Are there higher-order Betchov homogeneity constraints for incompressible isotropic turbulence?

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Incompressible and statistically homogeneous flows obey exact kinematic relations. The Betchov homogeneity constraints (Betchov, \textit{J. Fluid Mech.}, vol. 1, 1956, pp. 497–504) for the average principal invariants of the velocity gradient are among the most well known and extensively employed homogeneity relations. These homogeneity relations have far-reaching implications for the coupled dynamics of strain and vorticity, as well as for the turbulent energy cascade. Whether the Betchov homogeneity constraints are the only possible ones or whether additional homogeneity relations exist has not been proven yet. Here we show that the Betchov homogeneity constraints are the only homogeneity constraints for incompressible and statistically isotropic velocity gradient fields. We also extend our results to derive homogeneity relations involving the velocity gradient and other dynamically relevant quantities, such as the pressure Hessian.

\textbf{Key words:} homogeneous isotropic turbulence, statistical turbulence theory

\section{1. Introduction}

The velocity gradients, i.e. spatial derivatives of the velocity field, $\mathbf{A} = \nabla \mathbf{u}$, contain a wealth of information about small-scale turbulence, including the topology of vorticity and strain (Meneveau 2011). The moments of the velocity gradients of an incompressible and statistically homogeneous field obey exact kinematic relations (Betchov 1956). The two so-called Betchov constraints for the velocity gradient principal invariants, namely the matrix traces $\text{Tr}(\mathbf{A}^2)$ and $\text{Tr}(\mathbf{A}^3)$, are of central importance for a statistical description of the turbulent dynamics (Davidson 2004). The first Betchov constraint states that the second principal invariant of the velocity gradient is on average zero, $\langle \text{Tr}(\mathbf{A}^2) \rangle = 0$, which implies the proportionality between the mean dissipation rate and the mean squared vorticity,

$$\varepsilon = \nu \langle \omega^2 \rangle.$$  \hspace{1cm} (1.1)

Here $\nu$ is the kinematic viscosity of the fluid, $\varepsilon = 2\nu \langle \text{Tr}(\mathbf{S}^2) \rangle$ is the mean dissipation rate, $\mathbf{S}$ is the strain rate and $\omega = \nabla \times \mathbf{u}$ is the vorticity. The second Betchov relation $\langle \text{Tr}(\mathbf{A}^3) \rangle = 0$, connects strain self-amplification and vortex stretching,

$$4\langle \text{Tr}(\mathbf{S}^3) \rangle = -3\langle \omega \cdot (\mathbf{S}\omega) \rangle.$$  \hspace{1cm} (1.2)

While equation (1.1) constrains the strain-rate and vorticity magnitudes, relation (1.2) constrains their production rates. The latter relation was derived first by Townsend & Taylor

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(1951) and then rederived and extensively used by Betchov (1956). It allows to characterize the average turbulent energy cascade in physical space (Davidson 2004; Carbone & Bragg 2020; Johnson 2020) and to predict the preferential configuration of the strain-rate eigenvalues (Betchov 1956). It also implies that the vortex stretching has a positive average in presence of an average forward energy cascade, related to the negative skewness of the longitudinal velocity increment and gradient statistics. This positive average has implications, for example, on the vorticity magnitude and orientation relative to the strain rate (Tsinober 2009; Tom et al. 2021) and on the attenuation of extreme velocity gradients (Buaria et al. 2020). Additionally, relations (1.1, 1.2) have their analogues for the velocity structure functions (Hill 1997). Those relations for the velocity structure functions are related to the ones for the velocity gradients through a simple Taylor expansion at small scales, and at larger scales through a filtered velocity gradient corrected for compressible effects (Carbone & Bragg 2020).

Applications of the homogeneity relations (1.1, 1.2) are not limited to the theoretical understanding of turbulence, but they carry over to the modelling of turbulent flows. For example, stochastic models for the velocity gradient should in principle obey the constraints (1.1) and (1.2) (Johnson & Meneveau 2016) which help to reduce the number of free parameters in such models (Leppin & Wilczek 2020). The homogeneity relations can also be used to improve the performance of neural networks designed for machine learning of turbulent flows by including them into the training (Tian et al. 2021; Momenifar et al. 2021).

The Betchov relations (1.1, 1.2) follow by writing the matrix traces $\text{Tr}(\mathbf{A}^2)$ and $\text{Tr}(\mathbf{A}^3)$ as the divergence of a vector field. Then, because of statistical homogeneity, the average of such traces is zero since a spatial derivative can be factored out, and it acts on an average which does not depend on space explicitly. For example, due to incompressibility, the second principal invariant of the velocity gradient can be rewritten as

$$
\text{Tr}(\mathbf{A}^2) = \nabla_j u_i \nabla_i u_j = \nabla_i (u_j \nabla_j u_i),
$$

(1.3)

so that its average vanishes, and relation (1.1) follows. An analogous procedure applies to retrieve equation (1.2), since the third principal invariant can be expressed as

$$
\text{Tr}(\mathbf{A}^3) = \nabla_j u_i \nabla_k u_j \nabla_i u_k = \nabla_i \left( u_j \nabla_k u_i \nabla_j u_k - \frac{1}{2} u_i \nabla_k u_j \nabla_j u_k \right).
$$

(1.4)

However, while it is straightforward to check the validity of the Betchov homogeneity relations for the velocity gradient, it is more complicated to show whether those homogeneity relations are the only possible ones or if additional constraints exist. If there existed higher-order constraints we could, for example, improve the current reduced-order models of the velocity gradient dynamics just by imposing these additional homogeneity constraints.

Most of the previous attempts to find higher-order homogeneity relations were based on swapping the spatial derivatives: a scalar contraction of powers of the velocity gradient is manipulated by factoring out the spatial derivative, in order to rewrite the contraction as the divergence of some quantity (if possible). Attempts to obtain relations for the fourth-order moments of the velocity increments/gradients through this derivative-swapping procedure include Hill (1997); Hierro & Dopazo (2003); Bragg et al. (2021). However, it is very difficult to show the completeness of the homogeneity constraints for the incompressible gradient through this derivative-swapping approach. One would need to consider linear combinations of infinitely many contractions of the velocity gradients and try to recast them into the spatial derivative of some field. In this framework, Siggia (1981) showed that no homogeneity constraints exist on polynomials of fourth-order velocity gradient invariants.

The scenario is analogous to the search for inviscid invariants of the Navier-Stokes equations (Majda & Bertozzi 2001), which are central for the occurrence of cascades (Alexakis & Biferale 2018). While it is straightforward to check the conservation of kinetic
energy and helicity in the incompressible three-dimensional Euler equations by a derivative-swapping procedure (Majda & Bertozzi 2001), it is much more involved to show whether those conserved quantities are the only possible ones or if additional ones exist. This completeness question has been answered by Serre (1984) for the incompressible Euler equations and by Enciso et al. (2016) for volume-preserving diffeomorphisms.

In this work, we investigate the existence of higher-order homogeneity constraints for the incompressible velocity gradient using tensor function representation theory (Zheng 1994; Itskov 2015). The analysis allows identifying the homogeneity relations as the solutions of a system of partial differential equations, and it shows that no additional homogeneity constraints for the incompressible velocity gradient exist other than the ones already known from Betchov (1956).

2. An equation encoding the homogeneity constraints on the velocity gradient

We consider a three-dimensional, incompressible and statistically homogeneous and isotropic velocity field \( \mathbf{u}(x, t) \) together with its spatial gradient \( \mathbf{A} = \nabla \mathbf{u} \) (\( A_{ij} = \nabla_j u_i \) in Cartesian component notation). We search for homogeneity relations for scalar single-point statistics of the velocity gradient only.

Incompressibility implies that \( \text{Tr}(\mathbf{A}) = 0 \), while homogeneity implies translational invariance of ensemble averages, i.e. it renders them independent of the spatial coordinate \( x \). As a consequence, any scalar field \( \phi \) that is the divergence of a vector field, \( \phi = \nabla \cdot \mathbf{F} \), has zero ensemble/spatial average

\[
\langle \phi \rangle = \langle \nabla \cdot \mathbf{F} \rangle = \nabla \cdot \langle \mathbf{F} \rangle = 0.
\]

(2.1)

For example, in the first Betchov relation \( \langle \text{Tr}(\mathbf{A}^2) \rangle = 0 \), the vector field is \( \mathbf{F} = \mathbf{A} \mathbf{u} \) (\( F_i = A_{ij} u_j \) in component notation, see (1.3)). To generalize this, we search for scalar functions of the velocity gradient \( \phi(\mathbf{A}) \) which are the divergence of a vector field \( \mathbf{F} \).

The vector field \( \mathbf{F} \) is in general a functional of the velocity field \( \mathbf{u}(x, t) \). We restrict the analysis to functions of the velocity and its spatial derivatives, \( \mathbf{F}(\mathbf{u}, \mathbf{A}, \nabla \mathbf{A}, \nabla(\nabla \mathbf{A}), \ldots) \), because we search for homogeneity relations on the single-point statistics of the velocity gradient. By restricting the analysis to functions of the velocity and its spatial derivatives, we are implicitly assuming isotropy. Indeed, in a statistically isotropic flow, the governing equations and associated boundary conditions do not introduce any characteristic direction. Therefore, in that statistically isotropic situation, the velocity and velocity gradients are all the possible variables upon which the vector \( \mathbf{F} \) can depend. We thereby exclude, for example, rotations of the frame of the flow, anisotropic forcing, boundary layers, etc.

Focusing on single-point statistics of isotropic flows we have \( \mathbf{F} = \mathbf{F}(\mathbf{u}, \mathbf{A}, \nabla \mathbf{A}, \nabla(\nabla \mathbf{A}), \ldots) \), so that the corresponding \( \phi(\mathbf{A}) \) is, by chain rule and in component notation,

\[
\phi(\mathbf{A}) = \nabla_i F_i (\mathbf{u}(x, t), \ldots) = \frac{\partial F_i}{\partial u_p} A_{pi} + \frac{\partial F_i}{\partial A_{pq}} \nabla_i A_{pq} + \frac{\partial F_i}{\partial (\nabla_k A_{pq})} \nabla_i (\nabla_k A_{pq}) + \ldots
\]

(2.2)

The fact that the left-hand side of equation (2.2) depends only on the velocity gradient strongly constrains the functional form of the vector field \( \mathbf{F} \). Namely, the right-hand side of equation (2.2) should explicitly involve neither the velocity \( \mathbf{u} \) nor the gradients of the velocity gradient, \( \nabla \mathbf{A}, \nabla(\nabla \mathbf{A}) \), etc. This implies that all the terms on the right-hand side of (2.2) featuring gradients of the velocity gradient should identically cancel, while only \( \frac{\partial F_i}{\partial u_p} A_{pi} \) can contribute to \( \phi \). Moreover, the part of \( \frac{\partial F_i}{\partial u_p} A_{pi} \) that contributes to \( \phi \) can depend only on \( \mathbf{A} \). Therefore, we just need to consider vector functions of the velocity and velocity gradient, \( \mathbf{F}(\mathbf{u}, \mathbf{A}) \), that are linear in the velocity. Based on this, equation (2.2)
splits into

\[ \phi(\mathbf{A}) = \frac{\partial F_i}{\partial u_p}(\mathbf{u}, \mathbf{A}) A_{pi}, \quad (2.3a) \]

\[ \frac{\partial F_i}{\partial A_{pq}}(\mathbf{u}, \mathbf{A}) \nabla_i A_{pq} = 0. \quad (2.3b) \]

Equation (2.3b) yields the main differential equation to determine \( F \). The gradient of the gradient, \( \nabla_i A_{pq} = \nabla_i \nabla_q u_p \), is symmetric in \( i, q \), so that only the part of \( \frac{\partial F_i}{\partial A_{pq}} \) that is symmetric in \( i, q \) contributes to (2.3b). Additionally, the contractions \( i, p \) and \( q, p \) of \( \nabla_i A_{pq} \) are zero by incompressibility. Therefore, \( F \) solves equation (2.3b) only if, for some vector \( \mathbf{v} \),

\[ \frac{\partial F_i}{\partial A_{pq}} + \frac{\partial F_q}{\partial A_{pi}} = v_i \delta_{pq} + v_q \delta_{pi}. \quad (2.4) \]

Here, \( \delta_{ij} \) denotes the Kronecker delta, and the vector \( \mathbf{v} \) is easily determined by contracting two of the free indices, e.g. \( v_p = \frac{\partial F_k}{\partial A_{pk}} \).

Equations (2.3, 2.4) allow making the search for homogeneity constraints more systematic: instead of attempting to factor out the spatial derivatives in tensor contractions of velocity gradients, we need to solve a system of partial differential equations. Solving equation (2.4) for vectors \( \mathbf{F} \) that are linear in \( \mathbf{u} \) yields all possible vectors \( \mathbf{F}(\mathbf{u}, \mathbf{A}) \), whose divergence depends only on the velocity gradient, \( \phi(\mathbf{A}) = \nabla \cdot \mathbf{F}(\mathbf{u}, \mathbf{A}) \), as in (2.3a). Therefore, finding all solutions of (2.4) that are linear in the velocity amounts to deriving all possible homogeneity constraints on scalar functions of an incompressible and statistically isotropic velocity gradient.

**3. Tensor function representation of the homogeneity constraints**

In the following we construct the general isotropic tensor function \( \mathbf{F}(\mathbf{u}, \mathbf{A}) \). Tensor function representation theory (Weyl 1946; Rivlin & Ericksen 1955; Pennisi & Trovato 1987; Zheng 1994; Itskov 2015) allows writing all possible vector functions \( \mathbf{F} \) of the generating vector \( \mathbf{u} \) and tensor \( \mathbf{A} \) that transform consistently under any change of basis: when the arguments \( \mathbf{u} \) and \( \mathbf{A} \) undergo a rotation, \( \mathbf{F} \) rotates accordingly (Itskov 2015). The vector field \( \mathbf{F} \) will be finally determined by requiring that it is linear in \( \mathbf{u} \) and that it solves equation (2.4).

In general, \( \mathbf{F} \) can depend separately on the symmetric and anti-symmetric parts of the velocity gradient (Rivlin & Ericksen 1955)

\[ \mathbf{S} = \frac{1}{2} (\mathbf{A} + \mathbf{A}^\top), \quad \mathbf{W} = \frac{1}{2} (\mathbf{A} - \mathbf{A}^\top), \quad (3.1) \]

with \( \mathbf{A}^\top \) denoting the matrix transpose of \( \mathbf{A} \). Therefore, we consider all the vector functions \( \mathbf{F}(\mathbf{u}, \mathbf{S}, \mathbf{W}) \) constructed through the velocity and velocity gradients that, due to equation (2.3a), are linear in the velocity,

\[ \mathbf{F} = \sum_{n=0}^{8} f_n(I) \mathbf{B}^n \mathbf{u} \quad (3.2) \]

Here \( \mathbf{B}^n \) are the basis tensors that can be formed through \( \mathbf{S} \) and \( \mathbf{W} \) (Pennisi & Trovato 1987)

\[
\begin{align*}
\mathbf{B}^0 &= I & \mathbf{B}^1 &= \mathbf{S} & \mathbf{B}^2 &= \mathbf{W} \\
\mathbf{B}^1 &= \mathbf{S} & \mathbf{B}^4 &= \mathbf{S} \mathbf{W} - \mathbf{W} \mathbf{S} & \mathbf{B}^7 &= \mathbf{S} \mathbf{W} \mathbf{W} + \mathbf{W} \mathbf{W} \mathbf{S} \\
\mathbf{B}^2 &= \mathbf{W} & \mathbf{B}^5 &= \mathbf{S} \mathbf{W} + \mathbf{W} \mathbf{S} & \mathbf{B}^8 &= \mathbf{S} \mathbf{S} \mathbf{W} + \mathbf{W} \mathbf{S} \mathbf{S},
\end{align*}
\]

with \( I \) denoting the identity matrix and standard matrix product implied. Two additional
tensors would be necessary to fix degeneracies of the basis (3.3), which occur when the vorticity is an eigenvector of the strain-rate tensor or the strain-rate has two identical eigenvalues (Rivlin & Ericksen 1955). We ignore that zero-measure configuration of the gradients. Also, note that the superscript of $B^n$ serves to number the basis tensors rather than indicating powers of the tensor.

The components $f_n$ in (3.2) are functions of the set $I$ of independent invariants that can be formed though the velocity gradients (Pennisi & Trovato 1987)

\[
I_1 = \text{Tr}(SS) \quad \quad I_3 = \text{Tr}(SSS) \quad \quad I_5 = \text{Tr}(SSW),
\]

\[
I_2 = \text{Tr}(WW) \quad \quad I_4 = \text{Tr}(SWW),
\]

with standard matrix product implied. A sixth invariant would be necessary to fix the orientation/handedness of the vorticity with respect to the strain-rate eigenvectors. We do not consider the sixth invariant as an independent variable since it is determined by the invariants (3.4) up to a sign (Lund & Novikov 1992).

4. Solution of the equation encoding the homogeneity constraints

We use the general expression (3.2) combined with equation (2.4) in order to determine the components of $F, f_n(I)$. This will yield a vector field $F$ associated to the homogeneity constraints for the velocity gradient through $\phi(A) = \nabla \cdot F(u, A)$ and (2.1).

Inserting the general expression of $F$ (3.2) into equation (2.4) gives, in component notation,

\[
\frac{\partial f_i}{\partial I_k} B^l_{ij} + \frac{\partial f_i}{\partial A_{pq}} B^l_{ij} + \frac{\partial f_i}{\partial I_k} B_{ij}^l + \frac{\partial f_i}{\partial A_{pq}} B_{ij}^l + \frac{\partial f_i}{\partial A_{pq}} B_{ij}^l = v_i \delta_{pq} + v_q \delta_{pi}.
\]

(4.1)

Here and throughout repeated indices imply summation, unless otherwise specified. As shown in Appendix A, the derivatives of the invariants (3.4) can be written as

\[
\frac{\partial I_k}{\partial A_{pq}} = M_{km} B^m_{pq},
\]

(4.2)

while the derivatives of the basis tensors (3.3) can be expressed as

\[
\frac{\partial B^m_{ij}}{\partial A_{pq}} = \Gamma^{l,n}_{lm} B^l_{ip} B^m_{qj} + \Gamma^{2,n}_{lm} B^l_{iq} B^m_{pj} + \Gamma^{3,n}_{lm} B^l_{ij} B^m_{pq},
\]

(4.3)

with $0 \leq l, m, n \leq 8$ and $1 \leq k \leq 5$. The matrix entries $M_{km}$ featured in equation (4.2) are specified in (A.1). The symbols $\Gamma^{P,n}_{lm}$ in equation (4.3) play the role of Christoffel symbols (Grinfeld 2013) and their components are listed in (A.9, A.10, A.11). Inserting the derivatives expressions (4.2, 4.3) into equation (4.1) yields the following independent equations

\[
f_n \left( \Gamma^{2,n}_{lm} B^l_{ip} B^m_{qj} + \Gamma^{2,n}_{lm} B^l_{iq} B^m_{pj} \right) u_j = 0,
\]

(4.4a)

\[
\frac{\partial f_i}{\partial I_k} M_{km} B^l_{ij} B^m_{pq} + f_n \left( \Gamma^{3,n}_{lm} B^l_{ij} B^m_{pq} + \Gamma^{1,n}_{lm} B^l_{ip} B^m_{qj} \right) u_j = v_i B^0_{pq},
\]

(4.4b)

Equations (4.4) should hold for all $u$ and $A$, so that, separating out the basis tensors we have
the following equations for the components
\[
\sum_{n=0}^{8} \left( \Gamma_{lm}^{2,n} + t(l) \Gamma_{lm}^{1,n} \right) f_n = 0 \quad \forall 0 \leq l, m \leq 8, \tag{4.5a}
\]
\[
\sum_{k=1}^{5} \frac{\partial f_i}{\partial I_k} M_{km} = -\sum_{n=0}^{8} \left( \Gamma_{lm}^{3,n} + t(m) \Gamma_{ml}^{1,n} \right) f_n \quad \forall 0 \leq l \leq 8, 1 \leq m \leq 8, \tag{4.5b}
\]
where indices \( l, m \) are not contracted, \( t(l) = 1 \) if \( B^l \) is symmetric and \( t(l) = -1 \) if \( B^l \) is anti-symmetric. In the steps from equation (4.4b) to (4.5b), the components at \( m = 0 \) have been absorbed into the generic right-hand side \( v_t B_{pq}^0 \) of (4.4b), and therefore equation (4.5b) only concerns components with \( m \geq 1 \).

The linear system of 81 equations (4.5a) in the nine variables \( f_n, 0 \leq n \leq 8 \), can be solved using symbolic calculus (Meurer et al. 2017). This yields
\[
f_1 = f_2, \quad f_3 = f_5 = f_6, \quad f_4 = f_1 = f_8 = 0. \tag{4.6}
\]
Next, equation (4.5b) has a solution only if, for all \( l \), the right-hand side is orthogonal to the kernel of \( M \), but this condition imposes no further constraints on \( f_n \). Finally, with this orthogonality condition ensured, the derivatives \( \frac{\partial f_i}{\partial I_k} \) are obtained by multiplying equation (4.5b) by the Moore-Penrose inverse of \( M \), with components \( M_{mk}^{-1} \), thus yielding
\[
\frac{\partial f_0}{\partial I_1} = \frac{\partial f_0}{\partial I_2} = -\frac{1}{2} f_3, \tag{4.7a}
\]
\[
\frac{\partial f_0}{\partial I_k} = 0 \quad \forall 3 \leq k \leq 5, \tag{4.7b}
\]
\[
\frac{\partial f_n}{\partial I_k} = 0 \quad \forall 1 \leq k \leq 5, 1 \leq n \leq 8. \tag{4.7c}
\]
By solving the straightforward linear system (4.7) with the conditions (4.6), we obtain the components \( f_n \) of \( F \) that solves equation (2.4) and is linear in \( u \)
\[
F = \tilde{f}_1 u + \tilde{f}_2 A u + \tilde{f}_3 \left( A^2 - \frac{1}{2} \text{Tr}(A^2) I \right) u, \tag{4.8}
\]
where the \( \tilde{f}_n \) are arbitrary constants. The solution (4.8) of equation (2.4) encodes all the Betchov constraints since its divergence yields the gradient principal invariants
\[
\nabla \cdot F = \tilde{f}_2 \text{Tr}(A) + \tilde{f}_3 \text{Tr}(A^2), \tag{4.9}
\]
thus retrieving equations (1.3, 1.4) and, by homogeneity, \( \left< \tilde{f}_2 \text{Tr}(A) + \tilde{f}_3 \text{Tr}(A^2) \right> = 0 \).

The Betchov homogeneity relations, obtained by averaging equation (4.9), are all the possible homogeneous constraints on the single-point statistics of an incompressible and statistically isotropic gradient since they follow from all the independent solutions of equation (2.4). In other words, no scalar function of the velocity gradient invariants \( I_1, \ldots, I_5 \) can be written as the divergence of a vector field, other than the principal invariants \( \text{Tr}(A) = I_1 + I_2 \) and \( \text{Tr}(A^3) = I_3 + 3I_4 \).

5. Homogeneity constraints for the velocity gradient and additional quantities

Equation (4.8) shows that the homogeneity relations for the velocity gradient alone consist only of the two Betchov constraints. However, (4.8) easily generates homogeneity constraints concerning the velocity gradient together with additional variables. Indeed, the divergence
of $F$ in (4.8) does not depend on the gradient of the velocity gradient even when $u$ in (4.8) is replaced by any scalar, vector or tensor quantity $q$ that does not explicitly depend on the velocity gradient itself. This is because $F$ solves equation (2.4). Therefore, for any vector $q$, one can construct homogeneity relations for the scalar quantities

$$
\psi(A, \nabla q) = \nabla \cdot \left[ \vec{f}_1 q + \vec{f}_2 Aq + \vec{f}_3 \left( A^2 - \frac{1}{2} \text{Tr}(A^2)I \right) q \right],
$$

where standard matrix-vector product is implied and the left-hand side depends neither on $q$ nor on $\nabla A$. For example, using equation (5.1) with the pressure gradient divided by the fluid density, $q = \nabla P/\rho$, yields the homogeneity relations for the pressure Hessian in incompressible flows, namely $\langle A_{ij} \nabla_i \nabla_j P \rangle = 0$ and $\langle A_{ik} A_{kj} \nabla_i \nabla_j P \rangle = -\rho \langle (A_{ij} A_{ji})^2 \rangle /2$. Analogously, employing equation (5.1) with the velocity Laplacian, $q = \nabla^2 u$, gives the homogeneity relations for the Laplacian of a traceless gradient, $\langle A_{ij} \nabla^2 A_{ji} \rangle = 0$ and $\langle A_{ik} A_{kj} \nabla^2 A_{ji} \rangle = 0$. These relations are especially useful for the Lagrangian modelling of velocity gradients (Meneveau 2011; Tom et al. 2021).

6. Conclusions

We have shown that the Betchov homogeneity relations are all the possible homogeneity constraints for the velocity gradient in incompressible and statistically isotropic turbulence. Our conclusions apply to the single-point statistics of scalar functions of the velocity gradient. We have shown how our approach to searching for homogeneity relations on the velocity gradient generalizes to constraints involving additional quantities, like the pressure Hessian and the velocity gradient Laplacian. The presented methodology is also readily applicable to derive homogeneity constraints in less idealized flows (e.g. axisymmetric flows). More generally, the outcome of these calculations may help to deal with high-dimensional tensor equations, which are ubiquitous in fluid dynamics.

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Appendix A.

In this appendix, we compute the derivatives with respect to the gradient $A$ of the basis tensors $B^l_i$ (3.3) and invariants $I_k$ (3.4).

We start with the derivative of the invariants (3.4), which can be expressed as linear combinations of the basis tensors, as in equation (4.2). The matrix $M$ featuring the components of the derivatives of the invariants (3.4) in the employed basis (3.3) is computed by contracting equation (4.2) with the basis tensors,

$$
M_{km} = Z^{-1}_{ml} B^l_{pq} \frac{\partial I_k}{\partial A_{pq}} = \begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
-1/4 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\
-1/3 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix},
$$

where $Z^{-1}_{lm} = B^l_{pq} B^m_{pq}$ is the metric tensor and $Z^{-1}$ denotes its matrix inverse.

Next, we compute the derivatives of the basis tensors, for which we first introduce some notation. The basis tensors (3.3) are products of the symmetric and anti-symmetric parts of
the velocity gradient (3.1), which can in turn be expressed through the fourth-order tensors
\[ Q^{(t)}_{ijpq} = \frac{1}{2} \left( \delta_{ip} \delta_{jq} + t \delta_{iq} \delta_{jp} - \frac{1}{3} \delta_{ij} \delta_{pq} \right) \]  \hspace{1cm} (A 2)

contracted with the gradient itself, \( S_{ij} = Q^{(+1)}_{ijpq} A_{pq} \) and \( W_{ij} = Q^{(-1)}_{ijpq} A_{pq} \). Then, any basis tensor (3.3) of degree \( d \) consists of a linear combination of the products
\[ b^{t_1 \ldots t_d}_{ij} = Q^{(t_1)}_{i_1 k_1 p_1 q_1} Q^{(t_2)}_{k_1 k_2 p_2 q_2} \ldots Q^{(t_d)}_{k_d-1 j p_d q_d} (A_{p_1 q_1} A_{p_2 q_2} \ldots A_{p_d q_d}), \]  \hspace{1cm} (A 3)

with \( t_t = \pm 1 \) and summation over repeated indices, e.g. \( b^{+1,-1} = SW \). Also, the components of the basis tensors (3.3) with respect to the elementary products (A 3) are constant,
\[ B^n_{ij} = \sum_{t_1 \ldots t_d} c^n_{t_1 \ldots t_d} b_{ij}^{t_1 \ldots t_d}, \]  \hspace{1cm} (A 4)
e.g. the non-zero components of \( B^4 = SW - WS \) are \( c^4_{+1,-1} = -c^4_{-1,+1} = 1 \).

Using equations (A 3) and (A 4) we can take the derivative of any basis tensor of degree \( d \)
\[ \frac{\partial B^n_{ij}}{\partial A_{pq}} = \sum_{t_1 \ldots t_d} c^n_{t_1 \ldots t_d} Q^{(t_1)}_{i_1 k_1 p_1 q_1} Q^{(t_2)}_{k_1 k_2 p_2 q_2} \ldots Q^{(t_d)}_{k_d-1 j p_d q_d} \frac{\partial}{\partial A_{pq}} (A_{p_1 q_1} A_{p_2 q_2} \ldots A_{p_d q_d}) = \]
\[ = \sum_{t_1 \ldots t_d} c^n_{t_1 \ldots t_d} \sum_{m=1}^{d} \left[ b^{t_1 \ldots t_{m-1}}_{i k_{m-1}} Q^{(t_m)}_{k_{m-1} k_m p q} b^{t_{m+1} \ldots t_d}_{k_m j} \right], \]  \hspace{1cm} (A 5)

with \( t_t = \pm 1, k_0 = i, k_d = j \) and contraction over repeated indices. As expected, the derivative of any basis tensor consists of a linear combination of lower-order tensors. The derivative \( \partial(\cdot)/\partial A_{pq} \) is understood as a directional derivative in \( \mathbb{R}^{3,3} \) (Itskov 2015) and it is traceless due to incompressibility. In the steps to (A 5) this property is taken into account by the \( Q^{(t)}_{ijpq} \delta_{pq} = 0 \) (in fact, \( Q^{(t)} \) is the tensor derivative of \( b^t \) with respect to \( A \)).

Inserting the expression of the fourth-order tensors (A 2) into equation (A 5) we can write the derivatives of the basis tensors more explicitly. For the basis tensors (3.3) of degree one (i.e. for \( 1 \leq n \leq 2 \)) we have
\[ \frac{\partial B^n_{ij}}{\partial A_{pq}} = \frac{1}{2} \sum_{t} c^n_{t} \left( \delta_{ip} \delta_{qj} + t_1 \delta_{iq} \delta_{jp} - \frac{1+t_1}{3} \delta_{ij} \delta_{pq} \right). \]  \hspace{1cm} (A 6)

The derivatives of the basis tensors (3.3) of degree two (i.e. for \( 3 \leq n \leq 6 \)) read
\[ \frac{\partial B^n_{ij}}{\partial A_{pq}} = \frac{1}{2} \sum_{t_1 t_2} c^n_{t_1 t_2} \left( \delta_{ip} b^{t_2}_{qj} + t_1 \delta_{iq} b^{t_2}_{pj} - \frac{1+t_1}{3} b^{t_2}_{ij} \delta_{pq} + b^{t_1}_{ip} \delta_{qj} + t_2 b^{t_1}_{iq} \delta_{pj} - \frac{1+t_2}{3} b^{t_1}_{ij} \delta_{pq} \right). \]  \hspace{1cm} (A 7)

Finally, the derivatives of the basis tensors (3.3) of degree three (i.e. for \( 7 \leq n \leq 8 \)) read
\[ \frac{\partial B^n_{ij}}{\partial A_{pq}} = \frac{1}{2} \sum_{t_1 t_2 t_3} c^n_{t_1 t_2 t_3} \left( \delta_{ip} b^{t_2 t_3}_{qj} + t_1 \delta_{iq} b^{t_2 t_3}_{pj} - \frac{1+t_1}{3} b^{t_2 t_3}_{ij} \delta_{pq} + b^{t_1}_{ip} b^{t_3}_{qj} + b^{t_1}_{iq} b^{t_3}_{pj} \right) \]
\[ + t_2 b^{t_1}_{iq} b^{t_3}_{pj} - \frac{1+t_2}{3} b^{t_1}_{ij} \delta_{pq} + b^{t_1}_{ip} \delta_{qj} + t_3 b^{t_1}_{iq} \delta_{pj} - \frac{1+t_3}{3} b^{t_1}_{ij} \delta_{pq} \right). \]  \hspace{1cm} (A 8)
After replacing the symmetric and anti-symmetric parts of $\mathbf{b}^1$ and $\mathbf{b}^{1/2}$ with the corresponding $\mathbf{B}^n$, the tensor derivatives (A 6, A 7, A 8) can be compactly rewritten as contractions of the basis tensors with the Christoffel symbols $\Gamma_{lm}^{P,n}$, as in equation (4.3) in the main text. For the tensor basis (3.3) employed here, the non-zero elements of $\Gamma_{lm}^{1,n}$ are

$$
\begin{align*}
\Gamma_{00}^{1,1} &= \frac{1}{2}, & \Gamma_{02}^{1,4} &= \frac{1}{2}, & \Gamma_{10}^{1,5} &= \frac{1}{2}, & \Gamma_{05}^{1,7} &= \frac{1}{4}, & \Gamma_{50}^{1,7} &= \frac{1}{4}, & \Gamma_{12}^{1,8} &= \frac{1}{2}, \\
\Gamma_{00}^{1,2} &= \frac{1}{2}, & \Gamma_{10}^{1,4} &= \frac{1}{2}, & \Gamma_{20}^{1,5} &= \frac{1}{2}, & \Gamma_{12}^{1,7} &= \frac{1}{2}, & \Gamma_{21}^{1,8} &= \frac{1}{2}, \\
\Gamma_{01}^{1,3} &= \frac{1}{2}, & \Gamma_{20}^{1,4} &= -\frac{1}{2}, & \Gamma_{02}^{1,6} &= \frac{1}{2}, & \Gamma_{12}^{1,7} &= \frac{1}{2}, & \Gamma_{21}^{1,8} &= \frac{1}{2}, \\
\Gamma_{01}^{1,3} &= \frac{1}{2}, & \Gamma_{20}^{1,5} &= \frac{1}{2}, & \Gamma_{26}^{1,6} &= \frac{1}{2}, & \Gamma_{21}^{1,8} &= \frac{1}{2}, & \Gamma_{40}^{1,8} &= -\frac{1}{4}, \\
\Gamma_{02}^{1,4} &= \frac{1}{2}, & \Gamma_{12}^{1,7} &= \frac{1}{4}, & \Gamma_{40}^{1,7} &= \frac{1}{4}, & \Gamma_{50}^{1,8} &= \frac{1}{4}, \\
\Gamma_{02}^{1,5} &= \frac{1}{2}, & \Gamma_{14}^{1,7} &= -\frac{1}{4}, & \Gamma_{40}^{1,7} &= \frac{1}{4}, & \Gamma_{50}^{1,8} &= \frac{1}{4}.
\end{align*}
$$

the non-zero elements of $\Gamma_{lm}^{2,n}$ are

$$
\begin{align*}
\Gamma_{00}^{2,1} &= \frac{1}{2}, & \Gamma_{02}^{2,4} &= \frac{1}{2}, & \Gamma_{10}^{2,5} &= -\frac{1}{2}, & \Gamma_{05}^{2,7} &= -\frac{1}{4}, & \Gamma_{50}^{2,7} &= -\frac{1}{4}, & \Gamma_{12}^{2,8} &= \frac{1}{2}, \\
\Gamma_{00}^{2,2} &= \frac{1}{2}, & \Gamma_{10}^{2,4} &= -\frac{1}{2}, & \Gamma_{20}^{2,5} &= \frac{1}{2}, & \Gamma_{06}^{2,7} &= \frac{1}{2}, & \Gamma_{60}^{2,7} &= \frac{1}{2}, & \Gamma_{21}^{2,8} &= \frac{1}{2}, \\
\Gamma_{01}^{2,3} &= \frac{1}{2}, & \Gamma_{20}^{2,4} &= -\frac{1}{2}, & \Gamma_{02}^{2,6} &= \frac{1}{2}, & \Gamma_{12}^{2,7} &= -\frac{1}{2}, & \Gamma_{23}^{2,8} &= -\frac{1}{2}, & \Gamma_{30}^{2,8} &= -\frac{1}{2}, \\
\Gamma_{10}^{2,3} &= \frac{1}{2}, & \Gamma_{20}^{2,5} &= \frac{1}{2}, & \Gamma_{26}^{2,6} &= \frac{1}{2}, & \Gamma_{21}^{2,7} &= -\frac{1}{2}, & \Gamma_{40}^{2,8} &= -\frac{1}{4}, & \Gamma_{40}^{2,8} &= -\frac{1}{4}, \\
\Gamma_{12}^{2,4} &= \frac{1}{2}, & \Gamma_{20}^{2,5} &= \frac{1}{2}, & \Gamma_{40}^{2,7} &= -\frac{1}{4}, & \Gamma_{50}^{2,7} &= -\frac{1}{4}, & \Gamma_{50}^{2,8} &= \frac{1}{4}.
\end{align*}
$$

the non-zero elements of $\Gamma_{lm}^{3,n}$ are

$$
\begin{align*}
\Gamma_{00}^{3,1} &= -\frac{1}{3}, & \Gamma_{10}^{3,3} &= -\frac{2}{3}, & \Gamma_{20}^{3,5} &= -\frac{2}{3}, & \Gamma_{60}^{3,7} &= -\frac{2}{3}, & \Gamma_{50}^{3,8} &= -\frac{2}{3}.
\end{align*}
$$

To obtain equation (4.3) from the derivatives expressions (A 6, A 7, A 8), we computed first the Christoffel symbols relative to the independent elements of the set $\{\mathbf{b}^1, \mathbf{b}^{1/2}, \mathbf{b}^{1/2}\}$ and then changed the basis to the $\{\mathbf{B}^n\}$ listed in (3.3). Indeed, any set of basis tensors can be expressed through the transformation $\tilde{\mathbf{B}}^n = T^n_m(\mathcal{I}) \mathbf{B}^n$, with $\mathcal{I}$ an invertible matrix function of the invariants. Under this change of basis the Christoffel symbols transform as

$$
\begin{align*}
\tilde{\Gamma}_{lm}^{1,n'} &= T^n_m \Gamma_{lm}^{1,n} (T^{-1})^l_j (T^{-1})^m_r, & \tilde{\Gamma}_{lm}^{2,n'} &= T^n_m \Gamma_{lm}^{2,n} (T^{-1})^l_j (T^{-1})^m_r, \\
\tilde{\Gamma}_{lm}^{3,n'} &= \left( T^n_m \Gamma_{lm}^{3,n} + \frac{\partial T^n_l}{\partial I_k} M_{km} \right) (T^{-1})^l_j (T^{-1})^m_r.
\end{align*}
$$

REFERENCES

Alexakis, A. & Biferale, L. 2018 Cascades and transitions in turbulent flows. Phys. Rep. 767–769, 1–101.
Betchov, R. 1956 An inequality concerning the production of vorticity in isotropic turbulence. J. Fluid Mech. 1 (5), 497–504.
BRAGG, A. D., HAMMOND, A. L., DHARIWAL, R. & MENG, H. 2021 Hydrodynamic interactions and extreme particle clustering in turbulence. arXiv:2104.02758.

BUARIA, D., PUMIR, A. & BODENSCHATZ, E. 2020 Self-attenuation of extreme events in Navier–Stokes turbulence. Nat. Commun. 11.

CARBONE, M. & BRAGG, A. D. 2020 Is vortex stretching the main cause of the turbulent energy cascade? J. Fluid Mech. 883, R2.

DAVIDSON, P. A. 2004 Turbulence: an introduction for scientists and engineers. Oxford University Press.

ENCISO, A., PERALTA-SALAS, D. & DE LIZAUR, F. T. 2016 Helicity is the only integral invariant of volume-preserving transformations. Proc. Natl. Acad. Sci. U.S.A. 113 (8), 2035–2040.

GRINFIELD, P. 2013 Introduction to Tensor Analysis and the Calculus of Moving Surfaces. Springer New York.

HIERRO, J. & DOPAZO, C. 2003 Fourth-order statistical moments of the velocity gradient tensor in homogeneous, isotropic turbulence. Phys. Fluids. 15 (11), 3434–3442.

HILL, R. J. 1997 Applicability of Kolmogorov’s and Monin’s equations of turbulence. J. Fluid Mech. 353, 67–81.

ITYSKOV, M. 2015 Tensor Algebra and Tensor Analysis for Engineers. Switzerland: Springer International Publishing.

JOHNSON, P. L. 2020 Energy transfer from large to small scales in turbulence by multiscale nonlinear strain and vorticity interactions. Phys. Rev. Lett. 124, 104501.

JOHNSON, P. L. & MENEVEAU, C. 2016 A closure for lagrangian velocity gradient evolution in turbulence using recent-deformation mapping of initially Gaussian fields. J. Fluid Mech. 804, 387–419.

LEPPIN, L. A. & WILCZEK, M. 2020 Capturing velocity gradients and particle rotation rates in turbulence. Phys. Rev. Lett. 125, 224501.

LUND, T.S. & NOVIKOV, E. A. 1992 Parameterization of subgrid-scale stress by the velocity gradient tensor. Center for Turbulence Research (Stanford University and NASA).

MAIDA, A. J. & BERTOZZI, A. L. 2001 Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics. Cambridge University Press.

MENEVEAU, C. 2011 Lagrangian dynamics and models of the velocity gradient tensor in turbulent flows. Annu. Rev. Fluid Mech. 43 (1), 219–245.

MEURER, A., SMITH, C. P., PAPROCKI, M., ĆERTIK, O., KIRPICHEV, S. B., ROCKLIN, M., KUMAR, A., IVANOY, S., MOORE, J. K., SINGH, S., RATHNAYAKE, T., VIG, S., GRANGER, B. E., MULLER, R. P., BONAZZI, F., GUPTA, H., VATS, S., JOHANSSON, F., PEDREGOSA, F., CURRY, M. J., TERREL, A. R., ROUČKA, Š., SABOO, A., FERNANDO, I., KULAL, S., CIMMRAN, R. & SCOPATZ, A. 2017 Sympy: symbolic computing in Python. PeerJ Comput. Sci. 3, e103.

MOMEMIFAR, M., DIAO, E., TAROKH, V. & BRAGG, A. D. 2021 Dimension reduced turbulent flow data from deep vector quantizers, arXiv: 2103.01074.

PENNISI, S. & TROVATO, M. 1987 On the irreducibility of professor G.F. Smith’s representations for isotropic functions. Int. J. Eng. Sci. 25 (8), 1059–1065.

RIVLIN, R. S. & ERICKSEN, J. L. 1955 Stress-deformation relations for isotropic materials. J. Rat. Mech. Anal. 4, 323–425.

SERRE, D. 1984 Les invariants du premier ordre de l’équation d’Euler en dimension trois. Physica D 13 (1), 105–136.

SIGGIA, E. D. 1981 Invariants for the one-point vorticity and strain rate correlation functions. Phys. Fluids. 24 (11), 1934–1936.

TIAN, Y., LIVESCU, D. & CHERTKOV, M. 2021 Physics-informed machine learning of the Lagrangian dynamics of velocity gradient tensor. Phys. Rev. Fluid 6, 094607.

TOM, J., CARBONE, M. & BRAGG, A. D. 2021 Exploring the turbulent velocity gradients at different scales from the perspective of the strain-rate eigenframe. J. Fluid Mech. 910, A24.

TOWNSEND, A. A. & TAYLOR, G. I. 1951 On the fine-scale structure of turbulence. Proc. Math. Phys. Eng. Sci. 208 (1095), 534–542.

TSINOBER, A. 2009 An Informal Conceptual Introduction to Turbulence. Springer Netherlands.

WEYL, H. 1946 The Classical Groups: Their Invariants and Representations. Princeton Landmarks in Mathematics and Physics Nr. 1, Teil 1. Princeton University Press.

ZHENG, Q.-S. 1994 Theory of Representations for Tensor Functions—A Unified Invariant Approach to Constitutive Equations. Appl. Mech. Rev. 47 (11), 545–587.