THE BOUNDARY BEHAVIOUR OF $K$-QUASICONFORMAL
HARMONIC MAPPINGS

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Abstract. In this article, we first discuss the Lipschitz characteristic and the linear
measure distortion of $K$-quasiconformal harmonic mappings. Then we give some char-
acterizations of the radial John disks with the help of Pre-Schwarzian of harmonic mapp-
ings.

1. Preliminaries and the statement of main results

The purpose of this article is to continue our investigations of the boundary behavior of
$K$-quasiconformal harmonic mappings, using the Lipschitz continuity and Pre-Schwarzian
derivative defined in [10].

1.1. Notation. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For a sense-
preserving harmonic mapping $f = h + \overline{g}$ of $\mathbb{D}$, where $h$ and $g$ are analytic in $\mathbb{D}$, the
Jacobian of $f$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ and $\omega(z) = g'(z)/h'(z)$ denotes the
dilatation of $f$. Also, we let $\|Df\| = |f_z| + |f_{\overline{z}}|$ and $I(Df) = |f_z| - |f_{\overline{z}}|$, where $f_z$ and $f_{\overline{z}}$ are the usual partial derivatives. For $z \in \mathbb{D}$, let

$$B(z) = \{\zeta : |z| \leq |\zeta| < 1, \ |\arg z - \arg \zeta| \leq \pi(1 - |z|)\},$$

and

$$I(z) = \{\zeta \in \partial \mathbb{D} : \ |\arg z - \arg \zeta| \leq \pi(1 - |z|)\}.$$

Let $d_\Omega(z)$ be the Euclidean distance from $z$ to the boundary $\partial \Omega$ of $\Omega$. If $\Omega = \mathbb{D}$, then we
set $d(z) := d_\mathbb{D}(z)$. Throughout of this paper, we use the symbol $C$ to denote the various
positive constants, whose value may vary from one occurrence to another.

1.2. Preliminaries and Definitions.

Definition 1.1. A bounded simply connected plane domain $G$ is called a $c$-John disk for
$c \geq 1$ with John center $w_0 \in G$ if for each $w_1 \in G$ there is a rectifiable arc $\gamma$, called a
John curve, in $G$ with end points $w_1$ and $w_0$ such that

$$\sigma_\ell(w) \leq cd_G(w)$$

for all $w$ on $\gamma$, where $\gamma[w_1, w]$ is the subarc of $\gamma$ between $w_1$ and $w$, and $\sigma_\ell(w)$ is the
Euclidean length of $\gamma[w_1, w]$ (see [7, 9, 11]).
We can classify $c$-John disk according to some test mappings. More precisely, if $f$ is a complex-valued and univalent mapping ($f$ is not necessarily analytic) in $\mathbb{D}$, $G = f(\mathbb{D})$ and, for $z \in \mathbb{D}$, $\gamma = f([0, z])$ in Definition 1.1, then we call $c$-John disk a radial $c$-John disk, where $w_0 = f(0)$ and $w = f(z)$. In particular, if $f$ is univalent and analytic, then we call $c$-John disk a hyperbolic $c$-John disk with respect to $f$. It is well known that any point $w_0 \in G$ can be chosen as a John center by modifying the constant $c$ if necessary. When we do not wish to emphasize the role of $c$, then we regard the $c$-John disk simply as a John disk in the natural way (cf. [3, 4, 9, 11]).

A sense-preserving homeomorphism $f$ from a domain $\Omega$ onto $\Omega'$, contained in the Sobolev class $W_{1,2}^{1} (\Omega)$, is said to be a $K$-quasiconformal mapping if, for $z \in \Omega,$

$$\|Df(z)\|^2 \leq K |\text{det } Df(z)|,$$

i.e., $\|Df(z)\| \leq Kl(Df(z)),$

where $K \geq 1$ is a constant (cf. [12, 14]).

Let $S_H$ denote the family of sense-preserving planar harmonic univalent mappings $f = h + \overline{g}$ in $\mathbb{D}$ satisfying the normalization $h(0) = g(0) = h'(0) - 1 = 0$, where $h$ and $g$ are analytic in $\mathbb{D}$. Recall that $f$ is sense-preserving in $\mathbb{D}$ if $J_f > 0$ in $\mathbb{D}$. Thus, $f$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $J_f > 0$ in $\mathbb{D}$; or equivalently if $h' \neq 0$ in $\mathbb{D}$ and the dilatation $\omega = g'/h'$ has the property that $|\omega| < 1$ in $\mathbb{D}$ (see [5, 8]). The family $S_H$ together with a few other geometric subclasses, originally investigated in detail by [5, 17], became instrumental in the study of univalent harmonic mappings (see [8]) and has attracted much attention of many function theorists. If the co-analytic part $g$ is identically zero in the decomposition of $f = h + \overline{g}$, then the class $S_H$ reduces to the classical family $S$ of all normalized analytic univalent functions $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$. If $S^0_H = \{ f = h + \overline{g} \in S_H : g'(0) = 0 \}$, then the family $S^0_H$ is both normal and compact (cf. [5, 8]). Denote by $S_H(K)$ (resp. $S^0_H(K)$) if $f \in S_H$ (resp. $S^0_H$) and is a $K$-quasiconformal harmonic mapping in $\mathbb{D}$, where $K \geq 1$ is a constant. Also, we denote by $S_H(K, \Omega)$ (resp. $S^0_H(K, \Omega)$) if $f \in S_H(K)$ (resp. $f \in S^0_H(K)$) and $f$ maps $\mathbb{D}$ onto $\Omega$, where $\Omega$ is a subdomain of $\mathbb{C}$.

1.3. Statement of Main results. We now state our first main result which concerns the Lipschitz continuity on $K$-quasiconformal harmonic mappings of $\mathbb{D}$ onto a radial John disk.

Theorem 1.2. Let $f \in S^0_H(K, \Omega)$, where $\Omega$ is a radial John disk. Then, for $z \in \mathbb{D}$ and $\zeta_1, \zeta_2 \in B(z)$, there are constants $\delta \in (0, 1)$ and $C > 0$ such that

$$|f(\zeta_1) - f(\zeta_2)| \leq C d_{\Omega}(f(z)) \left( \frac{|\zeta_1 - \zeta_2|}{1 - |z|} \right)^{\delta}.$$ 

We would like to point out that Theorem 1.2 was established in [4, Theorem 4] but with an additional assumption that $|z| \geq \frac{1}{2}$, and thus, we see now that the condition "$|z| \geq \frac{1}{2}$" in [4, Theorem 4] is redundant. Moreover, by [4, Lemma 6] and [4, Inequality (2.3)], we obtain

$$\frac{1}{16K} \leq d_{\Omega}(f(0)) \leq \frac{2K}{1 + K} \|Df(0)\| = \frac{2K}{1 + K},$$

where $\Omega = f(\mathbb{D})$, $\|Df(0)\| = |f_{\bar{z}}(0)| + |f_z(0)|$ and $f \in S^0_H(K, \Omega)$. Therefore, by letting $z = 0$ in Theorem 1.2, we get the following result.
Corollary 1.3. Let \( f \in \mathcal{S}^0_H(K, \Omega) \), where \( \Omega \) is a radial John disk. Then, for all \( \zeta_1, \zeta_2 \in \mathbb{D} \), there are constants \( C > 0 \) and \( \delta \in (0,1) \) such that
\[
|f(\zeta_1) - f(\zeta_2)| \leq C|\zeta_1 - \zeta_2|^{\delta}.
\]

Our next result establishes the linear measure distortion on \( K \)-quasiconformal mappings of \( \mathbb{D} \) into a radial John disk.

Theorem 1.4. Let \( f \in \mathcal{S}^0_H(K, \Omega) \), where \( \Omega \) is a radial John disk. Then, for all \( z_1, z_2 \in \mathbb{D} \) with \( |z_2| \leq |z_1| \), there are constant \( C > 0 \) and \( \delta \in (0,1) \) such that
\[
\frac{\text{diam}(f(B(z_1)))}{\text{diam}(f(B(z_2)))} \leq C \left( \frac{\ell(I(z_1))}{\ell(I(z_2))} \right)^{\delta}.
\]

The Pre-Schwarzian derivative \( P_f \) of a sense-preserving harmonic mapping \( f = h + \overline{g} \) in \( \mathbb{D} \) is defined by
\[
P_f = (\log J_f)_z = \frac{h''\overline{g}' - g''h'}{|h'|^2 - |g'|^2} = T_h - \frac{\omega^2}{1 - |\omega|^2},
\]
where \( \omega = g'/h' \), and \( T_h = h''/h' \) denotes the Pre-Schwarzian of a locally univalent analytic function \( h \) in \( \mathbb{D} \). See [6, 10, 15] for recent investigations on Pre-Schwarzian derivatives of harmonic mappings.

Ahlfors and Weill [1] and, Becker and Pommerenke [2] characterized quasidisks by using the Pre-Schwarzian of analytic functions. On the basis of the works of Chuaqui, et al. [7], Kari Hag and Per Hag [9] discussed relationships between John disks and the Pre-Schwarzian of analytic functions. By analogy with [7, Theorem 4] and [9, Theorem 3.7], the present authors in [3, Theorem 5] showed that if \( f \in \mathcal{S}^0_H(K, \Omega) \) such that
\[
\limsup_{|z| \to 1^-} \left\{ (1 - |z|^2) \text{Re}(zP_f(z)) \right\} < 1,
\]
then \( \Omega \) is a radial John disk. Our final result improves this result in the following form.

Theorem 1.5. Let \( f = h + \overline{g} \in \mathcal{S}^0_H(K, \Omega) \) and \( \omega = g'/h' \). Then the following statements are true.

(a) If
\[
\limsup_{|z| \to 1^-} \left\{ (1 - |z|^2) \text{Re}(zP_f(z)) \right\} < 1 + k,
\]
then \( \Omega \) is a radial John disk, where \( k = \frac{K - 1}{K + 1} \leq \frac{1}{2} \).

(b) If \( h \) is univalent in \( \mathbb{D} \) and satisfies
\[
\limsup_{|z| \to 1^-} \left\{ (1 - |z|^2) \left| P_f(z) + \frac{\omega'(z)\overline{\omega}(z)}{1 - |\omega(z)|^2} \right| \right\} < 2,
\]
then \( \Omega \) is a radial John disk.

Corollary 1.6. Let \( f = h + \overline{g} \in \mathcal{S}^0_H(K, \Omega) \) and \( \omega = g'/h' \). If \( h \) is univalent in \( \mathbb{D} \) and satisfies
\[
\sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2) \left| P_f(z) + \frac{\omega(z)\overline{\omega}(z)}{1 - |\omega(z)|^2} \right| \right\} < 2,
\]
then \( \Omega \) is a radial John disk. The constant 2 is the best possible.
The proofs of Theorems 1.2, 1.4, 1.5 and Corollary 1.6 will be presented in Section 2.

2. PROOFS OF THE MAIN RESULTS

The hyperbolic plane is the unit disk $\mathbb{D}$ with the hyperbolic metric

$$\lambda_\mathbb{D}(z)|dz| = \frac{|dz|}{1-|z|^2}$$

which is indeed a mapping which associates to each smooth curves $\gamma$ in $\mathbb{D}$ its hyperbolic length $\ell_\mathbb{D}(\gamma)$ defined by

$$\ell_\mathbb{D}(\gamma) = \int_\gamma \lambda_\mathbb{D}(z)|dz| = \int_a^b \frac{|z'(t)|}{1-|z(t)|^2} dt,$$

where $\gamma$ is parameterized by $z(t)$, $a \leq t \leq b$. The hyperbolic distance (or Poincaré distance) $\lambda_\mathbb{D}(z_1, z_2)$ between points $z_1$ and $z_2$ in $\mathbb{D}$ is then defined by

$$\lambda_\mathbb{D}(z_1, z_2) = \inf_\gamma \ell_\mathbb{D}(\gamma) = \tanh^{-1} \frac{|z_1 - z_2|}{1-\overline{z_1}z_2},$$

where the infimum is taken over all smooth curves $\gamma$ in $\mathbb{D}$ that joins $z_1$ to $z_2$ in $\mathbb{D}$ (cf. [16]).

**Lemma A.** ([3, Lemma 1]) Let $f \in S_H$. Then for $z_1, z_2 \in \mathbb{D}$,

$$\frac{1}{2} \|D_f(z_1)\| e^{-(1+\alpha)|\lambda_\mathbb{D}(z_1, z_2)|} \leq \|D_f(z_2)\| \leq 2 \|D_f(z_1)\| e^{(1+\alpha)|\lambda_\mathbb{D}(z_1, z_2)|},$$

where $\alpha := \sup_{f \in S_H} \frac{|h'(0)|}{2} < +\infty$.

We remark that $2 \leq \alpha < +\infty$, but the sharp value of $\alpha$ is still unknown (cf. [5, 8, 17]).

**Theorem B.** ([3, Theorem 2]) Let $f \in S^0_H(K, \Omega)$, where $\Omega := f(\mathbb{D})$ is a bounded domain. Then the following conditions are equivalent:

1. $\Omega$ is a radial John disk;
2. There is a positive constant $C$ such that for all $z \in \mathbb{D}$,
   $$\text{diam}(B(z)) \leq C d_\Omega(f(z));$$
3. There are constants $C > 0$ and $\delta \in (0, 1)$ such that for all $z \in \mathbb{D}$ and $\zeta \in B(z),$
   $$\|D_f(\zeta)\| \leq C \|D_f(z)\| \left(\frac{1-|\zeta|}{1-|z|}\right)^{\delta^{-1}}.$$

**Lemma C.** ([3, Lemma 2]) Let $a_1, a_2$ and $a_3$ be positive constants and let $0 < |z_0| = 1 - \delta_0$, where $\delta_0 \in (0, 1)$. If $f = h + \overline{g} \in S_H$, $0 \leq 1 - a_2 \delta_0 \leq |z| \leq 1 - a_1 \delta_0$ and $|\arg z - \arg z_0| \leq a_3 \delta_0$, then

$$\frac{1}{M(a_1, a_2, a_3)} \|D_f(z_0)\| \leq \|D_f(z)\| \leq M(a_1, a_2, a_3) \|D_f(z_0)\|,$$

where $M(a_1, a_2, a_3) = 2e^{(1+\alpha)\left(a_3 + \frac{1}{2} \log \frac{2a_2-a_3}{a_1}\right)}$ and $\alpha := \sup_{f \in S_H} \frac{|h'(0)|}{2}$. 

Lemma D. ([4, Lemma 6]) If \( f \in S_H(K) \) and \( \Omega = f(\mathbb{D}) \), then for \( z \in \mathbb{D} \),
\[
d_\Omega(f(z)) \geq \frac{||Df(z)||(1 - |z|^2)}{16K}.
\]

2.1. Proof of Theorem 1.2. Let \( z = re^{i\theta}, \mu = |\zeta_1 - \zeta_2| \) and \( \zeta_j = r_j e^{i\theta_j} \) \((j = 1, 2)\) with \( r_1 \leq r_2 \).

Case I. If \( \rho = 1 - 2\mu < r \), then \( \frac{2\mu}{1-r} > 1 \) and, by Theorem B(2), we see that there is a positive constant \( C \) such that
\[
|f(\zeta_1) - f(\zeta_2)| \leq \text{diam}(B(z)) \leq Cd_\Omega(f(z)) \leq 2^\delta Cd_\Omega(f(z)) \left( \frac{|\zeta_1 - \zeta_2|}{1 - |z|} \right)^\delta.
\]

Case II. Suppose that \( \rho = 1 - 2\mu \geq r \) and \( r_1 < \rho \). In this case, for \( |\zeta - \zeta_1| \leq \mu \), we have
\[
\frac{|\zeta - \zeta_1|}{|1 - \zeta_1\zeta|} \leq \frac{\mu}{1-r_1} < \frac{\mu}{1-\rho} = \frac{1}{2},
\]
which implies that
\[
\lambda_\mathbb{D}(\zeta, \zeta_1) = \tanh^{-1} \left( \frac{\zeta - \zeta_1}{1 - \zeta_1\zeta} \right) \leq \tanh^{-1} \left( \frac{1}{2} \right) = \frac{1}{2} \log 3,
\]
where \( \lambda_\mathbb{D}(z_1, z_2) \) denotes the hyperbolic distance (or Poincaré distance) between points \( z_1 \) and \( z_2 \) in \( \mathbb{D} \) given by
\[
\lambda_\mathbb{D}(z_1, z_2) = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \overline{z_1}z_2} \right|.
\]

It follows from (2.2) and Lemma A that there is a positive constant \( C \) such that
\[
||Df(\zeta)|| \leq C||Df(\zeta_1)||,
\]
where \( |\zeta - \zeta_1| \leq \mu \). By (2.3), it follows that
\[
|f(\zeta_1) - f(\zeta_2)| \leq \int_{[\zeta_1, \zeta_2]} ||Df(\zeta)|||d\zeta| \leq C|\zeta_1 - \zeta_2||Df(\zeta_1)||,
\]
where \([\zeta_1, \zeta_2]\) is the line segment from \( \zeta_1 \) to \( \zeta_2 \).

By Theorem B(3) and Lemma D, there are constants \( C > 0 \) and \( \delta \in (0, 1) \) such that
\[
||Df(\zeta_1)|| \leq C||Df(\zeta)|| \left( \frac{1-r_1}{1-r} \right)^{\delta-1} \leq C||Df(\zeta)|| \left( \frac{1-\rho}{1-r} \right)^{\delta-1} \leq 16KCd_\Omega(f(z)) \left( \frac{1-\rho}{1-r} \right)^{\delta-1},
\]
which, together with (2.4), implies that there is a positive constant \( C \) such that
\[
|f(\zeta_1) - f(\zeta_2)| \leq Cd_\Omega(f(z)) \left( \frac{1-\rho}{1-r} \right)^{\delta-1} |\zeta_1 - \zeta_2| \leq 2^{\delta-1}C \left( \frac{|\zeta_1 - \zeta_2|}{1 - |z|} \right)^\delta.
\]
Case III. Suppose that $r \leq \rho = 1 - 2\mu \leq r_1$. Then, by Theorem B(3) and Lemma D, we see that there are constants $C > 0$ and $\delta \in (0, 1)$ such that

\begin{equation}
|f(\zeta_1) - f(\rho e^{i\theta_1})| \leq \int_{\rho}^{r_1} \|D_f(t e^{i\theta_1})\| dt
\end{equation}

\begin{align*}
&\leq C\|D_f(z)\| \int_{\rho}^{r_1} \left( \frac{1 - t}{1 - r} \right)^{\delta - 1} dt \\
&= \frac{C\|D_f(z)\|}{\delta} \frac{[(1 - \rho)^{\delta} - (1 - r_1)^{\delta}]}{(1 - r)^{\delta - 1}} \\
&\leq \frac{16KC}{2^\delta \delta} d_\Omega(f(z)) \left( \frac{|\zeta_1 - \zeta_2|}{1 - r} \right)^{\delta} \quad \text{(by Lemma D)}
\end{align*}

and

\begin{equation}
|f(\zeta_2) - f(\rho e^{i\theta_2})| \leq \int_{\rho}^{r_2} \|D_f(t e^{i\theta_2})\| dt
\end{equation}

\begin{align*}
&\leq C\|D_f(z)\| \int_{\rho}^{r_2} \left( \frac{1 - t}{1 - r} \right)^{\delta - 1} dt \\
&= \frac{C\|D_f(z)\|}{\delta} \frac{[(1 - \rho)^{\delta} - (1 - r_1)^{\delta}]}{(1 - r)^{\delta - 1}} \\
&\leq \frac{16KC}{2^\delta \delta} d_\Omega(f(z)) \left( \frac{|\zeta_1 - \zeta_2|}{1 - r} \right)^{\delta}.
\end{align*}

Let $\gamma$ be the smaller subarc of $\partial \mathbb{D}_\rho$ between $\rho e^{i\theta_1}$ and $\rho e^{i\theta_2}$. Then $|\theta_1 - \theta_2| \leq \pi$ and, since

\begin{align*}
|\zeta_1 - \zeta_2| &= \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right)} \\
&\geq 2\sqrt{r_1 r_2} \left| \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \right| \\
&\geq \frac{2\rho|\theta_1 - \theta_2|}{\pi},
\end{align*}

we see that

\begin{equation}
\ell(\gamma) = \rho|\theta_1 - \theta_2| \leq \frac{\pi}{2} \mu.
\end{equation}
Hence, we get

\[(2.9) \quad |f(\rho e^{i\theta_1}) - f(\rho e^{i\theta_2})| \leq \int_{\gamma} \rho \|D_f(\rho e^{i\tau})\|d\tau \]

\[\leq C \int_{\gamma} \|D_f(z)\| \left(\frac{1 - \rho}{1 - r}\right)^{\delta - 1} d\tau \quad \text{(by Theorem B(3))} \]

\[= C\|D_f(z)\| \left(\frac{1 - \rho}{1 - r}\right)^{\delta - 1} \ell(\gamma) \]

\[\leq \frac{\pi}{4} C\|D_f(z)\| \left(\frac{1 - \rho}{1 - r}\right)^{\delta} \quad \text{(by (2.8))} \]

\[\leq 4\pi KCd_\Omega(f(z)) \left(1 - \rho\right)^{\delta} \quad \text{(by Lemma D)} \]

\[= 2^{2+\delta}\pi KCd_\Omega(f(z)) \left(\frac{\zeta_1 - \zeta_2}{1 - r}\right)^{\delta}. \]

Therefore, by (2.6), (2.7) and (2.9), we conclude that there is a positive constant \(C\) such that

\[|f(\zeta_1) - f(\zeta_2)| \leq |f(\zeta_1) - f(\rho e^{i\theta_1})| + |f(\zeta_2) - f(\rho e^{i\theta_2})| + |f(\rho e^{i\theta_1}) - f(\rho e^{i\theta_2})| \]

\[\leq Cd_\Omega(f(z)) \left(\frac{\zeta_1 - \zeta_2}{1 - r}\right)^{\delta}. \]

The proof of the theorem is complete. \(\square\)

**Theorem E.** ([3, Theorem 1]) Let \(f \in S_1^H(K, \Omega)\), where \(\Omega := f(\mathbb{D})\) is a bounded domain. Then \(\Omega\) is a radial John disk if and only if there are constants \(M(K) > 0\) and \(\delta \in (0, 1)\) such that for each \(\zeta \in \partial \mathbb{D}\) and for \(0 \leq r \leq \rho < 1\),

\[\|D_f(\rho \zeta)\| \leq M(K)\|D_f(r \zeta)\| \left(\frac{1 - \rho}{1 - r}\right)^{\delta - 1}. \]

2.2. **Proof of Theorem 1.4.** Let \(f \in S_1^H(K, \Omega)\), where \(\Omega\) is a radial John disk. Suppose that \(z_1 = re^{i\theta}\) and \(r_1e^{i\theta_1}, r_2e^{i\theta_2} \in B(z_1)\) with \(r_2 \leq r_1\). Then, by Theorem B(3), we see that there are positive constants \(C\) and \(\delta \in (0, 1)\) such that

\[(2.10) |f(r_1e^{i\theta_1}) - f(re^{i\theta_1})| \leq \int_{r_1}^{r} \|D_f(\rho e^{i\theta_1})\|d\rho \leq C \int_{r}^{r_1} \|D_f(z_1)\| \left(\frac{1 - \rho}{1 - r}\right)^{\delta - 1} d\rho \]

\[= \frac{C}{\delta} \|D_f(z_1)\| \left[(1 - r)^{\delta} - (1 - r_1)^{\delta}\right] \]

\[\leq \frac{C}{\delta} \|D_f(z_1)\|(1 - r). \]

Similarly, we have

\[(2.11) \quad |f(r_2e^{i\theta_2}) - f(re^{i\theta_2})| \leq \frac{C}{\delta} \|D_f(z_1)\|(1 - r). \]
Let \( \gamma \) be the smaller subarc of \( \partial \mathbb{D} \) between \( re^{i\theta_1} \) and \( re^{i\theta_2} \). Since \( r_1e^{i\theta_1}, r_2e^{i\theta_2} \in B(z_1) \), we see that

\[
|\theta_1 - \theta_2| \leq |\theta_1 - \theta| + |\theta - \theta_1| \leq 2\pi (1 - r).
\]

(2.12)

It follows from (2.12) and Theorem B(3) that

\[
|f(re^{i\theta_1}) - f(re^{i\theta_2})| \leq r \int_\gamma \|D_f(re^{i\eta})\|d\eta \leq C \int_\gamma \|D_f(re^{i\eta})\|d\eta
\]

\[
= Cr\|D_f(re^{i\eta})\|\|\theta_1 - \theta_2\| \leq 2C\pi (1 - r)\|D_f(re^{i\eta})\|.
\]

(2.13)

Combining (2.10), (2.11) and (2.13) shows that

\[
|f(r_1e^{i\theta_1}) - f(r_2e^{i\theta_2})| \leq |f(r_1e^{i\theta_1}) - f(re^{i\theta_1})| + |f(r_2e^{i\theta_2}) - f(re^{i\theta_2})|
\]

\[
+|f(re^{i\theta_1}) - f(re^{i\theta_2})| 
\]

\[
\leq \left(2\pi C + \frac{2C}{\delta}\right)(1 - r)\|D_f(re^{i\eta})\|,
\]

which implies that there is a positive constant \( C \) such that

\[
\text{diam}(B(z_1)) \leq C(1 - |z_1|^2)\|D_f(z_1)\|.
\]

(2.14)

It follows from Theorem B(3), Lemma D and [3, Inequality (2.3)] that there is a positive constant \( C \) such that

\[
\text{diam}(B(z_2)) \geq d_\Omega(f(z_2)) \geq \frac{\|D_f(z_2)\|(1 - |z_2|^2)}{16K}.
\]

(2.15)

By (2.14), (2.15) and Theorem E, we conclude that there are constants \( M(K) > 0 \) and \( \delta \in (0, 1) \) such that

\[
\frac{\text{diam}_f(B(z_1))}{\text{diam}_f(B(z_2))} \leq 16KC\frac{\|D_f(z_1)\|(1 - |z_1|^2)}{\|D_f(z_2)\|(1 - |z_2|^2)}
\]

\[
\leq 32M(K)KC \left(\frac{1 - |z_1|}{1 - |z_2|}\right)^\delta
\]

\[
= 32M(K)KC \left(\frac{\ell(I(z_1))}{\ell(I(z_2))}\right)^\delta,
\]

which completes the proof. \( \square \)

2.3. Proof of Theorem 1.5. We first prove (a). It follows from (1.1) that there is a \( \nu \in (0, 1 + k) \) and \( r_0 \in (0, 1) \) such that, for \( r_0 \leq \eta < 1 \),

\[
\frac{\nu}{1 - \eta^2} \geq \text{Re}(\zeta P_f(\eta \zeta)) = \text{Re} \left(\frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)}\right) - \text{Re} \left(\frac{\zeta \omega'(\eta \zeta)\omega(\eta \zeta)}{1 - |\omega(\eta \zeta)|^2}\right),
\]

(2.16)

where \( \zeta \in \partial \mathbb{D} \). By Schwarz-Pick’s lemma, we obtain

\[
|\omega'(\eta \zeta)| \leq \frac{1 - |\omega(\eta \zeta)|^2}{1 - \eta^2}
\]

(2.17)

and, since \( f \) is a \( K \)-quasiconformal harmonic mapping, we see that,

\[
|\omega(z)| \leq k = \frac{K - 1}{K + 1}, \quad z \in \mathbb{D}.
\]

(2.18)
Thus, by (2.17) and (2.18), (2.16) gives
\[
\text{Re} \left( \frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)} \right) \leq \text{Re} \left( \frac{\zeta \omega'((\eta \zeta)\omega(\eta \zeta))}{1 - |\omega(\eta \zeta)|^2} \right) + \frac{\nu}{1 - \eta^2}
\]
\[
\leq \frac{\omega'(\eta \zeta) |\omega(\eta \zeta)|}{1 - |\omega(\eta \zeta)|^2} + \frac{\nu}{1 - \eta^2}
\]
\[
\leq \frac{\nu + k}{1 - \eta^2}.
\]
Choosing $\lambda \in (0, k + 1 - \nu)$, there is an $r_1 \in [r_0, 1)$ such that
\[
\text{Re} \left( \frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)} \right) < \frac{2\eta - (\lambda + 1 - 2k)}{1 - \eta^2}
\]
for all $\zeta \in \partial \mathbb{D}$, when $\eta \in [r_1, 1)$. For $0 \leq r_1 \leq r \leq \rho < 1$, by (2.19), we find that
\[
\log \left| \frac{h'(\zeta)}{h'(r \zeta)} \right| = \int_r^\rho \left[ \text{Re} \left( \frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)} \right) - \frac{2\eta}{1 - \eta^2} \right] d\eta
\]
\[
\leq -2 \left( \frac{\lambda + 1}{2} - k \right) \int_r^\rho \frac{d\eta}{1 - \eta^2}
\]
\[
= - \left( \frac{\lambda + 1}{2} - k \right) \log \left( \frac{1 + \rho}{1 + r} \cdot \frac{1 - r}{1 - \rho} \right),
\]
which implies that
\[
\frac{h'(\zeta)}{h'(r \zeta)} < \left( \frac{1 + r}{1 + \rho} \right)^{\frac{\lambda + 1}{2} - k + 1} \left( \frac{1 - \rho}{1 - r} \right)^{\frac{\lambda + 1}{2} - k} \leq \left( \frac{1 - \rho}{1 - r} \right)^{\frac{\lambda + 1}{2} - 1 - k}.
\]
By (2.20), we get
\[
\|D_f(\rho \zeta)\| \leq \frac{2K}{1 + K} |h'(\rho \zeta)| < \frac{2K}{1 + K} |h'(r \zeta)| \left( \frac{1 - \rho}{1 - r} \right)^{\frac{\lambda + 1}{2} - 1 - k}
\]
\[
\leq \frac{2K}{1 + K} \|D_f(\rho \zeta)\| \left( \frac{1 - \rho}{1 - r} \right)^{\frac{\lambda + 1}{2} - k - 1}.
\]
Next, we can use the similar approach as in the proof of [3, Theorem 5] to remove the restriction $r \geq r_1$ above. Hence, for $0 \leq r \leq \rho < 1$, there is a positive constant $C$ such that
\[
\|D_f(\rho \zeta)\| \leq C \|D_f(\rho \zeta)\| \left( \frac{1 - \rho}{1 - r} \right)^{\frac{\lambda + 1}{2} - k - 1},
\]
which, together with Theorem E, implies $\Omega$ is a radial John disk.

Now we prove the part of (b). Let $f = h + \overline{g} \in \mathcal{S}_H^0(K, \Omega)$ satisfy (1.2), where $h$ is univalent in $\mathbb{D}$. Then
\[
\limsup_{|z| \to 1^-} \left( 1 - |z|^2 \right) \left| \frac{h''(z)}{h'(z)} \right| = \limsup_{|z| \to 1^-} \left( 1 - |z|^2 \right) \left| P_f(z) + \frac{\omega'(z) \overline{\omega(z)}}{1 - |\omega(z)|^2} \right| < 2,
\]
which implies that
\[
\limsup_{|z| \to 1^-} \left( 1 - |z|^2 \right) \text{Re} \left( \frac{h''(z)}{h'(z)} \right) < 2.
\]
It follows from (2.21), [9, Theorem 3.7] and [9, Theorem 2.3] that there are constants \( C > 0 \) and \( \delta \in (0, 1) \) such that for each \( \zeta \in \partial D \) and for \( 0 \leq r \leq \rho < 1 \),

\[
|h'(\rho \zeta)| \leq C|h'(r \zeta)| \left( \frac{1 - \rho}{1 - r} \right)^\delta.
\]

Since \( f \) is a \( K \)-quasiconformal mapping, we see that

\[
\frac{2}{1 + K} |h'(z)| \leq \|D_f(z)\| \leq \frac{2K}{1 + K} |h'(z)|.
\]

By (2.22) and (2.23), there are constants \( C > 0 \) and \( \delta \in (0, 1) \) such that for each \( \zeta \in \partial D \) and for \( 0 \leq r \leq \rho < 1 \),

\[
\frac{K + 1}{2K} \|D_f(\rho \zeta)\| \leq |h'(\rho \zeta)| \leq C|h'(r \zeta)| \left( \frac{1 - \rho}{1 - r} \right)^\delta \leq \frac{(K + 1)C}{2} \|D_f(r \zeta)\| \left( \frac{1 - \rho}{1 - r} \right)^\delta,
\]

which, together with Theorem E, yields that \( \Omega \) is a radial John disk. The proof of the theorem is complete. \( \square \)

2.4. Proof of Corollary 1.6. By the assumption, we have

\[
\sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2)\left| \frac{h''(z)}{h'(z)} \right| \right\} = \sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2) \left| P_f(z) + \frac{\omega'(z)\overline{\omega(z)}}{1 - |\omega(z)|^2} \right| \right\} < 2,
\]

which implies that

\[
\limsup_{||z|\rightarrow 1^-} \left\{ (1 - |z|^2)\text{Re} \left( z\frac{h''(z)}{h'(z)} \right) \right\} \leq \sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2) \left| \frac{h''(z)}{h'(z)} \right| \right\} < 2.
\]

It follows from (2.24) and Theorem 1.5(b) that \( \Omega \) is a radial John disk.

Now we prove the sharpness part. For \( z \in \mathbb{D} \), let

\[
f(z) = \frac{1}{2} \log \frac{1 + z}{1 - z}.
\]

Then

\[
\sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| \right\} = 2,
\]

and \( f(\mathbb{D}) \) is an infinite strip and hence not a radial John disk. \( \square \)

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