Superfluid hydrodynamic in fractal dimension space

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Abstract. The complex behavior of such quantum fluids like liquid $^4$He and liquid $^3$He in nanoporous media is determined by influence of randomly distributed geometrical confinement as well as by significant contribution from the surface atoms. In the present paper Fractional Schrodinger equation has been used for deriving two-fluid hydrodynamical equations for describing the motion of superfluid helium in the fractal dimension space. Nonlinear equations for oscillations of pressure and temperature are obtained and coupling of pressure and temperature oscillations is observed. Moreover coupling should disappear at very low temperatures which provide an experimental test for this theory.

1. Introduction
Liquid of stable isotopes of helium $^4$He and $^3$He at very low temperature belong to the class of quantum fluids with strong correlations between atoms. The first one represents a Bose-system and shows superfluid transition at $\lambda$-point $T_\lambda = 2.17$ K (below this temperature it is called He-II). In recent years it has been recognized that the quantum fluids in the confined geometry at nanoscale length can be considered as a new state of quantum matter due to close values between characteristic lengths for these quantum liquids and the size of geometrical confinement and significant contribution from the surface atoms.

So the question of influence of geometrical factors (confinement dimension, dimension of nanopores space, etc) has been emerged[1–6]. And one has to apply new physics to describe such systems with taking into account their complex nature. For example, last two years the attempts to develop the fractionalized two-fluid hydrodynamics for nanoporous media with fractal dimensions have been made[7, 8].

In the present paper Fractional Schrodinger equation has been used for deriving two-fluid hydrodynamical equations for describing the motion of superfluid helium in the fractal dimension space (like aerogel) and nonlinear equations for oscillations of pressure and temperature are obtained and coupling of pressure and temperature oscillations is observed.

2. Two-fluid hydrodynamical model
The usual two-fluid hydrodynamic model [9], [10] of superfluid helium $^4$He is described by the system of four differential equations (1), (2), (3), (4) which represent continuity equation, entropy conservation law and Eiler equations for superfluid and normal components respectively

$$\frac{\partial \rho}{\partial t} + \nabla j = 0 \quad (1)$$
$$\frac{\partial \rho \sigma}{\partial t} + \nabla (\rho \sigma \mathbf{v}_n) = 0 \quad (2)$$
\[
\rho_s \frac{\partial \mathbf{v}_s}{\partial t} = -\frac{\rho_s}{\rho} \nabla p + \rho_s \sigma \nabla T
\]  
(3)

\[
\rho_n \frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_n}{\rho} \nabla p - \rho_s \sigma \nabla T,
\]  
(4)

where \(\rho\) – He-II density, \(\rho_s, \rho_n\) – superfluid and normal component density, \(\mathbf{v}_s, \mathbf{v}_n\) – superfluid and normal component velocity, \(j = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n\) – He-II flow density. Interaction between the fluid components, forces of viscous friction and energy dissipation are neglected because of the small velocities of both components. This model provide two type of oscillations - oscillations of pressure (first sound, both component move in phase) and oscillations of temperature (second sound, components move in antiphase). Neglecting the anomalously small thermal expansion coefficient of He-II, oscillations of pressure and temperature in bulk sample of superfluid helium are independent. But in superfluid helium in aerogel there is experimentally proven coupling of these two type of oscillations[11].

3. Fractional quantum mechanics

For taking into account complex nature of internal geometry of confinement (aerogel) one can propose to use formalism of fractional quantum mechanics. One of the point of view on quantum mechanical motion is the so-called Feynman formalism of quantum mechanics, which is focused on the concept of trajectory, and the particle can move along any possible trajectories. To move a particle from point A to point B one have to take into account the contribution from all possible trajectories with the corresponding weight (complex factor). Possible trajectories resemble Brownian trajectory of a free particle and have a fractal dimension of \(\alpha = 2\).

De Broglie thermal wavelength for the helium atom inside the aerogel at temperatures of about 1 K is about 10 Å, which is in proper relation with the characteristic length scale of the fractal structures formed aerogel. Thus, some quantum-mechanical trajectory of the helium atom will be forbidden due to the influence of the structure of aerogel. Realized trajectory will resemble the motion of a Brownian particle in a porous medium, where the mean square displacement depends on time as \(<x^2> \propto t^\alpha\). This phenomenon is called subdiffusion. To describe this type of diffusion an equation with fractional Riesz derivative is used. Probability density function for such case is written in terms of Levy function, which is a generalization of the Gaussian distribution.

Laskin proposed to generalize the Feynman’s path integrals to an arbitrary fractal dimension of trajectories \(\alpha\) [12]. From this type of path integrals one can obtain fractional Schrödinger equation which is written as

\[
i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = D_\alpha (\hbar \nabla)^\alpha \psi(\mathbf{r}, t) + V(\mathbf{r}, t)\psi(\mathbf{r}, t).
\]  
(5)

Here Riesz fractional derivative is introduced as

\[
(\hbar \nabla)^\alpha \psi(\mathbf{r}, t) = -\frac{1}{(2\pi \hbar)^3} \int d^3 p e^{i\mathbf{p} \cdot \mathbf{r}} |\mathbf{p}|^\alpha \varphi(\mathbf{p}, t),
\]  
(6)

\[
\varphi(\mathbf{p}, t) = \int d\mathbf{r} e^{-i\mathbf{p} \cdot \mathbf{r}} \psi(\mathbf{r}, t),
\]  
(7)

and \(D_\alpha\) is the generalized coefficient, the physical dimension of which is \([D_\alpha] = \text{erg}^{1-\alpha} \cdot \text{cm}^\alpha \cdot \text{sec}^{-\alpha}\). Thus fractional Hamiltonian can be written in the form

\[
\hat{H}_\alpha = D_\alpha (\hbar \nabla)^\alpha + V(\mathbf{r}, t) = D_\alpha |\mathbf{p}|^\alpha + V(\mathbf{r}, t).
\]  
(8)

Hamiltonian (8) is hermitian operator and provide probability conservation law. Also it obeys parity conservation law, so one divide particles at the fermions and bosons, as in usual case.
This type of Hamiltonian (8) already has been used to describe the specific heat of noncrystalline solids (glasses) associated with the unusual structure of these materials[13].

4. Galilean noninvariance

On other hand, using of nonlinearity in Hamiltonian results in breaking some other symmetries in equations of dynamics. One of them is Galilean invariance. In usual nonfractional case, transition from one inertial system to another will conserve general view of dynamical equations.

If we consider Schrödinger equation in two inertial reference frame $K$ (nonprimed variables) and $K'$ (primed variables), which move relative to each other with velocity $V$,

\[
\Psi(x, t) = \varphi(x', t) e^{i\frac{mV x' + mV^2 t}{2mh}}
\]

one can obtain Schrödinger equation in another inertial reference frame

\[
\frac{i\hbar}{\partial t} \frac{\partial \varphi}{\partial x} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2},
\]

On other hand, by making similar substitution in the form

\[
\Psi(x, t) = \varphi(x', t) e^{i\frac{mV x'}{\hbar} + \frac{D\alpha p'^2}{\hbar}},
\]

for fractional Schrödinger equation, one can obtain dynamical equation in new inertial reference frame

\[
\frac{i\hbar}{\partial t} \frac{\partial \varphi}{\partial x} - \varphi D\alpha |mV|^\alpha - i\hbar V \frac{\partial \varphi}{\partial x} + \varphi mV^2 = \frac{D\alpha}{2\pi\hbar} \int_{-\infty}^{\infty} dp e^{ipx / \hbar} \hat{\varphi}(p) |p + mV|^\alpha.
\]

The only possibility to reduce eqn. (9) to the form (5) is to set $V = 0$, i.e. do not change the inertial reference frame. As a consequence there is some special reference frame where, fractional Schrödinger equation has form (5). One can identify this reference frame with some reference frame where, for example, nanoporous media is in a rest.

5. Fractional two-fluid hydrodynamical model

By using Heisenberg representation one can obtain dynamical equations for operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{v}} = \frac{d\hat{\mathbf{r}}}{dt}$ by means of general dynamical equation for arbitrary operator $\hat{O}$

\[
\frac{d\hat{O}(t)}{dt} = \frac{\partial \hat{O}(t)}{\partial t} + \frac{i}{\hbar} [\hat{H}_\alpha, \hat{O}].
\]

Then for velocity and acceleration we will obtain next equations

\[
\hat{\mathbf{v}} = \alpha D\alpha |\mathbf{p}|^{\alpha-2} \mathbf{p},
\]

\[
\frac{d\hat{\mathbf{v}}}{dt} = -\frac{i}{\hbar} \alpha D\alpha \sum_{l=1}^{\infty} \frac{(-i\hbar)^l}{l!} \nabla^l V(\mathbf{r}) \frac{(\alpha - 1)!}{(\alpha - l - 1)!} |\mathbf{p}|^{\alpha-1-l}.
\]

Suppose that velocity is more general quantity than momentum [15], one can rewrite (12) in the form

\[
\frac{d\hat{\mathbf{v}}}{dt} = -\alpha D\alpha \frac{(\alpha - 1)}{(\alpha D\alpha)^{(\alpha-2)/\alpha-1}} |\mathbf{v}|^{\frac{\alpha-2}{2}} \nabla V(\mathbf{r}),
\]

where we left only first spatial derivation from potential $V$. From eqn. (5) one can obtain mass conservation law in the form

\[
\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \mathbf{J}(\mathbf{r}, t) + \mathbf{K}(\mathbf{r}, t) = 0,
\]
where $\rho = \Psi^*\Psi$ is density, $J = \frac{1}{\alpha} (\Psi^* \hat{\nabla} \Psi + \Psi \hat{\nabla} \Psi^*)$ is density flow and

$$K = \frac{i}{\alpha \hbar} (\hat{p} \Psi^* \hat{\nabla} \Psi - \hat{\nabla} \Psi^* \hat{p} \Psi)$$ (15)

is new term, which represent additional sources in fractal space. If we decompose wave function on the basis of plane waves $\Psi = \sum_p C_p \exp(i p r / \hbar)$ then substitution it in the eqn. (15) results in

$$K = \frac{i D_\alpha}{\hbar} \sum_{p_1 \neq p_2} p_1 p_2 C_{p_1}^* C_{p_2} e^{(p_2 - p_1) \cdot \hbar / \alpha} \left(|p_2|^{\alpha - 2} - |p_1|^{\alpha - 2}\right).$$

If we assume that density of superfluid helium is almost homogeneous, i.e. we have only set of plane waves with close values of $p$ in wave function and helium atoms is strongly delocalized, then one can suppose $K \approx 0$. In that case in fractional two-fluid hydrodynamical model one can left continuity equations in the form (1) and (2).

Instead of constant $D_\alpha$ one can introduce new constant with dimension of velocity as $\alpha D_\alpha = v_0^{2-\alpha} m^{1-\alpha}$. Then dynamical equation (13) for the superfluid component with taking into account thermodynamical relations [9] can be rewritten as

$$\rho_s \frac{d \mathbf{v}_s}{d t} = (\alpha - 1) \left| \frac{\mathbf{v}_s}{v_0} \right|^{\alpha - 2} \left( -\frac{\rho_s}{\rho} \nabla p + \rho_s \sigma \nabla T \right).$$ (16)

Also we can suppose the similar form for dynamical equation of density flow and write

$$\frac{d \mathbf{J}}{d t} = -(\alpha - 1) v_0^{2-\alpha} \left| \mathbf{j} \right|^{\alpha - 2} \frac{\rho_s}{\rho} \nabla p.$$ (17)

Thus equations (1), (2), (16) and (17) form fractional two-fluid hydrodynamical model of superfluid helium in nanoporous space. These equations set result in two oscillation equations

$$\frac{\partial^2 p}{\partial t^2} = (\alpha - 1) u_1^2 \nabla \left( \left| \frac{\mathbf{v}_s}{v_0} \right|^{\frac{\alpha - 2}{\alpha - 1}} \nabla p \right),$$ (18)

$$\frac{\partial^2 T}{\partial t^2} = (\alpha - 1) u_2^2 \nabla \left( \left| \frac{\mathbf{v}_s}{v_0} \right|^{\frac{\alpha - 2}{\alpha - 1}} \nabla T \right) - (\alpha - 1) \frac{u_2^2}{\rho_s \sigma} \nabla \left( \left( \left| \frac{\mathbf{v}_s}{v_0} \right|^{\frac{\alpha - 2}{\alpha - 1}} - \left| \frac{\mathbf{v}_s}{v_0} \right|^{\frac{\alpha - 2}{\alpha - 1}} \right) \nabla p \right),$$ (19)

where $u_1$ and $u_2$ is first and second sound velocity for bulk superfluid helium.

Let us consider approximate solution of equations (18) and (19). By using "weak fractality" approximation $\alpha - 2 \ll 1$ in low temperature region $T < 0.5K$ when $\rho_n / \rho_s \ll 1$ and $v_n / v_s \ll 1$ one can rewrite eqn.(19) as

$$\frac{\partial^2 T}{\partial t^2} = (\alpha - 1) u_2^2 \left( 1 + \gamma \ln \left| \frac{v_s}{v_0} \right| \right) \nabla^2 T + (\alpha - 2) u_2^2 \frac{\rho_n}{\rho_s \sigma} \left( \frac{v_n}{v_s} - 1 \right) \nabla^2 p,$$ (20)

where $\gamma = (\alpha - 2) / (\alpha - 1) \ll 1$. We will find solutions in the form

$$p = p_0 + p' e^{i(\alpha k t - kr)}, \quad T = T_0 + T' e^{i(\alpha k t - kr)}$$
and suppose that velocity has dependence \( v = v^0 \exp(\text{i}(ukt - kr)) \). As a result one can obtain two type of oscillations: temperature oscillations with sound speed \( u \approx u_2 \left( 1 + (\alpha - 2)(1 + \frac{1}{2} \ln \frac{v^0}{v^s}) \right) \) and pressure-temperature oscillations with sound speed \( u \approx u_1 \left( 1 + \frac{\alpha - 2}{2} (1 + \ln \frac{v^0}{v^s}) \right) \). The latter one has relation between pressure and temperature amplitude \( T' = \beta p' \), where coupling coefficient has form

\[
\beta = \frac{(\alpha - 2) u^2_2 \rho_n}{\rho \sigma v^0} \left( \frac{v^0_n}{v^0_s} - 1 \right) / (u^2_2 - u^2_1).
\]

At low temperatures, when \( u^2_1 \approx \text{const}, u^2_2 \approx \text{const} \) and \( \sigma \propto T^n \), this coupling coefficient should be linear on temperature \( \beta \propto T \).

6. Conclusion

It was proposed that for the microscopical description of superfluid in nanoporous media with complex fractal structure one can use fractional Schrödinger equation. But it is necessary to keep in mind that the fractal geometry of nanoporous media leads to the Galilean noninvariance of this equation and as a consequence one needs to choose the special frame of reference where, for example, nanoporous media is in a rest. One can interpret dynamical equations of fractional quantum operator in Heisnberg representations as a classical dynamical equations in fractal media and generalize them to obtain fractional hydrodynamical set of equations. From this two-fluid fractional hydrodynamical model one can obtain equations of pressure-temperature and pure temperature oscillation. It was shown that the pressure-temperature coupling constant has linear dependence on temperature at low temperature region which provide us with possibility of experimental proof of given model.

Acknowledgments

This work is supported in part by the Ministry of Education and Science of Russian Federation.

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