SPLITTING THEOREMS FOR PRO-\(p\) GROUPS
ACTING ON PRO-\(p\) TREES AND 2-GENERATED
SUBGROUPS OF FREE PRO-\(p\) PRODUCTS WITH
PROCYCLIC AMALGAMATIONS

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Abstract. Let \(G\) be a finitely generated infinite pro-\(p\) group acting on a pro-\(p\) tree such that the restriction of the action to some open subgroup is free. Then we prove that \(G\) splits as a pro-\(p\) amalgamated product or as a pro-\(p\) HNN-extension over an edge stabilizer.

Using this result we prove under certain conditions that free pro-\(p\) products with procyclic amalgamation inherit from its free factors the property of each 2-generated subgroup being free pro-\(p\). This generalizes known pro-\(p\) results, as well as some pro-\(p\) analogs of classical results in abstract combinatorial group theory.

1. Introduction

The main theorem of the Bass-Serre theory of groups acting on trees states that a group \(G\) acting on a tree \(T\) is the fundamental group of a graph of groups whose vertex and edge groups are the stabilizers of certain vertices and edges of \(T\). This tells that \(G\) can be obtained by successively forming amalgamated free products and HNN-extensions. The pro-\(p\) version of this theorem does not hold in general (cf. Example 3.10), namely a pro-\(p\) group acting on a pro-\(p\) tree does not have to be isomorphic to the fundamental pro-\(p\) group of a graph of finite \(p\)-groups (coming from the stabilizers). Moreover, the fundamental pro-\(p\) group of a profinite graph of pro-\(p\) groups does not have to split as an amalgamated free pro-\(p\) product or as a pro-\(p\) HNN-extension over some edge stabilizer (the reason is that by deleting an edge of the profinite graph one may destroy its compactness). These two facts are usually the main obstacles for proving subgroup theorems of free constructions in the category of pro-\(p\) groups.

We show that the two Bass-Serre theory principal results mentioned above hold for finitely generated infinite pro-\(p\) groups acting virtually
freely on pro-$p$ trees, i.e. such that the restriction of the action on some open subgroup is free. Such a group is then virtually free pro-$p$.

**Theorem 1.1.** Let $G$ be a finitely generated infinite pro-$p$ group acting virtually freely on a pro-$p$ tree $T$. Then

(a) $G$ splits either as an amalgamated free pro-$p$ product or as a pro-$p$ HNN-extension over some edge stabilizer;

(b) $G$ is isomorphic to the fundamental pro-$p$ group $\Pi_1(G, \Gamma)$ of a finite graph of finite $p$-groups.

One should say that in contrast to the classical theorem from Bass-Serre theory our $\Gamma$ in item (b) is not $T/G$. The graph $\Gamma$ is constructed in a special way by first modifying $T$ without loosing the essential information of the action.

As a corollary we deduce the following subgroup theorem.

**Theorem 1.2.** Let $H$ be a finitely generated subgroup of a fundamental pro-$p$ group $G$ of a finite graph of finite $p$-groups. Then $H$ is the fundamental pro-$p$ group of a finite graph of finite $p$-groups which are intersections of $H$ with some conjugates of vertex and edge groups of $G$.

Moreover, as an application of Theorem 1.1, we obtain the following result. It is a pro-$p$ analogue of a classical result of G. Baumslag [1, Thm. 2] that gave an impulse to the theory known now as the theory of limit groups. Note that our theorem also generalizes the pro-$p$ *ipsis litteris* version of [2], as well as [4, Thm. 7.3].

**Theorem 1.3.** Let $G = A \coprod C \times B$ be a free pro-$p$ product of $A$ and $B$ with procyclic amalgamating subgroup $C$. Suppose that the centralizer in $G$ of each non-trivial closed subgroup of $C$ is a free abelian pro-$p$ group and contains $C$ as a direct factor. If each 2-generated pro-$p$ subgroup of $A$ and each 2-generated pro-$p$ subgroup of $B$ is either a free pro-$p$ group or a free abelian pro-$p$ group then so is each 2-generated pro-$p$ subgroup of $G$.

The method of proof is to consider the standard pro-$p$ tree $T$ on which $G$ acts naturally; so $A$ and $B$ are stabilizers of vertices $v$ and $w$, and $C$ is the stabilizer of the edge connecting $v$ and $w$. Then we decompose the pair $(G, T)$ as a inverse limit of $(G_U, T_U)$ satisfying the hypothesis of Theorem 1.1.

**Notation.** Throughout this paper, $p$ is a fixed but arbitrary prime number. The additive group of the ring of $p$-adic integers is $\mathbb{Z}_p$; the natural numbers, $\mathbb{N}$. For $x, y$ in a group we shall write $y^x := x^{-1}yx$. All groups are pro-$p$, subgroups are closed and homomorphisms are continuous. For $A \subseteq G$ we denote by $\langle A \rangle$ the subgroup of $G$ (topologically) generated by $A$ and by $A^G$ the normal closure of $A$ in $G$.
i.e., the smallest closed normal subgroup of $G$ containing $A$. By $d(G)$ we denote the smallest cardinality of a generating subset of $G$. Recall that a cyclic profinite group is always finite. The Frattini subgroup of $G$ will be denoted by $\Phi(G)$. By $\text{tor}(G)$ we mean the set of all torsion elements of $G$.

For a pro-$p$ group $G$ acting continuously on a space $X$ we denote the set of fixed points of $G$ by $X^G$ and for each $x \in X$ the point stabilizer by $G_x$. We define $\tilde{G} = \langle G_x \mid x \in X \rangle$.

The rest of our notation is very standard and basically follows \[8\] and \[9\].

2. Preliminary Results

In this section we collect properties of amalgamated free pro-$p$ products, pro-$p$ HNN-extensions and pro-$p$ groups acting on pro-$p$ trees to be used in the paper. Further information on this subject can be found in \[8\] and \[9\]. Recall the following two notions. First, an amalgamated free pro-$p$ product $G := A \amalg_C B$ is non-fictitious if $C$ is a proper subgroup of both, $A$ and $B$. Unless differently stated we shall consider exclusively non-fictitious free amalgamated products and we shall make use of the fact from \[6\] that a free pro-$p$ product with either procyclic or finite amalgamating subgroup is always proper, i.e., the factors $A$ and $B$ embed in $G$ via the natural maps.

Second, a pro-$p$ HNN-extension $G = \text{HNN}(H, A, f, t)$ is proper if the natural map from $H$ to $G$ is injective. Only such free pro-$p$ products and pro-$p$ HNN-extensions will be used in this paper and they are therefore always proper.

We start with a simple general lemma.

**Lemma 2.1.** Let $G := \lim\downarrow G_i$ be the inverse limit of an inverse system \{\(G_i, \varphi_{ij}, I\)\} of pro-$p$ groups and $H_i \leq G_i$ so that $\varphi_{ij}(H_i) \leq H_j$ holds whenever $j \leq i$. Suppose that there is a constant $d$ with $d(G_i) = d$ for all $i \in I$. The following statements hold:

(a) If $d(G) = d$, then there exists $j \in I$ such that the projection $G \to G_j$ is surjective.

(b) For the induced inverse limit $H := \lim\downarrow H_i \leq G$, we have equality $H^G = \lim\downarrow H_i^{G_i}$.

**Proof.** For each $i \in I$, let $\varphi_i : G \to G_i$ be the projection.

(a) There is an induced inverse system of Frattini quotients with $G/\Phi(G) = \lim\downarrow G_i/\Phi(G_i)$. If $\varphi_{ij}(G_i)/\Phi(G_j)$ is a proper subgroup of $G_j/\Phi(G_j)$, for all $i, j$ belonging to a cofinal subset of $I$, then $G/\Phi(G) = \lim\downarrow \varphi_j(G_i)/\Phi(G_j)$ and so $G$ can be generated by $d - 1$ elements. Otherwise $\varphi_{ij}$ must be surjective for $i, j$ belonging to a cofinal subset of $I$, and so is $\varphi_j$ (cf. [9, Prop. 1.1.10]).
(b) Set, for the moment, $K := \lim_i H_i^G$. Since $H \leq K$, we have $H^G \leq K$, as $K \triangleleft G$. So, it suffices to establish $K \leq H$. This is certainly true when there is a bound on the orders of the $G_i$. Fix $n \in \mathbb{N}$. Then, as $d(G_i) = d$, there is a bound on the orders of all $G_i/\Phi^n(G_i)$. Then the statement reads $\lim_i H_i^G \Phi^n(G_i) \leq H^G \Phi^n(G)$ and therefore $K \leq H^G \Phi^n(G)$. Since, $d(G) \leq d$, $G$ is finitely generated, and so the set $(\Phi^n(G))_{n \geq 1}$ is a fundamental system of neighbourhoods of the identity in $G$ (cf. [9, Prop. 2.8.13]). Hence $K \leq H$, as needed. \hfill \Box

We recollect the following fundamental results from the theory of pro-$p$ groups acting on pro-$p$ trees and their consequences for an amalgamated free pro-$p$ product or a pro-$p$ HNN-extension.

Recall that for a pro-$p$ group $G$ acting on a pro-$p$ tree $T$, the closed subgroup generated by all vertex stabilizers is denoted by $\tilde{G}$; also, the (unique) smallest pro-$p$ subtree of $T$ containing two vertices $v$ and $w$ of $T$ is denoted by $[v, w]$ and called the geodesic connecting $v$ to $w$ in $T$ (cf. [8, p. 83]).

**Theorem 2.2.** Let $G$ be a pro-$p$ group acting on a pro-$p$ tree $T$.

(a) ([8, Prop. 3.5]) $T/\tilde{G}$ is a pro-$p$ tree.

(b) ([8, Cor. 3.6]) $G/\tilde{G}$ is a free pro-$p$ group.

(c) ([8, Cor. 3.8]) If $v$ and $w$ are two different vertices of $T$, then $E([v, w]) \neq \emptyset$ and $(G_v \cap G_w) \leq G_e$ for every $e \in E([v, w])$.

(d) ([8, Thm. 3.9]) If $G$ is finite, then $G = G_v$, for some $v \in V(T)$.

**Theorem 2.3.** Let $G = G_1 \coprod_H G_2$ be a proper amalgamated free pro-$p$ product of pro-$p$ groups.

(a) ([8, Thm. 4.2(b)]) Let $K$ be a finite subgroup of $G$. Then $K \leq G_i$ for some $g \in G$ and for some $i = 1$ or $2$.

(b) ([8, Thm. 4.3(b)]) Let $g \in G$. Then $G_i \cap G_j \leq H^b$ for some $b \in G_i$, whenever $1 \leq i \neq j \leq 2$ or $g \notin G_i$.

**Theorem 2.4.** Let $G = \text{HNN}(H, A, f)$ be a proper pro-$p$ HNN-extension.

(a) ([8, Thm. 4.2(c)]) Let $K$ be a finite subgroup of $G$. Then $K \leq H^g$ for some $g \in G$.

(b) ([8, Thm. 4.3(c)]) Let $g \in G$. Then

$H \cap H^g \leq A^b$

for some $b \in H \cup th$, whenever $g \notin H$.

The next technical results concern inverse systems that will play an essential role during the proof of Theorem 1.3 in section 4. Until the end of this section the directed set $I$ will be assumed to be order isomorphic to $\mathbb{N}$.

**Proposition 2.5.** Let $G$ be the inverse limit of a surjective inverse system $\{G_i, \varphi_{ij}, I\}$ of pro-$p$ groups. Suppose that each $G_i = \text{pro-p}$. Then...


HNN($H_i, A_i, t_i$) is an HNN-extension with $H_i$ finite and $\varphi_{ij}(H_i) \simeq H_j$. Then there are inverse systems of groups $\{H_i', \varphi_{ij}, I\}$ and $\{A_i', \varphi_{ij}, I\}$ such that $G = \text{HNN}(H, A, t)$ with $H := \varprojlim H_i'$, $A := \varprojlim A_i'$ where each $H_i'$ (resp. $A_i'$) is a conjugate of $H_i$ (resp. $A_i$) by an element of $G_i$.

**Proof.** Fix $k \in I$. By Theorem 2.4(a) there are $g_k \in G_k$ with $\varphi_{jk}(H_j) = H_k^{q_k}$ (remember they are isomorphic by the hypothesis). Pick $g_j \in \varphi_{jk}^{-1}(g_k)$ and define $H_j' := H_j^{q_j^{-1}}$, $A_j' := A_j^{q_j^{-1}}$, and, $t_j' := t_j^{q_j^{-1}}$ ; clearly $\varphi_{jk}(H_j') = H_k$. Since $A_j = H_j \cap H_j'$ and $\varphi_{jk}$ is surjective, we have

$$\varphi_{jk}(A_j') \leq H_k \cap H_j^{\epsilon_j}(t_j') \leq A_k^{t_k h_k},$$

for suitable $h_k \in H_k$ and $\epsilon_k = 0$ or 1, by Theorem 2.4(b). Choose $h_j \in \varphi_{jk}^{-1}(h_k) \cap H_j'$ and $x_j \in \varphi_{jk}^{-1}(t_k)$. Defining $A_j'' := A_j^{(x_j h_j)^{-1}}$ we obtain $\varphi_{jk}(A_j'') \leq A_k$ in both cases. Continuing inductively we obtain the desired inverse systems $\{H_i', \varphi_{ij}, I\}$ and $\{A_i', \varphi_{ij}, I\}$.

It is straightforward to check that the other associated subgroup also “fits” into the inverse system, that is $\varphi_{jk}(A_j''') \leq A_k$ where $t_j''' := t_j'(x_j^h_1)^{-1}$.

Now, let $H := \varprojlim H_i'$, $A := \varprojlim A_i''$ and $B := \varprojlim A_i''''$. For each $i \in I$ let us consider the subset

$$X_i := \{\tau_i \in G \mid A^\tau = B \text{ and } G_i := \langle H_i, \tau_i \rangle\}.$$ 

Clearly every $X_i$ is compact, and since $X_{i+1} \subseteq X_i$ for all $i \in I$, there exists $t \in \bigcap_i X_i$ so that $B = A^t$.

The desired isomorphism from HNN($H, A, t$) onto $G$ follows now from the universal property of HNN-extensions. \(\square\)

**Proposition 2.6.** Let $G$ be the inverse limit of a surjective inverse system $\{G_i, \varphi_{ij}, I\}$ of pro-$p$ groups $G_i$ each a free pro-$p$ product $G_i = A_i \ast B_i$ with $A_i$ cyclic and $B_i$ procyclic.

Then there are inverse systems of pro-$p$ groups, $\{A_i', \varphi_{ij}, I\}$ and $\{B_i', \varphi_{ij}, I\}$, where each $A_i'$ is a conjugate of $A_i$ by an element of $G_i$, and $B_i' \leq G_i$, and, $G \simeq \varprojlim_A A_i \ast \varprojlim_B B_i'$.

**Proof.** Suppose first that there exists $i_0 \in I$ such that $B_{i_0} \simeq \mathbb{Z}_p$. Then, since each $\varphi_{ij}$ is surjective, the induced homomorphism between the continuous abelianizations $A_j \times \mathbb{Z}_p \simeq G_j/[G_j, G_j] \rightarrow G_j/[G_j, G_j] \simeq A_j \times \mathbb{Z}_p$ is surjective for $i_0 \leq j \leq i$. Therefore $\varphi_{ij}(A_i) \leq A_j \Phi(G_j)$ and by Theorem 2.3(a) there is $g_j \in G_j$ with $\varphi_{ij}(A_i) \leq A_j^{g_j}$ showing that $A_i$ maps onto a conjugate of $A_j$ now, observing that $G = \varprojlim G_i$ with $G_i \simeq \text{HNN}(A_i, 1, t_i)$, where $t_i$ generates $B_i$, we can apply Proposition 2.5 to obtain the result.

Suppose that each $B_i$ is finite. Since $\varphi_{ij}$ are surjective, from Theorem 2.3(a), we obtain that distinct free factors of $G_i$ are mapped, up to
conjugation, to distinct free factors of $G_j$. So, there is $k_0$ in $I$ so that for all $i, j$ we have

$$\varphi_{ij}(A_i) = A_{ij}^x \text{ and } \varphi_{ij}(B_i) = B_{ij}^y,$$

for some $x_j, y_j \in G_j$. Then inductively the desired inverse systems $\{A_i, \varphi_{ij}, I\}$ and $\{B_i, \varphi_{ij}, I\}$, can be exhibited. The result follows now from [9, Lemma 9.1.5].

**Lemma 2.7.** Let $G$ be a 2-generated pro-$p$ group.

(a) If $G$ is a free pro-$p$ product with procyclic amalgamation, then one of its free factors is procyclic.

(b) If $G$ is a proper HNN-extension with procyclic associated subgroups, then its base subgroup $H$ is at most 2-generated. Moreover, if $d(H) = 2$ then $H$ is generated by the associated subgroups.

(c) If $G$ is the fundamental pro-$p$ group of a finite tree of finite groups such that all edge groups are cyclic, then either $|G| < \infty$ or $G = K \amalg C R$ with $K$ cyclic and $R$ finite, or $G = K \amalg C M \amalg D N$, with $K$ and $N$ cyclic and $M \leq \Phi(G)$.

**Proof.**

(a) Suppose that $G = A \amalg C B$ and let “bar” indicate passing to the Frattini quotient. We have an obvious epimorphism from $G$ to the induced pushout $\bar{P} := \bar{A} \amalg \bar{C} \bar{B}$. Let $n := d(A) + d(B)$. Since $C$ is procyclic, the image $M$ of the kernel of the canonical map $A \amalg B \to \bar{G}$ via the cartesian map $A \amalg B \to \bar{A} \times \bar{B}$ is also procyclic. The latter map induces an epimorphism from $\bar{G}$ to the at least $(n - 1)$-generated elementary abelian pro-$p$ group $(\bar{A} \times \bar{B})/M$. Therefore, $n \leq 3$ and the result follows.

(b) Suppose that $G = \text{HNN}(H, C, f, t)$ with $C = \langle c \rangle$. If $d(H) \geq 3$ then $d(G) \geq 3$ as can be seen by using the obvious epimorphism $G \to (H \times (t))/\langle t c^{-1} f(c)^{-1} \rangle$. Thus $d(H) \leq 2$.

Finally suppose that $d(H) = 2$. Now $G$ is the quotient of $Q := H \amalg \langle t \rangle$ modulo the relation $f(c)^{-1}c^t$. Since $d(Q) = 3$ we can conclude that $c \notin \Phi(G)$ and $f(c) \notin \Phi(G)$. Therefore neither $c \in \Phi(H)$ nor $f(c) \in \Phi(H)$. So we cannot have $f(c)^{-1}c \in \Phi(H)$ else $d(G/\Phi(G))$ turns out to be 3. Hence $H = \langle C, C^t \rangle$.

(c) Let $G = \Pi_1(G, \Gamma)$ with finite vertex groups $G(v)$ and cyclic edge groups $G(e)$. We claim that $|V(\Gamma)| \leq 3$. By assumption $|V(\Gamma)| \geq 2$, and therefore it has an edge $v$. Splitting $G$ over $v$, we can assume that $G(d_0(e))$ is procyclic by (a); hence $d_0(e)$ is a pending vertex of $\Gamma$. Suppose now that $\Gamma$ has at least 3 vertices, and let $a$ be an arbitrary edge $e$ of $\Gamma$ having initial or terminal vertex $v = d_1(e)$. Without loss of generality, suppose that $d_0(a) = v$. Then $d_1(a)$ is a pending vertex with procyclic vertex group $G(d_1(a))$; for, otherwise, by splitting $G$ over the edge $a$ we would obtain that $d(G) > 2$, a contradiction. Now, if we
have $r \geq 2$ edges with initial or terminal vertex $v$ then it follows from the pro-$p$ presentation of $G$ that it has a free pro-$p$ abelian group $\mathbb{Z}_p^r$ as a quotient; this implies $r = 2$, whence $|V(\Gamma)| \leq 3$.

If $|V(\Gamma)| = 2$ then $G = K \amalg C M$ with $K$ and $M$ finite, and, by (a), we can assume that $K$ is cyclic.

Suppose now that $|V(\Gamma)| = 3$. Then $G = K \amalg C M \amalg D N$ with $C$ and $D$ cyclic and $K$, $M$, and $N$ finite. By the properness of our decomposition we have $d(K \amalg C M) = d(M \amalg D N) = 2$ and, making use of (a), we can conclude that $K$ and $N$ must both be cyclic. Since $d(G) = 2$ then $M \leq \Phi(G)$ follows.

\[ \Box \]

**Proposition 2.8.** Let $G$ be the inverse limit of a surjective inverse system $\{G_i, \varphi_{ij}, I\}$ of pro-$p$ groups $G_i$. Suppose $G_i$ decomposes as an amalgamated free pro-$p$ product $G_i = K_i \amalg C_i R_i$ with $K_i$ cyclic and $R_i$ finite or $G_i = K_i \amalg C_i M_i \amalg D_i N_i$, with $K_i$ and $N_i$ cyclic and $M_i \leq \Phi(G_i)$. Then, passing to a cofinal subset of $I$, if necessary, there are inverse systems $\{K'_i, \varphi_{ij}, I\}$ and $\{C''_i, \varphi_{ij}, I\}$ such that $C''_i \leq K'_i$, $\varphi_{ij}(K'_i) = K'_j$ and $\varphi_{ij}(C''_i) \leq C''_j$ where each $K'_i$ (resp. $C''_i$) is a conjugate of $K_i$ (resp. $C_i$) by an element of $G_i$.

**Proof.** Using Theorem 2.3(a) in both cases we can pass to a cofinal subset $J$ of $I$ such that for all $i \geq j$ in $J$ we have $\varphi_{ij}(K_i) \leq K'_j$, for some $g_j \in G_j$. Indeed, in the first case $\varphi_{ij}$ sends factors to the factors up to conjugation and in the second case $\varphi_{ij}$ sends cyclic factors to cyclic factors up to conjugation. Then in fact, since $K_j$ is cyclic $\varphi_{ij}(K_i) = K'_j g_j$ (indeed, otherwise $\varphi_{ij}(K_i)G_j \neq K'_j G_j$ contradicting the surjectivity of $\varphi_{ij}$). Now selecting $g_i \in \varphi_{ij}^{-1}(g_j)$ and letting $K'_i := K'_i g_i^{-1}$, and using an induction argument, we obtain the desired inverse system $\{K'_i, \varphi_{ij}, J\}$. Next, letting $C''_i := C''_i g_i^{-1}$ we have $C''_i \leq K'_i \cap M_i g_i$; then, by Theorem 2.3(b), $\varphi_{ij}(C''_i) \leq K'_j \cap \varphi_{ij}(M_i) \leq C''_j$, for some $b_j \in K'_j$.

Choosing $b_i \in \varphi_{ij}^{-1}(b_j) \cap K'_i$ and letting $C''_i := C''_i b_i^{-1}$ we obtain the other inverse system $\{C''_i, \varphi_{ij}, I\}$.

\[ \Box \]

**Lemma 2.9.** Let $X$ be a $G$-space and $\{U_n\}_{n \geq 1}$ be a subset of normal subgroups of $G$ with $\bigcap U_n = 1$. Write $X_n := X/\overline{U_n}$ and $G_n := G/\overline{U_n}$. Let there be subgroups $S_n \leq G_n$, so that $\varphi_{nm}(S_n) \leq S_m$ and $S := \lim S_n$ be the inverse limit. If $X_n^{S_n} \neq \emptyset$ for all $n \in \mathbb{N}$ then $X^S \neq \emptyset$.

**Proof.** Let $\varphi_n$ denote the canonical projection from $X \amalg G$ onto $X_n \amalg G_n$. Then $X_n^{\varphi_n(S)} \supseteq X_n^{S_n} \neq \emptyset$. Therefore $Y_n := \varphi_n^{-1}(X_n^{\varphi_n(S)}) \neq \emptyset$. Now $Y_n = \{x \in X \mid x S \subseteq \overline{x U_n}\}$ so that $Y_{n+1} \subseteq Y_n$; by the compactness of $X$ we can deduce that $\emptyset \neq \bigcap Y_n \subseteq X^S$.

We end this section by quoting results to be used in the next section.

**Proposition 2.10** ([10, Thm. 1.1]). Let $G$ be a finitely generated pro-$p$ group which contains an open free pro-$p$ subgroup of index $p$. Then $G$
is isomorphic to a free pro-$p$ product

$$H_0 \amalg (S_1 \times H_1) \amalg \cdots \amalg (S_m \times H_m)$$

where $m \geq 0$, the $S_i$ are cyclic groups of order $p$ and the $H_i$ are free pro-$p$ groups of finite rank.

Corollary 2.11 ([10, Cor. 1.3(a)]). Every pro-$p$ group which contains an open free pro-$p$ subgroup of finite rank has, up to conjugation, only a finite number of finite subgroups.

Proposition 2.12. A pro-$p$ group $G$ acting on a pro-$p$ tree $T$ with trivial edge stabilizers such that there exists a continuous section $\sigma : V(T)/G \to V(T)$ is isomorphic to a free pro-$p$ product

$$\left( \prod_{v \in V(T)/G} G_{\sigma(v)} \right) \amalg (G/\langle G_w \mid w \in V(T) \rangle) .$$

Proof. This follows from the proof of [12, Thm. 3.6]. See also the last section of [5]. □

3. Groups acting virtually freely on trees

If a pro-$p$ group $G$ acts on a profinite graph $\Gamma$ we shall call sometimes $\Gamma$ a $G$-graph.

Lemma 3.1. Let $G$ be a non-trivial finitely generated pro-$p$ group, and let $\Gamma$ be a connected $G$-graph. Suppose that $\Delta$ is a connected subgraph of $\Gamma$ such that $\Delta G = \Gamma$. Then there exists a minimal set of generators $X$ of $G$ such that $\Delta \cap \Delta x \neq \emptyset$ holds for each $x \in X$.

Proof. It is enough to prove the lemma under the additional assumption that $G$ is elementary abelian. Indeed, using “bar” to denote passing to the quotient modulo $\Phi(G)$, making $Z$ a minimal generating set of $G$, suppose that for each $z \in Z$ there exists a vertex $v_z \in \Delta$ with $v_z \in \overline{\Delta} \cap \overline{\Delta z} \neq \emptyset$. Then there exists $f_z \in \Phi(G)$ with $v_z f_z \in \Delta f_z$, so that the set $X := \{zf_z^{-1} \mid z \in Z\}$ is a minimal set of generators of $G$ for which the assertion of the Lemma holds.

Suppose that the lemma is false for an elementary abelian group. Then there is a counterexample $G$ with minimal $d(G)$. Select a minimal generating subset $X$ of $G$. If $d(G) = 1$ then, due to the connectedness of $\Gamma$, there are $g_1, g_2 \in G$ with $g_1 \neq g_2$ such that $\Delta g_1 \cap \Delta g_2 \neq \emptyset$. Replacing $X$ by $\{g_1 g_2^{-1}\}$ shows that the conclusion of the lemma holds, a contradiction. Hence $d(G) \geq 2$. Select an element $z \in X$ and let “bar” denote passing to the quotient modulo $\langle z \rangle$. Since $d(\overline{G}) = d(G) - 1$, by the minimality assumption there is a subset $\overline{Y}$ of $\overline{G}$ which is a minimal set of generators of $\overline{G}$ such that $\overline{\Delta} \cap \overline{\Delta y} \neq \emptyset$ for all $\overline{y} \in \overline{Y}$. Let $Y$ denote a transversal of $\overline{Y}$ in $G$. Then there are elements $z_y \in \langle z \rangle$ such that $\Delta \cap \Delta y z_y \neq \emptyset$ for all $y \in Y$. Set $W = \{yz_y \mid y \in Y\}$. Since
$\Gamma$ can be viewed as a $\langle z \rangle$-graph, $\Delta W \cap \Delta W z \neq \emptyset$ can be assumed by the minimality assumption. This means that there exist $w_1, w_2 \in W$ such that $\Delta w_1 \cap \Delta w_2 \neq \emptyset$, so $X = W \cup \{w_1^{-1}w_2z\}$ would satisfy the assertion of the lemma, a contradiction.

**Lemma 3.2.** Let $G$ be a finitely generated infinite pro-$p$ group. Suppose that $G$ acts on a pro-$p$ tree $T$ containing a pro-$p$ subtree $D$ such that $DG = T$. Then there exists a minimal set of generators $X$ of a retract $H$ of $G$ such that $D \cap Dx \neq \emptyset$ for each $x \in X$.

**Proof.** Let “bar” denote passing to the quotient modulo $\widehat{G} = \langle G_v \mid v \in V(T) \rangle$. By Theorem 2.2(a) the quotient graph $\overline{T} := T/\widehat{G}$ is a pro-$p$ tree. Applying Lemma 3.1 to $\overline{G}$ acting on $\overline{T}$ yields a subset $Z$ of $G$ such that $\overline{Z}$ is a minimal set of generators of $\overline{G}$ and for each $z$ there exists a vertex $v_z \in D$ such that $v_z \in D \cap Dz \neq \emptyset$. Hence there exists $k_z \in \widehat{G}$ with $v_z k_z \in Dk_z \cap Dz$ and so $v_z z \in D \cap Dzk_z^{-1}$. Now set $X := \{zk_z^{-1} \mid z \in Z\}$ and $H := \langle X \rangle$. Finally observe that by Theorem 2.2(b), $\overline{G}$ is a free pro-$p$ group, so that $H$ is indeed a retract. \hfill $\Box$

**Lemma 3.3.** Let $G$ be a finitely generated pro-$p$ group acting on a pro-$p$ tree $T$. Suppose that all vertex stabilizers are finite and all edge stabilizers are pairwise conjugated. Assume further that there exist an edge $e \in T$ and a finite subset $V \subseteq T^{G_e}$ such that:

(i) for every $v_1, v_2 \in V$, $v_1 G = v_2 G$ implies $v_1 = v_2$,

(ii) $G$ is generated by the $G_v$, $v \in V$.

If $F$ is a free pro-$p$ open normal subgroup of $G$, then

$$\text{rank}(F) - 1 = [G : F] \left( \frac{|V| - 1}{|G_e|} - \sum_{v \in V} \frac{1}{|G_v|} \right).$$

**Proof.** We use induction on the index $[G : F]$. Obviously $F \neq G$, from hypothesis (ii); so, let us consider the preimage $N$ in $G$ of a central subgroup of order $p$ of $G/F$.

**Case 1.** $N \cap G_e = 1$.

It follows that each non-trivial torsion element $t$ of $N$ generates a self-centralized subgroup. Indeed, by Theorem 2.2(d) $t$ stabilizes some vertex $w$, so if $g$ centralizes $t$, the element $t$ also stabilizes $wg$. But then by Theorem 2.2(c) $t$ stabilizes the geodesic $\langle wg, w \rangle$. Since, however, $G_e \cap N = 1$, the element $t$ cannot stabilize any edge, so $wg = w$, and therefore $g$ is a power of $t$.

Thus the decomposition of $N$ according to Proposition 2.10 becomes $N = (\coprod_{i \in I} C_i) \amalg F_1$, with $F_1$ a free pro-$p$ subgroup of $F$. Taking into account that $G$ acts upon the conjugacy classes of subgroups of order $p$ we have

$$N = \prod_{v \in V} \left( \coprod_{r_v \in G/NG_v} (N \cap G_v)^{r_v} \right) \amalg F_1. \quad (3.1)$$
Set \( V_1 = \{ v \in V \mid N \cap G_v \neq 1 \} \). Since \( NG_v = FG_v \) for every \( v \in V_1 \) we can rewrite Eq. (3.1) as

\[
N = \prod_{v \in V_1} \left( \prod_{r_v \in G/FG_v} (N \cap G_v)^{r_v} \right) \prod F_1.
\]

Using this free decomposition and comparing it with Proposition 2.10 we find

\[
|I| = \sum_{v \in V_1} \frac{1}{|G/F G_v|} = [G : F] \sum_{v \in V_1} \frac{1}{|G_v|}, \tag{3.2}
\]

and

\[
\text{rank}(F) - 1 = p \text{rank}(F_1) + (p - 1)(|I| - 1) - 1. \tag{3.3}
\]

If \( N = G \) then \( F_1 = 1 \), since \( G_v, v \in V \) generate \( G \). Then \( G_e = 1 \) since otherwise \( G \) is finite. So \( |V| = |I| \) and the last equation becomes exactly the needed one. This gives the base of induction.

Suppose now that \( N \neq G \). Then the product \( p \text{rank}(F_1) \) can be computed by observing that passing to the quotient modulo \( \langle \text{tor}(N) \rangle \) and indicating it by “\( \bar{\text{tor}} \)" we have \( \text{rank}(\bar{F}) = \text{rank}(F_1) \), so that using \( [G : F] = p[\bar{G} : \bar{F}] \) the induction hypothesis yields

\[
p \text{rank}(F_1) = p \text{rank}(\bar{F})
\]

\[
= p[\bar{G} : \bar{F}] \left( \frac{|V| - 1}{|G_e|} - \sum_{v \in V} \frac{1}{|G_v|} \right) + p
\]

\[
= [G : F] \left( \frac{|V| - 1}{|G_e|} - \sum_{v \in V_1} \frac{1}{|G_v|} - \sum_{v \in V - V_1} \frac{1}{|G_v|} \right) + p
\]

\[
= [G : F] \left( \frac{|V| - 1}{|G_e|} - \sum_{v \in V_1} \frac{p}{|G_v|} - \sum_{v \in V - V_1} \frac{1}{|G_v|} \right) + p
\]

(we used \( G_e \cap N = 1 = G_v \cap N \) for all \( v \in V - V_1 \) and \( |G_v \cap N| = p \) for all \( v \in V_1 \) to obtain the last equality). Inserting this expression and the expression for \( |I| \) from Eq. (3.2) into Eq. (3.3) yields the claimed formula for \( \text{rank}(F) \).

**Case 2.** \( N \cap G_e \neq 1 \).

Since for all \( v \in V \) the edge group \( G_e \) is contained in \( G_v \) by the hypothesis, \( G_v \) centralizes \( N \cap G_e \). But \( G = \langle G_v \mid v \in V \rangle \) so \( N \cap G_e \) is a central subgroup of \( G \) of order \( p \). Then, using “\( \bar{\text{tor}} \)" to pass to the
quotient modulo $N \cap G_e$ and the inductive hypothesis, for $\overline{G}$ we have

\[
\text{rank}(F) - 1 = \text{rank}(\overline{F}) - 1
\]

\[
= [G : F] \left( \frac{|V| - 1}{|G_e|} - \sum_{v \in V} \frac{1}{|G_v|} \right)
\]

\[
= \frac{1}{p} [G : F] \left( \frac{p(|V| - 1)}{|G_e|} - \sum_{v \in V} \frac{p}{|G_v|} \right)
\]

\[
= [G : F] \left( \frac{|V| - 1}{|G_e|} - \sum_{v \in V} \frac{1}{|G_v|} \right)
\]

as needed. \(\square\)

Recall that a pro-$p$ group $G$ acts faithfully on a pro-$p$ tree $T$ if the kernel of the action is trivial; and $G$ acts irreducibly on $T$ if $T$ does not contain a proper $G$-invariant pro-$p$ subtree.

**Lemma 3.4.** Let a pro-$p$ group $G$ act faithfully and irreducibly on a pro-$p$ tree $T$. Suppose that $G_e$ is a minimal edge stabilizer and the set of edges $E(T^{G_e})G$ is open in $T$. Then $T' := T - E(T^{G_e})G$ is a subgraph having each connected component a pro-$p$ tree.

Let $\overline{T}$ be the quotient graph obtained by collapsing distinct connected components of $T'$ to distinct points. Then $\overline{T}$ is a pro-$p$ tree on which $G$ acts faithfully and irreducibly, and $\overline{T} = T^{G_e}G$.

**Proof.** Since $T'$ is closed and contains $V(T)$, it is a subgraph of $T$; hence its connected components are pro-$p$ trees. Moreover, $\overline{T}$ is a $G$-graph, and by [11, Proposition p. 486], it is simply connected and hence a pro-$p$ tree.

Now, we have $m \in T'$ if and only if there exists a subgroup $L \leq G_m$, an edge stabilizer, so that $Lg$ is not contained in $G_e$ for every $g \in G$. Therefore, since $G_e$ is a minimal edge stabilizer, we conclude that all edge stabilizers of edges in $T^{G_e}G$ are conjugates of $G_e$.

Let us show that $G_\tau$ is a conjugate of $G_e$ for every $\tau \in E(\overline{T})$. Let $f \in E(T)$ and $u, v$ be its end points which, by construction, belong to $T'$. Fix $g \in G_\tau = G_\tau \cap G_\tau$. Then $ug$ and $vg$ belong respectively to the same connected components as $u$ and $v$. The collapsing procedure induces a canonical epimorphism which is injective when restricted to $E(T^{G_e})G$. Since $\overline{T}$ is a pro-$p$ tree we find that after collapsing $e$ and $eg$ both map to $\tau$, and as edges of $eG$ under the collapsing procedure are not identified, $e = eg$ must follow. Hence $G_\tau$ is a conjugate of $G_e$ indeed.

Suppose that $G$ does not act irreducibly on $\overline{T}$. Since $\overline{T}$ is obtained by collapsing pro-$p$ subtrees, the preimage of a proper $G$-invariant pro-$p$ subtree of $\overline{T}$ is a proper $G$-invariant pro-$p$ subtree of $T$; a contradiction.
Suppose that \( g \in G \) acts trivially upon all of \( \overline{T} \). Then, in particular, \( \overline{eg} = \overline{e} \) and, as edges of \( eG \) under the collapsing procedure are not identified, we must have \( eg = e \), i.e., \( g \in G_e \). Therefore the kernel of the action of \( G \) upon \( \overline{T} \) is contained in \( G_e \) and so \( G_e \) contains a normal subgroup of \( G \) which, by [8, Thm. 3.12] must act trivially on \( T \). Hence \( G \) acts faithfully on \( T \). \( \square \)

Recall from the introduction that \( G \) acts virtually freely on a space \( X \) if some open subgroup \( H \) of \( G \) acts freely on \( X \).

**Lemma 3.5.** Let \( G \) be a finitely generated pro-\( p \) group acting faithfully, irreducibly and virtually freely on a pro-\( p \) tree \( T \). Then there are a pro-\( p \) tree \( D \), an edge \( e \in E(D) \), a finite subset \( V \subseteq \overline{D} \) and a finite subset \( X \subseteq G \) such that

(a) \( G \) acts faithfully upon \( D \).
(b) All edge stabilizers are pairwise conjugate; in particular, \( D = D^{G_e} G \).
(c) for every \( v_1, v_2 \in V \), \( v_1G = v_2G \) implies \( v_1 = v_2 \).
(d) \( X \) freely generates a free pro-\( p \) subgroup \( H \) such that for \( G_0 := \langle G_v \mid v \in V \rangle \) we have \( G = \langle G_0, H \rangle \) and \( H \cap G_0^G = 1 \).
(e) For each \( x \in X \), we have \( G_x e \subseteq \bigcup_{v \in V} G_v \).

**Proof.** Let \( e \in T \) be an edge with the stabilizer \( G_e \) of minimal order. Let \( \Sigma \) denote the set of all non-trivial finite subgroups \( L \) of \( G \) that are not conjugate to a subgroup of \( G_e \). Since \( G \) is finitely generated, Corollary 2.11 says that there exist up to conjugation only finitely many finite subgroups in \( G \); in particular there is a finite subset \( S \) of \( \Sigma \) such that \( \Sigma = \{ L^g \mid L \in S, g \in G \} \). Therefore the subset \( T_\Sigma := \{ m \in T \mid \exists L \in \Sigma, m \in T^L \} \), which is the union of all subtrees of fixed points \( T^L \) for subgroups \( L \in \Sigma \) can be represented in the form \( T_\Sigma = \bigcup_{L \in S} T^L G \) and is hence a closed \( G \)-invariant subgraph of \( T \). Therefore \( E(T^{G_e})G = T - T_\Sigma \) is open and we can apply Lemma 3.4 to obtain, by collapsing the connected components of \( T_\Sigma \), a pro-\( p \) tree \( D \) on which \( G \) acts irreducibly and faithfully with all edge stabilizers conjugate to \( G_e \). Thus \( D \) satisfies (a) and (b).

We come to proving (c),(d) and (e). Set \( N := \langle G_v \mid v \in V(D) \rangle \). By Lemma 3.2 there is finite minimal subset \( X \) of generators of a retract \( H \) in \( G \) of \( G/N \) such that \( D^{G_e} \cap D^G x \neq \emptyset \) for every \( x \) in \( X \); in fact, as \( G/N \) is free pro-\( p \) by Theorem 2.2(b), \( X \) freely generates \( H \). Moreover, by the construction of \( D \), there is only a finite subset \( V \) of vertices up to translation with stabilizers that are not conjugates of \( G_e \); to see this we observe that if vertices \( v, w \) are both stabilized by \( L \in \Sigma \), then \( L \) stabilizes the geodesic \( [v, w] \) (see Theorem 2.2(e)) and so \( v, w \) belong to the same connected component of \( T_\Sigma \).
It follows that \( G = \langle G_v, H \mid v \in V \rangle \) and \( H \cap \langle G_v \mid v \in V \rangle = 1 \). Moreover, since \( G \) is pro-\( p \) we can reduce \( V \) such that no two distinct vertices of it lie in the same orbit.

By construction, for every group element \( x \in X \) there is a vertex \( v_x \in D^{G_v} \) with \( v_x x^{-1} \in D^{G_v} \). When \( f \) is any edge in the geodesic \([v_x, v_x x^{-1}]\) then \( G_e = G_f = G_{v_x} \cap G_{v_x x}^{-1} \) (see Theorem 3.7(c)), so that, in particular, \( G_{v_x} \leq G_{v_x x} \). Finally modify \( V \) by replacing for every \( x \in X \) a vertex \( v \in V \) by the vertex \( v_x \) whenever \( v G = v_x G \). Then we see that (c), (d), and (e) all hold.

\[ \square \]

It is now convenient to introduce a notion of pro-\( p \) HNN-extension as a generalization of the construction described in [8, Sec. 4, p. 97].

**Definition 3.6.** Suppose that \( G \) is a pro-\( p \) group, and for a finite set \( X \), there are given monomorphisms \( \varphi_x : A_x \to G \) for subgroups \( A_x \) of \( G \). The HNN-extension \( \tilde{G} := \text{HNN}(G, A_x, \varphi_x, x \in X) \) is defined to be the quotient of \( G \amalg F(X) \) modulo the equations \( \varphi_x(a_x) = a_x^x \) for all \( x \in X \). We call \( \tilde{G} \) an HNN-extension and term \( G \) the base group, \( X \) the set of stable letters, and the subgroups \( A_x \) and \( B_x := \varphi_x(A_x) \) associated.

One can see that every HNN-extension in the sense of the present definition can be obtained by successively forming HNN-extensions, as defined in [8], each time defining the base group to be the just constructed group and then selecting a pair of associated subgroups and adding a new stable letter.

The HNN-extension \( \tilde{G} := \text{HNN}(G, A_x, \varphi_x, X) \) can also be defined by a universal property as follows. There are canonical maps \( \tilde{f} : G \to \tilde{G} \), \( \tilde{f}_x : A_x \to G \), \( \tilde{g} : X \to \tilde{G} \), with \( \tilde{f}_x(a_x)^{\tilde{g}(x)} = \tilde{f} \varphi_x(a_x) \) for all \( a_x \in A_x \), so that, given any pro-\( p \) group \( H \), any homomorphisms \( f : G \to H \), \( f_x : A_x \to H \) and a map \( g : X \to H \) such that for all \( x \in X \) and all \( a_x \in A_x \) we have \( f(\varphi_x(a_x)) = f_x(a_x)^{\tilde{g}(x)} \) there is a unique homomorphism \( \omega : \tilde{G} \to H \) with \( f = \omega \tilde{f} \), \( g = \omega \tilde{g} \), and, for all \( x \in X \), \( f_x = \omega \tilde{f}_x \).

**Theorem 3.7.** Let \( G \) be a finitely generated infinite pro-\( p \) group acting virtually freely on a pro-\( p \) tree \( T \). Then \( G \) splits either as an amalgamated free pro-\( p \) product or as a proper pro-\( p \) HNN-extension over some edge stabilizer.

**Proof.** We consider \( G \) to be a counterexample to the theorem with minimal index \( [G : F] \) where \( F \) is an open subgroup of \( G \) acting freely on \( T \).

**Claim 1:** \( G \) does not have a non-trivial finite normal subgroup. In particular, we can assume that \( G \) acts on \( T \) faithfully and irreducibly.
By [8, Lemma 3.11] there exists a unique minimal $G$-invariant subtree in $T$. Replacing $T$ by this subtree we may assume that the action of $G$ is irreducible.

Now, if $G$ contains a non-trivial finite normal subgroup, it contains a central subgroup $C$ of order $p$. By the minimality assumption on $[G : F]$ and as $[G/C : FC/C] < [G : F]$ the quotient group $\overline{G} := G/C$ satisfies the conclusion of the theorem, i.e. $\overline{G}$ is either an amalgamated free pro-$p$ product $\overline{G} = \overline{G}_1 \amalg \overline{G}_2$ with finite amalgamating subgroup or it is an HNN-extension $\overline{G} = \text{HNN}(G_1, H, t)$ with finite associated subgroups. Then $G$ is either an non-trivial amalgamated free pro-$p$ product $G = G_1 \amalg G_2$ or HNN($G_1, H, t$) with $G_1, G_2, H$ being preimages of $\overline{G}_1, \overline{G}_2, H$ in $G$, respectively, as needed. Hence $G$ does not possess a non-trivial finite normal subgroup. Since the vertex stabilizers are finite, the kernel of the action of $G$ upon $T$ is finite, hence it is trivial.

Thus, there exist $D, e, V, X$ and $G_0$ having the properties (a)-(e) of Lemma 3.5. Note that the stabilizers of vertices in $D$ may well be infinite.

Claim 2: The pro-$p$ subgroup $H$ of $G$ freely generated by $X$ must be trivial.

Suppose that $H \neq 1$. Let $\hat{G} = \text{HNN}(G_0, G_e, X)$ and $\lambda : \hat{G} \rightarrow G$ be the epimorphism given by the universal property. By induction on rank($F$) we show that $\lambda$ is an isomorphism.

It suffices to show that the rank of $F$ equals the rank of $\hat{F} := \lambda^{-1}(F)$. If $F_0 := G_0 \cap F \neq 1$ we can factor out the normal closure of $F_0$ in $G$ (and, if necessary, the kernel of the action as well) in order to obtain the quotient group $\overline{G}$ which acts on $D/F_0^G$ and satisfies rank($\overline{F}$) $< \text{rank}(F)$. Therefore the induced epimorphism $\overline{\lambda} : \text{HNN}(\overline{G}_0, \overline{G}_e, X) \rightarrow \overline{G}$ is an isomorphism, and it is not hard to see that $\text{HNN}(\overline{G}_0, \overline{G}_e, X)$ is isomorphic to $\hat{G}/\hat{F}_\overline{G}$, where $\hat{F}_0 := G_0 \cap \hat{F}$. This shows that the image $\overline{F}$ of $F$ in $\overline{G}$ is isomorphic to $\overline{\hat{F}}/\overline{\hat{F}_0}$. By Proposition 2.12 $F$ is a free pro-$p$ product $F \cong F_0 \amalg \overline{F}$ and $\overline{F} \cong \overline{F}_0 \amalg \overline{\hat{F}}/\overline{\hat{F}_0}$, so $F \cong \hat{F}$ and we are done. Thus we may assume that $G_0$ is finite. Now applying Lemma 3.3 to $\hat{G}$ and $G$ we deduce that rank($\hat{F}$) = rank($F$), so $\lambda|\hat{F}$ turns out to be an isomorphism, contradicting $G$ being a counterexample. This finishes the proof of the claim.

Claim 3: The natural epimorphism

$$\lambda : \prod_{v \in V} G_eG_v \rightarrow G$$

from the free pro-$p$ product of vertex stabilizers $G_v$ amalgamated along the single edge group $G_e$ onto $G$ is an isomorphism.
Let us use induction on $\text{rank}(F)$. Since $F$ acts freely on $E(D)$ by Proposition 2.12, for each $v \in V$ the intersection $F \cap G_v$ is a free factor of $F$, so similarly as in the proof of Claim 2 we can use the induction hypothesis, in order to achieve all $G_v$ to be finite.

Put $\tilde{F} = \lambda^{-1}(F)$. Since $\lambda$ restricted to all $G_v$ is injective, it suffices to prove that $\lambda_{|\tilde{F}}$ is an isomorphism. But by applying Lemma 3.3 to $\tilde{G}$ and $G$ we get that $F$ and $\tilde{F}$ have the same rank and therefore $\lambda$ is an isomorphism. The result follows.

Claim 3 shows that $G$ is not a counterexample, a final contradiction. □

**Theorem 3.8.** A finitely generated pro-$p$ group $G$ acting virtually freely on a pro-$p$ tree $T$ is isomorphic to the fundamental pro-$p$ group $\Pi_1(G, \Gamma)$ of a finite graph of finite $p$-groups whose edge and vertex groups are isomorphic to the stabilizers of some edges and vertices of $T$.

*Proof.* By induction on the rank of a maximal normal free pro-$p$ subgroup $F$ of $G$. If $\text{rank}(F) = 0$, that is $G$ is finite, take as graph of groups the single vertex $G$. In the general case, we apply Theorem 3.7 to split $G$ as an amalgamated free pro-$p$ product $G = G_1 \Pi_K G_2$ or as a pro-$p$ HNN-extension $G = \text{HNN}(G_1, K, t)$ over a finite subgroup $K$. Moreover, we are free to choose $K$ up to conjugation in $G_1$. Then every free factor, or the base group, satisfies the induction hypothesis and so exists the fundamental group of a finite graph of finite $p$-groups. By [14, Thm. 3.10] $K$ is conjugate to some vertex group of $G_1$ and so we may assume that $K$ is contained in a vertex group of $G_1$. Now in the case of an amalgamated product there is $g_2 \in G_2$ such that $K^{g_2}$ is contained in a vertex group of $G_2$, so $G$ admits a decomposition $G = G_1^{g_2} \Pi_{K^{g_2}} G_2$. Thus in both cases $G$ becomes the fundamental group of a finite graph of finite $p$-groups. □

**Theorem 3.9.** Let $H$ be a finitely generated subgroup of the fundamental pro-$p$ group $G$ of a finite graph of finite $p$-groups. Then $H$ is the fundamental pro-$p$ group of a finite graph of finite $p$-groups which are intersections of $H$ with some conjugates of vertex and edge groups of $G$.

*Proof.* The fundamental pro-$p$ group $G = \Pi_1(G, \Gamma)$ acts naturally on the standard pro-$p$ tree $T$ (cf. [14, Sec. 3]) and therefore so does $H$. Moreover, since the graph $\Gamma$ is finite, there exists an open normal subgroup $U$ of $G$ that intersects all vertex groups trivially and so acts freely on $T$. Thus Theorem 3.8 can be applied. □

**Example 3.10.** Let $A$ and $B$ be groups of order 2 and $G_0 = \langle A \times B, t \mid A^t = B \rangle$ be a pro-2 HNN-extension of $A \times B$ with associated subgroups $A$ and $B$. Note that $G_0$ admits an automorphism of order 2 that
swaps $A$ and $B$ and inverts $t$. Let $G = G_0 \rtimes \mathbb{C}$ be the holomorph. Set $H_0 = \langle \text{tor}(G_0) \rangle$ and $H = H_0 \rtimes \mathbb{C}$. Since $G_0$ acts on its standard pro-2 tree with finite vertex stabilizers, so does $H$. The main result in [3] shows that $H$ does not decompose as the fundamental group of a profinite graph of finite 2-groups. Its proof also shows that $H$ does not decompose as an amalgamated free pro-$p$ product or as a pro-$p$ HNN-extension over a finite group.

4. 2-generated subgroups

The final section is devoted to the proof of Theorem 1.3. So, henceforth, $G := A \amalg C B$ is a free pro-$p$ product of $A$ and $B$ with procyclic amalgamating subgroup $C$ satisfying the following assumptions:

(i) the centralizer in $G$ of each non-trivial closed subgroup of $C$ is a free abelian pro-$p$ group and contains $C$ as a direct factor.

(ii) each 2-generated pro-$p$ subgroup of $A$ and each 2-generated pro-$p$ subgroup of $B$ is either a free pro-$p$ group or a free abelian pro-$p$ group.

Lemma 4.1. For every subgroup $D \leq C$ we have $N_G(D) = C_G(D)$.

Proof. By the pro-$p$ version of [7, Cor. 2.7(ii)],

$$N_G(D) = N_A(D) \amalg_C N_B(D).$$

Since solvable 2-generated subgroups of $A$ and $B$ are abelian, $N_A(D) = C_A(D)$ and $N_B(D) = C_B(D)$; hence $N_G(D) = (C_A(D), C_B(D)) \subseteq C_G(D)$, as needed.

Theorem 4.2. Let $G = A \amalg C B$ be a free pro-$p$ product of $A$ and $B$ with procyclic amalgamating subgroup $C$. Suppose that the centralizer in $G$ of each non-trivial closed subgroup of $C$ is a free abelian pro-$p$ group and contains $C$ as a direct factor. If each 2-generated pro-$p$ subgroup of $A$ and each 2-generated pro-$p$ subgroup of $B$ is either a free pro-$p$ group or a free abelian pro-$p$ group then so is each 2-generated pro-$p$ subgroup of $G$.

Proof. Let $T$ be the standard pro-$p$ tree on which $G$ acts (cf. [8, Sec. 4]) and let $L$ be a 2-generated pro-$p$ subgroup of $G$. It follows from the definition of $T$ that if $L$ stabilizes a vertex of $T$, then $L$ is up to conjugation in one of the free factors of $G$; hence $L$ is either free pro-$p$ or free abelian pro-$p$, by hypothesis (ii).

Let us assume that $L$ fixes no vertex of $T$. Since $L$ is finitely generated, we have $L \cong \lim \leftarrow L/U_n$ where $\{U_n \mid n \in \mathbb{N}\}$ is a set of open normal subgroups of $L$ with $\bigcap U_n = 1$. Recall our notation $\widetilde{U_n}$ for the closed subgroup of $U_n$ generated by all vertex stabilizers with respect to the action of $U_n$ on $T$. We consider the infinite set $I$ of integers $n$ such that $U_n/\widetilde{U_n}$ is an infinite free pro-$p$ group (cf. Theorem
2.2(b)). So, defining \( L_n := L/\widetilde{U}_n \) we see that \( L_n \) acts virtually freely on a pro-\( p \)-tree \( T/\widetilde{U}_n \) (cf. Theorem 2.2(a)) and so we are in position to apply Theorem 3.8 to each of them. Thus \( L_n = \Pi_1(L_n, \Gamma_n) \) is the fundamental pro-\( p \)-group of a finite graph of finite \( p \)-groups whose edge and vertex groups are stabilizers of certain edges and vertices of \( T/\widetilde{U}_n \). Clearly we have \( L = \varprojlim \{L_n, \varphi_{nm}, I\} \) where each \( \varphi_{nm} \) is the canonical map.

Now, since \( L/\widetilde{L} \) is a free pro-\( p \)-group of rank at most 2, we need to consider only the two cases \( L = \widetilde{L} \) and \( L/\widetilde{L} \cong \mathbb{Z}_p \); in the remaining case, when \( d(L/\widetilde{L}) = 2 \), \( L \) is itself free pro-\( p \)-of rank 2 – by the Hopfian property. We can assume that \( \widetilde{L} \neq 1 \), otherwise there is nothing to prove.

Case 1. \( L = \widetilde{L} \).

We claim that \( \Gamma_n \) is a tree. If not then there is an edge \( e \in \Gamma_n \) so that \( L_n = \text{HNN}(P_n, G(e), t) \) for \( G(e) \) finite. But then there is a homomorphism from \( L_n \) onto \( \mathbb{Z}_p \) contradicting \( L_n = \langle \text{tor}(L_n) \rangle \).

Then in light of Lemma 2.7(c) and of Proposition 2.8, we have inverse systems of conjugates of \( K_n \) and \( D_n \); following the notation of the referred Proposition, we define two procyclic groups \( K := \varprojlim K_n' \) and \( D := \varprojlim D_n'' \).

We claim that \( D = 1 \). Note that since each \( D_n \) is an edge stabilizer with respect to the \( L_n \)-action, we have \( D = L \cap C^g \), for some \( g \in G \). Since \( C_L(D) = L \cap C_G(D) \), it follows from (i) that \( C^g \) is a direct factor of \( C_G(D) \), hence \( D \) is a direct factor of \( C_L(D) \). Suppose on the contrary that \( D \neq 1 \). Since the procyclic group \( K \) contains \( D \), it follows that \( D = K \). Now, the projection \( K \to K_n/\langle \text{tor}(K_n) \rangle \) is surjective for some sufficiently large \( n_0 \), by Lemma 2.1(a). Hence \( D_{n_0} = K_{n_0} \); a contradiction to the non-fictitious decomposition of \( L_{n_0} \).

Thus \( D = 1 \), and so \( \varprojlim D_n''L_n = 1 \), by Lemma 2.1(b). Then, writing \( L_n \cong K_n' \Pi_{D_n'} R_n' \) we have \( L \cong \varprojlim L_n/D_n''L_n \cong \varprojlim (K_n'/D_n'' \Pi R_n'/D_n''^n \Pi R_n^\prime) \).

Now, if \( d(L_n/D_n''L_n) = 1 \) for every \( n \), then \( L \) is procyclic; thus without loss of generality we may and do assume that each \( L_n/D_n''L_n \) is 2-generated. Since \( K_n'/D_n'' \) is 1-generated, so is \( R_n'/D_n''^n \). Therefore \( L \cong \mathbb{Z}_p \Pi \mathbb{Z}_p \), by Proposition 2.6. Our proof is finished for Case 1.

Case 2. \( L/\widetilde{L} \cong \mathbb{Z}_p \).

For \( n \in \mathbb{N} \) we have \( \mathbb{Z}_p \cong L/\widetilde{L} \cong L_n/(\widetilde{L}/\widetilde{U}_n) \) and therefore \( \Gamma_n \) cannot be a tree. Then we can select a suitable edge \( e_n \) of \( \Gamma_n \), set \( \Delta_n := \Gamma_n - \{e_n\} \), and present \( L_n = \text{HNN}(K_n, D_n, t_n) \) with cyclic edge group \( D_n \) of \( e_n \) and \( K_n = \Pi_1(\mathcal{G}_n(\Delta_n), \Delta_n) \).

Since \( \widetilde{L}/\widetilde{U}_n \) is generated by torsion, as a consequence of Theorem 2.4(a), it is contained in \( K_nL_n \); so, \( \langle \text{tor}(L_n) \rangle = K_nL_n \). By [13, Prop.
1.7(ii)], $K_n/\langle \text{tor}(K_n) \rangle$ is a free pro-$p$ group, whence $\langle \text{tor}(L_n) \rangle$ has trivial image in the quotient HNN($K_n/\langle \text{tor}(K_n) \rangle$, 1, $t_n$) of $L_n$. Thus $K_n = \langle \text{tor}(K_n) \rangle$. Since $K_n$ acts on the pro-$p$ tree $T/\sim$, we have $K_n = \tilde{K}_n$ (cf. Theorem 2.2(d)), so in particular, $\Delta_n$ must be a tree. Passing now to a cofinal subset of $\mathbb{N}$, if necessary, we may assume that for all $n$ either $\Delta_n$ is a single vertex or $\Delta_n$ contains an edge. We discuss the two subcases.

**Subcase 2(α).** For each $n \in \mathbb{N}$ the tree $\Delta_n$ is a single vertex.

Then $K_n$ is finite. Passing again to a cofinal subset of $\mathbb{N}$, if necessary, we can, making use of Theorem 2.4(a) and a projective limit argument, arrange that $\varphi_{n+1}(K_{n+1}) \leq K_n$ holds for all $n$. Passing again to a cofinal subset of $\mathbb{N}$, if necessary, and making use of Lemma 2.7(b) we can arrange that for all $n$ either $K_n$ is cyclic or that $d(K_n) = 2$. We shall discuss the situations when $K := \varprojlim K_n$ is procyclic and when $d(K) = 2$.

If $K$ is procyclic, then for every $m$ there exists $n > m$ such that the $\varphi_{nm}(K_n)$ is cyclic and so $\varphi_{nm}(D_n) = \varphi_{nm}(D_n^m)$. Hence $\varphi_{nm}(t_n)$ normalizes $\varphi_{nm}(D_n)$ and so $L_m = N_{L_m}(\varphi_{nm}(D_n))$. Since $L = \varprojlim L_m$ it follows that $D := \varprojlim D_m$ is normal in $L$. Since $E(T)$ is a compact $L$-space, setting in Lemma 2.9 $X := E(T)$, $G := L$, and $S_n := D_n$, we find $e \in E(T)$ with $D \leq G_e$. Therefore $D^g \leq C$ for some $g \in G$ and, if $D \neq 1$, making use of Lemma 4.1, we find that $L \cong \mathbb{Z}_p \times \mathbb{Z}_p$, by hypothesis (i), as needed.

Next assume that $D = 1$. It follows from Lemma 2.1(b) that $\varprojlim D_n^m = 1$ and so $L = \varprojlim L_m/D_n^m$. Observing that $L_m/D_n^m = \langle \tilde{K}_m/K_m \cap D_n^m \rangle \langle t_m \rangle$ Proposition 2.6 implies that $L \cong \mathbb{Z}_p \times \mathbb{Z}_p$, whence the result when $K$ is procyclic.

For finishing Subcase 2(α) we can now assume that $d(K) = 2$. Then Lemma 2.1(a) in conjunction with a projective limit argument implies that $\varphi_{n+1}(K_{n+1}) \cong K_n$ for every $n$. By virtue of Proposition 2.5, we have inverse systems of conjugates $K'_n$ and $D'_n$ of the finite $p$-groups $K_n$ and $D_n$, and $L = \text{HNN}(K, D, t)$ where $K := \varprojlim K'_n$ and $D := \varprojlim D'_n$ is procyclic. We must have $D \neq 1$, else $L \cong K \langle t \rangle$, and so $2 = d(K) = d(L) - 1 = 1$; a contradiction.

An application of Lemma 2.9 shows that $K$ stabilizes a vertex in $T$; it is therefore, up to conjugation, contained in either $A$ or $B$ and so by hypothesis (ii) is either free pro-$p$ or free abelian pro-$p$. In the first case we observe that Lemma 2.7(b) implies that $K = D \langle t \rangle$ and so $L = D \langle t \rangle$ is a free pro-$p$ group.

So assume in the sequel that $K$ is a free pro-$p$ abelian group. Note that $L = \text{HNN}(K, D, t)$ contains $H := K \langle t \rangle$, which is not abelian. On the other hand since $E(T)$ is a compact $L$-space, setting in Lemma 2.9 $X := E(T)$, $G := L$, and $S_n := D_n$ we find $e \in E(T)$ with $D \leq G_e$. Hence $D \leq C^g$ for suitable $g \in G$. Since $D \leq C^g$, by hypothesis (i)
$C_G(D)$ is abelian, and it contains $H$; a contradiction. Hence we are done with Subcase 2(a).

**Subcase 2(β).** For each $n \in \mathbb{N}$ the tree $\Delta_n$ contains an edge.

Lemma 2.7(c) and Proposition 2.8 imply that $K_n$ can be written as $K_n = X_n \amalg \mathbb{Z}_n \overline{W}_n$, with cyclic $p$-groups $X_n$, and there are inverse systems $\{X'_n\}$ and $\{Z''_n\}$ with $Z''_n \leq X'_n$ of conjugates of $X_n$ and $Z_n$ respectively. Define procyclic groups $X = \lim \limits_{\sim} X'_n$ and $Z = \lim \limits_{\sim} Z''_n$. We must have $Z \neq X$ else by Lemma 2.1(a) we could find $n$ with $Z_n = X_n$ and so the decomposition $K_n = X_n \amalg \mathbb{Z}_n \overline{W}_n$ would be fictitious; a contradiction. Setting in Lemma 2.9 $X := E(T)$, $G := L$, and $S_n := Z''_n$ we find $e \in E(T)$ with $Z \leq G_e$. Hence there is $g \in G$ with $Z \leq C^g$.

Now, since $Z \neq X$, hypothesis (i) implies $Z = 1$. Let $\overline{K}_n = K_n/Z_n^{K_n}$ and let $\overline{D}_n$ be the canonical image of $D_n$ in $\overline{K}_n$. Then, we consider

$$\overline{L}_n = \overline{L}/Z_n^L = \text{HNN}(\overline{K}_n, \overline{D}_n, \overline{e}) = \text{HNN}(X_n/Z_n X_n \amalg \overline{W}_n/Z_n \overline{W}_n, \overline{D}_n, \overline{e}_n).$$

By Lemma 2.7(b), each pro-$p$ group $K_n$ is at most 2-generated, hence considering $\overline{L}_n$ modulo its Frattini subgroup, we can conclude that $d(W_n/Z_n \overline{W}_n) = 1$. So, taking into account Lemma 2.7(b) we conclude that $X_n/Z_n X_n$ and $W_n/Z_n \overline{W}_n$ are isomorphic cyclic $p$-groups. Thus $\overline{L}_n \cong X_n/Z_n X_n \amalg \overline{L}_n$. By Lemma 2.1(b) and Proposition 2.6 we obtain that $L = \lim \limits_{\sim} \overline{L}_n \cong \mathbb{Z}_p \amalg \overline{L}_n$. This concludes the proof of the theorem. \hfill \Box

**Corollary 4.3.** Suppose that neither $A$ nor $B$ contains a 2-generated non-procyclic abelian subgroup. Then any 2-generated subgroup $L$ of $G$ is a free pro-$p$ group.

**Proof.** Suppose that $L$ is a free abelian pro-$p$ group of rank 2.

Let $T$ be the standard pro-$p$ tree on which $G$ acts. Then by [8, Thm. 3.18] either $L$ stabilizes a vertex or there is an edge $e$ of $T$ such that $L/L_e \cong \mathbb{Z}_p$. But $L$ cannot stabilize a vertex; else it is conjugate to a subgroup of one of the free factors of $G$, contradicting the supposition.

Therefore $L/L_e \cong \mathbb{Z}_p$ for some edge $e$. Since $d(L) = 2$ we must have $L_e \neq 1$. Conjugating $L$ by some element of $G$ we may assume that $L_e$ is contained in $C$. Then, $L = N_G(L_e) = C_G(L_e)$, by Lemma 4.1(a), and, by the centralizer condition of the theorem, $L = C \cong \mathbb{Z}_p$, a contradiction. Thus, by Theorem 4.2, $L$ must be free pro-$p$. \hfill \Box

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