The Dark Side of String Theory: 
Black Holes and Black Strings

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Abstract

Solutions to low energy string theory describing black holes and black strings are reviewed. Many of these solutions can be obtained by applying simple solution generating transformations to the Schwarzschild metric. In a few cases, the corresponding exact conformal field theory is known. Various properties of these solutions are discussed including their global structure, singularities, and Hawking temperature.

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1. INTRODUCTION

One of the most intriguing predictions of general relativity is the existence of black holes. There is now good observational evidence that black holes exist throughout the universe on scales from a solar mass (which are seen in binary star systems) up to millions of solar masses (which are seen in the center of galaxies and quasars). For these black holes, general relativity provides an adequate description at this time. However, it has been suggested that much smaller black holes could have been formed in the early universe. These black holes will become even smaller through the emission of Hawking radiation. Even the large black holes we see today will evaporate in the distant future if the temperature of the cosmic background radiation becomes less than their Hawking temperature.

In the late stages of this evaporation, general relativity is expected to break down and be replaced by a quantum theory of gravity. Since string theory is a promising candidate for a consistent quantum theory of gravity, it is of interest to examine black holes in string theory. As a first step, one should study the classical black holes solutions in this theory. This is what I plan to do in these lectures. We will concentrate on black holes with electric or magnetic charge. For these holes, the predictions of string theory differ from those of general relativity long before Planck scale curvatures are reached. The reason for this difference is the presence of a scalar field called the dilaton. We will see that the dilaton dramatically changes the properties of extremal black holes. For example, the extremal solutions can be completely free of curvature singularities. They can also repel each other. In addition to their possible astrophysical interest, charged black holes provide an ideal setting for studying the late stages of Hawking evaporation. The modifications predicted by string theory may help to resolve some of the puzzles associated with this process [1].

In addition to black holes, it turns out that string theory has solutions describing one-dimensional extended objects surrounded by event horizons i.e. black strings. We will see that these solutions can have unusual causal structure, and provide some insight into the properties of singularities in string theory. Most importantly, they are closely connected to fundamental strings themselves. A black string carries a charge per unit length, and in the extremal limit, the solution reduces to precisely the field outside a straight fundamental string. (There are other extended black hole solutions in string theory corresponding to black membranes or black \(p\)-branes [2] but I will not discuss them here.)

Let me clarify what I mean by classical solutions to string theory by distinguishing three different levels of approximation. In increasing importance (and difficulty) we have
1) Perturbative solutions of the low energy classical action
2) Exact solutions of the low energy classical action
3) Exact solutions of the full classical action

Solutions of the first type are obtained by considering (tree level) string scattering in flat spacetime. This was historically the first class of solutions discussed, but they are not very useful for describing black holes. Solutions of the third type are believed to correspond to two dimensional conformal field theories. These are ultimately what we after, but at this time, there are only a few black hole and black string solutions of this type known. Since the known solutions exist in unphysically low spacetime dimensions, I will focus mostly on solutions of the second type. These should be good approximations to exact solutions whenever the curvature is small compared to the Planck curvature. This can include the horizons as well as the region outside the black hole, but not of course a neighborhood of the singularity. It turns out that many of the solutions of the second type can be obtained by using solution generating techniques. These are similar to the transformations which have been found for the vacuum Einstein \cite{3}, and Einstein-Maxwell \cite{4} equations. Although the emphasis will be on the classical properties of these solutions, some basic aspects of Hawking evaporation such as the Hawking temperature will be discussed.

In Sec. 2, we will first review the standard black hole solutions in the Einstein-Maxwell theory. We then describe the low energy string field equations and discuss two methods of generating solutions to these equations. In Sec. 3 we begin our investigation of black holes in string theory by discussing the string analog of the Reissner-Nordstrøm solution. We describe several generalizations of this solution in Sec. 4. These include black holes with rotation, a (physically expected) mass term for the dilaton, and in other spacetime dimensions. The black string solutions are discussed in Sec. 5. Finally, in Sec. 6, we briefly consider the subject of singularities in exact solutions to the full classical string action, and conclude with some open problems.

In reviewing a subject of this type where there are a large number of known solutions, one faces a decision about how much information to include. My aim is to make this review self contained and useful as a reference, while keeping it clear and readable. Thus, I have not included the most general black hole solution known at this time. Instead, I describe the basic charged black hole solution in Sec. 3 and then discuss how it is modified when one includes rotation, other dimensions, etc. in Sec. 4. Similarly, in Sec. 2, I do not include the most general solution generating transformation, but restrict attention to the two which are most useful in obtaining the black hole and black string solutions. The reader interested in pursuing a topic further can use the references as a guide to the literature. For another recent review of this subject, see \cite{5}. 

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2. PRELIMINARIES

Before discussing the actual black holes solutions it is necessary to cover a few preliminaries. First we review the standard black hole solutions in Einstein-Maxwell theory, and the calculation of their Hawking temperature. Then we discuss the low energy string action and associated field equations. Finally, we describe two methods of generating solutions to these equations. We will see that many of the solutions of interest can be found by applying these methods to the Schwarzschild solution.

2.1. Black Holes in Einstein-Maxwell Theory

As is well known, uncharged static black holes are described by the Schwarzschild solution. This solution takes the form

\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega \]  \hspace{1cm} (2.1)

where \( M \) is the mass of the black hole. The global structure of Schwarzschild is conveniently described by a Penrose diagram in which light rays move along 45° lines and infinity has been brought to a finite distance by a conformal rescaling. This is shown in fig. 1. (One can interpret this two dimensional figure either as representing the \( r - t \) plane, or the entire spacetime where each point represents a two sphere of spherical symmetry.) The event horizon is located at \( r = 2M \) where \( g_{tt} = 0 \). Since Schwarzschild is time reversal invariant, the maximally extended spacetime contains a white hole as well as a black hole, and a second asymptotically flat region.

\[ \text{Fig. 1: The Penrose diagram for the maximally extended Schwarzschild solution.} \]

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\[ ^1 \text{For a more detailed discussion of Penrose diagrams see [6] or the lectures in this volume by Harvey and Strominger [7].} \]
A charged black hole in general relativity is described by the Reissner-Nordstrøm solution which has the metric

\[ ds^2 = - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega \quad (2.2) \]

together with a Maxwell field given by \( F_{rt} = Q/r^2 \) for an electrically charged hole and \( F_{\theta\phi} = Q \sin \theta \) for a magnetically charged hole. Its global structure is quite different from Schwarzschild and depends on the relative size of \( Q \) and \( M \). (We are using geometrical units \( G = c = 1 \) in which the charge on an electron is equivalent to \( 10^{-6} \) gm. Since this is much larger than the mass of an electron, it is relatively easy – in principle – to create a black hole with \( Q \) of order \( M \).) For \( 0 < |Q| < M \) there are now two zeros of \( g_{tt} \) at \( r = r_{\pm} \) where

\[ r_{\pm} \equiv M \pm \sqrt{M^2 - Q^2} \quad (2.3) \]

which correspond to two horizons. There is an event horizon at \( r = r_+ \) and an inner horizon at \( r = r_- \). The Penrose diagram is shown in fig. 2. The significance of the inner horizon is the following. Starting with initial data on an asymptotically flat spacelike surface, one is guaranteed a unique evolution only up to the inner horizon. After that, the evolution will be affected by boundary conditions at the singularity. Note that the singularity is now timelike, so an observer falling in is not forced to hit it. He can continue into another asymptotically flat region of spacetime. (The maximally extended spacetime contains an infinite number of such regions.) In fact, he must literally work hard to reach the singularity since freely falling observers avoid it: Reissner-Nordstrøm is timelike.
geodesically complete. In light of this, you might be tempted to carry a few charges with you in case you fall into a black hole. Until recently, it was widely believed that this would not help. The inner horizon was known to be unstable \[8\] and slightly nonspherical charged collapse was thought to result in a spacetime resembling Schwarzschild. However recent work \[9\] has indicated that the singularity at the inner horizon might be much more mild. (Although this is not yet settled \[10\].) In any case, a journey through the inner horizon would not be uneventful. Immediately after crossing the inner horizon, one would be face to face with a curvature singularity!

**Fig. 3:** The Penrose diagram for the Reissner-Nordstrøm solution with \( Q = M \). A \( t = \text{const} \) surface does not actually hit the singularity, but contains an infinite throat.

For \( Q = M \), the horizons coincide: \( r_+ = r_- = M \). The Penrose diagram is now shown in fig. 3. A \( t = \text{const} \) surface now looks like it hits the singularity, but this is just an artifact of the conformal rescaling. With \( Q = M \), the proper distance to the horizon \( r = M \) from a point \( r_0 > M \) along a \( t = \text{const} \), radial curve is

\[
L = \int_M^{r_0} \frac{dr}{(1 - M/r)} = \infty
\]  \hspace{1cm} (2.4)

So a \( t = \text{const} \) surface asymptotically resembles a cylinder as \( r \to M \). The geometry describes an infinite throat. In a sense, the horizon and other asymptotic flat region (which exist for \( Q < M \)) has been pushed off to infinity. However, even though the horizon is infinitely far away in spacelike directions, it is only a finite distance away in timelike or null directions. Observers can still fall into the black hole in a finite proper time.
Fig. 4: The Penrose diagram for the Reissner-Nordstrøm solution with $Q > M$.

Finally, for $Q > M$, the Reissner-Nordstrøm solution does not describe a black hole at all but rather a naked singularity. The Penrose diagram is shown in fig. 4. For this reason, the $Q = M$ solution is called the extremal black hole. It has the largest possible ratio of charge to mass.

The Hawking temperature of a static black hole can be calculated in several ways. Hawking’s original calculation [11] involved studying quantum matter fields in the black hole background. It was latter realized that one could compute this temperature by simply analytically continuing in $t$ and requiring that the resulting Riemannian space be nonsingular. This requires a periodic identification in imaginary time, and the temperature is one over this period. Physically, this instanton is related to a black hole in thermal equilibrium with a gas, in the approximation where the energy density of the gas is neglected. It will be convenient later to have a simple formula for the Hawking temperature of a general static, spherically symmetric black hole solution. Suppose the $r - t$ plane has metric

$$ds^2 = -\lambda dt^2 + dr^2/f$$

If there is an event horizon at $r = r_0$, then near this horizon, $\lambda \approx \lambda'(r_0)\xi$ and $f \approx f'(r_0)\xi$ where $\xi = r - r_0$. (We are assuming here that the event horizon is not degenerate.) We now set $\tau = it$ and $\rho = 2\sqrt{\xi/f'(r_0)}$. The resulting metric is

$$d\rho^2 + \frac{\lambda'(r_0)f'(r_0)}{4}\rho^2 d\tau^2$$

To avoid a conical singularity at $\rho = 0$ we must identify $\tau$ with period $4\pi/\sqrt{\lambda f'}$. Thus the Hawking temperature is

$$T_H = \frac{\sqrt{\lambda'(r_0)f'(r_0)}}{4\pi}$$
Applying this to the Reissner-Nordstrøm metric yields

\[ T_H = \frac{\sqrt{(M^2 - Q^2)}}{2\pi \left(M + \sqrt{M^2 - Q^2}\right)^2} \]  

(2.8)

A charged black hole will preferentially radiate away its charge. However the amount of charge that is actually radiated away depends on the charge to mass ratio of the particles in the theory. If this ratio is sufficiently small, most of the radiation will be in the form of neutral particles and \( Q \) will be essentially constant. This is likely to be true for magnetically charged black holes. In this case, the black hole will evolve toward its extremal limit. The Hawking temperature (2.8) vanishes as \( Q \to M \). This suggests that extremal charged black holes may be quantum mechanically stable. (There is still a possibility that extremal quantum black holes can bifurcate \( [12] \).) This is consistent with ideas of cosmic censorship. Although cosmic censorship is usually discussed in the context of classical general relativity, it is reassuring that as you approach the extremal limit, the Hawking radiation turns off. One does not continue to radiate to a naked singularity.

2.2. The Equations of Motion

We will work with part of the low energy action to heterotic string theory. The most general situation we will consider is described by a metric \( g_{\mu\nu} \), a dilaton \( \phi \), a Maxwell field \( F_{\mu\nu} \), and a three form \( H_{\mu\nu\rho} \). The Maxwell field is associated with a \( U(1) \) subgroup of the gauge group. We will set the rest of the gauge field to zero as well as the fermions. The three form \( H_{\mu\nu\rho} \) is related to a two-form potential \( B_{\mu\nu} \) and the gauge field \( A_\mu \) by \( H = dB - A \wedge F \) so that \( dH = -F \wedge F \). (Since we will keep only terms with two derivatives or less, it is consistent to drop the Lorentz Chern Simons term which also appears in the definition of \( H \).

These fields are governed by the following action\[2\] [14]:

\[ S = \int d^D x \sqrt{-g} \ e^{-2\phi} \left[ \Lambda + R + 4(\nabla \phi)^2 - F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] \]  

(2.9)

where \( \Lambda \) is a constant which is related to the spacetime dimension \( D \) and the central charge of a possible “internal” conformal field theory. (Since this internal part of the solution only affects black holes by changing the value of \( \Lambda \), we will not discuss it further.) It is clear from

\[ ^2 \text{Our conventions for the curvature follow those of Wald [13].} \]
this action that $e^\phi$ plays the role of a coupling constant. It governs the strength of quantum corrections. The complete string action includes higher order corrections $R^2, R^3, F^4$ etc. These can be neglected when discussing black hole solutions provided the size of the hole is much larger than the Planck length. In this case, the higher order corrections will induce, at most, small changes in the solution until one is well inside the event horizon and near the singularity. (The effect of some of these higher order corrections on the Schwarzschild solution has been discussed in [15][16].) It is, of course, important to find exact black hole solutions, but at the present time this is known only in two spacetime dimensions [17].

The equations of motion which follow from this action are

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - 2F_{\mu\lambda}F^\nu_\lambda - \frac{1}{4}H_{\mu\lambda\sigma}H^{\nu}_{\lambda\sigma} = 0 \quad (2.10a)$$

$$\nabla^\nu (e^{-2\phi} F_{\mu\nu}) + \frac{1}{12}e^{-2\phi} H_{\mu\nu\rho} F^{\nu\rho} = 0 \quad (2.10b)$$

$$\nabla^\mu (e^{-2\phi} H_{\mu\nu\rho}) = 0 \quad (2.10c)$$

$$4\nabla^2 \phi - 4(\nabla \phi)^2 + \Lambda + R - F^2 - \frac{1}{12}H^2 = 0 \quad (2.10d)$$

The dilaton appearing in these equations is massless. But it is expected that when supersymmetry is broken, the dilaton will acquire a mass. We will consider black hole solutions with a massive dilaton in Sec. 4.5. It turns out that black holes which are small compared to the Compton wavelength of the dilaton will resemble the massless dilaton solutions. So it is worthwhile to begin by studying the black hole solutions to (2.10) without the mass term.

2.3. Generating Solutions

At first sight it appears difficult to find exact solutions to (2.10). The presence of the exponential of the dilaton makes the field equations rather complicated. However it turns out that if one considers spacetimes with a symmetry, one can generate new solutions from old ones by a simple transformation [18][19][20]. For a solution with several symmetry directions there is a large class of new solutions, but here we will consider just the simplest case of a single symmetry. To illustrate the type of transformation we will use, let us first consider Kaluza-Klein theory. The five dimensional vacuum Einstein action, when restricted to spacetimes that are independent of one direction $x$, is equivalent to the following action for a four dimensional metric, Maxwell field, and scalar:

$$S = \int d^4x \sqrt{-g} \left(R - 2(\nabla \Phi)^2 - e^{-2\sqrt{3}\Phi} F^2\right) \quad (2.11)$$
The components \( g_{\mu 5} \) of the five dimensional metric are essentially the four dimensional vector potential. This theory also appears difficult to solve exactly. However, once the connection with five dimensional general relativity is understood, it is easy to generate nonvacuum solutions starting from a static vacuum solution. Given a static four dimensional vacuum metric, one can take its product with \( R \) to obtain a five dimensional solution with two symmetry directions. One can now boost this solution in the fifth direction. This clearly still satisfies the five dimensional field equations. However when reinterpreted in four dimensions, one obtains a solution with nonzero Maxwell field and dilaton. In particular, one can find charged black hole solutions to (2.11) starting from Schwarzschild this way \[21\] [22].

In heterotic string theory, the situation is slightly different. The extra spacetime coordinates are divided into left-moving and right-moving parts. Only half of these are added to the theory. This results in the low energy gauge fields. Nevertheless there is still a way to “boost” a static uncharged solution to obtain a charged one. To be explicit, we start with any static solution \( (g_{\mu \nu}, B_{\mu \nu}, \phi) \) to (2.10) with \( A_{\mu} = 0 \). Since the solution is static, rather than just stationary, \( g_{ti} = 0 \). (For simplicity, we will further assume \( B_{ti} = 0 \).) Then one can obtain a one parameter family of solutions with \( B_{\mu \nu} \) unchanged, \( g_{ij} \) unchanged and \[18\]

\[
\begin{align*}
\tilde{g}_{tt} &= \frac{g_{tt}}{[1 + (1 + g_{tt}) \sinh^2 \alpha]^2} \\
\tilde{A}_t &= -\frac{(1 + g_{tt}) \sinh 2\alpha}{2\sqrt{2}[1 + (1 + g_{tt}) \sinh^2 \alpha]} \\
e^{-2\tilde{\phi}} &= e^{-2\phi}[1 + (1 + g_{tt}) \sinh^2 \alpha] \tag{2.12}
\end{align*}
\]

where \( \alpha \) is an arbitrary parameter. When \( \alpha = 0 \), the transformation reduces to the identity. This formula can be generalized to include nonzero \( g_{ti} \) and \( B_{ti} \). There is an analogous transformation for spacelike symmetries.

There is a second transformation which will play an important role in our discussion of black strings. This is a discrete transformation which relates solutions of (2.10) with a symmetry\[3\]. For simplicity, we again set \( A_{\mu} = 0 \) and let \( (g_{\mu \nu}, B_{\mu \nu}, \phi) \) be a solution to

\[3\] The two transformations that we describe here are actually part of an \( O(2,1) \) symmetry group \[18\].
(2.10) which is independent of $x$. Then $(\tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\phi})$ is also a solution where

$$
\begin{align*}
\tilde{g}_{xx} &= 1/g_{xx}, \quad \tilde{g}_{x\alpha} = B_{x\alpha}/g_{xx} \\
\tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - (g_{x\alpha}g_{x\beta} - B_{x\alpha}B_{x\beta})/g_{xx} \\
\tilde{B}_{x\alpha} &= g_{x\alpha}/g_{xx}, \quad \tilde{B}_{\alpha\beta} = B_{\alpha\beta} - 2g_{x[\alpha}B_{\beta]x}/g_{xx} \\
\tilde{\phi} &= \phi - \frac{1}{2} \log g_{xx}
\end{align*}
$$

(2.13)

and $\alpha, \beta$ run over all directions except $x$. This transformation is sometimes called spacetime (or target space) duality. If $x$ is periodic, (2.13) is not just another solution to the field equations. It is known that string theory has the remarkable property that different spacetime geometries can correspond to the same conformal field theory [24][25]. The transformation (2.13) is a generalization of the $r \to 1/r$ symmetry of strings moving on a circle of radius $r$. As in that flat spacetime example, one can show [26] that if $x$ is compact, the tilded solution (2.13) is physically equivalent to the original solution. There are other examples of duality for solutions without a symmetry direction. It is conceivable that every solution has at least one dual description.

We now describe a general result on the effect of spacetime duality on asymptotically defined conserved quantities, and show how it can be applied to black strings. Let $g_{\mu\nu}, B_{\mu\nu}$ and $\phi$ be a solution to the low energy field equations which is independent of $x$, and is asymptotically flat in the transverse direction. Then one can define the mass per unit length, or more generally the ADM energy momentum per unit length $P_\mu$. We can also define a charge per unit length associated with the antisymmetric tensor field by $Q = \int e^{-2\phi} * H/V_{D-3}$ where $V_{D-3}$ is the volume of a unit $D-3$ sphere and the integral is over the $D-3$ sphere at fixed time, fixed $x$, and large transverse distance. Since the dual solution (2.13) is also translationally invariant and asymptotically flat one can define an energy momentum and charge per unit length associated with it. One can now ask what is the relation between these quantities. One finds the surprising result [27]

$$
\tilde{Q} = P_x, \quad \tilde{P}_x = Q, \quad \tilde{P}_\alpha = P_\alpha
$$

(2.14)

In other words, the effect of duality is simply to interchange the charge and the momentum in the symmetry direction. Since these solutions represent the same conformal field theory, one learns that the charge associated with $H$ is equivalent to momentum in string theory.

One can use this to add charge to any solution which is both static and translationally invariant as follows. One first boosts the solution to obtain $P_x \neq 0$ and then applies duality
to convert this momentum into charge. This result may have applications independent of black strings, but we use it in Sec. 5 to obtain black string solutions. Since this charge is equivalent to momentum, why bother constructing the charged solutions? The answer is that having an alternative description of the solution is very useful for making contact with other results in string theory. It also illustrates what properties of spacetime are well defined in string theory (i.e. duality invariant) and which are not.

The result (2.14) is somewhat reminiscent of Kaluza-Klein theory, where spacetime momentum in an internal direction gives rise to charge in the lower dimensional space. But there is a crucial difference. In the present case, the charge arises in the higher dimensional space and is associated with a separate field. It is not part of the higher dimensional metric.

3. STRING ANALOG OF REISSNER-NORDSTRØM

We can now begin our discussion of black hole solutions to the low energy string equations (2.10). In this section we will consider the most physical case of four spacetime dimensions. The appropriate boundary conditions are that the spacetime be asymptotically flat and the dilaton approach a constant at infinity which we will take to be zero. (We will see how to obtain other asymptotic values of the dilaton in Sec. 4.1.) These boundary conditions require \( \Lambda = 0 \). For simplicity, in this section we will also set \( H = 0 \). The metric in (2.9) is the natural one to use since it is the one that strings directly couple to. But in order to compare with general relativity, it is convenient to rescale \( g_{\mu \nu} \) by \( e^{-2\phi} \) to get a metric with the standard Einstein action: \( g_{\mu \nu}^E = e^{-2\phi} g_{\mu \nu} \). The action now becomes:

\[
S = \int d^4 x \sqrt{-g_E} \left( R_E - 2(\nabla \phi)^2 - e^{-2\phi} F^2 \right)
\] (3.1)

When \( F_{\mu \nu} = 0 \) this reduces to the standard Einstein-scalar field action. The “no hair” theorems \(^{28}\) show that the only black hole solutions of this theory are Schwarzschild with \( \phi = 0 \) everywhere. Thus uncharged black holes in low energy string theory are the same as general relativity. Since this is simultaneously the simplest and most physical black hole solution, it is extremely important to find the corresponding exact conformal field theory. The exact solution should agree with Schwarzschild until the curvature becomes of order the Planck scale. As we have remarked, for black holes with mass much larger than the Planck mass, this is well within the horizon. This shows that string theory has black hole solutions.
Since the dilaton $\phi$ couples to $F^2$, charged black holes are not Reissner-Nordstrøm with $\phi = 0$. One might worry that with the exponential coupling, the exact solutions would be very complicated. But, as we have seen, they are are easily found using the transformation (2.12). (Recall that this transformation yields the string metric.) Starting with a Schwarzschild solution with mass $m$ (and radial coordinate $\hat{r}$) one obtains

$$ ds^2 = - \left(1 - \frac{2m}{\hat{r}}\right) \left(1 + \frac{2m\sinh^2 \alpha}{\hat{r}}\right)^{-2} dt^2 + \left(1 - \frac{2m}{\hat{r}}\right)^{-1} d\hat{r}^2 + \hat{r}^2 d\Omega $$

$$ A_t = - \frac{m \sinh 2\alpha}{\sqrt{2[\hat{r} + 2m\sinh^2 \alpha]}} $$

$$ e^{-2\phi} = 1 + \frac{2m}{\hat{r}} \sinh^2 \alpha $$  \hspace{1cm} (3.2)

The causal structure of this spacetime is identical to Schwarzschild. There is an event horizon at $\hat{r} = 2m$ and a curvature singularity at $\hat{r} = 0$. (The vector potential $A_t$ is actually finite at $\hat{r} = 0$, although the invariant $F_{\mu\nu}F^{\mu\nu}$ diverges there.) Notice that unlike Schwarzschild, $g_{tt}$ vanishes at the singularity as well as the horizon. Of particular interest is the fact that there is no inner horizon. I used to think this was a result of the instability of the inner horizon: When the dilaton is included the inner horizon becomes singular. But as we will see in the next sections, there are several examples of solutions with dilaton which have a nonsingular inner horizon.

As $\hat{r} \to 0$, $e^\phi \to 0$, so the string coupling is becoming very weak near the singularity. As we have discussed, we have no right to trust this solution near the singularity, but its difficult to resist speculating about what it might mean if the exact classical solution had a similar behavior. It would suggest that, contrary to the usual picture of large quantum fluctuations and spacetime foam near the singularity, quantum effects might actually be suppressed. The singularity would behave classically.

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4 This solution was first found by Gibbons [29], and further discussed in [30]. It was independently found a few years later in a somewhat simpler form by Garfinkle et.al. [31]. All of these papers directly solved the field equations. The solution generating technique described here was discovered more recently.

5 This is not apparent when the action is expressed in terms of the Einstein metric because Newton’s constant has been suppressed. In string theory, Newton’s constant is not fundamental, but determined by the dilaton and the string tension.
The physical mass $M$ of a solution to (2.10) is independent of whether one expresses it in terms of the string metric or Einstein metric. Although the formula for the mass in terms of the asymptotic form of the metric and dilaton does depend on this choice. The easiest way to calculate the mass of (3.2) is to rescale to the Einstein metric and compare with Schwarzschild asymptotically. The physical charge $Q$ is of course related to the asymptotic form of $A_t$. One finds that $M$ and $Q$ are related to the original Schwarzschild mass $m$ and transformation parameter $\alpha$ by

$$M = m \cosh^2 \alpha, \quad Q = \sqrt{2} m \cosh \alpha \sinh \alpha$$

(3.3)

It follows that the charge to mass ratio depends only on $\alpha$ and is given by $Q^2/M^2 = 2 \tanh \alpha$. Thus for fixed $M$, one can increase the charge by increasing $\alpha$ and decreasing $m$. This results in the area of the event horizon becoming smaller.

**Fig. 5:** The extremal black hole with dilaton.

The largest possible charge for a given mass is $Q^2 = 2M^2$ and is obtained by taking the limit $m \to 0, \alpha \to \infty$ keeping $m \cosh^2 \alpha$ fixed. In this limit, the event horizon shrinks to zero size and becomes singular. The metric takes the extremely simple form:

$$ds^2 = -\left(1 + \frac{2M}{\hat{r}}\right)^{-2} dt^2 + d\hat{r}^2 + \hat{r}^2 d\Omega$$

(3.4)

The spatial part of the metric is now completely flat! Strictly speaking this spacetime does not represent a black hole since it does not possess a regular event horizon. Nevertheless, in analogy with the Reissner-Nordstrøm solution, we will call it the extremal charged black hole. The singularity at $\hat{r} = 0$ is not a typical naked singularity like the one shown in fig. 4.
It actually consists of two parts, each of which is null. The Penrose diagram is shown in fig. 5. This can be understood as follows. Near the singularity, radial null geodesics satisfy $\pm dt \propto d\hat{r}/\hat{r}$ which implies that as $\hat{r} \to 0$, the geodesics reach arbitrarily large values of $|t|$. This shows that an outgoing null geodesic must cross every ingoing null geodesic. Notice that the condition for the extremal limit has shifted from $Q^2 = M^2$ (without the dilaton) to $Q^2 = 2M^2$. We will see a way to understand this in Sec. 4.4.

To facilitate comparison with the standard black holes of general relativity, it is convenient to rescale to the Einstein metric. Performing this rescaling and introducing a new radial coordinate $r = \hat{r} + 2m \sinh^2 \alpha$ yields a remarkably simple form of the solution:

$$ds^2_E = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r \left(r - \frac{Q^2}{M}\right) d\Omega$$

$$F_{rt} = \frac{Q}{r^2}, \quad e^{2\phi} = 1 - \frac{Q^2}{Mr} \quad (3.5)$$

Note that the metric in the $r-t$ plane is identical to Schwarzschild! The only difference is that the area of the spheres is smaller. In fact, this area goes to zero when $r = Q^2/M$ and this surface is singular. Since $g_{tt}$ remains finite at the singularity, there is no “infinite stretching” analogous to what happens to an observer hitting the singularity in Schwarzschild. The Penrose diagram (for $Q^2 < 2M^2$) is identical to Schwarzschild.

As you increase $Q$, the singularity moves out in “$r$”. In the extremal limit $Q^2 = 2M^2$, the singularity coincides with the horizon. Since the causal structure in the $r-t$ plane is independent of $Q$, this shows that the spacetime is described by fig. 5 even in the Einstein metric. If you increase $Q$ farther, the singularity moves outside the horizon and becomes timelike.

The fact that the horizon shrinks to zero size in the extremal limit has an important consequence. It shows that there is no classical process which will cause a nearly extremal black hole to become extremal [32]. This is a result of the area theorem [33] which states that one cannot classically decrease the area of a black hole. This theorem depends on an energy condition which is indeed satisfied by (3.1).

So far we have discussed only electrically charged black holes. The magnetically charged black hole can be obtained from the electrically charged solution by an electromagnetic duality transformation. From (3.1), the equation of motion for $F_{\mu\nu}$ is

$$\nabla_{\mu} \left( e^{-2\phi} F^{\mu\nu} \right) = 0 \quad (3.6)$$

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This implies $\tilde{F}_{\mu\nu} \equiv e^{-2\phi} \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}$ is curl free. One can easily check that the equations of motion are invariant under $F \rightarrow \tilde{F}$, $\phi \rightarrow -\phi$, and $g_E \rightarrow g_E$. Starting with the electrically charged solution and applying this transformation, yields the magnetically charged solutions. Since the Einstein metric is unchanged, the Penrose diagrams are unchanged. However, since $\phi$ changes sign, the string coupling becomes strong near the singularity for these black holes.

Let us now consider the magnetically charged black hole in terms of the string metric. The Einstein metric doesn’t change when we go from electric to magnetic charged black hole, but since $\phi$ changes sign, the string metric does change. We get,

$$ds^2 = -\frac{(1 - \frac{2M}{r})}{(1 - \frac{Q^2}{Mr})} dt^2 + \frac{dr^2}{(1 - \frac{2M}{r}) (1 - \frac{Q^2}{Mr})} + r^2 d\Omega$$  \hspace{1cm} (3.7)

As one approaches the singularity $r = Q^2/M$, the area of the two spheres does not go to zero. Thus in this sense the singularity is larger than the previous case. Now the extremal limit is:

$$ds^2 = -dt^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 + r^2 d\Omega$$  \hspace{1cm} (3.8)

This metric (with $r > 2M$) is geodesically complete and has no curvature singularities! A $t =$-const surface is identical to a $t =$-const surface in extreme Reissner-Nordstrøm and resembles an infinite throat. There is an infinite proper distance from $r = 2M$ to any larger value of $r$. But unlike Reissner-Nordstrøm, $r = 2M$ is now also infinitely far away in null and timelike directions. The horizon has moved off to infinity taking the singularity with it. The dilaton is $e^{-2\phi} = 1 - \frac{Q^2}{Mr} = 1 - \frac{2M}{r}$ for the extremal case. In terms of proper distance along the infinite throat, $\rho \equiv 2Mln(r - 2M)$, we have $\phi = -\rho/4M$. So the dilaton is linear as $\rho \rightarrow -\infty$ and rolls to strong coupling.

Linearized perturbations around these black holes have been studied \cite{[34]}. One finds that they are classically stable, just like the more familiar Einstein-Maxwell black holes. It has also been shown that the extremal black holes are supersymmetric when embedded in an $N = 4$ supergravity theory \cite{[35]}.

The Hawking temperature of these black holes is easily determined as follows. Rescaling the metric by a function which is smooth and nonzero at the horizon, and goes to one at infinity does not affect $T_H$. So one can calculate the temperature using the Einstein metric. But the temperature only depends on the metric in the $r - t$ plane, and for this part of the metric $g_E$ is identical to Schwarzschild. Thus the Hawking temperature is same
as Schwarzschild $T_H = 1/8 \pi M$ independent of $Q$! Thus unlike Reissner-Nordstrøm, $T_H$ does not vanish in the extremal limit. This leads one to worry that a black hole may keep radiating past the extremal limit and become a naked singularity.

There are at least two ways to avoid this conclusion. The first is if there are large potential barriers outside the horizon which cause most of the radiation to be reflected back. If these barriers exist, and grow as the extremal limit is approached, the amount of energy radiated to infinity could go to zero and not result in a naked singularity. It turns out that there are potential barriers but they are not large enough, by themselves, to prevent the formation of naked singularities [34][36]. The second possibility is to include back reaction. Since the horizon moves from a finite to an infinite distance in the string metric, it is clear that the classical geometry is changing significantly as one approaches the extremal limit. This must be properly taken into account. Also, it has been shown that the thermal approximation breaks down near the extremal limit [37]. A complete calculation has not yet been done. A model two dimensional problem including backreaction is currently under investigation. (For a recent review, see [7].)

Although the temperature does not go to zero in the extremal limit, the extremal black hole itself has zero temperature. This is because it has no event horizon and is globally static. The analytic continuation to Euclidean space does not require periodically identifying imaginary time. Thus the Hawking temperature is discontinuous in this limit, which is further evidence that the thermal approximation is breaking down.

4. MORE GENERAL BLACK HOLE SOLUTIONS

In this section we will consider several generalizations of the basic black hole solution discussed above. These will include adding rotation, electric and magnetic charges, higher dimensions, etc. To keep the discussion manageable, we will consider these generalizations independently, starting each time with the black hole solution in Sec. 3. If desired, one can construct even more general solutions by combining several of these features simultaneously.
4.1. Nonzero Dilaton at Infinity

The solutions discussed so far all have vanishing dilaton at infinity. Since the value of the dilaton at infinity determines the string coupling constant at large distances from the black hole, one might wish to keep this a free parameter. Fortunately, it is easy to extend these solutions to allow $\phi$ to approach an arbitrary constant $\phi_0$. The action (3.1) is clearly invariant under $g_E \rightarrow g_E$, $\phi \rightarrow \phi + \phi_0$, $F \rightarrow e^{\phi_0}F$. Even though the Einstein metric is invariant under this transformation, when expressed in terms of the physical charge, it will depend on $\phi_0$. This is simply because the charge is rescaled by $e^{\phi_0}$. (One can obtain the same solution by keeping the Maxwell field fixed and rescaling the Einstein metric.) For example, applying this to the magnetically charged black hole, the solution becomes

$$ds^2_E = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r\left(r - \frac{Q^2e^{-2\phi_0}M}{r}\right)d\Omega$$

$$F_{\theta\phi} = Q \sin \theta \quad e^{-2\phi} = e^{-2\phi_0}\left(1 - \frac{Q^2e^{-2\phi_0}}{Mr}\right)$$ (4.1)

The extremal limit is now $Q^2 = 2M^2e^{2\phi_0}$. So for large $\phi_0$, there exist black holes with $Q$ much larger than $M$.

4.2. Both Electric and Magnetic Charges

We have seen that in terms of the string metric, the electric and magnetic black holes have very different properties in the extremal limit. One has an infinite throat and no curvature singularity, while the other has a singularity but flat spatial slices. What happens when both charges are present? One might think that since there is no reason to prefer one type of charge over another, the extremal limit should somehow combine the features of both. But that is not what happens. We will see that the magnetic charge dominates: If $Q_M \neq 0$, the extremal limit resembles the magnetic solution regardless of the value of $Q_E$!

In the Einstein-Maxwell theory, there is a continuous electromagnetic duality rotation $F \rightarrow \cos \theta F + \sin \theta \ast F$ (where $\ast$ denotes the dual) which interpolates between electric and magnetic charges. Since the stress tensor is left invariant, this will map solutions to solutions keeping the (Einstein) metric invariant. In string theory, life is complicated by both the dilaton and the antisymmetric tensor $H_{\mu\nu\rho}$. (A solution containing both electric and magnetic charge must also contain a nonzero $H_{\mu\nu\rho}$ because of the fact that $dH = -F \wedge F$.) Nevertheless, there is a generalization of this electromagnetic duality
rotation which can be used to generate solutions with both electric and magnetic charges. As before, the Einstein metric is left invariant but the charge $Q^2$ is now interpreted as $Q^2 \equiv Q_E^2 + Q_M^2$. The dilaton is:

$$e^{2\phi} = \frac{1}{Q^2} \left[ Q_E^2 e^{2\tilde{\phi}} + Q_M^2 e^{-2\tilde{\phi}} \right]$$  \hspace{1cm} (4.2)

where

$$e^{2\tilde{\phi}} = 1 - \frac{Q^2}{Mr}$$  \hspace{1cm} (4.3)

Notice that $e^{2\phi}$ is simply the sum of its electric and magnetic values. This linearity is quite surprising since the field equations are highly nonlinear. The fact that the magnetic charge dominates now follows immediately. Near $r = Q^2/M$, $e^{2\tilde{\phi}} \to 0$. Thus the contribution from $Q_E$ becomes negligible compared to that from $Q_M$. In particular, the string coupling will become strong near the singularity whenever $Q_M \neq 0$, and the rescaled string metric will be non-singular.

To complete the solution we must specify $F_{\mu\nu}$ and $H_{\mu\nu\rho}$. The Maxwell field is simply the sum of the fields for electric and magnetic charges. In four dimensions, the antisymmetric tensor field can be replaced by a scalar via $H = -e^{4\phi}(*d\chi)$. For the black hole solution, the scalar is given by

$$\chi = Q_EQ_M \frac{e^{2\tilde{\phi}} - e^{-2\tilde{\phi}}}{Q_E^2 e^{2\tilde{\phi}} + Q_M^2 e^{-2\tilde{\phi}}}$$  \hspace{1cm} (4.4)

This vanishes when either $Q_E$ or $Q_M$ is zero as it should. Since the Einstein metric is unchanged, the Hawking temperature is still $T_H = 1/8\pi M$ for these black holes.

The fact that the magnetic charge dominates is a consequence of the three form $H$. If one ignores the $H$ field, black hole solutions to (3.1) which have both electric and magnetic charge, are more symmetric in $Q_E$ and $Q_M$. The exact solution is known \[^{30},^{35}\] and is most conveniently expressed in terms of the following parameters:

$$r_0 \equiv \frac{Q_M^2 - Q_E^2}{2M} \quad r_{\pm} \equiv M \pm (M^2 + r_0^2 - Q_E^2 - Q_M^2)^{1/2}$$  \hspace{1cm} (4.5)

The Einstein metric is then

$$ds^2_E = -\frac{(r - r_+)(r - r_-)}{r^2 - r_0^2} dt^2 + \frac{r^2 - r_0^2}{(r - r_+)(r - r_-)} dr^2 + (r^2 - r_0^2) d\Omega$$  \hspace{1cm} (4.6)

\[^{6}\] A similar transformation had been noticed earlier in the context of supergravity theories \[^{38}\].
and the dilaton is
\[ e^{2\phi} = \frac{r + r_0}{r - r_0} \quad (4.7) \]

This solution is invariant under interchanging the charges \( Q_M \) and \( Q_E \) and changing the sign of \( \phi \). Near the singularity, the dilaton goes to strong or weak coupling depending on which of the two charges is larger. As expected, if \( Q_E \) or \( Q_M \) vanish, we recover our previous dilaton black hole solutions (3.5) (with the radial coordinate shifted by \( r_0 \)). If \( Q_E = Q_M \), then the dilaton vanishes and the metric reduces to the familiar Reissner-Nordstrøm solution. This is also what one should expect since the source of the dilaton is proportional to \( F^2 \) which vanishes when \( Q_E = Q_M \). Thus (4.6) provides an interesting interpolation between these two classes of solutions.

When \( Q_E \) and \( Q_M \) are both nonzero, the global structure of (4.6) is similar to Reissner-Nordstrøm with an event horizon at \( r_+ \), an inner horizon at \( r_- \), and a curvature singularity at \( r = |r_0| \). This is our first example of a solution with an inner horizon and a nontrivial dilaton. In the extremal limit, the two horizons coincide and the temperature vanishes. If one of the charges vanish, the extremal black hole now has a temperature which depends on the rate at which the charge goes to zero as the extremal hole is approached. Any temperature between 0 and \( 1/8\pi M \) can be obtained.

Although we have ignored the antisymmetric tensor \( H \), the solution (4.6) can still be viewed as a solution to low energy string theory if we interpret the charges as being associated with two different Maxwell fields \( F_i^{\mu\nu} \). Since string theory starts with a large gauge group, it is certainly possible to have two unbroken \( U(1) \) subgroups. In this case, the source of \( H \) would be \( F_i \wedge F_i \) which vanishes. (Solutions where both \( U(1) \) fields have electric and magnetic charges, and \( H \) is nonzero have recently been constructed [41].)

4.3. Rotation

So far we have considered only non-rotating black holes. When the charge is zero, the rotating black hole in string theory is the same as general relativity and is called the Kerr solution:
\[ ds^2_E = - \left( 1 - \frac{2mr}{\Sigma} \right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - \frac{4mra \sin^2 \theta}{\Sigma} dt d\phi \]
\[ + \left[ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right] \sin^2 \theta d\phi^2 \]
\[ \quad (4.8) \]
where
\[ \Sigma = r^2 + a^2 \cos^2 \theta \quad (4.9) \]

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\[ \Delta = r^2 + a^2 - 2mr, \quad (4.10) \]

and \( a \) is the angular momentum \( J \) divided by the mass. This solution is similar to Reissner-Nordstrøm in terms of its causal structure. When \( |a| < m \), there are two horizons at the zeros of \( \Delta \). When \( |a| = m \), these horizons coincide, and when \( |a| > m \), they disappear and the spacetime contains a naked singularity. Also, like Reissner-Nordstrøm, in the extremal limit (\( |a| = m \)) the Hawking temperature \( T_H \) vanishes and the event horizon remains nonsingular with non-zero area \( A \). This behavior is quite different from the extremal limit of the charged non-rotating black hole. We saw in Sec. 3 that as one approaches the extremal limit (\( Q^2 = 2M^2 \)) the event horizon in the Einstein metric becomes singular: \( A \to 0 \) and the dilaton diverges there. In addition, \( T_H \to 1/8\pi M \). Now consider a black hole which has both charge and rotation. We again have a situation like Sec. 4.2 where two special cases have different extremal limits. What is the behavior of the general extremal black hole with both charge and rotation in string theory? It turns out that angular momentum dominates over charge. If \( J \neq 0 \), then the extremal limit resembles the Kerr solution, independent of the value of \( Q \).

The solution for a rotating charged black hole in string theory was found by Sen [42], by applying a generalization of the transformation \( (2.12) \) to the Kerr solution. Since a rotating charged black hole has a magnetic dipole moment, \( F \wedge F \neq 0 \). So again one must include \( H \). In terms of the Einstein metric, the solution is

\[
d s_E^2 = - \left( 1 - \frac{2mr \cosh^2 \alpha}{\Upsilon} \right) dt^2 + \frac{\Upsilon}{\Delta} dr^2 + \Upsilon d\theta^2 - \frac{4mra \cosh^2 \alpha \sin^2 \theta}{\Upsilon} dtd\varphi \\
+ \left[ \frac{(r^2 + a^2 + 2mr \sinh^2 \alpha)^2 - \Delta a^2 \sin^2 \theta}{\Upsilon} \right] \sin^2 \theta d\varphi^2
\]

where

\[
\Upsilon = r^2 + a^2 \cos^2 \theta + 2mr \sinh^2 \alpha
\]

and \( \Delta \) is defined as before \((4.10)\). This is essentially the same as the Kerr metric with \( \Sigma \) replaced by \( \Upsilon \). The Maxwell field, dilaton, and antisymmetric tensor are

\[
A = -\frac{mr \sinh 2\alpha}{\sqrt{2\Upsilon}} (dt - a \sin^2 \theta d\varphi)
\]

\[
e^{-2\phi} = \frac{\Upsilon}{\Sigma}
\]

\[
B_{t\varphi} = \frac{2mra \sinh^2 \alpha \sin^2 \theta}{\Upsilon}
\]

(4.13)
One can easily verify that this solution has the correct limits when \( \alpha \to 0 \) or \( a \to 0 \). The mass \( M \) and charge \( Q \) are related to \( m \) and \( \alpha \) in exactly the same way as the nonrotating solution (3.3), and the angular momentum is given by \( J = Ma \). This solution has two horizons (at the zeros of \( \Delta \)) when \( m > a \) which corresponds to \( 2M^2 > Q^2 + 2|J| \). The presence of rotation does not have much affect on the behavior of the dilaton. The string coupling \( e^\phi \) still goes to zero at the singularity \( \Sigma = 0 \) as expected for an electrically charged black hole.

The extremal limit is \( 2M^2 = Q^2 + 2|J| \). (Since this corresponds to \( m = a \), the transformation generating this solution preserves the extremality of the black hole when it adds charge.) In this limit, one can show that the area of the event horizon is simply related to the angular momentum

\[
A = 8\pi |J|
\]

and is independent of \( Q \). This clearly shows how the zero area of the nonrotating black hole is modified by rotation. When \( J \) is nonzero, the horizon is perfectly regular in the extremal limit. In particular, the dilaton remains finite there. If one does a duality rotation to obtain a rotating magnetically charged black hole, the situation is similar. This shows that the string metric will be qualitatively the same as the Einstein metric. It will not have an infinite throat. Thus the “generic” black hole resembles Kerr. One can also show that the Hawking temperature goes to zero in the extremal limit whenever \( J \neq 0 \). However, since Hawking radiation carries away angular momentum, it is possible that an evaporating black hole will approach a nonrotating extremal limit.

A final comment about rotating black holes concerns the gyro-magnetic ratio. A rotating charged black hole has a magnetic dipole moment \( \mu \) so one can compute a \( g \)-factor from the ratio \( \mu/J \). For the Einstein-Maxwell theory, one has the remarkable result that black holes have \( g = 2 \) (the value for electrons), rather than \( g = 1 \) which one might have expected since this is the value for classical matter. It turns out that string black holes also have \( g = 2 \). One might be tempted to extrapolate from this that all black holes have \( g = 2 \), but this is incorrect. For instance, suppose one simply leaves out the \( H \) field and considers rotating charged black hole solutions to (3.1). Then one finds that \( g \) depends on the charge to mass ratio of the hole and varies from \( g = 2 \) for small charge to \( g = 3/2 \) in the extremal limit\(^7\). The value \( g = 2 \) is recovered only when \( H \) is included in exactly the manner predicted by string theory. The significance of this is not yet understood.

\(^7\) Black holes in Kaluza-Klein theory behave similarly.\[22\]
4.4. Multi-Black Holes

For extremal Reissner-Nordstrøm black holes, the gravitational attraction exactly balances the electromagnetic repulsion, and there exist static multi-black hole solutions. The same is true for the solutions described in Sec. 3. In fact, one can understand the fact that the extremal limit $Q^2 = 2M^2$ has a larger charge/mass ratio than general relativity as a result of the fact that the dilaton contributes an extra attractive force. So, for a given $M$ one needs a larger $Q$ to balance it. More explicitly, for a static asymptotically flat solution, one can define a dilaton charge

$$D = \frac{1}{4\pi} \int d^2 S^\mu \nabla_\mu \phi$$

where the integral is over the two-sphere at infinity. In agreement with “no hair” theorems, this dilaton charge is not an independent free parameter but is uniquely determined by the mass and charge. For the charged black hole solution (3.5), one finds $D = Q^2/2M$. For two widely separated black holes with mass and charge $M_i, Q_i$, the total force is thus

$$F = \left[ \frac{Q_1Q_2 - M_1M_2}{4M_1M_2} \right] \frac{1}{r^2}$$

(4.16)

So the force vanishes when $Q_1 Q_2 = 2M_1 M_2$. For black holes, $Q_i \leq \sqrt{2}M_i$ and the force vanishes in the extremal limit.

If $\vec{x}$ are Cartesian coordinates on $\mathbb{R}^3$, then the solution describing a collection of extremal electrically charged black holes of mass $M_i$ located at $\vec{x}_i$ is

$$ds^2 = -e^{4\phi} dt^2 + d\vec{x} \cdot d\vec{x}$$

$$e^{-2\phi} = 1 + \Sigma_i \frac{2M_i}{|\vec{x} - \vec{x}_i|}$$

(4.17)

This clearly reduces to the previous result (3.14) for one black hole. Space is again completely flat and there are singularities at the location of each black hole. For magnetically charged extremal black holes the solution is

$$ds^2 = -dt^2 + e^{4\phi} d\vec{x} \cdot d\vec{x}$$

$$e^{2\phi} = 1 + \Sigma_i \frac{2M_i}{|\vec{x} - \vec{x}_i|}$$

(4.18)

For a single black hole, this is simply our previous solution (3.8) reexpressed in isotropic coordinates. A spatial surface now looks like $\mathbb{R}^3$ with a finite number of throats branching off. The Maxwell field in each case is just the sum of the Maxwell fields for single black holes.
4.5. Massive Dilatons

In all the black hole solutions we have discussed so far, the dilaton was assumed to be strictly massless. While this is the prediction of classical low energy string theory, it is in conflict with experiment. In many respects the dilaton acts like a Brans-Dicke scalar, with a coupling that violates observational limits. Fortunately, there are strong theoretical arguments that the dilaton should have a mass. The dilaton must be massless as long as supersymmetry is unbroken, but when supersymmetry is broken at low energy it is likely to acquire a mass. We are not yet able to do the nonperturbative quantum calculations required to calculate the low energy dilaton potential. We will consider the simplest choice $m^2 \phi^2$.

The black hole solutions with a massive dilaton do not appear to be expressible in closed form. However by combining approximate solutions with numerical results one can obtain a fairly complete picture of their properties. They differ from the massless dilaton solutions in several respects. At large distances, the dilaton now falls off like $1/r^4$. This causes the dilaton contribution to the stress tensor to be negligible compared to the charge terms. Thus, at large distances, the solution always approaches Reissner-Nordstrøm. However the presence of the dilaton near the horizon still allows black holes to have $Q^2 > M^2$. Since the dilaton force is negligible at large distances, nearly extremal and extremal black holes repel each other. This may be the first example of gravitationally bound repulsive objects. One can show that for a large black hole, the extremal limit corresponds to $Q^2 = M^2 + 1/5m^2$. The energy of $n$ widely separated black holes with the same total charge is then

$$M_n = n \left[ \left( \frac{Q}{n} \right)^2 - \frac{1}{5m^2} \right]^{\frac{1}{2}} = \left[ Q^2 - \frac{n^2}{5m^2} \right]^{\frac{1}{2}}. \quad (4.19)$$

Clearly, $M_n$ is a decreasing function of $n$. In other words, when the dilaton is massive, it is energetically favorable for a large extremal black hole to split into several smaller black holes. This process cannot occur classically, but presumably can occur quantum mechanically.

The causal structure of extremal black holes in this theory depends on their charge. If $|Qm| > e/2$, then the extremal limit is similar to Reissner-Nordstrøm. There are two horizons which come together. This corresponds to an extremal black hole which is larger.

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For a discussion of black holes with more general dilaton potentials, see [44].
than the Compton wavelength of the dilaton. If $|Qm| < e/2$ then there is only one horizon. The extremal limit is then similar to the massless dilaton black holes. In particular, the string metric describing the extremal limit of a magnetically charged black hole will have an infinite throat. Physically, a large black hole with small charge, will start off close to the Reissner-Nordstrøm solution, but as it evaporates, it will begin to resemble the massless dilaton solution. If the mass of the dilaton is about $1 \text{TeV}$, the transition will occur when the black hole has a mass about $10^{11}$ gms. This is sufficiently large that other string corrections should still be negligible.

Another unusual property of black holes with a massive dilaton is the following. It is well known that the maximally extended Schwarzschild solution has a wormhole in the sense that a spacelike surface connecting the two asymptotically flat regions reaches a minimum size inside (or on) the black hole. However this wormhole cannot be transversed, since it quickly collapses to zero size. One can show [45] that the string metric describing certain black holes coupled to a massive dilaton have a wormhole outside the horizon. This wormhole is static and can be transversed. This only occurs when the size of the hole is of order the Compton wavelength of the dilaton. (In this case there is a slight possibility that the black hole solution will have three horizons [44].)

4.6. Lower Dimensions

There is no two dimensional analog of the Schwarzschild solution in general relativity for the simple reason that Einstein’s equation becomes trivial. However the low energy string action (2.9) is nontrivial even in two dimensions. (Although in this case, one can no longer rescale to the standard Einstein action.) Of course, the three form $H$ must vanish in two dimensions. Let us further assume that $F = 0$ to begin. Naive counting indicates that gravity in two dimensions has $-1$ degrees of freedom. This suggests that gravity plus the dilaton should have zero degrees of freedom. While this indicates that there are no propagating modes, there can still be nontrivial solutions. In fact, there are black hole solutions to (2.11) provided one includes the constant $\Lambda$ and allows the dilaton to grow linearly at infinity. The solution takes the form [46] [17]

$$ds^2 = -\left(1 - \frac{M}{r}\right)dt^2 + \frac{kdr^2}{4r(r-M)}$$

$$\phi = -\frac{1}{2} \ln r - \frac{1}{4} \ln k$$

(4.20)
where $M$ is the mass and $k$ is related to the constant $\Lambda$. One can show that these are the only classical solutions in two dimensions. To obtain charged black holes one can simply apply the transformation (2.12) [47].

What is the point of studying black holes in two dimensions with unusual boundary conditions, when more physical solutions are known in four dimensions? The answer is that by going to two dimensions, one can progress much farther than simply solving the low energy string equations of motion. For example, Witten has found the corresponding exact conformal field theory [17]. This is obtained by starting with a Wess-Zumino-Witten (WZW) model based on the noncompact group $SL(2, R)$ and gauging a one dimensional subgroup. One intriguing feature of the exact conformal field theory is that it includes a region of spacetime “beyond” the singularity corresponding to $r < 0$ in (4.20). However it appears unlikely that strings could propagate through the singularity. If so, there are potential causality problems since the light cones tip over on the other side. (Actually, the exact conformal field theory contains two copies of the entire maximally extended black hole spacetime which are joined at the singularity. It is not clear whether this has any physical significance.) In the supersymmetric case, the only higher order corrections to the solution (4.20) is an overall rescaling of the metric [48] [49]. In the purely bosonic case, there are other corrections. The corresponding exact metric has been found [50] [49] and is given by

$$ds^2 = -\beta^{-1}(r) \left( 1 - \frac{M}{r} \right) dt^2 + \frac{(k-2)dr^2}{4r(r-M)}$$

(4.21)

where

$$\beta(r) = 1 - \frac{2}{k} \left( 1 - \frac{M}{r} \right)$$

(4.22)

This agrees with the above metric for large $k$ which is equivalent to small curvature (since, by rescaling $t$, one can view $k$ as multiplying the entire metric). The metric is now regular at the former singularity $r = 0$. But it still has a curvature singularity at a negative value of $r$ where $\beta(r) = 0$. The exact dilaton has also been calculated and is

$$\phi = -\frac{1}{2} \ln r \sqrt{\beta(r)} + \phi_0$$

(4.23)

where $\phi_0$ is a constant. Notice that this still diverges at the original location of the singularity $r = 0$.

Another advantage of two dimensions is that one can study Hawking evaporation. One would like to do this in the context of the full quantum string theory, perhaps taking
advantage of the recent progress in nonperturbative solutions of string theory in two dimensions. However a more modest goal is to add matter to the low energy string action (2.9) and study Hawking evaporation in this theory. This has been the subject of extensive work over the past year and is reviewed in the lectures by Harvey and Strominger in this volume [7].

What about three dimensional black holes? Given the two dimensional black hole, one can clearly take its product with $S^1$ to obtain a three dimensional solution with an event horizon. Unfortunately, this is the best one can do. There do not exist any other static, axisymmetric three dimensional solutions of (2.10) with a regular horizon. This can be seen as follows*. For simplicity, we set $F = H = 0$. (If a charged black hole exists in three dimensions, then an uncharged one should exist as well.) The equation for the metric (2.10) is

$$R_{\mu\nu} = -2\nabla_\mu \nabla_\nu \phi \quad (4.24)$$

Let us assume a metric of the form

$$ds^2 = -\lambda dt^2 + f dr^2 + g d\theta^2 \quad (4.25)$$

By combining the $tt$ and $\theta\theta$ components of (4.24) one obtains

$$\frac{\lambda''}{\lambda'} - \frac{\lambda'}{\lambda} = \frac{g''}{g'} - \frac{g'}{g} \quad (4.26)$$

This equation is immediately integrated to yield:

$$g = c_1 \lambda^{c_2} \quad (4.27)$$

where $c_1$ and $c_2$ are constants. For a regular horizon, one needs $\lambda = 0$ with $g$ remaining finite and nonzero. This is possible only if $c_2 = 0$ which yields the simple product of a two dimensional solution with $S^1$. Although the curvature of this solution vanishes at infinity, it is not asymptotically flat in the usual sense of approaching the flat metric on $\mathbb{R}^2$ minus a ball. We are forced to conclude that there are no asymptotically flat three dimensional black holes in string theory.

Notice that this argument is independent of any boundary conditions on the dilaton at infinity. Since there are black hole solutions with the dilaton growing linearly at infinity in two dimensions, and going to a constant in four dimensions, one might have thought that there would be solutions with $\phi$ growing, say, logarithmically in three dimensions. This argument shows that such solutions do not exist.

It has recently been shown that there are three dimensional black holes in general relativity with negative cosmological constant [51]. It is not yet clear whether they have any significance for string theory.

* This argument was developed in collaboration with J. Horne.
4.7. Higher Dimensions

There is a straightforward generalization of the electrically charged black hole to higher dimensions \[30\]. One simply starts with the \(D\) dimensional Schwarzschild solution and applies the transformation (2.12). Since the \(D\) dimensional Schwarzschild solution is related to the four dimensional solution by essentially replacing \(r\) by \(r^n\) where \(n = D - 3\), the same is true for the stringy version

\[
ds^2 = - \left(1 - \frac{cm}{r^n}\right) \left(1 + \frac{cm \sinh^2 \alpha}{r^n}\right)^{-2} dt^2 + \left(1 - \frac{cm}{r^n}\right)^{-1} dr^2 + r^2 d\Omega_{n+1} \tag{4.28a}
\]

\[
A_t = - \frac{cm \sinh 2\alpha}{2\sqrt{2}[r^n + cm \sinh^2 \alpha]} \tag{4.28b}
\]

\[
e^{-2\phi} = 1 + \frac{cm}{r^n} \sinh^2 \alpha \tag{4.28c}
\]

where \(c\) is a dimension dependent constant. The mass and charge are given by

\[
M = m \left(1 + \frac{2n}{n+1} \sinh^2 \alpha\right) \quad Q = cmn \cosh \alpha \sinh \alpha / \sqrt{2} \tag{4.29}
\]

To obtain the Einstein metric one multiplies (4.28a) by \(e^{-4\phi/D - 2}\). These solutions have an event horizon at \(r^n = cm\) and a singularity at \(r = 0\). In the extremal limit the horizon shrinks to zero size and the spatial metric becomes flat. Note that the string coupling vanishes, \(g = e^{\phi} \to 0\) near the singularity in all dimensions.

Although the higher dimensional black holes resemble the four-dimension black holes in almost all respects, there are two important differences. In the extremal limit, the singularity is no longer null, but is now timelike (like the negative mass Schwarzschild solution). This can be seen from the fact that radial null rays satisfy \(\pm dt \propto dr/r^n\). For \(n > 1\), there is only a finite change in \(t\) as \(r \to 0\). The second difference is with the Hawking temperature. One can show from (2.7) that \(T_H \to 0\) as you approach the extremal black hole for all \(D > 4\). It is only for \(D = 4\) that the temperature approaches a non-zero limit.

Unlike the electrically charged case, there is no higher dimensional generalization of the magnetically charged black hole for the simple reason that there is no magnetic charge in higher dimensions. \(Q_M\) is defined by integrating \(F\) over the sphere at infinity and only in four spacetime dimensions is the sphere at infinity two dimensional. (The electric charge does not have a similar problem since \(Q_E \propto \int_{S_{\infty}} *F\) and \(*F\) is a \(D - 2\) form which can be integrated over the \(D - 2\) sphere at infinity for all \(D\).)
Even though there are no black holes with magnetic charge in higher dimensions, for $D = 5$ there is an analogous solution using the three form $H$. One can define a charge associated with $H$ in exactly the same way one defines magnetic charge in four dimensions:

$$\hat{Q} = \int_{S_3} H/V_3.$$  

The five-dimension black hole with $\hat{Q} \neq 0$ is

$$ds^2 = -\left[1 - \left(r_+/r\right)^2\right] \, dt^2 + \frac{dr^2}{1 - \left(r_+/r\right)^2} + r^2d\Omega_3$$

$$e^{-2\phi} = 1 - \left(r_-/r\right)^2 \quad H = \hat{Q}\epsilon_3$$  

(4.30)

where $\epsilon_3$ is the volume form on a unit three sphere. The constants $r_+, r_-$ are related to $M, \hat{Q}$ by

$$M = r_+^2 - \frac{1}{3} r_-^2, \quad \hat{Q} = 2r_+ r_-$$

This is very similar to the four-dimensional magnetically charged solution (3.7). The event horizon is at $r = r_+$ and the singularity is at $r = r_-$. The extremal limit ($r_+ = r_-$) is again completely nonsingular:

$$ds^2 = -dt^2 + \left[1 - \left(r_+/r\right)^2\right]^{-2} \, dr^2 + r^2d\Omega_3$$  

(4.31)

As in the previous case (3.7), the dilaton is again linear in proper distance along the throat and rolls to strong coupling. One can show [52] that this extremal limit is in fact an exact solution of the type II superstring theory. (To get a solution to the heterotic string one must add appropriate gauge fields.)

There is, in fact, a second way to take the extremal limit, in which one stays a finite distance from the horizon as the limit is taken [53]. The asymptotic region is now lost, and the solution becomes just a product of the two dimensional black hole with $S^3$. This is also an exact conformal field theory since the $H$ field has only components on the $S^3$ and is simply the SU(2) Wess-Zumino-Witten model.

This black hole has a magnetic type of $H$ charge. What about solutions with electric $H$ charge i.e. $Q \equiv \int_{S_3} \ast H/V_{D-3} \neq 0$. Since $\ast H$ is a $D-3$ form, it is clear that this charge is carried by a one-dimensional extended object i.e. a string. A few years ago Dabholkar [9] I know of no way to obtain it by a transformation of Schwarzschild. In five dimensions, the dual of $H$ is a two form, but it has a different coupling to the dilaton than the Maxwell field $F$. Thus one cannot obtain (4.30) as we did in four dimensions, by dualizing the electrically charged black hole. The above solution was found by explicitly solving the field equations.

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et. al. [54] found the solution describing a straight fundamental string i.e. they added a source term to the field equations (2.10) which was a $\delta$ function on the string worldsheet. These solutions had $Q \neq 0$ but did not have an event horizon. We will see in Sec. 5 that there are solutions to (2.10) with $Q \neq 0$ which describe one-dimensional extended objects surrounded by event horizons i.e. black strings.

5. BLACK STRINGS

Black strings can be constructed in various spacetime dimensions. As we will discuss in Sec. 5.3, for $D = 3$ the exact conformal field theory is known. However in higher dimensions, we only know the solutions to the low energy field equations (2.10).

5.1. Dimensions $D > 4$

Uncharged black strings in $D$ dimensions are simply the product of a $D-1$ dimensional Schwarzschild solution and $R$.

$$ds^2 = -\left(1 - \frac{cm}{r^n}\right)dt^2 + \left(1 - \frac{cm}{r^n}\right)^{-1}dr^2 + r^2d\Omega_{n+1} + dx^2$$

$$\phi = 0, \quad B = 0$$

where $n = D - 4$ and $c$ is, as before, a dimension dependent constant. Black strings with electric charge can similarly be obtained by simply taking the product of the $D-1$ dimensional charged black hole and $R$. It is more interesting to consider black strings with $H$ charge. These are not simple products. However, they can be found quite easily using spacetime duality as described in Sec. 2.3. First we apply a Lorentz boost $\hat{t} = t \cosh \alpha + x \sinh \alpha$, $\hat{x} = x \cosh \alpha + t \sinh \alpha$, and then a duality transformation (2.13) on $x$ to get [2][27]

$$ds^2 = -\frac{(1 - cm/r^n)}{(1 + cm \sinh^2 \alpha/r^n)}dt^2 + \frac{dx^2}{(1 + cm \sinh^2 \alpha/r^n)}$$

$$+ \frac{dr^2}{(1 - cm/r^n)} + r^2d\Omega_{n+1}$$

$$e^{-2\phi} = 1 + \frac{cm \sinh^2 \alpha}{r^n}$$

$$B_{xt} = \frac{cm \cosh \alpha \sinh \alpha}{r^n + cm \sinh^2 \alpha}$$

(5.2a)

(5.2b)

(5.2c)
Notice the similarity with the charged black hole solution \((4.28)\). The dilatons are identical, \(B_{tx}\) is a multiple of \(A_t\), and the only difference between the metrics is that the factor \((1 + cm \sinh^2 \alpha/r^n)\) does not appear squared in \(g_{tt}\) but is split evenly between \(g_{tt}\) and \(g_{xx}\). As before, the event horizon is at \(r^n = cm\) and the curvature singularity is at \(r = 0\). The metric is spherically symmetric, static, and translationally invariant in \(x\). The causal structure of this dual solution is exactly the same as for Schwarzschild, as one might have expected since (when \(x\) is periodic) they represent the same conformal field theory. However, like the charged black hole in string theory, it is very different from Reissner-Nordstrøm. In particular there is no inner horizon and the singularity is not timelike.

The parameters \(m\) and \(\alpha\) are related to the physical mass and charge by

\[
M = m(1 + \frac{n}{n+1} \sinh^2 \alpha) \quad Q = cmn \cosh \alpha \sinh \alpha.
\] (5.3)

Of particular interest is the extremal limit. As before, the extremal limit corresponds to \(m \to 0, \alpha \to \infty\) such that \(m \sinh^2 \alpha\) stays constant. In the extremal limit, the horizon shrinks down to zero size and becomes singular. The solution simplifies to

\[
ds^2 = \left(1 + \frac{\tilde{c}M}{r^n}\right)^{-1} \left(-dt^2 + dx^2\right) + dr^2 + r^2d\Omega_{n+1}^2
\]

\[
e^{-2\phi} = 1 + \frac{\tilde{c}M}{r^n}
\]

\[
B_{xt} = \frac{\tilde{c}M}{r^n + \tilde{c}M}
\] (5.4)

where \(\tilde{c} = \frac{n+1}{n} c\). Like the electrically charged black hole, the transverse space is flat in the extremal limit. The extremal solution has an extra symmetry which is not present in \((5.2)\): It is boost invariant in the \(x, t\) plane. Most importantly, \((5.4)\) is precisely the solution found by Dabholkar et.al. \([54]\) describing the field outside of a fundamental string. So a straight fundamental string can be viewed as an extremal black string \([4]\). This is not just a consequence of symmetry considerations. One expects the solution for a straight string to be static, translationally invariant and spherically symmetric. However, since it does not have a regular horizon, it did not have to be contained in the family of solutions \((5.2)\).

Even given that it is contained in this family of solutions, there is no a priori reason to expect that it would correspond to the extremal limit. Indeed, the analogous result is false \(10\)

\[10\) The general static, spherically symmetric solution to \((3.1)\) (without a horizon) is known in closed form \([43]\) and contains one extra parameter than the black hole solutions \((4.1)\).
in general relativity: An electron cannot be viewed as an extremal Reissner-Nordstrom black hole.

If \( x \) is compact, the straight string can be viewed as an unexcited string with winding number one. The fact that this string can be viewed as an extremal black string strongly suggests that excited string winding states are black strings and excited nonwinding states are black holes. In this regard, it is important to keep in mind the following simple observation. Given a particle of mass \( m \), there are two length scales that can be defined. One is the Compton wavelength \( \lambda_{QM} = h/mc \) which can be thought of as a quantum mechanical length scale. The second is the Schwarzschild radius \( \lambda_{G} = Gm/c^2 \) which can be thought of as a gravitational length scale. When \( \lambda_{QM} \gg \lambda_{G} \) i.e. when \( m \) is much less than the Planck mass, it is reasonable to ignore gravity and use quantum field theory in flat spacetime as one usually does. But for \( \lambda_{G} \gg \lambda_{QM} \) this is a terrible approximation. All massive states in string theory satisfy this second inequality. It suggests that a better way to treat them might be to start with the black hole solution and quantize about it.

In the dual picture, the extremal limit corresponds to boosting the uncharged string to the speed of light. The resulting metric takes the form

\[
ds^2 = -\left(1 - \frac{\tilde{c}M}{r^n}\right) dt^2 + \frac{2\tilde{c}M}{r^n} dt dx + \left(1 + \frac{\tilde{c}M}{r^n}\right) dx^2 + dr^2 + r^2 d\Omega_{n+1}. \tag{5.5}
\]

With new coordinates \( x = \frac{1}{2}(u - v) \) and \( t = \frac{1}{2}(u + v) \), the metric becomes

\[
ds^2 = -du dv + dr^2 + r^2 d\Omega_{n+1} + \frac{\tilde{c}M}{r^n} du^2. \tag{5.6}
\]

Metrics of this type are called plane fronted waves. Like most solutions we have discussed so far, the extremal black string (5.4) is only a solution to the low energy field equations. When higher powers of the curvature are included, it will have higher order corrections. Similarly, the duality transformation itself (2.13) has higher order corrections. Remarkably, these two corrections cancel each other! It has been shown \([55] [56]\) that (5.6) is a solution to string theory even including all higher order terms in the field equation. The reason is essentially that the curvature is null and so all powers of it vanish. Since this represents the same conformal field theory (for compact \( x \)) as (5.4), one can also view fundamental strings as strings moving at the speed of light.

It is known that spacetime duality is accompanied by interchanging the momentum and winding modes of test strings propagating in the background. Since the extremal black string can be viewed as the field generated by a pure winding state of the string, can
its dual be interpreted as the field generated by a pure momentum state? In other words, does spacetime duality, in this case, simply correspond to interchanging the momentum and winding of the source string? At first sight this interpretation looks promising since the classical constraint equations for a string show that for a pure momentum state, the momentum must be null: An unexcited string always moves at the speed of light. However an unexcited string is a pointlike object. The field outside of a point particle accelerated to the speed of light is given by (5.6) with an extra $\delta(u)$ added to $g_{uu}$ [57]. It does not have a spacelike translation symmetry and hence does not have a spacetime dual. The solution (5.6) describes an entire string boosted to the speed of light, not a point particle.

It is interesting to consider the stability of black strings. Perturbations of the simplest case, four dimensional Schwarzschild cross $R$, have been studied in detail [58]. The conclusion is that certain modes with sufficiently long wavelengths along the string grow exponentially with time. It appears that the black string is trying to split into a series of separate disconnected black holes. However this cannot happen since event horizons cannot bifurcate. If we consider the black strings with charge, there is another reason why they cannot split up. Recall that the charge is given by the integral of $\ast H$ over a $D - 3$ sphere at large transverse distances. This charge is conserved, but if the black string split into disjoint black holes, there would be a nonsingular surface spanning the $D - 3$ sphere, and the charge would vanish by Stokes’ theorem. At this time, the significance of the unstable modes is not yet clear. Are there stable black string solutions which are not translationally invariant along the string?

For five dimensional black strings, the extremal limit is shown in fig. 4. This is the familiar spacetime of a naked singularity. However, for $D > 5$, the extremal spacetime resembles the extremal charged black hole in four dimensions fig. 5. The singularity splits into two parts, both of which are null. Another difference is with the Hawking temperature. One can compute a Hawking temperature of a black string by the usual analytic continuation in time. In fact the formula derived earlier for black holes (2.7) is still applicable since $g_{xx}$ remains finite and nonzero at the event horizon. Applying this formula to (5.2) one finds

$$T_H = \frac{n}{4\pi m^{1/n} \cosh \alpha}$$

Since $m \to 0$ and $m \cosh \alpha^2$ stays constant in the extremal limit, we see that for $n = 1$ ($D = 5$), the Hawking temperature of black strings diverges in this limit, while for $n = 2$ ($D = 6$), it approaches a constant. For $n > 2$ ($D > 6$) the situation is similar.
to Reissner-Nordstrøm and the temperature goes to zero. The fact that the temperature diverges in the extremal limit for a five dimensional black string is quite worrisome for it appears that Hawking radiation will overshoot and end up with a naked singularity. But a similar situation occurs for black holes with a different coupling between the dilaton and the Maxwell field. The perturbations around these black holes have been studied by Holzhey and Wilczek [34] and they find that large potential barriers form outside the black black holes which go to infinity in the extremal limit. So even though the temperature is diverging, the energy radiated to infinity vanishes. If a similar thing happens here, the black string would only asymptotically reach its extremal limit.

It is interesting to note that it may actually be easier to do a complete calculation of Hawking evaporation for black strings than for ordinary black holes. This is because of two factors. First, like the electrically charged black holes, the coupling defined by the dilaton is becoming weak at the horizon which may suppress quantum effects in general. Second, the solution is approaching the field outside of a fundamental string and we know how to describe strings quantum mechanically. A black string is likely to approach its extremal limit since the charge $Q$ cannot be radiated away using point particles. This charge will change only if one radiates infinite strings.

More general black string solutions can be constructed along the lines of Sec. 4. For example, black strings with rotation [43] or electric and magnetic charge [59] have been found. Solutions describing waves travelling along an extremal black string have also been constructed [60].

5.2. Four dimensions

We now consider four dimensional black strings. Unfortunately this discussion will be very short since there aren’t any. To apply the construction in Sec. 2.3 one must start with a three dimensional black hole. But we saw in Sec. 4.6 that there are no three dimensional solutions of low energy string theory describing black holes. Even if one relaxes the field equations, one can show that there are no static black strings in any theory satisfying the dominant energy condition: $T_{\mu\nu} t^\mu \tilde{t}^\nu \geq 0$ for all future directed timelike vectors $t^\mu, \tilde{t}^\nu$. This is an immediate consequence of a theorem due to Hawking [4] which states that the event horizon of any stationary black hole in such a theory must be topologically $S^2$. If there was a static black string, one could periodically identify to obtain a black hole with topology $T^2$.  

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This raises the following puzzle. As we have discussed, there is a two dimensional black hole solution. One can always take its product with $T^2$ to obtain a four dimensional solution with a toroidal event horizon. Why doesn’t this contradict Hawking’s theorem? This spacetime is not asymptotically flat in the usual sense, but Hawking’s proof only involves a local calculation in the neighborhood of the horizon. It should apply to spacetimes with this asymptotic behavior. Since the dilaton is finite at the horizon, the Einstein metric will also contain a toroidal horizon. What about the dominant energy condition? Although this condition is not satisfied in the string metric, it is satisfied when the equations are reexpressed in terms of the Einstein metric, and the constant $\Lambda = 0$. But the two dimensional black hole only exists if $\Lambda$ is nonzero and positive. When reexpressed in terms of the Einstein metric, this corresponds to a negative potential for $\phi$ which violates the energy condition and allows the toroidal horizon.

5.3. Three dimensions

Finally we turn to three dimensions. Black strings do exist in three dimensions if one includes $\Lambda > 0$ and allows the dilaton to grow linearly at infinity. This is not surprising since one can simply take the product of the two dimensional black hole with $R$. This yields a black string without charge. To add charge, we can simply follow the above example of boosting and then dualizing. The result is [27][61]

\[
\begin{align*}
ds^2 &= -\left(1 - \frac{M}{r}\right)dt^2 + \left(1 - \frac{Q^2}{Mr}\right)dx^2 + \frac{kdr^2}{4(r-M)(r-Q^2/M)} \\
e^{-2\phi} &= r\sqrt{k} \\
B_{xt} &= Q/r
\end{align*}
\]

Setting $Q = 0$ we clearly recover the two dimensional black hole cross $R$. Unlike the higher dimensional examples, the exact conformal field theory is known [61]. Recall that Witten showed that the exact conformal field theory associated with the two dimensional black hole could be described in terms of a gauged WZW model in which one gauges a one dimensional subgroup of $SL(2,R)$. Similarly, the exact CFT associated with the black strings can be obtained by starting with the group $SL(2,R) \times R$ and gauging the same one dimensional subgroup of $SL(2,R)$ together with a translation of $R$.

This simple metric (and simple construction) has a very rich global structure. For $Q < M$, the spacetime is similar to the Reissner-Nordstrøm solution. There is an event
horizon at \( r = M \) and an inner horizon at \( r = Q^2/M \). The singularities are timelike and the spacetime is timelike geodesically complete. Note that the direction along the string becomes timelike near the singularity. There is also a region beyond the singularity corresponding to \( r < 0 \) in (5.8). In this case, the light cones do not turn over on the other side of the singularity. The WZW construction directly gives you only two copies of the spacetime with identifications along the inner horizon. But the universal covering space would have an infinite number of copies.

Thus, like the rotating black hole of Sec. 4.3, the presence of the dilaton does not cause the inner horizon to become singular. Does the exact solution have an inner horizon? It is not yet clear. On the one hand, (if \( x \) is compact) this solution is supposed to be equivalent to the boosted uncharged black string which does not have an inner horizon. This is consistent with the existence of an inner horizon in the low energy solution since this horizon is unstable. Even though the curvature can be made small (by taking \( k \) large) the higher order corrections may become large. On the other hand, the exact metric and dilaton (but not antisymmetric tensor) for the three dimensional black string has been found \[62\] and does have an inner horizon. Is it possible that an inner horizon might not be well defined in string theory? Equivalent (exact) solutions might differ on whether there is an inner horizon or not.

**Fig. 6:** The extremal three dimensional black string has a horizon but no singularity.

For \( Q = M \), the horizons in (5.8) coincide. The metric becomes boost invariant as in the higher dimensional cases. It looks like the coordinates \( t \) and \( x \) switch roles, but this is
misleading. One can show that geodesics never reach \( r < M \). The correct extension across \( r = M \) is in terms of a new radial coordinate \( \tilde{r}^2 = r - M \). The resulting metric is

\[
ds^2 = \frac{\tilde{r}^2}{\tilde{r}^2 + M} (-dt^2 + dx^2) + \frac{k}{\tilde{r}^2} d\tilde{r}^2
\]  

(5.9)

This metric has the unusual property of having a horizon but no singularity. (I know of no analog in general relativity.) This is just the opposite of the extremal black strings in higher dimensions which had a singularity but no horizon. The Penrose diagram looks like fig. 6. The regions on both sides of the horizon are identical. In the dual description, the extremal limit again corresponds to boosting the uncharged string up to the speed of light.

Finally, for \( Q > M \), the metric appears to change signature at \( r = Q^2/M \). But this is just another indication that an incorrect extension is being used. The point \( r = Q^2/M \) turns out to be a conical singularity which can be removed by making \( x \) periodic. The resulting space is completely nonsingular. Surfaces of constant \( t \) look like infinite cigars. The exact conformal field theory associated with this is obtained by gauging a different subgroup of \( SL(2, \mathbb{R}) \) together with translations in \( \mathbb{R} \) where \( \mathbb{R} \) is now timelike [61].

6. DISCUSSION

To summarize, we have investigated black hole and black string solutions to low energy string theory. Both classes of solutions have many unusual properties. For black holes, one of the most important is that the string metric describing an extremal magnetically charged black hole has neither a horizon nor a curvature singularity. The spacelike surfaces contain infinite throats. These throats remain when one adds electric as well as magnetic charge (provided that there is only one Maxwell field) and when one adds a mass to the dilaton (provided the black hole is sufficiently small). The infinite throats do not remain when rotation is included. An inner horizon is present only in certain cases including nonzero rotation or large black holes with a massive dilaton. Finally, extremal black holes (with charge of the same sign) have no force between them when the dilaton is massless, but become repulsive when the dilaton is massive.

Black string solutions are perhaps of less direct physical interest since, as we have seen, they do not exist in four dimensions. But the fact that in higher dimensions their extremal limit is equivalent to an elementary string, indicates that by studying them, one might gain a deeper understanding of the fundamental nature of strings. In three dimensions,
their extremal limit corresponds to a spacetime with the unexpected property of having an event horizon but no singularity.

Since most of the solutions we have discussed only solve the low energy equations of motion, one cannot use them to learn about singularities in string theory. However, there are a few exceptions. As we have discussed, the exact conformal field theory corresponding to the three dimensional black string is known. When $Q > M$, this solution is nonsingular. Yet, applying the duality transformation (2.13), one obtains a solution which has a curvature singularity \[61\]. (For a discussion of spacetime duality in the context of gauged WZW models see \[63\].) Since these are supposed to correspond to the same conformal field theory, one is led to the conclusion that certain curvature singularities do not adversely affect string theory\[11\]. String scattering in such a background is completely well defined (since it can be calculated in the equivalent nonsingular spacetime). A simpler example of this is to start with Minkowski space in cylindrical coordinates $ds^2 = -dt^2 + dx^2 + dr^2 + r^2 d\theta^2$. Applying the duality transformation to $\theta$ changes $r$ to $1/r$ and creates a curvature singularity at $r = 0$. Yet this solution is equivalent to flat spacetime.

However there are other examples of curvature singularities which do affect strings. These are gravitational plane waves with diverging amplitude. One can show that a string propagating through such a wave becomes infinitely excited \[56\]\[64\]. Physically, this is just a result of the gravitational tidal forces. The singularity in the exact two dimensional black hole is expected to be similar, but this has not yet been conclusively demonstrated.

We conclude with some open problems:

1) It would be of great interest to find the exact solution to string theory which approaches Schwarzschild (or the charged black holes of Sec. 3) when the higher order corrections to the field equations become unimportant. There have been several attempts to use the gauged WZW approach to find such a solution \[45\]\[49\]. Although new exact solutions have been obtained this way, so far none can be interpreted as an asymptotically flat four dimensional black hole.

2) As we have just mentioned, there are two types of curvature singularities in string theory. One affects strings and the other does not. What is the essential difference between them? How generic are they in solutions to string theory, and which type occurs in exact black hole solutions? The standard singularity theorems of general relativity do not apply to string theory. Is there an analogous theorem which does apply?

\[11\] Since the duality transformation is not exact there is a small possibility that the exact dual solution will not have a curvature singularity.
3) It was suggested in Sec. 5 that the massive states of a string should correspond to black holes or black strings depending on whether their winding number is zero or nonzero. Can this be rigorously established? If so, one could view the decay of a massive string state as analogous to Hawking evaporation (if the final states are massless) or quantum bifurcation of black holes (if the final states are massive).

4) We have seen that many of the black hole solutions can be obtained by solution generating techniques. Although we have discussed only a few special cases, for a spacetime with \( d \) symmetry directions, the general transformation (ignoring gauge fields) is \( O(d,d) \). However in general relativity, Geroch has shown [66] that there is an infinite dimensional group which relates solutions to the vacuum Einstein equations with two symmetry directions. Can one similarly extend the known \( O(2,2) \) symmetry of low energy string equations to an infinite dimensional group?

5) The full implications of spacetime duality have not yet been explored. Strictly speaking we should talk only about duality invariant concepts. For example, the three dimensional black string with \( Q < M \) indicates that the inner horizon is not duality invariant. Is the event horizon duality invariant? (If one applies the transformation (2.13) to time translations in Schwarzschild, one obtains another solution in which the horizon becomes a singularity. But since the symmetry is not spacelike and not compact, there is no proof that the two backgrounds correspond to the same conformal field theory.) There are also broader issues associated with the momentum-charge equivalence (2.14). For example we usually think of momentum as associated with a spacetime symmetry and \( Q \) as associated with an internal symmetry. The fact that they are equivalent is a concrete indication of the unification of these symmetries in string theory. As a second example, if \( x \) is compact, \( P_x \) should be quantized. This implies \( Q \) should be quantized as well. This appears to be a new argument for charge quantization.

6) Finally, there is the problem of calculating the Hawking evaporation of black holes and black strings in string theory. This appears to be beyond our current ability, although progress is being made.

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