Entanglement bounds of tripartite squeezed thermal states

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Abstract
I propose the multi-mode squeezed thermal state based on the multi-mode pure entangled state. The correlation matrix of the state is characterized by two parameters. I then analysis the separable condition for this state, and calculating the relative entropy of the state with respect to the same kind of fully separable state in order to provide an upper bound of the relative entropy of entanglement. The bound is compared with the other bounds which were obtained with reduced state.

1 Introduction
The quantification of entanglement is one of the most important problem in quantum information theory. The three most promising entanglement measures are the entanglement of formation, the entanglement of distillation [1] and the relative entropy of entanglement [2]. In the bipartite system, the meaning of both entanglement of formation and the entanglement of distillation is quite clear. However, these two measures do not have an entirely straightforward meaning when one considers multi-partite entanglement [3]. On the other hand, the relative entropy of entanglement can easily be generalized to multi-partite system. Recently, towards possible applications in quantum communication, both theoretical and experimental investigations increasingly focus on quantum states with a continuous spectrum defined in an infinite dimensional Hilbert space. These states can be relatively easily generated using squeezed light and beam splitters. Besides the research on the entanglement of bipartite continuous variable system, the non-locality of the multipartite entangled continuous-variable Greenberger-Horne-Zeilinger (CVGHZ) states was proved [4]. Hence the quantification of the entanglement of CVGHZ states comes into our sight. Now let us proceed to derive a simple CVGHZ mixed state and give out its upper bound of the relative entropy of entanglement.
2 Multi-mode squeezed thermal state

The continuous variable analogues to the multipartite entangled Greenberger-Horne-Zeilinger states was proposed\[4\]. The Wigner function of the pure entangled m-mode state is (with $\hbar = 1$)

$$W(x, p) = \left(\frac{1}{\pi}\right)^m \exp\left\{-e^{-2r} \left\{ \frac{1}{m} \left( \sum_{i=1}^{m} x_i \right)^2 + \frac{1}{2m} \sum_{i,j}^{m} (p_i - p_j)^2 \right\} - e^{2r} \left\{ \frac{1}{m} \left( \sum_{i=1}^{m} p_i \right)^2 + \frac{1}{2m} \sum_{i,j}^{m} (x_i - x_j)^2 \right\} \right\}$$

(1)

The correlation matrix (CM) of this state is $\alpha = \alpha_x \oplus \alpha_p$. The $m \times m$ matrices $\alpha_x$ and $\alpha_p$ are in a special kind of form. If we define a $m \times m$ matrix $A(x, y)$ with its all diagonal elements being equal to $\frac{1}{m} (x + (m - 1) y)$, all off-diagonal elements being equal to $\frac{1}{m} (x - y)$, then $\alpha_x = \frac{1}{2} A (e^{2r}, e^{-2r})$, $\alpha_p = \frac{1}{2} A (e^{-2r}, e^{2r})$. It is easy to check that

$$A(x, y)A(u, v) = A(xu, yv).$$

(2)

So that $\alpha_x \alpha_p = \alpha_p \alpha_x = \frac{1}{4} I_m$.

Now we introduce the m-mode squeezed thermal state (mST) based on the pure entangled m-mode state. The CM of mST state is $\alpha = \alpha_x \oplus \alpha_p$ with

$$\alpha_x = \left( N + \frac{1}{2} \right) A (e^{2r}, e^{-2r}), \quad \alpha_p = \left( N + \frac{1}{2} \right) A (e^{-2r}, e^{2r}).$$

(3)

where $N$ is the average photon number of thermal noise of every mode. Then we have

$$\alpha_x \alpha_p = \alpha_p \alpha_x = \left( N + \frac{1}{2} \right) I_m.$$  

(4)

In the calculation of relative entropy between these Gaussian states, We need the so called $M$ matrix which is the matrix in the exponential expression of density operator of Gaussian state\[5\][6]. It is related to the CM by a symplectic transformation $S$. The CM can be diagonalized by symplectic transformation such that $\alpha = S\left(\frac{1}{2} \coth \frac{1}{2} M\right)S^T = S\tilde{\alpha}S^T$ and $S^T MS = \tilde{M}$ (with $\tilde{\alpha}, \tilde{M}$ being diagonal). The procedure of obtaining the $M$ matrix is to explore the eigenfunctions of matrix $\alpha_x \alpha_p$ of the state\[5\]. But an unit matrix form of the matrix $\alpha_x \alpha_p$ like in Eq. 3 has not definite eigenfunctions. Here the $M$ matrix is constructed directly by making use of Eq. 2. Clearly the mST state we introduced is symmetric to all $m$ modes. So that $\tilde{\alpha}$ will be in the form of

$$\tilde{\alpha} = \left( N + \frac{1}{2} \right) (I_m \oplus I_m),$$

(5)

and $\alpha = \left( N + \frac{1}{2} \right) SS^T$. From Eq. 2, we can write Eq. 3 as

$$\alpha_x = \left( N + \frac{1}{2} \right) A (e^r, e^{-r}) A (e^r, e^{-r}).$$

(6)
and
\[ \alpha_p = \left( N + \frac{1}{2} \right) A(e^{-r}, e^r) A(e^{-r}, e^r). \]  
(7)

Hence
\[ S = A(e^r, e^{-r}) \oplus A(e^{-r}, e^r). \]  
(8)

The \( M \) matrix will be
\[ M = A(e^{-r}, e^r) \log \frac{1}{v} I_m A(e^{-r}, e^r) \oplus A(e^r, e^{-r}) \log \frac{1}{v} I_m A(e^r, e^{-r}), \]  
(9)

where \( v = \frac{N}{N + 1} \). The mST state we introduced is characterized by two parameters \( r \) and \( N \). When one of them is 0, the other one has a clear physical meaning: quantum correlation or noise. And when both of them are not 0, we will display with three mode state that it is the competition of the two parameters which will determine the state is authentically multipartite entangled or not.

3 Separability of the \( 1 \times 1 \times 1 \) squeezed thermal state

The separability problem of the three mode gaussian state was perfectly solved\[7\]. I in this paper apply it in order to derive a simple criterion in the special case of 3ST state\((1 \times 1 \times 1 \) squeezed thermal state). The three mode gaussian states were classified as 5 different entangled classes. But 3ST states can be classified as 3 different entangled classes: fully inseparable states, biseparable states, fully separable states. Following the notation of Ref. \[7\], the CM \( \alpha \) now is replaced with \( \gamma \), where \( \gamma = 2\alpha \), and denote by \( \Lambda_x = \Lambda \oplus I \) the partial transposition in \( x \)’s system only, \( x = A, B, C \) is one of the three modes. Denoting the partially transposed CM by \( \tilde{\gamma}_x = \Lambda_x \gamma \Lambda_x \), and
\[ J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}. \]  
(10)

The criterion for fully inseparable state is
\[ \tilde{\gamma}_x \npreceq iJ, \text{ for all } x = A, B, C. \]  
(11)

Because of the symmetry of the 3ST state, the criterion can be simplified to for example \( \tilde{\gamma}_A \npreceq iJ \). This will lead to the condition that
\[ \cosh^2 2r < \frac{9}{32} \left( 2N + 1 + \frac{1}{2N + 1} \right)^2 - \frac{1}{8}. \]  
(12)

While for \( \tilde{\gamma}_x \succeq iJ, (x = A, B, C) \). The state which we will call it PPT trimode state can be biseparable or fully separable. The criterion to distinguish the
biseparable and fully separable states is as follow. The CM $\gamma$ of PPT trimode state can be written of as

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

where $A$ is a $2 \times 2$ CM for the first mode, whereas $B$ is a $4 \times 4$ CM for the other two modes. Define the matrices $K$ and $\tilde{K}$ as

$$K \equiv A - C \frac{1}{B - iJ} C^T,$$

$$\tilde{K} \equiv A - C \frac{1}{B - i\tilde{J}} C^T,$$

where $\tilde{J} = J \oplus (-J)$ is the partially transposed $J$ for two modes.

Then the condition of the PPT trimode state to be fully separable is that if and only if there exists a point $(y, z) \in \mathbb{R}^2$ fulfilling the following inequality:

$$\min\{\text{tr}K, \text{tr}\tilde{K}\} \geq 2x,$$

$$\det K + 1 + L^T \left( y, z \right)^T \geq x \cdot \text{tr}K,$$

$$\det \tilde{K} + 1 + \tilde{L}^T \left( y, z \right)^T \geq x \cdot \text{tr}\tilde{K},$$

where $x = \sqrt{1 + y^2 + z^2}$, and $L = (a - c, 2\text{Re}(b))$ if we write $K$ as

$$K = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}.$$ 

Ineq.(15) restricts $(y, z)$ to a circular disk $C$, while Ineq.(16) and Ineq.(17) describe ellipses $E$ and $E'$ respectively. The existence of the point $(y, z)$ then turn out to be the intersection of the ellipses $E$ and $E'$ and the circular disk $C$.

For 3ST state, after lengthy algebra, Ineq.(16) and Ineq.(17) can be written of as

$$t + \frac{1}{t} \geq \left( s + \frac{1}{s} \right) x + \frac{1}{3} \left( s - \frac{1}{s} \right) y,$$

$$t + \frac{1}{t} \geq \left( s + \frac{1}{s} \right) x + \Delta \left( s - \frac{1}{s} \right) y,$$

respectively, where we denote $t = 2N + 1$, $s = e^{2r}$ and

$$\Delta = \frac{(t + \frac{1}{t})^2 - \frac{2}{3} \left( s + \frac{1}{s} \right)^2 - 1}{(t + \frac{1}{t})^2 - \frac{2}{3} \left( s + \frac{1}{s} \right)^2 + 5}.$$ 

Let us consider $\partial E$ and $\partial E'$, the borders of $E$ and $E'$, they are represented by the equality parts of Ineq.(16) and Ineq.(17).

$$t + \frac{1}{t} = \left( s + \frac{1}{s} \right) x + \frac{1}{3} \left( s - \frac{1}{s} \right) y,$$

$$t + \frac{1}{t} = \left( s + \frac{1}{s} \right) x + \Delta \left( s - \frac{1}{s} \right) y.$$
When $\Delta \neq 1$, the solution (intersection of $\partial E$ and $\partial E'$) to above equations will be $y = 0$, so that $x = \frac{(t + \frac{1}{s})}{(s + \frac{1}{s})}$. While for $\Delta = 1$, we have $s + \frac{1}{s} = 2$, hence $s = 1$, and we also have $x = \frac{(t + \frac{1}{s})}{(s + \frac{1}{s})}$. Because $x \geq 1$, Ineq. (19) and Ineq. (20) then will lead to the condition of fully separable state, $(t + \frac{1}{s}) \geq (s + \frac{1}{s})$, hence $t \geq s$. By denoting $v = \frac{N}{N + 1}$, $\lambda = \tanh r$, we have a very simple expression of fully separable condition for 3ST state,

\[ v \geq \lambda. \tag{23} \]

Clearly the two points in $(y, z)$ plane defined by intersection $\partial E$ and $\partial E'$ are the farthest points from the origin among all valid points defined by the intersection $E$ and $E'$. To verify that Ineq. (15) is also fulfilled, we just need that

\[ \min \{tr K, tr \tilde{K}\} \geq 2 \frac{t + \frac{1}{s}}{s + \frac{1}{s}}. \tag{24} \]

This can be verify numerically for all $v \geq \lambda$, although the analytical verification is possible. And the conclusion is that Ineq. ( ) fulfills for all $v \geq \lambda$. The districts for fully separable, biseparable and fully inseparable states are displayed in Fig.(1).

\section{Relative entropy and entanglement bound}

The relative entropy of entanglement for an arbitrary number of parties is defined by the following formula \cite{2, 3}

\[ E(\sigma) = \min_{\rho \in D} S(\sigma \| \rho) \tag{25} \]

where $D$ is a set of disentangled (separable) states and where $S(\sigma \| \rho) = tr\{\sigma \log \sigma - \sigma \log \rho\}$ is the quantum relative entropy. For the purpose of this paper, we assume that $D$ is the set of the states that can be created locally, i.e. it is fully separable \cite{3}. Also, by $E_n(\sigma)$ we will always denote the relative entropy of entanglement for n-party systems with respect to the set of fully separable states.

Now let us consider the relative entropy of a 3ST state $\sigma$ with respect to 3ST fully separable state $\rho$. It will be \cite{5}

\[ S(\sigma \| \rho) = -S(\sigma) - 3 \log (1 - v_\rho) + \frac{3}{2} \log v_\rho + \frac{1}{2} Tr \alpha_\sigma M_\rho. \]

where $S(\sigma) = -tr \sigma \log \sigma$ is the von Neumann entropy of $\sigma$, and it can be written of as the $S(\sigma) = 3g(N_\sigma)$. Here $g(x) = (x + 1) \log (x + 10 - x \log (x)$ is the bosonic entropy function and $N_\sigma = \frac{v_\sigma}{1 - v_\sigma}$. By using Eq. (2), Eq. (3) and Eq. (9) we have

\[ S(\sigma \| \rho) = -3g \left( \frac{v_\sigma}{1 - v_\sigma} \right) - 3 \log (1 - v_\rho) \]

\[ - \frac{3}{2} \left( \frac{1 + v_\sigma}{1 - v_\sigma} \cosh 2(r_\sigma - r_\rho) - 1 \right) \log v_\rho. \tag{26} \]
For a given 3ST entangled state \( \sigma \), the minimum of the relative entropy \( S(\sigma || \rho) \) will reach at \( \lambda_{\rho} (= \tanh r_{\rho}) = v_{\rho} \) for some \( v_{\rho} \), that is at the border of fully separable 3ST state set. The set of fully separable 3ST states is a subset of \( D \), and we obtain an upper bound \( E_{3ur}(\sigma) \) for the relative entropy of entanglement \( E_{3}(\sigma) \) of 3ST state \( \sigma \) as displayed in Fig.(2).

\[
E_{3ur}(\sigma) = \min_{r_{\rho}}\left\{-3g\left(\frac{v_{\sigma}}{1-v_{\sigma}}\right) - 3\log (1 - \tanh r_{\rho}) \right. \\
\left. - \frac{3}{2} \left[ \frac{1 + v_{\sigma}}{1 - v_{\sigma}} \cosh 2 (r_{\sigma} - r_{\rho}) - 1 \right] \log \tanh r_{\rho} \right\}. \tag{27}
\]

One the other hand, for any pure tripartite state \( \sigma \), the relative entropy of entanglement \( E_{3}(\sigma) \) will also be upper bounded by teleportation consideration \( 3 \), that is

\[
E_{3} (\sigma) \leq \min \{ S(\sigma_{A}) + S(\sigma_{B}), S(\sigma_{A}) + S(\sigma_{C}), S(\sigma_{B}) + S(\sigma_{C}) \}. \tag{28}
\]

where \( \sigma_{A} \) is the reduced state of \( \sigma \) with \( B, C \) parties traced out. For 3ST pure state \( \sigma \), it can be simplified to

\[
E_{3} (\sigma) \leq 2S (\sigma_{A}). \tag{29}
\]

Hence, one get the upper bound of \( E_{3} (\sigma) \),

\[
E_{3u} (\sigma) = \min \{ E_{3ur} (\sigma), 2S (\sigma_{A}) \}, \quad \text{for pure state} \ \sigma. \tag{30}
\]

The numerical result is displayed in Fig.(3). At high \( \lambda \) side, \( E_{3ur} (\sigma) \) is better than \( 2S (\sigma_{A}) \) as upper bound.

5 Conclusions and Discussions

The continuous variable analogues of the multipartite entangled Greenberger-Horne-Zeilinger states had been introduced are pure states. In this paper we have discussed how to extend these pure states to the multipartite squeezed thermal states. The such obtained mST state are highly symmetric and it is characterized by only two parameters, one is the squeezing parameter and the other is for thermal noise. I have got the \( M \) matrix (matrix in the exponential expression of density operator of the state) of the state in order that the relative entropy between the mST states can be calculated. The criterions of separability have been simplified for 3ST states. The fully separable condition now takes a very simple form, that is if the noise is stronger than the squeezing, the state will be fully separable, otherwise it will be entangled (biseparable or fully entangled). I have calculated the relative entropy between the 3ST entangled state and 3ST fully separable state and obtained an upper bound of the relative entropy of entanglement for 3ST entangled state. The bound was compared with the bound
Figure 1: The up left district for fully inseparable states, the narrow strip for biseparable states and the down right triangle district for fully separable states.

Figure 2: Upper bounds of relative entropy of entanglement, with $\lambda$ and $v \in [0.001, 0.999]$
obtained by teleportation consideration for pure 3ST state. Our result is better the other one at the high squeezing side. I in this paper mainly treated 3ST. But there are really some things need to be talked between the 3ST and 2ST. The bounds obtained for 3ST are exactly 1.5 times of the bounds for 2ST \[8\]. The separable (fully separable for 3ST) condition takes exactly the same form. I now wonder if these are true for mST. I will end this paper with the conjectures for mST: (1) The fully separable condition will be \( v \geq \lambda \). (2) The upper bound of relative entropy of entanglement for mST will be \( E_{mur} = \frac{m}{\lambda} E_{ur} (v, \lambda) \) where \( E_{ur} (v, \lambda) \) is the upper bound of relative entropy of entanglement for 2ST.

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