How to win some simple iteration games

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Abstract
We introduce two new iteration games: the game $G$, which is a strengthening of the weak iteration game, and the game $G^+$, which is somewhat stronger than $G$ but weaker than the full iteration game of length $\omega_1$. For a countable $M$ elementarily embeddable in some $V_\eta$, we can show that $II$ wins $G(M, \omega_1)$ and that $I$ does not win the $G^+(M)$.

1 Introduction

Iterability results, that is theorems ensuring the existence of wellfounded branches in iteration trees, are the main technical tool used in proving the comparison theorem for inner models for large cardinals. The main iterability result of [2], Theorem 4.3, shows that any countable iteration tree $\mathcal{T}$ on a countable $\mathcal{M} \preceq V_\alpha$, has a maximal wellfounded branch, and this is enough to prove a comparison theorem for the canonical inner model for one Woodin cardinal. In fact in [3] this result is used to prove a comparison theorem for countable $\textit{tame}$ premice $J_\alpha^{\vec{E}}$, i.e. structures in the sense of [3] satisfying “$\delta$ is not Woodin” for every $(\kappa, \lambda)$-extender $E$ on the coherent sequence $\vec{E}$, with $\kappa < \delta < \lambda$. Tame premice can have many Woodins, but cannot satisfy the sentence: \textit{there is $\kappa$ which is $\delta + 1$-strong and $\delta$ is Woodin}. On the other hand the absence of more powerful iterability results has been the main obstacle towards extending the existing theory to core models with larger cardinals. The Cofinal Branch Hypothesis (CBH) (for the definition of this or other notions see [4] or §2 below) is the single most important open question in this area, and a proof of it (if true) would almost certainly yield a comparison lemma for mice with, say, superstrong cardinals. Barring CBH, the next best thing we could hope to prove is the Strategic Branch Hypothesis (SBH), which is a weakening of CBH. As the name suggests SBH asserts that player $II$ has a winning strategy in the \textit{full iteration game} on $V$ of length $\nu$, $IG(V, \nu)$, for any $\nu$. In this game the two players cooperatively build in $\nu$ rounds an iteration tree on $V$, with $II$ on the move at limit rounds choosing a cofinal wellfounded branch. Just as with CBH, SBH is

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pretty much open. Theorem 4.3 of [2] implies that player II has a winning strategy in the weak iteration game on countable \( M \preceq V_\alpha \), \( \mathcal{WG}(M) \). This is a weaker game than \( \mathcal{IG}(M) \), in the sense that if II wins \( \mathcal{IG}(M) \) then he wins \( \mathcal{WG}(M) \). On the other hand the weak iteration game seems of little or no use in proving a comparison theorem for non-tame premice.

In this paper we prove two new iterability results which yield a comparison lemma for non-tame mice. The extent to which our results civilize these “wild” mice is not clear, but it should fall somewhere between the hypotheses: a strong cardinal below a Woodin and a Woodin limit of Woodins.

Our first result, proved in §4, says that player II wins a certain game which we call \( \mathcal{G}(M, \omega_1 + 1) \), when \( M \preceq V_\alpha \) is countable. The game \( \mathcal{G} \) is stronger than \( \mathcal{WG} \), but much weaker than \( \mathcal{IG} \). This is just about the best we are able to show in the line of proving directly that player II has a winning strategy for games approximating \( \mathcal{IG} \).

The second result, which takes up the rest of the paper §§5, 6 and 7, deals with an iteration game \( \mathcal{G}^+(M) \) which is a much closer approximation to \( \mathcal{IG} \). It is played like \( \mathcal{IG} \) except for the fact that I has to play distinct integers on the side. The game is over once I runs out of integers, provided none of the players has lost by that time. We are able to prove that I does not have a winning strategy in \( \mathcal{G}^+(M) \) for countable \( M \preceq V_\alpha \). (So perhaps this paper could have been more aptly entitled: How not to lose a short iteration game.) By results of Steel and Woodin, \( \mathcal{G}^+(M) \) is determined, modulo supercompact cardinals, hence II wins the game.

We think that both proofs present interesting new features. In a way these are more important than the statements of the theorems themselves. Both results seem likely to admit further generalizations, although at this time we do not know how to do it. One drawback to our present approach is the use of \( 2^{\aleph_0} \)-closed extenders in the proofs. In fact in the proof of Theorem [4] we must also assume that the iteration trees are non overlapping. This is not too great a restriction if the goal is to construct an inner model \( L[\vec{E}] \) with many Woodins assuming the existence of such cardinals in the universe, as the extenders witnessing Woodiness in \( V \) can be taken to be as closed as we want. Of course this would be a problem were we not to assume the existence of large cardinals in \( V \) in building \( L[\vec{E}] \), as done in core model theory.

This paper is fairly self-contained, but the reader is assumed to be acquainted with iteration trees and extenders. Sections §§1, 3 and parts of §5 of [2] would do. No knowledge of fine structure or inner model theory is required.

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2 Iteration games

The most general iteration game is the full iteration game and was defined in §5 of [2]. In the full iteration game of length $\nu$ on a premouse $M$, $\mathcal{I}G(M, \nu)$, players $I$ and $II$ cooperatively construct a plus-$2$ normal iteration tree ($\mathcal{T}, M$): $I$ plays at successor rounds, while $II$ plays at limit rounds. At round $\alpha + 1 < \nu$, $I$ plays an extender $E_\alpha \in M_\alpha$ and an ordinal $\rho_\alpha$ such that $M_\alpha \models \text{“}E_\alpha$ is $\rho_\alpha + 2$ strong.” Let $P = \text{ult}(M_\beta, E_\alpha)$, where $\beta$ is least such that $\text{crit}(E_\alpha) \leq \rho_\beta$. If $P$ is illfounded then $I$ wins, otherwise let $M_{\alpha+1} = P$ and we move to the next round $\alpha + 1$. At limit rounds $\lambda < \nu$, $II$ plays a cofinal wellfounded branch $b$ of the iteration tree built insofar, and set $M_\lambda = M_0$. (At round 0, neither player does anything.) The first player that cannot make a legal move loses. If neither player has lost by round $\nu$, then $II$ wins. [The reader should keep in mind that, as we are dealing with normal iteration trees, the game described above is slightly more restrictive than the game described in [4].]

The Strategic Branch Hypothesis (SBH) asserts that $V$ is strategically iterable, i.e. player $II$ has a winning strategy for $\mathcal{I}G(V, \nu)$, for all $\nu$. It is a weaker form of the Cofinal Branch Hypothesis (CBH), asserting that: if $\mathcal{T}$ is an iteration tree on $V$ then if $\mathcal{T}$ is of limit length it has a cofinal wellfounded branch, and if $\mathcal{T}$ is of successor length, we do not run into problems by taking an ultrapower and extending the tree one more step. Note that SBH is preserved by going to elementary substructures: if $M$ is a countable premouse elementarily embeddable in some $V_\alpha$ via $\pi : M \rightarrow V_\alpha$ and $\Sigma$ is a strategy for $II$ in $\mathcal{I}G(V, \nu)$, then a strategy for $II$ in $\mathcal{I}G(M, \nu)$ is obtained by copying via $\pi$ and following $\Sigma$.

The argument above does not apply, though, to CBH. Theorem 4.3 of [2] shows that every countable tree on a countable $M, M \preceq V_\alpha$ has a maximal wellfounded branch. On the other hand the analogous statement on $V$ is open even for trees of height $\omega$.

Open problem 1: Does every countable iteration tree $\mathcal{T}$ on $V$ have a maximal wellfounded branch? In particular: does every iteration tree of height $\omega$ have a (necessarily cofinal) wellfounded branch?

That the answer is affirmative for trees $\mathcal{T}$ where all extenders are $2^{\aleph_0}$-closed in the model they appear, is the content of

Theorem 5.6 of [2]: Suppose $\mathcal{T}$ is a countable iteration tree on a premouse $N$, $2^{\aleph_0} N \subseteq N$ and for all $\alpha + 1 < \text{lh}(\mathcal{T})$, $M_\alpha^\mathcal{T} \models \text{“ult}(V, E_\alpha) is $2^{\aleph_0}$-closed,” then there is a maximal wellfounded branch $b$ of $\mathcal{T}$.

This result will be used in §4 of this paper. A related conjecture is the Unique Branch Hypothesis (UBH) asserting that every iteration tree on $V$ of limit length has at most one cofinal wellfounded branch. Woodin, in unpublished work, has shown the consistency of $\neg(\text{UBH} + \text{CBH})$ assuming the existence of a non-trivial $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$, where $\lambda = \sup_\alpha j^\alpha(\kappa)$ and $\kappa = \text{crit}(j)$. Thus it is quite possible that CBH is consistently false, although at this point we have no reason to believe either way.

The weak iteration game of length $\nu$ on a premouse $M$, $\mathcal{W}G(M, \nu)$, is a weakening of the full iteration game, with player $I$ playing only at successor rounds and player $II$ playing at every round. At round $\alpha < \nu$, $\langle (\mathcal{T}_\beta, P_\beta) \mid \beta < \alpha \rangle$ and $\langle j_{\beta, \gamma} \mid \beta < \alpha \rangle$ are given such that
1. $P_0 = M$, each $(T_\beta, P_\beta)$ is an iteration tree of successor length $\theta_\beta + 1 < \omega_1$, $j_{\beta,\beta+1} = i_{\beta,\beta}$ and $j_{\gamma,\beta+1} = j_{\beta,\beta+1} \circ j_{\gamma,\beta}$.

2. $(T_{\beta+1}, P_{\beta+1}) \| (T_\beta, P_\beta)$; that means $P_{\beta+1} = M^{T_\beta}_{\theta_\beta}$ is the last model of $T_\beta$, $\rho_0^{T_{\beta+1}} \geq \sup \{ \rho_\gamma \mid \gamma + 1 \leq \theta_\beta \}$ and the first model $E_0^{T_{\beta+1}}$ can be applied to is $P_{\beta+1} = M_0^{T_{\beta+1}}$.

3. if $\gamma < \alpha$ is limit, then $P_\gamma$ is the direct limit of the $P_\beta$'s and $j_{\beta, \gamma}$ are the limit maps, for $\beta < \gamma$.

So $(T_\beta, P_\beta) \mid \beta < \alpha$ forms an iteration tree $(T, M)$, with $(T_\gamma, P_\gamma)$ stacked on top of $(T_\beta, P_\beta)$ for $\beta < \gamma < \alpha$. If $\alpha$ is limit, then $II$ is to move and must play a cofinal wellfounded branch of the tree constructed so far. There is not much choice in this case as there is only one cofinal branch of $T$: if the direct limit of the $P_\beta$'s is illfounded then $II$ loses, otherwise that will be $P_\alpha$. If $\alpha$ is successor, $\alpha = \beta + 1$, then $I$ plays a putative iteration tree $(S_\alpha, P_\alpha)$, with $P_\alpha = M^{T_\beta}_{\theta_\beta}$ such that, extending $T$ via $S_\alpha$, we still have a putative iteration tree on $M$. [A putative iteration tree is an object obeying all the usual rules for ordinary iteration trees except for the fact that the last model can be illfounded.] $II$ responds by playing either:

1. (accept), if $S_\alpha$ is of successor length and its last model is wellfounded, that is: $S_\alpha$ really is an iteration tree on $P_\alpha$; then set $T_\alpha = S_\alpha$. Or

2. (accept, $b$), if $S_\alpha$ is of limit length and $b$ is a cofinal wellfounded branch; then let $T_\alpha$ be $S_\alpha$ extended via $b$ and $\theta_\alpha = lh(S_\alpha)$. Or

3. (reject, $b$), where $b$ is a maximal wellfounded branch of $S_\alpha$; let $\theta_\alpha = sup(b)$, and $T_\alpha$ be $S_\alpha | \theta_\alpha$ extended via $b$.

It is easy to see that if $II$ wins $IG(M, \nu)$ for any countable $\nu$, then $II$ wins $WG(M, \omega_1 + 1)$. Let us recall the main result of §4 of [2].

**Theorem 4.3 of [2]:** If $N$ is a countable premouse, $\varphi : N \to V_\eta$ is elementary and $T$ is a countable iteration tree on $N$, then there exists a maximal branch $b$ and an elementary embedding $\tau$ such that $\tau \circ i^{T}_{0, b} = \varphi$.

This easily implies that $II$ wins $WG(M, \omega_1 + 1)$, for $M$ countable and embeddable in some $V_\eta$ via $\pi : M \to V_\eta$: it is enough to maintain inductively that at round $\alpha < \nu$ we have
elementary embeddings $\sigma_\gamma : P_\gamma \to V_\eta$, for $\gamma \leq \alpha$, $\sigma_0 = \pi$, so that, for $\beta < \gamma \leq \alpha$, the diagram

\[ \begin{array}{c}
\vdots \\
M = P_0 \quad \quad \quad \quad \quad V_\eta \\
\sigma_\gamma \\
\sigma_\beta \\
j_\beta, \gamma \\
j_\alpha, \beta \\
P_\gamma \\
P_\beta
\end{array} \]

commutes. If $\alpha$ is limit then $\sigma_\alpha$ is the limit map, and if $\alpha = \beta + 1$ then $\sigma_\alpha : P_\alpha = M^{T_\beta}_{\theta_\beta} \to V_\eta$ is the map $\tau$ given by Theorem 4.3 when $N = P_\beta$, $\varphi = \sigma_\beta$ and $\mathcal{T} = T_\beta$.

The weak iteration game described above will not suffice to ensure that the comparison process for non-tame mice terminates. The reason is that at some round $\alpha + 1 < \omega_1$ we might be forced by the comparison process to apply an extender $E \in M^{T_\alpha}_{\theta_\alpha}$ to an earlier model $M^{T_\beta}_\gamma$, $\gamma \leq \theta_\beta$ and $\beta < \alpha$. Suppose $P_{\alpha+1} = \text{ult}(M^{T_\beta}_{\theta_\beta}, E)$ and $S_{\alpha+1}$ is an iteration tree on $P_{\alpha+1}$ (rather than letting $P_{\alpha+1} = M^{T_\alpha}_{\theta_\alpha}$, as in $\mathcal{W}\mathcal{G}$). When this happens we write $(\mathcal{T}_{\alpha+1}, P_{\alpha+1}) \perp (\mathcal{T}_\alpha, P_\alpha)$. The technique used before, i.e. embedding the $P$’s back to $V_\eta$ does not apply here, because the embeddings $\sigma$’s do not agree enough with one another to ensure that $P_{\alpha+1}$ embeds back in $V_\eta$.

In §4 we introduce a new game $\mathcal{G}(M, \nu)$ in which $I$ is allowed to go back and construct $(\mathcal{T}_{\alpha+1}, P_{\alpha+1}) \perp (\mathcal{T}_\alpha, P_\alpha)$ infinitely often, and we will show that $II$ wins $\mathcal{G}(M, \omega_1 + 1)$, for countable $M$ embeddable in some $V_\eta$ (see Theorem 4.1). In order to highlight the ideas in that proof, we briefly describe the techniques needed to prove a simpler result.

Assume, as usual, that $M$ is countable and embeddable in $V_\eta$ via $\pi$. Suppose that the game considered is just like $\mathcal{W}\mathcal{G}$ except that player $I$ at any stage $\alpha + 1$ may play $(S_{\alpha+1}, P_{\alpha+1}, E, \beta, \gamma)$, where $E$ is an extender in $M^{T_\alpha}_{\theta_\alpha}$, $\beta \leq \alpha$ and $\gamma < \min(\theta_\beta + 1, \theta_\alpha)$, $P_{\alpha+1} = \text{ult}(M^{T_\beta}_{\theta_\beta}, E)$ and $S_{\alpha+1}$ is an iteration tree on $P_{\alpha+1}$. But from this point on the game proceeds as in the weak game. In other words: we can go back, if we want, but only once. The trick is to introduce an intermediate model $N$ between $M$ and $V_\eta$, so that $V_\eta$ is the background universe of $N$, and $N$ is the background universe of $M$. As long as we play the weak iteration game we just copy the trees on $N$ and then choose the branches by playing the weak game on $N$. If we do go back at some stage $\alpha + 1$ and take $P_{\alpha+1} = \text{ult}(M^{T_\beta}_{\theta_\beta}, E)$, we use the copy construction between $M$ and $N$ to embed $P_{\alpha+1}$ back into $N$, and hence into $V_\eta$. From this point on we simply play the weak game on $M$.

Formally, let $\kappa > \eta$ and let $\sigma_0 : N \to V_\kappa$, where $N$ is of size $2^{\aleph_0}$ and contains all reals. Suppose also $\pi_0 : M \to N \cap V_\eta$, some $\bar{\eta} \in N$, is such that $\pi = \sigma_0 \circ \pi_0$.

Let’s make it as a rule that the extender played are $2^{\aleph_0}$-closed. We now start playing the game. Suppose that until round $\alpha + 1$ the weak iteration game was played, so that
be the copy map. Suppose also that we are given embeddings \( \sigma_\beta : Q_\beta \to V_\kappa \) such that \( \sigma_\gamma \circ j_{\beta,\gamma}^N \), where \( j_{\beta,\gamma}^N : Q_\beta \to Q_\gamma \) are the embeddings given by the copied trees.

If \( I \) plays \((S_\alpha, P_\alpha)\), i.e. if he keeps on playing the weak game, then we choose \( \theta \) largest such that \( T_{\alpha+1} = S_{\alpha+1} \upharpoonright \theta \) can be copied on \( V_\kappa \) via \( \sigma_{\alpha+1} \circ \pi_{\alpha+1} \) and has no non-cofinal wellfounded branches. Theorem 5.6 of [2] guarantees the existence of a cofinal wellfounded branch. Let \( P_{\alpha+2} \) and \( Q_{\alpha+2} \) be the models \( M_{b_{\alpha+1}}^{T_{\alpha+1}} \) and \( M_{b_{\alpha+1}}^{\pi_{\alpha+1} T_{\alpha+1}} \) respectively. A tree argument enables us to replace the copy map from \( Q_{\alpha+2} \) to \((a \text{ rank of }) M_{b_{\alpha+1}}^{\pi_{\alpha+1} T_{\alpha+1}} \) with a similar embedding belonging to the latter model. So by elementarity we get \( \sigma_{\alpha+2} : Q_{\alpha+2} \to V_\kappa \).

If, otherwise, \( I \) decides to go back and take \( \text{ult}(M_\gamma^{T_\beta}, E) \) then a tree argument is used to replace the copy map from \( P_{\alpha+1} \) to \( Q_{\alpha+1} = \text{ult}(M_\gamma^{T_\beta}, \pi_\alpha(E)) \) with a similar map that belongs to \( Q_{\alpha+1} \) and then we pull it back to \( N \). In order to do this we need to know that \( Q_{\alpha+1} \) is wellfounded and that \( P_{\alpha+1} \) belongs to it. If we assume, as we do, that the iteration trees are non-overlapping, then \( Q_{\alpha+1} \) is wellfounded by Theorem 1.2 of [6] or Lemma 3.1. For \( P_{\alpha+1} \in Q_{\alpha+1} \) we seem to need that \( N \) contains \( HC \). This on the other hands forces \( N \) to be uncountable and so the usual tree argument will not apply to it. To overcome this difficulty, we have to resort to the concept of support (cf. Definition 3.5) and \( 2^{\aleph_0} \)-closed extenders.

To summarize: the copy maps \( \pi_\alpha \) are needed in order to be able to embed the ultrapower \( \text{ult}(M_\gamma^{T_\beta}, E) \) back into the \( V \)-like model \( N \), while the maps \( \sigma_\alpha \)'s are needed to ensure that the direct limit models of the tree copied on \( N \) are wellfounded. The \( \sigma_\alpha \)'s would be superfluous, were we able to prove the following instance of (CBH).

**Open problem 2:** Consider the following game. (For notational simplicity we state the length \( \omega \) case only.) \( I \) plays iteration trees \( T_\alpha \) and \( II \) plays cofinal wellfounded branches \( b_\alpha \) such that:

1. \( T_0 \) is on \( V \) and \( T_{\alpha+1} \) is on \( M_{b_\alpha}^{T_\alpha} \) and

2. each \( T_\alpha \) has no non-cofinal wellfounded branch and all extenders used in \( T_\alpha \) are \( 2^{\aleph_0} \)-closed in the model they appear.

The first player to violate the rules loses. If neither player has lost by the end of the game, then \( II \) wins iff the (only) cofinal branch of the resulting tree is wellfounded.

Does \( II \) win this game?

By Theorem 5.6 of [2] stated above, \( II \) does not lose at any finite position of the game, and with some extra work it can be shown that \( I \) does not have a winning strategy.

Although the game \( \mathcal{G}(M, \omega_1 + 1) \) of §4 ensures enough iterability to prove a comparison theorem for inner models with a cardinal strong past one Woodin cardinal (and slightly beyond), there seems to be genuine difficulties in generalizing \( \mathcal{G} \) to handle stronger hypotheses. As we always deal with countable iteration trees on countable premiss (hence objects that
can be coded as reals), the various iteration games can be studied from the point of view of descriptive set theory. In particular, rather than trying to prove outright that $II$ has a winning strategy in a given iteration game, one can try to show that $I$ does not have a winning strategy and then appeal to determinacy. Although real games (i.e. games in which the players play elements of $\omega$) of length $\omega_1$ are not determined, by work of Steel and Woodin it is consistent that variable length games of reasonable complexity are determined, assuming large cardinals. The expression “variable length” means that the length of the game varies with the play: for example we can stipulate that the game is over when we reach a position $p$ of length $\nu$, where $\nu$ is the least admissible in $p$ larger than $\omega$. A stronger game is obtained by letting $\nu$ be the second admissible in $p$. Another family of long games are the continuously coded ones: at stage $\alpha$, $I$ plays a real $x_\alpha$ and a natural number $n_\alpha$ such that $n_\alpha \notin \{n_\beta \mid \beta < \alpha\}$ and the game is over when $I$ runs out of integers. Continuously coded games are stronger than games ending at the first admissible, but weaker than the ones ending at the second admissible in the play.

Steel and Woodin proved that if there is a supercompact cardinal, then it is consistent that all continuously coded closed-$\Pi^1_1$ real games are determined (see [5] for a proof of this and other basic facts about long games). In §5 a new iteration game $G^+(M)$ is introduced. It is a continuously coded closed-$\Pi^1_1$ real game. In §6 and §7 it is shown (Theorem 5.1) that $I$ does not win $G^+(M)$ for countable $M$ elementarily embeddable in some $V_\eta$. Hence, modulo supercompact cardinals, $II$ wins $G^+(M)$.

The game $G^+(M)$ should yield enough iterability to give a comparison theorem for inner models with many strong cardinals overlapping Woodin cardinals, but it is still too weak for hypotheses like a Woodin limit of Woodins. In order to get a comparison theorem for inner models with large cardinals that powerful, we believe that progress must be made in two distinct areas. For one, Theorem 5.1 must be strengthened to non-continuously coded games: unfortunately our proof seems to use continuity in an essential way. The second area that needs to be further developed is more descriptive set theoretic in nature, as we need more powerful and sharper results concerning the determinacy of long games.

3 Preliminaries

In this section we define pseudo-iteration trees, which are a generalization of iteration trees ([2], [1]). Besides of being of independent interest, pseudo-iteration trees will be a key ingredient in the main part of the present paper §5, §6 and §7, where an iterability result about ordinary iteration trees is proved. Several basic facts about iteration trees hold also in this more general set-up, so we preferred to give a unified treatment to the subject, rather than repeating the arguments twice, first for ordinary iteration trees and then for their “pseudo” siblings. Pseudo-iteration trees will make no appearance until §5 so the reader only interested in §4 may skip some of the material in the present section. The reader should keep in mind, though, that the notions of support and chunk, and in particular Lemma 3.7 will be used in §4.

By a coarse premouse, or simply a premouse, we mean a transitive set or class $M$ with
a distinguished ordinal \( \delta = \delta(M) \in M \) such that \( M \) is power admissible, satisfies choice, comprehension and the collection schema for domains \( \subseteq V_\beta \). Whenever a \((\kappa, \lambda)\)-extender \( E \) is applied to a premouse \( M \), it will always be assumed that \( \kappa < \delta(M) \), so that Los’ theorem holds for \( \text{ult}(M, E) \) and the embedding \( i_E^M \) is fully elementary. An ordinal \( \gamma, \delta(M) < \gamma < M \cap \text{Ord} \) is a cut-off point of \( M \) iff \( M \cap V_\gamma \) is still a premouse with \( \delta(M \cap V_\gamma) = \delta(M) \). We say that two transitive sets or classes \( M \) and \( N \) agree through an ordinal \( \rho \) iff \( M \cap V_\rho = N \cap V_\rho \).

A tree ordering on \( \theta \) with \( \lambda + 1 \) roots, \( \lambda < \theta \), is a transitive, irreflexive, wellfounded relation \( <_T \) on \( \theta \) such that

1. \( \forall \alpha, \beta < \theta (\alpha <_T \beta \implies \alpha < \beta) \) and for all \( \beta < \theta \) the set \( \{ \alpha < \theta \mid \alpha <_T \beta \} \) is linearly ordered by \( <_T \).
2. \( \forall \alpha, \beta \leq \lambda (\alpha \neq \beta \implies \alpha, \beta \text{ are } <_T\text{-incomparable} ) \) and \( \forall \beta (\lambda < \beta < \theta \implies \exists! \alpha \leq \lambda (\alpha <_T \beta) \).

The ordinals \( \leq \lambda \) are called roots and root\( _T(\beta) \) is the unique \( \alpha \leq \lambda \) such that \( \alpha <_T \beta \) or \( \alpha = \beta \).
3. \( \forall \alpha (\lambda < \alpha < \theta) \)
   \( \alpha \text{ is a successor } \iff \alpha \text{ is a } <_T\text{-successor,} \)
   \( \alpha \text{ is a limit } \implies \{ \gamma \mid \gamma <_T \alpha \} \text{ is cofinal in } \alpha. \)

\( \alpha <_T \beta \) stands for \( \alpha <_T \beta \lor \alpha = \beta, \) and \( [\alpha, \beta]_T = \{ \gamma \mid \alpha \leq_T \gamma \leq_T \beta \} \). Similarly we define \( [\alpha, \beta)_T, (\alpha, \beta)_T, \) etc. If \( b \) is a branch, i.e. a maximal \( <_T\)–linearly ordered subset of \( \theta \), root\( _T(b) \) is the least \( \alpha \in b \). If \( \alpha + 1 > \lambda \), then \( <_T\)-pred\( (\alpha + 1) \) is the least \( \beta \) such that \( \beta <_T \alpha + 1 \).

**Definition 3.1:** A plus-\( n \) pseudo-iteration tree of length \((\theta, \lambda)\), with \( \lambda < \theta \), is a pair \((T, B)\) where

1. \( B = \langle B_\alpha \mid \alpha \leq \lambda \rangle \) is a sequence of premsice, called base models, together with a sequence of increasing ordinals \( \rho_\alpha \), for \( \alpha < \lambda \) such that \( B_\alpha \) and \( B_\beta \) agree through \( \rho_\alpha + n \), that is \( B_\alpha \cap V_{\rho_\alpha + n} = B_\beta \cap V_{\rho_\alpha + n} \), for \( \alpha < \beta \leq \lambda \);  
2. \( T \) is a tree ordering \( <_T \) on \( \theta \) with \( \lambda + 1 \) roots, together with a sequence

\[
\langle (E_\alpha, \rho_\alpha) \mid \lambda < \alpha + 1 < \theta \rangle
\]

of extenders and ordinals obeying the usual restrictions for iteration trees, that is: there are premsice \( M^{(T, B)}_\alpha \) and elementary embeddings \( i^{(T, B)}_{\alpha, \beta} : M^{(T, B)}_\alpha \to M^{(T, B)}_\beta, \delta(M^{(T, B)}_\alpha) = i^{(T, B)}_{\alpha, \beta}(\delta(M^{(T, B)}_\alpha)) \), for \( \alpha <_T \beta \), and such that

(a) the sequence \( \langle \rho_\alpha \mid \alpha + 1 < \theta \rangle \) is increasing and;
(b) \( M^{(T, B)}_\alpha = B_\alpha \), for \( \alpha \leq \lambda \);
(c) if $\lambda < \alpha + 1 < \theta$, then $M_{\alpha}^{(T,B)} \models \text{“$E$ is an extender $\rho + n$ strong”, $E \in V_{\beta}(M_{\alpha})$, and letting $\beta = \langle T \rangle \text{-pred}(\alpha + 1)$, then $\beta$ is least such that $\rho_{\beta} + n > \text{crit}(E_{\alpha}),$

$$M_{\alpha+1}^{(T,B)} = \text{ult}(M_{\beta}^{(T,B)}, E_{\alpha}),$$

$i_{\beta,\alpha+1}^{(T,B)}$ is the canonical ultrapower embedding $i_{E_{\alpha}}^{M_{\beta}}$, and $i_{\beta,\alpha+1}^{(T,B)} \circ i_{\gamma,\beta}^{(T,B)} = i_{\gamma,\alpha+1}^{(T,B)}$, for $\gamma < T \beta < T \alpha + 1$;

(d) if $\lambda < \alpha < \theta$ is limit, then $M_{\alpha}^{(T,B)}$ is the direct limit of $M_{\beta}^{(T,B)}$ for $\beta < T \alpha$ and the $i_{\beta,\alpha}^{(T,B)}$ are the direct limit maps.

Remarks.

1. For $\alpha < \beta < \theta$, $M_{\alpha}^{(T,B)}$ and $M_{\beta}^{(T,B)}$ agree through $\rho_{\alpha} + n$. When there is no danger of confusion the superscript will be dropped from the $M$’s as well as from the embeddings $i_{\alpha,\beta} : M_{\alpha} \rightarrow M_{\beta}$.

2. Iteration trees are pseudo-iteration trees $(T,B)$ of length $(\theta,0)$, that is $B = \langle B_{0} \rangle$ is a single premouse. In this case it is customary to denote its length by $\theta$, rather than $(\theta,0)$. On the other hand, any iteration tree $T$ of length $\theta$ on a model $M$ can be construed as a pseudo-iteration tree of length $(\theta,\lambda)$, any $\lambda < \theta$. (Just forget about the tree structure below $\lambda$ and take $B_{\alpha} = M_{\alpha}^{T}$.)

3. Implicit in 2.(c) of the above definition, is that $M_{\alpha}$ and $M_{\beta}$ agree through $\rho_{\beta} + n$, when $\beta < \alpha$. This is proved by induction on $\alpha$.

4. Note that plus-$n$ implies plus-$m$, for $n > m$. In this paper we will be mainly concerned with plus-1 and plus-2 trees.

5. The above definition, when restricted to ordinary iteration trees, is less general than the one in [2] as it covers only normal iteration trees. The reason we chose to eschew non-normal pseudo-iteration trees was to avoid awkward notation. On the other hand, the comparison process for models of the form $L[\bar{E}]$ entails normal trees only, so our present definition is not too restrictive.

In order to prove a few basic results about pseudo-iteration trees we must restrict our definition a bit.

**Definition 3.2:** Let $(T,B)$ be a plus-$n$ pseudo-iteration tree of length $(\theta,\lambda)$.

(a) $(T,B)$ is non-overlapping if $lh(E_{\alpha}) < \text{crit}(E_{\beta})$, whenever $\alpha + 1 = \langle T \rangle \text{-pred}(\beta + 1)$ and $\beta + 1 < \theta$.

(b) $(T,B)$ is internal if $\theta \in B_{0}$, $\langle B_{\alpha} \mid \alpha < \lambda \rangle \in B_{\lambda}$ and $B_{\lambda} \models \text{“$|B_{\alpha}| = |V_{\rho_{\alpha}+n}|$ and $B_{\alpha}$ is $2^{\aleph_{\alpha}}$-closed”}.$

(c) If all the extenders $E_{\alpha}$ are $2^{\aleph_{\alpha}}$-closed in the model they appear, i.e. $M_{\alpha} \models \text{“ult}(V,E_{\alpha})$ is $2^{\aleph_{\alpha}}$-closed”, then $(T,B)$ is said to be $2^{\aleph_{\alpha}}$-closed.
Notice that if \((T, B)\) is internal plus-\(n\), \(n \geq 1\), then \(\langle (B_\alpha, \rho_\alpha) \mid \alpha < \beta \rangle \in B_\beta\), for any \(\beta \leq \lambda\), as such sequence can be coded as a subset of \(V_{\rho_\beta + (n-1)} \cap B_\lambda\) and \(B_\lambda\) and \(B_\beta\) agree up to \(\rho_\beta + n\).

In the next two lemmata we derive some easy consequences of \(T\) being non-overlapping or \(2^{\aleph_0}\)-closed.

**Lemma 3.1:** Suppose we are given a countable, internal, non-overlapping, plus-1 pseudo-iteration tree \((T, B)\) of length \((\theta + 1, \lambda)\). Assume also that \(E \in M_\theta\) is an extender that can be applied to some earlier model \(M_\nu\) in a non-overlapping way. Then ult\((M_\nu, E)\) is wellfounded.

**Proof:** The proof is an obvious modification of Theorem 1.2 of \([6]\). Let \(\alpha = \text{root}(\nu)\), let \(M_{\theta+1} = \text{ult}(M_\nu, E)\) and let \(i_{\alpha, \theta+1} = i_{E}^{M_\nu} \circ i_{\alpha, \nu}\). As the pseudo-iteration tree is non-overlapping, every element in ult\((M_\nu, E)\) is of the form \(i_{\alpha, \theta+1}(f)(a)\), for some \(a \in [\beta]^{<\omega}\), where \(\beta = \text{lh}(E)\). [This follows from a straightforward induction on \(\nu\): the only place where the “non-overlapping” condition is used is when \(\nu\) is limit.] Suppose, towards a contradiction, that \(M_{\theta+1}\) is illfounded. As \(M_{\theta+1} = \text{ult}(M_\nu, E)\) agrees with ult\((M_\theta, E)\) through \(i_E(\kappa) + 1\), then \(V_{i_E(\kappa) + 1}^{M_{\theta+1}} \in \text{WFP}(M_{\theta+1})\). By absoluteness \(B_\lambda \models \text{“} M_{\theta+1} \text{ is illfounded”} \), hence there is a sequence of functions \(\langle f_n \mid n \in \omega \rangle \in B_\lambda\), with each \(f_n \in B_\alpha\), and \(a_n \in [\beta]^{<\omega}\) such that \(\langle i_{\alpha, \theta+1}(f_n)(a_n) \mid n \in \omega \rangle\) forms an infinite descending chain in \(M_{\theta+1}\). As \(B_\alpha\) is \(\omega\)-closed inside \(B_\lambda\), \(\langle f_n \mid n \in \omega \rangle \in B_\alpha\), hence the set \(Y = \{i_{\alpha, \theta+1}(f_n(b)) \mid n \in \omega, b \in [\beta]^{<\omega}\} \in M_{\theta+1}\). Working in \(M_{\theta+1}\), observe that \(|Y| \leq \beta \) so \(Z\), its transitive collapse, belongs to \(V_{i_E(\kappa) + 1}^{M_{\theta+1}}\). But \(Y\) is really illfounded (in \(V\)), and so must be \(Z\). Thus \(V_{i_E(\kappa) + 1}^{M_{\theta+1}}\) cannot be wellfounded: a contradiction.

\[\blacksquare\]

By inspecting the proof above we see that for the first \(\omega\) models, the non-overlapping condition is not needed.

**Corollary 3.1:** If \((T, B)\) is countable, internal, plus-1, of length \((\lambda + n + 1, \lambda)\), and such that \(E \in M_{\lambda+n}\) can be applied to some previous \(M_\alpha\) in a non-overlapping way, then ult\((M_\alpha, E)\) is wellfounded.

**Lemma 3.2:** If \((T, B)\) is plus-1, \(2^{\aleph_0}\)-closed, internal and of length \((\theta, \lambda)\), then every model \(M_\alpha^{(T,B)}\) is \(2^{\aleph_0}\)-closed in \(B_\lambda\), for \(\alpha < \min(\theta, \lambda + \omega)\).

**Remark:** In general, \(M_\alpha^{T}\) fails to be \(\omega\)-closed for \(\alpha \geq \lambda + \omega\), so the lemma cannot be improved.

**Proof:** By induction on \(\alpha\). We may assume \(\alpha = \beta + 1 > \lambda\) as when \(\alpha \leq \lambda\) the result follows at once. Let \(M_\alpha = \text{ult}(M_\gamma, E)\), where \(E = E_\beta\) is a \((\kappa, \nu)\)-extender, and \(\gamma = \text{-pred}(\alpha)\). Given \(\langle (a_\xi, f_\xi) \mid \xi < 2^{\aleph_0}\rangle \in B_\lambda\), with \([a_\xi, f_\xi]_{E}^{M_\gamma} \in M_\alpha\), we want to show that

\[\langle i_{E}^{M_\gamma}(f_\xi)(a_\xi) \mid \xi < 2^{\aleph_0}\rangle = \langle [a_\xi, f_\xi]_{E}^{M_\gamma} \mid \xi < 2^{\aleph_0}\rangle \in M_\alpha\]
[Here, and in the rest of this proof, $2^{\aleph_0}$ means $(2^{\aleph_0})^B$.]

First notice that $\langle a_\xi | \xi < 2^{\aleph_0} \rangle \in M_\alpha$: by the inductive hypothesis applied to $\beta$ and $2^{\aleph_0}$-closure of $E$, $\langle a_\xi | \xi < 2^{\aleph_0} \rangle$ belongs to ult$(M_\beta, E)$, which agrees with ult$(M_\gamma, E) = M_\alpha$ through $i_E(\kappa) + 1$. Hence $\langle a_\xi | \xi < 2^{\aleph_0} \rangle \in M_\alpha$.

As each $f_\xi \in M_\gamma$ and $M_\gamma$ is $2^{\aleph_0}$-closed inside $B_\lambda$, then $\langle f_\xi | \xi < 2^{\aleph_0} \rangle \in M_\gamma$, hence $F \in M_\gamma$ where we set

$$F((b_\xi | \xi < 2^{\aleph_0}))(\eta) = f_\eta(b_\eta)$$

for all sequences $\langle b_\xi | \xi < 2^{\aleph_0} \rangle \in M_\gamma$ with $b_\xi \in |\nu|^{\aleq_1}$. Thus

$$i_E^\alpha(F)((a_\xi | \xi < 2^{\aleph_0})) = i_E^\gamma((f_\xi)(a_\xi) | \xi < 2^{\aleph_0}) \in M_\alpha$$

and this is what we had to prove.

If $T$ is a pseudo-iteration tree on $B = \langle B_\alpha | \alpha \leq \lambda \rangle$ and $C = \langle C_\alpha | \alpha \leq \lambda \rangle$ are premice such that $B_\alpha \subset C_\alpha$ and $\delta(B_\alpha) = \delta(C_\alpha)$, then $T$ need not to be a pseudo-iteration tree on $C$: it is quite possible that for some $\lambda < \gamma < \lh(T)$, the $\gamma$th model $M_\gamma(T, C)$ is illfounded, while the corresponding model on the $B$-side is wellfounded, as required by our definition. Similarly, if $C_\alpha \subset B_\alpha$ and $\delta(C_\alpha) = \delta(B_\alpha)$, then again $(T, C)$ can fail to be a pseudo-iteration tree, as at some stage $\gamma > \lambda$, $E^\gamma_T$ might not belong to $M_\gamma(T, C)$. In order to find sufficient conditions on $C$ for $(T, C)$ to be a pseudo-iteration tree we introduce the notion of embedding.

**Definition 3.3:** Suppose $(T, B)$ and $(S, C)$ are pseudo-iteration trees of length $(\theta, \lambda)$, $(\theta, \nu)$, respectively, and $\lambda \leq \nu$. A family of maps $\Pi = \langle \pi_\alpha | \alpha < \theta \rangle$ is an embedding of pseudo-iteration trees, $\Pi : (T, B) \rightarrow (S, C)$, if there are ordinals $\eta_\alpha \leq M^{S}_\alpha \cap \Ord$ such that

1. each $\pi_\alpha : M^T_\alpha \rightarrow M^S_\alpha \cap V_{\eta_\alpha}$ is an elementary embedding, $\pi_\alpha(\delta(M^T_\alpha)) = \delta(M^S_\alpha)$, $\pi_\alpha(\rho^T_\alpha) = \rho^S_\alpha$ and $\pi_\alpha(E^T_\alpha) = E^S_\alpha$;

2. for $\alpha, \beta \geq \nu$, $\alpha <_T \beta \iff \alpha <_S \beta$ and, for $\alpha \leq \nu < \beta$,

   $\alpha <_S \beta \iff (\alpha <_T \beta \text{ and } \exists \alpha'(\alpha < \alpha' \leq \nu \land \alpha' <_T \beta))$;

3. if $\alpha <_S \beta$ then $\eta_\alpha \in M^S_\alpha \iff \eta_\beta \in M^S_\beta \iff \iota^S_{\alpha, \beta}(\eta_\alpha) = \eta_\beta$ and the diagram

\[
\begin{array}{ccc}
M^T_\alpha & \xrightarrow{\pi_\alpha} & M^S_\alpha \cap V_{\eta_\alpha} \\
\downarrow \iota^T_{\alpha, \beta} & & \downarrow \iota^S_{\alpha, \beta} \\
M^T_\beta & \xrightarrow{\pi_\beta} & M^S_\beta \cap V_{\eta_\beta}
\end{array}
\]

commutes.
If for each \( \alpha < \theta \), \( \eta_\alpha = M_\alpha^S \cap \text{Ord} \), then \( \Pi \) is an elementary embedding.

If for each \( \alpha < \theta \), \( \eta_\alpha \in M_\alpha^S \), then \( \Pi \) is a bounded embedding. Any sequence \( \langle \eta_\alpha \mid \alpha < \theta \rangle \) with \( \eta_\alpha \geq \eta_\alpha \), is called a bound for \( \Pi \).

Note that an embedding can be both bounded and elementary. Also if \( \Pi : (T, B) \rightarrow (S, C) \) is an embedding, \( (T, B) \) is plus-\( n \) (\( 2^{\aleph_0} \)-closed, non-overlapping) iff \( (S, C) \) is plus-\( n \) (\( 2^{\aleph_0} \)-closed, non-overlapping).

A particular kind of embedding is obtained via the copy construction (see [2]). Given \( (T, B) \), a plus-\( n \) pseudo-iteration tree of length \( (\theta, \lambda) \), and a family of preemice \( C = \langle C_\alpha \mid \alpha \leq \lambda \rangle \) and embeddings \( \Pi = \langle \pi_\alpha \mid \alpha \leq \lambda \rangle \), \( \pi_\alpha : B_\alpha \rightarrow C_\alpha \cap V_{\eta_\alpha} \) such that \( \pi_\alpha|_{\eta^\alpha} V_{\rho^\alpha + n} = \pi_\beta|_{\rho^\beta + n} \), for \( \alpha \leq \beta \leq \lambda \), we define the copied tree \( \Pi T = S \) by boot-strapping the definition of the \( \pi_\alpha \)'s for \( \alpha > \lambda \). For any \( \lambda \leq \nu < \theta \) we want \( \langle \pi_\alpha \mid \alpha \leq \nu \rangle \) to be an embedding of \( (T|_\nu, B) \) into \( (\Pi T|_\nu, C) \) such that for \( \eta \leq \xi < \nu \), \( \pi_\eta|_{M_\eta^T(B)} \cap V_{\rho_\eta + n} = \pi_\xi|_{M_\xi^T(B)} \cap V_{\rho_\xi + n} \). Thus if \( \nu = \xi + 1 \) and \( \gamma = \nu^T \)-pred(\( \nu \)) we let \( M_\nu^\Pi T = \text{ult} \left( M_\gamma^\Pi T, \pi_\xi(E_\xi^T) \right) \), if it is wellfounded and let \( \pi_\nu : M_\nu^T \rightarrow M_\nu^\Pi T \) be defined by

\[
\pi_\nu([a, f]|_E^M) = [\pi_\xi(a), \pi_\gamma(f)]|_F^N
\]

where \( M = M_\gamma^T \), \( E = E_\xi^T \), \( N = M_\nu^\Pi T \) and \( F = \pi_\xi(E_\xi^T) = E_\xi^\Pi T \). If \( \nu \) is limit, let \( M_\nu^\Pi T = \lim_{\gamma < \tau < \nu} M_\gamma^\Pi T \), if such direct limit is wellfounded, and \( \pi_\nu \) is the limit map. If at some stage \( \nu < \theta \) we hit an illfounded model \( M_\nu^\Pi T \), then we stop the construction and declare the length of \( \Pi T \) to be \( (\nu, \lambda) \).

If \( lh(\Pi T) = lh(T) \), then we say that \( T \) can be copied on \( C \) via \( \Pi \). Also, by a slight abuse of notation, the system of maps \( \langle \pi_\alpha \mid \alpha < \theta \rangle \) is still denoted by \( \Pi \). Observe also that if \( \Pi : (T, B) \rightarrow (S, C) \) is obtained from copying via \( \Pi \) and is a bounded embedding, then it is enough to specify the bounds on \( C \), i.e. it is enough to give \( \langle \eta_\alpha \mid \alpha < lh(C) \rangle \).

In the case \( (T, B) \) is internal and \( \Pi \) is elementary and \( lh(B) = \lambda \), then \( \Pi \) and \( C \) can be retrieved from \( \pi_\lambda \) and \( C_\lambda \) as \( C_\alpha = \pi_\lambda(B_\alpha) \) and \( \pi_\alpha = \pi_\lambda|B_\alpha \).

We should also notice that in order to run the copy construction the \( \pi_\alpha \)'s need not to be fully elementary. If, for example, \( B_\alpha \subseteq C_\alpha \), \( \delta(B_\alpha) = \delta(C_\alpha) = \delta_\alpha \) and \( B_\alpha \) and \( C_\alpha \) agree through \( \delta_\alpha \), then we can still try to copy \( T \) on \( C \) via the inclusion maps \( \pi_\alpha : B_\alpha \hookrightarrow C_\alpha \). (Of course \( lh(\Pi T) < lh(T) \) is possible.)

**Lemma 3.3:** Suppose \( (T, B) \) is a pseudo-iteration tree of length \( (\theta, \lambda) \) and let \( \delta_\nu = \delta(M_\nu^T(B)) \). Suppose also \( C = \langle C_\alpha \mid \alpha \leq \lambda \rangle \) are preemice with \( \delta(C_\alpha) = \delta_\alpha \) and \( \Pi = \langle \pi_\alpha \mid \alpha \leq \lambda \rangle \) are embeddings such that, for all \( \alpha \leq \lambda \),

\[
C_\alpha \cap V_{\delta_\alpha} = B_\alpha \cap V_{\delta_\alpha} \quad \pi_\alpha : C_\alpha \rightarrow B_\alpha \cap V_{\eta_\alpha} \quad \text{and} \quad \pi_\alpha|_{V_{\delta_\alpha}} \subseteq \text{id}
\]

where \( \eta_\alpha \leq B_\alpha \cap \text{Ord} \). Then \( T \) can be construed as a pseudo-iteration tree on \( C \) and \( \Pi \) copies \( (T, C) \) to \( (T, B) \). Moreover for \( \nu < \theta \),

\[
M_\nu^{T(C)} \cap V_{\delta_\nu} = M_\nu^{T(B)} \cap V_{\delta_\nu} \quad \text{and} \quad \pi_\nu|_{V_{\delta_\nu}} \subseteq \text{id}
\]
**Proof:** We verify by induction on $\nu$ that $M^{(T,C)}_\nu$ is wellfounded, that it agrees with $M^{(T,B)}_\nu$ through $\delta_\nu$ and that the copy map $\pi_\nu$ is the identity on $V_{\delta_\nu}$.

Suppose $\lambda < \nu + 1 < \theta$ and let $\xi = <_{\tau} \text{pred}(\nu + 1)$. By the agreement between $M^{(T,C)}_\nu$ and $M^{(T,B)}_\nu$, $E = E^T_\nu \in M^{(T,C)}_\nu$. Also $M^{(T,C)}_\xi$ and $M^{(T,B)}_\xi$ agree (at least) through $\rho_\xi + 1$, hence $E$ can be applied to $M^{(T,C)}_\xi$. Let $\pi_{\nu+1}: M^{(T,C)}_{\nu+1} \rightarrow M^{(T,B)}_{\nu+1} \cap V_{\eta_{\nu+1}}$ be given by

$$\pi_{\nu+1}([a, f]_{E^{M^{(T,C)}_\xi}}) = [\pi_\nu(a), \pi_\xi(f)]_{E^{M^{(T,B)}_\xi}}$$

$$= [a, \pi_\xi(f)]_{E^{M^{(T,B)}_\xi}}$$

$\pi_{\nu+1}$ is well-defined and elementary, as $\pi_\nu$ is the identity on $V_{\delta_\nu}$ and $E \in V_{\delta_\nu}$. Hence $M^{(T,C)}_{\nu+1}$ is wellfounded. As $M^{(T,C)}_\xi$ and $M^{(T,B)}_\xi$ agree through $\delta_\xi$, $M^{(T,C)}_{\nu+1}$ and $M^{(T,B)}_{\nu+1}$ agree through $i^{S}_{\xi,\nu+1}(\delta_\xi) = i^T_{\xi,\nu+1}(\delta_\xi) = \delta_{\nu+1}$. Similarly $\pi_{\nu+1}|_{V_{\delta_{\nu+1}}}$ is shown to be the identity.

The case when $\lambda < \nu < \theta$ is limit is left to the reader.

The very same argument shows that if $V_{\delta_\alpha} \cap B_\alpha \subseteq C_\alpha \subseteq B_\alpha$ and $\mathcal{T}$ is a pseudo-iteration tree on $\mathcal{B}$, then $\mathcal{T}$ can be construed on $\mathcal{C}$ and $M^{(T,C)}_\nu \cap V_{\delta_\nu} = M^{(T,B)}_\nu \cap V_{\delta_\nu}$, all $\nu < \theta$ and $\delta_\nu = \delta(M^{(T,B)}_\nu)$. In fact this is almost a corollary of the preceding lemma, except for the fact that the inclusion maps $\pi_\alpha : C_\alpha \rightarrow B_\alpha$ do not form an embedding in our official sense. [See also the remarks after the proof of the next lemma.]

The next result shows that we can truncate a $B_\alpha$ at a rank without affecting the illfoundedness of a given branch.

**Lemma 3.4:** Let $(\mathcal{T}, \mathcal{B})$ be internal, plus-1 pseudo-iteration tree of length $(\theta, \lambda)$, $\theta < \omega_1$, and let $\delta_\alpha = \delta(B_\alpha)$, for $\alpha \leq \lambda$.

(a) Suppose $b$ is an illfounded branch with root $\alpha$ and suppose $B_\alpha \models |V_{\delta_\alpha}| < \delta^\ast$. Then $b$ is illfounded below $\delta^\ast$, that is: the least ordinal of $B_\alpha$ sent by $i_{\alpha,b}$ into the illfounded part of $M^B_b$ is $< \delta^\ast$.

(b) Suppose $\theta = \nu + 1$, $\alpha = \text{root}(\beta)$, $\beta < \nu$ and $\delta^\ast$ is such that $B_\alpha \models |V_{\delta_\alpha}| < \delta^\ast$. Suppose also that $M^{(T,B)}_\nu \models " E \text{ is an extender with critical point } \leq \rho_\beta"$ and that $\text{ult}(M^B_b, E)$ is illfounded. Then the least ordinal sent by $i^B_{E} \circ i_{\alpha,\beta}$ into the illfounded part of the ultrapower is $< \delta^\ast$.

**Proof:** (a) Working inside $B_\lambda$ choose a cofinal sequence $\beta_n \in b$, with $\beta_0 = \alpha$, and ordinals $\xi_n$ such that $i^B_{\beta_n,\beta_{n+1}}(\xi_n) > \xi_{n+1}$, witnessing the illfoundedness of $M^B_b$. Pick $\zeta > \delta_\lambda$ large enough so that all the relevant stuff is in $V_\zeta$. We must consider whether or not $\alpha = \lambda$.

Suppose $\alpha = \lambda$. Let $C_\lambda$ be the transitive collapse of the Skolem hull, computed inside $B_\lambda$,

$$C_\lambda \cong \text{Hull}^{\zeta}(V_{\delta_\lambda} \cup \{(\xi_n, \beta_n) \mid n \in \omega\})$$
and let $\pi_\lambda$ be the inverse of the transitive collapse, $C_\beta = B_\beta$ and $\pi_\beta = \text{id} \upharpoonright C_\beta$, for $\beta < \lambda$. Lemma 3.3 implies that $(T, C)$ is a pseudo-iteration tree that copies to $(T, B)$. Moreover $M_0^{(T, C)}$ is illfounded via the ordinals $\pi_\lambda^{-1}(\xi_n) = \xi_n$. As $\xi_0 \in C_\lambda$, then $\xi_0 < |V_\delta|^{+} \leq \delta^*$. As the Skolem hull above was computed inside $B_\lambda$, then $C_\lambda \subset B_\lambda$, hence $\Phi$ copies $(T, C)$ to $(T, B)$, where $\phi_\beta = \pi_\beta$, $\beta \leq \lambda$, are the identity maps. By commutativity of the copy maps and the iteration embeddings

$$i_{\beta_n,\beta_n+1}^{(T, B)}(\phi_\beta(\xi_n)) > \phi_{\beta_n+1}(\xi_{n+1})$$

and $\phi_{\beta_0}(\xi_0) = \phi_\lambda(\xi_0) = \xi_0$. Thus the least ordinal mapped by $i_{\lambda,B}^{(T)}$ into the illfounded part is $\leq \xi_0 < \delta^*$. This completes the proof in the case when $\alpha = \lambda$.

Suppose now $\alpha < \lambda$. We cannot simply repeat word-by-word the argument above, as the sequence $\langle \xi_n \mid n \in \omega \rangle$ cannot be taken to be in $B_\alpha$. The plan is to get a countable copy $(T, B)$ of the tree belonging to $B_\alpha$ and then internalize the construction in $B_\alpha$. Let $B_\lambda$ be the collapse of the countable Skolem hull, computed inside $B_\lambda$,

$$B_\lambda \cong \text{Hull}^{V_\lambda}(\theta + 1 \cup \{ T, \langle (\xi_n, \beta_n) \mid n \in \omega \rangle \})$$

and let $\bar{\pi}_\lambda$ be the inverse of the collapsing function, $\bar{\pi}_\lambda(T) = T$. Set also $B_\beta = \bar{\pi}_\lambda^{-1}(B_\beta)$ and $\bar{\pi}_\beta = \bar{\pi}_\lambda[\bar{B}_\beta : \bar{B}_\beta \to B_\beta$ for all $\beta < \lambda$. By elementarity of $\bar{\pi}_\lambda$, $(\bar{T}, \bar{B})$ is a pseudo-iteration tree and $b$ is illfounded via the ordinals $\xi_n = \pi_\lambda^{-1}(\xi_n)$. As the Skolem hull was taken inside $B_\lambda$, then $\bar{\pi}_\alpha \in B_\lambda$, it is countable and $\bar{\pi}_\alpha \subseteq B_\alpha$, hence $\bar{\pi}_\alpha \in B_\alpha$. Similarly $\langle (\beta_n, \xi_n) \mid n < \omega \rangle, (\bar{T}, \bar{B}) \in B_\alpha$. Let $\gamma$ be large enough so that $\bar{\pi}_\alpha, (\bar{T}, \bar{B}) \in B_\alpha \cap V_\gamma$ and let $C_\alpha$ be transitive collapse of the following Skolem hull, computed inside $B_\alpha$

$$C_\alpha \cong \text{Hull}^{V_\gamma}(V_{\delta_\alpha} \cup \{ \bar{\pi}_\alpha \})$$

and let $h$ the collapsing map. Set $C_\beta = B_\beta, \pi_\beta = \bar{\pi}_\beta$, for $\beta \neq \alpha$ and $\pi_\alpha = h(\bar{\pi}_\alpha)$. Then $\Pi$ copies $(\bar{T}, \bar{B})$ to $(T, C)$, $M_0^{(T, C)}$ is illfounded as witnessed by $\langle \pi_\alpha(\xi_n) \mid n < \omega \rangle$. We now argue as in the case when $\alpha = \lambda$. Letting $\phi_\beta : C_\beta \to B_\beta$, $\beta \leq \lambda$, be the inclusion maps, then $\Phi$ copies $(T, C)$ to $(T, B)$ and

$$i_{\beta_n,\beta_n+1}^{(T, B)}(\phi_\beta(\pi_\alpha(\xi_n))) > \phi_{\beta_n+1}(\pi_\alpha(\xi_{n+1})).$$

Thus the least element mapped by $i_{\alpha,B}^{(T, B)}$ is $\leq \phi_{\beta_0}(\pi_\alpha(\xi_0)) = \pi_\alpha(\xi_0) < |C_\alpha|^+ < \delta^*$. This concludes the proof of part (a).

(b). The proof of this case is very similar to the one of (a), so we only indicate the main changes, leaving the details to the reader. Let $[a_n, f_n]_{E_\rho^{(T, B)}}$ witness the illfoundedness of the last ultrapower. By absoluteness the $f_n, a_n$ can be taken to be inside $B_\lambda$. By replacing $\beta_n, \xi_n$ with $a_n, f_n$, the proof adapts verbatim.
trees on them. One way to fix this problem would be to start with base premice \( B_\alpha \)'s with arbitrarily large cut-off points. The other way, which we implicitly followed, is to relax a bit our official definition of pseudo-iteration tree, so that \((T, C)\) makes sense even if the \( C_\alpha \)'s don’t satisfy replacement for domains of bounded rank. The only difference is that the tree embeddings \( i_{\alpha, \beta} \) are only \( \Sigma_0 \)-elementary which is enough, anyway, to show that the branch \( b \) is illfounded via the (images of the) \( \bar{\xi}_n \)'s.

**Corollary 3.2:** Suppose \((T, B)\) is internal, plus-1, of length \((\theta, \lambda)\), \( \theta < \omega_1 \). Suppose also that, for all \( \alpha \leq \lambda \), the \( \gamma_\alpha \)'s are cut-off points of the \( B_\alpha \)'s, and let \( C_\alpha = B_\alpha \cap V_{\gamma_\alpha} \). Then \((T, C)\) is a plus-1 pseudo-iteration tree and for any \( \nu < \theta \) with root \( \alpha \)

\[ M^{(T, C)}_{\nu} = M^{(T, B)}_{\nu} \cap V_{i_{\alpha, \nu}(\gamma_\alpha)}. \]

Moreover if \( b \) is a branch

\[ M^{(T, B)}_b \] is wellfounded \iff \[ M^{(T, C)}_b \] is wellfounded,

and if \( \theta = \tau + 1 \), \( E \) is an extender in \( M^{(T, B)}_\tau \) with critical point \( \leq \rho_\alpha \), then

\[ \text{ult}(M^{(T, B)}_\alpha, E) \] is wellfounded \iff \[ \text{ult}(M^{(T, C)}_\alpha, E) \] is wellfounded.

**Proof:** The result follows from the last two lemmata and the fact that for any premouse \( M \) and any cut-off point \( \gamma \), \( M \models |V_{i(M)}| < \gamma \).

So far we only really used that the \( B_\alpha \)'s are \( \omega \)-closed inside \( B_\lambda \), rather than \( 2^{\aleph_0} \)-closed. The reason for requiring the stronger closure property in the definition of “internal” will be clear from the proof of Lemma 3.7. In order to get to it we must first introduce the notion of support for pseudo-iteration trees. This is the generalization to our present set up of the notion defined in [4].

**Definition 3.4:** Let \( T \) be a tree ordering on \( \theta \) with \( \lambda + 1 \) roots. A set \( X \subseteq \theta \) is \( T \)-compatible iff

1. \( X \cap (\lambda + 1) \neq \emptyset. \)
2. If \( \alpha + 1 \in X \) and \( \alpha + 1 > \lambda \), then \( \alpha, <_T \text{-pred}(\alpha + 1) \in X. \)
3. Suppose \( \gamma \in X \) is limit, \( \gamma > \lambda \). Then \( \text{root}_T(\gamma) \in X. \)

If there is a largest \( \alpha \in X \) such that \( \alpha <_T \gamma, \) then \( \beta \in X, \) where \( \beta + 1 \) is least such that \( \alpha <_T \beta + 1 <_T \gamma. \)

It is easy to see that if \( X \subseteq \lambda \) or \( X \in \theta, \) then \( X \) is \( T \)-compatible. Clause (2) implies that if \( \beta + n \in X \) and \( \beta \geq \lambda, \) then \( \beta + i \in X, \) for all \( i \leq n. \) Clause (3) implies that, letting \( Y = (X \cap (\beta + 1)) \cup \{\beta + 1\} \) and \( Z = X \cap (\gamma + 1), \) \( (Y, T \cap Y \times Y) \cong (Z, T \cap Z \times Z). \)
Definition 3.5: Let \((\mathcal{T}, \mathcal{B})\) be of length \((\theta, \lambda)\). A set \(X \subseteq \theta\) is a support iff \(X\) is \(T\)-compatible and there are elementary substructures \((M^\alpha_{\mathcal{T}, \mathcal{B}})_X \prec M^\alpha_{\mathcal{T}, \mathcal{B}}\), for \(\alpha \in X\), such that

1. if \(\alpha \in X \cap (\lambda + 1)\) then \((M_\alpha)_X = M_\alpha = B_\alpha\);
2. if \(\alpha + 1 \in X\) and \(\alpha + 1 > \lambda\) and \(\beta \lessdot T\)-pred\((\alpha + 1)\), then \(E_\alpha, \rho_\alpha \in (M_\alpha)_X\) and

\[
(M_{\alpha + 1})_X = \{[a, f]^{M_\alpha}_{E_\alpha} \mid f \in (M_\beta)_X \wedge a \in (M_\alpha)_X\};
\]
3. if \(\alpha \in X\) then \(Y = X \cap (\alpha + 1)\) is a support and for all \(\beta \in Y\), \((M_\beta)_X = (M_\beta)_Y\);
4. suppose \(\gamma \in X\) is limit and \(\gamma > \lambda\) and let \(A = \{\nu \in X \mid \nu \lessdot T \gamma\}\):
   
   (a) if \(A\) has limit order type, let
   
   \[
   (M_\gamma)_X = \bigcup_{\nu \in A} i''_{\nu, \gamma}(M_\nu)_X ;
   \]
   (b) if \(A\) has a largest element \(\alpha\), let \(\beta + 1\) be least such that \(\alpha \lessdot T \beta + 1 \lessdot T \gamma\). Then \(Y = (X \cap (\beta + 1)) \cup \{\beta + 1\}\) is a support and
   
   \[
   (M_\gamma)_X = i''_{\beta + 1, \gamma}(M_{\beta + 1})_Y .
   \]

There is no suggestion that the \((M_\alpha)_X\)'s should be transitive: in fact, in general, they are not. The next lemma lists a few basic results about supports. The proof (a tedious but straightforward induction on \(\theta\)) is left to the reader.

Lemma 3.5: Fix \((\mathcal{T}, \mathcal{B})\) of length \((\theta, \lambda)\).

1. If \(X \setminus (\lambda + 1)\) is non-empty, \(X\) a support, then \(\lambda \in X\).
2. If \(\alpha \in X \subseteq Y\) and \(X, Y\) are supports then \((M_\alpha)_X \prec (M_\alpha)_Y\).
3. For any \(Y \subseteq \theta\) there is a smallest support \(X \supseteq Y\), called the support generated by \(Y\). Moreover if \(Y\) is finite, \(X\) is finite too.
4. For any \(y \in M_\alpha\) there is a finite support \(X \supseteq \{\alpha\}\) such that \(y \in (M_\alpha)_X\).

Given a pseudo-iteration tree \((\mathcal{T}, \mathcal{B})\) of length \((\theta, \lambda)\) and a support \(X\), a new pseudo-iteration tree \((\mathcal{T}, \mathcal{B})_X = (\mathcal{T}_X, \mathcal{B}_X)\) is defined as follows. Let \(h : \theta_X \to X\) be the enumerating function and let \(\lambda_X = \text{o.t.}(X \cap \lambda)\).

1. \(\mathcal{B}_X = \langle B_{h(\alpha)} \mid \alpha \leq \lambda_X\rangle\); 
2. the tree ordering \(<_{\mathcal{T}_X}\) on \(\theta_X\) is isomorphic to \(<_T \upharpoonright X\) via \(h\);
3. let

$$j^{-1}_{\alpha,B} : (M_{\rho(\alpha)})_{\alpha,X} \rightarrow M_{\rho(\alpha)}$$

be the transitive collapse and set $\rho_{\alpha}^X = j^{-1}_{\alpha,X}(\rho(\alpha))$ and $E_{\alpha} = j^{-1}_{\alpha,X}(E_{\alpha})$.

It is immediate to verify that $(T,B)_X$ is a pseudo-iteration tree and that, for $\alpha < \beta < \theta_X$, if $\alpha < T_X \beta$, then $h(\alpha) < T h(\beta)$ and

$$M_{\beta}^{(T,B)_X} \xrightarrow{j_{\beta,X}} M_{\beta}^{(T,B)}$$

$$M_{\alpha}^{(T,B)_X} \xrightarrow{j_{\alpha,X}} M_{\alpha}^{(T,B)}$$

commutes. We call $J_{X,Y} = \langle j_{\alpha,X} \mid \alpha \leq \theta_X \rangle$ an immersion of $(T,B)_X$ in $(T,B)$. Note that $J_{X,Y}$ is not an embedding in the sense of Definition 3.3, unless $X = \theta$, in which case it is the identity. Also if $(T,B)$ is internal (plus-$n$, non-overlapping, $2^{\aleph_0}$-closed), so is $(T,B)_X$.

If $X \subseteq Y$ are supports for $(T,B)$ and $\pi$ is the collapse of $Y$ and $W = \pi''X$, then, by a tedious but straightforward verification, it can be shown that

$$W$$

is a support for $(T,B)_Y$ and $(T,B)_Y|_W = (T,B)_X$.

There is also an immersion $J_{X,Y} : (T,B)_X \rightarrow (T,B)_Y$, such that $J_{X,Y} = J_{Y,Z} \circ J_{X,Y}$. Hence for supports $X \subseteq Y \subseteq Z$, $J_{X,Z} = J_{Y,Z} \circ J_{X,Y}$. Summarizing: any pseudo-iteration tree $(T,B)$ of length $(\theta, \lambda)$ is the direct limit of the system $\langle (T,B)_X, J_{X,Y} \mid X \subseteq Y \subseteq \theta \rangle$, with $J_{X,Y}$ the limit maps.

Suppose $\Pi : (T,B) \rightarrow (S,C)$ is an embedding of pseudo-iteration trees of length $(\theta, \lambda)$, $(\theta, \nu)$ respectively, and suppose $X$ is a support for $(T,B)$. Then $X$ need not be a support for $(S,C)$: in fact $X \setminus (\nu + 1)$ could be non-empty and yet $\nu \notin X$. Thus we set $\Pi X$ to be the support for $(S,C)$ generated by $X$, and for $\alpha \in X$ we have

$$\pi''(M^T_{\alpha})_X < (M^S_{\alpha})_{\Pi X}.$$  

On the other hand, if $\lambda = \nu$ then $\Pi X = X$. In particular, if $\Pi : (T,M) \rightarrow (S,N)$ is an embedding of ordinary iteration trees then a support for $T$ is also a support for $S$.

**Lemma 3.6:** Suppose $(T,B)$ of length $(\theta, \lambda)$, $\theta < \omega_1$, is plus-1, internal and $2^{\aleph_0}$-closed. Let $\langle S_n \mid n < \omega \rangle$ be an increasing sequence of finite supports such that $\bigcup_n S_n = \theta$. Then, for any $\alpha < \theta$, there is $n_\alpha = n_\alpha(\alpha)$ such that $\alpha \in S_{n_\alpha}$ and

$$M_\alpha = \bigcup_{n \geq n_\alpha} (M_\alpha)_{S_n}$$

and each $(M_\alpha)_{S_n}$ is $2^{\aleph_0}$-closed inside $B_\lambda$. 


Proof: As any element of $M_\alpha$ belongs to $(M_\alpha)_X$, for some finite support $X$ containing $\alpha$, and as $X \subseteq S_n$, for $n$ sufficiently large, $M_\alpha$ is the increasing union of the $(M_\alpha)_S$. As for $2^{\aleph_0}$-closure, note that $(M_\alpha)_S$ is isomorphic (via the transitive collapse) to a model of $(T, B)_S$. As $(T, B)_S$ is internal, plus-1, $2^{\aleph_0}$-closed and $B_\lambda$ is the last model of $B_{S_n}$, the result follows easily from Lemma 3.2.

The submodels $(M_\alpha)_S$ will be called sometimes chunks of $M_\alpha$. The next result will be a key ingredient in the main proofs of this paper.

**Lemma 3.7:** Suppose $\Pi : (T, B) \to (S, C)$ is a bounded embedding with bounds $\langle \eta_\alpha \mid \alpha < \theta \rangle$ and that $(T, B)$ and $(S, C)$ are internal, plus-1, $2^{\aleph_0}$-closed of length $(\theta, \nu)$ and $(\theta, \lambda)$, respectively, and $\theta < \omega_1$. Suppose also that $(T, B) \in C_\nu$ and $C_\nu = \forall \alpha \leq \lambda(|B_\alpha| \leq 2^{\aleph_0})$. Let $\langle S_n \mid n < \omega \rangle \in C_\nu$ be an increasing sequence of finite supports for $(T, B)$, such that $\bigcup_n S_n = \theta$. Then, for any $\alpha < \theta$, and $n$ such that $\alpha \in S_n$,

1. $\pi_\alpha(M_\alpha^T)_S \subseteq M_\alpha^S$ and
2. there is an elementary embedding

$$\varphi_\alpha : M_\alpha^T \to M_\alpha^S \cap V_{\eta_\alpha}$$

such that $\varphi_\alpha(M_\alpha^T)_S = \pi_\alpha(M_\alpha^T)_S$ and $\Phi$, the system of embeddings obtained from $\Pi$ by changing $\pi_\alpha$ to $\varphi_\alpha$, is an embedding of pseudo-iteration trees $\Phi : (T, B) \to (S, C)$ with the same bounds $\langle \eta_\alpha \mid \alpha < \theta \rangle$.

**Proof:** As $(T, B) \in C_\nu$, every model $M_\alpha^T$ belongs to every $M_\alpha^S$ and is of size $\leq 2^{\aleph_0}$. Thus, for fixed $\alpha < \theta$, and $n$ such that $\alpha \in S_n$,

$$M_\alpha^S \models (M_\alpha^T)_S \text{ is of size } \leq 2^{\aleph_0}.$$  

As $\pi^n = \pi_\alpha(M_\alpha^T)_S$ is an elementary embedding of $(M_\alpha^T)_S$ into $(M_\alpha^S)_{\Pi S_n} \cap V_{\eta_\alpha}$ and $C_\nu \models |\pi^n| \leq 2^{\aleph_0}$, by Lemma 3.6, $\pi^n \in (M_\alpha^S)_{\Pi S_n} \subseteq M_\alpha^S$, proving thus part (1).

Now for (2). Fix $n < \omega$ such that $\alpha \in S_n$ and let $V \in M_\alpha^S$ be the tree of attempts to find a sequence like $\langle \pi^n, \pi^{n+1}, \ldots \rangle$. That is, working inside $M_\alpha^S$, let

$$\langle \tau_0, \ldots, \tau_k \rangle \in V \iff \tau_0 = \pi^n, \tau_0 \subseteq \ldots \subseteq \tau_k$$

where $\tau_i : (M_\alpha^T)_{S^{n+i}} \to V_{\eta_\alpha}$ is an elementary embedding. By part (1), $\pi^n \in M_\alpha^S$, for any $m \geq n$, so $\langle \pi^i \mid n \leq i < \omega \rangle$ is a branch of $V$ in $V$. By absoluteness there is a branch $\langle \tau_i \mid i < \omega \rangle \in M_\alpha^S$ and let $\varphi_\alpha = \bigcup_{i<\omega} \tau_i$. 

\[\blacktriangleleft\]
4 Strong past one Woodin cardinal

In this section all iteration trees will be countable, \(2^{\aleph_0}\)-closed and non-overlapping. Suppose we are given an iteration tree \((\mathcal{T}, M)\) and a sequence \(\langle (\mathcal{T}_\alpha, P_\alpha) \mid \alpha < \nu \rangle\) of iteration trees \(\mathcal{T}_\alpha\) on \(P_\alpha\) of successor length \(\theta_\alpha + 1\), together with a last extender \(E^T_{\theta_\alpha} \in M^T_{\theta_\alpha}\) and such that \(\mathcal{T}_0 = \mathcal{T} | \theta_0 + 1\) and \(M = P_0\). We define, by induction on \(0 < \nu \leq lh(\mathcal{T})\), what it means for \(\langle (\mathcal{T}_\alpha, P_\alpha) \mid \alpha < \nu \rangle\) to be a decomposition of \((\mathcal{T}, M)\).

- If \(\nu = 1\), then \(\langle (\mathcal{T}, M) \rangle\) is the only possible decomposition of \((\mathcal{T}, M)\), hence \(lh(\mathcal{T}) = \theta_0 + 1\).
- If \(\nu\) is limit, then for every \(\xi < \nu\) there is an ordinal \(\theta < lh(\mathcal{T})\), such that \(\langle (\mathcal{T}_\alpha, P_\alpha) \mid \alpha < \xi \rangle\) is a decomposition of \((\mathcal{T}|_\theta, M)\).
- If \(\nu = \xi + 1 > 1\), then there is \(\theta < lh(\mathcal{T})\) such that \(\langle (\mathcal{T}_\alpha, P_\alpha) \mid \alpha < \xi \rangle\) is a decomposition of \((\mathcal{T}|_\theta, M)\), \(lh(\mathcal{T}) = \theta + \theta_\xi + 1\) and for \(\alpha < \theta_\xi\)

\[
M^T_{\alpha \xi} = M^T_{\theta_\alpha + \alpha} \quad \text{and} \quad E^T_{\alpha \xi} = E^T_{\theta_\alpha + \alpha}.
\]

Hence in particular \(P_\xi = M^T_{\theta_\xi}\). Moreover

1. if \(\xi\) is limit, then \(P_\xi\) is the direct limit of a (cofinal in \(\xi\)) sequence of \(P_\alpha\)'s,
2. if \(\xi = \eta + 1\), then \(\theta\) is a successor and either
   (a) \(P_\xi = M^T_{\theta_\eta}\) = the last model of \((\mathcal{T}_\eta, P_\eta)\), in which case we write \((\mathcal{T}_\xi, P_\xi) || (\mathcal{T}_\eta, P_\eta)\), or else
   (b) \(P_\xi = \text{ult}(M^T_{\beta}, E^T_{\gamma - 1})\), where \(\beta \leq \eta = \xi - 1\) and \(\gamma \leq \theta_\beta\), if \(\beta < \eta\), or \(\gamma < \theta_\beta\), if \(\beta = \eta\). In this case we write \((\mathcal{T}_\xi, P_\xi) \perp (\mathcal{T}_\eta, P_\eta)\).

(When there is no danger of confusion we simply drop the \(P_\alpha\)'s and write \(\mathcal{T}_{\alpha + 1} \parallel \mathcal{T}_\alpha\) or \(\mathcal{T}_{\alpha + 1} \perp \mathcal{T}_\alpha\).)

The idea here is that \((\mathcal{T}, M)\) can be written as a tree of trees \((\mathcal{T}_\alpha, P_\alpha)\): every model \(M^T_\alpha\) and extender \(E^T_\alpha\) are of the form \(M^T_\gamma\), \(E^T_\beta\), for some \(\gamma \leq \theta_\beta\), and \(\beta < \nu\). The pair \((\beta, \gamma)\) is unique except when \(\mathcal{T}_{\xi + 1} \parallel \mathcal{T}_\xi\). In this case \((\xi, \theta_\xi)\) and \((\xi + 1, 0)\) yield the same model \(M^T_{\theta_\xi} = P_{\xi + 1} = M^T_0\). Vice versa, for any pair \((\beta, \gamma)\) with \(\gamma \leq \theta_\beta\) and \(\beta < \nu\), there is a unique \(\alpha\), such that \(M^T_\alpha = M^T_\gamma\). Let \(\beta \otimes \gamma\) be such \(\alpha\), with

\[
\otimes : \{ (\beta, \gamma) \mid \gamma \leq \theta_\beta \text{ and } \beta < \nu \} \rightarrow lh(\mathcal{T}).
\]

**Definition 4.1:** Let \(S = \langle (\mathcal{T}_\alpha, P_\alpha) \mid \alpha < \nu \rangle\) be a decomposition of \((\mathcal{T}, M)\) and let \(\beta \otimes \gamma = \kappa < lh(\mathcal{T})\). By induction on \(\nu\), we will define:

1. what it means for \(S\) to be quasi-linear;
2. the ordinal $B(\kappa)$, the back-up point of $\kappa$, which depends on the tree ordering $<_T$ only;
3. the set $F_\nu \subseteq lh(T)$ of forbidden nodes.

- If $\nu = 1$, then $S = \langle (T, M) \rangle$ is quasi-linear, $B(\kappa) = 0$ and $F_1 = \emptyset$.
- Suppose $\nu$ is limit. $S$ is quasi-linear iff $\forall \xi < \nu (S \upharpoonright \xi$ is quasi-linear), $F_\nu = \cup_{\xi < \nu} F_\xi$, and $B(\kappa)$ is the back-up point of $\kappa$ as computed in $S \upharpoonright \beta + 1$.
- Suppose $\nu = \xi + 1$ and $\xi$ limit. Set $F_\nu = F_\xi$. If $\beta < \xi$, then $B(\kappa)$ has already been defined so we may assume $\beta = \xi$.

$S$ is quasi-linear iff $S \upharpoonright \xi$ is quasi-linear and either
- there is $\xi_0 < \xi$ such that for $\xi_0 \leq \eta < \xi$, $T_{\eta+1} \| T_\eta$ and $P_\xi$ is the direct limit of such $P_\eta$'s, and $\xi_0$ is least such. Set $B(\kappa) = B(\xi_0 \otimes 0)$. Or else
- there is an increasing sequence $\xi_n \rightarrow \xi$ such that $T_{\xi_n+1} \perp T_{\xi_n}$, $(\xi_n + 1) \otimes 0 <_T \xi \otimes 0$. Set $B(\kappa) = \xi \otimes 0$.
- Suppose $\nu = \xi + 2$. If $\beta < \xi + 1$, then $B(\kappa)$ has already been defined, so assume $\beta = \xi + 1$.

$S$ is quasi-linear iff $S \upharpoonright \xi + 1$ is quasi-linear and either
- $T_{\xi+1} \| T_\xi$, $B(\kappa) = B(\xi \otimes 0)$ and $F_\nu = F_{\xi+1}$, or else
- $T_{\xi+1} \perp T_\xi$, that is $P_{\xi+1} = \text{ult}(M_\xi^{T_\xi}, E_\xi^{T_\xi})$, and $\eta \otimes \xi \notin F_{\xi+1}$; set $F_\nu = F_{\xi+1} \cup [\xi \otimes \eta, \xi \otimes \theta_\xi]$ and $B(\kappa) = \eta \otimes \xi$.

Remarks.

1. The sets $F_\alpha$'s are increasing, i.e. $\alpha < \beta \implies F_\alpha \subseteq F_\beta$. If $\alpha \in F_\beta$, then we are not allowed to visit the models $M_\alpha$ past round $\beta$. In other words: there is no $\gamma \geq \beta$ such that $P_{\gamma+1} = \text{ult}(M_\alpha, E)$. This is the content of Lemma 4.1 to be proved shortly.

2. $B(\kappa) < \kappa$, unless $\kappa = 0$ or $\kappa = \xi \otimes 0$, where $\xi$ is limit of an increasing sequence $\xi_n$, with $T_{\xi_n+1} \perp T_{\xi_n}$. In this case $B(\kappa) = \kappa$.

3. If $T_{\alpha+1} \perp T_\alpha$, then $B((\alpha + 1) \otimes 0)$ is the immediate $<_T$-predecessor of $(\alpha + 1) \otimes 0$.

4. To make the notation a bit simpler we shall write $B(\beta, \gamma)$ rather than $B(\beta \otimes \gamma)$.

A few words on the motivations behind the notion of quasi-linearity are in order here. In the case of $W\mathcal{G}(M, \nu)$, the weak iteration game of length $\nu$, a tower of trees $\langle (T_\alpha, P_\alpha) \mid \alpha < \nu \rangle$ is built, with $P_0 = M$, $T_{\alpha+1} \| T_\alpha$ and for any limit $\lambda < \nu$, $P_\lambda$ is the direct limit of the $P_\alpha$'s, $\alpha < \lambda$. Thus the resulting iteration tree can be construed as a linear iteration of iteration trees. In order to consider a wider spectrum of trees for which we can still prove an iterability
result, we relax the “linearity” condition a bit. The trees \( T_\alpha \) are arranged themselves in a tree ordering, but this tree ordering is not too removed from a linear ordering: whenever we “go back” and take the ultrapower of the \( \gamma \)th model of the \( \beta \)th iteration tree to start a new \( T_{\alpha+1} \), then we give up the right to visit, from this point on, any model with index \((\beta', \gamma')\), with \( \beta \otimes \gamma < \beta' \otimes \gamma' < (\alpha + 1) \otimes 0 \). (For example: an “alternating chain of iteration trees” is not quasi-linear.)

**Lemma 4.1:** Suppose \( \langle (T_\alpha, P_\alpha) \mid \alpha < \nu \rangle \) is a quasi-linear decomposition of \( (T, M) \). If \( T_{\alpha+1} \downarrow T_\alpha \) and \( B(\alpha + 1, 0) = \beta \otimes \gamma \), then \( \forall \kappa \forall \xi (\beta \otimes \gamma \leq \kappa \leq \alpha \otimes \theta_\alpha \) and \( (\alpha + 1) \otimes 0 \leq \xi < \lh(T) \) implies \( \kappa \not\in T \xi \).

**Proof:** Deny. Choose counter-examples \( \kappa \) and \( \xi \), first minimizing \( \xi \), and then taking \( \kappa \) as large as possible (relative to this \( \xi \)). Then \( \kappa \) is the immediate \(<_T\)-predecessor of \( \xi \): otherwise, if \( \kappa < T \xi < T \xi \), then by maximality of \( \kappa \), \( \zeta \not\in \alpha \otimes \theta_\alpha \) hence \( \zeta \geq (\alpha \otimes \theta_\alpha) + 1 = (\alpha + 1) \otimes 0 \), contradicting the minimality of \( \xi \). As \( M_\xi^T \) is a model of some tree \( T_\eta \) with \( \eta > \alpha \), while \( M_\kappa^T \) appears in some \( T_\zeta \) with \( \zeta \leq \alpha \), then it must be the case that \( \xi = \alpha' + 1 \otimes 0 \), for some \( \alpha' > \alpha \), \( T_{\alpha'+1} \downarrow T_{\alpha'} \) and \( \kappa = B(\xi) \). So \( \kappa = \beta' \otimes \gamma' \) and \( \theta_{\alpha'+1} = \ult(M_\beta^{T_{\alpha'}}, E_{\theta_\alpha}^{T_{\alpha'}}) \). By quasi-linearity \( \kappa \not\in F_{\alpha'} \subseteq F_{\alpha+1} \). But \( F_{\alpha+1} \subseteq [\beta \otimes \gamma, \alpha \otimes \theta_\alpha] \), so \( \kappa \in F_{\alpha+1} \), a contradiction.

Given a quasi-linear decomposition \( \langle (T_\alpha, P_\alpha) \mid \alpha < \nu \rangle \) of \( (T, M) \), let \( A = \{ \alpha + 1 < \nu \mid T_{\alpha+1} \downarrow T_\alpha \} \). For \( \alpha + 1, \beta + 1 \in A \) with \( \alpha < \beta \), then \( B(\alpha + 1, 0) \in F_{\alpha+1} \subseteq F_\beta \) and \( B(\beta + 1, 0) \notin \) \( F_\beta \), hence \( B(\alpha + 1, 0) \neq B(\beta + 1, 0) \). In other words, the function \( A \ni \alpha + 1 \mapsto B(\alpha + 1, 0) \) is injective. We want to thin-down \( A \) so that this function is also increasing. Let \( B = \{ B(\alpha + 1, 0) \mid \alpha + 1 \in A \} \) and define \( f : B' \to A \), \( B' \subseteq B \) be such that \( f \) is increasing and \( B(f(\beta), 0) = \beta \). For \( \beta \in B \) let

\[
f(\beta) = \begin{cases} 
\alpha + 1 \in A & \text{if } \beta = B(\alpha + 1, 0) \text{ and } (\forall \gamma \in B \cap \beta \alpha > f(\gamma), \\
\text{undefined} & \text{otherwise}.
\end{cases}
\]

Let \( B' = \text{dom}(f) \). Let \( A' = \text{ran}(f) \subseteq A \) and let \( \langle \alpha_i + 1 \mid i < \lambda \rangle \) be the increasing enumeration of \( A', \lambda = \text{o.t.}(A') = \text{o.t.}(B') \). Note that \( B(\alpha_0 + 1, 0) = \min B' = \min B \) and \( i < j < \lambda \) implies \( B(\alpha_i + 1, 0) < B(\alpha_j + 1, 0) \). \( \langle \alpha_i + 1 \mid i < \lambda \rangle \) is the basic sequence of \( \langle (T_\alpha, P_\alpha) \mid \alpha < \nu \rangle \).

Note that \( \langle \alpha_i(\xi) + 1 \mid i < \lambda(\xi) \rangle \), the basic sequence of \( \langle (T_\alpha, P_\alpha) \mid \alpha < \xi \rangle \), \( \xi < \nu \), is not, in general, an initial segment of \( \langle \alpha_i + 1 \mid i < \lambda \rangle \). In fact it is not even true that \( \lambda(\xi) \leq \lambda \). Let us list a few facts whose proof is left to the reader.

1. If \( \xi \leq \eta + 1 < \nu \) implies \( T_{\eta+1} \upharpoonright T_\eta \), then \( \lambda(\xi) = \lambda \) and \( \langle \alpha_i(\xi) + 1 \mid i < \lambda(\xi) \rangle = \langle \alpha_i + 1 \mid i < \lambda \rangle \).
2. Suppose \( \xi = \eta + 1 < \nu \) and \( \mathcal{T}_{\eta+1} \perp \mathcal{T}_{\eta} \). Let

\[
\lambda' = \begin{cases} 
\lambda(\xi) & \text{if } B(\xi, 0) \geq \sup \{ B(\alpha_i(\xi), 1, 0) \mid i < \lambda(\xi) \}, \\
j & \text{otherwise, where } j < \lambda(\xi) \text{ is least such that } B(\xi, 0) < B(\alpha_j(\xi), 1, 0). 
\end{cases}
\]

Then \( \lambda(\xi + 1) = \lambda' + 1 \) and

\[
\langle \alpha_i(\xi + 1) + 1 \mid i < \lambda(\xi + 1) \rangle = \langle \alpha_i(\xi) + 1 \mid i < \lambda' \rangle \setminus \langle \xi \rangle.
\]

3. If \( \xi_n \to \xi \) and \( \mathcal{T}_{\xi_n+1} \perp \mathcal{T}_{\xi_n} \), then

\[
\lim_{n \to \infty} \langle \alpha_i(\xi_n + 1) + 1 \mid i < \lambda(\xi_n + 1) \rangle = \langle \alpha_i(\xi) + 1 \mid i < \lambda(\xi) \rangle
\]

meaning that \( \lambda(\xi_n + 1) \to \lambda(\xi) \) and for every \( i < \lambda(\xi) \), \( \alpha_i(\xi) = \lim_{n \to \infty} \alpha_i(\xi_n + 1) \). If we choose the \( \xi_n \)'s to be in \( \{ \alpha_i(\xi) + 1 \mid i < \lambda(\xi) \} \), then

\[
\langle \alpha_i(\xi) + 1 \mid i < \lambda(\xi) \rangle = \bigcup_{n<\omega} \langle \alpha_i(\xi_n + 2) + 1 \mid i < \lambda(\xi_n + 2) \rangle.
\]

**Lemma 4.2:** Suppose \( \langle (\mathcal{T}_\alpha, P_\alpha) \mid \alpha < \nu \rangle \) is a quasi-linear decomposition of \( (\mathcal{T}, M) \) and let \( \langle \alpha_i + 1 \mid i < \lambda \rangle \) be its basic sequence.

1. For \( i < \lambda \) and \( \kappa < \theta = lh(\mathcal{T}) \), if \( (\alpha_i + 1) \otimes 0 < \kappa \) then \( (\alpha_i + 1) \otimes 0 <_T \kappa \). In particular, \( i < j \) implies \( (\alpha_i + 1) \otimes 0 <_T (\alpha_j + 1) \otimes 0 \).

2. \( \mathcal{T} \) has exactly one cofinal branch \( b \) and \( (\alpha_i + 1) \otimes 0 \in b \), for every \( i < \lambda \).

**Proof:** (1). By induction on \( \kappa \), for fixed \( i < \lambda \). Let \( \beta = B(\kappa) \). If \( \beta = \kappa \), then \( \kappa = \xi \otimes 0 \) with \( \xi \) limit and for some increasing sequence \( \xi_n \to \xi \), \( \mathcal{T}_{\xi_n+1} \perp \mathcal{T}_{\xi_n} \). By inductive hypothesis, for \( n \) sufficiently large, \( (\alpha_i + 1) \otimes 0 <_T (\xi_n + 1) \otimes 0 \) and by quasi-linearity \( (\xi_n + 1) \otimes 0 <_T \xi \otimes 0 = \kappa \), so by transitivity \( (\alpha_i + 1) \otimes 0 <_T \kappa \). Assume now that \( \beta < \kappa \). By quasi-linearity, \( \beta \notin \mathbf{F}_{\alpha_i} \supseteq [B(\alpha_i + 1, 0), \alpha_i \otimes \theta_{\alpha_i}] \), so either \( \beta < B(\alpha_i + 1, 0) \) or \( \beta \geq \alpha_i \otimes \theta_{\alpha_i} + 1 = (\alpha_i + 1) \otimes 0 \). By definition of basic sequence, \( \beta < B(\alpha_i + 1, 0) \) is impossible, so \( \beta \geq (\alpha_i + 1) \otimes 0 \). As \( \beta <_T \kappa \), then, by inductive hypothesis, \( (\alpha_i + 1) \otimes 0 <_T \beta <_T \kappa \).

(2). If the \( (\alpha_i + 1) \otimes 0 \)'s are bounded in \( \theta = lh(\mathcal{T}) \), then for \( \sup_{i<\lambda} \alpha_i < \eta \leq \kappa < \nu \), \( \eta \otimes 0 <_T \kappa \otimes 0 \), so they yield a cofinal branch \( b \). If, otherwise, the \( \alpha_i \)'s are unbounded in \( \theta \), let \( b = \{ \alpha < \theta \mid (\exists i < \lambda) \alpha <_T (\alpha_i + 1) \otimes 0 \} \). By part (1) this is a branch. In both cases it is immediate to verify that \( b \) is the only cofinal branch of \( \mathcal{T} \).

\[\Diamond\]

Given a quasi-linear decomposition \( \langle (\mathcal{T}_\alpha, P_\alpha) \mid \alpha < \nu \rangle \) of \( (\mathcal{T}, M) \) and \( \beta < \nu \), it is not the case, in general, that \( \langle (\mathcal{T}_\alpha, P_\alpha) \mid \beta \leq \alpha < \nu \rangle \) yields an iteration tree: it is true if either we are dealing with a linear decomposition, i.e. \( \langle \forall \alpha + 1 < \nu \rangle \mathcal{T}_{\alpha+1} \perp \mathcal{T}_\alpha \), or else if \( \beta \) is chosen carefully.
Definition 4.2: Given \((T, M)\) of length \(\theta\), with quasi-linear decomposition \(\langle (T\alpha, P\alpha) \mid \alpha < \nu \rangle\) and basic sequence \(\langle \alpha_i + 1 \mid i < \lambda \rangle\), and given \(\beta \leq \omega_1\), let
\[
\Gamma(\beta) = \sup\{(\alpha_i + 1) \otimes 0 \mid i < \lambda \text{ and } (\alpha_i + 1) \otimes 0 \leq \beta\}.
\]
Remarks.
1. If \(\tilde{\alpha} = \sup\{\alpha_i + 1 \mid i < \lambda \text{ and } (\alpha_i + 1) \otimes 0 \leq \beta\}\), then \(\Gamma(\beta) = \tilde{\alpha} \otimes 0\). In other words, \(\Gamma(\beta)\) is always of the form \(\alpha \otimes 0\), for some \(\alpha\). In particular, if \(\langle (T\alpha, P\alpha) \mid \alpha < \nu \rangle\) is linear, then \(\Gamma(\beta) = 0 \otimes 0 = 0\), for all \(\beta\).
2. Note that \(\Gamma(\beta) \leq \min(\beta, \theta)\) and it is defined even for \(\beta \geq \theta\): if \(\beta = \sup\{(\alpha_i + 1) \otimes 0 \mid i < \lambda\}\), then \(\Gamma(\beta) = \alpha'\).
3. By part (1) of 4.2, \(\alpha \leq \beta\) implies \(\Gamma(\alpha) \leq T\Gamma(\beta) \leq T\beta\).
4. We will write \(\Gamma^\xi(\beta)\) when the basic sequence \(\langle \alpha_i(\xi) + 1 \mid i < \lambda(\xi)\rangle\), \(\xi \leq \nu\), is used.

Lemma 4.3: Let \(\langle (T\alpha, P\alpha) \mid \alpha < \nu \rangle\) be a quasi-linear decomposition of \((T, M)\). For \(\beta \leq \omega_1\)
\[
\mathcal{T}[\Gamma(\beta)] = \langle (E^T_\alpha, \rho^T_\alpha) \mid \Gamma(\beta) < \alpha + 1 < \theta \rangle
\]
is an iteration tree on \(M_{\Gamma(\beta)} = P\tilde{\alpha}\), where \(\Gamma(\beta) = \tilde{\alpha} \otimes 0\). Its tree ordering is \(<_T \mid [\Gamma(\beta), \theta)\) and \(\langle (T\alpha, P\alpha) \mid \tilde{\alpha} \leq \alpha < \nu \rangle\) is its quasi-linear decomposition.

Proof: We only need to check that, for a fixed \(\beta\), \(\Gamma(\beta) < \kappa\) implies \(\Gamma(\beta) <_T \kappa\). By part (1) of Lemma 4.2, for any \(i < \lambda\) such that \((\alpha_i + 1) \otimes 0 \leq \beta\), \((\alpha_i + 1) \otimes 0 <_T \kappa\), so taking the supremum, we have that \(\Gamma(\beta) <_T \kappa\).

Remarks:
1. The careful reader might object that, formally, \(\mathcal{T}[\Gamma(\beta)]\) is not an iteration tree as the field of its tree ordering is not an ordinal. On the other hand, by a trivial re-indexing it can be construed as an iteration tree of length
\[
\theta - \Gamma(\beta) = \text{o.t.}[\Gamma(\beta), \theta].
\]
It is convenient though, for our purposes, to think of the models and embeddings of \(\mathcal{T}[\Gamma(\beta)]\) as indexed by ordinals in the interval \([\Gamma(\beta), \theta)\).
2. If \(\gamma\) is an ordinal such that \(<_T \cap [\gamma, \theta] \times [\gamma, \theta)\) is a tree ordering on \(\xi\), then \(\mathcal{T}|[\gamma, \theta)\) makes sense. In a way, \(\Gamma(\beta)\) is the optimal such \(\gamma \leq \beta\): say that \(\gamma\) is minimal if there is no \(\alpha < \gamma\) such that for all \(\beta\) \((\alpha \leq \beta + 1 < \gamma \implies \mathcal{T}_{\beta+1}|\mathcal{T}_\beta)\). Then
\[
\Gamma(\beta) = \sup\{\gamma \leq \beta \mid \gamma\text{ is minimal and }\mathcal{T}|[\gamma, \theta)\text{ is defined}\}.\]
The game $G(M, \omega_1 + 1)$. This game lasts $\omega_1 + 1$ rounds, with $I$ playing only at successor rounds and $II$ playing at every round, in which $I$ and $II$ cooperatively construct an iteration tree on $M$. There are several constraints on the moves that $I$ and $II$ are allowed: the first player to violate these constraints loses. If $II$ has not lost by $\omega_1$, then he wins. At round $\nu + 1 < \omega_1$, a certain iteration tree $T$ is given and player $I$ plays an iteration tree of successor length $S_\nu$ with the intent of extending $T$ in a quasi-linear way. $II$ can either accept it, or reject it and play a maximal wellfounded branch $b$ of $S_\nu$: the tree $S_\nu$ is truncated at $\lambda = \sup(b)$ and then extended by $b$. The resulting tree $T_\nu$ is then used to extend $T$. At limit stages $II$ must play a cofinal wellfounded branch $b$ of the current $T$. Formally:

At round $\nu < \omega_1$ a quasi-linear decomposition $\langle (T_\alpha, P_\alpha) | \alpha < \nu \rangle$ of some $(T, M)$ of length $\theta < \omega_1$ has been played.

- If $\nu$ is limit, it is $II$’s turn to move: he must play a cofinal wellfounded branch $b$ of $(T, M)$. By Lemma 4.2, $II$ does not have much choice, so if the only cofinal branch $b$ is illfounded, $II$ loses. If $II$ does not lose at round $\nu$, set $P_\nu = M^{\nu \theta}$ and $T_\nu = \emptyset$. Thus $\langle (T_\alpha, P_\alpha) | \alpha < \nu + 1 \rangle$ is a quasi-linear decomposition of $T$ extended by $b$.

- Suppose $\nu = \xi + 1$. Player $I$ has two choices.

1. $I$ plays an iteration tree $(S_\nu, P_\nu)$, where $P_\nu = M^{\xi \theta}_{\theta}$, such that, extending $(T, M)$ via $S_\nu$, we still have a non-overlapping tree.

2. $I$ plays $(S_\nu, P_\nu, E_\theta, \beta, \gamma)$ such that
   
   (a) $M^{\xi \theta}_{\theta} \models \text{"}E_\theta\text{"}$ is a $2^{\kappa_0}$-closed extender", $\gamma \leq \theta_\beta, \beta \leq \xi$, the least model $E_\theta$ can be applied to, is $M^{\xi \theta}_{\beta \theta \gamma}$ and if $\beta = \xi$ then $\gamma < \theta_\xi$,
   
   (b) $S_\nu$ is an iteration tree on $P_\nu = \text{ult}(M^{\xi \theta}_{\beta \theta \gamma}, E_\theta)$,
   
   (c) the tree resulting from extending $T$ by $S_\nu$ is still non-overlapping and $\langle (T_\alpha, P_\alpha) | \alpha < \nu \rangle \sim \langle (S_\nu, P_\nu) \rangle$ satisfies the definition of quasi-linearity, except, possibly, for not having a last model.

Player $II$ responds by playing either

1. (accept). Then set $T_\nu = S_\nu$ and extend $T$ via $T_\nu$ (and $E_\theta$, if $I$ played as in case 2). This move is legal only if $S_\nu$ is of successor length, in which case we set $\theta_\nu = \text{lh}(T_\nu)$.

2. (accept, $b$), where $b$ is a cofinal wellfounded branch of $S_\nu$. Let $T_\nu$ be $S_\nu$ extended by $b$. Extend $T$ via $T_\nu$ (and $E_\theta$, if $I$ played as in case 2).

3. (reject, $b$), where $b$ is a maximal wellfounded branch of $S_\nu$, such that $\sup(b) < \text{lh}(S_\nu)$, i.e. $b$ is non-cofinal. Let $T_\nu$ be $S_\nu | \sup(b)$ extended by $b$. Extend $T$ via $T_\nu$ (and $E_\theta$, if $I$ played as in case 2).

This concludes the definition of $G(M, \omega_1 + 1)$.
**Theorem 4.1:** Suppose $M$ is a countable premouse elementarily embeddable in some $V_\eta$, $\pi : M \rightarrow V_\eta$. Then $\Pi$ has a winning strategy for $G(M, \omega_1 + 1)$.

**Proof:** Choose $\kappa > \eta$ large enough so that $V_\kappa$ is a premouse with $\delta(V_\kappa) = \eta$ and with $\omega_1 + 1$ cut-off points above $\eta$. We first construct premice $N_\alpha$, for $1 \leq \alpha \leq \omega_1$, ordinals $\eta(\alpha, \beta)$, $\eta(\alpha, \infty)$, and maps $\pi_{\alpha, \beta}$ and $\pi_{\alpha, \infty}$ for $0 \leq \alpha \leq \beta \leq \omega_1$ such that

1. $N_0 = M$ and, for $\alpha > 0$, each $N_\alpha$ contains $HC$ and $\langle (N_\beta, \eta(\gamma, \beta), \pi_{\gamma, \beta}) | 0 \leq \gamma \leq \beta < \alpha \rangle$, $\delta(N_\alpha) = \eta(0, \alpha)$, and for each $0 < \beta < \alpha$, $N_\alpha \models |N_\beta| = 2^{\aleph_0}$.

2. For $0 \leq \alpha < \beta \leq \omega_1$, $\eta(\alpha, \gamma) < \eta(\beta, \gamma)$, $N_\gamma \cap \text{Ord} = \eta(\gamma, \gamma)$. Moreover $\eta(0, \infty) = \eta$ and, for $\alpha > 0$, $\eta(\alpha, \infty)$ is the $\alpha$th cut-off point of $V_\kappa$ above $\eta$.

3. For $\alpha \leq \beta \leq \omega_1$, $\pi_{\alpha, \beta} : N_\alpha \rightarrow N_\beta \cap V_{\eta(\alpha, \beta)}$ is an elementary embedding such that, $\pi_{\alpha, \gamma} = \pi_{\beta, \gamma} \circ \pi_{\alpha, \beta}$ and $\pi_{\alpha, \alpha} = \text{id}|N_\alpha$.

4. For $\alpha \leq \beta \leq \omega_1$, $\pi_{\alpha, \infty} : N_\alpha \rightarrow V_{\eta(\alpha, \infty)}$ is an elementary embedding such that $\pi_{\beta, \infty} \circ \pi_{\alpha, \infty}$ and $\pi = \pi_{0, \infty}$.

To see this suppose $N_\beta$, $\eta(\gamma, \beta)$, $\pi_{\gamma, \beta}$, $\pi_{\gamma, \infty}$ have been defined for $\gamma \leq \beta < \alpha$, and let

$$H = \text{Hull}^{V_{\eta(\alpha, \infty)}}(HC \cup \bigcup_{\beta < \alpha} \text{ran}(\pi_{\beta, \infty}) \cup \langle \eta(\beta, \infty) | \beta < \alpha \rangle)$$

let $\pi_{\alpha, \infty} : N_\alpha \rightarrow V_{\eta(\alpha, \infty)}$ be the inverse of the transitive collapse and let $\pi_{\beta, \alpha} = (\pi_{\alpha, \infty})^{-1} \circ \pi_{\beta, \infty}$.

In order to show that $\Pi$ has a winning strategy $\Sigma$ in $G(M)$, we shall define a system of maps $\varphi_{\alpha, \beta}$, $\sigma_\alpha$ and trees $U_\alpha$ at round $\nu$. Suppose we are at a position of length $\nu$, according to $\Sigma$, and suppose we have built so far a tree $(T, M)$ of length $\theta$ with quasi-linear decomposition $\langle (T_\alpha, P_\alpha) | \alpha < \nu \rangle$ and basic sequence $\langle \alpha_i + 1 | i < \lambda \rangle$. The maps $\varphi_{\alpha, \beta}$ should be thought of as being the “stage $\nu$” versions of the $\pi_{\alpha, \beta}$: if $\xi+1 < \nu$, then $\varphi_{\alpha, \beta} = \pi_{\alpha, \beta}$ and $U_\alpha = \pi_{\alpha, \beta}T$. If, otherwise, there is $\xi + 1 < \nu$ such that $T_{\xi+1} \cap T_\xi$, and $\xi$ is least such, then the $\varphi$’s change: say that $P_{\xi+1} = \text{ult}(M_\xi^T, E)$ and let $\alpha = \beta \otimes \gamma$. Then $\varphi_{\alpha, \beta} = P_{\xi+1} \rightarrow N_{\alpha+1}$. On the other hand, the $\sigma^\nu$’s guarantee that, if $\nu$ is limit and for all sufficiently large $\xi + 1 < \nu$, $T_{\xi+1} \cap T_\xi$, then the direct limit model is wellfounded.

Here is our official definition. Suppose we are given:

1. elementary embeddings $\varphi_{0, \alpha} : M^T_{\Gamma(\alpha)} \rightarrow N_\alpha \cap V_{\eta(\alpha, \alpha)}$ such that, for $0 \leq \alpha \leq \beta \leq \omega_1$

$$\varphi_{0, \beta} \circ h_{\Gamma(\alpha),\Gamma(\beta)} = \pi_{\alpha, \beta} \circ \varphi_{0, \alpha}.$$ 

Note that $\Gamma(\alpha) = \Gamma(\beta)$ with $\alpha < \beta$ is possible: in this case $M^T_{\Gamma(\alpha)} = M^T_{\Gamma(\beta)}$ and $\varphi_{\alpha, \beta} = \pi_{\alpha, \beta} \circ \varphi_{0, \alpha}$.

2. $U_\alpha = \varphi_{0, \alpha}(T[\Gamma(\alpha)])$, the iteration tree $T[\Gamma(\alpha)]$ copied on $N_\alpha$ via $\varphi_{0, \alpha}$. So $\varphi_{0, \alpha} : (T[\Gamma(\alpha)], M^T_{\Gamma(\alpha)}) \rightarrow (U_\alpha, N_\alpha)$ is a bounded embedding with bound $\eta(0, \alpha)$. Following our convention above, all models and embeddings of $U_\alpha$ are indexed by ordinals in $[\Gamma(\alpha), \theta)$. 

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3. For \(0 \leq \alpha \leq \beta \leq \omega_1\), let \(\varphi_{\alpha,\beta} : M_{\Gamma(\beta)}^{U_\alpha} \to N_\beta \cap V_{\eta(\alpha,\beta)}\) be an elementary map such that \(\varphi_{\alpha,\beta} \circ i_{\Gamma(\alpha),\Gamma(\beta)} = \pi_{\alpha,\beta}\). [Recall that \(M_{\Gamma(\alpha)}^{U_\alpha}\) is the base model of \(U_\alpha\), that is \(N_\alpha\).] Moreover, letting \(\varphi^* : M_{\Gamma(\beta)}^T \to M_{\Gamma(\beta)}^{U_\alpha}\) be the copy map induced by \(\varphi_{0,\alpha}\), then \(\varphi_{\alpha,\beta} \circ \varphi^* = \varphi_{0,\beta}\).

4. For \(\sup_{i<\lambda} \alpha_i + 1 \leq \alpha \leq \beta < \nu\) the following is a commutative diagram of elementary maps

\[
\begin{array}{cccc}
M_{\Gamma(\beta)}^T & \xrightarrow{\varphi^*} & M_{\Gamma(\beta)}^{U_\alpha} & \xrightarrow{i_{\Gamma(\alpha),\Gamma(\beta)}} & M_{\Gamma(\alpha)}^T \\
\downarrow & & \downarrow & & \downarrow \\
M_{\Gamma(\alpha)}^T & & M_{\Gamma(\beta)}^{U_\alpha} & & M_{\Gamma(\beta)}^T \\
\downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{\varphi_{0,\alpha}} & N_\alpha & \xrightarrow{\pi_{\alpha,\beta}} & N_\beta \\
\end{array}
\]

In particular, \(\sigma_\nu\) is defined when \(\nu = \beta + 1\). Note that, for \(\beta \geq \sup_{i<\lambda} \alpha_i + 1\), \(T_{\beta+1} \parallel T_\beta\), so \(\beta \otimes \theta_\beta = (\beta + 1) \otimes 0\).

As the objects above change, as the play goes on, we should really write:

\[
U_\alpha^\nu, \quad \varphi_{\alpha,\beta}^\nu, \quad \sigma_\alpha^\nu, \quad \langle \alpha_i(\nu) + 1 \mid i < \lambda(\nu) \rangle \quad \text{and} \quad \Gamma^\nu(\beta).
\]

For the sake of legibility, we will drop the superscript \(\nu\) whenever possible. The objects defined at round \(\nu\) are related to the ones defined at round \(\xi < \nu\) as follows.
- Suppose $\exists \xi_0 < \nu$ such that for all $\xi_0 < \xi + 1 < \nu$, $\mathcal{T}_{\xi+1} \upharpoonright \mathcal{T}_\xi$. [In this case $\nu$ could be limit or successor.] Then, for such $\xi_0$ and $\xi$, $\lambda(\nu) = \lambda(\xi_0) = \lambda(\xi)$, and the basic sequences are the same, $\langle \alpha_i(\nu) + 1 \mid i < \lambda(\nu) \rangle = \langle \alpha_i(\xi_0) + 1 \mid i < \lambda(\xi) \rangle$. Also $\varphi^\nu_{\alpha,\beta} = \varphi^\xi_{\alpha,\beta}$ and $\mathcal{U}^\nu_\alpha$ extends $\mathcal{U}^\xi_\alpha$ via $\varphi^\nu_{0,\alpha} \mathcal{T}_\xi$, for $0 \leq \alpha \leq \beta \leq \omega_1$. Finally, if $\nu$ is limit, then $\sigma^\nu_\alpha = \sigma^\xi_\alpha$ for all $\alpha$ such that $\sup \{ \alpha_i(\nu) + 1 \mid i < \lambda(\nu) \} \leq \alpha < \xi + 1 \leq \nu$, and if $\nu = \xi + 1$, then $\sigma^\nu_\alpha = \sigma^\xi_\alpha$ for all $\alpha \leq \xi$.

- Suppose $\nu$ is limit and there is an increasing sequence $\xi_n \to \nu$ such that $\mathcal{T}_{\xi_{n+1}} \upharpoonright \mathcal{T}_{\xi_n}$. Without loss of generality we may assume that each $\xi_n + 1 \in \{ \alpha_i(\nu) + 1 \mid i < \lambda(\nu) \}$. Then, for $\gamma \leq \nu$, $\beta \leq \omega_1$ and $\nu \otimes 0 \leq \alpha \leq \omega_1$

$$\langle \alpha_i(\nu) + 1 \mid i < \lambda(\nu) \rangle = \bigcup_{n<\omega} \langle \alpha_i(\xi_n + 2) + 1 \mid i < \lambda(\xi_n + 2) \rangle,$$

$$\Gamma^\nu(\beta) = \sup_n \Gamma^{\xi_n + 2}(\beta), \quad \Gamma^\nu(\alpha) = \nu \otimes 0 \quad \text{and} \quad \sigma^\nu_\gamma = \pi_{\omega_1,\infty}.$$

[The “$+2$” in the indices above is because we want to consider quasi-linear systems where $\mathcal{T}_{\xi_n+1}$ is the last tree.] Note that for $\beta < \nu \otimes 0$ the sequence $\Gamma^{\xi_n + 2}(\beta)$ is eventually $= \Gamma^\nu(\beta)$. Thus, for $n$ larger than some fixed $m$,

$$\varphi^\nu_{\alpha,\beta} = \varphi^{\xi_n + 2}_{\alpha,\beta} \quad \text{and} \quad \mathcal{U}^\nu_\beta = \varphi^\nu_{0,\beta} (\mathcal{T}[\Gamma^\nu(\beta)]) = \bigcup_{m<n} \mathcal{U}^{\xi_n + 2}_\beta.$$

If, instead, $\beta \geq \nu \otimes 0$, then the ordinals $\Gamma^{\xi_n + 2}(\beta)$ are strictly increasing and converge to $\Gamma^\nu(\beta) = \nu \otimes 0$. In this case, for $\alpha < \beta = \nu \otimes 0$, $\varphi^\nu_{\alpha,\beta}$ is the direct limit map of the commutative system of embeddings

For $\nu \otimes 0 \leq \beta \leq \gamma$, $\varphi^\nu_{\beta,\gamma} = \pi_{\beta,\gamma}$ as $\mathcal{U}^\nu_\beta$ and $\mathcal{U}^\nu_\gamma$ are the empty iteration trees on $N_\beta$ and $N_\gamma$, respectively. Finally, if $\alpha < \nu \otimes 0 \leq \beta$, we set $\varphi^\nu_{\alpha,\beta} = \pi_\nu \otimes \alpha \circ \varphi^\nu_{\alpha,\nu \otimes 0}$.
• Suppose \( \nu = \xi + 1 \) and \( \xi \) is limit. By definition of the game \( \mathcal{G} \), the quasi-linear decomposition \( \langle (T_\alpha, P_\alpha) \mid \alpha < \xi + 1 \rangle \) of \( (T, M) \) is such that \( P_\xi \) is the direct limit of the \( P_{\xi_n} \)'s, for some cofinal sequence \( \xi_n \to \xi \), and \( T_\xi \) is the empty tree on \( P_\xi \). Then

\[
\langle \alpha_i(\nu) + 1 \mid i < \lambda(\nu) \rangle = \langle \alpha_i(\xi) + 1 \mid i < \lambda(\xi) \rangle
\]

and for \( 0 \leq \alpha \leq \beta \leq \omega_1 \) and \( \gamma < \xi \)

\[
\Gamma^{\nu}(\beta) = \Gamma^{\xi}(\beta), \quad \varphi_{\alpha,\beta}^{\nu} = \varphi_{\alpha,\beta}^{\xi}, \quad \sigma^{\nu}_\gamma = \sigma^{\xi}_\gamma, \quad U'_\beta \text{ extends } U_\beta^{\xi}
\]

and \( \sigma^{\nu}_\xi : M^T_{\xi \in \omega_1} \to V_\kappa \) is the direct limit map induced by the embeddings \( \sigma^{\nu}_\gamma \).

**Claim 4.1:** Suppose that \( \nu \leq \omega_1 \) is limit and that \( II \) has not lost by round \( \nu \), and that there are \( \varphi^{\nu} \)'s, \( U^{\nu} \)'s and \( \sigma^{\nu} \)'s as above, then \( II \) does not lose at round \( \nu \). In particular if \( II \) has not lost by round \( \omega_1 \), then the \( \varphi_{\omega_1}^{\nu}, U^{\omega_1} \)'s and \( \sigma^{\omega_1} \)'s witness that he wins \( \mathcal{G}(M, \omega_1 + 1) \).

**Proof:** As \( \nu \) is limit, it is \( II \)'s turn to move: he has to verify that the only cofinal branch \( b \) of the iteration tree \( T \) built so far, is wellfounded. By hypothesis we are given \( \varphi_{\alpha,\beta} = \varphi_{\alpha,\beta}^{\nu}, \quad U_\alpha = U_\alpha^{\nu} \) and \( \sigma_\alpha = \sigma_\alpha^{\nu} \) satisfying the conditions above.

**Case 1:** \( (\exists \xi_0 < \nu)(\forall \xi) \xi_0 < \xi + 1 < \nu \implies T_{\xi+1} \parallel T_\xi \).

Choosing \( \xi_0 \) least such, \( \Gamma(\omega_1) = \xi_0 \otimes 0 \) and \( \varphi_{0,\omega_1} \) copies \( T | \Gamma(\omega_1) \) to \( U_{\omega_1} \). Let \( \xi_n \to \nu \) be increasing and let

\[
f_n : M^{T}_{\xi_n \otimes 0} \to M^{U}_{\xi_n \otimes 0}
\]

be the copy map induced by \( \varphi_{0,\omega_1} \). Then \( \sigma_{\xi_n} \circ f_n \) witness \( M^T_{\xi_n} \) is wellfounded.

**Case 2:** There is an increasing sequence \( \xi_n \to \nu \) such that \( T_{\xi_n + 1} \not\parallel T_{\xi_n} \).

Without loss of generality we can assume that for all \( n < \omega \), \( \alpha_n = \Gamma(\xi_n + 1) \) belongs to the basic sequence, hence to \( b \). Let

\[
f_n = \pi_{\xi_n + 1, \omega_1} \circ \varphi_{0,\xi_n + 1} : M^{T}_{\alpha_n} \to N_{\omega_1}.
\]

Thus, for \( n < m \), \( f_n = f_m \circ i_{\alpha_n,\alpha_m} \) so that there is a limit map \( f_{\omega} : M^T_{\omega} \to N_{\omega_1} \) witnessing wellfoundedness.

\[\lor\]

So it is enough to show that \( \varphi^{\nu} \)'s, \( U^{\nu} \)'s and \( \sigma^{\nu} \)'s as above exist for every \( \nu \leq \omega_1 \).

If \( \nu = 0 \), then \( \varphi_{\alpha,\beta} = \pi_{\alpha,\beta}, U_\alpha \) is the empty tree on \( N_\alpha \) and \( \sigma_0 = \pi_{\omega_1, \infty} : N_{\omega_1} \to V_\kappa \).

If \( \nu \) is limit, then the remarks before the claim show how to define the \( \varphi^{\nu}, U^{\nu} \) and \( \sigma^{\nu} \) from the \( \varphi^{\xi}, U^{\xi} \) and \( \sigma^{\xi} \), for \( \xi < \nu \).

We now take care of round \( \nu + 1 \). Suppose we are given \( \varphi^{\nu} \)'s, \( \sigma^{\nu} \)'s and \( U^{\nu} \)'s, we must describe how \( II \) answers to \( I \)'s moves and how to build \( \varphi^{\nu+1} \)'s, \( \sigma^{\nu+1} \)'s and \( U^{\nu+1} \)'s. In order
to avoid making our formulæ exceedingly ornate, we shall drop the suffix $\nu$ for the objects at stage $\nu$ and use $\Gamma'$, $\psi_{\alpha,\beta}$, $\tau_{\alpha}$ and $\mathcal{V}_{\alpha}$ for $\Gamma^{\nu+1}$, $\varphi_{\alpha,\beta}^{\nu+1}$, $\sigma_{\alpha}^{\nu+1}$ and $\mathcal{U}_{\alpha}^{\nu+1}$.

- Suppose $I$ plays $(\mathcal{S}_{\nu}, P_{\nu})$.
  Then $P_{\nu} = M_{\theta}^{\mathcal{T}}$ and $II$ tries to copy $\mathcal{S}_{\nu}$ on $V_{\kappa}$. If he succeeds to do so and finds a branch $b$, then $II$ plays (accept, $b$). Otherwise he rejects $\mathcal{S}_{\nu}$. To be more precise.

As $\theta + 1 = \text{lh}(T)$, there is $\sigma_{\nu} : M_{\theta}^{\mathcal{T}_{\kappa-1}} \rightarrow V_{\eta(\omega_1,\infty)} \subset V_{\kappa}$ and let $\varphi : M_{\theta}^{\mathcal{T}} \rightarrow M_{\theta}^{\mathcal{T}_{\kappa-1}}$ be the copy map induced by $\varphi_{0,\omega_1}$. Let $\theta' \leq \text{lh}(\mathcal{S}_{\nu})$ be largest such that $\mathcal{S}_{\nu}\upharpoonright \theta'$ can be copied on $V_{\kappa}$ via $\sigma_{\nu} \circ \varphi$, and there are no wellfounded maximal branches cofinal in some $\gamma < \theta'$. If $\theta' = \text{lh}(\mathcal{S}_{\nu})$ is a successor ordinal, let $II$ play (accept) and set $T_{\nu} = \mathcal{S}_{\nu}$ and $\theta_{\nu} = \theta' - 1$.

By Lemma 3.1, if $\theta' < \text{lh}(\mathcal{S}_{\nu})$, then it must be a limit ordinal, so we may assume that $\theta' \leq \text{lh}(\mathcal{S}_{\nu})$ is limit. By $2^{\omega_0}$-closure of the extenders and Theorem 5.6 in [2], $\sigma_{\nu} \circ \varphi(\mathcal{S}_{\nu}\upharpoonright \theta')$ must have a cofinal wellfounded branch $b$. Let $II$ play (accept, $b$), if $\theta_{\nu} = \text{lh}(\mathcal{S}_{\nu})$, or (reject, $b$), if $\theta_{\nu} < \text{lh}(\mathcal{S}_{\nu})$, and in either case let $T_{\nu}$ be $\mathcal{S}_{\nu}\upharpoonright \theta_{\nu}$ extended by $b$.

In order to keep the induction going, the $\psi$'s, $\tau$'s and $\mathcal{V}$'s must be defined and shown to satisfy the inductive hypothesis. The tree $\mathcal{T}$ extended by $T_{\nu}$ will be denoted by $\mathcal{S}$. As $T_{\nu}\upharpoonright T_{\zeta}$, then $\langle (\mathcal{T}_{\alpha}, P_{\alpha}) \mid \alpha < \nu + 1 \rangle$ is quasi-linear, its basic sequence is still $\langle \alpha_i + 1 \mid i < \lambda \rangle$, hence $\Gamma' = \Gamma$. Set $\psi_{\alpha,\beta} = \varphi_{\alpha,\beta}$ and extend $U_{\alpha}$ to $\mathcal{V}_{\alpha}$, by tagging (an isomorphic copy of) $\mathcal{T}_{\nu}$ on top: $V_{\alpha} = \psi_{0,\alpha}(\mathcal{S}|\Gamma(\alpha))$. For $\xi \leq \nu$, $\tau_{\xi}$ can be taken to be $\sigma_{\xi}$, so we are only left to define $\tau_{\nu + 1} : M_{\nu_{\alpha}^{\nu+1}} \rightarrow V_{\eta(\omega_1,\infty)}$, which will be obtained using Lemma 3.7. [The argument will be presented in some detail and will serve as a template for other proofs in this paper.]

First observe that $\mathcal{V}_{\omega_1}\upharpoonright \theta = \psi_{0,\omega_1}(\mathcal{S}|\theta)$ is (isomorphic to) $T_{\nu}$ and, by construction, $\mathcal{V}_{\omega_1}$ can be copied on $V_{\kappa}$ via $\tau_{\nu}$. Let us denote by $W_{\alpha}$ and $Z_{\alpha}$ the $\alpha$th models of the tree $T_{\nu}$ copied on $V_{\kappa}$ and on $V_{\eta(\omega_1,\infty)}$, respectively, via $\tau \circ \psi_{0,\omega_1}$. In other words, $Z_{\alpha} = W_{\alpha} \cap V_{\eta'}$ where $\eta' = \eta_{0,\alpha}(\eta(\omega_1,\infty))$. Let $\tau^* : M_{\nu^{\nu+1}} \rightarrow Z_{\theta_{\nu}} \subset W_{\theta_{\nu}}$ be the copy map induced by $\tau_{\nu}$. The commutative diagram below may help to follow the argument. (The horizontal arrows come from the copy construction, while the vertical ones are the tree embeddings.)

Note that $|\mathcal{V}_{\omega_1}| = 2^{\omega_0}$, $\mathcal{S}$ is countable and $2^{\omega_0}$-closed, so the hypotheses of Lemma 3.7 hold. Thus $\tau^*$ can be taken to be an element of $W_{\theta_{\nu}}$, and by elementarity of the tree-embedding
from \( V_\kappa \) to \( W_\theta _\kappa \), there is \( \tau _{\nu + 1} \in V_\kappa \) such that

\[
M^S_{\nu \otimes \theta _\nu } \longrightarrow M^V_{\nu \otimes \theta _\nu }
\]

commutes, and this is what we had to prove.

**Suppose** \( I \) **plays** \((S_\nu , P_\nu , E_\theta , \beta _0, \gamma _0)\).

Let \( \alpha = \beta _0 \otimes \gamma _0 \). As \( I \)'s move is legal, then \( \alpha \) cannot belong to \( F_\nu \), the set of forbidden nodes at stage \( \nu \). The next Claim is crucial for the present construction.

**Claim 4.2:** \( \Gamma (\alpha + 1) \leq \alpha \).

**Proof:** Deny. As \( \Gamma (\alpha + 1) \leq \alpha + 1 \), it follows that \( \alpha + 1 = \Gamma (\alpha + 1) = (\alpha _i + 1) \otimes 0 \), for some \( i < \lambda (\nu) \). The definition of basic sequence implies that \( T_{\alpha _i + 1} \perp T_{\alpha _i} \), and \( \alpha _i = \alpha _i \otimes \theta _{\alpha _i} \). Thus \( \alpha \in F_{\alpha _i + 1} \subseteq F_\nu \): a contradiction.

As \( \Gamma (\alpha + 1) \leq \alpha \), the embedding \( \varphi _{\alpha, \alpha + 1} : M^T_{\Gamma (\alpha + 1)} \rightarrow N_{\alpha + 1} \cap V_{\eta (\alpha, \alpha + 1)} \) copies to bounded embeddings

\[
\varphi : M^T_{\alpha} \rightarrow M^T_{\alpha + 1} \quad \text{and} \quad \varphi ^* : M^T_{\theta} \rightarrow M^T_{\theta + 1}.
\]

Let also \( \varphi ': M^T_{\theta} \rightarrow M^T_{\theta} \) be the copy map induced by \( \varphi _{\alpha, \alpha} : M^T_{\Gamma (\alpha)} \rightarrow N_\alpha \). By Lemma 3.1,

\[
P_\nu = \operatorname{ult}(M^T_{\alpha} ; E_\theta ), \quad P = \operatorname{ult}(M^T_{\alpha}, \varphi ' (E_\theta )), \quad \text{and} \quad Q = \operatorname{ult}(M^T_{\alpha + 1}, \varphi ^* \circ \varphi ' (E_\theta ))
\]

are wellfounded, so the copy constructions yield bounded embeddings \( \psi ^* : P \rightarrow Q \) and \( \psi ': P_\nu \rightarrow P \). By Lemma 3.1, we get \( \psi _{\alpha, \alpha + 1} : P \rightarrow N_{\alpha + 1} \cap V_{\eta (\alpha, \alpha + 1)} \). Set

\[
\psi _{\beta, \gamma} = \begin{cases} 
\varphi _{\beta, \gamma} & \text{if } \beta \leq \gamma \leq \alpha, \\
\pi _{\alpha, \beta} & \text{if } \alpha + 1 \leq \beta \leq \gamma, \\
\pi _{\alpha + 1, \gamma} \circ \psi _{\alpha, \alpha + 1} \circ \varphi _{\beta, \alpha} & \text{if } \beta \leq \alpha < \gamma,
\end{cases}
\]

and \( \tau _\xi = \pi _{\omega _1, \omega _1} \), for all \( \xi \leq \nu \). Now the argument proceeds as before. Let \( \theta ' \leq \operatorname{lh}(S_\nu ) \) be largest such that \( S_\nu | \theta ' \) can be copied on \( V_\kappa \) via \( \tau _\nu \circ \psi _{\omega _1, \omega _1} \) and there are no wellfounded maximal branches cofinal in some \( \gamma < \theta ' \). If \( \theta ' = \operatorname{lh}(S_\nu ) \) is a successor ordinal, let \( II \) play
(accept) and set $T_\nu = S_\nu$. If, otherwise, $\theta' \leq \text{lh}(S_\nu)$ is limit, then let $b$ be a wellfounded branch of $\tau_\nu \circ \psi_{0,\alpha}(S_\nu, \theta')$ and let $II$ play (accept, $b$), or (reject, $b$), depending on whether $\theta' = \text{lh}(S_\nu)$ or not. In either case set $T_\nu = S_\nu$. By Lemma 3.7 there is $\tau_{\nu+1}$ such that the diagram

```
M^{S_\nu}_b \xrightarrow{\psi'} M^{\psi' S_\nu}_b
```

commutes. Let

$$
\Gamma'(\beta) = \begin{cases} 
\Gamma(\beta) & \text{for } \beta \leq \alpha = \beta_0 \otimes \gamma_0, \\
\alpha + 1 & \text{for } \alpha < \beta,
\end{cases}
$$

and $y_\beta = \psi_{0,\beta}(S[\Gamma'(\beta)])$. We leave to the reader the verification that the $\varphi^{\nu+1}$'s, $\sigma^{\nu+1}$'s, $U^{\nu+1}$'s so defined satisfy the inductive hypothesis.

As we have taken care of all possible cases, the Theorem is proved.

5 How not to lose

Let $M$ be a countable coarse premouse. We will consider the following iteration game on $M$, called $G^+(M)$. It is played like the ordinary full iteration game, with $II$ on the move at limits and $I$ on the move at the other rounds, except that $I$ must also play (besides the extenders $E_\alpha$'s) distinct natural numbers $n_\alpha$, with $n_\alpha \notin \{n_\beta \mid \beta < \alpha\}$. The game is over when $I$ runs out of integers. We also require that the iteration tree that $I$ and $II$ construct is $2^{\aleph_0}$-closed.

The length of the game thus depends on the play, but is always $< \omega_1$. On the other hand, it is easy to see that $G^+(M)$ is stronger than the full iteration game (for $2^{\aleph_0}$-closed extenders) of fixed countable length.

The rest of this paper is devoted to a proof of

**Theorem 5.1:** Player $I$ does not have a winning strategy for $G^+(M)$, for a countable premouse $M$ elementarily embeddable in some $V_\eta$. 

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If Σ is a strategy for I then we have a continuous coding of \( G^+(M) \) in the sense of \([3]\), i.e. a function \( c \) form the set of all legal positions of \( G^+(M) \) to \( \omega \) such that if \( p \) and \( q \) are positions and \( q \) extends \( p \), then \( c(p) \neq c(q) \): just take \( c(p) = \) the natural number \( n \) given by Σ at position \( p \). A position \( p \) of \( G^+(M) \) is, essentially, an iteration tree of successor length. To avoid confusions, we denote with \( lh(p) \) the length of \( p \) as a position, and with \( ht(p) \) the length (or height) of the associated iteration tree, so that \( ht(p) = lh(p) + 1 \).

For the reader’s benefit, here is a brief description the plan of the proof. In all previous iterability theorems (see \([2,3]\)) one argues by contradiction: given a “bad” tree, i.e. a tree without cofinal wellfounded branches, using the ordinals witnessing such badness, an infinite descending \( \in \)-chain is constructed. This contradiction forces us to conclude that such bad tree cannot exist in the first place, hence the theorem follows. To be more specific. Ordinals are assigned to the models of the b ad tree \( \mathcal{T} \), witnessing continuous illfoundedness. These ordinals are then used to build Skolem hulls of the models of the bad tree. Exercising proper care in the construction of such hulls, it can be shown that they form an inconsistent enlargement, that is a system of models resembling the original iteration tree, but containing an infinite descending \( \in \)-chain.

Back to our proof. If we try to argue by contradiction following the pattern above, we are immediately faced with the problem that we are not given a bad tree, but rather a bad strategy Σ, i.e. a winning strategy for I, hence we cannot first fix a tree and then get the ordinals for the construction. In other words, the ordinals should be given “continuously in the tree”. The cure for this is to construct positions \( p_1 \subset \ldots \subset p_n \ldots \) of the game \( G^+(M) \) together with enlargements \( \mathcal{P}^n = \langle P_\alpha \mid \alpha < \theta_n \rangle ^\sim (\mathcal{P}^*_n) \) of \( p_n \) (here \( \theta_n = lh(p_n) \)) such that \( \mathcal{P}^*_n \subset \mathcal{P}^{n+1} \), obtaining thus a contradiction. The enlargement \( \mathcal{P}^n \) will be constructed by taking hulls of the models of the (pseudo-)iteration tree obtained by copying the position \( p_n \) on \( \mathcal{P}^{n-1} \). The ordinals needed for this construction are ranks of certain nodes on a wellfounded tree \( \mathcal{U} \) on some \( V_\kappa \), searching for a defeat of Σ. Let’s take a closer look at \( \mathcal{U} \).

Fix towards a contradiction, \( \pi : M \to V_\eta \) and a winning strategy Σ for I in \( G^+(M) \). Let \( \kappa > \eta \) be such that \( V_\kappa \) is a premouse with \( \delta(V_\kappa) = \eta \). For any position \( p \), let

\[
I_p = \{ n \in \omega \mid \forall \beta \leq lh(p)(n \neq c(p \upharpoonright \beta)) \}
\]

that is, the set of all natural numbers not yet played by position \( p \). \( \mathcal{U} \) searches, among other things, for a sequence of positions according to I’s strategy Σ, \( p_1 \subset p_2 \subset \ldots \subset \bigcup_n p_n = \mathcal{T} \), a cofinal wellfounded branch \( b \) of \( \mathcal{T} \) and families of premice \( C^1 \subset \ldots \subset C^n \), with \( lh(C^n) = lh(p_n) \). The position \( p_n \) can be copied, as a pseudo-iteration tree, on \( C^{n-1} \backslash \langle V_\kappa \rangle \). Also if \( m \) is the least element of \( I_{p_n} \), we make sure that either \( m \notin I_{p_{n+1}} \), or there is no \( q \in \mathcal{U} \) extending \( p_n \) according to Σ that can be copied onto \( V_\kappa \) and that \( m \notin I_q \). We then add to the node of the tree \( \mathcal{U} \) a family of models \( \mathcal{B}^n \) of length \( ht(p_n) \), each model of size at most 2\(^{\aleph_0} \), and resembling enough to \( C^n \backslash \langle V_\kappa \rangle \), and witnessing that no such \( q \) can be copied on \( \mathcal{B}^n \).

Suppose \( \mathcal{U} \) had a branch, namely positions \( p_1 \subset p_2 \subset \ldots \subset \bigcup_n p_n = \mathcal{T} \), and a cofinal wellfounded branch \( b \) of \( \mathcal{T} \). This would determine a new position \( p = (\mathcal{T}, b) \), and let \( c(p) = m \). As the integer \( m \) was considered at some stage \( n \), while choosing position \( p_{n+1} \), and as \( m \in I_{p_{n+1}} \), it follows that no position \( q \) with \( m \notin I_q \) could have been copied onto \( \mathcal{B}^n \). But, as

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it turns out, $p$ is either such a position or a defeat for $\Sigma$. This shows that $U$ is wellfounded.

The proof now proceeds as follows. Positions $p_n$ and families of premice $P^n$ are built inductively so that:

1. the $p_n$'s are according to $\Sigma$, extend each other and $\langle p_n \mid n < \omega \rangle$ is a complete play of $G^+(M)$;
2. $P^n = \langle P_\alpha \mid \alpha < \theta_n \rangle \setminus \langle P^*_n \rangle$, $P^n|\theta_n = P^{n+1}|\theta_n$, where $\theta_n + 1 = ht(p_n)$, and $p_n$ together with $C^\downarrow = P^n|\theta_n$ make up for part of a node of $U$ of length $n + 1$;
3. $P^*_{n+1} \in P^n$.

The idea is to choose $p_{n+1}$ first, and then copy the iteration tree on the current enlargement $P^n$. For any $\theta_n \leq \alpha < \theta_{n+1}$, the $\alpha$th model on the pseudo-iteration tree on $P^n$ is replaced by the transitive collapse of a Skolem hull. Call these models $P_\alpha$. By a tree argument the $P_\alpha$'s can be taken to be in the last model (the one with index $\theta_{n+1}$) of the pseudo-iteration tree on $P^n$. By taking another hull and calling it $P^*_{n+1}$ we obtain $P^{n+1}$. The ordinal needed to take the hull in the $\alpha$th model is the rank of the node given by $p_{n+1}$, $P_n$ and an initial segment of a cofinal branch passing through $\alpha$.

This concludes our brief description of the structure of the proof.

### 6 The tree $U$

Let's first introduce some handy notation. Suppose $(T, B)$ is a pseudo-iteration tree of length $(\theta + 1, \lambda)$ and suppose that $C = \langle C_\alpha \mid \alpha \leq \theta \rangle$ is a family of premice and $\Pi = \langle \pi_\alpha \mid \alpha \leq \theta \rangle$

$$\Pi : (T, B) \rightarrow (\emptyset, C)$$

is a bounded embedding. If $C$ is internal, we call the pair $(C, \Pi)$ an **enlargement**.

**Definition 6.1:** Let $p$ be a position of $G^+(M)$, $\theta = ht(p)$, let $\alpha_0, \ldots, \alpha_k, < \theta$ and let $\Pi : (p, M) \rightarrow (\Pi p, B)$ be an embedding of pseudo-iteration trees.

- $e(p) : \theta \rightarrow \omega$ is the 1–1 function defined by $e(p)(\beta) = c(p|\beta)$.
- Let $X$ be the support for $(p, M)$ generated by $\{\alpha_0, \ldots, \alpha_k\} \cup e(p)^{-1}(k + 1)$.

$$[M^i_{\alpha_k}, \alpha_0, \ldots, \alpha_k] = (M^i_{\alpha_k})_{\Pi X}.$$
Remark. If \( q \) extends \( p \), then \( e(q) \supseteq e(p) \) so that, if \( \alpha_0, \ldots, \alpha_k < \text{ht}(p) \)

\[
[M_{\alpha_k}^{(P_{\alpha_k}, \mathcal{B})}; \alpha_0, \ldots, \alpha_k] \prec [M_{\alpha_k}^{(\Pi_{\alpha_k}, \mathcal{B})}; \alpha_0, \ldots, \alpha_k].
\]

We are now ready to define the tree \( \mathcal{U} \), a set of finite sequences from \( V_\kappa \) closed under initial segment. A node \( \mathbf{r} \), of length \( n+1 \), of \( \mathcal{U} \) is of the form

\[
\mathbf{r} = \langle (p_1, \mathcal{B}^0_1, \Pi^0_1, \nu_0, \sigma_0), (p_2, \mathcal{B}^1_1, \Pi^1_1, \Pi^0_1, \mathcal{C}^1_1, H^1_1, \Phi^1_1, \Psi^1_1, \nu_1, \sigma_1), \ldots \rangle
\]

\[
\ldots, (p_{n+1}, \mathcal{B}^n_n, \Pi^{n-1}_n, \mathcal{C}^n_n, H^n_n, \Phi^n_n, \Psi^n_n, \nu_n, \sigma_n) \rangle
\]

such that the following 6 clauses must hold.

1. \( p_1 \subset p_2 \subset \ldots \subset p_{n+1} \) are non-empty positions of the game \( \mathcal{G}^+(M) \), according to \( \Sigma \). Let \( \theta_i + 1 = \text{ht}(p_i) \) and, for notational convenience, let’s agree that \( p_0 = \emptyset \) = the empty position, hence \( I_{p_0} = \omega \) and \( \theta_0 = \text{lh}(p_0) = 0 \).

2. (a) \( \mathcal{C}^1 \subset \ldots \subset \mathcal{C}^n \),

\[
\mathcal{C}^n = \langle C_\alpha \mid \alpha < \theta_n \rangle
\]

and, for each \( \alpha < \theta_n \), \( C_\alpha \) is a premouse and \( H^{(2^{\theta_n})+}_n \subset C_\alpha \).

(b) \( \Phi^1 \subset \ldots \subset \Phi^n, H^1 \subset \ldots \subset H^n \),

\[
\Phi^n = \langle \varphi_\alpha \mid \alpha < \theta_n \rangle \quad H^n = \langle \eta_\alpha \mid \alpha < \theta_n \rangle
\]

such that, for each \( \alpha < \theta_n \), \( \varphi_\alpha : M^{p_n}_\alpha \rightarrow C_\alpha \cap V_{\theta_n} \) is an elementary embedding.

(c) There is an elementary embedding \( \varphi^n_* : M^{p_n}_{\theta_n} \rightarrow V_\eta \) such that

\[
(*\mathcal{C}^n, *\Phi^n) = (\mathcal{C}^n, \Phi^n, \langle \varphi^n_* \rangle)
\]

is an enlargement for \( (p_n, M) \) with bounds \( \langle \eta_\alpha \mid \alpha < \theta_n \rangle \rangle \langle \eta \rangle \).

Let’s agree to define \( (*\mathcal{C}^0, *\Phi^0) \) to be the pair \( (V_\kappa, \pi) \). There is a further constraint on the sequence of the positions of \( p_i \)’s.

Claim 6.1: For all \( n \geq 0 \) there is an \( k \in I_{p_n} \) and a \( q \supseteq p_n \) according to \( \Sigma \) such that \( k \notin I_q \) and \( q \) can be copied on \( (*\mathcal{C}^n, *\Phi^n) \).

Proof: \( I_{p_n} \neq \emptyset \) as \( c(p_{n+1}) \) belongs to this set. By Corollary 3.1, any \( q \) extending \( p_n \) of height \( \theta_n + 2 = \text{ht}(p_n) + 1 \) can be copied on \( (*\mathcal{C}^n, *\Phi^n) \), so we can take \( (q, k) \) to be the result of \( \Sigma \) applied to \( p_n \), hence \( k = c(q) \in I_{p_n} \).

Define \( N(p_n) \) to be the least integer \( k \) as in the Claim above. Thus \( N(p_0) \) is the least \( k \in \omega \) such that there is a non-empty position \( q \) that can be copied on \( V_\kappa \) via \( \pi \) and such that \( k = c(q|_\beta) \) for some \( \beta \leq \text{lh}(q) \).
3. For all $n \geq 0$, $N(p_n) \notin I_{p_{n+1}}$ and $p_{n+1}$ can be copied on $(^nC, ^n\Phi)$.

4. $B^0 \subseteq \ldots \subseteq B^n$, $\Pi^0 \subseteq \ldots \subseteq \Pi^n$ and

(a) $B^n = \langle B_\alpha \mid \alpha \leq \theta_m \rangle$, for some $m \leq n$, and for each $\alpha \leq \theta_m$, $B_\alpha$ is a premouse of cardinality $2^{\aleph_0}$ containing HC. Thus $B^0 = \langle B_0 \rangle$ is a single premouse.

(b) For $n > 0$, $N(p_n) \neq \min I_{p_n} \iff B^{n-1} \neq B^n$, and, if this is the case, $lh(B^n) = \theta_{n+1}$.

(c) $\Pi^n = \langle \pi_\alpha \mid \alpha < lh(B^n) \rangle$ and there are ordinals $\varepsilon_\alpha \in B_\alpha$ such that $\pi_\alpha : M^p_{\alpha} \to B_\alpha \cap V_{\varepsilon_\alpha}$ is an elementary embedding. The iteration tree $p_{n+1}$ can be copied on $(B^n, \Pi^n)$ hence $\Pi^n : (p_{n+1}, M) \to (\Pi^n_{p_{n+1}}, B^n)$ is a bounded embedding with bounds $\langle \varepsilon_\alpha \mid \alpha < lh(B^n) \rangle$.

(d) For $n > 0$, $\Pi^{n-1,n} = \langle \pi^{n-1,n}_\alpha \mid \alpha \leq \theta_{n+1} \rangle$, where the

$$\pi^{n-1,n}_\alpha : M^p_{\Pi^{n-1,n} p_{n+1}, B^{n-1}} \to M^p_{\Pi^n p_{n+1}, B^n}$$

are elementary, $\pi^{n-1,n}_\alpha = id | B_\alpha$, for $\alpha < lh(B^{n-1})$, and the diagram

$$
\begin{array}{ccc}
(p_{n+1}, M) & \xrightarrow{\Pi^n} & (\Pi^n p_{n+1}, B^n) \\
\Pi^{n-1} \downarrow & & \Pi^{n-1,n} \downarrow \\
(\Pi^{n-1} p_{n+1}, B^{n-1}) & & \\
\end{array}
$$

commutes.

(e) For $n \geq 0$,

$$\Psi^n = \langle \psi^n_\alpha \mid \alpha < \theta_n \rangle$$

$\psi^n_\alpha : M^p_{\alpha} \to C_\alpha$ are elementary embeddings, and there is a $\psi^*_n : M^p_{\theta_n} \to V_\kappa$ such that

$$^n\Psi = \Psi^n \circ (\psi^*_n) : (\Pi^n p_{n}, B^n) \to ^nC$$

is an elementary embedding of pseudo-iteration trees. Moreover

$$
\begin{array}{ccc}
(p_n, M) & \xrightarrow{^n\Phi} & ^nC \\
\Pi^n \downarrow & & \downarrow ^n\psi^n \\
(\Pi^n p_{n}, B^n) & & \\
\end{array}
$$

commutes and for $\alpha < lh(B^n)$, $\psi^n_\alpha(\varepsilon_\alpha) = \eta_\alpha$.

(f) For $n \geq 0$, if $N(p_n) > \min(I_{p_n})$, then for every $k \in I_{p_n} \cap N(p_n)$ and every position $q \supset p_n$ according to $\Sigma$ with $k \notin I_q$, $q$ cannot be copied on $B^n$. 

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Finally we take care of the $\nu$'s and $\sigma$'s.

5. $\nu_0 <_T \ldots <_T \nu_n$, where $<_T$ is the tree ordering of the largest iteration tree, $p_{n+1}$, and $\theta_n \leq \nu_n \leq \theta_{n+1}$.

6. $\sigma_0, \ldots, \sigma_n$ are elementary embeddings $\sigma_n : [M^{(\Pi^n_{p_{n+1}}, B^n)}_{\nu_n}; \nu_0, \ldots, \nu_n] \to V_\kappa$ such that, for $n > 0$, the diagram

\[
\begin{array}{ccc}
[M^{(\Pi^n_{p_{n+1}}, B^n)}_{\nu_n-1}; \nu_0, \ldots, \nu_{n-1}] & \overset{i}{\longrightarrow} & [M^{(\Pi^n_{p_{n+1}}, B^n)}_{\nu_n}; \nu_0, \ldots, \nu_n] \\
\uparrow \hat{i} & & \downarrow \sigma_n \\
[M^{(\Pi^{n-1}_{p_{n}}, B^{n-1})}_{\nu_n-1}; \nu_0, \ldots, \nu_{n-1}] & \longrightarrow & V_\kappa \\
\sigma_{n-1} \\
\end{array}
\]

commutes, where $\hat{i}$ and $\hat{\pi}$ are the restrictions of $i^{(\Pi^n_{p_{n+1}}, B^n)}_{\nu_n-1, \nu_n}$ and $\pi^{n-1, n}_{\nu_n-1}$. Moreover, $\pi = \sigma_0 \circ \pi_{0, \nu_0} \circ i_{0, \nu_0}$, where $\pi : M \to V_\eta$ is as in the hypothesis of our theorem. In other words

\[
\begin{array}{ccc}
[M^{(\Pi^1_{p_0}, B^0)}_{\nu_0}; \nu_0] & \overset{\hat{\pi}}{\longrightarrow} & [M^{(\Pi^0_{p_1}, B^0)}_{\nu_0}; \nu_0] \\
\uparrow \hat{i}^{0, \nu_0} & & \downarrow \sigma_0 \\
M & \longrightarrow & V_\kappa \\
\pi \\
\end{array}
\]

commutes, where $\hat{\pi}$ is the restriction of $\pi_{\nu_0}^0$ to $[M^{(\Pi^1_{p_0}, B^0)}_{\nu_0}; \nu_0]$.

This concludes the definition of $U$.

**Remarks.** The definition above has several awkward features that might appear unduly arbitrary. The only reason we have chosen this particular definition of $U$, rather than more natural ones, is that it will greatly simplify the construction in the next section. The remarks that follow should help the reader to understand some of the motivations behind the definition above.

1. The definition of $U$ involves $M, \Sigma, \pi, \eta$ and $\kappa$ as parameters. We will relativize $U$ to several models, all of which contain $HC$, hence only $\pi, \eta, \kappa$ will have to be changed.

2. Every model of the pseudo-iteration tree $((\Pi^n p_n, B^n))$ is of size $2^{R_0}$, hence belongs to every $C_\alpha$.

3. For $0 \leq n \leq m$ the elementary embedding $\Pi^{n,m} : (\Pi^n p_{m+1}, B^n) \to (\Pi^m p_{m+1}, B^m)$ mentioned in clause (6) is defined by induction: $\Pi^{n,n}$ is the identity embedding, $\Pi^{n,m+1} = \Pi^{n,m+1} \circ \Pi^{n,m}$.
4. Clauses (4.d) and (4.e) can be stated more concisely as
\[ \Pi^n = \Pi^{n-1} \circ \Pi^{n-1} \quad \text{and} \quad *\Phi^n = *\Psi^n \circ \Pi^n \]

(4.e) implies that for all \( q \supseteq p_n \), if \( q \) can be copied on \(*\mathcal{C}^n \) via \(*\Phi^n \) then \( q \) can be copied on \( \mathcal{B}^n \) via \( \Pi^n \). Using this and (4.d), every \( p_m \) can be copied on any \( (\mathcal{B}^k, \Pi^k) \).

5. Clause (4.f) says that \( (\mathcal{B}^n, \Pi^n) \) and \( (*\mathcal{C}^n, *\Phi^n) \) agree on the value of \( N(p_n) \). To be more specific: \( N(p_n) = \) the least \( k \in I_{p_n} \) such that there is a \( q \supseteq p_n \) according to \( \Sigma \), such that \( k \notin I_q \) and \( q \) can be copied on \( (\mathcal{B}^n, \Pi^n) \). Clause (3) and (4.f) also imply that \( N(p_{n+1}) > N(p_n) \) and \( I_{p_{n+1}} \subset I_{p_n} \).

6. The reason why the \( \sigma_n \)'s are defined on a chunks will become evident in the proof ofLemma 7.3 in the next section. The idea is that the \( \sigma_n \)'s will be obtained from the copying construction, and Lemma 6.7 will be used.

7. A few words on the commutative diagrams of clause (6). The embedding
\[ i_{\nu_{n-1},\nu_n}(\Pi^n \mathcal{B}^n) : M_{\nu_{n-1}}(\Pi^n \mathcal{B}^n) \rightarrow M_{\nu_n}(\Pi^n \mathcal{B}^n) \]

is well defined as \( \nu_{n-1} \) precedes \( \nu_n \) in the tree ordering of \( (\Pi^n \mathcal{B}^n) \) and, by Definition 6.7, it maps \( [M_{\nu_{n-1}}(\Pi^n \mathcal{B}^n); \nu_0, \ldots, \nu_n] \) elementarily into \([M_{\nu_n}(\Pi^n \mathcal{B}^n); \nu_0, \ldots, \nu_n] \), and

\[ [M_{\nu_{n-1}}(\Pi^n \mathcal{B}^n); \nu_0, \ldots, \nu_{n-1}] < [M_{\nu_n}(\Pi^n \mathcal{B}^n); \nu_0, \ldots, \nu_n]. \]

Thus \( i \) is well defined. Similarly, as

\[ [M_{\nu_{n-1}}(\Pi^n \mathcal{B}^n); \nu_0, \ldots, \nu_{n-1}] < [M_{\nu_n}(\Pi^n \mathcal{B}^n); \nu_0, \ldots, \nu_{n-1}], \]

\( \pi \) is well defined. Regarding the second commutative square, notice that 0 belongs to any support hence \( M = M_0^{\pi} = [M_0^{\pi}; \nu_0] \).

**Lemma 6.1:** \( \mathcal{U} \) is wellfounded.

**Proof:** Deny. A branch of \( \mathcal{U} \) is, essentially, a sequence
\[
\langle (p_n, \mathcal{B}^n, \Pi^n, \Pi_n, \Pi^{n+1}, \mathcal{C}^n, \Phi^n, \Psi^n, \nu_n, \sigma_n) \mid n < \omega \rangle
\]

Then \( \mathcal{T} = \cup_{n<\omega} p_n \) is a countable iteration tree according to \( \Sigma \), \( \theta = \sup \{ \theta_n + 1 \mid n < \omega \} = \text{lh}(\mathcal{T}) \) and \( b = \{ \beta < \theta \mid \exists n (\beta <_T \nu_n) \} \) a cofinal branch of \( \mathcal{T} \).

**Claim 6.2:** \( b \) is wellfounded branch of \( \mathcal{T} \) and letting \( \mathcal{T}^+ \) be the extension of \( \mathcal{T} \) via \( b \), \( \mathcal{T}^+ \) can be copied on \( (\mathcal{B}^n, \Pi^n) \), for all \( n \).
Proof: Fix \(0 \leq n < \omega\). By remark (4) \(\mathcal{T}\) can be copied on \((\mathcal{B}^n, \Pi^n)\). Also for \(m \geq n\), the ordinals \(\nu_m\) are linearly ordered in the tree ordering of \(\Pi^n\mathcal{T}\) and determine a cofinal branch of \(\Pi^n\mathcal{T}\). By a minor abuse of notation, such a branch will still be denoted by \(b\). The direct limit of \((\Pi^n\mathcal{T}, \mathcal{B}^n)\) along \(b\) will be shown to be wellfounded, proving thus the claim. By clause (6), for \(m \geq n\), the diagram

\[
\begin{array}{c}
\text{commutes, where } \tau_m = \sigma_m \circ \pi_{\nu_m}^{n,m}, \text{ and } \tau_\infty \text{ is the limit map: as every element } x \in M_b \text{ is of the form } i_{\nu_m,b}(y), \text{ pick } k \geq m \text{ large enough so that } i_{\nu_m,\nu_k}(y) \in [M_{\nu_k}(\Pi^m p_{k+1}, \mathcal{B}^m); \nu_0, \ldots, \nu_k], \text{ and set } \tau_\infty(x) = \tau_k(i_{\nu_m,\nu_k}(y)).
\end{array}
\]

Note that the game cannot be over once all the \(p_n\)'s have been played, i.e. \(\bigcap_n I_{p_n} \neq \emptyset\), as otherwise \(\Pi\) would win, \(M_b\) being wellfounded, thus contradicting the assumption that \(\Sigma\) is a winning strategy for \(I\).

Thus extending \(\mathcal{T}\) via \(b\) yields a legal move of \(\mathcal{G}^+(M)\), call it \(p\), that, by the claim, can be copied on any \(\mathcal{B}^n\). Let \(m = c(p)\) and let \(0 \leq i \leq m + 1\) be least such that \(N(p_i) > m\). As \(m \in \bigcap_n I_{p_n}\), then in particular \(m \in I_{p_i}\), hence \(N(p_i) > m \geq \min I_{p_i}\). Thus, by (3) in the definition of \(\mathcal{U}\), no extension \(q\) of \(p_i\) according to \(\Sigma\) with \(m \notin I_q\) could have been copied onto \((\mathcal{C}^i, \mathcal{D}^i)\), hence on \((\mathcal{B}^i, \Pi^i)\). In particular this should hold of \(p\), but the Claim shows that \(p\) can be copied on \(\mathcal{B}^i\): a contradiction.
7 The enlargement

From andretta@math.ucla.edu Mon Oct 4 18:53:13 1993 Received: from julia.math.ucla.edu by pianeta (4.1/SMI-4.1) id AA15464; Mon, 4 Oct 93 18:52:37 +0100 Received: from sonia.math.ucla.edu by julia.math.ucla.edu (Sendmail 4.1/1.07) id AA09092; Mon, 4 Oct 93 10:56:22 PDT Return-Path: jandretta@math.ucla.edu; Received: by sonia.math.ucla.edu (Sendmail 4.1/1.07) id AA07569; Mon, 4 Oct 93 10:56:21 PDT Date: Mon, 4 Oct 93 10:56:21 PDT From: Alessandro Andretta andretta@math.ucla.edu; Message-Id: ¡9310041756.AA07569@sonia.math.ucla.edu¿
To: andretta@di.unito.it Subject: enlargement.tex Status: RO

Let $r \in U$ be a node: $r^-$ is the finite sequence obtained from $r$ by dropping the $\alpha$’s and the $\sigma$’s and let $U^- = \{r^- \mid r \in U\}$. $U^-$ is still a tree on $V_k$, but it is not wellfounded. In fact we will construct a branch through it. Note that $U^-$ can also be defined using clauses (1)—(4), without mentioning $U$ at all. Obviously, different $r$’s may yield the same $r^-$, so there is no way to retrieve e.g. the $\nu$’s from $r^-$. Yet, for what we are going to do, we would like to be able to do this. To be more specific. Suppose $r \in U^-$ is of length $n + 1$ and that $p_1 \subset \ldots \subset p_{n+1}$ are its positions, and that $ht(p_n) \leq \alpha < ht(p_{n+1})$: can we find $\nu_0 < \ldots < \nu_n = \alpha$ so that for some $\sigma_0, \ldots, \sigma_n$, $(r, \nu, \sigma) \in U^-$? The answer is no, as $\alpha$ might not have $n + 1$ predecessors in the $p_{n+1}$ tree ordering. In fact $<_T \text{-pred}(\alpha)$ could be 0. The next two definition address to this problem.

**Definition 7.1:** Suppose we are given positions $p_1 \subset \ldots \subset p_{n+1}$, with $\theta_i + 1 = ht(p_i)$. For every $\alpha$ such that $\theta_n < \alpha < \theta_{n+1}$, if $n > 0$, or $0 \leq \alpha < \theta_1$, if $n = 0$, the backward sequence of $\alpha$ relative to $p_1, \ldots, p_{n+1}$, is the sequence $\langle (\alpha_0, m_1), \ldots, (\alpha_k, m_{k+1}) \rangle$ defined as follows.

- If $n = 0$, then $\langle (\alpha_0, m_1) \rangle = \langle (\alpha, 1) \rangle$ is the backward sequence of $\alpha$ relative to $p_1$.
- Suppose $n > 0$, then $k > 0$. Let $<_T$ be the tree ordering associated to $p_{n+1}$.

1. $0 \leq \alpha_0 < \ldots < \alpha_k = \alpha$.
2. $1 = m_1 < \ldots < m_{k+1} = n + 1$, (hence $k \leq n$).
3. $\alpha_{k-1}$ is the largest $\beta <_T \alpha$ such that $\beta < \theta_m$ for some $m < n + 1$. The least such $m$ is $m_k$.
4. $\langle (\alpha_0, m_1), \ldots, (\alpha_{k-1}, m_k) \rangle$ is the backward sequence of $\alpha_{k-1}$ relative to $p_1, \ldots, p_{m_k}$.

**Remarks.**

1. Note that a largest $\beta$ as in (3) always exists as the iteration trees $p_i$ have successor length $\theta_i + 1$.
2. For $0 < i \leq k$, $\alpha_i \in (\theta_{m_i+1})_{-1, \theta_{m_i+1}]}$.
3. If $\bar{p}_i = p_{m_i}$, the backward sequence of $\alpha$ relative to $\bar{p}_1, \ldots, \bar{p}_{k+1}$ is $\langle (\alpha_0, 1), \ldots, (\alpha_k, k+1) \rangle$.

Notice that if $n > 0$, then $k > 0$. 

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Lemma 7.1: Let \( r \in U^- \) be a node of length \( n + 1 \) and let \( p_1 \subset \ldots \subset p_{n+1} \) be its positions. Given \( \theta_n \leq \alpha < \theta_{n+1} \), let \( \langle (\alpha_0, m_1), \ldots, (\alpha_k, m_{k+1}) \rangle \) be the backward sequence of \( \alpha \) relative to \( (p_1, \ldots, p_{n+1}) \).

The contraction of \( r \) relative to \( \alpha \) is the node \( t \in U^- \) of length \( k + 1 \) obtained by rearranging some of the stuff contained in \( r \) in \( k + 1 \) pieces:

\[
t = \langle (p_{m_1}, B^0, \Pi^0), \ldots, (p_{m_k}, B^{m_k}, \Pi^{m_k}, C^{m_k}, H^{m_k}, \Phi^{m_k}, \Psi^{m_k}) \rangle = \langle (\bar{p}_1, \bar{B}^0, \bar{\Pi}^0), \ldots, (\bar{p}_{k+1}, \bar{B}^k, \bar{\Pi}^{k}, C^k, H^k, \Phi^k, \Psi^k) \rangle
\]

where we set \( m_0 = 0 \) so that \( \Pi^{m_0} m_1 = \Pi^{0,m_1} \).

Remarks.

1. In rearranging \( r \) into \( t \), a few of the \( B \)'s, \( C \)'s, \( \varphi \)'s, \( \pi \)'s and \( \psi \)'s may be lost if \( m_k < n \), but all the information coded by the \( p \)'s is preserved, as \( m_{k+1} = n + 1 \).

2. Suppose that for any \( m \leq n \) and any \( \theta_m < \alpha \leq \theta_{m+1} \) there is \( \theta_{m-1} < \beta \leq \theta_m \) such that \( \beta \leq \alpha \). That amounts to say that in extending \( p_m \) to \( p_{m+1} \), we never visit any model in the iteration trees \( p_l \), for \( l < m \). Then for \( \theta_n < \alpha \leq \theta_{n+1} \) the backward sequence of \( \alpha \) relative to \( p_1, \ldots, p_{n+1} \) is of length \( n + 1 \) and the contraction of \( r \) relative to \( \alpha \) is \( r \) itself. Unfortunately, we cannot assume this holds in general, and this is why we had to introduce this further complication. And, after all, life was not meant to be easy.

Before we move on, we still must verify that

Lemma 7.1: \( t \in U^- \)

Proof: By induction on \( k + 1 = \) the length of \( t \) = the length of the backward sequence of \( \alpha \).

Assume \( k = 0 \). Then \( t = \langle (p_{n+1}, B^0, \Pi^0) \rangle = \langle (\bar{p}_1, \bar{B}^0, \bar{\Pi}^0) \rangle \). By remark (4), \( \bar{p}_1 = p_{n+1} \) can be copied on \( (\bar{B}^0, \bar{\Pi}^0) = (B^0, \Pi^0) \), so clause (3) holds. The other clauses are immediate or vacuous.

So we may assume \( k > 0 \). The embedding \( \varphi_k : M_{\bar{\theta}_k}^\bar{p}_k = M_{\theta_{m_k}}^{p_{m_k}} \rightarrow V_\eta \) can be taken to be \( \varphi_{m_k} \), so clause (2.c) holds. As

\[
m_k + 1 \leq m_{k+1} \implies I_{p_{m_k+1}} \supseteq I_{p_{m_k+1}} \implies I_{\bar{p}_{k+1}}
\]

and \( N(\bar{p}_k) = N(p_{m_k}) \not\subseteq I_{p_{m_k+1}} \), then \( N(\bar{p}_k) \not\subseteq I_{\bar{p}_{k+1}} \). For any \( q \supseteq p_{m_k} \), \( q \) can be be copied on \( (\varphi^{\prime m_k}, \Phi^{\prime m_k}) = (C^k, \Phi^k) \) if and only if \( q \) can be copied on \( (B^{m_k}, \Pi^{m_k}) = (\bar{B}^k, \bar{\Pi}^k) \), so by clause (4.f) and remark (4), and taking \( q = \bar{p}_{k+1} = p_{m_{k+1}} \), clauses (3) and (4.f) hold. The other clauses are left to the reader.

A branch \( b \) through \( U^- \)

\[
b = \langle (p_1, B^0, \Pi^0), \ldots, (p_{n+1}, B^n, \Pi^{n}, C^n, H^n, \Phi^n, \Psi^n), \ldots \rangle
\]
will be constructed inductively together with sequences

\[ \langle (P_\alpha, \kappa_\alpha, r_\alpha) \mid \alpha < \theta \rangle \]
\[ \langle (P^*_n, \eta^*_n, \kappa^*_n, \varphi^*_n, \psi^*_n, r^*_n) \mid n < \omega \rangle \]

where \( \theta = \sup \theta_n = \text{ht}(T), T = \bigcup_n p_n \). Let \( \rho_\alpha = \rho_\alpha^T \).

**The conditions.** The following 9 conditions must hold for every \( 0 \leq n < \omega \).

1. (a) \( (P^n, \Phi^n) = \langle (P_\alpha, \varphi_\alpha) \mid \alpha < \theta_n \rangle \) is an enlargement for \( p_n \) with bounds \( \langle \eta_\alpha \mid \alpha < \theta_n \rangle \) is \( \langle \eta^*_n \rangle \).
   (b) \( (P^n, \Psi^n) = \langle (P_\alpha, \psi^*_\alpha) \mid \alpha < \theta_n \rangle \) is an enlargement for \( (\Pi^n_{p_n}, \mathcal{B}^n) \) with bounds \( \langle \kappa_\alpha \mid \alpha < \theta_n \rangle \) is \( \langle \kappa^*_n \rangle \).
2. \( \langle (P_\alpha, \varphi_\alpha) \mid \alpha < \theta_n \rangle \in P^*_n \) and for every \( \alpha < \theta_n \)
   \[ P^*_n \models P_\alpha \] is \( 2^{\aleph_0} \)-closed and of size \( |V^*_{\varphi_\alpha(p_\alpha)+1}| \)
   and
   \[ \varphi^*_n \upharpoonright V_{\rho_{\alpha}+1} = \varphi_{\theta_n} \upharpoonright V_{\rho_{\alpha}+1}. \]
3. \( p_{n+1} \) can be copied on \( (P^n, \Phi^n) \).
4. Let \( M(p_n) \in I_{p_n} \) be the least \( k \) such that \( \exists q \supset p_n \exists \beta(k = c(q \mid \beta), q \) is according to \( \Sigma \) and can be copied on \( (P^n, \Phi^n) \). Then \( M(p_n) \not\in I_{p_{n+1}} \).
5. (a) \( \delta(P_\alpha) = \eta_\alpha \) and \( P_\alpha \models "|V_{\eta_\alpha}| < \kappa_\alpha \) and \( \kappa_\alpha \) is a cut-off point.”
   (b) \( P_\alpha \cap V_{\kappa_\alpha} = C_\alpha \supset \text{H}(2^{\aleph_0})^+ \).  
6. For \( \theta_n \leq \alpha < \theta_{n+1} \), let \( U_\alpha \) be the tree \( U \) relativized to \( P_\alpha \), where the ordinals \( \eta, \kappa \) are interpreted as \( \eta_\alpha, \kappa_\alpha \) and \( \pi \) is replaced by \( \varphi_\alpha \circ \theta_{n+1} \). \( U_\alpha \) is wellfounded and \( \| \| \) denotes its rank function.

Then \( r_\alpha \) is a non-empty node of \( U_\alpha \) and

(a) the \( \nu \)'s of \( r_\alpha \) are \( \langle \nu_0, \ldots, \nu_k \rangle \), where \( \langle (\nu_0, m_1), \ldots, (\nu_k, m_{k+1}) \rangle \) is the backward sequence of \( \alpha \) relative to
   (i) \( p_1, \ldots, p_{n+1} \), if \( \alpha \neq \theta_n \) or \( n = 0 \);
   (ii) \( p_1, \ldots, p_n \), if \( \alpha = \theta_n \) and \( n > 0 \).
(b) \( (r_\alpha)^- \) is the contraction of
   (i) \( b \mid n + 1 \) relative to \( \alpha \), if \( \alpha \neq \theta_n \) or \( n = 0 \);
   (ii) \( b \mid n \) relative to \( \alpha \), if \( \alpha = \theta_n \) and \( n > 0 \).

In particular, \( r_0^- = \langle p_1, \mathcal{B}^0, \Pi^0 \rangle \).
Let $\zeta > \kappa$.

Base step.

Note that $\text{ran}(\sigma_k) \subseteq C_\alpha = P_\alpha \cap V_{\kappa\alpha}$, the relativization of $V_\kappa$ to $P_\alpha$.

(d) $P_\alpha$ has at least $|| r_\alpha || \cdot 2$ cut-off points above $\kappa_\alpha$.

7. (a) $\delta(P^*_n) = \eta^*_n$ and $P^*_n \models \langle V_{\eta^*_n} \rangle < \kappa^*_n$ and $\kappa^*_n$ is a cut-off point."

(b) $\varphi^*_n : M^{\eta^*_n}_{\theta_n} \to P^*_n \cap V_{\eta^*_n}$ and $\psi^*_n : M^{\Pi^n p_n}_{\theta_n} \to P^*_n \cap V_{\kappa^*_n}$ are elementary embeddings such that $\varphi^*_n = \pi^*_n \circ \psi^*_n$ and

$$P^*_n \models \langle (\mathcal{C}^n(\langle V_{\kappa^*_n} \rangle), \Phi^*(\langle \varphi^*_n \rangle) \rangle \rangle$$

and

$$P^*_n \models \langle (\mathcal{C}^n(\langle V_{\kappa^*_n} \rangle), \Psi^*\langle \psi^*_n \rangle) \rangle \rangle$$

is an enlargement of $(\Pi^n p_n, B^n)$.

(c) $P^*_n \cap V_{\kappa^*_n} \supset H(2^{\kappa^*_n} + 1)$. 

8. Let $U^*_n$ be the relativization of $U$ to $P^*_n$, with the ordinals $\eta$, $\kappa$, interpreted as $\eta^*_n$, $\kappa^*_n$, and $\pi$ interpreted as $\varphi^*_n \circ \psi^*_n$. $U^*_n$ is wellfounded and $|| \cdot ||$ denotes its rank function.

Then $r^*_n$ is a node of $U^*_n$ and

(a) The $\nu$’s of $r^*_n$ are $\langle \nu_0, \ldots, \nu_k \rangle$, where $\langle (\nu_0, m_1), \ldots, (\nu_k, m_{k+1}) \rangle$ is the backward sequence of $\theta_n$ relative to $p_1, \ldots, p_n$;

(b) $(r^*_n)^{-}$ is the contraction of $b \upharpoonright n$ relative to $\theta_n$. In particular $r^*_0 = \emptyset$.

(c) For $n > 0$, $\sigma_k = \psi^*_n \circ \pi^m_{\theta_n} \cap [M^{\Pi^n p_n, B^n}_{\theta_n}; \nu_0, \ldots, \nu_k]$, where $m = m_k$ and $n = m_{k+1}$.

Note that $\text{ran}(\sigma_k) \subseteq P^*_n \cap V_{\kappa^*_n}$, the interpretation of $V_\kappa$ in $P^*_n$.

(d) $P^*_n$ has at least $|| r^*_n || \cdot 2 + 1$ cut-off points above $\kappa^*_n$.

9. $P^*_{n+1} \in P^*_n$.

This contradiction will show that our assumption about $\Sigma$ being a winning strategy for $I$ in $\mathcal{G}^+(M)$ is false, hence the theorem will be proved.

• Base step.

Let $\zeta > \kappa$ be large enough so that $V_\zeta$ is a premouse with $\delta(V_\zeta) = \eta$ and such that there are $|| U || \cdot 2 + 1$ cut-off points above $\kappa = \kappa^*_0 > \eta^*_0 = \eta$. Let

$$P^*_0 = V_\zeta; \quad r^*_0 = \emptyset \quad \text{and} \quad \varphi^*_0 = \pi.$$

Now we have to define $B^0 = \langle B_0 \rangle$.

Case 1: $M(p_0) = \min I_{p_0}$.

Recall that $p_0 = \emptyset$ and $I_{p_0} = \omega$, so there is a position $q$ according to $\Sigma$ such that $0 = c(q \upharpoonright \beta)$ and $q$ can be copied on $(P^0, \Phi^0) = (V_\kappa, \pi)$. In this case, let $\psi^*_0 : B_0 \to V_\kappa$, such that $|B_0| = 2^{\kappa_0}$, and let $\psi^{-1}(\pi) = \pi_0$.

Case 2: $M(p_0) > \min I_{p_0}$.
For every integer \( k < M(p_0) \) and for every position \( q \) according to \( \Sigma \) such that \( c(q|\beta) = k \) for some \( \beta \leq lh(q) \), there are witnesses to the fact that such \( q \) cannot be copied on \((\mathcal{P}^0, *\Phi^0)\). That is, for any such \( q \), there is an ordinal \( \beta \) such that \( q|\beta \) can be copied on \((\mathcal{P}^0, *\Phi^0)\), but \( q|\beta + 1 \) cannot. Let \( S \) be \( q|\beta \) copied on \((\mathcal{P}^0, *\Phi^0)\).

If \( \beta \) is a limit ordinal, then fix an increasing sequence \( \beta_m \to \beta \) and ordinals \( \xi_m \in M_{\beta_m}^S \) such that \( i_{\beta_m, \beta_{m+1}}(\xi_m) > \xi_{m+1} \), and let \( w_m = (\xi_m, \beta_m) \). If \( \beta = \nu + 1 \) and \( \gamma = \prec_T \) -pred(\( \beta \)), then let \( w_m = (a_m, f_m) \) witness the illfoundedness of \( M_{\beta+1}^S \).

By absoluteness \( \langle w_m \mid m < \omega \rangle \) can be taken to be inside \( P_\nu^* = V_\xi \), and by Corollary 5.2, we can assume that \( \langle w_m \mid m < \omega \rangle \in V_\xi \). Repeating the argument for every position \( q \) as above, a set \( X \subseteq V_\xi \) of all such \( w_m(q) \)'s is obtained. \( |X| \leq 2^{\aleph_0} \), as there are at most \( 2^{\aleph_0} \) such \( q \)'s. Working inside \( V_\xi \), let

\[
H = \operatorname{Hull}_{V_\xi}(X \cup HC \cup \{ \eta, \pi \}).
\]

By construction \( |H| = 2^{\aleph_0} \). Let \( \psi_0^* : B_0 \to H \) be the inverse of the transitive collapse and let \( \pi, \eta \) be the images of \( \pi_0 \) and \( \varepsilon_0 \) via \( \psi_0^* \).

Thus in both cases \( \mathcal{B}^0, \Pi^0 \) and \( \psi_0^* \) are defined. Finally, choose a position \( p_1 \) according to \( \Sigma \) that can be copied on \( V_\xi \) via \( \pi \), and such that \( M(p_0) \notin I_{p_1} \).

- **Inductive step.**
  Let \( n \geq 0 \) and suppose we are given

\[
\begin{align*}
b|n + 1 & \in \mathcal{U}^- \\
\langle (P_\alpha, \eta_\alpha, \kappa_\alpha, \varphi_\alpha, r_\alpha) \mid \alpha < \theta_n \rangle \\
\langle \psi_\alpha^m \mid \alpha < \theta_n \rangle & \text{ and} \\
\langle (P_\alpha^m, \eta_\alpha^m, \kappa_\alpha^m, \varphi_\alpha^m, \psi_\alpha^m, r_\alpha^m) \mid m \leq n \rangle
\end{align*}
\]

satisfying conditions (1)—(9) above.

- **Construction of \( \mathcal{P}_n+1 \) and \( *\Phi^{n+1} \).**
  We will now build

\[
\langle (P_\alpha, \eta_\alpha, \kappa_\alpha, \varphi_\alpha, r_\alpha) \mid \theta_n \leq \alpha < \theta_{n+1} \rangle \quad \text{and} \quad (P_{n+1}^*, \eta_{n+1}^*, \kappa_{n+1}^*, \varphi_{n+1}^*, \psi_{n+1}^*, r_{n+1}^*).
\]

Let \( \mathcal{W} \) be the pseudo-iteration tree obtained by copying \( p_{n+1} \) on \((\mathcal{P}^n, *\Phi^n)\) and denote its \( \alpha \)th model by \( W_\alpha \). Let's also agree that for \( \beta \leq \theta_{n+1} \)

\[
G_\beta : M^\Pi_{\beta(n+1,B^n)} \to W_\beta, \quad f_\beta : M^{(p_{n+1}, M)}_{\beta} \to M^\Pi_{\beta(n+1,B^n)}
\]

are the copy map induced by \( *\Psi^n \) and \( \Pi^n \), respectively.
**Definition 7.3:** Let $\theta_n \leq \beta \leq \theta_{n+1}$ and let $\alpha \leq \theta_n$ be its root in $\mathcal{W}$. Let

$$
(\eta)^\beta, (\kappa)^\beta, (U)^\beta, t_\alpha = \begin{cases}
    i_{\alpha,\beta}(\eta_\alpha), i_{\alpha,\beta}(\kappa_\alpha), i_{\alpha,\beta}(U_\alpha), r_\alpha & \text{if } \alpha < \theta_n, \\
    i_{\alpha,\beta}(\eta^*_n), i_{\alpha,\beta}(\kappa^*_n), i_{\alpha,\beta}(U^*_n), r^*_n & \text{if } \alpha = \theta_n.
\end{cases}
$$

The node $s_\beta \in (U)^\beta$ is defined as follows.

1. If $n = 0$ and $\beta \geq \alpha = \theta_0 = 0$, then

$$
s_\beta = \langle (p_1, B_0^0, \Pi^0, \nu_0, \sigma_0) \rangle
$$

where $\nu_0 = \beta$ and $\sigma_0 = G_{\nu_0} \upharpoonright [M_{\nu_0}(\Pi^0_{p_1}, B^0); \nu_0]$ is the restriction of the copy map induced by $\psi_0 : B_0 \to V_\kappa$.

2. If $n > 0$ and $\alpha = \beta = \theta_n$, then $s_\beta = r^*_n$.

3. If $n > 0$ and $\beta > \theta_n$, then let $\langle (\nu_0, m_1), \ldots, (\nu_k, m_{k+1}), (\nu_{k+1}, m_{k+2}) \rangle$ be the backward sequence of $\beta$ relative to $p_1, \ldots, p_{n+1}$. Hence $\nu_k = \alpha < \theta_{m_{k+1}}$ and $\langle (\nu_0, m_1), \ldots, (\nu_k, m_{k+1}) \rangle$ is the backward sequence of $\alpha$ relative to $p_1, \ldots, p_{m_{k+1}}$. Then the $\nu$’s of $s_\beta$ are $\nu_0, \ldots, \nu_{k+1}$ and let

$$
\sigma_{k+1} = G_{\beta} \upharpoonright [M_{\beta}^{(\Pi^0_{n+1}, B^0_{m+1}); \nu_0, \ldots, \nu_{k+1}}]
$$

Let also

$$
s_\beta = i^{\mathcal{W}}_{\alpha,\beta}(t_\alpha)^- \langle (p_{n+1}, B_{m+1}^0, \Pi_{m+1}^0, \Pi_{m+1}^{m_{k+1}}, C_{m+1}^0, \Pi_{m+1}^{m_{k+1}}, \Phi_{m+1}^0, \Psi_{m+1}^0, \nu_{k+1}, \sigma_{k+1}) \rangle.
$$

**Lemma 7.2:** With $\alpha$ and $\beta$ as in the definition above, and $m = m_{k+1}$, $l = m_k$, if $k > 0$, or $l = 0$ otherwise

1. $t^-_\alpha \in W_\alpha \cap V_{\rho^l_{\alpha}^m + 1}$ and $i^{\mathcal{W}}_{\alpha,\beta}(t^-_\alpha) = t^-_\alpha$;

2. $s^-_\beta, b \upharpoonright n + 1 \in W_\beta \cap V_{\rho^l_{\alpha}^m + 1}$ and $s^-_\beta$ is the contraction of $b \upharpoonright n + 1$ relative to $\beta$.

**Proof:** If $n = 0$ then $t_\alpha = \emptyset$ and the Lemma is immediate. So we may assume $n > 0$, hence $k > 0$.

1. By the definition of backward sequence $\theta_l \leq \theta_{m-1} < \alpha \leq \theta_m$ and the critical point of $i = i^{\mathcal{W}}_{\alpha,\beta}$ is $\rho = \rho^l_{\theta_l}$, so it is enough to show that $t^-_\alpha \in W_\alpha \cap V_{\rho + 1}$. For any $j$, $p_j, B^j, \Pi^j, \nu_j \in H_{(2^k)^+}$, hence their rank is certainly less than $\rho$, so the only possible source of problems are the $H^2, H^j, \Phi^j$ and $C^j$, for $j \leq l$. Clause (2) in the definition of $\mathcal{U}$, implies that the rank of $C^j = \langle C_\alpha | \alpha < \theta_l \rangle$ is $\leq \rho$, so $t^-_\alpha \in W_{\theta_l} \cap V_{\rho + 1}$. But $W_{\theta_l}$ agrees with $W_\alpha$ through $\rho + 2$, so the result follows at once.

2. Of part (1) $s^-_\beta$ extends $t^-_\alpha$, so $(s^-_\beta)n \in W_\alpha \cap V_{\rho^l_{\alpha}^m}$. By an argument as the one in part (1) we can conclude that $(p_{n+1}, B^n, \Pi^n, \Pi_{m+1}^l, C^n, \Pi_{m+1}^l, \Phi^m, \Psi^m) \in W_\beta \cap V_{\rho^l_{\alpha}^m + 1}$. By the
agreement between $W_\alpha$ and $W_\beta$, $s_\beta^- \in W_\beta \cap V_{\vartheta_{\beta n}^m}$. Note that in the course of the proof we also managed to prove that $s_\beta^-$ is the contraction of $b|n+1$, and that $b|n+1 \in W_\beta \cap V_{\vartheta_{\beta n}^m}$.

\begin{lemma}
\begin{align*}
\forall \beta \geq \vartheta_n s_\beta^- \in (U)^\beta.
\end{align*}
\end{lemma}

\textbf{Proof:} We will use the same notation as in the proof of the previous lemma. If $n = 0$ or if $\alpha = \beta = \vartheta_n$ and $n > 0$, then the result follows easily, so we may assume $\beta > \vartheta_n$.

Let’s show first that $s_\beta^- \in (U)^\beta$. By part (1) of Lemma \ref{lem:main} and $t^\alpha_\beta \in (U^-)^\alpha$, it follows that $t^\alpha_\beta \in (U^-)^\beta$, so we only have to check clause (3) in the definition of $U$, relativized to $W_\beta$. By Corollary \ref{cor:main} and Remark (5), $M(p_m) = \leq k \in I_{\vartheta_m}$ such that there is $q \supset p_m$ according to $\Sigma$, and $k \notin I_q$ and $q$ can be copied on $(B^n, \Pi^n)$. Moreover $p_{n+1}$ is such a $q$. As $P_{n+1}, B^n, \Pi^n \in H(2^{\vartheta_n} \vee) \in W_\beta$, by Remark (5) relativized to $W_\beta$, we have that

\begin{align*}
W_\beta \models M(p_m) = N(p_m) \text{ and } s_\beta^- \in (U^-)^\beta.
\end{align*}

As $s_\beta$ extends $i_{\alpha, \beta}(t_\alpha) \in (U)^\beta$ we only have to care of $\beta = \nu_{k+1}$ and $\sigma_{k+1}$. Recall that $\langle (\nu_0, m_1), \ldots, (\nu_k, m_{k+1}) \rangle$ is the backward sequence of $\alpha = \nu_k$, $m = m_{k+1}$, $l = m_k$, and that $\theta_l \leq \theta_{m-1} < \alpha \leq \theta_m$. Let us verify clause (6) in the definition of $U$. Let $\sigma_k = \sigma_k(s_\beta)$ and $\tau = \sigma_k(t_\alpha)$. By definition of $s_\beta$, $\sigma_k = i_{\alpha, \beta}(\tau)$. By condition (6.c) or (8.c)

\begin{align*}
\tau = \psi \circ \pi_{l,m}^j \models [M^\Pi_{\vartheta_m} ; \nu_0, \ldots, \nu_k],
\end{align*}

where $\psi = \psi^s_\alpha$, if $\alpha = \vartheta_n$, or $\psi = \psi^m_\alpha$, otherwise. For $\gamma \leq \theta_{n+1}$, let $G_\gamma : M^\Pi_{\vartheta_{\varrho n+1}} \to W_\gamma$ be the copy map induced by the embedding $\ast \Psi^n : B^n \to \mathcal{P}^n$. Condition (1.d) implies that $\psi^m_\alpha = \psi^m_\alpha \circ \pi_{l,m}^\alpha$, when $\alpha < \vartheta_n$, thus the diagram

\begin{align*}
\begin{array}{cccccc}
[M^\Pi_{\vartheta_m} ; \nu_0, \ldots, \nu_k] & \xrightarrow{\tau} & W_\alpha \cap V_{(\alpha)} & \xrightarrow{i} & W_\beta \cap V_{(\alpha)} & \xrightarrow{G_\alpha = \psi^\alpha_\alpha} & M^\Pi_{\vartheta_{\varrho n+1}} \\
\pi_{l,n}^\alpha & & & & & & \\
M^\Pi_{\vartheta_{\varrho n+1}} & \xrightarrow{j} & M^\Pi_{\vartheta_{\varrho n+1}}
\end{array}
\end{align*}

commutes, where $i$ and $j$ are the embeddings of the pseudo-iteration trees $W$ and $\Pi^\alpha_{\vartheta_{\varrho n+1}}$, respectively. As $\sigma_k \in W_\alpha$, and $\text{dom}(\tau)$ is hereditarily countable, then $i \circ \tau = i(\tau)$. Thus letting $\sigma_{k+1} = \sigma_{k+1}(s_\beta)$ be $G_\beta$ restricted to the appropriate support, we have the commutative diagram

\begin{align*}
\begin{array}{cccccc}
[M^\Pi_{\vartheta_m} ; \nu_0, \ldots, \nu_k] & \xrightarrow{\sigma_k} & W_\beta \cap V_{(\alpha)} & \xrightarrow{\sigma_{k+1}} & M^\Pi_{\vartheta_{\varrho n+1}} + 1 \\
(\sigma_{k+1}) \circ \pi_{l,m}^\alpha & & & & & \\
M^\Pi_{\vartheta_{\varrho n+1}} + 1 & \xrightarrow{(\pi_{l,m}^\alpha) \circ \sigma_k} & M^\Pi_{\vartheta_{\varrho n+1}} + 1
\end{array}
\end{align*}
which is what we had to prove.

Thus $s_\beta$ is defined and belongs to $(U)^\beta$, for all $\theta_n \leq \beta \leq \theta_{n+1}$.

**Lemma 7.4:** For all $n \geq 0$,

1. $P_n^* = W_{\theta_n}$ has at least $||s_{\theta_n}|| \cdot 2 + 1$ cut-off points above $\kappa_n^*$. 

2. For $\theta_n < \beta \leq \theta_{n+1}$, $W_\beta$ has at least $||s_\beta|| \cdot 2 + 2$ cut-off points above $(\kappa)^\beta$.

**Proof:** Let $\alpha$, $\beta$ and $t_\alpha$ be as in Definition 7.3.

When $n = 0$ and $0 \leq \beta \leq \theta_n$, then $\alpha = 0$ and $t_\alpha = r_0 = \emptyset$. $V_\xi = P_0^* = W_{\theta_0}$ has at least $||r_0^*|| \cdot 2 + 1$ cut-off points above $\kappa = \kappa_0^* = (\kappa)^0$, hence $W_\beta$ has at least $i_{0,\beta}(||r_0^*|| \cdot 2 + 1)$ cut-off points above $(\kappa)^\beta$. As $s_\beta$ properly extends $i_{0,\beta}(r_0^*) = r_0^*$ for any $0 \leq \beta \leq \theta_1$, (1) and (2) follow at once.

Suppose now $n > 0$. Part (1) follows from condition (8.d) and the fact that $s_{\theta_n} = r_n^*$, so we may assume $\beta > \theta_n$. By condition (6.d), $W_\alpha$ has at least $||t_\alpha|| \cdot 2$ cut-off points above $(\kappa)^\alpha$, so $W_\beta$ has at least $i_{\alpha,\beta}(||t_\alpha|| \cdot 2) = ||i_{\alpha,\beta}(t_\alpha)|| \cdot 2$ cut-off points above $(\kappa)^\beta = i_{\alpha,\beta}((\kappa)^\alpha)$. As $s_\beta$ is a proper extension of $i_{\alpha,\beta}(t_\alpha)$, $||s_\beta|| \cdot 2 + 2 \leq ||i_{\alpha,\beta}(t_\alpha)|| \cdot 2$.

Let’s introduce one more piece of notation. For any $\theta_n < \beta \leq \theta_{n+1}$, let $q(s_\beta) = \sigma_k(s_\beta)$, where $k = k(\beta) = lh(s_\beta) - 1$. As

$$q(s_\beta) = G_\beta|[(M_\beta^{(\Pi^n p_{n+1}, \beta^\alpha)}; \nu_0, \ldots, \nu_k]$$

is the restriction to a chunk of the $\beta$th map of the embedding $^f\Psi : (\Pi^n p_{n+1}, B^n) \to (W, P^n)$, $q(s_\alpha)$ and $q(s_\beta)$ are compatible below $\rho_\alpha^W + 2$, for $\theta_n < \alpha \leq \beta \leq \theta_{n+1}$. (Recall that two functions $f$ and $g$ are compatible below an ordinal $\rho$ iff $f|V_\rho \cup g|V_\rho$ is still a function.)

Here is the plan of what comes next. First we construct $P_{\theta_n}$, $\rho_{\theta_n}$ and $\varphi_{\theta_n}$. Then fix $\beta$, $\theta_n < \beta < \theta_{n+1}$, and work inside $W_\beta$. Let $\xi$ be the $||s_\beta|| \cdot 2 + 1$st cut-off point above $(\kappa)^\beta$ and let $Q_\beta \supset V^W_{\rho_\beta^W}$ be the transitive collapse of a hull of $V_{\xi}$. By exercising proper care $Q_\beta$ can be taken to be $2^{\aleph_0}$-closed, of size $|V_{\rho_\beta^W+1}|$ and such that there are $||s_\beta|| \cdot 2$ cut-off points above $(\kappa)^\beta$, where $s_\beta$ and $(\kappa)^\beta$ are the collapses of $s_\beta$ and $(\kappa)^\beta$. We also want embeddings $q_\beta : M_\beta^{(\Pi^n p_{n+1})} \to Q_\beta \cap V_{(\kappa)^\beta}$ such that $q_\beta \supset q(s_\beta)$ and agree through $\pi_{\beta}(\rho_\beta) + 2$, for $\beta \leq \gamma < \theta_{n+1}$, $q_\beta$ and $q_\gamma$. As each $(Q_\beta, q_\beta, s_\beta)$ can be coded as an element of $V_{\rho_\beta^W+2} \cap W_\beta$, it belongs to $W_{\theta_{n+1}}$. Finally, working inside $W_{\theta_{n+1}}$, choose a $2^{\aleph_0}$-closed Skolem hull $H$ of $V_{\xi} \cap W_{\theta_{n+1}}$, where $\xi$ is the $|s_{\theta_{n+1}}| \cdot 2 + 2$nd cut-off point, so that $H$ contains all of the $Q_\beta$, $q_\beta$, $s_\beta$, etc. By letting $h : H \to P_{n+1}^*$ be the transitive collapse and $h(Q_\beta) = P_\beta$, $h(s_\beta) = r_\beta$ and $\varphi_\beta = h(q_\beta) \circ f_\beta$, the construction would be completed. Unfortunately we must be more ingenious than that as it is not clear that embeddings $q_\beta$ as above can be found inside $W_\beta$: the problem is that it is difficult to maintain the agreement between the
q’s past the first ω of them. And even if qβ’s as above were available, there is no guarantee that the sequence \(\langle (Q_β, q_β, s_β) \mid θ_n < β < θ_{n+1} \rangle\) belongs to \(W_{θ_{n+1}}\). In order to overcome these problems, a sequence of approximations

\[
\langle (Q_α^β, q_α^β, s_α^β) \mid θ_n < α ≤ β \rangle \in W_β
\]

will be built inductively, for \(θ_n ≤ β ≤ θ_{n+1}\).

We now construct \(P_{θ_n}, ϕ_{θ_n}\), and \(r_{θ_n}\). Working inside \(W_{θ_n} = P_n^*\) let \(ξ\) be the \(|s_θ_n|| \cdot 2 + 1\)st cut-off point above \((κ)_θ^α\), and let

\[
H_0 = \text{Hull}^V(\{V_{ρ_{θ_n}+1} \cup \{s_θ_n, ζ_θ_n, (κ)_θ^α, b|n\}\}),
\]

\[
H_γ = \text{Hull}^V(2^{σ_θ}(∪_{n<γ} H^γ)) \quad \text{for } γ ≤ (2^{κ_θ})^+
\]

and set \(H = H^{(2^{κ_θ})^+}\). It is easy to see that \(H\) is \(2^{κ_θ}\)-closed and of size \(|V_{ρ_{θ_n}+1}|\). (This is why plus-2 trees are used: had we taken hulls of size \(|V_{ρ_{θ_n}}|\) we could have run into problems with \(|H^γ|\)'s, if \(\text{cof}(|V_{ρ_{θ_n}}|) ≤ 2^ω\).) Let \(h : H → P_{θ_n}\) be the transitive collapse, let \(r_{θ_n} = h(s_θ_n)\), \(κ_{θ_n} = h(κ_θ^α)\), \(η_{θ_n} = h(η_θ^α)\). Also set \(g_{θ_n} = h(η_θ^α)\) and \(ϕ_{θ_n} = f_{θ_n} \circ g_{θ_n}\).

**Definition 7.4:** For \(θ_n ≤ β ≤ θ_{n+1}\), let \(R^β \in W_β\) be the set defined as follows.

\(\langle (Q_α^β, q_α^β, η_α^β, κ_α^β, s_α^β) \mid θ_n ≤ α ≤ β \rangle \in R^β\) if and only if:

(i) \(Q_α^β\) is a premouse, \(δ(Q_α^β) = η_α^β, κ_α^β\) is a cut-off point and \(Q_α^β \models |V_ρ^β| < κ_α^β\).

(ii) \(Q_α^β\) is a premouse, \(δ(Q_α^β) = η_α^β, κ_α^β\) is a cut-off point and \(Q_α^β \models |V_ρ^β| < κ_α^β\).

(iii) \(⟨(P^n|θ_n, Ψ^n)\rangle \prec \langle (Q_α^β, q_α^β) \mid θ_n ≤ α ≤ β \rangle\) is an enlargement for \((Π^n p_{n+1}|β + 1, B^n)\) with bounds \(⟨κ_α \mid α < θ_n⟩ ∪ ⟨κ_α^β \mid α ≤ β⟩\).

(iv) \(⟨(P^n|θ_n, B^n)⟩ \prec \langle (Q_α^β, q_α^β) \mid θ_n ≤ α ≤ β \rangle\) is an enlargement for \((p_{n+1}|β + 1, M)\) with bounds \(⟨η_α \mid α < θ_n⟩ ∪ ⟨η_α^β \mid θ_n ≤ α ≤ β⟩\).

(v) \(s_α^β ∈ (U)^α_β\), where \((U)^α_β\) is the relativization of \(U\) to \(Q_α^β\), with \(κ, η\) and \(π\) replaced by \(κ_α^β, η_α^β\) and \(q_α^β \circ f_α \circ i_{0, α}\); moreover for \(α > θ_n\), \((s_α^β)^-\) is the contraction of \(b|n + 1\) relative to \(α\) and \(p_1, ..., p_{n+1}\).

(vi) \(q_β^β|V_ρ^{β+2} ⊆ G_β\), the copy map, and \(q_α^β ⊇ q(s_α^β)\).

(vii) \(W_β \models “Q_α^β ⊇ V_ρ^{β+1}\) and is of size \(|V_ρ^{β+1}|\)”, where \(ρ = ρ_α^β = q_α^β(π(ρ_α))\).

(viii) \(Q_α^β\) has at least \(|s_α^β| \cdot 2 + 1\) cut-off points above \(κ_α^β\), and \(Q_θ^θ_{θ_n+1}\) has at least \(|s_θ_{θ_n+1}| \cdot 2 + 1\) cut-off points above \(κ_θ_{θ_n+1}\).
Lemma 7.5: For every $\beta$ with $\theta_n \leq \beta \leq \theta_{n+1}$, $R^\beta$ is non-empty. In fact, for any $\theta_n \leq \gamma < \beta$ and any sequence in $R^\gamma$ there is sequence in $R^\beta$ extending it.

Proof: By induction on $\beta$. Condition (vii) implies that every element of $R^\gamma$ can be coded as a subset of $V_{\rho^W_{\gamma+2}} \cap W_\gamma$, hence $R^\gamma \subseteq W_\beta$, for $\gamma \leq \beta$. If $\beta = \theta_n$ then (i) implies $R^\beta$ has one element only, namely $\langle (P_{\theta_n}, q_{\theta_n}, \eta_{\theta_n}, \kappa_{\theta_n}, r_{\theta_n}) \rangle$.

Assume the lemma holds for some $\beta > \theta_n$ and let’s prove it for $\beta + 1$. By the inductive hypothesis, it is enough to show that any $\langle (Q^\beta, q^\beta, \eta^\beta, \kappa^\beta, s^\beta) \mid \theta_n \leq \alpha \leq \beta \rangle \in R^\beta$ can be extended to a sequence in $R^{\beta+1}$. By compatibility of $G_\beta$ and $G_{\beta+1}$ below $\rho^W_\beta + 2$ and by (vi), $q^\beta \mid V_{\rho^W_{\beta+2}} \subseteq G_{\beta+1}$, and as $q(s_{\beta+1}) \subseteq G_{\beta+1}$, the maps $q^\beta \mid V_{\rho^W_{\beta+2}}$ and $q(s_{\beta+1})$ are compatible. First we must find an embedding $q \in W_\beta$ such that

$$q : M^{\Pi^m_n \rho_{\beta+1}}_{\beta+1} \rightarrow W_{\beta+1} \cap V_{(\kappa)\beta+1} \quad \text{and} \quad q \supseteq q^\beta \mid V_{\rho^W_{\beta+2}} \cup q(s_{\beta+1}).$$

Let $\langle S_m \mid m < \omega \rangle \in W_{\beta+1}$ be an increasing family of finite supports for $(\Pi^m_n, B^m)$ such that

$$\bigcup_m S_m = \theta_{n+1} + 1 \quad \text{and} \quad (M^{\Pi^m_n \rho_{\beta+1}, B^m})_{S_0} = \text{dom}(q(s_{\beta+1}))$$

and let $V$ be the tree, set of finite sequences closed under initial segment, whose nodes of length $m + 1$ are embeddings $q_0 \subseteq \ldots \subseteq q_m$

$$q_0 = q(s_{\beta+1}) \quad \text{and} \quad q_m : (M^{\Pi^m_n \rho_{\beta+1}, B^m})_{S_m} \rightarrow W_{\beta+1} \cap V_{(\kappa)\beta+1}$$

such that $q_m$ is compatible with $q^\beta_\beta$ below $V_{\rho^W_{\beta+2}} \cap V_{\beta+1}$ and it is ill-founded in $V$, hence there is a $q \in W_{\beta+1}$ as desired. We now proceed as in the construction of $P_{\theta_n}$. Working inside $W_{\beta+1}$ let $\xi$ be the $|s_{\beta+1}| \cdot 2 + i$th cut-off point above $(\kappa)\beta+1$, where $i = 1$ if $\beta + 1 < \theta_{n+1}$, or $i = 2$ if $\beta + 1 = \theta_{n+1}$. Let

$$H^0 = \text{Hull}V(\bigcup_{\nu < \mu} H^\nu \cap \{s_{\beta+1}, q, (\eta)\beta+1, (\kappa)\beta+1, B\beta+1\}),$$

$$H^\mu = \text{Hull}V(\bigcup_{\nu < \mu} H^\nu)$$

for $\mu \leq (2^n_0)^+$ and set $H = H^{(2^n_0)^+}$. Let $h : H \rightarrow Q_{\beta+1}^\beta$ be the transitive collapse, let $s_{\beta+1}^\beta = h(s_{\beta+1})$, $\kappa_{\beta+1}^\beta = h((\kappa)\beta+1)$, $\eta_{\beta+1}^\beta = h((\eta)\beta+1)$. Also set $q_{\beta+1}^\beta = h(q)$. It is easy to verify that

$$\langle (Q^\beta, q^\beta, \eta^\beta, \kappa^\beta, s^\beta) \mid \theta_n \leq \alpha \leq \beta \rangle \subseteq \langle (Q^\beta_{\beta+1}, q^\beta_{\beta+1}, \eta^\beta_{\beta+1}, \kappa^\beta_{\beta+1}, s^\beta_{\beta+1}) \rangle \in \mathcal{R}^{\beta+1}$$

and it extends $\langle (Q^\alpha, q^\alpha, \eta^\alpha, \kappa^\alpha, s^\alpha) \mid \theta_n \leq \alpha \leq \gamma \rangle$. The verification of (i) — (viii) is straightforward. As an example, let us check that each $Q^\gamma_{\beta+1}$ is $2^{(2^n_0)}$-closed inside $Q^\beta_{\beta+1}$. If $\alpha = \beta$, then, by inductive hypothesis $Q^\beta_{\beta+1} = Q^\beta_\beta$ is $2^{(2^n_0)}$-closed inside $W_\beta$. But any $2^{(2^n_0)}$-sequence of elements of $Q^\beta_{\beta+1}$ can be coded as an element of $V_{\rho^W_{\beta+2}} \cap W_{\beta} = V_{\rho^W_{\beta+2}} \cap W_{\beta+1}$, hence the
result follows. If $\alpha < \beta$, then $Q^\beta_\alpha = Q^\beta_{\alpha+1}$ is $2^{\aleph_0}$-closed inside $Q^\beta_\beta$, which is, as we just showed, $2^{\aleph_0}$-closed inside $Q^\beta_{\beta+1}$. Thus the lemma holds for $\beta + 1$.

Suppose now that $\beta > \theta_n$ is limit and that the result holds for $\gamma < \beta$. Fix a $\gamma$ with $\theta_n \leq \gamma < \beta$ and a sequence

$$\bar{Q} = \langle (Q^\gamma_\alpha, q^\gamma_\alpha, \eta^\gamma_\alpha, \kappa^\gamma_\alpha, s^\gamma_\alpha) \mid \theta_n \leq \alpha \leq \gamma \rangle \in R^\gamma.$$ 

As $R^\gamma \subseteq W_\beta$, this sequence belongs to $W_\beta$. Working inside $W_\beta$, choose a sequence $\beta_i \rightarrow \beta$ with $\gamma < \beta_i < \beta_{i+1}$ and let $\langle S_m \mid m < \omega \rangle \subseteq W_{\beta+1}$ be an increasing family of finite supports for $(\Pi^a p_{n+1}, B^n)$ such that

$$\bigcup_m S_m = \theta_{n+1} + 1 \quad \text{and} \quad (M^B_{\beta}(\Pi^a p_{n+1}, B^n))_{S_0} = \text{dom}(q(s_\beta)).$$

Let $V$ be the tree on $V^{\text{th}}_{\rho^W_\beta}$ searching for a sequence like

$$\langle (Q^\beta_\alpha, q^\beta_\alpha, \eta^\beta_\alpha, \kappa^\beta_\alpha, s^\beta_\alpha) \mid \theta_n \leq \alpha \leq \beta \rangle$$

such that

$$\bar{Q} \subseteq \langle (Q^\beta_\alpha, q^\beta_\alpha, \eta^\beta_\alpha, \kappa^\beta_\alpha, s^\beta_\alpha) \mid \theta_n \leq \alpha \leq \beta_i \rangle \in R^\beta_i,$$

together with embeddings $q_0 \subseteq \ldots \subseteq q_i$

$$q_i : (M^B_{\beta}(\Pi^a p_{n+1}))_{S_i} \rightarrow W_\beta \cap V(\kappa)^\beta$$

such that $q = \bigcup_i q_i \supseteq q(s_{\beta+1})$ and $q$ is compatible with $q^\beta_{\beta_i}$ below $\rho^W_\beta + 2$, for $i < \omega$. Using the inductive hypothesis we can choose

$$\bar{Q}' = \langle (Q^\beta_\alpha, q^\beta_\alpha, \eta^\beta_\alpha, \kappa^\beta_\alpha, s^\beta_\alpha) \mid \theta_n \leq \alpha \leq \beta_i \rangle \in R^\beta_i$$

such that $\bar{Q} \subseteq \bar{Q}' \subseteq \bar{Q}' + 1$. Thus

$$\langle (\bar{Q'}, G_{\beta i}(M^B_{\beta}(\Pi^a p_{n+1}, B^n))_{S_i}) \mid i < \omega \rangle$$

is an infinite branch of $V$. By absoluteness there is $Q^\omega \in W_\beta$ such that

$$\bar{Q} \subseteq Q^\omega = \langle (Q^\beta_\alpha, q^\beta_\alpha, \eta^\beta_\alpha, \kappa^\beta_\alpha, s^\beta_\alpha) \mid \theta_n \leq \alpha \leq \beta \rangle$$

and an embedding

$$q : M^B_{\beta}(\Pi^a p_{n+1}) \rightarrow W_\beta \cap V(\kappa)^\beta$$

compatible with $q^\beta_\alpha$ below $\rho^W_\beta + 2$ and such that $q \supseteq q(s_\beta)$. $Q^\omega$ fails to be an element of $R^\beta$ in that it has no $Q^\beta_\alpha, q^\beta_\alpha$, etc. We now proceed as before: working in $W_\beta$ we take $(2^{\aleph_0})^+$ many hulls of $V_\xi$, where $\xi$ is the $||s_\beta|| \cdot 2 + i$th cut-off point, $i = 1$ if $\beta < \theta_{n+1}$ or $i = 2$ if $\beta = \theta_{n+1}$. By collapsing we construct $(Q^\beta_{\beta_i}, q^\beta_{\beta_i}, \eta^\beta_{\beta_i}, \kappa^\beta_{\beta_i}, s^\beta_{\beta_i})$, so that $Q^\omega \sim \langle (Q^\beta_{\beta_i}, q^\beta_{\beta_i}, \eta^\beta_{\beta_i}, \kappa^\beta_{\beta_i}, s^\beta_{\beta_i}) \rangle \in R^\beta$ is the
desired sequence extending $Q$. This concludes the proof of the lemma.

We are now ready to define $P^{n+1}$ and $*\Phi^{n+1}$. Fix a sequence

$$
\langle (Q^\theta_{\alpha+1}, q^\theta_{\alpha+1}, \eta^\theta_{\alpha+1}, \kappa^\theta_{\alpha+1}, s^\theta_{\alpha+1}) \mid \theta_n \leq \alpha \leq \theta_{n+1} \rangle \in R^{\theta_{n+1}}
$$

and set

$$
P^*_{n+1} = Q^\theta_{n+1}, \quad g^* = q^\theta_{n+1}, \quad \varphi^*_{n+1} = g^* \circ f_{\theta_{n+1}}
$$

and

$$
\eta^*_{n+1} = \eta^\theta_{n+1}, \quad \kappa^*_{n+1} = \kappa^\theta_{n+1}, \quad r^*_{n+1} = s^\theta_{n+1}
$$

and for $\theta_n \leq \beta < \theta_{n+1}$

$$
P^s_{\beta} = Q^\theta_{\beta+1}, \quad g^s_{\beta} = q^\theta_{\beta+1}, \quad \varphi^s_{\beta} = g^s_{\beta} \circ f_{\beta}
$$

$$
\eta_s_{\beta} = \eta^\theta_{\beta+1}, \quad \kappa_s_{\beta} = \kappa^\theta_{\beta+1}, \quad r_s_{\beta} = s^\theta_{\beta+1}
$$

The verification of the conditions for $(P^{n+1}, *\Phi^n)$ is straightforward. As an example let us check (9). As $(W, P^n)$ is internal,

$$
P^*_{n+1} \in W_{\theta_{n+1}} = M^W_{\theta_{n+1}} \subseteq W_{\theta_n} = P^s_n
$$

so $P^*_{n+1} \in P^s_n$.

- **Construction of $p_{n+2}, B^{n+1}, \Pi^{n+1}, \Pi^{n,n+1}$ and $*\Psi^{n+1}$.**

The construction is very similar to what we did before. Pick $p_{n+2} \supset p_{n+1}$ according to $\Sigma$ such that $M(p_{n+1}) \notin I_{p_{n+2}}$ and $p_{n+2}$ can be copied on $(P^{n+1}, *\Phi^{n+1})$. Such a position must exist by Corollary 3.1. Thus clause (3) holds by fiat.

**Case 1:** $M(p_{n+1}) = \min I_{p_{n+1}}$.

Then set $B^{n+1} = B^n$, $\Pi^{n,n+1} = \Pi^n$ and let $*\Psi^{n+1} = \langle g^s_{\beta} \mid \beta < \theta_{n+1} \rangle \sim \langle g^s \rangle$.

**Case 2:** $M(p_{n+1}) > \min I_{p_{n+1}}$.

For every integer $k \in I_{p_{n+1}}$, $k < M(p_{n+1})$ and for every position $q$ according to $\Sigma$ such that $c(q \upharpoonright \beta) = k$ for some $\beta \leq lh(q)$, there are witnesses to the fact that such $q$ cannot be copied on $(P^{n+1}, *\Phi^{n+1})$. For any such $q$, there is an ordinal $\beta$ such that $q \upharpoonright \beta$ can be copied on $(P^{n+1}, *\Phi^{n+1})$, but $q \upharpoonright \beta + 1$ cannot. Let $S$ be the pseudo-iteration tree of height $\beta$ obtained by copying $q \upharpoonright \beta$ on $(P^{n+1}, *\Phi^{n+1})$.

If $\beta$ is a limit ordinal, then fix an increasing sequence $\beta_m \to \beta$ and ordinals $\xi_m \in M^{S}_{\beta_m}$ such that $i_{\beta_m, \beta_{m+1}}(\xi_m) > \xi_{m+1}$, and let $w_m = (\xi_m, \beta_m)$. If $\beta = \nu + 1$ and $\gamma = \text{-pred}(\beta)$, then let $w_m = (a_m, f_m)$ witness the illfoundedness of ult$(M^S_{\nu}, E_{\nu})$.

By absoluteness $\langle w_m \mid m < \omega \rangle$ can be taken to be inside $P^*_{n+1}$, and by Corollary 3.2 we can assume that $\langle w_m \mid m < \omega \rangle \in P^*_{n+1} \cap V_{\kappa_{n+1}}$. Repeating the argument for every position $q$ as above, a set $X \subseteq P^*_{n+1} \cap V_{\kappa_{n+1}}$ of all such $w_m(q)$ is obtained. $|X| \leq 2^{8_0}$, as there are at most $2^{8_0}$ such $q$’s.
Working inside \( P_{n+1}^* \), let
\[
H = \text{Hull}^{V_{\kappa+1}}(X \cup HC \cup \text{ran}(g^*) \cup \{C^{n+1}, H^{n+1}, \Phi^{n+1}, \langle g^*_\beta | \beta \leq \theta_{n+1} \rangle \}).
\]
By construction \(|H| = 2^{\aleph_0}\), and let \( h : H \rightarrow B_{\theta_{n+1}}^{n+1} \) be the transitive collapse. For \( \theta_n < \alpha \leq \theta_{n+1} \), let
\[
B_\alpha = h(C_\alpha) \quad \varepsilon_\alpha = h(\eta_\alpha) \quad \varepsilon_{\theta_{n+1}} = h(\eta_{n+1}^*) \quad \psi^{n+1}_\alpha = h^{-1} \upharpoonright B_\alpha
\]
\[
\pi^{n+1}_\alpha = h(g_\alpha) \quad \pi_{\theta_{n+1}}^{n+1} = h \circ g^* \quad \pi_\alpha = h(\varphi_\alpha) \quad \pi_{\theta_{n+1}} = h(\varphi^*_{n+1})
\]
This completes the construction of \( B^{n+1}, \Pi^{n+1}, \Pi^{n,n+1} \) and \( *\Psi^{n+1} = \langle \psi^{n+1}_\alpha | \alpha < \theta_{n+1} \rangle \setminus \langle h^{-1} \rangle \), hence Theorem 5.1 is proved.

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