Enumeration of chord diagrams

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Abstract

We determine the number of nonequivalent chord diagrams of order \( n \) under the action of two groups, \( C_{2n} \), a cyclic group of order \( 2n \), and \( D_{2n} \), a dihedral group of order \( 4n \). Asymptotic formulas are also established.

§1

Given \( 2n \) different points on a circle ask a question: in how many different ways may the points be joined by chords. The answer depends of course on our understanding of the word ”different”.

The configuration (actually a graph) consisting of the circle and \( n \) chords joining \( 2n \) different point is called a chord diagram of order \( n \) or, shortly, \( n \)-diagram. In the present paper, we let a group \( G \) act on the circle and consider two \( n \)-diagrams as indistinguishable or equivalent if the one is transformed into the other by a suitable element of the group.

With the identity group acting on the circle, all \( n \)-diagrams are distinct and there are altogether \( \frac{(2n)!}{2^n} = (2n - 1)!! \) diagrams with \( n \) chords. This case is well-studied: Errera [1] (see also A. M. Jaglom and I. M. Jaglom [2]) determined the number of \( n \)-diagrams with the additional requirement that no chords intersect inside the circle which equals \( \frac{(2n)!}{n!(n+1)!} \). Touchard [3] and later Riordan [4] extended that result to enumeration of \( n \)-diagrams by the number of crossings of the chords, which is given by a generating function \( T_n(x) \) satisfying the following relation

\[
(1 - x)^n T_n(x) = \sum_{j=0}^{n} (-1)^j t_{nj} x^j
\]

with

\[
J = \binom{j + 1}{2}, \quad t_{nj} = \frac{2j + 1}{2n + 1} \binom{2n + 1}{n - j}.
\]

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Generally chord diagrams are not strict graphs as they may have double edges. Chord diagrams which are strict graphs were considered by Hazewinkel and Kalashnikov [5]. Let $b_{2n}$ be the number of strict $n$-diagrams under the identity group. They proved that $a_{2n} = \sum_{i=1}^{n} b_{2i}$ satisfies the recurrence $a_{2n} = (2n - 1) a_{2n-2} + a_{2n-4}$.

Chord diagrams considered to within an equivalence induced by a cyclic group action appear in different contexts. In the theory of knots they are used to describe Vassiliev knot invariants in a purely combinatorial way by considering the algebra of functions defined on the set of chord diagrams satisfying certain linear equations [6], [7]. Chord diagrams also appear in the classification of vector fields and smooth functions on surfaces up to a homeomorphism where stable separatrices of critical points together with a component of the surface boundary form a chord diagram embedded into the surface [8].

§2

Definition 1. A chord diagram of order $n$ is a 3-regular graph with the vertex set $[2n] = \{1, 2, \ldots, 2n\}$ containing the 2n-circuit $\Delta_{2n} = (1 \ 2 \ \ldots \ 2n)$ as a subgraph. The circuit is called a circle, the edges not belonging to the circuit are called chords.

Definition 2. Let a group $G$ act on the circuit $\Delta_{2n}$. Two $n$-diagrams $\Gamma_1$ and $\Gamma_2$ are said to be equivalent if there is a $g \in G$ which takes the chords of $\Gamma_1$ into the chords of $\Gamma_2$.

Our first result is the following

Theorem 1. The number of nonequivalent $n$-diagrams under the action of a group $G$ equals

$$\frac{1}{|S_n \wr S_2| \cdot |G|} \sum_{\pi \in S_n \wr S_2 \ \eta \in G} \prod_i \eta_i \ (\eta_i - 1) \ldots (\eta_i - \pi_i + 1). \quad (1)$$

Here $S_n \wr S_2$ is the wreath product of two symmetric groups $S_n$ and $S_2$, $\pi \in S_n \wr S_2$ has cycle type $1^{\pi_1} 2^{\pi_2} \ldots (2n)^{\pi_{2n}}$, $\eta \in G$ cycle type $1^{\eta_1} 2^{\eta_2} \ldots (2n)^{\eta_{2n}}$ and the product is taken over all $i \in [2n]$ such that $\pi_i > 0$, the product being equal to zero if $\pi_i > \eta_i$ for some $i$.

Proof. For each chord diagram, a subgraph consisting of its chords is a 1-factor. The chords of all $n$-diagrams constitute the complete graph $K_{2n}$, the chords of each single $n$-diagram being again a 1-factor of $K_{2n}$. The action of
G on $\Delta_{2n}$ induces an action on the set $\mathcal{F}_1$ of all 1-factors of $K_{2n}$. The orbits of $\mathcal{F}_1$ under that action are in a one-one correspondence with the nonequivalent $n$-diagrams.

A 1-factor of $K_{2n}$ can be represented by a $n \times 2$ matrix whose entries belong to $[2n]$ and are all distinct: each row corresponds to an edge, two row entries being the end points of the edge. This correspondence is not unique. It is defined to within an equivalence induced on the set of such matrices by independently permuting entries in each row and permuting the rows bodily. This amounts to the action of the wreath product $S_n \wr S_2$. The action of $G$ on the set of 1-factors is equivalent to the action of $G$ on the set of $2 \times n$-matrices.

We thus arrive at the following setting: given two sets $[n] \times [2]$ and $[2n]$, consider the set of bijective mappings $[n] \times [2] \rightarrow [2n]$. The wreath product $S_n \wr S_2$ acts on the set $[n] \times [2]$ by the rule $(\tau, \bar{\sigma}) \cdot (i,j) = (\tau(i), \sigma_i(j))$ where $\tau \in S_n$, $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_2^n$, and the group $G$ acts on the set $[2n]$. Two mappings $f_1, f_2 : [n] \times [2] \rightarrow [2n]$ are equivalent if there exist a $\pi \in S_n \wr S_2$ and an $\eta \in G$ such that

$$f_1(\pi(i,j)) = \eta f_2((i,j))$$

for all $(i,j) \in [n] \times [2]$. The equivalence classes of mappings are in a one-one correspondence with the orbits of $\mathcal{F}_1$. In this setting, an argument of de Bruijn [9] applies which he used to prove a theorem on the number of classes of bijective mappings. The proof is complete.

\[\square\]

§3

We now specialize $G$ to a cyclic group $C_{2n}$ of order $2n$ and obtain a much simpler expression for the number of nonequivalent $n$-diagrams.

**Theorem 2.** The number $c_n$ of nonequivalent $n$-diagrams under the action of a cyclic group $C_{2n}$ equals

$$c_n = \frac{1}{2n} \sum_{i \mid 2n} \varphi(i) \nu_n(i),$$

(2)

where $\varphi(i)$ is the Euler function and

$$\nu_n(i) = \begin{cases} 
\frac{i^n}{i} \ (2n/i - 1)!! & i \text{ odd}, \\
\sum_{k=0}^{[i/2]} \left(\binom{2n/i}{2k}\right) i^k (2k - 1)!! & i \text{ even}
\end{cases}$$

for $i \mid 2n$. 

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Proof. Each permutation $\eta \in C_{2n}$ has cycle type $i^{2n/i}$, $i \mid 2n$. If $\pi \in S_n \wr S_2$ has a cycle of length $\neq i$ the product in (1) equals zero, otherwise it reduces to a single term $i^{\pi_i} \eta_i (\eta_i - 1) \ldots (\eta_i - \pi_i + 1) = i^{2n/i} (2n/i)!$ since $\pi_i = \eta_i = 2n/i$. The double sum in (1) can then be replaced with a single sum over all $i \mid 2n$. The group $C_{2n}$ contains $\varphi(i)$ permutations of cycle type $i^{2n/i}$. Denoting by $\psi_n(i)$ the number of permutations of the same cycle type $i^{2n/i}$ in $S_n \wr S_2$ we rewrite (1) as follows

$$c_n = \frac{1}{2^n n! 2n} \sum_{i \mid 2n} i^{2n/i} (2n/i)! \varphi(i) \psi_n(i).$$

(4)

To determine $\psi_n(i)$ we first establish a relationship between the cycles of $\tau \in S_n$ and the cycles of $(\tau, \bar{\sigma}) \in S_n \wr S_2$. If $K_\pi \subset [n] \times [2]$ is a cycle of $\pi = (\tau, \bar{\sigma})$ then its projection onto $[n]$ is a cycle of $\tau$. If $K_\tau \subset [n]$ is a cycle of $\tau \in S_n$ then, for any $\bar{\sigma} \in S_2^n$, the length of a cycle of $(\tau, \bar{\sigma}) \in S_n \wr S_2$ induced by $K_\tau$ depends only on $\sigma_k \in S_2$, $k \in K_\tau$. Let $L = \{k \in K_\tau : \sigma_k \neq e\}$ where $e$ is the identity permutation. If $|L|$ is odd then $K_\tau$ induces one cycle of length $2 |K_\tau|$. If $|L|$ is even then $K_\tau$ induces two cycles both of length $|K_\tau|$. Let $i \mid 2n$ be odd. For $\pi = (\tau, \bar{\sigma})$ to have cycle type $i^{2n/i}$ the permutation $\tau$ must have cycle type $i^{n/i}$. The number of such $\tau \in S_n$ equals

$$\frac{n!}{i^{n/i} (n/i)!}.$$

Now we fix $\tau$ and count the number of $\bar{\sigma} \in S_2^n$ such that $(\tau, \bar{\sigma})$ has cycle type $i^{2n/i}$. Each cycle $K_\tau$ of $\tau$ induces two cycles of $(\tau, \bar{\sigma})$ of the same length $i$. Hence $|L|$ must be even for each $K_\tau$. There are

$$\sum_{m \leq i, \text{ even}} \binom{i}{m} = 2^{i-1}$$

choices for $L \subset K_\tau$. Clearly, $L$ uniquely determines $\sigma_k$ for all $k \in K_\tau$. For different cycles of $\tau$ the choices are independent, so we have $\binom{2^{i-1}}{n/i} = 2^{n-n/i}$ different $\bar{\sigma} \in S_2^n$. Multiplying the expressions for $\tau$ and $\bar{\sigma}$ we get

$$\psi_n(i) = \frac{2^n n!}{2^{n/i} i^{n/i} (n/i)!}$$

(5)

for $i$ odd.
Let now \( i | 2n \) be even. For \( \pi = (\tau, \bar{\sigma}) \) to have cycle type \( i^{2n/i} \) the permutation \( \tau \) must have cycle type \( (i/2)^l i^k \) with \( l \geq 0, k \geq 0, l \cdot i/2 + k \cdot i = n \). The number of such \( \tau \in S_n \) equals

\[
\frac{n!}{(i/2)^l! i^k k!}.
\]

We fix \( \tau \) and count the number of corresponding \( \bar{\sigma} \). Each \( i/2 \)-cycle of \( \tau \) induces one cycle of \( (\tau, \bar{\sigma}) \) of the length \( i \). Hence \( |L| \) must be odd for each \( i/2 \)-cycle of \( \tau \). There are

\[
\sum_{m \leq i/2, m \ odd} \binom{i/2}{m} = 2^{i/2-1}
\]

choices for \( L \). Each \( i \)-cycle of \( \tau \) induces two cycles of \( (\tau, \bar{\sigma}) \), both having length \( i \). Hence \( |L| \) must be even for each \( i \)-cycle of \( \tau \), and we again have \( 2^{i-1} \) choices for \( L \). It follows, there are \( (2^{i/2-1})^l \cdot (2^{i-1})^k = 2^{n-l-k} \) different \( \bar{\sigma} \in S_n^2 \) such that \( (\tau, \bar{\sigma}) \) has cycle type \( i^{2n/i} \). Multiplying the expressions for \( \tau \) and \( \bar{\sigma} \) and summing up over all admissible \( l, k \) we obtain

\[
\psi_n(i) = \sum_{l \geq 0, k \geq 0, l \cdot i/2 + k \cdot i = n} \frac{2^n n!}{(i/2)^l! 2^l i^k k!} = \frac{2^n n!}{i^{2n/i}} \sum_{k=0}^{[n/i]} \frac{i^k}{(2n/i - 2k)! 2^k k!} \tag{6}
\]

for \( i \) even.

It remains to substitute (5) and (6) into (4). Setting

\[
\nu_n(i) = \frac{i^{2n/i} (2n/i)!}{2^n n!} \psi_n(i)
\]

we have

\[
\nu_n(i) = \frac{i^n/n! (2n/i)!}{2^n/n! (n/i)!} = i^n/n! (2n/i - 1)!!
\]

for \( i \) odd and

\[
\nu_n(i) = \sum_{k=0}^{[n/i]} \frac{i^k (2n/i)!}{(2n/i - 2k)! 2^k k!} \sum_{k=0}^{[n/i]} \frac{i^k (2n/i)!}{2^k k!} (2k - 1)!!
\]

for \( i \) even which completes the proof. \( \square \)

As the terms in (4) are all positive it is clear that \( \omega_n = (2n)^{-1} (2n - 1)!! \) is a lower bound for \( c_n \) for \( n \geq 1 \). It is an easy matter to show that \( \omega_n \) is actually an asymptotic estimate for \( c_n \) as \( n \to \infty \).
Corollary 1.

\[ c_n \sim \mathcal{L}_n \quad \text{as } n \to \infty \quad (7) \]

*Proof.* Dividing out the first term in (2) gives

\[ 2nc_n = (2n - 1)!! + \sum_{i \mid 2n, i > 1} \varphi(i)\nu_n(i). \quad (8) \]

We begin with establishing an upper bound for \( \nu_n(i) \) using Stirling’s formula in the following form

\[ n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{\theta}{12n}} \]

where \( \theta = \theta(n) \) satisfies \( 0 < \theta < 1 \) (see [10]). We have

\[ (2k - 1)!! = \sqrt{2} \left( \frac{2k}{e} \right)^k e^{(\frac{1}{2\theta_1 - \theta_2})\frac{1}{12k}} < \sqrt{2e} \left( \frac{2k}{e} \right)^k \]

for \( k \geq 1, 0 < \theta_1 < 1, 0 < \theta_2 < 1 \). Then we find

\[ \nu_n(i) = (2n/i - 1)!! < \sqrt{2e} \left( \frac{2n}{ei} \right)^{n/i} \quad (9) \]

for \( i \mid 2n \) odd. Using the following estimate

\[ \binom{n}{k} \leq \left( \frac{nc}{k} \right)^k \]

(see [11]) for the binominal coefficients we obtain

\[ \binom{2n/i}{2k} \varphi^k (2k - 1)!! < \sqrt{2e} \left( \frac{2n^2e}{ik} \right)^k < \sqrt{2e}(2en)^{n/i} \]

for \( k \in \{1, \ldots, \lfloor n/i \rfloor \} \) whence

\[ \nu_n(i) = 1 + \sum_{k=1}^{\lfloor n/i \rfloor} \binom{2n/i}{2k} \varphi^k (2k - 1)!! < \sqrt{2e} n(2en)^{n/i} \quad (10) \]

for \( i \mid 2n \) even. Comparing upper bounds (9) and (10) we conclude that

\[ \nu_n(i) < \bar{\nu}_n = \sqrt{2e} n(2en)^{n/2} \]

for \( i > 1, i \mid 2n \). Going back to (8) we see that

\[ \sum_{i \mid 2n, i > 1} \varphi(i)\nu_n(i) < 2n \bar{\nu}_n = o ((2n - 1)!!) \]

as \( n \to \infty \) and the result follows. \( \square \)

Corollary 1 shows that asymptotically each equivalence class contains \( 2n \) diagrams, which is equivalent to saying that the fraction of 1-factors of \( K_{2n} \) with a nontrivial stabilizer in \( C_{2n} \) tends to zero as \( n \to \infty \).
The analysis done in the proof of Theorem 2 allows us to handle the case of a dihedral group $D_{2n}$.

**Theorem 3.** The number $d_n$ of nonequivalent $n$-diagrams under the action of a dihedral group $D_{2n}$ equals

$$d_n = \frac{1}{2} \left( c_n + \frac{1}{2} (\kappa_{n-1} + \kappa_n) \right)$$  \hspace{1cm} (11)

where

$$\kappa_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{k! (n-2k)!}.$$  \hspace{1cm} (12)

**Proof.** As $C_{2n} < D_{2n}$ it follows from (1) that

$$d_n = \frac{1}{2} \left( c_n + \frac{1}{2n^! 2n} \gamma_n \right)$$  \hspace{1cm} (13)

where $\gamma_n$ represents the contribution of those permutations of $D_{2n}$ which are not in $C_{2n}$. Such permutations have cycle type either $2^n$ or $1^2 2^{n-1}$ and there are $n$ permutations of each type in $D_{2n}$. The product in (1) equals $2^n n!$ for permutations of cycle type $2^n$ and $2^n (n-1)!$ for permutations of cycle type $1^2 2^{n-1}$. So we can write

$$\gamma_n = 2^n n! n \psi_n(2) + 2^n (n-1)! n \Psi_n$$  \hspace{1cm} (14)

where $\Psi_n$ is the number of $\pi \in S_n \wr S_2$ of cycle type $1^2 2^{(n-1)}$ and $\psi_n(2)$ is the number of $\pi$ of cycle type $2^n$. Applying the analysis in the proof of Theorem 2 we see that for each $\tau \in S_n$ of cycle type $1^l 2^k$, $l + 2k = n$, $l \geq 1$, $k \geq 0$ there are $l \cdot 2^k$ permutations $\bar{\sigma} \in S_2^n$ such that $\pi = (\tau, \bar{\sigma}) \in S_n \wr S_2^n$ has cycle type $1^2 2^{(n-1)}$. Multiplying and summing up over all admissible $l, k$ and simplifying we get

$$\Psi_n = \sum_{l \geq 1, k \geq 0 \atop l + 2k = n} \frac{l n!}{l! k!} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{(n-1-2k)! k!}.$$  \hspace{1cm} (15)

Substituting the expressions for $\psi_n(2)$ and $\Psi_n$ into (13) we obtain

$$\gamma_n = 2^n n! n \left( \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{(n-2k)! k!} + \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(n-1)!}{(n-1-2k)! k!} \right).$$  \hspace{1cm} (16)
Denoting
\[ \kappa_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{k! (n-2k)!} \]
and substituting (14) into (12) we get the statement of the theorem.

As in the case of the cyclic group, \( d_n = (4n)^{-1} (2n-1)!! \) is a lower bound for \( d_n \) for \( n \geq 1 \) and in fact an asymptotic estimate.

Corollary 2.

\[ d_n \sim d_n \quad \text{as } n \to \infty \]  
(15)

Proof. From (11) we get
\[ 4nd_n = 2nc_n + n(\kappa_{n-1} + \kappa_n). \]

Clearly
\[ n(\kappa_{n-1} + \kappa_n) < 2n\kappa_n < 2n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{k!} < 2n^2 n! \]

for \( n \geq 1 \). Stirling’s formula shows that \( 2n^2 n! = o((2n-1)!!) \) and hence \( n(\kappa_{n-1} + \kappa_n) = o((2n-1)!!) \). But \( 2nc_n \sim (2n-1)!! \) which completes the proof.

Corollary 2 shows that asymptotically each equivalence class contains \( 4n \) diagrams.

§5

The following table gives an idea of the growth rate of \( c_n \) and \( d_n \) along with the integral parts of the corresponding asymptotic estimates.

| \( n \) | \( c_n \) | \( \lfloor c_n \rfloor \) | \( d_n \) | \( \lfloor d_n \rfloor \) |
|---|---|---|---|---|
| 3  | 5  | 2  | 5  | 1  |
| 4  | 18 | 13 | 17 | 6  |
| 5  | 105| 94 | 79 | 47 |
| 6  | 902| 866| 554| 433|
| 7  | 9749| 9652| 5283| 4826|
| 8  | 127072| 126689| 65346| 63344|
| 9  | 1915951| 1914412| 966156| 957206|
| 10 | 32743182| 32736453| 16411700| 16368226|
| 11 | 625002933| 624968662| 312702217| 312484331|
Below are shown all nonequivalent (under the cyclic group) 3- and 4-diagrams. Except for the two diagrams 12 and 13 all of them are also nonequivalent under the dihedral group.

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