COHERENCE FOR MONOIDAL $G$-CATEGORIES AND BRAIDED $G$-CROSSED CATEGORIES

CÉSAR GALINDO

Abstract. We prove a coherence theorem for actions of groups on monoidal categories. As an application we prove coherence for arbitrary braided $G$-crossed categories.

1. Introduction

Given a group $G$ and a monoidal category $\mathcal{C}$, a strict action of $G$ on $\mathcal{C}$ is a group morphisms from $G$ to $\text{Aut}^{\text{Strict}}(\mathcal{C})$ (the group of all strict monoidal automorphisms of $\mathcal{C}$). For almost all situations where symmetries of monoidal categories arise, strict actions are not sufficient, mainly because the natural notion of symmetry in monoidal category theory is not a strict monoidal automorphism. Rather, a symmetry in a monoidal category is a strong monoidal auto-equivalence. The categorical symmetries of a monoidal category form a monoidal category. Since every group $G$ defines a discrete monoidal category $\overline{G}$, the appropriate definition of action is a monoidal functor from $\overline{G}$ to $\text{End}_{\otimes}(\mathcal{C})$, where $\text{End}_{\otimes}(\mathcal{C})$ is the monoidal category of strong monoidal endofunctors as objects and morphisms given by monoidal natural isomorphism). For simplicity, a monoidal category with a $G$-action will be called a monoidal $G$-category. If the action is strict we will say that the monoidal $G$-category is strict. Thus, a monoidal $G$-category is a monoidal category $\mathcal{C}$ with a family of monoidal auto-equivalences $\{(g_*, \psi^g) : \mathcal{C} \to \mathcal{C}\}_{g \in G}$ and a family of monoidal isomorphisms $\{\phi(g, h) : (gh)_* \to g_*h_*\}_{g, h \in G}$ satisfying certain coherence axioms (see Section 3.1).

The distinction between strict monoidal $G$-categories and general monoidal $G$-categories is analogous to the relation between strict monoidal categories and general monoidal categories. In monoidal category theory, the Mac Lane coherence theorem says that for two expressions $S_1, S_2$ obtained from $X_1 \otimes X_2 \otimes \ldots \otimes X_n$ by inserting 1’s and brackets, any pair of isomorphisms $\Phi : S_1 \to S_2$, composed of the associativity and unit constraints and their inverses, are equal. A similar presentation of coherence for arbitrary monoidal $G$-categories can be stated. However, an equivalent statement and convenient way of expressing the coherence theorem for monoidal $G$-categories is the following:

Theorem 1.1 (Coherence for monoidal $G$-categories). Let $G$ be a group. Every monoidal $G$-category is equivalent to a strict monoidal $G$-category.
In essence, Theorem 1.1 says that in order to prove a general statement for monoidal $G$-categories, we may assume without loss of generality to assume that we are working with strict $G$-categories.

The main result of this paper is to prove Theorem 1.1. For this, given a group $G$ and a monoidal $G$-category $\mathcal{C}$, we construct a strict monoidal $G$-category $\mathcal{C}(G)$ and a adjoint monoidal $G$-equivalence $\mathcal{F} : \mathcal{C} \to \mathcal{C}(G)$. In fact, we construct a strict left adjoint 2-functor to the forgetful 2-functor from the 2-category of strict monoidal $G$-categories to the 2-category of monoidal $G$-categories.

Braided $G$-crossed categories are interesting because they have applications to mathematical physics [2, 6, 8] and low-dimensional topology [12, 13, 14]. As an application of Theorem 1.1, we prove coherence theorems for general $G$-crossed categories and braided $G$-crossed categories. This coherence theorem generalizes the Müger’s coherence theorem for braided $G$-crossed fusion categories over algebraically closed field of characteristic zero and $G$ finite, [12, Appendix 5, Theorem 4.3]. Müger’s coherence theorem is obtained as corollary of depth and difficult to prove characterization of braided $G$-crossed fusion categories. The inconveniences of [12, Appendix 5, Theorem 4.3] are that the construction of the strictification is not direct, $G$ must be finite and the conditions on the braided $G$-crossed category is very restrictive. In loc. cit. Michael Müger asked for a proof of the coherence in a more direct way, extending its domain of validity. Theorem 5.6 has no restriction on $G$ or the underlying category $\mathcal{C}$ and the constructions of the strictification $\mathcal{C}(G)$ is very explicit. Thus, Theorem 5.6 answers Müger’s question.

The paper is organized as follows. Section 2 contains preliminaries and notations. In Section 3, we define the 2-category of monoidal $G$-categories. Section 4 contains the proof of coherence theorem for monoidal $G$-categories. In Sec. 5, we apply the main result to crossed $G$-categories and braided $G$-crossed categories.

Acknowledgements. The author is grateful to Paul Bressler for useful discussions. This research was partially supported by the FAPA funds from Vicerrectoría de Investigaciones de la Universidad de los Andes.

2. PRELIMINARIES AND NOTATIONS

Let $\mathcal{C}$ be a category. We denote by Obj$(\mathcal{C})$ the class of objects of $\mathcal{C}$ and by Hom$_C(X,Y)$ the set of morphisms in $\mathcal{C}$ from an object $X$ to an object $Y$. Also, by abuse of notation $X \in \mathcal{C}$ means that $X$ is an object of $\mathcal{C}$.

The symbols $\mathcal{C}$ and $\mathcal{D}$ will denote monoidal categories with unit objects $1_\mathcal{C}$ and $1_\mathcal{D}$ respectively. If no confusion arise, we will denote the unit object of a monoidal category just by 1. In order to simplify computations and statements, by monoidal category we will mean a strict monoidal category. This is justified by the coherence theorem of S. MacLane [9, 10].
2.1. Monoidal functors. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, F_2, F_0)$, where $F : \mathcal{C} \to \mathcal{D}$ is a functor, 
\[ F_2 = \{ F_2(X, Y) : F(X \otimes Y) \to F(X) \otimes F(Y) \} \]
are natural isomorphisms, $F_0 : F(1) \to 1$ is an isomorphism, such that the diagrams
\[
\begin{array}{ccc}
F(X \otimes Y \otimes Z) & \xrightarrow{F_2(X, Y \otimes Z)} & F(X \otimes Y) \otimes F(Z) \\
\downarrow F_2(X, Y \otimes Z) & & \downarrow F_2(X,Y) \otimes F(Z) \\
F(X) \otimes F(Y \otimes Z) & \xrightarrow{\text{id}_F(X) \otimes F_2(Y, Z)} & F(X) \otimes F(Y) \otimes F(Z)
\end{array}
\]
\[
\begin{array}{ccc}
F(X) & \xrightarrow{F_2(1, X)} & F(X) \otimes F(1) \\
\downarrow F_2(1, X) & & \downarrow \text{id}_F(X) \otimes F_0 \\
F(1) \otimes F(X) & \xrightarrow{F_0 \otimes \text{id}_F(X)} & F(X)
\end{array}
\]
commute for all objects $X, Y, Z \in \mathcal{C}$.

A monoidal functor $(F, F_2, F_0)$ is called unital if $F_0$ is the identity morphisms and strict if $F_2$ and $F_0$ are identity morphisms.

Remark 2.1. A monoidal functor $(F, F_2, F_0)$ is unital if and only if
\[ F_2(X, 1) = F_2(1, X) = \text{id}_X \]
for all $X \in \text{Obj}(X)$.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two monoidal functors. A natural transformation $\varphi = \{ \varphi_X : F(X) \to G(X) \} \} \x_{\mathcal{C}}$ from $F$ to $G$ is monoidal if the diagrams
\[
\begin{array}{ccc}
F(1) & \xrightarrow{\varphi_1} & G(1) \\
\downarrow F_0 & & \downarrow G_0 \\
1 & \xleftarrow{G_0} & 1
\end{array}
\]
and
\[
\begin{array}{ccc}
F(X \otimes Y) & \xrightarrow{F_2(X, Y)} & F(X) \otimes F(Y) \\
\downarrow \varphi_{X \otimes Y} & & \downarrow \varphi_X \otimes \varphi_Y \\
G(X \otimes Y) & \xrightarrow{G_2(X, Y)} & G(X) \otimes G(Y)
\end{array}
\]
commute for all objects $X, Y$ of $\mathcal{C}$.

Remark 2.2. If $F, G : \mathcal{C} \to \mathcal{D}$ are two unital monoidal functors then condition of (2.3) is just $\varphi_1 = \text{id}_1$. 

If $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ are two monoidal functors, then their composition $GF : \mathcal{C} \rightarrow \mathcal{E}$ is a monoidal functor with $(GF)_0 = F(G_0)F_0$ and

$$(GF)_2 = \{G_2(F(X), F(Y))G(F_2(X, Y))\}_{X, Y \in \mathcal{C}}.$$ 

A monoidal functor $(F, F_2, F_0) : \mathcal{C} \rightarrow \mathcal{D}'$ is called a monoidal equivalence if the functor $F$ is an equivalence of categories. If $(F, F_2, F_0)$ is a monoidal equivalence, the adjoint functor of $F$ has a canonical monoidal structure, and the unit and counit of the adjointion are monoidal natural isomorphisms, see [III, Proposition 4.4.2].

3. $G$-categories

Given monoidal categories $\mathcal{C}$ and $\mathcal{D}$, we will denote by $\text{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})$ the category of monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. The objects are monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms are monoidal natural isomorphisms. If $\mathcal{C} = \mathcal{D}$, we denote $\text{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})$ just by $\text{End}_{\otimes}(\mathcal{C})$. The category $\text{End}_{\otimes}(\mathcal{C})$ is a strict monoidal category with tensor product given by composition of monoidal functors and unit object the identity endofunctor $\text{Id}_\mathcal{C}$.

Analogously, given monoidal categories $\mathcal{C}$ and $\mathcal{D}$, we define $\text{Fun}_{\otimes}^u(\mathcal{C}, \mathcal{D})$ (respectively $\text{End}_{\otimes}^u(\mathcal{C})$) as the full subcategory of unital monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ (respectively the monoidal category of unital monoidal endofunctors of $\mathcal{C}$).

3.1. Definition of monoidal $G$-categories. Let $G$ be a group (or a monoid). We will denote by $\overline{G}$ the discrete monoidal category with $\text{Obj}(\overline{G}) = G$ and monoidal structure defined by the multiplication of $G$.

A monoidal $G$-category is pair $(\psi, \mathcal{C})$, where $\mathcal{C}$ is a monoidal category and $\psi : \overline{G} \rightarrow \text{End}_{\otimes}(\mathcal{C})$ is a monoidal functor. Two monoidal $G$-categories $(\psi, \mathcal{C})$ and $(\psi', \mathcal{C})$ are called strongly monoidally $G$-equivalent if $\psi$ and $\psi'$ are monoidal equivalent.

A monoidal $G$-category is called unital if $\psi$ is a unital monoidal functor from $\overline{G}$ to $\text{End}_{\otimes}^u(\mathcal{C})$.

**Proposition 3.1.** Every monoidal $G$-category $(\psi, \mathcal{C})$ is strongly $G$-equivalent to a unital monoidal $G$-category $(\psi', \mathcal{C})$.

**Proof.** It is clear that the proposition follows from the following statement: Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. Then, the inclusion functors $\iota : \text{Fun}_{\otimes}^u(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})$ and $\iota : \text{End}_{\otimes}^u(\mathcal{C}) \rightarrow \text{End}_{\otimes}(\mathcal{C})$ are equivalences of categories and of monoidal categories, respectively.

Recall that a functor is an equivalence if and only if it is essentially surjective and fully faithful. It is clear that the functors $\iota$ are fully faithful. We only need to see that they are surjective.

By (2.2), $F_2(X, 1_\mathcal{C}) = \text{id}_{F(X)} \otimes F_0^{-1}$, $F_2(1_\mathcal{C}, Y) = F_0^{-1} \otimes \text{id}_{F(Y)}$. Thus, defining

$$F'(X) = \begin{cases} F(X), & X \neq 1_\mathcal{C}; \\ 1_\mathcal{D}, & X = 1_\mathcal{C}. \end{cases}$$
we have a unital monoidal functor \((F', F'_2) : C \to D\). Finally, a monoidal natural isomorphisms \(\sigma : F \to F'\) is defined by

\[
\sigma(X) = \begin{cases} 
\text{id}_{F(X)}, & X \neq 1_C; \\
F_0, & X = 1_C.
\end{cases}
\]

In fact, \(\sigma\) is a natural isomorphism since \(\text{Hom}_D(1_D, 1_D)\) is a commutative monoid and \(hh' = h \otimes h'\) for all \(h, h' \in \text{Hom}_D(1_D, 1_D)\).

As a consequence of Proposition 3.1, from now on we will consider only unital monoidal \(G\)-categories and we just call them monoidal \(G\)-categories.

A monoidal \(G\)-category \(C\) consists of the following data:

- functors \(g_* : C \to C\) for each \(g \in G\),
- natural isomorphisms \(\phi(g, h) : (gh)_* \to g_* \circ h_*\) for each pair \(g, h \in G\),
- natural isomorphisms \(\psi^g(X, Y) : g_*(X \otimes Y) \to g_*(X) \otimes g_*(Y)\) for all \(X, Y \in \text{Obj}(C)\),

such that

1. \(g_*(1) = 1\)
2. \(\psi^g(X, 1) = \psi^g(1, X) = \text{id}_X\),
3. \(e_* = \text{Id}_C\),
4. \(\phi(e, g) = \phi(g, e) = \text{Id}_{g_*}\)

for all \(g \in G\), \(X \in \text{Obj}(C)\) and, for all \(g, h, k \in G\) and \(X, Y, Z \in \text{Obj}(C)\), the following diagrams commute:

\[
\begin{array}{cccc}
(ghk)_*(X) & \xrightarrow{\phi(ghk)(X)} & (gh)_*k_*(X) \\
\downarrow_{\phi(ghk)(X)} & & \downarrow_{\phi(ghk)(X)} \\
g_*(hk)_*(X) & \xrightarrow{g_*(\phi(hk)(X))} & g_*h_*k_*(X)
\end{array}
\]

\[
\begin{array}{cccc}
g_*(X \otimes Y \otimes Z) & \xrightarrow{\psi^g(X \otimes Y, Z)} & g_*(X \otimes Y) \otimes g_*(Z) \\
\downarrow_{\psi^g(X, Y \otimes Z)} & & \downarrow_{\psi^g(X, Y) \otimes \text{id}_{g_*(Z)}} \\
g_*(X) \otimes g_*(Y \otimes Z) & \xrightarrow{\text{id}_{g_*(X)} \otimes \psi^g(Y, Z)} & g_*(X) \otimes g_*(Y) \otimes g_*(Z)
\end{array}
\]
A monoidal $G$-category is called strict if $\psi^g$ and $\phi(g, h)$ are identities for all $g, h \in G$.

Given $\mathcal{C}$ and $\mathcal{D}$ monoidal $G$-categories, a monoidal $G$-functor is a pair $(F, \gamma)$, where $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor and $\gamma(g) : g_* \circ F \to F \circ g_*$ is a family of monoidal natural isomorphisms indexed by $G$, such that $\eta(e) = \text{Id}_F$ and for all $X \in \text{Obj}(\mathcal{C})$, $g, h \in G$ the diagrams

$$(3.4)$$

commute.

We say that $(F, \eta)$ is an equivalence of monoidal $G$-categories if the functor $F$ is an equivalence of categories. If $\mathcal{C} = \mathcal{D}$, a strongly equivalence is just an equivalence of monoidal $G$-categories of the form $(\text{Id}_\mathcal{C}, \eta)$.

If $(F, \eta), (L, \chi) : \mathcal{C} \to \mathcal{D}$ are monoidal $G$-functors, a monoidal natural transformation $\varphi : \mathcal{C} \to \mathcal{D}$ is called a monoidal natural transformation of $G$-categories if the diagrams

$$(3.5)$$

commute for all $X \in \mathcal{C}$ and $g \in G$. 
3.2. **Weak actions on Crossed modules.** The goal of this section is to present some examples of monoidal \( G \)-categories associated to crossed modules.

Recall that a crossed module is a pair of groups \( P \) and \( H \), a left action of \( P \) on \( H \) (by group automorphisms)

\[
P \times H \rightarrow H
\]

\((g,h) \mapsto gh,\)

and a group homomorphism \( \partial : H \rightarrow P \), such that \( \partial \) is \( G \)-equivariant:

\[
\partial(hg) = g\partial(h)g^{-1}
\]

and \( \partial \) satisfies the so-called Peiffer identity:

\[
\partial(h_1h_2) = h_1h_2h_1^{-1}.
\]

Note that \( \text{Im}(\partial) \subset P \) is a normal subgroup, then \( \text{coker}(\partial) \) is a group.

**Example 3.2.**

- Let \( H \) be a normal subgroup of a group \( P \). The group \( P \) acts by conjugation on \( H \) and \( \partial \) given by the inclusion defines a crossed module \((H, P, \partial)\).
- Let

\[
1 \rightarrow A \rightarrow H \xrightarrow{\partial} P \rightarrow 1,
\]

be a central extension of groups. If \( \iota : P \rightarrow H \) is a section (of sets) of \( \partial \), the group \( P \) acts on \( H \) by \( \iota(h) \iota(p)h(h)^{-1} \). Since \( A \) is central the action does not depend of the choice of \( \iota \). The projection \( \partial : H \rightarrow P \), defines a crossed module \((H, P, \partial)\).

If \((H, P, \partial)\) and \((H', P', \partial')\) are crossed modules, a morphism \((\alpha, \phi) : (H, P, \partial) \rightarrow (H', P', \partial')\) of a crossed modules is a commutative diagram of group morphisms

\[
\begin{array}{ccc}
H & \xrightarrow{\alpha} & H' \\
\downarrow{\partial} & & \downarrow{\partial'} \\
P & \xrightarrow{\phi} & P'
\end{array}
\]

such that \( \alpha(\iota h) = \phi(\iota)\alpha(h) \), for all \( h \in H, g \in P \). A morphism \((\alpha, \phi) : (H, P, \partial) \rightarrow (H', P', \partial')\) is called a weak equivalence if induces group isomorphisms

\[
\ker(\partial) \cong \ker(\partial'), \quad \text{coker}(\partial) \cong \text{coker}(\partial).
\]

Given two morphisms \((\alpha, \phi), (\alpha', \phi') : (H, P, \partial) \rightarrow (H', P', \partial')\) we define a natural transformation \( \theta : (\alpha, \phi) \Rightarrow (\alpha', \phi') \) as a map

\[
\theta : P \rightarrow H^\text{Im}(\phi) = \{h \in H : \iota h = h, \forall g \in \text{Im}(\phi)\}
\]

such that

\[
\phi'(g) = \partial(\theta(g))\phi(g), \quad \theta(g)\alpha'(h) = \alpha(h)\theta(g),
\]
for all $g \in P, h \in H$.

Let $G$ be a group and $(H, P, \partial)$ a crossed module. A weak action of $G$ on $(H, P, \partial)$ consists of the following data:

- a morphism $(\alpha_x, \phi_x)$ for each $x \in G$,
- natural transformations $\theta_{x,y} : (\alpha_{xy}, \phi_{xy}) \to (\alpha_x \alpha_y, \phi_x \phi_y)$ for each pair $x, y \in G$,

such that

1. $(\alpha_e, \phi_e) = (\text{Id}_H, \text{Id}_G)$
2. $\theta_{e,x} = \theta_{x,e}$ are the constant function $e$.
3. $\theta_{x,y,z}(g)\theta_{x,y}(\phi_z(g)) = \theta_{x,yz}(g)\alpha_x(\theta_{y,z}(g))$

for all $x, y, z \in G, g \in P$

**Example 3.3.** Let $1 \to H \overset{\partial}{\to} P \overset{\pi}{\to} G \to 1,$

be an exact sequence of groups and $\iota : Q \to G$ a section (of sets) of $\pi$. The group $G$ acts on $(H, P, \partial)$ by

$$
\phi_g(x) = \iota(g)x\iota(g)^{-1}, \quad \alpha_g(h) = \iota(g)\iota(h)\iota(g)^{-1},
$$

$$
\theta_{g,h}(x) = \alpha_g \circ \alpha_h(x)\alpha_{gh}(x)^{-1},
$$

for all $h \in H, x \in P, g, h \in G$

We can build a small strict monoidal category $\mathcal{C}(H, P, \partial)$ (in fact, a strict categorical group) from a crossed module $(H, P, \partial)$ as follows. First we let the set of object by $P$ and the set of arrow the semidirect product $H \rtimes P$ in which tensor product is given by the multiplication

$$(h, g)(h', g') = (h^{\circ}h', gg').$$

We define source and target maps $s, t : H \rtimes P \to P$ by:

$$s(h, g) = g, \quad t(h, g) = \partial(h)g,$$

define the identity-assigning map

$$i : P \to H \rtimes P$$

$$g \mapsto (1, g),$$

and define the composite of morphisms

$$(h, g) : g \to g', \quad (h', g') : g' \to g'',$$

to be

$$(hh', g) : g \to g''.$$
Every weak action \( \{\alpha_x, \phi_x, \theta_{x,y}\}_{x,y \in G} \) of a group \( G \) on a crossed module \((H, P, \partial)\) defines an action of \( G \) on the monoidal category \( \mathcal{C}(H, P, \partial) \). In fact, every morphism \((\alpha_x, \phi_x)\) defines a strict monoidal functor \( F_{(\alpha_x, \phi_x)} : \mathcal{C}(H, P, \partial) \rightarrow \mathcal{C}(H, P, \partial) \), \( F_{(\alpha_x, \phi_x)}(g) = \phi_x(g), \quad F_{(\alpha_x, \phi_x)}(h, g) = (\alpha_x(h), \phi_x(g)) \), and every natural transformation \( \theta_{x,y} \) defines a monoidal natural isomorphism \( \theta_{x,y} : F_{(\alpha_x, \phi_x)} \rightarrow F_{(\alpha_y, \phi_y)} \) by \( (\theta_{x,y}(g), \phi_{xy}(g)) : \phi_{xy}(g) \rightarrow \phi_x \circ \phi_y(g) \).

4. COHERENCE FOR MONOIDAL \( G \)-CATEGORIES

Let \( G \) be a group and let \( \mathcal{C} \) be a monoidal \( G \)-category. We define the category \( \mathcal{C}(G) \) as follows: the objects of \( \mathcal{C}(G) \) are pairs \((L, \eta)\), where \( L = \{L_g\}_{g \in G} \) is a family of objects of \( \mathcal{C} \) and
\[
(4.1) \quad \eta = \{\eta_{g,h} : g_* (L_h) \rightarrow L_{gh}\}_{(g,h) \in G \times G}
\]
is a family of isomorphisms such that \( \eta_{e,g} = \text{id}_{L_g} \) for all \( g \in G \) and the diagrams
\[
(4.2)
\begin{align*}
& (gh)_* (L_k) \xrightarrow{\eta_{gh,k}} L_{ghk} \\
& \downarrow \phi_{(gh)L_k} \quad \quad \quad \uparrow \eta_{g,h,k} \\
& g_* h_* (L_k) \xrightarrow{g_*(\eta_{h,k})} g_* (L_{hk})
\end{align*}
\]
commute for all \( g, h, k \in G \).

A morphism from \((L, \eta)\) to \((T, \chi)\) is a family of morphisms
\[
f = \{f_g : L_g \rightarrow T_g\}_{g \in G},
\]
such that the diagrams
\[
(4.3)
\begin{align*}
& g_* (L_h) \xrightarrow{\eta_{g,h}} L_{gh} \\
& \downarrow g_* (f_h) \quad \quad \quad \quad \downarrow f_{gh} \\
& g_* (T_h) \xrightarrow{\chi_{g,h}} T_{gh}
\end{align*}
\]
commute for all \( g, h \in G \). The composition of \( f : (L, \eta) \rightarrow (L', \eta') \) and \( l : (L', \eta') \rightarrow (L'', \eta'') \) is \( lf := \{l_g f_g\}_{g \in G} \). This composition is well defined because of functoriality of \( g_* \).

Consider now the functor \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}(G) \) defined as follows: For every object \( X \) in \( \mathcal{C} \), \( \mathcal{F}(X) = (\overline{X}, \phi_X^{-1}) \), where \( \overline{X}_g = g_*(X) \) and
\[
(\phi_X^{-1})_{g,h} := \phi(g, h)^{-1} : g_*(\overline{X}_h) = g_* h_* (X) \rightarrow \overline{X}_{gh} = (gh)_* (X).
\]
The pair \((\overline{X}, \phi_X^{-1})\) is an object in \( \mathcal{C}(G) \) by the commutativity of diagram (3.1). Given \( l : X \rightarrow Y \) a morphism in \( \mathcal{C} \), we define \( \mathcal{F}(l) : (\overline{X}, \phi_X^{-1}) \rightarrow (\overline{Y}, \phi_Y^{-1}) \) as \( \mathcal{F}(l) = \{g_*(l)\}_{g \in G} \). The family of morphisms \( \mathcal{F}(l) \) is a morphism in \( \mathcal{C}(G) \) because
\[ \phi^{-1}(g, h) : g_* h_* \rightarrow (gh)_* \] are natural isomorphisms for each pair \( g, h \in G \) and \( \mathcal{F}(l) \circ \mathcal{F}(f) = \mathcal{F}(l \circ f) \) because \( g_* \) are functors for each \( g \in G \).

**Proposition 4.1.** The functor \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}(G) \) is an adjoint equivalence of categories with adjoint functor \( U_e : \mathcal{C}(G) \rightarrow \mathcal{C}, (L, \eta) \mapsto L_e, f \mapsto f_e \).

**Proof.** We define the unit isomorphism of the adjoint equivalence \( \eta : \mathcal{F} \circ U_e \rightarrow \text{Id}_\mathcal{C} \) as
\[
\eta_{(L, \eta)} := \{ \eta_{g,e} \}_{g \in G}.
\]
The naturality of \( \eta \) follows from the diagrams (4.3), taking \( h = e \). The counit isomorphism of the adjoint equivalence is the identity since \( U_e \circ \mathcal{F} = \text{Id}_\mathcal{C} \). The counit-unit equations follow immediately from the definitions. \( \square \)

**Proposition 4.2.** The category \( \mathcal{C}(G) \) is a strict monoidal category with tensor product of objects \( (L, \eta) \otimes (L', \eta') = (LL', \eta\eta') \), where \( (LL')_g := L_g \otimes L'_g \).

\[
\begin{array}{ccc}
g_* (L_h \otimes L'_h) & \xrightarrow{(\eta\eta')_{g,h}} & L_{gh} \otimes L'_{gh} \\
\psi^g(\phi(L_h, L'_h)) & & \eta_{g,h} \otimes \eta'_{g,h} \\
& \downarrow & \\
g_* (L_h) \otimes g_* (L'_h)
\end{array}
\]

\[ (\eta\eta')_{gh,k} = \eta_{g,h,k} \circ g_* ((\eta\eta')_{h,k}) \phi(g, k) L_k \otimes L'_k, \]
for all \( g, h, k \in G \).

We will use the following equations:
\[
\psi^g(L_{hk}, L_{hk}) \circ g_*(\eta_{h,k} \otimes \eta'_{h,k}) = (g_*(\eta_{h,k}) \otimes g_*(\eta'_{h,k})) \circ \psi^g(h_*(L_k), h_*(L'_k))
\]
\[
\psi^g(h_*(L_k), h_*(L'_k)) \circ g_*(\psi^h(L_k, L'_k)) \circ \phi(g, h) L_k \otimes L'_k = (\phi(g, h) L_k \otimes \phi(g, h) L'_k) \circ \psi^g(L_K, L'_K)
\]
for all \( g, k, k \in G \). Equation (4.5) follows because \( \psi^g \) is a monoidal natural isomorphisms and equation (4.6) follows from diagram (3.3).
Thus,

\[ \eta_{g,hk} \circ g_*(\eta'_{g,hk}) \circ \phi(g,k) \circ L_h \otimes L'_k = \]
\[ = (1) \left( \eta_{g,hk} \otimes \eta'_{g,hk} \right) \circ \psi^g(L_{hk}, L'_{hk}) \]
\[ \circ g_\ast \left[ \eta_{h,k} \otimes \eta'_{h,k} \right] \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ = (2) \left( \eta_{g,hk} \otimes \eta'_{g,hk} \right) \circ \psi^g(L_{hk}, L'_{hk}) \circ g_\ast \left( \eta_{h,k} \otimes \eta'_{h,k} \right) \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ = (3) \left( \eta_{g,hk} \otimes \eta'_{g,hk} \right) \circ g_\ast \left( \eta_{h,k} \otimes \eta'_{h,k} \right) \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ \circ \left[ \eta_{g,hk} \otimes \eta'_{g,hk} \right] \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ = (4) \left( \eta_{g,hk} \otimes \eta'_{g,hk} \right) \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ \circ \left[ \psi^g(h_*(L_k), h_*(L'_k)) \right] \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ = (5) \left( \eta_{g,hk} \otimes \eta'_{g,hk} \right) \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ \circ \left[ \psi^g(h_*(L_k), h_*(L'_k)) \right] \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ \circ \left[ \psi^g(h_*(L_k), h_*(L'_k)) \right] \circ \phi(g,h) \circ L_k \otimes L'_k \]
\[ = (7) \left( \eta_{g,hk} \otimes \eta'_{g,hk} \right) \circ \phi(g,h) \circ L_k \otimes L'_k \circ \psi^g(L_{hk}, L'_{hk}) \]
\[ = (8) \left( \eta'_{g,hk} \right) \circ \psi^g(L_{hk}, L'_{hk}) \]

The equality (1) from definition, the equality (2) by the functoriality of \( g_* \), the equality (3) from equation (4.5), the equality (4) from bifunctoriality of \( \otimes \), the equality (5) from equation (4.6), the equality (6) from bifunctoriality of \( \otimes \), the equality (7) from diagram (4.3) and the equality (8) from definition.

Let \( f : (L, \eta) \rightarrow (T, \chi) \) and \( l : (L', \eta') \rightarrow (T', \chi') \) be morphisms in \( C(G) \). In order to show that \( f \otimes g := \{ f_g \otimes l_g \}_{g \in G} \) is a morphism \( C(G) \) from \( (L, \eta) \otimes (L', \eta') \) to \( (T, \chi) \otimes (T', \chi') \) we need to check the equation

\[ (4.7) \left( g_*(f_h) \otimes g_*(l_h) \right) \circ \psi^g(L_h, L'_h) = \psi^g(T_h, T'_h) \circ g_*(f_h \otimes l_h) \]

which follows from the monoidal naturality of \( \psi^g \).

Then,

\[ (f_{gh} \otimes l_{gh}) \circ (\eta'_{g,h}) = (1) \left( f_{gh} \otimes l_{gh} \right) \circ \left( \eta_{g,h} \otimes \eta'_{g,h} \right) \circ \psi^g(L_h, L'_h) \]
\[ = (2) \left( f_{gh} \eta_{g,h} \otimes l_{gh} \eta'_g \right) \circ \psi^g(L_h, L'_h) \]
\[ = (3) \left( \chi_{g,h} g_*(f_h) \otimes \chi'_{g,h} g_*(l_h) \right) \circ \psi^g(L_h, L'_h) \]
\[ = (4) \left( \chi_{g,h} \otimes \chi'_{g,h} \right) \left( g_*(f_h) \otimes g_*(l_h) \right) \circ \psi^g(L_h, L'_h) \]
\[ = (5) \left( \chi_{g,h} \otimes \chi'_{g,h} \right) \circ \psi^g(T_h, T'_h) \circ g_*(f_h \otimes l_h) \]
\[ = (6) \left( \chi \chi'_{g,h} \right) \circ g_*(f_h \otimes l_h) \]

The equality (3) follows from diagram (4.3) and the equality (5) from equation (4.7).

It follows from the definition that \( (T, \eta id_1) \) is a strict unit object.
Finally, we will show that the tensor product is strictly associative. Let \((L, \eta), (L', \eta'),\) and \((L'', \eta'')\), objects in \(\mathcal{C}(G)\). Then,

\[
((\eta' \cdot \eta'')(g,h))_{g,h} = (\eta'')_{g,h} \otimes \eta_{g,h} \circ \psi^g(L_h \otimes L'_h, L''_h) = (\eta''(g \cdot h))_{g,h} \circ (\psi^g(L_h \otimes L'_h, L''_h) \otimes \text{id}_{\eta''(L'_h)}) \circ \psi^g(L_h \otimes L'_h, L''_h) = \eta''(g \cdot h) \otimes (\eta''(g \cdot h))_{g,h} \circ \psi^g(L_h \otimes L'_h, L''_h) = (\eta(\eta'(\eta''))(g,h).
\]

The equality (3) follows from the commutativity of diagram (3.2). Hence, the tensor product is associative and \(\mathcal{C}(G)\) is a strict monoidal category. \(\square\)

The monoidal category \(\mathcal{C}(G)\) is a strict monoidal \(G\)-category with action on objects \(g_*(L, \eta) := (gL, g\eta)\), where \((gL)_{g,h} := L_{hg}\) and \((g\eta)_{x,y} := \eta_{x,gy}\) for all \(g, h, x, y \in G\) and action on morphisms \(g_*(f)_h = f_{hg}\), for all \(g, h \in G\).

The following theorem implies Theorem 1.1.

**Theorem 4.3.** Let \(\mathcal{C}\) be a monoidal \(G\)-category. The strict monoidal functor \(U_e: \mathcal{C}(G) \to \mathcal{C}\) with the natural isomorphisms

\[
\gamma(g)_{(L, \eta)} := \eta_{g,e} : g_*(U_e(L, \eta)) = U_e(g_*(L, \eta))
\]

is an equivalence of monoidal \(G\)-categories.

**Proof.** It follows immediately from the definitions that \(\mathcal{C}(G)\) is a strict monoidal \(G\)-category and \(U_e\) is an strict monoidal functor.

By Proposition 4.1, \(U_e\) is an equivalence of categories. Thus, we only need to prove that \((U_e, \gamma)\) is a functor of monoidal \(G\)-categories. The naturality of the family of isomorphism

\[
\gamma(g) := \{\gamma(g)_{(L, \eta)} := \eta_{g,e}\}_{(L, \eta) \in \text{Obj} \mathcal{C}(G)}
\]

follows from the diagrams (4.3), taking \(h = e\). Let \((L, \eta)\) and \((T, \chi)\) be objects in \(\mathcal{C}(G)\). The monoidality of \(\gamma(g)\) is equivalent to the equation \((\eta\chi)_{g,e} = \psi^g_{L,e,T} \circ (\eta_{g,e} \otimes \chi_{g,e})\), that follows from the definition of \((\eta\chi)_{g,h}\). Finally, the diagrams (3.2) for \((U_e, \eta)\) are just the diagrams (4.2) with \(k = e\). \(\square\)

If \((F, \gamma): \mathcal{C} \to \mathcal{D}\) is a monoidal \(G\)-functor, we define a monoidal \(G\)-functor \((F, \gamma)(G) : \mathcal{C}(G) \to \mathcal{D}(G)\) as follows: if \((L, \eta) \in \mathcal{C}(G)\), then \(F(L) = \{F(L_g)\}_{g \in G}\) and \(F(\eta_h) = g_*(F(L_h)) = g_*(F(L_h))\) for all \(g, h \in G\).

**Remark 4.4.** If the monoidal \(G\)-category is fusion category (or more generally a finite tensor category) over a field \(k\) and every monoidal equivalence \(g_*\) is \(k\)-linear then \(U_e\) is an equivalence of fusion categories (or more generally an equivalence of finite tensor categories). Thus, Theorem 4.3 immediately implies coherence for fusion categories with (not necessarily finite) group actions.
The following statement is equivalent to Theorem 1.1.

**Corollary 4.5.** The construction \( C \mapsto C(G) \) defines a strict left adjoint to the forgetful 2-functor from the 2-category of strict monoidal \( G \)-categories to the 2-category of monoidal \( G \)-categories and the components of the unit are equivalences of monoidal \( G \)-categories.

\[ \square \]

Let \((H, P, \vartheta)\) be a crossed module, \(G\) a group and \( \{(\alpha_x, \phi_x, \theta_{x,y})\}_{x,y \in G} \) a weak action of \(G\) on \((H, P, \vartheta)\) (see Example 3.2).

The strict monoidal \( G \)-category \( C(H,P,\vartheta)(G) \) is a strict monoidal category where every arrow is invertible and every object has a strict inverse. This kind of monoidal categories are called strict categorical group or strict 2-groups (see [1]). Since the notation of strict categorical group and crossed module are essentially equivalents (see [7] for details), we can construct a new crossed module \((H(G), P(G), \vartheta')\) with a strict \(G\) action such that is is weak equivalent to \((H, P, \vartheta)\).

In fact, \(P(G)\) consist of the group of objects of \(C(H,P,\vartheta)(G)\), \(H(G)\) is the group of morphisms \(X \to \mathbf{1}\) into the identity object of \(C(H,P,\vartheta)(G)\) with product:

\[
(X \xrightarrow{a} \mathbf{1}) \cdot (Y \xrightarrow{b} \mathbf{1}) = X \otimes Y \xrightarrow{a \otimes b} \mathbf{1} \otimes \mathbf{1} = \mathbf{1},
\]

action

\[
y(X \xrightarrow{a} \mathbf{1}) = Y \otimes X \otimes Y^{-1} \xrightarrow{\text{id}_Y \otimes a \otimes \text{id}_{Y^{-1}}} Y \otimes \mathbf{1} \otimes Y^{-1} = \mathbf{1}.
\]

and the homomorphism \(\vartheta' : H(G) \to P(G)\) sending \(X \to \mathbf{1}\) into \(X\).

5. **Coherence for Braided \( G \)-crossed categories**

5.1. **\( G \)-crossed monoidal categories.** A \(G\)-graded monoidal category is a monoidal category \(\mathcal{C}\) endowed with a decomposition \(\mathcal{C} = \bigsqcup_{g \in G} \mathcal{C}_g\) (coproduct of categories) such that

- \(\mathbf{1} \in \mathcal{C}_e\),
- \(\mathcal{C}_g \otimes \mathcal{C}_h \subset \mathcal{C}_{gh}\) for all \(g, h \in G\).

If \(k\) is a commutative ring and \(\mathcal{C}\) is a \(k\)-linear abelian category, the coproduct \(\mathcal{C} = \bigsqcup_{g \in G} \mathcal{C}_g\) is taken in the category of \(k\)-linear abelian categories.

**Definition 5.1.** A \(G\)-crossed monoidal categories is a \(G\)-graded monoidal category with a structure of \(G\)-category such that \(g_s(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}\) for all \(g, h \in G\).

A \(G\)-crossed monoidal category is called \(k\)-linear if \(\mathcal{C}\) is a \(k\)-linear category and the functors \(g_s\) are \(k\)-linear for each \(g \in G\).

Let \(\mathcal{C}\) and \(\mathcal{D}\) be \(G\)-crossed monoidal categories. An equivalence of monoidal \(G\)-categories \(F : \mathcal{C} \to \mathcal{D}\) is an equivalence of \(G\)-crossed monoidal categories if \(F(\mathcal{C}_g) \subset \mathcal{D}_g\) for all \(g \in G\).

In Example 3.2 we define the notion of a weak action of a group \(G\) on a crossed module \((H, P, \vartheta)\) and the associated monoidal \(G\)-category \(\mathcal{C}(H, P, \vartheta)\).
Lemma 5.2. Let $G$ be a group, $(H, P, \partial)$ be a crossed module and $\{(\alpha_x, \phi_x, \theta_{x,y})\}_{x,y \in G}$ a weak action of $G$ on $(H, P, \partial)$.

1. The $G$-gradings of $\mathcal{C}(H,P,\partial)$ are in correspondence with group homomorphisms

\[ \text{gr} : P \to G \]

such that $\text{Im}(\partial) \subset \ker(\text{gr})$.

2. A $G$-grading given by a group homomorphism $\text{gr} : P \to G$, defines a $G$-crossed monoidal structure on $\mathcal{C}(H,P,\partial)$ if and only if

\[ \text{gr}(\phi_x(g)) = x \text{gr}(g)x^{-1}, \]

for all $x \in G, g \in P$.

Proof. The group of isomorphism classes of objects of $\mathcal{C}(H,P,\partial)$ is $P/\text{Im}(\partial)$. Since $\mathcal{C}(H,P,\partial)$ is a groupoid, $G$-gradings correspond with group morphisms $P/\text{Im}(\partial) \to G$. Thus, $G$-gradings correspond with groups morphism $\text{gr} : P \to G$ such that $\text{Im}(\partial) \subset \ker(\text{gr})$.

The second part of the lemma follows immediately from the definition.

Let $\mathcal{C}$ and $\mathcal{D}$ be $G$-crossed monoidal categories. A monoidal $G$-functor $(F, \eta) : \mathcal{C} \to \mathcal{D}$ is a functor of $G$-crossed monoidal categories if $F(C^g) \subset D^g$ for all $g \in G$. We say that $(F, \eta)$ is an equivalence of $G$-crossed monoidal categories if $F$ is an equivalence of categories.

A $G$-crossed monoidal categories is called strict if it is strict as a monoidal $G$-category.

Corollary 5.3 (Coherence for $G$-crossed monoidal categories). Let $G$ be a group. If $\mathcal{C}$ is a $G$-crossed monoidal category, $\mathcal{C}(G)$ is a strict $G$-crossed monoidal category equivalent to $\mathcal{C}$. If $\mathcal{C}$ is $k$-linear, $\mathcal{C}(G)$ is $k$-linear and equivalent to $\mathcal{C}$ as $k$-linear categories.

Proof. The strict monoidal $G$-category $\mathcal{C}(G)$ is a $G$-crossed monoidal category with $\mathcal{C}(G)^g = \{(L, \eta) : L_e \in C^g\}$ for all $g \in G$.

If each $g_*$ is $k$-linear, the strict monoidal $G$-category $\mathcal{C}(G)$ is $k$-linear and the equivalence of categories $U_e : \mathcal{C}(G) \to \mathcal{C}$ is a $k$-linear equivalence such that $U_e(C^g) \subset C^g$ for all $g \in G$. Thus, the corollary follows from Theorem 4.3.

5.2. Braided $G$-crossed categories.

Definition 5.4. Let $\mathcal{C}$ be a $G$-crossed monoidal category. A $G$-braiding is a family of isomorphisms

\[ c : \{c_{X,Y} : X \otimes Y \to g_*(Y) \otimes X\}_{Y \in \mathcal{C}, x \in C^g, g \in G} \]

natural in $X$ and $Y$, such that the diagrams
(5.1)

\[
g_*(X \otimes Z) \xrightarrow{g_*(c_{X,Z})} g_*(Z) \otimes g_*(X)
\]

\[
\begin{array}{ccc}
g_*(X) \otimes g_*(Z) & \xrightarrow{\psi^g(X,Z)} & g_*(X) \otimes g_*(Z) \\
\end{array}
\]

\[
\begin{array}{ccc}
g_*(X) \otimes g_*(Z) & \xrightarrow{\phi(g,h) \otimes \text{id}_{g_*(X)}} & (gh)_*(Z) \otimes g_*(X)
\end{array}
\]


\[
(ghg^{-1})_* g_*(Z) \otimes g_*(X) \xrightarrow{\phi(ghg^{-1},g) \otimes \text{id}_{g_*(X)}} (gh)_*(Z) \otimes g_*(X)
\]


\[
\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{c_{X,Y} \otimes Z} & g_*(Y) \otimes (Z \otimes X)
\end{array}
\]

\[
\begin{array}{ccc}
g_*(Y) \otimes X \otimes Z & \xrightarrow{\text{id}_{g_*(Y)} \otimes c_{X,Y}} & g_*(Y) \otimes (g_*(Z) \otimes X)
\end{array}
\]


\[
\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{c_{X,Y} \otimes Z} & (gh)_*(Z) \otimes X \otimes Y
\end{array}
\]

\[
\begin{array}{ccc}
X \otimes h_*(Z) \otimes Y & \xrightarrow{c_{X,h_*(Z)} \otimes \text{id}_Y} & g_*(h_*(Z) \otimes X) \otimes Y
\end{array}
\]

A braided $G$-crossed category is a $G$-crossed category with a $G$-braiding. A braided $G$-crossed monoidal category $(\mathcal{C},c)$ is called strict if $\mathcal{C}$ is a strict monoidal $G$-category. In a strict braided $G$-crossed monoidal category we have that

- $g_*(c_{X,Z}) = c_{g_*(X),g_*(Z)}$
- $c_{X,Y} \otimes Z = (\text{id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \text{id}_Z)$
- $c_{X \otimes Y,Z} = (c_{X,h_*(Z)} \otimes \text{id}_Y) \circ (\text{id}_X \otimes c_{Y,Z})$

for all $X \in \mathcal{C}$, $Y \in \mathcal{C}$, $Z \in \mathcal{C}$, $g, h \in G$. Let $\mathcal{C}$ and $\mathcal{D}$ be braided $G$-crossed categories.

**Example 5.5.** (1) Let $(H, P, \partial)$ be a crossed module. Since $\partial(h)h'h = hh'$ for all $h, h' \in H$, the discrete monoidal category $\overline{H}$ is a strict braided $P$-crossed category.

(2) Let $(\mathcal{B}, c)$ be a braided monoidal category and $G$ a group with an action on $\mathcal{B}$ by braided autoequivalences. Then $\mathcal{B}$, with all objects graded only by $e \in G$ and $G$-braiding $c$ is a braided $G$-crossed monoidal category.
A functor of $G$-crossed monoidal categories $(F, \eta) : \mathcal{C} \to \mathcal{D}$ is a functor of braided $G$-crossed monoidal categories if for all $X \in \mathcal{C}_g, Y \in \mathcal{C}, g \in G$ the diagrams

$$
\begin{array}{ccc}
F(X \otimes Y) & \xrightarrow{F(c_{X,Y})} & F(g_* (Y) \otimes X) \\
\downarrow F_2(X,Y) & & \downarrow F_2(g_* (Y),X) \\
F(X) \otimes F(Y) & \xrightarrow{c_{F(X),F(Y)}} & F(g_* (Y)) \otimes F(X) \\
\downarrow \eta_Y \otimes \text{id}_{F(X)} & & \downarrow \eta_{g_* (Y)} \otimes \text{id}_{F(X)} \\
g_* (F(Y)) \otimes F(X) & \xrightarrow{g_* (\eta_Y) \otimes \text{id}_{F(X)}} & \end{array}
$$

commute.

We say that $(F, \eta)$ is an equivalence of braided $G$-crossed monoidal categories if $F$ is an equivalence of categories.

**Theorem 5.6** (Coherence for braided $G$-crossed categories). Let $G$ be a group. Every braided $G$-crossed category is equivalent to a strict braided $G$-crossed category.

**Proof.** Using the adjoint equivalence of monoidal $G$-crossed categories, $(U_e, \mathcal{F})$, we will transport the $G$-braiding of $\mathcal{C}$ to a $G$-braiding on $\mathcal{C}(G)$. The $G$-braiding on $\mathcal{C}(G)$ is defined by the commutativity of the diagram

$$
\begin{array}{ccc}
L_h \otimes T_h & \xrightarrow{c_{L_h,T_h}} & T_{hg} \otimes L_h \\
\downarrow \eta_{h,e} \otimes \chi_{h,e} & & \downarrow \eta_{h,g} \otimes \chi_{h,e} \\
h_* (L_e) \otimes h_* (T_e) & & h_* (T_g) \otimes h_* (L_e) \\
\downarrow \psi^h (L_e,T_e) & & \downarrow \psi^h (L_g,T_e) \\
h_* (L_e \otimes T_e) & & h_* (T_g \otimes L_e) \\
\downarrow h_* (c_{L_e,T_e}) & & h_* (\eta_{g,e} \otimes \text{id}_{L_e}) \\
h_* (g_* (T_e) \otimes L_e) & & h_* (g_* (T_e) \otimes L_e)
\end{array}
$$

where $(L, \eta) \in \mathcal{C}(G)g, (T, \chi) \in \mathcal{C}(G), g \in G$. The monoidal $G$-functor $(U_e, \gamma) : \mathcal{C}(G) \to \mathcal{C}$ is a functor of braided $G$-categories. Thus, $(U_e, \gamma)$ is an equivalence of braided $G$-crossed categories. \qed
Remark 5.7. Using the same ideas in the proof of Theorem 5.4, without significant changes a coherence theorem for $G$-ribbon crossed categories can be proved.

Example 5.8. Let $(H, P, \partial)$ be a crossed module, $G$ be a group, $\{ (\alpha_x, \phi_x, \theta_{x,y}) \}_{x,y \in G}$ be a weak $G$-action and $\text{gr} : P \to G$ be a group homomorphism such that

$$\text{Im}(\text{gr}) \subset \ker(\partial), \quad \text{gr}(\phi_x(g)) = x \text{gr}(g)x^{-1},$$

for all $x \in G, g \in P$.

A $G$-braiding is a map

$$\{-,-\} : P \times P \to H$$

satisfying the following axioms:

(a) $\partial(\{ x, y \})xy = \phi_{\text{gr}(x)}(y)x$,
(b) $\{ x, y \} \phi_{\text{gr}(x)}(g)h = h \{ \partial(h)x, y \}$,
(c) $x \text{h} \{ x, \partial(h)y \} = \{ x, y \} \alpha_{\text{gr}(x)}(h)$,
(d) $\{ \phi_g(x), \phi_g(y) \} = \alpha_g(\{ x, y \}) \theta_{g,\text{gr}(x)}^{g^{-1}} \theta_{g,\text{gr}(x)g^{-1},g}$,
(e) $\{ x, yz \} = \{ x, y \} \phi_{\text{gr}(x)}(y) \{ x, z \}$,
(f) $\{ xy, z \} \theta_{\text{gr}(x),\text{gr}(y)}(z) = \{ x, y \} \{ x, \phi_{\text{gr}(y)}(z) \}$,

for all $x, y, z \in P, g \in G$.

If $G$ is a trivial group, we obtain the notion of braiding of crossed module, see [3].

If the $G$-action is strict, that is, $\theta_{x,y}(g) = e$, for all $x, y \in G, g \in P$, then we obtain the notion of 2-crossed module of Conduché [5].

Every $G$-braiding in a crossed module $(P, H, \partial)$ with a weak action of $G$, induces a $G$-braiding

$$c_{x,y} := (\{ x, y \}, xy) : xy \to \phi_{\text{gr}(x)}(y)x, \quad x, y \in P$$

of $C(P, H, \partial)$. In fact, (a) says that the target of $(\{ x, y \}, xy)$ is $\phi_{\text{gr}(x)}(y)x$. Condition (b) and (d) are naturality of $c_{x,y}$. Commutativity of diagrams (5.1), (5.2) and (5.3) are equivalent to axioms (d), (e) and (f), respectively.

Recall that a categorical groups is a rigid monoidal groupoid, (see [1] for more details). Every categorical groups is equivalent to a strict categorical groups, that is, to a categorical groups where every object has a strict inverse respect to the tensor product.

Applying Theorem 5.6 we have that every braided $G$-crossed categorical groups is equivalent to a strict braided $G$-crossed strict categorical groups $\tilde{C}$. Since every strict braided $G$-crossed strict categorical group defines a 2-crossed module, (see [4] Example 2.5 (ii)), associated to every braided $G$-crossed categorical group there is a 2-crossed module.
REFERENCES

[1] J. C. Baez and A. D. Lauda. Higher-dimensional algebra. V. 2-groups. Theory Appl. Categ., 12:423–491, 2004.
[2] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang. Symmetry, defects, and gauging of topological phases. arXiv preprint arXiv:1410.4540, 2014.
[3] R. Brown and N. D. Gilbert. Algebraic models of 3-types and automorphism structures for crossed modules. Proc. London Math. Soc. (3), 59(1):51–73, 1989.
[4] P. Carrasco and J. Martínez Moreno. Categorical $G$-crossed modules and 2-fold extensions. J. Pure Appl. Algebra, 163(3):235–257, 2001.
[5] D. Conduché. Modules croisés généralisés de longueur 2. In Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983), volume 34, pages 155–178, 1984.
[6] S. X. Cui, C. Galindo, J. Y. Plavnik, and Z. Wang. On gauging symmetry of modular categories. arXiv preprint arXiv:1510.03475. To appear in Communications in Mathematical Physics, 2015.
[7] M. Forrester-Barker. Group objects and internal categories. arXiv preprint math/0212065, 2002.
[8] T. Lan, L. Kong, and X.-G. Wen. Modular extensions of unitary braided fusion categories and 2+ 1d topological/spt orders with symmetries. arXiv preprint arXiv:1602.05936, 2016.
[9] S. Mac Lane. Categories for the working mathematician, volume 5. Springer Science & Business Media, 1978.
[10] S. MacLane. Natural associativity and commutativity. Rice Institute Pamphlet-Rice University Studies, 49(4), 1963.
[11] N. Saavedra Rivano. Catégories Tannakiennes. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972.
[12] V. Turaev. Homotopy quantum field theory, volume 10 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2010. Appendix 5 by Michael Müger and Appendices 6 and 7 by Alexis Virelizier.
[13] V. Turaev and A. Virelizier. On 3-dimensional homotopy quantum field theory, I. Internat. J. Math., 23(9):1250094, 28, 2012.
[14] V. Turaev and A. Virelizier. On 3-dimensional homotopy quantum field theory II: The surgery approach. Internat. J. Math., 25(4):1450027, 66, 2014.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LOS ANDES, CARRERA 1 N. 18A - 10, BOGOTÁ, COLOMBIA
E-mail address: cn.galindo1116@uniandes.edu.co