Abstract. We will discuss some sharp estimates for CMC graphs $\Sigma$ in a Riemannian 3-manifold $M \times \mathbb{R}$ whose boundary $\partial \Sigma$ is contained in a slice $M \times \{t_0\}$. We will start by giving sharp lower bounds for the geodesic curvature of the boundary and improve these bounds when assuming additional restrictions on the maximum height that such a surface reaches in $M \times \mathbb{R}$. We will also give a bound for the distance from an interior point to the boundary in terms of the height at that point, and characterize when these bounds are attained.

1. Introduction

Constant mean curvature surfaces in several 3-manifolds have been extensively studied in recent times. One of the most important families of such 3-manifolds are product spaces $M \times \mathbb{R}$, $M$ being a Riemannian surface, which includes some homogeneous spaces as $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$. It was Rosenberg in [12] who started the study of minimal surfaces in $M \times \mathbb{R}$ and, since then, many papers in this setting have appeared.

We will focus on constant mean curvature $H > 0$ graphs $\Sigma$ in $M \times \mathbb{R}$ whose boundary $\partial M$ lies in some slice $M \times \{t_0\}$ and, if we denote by $c$ the infimum of the Gaussian curvature of the domain of $M$ over which $\Sigma$ is a graph, we will assume the hypothesis

$$4H^2 + c > 0.$$  

As CMC graphs are stable, it is possible to apply Theorem 2.8 in [6] to conclude that the distance function $d(p, \partial \Sigma)$, $p \in \Sigma$, is bounded, so the height function is also bounded. In the case $4H^2 + c \leq 0$, this property fails to be true as invariant examples in $\mathbb{H}^2 \times \mathbb{R}$ given in [7] and [13] show.

To fix notation, let $M$ be a Riemannian surface without boundary and consider $\Sigma \subseteq M \times \mathbb{R}$ an embedded constant mean curvature $H > 0$ surface (an $H$-surface in the sequel). Let us also consider the height function $h \in C^\infty(\Sigma)$ given by $h(p, t) = t$ and the angle function $\nu \in C^\infty(\Sigma)$ given by $\nu = \langle N, E_3 \rangle$, where $N$ is the unit normal vector field to $\Sigma$ for which the mean curvature of $\Sigma$ is $H$, and $E_3 = \partial_t$ is the vertical Killing vector field. Throughout this paper, we will denote by $K$ the intrinsic curvature of $\Sigma$ and $K_M$ will stand for the intrinsic curvature of $M$ extended to $M \times \mathbb{R}$ by making it constant along the vertical geodesics. Besides, $\sigma$ and $A$ will be the second fundamental form and the shape operator of $\Sigma$, respectively. In this situation, the Gauss equation reads $\det(A) = K - K_M \nu^2$.

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Aledo, Espinar and Gálvez proved in [1] that if \( \Sigma \subseteq M \times \mathbb{R} \) is a constant mean curvature \( H > 0 \) graph (or \( H \)-graph for short) over a compact open domain that extends to its boundary with \( h = 0 \) and \( v = v_0 \) in \( \partial \Omega \), and we denote by \( c = \inf \{ K_M(p) : p \in \Omega \} > -4H^2 \), then \( \Sigma \) can reach at most height \( \alpha(c, H, v_0) \), where

\[
\alpha(c, H, v_0) = \begin{cases} 
\frac{4H}{\sqrt{-4cH^2 - c}} \left( \arctan \left( \frac{\sqrt{-c}}{\sqrt{c + 4H^2}} \right) + \arctan \left( \frac{v_0 \sqrt{-c}}{\sqrt{c + 4H^2}} \right) \right) & \text{if } c < 0, \\
\frac{2H}{\sqrt{4cH^2 + c}} \left( \arctan \left( \frac{\sqrt{-c}}{\sqrt{c + 4H^2}} \right) + \arctan \left( \frac{v_0 \sqrt{-c}}{\sqrt{c + 4H^2}} \right) \right) & \text{if } c = 0, \\
\frac{4H}{\sqrt{4cH^2 + c}} \left( \arctan \left( \frac{\sqrt{-c}}{\sqrt{c + 4H^2}} \right) + \arctan \left( \frac{v_0 \sqrt{-c}}{\sqrt{c + 4H^2}} \right) \right) & \text{if } c > 0.
\end{cases}
\]

Indeed, they gave the estimate for the case \( v_0 = 0 \) but their argument can be directly generalized to this more general case. They also proved that this bound is the best one in terms of \( c \) and \( H \) in the sense that in the homogeneous space \( \mathbb{M}^2(c) \times \mathbb{R} \) the only such \( H \)-graphs for which equality holds are rotationally invariant spheres for \( v_0 = 0 \), and in the general case they are spherical caps of rotationally invariant spheres which meet the boundary with constant angle function \( v_0 \).

We will restrict ourselves to the capillarity problem, i.e. when the surface has constant angle function \( v = v_0 \) for some \(-1 < v_0 \leq 0\) along its boundary. This situation includes compact embedded CMC surfaces for \( v_0 = 0 \) because of the Alexandroff reflection principle, and the more general case of embedded \( H \)-bigraphs, that is to say, (not necessarily compact) connected embedded \( H \)-surfaces which are made up of two graphs, symmetric with respect to some slice \( M \times \{ t_0 \} \). Ritoré [10] and Große-Brauckmann [2] constructions are examples of this kind of surfaces in \( \mathbb{R}^3 \). We will prove the following results, where we denote by \( \mathbb{M}^2(c) \) the simply-connected constant curvature \( c \) surface.

- The geodesic curvature \( \kappa_g \) of \( \partial \Omega \) in \( M \), with respect to the outer conormal vector field, satisfies the lower bound

\[
\kappa_g \geq \frac{-4H^2 + c(1 - v_0)^2}{4H\sqrt{1 - v_0^2}},
\]

and, when \( M = \mathbb{M}^2(c) \), equality holds only for rotationally invariant spheres.

- If we additionally suppose that \( |h| \leq m \cdot \alpha(c, H, v_0) \) for some constant \( 0 < m \leq \frac{1}{2} \), then the previous bound is improved to the following one:

\[
\kappa_g \geq \frac{(4 - 8m)H^2 + c(1 - v_0)^2}{4mH\sqrt{1 - v_0^2}}.
\]

In this case, when \( M = \mathbb{M}^2(c) \), equality holds if, and only if, \( m = \frac{1}{2} \) and \( \Sigma \) is the boundary of some neighborhood of a geodesic of \( \mathbb{M}^2(c) \times \{ 0 \} \), examples which will be fully described in section [2].

- In the last section, we will give an application of the techniques used in the above two items to obtain a sharp lower bound for the distance from a point in \( \Sigma \) to \( \partial \Sigma \). As in the results above, equality holds when the surface is rotationally invariant.

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2. Invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$

In this section, we will study surfaces that are invariant by 1-parameter groups of isometries in $\mathbb{H}^2(c) \times \mathbb{R}$ which act trivially on the vertical lines. In fact, among these, we are interested in surfaces which are $H$-bigraphs (i.e. embedded $H$-surfaces symmetric with respect to a horizontal slice), for $H > 0$ and $4H^2 + c > 0$. Thus, these groups of isometries can be identified with 1-parameter groups of isometries of the base $\mathbb{H}^2(c)$.

In $\mathbb{H}^2$, there exist three different types of 1-parameter groups of isometries, namely, rotations around a point, parabolic translations (i.e. rotations about a point at infinity) and hyperbolic translations. The family of rotationally invariant CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ was studied by Hsiang and Hsiang [5] and those invariant by the other two families (including screw motion) were also studied by Sa Earp [13] but it was Onnis [7] who gave a full classification of all invariant CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The case of $S^2$ is quite different, because the only 1-parameter groups of isometries of $S^2$ are the rotations around a certain point and, up to conjugation, this point can be supposed to be the north pole. Such rotationally invariant $H$-surfaces were classified by Pedrosa [9].

Finally, the only 1-parameter groups of isometries of $\mathbb{R}^2$ are rotations around a point and translations; the former give rise in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ to Euclidean spheres of radius $\frac{1}{H}$, the latter to horizontal cylinders of radius $\frac{1}{2H}$.

For the sake of completeness, we will now derive the parametrizations and formulas that we will need in each of these situations. We will begin with rotations in both $\mathbb{H}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$ and then proceed to parabolic and hyperbolic translations in $\mathbb{H}^2 \times \mathbb{R}$. Let us recall that, up to a homothety, we can suppose $c \in \{-1, 0, 1\}$ and, in the cases $c = 1$ and $c = 0$, the condition $4H^2 + c > 0$ is meaningless (as $H > 0$) but, for $c = -1$, it implies that $H > \frac{1}{2}$.

2.1. Rotationally invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$. To start with, let us consider the model $\mathbb{H}^2 \times \mathbb{R} = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 - z^2 = -1, z > 0\}$ endowed with the metric $dx^2 + dy^2 - dz^2 + dt^2$. It was shown by Hsiang and Hsiang that, for any $H > \frac{1}{2}$, the only rotationally invariant $H$-bigraphs are the rotationally invariant CMC spheres.

If we suppose the axis of rotation to be $\{(0, 0, 1)\} \times \mathbb{R}$, the upper half of such a sphere is parametrized by $X(r, u) = (\sinh r \cos u, \sinh r \sin u, \cosh r, h(r))$, where $u \in \mathbb{R}$ and

$$h(r) = \frac{4H}{\sqrt{4H^2 - 1}} \arcsin \sqrt{\frac{1 - (4H^2 - 1) \sinh^2 \frac{r}{2}}{4H^2}}, \quad r \in \left[0, 2 \arcsin \frac{1}{\sqrt{4H^2 - 1}}\right]$$

(see figure where some examples have been depicted).

On the other hand, we will consider the standard model of $S^2 \times \mathbb{R}$ as a submanifold of $\mathbb{R}^4$, given by $S^2 \times \mathbb{R} = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 = 1\}$ with the induced Riemannian metric. It is well-known that every 1-parameter group of ambient isometries consists only of rotations so, up to an isometry, they may be supposed to be rotations around the axis $\{(0, 0, 1, t) : t \in \mathbb{R}\}$. Hence, the orbit space can be identified with the totally geodesic surface $\{(x, y, z, t) \in S^2 \times \mathbb{R} : x = 0\} \cong S^1 \times \mathbb{R}$, and we will take the generating curve as

$$\gamma(t) = (0, \sin r(t), \cos r(t), h(t))$$
Figure 1. On the left, rotationally invariant CMC spheres (the horizontal axis represents the intrinsic length in $\mathbb{H}^2$ and the vertical one is the real line) and, on the right, CMC cylinders invariant under hyperbolic translations, where we see their intersection with the plane $y = 1$ in the halfspace model. In both cases, the represented values of $H$ are 0.54, 0.6, 0.7, 0.8, 0.9 and 1.

for some functions $r, h$ defined on some interval of the real line. Pedrosa [9] showed that the generated surface has constant mean curvature $H \in \mathbb{R}$ if, and only if, certain ODE system is satisfied. In fact, he proved that in the intervals where $r$ is invertible, we can take it as the parameter and the corresponding ODE system becomes

\[
\begin{align*}
  h'(r) &= \cot(\sigma(r)), \\
  \sigma'(r) &= \frac{2H+\cot(r)\cos(\sigma(r))}{\sin(\sigma(r))},
\end{align*}
\]

for an auxiliary function $\sigma$. The second equation can be easily solved as it only depends on $r$ and $\sigma$ and we obtain

\[
\sigma(r) = \arccos(2H(c_0 + \cos r) \csc r)
\]

for some $c_0 \in \mathbb{R}$, where $r \in [a(c_0), b(c_0)] \subseteq [-\pi, \pi]$ is the maximal interval in which $\sigma$ is defined. By plugging this expression into the first equation in (2), we arrive to

\[
h(r) = \int_{a(c_0)}^{r} \frac{2H(c_0 + \cos s) \csc s}{\sqrt{1 - 4H^2(c_0 + \cos s)^2 \csc^2 s}} \, ds.
\]

The only two cases which lead to $H$-bigraphs are the following:

- For $c_0 = -1$, rotationally invariant spheres are obtained. More explicitly,

\[
h(r) = \frac{4H}{\sqrt{1 + 4H^2}} \arccosh \left( \frac{\sqrt{1 + 4H^2}}{4H} \cos \frac{r}{2} \right)
\]

where $r$ lies in the interval $[-2 \arctan \frac{1}{4H}, 2 \arctan \frac{1}{4H}]$. Thus, the maximum height is attained for $r = 0$ and the sphere is a bigraph over a domain whose boundary has constant geodesic curvature in $\mathbb{S}^2$ with respect to the outer conormal vector field, equal to $-H + \frac{1}{4H}$. 
ESTIMATES FOR CONSTANT MEAN CURVATURE GRAPHS IN $M \times \mathbb{R}$

Figure 2. On the left, rotationally invariant CMC spheres and, on the right, rotationally invariant CMC tori, in $S^2 \times \mathbb{R}$. In both cases, the represented values of $H$ are 0.05, 0.12, 0.331372, 0.6, 1 and 2. The horizontal axis measures the intrinsic distance in $S^2$ while the vertical one is the real line. The maximum height is attained for $H \approx 0.331372$, which is drawn as a dashed line.

For $c_0 = 0$, we obtain rotationally invariant tori instead. In this case,

$$h(r) = \frac{2H}{\sqrt{1 + 4H^2}} \arccosh \left( \frac{\sqrt{1 + 4H^2}}{2H} \sin r \right)$$

where $r \in \left[ \frac{\pi}{2} - \arctan \frac{1}{2H}, \frac{\pi}{2} + \arctan \frac{1}{2H} \right]$. The maximum height is attained when $r = \frac{\pi}{2}$ and the boundary of the domain over which the torus is a bigraph has two connected components which have constant geodesic curvature $\frac{1}{2H}$ in $S^2 \times \mathbb{R}$ (with respect to the outer conormal vector field).

These two families are represented in figure 2. We remark that the maximum height of a CMC torus is exactly a half of that of the corresponding sphere for the same mean curvature.

2.2. Invariant surfaces under hyperbolic translations in $H^2 \times \mathbb{R}$. In this section, we will work in the upper halfplane model $H^2 \times \mathbb{R} = \{(x, y, t) \in \mathbb{R}^3 : y > 0\}$ endowed with the metric $(dx^2 + dy^2)/y^2 + dt^2$. Up to conjugation by an ambient isometry, the 1-parameter group of hyperbolic translations may be considered to be $\{\Phi_s^h\}_{s \in \mathbb{R}}$, where

$$\Phi_s^h : H^2 \times \mathbb{R} \rightarrow H^2 \times \mathbb{R}, \quad \Phi_s^h(x, y, t) = (xe^s, ye^s, t).$$

First of all, we observe that, as the orbit of any point is the horizontal Euclidean straight line which joins the point to a point in the axis $x = y = 0$, we can consider the plane $y = 1$ as the orbit space of this group of transformations.

Let us take a curve $\gamma(t) = (x(t), 1, h(t))$ for some $C^2$ functions $x, h$ defined in some interval of the real line. Thus, a surface invariant by $\{\Phi_s^h\}_{s \in \mathbb{R}}$ can be parametrized as

$$X(u, t) = (x(t)e^u, e^u, h(t)).$$

It is a straightforward computation to check that the mean curvature of this parametrization is given by

$$H = \frac{-x^3(h')^3 - xh'((h')^2 + 2(x')^2) + x^2(h'x'' - x'h'') - x'h'' + h'x''}{2((1 + x^2)(h')^2 + (x')^2)^{3/2}}.$$
so there exists a $C^1$ function $\alpha$ such that $h' = \cos \alpha$ and $x' = \sqrt{1+x^2}\sin \alpha$. Now, we can obtain expressions for $x''$ and $h''$ just by taking derivatives in these identities. If we substitute the results in equation (5), we get

$$H = \frac{\sqrt{1+x^2} \alpha' - x \cos \alpha}{2\sqrt{1+x^2}}.$$ 

The proof of the following lemma is now trivial.

**Lemma 2.1.** The parametrized surface defined in (4) has constant mean curvature $H \in \mathbb{R}$ if, and only if, the functions $(x, h, \alpha)$ satisfy the following ODE system

$$\begin{align*}
h' &= \cos \alpha \\
x' &= \sqrt{1+x^2}\sin \alpha \\
\alpha' &= 2H + \frac{x \cos \alpha}{\sqrt{1+x^2}}
\end{align*}$$

Furthermore, the energy function $E = -2Hx - \sqrt{1+x^2}\cos \alpha$ is constant along any solution.

We will restrict ourselves to the case $H > 1/2$. Plugging the expression of the energy into the second equation in (6), it is not difficult to conclude that $x$ verifies the equation

$$(x')^2 = (1 - E^2) + 4HEx + (1 - 4H^2)x^2.$$ 

As $H > \frac{1}{2}$, the RHS has two different real roots as a polynomial in $x$ and, if we factor it, the equation can be expressed, up to a sign, as

$$x' = \frac{\pm 1}{\sqrt{4H^2 - E^2 - 1}} \cdot \sqrt{((4H^2 - 1)x - 2HE)^2 - (4H^2 + E^2 - 1)}.$$ 

from where it is easy to deduce that there exists $c_0 \in \mathbb{R}$ such that

$$(8) \quad x(t) = \frac{2HE}{4H^2 - 1} + \frac{\sqrt{4H^2 + E^2 - 1}}{4H^2 - 1} \sin \left( \pm t \sqrt{4H^2 - 1} + c_0 \right).$$

After a translation and a reflection in the parameter $t$, we will suppose without loss of generality that $c_0 = 0$ and the $\pm$ sign is positive. Now, we can integrate $h$ by taking into account the identity $h' = \cos \alpha = (E + 2H) / \sqrt{1 + x^2}$, and we get

$$(9) \quad h(t) = h(0) + \int_0^t \frac{(8H^2 - 1)E + 2H\sqrt{4H^2 + E^2 - 1} \sin \left( s\sqrt{4H^2 - 1} \right)}{\sqrt{(1 - 4H^2)^2 + \left( \sqrt{4H^2 + E^2 - 1} \sin(s\sqrt{4H^2 - 1}) + 2HE \right)^2}} \, ds.$$ 

Finally, we are able to characterize the surfaces we were looking for. Some pictures of them are drawn in Figure 1.

**Proposition 2.2.** Let $(x, h, \alpha)$ be a solution of (6) with energy $E \in \mathbb{R}$ for some $H > \frac{1}{2}$. Then, the generated invariant surface can be extended to an $H$-bigraph if, and only if, $E = 0$. In this...
case, the generating curve can be reparametrized, up to an ambient isometry, as
\[
\begin{align*}
    x(r) &= \frac{1}{\sqrt{4H^2 - 1}} \sin r \\
    h(r) &= \frac{\cos r}{\sqrt{4H^2 - 1 + \sin^2 r}}
\end{align*}
\], \quad r \in \mathbb{R}.
\]

**Proof.** Let us split \( h(t) = h_1(t) + h_2(t) \) in (9) by splitting the integrand in two additive terms which correspond to the two terms in its numerator. The first term does not vanish unless \( E = 0 \) so \( h_1 \) is monotonic and the second one is an odd periodic function in \( s \) which vanishes at \( s = k\pi/\sqrt{4H^2 - 1} \) for any \( k \in \mathbb{Z} \). On the other hand, if the parametrization interval contains \( t = 0 \), then from (8) we deduce that \( |t| \leq \pi/(2\sqrt{4H^2 - 1}) \) so the surface is a graph and, furthermore, the points at which the normal vector field is horizontal must satisfy \( x' = 0 \), so the parametrization interval must be \( |t| \leq \pi/(2\sqrt{4H^2 - 1}) \). Now, as the integral of \( h_2' \) over \( [-\pi/(2\sqrt{4H^2 - 1}), \pi/(2\sqrt{4H^2 - 1})] \) vanishes, we have
\[
    h_1 \left( \frac{\pi}{2\sqrt{4H^2 - 1}} \right) - h_1 \left( -\frac{\pi}{2\sqrt{4H^2 - 1}} \right) = h_2 \left( \frac{\pi}{2\sqrt{4H^2 - 1}} \right) - h_2 \left( -\frac{\pi}{2\sqrt{4H^2 - 1}} \right).
\]
The RHS term vanishes if, and only if, \( h_1 \) identically vanishes as it is monotonic and \( h_1 \) vanishes if, and only if, \( E = 0 \). The expressions given in the statement follow from a direct computation in (8) and (9) for \( E = 0 \) and from the substitution \( r = t\sqrt{4H^2 - 1} \). Observe that there is no restriction in taking \( r \in \mathbb{R} \) because this parametrization generates the whole bigraph.

In the parametrization given in the statement of Proposition 2.2 observe that the maximum height is attained for \( r = 0 \) and the surface is a bigraph over a domain whose boundary consists in two hypercycles which have constant geodesic curvature in \( \mathbb{H}^2 \) with respect to the outer conormal vector field, equal to \( \pi/2 \). Furthermore, the maximum height is exactly a half of that of the corresponding CMC sphere.

### 2.3. Invariant surfaces under parabolic translations.

In this case, we will also consider the upper halfplane model for \( \mathbb{H}^2 \) so, up to conjugation by an ambient isometry, the 1-parameter group of parabolic translations is \( \{ \Phi_s^p \}_{s \in \mathbb{R}} \), where
\[
    \Phi_s^p : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}, \quad \Phi_s^p(x, y, t) = (x + s, y, t).
\]

Hence, the orbit of any point in \( \mathbb{H}^2 \) is a horizontal Euclidean line parallel to the plane \( y = 0 \). Thus, the orbit space may be considered to be the Euclidean plane \( x = 0 \) so the generating curve can be thought as \( \gamma(t) = (0, y(t), h(t)) \) and a surface invariant by \( \{ \Phi_s^p \}_{s \in \mathbb{R}} \) can be parametrized as
\[
    X(u, t) = (s, y(t), h(t)).
\]

It is straightforward to check that the mean curvature of this parametrization is
\[
    H = -\frac{y^2 (-hh'y'' + h'y'' + y(h')^3)}{2(y^2(h')^2 + (y')^2)^{3/2}}.
\]

Furthermore, there is no loss of generality in supposing that the curve \( \gamma \) is parametrized by its arc-length, i.e. \( 1 = \|a'||^2 = (y')^2/y^2 + (h')^2 \). Hence, we can take an auxiliary
function $\alpha$, determined by $y' = y \sin \alpha$, $h' = \cos \alpha$. Substituting these equalities in (10), it simplifies to the following ODE system

\[
\begin{cases}
y' = y \sin \alpha \\
h' = \cos \alpha \\
\alpha' = -2H - \cos \alpha.
\end{cases}
\]

Observe that, if we assume an initial condition $\alpha(0) = \alpha_0 \in [0, 2\pi]$, the third equation has a unique solution. Let us focus in the case $H > \frac{1}{2}$ which is the most interesting for our purposes and allows us to integrate the function $\alpha$ as

\[
\alpha(t) = 2 \arctan \left( \frac{(2H + 1)}{\sqrt{4H^2 - 1}} \tan \left( \frac{1}{2} \sqrt{4H^2 - 1}(t - c_0) \right) \right)
\]

for some $c_0 \in \mathbb{R}$ depending on $\alpha_0$. We emphasize that this formula defines $\alpha : \mathbb{R} \to \mathbb{R}$ as a strictly increasing diffeomorphism by considering all the branches of the function arctan and extending it by continuity, so the uniqueness of solution guarantees that every solution is considered in (12). We will suppose, after a translation in the parameter $t$, that $c_0 = 0$. By plugging expression (12) into the first two equations in (11), we can integrate $h$ and $y$ to obtain

\[
\begin{align*}
y(t) &= c_1 \left( \cos \left( t\sqrt{4H^2 - 1} \right) + 2H \right) \\
h(t) &= \alpha(t) + 2Ht + c_2,
\end{align*}
\]

for some constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ which can be supposed to be $c_1 = 1$ (after a hyperbolic translation) and $c_2 = 0$ (after a vertical translation).

**Proposition 2.3.** There are no invariant embedded bigraphs under parabolic translations with constant mean curvature $H > \frac{1}{2}$.

**Proof.** Observe that such a graph must be given by a triple $(y, h, \alpha)$ satisfying (11) so (12) and (13) are also satisfied. The values of $t \in \mathbb{R}$ for which $y' = 0$ are $t_k = k\pi/\sqrt{4H^2 + 1}$ for any $k \in \mathbb{Z}$ (these ones correspond to the points in the surface whose tangent plane is vertical). Now from (12) and (13) it is easy to check that $h(t_k) \neq h(t_{k+1})$ for every $k \in \mathbb{Z}$, which makes impossible the surface to be a bigraph.

\[\square\]

### 3. Boundary curvature estimates

Let us suppose along this section that $\Sigma \subseteq M \times \mathbb{R}$ is a graph over a domain $\Omega \subseteq M$ with constant mean curvature $H > 0$. Following the ideas given in (11), for any given $c \in \mathbb{R}$ with $c + 4H^2 > 0$, we will consider the function $g : [-1, 1] \to \mathbb{R}$ determined by

\[
g'(t) = \frac{4H}{4H^2 + c(1 - t^2)}, \quad g(0) = 0,
\]

which is strictly increasing and allows us to define the smooth function $\psi = h + g(\nu) \in C^\infty(\Sigma)$, where $h$ and $\nu$ are the height and angle functions, respectively. We are interested in applying the boundary maximum principle for the laplacian to $\psi$ so we will need to work out $\Delta \psi$ (where the laplacian is computed in the surface $\Sigma$) and $\frac{\partial \psi}{\partial \eta}$, where $\eta$ is some outer conormal vector field to $\partial \Sigma$. The following formulas will be useful.
Lemma 3.1. In the previous situation, the following equalities hold.

i) \( \nabla h = E_3^T \)

ii) \( \Delta h = 2H \nu \)

iii) \( \nabla v = -AE_3^T \)

iv) \( \Delta v = (2K - 4H^2 - K_M(1 + \nu^2)) \nu \)

Proof. The identities for the gradient and the laplacian of \( h \) are easy to check as \( h \) is the restriction to \( \Sigma \) of the height function in \( M \times \mathbb{R} \) (see also [12] Lemma 3.1). On the other hand, the gradient of \( \nu \) is

\[ \langle \nabla \nu, X \rangle = X(\langle N, E_3 \rangle) = \langle \nabla_X N, E_3 \rangle = \langle AX, E_3 \rangle = \langle X, -AE_3^T \rangle \]

for any vector field \( X \) on \( \Sigma \), so \( \nabla \nu = -AE_3^T \). Finally, since the vertical translations are isometries of \( M \times \mathbb{R} \), \( \nu \) is a Jacobi function, i.e. \( \nu \) lies in the kernel of the linearized mean curvature operator \( L = \Delta - \|\sigma\|^2 + \text{Ric}(\nu) \) on \( \Sigma \) so we can compute its laplacian from \( L \nu = 0 \) and obtain

\[ \Delta \nu = (\|\sigma\|^2 + \text{Ric}(\nu)) \nu = (2K - 4H^2 - K_M(1 + \nu^2)) \nu, \]

where we have used the Gauss equation and the well-known identities \( \|\sigma\|^2 = 4H^2 - 2\det(A) \) and \( \text{Ric}(\nu) = K_M(1 - \nu^2) \).

On the other hand, we need to obtain some suitable expression for the modulus of the Abresch-Rosenberg differential, in the case \( M = M^2(\mathbb{C}) \). If we take a conformal parametrization \((U, z)\) in \( \Sigma \), this quadratic differential can be written as

\[ Q = (2Hp - ch_2^2) \, dz^2 \]

(see [4]), where \( p \, dz^2 = \langle -\nabla \partial_{\bar{n}} N, \partial_z \rangle \, dz^2 \) is the Hopf differential and \( h_2 = \frac{\partial h}{\partial \bar{z}} \). Although this expression depends on the parametrization, we may consider the function

\[ q = 4\frac{1}{\lambda^2} |Q|^2 = \frac{4}{\lambda^2} (4H^2|p|^2 + c^2|h_2|^4 - 2cH(p h_2^2 - \bar{p}h_2^2)) \]

\[ = 4H^2(H^2 - \det(A)) + \frac{c^2}{4}(1 - \nu^2)^2 - c(\|\nabla \nu\|^2 - (2H^2 - \det(A))(1 - \nu^2)), \]

which is well-defined and smooth on the whole \( \Sigma \).

Back to the computation of \( \Delta \psi \) and taking into account the formulas in Lemma 3.1 and identity (15), we get

\[ \Delta \psi = \Delta h + g'(\nu)\Delta \nu + g''(\nu)\|\nabla \nu\|^2 = \frac{-8q
u}{(4H^2 + c(1 - \nu^2))^2} - \frac{4H
u (1 - \nu^2)(K_M - c)}{4H^2 + c(1 - \nu^2)}. \]

Finally, we are interested in working out \( \frac{\partial \psi}{\partial \eta} \) along \( \partial \Sigma \), where we have considered the outer conormal vector field to \( \partial \Sigma \) in \( \Sigma \) given by \( \eta = -E_3^T \) (it does not matter which outer conormal vector field is chosen as the only needed information is the sign of \( \frac{\partial \psi}{\partial \eta} \)). Hence,

\[ \frac{\partial h}{\partial \eta} = \langle \nabla h, \eta \rangle = \langle E_3^T, -E_3^T \rangle = -\|E_3^T\|^2, \]

\[ \frac{\partial \nu}{\partial \eta} = \langle \nabla \nu, \eta \rangle = \langle -AE_3^T, -E_3^T \rangle = \langle \nabla_{E_3^T} E_3^T, N \rangle. \]
However, if we parametrize $\partial \Sigma$ by $\gamma$ with $\|\gamma'\| = 1$, then $\{E_3^\top / \|E_3\|, \gamma'\}$ is an orthonormal basis of $T\Sigma$, and it is clear that

$$2H = \left\langle \frac{1}{\|E_3\|^2} \nabla_{E_3^\top} E_3^\top + \nabla_{\gamma'} \gamma', N \right\rangle$$

so $\frac{\partial \psi}{\partial \eta} = 2H\|E_3\|^2 + \|E_3\|^3\kappa_\Sigma$. Here, $\kappa_\Sigma$ denotes the geodesic curvature of $\partial \Omega = \partial \Sigma$ in the base $M$ with respect to $\|E_3\|^{-1}(N - vE_3)$, the outer conormal vector field to $\partial \Omega$ in $M$. Finally, we obtain

$$\frac{\partial \psi}{\partial \eta} = \frac{\partial h}{\partial \eta} + g'(v)\frac{\partial v}{\partial \eta} = \|E_3\|^2 \left( -1 + g'(v)(2H + \|E_3\|\kappa_\Sigma) \right).$$

Note that any outer conormal vector field to $\partial \Omega$ in $M$ is a linear combination of $N$ and $E_3$, which is the key property to relate the geometries of $\Sigma$ and $M$. Moreover, these computations allow us to give an optimal bound for the geodesic curvature of the boundary of the domain of a compact $H$-graph with a capillarity boundary condition.

**Theorem 3.2.** Let $\Sigma \subseteq M \times \mathbb{R}$ be a constant mean curvature $H > 0$ graph over a compact regular domain $\Omega \subseteq M$ with zero values in $\partial \Omega$. Let us consider $c = \inf\{K_M(p) : p \in \Omega\}$ and suppose that $c + 4H^2 > 0$ and $v = v_0$ in $\partial \Omega$ for some $-1 < v_0 \leq 0$. Then

$$\kappa_\Sigma \geq \frac{-4H^2 + c(1 - v_0^2)}{4H\sqrt{1 - v_0^2}},$$

where $\kappa_\Sigma$ is the geodesic curvature of $\partial \Omega$ in $M$ with respect to the outer conormal vector field.

Furthermore, if there exists $p \in \partial \Omega$ such that equality holds in (18), then $\Omega$ has constant curvature and $\Sigma$ is invariant by a 1-parameter group of isometries.

**Proof.** Let us consider the function $\psi = h + g(v)$, defined in terms of (14). Since $v \leq 0$ and $K_M \geq c$ in $\Sigma$, equation (16) insures that $\Delta \psi \geq 0$ in $\Sigma$. Let $p_0 \in \Sigma$ a point where $\psi$ attains its maximum. If $p_0$ is interior to $\Sigma$, then the maximum principle for the laplacian guarantees that $\psi$ is constant, so from (16), we get that $q = 0$ and $K_M = c$ in $\Omega$. These conditions imply that $\Omega$ has constant curvature $c$ and that $\Sigma$ is invariant by a 1-parameter subgroup of isometries which acts trivially on the vertical lines (see [3, Lemma 6.1]).

If, on the contrary, $p_0 \in \partial \Omega$ and $\psi(p_0) > \psi(p)$ for every interior point $p \in \Omega$, then such maximum is attained in the whole boundary as $\psi$ is constant along it, so the maximum principle in the boundary insures that $\frac{\partial \psi}{\partial \eta} > 0$ in $\partial \Omega$. Then the desired strict inequality for $\kappa_\Sigma$ can be deduced from (17). \qed

**Remark 3.3.** In the situation of the statement of Theorem 3.2, if we suppose that $-1 < v \leq v_0$ in $\partial \Omega$ for some $v_0 \leq 0$ (instead of $v = v_0$ in $\partial \Omega$) and there exists a point $p \in \partial \Omega$ such that $v(p) = v_0$ and at which the inequality for $\kappa_\Sigma$ becomes and equality, then $\Omega$ has constant curvature and $\Sigma$ is a spherical cap of a standard rotational sphere. This can be easily seen as a consequence of the maximum principle in the boundary.
Observe that, in case \( \Sigma \subseteq M \times \mathbb{R} \) is a compact embedded constant mean curvature \( H > 0 \) surface, it is possible to apply the Alexandrov reflection principle to vertical reflections and conclude that \( \Sigma \) is a symmetric bigraph with respect to some slice \( M \times \{t_0\} \) which, after a vertical translation, may be supposed to be \( M \times \{0\} \). In this setting, it is obvious that \( \Sigma \) intersects orthogonally such a slice.

**Corollary 3.4.** Let \( H > 0 \) and \( \Sigma \subseteq M \times \mathbb{R} \) be a compact embedded \( H \)-bigraph over a domain \( \Omega \subseteq M \), symmetric with respect to \( M \times \{0\} \). If we suppose that \( c = \inf \{K_M(p) : p \in \Omega\} > -4H^2 \), then \( \partial \Omega \) is a curve whose geodesic curvature in \( M \) with respect to the outer conormal vector field satisfies
\[
\kappa_g \geq -H + \frac{c}{4H}.
\]
Furthermore, if equality holds at some point in \( \partial \Omega \), then \( \Omega \) has constant curvature and \( \Sigma \) is invariant under a 1-parameter isometry group.

We now adjust the value of \( H \) for which the lower bound is exactly zero.

**Corollary 3.5.** Let \( M \) be a orientable complete Riemannian surface with \( K_M \geq c > 0 \) in \( M \). Then, each compact embedded \( H \)-surface in \( M \times \mathbb{R} \) with \( 0 < H < \frac{1}{4} \sqrt{c} \) is an \( H \)-bigraph over a connected domain \( \Omega \subseteq M \) and \( M \setminus \Omega \) is a finite union of disjoint convex disks. Furthermore, either
- these disks are strictly convex or
- \( M = S^2(\sqrt{c}) \), \( \Omega \) is a closed hemisphere and \( \Sigma \) is a rotationally invariant \( H \)-sphere for \( H = \frac{1}{2} \sqrt{c} \) (in this case \( \kappa_g \) identically vanishes).

We recall that, in the case \( M = S^2(\sqrt{c}) \), each of these disks must lie in an open hemisphere because they are convex.

### 4. Further boundary curvature estimates

In this section, we will obtain better estimates for the geodesic curvature of the boundary by assuming restrictions on the maximum height that the surface can reach. In order to achieve this, we will use a technique which has its origins in a paper by Payne and Philippin [8] and which has also been used by Ros and Rosenberg in [11].

Let \( \Sigma \subseteq M \times \mathbb{R} \) be a constant mean curvature \( H > 0 \) surface which is a graph over a domain \( \Omega \subseteq M \) and extends continuously to the boundary of \( \Omega \) with zero values. For any given \( m > 0 \), let consider the function \( g_m : [-1, 1] \to \mathbb{R} \) determined by
\[
g'_m(t) = \frac{4mH}{4H^2 + c(1-t^2)} - \frac{2Hv g_m'(v)}{m(1-v^2)} A E_3^\top,
\]
where
\[
X = \frac{2Hv(2m - 1)}{m(1-v^2)} E_3^\top - \frac{2Hv g_m'(v)}{m(1-v^2)} A E_3^\top,
\]
which is a smooth vector field defined on \( \Sigma \setminus V \), where \( V = \{p \in \Sigma : v(p) = -1\} \) is the subset of \( \Sigma \) with vertical Gauss map. We will consider the second order elliptic operator
L on $C^\infty(\Sigma \setminus V)$ given by $Lf = \Delta f + X(f)$, and the function $\psi_m = h + g_m(v) \in C^\infty(\Sigma)$. We are now interested in working out $L\psi_m$. By using Lemma 3.1, we obtain

$$L\psi_m = -\frac{(m-1)(m-\frac{1}{2})}{m} HV - \frac{4Hmv(1-v^2)(K_M - c)}{4H^2 + c(1-v^2)}.$$  

We observe that the second term in the RHS is positive because $K_M \geq c$ is satisfied. Moreover, for $m \geq 1$ or $m \leq \frac{1}{2}$, the first term is also positive so the function $\psi_m$ verifies $L\psi_m \geq 0$ in $\Sigma \setminus V$. Thus, it is possible to apply the maximum principle for the operator $L$ in $\Sigma \setminus V$, which insures that $\psi_m$ cannot achieve an interior maximum in $\Sigma \setminus V$ unless it is constant.

**Lemma 4.1.** Let $\Sigma \subseteq M \times \mathbb{R}$ be a constant mean curvature $H > 0$ graph over a (not necessarily compact) domain $\Omega \subseteq M$ which extends continuously to $\partial \Omega$ with zero values and suppose that $c = \inf\{K_M(p) : p \in \Omega\} > -4H^2$. Additionally, if $\psi_m$ is constant in $\Sigma$ for some $m \leq \frac{1}{2}$, then

a) $m = \frac{1}{2}$ and $K_M$ is constant in $\Omega$,

b) $\Sigma$ is invariant by a 1-parameter group of isometries.

In particular, if $c > 0$ and $M = S^2(c)$, then $\Sigma$ is a compact rotationally invariant torus and, if $c \leq 0$ and $M = H^2(c)$, then $\Sigma$ is an invariant horizontal cylinder, both described in Section 2.

**Proof.** If $\psi_m$ is constant, then $L\psi_m = 0$ so from (20) we get that $(m-1)(m-\frac{1}{2}) \leq 0$ which is only possible if $m = \frac{1}{2}$. Then, as equality in (20) holds, $K_M$ must be constant in $\Sigma \setminus V$ so it is constant in $\Sigma$ as $K_M$ is continuous and $V$ has empty interior.

Now, suppose that $m = \frac{1}{2}$ and $\psi_m$ is constant. On one hand, from $\nabla \psi_m = 0$ we obtain $AE_3^\top = \frac{1}{g'(v)}E_3^\top$ so $E_3^\top$ must be a principal direction and $\frac{1}{g'(v)}$ its corresponding principal curvature. We also deduce the following expressions:

$$\det(A) = \frac{1}{g'(v)} \left(2H - \frac{1}{g'(v)}\right), \quad \|\nabla v\|^2 = \langle AE_3^\top, AE_3^\top \rangle = \frac{1-v^2}{g'(v)^2}.$$  

If we consider the differentiable function $f : ]-1,1[ \to \mathbb{R}$ determined by

$$f'(t) = \frac{1}{\sqrt{(1-t^2)(4H^2 + c(1-t^2))}}, \quad f(0) = 0,$$

and take into account (21) and Lemma 3.1, it is easy to check that

$$\Delta(f(v)) = f''(v)\|\nabla v\|^2 + f'(v)\Delta v$$

$$= \frac{f''(v)}{g'(v)^2} (1-v^2) + f'(v)(2\det(A) - 4H^2 - c(1-v^2))v = 0$$

where we have also used that $K_M$ is constant by item (a). As $f(v)$ is a non-constant harmonic function on $\Sigma \setminus V$, we can (at least locally) take a conformal parameter $z$ on $\Sigma \setminus V$ such that $\text{Re}(z) = f(v)$. Now, we can repeat the arguments given in [3] Lemma 6.1] to conclude that $\Sigma$ is invariant by a 1-parameter group of isometries of $\mathbb{M}^2(c) \times \mathbb{R}$. As $\Sigma$ is an embedded $H$-graph over a domain $\Omega \subseteq M$ which continuously extends to $\partial \Omega$ with zero values, the isometries in this group act trivially on the vertical lines so, if $c \neq 0$, we are in the situation studied in Section 2 and the only possibilities for $\Sigma$ are...
those mentioned in the statement. If \( c = 0 \), it is well-known that \( \Sigma \) must be a horizontal cylinder. \( \square \)

**Theorem 4.2.** Let \( \Sigma \subseteq M \times \mathbb{R} \) be a constant mean curvature \( H > 0 \) graph over a compact regular domain \( \Omega \) with zero values in \( \partial \Omega \) and suppose that \( c = \inf \{ K_m(p) : p \in \Omega \} \) satisfies that \( 4H^2 + c > 0 \) and \( v = v_0 \) in \( \partial \Omega \) for some \(-1 < v_0 \leq 0\).

If there exists \( 0 < m \leq \frac{1}{2} \) such that \( |h| \leq m \cdot \alpha(c, H, v_0) \), then the following lower bound for the geodesic curvature of \( \partial \Omega \) in \( M \) (with respect to the outer conormal vector field) holds:

\[
\kappa_{\Sigma} \geq \frac{(4 - 8m)H^2 + c(1 - v_0^2)}{4mH\sqrt{1 - v_0^2}}.
\]

**Proof.** Let us consider the function \( \psi_m = h + g_m(v) \in C^\infty(\Sigma) \), which verifies that \( L \psi_m \geq 0 \) in view of (20). As \( \Sigma \) is compact, there exists a point \( p_0 \in \Sigma \) where \( \psi_m \) attains its maximum. We distinguish three possibilities:

- **If** \( p_0 \) **is an interior point of** \( \Sigma \cap V \), **then** \( \psi_m \) **is constant in** \( \Sigma \), **which implies that** \( m = \frac{1}{2} \), \( K_M \) **is constant in** \( \Omega \) **and** \( \Sigma \) **is invariant by a 1-parameter isometry group because of Lemma 3.1.**

- **If** \( p_0 \in \partial \Omega \), **then such a maximum is attained in the whole boundary** \( \partial \Omega \) **since** \( (\psi_m)_{|\partial \Omega} \) **is constant.** Then the boundary maximum principle for the operator \( L \) guarantees that \( \frac{\partial \psi_m}{\partial \eta} \geq 0 \) **along** \( \partial \Omega \). **It is straightforward to check from (17) that this is equivalent to the inequality in the statement above.**

- **If** \( p_0 \in V \), **then** \( v(p_0) = -1 \). **The inequality we are looking for would be proved if we discarded this case, but it turns out that** \( h \leq m \cdot \alpha(c, H, v_0) = -g_m(-1) \) **so** \( \psi_m \leq \psi_m(p_0) = h(p_0) + g_m(-1) = h(p_0) - m \cdot \alpha(c, H, v_0) + g_m(v_0) \leq g_m(v_0) \) **and, since** \( \psi_m \) **is equal to** \( g_m(v_0) \) **in** \( \partial \Omega \), **the maximum is also attained in the boundary, which reduces this case to the previous one.** \( \square \)

We now apply the theorem to the compact embedded case, where we lay in the same situation of Corollary 3.4.

**Corollary 4.3.** Let \( \Sigma \subseteq M \times \mathbb{R} \) be a compact constant mean curvature \( H > 0 \) surface, symmetric with respect to \( M \times \{0\} \), and suppose that \( c = \inf \{ K_m(p) : p \in \Omega \} \) satisfies \( 4H^2 + c > 0 \).

If there exists \( 0 < m \leq \frac{1}{2} \) such that \( h \leq m \cdot \alpha(c, H, 0) \), then the following lower bound for the geodesic curvature of \( \partial \Omega \) in \( M \) (with respect to the outer conormal vector field) holds:

\[
\kappa_{\Sigma} \geq \frac{(4 - 8m)H^2 + c}{4mH}.
\]

We now adjust the constant \( 0 < m < \frac{1}{2} \) to guarantee the convexity of the boundary, as we did in Corollary 3.5.
Corollary 4.4. Let $\Sigma \subseteq M \times \mathbb{R}$ be an $H$-graph over a compact regular domain $\Omega$ with zero boundary values. Suppose that $c = \inf \{ K_M(p) : p \in \Omega \}$ satisfies that $4H^2 + c > 0$ and $v = v_0$ in $\partial \Omega$ for some $-1 < v_0 \leq 0$. In any of the following two situations,

i) $c \geq 0$ and $h \leq \frac{1}{2} \alpha(c, H, v_0)$ in $\Sigma$ or

ii) $c < 0$ and $h \leq \frac{4H^2 + (1-v')^2}{8H^2} \alpha(c, H, v_0)$ in $\Sigma$,

the boundary $\partial \Omega$ is convex in $M$ with respect to the outer conormal vector field.

We finally wonder if the compactness hypothesis for the domain of the graph can be removed. In order to achieve this, we will restrict ourselves to the ambient space $M^2(c) \times \mathbb{R}$ and $v_0 = 0$ (that is, $\Sigma$ is an $H$-bigraph), where the technique developed by Ros and Rosenberg in [11] can be easily adapted.

Theorem 4.5. Let $\Sigma \subseteq M(c) \times \mathbb{R}^+$ be a properly embedded $H$-bigraph over a domain $\Omega \subseteq M$ with $4H^2 + c > 0$, symmetric with respect to $M \times \{0\}$, and suppose that there exists $0 < m \leq \frac{1}{2}$ such that $|h| \leq m \cdot \alpha(c, H, v_0)$ in $\Sigma$. Then, the following lower bound for the geodesic curvature of $\partial \Omega$ in $M$ (with respect to the outer conormal vector field) holds:

$$\kappa_8 \geq \frac{(4 - 8m)H^2 + c}{4mH}.$$ 

Furthermore, if there exists $p \in \partial \Omega$ for which equality holds, then

i) $\Sigma$ is a rotationally invariant torus (described in Section 2.1) if $c > 0$,

ii) $\Sigma$ is an invariant cylinder under horizontal translations if $c = 0$, and

iii) $\Sigma$ is an invariant cylinder under hyperbolic translations (described in Section 2.2) if $c < 0$.

Proof. Let us consider the same function $\psi_m \in C^\infty(\Sigma)$ as before. If $\psi_m$ attained its maximum or $\sup_\Sigma \psi_m \leq 0$, we could reason in the same way we did for the compact case and the proof would be finished. Otherwise, let us take a sequence $\{p_n\} \subseteq \Sigma$ such that $\{\psi_m(p_n)\}$ converges to $\sup_\Sigma \psi_m$, and distinguish two cases.

- If $\lim \{h(p_n)\} = 0$, then $\psi_m(p_n) = h(p_n) + g_m(v(p_n)) \leq h(p_n) \rightarrow 0$ from where $\sup_\Sigma \psi_m \leq 0$ and we are done.

- If $\{h(p_n)\}$ does not converge to zero, we can suppose that $\{h(p_n)\} \rightarrow a > 0$ without loss of generality. Now, ambient isometries allow us to translate $\Sigma$ horizontally so that $p_n$ is over some fixed point $q_0 \in M$ and standard convergence arguments make possible to consider $\Sigma_{\infty}$, the limit $H$-graph of these translated surfaces (note that $\Sigma$ has uniform curvature estimates around $p_n$ by stability, since the distance from $p_n$ to $\partial \Sigma$ is bounded away from zero). The corresponding function in $\Sigma_{\infty}$, given by $\psi_{m,\infty} = h_{\infty} + g_m(v_{\infty}) \in C^\infty(\Sigma_{\infty})$, attains its maximum at the interior point $p_{\infty} = \lim \{p_n\}$ (which is not in the boundary of $\Sigma_{\infty}$ because $h_{\infty}(p_{\infty}) = a > 0$), so it identically vanishes since $\psi_{\infty}$ vanishes at $\partial \Sigma_{\infty}$. This implies that $\sup_\Sigma \psi_m = \psi_{m,\infty}(p_{\infty}) = 0$ and we are also done.

In case that equality holds, the description in the statement follows from Lemma 4.1. □

Observe that, if the maximum heights of a sequence $\{\Sigma_n\}$ of such $H$-bigraphs tend to zero, then Theorem 4.5 insures that the geodesic curvatures of the boundaries diverge uniformly, in the sense that the bound only depends on that maximum height. Thus,
the sequence of domains $\Omega_n \subseteq M$ over which $\Sigma_n$ is a graph cannot eventually omit any set in $M$ with nonempty interior.

5. Intrinsic length estimates

Let $\Sigma \subseteq M \times \mathbb{R}$ be an $H$-graph over a compact domain $\Omega \subseteq M$ which extends continuously to the boundary with zero values. Suppose that $K_M \geq c > -4H^2$ in $\Sigma$ for some $c > 0$. In Section 3 we proved that $\psi = h + g(v)$ is subharmonic in $\Sigma$, where $g$ is defined in (14) so, if we suppose that $v \leq v_0$ along $\partial \Sigma$, then $h + g(v) \leq g(v_0)$, as a consequence of that $g$ is strictly increasing and $h$ vanishes on $\partial \Sigma$. Therefore, as $g$ is also an odd function, we derive that $g(-v) \geq h - g(v_0)$. Now we can invert the function $g$ and square both sides to obtain

$$v^2 \geq \zeta(h, v_0) := \begin{cases} \frac{c+4H^2}{c} \tanh^2 \left( \frac{\sqrt{c^2 + 4H^2}}{4H} (h - g(v_0)) \right) & \text{if } c < 0, \\ H^2 (h - g(v_0))^2 & \text{if } c = 0, \\ \frac{c+4H^2}{c} \tanh^2 \left( \frac{\sqrt{c^2 - 4H^2}}{4H} (h - g(v_0)) \right) & \text{if } c > 0. \end{cases}$$

(22)

Let $\gamma : [a, b] \to \Sigma$ be a smooth curve which is parametrized by arc-length and let $\eta$ be a smooth unit vector field along $\gamma$, orthogonal to $\gamma'$. Then, as $\{N, a', \eta\}$ is an orthonormal frame, we have

$$E_3 = \langle N, E_3\rangle E_3 + \langle \gamma', E_3\rangle \gamma' + \langle \eta, E_3\rangle \eta,$$

and, since $\langle N, E_3\rangle = v$ and $\langle \gamma', E_3\rangle = h'(\gamma)$, we deduce that $1 = v^2 + h'(\gamma)^2 + \langle \eta, E_3^\top \rangle^2$. Taking into account that $\langle \eta, E_3^\top \rangle^2 \geq 0$, we finally get $|h'| \leq 1 - v^2$. Thus, plugging (22) into this inequality, we have

$$\text{Long}(\gamma) \geq \int_0^a \frac{|h'|}{\sqrt{1 - v^2}} \, dt \geq \int_0^a \frac{-h'}{\sqrt{1 - \zeta(h, v_0)}} \, dt = \int_{h(a)}^{h(0)} \frac{ds}{\sqrt{1 - \zeta(s, v_0)}}.$$

(23)

Considering all the curves that join a point $p$ with the boundary (along which the height vanishes), we obtain the following result:

**Theorem 5.1.** Let $\Sigma \subseteq M \times \mathbb{R}$ be a constant mean curvature $H > 0$ graph over a compact domain $\Omega \subseteq M$ which extends continuously to the boundary with zero values and suppose that $c = \inf\{K_M(p) : p \in \Sigma\} > -4H^2$. If $v \leq v_0$ in $\partial \Omega$ for some $-1 < v_0 \leq 0$, then

$$\text{dist}(p, \partial \Sigma) \geq \int_0^{h(p)} \frac{ds}{\sqrt{1 - \zeta(s, v_0)}}.$$

Furthermore, if there exists $p \in \Sigma$ such that equality holds, then $\Omega$ has constant curvature and $\Sigma$ is a spherical cap of a rotationally invariant sphere.

In other words, Theorem 5.1 is a comparison result which claims that rotationally invariant spheres in the corresponding homogeneous space $\mathbb{M}^2(c) \times \mathbb{R}$ minimize the distance from a point to the boundary in terms of the height of that point.

We remark that the bound given in the statement can be worked out explicitly in terms of elementary functions, but the result of that computation is a large formula which does not contribute to a better understanding so we have preferred to leave it in this way.
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