UNIQUENESS FOR TWO DIMENSIONAL INCOMPRESSIBLE IDEAL FLOW ON SINGULAR DOMAINS

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Abstract. The existence of a solution to the two dimensional incompressible Euler equations in singular domains was established in [Gérard-Varet and Lacave, The 2D Euler equation on singular domains, submitted]. The present work is about the uniqueness of such a solution when the domain is the exterior or the interior of a simply connected set with corners, although the velocity blows up near these corners. In the exterior of a curve with two end-points, it is showed in [Lacave, Two Dimensional Incompressible Ideal Flow Around a Thin Obstacle Tending to a Curve, Ann. IHP, Anl 26 (2009), 1121-1148] that this solution has some interesting properties, as to be seen as a special vortex sheet. Therefore, we prove the uniqueness, whereas the problem of general vortex sheets is open.

1. Introduction

The motion of a two dimensional flow can be described by the velocity \( u(t, x) = (u_1, u_2) \) and the pressure \( p \). Concerning incompressible ideal flow in an open set \( \Omega \), the pair \((u, p)\) verifies the Euler equations:
\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= 0, & t > 0, x \in \Omega \\
\text{div} u &= 0, & t > 0, x \in \Omega
\end{align*}
\]
endowed with an initial condition and an impermeability condition at the boundary \( \partial \Omega \):
\[
\begin{align*}
u|_{t=0} &= u_0, & u \cdot \hat{n}|_{\partial \Omega} = 0.
\end{align*}
\]
The vorticity \( \omega \) defined by
\[
\omega := \text{curl} u = \partial_1 u_2 - \partial_2 u_1
\]
plays a crucial role in the study of the ideal flow, thanks to the transport nature governing it:
\[
\partial_t \omega + u \cdot \nabla \omega = 0.
\]

When \( \Omega \) and \( u_0 \) are smooth, the well-posedness of system (1.1)-(1.2) has been of course the matter of many works. Starting from the paper of Wolibner in bounded domains [21], McGrath treated the case of the full plane [17], and finally Kikuchi studied the exterior domains [8]. In the case where the vorticity is only assumed to be bounded, existence and uniqueness of a weak solution has been established by Yudovich in [22]. We quote that the well-posedness result of Yudovitch applies to smooth bounded domains, and to unbounded ones under further decay assumptions.

We stress that all above studies require \( \partial \Omega \) to be at least \( C^{1,1} \). Roughly, the reason is the following: due to the non-local character of the Euler equation, these works rely on global in space estimates of \( u \) in terms of \( \omega \). These estimates \textit{up to the boundary} involve Biot and Savart type kernels, corresponding to operators such as \( \nabla \Delta^{-1} \). Unfortunately, such operators are known to behave badly in general non-smooth domains. This explains why well-posedness results are dedicated to regular domains.

However the case of a singular obstacle is physically relevant. For example, the study of the perturbation created by a plane wing stays a capital issue to determine the safety time between two landings in big airports.

Without solving the question of uniqueness, Taylor established in [19] the existence of a global weak solution of (1.1)-(1.2) in a bounded sharp convex domain. He used that \( \Omega \) convex implies that the solution \( v \) of the Dirichlet problem
\[
\Delta v = f \text{ in } \Omega, \quad v|_{\partial \Omega} = 0
\]
belongs to $H^2(\Omega)$ when the source term $f$ belongs to $L^2(\Omega)$, irrespective of the domain regularity. Nevertheless, this interesting result still leaves aside many situations of practical interest, notably flows around irregular obstacles. Recently, the article \[10\] gave such a result in the exterior of a $C^2$ Jordan arc, where it is noted that the velocity blows up near the end-points of the arc. In particular, it shows that the previous property on the Dirichlet problem is false in domains with some bad corners.

The question of the existence of global weak solutions is now solved for a large class of singular domains in \[8\]. The authors therein considered two kinds of domains: any open bounded domain where we retrieve a fixed (possibly zero) number of closed sets with positive capacity, and any exterior domain of one connected closed set with positive capacity.

Our goal here is to prove that such a solution is unique if the domain is bounded, simply connected with some corners, or if it is the complementary of a closed simply connected bounded set with some corners. We prove the uniqueness for an initial vorticity which is bounded, compactly supported in $\Omega$ and having a definite sign.

More precisely, we consider two kinds of domains. On one hand, we denote by $\Omega$ a bounded, simply connected open set, such that $\partial \Omega$ has a finite number of corners $z_i$ with angles $\alpha_i$ (i.e. locally, $\Omega$ coincides with the sector $\{ z_i + (r \cos \theta, r \sin \theta) ; r > 0, \theta_i < \theta < \theta_i + \alpha_i \}$). On the other hand, we denote by $\Omega := \mathbb{R}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is a bounded, simply connected closed set, such that $\partial \Omega$ has a finite number of corners.

To define a global weak solution to the Euler equation, let us point out that the space $L^2(\Omega)$ is not suitable for weak solutions in unbounded domain. Working with square integrable velocities in exterior domains is too restrictive (see page \[8\] to note that $u$ behaves in general like $1/|x|$ at infinity), so we consider initial data satisfying

$$u_0 \in L^2_{\text{loc}}(\Omega), \quad u_0 \to 0 \quad \text{as} \ |x| \to +\infty, \quad \text{curl} \ u_0 \in L^\infty(\Omega), \quad \text{div} \ u_0 = 0, \quad u_0 \cdot \hat{n}|_{\partial \Omega} = 0. \quad (1.4)$$

Note that the divergence free condition and this last impermeability condition have to be understood in the weak sense: for any $\varphi \in C^1_c(\Omega)$,

$$\int_{\Omega} u_0 \cdot \nabla \varphi = - \int_{\Omega} \text{div} \ u_0 \varphi = 0. \quad (1.5)$$

Let us stress that this set of initial data is large: we will show later that for any function $\omega_0 \in L^\infty(\Omega)$, there exists $u_0$ verifying (1.4) with $\text{curl} \ u_0 = \omega_0$.

Similarly, the weak form of the divergence free and tangency conditions on the Euler solution $u$ will read:

$$\forall \varphi \in \mathcal{D} \left( [0, +\infty); C^1_c(\mathbb{R}^2) \right), \quad \int_{\mathbb{R}^+} \int_{\Omega} u \cdot \nabla \varphi = 0. \quad (1.6)$$

Finally, the weak form of the momentum equation on $u$ will read:

$$\forall \varphi \in \mathcal{D} \left( [0, +\infty]\times \Omega \right) \text{ with div } \varphi = 0, \quad \int_0^\infty \int_{\Omega} (u \cdot \partial_t \varphi + (u \otimes u) : \nabla \varphi) = - \int_{\Omega} u_0 \cdot \varphi(0, \cdot). \quad (1.7)$$

For $\Omega$ an open bounded simply connected domain, or $\Omega$ the complementary of a compact simply connected domain $\mathcal{C}$, we get the existence of a weak solution from \[3\]:

**Theorem 1.1.** Assume that $u_0$ verifies (1.4). Then there exists

$$u \in L^\infty([0, +\infty); L^2_{\text{loc}}(\Omega)), \quad \text{curl} \ u \in L^\infty([0, +\infty); L^1 \cap L^\infty(\Omega)), \quad (1.1)-(1.2)$$

which is a global weak solution of (1.1)-(1.2) in the sense of (1.6) and (1.7).

In a few words, this existence result follows from a compactness argument, performed on a sequence of solutions $u_n$ of the Euler equations on the sequence of approximating domains $\Omega_n$. A key ingredient of the proof is the so-called $\Gamma$-convergence of $\Omega_n$ to $\Omega$ (see \[3\] for the details).

The main result of this article concerns the uniqueness of global weak solutions, when the initial vorticity has definite sign.
Theorem 1.2. Let $\Omega$ be a bounded, simply connected open set, such that $\partial \Omega$ has a finite number of corners with angles greater than $\pi/2$ and let $u_0$ verifying (1.4). If $\text{curl} u_0$ is non-positive (respectively non-negative), then there exists a unique global weak solution of the Euler equations on $\Omega$ verifying

$$u \in L_\text{loc}^\infty(\mathbb{R}^+; L^2_\text{loc}(\Omega)), \text{curl} u \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega)) \bigg).$$

In exterior domains, the vorticity is not sufficient to uniquely determine the velocity. We need the circulation around $C$. As we will see in Subsection 2.3 for $u_0$ verifying (1.4), we can define the initial circulation:

$$\gamma_0 := \oint_{\partial C} u_0 \cdot \tau \, ds.$$

Inversely, let us mention that we can fix independently the vorticity and the circulation: we will show that for any function $\omega_0 \in L^\infty(\Omega)$ and any real number $\gamma \in \mathbb{R}$, there exists a unique $u_0$ verifying (1.4) with $\text{curl} u_0 = \omega_0$ and with circulation around $C$ equal to $\gamma$.

Assuming a sign condition on $\gamma_0$, we will prove a uniqueness theorem in exterior domains.

Theorem 1.3. Let $\Omega := \mathbb{R}^2 \setminus C$, where $C$ is a compact, simply connected set, such that $\partial \Omega$ has a finite number of corners with angles greater than $\pi/2$. Let $u_0$ verifying (1.4). If $\text{curl} u_0$ is non-positive and $\gamma_0 \geq -\int \text{curl} u_0$ (respectively $\text{curl} u_0$ non-negative and $\gamma_0 \leq -\int \text{curl} u_0$), then there exists a unique global weak solution of the Euler equations on $\Omega$, verifying

$$u \in L_\text{loc}^\infty(\mathbb{R}^+; L^2_\text{loc}(\Omega)), \text{curl} u \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega)) \bigg).$$

In particular, we will also prove that the velocity blows up near the obtuse corners: if $\partial \Omega$ admits at $z_0$ a corner of angle $\alpha$, then the velocity behaves near $z_0$ like $1/|x - z_0|^{1-\pi}$.

The remainder of this work is organized in six sections. We introduce in Section 2 the biholomorphism $T$ and the Biot-Savart law (law giving the velocity in terms of the vorticity) in the interior or the exterior of one simply connected domain. We will recall the existence of weak solution in this section, and derive some formulations (on vorticity and on extensions in $\mathbb{R}^2$). We will take advantage of this section to show that the weak solution is a renormalized solution in the sense of DiPerna-Lions [1], which will allow us to prove that the $L^p$ norm of the vorticity for $p \in [1, \infty]$, the total mass $\int_\Omega \omega(t, \cdot)$ and the circulation of the velocity around $C$ are conserved quantities.

Let us mention that the explicit form of the Biot-Savart law is one of the key of this work, and it explains why $\Omega$ is assumed to be the interior or the exterior of a simply connected domain. This law will read

$$u(t, x) = DT(x)^T R[\omega]$$

where $R[\omega]$ is an integral operator. Using classical elliptic theory, we will obtain the exact behavior of the biholomorphism $T$ near the corners, and then the behavior of the velocity. We note that the blow-up is stronger if the angle $\alpha$ is bigger. Unfortunately, the following study needs sometimes that the integral operator $R[\omega]$ verifies good estimates, which are possible only if we assume that all the angles $\alpha_i$ are greater than $\pi/2$ (namely in Proposition 2.3 to prove that $R[\omega]$ is bounded, in Lemma 2.7 to establish the equation verified by the extended functions, in Lemma 2.8 to use the renormalization theory).

Section 3 is the central part of this paper: we will prove that the support of $\omega$ never meets the boundary if we assume that the characteristics corresponding to (1.3) exist and are differentiable. The idea is to introduce a good Liapounov function, which blows up if the trajectories meet the boundary. Next, we will establish some estimates implying that this Liapounov energy is bounded which will
give the result. Although we cannot say that the characteristics are regular for weak solutions, this computation gives us an excellent intuition.

In light of this proof, we rigorously prove in Section 4 thanks to the renormalization theory, that we have the same property, even if we do not consider the characteristics.

Finally, we prove Theorems 1.2, 1.3 in Section 5. We will introduce \( v := K_{\mathbb{R}^2} * \omega \), where \( K_{\mathbb{R}^2} \) is the Biot-Savart kernel in the full plane. As \( \omega \) does not meet the boundary, it means that \( \text{div} \, v = \text{curl} \, v \equiv 0 \) in a neighborhood of the boundary, i.e. \( v \) is harmonic therein. This provides in particular a control of its \( L^\infty \) norm (as well as the \( L^2 \) norm for the gradient) by its \( L^2 \) norm. Although the total velocity is not bounded near the boundary, but just integrable, this argument allows us to yield a Gronwall-type estimate, as Yudovich did.

Therefore, the fact that the support of the vorticity stays far from the boundary will imply the uniqueness result. This idea was already used in [13], in the case of one Dirac mass in the vorticity. In this article, we consider the Euler equations in \( \mathbb{R}^2 \) and we have to take care of integrability at infinity, to control the size of the support of the vorticity. This explicit formula will be used to construct the Liapounov function. We give in the following subsection the properties of this Riemann mapping.

Therefore, the fact that the support of the vorticity stays far from the boundary will imply the uniqueness result. This idea was already used in [13], in the case of one Dirac mass in the vorticity. In this article, we consider the Euler equations in \( \mathbb{R}^2 \) and a Dirac mass. The equation is called the system mixed Euler/point vortex and derived in [16]. When trajectories exist, it is proved that they do not meet the point vortex in [16] if the point vortex moves on the influence of the regular part, and in [15] if the Dirac is fixed. The method is also constructed on Liapounov functions. An important issue in [13] is to generalize this result when trajectories are not regular. The Lagrangian formulation gives us a helpful intuition, it is the reason why we choose first to present the proof of uniqueness assuming the differentiability of trajectories (Section 3). Moreover, proving in Section 4 that the vorticity never meets the boundary, we state that the “weak” Lagrangian flow coming from the renormalization theory evolves in the area far from the corners. As the velocity is regular enough in this region, we can conclude that the flow is actually classical and regular.

Section 6 is devoted to the proofs of some technical lemmas.

We finish this article by Section 7 with some final comments. In the exterior of the wing plane, we will try to give a mathematical justification of the Kutta-Joukowski condition. In the exterior of the Jordan arc (see [10]), we will make a parallel with the vortex sheet problem. We will also give some explanations about the sign assumptions in the main theorems.

We warn the reader that we write in general the proofs in the case of exterior domains. In this kind of domain, we have to take care of integrability at infinity, to control the size of the support of the vorticity, and we have also to consider harmonic vector fields and circulations of velocities around \( C \). The proofs in the case of bounded domains are strictly easier, without additional arguments. We will make sometimes some remarks about that.

2. BIOT-SAVART LAW AND EXISTENCE

As in [7, 10, 12], the crucial assumption is that we work in dimension two outside (or inside) one simply connected domain. Identifying \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \), there exists a biholomorphism \( T \) mapping \( \Omega \) to the exterior (resp. to the interior) of the unit disk. Thanks to this biholomorphism, we will obtain an explicit formula for the Biot-Savart law: the law giving the velocity in terms of \( \omega \). This explicit formula will be used to construct the Liapounov function. We give in the following subsection the properties of this Riemann mapping.

2.1. Conformal mapping.

Let \( \Omega \) as in Theorem 1.3 (resp. as in Theorem 1.2), then the Riemann mapping theorem states that there exists a unique biholomorphism \( T \) mapping \( \Omega \) to \( B(0,1) \) (resp. to \( B(0,1) \)) such that \( T(\infty) = \infty \) and \( T'(\infty) \in \mathbb{R}_+^* \) (resp. \( T(0) = 0 \) and \( T'(0) \in \mathbb{R}_+^* \), for a \( z_0 \in \Omega \)). We remind that the last two conditions mean

\[
T(z) \sim \lambda z, \quad \lim_{z \to +\infty} z \sim +\infty, \quad \text{for some } \lambda > 0.
\]

**Theorem 2.1.** Let assume that \( \partial \Omega \) is a \( C^\infty \) Jordan curve, except in a finite number of point \( z_1, z_2, ..., z_n \) where \( \partial \Omega \) admits corner of angle \( \alpha_i > \frac{\pi}{2} \) (i.e. \( \Omega \) coincides locally with the sector \( \{z_i + (r \cos \theta, r \sin \theta); r > 0, \theta_i < \theta < \theta_i + \alpha_i\} \)). Then the biholomorphism \( T \) defined above satisfies
\[ \begin{align*}
\textbullet & \quad \mathcal{T}^{-1} \text{ and } \mathcal{T} \text{ extend continuously up to the boundary;} \\
\textbullet & \quad DT^{-1} \text{ extends continuously up to the boundary, except at the points } \mathcal{T}(z_i) \text{ with } \alpha_i < \pi \text{ where } DT^{-1} \text{ behaves like } 1/|y - \mathcal{T}(z_i)|^{1 - \frac{\alpha_i}{\pi}}; \\
\textbullet & \quad DT \text{ extends continuously up to the boundary, except at the points } z_i \text{ with } \alpha_i > \pi \text{ where } DT \text{ behaves like } 1/|x - z_i|^{1 - \frac{\alpha_i}{\pi}}; \\
\textbullet & \quad D^2\mathcal{T} \text{ belongs to } L^p_{\text{loc}}(\Omega) \text{ for any } p < 4/3.
\end{align*} \]

Proof. As \( \partial \Omega \) is \( C^{0,\alpha} \), the Kellogg-Warschawski theorem (Theorem 3.6 in [18]) states directly that \( \mathcal{T} \) and \( \mathcal{T}^{-1} \) is continuous up to the boundary. For the behavior of the derivatives, we use the classical elliptic theory: let 
\[ u(x) := \ln |\mathcal{T}(x)|. \]
As \( \mathcal{T} \) is holomorphic, we have that 
\[ \Delta u = 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial \Omega. \]
To localize near each corners, we can introduce a smooth cutoff function \( \chi \) supported in a small neighborhood of \( z_i \). Therefore, we are exactly in the setting of elliptic studies:
\[ \Delta(u\chi) = f \in C^\infty \text{ in } O_i \text{ and } u = 0 \text{ on } \partial O_i, \]
where \( O_i \) is the sector \( \{ z_i + (r \cos \theta, r \sin \theta); r > 0, \theta_i < \theta < \theta_i + \alpha_i \} \). The standard idea is to compose by \( z^{\pi/\alpha_i} \) in order to maps the sector on the half plane, where the solution of the elliptic problem \( g \) is smooth. Therefore, we have that 
\[ u\chi = g \circ z^{\pi/\alpha_i}, \]
which implies that 
\[ \nabla u \approx r^{\pi/\alpha_i - 1} \text{ and } \nabla^2 u \approx r^{\pi/\alpha_i - 2}. \]

More precisely, we used the so-called shift theorem in non-smooth domain (see the preface of [14]): there exist numbers \( c_k \) such that 
\[ u\chi - \sum c_k v_k \in W^{m+2,p}(O_i \cap B(0,R)), \quad \forall R > 0 \]
where the \( k \) in the summation ranges over all integers such that 
\[ \pi/\alpha_i \leq k\pi/\alpha_i < m + 2 - 2/p \]
and with 
\[ \textbullet v_k = r^{k\pi/\alpha_i} \sin(k\pi\theta/\alpha_i) \text{ if } k\pi/\alpha_i \text{ is not an integer}; \]
\[ \textbullet v_k = r^{k\pi/\alpha_i} \lfloor r \sin(k\pi\theta/\alpha_i) + \theta \cos(k\pi\theta/\alpha_i) \rfloor \text{ if } k\pi/\alpha_i \text{ is an integer.} \]
In this theorem, \( r \) denotes the distance between \( x \) and \( z_i \): \( r := |x - z_i| \).

We apply it for \( m = 1 \) and \( p = 2 \). As \( H^1_{\text{loc}}(\mathbb{R}^2) \) embeds in \( C^0 \), we see again that \( u \) is continuous up to the boundary.

If \( \pi < \alpha_i \leq 2\pi \) then \( 1/2 \leq \pi/\alpha_i < 1 \), which gives that \( \pi/\alpha_i \) cannot be an integer. Then, the shift theorem states that \( D(u\chi) - \sum c_k Dv_k \) belongs to \( H^1_{\text{loc}}(\Omega_i) \), so it belongs to \( C^0 \). Thanks to formula of \( v_k \), we see that \( Du \) is continuous up to the boundary, except near \( z_i \) where \( Du = O(r^{\pi/\alpha_i - 1}) \). Next, we derive once more to obtain that \( D^2(u\chi) - \sum c_k D^2v_k \) belongs to \( H^1_{\text{loc}}(\Omega_i) \), so it belongs to \( L^p_{\text{loc}}(\Omega_i) \) for any \( p \). As \( \sum c_k D^2v_k = O(r^{\pi/\alpha_i - 2}) \), with \( 2 - \pi/\alpha_i < 3/2 \), then \( D^2u \) belongs to \( L^p_{\text{loc}}(\Omega_i) \) for any \( p < 4/3 \).

The case \( \alpha_i = \pi \) is not interesting because we assume that \( z_i \) is a singular point.
If \( \pi/2 < \alpha_i < \pi \), then we note that \( \pi/\alpha_i \) is not an integer and that \( k\pi/\alpha_i < 2 \) is obtained only for \( k = 1 \). We apply the above argument to see that \( u \) and \( Du \) is continuous up to the boundary, and \( D^2u \) belongs to \( L^p_{\text{loc}}(\Omega_i) \) for any \( p < 2 \).

Therefore, the shift theorem establishes rigorously that \( u = O(r^{\pi/\alpha_i}) \) and \( Du = O(r^{\pi/\alpha_i - 1}) \) if all the angles are greater than \( \pi/2 \). We show now that \( Du \) and \( DT \) have the same behavior.
On one hand, differentiating \( u \), we have
\[
\nabla u(x) = \frac{T(x)}{|T(x)|^2} DT(x)
\]
hence
\[
|\nabla u(x)|_\infty \leq 4|DT(x)|_\infty
\] (2.3)
where \(|A|_\infty = \max |a_{ij}|. Indeed, by continuity of \( T \), we have that \(|T(x)| = \sqrt{T_1^2(x) + T_2(x)^2} \geq 1/2\) near the boundary.

On the other hand,
\[
\frac{T(x)}{|T(x)|^2} = \nabla u(x) DT(x)^{-1}.
\]
By continuity of \( T \), there exists a neighborhood of \( \partial \Omega \) such that \(|T(x)| \leq 2\). Then, near the boundary, we have
\[
\frac{1}{2} \leq \frac{1}{|T(x)|} \leq 2\sqrt{2} |\nabla u(x)|_\infty |DT(x)^{-1}|_\infty.
\]
Moreover, as \( T \) is holomorphic, \( DT \) is a matrix \( 2 \times 2 \) on the form \( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \). We deduce from this form that \( DT(x)^{-1} = \frac{1}{\det DT(x)} DT(x)^T \). We use that \( \det DT(x) = a^2 + b^2 \geq |DT(x)|_\infty^2 \) to get
\[
|DT(x)|_\infty \leq 4\sqrt{2} |\nabla u(x)|_\infty.
\] (2.4)
Putting together (2.2), (2.3) and (2.4), we can conclude on the behavior of \( DT \).

Differentiating once more, we obtain the result for \( D^2 T \).

Finally, as \( u = O(r^{\pi/\alpha_i}) \), we state that
\[
|T(x)| = 1 + O(r^{\pi/\alpha_i}), \quad T(x) = T(z_i) + O(|x - z_i|^{\pi/\alpha_i}), \quad T^{-1}(y) = z_i + O(|y - T(z_i)|^{\alpha_i/\pi}).
\]
Next, we use the fact that \( DT(x) = O(|x - z_i|^{\pi/\alpha_i^{-1}}) \) to write
\[
DT^{-1}(y) = \left( DT(T^{-1}(y)) \right)^{-1} = O\left( \frac{1}{(|y - T(z_i)|^{\alpha_i/\pi})^{1/\alpha_i^{-1}}} \right) = O(|y - T(z_i)|^{\alpha_i/\pi})
\]
which ends the proof.

We refine the result of the exterior of the curve (see [10]): \( \alpha = 2\pi \) gives that \( DT \) behaves like \( 1/\sqrt{|x|} \). In that paper, we found the behavior of \( DT \) thanks to the explicit formula of \( T \). The Joukowski function \( G(z) = \frac{1}{2}(z + \frac{1}{z}) \) maps the exterior of the unit disk to the exterior of the segment \([-1,0), (1,0)]\). Then, in the case of this segment \( T = G^{-1} \) and we can compute that
\[
DT(z) = z \pm \frac{z}{\sqrt{z^2 - 1}}.
\]
We also note that \( DT \) near a corner (\( \alpha > \pi \)) is less singular than around a cusp (as the intuition).

Remark 2.2: This kind of theorem will be useful to remark that the velocity in the exterior of a square blows-up like \( 1/|x|^{1/3} \) near the corner. However, the only things that we need in the sequel are:

- There exists \( p_0 > 2 \) such that \( \det DT^{-1} \) belongs to \( L^{p_0}_{\text{loc}}(\Omega) \): property holding true if all the corners \( z_i \) have angles \( \alpha_i \) greater than \( \pi/2 \) (as in Theorems [12],[13],[14]);
- \( DT \) belongs to \( L^p_{\text{loc}}(\Omega) \) for any \( p < 4 \) and \( D^2 T \) belongs to \( L^p_{\text{loc}}(\Omega) \) for any \( p < 4/3 \).

Therefore, Theorems [12],[13] can be applied for any simply connected domain (or exterior of a simply connected set) such that the two previous points hold true. For a sake of clarity, we express the theorems when the boundary is locally a corner at \( z_i \), but we can generalize for \( \partial \Omega \) such that \( \partial \Omega \) is a \( C^{1,1} \) Jordan curve except at a finite number of points \( z_i \). In these points, we would define
\[
\alpha_i := \lim_{s \to 0} \arg(\Gamma'(s_i + s), \Gamma'(s_i - s)) + \pi,
\]
where \( \Gamma \) is a parametrization of \( \partial \Omega \) and \( z_i = \Gamma(s_i) \). Indeed, up to a smooth change of variable, the Laplace equation in \( \Omega \) [2] turns into a divergence form elliptic equation in the exterior of a corner, and we would use results related to elliptic equations in polygons, see [9].
The previous theorem is about the behavior near the obstacle. In the case of an unbounded domain (as in Theorem [33]), we will need the following proposition about the behavior of $\mathcal{T}$ at infinity.

**Proposition 2.3.** If $\mathcal{T}$ is a biholomorphism from $\Omega$ to the exterior of the unit disk such that $\mathcal{T}(\infty) = \infty$ and $\mathcal{T}'(\infty) \in \mathbb{R}_{+}^{*}$, then there exist $(\beta, \tilde{\beta}) \in \mathbb{R}_{+}^{*} \times \mathbb{C}$ and a holomorphic function $h : \Omega \rightarrow \mathbb{C}$ such that

$$\mathcal{T}(z) = \beta z + \tilde{\beta} + h(z)$$

with

$$h(z) = \mathcal{O}\left(\frac{1}{|z|}\right) \text{ and } h'(z) = \mathcal{O}\left(\frac{1}{|z|^2}\right), \text{ as } |z| \to \infty.$$ 

Moreover, $\mathcal{T}^{-1}$ admits a similar development.

**Proof.** We consider $E := \mathcal{T}^{-1}(B(0,0) \setminus B(0,1)) \cup C$, which is an open, bounded, connected, simply connected and smooth subset of the plane. Then, the map $H := \mathcal{T}/2$ is a biholomorphism between $E^c$ and $B(0,1)^c$, and we can apply Remark 2.5 of [10] to end this proof. $\Box$

### 2.2. Biot-Savart Law.

One of the keys of the study for two dimensional ideal flow is to work with the vorticity equation, which is a transport equation. For example, in the case of a smooth obstacle, if we have initially $\omega_0 := \text{curl} u_0 \in L^1 \cap L^\infty$, then $||\omega(t,\cdot)||_{L^p} = ||\omega_0||_{L^p}$ for all $t, p$. So, we have some estimates for the vorticity, and the goal is to establish estimates for the velocity. For that, we introduce the Biot-Savart law, which gives the velocity in terms of the vorticity. Another advantage of the two dimensional space is that we have explicit formula in the exterior of one obstacle, thanks to complex analysis and the identification of $\mathbb{R}^2$ and $\mathbb{C}$.

Let $\Omega$ be the exterior (resp. the interior) of a bounded, closed, connected, simply connected subset of the plane, the boundary of which is a Jordan curve. Let $\mathcal{T}$ be a biholomorphism from $\Omega$ to $(\overline{B(0,1)})^c$ (resp. $B(0,1))^c$) such that $\mathcal{T}(\infty) = \infty$ (resp. $\mathcal{T}(z_0) = 0$).

We denote by $G_{\Omega} = G_{\Omega}(x,y)$ the Green’s function, whose the formula is:

$$G_{\Omega}(x,y) = \frac{1}{2\pi} \ln \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|}$$

writing $x^* = \frac{x}{|x|^2}$. The Green’s function verifies:

$$\Delta_y G_{\Omega}(x,y) = \delta(y-x) \ \forall x,y \in \Omega, \quad G_{\Omega}(x,y) = 0 \ \forall (x,y) \in \Omega \times \partial \Omega, \quad G_{\Omega}(x,y) = G_{\Omega}(y,x) \ \forall x,y \in \Omega.$$ 

The kernel of the Biot-Savart law is $K_{\Omega} = K_{\Omega}(x,y) := \nabla_{x}G_{\Omega}(x,y)$. With $(x_1, x_2)^\perp = \left(\frac{-x_2}{x_1}\right)$, the explicit formula of $K_{\Omega}$ is given by

$$K_{\Omega}(x,y) = \frac{1}{2\pi} \mathcal{D} \mathcal{T}(x) \left( \frac{(\mathcal{T}(x) - \mathcal{T}(y))^\perp}{|\mathcal{T}(x) - \mathcal{T}(y)|^2} - \frac{(\mathcal{T}(x) - \mathcal{T}(y)^*)^\perp}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2} \right)$$

and we introduce the notation

$$K_{\Omega}[f] = K_{\Omega}[f](x) := \int_{\Omega} K_{\Omega}(x,y) f(y) dy,$$

with $f \in C^\infty_c(\Omega)$.

In unbounded domain, we require information on far-field behavior of $K_{\Omega}$. We will use several times the following general relation:

$$\left| \frac{a}{|a|^2} - \frac{b}{|b|^2} \right| = \frac{|a - b|}{|a||b|},$$

which can be easily checked by squaring both sides. Using the behavior of $\mathcal{D} \mathcal{T}$ at infinity (Proposition 2.3), we obtain for large $|x|$ that

$$|K_{\Omega}[f](x)| \leq \frac{C_2}{|x|^2},$$

where $C_2$ depends on the size of the support of $f$. 

The vector field $u = K_\Omega[f]$ is a solution of the elliptic system:
\[
\begin{align*}
\text{div } u &= 0 \text{ in } \Omega, \\
\text{curl } u &= f \text{ in } \Omega, \\
 u \cdot \hat{n} &= 0 \text{ on } \partial \Omega, \\
\lim_{|x| \to \infty} |u| &= 0.
\end{align*}
\]

If we consider a non-simply connected domain (as in Theorem 1.3), the previous system has several solutions. To uniquely determine the solution, we have to take into account the circulation. Let $\hat{n}$ be the unit normal exterior to $\Omega$. In what follows all contour integrals are taken in the counter-clockwise sense, so that $\oint_{\partial \mathcal{C}} F \cdot \hat{\tau} \, ds = -\oint_{\partial \mathcal{C}} F \cdot \hat{n} \, ds$. Then the harmonic vector field

\[H_\Omega(x) = \frac{1}{2\pi} \nabla \ln |T(x)| = \frac{1}{2\pi} DT^T(x) \frac{T(x) \perp}{|T(x)|^2}\]

is the unique vector field verifying

\[\text{div } H_\Omega = \text{curl } H_\Omega = 0 \text{ in } \Omega, \quad H_\Omega \cdot \hat{n} = 0 \text{ on } \partial \mathcal{C}, \quad H_\Omega(x) \to 0 \text{ as } |x| \to \infty, \quad \oint_{\partial \mathcal{C}} H_\Omega \cdot \hat{\tau} \, ds = 1.\]

Using Proposition 2.3 we see that $H_\Omega(x) = \mathcal{O}(1/|x|)$ at infinity. Therefore, putting together the previous properties, we obtain the existence part of the following.

**Proposition 2.4.** Let $\omega \in L^\infty(\Omega)$ and $\gamma \in \mathbb{R}$. If $\Omega$ is an open simply connected bounded subset of $\mathbb{R}^2$, then there is a unique solution $u$ of

\[
\begin{align*}
\text{div } u &= 0 \quad \text{in } \Omega \\
\text{curl } u &= \omega \quad \text{in } \Omega \\
u \cdot \hat{n} &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

which is given by

\[u(x) = K_\Omega[\omega](x).\] (2.8)

If $\mathcal{C}$ is a closed simply connected bounded subset of $\mathbb{R}^2$ and $\Omega = \mathbb{R}^2 \setminus \mathcal{C}$, then there is a unique solution $u$ of

\[
\begin{align*}
\text{div } u &= 0 \quad \text{in } \Omega \\
\text{curl } u &= \omega \quad \text{in } \Omega \\
u \cdot \hat{n} &= 0 \quad \text{on } \partial \mathcal{C} \\
u(x) &= 0 \text{ as } |x| \to \infty \\
\oint_{\partial \mathcal{C}} u \cdot \hat{\tau} \, ds &= \gamma
\end{align*}
\]

which is given by

\[u(x) = K_\Omega[\omega](x) + (\gamma + \int \omega) H_\Omega(x).\] (2.9)

Concerning the uniqueness, we can see e.g. [8, Lemma 2.14] (see also [7, Proposition 2.1]).

We take advantage of this explicit formula to give estimates on the kernel. We introduce

\[R[\omega](x) := \int_\Omega \left( \frac{(T(x) - T(y)) \perp}{|T(x) - T(y)|^2} - \frac{(T(x) - T(y)) \perp}{|T(x) - T(y)|^2} \right) \omega(y) \, dy,
\]

so that (2.9) reads

\[u(x) = \frac{1}{2\pi} \int DT^T(x) \left( R[\omega](x) + (\gamma + \int \omega) \frac{T(x) \perp}{|T(x)|^2} \right).
\]

**Proposition 2.5.** Let assume that $\omega$ belongs to $L^1 \cap L^\infty(\Omega)$. If all the angles of $\Omega$ are greater than $\pi/2$, then there exist $(C, a) \in \mathbb{R}^+ \times (0, 1/2]$ depending only on the shape of $\Omega$ such that

\[\|R[\omega]\|_{L^\infty(\Omega)} \leq C(\|\omega\|_{L^1}^{1/2} \|\omega\|_{L^\infty}^{1/2} + \|\omega\|_{L^1} \|\omega\|_{L^\infty}^{1-a} + \|\omega\|_{L^1}).
\]

Moreover, $R[\omega]$ is continuous up to the boundary.

---

[1] see e.g. [7].
In the case where \( C \) is a Jordan arc, the uniform bound is proved in [10, Lemma 4.2] and the continuity in [10, Proposition 5.7]. The proof here is almost the same, except that we have to take care that \( DT^{-1} \) is not bounded if there is an angle less than \( \pi \) (see Theorem 2.1). For completeness, we write the details in Section 6. In this proof, we can understand why we assume that the angles are greater than \( \pi/2 \): we need that \( \det DT^{-1} \) belongs in \( L^{p_0} \) for some \( p_0 > 2 \) (see Remark 2.2).

### 2.3. Existence and properties of weak solutions.

The goal of this subsection is to derive some properties about a weak solution obtained in Theorem 2.1 from [3]. We will also establish similar formulations verified by extensions on the full plane.

#### a) Weak solution in an unbounded domain.

We begin by the hardest case: let \( \Omega := \mathbb{R}^2 \setminus C \), where \( C \) is a bounded, simply connected closed set, such that \( \partial C \) is a \( C^\infty \) Jordan curve, except in a finite number of point \( z_1, z_2, \ldots, z_n \) where \( \partial \Omega \) admits corner of angle \( \alpha_i \). Then, there exists some pieces of the boundary which are smooth, implying that the capacity of \( C \) is greater than \( 0 \) (see e.g. [3, Proposition 6]). Therefore, Theorem 2.1 with our exterior domains is a direct consequence of [3, Theorem 2].

We know the existence of a global weak solution. We search now some features of such a solution. Let \( u_0 \) satisfying (1.4) and \( u \) be a global weak solution of (1.1)-(1.2) in the sense of (1.6) and (1.7) such that

\[
\gamma_0 := \int_J u_0 \cdot \hat{\tau} \, ds.
\]

Let us remind that \( u_0 \) satisfies (1.4), so that it belongs to \( W^{1,q}_\text{loc} \) for all finite \( q \), and so that the integral at the r.h.s. is well-defined. Moreover, \( \gamma_0 \) does not depend on the curve separating \( C \) and \( \text{supp} \, \omega_0 \) (thanks to the curl free condition near \( C \)). Passing to the limit, we obtain

\[
\gamma_0 = \int_{\partial C} u_0 \cdot \hat{\tau} \, ds. \tag{2.10}
\]

We have proven in the previous subsection that we can reconstruct the velocity in terms of the vorticity and the circulation:

\[
u_0(x) = K_\Omega[\omega_0](x) + (\gamma_0 + \int \omega_0)H_\Omega(x).
\]

From the definition of weak solution, we know that the quantities \( \|\omega(t, \cdot)\|_{L^1 \cap L^\infty(\Omega)} \) and \( \int \omega(t, \cdot) \) are bounded in \( \mathbb{R}^+ \). Moreover, we infer that the circulation

\[
\gamma(t) := \int_{\partial C} u(t, \cdot) \cdot \hat{\tau} \, ds
\]

is bounded locally in time. To show this estimate, first, we note that the previous integral is well defined putting

\[
\gamma(t) := \int_J u(t, \cdot) \cdot \hat{\tau} \, ds - \int_A \omega(t, \cdot) \, dx,
\]

with \( A = \Omega \cap (\text{bounded connected component of } \mathbb{R}^2 \setminus J) \). Indeed, thanks to the uniqueness part of Proposition 2.4 with \( \int_J u(t, \cdot) \cdot \hat{\tau} \, ds = g_0 \), we state that \( u \) can be written as in (2.9), and we deduce from Theorem 2.1 and Proposition 2.4 that \( \int_{\partial C} u(t, \cdot) \cdot \hat{\tau} \, ds \) is well defined.

Next, let \( K \) be a compact subset of \( \Omega \). In this subset, we know by the definition of \( T \) and Proposition 2.5 that \( K_\Omega[\omega(t, \cdot)](x) \) is uniformly bounded in \( \mathbb{R}^+ \times K \). Then there exist \( C_1, C_2 \) such that (2.9) implies

\[
C_1|\gamma(t)| \leq \|u(t, \cdot)\|_{L^2(K)} + C_2 + C_1\|\omega(t, \cdot)\|_{L^1(\Omega)},
\]
for any $t \in \mathbb{R}^+$ (we have $C_1 = \|H_\Omega\|_{L^2(K)}$). As $u$ belongs to $L^\infty_{\text{loc}}(\mathbb{R}^+; L^2(K))$ (see the definition of weak solution), then we have that

\[
\gamma \in L^\infty_{\text{loc}}([0, \infty)).
\]

Moreover, putting together this estimate of $\gamma$, Remark 2.2 and Proposition 2.5, then (2.9) gives that

\[
u \in L^p_{\text{loc}}([0, \infty); L^p(\bar{\Omega})), \quad \forall p < 4,
\]

which is an improvement compared to the definition of weak solution, because we control up to the boundary.

Let us derive a formulation verified by $\omega$.

First, we note that for any test function $\phi \in \mathcal{D}([0, \infty) \times \Omega; \mathbb{R})$, then $\psi := \nabla \phi$ belongs to the set of admissible test functions, and (1.7) reads

\[
\forall \phi \in \mathcal{D}([0, \infty) \times \Omega; \mathbb{R}), \quad \int_0^\infty \int_\Omega (\omega \cdot \partial_t \phi + \omega u \cdot \nabla \phi) = -\int_\Omega \omega_0 \phi(0, \cdot).
\]

Then, $(\omega, u)$ verifies the transport equation

\[
\partial_t \omega + u \cdot \nabla \omega = 0
\]

in the sense of distribution (2.13) in $\Omega$. We need a formulation on $\mathbb{R}^2$. For that, we denote by $\tilde{\omega}$ (respectively $\tilde{u}$) the extension of $\omega$ (respectively $\tilde{u}$) to $\mathbb{R}^2$ by zero in $\Omega^c$. Let us check that it verifies the transport equation for any test function $\phi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^2)$.

**Proposition 2.6.** Let $(\omega, u)$ a weak solution to the Euler equations in $\Omega$. Then, the pair of extension verifies in the sense of distribution

\[
\partial_t \omega + \tilde{u} \cdot \nabla \omega = 0, \quad \text{in } \mathbb{R}^2 \times (0, \infty),
\]

\[
\text{div } \tilde{u} = 0 \text{ and curl } \tilde{u} = \frac{g_{\omega, \gamma}(s) \delta_{\partial C}}{2}
\]

\[
\text{as } |x| \to \infty
\]

where $\delta_{\partial C}$ is the Dirac function along the curve and with

\[
g_{\omega, \gamma}(x) = \frac{u \cdot \nabla \tilde{\omega}}{|\tau|}
\]

\[
= \lim_{\rho \to 0^+} K_\Omega[\tilde{\omega}](x - \rho \hat{n}) + (\gamma + \int \tilde{\omega} H_\Omega(x - \rho \hat{n}) \cdot \hat{n})
\]

(2.16)

**Proof.** The third and fourth points are obvious. The second point is a classical computation concerning tangent vector fields: there is no additional term on the divergence, whereas it appears on the curl the jump of the tangential velocity (see e.g. the proof of Lemma 5.8 in [10]).

Concerning the first point, we have to consider the case of a test function whose the support meet the boundary. Let $\varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^2)$. We introduce $\Phi$ a non-decreasing function which is equal to 0 if $s \leq 1$ and to 1 if $s \geq 2$. Let

\[
\Phi^\varepsilon(x) := \Phi\left(\frac{|T(x)| - 1}{\varepsilon}\right).
\]

We note that

- it is a cutoff function of an $\varepsilon$-neighborhood of $C$, because $T$ is continuous up to the boundary (see Theorem 2.1),
- we have $\nabla \Phi^\varepsilon \cdot H_\Omega \equiv 0$, because $H_\Omega(x) = \sum_{i=1}^2 |T(x)| |T_i(x)|$ (see Subsection 2.2),
- the Lebesgue measure of the support of $\nabla \Phi^\varepsilon$ is $o(\sqrt{\varepsilon})$. Indeed the support of $\nabla \Phi^\varepsilon$ is contained in the subset $\{x \in \Omega \varepsilon | 1 + \varepsilon \leq |T(x)| \leq 1 + 2\varepsilon\}$. The Lebesgue measure can be estimated thanks to Remark 2.2

\[
\int_{1+\varepsilon \leq |T(x)| \leq 1+2\varepsilon} dx = \int_{1+\varepsilon \leq |z| \leq 1+2\varepsilon} |\det(\mathcal{D}^{-1})(z)|dz \leq \sqrt{\varepsilon} \|\det(\mathcal{D}^{-1})\|_{L^2(B(0,1+\varepsilon) \setminus B(0,1))},
\]

where the norm in the right hand side term tends to zero as $\varepsilon \to 0$ (by the dominated convergence theorem).
Another interesting property is the fact that the velocity is tangent to the boundary whereas $\nabla \Phi^\varepsilon$ is normal. Indeed, we claim the following.

**Lemma 2.7.** As $\omega$ belongs to $L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega))$ then

$$u \cdot \nabla \Phi^\varepsilon \to 0$$

strongly in $L^1(\mathbb{R}^2)$,

uniformly in time, when $\varepsilon \to 0$.

This property is not so obvious, because $|u \cdot \nabla \Phi^\varepsilon| \approx |DT|^2 \varepsilon R[\omega] \Phi^\varepsilon \left(\frac{|T(x)| - 1}{\varepsilon}\right)$ with $\|\Phi^\varepsilon \left(\frac{|T(x)| - 1}{\varepsilon}\right)\|_{L^1} = O(\varepsilon)$ (in the case where $DT^{-1}$ is bounded) and $DT$ blowing up. The perpendicular argument is crucial here and we use the explicit formula (2.3) to show the cancellation effect. This lemma is proved in the case where $C$ is a Jordan arc in [10] Lemma 4.6. For a sake of completeness, we give the general proof in Section 6.

As $\Phi^\varepsilon \varphi$ belongs to $C^{\infty}_c(\mathbb{R} \times \Omega)$ for any $\varepsilon > 0$, we can write that $(\omega, u)$ is a weak solution in $\Omega$:

$$\int_0^\infty \int_{\mathbb{R}^2} (\Phi^\varepsilon \varphi) \omega \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla (\Phi^\varepsilon \varphi) \cdot u \omega \, dx \, dt + \int_{\mathbb{R}^2} (\Phi^\varepsilon \varphi)(0, x) \omega_0(x) \, dx = 0.
$$

As $\omega \in L^\infty(L^1 \cap L^\infty)$, it is obvious that the first and the third integrals converge to

$$\int_0^\infty \int_{\mathbb{R}^2} \varphi \tilde{\omega} \, dx \, dt \text{ and } \int_{\mathbb{R}^2} \varphi(0, x) \tilde{\omega}_0(x) \, dx$$

as $\varepsilon \to 0$. Concerning the second integral, we have

$$\int_0^\infty \int_{\mathbb{R}^2} \nabla (\Phi^\varepsilon \varphi) \cdot u \omega \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^2} \varphi (\nabla \Phi^\varepsilon \cdot u) \omega \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \Phi^\varepsilon \nabla \varphi \cdot u \omega \, dx \, dt.$$

The first right hand side term tends to zero because $\nabla \Phi^\varepsilon \cdot u \to 0$ in $L^1(\mathbb{R}^2)$ and $\omega \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$. The second right hand side term converges to

$$\int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot \tilde{u} \tilde{\omega} \, dx \, dt$$

because $u$ belongs to $L^2(\text{supp } \varphi \cap (\mathbb{R}^+ \times \overline{\Omega}))$ (see (2.12)). Putting together these limits, we obtain that:

$$\int_0^\infty \int_{\mathbb{R}^2} \varphi \tilde{\omega} \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot \tilde{u} \tilde{\omega} \, dx \, dt + \int_{\mathbb{R}^2} \varphi(0, x) \tilde{\omega}_0(x) \, dx = 0,$$

which ends the proof. \hfill \Box

The goal of the following is to prove that the $L^p$ norm, the total mass of the vorticity and the circulation are conserved quantities.

In a domain with smooth boundaries, the pair $(\omega, u)$ is a strong solution of the transport equation, and the conservation of the previous quantities is classical. The main point here is to remark that this pair in our case is a renormalized solution in the sense of DiPerna and Lions (see [1]) of the transport equation. We consider equation (2.14) as a linear transport equation with given velocity field $\tilde{u}$. Our purpose here is to show that if $\tilde{\omega}$ solves this linear equation, then so does $\beta(\tilde{\omega})$ for a suitable smooth function $\beta$. This follows from the theory developed in [1] where they need that the velocity field belongs to $L^1_{loc}(\mathbb{R}^+, W^{1,1}_{loc}(\mathbb{R}^2)) \cap L^1_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2))$ and that $\text{div } u$ is bounded. Let us check that we are in this setting.

**Lemma 2.8.** Let $(\omega, u)$ be a global weak solution in $\Omega$. Then

$$\tilde{u} \in L^\infty_{loc}(\mathbb{R}^+, W^{1,1}_{loc}(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)) .$$

We use the explicit form of the velocity (2.9): $u(x) = DT(x) f(T(x))$, where $f$ looks like the Biot-Savart operator in $\mathbb{R}^2$. Therefore, the result follows from the fact that $DT$ belongs to $W^{1,p}_{loc}(\overline{\Omega})$ for any $p < 4/3$ (see Theorem 2.1), and thanks to Proposition 2.5 and the Calderon-Zygmund inequality. The proof is written in [12] in the case where $C$ is a Jordan arc. We generalize it in Section 6.

Therefore, [1] implies that $\tilde{\omega}$ is a renormalized solution.
Lemma 2.9. For \( \bar{u} \) fixed. Let \( \bar{\omega} \) be a solution of the linear equation (2.14) in \( \mathbb{R}^2 \). Let \( \beta : \mathbb{R} \to \mathbb{R} \) be a smooth function such that

\[
|\beta'(t)| \leq C(1 + |t|^p), \quad \forall t \in \mathbb{R},
\]

for some \( p \geq 0 \). Then \( \beta(\bar{\omega}) \) is a solution of (2.14) in \( \mathbb{R}^2 \) (in the sense of distribution) with initial datum \( \beta(\bar{\omega}_0) \).

We recall that \( \bar{u} \) denotes the extension of \( u \) by zero in \( C \), and the previous lemma means that for any \( \Phi \in C^\infty_c([0, \infty) \times \mathbb{R}^2) \) we have

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \beta(\bar{\omega})\Phi(t, x) \, dx = \int_{\mathbb{R}^2} \beta(\bar{\omega})(\partial_t \Phi + u \cdot \nabla \Phi) \, dx \tag{2.17}
\]

in the sense of distributions on \( \mathbb{R}^+ \). Now, we write a remark from [13] in order to establish some desired properties for \( \omega \).

Remark 2.10. (1) Since the right-hand side in (2.17) belongs to \( L^1_{\text{loc}}(\mathbb{R}^+) \), the equality holds in \( L^1_{\text{loc}}(\mathbb{R}^+) \). With this sense, (2.17) actually still holds when \( \Phi \) is smooth, bounded and has bounded first derivatives in time and space. In this case, we have to consider smooth functions \( \beta \) which in addition satisfy \( \beta(0) = 0 \), so that \( \beta(\omega) \) is integrable. This may be proved by approximating \( \Phi \) by smooth and compactly supported functions \( \Phi_n \) for which (2.17) applies, and by letting then \( n \) go to \( +\infty \).

(2) We apply the point (1) for \( \beta(t) = t \) and \( \Phi \equiv 1 \), which gives

\[
\int_\Omega \omega(t, x) \, dx = \int_\Omega \omega_0(x) \, dx \quad \text{for all} \quad t > 0. \tag{2.18}
\]

(3) We let \( 1 \leq p < +\infty \). Approximating \( \beta(t) = |t|^p \) by smooth functions and choosing \( \Phi \equiv 1 \) in (2.17), we deduce that for a solution \( \omega \) to (2.14), the maps \( t \mapsto \|\omega(t)\|_{L^p(\Omega)} \) are continuous and constant. In particular, we have

\[
\|\omega(t)\|_{L^1(\Omega)} + \|\omega(t)\|_{L^\infty(\Omega)} \equiv \|\omega_0\|_{L^1(\Omega)} + \|\omega_0\|_{L^\infty(\Omega)}. \tag{2.19}
\]

In the case of unbounded domain, we will require that \( \omega \) stays compactly supported. Specifying our choice for \( \Phi \) in (2.17), we are led to the following.

Proposition 2.11. Let \( \omega \) be a weak solution of (2.14) such that

\[
\omega_0 \quad \text{is compactly supported in} \quad B(0, R_0)
\]

for some positive \( R_0 \). For any \( T^* \) fixed, then there exists \( C > 0 \) such that

\[
\omega(t, \cdot) \quad \text{is compactly supported in} \quad B(0, R_0 + Ct),
\]

for any \( t \in [0, T^*] \).

The main computation of this proof can be found in [13] or in [12]. For a sake of self-containedness we write the details in Section 6.

Therefore, for \( T^* \) fixed, there exists \( R_1 \) such that the support of the vorticity is included in \( B(0, R_1) \) for all \( t \in [0, T^*] \). It implies that \( u \) is harmonic in \( B(0, R_1)^c \) (\( \text{div} \ u = \text{curl} \ u = 0 \)), and (1.1) is verified in the strong way on this set. With strong solution, the Kelvin’s circulation theorem can be used, which states that the circulation at infinity is conserved:

\[
\gamma(t) + \int_\Omega \omega(t, \cdot) = \gamma^\infty(t) \equiv \gamma_0^\infty = \gamma_0 + \int_\Omega \omega_0.
\]

Using the conservation of the total mass (2.13), we obtain that the circulation of the velocity around the obstacle is conserved:

\[
\gamma(t) \equiv \gamma_0, \quad \forall t \in [0, T^*]. \tag{2.20}
\]

b) Weak solution in a bounded domain.

The previous part can be adapted easily to the bounded case. In simply connected domain, we do not consider the circulation:

\[
u_0(x) = K\Omega[\omega_0].\]
As Proposition 2.6 is about the behavior near the boundary, we can check that we obtain exactly the same.

**Proposition 2.12.** Let \((\omega, u)\) a weak solution to the Euler equations in \(\Omega\) bounded. Then, the pair of extension verifies in the sense of distribution

\[
\begin{aligned}
\omega_t + uu_x &= 0, & \text{in } \mathbb{R}^2 \times (0, \infty), \\
\text{div } u &= 0 \text{ and curl } u = \omega + g(x)\delta_{\partial \Omega}, & \text{in } \mathbb{R}^2 \times [0, \infty), \\
\omega(x, 0) &= \omega_0(x), & \text{in } \mathbb{R}^2.
\end{aligned}
\]

(2.21)

where \(\delta_{\partial \Omega}\) is the Dirac function along the curve and \(g(x)\) is :

\[
g(x) = -u \cdot \hat{r} = -\left[\lim_{\rho \to 0^+} K_{\Omega}[\hat{w}] (x - \rho \hat{n})\right] \cdot \hat{r}
\]

(2.22)

Moreover, we have a term less compared of the unbounded case, then we can also check that \(\bar{\omega}\) is a renormalized solution and that

\[
\int_{\Omega} \omega(t, x) \, dx = \int_{\Omega} \omega_0(x) \, dx \text{ for all } t > 0
\]

(2.23)

and

\[
||\omega(t)||_{L^1(\Omega)} + ||\omega(t)||_{L^\infty(\Omega)} \equiv ||\omega_0||_{L^1(\Omega)} + ||\omega_0||_{L^\infty(\Omega)}.
\]

(2.24)

### 3. Liapounov Method

In this section, we present the proof for a Lagrangian solution. When the velocity \(u\) is smooth, it gives rise to a flow \(\phi_x(t)\) defined by

\[
\begin{aligned}
\frac{d}{dt} \phi_x(t) &= u(t, \phi_x(t)) \\
\phi_x(0) &= x \in \mathbb{R}^2.
\end{aligned}
\]

(3.1)

In view of (3.1), we then have

\[
\frac{d}{dt} \omega(t, \phi_x(t)) \equiv 0,
\]

(3.2)

which gives that \(\omega\) is constant along the characteristics. We assume here that these trajectories exist and are differentiable in our case, and we prove by the Liapounov method that the support of the vorticity never meets the boundary \(\partial \Omega\). Although we do not know that the flow is smooth, the following computation is the main idea of this article, and it will be rigourously applied in Section 4.

The Liapounov method to prove this kind of result is used by Marchioro and Pulvirenti in [16] in the case of a point vortex which moves under the influence of the regular part of the vorticity, and by Marchioro in [15] when the dirac is fixed. In both articles, the authors use the explicit formula of the velocity associated to the dirac centered at \(z(t)\): \(H(x) = (x - z)^2/(2\pi|x - z|^3)\). The geometrical structure is the key of their analysis. Indeed, choosing \(L(t) = -\ln|\phi_x(t) - z(t)|\) they have that

a) \(L(t) \to \infty\) if and only if the trajectory meets the dirac. Then, it is sufficient to prove that \(L(t)\) stays bounded in order to prove the result.

b) \(H(\phi_x(t)) \cdot (\phi_x(t) - z(t)) \equiv 0\), which implies that the singular term in the velocity does not appear.

Therefore, the explicit blow up in the case of the dirac point is crucial in two points of view: for the symmetry cancelation (point b) and for the fact that the primitive of \(1/x\) is \(\ln x\) which blows up near the origin (point a). In our case, we do not have such an explicit form of the blow up near the corners and the primitive of \(1/\sqrt{x}\) is \(\sqrt{x}\) which is bounded near 0. The idea is to add a logarithm. When \(C\) is a Jordan arc, \(|C| \approx 1 + \sqrt{z^2 - 1}\) and we note that \(\ln(1 + \sqrt{z^2 - 1})\) blows up near the end-points \(\pm 1\).

However, the problem with Liapounov function is that it is very specific on the case studied. For example, this function is different if the dirac point is fixed or if it moves with the fluid (for more details and explanations, see the discussion on Liapounov functions in Section 7).
We fix \( x_0 \in \Omega \) and we consider \( \phi = \phi_{x_0}(t) \) the trajectory which comes from \( x_0 \) (see \( 3.1 \)). We denote 
\[ L(t) := -\ln |L_1(t, \phi(t))| \]
with \( L_1 \) depending on the geometric property of \( \Omega \):

1. if \( \Omega \) is a bounded, simply connected open set, such that \( \partial \Omega \) has a finite number of corner with angles greater than \( \pi/2 \) (as in Theorem 1.2), then we choose 
\[ L_1(t, x) := \frac{1}{2\pi} \int_{\Omega} \ln \left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)||} \right) \omega(t, y) \, dy; \]
\[ (3.3) \]
2. if \( \Omega := \mathbb{R}^2 \setminus C \), where \( C \) is a compact, simply connected set, such that \( \partial \Omega \) has a finite number of corner with angles greater than \( \pi/2 \) (as in Theorem 1.3), then we choose 
\[ L_1(t, x) := \frac{1}{2\pi} \int_{\Omega} \ln \left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)||} \right) \omega(t, y) \, dy + \frac{\alpha}{2\pi} \ln |\mathcal{T}(x)|, \]
\[ (3.4) \]
where \( \alpha := \gamma_0 + \int \omega_0. \)

When trajectories exist, it is obvious (without renormalization) that \( \bullet 4 \) and \( \bullet 2 \) imply that 
\[ \|\omega(t, \cdot)\|_{L^p} = \|\omega_0\|_{L^p} \text{ and } \int_{\Omega} \omega(t, \cdot) = \int_{\Omega} \omega_0, \forall t > 0, \forall p \in [1, \infty]. \]
\[ (3.5) \]
We assume that \( \omega_0 \) is compactly supported, then included in \( B(0, R_0) \) for some \( R_0 > 0 \). Thanks to Propositions 2.3 and 2.5, we see that the velocity \( u \) is bounded outside this ball by a constant \( C_0 \), and \( \bullet 1 \) and \( \bullet 2 \) give 
\[ \text{supp } \omega(t, \cdot) \subset B(0, R_0 + C_0 t), \forall t \geq 0. \]
\[ (3.6) \]
We also have that the circulation is conserved.

If we assume that \( \omega_0 \) is non positive, then it follows from \( \bullet 2 \) that 
\[ \omega(t, x) \leq 0, \forall t \geq 0, \forall x \in \Omega. \]
\[ (3.7) \]

**3.1. Blow up of the Liapounov function near the curve.**

The first required property is that \( L \) goes to infinity iff the trajectory meets the boundary. Next, if we prove that \( L \) is bounded, then it will follow that the trajectory stays far away the boundary. We fix \( T^* > 0 \), using \( \bullet 3.6 \) we denote by \( R_{T^*} := R_0 + C_0 T^* \), such that \( \text{supp } \omega(t, \cdot) \subset B(0, R_{T^*}) \) for all \( t \in [0, T^*] \).

**Lemma 3.1.** For any case (1)-(2), there exists \( C_1 = C_1(T^*, \omega_0, \gamma_0) \) such that 
\[ |L_1(t, x)| \leq C_1 |\mathcal{T}(x)| - 1|^{1/2}, \forall x \in B(0, R_{T^*}), \forall t \in [0, T^*]. \]

**Proof.** For a sake of shortness, we write the proof in the hardest case: case (2). The other case follows easily. Recalling the notation \( z^* = z/|z|^2 \), we can compute 
\[ \frac{|\mathcal{T}(x) - \mathcal{T}(y)|^2}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2|\mathcal{T}(y)|^2} = 1 - \frac{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2|\mathcal{T}(y)|^2}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2|\mathcal{T}(y)|^2} \]
\[ = 1 - \frac{|\mathcal{T}(x)|^2 - 2 T(x) \cdot T(y) + |T(y)|^2}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2|\mathcal{T}(y)|^2} \]
\[ = 1 - \frac{(|T(x)|^2 - 1)(|T(y)|^2 - 1)}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2|\mathcal{T}(y)|^2}. \]

Therefore, we have 
\[ \ln \left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)||} \right) = \frac{1}{2} \ln \left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|^2}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2|\mathcal{T}(y)|^2} \right) \]
\[ = \frac{1}{2} \ln \left( 1 - \frac{(|T(x)|^2 - 1)(|T(y)|^2 - 1)}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2|\mathcal{T}(y)|^2} \right), \]
and we need an estimate of \( \ln(1 - r) \) when \( r \in (0, 1) \), because we recall that \( |\mathcal{T}(z)| > 1 \) for any \( z \in \Omega \). It is easy to see (studying the difference of the functions) that

\[
|\ln(1 - r)| = -\ln(1 - r) \leq \left( \frac{r}{1 - r} \right)^{1/2}, \quad \forall r \in [0, 1).
\]

Applying this inequality, we have for any \( y \neq x \) that

\[
\left| \ln\left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|} \right) \right| \leq \frac{1}{2} \left( \frac{(|\mathcal{T}(x)|^2 - 1)(|\mathcal{T}(y)|^2 - 1)}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|^2} \right)^{1/2} \leq \frac{1}{2} \sqrt{\frac{(|\mathcal{T}(x)|^2 - 1)(|\mathcal{T}(y)|^2 - 1)}{|\mathcal{T}(x) - \mathcal{T}(y)|^2}}.
\]

By continuity of \( \mathcal{T} \), we denote by \( C_{T^*} \) a constant such that \( \mathcal{T}(B(0,R_{T^*})) \subset B(0,C_{T^*}) \). Finally, we apply the previous inequality to \( L_1 \) and we find for all \( x \in B(0,R_{T^*}) \) and \( t \in [0,T^*] \):

\[
|L_1(t,x)| \leq C_{T^*}(C_{T^*} + 1)^{1/2}(|\mathcal{T}(x)| - 1)^{1/2} \int_{\Omega} \frac{|\omega(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)|} dy + \frac{|\alpha|}{2\pi} \ln|\mathcal{T}(x)|
\]

\[
\leq \frac{\sqrt{2}C_{T^*}^{3/2}}{4\pi} (|\mathcal{T}(x)| - 1)^{1/2} C(\|\omega\|_{L^1}\|\omega\|_{L^\infty}^{1/2} + \|\omega\|_{L^1}^2 + \|\omega\|_{L^\infty}^{1/2} + \|\omega\|_{L^1}^2 + \|\omega\|_{L^\infty}^{1/2}) + \frac{|\alpha|}{2\pi} (|\mathcal{T}(x)| - 1).
\]

For the last inequality, we used a part of Proposition 2.3. As \( (|\mathcal{T}(x)| - 1) \leq C_{T^*}^2 (|\mathcal{T}(x)| - 1)^{1/2} \), the conclusion follows from \( (3.5) \). \( \square \)

Concerning the lower bound for the case (1)-(2), we need some conditions on the sign.

**Lemma 3.2.** If \( \omega_0 \) is non-positive and \( \gamma_0 \geq -\int \omega_0 \) (only in case (2)), then there exists \( C_2 = C_2(T^*,\omega_0) \) such that

\[
L_1(t,x) \geq C_2 |\mathcal{T}(x)| - 1, \quad \forall x \in B(0,R_{T^*}), \quad \forall t \in [0,T^*].
\]

**Proof.** Again, we write the details in the case (2). We denote by \( r_\infty := \|\omega_0\|_{L^\infty} \) and \( r_1 := \|\omega_0\|_{L^1} \).

For \( \rho > 0 \), we denote by

\[
V_1 := (C + B(0,\rho)) \cap \Omega = \{ x \in \Omega; \text{dist}(x,C) < \rho \}, \quad V_2 := \Omega \setminus V_1.
\]

We fix \( \rho \) such that the lebesgue measure of \( V_1 \) is equal to \( r_1/(2r_\infty) \).

We deduce from \( (3.5) \) that

\[
r_1 = \|\omega(t,\cdot)\|_{L^1(V_1)} + \|\omega(t,\cdot)\|_{L^1(V_2)}
\]

with \( \|\omega(t,\cdot)\|_{L^1(V_1)} \leq r_\infty r_1/(2r_\infty) = r_1/2 \) which implies that \( \|\omega(t,\cdot)\|_{L^1(V_2)} \geq r_1/2 \).

As the logarithm of the fraction is negative (see the proof of Lemma 3.1), we have, with the sign condition, that:

\[
L_1(t,x) \geq \frac{1}{2\pi} \int_{V_2} \ln\left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|} \right) \omega(y) dy.
\]

Moreover, thanks to the computation made in the proof of Lemma 3.1 we have

\[
\ln\left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|} \right) = \frac{1}{2} \ln\left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|^2}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|^2} \right) = \frac{1}{2} \ln\left( 1 - \frac{(|\mathcal{T}(x)|^2 - 1)(|\mathcal{T}(y)|^2 - 1)}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|^2} \right) \leq \frac{1}{2} \frac{(|\mathcal{T}(x)|^2 - 1)(|\mathcal{T}(y)|^2 - 1)}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|^2}.
\]

because \( \ln(1 + x) \leq x \) for any \( x > -1 \).
As \( \rho > 0 \) and \( T \) is continuous, there exists \( C_{\rho} > 0 \) such that \( |T(y)| \geq 1 + C_{\rho} \), for all \( y \in V_2 \). Moreover, there exists also \( \bar{R}_{T^*} > 1 \) such that \( T(B(0, \bar{R}_{T^*})) \subset B(0, \bar{R}_{T^*}) \). Adding the fact that \( \omega \) is non positive, we have for all \( y \in V_2 \cap \text{supp } \omega \) and \( x \in B(0, \bar{R}_{T^*}) \):

\[
\ln\left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|^* |T(y)|} \right) \omega(y) \geq \frac{1}{2} \frac{(|T(x)|^2 - 1) (|T(y)|^2 - 1)}{|T(x) - T(y)|^* |T(y)|^2} |\omega(y)| \\
\geq \frac{1}{2} \frac{(|T(x)| - 1)(|T(x)| + 1)(|T(y)| - 1)(|T(y)| + 1)}{(|T(x)| + 1)^2 |T(y)|^2} |\omega(y)| \\
\geq \frac{1}{2} \frac{(|T(x)| - 1)C_{\rho} |\omega(y)|}{(\bar{R}_{T^*} + 1)\bar{R}_{T^*}}.
\]

Integrating this last inequality over \( V_2 \), we obtain that

\[
L_1(t, x) \geq \frac{1}{2\pi} \int_{V_2} \ln\left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|^* |T(y)|} \right) \omega(y) \, dy \geq \frac{|T(x)| - 1}{4\pi(\bar{R}_{T^*} + 1)\bar{R}_{T^*}} C_{\rho} ||\omega||_{L^1(V_2)} \\
\geq \frac{C_{\rho}}{8\pi(\bar{R}_{T^*} + 1)\bar{R}_{T^*}} r_1(|T(x)| - 1),
\]

which ends the proof. \( \square \)

Multiplying by \(-1\) the expression of \( L_1 \), we can establish the same result with opposite sign condition:

**Remark 3.3.** If \( \omega_0 \) is non-negative and \( \gamma_0 \leq -\int \omega_0 \), then there exists \( C_2 \) such that

\[-L_1(t, x) \geq C_2||T(x)|| - 1, \quad \forall x \in B(0, \bar{R}_{T^*}), \quad \forall t \in [0, T^*].\]

From these two lemmas, it follows obviously the following.

**Corollary 3.4.** If \( \omega_0 \) is non-positive and \( \gamma_0 \geq -\int \omega_0 \) (only for (2)), then we have that

- \( L_1(x) > 0 \) for all \( x \in \Omega \);
- \( L_1(x) \rightarrow 0 \) if and only if \( x \rightarrow \partial \Omega \).

If \( \omega_0 \) is non-negative and \( \gamma_0 \leq -\int \omega_0 \) (only for (2)), then we have that

- \( L_1(x) < 0 \) for all \( x \in \Omega \);
- \( L_1(x) \rightarrow 0 \) if and only if \( x \rightarrow \partial \Omega \).

Indeed, \( |T(x)| \rightarrow 1 \) iff \( x \rightarrow \partial \Omega \).

### 3.2. Estimates of the Liapounov.

The issue of this part is to prove that the trajectory never meets the obstacle in finite time. In other word, let \( x_0 \in \text{supp } \omega_0 \) (then \( L_1(0, x_0) \neq 0 \)) and \( T^* > 0 \), we will prove that \( L(t) \) stays bounded in \([0, T^*]\). Then, we differentiate \( L \):

\[
L'(t) = -\left( \partial_t L_1(t, \phi(t)) + \phi'(t) \cdot \nabla L_1(t, \phi(t)) \right) / L_1(t, \phi(t))
\]

and we want to estimate the right hand side term.

As usual, we write the details for the case (2).

On one hand, we note that

\[
u(t, x) \cdot \nabla L_1(t, x) = u(t, x) \cdot \left[ \frac{1}{2\pi} \int_{\Omega} \left( \frac{T(x) - T(y)}{|T(x) - T(y)|^2} - \frac{T(y) - T(x)^*}{|T(y) - T(x)^*|^2} \right) \omega(y) \, dy \, DT(x) \right] \\
+ \frac{\alpha}{2\pi} \frac{T(x)}{|T(x)|^2} DT(x) \equiv 0
\]

thanks to the explicit formula of \( u \) (see (2.9)).
On the other hand, we use the equation verified by \( \omega \) to have
\[
\partial_t L_1(t, x) = \frac{1}{2\pi} \int_{\Omega} \ln \left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|} \right) \partial_t \omega(y) dy \\
= -\frac{1}{2\pi} \int_{\Omega} \ln \left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|} \right) \text{div} \left( u(y) \omega(y) \right) dy \\
= \frac{1}{2\pi} \int_{\Omega} \nabla_y \left[ \ln \left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|} \right) \right] u(y) \omega(y) dy.
\]

Now, we use the symmetry of the Green kernel (see Subsection 2.2)
\[
\nabla_y \ln \left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|} \right) = \nabla_y \ln \left( \frac{|T(y) - T(x)|}{|T(y) - T(x)|} \right)
\]
and the explicit formula of \( u(y) \) to write
\[
\partial_t L_1(t, x) = \frac{1}{2\pi} \int_{\Omega} \left[ \left( \frac{T(y) - T(x)}{|T(y) - T(x)|^2} - \frac{T(y) - T(x)^*}{|T(y) - T(x)^*|^2} \right) \right.
\]
\[
\left. \left( D\mathcal{T}(y) \frac{1}{2} D\mathcal{T}^T(y) \left( R[\omega](y) + \frac{T(y)\perp}{|T(y)|^2} \right) \right) \right|_T \omega(y) dy.
\]

As \( \mathcal{T} \) is holomorphic, \( D\mathcal{T} \) is of the form \( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) and we can check that \( D\mathcal{T}(y) D\mathcal{T}^T(y) = (a^2 + b^2)I_2 = |\text{det}(D\mathcal{T}(y))| I_2 \), so
\[
\partial_t L_1(t, x) = \frac{1}{(2\pi)^2} \int_{\Omega} \left[ \left( \frac{T(y) - T(x)}{|T(y) - T(x)|^2} - \frac{T(y) - T(x)^*}{|T(y) - T(x)^*|^2} \right) \right.
\]
\[
\left. \left( R[\omega](y) + \frac{T(y)\perp}{|T(y)|^2} \right) \right|_T \text{det}(D\mathcal{T}(y))| \omega(y) dy.
\]

The goal is to estimate \( \partial_t L_1 / L_1 \). However, Corollary \ref{Cor_3.4} states that \( L_1 \) goes to zero if and only if \( x \to \partial \Omega \). Then it is important to show that \( \partial_t L_1 \) tends to zero as \( x \to \partial \Omega \), and to prove that it goes to zero faster than \( L_1 \).

We will need the following general lemma.

**Lemma 3.5.** Let \( h \) be a bounded function, compactly supported in \( B(0, R_h) \) for some \( R_h > 1 \). Then, there exists \( C_h = C(\|h\|_{L^\infty}, R_h) \) such that
\[
\int_{D^c} \frac{|h(y)|}{|y - x||y - x|^*} dy \leq C_h \left( \ln(\|x| - 1 + |x|) \right), \quad \forall x \in D^c
\]
with the notation \( x^* = x/|x|^2 \) and \( D = B(0, 1) \).

**Proof.** We fix \( x \in D^c \) and we denote
\[
\rho = |x| - 1 \quad \text{and} \quad \rho^* = 1 - |x^*| = 1 - \frac{1}{1 + \rho} = \frac{\rho}{1 + \rho}.
\]
We compute
\[
\int_{D^c} \frac{|h(y)|}{|y - x||y - x|^*} dy = \int_{D^c \cap B(x, \rho)} \frac{|h(y)|}{|y - x||y - x|^*} dy + \int_{D^c \cap B(x, \rho^*)} \frac{|h(y)|}{|y - x||y - x|^*} dy =: I_1 + I_2.
\]
For \( I_1 \), we know that \( |y - x^*| \geq |y| - |x^*| \geq \rho^* \), hence
\[
I_1 \leq \frac{1}{\rho^*} \int_{D^c \cap B(x, \rho^*)} \frac{|h(y)|}{|y - x|} dy \leq \frac{\|h\|_{L^\infty}}{\rho^*} \int_{B(x, \rho^*)} \frac{1}{|y - x|} dy
\]
\[
\leq \frac{(1 + \rho)}{\rho} \frac{1}{\rho^*} \frac{\|h\|_{L^\infty}}{2\pi \rho^*}
\]
which gives that \( I_1 \leq C_1 |x| \).

\footnote{To justify that it works even for a weak solution, the reader can see the first lines of the proof of Proposition \ref{Prop_2.11}.}
Concerning $I_2$, we note that

\[ |x - x^*| = \rho + \rho^* = \rho + \frac{\rho}{1 + \rho} \leq 2\rho \leq \frac{1}{2}|y - x| \]

for any $y \in B(x, 4\rho)^c$. Hence,

\[ |y - x^*| \geq |y - x| - |x - x^*| \geq \frac{1}{2}|y - x|, \]

and we have

\[
I_2 \leq \int_{D^c \cap B(x, 4\rho)^c} \frac{2|h(y)|}{|y - x|^2} \, dy \leq 2\|h\|_{L^\infty} \int_0^{2\pi} \int_{4\rho}^{\infty} \frac{1}{r} \, dr \, d\theta
\]

\[
\leq 4\pi \|h\|_{L^\infty} \ln \left( \frac{|x| + R_h}{4\rho} \right)
\]

which implies that $I_2 \leq C_2 \left( |\ln(|x| - 1)| + \ln \frac{|x| + R_h}{4} \right)$.

We conclude because there exists $C_3 = C_3(R_h)$ such that $\ln \frac{|x| + R_h}{4} \leq C_3 |x|$ for any $x \in D^c$. \qed

We recall that we have fixed $T^* > 0$ and $x_0 \in \text{supp} \, \omega_0$. Using \eqref{3.5}, we denote by $\rho_T := R_0 + C_0 T^*$, such that $\text{supp} \, \omega(t, \cdot) \subset B(0, \rho_T)$ for all $t \in [0, T^*]$. Finally, we estimate $\partial_t L_1$ without sign conditions.

**Lemma 3.6.** There exists $C_3 = C_3(T^*)$ such that

\[ |\partial_t L_1(t, x)| \leq C_3 ||T(x)| - 1|| \left( 1 + \left| \ln ||T(x)|| - 1 \right| \right), \forall x \in B(0, \rho_T), \forall t \in [0, T^*]. \]

**Proof.** Using \eqref{2.6} we know that

\[ \left| \frac{T(y) - T(x)}{|T(y) - T(x)|^2} - \frac{T(y) - T(x)^*}{|T(y) - T(x)^*|^2} \right| = \left| \frac{T(x) - T(x)^*}{|T(y) - T(x)||T(y) - T(x)^*|} \right|. \]

Then, Proposition \ref{2.5} and \eqref{3.5} allow us to estimate \eqref{3.8}

\[ |\partial_t L_1(t, x)| \leq C_3 T(x) - T(x)^* \int_{D} \frac{|\omega(y)|}{|T(y) - T(x)||T(y) - T(x)^*|} |\det(DT)(y)| \, dy. \]

On one hand, we have for all $x \in B(0, \rho_T)$

\[ |T(x) - T(x)^*| = \frac{|T(x)|^2 - T(x) |T(x)|^2 - T(x) |T(x)|^2 - 1}{|T(x)|^2} \leq 2(|T(x)| - 1). \]

On the other hand, we change variables $\eta = T(y)$ and we compute

\[ \int_{D^c} \frac{|\omega(y)|}{|T(y) - T(x)||T(y) - T(x)^*|} |\det(DT)(y)| \, dy = \int_{D^c} \frac{|\omega(T^{-1}(\eta))|}{|\eta - T(x)||\eta - T(x)^*|} \, d\eta. \]

As $||\omega \circ T^{-1}||_{L^\infty} = ||\omega_0||_{L^\infty}$ and as

\[ \text{supp} \, \omega \circ T^{-1} = T(\text{supp} \, \omega) \subset T(B(0, \rho_T)) \subset B(0, \check{\rho}_T), \]

we apply Lemma \ref{3.5} to establish that

\[ \int_{D} \frac{|\omega(y)|}{|T(y) - T(x)||T(y) - T(x)^*|} |\det(DT)(y)| \, dy \leq C \left( |\ln(|T(x)| - 1)| + \check{\rho}_T \right), \forall x \in B(0, \rho_T). \]

This finishes the proof. \qed
Remark 3.7. In the bounded case, there is a tricky difference in the previous proof. We note that

$$|T(x) - T(x^*)| = \frac{(1 - |T(x)|)(|T(x)| + 1)}{|T(x)|} \leq 2 \frac{(1 - |T(x)|)}{|T(x)|}$$

with $|T(x)|$ which can go to zero. To fix this problem, we can prove a similar result to Lemma 3.5: there exists $C_h = C(\|h\|_{L^\infty})$ such that

$$\frac{1}{|x|} \int_D \frac{|h(y)|}{|y - x||y - x^*|} dy \leq C_h \left( \ln(1 - |x|) + 1 \right), \forall x \in D.$$

Indeed, we can write $|x||y - x^*| = |y|x| - \frac{x}{|x|}$ and we deduce putting $\rho := 1 - |x|$ that:

- for $y \in B(x, 4\rho) \cap D$, $|y|x| - \frac{x}{|x|} \geq \left| \frac{x}{|x|} \right| - |y||x| \geq 1 - |x| = \rho$;
- for $y \in B(x, 4\rho)^c \cap D$, $\left| y|x| - \frac{x}{|x|} \right|^2 - |y - x|^2 = (1 - |y|^2)(1 - |x|^2) \geq 0$.

Using this two inequality, we follow exactly the proof of Lemma 3.5 which allows us to establish Lemma 3.6 in the bounded case.

In light of Lemmas 3.2 and 3.6, we see that we have an additional logarithm which implies that $\frac{\partial_t L_1}{L_1} \to \infty$ if $x \to C$. However, the logarithm is exactly what we can estimate by Gronwall inequality: $L'(t) = \frac{\partial_t L_1}{L_1} \approx \ln L_1 = L(t)$. It is the general idea to establish the main result of this section.

**Proposition 3.8.** We assume that $\omega_0$ is non-positive, compactly supported in $\Omega$ and $\gamma_0 \geq -\int \omega_0$. Then, for any $T^* > 0$, there exists $C_{T^*}$ such that

$$L(t) \leq C_{T^*}, \forall x_0 \in \text{supp} \omega_0, \forall t \in [0, T^*].$$

**Proof.** As the support of $\omega_0$ does not intersect $\partial \Omega$, we have by continuity of $T$ and by Lemma 3.2 that

$$L(0) = -\ln L_1(0, x_0) \leq -\ln C_2(|T(x_0)| - 1)$$

is bounded uniformly in $x_0 \in \text{supp} \omega_0$.

For any, $x_0 \in \text{supp} \omega_0$, (3.9) gives that $\phi(t) \in B(0, R_{T^*})$, for all $t \in [0, T^*]$. Therefore, the computation made in the begin of this subsection gives

$$L'(t) = -\partial_t L_1(t, \phi(t))/L_1(t, \phi(t)).$$

As $L_1$ is positive, we have

$$L'(t) = -\partial_t L_1(t, \phi(t))/L_1(t, \phi(t)) \leq |\partial_t L_1(t, \phi(t))|/L_1(t, \phi(t)).$$

Lemma 3.2 states that there exists $C_2$ such that

$$L_1(t, \phi(t)) \geq C_2(|T(\phi(t))| - 1).$$

Moreover, thanks to Lemma 3.1 it is easy to find $C_4$ such that

$$L_1(t, x) \leq C_4, \forall x \in B(0, R_{T^*} \cap \Omega), \forall t \in [0, T^*].$$

Finally, we proved in Lemma 3.6 that there exists $C_3$ such that

$$|\partial_t L_1(t, \phi(t))| \leq C_3(|T(\phi(t))| - 1) \left( 1 + |\ln(|T(\phi(t))| - 1)| \right).$$

We can easily check that in the interval $(0, e^{-1})$ the function $x \mapsto x|\ln x|$ is equal to the map $x \mapsto -x \ln x$, which is increasing. By (3.9) and (3.10), we use the fact that

$$0 \leq \frac{C_2(|T(\phi(t))| - 1)}{eC_4} \leq \frac{L_1(t, \phi(t))}{eC_4} \leq e^{-1}$$
Proposition 4.1. Let \( \omega \) be a global weak solution of \( \square \) such that \( \omega_0 \) is compactly supported in \( \Omega \). If \( \omega_0 \) is non-positive and \( \gamma_0 \geq -\int \omega_0 \) (only for exterior domain), then, for any \( T^* > 0 \), there exists a neighborhood \( U_{T^*} \) of \( \partial \Omega \) such that

\[
\omega(t) \equiv 0 \quad \text{on} \quad U_{T^*}, \quad \forall t \in [0, T^*].
\]
Proof. According to Proposition [2.11], we have
\[ \text{supp } \omega(t) \subset B(0, R_0 + C_0t)) , \quad \forall t \geq 0. \] (4.1)

We note \( R_{T^*} := R_0 + C_0 T^* \).

Thanks to Lemma [3.1], it is easy to find \( C_4 \) such that
\[ L_1(t,x) \leq C_4, \quad \forall x \in B(0, R_{T^*} \cap \Omega), \quad \forall t \in [0, T^*]. \] (4.2)

We also deduce from the conservation of the vorticity sign that Corollary 3.4 holds true.

We aim to apply (2.17) with the choice \( \beta(t) = t^2 \) and we set
\[ \Phi(t,x) = \chi_0 \left( -\ln L_1(t,x) + \ln C_4 \right) \frac{R(t)}{R}, \]
where \( \chi_0 \) is a smooth function: \( \mathbb{R} \to \mathbb{R}^+ \) which is identically zero for \( |x| \leq 1/2 \) and identically one for \( |x| \geq 1 \) and increasing on \( \mathbb{R}^+ \), \( L_1 \) is defined in (3.3) and \( R(t) \) is an increasing continuous function to be determined later on.

As \( L_1(t,x) \leq C_4 \), we have that \( -\ln L_1(t,x) + \ln C_4 \) is positive \( \forall x \in B(0, R_{T^*} \cap \Omega), \quad \forall t \in [0, T^*]. \)

On one hand, Lemma 3.2 states that there exists \( C_2 \) such that
\[ L_1(t,x) \geq C_2^2(\sqrt{t} - 1), \quad \forall x \in B(0, R_{T^*} \cap \Omega), \quad \forall t \in [0, T^*]. \] (4.3)

Finally, we proved in Lemma [5.5] that there exists \( C_3 \) such that
\[ |\partial_t L_1(t,x)| \leq C_3(|\sqrt{t} - 1| + |\ln(|\sqrt{t} - 1|)|), \quad \forall x \in B(0, R_{T^*} \cap \Omega), \quad \forall t \in [0, T^*]. \] (4.4)

Then, using the fact that \( x \mapsto -x \ln x \) is increasing in \( [0,e^{-1}] \) (see the proof of Proposition 3.8), we have that
\[ |\partial_t L_1(t,x)| \leq L_1(t,x) C_3 - C_6 \ln \frac{L_1(t,x)}{C_4}, \quad \forall x \in B(0, R_{T^*} \cap \Omega), \quad \forall t \in [0, T^*]. \] (4.5)

On the other hand, we have
\[ \nabla_x L_1(t,x) = -u^1(t,x), \]
therefore
\[ u \cdot \nabla \Phi = u \cdot u^1 \frac{\chi_0'}{RL_1} = 0. \]

Besides,
\[ \partial_t \Phi(t,x) = \left( \frac{R'(t)}{R^2(t)} \ln \frac{L_1(t,x)}{C_4} - \frac{1}{R} \frac{\partial_t L_1(t,x)}{L_1(t,x)} \right) \chi_0' \left( -\ln \frac{L_1(t,x)}{C_4} \right) \frac{R(t)}{R}. \]

In view of (2.17), this yields for any \( T \in [0, T^*] \)
\[ \int_{\mathbb{R}^2} \Phi(T,x) \omega^2(T,x) \, dx - \int_{\mathbb{R}^2} \Phi(0,x) \omega_0^2(x) \, dx = \int_0^T \int_{\mathbb{R}^2} \omega^2(t,x) \chi_0' \left( -\ln \frac{L_1(t,x)}{R} + \ln C_4 \right) \left( \frac{R'}{R} \ln \frac{L_1(t,x)}{C_4} - \frac{\partial_t L_1(t,x)}{L_1(t,x)} \right) \, dx \, dt. \]

Since \( -\ln \frac{L_1(t,x)}{C_4} \geq 0 \), the term \( \chi_0' \left( \frac{-\ln L_1(t,x) + \ln C_4}{R} \right) \) is non negative and non zero provided \( \frac{1}{2} \leq \frac{-\ln L_1(t,x) + \ln C_4}{R} \leq 1 \), so we obtain
\[ \int_{\mathbb{R}^2} \Phi(T,x) \omega^2(T,x) \, dx - \int_{\mathbb{R}^2} \Phi(0,x) \omega_0^2(x) \, dx \leq \int_0^T \int_{\mathbb{R}^2} \omega^2(t,x) \chi_0' \left( -\frac{R'}{2} + C_5 + C_6 R \right) \, dx \, dt. \]

In the last inequality, we have used (4.5), which is allowed because \( \text{supp } \omega \subset B(0, R_{T^*} \cap \Omega) \) for all \( t \in [0, T^*] \).

We now choose
\[ R(t) = \lambda_0 e^{2C_6 t} - \frac{C_6}{C_0}, \]
\[ \text{see the proof of Proposition 2.11 to check that this equality holds for all } T. \]
with $\lambda_0$ to be determined later on, so that

$$
\int_{\mathbb{R}^2} \Phi(T, x) \omega^2(T, x) \, dx \leq \int_{\mathbb{R}^2} \Phi(0, x) \omega_0^2(x) \, dx.
$$

Since the support of $\omega_0$ does not intersect some neighborhood of $C$, the continuity of $T$ implies that there exists $\mu_0 > 0$ such that $T(\text{supp } \omega_0) \subset B(0, \mu_0 + 1)^c$. Then,

$$
0 \leq -\ln L_1(0, x) + \ln C_4 \leq -\ln \left( C_2(\|T(x)| - 1) \right) + \ln C_4 \leq -\ln (C_2 \mu_0 + \ln C_4)
$$

for all $x$ in the support of $\omega_0$. We finally choose $\lambda_0$ so that

$$
0 < \frac{-\ln (C_2 \mu_0) + \ln C_4}{\lambda_0 - \frac{\xi}{C_6}} \leq \frac{1}{2}.
$$

For this choice, we have

$$
\Phi(0, x) \omega_0^2(x) = \chi_0 \left( \frac{-\ln L_1(0, x) + \ln C_4}{\lambda_0 - \frac{\xi}{C_6}} \right) \omega_0^2(x) \equiv 0.
$$

We deduce that for all $T \in [0, T^*)$, $\Phi(T, x) \omega^2(T, x) \equiv 0$. Thanks to Lemma 3.1, we know that there exists $C_1$ such that

$$
L_1(T, x) \leq C_1 (\|T(x)| - 1)^{1/2}, \quad \forall x \in B(0, R_{T^*}), \quad \forall T \in [0, T^*].
$$

Therefore, for any $x \in T^{-1}\left( B(0, 1 + e^{-\frac{C_1}{2}(R(T^*) - \ln C_4)}) \setminus B(0, 1) \right)$ and any $T \in [0, T^*)$, we have that

$$
\begin{align*}
|T(x)| &\leq 1 + e^{-\frac{C_1}{2}(R(T^*) - \ln C_4)} \\
\ln(\|T(x)| - 1) &\leq -\frac{2}{C_1} (R(T^*) - \ln C_4) \\
-\frac{C_1}{2} \ln(\|T(x)| - 1) &\geq (R(T^*) - \ln C_4)
\end{align*}
$$

which implies that

$$
-\frac{C_1}{2} \ln(\|T(x)| - 1) + \ln C_4 \geq 1. \quad (4.6)
$$

Moreover, for any $x \in B(0, R_{T^*})$ and $T \in [0, T^*)$ we have that

$$
\ln L_1(T, x) \leq \frac{C_1}{2} \ln(\|T(x)| - 1) \\
- \ln L_1(T, x) + \ln C_4 \geq -\frac{C_1}{2} \ln(\|T(x)| - 1) + \ln C_4
$$

which gives (using that $R$ is an increasing function and that $- \ln L_1(T, x) + \ln C_4 \geq 0$):

$$
\frac{-\ln L_1(T, x) + \ln C_4}{R(T)} \geq \frac{-\ln L_1(T, x) + \ln C_4}{R(T^*)} \geq \frac{-\frac{C_1}{2} \ln(\|T(x)| - 1) + \ln C_4}{R(T^*)}.
$$

Putting together the last inequality and (4.6), $\Phi(T, x) \omega^2(T, x) \equiv 0$ for any $T \in [0, T^*)$ implies that

$$
\omega(T, x) \equiv 0, \quad \forall x \in T^{-1}\left( B(0, 1 + e^{-\frac{C_1}{2}(R(T^*) - \ln C_4)}) \setminus B(0, 1) \right), \quad \forall T \in [0, T^*)
$$

and the conclusion follows.

**Remark 4.2.** Of course, as in Remarks 3.3 and 3.9, the previous proposition holds true for the opposite sign condition:

$$
\omega_0 \text{ non negative and } \gamma_0 \leq -\int \omega_0.
$$

Actually, we can prove Propositions 2.11 and 3.1 without the renormalized solutions. Indeed, as we proved in Remark 2.10 that $\omega$ stays definite sign (thanks to the renormalization theory), then we can use $\omega$ instead of $\omega^2$ in the proofs. In this case, we just need that $\omega$ is a weak solution in the sense of distribution. However, we have presented here the proofs with $\beta(\omega) = \omega^2$ in order to extend the theorems in the case where $\omega_0$ is constant near the boundary (see Section 7).
5. Uniqueness of Eulerian solutions

5.1. Velocity formulation.

In order to follow the proof of Yudovich, we give a velocity formulation\(^4\) of the extension \(\bar{u}\).

We begin by introducing

\[
v(x) := \int_{\mathbb{R}^2} K_{\mathbb{R}^2}(x-y)\bar{\omega}(y)dy
\]

with \(K_{\mathbb{R}^2}(x) = \frac{1}{2\pi} \frac{1}{|x|^2}\). the solution in the full plane of

\[
div v = 0 \text{ on } \mathbb{R}^2, \quad \text{curl } v = \bar{\omega} \text{ on } \mathbb{R}^2, \quad \lim_{|x| \to \infty} |v| = 0.
\]

This velocity is bounded, and we denote the perturbation by \(w = \bar{u} - v\), which belongs to \(L^p_{\text{loc}}(\mathbb{R}^+; L^p_{\text{loc}}(\mathbb{R}^2))\) for \(p < 4\), and verifying

\[
div w = 0 \text{ on } \mathbb{R}^2, \quad \text{curl } w = g_{\omega,\gamma}(s)\delta_{\partial\Omega} \text{ on } \mathbb{R}^2, \quad \lim_{|x| \to \infty} |w| = 0.
\]

We infer that \(v\) verifies the following equation:

\[
\begin{aligned}
|v_t + v \cdot \nabla v + v \cdot \nabla w + w \cdot \nabla v - v(s)^{1/2}g_{\omega,\gamma}(s) \cdot \delta_{\partial\Omega} = -\nabla p, & \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\
\text{div } v = 0, & \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\
w(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-s)^{1/2}}{|x-s|}g_{\omega,\gamma}(s)ds, & \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\
v(x,0) = K_{\mathbb{R}^2}[\bar{\omega}], & \quad \text{in } \mathbb{R}^2.
\end{aligned}
\]

with \(g_{\omega,\gamma} := g_{\text{curl } v, \gamma}\) (see (2.13)).

In order to prove the equivalence of (2.13) and (5.1), it is sufficient to show that

\[
\text{curl } [v \cdot \nabla w + w \cdot \nabla v - v(s)^{1/2}g_{\omega,\gamma}(s) \cdot \delta_{\partial\Omega}] = \text{div } (\bar{\omega} w)
\]

for all divergence free fields \(v \in W^{1,p}_{\text{loc}}\), with some \(p > 2\). Indeed, if (5.2) holds, then we get for \(\bar{\omega} = \text{curl } v\)

\[
0 = -\text{curl } \nabla p = \text{curl } [v_t + v \cdot \nabla v + v \cdot \nabla w + w \cdot \nabla v - v(s)^{1/2}g_{\omega,\gamma}(s) \cdot \delta_{\partial\Omega}]
\]

\[
= \partial_t \bar{\omega} + v \cdot \nabla \bar{\omega} + w \cdot \nabla \bar{\omega} = \partial_t \bar{\omega} + \bar{u} \cdot \nabla \bar{\omega} = 0
\]

so relation (2.13) holds true. And vice versa, if (2.13) holds then we deduce that the left hand side of (5.1) has zero curl so it must be a gradient.

We now prove (5.2). As \(W^{1,p}_{\text{loc}} \subset C^0\), \(v(s)\) is well defined. Next, it suffices to prove the equality for smooth \(v\), since we can pass to the limit on a subsequence of smooth approximations of \(v\) which converges strongly in \(W^{1,p}_{\text{loc}}\) and \(C^0\). Now, it is trivial to check that, for a \(2 \times 2\) matrix \(A\) with distribution coefficients, we have

\[
\text{curl div } A = \text{div } \begin{pmatrix} \text{curl } C_1 \\ \text{curl } C_2 \end{pmatrix}
\]

where \(C_i\) denotes the \(i\)-th column of \(A\). For smooth \(v\), we deduce

\[
\text{curl } [v \cdot \nabla w + w \cdot \nabla v] = \text{curl div } (v \otimes w + w \otimes v)
\]

\[
= \text{div } \begin{pmatrix} \text{curl } (v w_1) + \text{curl } (w v_1) \\ \text{curl } (v w_2) + \text{curl } (w v_2) \end{pmatrix}
\]

\[
= \text{div } (w \text{ curl } v + v \cdot \nabla w + v \cdot \nabla v).
\]

It is a simple computation to check that

\[
\text{div } (v \cdot \nabla w + w \cdot \nabla v) = v \cdot \nabla \text{div } w + w \cdot \nabla \text{div } v + \text{curl } v \text{ div } w + \text{curl } w \text{ div } v.
\]

Taking into account that we have free divergence fields, we can finish by writing

\[
\text{curl } [v \cdot \nabla w + w \cdot \nabla v] = \text{div } (w \text{ curl } v + v g_{\omega,\gamma}(s) \cdot \delta_{\partial\Omega}) = \text{div } (w \text{ curl } v) + \text{curl } [v(s)^{1/2}g_{\omega,\gamma}(s) \cdot \delta_{\partial\Omega}].
\]

\(^4\)The original proof comes from [7] and we copy it for a sake of clarity.
which proves \((5.2)\).

### 5.2. Proof of Theorems 1.2,1.3

The goal is to adapt the proof of Yudovich: let \(u_1\) and \(u_2\) be two weak solutions of \((1.1)\) (Theorem 1.1) from the same initial data \(u_0\) verifying \((1.4)-(1.5)\). We define as above \(v_1\), \(v_2\) (resp. \(w_1\), \(w_2\)) associated to \(\omega_1 := \text{curl} \, u_1\) (resp. \(\omega_2 := \text{curl} \, u_2\)) and \(\gamma_0\) (see (2.10) and (2.20)). We denote

\[
\tilde{\omega} := \omega_1 - \omega_2
\]

where the bar means that we extend by zero outside \(\Omega\) and

\[
\tilde{v} := v_1 - v_2,
\]

which verifies

\[
\partial_t \tilde{v} + \tilde{v} \cdot \nabla v_1 + v_2 \cdot \nabla \tilde{v} + \text{div} \, (\tilde{v} \otimes w_1 + v_2 \otimes \tilde{w} + w_1 \otimes \tilde{v} + \tilde{w} \otimes v_2) = (v_1(s) - \tilde{v}(s) - \tilde{w}(s)) \cdot \delta_{\gamma_0} = -\nabla \tilde{p} \tag{5.3}
\]

Next, we will multiply by \(\tilde{\omega}\) and integrate. The difficulty compared with the Yudovich’s original proof is that we have some terms as \(\int_{\mathbb{R}^2} |w_1| ||\tilde{v}|| \nabla \tilde{v}|\) with \(w_1\) blowing up near the corners. The general idea is to divide such an integral in two parts: on \(U\) a small neighborhood of the boundary where the vorticity vanishes (see Proposition 4.1) and on \(\mathbb{R}^2 \setminus U\) where the velocity \(w_1\) is regular. Far from the boundary, we follow what Yudovich did, and near the boundary we compute

\[
\int_U |w_1| ||\tilde{v}|| \nabla \tilde{v}| \leq ||w_1||_{L^1(U)} ||\tilde{v}||_{L^\infty(U)} ||\nabla \tilde{v}||_{L^\infty(U)}
\]

Indeed \(w_1\) is integrable near the boundary, and as \(\tilde{v}\) is harmonic in \(U\) (\(\text{div} \, \tilde{v} = \text{curl} \, \tilde{v} = 0\)), then we have

\[
\int_U |w_1| ||\tilde{v}|| \nabla \tilde{v}| \leq C ||\tilde{v}||_{L^2(U)}^2
\]

which will allow us to conclude by the Gronwall’s lemma. We see here why Proposition 4.1 is the main key of the uniqueness proof.

This idea was used in [13] in order to prove the uniqueness of the vortex-wave system, and we follow the same plan.

We denote by \(W^{1,4}_\sigma(\mathbb{R}^2)\) the set of functions belonging to \(W^{1,4}(\mathbb{R}^2)\) and which are divergence-free in the sense of distributions, and by \(W^{-1,4/3}_\sigma(\mathbb{R}^2)\) its dual space.

First, we prove that we can multiply by \(\tilde{\omega}\) and integrate. As a consequence of \((5.1)\) and \((5.3)\), we obtain the following properties for \(\tilde{v}\).

**Proposition 5.1.** Let \(u_0\) verifying \((1.3)\), \(u_1\), \(u_2\) be two weak solutions of \((1.1)\) with initial condition \(u_0\). Let \(\tilde{v} = v_1 - v_2\). Then we have

\[
\tilde{v} \in L^2_{\text{loc}}(\mathbb{R}^+, W^{1,4}_{\sigma}(\mathbb{R}^2)), \quad \partial_t \tilde{v} \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4/3}_{\sigma}(\mathbb{R}^2)).
\]

In addition, we have \(\tilde{v} \in C(\mathbb{R}^+, L^2(\mathbb{R}^2))\) and for all \(T \in \mathbb{R}^+\),

\[
||\tilde{v}(T)||_{L^2(\mathbb{R}^2)}^2 = 2 \int_0^T \langle \partial_t \tilde{v}, \tilde{v} \rangle_{W^{-1,4/3}_{\sigma}, W^{1,4}_{\sigma}} \, ds, \quad \forall T \in \mathbb{R}^+.
\]

The proof follows easily from the estimates established in Section 2. The reader can find the details in Section 6.

Now, we take advantage of the fact that \(\omega_i\) is equal to zero near \(\partial \Omega\) (Proposition 4.1) to give harmonic regularity estimates on \(\tilde{v}(t)\).

**Lemma 5.2.** Let \(T^* > 0\). We assume that \(\omega_0\) is compactly supported in \(\Omega\) and has the sign conditions of Proposition 4.7 (or of Remark 4.3). Then, there exists a neighborhood \(U_{T^*}\) of \(\partial \Omega\) such that for all \(t \leq T^*\), \(\tilde{v}(t, \cdot)\) is harmonic on \(U_{T^*}\). In particular, for \(\partial \Omega\), an open set such that \(\partial \Omega \subset U_{T^*} \subset U_{T^*}\), we have the following estimates:

1. \(\|\tilde{v}(t, \cdot)\|_{L^\infty(U_{T^*})} \leq C \|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)},\)
(2) \(\|\nabla \tilde{v}(t, \cdot)\|_{L^\infty(O_{T^*})} \leq C\|\tilde{v}(t, \cdot)\|_{L^2(R^2)},\)
where \(C\) only depends on \(O_{T^*}\).

The proof is a direct consequence of the mean-value formula (see e.g. the proof of Lemma 3.9 in [13]). In order to prepare the Gronwall estimate, we establish the following estimates on \(w_1 - w_2\).

**Lemma 5.3.** Let \(T^* > 0\) and \(\partial \Omega \subset O_{T^*} \subset U_{T^*}\) as Lemma 5.2 Then \(\tilde{w} := w_1 - w_2\) verifies the following estimates for any \(t \in [0, T^*]::
\begin{enumerate}
\item \(\|\tilde{w}(t, \cdot)\|_{L^2(R^2)} \leq 2\|\bar{v}(t, \cdot)\|_{L^2(R^2)},\)
\item \(\|\bar{w}(t, \cdot)\|_{L^\infty(O_{T^*})} \leq C\|\tilde{v}(t, \cdot)\|_{L^2(R^2)},\)
\item \(\|\nabla \bar{w}(t, \cdot)\|_{L^2(O_{T^*})} \leq C\|\tilde{v}(t, \cdot)\|_{L^2(R^2)},\)
\end{enumerate}
where \(C\) only depends on \(O_{T^*}\).

**Proof.** We fix \(t \in [0, T^*]\) and we denote \(\bar{u} := \bar{u}_1 - \bar{u}_2\). From the explicit formula and the conservation law, we have that
\[
\begin{cases}
\text{div} \bar{u} = 0 & \text{on} \ \Omega, \\
\text{curl} \bar{u} = \bar{\omega} & \text{on} \ \Omega, \\
\bar{u} \cdot \bar{n} = 0 & \text{on} \ \partial \Omega, \\
\int \bar{u} \cdot \bar{\tau} = 0 & \text{only if} \ \Omega \ \text{is an exterior domain}, \\
\lim_{|x| \to \infty} |\bar{u}| = 0 & \text{only if} \ \Omega \ \text{is an exterior domain},
\end{cases}
\]
and
\[
\begin{cases}
\text{div} \bar{v} = 0 & \text{on} \ \Omega, \\
\text{curl} \bar{v} = \bar{\omega} & \text{on} \ \Omega, \\
\int \bar{v} \cdot \bar{\tau} = 0 & \text{only if} \ \Omega \ \text{is an exterior domain}, \\
\lim_{|x| \to \infty} |\bar{v}| = 0 & \text{only if} \ \Omega \ \text{is an exterior domain}.
\end{cases}
\]
Indeed, in the case of exterior domain, \(\bar{\omega} \equiv 0\) on \(C\) which implies that the circulation of \(\bar{v}\) around \(C\) is equal to zero. Therefore, we have the following.

**Lemma 5.4.** \(\bar{u}\) is the orthogonal projection of \(\bar{v}\) on the set of the vector field defined on \(\Omega\) square integrable, divergence free and tangent to the boundary. Therefore we have:
\[
\|\bar{u}(t, \cdot)\|_{L^2(\Omega)} \leq \|\bar{v}(t, \cdot)\|_{L^2(\Omega)}.
\]

This lemma is a classical property of the Leray projector in arbitrary domains (see [2] Theorem 1.1 in Chapter III.1.1). Then the first point is a direct consequence of this lemma:
\[
\|\bar{w}(t, \cdot)\|_{L^2(R^2)} \leq \|\bar{u}(t, \cdot)\|_{L^2(\Omega)} + \|\bar{v}(t, \cdot)\|_{L^2(\Omega)} \leq \|\bar{v}(t, \cdot)\|_{L^2(\Omega)} + \|\tilde{v}(t, \cdot)\|_{L^2(R^2)} \leq 2\|\tilde{v}(t, \cdot)\|_{L^2(R^2)}.
\]

The second point is exactly the same thing as in Lemma 5.2 \(\tilde{w}\) is harmonic in \(\Omega\) then there exists \(C\) depending on \(O_{T^*}\) such that
\[
\|\bar{w}(t, \cdot)\|_{L^\infty(O_{T^*})} \leq C\|\tilde{w}(t, \cdot)\|_{L^2(\Omega)} \leq 2C\|\tilde{v}(t, \cdot)\|_{L^2(R^2)}.
\]

Another consequence of the mean-value Theorem is that
\[
\|\nabla \bar{w}(t, \cdot)\|_{L^2(O_{T^*})} \leq C\|\tilde{w}(t, \cdot)\|_{L^2(\Omega)} \leq 2C\|\tilde{v}(t, \cdot)\|_{L^2(R^2)}.
\]

Indeed, there is \(R_1\) such that \(\text{dist}(\partial \Omega, \partial O_{T^*}) > R_1\), then
\[
\|\nabla \tilde{w}(t, x)\|_{L^2(O_{T^*})} = \left\| \frac{1}{\pi R_1^2} \int_{B(x, R_1)} \nabla \tilde{w}(t, y) \, dy \right\|_{L^2(\Omega)} \leq \frac{1}{\pi R_1} \int_0^{2\pi} \|\tilde{w}(t, x + R_1 e^{i\theta})\|_{L^2(\Omega)} \, d\theta \leq \frac{2\|\tilde{v}(t, \cdot)\|_{L^2(\Omega)}}{R_1}.
\]
\[\square\]
Remark 5.5. We remark that the result from Galdi’s book does not require regularity of $\partial \Omega$ when we consider the $L^2$ norm (thanks to the Hilbert structure). In contrast for $p \neq 2$, he states that the Leray projector is continuous from $L^p$ to $L^p$ if the boundary $\partial \Omega$ is $C^2$. Indeed, in our case we see that $\tilde{v}$ belongs to $L^p$ for any $p > 1$, whereas $\tilde{u} = \mathbb{P}\tilde{v}$ does not belongs in $L^p(\Omega)$ for some $p > 4$ (if there is an angle greater than $\pi$, see Remark 2.2).

We can adapt now the Yudovich proof, as it is done in [13]. We fix $T^* > 0$ in order to fix $O_T^*$ in Lemmata 5.2 and 5.3. We consider smooth and divergence-free functions $\Phi_n \in C_c^\infty (\mathbb{R}^+ \times \mathbb{R}^2)$ converging to $\tilde{v}$ in $L^2_{\text{loc}} (\mathbb{R}^+ , W^{1,4}(\mathbb{R}^2))$ as test functions in (5.3), and let $n$ goes to $+\infty$. First, we have for all $T \in [0, T^*]$

$$\int_0^T \langle \partial_t \tilde{v} , \Phi_n \rangle_{W^{-1,4/3}_{\sigma}, W^{1,4}_{\sigma}} ds \to \int_0^T \langle \partial_t \tilde{v} , \tilde{v} \rangle_{W^{-1,4/3}_{\sigma}, W^{1,4}_{\sigma}} ds,$$

and we deduce the limit in the other terms from the several bounds for $v_i$ stated in the proof of Proposition 5.1. This yields

$$\frac{1}{2} \| \tilde{v}(T, \cdot) \|_{L^2}^2 = I + J + K, \quad (5.4)$$

where

$$I = - \int_0^T \int_{\mathbb{R}^2} \tilde{v} \cdot (\tilde{v} \cdot \nabla v_1 + v_2 \cdot \nabla \tilde{v}) \, dx \, dt,$$

$$J = \int_0^T \int_{\mathbb{R}^2} (\tilde{v} \otimes w_1 + v_2 \otimes \tilde{w} + w_1 \otimes \tilde{v} + \tilde{w} \otimes v_2) : \nabla \tilde{v} \, dx \, dt,$$

$$K = \int_0^T \int_{\partial \Omega} v_1(s) \frac{1}{\tilde{v}^2} \tilde{v} \cdot \tilde{v}(s) \, ds.$$

The goal is to estimate all the terms in the right-hand side in order to obtain a Gronwall-type inequality.

For the first term $I$ in (5.4), we begin by noticing that

$$\int_{\mathbb{R}^2} (v_2 \cdot \nabla \tilde{v}) \cdot \tilde{v} \, dx = \frac{1}{2} \int_{\mathbb{R}^2} v_2 \cdot \nabla |\tilde{v}|^2 \, dx = -\frac{1}{2} \int_{\mathbb{R}^2} |\tilde{v}|^2 \div v_2 \, dx = 0,$$

where we have used that $v_2 = O(1/|x|)$ and $\tilde{v} = O(1/|x|^2)$ at infinity. Moreover, Hölder’s inequality gives

$$\left| \int_{\mathbb{R}^2} (\tilde{v} \cdot \nabla v_1) \cdot \tilde{v} \, dx \right| \leq \|\tilde{v}\|_{L^2} \|\tilde{v}\|_{L^p} \|\nabla v_1\|_{L^q},$$

with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. On one hand, Calderón-Zygmung inequality states that $\|\nabla v_1\|_{L^p} \leq Cp\|\omega_1\|_{L^p}$ for $p \geq 2$. On the other hand, we write by interpolation $\|\tilde{v}\|_{L^q} \leq \|\tilde{v}\|_{L^a}^{a/q} \|\tilde{v}\|_{L^\infty}^{1-a/q}$ with $\frac{1}{q} = \frac{a}{2} + \frac{1-a}{\infty}$. We have that $a = 1 - \frac{2}{p}$, so we are led to

$$|I| \leq Cp \int_0^T \|\tilde{v}\|_{L^2}^{2-2/p} \, dt. \quad (5.5)$$

We now estimate $J$. We have

$$\int_{\mathbb{R}^2} (\tilde{v} \otimes w_1) : \nabla \tilde{v} \, dx = \int_{\mathbb{R}^2} \sum_{i,j} \tilde{v}_i w_{1,j} \partial_j \tilde{v}_i \, dx = \frac{1}{2} \sum_i \int_{\mathbb{R}^2} \sum_j w_{1,j} \partial_j \tilde{v}_i^2 \, dx$$

$$= -\frac{1}{2} \sum_i \int_{\mathbb{R}^2} \tilde{v}_i^2 \div w_1 \, dx = 0,$$

since $w_1$ is divergence-free, and

$$\left| \int_0^T \int_{\mathbb{R}^2} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right| \leq \left| \int_0^T \int_{O_T^*} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right| + \left| \int_0^T \int_{O_T^*} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right|. \quad (5.6)$$
We perform an integration by part for the second term in the right-hand side of (5.6). Arguing that $\text{div} \tilde{v} = 0$, we obtain
\[
\left| \int_0^T \int_{\mathbb{R}^2} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right| \leq \left| \int_0^T \int_{\Omega_T} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right|
+ \left| \int_0^T \left( \int_{\partial \Omega_T} (\tilde{v} \cdot \nabla w_1) \cdot \tilde{v} \, ds \right) \right| \leq \int_0^T \| w_1 \|_{L^1(\Omega_T)} \| \tilde{v} \|_{L^\infty(\partial \Omega_T)} \| \nabla \tilde{v} \|_{L^\infty(\partial \Omega_T)} \, dt
+ \int_0^T \| \nabla w_1 \|_{L^\infty(\partial \Omega_T)} \| \tilde{v} \|_{L^2} \, dt
+ \int_0^T \| w_1 \|_{L^\infty(\partial \Omega_T)} \| \tilde{v} \|_{L^\infty(\partial \Omega_T)} \| \nabla \tilde{v} \|_{L^\infty(\partial \Omega_T)} \, dt.
\]
As we remarked when we introduce $w$: $\| w_1 \|_{L^1(\Omega_T)} \leq C$ with $C$ depending only on $\Omega$, $T^*$ and $u_0$. Moreover, using the harmonicity of $w_1$, we know that $\| \nabla w_1 \|_{L^\infty(\partial \Omega_T)}$ is bounded by a constant times $\| w_1 \|_{L^\infty(V_{T^*})}$, with $\partial \Omega \subset V_{T^*} \subset \Omega_T$. Using the behavior of $DT$ at infinity (Proposition 2.3), Proposition 2.5, conservation laws (2.18), (2.19), (2.20), then (2.9) allows us to state that $\| w_1 \|_{L^\infty(0, T^*)}$ and $C_0$ depending only on $\Omega$, $T^*$ and $u_0$. As $v_1$ is uniformly bounded, we obtain that $\| \nabla w_1 \|_{L^\infty(\partial \Omega_T)}$ and $\| w_1 \|_{L^\infty(\partial \Omega_T)}$ is bounded uniformly in $(0, T^*)$. Then, according to Lemma 5.2 this gives
\[
\left| \int_0^T \int_{\mathbb{R}^2} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right| \leq C \int_0^T \| \tilde{v} \|_{L^2}^2 \, dt.
\]
In the same way, we obtain by integration by part
\[
\left| \int_0^T \int_{\mathbb{R}^2} (v_2 \otimes \tilde{w}) : \nabla \tilde{v} \, dx \, dt \right| \leq \left| \int_0^T \int_{\Omega_T} (v_2 \otimes \tilde{w}) : \nabla \tilde{v} \, dx \, dt \right|
+ \left| \int_0^T \left( \int_{\partial \Omega_T} (\tilde{w} \cdot \nabla v_2) \cdot \tilde{v} \, ds \right) \right| \leq \int_0^T \| v_2 \|_{L^2(\Omega_T)} \| \tilde{v} \|_{L^2(\Omega_T)} \| \nabla \tilde{v} \|_{L^\infty(\partial \Omega_T)} \, dt
+ \int_0^T \| \tilde{w} \|_{L^\infty(\partial \Omega_T)} \| \tilde{v} \|_{L^2} \| v_2 \|_{L^2} \, dt
+ \int_0^T \| \tilde{w} \|_{L^\infty(\partial \Omega_T)} \| \tilde{v} \|_{L^\infty(\partial \Omega_T)} \| v_2 \|_{L^\infty(\partial \Omega_T)} \| \nabla \tilde{v} \|_{L^\infty(\partial \Omega_T)} \, dt.
\]
Using again Calderón-Zygmund inequality for $v_2$ and Lemmata 5.2 and 5.3 we get
\[
\left| \int_0^T \int_{\mathbb{R}^2} (v_2 \otimes \tilde{w}) : \nabla \tilde{v} \, dx \, dt \right| \leq C \int_0^T \| \tilde{v} \|_{L^2}^2 \, dt.
\]
A very similar computation yields
\[
\left| \int_0^T \int_{\mathbb{R}^2} (\tilde{w} \otimes v_2) : \nabla \tilde{v} \, dx \, dt \right| \leq C \int_0^T \left( \| \tilde{w} \|_{L^2(\Omega_T)} + \| \nabla \tilde{w} \|_{L^2(\Omega_T)} + \| \tilde{w} \|_{L^\infty(\partial \Omega_T)} \right) \| \tilde{v} \|_{L^2} \, dt
\leq C \int_0^T \| \tilde{v} \|_{L^2}^2 \, dt.
\]
Therefore, we arrive at
\[
|J| \leq 3C \int_0^T \| \tilde{v} \|_{L^2}^2 \, dt. \quad (5.7)
\]
Finally, using (2.16) we write the third term $K$ in (4.4) as follows:

$$K = \pm \int_0^T \int_{\partial \Omega} (\bar{u} \cdot \hat{\tau})(v_1^+ \cdot \bar{\nu}) \, ds$$

$$= \pm \int_0^T \int_{\Omega} \text{curl } \bar{u} (v_1^+ \cdot \bar{\nu}) \, dx \pm \int_0^T \int_{\Omega} \bar{u} \cdot \nabla^\perp (v_1^+ \cdot \bar{\nu}) \, dx,$$

where $\pm$ depends if we treat exterior or interior domains. Using that $\text{curl } \bar{u} = \text{curl } \bar{\nu}$ in $\Omega$, $\text{div } \bar{\nu} = 0$ and the behaviors at infinity, we obtain by several integrations by parts:

$$\int_{\Omega} \text{curl } \bar{u} (v_1^+ \cdot \bar{\nu}) \, dx = \int_{\Omega} \text{curl } \bar{\nu} (v_1^+ \cdot \bar{\nu}) \, dx = \int_{\mathbb{R}^2} \text{curl } \bar{\nu} (v_1^+ \cdot \bar{\nu}) \, dx$$

$$= \int_{\mathbb{R}^2} (-v_{1,2} \hat{\nu}_1 \partial_1 \bar{\nu}_2 + v_{1,2} \partial_2 \hat{\nu}_1^+ |\hat{\nu}_1|^2 + v_{1,1} \partial_1 \hat{\nu}_2 |\hat{\nu}_2|^2 - v_{1,1} \bar{\nu}_2 \partial_2 \hat{\nu}_1) \, dx$$

$$= \int_{\mathbb{R}^2} (\partial_1 v_{1,2} \hat{\nu}_1 \partial_1 \bar{\nu}_2 + v_{1,2} \partial_1 \hat{\nu}_1 |\hat{\nu}_2|^2 - v_{1,1} \partial_1 \hat{\nu}_2 |\hat{\nu}_1|^2 + v_{1,1} \partial_2 \hat{\nu}_2 \bar{\nu}_1 + v_{1,1} \partial_2 \bar{\nu}_2 \bar{\nu}_1) \, dx$$

$$= \int_{\mathbb{R}^2} (\partial_1 v_{1,2} \hat{\nu}_1 \partial_1 \bar{\nu}_2 + v_{1,2} \partial_1 \hat{\nu}_2 |\hat{\nu}_2|^2 - v_{1,1} \partial_1 \hat{\nu}_1 |\hat{\nu}_1|^2 + v_{1,1} \partial_2 \hat{\nu}_2 \bar{\nu}_1 + v_{1,1} \partial_2 \bar{\nu}_2 \bar{\nu}_1) \, dx.$$

Hence,

$$\int_0^T \int_{\Omega} \text{curl } \bar{u} (v_1^+ \cdot \bar{\nu}) \, dx dt \leq 4 \int_0^T \int_{\mathbb{R}^2} |\nabla v_1| |\bar{\nu}|^2 \, dx dt$$

which gives by Calderón-Zygmund inequality (as for $I$):

$$\int_0^T \int_{\Omega} \text{curl } \bar{u} (v_1^+ \cdot \bar{\nu}) \, dx dt \leq C_p \int_0^T \|\bar{\nu}\|_{L_{2/p}}^{2-2/p} \, dt.$$
As we choose $p > 2$ and as $\|\tilde{v}\|_{L^2} \leq C_0$ for all $t \in [0, T^*)$ (see Proposition 5.11), we have $\|\tilde{v}\|^{2/p}_{L^2} \leq C_0^{2/p}$ which implies that for $p$ large enough, the previous inequality gives

$$\|\tilde{v}(T, \cdot)\|^2_{L^2} \leq 2C_p \int_0^T \|\bar{v}\|^{2-2/p}_{L^2} \, dt.$$  

Using a Gronwall-like argument, this implies

$$\|\tilde{v}(T, \cdot)\|^2_{L^2} \leq (2CT)^p, \quad \forall p \geq 2.$$  

Letting $p$ tend to infinity, we conclude that $\|\tilde{v}(T, \cdot)\|_{L^2} = 0$ for all $T < \min(T^*, 1/(2C))$. Finally, we consider the maximal interval of $[0, T^*)$ on which $\|\tilde{v}(T, \cdot)\|_{L^2} \equiv 0$, which is closed by continuity of $\|\tilde{v}(T, \cdot)\|_{L^2}$. If it is not equal to the whole of $[0, T^*)$, we may repeat the proof above, which leads to a contradiction by maximality. Therefore uniqueness holds on $[0, T^*)$, and this concludes the proof of Theorems 1.2 and 1.3. Indeed, Lemma 5.3 implies that $\|u_1 - u_2\|_{L^2} \leq \|\tilde{w}\|_{L^2} + \|\tilde{v}\|_{L^2} \leq 2\|\tilde{v}\|_{L^2}$.

6. Technical results

We will use several times the following from [6]:

**Lemma 6.1.** Let $S \subset \mathbb{R}^2$, $\alpha \in (0, 2)$ and $g : S \to \mathbb{R}^+$ be a function belonging in $L^1(S) \cap L^r(S)$, for $r > \frac{2}{2-\alpha}$. Then

$$\int_S \frac{g(y)}{|x-y|^{\alpha}} \, dy \leq C \|g\|_{L^1(S)}^{2-2/r} \|g\|_{L^r(S)}^{2/r}.$$  

**6.1. Proof of Proposition 2.5.** We make the proof in the unbounded case (which is the hardest case). We decompose $R[\omega]$ in two parts:

$$R_1(x) := \int_{\Omega} \frac{(T(x) - T(y))_{\perp}}{|T(x) - T(y)|^2} \omega(y) \, dy \quad \text{and} \quad R_2(x) := \int_{\Omega} \frac{(T(x) - T(y))_{\perp}}{|T(x) - T(y)|^2} \omega(y) \, dy.$$  

a) Estimate and continuity of $R_1$.

Let $z := T(x)$ and $f(\eta) := \omega(T^{-1}(\eta)) \det(DT^{-1}(\eta)) \chi_{\{|\eta| \geq 1\}}$, with $\chi_E$ the characteristic function of the set $E$. Making the change of variables $\eta = T(y)$, we find

$$R_1(T^{-1}(z)) = \int_{\mathbb{R}^2} \frac{(z - \eta)_{\perp}}{|z - \eta|^2} f(\eta) \, d\eta.$$  

Changing variables back, we get

$$\|f\|_{L^1(\mathbb{R}^2)} = \|\omega\|_{L^1}.$$  

We choose $p_0 > 2$ such that $\det(DT^{-1})$ belongs to $L^{p_0}_\infty(\Omega)$ (see Remark 2.2). If all the angles are greater than $\pi$, we can choose $p_0 = \infty$ (thanks to Theorem 2.1 and Proposition 2.3) and we would have $\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|\omega\|_{L^\infty}$. However, if there is one angle less than $\pi$, we have to decompose the integral in two parts:

$$R_1(T^{-1}(z)) = \int_{|\eta| \geq 2} \frac{(z - \eta)_{\perp}}{|z - \eta|^2} f(\eta) \, d\eta + \int_{|\eta| \leq 2} \frac{(z - \eta)_{\perp}}{|z - \eta|^2} f(\eta) \, d\eta$$  

with

$$\|f\|_{L^\infty(\mathbb{R}^2 \setminus B(0, 2))} \leq C_1 \|\omega\|_{L^\infty}$$  

by Proposition 2.3 and

$$\|f\|_{L^{p_0}(B(0, 2))} \leq C_2 \|\omega\|_{L^{p_0}}$$  

by Remark 2.2. Then we use the classical estimate for the Biot-Savart kernel in $\mathbb{R}^2$ (see Lemma 6.1):

$$\int_{|\eta| \geq 2} \frac{(z - \eta)_{\perp}}{|z - \eta|^2} f(\eta) \, d\eta \leq C_0 \|f\|_{L^1(B(0, 2))}^{1/2} \|f\|_{L^2(\mathbb{R}^2 \setminus B(0, 2))}^{1/2} \leq C_4 \|\omega\|^{1/2}_{L^1} \|\omega\|^{1/2}_{L^\infty}$$  

and

$$\int_{|\eta| \leq 2} \frac{(z - \eta)_{\perp}}{|z - \eta|^2} f(\eta) \, d\eta \leq C_0 \|f\|_{L^{p_0}(B(0, 2))}^{p_0/2} \|f\|_{L^{p_0}(\mathbb{R}^2 \setminus B(0, 2))}^{p_0/2} \leq C_5 \|\omega\|^{p_0/2}_{L^1} \|\omega\|^{p_0/2}_{L^{p_0}}.$$  


which gives the uniform estimate
\[ \| R_1 \|_{L^\infty(\Omega)} \leq C(\| \omega \|_{L^1}^{1/2} \| \omega \|_{L^\infty}^{1/2} + \| \omega \|_{L^1} \| \omega \|_{L^\infty}^{1-a}) \]
with \( a = \frac{p_0 - 2}{2(p_0 - 1)} \) including in \((0, 1/2] \). Concerning the continuity, we approximate \( f_{\chi B(0,2)} \) by \( f_n \in C_0^\infty(B(0,2)) \) and \( f_{\chi B(0,2)^c} \) by \( g_n \in C_0^\infty(B(0,2)^c) \) such that
\[ \| f_n - f \|_{L^1(B(0,2))} \to 0, \quad \| g_n - f \|_{L^1(B(0,2)^c)} \to 0, \quad \| g_n \|_{L^\infty} \leq C(f) \text{ as } n \to \infty. \]
As \( f_n \) and \( g_n \) are smooth, we infer that the functions
\[ z \mapsto \int_{\mathbb{R}^2} \frac{\xi^1}{|\xi|^2} f_n(z - \xi) \, d\xi \quad \text{and} \quad t \mapsto \int_{\mathbb{R}^2} \frac{\xi^1}{|\xi|^2} g_n(z - \xi) \, d\xi \]
are continuous. Moreover, we deduce from the previous estimates that
\[ \| R_1(\mathcal{T}^{-1}(z)) - \int_{|\eta| \geq 1} \frac{(z - \eta^*)^\perp}{|z - \eta^*|^2} f(\eta) \, d\eta - \int_{|\eta| \geq 2} \frac{(z - \eta^*)^\perp}{|z - \eta^*|^2} f(\eta) \, d\eta \leq C_0 \left( \| f - g_n \|_{L^1(B(0,2)^c)}^1 \right. \]
\[ \left. + \| f - g_n \|_{L^1(B(0,2)^c)}^1 \right) \]
Thanks to the limit \( n \to \infty \), we prove the continuity of \( R_1 \circ \mathcal{T}^{-1} \). Using Theorem 2.1, we conclude that \( R_1 \) is continuous up to the boundary.

b) Estimate and continuity of \( R_2 \).
We use, as before, the notations \( f, z \) and the change of variables \( \eta \)
\[ R_2(\mathcal{T}^{-1}(z)) = \frac{\int_{|\eta| \geq 1} (z - \eta^*)^\perp f(\eta) \, d\eta}{|z - \eta^*|^2} \]
\[ = \frac{\int_{|\eta| \geq 2} (z - \eta^*)^\perp f(\eta) \, d\eta + \int_{1 \leq |\eta| \leq 2} (z - \eta^*)^\perp f(\eta) \, d\eta}{|z - \eta^*|^2} \]
\[ = R_{21}(z) + R_{22}(z). \]
If \( |\eta| \geq 2, |z - \eta^*| \geq 1/2 \) because \( |z| \geq 1 \) (see the definition of \( \mathcal{T} \)). Therefore, we obtain obviously that
\[ \| R_{21} \|_{L^\infty(B(0,1)^c)} \leq 2 \| f \|_{L^1(B(0,2)^c)} \leq 2 \| \omega \|_{L^1}. \]
The continuity is easier than above:
- we approximate \( f_{\chi B(0,2)^c} \) by \( g_n \in C_0^\infty(B(0,2)^c) \) such that \( \| g_n - f \|_{L^1(B(0,2)^c)} \to 0 \) as \( n \to \infty \);
- the functions
\[ z \mapsto \int_{|\eta| \geq 2} \frac{(z - \eta^*)^\perp}{|z - \eta^*|^2} g_n(\eta) \, d\eta \]
is continuous up to the boundary \( \partial B(0,1) \) because \( |z - \eta^*| \geq 1/2 \);
- the previous estimates gives
\[ \| R_{21}(z) \|_{L^\infty(B(0,1)^c)} \leq 2 \| f - g_n \|_{L^1(B(0,2)^c)}; \]
which gives the continuity of \( R_{21} \).
Concerning \( R_{22} \), we again change variables writing \( \theta = \eta^* \), to obtain:
\[ R_{22}(z) = \frac{\int_{1/2 \leq |\theta| \leq 1} (z - \theta^*)^\perp f(\theta^*) \, d\theta}{|z - \theta^*|^2}. \]
Let \( g(\theta) := f(\theta^*) \). As above, we deduce by changing variables back that
\[ \| g \|_{L^1(1/2 \leq |\theta| \leq 1)} \leq \| \omega \|_{L^1}. \]
It is also easy to see that
\[ \| g \|_{L^p(1/2 \leq |\theta| \leq 1)} \leq 2^{4(p_0 - 1)} \left[ \frac{p_0}{p_0} \right] \| f \|_{L^p(B(0,2))} \leq C_0 \| \omega \|_{L^\infty}. \]
Then, by the classical estimates of the Biot-Savart law in \( \mathbb{R}^2 \), we have
\[
\|R_{22}\|_{L^\infty(B(0,1)^c)} \leq C\|\omega\|_{L^1}^2\|\omega\|_{L^\infty}.
\]
Reasoning as for \( R_1 \), where we approximate \( g \), we get that \( R_{22} \) is continuous.

The continuity of \( T \) allows us to conclude that \( R_2 \) is continuous up to the boundary, which ends the proof in the case of \( \Omega \) unbounded.

**Remark about the bounded case.**
Concerning \( R_1 \), we do not need to decompose the integral in two parts:
\[
\|f\|_{L^p_0(B(0,1))} \leq C_2\|\omega\|_{L^\infty},
\]
where \( f(\eta) := \omega(T^{-1}(\eta))|\det(DT^{-1}(\eta))|\chi(|\eta| \leq 1) \).

Even of \( R_2 \), we directly have
\[
R_2(T^{-1}(z)) = \int_{|\eta| \geq 1} \frac{(z - \theta)^T}{|z - \theta|^2} f(\theta^*) \frac{d\theta}{|\theta|^4}
\]
and we conclude following the proof concerning \( R_{22} \).

### 6.2. Proof of Lemma 2.7
Using the explicit formula of \( \Phi \) and (2.9), we write
\[
u(x) \cdot \nabla \Phi^\varepsilon(x) = u^\varepsilon(x) \cdot \nabla \Phi^\varepsilon(x)
= -\frac{1}{2\pi\varepsilon} \Phi^\varepsilon\left(\frac{|T(x)| - 1}{\varepsilon}\right) \int_\Omega \left(\frac{T(x) \cdot T(y)}{|T(x) - T(y)|^2} - \frac{T(x) \cdot T(y)^*}{|T(x) - T(y)^*|^2}\right) \omega(t, y) dy
\times DT(x)DT^T(x)T(x)^{-1} |T(x)|.
\]
As \( T \) is holomorphic, \( DT \) is of the form \( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) and we can check that \( DT(x)DT^T(x) = (a^2 + b^2)Id = |\det(DT)(x)|Id \), so
\[
u(x) \cdot \nabla \Phi^\varepsilon(x) = \frac{\Phi^\varepsilon\left(\frac{|T(x)| - 1}{\varepsilon}\right) |\det(DT)(x)|}{2\pi\varepsilon|T(x)|} \int_\Omega \left(\frac{T(y) \cdot T(x)}{|T(x) - T(y)|^2} - \frac{T(y)^* \cdot T(x)^*}{|T(x) - T(y)^*|^2}\right) \omega(t, y) dy.
\]
We compute the \( L^1 \) norm, next we change variables twice \( \eta = T(y) \) and \( z = T(x) \), to have
\[
\|\nu \cdot \nabla \Phi^\varepsilon\|_{L^1} = \frac{1}{2\pi\varepsilon} \int_{|z| \geq 1} \left|\Phi^\varepsilon\left(\frac{|z| - 1}{\varepsilon}\right)\right| \int_{|\eta| \geq 1} \left|\frac{(\eta \cdot z^\perp/|z| - \eta^* \cdot z^\perp/|z|)\omega(t, \eta)}{|z - \eta|^2} \right| f(t, \eta) d\eta |dz|
\]
where \( f(t, \eta) = \omega(t, T^{-1}(\eta))|\det(DT^{-1}(\eta))| \).

Thanks to the definition of \( \Phi \), we know that \( \|\frac{1}{\varepsilon} \Phi^\varepsilon\left(\frac{|z| - 1}{\varepsilon}\right)\|_{L^1} \leq C \). So it is sufficient to prove that
\[
\left\|\int_{|\eta| \geq 1} \left|\frac{(\eta \cdot z^\perp/|z| - \eta^* \cdot z^\perp/|z|)\omega(t, \eta)}{|z - \eta|^2} \right| f(t, \eta) d\eta \right\|_{L^\infty(1+\varepsilon \leq |z| \leq 1+2\varepsilon) \rightarrow 0 \quad (6.1)}
\]
as \( \varepsilon \to 0 \), uniformly in time.

Let
\[
A := \eta \cdot z^\perp/|z| - \eta^* \cdot z^\perp/|z| \frac{|\eta|}{|z - \eta|^2}.
\]
We compute
\[
A = \left(\frac{|z|^2 - 2z \cdot \eta/|\eta|^2 + 1/|\eta|^2 - 1/|\eta|^2(2z^2 - 2z \cdot \eta + |\eta|^2)}{|z - \eta|^2z - \eta^*|^2}\right)\eta \cdot z^\perp/|z|
= \left(\frac{|z|^2 - 1(1 - 1/|\eta|^2)}{|z - \eta|^2z - \eta^*|^2}\right)\eta \cdot z^\perp/|z|.
\]
We now use that $|z| \geq 1$, to write
\[ |z - \eta^*| \geq 1 - \frac{1}{|\eta|}. \]
Moreover, $|\eta^*| \leq 1$ allows to have
\[ |z - \eta^*| \geq |z| - 1. \]

We can now estimate $A$ by:
\[ |A| \leq \frac{(|z| + 1)(1 + 1/|\eta|)(|z| - 1)^b}{|z - \eta|^2|z - \eta^*|^b}|\eta|^\frac{1}{|z|} \]
with $0 \leq b \leq 1$, to be chosen later. We remark also that $\eta \cdot \frac{z^*}{|z|} = (\eta - z) \cdot \frac{z^*}{|z|}$ and the Cauchy-Schwarz inequality gives
\[ |\eta \cdot \frac{z^*}{|z|}| \leq |\eta - z|. \]

We now use the fact that $|z| - 1 \leq 2\varepsilon$, to estimate (6.1):
\[ \left| \int_{|\eta| \geq 1} Af(t, \eta) \, d\eta \right| \leq (2 + 2\varepsilon) \cdot 2 \cdot (2\varepsilon)^b \int_{|\eta| \geq 1} \frac{|f(t, \eta)|}{|z - \eta|} \, d\eta, \]
hence, the Hölder inequality gives
\[ \left| \int_{|\eta| \geq 1} Af(t, \eta) \, d\eta \right| \leq (2 + 2\varepsilon) \cdot 2 \cdot (2\varepsilon)^b \left\| \frac{|f(t, \eta)|^{1/p}}{|z - \eta|^b} \right\|_{L^p} \left\| \frac{|f(t, \eta)|^{1/q}}{|z - \eta^*|^b} \right\|_{L^q}, \]
with $1/p + 1/q = 1$ chosen later.

In the same way we estimate $R_2$ in the proof of Proposition 2.5, we obtain for $bq = 1$:
\[ \left\| \frac{|f(t, \eta)|^{1/q}}{|z - \eta^*|^b} \right\|_{L^q} = \left( \int_{|\eta| \geq 1} \frac{|f(t, \eta)|}{|z - \eta|^b} \, d\eta \right)^{1/q} \leq C_q, \]
where we have used that $\omega$ belongs to $L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega))$.

Now we use Lemma 6.1 for $f \in L^1 \cap L^{p_0}$, with $p_0 > 2$ and for $f \in L^1 \cap L^\infty$ (see the proof of Proposition 2.5). Then, we choose $p \in (1, 2)$ such that $p_0 > \frac{2}{2 - p}$ and we follow the estimate of $R_1$ in the proof of Proposition 2.5 to obtain:
\[ \left\| \frac{|f(t, \eta)|^{1/p}}{|z - \eta|} \right\|_{L^p} = \left( \int_{|\eta| \geq 1} \frac{|f(t, \eta)|}{|z - \eta|^p} \, d\eta \right)^{1/p} \leq C_p. \]
We have used again that $\omega$ belongs to $L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega))$.

Fixing a $p \in (1, 2)$ such that $p_0 > \frac{2}{2 - p}$, it gives $q \in (2, \infty)$ and $b \in (0, 1/2)$ and it follows
\[ \|u \cdot \nabla \Phi^\varepsilon\|_{L^1} \leq C(2 + 2\varepsilon) \cdot 2 \cdot (2\varepsilon)^b C_p C_q \]
which tends to zero when $\varepsilon$ tends to zero, uniformly in time.

6.3. Proof of Lemma 2.8 Let $T > 0$ fixed. We rewrite (2.9):
\[ u(x) = \frac{1}{2\pi} DT^T(x) \left( \int_\Omega \left( \frac{T(x) - T(y)}{|T(x) - T(y)|^2} - \frac{T(x) - T(y)^*}{|T(x) - T(y)^*|^2} \right)^\perp \omega(y) \, dy + \alpha \frac{T(x)^\perp}{|T(x)|^2} \right) \]
\[ := \frac{1}{2\pi} DT^T(x) h(T(x)) \]
where $\alpha$ is bounded by $\|\gamma\|_{L^\infty([0,T])} + \|\omega\|_{L^\infty(L^1)}$ in $[0,T]$ (see (2.11)).
We start by treating $h$. We change variable $\eta = \mathcal{T}(y)$, and we obtain

$$h(z) = \int_{B(0,1)^c} \left( \frac{z - \eta}{|z - \eta|^2} - \frac{z - \eta^*}{|z - \eta^*|^2} \right) \omega(\mathcal{T}^{-1}(\eta)) |\det D\mathcal{T}^{-1}(\eta)| d\eta + \alpha \frac{|z|}{|z|^2}$$

$$= \int_{B(0,2)^c} \frac{(z - \eta)^+}{|z - \eta|^2} f(t, \eta) d\eta + \int_{B(0,2) \setminus B(0,1)} \frac{(z - \eta^*)^+}{|z - \eta^*|^2} f(t, \eta) d\eta - \int_{B(0,2)^c} \frac{(z - \eta^*)^+}{|z - \eta^*|^2} f(t, \eta) d\eta$$

$$= h_1(z) + h_2(z) - h_3(z) - \alpha h_5(z),$$

with $f(t, \eta) = \omega(t, \mathcal{T}^{-1}(\eta)) |\det D\mathcal{T}^{-1}(\eta)|$ belongs to $L^\infty(B(0,2) \setminus B(0,1))$ with some $p_0 > 2$ and to $L^\infty(B(0,2)^c)$ (see the proof of Proposition 2.5). As $|z| = |\mathcal{T}(x)| \geq 1$, we are looking for estimates in $B(0,1)^c$. Obviously we have that

$$h_5 \text{ belongs to } L^\infty(B(0,1)^c) \text{ and } Dh_5 \text{ belongs to } L^\infty(B(0,1)^c).$$

Concerning $h_1$, we introduce $f_1 := f \chi_{B(0,2)}$ where $\chi_S$ denotes the characteristic function on $S$. Hence

$$h_1(z) = \int_{\mathbb{R}^2} \frac{(z - \eta)^+}{|z - \eta|^2} f_1(\eta) d\eta \text{ with } f_1 \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)).$$

We have used the work made in the proof of Proposition 2.5 about the computation of the $L^p$ norm of $f$ in terms of $\omega$. The standard estimates on Biot-Savart kernel in $\mathbb{R}^2$ and Calderon-Zygmund inequality give that

$h_1$ belongs to $L^\infty(\mathbb{R}^+ \times B(0,1)^c)$ and $Dh_1$ belongs to $L^\infty(\mathbb{R}^+; L^p(B(0,1)^c))$, $\forall p \in (1, \infty)$.

For $h_2$, is almost the same argument: we introduce $f_2 := f \chi_{B(0,2) \setminus B(0,1)}$, hence

$$h_2(z) = \int_{\mathbb{R}^2} \frac{(z - \eta)^+}{|z - \eta|^2} f_2(\eta) d\eta \text{ with } f_2 \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2)).$$

The standard estimates on Biot-Savart kernel in $\mathbb{R}^2$ and Calderon-Zygmund inequality give that

$h_2$ belongs to $L^\infty(\mathbb{R}^+ \times B(0,1)^c)$ and $Dh_2$ belongs to $L^\infty(\mathbb{R}^+; L^{p_0}(B(0,1)^c))$.

For $h_3$, we can remark that for any $\eta \in B(0,2)^c$ we have $|z - \eta^*| \geq \frac{1}{2}$. Therefore, the function $(z, \eta) \mapsto \frac{1}{|z - \eta^*|^2}$ is smooth in $B(0,1)^c \times B(0,2)^c$, which gives us, by a classical integration theorem, that

$h_3$ belongs to $L^\infty(\mathbb{R}^+ \times B(0,1)^c)$ and $Dh_3$ belongs to $L^\infty(\mathbb{R}^+ \times B(0,1)^c)$.

To treat the last term, we change variables $\theta = \eta^*$

$$h_4(z) = \int_{B(0,1) \setminus B(0,1/2)} \frac{(z - \theta)^+}{|z - \theta|^2} f(\theta^*) \frac{d\theta}{|\theta|^2} := \int_{\mathbb{R}^2} \frac{(z - \theta)^+}{|z - \theta|^2} f_4(\theta) d\theta,$$

with $f_4(\theta) := \frac{f(t, \theta^*)}{|\theta|^2} \chi_{B(0,1) \setminus B(0,1/2)}(\theta)$ which belongs to $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2))$. Therefore, standard estimates on Biot-Savart kernel and Calderon-Zygmund inequality give that

$h_4$ belongs to $L^\infty(\mathbb{R}^+ \times B(0,1)^c)$ and $Dh_4$ belongs to $L^\infty(\mathbb{R}^+; L^{p_0}(B(0,1)^c))$.

Now, we come back to $u$. As $u(x) = \frac{1}{2\pi} DT^T(x) h(\mathcal{T}(x))$, with $DT$ belonging to $L^1_{\text{loc}}(\overline{\Omega})$ (see Remark 2.2) and $h \circ \mathcal{T}$ uniformly bounded, we have that

$$u \text{ belongs to } L^\infty([0, T]; L^1_{\text{loc}}(\overline{\Omega})).$$

Adding the bounded behavior of $DT$ at infinity, we have that

$$\bar{u} \text{ belongs to } L^\infty([0, T]; L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)).$$
Moreover, we have
\[
|Du(x)| \leq \frac{1}{2\pi} \left( |D^2T(x)||h(T(x))| + |DT(x)|^2|(-Dh_3 + \alpha Dh_5)(T(x))| \right) \\
+ |DT(x)|^2|Dh_1 + Dh_2 - Dh_4(T(x))|.
\]
For the right first hand side term, we know that $h \circ T$ is uniformly bounded and that $D^2T$ belongs to $L^1_{\text{loc}}(\Omega)$ (see Theorem 2.11). We see that the second right hand side term belongs to $L^\infty([0, T]; L^1_{\text{loc}}(\Omega))$ because $DT$ belongs to $L^2_{\text{loc}}(\Omega)$ and $(-Dh_3 - \alpha Dh_5)(T(x))$ belongs to $L^\infty([0, T] \times \Omega)$.

Concerning the third right hand side term, we use that $T$ holomorphic implies that $DT$ is of the form \( \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \). Hence, we get easily that
\[
|DT(x)|^2_c = (\sup(|a|, |b|))^2 \leq a^2 + b^2 = |\det DT(x)|.
\]
Therefore, changing variables, we have for $i = 1, 2, 4$ and $K$ any compact set of $\Omega$:
\[
|||DT|^2|Dh_i \circ T||_{L^1(K)} \leq ||Dh_i||_{L^1(K)}
\]
with $\tilde{K} := T(K)$ a compact set (by the continuity of $T$). Therefore, the estimates obtained for $Dh_i$ allow us to conclude.

6.4. Proof of Proposition 2.11 We set $\beta(t) = t^2$ and use (2.17) with this choice. Let $\Phi \in D(\mathbb{R}^+ \times \mathbb{R}^2)$
\[
\int_{\mathbb{R}^2} \Phi(T(x), \omega)^2(T, x) \, dx - \int_{\mathbb{R}^2} \Phi(0, x)(\bar{\omega})^2(0, x) \, dx = \int_0^T \int_{\mathbb{R}^2} (\bar{\omega})^2(\partial_t \Phi + \bar{\omega} \cdot \nabla \Phi) \, dx \, dt.
\]
This is actually an improvement of (2.17), in which the equality holds in $L^1_{\text{loc}}(\mathbb{R}^+)$. Indeed, we have $\partial_t \bar{\omega} = -\text{div} (\bar{\omega} \bar{\omega})$ (in the sense of distributions) with $\bar{\omega} \in L^\infty$ and $\bar{\omega} \in L^\infty(\mathbb{R}^+, L^p_{\text{loc}}(\mathbb{R}^2))$ for all $p < 4$ (see (2.12)), which implies that $\partial_t \bar{\omega}$ belongs to $L^1_{\text{loc}}(\mathbb{R}^+, W^{-1,p}_{\text{loc}}(\mathbb{R}^2))$. Hence, $\bar{\omega}$ belongs to $C(\mathbb{R}^+, W^{-1,p}_{\text{loc}}(\mathbb{R}^2)) \subset C_w(\mathbb{R}^+, L^2_{\text{loc}}(\mathbb{R}^2))$, where $C_w L^2_{\text{loc}}$ stands for the space of maps $f$ such that for any sequence $t_n \to t$, the sequence $f(t_n)$ converges to $f(t)$ weakly in $L^2_{\text{loc}}$. Since on the other hand $t \mapsto \|\bar{\omega}(t)\|_{L^2}$ is continuous by Remark 2.10, we have $\bar{\omega} \in C(\mathbb{R}^+, L^2(\mathbb{R}^2))$. Therefore the previous integral equality holds for all $T$.

Now, we choose a good test function. We let $\Phi_0$ be a non-decreasing function on $\mathbb{R}$, which is equal to $1$ for $s \geq 2$ and vanishes for $s \leq 1$ and we set $\Phi(t, x) = \Phi_0(|x|/R(t))$, with $R(t)$ a smooth, positive and increasing function to be determined later on, such that $R(0) = R_0$. For this choice of $\Phi$, we have $\omega(x)^2 \Phi_0(0, x) \equiv 0$.

We compute then
\[
\nabla \Phi = \frac{x}{|x|} \frac{\Phi_0'}{R(t)}
\]
and
\[
\partial_t \Phi = -\frac{R'(t)}{R^2(t)} |x| \Phi_0'.
\]
We obtain
\[
\int_{\mathbb{R}^2} \Phi(T(x), \omega)^2(T, x) \, dx = \int_0^T \int_{\mathbb{R}^2} (\bar{\omega})^2 \frac{\Phi_0'(|x|)}{R} \left( \bar{\omega}(x) \cdot \frac{x}{|x|} - \frac{R'}{R} |x| \right) \, dx \, dt \\
\leq \int_0^T \int_{\mathbb{R}^2} (\bar{\omega})^2 \frac{\Phi_0'(|x|)}{R} (C - R') \, dx \, dt,
\]
where $C$ is independent of $t$ and $x$. Indeed, we have that
\[
u(t, x) = \frac{1}{2\pi} \frac{D^2T(x)}{T(x)} \left( R[\omega](x) + \gamma + \int_0^{\omega(x)} \frac{T(x)(\omega)}{|T(x)|^2} \right)
\]
Moreover, Proposition 2.11 states that there exists $C_\beta$ such that

\[ |D\mathcal{T}(x)| \leq C_\beta |\beta|, \quad \forall |x| \geq R_0. \]

Putting together all these inequalities with (2.11), we obtain

\[
C = \frac{1}{2\pi} C_\beta \left( C_1 + \|\gamma\|_{L^\infty([0,T^*])} + \|\omega_0\|_{L^1} \right).
\]

Taking $R(t) = R_0 + Ct$, we arrive at

\[
\int_{\mathbb{R}^2} \Phi(T,x)(\omega)^2(T,x) \, dx \leq 0,
\]

which ends the proof.

6.5. **Proof of Proposition 5.1.** By the conservation of the total mass of $\omega_i$ (2.18), we have that

\[
\int_{\mathbb{R}^2} \tilde{\omega}(t,\cdot) = 0, \quad \forall t \geq 0.
\]

Moreover, Proposition 2.11 states that there exists $C_1(\omega_0, \Omega, \gamma)$ such that $\omega_1(t,\cdot)$ and $\omega_2(t,\cdot)$ are compactly supported in $B(0, R_0 + C_1 t)$. So we first infer that $\tilde{v}(t) \in L^2(\mathbb{R}^2)$ for all $t$ (see e.g. [14]). Using that $\|\omega_i\|_{L^1(\Omega)} \in L^\infty(\mathbb{R}^+)$, we even obtain

\[
\tilde{v} \in L^\infty_{\text{loc}}(\mathbb{R}^+, L^2(\mathbb{R}^2)).
\]

We now turn to the first assertion in Proposition 5.1. By Lemma 6.1 and Calderon-Zygmund inequality we state that (2.19) implies that $v_i = K_{\mathbb{R}^2} * \tilde{\omega}_i$ belongs to $L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ and its gradient $\nabla v_i$ to $L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))$. On the other hand, since the vorticity $\omega_i$ is compactly supported, we have for large $|x|

\[
|v_i(t,x)| \leq \frac{C_i}{|x|} \int_{\mathbb{R}^2} |\tilde{\omega}_i(t,y)| \, dy,
\]

hence $v_i$ belongs to $L^\infty_{\text{loc}}(\mathbb{R}^+, L^p(\mathbb{R}^2))$ for all $p > 2$. It follows in particular that

\[
v_i \in L^\infty_{\text{loc}}(\mathbb{R}^+, W^{1,4}(\mathbb{R}^2))
\]

and also that $v_i \otimes v_i$ belongs to $L^\infty_{\text{loc}}(L^{4/3})$. Since $v_i$ is divergence-free, we have $v_i \cdot \nabla v_i = \text{div} (v_i \otimes v_i)$, and so $v_i \cdot \nabla v_i \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4}(\mathbb{R}^2))$.

Thanks to (2.12), we know that $v_i(t) \otimes w_i(t)$ belongs to $L^{4/3}_{\text{loc}}$. At infinity, we use the explicit formula of $u$ (2.3), the compact support of the vorticity and the behavior of $T$ at infinity (Proposition 2.3) to note that $w_i$ is bounded by $C/|x|$. $v_i$ has the same behavior at infinity, which belongs to $L^{8/3}$. This yields

\[
\text{div} (v_i \otimes w_i), \quad \text{div} (w_i \otimes v_i) \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4}(\mathbb{R}^2)).
\]

Besides, we can infer from the behavior of $T$ on the boundary (Theorem 2.1 and Proposition 2.5) that $\tilde{g}_{v_i, \gamma_0}$, defined in (2.10), is uniformly bounded in $L^1(\partial \Omega)$. Then we deduce from the embedding of $W^{1,4}(\mathbb{R}^2)$ in $C^0(\mathbb{R}^2)$ that $\tilde{g}_{v_i, \gamma_0} \delta_0$ belongs to $L^2_{\text{loc}}(W^{-1,4}(\mathbb{R}^2))$. Therefore, $v_i \tilde{g}_{v_i, \gamma_0} \delta_0 \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4}(\mathbb{R}^2))$.

According to (5.1), we finally obtain

\[
\langle \partial_t v_i, \Phi \rangle = \langle \partial_t v_i - \nabla p_i, \Phi \rangle \leq C \|\Phi\|_{L^2(W^{1,4})}
\]

for all divergence-free smooth vector field $\Phi$. This implies that

\[
\partial_t v_i \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4/3}(\mathbb{R}^2)), \quad i = 1, 2,
\]

and the same holds for $\partial_t \tilde{v}$. Now, since $\tilde{v}$ belongs to $L^2_{\text{loc}}(\mathbb{R}^+, W^{1,4})$, we deduce from (6.2) and Lemma 1.2 in Chapter III of [20] that $\tilde{v}$ is almost everywhere equal to a function continuous from $\mathbb{R}^+$ into $L^2$ and we have in the sense of distributions on $\mathbb{R}^+$:

\[
\frac{d}{dt} \|\tilde{v}\|_{L^2(\mathbb{R}^2)} = 2 \langle \partial_t \tilde{v}, \tilde{v} \rangle_{W^{-1,4/3}, W^{1,4}}.
\]

We finally conclude by using the fact that $\tilde{v}(0) = 0$. 

7. Final remarks and comments

7.1. Application: safety time in airports. One of the crucial issues in biggest airports is the following: the wings of a plane create (at landing and take off) an important vortex where another plane should not pass through. We can find in history some examples of crashes due to these turbulences. Consequently, airports decided to define a safety time between two planes, in order that the wake vortex dissipates. However, this safety time have to be redefined because of “very heavy” new airplanes as the Airbus A380. Increasing the time between two planes for security is in contradiction with the traffic jam in the main airports. Several programs were created in order to understand and optimize this phenomena. One of the solutions investigated is the creation of big radars which try to detect if the vortex (of course invisible) has vanished. However, engineers and physicists are not able to interpret the data given by the radar, because we do not know the exact shape of the vortex behind a plane wing.

Air around an infinite airplane wing can be modeled by Euler equation in dimension two around a singular obstacle: smooth except in one point where we have a cusp ($\alpha = 2\pi$). The consequence of this article and of [3] is to show that we have existence and uniqueness of a solution to such a problem, and that we have an explicit form of the velocity in terms of the vorticity and the shape of the wing. In particular, we show that the velocity blows up like $1/\sqrt{|x|}$. Thanks to this explicit formula (see (2.9)), it should be possible to plug it in order to obtain the shape of the stream line.

Actually, in order to apply exactly this article, we consider that the wing is fixed in the plane frame, but the velocity at infinity is equal to $-l$ where $l$ is the velocity of the plane. Having vector field constant at infinity instead of vanishing, change a bit the Biot-Savart law but not the final result and the behavior near the obstacle. To be convinced, the reader can read [11, Section 3.4], where we adapt [10] with constant velocity at infinity.

7.2. The Kutta condition. As the velocity stays tangent to the obstacle, we know that the trajectory of a flow particle which arrives to the solid will be deformed, and the particle will leave behind the solid. Indeed, we have proved that this particle will never meet the boundary. By reversibility of time, it also means that a particle near the boundary stays close to the boundary, even if the particle goes closer and closer to a corner. Therefore, near the trailing edge of a plane wing, the particle has a velocity which blows up, but it stays close to the boundary. For the Euler equation, this particle should take the turn of the corner with a huge tangential velocity, but moving in a small neighborhood of the boundaries.

Such a kind of behavior can justify that for infinite velocity, the Euler equations are no longer relevant for modeling the air near the corners. This observation is well known by engineers and they use an unproved principle, the so-called Kutta condition: “A body with a sharp trailing edge which is moving through a fluid will create about itself a circulation of sufficient strength to hold the rear stagnation point at the trailing edge”. In other word, even if initially $\gamma = 0$, they assume that there is a creation of circulation or vorticity such that the fluid particles near the edge go away from the obstacle. Concerning Euler equations, we see that particles which arrives to the wing can go away from opposite side, and not necessarily from the cups.
7.3. No extraction in convergence results. In [10, 12, 3], the existence of a weak solution is a consequence of a compactness argument. Indeed, we consider therein the unique solutions $u_n$ of the Euler equations on the smooth domain $\Omega_n$, which converges to $\Omega$ in some senses. Then, in these articles, we extract a subsequence such that $u_{\varphi(n)} \rightarrow u$ and we check that $u$ is solution of the Euler equations in $\Omega$. Putting together the present result with [3], we can state the following.

**Theorem 7.1.** Let $\omega_0, \gamma_0, \Omega$ as in Theorems 7.2 or 7.3. For any sequence of smooth open simply connected domains (or exterior of simply connected domains) $\Omega_n$ converging to $\Omega$ in the Hausdorff sense, then the unique solution $u_n$ of the Euler equations on $\Omega_n$, with initial datum $u_0^n$ such that
\[
\text{div } u_0^n = 0, \text{ curl } u_0^n = \omega_0, \text{ } u_0^n \cdot \hat{n}|_{\partial \Omega_n} = 0, \quad \lim_{|x| \to +\infty} u_0^n = 0, \quad \oint_{\partial \Omega_n} u_0^n \cdot \hat{\tau} \, ds = \gamma_0 \text{ (only for exterior domains)},
\]
converges in $L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)$ to the unique solution $u$ of the Euler equations on $\Omega$ with initial datum $u^0$ such that
\[
\text{div } u^0 = 0, \text{ curl } u^0 = \omega_0, \text{ } u^0 \cdot \hat{n}|_{\partial \Omega} = 0, \quad \lim_{|x| \to +\infty} u^0 = 0, \quad \oint_{\partial \Omega} u^0 \cdot \hat{\tau} \, ds = \gamma_0 \text{ (only for exterior domains)}.
\]

7.4. Special vortex sheet. In [10], we consider some smooth domains $\Omega_\varepsilon$ which shrink to a $C^2$ Jordan arc $\Gamma$ as $\varepsilon$ tends to zero. For $\omega_0 \in L^\infty(\Gamma^c)$ and $\gamma \in \mathbb{R}$ given, we denote by $(u_\varepsilon, \omega_\varepsilon)$ the corresponding regular solutions of the Euler equations on $\Pi_\varepsilon := \mathbb{R}^2 \setminus \Omega_\varepsilon$. Up to a truncated smoothly over a size $\varepsilon$ around the obstacle, it is proved therein that the resulting truncations $\hat{u}_0$ and $\hat{\omega}_0$, defined over the whole of $\mathbb{R}^2$, converge in appropriate topologies to the solutions $\hat{u}$, $\hat{\omega}$ of the system
\[
\begin{cases}
\partial_t \hat{\omega} + \hat{u} \cdot \nabla \hat{\omega} = 0, & t > 0, \quad x \in \mathbb{R}^2, \\
\text{div } \hat{u} = 0, & t > 0, \quad x \in \mathbb{R}^2, \\
\text{curl } \hat{u} = \hat{\omega} + g_{\hat{\omega}, \gamma} \delta_1, & t > 0, \quad x \in \mathbb{R}^2.
\end{cases}
\]

This is an Euler like equation, modified by a Dirac mass along the arc. The density function $g_{\hat{\omega}, \gamma}$ is given explicitly in terms of $\hat{\omega}$ and $\Gamma$. Moreover, it is shown that it is equal to the jump of the tangential component of the velocity across the arc. We refer to [10] for all necessary details.

Actually, the presence of this additional measure is mandatory in order that the velocity $\hat{u}$ is tangent to the curve, with circulation $\gamma$ around it.

Therefore, in the exterior of a Jordan arc, (7.1) appears to be a special vortex sheet, “special” because the support of the dirac mass does not move (staying to be $\Gamma$) and because the normal component of the velocity on the curve is equal to zero. For a general vortex sheet, we can prove that the normal component is continuous, but not necessarily zero. In both case, we have a jump of the tangential component.

A consequence of the present work is the uniqueness of a solution of (7.1), with the good sign conditions for $\omega_0$ and $\gamma$ (see Theorem 7.3).

For instance, if we assume that $\Gamma$ is the segment $[(-1, 0); (1, 0)]$, then we have the explicit expression of the harmonic vector field thanks to the Joukowski function, and we can find in [10] p. 1144] the following:
\[
\text{curl } H_\Gamma = \frac{1}{\pi} \frac{1}{\sqrt{1 - x_1^2}} \chi(-1, 1) (x_1) \delta_0(x_2).
\]

Then, choosing $\omega_0 \equiv 0$ and $\gamma = 1$, we have proven that the stationary function $u(t, x) = H_\Gamma(x)$ is the unique solution of the Euler equations with initial vorticity $\frac{1}{\pi} \frac{1}{\sqrt{1 - x_1^2}} \chi(-1, 1) (x_1) \delta_0(x_2)$.

Adding a vorticity or considering other shape for $\Gamma$ complicates a lot the expression of $g_{\hat{\omega}, \gamma}$ (see [10]). In particular, we do not prove the uniqueness for the so-called Prandtl-Munk vortex sheet: $\frac{1}{\pi} \frac{x_1}{\sqrt{1 - x_1^2}} \chi(-1, 1) (x_1) \delta_0(x_2)$.

7.5. Extension for constant vorticity near the boundary. As it is remarked several times, the crucial point is to prove that the vorticity never meets the boundary if we consider an initial vorticity compactly supported in $\Omega$. However, we can extend easily this result to the case of an initial vorticity
constant to the boundary. Indeed, for $\alpha \in \mathbb{R}$ given, choosing $\beta(t) = (t - \alpha)^2$ in the proof of Proposition 4.11 gives in the same way the following.

**Proposition 7.2.** Let $\omega$ be a global weak solution of (2.11) such that $\omega_0$ is compactly supported in $\overline{\Omega}$ and such that $\omega_0 \equiv \alpha$ in a neighborhood of the boundary. If $\omega_0$ is non-positive and $\gamma_0 \geq -\int \omega_0$ (only for exterior domain), then, for any $T^* > 0$, there exists a neighborhood $U_{T^*}$ of $\partial \Omega$ such that $\omega(t) \equiv \alpha$ on $U_{T^*}$, $\forall t \in [0, T^*]$. Therefore, in the proof of the uniqueness, we still have on $U$

$$\text{curl} \tilde{v} = \text{curl} v_1 - \text{curl} v_2 = \alpha - \alpha = 0,$$

which implies that the velocity $\tilde{v}$ is harmonic near the boundary, allowing us to follow exactly the proof made in Section 3.

7.6. Liapounov and sign conditions. Let us present in this subsection the different Liapounov functions, the advantage of each, and why it is specific to the case studied.

**Vortex wave system in $\mathbb{R}^2$.** Let us consider that the initial vorticity is composed on a regular part plus a dirac mass centered at the point $z(t)$. Then Marchioro and Pulvirenti proved in [16] that there exists one solution to the following system:

$$
\begin{align*}
&v(t, \cdot) = (K_{\mathbb{R}^2} * \omega)(\cdot, t), \\
&\dot{z}(t) = v(t, z(t)), \\
&\dot{\phi}_x(t) = v(t, \phi_x(t)) + \frac{(\phi_x(t) - z(t))}{2\pi|\phi_x(t) - z(t)|^2}, \\
&\phi_x(0) = x, \ x \neq z_0, \\
&\omega(t, \phi_x(t)) = \omega_0(x),
\end{align*}
$$

which means that the point vortex $z(t)$ moves under the velocity field $v$ produced by the regular part $\omega$ of the vorticity, whereas the regular part and the vortex point give rise to a smooth flow $\phi$ along which $\omega$ is constant. In this case, we can prove that the trajectories never meet the point vortex considering the following Liapounov function:

$$L(t) := -\ln|\phi_x(t) - z(t)|,$$

for $x \neq z_0$ fixed. We note that $L$ goes to $+\infty$ iff $\phi_x(t) \to z(t)$, so we want to prove that $L$ stays bounded. Next we compute:

$$L'(t) = -\frac{(\phi_x(t) - z(t)) \cdot (\dot{\phi}_x(t) - \dot{z}(t))}{|\phi_x(t) - z(t)|^2} = -\frac{(\phi_x(t) - z(t)) \cdot (v(t, \phi_x(t)) - v(t, z(t)))}{|\phi_x(t) - z(t)|^2}.$$

Next, we use that the regular part $v$ is log-lipschitz in order to obtain a Gronwall-type inequality. To summarize, we remark that in the case, the important points are:

$$L(t) \to \infty \text{ iff } \phi_x(t) \to z(t) \quad \text{and} \quad (\phi_x(t) - z(t)) \cdot \frac{(\phi_x(t) - z(t))}{2\pi|\phi_x(t) - z(t)|^2} \equiv 0$$

removing the singular part.

**Dirac mass fixed in $\mathbb{R}^2$.** Marchioro in [15] studied exactly the same problem as above, assuming that the vortex mass cannot move. Therefore, the previous Liapounov does not work, because we do not have a difference of two velocities and we cannot use the log-lipschitz regularity. In this article, the author introduced a new Liapounov:

$$L(t) := -\int_{\mathbb{R}^2} \left(\ln |\phi_x(t) - y|\right) \omega(t, y) \, dy - \ln |\phi_x(t) - z_0|,$$
where the first integral is the stream function associated to \( v \). Then, the first step was to prove that this integral is bounded, which implies that \( L \) goes to \( +\infty \) iff \( \phi_x(t) \to z_0 \). Next, he computed:

\[
L'(t) = -\left( \int_{\mathbb{R}^2} \frac{\phi_x(t) - y}{|\phi_x(t) - y|^2} \omega(t, y) dy + \frac{\phi_x(t) - z_0}{|\phi_x(t) - z_0|^2} \right) \phi_x(t) - \int_{\mathbb{R}^2} \left( \ln |\phi_x(t) - y| \right) \partial_t \omega(t, y) dy
\]

\[
= -\int_{\mathbb{R}^2} \left( \ln |\phi_x(t) - y| \right) \partial_t \omega(t, y) dy = -\int_{\mathbb{R}^2} \nabla \left( \ln |\phi_x(t) - y| \right) \cdot \left( v(t, y) + \frac{(y - z_0) \cdot \omega(t, y)}{2\pi |y - z_0|^2} \right) dy
\]

Next, the second step was to prove some good estimate for the right hand side integral in order to conclude by the Gronwall lemma. Here, we see that the singular term is now passed in a integral, which is bounded. Similarly, we note that the important points in this case are:

\[
L(t) \to \infty \text{ iff } \phi_x(t) \to z_0 \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{\phi_x(t) - y}{|\phi_x(t) - y|^2} \omega(t, y) dy + \frac{\phi_x(t) - z_0}{|\phi_x(t) - z_0|^2} \phi_x(t) \equiv 0.
\]

**Interior or exterior of simply connected domains.** In our case, we have again an explicit formula of the velocity by the Biot-Savart law (see (2.8) and (2.9)). As the velocity near the boundary blows up, we have to make appear some cancellation as Marchioro did, in order that the singular part goes in an integral. To do that, we introduce the stream function associated to the velocity:

\[
L_1(t, x) := \frac{1}{2\pi} \int_{\Omega} \ln \left( \frac{|T(x) - T(y)|}{|T(x) - T(y)| + |T(y)|} \right) \omega(y) dy + \frac{\alpha}{2\pi} \ln |T(x)|
\]

with \( \alpha = 0 \) in the bounded case. However, as this function tends to zero (instead to \( \infty \)) when \( x \to \partial \Omega \) (see Lemma 3.1), we add a logarithm:

\[
L(t) := -\ln |L_1(t, \phi_x(t))|,
\]

and the goal is to prove that \( L \) stays bounded. Then, we computed in Section 3

\[
L'(t) = -\frac{\partial_t L_1(t, \phi_x(t))}{|L_1(t, \phi_x(t))|},
\]

and we proved that \( \partial_t L_1 \) tends to zero as \( \phi_x(t) \to \partial \Omega \), comparing the rate with \( L_1 \). Then, we see here that it is important that \( \partial_t L_1 \) goes to zero where \( L_1 \) tends to zero. We managed to prove that \( \partial_t L_1 \) tends to zero near the boundary, and the sign condition allows us to state that the boundary is the only set where \( L_1 \) vanishes (see Lemma 3.2). For instance, in bounded domain (i.e. \( \alpha = 0 \)) we see that a vorticity with different sign can imply that \( L_1 = 0 \) somewhere else than on \( \partial \Omega \). This last remark is the main reason of the sign condition of the vorticity. Next, the sign condition on the circulation follows from the fact that we want the same sign for both terms in \( L_1 \).

Therefore, one difference with the case studied by Marchioro is that the stream function of the harmonic vector field does not blow-up. To conclude, let us mention that the Liapounov method is specific to the case studied and it is hard to adapt for other cases. For example, we have presented here the case of dirac mass when \( \dot{z}(t) = v(t, z(t)) \), when \( \dot{z}(t) = 0 \), but we do not know how to prove for other dynamics, like e.g. \( \dot{z}(t) = (1, 0) \).

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