Spectral estimates for finite combinations of Hermite functions and null-controllability of hypoelliptic quadratic equations

by

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Abstract. Some recent works have shown that the heat equation on the whole Euclidean space is null-controllable in any positive time if and only if the control subset is a thick set. This necessary and sufficient condition for null-controllability is linked to some uncertainty principles, such as the Logvinenko–Sereda theorem, which give limitations on the simultaneous concentration of a function and its Fourier transform. In the present work, we prove new uncertainty principles for finite combinations of Hermite functions. We establish an analogue of the Logvinenko–Sereda theorem with an explicit control of the constant with respect to the energy level of the Hermite functions as eigenfunctions of the harmonic oscillator for thick control subsets. This spectral inequality allows us to derive null-controllability in any positive time from thick control regions for parabolic equations associated with a general class of hypoelliptic non-selfadjoint quadratic differential operators. More generally, the spectral estimate for finite combinations of Hermite functions is actually shown to hold for any measurable control subset of positive Lebesgue measure, and some quantitative estimates of the constant with respect to the energy level are given for another two classes of control subsets including the case of non-empty open control subsets.

1. Introduction. Heisenberg’s classical uncertainty principle in quantum mechanics states that the position and the momentum of particles cannot both be determined explicitly, but only in a probabilistic sense with an uncertainty. More generally, uncertainty principles are mathematical results that give limitations on the simultaneous concentration of a function and its Fourier transform. When using the normalization

\[ \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi} \, dx, \quad \xi \in \mathbb{R}^n, \]

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the mathematical formulation of Heisenberg’s uncertainty principle can be stated as

\[
\inf_{a \in \mathbb{R}} \left( \int_{\mathbb{R}^n} (x_j - a)^2 |f(x)|^2 \, dx \right) \inf_{b \in \mathbb{R}} \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\xi_j - b)^2 |\hat{f}(\xi)|^2 \, d\xi \right) \geq \frac{1}{4} \|f\|_{L^2(\mathbb{R}^n)}^4
\]

for all \( f \in L^2(\mathbb{R}^n) \) and \( 1 \leq j \leq n \). It shows that a function and its Fourier transform cannot both be arbitrarily localized. Moreover, (1.2) is an equality if and only if

\[ f(x) = g(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)e^{-ibx_j}e^{-\alpha(x_j-a)^2}, \]

where \( g \in L^2(\mathbb{R}^{n-1}), \alpha > 0, \) and \( a \) and \( b \) are real constants for which the two infima in (1.2) are achieved. There are various uncertainty principles of different nature. For details and references, we refer the reader to the survey article by Folland and Sitaram [18] and the book of Havin and Jöricke [22].

Another formulation of uncertainty principles is that a non-zero function and its Fourier transform cannot both have small supports. For instance, a non-zero \( L^2(\mathbb{R}) \)-function whose Fourier transform is compactly supported must be an analytic function with a discrete zero set and therefore a full support. This leads to the notion of weak annihilating pairs as well as the corresponding quantitative notion of strong annihilating pairs:

**Definition 1.1 (Annihilating pairs).** Let \( S, \Sigma \) be measurable subsets of \( \mathbb{R}^n \).

- \((S, \Sigma)\) is said to be a **weak annihilating pair** if the only function \( f \in L^2(\mathbb{R}^n) \) with \( \text{supp } f \subset S \) and \( \text{supp } \hat{f} \subset \Sigma \) is \( f = 0 \).
- \((S, \Sigma)\) is said to be a **strong annihilating pair** if there exists a constant \( C = C(S, \Sigma) > 0 \) such that for all \( f \in L^2(\mathbb{R}^n) \),

\[
(1.3) \quad \int_{\mathbb{R}^n} |f(x)|^2 \, dx \leq C \left( \int_{\mathbb{R}^n \setminus S} |f(x)|^2 \, dx + \int_{\mathbb{R}^n \setminus \Sigma} |\hat{f}(\xi)|^2 \, d\xi \right).
\]

It can be readily checked that \((S, \Sigma)\) is a strong annihilating pair if and only if there exists \( D = D(S, \Sigma) > 0 \) such that for all \( f \in L^2(\mathbb{R}^n) \) with \( \text{supp } \hat{f} \subset \Sigma \),

\[
(1.4) \quad \|f\|_{L^2(\mathbb{R}^n)} \leq D \|f\|_{L^2(\mathbb{R}^n \setminus S)}.
\]

As already mentioned above in the one-dimensional setting, a pair \((S, \Sigma)\) is weak annihilating if \( S \) and \( \Sigma \) are compact sets. More generally, Benedicks [5] has shown that \((S, \Sigma)\) is weak annihilating if \( S \) and \( \Sigma \) are of finite Lebesgue measure, \(|S|, |\Sigma| < +\infty\). Under this assumption, the result of Amrein–Berthier [3] actually shows that \((S, \Sigma)\) is strong annihilating. The estimate \( C(S, \Sigma) \leq \kappa e^{\kappa |S||\Sigma|} \) (which is sharp up to a numerical constant \( \kappa > 0 \)) has
been established by Nazarov [40] in dimension $n = 1$. This result was extended to the multi-dimensional case by the second author [27], with the quantitative estimate $C(S, \Sigma) \leq \kappa e^{\kappa(|S| \Sigma)^{1/n}}$ holding if in addition $S$ or $\Sigma$ is convex.

A description of all strong annihilating pairs seems out of reach for now. We refer the reader to [2, 8, 9, 12, 14, 45] for a large variety of results and techniques available as well as for examples of weak annihilating pairs that are not strong annihilating. However, there is a complete description of all the support sets $S$ forming a strong annihilating pair with any bounded spectral set $\Sigma$, provided by the Logvinenko–Sereda theorem [35]:

**Theorem 1.2 (Logvinenko–Sereda).** Let $S, \Sigma \subset \mathbb{R}^n$ be measurable subsets with $\Sigma$ bounded. Set $\tilde{S} = \mathbb{R}^n \setminus S$. The following assertions are equivalent:

- The pair $(S, \Sigma)$ is strong annihilating.
- The subset $\tilde{S}$ is thick, that is, there exists a cube $K \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and a constant $0 < \gamma \leq 1$ such that
  $$\forall x \in \mathbb{R}^n, \quad |(K + x) \cap \tilde{S}| \geq \gamma |K| > 0.$$  

It is interesting to observe that if $(S, \Sigma)$ is a strong annihilating pair for some bounded subset $\Sigma$, then it is so for every bounded $\Sigma$, but the constants $C(S, \Sigma)$ and $D(S, \Sigma)$ depend on $\Sigma$. In order to use this result in the control theory of partial differential equations, it is essential to understand how $D(S, \Sigma)$ depends on $|\Sigma|$. This question was answered by Kovrijkine [28, 29] who established the following quantitative estimates:

**Theorem 1.3 (Kovrijkine).** There exists a universal constant $C_n > 0$ depending only on the dimension $n$ such that if $\tilde{S}$ is a $\gamma$-thick set at scale $L > 0$ with $0 < \gamma \leq 1$, that is, for all $x \in \mathbb{R}^n$,

$$|\tilde{S} \cap (x + [0, L]^n)| \geq \gamma L^n,$$

then for all $R > 0$ and $f \in L^2(\mathbb{R}^n)$ with $\text{supp } \hat{f} \subset [-R, R]^n$,

$$\|f\|_{L^2(\mathbb{R}^n)} \leq (C_n/\gamma)^{C_n(1+LR)}\|\hat{f}\|_{L^2(\tilde{S})}.$$  

Thanks to this explicit dependence of the constant on $R$, Egidi and Veselić [15] and Wang, Wang, Zhang and Zhang [49] have independently established that the heat equation

$$\begin{cases}
\left(\partial_t - \Delta_x\right)f(t, x) = u(t, x)\mathbb{1}_\omega(x), & x \in \mathbb{R}^n, t > 0, \\
f|_{t=0} = f_0 \in L^2(\mathbb{R}^n)
\end{cases}$$  

is null-controllable in any positive time $T$ from a measurable control subset $\omega \subset \mathbb{R}^n$ if and only if $\omega$ is thick in $\mathbb{R}^n$. The notion of null-controllability is defined as follows:
DEFINITION 1.4 (Null-controllability). Let $P$ be a closed operator on $L^2(\mathbb{R}^n)$ which is the infinitesimal generator of a strongly continuous semigroup $(e^{-tP})_{t \geq 0}$ on $L^2(\mathbb{R}^n)$, and let $T > 0$ and $\omega$ be a measurable subset of $\mathbb{R}^n$. The equation
\begin{equation}
(\partial_t + P)f(t, x) = u(t, x)1_\omega(x), \quad x \in \mathbb{R}^n, \ t > 0,
\end{equation}
is said to be null-controllable from $\omega$ in time $T > 0$ if, for any initial datum $f_0 \in L^2(\mathbb{R}^n)$, there exists $u \in L^2((0, T) \times \mathbb{R}^n)$, supported in $(0, T) \times \omega$, such that the mild (or semigroup) solution $f$ of (1.8) satisfies $f|_{t=T} = 0$.

By the Hilbert Uniqueness Method (see [11, Theorem 2.44] or [34]), the null-controllability of (1.8) is equivalent to the (final state) observability of the adjoint system
\begin{equation}
(\partial_t + P^*)g(t, x) = 0, \quad x \in \mathbb{R}^n, \ t > 0,
\end{equation}
where $P^*$ stands for the $L^2(\mathbb{R}^n)$-adjoint of $P$. The notion of observability is defined as follows:

DEFINITION 1.5 (Observability). Let $T > 0$ and $\omega$ be a measurable subset of $\mathbb{R}^n$. Equation (1.9) is said to be observable from $\omega$ in time $T > 0$ if there exists a constant $C_T > 0$ such that, for any initial datum $g_0 \in L^2(\mathbb{R}^n)$, the mild (or semigroup) solution $g$ of (1.9) satisfies
\begin{equation}
\int_{\mathbb{R}^n} |g(T, x)|^2 \, dx \leq C_T \int_0^T \left( \int_{\omega} |g(t, x)|^2 \, dx \right) \, dt.
\end{equation}

Following [15] or [49], the necessity of the thickness of the control subset for the null-controllability in any positive time is a consequence of a quasimodes construction, whereas the sufficiency is derived in [15] from an abstract observability result obtained by an adapted Lebeau–Robbiano method [31] and established by the first and third authors with some contributions from Luc Miller of Université Paris-Ouest.

THEOREM 1.6 ([4, Theorem 2.1]). Let $\Omega$ be an open subset of $\mathbb{R}^n$, $\omega$ be a measurable subset of $\Omega$, $(\pi_k)_{k \in \mathbb{N}^*}$ be a family of orthogonal projections defined on $L^2(\Omega)$, $(e^{-tA})_{t \geq 0}$ be a strongly continuous contraction semigroup on $L^2(\Omega)$, and $c_1, c_2, a, b, t_0, m > 0$ be constants with $a < b$. If the spectral inequality
\begin{equation}
(1.11) \quad \forall g \in L^2(\Omega), \ \forall k \geq 1, \quad \|\pi_k g\|_{L^2(\Omega)} \leq e^{c_1 k^a} \|\pi_k g\|_{L^2(\omega)}
\end{equation}
and the dissipation estimate

\((1.12)\) \hspace{1cm} \forall g \in L^2(\Omega), \forall k \geq 1, \forall 0 < t < t_0, \hspace{1cm} \\
\|(1 - \pi_k)(e^{-tA}g)\|_{L^2(\Omega)} \leq \frac{1}{c_2} e^{-c_2m^k} \|g\|_{L^2(\Omega)} \hspace{1cm}

hold, then there exists a constant \(C > 1\) such that the following observability estimate holds:

\((1.13)\) \hspace{1cm} \forall T > 0, \forall g \in L^2(\Omega), \hspace{1cm} \\
\|e^{-TA}g\|^2_{L^2(\Omega)} \leq C \exp \left( \frac{C}{T^{\frac{a}{b-a}}} \right) \int_0^T \|e^{-tA}g\|^2_{L^2(\omega)} dt. \hspace{1cm}

The proof of the above result is inspired by \[37, 38\]. In the statement of \[4, Theorem 2.1\], the subset \(\omega\) is supposed to be open in \(\Omega\). However, the proof in \[4\] works when \(\omega\) is only assumed to be measurable. Notice that the assumptions in the above statement do not require that the orthogonal projections \((\pi_k)_{k \geq 1}\) be spectral projections onto the eigenspaces of the infinitesimal generator \(A\), which is allowed to be non-selfadjoint.

According to Theorem 1.6, there are two key ingredients to derive null-controllability, or equivalently an observability inequality, while using that theorem: the spectral inequality \((1.11)\) and the dissipation estimate \((1.12)\). For the heat equation, the orthogonal projections used are the frequency cutoff operators given by the orthogonal projections onto the closed vector subspaces

\((1.14)\) \hspace{1cm} E_k = \{ f \in L^2(\mathbb{R}^n) : \text{supp } \hat{f} \subset [-k, k]^n \} \hspace{1cm}

for \(k \geq 1\). With this choice, the dissipation estimate readily follows from the explicit formula

\((1.15)\) \hspace{1cm} (e^{t\Delta_x}g)(t, \xi) = \hat{g}(\xi)e^{-t|\xi|^2}, \hspace{1cm} t \geq 0, \xi \in \mathbb{R}^n, \hspace{1cm}

whereas the spectral inequality is given by the sharp formulation of the Logvinenko–Sereda theorem \((1.6)\). Notice that the power 1 for the parameter \(R\) in \((1.6)\) and the power 2 for the term \(|\xi|\) in \((1.15)\) account for the fact that Theorem 1.6 can be applied with \(a = 1, b = 2\), which satisfy the required condition \(0 < a < b\). It is therefore essential that the power of \(R\) in the exponent of \((1.6)\) is strictly less than 2. As there is still a gap between the cost of the localization \((a = 1)\) given by the spectral inequality and its compensation by the dissipation estimate \((b = 2)\), it is interesting to notice that we could have expected that the null-controllability of the heat equation could have held under weaker assumptions than the thickness of the control subset, by allowing some higher costs for localization with some parameters \(1 < a < 2\), but the Logvinenko–Sereda theorem actually shows that this is not the case. Indeed, if the spectral inequality holds with \(1 < a < 2\), the
pair \((\mathbb{R}^n \setminus \omega, [-k, k]^n)\) has to be strong annihilating and \(\omega\) has to be thick according to Theorem 1.2.

Theorem 1.6 does not only apply with the use of frequency cutoff projections and a dissipation estimate induced by some Gevrey type regularizing effects \(^{(1)}\). Other regularities than the Gevrey one can be taken into account. In the previous work \cite{Beauchard2017} by the first and third authors, Theorem 1.6 is used for a general class of accretive hypoelliptic quadratic operators \(q^w\) generating some strongly continuous contraction semigroups \((e^{-tq^w})_{t \geq 0}\) enjoying some Gelfand–Shilov regularizing effects. The definition and standard properties related to the Gelfand–Shilov regularity \(^{(2)}\) are recalled in the Appendix (Section 5.1). As stated there, the Gelfand–Shilov regularity is characterized by specific exponential decays of the functions and their Fourier transforms; and in the symmetric case, can be read off from the exponential decay of the Hermite coefficients of the functions in their expansions in the \(L^2(\mathbb{R}^n)\)-Hermite basis \((\Phi_\alpha)_{\alpha \in \mathbb{N}^n}\). Explicit formulas and some reminders of basic facts about Hermite functions are given in Section 3.1.1.

The class of hypoelliptic quadratic operators whose description will be given in Section 4 enjoys some Gelfand–Shilov regularizing effects ensuring that the following dissipation estimate holds \cite[Proposition 4.1]{Beauchard2017}:

\[
\exists C_0 > 1, \exists t_0 > 0, \forall t > 0, \forall k \geq 0, \forall f \in L^2(\mathbb{R}^n), \quad \|(1 - \pi_k)(e^{-tq^w}f)\|_{L^2(\mathbb{R}^n)} \leq C_0 e^{-\delta(t)k} \|f\|_{L^2(\mathbb{R}^n)},
\]

with

\[
\delta(t) = \inf\left(\frac{(t, t_0)^{2k_0+1}}{C_0}\right) > 0, \quad t > 0, \quad 0 \leq k_0 \leq 2n - 1,
\]

where

\[
P_k g = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} \langle g, \Phi_\alpha \rangle_{L^2(\mathbb{R}^n)} \Phi_\alpha, \quad k \geq 0,
\]

with \(|\alpha| = \alpha_1 + \cdots + \alpha_n\), denotes the orthogonal projection onto the \(k\)th

\(^{(1)}\) Following \cite{Gevrey1930}, the Gevrey type spaces \(A^s(\mathbb{R}^n)\), with \(s > 0\), are defined as the spaces of smooth functions \(f \in C^\infty(\mathbb{R}^n)\) satisfying

\[
\exists C > 1, \forall \alpha \in \mathbb{N}^n, \quad \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)} \leq C^{1+|\alpha|!}(\alpha!)^s.
\]

Thanks to Sobolev embeddings, the \(L^2\)-norm can be replaced by the \(L^\infty\)-norm in the above estimates.

\(^{(2)}\) As explained in Section 4, this notion of regularity plays a key role in obtaining the null-controllability results in Theorem 4.1 as an application of abstract results established in \cite{Beauchard2017}, but is not used to derive the uncertainty principles in Theorem 2.1.
energy level associated with the harmonic oscillator
\[ \mathcal{H} = -\Delta_x + |x|^2 = \sum_{k=0}^{+\infty}(2k + n)\mathbb{P}_k, \]
and
\[ (1.19) \quad \pi_k = \sum_{j=0}^{k} \mathbb{P}_j, \quad k \geq 0, \]
denotes the orthogonal projection onto the \((k + 1)\)th first energy levels. The above integer \(0 \leq k_0 \leq 2n - 1\), whose definition is given in (4.11), is a structural parameter intrinsically associated to the Weyl symbol of the quadratic operator.

In order to apply Theorem 1.6, we need a spectral inequality for finite combinations of Hermite functions of the type
\[ (1.20) \quad \exists C > 1, \forall k \geq 0, \forall f \in L^2(\mathbb{R}^n), \quad \|\pi_k f\|_{L^2(\mathbb{R}^n)} \leq C e^{Ck^{1/2}} \|\pi_k f\|_{L^2(\mathbb{R}^n)}, \]
with \(a < 1\), where \(\pi_k\) is the orthogonal projection (1.19). In [4, Proposition 4.2], such a spectral inequality is established with \(a = 1/2\) when the control subset \(\omega\) is an open subset of \(\mathbb{R}^n\) satisfying the geometric condition
\[ (1.21) \quad \exists \delta, r > 0, \forall y \in \mathbb{R}^n, \exists y' \in \omega, \quad \left\{ \begin{array}{l} B(y', r) \subset \omega, \\ |y - y'| < \delta, \end{array} \right. \]
where \(B(y', r)\) denotes the open Euclidean ball in \(\mathbb{R}^n\) centered at \(y'\) with radius \(r > 0\). It allows one to derive the null-controllability of parabolic equations associated to accretive quadratic operators with zero singular spaces in any positive time \(T\) from any open subset \(\omega\) of \(\mathbb{R}^n\) satisfying (1.21). The above geometric condition was introduced by Le Rousseau and Moyano [33] and was shown to be sufficient for the null-controllability of the Kolmogorov equation in any positive time.

In the present work, we study under which geometric conditions on the control subset \(\omega \subset \mathbb{R}^n\) the spectral inequality
\[ (1.22) \quad \forall k \geq 0, \exists C_k(\omega) > 0, \forall f \in L^2(\mathbb{R}^n), \quad \|\pi_k f\|_{L^2(\mathbb{R}^n)} \leq C_k(\omega)\|\pi_k f\|_{L^2(\mathbb{R}^n)} \]
holds and how the geometric properties of the set \(\omega\) relate to the possible growth of the constant \(C_k(\omega)\) with respect to the energy level as \(k \to +\infty\). The main results contained in this article provide some quantitative upper bounds on \(C_k(\omega)\) with respect to the energy level for three different classes of measurable subsets:

- non-empty open subsets in \(\mathbb{R}^n\),
- measurable sets in \(\mathbb{R}^n\) satisfying
\[ (1.23) \quad \lim_{R \to +\infty} \frac{\inf_{r \geq R} \frac{|\omega \cap B(0, r)|}{|B(0, r)|}}{\frac{|B(0, R)|}{|B(0, r)|}} = \lim_{R \to +\infty} \left( \inf_{r \geq R} \frac{|\omega \cap B(0, r)|}{|B(0, r)|} \right) > 0, \]
- thick measurable sets in \(\mathbb{R}^n\).
We observe that in the first two classes, the measurable control subsets are allowed to have gaps containing balls with radii tending to infinity, whereas in the last class there must be a bound on such radii. We shall see that the quantitative upper bounds obtained for the first two classes (Theorem 2.1(i, ii)) are not sufficient to obtain any null-controllability result for the class of hypoelliptic quadratic operators studied in Section 4. Regarding the third one, the quantitative upper bound (Theorem 2.1(iii)) is an analogue of the Logvinenko–Sereda theorem for finite combinations of Hermite functions. As an application of this third result, in Theorem 4.1 we extend the null-controllability result to parabolic equations associated to accretive quadratic operators with zero singular spaces from any thick set $\omega \subset \mathbb{R}^n$ in any positive time $T$.

2. Uncertainty principles for finite combinations of Hermite functions. Let $(\Phi_\alpha)_{\alpha \in \mathbb{N}^n}$ be the $n$-dimensional Hermite functions whose definitions are recalled in Section 3.1.1 and

\begin{equation}
E_N = \text{Span}_\mathbb{C} \{ \Phi_\alpha \}_{\alpha \in \mathbb{N}^n, |\alpha| \leq N}.
\end{equation}

As the Lebesgue measure of the zero set of a non-zero analytic function on $\mathbb{C}$ is zero, the $L^2$-norm $\| \cdot \|_{L^2(\omega)}$ on any measurable set $\omega \subset \mathbb{R}$ of positive measure defines a norm on the finite-dimensional vector space $E_N = \pi_N(L^2(\mathbb{R}))$, with $\pi_N$ the orthogonal projection defined in (1.19). As a consequence of the Remez inequality, we check in the Appendix (Section 5.2) that this result holds true as well in the multi-dimensional case when $\omega \subset \mathbb{R}^n$ is a measurable subset of positive Lebesgue measure. By equivalence of norms in finite dimension, for any measurable set $\omega \subset \mathbb{R}^n$ of positive Lebesgue measure and for all $N \in \mathbb{N}$, there exists a constant $C_N(\omega) > 0$ depending on $\omega$ and $N$ such that the following spectral inequality holds:

\begin{equation}
\forall f \in L^2(\mathbb{R}^n), \quad \|\pi_N f\|_{L^2(\mathbb{R}^n)} \leq C_N(\omega)\|\pi_N f\|_{L^2(\omega)}.
\end{equation}

We aim to study how the geometric properties of $\omega$ relate to the possible growth of $C_N(\omega)$ with respect to the energy level.

Our main results are the following uncertainty principles for finite combinations of Hermite functions:

**Main Theorem 2.1.** The following spectral estimates hold:

(i) If $\omega$ is a non-empty open subset of $\mathbb{R}^n$, then there exists a constant $C = C(\omega) > 1$ such that

\begin{equation*}
\forall N \in \mathbb{N}, \forall f \in L^2(\mathbb{R}^n), \quad \|\pi_N f\|_{L^2(\mathbb{R}^n)} \leq C e^{\frac{1}{2} N \ln(N+1)+CN}\|\pi_N f\|_{L^2(\omega)}.
\end{equation*}

(ii) If the measurable subset $\omega \subset \mathbb{R}^n$ satisfies condition (1.23), then there exists a constant $C = C(\omega) > 1$ such that

\begin{equation*}
\forall N \in \mathbb{N}, \forall f \in L^2(\mathbb{R}^n), \quad \|\pi_N f\|_{L^2(\mathbb{R}^n)} \leq C e^{CN}\|\pi_N f\|_{L^2(\omega)}.
\end{equation*}
(iii) If the measurable subset $\omega \subset \mathbb{R}^n$ is $\gamma$-thick at scale $L$ in the sense defined in (1.5), then there exist a constant $C = C(L, \gamma, n) > 0$ and a universal constant $\kappa = \kappa(n) > 0$ only depending on the dimension such that

$$\forall N \in \mathbb{N}, \forall f \in L^2(\mathbb{R}^n), \quad \|\pi_N f\|_{L^2(\mathbb{R}^n)} \leq C \left( \frac{\kappa}{\gamma} \right)^{\kappa L \sqrt{N}} \|\pi_N f\|_{L^2(\omega)}.$$ 

According to the above result, control of the growth of $C_N(\omega)$ with respect to the energy level for an arbitrary non-empty open subset $\omega$ of $\mathbb{R}^n$, or when the measurable subset $\omega \subset \mathbb{R}^n$ satisfies (1.23), is not sufficient to guarantee the estimates (1.20) needed to obtain some null-controllability and observability results for the parabolic equations associated to the class of hypoelliptic quadratic operators studied in Section 4. As is known from [13, Proposition 5.1] (see also [36]) the one-dimensional harmonic heat equation not null-controllable, or observable, in any time $T > 0$ from a half-line, and as the harmonic oscillator obviously belongs to the class of hypoelliptic quadratic operators studied in Section 4 spectral estimates of the type

$$\exists 0 < a < 1, \exists C > 1, \forall N \in \mathbb{N}, \forall f \in L^2(\mathbb{R}^n), \quad \|\pi_N f\|_{L^2(\mathbb{R}^n)} \leq C e^{CN^a} \|\pi_N f\|_{L^2(\omega)}$$

cannot hold for an arbitrary non-empty open subset $\omega$ of $\mathbb{R}^n$, nor when the measurable subset $\omega \subset \mathbb{R}^n$ satisfies (1.23), since Theorem 1.6 together with (1.16) would then imply the null-controllability and observability of one-dimensional harmonic heat equation from a half-line.

On the other hand, when $\omega \subset \mathbb{R}^n$ is $\gamma$-thick at scale $L > 0$, the above spectral estimate (iii) is an analogue for finite combinations of Hermite functions of the sharpened version of the Logvinenko–Sereda theorem proved by Kovrijkine [28, 29] with a similar dependence of the constant on $0 < \gamma \leq 1$ and $L > 0$ as in (1.6). Notice that the growth in $\sqrt{N}$ is of the order of the square root of the largest eigenvalue of the harmonic oscillator $\mathcal{H} = -\Delta_x + |x|^2$ on the spectral vector subspace $\mathcal{E}_N$, whereas the growth in $R$ in (1.6) is also of the order of the square root of the largest spectral value of the Laplace operator $-\Delta_x$ on the spectral vector subspace

$$E_R = \{ f \in L^2(\mathbb{R}^n) : \text{supp} \hat{f} \subset [-R, R]^n \}.$$ 

This is in agreement with what is usually expected to hold for that type of spectral estimates (see [32]).

The spectral estimate (i) for arbitrary non-empty open subsets is proved in Section 3.2.1. Its proof uses some estimates on Hermite functions together with the Remez inequality. The spectral estimate (ii) for measurable subsets satisfying (1.23) is proved in Section 3.2.2 and follows from similar arguments to the ones used in Section 3.2.1. The spectral estimate (iii) for thick sets is proved in Section 3.2.3.
3. Proof of Theorem 2.1

3.1. Preliminary results

3.1.1. Hermite functions. This section is devoted to setting some notations and recalling basic facts about Hermite functions. The standard Hermite functions \((\phi_k)_{k \geq 0}\) are defined for \(x \in \mathbb{R}\) by

\[
\phi_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} e^{x^2/2} \frac{d^k}{dx^k} (e^{-x^2}) = \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} \left(x - \frac{d}{dx}\right)^k (e^{-x^2/2}) = a_+^k \phi_0 \sqrt{k!},
\]

where \(a_+\) is the creation operator

\[
a_+ = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx}\right).
\]

They satisfy the identity

\[
\forall k \geq 0, \quad \hat{\phi}_k = (-i)^k \sqrt{2\pi} \phi_k,
\]

when using the normalization (1.1) of the Fourier transform. The \(L^2\)-adjoint of the creation operator is the annihilation operator

\[
a_- = a_+^* = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx}\right).
\]

The following identities hold:

\[
[a_-, a_+] = 1, \quad -\frac{d^2}{dx^2} + x^2 = 2a_+ a_- + 1
\]

and

\[
\forall k \in \mathbb{N}, \quad a_+ \phi_k = \sqrt{k+1} \phi_{k+1},
\]

\[
\forall k \in \mathbb{N}, \quad a_- \phi_k = \sqrt{k} \phi_{k-1} \quad (= 0 \text{ if } k = 0)
\]

and

\[
\forall k \in \mathbb{N}, \quad \left(-\frac{d^2}{dx^2} + x^2\right) \phi_k = (2k+1) \phi_k,
\]

where \(\mathbb{N} = \{0,1,2,\ldots\}\). The family \((\phi_k)_{k \in \mathbb{N}}\) is an orthonormal basis of \(L^2(\mathbb{R})\). For \(\alpha = (\alpha_j)_{1 \leq j \leq n} \in \mathbb{N}^n\), \(x = (x_j)_{1 \leq j \leq n} \in \mathbb{R}^n\) we set

\[
\Phi_\alpha(x) = \prod_{j=1}^n \phi_{\alpha_j}(x_j).
\]
The family \((\Phi_\alpha)_{\alpha \in \mathbb{N}^n}\) is an orthonormal basis of \(L^2(\mathbb{R}^n)\) composed of the eigenfunctions of the \(n\)-dimensional harmonic oscillator
\[
(3.7) \quad \mathcal{H} = -\Delta_x + |x|^2 = \sum_{k \geq 0} (2k + n) \mathbb{P}_k, \quad \text{Id} = \sum_{k \geq 0} \mathbb{P}_k,
\]
where \(\mathbb{P}_k\) is the orthogonal projection onto \(\text{Span}_\mathbb{C} \{\Phi_\alpha\}_{\alpha \in \mathbb{N}^n, |\alpha| = k}\), with \(|\alpha| = \alpha_1 + \cdots + \alpha_n\).

The following estimates on Hermite functions are a key ingredient for the proof of the spectral inequalities (i) and (ii) in Theorem 2.1. This result was established by Bonami, Karoui and the second author [6, proof of Theorem 3.2], and is recalled here for completeness:

**Lemma 3.1.** The one-dimensional Hermite functions \((\phi_k)_{k \in \mathbb{N}}\) defined in (3.1) satisfy the estimates
\[
\forall k \in \mathbb{N}, \forall a \geq \sqrt{2k + 1}, \quad \int_{|x| \geq a} |\phi_k(x)|^2 \, dx \leq \frac{2^{k+1}}{k! \sqrt{\pi}} a^{2k-1} e^{-a^2}.
\]

**Proof.** For any \(k \in \mathbb{N}\), the \(k\)th Hermite polynomial function
\[
(3.8) \quad H_k(x) = (-1)^k e^{x^2} \left( \frac{d}{dx} \right)^k (e^{-x^2}),
\]
has degree \(k\) and is an even (respectively odd) function when \(k\) is an even (respectively odd) non-negative integer. In particular,
\[
(3.9) \quad H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2.
\]

The \(k\)th Hermite polynomial function \(H_k\) admits \(k\) distinct real simple roots. More specifically, we recall from [47, Section 6.31] that for \(k \geq 2\) even the \(k\) roots of \(H_k\), denoted \(-x_{k/2,k}, \ldots, -x_{1,k}, x_{1,k}, \ldots, x_{k/2,k}\), satisfy
\[
(3.10) \quad -\sqrt{2k + 1} \leq -x_{k/2,k} < \cdots < -x_{1,k} < 0
\]
\[
< x_{1,k} < \cdots < x_{k/2,k} \leq \sqrt{2k + 1}.
\]

On the other hand, for \(k \geq 1\) odd, the \(k\) roots of \(H_k\), denoted \(-x_{[k/2],k}, \ldots, -x_{1,k}, x_{0,k}, x_{1,k}, \ldots, x_{[k/2],k}\), satisfy
\[
(3.11) \quad -\sqrt{2k + 1} \leq -x_{[k/2],k} < \cdots < -x_{1,k} < x_{0,k} = 0
\]
\[
< x_{1,k} < \cdots < x_{[k/2],k} \leq \sqrt{2k + 1}.
\]

We denote by \(z_k\) the largest non-negative root of \(H_k\), that is, with the above notations as \(z_k = x_{[k/2],k}\), when \(k \geq 1\).

Relabel temporarily the \(k\) roots of \(H_k\) as \((a_j)_{1 \leq j \leq k}\) such that
\[
a_1 < \cdots < a_k.
\]

The classical formula
\[
(3.12) \quad \forall k \in \mathbb{N}^*, \quad H'_k(x) = 2kH_{k-1}(x)
\]
(see e.g. [47, Section 5.5]) together with Rolle’s theorem imply that $H_{k-1}$ admits exactly one root in each of the $k-1$ intervals $(a_j, a_{j+1})$, with $1 \leq j \leq k - 1$, when $k \geq 2$. According to (3.9)–(3.11), it implies in particular that for all $k \geq 1$,

$$0 = z_1 < z_2 < \cdots < z_k \leq \sqrt{2k + 1}. \tag{3.13}$$

Next, we claim that

$$\forall k \geq 1, \forall |x| \geq z_k, \quad |H_k(x)| \leq 2^k |x|^k. \tag{3.14}$$

To prove this we first observe that

$$\forall k \geq 1, \forall x \geq z_k, \quad H_k(x) \geq 0, \tag{3.15}$$

since the leading coefficient of $H_k \in \mathbb{R}[X]$ is $2^k > 0$. As $H_k$ is an even or odd function, we notice from (3.15) that it is actually sufficient to establish that

$$\forall k \geq 1, \forall x \geq z_k, \quad H_k(x) \leq 2^k x^k. \tag{3.16}$$

The estimates (3.16) are proved by recurrence on $k \geq 1$. Indeed, we observe from (3.9) that

$$\forall x \geq z_1 = 0, \quad H_1(x) = 2x.$$

Let $k \geq 2$ be such that the estimate (3.16) is satisfied at rank $k-1$. It follows from (3.12) that for all $x \geq z_k$,

$$H_k(x) = H_k(x) - H_k(z_k) = \int_{z_k}^{x} H_k'(t) \, dt = 2k \int_{z_k}^{x} H_{k-1}(t) \, dt$$

$$\leq 2k \int_{z_k}^{x} 2^{k-1} t^{k-1} \, dt = 2^k (x^k - z_k^k) \leq 2^k x^k,$$

since $0 \leq z_{k-1} < z_k$. This ends the proof of the claim (3.14).

We deduce from (3.9), (3.13) and (3.14) that

$$\forall k \in \mathbb{N}, \forall |x| \geq \sqrt{2k + 1}, \quad |H_k(x)| \leq 2^k |x|^k. \tag{3.18}$$

It follows from (3.1), (3.8) and (3.18) that

$$\forall k \in \mathbb{N}, \forall |x| \geq \sqrt{2k + 1}, \quad |\phi_k(x)| \leq \frac{2^{k/2}}{\sqrt{k! \pi^{1/4}}} |x|^k e^{-x^2/2}. \tag{3.19}$$

Now,

$$\forall a > 0, \quad \int_{a}^{+\infty} e^{-t^2} \, dt \leq a^{-1} e^{-a^2/2} \int_{a}^{+\infty} te^{-t^2/2} \, dt = a^{-1} e^{-a^2}$$
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and

\( (3.21) \quad \forall \alpha > 1, \forall a > \sqrt{\alpha - 1}, \)
\[
\int_{a}^{+\infty} t^{\alpha} e^{-t^2} dt \leq a^{\alpha - 1} e^{-a^2/2} \int_{a}^{+\infty} t e^{-t^2/2} dt = a^{\alpha - 1} e^{-a^2},
\]

as the function \((a, +\infty) \ni t \mapsto t^{\alpha - 1} e^{-t^2/2} \in (0, +\infty)\) is decreasing on \((a, +\infty)\). We deduce from (3.19)–(3.21) that

\( (3.22) \quad \forall \alpha \in \mathbb{N}, \forall a \geq \sqrt{\sqrt{2}k + 1}, \quad |x| \geq a \leq \left| \Phi_{\alpha}(x) \right| \leq \left| \Phi_{\beta}(x) \right| dx \leq \frac{2k}{k! \pi^{1/2}} \int_{x \geq a} x^{2k} e^{-x^2} dx \leq \frac{2k}{k! \pi^{1/2}} a^{2k-1} e^{-a^2}.
\]

This ends the proof of Lemma 3.1.

The following lemma is also instrumental in the proof of Theorem 2.1:

**Lemma 3.2.** With \( E_N = \text{Span}_\mathbb{C} \{ \Phi_\alpha \}_{\alpha \in \mathbb{N}^n, |\alpha| \leq N} \), there exists a positive constant \( c_n > 0 \) depending only on \( n \) such that

\( (3.22) \quad \forall \alpha, \beta \in \mathbb{N}^n, \forall f \in E_N, \quad \int_{|x| \geq c_n \sqrt{N+1}} |f(x)|^2 dx \leq \frac{1}{4} \| f \|_{L^2(\mathbb{R}^n)}^2, \)

where \(| \cdot |\) denotes the Euclidean norm on \( \mathbb{R}^n \).

**Proof.** Let \( N \in \mathbb{N} \). We deduce from Lemma 3.1 and the Cauchy–Schwarz inequality that the one-dimensional Hermite functions \((\phi_k)_{k \in \mathbb{N}}\) satisfy, for all \( 0 \leq k, l \leq N \) and \( a \geq \sqrt{2N + 1}, \)

\( (3.23) \quad \int_{|t| \geq a} |\phi_k(t)\phi_l(t)| dt \leq \left( \int_{|t| \geq a} |\phi_k(t)|^2 dt \right)^{1/2} \left( \int_{|t| \geq a} |\phi_l(t)|^2 dt \right)^{1/2} \leq \frac{2^{(k+l)/2+1}}{\sqrt{\pi} \sqrt{k! \sqrt{l!}}} a^{k+l-1} e^{-a^2}.
\]

In order to extend these estimates to the multi-dimensional setting, we first notice that for all \( a > 0 \) and \( \alpha, \beta \in \mathbb{N}^n \) with \(|\alpha|, |\beta| \leq N\),

\( (3.24) \quad \int_{|x| \geq a} |\Phi_\alpha(x)\Phi_\beta(x)| dx \leq \sum_{j=1}^{n} \int_{|x_j| \geq a/\sqrt{n}} |\Phi_\alpha(x)\Phi_\beta(x)| dx.
\]

On the other hand, we notice from (3.23) and (3.24) that
\[ \int_{|x_j| \geq a/\sqrt{n}} |\Phi_\alpha(x)\Phi_\beta(x)| \, dx \]
\[ = \left( \int_{|x_j| \geq a/\sqrt{n}} |\phi_{\alpha_j}(x_j)\phi_{\beta_j}(x_j)| \, dx_j \right) \prod_{1 \leq k \leq n \atop k \neq j} \left( \int |\phi_{\alpha_k}(x_k)\phi_{\beta_k}(x_k)| \, dx_k \right) \]
\[ \leq \left( \int_{|x_j| \geq a/\sqrt{n}} |\phi_{\alpha_j}(x_j)\phi_{\beta_j}(x_j)| \, dx_j \right) \prod_{1 \leq k \leq n \atop k \neq j} \|\phi_{\alpha_k}\|_{L^2(\mathbb{R})} \|\phi_{\beta_k}\|_{L^2(\mathbb{R})} \]

implies that for all \( a \geq \sqrt{n}/\sqrt{2N+1} \) and \( \alpha, \beta \in \mathbb{N}^n \) with \( |\alpha|, |\beta| \leq N \),

\[ (3.25) \quad \int_{|x| \geq a} |\Phi_\alpha(x)\Phi_\beta(x)| \, dx \leq \sum_{j=1}^n \int_{|x_j| \geq a/\sqrt{n}} |\phi_{\alpha_j}(x_j)\phi_{\beta_j}(x_j)| \, dx_j \]
\[ \leq 2 \sqrt{\frac{n}{\pi}} \frac{e^{-a^2/2n}}{a} \sum_{j=1}^n \frac{1}{\sqrt{\alpha_j!}\sqrt{\beta_j!}} \left( \sqrt{\frac{2}{n}} a \right)^{\alpha_j+\beta_j} \]

since \((\phi_k)_{k \in \mathbb{N}}\) is an orthonormal basis of \( L^2(\mathbb{R}) \). For any \( f = \sum_{|\alpha| \leq N} \gamma_\alpha \Phi_\alpha \in \mathcal{E}_N \) and \( a \geq \sqrt{n}/\sqrt{2N+1} \), we deduce from (3.25) that

\[ (3.26) \quad \int_{|x| \geq a} |f(x)|^2 \, dx \]
\[ = \sum_{|\alpha| \leq N \atop |\beta| \leq N} \gamma_\alpha \gamma_\beta \int_{|x| \geq a} \Phi_\alpha(x)\overline{\Phi_\beta(x)} \, dx \]
\[ \leq \sum_{|\alpha| \leq N \atop |\beta| \leq N} |\gamma_\alpha| |\gamma_\beta| \int_{|x| \geq a} |\Phi_\alpha(x)\Phi_\beta(x)| \, dx \]
\[ \leq 2 \sqrt{\frac{n}{\pi}} \frac{e^{-a^2/2n}}{a} \sum_{|\alpha| \leq N \atop |\beta| \leq N} \frac{|\gamma_\alpha| |\gamma_\beta|}{\sqrt{\alpha_j!}\sqrt{\beta_j!}} \left( \sqrt{\frac{2}{n}} a \right)^{\alpha_j+\beta_j} \]

For any \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we write \( \alpha' = (\alpha_2, \ldots, \alpha_n) \in \mathbb{N}^{n-1} \) when \( n \geq 2 \). We observe that

\[ (3.27) \quad \sum_{|\alpha| \leq N \atop |\beta| \leq N} \frac{|\gamma_\alpha| |\gamma_\beta|}{\sqrt{\alpha_1!}\sqrt{\beta_1!}} \left( \sqrt{\frac{2}{n}} a \right)^{\alpha_1+\beta_1} \]
\[ = \sum_{|\alpha'| \leq N \atop |\beta'| \leq N} \left( \sum_{0 \leq \alpha_1 \leq N-|\alpha'| \atop 0 \leq \beta_1 \leq N-|\beta'|} \frac{|\gamma_{\alpha_1,\alpha'}| |\gamma_{\beta_1,\beta'}|}{\sqrt{\alpha_1!}\sqrt{\beta_1!}} \left( \sqrt{\frac{2}{n}} a \right)^{\alpha_1+\beta_1} \right) \]
and

\begin{equation}
(3.28) \quad \sum_{0 \leq \alpha_1 \leq N-|\alpha'| \atop 0 \leq \beta_1 \leq N-|\beta'|} \frac{|\gamma_{\alpha_1,\alpha'}| |\gamma_{\beta_1,\beta'}|}{\sqrt{\alpha_1! \sqrt{\beta_1!}}} \left(\sqrt{\frac{2}{n}} a\right)^{\alpha_1+\beta_1} \leq \left( \sum_{0 \leq \alpha_1 \leq N-|\alpha'| \atop 0 \leq \beta_1 \leq N-|\beta'|} |\gamma_{\alpha_1,\alpha'}|^2 |\gamma_{\beta_1,\beta'}|^2 \right)^{1/2} \left( \sum_{0 \leq \alpha_1 \leq N-|\alpha'| \atop 0 \leq \beta_1 \leq N-|\beta'|} \frac{(2a^2/n)^{\alpha_1+\beta_1}}{\alpha_1! \beta_1!} \right)^{1/2},
\end{equation}

thanks to the Cauchy–Schwarz inequality. On the other hand, we notice that

\begin{equation}
(3.29) \quad \left( \sum_{0 \leq \alpha_1 \leq N-|\alpha'| \atop 0 \leq \beta_1 \leq N-|\beta'|} \frac{(2a^2/n)^{\alpha_1+\beta_1}}{\alpha_1! \beta_1!} \right)^{1/2} \leq 4^N \left( \sum_{0 \leq \alpha_1 \leq N-|\alpha'| \atop 0 \leq \beta_1 \leq N-|\beta'|} \frac{(a^2/n)^{\alpha_1+\beta_1}}{\alpha_1! \beta_1!} \right)^{1/2} \leq 4^N e^{a^2/(2n)}.
\end{equation}

It follows from \([3.27]–[3.29]\) that

\begin{equation}
(3.30) \quad \sum_{|\alpha| \leq N \atop |\beta| \leq N} \frac{|\gamma_{\alpha}| |\gamma_{\beta}|}{\sqrt{\alpha_1! \sqrt{\beta_1!}}} \left(\sqrt{\frac{2}{n}} a\right)^{\alpha_1+\beta_1} \leq 4^N e^{a^2/(2n)} \sum_{|\alpha'| \leq N \atop |\beta'| \leq N} \left( \sum_{0 \leq \alpha_1 \leq N-|\alpha'| \atop 0 \leq \beta_1 \leq N-|\beta'|} |\gamma_{\alpha_1,\alpha'}|^2 |\gamma_{\beta_1,\beta'}|^2 \right)^{1/2}.
\end{equation}

The Cauchy–Schwarz inequality implies that

\begin{equation}
(3.31) \quad \left( \sum_{|\alpha'| \leq N \atop |\beta'| \leq N} |\gamma_{\alpha_1,\alpha'}|^2 |\gamma_{\beta_1,\beta'}|^2 \right)^{1/2} \leq \left( \sum_{|\alpha'| \leq N \atop |\beta'| \leq N} \left( \sum_{0 \leq \alpha_1 \leq N-|\alpha'| \atop 0 \leq \beta_1 \leq N-|\beta'|} |\gamma_{\alpha_1,\alpha'}|^2 |\gamma_{\beta_1,\beta'}|^2 \right) \right)^{1/2} \left( \sum_{|\alpha'| \leq N \atop |\beta'| \leq N} 1 \right)^{1/2}.
\end{equation}

Since \((\Phi_\alpha)_{\alpha \in \mathbb{N}_n}\) is an orthonormal basis of \(L^2(\mathbb{R}^n)\) and the number of solutions to \(\alpha_2 + \cdots + \alpha_n = k\) with \(k \geq 0, n \geq 2\) and unknown \(\alpha' = (\alpha_2, \ldots, \alpha_n) \in \mathbb{N}^{n-1}\) is \(\binom{k+n-2}{n-2}\), we deduce from \([3.31]\) that
It follows from (3.34) that there exists a constant
\[ \|f\|_{L^2(\mathbb{R}^n)} \]
from (3.26) and (3.33) that for all
\[ n \]
when \( n \geq 2 \). Notice that the very same estimate holds for \( n = 1 \). We deduce from (3.26) and (3.33) that for all \( N \in \mathbb{N} \), \( f \in \mathcal{E}_N \), \( a \geq \sqrt{n} \sqrt{2N + 1} \),
\[ \int_{|x| \geq a} |f(x)|^2 \, dx \leq \frac{2^{n-1} 8^N}{\sqrt{n}} \frac{e^{a^2/(2n)}}{a} \|f\|_{L^2(\mathbb{R}^n)}^2. \] (3.34)
It follows from (3.34) that there exists a constant \( c_n > 0 \) depending only on \( n \) such that
\[ \forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \int_{|x| \geq c_n \sqrt{N+1}} |f(x)|^2 \, dx \leq \frac{1}{4} \|f\|_{L^2(\mathbb{R}^n)}^2. \]
This ends the proof of Lemma 3.2. 

3.1.2. Bernstein type estimates for Hermite functions. This section is devoted to the proof of the following Bernstein type estimates for Hermite functions:

**Proposition 3.3.** With \( \mathcal{E}_N = \text{Span}_{\mathbb{C}} \{ \Phi_\alpha \}_{\alpha \in \mathbb{N}^n, |\alpha| \leq N} \), we have:

(i) \( \forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \forall 0 < \delta \leq 1, \forall \beta \in \mathbb{N}^n, \)
\[ \|\partial_\beta f\|_{L^2(\mathbb{R}^n)} \leq e^{\delta/(2\delta^2)}(2\delta)^{|\beta|} \|f\|_{L^2(\mathbb{R}^n)}. \]
(ii) \( \forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \forall 0 < \delta < \frac{1}{32n}, \forall \beta \in \mathbb{N}^n, \)
\[ e^{\delta |x|^2} \|x^\beta f\|_{L^2(\mathbb{R}^n)} + e^{\delta |D_x|^2} \|x^\beta f\|_{L^2(\mathbb{R}^n)} \leq \frac{2^n}{1-32n\delta} \frac{2^N}{2^\delta} 2^{\frac{3}{2} |\beta|} \sqrt{||\beta||} \|f\|_{L^2(\mathbb{R}^n)}. \]

**Proof.** We notice that
\[ x_j = \frac{1}{\sqrt{2}}(a_{j,+} + a_{j,-}), \quad \partial_{x_j} = \frac{1}{\sqrt{2}}(a_{j,-} - a_{j,+}), \] (3.35)
with

\[ a_{j,+} = \frac{1}{\sqrt{2}} (x_j - \partial x_j), \quad a_{j,-} = \frac{1}{\sqrt{2}} (x_j + \partial x_j). \]

Letting \((e_j)_{1 \leq j \leq n}\) be the canonical basis of \(\mathbb{R}^n\), we deduce from (3.4) and (3.35) that for all \(N \in \mathbb{N}\) and \(f \in \mathcal{E}_N\),

\[
\|a_{j,+}f\|_{L^2(\mathbb{R}^n)}^2 = \left\| a_{j,+} \left( \sum_{|\alpha| \leq N} \langle f, \Phi_\alpha \rangle L_2 \Phi_\alpha \right) \right\|_{L^2(\mathbb{R}^n)}^2
\]

\[
= \left\| \sum_{|\alpha| \leq N} \sqrt{\alpha_j + 1} \langle f, \Phi_\alpha \rangle L_2 \Phi_\alpha + e_j \right\|_{L^2(\mathbb{R}^n)}^2 = \sum_{|\alpha| \leq N} (\alpha_j + 1) |\langle f, \Phi_\alpha \rangle L_2|^2
\]

\[
\leq (N + 1) \sum_{|\alpha| \leq N} |\langle f, \Phi_\alpha \rangle L_2|^2 = (N + 1) \|f\|_{L^2(\mathbb{R}^n)}^2
\]

and

\[
\|a_{j,-}f\|_{L^2(\mathbb{R}^n)}^2 = \left\| a_{j,-} \left( \sum_{|\alpha| \leq N} \langle f, \Phi_\alpha \rangle L_2 \Phi_\alpha \right) \right\|_{L^2(\mathbb{R}^n)}^2
\]

\[
= \left\| \sum_{|\alpha| \leq N} \sqrt{\alpha_j} \langle f, \Phi_\alpha \rangle L_2 \Phi_\alpha - e_j \right\|_{L^2(\mathbb{R}^n)}^2 = \sum_{|\alpha| \leq N} \alpha_j |\langle f, \Phi_\alpha \rangle L_2|^2
\]

\[
\leq N \sum_{|\alpha| \leq N} |\langle f, \Phi_\alpha \rangle L_2|^2 = N \|f\|_{L^2(\mathbb{R}^n)}^2.
\]

It follows that for all \(N \in \mathbb{N}\) and \(f \in \mathcal{E}_N\),

\[
(3.36) \quad \|x_j f\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{\sqrt{2}} (\|a_{j,+}f\|_{L^2(\mathbb{R}^n)} + \|a_{j,-}f\|_{L^2(\mathbb{R}^n)})
\]

\[
\leq \sqrt{2N + 2} \|f\|_{L^2(\mathbb{R}^n)},
\]

\[
(3.37) \quad \|\partial x_j f\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{\sqrt{2}} (\|a_{j,+}f\|_{L^2(\mathbb{R}^n)} + \|a_{j,-}f\|_{L^2(\mathbb{R}^n)})
\]

\[
\leq \sqrt{2N + 2} \|f\|_{L^2(\mathbb{R}^n)}.
\]

We notice from (3.4) and (3.35) that

\[
\forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \forall \alpha, \beta \in \mathbb{N}^n, \quad x^\alpha \partial^\beta f \in \mathcal{E}_{N + |\alpha| + |\beta|},
\]

with \(x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}\) and \(\partial^\beta = \partial_{x_1}^{\beta_1} \ldots \partial_{x_n}^{\beta_n}\). We deduce from (3.36) that for all \(N \in \mathbb{N}\), \(f \in \mathcal{E}_N\), and \(\alpha, \beta \in \mathbb{N}^n\) with \(\alpha_1 \geq 1\),

\[
\|x^\alpha \partial^\beta f\|_{L^2(\mathbb{R}^n)} = \|x_1 \left( \sum_{\mathcal{E}_N + |\alpha| + |\beta| - 1} \ x^{\alpha - e_1} \partial^\beta \right) f\|_{L^2(\mathbb{R}^n)}
\]

\[
\leq \sqrt{2} \sqrt{N + |\alpha| + |\beta|} \|x^{\alpha - e_1} \partial^\beta f\|_{L^2(\mathbb{R}^n)}.
\]
By iterating the previous estimates, we readily conclude from \((3.36)\) and \((3.37)\) that for all \(N \in \mathbb{N}, f \in E_N\) and \(\alpha, \beta \in \mathbb{N}^n\),

\[
\|x^\alpha \partial_x^\beta f\|_{L^2(\mathbb{R}^n)} \leq 2^{\frac{|\alpha| + |\beta|}{2}} \sqrt{\frac{(N + |\alpha| + |\beta|)!}{N!}} \|f\|_{L^2(\mathbb{R}^n)}.
\]

We recall the following basic estimates:

\[
\forall k \in \mathbb{N}^*,\quad k^k \leq e^k k!,\quad \forall t, A > 0,\quad t^A \leq A^A e^{-A},
\]

(see e.g. [41 (0.3.12) and (0.3.14)]). Let \(0 < \delta \leq 1\) be a constant. When \(N \leq |\alpha| + |\beta|\), we deduce from \((3.39)\) that

\[
2^{\frac{|\alpha| + |\beta|}{2}} \sqrt{\frac{(N + |\alpha| + |\beta|)!}{N!}} \leq 2^{\frac{|\alpha| + |\beta|}{2}} (N + |\alpha| + |\beta|)^{\frac{|\alpha| + |\beta|}{2}}
\]

\[
\leq 2^{\frac{|\alpha| + |\beta|}{2}} (N + |\alpha| + |\beta|)^{\frac{|\alpha| + |\beta|}{2}} (2\sqrt{e})^{\frac{|\alpha| + |\beta|}{2}} \sqrt{(|\alpha| + |\beta|)!}
\]

\[
= (2\sqrt{e})^{\frac{|\alpha| + |\beta|}{2}} \left(\frac{|\alpha| + |\beta|}{2}\right)! \leq e^{(2\delta^2)} (2\delta)^{|\alpha| + |\beta|} (|\alpha| + |\beta|)!
\]

On the other hand, when \(N \geq |\alpha| + |\beta|\), we deduce from \((3.39)\) that

\[
2^{\frac{|\alpha| + |\beta|}{2}} \sqrt{\frac{(N + |\alpha| + |\beta|)!}{N!}} \leq 2^{\frac{|\alpha| + |\beta|}{2}} (N + |\alpha| + |\beta|)^{\frac{|\alpha| + |\beta|}{2}}
\]

\[
\leq (2\delta)^{|\alpha| + |\beta|} \left(\delta^{-1}\sqrt{N}\right)^{|\alpha| + |\beta|} \leq (2\delta)^{|\alpha| + |\beta|} (|\alpha| + |\beta|)^{|\alpha| + |\beta|} e^{\delta^{-1}\sqrt{N} - |\alpha| - |\beta|}
\]

\[
\leq (2\delta)^{|\alpha| + |\beta|} (|\alpha| + |\beta|)! e^{\delta^{-1}\sqrt{N}}.
\]

It follows from \((3.38)\), \((3.40)\) and \((3.41)\) that for all \(N \in \mathbb{N}, f \in E_N\) and \(\alpha, \beta \in \mathbb{N}^n\),

\[
\|x^\alpha \partial_x^\beta f\|_{L^2(\mathbb{R}^n)} \leq e^{(2\delta^2)} (2\delta)^{|\alpha| + |\beta|} (|\alpha| + |\beta|)! e^{\delta^{-1}\sqrt{N}} \|f\|_{L^2(\mathbb{R}^n)}.
\]

This provides in particular the following Bernstein type estimates:

\[
\forall N \in \mathbb{N}, \forall f \in E_N, \forall 0 < \delta \leq 1, \forall \beta \in \mathbb{N}^n,
\]

\[
\|\partial_x^\beta f\|_{L^2(\mathbb{R}^n)} \leq e^{(2\delta^2)} (2\delta)^{|\beta|} (|\beta|)! e^{\delta^{-1}\sqrt{N}} \|f\|_{L^2(\mathbb{R}^n)}.
\]

On the other hand, we deduce from \((3.38)\) that for all \(N \in \mathbb{N}, f \in E_N\) and \(\alpha, \beta \in \mathbb{N}^n\),

\[
\|x^\alpha \partial_x^\beta f\|_{L^2(\mathbb{R}^n)} \leq 2^{\frac{|\alpha| + |\beta|}{2}} \sqrt{\frac{(N + |\alpha| + |\beta|)!}{N!}} \|f\|_{L^2(\mathbb{R}^n)}
\]

\[
\leq 2^{\frac{N}{2}} 2^{\frac{|\alpha| + |\beta|}{2}} \sqrt{(|\alpha| + |\beta|)!} \|f\|_{L^2(\mathbb{R}^n)},
\]
since
\[
\frac{(k_1 + k_2)!}{k_1!k_2!} = \binom{k_1 + k_2}{k_1} \leq \sum_{j=0}^{k_1+k_2} \binom{k_1 + k_2}{j} = 2^{k_1+k_2}.
\]

We observe from (3.44) that for all \( N \in \mathbb{N} \), \( f \in \mathcal{E}_N \), \( \delta > 0 \) and \( \alpha, \beta \in \mathbb{N}^n \),
\[
\left(\frac{\delta^{|\alpha|} x^{2\alpha}}{|\alpha|!} \partial_x^\beta f\right)_{L^2(\mathbb{R}^n)} \leq \frac{2^N \delta^{|\alpha|} 2^{2|\alpha|+|\beta|}}{|\alpha|!} \sqrt{(2|\alpha| + |\beta|)!} \| f \|_{L^2(\mathbb{R}^n)}
\leq 2^N \frac{\delta^{|\alpha|} 2^{|\alpha|+\frac{3}{2}|\beta|}}{|\alpha|!} \sqrt{|\beta|!} \| f \|_{L^2(\mathbb{R}^n)}
\leq 2^N (16n\delta)^{|\alpha|} 2^\frac{3}{2}|\beta| \sqrt{|\beta|!} \| f \|_{L^2(\mathbb{R}^n)},
\]

since
\[
(2|\alpha| + |\beta|)! \leq 2^{2|\alpha|+|\beta|} (2|\alpha|)! |\beta|! \leq 2^{4|\alpha|+|\beta|} (|\alpha|!)^2 |\beta|!
\]

and
\[
(3.46) \quad |\alpha|! \leq n^{|\alpha|} |\alpha|!.
\]

The last estimate is a direct consequence of the generalized Newton formula
\[
\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \forall N \in \mathbb{N}, \quad \left(\sum_{j=1}^{n} x_j\right)^N = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = N} \frac{N!}{\alpha!} x^\alpha.
\]

Since the number of solutions to \( \alpha_1 + \cdots + \alpha_n = k \) with \( k \geq 0 \), \( n \geq 1 \) and unknown \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) is \( \binom{k+n-1}{n-1} \), it follows from (3.45) that for all \( N \in \mathbb{N} \), \( f \in \mathcal{E}_N \), \( 0 < \delta < \frac{1}{32n} \) and \( \beta \in \mathbb{N}^n \),
\[
(3.47) \quad \| e^{\delta|x|^2} \partial_x^\beta f \|_{L^2(\mathbb{R}^n)} \leq \sum_{\alpha \in \mathbb{N}^n} \left\| \frac{\delta^{|\alpha|} x^{2\alpha}}{|\alpha|!} \partial_x^\beta f \right\|_{L^2(\mathbb{R}^n)}
\leq 2^N \left( \sum_{\alpha \in \mathbb{N}^n} (16n\delta)^{|\alpha|} \right) 2^\frac{3}{2}|\beta| \sqrt{|\beta|!} \| f \|_{L^2(\mathbb{R}^n)}
\leq 2^N \left( \sum_{k=0}^{+\infty} \binom{k+n-1}{n-1} (16n\delta)^k \right) 2^\frac{3}{2}|\beta| \sqrt{|\beta|!} \| f \|_{L^2(\mathbb{R}^n)}
\leq \frac{2^{n-1}}{1-32n\delta} 2^N 2^\frac{3}{2}|\beta| \sqrt{|\beta|!} \| f \|_{L^2(\mathbb{R}^n)},
\]

since \( \binom{k+n-1}{n-1} \leq \sum_{j=0}^{k+n-1} \binom{k+n-1}{j} = 2^{k+n-1} \). By noticing from (3.2) that \( f \in \mathcal{E}_N \) if and only if \( \hat{f} \in \mathcal{E}_N \), from the Parseval formula and (3.47) we
deduce that for all $N \in \mathbb{N}, f \in \mathcal{E}_N$, $0 < \delta < \frac{1}{32n}$ and $\beta \in \mathbb{N}^n$,
\begin{equation}
(3.48) \quad \|e^{\delta|D_x|^2} x^\beta f\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{\frac{n}{2}}} \|e^{\delta|\xi|^2} \partial^\beta \hat{f}\|_{L^2(\mathbb{R}^n)} \\
\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{2^{n-1}}{1 - 32n\delta} 2^{\frac{n}{2}} 2^\frac{3}{2} |\beta|! \|\hat{f}\|_{L^2(\mathbb{R}^n)} \\
= \frac{2^{n-1}}{1 - 32n\delta} 2^{\frac{n}{2}} 2^\frac{3}{2} |\beta|! \|f\|_{L^2(\mathbb{R}^n)}.
\end{equation}
This ends the proof of Proposition 3.3.

3.2. Proofs of the uncertainty principles for Hermite functions.
This section is devoted to the proof of Theorem 2.1.

3.2.1. Case when the control subset is a non-empty open set. Let $\omega \subset \mathbb{R}^n$ be a non-empty open set. There exist $x_0 \in \mathbb{R}^n$ and $r > 0$ such that
\begin{equation}
B(x_0, r) \subset \omega.
\end{equation}
We recall from (2.2) that
\begin{equation}
(3.50) \quad \forall N \in \mathbb{N}, \exists C_N(\omega) > 0, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq C_N(\omega) \|f\|_{L^2(\omega)}
\end{equation}
with $\mathcal{E}_N = \pi_N(L^2(\mathbb{R}^n))$. On the other hand, it follows from Lemma 3.2 that
\begin{equation}
(3.51) \quad \forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{\sqrt{3}} \|f\|_{L^2(B(0, c_n \sqrt{N+1}))}.
\end{equation}
Let $N \in \mathbb{N}$ and $f \in \mathcal{E}_N$. By (3.1) and (3.6), there exists a complex polynomial function $P \in \mathbb{C}[X_1, \ldots, X_n]$ of degree at most $N$ such that
\begin{equation}
(3.52) \quad \forall x \in \mathbb{R}^n, \quad f(x) = P(x)e^{-|x|^2/2}.
\end{equation}
We observe from (3.51) and (3.52) that
\begin{equation}
(3.53) \quad \|f\|^2_{L^2(\mathbb{R}^n)} \leq \frac{4}{3} \int_{B(0, c_n \sqrt{N+1})} |P(x)|^2 e^{-|x|^2} \, dx \\
\leq \frac{4}{3} \|P\|^2_{L^2(B(0, c_n \sqrt{N+1}))}
\end{equation}
and
\begin{equation}
(3.54) \quad \|P\|^2_{L^2(B(x_0, r))} = \int_{B(x_0, r)} |P(x)|^2 e^{-|x|^2} c|x|^2 \, dx \\
\leq e^{(|x_0|^2+1)^2} \|f\|^2_{L^2(B(x_0, r))}.
\end{equation}
We aim at deriving an estimate of $\|P\|_{L^2(B(0, c_n \sqrt{N+1}))}$ by $\|P\|_{L^2(B(x_0, r))}$ when $N \gg 1$ is sufficiently large.

Let $N$ be an integer such that $c_n \sqrt{N+1} > 2|x_0| + r$. This implies $B(x_0, r) \subset B(0, c_n \sqrt{N+1})$. We may assume that $P$ is a non-zero poly-
nominal function. By using polar coordinates centered at $x_0$, we notice that

$$B(x_0, r) = \{x_0 + t\sigma : 0 \leq t < r, \sigma \in S^{n-1}\}$$

and

$$\|P\|_{L^2(B(x_0, r))}^2 = \int_{S^{n-1}} \int_0^r |P(x_0 + t\sigma)|^2 t^{n-1} \, dt \, d\sigma.$$  \hspace{1cm} (3.55)

As $c_n\sqrt{N + 1} > 2|x_0| + r$, there is a continuous function $\rho_N : S^{n-1} \to (0, +\infty)$ such that

$$B(0, c_n\sqrt{N + 1}) = \{x_0 + t\sigma : 0 \leq t < \rho_N(\sigma), \sigma \in S^{n-1}\}$$

and

$$0 < |x_0| + r < c_n\sqrt{N + 1} - |x_0| < \rho_N(\sigma) < c_n\sqrt{N + 1} + |x_0|$$

for all $\sigma \in S^{n-1}$. It follows from (3.56) and (3.57) that

$$\|P\|_{L^2(B(0, c_n\sqrt{N + 1}) \setminus B(x_0, r/2))}^2 = \int_{S^{n-1}} \int_{r/2}^{\rho_N(\sigma)} |P(x_0 + t\sigma)|^2 t^{n-1} \, dt \, d\sigma$$

$$\leq (c_n\sqrt{N + 1} + |x_0|)^{n-1} \int_{S^{n-1}} \int_{r/2}^{\rho_N(\sigma)} |P(x_0 + t\sigma)|^2 \, dt \, d\sigma.$$  \hspace{1cm} (3.58)

By noticing that

$$t \to P\left(x_0 + \left(\frac{\rho_N(\sigma)}{2} + \frac{r}{4}\right)\sigma + t\sigma\right)$$

is a polynomial function of degree at most $N$, we deduce from (3.57) and Lemma 5.1 (see the Appendix) used in the case $n = 1$ that

$$\int_{r/2}^{\rho_N(\sigma)} |P(x_0 + t\sigma)|^2 \, dt$$

$$\leq \frac{2^{4N+2}}{3} \left(\frac{r/2}{4(\rho_N(\sigma)/2 - r/4)}\right)^{2N} \int_{-\rho_N(\sigma)/2 + 3r/4}^{\rho_N(\sigma)/2 - 2r/4} \left|P\left(x_0 + \left(\frac{\rho_N(\sigma)}{2} + \frac{r}{4}\right)\sigma + t\sigma\right)\right|^2 \, dt$$  \hspace{1cm} (3.59)
It follows that there exist some constants $c, \omega > 0$.

The estimates (3.50) and (3.63) yield assertion (i) in Theorem 2.1.

It follows from (3.53), (3.54) and (3.62) that for all $n \in \mathbb{N}$,

$$\|P\|_{L^2(B(0,c_n\sqrt{N+1}\setminus B(x_0,r/2))} \leq \left(c_n\sqrt{N + 1} + |x_0|\right)^{n-1}$$

implying that

$$\|P\|_{L^2(B(0,c_n\sqrt{N+1}))} \leq \left(1 + (c_n\sqrt{N + 1} + |x_0|)\right)^{n-1}$$

thanks to (3.55). We deduce from (3.61) that there exists a constant $C = C(x_0, r, n) > 1$ independent of $N$ such that

$$\|P\|_{L^2(B(0,c_n\sqrt{N+1}))} \leq Ce^{\frac{1}{2}N\ln(N+1)+CN}\|P\|_{L^2(B(x_0,r))}.$$ 

It follows from (3.53), (3.54) and (3.62) that for all $N \in \mathbb{N}$ such that $c_n\sqrt{N + 1} > 2|x_0| + r$ and all $f \in \mathcal{E}_N$,

$$\|f\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{\sqrt{3}}Ce^{\frac{1}{2}(|x_0|+r)^2}e^{\frac{1}{2}N\ln(N+1)+CN}\|f\|_{L^2(B(x_0,r))}.$$ 

The estimates (3.50) and (3.63) yield assertion (i) in Theorem 2.1

**3.2.2. Case when the control subset is a measurable set satisfying (1.23).**

Let $\omega \subset \mathbb{R}^n$ be a measurable subset satisfying

$$\liminf_{R \to +\infty} \frac{|\omega \cap B(0,R)|}{|B(0,R)|} = \lim_{R \to +\infty} \left(\inf_{r \geq R} \frac{|\omega \cap B(0,r)|}{|B(0,r)|}\right) > 0.$$ 

It follows that there exist some constants $R_0, \delta > 0$ such that

$$\forall R \geq R_0, \quad \frac{|\omega \cap B(0,R)|}{|B(0,R)|} \geq \delta > 0.$$ 

We recall from (2.2) that

$$\forall N \in \mathbb{N}, \exists C_N(\omega) > 0, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq C_N(\omega)\|f\|_{L^2(\omega)},$$ 

and as in the previous section, it follows from Lemma 3.2 that

$$\forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{\sqrt{3}}\|f\|_{L^2(B(0,c_n\sqrt{N+1}))}.$$
where \( c_n > 0 \) is a constant depending only on \( n \). Let \( N \in \mathbb{N} \) satisfy \( c_n \sqrt{N+1} \geq R_0 \) and \( f \in \mathcal{E}_N \). It follows from (3.65) that

\[
|\omega \cap B(0, c_n \sqrt{N+1})| \geq \delta |B(0, c_n \sqrt{N+1})| > 0.
\]

According to (3.1) and (3.6), there exists \( P \in \mathbb{C}[X_1, \ldots, X_n] \) of degree at most \( N \) such that

\[
\forall x \in \mathbb{R}^n, \quad f(x) = P(x)e^{-|x|^2/2}.
\]

We observe from (3.67) and (3.69) that

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{4}{3} \int_{B(0, c_n \sqrt{N+1})} |P(x)|^2 e^{-|x|^2} \, dx \leq \frac{4}{3} \|P\|_{L^2(B(0, c_n \sqrt{N+1}))}^2,
\]

and

\[
\|P\|_{L^2(\omega \cap B(0, c_n \sqrt{N+1}))}^2 = \int_{\omega \cap B(0, c_n \sqrt{N+1})} |P(x)|^2 e^{-|x|^2} \, dx \leq e^{c_n^2(N+1)} \|f\|_{L^2(\omega \cap B(0, c_n \sqrt{N+1}))}^2.
\]

We deduce from Lemma 5.1 and (3.68) that

\[
\|P\|_{L^2(B(0, c_n \sqrt{N+1}))}^2 \leq \frac{2^{4N+2}}{3} \frac{4|B(0, c_n \sqrt{N+1})|}{|\omega \cap B(0, c_n \sqrt{N+1})|}
\times \left[ F\left( \frac{|\omega \cap B(0, c_n \sqrt{N+1})|}{4|B(0, c_n \sqrt{N+1})|} \right) \right]^{2N} \|P\|_{L^2(\omega \cap B(0, c_n \sqrt{N+1}))}^2,
\]

with \( F \) the decreasing function

\[
\forall 0 < t \leq 1, \quad F(t) = \frac{1 + (1 - t)^{1/n}}{1 - (1 - t)^{1/n}} \geq 1.
\]

Since \( F \) is decreasing, it follows from (3.68) and (3.72) that

\[
\|P\|_{L^2(B(0, c_n \sqrt{N+1}))}^2 \leq \frac{2^{4N+4}}{3\delta} \left[ F\left( \frac{\delta}{4} \right) \right]^{2N} \|P\|_{L^2(\omega \cap B(0, c_n \sqrt{N+1}))}^2.
\]

Putting together (3.70), (3.71) and (3.73), we deduce that there exists a constant \( C = C(\delta, n) > 0 \) such that for all \( N \in \mathbb{N} \) with \( c_n \sqrt{N+1} \geq R_0 \) and all \( f \in \mathcal{E}_N \),

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{2^{4N+6}}{9\delta} \left[ F\left( \frac{\delta}{4} \right) \right]^{2N} e^{c_n^2(N+1)} \|f\|_{L^2(\omega \cap B(0, c_n \sqrt{N+1}))}^2
\leq C^2 e^{2CN} \|f\|_{L^2(\omega)}^2,
\]

The estimates (3.66) and (3.74) yield assertion (ii) in Theorem 2.1.
3.2.3. Case when the control subset is a thick set. Let \( \omega \) be a measurable subset of \( \mathbb{R}^n \). We assume that \( \omega \) is \( \gamma \)-thick \( (0 < \gamma \leq 1) \) at scale \( L > 0 \):

\[
\forall x \in \mathbb{R}^n, \quad |\omega \cap (x + [0, L]^n)| \geq \gamma L^n.
\]

The following proof is an adaptation of the proof of the sharpened version of the Logvinenko–Sereda theorem given by Kovrijkine [28, 29].

**Step 1: Bad and good cubes.** Let \( N \in \mathbb{N} \) and \( f \in \mathcal{E}_N \setminus \{0\} \). For each \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (L\mathbb{Z})^n \), let

\[
Q(\alpha) = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \forall 1 \leq j \leq n, |x_j - \alpha_j| < L/2 \}.
\]

Notice that \( \forall \alpha, \beta \in (L\mathbb{Z})^n, \alpha \neq \beta, Q(\alpha) \cap Q(\beta) = \emptyset \), \( \mathbb{R}^n = \bigcup_{\alpha \in (L\mathbb{Z})^n} \overline{Q(\alpha)} \), where the bar denotes closure. It follows that

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 = \sum_{\alpha \in (L\mathbb{Z})^n} \int_{Q(\alpha)} |f(x)|^2 \, dx.
\]

Let \( \delta > 0 \) be a constant to be chosen later on. We divide the family \( (Q(\alpha))_{\alpha \in (L\mathbb{Z})^n} \) into good and bad cubes. A cube \( Q(\alpha) \) with \( \alpha \in (L\mathbb{Z})^n \) is said to be **good** if for all \( \beta \in \mathbb{N}^n \),

\[
\int_{Q(\alpha)} |\partial_x^\beta f(x)|^2 \, dx \leq e^{e\delta - 2(8\delta^2(2^n + 1))|\beta|(|\beta|!)^2 e^{2\delta - 1}\sqrt{N}} \int_{Q(\alpha)} |f(x)|^2 \, dx.
\]

A cube \( Q(\alpha) \) which is not good is said to be **bad**; then

\[
\int_{Q(\alpha)} |\partial_x^\beta f(x)|^2 \, dx > e^{e\delta - 2(8\delta^2(2^n + 1))|\beta|(|\beta|!)^2 e^{2\delta - 1}\sqrt{N}} \int_{Q(\alpha)} |f(x)|^2 \, dx.
\]

If \( Q(\alpha) \) is bad, it follows from (3.77) that there exists \( \beta_0 \in \mathbb{N}^n \) with \( |\beta_0| > 0 \) such that

\[
\int_{Q(\alpha)} |f(x)|^2 \, dx \leq \frac{e^{-e\delta - 2}}{(8\delta^2(2^n + 1))|\beta_0|(|\beta_0|!)^2 e^{2\delta - 1}\sqrt{N}} \int_{Q(\alpha)} |\partial_x^\beta_0 f(x)|^2 \, dx
\]

\[
\leq \sum_{\beta \in \mathbb{N}^n, |\beta| > 0} e^{-e\delta - 2} \frac{(8\delta^2(2^n + 1))|\beta|(|\beta|!)^2 e^{2\delta - 1}\sqrt{N}} \int_{Q(\alpha)} |\partial_x^\beta f(x)|^2 \, dx.
\]
By summing over all the bad cubes, we deduce from (3.78) and the Fubini–Tonelli theorem that

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 = \sum_{\beta \in \mathbb{N}^n, |\beta| > 0} e^{-\epsilon \delta^2} \left( \frac{1}{2(2^n + 1)} \right)^{|\beta|} \int_{\mathbb{R}^n} |\partial_x^\beta f(x)|^2 \, dx
\]

and

\[
\leq \sum_{\beta \in \mathbb{N}^n, |\beta| > 0} e^{-\epsilon \delta^2} \left( \frac{1}{2(2^n + 1)} \right)^{|\beta|} \int_{\mathbb{R}^n} |\partial_x^\beta f(x)|^2 \, dx.
\]

Since the number of solutions to \(\beta_1 + \cdots + \beta_n = k\) with \(k \geq 0, n \geq 1\) and unknown \(\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n\) is \((k+n-1)\), we deduce from the Bernstein type estimates in Proposition 3.3(i) and (3.79) that

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 \leq \left( \sum_{\beta \in \mathbb{N}^n, |\beta| > 0} \frac{1}{(2(2^n + 1))^{|\beta|}} \right) \|f\|_{L^2(\mathbb{R}^n)}^2
\]

and

\[
\leq 2^{n-1} \left( \sum_{k=1}^{\infty} \frac{1}{(2^n + 1)^k} \right) \|f\|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{2} \|f\|_{L^2(\mathbb{R}^n)}^2,
\]

since

\[
\binom{k+n-1}{k} \leq \sum_{j=0}^{k+n-1} \binom{k+n-1}{j} = 2^{k+n-1}.
\]

By writing

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 = \int_{\text{good cubes } Q(\alpha)} |f(x)|^2 \, dx + \int_{\text{bad cubes } Q(\alpha)} |f(x)|^2 \, dx,
\]

it follows from (3.80) that

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 \leq 2 \int_{\text{good cubes } Q(\alpha)} |f(x)|^2 \, dx.
\]

**Step 2: Properties on good cubes.** As any cube \(Q(\alpha)\) satisfies the cone condition, the Sobolev embedding

\[
W^{n,2}(Q(\alpha)) \hookrightarrow L^\infty(Q(\alpha))
\]
implies that there exists a universal constant $C_n > 0$ depending only on $n$ such that
\begin{equation}
\forall u \in W^{n,2}(Q(\alpha)), \quad \|u\|_{L^\infty(Q(\alpha))} \leq C_n\|u\|_{W^{n,2}(Q(\alpha))}.
\end{equation}

By translation invariance of the Lebesgue measure, the constant $C_n$ does not depend on $\alpha \in (LZ)^n$. Let $Q(\alpha)$ be a good cube. We deduce from (3.76) and (3.83) that for all $\beta \in \mathbb{N}^n$,
\begin{equation}
\|\partial_\beta f\|_{L^\infty(Q(\alpha))} \leq C_n \left( \sum_{\tilde{\beta} \in \mathbb{N}^n, |\tilde{\beta}| \leq n} (32\delta^2(2^n + 1)|\tilde{\beta}|^2)!e^{\delta^{-1}\sqrt{n}}\|f\|_{L^2(Q(\alpha))} \right)^{1/2} > 0,
\end{equation}

with
\begin{equation}
\tilde{C}_n(\delta) = C_n e^{\delta^{-2}/2} e^{\sqrt{n}/\delta} \left( \sum_{\tilde{\beta} \in \mathbb{N}^n, |\tilde{\beta}| \leq n} (32\delta^2(2^n + 1)|\tilde{\beta}|^2)!e^{\delta^{-1}\sqrt{n}}\right)^{1/2} > 0,
\end{equation}

since
\begin{equation*}
(|\beta| + |\tilde{\beta}|)! \leq 2^{|\beta| + |\tilde{\beta}|} |\beta|! |\tilde{\beta}|!.
\end{equation*}

Recalling that $f$ is a finite combination of Hermite functions, we deduce from the continuity of $f$ and the compactness of $Q(\alpha)$ that there exists $x_\alpha \in Q(\alpha)$ such that
\begin{equation}
\|f\|_{L^\infty(Q(\alpha))} = |f(x_\alpha)|.
\end{equation}

By using spherical coordinates centered at $x_\alpha \in Q(\alpha)$ and the fact that the Euclidean diameter of $Q(\alpha)$ is $\sqrt{n} L$, we find that
\begin{equation}
|\omega \cap Q(\alpha)| = \int_0^{+\infty} \left( \int_{S^{n-1}} 1_{\omega \cap Q(\alpha)}(x_\alpha + r\sigma) \, d\sigma \right) r^{n-1} \, dr
= \int_0^{\frac{\sqrt{n} L}{n/2}} \left( \int_{S^{n-1}} 1_{\omega \cap Q(\alpha)}(x_\alpha + r\sigma) \, d\sigma \right) r^{n-1} \, dr
= n^{n/2} L^n \int_0^1 \left( \int_{S^{n-1}} 1_{\omega \cap Q(\alpha)}(x_\alpha + \sqrt{n} Lr\sigma) \, d\sigma \right) r^{n-1} \, dr,
\end{equation}

where $1_{\omega \cap Q(\alpha)}$ stands for the characteristic function of the measurable set $\omega \cap Q(\alpha)$. By using Fubini’s theorem, we deduce from (3.87) that
Spectral estimates and null-controllability

\[ |\omega \cap Q(\alpha)| \leq n^{n/2} L^n \int_0^1 \left( \int_{S^{n-1}} 1_{\omega \cap Q(\alpha)}(x, \sigma) \, d\sigma \right) \, dr \]

\[ = n^{n/2} L^n \int_{S^{n-1}} \left( \int_0^1 1_{\omega \cap Q(\alpha)}(x, \sigma) \, d\sigma \right) \, dr \]

\[ = n^{n/2} L^n \int_{S^{n-1}} \left( \int_0^1 I_\sigma(r) \, dr \right) \, d\sigma = n^{n/2} L^n \int_{S^{n-1}} |I_\sigma| \, d\sigma, \]

where

\[ I_\sigma = \{ r \in [0, 1] : x, \sigma = x + \sqrt{n} L r \sigma \in \omega \cap Q(\alpha) \}. \]

The estimate (3.88) implies that there exists \( \sigma_0 \in S^{n-1} \) such that

\[ |\omega \cap Q(\alpha)| \leq n^{n/2} L^n |S^{n-1}| |I_{\sigma_0}|. \]

By using the thickness property (3.75), it follows from (3.90) that

\[ |I_{\sigma_0}| \geq \frac{\gamma}{n^{n/2} |S^{n-1}|} > 0. \]

**Step 3: Recovery of the \( L^2(\mathbb{R}^n) \)-norm.** First notice that \( \|f\|_{L^2(Q(\alpha))} \neq 0 \), since \( f \) is a non-zero entire function. We consider the entire function

\[ \forall z \in \mathbb{C}, \quad \phi(z) = L^{n/2} f(x, \sigma_0) \frac{f(x, \sigma_0)}{\|f\|_{L^2(Q(\alpha))}}. \]

We observe from (3.86) that

\[ |\phi(0)| = \frac{L^{n/2} |f(x, \sigma_0)|}{\|f\|_{L^2(Q(\alpha))}} = \frac{L^{n/2} \|f\|_{L^\infty(Q(\alpha))}}{\|f\|_{L^2(Q(\alpha))}} \geq 1. \]

Instrumental the proof is the following lemma proved by Kovrijkine:

**Lemma 3.4 ([29, Lemma 1]).** Let \( I \subset \mathbb{R} \) be an interval of length 1 such that \( 0 \in I \), and \( E \subset I \) be a subset of positive measure. There exists a constant \( C > 1 \) such that if \( \Phi \) is an analytic function on the open disc \( D(0, 5) \) such that \( |\Phi(0)| \geq 1 \), then

\[ \sup_{x \in I} |\Phi(x)| \leq \left( \frac{C}{|E|} \right)^{\frac{\ln M}{2}} \sup_{x \in E} |\Phi(x)|, \]

with \( M = \sup_{|z| \leq 4} |\Phi(z)| \geq 1. \)

Applying Lemma 3.4 with \( I = [0, 1], \ E = I_{\sigma_0} \subset [0, 1] \) satisfying \( |E| = |I_{\sigma_0}| > 0 \) as in (3.91), and the analytic function \( \Phi = \phi \) defined in (3.92)
satisfying $|\phi(0)| \geq 1$, we obtain

$$
L^{n/2} \sup_{x \in [0,1]} |f(x_\alpha + \sqrt{n} L x \sigma_0)| \frac{\|f\|_{L^2(Q(\alpha))}}{\|f\|_{L^2(Q(\alpha))}} \\
\leq \left( \frac{C}{|I_{\sigma_0}|} \right)^{\frac{\ln M}{\ln 2}} L^{n/2} \sup_{x \in I_{\sigma_0}} |f(x_\alpha + \sqrt{n} L x \sigma_0)| \frac{\|f\|_{L^2(Q(\alpha))}}{\|f\|_{L^2(Q(\alpha))}},
$$

with

$$
M = L^{n/2} \sup_{|z| \leq 4} |f(x_\alpha + \sqrt{n} L z \sigma_0)| \frac{\|f\|_{L^2(Q(\alpha))}}{\|f\|_{L^2(Q(\alpha))}}.
$$

It follows from (3.91) and (3.93) that

$$
\sup_{x \in [0,1]} |f(x_\alpha + \sqrt{n} L x \sigma_0)| \leq \left( \frac{C n^{n/2} |S^{n-1}|}{\gamma} \right)^{\frac{\ln M}{\ln 2}} \sup_{x \in I_{\sigma_0}} |f(x_\alpha + \sqrt{n} L x \sigma_0)| \leq M^{\frac{1}{\ln 2} \ln(C n^{n/2} |S^{n-1}|/\gamma)} \sup_{x \in I_{\sigma_0}} |f(x_\alpha + \sqrt{n} L x \sigma_0)|.
$$

According to (3.89),

$$
\sup_{x \in I_{\sigma_0}} |f(x_\alpha + \sqrt{n} L x \sigma_0)| \leq \|f\|_{L^\infty(Q(\alpha))}.
$$

On the other hand, we deduce from (3.86) that

$$
\|f\|_{L^\infty(Q(\alpha))} = |f(x_\alpha)| \leq \sup_{x \in [0,1]} |f(x_\alpha + \sqrt{n} L x \sigma_0)|.
$$

It follows from (3.95)–(3.97) that

$$
\|f\|_{L^\infty(Q(\alpha))} \leq M^{\frac{1}{\ln 2} \ln(C n^{n/2} |S^{n-1}|/\gamma)} \|f\|_{L^\infty(\omega \cap Q(\alpha))}.
$$

By using the analyticity of $f$, we observe that

$$
\forall z \in \mathbb{C}, \quad f(x_\alpha + \sqrt{n} L z \sigma_0) = \sum_{\beta \in \mathbb{N}^n} \frac{\partial^\beta f(x_\alpha)}{\beta!} \sigma_\beta n^{||\beta||/2} L^{||\beta||} |z|^{|\beta|}.
$$

Since $Q(\alpha)$ is a good cube and $x_\alpha \in Q(\alpha)$, and since the functions $\partial^\beta f$ are continuous, we deduce from (3.84) and (3.99) that for all $|z| \leq 4$,

$$
|f(x_\alpha + \sqrt{n} L z \sigma_0)| \leq \sum_{\beta \in \mathbb{N}^n} \frac{|(\partial^\beta f)(x_\alpha)|}{\beta!} \left( 4\sqrt{n} L \right)^{|\beta|} \\
\leq \tilde{C}_n(\delta)e^{\delta^{-1}\sqrt{N}} \left( \sum_{\beta \in \mathbb{N}^n} \frac{|\beta||\beta|}{\beta!} \left( \delta L \sqrt{2^{\beta n}(2^n + 1)} \right)^{|\beta|} \right) \|f\|_{L^2(Q(\alpha))}.
$$
Since the number of solutions to $\beta_1 + \cdots + \beta_n = k$, with $k \geq 0$, $n \geq 1$ and unknown $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ is $\binom{k+n-1}{k}$, and since

$$|\beta|! \leq n|\beta|!$$

(see e.g. (3.46)), we notice from (3.81) that

$$\sum_{\beta \in \mathbb{N}^n} \frac{|\beta|!}{\beta!} (\delta L \sqrt{2^9 n (2^n + 1)})^{|\beta|} \leq \sum_{\beta \in \mathbb{N}^n} (\delta L \sqrt{2^9 n^3 (2^n + 1)})^{|\beta|}$$

$$= \sum_{k=0}^{+\infty} \binom{k+n-1}{k} (\delta L \sqrt{2^9 n^3 (2^n + 1)})^k \leq 2^{n-1} \sum_{k=0}^{+\infty} (\delta L \sqrt{2^{11} n^3 (2^n + 1)})^k.$$ 

From now on, we fix

$$(3.102) \quad \delta = \frac{1}{\delta_n L} > 0$$

with

$$\delta_n = 2\sqrt{2^{11} n^3 (2^n + 1)} > 0.$$ 

With this choice, it follows from (3.94) and (3.100)–(3.102) that

$$(3.103) \quad M \leq (4L)^{n/2} \tilde{C}_n (\delta_n^{-1} L^{-1}) e^{\delta_n L \sqrt{N}}.$$ 

The constant $C > 1$ of Lemma 3.4 may be chosen such that

$$(3.104) \quad Cn^{n/2} |S^{n-1}| > 1.$$ 

With this choice, we deduce from (3.98) and (3.103) that

$$(3.105) \quad \|f\|_{L^\infty(Q(\alpha))} \leq \left( \frac{Cn^{n/2} |S^{n-1}|}{\gamma} \right)^{\ln((4L)^{n/2} \tilde{C}_n (\delta_n^{-1} L^{-1}))/\ln 2 + (\delta_n / \ln 2) L \sqrt{N}} \|f\|_{L^\infty(\omega \cap Q(\alpha))}.$$ 

Recalling from the thickness property (3.75) that $|\omega \cap Q(\alpha)| \geq \gamma L^n > 0$ and setting

$$(3.106) \quad \tilde{\omega}_\alpha = \left\{ x \in \omega \cap Q(\alpha) : |f(x)| \leq \frac{2}{|\omega \cap Q(\alpha)|} \int_{\omega \cap Q(\alpha)} |f(t)| \, dt \right\},$$

we observe that

$$(3.107) \quad \int_{\omega \cap Q(\alpha)} |f(x)| \, dx \geq \int_{(\omega \cap Q(\alpha)) \setminus \tilde{\omega}_\alpha} |f(x)| \, dx$$

$$\geq \frac{2|\omega \cap Q(\alpha)| \setminus \tilde{\omega}_\alpha}{|\omega \cap Q(\alpha)|} \int_{\omega \cap Q(\alpha)} |f(x)| \, dx.$$
Since $\int_{\omega \cap Q(\alpha)} |f(x)| \, dx > 0$ because $f$ is a non-zero entire function and $|\omega \cap Q(\alpha)| > 0$, we obtain

$$|\omega \cap Q(\alpha) \setminus \bar{\omega}_\alpha| \leq \frac{1}{2} |\omega \cap Q(\alpha)|,$$

which implies that

(3.108) \hspace{2cm} |\tilde{\omega}_\alpha| = |\omega \cap Q(\alpha)| - |(\omega \cap Q(\alpha)) \setminus \tilde{\omega}_\alpha| \geq \frac{1}{2} |\omega \cap Q(\alpha)| \geq \frac{1}{2} \gamma L^n > 0,

thanks anew to the thickness property (3.75). By using again spherical coordinates as in (3.87) and (3.88), we observe that

(3.109) \hspace{2cm} |\tilde{\omega}_\alpha| = |\tilde{\omega}_\alpha \cap Q(\alpha)|

$$= n^{n/2} L^n \left( \int_{\mathbb{S}^{n-1}} I_{\omega_\alpha \cap Q(\alpha)}(x_\alpha + \sqrt{n} Lr \sigma) \, d\sigma \right) r^{n-1} \, dr \leq n^{n/2} L^n \int_{\mathbb{S}^{n-1}} |I_\sigma| \, d\sigma,$$

where

(3.110) \hspace{2cm} \tilde{I}_\sigma = \{ r \in [0,1] : x_\alpha + \sqrt{n} Lr \sigma \in \tilde{\omega}_\alpha \cap Q(\alpha) \}.

As in (3.90), the estimate (3.109) implies that there exists $\sigma_0 \in \mathbb{S}^{n-1}$ such that

(3.111) \hspace{2cm} |\tilde{\omega}_\alpha| \leq n^{n/2} L^n |\mathbb{S}^{n-1}| |\tilde{I}_{\sigma_0}|.

We deduce from (3.108) and (3.111) that

(3.112) \hspace{2cm} |\tilde{I}_{\sigma_0}| \geq 2 \gamma n^{n/2} |\mathbb{S}^{n-1}| > 0.

Applying anew Lemma 3.4 with $I = [0,1]$, $E = \tilde{I}_{\sigma_0} \subset [0,1]$ satisfying $|E| = |\tilde{I}_{\sigma_0}| > 0$, and the analytic function $\Phi = \phi$ defined in (3.92) satisfying $|\phi(0)| \geq 1$, we obtain

(3.113) \hspace{2cm} L^{n/2} \frac{\sup_{x \in [0,1]} |f(x_\alpha + \sqrt{n} Lx_{\sigma_0})|}{\| f \|_{L^2(Q(\alpha))}} \leq \left( \frac{C}{|\tilde{I}_{\sigma_0}|} \right)^{\frac{\ln M}{\ln 2}} \frac{\sup_{x \in \tilde{I}_{\sigma_0}} |f(x_\alpha + \sqrt{n} Lx_{\sigma_0})|}{\| f \|_{L^2(Q(\alpha))}},

where $M$ denotes the constant defined in (3.94). By (3.112) and (3.113) it follows that

(3.114) \hspace{2cm} \sup_{x \in [0,1]} |f(x_\alpha + \sqrt{n} Lx_{\sigma_0})| \leq \left( \frac{2C n^{n/2} |\mathbb{S}^{n-1}|}{\gamma} \right)^{\frac{\ln M}{\ln 2}} \sup_{x \in \tilde{I}_{\sigma_0}} |f(x_\alpha + \sqrt{n} Lx_{\sigma_0})| \leq M \frac{1}{\ln 2} \ln(2C n^{n/2} |\mathbb{S}^{n-1}|/\gamma) \sup_{x \in \tilde{I}_{\sigma_0}} |f(x_\alpha + \sqrt{n} Lx_{\sigma_0})|.$
According to (3.110),
\[
(3.115) \quad \sup_{x \in \tilde{I}_{\sigma_0}} |f(x_{\alpha} + \sqrt{n} \mathbf{L} x_{\sigma_0})| \leq \|f\|_{L^\infty(\tilde{\omega}_\alpha \cap Q(\alpha))}.
\]

It follows from (3.97), (3.114) and (3.115) that
\[
(3.116) \quad \|f\|_{L^\infty(Q(\alpha))} \leq M \frac{1}{\ln 2} \ln(2Cn^{n^2} / |\mathbb{S}^{n-1}| / \gamma) \|f\|_{L^\infty(\tilde{\omega}_\alpha \cap Q(\alpha))}.
\]

On the other hand, (3.106) implies
\[
(3.117) \quad \|f\|_{L^\infty(\tilde{\omega}_\alpha \cap Q(\alpha))} \leq \frac{2}{|\omega \cap Q(\alpha)|} \int_{\omega \cap Q(\alpha)} |f(x)| \, dx.
\]

We deduce from (3.116), (3.117) and the Cauchy–Schwarz inequality that
\[
(3.118) \quad \|f\|_{L^2(Q(\alpha))} \leq L^{n/2} \|f\|_{L^\infty(Q(\alpha))}
\]
\[
\leq \frac{2L^{n/2}}{|\omega \cap Q(\alpha)|} M \frac{1}{\ln 2} \ln(2Cn^{n^2} / |\mathbb{S}^{n-1}| / \gamma) \int_{\omega \cap Q(\alpha)} |f(x)| \, dx
\]
\[
\leq \frac{2L^{n/2}}{|\omega \cap Q(\alpha)|^{1/2}} M \frac{1}{\ln 2} \ln(2Cn^{n^2} / |\mathbb{S}^{n-1}| / \gamma) \|f\|_{L^2(\omega \cap Q(\alpha))}.
\]

By the thickness property (3.75), it follows from (3.103), (3.104) and (3.118) that
\[
(3.119) \quad \|f\|_{L^2(Q(\alpha))}^2 \leq \frac{4}{\gamma} M \frac{2}{\ln 2} \ln(2Cn^{n^2} / |\mathbb{S}^{n-1}| / \gamma) \|f\|_{L^2(\omega \cap Q(\alpha))}^2
\]
\[
\leq \frac{4}{\gamma} \left(4L^{n/2} \tilde{c}_n (\delta_n^{-1} L^{-1}) e^{\delta_n L \sqrt{N}} \right)^2 M \frac{1}{\ln 2} \ln(2Cn^{n^2} / |\mathbb{S}^{n-1}| / \gamma) \|f\|_{L^2(\omega \cap Q(\alpha))}^2.
\]

With
\[
(3.120) \quad \kappa_n(L, \gamma) = \frac{2L^{n/2}}{\gamma^{1/2}} \left( \frac{2Cn^{n^2} / |\mathbb{S}^{n-1}|}{\gamma} \right)^{\frac{\ln(4L) / \ln(2Cn^{n^2} / |\mathbb{S}^{n-1}| / \gamma)}{\ln 2} \frac{\ln(\tilde{c}_n (\delta_n^{-1} L^{-1}))}{\ln 2}} > 0,
\]
we deduce from (3.119) that there exists a universal constant \( \tilde{\kappa}_n > 0 \) such that for any good cube \( Q(\alpha) \),
\[
(3.121) \quad \|f\|_{L^2(Q(\alpha))}^2 \leq \frac{1}{2} \kappa_n(L, \gamma) (\tilde{\kappa}_n / \gamma)^2 \tilde{\kappa}_n L \sqrt{N} \|f\|_{L^2(\omega \cap Q(\alpha))}^2.
\]

It follows from (3.82) and (3.121) that
\[
\|f\|_{L^2(\mathbb{R}^n)}^2 \leq 2 \int_{\bigcup_{\text{good cubes}} Q(\alpha)} |f(x)|^2 \, dx = 2 \sum_{\text{good cubes}} \|f\|_{L^2(Q(\alpha))}^2
\]
\[
\leq \kappa_n(L, \gamma) (\tilde{\kappa}_n / \gamma)^2 \tilde{\kappa}_n L \sqrt{N} \sum_{\text{good cubes}} \|f\|_{L^2(\omega \cap Q(\alpha))}^2.
\]
and
\[ \|f\|_{L^2(\mathbb{R}^n)}^2 \leq \kappa_n(L,\gamma)^2(\bar{\kappa}_n/\gamma)^2\bar{\kappa}_nL \sqrt{N} \int_{\omega\cup\text{good cubes } Q(\alpha)} |f(x)|^2 \, dx \]
\[ \leq \kappa_n(L,\gamma)^2(\bar{\kappa}_n/\gamma)^2\bar{\kappa}_nL \sqrt{N} \|f\|_{L^2(\omega)}^2. \]
This ends the proof of assertion (iii) in Theorem 2.1.

4. Applications to null-controllability of quadratic equations.
This section presents a null-controllability result for parabolic equations associated to a general class of hypoelliptic non-selfadjoint accretive quadratic operators from any thick set \( \omega \) of \( \mathbb{R}^n \) in any positive time \( T \). We begin by recalling a few facts about quadratic operators.

4.1. Miscellaneous facts about quadratic differential operators.
Quadratic operators are pseudodifferential operators defined in the Weyl quantization by
\[ q^w(x,D_x)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi} q\left(\frac{x+y}{2},\xi\right) f(y) \, dy \, d\xi, \]
with symbols \( q(x,\xi) \), for \( (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \), which are complex-valued quadratic forms
\[ q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}, \quad (x,\xi) \mapsto q(x,\xi). \]
These operators are non-selfadjoint differential operators in general, with simple and fully explicit expression since the Weyl quantization of the quadratic symbol \( x^\alpha \xi^\beta \), with \( (\alpha,\beta) \in \mathbb{N}^{2n}, |\alpha+\beta|=2 \), is the differential operator
\[ \frac{x^\alpha D^\beta_x + D^\beta_x x^\alpha}{2}, \quad D_x = i^{-1} \partial_x. \]
The maximal closed realization of the quadratic operator \( q^w(x, D_x) \) on \( L^2(\mathbb{R}^n) \), that is, the operator with domain
\[ D(q^w) = \{ f \in L^2(\mathbb{R}^n) : q^w(x, D_x)f \in L^2(\mathbb{R}^n) \}, \]
where \( q^w(x, D_x)f \) is defined in the distribution sense, is known to coincide with the graph closure of its restriction to the Schwartz space \( [26] \) pp. 425–426,
\[ q^w(x, D_x) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n). \]
Let \( q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C} \) be a quadratic form defined on the phase space and write \( q(\cdot,\cdot) \) for its associated polarized form. Classically, one associates to \( q \) a matrix \( F \in M_{2n}(\mathbb{C}) \) called its Hamilton map, or its fundamental matrix.
With $\sigma$ standing for the standard symplectic form
\begin{equation}
\sigma((x,\xi),(y,\eta)) = \langle \xi, y \rangle - \langle x, \eta \rangle = \sum_{j=1}^{n} (\xi_j y_j - x_j \eta_j),
\end{equation}
with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n)$ $\in \mathbb{C}^n$, the Hamilton map $F$ is defined as the unique matrix satisfying
\begin{equation}
\forall (x, \xi) \in \mathbb{R}^{2n}, \forall (y, \eta) \in \mathbb{R}^{2n}, \quad q((x, \xi), (y, \eta)) = \sigma((x, \xi), F(y, \eta)).
\end{equation}
We observe from the definition that
\[
F = \frac{1}{2} \begin{pmatrix}
\nabla_{\xi} \nabla_{x} q & \nabla_{\xi}^2 q \\
-\nabla_{x}^2 q & -\nabla_{x} \nabla_{\xi} q
\end{pmatrix},
\]
the matrices $\nabla_{\xi}^2 q = (a_{i,j})_{1 \leq i,j \leq n}$, $\nabla_{x}^2 q = (b_{i,j})_{1 \leq i,j \leq n}$, $\nabla_{\xi} \nabla_{x} q = (c_{i,j})_{1 \leq i,j \leq n}$, $\nabla_{x} \nabla_{\xi} q = (d_{i,j})_{1 \leq i,j \leq n}$ being defined by
\[
a_{i,j} = \partial^2_{x_i,x_j} q, \quad b_{i,j} = \partial^2_{\xi_i,\xi_j} q, \quad c_{i,j} = \partial^2_{\xi_i,x_j} q, \quad d_{i,j} = \partial^2_{x_i,\xi_j} q.
\]

The notion of singular space was introduced in \cite{23} by Hitrik and the third author. This vector subspace of $\mathbb{R}^{2n}$ is defined by
\begin{equation}
S = \left( \bigcap_{j=0}^{2n-1} \text{Ker}[(\text{Re } F)(\text{Im } F)^j] \right) \cap \mathbb{R}^{2n},
\end{equation}
where $F$ is the Hamilton map associated with the quadratic symbol $q$ and
\[
\text{Re } F = \frac{1}{2} (F + \overline{F}), \quad \text{Im } F = \frac{1}{2i} (F - \overline{F}).
\]
As pointed out in \cite{23,43}, the notion of singular space plays a basic role in the understanding of the spectral and hypoelliptic properties of the (possibly) non-elliptic quadratic operator $q^u(x, D_x)$, as well as the spectral and pseudospectral properties of certain classes of degenerate doubly characteristic pseudodifferential operators \cite{24}. In particular, \cite{23} Theorem 1.2.2] gives a complete description for the spectrum of any non-elliptic quadratic operator $q^u(x, D_x)$ whose Weyl symbol $q$ has a non-negative real part, and satisfies a condition of partial ellipticity along its singular space $S$:
\begin{equation}
(x, \xi) \in S, \quad q(x, \xi) = 0 \implies (x, \xi) = 0.
\end{equation}
Under these assumptions, the spectrum of $q^u(x, D_x)$ is shown to be composed of a countable number of eigenvalues with finite algebraic multiplicities. The structure of this spectrum is similar to that of elliptic quadratic operators \cite{46}. The condition of partial ellipticity is generally weaker than ellipticity (i.e. $S \subsetneq \mathbb{R}^{2n}$) and allows one to deal with more degenerate situations.

An important class of quadratic operators satisfying (4.6) is those with $S = \{0\}$. In this case, partial ellipticity trivially holds. More specifically,
these quadratic operators have been shown in [43, Theorem 1.2.1] to be hypoelliptic and to enjoy global subelliptic estimates of the type

\[
\exists C > 0, \forall f \in \mathcal{S}(\mathbb{R}^n), \quad \|\langle (x, D_x) \rangle^{2(1-\delta)} f \|_{L^2(\mathbb{R}^n)} \leq C(\|q^w(x, D_x)f\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)}),
\]

where \(\langle (x, D_x) \rangle^2 = 1 + |x|^2 + |D_x|^2\), with a sharp loss of derivatives \(0 \leq \delta < 1\) with respect to the elliptic case (case \(\delta = 0\)), which can be explicitly derived from the structure of the singular space.

When \(\text{Re} q \geq 0\), the singular space can also be defined in an equivalent way as the subspace in the phase space where all the Poisson brackets

\[
H^k_{\text{Im} q} \text{Re} q = \left( \frac{\partial \text{Im} q}{\partial \xi} \cdot \frac{\partial}{\partial x} - \frac{\partial \text{Im} q}{\partial x} \cdot \frac{\partial}{\partial \xi} \right)^k \text{Re} q, \quad k \geq 0,
\]

are vanishing:

\[
S = \{ X = (x, \xi) \in \mathbb{R}^{2n} : (H^k_{\text{Im} q} \text{Re} q)(X) = 0, k \geq 0 \}.
\]

This dynamical definition shows that the singular space corresponds exactly to the set of points \(X \in \mathbb{R}^{2n}\) where

\[
(4.8) \quad t \mapsto (\text{Re} q)(e^{tH_{\text{Im} q} X}),
\]

vanishes to infinite order at \(t = 0\). This is also equivalent to the fact that the function (4.8) is identically zero on \(\mathbb{R}\).

### 4.2. Null-controllability of hypoelliptic quadratic equations.

We study the class of quadratic operators with Weyl symbols having \(\text{Re} q \geq 0\), and with \(S = \{0\}\). According to the above description of the singular space, these quadratic operators are exactly those whose Weyl symbols have \(\text{Re} q \geq 0\) and become positive definite after averaging by the linear flow of the Hamilton vector field associated with its imaginary part:

\[
(4.9) \quad \forall T > 0, \quad \langle \text{Re} q \rangle_T(X) = \frac{1}{2T} \int_{-T}^{T} (\text{Re} q)(e^{tH_{\text{Im} q} X}) \, dt \gg 0.
\]

Here \(a \gg 0\) means that the quadratic form \(a\) is positive definite. These quadratic operators are also known [23, Theorem 1.2.1] to generate strongly continuous contraction semigroups \((e^{-tq^w})_{t \geq 0}\) on \(L^2(\mathbb{R}^n)\) which are smoothing in the Schwartz space for any positive time:

\[
\forall t > 0, \forall f \in L^2(\mathbb{R}^n), \quad e^{-tq^w} f \in \mathcal{S}(\mathbb{R}^n).
\]

In [25, Theorem 1.2], these regularizing properties were sharpened and these contraction semigroups were shown to be actually smoothing for any positive
time in the Gelfand–Shilov space $S^{1/2}(\mathbb{R}^n)$:

\begin{equation}
\exists C, t_0 > 0, \forall f \in L^2(\mathbb{R}^n), \forall \alpha, \beta \in \mathbb{N}^n, \forall 0 < t \leq t_0, \quad \| x^\alpha \partial_x^\beta (e^{-tq^w} f) \|_{L^\infty(\mathbb{R}^n)} \leq \frac{C^{1+|\alpha|+|\beta|}}{t^{2k_0+1/2}} (\alpha!)^{1/2} (\beta!)^{1/2} \| f \|_{L^2(\mathbb{R}^n)},
\end{equation}

where $s$ is a fixed integer satisfying $s > n/2$, and where $0 \leq k_0 \leq 2n - 1$ is the smallest integer with

\begin{equation}
\left( \bigcap_{j=0}^{k_0} \ker [(\text{Re } F)(\text{Im } F)]^j \right) \cap \mathbb{R}^{2n} = \{0\}.
\end{equation}

Thanks to the smoothing effect (4.10), the first and third authors [4, Proposition 4.1] have established that, for any quadratic form $q : \mathbb{R}_{x, \xi}^{2n} \to \mathbb{C}$ with $\text{Re } q \geq 0$ and $S = \{0\}$, the dissipation estimate (1.16) holds with $0 \leq k_0 \leq 2n - 1$ being the smallest integer satisfying (4.11). Let $\omega \subset \mathbb{R}^n$ be a measurable $\gamma$-thick set at scale $L > 0$ and choose the parameters as follows:

- $\Omega = \mathbb{R}^n$,
- $A = -q^w(x, D_x)$,
- $a = \frac{1}{2}$, $b = 1$,
- $t_0 > 0$ as in (1.16) and (1.17),
- $m = 2k_0 + 1$, where $k_0$ is defined in (4.11),
- any constant $c_1 > 0$ satisfying
  \[
  \forall k \geq 1, \quad C(\kappa/\gamma)^\kappa L^{\sqrt{k}} \leq e^{c_1 \sqrt{k}},
  \]
  where $C = C(L, \gamma, n) > 0$ and $\kappa = \kappa(n) > 0$ are defined in Theorem 2.1(iii),
- $c_2 = 1/C_0 > 0$, where $C_0 > 1$ is defined in (1.16) and (1.17).

We can then deduce from Theorem 1.6 the following observability estimate in any positive time:

\[ \exists C > 1, \forall T > 0, \forall f \in L^2(\mathbb{R}^n), \]
\[ \| e^{-Tq^w} f \|_{L^2(\mathbb{R}^n)}^2 \leq C \exp \left( \frac{C}{T^{2k_0+1}} \right) \int_0^T \| e^{-tq^w} f \|_{L^2(\omega)}^2 dt. \]

We therefore obtain the following null-controllability result:

**Theorem 4.1.** Let $q : \mathbb{R}_{x, \xi}^{2n} \to \mathbb{C}$ be a complex-valued quadratic form with $\text{Re } q \geq 0$ and $S = \{0\}$. If $\omega$ is a measurable thick subset of $\mathbb{R}^n$, then the parabolic equation

\[
\begin{aligned}
\partial_t f(t, x) + q^w(x, D_x) f(t, x) &= u(t, x) \mathbb{1}_\omega(x), \quad x \in \mathbb{R}^n, t > 0, \\
f|_{t=0} &= f_0 \in L^2(\mathbb{R}^n),
\end{aligned}
\]
with \( q^w(x, D_x) \) defined by the Weyl quantization of \( q \), is null-controllable from \( \omega \) in any positive time \( T \).

As in [4], the above theorem applies in particular to the parabolic equation associated to the Kramers–Fokker–Planck operator

\[
(4.12) \quad K = -\Delta_v + v^2/4 + v\partial_x - \partial_x V(x)\partial_v, \quad (x, v) \in \mathbb{R}^2,
\]

with a quadratic potential

\[ V(x) = \frac{1}{2}ax^2, \quad a \in \mathbb{R}^*, \]

which is an example of an accretive quadratic operator with \( S = \{0\} \). The theorem also applies in the very same way to hypoelliptic Ornstein–Uhlenbeck equations in \( L^2(\mathbb{R}^n, \rho(x)dx) \)-spaces, or to hypoelliptic Fokker–Planck equations in \( L^2(\mathbb{R}^n, \rho(x)^{-1}dx) \)-spaces with respect to (Gaussian) invariant measures \( \rho \). Indeed, as explained in [4] Sections 5 and 6 and after conjugation by \( \sqrt{\rho} \) or \( \sqrt{\rho}^{-1} \), the problem of null-controllability in weighted \( L^2 \)-spaces can be rephrased as a problem of null-controllability in the flat \( L^2(\mathbb{R}^n, dx) \)-space, to which Theorem [4.1] applies. We refer the reader to [4, 42] for detailed discussions of various physical models whose evolution is ruled by accretive quadratic operators with zero singular spaces and to which therefore the above null-controllability result applies.

The notion of thickness is a sufficient geometric condition for control subsets to guarantee null-controllability for a general class of evolution equations associated to hypoelliptic non-selfadjoint quadratic operators that includes the harmonic heat equation. It is therefore a natural question whether this condition is also sufficient. To the best of our knowledge, there is no known necessary and sufficient geometric condition to ensure the null-controllability of these evolution equations even in the case of the harmonic heat equation. Nevertheless, one can expect that the thickness condition is not sharp. Indeed, in contrast to the heat equation, the solutions of the above evolution equations do enjoy specific decay properties at infinity and one can conjecture that control subsets need not be distributed as much at infinity as required by the thickness condition. This conjecture will be investigated in future work.

5. Appendix

5.1. Gelfand–Shilov regularity. We refer the reader to [20, 21, 41, 48] and the references therein for extensive expositions of the Gelfand–Shilov regularity theory. The Gelfand–Shilov spaces \( S^\mu_0(\mathbb{R}^n) \) with \( \mu, \nu > 0, \mu + \nu \geq 1 \), are the spaces of functions \( f \in C^\infty(\mathbb{R}^n) \) satisfying

\[ \exists A, C > 0, \quad |\partial^\alpha_x f(x)| \leq CA^{[\alpha]}(\alpha!)^\mu e^{-\frac{A}{2}|x|^{2\nu}}, \quad x \in \mathbb{R}^n, \alpha \in \mathbb{N}^n, \]
or equivalently

\[ \exists A, C > 0, \sup_{x \in \mathbb{R}^n} |x^\beta \partial_x^\alpha f(x)| \leq CA^{\alpha + |\beta|}(\alpha!)^\mu (\beta!)^\nu, \quad \alpha, \beta \in \mathbb{N}^n. \]

These spaces may also be characterized as the spaces of functions \( f \in \mathcal{F}(\mathbb{R}^n) \) satisfying

\[ \exists C, \varepsilon > 0, |f(x)| \leq Ce^{-\varepsilon|x|^{1/\mu}}, \quad x \in \mathbb{R}^n, \quad |\hat{f}(\xi)| \leq Ce^{-\varepsilon|\xi|^{1/\mu}}, \quad \xi \in \mathbb{R}^n. \]

In particular, the Hermite functions belong to the symmetric Gelfand–Shilov space \( S_{1/2}^{1/2}(\mathbb{R}^n) \). More generally, the symmetric Gelfand–Shilov spaces \( S_{\mu}^{\mu}(\mathbb{R}^n) \), with \( \mu \geq 1/2 \) can be nicely characterized through the decomposition into the Hermite basis \( (\Phi_\alpha)_{\alpha \in \mathbb{N}^n} \) (see e.g. [48, Proposition 1.2])

\[ f \in S_{\mu}^{\mu}(\mathbb{R}^n) \iff f \in L^2(\mathbb{R}^n), \exists t_0 > 0, \|\langle f, \Phi_\alpha \rangle_{L^2} \exp(t_0|\alpha|^{1/\mu})\|_{l^2(\mathbb{N}^n)} < +\infty \]

\[ \iff f \in L^2(\mathbb{R}^n), \exists t_0 > 0, \left\|e^{t_0\mathcal{H}^{1/2}_1}f\right\|_{L^2(\mathbb{R}^n)} < +\infty, \]

where \( \mathcal{H} = -\Delta_x + |x|^2 \) stands for the harmonic oscillator.

### 5.2. Remez inequality.

The classical Remez inequality [44] (see also [16, 17]) is the following estimate providing a bound on the maximum of the absolute value of a polynomial function \( P \in \mathbb{R}[X] \) of degree \( d \) on \([-1, 1]\) by the maximum of its absolute value on any measurable subset \( E \subset [-1, 1] \) with \( 0 < |E| < 2 \):

\[ \sup_{[-1,1]} |P| \leq T_d \left( \frac{4 - |E|}{|E|} \right) \sup_E |P|, \tag{5.1} \]

where

\[ T_d(X) = \frac{d}{2} \sum_{k=0}^{[d/2]} (-1)^k \frac{(d - k - 1)!}{k!(d - 2k)!} 2^{d-2k} X^{d-2k} \]

\[ = \sum_{k=0}^{[d/2]} \binom{d}{2k} (X^2 - 1)^k X^{d-2k} \tag{5.2} \]

denotes the \( d \)th Chebyshev polynomial function of the first kind (see e.g. [7, Chapter 2]). We also recall from [7, Chapter 2] the definition of the Chebyshev polynomial functions of the second kind:

\[ \forall d \in \mathbb{N}, \quad U_d(X) = \sum_{k=0}^{[d/2]} (-1)^k \binom{d - k}{k} 2^{d-2k} X^{d-2k} \tag{5.3} \]

and

\[ \forall d \in \mathbb{N}^*, \quad U_{d-1}(X) = \frac{1}{d} T'_d(X). \tag{5.4} \]
The Remez inequality was extended to the multi-dimensional case in [10] (see also [19, formula (4.1)] and [30]) as follows: for all convex bodies \( K \subseteq \mathbb{R}^n \), measurable subsets \( E \subseteq K \) with \( 0 < |E| < |K| \) and \( P \in \mathbb{R}[X_1, \ldots, X_n] \) of degree \( d \),

\[
(5.5) \quad \sup_K |P| \leq T_d \left( \frac{1 + (1 - |E|/|K|)^{1/n}}{1 - (1 - |E|/|K|)^{1/n}} \right) \sup_E |P|.
\]

By recalling that all the zeros of Chebyshev polynomial functions of the first and second kind are simple and contained in \((-1, 1)\), we observe from (5.2) and (5.4) that \( T_d \) is increasing on \([1, +\infty)\) and for all \( d \in \mathbb{N}, x \geq 1 \),

\[
(5.6) \quad 1 = T_d(1) \leq T_d(x) = \sum_{k=0}^{[d/2]} \binom{d}{2k} (x - 1)^k (x + 1)^{d - 2k} \leq \sum_{k=0}^{[d/2]} \left( \frac{d}{2k} \right) x^k (x + 1)^{d - 2k} = \sum_{k=0}^{[d/2]} \left( \frac{d}{2k} \right) 2^k x^d \leq (2x)^d \sum_{k=0}^{[d/2]} 2^k \leq (4x)^d,
\]

because \( \binom{d}{2k} \leq \sum_{j=0}^{d} \binom{d}{j} = 2^d \). Since

\[
\sup_K |Q| \leq \sup_K |\text{Re} \, Q| + \sup_K |\text{Im} \, Q|,
\]

\[
\sup_E |\text{Re} \, Q| + \sup_E |\text{Im} \, Q| \leq 2 \sup_E |Q|,
\]

we deduce from (5.5) and (5.6) that for all convex bodies \( K \subseteq \mathbb{R}^n \), measurable subsets \( E \subseteq K \) with \( 0 < |E| < |K| \), and \( Q \in \mathbb{C}[X_1, \ldots, X_n] \) of degree \( d \),

\[
(5.7) \quad \sup_K |Q| \leq 2^{2^{d+1}} \left( \frac{1 + (1 - |E|/|K|)^{1/n}}{1 - (1 - |E|/|K|)^{1/n}} \right) \sup_E |Q|.
\]

Thanks to this estimate, we can prove that the \( L^2 \)-norm \( \| \cdot \|_{L^2(\omega)} \) on any measurable subset \( \omega \subseteq \mathbb{R}^n \) with \( |\omega| > 0 \) defines a norm on the finite-dimensional vector space \( \mathcal{E}_N \) defined in (2.1). Indeed, let \( f \in \mathcal{E}_N \) satisfy \( \|f\|_{L^2(\omega)} = 0 \). According to (3.1) and (3.6), there exists \( Q \in \mathbb{C}[X_1, \ldots, X_n] \) such that

\[
\forall (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad f(x_1, \ldots, x_n) = Q(x_1, \ldots, x_n) e^{- (x_1^2 + \cdots + x_n^2)/2}.
\]

The condition \( \|f\|_{L^2(\omega)} = 0 \) first implies that \( f = 0 \) almost everywhere in \( \omega \), and therefore \( Q = 0 \) almost everywhere in \( \omega \). We deduce from (5.7) that \( Q \) has to be zero on any convex body \( K \) with \( |K \cap \omega| > 0 \), and therefore is zero everywhere. We conclude that \( \| \cdot \|_{L^2(\omega)} \) indeed defines a norm on \( \mathcal{E}_N \).

\(^{(3)}\) A compact convex subset of \( \mathbb{R}^n \) with non-empty interior.
On the other hand, the Remez inequality is a key ingredient in the proof of the following lemma needed for the proof of Theorem 2.1.

**Lemma 5.1.** Let $R > 0$ and let $\omega \subset \mathbb{R}^n$ be a measurable subset satisfying $|\omega \cap B(0, R)| > 0$. Then for all $P \in \mathbb{C}[X_1, \ldots, X_n]$ of degree $d$,

$$
\| P \|_{L^2(B(0,R))} \leq \frac{2^{2d+1}}{\sqrt{3}} \sqrt{\frac{4|B(0,R)|}{|\omega \cap B(0,R)|}} \times \left( \frac{1}{1 - \left( 1 - \frac{|\omega \cap B(0,R)|}{4|B(0,R)|} \right)^{1/n}} \right)^d \| P \|_{L^2(\omega \cap B(0,R))}.
$$

**Proof.** Let $P \in \mathbb{C}[X_1, \ldots, X_n] \setminus \{0\}$ be of degree $d$ and let $R > 0$. We consider the subset

$$
(5.8) \quad E_{\varepsilon} = \left\{ x \in B(0, R) : |P(x)| \leq 2^{-2d-1} F\left( \frac{\varepsilon}{|B(0,R)|} \right)^{-d} \sup_{B(0,R)} |P| \right\}
$$

for all $0 < \varepsilon \leq B(0, R)$, and the decreasing function

$$
(5.9) \quad \forall 0 < t \leq 1, \quad F(t) = \frac{1 + (1-t)^{1/n}}{1 - (1-t)^{1/n}} \geq 1.
$$

The estimate

$$
2^{-2d-1} F\left( \frac{\varepsilon}{|B(0,R)|} \right)^{-d} < 1
$$

implies that $|E_{\varepsilon}| < |B(0, R)|$. We first check that $|E_{\varepsilon}| \leq \varepsilon$. If $|E_{\varepsilon}| > 0$, it follows from (5.7) that

$$
(5.10) \quad \sup_{B(0,R)} |P| \leq 2^{2d+1} F\left( \frac{|E_{\varepsilon}|}{|B(0,R)|} \right)^d \sup_{E_{\varepsilon}} |P| \leq F\left( \frac{|E_{\varepsilon}|}{|B(0,R)|} \right)^d F\left( \frac{\varepsilon}{|B(0,R)|} \right)^{-d} \sup_{B(0,R)} |P|.
$$

From (5.10) we obtain

$$
(5.11) \quad F\left( \frac{\varepsilon}{|B(0,R)|} \right) \leq F\left( \frac{|E_{\varepsilon}|}{|B(0,R)|} \right).
$$

As $F$ is a decreasing function, we deduce from (5.11) that indeed

$$
(5.12) \quad \forall 0 < \varepsilon \leq B(0, R), \quad |E_{\varepsilon}| \leq \varepsilon.
$$

Let $\omega \subset \mathbb{R}^n$ be a measurable subset satisfying $|\omega \cap B(0, R)| > 0$. We consider the parameter

$$
(5.13) \quad 0 < \varepsilon_0 = \frac{1}{4} |\omega \cap B(0,R)| < |B(0,R)|.
$$
Setting

\[(5.14) \quad G_{\varepsilon_0} = \left\{ x \in B(0, R) : |P(x)| > 2^{-2d-1} F\left( \frac{\varepsilon_0}{|B(0, R)|} \right)^{-d} \sup_{B(0, R)} |P| \right\}, \]

we observe that

\[(5.15) \quad \int_{\omega \cap B(0, R)} |P(x)|^2 \, dx \geq \int_{\omega \cap B(0, R)} 1_{G_{\varepsilon_0}}(x) |P(x)|^2 \, dx \]

\[\geq 2^{-4d-2} F\left( \frac{\varepsilon_0}{|B(0, R)|} \right)^{-2d} \left( \sup_{B(0, R)} |P| \right)^2 |\omega \cap G_{\varepsilon_0}|.\]

We deduce from (5.8), (5.12) and (5.14) that

\[|\omega \cap G_{\varepsilon_0}| = |G_{\varepsilon_0}| - \left\{ x \in B(0, R) \setminus \omega : |P(x)| > \frac{\sup_{B(0, R)} |P|}{2^{2d+1} F(\varepsilon_0/|B(0, R)|)^d} \right\} \]

\[\geq (|B(0, R)| - |E_{\varepsilon_0}|) - |B(0, R) \setminus \omega| \]

\[\geq |B(0, R)| - \frac{1}{4}|\omega \cap B(0, R)| - (|B(0, R)| - |\omega \cap B(0, R)|),\]

that is,

\[(5.16) \quad |\omega \cap G_{\varepsilon_0}| \geq \frac{3}{4} |\omega \cap B(0, R)| > 0.\]

It follows from (5.13), (5.15) and (5.16) that

\[\|P\|^2_{L^2(B(0, R))} \leq |B(0, R)| \left( \sup_{B(0, R)} |P| \right)^2 \]

\[\leq 2^{4d+2} \frac{4|B(0, R)|}{3|\omega \cap B(0, R)|} F\left( \frac{|\omega \cap B(0, R)|}{4|B(0, R)|} \right)^{2d} \int_{\omega \cap B(0, R)} |P(x)|^2 \, dx.\]

Therefore

\[\|P\|^2_{L^2(B(0, R))} \leq 2^{2d+1} \frac{\sqrt{3}}{\sqrt[4d+2]{4|B(0, R)|}} F\left( \frac{|\omega \cap B(0, R)|}{4|B(0, R)|} \right)^d \|P\|_{L^2(\omega \cap B(0, R))}.\]

This ends the proof of Lemma 5.1.

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