Explanation of Rotation Curves in Galaxies and Clusters of them, by Generalization of Schwarzschild Metric and Combination with MOND, eliminating Dark Matter

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Abstract. Schwarzschild Metric is the first and the most important solution of Einstein vacuum field equations. This is associated with Lorentz metric of flat spacetime and produces the relativistic potential (\(\Phi\)) and the field strength (\(g\)) outside a spherically symmetric mass or a non-rotating black hole. It has many applications such as gravitational red shift, the precession of Mercury’s orbit, Shapiro time delay etc. However, it is inefficient to explain the rotation curves in large galaxies and clusters of them, causing the necessity for dark matter. On the other hand, Modified Newtonian Dynamics (MOND) has already explained these rotation curves in many cases, using suitable interpolating function (\(\mu\)) in Milgrom’s Law. In this presentation, we initially produce a Generalized Schwarzschild potential and the corresponding Metric of spacetime, in order to be in accordance with any isotropic metric of flat spacetime (including Galilean Metric of spacetime which is associated with Galilean Transformation of spacetime). From this Generalized Schwarzschild potential (\(\Phi\)), we calculate the corresponding field strength (\(g\)), which is associated with the interpolating function (\(\mu\)). In this way, a new relativistic potential is obtained (let us call 2\(^{nd}\) Generalized Schwarzschild potential) which describes the gravitational interaction at any distance and for any metric of flat spacetime. Thus, not only the necessity for Dark Matter is eliminated, but also MOND becomes a pure Relativistic Theory of Gravitational Interaction. Then, we pass to the case of flat spacetime with Lorentz metric (Minkowski space), because the experimental data have been extracted

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using the Relativistic Doppler Shift and the gravitational red shift of Classic Relativity (CR). Thus, we explain the Rotation Curves in Galaxies (e.g. NGC 3198) and Clusters of them as well as the Solar system, eliminating Dark Matter. This relativistic potential and the corresponding metric of spacetime have been obtained by the light of Euclidean Closed Linear Transformations of Complex Spacetime endowed with the Corresponding Metric. Of course, may also be applied by scientists who prefer the hyperbolic geometry of Classic Relativity (CR).

1. Introduction

1.1. Schwarzschild Metric, Relativistic Potential and Field Strength

**Schwarzschild Metric**

\[
g_{00} = -\left(1 - \frac{r_s}{r}\right); \quad g_{rr} = \frac{1}{\left(1 - \frac{r_s}{r}\right)}
\]

is the first and the most important solution of *Einstein vacuum field equations*. It is associated with *Lorentz metric* of flat spacetime, e.g.

\[
\eta = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

if a contravariant four-vector is expressed in *Cartesian Coordinates* (CCs)

\[
X = \begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix}
\]

and produces the corresponding Relativistic Potential

\[
\Phi = \frac{c^2}{2} \ln \left(1 - \frac{r_s}{r}\right)
\]

where

\[
r_s = \frac{2GM}{c^2}
\]

is *Schwarzschild radius*. It is referred to the space outside a spherically symmetric mass or a non-rotating black hole. Thus, it has many applications such as gravitational red shift, the precession of Mercury’s orbit, Shapiro time delay etc [1]. According *Schwarzschild Metric* the field strength is

\[
g = \frac{GM}{r^2} \left(1 - \frac{r_s}{r}\right)^{\frac{1}{2}}
\]

which far away from *Schwarzschild radius* becomes

\[
g = \frac{GM}{r^2} \left(1 + \frac{r_s}{2r}\right)
\]

and at even larger distances is equal to the *Newtonian field strength*

\[
g = \frac{GM}{r^2}
\]

In case of Uniform Circular Motion (UCM), it is
\[ g = \frac{\nu^2}{r} \]  

so 

\[ \nu^2 = \frac{GM}{r} \]  

However, the above field strength is inefficient to explain the rotation curves in large galaxies and clusters of them, causing the necessity for dark matter [2-6].

1.2. Modified Newtonian Dynamics (MOND)

On the other hand, Modified Newtonian Dynamics (MOND) has already explained these rotation curves in many cases, using suitable interpolating function (\( \mu \)) in Milgrom’s Law [7-11]. The Modified Newtonian field strength is 

\[ g = \frac{GM}{r^2} \frac{1}{\mu} \]  

The interpolating function (\( \mu \)) is an unspecified function having the properties

\[ \lim_{a \to 0} \mu = \frac{a}{a_0} \]  

where \( a \) is the acceleration of a body and \( a_0 \) is a new fundamental constant with value (see [8] p. 1)

\[ a_0 \approx 1.2 \times 10^{-10} \frac{m}{s^2} \]  

and \( \mu = 1 \), if \( a >> a_0 \). Two common choices are the Simple interpolating function

\[ \mu = \frac{1}{1 + \frac{a_0}{a}} \]  

and the Standard interpolating function

\[ \mu = \frac{1}{\sqrt{1 + \left( \frac{a_0}{a} \right)^2}} \]  

Using Simple \( \mu \) of MOND (14) and (11), for \( \alpha = g \), it emerges

\[ g = \frac{GM}{r^2} \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4a_0 r^2}{GM}} \right) \]  

Let us define Milgrom radius

\[ r_0 = \sqrt{\frac{GM}{4a_0}} \]  

We may return to the Newtonian results, if we take the limit \( \alpha \to 0 \) or \( r \to \infty \). With that (16) becomes

\[ g = \frac{GM}{r^2} \frac{1}{2} \left( 1 + \sqrt{1 + \left( \frac{r}{r_0} \right)^2} \right) \]  

and we obtain Simple interpolating function

\[ \mu = \frac{2}{1 + \left( \frac{r}{r_0} \right)^2} \]  

In case of Uniform Circular Motion (UCM), it is
\[ v^2 = \frac{GM}{2r} \left( 1 + \sqrt{1 + \frac{r^2}{r_0^2}} \right) \]  

(20)

Using Standard \( \mu \) of MOND (15) and (11), for \( \alpha = g \), it emerges

\[ g = \frac{GM}{r^2} \frac{1}{\sqrt{2}} \left( 1 + \sqrt{1 + \frac{1}{4} \left( \frac{r}{r_0} \right)^4} \right) \]  

(21)

and we obtain Standard interpolating function

\[ \mu = \frac{\sqrt{2}}{\sqrt{1 + \sqrt{1 + \frac{1}{4} \left( \frac{r}{r_0} \right)^4}}} \]  

(22)

In case of Uniform Circular Motion (UCM), it is

\[ v^2 = \frac{GM}{r} \frac{1}{\sqrt{2}} \left( 1 + \sqrt{1 + \frac{1}{4} \left( \frac{r}{r_0} \right)^4} \right) \]  

(23)

In Figure 1, we show the plots of \( 1/\mu_{\text{Simpl}} \) and \( 1/\mu_{\text{Stand}} \) wrt the distance \( (r/r_0) \) from the center of gravity, where \( \mu_{\text{Simpl}} \) and \( \mu_{\text{Stand}} \) are simple and standard interpolating functions, respectively. We observe that they are similar to the contribution of dark matter (see e.g. Figure 3). Besides Simple \( \mu \) produces stronger gravity than Standard \( \mu \).

![Figure 1. Plot of \( 1/\mu_{\text{Simpl}} \) and \( 1/\mu_{\text{Stand}} \) wrt the distance \( (r/r_0) \) from the center of gravity, where \( \mu_{\text{Simpl}} \) and \( \mu_{\text{Stand}} \) are simple and standard interpolating functions, respectively.](image)

Both Interpolating functions give the same velocity at infinite distance from the center of gravity

\[ v_{\infty}^2 = \sqrt{GMa_0} \]  

(24)
so

\[ \beta_\infty = \frac{v_\infty}{c} = \frac{4\sqrt{GMa_0}}{c} \]  

(25)

Milgrom radius is connected with Schwarzschild radius via the formula

\[ r_0 = \sqrt[4]{\frac{GM}{2GM}} \frac{r_\Sigma}{c^2} = \frac{c^2}{4\sqrt{GMa_0}} r_5 = \frac{c^2}{4\beta_\infty} r_5 \]  

(26)

Thus, we have another characteristic parameter of a body or cluster

\[ N_0 = \frac{c^2}{4\sqrt{GMa_0}} = \frac{c^2}{4\sqrt{Ga_0}} \frac{1}{\sqrt{M}} \]  

(27)

Three fundamental physical constants are combined giving an extremely large value

\[ N = \frac{c^2}{4\sqrt{Ga_0}} = 2.510664 \times 10^{26} \text{ Kg}^{-\frac{1}{2}} \]  

(28)

Thus, only Clusters of Galaxies and entire Universe have mass that makes \( r_0 \) having similar value to \( r_\infty \). In any other case

\[ r_0 << r \]  

(29)

Some values of the aforementioned characteristic parameters for the Earth, the Sun (see [12] pp 8 and [13] pp 1-1, 14-2) and galaxy NGC3198 (see [4] p 56 and [2] p 3) are contained in Table 1.

Table 1. Characteristic parameters (mass \( M \), radius \( R \), Schwarzschild radius \( r_\Sigma \), Milgrom radius \( r_0 \), \( r_r \), velocity at infinite distance \( v_\infty \) and \( \beta_\infty \)) for the Earth, the Sun and galaxy NGC 3198.

| Parameter | Earth | Sun | NGC 3198 |
|-----------|-------|-----|----------|
| \( M / \text{Kg} \) | 5.9742\times10^5 | 1.9891\times10^5 | 6.76294\times10^5 |
| \( R / \text{m} \) | 6378140 | 6.9599\times10^5 | 2.47\times10^6 |
| \( R / \text{Kpc} \) | 2.066999\times10^{-6} | 2.2555\times10^{-6} | 8 |
| \( r_r / \text{m} \) | 8.873578523\times10^{-1} | 2.954.44328 | 1.0044505\times10^{-1} |
| \( r_r / \text{Kpc} \) | 2.875710057\times10^{-10} | 9.57462903\times10^{-10} | 0.0000680703 |
| \( r_\Sigma / \text{m} \) | 9.114261212\times10^{-1} | 5.25908150\times10^{-1} | 9.6972671\times10^{-1} |
| \( r_\Sigma / \text{Kpc} \) | 2.953709437\times10^{-10} | 0.0000170434 | 3.1426474 |
| \( r_0 / r_r \) | 1.027123520\times10^{-10} | 1.78005837\times10^{-10} | 965,430 |
| \( v_\infty / \text{m s}^{-1} \) | 14.7899 | 355.272 | 152,556 |
| \( \beta_\infty \) | 4.933392321\times10^{-10} | 1.18506\times10^{-10} | 0.000508873 |

The work that is needed for moving one unit of mass from a point A with distance \( r \) from the center of gravity to another point B with larger distance \( r \), (the Potential Difference between point A and point B), is

\[ w = V_{(r)} - V_r = \int_0^r g \, dr = GM \int_0^r \frac{dr}{r^2 \mu(r)} \]  

(30)

In case of Simple \( \mu \), we have

\[ w = \Delta V = 2a_0 \int_0^r \frac{1}{r} \left[ 1 + \left( \frac{r}{r_0} \right)^2 \right] \left[ 1 + \left( \frac{r}{r_0} \right)^2 \right] \frac{dr}{r^2 r_0} = \frac{GM}{2} \left[ 1 + \frac{1}{r} \right] \left[ 1 + \left( \frac{r}{r_0} \right)^2 \right] + \frac{GM}{4a_0} \text{ArcSin}\left( \frac{r}{r_0} \right) \]  

(31)

because the integral
\[ I_{\text{Simp}} = \int \frac{1}{x^2} \left( 1 + \sqrt{1 + x^2} \right) \, dx = \frac{1}{x} - \frac{\sqrt{1 + x^2}}{x} + \text{ArcSinh} \ x = \frac{1}{x} - \frac{\sqrt{1 + x^2}}{x} + \ln \left( x + \sqrt{1 + x^2} \right) \] (32)

Taking the limit \( r \to \infty \), we find that if we wish to move the mass at infinite distance from the center of gravity, we must pay infinite work. So, we cannot define the infinite as the point with zero potential. In addition, it appears that nothing cannot be moved out of the Universe unless we pay infinite work.

In case of \textit{Standard} \( \mu \), the work that is needed for moving one unit of mass from a point \( A \) with distance \( r \) from the center of gravity to another point \( B \) with larger distance \( r_1 \), is

\[ w = \Delta V = 2\sqrt{2} a_0 r_0 \int \frac{1}{\left( \frac{r}{r_0} \right)^2} \sqrt{1 + \frac{1}{4} \left( \frac{r}{r_0} \right)^4} \, d \left( \frac{r}{r_0} \right) \] (33)

Unfortunately, the integral

\[ I_{\text{Stand}} = \int \frac{1}{x^2} \sqrt{1 + \frac{1}{4} x^4} \, dx \] (34)

has no analytical solution.

The values of Circular Velocities are contained in Table 1 and the Mass of the Galaxy NGC 3198 that is enclosed within the circular orbit wrt the distance from the center of Galaxy are contained in Table 2 (see [5] p 2). In addition, the corresponding Rotation Curves and Mass Distribution in Galaxy NGC 3198 are shown in Figure 2 and Figure 3 respectively.

\[ \text{NGC 3198 Rotation Curves} \]

**Figure 2.** Universal Rotation Curves in Galaxy NGC 3198. Circular Velocities (experimental \( V_{\text{exp}} \), calculated by Schwarzschild or Newtonian field strength \( V_{\text{d}} \), calculated by Simple \( \mu \) in Milgrom’s Law \( V_{\text{d}} \), calculated by Standard \( \mu \) in Milgrom’s Law \( V_{\text{d}} \)) wrt the distance from the center of Galaxy NGC 3198.
Table 2. Circular Velocities (experimental $V_\text{circ}$, calculated using Schwarzschild or Newtonian field strength $V_{\text{c}}$, Simple $\mu$ in Milgrom's Law $V_{\text{c simp}}$, Standard $\mu$ in Milgrom's Law $V_{\text{c exp}}$) and Luminous Mass Distribution (Luminous Mass of the galaxy that is enclosed within the circular orbit $M$) wrt the distance from the center of Galaxy NGC 3198. $(\Delta V_{\text{c}})\approx8\%$ $(\Delta V_{\text{c simp}})$, $(\Delta V_{\text{c exp}})$, and $(\Delta V_{\text{c}})$ are the corresponding relative errors.

| $r$ | $M_\text{c}$ | $V_{\text{c}}$ | $V_{\text{c simp}}$ | $V_{\text{c exp}}$ | $(\Delta V_{\text{c simp}})$% | $V_{\text{c}}$% | $(\Delta V_{\text{c exp}})$% | $V_{\text{c}}$% |
|-----|--------------|----------------|---------------------|-------------------|-----------------------------|---------------|-----------------------------|---------------|
| 2.0 | 0.687        | 79.0           | 86.2                | 9                 | 109.6                      | 39            | 97.2                        | 23            |
| 3.0 | 1.012        | 97.8           | 85.4                | -13               | 115.6                      | 18            | 102.8                       | 5             |
| 4.0 | 1.620        | 118.0          | 93.6                | -21               | 128.8                      | 9             | 114.8                       | -3            |
| 5.5 | 3.404        | 139.4          | 115.7               | -17               | 156.6                      | 12            | 139.3                       | 0             |
| 6.0 | 3.698        | 144.2          | 120.8               | -16               | 163.4                      | 13            | 145.5                       | 1             |
| 7.0 | 5.089        | 143.3          | 125.4               | -12               | 171.8                      | 20            | 153.1                       | 7             |
| 8.0 | 5.825        | 150.3          | 125.5               | -17               | 175.76                     | 17            | 157.0                       | 4             |
| 9.0 | 6.346        | 149.9          | 123.5               | -18               | 177.3                      | 18            | 158.9                       | 6             |
| 10.1| 6.735        | 152.1          | 120.1               | -21               | 177.6                      | 17            | 159.9                       | 5             |
| 11.0| 6.914        | 151.1          | 116.6               | -23               | 176.9                      | 17            | 159.9                       | 6             |
| 12.1| 7.093        | 156.2          | 112.6               | -28               | 176.14                     | 13            | 159.9                       | 2             |
| 14.1| 7.214        | 161.0          | 105.2               | -35               | 173.81                     | 8             | 159.2                       | -1            |
| 16.1| 7.237        | 155.3          | 98.6                | -37               | 171.5                      | 10            | 158.4                       | 2             |
| 18.1| 7.191        | 148.7          | 92.7                | -38               | 169.3                      | 14            | 157.4                       | 6             |
| 20.1| 7.115        | 149.1          | 87.5                | -41               | 167.3                      | 12            | 156.5                       | 5             |
| 22.1| 7.005        | 148.4          | 82.8                | -44               | 165.4                      | 11            | 155.5                       | 5             |
| 24.1| 6.901        | 146.2          | 78.7                | -46               | 163.7                      | 12            | 154.7                       | 6             |
| 26.1| 6.806        | 145.5          | 75.1                | -48               | 162.3                      | 12            | 153.9                       | 6             |
| 28.1| 6.716        | 147.3          | 71.9                | -51               | 161.0                      | 9             | 153.2                       | 4             |
| 30.2| 6.628        | 146.5          | 68.9                | -53               | 159.8                      | 9             | 152.6                       | 4             |
| 32.2| 6.544        | 148.4          | 66.3                | -55               | 158.7                      | 7             | 152.0                       | 2             |
| 34.2| 6.456        | 149.3          | 63.9                | -57               | 157.7                      | 6             | 151.4                       | 1             |
| 36.2| 6.392        | 149.9          | 61.8                | -59               | 156.9                      | 5             | 151.0                       | 1             |
| 38.2| 6.316        | 149.3          | 59.8                | -60               | 156.0                      | 5             | 150.4                       | 1             |
| 40.2| 6.252        | 150.0          | 58.0                | -61               | 155.3                      | 4             | 150.0                       | 0             |
| 42.1| 6.191        | 147.6          | 56.4                | -62               | 154.6                      | 5             | 149.6                       | 1             |
| 44.2| 6.159        | 149.8          | 54.9                | -63               | 154.2                      | 3             | 149.5                       | 0             |
| 46.2| 6.114        | 151.5          | 53.5                | -65               | 153.6                      | 1             | 149.1                       | -2            |
| 48.2| 6.072        | 151.9          | 52.2                | -66               | 153.2                      | 1             | 148.8                       | -2            |

The Circular Velocities $V_c$ have been calculated using (10) of Schwarzschild or Newtonian field strength $V_c$, Simple $\mu$ in Milgrom's Law and $V_{c simp}$ using (20) of Simple $\mu$ in Milgrom's Law and $V_{c exp}$ using (23) of Standard $\mu$ in Milgrom's Law. We observe that in case of galaxy NGC 3198, Schwarzschild or Newtonian field strength produces maximum relative error about 66% at extra large distances, while Simple $\mu$ in Milgrom's Law gives better results, producing maximum relative error about 39% near to the galactic center and Standard $\mu$ in Milgrom's Law gives even better results, producing maximum relative error about 23%
at the center of the galaxy. It is noted that the relative error of experimental Circular Velocities is \(\Delta V_r \approx 8\%\). Thus Standard \(\mu\) in Milgrom’s Law gives the best results of MOND.

**Table 3.** Mass Distribution (Luminous Mass \(M_\text{d}\), Total Mass \(M_\text{tot}\) and Dark Mass \(M_\text{dark}\) of the galaxy that is enclosed within the circular orbit) wrt the distance from the center of Galaxy NGC 3198.

| \(r\) / Kpc | \(M_\text{d}/10^4\) Kg | \(M_\text{tot}/10^4\) Kg | \(M_\text{dark}/10^4\) Kg | \(R\) / Kpc | \(M_\text{d}/10^4\) Kg | \(M_\text{tot}/10^4\) Kg | \(M_\text{dark}/10^4\) Kg |
|----------|-----------------|-----------------|-----------------|----------|-----------------|-----------------|-----------------|
| 2.0      | 0.687           | 0.577           | -0.110          | 22.1     | 7.005           | 22.501          | 15.496          |
| 3.0      | 1.012           | 1.327           | 0.315           | 24.1     | 6.901           | 23.816          | 16.915          |
| 4.0      | 1.620           | 2.575           | 0.955           | 26.1     | 6.806           | 25.546          | 18.740          |
| 5.5      | 3.404           | 4.941           | 1.537           | 28.1     | 6.716           | 28.188          | 21.472          |
| 6.0      | 3.698           | 5.287           | 1.589           | 30.2     | 6.628           | 29.966          | 23.338          |
| 7.0      | 5.089           | 6.646           | 1.557           | 32.2     | 6.544           | 32.785          | 26.241          |
| 8.0      | 5.825           | 8.355           | 2.530           | 34.2     | 6.456           | 35.245          | 28.789          |
| 9.0      | 6.346           | 9.350           | 3.003           | 36.2     | 6.392           | 37.606          | 31.214          |
| 10.1     | 6.735           | 10.803          | 4.067           | 38.2     | 6.316           | 39.367          | 33.051          |
| 11.0     | 6.914           | 11.611          | 4.979           | 40.2     | 6.252           | 41.817          | 35.565          |
| 12.1     | 7.093           | 13.649          | 6.556           | 42.1     | 6.191           | 42.404          | 36.212          |
| 14.1     | 7.214           | 16.897          | 9.683           | 44.2     | 6.159           | 45.856          | 39.697          |
| 16.1     | 7.237           | 17.952          | 10.716          | 46.2     | 6.114           | 49.025          | 42.911          |
| 18.1     | 7.191           | 18.503          | 11.312          | 48.2     | 6.072           | 51.418          | 45.346          |
| 20.1     | 7.115           | 20.659          | 13.544          |          |                 |                 |                 |

**Figure 3.** Mass Distribution in Galaxy NGC 3198. Luminous Mass \(M_\text{d}\), Total Mass \(M_\text{tot}\) and Dark Mass \(M_\text{dark}\) of the galaxy that is enclosed within the circular orbit wrt the distance from the center of Galaxy NGC 3198.
2. Euclidean Closed Linear Transformations of Complex Spacetime endow with the Corresponding Metric

2.1. Theoretical Background

In case of Closed Isometric Linear Spacetime Transformation (LSTT), 3D-space is isotropic [12]. So for Relativistic Inertial observers (RIOs), the representation of the non-degenerate inner product in basis $[c\hat{t}, \hat{x}, \hat{y}, \hat{z}]$ is the real matrix

$$g_1 = \begin{bmatrix} g_{100} & 0 & 0 & 0 \\
0 & g_{111} & 0 & 0 \\
0 & 0 & g_{122} & 0 \\
0 & 0 & 0 & g_{133} \end{bmatrix} = g_{111} \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}$$

(35)

where

$$\omega_1^2 = \frac{g_{111}}{g_{100}}$$

(36)

The Universal Speed ($U$) of the corresponding SR is given by the equation

$$U_1^2 = -\frac{c^2}{\omega_1^2}$$

(37)

and the transformation of a contravariant four-vector is

$$dX' = \Lambda_{(\omega_1, \beta)} dX$$

(38)

where

$$\Lambda_{(\omega_1, \beta)} = \gamma_{(\omega_1 \beta)} \begin{bmatrix} 1 & \omega_1^2 \beta_x & \omega_1^2 \beta_y & \omega_1^2 \beta_z \\
-\beta_x & 1 & \omega_1 \beta_z & -\omega_1 \beta_y \\
-\beta_y & -\omega_1 \beta_z & 1 & \omega_1 \beta_x \\
-\beta_z & \omega_1 \beta_y & -\omega_1 \beta_x & 1 \end{bmatrix} = \gamma_{(\omega_1 \beta)} \begin{bmatrix} 1 & \omega_1^2 \beta^T \\
-\beta & I_3 + \omega_1 A_{(\beta)} \end{bmatrix}$$

(39)

$$\beta^i = \frac{dx^i}{dx^0}; \quad \beta = \begin{bmatrix} \beta_x \\
\beta_y \\
\beta_z \end{bmatrix}; \quad A_\gamma = \begin{bmatrix} 0 & \beta_z & -\beta_y \\
-\beta_z & 0 & \beta_x \\
\beta_y & -\beta_x & 0 \end{bmatrix}$$

(40)

and $\gamma$-function is

$$\gamma_{(\delta)} = \frac{1}{\sqrt{1 - \beta^2}}$$

(41)

The specific value $\omega = 0$ ($g_{10}, g_{11}, g_{12} \neq 0$) gives Galilean Transformation (GT) with Infinite Universal Speed ($U$) and the corresponding metric of the spacetime (let us call Galilean metric)

$$g_\Gamma = \begin{bmatrix} g_{100} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}$$

(42)

If the coefficients of metric of time and space have different signs, then $\omega$ is an imaginary number

$$\omega = \pm i$$

(43)

where
\[ \xi_I = \sqrt{\frac{g_{111}}{-g_{100}}} \]  

is positive number. The corresponding SR has real Universal Speed \( U \)

\[ U_I = \frac{1}{\xi_I} c \]  

The specific value \( \omega = \pm i \) or equivalently \( \xi = 1 \) \((g_\omega = -g_\omega)\) gives Vossos Transformation (VT) with Universal Speed

\[ U = c \]  

and the corresponding metric of the spacetime (let us call Vossos metric)

\[ g_B = g_{ii} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g_{ii} \eta \]  

which for \( g_\omega = 1 \) becomes the Lorentz metric. Thus, we have the SR theory of Euclidean Complex Relativistic Mechanics (ECRMs) [14].

In case that the metric depends on the position of the event in spacetime (GR), the Closed LSTT is applied locally, not globally. The limit of vanishing acceleration leads to the corresponding SR. Thus

\[ \lim_{\omega \to 0} g_{111} = g_{111} \]

\[ \lim_{\omega \to 0} g_{00} = \lim_{\omega \to 0} \frac{g_{111}}{g_{00}} = g_{100} = \frac{g_{111}}{\omega^2} \]  

where

\[ \omega^2 = \frac{g_{11}}{g_{00}} \]  

The Local Invariant Speed \( (U) \) of the corresponding GR is given by the equation

\[ U^2 = -\frac{c^2}{\omega^2} \]  

and the transformation of a contravariant four-vector is

\[ dX' = \Lambda_{(\omega, \beta)} dX \]  

where

\[ \Lambda_{(\omega, \beta)} = \gamma_{(\omega, \beta)} \begin{pmatrix} 1 & \omega^2 \beta_x & \omega^2 \beta_y & \omega^2 \beta_z \\ -\beta_x & 1 & \omega \beta_z & -\omega \beta_y \\ -\beta_y & -\omega \beta_z & 1 & \omega \beta_x \\ -\beta_z & \omega \beta_y & -\omega \beta_x & 1 \end{pmatrix} \]  

In case of GR, the coefficients of metric of time and space combined producing \( \omega \) which is a local parameter of GR. Thus, parameter \( \omega \) is a real or an imaginary number.

If the metric coefficients of time and space have different signs, then \( \omega \) is an imaginary number 

\[ \omega = \pm i \xi \]  

where

\[ \xi = \sqrt{\frac{g_{11}}{-g_{00}}} \]  

is positive number. The corresponding GR has real Local Invariant Speed \( (U) \)

\[ U = \frac{1}{\xi} c \]
In addition, any complex *Cartesian Coordinates* (CCs) of the theory may be turned to the corresponding real CCs, in order to be perceived by human senses.

2.2. Relativistic potential around a center of gravity (1st Generalized Schwarzschild potential)

The purpose is the creation of a spacetime metric around a center of gravity that is in accordance with any acceptable SR having Real, Imaginary or Infinite universal speed ($U_I$) (let us call 1st Generalized Schwarzschild metric). Thus, we define a relativistic potential $\Phi$ around a center of gravity (let us call 1st Generalized Schwarzschild potential)

$$\Phi = \frac{U_I^2}{2} \ln \left( 1 - \frac{r_{SI}}{r} \right)$$

where $r$ is the distance between the center of the gravity and the spatial position of the event and $r_{SI}$ is the radius (let us call 1st Generalized Schwarzschild radius)

$$r_{SI} = \frac{2GM}{U_I^2}$$

Thus the above equation combined with (5) and (37) gives

$$r_{SI} = -\omega_I^2 r_S$$

and (34) becomes

$$\Phi = -\frac{c^2}{2\omega_I^2} \ln \left( 1 + \frac{\omega_I^2 r_S}{r} \right) = -\frac{c^2}{2} \frac{r_S}{r} + \ldots = -\frac{GM}{r} + \ldots$$

**Figure 4.** Parametric plot of 1st Generalized Schwarzschild potential ($\Phi$) wrt the distance ($r/r_S$), for different values of characteristic parameter ($\omega_I$) of the specific SR.

We observe that if $\omega_I = \pm i$ (VT), it emerges the original Schwarzschild potential. Moreover, if $\omega_I \to 0$ (GT), we compute
In case that the specific SR. We observe that for $\omega \rightarrow \pm \infty$, the relativistic potential becomes zero–function. The curve with $\omega = 0$ (g. = 0) describes the Relativistic Potential of Newtonian Physics (NP) which corresponds to Galilean metric and Galilean Transformation (GT), while the curve with $\omega = \pm i$ (g. = g.) describes Schwarzschild potential, which corresponds to Vossos metric of flat spacetime and Vossos Transformation (VT).

2.3. Metric of spacetime around a center of gravity (1- Generalized Schwarzschild metric)

The 1- Type of generalized Schwarzschild metric is

\[
dS^2 = dX^T g dX = \begin{bmatrix} cd^t & dr & d\theta & d\phi \end{bmatrix} \begin{bmatrix} -\left(\frac{\xi}{g_{100}}\right) & 0 & 0 & 0 \\
0 & g_{111}e^\theta & 0 & 0 \\
0 & 0 & g_{11}^2r^2 & 0 \\
0 & 0 & 0 & g_{11}^2r^2\sin^2\theta \end{bmatrix} \begin{bmatrix} cd^t \\
 dr \\
 d\theta \\
 d\phi \end{bmatrix}
\]

Where

\[
A = \frac{2}{U^2}\Phi
\]

This combined with Einstein equations in vacuum gives

\[
dS^2 = g_{100}\left(1 - \frac{\xi}{r}\right)c^2dt^2 + \frac{g_{111}}{1 - \frac{\xi}{r}}dr^2 + g_{11}^2 r^2 d\theta^2 + g_{11}^2 r^2\sin^2\theta d\phi^2
\]

Thus, the isotropic form of 1- generalized Schwarzschild metric has

\[
g_{00} = g_{100}\left(1 - \frac{\xi^2}{4\tilde{r}^2}\right)^2; \quad \tilde{g}_{11} = g_{111}\left(1 + \frac{\xi^2}{4\tilde{r}^2}\right)^4
\]

Where

\[
\tilde{r} = \frac{\xi^2}{4}\left(\frac{2\tilde{r}}{\xi^2}\right)\left(1 + \sqrt{1 - \frac{\xi^2}{4\tilde{r}^2}} - 1\right); \quad r = \tilde{r}\left(1 + \frac{\xi^2}{4\tilde{r}^2}\right)^2
\]

In case that $\xi \rightarrow 0$, it emerges $\tilde{r} \rightarrow r \cdot g. \rightarrow g. \rightarrow \tilde{g}_{11} \rightarrow 0$. We also have

\[
\tilde{\xi} = \frac{\xi^2}{4\tilde{r}}\left(1 + \frac{\xi^2}{4\tilde{r}^2}\right)^3
\]

and the Local Invariant speed is
\[
\hat{U} = \left( 1 - \frac{\xi^2 r_s^2}{4r^2} \right)^3 \left( 1 + \frac{\xi^2 r_s^2}{4r^2} \right) c
\]

Besides

\[g_{00} = g_{110} \left( 1 - \frac{\xi^2 r_s^2}{r} \right); \quad g_{rr} = \frac{g_{111}}{1 - \frac{\xi^2 r_s^2}{r}} \]

2.4. Field strength (acceleration) around a center of gravity with 1st Generalized Schwarzschild potential for SR having real, imaginary or infinite universal speed.

From (60), we can calculate the field strength \( g \) (see e.g. [1] p. 230). The field is radial and we define

\[g = \sqrt{g_{111}} \left| \text{grad} \Phi \right| = \sqrt{g_{111}} \frac{d\Phi}{dl} = \sqrt{g_{111}} \frac{d\Phi}{dr} \frac{dr}{dl} \]

where \( l \) is the radial ruler distance (see eq. 64, 59).

\[dl^2 = \frac{g_{111}}{1 + \omega^2 \frac{r_s^2}{r}} dr^2 + g_{111} r^2 d\theta^2 + g_{111} r^2 \sin^2 \theta d\phi^2 \]

So

\[\frac{dr}{dl} = \frac{1}{\sqrt{g_{111}}} \left( 1 + \omega^2 \frac{r_s^2}{r} \right)^{-\frac{1}{2}} \]

and

\[g = \frac{GM}{r^2} \left( 1 + \omega^2 \frac{r_s^2}{r} \right)^{-\frac{1}{2}} \]

In case that \( \omega = i \) (\( g_\omega = g_u \)), we have the SR which corresponds to VT and it emerges the original Schwarzschild metric, potential and field strength.

3. 2nd Generalized Schwarzschild metric and the corresponding Relativistic potential and Field strength and Combination with MOND

3.1. 2nd Generalized Schwarzschild metric and the corresponding Relativistic potential and Field strength

Now, we can modify (60). Thus, we define a new relativistic potential \( \Phi \) around a center of gravity (let us call Modified Generalized Schwarzschild potential) as

\[\Phi = h_{(r)} \frac{-c^2}{2\omega^2} \ln \left( 1 + f_{(r)} \frac{\omega^2 r_s^2}{r} \right) \]

In case that

\[h_{(r)} = 1; \quad f_{(r)} = 1\]

it emerges 1st Generalized Schwarzschild potential. Now, we examine the case that only \( f_{(r)} = 1 \). Let us call this potential as 2nd Generalized Schwarzschild potential

\[\Phi = h_{(r)} \frac{-c^2}{2\omega^2} \ln \left( 1 + \frac{\omega^2 r_s^2}{r} \right) \]

We demand the above relativistic potential having value equal to the 1st Generalized Schwarzschild potential at the distance of 1st Generalized Schwarzschild radius. Thus, we have
Taking into account
So
or
This
We may define
Thus, we have
The coefficients \(( -g^{00} )\) and \(g^{1\alpha} \) may be written as
Thus, we have
We may define
So, we obtain the formula of the original Schwarzschild metric except for containing \(A\) and \(B\).
This metric combined with Einstein equations in vacuum (see e.g. [1] p 229, eq 11.8) gives
or
So
Taking into account (79), we have
\[ B = \ln \left( \frac{e^{C_i}}{-g_{100}g_{111} \left( 1 + \frac{\omega_i^2 r_S}{r} \right)^{h_{(r)}}} \right) \]  

(88)

and

\[ g_{rr} = \frac{e^{C_i}}{-g_{100} \left( 1 + \frac{\omega_i^2 r_S}{r} \right)^{h_{(r)}}} \]  

(89)

At infinite distance, we have metric of the corresponding SR. So

\[ g_{111} = \frac{e^{C_i}}{-g_{100}} \]  

and

\[ g_{rr} = \frac{g_{111}}{\left( 1 + \frac{\omega_i^2 r_S}{r} \right)^{h_{(r)}}} \]  

(90)

Thus, the 2-\textit{generalized Schwarzschild metric} in spherical coordinates is

\[ dS^2 = g_{100} \left( 1 + \frac{\omega_i^2 r_S}{r} \right)^{h_{(r)}} c^2 dt^2 \frac{g_{111}}{\left( 1 + \frac{\omega_i^2 r_S}{r} \right)^{h_{(r)}}} dr^2 + g_{111} r^2 d\theta^2 + g_{111} r^2 \sin^2 \theta d\phi^2 \]  

(91)

with spatial part

\[ dl^2 = \frac{g_{111}}{\left( 1 + \frac{\omega_i^2 r_S}{r} \right)^{h_{(r)}}} dr^2 + g_{111} r^2 d\theta^2 + g_{111} r^2 \sin^2 \theta d\phi^2 \]  

(92)

Now, we can calculate this radial field strength defining

\[ \ddot{g} = -\sqrt{g_{111}} \nabla \Phi = -\sqrt{g_{111}} \frac{d\Phi}{dl} \ddot{r} = -\sqrt{g_{111}} \frac{d\Phi}{dr} \frac{dr}{dl} \ddot{r} \]  

(93)

and

\[ g = \sqrt{g_{111}} \frac{d\Phi}{dr} \frac{dr}{dl} \]  

(94)

The positive value of field strength means gravity, while negative value means antigravity. Moreover, it is

\[ \frac{dr}{dl} = \frac{1}{\sqrt{g_{111}}} \left( 1 + \frac{\omega_i^2 r_S}{r} \right)^{\frac{h_{(r)}}{2}} \]  

(95)

and

\[ \frac{d\Phi}{dr} = -\frac{c^2}{2\omega_i^2} \left( \frac{\omega_i^2 r_S}{r^2} \frac{h_{(r)}}{1 + \frac{\omega_i^2 r_S}{r}} + \frac{dh}{dr} \ln \left( 1 + \frac{\omega_i^2 r_S}{r} \right) \right) \]  

(96)

So
\[
g = \frac{-c^2}{2\omega_i^2} \left( -\frac{\omega_i^2 r_S h_{(r)}}{r^3} + \left(1 + \frac{\omega_i^2 r_S}{r} \right)^{\frac{h_{(r)}}{2}} \frac{dh}{dr} \left(1 + \frac{\omega_i^2 r_S}{r} \right)^{\frac{h_{(r)}}{2}} \ln \left(1 + \frac{\omega_i^2 r_S}{r} \right) \right) \tag{97}
\]

or
\[
g = \frac{GM}{r^2} h_{(r)} \left(1 + \frac{\omega_i^2 r_S}{r} \right)^{\frac{h_{(r)}}{2}} - \frac{c^2}{2\omega_i^2} \frac{dh}{dr} \left(1 + \frac{\omega_i^2 r_S}{r} \right)^{\frac{h_{(r)}}{2}} \ln \left(1 + \frac{\omega_i^2 r_S}{r} \right) \tag{98}
\]

In case of UCM, it is
\[
\nu^2 = \frac{GM}{r} h_{(r)} \left(1 + \frac{\omega_i^2 r_S}{r} \right)^{\frac{h_{(r)}}{2}} - \frac{c^2}{2\omega_i^2} \frac{dh}{dr} \left(1 + \frac{\omega_i^2 r_S}{r} \right)^{\frac{h_{(r)}}{2}} \ln \left(1 + \frac{\omega_i^2 r_S}{r} \right) \tag{99}
\]

3.2. Combination of Galilean-Newtonian 2- Generalized Schwarzschild metric with MOND

We firstly examine the case which flat spacetime follows Galilean Metric \((\omega \to 0)\). Thus
\[
\Phi = -\frac{GM}{r} h_{(r)} \tag{100}
\]

or
\[
\Phi = -\frac{GM}{r} \sqrt{4d_0 r_0} h_{(r)} = -\frac{2}{r_0} \sqrt{G \mu_0 h_{(r)}} = -\frac{2\nu^2}{r_0} h_{(r)} \tag{101}
\]

Besides
\[
g = \frac{GM}{r^2} h_{(r)} - \frac{GM}{r} \frac{dh}{dr} = \frac{GM}{r} \left( h_{(r)} - \frac{dh}{dr} \right) \tag{102}
\]

and
\[
\nu^2 = \frac{GM}{r} h_{(r)} - GM \frac{dh}{dr} \tag{103}
\]

Using Interpolating function of MOND and (11), it emerges
\[
\frac{1}{\mu_{(r)}} = h_{(r)} - r \frac{dh}{dr} \tag{104}
\]

Thus, if we know function \(h_{(r)}\), we may calculate Interpolating function \(\mu_{(r)}\), using the above equation. But, if we know Interpolating function \(\mu_{(r)}\), we may calculate \(h_{(r)}\), solving the above differential equation and it emerges
\[
\frac{h_{(r)}}{r} = \int \frac{dr}{r^2 \mu_{(r)}} \tag{105}
\]

So, calculating the above integral, we easily obtain function \(h_{(r)}\). The value of integration constant is obtained using the condition (76). Besides (105) may be written as
\[
GM \frac{h_{(r)}}{r} = -\int \frac{GM dr}{r^2 \mu_{(r)}} = -\int gdr \tag{106}
\]

It emerges that the work we need to pay for moving one mass unit from a point A with distance \(r\) from the center of gravity to another point B with larger distance \(r\), (the Potential Difference between point A and point B), is
\[ w = \Delta V = V_{(r)} - V_{(r)} = \int_0^r g dr = -GM \frac{h_{(r)}}{r_1} + GM \frac{h_{(r)}}{r} \] 

(107)

We observe that

\[ V_{(r)} = \Phi = -\frac{GM}{r} h_{(r)} \] 

(108)

Finally, from (91) we have the (Modified) Galilean metric

\[ dS^2 = -c^2 dt^2 + 0 dr^2 + 0 r^2 d\theta^2 + 0 r^2 \sin^2 \theta d\phi^2 \] 

(109)

This is independent by distance from the center of gravity.

3.2.1 Combination of Galilean-Newtonian 2- Generalized Schwarzschild metric with MOND Simple \( \mu \)

Replacing (19) to (105), we obtain

\[ \frac{h_{(r)}}{r} = -\frac{1}{2r_0} \int \frac{1}{2r_0} \left( 1 + \left( \frac{r}{r_0} \right)^2 \right) dr \] 

(110)

or

\[ \frac{h_{(r)}}{r} = -\frac{1}{2r_0} \int \frac{1}{r_0} \left( 1 + \left( \frac{r}{r_0} \right)^2 \right) dr \] 

(111)

and using (32), we have

\[ h_{(r)} = \frac{1}{2} \left( 1 + \left( \frac{r}{r_0} \right)^2 \right) - \frac{r}{r_0} \text{ArcSinh} \left( \frac{r}{r_0} \right) \] 

(112)

Thus

\[ \frac{dh}{dr} = C_1 - \frac{1}{2r_0} \text{ArcSinh} \left( \frac{r}{r_0} \right) \] 

(113)

Because 1- Generalized Schwarzschild radius is equal to the zero in Galilean-Newtonian Physics, the value of integration constant \( C_1 \) is calculated as following

\[ h_{(-0^2 \gamma_0)} = \frac{1}{2} \left( 1 + \left( \frac{-\omega_0^2 r_0^2}{r_0} \right)^2 \right) + \frac{\omega_0^2 r_0^2}{r_0} \text{ArcSinh} \left( \frac{-\omega_0^2 r_0^2}{r_0} \right) \] 

Thus

\[ h_{(-0^2 \gamma_0)} = \frac{1}{2r_0 \omega_0^2} \left( -1 + \left( \frac{-\omega_0^2 r_0^2}{r_0} \right)^2 \right) + \frac{\omega_0^2 r_0^2}{r_0} \text{ArcSinh} \left( \frac{-\omega_0^2 r_0^2}{r_0} \right) \] 

(114)

Taking the limit \( \omega_0 \to 0 \), it emerges

\[ C_1 = 0 \] 

(115)

So

\[ h_{(r)} = \frac{1}{2} \left( 1 + \left( \frac{r}{r_0} \right)^2 \right) - \frac{r}{r_0} \text{ArcSinh} \left( \frac{r}{r_0} \right) \] 

(116)

and
In Figure 5, we show the plot of function $h$ wrt the distance $(r/r_0)$ for the Combination of Galilean-Newtonian 2- Generalized Schwarzschild metric with MOND Simple $\mu$.

We observe that function $h$ is decrescent, its graph cuts the distance axis at $r \approx 2.2 r_0$ and goes to the minus infinite ($-\infty$) at infinite distance from the center of gravity.

Now, using (101) we calculate the corresponding Relativistic Potential

$$\Phi = -\frac{\nu_\infty^2}{r} \left( 1 + \sqrt{1 + \left( \frac{r}{r_0} \right)^2} - \frac{r}{r_0} \text{ArcSinh} \left( \frac{r}{r_0} \right) \right)$$

In Figure 6, we show the plot of Relativistic Potential $(\Phi/\nu_\infty^2)$ wrt the distance $(r/r_0)$ for the Combination of Galilean-Newtonian 2- Generalized Schwarzschild metric with MOND Simple $\mu$.

We observe that this Modified Relativistic Potential $\Phi$ is increscent, its graph cuts the distance axis at $r \approx 2.3 r_0$ and goes slightly to the plus infinite ($+\infty$) at infinite distance from the center of gravity, in contrast to the pure Newtonian Potential that goes to zero.
3.2 Combination of Galilean-Newtonian 2nd Generalized Schwarzschild metric with MOND

Replacing (22) to (105) we have results

$$\frac{h_{(r)}}{r} = -\int \frac{1}{\sqrt{2}r^2} \sqrt{1 + \frac{1}{4} \left( \frac{r}{r_0} \right)^4} \, dr$$  \hspace{1cm} (119)$$

or

$$\frac{h_{(r)}}{r} = -\frac{1}{\sqrt{2}r_0} \int \frac{1}{\sqrt{\frac{r}{r_0}}} \sqrt{1 + \frac{1}{4} \left( \frac{r}{r_0} \right)^4} \left( \frac{r}{r_0} \right) \, d\left( \frac{r}{r_0} \right)$$  \hspace{1cm} (120)$$

Unfortunately, the integral (34) has not yet analytical solution. So, we cannot go further our analysis.

3.3. Combination of Lorentzian-Einsteinian 2nd Generalized Schwarzschild metric with MOND

In case of Lorentzian-Einsteinian Physics the 2nd generalized Schwarzschild metric in spherical coordinates (91) becomes

$$ds^2 = -\left( 1 - \frac{r_S}{r} \right)^{h_{(r)}} c^2 \, dt^2 + \frac{1}{\left( 1 - \frac{r_S}{r} \right)^{h_{(r)}}} \left( dr + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \right)$$  \hspace{1cm} (121)$$

2nd Generalized Schwarzschild potential (75) is written as

![Figure 6. Plot of Relativistic Potential (\Phi/\nu, \xi') wrt the distance (r/r_0) for the Combination of Galilean-Newtonian 2nd Generalized Schwarzschild metric with MOND Simple \mu.](image)
because of this additional term, we observe that this

\[ \Phi = h_{(r)} \frac{c^2}{2} \ln \left( 1 - \frac{r}{r_0} \right) \] (122)

and field strength

\[ g = \frac{GM}{r^2} h_{(r)} \left( 1 - \frac{r}{r_0} \right)^{\frac{h_{(r)}}{2} - 1} + \frac{c^2}{2} \frac{d h}{d r} \left( 1 - \frac{r}{r_0} \right)^{\frac{h_{(r)}}{2} - 1} \ln \left( 1 - \frac{r}{r_0} \right) \] (123)

as well as \( \nu^2 \) in UCM

\[ \nu^2 = \frac{GM}{r} h_{(r)} \left( 1 - \frac{r}{r_0} \right)^{\frac{h_{(r)}}{2} - 1} + \frac{c^2}{2} r \frac{d h}{d r} \left( 1 - \frac{r}{r_0} \right)^{\frac{h_{(r)}}{2} - 1} \ln \left( 1 - \frac{r}{r_0} \right) \] (124)

The experimental data have been extracted using the Relativistic Doppler Shift and the gravitational red shift of Classic Relativity (CR)

\[ f_E = f_S \gamma (1 - \beta \cos \phi) \]

\[ \frac{f_S}{f_E} = \sqrt{\frac{g_{00(S)}}{g_{00(E)}}} \] (125) (126)

where \( f \) and \( f_s \) is the frequency of the radiation which has emitted by the Star, measured on the Star and on the Earth respectively and \( \beta \) is the velocity factor of the Star wrt the Earth and \( \phi \) is the angle between the position vector \( (\vec{r}) \) and the velocity of the Star wrt Earth \( (\vec{v}) \).

In case of Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple \( \mu \), it is

\[ g_{00} = -\left( 1 - \frac{r}{r_0} \right)^{\frac{h_{(r)}}{r}} \] (127)

but there is not significant change to the corresponding values (see Table 5). So, the experimental data may be used without problem. Moreover, for \( h > 0 \), there is time dilation of the accelerated observer wrt unmoved RIO at infinite distance form the center of gravity, while time contraction happens for \( h < 0 \).

3.3.1 Combination of Lorentzian-Einsteinian 2- Type of Generalized Schwarzschild metric with MOND Simple \( \mu \)

We firstly use function \( h_{(r)} \) that has produced by the combination of Galilean-Newtonian 2- Type of Generalized Schwarzschild metric with MOND Simple \( \mu \). We simply change integration constant \( C \), in order to correspond to Lorentzian-Einsteinian Physics. Thus, if we replace \( \omega_r = -1 \) to (114), we obtain

\[ C_i = -\frac{1}{2r_0} \left[ -1 + 1 + \left( \frac{r}{r_0} \right) + \frac{r}{r_0} \right] - \frac{r}{r_0} \] (128)

Moreover (112) becomes

\[ h_{(r)} = \frac{1}{2} \left[ \frac{r}{r_0} + 1 + \left( \frac{r}{r_0} \right)^{-2} - \frac{r}{r_0} \right] - \frac{r}{r_0} \] (129)

Due to (26), the above equation may be written as

\[ h_{(r)} = \frac{1}{2} \left[ \frac{1 + 4 \beta_0^2}{4 \beta_0^2} \right] \frac{\text{ArcSinh} \left( 4 \beta_0^2 \right)}{\text{ArcSinh} \left( 4 \beta_0^2 \right)} + 1 + \left( \frac{r}{r_0} \right)^2 - \frac{r}{r_0} \] (130)

We observe that this function \( h_{(r)} \) has a different value than the corresponding Galilean-Newtonian because of this additional term.
the distance of gravitation of the star which is very gravitational bleu shift to the radiation emitted by the star surface, because there exist the self-gravitation of the star which is very stronger. Finally, the values at distance 13.8 Mpc which is the distance of Galaxy NGC 3198 from Earth give us the image of what happens at extremely large distances (compare with the corresponding values in Table 1).

\[
\frac{1 + \text{ArcSinh}(4\beta_w^2)}{4\beta_w^2} - \frac{\sqrt{1 + 16\beta_w^4}}{r} \quad (131)
\]

Besides (113) becomes

\[
\frac{dh}{dr} = -\frac{1}{2} \left( -\frac{1}{r_s} + \frac{1}{r_s} \sqrt{1 + \left(\frac{r_s}{r_0}\right)^2} - \frac{1}{r_0} \text{ArcSinh}\left(\frac{r_s}{r_0}\right) + \frac{1}{r_0} \text{ArcSinh}\left(\frac{r}{r_0}\right) \right) \quad (132)
\]

Due to (26) the above equation may be written as

\[
\frac{dh}{dr} = -\frac{1}{2} \left( -\frac{1}{r_s} + \frac{1}{r_s} \sqrt{1 + 16\beta_w^4} - \frac{1}{r_0} \text{ArcSinh}(4\beta_w^2) + \frac{1}{r_0} \text{ArcSinh}\left(\frac{r}{r_0}\right) \right) \quad (133)
\]

3.3.2 Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Standard Interpolating Function

The Standard \( \mu \) in Milgrom’s Law gives the best results, explaining the rotation curves in galaxies. Unfortunately, the integral (34) has not yet analytical solution and we cannot go further our analysis.

4. Experimental Validation

4.1. The Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple Interpolating Function in Galaxy NGC 3198

The values of Circular Velocities (experimental \( V_\star \) and calculated by the Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple \( \mu \), \( V_{\text{sim,Lor}} \)), the Luminous Mass of the galaxy that is enclosed within the circular orbit (\( M \)), the corresponding Schwarzschild radius (\( r_s \)), Milgrom radius (\( r_0 \)), the value of function \( h \) and the value of time coefficient of metric (\( g_\star \)) wrt the distance from the center of Galaxy NGC 3198, are contained in Table 5 (see [5] p 2). The Circular Velocities \( V_{\text{sim,Lor}} \) have been calculated using (124) and the values of time coefficient of metric (\( g_\star \)) using (127).

In Figure 7, we show the plot of function \( h \), wrt the distance from the center of NGC 3198 for the Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple \( \mu \).

In Figure 8, we show the plot of Relativistic Potential \( (\Phi/V_{\text{sim,Lor}}) \) wrt the distance from the center of NGC 3198 (\( r/r_s \)) for the Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple \( \mu \), as well as the original Schwarzschild potential.

In Figure 9, we show the plot of coefficients of metric \( g_\star \), \( g_w \) wrt the distance from the center of NGC 3198 for the Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple \( \mu \).

We observe that in case of Galaxy NGC 3198, the Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple \( \mu \) gives almost the same Circular Velocities as those calculated by Simple \( \mu \) in Milgrom’s Law (16). It is noted that the relative error of experimental Circular Velocities is \( (\Delta V_\star) \approx 8\% \). In addition, the values of time coefficient metric (\( g_\star \)) show that it emerges time contraction at distances larger than 3.0 Kpc. this does not produce gravitational bleu shift to the radiation emitted by the star surface, because there exist the self-gravitation of the star which is very-very stronger. Finally, the values at distance 13.8 Mpc which is the distance of Galaxy NGC 3198 from Earth give us the image of what happens at extremely large distances (compare with the corresponding values in Table 1).
| $r$ (Kpc) | $M$ /10$^{-11}$ kg | $V_{\text{c}}$ /Km s$^{-1}$ | $r_{1}$ /10$^{-1}$ m | $r_{2}$ /10$^{-1}$ m | $r_{3}$ /10$^{-1}$ m | $h$ /10$^{-1}$ m | $g_{m}$ /10$^{-1}$ m | $V_{\text{exp}}$ /Km s$^{-1}$ |
|---------|-------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2.0     | 0.687             | 79.0            | 0.1020          | 0.3091          | 0.1768          | -0.9999999708   | 109.6           |
| 3.0     | 1.012             | 97.8            | 0.1502          | 0.3750          | -0.1867         | -1.0000000303   | 115.6           |
| 4.0     | 1.620             | 118.0           | 0.2406          | 0.4746          | -0.2964         | -1.00000057     | 128.8           |
| 5.5     | 3.404             | 139.4           | 0.5056          | 0.6880          | -0.1856         | -1.0000005553   | 156.6           |
| 6.0     | 3.698             | 144.2           | 0.5493          | 0.7171          | -0.1863         | -1.000000605    | 163.4           |
| 7.0     | 5.089             | 143.3           | 0.7558          | 0.8412          | -0.2690         | -1.000000941    | 171.8           |
| 8.0     | 5.825             | 150.3           | 0.8652          | 0.9000          | -0.4181         | -1.000001465    | 175.7           |
| 9.0     | 6.346             | 149.9           | 0.9426          | 0.9394          | -0.6070         | -1.000002060    | 177.3           |
| 10.1    | 6.735             | 152.1           | 1.0003          | 0.9677          | -0.8507         | -1.000002731    | 177.6           |
| 11.0    | 6.914             | 151.1           | 1.0269          | 0.9805          | -1.0825         | -1.000003275    | 176.9           |
| 12.1    | 7.093             | 156.2           | 1.0534          | 0.9931          | -1.3798         | -1.000003893    | 176.1           |
| 14.1    | 7.214             | 161.0           | 1.0715          | 1.0016          | -1.9952         | -1.000004914    | 173.8           |
| 16.1    | 7.237             | 155.3           | 1.0748          | 1.0031          | -2.6769         | -1.000005791    | 171.5           |
| 18.1    | 7.191             | 148.7           | 1.0680          | 0.9999          | -3.4247         | -1.000006549    | 169.3           |
| 20.1    | 7.115             | 149.1           | 1.0567          | 0.9946          | -4.2299         | -1.000007207    | 167.3           |
| 22.1    | 7.005             | 148.4           | 1.0404          | 0.9869          | -5.0999         | -1.000007781    | 165.4           |
| 24.1    | 6.901             | 146.2           | 1.0250          | 0.9796          | -6.0130         | -1.000008288    | 163.7           |
| 26.1    | 6.806             | 145.5           | 1.0108          | 0.9728          | -6.9645         | -1.000008741    | 162.3           |
| 28.1    | 6.716             | 147.3           | 0.9975          | 0.9664          | -7.9533         | -1.000009149    | 161.0           |
| 30.2    | 6.628             | 146.5           | 0.9844          | 0.9600          | -9.0290         | -1.000009538    | 159.8           |
| 32.2    | 6.544             | 148.4           | 0.9719          | 0.9539          | -10.0944        | -1.000009874    | 158.7           |
| 34.2    | 6.456             | 149.3           | 0.9589          | 0.9475          | -11.2033        | -1.000010180    | 157.7           |
| 36.2    | 6.392             | 149.9           | 0.9494          | 0.9428          | -12.3175        | -1.000010469    | 156.9           |
| 38.2    | 6.316             | 149.3           | 0.9380          | 0.9371          | -13.4845        | -1.000010731    | 156.0           |
| 40.2    | 6.252             | 150.0           | 0.9286          | 0.9324          | -14.6652        | -1.000010978    | 155.3           |
| 42.1    | 6.191             | 147.6           | 0.9196          | 0.9278          | -15.8182        | -1.000011197    | 154.6           |
| 44.2    | 6.159             | 149.8           | 0.9148          | 0.9254          | -17.0560        | -1.000011440    | 154.2           |
| 46.2    | 6.114             | 151.5           | 0.9080          | 0.9220          | -18.2900        | -1.000011650    | 153.6           |
| 48.2    | 6.072             | 151.9           | 0.9018          | 0.9189          | -19.5417        | -1.000011849    | 153.2           |
| 13800   | 6.763             | -               | 1.0045          | 0.9697          | -17.7411        | -1.0000041848   | 152.6           |
Function $h$ for the Combination of Lorentzian-Einsteinian 2nd type of Generalized Schwarzschild potential with Simple $\tilde{\mu}$ in NGC 3198

Figure 7. Plot of function $h$, wrt the distance ($r$) from the center of NGC 3198 for the Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple $\mu$.

Figure 8. Plot of Relativistic Potential ($\Phi/\upsilon^2$) wrt the distance ($r/r_0$) from the center of NGC 3198 for the Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple Interpolating Function (M), as well as the original Schwarzschild potential (S).
4.2. The Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with MOND Simple $\mu$ in the Solar System

In order to find out what is the effect of the modification at medium mass and size systems, we now examine our Solar System.

The mean values of Rotational Velocities are contained in Table 6 and the Mass of the Solar System that is enclosed within the orbit wrt the mean distance the planet from the Sun, are contained in Table 7 (see [13] p 14-3). The Circular Velocities $V_{\text{circ}}$, have been calculated using (124) and the values of time coefficient of metric ($g_{00}$) using (127). In addition, the corresponding Rotation Curves and Mass Distribution in the Solar System are shown in Figure 10 and Figure 11 respectively.

In Figure 12, we show the plot of function $h$, wrt the distance of the planet from the Sun for the Combination of Lorentzian-Einsteinian 2- Type of Generalized Schwarzschild metric with MOND Simple $\mu$.

In Figure 13, we show the plot of coefficients of metric $g_{00}$, $g_{rr}$ the distance from the center of NGC 3198 for the Combination of Lorentzian-Einsteinian 2- Generalized Schwarzschild metric with Simple $\mu$ in NGC 3198.

Finally, the values at distance 13.8 Mpc which is the distance of Galaxy NGC 3198 from Earth give us...
Table 5. Rotational Velocities (experimental $V_\infty$ and calculated by the Combination of Lorentzian-Einsteinian $2\textsuperscript{nd}$ Type of Generalized Schwarzschild metric with MOND Simple Interpolating Function $V_{\text{simp,Lor}}$), the Luminous Mass of the Solar System that is enclosed within the circular orbit ($M_d$), the corresponding Schwarzschild radius ($r_S$), Milgrom radius ($r_0$), the value of function $h$ and the value of time coefficient of metric ($g_{00}$) wrt the mean distance from the Sun.

| Name      | $r$ / ua | $M_d$ / $10^2$ kg | $V_{\text{Schwar}}$ / Km s$^{-1}$ | $h$                      | $g_{00}$       | $V_{\text{simp,Lor}}$ / Km s$^{-1}$ |
|-----------|----------|-------------------|---------------------------------|--------------------------|---------------|-----------------------------------|
| Sun Surface | 0.00465  | 1,989,100         | 436.747                         | 0.9999999999991         | -0.9999957553 | 436.747                           |
| Mercury    | 0.38710  | 1,989,100.0000    | 47.880                          | 0.99999999976629        | -0.999999490  | 47.880                            |
| Venus      | 0.72333  | 1,989,100.3302    | 35.027                          | 0.99999999900082        | -0.999999727  | 35.027                            |
| Earth      | 1.00000  | 1,989,105.1992    | 29.790                          | 0.9999999781689         | -0.999999803  | 29.790                            |
| Mars       | 1.52369  | 1,989,111.1715    | 24.134                          | 0.9999999507762         | -0.999999870  | 24.134                            |
| Jupiter    | 5.20283  | 1,989,111.8134    | 13.060                          | 0.9999994561830         | -0.999999962  | 13.060                            |
| Saturn     | 9.53876  | 1,991,010.6134    | 9.650                           | 0.9999981701195         | -0.999999979  | 9.650                             |
| Uranus     | 19.19139 | 1,991,579.1134    | 6.804                           | 0.9999925682555         | -0.999999990  | 6.804                             |
| Neptune    | 30.06107 | 1,991,665.7384    | 5.437                           | 0.999981262519          | -0.999999993  | 5.437                             |
| Pluto      | 39.52940 | 1,991,768.5184    | 4.741                           | 0.999684583836          | 0.999999957   | 4.741                             |
| NGC 3198   | 2.846×10$^{12}$ | 1,991,768.5334  | 0.0177                          | -8,174,346,104.103      | -1,0000000000 | 0.00000                          |

Figure 10. Rotation Curves in the Solar System. Rotational Velocities (calculated by Schwarzschild field strength $V_\infty$, and by the Combination of Lorentzian-Einsteinian $2\textsuperscript{nd}$ Generalized Schwarzschild metric with MOND Simple Interpolating Function $V_{\text{simp,Lor}}$) wrt the distance from the Sun (ua=astronomical unit).
**Figure 11.** Mass Distribution in the Solar System. Mass $M_d$ that is enclosed within the orbit of the planet wrt the distance from the Sun (ua=astronomical unit).

**Figure 12.** Plot of function $h$, wrt the distance of the planet ($r$) from the Sun, for the Combination of Lorentzian-Einsteinian 2$^nd$ type of Generalized Schwarzschild potential with Simple $\mu$ in Solar System.
5. Conclusion

A new modified relativistic potential (2- Generalized Schwarzschild potential) is obtained. This describes the gravitational interaction at any distance and for any metric of flat spacetime (any theory of physics). In case of Galilean-Newtonian physics, the Modified Newtonian Dynamics (MOND) has already been developed, explaining the Rotation Curves in many Galaxies, using suitable Interpolating function ($\mu$) in Milgrom’s Law. Solving suitable integral, we may obtain the corresponding new Interpolating function $h$, that is contained in 2- Generalized Schwarzschild potential and metric. In this paper, we present the results of simple interpolating function, although the results of standard interpolating function are better. Unfortunately, we have not yet analytical solution of the corresponding integral, but also a completely new $h$, may be proposed.

We then studied the case of flat spacetime with Lorentz metric (Minkowski space), because the experimental data have been extracted using the Relativistic Doppler Shift and the gravitational red shift of Lorentz-Einsteinian Relativity. We use $h$, which has developed by Galilean-Newtonian physics by adding suitable term, in order to achieve Schwarzschild potential near to Schwarzschild radius. The experimental validation was performed on the Rotation Curves of both the Galaxy NGC 3198 and the Solar system.

This new modified relativistic potential and the corresponding metric of spacetime have been obtained by the light of Euclidean Closed Linear Transformations of Complex Spacetime endowed with the Corresponding Metric. Of course, may also be applied by scientists who prefer the hyperbolic geometry of Classic Relativity (CR).

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