FRAGILE CYCLES

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Abstract. We study diffeomorphisms $f$ with heterodimensional cycles, that is, heteroclinic cycles associated to saddles $p$ and $q$ with different indices. Such a cycle is called fragile if there is no diffeomorphism close to $f$ with a robust cycle associated to hyperbolic sets containing the continuations of $p$ and $q$. We construct a codimension one submanifold of Diff$^r(S^2 \times S^1)$, $r \geq 1$, that consists of diffeomorphisms with fragile heterodimensional cycles. Our construction holds for any manifold of dimension $\geq 4$.

1. Introduction

In the late sixties, Newhouse constructed the first examples of $C^2$-open sets of non-hyperbolic surface diffeomorphisms. Any such set $U$ consists of diffeomorphisms with $C^2$-robust homoclinic tangencies: every diffeomorphism $f \in U$ has a hyperbolic set $K_f$ (depending continuously on $f$) whose stable and unstable manifolds have non-transverse intersections, see [17].

Later, in [18], Newhouse proved that homoclinic tangencies of surface diffeomorphisms can be stabilized: given a diffeomorphism $f$ with a homoclinic tangency associated to a saddle $p_f$, there is a $C^2$-open set whose closure contains $f$ and which consists of diffeomorphisms $g$ with robust homoclinic tangencies associated to hyperbolic sets $K_g$ containing the continuation $p_g$ of $p_f$. In particular, these results show that homoclinic tangencies always generate $C^2$-robust homoclinic tangencies. In fact, robust homoclinic tangencies are present in all known examples of open sets of non-hyperbolic surface diffeomorphisms. Let us observe that homoclinic tangencies of $C^1$-diffeomorphisms defined on surfaces cannot be stabilized, see [15].

Similarly, all known examples of $C^1$-open sets formed by non-hyperbolic diffeomorphisms exhibit $C^1$-robust heterodimensional cycles, that is, cycles relating the invariant manifolds of two hyperbolic sets of different $s$-indices (dimension of the stable bundle). Note that the existence of such cycles can only occur in dimension $\geq 3$.

We wonder if, as in the case of homoclinic tangencies of $C^2$-diffeomorphisms, heterodimensional cycles can be made $C^1$-robust and can be $C^1$-stabilized. A first partial answer to this question is given in [8]: heterodimensional cycles associated to periodic saddles whose indices differ by one generate (by arbitrarily small $C^1$-perturbations) $C^1$-robust heterodimensional cycles. In some extend, the results in [8] are a version of the ones by Newhouse in [17, 18] in the context of $C^1$-heterodimensional cycles.

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However, compared with Newhouse’s results for homoclinic tangencies, the ones in [8] have an important disadvantage: While the hyperbolic sets with the robust homoclinic tangencies in [17, 18] contain continuation of the saddle with the initial tangency, the hyperbolic sets involved in robust cycles in [8] do in general not contain the continuations of the saddles in the initial cycle. However this precise question can be important for understanding the global dynamics of non-hyperbolic diffeomorphisms. Let us discuss more this question in more detail.

Following Conley theory [11] and motivated by spectral decomposition theorems [16, 1], this global dynamics is structured using homoclinic or/and chain recurrence classes as “elementary” pieces of dynamics, see the definitions below. One aims to describe the dynamics of each piece and the relations between different pieces (cycles), for further details see [10, Chapter 10.3-4] and [5].

In general, the homoclinic class of a hyperbolic periodic point is contained in its chain recurrence class. An important property is that for $C^1$-generic diffeomorphisms homoclinic classes and chain recurrence classes of periodic points coincide, [6, Remarque 1.10]. However, in non-generic situations, two different homoclinic classes (even of saddles of different indices) may be joined by a cycle, hence they are contained in the same chain recurrence class. A question is when one can join them in a $C^1$-robust way by small perturbations. This occurs if the cycle can be stabilized. For instance, this is specially important for understanding the indices of the periodic points in an elementary piece of dynamics.

While the above explains why the stabilization of cycles is relevant, let us now provide the precise definitions of the concepts involved. First, recall that a hyperbolic basic set $K$ of a diffeomorphism $f$ has a (uniquely defined) continuation $K_g$ for all $g$ close to $f$: $K_g$ is a hyperbolic basic set, close to $K$, and the dynamics of $f|_K$ and $g|_{K_g}$ are conjugate. The $s$-index of a hyperbolic transitive set is the dimension of its stable bundle.

Definition 1.1 (Robust continuations of cycles).

- The diffeomorphism $f$ is said to have a heterodimensional cycle associated to hyperbolic basic sets $K$ and $L$ if these sets have different $s$-indices and their stable and unstable manifolds meet cyclically:

$$ W^s(K, f) \cap W^u(L, f) \neq \emptyset \quad \text{and} \quad W^u(K, f) \cap W^s(L, f) \neq \emptyset. $$

- The cycle associated to $K$ and $L$ is $C^1$-robust if there is a $C^1$-neighborhood $U$ of $f$ such that for all $g \in U$ the hyperbolic continuations $K_g$ and $L_g$ of $K$ and $L$ have a heterodimensional cycle.

- A heterodimensional cycle associated to a pair of saddles $p$ and $q$ of $f$ can be $C^1$-stabilized if every $C^1$-neighborhood $U$ of $f$ contains a diffeomorphism $g$ with hyperbolic basic sets $K_g \ni p_g$ and $L_g \ni q_g$ having a robust heterodimensional cycle. Here $p_g$ and $q_g$ are the continuations of $p$ and $q$ for $g$.

Definition 1.2 (Fragile cycle). A heterodimensional cycle associated to a pair of saddles is $C^1$-fragile if it cannot be $C^1$-stabilized.

The previous discussion leads to the following question that we address in this paper: Can every heterodimensional cycle be $C^1$-stabilized?

The results in [9] give a positive answer to this question for “most” types of coindex one heterodimensional cycles, that is, related to saddles whose $s$-indices differ by one. Indeed in [9] it is proved that fragile coindex one cycles associated to saddles $p$ and $q$ exhibit a quite specific geometry:
The homoclinic classes of \( p \) and \( q \) are both trivial.

The central eigenvalues of \( p \) and \( q \) are all real and positive.\(^1\)

There is a well-defined one-dimensional orientable central bundle \( E^c \) along the cycle (i.e. defined on some closed set containing the saddles \( p \) and \( q \) in the cycle and a pair of heteroclinic orbits \( x \in W^s(p) \cap W^u(q) \) and \( y \in W^u(p) \cap W^s(q) \)), but the cycle diffeomorphism does not preserve the orientation of \( E^c \). The cycle is twisted by the terminology in \( \mathbb{H} \).

In this paper we provide examples of fragile coindex one cycles, see Theorem \( \mathbb{H} \).

1.1. Definitions and statement of results. Recall that the homoclinic class of a hyperbolic periodic point \( p \), denoted by \( H(p, f) \), is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of \( p \). The homoclinic class \( H(p, f) \) coincides with the closure of the set of all saddles \( q \) homoclinically related with \( p \), i.e. the stable manifold of the orbit of \( q \) transversely meets the unstable manifold of the orbit of \( p \) and vice-versa. A homoclinic class is non-trivial if it contains at least two different orbits.

Let us now recall the definition of a chain recurrence class. A finite sequence of points \( (x_i)_{i=0}^n \) is an \( \epsilon \)-pseudo-orbit of a diffeomorphism \( f \) if \( \text{dist}(f(x_i), x_{i+1}) < \epsilon \) for all \( i = 0, \ldots, n-1 \). A point \( x \) is chain recurrent for \( f \) if for every \( \epsilon > 0 \) there is an \( \epsilon \)-pseudo-orbit \( (x_i)_{i=0}^n \), \( n \geq 1 \), starting and ending at \( x \) (i.e. \( x = x_0 = x_n \)). The chain recurrent set \( R(f) \) of \( f \) is the set of all chain recurrent points. This set splits into disjoint chain recurrence classes: the class \( C(x, f) \) of \( x \in R(f) \) is the set of points \( y \) such that for every \( \epsilon > 0 \) there are \( \epsilon \)-pseudo-orbits joining \( x \) to \( y \) and \( y \) to \( x \). A periodic point \( p \) of \( f \) is isolated if its chain recurrence class coincides with its orbits. In this case, the orbit of \( p \) is the maximal invariant set in some filtrating neighborhood. This implies that the homoclinic class of \( p \) is \( C^1 \)-robustly trivial (i.e. the homoclinic class of \( p \) is trivial for every \( g \) close to \( f \)).

We are now ready to state our main result.

**Theorem 1.** There is an open set \( U \) of \( \text{Diff}^1(S^2 \times S^1) \) and a codimension one submanifold \( \Sigma \) contained in \( U \) with the following property: For every \( f \in U \) there are hyperbolic saddles \( p_f \) and \( q_f \) with different s-indices depending continuously on \( f \) such that

1. every \( f \in \Sigma \) has a heterodimensional cycle associated to \( p_f \) and \( q_f \),
2. the set \( U \setminus \Sigma \) is the union of two connected sets \( U^+ \) and \( U^- \) such that
   - for every \( f \in U^+ \) the saddle \( p_f \) is isolated,
   - for every \( f \in U^- \) the saddle \( q_f \) is isolated.

Note that if two hyperbolic basic sets \( K_f \) and \( L_f \) have a heterodimensional cycle then the chain recurrence classes of any pair of saddles \( a_f \in K_f \) and \( b_f \in L_f \) coincide. In particular, if the cycle associated to \( K_f \) and \( L_f \) is robust then the chain recurrence classes of \( a_g \) and \( b_g \) are the same for all \( g \) in some neighborhood of \( f \). In particular, the chain recurrence class \( C(a_g, g) = C(b_g, g) \) is non-trivial and the saddles \( a_g \) and \( b_g \) are both non-isolated for \( g \). Thus Theorem \( \mathbb{H} \) implies that the cycles in \( \Sigma \) cannot be made robust. This implies the following:

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\(^1\)The definition of central eigenvalues is a little intricate. Assuming that the s-index of \( p \) is bigger than the one of \( q \), the central eigenvalues correspond to the weakest contracting direction of \( p \) and the weakest expanding direction of \( q \).
Corollary 1. The submanifold $\Sigma$ in Theorem 1 consists of diffeomorphisms having $C^1$-fragile heterodimensional cycles.

As for any $n > 3$ the set $\mathbb{S}^2 \times \mathbb{S}^1$ can be embedded as a normally contracting manifold in a ball $\mathbb{B}^n$, we obtain the following.

Corollary 2. Any compact manifold $M$ with $\dim M > 3$ supports diffeomorphisms with $C^1$-fragile cycles.

As our examples demand a somewhat specific topological configuration, the following question arises naturally.

Question 1. Does every 3-manifold admit diffeomorphisms with $C^1$-fragile heterodimensional cycles?

The examples presented in this paper display many interesting and somehow unexpected properties. There are also many important aspects of their dynamics yet unexplored. Thus, after completing our construction, in Section 6 we conclude with a discussion about the properties of our examples.

This paper is organized as follows. In the first step of our construction, in Section 2, we build an auxiliary Morse-Smale vector field $X$ on the 3-sphere $\mathbb{S}^3$. In Section 3 we consider a surgery in $\mathbb{S}^3$ (associated to some identifications by a local diffeomorphism $\Psi$ of $\mathbb{S}^3$). This surgery provides a diffeomorphism $F_\Psi$ defined on $\mathbb{S}^2 \times \mathbb{S}^1$ induced by the time-one map $F_0 = X_1$ of the vector field $X$ and the gluing map $\Psi$. We also see how the dynamics of $F_\Psi$ depends on the gluing map $\Psi$. In Section 4 we study the dynamics of diffeomorphisms close to $F_\Psi$. Finally, in Section 5 we choose the gluing map $\Psi$ to get a diffeomorphism $F_\Psi$ with a fragile cycle and construct the submanifold $\Sigma$ consisting of diffeomorphisms with fragile cycles. The paper is closed with a discussion section.

2. An auxiliary vector field on $\mathbb{S}^3$

In this section we construct a Morse-Smale vector field defined on the three sphere $\mathbb{S}^3$ whose non-wandering set consists of singular points. This vector field also satisfies some normally hyperbolic properties. We now go to the details of this construction.

We consider the sphere $\mathbb{S}^3$ as the union of two solid tori $T_1$ and $T_2$ with the same boundary $\partial T_1 = \partial T_2 = \mathbb{T}^2$. A simple closed curve of $\mathbb{T}^2$ is a $T_1$-meridian if it is not 0-homotopic in $\mathbb{T}^2$ but is 0-homotopic in $T_1$. We consider an identification of the boundaries of these solid tori that does not preserve the meridians: the $T_2$-meridians are isotopic in $T_1$ to the “central circle” of $T_1$ and are classically called $T_1$-parallels. Similarly, $T_1$-meridians are $T_2$-parallels.

2.1. An auxiliary Morse-Smale vector field $X$ in $\mathbb{S}^3$. Consider a Morse-Smale vector field $X$ defined on $\mathbb{S}^3$ such that (see Figure 2.1):

(1) $X$ is transverse to $\partial T_1 = \partial T_2 = \mathbb{T}^2$.
(2) The solid torus $T_1$ is attracting and the solid torus $T_2$ is repelling: The positive orbit of any point $x \in \mathbb{T}^2$ enters (and remains) in the interior of $T_1$ and its negative orbits enters (and remains) in the interior of $T_2$.
(3) The maximal invariant set of $X$ in $T_1$ is a normally hyperbolic (contracting) circle $\sigma^s$. Analogously, the maximal invariant set of $X$ in $T_2$ is a normally hyperbolic (repelling) circle $\sigma^u$.
The limit set of $X$ is $\{s_1, s_2, r_1, r_2, p_1, p_2, q_1, q_2\}$, where
- $s_1, s_2 \in \sigma^s$ are attracting singularities,
- $r_1, r_2 \in \sigma^u$ are repelling singularities,
- $p_1, p_2 \in \sigma^s$ are saddle singularities of $s$-index 2, and
- $q_1, q_2 \in \sigma^u$ are saddle singularities of $s$-index 1.

The two (one-dimensional) separatrices of the unstable manifold of the singularity $p_i, i = 1, 2$, are contained in $W^s(s_1)$ and in $W^u(s_2)$. A similar assertion holds for the separatrices of the stable manifold of $q_i, i = 1, 2$, that are contained in $W^u(r_1)$ and $W^u(r_2)$.

The local stable manifold of $p_i, i = 1, 2$, is a 2-disk contained in $T_1$ whose boundary
$$\partial W^s_{\text{loc}}(p_i) = W^s_{\text{loc}}(p_i) \cap \partial T_1 \overset{\text{def}}{=} \gamma^s_i$$
is a $T_1$-meridian. Similarly, the local unstable manifold of $q_i, i = 1, 2$, is a 2-disk contained in $T_2$ whose boundary
$$\partial W^u_{\text{loc}}(q_i) = W^u_{\text{loc}}(q_i) \cap \partial T_2 \overset{\text{def}}{=} \gamma^u_i,$$
is a $T_2$-meridian.

For every $i, j \in \{1, 2\}$, the curve $\gamma^s_i$ is transverse to $\gamma^u_j$ and the intersection $\gamma^s_i \cap \gamma^u_j$ is exactly one point $x_{ij}$.

**Remark 2.1** (Dynamics of the vector field $X$).

(1) The boundary $T^2$ of the solid torus $T^1$ is the union of two cylinders $C_1^s$ and $C_2^s$ with disjoint interiors and the same boundary $\gamma^s_1 \cup \gamma^s_2$. The notation is chosen such that $W^s(s_i) \cap T^2$ is the interior of the cylinder $C_i^s$.

Similarly, $T^2 = \partial T_2$ is the union of the cylinders $C_1^u$ and $C_2^u$ bounded by $\gamma^u_1$ and $\gamma^u_2$ and whose interiors are the intersections $T^2 \cap W^u(r_1)$ and $T^2 \cap W^u(r_2)$, respectively. See Figure 2.2.
(2) As a consequence of item (7) in the definition of the vector field $X$, the intersection $C^u_1 \cap C^s_1$ is a “rectangle” $R$ such that

$$R = W^s(s_1) \cap W^u(r_1) \cap \mathbb{T}^2$$

and its boundary $\partial R$ of $R$ is the union of four curves $a^s_1$, $a^s_2$, $b^u_1$, $b^u_2$ with disjoint interiors such that

$$a^s_1 \subset \gamma^s_1, \quad a^s_2 \subset \gamma^s_2, \quad b^u_1 \subset \gamma^u_1, \quad b^u_2 \subset \gamma^u_2.$$  

See Figure 2.2. Note that the interiors of $b^u_1$ and $b^u_2$ are contained in $W^s(s_1)$ and the interiors of $a^s_1$ and $a^s_2$ are contained in $W^u(r_1)$.

![Figure 2.2. The rectangle $R$.](image)

2.2. Partially hyperbolicity of $X$. We also assume that the vector field $X$ satisfies the following partially hyperbolic conditions:

1. There is a partially hyperbolic splitting of $X$ over the circle $\sigma^s$ (recall (3) in Section 2.1) of the form

$$T_{\sigma^s}S^3 = E^{ss} \oplus E^{cs},$$

where $E^{cs}$ is a 2-dimensional central bundle containing the $X$ direction, and $E^{ss}$ is a strong stable bundle that is oriented along the circle $\sigma^s$.

A similar condition holds for the circle $\sigma^u$: There is a partially hyperbolic splitting of $X$ over $\sigma^u$ of the form

$$T_{\sigma^u}S^3 = E^{cu} \oplus E^{uu},$$

where $E^{cu}$ is a 2-dimensional center bundle containing the $X$-direction and $E^{uu}$ is a strong unstable bundle that is oriented along $\sigma^u$.

2. Consider the two-dimensional strong stable manifold $W^{ss}(\sigma^s)$ of $\sigma^s$ that is tangent to $X \oplus E^{ss}$ along $\sigma^s$. Define the local strong stable manifold of $\sigma^s$ by $W^{ss}_{loc}(\sigma^s) = W^{ss}(\sigma^s) \cap T_1$. Then the intersection between $W^{ss}_{loc}(\sigma^s)$ and $\mathbb{T}^2$ consists of two disjoint $T^1$-parallels $\sigma^s_-$ and $\sigma^s_+$. Similarly, the intersection $W^{uu}_{loc}(\sigma^u) \cap \mathbb{T}^2$ is the disjoint union of two $T^2$-parallels $\sigma^u_-$ and $\sigma^u_+$. We require that

$$C^u_1 \cap (\sigma^s_- \cup \sigma^s_+) = \emptyset \quad \text{and} \quad C^s_1 \cap (\sigma^u_- \cup \sigma^u_+) = \emptyset.$$
Let us explain how this property can be obtained. Recall that $W^u(r_1) \cap \mathbb{T}^2$ is the interior of the cylinder $\mathbb{C}_1$ bounded by $\gamma_1^u$ and $\gamma_2^u$. Thus, since $\gamma_1^u$, $\gamma_2^u$, $\sigma_+^s$, and $\sigma_+^u$ are $T_1$-parallels (or equivalently $T_2$-meridians), we can assume that $\mathbb{C}_1 \cap (\sigma_+^s \cup \sigma_+^u) = \emptyset$. See Figure 2.3. The condition for the cylinder $\mathbb{C}_1$ and the circles $\sigma_+^s$ and $\sigma_+^u$ follows identically noting that $\gamma_1^s$, $\gamma_2^s$, $\sigma_+^s$, and $\sigma_+^u$ are $T_2$-parallels.

By the partially hyperbolic conditions, the strong stable manifolds $W^s(p_i)$, $i = 1, 2$, (tangent to $E^u$ at $p_i$) are well defined and has dimension one. Similarly, the strong unstable manifold $W^u(q_1)$ and $W^u(q_2)$ are well defined and have dimension one. As a consequence of item (2) above (see Figure 2.3) we have the following:

\[
\begin{align*}
    a_1^s \cap W^s(p_1) &= \emptyset, & a_2^s \cap W^s(p_2) &= \emptyset, \\
    b_1^u \cap W^u(q_1) &= \emptyset, & b_2^u \cap W^u(q_2) &= \emptyset.
\end{align*}
\]

2.3. Transverse heteroclinic intersection. Consider the “corner” points of the rectangle $R$,

\[
\begin{align*}
    a_1^s \cap b_1^u &\overset{\text{def}}{=} x_{1,1}, & a_2^s \cap b_2^u &\overset{\text{def}}{=} x_{1,2}, & a_2^s \cap b_1^u &\overset{\text{def}}{=} x_{2,1}, & a_1^s \cap b_2^u &\overset{\text{def}}{=} x_{2,2}.
\end{align*}
\]

By definition, the $\omega$ and $\alpha$-limits of the point $x_{i,j}$ are the singularities $p_i$ and $q_j$, respectively. Denote by $\Gamma_{i,j}$ the closure of the orbit of $x_{i,j}$ (see Figures 2.3 and 2.5). As the intersection between $W^s(p_i)$ and $W^u(q_j)$ is exactly the orbit of $x_{i,j}$, we have

\[
\Gamma_{i,j} \overset{\text{def}}{=} W^s(p_i) \cap W^u(q_j) = \{p_i\} \cup \{q_j\} \cup (W^s(p_i) \cap W^u(q_j)).
\]

Remark 2.2. The curve $\Gamma_{i,j}$ is a $C^1$-invariant normally hyperbolic compact segment.

This remark is a standard consequence of the following facts:

- The point $x_{i,j}$ is a transverse heteroclinic intersection associated to the singularities $p_i$ and $q_j$.
- The partial hyperbolicity hypothesis at the singularities $p_i$ and $q_j$ implies that $W^s(p_i)$ and $W^u(q_j)$ are well defined.
- By construction, recall equation (2.2), $x_{i,j} \not\in W^s(p_i) \cup W^u(q_j)$.
- The curve $\Gamma_{i,j}$ is the closure of the orbit of $x_{i,j}$. 
2.4. Invariant manifolds of the segments $\Gamma_{i,j}$. For each singularity $q_j$, we have that $(W^u(q_j) \setminus W^{uu}(q_j))$ is the disjoint union of two connected invariant surfaces

$$(W^u(q_j) \setminus W^{uu}(q_j)) = W^{u,+}(q_j) \cup W^{u,-}(q_j),$$

where $W^{u,+}(q_j)$ contains the interior of the curves $\Gamma_{1,j}$ and $\Gamma_{2,j}$. With this notation, the invariant manifolds of the curve $\Gamma_{i,j}$ are

$$W^u(\Gamma_{i,j}) = W^u(p_i) \cup W^{u,+}(q_j) \cup W^{uu}(q_j),$$

$$W^s(\Gamma_{i,j}) = W^s(q_j) \cup W^{s,+}(p_i) \cup W^{ss}(p_i).$$

See Figures 2.5 and 2.6. Note that these manifolds are injective $C^1$-immersions of $[0, 1] \times \mathbb{R}$.

Figures 2.4, 2.5, and 2.6 suggest that the curves $\Gamma_{1,j}$ and $\Gamma_{2,j}$ form a “cusp” at the point $q_j$. This geometric configuration will play a key role in our construction. So let us define precisely what we mean by a cusp.

A topological two-disk $G$ contained in the interior of a smooth surface $S$ has a cusp at a point $p \in \partial G$ if for every $\varepsilon > 0$ there are a convex cone $C_\varepsilon$ of angle $\varepsilon$ at $p$ and a neighborhood $U_\varepsilon$ of $p$ in $S$ such that $G \cap U_\varepsilon \subset C_\varepsilon$. Given a curve $\gamma$ contained
where the curves $\Gamma_{i}$, $(2.8)$ and its stable manifold $(2.9)$

there is a unique invariant "central" bundle $E$.

Central bundles.

2.5. $(2.7)$ $\Gamma(\cdot)$ type with a strong stable direction and $\Gamma_{i}$,

We observe that the set $(2.5)$ $\Gamma(p_{i}) = \Gamma_{1,i} \cap \Gamma_{2,i}$, $i = 1, 2,$

is a curve with a cusp singularity at $q_{1}$, see Figures 2.4, 2.5, and 2.6.

The unstable manifold $W^{u}(\Gamma(q_{i}))$ of $\Gamma(q_{i})$ is the set $W^{u}(\Gamma_{1,i}) \cup W^{u}(\Gamma_{2,i})$. Noting that the interior of the "strip" $W^{u}(\Gamma(q_{i}))$ is $W^{u,+}(q_{i}),$ $i = 1, 2,$ we get

$W^{u}(\Gamma(q_{i})) = W^{u}(p_{1}) \cup W^{u}(p_{2}) \cup W^{u,+}(q_{i}) \cup \Gamma^{uu}(q_{i}).$

We observe that the set $W^{u}(\Gamma(q_{i}))$ is an injective $C^{1}$-immersion of a connected surface with boundary. Equivalent statements hold for

$\Gamma(p_{i}) = \Gamma_{1,i} \cup \Gamma_{2,i},$ $i = 1, 2,$

and its stable manifold

$W^{s}(\Gamma(p_{i})) = W^{s}(q_{1}) \cup W^{s}(q_{2}) \cup W^{s,+}(p_{i}) \cup W^{ss}(p_{i}),$

where $W^{s,+}(p_{i})$ is the component of $(W^{s}(p_{i}) \setminus W^{ss}(p_{i}))$ containing the interior of the curves $\Gamma_{1,i}$ and $\Gamma_{1,2}$.

With this notation, the sides $a_{i}^{s}$ and $b_{j}^{u}$ of the rectangle $R$ (see (2.1)) satisfy the following property

$a_{i}^{s} \subset W^{s}(\Gamma(p_{i})) \cap T^{2}$ and $b_{j}^{u} \subset W^{u}(\Gamma(q_{j})) \cap T^{2},$ $i, j \in \{1, 2\}.$

2.5. Central bundles. In this section, we see that the unstable manifolds of $\Gamma_{1,1}$ and $\Gamma_{1,2}$ touch each other at $W^{u}(p_{i})$ tangentially "coming from the same side" of $W^{u}(p_{i})$, see Figure 2.7. In the following we will precise what this means.

Lemma 2.4 (Center stable/unstable bundles). Given any singularity $p$ of saddle type with a strong stable direction $W^{ss}(p)$ (tangent to a strong stable bundle $E^{ss}(p)$) there is a unique invariant "central" bundle $E$ defined over the unstable manifold $W^{u}(p)$ of $p$ that is transverse at $p$ to the bundle $E^{ss}$ and has codimension dim$(E^{ss})$. 

Figure 2.6. The unstable manifold of $\Gamma_{i,j}$, 

in the interior of the surface $S$, a point $q$ in the interior of $\gamma$ is a cusp of $\gamma$ if there is a topological disk $F$ whose boundary contains $\gamma$ and has a cusp at $q$.

Remark 2.3. Recall that the interior of the curves $\Gamma_{1,i}$ and $\Gamma_{2,i}$ are disjoint from $W^{uu}(q_{i})$. Moreover, the interior of these curves are the orbits of the points $x_{1,i}$ and $x_{2,i}$, respectively. These two curves are connected by the segment $b_{j}^{u} \subset W^{u}(q_{i})$ that is disjoint from $W^{uu}(q_{i})$, recall (2.1) and see Figure 2.6. The partial hyperbolicity hypothesis now implies that $\Gamma_{1,i}$ and $\Gamma_{2,i}$ are “central curves” arriving to $q_{1}$ from the same side of $W^{uu}(q_{i})$. These two conditions imply that

(2.5) $\Gamma(q_{i}) = \Gamma_{1,i} \cap \Gamma_{2,i},$ $i = 1, 2,$

is a curve with a cusp singularity at $q_{1}$, see Figures 2.4, 2.5, and 2.6.

The unstable manifold $W^{u}(\Gamma(q_{i}))$ of $\Gamma(q_{i})$ is the set $W^{u}(\Gamma_{1,i}) \cup W^{u}(\Gamma_{2,i})$. Noting that the interior of the “strip” $W^{u}(\Gamma(q_{i}))$ is $W^{u,+}(q_{i}),$ $i = 1, 2,$ we get

(2.6) $W^{u}(\Gamma(q_{i})) = W^{u}(p_{1}) \cup W^{u}(p_{2}) \cup W^{u,+}(q_{i}) \cup W^{uu}(q_{i}).$

We observe that the set $W^{u}(\Gamma(q_{i}))$ is an injective $C^{1}$-immersion of a connected surface with boundary. Equivalent statements hold for

(2.7) $\Gamma(p_{i}) = \Gamma_{1,i} \cup \Gamma_{2,i},$ $i = 1, 2,$

and its stable manifold

(2.8) $W^{s}(\Gamma(p_{i})) = W^{s}(q_{1}) \cup W^{s}(q_{2}) \cup W^{s,+}(p_{i}) \cup W^{ss}(p_{i}),$

where $W^{s,+}(p_{i})$ is the component of $(W^{s}(p_{i}) \setminus W^{ss}(p_{i}))$ containing the interior of the curves $\Gamma_{1,i}$ and $\Gamma_{1,2}$.

With this notation, the sides $a_{i}^{s}$ and $b_{j}^{u}$ of the rectangle $R$ (see (2.1)) satisfy the following property

(2.9) $a_{i}^{s} \subset W^{s}(\Gamma(p_{i})) \cap T^{2}$ and $b_{j}^{u} \subset W^{u}(\Gamma(q_{j})) \cap T^{2},$ $i, j \in \{1, 2\}.$
A similar property holds for saddle singularities $q$ with a strong unstable manifold tangent to some strong unstable bundle $E^{uu}$. In this case, there is a central bundle defined over $W^s(q)$ that is transverse to $E^{uu}$ at $q$ and has codimension $\dim(E^{uu})$.

**Proof.** We can assume that for every point $x \in W^u_{\text{loc}}(p)$ there is defined a negatively invariant cone-field $C^{ss}$ around the strong stable direction $E^{ss}$. Consider the complement $C^c$ of $C^{ss}$. Given $y \in W^u(p)$ there is $t(y) > 0$ such that $X_{-t}(y) \in W^u_{\text{loc}}(p)$ for all $t \geq t(y)$. Given $y \in W^u(p)$ it is enough to define

$$E^c(y) \overset{\text{def}}{=} \{ v : D_y X_{-t}(v) \in C^c(X_{-t}(y)) \text{ for all } t \geq t(y) \}.$$ 

By construction, the bundle $E^c(y)$ is transverse to $E^{ss}$ and its dimension is the codimension of $E^{ss}$. This completes the proof of the lemma. \hfill $\square$

**Remark 2.5.** Applying Lemma 2.4 to the singularity $p_i$, we get a two dimensional bundle $E^c(p_i)$, $i = 1, 2$, coinciding with the bundle $E^{cs}$ defined along the curve $\sigma^s$ in Section 2.2. Analogously, the bundle $E^c$ defined along $W^s(q_j)$ coincides with the bundle $E^{cu}$ along $\sigma^u$.

**Lemma 2.6.** The surfaces $W^u(\Gamma(q_1))$ and $W^u(\Gamma(q_2))$ are tangent to the bundle $E^{ss}$ along their intersection $W^u(p_1) \cup W^u(p_2)$. Similarly, the surfaces $W^s(\Gamma(p_1))$ and $W^s(\Gamma(p_2))$ are tangent to the bundle $E^{cu}$ along their intersection $W^s(q_1) \cup W^s(q_2)$.

**Proof.** The boundary part of $W^u(\Gamma(q_1))$ has three components, $W^u(p_1)$, $W^u(p_2)$, and $W^{uu}(q_1)$, recall equation (2.6) and see Figure 2.6. Moreover, the surface $W^u(\Gamma(q_1))$ is transverse to $W^{ss}(p_i)$. The uniqueness of the central bundle $E^c$ in Lemma 2.4 implies that for each $x \in W^u(p_i) \subset \sigma^s$ the fiber $E^c(x) = E^{cs}(x)$ (recall Remark 2.5) is the tangent space $T_x(W^u(\Gamma(q_1)))$. This implies the lemma. \hfill $\square$

**Figure 2.7.** The open “half-planes” $E^{cu}_+$. 

**Remark 2.7** (The open “half-planes” $E^{cs}_+$ and $E^{cu}_+$). The normally hyperbolic curves $\Gamma_{1,1}$ and $\Gamma_{1,2}$ are “central curves” contained in $W^s(p_i)$ arriving to $p_i$ from the same side of $W^s(p_i) \setminus W^{ss}(p_i)$. Furthermore, the boundary surfaces $W^u(\Gamma_{1,1})$ and $W^u(\Gamma_{1,2})$ are tangent to $E^{cs}$ along $W^s(p_i)$. This implies that, for every $x \in W^u(p_i)$, the vectors in $E^{cs}(x)$ pointing to the interior of $W^u(\Gamma_{1,1})$ form an open half-plane $E^{cs}_+(x)$. This half-plane coincides with the vectors of $E^{cs}(x)$ pointing to the interior of $W^u(\Gamma_{1,1})$ or (equivalently) of $W^u(\Gamma_{1,2})$. See Figure 2.7.

For points $y \in W^u(q_j)$, we similarly define the half-plane $E^{cu}_+(y)$ as the vectors in $E^{cu}$ pointing to the interior of $W^s(\Gamma(p_i))$, $i = 1, 2$. 
2.6. Position of the invariant manifolds in the basins of $r_1$ and $s_1$. Consider a “small” two-sphere $S^s$ contained in the interior of the solid torus $T_1$ that is transverse to the vector field $X$ and bounds a three-ball $B^s \subset W^s(s_1) \cap T_1$ whose interior contains the singularity $s_1$. Let

$$\eta^s \equiv S^s \cap W^{ss}(\sigma^s).$$

Note that $\eta^s$ is a circle that contains the points

$$y^u_1 = W^u(p_1) \cap S^s \text{ and } y^u_2 = W^u(p_2) \cap S^s, \quad y^s_1, y^s_2 \in \eta^s.$$

We similarly define a “small” two-sphere $S^u \subset T_2$ transverse to $X$ bounding a three-ball $B^u \subset W^u(r_1)$ whose interior contains $r_1$. We define the circle

$$\eta^u \equiv S^u \cap W^{uu}(\sigma^u)$$

and the intersection points

$$y^s_1 = W^s(q_1) \cap S^u \text{ and } y^s_2 = W^s(q_2) \cap S^u, \quad y^s_1, y^s_2 \in \eta^s.$$

**Remark 2.8.** Choosing the balls $B^s$ and $B^u$ small enough, we can assume that the minimum time that a point takes to go from $B^u$ to $B^s$ is arbitrarily large. In particular, this time is bigger than 10: $X_t(B^u) \cap B^s = \emptyset$ for all $t \in [0, 10]$.

Consider the sets $G^u$ and $G^s$ (the set $G^u$ is depicted in Figure 2.8),

$$G^u \equiv \overline{W^u(r_1) \cap S^u} \text{ and } G^s \equiv \overline{W^s(s_1) \cap S^u}.$$

![Figure 2.8. The set $G^u$.](image)

**Lemma 2.9.** The set $G^u$ is a topological two-disk bounded by $\ell^u_1 \cup \ell^u_2 \cup \{y^u_1\} \cup \{y^u_2\}$, where

$$\ell^u_1 \subset W^u(q_1) \text{ and } \ell^u_2 \subset W^u(q_2).$$
are disjoint (open) simple curves whose endpoints are \( y_1^u \) and \( y_2^u \). The closed curves \( \overline{\ell}_1^u \) and \( \overline{\ell}_2^u \) have the same tangent direction at their endpoints \( y_1^u \) and \( y_2^u \). The disk \( G^u \) has two cusps at the points \( y_1^u \) and \( y_2^u \).

Similarly, the set \( G^s \) is a topological two-disk bounded by \( \ell_1^s \cup \ell_2^s \cup \{ y_1^s \} \cup \{ y_2^s \} \), where

\[
\ell_1^s \subset W^s(p_1) \quad \text{and} \quad \ell_2^s \subset W^s(p_2)
\]

are disjoint (open) simple curves whose endpoints are \( y_1^s \) and \( y_2^s \). The closed curves \( \overline{\ell}_1^s \) and \( \overline{\ell}_2^s \) have the same tangent direction at their endpoints \( y_1^s \) and \( y_2^s \). The disk \( G^s \) has two cusps at \( y_1^s \) and \( y_2^s \).

We consider the following notation, given an interval \([t_1, t_2]\) and a set \( A \) we let

\[
X_{[t_1, t_2]}(A) \overset{\text{def}}{=} \bigcup_{t \in [t_1, t_2]} X_t(A).
\]

**Proof.** We only prove the lemma for the set \( G^u \), the proof for \( G^s \) is identical.

Note that the sphere \( S^s \) intersects every orbit of the set \( W^s(s_1) \setminus \{ s_1 \} \) in exactly one point. Thus, since the rectangle \( R \) in Remark 2.1 is the closure of \( W^u(r_1) \cap W^s(s_1) \cap T^2 \), the positive orbit of any point in the interior of \( R \) intersects \( S^s \) in exactly one point. Hence the set \( G^u \) is the closure of the “projection” along the orbits of \( X \) of the interior of \( R \) into \( S^s \), that is,

\[
G^u = \overline{\bigcup_{[0, \infty)} (\text{int}(R)) \cap S^s}.
\]

By construction the set \( G^u \) is a topological two-disk. We next describe its boundary.

By equation (2.9) the boundary of \( R \) consists of the segments \( a_i^u \subset W^u(p_i) \) and \( b_i^u \subset W^u(q_i) \), \( i = 1, 2 \). Furthermore, the interiors of the segments \( a_1^u \) and \( b_1^u \) are contained in \( W^s(s_1) \). Denote by \( \ell_1^u \) and \( \ell_2^u \) the “projections” by the flow of \( X \) of these interiors into \( S^u \), that is, \( \ell_i^u \overset{\text{def}}{=} X_{[0, \infty)}(\text{int}(a_i^u)) \cap S^s \).

Consider any sequence \((x_n)\) of points in the interior of \( R \) accumulating to the side \( a_i^u \) of \( R \). Note that the (positive) orbit of \( x_n \) by the flow of \( X \) goes arbitrarily close to the saddle singularity \( p_i \) before intersecting \( S^a \) at a point \( y_n \). By construction, the sequence \((y_n)\) converges to \( y_u^u = W^u(p_i) \cap S^u \). Indeed, for any given curve \( b \subset R \) transverse to \( X \) joining the sides \( a_1^u \) and \( a_2^u \) of \( R \) the intersection of the sphere \( S^a \) and the positive orbit of \( b \) by the flow \( X \) (i.e., the “projection” of \( b \) into \( S^a \) by the flow) is a curve \( \ell_b \) joining \( y_1^u \) and \( y_2^u \) (these points are in the closure of \( \ell_b \)). In particular, \( y_1^u \) and \( y_2^u \) are the endpoints of \( \ell_i^u \), \( i = 1, 2 \).

Bearing in mind equation (2.9) and the definitions of \( y_i^u \), \( \ell_i^u \), and \( \Gamma(q_i) \), \( i = 1, 2 \), we get the following:

\[
\ell_i^u = \ell_i^u \cup \{ y_1^u, y_2^u \} = W^u(\Gamma(q_i)) \cap S^s,
\]

\[
\partial G^u = (W^u(\Gamma(q_1)) \cap S^s) \cup (W^u(\Gamma(q_2)) \cap S^s) = \ell_1^u \cup \ell_2^u \cup \{ y_1^u, y_2^u \}.
\]

This completes the description of the set \( \partial G^u \).

It remains to see that \( y_1^u \) and \( y_2^u \) are cusps of \( G^u \). By Lemma 2.6 the curves \( \overline{\ell}_1^u \) and \( \overline{\ell}_2^u \) are tangent at \( y_1^u \) to \( E^{cs}(y_1^u) \cap T_{y_1^u}(S^s) \). To see that the point \( y_1^u \) is a cusp of \( G^u \) it is enough to note that the interior of \( G^u \) is disjoint from the circle \( \eta^u \). Thus the disk \( G^u \) is the “thin component” of \( S^s \setminus \partial G^u \). This ends the proof of the lemma. \( \square \)
Remark 2.10. With the notations above, the following inclusions hold
\[(S^s \setminus G^u) \subset W^u(r_2) \quad \text{and} \quad (S^u \setminus G^s) \subset W^s(s_2).\]

2.7. The diffeomorphism time-one map \(X_1\). Let \(X_1\) denote the time-one map of the vector field \(X\) and define the diffeomorphism \(F_0 \overset{\text{def}}{=} X_1\). Note that \(F_0\) is a Morse-Smale diffeomorphism whose non-wandering set consists of the sinks \(s_1\) and \(s_2\), the saddles of \(s\)-index two \(p_1\) and \(p_2\), the saddles of \(s\)-index one \(q_1\) and \(q_2\), and the sources \(r_1\) and \(r_2\). Note that the invariant manifolds of these points for the vector field \(X\) and for the diffeomorphism \(F_0\) coincide. We only write \(W^s(x, X)\) or \(W^s(x, F_0)\) to emphasize the role of \(X\) or \(F_0\), otherwise we just write \(W^s(x)\).

Consider the fundamental domain \(\Delta^s\) of \(W^s(s_1)\) for \(F_0\) bounded by \(S^s\) and \(F_0(S^s)\). Note that
\[
\Delta^s \overset{\text{def}}{=} X_{[0,1]}(S^s) = B^s \setminus \text{int}(F_0(B^s)) \simeq S^s \times [0, 1].
\]

Let
\[
\mathcal{E}^u \overset{\text{def}}{=} X_{[0,1]}(G^u), \quad \mathcal{L}^u_i \overset{\text{def}}{=} X_{[0,1]}(\ell^u_i), \quad \mathcal{Y}^u_i \overset{\text{def}}{=} X_{[0,1]}(y^u_i).
\]

These sets are depicted in Figure 2.9.

![Figure 2.9. The sets \(\mathcal{E}^u, \mathcal{L}^u_i,\) and \(\mathcal{Y}^u_i\).](image)

By construction, the set \(\mathcal{E}^u\) is a topological ball bounded by the disks \(G^u\) and \(F_0(G^u)\), the “rectangles” \(\mathcal{L}^u_1\) and \(\mathcal{L}^u_2\), and the curves \(\mathcal{Y}^u_1\) and \(\mathcal{Y}^u_2\). By the definition of \(G^u\) and of the cylinders \(C^u_1\) and \(C^u_2\) and by Lemma 2.9 (see also Remark 2.10), we have that
\[
W^u(r_1) \cap \Delta^s = \mathcal{E}^u \quad \text{and} \quad \Delta^s \setminus \mathcal{E}^u \subset W^u(r_2).
\]

Since \(\ell^u_1 \subset W^u(q_1, X)\) and \(y^u_1 \subset W^u(p_1, X)\) we have
\[
L^u_1 \subset W^u(q_1, F_0) \quad \text{and} \quad \mathcal{Y}^u_1 \subset W^u(p_1, F_0).
\]

Note that the common boundary of the rectangles \(L^u_1\) and \(L^u_2\) are the curves \(\mathcal{Y}^u_1\) and \(\mathcal{Y}^u_2\). Moreover, the closure of \(L^u_1\) is tangent to the center unstable bundle \(E^u\) along the curves \(\mathcal{Y}^u_1\) and \(\mathcal{Y}^u_2\).

We similarly define the fundamental domain \(\Delta^u\) of \(W^u(r_1)\) for \(F_0\) bounded by \(S^u\) and \(F_0(S^u)\). As before
\[
\Delta^u \overset{\text{def}}{=} X_{[0,1]}(S^u) \simeq S^u \times [0, 1].
\]

We let
\[
\mathcal{E}^s \overset{\text{def}}{=} X_{[0,1]}(G^s), \quad \mathcal{L}^s_i \overset{\text{def}}{=} X_{[0,1]}(\ell^s_i), \quad \mathcal{Y}^s_i \overset{\text{def}}{=} X_{[0,1]}(y^s_i)
\]
and as above we have
\begin{equation}
L_i^s \subset W^s(p_i) \quad \text{and} \quad Y_i^s \subset W^s(q_i).
\end{equation}
Moreover, the “rectangles” $L_i^s$ and $L_2^s$ are tangent at the curves $Y_1^s$ and $Y_2^s$ to the center stable bundle $E^c$. By construction, $E^s$ is bounded by the disks $G^s$ and $F_0(G^s)$, the “rectangles” $L_i^s$ and $L_2^s$, and the curves $Y_1^s$ and $Y_2^s$. Finally,
\begin{equation}
W^s(s_1) \cap \Delta^u = E^s \quad \text{and} \quad \Delta^u \setminus E^s \subset W^s(s_2).
\end{equation}

3. A DIFFEOMORPHISM ON $S^2 \times S^1$ OBTAINED BY A SURGERY

3.1. The surgery. In this section identify some regions of $S^3$ by a local diffeomorphism $\Psi$. This surgery provides a diffeomorphism $F_\Psi$ defined on $S^2 \times S^1$ induced by $F_0 = X_1$ and the quotient of $S^3$ by $\Psi$. We will see in Section 5 that for an appropriate choice of $\Psi$ the diffeomorphism $F_\Psi$ has fragile cycles.

With the notation in Section 2 consider a diffeomorphism $\Psi: \Delta^s \to \Delta^u$ such that
- $\Psi(S^s) = S^u$ and $\Psi(F_0(S^s)) = F_0(S^u)$,
- $\Psi \circ F_0|_{S^u} = F_0|_{S^u} \circ \Psi$,
- $D\Psi \circ D\Psi|_{S^u} = D\Psi|_{S^u} \circ D\Psi$.

Recall that $B^s \subset W^s(s_1)$ and $B^u \subset W^u(r_1)$ are small balls containing $s_1$ and $r_1$, respectively. In the set $S^3 \setminus \left(\text{int}(F_0(B^s)) \cup \text{int}(B^u)\right)$ we identify the points $x \in \Delta^s$ and $\Psi(x) \in \Delta^u$ obtaining the quotient space
\[ M \overset{\text{def}}{=} \left(S^3 \setminus \left(\text{int}(F_0(B^s)) \cup \text{int}(B^u)\right)\right) / \Psi. \]
The set $M$ is a $C^1$-manifold diffeomorphic to $S^2 \times S^1$ and the diffeomorphism $F_0$ induces a diffeomorphism $F_\Psi: M \to M$.

Denote by $\pi$ the projection $\pi: S^3 \setminus \left(\text{int}(F_0(B^s)) \cup \text{int}(B^u)\right) \to M$ that associates to $x$ its class $\pi(x) = [x]$ by the equivalence relation induced by $\Psi$. For notational simplicity, if $x \notin \Delta^s \cup \Delta^u$ we simply write $x$ instead of $\pi(x)$. Write
\[ \Delta = (\Delta^s \cup \Delta^u) / \Psi = \pi(\Delta^s) = \pi(\Delta^u). \]
In what follows we write $K_i^u(F_\Psi) \overset{\text{def}}{=} \pi(K_i^u)$, where $K = Y, L, G, E$ and $i = 1, 2, 0$.

Remark 3.1. The set
\[ \Delta \setminus (L_1^s(F_\Psi) \cup L_2^s(F_\Psi) \cup Y_1^s(F_\Psi) \cup Y_2^s(F_\Psi) \cup G^s(F_\Psi) \cup F_\Psi(G^u(F_\Psi))) \]
has two connected components. The set $E^u(F_\Psi)$ is the component that is diffeomorphic to $G^u \times [0, 1]$. This set is a topological ball. There is a similar characterization for $E^s(F_\Psi)$.

3.2. The dynamics of $F_\Psi$. Note that the orbits of $F_\Psi$ disjoint from $\Delta$ are the projection of the orbits of the diffeomorphism $F_0$. Thus $s_2$ is a sink and $r_2$ is a source of $F_\Psi$ (note that the sink $s_1$ and the source $r_1$ are removed in the construction of $M$). Similarly, the points $p_1$ and $p_2$ are saddles of $s$-index 2 of $F_\Psi$ and the points $q_1$ and $q_2$ are saddles of $s$-index 1 of $F_\Psi$. Observe that $F_\Psi$ can have further periodic points, but by Remark 2.8, these points have period larger than 10.
Lemma 3.3 (Invariant manifolds and their intersections)

characterization of the sets $L_i$ is a connected component of $W$ disjoint from $B$.

Proof. An immediate consequence of the first two items of Lemma 3.3 is the following:

Using the identification by $\Psi$ and the properties of $F_0$, we have the following characterization of the sets $L_i^u(F_\Psi)$ and $Y_i^u(F_\Psi)$, $i = 1, 2$:

\begin{align*}
L_i^u(F_\Psi) &= \{ x \in \Delta : x \in W^u(q_i, F_\Psi) \text{ and } F_\Psi^{-j}(x) \notin \Delta \text{ for all } j \geq 2 \}, \\
Y_i^u(F_\Psi) &= \{ x \in \Delta : x \in W^u(p_i, F_\Psi) \text{ and } F_\Psi^{-j}(x) \notin \Delta \text{ for all } j \geq 2 \}, \\
L_i^s(F_\Psi) &= \{ x \in \Delta : x \in W^s(p_i, F_\Psi) \text{ and } F_\Psi^j(x) \notin \Delta \text{ for all } j \geq 2 \}, \\
Y_i^s(F_\Psi) &= \{ x \in \Delta : x \in W^s(q_i, F_\Psi) \text{ and } F_\Psi^j(x) \notin \Delta \text{ for all } j \geq 2 \}.
\end{align*}

(3.1)

Remark 3.2. The normally hyperbolic curves $\Gamma_{i,j}$ of $F_0$ (recall Remark 2.2) are disjoint from $E^u \cap B^s$, thus their projections on $M$ (also denoted by $\Gamma_{i,j}$) are normally hyperbolic curves of $F_\Psi$. Observe also that by construction, the interior of $\Gamma_{i,j}$ is contained in $W^s(p_i, F_\Psi) \cap W^u(q_j, F_\Psi)$, recall equation (2.4).

We continue to use the notation $\Gamma(q_j) = \Gamma_{1,j} \cup \Gamma_{2,j}$ and $\Gamma(p_i) = \Gamma_{i,1} \cup \Gamma_{i,2}$.

With the previous notation we have that

\[ L_i^u(F_\Psi) \cup Y_i^u(F_\Psi) \cup Y_i^s(F_\Psi) = L_i^u(F_\Psi) \]

is a connected component of $W^u(\Gamma(q_j), F_\Psi) \cap \Delta$.

Lemma 3.3 (Invariant manifolds and their intersections). Consider $x \in \Delta$.

(1) if $x \notin E^u(F_\Psi)$ then $x \in W^u(r_2, F_\Psi)$ and thus it is not chain recurrent,

(2) if $x \notin E^s(F_\Psi)$ then $x \in W^s(s_2, F_\Psi)$ and thus it is not chain recurrent,

(3) if $x \in L_i^u(F_\Psi)$ then $x \in W^u(q_i, F_\Psi)$,

(4) if $x \in Y_i^u(F_\Psi)$ then $x \in W^u(p_i, F_\Psi)$,

(5) if $x \in L_i^s(F_\Psi)$ then $x \in W^s(p_i, F_\Psi)$, and

(6) if $x \in Y_i^s(F_\Psi)$ then $x \in W^s(q_i, F_\Psi)$.

Proof. The first item follows immediately from equation (2.13) and the definition of $F_\Psi$. Similarly, the second item follows from (2.16). Items (3)-(6) follow from equation (3.1). \qed

**Figure 3.1.** The recurrence region of $F_\Psi$.

An immediate consequence of the first two items of Lemma 3.3 is the following:
Corollary 3.4. Let \( x \in \Delta \) be a chain recurrent point for \( F_\Psi \). Then \( x \in E^s(F_\Psi) \cap E^u(F_\Psi) \).

A straightforward consequence of Lemma 3.3 and Remark 3.2 is the following:

Corollary 3.5. Consider \( x \in \Delta \).

(1) If \( x \in L^u_1(F_\Psi) \cap L^s_1(F_\Psi) \) then \( x \in W^u(q_1, F_\Psi) \cap W^s(p_j, F_\Psi) \).

(2) If \( x \in L^u_1(F_\Psi) \cap Y^u_1(F_\Psi) \) then \( x \in W^u(q_1, F_\Psi) \cap W^s(q_j, F_\Psi) \).

(3) If \( x \in Y^u_1(F_\Psi) \cap L^u_1(F_\Psi) \) then \( x \in W^u(p_i, F_\Psi) \cap W^s(p_j, F_\Psi) \).

(4) If \( x \in Y^u_1(F_\Psi) \cap Y^u_1(F_\Psi) \) then \( x \in W^u(p_i, F_\Psi) \cap W^s(q_j, F_\Psi) \).

(5) The interior of \( \Gamma_{i,j} \) is contained in \( W^s(p_i, F_\Psi) \cap W^u(q_j, F_\Psi) \).

Recall that a periodic point is called isolated if its chain recurrent class coincides with its (finite) orbit.

Lemma 3.6 (Heterodimensional cycles and trivial homoclinic classes).

(1) If \( Y^u_1(F_\Psi) \cap Y^s_1(F_\Psi) \neq \emptyset \) then \( F_\Psi \) has a heterodimensional cycle associated to \( p_1 \) and \( q_1 \).

(2) If \( Y^u_1(F_\Psi) \cap L^u_1(F_\Psi) \neq \emptyset \) then the homoclinic class of \( q_1 \) is non trivial.

(3) If \( Y^s_1(F_\Psi) \cap L^u_1(F_\Psi) \neq \emptyset \) then homoclinic class of \( p_1 \) is non trivial.

(4) If \( Y^u_1(F_\Psi) \cap E^s(F_\Psi) = \emptyset \) then \( p_1 \) is an isolated saddle.

(5) If \( Y^s_1(F_\Psi) \cap E^u(F_\Psi) = \emptyset \) then \( q_1 \) is an isolated saddle.

Proof. To prove the first item just recall that by item (i) in Corollary 3.5 the interior of \( \Gamma_{1,1} \) is contained in \( W^s(p_1, F_\Psi) \cap W^u(q_1, F_\Psi) \). By item (ii) in Corollary 3.5 if \( Y^u_1(F_\Psi) \cap Y^s_1(F_\Psi) \neq \emptyset \) then \( W^u(p_1, F_\Psi) \cap W^s(q_1, F_\Psi) \neq \emptyset \) and thus there is a heterodimensional cycle associated to \( p_1 \) and \( q_1 \).

Items (2) and (3) follow from Lemma 3.3.

To prove item (4) we use the following simple fact whose proof we omit.

Remark 3.7. Let \( p \) be a hyperbolic saddle that is non-isolated. Then its stable/unstable manifold contains points of its chain recurrence class that do not belong to its orbit. In particular, if a hyperbolic fixed point \( p \) is such that \( W^u(p) \setminus \{p\} \) (respectively, \( W^s(p) \setminus \{p\} \)) is contained in the stable (resp. unstable) manifolds of some sinks (resp. sources) then it is isolated.

Note that every point \( x \in W^u(p_1, F_\Psi), x \neq p_1 \), in the separatrix of \( W^u(p_1, F_\Psi) \) that does not contain \( y^1_i \) is contained in \( W^s(s_2, F_\Psi) \). Thus this separatrix does not contain chain recurrent points. Hence it is enough to consider points in the separatrix of \( W^u(p_1, F_\Psi) \) containing \( y^1_i \). Note that \( W^u(p_1, F_\Psi) \cap \Delta \) contains a fundamental domain of \( W^u(p_1, F_\Psi) \) that is contained in \( Y^u_1(F_\Psi) \). Thus, by Remark 3.7 if \( p_1 \) is not isolated then \( Y^u_1(F_\Psi) \) must contain some point of the chain recurrence class of \( p_1 \). Thus in such a case \( Y^u_1(F_\Psi) \) cannot be contained in \( W^s(s_2, F_\Psi) \).

Suppose that \( Y^u_1(F_\Psi) \cap E^s(F_\Psi) = \emptyset \). Then, by item (ii) in Lemma 3.3 one has that \( Y^u_1(F_\Psi) \subset W^s(s_2) \). By the above discussion this implies that \( p_1 \) is isolated, proving item (4).

The proof of item (5) is identical to the previous one and thus it is omitted. \( \square \)

4. Dynamics in a neighborhood of \( F_\Psi \)

In this section we consider diffeomorphisms \( F \) in a small \( C^1 \)-neighborhood of \( F_\Psi \). The hyperbolic-like properties of the objects introduced in Section 3 for the
diffeomorphism $F_\Psi$ allows us to define their continuations for nearby diffeomorphisms and thus to repeat these constructions. In particular, the continuations of the hyperbolic points $s_2,r_2,p_1,p_2,q_1$, and $q_2$ of $F_\Psi$ are defined. We will omit the dependence on $F$ of these continuations.

As the arguments in this section are similar to those in Sections 2 and 3 some of these constructions will be just sketched. We now go to the details of our constructions.

Consider the spheres $\pi(S^*)$ and $F(\pi(S^*))$ and denote by $\Delta_F$ the closure of the connected component of $M \setminus (\pi(S^*) \cup F(\pi(S^*)))$ which is close to $\Delta$ (i.e., the component that is in the same side of $\pi(S^*)$ as $\Delta$). The set $\Delta_F$ is diffeomorphic to $S^2 \times [0,1]$ and varies continuously with $F$ in the $C^1$-topology. In particular, by Remark 2.8, if $x \in \Delta_F$ and $F^i(x) \in \Delta_F$ then $|i| \geq 9$.

Bearing in mind the definitions of the sets $\mathbb{L}^a_i(F_\Psi)$ and $\mathbb{Y}^a_i(F_\Psi)$ in (3.1), we define their “continuations” $\mathbb{L}^a_i(F)$ and $\mathbb{Y}^a_i(F)$ for $F$ close to $F_\Psi$ by

\[
\mathbb{L}^a_i(F) = \{ x \in \Delta_F : x \in W^u(q_i,F) \text{ and } F^{-i}(x) \notin \Delta_F \text{ for all } i \geq 2 \},
\]

\[
\mathbb{Y}^a_i(F) = \{ x \in \Delta_F : x \in W^u(p_i,F) \text{ and } F^{-i}(x) \notin \Delta_F \text{ for all } i \geq 2 \},
\]

\[
\mathbb{L}^a_1(F) = \{ x \in \Delta_F : x \in W^s(p_i,F) \text{ and } F^i(x) \notin \Delta_F \text{ for all } i \geq 2 \},
\]

\[
\mathbb{Y}^a_1(F) = \{ x \in \Delta_F : x \in W^s(q_i,F) \text{ and } F^i(x) \notin \Delta_F \text{ for all } i \geq 2 \}.
\]

**Remark 4.1.** The sets $\mathbb{L}^a_i(F)$ and $\mathbb{Y}^a_i(F)$, $i = 1, 2$, depend continuously on $F$.

Note that the closed curves $\Gamma_{i,j}$ are normally hyperbolic for $F_\Psi$ (recall Remark 3.2). Thus for every $F$ close to $F_\Psi$ there are defined their continuations, denoted by $\Gamma_{i,j}(F)$, that depend continuously on $F$. These curves join the saddles $p_i$ and $q_j$ and their interiors are center stable manifolds of $p_i$ and center unstable manifolds of $q_j$. Finally, from the normal hyperbolicity of $\Gamma_{i,j}(F)$, compact parts of the invariant manifolds $W^s(\Gamma_{i,j}(F))$ and $W^u(\Gamma_{i,j}(F))$ depend continuously on $F$.

Observe that $W^u(q_i,F) \setminus W^{uu}(q_i,F)$ (resp. $W^s(p_i,F) \setminus W^{ss}(p_i,F)$) has two connected components (separatrices), denoted by $W^{u,+}(q_i,F)$ and $W^{u,-}(q_i,F)$ (resp. $W^{s,+}(p_i,F)$). We choose these components such that the following holds:

**Remark 4.2.** (Invariant manifolds of $\Gamma_{i,j}(F)$).

- $W^s(\Gamma_{1,1}(F)) \setminus W^u(p_1,F) = W^s(\Gamma_{2,1}(F)) \setminus W^u(p_2,F) = W^{s,+}(q_j,F) \cup W^{uu}(q_j,F)$.
- $W^s(\Gamma_{1,2}(F)) \setminus W^s(q_1,F) = W^s(\Gamma_{1,2}(F)) \setminus W^s(q_2,F) = W^{s,+}(p_i,F) \cup W^{ss}(p_i,F)$.
- $W^s(\Gamma_{1,1}(F)) \cap W^u(\Gamma_{1,2}(F)) = W^u(p_i,F)$.
- $W^s(\Gamma_{1,2}(F)) \cap W^s(\Gamma_{2,2}(F)) = W^s(q_j,F)$.

By Lemma 2.4 and using the notation in Remark 2.4, for every $F$ close to $F_\Psi$ there is a unique invariant central bundle $E_F^{cs}$ (resp. $E_F^{cu}$) defined on $W^u(p_i,F)$ (resp. $W^s(q_j,F)$) and transverse to the strong stable (resp. unstable) direction at $p_i$ (resp. $q_j$). The central bundles $E_F^{cs}$ and $E_F^{cu}$ depend continuously on $F$.

**Remark 4.3.** The manifolds with boundary $W^u(\Gamma_{1,1}(F))$ and $W^u(\Gamma_{1,2}(F))$ are tangent along $W^u(p_i,F)$ to the plane field $E_F^u$. As in Remark 2.7 for $x \in W^u(p_i,F)$, the vectors of $E_F^{cs}$ entering in the interior of $W^u(\Gamma_{i,j}(F))$ define an half plane $E_+^{cs,F}(x)$. 
Analogously, the manifolds with boundary $W^s(\Gamma_{1,j}(F))$ and $W^u(\Gamma_{2,j}(F))$ are tangent along $W^s(q_j, F)$ to the plane field $E^u_F$. For $x \in W^s(q_j, F)$, the vectors of $E^s_F$ entering in the interior of $W(\Gamma_{1,j}(F))$ define a half plane $E^s_{+,F}$.

As in \cite{2} and \cite{27}, we define the sets

$$\Gamma(q_j, F) \overset{\text{def}}{=} \Gamma_{1,j}(F) \cup \Gamma_{2,j}(F) \quad \text{and} \quad \Gamma(p_i, F) \overset{\text{def}}{=} \Gamma_{1,1}(F) \cup \Gamma_{1,2}(F).$$

Then

$$W^u(\Gamma(q_j, F)) = W^u(\Gamma_{1,j}(F)) \cup W^u(\Gamma_{2,j}(F))$$

is a $C^1$-surface with boundary whose compact parts depend continuously on $F$. Moreover,

$$W^u(\Gamma(q_1, F)) \cap W^u(\Gamma(q_2, F)) = W^u(p_1, F) \cup W^u(p_2, F)$$

The surfaces $W^u(\Gamma(q_1, F))$ and $W^u(\Gamma(q_2, F))$ depend continuously on $F$ and are tangent to $E^u_F$ along this intersection. This last assertion is just a version of Lemma \ref{2.0} for $F$ close to $F_0$.

Similarly, the compact parts of the surfaces $W^s(\Gamma(p_1, F))$ and $W^s(\Gamma(p_2, F))$ depend continuously on $F$ and they are tangent to $E^u_F$ along their intersection $W^s(q_1, F) \cup W^s(q_2, F)$.

Using the previous notation we get that

$$\overline{L^u_j(F)} = L^u_j(F) \cup \overline{\mathcal{Y}^u_1(F) \cup \mathcal{Y}^u_2(F)},
\overline{L^s_j(F)} = L^s_j(F) \cup \overline{\mathcal{Y}^s_1(F) \cup \mathcal{Y}^s_2(F)}.$$

The set $\overline{L^u_j(F)}$ is the connected component of $W^u(\Gamma(q_j, F)) \cap \Delta_F$ whose negative iterates $F^{-i}(\overline{L^u_j(F)})$, $i \geq 2$, are disjoint from $\Delta_F$. Similarly, the set $\overline{L^s_j(F)}$ is the connected component of $W^u(\Gamma(p_j, F) \cap \Delta_F$ whose positive iterates larger than 2 are disjoint from $\Delta_F$.

As a consequence of the previous constructions, we get

**Lemma 4.4.** The sets $\overline{L^u_j(F)}$ and $\overline{L^s_j(F)}$ are “rectangles” depending continuously on $F$ (for the $C^1$ topology).

The sets $\mathcal{G}^s(F)$ and $\mathcal{E}^u(F)$ are defined similarly as in the case $F_\Psi$. The set $\mathcal{G}^s(F)$ is the topological disk with two cusps (these cuspidal points are in $W^u(p_1, F)$ and $W^u(p_2, F)$) whose boundary is the union of $W^u(\Gamma(q_1, F)) \cap \Delta_F$ and $W^u(\Gamma(q_2, F)) \cap \Delta_F$. There is an analogous definition for $\mathcal{G}^u(F)$. Note that by construction these sets depend continuously on $F$.

Finally, the set $\mathcal{E}^u(F)$ is the topological ball bounded by $\overline{L^u_1(F)}$, $\overline{L^u_2(F)}$, $\mathcal{G}^u(F)$ and $F(\mathcal{G}^u(F))$ that is close to $\mathcal{E}^u(F_\Psi)$. There is a similar definition for the set $\mathcal{E}^s(F)$. By construction the sets $\mathcal{G}^s(F)$ and $\mathcal{E}^u(F)$ depend continuously on $F$.

There is the following reformulation of Corollary \ref{3.3} and Lemma \ref{3.6} for diffeomorphisms $F$ close to $F_\Psi$.

**Lemma 4.5.** Consider a diffeomorphism $F$ close to $F_\Psi$.

1. If $x \in \Delta_F$ is a chain recurrent point for $F$ then $x \in \mathcal{E}^s(F) \cap \mathcal{E}^u(F)$.
2. If $\mathcal{Y}^s_1(F) \cap \mathcal{Y}^u_1(F) \neq \emptyset$ then $F$ has a heterodimensional cycle associated to $p_1$ and $q_1$.
3. If $\mathcal{Y}^s_1(F) \cap \mathcal{L}^s_1(F) \neq \emptyset$ then the homoclinic class of $q_1$ is non-trivial.
4. If $\mathcal{Y}^u_1(F) \cap \mathcal{L}^u_1(F) \neq \emptyset$ then homoclinic class of $p_1$ is non-trivial.
5. If $\mathcal{Y}^u_1(F) \cap \mathcal{E}^s(F) = \emptyset$ then $p_1$ is isolated.
(6) If \( Y^s_1(F) \cap E^u(F) = \emptyset \) then \( q_1 \) is isolated.

Observe also that by construction

\[
\Delta_F \subset E^s(F) \subset W^s(s_2, F) \quad \text{and} \quad \Delta_F \setminus E^u(F) \subset W^u(r_2, F).
\]

As in Corollary 3.3, a consequence of these inclusions is the following.

**Lemma 4.6.** For every \( F \) close to \( F_\psi \), every point of \( \Delta_F \) that is chain recurrent is contained in \( E^s(F) \cap E^u(F) \).

5. **Choice of the local diffeomorphism \( \Psi \)**

Lemmas 3.3 means that, for the diffeomorphism \( F_\psi \), the existence of heterodimensional cycles and homoclinic intersections for \( p_1, p_2, q_1, \) and \( q_2 \) depend on the intersections of the sets \( L^u_{i,s}(F_\psi), Y^u_{i,s}(F_\psi), E^u(F_\psi), \) and \( E^s(F_\psi) \). The choice of the identification map \( \Psi \) determines these intersections. Lemma 4.5 explains how these properties are translated for diffeomorphisms close to \( F_\psi \).

We assume that the local diffeomorphism \( \Psi \) is such that the diffeomorphism \( F_\psi \) satisfies the following two conditions, see Figure 5.1:

- **(T) Topological hypothesis:**
  - There is a point \( z \in \Delta \) such that
    \[
    Y^s_1(F_\psi) \cap Y^u_1(F_\psi) = Y^s_1(F_\psi) \cap E^u(F_\psi) = Y^u_1(F_\psi) \cap E^s(F_\psi) = \{ z \}.
    \]

- **(D) Differentiable hypothesis:**
  - The intersection of the semi-planes \( E^{cs}_u(z) \cap E^{cu}_u(z) \) is a half straight line. This implies that the intersection \( Y^u_1(F_\psi) \cap Y^s_1(F_\psi) \) at \( z \) is quasi-transverse.
  - In a neighborhood of \( z \), the sets \( E^s(F_\psi) \) and \( E^u(F_\psi) \) are locally in the same side of any locally defined surface containing the curves \( Y^s_1(F_\psi) \) and \( Y^u_1(F_\psi) \).

![Figure 5.1. Conditions T and D](image)

**Proposition 5.1.** Suppose that that \( F_\psi \) satisfies conditions (T) and (D). Then there are a \( C^1 \)-neighborhood \( U_{F_\psi} \) of \( F_\psi \) and a codimension one submanifold \( \Sigma \) containing \( F_\psi \) such that:

1. The set \( U_{F_\psi} \setminus \Sigma \) is the union of two disjoint open sets \( U^+_{\Sigma} \) and \( U^-_{\Sigma} \) such that:
   - for every \( G \in U^+_{\Sigma} \) the saddle \( p_1 \) is isolated and the homoclinic class of \( q_1 \) is non-trivial, and
   - for every \( G \in U^-_{\Sigma} \) the saddle \( q_1 \) is isolated and the homoclinic class of \( p_1 \) is non-trivial.
(2) Every diffeomorphism \( G \in \Sigma \) has a heterodimensional cycle associated to \( p_1 \) and \( q_1 \).

We have the following corollary:

**Corollary 5.2.** The submanifold \( \Sigma \) consists of diffeomorphisms \( F \) having fragile cycles associated to \( p_1 \) and \( q_1 \).

Note that Theorem 1 follows immediately from Proposition 5.1 and Corollary 5.2.

Corollary 5.2 is a consequence of Proposition 5.1 and the following simple fact about cycles and chain recurrence classes:

**Remark 5.3 (Cycles and chain recurrence classes).** Let \( F \) be a diffeomorphism with a heterodimensional cycle associated to two transitive hyperbolic sets \( L \) and \( K \). Then every pair of saddles \( p \in L \) and \( q \in K \) are in the same chain recurrence class of \( F \). In particular, both saddles \( p \) and \( q \) are not isolated.

**Proof of Corollary 5.2.** We argue by contradiction. If there is \( F \in \Sigma \) such that the cycle is not fragile then there is a diffeomorphism \( G \) close to \( F \) with a robust cycle associated to a pair of transitive hyperbolic sets \( L \ni p_1 \) and \( K \ni q_1 \). Then by Remark 5.3 the saddles \( p_1 \) and \( q_1 \) are not isolated for \( G \). Finally, as the cycle is robust, we can assume that \( G \notin \Sigma \), that is, \( G \in \mathcal{U}_\Sigma^+ \cup \mathcal{U}_\Sigma^- \). Since \( p_1 \) is isolated if \( G \in \mathcal{U}_\Sigma^+ \) this implies that \( G \notin \mathcal{U}_\Sigma^+ \). Similarly, as \( q_1 \) is isolated if \( G \in \mathcal{U}_\Sigma^- \) we have that \( G \notin \mathcal{U}_\Sigma^- \). This contradicts the fact that the cycle associated to \( G \) is robust. □

5.1. **Proof of Proposition 5.1.** To define the submanifold \( \Sigma \) we take the unitary vector \( \vec{n} \) normal to the plane \( T_z(Y_u^1(F)) \oplus T_z(Y_s^2(F)) \) pointing to the “opposite” direction of \( E_u(F) \) and \( E_s(F) \). Fix small \( \epsilon > 0 \) and for \( F \) close to \( F_\Psi \) consider the family of one-dimensional disks

\[ \{Y_s^1(F) + t\vec{n}\}_{t \in [-\epsilon, \epsilon]} \]

The main step of the proof of the proposition if the following lemma whose proof we postpone.

**Lemma 5.4.** There is a small neighborhood \( \mathcal{U}_{F_\Psi} \) of \( F_\Psi \) such that for every \( F \in \mathcal{U}_{F_\Psi} \) there is a unique parameter \( t = \tau_F \), depending continuously on \( F \), such that

\[ (Y_s^1(F) + \tau_F \vec{n}) \cap Y_u^2(F) \neq \emptyset. \]

There are the following three possibilities according to the value of \( \tau_F \):

- If \( \tau_F = 0 \) then the diffeomorphism \( F \) has a heterodimensional cycle associated to \( p_1 \) and \( q_1 \).
- If \( \tau_F > 0 \) then \( p_1 \) is isolated and the homoclinic class of \( q_1 \) is non-trivial.
- If \( \tau_F < 0 \) then \( q_1 \) is isolated and the homoclinic class of \( p_1 \) is non-trivial.

**Proof of Proposition 5.1.** In view of Lemma 5.4 we let

\[ \Sigma \overset{\text{def}}{=} \{ F \in \mathcal{U}_{F_\Psi} \text{ such that } \tau_F = 0 \}. \]

Then the set \( \mathcal{U}_{F_\Psi} \setminus \Sigma \) has two components \( \mathcal{U}_{\Sigma}^+ \) and \( \mathcal{U}_{\Sigma}^- \). The component \( \mathcal{U}_{\Sigma}^+ \) consists of the diffeomorphisms \( F \) such that \( \tau_F > 0 \) and \( \mathcal{U}_{\Sigma}^- \) consists of the diffeomorphisms \( F \) with \( \tau_F < 0 \). The proposition now follows immediately from Lemma 5.4. □
As above item (6) in Lemma 4.5 implies that the point \( q \) is isolated and \( H(p_1) \) is non-trivial for every small \( t > 0 \) the saddle \( p_1 \) escapes from the non-trivial homoclinic class of \( q_2 \). Moreover, the parameter \( \tau_F \) depends continuously on \( F \).

If \( \tau_F = 0 \) then \( Y_1^s(F) \cap Y_2^u(F) \neq \emptyset \). Thus by item (2) in Lemma 4.5 the diffeomorphism \( F \) has a heterodimensional cycles associated to \( p_1 \) and \( q_1 \).

The choice of \( \eta \) implies that for \( \tau_F > 0 \) the sets \( Y_1^s(F) \) and \( E^u(F) \) are disjoint. Thus by item (3) in Lemma 4.5, the point \( p_1 \) is isolated for \( \tau_F > 0 \) one has that \( Y_1^s(F) \) intersects \( E^u(F) \). Thus, if \( \tau_F \) is small enough, one has that \( Y_1^s(F) \cap L_1^u(F) \). Hence by item (3) in Lemma 4.5 the homoclinic class \( H(q_1,F) \) is non-trivial. A similar argument shows that for \( \tau_F < 0 \) the sets \( Y_1^u(F) \) and \( E^s(F) \) are disjoint. As above item (1) in Lemma 4.5 implies that the point \( q_1 \) is isolated. Also for \( \tau_F < 0 \) one has that \( Y_1^u(F) \cap L_1^u(F) \) and item (1) in Lemma 4.5 implies that the homoclinic class \( H(p_1,F) \) is non-trivial. This completes the proof of the lemma.

**Remark 5.5.** Our construction can be done such that the saddles \( q_2 \) and \( q_1 \) are homoclinically related for every small \( t > 0 \). For that it is enough to choose the diffeomorphisms \( \Psi \) such that \( Y_2^s(F_\psi) \cap L_2^u(F_\psi) \), see Corollary 3.6.

Observe that in this case one has \( H(q_2,F_t) = H(q_1,F_t) \) for \( t > 0 \). However, for \( t = 0 \) the saddle \( q_1 \) escapes from the non-trivial homoclinic class of \( q_2 \). Surprisingly, in such a case one can generate robust cycles associated to \( q_2 \) and \( p_1 \) (this follows from [3]) but not associated to \( q_1 \) and \( p_1 \).

6. Discussion

Our construction provides examples of fragile cycles relating two saddles \( p_1 \) and \( q_1 \) of different indices (for simplicity we will omit the dependence on the diffeomorphisms). This construction has a “prescribed” part concerning the relative positions of the invariant manifolds of these saddles. But this prescribed dynamics involves only the “cuspidal” regions \( Y_1^s \) and \( Y_1^u \) of \( E^s \) and \( E^u \), respectively. The rest of the...
intersection $E^s \cap E^u$ can be chosen arbitrarily. This provides a lot of “freedom” for the global dynamics, for instance, for the behavior of the other saddles of the diffeomorphism and their invariant manifolds (recall Remark 5.5). Hence, without further assumptions on the dynamics outside the cycle the global dynamics cannot be described.

6.1. Partial hyperbolicity, wild dynamics, and fragile cycles. A first ingredient of our construction is the gluing map $\Psi$ that plays a key role for determining the resulting global dynamics.

**Question 6.1.** Can the resulting dynamics be partially hyperbolic? More precisely, does it exist a gluing map $\Psi$ so that the chain recurrence class of $p_1$ for $F_\Psi$ has a partially hyperbolic splitting with three 1-dimensional bundles?

We expect a positive answer to this question. This would imply that the phenomena associated to these fragile cycles would also occur in the most rigid nonhyperbolic setting of partial hyperbolicity with one-dimensional center.

We next discuss how fragile cycles can be involved in the generation of wild dynamics (roughly, persistent coexistence of infinitely many homoclinic classes, see § for further details).

First note that perturbations of the diffeomorphism $F$ in the fragile cycles submanifold $\Sigma$ in Theorem 1 expels periodic points in the chain recurrence class that simultaneously contains $p_1$ and $q_1$: the saddle $p_1$ is expelled by diffeomorphisms in the component $U^s_{\Sigma}$ and the saddle $q_1$ by diffeomorphisms in the component $U^u_{\Sigma}$. Thus the chain recurrence class fall into pieces. On the other hand, there is a natural question about whether fragile cycles may generate a “cascade” of fragile cycles:

**Question 6.2.** Can the submanifold of fragile cycles $\Sigma$ be accumulated by (co-dimension one) submanifolds consisting of fragile cycles?

If the answer to this question would be positive then a “fragile cycles configuration” could be repeated generating infinitely many different chain recurrence classes. Being very optimistic, positive answers to the two questions above could provide the first examples of wild dynamics in the partially hyperbolic (with one-dimensional center) setting. We will discuss further questions about wild dynamics later.

6.2. Chain recurrence classes. Consider a diffeomorphism $F$ in the submanifold $\Sigma$ of fragile cycles. By construction, the intersection $W^s(p_1) \cap W^u(q_1)$ contains a curve $\Gamma_{1,1}$ joining $p_1$ to $q_1$. Moreover, the intersection $W^u(p_1) \cap W^u(q_1)$ is exactly the orbit of a quasi-transverse heteroclinic point $x_F$. This cyclic configuration implies that the chain recurrence classes of $p_1$ and $q_1$ coincide and contain the curve $\Gamma_{1,1}$ and the orbit of $x_F$, recall also Remark 5.3.

We can now perform perturbations of $F \in \Sigma$ preserving the cycle, that is, the resulting diffeomorphisms continue to belong to $\Sigma$. In this way, and using for instance the arguments in §, one can slightly modify the central eigenvalues of the saddles $p_1$ and $q_1$ in the cycle (corresponding to the tangent direction of the connection $\Gamma_{1,1}$) to generate “new” periodic saddles $r$ whose orbits pass arbitrarily close to $p_1$ and $q_1$ and belong to the chain recurrence class of $C(p_1) = C(q_1)$. The latter fact is a consequence of the geometry of the cycle that guarantees that indeed $W^u(r) \cap W^s(p_1) \neq \emptyset$ and $W^s(r) \cap W^u(q_1) \neq \emptyset$, see Figure 6.1.
**Question 6.3.** For $C^1$-generic diffeomorphisms $F \in \Sigma$, is the heteroclinic curve $\Gamma_{1,1}$ contained in the closure of a set of periodic points? Can the periodic points in that set be chosen in the same homoclinic class or (even more) homoclinically related?

Concerning this question, observe that our construction provides saddles that are in the same recurrence class (the one of $p_1$ and $q_1$). Note also that by [5], for $C^1$-generic diffeomorphisms, the chain recurrence classes of periodic points always coincide with their homoclinic classes. The difficulty here is that we consider $C^1$-generic diffeomorphisms in the codimension one submanifold $\Sigma$ (which is a meager set). Thus the result in [6] cannot be applied. Indeed, for $F \in \Sigma$ the homoclinic classes of $p_1$ and $q_1$ are both trivial and hence different from their non-trivial chain recurrence classes since $p_1, q_1 \in C(p_1) = C(q_1)$.

We believe that the answer to Question 6.3 is positive. In such a case, it would be interesting to consider the central Lyapunov exponents of the periodic orbits in the chain recurrence class $C(p_1) = C(q_1)$. This is our next topic.

**6.3. Lyapunov exponents.** The papers [13, 14, 12] consider examples of diffeomorphisms having twisted heterodimensional cycles (recall Section 1) and analyze the spectrum of central Lyapunov exponents of the (non-trivial) homoclinic classes involved in the cycle. In the examples in [14, 12] this spectrum has a gap. The existence of this gap is related to the fact that the homoclinic class $H(p)$ considered contains a saddle $q$ of different index that looks like a “cuspidal corner point” (exactly as the points $p_1$ and $q_1$ of our construction) and satisfies $q \in H(p)$ but $H(q) = \{q\}$. So the saddle $q$ is topologically an extreme point but also dynamically is also an extreme point as it is not homoclinically related to other saddles in the class $H(p)$. We believe that precisely this is reflected by the Lyapunov spectrum that has one gap.

The previous comments and the fact that the chain recurrence classes considered in this paper have “two cuspidal corner points” (the saddles $p_1$ and $q_1$) that are also dynamically extremal lead to the following question.

**Question 6.4.** Does there exist a gluing map $\Psi$ such that there is a neighborhood $U$ of $F_\Psi$ such that “persistently” in $U \cap \Sigma$ the diffeomorphisms have two (or more) gaps in the spectrum of central Lyapunov exponents of the periodic orbits in the chain recurrence class $C(p_1)$? Here the term “persistent” is purposely vague and may mean $C^1$-generic or $C^1$-dense, for instance.
Observe that by \[2\] for a \(C^1\)-generic diffeomorphism the spectrum of Lyapunov exponents of homoclinic classes has no gaps. Hence, one would need to consider diffeomorphisms of non-generic type.

6.4. Collision, collapse, and birth of classes. Consider an arc of diffeomorphisms \((F_t)_{t \in [-1,1]}\) that intersects \(\Sigma\) transversely at \(t = 0\). Assume that \(F_t \in \mathcal{U}_\Sigma^+\) for \(t > 0\) and \(F_t \in \mathcal{U}_\Sigma^-\) for \(t < 0\). Our construction implies that for \(t > 0\) the class \(H(q_1, F_t)\) is non-trivial, \(H(p_1, F_t) = \{p_1\}\), and both classes are disjoint. Similarly, for \(t < 0\) the class \(H(p_1, F_t)\) is non-trivial, \(H(q_1, F_t) = \{q_1\}\), and both classes are disjoint. Since \(F_0 \in \Sigma\) and \(H(q_1, F_0) = \{q_1\}\) and \(H(p_1, F_0) = \{p_1\}\), each of these classes collapses to a single point at \(t = 0\). Similarly (or symmetrically), for \(t = 0\) the chain recurrence classes of \(p_1\) and \(q_1\) collide at \(t = 0\) and contain the heteroclinic segment \(\Gamma_{1,1}\), when one of these classes collapses to a point for \(t \neq 0\). This illustrates the lower semi-continuous dependence of homoclinic classes and the upper semi-continuity of chain recurrence classes.

Let us observe that \[7\] provides a locally \(C^1\)-dense set of diffeomorphisms where homoclinic classes are properly contained in a robustly isolated chain recurrence class. This construction is somewhat similar to the one in this paper and involves a heterodimensional cycle relating cuspidal corner points of the class.

Coming back to our construction, it would be interesting to understand how for \(t > 0\) the points of \(H(q_1, F_t)\) escape from the class or “disappear” as \(t \to 0^+\).

As we discussed above, in some cases the saddle \(q_1\) is accumulated by saddles of the same index when \(t = 0\). These saddles are not homoclinically related to \(q_1\) but they are in its chain recurrence class. This indicates that there are (infinitely many) saddles that “escape” from the homoclinic class of \(q_1\) but not from the chain recurrence class of \(q_1\) as \(t\) evolves. Though we do not know how these saddles are homoclinically related. Note that there is completely symmetric scenery for the saddle \(p_1\).

**Question 6.5.** Is there a diffeomorphism \(F \in \Sigma\) with infinitely many different homoclinic classes, all of them contained in the chain recurrence class of \(p_1\) and \(q_1\)?

In view of the previous discussion, a simpler question is if the diffeomorphisms in \(\Sigma\) can be chosen having wild dynamics close to the cycle. More precisely,

**Question 6.6.** Does there exist \(F \in \Sigma\) with infinitely many different chain recurrence classes accumulating to \(p_1\) or to \(q_1\)?

Finally, let us observe that the previous setting is somehow reminiscent to the setting of the geometrical Lorenz attractor: at the point of “bifurcation”, when a singular cycle occurs, infinitely many orbits of the vector field transform into heteroclinic orbits of a singularity, see for instance \[3\].

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