Abstract

We propose a new method to apply the Lipschitz functional calculus of local Dirichlet forms to Poisson random measures. To cite this article: N. Bouleau, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

Résumé

Calcul d’erreur et régularité des fonctionnelles de Poisson : la méthode de la particule prêtée. Nous proposons une nouvelle méthode pour appliquer le calcul fonctionnel lipschitzien des formes de Dirichlet locales aux mesures aléatoires de Poisson. Pour citer cet article : N. Bouleau, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

1. Notation and basic formulae

Let us consider a local Dirichlet structure with carré du champ \((X, \mathcal{X}, \nu, d, \gamma)\) where \((X, \mathcal{X}, \nu)\) is a \(\sigma\)-finite measured space called bottom-space. Singletons are in \(\mathcal{X}\) and \(\nu\) is diffuse, \(d\) is the domain of the Dirichlet form \(\epsilon[u] = \frac{1}{2} \int \gamma[u] \, dv\). We denote \((a, D(a))\) the generator in \(L^2(\nu)\) (cf. [3]).

A random Poisson measure associated to \((X, \mathcal{X}, \nu)\) is denoted \(N\). \(\Omega\) is the configuration space of countable sums of Dirac masses on \(X\) and \(\mathcal{A}\) is the \(\sigma\)-field generated by \(N\), flow \(\mathbb{P}\) on \(\Omega\). The space \((\Omega, \mathcal{A}, \mathbb{P})\) is called the up-space. We write \(N(f)\) for \(\int f \, dN\). If \(p \in [1, \infty]\) the set \(\{e^{iN(f)} : f \text{ real, } f \in L^1 \cap L^2(\nu)\}\) is total in \(L^p_\mathbb{C}(\Omega, \mathcal{A}, \mathbb{P})\). We put \(\tilde{N} = N - \nu\). The relation \(\mathbb{E}(\tilde{N} f)^2 = \int f^2 \, dv\) extends and gives sense to \(\tilde{N}(f), f \in L^2(\nu)\). The Laplace functional and the differential calculus with \(\gamma\) yield

\[
\forall f \in d, \forall h \in D(a) \quad \mathbb{E}\left[e^{i\tilde{N}(f)} \left(\tilde{N}(a[h]) + \frac{i}{2} \gamma(f, h)\right)\right] = 0. \tag{1}
\]
2. Product, particle by particle, of a Poisson random measure by a probability measure

Given a probability space \((R, \mathcal{R}, \rho)\), let us consider a Poisson random measure \(N \circ \rho\) on \((X \times R, \mathcal{X} \times \mathcal{R})\) with intensity \(v \times \rho\) such that for \(f \in L^1(v)\) and \(g \in L^1(\rho)\) if \(N(f) = \sum f(x_n)\) then \((N \circ \rho)(fg) = \sum f(x_n)g(r_n)\) where the \(r_n\)'s are i.i.d. independent of \(N\) with law \(\rho\). Calling \((\hat{\mathcal{O}}, \hat{\mathcal{A}}, \hat{\mathbb{P}})\) the product of all the factors \((R, \mathcal{R}, \rho)\) involved in the construction of \(N \circ \rho\), we obtain the following properties: For an \(\mathcal{A} \times \mathcal{X} \times \mathcal{R}\)-measurable and positive function \(F\),

\[
\hat{\mathbb{E}} \left( \int F \, d(N \circ \rho) \right)^2 = \int F^2 \, dN \, d\rho \quad \mathbb{P}\text{-a.s.}
\]

Lemma 2.1. Let \(F\) be \(\mathcal{A} \times \mathcal{X} \times \mathcal{R}\)-measurable, \(F \in L^2(\mathbb{P} \times N \times \rho)\) and such that \(\int F(\omega, x, r) \rho(\mathrm{dr}) = 0 \mathbb{P}_N\text{-a.s.}, then \(\int F \, d(N \circ \rho)\) is well defined, belongs to \(L^2(\mathbb{P} \times \hat{\mathbb{P}})\) and

\[
\hat{\mathbb{E}} \left( \int F \, d(N \circ \rho) \right)^2 = \int F^2 \, dN \, d\rho \quad \mathbb{P}\text{-a.s.}
\]

3. Construction by Friedrichs' method and expression of the gradient

(a) We suppose the space by \(d\) of the bottom structure is separable, then a gradient exists (cf. [3] Chap. V, p. 225 et seq.). We denote it \(\flat\) and choose it with values in the space \(L^2(\mathbb{R}, \mathcal{X})\) involved in the subspace orthogonal to the constant 1, i.e.

\[
\forall u \in \mathbb{d} \quad \int u^2 \, d\rho = 0 \quad \nu\text{-a.s.}
\]

This hypothesis is important here as in many applications (cf. [2] Chap. V §4.6). We suppose also, but this is not essential (cf. [3] p. 44) \(1 \in \mathbb{d}_{\text{loc}}\ \gamma[1] = 0\) so that \(1^2 = 0\).

(b) We define a pre-domain \(D_0\) dense in \(L^2(\mathbb{P})\) by

\[
D_0 = \left\{ \sum_{p=1}^m \lambda_p e^{iN(f_p)} ; m \in \mathbb{N}^\ast, \lambda_p \in \mathbb{C}, f_p \in D(\omega) \cap L^1(v) \right\}
\]

(c) We introduce the creation operator inspired from quantum mechanics (see [7–9,1,5,6] and [10] among others) defined as follows

\[
\varepsilon^+_x(\omega) \text{ equals } \omega \text{ if } x \in \text{supp}(\omega), \quad \text{and} \quad \varepsilon^+_x(\omega) \text{ equals } \omega + \varepsilon_x \text{ if } x \notin \text{supp}(\omega)
\]

so that

\[
\varepsilon^+_x(\omega) = \omega \quad \mathcal{N}_\omega\text{-a.e. } x \quad \text{and} \quad \varepsilon^+_x(\omega) = \omega + \varepsilon_x \quad \nu\text{-a.e. } x.
\]

This map is measurable and the Laplace functional shows that for an \(\mathcal{A} \times \mathcal{X}\)-measurable \(H \geq 0\),

\[
\hat{\mathbb{E}} \int \varepsilon^+ H \, d\nu = \mathbb{E} \int H \, dN.
\]

Let us remark also that by (5), for \(F \in L^2(\mathbb{P} \times N \times \rho)\)

\[
\int \varepsilon^+ F \, d(N \circ \rho) = \int F \, d(N \circ \rho) \quad \mathbb{P} \times \hat{\mathbb{P}}\text{-a.s.}
\]

(d) We defined a gradient \(\sharp\) for the up-structure on \(D_0\) by putting for \(F \in D_0\)

\[
F^\sharp = \int (\varepsilon^+ F)\flat \, d(N \circ \rho)
\]
this definition being justified by the fact that for \( \mathbb{P} \)-a.e. \( \omega \) the map \( y \mapsto F(\varepsilon^+_y(\omega)) - F(\omega) \) is in \( d \), \( \varepsilon^+_F \) belongs to \( L^\infty(\mathbb{P}) \otimes d \) algebraic tensor product, and \( (\varepsilon^+_F - F^\circ) = (\varepsilon^+_F)^\circ \in L^2(\mathbb{P}_N \times \rho) \).

For \( F, G \in D_0 \) of the form

\[
F = \sum_p \lambda_p e^{i\tilde{N}(fp)} = \Phi(\tilde{N}(f_1), \ldots, \tilde{N}(f_m)), \quad G = \sum_q \mu_q e^{i\tilde{N}(g_q)} = \Psi(\tilde{N}(g_1), \ldots, \tilde{N}(g_n))
\]

we compute using (2), (3) and (7) (in the spirit of Prop. 1 of [9] or Lemma 1.2 of [6])

\[
\mathbb{E}[F^\ast G^\ast] = \sum_{p,q} \lambda_p \mu_q e^{i\tilde{N}(fp) - i\tilde{N}(g_q)} N(\gamma(fp, g_q))
\]

and we have:

**Proposition 3.1.** If we put \( A_0[F] = \sum_p \lambda_p e^{i\tilde{N}(fp)}(i\tilde{N}(a[fp]) - 1/2 N(\gamma(fp))) \) it comes

\[
\mathbb{E}[A_0[F]G] = -\frac{1}{2} \mathbb{E} \sum_{p,q} \Phi_p^* \Psi_q N(\gamma(fp, g_q)).
\]

In order to show that \( A_0[F] \) does not depend on the form of \( F \), by (10) it is enough to show that the expression \( \sum_{p,q} \Phi_p^* \Psi_q N(\gamma(fp, g_q)) \) depends only on \( F \) and \( G \). But this comes from (9) since \( F^\zeta \) and \( G^\zeta \) depend only on \( F \) and \( G \).

By this proposition, \( A_0 \) is symmetric on \( D_0 \), negative, and the argument of Friedrichs applies (cf. [3] p. 4), \( A_0 \) extends uniquely to a selfadjoint operator \( (A, D(A)) \) which defines a closed positive (hermitian) quadratic form \( \mathcal{E}[F] = -\mathbb{E}[A[F]^2] \). By (10) contractions operate and (cf. [3]) \( \mathcal{E} \) is a Dirichlet form which is local with carré du champ denoted \( \Gamma \) and the up-structure obtained \( (\Omega, A, \mathbb{P}, D, \Gamma) \) satisfies

\[
\forall f \in d, \quad \tilde{N}(f) \in D \quad \text{and} \quad \Gamma(\tilde{N}(f)) = N(\gamma[f]).
\]

The operator \( \tilde{\zeta} \) extends to a gradient for \( \Gamma \) as a closed operator from \( L^2(\mathbb{P}) \) into \( L^2(\mathbb{P} \times \hat{\mathbb{P}}) \) with domain \( D \) which satisfies the chain rule and may be computed on functionals \( \Phi(\tilde{N}(f_1), \ldots, \tilde{N}(f_m)), \Phi \) Lipschitz and \( C^1 \) and their limits in \( D \) (as done in [4]).

Formula (8) for \( \tilde{\zeta} \) can be extended from \( D_0 \) to \( D \). Let us introduce the space \( D \) closure of \( D_0 \otimes d \) for the norm

\[
\|H\|_D = \left( \mathbb{E} \int \gamma[H(\omega, \cdot)](x)N(dx) \right)^{1/2} + \mathbb{E} \int \|H(\omega, x)|\xi(x)N(dx)
\]

where \( \xi > 0 \) is a fixed function such that \( N(\xi) \in L^2(\mathbb{P}) \).

**Theorem 3.2.** The formula \( F^\zeta = \int(\varepsilon^+_F)^\circ d(N \otimes \rho) \) decomposes as follows

\[
F \in D \otimes \varepsilon^+_F \in D_0 \otimes (\varepsilon^+_F)^\circ \in L^2(\mathbb{P}_N \times \rho) \xrightarrow{d(N \otimes \rho)} F^\zeta \in L^2(\mathbb{P} \times \hat{\mathbb{P}})
\]

where each operator is continuous on the range of the preceding one, \( L^2(\mathbb{P}_N \times \rho) \) denoting the closed subspace of \( L^2(\mathbb{P}_N \times \rho) \) of \( \rho \)-centered elements, and we have

\[
\Gamma[F] = \mathbb{E}[F^\ast]^2 = \int \gamma(\varepsilon^+_F) dN.
\]

4. The lent particle method

Let us consider, for instance, a real process \( Y_t \) with independent increments and Lévy measure \( \sigma \) integrating \( x^2, Y_t \) being supposed centered without Gaussian part. We assume that \( \sigma \) has an l.s.c. density so that a local Dirichlet structure may be constructed on \( \mathbb{R} \setminus \{0\} \) with carré du champ \( \gamma[f] = x^2 f''(x) \). If \( N \) is the random Poisson measure with intensity \( d \times \sigma \) we have \( \int_0^t h(s) Y_s \, ds = \int \chi_{[0,t]}(s) h(s) x \tilde{N}(ds \, dx) \) and the choice done for \( \gamma \) gives \( \Gamma[\int_0^t h(s) Y_s \, ds] = \int_0^t h^2(s) \, d\gamma(Y_t, Y_t) \) for \( h \in L^2_{\text{loc}}(d) \). In order to study the regularity of the random variable \( V = \int_0^T \varphi(Y_s) \, ds \) where \( \varphi \) is Lipschitz and \( C^1 \), we have two ways:
(a) We may represent the gradient \( \nabla \) as
\[
\nabla Y_t^x = B \left[ Y_t \right]
\]
where \( B \) is a standard auxiliary independent Brownian motion. Then by the chain rule
\[
\nabla V^x = \int_0^t \varphi'(Y_s) dY_s + \int_0^t \varphi(Y_s) dB \left[ Y_s \right]
\]
now, using \( (Y_s^x)^{\prime} = (Y_s^x)^{\prime} \), a classical but rather tedious stochastic computation yields
\[
\Gamma[V] = \hat{\mathbb{E}}[V^2] = \sum_{0 \leq \alpha \leq t} \Delta Y^2_\alpha \left( \int_{[\alpha]} \varphi'(Y_s) dY_s + \varphi(Y_\alpha) \right)^2.
\]
(13)
Since \( V \) has real values the energy image density property holds, and \( V \) has a density as soon as \( \Gamma[V] \) is strictly positive a.s. what may be discussed using the relation (13).

(b) Another more direct way consists in applying the theorem. For this we define \( \flat \) by choosing \( \eta \) such that
\[
\int_0^1 \eta(r) dr = 0 \text{ and } \int_0^1 \eta^2(r) dr = 1
\]
and putting \( f^{\flat} = xf'(x) \eta(r) \).

1°. First step. We add a particle \( (\alpha, x) \) i.e. a jump to \( Y \) at time \( \alpha \) with size \( x \) what gives
\[
\varepsilon + V = \varphi(Y_\alpha) x + \int_{[\alpha]} \varphi'(Y_s) dY_s + \varphi(Y_\alpha)
\]
2°. \( V^x = 0 \) since \( V \) does not depend on \( x \), and \( (\epsilon + V)^x = (\varphi(Y_\alpha) x + \int_{[\alpha]} \varphi'(Y_s + x) dY_s)\eta(r) \) because \( x^{\flat} = x\eta(r) \).

3°. We compute \( \gamma[\epsilon + V] = \int (\epsilon + V)^2 dr = (\varphi(Y_\alpha) x + \int_{[\alpha]} \varphi'(Y_s + x) dY_s)^2 \).

4°. We take back the particle we gave, because in order to compute \( \int \gamma[\epsilon + V] dN \) the integral in \( N \) confuses \( \epsilon + \omega \) and \( \omega \). That gives \( \int \gamma[\epsilon + V] dN = \int (\varphi(Y_\alpha) + \int_{[\alpha]} \varphi'(Y_s) dY_s)^2 N(1) d(\alpha, x) \) and (13).

We remark that both operators \( F \mapsto \epsilon + F, \ F \mapsto (\epsilon + F)^x \) are non-local, but instead \( F \mapsto \int (\epsilon + F)^{\flat} d(N \odot \rho) \) and \( F \mapsto \int \gamma[\epsilon + F] dN \) are local: taking back the lent particle gives the locality.

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