delta_\varepsilon\), i.e., \(3.13\) This shows that \((x_\varepsilon - \tilde{x}_\varepsilon)\) is negligible (by [GKOS01 Thm. 1.2.3]). Furthermore since \((v_\varepsilon - \tilde{v}_\varepsilon)\) can be obtained by integrating \((x_\varepsilon - \tilde{x}_\varepsilon)\) we conclude that it is negligible too.

Finally we briefly discuss what can be said classically about the geodesics, that is we provide associated distributions for the unique global geodesics \((u, x) \in \mathcal{G}[\mathbb{R}, \mathbb{R} \times N]\) obtained in Theorem 3.3 In [SS12 Sec. 4] it was shown that
\[
x_\varepsilon \approx y, \quad v_\varepsilon \approx w - \frac{1}{2} f(x(0))H - \sum_{j=1}^{n} \left( \dot{x}^j(0) + \frac{1}{4} \text{grad}^h(f)(x(0)) \right) \partial_j f(x(0))u_+,
\]
where the first relation even holds in the sense of 0-association, i.e., the convergence is locally uniformly. Here the limit $y$ is given by pasting together appropriate (unperturbed) geodesics of the background $(N, h)$, i.e.,

$$y(u) := \begin{cases} x(u) & u \leq 0, \\ \hat{x}(u) & u \geq 0, \end{cases}$$  \hspace{1cm} (15)$$

where $x$ and $\hat{x}$ are solution of $\nabla^h_\dot{x} \dot{x} = 0$ with data $x(-1) = x_0, \dot{x}(-1) = \dot{x}_0$, and $\hat{x}(0) = x(0), \hat{\dot{x}}(0) = \hat{x}(0) + \frac{1}{2} \text{grad}^h(f)(x(0))$ respectively. Moreover $w(u) = v_0 + \dot{v}_0(1 + u)$, $H$ is the Heaviside function and $u_+(u) = uH(u)$ denotes the kink function.

Hence the $x$-component is continuous, while in general it is not differentiable at the point where it hits the impulse, i.e., at $u = 0$, with its derivative having a jump there. Furthermore the $v$-component is discontinuous, it has a jump at the impulse and a $\delta$ in its derivative. We also observe that the parameters of the jump and the refraction at $u = 0$ are given in terms of $f$ and its derivative at the point where the geodesic hits the impulse.

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