Conformal supergravity in three dimensions: new off-shell formulation

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ABSTRACT: We propose a new off-shell formulation for $\mathcal{N}$-extended conformal supergravity in three spacetime dimensions. Our construction is based on the gauging of the $\mathcal{N}$-extended superconformal algebra in superspace. Covariant constraints are imposed such that the algebra of covariant derivatives is given in terms of a single curvature superfield which turns out to be the super Cotton tensor. An immediate corollary of this construction is that the curved superspace is conformally flat if and only if the super Cotton tensor vanishes. Upon degauging of certain local symmetries, our formulation is shown to reduce to the conventional one with the local structure group $\text{SL}(2,\mathbb{R}) \times \text{SO}(\mathcal{N})$.

KEYWORDS: Extended Supersymmetry, Superspaces, Supergravity Models

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1 Introduction

Inspired by the construction of topologically massive $\mathcal{N} = 1$ supergravity in three dimensions (3D) [1, 2], conformal supergravity theories in 3D were formulated as supersymmetric Chern-Simons theories for $\mathcal{N} = 1$ [3], $\mathcal{N} = 2$ [4],¹ and finally for arbitrary $\mathcal{N}$ [5, 6].² The constructions in [3–6] are based on the gauging of the $\mathcal{N}$-extended superconformal algebra

¹In the $\mathcal{N} = 1$ case, the superconformal tensor calculus was independently developed in [7, 8]. Early superspace approaches to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravity theories were given in [9–12].
²We are grateful to Jim Gates for bringing ref. [6] to our attention.
in ordinary spacetime. The important point is that the formulation for extended conformal supergravity given in [5, 6] is on-shell for \( N > 2 \). This means that alternative approaches are required if one is interested in deriving off-shell actions for extended conformal supergravity, especially in the presence of matter.

In 1995 Howe et al. [13] proposed a curved superspace geometry with local structure group \( \text{SL}(2, \mathbb{R}) \times \text{SO}(N) \), which is suitable for the description of off-shell \( N \)-extended conformal supergravity in three dimensions.\(^3\) Specifically, ref. [13] postulated the superspace constraints, determined all components of the superspace torsion of dimension-1, and identified the component \( N \)-extended Weyl supermultiplet. At the same time, some crucial elements of the formalism (including the explicit structure of super-Weyl transformations and the solution of the dimension-3/2 and dimension-2 Bianchi identities) did not appear in [13]. The geometry of \( N \)-extended conformal supergravity has recently been fully developed in [16]\(^4\) and applied to the construction of general supergravity-matter couplings in the cases \( N \leq 4 \) (the simplest extended case \( N = 2 \) was studied in more detail in [20]).

It turns out that the problem of constructing off-shell superspace actions for pure extended conformal supergravity theories is rather nontrivial. The action for \( N = 1 \) conformal supergravity can readily be derived in terms of the superfield connection as a superspace integral [10–12] (although the results in [10–12] are incomplete, and the conformal supergravity action has only recently been given in [21]). However, such a construction becomes impossible starting from \( N = 2 \).\(^5\) As discussed in [21], this is because (i) the spinor and vector sectors of the superfield connection have positive dimension equal to 1/2 and 1 respectively; and (ii) the dimension of the full superspace measure is \( (N - 3) \). This implies that it is not possible to construct contributions to the action that are cubic in the superfield connection for \( N \geq 2 \).

Nevertheless, it was argued by two of us [21] that \( N \)-extended conformal supergravity can be realized in terms of the off-shell Weyl supermultiplet [13] and the associated curved superspace geometry [13, 16]. Such a realization was explicitly worked out in [21] for the case \( N = 1 \), and a general method of constructing conformal supergravity actions for \( N > 1 \) was outlined. It should be pointed out that the approach of [21] is a generalization of the superform formulation for the linear multiplet in four-dimensional \( N = 2 \) conformal supergravity given in [22]. Both works [21, 22] make use of the superform approach for the construction of supersymmetric invariants [23–26], also known as the ectoplasm formalism [24–26].

It is worth recalling the method sketched in [21]. Let \( \mathcal{D}_A = \left( \mathcal{D}_a, \mathcal{D}_\alpha^I \right) \) be the superspace covariant derivatives, with \( I = 1, \ldots, N \), which describe the off-shell \( N \)-extended Weyl supermultiplet [13, 16]. Following the conventions of [16], one should start with a two-

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\(^3\)This construction is a natural generalization of Howe’s superspace formulation for 4D \( N \)-extended conformal supergravity in four dimensions [14, 15].

\(^4\)The special cases of \( N = 8 \) and \( N = 16 \) conformal supergravity theories were independently worked out in [17, 18] and [19] respectively.

\(^5\)If a prepotential formulation is available, the conformal supergravity action may be written as a superspace integral in terms of the prepotentials.
parameter deformation of the vector covariant derivative

$$D_{\alpha\beta} \rightarrow D_{\alpha\beta} = D_{\alpha\beta} + \lambda S_{\alpha\beta} + \rho C_{\alpha\beta}^{KL} N_{KL},$$  \hspace{1cm} (1.1)

where $\lambda$ and $\rho$ are real parameters, and $S$ and $C_{\alpha\beta}^{KL}$ are certain dimension-1 torsion tensors. The deformed covariant derivatives $D_A = (D_a, D^I_{\alpha}) := (D_a, D^I_{\alpha})$ obey the algebra

$$[D_A, D_B] = -T_{AB}^C D_C - \frac{1}{2} R_{AB}^{cd} M_{cd} - \frac{1}{2} R_{AB}^{KL} N_{KL},$$  \hspace{1cm} (1.2)

with $T_{AB}^C$ the torsion, $R_{AB}^{cd}$ the Lorentz curvature and $R_{AB}^{KL}$ the $SO(N)$ curvature.\footnote{Our conventions for sign of the torsion, curvatures and connections differ from those in \cite{16, 21}.}

As a next stage, one has to consider the superform equation

$$d\Sigma = \frac{1}{2} R^{ab} \wedge R_{ab} + \frac{\kappa}{2} R^{IJ} \wedge R_{IJ},$$  \hspace{1cm} (1.3)

with $\kappa$ a real parameter, and look for two solutions $\Sigma_T$ and $\Sigma_{CS}$. Here $\Sigma_T$ is a three-form constructed in terms of the torsion and curvature tensors and their covariant derivatives, while $\Sigma_{CS}$ is a standard Chern-Simons three-form. Now, the three-form $\Sigma := \Sigma_T - \Sigma_{CS}$ has the following properties (i) $\Sigma$ is closed; and (ii) $\Sigma$ is a polynomial in two variables $\lambda$ and $\rho$. By differentiating $\Sigma$ with respect to $\lambda$ and $\rho$, we will generate a number of closed three-forms. Finally, one has to look for a linear combination $\mathfrak{J}$ of these closed three-forms, which is super-Weyl invariant modulo exact contributions. The parameter $\kappa$ is expected to be fixed by this requirement. It is also expected that $\mathfrak{J}$ is independent of $\lambda$ and $\rho$, due to its uniqueness. The closed three-form $\mathfrak{J}$ generates the action for $\mathcal{N}$-extended conformal supergravity.

In the previous paper \cite{21}, the above method was applied only in the case $\mathcal{N} = 1$. In this and only this case, there is no $\rho$-deformation. In spite of this simplification, the calculation of $\mathfrak{J}$ was rather long and tedious. Two of us (SMK and GT-M) have tried to apply the same method in order to construct the action for $\mathcal{N} = 2$ conformal supergravity. The computation required turned out to be extremely involved. This essentially means that the curved superspace geometry of \cite{13, 16} is not well adapted for the construction of conformal supergravity actions, and we should look for an alternative formulation for $\mathcal{N}$-extended conformal supergravity.

In the present paper, we propose a new off-shell formulation for $\mathcal{N}$-extended conformal supergravity in three dimensions. It is inspired by the recently developed formulations for $\mathcal{N} = 1$ \cite{27} and $\mathcal{N} = 2$ \cite{28} conformal supergravities in four dimensions. These formulations are obtained by gauging the superconformal algebra in superspace. Conceptually such a gauging is similar to the superconformal tensor calculus in the component setting (see, e.g., \cite{29, 30} for reviews). The crucial new point of the 4D superspace approaches in \cite{27, 28} is that covariant constraints are imposed such that the algebra of covariant derivatives is given in terms of a single curvature superfield which coincides with the super Weyl tensor. This turns out to lead to dramatic computational simplifications.

This paper is organized as follows. Section 2 describes the geometric setup of $\mathcal{N}$-extended conformal superspace in three dimensions. We present the superconformal algebra
and the procedure in which it is gauged within superspace. In section 3 we provide a warm-up construction. As a straightforward extension of the gauging procedure for the bosonic case, we describe the geometry of conformal gravity for \( D \geq 3 \). In section 4 we show how to constrain the geometry of section 2 to describe \( \mathcal{N} \)-extended conformal supergravity, thus providing a new off-shell formulation in superspace. Section 5 is dedicated to showing how the conventional superspace formulation of \([13, 16]\) may be viewed as a degauged version of the \( \mathcal{N} \)-extended conformal superspace. Finally, section 6 concludes the paper by discussing the newly obtained results.

We have included a couple of technical appendices. In appendix A we include a summary of our notation and conventions. Appendix B shows how to couple an Abelian \( \mathcal{N} \)-extended vector multiplet to conformal supergravity with our superspace formulation.

2 Setup for \( \mathcal{N} \)-extended conformal superspace

In this section we present a geometric setup for \( \mathcal{N} \)-extended conformal superspace in three dimensions (3D), which arises from gauging the \( \mathcal{N} \)-extended superconformal algebra. We begin our discussion by giving the \( \mathcal{N} \)-extended superconformal algebra in our notation and conventions. We then present the gauging procedure for the construction of conformal superspace, which parallels the previous work in four dimensions \([27, 28]\).

2.1 \( \mathcal{N} \)-extended superconformal algebra in three dimensions

The bosonic part of the 3D \( \mathcal{N} \)-extended superconformal algebra \([31]\), \( \mathfrak{osp}(\mathcal{N}|4, \mathbb{R}) \), contains the translation \((P_a)\), Lorentz \((M_{ab})\), special conformal \((K_a)\), dilatation \((\mathbb{D})\) and \(\text{SO}(\mathcal{N})\) \((N_{KL})\) generators, where \(K, L = 1, \ldots, \mathcal{N}\). Their algebra is

\[
\begin{align*}
[M_{ab}, M_{cd}] &= 2\eta_{[a}M_{b]d} - 2\eta_{[a}M_{b]c}, \\
[M_{ab}, P_c] &= 2\eta_{[a}P_{b]}, \\
[M_{ab}, K_c] &= 2\eta_{[a}K_{b]}, \\
[K_a, P_b] &= 2\eta_{ab}\mathbb{D} + 2M_{ab}, \\
[N_{KL}, N^{IJ}] &= 2\delta_{[K}^{[I}N_{L]J]} - 2\delta_{[K}^{[J}N_{L]I},
\end{align*}
\]

where all other commutators vanish. The extension to the superconformal case is achieved by extending the translation generator to \(P_A = (P_a, Q_I^\alpha)\) and the special conformal generator to \(K_A = (K_a, S_I^\alpha)\), where \(Q^\alpha_I\) and \(S^\alpha_I\) are 3D spinors with respect to the index \(\alpha\) and \(\text{SO}(\mathcal{N})\) vectors with respect to the index \(I\) (see appendix A).\(^7\) The fermionic generator \(Q^I_\alpha\) obeys the algebra

\[
\begin{align*}
\{Q^I_\alpha, Q^J_\beta\} &= 2i\delta^{IJ}(\gamma^c)_{\alpha\beta}P_c = 2i\delta^{IJ}P_{\alpha\beta}, \\
\{Q^I_\alpha, P_b\} &= 0, \\
[M_{\alpha\beta}, Q^I_\gamma] &= \varepsilon_{\gamma(\alpha}Q^J_{\beta)\gamma}, \\
[\mathbb{D}, Q^I_\alpha] &= \frac{1}{2}Q^I_\alpha, \\
[N_{KL}, Q^I_\alpha] &= 2\delta^I_{[K}Q_{\alpha L]},
\end{align*}
\]

\(^7\)In line with usual nomenclature we refer to \(S^I_\alpha\) as the \(S\)-supersymmetry generator and \(K_a\) as the special conformal boost. We will also frequently refer to the full set \(K_A = (K_a, S^I_\alpha)\) as the special conformal generator where there is little ambiguity.
while the generator $S^I_\alpha$ obeys the algebra
\begin{equation}
\{S^I_\alpha, S^J_\beta\} = 2i\delta^{IJ}(\gamma^c)_{\alpha\beta}K_c, \quad [S^I_\alpha, K] = 0, \quad [S^I_\alpha, P_A] = i(\gamma_A)_{\alpha\beta}Q^I_\beta, \quad [S^I_\alpha, N^{KL}] = 2\delta^I_KS^L_\alpha.
\end{equation}

Finally, the remainder of the algebra of $K_A$ with $P_A$ is given by
\begin{align}
[K_a, Q^I_\alpha] &= -i(\gamma_a)_{\alpha\beta}S^I_\beta, \quad [S^I_\alpha, P_A] = i(\gamma_A)_{\alpha\beta}Q^I_\beta, \quad [S^I_\alpha, N^{KL}] = 2\delta^I_KS^L_\alpha.
\end{align}

For a matrix realization of the $\mathcal{N}$-extended superconformal algebra, see e.g. [32].

The superconformal algebra must obey the Jacobi identities. If we denote the generators of the algebra by $X_\tilde{a}$ then the Jacobi identities may be written as
\begin{equation}
[X_\tilde{a}, [X_\tilde{b}, X_\tilde{c}]] + \epsilon(\alpha + \epsilon\gamma_5)[X_\tilde{b}, [X_{\tilde{a}}, X_{\tilde{c}}]] + \epsilon(\beta + \delta)[X_\tilde{c}, [X_{\tilde{a}}, X_{\tilde{b}}]] = 0,
\end{equation}
where $\epsilon_\tilde{a}$ is the Grassmann parity of $X_{\tilde{a}}$. If we further denote the algebra (2.1) by
\begin{equation}
[X_{\tilde{a}}, X_{\tilde{b}}] = -f_{\tilde{a}\tilde{b}} X_{\tilde{c}},
\end{equation}
where $f_{\tilde{a}\tilde{b}} = \epsilon(\alpha + \delta)[X_{\tilde{a}}, X_{\tilde{b}}]$ are the structure constants, then we may equivalently write the Jacobi identities as
\begin{equation}
f_{\tilde{a}\tilde{b} \tilde{c}} f_{\tilde{d} \tilde{c}} = 0.
\end{equation}

The remainder of our notation and conventions follow closely those of [16] and are summarized in appendix A.

### 2.2 Gauging the superconformal algebra

To perform our gauging procedure, we begin with a curved 3D $\mathcal{N}$-extended superspace $\mathcal{M}^{3|2\mathcal{N}}$ parametrized by local bosonic $(x)$ and fermionic coordinates $(\theta_I)$:
\begin{equation}
z^M = (x^m, \theta^I),
\end{equation}
where $m = 0, 1, 2$, $\mu = 1, 2$ and $I = 1, \cdots, \mathcal{N}$. In order to describe supergravity it is necessary to have built into the theory a vielbein and appropriate connections. However the gauging of the superconformal algebra is made non-trivial due to the fact that the graded commutator of $K_A$ with $P_A$ contains generators other than $P_A$. This requires some of the connections to transform under $K_A$ into the vielbein. To perform the gauging we will follow closely the approach given in [28].

In order to gauge the superconformal algebra it is useful to denote by $X_{\tilde{a}}^\alpha$ the subset of the generators which do not contain the $P_A$ generators. The superconformal algebra

\[\ldots\]

\[\ldots\]
may be written as

\[ [X_a, X_b] = -f_{ab}^c X^c, \]
\[ [X_a, P_B] = -f_{ab}^c X^c - f_{ab}^C P_C := -f_a P_b X^c, \]
\[ [P_A, P_B] = -f_{AB}^C P_C := -f_{AB} P_a P_b P_C, \]

where \( f_{AB}^C \) contains only the constant torsion tensor \( f_{\alpha \beta}^c = T_{\alpha \beta}^c = -2i \delta^{IJ} (\gamma^c)_{\alpha \beta} \). It is seen that the generators \( X_a \) form a superalgebra. The gauge group associated with the superalgebra will be denoted \( G \).

In order to gauge the algebra (2.1) we associate with each generator \( X_a \) a connection one-form \( \omega^a = d z^M \omega^M_a \) and with \( P_A \) the vielbein \( E^A = d z^M E_M^A \). Their gauge transformations are postulated to be

\[ \delta G E^A = E^B \Lambda^a f_{ab}^B, \]
\[ \delta G \omega^a = d \Lambda^a + E^B \Lambda c f_{bc}^a + \omega^b \Lambda^c f_{bc}^a, \]

with \( \Lambda^a \) the gauge parameters. A superfield \( \Phi \) is said to be covariant if it transforms under \( G \) with no derivative of the parameter \( \Lambda^a \)

\[ \delta G \Phi = \Lambda^a X_a \Phi. \]

If \( \Phi \) transforms in some tensor representation of \( G \) we have matrix realizations

\[ M_{ab} \Phi = m_{ab} \Phi, \quad N_{IJ} \Phi = n_{IJ} \Phi, \quad \Box \Phi = \Delta \Phi, \]

where \( \Delta \) is a real number corresponding to the conformal dimension, and \( m_{ab} \) and \( n_{IJ} \) are the Lorentz and isospin matrices associated with \( \Phi \).

The final generators \( K_A = (K_a, S^I_a) \) are used to define conformal primary superfields:

\[ K_A \Phi = 0 \]

From the algebra, we note that if a superfield is annihilated by \( S \)-supersymmetry, then it is necessarily primary.

It is obvious that \( \partial_M \Phi \) is not itself covariant. We are led to introduce the covariant derivative

\[ \nabla = d - \omega^a X_a, \quad \nabla = E^A \nabla_A. \]

Its transformation is found to be

\[ \delta G (\nabla_A \Phi) = (-1)^{\epsilon \alpha \beta} \Lambda^\alpha \nabla_A X^\beta \Phi - \Lambda^\beta f_{\beta \alpha}^C \nabla_C \Phi - \Lambda^\alpha f_{\alpha \beta} \nabla^\beta \Phi, \]

with no derivatives on the gauge parameter \( \Lambda^a \). Rewriting this as \( \delta_G (\nabla_A \Phi) = \Lambda^\alpha \nabla_X \nabla_A \Phi \), we immediately find the operator relation

\[ [X_b, \nabla_A] = -f_{bA}^C \nabla_C - f_{bA}^C \nabla_X. \]
The curvature and torsion tensors appear in the commutator of two covariant derivatives,
\[ [\nabla_A, \nabla_B] = -T_{AB}^C \nabla_C - R_{AB}^\xi \xi. \tag{2.14} \]

The explicit expressions for these tensors are most compactly given in terms of two-forms
\[ T^A := \frac{1}{2} E^C \wedge E^B T_{BC}^A = dE^A - E^C \wedge \omega^b \varepsilon_{BC}^A, \tag{2.15a} \]
\[ R^n := \frac{1}{2} E^C \wedge E^B R_{BC}^a = d\omega^a - E^C \wedge \omega^b \varepsilon_{BC}^a - \frac{1}{2} \omega^a \wedge \omega^b \varepsilon_{BC}^a. \tag{2.15b} \]

Using the definition of curvature and torsion together with the vielbein and connection transformation rules (2.7) we find
\[ \delta \Phi^T = \Lambda^B \varepsilon_{BC} T^D, \tag{2.16a} \]
\[ \delta \Phi^R = \Lambda^B \varepsilon_{BC} R^D, \tag{2.16b} \]
\[ \delta \Phi^E = \Lambda^B \varepsilon_{BC} T^D, \tag{2.17a} \]
\[ \delta \Phi^D = -\Lambda^B \varepsilon_{BC} R^D, \tag{2.17b} \]

One can show the above results are the necessary conditions for the Jacobi identity involving two \( \nabla \)'s
\[ 0 = [X_2, [\nabla_B, \nabla_C]] + \text{cycles} \tag{2.18} \]

to be identically satisfied. The Bianchi identities
\[ 0 = [\nabla_A, [\nabla_B, \nabla_C]] + \text{cycles} \tag{2.19} \]
can also be shown to be satisfied identically. Therefore, we have a consistent algebraic structure
\[ [X_2, X_2] = -f_{[2]} \xi \xi X_\xi, \tag{2.20a} \]
\[ [X_2, \nabla_B] = -f_{[2]} B^C \nabla_C - f_{[2]} B^\xi \xi X_\xi, \tag{2.20b} \]
\[ [\nabla_A, \nabla_B] = -T_{AB}^C \nabla_C - R_{AB}^\xi \xi X_\xi, \tag{2.20c} \]

which satisfies all the Jacobi identities. In the flat space limit the curvature vanishes and the torsion becomes the usual constant torsion, so that the algebra (2.20) exactly matches the superconformal algebra that we started with, in which \( P_A \) is replaced with \( \nabla_A \). The curved case requires a deformation via the introduction of torsion and curvature. The superconformal algebra is then said to be “gauged” in this sense.

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12The reason why the sign of the structure constants was chosen was so that in the flat limit the torsion becomes the usual structure constant for the \( [P_A, P_B] \) (anti-)commutator.
The full set of operators \((\nabla_A, X^A_\underline{a})\) generates the conformal supergravity gauge group \(\mathcal{G}\). The form of the covariant derivative suggests that we should extend the usual diffeomorphisms \(\delta_{\text{get}}\) into covariant diffeomorphisms\(^{13}\)

\[
\delta_{\text{get}}(\xi^A) := \delta_{\text{get}}(\xi^A E_M^A) - \delta_{\mathcal{H}}(\xi^A \omega_{A}^\underline{a}),
\]

where \(\delta_{\text{get}}(\xi^M)\) acts on scalars under diffeomorphisms as

\[
\delta_{\text{get}} \Phi = \xi^M \partial_M \Phi .
\]

The full conformal supergravity gauge group \(\mathcal{G}\) is then generated by

\[
\mathcal{K} = \xi^C \nabla_C + \Lambda^\underline{a} X^\underline{a} .
\]

If a superfield \(\Phi\) is a scalar under diffeomorphisms and covariant under the group \(\mathcal{H}\), then its transformation under the full supergravity gauge group \(\mathcal{G}\) is

\[
\delta_{\mathcal{G}} \Phi = \mathcal{K} \Phi = \xi^C \nabla_C \Phi + \Lambda^\underline{a} X^\underline{a} \Phi .
\]

It is a straightforward exercise to show that the vielbein and connection one-forms transform as

\[
\delta_{\mathcal{G}} E^A = d\xi^A + E^B \Lambda^c f_{cB}^A + \omega^b \xi^C f_{Cb}^A + E^B \xi^C T_{CB} A , \quad (2.25a)
\]

\[
\delta_{\mathcal{G}} \omega^a = d\Lambda^a + \omega^b \Lambda^c f_{cB}^a + \omega^b \Lambda^C f_{CB}^a + E^B \Lambda^c f_{cB}^a + E^B \xi^C R_{CB} a . \quad (2.25b)
\]

From this definition, one can check that the covariant derivative transforms as

\[
\delta_{\mathcal{G}} \nabla_A = [\mathcal{K}, \nabla_A] \quad (2.26)
\]

provided we interpret\(^{14}\)

\[
\nabla_A \xi^B := E_A \xi^B + \omega_A \xi^D f_{Dc}^B , \quad (2.27a)
\]

\[
\nabla_A \Lambda^b := E_A \Lambda^b + \omega_A \xi^D f_{Dc}^b + \omega_A \Lambda^c f_{dc}^b . \quad (2.27b)
\]

We can summarize the superspace geometry of conformal supergravity as follows. The covariant derivatives have the form\(^{15}\)

\[
\nabla_A = E_A - \omega_A \xi^B f_{Bc}^A X^c_\underline{a} = E_A - \frac{1}{2} \Omega_A^{ab} M_{ab} - \frac{1}{2} \Phi_A^{PQ} N_{PQ} - B_A D - \mathfrak{S}_A B K_B . \quad (2.28)
\]

The action of the generators on the covariant derivatives, eq. (2.20b), resembles that for the \(P_A\) generators given in (2.1). The supergravity gauge group is generated by local transformations of the form (2.26) where

\[
\mathcal{K} = \xi^C \nabla_C + \frac{1}{2} \Lambda^{cd} M_{cd} + \frac{1}{2} \Lambda^{PQ} N_{PQ} + \sigma D + \Lambda^A K_A \quad (2.29)
\]

\(^{13}\)These transformations are also known as covariant general coordinate transformations. Their use is standard, see e.g. [30].

\(^{14}\)One must take care in applying the formulae (2.26) and (2.27). Observe that we can have \(\Lambda^\underline{a} = 0\) but \(\nabla_A \Lambda^\underline{a} \neq 0\) if either \(\xi^D f_{Dc}^\underline{a}\) or \(\Lambda^c f_{dc}^\underline{a}\) is non-vanishing.

\(^{15}\)Note that the complex conjugation rule (A.13) induces a natural reality condition on the vielbein and the connections.
and the gauge parameters satisfy natural reality conditions. The covariant derivatives satisfy the (anti-)commutation relations

\[
\begin{align*}
[\nabla_A, \nabla_B] &= -T^a_{AB} C^C \nabla_C - \frac{1}{2} R(M)_{AB} a^{cd} M_{cd} - \frac{1}{2} R(N)_{AB} P^M_{PQ} N_{PQ} \\
&- R(\mathcal{D})_{AB} \mathcal{D} - R(S)_{AB} \gamma^K S^K_{\gamma} - R(K)_{AB} e^K c^e,
\end{align*}
\]

where the torsion and curvature tensors are given by\(^\text{16}\)

\[
\begin{align*}
T^a &= dE^a + E^a \land B + E^b \land \Omega_b^a, \\
T^a_{ij} &= dE^a + \frac{1}{2} E^b_{ij} \land \Omega^c_b \land \Theta_c^a_{\alpha} + \frac{1}{2} E^a_{ij} \land B + E^a_{ij} \land \Phi_J J_i^a + i E^c \land \tilde{\gamma}_j^a (\gamma_c)_{\beta}^a, \\
R(\mathcal{D}) &= dB + 2 E^a \land \tilde{\gamma}_a - 2 E^a \land \tilde{\gamma}_a^j, \\
R(M)^{ab} &= d \Omega^{ab} + \Omega^{ac} \land \Omega_c^b - 4 E^a \land \tilde{\gamma}_b - 2 E^a_{ij} \land \tilde{\gamma}^j (\gamma_c)_{\alpha}^a c^{ab}, \\
R(N)^{IJ} &= d \Phi^I J^J + \Phi^I K^\gamma \land \Phi_K J^{ij} - 4 E^{[a]} \land \tilde{\gamma}^{b]} (\gamma_c)_{\alpha}^a c^{ab}, \\
R(K)^a &= d \tilde{\gamma}^a - \tilde{\gamma}^a \land B + \tilde{\gamma}^b \land \Omega_b^a + i \tilde{\gamma}^a \land \tilde{\gamma}^j (\gamma_c)_{\alpha}^a, \\
R(S)^{ij} &= d \tilde{\gamma}_i - i E^a_{ij} \land \tilde{\gamma}^a (\gamma_c)_{\alpha}^a - \frac{1}{2} \tilde{\gamma}^a \land B + \frac{1}{2} \tilde{\gamma}^{j} \land \Omega^c c^{ij} (\gamma_c)_{\alpha}^a + \tilde{\gamma}^{a} \land \Phi_J J_i^a.
\end{align*}
\]

3 Conformal gravity in \(D \geq 3\) dimensions

Before we turn to 3\text{D} conformal supergravity we will first discuss conformal gravity in \(D \geq 3\) dimensions.\(^\text{17}\) To do so we note that the bosonic part of the superconformal algebra (2.1) without the SO(\(N\)) generator can be straightforwardly extended to \(D\) dimensions. The algebra is

\[
\begin{align*}
[M_{ab}, M_{cd}] &= 2 \eta_{[a} M_{b]d} - 2 \eta_{d[a} M_{b]c}, \tag{3.1a} \\
[M_{ab}, P_c] &= 2 \eta_{[a} P_{b]c}, \quad [\mathcal{D}, P_a] = P_a, \tag{3.1b} \\
[M_{ab}, K_c] &= 2 \eta_{[a} K_{b]c}, \quad [\mathcal{D}, K_a] = -K_a, \tag{3.1c} \\
[K_a, P_b] &= 2 \eta_{a[b} \mathcal{D} + 2 M_{ab}. \tag{3.1d}
\end{align*}
\]

where all other commutators vanish and \(\eta_{ab}\) is the the \(D\)-dimensional Minkowski metric. It is clear that the gauging procedure of section 2.2 may be straightforwardly extended to conformal gravity in \(D\) dimensions, while the restriction to a bosonic manifold is trivial.

The covariant derivatives have the form

\[
\nabla_a = e_a - \frac{1}{2} \omega_a^{bc} M_{bc} - b_a \mathcal{D} - f_a^b K_b, \quad e_a := e_a^m \partial_m,
\]

where

\[
\begin{align*}
\omega_a^{bc} &= e_a^m \omega_m^{bc}, \quad b_a := e_a^m b_m, \quad f_a^b := e_a^m f_m^b.
\end{align*}
\]

\(^\text{16}\)Since SO(\(N\)) vector indices are raised and lowered using the Kronecker delta, there is no need to distinguish between upper and lower SO(\(N\)) vector indices.

\(^\text{17}\)Conformal gravity has been discussed elsewhere in many places, e.g.\(^\text{[30]}\). Here we review conformal gravity emphasizing some points relevant to our paper. The important feature is that the algebra of covariant derivatives may be constructed entirely in terms of a primary superfield.
The covariant derivatives satisfy the same algebra as $P_a$, except for the introduction of curvatures and torsion

$$\left[\nabla_a, \nabla_b\right] = -T_{ab}^c \nabla_c - \frac{1}{2} R(M)_{ab}^{\quad cd} M_{cd} - R(\mathbb{D})_{ab} \mathbb{D} - R(K)_{ab}^c K_c , \quad (3.4)$$

where the curvatures and torsion are given by the form expressions:

$$T^a = de^a + e^a \wedge b + e^b \wedge \omega^a_b , \quad (3.5a)$$
$$R(M)^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^c_b - 4e^{[a} \wedge f^{b]} , \quad (3.5b)$$
$$R(\mathbb{D}) = db + 2e^a \wedge f_a , \quad (3.5c)$$
$$R(K)^a = df^a - f^a \wedge b + f^b \wedge \omega^b_a . \quad (3.5d)$$

In order to define the spin connection (as a composite object) it is necessary to impose some covariant constraint. The appropriate constraint is

$$T_{ab}^c = 0 . \quad (3.6)$$

It is clear that the constraint is Lorentz and dilatation invariant, while the conformal invariance may be checked by making use of the Jacobi identity

$$[[K_a, \nabla_b], \nabla_c] - [[K_a, \nabla_c], \nabla_b] + [[\nabla_b, \nabla_c], K_a] = 0 \quad \implies [K_a, [\nabla_b, \nabla_c]] = 2[[K_a, \nabla_b], \nabla_c] . \quad (3.7)$$

The right hand side is identically zero as a result of the conformal algebra, so that

$$[K_a, [\nabla_b, \nabla_c]] = 0 . \quad (3.8)$$

From here we see that $K_a T_{bc}^d = 0$.\footnote{Torsion has dimension 1 and $K_a$ carries dimension $-1$; therefore, $K_a T_{bc}^d$ has nothing of lower dimension to transform into.} As a result the constraint (3.6) is conformally covariant.

The $K$-gauge transformation of $b_a$ is

$$\delta_K(\Lambda) b_a = -2\Lambda_a . \quad (3.9)$$

It is clear that the $K$-gauge transformations can be completely used up to make the gauge choice

$$b_a = 0 . \quad (3.10)$$

In what follows we make use of this gauge choice.

It is necessary to constrain the curvatures to correspond to the structure of conformal gravity. Now constraining the torsion

$$T_{ab}^c = -C_{ab}^c + 2\omega_{[ab]}^c , \quad C_{ab}^c = -2\epsilon_a^m \epsilon_b^m \epsilon_c^m \partial_{[m} \epsilon_{n]}^c = 2\epsilon_a^m \epsilon_b^m \epsilon_c^m \quad (3.11)$$

to vanish (3.6) allows one to solve for the Lorentz connection in the usual way,

$$\omega_{abc} = \frac{1}{2} (C_{abc} - C_{acb} - C_{bca}) . \quad (3.12)$$
Next we note that the Lorentz curvature is given by
\[
R(M)_{ab}^{cd} = \mathcal{R}_{ab}^{cd} + 8\delta_{[d}^c\delta_{b]}^d ,
\] (3.13)
where
\[
\mathcal{R}_{ab}^{cd} := 2\epsilon_{[a}^m\epsilon_b^n\partial_m\omega_n^{cd} - 2\omega_{[a}^c\omega_b]^d ,
\] (3.14)
is the standard Riemann tensor constructed from \( \omega \). Since we want the special conformal connection to be a composite field we impose the conformal gravity constraint\(^{19}\)
\[
\eta^{bd}R(M)_{abcd} = 0 .
\] (3.15)
This constraint gives
\[
f_{ab} = -\frac{1}{2(D-2)}\mathcal{R}_{ab} + \frac{1}{4(D-1)(D-2)}\eta_{ab}\mathcal{R} ,
\] (3.16)
where
\[
\mathcal{R}_{ac} := \eta^{bd}\mathcal{R}_{abcd} , \quad \mathcal{R} := \eta^{ab}\mathcal{R}_{ab} .
\] (3.17)
Putting our solution for \( f_{ab} \) into our expression for \( R(M)_{ab}^{cd} \) leads us to the result that
\[
R(M)_{ab}^{cd} \text{ coincides with the conformal Weyl tensor}
\]
\[
R(M)_{ab}^{cd} = C_{abcd} = \mathcal{R}_{abcd} - \frac{2}{D-2}(\eta_{a[c}R_{d]b} - \eta_{b[c}R_{d]a}) + \frac{2}{(D-1)(D-2)}\mathcal{R}\eta_{a[c}\eta_{d]b} .
\] (3.18)
Furthermore, since \( f_{ab} \) is symmetric we also have
\[
R(D)_{ab} = 4f_{[ab]} = 0 .
\] (3.19)
We can also infer information about the conformal curvature \( R(K)_{ab}^{ce} \). Due to the constraint (3.6) the Bianchi identity
\[
\nabla_{[a}R_{b]c} = 0
\] (3.20)
may be expanded as
\[
0 = \frac{1}{2}\nabla_{[a}R(M)_{bc]}^{de}M_{de} + 2R(K)_{[ab}^{d}M_{c]d} - R(M)_{[abc]}^{d}\nabla_{d} - R(D)_{[ab}^d\nabla_c]
+ \nabla_{[a}R(D)_{bc]}^d - 2R(K)_{[abc]}^{d} + \nabla_{[a}R(K)_{bc]}^{d}K_{d}
= \frac{1}{2}\nabla_{[a}R(M)_{bc]}^{de}M_{de} + 2R(K)_{[ab}^{d}M_{c]d} - R(M)_{[abc]}^{d}\nabla_{d}
- 2R(K)_{[abc]}^{d} + \nabla_{[a}R(K)_{bc]}^{d}K_{d} ,
\] (3.21)
where we used the fact that \( R(D) = 0 \). This result leads to the identities
\[
R(K)_{[abc]} = 0 ,
\] (3.22a)
\[
R(M)_{[abc]}^{d} = 0 ,
\] (3.22b)
\[
\nabla_{[a}R(K)_{bc]}^{d} = 0 ,
\] (3.22c)
\[
\nabla_{[a}R(M)_{bc]}^{d} - 4R(K)_{[ab}^{d}e_c] = 0 .
\] (3.22d)

\(^{19}\)From the transformations of the curvatures in the last section it is easy to see that when the torsion vanishes it holds that \( K_c R(M)_{abcd} = 0 \), which makes eq. (3.15) a conformally invariant constraint.
Contracting $c$ with $d$ in eq. (3.22d), and using the constraint (3.15), gives
\[ \frac{1}{2} \nabla_c R(M)_{ab}^{ce} + (D - 3) R(K)_{ab}^{e} - 2 R(K)_{c[a}^{e} \delta_{b]}^{e} = 0 . \] (3.23)

From here we deduce that for $D \geq 3$ we have
\[ R(K)_{ab}^{b} = 0 \] (3.24)
and
\[ 2(D - 3) R(K)_{abc} = \nabla^d R(M)_{abcd} = \nabla^d C_{abcd} . \] (3.25)
Thus all the curvatures may be expressed in terms of the Weyl tensor $C_{abcd}$ for $D \geq 4$. Therefore, the vanishing of the Weyl tensor $C_{abcd}$ implies the vanishing of all the conformal gravity curvatures and hence conformal flatness.

The $D = 3$ case is special because the Weyl tensor (the traceless part of $R(M)_{abcd}$) vanishes for the choice of $\omega$ which solves the torsion constraint (3.6).$^{20}$ Due to the constraint (3.15) it must also correspond to the traceless part of $R(M)_{abcd}$. Thus we automatically have $R(M)_{abcd} = 0$.

In 3D the commutator of two covariant derivatives only involves the special conformal connection$^{21}$
\[ [\nabla_a, \nabla_b] = - R(K)_{ab}^{e} K_e . \] (3.26)

One can show that
\[ R(K)_{mn}^{c} := \epsilon_m^{a} \epsilon_n^{b} R(K)_{ab}^{c} = 2 \partial_{[m} \delta_{n]}^{c} + \omega_{[m}^{b} \delta_{n]}^{b} \]
\[ = 2 D_{[m}^{c}, \] (3.27)
where we introduce the Lorentz-covariant derivative
\[ D_m = \partial_m - \frac{1}{2} \omega_m^{ab} M_{ab}, \quad D_a := \epsilon_a^{m} D_m . \] (3.28)

Since the torsion vanishes, the curvature may also be written as
\[ - \frac{1}{2} W_{abc} := R(K)_{abc} = 2 D_{[a} \delta_{b]}^{c} - \nabla_d R_{d[c}^{b]} - \frac{1}{4} \eta_{c[a} D_{b]}^{b]b} , \] (3.29)
where $W_{abc}$ is the Cotton tensor. Furthermore, it is easy to see that when the Cotton tensor vanishes the space is conformally flat.

Due to the symmetry properties satisfied by the Cotton tensor,$^{22}$
\[ W_{abc} = - W_{bac} , \quad W_{[abc]} = 0 , \quad W_{ab}^{b} = 0 , \] (3.30)
we can instead view it as a traceless symmetric rank 2 tensor
\[ W_{ab} := \frac{1}{2} \epsilon_a^{cd} W_{cdb} , \quad W_{ab} = W_{ba} , \quad W_{a}^{a} = 0 . \] (3.31)

$^{20}$3D has the unique property that the Riemann tensor is completely determined by the Ricci tensor.
$^{21}$ $R(K)_{ab}^{c}$ is a conformal primary, $K_d R(K)_{bc}^{d} = 0$, as a result of eq. (3.8).
$^{22}$Keep in mind the eqs. (3.22a) and (3.24).
The Cotton tensor also satisfies a divergenceless condition as a result of eq. (3.22c)

$$\nabla^b W_{ab} = 0 .$$

(3.32)

Note that we could have also chosen $b_m \neq 0$. However, in this case $R(\mathbb{D})_{ab}$ would still vanish because it is invariant under the $K$-gauge transformations, $K_c R(\mathbb{D})_{ab} = 0$. In fact, in order to derive the geometry all that is required is to impose the constraints

$$T_{ab}^c = 0 , \quad R(M)_{abcd} = 0 .$$

(3.33)

Considering the term appearing in front of the covariant derivative in the Bianchi identity

$$\nabla [a \nabla b \nabla c] = 0 ,$$

(3.34)

we see that under the constraints (3.33) $R(\mathbb{D})_{ab}$ vanishes also. The Cotton tensor is again given by the only surviving curvature, the special conformal curvature. We note that these constraints are conformally invariant, and so the composite expressions for $\omega_{abc}$ and $f_{ab}$, which depend on $b_m$ in general, retain their original transformation laws.

4 $\mathcal{N}$-extended conformal supergravity

We saw in the last subsection that in the conformal gravity approach the covariant derivative algebra may be expressed in terms of a single primary superfield: the Weyl tensor for $D \geq 4$ and the Cotton tensor in $D = 3$. Therefore in the 3D $\mathcal{N}$-extended case we look for a formulation in which the entire covariant derivative algebra is expressed in terms of a single primary superfield, the $\mathcal{N}$-extended super Cotton tensor. A feature of such a setting is that the vanishing of the super Cotton tensor implies trivially that the space is conformally flat.

The super Cotton tensor possesses a different index structure for various values of $\mathcal{N}$. In the bosonic case, $\mathcal{N} = 0$, the Cotton tensor may be expressed in terms of spinor indices as

$$W_{\alpha\beta\gamma\delta} := (\gamma^a)_{\alpha\beta} (\gamma^b)_{\gamma\delta} W_{ab} = W_{(\alpha\beta\gamma\delta)} ,$$

(4.1)

which is totally symmetric since $W_{ab}$ is both symmetric and traceless. The super Cotton tensors for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ are described by superfields $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ and $W_{\alpha\beta} = W_{(\alpha\beta)}$ and were given in [21] and [33] respectively. For $\mathcal{N} > 3$ it is known [13] that the super Cotton tensor may be described by a totally antisymmetric SO($\mathcal{N}$) superfield $W^{IJKL} = W^{[IJKL]}$, while for $\mathcal{N} = 3$ we will see that the super Cotton tensor is described by a real spinor superfield $W_\alpha$.

For the known formulations of conformal superspace in 4D the constrained geometry describing conformal supergravity surprisingly takes a simple form [27, 28], despite gauging the entire structure group. More precisely, the curvature structure of the theory resembles super Yang-Mills. As we will demonstrate below, the corresponding ansatz in 3D turns out to be a very economical means of constraining the curvatures of the theory. In what follows we will proceed case by case with increasing values of $\mathcal{N}$.
4.1 The $\mathcal{N} = 1$ case

We begin by first considering the $\mathcal{N} = 1$ case. It is necessary to constrain the curvatures so as to describe conformal supergravity. We constrain the curvatures by

$$\{\nabla_\alpha, \nabla_\beta\} = 2i \nabla_{\alpha\beta}.$$  (4.2)

It then follows from the Bianchi identities that the remaining commutation relations may be written entirely in terms of the operator

$$W_\alpha = W(P)_\alpha^a \nabla_a + W(Q)_\alpha^\beta \nabla_\beta + \frac{1}{2} W(M)_\alpha^{ab} M_{ab}$$

$$+ W(\mathcal{D})_\alpha \mathcal{D} + W(K)_\alpha^a K_a + W(S)_\alpha^\beta S_\beta.$$  (4.3)

The remaining commutation relations are

$$[\nabla_a, \nabla_\alpha] = -\frac{1}{2} (\gamma_a)_\alpha^\beta W_\beta;$$  (4.4)

$$[\nabla_a, \nabla_b] = \frac{i}{4} \epsilon_{abc} (\gamma^c)^{\alpha\beta} \{\nabla_\alpha, W_\beta\} ,$$  (4.5)

where $W_\alpha$ must satisfy the Bianchi identity

$$\{\nabla^\alpha, W_\alpha\} = 0.$$  (4.6)

Moreover, as a result of the Jacobi identities, $W_\alpha$ must be of dimension-3/2 and a conformal primary:

$$[\mathcal{D}, W_\alpha] = \frac{3}{2} W_\alpha , \quad \{S_\alpha, W_\beta\} = 0.$$  (4.7)

We now make the following simple ansatz for $W_\alpha$:

$$W_\alpha = W(K)_\alpha^a K_a + W(S)_\alpha^\beta S_\beta.$$  (4.8)

Then the Bianchi identity (4.6) implies

$$W(S)_\alpha^\beta = 0, \quad W_\alpha^\beta_\gamma := W(K)_\alpha^a (\gamma_a)_\beta_\gamma = W_\alpha^{(\beta_\gamma)}$$  (4.9)

and the conformally invariant constraint

$$\nabla^\alpha W_\alpha^\beta_\gamma = 0.$$  (4.10)

The conditions (4.7) give

$$\mathcal{D} W_\alpha^\beta_\gamma = \frac{5}{2} W_\alpha^\beta_\gamma , \quad S_\delta W_\alpha^\beta_\gamma = 0.$$  (4.11)

The covariant derivative algebra takes the simple form

$$\{\nabla_\alpha, \nabla_\beta\} = 2i \nabla_{\alpha\beta} ,$$  (4.12a)

$$[\nabla_a, \nabla_\alpha] = \frac{1}{4} (\gamma_a)^{\alpha\beta} W_\beta_\gamma_\delta K^{\gamma\delta} ,$$  (4.12b)

$$[\nabla_a, \nabla_b] = -\frac{1}{8} \epsilon_{abc} (\gamma^c)^{\alpha\beta} \nabla_\alpha W_\beta_\gamma_\delta K^{\gamma\delta} - \frac{1}{4} \epsilon_{abc} (\gamma^c)^{\alpha\beta} W_\alpha^\beta_\gamma S_\gamma.$$  (4.12c)

We are motivated by the fact that the torsion and Lorentz and dilatation curvatures vanish in the bosonic case.
The above algebra has the property that it may be written in terms of a primary superfield $W_{\alpha\beta\gamma}$ with the symmetry properties of the $\mathcal{N} = 1$ super Cotton tensor. In particular, we see that the only curvatures in (4.12c) which arise in the algebra are $R(K)^{a}_{\ b}$ and $R(S)^{a}_{\ b}\gamma$, which should correspond to the component Cotton and Cottino tensors. In section 5 we will see that $W_{\alpha\beta\gamma}$ is indeed proportional to the super Cotton tensor in the formulation of [16].

4.2 The $\mathcal{N} = 2$ case

In the $\mathcal{N} = 2$ case we take

$$\{\nabla^I_\alpha, \nabla^J_\beta\} = 2i\delta^{IJ}\nabla_\alpha + 2i\varepsilon_{\alpha\beta}\varepsilon^{IJ}W, \quad (4.13)$$

where

$$W = W(P)^a\nabla_a + W(Q)^j\nabla^j_a + \frac{1}{2}W(M)^{ab}M_{ab} + W(D)\nabla,$$

$$+ W(K)^aK_a + W(S)^{\gamma I}S^I_{\alpha} + \frac{1}{2}W(N)^{IJ}N_{IJ}. \quad (4.14)$$

The remaining commutation relations are

$$[\nabla_a, \nabla^I_\beta] = -\varepsilon^{JK}(\gamma_a)_{\beta}\gamma^\gamma_\gamma\nabla_K, W, \quad (4.15)$$

$$[\nabla_a, \nabla_b] = \frac{i}{4}\varepsilon_{abc}(\gamma^c)\gamma^\delta\varepsilon^{KL}\{\nabla_K, [\nabla_L, W]\}, \quad (4.16)$$

with $W$ satisfying the Bianchi identity

$$\varepsilon^{K(I}\{\nabla^{\gamma J)}, [\nabla_\gamma K, W]\} = 0. \quad (4.17)$$

Moreover, $W$ must be of dimension-1 and a conformal primary:

$$[\nabla, W] = W, \quad [S^I_{\alpha}, W] = 0. \quad (4.18)$$

The Bianchi identity (4.17) may be solved by the ansatz

$$W = W(K)^aK_a. \quad (4.19)$$

Introducing the notation $W_{\alpha\beta} := W(K)^a(\gamma_a)_{\alpha\beta}$, we find the following conformally invariant constraint

$$\nabla^{\alpha I}W_{\alpha\beta} = 0, \quad (4.20)$$

while the conditions (4.18) give

$$\nabla W_{\alpha\beta} = 2W_{\alpha\beta}, \quad S^K_{\gamma}W_{\alpha\beta} = 0. \quad (4.21)$$

Hence $W_{\alpha\beta}$ is a primary superfield of dimension-2. We will verify in section 5 that $W_{\alpha\beta}$ corresponds to the $\mathcal{N} = 2$ super Cotton tensor.

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24 The antisymmetric tensors $\varepsilon^{IJ} = \varepsilon_{IJ}$ are normalized as $\varepsilon^{12} = \varepsilon_{12} = 1$. 
We then find the algebra to be
\[
\{\nabla^I, \nabla^J\} = 2i\delta^{IJ}\nabla_{\alpha\beta} - i\varepsilon^{IJ}\varepsilon_{\alpha\beta}W_{\gamma\delta}K^{\gamma\delta}, \tag{4.22a}
\]
\[
[\nabla_a, \nabla^J] = \frac{1}{2}\varepsilon(a)_{\beta\gamma}e^{J\gamma K}W_{\alpha\beta}K^{\alpha\delta} + i(\varepsilon(a)_{\beta\gamma}e^{J\gamma K}W_{\alpha\beta}K^{\alpha\delta} - 8W_{\gamma\delta}J), \tag{4.22b}
\]
\[
[\nabla_a, \nabla_b] = -i8\varepsilon^{abc}(\varepsilon(c)_{\gamma\delta})^{\gamma\delta}(2e^{KL}(\nabla_{\gamma K}\nabla_{\delta L}W_{\alpha\beta}K^{\alpha\delta} + 4i\nabla_{\gamma K}W_{\delta\beta}S_{\delta L} - 8W_{\gamma\delta}J), \tag{4.22c}
\]
where we conveniently introduce the U(1) generator \(J\) defined by
\[
N_{KL} = i\varepsilon_{KL}J, \quad J := -i\frac{1}{2}\varepsilon_{KL}N_{KL}. \tag{4.23}
\]
The operator \(J\) acts on the covariant derivatives as
\[
[J, \nabla^I] = -i\varepsilon^{IJ}\nabla_J. \tag{4.24}
\]

It is often easier to work in a complex basis for the spinor covariant derivatives:
\[
\nabla_\alpha = \frac{1}{\sqrt{2}}(\nabla^1_\alpha - i\nabla^2_\alpha), \quad \nabla_\alpha = -\frac{1}{\sqrt{2}}(\nabla^1_\alpha + i\nabla^2_\alpha), \tag{4.25}
\]
with definite U(1) charges:
\[
[J, \nabla_\alpha] = \nabla_\alpha, \quad [J, \nabla_\bar{\alpha}] = -\nabla_{\bar{\alpha}}. \tag{4.26}
\]
The SO(2) connection and curvature then take the form
\[
\frac{1}{2}\Phi_{A}^{KL}N_{KL} = i\Phi_{A}J, \quad \frac{1}{2}R(N)_{AB}^{KL}N_{KL} = iR(J)_{AB}J. \tag{4.27}
\]
The conjugation rule in the complex basis is
\[
(\nabla_\alpha F)^* = (-1)^{\varepsilon(F)}\nabla_\alpha \bar{F}, \tag{4.28}
\]
where \(F\) is a complex superfield and \(\bar{F} = (F)^*\) is its complex conjugate.

In the new basis \((\nabla_\alpha, \nabla_{\bar{\alpha}})\), the covariant derivative algebra (4.22) takes the form
\[
\{\nabla_\alpha, \nabla_\beta\} = 0, \quad \{\nabla_\alpha, \nabla_{\bar{\beta}}\} = 0, \tag{4.29a}
\]
\[
\{\nabla_\alpha, \nabla_{\bar{\beta}}\} = -2i\nabla_{\alpha\beta} - \varepsilon_{\alpha\beta}W_{\gamma\delta}K^{\gamma\delta}, \tag{4.29b}
\]
\[
[\nabla_\alpha, \nabla_{\bar{\beta}}] = \frac{1}{2}(\varepsilon(a)_{\beta\gamma})\gamma_{\gamma\delta}W_{a\delta}K^{\alpha\delta} - (\varepsilon(a)_{\beta\gamma})W_{\gamma\delta}S_{\delta}, \tag{4.29c}
\]
\[
[\nabla_\alpha, \nabla_{\bar{\beta}}] = -\frac{1}{8}\varepsilon^{abc}(\varepsilon(c)_{\gamma\delta})^{\gamma\delta}(i[\nabla_\gamma, \nabla_{\bar{\delta}}]W_{a\beta}K^{a\delta} + 4\nabla_\gamma W_{a\beta}S_{a\delta} + 4\nabla_\gamma W_{\bar{a}\beta}S_{\bar{a}\delta} - 8W_{\gamma\delta}J), \tag{4.29d}
\]
where we define
\[
S_{a} := \frac{1}{\sqrt{2}}(S^1_{a} + iS^2_{a}), \quad S_{\bar{a}} := \frac{1}{\sqrt{2}}(S^1_{\bar{a}} - iS^2_{\bar{a}}). \tag{4.30}
\]
In the complex basis the generators act on the covariant derivatives as
\[
[M_{\alpha\beta}, \nabla_\gamma] = \varepsilon_{\gamma(\alpha} \nabla_{\beta)} , \quad [M_{\alpha\beta}, \nabla_{\bar{\gamma}}] = \varepsilon_{\gamma(\alpha} \nabla_{\bar{\beta)}}, \tag{4.31a}
\]
\[ [\mathcal{D}, \nabla_\alpha] = \frac{1}{2} \nabla_\alpha, \quad [\mathcal{D}, \bar{\nabla}_\alpha] = \frac{1}{2} \bar{\nabla}_\alpha, \quad (4.31b) \]
\[ [J, \nabla_\alpha] = \nabla_\alpha, \quad [J, \bar{\nabla}_\alpha] = -\bar{\nabla}_\alpha, \quad (4.31c) \]
\[ \{S_\alpha, S_\beta\} = 0, \quad \{\bar{S}_\alpha, \bar{S}_\beta\} = 0, \quad \{S_\alpha, \bar{S}_\beta\} = 2iK_{\alpha\beta}, \quad (4.31d) \]
\[ [S_\alpha, K_b] = 0, \quad (4.31e) \]
\[ [\bar{S}_\alpha, \nabla_\beta] = -2\varepsilon_{\alpha\beta} \mathcal{D}, \quad [S_\alpha, \nabla_\beta] = -2\varepsilon_{\alpha\beta} \mathcal{J}, \quad (4.31f) \]
\[ [J, \nabla_\alpha] = \nabla_\alpha, \quad [J, \bar{\nabla}_\alpha] = -\bar{\nabla}_\alpha, \quad (4.31g) \]
\[ [\bar{S}_\alpha, \nabla_\beta] = -2\varepsilon_{\alpha\beta} \mathcal{D} + 2M_{\alpha\beta} - 2\varepsilon_{\alpha\beta} \mathcal{J}, \quad \{S_\alpha, \nabla_\beta\} = 2\varepsilon_{\alpha\beta} \mathcal{D} - 2M_{\alpha\beta} - 2\varepsilon_{\alpha\beta} \mathcal{J}. \quad (4.31h) \]

One may compare the equations (4.31) with the algebra given in four-dimensional \( N = 1 \) conformal superspace [27].

### 4.3 The \( N = 3 \) case

In the \( N = 3 \) case we take
\[
\{\nabla_\alpha^I, \nabla_\beta^J\} = 2i\delta^{IJ}\nabla_{\alpha\beta} + 2i\varepsilon_{\alpha\beta} W^{IJ}, \quad (4.32)\]
where we require \( W^{IJ} \) to have dimension 1 and be a conformal primary
\[
[\mathcal{D}, W^{IJ}] = W^{IJ}, \quad [S_\alpha^I, W^{JK}] = 0. \quad (4.33)\]

We find the remaining commutation relations
\[
[\nabla_\alpha, \nabla_\alpha^I] = -\frac{1}{2}(\gamma_a)_{\alpha}^\beta [\nabla_\beta J, W^{IJ}], \quad (4.34)\]
\[
[\nabla_\alpha, \nabla_\beta] = \frac{i}{12} \varepsilon_{abc}(\gamma^c)^{\alpha\beta} \{\nabla_\alpha K, [\nabla_\beta L, W^{KL}]\}, \quad (4.35)\]
and the Bianchi identity
\[
[\nabla_\gamma^I, W^{JK}] = [\nabla_\gamma^I, W^{JK}] - \frac{1}{2}(\delta^{IJ}[\nabla_\gamma L, W^{KL}] - \delta^{IK}[\nabla_\gamma L, W^{JL}]]. \quad (4.36)\]

Based on our experience with the previous cases we expect that the covariant derivative algebra should be expressed entirely in terms of the \( N = 3 \) super Cotton tensor, \( W_\alpha \). We therefore conjecture
\[
W^{IJ} := \varepsilon^{IJK} W_K, \quad W^K = iW^\gamma S^K_\gamma + A(\gamma^c)^{\alpha\beta}(\nabla^K_\alpha W_\beta)K_c \quad (4.37)\]
and \( A \) is some constant to be determined. Requiring \( W^I \) to be a conformal primary fixes the coefficient as
\[
W^K = iW^\gamma S^K_\gamma + \frac{1}{2}(\gamma^c)^{\gamma\delta}(\nabla^K_\gamma W_\delta)K_c. \quad (4.38)\]
Furthermore, the Bianchi identity (4.36) is identically satisfied if we demand the conformally invariant constraint
\[ \nabla^\gamma W_\gamma = 0. \] (4.39)

We find the algebra to be
\[
\{\nabla^I_\alpha, \nabla^J_\beta\} = 2i\delta^{IJ}\nabla_{\alpha\beta} - 2\varepsilon_{\alpha\beta\varepsilon} \epsilon^{IJK} W^{\gamma} S_{\gamma L} + i\varepsilon_{\alpha\beta}(\gamma^c)^{\gamma\delta} \epsilon^{IJK}(\nabla_\gamma K W_\delta) K_c, \] (4.40a)
\[
[\nabla_a, \nabla_b^I] = \epsilon^{JKL}(\gamma_a)_{\beta\gamma} \left[iW^\gamma N_{KL} + i(\nabla_\gamma W^\delta) S_{\delta L}\right. \\
+ \frac{1}{4}(\gamma^c)_{\delta\rho}(\nabla_\gamma W_\delta^\mu)K_c], \] (4.40b)
\[
[\nabla_a, \nabla_b^I] = -\frac{1}{2}\varepsilon_{abc}(\gamma^c)^{\alpha\beta}\epsilon^{IJK} \left[(\nabla_\gamma W^\mu) N_{JK} + \frac{1}{2}(\nabla_\nu W_\gamma^\beta) S_{\mu L}\right. \\
- \frac{1}{12}(\gamma^d)_{\gamma\delta}(\nabla_\gamma W_\delta^\beta S_{\gamma K}) K_d]. \] (4.40c)

4.4 The \( N > 3 \) case

For the \( N > 3 \) case we again take
\[
\{\nabla^I_\alpha, \nabla^J_\beta\} = 2i\delta^{IJ}\nabla_{\alpha\beta} + 2i\varepsilon_{\alpha\beta} W^{IJ}, \] (4.41)
and require \( W^{IJ} \) to be of dimension-1 and a conformal primary
\[ [\mathbb{D}, W^{IJ}] = W^{IJ}, \quad [S^I_{\alpha\beta}, W^{JK}] = 0. \] (4.42)

Then we find the remaining commutation relations to be
\[
[\nabla_\alpha, \nabla_\alpha^I] = -\frac{1}{(N-1)}(\gamma_a)_{\alpha\beta}[\nabla_\beta J, W^{IJ}], \] (4.43)
\[
[\nabla_\alpha, \nabla_\beta^I] = \frac{i}{2N(N-1)}\varepsilon_{abc}(\gamma^c)^{\alpha\beta} \{\nabla_\alpha K, [\nabla_\beta L, W^{KL}]\}, \] (4.44)
where \( W^{IJ} \) satisfies the Bianchi identity
\[ [\nabla^I_\gamma, W^{JK}] = [\nabla^J_\gamma, W^{KL}] - \frac{1}{N-1}(\delta^{IJ}[\nabla_\gamma L, W^{KL}] - \delta^{IK}[\nabla_\gamma L, W^{JL}]). \] (4.45)

The above algebra and constraints are modeled on those describing a vector multiplet, see appendix B.

Now we expect that the covariant derivative algebra should be expressed entirely in terms of the \( N > 3 \) super Cotton tensor, \( W^{IJKL} \) and at the lowest dimension we expect it will appear in front of the \( SO(N) \) generator (see [16] or section 5). We therefore conjecture that \( W^{IJ} \) takes the form
\[ W^{IJ} = \frac{1}{2}W^{IJKL} N_{KL} + A(\nabla_\gamma W^{IJKL}) S_{\alpha L} + Bi(\gamma^c)^{\alpha\beta}(\nabla_\alpha K W^{IJKL}) K_c, \] (4.46)
with \( A \) and \( B \) some constants to be determined. Requiring \( W^{IJ} \) to be a conformal primary fixes the coefficients as
\[
W^{IJ} = \frac{1}{2}W^{IJKL} N_{KL} - \frac{1}{2(N-3)}(\nabla_\gamma W^{IJKL}) S_{\alpha L} \\
- \frac{i}{4(N-2)(N-3)}(\gamma^c)^{\alpha\beta}(\nabla_\alpha K W^{IJKL}) K_c, \] (4.47)
while the Bianchi identity (4.45) for $N > 4$ is identically satisfied if we demand the conformally invariant constraint

$$\nabla^I_a W^{JKLP} = \nabla^I_a W^{[JKLP]} - \frac{4}{N-3} \nabla_{aQ} W^{Q[JKL]5P]}I .$$  

(4.48)

In the $N = 4$ case, the equation (4.48) is trivially satisfied, and instead a fundamental Bianchi identity occurs at dimension-2. The super Cotton tensor is equivalently described by a scalar primary superfield in this case, $W^{IJKL} := \epsilon^{IJKL}W$, and eq. (4.45) is solved by

$$\nabla^a I \nabla^J_a W = \frac{1}{4} \epsilon^{IJK} \nabla^P_a \nabla^P W .$$  

(4.49)

The algebra of covariant derivatives for $N > 3$ may be found to be

$$\{\nabla^I_a, \nabla^J_{a\beta}\} = 2i\epsilon^{IJ} \nabla^a_{\alpha\beta} + i\epsilon_{\alpha\beta} W^{IJKL} N_{KL} - \frac{1}{N-3} \epsilon_{\alpha\beta} (\nabla^K W^{IJKL}) S_{KL}$$

$$+ \frac{1}{2(N-2)(N-3)} \epsilon_{\alpha\beta} (\gamma_K \nabla^K W^{IJKL}) S_{KL}$$

$$- \frac{1}{4(N-1)(N-2)(N-3)} \epsilon_{\alpha\beta} (\gamma_L \nabla^K W^{IJKL}) S_{KL}$$

$$- \frac{1}{N-1} (\gamma^K \nabla^K W^{IJKL}) S_{KL}$$

$$+ \frac{1}{2N-1} (\gamma^K \nabla^K W^{IJKL}) S_{KL}$$

(4.50a)

$$[\nabla_a, \nabla_b] = \frac{1}{2(N-3)} (\gamma_a)_{\beta\gamma} (\nabla^K W^{JPQK}) N_{PQ}$$

$$- \frac{1}{2(N-2)(N-3)} (\gamma_a)_{\beta\gamma} (\nabla^K \nabla^K W^{IJKL}) S_{KL}$$

$$+ \frac{1}{4(N-1)(N-2)(N-3)} (\gamma_a)_{\beta\gamma} (\nabla^K \nabla^K W^{IJKL}) S_{KL}$$

$$+ \frac{1}{2N-1} (\gamma^K \nabla^K W^{IJKL}) S_{KL}$$

(4.50b)

$$[\nabla_a, \nabla_b] = \frac{1}{4(N-2)(N-3)} \epsilon_{ab} (\gamma^c)_{\alpha\beta} (i(\nabla^K \nabla^K W^{IJKL}) N_{PQ}$$

$$+ \frac{1}{N-1} (\nabla^K \nabla^K W^{IJKL}) S_{KL}$$

$$+ \frac{1}{2N-1} (\gamma^K \nabla^K W^{IJKL}) S_{KL}$$

(4.50c)

It is worth mentioning that although we considered the $N > 3$ case separately, its covariant derivative algebra contains information about the lower $N$ cases. To see this let us consider each value of $N$ separately.

For the $N = 3$ case we may formally rewrite all terms in the algebra (4.50) involving spinor derivatives of $W^{IJKL}$ in terms of $W_a$,

$$\epsilon^{IJK} W_a := - \frac{i}{2(N-3)} \nabla_a W^{IJKL} .$$  

(4.51)

Then by independently switching off the remaining super Cotton tensor $W^{IJKL}$ we recover the algebra (4.40).

For the $N = 2$ case we similarly rewrite all terms involving two or more spinor derivatives in the algebra (4.50) in terms of $W_{a\beta}$:

$$\epsilon^{IJ} W_{a\beta} := - \frac{i}{2(N-2)(N-3)} \nabla_a W^{IJKL} .$$  

(4.52)

Independently switching off the remaining terms produces the algebra (4.22).
Finally, the $\mathcal{N} = 1$ case may be recovered similarly. To do so we introduce $W_{\alpha\beta\gamma}$ as

$$W_{\alpha\beta\gamma} := \frac{i}{(N - 1)(N - 2)(N - 3)} \nabla(\alpha K \nabla K L \nabla \gamma) \eta W^{JLK}$$

and set to zero the lower dimension fields in the algebra (4.50). This precisely recovers the $\mathcal{N} = 1$ algebra (4.12).

Thus we may recover all cases from the algebra (4.50). The corresponding Bianchi identities (4.10), (4.20) and (4.39) can be similarly deduced from the consequences of the Bianchi identity (4.48).

To conclude this section, we note that the $\mathcal{N} = 8$ case is special since the super Cotton tensor is a reducible tensor. We can consistently constrain $W_{IJKL}$ to be self-dual or anti-self-dual. The resulting conformal supergeometry may be shown to reduce, upon degauging spelled out in the next section, to the $\mathcal{N} = 8$ Weyl supermultiplet described in [17].

5 Degauging $\mathcal{N}$-extended conformal superspace

Although the conformal superspace constructed in the previous section involves gauging the entire superconformal algebra, this has traditionally not been the case for conventional superspace formulations. This is because it was seen as unnecessary, since with a smaller structure group the local scale and the special conformal transformations may be realized economically as special gauge transformations, known as super-Weyl transformations. This is exactly the approach adopted in [13, 16] where the $\mathcal{N}$-extended case in superspace was addressed by gauging $\text{SL}(2, \mathbb{R}) \times \text{SO}(\mathcal{N})$. We will refer to the construction of [13, 16] as $\text{SO}(\mathcal{N})$ superspace.

In this section we will show how the conventional gauging of [16] may be seen to originate within the conformal superspace formulated in the last section. We begin with a discussion of some of the salient facts of $\text{SO}(\mathcal{N})$ superspace and then show how the superspace may be derived via gauge-fixing some of the symmetries of conformal superspace. Furthermore, by using the degauging procedure of this section we verify our claim that the primary superfields appearing in each of the covariant derivative algebras are the corresponding super Cotton tensors. Finally, we derive the super-Weyl transformations of $\text{SO}(\mathcal{N})$ superspace entirely from our conformal superspace.

5.1 $\text{SO}(\mathcal{N})$ superspace

The superspace geometry of [13, 16] has the structure group $\text{SL}(2, \mathbb{R}) \times \text{SO}(\mathcal{N})$. The covariant derivatives are given by

$$D_A = E_A^M \partial_M - \frac{1}{2} \Omega_A^{bc} M_{bc} - \frac{1}{2} \Phi_A^{P Q} N_{P Q},$$

with the algebra

$$[D_A, D_B] = -T_{AB}^{C} D_C - \frac{1}{2} R_{AB}^{cd} M_{cd} - \frac{1}{2} R_{AB}^{P Q} N_{P Q}.$$
The torsion is subject to the conventional constraints [13]:

\begin{align*}
T^{IJc}_{\alpha\beta} &= -2i\delta^{IJ}(\gamma^c)_{\alpha\beta}, & \text{(dimension 0)} \\
T^{J}_{I\beta K} &= 0, & \text{(dimension 1/2)} \\
T^{I}_{ab\beta c} &= 0, & \text{(dimension 1)} \\
\varepsilon^{\beta\gamma} T_{a\beta\gamma} &= 0.
\end{align*}

(5.3a) (5.3b) (5.3c)

The solution to the constraints (5.3) is given in terms of the superfields

\begin{align*}
W^{IJKL} &= W^{[IJKL]}, & S^{IJ} &= S^{(IJ)}, & C^a_{IJ} &= C^a_{[IJ]},
\end{align*}

(5.4)

which appear at dimension-1 in the covariant derivative algebra\textsuperscript{27}

\begin{align*}
\{D^I_{\alpha}, D^J_{\beta}\} &= 2i\delta^{IJ}(\gamma^c)_{\alpha\beta}D^c I - 2i\varepsilon_{\alpha\beta\gamma}C^{I\gamma J}M_{\gamma\delta} - 4iS^{IJ}M_{\alpha\beta} \\
&\quad + \left(i\varepsilon_{\alpha\beta}W^{IJKL} - 4i\varepsilon_{\alpha\beta}S^K[I\delta^L] + iC_{\alpha\beta}^{KL\delta IJ} - 4iC_{\alpha\beta}^{K[I\delta^J]}\right)N_{KL}.
\end{align*}

(5.5)

The Bianchi identities imply the dimension-3/2 constraints \textsuperscript{16}

\begin{align*}
D^I_{\alpha}S^{JK} &= 2T^{I(JK)} + S^{I(JK)} - \frac{1}{N}S^{I\delta JK}, \\
D^I_{\alpha}C^{\beta J K} &= \frac{2}{3}\varepsilon^{\alpha(\beta}C^{IJKLM} + 3T^{IJKLM} + 4(D^{IJ}[S]K[I\delta^J] + \frac{N}{4}S^{[I}\delta^K[I\delta^J]}

+ C_{\alpha\beta\gamma}^{IJ} - 2C_{\alpha\beta\gamma}^{I[K\delta^J]}I, \\
D^I_{\alpha}W^{IJKLP} &= W^{aIJKLP} - 4C_{a[IJKL\delta^P]}I, \quad \text{(5.6b)}
\end{align*}

(5.6a) (5.6b) (5.6c)

where the symmetry properties of the superfields \(T^a_{IJK}, C^{a\beta\gamma}_{IJK}, C^{a\beta\gamma}_I\) and \(W^a_{IJKLP}\) are

\begin{align*}
T^a_{IJK} &= T^a_{[IJK]}, & \delta_{JK}T^a_{IJK} &= T^a_{[IJK]} = 0, \\
C^{a\beta\gamma}_{IJK} &= C^{a\beta\gamma}_{[IJK]}, & C^{a\beta\gamma}_I &= C^{a\beta\gamma}_I, \quad \text{and} \\
C^a_{IJK} &= C^a_{[IJK]}, & W^a_{IJKQP} &= W^a_{[IJKPQ]}. \quad \text{\(5.7a\) \(5.7b\) \(5.7c\)}
\end{align*}

(5.7a) (5.7b) (5.7c)

The superspace formulation of \textsuperscript{[13, 16]} describes conformal supergravity since the torsion constraints admit super-Weyl transformations. The constraints (5.3) can be shown to be invariant under arbitrary super-Weyl transformations of the form \textsuperscript{16}\textsuperscript{28}

\begin{align*}
\delta\sigma D^I_{\alpha} &= \frac{1}{2}\sigma D^I_{\alpha} + (D^I_{\alpha}\sigma)M_{\alpha\beta} + (D_{\alpha}J\sigma)N^{IJ}, \\
\delta\sigma D'_I &= \sigma D'_I + \frac{1}{2}(\gamma^c)\gamma^d(D^K_{\gamma\sigma}D_{\delta K} + \varepsilon_{abc}(D^b_{\gamma\sigma})M^c

+ \frac{1}{16}(\gamma^c)\gamma^d((D^K_{\gamma}, D^L_{\delta})\sigma)N_{KL}.
\end{align*}

\textsuperscript{27}We have placed a prime on the vector covariant derivative since it will differ from the one we find from straightforward degauging. We have also denoted the super Cotton tensor by \(W^{IJKL}\) instead of \(X^{IJKL}\).  
\textsuperscript{28}In the case \(N = 8\), the super-Weyl transformations were first given in \textsuperscript{[17]}. 

\textsuperspace
where $\sigma$ is a real unconstrained superfield. This requires the torsion and curvature components to transform as\(^{29}\)

$$
\delta_\sigma S^{IJ} = \sigma S^{IJ} - \frac{i}{8}[D^\gamma(I), D^\delta(J)]\sigma, \quad (5.8c)
$$

$$
\delta_\sigma C_a^{IJ} = \sigma C_a^{IJ} - \frac{1}{8}(\gamma_a)_{\gamma\delta}[D^{IJ}, D^{\delta\gamma}]\sigma, \quad (5.8d)
$$

$$
\delta_\sigma W^{IJKL} = \sigma W^{IJKL}. \quad (5.8e)
$$

Remarkably, the formulation of \([13, 16]\) treats all cases simultaneously and possesses the simple torsion constraints (5.3).

### 5.2 Conventional deauging

The structure of conformal superspace differs from that of \([13, 16]\) by the addition of dilatation and special conformal symmetry in the structure group. To fix these additional symmetries we follow the procedure given in \([27, 28]\).

We first note that under a $K_A$-transformation the dilatation gauge field $B = E^a B_a + E^\alpha I^\alpha$ transforms as

$$
\delta_K(\Lambda) B = -2E^a \Lambda_a + 2E^\alpha I^\alpha, \quad (5.9)
$$

which permits the gauge choice

$$
B_A = 0. \quad (5.10)
$$

This completely removes the dilatation connection from all the covariant derivatives.

The special conformal connection $\tilde{\mathfrak{g}}^A$ still remains. However, its symmetry has been fixed and as a result we introduce deauged covariant derivatives with no special conformal connection

$$
D_A := \nabla_A + \tilde{\mathfrak{g}}^B_A K_B, \quad (5.11)
$$

where $D_A$ corresponds to the structure group $\text{SL}(2, \mathbb{R}) \times \text{SO}(\mathcal{N})$ and possesses the algebra\(^{30}\)

$$
[D_A, D_B] = -\tilde{T}_{AB}^C D_C - \frac{1}{2} \tilde{R}_{AB}^{cd} M_{cd} - \frac{1}{2} \tilde{R}_{AB}^{IJ} N_{IJ}. \quad (5.12)
$$

In fact, it is possible to show that up to a redefinition of the deauged vector covariant derivatives, the torsion and curvature correspond to those of \([16]\). To see this we first note that the torsion tensors are related by

$$
T^a = \hat{T}^a, \quad T_I^\alpha = \hat{T}_I^\alpha + iE^a \wedge \tilde{\mathfrak{g}}_I^\beta (\gamma_a)_{\beta\gamma}. \quad (5.13)
$$

We then see that the torsion is constrained as in eqs. (5.3) except that\(^{31}\)

$$
\varepsilon^{\beta\gamma} \hat{T}^{[JK]}_{\alpha\beta} \neq 0. \quad (5.14)
$$

This is due to the fact that the deauged covariant derivatives are defined slightly differently to those of \([16]\). We now turn to explaining this point by explicitly deriving the constraints obeyed by the special conformal connection coefficients $\tilde{\mathfrak{g}}_A^B$.

\(^{29}\)Notice that $W^{IJKL}$ is the only superfield which transforms homogeneously. For $\mathcal{N} > 3$ it is the super Cotton tensor.

\(^{30}\)The hatted objects denote the deauged versions of the torsion and curvatures.

\(^{31}\)This torsion component only vanishes for $\mathcal{N} = 1$. 

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5.3 The degauged special conformal connection

In the gauge (5.10) the dilatation curvature is

\[ R(\mathcal{D})_{AB} = 2\mathcal{F}_{AB}(-1)^{\varepsilon_B} - 2\mathcal{F}_{BA}(-1)^{\varepsilon_A + \varepsilon_B} . \] (5.15)

The vanishing of the dilatation curvature constrains the special conformal connection as

\[ \mathcal{F}_{AB} = \mathcal{F}_{BA}(-1)^{\varepsilon_A \varepsilon_B + \varepsilon_A + \varepsilon_B} \] (5.16)

which implies

\[ \mathcal{F}^I_{\alpha} = -\mathcal{F}_{\alpha}^I \delta^I_{\alpha}, \]
\[ \mathcal{F}^K_{\alpha\beta} = \mathcal{C}^{(K}_{\alpha\beta} + \frac{2}{3} \varepsilon_{\gamma(\alpha} C_{\beta)}^K, \]
\[ \mathcal{F}_{ab} = \mathcal{F}_{ba} , \] (5.17)

where the superfields \( S^{IJ}, C^{I}_{\alpha\beta}, C^{I}_{\alpha\beta\gamma} \) and \( C_{\alpha}^I \) satisfy the symmetry properties

\[ S^{IJ} = S^{(IJ)}, \quad C_{\alpha}^I = C_{\alpha}^{(I)}, \quad C_{\alpha\beta}^I = C_{(\alpha\beta)}^I . \] (5.18)

From here it is possible to derive the degauged covariant derivative algebra by computing \([D_A, D_B]\). An efficient way to do this is to consider a conformal primary tensor superfield \( \Phi \) transforming in some representation of the remainder of the superconformal algebra (compare with [28]). For example, to determine the dimension-1 covariant derivative algebra we consider

\[ \{ D^I_{\alpha}, D^J_{\beta} \} \Phi = \{ \nabla^I_{\alpha}, \nabla^J_{\beta} \} \Phi + \bar{\mathcal{F}}^{I}_{\alpha} C^{[K}_{\gamma} \nabla^J_{\beta}] \Phi + \bar{\mathcal{F}}^{J}_{\beta} C^{[K}_{\gamma} \nabla^I_{\alpha}] \Phi . \] (5.19)

Making use of the form of \( \mathcal{F} \) and of the superconformal algebra we find

\[ \{ D^I_{\alpha}, D^J_{\beta} \} = 2i\delta^{IJ}(\gamma^c)_{\alpha\beta} D_c - 2i\varepsilon_{\alpha\beta} C^{\gamma\delta} I J M_{\gamma\delta} - 4iS^{IJ} M_{\alpha\beta} \]
\[ + \left( i\varepsilon_{\alpha\beta} W^{IJKL} - 4i\varepsilon_{\alpha\beta} S^{[I|J]} |L K L - 4iC_{\alpha\beta} K^{(I} |J)| L \right) N_{KL} , \]

\[ = 2i\delta^{IJ}(\gamma^c)_{\alpha\beta} D_c - 2i\varepsilon_{\alpha\beta} C^{\gamma\delta} I J M_{\gamma\delta} - 4iS^{IJ} M_{\alpha\beta} \]
\[ + \left( i\varepsilon_{\alpha\beta} W^{IJKL} - 4i\varepsilon_{\alpha\beta} S^{[I|J]} |L K L + iC_{\alpha\beta} K^{(I} |J)| L - 4iC_{\alpha\beta} K^{(I} |J)| L \right) N_{KL} , \] (5.20)

where

\[ D'_a = D_a - \frac{1}{2} C^I_{\alpha} N_{IJ} . \] (5.21)

The degauged covariant derivative algebra agrees with the one given in [16], with the vector covariant derivative defined above. The reason for the difference in the vector covariant derivative can be attributed to the appearance of the non-zero torsion component,

\[ \varepsilon^{J\gamma} T^{[J \gamma}_{a\beta} = -2C^{JK}_{a} , \] (5.22)

\[ 32 \text{We have lowered the index on the } K\text{-connection as } \mathcal{F}_{AB} = \eta_{cd} \mathcal{A}^c_A \text{ and } \mathcal{F}_{A\beta} = \varepsilon_{\gamma\delta} A^{[J \gamma}_{a\beta} \mathcal{A}_{\delta]} . \]
\[ 33 \text{The reason for the chosen coefficients will be clear later.} \]
which can be removed if one redefines the vector covariant derivative as in eq. (5.21).\textsuperscript{34} Therefore, the degauged version of conformal superspace is constrained in such a way so as to correspond to the formulation of [13, 16].

The torsion and curvature in [16] are constrained by a set of dimension-3/2 Bianchi identities. These must follow directly from the degauging procedure. To derive these explicitly, we analyze the constraints imposed on the special conformal curvatures

\begin{align}
R(S)_{\alpha\beta} &= 2D_{(\alpha}F_{\beta)\gamma} + \dot{T}_{\alpha\beta}D_{\gamma} + i\delta A^A\delta B^F(\gamma_c)\delta(-1)^{\varepsilon F} \\
R(K)_{\alpha\beta} &= 2D_{(\alpha}F_{\beta)\gamma} + \dot{T}_{\alpha\beta}D_{\gamma} + i\delta A^A\delta B^F(\gamma_c)\delta(-1)^{\varepsilon F} \\
&\quad - i\delta B^F\delta A^A(\gamma_c)\delta(-1)^{\varepsilon F A + \varepsilon A}, \\
&\quad - i\delta A^A\delta B^F(\gamma_c)\delta(-1)^{\varepsilon F A + \varepsilon A},
\end{align}

which appear in the covariant derivative algebra of the conformal covariant derivatives \( \nabla_A \).

We will consider each case in turn.

### 5.3.1 The \( \mathcal{N} = 1 \) case

In the \( \mathcal{N} = 1 \) case the special conformal connection is given by

\begin{align}
\tilde{\delta}_{\alpha\beta} &= -\tilde{\delta}_{\beta\alpha} = -i\varepsilon_{\alpha\beta}S, \\
\tilde{\delta}_{\alpha\beta,\gamma} &= -\tilde{\delta}_{\gamma,\alpha\beta} = C_{\alpha\beta\gamma} + \frac{2}{3}\varepsilon_{(\alpha C_{\beta)}}. 
\end{align}

From the \( \mathcal{N} = 1 \) algebra (4.12), we find \( R(S)_{\alpha\beta} = 0 \), which together with (5.23a) implies

\begin{align}
2D_{(\alpha}F_{\beta)\gamma} - 2i\tilde{\delta}_{\alpha\beta,\gamma} + 2i\tilde{\delta}_{\gamma,(\alpha,\beta)} &= 0 \\
\Longrightarrow C_{\alpha} &= D_{\alpha}S.
\end{align}

The constraint \( R(S)_{\alpha\beta} = 0 \) gives

\begin{align}
D_{\alpha}F_{\beta\alpha} - D_{\beta}F_{\alpha\alpha} + \dot{T}_{\alpha\beta}\tilde{\delta}_{\gamma\alpha} - i\tilde{\delta}_{\alpha,\alpha\beta} &= 0.
\end{align}

Then using the degauged torsion

\begin{align}
\dot{T}_{\alpha\beta} &= -(\gamma_\alpha)_{\beta}^\gamma S
\end{align}

we deduce the constraint

\begin{align}
D^\alpha C_{\alpha\beta\gamma} = -\frac{4i}{3}D_{\beta\gamma}S \quad \Longrightarrow \quad D_{\alpha}C_{\beta\gamma\delta} = D_{(\alpha}C_{\beta\gamma\delta)} - i\varepsilon_{\alpha(\beta D_{\gamma\delta})} S
\end{align}

and the final expression for the remaining component of \( \tilde{\delta}_{\alpha\beta} \):

\begin{align}
\tilde{\delta}_{\alpha\beta} &= -\frac{1}{4}(\gamma_\alpha)_{\alpha\beta}^\gamma(\gamma_\beta)\delta D_{(\alpha}C_{\beta\gamma\delta)} + \frac{i}{6}\eta_{\alpha\beta}D^2S + \eta_{\alpha\beta}S^2.
\end{align}

The above results show that we recover the \( \mathcal{N} = 1 \) superspace geometry of [16]. Moreover, the results for \( \tilde{\delta}_{\alpha\beta} \) are important because they enable us to take a superfield expression
in conformal superspace and degauge to the corresponding result in the superspace formulation of \[16\].

With the degauging procedure outlined above we can go one step further. First note that the superfield \( W_{\alpha\beta\gamma} \) has the appropriate index structure and dimension to correspond to the \( \mathcal{N} = 1 \) supersymmetric super Cotton tensor. To verify this we can derive an expression for \( W_{\alpha\beta\gamma} \) in the degauged superspace and show that it is proportional to the expression given in \[21\]. Using the constraint

\[
R(K)_{ab} = -\frac{1}{4} (\gamma_a^\alpha \beta (\gamma_b)^\gamma \delta) W_{\beta\gamma\delta}
\]

and the corresponding definition for \( R(K)_{aa} \) in \(5.23\), we find

\[
\frac{1}{4} (\gamma_a^\alpha \beta (\gamma_b)^\gamma \delta) W_{\beta\gamma\delta} = -\mathcal{D}_a \mathbf{\bar{\delta}}_{ab} + \mathcal{D}_a \mathbf{\bar{\delta}}_{ab} - \mathcal{D}_a \mathbf{\bar{\delta}}_{ab} + 2i \mathbf{\bar{\delta}}_{a} \gamma \delta_{(b)\gamma}\delta ,
\]

which gives

\[
W_{\alpha\beta\gamma} = -i\mathcal{D}^2 C_{\alpha\beta\gamma} - 2\mathcal{D}^2\gamma (\gamma_b)\delta S - 8SC_{\alpha\beta\gamma} .
\]

This is indeed proportional to the super Cotton tensor given in \[21\]. The divergenceless condition \(4.10\) reduces to

\[
\mathcal{D}^\alpha W_{\alpha\beta\gamma} = 0 .
\]

Since the degauged special conformal connection is important for comparing the results of conformal superspace with those derived in the formulation of \[16\], we summarize its components below:

\[
\mathbf{\bar{\delta}}_{\alpha\beta} = -i\varepsilon_{\alpha\beta} S , \quad (5.34a)
\]

\[
\mathbf{\bar{\delta}}_{\alpha\beta\gamma} = -i\varepsilon_{\alpha\beta\gamma} C_{\alpha\beta\gamma} + \frac{2}{3} \varepsilon_{\gamma(\alpha} \mathcal{D}_{\beta) S} , \quad (5.34b)
\]

\[
\mathbf{\bar{\delta}}_{ab} = -\frac{1}{4} (\gamma_a^\alpha \beta (\gamma_b)^\gamma \delta) \mathcal{D}_{(a} C_{\beta\gamma)} + \frac{1}{6} \eta_{ab} \mathcal{D}^2 S + \eta_{ab} S^2 . \quad (5.34c)
\]

### 5.3.2 The \( \mathcal{N} > 1 \) case

For \( \mathcal{N} > 1 \) we will need the lowest dimension component of \(5.23\),

\[
R(S)^{IJK}_{J\alpha\beta\gamma} = \mathcal{D}^I_\alpha \mathbf{\bar{\delta}}_{J\alpha\beta\gamma} + \mathcal{D}^J_\beta \mathbf{\bar{\delta}}_{I\gamma\delta} - 2i\mathbf{\bar{\delta}}_{IJ} \mathbf{\bar{\delta}}_{\alpha\beta\gamma} + i\mathbf{\bar{\delta}}_{I} \mathbf{\bar{\delta}}_{\alpha\beta\gamma} - 2\mathbf{\bar{\delta}}_{\alpha\beta\gamma} \mathbf{\bar{\delta}}_{IJ} \quad (5.35)
\]

and the constraints \( R(S)^{IJK}_{J\alpha\beta\gamma} = R(S)^{IJK}_{J\alpha\beta\gamma} = 0 \), which hold for arbitrary \( \mathcal{N} \). First we decompose \( \mathcal{D}^I_\alpha S^{JK} \) and \( \mathcal{D}^I_\alpha C_{\beta\gamma}^{JK} \) as:

\[
\mathcal{D}^I_\alpha S^{JK} = S^{IJ} + 2T^{I(JK)} + S^{(J\delta K)} I - \frac{1}{\mathcal{N}} S_{(J}^{I} \delta K) ,
\]

\[
\mathcal{D}^I_\alpha C_{\beta\gamma}^{JK} = C_{\alpha\beta\gamma}^{IJ} + 2T_{\alpha\beta\gamma}^{I(JK)} - 2C_{(\alpha\beta\gamma}^{(J\delta K)} I
\]

\[
+ \frac{2}{3} \varepsilon_{\alpha(\beta} \left( C_{\gamma)}^{J} IK + 2D_{\gamma}^{I(JK)} + T_{\gamma)}^{J\delta K} I \right) , \quad (5.36)
\]

where we define \( S^{IJ} \) by the decomposition

\[
S^{IJ} = S^{IJ} + S^{IJ} , \quad S = \frac{1}{\mathcal{N}} \delta_{IJ} S^{IJ} , \quad \delta_{IJ} S^{IJ} = 0 \quad (5.37)
\]
and we introduce superfields which satisfy the properties

\[ S_{\alpha}^{IJK} = S_{\alpha}^{(IJK)}, \quad \delta_{JK} S_{\alpha}^{IJK} = 0, \quad (5.38a) \]

\[ T_{\alpha}^{IJK} = T_{\alpha}^{(IJK)}, \quad \delta_{JK} T_{\alpha}^{IJK} = T_{\alpha}^{[IJK]} = 0, \quad (5.38b) \]

\[ D_{\alpha J} S^{IJ} = \frac{(N+2)(N-1)}{2N} S_{\alpha}^{I}, \quad (5.38c) \]

\[ C_{\alpha \beta \gamma}^{IJK} = C_{(\alpha \beta \gamma)}^{(IJK)}, \quad \tilde{C}_{\alpha \beta \gamma}^{I} = \tilde{C}_{(\alpha \beta \gamma)}^{I}, \quad (5.38d) \]

\[ T_{\alpha \beta \gamma}^{IJK} = T_{(\alpha \beta \gamma)}^{(IJK)}, \quad \delta_{JK} T_{\alpha \beta \gamma}^{IJK} = T_{\alpha \beta \gamma}^{[IJK]} = 0, \quad (5.38e) \]

\[ D_{\alpha}^{IJK} = D_{\alpha}^{[IJK]}, \quad \delta_{JK} D_{\alpha}^{IJK} = D_{\alpha}^{[IJK]} = 0, \quad (5.38f) \]

\[ D_{\gamma}^{I} C_{\beta \gamma}^{IJ} = (N-1) T_{\beta}^{J}. \quad (5.38g) \]

Symmetrizing the indices \( I, J \) and \( K \) in eq. (5.35) gives

\[ 0 = 2D_{\alpha}^{(I} \tilde{S}_{\beta)}^{J K} - 2i \delta^{(I J} \tilde{S}_{\alpha \beta \gamma}^{K)} + 2i \delta^{(I J} \tilde{S}_{\alpha, \beta}^{K)}, \quad (5.39) \]

which implies

\[ D_{\beta}^{(I} S_{\gamma)}^{J K} = \delta^{(I J} C_{\beta}^{K)}. \quad (5.40) \]

From here we find

\[ S_{\alpha}^{IJK} = 0, \quad C_{\beta}^{J} = \frac{N-1}{N} S_{\beta}^{J} + \frac{N}{N+2} D_{\beta}^{J} S. \quad (5.41) \]

Now symmetrizing the indices \( \alpha, \beta \) and \( \gamma \) in eq. (5.35) gives

\[ 0 = 2D_{\alpha}^{(I} \tilde{S}_{\beta \gamma)}^{J K} - 2i \delta^{(I J} \tilde{S}_{\alpha \beta \gamma}^{K)} + 2i \delta^{K(I} \tilde{S}_{\alpha \beta \gamma)}^{J)}, \quad (5.42) \]

and then deduce

\[ \tilde{C}_{\alpha \beta \gamma}^{J} = C_{\alpha \beta \gamma}^{J}, \quad T_{\alpha \beta \gamma}^{IJK} = 0. \quad (5.43) \]

Contracting the indices \( \alpha \) with \( \beta \) in eq. (5.35) and using \( R(S)^{I JK}_{(\alpha \beta) \gamma} = 0 \) leads to

\[ 0 = 2D^{\alpha[I} \tilde{S}_{\alpha \beta \gamma)}^{J K} + 2i \delta^{K[I} \tilde{S}_{\alpha \beta \gamma)}^{J). \quad (5.44) \]

From here it is easy to see that

\[ D_{\alpha}^{IJK} = 0. \quad (5.45) \]

Furthermore, we also find

\[ D^{\alpha I} C_{\alpha \beta}^{JK} = 3T_{\beta}^{JK I} + 4(D_{\beta}^{[I} S_{\gamma)}^{J K} + \frac{N-4}{N} S_{\beta}^{J} \delta^{K[I} \tilde{S}_{\gamma \alpha)}^{J)}. \quad (5.46) \]

Putting all these constraints together precisely recovers the Bianchi identities (5.6) except for the one involving \( W_{IJKL} \), which only appears for \( N > 3 \). In this case we can make use of

\[ R(S)^{a I J}_{\alpha K} = \frac{2i}{N-3} \nabla_{\gamma} L W_{IJ LK} = \frac{2i}{N-3} D_{\gamma}^{L} W_{IJ LK}. \quad (5.47) \]
and eq. (5.35) to derive
\[
C_\alpha^{IJK} = \frac{1}{N-3} \nabla_{\alpha L} W^{L IJK} .
\] (5.48)

Since \( W^{IJKL} \) is primary we recover the final Bianchi identity from eq. (4.48)
\[
D^I_\alpha W^{JKLP} = D^{[I}_\alpha W^{JKLP]} - 4 C_\alpha^{[JKL} \delta^P_I] .
\] (5.49)

So far we have obtained the special conformal connection components:
\[
F^{I \alpha}_{J \beta} = - F^{J \beta}_{I \alpha} = i C^{I \beta}_{J \alpha} - i \varepsilon^{I \alpha \beta} S^{KL} ,
\] (5.50a)
\[
F_{\alpha \beta, K} = - F_{K, \alpha \beta} = C_{\alpha \beta \gamma}^K + \frac{2}{3} \varepsilon_{\gamma (\alpha} \left( \frac{N-1}{N} S_{\beta) J} + \frac{N}{N+2} D^J_{\beta)} S \right) .
\] (5.50b)

The final component \( F^{ab} \) may also be found by considering each value of \( N \) separately. We have already shown how to do this for \( N = 1 \). Below we illustrate the higher \( N \) cases and derive the corresponding super Cotton tensors directly from the degauging of conformal superspace in the \( N = 2 \) and \( N = 3 \) cases.

**The \( N = 2 \) case.** In the \( N = 2 \) case the torsion component \( C_{\alpha}^{KL} \) takes the form
\[
C_{\alpha}^{KL} = C_{\alpha}^K L
\] (5.51)
and the remaining constraints become
\[
D^I_\alpha S^{JK} = S_\alpha^{(J \delta K)I} - \frac{1}{2} S^{I JK} ,
\] (5.52a)
\[
D^I_\alpha C_{\beta \gamma} = \varepsilon^{I J} C_{\alpha \beta \gamma}^J - \frac{1}{3} \varepsilon_{\alpha (\beta} \varepsilon^{IJ} (4 D^I_{\gamma}) S - S_{\gamma J}) ,
\] (5.52b)
\[
S^{IJ} = S^{\delta I J} + S^{I J} , \quad S := \frac{1}{2} \delta_{IJ} S^{IJ} , \quad \delta_{IJ} S^{IJ} = 0 .
\] (5.52c)

To construct both \( S^{ab} \) and the \( N = 2 \) super Cotton tensor in the formulation of [16] we make use of the special conformal curvature component (see the algebra (4.22))
\[
R(K)^I_{\alpha \beta} = i \varepsilon^{IJ} C_{\alpha \beta} W_{\gamma \delta} (\gamma^a)^{\gamma \delta} .
\] (5.53)

Plugging this result into eq. (5.23b) for the \( N = 2 \) case yields
\[
i \varepsilon^{IJ} C_{\alpha \beta} W_{\gamma \delta} (\gamma^a)^{\gamma \delta} = D^I_\alpha \delta^J_{\beta} + D^J_\beta \delta^I_{\alpha} - 2 i \delta^{IJ} \delta^{a}_{\alpha \beta} - i \delta^{\gamma \delta}_{\alpha K} \delta^J_{\beta} (\gamma^a)^{\gamma \delta} - i \delta^{\gamma \delta}_{\beta K} \delta^I_{\alpha} (\gamma^a)^{\gamma \delta} .
\] (5.54)

We then find the super Cotton tensor by antisymmetrizing \( \alpha \) with \( \beta \) and \( I \) with \( J \)
\[
W_{\alpha \beta} = - \frac{i}{4} \varepsilon_{IJ} D^J_\gamma \delta^I_{\gamma \alpha \beta} + 2 S C_{\alpha \beta}
\]
\[
= \frac{1}{8} \left[ D^I_\alpha , D^J_\beta \right] C_{\alpha \beta} - \frac{i}{4} \varepsilon_{IJ} \left[ D^I_\alpha , D^J_\beta \right] S + 2 S C_{\alpha \beta} ,
\] (5.55)
which reads in the complex basis
\[
W_{\alpha \beta} = - \frac{i}{4} \left[ D^I_\alpha , \bar{D}^J_\gamma \right] C_{\alpha \beta} + \frac{1}{2} \left[ D^I_\alpha , D^J_\beta \right] S + 2 S C_{\alpha \beta} .
\] (5.56)

This is proportional to the super Cotton tensor constructed in [33].

\(^{35}\)One may always choose a super-Weyl gauge \( S = 0 \) in which the expression (5.56) reduces to that given for the first time by Zupnik and Pak [11].
On the other hand symmetrizing $\alpha$ with $\beta$ and contracting $I$ with $J$ in (5.54) gives

$$\tilde{g}_{ab} = \frac{i}{4} \delta_{IJ} (\gamma_a)^{\alpha\beta} D_\alpha D_\beta \tilde{I}_{a,b} + \frac{i}{4} \delta_{IJ} (\gamma_a)^{\alpha\beta} (\gamma_b)^{\gamma\delta} \eta^{IJK} \tilde{I}_{\alpha\beta} \tilde{g}_{a,b}^{IJK}$$

$$= -i \frac{1}{4} (\gamma_a)^{\alpha\beta} (\gamma_b)^{\gamma\delta} \eta^{IJK} [D_\alpha D_\beta] C_{\gamma\delta} - \frac{i}{2} \eta_{ab} D_\gamma D_\delta S + \eta_{ab} S_{IJK} + 2 \eta_{ab} S^2.$$  (5.57)

It should be mentioned that degauging the constraint (4.20) gives

$$D^{\alpha I} W_{\alpha\beta} = 0.$$  (5.58)

The $\mathcal{N} = 3$ case. In the $\mathcal{N} = 3$ case the super Cotton tensor appears in the special conformal curvature component

$$R(S)^{IJK}_{\alpha\beta\gamma} = 2 \varepsilon_{\alpha\beta\gamma} W_{IJK}.$$  (5.59)

Therefore, using eq. (5.35), we may derive an expression for the super Cotton tensor in the formulation of [16]. We find

$$2 \varepsilon^{IJK} W_{\gamma} = D^{\alpha I} \tilde{g}_{\alpha I}^{\gamma} + i \varepsilon^{IJK} [D_\alpha C_{\beta\gamma}^{IJK}],$$  (5.60)

which gives

$$W_\alpha = \frac{1}{12} \varepsilon^{IJK} D_\beta I \tilde{g}_{\alpha I}^{\gamma} = \frac{1}{12} \varepsilon^{IJK} D_\beta I C_{\alpha\beta}^{IJK}.$$  (5.61)

As a check one can show that $W_\alpha$ transforms homogeneously under the super-Weyl transformations (5.8). One can also show that the constraint (4.39) degauges to

$$D^{\gamma I} W_\gamma = 0.$$  (5.62)

To construct $\tilde{g}_{ab}$ we use the special conformal curvature component (see the algebra (4.40))

$$R(K)^{IJa}_{\alpha\beta} = -i \varepsilon_{\alpha\beta} (\gamma^a)^{\gamma\delta} \varepsilon^{IJK} \nabla_{\gamma} W_\delta.$$  (5.63)

Making use of eq. (5.23b) and the fact that $R(K)^{IJa}_{(\alpha\beta)a} = 0$ gives us

$$0 = 2 D_\gamma (\tilde{g}_{K})^{IJa} - 6 i \varepsilon_{\alpha\beta} (\gamma^a)^{\gamma\delta} \tilde{g}_{(\alpha K \delta)}^{\gamma} (\gamma^a)^{\gamma\delta},$$  (5.64)

which yields

$$\tilde{g}_{ab} = -\frac{i}{12} (\gamma_a)^{\alpha\beta} (\gamma_b)^{\gamma\delta} D_\alpha D_\beta C_{\gamma\delta}^{IJa} - \frac{i}{27} \eta_{ab} D_\gamma D_\delta S_{IJa} - \frac{i}{30} \eta_{ab} D_\gamma D_\delta D_\gamma D_\delta S$$

$$- \frac{1}{6} (\gamma_a)^{\alpha\beta} (\gamma_b)^{\gamma\delta} C_{\alpha\gamma}^{IJK} C_{\beta\delta}^{IJK} + \frac{1}{3} \eta_{ab} S_{IJK} S_{IJK} + \eta_{ab} S^2.$$  (5.65)
The $\mathcal{N} > 3$ case. Similarly, using eq. (5.23b) and the fact that $R(K)^I_{I(\alpha \beta)I} = 0$ for $\mathcal{N} > 3$ gives us
\begin{equation}
0 = 2D^I_{(a} \delta^{\beta)} l^a - 2i\mathcal{N} \delta^{\alpha}_a l^a + 2i\delta I_{a} \gamma I \delta(\gamma^a)_{\gamma^b},
\end{equation}
which yields
\begin{equation}
\mathcal{N} \delta_{ab} = -\frac{i}{4}(\gamma_{(a})^{\alpha \beta}(\gamma_{b)}^{\gamma \delta} D_{\alpha I} C_{\beta \gamma \delta I} - \frac{i}{6} \frac{\mathcal{N} - 1}{\mathcal{N}} \eta_{ab} D^\gamma D^I_s l^I - \frac{i}{6} \frac{\mathcal{N}}{\mathcal{N} + 2} \eta_{ab} D^\gamma D^I_s l^I S
- \frac{1}{2}(\gamma_{(a})^{\alpha \beta}(\gamma_{b)}^{\gamma \delta} C_{\alpha \gamma I} C_{\beta \delta I I} + \eta_{ab} S^I_{I I} S_I + \mathcal{N} \eta_{ab} S^2).
\end{equation}

We may summarize the components of $\mathcal{F}_{AB}$ for $\mathcal{N} > 1$ as:
\begin{equation}
\mathcal{F}_{\alpha \beta} = -\delta_{\alpha \beta} = i \varepsilon_{I J} C_{\alpha \beta I J} - i \varepsilon_{\alpha \beta I J} S_{I J},
\end{equation}
\begin{equation}
\mathcal{F}_{\alpha \gamma} = -\delta_{\alpha \gamma} = C_{\alpha \gamma I} I_J C_{\beta \delta I J} + \frac{2}{3} \varepsilon_{\gamma(a}(\frac{\mathcal{N} - 1}{\mathcal{N}} S_{\beta)}^I J + \frac{\mathcal{N}}{\mathcal{N} + 2} D^I_s l^I S),
\end{equation}
\begin{equation}
\mathcal{F}_{ab} = -\frac{i}{4\mathcal{N}}(\gamma_{(a})^{\alpha \beta}(\gamma_{b)}^{\gamma \delta} D_{\alpha I} C_{\beta \gamma \delta I} - \frac{i}{6} \frac{\mathcal{N} - 1}{\mathcal{N}} \eta_{ab} D^\gamma D^I_s l^I - \frac{i}{6(\mathcal{N} + 2)} \eta_{ab} D^\gamma D^I_s l^I S
- \frac{1}{2\mathcal{N}}(\gamma_{(a})^{\alpha \beta}(\gamma_{b)}^{\gamma \delta} C_{\alpha \gamma I} C_{\beta \delta I I} + \frac{1}{\mathcal{N}} \eta_{ab} S^I_{I I} S_I + \eta_{ab} S^2).
\end{equation}

5.4 The conformal origin of the super-Weyl transformations

In [13, 16] a formulation for conformal supergravity was given in which the dilatations and special conformal transformations were not realized manifestly. As we have just demonstrated, this SO($\mathcal{N}$) superspace can be viewed as a “degauged” version of our conformal superspace, where the special conformal symmetry has been fixed by the gauge condition $B_A = 0$. As we have left the dilatational symmetry unfixed, it must survive as an additional nonlinear transformation not residing in the remaining structure group or the general coordinate transformations. This is precisely the super-Weyl transformation\(^{36}\) and is what ensures the superspace formulation of [16] describes conformal supergravity. We may now show explicitly how these super-Weyl transformations originate in the degauging of conformal superspace.

Suppose we have gauge fixed the dilatation connection to vanish by using the special conformal symmetry. If we now perform a dilatation with parameter $\sigma$, we must accompany it with an additional $K_A$ transformation with $\sigma$-dependent parameters $\Lambda^A(\sigma)$ to maintain the gauge $B_A = 0$. With respect to the covariant derivatives this means
\begin{equation}
\delta K^A(\Lambda(\sigma)) \nabla_A + \delta D(\sigma) \nabla_A
\end{equation}
cannot contain any terms proportional to the dilatation generator $\mathbb{D}$. Using eq. (2.26), we find
\begin{equation}
\Lambda^a(\sigma) = \frac{1}{2} D^a(\sigma), \quad \Lambda^K_I(\sigma) = -\frac{1}{2} D^I_s(\sigma).
\end{equation}
Then the super-Weyl transformations may be simply read off from
\begin{equation}
\delta_N \nabla_A := \delta K^I(\Lambda(\sigma)) \nabla_A + \delta D(\sigma) \nabla_A.
\end{equation}
\(^{36}\)This is exactly the same origin as the Weyl transformation in conformal gravity as well as the super-Weyl transformations for $4D \mathcal{N} = 1$ [27] and $\mathcal{N} = 2$ [28] conformal supergravity.
The super-Weyl transformations of the degauged covariant derivatives $D_A$ and the special conformal connection can be read from
\[ \delta_\sigma \nabla_A = \delta_\sigma D_A - \delta_\sigma \tilde{A}^B K_B. \] (5.72)

The super-Weyl transformations of $D_A$ are found to be
\[ \delta_\sigma D^I_\alpha = \frac{1}{2} \sigma D^I_\alpha + \frac{1}{2} (\gamma^a)_\alpha \beta (D^I \gamma^a) D^K_\beta N^{KJ}, \] (5.73a)
\[ \delta_\sigma D_a = \sigma D_a + \frac{i}{2} (\gamma^a)_{\alpha \beta} (D^K \gamma^a) D^K_\beta + \varepsilon_{abc} (D^K \gamma^a) M^c, \] (5.73b)
while the super-Weyl transformation of, for example, $\tilde{F}_{IJ}^\alpha$ is
\[ \delta_\sigma \tilde{F}_{IJ}^\alpha = \sigma \tilde{F}_{IJ}^\alpha - \frac{1}{4} [D^I_\alpha, D^J_\beta] \sigma. \] (5.74)

Eq. (5.74) recovers the super-Weyl transformations of the torsion components in the formulation of [16], while the super-Weyl transformation of the degauged vector covariant derivative does not exactly match that of eq. (5.8), since it does not contain an $\text{SO}(N)$ contribution. However, the redefined vector covariant derivatives
\[ D'_a = D_a - \frac{1}{2} C_a^{IJ} N_{IJ} \] (5.75)
possess the appropriate transformation law as expected.

Note that all primary superfields $\Phi$ transform homogeneously
\[ \delta_K (\Lambda (\sigma)) \Phi + \delta_D (\sigma) \Phi = \delta_D (\sigma) \Phi = \sigma \Delta \Phi, \] (5.76)
where $\Delta$ is the dimension of $\Phi$
\[ \mathcal{D} \Phi = \Delta \Phi. \] (5.77)

Therefore we also have
\[ \delta_\sigma W^{IJKL} = \sigma W^{IJKL}. \] (5.78)

6 Discussion and outlook

In this paper we have constructed a new off-shell formulation for $\mathcal{N}$-extended conformal supergravity in three dimensions, which possesses a number of key properties. Firstly, it gauges the entire superconformal algebra without the need to introduce the super-Weyl transformations as in the conventional approach [13, 16], which is based on the local structure group $\text{SL}(2,R) \times \text{SO}(\mathcal{N})$. Secondly, it possesses a simple covariant derivative algebra, the structure of which resembles that of the super Yang-Mills algebra. Thirdly, the entire algebra of covariant derivatives is expressed in terms of a single primary superfield, the $\mathcal{N}$-extended super Cotton tensor, which makes our formulation quite geometrical.

Upon degauging of the local special conformal and $S$-supersymmetry transformations, the conformal superspace constructed in this paper reduces to the conventional formulation for conformal supergravity [13, 16], with the local scale transformation turning into the
super-Weyl transformation. This means that there is no need to carry out a thorough component analysis to justify that our formalism is indeed suitable to describe conformal supergravity.

Although the suitability to describe conformal supergravity is justified, it is worth mentioning that the conformal superspace may be shown to reduce in components to the superconformal framework of \([3, 4]\) for the \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) cases. Recall that a supersymmetry transformation with parameter \(\xi^\alpha_I\) should be identified with a supergravity gauge transformation (2.26) with \(K = \xi^I_l \nabla_a\). For the \(\mathcal{N} = 1\) case and with the general ansatz (4.3), we may derive the supersymmetry transformations of the connection fields using (2.25):\(^{37}\)

\[
\delta Qe^a_m = -i(\xi^a_m \psi_m) - \frac{1}{2}(\xi\gamma_m)^\beta W(P)_{\beta}^a |, \\
\frac{1}{2} \delta Q\psi_m^\alpha = (\partial_m - \frac{1}{2} \omega^a_m M_{ab} + \frac{1}{2} b_m)\xi^\alpha - \frac{1}{2}(\xi\gamma_m)^\beta W(Q)_{\beta}^a |, \\
\delta Qb_m = -(\xi\phi_m) - \frac{1}{2}(\xi\gamma_m)^\beta W(\mathcal{D})_{\beta} |, \\
\delta Q\omega^a_{mb} = -e^{abc}(\gamma_c)_\alpha\beta \xi^\alpha_m \phi_{m\beta} - \frac{1}{2}(\xi\gamma_m)^\beta W(M)_{\beta}^{ab} |, \\
\frac{1}{2} \delta Q\phi_m^\alpha = -i(\xi\gamma_m)^\beta f_m^b - \frac{1}{2}(\xi\gamma_m)^\beta W(S)_{\beta}^a |, \\
\delta Qf_m^a = -\frac{1}{2}(\xi\gamma_m)^\beta W(K)_{\beta}^a |. 
\]

Now comparing with \([3]\) we see that we must set \(W_\alpha = W(K)_\alpha^a K_a\), which was what we used in superspace. Therefore, at least for the \(\mathcal{N} = 1\) case the conformal superspace correctly reduces in components (up to conventions) to those derived within the superconformal tensor calculus. As a result our formulation may provide a useful bridge between the two approaches.

As compared with the conventional formulation \([13, 16]\), conformal superspace has a larger gauge group. A nontrivial manifestation of this enlarged gauge symmetry is a dramatic reduction of dimension-1 curvature tensors. In the conventional setting, there are several such tensors: \(S^{IJ} = S^{(IJ)}, C_a^{IJ} = C_a^{[IJ]}\) and \(W^{IJKL} = W^{[IJKL]}\). Their presence makes the algebra of covariant derivatives rather involved and somewhat cumbersome from the point of view of practical calculations. On the other hand, conformal superspace has no dimension-1 curvature for the cases \(\mathcal{N} = 1, 2, 3\), while for \(\mathcal{N} > 3\) the entire algebra of covariant derivatives is constructed entirely in terms of the super Cotton tensor \(W^{IJKL}\).

The fact that the dimension-1 tensors \(S^{IJ}\) and \(C_a^{IJ}\) do not show up in conformal superspace is of primary importance for the explicit construction of conformal supergravity actions. In section 1, we briefly discussed the method proposed in \([21]\) to construct off-shell supergravity actions in superspace and the technical difficulty in implementing this method for the case \(\mathcal{N} \geq 2\). Let us recall that the main technical problem is the existence of a two-parameter freedom to choose the vector covariant derivative, eq. (1.1). This leads to a two-parameter family of closed three-forms that should be considered as candidates to

\(^{37}\)Per the usual convention, we have identified \(\psi_m^\alpha = 2 E_m^\alpha |\) and \(\phi_m^\alpha = 2 \tilde{E}_m^\alpha |\).
generate the action for conformal supergravity. This two-parameter family has only one true candidate subject to the condition of super-Weyl invariance modulo exact terms. The explicit construction of such a form is highly nontrivial. In conformal superspace, however, both of these problems do not occur by construction. Firstly, the vector covariant derivative is uniquely defined. Secondly, the super-Weyl invariance is built in \textit{ab initio}. The problem of the explicit construction of off-shell actions for conformal supergravities will be addressed in an accompanying paper \cite{34}.

In this paper we only considered the vector supermultiplets in conformal superspace. The rigid \( \mathcal{N} = 3 \) and \( \mathcal{N} = 4 \) projective hypermultiplets introduced in \cite{32} may naturally be lifted to conformal superspace. This will be discussed elsewhere.

Using the explicit structure of the super Cotton tensors discussed above, we can predict the superfield types of conformal supergravity prepotentials for \( 1 \leq \mathcal{N} \leq 4 \) without working out unconstrained prepotential formulations for conformal supergravity theories.\footnote{On general grounds, such formulations should exist at least in the cases \( 1 \leq \mathcal{N} \leq 4 \), and they have been constructed in the cases \( \mathcal{N} = 1 \) \cite{10} and \( \mathcal{N} = 2 \) \cite{33}.} Indeed, given the off-shell action for conformal supergravity, \( S_{CSG} \), we expect that

\[
W \propto \frac{\delta S_{CSG}}{\delta H},
\]

with \( W \) and \( H \) being respectively the super Cotton tensor and the conformal prepotential (with all indices suppressed). This implies that the unconstrained conformal prepotentials should be as follows: \( H_{\alpha\beta\gamma} \) for \( \mathcal{N} = 1 \) \cite{10}, \( H_{\alpha\beta} \) for \( \mathcal{N} = 2 \) \cite{11,33}, \( H_{\alpha} \) for \( \mathcal{N} = 3 \), and \( H \) for \( \mathcal{N} = 4 \). Using the harmonic superspace techniques \cite{35}, one may derive the \( \mathcal{N} = 3 \) and \( \mathcal{N} = 4 \) prepotentials by generalizing the four-dimensional \( \mathcal{N} = 2 \) analysis of \cite{36} (see also \cite{37}).

In the component approach, there is a remarkable (AdS/CFT inspired) construction \cite{38} of the \( \mathcal{N} = 8 \) off-shell conformal supergravity in three dimensions starting from the \( \mathcal{N} = 8 \) SO(8) gauge supergravity in four dimensions \cite{39}, which has an AdS\(_4\) solution. It would be interesting to derive a superspace analog of this construction.

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\appendix

\section{Notation and conventions}

Our conventions for spinors in three spacetime dimensions (3D) follow closely those of \cite{16}.\footnote{In particular they are compatible with the 4D two-component spinor formalism used in \cite{40,41}.}

We summarize them here.
Spinor indices are raised and lowered using the $\SL(2,\mathbb{R})$ invariant tensors

$$
\varepsilon_{\alpha \beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha \beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha \gamma} \varepsilon_{\gamma \beta} = \delta_{\beta}^{\alpha} \quad \text{(A.1)}
$$

as follows:

$$
\psi^\alpha = \varepsilon^{\alpha \beta} \psi_{\beta}, \quad \psi_{\alpha} = \varepsilon_{\alpha \beta} \psi^\beta. \quad \text{(A.2)}
$$

We use a Majorana representation in which all the $\gamma$-matrices are real and any Majorana spinor $\psi^\alpha$ is real,

$$
(\psi^\alpha)^* = \psi^\alpha, \quad (\psi_{\alpha})^* = \psi_{\alpha}. \quad \text{(A.3)}
$$

In such a representation the 3D gamma matrices $(\gamma_{a})_{\alpha \beta}$ and $(\gamma_{a})^{\alpha \beta}$ are real and symmetric.

The matrices

$$
\gamma_{a} := (\gamma_{a})_{\alpha \beta} = \varepsilon^{\beta \gamma} (\gamma_{a})_{\alpha \gamma} \quad \text{(A.4)}
$$

satisfy the relations

$$
\{ \gamma_{a}, \gamma_{b} \} = 2 \eta_{ab}, \quad \text{(A.5a)}
$$

$$
\gamma_{a} \gamma_{b} = \eta_{ab} + \varepsilon_{abc} \gamma_{c}, \quad \text{(A.5b)}
$$

where the 3D Minkowski metric is $\eta_{ab} = \eta^{ab} = \text{diag}(-1, 1, 1)$ and the Levi-Civita tensor is normalized as $\varepsilon_{012} = -\varepsilon^{012} = -1$. Some useful relations involving $\gamma$-matrices and the Levi-Civita tensor are

$$
(\gamma^{a})_{\alpha \beta} (\gamma_{a})_{\gamma \delta} = 2 \varepsilon_{\alpha (\gamma} \varepsilon_{\delta) \beta}, \quad \text{(A.6a)}
$$

$$
\varepsilon_{abc} (\gamma^{b})_{\alpha \beta} (\gamma^{c})_{\gamma \delta} = \varepsilon_{\gamma (\alpha (\gamma} \varepsilon_{\delta) \beta) \delta + \varepsilon_{\delta (\alpha (\gamma} \varepsilon_{\beta) \gamma)}, \quad \text{(A.6b)}
$$

$$
\text{tr} [\gamma_{a} \gamma_{b} \gamma_{c} \gamma_{d}] = 2 \eta_{ab} \eta_{cd} - 2 \eta_{ac} \eta_{bd} + 2 \eta_{ad} \eta_{bc}, \quad \text{(A.6c)}
$$

$$
\varepsilon_{abc} \varepsilon^{def} = -6 \delta_{d}^{f} \delta_{e}^{d} \delta_{c}^{f} \quad \text{(A.6d)}
$$

Given a three-vector, $V_{a}$, it can equivalently be realized as a symmetric spinor $V_{\alpha \beta} = V_{\beta \alpha}$. The relationship between $V_{a}$ and $V_{\alpha \beta}$ is as follows:

$$
V_{\alpha \beta} := (\gamma^{a})_{\alpha \beta} V_{a} = V_{\beta \alpha}, \quad V_{a} = -\frac{1}{2} (\gamma_{a})^{\alpha \beta} V_{\alpha \beta}. \quad \text{(A.7)}
$$

In three dimensions an antisymmetric tensor $F_{ab} = -F_{ba}$ is Hodge-dual to a three-vector $F_{a}$:

$$
F_{a} = \frac{1}{2} \varepsilon_{abc} F^{bc}, \quad F_{ab} = -\varepsilon_{abc} F^{c}. \quad \text{(A.8)}
$$

The symmetric spinor $F_{\alpha \beta} = F_{\beta \alpha}$ associated with $F_{a}$, can equivalently be defined in terms of $F_{ab}$:

$$
F_{\alpha \beta} := (\gamma^{a})_{\alpha \beta} F_{a} = \frac{1}{2} (\gamma^{a})_{\alpha \beta} \varepsilon_{abc} F^{bc}. \quad \text{(A.9)}
$$
It follows that the three algebraic objects, $F_a$, $F_{ab}$ and $F_{\alpha\beta}$, are in one-to-one correspondence with each other, $F_a \leftrightarrow F_{ab} \leftrightarrow F_{\alpha\beta}$. Their corresponding inner products are related to each other as follows:

$$- F^a G_a = \frac{1}{2} F^{ab} G_{ab} = \frac{1}{2} F^{\alpha\beta} G_{\alpha\beta} .$$  \hspace{1cm} (A.10)

The spinor covariant derivatives in Minkowski superspace $\mathbb{R}^{3|2\mathcal{N}}$ satisfy the anticommutation relations

$$\{ D^I_\alpha, D^J_\beta \} = 2i \delta^{IJ}_c (\gamma^c)_{\alpha\beta} \partial_c.$$ \hspace{1cm} (A.11)

They may be realized as

$$D^I_\alpha = \frac{\partial}{\partial \theta^I_\alpha} + i (\gamma^b)_{\alpha\beta} \theta^b_\beta \partial_b ,$$ \hspace{1cm} (A.12)

where $SO(\mathcal{N})$ indices may be equivalently written in the upper or lower position.

Due to the reality of the 3D spinors we have the conjugation rule

$$(D^I_\alpha F)^* = -(-1)^{\varepsilon(F)} D^I_\alpha \bar{F} ,$$ \hspace{1cm} (A.13)

with $F$ a superfield of Grassmann parity $\varepsilon(F)$ and $\bar{F} = (F)^*$.  

For $\mathcal{N} > 1$ it is useful to introduce the totally antisymmetric invariant $SO(\mathcal{N})$ tensor

$$\varepsilon_{I_1 \cdots I_N} = \varepsilon^{[I_1 \cdots I_N]} ,$$ \hspace{1cm} (A.14)

normalized as

$$\varepsilon^{12 \cdots \mathcal{N}} = \varepsilon_{12 \cdots \mathcal{N}} = 1 .$$ \hspace{1cm} (A.15)

In the $\mathcal{N} = 2$ case $\varepsilon^{IJ}$ should not be confused with $\varepsilon_{\alpha\beta}$ since it is normalized differently.

**B Coupling to a vector multiplet**

Here we do not consider general matter couplings within our formulation, however the constraints imposed on the geometry of section 4 were modeled on an abelian vector multiplet.

It is therefore natural to discuss the coupling of an Abelian $\mathcal{N}$-extended vector multiplet

$$V = dz^M V_M = E^A V_A , \quad V_A := E_A^M V_M$$ \hspace{1cm} (B.1)

to conformal supergravity, both for completeness and as a straightforward extension of the results in [16]. To do so we introduce the gauge covariant derivatives

$$\nabla_A := \nabla_A - V_A Z , \quad [Z, \nabla_A] = 0 ,$$ \hspace{1cm} (B.2)

with $V_A(z)$ the gauge connection associated with the generator $Z$. The gauge transformation of $V_A$ is

$$\delta V_A = \nabla_A \tau ,$$ \hspace{1cm} (B.3)

with $\tau(z)$ an arbitrary scalar superfield.
The algebra of covariant derivatives is found to be
\[ [\nabla_A, \nabla_B] = -T^{\phantom{CD}C}_{AB} \nabla_C - \frac{1}{2} R(M)_{AB \alpha \beta} M_{\alpha \beta} - \frac{1}{2} R(N)_{AB \alpha \beta} N_{\alpha \beta} - R(D)_{AB \alpha \beta} \]
\[ - R(S)_{AB \alpha \beta} S^\gamma_{\gamma \alpha \beta} - R(K)_{AB \alpha \beta} K_{\alpha \beta} - F_{AB \gamma} Z, \]  
(B.4)
where the torsion and curvatures are those of conformal superspace but with \( F_{AB} \) the gauge-invariant field strength. The field strength \( F_{AB} \) satisfies the Bianchi identity
\[ \nabla_{[A} F_{B]} + T_{AB}^\gamma S^\gamma_{\gamma \alpha \beta} = 0, \]  
(B.5)
and must be subject to covariant constraints to describe an irreducible vector multiplet. The structure of the constraints and their consequence is different for \( \mathcal{N} = 1 \) and for \( \mathcal{N} > 1 \).

B.1 The \( \mathcal{N} = 1 \) case
In the \( \mathcal{N} = 1 \) case, one imposes the covariant constraint \([10, 42]\)
\[ F_{\alpha \beta} = 0. \]  
(B.6)
Then one derives from the Bianchi identities the remaining components
\[ F_{\alpha \beta} = \frac{1}{2} (\gamma a)_{\gamma} G_{\gamma}, \]  
(B.7a)
\[ F_{ab} = -\frac{1}{4} \varepsilon_{abc} (\gamma^c)_{\gamma \delta} \nabla_\gamma G_{\delta}, \]  
(B.7b)
together with the dimension-2 differential constraint on the spinor field strength
\[ \nabla^\alpha G_{\alpha} = 0. \]  
(B.8)
Furthermore, the Jacobi identities require \( G_{\alpha} \) to be primary and of dimension-3/2:
\[ S_{\beta} G_{\alpha} = 0, \quad \mathbb{D} G_{\alpha} = \frac{3}{2} G_{\alpha}. \]  
(B.9)

B.2 The \( \mathcal{N} > 1 \) case
For \( \mathcal{N} > 1 \) one imposes the following dimension-1 covariant constraint \([11, 43, 44]\)
\[ F_{\alpha \beta} = -2i \varepsilon_{\alpha \beta} G^{IJ}, \]  
(B.10)
where \( G^{IJ} \) is antisymmetric, primary and of dimension-1
\[ G^{IJ} = -G^{JI}, \quad S^I_{JK} G^{IJ} = 0, \quad \mathbb{D} G^{IJ} = G^{IJ}. \]  
(B.11)
Note that these constraints are a natural generalization of the \( \mathcal{N} > 1 \) constraints in four dimensions \([45, 46]\). The Bianchi identities then give the remaining field strength components:
\[ F^{I}_{\alpha} = \frac{1}{(\mathcal{N} - 1)} (\gamma a)_{\alpha} \nabla_{a} G^{IJ}, \]  
(B.12a)
\[ F_{ab} = -\frac{i}{4 \mathcal{N} (\mathcal{N} - 1)} \varepsilon_{abc} (\gamma^c)_{\alpha \beta} [\nabla_{a} G^{JK} , \nabla_{b} L_{KL}] . \]  
(B.12b)
The $\mathcal{N} = 2$ case is special because $G^{IJ}$ becomes proportional to the antisymmetric tensor $\varepsilon^{IJ}$

$$G^{IJ} = \varepsilon^{IJ} G . \quad \text{(B.13)}$$

The components of $F_{AB}$ then become

$$F_{\alpha \beta}^{IJ} = -2i \varepsilon_{\alpha \beta} \varepsilon^{IJ} G , \quad \text{(B.14a)}$$

$$F_{\alpha}^{J} = \varepsilon^{JK} (\gamma_{\alpha})_{\beta} \partial \gamma K G , \quad \text{(B.14b)}$$

$$F_{ab} = -i \frac{1}{4} \varepsilon_{abc} (\gamma^{c})^{\gamma} \delta^{KL} \partial \gamma K \partial \delta L G . \quad \text{(B.14c)}$$

The Bianchi identities imply a constraint on $G$ at dimension-2

$$\varepsilon^{K(I} \partial \gamma J) \partial \gamma K G = 0 . \quad \text{(B.15)}$$

In the complex basis, this constraint means that $G$ is covariantly linear,

$$\nabla^{2} G = \bar{\nabla}^{2} G = 0 . \quad \text{(B.16)}$$

Unlike for $\mathcal{N} = 2$, in the case $\mathcal{N} > 2$ the field strength $G^{IJ}$ is constrained by the dimension-3/2 Bianchi identity

$$\nabla^{I} G^{JK} = \nabla^{I} [G^{JK}] - \frac{1}{(N - 1)} (\delta^{IJ} \nabla \gamma L G^{KL} - \delta^{IK} \nabla \gamma L G^{JL}) . \quad \text{(B.17)}$$

This constraint may be shown to define an off-shell supermultiplet [47].\textsuperscript{40} This is in contrast with the four-dimensional case where the standard superspace constraints define an on-shell vector multiplet for $\mathcal{N} > 2$ [46].

It is worth remarking briefly on why this difference should arise between the three and four-dimensional cases. Following Sohnius [46], in four-dimensional $\mathcal{N}$-extended Minkowski superspace, the Abelian vector multiplet is described by the complex field strength $\bar{W}^{jk} = -\bar{W}^{kj}$ with SU($\mathcal{N}$) indices. The field strength obeys the constraints

$$D_{a}^{j} \bar{W}^{jk} = -D_{a}^{j} \bar{W}^{kj} , \quad \text{(B.18a)}$$

$$\bar{D}_{a}^{i} \bar{W}^{jk} = \frac{1}{N - 1} \left( \delta_{i}^{j} \bar{D}_{a}^{i} \bar{W}^{lk} - \delta_{i}^{k} \bar{D}_{a}^{i} \bar{W}^{lj} \right) , \quad \text{(B.18b)}$$

which (one can check) are conformally invariant. As a consequence of these constraints, one can show for $\mathcal{N} > 2$ that

$$\Box \bar{W}^{jk} = 0 . \quad \text{(B.19)}$$

which places the multiplet on-shell [46]. Since the original constraints are conformally invariant, any equation derived from them must also be conformally invariant or transform

\textsuperscript{40}It was claimed in [16] that the constraint (B.17) defines an on-shell vector supermultiplet for $\mathcal{N} > 4$. This claim had been based on a harmonic-superspace analysis by one of us (SMK), which turned out to be erroneous.
under special conformal transformations back into the original constraints. One easily observes, for example, that eq. (B.19) is invariant under $K_a$ because $W_{jk}$ has conformal dimension $1$.

In the three-dimensional case, one might expect that one could similarly prove $\Box G^{JK} = 0$ for $N > 4$. But in three dimensions, the massless Klein-Gordon equation is conformally invariant only for Lorentz scalars of dimension-$1/2$. Since $G^{JK}$ has dimension $1$, one can prove that under successive applications of $K_a$,

$$\Box G^{JK} = 0 \quad \Rightarrow \quad \nabla_a G^{JK} = 0 \quad \Rightarrow \quad G^{JK} = 0 .$$

(B.20)

In other words, provided superconformal invariance is maintained, $\Box G^{JK}$ can vanish only if $G^{JK}$ also vanishes.

Similar arguments may be used to argue that a linearized version of the constraint (4.48) defines an off-shell supermultiplet for $N > 4$. This is in agreement with the statement [13] that the $N$-extended Weyl multiplet is off-shell in three dimensions.

References

[1] S. Deser and J. Kay, Topologically massive supergravity, Phys. Lett. B 120 (1983) 97 [SPIRE].

[2] S. Deser, Cosmological topological supergravity, in Quantum Theory Of Gravity, S.M. Christensen ed., Adam Hilger, Bristol, U.K. (1984), pg. 374–381.

[3] P. van Nieuwenhuizen, $D = 3$ conformal supergravity and Chern-Simons terms, Phys. Rev. D 32 (1985) 872 [SPIRE].

[4] M. Roček and P. van Nieuwenhuizen, $N \geq 2$ supersymmetric Chern-Simons terms as $D = 3$ extended conformal supergravity, Class. Quant. Grav. 3 (1986) 43 [SPIRE].

[5] U. Lindström and M. Roček, Superconformal gravity in three-dimensions as a gauge theory, Phys. Rev. Lett. 62 (1989) 2905 [SPIRE].

[6] H. Nishino and S.J. Gates Jr., Chern-Simons theories with supersymmetries in three-dimensions, Int. J. Mod. Phys. A 8 (1993) 3371 [SPIRE].

[7] T. Uematsu, Structure of $N = 1$ conformal and Poincaré supergravity in $(1 + 1)$-dimensions and $(2 + 1)$-dimensions, Z. Phys. C 29 (1985) 143 [SPIRE].

[8] T. Uematsu, Constraints and actions in two-dimensional and three-dimensional $N = 1$ conformal supergravity, Z. Phys. C 32 (1986) 33 [SPIRE].

[9] M. Brown and S.J. Gates Jr., Superspace Bianchi Identities and the Supercovariant Derivative, Annals Phys. 122 (1979) 443 [SPIRE].

[10] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, Superspace Or One Thousand and One Lessons in Supersymmetry, Front. Phys. 58 (1983) 1 [hep-th/0108200] [SPIRE].

[11] B. Zupnik and D. Pak, Superfield formulation of the simplest three-dimensional gauge theories and conformal supergravities, Theor. Math. Phys. 77 (1988) 1070 [SPIRE].

[12] B. Zupnik and D. Pak, Differential and integral forms in supergauge theories and supergravity, Class. Quant. Grav. 6 (1989) 723 [SPIRE].
[13] P.S. Howe, J. Izquierdo, G. Papadopoulos and P. Townsend, New supergravities with central charges and Killing spinors in (2+1)-dimensions, Nucl. Phys. B 467 (1996) 183 [hep-th/9505032] [inSPIRE].
[14] P.S. Howe, A superspace approach to extended conformal supergravity, Phys. Lett. B 100 (1981) 389 [inSPIRE].
[15] P.S. Howe, Supergravity in Superspace, Nucl. Phys. B 199 (1982) 309 [inSPIRE].
[16] S.M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, Off-shell supergravity-matter couplings in three dimensions, JHEP 03 (2011) 120 [arXiv:1101.4013] [inSPIRE].
[17] P.S. Howe and E. Sezgin, The Supermembrane revisited, Class. Quant. Grav. 22 (2005) 2167 [hep-th/0412245] [inSPIRE].
[18] M. Cederwall, U. Gran and B.E. Nilsson, D = 3, N = 8 conformal supergravity and the Dragon window, JHEP 09 (2011) 101 [arXiv:1103.4530] [inSPIRE].
[19] J. Greitz and P.S. Howe, Maximal supergravity in three dimensions: supergeometry and differential forms, JHEP 07 (2011) 071 [arXiv:1103.2730] [inSPIRE].
[20] S.M. Kuzenko and G. Tartaglino-Mazzucchelli, Three-dimensional N = 2 (AdS) supergravity and associated supercurrents, JHEP 12 (2011) 052 [arXiv:1109.0496] [inSPIRE].
[21] S.M. Kuzenko and G. Tartaglino-Mazzucchelli, Conformal supergravities as Chern-Simons theories revisited, JHEP 03 (2013) 113 [arXiv:1212.6852] [inSPIRE].
[22] D. Butter, S.M. Kuzenko and J. Novak, The linear multiplet and ectoplasm, JHEP 09 (2012) 131 [arXiv:1205.6981] [inSPIRE].
[23] M.F. Hasler, The Three form multiplet in N = 2 superspace, Eur. Phys. J. C 1 (1998) 729 [hep-th/9606076] [inSPIRE].
[24] S.J. Gates Jr., Ectoplasm has no topology: The prelude, in Supersymmetries and Quantum Symmetries, J. Wess and E.A. Ivanov eds., Springer, Berlin (1999), pg. 46 [hep-th/9709104] [inSPIRE].
[25] S.J. Gates Jr., Ectoplasm has no topology, Nucl. Phys. B 541 (1999) 615 [hep-th/9809056] [inSPIRE].
[26] S.J. Gates Jr., M.T. Grisaru, M.E. Knutt-Wehlau and W. Siegel, Component actions from curved superspace: Normal coordinates and ectoplasm, Phys. Lett. B 421 (1998) 203 [hep-th/9711151] [inSPIRE].
[27] D. Butter, N=1 Conformal Superspace in Four Dimensions, Annals Phys. 325 (2010) 1026 [arXiv:0906.4399] [inSPIRE].
[28] D. Butter, N=2 Conformal Superspace in Four Dimensions, JHEP 10 (2011) 030 [arXiv:1103.5914] [inSPIRE].
[29] E. Fradkin and A.A. Tseytlin, Conformal supergravity, Phys. Rept. 119 (1985) 233 [inSPIRE].
[30] D.Z. Freedman and A. Van Proeyen, Supergravity, Cambridge University Press, Cambridge, U.K. (2012).
[31] W. Nahm, Supersymmetries and their Representations, Nucl. Phys. B 135 (1978) 149 [inSPIRE].
[32] S.M. Kuzenko, J.-H. Park, G. Tartaglino-Mazzucchelli and R. Unge, Off-shell superconformal nonlinear $\sigma$-models in three dimensions, *JHEP* 01 (2011) 146 [arXiv:1011.5727] [inSPIRE].

[33] S.M. Kuzenko, Prepotentials for $N=2$ conformal supergravity in three dimensions, *JHEP* 12 (2012) 021 [arXiv:1209.3894] [INspire].

[34] D. Butter, S.M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, Conformal supergravity in three dimensions: Off-shell actions, arXiv:1306.1205 [INspire].

[35] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic Superspace*, Cambridge University Press, Cambridge, U.K. (2001).

[36] S.M. Kuzenko and S. Theisen, Correlation functions of conserved currents in $N=2$ superconformal theory, *Class. Quant. Grav.* 17 (2000) 665 [hep-th/9907107] [INspire].

[37] W. Siegel, Curved extended superspace from Yang-Mills theory a la string, *Phys. Rev. D* 53 (1996) 3324 [hep-th/9510150] [INspire].

[38] M. Nishimura and Y. Tanii, Coupling of the BLG theory to a conformal supergravity background, *JHEP* 01 (2013) 120 [arXiv:1206.5388] [INspire].

[39] B. de Wit and H. Nicolai, N=8 Supergravity, *Nucl. Phys. B* 208 (1982) 323 [INspire].

[40] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, Princeton, U.S.A. (1992).

[41] I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity, Or a Walk Through Superspace*, IOP, Bristol, U.K. (1998).

[42] W. Siegel, Unextended Superfields in Extended Supersymmetry, *Nucl. Phys. B* 156 (1979) 135 [INspire].

[43] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, HyperKähler Metrics and Supersymmetry, *Commun. Math. Phys.* 108 (1987) 535 [INspire].

[44] B. Zupnik and D. Khetselius, Three-dimensional extended supersymmetry in the harmonic superspace (in russian), *Sov. J. Nucl. Phys.* 47 (1988) 730 [INspire].

[45] R. Grimm, M. Sohnius and J. Wess, Extended Supersymmetry and Gauge Theories, *Nucl. Phys. B* 133 (1978) 275 [INspire].

[46] M.F. Sohnius, Bianchi Identities for Supersymmetric Gauge Theories, *Nucl. Phys. B* 136 (1978) 461 [INspire].

[47] U. Gran, J. Greitz, P.S. Howe and B.E. Nilsson, Topologically gauged superconformal Chern-Simons matter theories, *JHEP* 12 (2012) 046 [arXiv:1204.2521] [INspire].