BLOW-UP FOR THE NONLINEAR SCHRÖDINGER EQUATION
WITH A POINT INTERACTION IN DIMENSION TWO

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Abstract. In the present note we study the focusing NLS equation in dimension two with a point interaction and in the supercritical regime, showing two results. After obtaining the (nonstandard) virial formula, we exhibit a set of initial data that blow-up. Moreover we show the standing waves \( e^{i\omega t} \varphi_\omega \) corresponding to ground states \( \varphi_\omega \) of the action are strongly unstable, at least for sufficiently high \( \omega \).

Keywords: Non-linear Schrödinger equation; Singular solutions; Point interactions. Blow-up.

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1. INTRODUCTION

In the present paper we study the blow-up of solutions of a focusing Nonlinear Schrödinger equation (NLS) with a power nonlinearity in two dimension and in the \( L^2 \) supercritical regime, perturbed by a point defect. The point defect is represented as a point interaction, sometimes improperly called delta potential. Namely, we consider the model

\[
\begin{cases}
  i\dot{\psi}(t) = \mathcal{H}_\alpha \psi(t) - |\psi|^{p-1} \psi \\
  \psi(0) = \psi_0
\end{cases}
\]

where \( \mathcal{H}_\alpha \) is defined as a self-adjoint extension of the symmetric operator \(-\Delta\) starting from the domain \( C^\infty(\mathbb{R}^2 \setminus \{0\})\), and \( \alpha \) is a parameter classifying the self-adjoint extension (see Section 2 for details). It is a noteworthy feature of the 2d delta interaction \( \mathcal{H}_\alpha \) that it admits a negative eigenvalue \( E_\alpha \) irrespectively of the parameter \( \alpha \), so that the parameter \( \alpha \) cannot be considered as a measure of strength and its sign it is not related to the attractive or repulsive character of the interaction. This model has been studied extensively in one dimension, as regards well posedness, blow-up, existence of standing waves and their orbital and asymptotic stability, with several variation on the theme ([12, 14, 17, 8, 13, 15] and references therein).

The well posedness of the 2d model has been given only quite recently, first in the strong setting in [6] and then in the energy setting in [9]. The critical nonlinearity power, as in the unperturbed model, is \( p = 3 \) (notice in this respect the rather different behavior of the model studied in [3]). We want to give information about the blow-up of solutions for \( p > 3 \), i.e. showing that for definite classes of initial data \( \psi_0 \) one has a finite existence time \( T^*(\psi_0) \). The starting point is the formula for the second derivative of the variance, or virial identity, obtained in Section 3 (see Lemma 3.3). Such a formula contains an anomalous term, which is positive definite and not conserved by the evolution, and that prevents a simple identification of an invariant set of initial data that blow-up. To overcome the issue, we adopt a strategy originally developed in the classical paper [5] for the unperturbed model (see for more details Section 8.2 in [7]) and that has to be suitably modified, relying on variational properties of certain constrained functionals, here the action functional \( S_\omega \) on the Nehari manifold (see Section 2). Existence and properties of the ground state \( \varphi_\omega \) of the action have been studied in [2] and [9]. In particular \( \varphi_\omega \) exists for any \( \alpha \) and for any \( \omega > -E_\alpha \). Our first main result gives a class of initial data (containing an open set in the phase space) that undergo blow-up (\( E \) is the total energy \( (2.4) \) and \( Q \) is the functional defined in \( (3.4) \)).

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**Theorem 1.1.** Let $p > 3$ and $\psi_0 \in \Sigma$ such that $S(\psi_0) < S_\omega(\phi_\omega)$, $E(\psi_0) \geq 0$, and $Q(\psi_0) < 0$. Then $T^*(\psi_0) < +\infty$.

We notice that analogous results with an analogous strategy have been already obtained in different models, including the one dimensional delta interaction (see in particular [10, 16]); the latter however is a form perturbation of the Laplacian, and in this sense it can be considered a standard potential, allowing for an easier treatment in comparison to the present model. Notice also that the virial identity (see [3, 3]) needs a somewhat different treatment than the standard formula, as well as the transformation properties of the mass preserving map $T^*$ needed to the analysis (see Proposition 3.7). The second result concerns the strong instability of the standing waves, i.e. the fact that in the vicinity of any standing waves there are solutions that blow-up (see Definition 2.3). This fact, again following the ideas contained in [5] and in the cited papers related to more standard potential perturbation of the laplacian, is contained in the second main result.

**Theorem 1.2.** Let $p > 3$, $\omega > |E_\alpha|$ and $\phi_\omega$ a ground state of the action $S_\omega$ with $E(\phi_\omega) > 0$. Then the standing wave $\phi_\omega e^{i\omega t}$ is strongly unstable.

It is well known that in the unperturbed NLS equation, the ground states with $p > 3$ have positive energy, while in the present case one expects positive energy only for sufficiently big $\omega$ in analogy with the known case of the presence of external potentials (See Remark 3.11).

In the last Section we consider a different condition for the strong instability of standing waves, replacing the positive total energy with the more general $\frac{d^2}{d\omega^2}S_\omega(\varphi_\omega) \leq 0$. It is known that this condition is sufficient to guarantee strong instability in the case of NLS with a delta potential in one dimension and with the generalized Coulomb potential with general dimension (see [10, 16]). We show first that a an invariant set of blowing-up initial data exist (see Theorem 4.3), and then that the action ground states with $\frac{d^2}{d\omega^2}S_\omega(\varphi_\omega) \leq 0$ belong to the norm closure of this set and so they are strongly unstable (see Theorem 4.4).

2. Preliminaries

2.1. Point interaction in 2d. In the following we shall denote with a boldface points in $\mathbb{R}^2$. Let us recall, see [3], that for $n = 2$ the operator $\mathcal{H}_\alpha$ has the domain:

$$D(\mathcal{H}_\alpha) = \{\psi \in L^2(\mathbb{R}^n) | \psi = \phi^\lambda + q G^\lambda, \phi^\lambda \in H^2(\mathbb{R}^2), q = (\Gamma^\lambda)^{-1} \phi^\lambda(0)\}$$

with $G^\lambda$ fundamental solution of the laplacian and $\Gamma^\lambda$ a certain fixed constant. Explicitly ($\mathcal{F}$ is the Fourier transform)

$$G^\lambda := (-\Delta + \lambda)^{-1}\delta_0 = \frac{1}{2\pi} \mathcal{F}^{-1} \left[ \frac{1}{|k|^2 + \lambda} \right] = \frac{1}{2\pi} K_0(\sqrt{\lambda}|x|)$$

$$\Gamma^\lambda_\alpha := \alpha + \frac{1}{2\pi} \gamma + \frac{1}{2\pi} \ln(\sqrt{\lambda}/2), \quad \alpha \in \mathbb{R} \cup \{+\infty\}.$$

Here $K_0$ is the MacDonald function of order zero and $\gamma$ is the Euler-Mascheroni constant. The constant $\alpha$ is real (nontrivial interaction) or $+\infty$ ($q = 0$, corresponding to the standard Laplacian). It enters in the relation $\phi^\lambda(0) = \Gamma^\lambda_\alpha q$, playing the role of a boundary condition at the singularity and more concretely it is related to the s-wave scattering length $a_0$ through the relation $a_0 = (-2\pi\alpha)^{-1}$. The number $\lambda$ can be any number in $\mathbb{R}^+ \setminus \{-E_\alpha\}$, where $E_\alpha$ is the negative eigenvalue of $\mathcal{H}_\alpha$, always existing when $\alpha \in \mathbb{R}$ (see later). The action of the operator is given by

$$(\mathcal{H}_\alpha + \lambda)\psi = (-\Delta + \lambda)\phi^\lambda \quad (\iff \mathcal{H}_\alpha \psi = -\Delta \phi^\lambda - \lambda q G^\lambda) \quad \forall \psi \in D(\mathcal{H}_\alpha)$$

It is easily seen that while the decomposition in regular part $\phi^\lambda$ and singular part $q G^\lambda$ of any element $\psi \in D(\mathcal{H}_\alpha)$ depends on the choice of $\lambda$, the definition of $\mathcal{H}_\alpha$ does not. We often use the short notation
\[ \mathcal{D}_\alpha := D(\mathcal{H}_\alpha). \] One has \( \sigma_c(\mathcal{H}_\alpha) = \sigma_{ac}(\mathcal{H}_\alpha) = [0, \infty); \mathcal{H}_\alpha \) has a simple negative eigenvalue \( \{ E_\alpha \} \) for any \( \alpha \in \mathbb{R} \) and \( \psi_\alpha \) is the corresponding eigenvector. Explicitly
\[
E_\alpha = -4e^{-2(2\pi \alpha + \gamma)}, \quad \psi_\alpha(x) = \frac{1}{2\pi}K_0(2e^{-(2\pi \alpha + \gamma)}x).
\]
Let us also introduce the quadratic form \( \mathcal{F}_\alpha \) on \( L^2(\mathbb{R}^n) \) with domain and action
\[
\mathcal{D}(\mathcal{F}_\alpha) = \{ \psi \in L^2(\mathbb{R}^n) \mid \exists q \in \mathbb{C}, \phi^\lambda \in H^1(\mathbb{R}^n) : \psi = \phi^\lambda + qG^\lambda \}
\]
\[
\mathcal{F}_\alpha(\psi) = \mathcal{F}_\lambda(\psi) + \Gamma^\lambda q^2 \quad \text{and} \quad \mathcal{F}_\lambda(\psi) = ||\nabla \phi^\lambda||^2 + \lambda(||\phi^\lambda||^2 - ||\psi||^2)
\]
It does not depend on \( \lambda \), it is symmetric, closed, bounded from below. The map \( \psi \mapsto \mathcal{F}_\alpha(\psi) + \lambda||\psi||^2 \) is positive for every \( \lambda > -E_\alpha \) and it coincides with \( \langle \psi, (\mathcal{H}_\alpha + \lambda)\psi \rangle \forall \psi \in D(\mathcal{H}_\alpha) \). This allows to interpretate the form domain \( D(\mathcal{F}_\alpha) \) as the domain of the square root of the positive self-adjoint operator \( \mathcal{H}_\alpha + \lambda \), so that we make use of the notation
\[
\mathcal{D}(\mathcal{F}_\alpha) = D((\mathcal{H}_\alpha + \lambda)^{\frac{1}{2}}) := D_{\alpha}^{s}, \quad \lambda > -E_\alpha.
\]
Notice that algebraically and topologically one has \( \mathcal{D}(\mathcal{F}_\alpha) \cong H^1(\mathbb{R}^2) \oplus \mathbb{C} \) and the form domain is in a natural way a Hilbert space. By functional calculus we can introduce the scale of Hilbert spaces
\[
\mathcal{D}_\alpha^s := D((\mathcal{H}_\alpha + \lambda)^s), \quad s \in \mathbb{R}, \quad \lambda > -E_\alpha.
\]
\( \mathcal{D}_\alpha^s \) is equipped with the norm \( ||\psi||_{\mathcal{D}_\alpha^s} := ||(\mathcal{H}_\alpha + \lambda)^s\psi|| \), equivalent to the graph norm of the operator \( (\mathcal{H}_\alpha + \lambda)^s \). In particular, the spaces \( \mathcal{D}_\alpha^s \) and \( \mathcal{D}_{-\alpha}^{-s} \) are in duality and
\[
\mathcal{D}_\alpha^s \hookrightarrow L^2(\mathbb{R}^2) \hookrightarrow \mathcal{D}_{-\alpha}^{-s}
\]
is a Hilbert triplet. We denote the duality product by \( \langle \cdot, \cdot \rangle_{-s,s} \). In the following we will only consider the case \( s = \frac{1}{2} \) and we stress that \( ||\psi||_{\mathcal{D}_\alpha^{\frac{1}{2}}} \cong ||\phi||_{H^1} \oplus ||q|| \). Finally, we recall that the fundamental solution \( G_\lambda \) is positive, radial, strictly decreasing and moreover it has the following asymptotic behavior (see [1], formulae 9.6.12 and 9.6.13 for the first asymptotic and 9.7.2 for the second)
\[
(2.2) \quad G_\lambda = -\frac{1}{2\pi} \ln(\sqrt{\frac{\lambda}{2}}|x|) - \frac{\gamma}{2\pi} + o(1) \quad \text{as} \quad x \to 0, \quad G_\lambda \sim \frac{1}{\sqrt{8\pi \sqrt{\lambda}}e^{-\sqrt{\lambda}|x|}} \quad \text{as} \quad x \to \infty.
\]

### 2.2. The NLS equation with a point interaction.

We are interested in solutions of the Cauchy problem for the NLS equation
\[
(2.3) \quad \begin{cases}
  i\partial_t \psi(t) = \mathcal{H}_\alpha \psi(t) + f(\psi(t)) \\
  \psi(0) = \psi_0 \in \mathcal{D}_\alpha \quad \text{or} \quad \mathcal{D}_\alpha^{\frac{1}{2}}
\end{cases}
\]
where \( f(\psi) = |\psi|^{p-1}\psi \), and \( g = \pm 1 \).

The following theorem collects the known results about well posedness in the energy domain (mild solution) and operator domain (strong solution) for the equation (2.3) (see [6] where a detailed analysis of the well posedness of strong solutions is given, also for the three dimensional case, and [9] where a treatment of the solutions in the energy domain is given).

**Theorem 2.1** (Well-Posedness in \( \mathcal{D}_\alpha^{\frac{1}{2}} \) and \( \mathcal{D}_\alpha \)). Assume \( p > 1 \) and \( \psi_0 \in \mathcal{D}_\alpha^{\frac{1}{2}} \). Then the following properties hold true.

1) There exists \( T > 0 \) and a unique weak solution of (2.3) in \( C([0, T]; \mathcal{D}_\alpha^{\frac{1}{2}}) \cap C^1([0, T]; \mathcal{D}_\alpha^{-\frac{1}{2}}) \).

2) The following blow-up alternative holds. Let the maximal existence time be defined as
\[
T^* = \sup_{T > 0} \left\{ \psi \in C([0, T], \mathcal{D}_\alpha^{\frac{1}{2}}) \cap C^1([0, T], \mathcal{D}_\alpha^{-\frac{1}{2}}) \text{ solves mildly } (2.3) \right\}.
\]
then
\[ \lim_{t \to T^*} \| \psi(t) \|_{D^\frac{1}{2}_a} < \infty \implies T^* = \infty. \]

3) \( L^2 \)-mass is conserved along the evolution: \( \| \psi(t) \|^2 = \| \psi_0 \|^2 \ \forall t \in [0, T^*) \).

4) Energy is conserved along the evolution: \( E(\psi(t)) = E(\psi_0) \ \forall t \in [0, T^*) \)

where
\[ (2.4) \quad E(\psi) = \frac{1}{2} F_\omega(\psi) + \frac{g}{p+1} \| \psi \|_{p+1}^{p+1} \quad \forall \psi \in D^\frac{1}{2}_a. \]

5) Let \( \psi_0 \in D_\alpha \). Then there exists \( T > 0 \) and a unique strong solution \( \psi \) of \( (2.3) \) in \( C([0, T]; D_\alpha) \cap C^1([0, T]; L^2(\mathbb{R}^2)) \).

6) Let \( \psi_0 \in D_\alpha \) and the maximal existence time be defined as
\[ \tilde{T}^* = \sup \{ \psi \in C([0, T], D(H_\alpha)) \cap C^1([0, T], L^2) \ |	ext{ solves strongly (2.3)} \}; \]

then \( \lim_{T \to \tilde{T}^*} \| \psi(t) \|_{D_\alpha} < \infty \implies \tilde{T}^* = \infty. \)

7) \( \tilde{T}^* = T^* \).

8) \( T^* = +\infty \) if \( g = +1 \) and \( p > 1 \) or if \( g = -1 \) and \( 1 < p < 3 \).

In the following we will denote as \( T^*(\psi_0) \) the maximal existence time of the solution of \( (2.3) \). When \( T^*(\psi_0) < +\infty \) we say that the solution \( \psi(t) \) corresponding to the initial datum \( \psi_0 \) blows-up in a finite time (in the future, analogous definition for blow-up in the past). We will omit the dependence of \( T^* \) on \( \psi_0 \) when it is clear from the context.

Recall that a standing wave of \( (2.3) \) is a solution of the form \( \psi(t) = e^{i\omega t} \varphi \). The profile \( \varphi_\omega \) is a solution of the stationary equation
\[ (2.5) \quad H_\omega \varphi + \omega \varphi + f(\varphi) = 0 \]
equivalent to \( S'_\omega(\varphi) = 0 \), where the action functional \( S_\omega \) is defined as
\[ S_\omega(\psi) = E(\psi) + \frac{\omega}{2} \| \psi \|^2 \quad \forall \psi \in D^\frac{1}{2}_\alpha. \]

The set of ground states of the action \( S_\omega \) is defined as
\[ \mathcal{G} = \{ \varphi_\omega \in D^\frac{1}{2}_\alpha \mid \text{s.t. } S_\omega(\varphi_\omega) \leq S_\omega(\varphi) \ \forall \varphi \in D^\frac{1}{2}_\alpha \text{ satisfying } S'(\varphi) = 0 \}. \]

Recently in [9] and [2] existence and properties of ground states of the action \( S_\omega \) for the case of attractive nonlinearity (i.e., \( g = -1 \)) in (2.3) have been proved by variational methods. In particular, a ground state exists for every \( \omega > -E_\alpha \) and if \( \varphi_\omega \in \mathcal{G} \) then
\[ d(\omega) = \inf \{ S_\omega(\psi) \mid \psi \in D^\frac{1}{2}_\alpha, \ \psi \neq 0, \ N_\omega(\psi) = 0 \} = S_\omega(\varphi_\omega) \]
where \( N_\omega \) is the Nehari functional
\[ N_\omega(\psi) = F_\alpha(\psi) + \omega \| \psi \|^2 - \| \psi \|_{p+1}^{p+1}. \]

The following fact is an immediate consequence of the results in [9] and [2] and it will be useful later (see also the analogous Lemma in [10]).

**Proposition 2.2.** Let \( \varphi_\omega \in \mathcal{G} \) a ground state of the action \( S_\omega \) and \( \psi \in D^\frac{1}{2}_\alpha \) s.t. \( \| \psi \|_{p+1}^{p+1} = \| \varphi_\omega \|_{p+1}^{p+1}. \) Then
a) \( N_\omega(\psi) \geq 0 \)
b) \( S_\omega(\psi) \geq S_\omega(\varphi_\omega). \)
Proof. From Lemma 3.3 in [9] and $d(\omega) = \inf \{ \frac{p}{2(p+1)} \| \varphi \|_{p+1}^p : \varphi \in \mathcal{N}_\omega \} = \frac{p}{2(p+1)} \| \varphi_\omega \|_{p+1}^p = S_\omega(\varphi_\omega)$ property a) follows. Taking into account a) one has $S_\omega(\varphi_\omega) = \frac{p}{2(p+1)} \| \varphi_\omega \|_{p+1}^p = \frac{p}{2(p+1)} \| \varphi_\omega \|_{p+1}^p + N_\omega(\varphi_\omega) \leq \frac{p}{2(p+1)} \| \varphi \|_{p+1}^p + \frac{1}{2} N_\omega(\varphi_\omega) = S(\varphi)$. □

Definition 2.3. The standing wave $\psi(t) = e^{i\omega t} \varphi_\omega$ is said to be strongly unstable if for every $\varepsilon > 0$ there exist $\psi_0 \in D^\frac{1}{2}_\alpha$ such that $\| \psi_0 - \varphi_\omega \|_{D^\frac{1}{2}_\alpha} < \varepsilon$ with $T^*(\psi_0) < +\infty$.

3. Blow-up and strong instability.

3.1. Virial identity. The space $\Sigma$ adapted to the point interaction framework is defined as follows.

Definition 3.1. We put $\Sigma_\alpha := \{ \psi \in D^\frac{1}{2}_\alpha(\mathbb{R}^2) \mid x \psi \in L^2(\mathbb{R}^2) \}$

Lemma 3.2. Let $\psi_0 \in \Sigma$ and $\psi \in C([0, T^*) ; D^\frac{1}{2}_\alpha(\mathbb{R}^2))$ the corresponding weak maximal solution of (2.3). Then $\psi \in C([0, T^*) ; \Sigma)$. Moreover for any fixed $\psi \in D^\frac{1}{2}_\alpha(\mathbb{R}^2)$ the variance

$$I(t) := \int_{\mathbb{R}^2} |x|^2 |\psi(t, x)|^2 \, dx$$

defines a $C^1([0, T^*) ; \mathbb{R})$ function and

$$\frac{d}{dt} I(t) = 4 \Im \int_{\mathbb{R}^2} \bar{\psi}(t, x) x \cdot \nabla \psi(t, x) \, dx. \tag{3.1}$$

When we want to emphasize the dependence on $\psi(t)$ we use the notation $I_\psi(t)$.

Proof. We firstly show that $t \mapsto x \psi(t, x) \in C^0([0, T^*) ; L^2(\mathbb{R}^2))$. Let $\chi_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$, $\chi_\varepsilon(x) = e^{-\varepsilon |x|^2}$. Let $\psi_0 \in \Sigma$ and $\psi \in C([0, T^*), D^\frac{1}{2}_\alpha) \cap C^1([0, T^*) ; D^\frac{1}{2}_\alpha(\mathbb{R}^2))$ the weak solution of the (2.3). One has $x \chi_\varepsilon \psi \in C([0, T^*), D^\frac{1}{2}_\alpha)$ and for any $t \in [0, T^*)$ we have

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x\chi_\varepsilon \psi(t, x)|^2 \, dx = 2 \Re \langle |x|^2 \chi_\varepsilon^2 \psi, \partial_t \psi \rangle - \frac{1}{2}$$

$$= 2 \Re \langle |x|^2 \chi_\varepsilon^2 \psi, -i \mathcal{H}_\alpha \psi - ig |\psi|^{-1} \psi \rangle - \frac{1}{2}$$

$$= 2 \Im \langle \mathcal{H}_\alpha (|x|^2 \chi_\varepsilon^2 \psi), \psi \rangle - \frac{1}{2}$$

$$= 2 \Im (\Delta (|x|^2 \chi_\varepsilon^2 \psi), \psi) - \frac{1}{2}$$

$$= -2 \Im \int_{\mathbb{R}^2} \psi \nabla \cdot \nabla (|x|^2 \chi_\varepsilon^2 \psi) \, dx$$

$$= -2 \Im \int_{\mathbb{R}^2} \psi \nabla \cdot [\chi_\varepsilon^2 (|x|^2 \nabla \overline{\psi} + 2 x \overline{\psi} - 2 \varepsilon \varepsilon x |x|^2 \overline{\psi})] \, dx.$$ 

Now we can integrate by parts noticing that $x \nabla \psi \in L^2_{\text{loc}}(\mathbb{R}^2)$ and after suppressing a real term in the integrand we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x\chi_\varepsilon \psi(t, x)|^2 \, dx = 4 \Im \int \chi_\varepsilon^2 (1 - |x|^2) \frac{\psi}{\overline{\psi}} x \cdot \nabla \psi \, dx$$
and integrating in time
\[(3.2) \quad I_\varepsilon(t) = I_\varepsilon(0) + 4 \text{Im} \int_0^t \int \chi_\varepsilon^2(1 - \varepsilon |x|^2) \overline{\psi} \cdot \nabla \psi \, dx \, ds \]

Notice now that \( \mathbf{x} \cdot \nabla \psi = \mathbf{x} \cdot \nabla \phi^\lambda + q \mathbf{x} \cdot \nabla G^\lambda \) and taking into account that \( \| \nabla \phi^\lambda \| \leq c \| \psi \|_{D_{\alpha}^1}, \)
\( \| \mathbf{x} \cdot q \nabla G^\lambda \| \leq c \| \psi \|_{D_{\alpha}^1} \) one gets
\[
I_\varepsilon(t) \leq I_\varepsilon(0) + c(m) \int_0^t \| \psi(s) \|_2^2 \, ds + c \int_0^t \| \psi(s) \|_2^2 \, I_\varepsilon^{1/2}(s) \, ds
\]

where \( c(m) \) is a constant depending on the mass. From Grönwall inequality it follows that there exists a constant \( c \) independent on \( \varepsilon \) such that
\[
I_\varepsilon(t) \leq c \quad t \in [0, T^*)
\]

From Fatou lemma one finally concludes that
\[
I(t) = \int \liminf \chi_\varepsilon^2|x|^2|\psi(t, x)|^2 \, dx \leq \liminf \chi_\varepsilon^2|x|^2|\psi(t, x)|^2 \, dx \leq c \quad t \in [0, T^*)
\]
which gives \( I(t) \in L^\infty \quad \forall t \in [0, T^*) \) , the map \( t \mapsto \| | \cdot |u(t, \cdot)| \| \) is bounded on any \( (0, T) \) with \( T < T^* \) and consequently weakly continuous as a map \( (0, T^*) \to L^2(\mathbb{R}^2) \). From (3.2), the fact that \( \psi \mathbf{x} \cdot \nabla \psi \in C_t L^1_x \) and the dominated convergence theorem, we also obtain
\[
I(t) = I(0) + 4 \text{Im} \int_0^t \int \overline{\psi} \mathbf{x} \cdot \nabla \psi \, dx \, ds \quad \forall t \in [0, T^*)
\]

which gives at once that the \( \psi \in C^0((0, T^*); \Sigma) \) and validity of (3.1) \( \square \)

The crucial information is contained in the following lemma

**Lemma 3.3. [Virial identity]** Let \( \psi_0 \in \Sigma \) and \( \psi \in C((0, T^*); D_{\alpha}^1) \) the corresponding maximal weak solution of (2.3). Then the function
\[
(3.3) \quad t \mapsto I(t) = \int_{\mathbb{R}^2} |x|^2 |\psi(t, x)|^2 \, dx
\]
is in \( C^2([0, T^*]; \mathbb{R}) \) and the following identity holds
\[
\frac{d^2}{dt^2} I(t) = 16E(\psi) + 8g \frac{(p-3)}{p+1} \| \psi(t) \|_{\dot{H}^{p+1}}^2 + \frac{2}{\pi} |q|^2
\]
where
\[
(3.4) \quad Q(\psi) = \mathcal{F}_\alpha(\psi_0) + g \frac{(p-1)}{p+1} \| \psi(t) \|_{\dot{H}^{p+1}}^2 + \frac{1}{4\pi} |q|^2.
\]

**Proof.** Let us show the result first assuming that \( \psi_0 \in \Sigma \cap \mathcal{D}_\alpha \) and considering the corresponding strong solution \( \psi \in C((0, T^*); D_{\alpha}) \cap C^1((0, T^*); L^2(\mathbb{R}^2)) \). We need to derive in time the r.h.s. of (3.1). We regularize it writing
\[
(3.5) \quad h_\varepsilon(t) := \text{Im} \int_{\mathbb{R}^2} e^{-\varepsilon |x|^2} \overline{\psi} \mathbf{x} \cdot \nabla \psi \, dx.
\]
Admitting that \( \psi \in C^1([0, T^*), D_{\alpha}^{3/2}) \) one can safely derive in time (3.5), obtaining

\[
\dot{h}_\varepsilon(t) := \text{Im} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \bar{\psi} \cdot x \cdot \nabla \psi \, dx + \text{Im} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \bar{\psi} \cdot x \cdot \nabla \psi \, dx.
\]

Both addenda are well defined, and more precisely the map \( t \mapsto e^{-\varepsilon|x|^2} \cdot \nabla \psi \) is in \( C^1([0, T^*), L^2(\mathbb{R}^2)) \) because

\[
\| e^{-\varepsilon|x|^2} \cdot \nabla \psi \| = \| e^{-\varepsilon|x|^2} \cdot \nabla \phi + e^{-\varepsilon|x|^2} \psi(t)x \cdot \nabla G^\lambda \| \leq c_\varepsilon (\| \nabla \phi \| + |q(t)|) \leq \| \psi \|_{D_\alpha^{3/2}},
\]

Now, integrating by part the second addendum one has

\[
(3.6) \quad \dot{h}_\varepsilon(t) := \text{Im} \left\{ \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \left( \bar{\psi} \cdot x \cdot \nabla \psi - \dot{\psi} x \cdot \nabla \psi \right) \, dx - 2 \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \left( \bar{\psi} \cdot \nabla \psi - \varepsilon|x|^2 \bar{\psi} \right) \, dx \right\}
\]

and the r.h.s. is well defined and continuous in time only assuming \( \psi \in C^1([0, T^*), D_{\alpha}^{3/2}) \). By density of \( C^1([0, T^*), D_{\alpha}^{3/2}) \) in \( C((0, T^*), D_{\alpha}^{3/2}) \cap C^1([0, T^*), L^2) \) (which is proven as in the case of standard Sobolev case), formula (3.6) still holds in these hypotheses. Now we perform a second regularization considering \( \psi \in C^0([0, T^*), D_{\alpha}) \cap C^1([0, T^*), L^2) \), so that we can apply the equation in strong form. From (3.6), by using \( \text{Im} \, z = - \text{Im} \, \bar{z} \) and then the equation in (2.3) one obtains

\[
\dot{h}_\varepsilon(t) = 2 \text{Im} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \bar{\psi} (x \cdot \nabla \psi + \psi) \, dx + 2 \text{Im} \varepsilon \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} |x|^2 \bar{\psi} \, dx
\]

\[
= 2 \text{Im} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \left( \bar{H} \psi + f(\psi) \right) (x \cdot \nabla \psi + \psi) \, dx + 2 \varepsilon \text{Im} (\text{Re} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} |x|^2 \psi \, dx) \]

\[
= 2 \text{Re} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \left( \bar{H} \psi + f(\psi) \right) \psi \, dx - 2 \text{Re} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} |x|^2 \bar{\psi} \, dx + \text{Re} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \bar{f}(\psi) (x \cdot \nabla \psi) \, dx = I + II + III + IV
\]

Thanks to the dominated convergence theorem the term I converges to

\[
2 \left( \langle H \psi, \psi \rangle + \| \psi \|^2_{p+1} \right) = 4E(\psi) - 4 \int_{\mathbb{R}^2} F(\psi) \, dx + (2p + 1) \int_{\mathbb{R}^2} \psi^p \, dx
\]

(3.7)

\[
= 4E(\psi) + 2(p - 1) \int_{\mathbb{R}^2} F(\psi) \, dx
\]

where we have denoted

\[
F(\psi) = \frac{9}{p + 1} |\psi|^p \cdot \psi^{p - 1}.
\]

The term II is vanishing and now let us consider III and IV. To treat IV we make use of the identity

\[
2 \text{Re} e^{-\varepsilon|x|^2} \bar{f}(\psi) x \cdot \nabla \psi = \nabla \cdot (2x e^{-\varepsilon|x|^2} F(\psi)) + 4 \varepsilon |x|^2 e^{-\varepsilon|x|^2} F(\psi) - 4 e^{-\varepsilon|x|^2} F(\psi)
\]

and it follows by the divergence theorem and dominated convergence that

\[
(3.8) \quad 2 \text{Re} \int_{\mathbb{R}^2} e^{-\varepsilon|x|^2} \bar{f}(\psi) x \cdot \nabla \psi \, dx = \int_{\mathbb{R}^2} 4 \varepsilon |x|^2 e^{-\varepsilon|x|^2} F(\psi) - 4 e^{-\varepsilon|x|^2} F(\psi) \, dx \rightarrow -4 \int_{\mathbb{R}^2} F(\psi) \, dx
\]
For $III$ we preliminarily decompose the domain element in regular and singular part, obtaining

$$2 \operatorname{Re} \int_{\mathbb{R}^2} e^{-|x|^2} \overline{\psi} \cdot \nabla \psi \, dx =$$

$$2 \operatorname{Re} \int_{\mathbb{R}^2} e^{-|x|^2} \left( -\overline{\Delta \phi^\lambda} \cdot \nabla \phi^\lambda - \lambda \overline{q G^\lambda} \cdot \nabla \phi^\lambda - q \overline{\Delta \phi^\lambda} \cdot \nabla G^\lambda - \lambda |q|^2 G^\lambda \right) \, dx =$$

$$III_a + III_b + III_c + III_d$$

Now we treat separately the various addenda. Integrating by parts the following identity, which holds true in the two dimensional case,

$$\int_{\mathbb{R}^2} \overline{\Delta \phi^\lambda} \cdot \nabla (\phi^\lambda) e^{-|x|^2} \, dx - 4 \operatorname{Re} \int_{\mathbb{R}^2} e^{-|x|^2} |x| \cdot \nabla \phi^\lambda|^2 \, dx$$

The second term vanishes by dominated convergence and the first term vanishes as well exploiting the following identity, which holds true in the two dimensional case,

$$2 \operatorname{Re} e^{-|x|^2} \overline{\Delta \phi^\lambda} \cdot \nabla (x \cdot \nabla \phi^\lambda) = 2 |x| e^{-|x|^2} |\nabla \phi^\lambda|^2 + \nabla \cdot (x e^{-|x|^2} |\nabla \phi^\lambda|^2)$$

and then integrating and applying the divergence theorem and dominated convergence again.

To proceed, let us note preliminary the following identities (where formula (2.1) is used and it is essential the dimension 2 in the first):

$$\mathcal{F}(x \cdot \nabla G^\lambda) = -\frac{1}{2\pi} \nabla \cdot \frac{k}{|k|^2 + \lambda} = \frac{1}{2\pi} \frac{2\lambda}{|k|^2 + \lambda}$$

$$\mathcal{F}(\nabla \cdot x G^\lambda) = -\frac{1}{2\pi} k \cdot \nabla \frac{1}{|k|^2 + \lambda} = \frac{1}{2\pi} \frac{2|k|^2}{|k|^2 + \lambda}$$

In particular one sees that $x \cdot \nabla G^\lambda \in H^2(\mathbb{R}^2)$ and $x G^\lambda \in H^1(\mathbb{R}^2)$, and we can integrate by parts in

$$III_b + III_c = 2 \operatorname{Re} \int_{\mathbb{R}^2} e^{-|x|^2} \left( -\lambda \overline{q G^\lambda} \cdot \nabla \phi^\lambda - q \overline{\Delta \phi^\lambda} \cdot \nabla G^\lambda \right) \, dx$$

$$= 2 \operatorname{Re} \int_{\mathbb{R}^2} \left( \lambda G^\lambda \overline{\phi^\lambda} \cdot \nabla e^{-|x|^2} - q \overline{\phi^\lambda} \cdot \nabla G^\lambda (\Delta e^{-|x|^2}) - 2 q \overline{\phi^\lambda} \cdot \nabla (x \cdot \nabla G^\lambda) \cdot \nabla e^{-|x|^2} \right) \, dx$$

$$+ 2 \operatorname{Re} \int_{\mathbb{R}^2} e^{-|x|^2} (\lambda \overline{\phi^\lambda} \cdot \nabla (x G^\lambda) - q \overline{\phi^\lambda} \Delta (x \cdot \nabla G^\lambda) ) \, dx$$

The last integral identically vanishes and from anyone of the terms in the first integral can be extracted a factor $\varepsilon$; one concludes that $III_b + III_c \to 0$ by dominated convergence.

It remains to consider the limit for $\varepsilon \to 0$ of $III_d$, which can be computed explicitly thanks to the Plancherel theorem and identities (3.9):

$$III_d = 2 \operatorname{Re} \int_{\mathbb{R}^2} e^{-|x|^2} \left( -\lambda |q|^2 G^\lambda \cdot \nabla G^\lambda \right) \, dx \to -2 \lambda |q|^2 \operatorname{Re} \int_{\mathbb{R}^2} G^\lambda \cdot \nabla G^\lambda \, dx$$

$$+ \frac{2\lambda^2 |q|^2}{\pi} \int_{0}^{\infty} \frac{r}{(r^2 + \lambda)^2} \, dr = \frac{1}{2\pi} |q|^2.$$
Finally, collecting (3.7), (3.8), (3.10) and taking into account that the other terms involved vanish, we obtain
\[ \tilde{I}(t) = 4\lim_{\varepsilon \to 0} \tilde{h}_\varepsilon(t) = 16E(\psi) + (8p - 24) \int_{\mathbb{R}^2} F(\psi) \, dx + \frac{4}{\pi}|q|^2 \]
\[ = 16E(\psi) + 8g \int_{\mathbb{R}^2} |\psi|^{p+1} \, dx + \frac{2}{\pi}|q|^2 \]
\[ = 8F_\alpha(\psi) + 8g \frac{(p-1)}{p+1}||\psi(t)||_{p+1}^{p+1} + \frac{2}{\pi}|q|^2 \]
Having proved the identity (3.3) for strong solutions, the same identity follows for weak solutions exploiting continuous dependence and density, and this ends the proof of the Lemma. □

3.2. Mass preserving scaling and its properties. From now on, we will only consider the attractive nonlinearity, i.e. the case \( g = -1 \).

Definition 3.4. Let us introduce the mass preserving scaling map \( T^\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \)
\[ T^\sigma(\psi) \equiv \psi(\mathbf{x}) = \sigma \psi(\mathbf{x}) \quad \forall \psi \in L^2(\mathbb{R}^2) \]

Remark 3.5. By using \( G^\lambda(\psi) = \frac{1}{2\pi}K_0(\sqrt{\lambda}|\mathbf{x}|) = \frac{1}{2\pi}K_0(\sqrt{\lambda}\sigma^2|x|) = G^{\lambda\sigma^2}(\mathbf{x}) := G^{\lambda\sigma}(\mathbf{x}) \) one obtains
\[ \psi^\sigma(\mathbf{x}) = \sigma \phi(\sigma \mathbf{x}) + q \sigma^2 G^\lambda(\sigma \mathbf{x}) = \psi^\lambda(\mathbf{x}) + q^\sigma G^{\lambda\sigma}(\mathbf{x}) \]
and the map \( \psi \to \psi^\sigma \) leaves invariant \( D^\dagger_\alpha \). It also follows that \( q^\sigma = \sigma q \).

Remark 3.6. One has \( G^\lambda - G^{\lambda\sigma^2} \in H^{3-\varepsilon} \quad \forall \sigma > 0 \) and exploiting the first asymptotic relation in (2.2) one obtains \( (G^{\lambda\sigma^2} - G^\lambda)(0) = -\frac{1}{2\pi}\log \sigma \). From this one concludes that \( T^\sigma \) does not preserve \( D^\dagger_\alpha \). Instead, \( T^\sigma : D_\alpha \to D_{\alpha - \frac{1}{2\pi}\log \sigma} \). In fact, from \( \psi = \phi^\lambda + q G^\lambda, \phi^\lambda(0) = \Gamma^\lambda_0 q \), it follows
\[ \psi^\sigma(\mathbf{x}) = \sigma^2 \phi^\lambda(\mathbf{x}) + \sigma q G^\lambda(\mathbf{x}) = \sigma^2 \phi^\lambda(\mathbf{x}) + \sigma q(G^{\lambda\sigma^2} - G^\lambda)(\mathbf{x}) + \sigma q G^\lambda(\mathbf{x}) \]
\[ = (\phi^\lambda)^\sigma(\mathbf{x}) + q^\sigma G^\lambda(\mathbf{x}) \]
and \( (\phi^\lambda)^\sigma(0) = \sigma \phi^\lambda(0) - \sigma \frac{1}{2\pi}\log \sigma = \sigma q(\Gamma^\lambda_0 - \frac{1}{2\pi}\log \sigma) = q^\sigma \Gamma^\lambda_{\alpha - \frac{1}{2\pi}\log \sigma} \); hence \( \psi^\sigma \in D_{\alpha - \frac{1}{2\pi}\log \sigma} \).

Proposition 3.7. Let \( \psi \in D^\dagger_\alpha \). Then
\[ F_\alpha(\psi^\sigma) = \sigma^2 F_\alpha(\psi) + \frac{|q|^2}{2\pi}\sigma^2 \log \sigma \]
\[ \|\psi^\sigma\|_{p+1}^{p+1} = \sigma^{p-1}\|\psi\|_{p+1}^{p+1} \]
\[ \frac{d}{d\sigma}F_\alpha(\psi^\sigma)|_{\sigma=1} = 2F_\alpha(\psi) + \frac{1}{2\pi}|q|^2 \]
\[ \frac{d}{d\sigma}\|\psi^\sigma\|_{p+1}^{p+1}|_{\sigma=1} = (p-1)\|\psi\|_{p+1}^{p+1} \]
\[ \frac{d}{d\sigma}E(\psi^\sigma)|_{\sigma=1} = \frac{d}{d\sigma}S(\psi^\sigma)|_{\sigma=1} = Q(\psi) \]

Proof. From \( \psi^\sigma = \sigma \phi(\sigma \mathbf{x}) + \sigma q G^{\lambda\sigma^2} \) and the identity \( \Gamma^{\lambda\sigma^2} = \Gamma^\lambda + \frac{1}{2\pi} \log \sigma \) one obtains immediately \( F_\alpha(\psi^\sigma) = \sigma^2 \|\nabla \phi\|^2 + \sigma^2 \Gamma^\lambda_{\alpha} |q|^2 + \lambda \sigma^2 (\|\phi\|^2 - \|\psi\|^2) = \sigma^2 F_\alpha(\psi) + \frac{|q|^2}{2\pi}\sigma^2 \log \sigma \). The other identities are obtained by direct computation without difficulty. □
For fixed $\varphi_\omega \in \mathcal{G}$ let us collect several properties, useful in the following, of the functions $\sigma \mapsto S_\omega(\varphi_\omega^\sigma)$ and $\sigma \mapsto Q_\omega(\varphi_\omega^\sigma)$ given by

\begin{equation}
S_\omega(\varphi_\omega^\sigma) = \frac{\sigma^2}{2} F_\alpha(\varphi_\omega) + \frac{\omega}{2} \|\varphi_\omega\|^2 + \frac{\sigma^2}{4\pi} \log \sigma |q_\omega|^2 - \frac{1}{p+1} \sigma^{p-1} \|\varphi_\omega\|_{p+1}^{p+1}
\end{equation}

\begin{equation}
Q(\varphi_\omega^\sigma) = \sigma^2 F_\alpha(\varphi_\omega) + \frac{\sigma^2}{2\pi} \log \sigma |q_\omega|^2 - \frac{p-1}{p+1} \sigma^{p-1} \|\varphi_\omega\|_{p+1}^{p+1} + \frac{\sigma^2}{4\pi} |q_\omega|^2
\end{equation}

It is immediate that $S_\omega(\varphi_\omega^\sigma), Q(\varphi_\omega^\sigma) \in C^\infty(\mathbb{R}^+)$. Let us denote

\begin{equation}
A = F_\alpha(\varphi_\omega) + \frac{1}{4\pi} |q_\omega|^2, \quad B = \frac{1}{2\pi} |q_\omega|^2, \quad C = \frac{p-1}{p+1} \|\varphi_\omega\|_{p+1}^{p+1}.
\end{equation}

Then we have

**Proposition 3.8.**

\begin{align}
\frac{d}{d\sigma} S_\omega(\varphi_\omega^\sigma) &= A\sigma + B\sigma \log \sigma - C\sigma^{p-2} \\
\frac{d^2}{d\sigma^2} S_\omega(\varphi_\omega^\sigma) &= (A + B) + B \log \sigma - C(p-2)\sigma^{p-3} \\
\frac{d^3}{d\sigma^3} S_\omega(\varphi_\omega^\sigma) &= \frac{B}{\sigma} - C(p-2)(p-3)\sigma^{p-4} \\
\frac{d}{d\sigma} S_\omega(\varphi_\omega^\sigma)|_{\sigma=1} &= 0 \text{ or equivalently } A = C \\
Q(\varphi_\omega^\sigma) &= \sigma \frac{d}{d\sigma} S_\omega(\varphi_\omega^\sigma) \\
\frac{d}{d\sigma} Q(\varphi_\omega^\sigma) &= \frac{d}{d\sigma} S_\omega(\varphi_\omega^\sigma) + \sigma \frac{d^2}{d\sigma^2} S_\omega(\varphi_\omega^\sigma)
\end{align}

**Proof.** The proof of (3.13), (3.14), (3.15) is obtained by direct computation of the derivatives. Property (3.16) is obtained just exploiting $\varphi_\omega \in \mathcal{G}$ i.e. $S_\omega'(\varphi_\omega) = 0$. Property (3.17) is a reformulation of (3.16). Properties (3.18) and (3.19) are based on the previously proven identities. For (3.18),

$$\sigma \frac{d}{d\sigma} S_\omega(\varphi_\omega^\sigma) = \sigma^2 F(\varphi_\omega) + \frac{\sigma^2}{2\pi} \log \sigma |q_\varphi|^2 + \frac{\sigma^2}{4\pi} |q_\varphi|^2 - \frac{p-1}{p+1} \sigma^{p-1} \|\varphi_\omega\|_{p+1}^{p+1} = Q(\varphi_\omega^\sigma).$$

Identity (3.19) is obtained by deriving (3.18). \qed

**Remark 3.9.** In the previous proposition, properties (3.13), (3.14), (3.15), (3.18), (3.19) do not depend from $\varphi_\omega$ being a stationary state and they hold for every $\varphi \in \mathcal{D}_\alpha^1$.  

3.3. Blow-up and strong instability. The next result is crucial for the analysis.

**Lemma 3.10.** Let $p > 3$, $\varphi \in \mathcal{D}_\alpha^1$, $\varphi \neq 0$, $E(\varphi) \geq 0$, $Q(\varphi) \leq 0$ and $\varphi_\omega \in \mathcal{G}$; then

$$S_\omega(\varphi_\omega) < S_\omega(\varphi) - \frac{1}{2} Q(\varphi).$$

**Proof.** Let

$$\sigma_0 = \left( \frac{\|\varphi_\omega\|_{p+1}^{p+1}}{\|\varphi\|_{p+1}^{p+1}} \right)^{\frac{1}{p-1}}.$$
Then $\|\varphi^{n_0}\|_{p+1} = \|\varphi_\omega\|_{p+1}$ and thanks to Lemma [2.2] it follows $S_\omega(\varphi_\omega) \leq S_\omega(\varphi^{n_0})$. Now consider the real function

$$g(\sigma) := S_\omega(\varphi^\sigma) - \frac{\sigma^2}{2}Q(\varphi) = \frac{\omega}{2}\|\varphi\|^2 + \frac{\sigma^2}{4\pi}(\log\sigma - \frac{1}{2})|q_\sigma|^2 - \frac{\sigma^2}{p+1}\left(\sigma^{p-3} - \frac{p - 1}{2}\right)\|\varphi\|_{p+1}^2$$

Suppose that $g(\sigma_0) \leq g(1)$; then, from the variational characterization [2.2] of $\varphi_\omega$ and $Q(\varphi) \leq 0$ it follows

$$S_\omega(\varphi_\omega) \leq S_\omega(\varphi^{n_0}) - \frac{\sigma_0^2}{2}Q(\varphi) \leq S_\omega(\varphi) - \frac{1}{2}Q(\varphi)$$

which is the thesis. So it is enough to show that $g(\sigma_0) \leq g(1)$. Actually we will show that $\sigma = 1$ is the unique point of absolute maximum of $g$. One has

$$g'(\sigma) = B\sigma\log\sigma - A\sigma(\sigma^{p-3} - 1).$$

It is immediate that $\sigma = 1$ is a root. An elementary analysis shows that a second root $\sigma^*$ exists in $(0, 1]$. It is an easy check that in $\sigma = 1$ there is a maximum and in $\sigma^* \in (0, 1)$ there is a minimum whatever are $A$ and $B$. Moreover, being $Q(\varphi) \leq 0$ and $E(\varphi) > 0$, one has that

$$g(1) = S_\omega(\varphi) - \frac{1}{2}Q(\varphi) \geq S_\omega(\varphi) \geq \frac{\omega}{2}\|\varphi\|^2 = g(0^+) .$$

Finally, thanks to $p > 3$ one has $\lim_{\sigma \to +\infty} g(\sigma) = -\infty$ and this ends the proof. \hfill \Box

**Proof of Theorem [1.1]**. Let us set

$$U_\omega = \left\{ \varphi \in D^2_0(\mathbb{R}^2) \text{ s.t. } S_\omega(\varphi) < S_\omega(\varphi_\omega), \ E(\varphi) \geq 0, \ Q(\varphi) < 0 \right\} .$$

We firstly show that the set $U_\omega$ is invariant for the flow of (2.3). Thanks to the conservation law of mass and energy, one has that $S_\omega(\psi(t)) < S_\omega(\varphi_\omega)$ and $E(\psi(t)) \geq 0 \ \forall t \in (0, T^*)$. It remains to show that $Q(\psi(t)) < 0$. Suppose, by absurd that there exist a time $\overline{t} \in (0, T^*)$ such that $Q(\psi(\overline{t})) = 0$. Being necessarily $\psi(\overline{t}) \neq 0$, applying Lemma [3.10] one obtains

$$S_\omega(\varphi_\omega) < S_\omega(\psi(\overline{t})) - \frac{1}{2}Q(\psi(\overline{t})) = S_\omega(\psi(\overline{t}))$$

against the hypotheses. So $Q(\psi(t)) < 0 \ \forall t \in (0, T^*)$. Now let $\psi_0 \in U_\omega \cap \Sigma$, it follows from Lemma [3.2] and the invariance of $U_\omega$ that the solution $\psi(t) \in U_\omega \cap \Sigma \ \forall t \in (0, T^*)$. From Lemma [3.3] and in particular [3.3] exploiting conservation laws of mass and energy, it follows that

$$\frac{1}{8}d^2 dt^2 I_\psi(t) = Q(\psi(t)) < 2(S_\omega(\psi(t)) - S_\omega(\varphi_\omega)) = 2(S_\omega(\psi_0) - S_\omega(\varphi_\omega)) < 0 \ \forall t \in (0, T^*(\psi_0))$$

and this implies $T^*(\psi_0) < +\infty$ by the classical elementary concavity estimate. \hfill \Box

**Proof of Theorem [1.2]**. By elliptic regularity it follows that $\varphi_\omega \in \Sigma$. Now consider $\varphi_\omega^n(\mathbf{x}) = \sigma\varphi_\omega(\sigma\mathbf{x}) \in \Sigma$. Notice that from formula (3.12) and formula (3.16)

$$E(\varphi_\omega) > 0 \iff \frac{1}{2}(A - B) - \frac{1}{p - 1}C > 0 \iff \frac{1}{2\pi}|q_\sigma|^2 < 2\frac{p - 3}{p + 1}\|\varphi_\omega\|_{p+1}^2$$

As already known, $\sigma = 1$ is a stationary point of $\sigma \mapsto S_\omega(\varphi_\omega^n)$, Moreover, $\frac{d^2}{d\sigma^2}S_\omega(\varphi_\omega^n)|_{\sigma=1} = \{(A + B) + B\log\sigma - A(p - 2)\sigma^{p-3}\}|_{\sigma=1} = A(3 - p) + B$ so that

$$\frac{d^2}{d\sigma^2}S_\omega(\varphi_\omega^n)|_{\sigma=1} < 0 \iff B < (p - 3)A \iff \frac{1}{2\pi}|q_\sigma|^2 < \frac{(p - 1)(p - 3)}{p + 1}\|\varphi_\omega\|_{p+1}^2$$
This means that \( p > 3 \) and \( E(\varphi_\omega) > 0 \) imply that \( \sigma = 1 \) is a local maximum for \( \sigma \mapsto S_\omega(\varphi^\sigma) \) and actually the absolute maximum, thanks to \( S(\varphi_\omega) > \omega \|\varphi_\omega\|^2 = S(\varphi^\sigma)|_0 \). Consequently \( S_\omega(\varphi^\sigma) < S_\omega(\varphi_\omega) \ \forall \sigma > 1 \). Finally from formula (3.18) and \( \sigma > 1 \) one has

\[
Q(\varphi^\sigma) = \sigma \frac{d}{d\sigma} S_\omega(\varphi^\sigma) < 0.
\]

To summarize, \( \varphi^\sigma \in U_\omega \cap \Sigma \ \forall \sigma > 1 \). Being \( \|\varphi^\sigma - \varphi_\omega\|_{D^\alpha_{\sigma}} \to 0 \) as \( \sigma \to 1 \) the proof is complete. \( \square \)

**Remark 3.11.** The condition \( E(\varphi_\omega) > 0 \) is expected to be true for \( \omega \geq \omega^* \) great enough. That this should be true can be understood by means of the scaling \( \varphi_\omega(x) \to \hat{\varphi}_\omega(x) = \omega^{-\frac{1}{2}} \varphi_\omega(\frac{x}{\omega}) \). One has

\[
(3.20) \quad \mathcal{H}_\alpha \hat{\varphi}_\omega + \hat{\varphi}_\omega - |\hat{\varphi}_\omega|^{p-1} \hat{\varphi}_\omega = 0
\]

with the modified parameter \( \hat{\alpha} = \alpha + \frac{1}{2} \log \omega \). Formally, \( \hat{\alpha} \to +\infty \) as \( \omega \to \infty \) and the operator \( \mathcal{H}_\alpha \to -\Delta \) so that (3.20) reduces to the standard NLS, for which it is well known that the ground state has positive energy if \( p > 3 \) (see for example Corollary 8.1.3 in [7]). The previous formal argument works rigorously for fairly general Schrödinger operators \(-\Delta + V\) (see [11]).

4. **Strong instability with \( \frac{d}{d\sigma} S_\omega(\varphi_\omega) \leq 0 \).**

It appears from the proof of the previous result that the condition \( E(\varphi_\omega) \geq 0 \) is in general stronger than the condition \( \frac{d}{d\sigma} S_\omega(\varphi_\omega) \leq 0 \). So is a natural generalization of the result given in the previous section consists in assuming \( \frac{d}{d\sigma} S_\omega(\varphi_\omega) \leq 0 \) as the condition selecting the frequencies of the ground waves the instability of which we want to prove. This more general condition has been advocated by M. Ohta in several papers with various collaborators ([11, 13], see also [16]).

**Definition 4.1.** Let \( \varphi_\omega \in \mathcal{G} \) and set

\[
V_\omega = \left\{ \varphi \in D^\frac{1}{2}_{\alpha}(\mathbb{R}^2) \text{ s.t. } S_\omega(\varphi) < S_\omega(\varphi_\omega), \ Q(\varphi) < 0, \ \|\varphi\| \leq \|\varphi_\omega\|, \ \|\varphi\|_{p+1} > \|\varphi_\omega\|_{p+1} \right\}.
\]

**Lemma 4.2.** Let \( p > 3 \), \( \varphi_\omega \in \mathcal{G} \) with \( \omega \) s.t. \( \frac{d}{d\sigma} S_\omega(\varphi_\omega) \leq 0 \), and let \( \varphi \in D^\frac{1}{2}_{\alpha} \) such that \( \varphi \neq 0 \), \( Q(\varphi) \leq 0 \), \( \|\varphi\| \leq \|\varphi_\omega\| \), \( \|\varphi\|_{p+1} > \|\varphi_\omega\|_{p+1} \); then

\[
S_\omega(\varphi_\omega) < S_\omega(\varphi) - \frac{1}{2} Q(\varphi).
\]

**Proof.** Let

\[
\sigma_0 = \left( \frac{\|\varphi_\omega\|_{p+1}^{p+1}}{\|\varphi\|_{p+1}^{p+1}} \right)^{\frac{1}{p+1}}.
\]

Then \( \sigma_0 \in (0, 1] \), \( \|\varphi_\sigma\|_{p+1} = \|\varphi_\omega\|_{p+1} = \sigma_0^{p+1} \|\varphi\|_{p+1} \). Now consider the real function

\[
\sigma \mapsto S_\omega(\varphi^\sigma) - \frac{\sigma^2}{2} Q(\varphi) = \frac{\sigma^2}{2} \|\varphi\|^2 + \frac{\sigma^2}{4\pi} (\log \sigma - \frac{1}{2}) |g_\sigma|^2 - \frac{\sigma^2}{p+1} (\sigma^{p-3} - \frac{p-1}{2}) |\varphi|_{p+1}^{p+1}
\]

Suppose that \( g(\sigma_0) \leq g(1) \); then, thanks to lemma 2.2 it follows \( S_\omega(\varphi_\omega) \leq S_\omega(\varphi_\sigma) \) and being \( Q(\varphi) \leq 0 \) one has

\[
S_\omega(\varphi_\omega) < S_\omega(\varphi_\sigma) \leq S_\omega(\varphi_\sigma) - \frac{\sigma_0^2}{2} Q(\varphi) \leq S_\omega(\varphi) - \frac{1}{2} Q(\varphi)
\]

which is the thesis. So it is enough to show that \( g(\sigma_0) \leq g(1) \). This inequality is equivalent to

\[
\frac{\sigma_0^2}{4\pi} (\log \sigma_0 - \frac{1}{2}) |g_\sigma|^2 - \frac{\sigma_0^2}{p+1} \left( \sigma_0^{p-3} - \frac{p-1}{2} \right) |\varphi|_{p+1}^{p+1} \leq \frac{1}{8\pi} |g_\sigma|^2 + \frac{p-3}{2(p+1)} |\varphi|_{p+1}^{p+1}
\]
\[
\left(\frac{\sigma_0^2 \log \sigma_0}{8\pi} + \frac{1}{8\pi}\right)|q_\varphi|^2 \leq \frac{1}{p+1} \left(\sigma_0^{p-1} - \frac{(p-1)}{2} \sigma_0^2 + \frac{p-3}{2}\right) \|\varphi\|_{p+1}^{p+1}
\]

or also
\[
|q_\varphi|^2 \leq \frac{4\pi}{p+1} \frac{2\sigma_0^{p-1} - (p-1)\sigma_0^2 + (p-3)}{2\sigma_0^2 \log \sigma_0 - \sigma_0^2 + 1} \|\varphi\|_{p+1}^{p+1}.
\]

The idea is to find an estimate of the type \( |q_\varphi|^2 \leq h(\sigma_0)\|\varphi\|_{p+1}^{p+1} \) by making use of the hypotheses on \( \varphi \), and then to verify that
\[
h(\sigma_0) \leq \frac{4\pi}{p+1} \frac{2\sigma_0^{p-1} - (p-1)\sigma_0^2 + (p-3)}{2\sigma_0^2 \log \sigma_0 - \sigma_0^2 + 1} \quad \forall \sigma \in (0, 1)
\]
that would prove \( g(\sigma_0) \leq g(1) \).

Notice that from (3.14) and (3.16),
\[
\frac{d^2}{d\sigma^2} S_\omega(\varphi_\omega)|_{\sigma=1} \leq 0 \iff \mathcal{F}(\varphi_\omega) + \frac{3}{4\pi} |q_\varphi|^2 \leq \frac{(p-1)(p-2)}{p+1} \|\varphi_\omega\|_{p+1}^{p+1}
\]
\[
\iff \frac{1}{2\pi} |q_\varphi|^2 \leq \frac{(p-3)(p-1)}{p+1} \|\varphi\|_{p+1}^{p+1}
\]
(4.1)

The following Pohozaev identity is obtained applying the computations done in Lemma 3.3 to the stationary equation (2.5), or equivalently combining the constraints \( N_\omega(\varphi_\omega) = 0 \) and \( Q(\varphi_\omega) = 0 \):
\[
\omega \|\varphi\|_{p+1}^2 = \frac{|q_\varphi|^2}{4\pi} + 2 \frac{|\varphi_\omega|_{p+1}^p}{p+1} \|\varphi\|_{p+1}^{p+1}
\]
and making use of inequality (4.1) one gets
\[
\omega \|\varphi\|_{p+1}^2 = \frac{|q_\varphi|^2}{4\pi} + 2 \frac{|\varphi_\omega|_{p+1}^p}{p+1} \|\varphi\|_{p+1}^{p+1} \leq \left( \frac{2}{p+1} + \frac{1}{2} \frac{(p-3)(p-1)}{p+1} \right) \|\varphi_\omega\|_{p+1}^{p+1} = \frac{p^2 - 4p + 7}{2(p+1)} \|\varphi_\omega\|_{p+1}^{p+1}
\]
(4.2)

Now, from \( \|\varphi\| \leq \|\varphi_\omega\|, \sigma_0^{p-1}\|\varphi\|_{p+1} = \|\varphi_\omega\|_{p+1} \) one obtains
\[
\omega \|\varphi\|_{p+1}^2 \leq \frac{p^2 - 4p + 7}{2(p+1)} \sigma_0^{p-1}\|\varphi_\omega\|_{p+1}^{p+1}.
\]

The condition \( N(\varphi^{\sigma_0}) \geq 0 \) which holds thanks to proposition 2.2 is equivalent to
\[
\sigma_0^2 \mathcal{F}(\varphi) + \frac{1}{2\pi} \sigma_0^2 \log \sigma_0 |q_\varphi|^2 + \omega |\varphi|^2 - \sigma_0^{p-1} |\varphi|_{p+1}^{p+1} \geq 0
\]
and exploiting \( Q(\varphi) < 0 \) one arrives to
\[
-\frac{1}{2\pi} \sigma_0^2 \log \sigma_0 |q_\varphi|^2 < \sigma_0^{p-1} \frac{p-1}{p+1} |\varphi|_{p+1}^{p+1} - \sigma_0^2 |q_\varphi|^2 + \omega |\varphi|^2 - \sigma_0^{p-1} |\varphi|_{p+1}^{p+1}
\]
that combined with (4.2) yields
\[
\frac{1}{4\pi} \sigma_0^2 (1 - 2 \log \sigma_0) |q_\varphi|^2 < \left( \frac{p^2 - 4p + 7}{2(p+1)} \sigma_0^2 - \sigma_0^{p-1} - \sigma_0^{p-1} \right) |\varphi|_{p+1}^{p+1}
\]
or also
\[
|q_\varphi|^2 < \frac{4\pi}{p+1} \frac{(2p-1) + (p^2 - 4p + 7)\sigma_0^{p-3} - 2(p+1)\sigma_0^{p-3}}{2(1 - 2 \log \sigma_0)} \|\varphi\|_{p+1}^{p+1}
\]
so that the inequality that has to be checked is
\[
\frac{2(p-1) + (p^2 - 4p + 7)\sigma_0^{p-3} - 2(p+1)\sigma_0^{p-3}}{2(1 - 2 \log \sigma_0)} \leq \frac{2\sigma_0^{p-1} - (p-1)\sigma_0^2 + (p-3)}{2\sigma_0^2 \log \sigma_0 - \sigma_0^2 + 1}.
\]
The previous inequality is equivalent to \( f(\sigma_0) \geq 0 \) where
\[
f(\sigma) = (p-3)^2 (\sigma^{p-1} - 2\sigma^{p-1} \log \sigma) - 4(p-3) \log \sigma - 4 - (p^2 - 6p + 5)\sigma^{p-3}
\]
Notice that \( f(0^+) = +\infty, \ f(1) = 0 \), so that to prove the inequality it is sufficient to prove that \( f \) is decreasing. That this is indeed the case is a lengthy but elementary check based on the analysis of the derivatives of the function \( f \) until the third included. We omit the details.

**Theorem 4.3.** Let \( g = -1 \) and \( p > 3 \) and \( \psi_0 \in \Sigma \cap V_\omega \). Then \( T^*(\psi_0) < +\infty \).

**Proof.** It is already known that \( \psi(t) \in \Sigma \) and by the conservation laws that \( S_\omega(\psi(t)) < S_\omega(\varphi_\omega) \) and \( \|\psi(t)\| \leq \|\varphi_\omega\| \). Thanks to Proposition 2.2 and \( S_\omega(\psi(t)) < S_\omega(\varphi_\omega) \) necessarily \( \|\psi(t)\|_{p+1} \neq \|\varphi_\omega\|_{p+1} \) \( \forall t \in (0,T^*(\psi_0)) \); being \( \|\psi_0\|_{p+1} > \|\varphi_\omega\|_{p+1} \), by continuity \( \|\psi(t)\|_{p+1} > \|\varphi_\omega\|_{p+1} \) \( \forall t \in (0,T^*(\psi_0)) \). Finally, \( Q(\psi(t)) < 0 \) is a consequence of Lemma 4.2. Now, from 4.2 and the virial identity 3.3 one gets
\[
\frac{1}{8} \frac{d^2}{dt^2} I_\psi(t) = Q(\psi(t)) \leq 2(S_\omega(\psi(t)) - S_\omega(\varphi_\omega)) = 2(S_\omega(\psi_0) - S_\omega(\varphi_\omega)) < 0 \quad \forall t \in (0,T^*)
\]
This gives the thesis by the classical concavity argument.

Now we will consider the standing waves and we will show that they are strongly unstable.

**Theorem 4.4.** Let \( g = -1 \) and \( p > 3 \). Let \( \omega > -E_\alpha \) and \( \varphi_\omega \in \mathcal{G} \) such that \( \frac{dS(\varphi_\omega)}{d\sigma}|_{\sigma=1} \leq 0 \). Then the standing wave \( \varphi_\omega e^{it\omega} \) is strongly unstable.

**Proof.** One has \( \|\varphi_\omega^\sigma\| = \|\varphi_\omega\|, \|\varphi_\omega^\sigma\|_{p+1} = \frac{d}{d\sigma} \|\varphi_\omega\|_{p+1} > \|\varphi_\omega\|_{p+1} \) \( \forall \sigma > 1 \). Now consider the function \( S(\varphi_\omega^\sigma) \) given in 3.11. We want to show that \( S_\omega(\varphi_\omega^\sigma) < S_\omega(\varphi_\omega) \) \( \forall \sigma > 1 \). Thanks to (3.15) in Proposition 3.8 we have \( \frac{d^2}{d\sigma^2} S_\omega(\varphi_\omega^\sigma) < 0 \) \( \forall \sigma > 1 \) from which we deduce that \( \frac{d}{d\sigma} S_\omega(\varphi_\omega^\sigma) \) is decreasing for \( \sigma > 1 \). Exploiting the hypothesis \( \frac{dS(\varphi_\omega^\sigma)}{d\sigma}|_{\sigma=1} \leq 0 \) we obtain \( \frac{d^2}{d\sigma^2} S(\varphi_\omega^\sigma) < 0 \) \( \forall \sigma > 1 \) and consequently \( \frac{dS(\varphi_\omega^\sigma)}{d\sigma}\) decreasing. Being \( \frac{dS(\varphi_\omega^\sigma)}{d\sigma}|_{\sigma=1} = 0 \) we finally obtain that \( S_\omega(\varphi_\omega^\sigma) < S_\omega(\varphi_\omega) \) \( \forall \sigma > 1 \) as claimed. Finally, using properties 3.17 and 3.19 of Proposition 3.8 we get \( \frac{d}{d\sigma} Q(\varphi_\omega^\sigma) = \frac{d}{d\sigma} S_\omega(\varphi_\omega^\sigma) + \sigma \frac{d^2}{d\sigma^2} S_\omega(\varphi_\omega^\sigma) < 0 \) by using the monotonicity properties just proved. By 3.17 one finally gets \( Q(\varphi_\omega^\sigma) < Q(\varphi_\omega) = 0 \). The proof is completed thanks to \( \lim_{\sigma \to 1} \|\varphi_\omega^\sigma - \varphi_\omega\|_{p+1} = 0 \).

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