Inhomogeneous quenches in a free fermionic chain: Exact results

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Abstract – We consider the non-equilibrium physics induced by joining together two tight-binding fermionic chains to form a single chain. Before being joined, each chain is in a many-fermion ground state. The fillings (densities) in the two chains might be different. We present a number of exact results, focusing on two-point correlators and the Loschmidt echo (return probability). For the non-interacting case, we identify through an exact derivation the regime in which a semiclassical ansatz is valid. We present a number of analytical results beyond semiclassics, such as the approach to the non-equilibrium steady state and the appearance of Tracy-Widom distributions at the front of the light cone. The light cone behavior is quantified through a series expansion in time, and this description is shown to be valid for interacting systems as well. Results on the Loschmidt echo, presented for finite and zero interactions, illustrate that the physics is different from both local and global quenches.

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Introduction. – Non-equilibrium evolution of many-body quantum systems in the absence of external baths, i.e., unitary dynamics, is the focus of an explosive growth of interest [1,2]. In particular, quantum quenches (both global or local) are by now a widely studied paradigm for understanding collective quantum dynamics. Concepts of many-body quantum physics often take on new significance in the absence of dissipation. For example, transport under unitary dynamics, now starting to be explored through cold-atom experiments [3–5], is quite different from the traditional solid-state situation with near-ideal baths and leads.

In this work, we study a quantum quench which, although injecting an extensive amount of energy into the system, has mostly local effects, and at the same time can lead to a current-carrying steady state. We take two equal long tight-binding chains, each of length $L/2$, in their respective ground states, but with a possibly different number of spinless fermions. At time $t = 0$ the two edges are connected so that they turn into a single chain of length $L$.

We denote by $k_F^l$ ($k_F^r$) the Fermi momenta on the left (right), so that the particle number is $k_F^{\ell(r)}L/(2\pi)$ on the left (right). We focus on the symmetric case $k_F^l + k_F^r = \pi$. We consider fermions with nearest-neighbor interactions; this is equivalent (for many purposes) to the anisotropic Heisenberg (XXZ) chain.

Particular instances of this quench protocol have been the subject of recurrent interest for more than a decade. When the fillings are equal, $k_F^l = k_F^r = \pi/2$, this is a true local quench. This situation was studied in [6–11] using low-energy field theory and other techniques, and belongs to the class of Fermi-edge problems [12,13]. Reference [14] addressed dynamical effects beyond the field-theory description. The other extreme limit [15–18] is $k_F^l = \pi$, with a domain wall (DW) initial state $|\psi_0\rangle = \prod_{x<0} c_x^\dagger|0\rangle$, where $|0\rangle$ is the fermion vacuum. For intermediate filling, $k_F^l \in (\frac{\pi}{2}, \pi)$, a current-carrying non-equilibrium steady state (NESS) develops in the middle of the chain, in which the density is the average of the two initial densities [16,19]. In this region, correlations have been found to be that of a Fermi sea $k \in [-k_F^l, k_F^r]$ with shifted momenta [16,19]. Some properties of the NESS can be addressed using bosonization [20]. Related situations have been studied in refs. [21–25]. A parallel line of current research addresses energy-carrying steady states, e.g., when the two chains have different initial temperatures [26–32].

The protocol of connecting two initially isolated chains is thus worth studying in some depth, in general for
quantum quenches, and in particular for current-carrying steady states in a unitary dynamics setting. While an overall picture is emerging, many important aspects require clarification.

An important issue is the applicability of a semiclassical description. Semiclassical approaches are based on the notion of assigning momentum and position simultaneously to single-particle quantum states, in violation of the uncertainty principle. For continuum problems, the use of semiclassics for fermionic transport problems [33–35] may be justified exactly in some cases, using Wigner function techniques. However, on a lattice, the justification of semiclassics or low-energy field theories. In addition, using short-time expansions directly in the L → ∞ limit, we can write analytic power-series expressions for correlators, and we have developed techniques to efficiently evaluate the associated nested commutators. For V ≠ 0, we use numerical exact diagonalization.

Integral representation. – In the non-interacting (V → 0) and large-size (L → ∞) limit the final Hamiltonian is diagonal in momentum space, H = \( \int dk \varepsilon(k) f^\dagger(k) f(k) \), with \( \varepsilon(k) = -\cos k \). The two-point function at time t is then

\[
\langle c^\dagger_j(t) c_y(t) \rangle = \int \frac{dk dq}{2\pi} e^{i(\varepsilon(k) - \varepsilon(q)) t - ikx + iqy} f(k, q),
\]

where f(k, q) = ⟨|ψ0⟩f^l(k)f(q)|ψ0⟩. The integral is taken over \([-\pi, \pi]^2\]. The importance of the structure of the integrand for long-time behaviors has been pointed out in the general context of thermalization [37], but to our knowledge this representation has never been utilized to obtain semiclassics or asymptotic time dependence. We will show that physical insights into several asymptotic regimes follow from careful stationary phase treatments of the double integral (1).

For our initial state, we find f(k, q) = f^l(k, q) + f^r(k, q), where f^l/r(k, q) = f^l/r_H(k, q) and

\[
f^l_H(k, q) = \frac{\chi_l(q) + g_l(e^{ik}) - g_l(e^{iq})}{2\pi(1 - e^{-i\eta(k + \alpha r)})},
\]

\[
f^r_H(k, q) = \frac{e^{-i\eta} g_l(e^{-iq}) - e^{i\eta} g_l(e^{ik})}{2\pi(e^{-ik} - e^{iq})},
\]

where \( g_l(z) = \frac{i}{2\pi} \text{log} \frac{e^{iky} - z}{e^{iky} - z^*} \). Here \( \chi_l(k) = 1 \) if k ∈ \([-k_F^l, k_F^l]\) and zero otherwise; similar expressions hold for f^r(k, q). Note that the f^l/r_H have a pole and are not analytic at \( \pm k_F^r \).

Semiclassics from a stationary phase treatment. – The stationary points of the phase in (1) satisfy the equations v(k_n) - x/t \( = 0 \) and v(q_n) - y/t \( = 0 \) for general x and y, where v(k) = \( \frac{d\varepsilon(k)}{dk} \) is the group velocity. In the limit x/t, y/t finite and |x - y|/t \( \ll 1 \) the two stationary points almost coincide and f(k, q) is singular. The integral is then dominated by the region where k - q is small. Introducing K = (k + q)/2 and Q = k - q, the stationary

\[V_{n_j} n_{j+1}\] across each pair of neighboring sites. (Here \( n_{j} = c^\dagger_j c_j \) is the site occupancy.) The hopping and interactions between sites \( j = 0 \) and \( j = 1 \) are initially absent; the initial state |ψ0⟩ = |ψ1⟩|ψr⟩ is the tensor product of the ground states (with specified occupancies) of the two decoupled chains.

We will study the time evolution of two-point correlators, ⟨c^\dagger_j c_y⟩, and of the Loschmidt echo. For V = 0, the time evolution can be performed numerically exactly for large systems L \( \sim 10^3 \). In addition, using short-time expansions directly in the L → ∞ limit, we can write analytic power-series expressions for correlators, and we have developed techniques to efficiently evaluate the associated nested commutators. For V ≠ 0, we use numerical exact diagonalization.
point is located at $Q_s = 0$, irrespectively of $K$. Expanding around it and using $\int_0^L \frac{dQ}{2\pi} \Theta^2 \Theta^2 = \Theta(x)$ we obtain

$$\langle c_{s}^\dagger(t) c_y(t) \rangle = \int_{-k^F}^{k^F} \frac{dk}{2\pi} e^{-iKx} e^{iKy} \Theta \left(-\frac{x+y}{2} + v(K)t\right) + \int_{-k^F}^{k^F} \frac{dk}{2\pi} e^{-iKx} e^{iKy} \Theta \left(\frac{x+y}{2} - v(K)t\right),$$

where $\Theta(x)$ is the Heaviside step function.

The form obtained above is exactly what one would obtain from a semiclassical ansatz: Assuming each fermion at momentum $k$ in the initial state to be propagating ballistically at speed $v(k)$, one obtains a picture like fig. 1(a), which readily yields the $x = y$ correlator, i.e., the density [16,17,19]. Here, we have derived the expression, also including $x \neq y$ correlators, showing that the semiclassical approximation becomes exact in the scaling limit with $x/t, y/t$ finite but $|x - y|/t \ll 1$. This derivation can be readily modified to the case where the two halves have different initial temperatures, thus justifying the use of semiclassics for energy currents.

In fig. 1(b), a numerical comparison with the exact correlators is shown for $|x - y| = 8$. The small deviations are due to finite $t$ and finite $L$. There are three regions. In the outer ($|x/t| > 1$) region, one of the step functions of (4) is identically zero while the other is identically one. The correlations are locally those in the bulk of the initial left (or right) ground state. In the central region $|x/t| < \sin k^F$ the correlations are those of a shifted Fermi sea with momenta in $[-k^F, k^F]$. This is the NESS. Intermediate between the two is the “front”, $\sin k^F < |x/t| < 1$, where the correlations have non-trivial structure.

**Front structure beyond semiclassics.** – The edge of the front, $x/t \sim 1$, corresponding to the tip of the $v(x,t)$ curve in fig. 1(a), is particularly interesting: Here the stationary points $k_{\pm}^F$ coalesce, $k_+^F = k_-^F = k_T$. The evaluation of the stationary phase now requires higher order terms (the second derivative vanishes), resulting in corrections to the semiclassic formula, fig. 1(c). For the DW initial state ($k_F = \pi$), the density near the front has oscillations described by the Airy kernel [15]. Using our stationary phase analysis, we find that the correlators (also for $x \neq y$) have oscillations of Airy kernel form for any filling $k^F \neq \pi/2$ (see footnote 1): in the front region, $\langle c_{s}^\dagger(t) c_y(t) \rangle \sim 2^{1/3} e^{-1/3e^{-ix(y)/2}K(X,Y)}$, where $K(X,Y)$ is the Airy kernel [36], and $X = (x-t)/(2/t)^{1/3}, Y = (y-t)/(2/t)^{1/3}$. We conjecture the Airy kernel form to be universal for the entire $V \in [0,1]$ range, but currently a general proof is unavailable.
Reaching the steady state. – At long times and finite $x$, $y$, the correlators are given by the NESS limit, the inner region of fig. 1(a), (b). We quantify how this long-time limit is reached, by extending the stationary phase treatment to higher orders [38]. The NESS asymptote and (4) are entirely determined by the pole contribution of $f(k,q)$, but the subleading terms depends on its full analytic form. This calculation requires special treatment because of singularities in $f(k,q)$ and the presence of sharp boundaries in $k$-space when $k_F \neq \pi$. We have solved this problem by considering the time derivative of (1). The extra factor $\varepsilon(k) - \varepsilon(q)$ gets rid of the denominator in (2), (3) (see footnote 1). The resulting integrals can be evaluated by stationary phase in the large $t$ limit and the result integrated back to derive the asymptotic expansion of the correlators. As an example, we display here the imaginary part of $\langle \epsilon_0(t) \epsilon_1(t) \rangle$, which is the current $J(t)$ across the link between the connected halves ($t_{\pm} = t \pm \pi/4$):

$$J(t) \overset{t \gg 1}{=} \frac{1}{\pi} \frac{\cos(\pi - k_F)}{2\pi t^2} + \frac{\cos(k_F - t \cos k_F) \sin(t_{+})}{2^{-1/2} \sin(k_F)(\pi)^{3/2}}.$$  \hspace{1cm} (5)

Half-integer powers of $1/t$ appear in all correlators, for all $k_F \neq 0, \pi$. These are due to the aforementioned sharp boundaries in $k$-space, and are universal. The frequencies of oscillations are $\omega(k_s,k_s') = |\varepsilon(k_s) - \varepsilon(k_s')|$, where $k_s$ and $k_s'$ belong to the set of stationary points $\{0, \pi, k_F\}$. Figure 2(a), (b) compares asymptotic expressions to exact curves.

Loschmidt echo. – The Loschmidt echo is defined as the overlap $\mathcal{L}(t) = |\langle \psi_0 | e^{-iHt} | \psi_0 \rangle |^2$. For short times, a power-series expansion gives $\mathcal{L}(t) \approx 1 - \bar{\gamma} t^2$, with $\bar{\gamma} = (H^2) - \langle H \rangle^2$ the variance of the post-quench Hamiltonian in the initial state.

For the equal-filling ($k_F = \pi/2$) case, conformal field theory (CFT) techniques were previously used [9,10] to show at large times a universal power-law decay $\mathcal{L}(t) \sim t^{-\gamma/4}$. We show here that this behavior is non-generic and does not occur at other initial fillings.

We first consider the DW initial state ($k_F^l = \pi$). We are able to compute $\mathcal{L}(t)$ exactly for $V = 0$. By application of Wick’s theorem, $\mathcal{L}(t) = \lim_{t \rightarrow \infty} \det_{1 \leq i,j \leq N}(g_{i,j} - g_{i+})$, where $g_{i,j} = \frac{1}{2\pi} \int_{\theta_{0}}^{\pi} d\theta e^{-i\theta(t_{+} + \pi t_{-})}$. The determinant is of Toeplitz+Hankel form. Asymptotic results for such determinants have been obtained in the mathematical literature. Using a generalization [39] of the strong Szegő limit theorem [40], we find (see footnote 1)

$$\mathcal{L}(t) = e^{-\bar{\gamma} t^2/4}.$$  \hspace{1cm} (6)

Alternatively, this result can be derived (see footnote 1) exploiting combinatorial techniques [41–43]. Remarkably, this simple formula is exact at all times in an infinite system: it appears as a straight line in fig. 3(a). For finite $L$, it is exponentially accurate until $t \approx L/2$.

For finite $V$, throughout the gapless phase ($V < 1$), the large-time Gaussian behavior $-\ln \mathcal{L}(t) \propto t^2$ persists. This can be understood using the following correspondence: for the DW initial state, the imaginary-time version $-\ln \mathcal{L}(it)$ can be interpreted as the free energy of a statistical mechanical system where all degrees of freedom outside a circle are frozen [44,45], analogous to the celebrated “arctic circle theorem” [46] for classical dimers. For $V = 0$, the area of the circle, which gives $-\ln \mathcal{L}(it)$, is $\propto \tau^2$ for any $\tau$. For $V \in (0,1)$, existing results on classical free energy [47] can be translated using this correspondence to predict $-\ln \mathcal{L}(t) \propto \tau^2$ for large times. Our prediction is consistent with the numerical data in fig. 3(b). For $V > 1$ (gapped phase), $\mathcal{L}(t)$ relaxes at large times to a finite $L$-dependent value.

The short-time behavior for $k_F^l = \pi$ is $\mathcal{L}(t) \sim 1 - t^2/4$ for all $V$ [48,49]; see fig. 3(b). This can be understood by analyzing the cumulant $(H^2) - \langle H \rangle^2$; the interaction contributions cancel, and the only contributing hopping process involves moving the rightmost particle one step to the right and then back, giving $\bar{\gamma} = \frac{1}{4}$. 40011-p4
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For intermediate fillings, \( k^j_L \in (\pi/2, \pi) \), we have the following results for \( V = 0 \). The short-time coefficient \( \tilde{\gamma} \) can be computed analytically as a function of \( k^j_L \); it decreases from \( \frac{1}{2} \) for \( k^j_L = \frac{\pi}{2} \) to \( \frac{1}{2} \) for \( k^j_L = \pi/2 \). There is Gaussian decay at long times for any \( k^j_L > \pi/2 \), i.e., \( \mathcal{L}(t) \sim e^{-\gamma t^2} \), with a prefactor well described by the formula \( \gamma = \frac{1}{2} \cos^2 k^j_L \); an analytic derivation is currently lacking. Figure 3(a) displays these behaviors numerically.

**Power-series characterization of the light cone.** – The correlators \( G_j(t) \equiv \langle c_{-j}^\dagger(t)c_{j+1}(t) \rangle \) between points symmetrically placed around the connecting link should have significant dynamics only for time \( t \gg j \). Power-series expansions in \( t \) allow us to quantify this effect for \( V = 0 \): we find that for generic \( k^j_L \), the leading term in this expansion for \( G_j(t) \) is \( t^{j+1} \) (see footnote 1). Figure 2(c) demonstrates that this holds also for \( V \neq 0 \). The DW initial state is anomalous; the leading term is \( t^{2j+1} \) (see footnote 1).

**Discussion.** – Inhomogeneous quantum quenches involve physics beyond the two standard paradigms of quantum quenches (local and global quenches). This is exemplified, e.g., by our finding that the short-time coefficient \( \tilde{\gamma} \) for \( \mathcal{L}(t) \) is \( L \)-independent, in sharp contrast to global quenches, and that the long-time decay is not power-law but Gaussian, in contrast to local quenches in homogeneous systems. This new class of non-equilibrium situation is now beginning to attract extensive attention, and we have provided a collection of exact results in this context. Semiclassic arguments are attractive and widely used in inhomogeneous situations: we have explored this through explicit derivation where semiclassic predictions are exact. We have shown subleading corrections to be described by the Airy kernel, providing a connection between large deviation functions in fermionic transport and the Tracy-Widom distribution.

Our work raises a number of interesting questions, e.g., the possibility of a field-theory formulation of the scaling behavior, the study of the entanglement spreading in an inhomogeneous background [50], a deeper understanding of interaction effects (which might be achieved by engineering a tailored Bethe rapidity distribution for the NESS), and explicit calculations of large distance correlators, for \( |x - y|/t \) of order one.

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