Exact Relations for Heavy-Light Quark Systems

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Abstract

We derive general relations among hadronic form factors involving one heavy meson $b\bar{q}$ and another, not necessarily heavy, meson $Q\bar{q}$. The relations are valid to all orders of mass corrections of $m_Q$.
The inclusive rare decay $B \to X_s \gamma$ is now well understood in the context of the standard model [1] and the new experimental upper bound [2] of $5.4 \times 10^{-4}$ for the branching ratio is already playing an important role [3] in constraining the parameters of various models other than the standard model. The first experimental observation of the exclusive decay $B \to K^{*}\gamma$ has been reported from the CLEO collaboration [4] which gives a branching ratio for this mode of $(4.5 \pm 1.5 \pm 0.9) \times 10^{-5}$. It is this exclusive rare decay $B \to K^{*}\gamma$, however, which is the least well known theoretically due to the large recoil momentum of the $K^*$ meson [5]. We have recently shown [6] that heavy-quark symmetry together with $SU(3)$ flavor symmetry could relate the rare decay $B \to K^{*}\gamma$ to a measurement of the semileptonic decay $B \to \rho e \bar{\nu}$. This work made use of the heavy-quark limit for the $B$ meson and the weak binding limit for the $K^*$ meson. However, we also showed that the results were not an artifact of a particular quark model by demonstrating that the agreement of the form factor relations also held for the BSW model [7].

In analysing the relations among the form factors, we noted that there were a number of relations that only depend on the heavy-quark limit for the $b$ quark and not on the weak binding limit. It is these relations which we describe here. Although the interest is in the $K^*$ meson, we shall describe the decays from a $B(b\bar{q})$ or $B^*$ to an arbitrary vector meson $V(Q\bar{q})$, so that our results are also applicable to the charm system.

First, we define the hadronic form factors of interest as in Ref. [8]:-
\[ \langle V(v', \epsilon) | \bar{Q} \gamma_\mu b | B(v) \rangle = \sqrt{m_B m_V} h_V i \varepsilon_{\mu\nu\lambda\sigma} \epsilon^{\nu\lambda} v'^\lambda v^\sigma, \quad (1) \]

\[ \langle V(v', \epsilon) | \bar{Q} \gamma_\mu \gamma_5 b | B(v) \rangle = \sqrt{m_B m_V} \left[ h_{A_1} (1 + w) \epsilon^*_\mu - h_{A_2} (\epsilon^* \cdot v) v_\mu - h_{A_3} (\epsilon^* \cdot v) v'_\mu \right], \quad (2) \]

\[ \langle V(v', \epsilon) | \bar{Q} \gamma_\mu b | B^*(v, \zeta) \rangle = \sqrt{m_B m_V} \left\{ - (\zeta \cdot \epsilon^*) [h_1 (v + v')_\mu + h_2 (v - v')_\mu] \right. \]
\[ + \left. h_3 (\epsilon^* \cdot v) \zeta_\mu + h_4 (\zeta \cdot v') \epsilon^*_\mu - (\zeta \cdot v') (\epsilon^* \cdot v) [h_5 v_\mu + h_6 v'_\mu] \right\}, \quad (3) \]

\[ \langle V(v', \epsilon) | \bar{Q} \gamma_\mu \gamma_5 b | B^*(v, \zeta) \rangle = \sqrt{m_B m_V} i \varepsilon_{\mu\nu\lambda\sigma} \left\{ \zeta^\lambda \epsilon^{\nu\sigma} [h_7 (v + v')^\nu + h_8 (v - v')^\nu] \right. \]
\[ + \left. v'^\lambda v^\sigma [h_9 (\epsilon^* \cdot v) \zeta^\nu + h_{10} (\zeta \cdot v') \epsilon^{\nu\sigma}] \right\}. \quad (4) \]

The terms \( \zeta \) and \( \epsilon^* \) are the polarization vectors of \( B^* \) and \( V \), respectively. The variables \( v \) and \( v' \) are the four velocities of \( B^* \) (or \( B \)) and \( V \), and we define \( w = v \cdot v' \). We show below that the form factors for the decays \( B \to V \) and \( B^* \to V \) can be related using the spin symmetry and static limit of the heavy \( b \) quark.

In the heavy \( b \) limit, the spin of the \( b \) quark is decoupled from all other light fields in the \( B \) meson [3]. We can therefore construct the spin operator \( S_b^Z \) for the \( b \) quark such that

\[ S_b^Z | B(b\bar{q}) \rangle = \frac{1}{2} | B^*_1 (b\bar{q}) \rangle, \quad S_b^Z | B^*_1^*(b\bar{q}) \rangle = \frac{1}{2} | B (b\bar{q}) \rangle, \]

where \( B^*_1 \) stands for a longitudinal vector \( B^* \) meson. In \( |B\rangle \) and \( |B^*_1\rangle \), the spatial momentum of the \( b \) quark is defined in the \( z \)-direction for the \( b \) spinor to be an eigenstate of \( S_b^Z \). Using the relation \( \langle V | \bar{Q} \Gamma b | B \rangle = -2 \langle V | [S_b^Z, \bar{Q} \Gamma b] | B^*_1 \rangle \), for \( \Gamma \) any product of \( \gamma \) matrices, we have the following identities between the \( B \to V \) and \( B^*_1 \to V \) matrix elements:

\[ \langle V | A_0 | B \rangle = -\langle V | V_3 | B^*_1 \rangle, \quad (5) \]
\[ \langle V | A_3 | B \rangle = -\langle V | V_0 | B^*_1 \rangle, \quad (6) \]
\[ \langle V|V_{\pm}|B\rangle = \mp \langle V|V_{\pm}|B_i^*\rangle, \quad (7) \]
\[ \langle V|V_0|B\rangle = -\langle V|A_3|B_i^*\rangle, \quad (8) \]
\[ \langle V|V_3|B\rangle = -\langle V|A_0|B_i^*\rangle, \quad (9) \]
\[ \langle V|A_{\pm}|B\rangle = \mp \langle V|A_{\pm}|B_i^*\rangle, \quad (10) \]

where \( V_{\mu} = \bar{Q}\gamma_{\mu}b \) and \( A_{\mu} = \bar{Q}\gamma_{\mu}\gamma_5b \).

Using the matrix identities in Eqs. (5-10), we can relate the form factors of \( B \to V \) to those of \( B^* \to V \). Since the spatial momentum of the \( b \) quark is defined in the \( z \)-direction, we should work in the \( B \) rest frame and choose the longitudinal polarization vector for \( B_i^* \) to be \( \zeta_{i\mu} = (0; 0, 0, 1) \). The matrix identities are evaluated for both transverse and longitudinal polarizations of the vector meson \( V(Q\bar{q}) \). This gives the following relations among the form factors:-

\[
\begin{align*}
    h_4 &= h_1 - h_2, \\
    h_5 &= h_9, \\
    h_6 &= 0, \\
    h_7 &= h_1, \\
    h_8 &= h_2, \\
    h_{10} &= 0, \\
    h_V &= h_1 - h_2, \\
    h_{A_1} &= (h_1 - h_2) + \frac{2h_2}{1 + w}, \\
    h_{A_2} &= (h_1 + h_2 - h_3) + wh_9.
\end{align*}
\]
\[ h_{A_3} = (h_1 - h_2) - h_9. \]  

In the recent literature [8, 10] these relations are obtained using an effective Lagrangian approach. While this eases the burden in calculating the symmetry limit when both mesons contain heavy constituent quarks (and allows for a systematic inclusion of mass corrections [8]), we shall see that Eqs. (5 - 10) are simpler to use when the symmetry may be badly broken, or even is not applicable.

In the decay \( B \to K^*\gamma \) in particular, we consider also the hadronic matrix element for the current \( \bar{Q}\sigma_{\mu\nu}q^\nu b_R \), where \( q = p_B - k \) is the momentum of the outgoing photon. The covariant expansion of the matrix element is given by

\[
\langle V(k, \epsilon)|\bar{Q}\sigma_{\mu\nu}q^\nu b_R|B(p_B)\rangle = f_1(q^2)i\varepsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}p_B^\lambda k^\sigma \\
+ \left[(m_B^2 - m_V^2)\epsilon^*_\mu - (\epsilon^* \cdot q)(p_B + k)_\mu\right] f_2(q^2) \\
+ (\epsilon^* \cdot q) \left[(p_B - k)_\mu - \frac{q^2}{(m_B^2 - m_V^2)}(p_B + k)_\mu\right] f_3(q^2). \tag{12}
\]

We can relate the form factors \( f_{1,2,3} \) to \( h_s \) defined in Eq. (4) using the static limit of the \( b \) quark. In the \( B \) rest frame, the static \( b \) quark spinor satisfies the equation of motion \( \gamma_0 b = b \).

We then have the relations between the \( \gamma_\mu \) and \( \sigma_{\mu\nu} \) matrix elements [11]-

\[
\langle V|\bar{Q}\gamma_\mu b|B\rangle = \langle V|\bar{Q}i\sigma_{0\mu} b|B\rangle, \tag{13}
\]

\[
\langle V|\bar{Q}\gamma_\mu \gamma_5 b|B\rangle = -\langle V|\bar{Q}i\sigma_{0\mu} \gamma_5 b|B\rangle. \tag{14}
\]

This gives the form-factor relations

\[
h_{f_1} = (m_B + m_V)(h_1 - h_2) + 2m_V h_2,
\]

\[
h_{f_2} = \frac{m_B m_V}{(m_B + m_V)}(1 + w)(h_1 - h_2) + \frac{(m_B - w m_V)}{(m_B^2 - m_V^2)} 2m_B m_V h_2,
\]

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where \( h_{f_1} = \sqrt{4m_B m_V f_1} \), \( h_{f_2} = \sqrt{4m_B m_V f_2} \), and \( h_{f_3} = \sqrt{4m_B m_V f_3} \). Thus, using only the spin symmetry and static limit of the heavy \( b \) quark, we can express the \( B \to V \) and \( B^* \to V \) hadronic form factors in terms of four independent form factor combinations \( h_1 - h_2 \), \( h_2 \), \( h_1 + h_2 - h_3 \), and \( h_9 \) as shown in Eqs. (11) and (15). Note that we have made no assumptions about the mass of the quark \( Q \) at any point in the above discussion. The form factor relations are therefore valid for heavy or light \( Q \).

In terms of the factors \( \epsilon, \bar{\epsilon}, \rho \) defined as

\[
- \frac{(h_1 + h_2 - h_3)}{(h_1 - h_2)} \equiv (1 + w)\bar{\epsilon} ,
- \frac{2h_2}{(h_1 - h_2)} \equiv (1 + w)\epsilon ,
- \frac{h_9}{(h_1 - h_2)} \equiv \rho ,
\]

we can rewrite the form factor relations in Eqs. (11) and (15) in terms of \( h_V = h_1 - h_2 \) and the three newly defined factors as,

\[
\begin{align*}
  h_3 &= h_V[1 + (1 + w)(\bar{\epsilon} - \epsilon)] , \\
  h_4 &= h_V , \\
  h_9 &= h_5 = -\rho h_V , \\
  h_6 &= 0 , \\
  h_1 &= h_7 = h_V \left(1 - \frac{1+w}{2}\epsilon\right) , \\
  h_2 &= h_8 = -\frac{1+w}{2}\epsilon h_V ,
\end{align*}
\]
\[ h_{10} = 0 , \]
\[ h_{A_1} = h_V (1 - \varepsilon) , \]
\[ h_{A_2} = -h_V [(1 + w)\bar{\varepsilon} + w\rho] , \]
\[ h_{A_3} = h_V (1 + \rho) , \]
\[ h_{f_1} = h_V [(m_B + m_V) - m_V (1 + w)\varepsilon] , \]
\[ h_{f_2} = h_V m_B m_V (1 + w) \left[ \frac{1}{(m_B + m_V)} - \frac{(m_B - w m_V)}{(m_B^2 - m_V^2)} \varepsilon \right] , \]
\[ h_{f_3} = h_V \left[ \frac{1}{2} (m_B - m_V) + \frac{1}{2} m_V (1 + w) \varepsilon + \frac{(m_B^2 - m_V^2)}{2m_B} \rho \right] . \]

(16)

In the heavy \( Q \) limit [8], we have \( h_1 = h_3 \) and \( h_2 = h_9 = 0 \); thus, the factors \( \varepsilon, \bar{\varepsilon}, \) and \( \rho \) all vanish in this limit. For finite quark mass \( m_Q \), however, they should represent the full \( 1/m_Q \) corrections to the heavy-quark symmetry relations. By simple inspection we see in particular that the symmetry relations in Eq. (16) are consistent with the order \( 1/m_Q^2 \) result in Ref. [8].

For the form factors \( h_{A_1}, h_1 \) and \( h_7 \) we can see that at the point where Luke’s [12] theorem would set in, i.e. \( \omega = 1 \) and for heavy \( m_b \) and \( m_Q \), the renormalization is the same to all orders in the mass of \( Q \). (By inspection, the same results holds true for all orders in the \( b \) quark for large \( Q \) but the mixed expansion of the masses is different).

We can estimate the size of the correction factors \( \varepsilon, \bar{\varepsilon}, \) and \( \rho \) using the nonrelativistic quark model [13]. In the quark model, we have in the \( B \) rest frame

\[- \rho \approx \varepsilon = 1 - \left(E_V - m_V\right) \frac{H_1}{H_2} , \]

(17)

where \( E_V = (m_B^2 + m_V^2 - q^2)/(2m_B) \) is the energy of \( V \). The terms \( H_1 \) and \( H_2 \) are overlapping integrals of the momentum wave functions in the quark model, they are given by Eq. (22) of
our recent paper [8]. In Ref. [6], we have shown that in the weak binding limit of $V$ the factor $1 - (E_V - m_V)H_1/H_2$ in Eq. (17) is much smaller than 1 throughout the whole kinematic range. In an numerical calculation using the ISGW parameterization [13] of the momentum wave function, we show explicitly that $1 - (E_V - m_V)H_1/H_2 < 0.05$ for $B \rightarrow K^*$ and 0.11 for $B \rightarrow \rho$ throughout the full kinematic range. Accordingly, the factors $\epsilon$, $\bar{\epsilon}$, and $\rho$ can be treated as small corrections to the heavy-quark symmetry relations even for light $Q$.

It is not surprising that the above result coincides with the one that emerges in the heavy-quark limit, since the spin symmetry for $Q$ should approximately hold in the quark model. While the validity of estimating the size of correction factors using the nonrelativistic quark model may be questioned, we will show that the same result can be obtained using a relativistic quark model. In the BSW model [7], we have at the maximum recoil of $V$ ($q^2 = 0$ or $v \cdot v' = (m_B^2 + m_V^2)/(2m_Bm_V)$) the expressions for $\epsilon$, $\bar{\epsilon}$, and $\rho$ given by

\begin{align*}
\varepsilon(0) &= 1 - \left(\frac{m_B - m_V}{m_B + m_V}\right)\left(\frac{m_b + m_Q}{m_b - m_Q}\right), \quad (18) \\
\bar{\varepsilon}(0) &= \frac{1}{(m_b - m_Q)(m_B + m_V)^2} \left[\frac{g_1}{g_2} - \frac{1}{2} \left(\frac{m_b + m_Q}{m_B + m_V}\right) \left(1 + \frac{m_V}{m_B}\right)\right], \quad (19) \\
\rho(0) &= \frac{1}{(m_b - m_Q)(m_B - m_V)} \left[\frac{m_b}{m_B} - \frac{m_Q}{m_V} \left(\frac{m_B - m_V}{m_B + m_V}\right)\right]. \quad (20)
\end{align*}

The terms $g_1$ and $g_2$ are overlap integrals given by

\begin{align*}
g_1 &= \int d\mathbf{p}_T \int_0^1 dx \phi_V^*(\mathbf{p}_T, x) \phi_B(\mathbf{p}_T, x), \quad (21) \\
g_2 &= \int d\mathbf{p}_T \int_0^1 \frac{dx}{x} \phi_V^*(\mathbf{p}_T, x) \phi_B(\mathbf{p}_T, x). \quad (22)
\end{align*}

In the BSW model, the orbital wave function $\phi_B$ and $\phi_V$ are solutions to a relativistic scalar harmonic oscillator potential. Notice that only $\bar{\varepsilon}(0)$ depends on the overlapping effects in the
BSW model, and $\bar{\epsilon}$ enters only into the expressions for $h_{A_2}$ and $h_3$ in the symmetry relations of Eq. (16). All other relations in (16) are overlap independent. It can be shown that the ratio $g_1/g_2$ is very stable with respect to the parameter changes in the orbital wave functions in the BSW model. Numerically, for the decay $B \rightarrow K^*$, we have the values of the correction factors given by \( \bar{\epsilon}^{B\rightarrow K^*}(0) = -0.023 \), \( \bar{\epsilon}^{B\rightarrow K^*}(0) = 0.11 \), and \( \rho^{B\rightarrow K^*}(0) = -0.25 \). For the decay $B \rightarrow \rho$, we have \( \bar{\epsilon}^{B\rightarrow \rho}(0) = 0.042 \), \( \bar{\epsilon}^{B\rightarrow \rho}(0) = 0.15 \), and \( \rho^{B\rightarrow \rho}(0) = -0.24 \).

In a recent paper [6], we have discussed a method of relating the decay $B \rightarrow K^*\gamma$ to the semileptonic decay $B \rightarrow \rho e\bar{\nu}$ using $SU(3)$ [14] symmetry and the symmetry relations in Eq. (16). The ratio of the decay $B \rightarrow K^*\gamma$ to the semileptonic decay $B \rightarrow \rho e\bar{\nu}$ is shown to be proportional to the factor $I$ which is equal to 1 in the heavy-quark limit. For finite quark mass $m_Q$, the factor $I$ takes into account the corrections to the symmetry relations coming from $\epsilon$, $\bar{\epsilon}$, and $\rho$. In Ref. [6], however, the hadronic matrix element $\langle V|\bar{Q}\gamma_\mu\gamma_5b|B^*\rangle$ does not include the $h_9$ and $h_{10}$ terms as in Eq. (4). Including the form factors $h_9$ and $h_{10}$, we have the correct expression for $I$ given by

\[
I = \frac{1 - \frac{(m_B + m_{K^*})}{2m_B} \bar{\epsilon}^{B\rightarrow K^*}(0)}{1 + \frac{(m_B - m_\rho)(m_B + m_\rho)}{4m_B^2m_\rho} \bar{\epsilon}^{B\rightarrow \rho}(0) - \frac{(m_B - m_\rho)}{2m_\rho} \bar{\epsilon}^{B\rightarrow \rho}(0) - \frac{(m_B - m_\rho)^2(m_B + m_\rho)}{4m_B^2m_\rho} \rho^{B\rightarrow \rho}(0)}.
\]

(23)

and the value of $I = 1.12$ which remains close to 1.

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