LARGE N RENORMALIZATION GROUP APPROACH TO MATRIX MODELS

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We summarize our recent results on the large $N$ renormalization group (RG) approach to matrix models for discretized two-dimensional quantum gravity. We derive exact RG equations by solving the reparametrization identities, which reduce infinitely many induced interactions to a finite number of them. We find a nonlinear RG equation and an algorithm to obtain the fixed points and the scaling exponents. They reproduce the spectrum of relevant operators in the exact solution. The RG flow is visualized by the linear approximation.
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We summarize our recent results on the large $N$ renormalization group (RG) approach to matrix models for discretized two-dimensional quantum gravity. We derive exact RG equations by solving the reparametrization identities, which reduce infinitely many induced interactions to a finite number of them. We find a nonlinear RG equation and an algorithm to obtain the fixed points and the scaling exponents. They reproduce the spectrum of relevant operators in the exact solution. The RG flow is visualized by the linear approximation.

1. Introduction
Matrix models have been a powerful tool to study two-dimensional quantum gravity (2D QG) via simplicial decomposition of the spacetime. We are particularly interested in the 2D QG both as a string theory and a toy model for the QG in higher dimensions. Exact solutions of the matrix model have been successfully obtained for 2D QG coupled to minimal conformal matter with central charge $c \leq 1$, but unsuccessful for $c > 1$ cases which correspond to realistic string theories. We need to obtain approximation schemes which provide us with correct results for the exactly solved cases and enable us to calculate critical coupling constants and critical exponents for unsolved matrix models, especially for $c > 1$.

A large $N$ renormalization group\(^2\) (RG) has been proposed by Brézin and Zinn-Justin as such an approximation method\(^3\), identifying the size of the matrix with the inverse lattice constant of the simplices, and was discussed by several groups\(^4\). We have succeeded to derive explicitly exact RG equations for $O(N)$-vector models\(^5\) and one-\(^6\) and two-matrix models\(^7\). We found that it is crucial to take account of the reparametrization identities in order to obtain a meaningful RG equation. The RG equation for the matrix model turned out to be nonlinear in contrast to the linear RG equation for the vector model. Moreover, we found that the global picture of the RG flow can be drawn practically by a linear approximation to the nonlinear RG equation\(^7\).

2. Nonlinear RG equation for the one-matrix model
The partition function $Z_N(g_j)$ of a one-matrix model with a general potential $V(\phi) = \sum_{k \geq 1} \frac{g_k}{k!} \phi^k$ is defined by an integral over an $N \times N$ hermitian matrix $\phi$, $Z_N(g_j) = \int d\phi \ e^{-N \text{tr} V(\phi)}$. Starting from an $(N+1) \times (N+1)$ hermitian matrix variable $\Phi$, we decompose it into an $N \times N$ hermitian matrix $\phi$, a complex $N$-vector $v$ and a real scalar $\alpha$. Taking cubic interaction $V(\phi) = \frac{1}{2} \phi^2 + \frac{g}{3} \phi^3$,
we integrate over \( v \) exactly to obtain

\[
Z_{N+1}(g) = \left( \frac{\pi}{N + 1} \right)^N \int d\phi \ e^{-(N+1)\text{tr} V(\phi)} \cdot \int d\alpha \ e^{-(N+1)V(\alpha) - \text{tr} \log(1+g(\phi+\alpha))}. \tag{1}
\]

We can evaluate the \( \alpha \)-integral by the saddle point method, systematically in \( 1/N \)-expansion. If we explicitly retain only the leading part of the \( 1/N \)-expansion, we obtain a difference equation relating \( Z_{N+1} \) and \( Z_N \). We shall denote the average \( Z_N^{-1} \int d\phi \cdots e^{-N\text{tr} V(\phi)} \) by \( \langle \cdots \rangle \). Factorization holds in the large-\( N \) limit \( \langle \mathcal{O} \mathcal{O}' \rangle = \langle \mathcal{O} \rangle \langle \mathcal{O}' \rangle + O(N^{-2}) \) for a multi-point function of \( U(N) \)-invariants \( \mathcal{O} \), \( \mathcal{O}' \). By using this property we obtain for the free energy \( F(N, g) = -N^{-2} \log(Z_N(g)/Z_N(0)) \)

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g) = -\frac{1}{2} + \left( \frac{1}{N} \text{tr} V(\phi) \right) + V(\langle \alpha_s \rangle) + \left( \frac{1}{N} \text{tr} \log(1 + g(\phi + \langle \alpha_s \rangle )) \right) + O\left( \frac{1}{N} \right). \tag{2}
\]

Here \( \alpha_s = \alpha_s(\phi) \) is determined as a \( U(N) \)-invariant function by the saddle point equation

\[
V'(\alpha_s) + \frac{1}{N} \text{tr} \left( \frac{1}{1/g + \alpha_s} + \phi \right) = 0. \tag{3}
\]

The right hand side of eq. (2) consists of products of \( \langle 1/N \text{tr} \phi^j \rangle = j \partial F/\partial g_j \), \( j = 1, 2, \ldots \). The crucial observation we make is that there is an ambiguity to define a flow in the coupling constant space, since we can make an arbitrary reparametrization of matrix variable \( \phi \) in the partition function. This will enable us to express the \( \partial F/\partial g_j \)-dependence \( (j \neq 3) \) in terms of \( \partial F/\partial g_3 \), and to reduce all the induced interactions to those in the original potential. The resulting identities are a part of the discrete Schwinger-Dyson equation for the system.

More concretely, if we perform reparametrizations regular at the origin, \( \phi' = \phi + \epsilon \phi^{n+1} \) \( (n \geq -1) \) for the partition function, we obtain identities

\[
\sum_{j=0}^{n} \left( \frac{1}{N} \text{tr} \phi^j \frac{1}{N} \text{tr} \phi^{n-j} \right) = \left( \frac{1}{N} \text{tr} \left( \phi^{n+1} V'(\phi) \right) \right) \quad (n \geq -1). \tag{4}
\]

These identities imply relationship among \( \partial F/\partial g_j \)'s in the large \( N \) limit,

\[
2n \frac{\partial F}{\partial g_n} + \sum_{j=1}^{n-1} j(n-j) \frac{\partial F}{\partial g_j} \frac{\partial F}{\partial g_{n-j}} = \sum_{k \geq 1} (n+k)g_k \frac{\partial F}{\partial g_{n+k}}. \tag{5}
\]

By introducing the resolvent \( \tilde{W}(z) = (1/N) \text{tr} (z - \phi)^{-1} \), eq. (4) is neatly summarized into a single equality called the loop equation

\[
\left\langle \tilde{W}(z) \right\rangle^2 - V'(z) \left\langle \tilde{W}(z) \right\rangle + Q(z) = 0, \quad Q(z) \equiv \sum_{k=1}^{m-1} \frac{V^{(k+1)}(z)}{k!} \left( \frac{1}{N} \text{tr} (\phi - z)^{k-1} \right). \tag{6}
\]

Eqs. (4)–(6) hold for a generic polynomial potential \( V(\phi) = \sum_{k=1}^{m} \frac{\alpha_k}{k!} \phi^k \); for our case \( m = 3 \), \( Q(z) \) is given by \( Q(z) = 1 + gz - g^2 + 3g^3 \partial F/\partial g \). Consequently we obtain a differential equation obeyed by the free energy \( F(N, g) \) of the one-matrix model with the cubic coupling,

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g) = G \left( g, \frac{\partial F}{\partial g} \right) + O\left( \frac{1}{N} \right), \quad \tilde{\alpha}(g, a) \equiv \langle \alpha_s \rangle = -g + 3g^2a,
\]

\[
G(g, a) = -\frac{g}{3} a + \frac{1}{3} a^2 + \frac{g}{2} a^3 + \log \left( 1 + g \tilde{\alpha} \right) + \int_{-\frac{1}{g-\tilde{\alpha}}}^{1} dz \left( \left\langle \tilde{W}(z, g, a) \right\rangle - \frac{1}{N} \right). \tag{7}
\]
To establish an algorithm for fixed points and scaling exponents, we first concentrate on the leading part $F^0(g)$ of the free energy in the $1/N$-expansion. It is easy to see that $F^0(g)$ satisfies $2F^0(g) = G(g, \partial F^0(g)/\partial g)$. We assume that $F^0$ consists of regular and singular parts around a fixed point $g_*$, $F^0(g) = \sum_{k=0}^{\infty} a_k (g - g_*)^k + \sum_{k=0}^{\infty} b_k (g - g_*)^{k+2-\gamma_0}$ ($2 - \gamma_0 \not\in \mathbb{N}$), and that $G(g, a)$ is regular around $(g_*, a_1)$. $\gamma_0$ is referred to as the susceptibility exponent. By comparing the coefficients of various powers of $g - g_*$ on both sides of the RG equation, we have a set of equations to determine unknowns $\gamma_0$, $g_*$, $a_k$ and $b_k$. We can also derive the RG equation for higher genus contributions and find that they exhibit the double scaling behavior under a plausible assumption.

We can as well use the eigenvalue representation for the RG equation which is particularly useful for multi-coupling cases. We find four fixed points (Yang-Lee edge singularity, pure gravity and Gaussian) in the two-coupling ($g_3$ and $g_4$) case; there the spectrum of relevant operators of the exact solution ($\gamma_0 = -1/3, -3/2$ for Yang-Lee and $\gamma_0 = -1/2$ for pure gravity) are exactly reproduced.

3. The two-matrix model
The partition function for the two-matrix model with $V(\phi, g) = \frac{1}{2} \phi^2 + \frac{g}{3} \phi^3$ is defined as

$$Z_N(g_+, g_-, c) = \int d\phi_+ d\phi_- \exp \left\{ - N \text{tr} \left( V(\phi_+, g_+) + V(\phi_-, g_-) + c\phi_+\phi_- \right) \right\}. \quad (8)$$

Similarly to the one-matrix case, we integrate the $N+1$-th row and column exactly and evaluate the singlet integral by the saddle point method. We consider the case $g_+ = g_- \equiv g$ which describes the Ising model coupled to 2D QG in the absence of an external magnetic field. We assume that $\langle \text{tr} \phi_j^+ \rangle = \langle \text{tr} \phi_j^- \rangle$ holds for any $j$, and solve the reparametrization identities (a fourth-order algebraic equation for the resolvent in need), $\hat{W}_0(z) = \frac{1}{N} \text{Tr} \left( \frac{z + \phi_+}{c/g} \frac{c/g}{z + \phi_-} \right)^{-1}$. Then we find a nonlinear RG equation for the two-matrix model

$$\left[ N \frac{\partial}{\partial N} + 2 \right] F = G\left(g, c; \frac{\partial F}{\partial g}, \frac{\partial F}{\partial c}\right) + \mathcal{O}\left(\frac{1}{N}\right),$$

$$G(g, c; a_g, a_c) = \frac{g}{2} a_g + (1 + c) \bar{\alpha}^2 + \frac{2g}{3} \bar{\alpha}^3 + 2 \log(1 + g \bar{\alpha})$$

$$+ \int_{-\infty}^{1/\gamma + \alpha} dz \left( \langle \hat{W}_0(z; g, c; a_g, a_c) \rangle - \frac{2}{z} \right),$$

$$\bar{\alpha}(g, c; a_g, a_c) \equiv \langle \alpha_{\pm,s} \rangle = \frac{g}{1 + c} \left( -1 + c a_c + \frac{3g}{2} a_g \right) \quad (9)$$

We find three fixed points (critical Ising, pure gravity and Gaussian) in the physical region of the coupling constant space; there the spectrum of relevant operators of the exact solution ($\gamma_0 = -1/3, -5$ for critical Ising and $\gamma_0 = -1/2$ for pure gravity) are again exactly reproduced.

4. Linearized RG flow
We observe generically that the coefficients of the nonlinear terms in the $G$-function, $\partial^n G/\partial a^n(g, 0)$ for $n \geq 2$, are suppressed by increasing powers of $g$. Therefore we expect that within a region $g \ll 1$ where most fixed points lie, the RG flow should be well approximated by the linearized $\beta$-function, $\beta_{\text{lin}}(g) \equiv \partial G/\partial a(g, 0)$. Here we exhibit the RG flow in one- (Fig.1) and two-matrix models (Fig.2) described by the real parts of $\beta_{\text{lin}}$. We find that the critical lines (real lines) emanating from the multicritical fixed points toward the pure gravity fixed point are characterized as
the RG trajectories corresponding to the least relevant perturbations of the UV theories. These flows are consistent with those for matter fields over a fixed background, and suggest a gravitational analogue of the celebrated $c$-theorem$^8$ stating the decrement of the degrees of freedom of the matter sector along the RG flow.

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Figure 1: RG flow for the one-matrix model. Figure 2: RG flow for the two-matrix model. The arrows represent a flow from the UV ($N = \infty$) to the IR ($N = 0$).