K-WEIGHT BOUNDS FOR $\gamma$-HYPERELLIPTIC SEMIGROUPS

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Abstract. In this note, we show that $\gamma$-hyperelliptic numerical semigroups of genus $g \gg \gamma$ satisfy a refinement of a well-known characteristic weight inequality due to Torres. The refinement arises from substituting the usual notion of weight by an alternative version, the $K$-weight, which we previously introduced in the course of our study of unibranch curve singularities.

1. $K$-WEIGHTS OF NUMERICAL SEMIGROUPS

Let $S \subset \mathbb{N}$ denote a numerical semigroup of genus $g$. Recall that this means that the complement $G_S = \mathbb{N} \setminus S$ is of cardinality $g$; say

$$G_S = \{\ell_1, \ldots, \ell_g\}$$

where $\ell_i < \ell_j$ whenever $1 \leq i < j \leq g$. Following [3], we define the $K$-weight of $S$ to be the quantity

$$W_K := \sum_{i=1}^{g-1} (\ell_i - i) + g - 1.$$

The definition of the $K$-weight was motivated by the study of unibranch curve singularities. It is also closely related to a more familiar notion of weight, which we will call the S-weight, namely:

$$W_S := \sum_{i=1}^{g} (\ell_i - i).$$

It is not hard to show that $W_K = W_S$ whenever $S$ is the value semigroup of a Gorenstein singularity. In purely combinatorial terms, the $K$- and $S$-weights agree if and only if $S$ is a symmetric semigroup.

2. $K$-WEIGHTS OF $\gamma$-HYPERELLIPTIC SEMIGROUPS

Now let $\gamma \geq 0$ be an integer. Recall from [5] that $S$ is $\gamma$-hyperelliptic if it satisfies the following conditions:

1. $S$ has $\gamma$ even elements in $[2, 4\gamma]$; and
2. The $(\gamma + 1)$st positive element of $S$ is $4\gamma + 2$.

In [3], we conjectured that any $\gamma$-hyperelliptic numerical semigroup $S$ satisfies

$$\left(\frac{g - 2\gamma}{2}\right) + 2\gamma \leq W_K \leq \left(\frac{g - 2\gamma}{2}\right) + 2\gamma + 2.$$

The inequalities (1) are not far-removed from the inequalities

$$\left(\frac{g - 2\gamma}{2}\right) \leq W_S \leq \left(\frac{g - 2\gamma}{2}\right) + 2\gamma + 2.$$

proved by Torres in [4, 5]. As the disparity between upper and lower bounds in (1) is in general smaller than that of (2), we regard the K-weight inequalities as a refinement of the S-weight inequalities.

In this note, we will prove the K-weight inequalities (1) are satisfied by any \( \gamma \)-hyperelliptic semigroup of sufficiently large genus. To do so, we adapt the geometric interpretation of the S-weight given in [2]. Namely, each numerical semigroup may be represented as a Dyck path \( \tau = \tau(S) \) on a \( g \times g \) square grid \( \Gamma \) with axes labeled by \( 0, 1, \ldots, g \). Each path starts at \((0,0)\), ends at \((g,g)\), and has unit steps upward or to the right. The \( i \)th step of \( \tau \) is up if \( i \notin S \), and is to the right otherwise. The weight \( W_S \) of \( S \) is then equal to the total number of boxes in the Young tableau \( T_S \) traced by the upper and left-hand borders of the grid and the Dyck path \( \tau \). The contribution of each gap \( \ell \) of \( S \) to \( W_S \) is then computed by the number of boxes inside the grid and to the left of the corresponding path edge.

**Theorem 2.1.** Fix a choice of non-negative integer \( \gamma \), and let \( S \) denote a \( \gamma \)-hyperelliptic semigroup of genus \( g \gg \gamma \). The K-weight of \( S \) satisfies the inequalities (1).

**Proof.** Suppose \( S \) is a \( \gamma \)-hyperelliptic semigroup of genus \( g \). For the calculation of the K-weight of \( S \), the largest gap \( \ell_g \) of \( S \) is irrelevant, so we focus on the subdiagram of \( \Gamma \) given by omitting the uppermost row of boxes in the grid, and on the corresponding subtableau \( T_K \), which we will denote by \( T_K \).

It is well-known (and follows easily from the semigroup structure in any case) that every even number greater than or equal to \( 4\gamma \) belongs to \( S \). So the weight contributed by elements \( m \geq 4\gamma \) of \( S \) will be minimized when there are no such odd elements \( m \) strictly less than \( \ell_g \). Combinatorially, this means that \( T_K \) stabilizes to a *staircase*, i.e. a path in which up- and rightward steps alternate. It follows easily \( T_K \) will be of minimal weight precisely when it is a staircase for which the \( \gamma \) even numbers \( P_i \in [2, 4\gamma], 1 \leq i \leq \gamma \) that belong to \( S \) are maximal, namely when

\[
P_i = 2\gamma + 2j, 1 \leq j \leq \gamma.
\]

Since the first column of \( T_K \) has precisely \( (g - 1) - (2\gamma + 2 - 1) = g - 2\gamma - 2 \) boxes, we deduce that its total weight is

\[
W(T_K) = \binom{g - 2\gamma - 1}{2}
\]

and it follows that

\[
W_K = W(T_K) + g - 1 = \binom{g - 2\gamma}{2} + 2\gamma.
\]

It is clear, moreover, that there are \( \gamma \)-hyperelliptic semigroups of genus \( g \) whose K-weights realize the minimum value of \( W_K \).

We will now argue that the maximum possible \( \gamma \)-hyperelliptic K-weight is achieved precisely when

\[
S = S_0 := \langle 4, 4\gamma + 2, 2g - 4\gamma + 1 \rangle
\]

just as is the case for S-weights [5]. See Figure 1 for the K-tableau associated with \( S_0 \) when \( \gamma = 3 \) and \( g = 20 \).
Figure 1. Tableau $T_K$ associated with the weight-maximizing $\gamma$-hyperelliptic semigroup $S_0 = \langle 4, 4\gamma + 2, 2g - 4\gamma + 1 \rangle$ when $\gamma = 3$ and $g = 20$. The (irrelevant) uppermost line is left empty, while the disparity in weights between the maximizing and minimizing semigroups is in red.

To do so, we argue in two stages, depending upon where $S$ is symmetric or not. Accordingly, assume that $S$ is $\gamma$-hyperelliptic, and that $g \gg \gamma$. Let the K-antidiagonal denote the line in $\Gamma$ passing through the points $(0, g-1)$ and $(g-1, 0)$. The K-antidiagonal splits the tableau $T_K$ associated with any given $S$ into two pieces, and $S$ is symmetric precisely when these are reflected images of one another.

Case: $S$ is symmetric. Let $T^L_K$ and $T^R_K$ denote the pieces of $T_K$ to the left and right of the K-antidiagonal, respectively. A result of Bernardini–Torres [1, Cor. 2.4] establishes that the smallest odd element in $S$ is at least $2g - 4\gamma + 1$. As $g \gg \gamma$, this implies in particular that there are no odd elements in $S$ to the left of the K-antidiagonal, and that $T^L_K$ has maximal weight precisely when the $\gamma$-tuple of even elements $(P_1, \ldots, P_\gamma)$ is minimized, i.e. when

$$P_j = 4j, 1 \leq j \leq \gamma.$$

Note that it also follows from [1, Cor. 2.4] that

- the largest $\gamma$ gaps $\ell_{g-\gamma+1}, \ldots, \ell_g$ of $S$ lie in $[2g - 4\gamma + 1, 2g]$; and
- The conductor $c$ of $S$ satisfies

$$c \geq 2g - 4\gamma + 2\gamma = 2g - 2\gamma$$

with equality if and only if $T_S$ stabilizes to a staircase.

On the other hand, each of the $\gamma$ differences

$$c - 1 - P_j, 1 \leq j \leq \gamma$$

belongs to $G_\delta$. Because $S$ is symmetric, we have $c = 2g$, and accordingly $\gamma - 1$ of these, namely

$$c - 1 - P_j, 1 \leq j \leq \gamma - 1,$$
lie in \([2g - 4\gamma + 1, 2g - 3]\). It follows that these are precisely the gaps \(\ell_{g-\gamma+1}, \ldots, \ell_g\) of \(S\). We conclude easily that \(S_0\) has maximal K-weight among symmetric \(\gamma\)-hyperelliptic semigroups of genus \(g\).

**Case:** \(S\) is nonsymmetric. It seems difficult to extend the preceding argument to (bound the weight of) nonsymmetric \(\gamma\)-hyperelliptic semigroups. Instead, we exploit the dual relationship between ramification and weight already used in [5] to prove that \(S_0\) is of maximal (traditional) weight \(W_S\) among \(\gamma\)-hyperelliptic semigroups of genus \(g\). (Indeed, our argument is a modification of that used in [5].)

Namely, let \(R = R(S)\) denote the total ramification of \(S\), given by

\[
R := \sum_{i=1}^{g}(m_i - i) = \sum_{i=1}^{g} m_i - \left(\frac{g + 1}{2}\right)
\]

where \(m_1 < \cdots < m_g\) are the \(g\) smallest nonzero elements of \(S\). Using the structure theory for \(\gamma\)-hyperelliptic semigroups presented in [4, 5], we may rewrite the total ramification (3) of a \(\gamma\)-hyperelliptic semigroup \(S\) in the following form:

\[
R = \sum_{i=1}^{\gamma}(n_i + u_i) + \sum_{i=1}^{g-2\gamma}(4\gamma + 2i) - \left(\frac{g + 1}{2}\right)
\]

where \(n_1 < \cdots < n_\gamma\) are the smallest nonzero even elements and \(u_1 > \cdots > u_\gamma\) are the odd elements of \(S\), respectively.

Maximizing \(W_S\) is then equivalent to minimizing \(R(S)\). Similarly, maximizing \(W_K\) is equivalent to minimizing

\[
R_K := \sum_{i=1}^{\gamma}(n_i + u_i) + \sum_{i=1}^{g-2\gamma-1}(4\gamma + 2i) - \left(\frac{g + 1}{2}\right) - 2k
\]

where \(k = k(S)\) is the number of odd elements of \(S\) greater than or equal to the conductor. Note that (4) computes the area of the complement of \(T_K\) in its minimal \((g-1) \times (g-1)\) bounding box inside of \(\Gamma\).

On the other hand, when \(S = S_0\) the sum \(\sum_{i=1}^{\gamma} n_i\) is minimized, while \(k(S_0) = 0\). Moreover (and this is the key point of Torres’ argument in [5]) by construction we have

\[
n_i + u_i \geq 2g + 1
\]

for every \(1 \leq i \leq \gamma\), for every \(\gamma\)-hyperelliptic semigroup \(S\) of genus \(g\), and when \(S = S_0\) there is inequality in (5) for every \(i\). So in light of (4), it suffices to check that

\[
\sum_{i=1}^{k}(u_i(S_0) - u_i(S)) - 2k(S)
\]

is nonpositive for every \(\gamma\)-hyperelliptic semigroup \(S\), for which it suffices in turn to show that

\[
u_i(S) - u_i(S_0) \geq 2
\]

for all \(1 \leq i \leq k\). The inequality (6), however, is obvious, as \(S_0\) is symmetric and \(S\) is not. \(\square\)
References

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