VECTORS BUNDLES ON HIRZEBRUCH SURFACES WHOSE
TWISTS BY A NON-AMPLE LINE BUNDLE HAVE NATURAL
COHOMOLOGY

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ABSTRACT. Here we study vector bundles $E$ on the Hirzebruch surface $F_{e}$ such
that their twists by a spanned, but not ample, line bundle $M = \mathcal{O}_{F_{e}}(h + ef)$
have natural cohomology, i.e. $h^0(F_{e}, E(tM)) > 0$ implies $h^1(F_{e}, E(tM)) = 0$.

1. INTRODUCTION

Let $F_{e}$, $e > 0$, denote the Hirzebruch surface with a section with self-intersection
$-e$. For any $L \in \text{Pic}(F_{e})$ and any vector bundle $E$ on $F_{e}$ we will say that $E$
has property $\mathcal{P} \mathcal{L}$ (resp. $\mathcal{P} \mathcal{E}$) with respect to $L$ if $h^1(F_{e}, E \otimes L^{\otimes m}) = 0$ for all $m \in \mathbb{Z}$
(resp. for all $m \in \mathbb{Z}$ such that $h^0(F_{e}, E \otimes L^{\otimes m}) \neq 0$). We think that property $\mathcal{P}$ is
nicer for reasonable $L$. We take as a basis of $\text{Pic}(F_{e}) \cong \mathbb{Z}^2$ a fiber $f$ of the ruling
$\pi : F_{e} \to \mathbb{P}^1$ and the section $h$ of $\pi$ with negative self-intersection. Thus $h^2 = -e,
h \cdot f = 1$ and $f^2 = 0$. We have $\omega_{F_{e}} \cong \mathcal{O}_{F_{e}}(-2h - (e + 2)f)$. $\mathcal{O}_{F_{e}}(\alpha h + \beta f)$ is spanned
(resp. ample) if and only if $\alpha \geq 0$ and $\beta \geq 0$ (resp. $\alpha > 0$ and $\beta > e\alpha$). The Leray
spectral sequence of $\pi$ and Serre duality give that $h^1(F_{e}, \mathcal{O}_{F_{e}}(\gamma h + \delta f)) = 0$ if and
only if either $\gamma \geq 0$ and $\delta \geq e\gamma$ or $\gamma = -1$ and $\delta \leq -2$ and $-\delta - e - 2 \geq e(\gamma - 2) - 1$
(i.e. $\delta \leq e\gamma + e - 1$). We consider as the test line bundle the spanned, but not ample,
line bundle $M := \mathcal{O}_{F_{e}}(h + ef)$. Notice that the linear system $|M^{\otimes 2}|$ contains the
sum of the effective divisor $h$ and the ample divisor $h + 2ef$. Thus for every vector
bundle $E$ on $F_{e}$ there is an integer $m_{0}(E)$ such that $h^0(F_{e}, E \otimes M^{\otimes m}) \neq 0$ for all
$m \geq m_{0}(E)$. We will see that property $\mathcal{P} \mathcal{E}$ is too strong and not interesting (see
Remarks 1 and 2). We stress the property $\mathcal{P}$ with respect to $M$ is quite different
from similar looking properties (e.g. natural cohomology) with respect to an ample
line bundle. (see Remarks 2 and 3 for the rank 1 case). Obviously, properties $\mathcal{P}$
and $\mathcal{P} \mathcal{E}$ may be stated for arbitrary projective varieties. In dimension $n \geq 3$, one
need to choose between vanishing of $h^i$ or vanishing of all $h^i$, $1 \leq i \leq n - 1$. We
considered here the example $(F_{e}, M)$, because it is geometrically significant. Indeed,
let $\phi_{M}$ denote the morphism associated to the base point free linear system $|M|$. If
$e = 1$ the morphism $\phi_{M}$ is the blowing up $F_{1} \to \mathbb{P}^2$. If $e \geq 2$, then $\phi_{M} : F_{e} \to \mathbb{P}^{e+1}$
contracts $h$ and its image is a cone over the rational normal curve of $\mathbb{P}^{e}$. Moreover,
for any spanned and non-trivial line bundle $L$ on $F_{e}$ there is an effective divisor $D$
such that $L \cong M(D)$. For any spanned, but not ample line bundle $A$ on $F_{e}$ there
is an integer $c \geq 0$ such that $A \cong M^{\otimes c}$. We prove the following results.

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Theorem 1. Fix integers $e \geq 1$, $r \geq 1$, $u,v$ such that $v \leq e(u-r+1) - 2$. Then there is no rank $r$ vector bundle $E$ on $F_e$ with property $\mathcal{L}$ with respect to $M$ such that $c_1(E) = \mathcal{O}_{F_e}(uh + vf)$.

Theorem 2. Fix integers $e,m,u,v$ such that $e \geq 1$, and $v \geq e(u-1) - 1$ and $m \geq 0$. Set $\overline{a} := \sum_{i=0}^{u+2m-2} v + 2m - 1 - ie$ and $\overline{b} := \sum_{i=0}^{u+2m-1} v + 2m - ie$. Fix any integer $s$ such that $\overline{a} \leq s \leq \overline{b}$. Then there exists a rank 2 vector bundle $E$ on $F_e$ with property $\mathcal{L}$ with respect to $M$ such that $c_1(E) = \mathcal{O}_{F_e}(uh + vf)$ and $c_2(E) := s - e(u+m-1) + (1-m)(v+em)$. Set $R := \mathcal{O}_{F_e}(h+(e+1)f)$ and assume $m = 0, u \geq 3, v < 2eu$. Then we may find $E$ as above which is $R$-stable in the sense of Mumford-Takemoto and (under the additional condition $v \leq 2eu - 3$) such that $N \cdot M < c_1(E) \cdot M/2$ for all rank 1 subsheaves $N$ of $E$.

The case “$r = 1$” of Theorem 1 is obviously true (use the cohomology of line bundles on $F_e$, i.e. Remark 2 below). In this case the converse is true, i.e. $\mathcal{O}_{F_e}(uh + vf)$ has property $\mathcal{L}$ with respect to $M$ if and only if $v \geq eu - 1$ (Remark 2). We were surprised that for $r \geq 2$ there is no way to overcome this $c_1$-obstruction.

The assumptions of the last part of Theorem 2 may be relaxed and instead of $R$ we may take an arbitrary ample divisor $H$. An interesting offshot of our proof of Theorem 2 is that our examples are given by an extension $\mathcal{H}$ and all locally free sheaves fitting in $\mathcal{H}$ have property $\mathcal{L}$ with respect to $M$ and (under the additional conditions listed in Theorem 2) are $R$-stable and $N \cdot M < c_1(E) \cdot M/2$ for all rank 1 subsheaves $N$ of $E$.

In the case of direct sums of line bundles we will prove the following result.

Proposition 1. Fix integers $e \geq 1$, $r \geq 2$ and $L_i \in \text{Pic}(F_e)$, $1 \leq i \leq r$, say $L_i \cong \mathcal{O}_{F_e}(u_ih + v_if)$. Set $E := L_1 \oplus \cdots \oplus L_r$. Up to a permutation of the factors of $E$ we may assume $u_1 \geq \cdots \geq u_r$ and that if $u_i = u_j$ for some $i < j$, then $v_i \geq v_j$. Set $m := -u_1$ if $v_1 \geq eu_1$ and $m := -u_1 + 1$ if $v_1 \geq eu_1 - 1$. The vector bundle $E$ has property $\mathcal{L}$ with respect to $M$ if and only if $v_i \geq eu_i - 1$ for all $i$, and for each $i \in \{2, \ldots, r\}$ either $u_i - m \geq -1$ or $-1 \leq v_i - eu_i \leq e - 1$.

We raise the following question.

Question 1. Assume $e = 1$ or $e = 2$. Is it possible to describe all invariants $r,c_1,c_2$ of vector bundles on $F_e$ with property $\mathcal{L}$ with respect to $M$?

2. The proofs

For any sheaf $F$ we will often write $F(mM)$ instead of $F \otimes M^{\otimes m}$.

Remark 1. The line bundle $\mathcal{O}_{F_e}(ch + df)$ is ample if and only if $c > 0$ and $d > ec$. Hence any ample line bundle is spanned. Assume that $H := \mathcal{O}_{F_e}(ch + df)$ is ample. The cohomology of line bundles on $F_e$ shows that for every $t \in \mathbb{Z}$ the line bundle $H^{\otimes t}$ has property $\mathcal{L}$ with respect to $H$. Hence $\mathcal{O}_{F_e}$ has property $\mathcal{L}$ with respect to any ample line bundle. Set $H' := \mathcal{O}_{F_e}(ch + (d+2)f)$. Taking $m_t := -tc$ we see that no $H^{\otimes t}$, $t > 0$, has property $\mathcal{L}$ with respect to the ample line bundle $H'$. Taking $m_t := -tc$ we see that no $H^{\otimes t}$, $t > 0$, has property $\mathcal{L}$ with respect to $M$.

Remark 2. Here we study properties $\mathcal{L}$ and $\mathcal{L}$ with respect to $M$ for line bundles on $F_e$. Fix $L \in \text{Pic}(F_e)$, say $L \cong \mathcal{O}_{F_e}(uh + vf)$. First assume $u \geq eu$. We have $h^0(F_e, L(xM)) > 0$ if and only if $x \geq -u$. Since $h^1(F_e, \mathcal{O}_{F_e}(ch + df)) = 0$ if $c \geq 0$ and $d \geq ec$, $L$ has property $\mathcal{L}$ with respect to $M$. Now assume $v < eu$. We have...
\( h^0(F_x, L(xM)) > 0 \) if and only if \( ex \geq v \). Since \( h^1(F_x, O_{F_x}(ch + df)) = 0 \) if \( c \geq 0 \) and only if \( d \geq ec - 1 \), we get that \( L \) has property \( \mathcal{L} \) with respect to \( M \) if and only if \( v \geq eu - 1 \). Take \( m := -u \). If \( v = eu \), then we saw in the introduction that \( L \) has property \( \mathcal{L} \) with respect to \( M \) if and only if \( e = 1 \). Notice that

\[
\begin{align*}
\text{if and only if} \\
\text{L has property L with respect to M if and only if e = 1. Notice that} \\
\text{h}^1(F_x, O_{F_x}((u-x)h + (v-ex)f)) = h^1(F_x, O_{F_x}((x-u-2)h + (ex-v-e-2)) > 0 \\
\text{when x} \geq u-2 \text{if and only if} -eu Faster than -v-e-1, i.e. if and only if v} \leq eu+e-1.
\end{align*}
\]

\( \text{Remark 3. Here we look at property L with respect to the ample line bundle } R := O_{F_x}(h+(e+1)f) \text{ for line bundles on } F_x. \text{ Fix } L \in \text{Pic}(F_x), \text{ say } L \cong O_{F_x}(uh+v). \text{ We have} \ h^0(F_x, L(xR)) = 0 \text{ if and only if } x \geq u \text{ and } x(e+1) \geq v. \text{ We immediately see that if } v \geq (e+1)u, \text{ then } L \text{ has property L with respect to } M. \text{ Now assume } v < (e+1)u. \text{ Set } y := \lfloor -v/(e+1) \rfloor. \text{ We have} \ h^0(F_x, L(xR)) > 0 \text{ if and only if } x \geq y. \text{ Fix an integer } x \geq y. \text{ Since } u+x \geq u+y \geq 0, \ h^1(F_x, O_{F_x}(u+xh + (v+(e+1)x))) > 0 \text{ if and only if } v \geq (e+1)x \leq eu + ex - 2. \text{ The strongest condition is obtained when } x = y. \text{ We get that } L \text{ has property } \alpha \text{ with respect to } R \text{ if and only if either } v \geq (e+1)u \text{ or } v + ey \geq eu - 1, \text{ where } y := \lfloor -v/(e+1) \rfloor. \]

\( \text{Remark 4. } E_1 \oplus E_2 \text{ has property L with respect to } L \text{ if and only if both } E_1 \text{ and } E_2 \text{ have property L with respect to } L. \text{ If } E_1 \oplus E_2 \text{ has property L with respect to } L, \text{ then the same is true for } E_1 \text{ and } E_2. \text{ Now we check that the converse is not true. Both } O_{F_x} \text{ and } O_{F_x}(-2h + (-e+4)f) \text{ have property L with respect to } M \text{ (Remark 2). Since } h^1(F_x, O_{F_x}(-2h + (-e+4)f)) = h^1(F_x, O_{F_x}(-2f)) = 1, \text{ } O_{F_x} \oplus O_{F_x}(-2h + (-e+4)f) \text{ has not property L with respect to } M. \]

\( \text{Remark 5. The definition of property } \mathcal{L} \text{ may be given for an arbitrary torsion free sheaf, but not much may be said in the general case. Here we look at the rank 1 case, because we will need it in the proofs of Theorems 1 and 2. Let } A \text{ be a rank 1 torsion free sheaf on } F_x. \text{ Hence } A \cong I_Z(uh+vf) \text{ for some zero-dimensional scheme } Z \text{ and some integers } u, v. \text{ Since } Z \text{ is zero-dimensional, } h^1(F_x, O_{F_x}((u+t)h + (v+ct)f)) \leq h^1(F_x, I_Z((u+t)h + (v+ct)f)) \text{ for all } t \in \mathbb{Z}. \text{ Taking } t \gg 0 \text{ we see that if } A \text{ has property L with respect to } M, \text{ then } v \geq eu - 1. \text{ When } v \geq eu - 1, \text{ for a general } Z \text{ (in the following sense) } A \text{ has property L with respect to } M \text{ for the following reason. Fix an integer } z > 0. \text{ Since } F_x \text{ is a smooth surface, the Hilbert scheme } \text{Hilb}^2(F_x) \text{ of all length } z \text{ zero-dimensional subschemes of } F_x \text{ is irreducible and of dimension } 2z \text{ (2). Take a general } S \in \text{Hilb}^2(F_x), \text{ i.e. take a general points of } F_x. \text{ Since } h^0(F_x, I_S \otimes L) = \max(0, h^0(F_x, L) - z) \text{ for every } L \in \text{Pic}(F_x), \text{ it is easy to check that if } v \geq eu - 1, \text{ then } I_S(uh+vf) \text{ has property L with respect to } M. \text{ Now take } v = eu - 1, \text{ any integer } z > 0 \text{ and any zero-dimensional length } z \text{ subscheme } \mathcal{B} \text{ of } h. \text{ Twisting with } -(u+1)M \text{ we see that } I_B(uh, (eu-1)f) \text{ has not property L with respect to } M. \text{ Now assume } v > eu. \text{ Take a zero-dimensional length } z \geq 2 \text{ scheme } W \text{ of a fiber of } \pi. \text{ Twisting with } -uM \text{ we see that } I_W(uh+vf) \text{ has not property L with respect to } M. \text{ If } z \geq 3 \text{ and } v = eu, \text{ twisting with } -(u+1)M \text{ and using the same W we get a sheaf without property L with respect to } M. \]

Property L with respect to M has the following open property.

**Proposition 2.** Let \( \{ E_t \}_{t \in T} \) be a flat family of vector bundles on \( F_x \), parametrized by an integral variety \( T \). Assume the existence of \( s \in T \) such that \( E_s \) has property
$\mathcal{L}$ with respect to $M$. Then there exists an open neighborhood $U$ of $s$ in $T$ such that $E_t$ has property $\mathcal{L}$ with respect to $M$ for all $t \in U$.

Proof. Let $m$ be the minimal integer such that $h^0(F, e, E_\pi(mM)) > 0$. Thus $h^1(F, e, E_\pi(xM)) = 0$ for all $x \geq m$. By semicontinuity there is an open neighborhood $V$ of $s$ in $T$ such that $h^0(F, E_t((m-1)M)) = 0$ for all $t \in V$. By semicontinuity for every integer $x \geq m$ there is an open neighborhood $V_x$ of $s$ in $T$ such that $h^1(F, E_t(xM)) = 0$ for all $t \in V_x$. Fix an irreducible $D \in |M|$. Hence $D \cong \mathbf{P}^1$. Since $D^2 > 0$, there is an integer $a$ such that $h^1(D, E_\pi(aM)|D) = 0$. By semicontinuity there is an open neighborhood $V$ of $s$ in $T$ such that $h^1(D, E_t(aM)|D) = 0$ for every $t \in V$. Since $D^2 > 0$, $h^1(D, E_t(xM)|D) = 0$ for every $t \in V$ and every integer $x \geq a$. Fix an integer $x \geq a$. From the exact sequence

$$0 \to E_t((x-1)M) \to E_t(xM) \to E_t(xM)|D \to 0$$

we get that if $h^1(F, E_t((x-1)M)) = 0$, then $h^1(F, E_t(xM))$. Hence we may take $U := V \cap \bigcap_{x=m}^{\max\{a, m\}} V_x$.  

Proof of Proposition 4 If $E$ has property $\mathcal{L}$ with respect to $M$, then each $L_i$ has property $\mathcal{L}$ with respect to $M$ (Remark 1) and hence $v_i \geq eu_i - 1$ for all $i$. Now we assume $v_i \geq eu_i - 1$ for all $i$. Notice that $m$ is the minimal integer $t$ such that $h^0(F, E_t) \neq 0$. Since $L_1$ has property $\mathcal{L}$ with respect to $M$, $E$ has property $\mathcal{L}$ with respect to $E$ if and only if $h^1(F, E_t(tM)) = 0$ for all $t \geq m$ and all $i = 2, \cdots, r$. If $u_i - m \geq -1$, then $h^1(F, L_i(tM)) = 0$ for all $t \geq m$ because $v_i \geq eu_i - 1$. Now assume $u_i - m \leq -2$. We get $h^1(F, E_t(tM)) = 0$ for any $t \geq m$ if and only if $-1 \leq v_i - eu_i \leq e - 1$. 

Here we discuss the set-up for the rank 2 case. Consider an exact sequence

$$(1) \quad 0 \to \mathcal{O}_F(D) \to E(mM) \to \mathcal{I}_Z(c_1 + 2mM - D) \to 0$$

in which $Z$ is a zero-dimensional scheme with length $s$ and either $D = 0$ or $D = h$ or $D \in |f|$ for some $1 \leq z \leq e$ or $e \geq 2$ and $D \in |h + w|f$ for some $1 \leq w \leq e - 1$. We have $c_1(E(mM)) = c_1 + 2mM$ and $c_2(E(mM)) = s + D \cdot c_1 + 2mM \cdot D - D^2$. Thus $c_1(E) = c_1$ and $c_2(E) = c_2$ by the choice of $s$ (4, Lemma 2.1). Each $E$ fitting in (1) is torsion free. To have some locally free $E$ fitting in (1) a necessary condition is that $Z$ is a locally complete intersection. Notice that $h^1(F, \mathcal{O}_F(D)(-M)) = 0$ if $h$ is not a component of $D$. Hence a sufficient condition to have $h^0(F, E(mM)) > 0$ and $h^0(F, E((m-1)M)) = 0$ is the equality

$$(2) \quad h^0(F, \mathcal{I}_Z(c_1 + (2m - 1)M - D)) = 0$$

and (2) is a necessary condition if $h$ is not a component of $D$. Assume that $Z$ is a locally complete intersection. The Cayley-Bacharach condition associated to (1) is satisfied if

$$(3) \quad h^0(F, \mathcal{I}_Z(c_1 + 2mM - 2D - 2h - (e + 2)f)) = 0$$

for every length $s - 1$ closed subscheme of $s$. This condition is satisfied if $h^0(F, \mathcal{I}_Z(c_1 + (2m - 1)M - D)) = 0$, $Z_{\text{red}} \cap h = \emptyset$ and no connected component of $Z$ is tangent to a fiber of the fiber of $\pi$, because the line bundle $\mathcal{O}_F(h + 2f)$ is base point free outside $h$ and the morphism associated to $|f|$ is the ruling; if $e = 12$, then (3) is satisfied if (2) is satisfied, because $\mathcal{O}_F(h + 2f)$ is
very ample; if \( e = 2 \) it is sufficient to assume \( Z_{\text{red}} \cap h = \emptyset \), because the morphism associated to \( O_{F_2}(h+2f) \) is an embedding outside \( h \).

**Proof of Theorem 4** for \( r \leq 2 \). If \( r = 1 \), then use Remark 2. Assume the existence of a rank two vector bundle \( E \) with property \( \mathcal{L} \) with respect to \( M \) and \( c_1(E) = O_{F_2}(uh+vf) \). Let \( m \) be the first integer such that \( h^0(F_e, E(mM)) > 0 \). We get an exact sequence (1) with \( D \in |O_{F_2}(xh+fy)| \) with the convention \( (x, y) = (0, 0) \) if \( D = \emptyset \). Hence either \( (x, y) = (0, 0) \) or \( (x, y) = (1, 0) \) or \( x = 0 \) and \( 1 \leq y \leq e \) or \( e \geq 2 \), \( x = 1 \), and \( 1 \leq y \leq e - 1 \). Since \( h^2(F_e, M^{\otimes z}(D)) = 0 \) for all \( z \geq 0 \), (1) and property \( \mathcal{L} \) for \( E \) imply \( h^1(F_e, I_S((u+2m-x+z)h+(v+2me-y+ze)f)) = 0 \) for all \( z \geq 0 \). As in Remark 5, we see that when \( z > 0 \) the last inequality implies \( v-y \geq e(u-x)-1 \). If \( v \leq e(u-1)-2 \) the last inequality is not satisfied for any choice of the pair \( (x, y) \) in the previous list.

**Proof of Theorem 5**. Fix a general \( S \subset F_e \) such that \( \mathfrak{i}(S) = s \). Let \( E \) be any torsion free sheaf fitting in the following exact sequence:

\[
(4) \quad 0 \to O_{F_2}((1-m)h-emf) \to E \to I_S((u+m-1)h+(v+em)f) \to 0
\]

We have \( c_1(E) = O_{F_2}(uh+vf) \) and \( c_2(E) = s - e(u+m-1) + (1-m)(v+em) \).

By construction \( h^0(F_e, E(mM)) \neq 0 \). We have \( h^0(F_e, E((m-1)M)) = 0 \). If \( h^0(F_e, I_S((u+2m-2)h+(v+2em-e)f)) = 0 \). Since \( S \) is general, \( h^0(F_e, I_S((u+2m-2)h+(v+2em-e)f)) = 0 \) if and only if

\[
(5) \quad h^0(F_e, O_{F_2}((u+2m-2)h+(v+2em-e)f)) \leq s
\]

Since \( S \) is general, every subset of it is general. Hence to check the Cayley-Bacharach condition and hence show the local freeness of a general \( E \) given by the extension (5) it is sufficient to prove check the following inequality:

\[
(6) \quad h^0(F_e, O_{F_2}((u+2m-5)h+(v+2em-2e-2)f)) \leq s - 1
\]

This is true, because we assumed \( s \geq a \) and \( a > h^0(F_e, O_{F_2}((u+2m-5)h+(v+2em-2e-2)f)) \). Hence a general \( E \) fitting in the extension (5) is locally free. Since \( O_{F_2}(3h+e+2) \) has a subsheaf the very ample line bundle \( O_{F_2}(h+e+2) \), (5) is satisfied if (5) is satisfied. The generality of \( S \) implies that \( h^1(F_e, I_S((u+m-1+t)h+(v+em+et)f)) = 0 \) if and only if \( h^1(F_e, O_{F_2}((u+m-1+t)h+(v+em+et)f)) = 0 \) and \( h^0(F_e, O_{F_2}((u+m-1)t+h+(v+em+et)f)) = s \). Notice that \( a = h^0(F_e, O_{F_2}((u+2m-2)h+(v+2me-e)f)) \) and \( b = h^0(F_e, O_{F_2}((u+2m-1)h+(v+2mef)) \).

Since \( h^1(F_e, (u+m-1+t)u+(v+me+te)f)) = 0 \) for all \( t \geq 0 \), \( a \leq s \leq b \) and \( S \) is general, any sheaf \( E \) in (5) has property \( \mathcal{L} \) with respect to \( M \). Since a general extension (4) has locally free middle term \( E \), the proof of the first part of Theorem 2 is over. Now assume \( m = 0 \), \( u \geq 3, v < 2ru \), and that \( E \) is not \( R \)-stable, i.e. assume the existence of \( N \in \text{Pic}(F_e) \) such that \( N \cdot R \geq c_1(E) \cdot R/2 \) and an inclusion \( j : N \to E \); here to have \( N \) locally free we use that \( E \) is reflexive. Since \( m = 0 \) and \( u \geq 3, c_1(E) \cdot R > 2(O_{F_2}(h)) \cdot R \). Hence \( j \) induces a non-zero map \( N \to I_S((u-1)h+vf) \). Any non-zero map \( N \to O_{F_2}((u-1)h+vf) \) is associated to a unique non-negative divisor \( \Delta \in |O_{F_2}((u-1)h+vf)| \otimes \mathbb{N}^* \). Since \( j \) factors through \( I_S((u-1)h+vf) \), \( h^0(F_e, I_S(\Delta)) > 0 \). We fixed \( R \) and the integers \( m, u, v \). There are only finitely many possibilities for the line bundle \( O_{F_2}(\Delta) \). Since \( S \) is general, we get \( h^0(F_e, O_{F_2}(\Delta)) \geq s \). Write \( N = O_{F_2}((\gamma h+\delta f) \) for some integers
The inequality $N \cdot R \geq c_1(E) \cdot R/2$ is equivalent to the inequality
\begin{equation}
2\gamma + 2\delta \geq u + v
\end{equation}
We have $\mathcal{O}_{E_i}(\Delta) = \mathcal{O}_{E_i}((u - 1 - \gamma)h + (v - \delta)f)$. Since $h^0(F_e, \mathcal{O}_{E_i}(\Delta)) \geq s$ and $s \leq \bar{s} = h^0(F_e, \mathcal{O}_{E_i}((u - 1)h + (v)f))$, either $\gamma \leq 0$ or $\delta \leq 0$. Since $\Delta$ is effective, we also have $\gamma \leq u - 1$ and $\delta \leq v$. First assume $\delta \leq 0$. Hence $\gamma \geq (u + v)/2$. Since $\gamma \leq u - 1$, we get $v \leq u - 2$. Since $v \geq eu - e$, we get a contradiction. Now assume $\gamma \leq 0$. We get $\delta \geq (u + v)/2$. Consider the exact sequence
\begin{equation}
0 \to N \to E \to \text{Coker}(j) \to 0
\end{equation}
Notice that $\text{Coker}(j)^{**} \cong \mathcal{O}_{E_i}((u - \gamma)h + (v - \delta)f)$. Since $\gamma \leq 0$, $\delta \geq (u + v)/2$, and $v < 2eu$, we have $v - \delta \leq e(u - \gamma) - 2$. In Remark 6 we checked that $h^1(F_e, \text{Coker}(j)(tM)) > 0$ for $t \gg 0$. Since $h^2(F_e, L(tM)) = 0$ for $t \gg 0$ and any $L \in \text{Pic}(F_e)$, the exact sequence gives that $E$ has not property $L$ with respect to $M$, contradicting the already proved part of Theorem 2. If instead of $R$ we use $M$ for the intersection product, instead of (7) we only have the inequality $2\delta \geq v$. Everything works in the same way with only minor numerical modifications. □

Remark 6. There are at least 2 well-known and related ways to obtain rank $r \geq 3$ vector bundles as extensions. Instead of (11) we may take the exact sequence
\begin{equation}
0 \to \oplus_{r-1}^r \mathcal{O}_{E_i}(D_i - m_iM) \to \mathcal{I}_Z(uh + vf) \to 0
\end{equation}
In [4], proof of Theorem 5.1.6, the following extension is used:
\begin{equation}
0 \to L_1 \to E \to \oplus_{r-1}^r \mathcal{I}_{Z_i}(u_ih + v_i f) \to 0
\end{equation}
The latter extension was behind the proof of Proposition [1]. Both extensions can give several examples of vector bundles with or without property $L$ with respect to $M$. To prove Theorem 4 we will use iterated extensions, i.e. increasing filtrations $E_i$, $1 \leq i \leq r$, of $E$ such that $E_1$ is a line bundle, $E_r = E$ and each $E_i/E_{i-1}$ is a rank 1 torsion free sheaf.

Proof of Theorem 4 for $r \geq 3$. Assume the existence of a rank $r$ vector bundle $E$ with property $L$ with respect to $M$ and $c_1(E) = \mathcal{O}_{E_i}(uh + vf)$. Let $m_1$ be the first integer such that $h^0(F_e, E(m_1M)) > 0$. Fix a general $\sigma \in H^0(F_e, E(m_1M)))$. Since $h^0(F_e, E((m_1 - 1)M)) = 0$, $\sigma$ induces an exact sequence
\begin{equation}
0 \to \mathcal{O}_{F_e}(-m_1M + D_1) \to E \to G_1 \to 0
\end{equation}
with $F_1$ torsion free, $D_1$ of type $(x_1, y_1)$ and either $(x_1, y_1) = (0, 0)$ or $(x_1, y_1) = (1, 0)$ or $x_1 = 0$ and $1 \leq y_1 \leq e$ or $e \geq 2$, $x_1 = 1$, and $1 \leq y_1 \leq e - 1$. Notice that $c_1(G_1) = \mathcal{O}_{F_e}((u + m_1 - x_1)h + (v + em_1 - y_1)f)$. Set $E_1 := \mathcal{O}_{F_e}(-m_1M + D_1)$. Since $h^2(F_e, \mathcal{O}_{F_e}((t - m_1)D + D_1)) = 0$ for all $t \geq m_1$, property $L$ for $m$ with respect to $M$ implies $h^1(F_e, F_1(tM)) = 0$ for all $t \geq m_1$. Let $m_2$ be the first integer such that $m_2 \geq m_1$ and $h^0(F_e, F_1(m_2M)) > 0$. A non-zero section of $H^0(F_e, G_1(m_2M))$ induces an exact sequence
\begin{equation}
0 \to \mathcal{I}_{Z_1}(m_2M + D_2) \to G_1 \to G_2 \to 0
\end{equation}
with $\mathcal{I}_{Z_1}$ zero-dimensional, $G_2$ torsion free and $D_2$ an effective divisor of type $(x_2, y_2)$ and either $(x_2, y_2) = (0, 0)$ or $(x_2, y_2) = (1, 0)$ or $x_2 = 0$ and $1 \leq y_2 \leq e$ or $e \geq 2$, $x_2 = 1$, and $1 \leq y_2 \leq e - 1$. Here we cannot claim that $Z_1 = \emptyset$, because $G_1$ is not assumed to be locally free. Notice that $c_1(G_2) = \mathcal{O}_{F_e}((u + m_1 + m_2 - x_1 - x_2)h + (v + em_1 + em_2 - y_1 - y_2)f)$. Since $Z_1$ is zero-dimensional,
$h^2(F_e, \mathcal{I}_Z \otimes L) = h^2(F_e, L)$ for every $L \in \text{Pic}(F_e)$. Hence as in the first step we get $h^3(F_e, G_2(tM)) = 0$ for all $t \geq m_2$. If $r = 3$, we are done as in the proof of the case $r = 2$. If $r \geq 4$, we iterate the last step $r - 3$ times. \hfill \square

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