Wave Packets of Mandelbrot Type in Boundary Problems of Quantum Mechanics

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Abstract
An initial value boundary problem for the linear Schrödinger type equation with nonlinear functional boundary conditions is considered. It is shown that attractor of the problem contains periodic piecewise constant functions on a complex plane with infinite points of discontinuities on a period. It has been applied the method of reduction of the problem to a system of integro-difference equations. Further, it is proved that for large time integro-difference equations tends to a system of difference equations asymptotic of which is known. It has been done application to boundary problems of nonlinear optics, where impulse periodic distributions of light are interpreted as white and black solitons.

Keywords: The Schrödinger equation, The functional two points boundary conditions; Asymptotic periodic piecewise constant distributions of relaxation type

Introduction
In last years, in physics studied the nonlinear interaction of light which can mimic the physics of so called an event horizon. As shown in a study [1], this analogue arises when a weak probe wave is unable to pass through an intense soliton, despite propagating at a different velocity. These dynamics arises as a soliton-induced refractive index barrier. In all paper this barrier characterize the volume optic properties of a fibre with linear boundary conditions. In this paper, we consider the opposite problem when the optical medium is ideal or linear, but boundaries of the medium have the nonlinear optic properties, and describes, for example, the all-optical transistor [2]. It may be also a case when a bright soliton is passing through the soliton. In this case, the intensity of light depends on the fibre refractive index (that is take place the Kerr effect). Thus, the soliton creates a moving refractive index perturbations which passage through the another soliton [3-6]. This interaction between such surface solitons plays the main role of distributions of the light in the linear medium with nonlinear surface interaction. Thus for the ideal medium the main role plays the fibre surface nonlinear refractive index. In a study [2], it has been mimicked two spectral modes of solitons when the mode-locked laser diode generate picosecond solitons. This generation will be described as a functional boundary conditions generated these solitons. The simulation is described by the hyperbolic type equations which models an evolution of amplitude of electric field \( A(x, t) \) [7]. These equations depends on the dispersion coefficients and on the nonlinear interaction coefficient of the fibre, depending on \( A(x, t) \), where \( x \) is a coordinate of a beam and \( t \) is the time. Of course, these equations must be follows from the Schrodinger type of equations. Thus phases and amplitudes of lights describes by systems of linear quantum equations. Boundary condition describes surface beams in an optical resonator with feedback. Thus the equations are linear without interaction potential, but the feedback is modeled by the nonlinear boundary conditions.

The problem of the coherent interaction of impulses in a nonlinear medium is well-known. As noted in a study [8], an interaction may be mimicked as a flight of an event horizon. An interaction may be utilized for the transmission of information, for frequency conversion, and for the description of processes which proceed in more intensive fields and at times shorter than to relaxation time" [9].

where \( \psi = \psi(x; t) : \mathbb{R}^2 \to C \), where \( C \) is a complex space, is an unknown function, \( h \) is the Planck constant. Let us divide the two parts of equation on a value \( mv^2 \), where \( m \) is the mass of particles, \( v \) is their velocity and \( p=mv \) is an impulse. Further, we introduce \( T = \frac{t}{\tau} \), \( \tau \) is a relaxation time of a wave function to some equilibrium, and we consider a dimensionless constant \( \lambda = \frac{\lambda}{\tau} \). As a result, we obtain the dimensionless equation:

\[
-i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{\nabla^2}{2m} \psi \tag{1}
\]

We can consider the functional two-point boundary conditions

\[
\psi(0, t, h) = \Phi[\psi(1, t, h)]
\]

with the real or complex parameter \( \theta \). For these conditions there is a theorem of existent and uniqueness of solutions the problem [10-13]. But our aim is instead of the linear boundary conditions to consider nonlinear conditions

\[
\psi(x; 0, h) = \psi_{h}(x; 0), \quad 0 \leq x \leq 1
\]

\[
\psi(0, 0, h) = \Phi[\psi(1, 0, h)], \quad \psi(0, 1, h) = \Phi[\psi(1, 1, h)]
\]

and the similar conditions for second derivatives at points (0, 0) and (0, 0) are satisfied. This ensures the existence of solutions of \( C \) -class. Of course, real and imaginary parts of the map \( \Phi \) are from \( C \) -class. This ensures that the map \( \Phi : C \to C \) is stable. It means that a corresponding map \( G : \mathbb{R}^2 \to \mathbb{R}^2 \) is stable. It means that

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that spectrum $\sigma(TG') \cap \{x: |x|=1\}=\emptyset$, where $\emptyset$ is empty set.

Further we assume that $h>0$ is a small parameter, and we consider the problem with accuracy $O(h^2)$, where $O(h^2) \to 0$ as $t \to +\infty$. We find solutions in the form:

$$
\psi(x, t, h) = e^{ikx} \phi(x, t, h)
$$

where $S(x, t)$ is a real phase, and $\phi(x, t, h)$ is real amplitude. (Below, where it will not cause misunderstandings, the parameter $h$ will be omitted).

We assume that $C([0, l] \times [0, +\infty))$ is the space of bounded continuous functions, and $C^2$ is the space of twice differentiable functions with the norm:

$$
|f|^2 = \sum_{i=0}^{2} \sup_{[0, l]} |f_i|,
$$

where $\|f\|$ is the norm in $C([0, l] \times [0, +\infty))$. The function $\psi \in C^2$ belongs to its real and imaginary parts belong $C([0, l] \times [0, +\infty))$. Then in $C^2$ - norm there is the following convergence:

$$
\|\phi(x, t)\|^2 \Rightarrow \|\phi_1(S, \phi), \phi_2(S, \phi)\|^2
$$

where $\phi_1$ and $\phi_2$ are $2\pi$ - periodic piecewise constant distribution with finite number $\Gamma$ of points of discontinuities. Further

$$
\|\phi(x, t)\|^2 \Rightarrow \|\phi_1(S, \phi), \phi_2(S, \phi)\|^2
$$

where $\phi_1$ and $\phi_2$ are $2\pi$ - periodic piecewise constant distribution with finite number $\Gamma$ of points of discontinuities. Here, $\Gamma := \mathbb{C}(D)$, where $D = \bigcup_{i=1}^{n} C^{-\infty}(A'_{i})$ and $A'_{i}$ is a set of saddle points of codimension one, and $\phi(\phi) = (S_0(\phi), \Phi(S, \phi))$ is an initial curve in $\mathbb{R}^2$, which is determined by initial data of the boundary problem, and $N$ is least common multiple of the map $G: (S, \phi) \to (\Phi_1(S, \phi), \Phi_2(S, \phi))$.

**Method of Reduction of Problem to System of Integro-Difference Equations**

Now we return to the problem. Indeed, substituting (7) in eqn. (2), we obtain that

$$
\left\{ \begin{array}{l}
\frac{\partial S}{\partial t} + \frac{1}{2} \left(VS^2\right)\phi + (i\theta)(\frac{\partial S}{\partial x} + \frac{1}{2} \frac{\partial \phi}{\partial x} + \frac{1}{2} \frac{\partial S}{\partial \phi}) + (-i\theta)\frac{1}{2} \frac{\partial S}{\partial \phi} + \frac{1}{2} \phi \Delta S
\end{array} \right. = 0
$$

We find solution with accuracy $O(h^2)$ so that

$$
\left\{ \begin{array}{l}
\frac{\partial S}{\partial t} + \frac{1}{2} \left(VS^2\right)\phi + (i\theta)(\frac{\partial S}{\partial x} + \frac{1}{2} \frac{\partial \phi}{\partial x} + \frac{1}{2} \frac{\partial S}{\partial \phi}) + (-i\theta)\frac{1}{2} \frac{\partial S}{\partial \phi} + \frac{1}{2} \phi \Delta S
\end{array} \right. = 0
$$

The boundary condition can be written in form:

$$
\cos S_{t=0} = -\Phi(S, \phi), \sin S_{t=0} = \Phi(S, \phi)
$$

where $\mathbb{R}$ and $\mathbb{I}$ real and imaginary parts of a complex number.

Let us denote $F_i := \Phi_i(S, \phi)$ and $F_i := \Phi_i(S, \phi)$. Then from eqn. (13) it follows that:

$$
\phi_{t=0} = F_1(S, \phi)_{t=0} + F_2(S, \phi)_{t=0}
$$

$$
\tan S_{t=0} = F_1(S, \phi)_{t=0} - F_2(S, \phi)_{t=0}
$$

We define the function $\Phi_1 := \sqrt{F_1^2 + F_2^2}$ and $\Phi_2 := \arctan(F_2/F_1)$, then the boundary condition (4) can be written as

$$
\Phi_1 = \Phi_2(\phi), \quad \Phi_2 = \Phi_2(\phi)
$$

For simplicity, we consider, initially, the case, when $\Phi_2 = \Phi_2(\phi)$ and $\Phi_1 = \Phi_1(\phi)$. Then general case can be treated similarly.

Thus for the Hamilton-Jacobi equation we have the boundary condition

$$
S(0, t) = \Phi_1([S(01)], 18)
$$

And, for the transport equation

$$
\frac{\partial \phi}{\partial t} + p \frac{\partial \phi}{\partial x} + 2 S \phi = 0
$$

We have the boundary condition

$$
\phi(0, t) = \Phi_1(\phi(0, t))
$$

Here maps $\Phi_1, \Phi_2 \in C(1 \to I)$ are structural stable, where $I$ is an open closed interval. The structural stable maps form an open dense subset [5].

In order to solve equations, we use the method of characteristics. To do this, we consider the Hamilton system of ordinary difference equations with hamiltonian $H(p) = \frac{1}{2} p^2$ as

$$
\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}
$$

With the initial condition

$$
x(0, t) = x_0, \quad \dot{x}(0) = -\frac{\partial S}{\partial x}(x_0) = p
$$

For the given constant $p$ the function $x := x(p, t)$ is the solution of equation

$$
p - \frac{\partial S}{\partial x}(x, t, x) = 0
$$

Where $p$ can be considered as additional coordinate in $(x, p, t)$-space.

Interpretation of the equation follow from the relation

$$
\Phi = S + xS_t,
$$

Here, $L = \Phi_1$ along a curve $x := x(t, x, p)$ which is determined from a solution of equation (R31). Then $L \geq \Phi_1$ along each other curve and the curve $\theta_{0}$ minimizes the functional $\int L(\dot{x}(t), \dot{x}(t), t)dt$. In other words, the difference $L - \Phi_1$, as a function of $\dot{x}$ has a minimal value $0$ as $\dot{x} = p$. It follows that

$$
S_{t=0} = \int L_{t=0}, \quad L = L - xS_{t=0}
$$

Let us define $y = L_{t=0}$ and $H = L - yx$. These values are called by spatial and temporal impulses in a space $t, x$. The the differential form $dx - H dt$ is differential ds(x, t) if $x := x(p, t)$ [14]. The same result follows from the Euler equation

$$
\frac{d}{dt} L_{t=0} = L
$$

Where $L_{t=0} = L - H(t, x, dx)$ is differential of the functional. As shown above, this extremal is the required minimum.

From the variational problem for a lagrangian $L(t, x, \dot{x})$ we can obtain a local Hamiltonian $L(t, x, y)$ and the corresponding local Hamilton-Jacobi equation. Then on characteristics $dx(p, t) = p$ we have the equation

$$
\frac{dS(x(t), p, t)}{dt} = \frac{\partial S(x(t), p, t)}{\partial t} + \frac{\partial S(x(t), p, t)}{\partial x} \frac{dx(t)}{dt} - \frac{\partial H}{\partial p} \frac{dx(t)}{dt}
$$

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By integration along characteristics $dx/dt=p$ and with help of boundary conditions the problem can be reduced to the system of integro-difference equations:

$$s(x,t) = \Phi_1[S(x,t-l/p)] + \frac{pl}{2} (t-x)$$  \hspace{0.5cm} (28)

$$\varphi(x,t) = \varphi(0,t-x/l) + \int_{-\infty}^{\infty} \frac{d^2S}{dx^2} p(s-t)+x,x \varphi(p(s-t)+x,x)ds$$ \hspace{0.5cm} (29)

$$\Phi_1[\varphi(l,t-x/l)] = \int_{s-l/p}^{s+l/p} \frac{d^3S}{dx^3} [p(s-t)+l/p+x,x] \varphi(p(s-t)+l/p+x,x)ds$$

$$+ \int_{s-l/p}^{s+l/p} \frac{d^3S}{dx^3} [p(s-t)+x,x] \varphi(p(s-t)+x,x)ds$$

The first equation follow from the relation

$$S(x,t) = S(0,t-x/l) + \frac{pl}{2} \Phi_1[S(l,t-x/l)] + \frac{pl}{2}$$ \hspace{0.5cm} (30)

$$\Phi_1[S(x,t-l/p)] = \Phi_1[S(0,t-x/l)] + \frac{1}{2} (l-x) + \frac{1}{2} pl,$$

where we used the boundary conditions for the phase. Integro-difference eqn. (33) follows from the transport eqn. (19) which can be written as

$$\frac{d\varphi}{dt} = -\frac{1}{2} \frac{d^2S}{dx^2} \varphi$$ \hspace{0.5cm} (31)

along characteristics $dx/dt=p$. Integrating this equation from a point $(x, t)$ to a point $(x, t - l/p)$ (Figure 1), and using functional boundary conditions, we obtain eqn. (33). Let us consider this system at a point $x=1$. Then we obtain

$$S(l,t) = \Phi_1[S(l,t-l/p)] + \frac{pl}{2}$$ \hspace{0.5cm} (32)

$$\varphi(t) = \varphi(l,t-l/p) + \int_{s-l/p}^{s} \frac{d^2S}{dx^2} [p(s-t)+l,s] \varphi(p(s-t)+l,s)ds$$ \hspace{0.5cm} (33)

Here,

$$S(x,t) = S(0,t-x/l) + \frac{1}{2} px = \Phi_1[S(l,t-x/l)] + \frac{1}{2} px$$ \hspace{0.5cm} (34)

Then,

$$\frac{dS}{dx}[0,t-x/l] = -\frac{1}{p} \Phi_1[S(l,t-x/l)] \frac{S'(l,t-x/l)}{2}$$ \hspace{0.5cm} (35)

Hence,

$$\frac{dS}{dx}(0,t-x/l) = -\frac{1}{p} \Phi_1[S(l,t-x/l)] \frac{S'(l,t-x/l)}{2}$$ \hspace{0.5cm} (36)

From eqn. (36) we obtain that

$$\frac{dS}{dx}(0,t-x/l) = -\frac{1}{p} \Phi_1[S(l,t-l/p)] \frac{S'(l,t-l/p)}{2}$$ \hspace{0.5cm} (37)

Here, a function $S(l,t)$ can be found from difference eqn. (32). It is known [5] that, in typical cases, asymptotic solutions of difference equations tends to elements of attractor of corresponding dynamic system which are piecewise constant periodic functions $p_1(t)$ such that $p_1(t) \in \mathbb{A}_1$, where $\mathbb{A}_1$ is a set of attractive fixed points of the map $\Phi_1 = \Phi + \Phi_1$, for almost all points $t \in \mathbb{R}^2$. Let $s = a + zS$ where $a \in \mathbb{A}_1$, and $z>0$ is a small parameter. Then, with accuracy $O(z)$, we have

$$\tilde{S}(l,t) = \Phi_1(a) \tilde{S}(l,t-l/p)$$ \hspace{0.5cm} (38)

Where

$$\lambda_1 \geq 1.$$ Then solutions of equation are

$$\tilde{S}(l,t) = e^{\lambda_1 t},$$ where $k_1 = \frac{pl}{2} \lambda_1, \text{ so that } S'(l,t) = k_1 e^{\lambda_1 t}.$ Since, for parametrisation, $x(p, s, t) = p(s + t), x(t) = s$, we have

$$\frac{d^2S}{dx^2}[p(s-t)+x,s] = S'(0,t-x/l)$$ \hspace{0.5cm} (39)

Integro-difference equation can be written as

$$\varphi(t) = \Phi_1[S(l,t-l/p)] - \frac{1}{2} k_1 \int_{s-l/p}^{s+l/p} \varphi(p(s-t)+x,s)ds$$ \hspace{0.5cm} (40)

$$\Phi_1[\varphi(l,t-l/p)] - \frac{1}{2} k_1 \int_{s-l/p}^{s+l/p} \varphi(p(s-t)+x,s)ds$$

Let $\varphi = b + \varepsilon \tilde{\varphi}$, where $b \in \mathbb{A}_1$, and $\lambda_1$ is a set of attractive fixed points of the map $\Phi_1$. Let us also define $y(t) = e^{\lambda_1(t-l/p)}$. Then eqns. (39) and (42) can be reduced, with accuracy $O(\varepsilon^2)$, to the system of equations:

$$\tilde{S}(l,t) = \lambda_1 \tilde{S}(l,t-l/p)$$ \hspace{0.5cm} (41)

$$\tilde{y}(t) = \lambda_1 \tilde{y}(t-l/p) - \frac{1}{2} k_1 b \tilde{y}(t-l/p)$$ \hspace{0.5cm} (42)

$$y(t) = e^{\lambda_1(t-l/p)}$$ \hspace{0.5cm} (43)

Where $\lambda_1 = (l/p)^2 e^{l/p} / 24 + \frac{2}{l/p}$. The eigenvalues of these equations satisfies to the cubic equation

$$(\lambda_1 - \lambda_1)(\lambda_1 - \lambda_1) = 0$$ \hspace{0.5cm} (44)

Since $k<0$, all trajectories are attracted by a plane $S \varphi$. But on this plane we obtain two independent differential equations for the phase and amplitude. asymptotic solutions of the boundary problem are determined by asymptotic solutions of the system of difference equations.

Let as define $P$ a set of fixed points of the map $\Phi_1(S, \varphi) \in \mathbb{R}$. It means that asymptotic behaviour of trajectories is determined by

$$G(x, \varphi, y) \rightarrow \Phi_1[S + \frac{1}{2} k_1 \Phi[S] - \frac{1}{2} k_1 \Phi[\Phi[S]] \cdot y(t) = e^{\lambda_1 y(t-l/p)}$$ \hspace{0.5cm} (45)

Hence, for $k<0$ all trajectories in $R^2$ are attracted by a plane $(S, \varphi) \in \mathbb{R}$. It means that asymptotic behaviour of trajectories is determined by

\[ G(x, \varphi, y) \rightarrow \Phi_1[S + \frac{1}{2} k_1 \Phi[S] - \frac{1}{2} k_1 \Phi[\Phi[S]] \cdot y(t) = e^{\lambda_1 y(t-l/p)} \]
by solutions of two difference equations. Let us a map $G:=(\Phi_1, \Phi_2): \mathbb{R}^2 \to \mathbb{R}^2$ has two attractive fixed points $(a_1^+, a_2^+)$ and one saddle point $a_1^-$ of codimension 1 as shown on Figure 1. Then from [15-17] it follows that that $S(l, t) \Rightarrow \gamma_1(t) \Rightarrow p_1(t)$ and $\varphi(l, t) \Rightarrow p_2(t)$ as $t \to \infty$, where $p_1(t) \in A_1^+$ and $p_2(t) \in A_2^+$. Here, $A_1^+, A_2^+$ are sets of attractive fixed points of maps $(\varphi, \varphi)$, correspondingly. The functions $p_1(t), p_2(t)$ are periodic with periods $2\pi$ is least common multiple of periods of attractive circles of the map $G$.

Example 1

For example, we consider a quadratic map $\Phi: z \to z^2+\mu$, where $\mu, z \in \mathbb{C}$ and $\mathbb{C}$ is a complex space. Then from the boundary conditions follows the relations:

\[
\cos \psi \varphi = 0 = \cos 2\sigma \varphi + \mu \varphi = l
\]

(46)

\[
\sin \psi \varphi = 0 = \sin 2\sigma \varphi + \mu \varphi = l
\]

(47)

Where $\mu = \Re \mu$ and $\mu_0 = \Im \mu$. Next, from eqns. (46) and (47), we obtain that

\[
\tan \psi \varphi = \frac{\sin 2\varphi + \mu \varphi}{\cos 2\varphi + \mu \varphi} \quad \text{as} \quad l \to \infty
\]

(48)

\[
\psi \varphi = \sqrt{2\mu \varphi + \mu^2} \quad \text{as} \quad l \to \infty
\]

(49)

Where $\psi = \varphi^{p_2}$. Now linearising relation (48), (52) at a point $S=0$, we obtain that

\[
\tan \psi \varphi = \frac{2\mu \varphi + \mu^2}{\mu} \quad \text{as} \quad l \to \infty
\]

(50)

\[
\psi \varphi = \sqrt{\mu \varphi + \mu^2} \quad \text{as} \quad l \to \infty
\]

(51)

We assume that $\mu_0 = 0$, then

\[
\tan \psi \varphi = \frac{2\mu \varphi + \mu^2}{\mu} \quad \text{as} \quad l \to \infty
\]

(52)

\[
\psi \varphi = \sqrt{\mu \varphi + \mu^2} \quad \text{as} \quad l \to \infty
\]

(53)

If $\psi = \mu \varphi$, then

\[
\psi \varphi = \sqrt{\mu \varphi + \mu^2} \quad \text{as} \quad l \to \infty
\]

(54)

Where $\mu = \mu^2 - \mu_0$. Hence,

\[
\psi \varphi (0, t) \Rightarrow \psi \varphi \sim \mu t
\]

(55)

As $t \to \infty$, and we obtain the difference equation

\[
\sqrt{\mu \varphi + \mu^2} = \mu t + \text{const}
\]

(56)

If $\mu > 0$, then the map $\Theta_1: u \to u^2$ has no fixed points, so that for each $u \in \mathbb{R}$ We have $\Theta_1(u) \to \infty$ as $t \to \infty$. If $\mu \leq 0$, then the map $\Theta_1$ has fixed points

\[
\beta_0 = \frac{1}{2} \pm \frac{1}{2} \mu
\]

(57)

Where $\beta_0$ is an attractive point. In this case $\Theta_1(u) \to \infty$ for each $u \in \mathbb{R}$, where $J=[-\beta_0, \beta_0]$. If $-2 \leq \mu \leq 1$, then $\Theta(J) \subset J$. If $\mu \leq -2$, then each point $u \in J \cap \Theta(J)$ is a point of non wandering (in our case this set of attractive fixed point). For $\mu \leq -2$, the set $\Omega(\Theta)$ is a cantor set on $J$. A solution of the bounded on $[0,\infty]$ if and only if $u(\infty)$ for each $t \in [-1,0]$. We assume that $-2 < \mu < -1$.

If interval $[\mu, -\mu]$, then interval $[\mu, -\mu]$, is invariant and $\Theta_1(u)$ is closure of a set. Hence, for $-3/4 < \mu < 1/4$ on $J$ is unique attractive fixed point $\beta_1$. Then each solution $u(t)$ tends to $\beta_1$ as $t \to \infty$, where $\beta_1 = \frac{1}{2} \frac{1}{\sqrt{4 - \mu^2}}$. This point exists if $\mu > 1/2$.
of fixed points, and $N=1; 2;\ldots$. Let $W=\bigcup_{a \in \text{Fix}(G)} W^{\ast}(a)$ where $W^{\ast}(a)=\{u \in W: \lim_{j \to \infty} G^{n}(u)=a\}$ is a stable manifold of a fixed point $a$ of the map $G$. Particularly, for any point $u \in W$ there is the finite limit

$$\lim_{j \to \infty} G^{j}(u)=G^{j}(u)$$

(60)

Let $u(t)=(S(t), \varphi(t)) \in C([0, \infty) \to \mathbb{R}^{2})$. We define a set of initial functions

$$H_{t}^{\ast} \left( h(t) \right) \subset \left[ \mathbb{R}^{+} \right]$$

(61)

Then for each function $h(t) \in H_{t}^{\ast}$ there is periodic piecewise constant function $p^{\ast}[h(t)]_{\mathbb{R}^{+}}$ such that

$$p^{\ast}[h(t)]_{\mathbb{R}^{+}} = G^{\ast}\left[ p^{\ast}[h(t)]_{\mathbb{R}^{+}} \right]$$

(62)

Let $u(t)=\left( S(t), \varphi(t) \right) \in C([0, \infty) \to \mathbb{R}^{2})$. We define a set of initial functions

$$H_{t}^{\ast} \left( h(t) \right) = \left\{ h(t) \in C([0, \infty) \to \mathbb{R}^{2}): h(0)=G^{\ast}\left[ h(0) \right]\right\}$$

(63)

and $H_{t} \neq \emptyset$ where $\emptyset$ is empty set, if and only if $a \in \text{Fix}(G)$. Then each solution $u(t)$ of the system of difference equations with an initial function $u(t)_{0} \in H_{t}^{\ast}$ tends to a constant $a$ if $t \to +\infty$. Each solution $u(t)$ of the system of difference equations with an initial function $u(t)_{0} \in H_{t}^{\ast}$ is asymptotic periodic function, and

$$\lim_{j \to \infty} \left\| u(t+Nj)-p^{\ast}[h(t')]_{\mathbb{R}^{+}} \right\|=0$$

(64)

where $t' \in \mathbb{R}^{+}$. Each solution $u(t)$ of the system of difference equations with an initial function $u(t)_{0} \in H_{t}^{\ast}$ is asymptotic periodic function, and

$$\lim_{j \to \infty} \left\| u(t+Nj)-p^{\ast}[h(t')]_{\mathbb{R}^{+}} \right\|=0$$

(65)

It must be noted that

$$\lim_{j \to \infty} \left\| u(t+Nj)-p^{\ast}[u(t')]_{\mathbb{R}^{+}} \right\|=0$$

(66)

Thus asymptotic solution has the form

$$\psi(x,t)=e^{(\psi(-t)/p)}p_{1}(t-x/p)+O(h^2)$$

(67)

where $O(h^2) \to 0$ as $h^2 \to 0$. For example, such type solutions describe distributions of order parameter has been obtained for the Ginzburg-Landau equation with some boundary conditions for the phases and amplitudes of order parameter.

Thus it has been proved that the initial boundary value problem can be reduced to a system of integro-difference equations. This system can be, in its turn, reduced to a system of non-autonomous difference equations in $\mathbb{R}^{2}$, where non-autonomous perturbations are produced by the phase $S(x,t)$ on characteristics $dx/dt=\psi$. These perturbations tends to zero as $t \to +\infty$. As a result, the difference equations in $\mathbb{R}^{2}$ are reduced to a difference equations in $\mathbb{R}^{2}$ at $(S,\psi)$ - plane. Solutions of these equations $(S,\psi)$ are piecewise constant asymptotic periodic functions $(p_{1}, p_{2})$ with finite ‘points’ of discontinuities on a period. It should be noted that for difference equations we get asymptotic distributions from some class of initial data from $C(\mathbb{R}, I)$, but for integro-difference equations it has been proved only that if solutions of integro-difference equations exists, then these solutions tends to functions $(p_{1}, p_{2})$ for a special class of initial data, which belongs to a small neighbourhood of limit functions $(p_{1}, p_{2})$. It should be noted also that if asymptotic invariant distributions of difference equations are asymptotic invariant solutions of integro-difference equations, becomes functions $(p_{1}, p_{2})$ are piecewise constant asymptotic periodic functions. It should be noted also that if a limit function $p_{2}(t-x/p)$ is piecewise constant, the a function $(\psi(-t)/p)p_{1}(t-x/p)$ in eqn. (10) tends to zero as $t \to +\infty$. It means that obtained solutions of the initial boundary value problem are accurate.

Figure 2: Limit solution of relaxation type.
Physical sense of the boundary conditions

The boundary conditions describes the changing of phases between input $S(0, t)$ and output phase in the electronic device the multiplier of the phase $N$ and the amplifier $J$ of a frequency of the input signal at a point $x=\lambda$. Additionally, there is the resistor $R$. The device is designed so that we can change independently the phase and amplitude. The map $\Phi^t$ describes actions of the amplifier $J$ and the multiplier on the input phases independently on the values of the amplitude of input signal [14, 15].

Physical Applications for Mandelbrot and Julia Sets

In this section, we use the paper [19] where it the dynamics of a kicked particle which moves in a double-well potential. In this paper, it is shown that the stroboscopic dynamics of the particle can be reduced to the research of dynamics of orbits of the known complex logistic map. The logistic map is homeomorphic to the quadratic map $Q(z):=z^2 + c$ where $C \in \mathbb{C}$ is a complex parameter. But this map has well-known Julia and Mandelbrot sets. Indeed, the map $Q(z): C \to C$ is connected or totally disconnected [20, 21]. The Mandelbrot set represent is a 'delimiter' between $c$ - values with connected Julia and totally disconnected Julia sets [22]. The Mandelbrot set is connected [23]. They also discussed also the non-solved hypotheses that the Mandelbrot set is locally connected [20]. Next, by Brunner and Hubbard it has been studied the Mandelbrot set for cubic. There are two critical orbits for cubics [21].

For the map $\tilde{Q}(z):=z^2 + c$, $n=2,3,4,...$. Roshon generalize the study of the case $n=3,4$. He proved that in a case $n=4$ the Mandelbrot set is connected. In a study [20] the cases $n=2,3,4,5$ has been considered. Let $EX \to X$ any function, where $X$ is a topological space. Let us define the Picard orbit by the recurrent formula:

$$u_n=f(u_{n-1}), n=1,2,...$$

(68)

Let $M$ be the Mandelbrot set for the map $Q_z(z):=z^2 + c$ such that $c \in C$ such that the orbit of the point $0$ is bounded so that

$$M: = \{c \in C \mid Q_z(0)=0, k=0,1,... \text{ is bounded}\}$$

(69)

where $Q_z(0)$ are iterations of the function with a condition $Q_z(0)=0$ which produce a critical orbit. Then is known for the quadratic map that if $f(c)<-2$ and $z:=z$, then $P(z) \to \infty$ as $n \to \infty$. For example, the point $c$ is not in the set $M$. As shown in a study [20], if $c \in [-2,1/4]$, then $P(z) \to \infty$. In this case, the Julia set is connected.

Conclusion

Thus an initial boundary problem for the linear Schrodinger equation with functional nonlinear two-points boundary conditions is considered. A structure of attractors of the problem has been constructed. It is shown that the attractor contains piecewise constant periodic functions with finite 'points' of discontinuities on a period. Thus phases and amplitudes of wave functions of the Shrdinger equation are piecewise constant periodic functions. The method of reduction of the problem to a system of integro-difference equations has been applied. It is shown that these equations have invariant piecewise constant periodic solution. It is proved that perturbations of such solutions are asymptotically stable.

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