All WZW models in $D \leq 5$

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Abstract

We present here all the real algebras $\mathcal{A}$ with $\dim \mathcal{A} \leq 5$ and all 6-dimensional nilpotent ones with symmetric, invariant and non-degenerate metrics for which a WZW model can be constructed. In three and four dimensions there are no other algebras than the well known $SU(2)$, $SU(1,1)$, $E_6^c$ and $H_4$. There exist only one five-dimensional and one six-dimensional nilpotent algebra with invariant non-degenerate metric and central charge $c = 5, 6$, respectively. We examine in details the five-dimensional case and, by gauging an appropriate subgroup, four-dimensional plane-wave string backgrounds are obtained. The corresponding background for the six-dimensional case is flat.

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1 Introduction

An important problem in string theory is to find exact string backgrounds. These will determine the short distance structure of space-time and will provide informations about string-generated gravitational interactions. One may distinguishes roughly the following classes of exact solutions, flat space with linear dilaton [1], plane wave [2] and N = 4 supersymmetric string backgrounds [3], WZW models [4] and gauged WZW models [5, 6]. We will examine here solutions in the last two classes writing down all WZW models based on groups with dimension up to five as well as on six-dimensional nilpotent groups.

Conformal field theories can be constructed, in this or in the other way, using current algebras. The simplest models are the WZW models based on a group G where the full symmetry of the action is realized in terms of current algebras [4]. Gauging appropriate (anomaly free) subgroups we obtain new conformal field theories, the coset models [7]. WZW models based on compact groups have been employed in string compactification while those based on non-compact ones may regarded as curved backgrounds [8]. Recently, attention has been given in models based on non-semisimple groups [9]–[19]. The non-invertibility of the Killing form makes impossible in this case to construct the corresponding WZW models in the standard fashion. However, there exist groups which, although non-semisimple, they have an invariant, symmetric and non-degenerate metric. In this case, an affine Sugawara construction can be carried out leading to integer central charge. Attempts to find non-semisimple affine Sugawara constructions with non-integer central charge [12] show that in this case the construction factorizes into a semisimple standard Sugawara construction and a non-semisimple one with integer central charge.

The first of these WZW models was based on the group $E_2^c$, a central extension of the 2-dim Euclidean group and the corresponding $\sigma$-model describes string propagation on a four-dimensional gravitational plane wave background [1]. This construction was subsequently extended to other non-semisimple groups [10]–[13], [14], the representations of the affine $E_2^c$ were constructed in terms of free fields and various gaugings have been considered [14, 15, 16]. The basic feature of all these models is that they describe string propagation in backgrounds which admit one or more null Killing vectors. They are geodesically complete (free of singularities) [17] and, furthermore, it was shown that any gravitational wave can be extended to an exact string background [18] where the underlying conformal field theory remains to be discovered [19].

Here we examine algebras with invariant symmetric and non-degenerate metric. Since all real algebras with dimension up to five and nilpotent six dimensional ones are known, it is straightforward to find those algebras for which such a metric exists. We found that the first non-trivial case is the well-known three-dimensional SU(2) and SU(1,1) and in four dimensions the centrally extended Euclidean and the Heisenberg algebra. Finally, there exists only one five-dimensional
and one six-dimensional nilpotent algebra for which a WZW model can be constructed. We examine in details the five-dimensional case and, by gauging an appropriate subgroup, a four-dimensional string background is obtained. The corresponding background for the six-dimensional case is flat.

2 Invariants of Lie algebras

A central problem in representation theory of Lie groups and Lie algebras is the determination of their invariants, i.e., functions of the generators which commute with all generators. From the mathematical point of view, their importance follows from the fact that they label representations, split reducible representations into irreducible ones, produce all special functions as eigenfunctions of the corresponding invariant operators etc.. On the other hand, from the physical point of view, the invariants of the symmetry group of a physical system provide quantum numbers, mass formulae, energy spectra and so on. Let us stress, however, that the invariants of a Lie group are not necessarily Casimir operators (polynomials of the generators). There are groups with polynomial invariants which give rise to Casimir operators, as well as groups with rational invariants (ratio of polynomials) or even groups with no invariants at all [20, 21].

The invariants of semisimple groups are known and their number equals the rank of the group. For non-semisimple groups, there exists a systematic way to compute their invariants. The most well studied group in this category is the Poincaré group the invariants of which provide the particle states with mass and spin.

An invariant of fundamental importance is the quadratic Casimir. It is quadratic in the generators $J_i$, $(i = 1, 2, ..., \text{dim}G)$ of the group $G$ and can collectively be written as

$$C^{(2)} = \Omega^{ij} J_i J_j ,$$

where $\Omega^{ij}$ may be viewed as elements of a symmetric and possibly degenerate matrix. The generators $J_i$ satisfy

$$[J_i, J_j] = f_{ij}^k J_k ,$$

where $f_{ij}^k$ are the structure constants. From the invariance of $C^{(2)}$ it follows that $\Omega^{ij}$ must obey

$$\Omega^{ij} f_{ki}^\ell + \Omega^{i\ell} f_{k\ell}^i = 0 .$$

If we consider $\Omega^{ij}$ as a symmetric matrix and if its determinant is not zero, one can form the inverse matrix $\Omega_{ij}$ which is also symmetric, non-degenerate and invariant under the adjoint action of the group

$$\Omega_{ij} f_{k\ell}^n + \Omega_{i\ell} f_{kj}^n = 0 .$$
One may then use $\Omega_{ij}$ as the metric on the group manifold and for semisimple groups $\Omega_{ij}$ is proportional to the Killing form $g_{ij} = f_{ik} f_{j\ell}$. Non-semisimple groups may also have quadratic Casimirs with non-degenerate $\Omega_{ij}$ although their Killing form is degenerate. We recall the Poincaré group in three dimensions which, apart from the usual mass$^2 = \vec{P} \cdot \vec{P}$ invariant, it has an additional one, the helicity defined as $\vec{J} \cdot \vec{P}$. As a result, the Poincaré group in three dimensions is a six-dimensional non-semisimple group with quadratic Casimir

$$C^{(2)} = k_1 \vec{P} \cdot \vec{P} + k_2 \vec{J} \cdot \vec{P}$$

(5)

where $k_1, k_2$ are constants. In this case, although the Killing form is degenerate the corresponding $\Omega_{ij}$ is not and can be considered as the bi-invariant metric on the group manifold. One may observe the existence of parameters in $C^{(2)}$. The reason is that there are two linearly independent quadratic Casimirs. Generally speaking, the number $\tau$ of independent Casimirs of an algebra $A$ satisfies $\tau \leq \text{dim} A - r(A)$ where $r(A)$ is the rank of the matrix $R_{ij} = [J_i, J_j]$.

The equality holds for semisimple and nilpotent groups which have only polynomial invariants and fails for an abstract algebra since there exist in general and non-polynomial invariants [22].

There are also exist other non-semisimple groups with a non-degenerate metric, as the centrally extended Euclidean group $E_d^c$ [9]–[11], the Heisenberg group [15] and so on. However, we do not know all the groups with bi-invariant metric $\Omega_{ij}$, although theorems for the structure of these groups have been proved [23]. On the other hand, all real Lie algebras with dimension less than five ($\text{dim} G \leq 5$) [24] and all nilpotent six-dimensional ones [25] are known. Thus, in this case it is straightforward to write down the algebras with non-degenerate metric. We will follow the list presented in [21] where the independent Casimirs are given as well.

a) 1-dimensional real Lie algebras

The one-dimensional algebras, denoted by $A_1$, are abelian and the quadratic Casimir is just the square of their generators.

b) 2-dimensional real Lie algebras

There exist two two-dimensional Lie groups, the $A_{2,1}$ and the $A_{2,2}$. The former is abelian $A_{2,1} = A_1 \oplus A_1$ with Casimir

$$C^{(2)} = J_1^2 + k_2 J_2^2.$$  

(6)

The latter $A_{2,2}$ is solvable and it is defined by the commutation relation

$$[J_1, J_2] = J_1.$$  

(7)
This group is the affine group in one dimension and it has no invariants at all.

\(\gamma\) 3-dimensional real Lie algebras

The three-dimensional Lie algebras have been classified by Bianchi and are known as Bianchi-type. There exist nine Bianchi-type algebras (one of which depends on a parameter and it is actually a continuous family) which together with the \(A_1 \oplus A_1 \oplus A_1\) and \(A_2,2 \oplus A_1\) consist the 11 different three-dimensional real Lie algebras. From these, only the Bianchi-type VII and IX (denoted by \(A_{3,8}, A_{3,9}\), respectively,) have quadratic Casimirs. These algebras are defined by the commutation relations

\[
\begin{align*}
[J_1, J_2] &= J_3, & [J_2, J_3] &= J_1, & [J_3, J_1] &= J_2, \\
[J_1, J_2] &= -2J_3, & [J_2, J_3] &= J_1, & [J_3, J_1] &= J_2,
\end{align*}
\]

and the corresponding quadratic Casimirs are

\[
\begin{align*}
C^{(2)} &= J_1^2 + J_2^2 + J_3^2, \\
C^{(2)} &= J_1J_3 + J_3J_1 + 2J_2^2.
\end{align*}
\]

\(A_{3,8}, A_{3,9}\) are semisimple and isomorphic to \(SU(2), SL(2, R)\), respectively.

\(\delta\) 4-dimensional real Lie algebras

There exist 12 real four-dimensional Lie algebras. Five of them depend on parameters and thus they form continuous families of algebras. There are two algebras which have invariant metric, the solvable algebras \(A_{4,8}\) and \(A_{4,10}\). The former is defined by the commutation relations

\[
\begin{align*}
[J_2, J_3] &= J_1, \\
[J_2, J_4] &= J_2, \\
[J_3, J_4] &= -J_3.
\end{align*}
\]

It is isomorphic to the real Heisenberg algebra \(H(4, R)\). The quadratic Casimir is

\[
C^{(2)} = k_1J_1^2 + k_2(J_2J_3 + J_3J_2 - 2J_1J_4),
\]

and thus \(\Omega^{ij}, \Omega_{ij}\) are given by

\[
\begin{align*}
\Omega^{ij} &= \begin{pmatrix}
k_1 & 0 & 0 & -k_2 \\
0 & 0 & k_2 & 0 \\
0 & k_2 & 0 & 0 \\
-k_2 & 0 & 0 & 0
\end{pmatrix}, & \Omega_{ij} &= \begin{pmatrix}
0 & 0 & 0 & -q_2 \\
0 & 0 & q_2 & 0 \\
0 & q_2 & 0 & 0 \\
-q_2 & 0 & 0 & q_1
\end{pmatrix},
\end{align*}
\]

where \(q_1 = -k_1/k_2^2, q_2 = 1/k_2\).
The algebra $A_{4,10}$ is defined by the commutation relations

\[
\begin{align*}
[J_2, J_3] &= J_1, \\
[J_2, J_4] &= -J_3, \\
[J_3, J_4] &= J_2.
\end{align*}
\]

and it is isomorphic to the centrally extended Euclidean algebra $E_6^\pm$. The quadratic Casimir is

\[
C^{(2)} = k_1 J_1^2 + k_2 (J_2^2 + J_3^2 + 2J_1J_4),
\]

and the metric is found to be

\[
\Omega^{ij} = \begin{pmatrix} k_1 & 0 & 0 & k_2 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ -k_2 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 & q_2 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_2 & 0 \\ q_2 & 0 & 0 & q_1 \end{pmatrix},
\]

where $q_1 = -k_1/k_2^2$, $q_2 = 1/k_2$. The two algebras above are the only indecomposable ones with quadratic Casimirs. There also exist decomposable algebras with quadratic Casimirs, namely, $A_1 \oplus A_4 \oplus A_1 \oplus A_1$, $A_{3,8} \oplus A_1$ and $A_{3,9} \oplus A_1$. Note that $A_{4,8}$ over the complex is isomorphic to $A_{4,10}$ (they have the same complexification).

e) 5-dimensional real Lie algebras

There exist 40 five-dimensional real indecomposable algebras eighteen of which depend on one or more parameters and only one of these has an invariant metric. It is the nilpotent algebra $A_{5,3}$ which is defined by the commutation relations

\[
\begin{align*}
[J_3, J_4] &= J_2, \\
[J_3, J_5] &= J_1, \\
[J_4, J_5] &= J_3.
\end{align*}
\]

The quadratic Casimir for this algebra is

\[
C^{(2)} = k_1 J_1^2 + k_2 J_2^2 + k_3 (J_3^2 + 2J_2J_5 - 2J_1J_4),
\]

and the invariant metric is then found to be

\[
\Omega^{ij} = \begin{pmatrix} k_1 & 0 & 0 & -k_3 & 0 \\ 0 & k_2 & 0 & 0 & k_3 \\ 0 & 0 & k_3 & 0 & 0 \\ -k_3 & 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 & -q_1 & 0 \\ 0 & 0 & 0 & 0 & q_1 \\ 0 & 0 & q_1 & 0 & 0 \\ -q_1 & 0 & 0 & q_2 & 0 \\ 0 & q_1 & 0 & 0 & q_3 \end{pmatrix},
\]

where $q_1 = 1/k_3$, $q_2 = -k_1/k_3^2$, $q_3 = -k_2/k_3^2$. The full list of real five-dimensional algebras with invariant metric contains also the decomposable algebras $\oplus^5 A_1$, $A_{3,8} \oplus A_1 \oplus A_1$, $A_{3,9} \oplus A_1 \oplus A_1$, $A_{4,8} \oplus A_1$ and $A_{4,10} \oplus A_1$. 
στ) 6-dimensional nilpotent Lie algebras

There is no complete classification of the six-dimensional real algebras. However, all nilpotent six-dimensional algebras are known. There exist 22 of them and only one has an invariant metric. This is the algebra $A_{6,3}$ defined by the commutation relations

$$
[J_1, J_2] = J_6,
[J_1, J_3] = J_4,
[J_2, J_3] = J_5.
$$

(19)

The quadratic Casimir is

$$
C^{(2)} = k_1 J_1^2 + k_2 J_5^2 + k_3 J_6^2 + k_4 (J_1 J_5 + J_3 J_6 - J_2 J_4)
$$

(20)

and the metric is given by

$$
\Omega^{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 & k_4 & 0 \\
0 & 0 & 0 & -k_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & k_4 \\
0 & -k_4 & 0 & k_1 & 0 & 0 \\
k_4 & 0 & 0 & 0 & k_2 & 0 \\
0 & 0 & k_4 & 0 & 0 & k_3
\end{pmatrix}
,$$

$$
\Omega_{ij} = \begin{pmatrix}
q_1 & 0 & 0 & 0 & q_2 & 0 \\
0 & q_3 & 0 & -q_2 & 0 & 0 \\
0 & 0 & q_4 & 0 & 0 & q_2 \\
0 & -q_2 & 0 & 0 & 0 & 0 \\
q_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_2 & 0 & 0 & 0
\end{pmatrix},
$$

(21)

where $q_1 = -k_2/k_4^2$, $q_2 = 1/k_4$, $q_3 = -k_1/k_4^2$, $q_4 = -k_3/k_4^2$. The algebra $A_{6,3}$ together with $\oplus^6 A_1$ and $A_{5,3} \oplus A_1$ completes the list of all real six-dimensional Lie algebras with an invariant metric.

3 WZW models

A WZW model based on a simple group $G$ is defined on a two surface $\Sigma$ by the action

$$
S_{WZW} = \frac{\kappa}{4\pi} \int_{\Sigma} d\sigma^2 Tr(g^{-1} \partial g g^{-1} \partial g) - i \frac{\kappa}{6\pi} \int_B d\sigma^3 \epsilon^{abc} Tr(g^{-1} \partial_a g \wedge g^{-1} \partial_b g \wedge g^{-1} \partial_c g),
$$

(22)

where $B$ is a three-manifold bounded by $\Sigma$. If the traces are in some representation $R$ of the group, then the normalization factor $\kappa$ is given by

$$
\kappa = \frac{k c_R d_R}{4 \tilde{h}_G c_R d_R}
$$

where $k \in \mathbb{Z}$ is an integer, $d_R(d_G)$ and $c_R(c_G)$ are the dimension and the quadratic Casimir of the $R$ ($adj$) representation and $\tilde{h}_G$ is the dual Coxeter number.
It is also possible to write down WZW models for non-semisimple groups as well. The necessary condition is the existence of an (bi-)invariant metric. This can be done as follows. One defines on the group manifold the left and right invariant forms \( g^{-1}dg \) and \( dgg^{-1} \), respectively. Since these forms are elements of the corresponding algebra, they may be expressed as

\[
    g^{-1}dg = L^i J_i = L^i_a J_i dz^a, \quad (23)
\]
\[
    dgg^{-1} = R^i J_i = R^i_a J_i dz^a, \quad (24)
\]
where \((z^a)\) parametrizes the Euclidean 2d world sheet. If the algebra has an invariant metric \( \Omega_{ij} \), then the corresponding WZW model can be written as

\[
    S_{WZW} = \frac{k}{4\pi} \int_{\Sigma} d^2 \sigma L^i_a L^j_a \Omega_{ij} + \frac{ik}{12\pi} \int_B d^3 \sigma \epsilon^{abc} L^i_a L^j_b L^k_c f_{ij} f_{kl} \Omega_{kl}. \quad (25)
\]

The currents associated with the above action satisfy the current algebra of \( G \) which is specified by the OPEs

\[
    J_i(z) J_j(w) = \frac{\Omega_{ij}}{(z-w)^2} + f_{ij}^k \frac{J_k(z)}{(z-w)} + \text{regular} . \quad (26)
\]

Starting with the current algebra one can construct stress tensors which are bilinear in the currents, \( T_L = L^{ij} : J_i J_j : \) where \( L^{ij} \) is the inverse inertial tensor \textsuperscript{20}.

The condition for the \( J_i \)'s to be primary fields of weight 1 is written as

\[
    T_L(z) J_i = \frac{J_i(z)}{(z-w)^2} + \frac{\partial J_i(w)}{(z-w)} + \text{regular} . \quad (27)
\]

By using eqs.\textsuperscript{26,27} and the definition of \( T_L \), we find that \( L^{ij} \) must satisfy \textsuperscript{13, 26}

\[
    L^{ij} f_{kj}^\ell + L^{j} f_{kj}^i = 0, \quad (28)
\]
\[
    2L^{ij} \Omega_{kj} + L^{mn} f_{km} f_{tn}^i = \delta^i_k. \quad (29)
\]

Thus, \( L^{ij} \) is an invariant symmetric tensor as follows from eq.\textsuperscript{28} and by employing the latter in eq.\textsuperscript{29} we get

\[
    L^{ij} (2\Omega_{kj} + g_{kj}) = \delta^i_k, \quad (30)
\]

where \( g_{kj} \) is the Killing form. Thus, \( L^{ij} \) is the inverse of the matrix \((2\Omega_{kj} + g_{kj})\) and the central charge is

\[
    c = 2L^{ij} \Omega_{ij} . \quad (31)
\]

By using eqs.\textsuperscript{12,13,16,21}, we find that the \( L^{ij} \) for the algebras \( A_{4,8}, A_{4,10}, A_{5,3} \) and \( A_{6,3} \) are

\[
    L^{ij}_{4,8} = \frac{1}{2} \begin{pmatrix}
        k_2^2 + k_1 & 0 & 0 & -k_2 \\
        0 & 0 & k_2 & 0 \\
        0 & k_2 & 0 & 0 \\
        -k_2 & 0 & 0 & 0
    \end{pmatrix}, \quad L^{ij}_{4,10} = \frac{1}{2} \begin{pmatrix}
        k_2^2 + k_1 & 0 & 0 & k_2 \\
        0 & k_2 & 0 & 0 \\
        0 & 0 & k_2 & 0 \\
        k_2 & 0 & 0 & 0
    \end{pmatrix},
\]
\[
L_{5,3}^{ij} = \frac{1}{2} \begin{pmatrix}
  k_1 & 0 & 0 & -k_3 & 0 \\
  0 & k_2 & 0 & 0 & k_3 \\
  0 & 0 & k_3 & 0 & 0 \\
-k_3 & 0 & 0 & 0 & 0 \\
  0 & k_3 & 0 & 0 & 0
\end{pmatrix}, \quad L_{6,3}^{ij} = \frac{1}{2} \begin{pmatrix}
  0 & 0 & 0 & 0 & k_4 & 0 \\
  0 & 0 & 0 & -k_4 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & k_4 \\
  0 & -k_4 & 0 & k_1 & 0 & 0 \\
 k_4 & 0 & 0 & 0 & k_2 & 0 \\
  0 & 0 & k_4 & 0 & 0 & k_3
\end{pmatrix}
\]

and the central charge is \( c = 4, 4, 5, 6 \), respectively, i.e. equals the dimension of the corresponding algebras \([12]\). The same result could be obtained by solving directly the Virasoro master equation \([26]\).

One may read off a space-time and antisymmetric tensor field background by identifying the WZW action with the \( \sigma \)-model action

\[
S = \int d^2 \sigma (G_{MN} \partial_a X^M \partial^a X^N + iB_{MN} \epsilon^{ab} \partial_a X^M \partial_b X^N) .
\]

We will examine only the cases of \( A_{5,3} \) and \( A_{6,2} \) since all the rest have been studied so far. To begin with the algebra \( A_{5,3} \), we form the corresponding group by exponentiation and we parametrize the group manifold with coordinates \((\alpha_i, i = 1, \ldots, 5)\) by writing the elements of the group as

\[
g = e^{a_1 J_1 + a_2 J_2} e^{a_3 J_3} e^{a_4 J_4} e^{a_5 J_5} .
\]

By using eqs \([23,24]\), we find that the left invariant forms are

\[
L^1 = d\alpha_1 + \frac{\alpha_5^2}{2} d\alpha_3 , \\
L^2 = d\alpha_2 + \alpha_4 \alpha_5 d\alpha_3 - \frac{\alpha_5^2}{2} d\alpha_5 , \\
L^3 = \alpha_5 d\alpha_3 - \alpha_4 d\alpha_5 , \\
L^4 = d\alpha_3 + d\alpha_4 , \\
L^5 = d\alpha_5 ,
\]

and the right invariant ones are

\[
R^1 = d\alpha_1 + \frac{\alpha_5^2}{2} d\alpha_4 , \\
R^2 = d\alpha_2 + \alpha_3 \alpha_5 d\alpha_4 - \frac{\alpha_5^2}{2} d\alpha_5 , \\
R^3 = \alpha_3 d\alpha_5 - \alpha_5 d\alpha_4 , \\
R^4 = d\alpha_3 + d\alpha_4 , \\
R^5 = d\alpha_5 .
\]

Thus, the terms which are integrated in the action \([25]\) are calculated to be

\[
L^i L^a \Omega_{ij} = 2q_1 \partial_a \alpha_1 \partial^a \alpha_3 - 2q_1 \partial_a \alpha_1 \partial^a \alpha_4 - q_1 \alpha_5^2 \partial_a \alpha_3 \partial^a \alpha_4 + 2q_1 \partial_a \alpha_2 \partial^a \alpha_5 + q_2 (\partial_a \alpha_3 + \partial_a \alpha_4)^2 + q_3 \partial_a \alpha_5 \partial^a \alpha_5 ,
\]

\[9\]
\[ \epsilon^{abc} L_a L_b L_c f_{ij} \Omega_{jk} = 6 q_1 \epsilon^{abc} \partial_a \alpha_3 \partial_b \alpha_4 \partial_c \alpha_5 \]
\[ = 3 q_1 \epsilon^{abc} \partial_c (\alpha_2^2 \partial_a \alpha_3 \partial_b \alpha_4), \]  
(38)

and the WZW action is written as
\[ S = \frac{k}{4\pi} \int d^2 \sigma \left( 2 q_1 \partial_a \alpha_1 \partial^a \alpha_3 - 2 q_1 \partial_a \alpha_1 \partial^a \alpha_4 - q_1 \alpha_2^2 \partial_a \alpha_3 \partial^a \alpha_4 + 2 q_1 \partial_a \alpha_2 \partial^a \alpha_5 + q_2 (\partial_a \alpha_3 + \partial_a \alpha_4)^2 + q_3 \partial_a \alpha_5 \partial^a \phi + i q_1 \epsilon^{ab} \partial_c \alpha_2^2 \partial_a \alpha_3 \partial_b \alpha_4 \right). \]  
(39)

By identifying the WZW action above with the \(\sigma\)-model action (33) \((X^M = (\alpha_1, \ldots, \alpha_5))\), we may read off the space-time metric and the antisymmetric tensor field
\[
G_{ij} = \begin{pmatrix} 0 & 0 & -q_1 & -q_1 & 0 \\ 0 & 0 & 0 & 0 & q_1 \\ -q_1 & 0 & q_2 & \frac{1}{2}(q_2 - q_1 \alpha_5^2) & 0 \\ -q_1 & 0 & \frac{1}{2}(q_2 - q_1 \alpha_5^2) & q_2 & 0 \\ 0 & q_1 & 0 & 0 & q_3 \end{pmatrix},
\]

\[
B_{34} = \frac{\alpha_5^2}{2}. \quad (40)
\]

The corresponding space-time line element and the antisymmetric field \(H_{MNL} = \partial_i [B_{NL}]\) are therefore
\[
ds^2 = -2 q_1 d\alpha_1 d\alpha_3 - 2 q_1 d\alpha_1 d\alpha_4 - q_1 \alpha_2^2 d\alpha_3 d\alpha_4 + q_2 (d\alpha_3 + d\alpha_4)^2 \\
+ 2 q_1 d\alpha_2 d\alpha_5 + q_3 \alpha_5^2, \quad (41)
\]

\[
H_{345} = \alpha_5. \quad (42)
\]

We observe that the metric (41) has signature +1 since it has two null Killing vectors. Thus, it cannot be considered as a physical space-time background, not even in a higher dimensional Kaluza-Klein setting. However, one may gauge an anomaly-free subgroup in such a way to remove one time-like direction \([5, 6]\). This will be done in the next section.

In view of the conformal invariance of the WZW model, the one-loop beta-function equations
\[
R_{MN} - \frac{1}{4} H_{MN}^2 - \nabla_M \nabla_N \phi = 0, \quad (43)
\]
\[
\nabla^M (e^\phi H_{MNL}) = 0, \quad (44)
\]
\[
-R + \frac{1}{12} H^2 + 2 \Delta \phi + \nabla_M \phi \nabla^M \phi + \frac{2 \delta c}{3} = 0, \quad (45)
\]

must be satisfied. The shorthand notation \(H_{MN}^2 = H_M^{KL} H_N^{KL}, H^2 = H_{MNL} H^{MNL}\) have been used and \(\delta c\) is the central-charge deficit. One can verify that the metric (41) is Ricci flat \((R_{MN} = 0)\), \(H_{MN}^2 = 0\) and thus, the beta-function equations are indeed satisfied with \(c = 5\) and constant dilaton field.
Let us now turn into the six-dimensional algebra $A_{6,3}$. We parametrize the corresponding group with coordinates $(\alpha_i, i = 1, \ldots, 6)$ so that its elements can be written as:

$$g = e^{\alpha_1 J_1} e^{\alpha_2 J_2} e^{\alpha_3 J_3} e^{\alpha_4 J_4 + \alpha_5 J_5 + \alpha_6 J_6}.$$  

(46)

By using eqs (23, 24), we find that the left invariant forms are

$$L^1 = d\alpha_1,$$

$$L^2 = d\alpha_2,$$

$$L^3 = d\alpha_3,$$

$$L^4 = d\alpha_4 + \alpha_3 d\alpha_1,$$

$$L^5 = d\alpha_5 + \alpha_3 d\alpha_2,$$

$$L^6 = d\alpha_6 + \alpha_2 d\alpha_1,$$

(47)

and the right invariant ones

$$R^1 = d\alpha_1,$$

$$R^2 = d\alpha_2,$$

$$R^3 = d\alpha_3,$$

$$R^4 = d\alpha_4 + \alpha_1 d\alpha_3,$$

$$R^5 = d\alpha_5 + \alpha_2 d\alpha_3,$$

$$R^6 = d\alpha_6 + \alpha_1 d\alpha_2.$$  

(48)

Thus we have

$$L^i_a L^j_b \Omega_{ij} = q_1 \partial_a \alpha_1 \partial^a \alpha_1 + q_3 \partial_a \alpha_2 \partial^a \alpha_2 + q_4 \partial_a \alpha_3 \partial^a \alpha_3$$

$$+ 2q_2 \partial_a \alpha_1 \partial^a \alpha_5 - 2q_2 \partial_a \alpha_2 \partial^a \alpha_4 + 2q_2 \partial_a \alpha_3 \partial^a \alpha_6$$

$$+ 2q_2 \partial_a \alpha_2 \partial^a \alpha_3 \partial^a \alpha_1,$$  

(49)

$$e^{abc} L^i_a L^j_b L^k_c f_{ij} f_{kl} = 6q_2 e^{abc} \partial_a \alpha_1 \partial_b \alpha_2 \partial_c \alpha_3$$

$$= 6q_2 e^{abc} \partial_a (\alpha_1 \partial_b \alpha_2 \partial_c \alpha_3).$$  

(50)

Substituting the above expressions in the WZW action and comparing with the $\sigma$-model action (33), the metric and the non-zero components of the $H_{MNL}$-field are given by

$$ds^2 = q_1 d\alpha_1^2 + q_3 d\alpha_2^2 + q_4 d\alpha_3^2 + 2q_2 d\alpha_1 d\alpha_5 - 2q_2 d\alpha_2 d\alpha_4$$

$$+ 2q_2 d\alpha_3 d\alpha_6 + 2q_2 \alpha_2 d\alpha_3 d\alpha_1,$$  

(51)

$$H_{123} = q_2.$$  

(52)
One may easily verify that, by defining coordinates

\[
\begin{align*}
  x_1 &= \sqrt{q_1} \alpha_1 + \frac{q_2}{\sqrt{q_1}} (\alpha_5 + \frac{1}{2} \alpha_2 \alpha_3), \\
  x_2 &= \sqrt{q_3} \alpha_2 - \frac{q_2}{\sqrt{q_3}} (\alpha_4 + \frac{1}{2} \alpha_3 \alpha_1), \\
  x_3 &= \sqrt{q_4} \alpha_3 + \frac{q_2}{\sqrt{q_4}} (\alpha_6 + \frac{1}{2} \alpha_1 \alpha_2), \\
  x_4 &= \frac{q_2}{\sqrt{q_3}} (\alpha_4 + \frac{1}{2} \alpha_1 \alpha_3), \\
  x_5 &= \frac{q_2}{\sqrt{q_1}} (\alpha_5 + \frac{1}{2} \alpha_2 \alpha_3), \\
  x_6 &= \frac{q_2}{\sqrt{q_4}} (\alpha_6 + \frac{1}{2} \alpha_1 \alpha_2),
\end{align*}
\]

the metric takes the form

\[
ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 - dx_5^2 - dx_6^2,
\]

while the non-vanishing components of the $H_{MNL}$-field are

\[
H_{123} = H_{162} = H_{134} = H_{164} = H_{532} = H_{526} = H_{543} = H_{564} = \frac{q_2}{\sqrt{q_1 q_2 q_3}}.
\]

Thus, the corresponding $\sigma$-model is flat with three time-like directions and the beta-function equations are satisfied (with $c = 6$) in view of $H_{MN}^2 = 0$.

## 4 Gauged WZW models

The WZW action is invariant under global $g \rightarrow h_L^{-1}gh_R$ transformations. We can make this transformation local by introducing gauge fields $A$, $\overline{A}$ with transformation properties $A \rightarrow h_L^{-1}(A + \d)h_L$, $\overline{A} \rightarrow h_R^{-1}(\overline{A} + \overline{\d})h_R$. The choice $h_R = h_L$ corresponds to vector gauging while $h_R = h_L^{-1}$ to axial one. Anomaly considerations allow axial and vector gauging for abelian subgroups and vector gauging for non-abelian ones. The gauged action takes the form

\[
S[g, A] = S[g] + \frac{k}{2\pi} \int d^2z (< A, \overline{\d} g g^{-1} > + < \overline{\d} \overline{A} g ^{-1} > + A \overline{A} + < g^{-1} A g, \overline{A} >),
\]

where $<,>$ denotes the inner product defined by $\Omega_{ij}$ and upper and lower signs correspond to vector and axial gauging, respectively [3, 4].

Let us now consider axial gauging for the five-dimensional case choosing to gauge the abelian subgroup generated by $J_4$. The gauged WZW action is written,
in complex coordinates, as

\[ S_{\text{axial}} = \frac{k}{4\pi} \int d^2z \{ -q_1 \partial \alpha_1 \bar{\partial} \alpha_3 - q_1 \partial \alpha_3 \bar{\partial} \alpha_1 - q_1 \partial \alpha_1 \bar{\partial} \alpha_4 - q_1 \bar{\partial} \alpha_1 \partial \alpha_4 \\
- \frac{1}{2} q_1 \alpha_5^2 (\partial \alpha_3 \bar{\partial} \alpha_4 + \partial \alpha_4 \bar{\partial} \alpha_3) + q_1 (\partial \alpha_5 \bar{\partial} \alpha_2 + \partial \alpha_2 \bar{\partial} \alpha_5) \\
+ q_2 \partial \alpha_3 \bar{\partial} \alpha_3 + q_2 (\partial \alpha_3 \bar{\partial} \alpha_4 + \partial \alpha_4 \bar{\partial} \alpha_3) + q_2 \partial \alpha_4 \bar{\partial} \alpha_4 \\
+ \frac{1}{2} q_1 \alpha_5^2 (\partial \alpha_4 \bar{\partial} \alpha_3 - \partial \alpha_3 \bar{\partial} \alpha_4) \\
+ A \left[ -2q_1 (\bar{\partial} \alpha_1 + \frac{1}{2} \alpha_5^2 \bar{\partial} \alpha_4) + 2q_2 (\bar{\partial} \alpha_3 + \bar{\partial} \alpha_4) \right] \\
+ \bar{A} \left[ -2q_1 (\bar{\partial} \alpha_1 + \frac{1}{2} \alpha_5^2 \partial \alpha_3) + 2q_2 (\partial \alpha_3 + \partial \alpha_4) \right] \\
+ A\bar{A} (4q_2 - q_1 \alpha_5^2) \} \tag{57} \]

and it is invariant under the transformations

\[ \delta \alpha_1 = \delta \alpha_2 = \delta \alpha_5 = 0, \]
\[ \delta \alpha_3 = \delta \alpha_4 = \epsilon, \]
\[ \delta A = -\partial \epsilon. \tag{58} \]

One can obtain the \( \sigma \)-model by fixing the gauge and then integrate over the gauge field. A convenient gauge choice is \( \alpha_3 + \alpha_4 = 0 \) and the resulting \( \sigma \)-model action is

\[ S = \frac{k}{4\pi} \int d^2z \left( \frac{4q_1 q_2 \alpha_5^2}{4q_2 - q_1 \alpha_5^2} \partial \alpha \bar{\partial} \alpha + q_1 \partial \alpha_5 \bar{\partial} \alpha_2 + q_1 \bar{\partial} \alpha_5 \partial \alpha_2 + q_3 \partial \alpha_5 \bar{\partial} \alpha_5 \\
- \frac{4q_1^2}{4q_2 - q_1 \alpha_5^2} \partial \alpha_1 \bar{\partial} \alpha_1 - \frac{2q_1^2 \alpha_5^2}{4q_2 - q_1 \alpha_5^2} (\bar{\partial} \alpha \partial \alpha_1 - \partial \alpha \bar{\partial} \alpha_1) \right). \tag{59} \]

By comparing (58) with the most general class of conformal invariant \( \sigma \)-models

\[ S = \int d^2z [(G_{\mu\nu} + B_{\mu\nu}) \partial X^\nu \bar{\partial} X^\mu + \alpha' R \Phi (X)], \tag{60} \]

we may read off the background metric \( G_{\mu\nu} \), the antisymmetric field \( B_{\mu\nu} \) and the dilaton \( \Phi \) (coming from the \( A\bar{A} \) term in the action)

\[ ds^2 = \frac{1}{u^2 + \mu^2} dx^2 + \frac{u^2}{u^2 + \mu^2} dy^2 + 2dudv, \]
\[ B_{xy} = -\frac{u^2}{\mu (u^2 + \mu^2)}, \]
\[ \Phi = \ln (u^2 + \mu^2) + \text{const.}. \tag{61} \]

We have defined \( x = \sqrt{q_1} \alpha_1, \ y = \sqrt{q_2} \alpha_2, \ u = \alpha_5, \ v = q_3 \alpha_2 + \frac{q_4}{2} \alpha_5, \ \mu^2 = \frac{4q_2}{q_1} \) and we have replaced \( k \) by \( (-k) \) and \( q_3 \) by \( (-q_3) \) in the action (59). This plane-wave
solution has been found (with a different $B_{xy}$ but still the same $H_{uxy}$ field) in [19] as a special limit of the $E_2^c \times U(1)/U(1)$ coset model.

Let us now consider the vector gauging of the same abelian subgroup generated by $J_4$. The gauged WZW action in complex coordinates in this case is

$$S_{\text{vect}} = \frac{1}{4\pi} \int d^2 z \left\{ -q_1 \partial \alpha_1 \overline{\partial} \alpha_3 - q_1 \partial \alpha_3 \overline{\partial} \alpha_1 - q_1 \partial \alpha_4 \overline{\partial} \alpha_4 \\
- \frac{1}{2} q_1 \alpha_5^2 (\partial \alpha_3 \overline{\partial} \alpha_4 + \partial \alpha_4 \overline{\partial} \alpha_3) + q_1 (\partial \alpha_5 \overline{\partial} \alpha_2 + \partial \alpha_2 \overline{\partial} \alpha_5) \\
+ q_2 \partial \alpha_3 \overline{\partial} \alpha_3 + q_2 (\partial \alpha_3 \overline{\partial} \alpha_4 + \partial \alpha_4 \overline{\partial} \alpha_3) + q_2 \partial \alpha_4 \overline{\partial} \alpha_4 \\
+ \partial \alpha_5 \overline{\partial} \alpha_5 + \frac{1}{2} q_1 \alpha_5^2 (\partial \alpha_4 \overline{\partial} \alpha_3 - \partial \alpha_3 \overline{\partial} \alpha_4) \\
+ A \left[ -2q_1 (\overline{\partial} \alpha_1 + \frac{1}{2} \alpha_5^2 \overline{\partial} \alpha_4) + 2q_2 (\overline{\partial} \alpha_3 + \overline{\partial} \alpha_4) \right] \\
- \overline{A} \left[ -2q_1 (\partial \alpha_1 + \frac{1}{2} \alpha_5^2 \partial \alpha_4) + 2q_2 (\partial \alpha_3 + \partial \alpha_4) \right] + q_1 \alpha_5^2 A \overline{A} \right\} \quad (62)$$

This action is invariant under the transformations

$$\delta \alpha_1 = \delta \alpha_2 = \delta \alpha_5 = 0, \\
- \delta \alpha_3 = \delta \alpha_4 = \epsilon, \\
\delta A = \partial \epsilon. \quad (63)$$

As in the case of axial gauging, by fixing the gauge and integrating over the gauge field one can obtain the $\sigma$-model action which, choosing $\alpha_3 = \alpha_4 (= \alpha)$ is found to be

$$S = \frac{k}{4\pi} \int d^2 z \left[ \frac{4q_2 (4q_2 - q_1 \alpha_5^2)}{q_1 \alpha_5^2} \partial \alpha \overline{\partial} \alpha + \frac{4q_1}{\alpha_5^2} \partial \alpha_1 \overline{\partial} \alpha_1 + q_3 \partial \alpha_5 \overline{\partial} \alpha_5 \\
+ q_1 (\partial \alpha_5 \overline{\partial} \alpha_2 + \partial \alpha_2 \overline{\partial} \alpha_5) - \frac{8q_2}{\alpha_5^2} (\partial \alpha \overline{\partial} \alpha_1 + \partial \alpha_1 \overline{\partial} \alpha) \right]. \quad (64)$$

By defining coordinates $x = \sqrt{q_1} (\alpha_1 - \frac{2q_2}{q_1} \alpha), \; y = \sqrt{q_2} \alpha, \; u = \alpha_5, \; v = \frac{q_1}{4} \alpha_2 + \frac{q_3}{8} \alpha_5$ and comparing (64) with (60), we may read off the space-time metric and the dilaton

$$ds^2 = \frac{1}{u^2} dx^2 + dy^2 + 2dudv, \quad \Phi = \ln u^2 + \text{const.} \quad (65)$$

One can verify that the one-loop beta-function equations for conformal invariance are indeed satisfied with central charge deficit $\delta c = 0$.

Finally, let us stress that the solutions (61,65) are related to the 4-dim Minkowski space-time by a duality transformation [27]. The duality transformation for the
case of an abelian isometry $x^0 \to x^0 + \text{const.}$ in the coordinate system $(x^0, x^i)$ reads

$$
\tilde{G}_{00} = \frac{1}{G_{00}}, \quad \tilde{G}_{0i} = \frac{B_{0i}}{G_{00}}, \quad \tilde{G}_{ij} = G_{ij} - \frac{G_{0i}G_{0j} - B_{0i}B_{0j}}{G_{00}} \\
\tilde{B}_{0i} = \frac{G_{0i}}{G_{00}}, \quad \tilde{B}_{ij} = B_{ij} - \frac{G_{0i}B_{0j} - G_{0j}B_{0i}}{G_{00}}, \\
\tilde{\Phi} = \Phi + \ln G_{00}.
$$

(66)

Applying this transformation to the solution (61) for the isometry $x \to x + \text{const.}$, and defining coordinates $x_1 = \mu x$, $x_2 = x - \frac{\mu}{\mu}$, we get

$$
d\tilde{s}^2 = dx_1^2 + u^2 dx_2^2 + 2dudv, \\
\tilde{B}_{xy} = 0, \quad \tilde{B}_{ij} = 0 \\
\tilde{\Phi} = \text{const.}
$$

(67)

while the same isometry for the solution (65) leads similarly to

$$
d\tilde{s}^2 = u^2 dx^2 + dy^2 + 2dudv, \\
\tilde{\Phi} = \text{const.}
$$

(68)

One may verify that the Riemann tensor for both the above metrics vanishes and thus, they describe flat spacetimes.

5 Conclusions

Although many theorems about the structure of algebras with invariant and non-degenerate metric exist, we do not know these algebras explicitly. In particular, non-semisimple affine Sugawara constructions leads to integer central charge and the corresponding exact background can be considered as a physical space-time (as long as there exist one time-like direction) for string propagation. Thus, the non-semisimple WZW models are particularly interesting since they provide exact string backgrounds but, in view of the absence of any classification of non-semisimple groups, their construction is an open problem.

Starting from the fact that all real algebras with dimension up to five and all nilpotent six dimensional algebras are known, we examined for which of these an invariant, symmetric and non-degenerate metric exists. We found that, up to four dimensions, there are no other such algebras except the already known $SU(2)$, $SU(1, 1)$, $E_6$ and $H_4$. In five dimensions there exist only one such algebra. We constructed the corresponding WZW model and the resulting five-dimensional background is of plane-wave type but with two null Killing vectors. Thus, the resulting space-time cannot be considered as a physical background for string propagation. However, by gauging appropriate subgroups, gauged WZW were
obtained corresponding to 4-dimensional plane-wave space-time with physical Lorentz signature. These gauged models are related to flat Minkowski space-time by a duality transformation. The six dimensional case corresponds to flat space with three time-like directions.

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