EQUIVARIANT PICARD GROUPS OF $C^*$-ALGEBRAS WITH
FINITE DIMENSIONAL $C^*$-HOPF ALGEBRA COACTIONS

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Abstract. Let $A$ be a $C^*$-algebra and $H$ a finite dimensional $C^*$-Hopf algebra with its dual $C^*$-Hopf algebra $H^0$. Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. We shall define the $(\rho, u, H)$-equivariant Picard group of $A$, which is denoted by $\text{Pic}^{\rho, u}_H(A)$, and discuss basic properties of $\text{Pic}^{\rho, u}_H(A)$. Also, we suppose that $(\rho, u)$ is the coaction of $H^0$ on the unital $C^*$-algebra $A$, that is, $u = 1 \otimes 1^0$. We investigate the relation between $\text{Pic}(A^*)$, the ordinary Picard group of $A^*$ and $\text{Pic}^{\rho, u}_H(A^*)$ where $A^*$ is the stable $C^*$-algebra of $A$ and $\rho^*$ is the coaction of $H^0$ on $A^*$ induced by $\rho$. Furthermore, we shall show that $\text{Pic}^{\rho, u}_H(A \rtimes_{\rho, u} H)$ is isomorphic to $\text{Pic}^{\rho, u}_H(A)$, where $\hat{\rho}$ is the dual coaction of $H$ on the twisted crossed product $A \rtimes_{\rho, u} H$ of $A$ by the twisted coaction $(\rho, u)$ of $H^0$ on $A$.

1. Introduction

Let $A$ be a $C^*$-algebra and $H$ a finite dimensional $C^*$-Hopf algebra with its dual $C^*$-Hopf algebra $H^0$. Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. We shall define the $(\rho, u, H)$-equivariant Picard group of $A$, which is denoted by $\text{Pic}^{\rho, u}_H(A)$. Also, we shall give a similar result to the ordinary Picard group as follows: Let $\text{Aut}^{\rho, u}_H(A)$ be the group of all automorphisms $\alpha$ of $A$ satisfying that $(\alpha \otimes \text{id}) \circ \rho = \rho \circ \alpha$ and let $\text{Int}^{\rho, u}_H(A)$ be the normal subgroup of $\text{Aut}^{\rho, u}_H(A)$ consisting of all generalized inner automorphisms $\text{Ad}(v)$ of $A$ satisfying that $\rho(v) = v \otimes 1^0$, where $v$ is a unitary element in the multiplier algebra $M(A)$ of $A$. Then we have the following exact sequence:

$$1 \longrightarrow \text{Int}^{\rho, u}_H(A) \longrightarrow \text{Aut}^{\rho, u}_H(A) \longrightarrow \text{Pic}^{\rho, u}_H(A).$$

Especially, let $A^*$ be a stable $C^*$-algebra of a unital $C^*$-algebra $A$ and $\rho$ a coaction of $H^0$ on $A$. Also, let $\rho^*$ be the coaction of $H^0$ on $A^*$ induced by a coaction $\rho$ of $H^0$ on $A$. Then under a certain condition, we can obtain the exact sequence

$$1 \longrightarrow \text{Int}^{\rho}_H(A^*) \longrightarrow \text{Aut}^{\rho}_H(A^*) \longrightarrow \text{Pic}^{\rho}_H(A^*) \longrightarrow 1.$$

In order to do this, we shall extend definitions and results in the case of unital $C^*$-algebras to ones in the case of non-unital $C^*$-algebras in the section of Preliminaries. Using this result, we shall investigate the relation between $\text{Pic}(A^*)$, the ordinary Picard group of $A^*$ and $\text{Pic}^{\rho}_H(A^*)$, the $(\rho^*, H)$-equivariant Picard group of $A^*$. Furthermore, we shall show that $\text{Pic}^{\rho, u}_H(A \rtimes_{\rho, u} H)$ is isomorphic to $\text{Pic}^{\rho, u}_H(A)$, where $\hat{\rho}$ is the dual coaction of $H$ on the twisted crossed product $A \rtimes_{\rho, u} H$ of $A$ by the twisted coaction $(\rho, u)$.

2. Preliminaries

Let $H$ be a finite dimensional $C^*$-Hopf algebra. We denote its comultiplication, counit and antipode by $\Delta, \epsilon$ and $S$, respectively. We shall use Sweedler’s notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for any $h \in H$ which suppresses a possible summation when we write comultiplications. We denote by $N$ the dimension of $H$. Let $H^0$ be the dual $C^*$-Hopf algebra of $H$. We denote its comultiplication, counit and antipode by $\Delta^0, \epsilon^0$ and $S^0$, respectively. There is the distinguished projection $e$ in $H$. We denote
that \( e \) is the Haar trace on \( H^0 \). Also, there is the distinguished projection \( \tau \) in \( H^0 \), which is the Haar trace on \( H \). Since \( H \) is finite dimensional, \( H \cong \bigoplus_{k=1}^L M_{d_k}(C) \) and \( H^0 \cong \bigoplus_{k=1}^L M_{d_k}(C) \) as \( C^* \)-algebras. Let \( \{ \phi^k_{ij} : k = 1, 2, \ldots, L, i, j = 1, 2, \ldots, f_k \} \) be a system of matrix units of \( H \). Also let \( \{ \omega^k_{ij} : k = 1, 2, \ldots, L, i, j = 1, 2, \ldots, d_k \} \) be systems of matrix units and comatrix units of \( H^0 \), respectively.

Let \( A \) be a \( C^* \)-algebra and \( M(A) \) its multiplier algebra. Let \( p, q \) be projections in \( A \). If \( p \) and \( q \) are Murray-von Neumann equivalent, then we denote it by \( p \sim q \) in \( A \). We denote by \( \text{id}_A \) and \( 1_A \) the identity map on \( A \) and the unit element in \( A \), respectively. We simply denote them by \( \text{id} \) and \( 1 \) if no confusion arises. Modifying Blattner, Cohen and Montgomery [10, Definition 2.1], we shall define a weak coaction of \( H^0 \) on \( A \).

**Definition 2.1.** By a weak coaction of \( H^0 \) on \( A \), we mean a \( * \)-homomorphism \( \rho : A \to A \otimes H^0 \) satisfying the following conditions:

1. \( \rho(A)(A \otimes H^0) = A \otimes H^0 \),
2. \( (\text{id} \otimes \Delta^0)(\rho(x)) = x \) for any \( x \in A \).

By a coaction of \( H^0 \) on \( A \), we mean a weak coaction \( \rho \) such that

3. \( (\rho \otimes \text{id}) \circ \rho = (\text{id} \otimes \Delta^0) \circ \rho \).

By Condition (1) in Definition 2.1 for any approximate unit \( \{ u_n \} \) of \( A \) and \( x \in A \otimes H^0 \), \( \rho(u_n)x \to x \) \((n \to \infty)\). Hence \( \rho(1) = 1 \otimes 1^0 \) when \( A \) is unital. Since \( H^0 \) is finite dimensional, \( M(A \otimes H^0) \cong M(A) \otimes H^0 \). We identify \( M(A \otimes H^0) \) with \( M(A) \otimes H^0 \). We also identify \( M(A \otimes H^0 \otimes H^0) \) with \( M(A) \otimes H^0 \otimes H^0 \). Let \( \rho \) be a weak coaction of \( H^0 \) on \( A \). By Jensen and Thomsen [12, Corollary 1.1.15], there is the unique strictly continuous homomorphism \( \overline{\rho} : M(A) \to M(A) \otimes H^0 \) extending \( \rho \).

**Lemma 2.1.** With the above notations, \( \overline{\rho} \) is a weak coaction of \( H^0 \) on \( M(A) \).

**Proof.** Clearly \( \overline{\rho} \) is a \( * \)-homomorphism of \( M(A) \) to \( M(A) \otimes H^0 \). Let \( \{ u_n \} \) be an approximate unit of \( A \). Then by Condition (1) in Definition 2.1 \( \{ \rho(u_n) \} \) is an approximate unit of \( A \otimes H^0 \). Hence \( \overline{\rho}(1) = 1 \otimes 1^0 \). Since \( H^0 \) is finite dimensional, \( \text{id} \otimes \Delta^0 \) is strictly continuous. Hence \( \overline{\rho} \) satisfies Condition (2) in Definition 2.1. \( \square \)

Let \( \rho \) be a weak coaction of \( H^0 \) on \( A \) and \( u \) a unitary element in \( M(A) \otimes H^0 \). Following Masuda and Tomatsu [19, Section 3], we shall define a twisted coaction of \( H^0 \) on \( A \).

**Definition 2.2.** The pair \( (\rho, u) \) is a twisted coaction of \( H^0 \) on \( A \) if the following conditions hold:

1. \( (\rho \otimes \text{id}) \circ \rho = \text{Ad}(u) \circ (\text{id} \otimes \Delta^0) \circ \rho \),
2. \( (u \otimes 1^0)(\text{id} \otimes \Delta^0 \otimes \text{id})(u) = (\rho \otimes \text{id} \otimes \text{id})(u)(\text{id} \otimes \text{id} \otimes \Delta^0)(u) \),
3. \( (\text{id} \otimes \text{id} \otimes \Delta^0)(u) = (\text{id} \otimes \text{id} \otimes \text{id})(u) = 1 \otimes 1^0 \).

**Remark 2.2.** Let \( (\rho, u) \) be a twisted coaction of \( H^0 \) on \( A \). Since \( H^0 \) is finite dimensional, \( \text{id}M(A) \otimes \Delta^0 \) is strictly continuous. Hence by Lemma 2.1 \( (\rho, u) \) satisfies Definition 2.2. Therefore, \( (\rho, u) \) is a twisted coaction of \( H^0 \) on \( M(A) \). Hence if \( \rho \) is a coaction of \( H^0 \) on \( A \), \( \overline{\rho} \) is a coaction of \( H^0 \) on \( M(A) \).

Let \( \text{Hom}(H, M(A)) \) be the linear space of all linear maps from \( H \) to \( M(A) \). Then by Sweedler [24, pp69-70], it becomes a unital convolution \( * \)-algebra. Similarly, we
define \( \text{Hom}(H \times H, M(A)) \). We note that \( \epsilon \) and \( \epsilon \otimes \epsilon \) are the unit elements in \( \text{Hom}(H, M(A)) \) and \( \text{Hom}(H \times H, M(A)) \), respectively.

Modifying Blattner, Cohen and Montgomery, [3, Definition 1.1] we shall define a weak action of \( H \) on \( A \).

**Definition 2.3.** By a weak action of \( H \) on \( A \), we mean a bilinear map \((h, x) \mapsto h \cdot x\) of \( H \times A \) to \( A \) satisfying the following conditions:

1. \( h \cdot (xy) = [h(1) \cdot x][h(2) \cdot y] \) for any \( h \in H, x, y \in A \),
2. \( [h \cdot u_{\alpha}x \mapsto \epsilon(h)x] \) for any approximate unit \( \{u_{\alpha}\} \) of \( A \) and \( x \in A \).
3. \( 1 \cdot x = x \) for any \( x \in A \),
4. \( [h \cdot x^* = S(h)^* \cdot x^*] \) for any \( h \in H, x \in A \).

By an action of \( H \) on \( A \), we mean a weak action of such that
5. \( h \cdot [l \cdot x] = (hl) \cdot x \) for any \( x \in A \) and \( h, l \in H \).

Since \( H \) is finite dimensional, as mentioned in [3, pp. 163], there is an isomorphism \( \iota \) of \( A \otimes H^0 \) onto \( \text{Hom}(H, M(A)) \) defined by \( \iota(x \otimes \phi)(h) = \phi(h)x \) for any \( x \in M(A), h \in H, \phi \in H^0 \). Also, we can define an isomorphism \( \jmath \) of \( A \otimes H^0 \otimes H^0 \) onto \( \text{Hom}(H \times H, M(A)) \) in the similar way to the above. We note that \( \iota(A \otimes H^0) = \text{Hom}(H, A) \) and \( \iota(A \otimes H^0 \otimes H^0) = \text{Hom}(H \times H, A) \).

For any \( x \in M(A) \otimes H^0 \) and \( y \in M(A) \otimes H^0 \), we denote \( \iota(x) \) and \( \iota(y) \) by \( \hat{x} \) and \( \hat{y} \), respectively.

Let a bilinear map \((h, x) \mapsto h \cdot x\) from \( H \times A \) to \( A \) be a weak action. For any \( x \in A \), let \( f_x \) be the linear map from \( H \) to \( A \) defined by \( f_x(h) = h \cdot x \) for any \( h \in H \). Let \( \rho \) be the linear map from \( A \) to \( A \otimes H^0 \) defined by \( \rho(x) = \iota^{-1}(f_x) \) for any \( x \in A \).

**Lemma 2.3.** With the above notations, \( \rho \) is a weak coaction of \( H^0 \) on \( A \).

**Proof.** By its definition, \( \rho \) is a \( \ast \)-homomorphism of \( A \) to \( A \otimes H^0 \) satisfying Conditions (2) in Definition 2.1. So, we have only to show that \( \rho \) satisfies Condition (1) in Definition 2.1. Let \( \{u_{\alpha}\} \) be an approximate unit of \( A \). We can write that \( \rho(u_{\alpha}) = \sum_j u_{\alpha j} \otimes \phi_j \), where \( u_{\alpha j} \in A \) and \( \{\phi_j\} \) is a basis of \( H^0 \) with \( \sum_j \phi_j = 1^0 \). Let \( \{h_j\} \) be the dual basis of \( H \) corresponding to \( \{\phi_j\} \). Then for any \( x \in A \) and \( j \), \( [h_j \cdot u_{\alpha}x \mapsto \epsilon(h_j)x] \) by Condition (2) in Definition 2.1. Since \( [h_j \cdot u_{\alpha}x = (id \otimes h_j)(\rho(u_{\alpha}))x = u_{\alpha j}x, u_{\alpha j}x \mapsto \epsilon(h_j)x] \) for any \( j \). Also, since \( \sum_j \phi_j = 1^0 \),

\[
1 = \phi_j(h_j) = \sum_i \phi_i(h_j) = 1^0(h_j) = \epsilon(h_j)
\]

for any \( j \). Hence \( u_{\alpha j}x \mapsto x \) for any \( j \). Therefore, for any \( x \in A \) and \( \phi \in H^0 \),

\[
\rho(u_{\alpha})(x \otimes \phi) = \sum_j u_{\alpha j}x \otimes \phi_j \phi = \sum_j x \otimes \phi_j \phi = x \otimes \phi.
\]

Thus \( \rho(A)(A \otimes H^0) = A \otimes H^0 \). \( \square \)

For any weak coaction \( \rho \) of \( H^0 \) on \( A \), we can define the bilinear map \((h, x) \mapsto h \cdot \rho x\) from \( H \times A \) to \( A \) by

\[
h \cdot \rho x = (id \otimes h)(\rho(x)) = \rho(\hat{x})(h).
\]

We shall prove that the above map is a weak action of \( H \) on \( A \).

**Lemma 2.4.** With the above notations, the linear map \((h, x) \mapsto h \cdot \rho x\) from \( H \times A \) to \( A \) is a weak action of \( H \) on \( A \).

**Proof.** We have only to show that the above linear map satisfies Condition (2) in Definition 2.1. Let \( \{u_{\alpha}\} \) be an approximate unit of \( A \). Then for any \( x \in A \otimes H^0 \),
Lemma 2.1. There is the weak coaction $H$ on $A$ and a basis of $H^0$. Then for any $a \in A$,

$$[h \cdot \rho u_a]a = (\text{id} \otimes h)(\rho(u_a))a = \sum_j u_{aj} \phi_j(h)a$$

$$= (\text{id} \otimes h)(\rho(u_a)(a \otimes 1^0)) \rightarrow \epsilon(h)a$$

since $\text{id} \otimes h$ is a bounded operator from $A \otimes H^0$ to $A$.

Remark 2.5. By the proofs of Lemmas 2.3 and 2.4. Condition (2) in Definition 2.3 is equivalent to the following Condition (2)’:

$$(2)' [h \cdot u_a]x \rightarrow \epsilon(h)x$$

for some approximate unit of $A$ and any $x \in A$. Also, if $A$ is unital, Condition (2) in Definition 2.3 means that $h \cdot 1 = \epsilon(h)$ for any $h \in H$.

Let $\rho$ be a weak coaction of $H^0$ on $A$. Then by Lemma 2.4 there is a weak action of $H$ on $A$. We call it the weak action of $H$ on $A$ induced by $\rho$. Also, by Lemma 2.7 there is the weak coaction $\rho$ of $H^0$ on $M(A)$, which is an extension of $\rho$ to $M(A)$. Hence we can obtain the action of $H^0$ on $M(A)$ induced by $\rho$. We can see that this action is an extension of the action induced by $\rho$ to $M(A)$.

Definition 2.4. Let $\sigma : H \times H \rightarrow M(A)$ be a bilinear map. $\sigma$ is a unitary cocycle for a weak action of $H$ on $A$ if $\sigma$ satisfies the following conditions:

1. $\sigma$ is a unitary element in $\text{Hom}(H \times H, M(A))$,
2. $\sigma$ is normal, that is, for any $h \in H$, $\sigma(h, 1) = \sigma(1, h) = \epsilon(h)1$,
3. (cocycle condition) For any $h, l, m \in H$,

$$[h(1), \sigma(l, m)] = \sigma(h(1), l)\sigma(h, m),$$

4. (twisted modular condition) For any $h, l \in H$, $x \in A$,

$$[h(1), \sigma(l, x)] = \sigma(h(1), l)\sigma(h, x),$$

where if necessary, we consider the extension of the weak action to $M(A)$.

We call a pair which is consisting of a weak action of $H$ on $A$ and its unitary cocycle, a twisted action of $H$ on $A$.

Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. Then we can consider the twisted action of $H$ on $A$ and its unitary cocycle $\hat{u}$ defined by

$$h \cdot \rho u_a x = \rho(x)(h) = (\text{id} \otimes h)(\rho(x))$$

for any $a \in A$ and $h \in H$. We call it the twisted action induced by $(\rho, u)$. Also, we can consider the twisted coaction $(\rho, u)$ of $H^0$ on $M(A)$ and the twisted action of $H$ on $M(A)$ induced by $(\rho, u)$. Let $\tilde{M}(A) \rtimes_{\rho, u} H$ be the twisted crossed product by the twisted action of $H$ on $M(A)$ induced by $(\rho, u)$. Let $x \rtimes_{\rho, u} h$ be the element in $M(A) \rtimes_{\rho, u} H$ induced by elements $x \in M(A)$, $h \in H$. Let $\tilde{A} \rtimes_{\rho, u} H$ be the set of all finite sums of elements in the form $x \rtimes_{\rho, u} h$, where $x \in A$, $h \in H$. By easy computations, we can see that $A \rtimes_{\rho, u} H$ is a closed two-sided ideal of $M(A) \rtimes_{\rho, u} H$.

We call it the twisted crossed product by $(\rho, u)$ and its element denotes by $x \rtimes_{\rho, u} h$, where $x \in A$ and $h \in H$. Let $E_{\tilde{A} \rtimes_{\rho, u} H}^0$ be the canonical conditional expectation from $M(A) \rtimes_{\rho, u} H$ onto $M(A)$ defined by

$$E_{\tilde{A} \rtimes_{\rho, u} H}^0(x \rtimes_{\rho, u} h) = \tau(h)x$$

for any $x \in M(A)$ and $h \in H$. Let $A$ be the set of all triplets $(i, j, k)$, where $i, j = 1, 2, \ldots, d_k$ and $k = 1, 2, \ldots, N$ with $\sum_{k=1}^K d_k^2 = N$. Let $W_i = \sqrt{d_k} \rtimes_{\rho, u} w_{ij}$ for any $i = (i, j, k) \in A$. By [13] Proposition 3.18, $(W_i, W_j)_{I \in A}$ is a quasi-basis for $E_{\tilde{A} \rtimes_{\rho, u} H}^0$. We suppose that $A$ acts on a Hilbert space faithfully and nondegenerately.

Lemma 2.6. With the above notations, $M(A) \rtimes_{\rho, u} H = M(A \rtimes_{\rho, u} H)$. 

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Proof. By the definition of the multiplier algebras $M(A)$ and $M(A \rtimes_{\rho,u} H)$, it is clear that $M(A) \rtimes_{\rho,u} H \subset M(A \rtimes_{\rho,u} H)$ since $M(A) \rtimes_{\rho,u} H$ and $M(A \rtimes_{\rho,u} H)$ act on the same Hilbert space. We show that another inclusion. Let $x \in M(A \rtimes_{\rho,u} H)$. Then there is a bounded net $\{x_\alpha\}_{\alpha \in \Gamma} \subset M(A \rtimes_{\rho,u} H)$ such that $\{x_\alpha\}_{\alpha \in \Gamma}$ converges to $x$ strictly. Since $x_\alpha \in A \rtimes_{\rho,u} H$, $x_\alpha = \sum_I \hat{E}^\rho_{\alpha}(x_\alpha W^*_I)W_I$. By the definition of $\hat{E}^\rho_{\alpha}(x_\alpha W^*_I)$, we can see that $(1)' \cdot \cdot \cdot (n)'$ completely strictly continuous. Since $\hat{E}^\rho_{\alpha}(x_\alpha W^*_I)$, $\sum_I \hat{E}^\rho_{\alpha}(x_\alpha W^*_I)W_I$ converges to $x \in M(A \rtimes_{\rho,u} H)$. Therefore, $x = \sum_I \hat{E}^\rho_{\alpha}(x_\alpha W^*_I)W_I$ since $x_\alpha = \sum_I \hat{E}^\rho_{\alpha}(x_\alpha W^*_I)W_I$. It follows that $x \in M(A) \rtimes_{\rho,u} H$.

Remark 2.7. Let $(\hat{\rho})$ be the dual coaction of $\rho$ of $H$ on $M(A) \rtimes_{\rho,u} H$ and $(\hat{\rho})$ be the coaction of $H$ on $M(A \rtimes_{\rho,u} H)$. By Lemma 2.6 we can see that $(\hat{\rho}) = (\hat{\rho})$. Indeed, by Lemma 2.6 it suffices to show that $(\hat{\rho})(x \rtimes_{\rho,u} h) = (\hat{\rho})(x \rtimes_{\rho,u} h)$ for any $x \in M(A)$ and $h \in H$. Since $x \in M(A)$, there is a bounded net $\{x_\alpha\} \subset A$ such that $x_\alpha$ converges to $x$ strictly in $M(A)$. Then since $x_\alpha \rtimes_{\rho,u} h$ converges to $x \rtimes_{\rho,u} h$ strictly in $M(A) \rtimes_{\rho,u} H$ and $(\hat{\rho})$ is strictly continuous, $(\hat{\rho})(x \rtimes_{\rho,u} h) = \lim_{\alpha \to \infty} \hat{\rho}(x_\alpha \rtimes_{\rho,u} h) = \lim_{\alpha \to \infty} (x_\alpha \rtimes_{\rho,u} h(1)) \otimes h(2) = (\hat{\rho})(x \rtimes_{\rho,u} h)$.

Next, we extend [16] Theorem 3.3 to a twisted coaction of $H^0$ on a (non-unital) $C^*$-algebra $A$. Before doing it, we define the exterior equivalence for twisted coactions of a finite dimensional $C^*$-Hopf algebra $H^0$ on a $C^*$-algebra $A$.

Definition 2.5. Let $(\rho, u)$ and $(\sigma, v)$ be twisted coactions of $H^0$ on $A$. We say that $(\rho, u)$ is exterior equivalent to $(\sigma, v)$ if there is a unitary element $w \in M(A) \otimes H^0$ satisfying the following conditions:

1. $\sigma = \text{Ad}(w) \circ \rho$.
2. $v = (w \otimes 1^0)(\rho \otimes \text{id})(w)u(\text{id} \otimes \Delta^0)(w^*)$.

The above conditions (1), (2) are equivalent to the following, respectively:

1. $h \cdot (\cdot, \cdot)_\rho \cdot \cdot \cdot (n) = \hat{\omega}(h(1))h(2)(\cdot, \cdot)_\rho \cdot \cdot \cdot (n)(\hat{\omega}(h(3)))$ for any $a \in A$ and $h \in H$.
2. $\hat{\sigma}(h, l) = \hat{\omega}(h(1))h(2)(\cdot, \cdot)_\rho \cdot \cdot \cdot (n)(\hat{\omega}(h(3)))h(l(3))h(l(3)) \hat{\omega}(h(4))h(l(3))$ for any $h, l \in H^0$.

If $\rho$ and $\sigma$ are coactions of $H^0$ on $A$, Conditions (1), (2) and (1)', (2)' are following:

1. $\sigma = \text{Ad}(w) \circ \rho$. 

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So, we show the equation. By [16, Section 3],

That is, the coaction

Proposition 2.8. Let $A$ be a $C^*$-algebra and $H$ a finite dimensional $C^*$-Hopf algebra with its dual $C^*$-Hopf algebra $H^0$. Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. Then there are an isomorphism $\Psi$ of $M(A) \otimes M_N(C)$ onto $M(A) \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$ and a unitary element $U \in (M(A) \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0) \otimes H^0$ such that

That is, the coaction $\hat{\rho}$ of $H^0$ on $A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$ is exterior equivalent to the twisted coaction

$$(\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(C)}) \circ \Psi^{-1},$$

where we identify $A \rtimes H^0 \otimes H^0 \otimes M_N(C)$ with $A \rtimes M_N(C) \otimes H^0 \otimes H^0$.

Proof. By [16] Theorem 3.3, there are an isomorphism $\Psi$ of $M(A) \otimes M_N(C)$ onto $M(A) \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$ and a unitary element $U \in (M(A) \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0) \otimes H^0$ satisfying the required conditions except for the equation

$$\Psi(A \rtimes M_N(C)) = A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0.$$

So, we show the equation. By [16] Section 3,

$$\Psi([a_{IJ}]) = \sum_{I,J} V_I^J (a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} 1^0) V_J$$

for any $[a_{IJ}] \in A \otimes M_N(C)$, where $V_I = (1 \rtimes_{\hat{\rho}} \tau)(W_I \rtimes_{\hat{\rho}} 1^0)$ for any $I \in \Lambda$. Since $V_I \in M(A) \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$ for any $I \in \Lambda$, $\Psi(A \rtimes M_N(C)) \subset A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$. For any $z \in A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$, we can write that

$$z = \sum_{i=1}^n (x_i \rtimes_{\hat{\rho}} 1^0)(1 \rtimes_{\hat{\rho}} \tau)(y_i \rtimes_{\hat{\rho}} 1^0),$$

where $x_i, y_i \in M(A) \rtimes_{\rho,u} H$ for any $i$. Let $\{u_\alpha\}$ be an approximate unit of $A$. Then $(u_\alpha \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} 1^0)(x_i \rtimes_{\hat{\rho}} 1^0)$ and $(y_i \rtimes_{\hat{\rho}} 1^0)(u_\alpha \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} 1^0)$ are in $A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$ for any $i$ and $\alpha$. Hence $(A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} 1^0)(1 \rtimes_{\hat{\rho}} \tau)(A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} 1^0)$ is dense in $A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$. On the other hand, for any $x, y \in A \rtimes_{\rho,u} H$,

$$\Psi([E_{T}^{\alpha}(W_I x) E_{T}^{\alpha}(y W_I^*)]_{I,J}) = (x \rtimes_{\hat{\rho}} 1^0)(1 \rtimes_{\hat{\rho}} \tau)(y \rtimes_{\hat{\rho}} 1^0)$$

by the proof of [16] Theorem 3.3. Since $E_{T}^{\alpha}(A \rtimes_{\rho,u} H) = A$ and $E_{T}^{\alpha}$ is continuous by its definition, $A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0 \subset \Psi(A \otimes M_N(C))$. □
We extend [15, Theorem 6.4] to coactions of $H^0$ on a (non-unital) $C^*$-algebra. First, we recall a saturated coaction. We say that a coaction $\rho$ of $H^0$ on a unital $C^*$-algebra $A$ is saturated if the induced action from $\rho$ of $H$ on $A$ is saturated in the sense of [23, Definition 4.2].

Let $B$ be a $C^*$-algebra and $\sigma$ a coaction of $H^0$ on $B$. Let $B^\sigma = \{ b \in B \mid \sigma(b) = b \otimes 1^0 \}$, the fixed point $C^*$-subalgebra of $B$ for the coaction $\sigma$. We suppose that $B$ acts on a Hilbert space $\mathcal{H}$ non-degenerately and faithfully. Also, we suppose that $\sigma$ is saturated. Then the canonical conditional expectation $E_\sigma$ from $M(B)$ onto $M(B)_\sigma$ defined by $E_\sigma(x) = \epsilon \cdot \tau \cdot x$ for any $x \in M(B)$ is of Watatani index-finite type by [25, Theorem 4.3]. Thus there is a quasi-basis $\{(u_\alpha, u_\alpha^*)\}_{\alpha=1}^{\infty}$ of $E_\sigma$. Let $\{ v_\alpha \}$ be an approximate unit of $B^\sigma$. For any $x \in B$,

$$v_\alpha x = v_\alpha \sum_{i=1}^{n} E_\sigma(xu_i)u_i^* \rightarrow \sum_{i=1}^{n} E_\sigma(xu_i)u_i^* = x \quad (\alpha \rightarrow \infty)$$

since $E_\sigma(xu_i) \in B^\sigma$. Similarly $xv_\alpha \rightarrow x$ since $x = \sum_{i=1}^{n} u_i E_\sigma(u_i^* x)$. Thus $\{ v_\alpha \}$ is an approximate unit of $B$. Hence $B^\sigma$ acts on $\mathcal{H}$ non-degenerately and faithfully.

**Lemma 2.9.** With the above notations, we suppose that $\sigma$ is saturated. Then $M(B^\sigma) = M(B)_\sigma$.

**Proof.** By the above discussions, we may suppose that $B$ and $B^\sigma$ act on a Hilbert space non-degenerately and faithfully. Let $x \in M(B^\sigma)$. Then there is a bounded net $\{ a_\alpha \} \subset B^\sigma$ such that $a_\alpha \rightarrow x$ ($\alpha \rightarrow \infty$) strictly in $M(B^\sigma)$. Since any approximate unit of $B^\sigma$ is an approximate unit of $B$ by the above discussion, for any $y \in B^\sigma$,

$$\sigma(x)(y \otimes 1^0) = \sigma(xy) = \sigma(\lim_{\alpha \rightarrow \infty} a_\alpha y) = \lim_{\alpha \rightarrow \infty} \sigma(a_\alpha y) = \lim_{\alpha \rightarrow \infty} a_\alpha y \otimes 1^0 = xy \otimes 1^0.$$ 

Thus $x \in M(B)_\sigma$. Next, let $x \in M(B)_\sigma$. Then for any $b \in B^\sigma$, $xb$ and $bx$ are in $B$. Thus

$$\sigma(xb) = \sigma(x)\sigma(b) = (x \otimes 1^0)(b \otimes 1^0) = xb \otimes 1^0,$$

$$\sigma(bx) = \sigma(b)\sigma(x) = (b \otimes 1^0)(x \otimes 1^0) = bx \otimes 1^0.$$ 

Hence $x \in M(B^\sigma)$.

We suppose that $\mathcal{E}\!1_{\times_{\mathbb{C}}^{}\epsilon}$ $(\mathcal{E}\!1_{\times_{\mathbb{C}}^{}\epsilon} \otimes 1) \in (M(B)_\sigma \otimes \mathcal{H}) \otimes \mathcal{H}$. As mentioned in [15, Section 2], without the assumption of saturation for an action, all the statements in Sections 4, 5 and 6 in [15] hold. Hence by [15, Sections 4 and 5], $\sigma$ is saturated and there is a unitary element $w^\sigma \in M(B) \otimes \mathcal{H}$ satisfying that

$$w^\sigma((1 \times_{\mathbb{C}}^{} \epsilon) \otimes 1)w^\sigma = \mathcal{E}\!1_{\times_{\mathbb{C}}^{}\epsilon}.$$ 

Let $U^\sigma = w^\sigma(z^\sigma \otimes 1)$, where $z^\sigma = (\text{id}_{M(B)} \otimes \epsilon)(w^\sigma) \in M(B)_\sigma$. Then $U^\sigma \in M(B) \otimes \mathcal{H}$ and satisfies that

$$\tilde{U}^\sigma(1^0) = 1, \quad \tilde{U}^\sigma(\phi_{(1)}(a)\tilde{U}^\sigma(\phi_{(2)}) \in M(B)_\sigma$$

for any $a \in M(B)_\sigma$, $\phi \in H^0$. Let $\tilde{\omega}^\sigma$ be a bilinear map from $H^0 \times H^0$ to $M(B)$ defined by

$$\tilde{\omega}^\sigma(\phi, \psi) = \tilde{U}^\sigma(\phi_{(1)}(a)\tilde{U}^\sigma(\psi_{(1)}))\tilde{U}^\sigma(\phi_{(2)}(\psi_{(2)}))$$

for any $\phi, \psi \in H^0$. Then by [15, Lemma 5.4], $\tilde{\omega}^\sigma(\phi, \psi) \in M(B)_\sigma$ for any $\phi, \psi \in H^0$ and by [15, Corollary 5.3], the map

$$H^0 \times M(B)_\sigma \rightarrow M(B)_\sigma; (\phi, a) \mapsto \tilde{U}^\sigma(\phi_{(1)}(a)\tilde{U}^\sigma(\phi_{(2)}))$$

is a weak action of $H^0$ on $M(B)_\sigma$. Furthermore, by [15, Proposition 5.6] $\tilde{\omega}^\sigma$ is a unitary cocycle for the above weak action. Let $u^\sigma$ be the unitary element in $M(B)_\sigma \otimes \mathcal{H}$ induced by $\tilde{\omega}^\sigma$ and $\rho'$ the weak coaction of $H$ on $M(B)_\sigma$ induced
by the above weak action. Thus we obtain a twisted coaction \((\rho', u^\sigma)\) of \(H\) on \(M(B)^\varnothing\). Let \(\pi'\) be the map from \(M(B)^\varnothing \times_{\rho', u^\sigma} H^0\) to \(M(B)\) defined by 
\[
\pi'(a \rtimes_{\rho', u^\sigma} \phi) = a \tilde{\varnothing}^\sigma(\phi)
\]
for any \(a \in M(B)^\varnothing, \phi \in H^0\). Then by [15] Proposition 6.1 and Theorem 6.4, \(\pi'\) is an isomorphism of \(M(B)^\varnothing \times_{\rho', u^\sigma} H^0\) onto \(M(B)\) satisfying that 
\[
\pi' \circ \rho' = (\pi' \otimes \text{id}_{H^0}) \circ \tilde{\varnothing}^\sigma, \quad E^\sigma_{\check{\varnothing}^\sigma} = E_{\check{\varnothing}^\sigma} \circ \pi',
\]
where \(E^\sigma_{\check{\varnothing}^\sigma}\) is the canonical conditional expectation from \(M(B)^\varnothing \times_{\rho', u^\sigma} H^0\) onto \(M(B)^\varnothing\) and \(E_{\check{\varnothing}^\sigma}\) is the canonical conditional expectation from \(M(B)\) onto \(M(B)^\varnothing\).

Let \(\rho = \rho'|_{B^\sigma}\).

**Lemma 2.10.** With the above notations, \((\rho, u^\sigma)\) is a twisted coaction of \(H\) on \(B^\sigma\) and \(\rho = \rho'\).

**Proof.** By the definition of \(\rho\), for any \(a \in B^\sigma\), 
\[
\rho(a) = U^\sigma(a \otimes 1)U^{\sigma^*}.
\]
Since \(a \in B^\sigma \subset M(B^\sigma) = M(B)^\varnothing\) by Lemma 2.3, \(\rho(a) \in M(B)^\varnothing \otimes H\). On the other hand, since \(U^\sigma \in M(B) \otimes H\), \(\rho(a) \in B \otimes H\). Thus \(\rho(a) \in (M(B)^\varnothing \otimes H) \cap (B \otimes H) = B^\sigma \otimes H\). Hence \(\rho\) is a homomorphism of \(B^\sigma\) to \(B^\sigma \otimes H\). Since \((\rho \otimes \text{id}) \circ \rho' = \text{Ad}(u^\sigma) \circ (\text{id} \otimes \Delta) \circ \rho'\) and \(\rho(a) \in B^\sigma \otimes H\) for any \(a \in B^\sigma\), we can see that \((\rho \otimes \text{id}) \circ \rho = \text{Ad}(u^\sigma) \circ (\text{id} \otimes \Delta) \circ \rho\). By the definition of \(\rho', \rho'\) is strictly continuous on \(M(B)^\varnothing\). Hence for any approximate unit \(\{u_\alpha\}\) of \(B^\sigma\), 
\[
1 \otimes 1 = \rho'(1) = \rho'\left( \lim_{\alpha \to \infty} u_\alpha \right) = \lim_{\alpha \to \infty} \rho'(u_\alpha) = \lim_{\alpha \to \infty} \rho(u_\alpha),
\]
where the limits are taken under the strict topologies in \(M(B^\sigma)\) and \(M(B^\sigma) \otimes H\), respectively. This means that 
\[\rho(B^\sigma)(B^\sigma \otimes H) = B^\sigma \otimes H.\]
It follows that \((\rho, u^\sigma)\) is a twisted coaction of \(H\) on \(B^\sigma\). Furthermore, since \(\rho'\) is strictly continuous, \(\rho' = \rho\) on \(M(B^\sigma)\). \(\square\)

Let \(\pi = \pi'|_{B^\sigma \times_{\rho, u^\sigma} H^0}\).

**Lemma 2.11.** With the above notations, \(\pi\) is an isomorphism of \(B^\sigma \times_{\rho, u^\sigma} H^0\) onto \(B\) satisfying that 
\[
\sigma \circ \pi = (\pi \otimes \text{id}_{H^0}) \circ \tilde{\varnothing}^\sigma, \quad E^\sigma_{\check{\varnothing}^\sigma} = E_{\check{\varnothing}^\sigma} \circ \pi,
\]
where \(E^\sigma_{\check{\varnothing}^\sigma}\) is the canonical conditional expectation from \(B^\sigma \times_{\rho, u^\sigma} H^0\) onto \(B^\sigma\) and \(E_{\check{\varnothing}^\sigma}\) is the canonical conditional expectation from \(B^\sigma\) onto \(B^\sigma\). Furthermore, \(\pi' = \pi\).

**Proof.** Let \(E_{\check{\varnothing}^\sigma}\) be the canonical conditional expectation from \(M(B)\) onto \(M(B)^\varnothing\). By [15] Proposition 4.3 and Remark 4.9, \(\{(\sqrt{\mathcal{T}_k} \check{U}^\sigma(\omega_{ij}^k))^*, \sqrt{\mathcal{T}_k} \check{U}^\sigma(\omega_{ij}^k)\}_{i,j,k}\) is a quasi-basis for \(E_{\check{\varnothing}^\sigma}\). Hence for any \(b \in B\), 
\[
b = \sum_{i,j,k} f_k E_{\check{\varnothing}^\sigma}(b \check{U}^\sigma(\omega_{ij}^k))^* \check{U}^\sigma(\omega_{ij}^k).
\]
Since \(\check{U}^\sigma(\omega_{ij}^k) \in M(B)\) for any \(i, j, k\), 
\[
E_{\check{\varnothing}^\sigma}(b \check{U}^\sigma(\omega_{ij}^k))^* \in B^\sigma
\]
for any \(i, j, k\) and \(b \in B\). Let \(a = \sum_{i,j,k} f_k E_{\check{\varnothing}^\sigma}(b \check{U}^\sigma(\omega_{ij}^k))^* \times_{\rho, u^\sigma} \omega_{ij}^k\). Then \(a \in B^\sigma \times_{\rho, u^\sigma} H^0\) and \(\pi(a) = b\). Thus \(\pi\) is surjective. Since \(\pi'\) is an isomorphism of
$M(B)^0 \times_{\rho,u^*} H^0$ onto $M(B)$, we can see that $\pi$ is an isomorphism of $B^* \times_{\rho,u^*} H^0$ onto $B$. Also, since $a \circ \pi' = (\pi' \circ \id) \circ \hat{\rho}$ and $E_{1}^{\rho,u^*} = E_2 \circ \pi'$, we can see that

$$\sigma \circ \pi' = (\pi \circ \id) \circ \hat{\rho}, \quad E_{1}^{\rho,u^*} = E_2 \circ \pi'.$$

Furthermore, by the definition of $\pi'$, $\pi'$ is strictly continuous. Thus $\pi' = \mathfrak{A}$.

Combining Lemmas 2.9, 2.10 and 2.11, we obtain the following proposition:

**Proposition 2.12.** Let $B$ be a C$^*$-algebra and $\sigma$ a coaction of $H^0$ on $B$. We suppose that $\hat{\sigma}(1 \otimes a) \sim (1 \otimes a \otimes 1$ in $(M(B) \otimes_{\mathbb{A}} H) \otimes H$. Then there is a twisted coaction $(\rho,\pi^0)$ of $H$ on $B^*$ and an isomorphism $\pi$ of $B^* \times_{\rho,u^*} H^0$ onto $B$ satisfying that

$$\sigma \circ \pi' = (\pi \circ \id) \circ \hat{\rho}, \quad E_{1}^{\rho,u^*} = E_2 \circ \pi',$$

where $B^*$ is the fixed point C$^*$-subalgebra of $B$ for $\sigma$ and $E_{1}^{\rho,u^*}$ and $E_2$ are the canonical conditional expectations from $B$ and $B^* \times_{\rho,u^*} H^0$ onto $B^*$, respectively.

3. Twisted coactions on a Hilbert C$^*$-bimodule and strong Morita equivalence for twisted coactions

First, we shall define crossed products of Hilbert C$^*$-bimodules in the sense of Brown, Mingo and Shen [2] and show their duality theorem, which is similar to [17, Theorem 5.7]. We give the definition of a Hilbert C$^*$-bimodule in the sense of [2].

Let $A$ and $B$ be C$^*$-algebras. Let $X$ be a left pre-Hilbert $A$-bimodule and a right pre-Hilbert $B$-module. Its left $A$-valued inner product and right $B$-valued inner product denote by $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$, respectively.

**Definition 3.1.** We call $X$ a pre-Hilbert $A-B$-bimodule if $X$ satisfies the condition

$$A(x, y)z = x\langle y, z \rangle_B$$

for any $x, y, z \in X$. We call $X$ a Hilbert $A-B$-bimodule if $X$ is complete with the norms.

**Remark 3.1.** We suppose that $X$ is a pre-Hilbert $A-B$-bimodule. Then by [2, Remark 1.9], we can see the following:

1. For any $x \in X$, $\|A(x, x)\| = \|\langle x, x \rangle_B\|$
2. For any $a \in A, b \in B$ and $x, y \in X$,

$$A(x, yb) = A(xb^*, y), \quad \langle ax, y \rangle_B = \langle x, a^*y \rangle_B.$$  

3. If $X$ is complete with the norm and full with the both-sided inner products, then $X$ is an $A-B$-equivalence bimodule.

In this paper, by the words “pre-Hilbert C$^*$-bimodules” and “Hilbert C$^*$-bimodules”, we mean pre-Hilbert C$^*$-bimodule and Hilbert C$^*$-bimodules in the sense of [2], respectively.

Let $A$ and $B$ be C$^*$-algebras. Let $X$ be a Hilbert $A-B$-bimodule and let $B_B(X)$ be the C$^*$-algebra of all right $B$-linear operators on $X$ for which there is a right adjoint $B$-linear operator on $X$. We note that a right $B$-linear operator on $X$ is bounded. For each $x, y \in X$, let $\theta_{x,y}$ be a rank-one operator on $X$ defined by $\theta_{x,y}(z) = xy^*z_B$ for any $z \in X$. Then $\theta_{x,y}$ is a right $B$-linear operator on $X$. Let $K_B(X)$ be the closure of all linear spans of such $\theta_{x,y}$. Then $K_B(X)$ is a closed two-sided ideal of $B_B(X)$. Similarly, we define $A_B(X)$ and $A_K(X)$. If $X$ is an $A-B$-equivalence bimodule, we identify $A$ and $M(A)$ with $K_B(X)$ and $B_B(X)$, respectively and identify $B$ and $M(B)$ with $A_B(X)$ and $A_K(X)$, respectively. For any $a \in M(A)$, we regard $a \in M(A)$ as an element in $B_B(X)$ as follows: For any $b \in A, x \in X$,

$$a(bx) = (ab)x.$$  

Since $X = \overline{AX}$ by [6] Proposition 1.7], we can obtain an element in $B_B(X)$ induced by $a \in M(A)$. Similarly, we can obtain an element in $B_B(X)$ induced by any $b \in M(B)$.

**Lemma 3.2.** With the above notations, we suppose that $X$ is a Hilbert $A - B$-bimodule. For any $a \in M(A)$, there is a bounded net $\{a_\alpha\}_{\alpha \in \Gamma} \subset A$ such that $ax = \lim_{\alpha \to \infty} a_\alpha x$ for any $x \in X$.

**Proof.** Since $a \in M(A)$, there is a bounded net $\{a_\alpha\}_{\alpha \in \Gamma} \subset A$ such that $\{a_\alpha\}_{\alpha \in \Gamma}$ converges to $a$ strictly. We can prove that $ax = \lim_{\alpha \to \infty} a_\alpha x$ for any $x \in X$ in a routine way since $X = \overline{AX}$ by [6] Proposition 1.7. □

Let $(\rho, u)$ and $(\sigma, v)$ be twisted coactions of $H^0$ on $A$ and $B$, respectively.

**Definition 3.2.** Let $\lambda$ be a linear map from a Hilbert $A - B$-bimodule $X$ to $X \otimes H^0$. Then we say that $\lambda$ is a twisted coaction of $H^0$ on $X$ with respect to $(A, B, \rho, u, \sigma, v)$ if the following conditions hold:

1. $\lambda(ax) = \rho(a)\lambda(x)$ for any $a \in A$, $x \in X$,
2. $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B$, $x \in X$,
3. $\rho(A(x, y)) = A \otimes H^0(\lambda(x), \lambda(y))$ for any $x, y \in X$,
4. $\sigma(\lambda(x)B) = (\lambda(x), \lambda(y))_{B \otimes H^0}$ for any $x, y \in X$,
5. $(id \otimes \delta^0) \circ \lambda = id_X$,
6. $(\lambda \otimes id)\lambda(x) = u(id \otimes \Delta^0)(\lambda(x)v^*)$ for any $x \in X$,

where $u$ and $v$ are regarded as elements in $B_B(X)$ and $A_B(X)$, respectively.

We note that the twisted coaction $\lambda$ of $H^0$ on the Hilbert $A - B$-bimodule $X$ induced by $\lambda$ is isometric. Indeed, for any $x \in X$,

$$||\lambda(x)|| = ||A \otimes H^0(\lambda(x), \lambda(x))|| = ||\rho(x, y)|| = ||A(x, y)|| = ||x||^2.$$

Let $\lambda$ be a twisted coaction of $H^0$ on a Hilbert $A - B$-bimodule $X$ with respect to $(A, B, \rho, u, \sigma, v)$. We define the **twisted action** of $H$ on $X$ induced by $\lambda$ as follows:

For any $x \in X$, $h \in H$,

$$h \cdot \lambda x = (id \otimes h)(\lambda(x)) = \lambda(x)(h),$$

where $\lambda(x)$ is the element in $\text{Hom}(H, X)$ induced by $\lambda(x)$ in $X \otimes H^0$. Then we obtain the following conditions which are equivalent to Conditions (1)-(6) in Definition 3.2, respectively:

1. $^\dagger h \cdot \lambda ax = [h(1), \rho, u][h(2), \lambda x]$ for any $a \in A$, $x \in X$,
2. $^\dagger h \cdot \lambda xb = [h(1), \lambda x][h(2), \sigma, v, b]$ for any $b \in B$, $x \in X$,
3. $^\dagger h \cdot A(x, y) = A([h(1), \lambda x], [S(h(2)), \lambda y])$ for any $x, y \in X$,
4. $^\dagger h \cdot \sigma(\lambda(x)B) = ([S(h(1)), \lambda x], [h(2), \lambda y])B$ for any $x, y \in X$,
5. $1_H \cdot \lambda x = x$ for any $x \in X$,
6. $^\dagger h \cdot \lambda [l, \lambda x] = \widehat{\mu}(h(1), l(1))[h(2), l(2), \lambda x]\widehat{\mu}(h(3), l(3))$ for any $x \in X$, $h, l \in H$, where $\mu$ and $\nu$ are elements in $\text{Hom}(H \times H, M(A))$ and $\text{Hom}(H \times H, M(B))$ induced by $u \in M(A) \otimes H^0 \otimes H^0$ and $v \in M(B) \otimes H^0 \otimes H^0$, respectively.

**Remark 3.3.** In Definition 3.2, if $\rho$ and $\sigma$ are coactions of $H^0$ on $A$ and $B$, respectively, then Condition (6) in Definition 3.2 and its equivalent Condition (6)$^\dagger$ are following, respectively:

6. $(\lambda \otimes id) \circ \lambda = (id \otimes \Delta^0) \circ \lambda$,

6.$^\dagger$ $h \cdot \lambda [l, \lambda x] = hl \cdot \lambda x$ for any $x \in X$.

In this case, we call $\lambda$ a coaction of $H^0$ on $X$ with respect to $(A, B, \rho, \sigma)$.

Next, we shall define crossed products of Hilbert $C^*$-bimodules by twisted coactions in the same way as in [17] Section 4 and give a duality theorem for them.

Let $(\rho, u)$ and $(\sigma, v)$ be twisted coactions of $H^0$ on $C^*$-algebras $A$ and $B$, respectively. Let $\lambda$ be a twisted coaction of $H^0$ on a Hilbert $A - B$-bimodule $X$ with
respect to \((A,B,\rho,u,\sigma,v)\). We define \(X \rtimes_{\lambda} H\), a Hilbert \(A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H\)-bimodule as follows: Let \((X \rtimes_{\lambda} H)_0\) be just \(X \otimes H\) (the algebraic tensor product) as vector spaces. Its left and right actions are given by
\[
(a \rtimes_{\rho,u} h)(x \rtimes_{\lambda} l) = a[h(1) \cdot x] \tilde{\rho}(h(2), (l(1)) \rtimes_{\lambda} h(3) l(2)),
(x \rtimes_{\lambda} l)(b \rtimes_{\sigma,v} m) = x[l(1) \cdot_{\sigma,v} b] \tilde{\sigma}(l(2), m(1)) \rtimes_{\lambda} l(3) m(2)
\]
for any \(a \in A, b \in B, x \in X\) and \(h, l, m \in H\). Also, its left \(A \rtimes_{\rho,u} H\)-valued and right \(B \rtimes_{\sigma,v} H\)-valued inner products are given by
\[
A_{\rtimes_{\rho,u}} H \langle x \rtimes_{\lambda} h, y \rtimes_{\lambda} l \rangle = A(x, [S(h(2)) l(3)] \cdot_{\lambda} y) \tilde{\rho}(S(h(1)) l(2)), l(1)) \rtimes_{\rho,u} h(3) l(4),
\]
\[
\langle x \rtimes_{\lambda} h, y \rtimes_{\lambda} l \rangle_{B \rtimes_{\sigma,v} H} = \tilde{\sigma}(h(2)), S(h(1)) \cdot_{\sigma,v} x, y) B \tilde{\sigma}(h(4), l(1)) \rtimes_{\sigma,v} h(5) l(2)
\]
for any \(x, y \in X\) and \(h, l \in H\). In the same way as in [17 Section 4], we can see that \((X \rtimes_{\lambda} H)_0\) is a pre-Hilbert \(A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H\)-bimodule. Let \(X \rtimes_{\lambda} H\) be the completion of \((X \rtimes_{\lambda} H)_0\). It is a Hilbert \(A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H\)-bimodule. Let \(\hat{\lambda}\) be a linear map from \((X \rtimes_{\lambda} H)_0\) to \((X \rtimes_{\lambda} H)_0 \otimes H\) defined by
\[\hat{\lambda}(x \rtimes_{\lambda} h) = (x \rtimes_{\lambda} h(1)) \otimes h(2)\]
for any \(x \in X, h \in H\). By easy computations, we can see that \(\hat{\lambda}\) is a linear map from \(H\) to \((X \rtimes_{\lambda} H)_0 \otimes H\) satisfying Conditions (1)-(6) in Definition 5.2. Thus for any \(x \in (X \rtimes_{\lambda} H)_0\),
\[
||\hat{\lambda}(x)||^2 = ||(A \rtimes_{\rho,u} H) \otimes H(\hat{\lambda}(x), \hat{\lambda}(x))|| = ||\tilde{\rho}(A \langle x, x \rangle)|| = ||A \langle x, x \rangle|| = ||x||^2.
\]
Hence \(\hat{\lambda}\) is an isometry. We extend \(\hat{\lambda}\) to \(X \rtimes_{\lambda} H\). We can see that the extension of \(\hat{\lambda}\) is a coaction of \(H\) on \(X \rtimes_{\lambda} H\) with respect to \((A \rtimes_{\rho,u} H, B \rtimes_{\sigma,v} H, \tilde{\rho}, \tilde{\sigma})\). We also denote it by the same symbol \(\hat{\lambda}\) and call it the dual coaction of \(\lambda\). Similarly we define the second dual coaction of \(\lambda\), which is a coaction of \(H^0\) on \(X \rtimes_{\lambda} H \rtimes_{\chi} H^0\). Let \(A \Lambda\) be as in Section 2. For any \(I = (i, j, k) \in A\), let \(W^I_1, V^I_1\) be elements in \(M(A) \rtimes_{\rho,u} H \rtimes_{\chi} H^0\) defined by
\[W^I_1 = \sqrt{a_k} \rtimes_{\rho,u} w^k_{ij}, V^I_1 = (1 \rtimes_{\rho,u} 1 \rtimes_{\chi} \tau)(W^I_1 \rtimes_{\chi} 1\rangle).\]
Similarly for any \(I = (i, j, k) \in A\), we define elements
\[W^I_2 = \sqrt{a_k} \rtimes_{\rho,u} w^k_{ij}, V^I_2 = (1 \rtimes_{\rho,u} 1 \rtimes_{\chi} \tau)(W^I_2 \rtimes_{\chi} 1\rangle)\]
in \(M(B) \rtimes_{\rho,v} H \rtimes_{\chi} H^0\). We regard \(M(A)\) as an equivalence \(M_N(C) - M_N(C)\)-bimodule in the usual way. Let \(X \otimes M_N(C)\) be the exterior tensor product of \(X\) and \(M_N(C)\), which is a Hilbert \(A \otimes M_N(C) - B \otimes M_N(C)\)-bimodule. Let \(\{f_{IJ}\}_{I,J} \in A\) be a system of matrix units of \(M_N(C)\). Let \(\Psi_X\) be a linear map from \(X \otimes M_N(C)\) to \(X \rtimes_{\lambda} H \rtimes_{\chi} H^0\) defined by
\[
\Psi_X \left( \sum_{I,J} x_{IJ} \otimes f_{IJ} \right) = \sum_{I,J} V^0_{IJ}^* (x_{IJ} \rtimes_{\lambda} 1 \rtimes_{\chi} 1\rangle V^0_{IJ}^\dag).
\]
Let \(\Psi_A\) and \(\Psi_B\) be the isomorphisms of \(A \otimes M_N(C)\) and \(B \otimes M_N(C)\) onto \(A \rtimes_{\rho,u} H \rtimes_{\chi} H^0\) and \(B \rtimes_{\rho,v} H \rtimes_{\chi} H^0\) defined in Proposition 2.8 respectively. Then we have the same lemmas as [17 Lemmas 5.1 and 5.5]. Hence \(\Psi_X\) is an isometry from \(X \otimes M_N(C)\) to \(X \rtimes_{\lambda} H \rtimes_{\chi} H^0\) whose image is \((X \rtimes_{\lambda} H)_0 \rtimes_{\chi} H^0\), the linear span of the set
\[
\{x \rtimes_{\lambda} h \rtimes_{\chi} \phi | x \in X, h \in H, \phi \in H^0\}.
\]
Since \(X \otimes M_N(C)\) is complete, so is \((X \rtimes_{\lambda} H)_0 \rtimes_{\chi} H^0\). Furthermore, we claim that \((X \rtimes_{\lambda} H)_0\) is also complete. In order to show it, we need the following lemma: Let \(E^\lambda_1\) be a linear map from \((X \rtimes_{\lambda} H)_0\) onto \(X\) defined by
\[
E^\lambda_1(x \rtimes_{\lambda} h) = \tau(h) x
\]
for any $x \in X$, $h \in H$.

**Lemma 3.4.** With the above notations, $E_2^\lambda$ is continuous.

Proof. In the same way as in the proof of [17, Lemma 5.6], we can see that
$$
E_2^\lambda(x \times_\Lambda h) = \tau_{(1)}(x \times_\Lambda h) = \hat{V}_\rho \tau_{(1)}((x \times_\Lambda h \times_\Lambda 1^0)\hat{V}_\rho^* \tau_{(2)}),
$$
where we identify $X \times_\Lambda H \times_\Lambda 1^0$ with $X \times_\Lambda H$ and
$$
\hat{V}_\rho(\phi) = 1 \times \rho \times_\Lambda \bar{\phi}, \quad \hat{V}_\rho^*(\phi) = 1 \times_\Lambda \rho \times \bar{\phi}
$$
for any $\phi \in H^0$. Hence $E_2^\lambda$ is continuous. \qed

Let $E_2^\lambda$ be a linear map from $(X \times_\Lambda H \times_\Lambda H^0)_0$ to $X \times_\Lambda H$ defined by
$$
E_2^\lambda(x \times_\Lambda \phi) = \phi(x)x
$$
for any $x \in X \times_\Lambda H$, $\phi \in H^0$.

**Lemma 3.5.** With the above notations, $(X \times_\Lambda H)_0$ is complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(X \times_\Lambda H)_0$. Using Lemma 3.3 and the linear map $E_2^\lambda$, we can see that $\{x_n\}$ is convergent in $(X \times_\Lambda H)_0$. \qed

By Lemma 3.5, $X \times_\Lambda H = (X \times_\Lambda H)_0$. In the same way as in the proof of [17, Theorem 5.7], we obtain the following proposition using Lemma 3.5.

**Proposition 3.6.** Let $A$, $B$ be $C^*$-algebras and $H$ a finite dimensional $C^*$-Hopf algebra with its dual $C^*$-Hopf algebra $H^0$. Let $(\rho, u)$ and $(\sigma, v)$ be twisted coactions of $H^0$ on $A$ and $B$, respectively. Let $\lambda$ be a twisted coaction of $H^0$ on a Hilbert $A - B$-bimodule $X$ with respect to $(A, B, \rho, u, \sigma, v)$. Then there is an isomorphism $\Psi_X$ from $X \otimes M_N(C)$ onto $X \times_\Lambda H \times_\Lambda H^0$ satisfying that

1. $\Psi_X(\sum_{i,j} a_{iJ} \otimes f_{iJ})(\sum_{i,j} x_{iJ} \otimes f_{iJ}^2) = \Psi_A(\sum_{i,j} a_{iJ} \otimes f_{iJ}) \Psi_X(\sum_{i,j} x_{iJ} \otimes f_{iJ})$, 
2. $\Psi_X((\sum_{i,j} x_{iJ} \otimes f_{iJ}) (\sum_{i,j} b_{iJ} \otimes f_{iJ}^2)) = \Psi_X(\sum_{i,j} x_{iJ} \otimes f_{iJ}) \Psi_B(\sum_{i,j} b_{iJ} \otimes f_{iJ})$, 
3. $A \times_{\rho, u, H \times_\Lambda H^0} (\Psi_X(\sum_{i,j} x_{iJ} \otimes f_{iJ}), \Psi_X(\sum_{i,j} y_{iJ} \otimes f_{iJ})) = \Psi_A((\sum_{i,j} x_{iJ} \otimes f_{iJ}, \sum_{i,j} y_{iJ} \otimes f_{iJ}))$, 
4. $\Psi_X(\sum_{i,j} x_{iJ} \otimes f_{iJ}), \Psi_X(\sum_{i,j} y_{iJ} \otimes f_{iJ}))_{B \times_{\sigma, v, H \times_\Lambda H^0}} = \Psi_B((\sum_{i,j} x_{iJ} \otimes f_{iJ}, \sum_{i,j} y_{iJ} \otimes f_{iJ}))_{B \otimes M_N(C)}$

for any $a_{iJ} \in A$, $b_{iJ} \in B$, $x_{iJ}, y_{iJ} \in X$, $i, j \in I$, where $X \times_\Lambda H \times_\Lambda H^0$ is a Hilbert $A \times_{\rho, u, H \times_\Lambda H^0} B \times_{\sigma, v, H \times_\Lambda H^0}$-bimodule and $X \otimes M_N(C)$ is an exterior tensor product of $X$ and the Hilbert $M_N(C) - M_N(C)$-bimodule $M_N(C)$. Furthermore, there are unitary elements $U \in (M(A) \times_{\rho, u, H} H \times_{\sigma, v, H^0}) \otimes H^0$ and $V \in (M(B) \times_{\sigma, v, H} H \times_{\sigma, v, H^0}) \otimes H^0$ such that

$$
\hat{U}(x)V = \left(\Psi_X \otimes \text{id} \circ (\lambda \otimes \text{id}_{M_N(C)}) \circ \Psi_X^{-1}\right)(x)
$$
for any $x \in X \otimes M_N(C)$.

The above proposition has already obtained in the case of Kac systems by Guo and Zhang [10], which is a generalization of the above result. Also, we have the following lemmas:
Lemma 3.7. With the above notations, if $X$ is full with the both-sided inner products, then so is $X \rtimes H$.

Proof. Modifying the proof of [17, Lemma 4.5], we can prove the lemma.

Lemma 3.8. With the above notations, if $X \rtimes H$ is full with the both-sided inner products, then so is $X$.

Proof. Since $X \rtimes H$ is full with the both-sided inner products, so is $X \rtimes H \rtimes H^0$ by Lemma 3.7. Thus $X \otimes M_N(\mathbb{C})$ is full with the both-sided inner products by Proposition 3.6. Let $f$ be a minimal projection in $M_N(\mathbb{C})$. Then

$$A \otimes f = (1_{M(A)} \otimes f)(A \otimes M_N(\mathbb{C}))(1_{M(A)} \otimes f)$$

$$= (1 \otimes f)_{A \otimes M_N(\mathbb{C})}(X \otimes M_N(\mathbb{C}), X \otimes M_N(\mathbb{C}))(1 \otimes f)$$

$$= A(X, X) \otimes f M_N(\mathbb{C})f = A(X, X) \otimes f.$$

Hence $X$ is full with the left-sided inner product. Similarly, we can see that $X$ is full with the right-sided inner product. Therefore, we obtain the conclusion.

Definition 3.3. Let $(\rho, u)$ and $(\sigma, v)$ be twisted coactions of $H^0$ on $C^*$-algebras $A$ and $B$, respectively. Then $(\rho, u)$ is strongly Morita equivalent to $(\sigma, v)$ if there are an $A-B$-equivalence bimodule $X$ and a twisted coaction $\lambda$ of $H^0$ on $X$ with respect to $(A, B, \rho, u, \sigma, v)$.

In the same way as in [17, Section 3], we can see that the strong Morita equivalence for twisted coactions of $H^0$ on $C^*$-algebras is an equivalence relation. Also, we can obtain the following lemma in the similar way to [17, Lemma 3.12] using approximate units in a $C^*$-algebra. We give it without its proof.

Lemma 3.9. Let $(\rho, u)$ and $(\sigma, v)$ be twisted coactions of $H^0$ on $A$. Then the following conditions are equivalent:

1. The twisted coactions $(\rho, u)$ and $(\sigma, v)$ are exterior equivalent,
2. The twisted coactions $(\rho, u)$ and $(\sigma, v)$ are strongly Morita equivalent by a twisted coaction $\lambda$ of $H^0$ on $A$, which is a linear map from $\lambda AA_A$ to $\lambda \otimes H^0 A \otimes H^0 \otimes A_H$, where $AA_A$ and $A \otimes H^0 \otimes A_H$ are regarded as an $A - \lambda$-equivalence bimodule and an $A \otimes H^0 - \lambda$-equivalence bimodule in the usual way.

Remark 3.10. Let $A$ and $B$ be $C^*$-algebras and $\sigma$ a coaction of $H^0$ on $B$. Let $X$ be an $A - \lambda$-equivalence bimodule and $\lambda$ a linear map from $X$ to $X \otimes H^0$ satisfying that

1. $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B$, $x \in X$,
2. $\sigma(\langle x, y \rangle_B) = \langle \lambda(x), \lambda(y) \rangle_{B \otimes H^0}$ for any $x, y \in X$,
3. $(\lambda \otimes \lambda^0) \circ \lambda = \lambda \chi$,
4. $(\lambda \otimes \text{id}) \circ \lambda = (\text{id} \otimes \Delta^0) \circ \lambda$.

We call $(B, X, \sigma, \lambda, H^0)$ a right covariant system (See [17, Definition 3.4]). Then we can construct an action “$\cdot$” of $H$ on $K_B(X)$ as follows: For any $a \in B_B(X)$, $h \in H$ and $x \in X$,

$$[h \cdot a]x = h_{(1)} \cdot \lambda a[S(h_{(2)}) \cdot \lambda x].$$

If $a \in K_B(X)$, we can see that $h \cdot a \in K_B(X)$. Thus identifying $A$ with $K_B(X)$, we can obtain an action of $H$ on $A$.

4. Linking $C^*$-algebras and coactions on $C^*$-algebras

Let $(\rho, u)$ and $(\sigma, v)$ be twisted coactions of $H^0$ on $C^*$-algebras $A$ and $B$, respectively. We suppose that there are a Hilbert $A - B$-bimodule $X$ and a twisted
coaction \( \lambda \) of \( H^0 \) on \( X \) with respect to \((A, B, \rho, u, \sigma, v)\). Let \( C \) be the linking \( C^*\)-algebra for \( X \) defined in Brown, Mingo and Shen [6]. By [6] Proposition 2.3, \( C \) is the \( C^*\)-algebra which is consisting of all \( 2 \times 2 \)-matrices

\[
\begin{bmatrix}
a & x \\
y & b
\end{bmatrix}, \quad a \in A, \quad b \in B, \quad x, y \in X,
\]

where \( \tilde{y} \) denotes \( y \) viewed as an element in \( \tilde{X} \), the dual Hilbert \( C^*\)-bimodule of \( X \). Before we define the coaction of \( H^0 \) on \( C \) induced by \( \lambda \), we give a remark.

**Remark 4.1.** We identify the \( H^0 - H^0\)-equivalence bimodule \( \tilde{H}^0 \) with \( H^0 \) as \( H^0 - H^0\)-equivalence bimodule by the map

\[
\tilde{H}^0 \rightarrow H^0: \tilde{\phi} \mapsto \phi^*.
\]

Also, we identify the Hilbert \( B \otimes H^0 - A \otimes H^0 \)-bimodule \( \tilde{X} \otimes H^0 \) with \( \tilde{X} \otimes H^0 \) by the map

\[
\tilde{X} \otimes H^0 \rightarrow \tilde{X} \otimes H^0: x \otimes \tilde{\phi} \mapsto \tilde{x} \otimes \phi^*.
\]

Furthermore, we identify the linking \( C^*\)-algebra for \( X \otimes H^0 \), the Hilbert \( A \otimes H^0 - B \otimes H^0 \)-bimodule with \( C \otimes H^0 \) by the isomorphism defined by

\[
\Phi(\begin{bmatrix}
a \otimes \phi_{11} & x \otimes \phi_{12} \\
y \otimes \phi_{21} & b \otimes \phi_{22}
\end{bmatrix}) = \begin{bmatrix}
a & 0 \\
y & 0
\end{bmatrix} \otimes \phi_{11} + \begin{bmatrix}0 & x
\end{bmatrix} \otimes \phi_{12} + \begin{bmatrix}0 & 0
\end{bmatrix} \otimes \phi_{21} + \begin{bmatrix}0 & 0
\end{bmatrix} \otimes \phi_{22},
\]

where \( a \in A, b \in B, x, y \in X \) and \( \phi_{ij} \in H^0 \) \((i, j = 1, 2)\).

Let \( \gamma \) be the homomorphism of \( C \) to \( C \otimes H^0 \) defined by for any \( a \in A, b \in B, x, y \in X \),

\[
\gamma(\begin{bmatrix}
a \\
y
\end{bmatrix}) = \begin{bmatrix}
\rho(a) & \lambda(x) \\
\lambda(y) & \sigma(b)
\end{bmatrix}.
\]

Let \( \tilde{w} \) be the unitary element in \( M(C) \) defined by \( \tilde{w} = \begin{bmatrix}u & 0 \\
0 & v
\end{bmatrix} \). By routine computations, \((\gamma, \tilde{w})\) is a twisted coaction of \( H^0 \) on \( C \).

**Remark 4.2.** (1) We note that the twisted action of \( H \) on \( C \) induced by \((\gamma, \tilde{w})\) as follows: For any \( a \in A, b \in B, x, y \in X \) and \( h \in H \),

\[
h \cdot \gamma \left(\begin{bmatrix}
a \\
y
\end{bmatrix}
\right) = \begin{bmatrix}
h \rho(a) & h \lambda x \\
S(h)^* \lambda y & h \sigma(b)
\end{bmatrix}.
\]

(2) Let \( \tilde{\lambda} \) be a linear map from \( X \) to \( X \otimes H^0 \) defined by for any \( x \in X \),

\[
\tilde{\lambda}(\tilde{x}) = \tilde{\lambda(}x\).
\]

Then \( \tilde{\lambda} \) is the coaction of \( H^0 \) on \( \tilde{X} \) induced by \( \lambda \). Also, the twisted action of \( H \) on \( \tilde{X} \) induced by \( \tilde{\lambda} \) is as follows: For any \( x \in X, h \in H \),

\[
h \cdot \tilde{\lambda} \left(\begin{bmatrix}
a \\
y
\end{bmatrix}
\right) = S(h)^* \lambda y.
\]

Let \( C_1 \) be the linking \( C^*\)-algebra for the Hilbert \( A \rtimes_{\gamma, \tilde{w}} (B \rtimes_{\sigma} H) \)-bimodule \( X \rtimes_{\lambda} H \). Then we obtain the following lemma by Remarks 4.1 and 4.2.

**Lemma 4.3.** With the above notations, there is an isomorphism \( \pi_1 \) of \( C \rtimes_{\gamma, \tilde{w}} H \) onto \( C_1 \).
Proof. Let \( \pi_1 \) be the map from \( C \rtimes_{\gamma,w} H \) to \( C \) defined by
\[
\pi_1\left(\frac{a}{b} \right)_{\times_{\gamma,w} h} = \left[ a \times_{\rho,w} h, b \times_{\sigma,v} h \right]
\]
for any \( a \in A, b \in B, x, y \in X \) and \( h \in H \). Let \( \theta_1 \) be the map from \( C \) to \( C \rtimes_{\gamma,w} H \) defined by
\[
\theta_1\left(\frac{a}{b} \right)_{\times_{\gamma,w} l} = \frac{a}{b} \times_{\gamma,w} m
\]
for any \( a \in A, b \in B, x, y \in X \) and \( h, k, l, m \in H \). Then by routine computations, \( \pi_1 \) is a homomorphism of \( C \rtimes_{\gamma,w} H \) to \( C \) and \( \theta_1 \) is a homomorphism of \( C \) to \( C \rtimes_{\gamma,w} H \). Moreover, we can see that \( \theta_1 \) is the inverse map of \( \pi_1 \). Therefore, we obtain the conclusion.

By the proof of the above lemma, we obtain the following corollary:

**Corollary 4.4.** With the above notations, there is a Hilbert \( \rtimes_{\theta} A \rtimes_{\rho} H \)-bimodule isomorphism \( \pi \) of \( X \rtimes_{\gamma} H \) onto \( \tilde{X} \rtimes_{\tilde{\gamma}} H \).

**Remark 4.5.** Let \( \gamma_1 \) be a coaction of \( H \) on \( C \) defined by
\[
\gamma_1 = (\pi_1 \otimes \text{id}_H) \circ \tilde{\gamma} \circ \pi_1^{-1}.
\]
Then by routine computations, for any \( a \in A, b \in B, x, y \in X \) and \( h, l, k, m \in H \),
\[
\gamma_1\left(\frac{a}{b} \right)_{\times_{\gamma,w} h} = \left[ a \times_{\rho,w} h, b \times_{\sigma,v} h \right]
\]
for any \( a \in A, b \in B, x, y \in X \) and \( h, k, l, m \in H \). We give a result similar to [15, Theorem 6.4] for coactions of \( H \) on a Hilbert \( C^* \)-bimodule applying Proposition [2,12] to a linking \( C^* \)-algebra. Let \( \rho \) and \( \sigma \) be coactions of \( H \) on \( A \) and \( B \), respectively and let \( X \) be a Hilbert \( A \times B \)-bimodule.

We give a result similar to [15, Theorem 6.4] for coactions of \( H \) on a Hilbert \( C^* \)-bimodule applying Proposition [2,12] to a linking \( C^* \)-algebra. Let \( \rho \) and \( \sigma \) be coactions of \( H \) on \( A \) and \( B \), respectively and let \( X \) be a Hilbert \( A \times B \)-bimodule. Let \( \lambda \) be a coaction of \( H \) on \( X \) with respect to \( (A,B,\rho,\sigma) \). Let \( C \) be the linking \( C^* \)-algebra and \( \gamma \) the coaction of \( H \) on \( C \) induced by \( \rho,\sigma \) and \( \lambda \). As defined in Section [3], let
\[
X^\lambda = \{ x \in X \mid \lambda(x) = x \otimes 1 \}.
\]
Then by Lemma [3,3] \( X^\lambda \) is an Hilbert \( A^\rho \times B^\sigma \)-bimodule. Let \( C_0 \) be the linking \( C^* \)-algebra for \( X^\lambda \). We can prove the following lemma in the straightforward way.

**Lemma 4.6.** With the above notations and assumptions, \( C^\gamma = C_0 \), where \( C^\gamma \) is the fixed point \( C^* \)-subalgebra of \( C \) for \( \gamma \).

**Lemma 4.7.** With the above notations, if \( \widehat{\theta}(1 \times_{\theta} e) \sim (1 \times_{\theta} e) \otimes 1 \) in \( (M(A) \times_{\rho} H) \otimes H \) and \( \widehat{\sigma}(1 \times_{\sigma} e) \sim (1 \times_{\sigma} e) \otimes 1 \) in \( (M(B) \times_{\sigma} H) \otimes H \), then \( \widehat{\gamma}(M(C) \times_{\gamma} e) \sim (1 \times_{\theta} H) \otimes (1 \times_{\rho} H) \).
Proof. By Remark 4.3, we identify $C \rtimes_r H$ with $C_1$, the linking $C^*$-algebra for the Hilbert $A \rtimes_p H \sim A \rtimes_p H$-bimodule $X \rtimes H$. Also, we identify $\hat{\gamma}$ with $\gamma_1$, the coaction of $H$ on $C_1$ defined in Remark 4.5. Hence

$$\hat{\gamma}(1 \times e) = \begin{bmatrix} 1 \times_p e(1) & 0 \\ 0 & 0 \end{bmatrix} \otimes e(2) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \times_p e(1) \end{bmatrix} \otimes e(2).$$

By the assumptions,

$$\begin{bmatrix} 1 \times_p e(1) & 0 \\ 0 & 0 \end{bmatrix} \otimes e(2) \sim \begin{bmatrix} 1 \times_p e & 0 \\ 0 & 0 \end{bmatrix} \otimes 1 \text{ in } \begin{bmatrix} M(A) \rtimes_p H & 0 \\ 0 & 0 \end{bmatrix} \otimes H,$$

$$\begin{bmatrix} 0 & 0 \\ 1 \times_p e(1) \end{bmatrix} \otimes e(2) \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \times_p e \end{bmatrix} \otimes 1 \text{ in } \begin{bmatrix} 0 & 0 \\ 0 & M(A) \rtimes_p H \end{bmatrix} \otimes H.$$

Since $\begin{bmatrix} M(A) \rtimes_p H & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & M(A) \rtimes_p H \end{bmatrix}$ are $C^*$-subalgebras of $M(C_1)$ by the proof of Echterhoff and Raeburn [9, Proposition A.1],

$$\begin{bmatrix} 1 \times_p e(1) & 0 \\ 0 & 0 \end{bmatrix} \otimes e(2) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \times_p e(1) \end{bmatrix} \otimes e(2) \sim \begin{bmatrix} 1 \times_p e & 0 \\ 0 & 1 \times_p e \end{bmatrix} \otimes 1$$
in $M(C_1) \otimes H$. Therefore, we obtain the conclusion since $M(C_1) \otimes H$ is identified with $(M(C) \rtimes_m H) \otimes H$. \qed

By [15 Section 4], there is a unitary element $w^\rho \in M(A) \otimes H$ satisfying that

$$w^\rho((1 \times_p e) \otimes 1)w^\rho = \hat{\rho}(1 \times_p e),$$

$$U^\rho = w^\rho(z^s \otimes 1), \quad z^\rho = (id_{M(A)} \otimes e)(w^\rho) \in M(A)\mathcal{L}.$$ 

Also, there is a unitary element $w^\sigma \in M(B) \otimes H$ satisfying that

$$w^{s^*}(1 \times_p e) w^\sigma = \hat{\rho}(1 \times_p e),$$

$$U^\sigma = w^\sigma(z^{s^*} \otimes 1), \quad z^\sigma = (id_{M(A)} \otimes e)(w^\sigma) \in M(B)\mathcal{L}.$$ 

Let $w^\gamma = \begin{bmatrix} w^\rho & 0 \\ 0 & w^\sigma \end{bmatrix} \in M(C) \otimes H$. Then $w^\gamma$ is a unitary element satisfying that

$$w^{\gamma^*}(1 \times_p e) w^\gamma = \hat{\gamma}(1 \times_p e).$$

Let $U^\gamma = w^\gamma(z^{\gamma^*} \otimes 1)$, where $z^\gamma = (id_{M(C)} \otimes e)(w^\gamma) \in M(C)\mathcal{L}$. Then by Section 2 $U^\gamma$ satisfies that

$$\tilde{U}^\gamma(1^0) = 1, \quad \tilde{U}^\gamma(\phi(1)) c\tilde{U}^\gamma(\phi(2)) \in M(C)\mathcal{L}$$

for any $c \in M(C)\mathcal{L}, \phi \in H^0$. Let $(\eta, u^\gamma)$ be the twisted coaction of $H$ on $C^\gamma$ induced by $U^\gamma$ which is defined in Section 2. Then by the proof of Proposition 2.12 there is the isomorphism $\pi_C$ of $C^\gamma \rtimes_{\eta,u^\gamma} H^0$ onto $C$ defined by

$$\pi_C(c \rtimes_{\eta,u^\gamma} \phi) = c\tilde{U}^\gamma(\phi)$$

for any $c \in C^\gamma, \phi \in H^0$, which satisfies that

$$\gamma \circ \pi_C = (\pi_C \circ id_H) \circ \hat{\gamma}, \quad E^{\eta,u^\gamma} = E^\gamma \circ \pi_C,$$

where $E^{\eta,u^\gamma}$ and $E^\gamma$ are the canonical conditional expectations from $C^\gamma \rtimes_{\eta,u^\gamma} H^0$ and $C$ onto $C^\gamma$, respectively. Let $p = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}, q = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$. Then $p$ and $q$ are projection in $M(C^\gamma)$. We note that $M(\overline{C}) = M(C)\mathcal{L}$ by Lemma 2.9.

Lemma 4.8. With the above notations and assumptions,

$$\pi_C(p \rtimes_{\eta,u^\gamma} 1^0) = p, \quad u^\gamma(p \otimes 1 \otimes 1) = (p \otimes 1 \otimes 1)u^\gamma,$$

$$\pi_C(q \rtimes_{\eta,u^\gamma} 1^0) = q, \quad u^\gamma(q \otimes 1 \otimes 1) = (q \otimes 1 \otimes 1)u^\gamma.$$
Proof. We note that $C^\gamma$ is identified with the $C^*$-subalgebra $C^\gamma \ltimes_{\eta,u^\gamma} H^0$ of $C^\gamma \ltimes_{\eta,u^\gamma} H^0$. Then by Proposition 2.12,
\[
p = E^0_C(p \ltimes_{\eta,u^\gamma} 1^0) = E^\gamma(\pi_C(p \ltimes_{\eta,u^\gamma} 1^0)) = e^\gamma \pi_C(p \ltimes_{\eta,u^\gamma} 1^0)
\]
\[
= \pi_C(e^{-\eta}(p \ltimes_{\eta,u^\gamma} 1^0)) = \pi_C(p) = \pi_C(p \ltimes_{\eta,u^\gamma} 1^0)
\]
since $\gamma \circ \pi_C = (\pi_C \circ \text{id}_H) \circ \eta$. Similarly, we can obtain that $\pi_C(q \ltimes_{\eta,u^\gamma} 1^0) = q$.
Furthermore, by the definition of $U^\gamma$, $U^\gamma = \begin{bmatrix} U^0 & 0 \\ 0 & U^\sigma \end{bmatrix} \in M(C) \otimes H$. Hence $U^\gamma(p \otimes 1) = (p \otimes 1)U^\gamma$. Since
\[
\hat{\gamma}(\phi, \psi) = \hat{U}^\gamma(\phi(1))\hat{U}^\gamma(\psi(1))\hat{U}^\gamma(\phi(2))\psi(2)
\]
for any $\phi, \psi \in H^0$, we can see that $u^\gamma(p \otimes 1 \otimes 1) = (p \otimes 1 \otimes 1)u^\gamma$. Similarly $u^\gamma(q \otimes 1 \otimes 1) = (q \otimes 1 \otimes 1)u^\gamma$. □

Let $\alpha = \eta|_{A^\sigma}$, $\beta = \eta|_{B^\sigma}$ and $\mu = \eta|_{X^\lambda}$. Let $u^\alpha = u^\gamma(p \otimes 1 \otimes 1)$, $u^\sigma = u^\gamma(q \otimes 1 \otimes 1)$.
Furthermore, let $\pi_A = \pi_C|_A$, $\pi_B = \pi_C|_B$, $\pi_X = \pi_C|_X$. Then $(\alpha, u^\alpha)$ and $(\beta, u^\sigma)$ are twisted coactions of $H^0$ on $A^\sigma$ and $B^\sigma$, respectively, and $\mu$ is a twisted coaction of $H^0$ on $X^\lambda$ with respect to $(A, B, \alpha, u^\alpha, \beta, u^\sigma)$. Also, $\pi_A$ and $\pi_B$ are isomorphisms of $A^\sigma \ltimes_{\alpha,u^\alpha} H^0$ and $B^\sigma \ltimes_{\beta,u^\sigma} H^0$ onto $A$ and $B$ satisfying the results in Proposition 2.12, respectively. Furthermore, we obtain the following:

**Theorem 4.9.** Let $A$ and $B$ be $C^*$-algebras and $H$ a finite dimensiona $C^*$-Hopf algebra with its dual $C^*$-Hopf algebra $H^0$. Let $\rho$ and $\sigma$ be coactions of $H^0$ on $A$ and $B$, respectively. Let $X$ be a coaction of $H^0$ on a Hilbert $A-B$-bimodule $X$ with respect to $(A, B, H, \rho, \sigma)$. We suppose that $\hat{\rho}(1 \ltimes_{A^\lambda} e) \sim (1 \ltimes_{A^\lambda} e) \otimes 1$ in $M(A) \ltimes_{A^\lambda} H$ and that $\hat{\sigma}(1 \ltimes_{B^\lambda} e) \sim (1 \ltimes_{B^\lambda} e) \otimes 1$ in $M(B) \ltimes_{B^\lambda} H$. Then there are a twisted coaction $\mu$ of $H^0$ on $\check{X}^\lambda$ and a bijective linear map $\pi_X$ from $X^\lambda \ltimes_{\mu} H^0$ onto $X$ satisfying the following conditions:

(1) $\pi_X((a \ltimes_{\alpha,u^\alpha} \phi)(x \ltimes_{\mu} \psi)) = \pi_A(a \ltimes_{\alpha,u^\alpha} \phi)\pi_X(x \ltimes_{\mu} \psi)$,
(2) $\pi_X((x \ltimes_{\mu} \phi)(b \ltimes_{\beta,u^\sigma} \psi)) = \pi_X(x \ltimes_{\mu} \phi)\pi_B(b \ltimes_{\beta,u^\sigma} \psi)$,
(3) $\pi_A(A^\sigma \ltimes_{\alpha,u^\alpha} H^0 \ltimes_{\mu} \phi, y \ltimes_{\mu} \psi) = A(\pi_X(x \ltimes_{\mu} \phi), \pi_X(y \ltimes_{\mu} \psi))$,
(4) $\pi_B((x \ltimes_{\mu} \phi, y \ltimes_{\mu} \psi) B^\sigma \ltimes_{\beta,u^\sigma} H^0) = (\pi_X(x \ltimes_{\mu} \phi), \pi_X(y \ltimes_{\mu} \psi))$,
(5) $\rho \cdot \pi_X(x \ltimes_{\mu} \phi) = \pi_X(\rho(f)(x \ltimes_{\mu} \phi))$, for any $x, y \in X^\lambda$, $a \in A^\sigma$, $b \in B^\sigma$, $h \in H$, $\phi, \psi \in H^0$.

Proof. Using the above discussion, we can prove the theorem in a straightforward way. □

Let $A$ be a unital $C^*$-algebra and $\rho$ a coaction of $H^0$ on $A$. Let $K$ be the $C^*$-algebra of all compact operators on a countably infinite dimensiona Hilbert space. Let $A^\ast = A \otimes K$ and $\rho^\ast = \rho \otimes \text{id}$. We identify $H^0 \otimes K$ with $K \otimes H^0$. Then $\rho^\ast$ is a coaction of $H^0$ on $A^\ast$.

**Lemma 4.10.** With the above notations, $\rho$ and $\rho^\ast$ are strongly Morita equivalent.

Proof. This is immediate by routine computations. □

Let $A$ and $B$ be unital $C^*$-algebras. Let $\rho$ and $\sigma$ be coactions of $H^0$ on $A$ and $B$, respectively. We suppose that $\rho$ and $\sigma$ are strongly Morita equivalent. Also, we suppose that there are an $A-B$-equivalence bimodule $X$ and a coaction $\lambda$ of $H^0$ on $X$ with respect to $(A, B, \rho, \sigma)$. Let $C$ be the linking $C^*$-algebra for $X$ and $\gamma$ the coaction of $H^0$ on $C$ induced by $\rho, \sigma$ and $\lambda$, which is defined in the above. Let $A^\ast = A \otimes K$, $B^\ast = B \otimes K$ and $C^\ast = C \otimes K$. Let $X^\ast = X \otimes K$, the exterior tensor product of $X$ and $K$, which is an $A^\ast-B^\ast$-equivalence bimodule in the usual way.
Let $\rho^* = \rho \otimes \text{id}$, $\sigma^* = \sigma \otimes \text{id}$ and $\gamma^* = \gamma \otimes \text{id}$. Let $\lambda^* = \lambda \otimes \text{id}$, which is a coaction of $H^0$ on $X^*$. Let

$$p = \begin{bmatrix} 1_A \otimes 1_M(K) & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \otimes 1_M(K) \end{bmatrix}.$$

Then $p$ and $q$ are full projections in $M(C^*)$ and $A^* \cong pC^*p$, $B^* \cong qC^*q$. We identify $A^*$ and $B^*$ with $pC^*p$ and $qC^*q$, respectively. By Brown [4, Lemma 2.5], there is a partial isometry $w \in M(C^*)$ such that $w^*w = p$, $ww^* = q$. Let $\theta$ be a map from $A^*$ to $C^*$ defined by

$$\theta(a) = wav^* = w \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} w^*$$

for any $a \in A$. Since $w^*w = p$ and $ww^* = q$, by easy computations, we can see that $\theta$ is an isomorphism of $A^*$ onto $B^*$.

**Proposition 4.11.** With the above notations, there is a unitary element $u \in M(B^*) \otimes H^0$ such that

$$(\theta \otimes \text{id}_{H^0}) \circ \rho^* \circ \theta^{-1} = \text{Ad}(u) \circ \sigma^*,$$

$$(u \otimes 1^0)(\sigma^* \otimes \text{id}_{H^0})(u) = (\text{id}_{M(B^*)} \otimes \Delta^0)(u),$$

where $\sigma^*$ is the strictly continuous coaction of $H^0$ on $M(B^*)$ extending the coaction $\sigma^*$ on $H^0$ to $B^*$.

**Proof.** We note that $\theta = \text{Ad}(w)$. Since $\rho^* = \gamma^*|_A$, and $\sigma^* = \gamma^*|_B$, we can obtain that

$$(\theta \otimes \text{id}_{H^0}) \circ \rho^* \circ \theta^{-1} = \text{Ad}((w \otimes 1^0)\gamma^*(w^*)) \circ \sigma^*,$$

where $\gamma^*$ is the strictly continuous coaction of $H^0$ on $M(C^*)$ extending the coaction $\gamma^*$ on $H^0$ to $C^*$. Let $u = (w \otimes 1^0)\gamma^*(w^*)$. By routine computations, we can show that $u$ is a desired unitary element in $M(B^*) \otimes H^0$. \qed

5. Equivariant Picard groups

Following Jansen and Waldmann [11], we shall define the equivariant Picard group of a $C^*$-algebra.

Let $A$ be a $C^*$-algebra and $H$ a finite dimensional $C^*$-Hopf algebra with its dual $C^*$-Hopf algebra $H^0$. Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. We denote by $(X, \lambda)$, a pair of an $A - A$-equivalence bimodule $X$ and a twisted coaction $\lambda$ of $H^0$ on $X$ with respect to $(A, A, \rho, u, \rho, u)$. Let $\text{Equiv}^{\rho, u}_H(A)$ be the set of all such pairs $(X, \lambda)$ as above. We define an equivalence relation $\sim$ in $\text{Equiv}^{\rho, u}_H(A)$ as follows: For $(X, \lambda), (Y, \mu) \in \text{Equiv}^{\rho, u}_H(A)$, $(X, \lambda) \sim (Y, \mu)$ if and only if there is an $A - A$-equivalence bimodule isomorphism $\pi$ of $X$ onto $Y$ such that $\mu \circ \pi = (\pi \otimes \text{id}_{H^0}) \circ \lambda$, that is, for any $x \in X$ and $h \in H$, $\pi(h \cdot x) = h \cdot \mu \pi(x)$. We denote by $[X, \lambda]$ the equivalence class of $(X, \lambda)$ in $\text{Equiv}^{\rho, u}_H(A)$. Let $\text{Pic}^{\rho, u}_H(A) = \text{Equiv}^{\rho, u}_H(A)/\sim$. We define the product in $\text{Pic}^{\rho, u}_H(A)$ as follows: For $(X, \lambda), (Y, \mu) \in \text{Equiv}^{\rho, u}_H(A)$,

$$[X, \lambda][Y, \mu] = [X \otimes_A Y, \lambda \otimes \mu],$$

where $\lambda \otimes \mu$ is the twisted coaction of $H^0$ on $X$ induced by the action $\cdot$ of $\lambda \otimes \mu$ of $H$ on $X$ defined in [11, Proposition 3.1]. By easy computations, we can see that the above product is well-defined. We regard $A$ as an $A - A$-equivalence bimodule in the usual way. We sometimes denote it by $A_A$. Also, we can regard a twisted coaction $\rho$ of $H^0$ on $C^*$-algebra $A$ as a twisted coaction of $H^0$ on the $A - A$-equivalence bimodule $A_A$ with respect to $(A, A, \rho, u, \rho, u)$. Then $[A_A, \rho]$ is the unit element in $\text{Pic}^{\rho, u}_H(A)$. Let $\tilde{\lambda}$ be the coaction of $H^0$ on $X$ defined by $\tilde{\lambda}(x) = \lambda(x)$ for any $x \in X$, which is also defined in Remark 4.2 (2). Then we can see that $[X, \tilde{\lambda}]$
is the inverse element of $[X, \lambda]$ in $\text{Pic}_H^{\rho,u}(A)$. By the above product, $\text{Pic}_H^{\rho,u}(A)$ is a group. We call it the $(\rho, u, H)$-equivariant Picard group of $A$.

Let $\text{Aut}_H^{\rho,u}(A)$ be the group of all automorphisms $\alpha$ of $A$ satisfying that $(\alpha \otimes \text{id}_H) \circ \rho = \rho \circ \alpha$ and $\text{Int}_H^{\rho,u}(A)$ the set of all generalized inner automorphisms $\text{Ad}(v)$ of $A$ satisfying that $\rho(v) = v \otimes 1^0$, where $v$ is a unitary element in $M(A)$. By easy computations $\text{Int}_H^{\rho,u}(A)$ is a normal subgroup of $\text{Aut}_H^{\rho,u}(A)$. Modifying \cite{9}, for each $\alpha \in \text{Aut}_H^{\rho,u}(A)$, we shall prove this Proposition 3.1.

With the above notations, we have the exact sequence

$$
\text{Pic}_H^{\rho,u}(A) \to \text{Aut}_H^{\rho,u}(A) \overset{\Phi}{\to} \text{Pic}_H^{\rho,u}(A),
$$

where $\Phi : \text{Aut}_H^{\rho,u}(A) \to \text{Pic}_H^{\rho,u}(A) : \alpha \mapsto [X_\alpha, \lambda_\alpha].$

Modifying \cite{9}, we can see that the map $\Phi$ is a homomorphism of $\text{Aut}_H^{\rho,u}(A)$ to $\text{Pic}_H^{\rho,u}(A)$. We have the similar result to \cite{9} Proposition 3.1.

**Proposition 5.1.** With the above notations, we have the exact sequence

$$
1 \to \text{Int}_H^{\rho,u}(A) \to \text{Aut}_H^{\rho,u}(A) \overset{\Phi}{\to} \text{Pic}_H^{\rho,u}(A),
$$

where $\iota$ is the inclusion map of $\text{Int}_H^{\rho,u}(A)$ to $\text{Aut}_H^{\rho,u}(A)$.

**Proof.** Modifying the proof of \cite{9} Proposition 3.1, we shall prove this Proposition. Let $v$ be a unitary element in $M(A)$ with $\rho(v) = v \otimes 1^0$. We show that $[X_{\text{Ad}(v)}, \lambda_{\text{Ad}(v)}] = [\lambda_A, \rho]$ in $\text{Pic}_H^{\rho,u}(A)$. Let $\pi$ be the map from $\lambda_A$ to $X_{\text{Ad}(v)}$ defined by $\pi(a) = av^*$ for any $a \in \lambda_A$. Then $\pi$ is an $A - A$-equivariance bimodule isomorphism. Also, for any $a \in \lambda_A$ and $h \in H$,

$$
\lambda_{\text{Ad}(v)} \pi(a) = h \cdot \lambda_{\text{Ad}(v)} (av^*) = [h(1) \cdot \rho a][h(2) \cdot \rho v^*] = [h \cdot \rho a]v^* = \pi(h \cdot \rho a).
$$

Thus $[X_{\text{Ad}(v)}, \lambda_{\text{Ad}(v)}] = [\lambda_A, \rho]$ in $\text{Pic}_H^{\rho,u}(A)$. Conversely, let $\alpha \in \text{Aut}_H^{\rho,u}(A)$ with $[X_\alpha, \lambda_\alpha] = [\lambda_A, \rho]$ in $\text{Pic}_H^{\rho,u}(A)$. Then there is an $A - A$-equivariance bimodule isomorphism $\pi$ of $\lambda_A$ onto $X_\alpha$ such that

$$
\lambda_\alpha \circ \pi = (\pi \otimes \text{id}) \circ \rho.
$$

By the proof of \cite{9} Proposition 3.1, $(\pi \circ \alpha^{-1}, \pi)$ is a double centralizer of $A$. Hence $(\pi \circ \alpha^{-1}, \pi) \in M(A)$. Let $v = (\pi \circ \alpha^{-1}, \pi)$. Then $v$ is a unitary element in $M(A)$ such that $\alpha = \text{Ad}(v^*)$. Furthermore, since $\lambda_\alpha \circ \pi = (\pi \otimes \text{id}) \circ \rho$, for any $a \in A$, $\lambda_\alpha(\pi(a)) = (\pi \otimes \text{id})(\rho(a))$. It follows that $\rho(av^*) = \rho(a)(v \otimes 1^0)^* \text{ for any } a \in A$. That is, $\rho(v) = v \otimes 1^0$. Therefore, we obtain the conclusion. \hfill \Box

Next, we shall show a similar result to \cite{9} Corollary 3.5. Let $A$ be a $C^*$-algebra and $X$ an $A - A$-equivariance bimodule. Let $\rho$ be a coaction of $H^0$ on $A$ and $\lambda$ a coaction of $H^0$ on $X$ with respect to $(A, A, \rho, \mu)$. Let $C$ be the linking $C^*$-algebra for $X$ and $\gamma$ the coaction of $H^0$ on $C$ induced by $\rho$ and $\lambda$ which is defined in Section 4. Furthermore, we suppose that $A$ is unital and that $\tilde{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$. Then $\rho$ is saturated by \cite{13} Section 4. Let $(\tilde{\rho})^*$ be the coaction of $H$ on $(A \rtimes_\rho H)^* \otimes H$ induced by the dual coaction $\tilde{\rho}$ of $H$ on $A \rtimes_\rho H$. Also, let $(\rho^*)^\gamma$ be the dual coaction of $\rho^*$ which is a coaction of $H$ on $A^* \rtimes_\rho H$. By their definitions, we can see that $(\tilde{\rho})^* = (\rho^*)^\gamma$, where we identify $(A \rtimes_\rho H)^* \cong A^* \rtimes_\rho H$. We denote them by $\tilde{\rho}^*$.  

**Lemma 5.2.** With the above notations, if $\tilde{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$, then $\tilde{\rho}^*(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(M(A^*)) \rtimes_\rho H \otimes H$. 

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Proof. This is immediate by straightforward computations.  

Let $C$ be the linking $C^*$-algebra for an $A^* - A^*$-equivalence bimodule $X^*\gamma$ and $\gamma$ the coaction of $H$ on $C$ induced by $\rho^s$ and $\lambda^s$.

**Lemma 5.3.** With the above notations, if $\hat{\rho}(1\times_\rho e) \sim (1\times_\rho e) \otimes 1$ in $(A\times_\rho H) \otimes H$, then $\hat{\gamma}(1_{M(C)} \times_\gamma e) \sim (1_{M(C)} \times_\gamma e) \otimes 1$ in $(M(C) \times_\gamma H) \otimes H$.

**Proof.** This is immediate by Lemmas 4.7 and 5.2. □

**Lemma 5.4.** With the above notations, we suppose that $\hat{\rho}(1\times_\rho e) \sim (1\times_\rho e) \otimes 1$ in $(A\times_\rho H) \otimes H$. Let $\Phi$ be the homomorphism of $\text{Aut}_{H}^{\rho^s}(A^*) \otimes \text{Pic}_{H}^{\rho^s}(A^*)$ defined by $\Phi(\alpha) = [X_\alpha, \lambda_\alpha]$ for any $\alpha \in \text{Aut}_{H}^{\rho^s}(A^*)$. Then $\Phi$ is surjective.

**Proof.** Let $[X, \lambda]$ be any element in $\text{Pic}_{H}^{\rho^s}(A^*)$. Let

$$X^\lambda = \{ x \in X \mid \lambda(x) = x \otimes 1^0 \}.$$

Since $\hat{\rho}(1\times_\rho e) \sim (1\times_\rho e) \otimes 1$ in $(A\times_\rho H) \otimes H$, by Lemma 5.3, $\hat{\rho}(1\times_\rho e) \sim (1\times_\rho e) \otimes 1$ in $(M(A^*) \times_\rho H) \otimes H$. Since $X$ is an $A^* - A^*$-equivalence bimodule, by Lemma 5.8 and Theorem 4.9, $X^\lambda$ is an $(A^*)^\rho^s - (A^*)^\rho^s$-equivalence bimodule, where $(A^*)^\rho^s$ is the fixed point $C^*$-subalgebra of $A^*$ for the coaction $\rho^s$. Let $C$ be the linking $C^*$-algebra for $X$ and $\gamma$ the coaction of $H^0$ on $C$ induced by $\rho^s$ and $\lambda$. Let $C^\gamma$ be the fixed point $C^*$-algebra of $C$ for $\gamma$. Then by Lemma 4.6, $C^\gamma$ is isomorphic to $C_0$, the linking $C^*$-algebra for $X^\lambda$. We identify $C^\gamma$ with $C_0$. Let

$$p = \begin{bmatrix} 1_A \otimes 1_M(K) & 0 \\ 0 & 1_{A} \otimes 1_M(K) \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1_{A} \otimes 1_M(K) \end{bmatrix}.$$

Then $p$ and $q$ are projections in $M(C)^\gamma$. Since $M(C)^\gamma = M(C^\gamma)$ by Lemmas 2.9 and 4.7, $p$ and $q$ are full for $C^\gamma$. By the proof of [5, Theorem 3.4], there is a partial isometry $w \in M(C)^\gamma$ such that

$$w^*w = p, \quad q = ww^*.$$

Hence $w \in M(C)$. Let $\alpha$ be the map on $A^*$ defined by

$$\alpha(a) = w^*aw = w^*\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}w$$

for any $a \in A^*$. By routine computations, $\alpha$ is an automorphism of $A^*$. Let $\pi$ be a linear map from $X$ to $X_\alpha$ defined by

$$\pi(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}w = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}wp$$

for any $x \in X$. In the same way as in the proof of [5, Lemma 3.3], we can see that $\pi$ is an $A^* - A^*$-equivalence bimodule isomorphism of $X$ onto $X_\alpha$. For any $a \in A^*$,

$$(\rho^s \circ \alpha)(a) = \rho^s(w^*aw) = \gamma(w^*)\begin{bmatrix} 0 & 0 \\ 0 & \rho^s(a) \end{bmatrix}\gamma(w) = (\alpha \otimes \text{id}_{H^0})(\rho^s(a))$$

since $w \in M(C)^\gamma$. Hence $\alpha \in \text{Aut}_{H}^{\rho^s}(A^*)$. Furthermore, for any $x \in X$,

$$(\lambda_\alpha \circ \pi)(x) = \lambda_\alpha(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix})w = \rho^s(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix})w)$$

$$= \gamma(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix})w = \begin{bmatrix} \lambda(x) & 0 \\ 0 & 0 \end{bmatrix}(w \otimes 1^0) = (\pi \otimes \text{id}_{H^0})(\lambda(x)),$$

where we identify $K \otimes H^0$ with $H^0 \otimes K$. Thus $\Phi(\alpha) = [X, \lambda]$. Therefore, we obtain the conclusion. □
respectively. We suppose that 

\( a \) for any \( \lambda \)

Lemma 5.6.

Proof. This is immediate by Proposition 5.1 and Lemma 5.4.

Since the following lemma is obtained in a straightforward way, we omit its proof:

Lemma 5.6. Let \((\rho,u)\) and \((\sigma,v)\) be twisted coactions on \(C^*\)-algebras \(A\) and \(B\), respectively. We suppose that \((\rho,u)\) is strongly Morita equivalent to \((\sigma,v)\). Then \(\text{Pic}^\rho_H(A) \cong \text{Pic}^\rho_H(B)\).

6. Ordinary Picard groups and equivariant Picard groups

In this section, we shall investigate the relation between ordinary Picard groups and equivariant Picard groups. Let \(\rho\) be a coaction of \(H^0\) on a \(C^*\)-algebra \(A\) and let \(f_{\rho}\) be the map from \(\text{Pic}^\rho_H(A)\) to \(\text{Pic}(A)\) defined by

\[ f_{\rho} : \text{Pic}^\rho_H(A) \to \text{Pic}(A) : [X,\lambda] \mapsto [X], \]

where \(\text{Pic}(A)\) is the ordinary Picard group of \(A\). Clearly \(f_{\rho}\) is a homomorphism of \(\text{Pic}^\rho_H(A)\) to \(\text{Pic}(A)\). Let \(\text{Aut}(A)\) be the group of all automorphisms of \(A\) and let \(\alpha \in \text{Aut}(A)\). Let \(X_\alpha\) be the \(A - A\)-equivalence bimodule induced by \(\alpha\) defined in Section 3. Let \(\lambda\) be a coaction of \(H^0\) on \(X_\alpha\) with respect to \((A,A,\rho,\rho)\). Then for any \(a \in A\) and \(x,y \in X_\alpha\),

1. \(\lambda(ax) = \lambda(a \cdot x) = \rho(a) \cdot \lambda(x) = \rho(a)\lambda(x)\),
2. \(\lambda(x(\alpha)) = \lambda(x \cdot a) = \rho(a) = \lambda(\alpha)(\alpha \otimes \text{id})(\rho(a))\),
3. \(\rho(x^*) = \rho(A(x,y)) = A_{\rho,H^0}(\lambda(x),\lambda(y)) = \lambda(x^*)\lambda(y)^*\),
4. \(\rho(\alpha^{-1}(x^*)y) = \rho(x^*)\rho(y) = \lambda(x)\lambda(y))\lambda_{\rho,H^0} = (\alpha^{-1} \otimes \text{id})(\lambda(x^*)\lambda(y))\),
5. (\text{id} \oplus \Delta^0)(\lambda(x)) = x,
6. (\lambda \otimes \text{id})(\lambda(x)) = (\text{id} \oplus \Delta^0)(\lambda(x)).

Let \(\{u_\gamma\}\) be an approximate unit of \(A\). Then \(\lambda(u_\gamma) \in X_\alpha \otimes H^0\). Since \(X_\alpha = A\) as vector spaces, we regard \(\lambda(u_\gamma)\) as an element in \(A \otimes H^0\).

Lemma 6.1. With the above notations, we regard \(\lambda(u_\gamma)\) as an element in \(A \otimes H^0\). Then \(\{\lambda(u_\gamma)\}\) converges to a unitary element in \(M(A \otimes H^0)\) strictly and the unitary element does not depend on the choice of an approximate unit of \(A\).

Proof. Let \(a \in A\) and \(x \in A \otimes H^0\). Then by Equation (2),

\[
||\lambda(u_\gamma) - \lambda(u_{\gamma'})||(\alpha \otimes \text{id})(\rho(a))x) = ||\lambda(u_\gamma - u_{\gamma'})\alpha(a)(\alpha \otimes \text{id})(x)||
\leq ||\lambda(u_\gamma - u_{\gamma'})\alpha(a)|| ||x||
\]

since \(\lambda\) is isometric. Since \(\rho(A)(A \otimes H^0)\) is dense in \(A \otimes H^0\), \(\{\lambda(u_\gamma)\}\) is a Cauchy net for any \(y \in A \otimes H^0\). Similarly by Equation (1), \(\{y\lambda(u_\gamma)\}\) is also a Cauchy net for any \(y \in A \otimes H^0\). Thus \(\{\lambda(u_\gamma)\}\) converges to some element \(u \in M(A \otimes H^0)\) strictly. We note

\[
\lim_{\gamma \to \infty} \rho(u_\gamma) = \lim_{\gamma \to \infty} \rho(u_\gamma) = \rho(\lim_{\gamma \to \infty} u_\gamma) = \rho(1) = 1,
\]

\[
\lim_{\gamma \to \infty} \alpha^{-1}(u_\gamma) = \lim_{\gamma \to \infty} \alpha^{-1}(u_\gamma) = \alpha^{-1}(\lim_{\gamma \to \infty} u_\gamma) = \alpha^{-1}(1) = 1,
\]

where the limits are taken under the strict topologies in \(M(A \otimes H^0)\) and \(M(A)\), respectively and \(\alpha^{-1}\) is an automorphism of \(M(A)\) extending \(\alpha^{-1}\) to \(M(A)\), which is strictly continuous on \(M(A)\). Hence by Equations (3), (4), we can see that \(u\) is a unitary element in \(M(A \otimes H^0)\). Let \(\{v_\beta\}\) be another approximate unit of \(A\) and
let \( v \) be the limit of \( \lambda(v_\beta) \) under the strict topology in \( M(A \otimes H^0) \). Then by the above discussion, we have that
\[
\| \lambda(u) - \lambda(v_\beta) \| (\alpha \otimes \text{id})(\rho(a)x) \leq \| (u_\gamma - v_\beta) \alpha(a) \| \| x \|
\]
for any \( a \in A \) and \( x \in A \otimes H^0 \). Since \( \rho(A)(A \otimes H^0) \) is dense in \( A \otimes H^0 \), \( u = v \). □

**Lemma 6.2.** Let \( u \) be as in the proof of Lemma 6.1. Then \( u \) satisfies that \( \lambda(x) = \rho(x)u \) for any \( x \in X_\alpha \), \( \rho(\alpha(a)) = u(\alpha \otimes \text{id})(a)u^* \) for any \( a \in A \) and that \( (\rho \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u) \).

**Proof.** Let \( \{ u_\gamma \} \) be an approximate unit of \( A \). By Equation (1), for any \( x \in X_\alpha \), \( \lambda(xu_\gamma) = \rho(x)u_\gamma \). Thus \( \lambda(x) = \rho(x)u \). Also, by Equation (2) for any \( a \in A \),
\[
\lambda(u_\gamma \alpha(a)) = \lambda(u_\gamma)(\alpha \otimes \text{id})(\rho(a)).
\]
Hence \( \lambda(\alpha(a)) = u(\alpha \otimes \text{id})(\rho(a)) \). Since \( \lambda(\alpha(a)) = \rho(\alpha(a))u \) for any \( a \in A \) by the above discussion, for any \( a \in A \),
\[
\rho(\alpha(a))u = u(\alpha \otimes \text{id})(\rho(a)).
\]
for any \( a \in A \). Since \( u \) is a unitary element in \( M(A \otimes H^0) \),
\[
\rho(u) = u(\alpha \otimes \text{id})(\rho(a))u^*
\]
for any \( a \in A \). Furthermore, for any \( a \in A 
\[
(\lambda \otimes \text{id})(\lambda(u_\gamma a)) = (\lambda \otimes \text{id})(\lambda(u_\gamma)(\alpha \otimes \text{id})(\rho(\alpha^{-1}(a)))
\]
by Equations (1), (2). Thus Equation (2)
\[
(\lambda \otimes \text{id})(\lambda(a)) = \lim_{\gamma \to \infty} (\rho \otimes \text{id})(u)(\lambda \otimes \text{id})(\alpha \otimes \text{id})(\rho \circ \alpha^{-1})(a)
\]
\[
= \lim_{\gamma \to \infty} (\rho \otimes \text{id})(u)(\lambda(u_\gamma)(\alpha \otimes \text{id})(\rho \circ \alpha^{-1})(a))
\]
\[
= \lim_{\gamma \to \infty} (\rho \otimes \text{id})(u)(u \otimes 1^0)(\alpha \otimes \text{id})(\rho \circ \alpha^{-1})(a)
\]
\[
= (\rho \otimes \text{id})(u)(u \otimes 1^0)(\text{id} \otimes \Delta^0)(\rho \circ \alpha^{-1})(a)
\]
Also, by Equation (2)
\[
(\text{id} \otimes \Delta^0)(\lambda(u_\gamma a)) = (\text{id} \otimes \Delta^0)(\lambda(u_\gamma)(\alpha \otimes \text{id})(\rho \circ \alpha^{-1})(a))
\]
\[
= (\text{id} \otimes \Delta^0)(\lambda(u_\gamma)(\text{id} \otimes \Delta^0)(\rho \circ \alpha^{-1})(a))
\]
Thus
\[
(\text{id} \otimes \Delta^0)(\lambda(a)) = (\text{id} \otimes \Delta^0)(u)(\text{id} \otimes \Delta^0)(\rho \circ \alpha^{-1})(a).
\]
By Equation (6)
\[
[(\rho \otimes \text{id})(u)(u \otimes 1^0) - (\text{id} \otimes \Delta^0)(u)((\text{id} \otimes \Delta^0)(\rho \circ \alpha^{-1})(a) = 0
\]
for any \( a \in A \). Therefore,
\[
(\rho \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u).
\]
□

**Remark 6.3.** By Lemma 6.2, we can see that the coaction \( (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1} \) of \( H^0 \) on \( A \) is exterior equivalent to \( \rho \).
Conversely, let $u$ be a unitary element in $M(A \otimes H^0)$ satisfying that
\[ \rho = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \alpha^{-1}, \quad (\rho \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u). \]
Let $\lambda_u$ be the linear map from $X_\alpha$ to $X_\alpha \otimes H^0$ defined by
\[ \lambda_u(x) = \rho(x)u \]
for any $x \in X_\alpha$. Then by routine computations, we can see that $\lambda_u$ is a coaction of $H^0$ on $X_\alpha$ with respect to $(A, A, \rho, \rho)$.

**Proposition 6.4.** With the above notations, the following conditions are equivalent:

1. $[X_\alpha] \in \text{Im}f_\rho$.
2. There is a unitary element $u \in M(A \otimes H^0)$ such that
\[ \rho = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \alpha^{-1}, \quad (\rho \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u). \]

**Proof.** This is immediate by Lemma 6.2 and the above discussion. \(\square\)

Let $u$ be a unitary element in $M(A \otimes H^0)$ satisfying Condition (2) in Proposition 6.4. Let $\lambda_u$ be as above. We call $\lambda_u$ the coaction of $H^0$ on $X_\alpha$ with respect to $(A, A, \rho, \rho)$ induced by $u$.

Let $\alpha, \beta \in \text{Aut}(A)$ satisfying that there are unitary elements $u, v \in M(A \otimes H^0)$ such that
\[ \rho = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}, \quad (\rho \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u), \]
\[ \rho = \text{Ad}(v) \circ (\beta \otimes \text{id}) \circ \rho \circ \beta^{-1}, \quad (\rho \otimes \text{id})(v)(v \otimes 1^0) = (\text{id} \otimes \Delta^0)(v). \]

**Lemma 6.5.** With the above notations, we have the following:
\[ (\rho \otimes \text{id})(u(\alpha \otimes \text{id})(v))(u(\alpha \otimes \text{id})(v) \otimes 1^0) = (\text{id} \otimes \Delta^0)(u(\alpha \otimes \text{id})(v)). \]

**Proof.** By routine computations, we can see that
\[ ((\alpha \circ \beta) \otimes \text{id}) \circ \rho \circ (\alpha \circ \beta)^{-1} = \text{Ad}((\alpha \otimes \text{id})(v^*)) \circ \text{Ad}(u^*) \circ \rho. \]
Thus we obtain that
\[ \rho = \text{Ad}(u(\alpha \otimes \text{id})(v)) \circ ((\alpha \circ \beta) \otimes \text{id}) \circ \rho \circ (\alpha \circ \beta)^{-1}. \]
Since $\rho \circ \alpha = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho$,
\[ (\rho \otimes \text{id})(u(\alpha \otimes \text{id})(v)) = (u \otimes 1^0)(\alpha \otimes \text{id})(\rho(\alpha \otimes \text{id})(v))(u \otimes 1^0)^*. \]
Hence by routine computations, we can see that
\[ (\rho \otimes \text{id})(u(\alpha \otimes \text{id})(v))(u(\alpha \otimes \text{id})(v) \otimes 1^0) = (\text{id} \otimes \Delta^0)(u(\alpha \otimes \text{id})(v)). \]
\(\square\)

Let $\alpha, \beta$ and $u, v$ be as above. Let $\lambda_u$ and $\lambda_v$ be coactions of $H^0$ on $X_\alpha$ and $X_\beta$ with respect to $(A, A, \rho, \rho)$ induced by $u$ and $v$, respectively. Let $u\sharp v = u(\alpha \otimes \text{id})(v) \in M(A \otimes H^0)$. By Lemma 6.3, we can define the coaction $\lambda_{u\sharp v}$ of $H^0$ on $X_{\alpha \otimes \beta}$ with respect to $(A, A, \rho, \rho)$ induced by $u\sharp v$. By easy computations, we can see that $X_\alpha \otimes X_\beta$ is isomorphic to $X_{\alpha \otimes \beta}$ by an $A - A$-equivalence bimodule isomorphism $\pi$
\[ \pi : X_\alpha \otimes_A X_\beta \to X_{\alpha \otimes \beta} : x \otimes y \mapsto x\alpha(y). \]
We identify $X_\alpha \otimes_A X_\beta$ with $X_{\alpha \otimes \beta}$ by the above $A - A$-equivalence bimodule isomorphism $\pi$.

**Lemma 6.6.** With the above notations, for $[X_\alpha, \lambda_u], [X_\beta, \lambda_v] \in \text{Pic}^H_\rho(A)$,
\[ [X_\alpha, \lambda_u][X_\beta, \lambda_v] = [X_{\alpha \otimes \beta}, \lambda_{u\sharp v}] \in \text{Pic}^H_\rho(A), \]
where $u\sharp v = u(\alpha \otimes \text{id})(v) \in M(A \otimes H^0)$. 

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Proof. By the definition of the product in $\text{Pic}_H^0(A)$,
\[ [X_\alpha, \lambda_u][X_\beta, \lambda_v] = [X_\alpha \otimes_A X_\beta, \lambda_u \otimes \lambda_v]. \]
Hence it suffices to show that
\[ \pi(h \cdot \lambda_{x \cdot h}, x \otimes y) = h \cdot \lambda_{x \otimes y} \pi(x \otimes y) \]
for any $x \in X_\alpha, y \in X_\beta$ and $h \in H$. For any $x \in X_\alpha, y \in X_\beta$ and $h \in H$,
\[
\begin{align*}
\pi(h \cdot \lambda_{x \cdot h}, x \otimes y) &= \pi((h(1) \cdot \lambda_x, x) \otimes [h(2) \cdot \lambda_y]) \\
&= \pi(h(1) \cdot \rho x)[\bar{\alpha}(h(3) \cdot \rho y)]
\end{align*}
\]
\[
\begin{align*}
&= [h(1) \cdot \rho x]\bar{\alpha}(h(3) \cdot \rho y) \pi(h(4)) \\
&= [h(1) \cdot \rho x]\bar{\alpha}(h(3) \cdot \rho y) \bar{\alpha}(h(4)).
\end{align*}
\]
Since $\rho \circ \alpha = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho$, 
\[
\begin{align*}
\pi(h \cdot \lambda_{x \cdot h}, x \otimes y) &= [h(1) \cdot \rho x][\bar{\alpha}(h(3) \cdot \rho y)]
\end{align*}
\]
\[
\begin{align*}
&= [h(1) \cdot \rho x]\bar{\alpha}(h(3) \cdot \rho y)(u(\alpha \otimes \text{id})(v)(h(2)) \\
&= h \cdot \lambda_{x \otimes y} \pi(x \otimes y).
\end{align*}
\]
Therefore, we obtain the conclusions. \qed

Corollary 6.7. With the above notations, for any $[X_\alpha, \lambda_u] \in \text{Pic}_H^0(A)$,
\[ [X_\alpha, \lambda_u]^{-1} = [X_{\alpha^{-1}}, \lambda_{U_{\alpha^{-1}}(\alpha \otimes \text{id})}] \in \text{Pic}_H^0(A). \]

Proof. This is immediate by Lemma 6.6 and routine computations. \qed

For any $\alpha \in \text{Aut}(A)$, let $U_0^p(M(A \otimes H^0))$ be the set of all unitary elements $u \in M(A \otimes H^0)$ satisfying that
\[ \rho = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}, \quad (\rho \otimes \text{id})(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u). \]

Lemma 6.8. With the above notations, for any $\alpha \in \text{Aut}(A)$, we have the following:
(1) For any $u \in U_0^p(M(A \otimes H^0))$ and $v \in U_0^p(M(A \otimes H^0))$, $uv \in U_0^p(M(A \otimes H^0))$,
(2) For any $u, v \in U_0^p(M(A \otimes H^0))$, $uv^* \in U_0^p(M(A \otimes H^0))$.

Proof. (1) This is immediate by Lemma 6.6.
(2) By Corollary 6.7, $\alpha^{-1}(\alpha \otimes \text{id})(v^*) \in U_0^p(M(A \otimes H^0))$. Hence $uv^* \in U_0^p(M(A \otimes H^0))$. \qed

Lemma 6.9. Let $u \in U_0^p(M(A \otimes H^0))$. Then the following conditions are equivalent:
(1) $[A_A, \lambda_u] = [A_A, \rho]$ in $\text{Pic}_H^0(A)$,
(2) There is a unitary element $w \in M(A) \cap A'$ such that $u = (w^* \otimes 1^0)p(w)$.

Proof. We suppose Condition (1). Then there is an $A - A'$-equivalence bimodule automorphism $\pi$ of $A_A$ such that
\[ \rho(\pi(x)) = (\pi \otimes \text{id})(\lambda_u(x)) = (\pi \otimes \text{id})(\rho(x)u) \]
for any $x \in A_A$. We note that $\pi \in \text{Aut}_{A_A}(A_A)$ and that
\[ A_A \cong A' \cap B_A(A_A) \cong A' \cap M(A). \]
Hence there is a unitary element $w \in A' \cap M(A)$ such that $\pi(x) = wx$ for any $x \in A$. Thus for any $x \in A$
\[ \rho(wx) = (w \otimes 1^0)p(x)u. \]
Therefore $u = (w^* \otimes 1^0)\rho(w)$. Next we suppose Condition (2). Let $\pi$ be the $A - A$-equivalence bimodule automorphism of $AA_A$ defined by $\pi(x) = wx$ for any $x \in AA_A$. Then for any $x \in AA_A$

$$\rho(\pi(x)) = \rho(wx) = \rho(xw) = \rho(x)\rho(w) = \rho(x)(w \otimes 1^0)u = (w \otimes 1^0)\rho(x)u = (\pi \otimes \text{id})(\lambda_u(x)).$$

Thus we obtain Condition (1). \hfill \Box

Corollary 6.10. Let $\alpha \in \text{Aut}(A)$ and $u, v \in U^0_{\text{id}}(M(A \otimes H^0))$. Then the following conditions are equivalent:

1. $[X_\alpha, \lambda_u] = [X_\alpha, \lambda_v]$ in $\text{Pic}^0_H(A)$.
2. There is a unitary element $w \in M(A) \cap A'$ such that $u = (w^* \otimes 1^0)\rho(w)v$.

Proof. We suppose Condition (1). By Lemma 6.6 and Corollary 6.7 we can see that $[AA_A, \lambda_{uv^*}] = [AA_A, \rho]$ in $\text{Pic}^0_H(A)$. Thus by Lemma 6.9 there is a unitary element in $w \in M(A) \cap A'$ such that $wv^* = (w^* \otimes 1^0)\rho(w)$. Hence we obtain Condition (2). Conversely we suppose Condition (2). Then there is a unitary element $w \in M(A) \cap A'$ such that $wv^* = (w^* \otimes 1^0)\rho(w)$. Hence $[AA_A, \lambda_{uv^*}] = [AA_A, \rho]$ in $\text{Pic}^0_H(A)$. Since $[X_\alpha, \lambda_u] [X_\alpha, \lambda_v]^{-1} = [AA_A, \lambda_{uv^*}]$ in $\text{Pic}^0_H(A)$ by Lemma 6.6 and Corollary 6.7 $[X_\alpha, \lambda_u] = [X_\alpha, \lambda_v]$ in $\text{Pic}^0_H(A)$.

We shall compute Ker$f_\rho$, the kernel of $f_\rho$. Let $[X, \lambda] \in \text{Pic}^0_H(A)$. Then by Proposition 6.3 we can see that $[X] = [AA_A]$ in Pic$(A)$ if and only if there is a unitary element $u \in U^0_{\text{id}}(M(A \otimes H^0))$ such that $[X, \lambda] = [AA_A, \lambda_u]$ in $\text{Pic}^0_H(A)$. Furthermore by Corollary 6.10 $[AA_A, \lambda_u] = [AA_A, \lambda_v]$ in $\text{Pic}^0_H(A)$ if and only if there is a unitary element $w \in M(A) \cap A'$ such that $u = (w^* \otimes 1^0)\rho(w)v$, where $u, v \in U^0_{\text{id}}(M(A \otimes H^0))$. We define an equivalence relation in $U^0_{\text{id}}(M(A \otimes H^0))$ as follows: Let $u, v \in U^0_{\text{id}}(M(A \otimes H^0))$, written $u \sim v$ if there is a unitary element $w \in M(A) \cap A'$ such that

$$u = (w^* \otimes 1^0)\rho(w)v.$$

Let $U^0_{\text{id}}(M(A \otimes H^0))/\sim$ be the set of all equivalence classes in $U^0_{\text{id}}(M(A \otimes H^0))$. We denote by $[u]$ the equivalence class of $u \in U^0_{\text{id}}(M(A \otimes H^0))$. By Lemma 6.8 $U^0_{\text{id}}(M(A \otimes H^0))$ is a group. Hence $U^0_{\text{id}}(M(A \otimes H^0))/\sim$ is a group by easy computations.

Proposition 6.11. With the above notations, Ker$f_\rho \cong U^0_{\text{id}}(M(A \otimes H^0))/\sim$ as groups.

Proof. Let $\pi$ be a map from $U^0_{\text{id}}(M(A \otimes H^0))/\sim$ to Ker$f_\rho$ defined by

$$\pi([u]) = [AA_A, \lambda_u]$$

for any $u \in U^0_{\text{id}}(M(A \otimes H^0))$. By the above discussions, we can see that $\pi$ is well-defined and bijective. For any $u, v \in U^0_{\text{id}}(M(A \otimes H^0))$

$$\pi([u])\pi([v]) = [AA_A, \lambda_u][AA_A, \lambda_v] = [AA_A, \lambda_{uv}] = \pi([uv])$$

by Lemma 6.6. Therefore, we obtain the conclusion. \hfill \Box

We recall that there is a homomorphism $\Phi$ of $\text{Aut}^0_H(A^*)$ to $\text{Pic}^0_H(A^*)$ defined by

$$\Phi(\alpha) = [X_\alpha, \lambda_\alpha]$$

for any $\alpha \in \text{Aut}^0_H(A^*)$, where $\lambda_\alpha$ is a coaction of $H^0$ on $X_\alpha$ induced by $\rho^*$ (See Section 5). Then the following results hold:
Lemma 6.12. With the above notations, for any $\alpha \in \text{Aut}_H^\rho(A^s)$

$$(f_{\rho^s} \circ \Phi)(\alpha) = [X_\alpha]$$

in $\text{Pic}(A^s)$. Furthermore, if $\hat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$, then

$$\text{Im}f_{\rho^s} = \{ [X_\alpha] \in \text{Pic}(A^s) \mid \alpha \in \text{Aut}_H^\rho(A^s) \}.$$  

Proof. This is immediate by Proposition 6.11 and Lemma 6.12. \hfill \square

Let $G$ be a subgroup of $\text{Pic}(A^s)$ defined by

$$G = \{ [X_\alpha] \in \text{Pic}(A^s) \mid \alpha \in \text{Aut}_H^\rho(A^s) \}.$$  

Theorem 6.13. Let $H$ be a finite dimensional $C^\ast$-Hopf algebra with its dual $C^\ast$-algebra $H^0$. Let $A$ be a unital $C^\ast$-algebra and $\rho$ a coaction of $H^0$ on $A$ with $\hat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$. Let $A^s = A \otimes K$ and $\rho^s$ the coaction of $H^0$ on $A^s$ induced by $\rho$. Let $U_{id}^\rho(M(A^s \otimes H^0))$ be the group of all unitary elements $u \in M(A^s \otimes H^0)$ satisfying that

$$\rho^s = \text{Ad}(u) \circ \rho^s, \quad (\rho^s \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u).$$  

Then we have the following exact sequence:

$$1 \longrightarrow U_{id}^\rho(M(A^s \otimes H^0))/\sim \longrightarrow \text{Pic}^\rho_\rho(A^s) \longrightarrow G \longrightarrow 1,$$

where “$\sim$” is the equivalence relation in $U_{id}^\rho(M(A^s \otimes H^0))$ defined in this section.

Proof. This is immediate by Proposition 6.11 and Lemma 6.12. \hfill \square

Let $A$ be a UHF-algebra of type $N^\infty$, where $N = \dim H$. Let $\rho$ be the coaction of $H^0$ on $A$ defined in \[16\] Section 7, which has the Rohlin property. We note that $\hat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$ by \[16\] Definition 5.1.

Corollary 6.14. With the above notations, we have the following exact sequence:

$$1 \longrightarrow U_{id}^\rho(M(A^s \otimes H^0)) \longrightarrow \text{Pic}^\rho_\rho(A^s) \longrightarrow G \longrightarrow 1.$$  

Proof. Since $A^s$ is simple, $M(A^s) \cap (A^*)' = C1$ by Pedersen \[21\] Corollary 4.4.8]. Therefore by Theorem 6.13 we obtain the conclusion. \hfill \square

7. Equivariant Picard groups and crossed products

Let $(\rho, u)$ be a twisted coaction of $H^0$ on a unital $C^\ast$-algebra $A$. Let $f$ be a map from $\text{Pic}_H^{\rho,u}(A)$ to $\text{Pic}_H^\rho(A \rtimes_{\rho,u} H)$ defined by

$$f([X, \lambda]) = [X \rtimes \lambda H, \hat{\lambda}]$$  

for any $[X, \lambda] \in \text{Pic}_H^{\rho,u}(A)$. In this section, we shall show that $f$ is an isomorphism of $\text{Pic}_H^{\rho,u}(A)$ onto $\text{Pic}_H^\rho(A \rtimes_{\rho,u} H)$. We can see that $f$ is well-defined in a straightforward way. We show that $f$ is a homomorphism of $\text{Pic}_H^{\rho,u}(A)$ to $\text{Pic}_H^\rho(A \rtimes_{\rho,u} H)$. Let $A$, $B$ and $C$ be unital $C^\ast$-algebras and $(\rho, u), (\sigma, v)$ and $(\gamma, w)$ be twisted coactions of $H^0$ on $A$, $B$ and $C$, respectively. Let $\lambda$ be a twisted coaction of $H^0$ on an $A - B$-equivalence bimodule $X$ with respect to $(A, B, \rho, u, \sigma, v)$. Also, let $\mu$ be a twisted coaction of $H^0$ on a $B - C$-equivalence bimodule $Y$ with respect to $(B, C, \sigma, v, \gamma, w)$. Let $\Phi$ be a linear map from $(X \otimes_B Y) \rtimes_{\lambda \otimes \mu} H$ to $(X \rtimes_\lambda H) \otimes_B \rtimes_{\sigma, \gamma, \mu} H$ defined by

$$\Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h) = (x \rtimes_{\lambda} H) \otimes_{B \rtimes_{\sigma, \gamma, \mu}} (y \rtimes_{\mu} H)$$  

for any $x \in X$, $y \in Y$ and $h \in H$. By routine computations, $\Phi$ is well-defined. We note that $(X \rtimes_\lambda H) \otimes_B \rtimes_{\sigma, \gamma, \mu} H$ is consisting of finite sums of elements in the form $(x \rtimes_{\lambda} 1) \otimes (y \rtimes_{\mu} h)$ by the definition of $(X \rtimes_\lambda H) \otimes_B \rtimes_{\sigma, \gamma, \mu} H$, where
On the other hand, we obtain that \( s \), satisfying that \( x, y \in X, y \in Y \) and \( h \in H \). Hence we can see that \( \Phi \) is bijective and its inverse map \( \Phi^{-1} \) is:

\[
(x \rtimes_{\lambda} H) \otimes_{B \rtimes_{\sigma,U}} (Y \rtimes_{\mu} H) = (X \rtimes_{\lambda} H) : (x \rtimes_{\lambda} 1) \otimes (y \rtimes_{\mu} h) \mapsto x \rtimes_{\lambda} y \rtimes_{\lambda} \mu h.
\]

Furthermore, we have the following lemmas:

**Lemma 7.1.** With the above notations,

\[
\langle \Phi(x \otimes y \rtimes_{\lambda} \mu h), \Phi(z \otimes r \rtimes_{\lambda} \mu l) \rangle = \langle A \rtimes_{\sigma,U} H \rangle \langle x \rtimes_{\lambda} H, y \rtimes_{\mu} H \rangle \rtimes_{\sigma,U} \mu h.
\]

**Proof.** We can prove this lemma by routine computations. Indeed,

\[
\langle A \rtimes_{\sigma,U} H \rangle \langle x \rtimes_{\lambda} H, y \rtimes_{\mu} H \rangle \rtimes_{\sigma,U} \mu h = \langle A \rtimes_{\sigma,U} H \rangle \langle x \rtimes_{\lambda} H, y \rtimes_{\mu} H \rangle \rtimes_{\sigma,U} \mu h,
\]

On the other hand,

\[
\langle A \rtimes_{\sigma,U} H \rangle \langle x \rtimes_{\lambda} H, y \rtimes_{\mu} H \rangle \rtimes_{\sigma,U} \mu h = \langle A \rtimes_{\sigma,U} H \rangle \langle x \rtimes_{\lambda} H, y \rtimes_{\mu} H \rangle \rtimes_{\sigma,U} \mu h.
\]

Thus we obtain that

\[
\langle A \rtimes_{\sigma,U} H \rangle \langle x \rtimes_{\lambda} H, y \rtimes_{\mu} H \rangle \rtimes_{\sigma,U} \mu h = \langle A \rtimes_{\sigma,U} H \rangle \langle x \rtimes_{\lambda} H, y \rtimes_{\mu} H \rangle \rtimes_{\sigma,U} \mu h.
\]

**Lemma 7.2.** With the above notations, \( \Phi \) is an \( A \rtimes_{\sigma,U} H \rtimes C \rtimes_{\sigma,U} H \)-equivalence bimodule isomorphism of \( (X \rtimes_{\lambda} H) \otimes_{B \rtimes_{\sigma,U} H} (Y \rtimes_{\mu} H) \) satisfying that

\[
\Phi(\phi \rtimes_{\lambda \rtimes_{\mu}} (x \rtimes y \rtimes_{\lambda} \mu h)) = \Phi(x \rtimes y \rtimes_{\lambda} \mu h)
\]

for any \( x \in X, y \in Y, h \in H \) and \( \phi \in H^0 \).

**Proof.** By Lemma 7.1 and the remark after Jensen and Thomsen [12, Definition 1.1.18], we see that \( \Phi \) is an \( A \rtimes_{\sigma,U} H \rtimes C \rtimes_{\sigma,U} H \)-equivalence bimodule isomorphism of \( (X \rtimes_{\lambda} H) \otimes_{B \rtimes_{\sigma,U} H} (Y \rtimes_{\mu} H) \). Furthermore, for any \( x \in X, y \in Y, h \in H \) and \( \phi \in H^0 \),

\[
\Phi(\phi \rtimes_{\lambda \rtimes_{\mu}} (x \rtimes y \rtimes_{\lambda} \mu h)) = \Phi(x \rtimes y \rtimes_{\lambda} \mu h) = \Phi(x \rtimes y \rtimes_{\lambda} \mu h).
\]

Therefore, we obtain the conclusion.

\square
Corollary 7.3. Let $f$ be a map from $\Pic^\rho_u(A)$ to $\Pic^\rho_u(A \rtimes_{\rho,u} H)$ defined by $f([X,\lambda]) = [X \rtimes_{\rho} H, \lambda]$ for any $[X,\lambda] \in \Pic^\rho_u(A)$. Then $f$ is a homomorphism of $\Pic^\rho_u(A)$ to $\Pic^\rho_u(A \rtimes_{\rho,u} H)$.

Proof. This is immediate by Lemma 7.2.

Next, we construct the inverse homomorphism of $f$ of $\Pic^\rho_u(A \rtimes_{\rho,u} H)$ to $\Pic^\rho_u(A)$. First, we note the following: Let $(\alpha, v)$ and $(\beta, z)$ be twisted coactions of $H^0$ on unital $C^*$-algebras $A$ and $B$, respectively. We suppose that there is an isomorphism $\Phi$ of $B$ onto $A$ such that $(\Phi \otimes \text{id}) \circ \beta = \alpha \circ \Phi$ and $v = (\Phi \otimes \text{id})(z)$.

Let $(X,\lambda) \in \Equiv(H^u_u(A))$. We construct an element $(X_\Phi, \lambda_\Phi)$ in $\Equiv(H^u_u(B))$ from $(X,\lambda) \in \Equiv(H^u_u(A))$ and $\Phi$ as follows: Let $X_\Phi = X$ as vector spaces. For any $x,y \in X_\Phi$ and $a \in B$,

$$b \cdot x = \Phi(b)x, \quad x \cdot b = x\Phi(b)$$

$$b(x,y) = \Phi^{-1}(a(x,y)), \quad \langle x,y \rangle_B = \Phi^{-1}(\langle x,y \rangle_A).$$

We regard $\Lambda$ as a linear map from $X_B$ to $X_\Phi \otimes H^0$. We denote it by $\lambda_\Phi$. Then $(X_\Phi, \lambda_\Phi)$ is an element in $\Equiv(H^u_u(B))$. By easy computations, the map

$$\Pic^\rho_u(A) \to \Pic^\rho_u(B) : [X,\lambda] \mapsto [X_\Phi, \lambda_\Phi]$$

is well-defined and it is an isomorphism of $\Pic^\rho_u(A)$ onto $\Pic^\rho_u(B)$. By Corollary 7.3, there is the homomorphism $\hat{f}$ of $\Pic^\rho_u(A \rtimes_{\rho,u} H)$ to $\Pic^\rho_u(A \rtimes_{\rho,u} H \rtimes\rho H^0)$ defined by

$$\hat{f}([Y,\mu]) = [Y \rtimes_{\rho} H^0, \hat{\mu}]$$

for any $[Y,\mu] \in \Pic^\rho_u(A \rtimes_{\rho,u} H)$. By Proposition 2.8 there are an isomorphism $\Psi_A$ of $A \otimes M_N(C)$ onto $A \rtimes_{\rho,u} H \rtimes\rho H^0$ and a unitary element $U \in (A \rtimes_{\rho,u} H \rtimes\rho H^0) \otimes H^0$ such that

$$\Ad(U) \circ \hat{\rho} = (\Psi_A \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(C)}) \circ \Psi_A^{-1},$$

$$(\Psi_A \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(U \otimes I_N) = (U \otimes 1^0(\hat{\rho} \otimes \text{id}_{H^0})(U)\text{id} \otimes \Delta^0(U^*)) = 0.$$}

Let $\overline{g} = (\Psi_A^{-1} \otimes \text{id}_{H^0}) \circ \hat{\rho} \circ \Psi_A$. By the above discussions, there is the isomorphism $g_1$ of $\Pic^\rho_u(A \rtimes_{\rho,u} H \rtimes\rho H^0)$ onto $\Pic^\rho_u(A \rtimes M_N(C))$ defined by

$$g_1([X,\lambda]) = [X_{\Phi_A}, \lambda_{\Phi_A}]$$

for any $[X,\lambda] \in \Pic^\rho_u(A \rtimes_{\rho,u} H \rtimes\rho H^0)$. Furthermore, the coaction $\overline{g}$ of $H^0$ on $A \rtimes M_N(C)$ is exterior equivalent to the twisted coaction $(\rho \otimes \text{id}, u \otimes I_N)$. Indeed,

$$\rho \otimes \text{id}_{M_N(C)} = (\Psi_A^{-1} \otimes \text{id}_{H^0}) \circ \Ad(U) \circ \hat{\rho} \circ \Psi_A = \Ad(U_1) \circ \overline{g},$$

$$u \otimes I_N = (U_1 \otimes 1^0(\overline{g} \otimes \text{id})(U_1)\text{id} \otimes \Delta^0(U_1^*).$$

We also note the following: We consider twisted coactions $(\alpha, v)$ and $(\beta, z)$ of $H^0$ on a unital $C^*$-algebra $A$. We suppose that $(\alpha, v)$ and $(\beta, z)$ are exterior equivalent. Then there is a unitary element $w$ in $A \otimes H^0$ such that

$$\beta = \Ad(w) \circ \alpha, \quad z = (w \otimes 1^0)(\rho \otimes \text{id})(w)v(\text{id} \otimes \Delta^0)(w^*).$$

By Lemmas 3.9 and 5.9 and their proofs, there is the isomorphism $g_2$ of $\Pic^\rho_u(A)$ onto $\Pic^\rho_u(A)$ defined by $g_2([X,\lambda]) = [X, \Ad(w) \circ \lambda]$ for any $[X,\lambda] \in \Pic^\rho_u(A)$, where $\Ad(w) \circ \lambda$ means a linear map from $X$ to $X \otimes H^0$ defined by $(\Ad(w) \circ \lambda)(x) = w\lambda(x)w^*$ for any $x \in X$, which is a coaction of $H^0$ on $X \otimes H^0$ with
respect to \((A, A, \beta, z, \beta, z)\). Since \(\mathcal{P}\) and \((\rho \otimes \text{id}, u \otimes I_N)\) are exterior equivalent, by the above discussions, there is the isomorphism \(g_2\) of \(\text{Pic}^\mathcal{P}_H(A \otimes M_N(C))\) onto \(\text{Pic}^{\rho \otimes \text{id}}(A \otimes M_N(C))\) defined by

\[ g_2([X, \lambda]) = [X, \text{Ad}(U_1) \circ \lambda] \]

for any \([X, \lambda] \in \text{Pic}^\mathcal{P}_H(A \otimes M_N(C))\). By easy computations, \((\rho, u)\) is strongly Morita equivalent to \((\rho \otimes \text{id}_{M_N(C)}, u \otimes I_N)\). Hence by Lemma 5.6 and its proof, there is the isomorphism \(g_3\) of \(\text{Pic}^{\rho, u}_H(A)\) onto \(\text{Pic}^{\rho \otimes \text{id}}(A \otimes M_N(C))\) defined by

\[ g_3([X, \lambda]) = [X \otimes M_N(C), \lambda \otimes \text{id}_{M_N(C)}] \]

for any \([X, \lambda] \in \text{Pic}^{\rho, u}_H(A)\). Let \(g = g_3^{-1} \circ g_2 \circ g_1 \circ \tilde{f}\). Then \(g\) is a homomorphism of \(\text{Pic}^{\rho, u}_H(A \rtimes_{\rho, u} H)\) to \(\text{Pic}^{\rho, u}_H(A)\).

**Proposition 7.4.** With the above notations, \(g \circ f = \text{id}\) on \(\text{Pic}^{\rho, u}_H(A)\).

**Proof.** Let \([X, \lambda] \in \text{Pic}^{\rho, u}_H(A)\). By the definitions of \(f, \tilde{f}, g_1\) and \(g_2\),

\[(g_2 \circ g_1 \circ \tilde{f} \circ f)([X, \lambda]) = [(X \rtimes A H \rtimes \hat{X} H^0)^{\Psi_A}, \text{Ad}(U_1) \circ (\hat{\lambda})^{\Psi_A}]\]

Let \(\Psi\) be the linear map from \(X \otimes M_N(C)\) to \(X \rtimes A H \rtimes \hat{X} H^0\) defined in Proposition 3.6 and we regard \(\Psi\) as an \(A \otimes M_N(C)\) - equivalence relation between \(X \rtimes A H \rtimes \hat{X} H^0\) and \(X \otimes M_N(C)\) – equivalence relation between \(X \times A C\), \(\nu \otimes I_N\) and \(X \otimes M_N(C)\), \(\lambda \otimes \text{id}_{M_N(C)}\). Also, since \(\text{Ad}(U) \circ \lambda = (\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}) \circ \Psi_X^{-1}\) by Proposition 5.6 for any \(x \in A \otimes_{M_N(C)}\),

\[ (\text{Ad}(U_1) \circ (\lambda)^{\Psi_A})(x) = U_1 \cdot (\hat{\lambda})^{\Psi_A}(x) \cdot U_1^* = U \lambda(x) U^* = (\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}) \circ \Psi_X^{-1}(x) \]

Thus

\[ [(X \rtimes A H \rtimes \hat{X} H^0)^{\Psi_A}, \text{Ad}(U_1) \circ (\hat{\lambda})^{\Psi_A}] = [X \otimes M_N(C), \lambda \otimes \text{id}] \]

in \(\text{Pic}^{\rho \otimes \text{id}}(A \otimes M_N(C))\). Since \(g_3([X, \lambda]) = [X \otimes M_N(C), \lambda \otimes \text{id}_{M_N(C)}]\), we obtain the conclusion. \(\square\)

**Theorem 7.5.** Let \((\rho, u)\) be a twisted coaction of \(H^0\) on a unital \(C^*\)-algebra \(A\). Then \(\text{Pic}^{\rho, u}_H(A) \cong \text{Pic}^{\rho \otimes \text{id}}_H(A \rtimes_{\rho, u} H)\).

**Proof.** Let \(f, \tilde{f}, g_i, (i = 1, 2, 3)\) and \(g\) as in the proof of Proposition 7.3. By Proposition 7.4, \(g \circ f = \text{id}\) on \(\text{Pic}^{\rho, u}_H(A)\). Hence \(f\) is injective and \(g\) is surjective. Furthermore, we can see that \(f\) is injective by Proposition 7.4. Since \(g = g_3^{-1} \circ g_2 \circ g_1 \circ f\) and \(g_i, (i = 1, 2, 3)\) are bijective, \(g\) is injective. It follows that \(g\) is bijective. Therefore, \(f\) is an isomorphism of \(\text{Pic}^{\rho, u}_H(A)\) onto \(\text{Pic}^{\rho \otimes \text{id}}_H(A \rtimes_{\rho, u} H)\). \(\square\)

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