Shadowing of Virtual Photons in Nuclei at Small $x_{\text{Bj}}$ in the QCD Dipole Picture

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Abstract

Compact and well defined formulae for the shadow of the virtual photon interacting with a large nucleus at small $x_{\text{Bj}}$ are given in the QCD dipole picture. Two classes of contributions are considered: (a) quasi-elastic interaction of the $q\bar{q}$ dipole and (b) multi-pomeron coupling.

1 Introduction

We have now at our disposal the QCD dipole picture of virtual photon interactions [1, 2, 3] with hadrons [4, 5] which is able to account for existing data on proton structure function $F_2$ (i.e. on total virtual photon–proton cross-section) at small $x_{\text{Bj}}$. It is therefore interesting to investigate whether the model can be also applied to nuclear targets, in particular if it can describe the shadowing of the virtual photons. This becomes especially interesting in view of a possible extension of the HERA experimental program to nuclear targets [6].

In our recent paper [7] we have calculated in this model the double scattering contribution to the virtual photon shadowing. The multiple scattering corrections, which are more difficult to handle, were merely estimated, however. As the resulting shadowing at small $x_{\text{Bj}}$ turned out to be rather

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1The subject of photon shadowing in nuclei has a vast literature. For recent reviews, see e.g. [6] and [8].
substantial and not too far from experimental data, it seems worthwhile to pursue the problem further and to evaluate more precisely the multiple scattering terms. This is the purpose of the present paper. As in [7] (c.f. also [8, 10]) we are dealing with two types of contributions:

(a) The “quasi-elastic” interaction of the $q\bar{q}$ dipole (first introduced in the dipole picture in [10]) contributing mostly at small mass excitations of the virtual photon.

(b) The “direct” interaction of the dipole cascade (corresponding to the so-called multiple pomeron coupling in the Regge language) which dominates the high-mass excitations.

The amplitudes describing these processes were derived and discussed in [1, 2, 3, 9, 10]. When applied to the interaction with nuclear targets, however, they must be modified in two important points:

(i) a phase difference related to the change in the longitudinal momentum of system travelling through the nucleus must be taken into account. Consequently, at some point it is necessary to transform the amplitudes into the momentum representation. This makes the calculation substantially more complicated.

(ii) Since the transverse size of the gluon cascade is much smaller than that of a large nucleus, we can use the standard approximation and integrate over the impact-parameter dependence of the elementary amplitudes, i.e. we need to deal with forward scattering amplitudes only. Indeed, the convolution of the onium and the nucleus transverse profiles is approximately

$$\int T_A(\mathbf{s}) T(\mathbf{b} - \mathbf{s}) \, d^2s \approx T_A(b) \int T'(b') \, d^2b' = \tilde{T}(k = 0) T_A(b),$$  

where $\tilde{T}(k = 0)$ is the forward scattering amplitude and $T_A(b)$ is the nucleus profile.

This simplifies the calculation in an essential way. Our ultimate goal is to give an explicit expression for the shadow, defined as

$$C(A, x_{Bj}, Q^2) = \frac{\sigma_{tot}(A)}{A\sigma_{tot}(n)} - 1,$$  

where $\sigma_{tot}(A)$ is the total photon-nucleus cross-section and $\sigma_{tot}(n)$ is the total photon-nucleon cross-section. $\sigma_{tot}(n)$ has been measured [12] and successfully computed in the dipole picture [4]. This will be our input.

To compute $\sigma_{tot}(A)$ we have to know the forward amplitudes for (a) quasi-elastic scattering of the dipole and (b) direct scattering of the dipole cascade from a given number, $N$, of nucleons. We then dress them up with
the light-cone photon wave functions [13] and with single nucleon densities characterizing the nuclear shape. Finally we add them up for \(2 \leq N \leq A\). This calculation is described in the next two sections. In Sec. 4 we discuss the obtained results and formulate our conclusions.

2 Forward \(\gamma^*\)-nucleus amplitude from the direct interaction of the dipole cascade

We start by giving the final result for the forward amplitude for scattering on \(N\) nucleons of the dipole cascade emerging from an onium. It reads

\[
F_{N}^{\text{dir}}(r, r_{0}; Y, y) = (2N - 3)!! \pi \alpha^2 (\alpha^3 N_{c})^{N-1} r_{0}^{2N} n_{\text{eff}} e^{i(z_{1} - z_{N})m_{\text{XP}}} \times \prod_{k=1}^{N} \left( \int_{c_{k} - i\infty}^{c_{k} + i\infty} \frac{d\lambda_{k}}{2\pi i} h(\lambda_{k}) e^{\Delta(\lambda_{k})} G(\lambda_{k}) \right) e^{\Delta(\gamma)(Y - y)} \left( \frac{r}{r_{0}} \right)^{\gamma} \frac{1}{G(\gamma)}, \tag{3}
\]

where

\[
\Delta(\lambda) = \frac{\alpha N_{c}}{\pi} \chi(\lambda), \quad \chi(\lambda) = 2\psi(1) - \psi \left( 1 - \frac{\lambda}{2} \right) - \psi \left( \frac{\lambda}{2} \right), \tag{4}
\]

\(\alpha\) is the strong coupling constant, \(N_{c}\) is the number of colours, \(r_{0}\) is a parameter of the order of the effective size of the nucleon, \(r\) is the transverse size of the incident onium. The remaining symbols denote:

\[
h(\lambda) = \frac{4}{\lambda^{2}(2 - \lambda)^{2}}, \quad G(\lambda) = \frac{\Gamma \left( \frac{\lambda}{2} \right)}{\Gamma \left( 1 - \frac{\lambda}{2} \right)}, \quad \gamma = \sum_{k=1}^{N} \lambda_{k},
\]

\[Y = -\log \frac{x_{Bj}}{c}, \quad y = -\log \frac{x_{P}}{\beta}, \quad x_{P} = \frac{x_{Bj}}{\beta}, \quad \beta = \frac{Q^{2}}{Q^{2} + M^{2}}, \tag{5}\]

and \(M\) is the mass of the dipole cascade; \(c\) is a constant giving the energy scale, \(n_{\text{eff}}\) is an effective number of onia in one nucleon (estimated in [4], c.f. also [7]).

The origin of the phase factor \(e^{i(z_{1} - z_{N})m_{\text{XP}}} = e^{i(z_{1} - z_{N})\frac{Q^{2} + M^{2}}{2\nu}}\) where \(z_{1}\) and \(z_{N}\) are the longitudinal positions of the first and the last scattering (\(m\) is nucleon mass and \(\nu\) is the energy of the photon ) was discussed in detail already by Gottfried and Yennie [14].

\(^{2}\)When only one mass is excited, as it is in the present case, the phase factor depends only on the first and last points of the interaction.
The starting point in the derivation of Eq. (3) is the generalization of the Mueller–Patel formula \([2]\) for triple-pomeron coupling. In the case of \(N\) nucleons it reads
\[
F^{\text{dir}}_N(r, r_1, \ldots, r_N; Y, y) = e^{i(z_1 - z_N)\alpha_p} \int \left( \prod_{k=1}^{N} \frac{dx_k dx'_k}{x_k x'_k} \tau(x_k, x'_k) n_1(r_k, x_k, y^*) \right) \times n_N(r, x'_1, \ldots, x'_N; Y - y^*, y - y^*). \tag{6}
\]

Here
\[
n_1(r_k, x_k, y^*) = \int_{c-i\infty}^{c+i\infty} \frac{d\lambda}{2\pi i} e^{\Delta(\lambda)y^*} \left( \frac{r_k}{x_k} \right)^\lambda \tag{7}
\]
are the densities of the dipoles of the transverse size \(x_k\) integrated over the transverse positions of the dipoles within an onium of size \(r_k\) moving with rapidity \(y^*\). \(n_N(r, x_1, \ldots, x_N; Y - y^*, y - y^*)\) is the density of \(N\) dipoles of the transverse sizes \(x_1, \ldots, x_N\) again integrated over the transverse positions of the dipoles within the cascade of the transverse size \(r\) evolved from the incident onium. \(\tau(x, x')\) is the forward scattering amplitude of two dipoles of transverse sizes \(x\) and \(x'\) (in the two-gluon exchange approximation it is energy independent \([2]\)). The arguments \(r_1, \ldots, r_N\) are the transverse sizes of the onia representing the target nucleons. \(Y, y\) are defined above in Eq. (5).

In the terminology employing the pomeron concept, \(y\) is the rapidity which divides the process into the region of one pomeron (rapidities between \(Y\) and \(y\)) and \(N\) pomerons (rapidities smaller than \(y\)) in the \(N + 1\) pomeron coupling.

To obtain a formula for \(n_N(r, x_1, \ldots, x_N; Y - y^*, y - y^*)\) one has to solve an integro-differential equation which one obtains by \(N\)-fold differentiation of the generating functional given in \([3]\). We do this by the method described in \([10]\) for double-dipole scattering. An extension of this procedure to arbitrary \(N\) is described in Appendix A. At this point it should be emphasized that a solution in compact form, necessary to derive the result of Eq. (3), can be obtained only for distributions integrated over the transverse position of the dipoles with respect to the original onium. The solution for arbitrary dipole position is much more involved. Fortunately, as already mentioned in the Introduction, for large nuclei the transverse size of the incident onium is much smaller than the nuclear diameter and thus we only need the distributions integrated over the transverse positions inside the onium.

In order to obtain the contribution to the shadow (Eq. (2)) one has to add the contributions from \(N = 2, 3, \ldots A\) nucleons and average them.
over the positions of the nucleon inside the nucleus and over the virtual photon wave functions. Finally we have to integrate over the excited mass $M$ (i.e. the rapidity $y$). Using (3) and 

$$\sigma_{\text{tot}} = 2\Re F$$  \hspace{1cm} (8)$$

one obtains

$$C^{\text{dir}}(A, x_{Bj}, Q^2) = -\frac{2}{A\sigma_{\text{tot}}(n)}$$

$$\times \Re \left(Y \int_0^Y dy < \sum_{N=2}^A (-1)^N \frac{A!}{(A-N)!} n_{\text{eff}}^N \langle \langle \psi_Q | F_N^{\text{dir}} | \psi_Q \rangle \rangle \right). \hspace{1cm} (9)$$

The average $\langle \langle \psi_Q | F_N^{\text{dir}} | \psi_Q \rangle \rangle$ is the integral of $F_N^{\text{dir}}$ multiplied by light-cone photon densities [13] given by

$$| \Psi_Q(r, \eta) |^2 = \frac{N_c \alpha_{\text{em}} e_f^2}{\pi^2} W(r, \eta, Q), \hspace{1cm} (10)$$

$$W^T(r, \eta, Q) = \frac{1}{2}[\eta^2 + (1 - \eta)^2] \hat{Q}^2 K^2_0(\hat{Q}r), \hspace{1cm} (11)$$

$$W^L(r, \eta, Q) = 2\eta(1 - \eta) \hat{Q}^2 K^2_0(\hat{Q}r), \hspace{1cm} (12)$$

where $\hat{Q} = [\eta(1 - \eta)]^{1/2} Q$, $\alpha_{\text{em}} = \frac{1}{137}$, $e_f^2$ is the sum of the squares of the quark charges. $\eta$ is the light-cone momentum fraction of one of the quarks in the photon.

The relevant integrals $\int d^2rd\eta \ldots$ can be found in [16] and they result in the following prescription: $r^\gamma$ in (3) should be replaced by the expressions

$$r^\gamma \rightarrow \frac{N_c \alpha_{\text{em}} e_f^2}{\pi^2} \frac{\Gamma^2(2 - \frac{\gamma}{2}) \Gamma^4(1 + \frac{\gamma}{2})}{\Gamma(4 - \gamma) \Gamma(2 + \gamma)} \left(\frac{Q}{2}\right)^{-\gamma} I^{T, L}(\gamma),$$  \hspace{1cm} (13)$$

where

$$I^T(\gamma) = \frac{(2 - \frac{\gamma}{2})(1 + \frac{\gamma}{2})}{(\frac{\gamma}{2})(1 - \frac{\gamma}{2})}, \hspace{1cm} (14)$$

$$I^L(\gamma) = 2 \hspace{1cm} (15)$$

Averaging over the nucleon positions reduces to multiplying $\langle \langle \psi_Q | F_N^{\text{dir}} | \psi_Q \rangle \rangle$ by $N$ single particle nucleon densities $\rho(b, z_k)$ and integrating over $z_k$'s kept ordered:

$$\langle \langle \langle \psi_Q | F_N^{\text{dir}} | \psi_Q \rangle \rangle \rangle = \langle \psi_Q | F_N^{\text{dir}} | \psi_Q \rangle$$
\[ \times \int d^2b \int_{-\infty}^{+\infty} dz_1 \int_{z_1}^{+\infty} dz_N e^{i(z_1-z_N)m} \rho(b,z_1) \rho(b,z_N) \]
\[ \times \int_{z_1}^{z_N} dz_2 \rho(b,z_2) \ldots \int_{z_{N-2}}^{z_N} dz_{N-1} \rho(b,z_{N-1}) \]
\[ = \langle \psi_Q | F^\text{dir}_N | \psi_Q \rangle \int d^2b \int_{-\infty}^{+\infty} dz_1 \int_{z_1}^{+\infty} dz_N e^{i(z_1-z_N)m} \rho(b,z_1) \rho(b,z_N) \]
\[ \times \frac{1}{(N-2)!} \left( \int_{z_1}^{z_N} dz \rho(b,z) \right)^{N-2} \] \hfill (16)

Inserting (16) into (9) we obtain the required formula for the shadow.

For the sake of completeness let us also give the explicit formula for \( \sigma_{\text{tot}}(n) \):

\[ \sigma_{\text{tot}}(n) = \sigma(r_0,Y) = 2\pi\alpha^2 r_0^2 n_{\text{eff}} \int \frac{d\gamma}{2\pi i} \left( \frac{r}{r_0} \right)^\gamma e^{\Delta(\gamma)\gamma} h(\gamma) , \] \hfill (17)

where \( r^\gamma \) is given by Eq. (13).

3 Forward \( \gamma^* \)-nucleus amplitude from the quasi-elastic scattering of a QCD dipole in the nucleus

As is well-known since the seminal paper of Stodolsky [15] and of Gottfried and Yennie [14], the multiple elastic scattering of a photon fluctuation inside a nucleus contributes in an essential way to the nuclear shadowing. In the QCD dipole picture this corresponds to quasi-elastic scattering of an onium whose transverse size is distributed according to the light-cone wave functions of the photon given in [13]. In the present section we derive the formula for this contribution to the shadow.

The \( N \)-fold scattering contribution to the forward \( \gamma^* \)-nucleus amplitude is built as a product of \( N \) onium-nucleon amplitudes (including the eikonal phases) sandwiched between the initial and final state of the virtual photon and summed over all intermediate states of the onium as follows

\[ \langle \psi_Q | F^\text{qel}_N | \psi_Q \rangle = n_{\text{eff}}^N \langle Q | T(z_N) | k_{N-1} \rangle \]
\[ \times \frac{d^2k_{N-1}}{4\pi^2} \langle k_{N-1} | T(z_{N-1}) | k_{N-2} \rangle \frac{d^2k_{N-2}}{4\pi^2} \]
Here $k_1, \ldots k_{N-1}$ are the relative transverse momenta of the quark and antiquark forming the onium. To shorten the expression, we skip the integral symbols (and we shall continue to so doing).

At this point it is important to realize that the amplitudes entering (18) depend on the longitudinal position, acquiring the eikonal phase. So we can write

$$\langle k' \mid T(z) \mid k \rangle = e^{iz k'_L} \langle k' \mid T \mid k \rangle e^{-iz k_L},$$

where $k_L$ and $k'_L$ are the longitudinal momenta of the onium before and after scattering.
The first step is to transform the transverse momentum amplitudes 
\( \langle k' \mid T \mid k \rangle \) into transverse position amplitudes, so that we can use the explicit expression for the onium-onium forward elastic amplitude, \( T(r) \), derived in [2, 3]. We shall use here the path integral representation, given by:

\[
T(r, r_0, Y) = \pi\alpha^2 r r_0 \int_{c-i\infty}^{c+i\infty} \frac{d\lambda}{2\pi i} \left( \frac{r}{r_0} \right)^{\lambda-1} e^{\Delta(\lambda)} Y h(\lambda),
\]

where \( r \) and \( r_0 \) are the transverse sizes of the colliding onia. In terms of \( T(r, r_0, Y) \equiv T(r) \) we thus obtain

\[
\langle k' \mid T(z) \mid k \rangle = e^{iz(k^2-k'^2)} \xi d^2 \rho e^{i(k'-k)\rho} T(\rho)
\]

with

\[
\xi = \frac{1}{2\nu\eta(1-\eta)}.
\]

In arriving at (22) we have used the high energy approximation for \( k_L \), viz.

\[
k_L = \nu - \frac{M^2}{2\nu} = \nu - \frac{k^2}{2\nu\eta(1-\eta)}.
\]

Similarly we obtain

\[
\langle k \mid T(z) \mid Q \rangle = e^{iz(x_{Bj} m - \xi k^2)} d^2 \rho e^{ik\rho} \langle \rho \mid T \mid Q \rangle,
\]

where \( \langle \rho \mid T \mid Q \rangle \) includes the photon wave function:

\[
\langle \rho \mid T \mid Q \rangle = T(\rho)\Psi_Q(\rho).
\]

Putting this in (18) and rescaling the variables

\[
k = Q\kappa, \quad \rho = \frac{r}{Q}, \quad \zeta = \frac{mz}{\eta(1-\eta)}
\]

we obtain

\[
\langle \psi_Q \mid F^{qg}_{r} \mid \psi_Q \rangle = \frac{1}{Q^2} e^{i(x_{Bj}(z_{N-1})m_{Bj})} \eta_{\text{eff}}(Q \mid r_N) d^2 r_N
\]

\[
\times \Phi[r_{N-1} - r_N, x_{Bj}(\zeta_{N-1} - \zeta_{N-1})] T(r_{N-1}) d^2 r_{N-1}
\]

\[
\times \Phi[r_{N-2} - r_{N-1}, x_{Bj}(\zeta_{N-2} - \zeta_{N-2})] T(r_{N-2}) \ldots \Phi[r_2 - r_3, x_{Bj}(\zeta_3 - \zeta_2)]
\]

\[
\times T(r_2) d^2 r_2 \Phi[r_1 - r_2, x_{Bj}(\zeta_2 - \zeta_1)] T(r_1) d^2 r_1 \langle r_1 \mid T \mid Q \rangle,
\]

with

\[
\Phi(\Delta, a) \equiv \int \frac{d^2 \kappa}{(2\pi)^2} e^{i\kappa a} e^{i\Delta a^2} = \frac{i}{4\pi a} e^{-\frac{\Delta^2}{4a}},
\]
where we have assumed that \( a = \text{Re}(a) + i \varepsilon \) to give a definite meaning to the integral.

Further calculations are continued in the limit \( x_{Bj} \ll 1 \). In this limit the following formula can be derived by the saddle point method for \( a, a' \ll 1 \)

\[
\int d^2r \Phi(s-r,a)F(r)\Phi(r-s',a') = \Phi(s-s', a + a')F(\bar{s}) \exp \left[ i\bar{a} \left( \nabla_F(\bar{s}) / F(\bar{s}) \right)^2 \right],
\]

(29)

where

\[
\bar{s} = \frac{a's + as'}{a + a'},
\]

(30)

and

\[
\bar{a} = \frac{aa'}{a + a'}.
\]

(31)

By repeated application of this formula we finally arrive at the following result

\[
\langle \psi_Q | F_{N}^{\text{qel}} | \psi_Q \rangle = \frac{1}{\int d\eta \int d^2\rho | \Psi_Q(\rho, \eta) |^2 \left[ n_{\text{eff}}T(\rho, r_0, x_{Bj}) \right]^N}
\]

\[
\times e^{-\frac{1}{2\nu} \sum_{j=1}^{N} z_j (M_j^2 - M_{j+1}^2)},
\]

(32)

where \( T \) is given by (20) and

\[
M_1^2 = M_{N+1}^2 = -Q^2, \quad M_j^2 = \left( \frac{N - j + 1}{(\eta(1 - \eta))^{1/2} \rho^2 + \nabla \log[\Psi(\rho)]} \right)^2, \quad j = 2, \ldots, N.
\]

(33)

Some details of these calculations are given in the Appendix B.

The shadow for the quasi-elastic process is now

\[
C_{\text{qel}}^{A} (A, x_{Bj}, Q^2) = \frac{-2}{A\sigma_t(n)} \text{Re} \left( \sum_{N=2}^{A} (-1)^N \frac{A!}{(A-N)!} \langle \langle \psi_Q | F_{N}^{\text{qel}} | \psi_Q \rangle \rangle \right)
\]

(34)

with the following averaging over the nuclear densities

\[
\langle \langle \psi_Q | F_{N}^{\text{qel}} | \psi_Q \rangle \rangle = \frac{1}{\int d\eta d^2\rho | \Psi_Q(\rho, \eta) |^2 \left[ n_{\text{eff}}T(\rho, r_0, x_{Bj}) \right]^N}
\]

\[
\times \int d^2b \int \ldots \int dz_N dz_{N-1} \ldots dz_1 \rho(b, z_N)e^{-\frac{1}{2\nu} z_N (M_N^2 - M_{N+1}^2)}
\]

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\begin{align*}
\times \rho(b, z_{N-1})e^{-\frac{1}{2}z_{N-1}(M_{N-1}^2-M_2^2)} \cdots \\
\times \cdots \rho(b, z_2)e^{-\frac{1}{2}z_2(M_2^2-M_1^2)}\rho(b, z_1)e^{-\frac{1}{2}z_1(M_1^2-M_2^2)},
\end{align*}

(35)

4 Conclusions and outlook

We have shown that the QCD dipole picture can provide a well defined and compact formulae for nuclear shadowing of the virtual photons. The formulae are well suited for numerical evaluation. They contain the following parameters:

(i) The pomeron intercept \( \Delta_P \);

(ii) The nucleon size parameter \( r_0 \);

(iii) The effective number of dipoles in the proton \( n_{\text{eff}} \);

(iv) The effective number of flavours (reflecting on \( e_2^2 \));

(v) The scales in the elastic and in the diffractive \( \gamma^*\)-proton scattering.

All these parameters can, in principle, be determined from the fit of the dipole picture to the proton data. Such a fit has been completed for the total \( \gamma^*\)-proton cross-section \([4]\). That gave \( \Delta_P = 0.285, Q_0 \equiv \frac{2}{r_0} = 0.622 \text{ GeV}; \)

\( n_{\text{eff}}e_2^2 = 3.8; Y = \log \frac{1.65}{x_{\text{ Bj}}} \). So, as long as the fits to the diffractive production are not available, we are left with two not fully determined parameters: the effective number of flavours and the scale in the diffractive dissociation.

It is important to remember that the nuclear shadowing effects are determined by the forward diffractive amplitudes. Our formulae can thus be employed to obtain the cross section for the forward diffractive dissociation of the virtual photon on one nucleon. Indeed, such cross sections on one nucleon can be obtained from the formulae of Sections 2 and 3 for \( F_{N=2}^{\text{dir}} \) and \( F_{N=2}^{\text{qel}} \):

\[
\frac{d\sigma^{\text{dir}}}{dyd^2p_t}|_{p_t=0} = \frac{1}{(2\pi)^2} \langle \Psi_Q| F_{N=2}^{\text{dir}}(z_1 = z_2)|\Psi_Q \rangle ,
\]

(36)

and

\[
\frac{d\sigma^{\text{qel}}}{d^2p_t}|_{p_t=0} = \frac{1}{(2\pi)^2} \langle \Psi_Q| F_{N=2}^{\text{qel}}(z_1 = z_2)|\Psi_Q \rangle ,
\]

(37)

where \( p_t \) is the transverse momentum of the final proton, and \( y \) defined in Eq. (3) contains \( M \), the diffractively produced mass. Therefore (36) is, in fact, a differential cross section for production from the incident virtual photon of an object of mass \( M \). On the other hand (37) is the total diffractive
dissociation cross section \textit{integrated over diffractively excited masses}. One can resolve (37) into the differential cross sections \( \frac{d\sigma^{q\bar{q}l}}{dM^2 dq^2} \big|_{p_t=0} \) applying a straightforward procedure. This will be discussed elsewhere [19].

Our formulae, given in Sections 2 and 3 refer to two mechanisms described in the Introduction: (a) the quasi-elastic interaction of the \( q\bar{q} \) dipole and (b) the multiple-pomeron coupling. One must remember, however, that these two mechanisms are not mutually exclusive: they mix in the multiple scattering terms, i.e. there are collisions in which they both take place. Such mixed amplitudes are also readily calculable by the methods developed in the present paper.

Any further discussion strongly depends on the outcome of the numerical estimates. Before this is available let us simply list the problems which, in our opinion, seem interesting.

(a) The relative importance of the multi-pomeron and the quasi-elastic interactions. Apart from its primary interest, it is essential for an estimation of the importance of the mixing terms.

(b) An extrapolation for very small \( x_{Bj} \) where the unitarity is expected to break down. It is likely that for nuclear targets this effect will occur much earlier (i.e. for larger \( x_{Bj} \)) than estimated in [3, 17].

(c) The dependence of the kind \( x_{Bj}^{-\Delta_p} \) which appear in BFKL forward amplitudes imply the existence of a large ratio of the real to imaginary part of the forward amplitudes. We have shown already [7, 18] that this influences significantly the predicted amount of shadowing. It will thus be interesting to discuss this problem again with the present, more precise, formulation.

\textbf{Appendix A}

In this Appendix we indicate some details of the derivation of the formula for \( n_N \), Eq. (3).

We start with \( n_2 \).

Using the generating function given in [3] and assuming that evolution of the two exchanged pomerons goes until the rapidity \( y \) is reached, we obtain the following equation for \( n_2 \) (\textit{c.f. also Eq. (52) of [2]}):
\[
\frac{dn_2(x_{01}, Y, y, x, x')}{dY} = \frac{\alpha N_c}{\pi^2} \int \frac{x_{01}^2 d^2 x_2}{x_{02}^2 x_{12}} n_1(x_{12}, y, x)n_1(x_{02}, y, x')\delta(Y - y)
+ \frac{2\alpha N_c}{\pi} \int dx_{12} K(x_{01}, x_{12}) n_2(x_{12}, Y, y, x, x'),
\]
where \( n_1 \) is the single dipole density and
\[
K(x_{01}, x_{12}) = \frac{1}{2\pi} \int \frac{x_{01}^2 d^2 x_2}{x_{02}^2 x_{12}} - \delta(x_{01} - x_{12}) \log(x_{01}/\rho)
\]
is the Lipatov kernel (c.f. [1]) which satisfies the eigenvalue equation:
\[
\int K(x_{01}, x_{12}) x_{12}^\lambda dx_{12} = x_{01}^\lambda \chi(\lambda)
\]
with \( \chi(\lambda) \) given by (4). We now introduce Mellin transforms to take advantage of (40):
\[
n_2(x_{01}, Y, y, x, x') = \int_{c-i\infty}^{c+i\infty} d\gamma \frac{\tilde{n}_2(\gamma, Y, y, x, x') x_{01}^\gamma}{2\pi i}
\]
and
\[
\tilde{n}_2(\gamma, Y, y, x, x') = \int_0^\infty dx_{01} x_{01}^{-1-\gamma} n_2(x_{01}, Y, y, x, x').
\]
Substituting (41) into (38) and using (40) we obtain a differential equation for \( \tilde{n}_2 \) which can be easily solved giving
\[
\tilde{n}_2(\gamma, Y, y, x, x') = e^{\Delta(\gamma)(Y-y)} g_2(\gamma, y, x, x'),
\]
where \( \Delta(\gamma) \) is given by (4) and
\[
g_2(\gamma, y, x, x') = \frac{\alpha N_c}{\pi^2} \int_0^\infty dx_{01} x_{01}^{-1-\gamma} \int \frac{x_{01}^2 d^2 x_2}{x_{02}^2 x_{12}} n_1(x_{12}, y, x)n_1(x_{02}, y, x').
\]
Substituting (44) into (41) we finally have
\[
n_2(r, Y, y, x, x') = \frac{\alpha N_c}{\pi^2} \int_{c-i\infty}^{c+i\infty} d\gamma \frac{r^\gamma e^{\Delta(\gamma)(Y-y)}}{2\pi i}
\times \int_0^\infty dx_{01} x_{01}^{-1-\gamma} \int \frac{x_{01}^2 d^2 x_2}{x_{02}^2 x_{12}} n_1(x_{12}, y, x)n_1(x_{02}, y, x').
\]
We now turn to $n_3$.

Using the generating function given in [3] and assuming that evolution of the three exchanged pomerons goes until the rapidity $y$ is reached, we obtain the following equation for $n_3$:

$$\frac{d n_3(x_{01}, Y, y, x, x', x'')}{dY} = \frac{\alpha N_c}{\pi^2} \int \frac{x_{01}^2 d^2 x_2}{x_{02}^2 x_{12}^2} [n_1(x_{12}, y, x)n_2(x_{02}, Y, y, x', x'')]_{\text{sym}} \delta(Y - y)$$

$$+ \frac{2\alpha N_c}{\pi} \int dx_{12} K(x_{01}, x_{12}) n_3(x_{12}, Y, y, x, x', x''),$$  \hspace{1cm} (46)

where $n_2(x_{02}, Y, y, x', x'')$ is given by (45). The symbol $\text{sym}$ denotes a sum of three terms, symmetrized with respect to $x, x', x''$.

Introducing Mellin transforms, as in (41), (42), substituting them into (38) and using (40) we obtain a differential equation for $\tilde{n}_3$. The solution is found again in the form

$$\tilde{n}_3(\gamma, Y, y, x, x', x'') = e^{\Delta(\gamma)(Y-y)} g_3(\gamma, y, x, x', x''),$$  \hspace{1cm} (47)

where

$$g_3(\gamma, y, x, x', x'') = \frac{\alpha N_c}{\pi^2} \int_0^\infty dx_{01} x_{01}^{-1-\gamma}$$

$$\times \int \frac{x_{01}^2 d^2 x_2}{x_{02}^2 x_{12}^2} [n_1(x_{12}, y, x)n_2(x_{02}, y, y, x, x', x'')]_{\text{sym}},$$  \hspace{1cm} (48)

and $n_2(x_{02}, y, y, x', x'')$ is given by (45).

The formulae for $n_4, n_5, \ldots$ are derived in a similar way.

The integrals over $x_1, \ldots, x_N; x'_1, \ldots, x'_N$ in (41) are performed using the formula for the forward onium-onium amplitude $T(r, r', y)$

$$T(r, r', y) = \int \frac{dx}{x} \frac{dx'}{x'} r(x, x') n_1(r, y*, x)n_1(r', y - y*, x')$$

$$= \pi \alpha^2 r r' \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2\pi i} e^{\Delta(\gamma)y} \left( \frac{r}{r'} \right)^{1-\gamma} h(\gamma)$$  \hspace{1cm} (49)

which was first derived in [2] and [1, 4] in a slightly more simplified form.
Using (49) and (45), (6) gives

\[ F_{\text{dir}}^2(r, r_1, r_2; Y, y) = e^{i(z_1 - z_2)\alpha N_c^c \pi^2} e^{\Delta(\gamma)(Y - y)} \]

\[ \times \int_{c - i\infty}^{c + i\infty} \frac{d\gamma}{2\pi i} r_1^{\gamma - 1} e^{\Delta(\gamma)y} h(\lambda_1) \]

\[ \times \int_{c - i\infty}^{c + i\infty} \frac{d\gamma}{2\pi i} r_2^{\gamma - 1} e^{\Delta(\gamma)y} h(\lambda_2) \Omega(\gamma, \lambda_1, \lambda_2), \]

where

\[ \Omega(\gamma, \lambda_1, \lambda_2) = \int_0^\infty dx_01 x_01^{1-\gamma} \int_R \frac{x_01^2 d^2 x_2}{x_02^2 x_{12}^2} x_02^{2-\lambda_1} x_{12}^{2-\lambda_2}, \]

(50)

\[ \Omega(\gamma, \lambda_1, \lambda_2) \]

\[ \text{can be calculated using } \]

\[ \frac{x_01^2 d^2 x_2}{x_02^2 x_{12}^2} = 2\pi x_02 \int_0^\infty dx_02 \int_0^\infty dx_{12} \int_0^\infty dk J_0(k x_01) J_0(k x_02) J_0(k x_{12}), \]

(51)

from which the following useful identity can be obtained:

\[ \int_0^\infty dx_01 x_01^{1-\gamma} \int_R \frac{x_01^2 d^2 x_2}{x_02^2 x_{12}^2} x_02^{\lambda_2} x_{12}^{\lambda'} = \pi G(\lambda) G(\lambda') G(\gamma) 2\pi i \delta(\gamma - \lambda - \lambda'). \]

(52)

Using (53) in (54) and setting \( r_1 = r_2 = r_0 \) we obtain the formula (3) for \( F_{\text{dir}}^2 \). With the same technique one can obtain (3) for more than two collisions. We indicate below the derivation for three collisions.

Substituting (53) into (54) and using (47) we obtain

\[ n_3(x_01, Y, y, x, x', x'') = \left( \frac{\alpha N_c^c \pi^2}{2} \right)^2 \int_0^\infty \frac{d\gamma}{2\pi i} x_01 e^{\Delta(\gamma)(Y - y)} \int_0^\infty dx_01 x_01^{1-\gamma} \]

\[ \times \int_R \frac{x_01^2 d^2 x_2}{x_02^2 x_{12}^2} n_1(x_{12}, y, x) \int_0^\infty \frac{d\gamma'}{2\pi i} x_02^{\gamma'} \]

\[ \times \int_0^\infty dx_{01}' (x_{01}')^{1-\gamma'} \int_R \frac{(x_{01}')^2 d^2 x_{12}'}{(x_{02}')^2 (x_{12}')^2} n_1(x_{12}', y, x') n_1(x_{02}', y, x'') \]

\[ \text{sym}. \]
When (54) is substituted into (1) with $r_1 = r_2 = r_3 = r_0$ we obtain

$$F_{\text{dir}}^3(r, r_0; Y, y) = e^{i(z_1 - z_N)\text{mexp}}$$

$$\times 3 \left( \frac{\alpha N e}{\pi^2} \right)^2 \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2\pi i} x_{01}^\gamma e^{\Delta(Y - y)} \int_0^\infty dx_{01} x_{01}^{1-\gamma}$$

$$\times \int_R \frac{x_{01}^2 d^2 x_{12}}{x_{02}^2 x_{12}} T(x_{12}, r_0, y) \int_{c-i\infty}^{c+i\infty} \frac{d\gamma'}{2\pi i} x_{02}^\gamma'$$

$$\times \int_0^\infty dx_{01}' (x_{01}')^{-1-\gamma'} \int_R \frac{(x_{01}')^2 d^2 x_{12}'}{(x_{03}')^2 (x_{12}')^2} T(x_{12}', r_0, y) T(x_{02}', r_0, y),$$

(55)

where $T(r, r', y)$ is the dipole–dipole forward scattering amplitude given by (49).

The integrals over $dx_{01}, d^2 x_{2}$ and $dx_{01}' d^2 x_{2}'$ can be performed using again the identity (53) and taking into account the fact that the dependence of $T(r, r', y)$ on $r$ and $r'$ is in the form of a power law. The result of these operations is the Eq. (3) for $N = 3$.

### Appendix B

In this Appendix we give a few intermediate steps of the calculations of Section 3 which lead to Eq. (32).

First, by repeated application of (29) we obtain

$$\langle \psi_Q | F_{\text{qel}}^N | \psi_Q \rangle = n_{\text{eff}}^N Q^{-2} e^{i(z_1 - z_N)\text{mexp}} (Q | T | r_N) d^2 r_N$$

$$\times \Phi(r_1 - r_N, x_{Bj}(\zeta_N - \zeta_{N-1})) S(r_N, r_1; \zeta_N, \ldots, \zeta_1) d^2 r_1 \langle r_1 | T | Q \rangle,$$

(56)

where

$$S(r_N, r_1; \zeta_N, \ldots, \zeta_1) = \prod_{j=2}^{N-1} \frac{d\lambda_j}{2\pi i} A(\lambda_j, r_0, Y) r_j^{\lambda_j}$$

$$\times \exp \left( i \frac{a_j \lambda_j}{r_j^2} + i 2 \sum_{j,k=2}^{N-1} \frac{\lambda_j \lambda_k}{r_j^2 r_k^2} a_{jk} \right)$$

(57)

with

$$\vec{r}_j = \frac{(z_N - z_j) \vec{r}_1 + (z_j - z_1) \vec{r}_N}{z_n - z_1},$$

(58)
\[ a_j = \frac{(z_N - z_j)(z_j - z_1)}{z_n - z_1}, \quad (59) \]
\[ a_{jk} = \frac{(z_N - z_j)(z_k - z_1)}{z_n - z_1}. \quad (60) \]

\[ A(\lambda, r_0, Y) = n_{\text{eff}} \pi \alpha \frac{r_0^2}{2} \lambda e^{\Delta(\lambda)} Y h(\lambda). \quad (61) \]

is taken from the integrand of Eq. (21) and \( n_{\text{eff}} \) added for the reasons explained in the main text.

We finally perform the integral over \( d^2 r_N \), again in the saddle point approximation which is valid for small \( x_{\text{BJ}} \). Then \( \Phi(r_1 - r_N, x_{\text{BJ}}(\zeta_N - \zeta_1)) \) is very strongly peaked around \( r_n \approx r_1 \), so that in leading order all \( r_j \)'s become \( r_1 \). This approximation puts a new kind of factor, \( \nabla \psi_Q/\psi_Q \), into the over all phase of the expression and we obtain (returning to \( z_j \)'s and \( \rho_j \)'s)

\[
\langle \psi_Q | F_{N}^{\text{qel}} | \psi_Q \rangle = n_{\text{eff}} e^{i x_{\text{BJ}}(z_1 - z_N)} \int_0^1 d\eta \int d^2 \rho \left| \psi_Q(\rho, \eta) \right|^2 \times \exp \left[ \frac{i}{2 \nu} \left( \frac{1}{\eta(1 - \eta) \rho^2} \sum_{j=2}^{N} [2(N - j) + 1](z_j - z_1) \right. \right.
\]
\[
+ \left. \left. 2 \sum_{j=2}^{N} z_j - z_1 | \right) \frac{\vec{\rho} \cdot \nabla \psi_Q(\rho)}{[\eta(1 - \eta)]^{1/2} \rho^2 \psi_Q(\rho)} + (z_n - z_1) \left[ \frac{\nabla \psi_Q(\rho)^2}{\psi_Q(\rho)^2} \right] \right]\]
\[
\times \prod_{j=1}^{N} \int \frac{d\lambda_j}{2 \pi i} A(\lambda_j, r_0, Y) \rho^\lambda_j. \quad (62) \]

One can check by a direct substitution of \( M_j^2 \) given by (33) into (32) that it reduces to (62). The version of \( \langle \psi_Q | F_{N}^{\text{qel}} | \psi_Q \rangle \) given in (33) has, however, a clearer physical interpretation.

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References

[1] A.H. Mueller, *Nucl. Phys.* **B415**, 373 (1994).

[2] A.H. Mueller, B. Patel, *Nucl. Phys.* **B425**, 471 (1994).

[3] A.H. Mueller, *Nucl. Phys.* **B437**, 107 (1995).

[4] H. Navelet, R. Peschanski, Ch. Royon, *Phys. Lett.* **B366**, 329 (1995).

[5] H. Navelet, R. Peschanski, Ch. Royon, S. Wallon, *Phys. Lett.* **B385**, 357 (1996); and private communication from Ch. Royon.

[6] M. Arneodo et al., “Nuclear Beams at HERA”, in “Future Physics at HERA”, Proc. of the Workshop 1995/96 Vol.2, p. 887, ed. by G.Ingelman, A.De Roeck and R. Kalmer, DESY (1996).

[7] A. Bialas, W. Czyz, W. Florkowski, *Phys. Rev* **D55**, 6830 (1997).

[8] M. Strikman, Acta Phys. Pol. **B27**, 3431 (1996).

[9] A. Bialas, R. Peschanski, *Phys. Lett.* **B378**, 302 (1996).

[10] A. Bialas, R. Peschanski, *Phys. Lett.* **B387**, 405 (1996).

[11] A. Bialas, *Acta Phys. Pol.* **B27**, 1263 (1996).

[12] H1 Coll. T. Ahmed et al., *Phys. Lett.* **B348**, 681 (1995); ZEUS Coll. M. Derrick et al., *Z. Phys.* **C68**, 569 (1995).

[13] J.D. Bjorken, J. Kogut, D. Soper, *Phys. Rev.* **D3**, 1382 (1971); N.N. Nikolaev, B.G. Zakharov, *Z. Phys.* **C49**, 607 (1991).

[14] K. Gottfried, D.R. Yennie, *Phys. Rev.* **182**, 1595 (1969).

[15] L. Stodolsky, *Phys. Rev. Lett.* **18**, 135 (1967).

[16] P.S. Gradstein, I.M. Rhyzhik, *Tablitsy integralov, sum, riadov i proizvedenii*, Moscow 1971, 5th ed., p.707.

[17] G.P. Salam, *Nucl. Phys.* **B461**, 512 (1996); A.H. Mueller, G.P. Salam, *Nucl. Phys.* **B475**, 293 (1996).

[18] A. Bialas, W. Czyz, *Phys. Lett.* **B328**, 172 (1994).

[19] A. Bialas, W. Czyz, to be published.