Resolving the mystery of electron perpendicular temperature spike in the plasma sheath

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A large family of plasmas has collisional mean-free-path much longer than the non-neutral sheath width, which scales with the plasma Debye length. The plasmas, particularly the electrons, assume strong temperature anisotropy in the sheath. The temperature in the sheath flow direction (\(T_{e\parallel}\)) is lower and drops towards the wall as a result of the decompressional cooling by the accelerating sheath flow. The electron temperature in the transverse direction of the flow field (\(T_{e\perp}\)) not only is higher but also spikes up in the sheath. This abnormal behavior of \(T_{e\perp}\) spike is found to be the result of a negative gradient of the parallel heat flux of transverse degrees of freedom (\(q_{es}\)) in the sheath. The non-zero heat flux \(q_{es}\) is induced by pitch-angle scattering of electrons via either their interaction with self-excited electromagnetic waves in a nearly collisionless plasma or Coulomb collision in a collisional plasma, or both in the intermediate regime of plasma collisionality.
I. INTRODUCTION

When a plasma is in contact with solid boundaries, due to the greater mobility of electrons, a non-neutral plasma sheath forms next to the wall\cite{1,2,3,4}. In the absence of the copious amount of electron emission from the wall, a negative electrical potential is established at the boundary, which is promptly shielded out over a few Debye lengths $\lambda_D$ so that the bulk plasma remains quasi-neutral. The resulting sheath electric field is essential for maintaining ambipolar transport, through which the particle and heat losses from the plasma to the solid boundary are regulated.

In most low-density plasmas of interest, the collisional mean-free-path ($\lambda_{mf_p}$) is larger or much larger than the Debye length, so the sheath Knudsen number $Kn^{sh}$ $\equiv \lambda_{mf_p}/\lambda_D$, which is defined as the ratio between the plasma mean free path and the Debye length at the sheath entrance, satisfies $Kn^{sh} > 1$ or $Kn^{sh} \gg 1$. Remarkably, despite $Kn^{sh} \gg 1$, large gradients of plasma temperature, density, and flow can be sustained in the narrow sheath region on the order of a few Debye lengths. This is fundamentally the result of the large sheath electric field, itself a large gradient of the sheath electrostatic potential, which is independent of the collisional mean-free-path, which would otherwise set the gradient length scale in the quasi-neutral plasma away from the sheath region.

The lack of plasma collisions in the narrow sheath region allows strong temperature anisotropy to develop. The driver is the sheath plasma flow into the wall, which has a large gradient along the streamline direction due to the sheath electric field acceleration of the mostly collisionless ions. Let’s label the wall-bound plasma flow direction as parallel and the cross-flow plane as perpendicular, and define two temperatures $T_\parallel$ and $T_\perp$. The accelerating sheath flow would decompressional cool $T_\parallel$, so a temperature anisotropy of $T_\parallel < T_\perp$ would naturally develop in the sheath region\cite{5}. In a strongly magnetized plasma where the magnetic field intercepts the wall at a large angle, the plasma mostly flows along the magnetic field line. This translates into a much lower parallel electron temperature $T_{e\parallel}$ compared with the perpendicular electron temperature $T_{e\perp}$, all with respect to the magnetic field $B$. The situation becomes more complicated when the magnetic field line intercepts the wall at an oblique angle\cite{6} for which the plasma flow in the Debye sheath would be non-aligned with the magnetic field as it meets the wall. In that case, as $T_{e\parallel,\perp}$ originally defined with respect to the flow direction, they become non-aligned with the magnetic field as well.

It can be noted that the interesting physics of anisotropic temperature for the sheath plasma is usually ignored in the vast plasma sheath literature (for recent reviews, see Refs.\cite{7,8,9}). This is simply the result of deploying physical models with isotropic plasma temperatures. First principle
kinetic simulations of plasma sheath, using either particle-in-cell (e.g., Ref. 5) or continuum discretization (e.g., Ref. 10) would be able to capture the temperature anisotropy physics, but only if the perpendicular degrees of freedom in the momentum space are accounted for. For strictly 1D1V kinetic modeling, such as that reported in Ref. 11, the physics of sheath plasma temperature anisotropy would still be excluded.

While the deep drop of $T_e\parallel$ in the sheath region is well understood as the result of decompressional cooling by the accelerating sheath flow\textsuperscript{5,12}, there is a long-standing mystery in the behavior of $T_e\perp$ in the plasma sheath. Instead of staying flat or slowly dropping, $T_e\perp$ spikes up in the non-neutral plasma sheath in an unmagnetized plasma or a magnetized plasma with a large angle between the magnetic field and the wall. This is most clearly demonstrated in first-principles kinetic simulations\textsuperscript{13–15}, as all physical quantities are readily diagnosed from simulation data. We shall note that although the evidence to such effect was explicitly reported in Ref.\textsuperscript{13} (see Figs. 1-2 in the paper) as an unresolved mystery and more recently in Refs.\textsuperscript{14 and 15} (see Fig. 1 in the Supplemental Material of Ref.\textsuperscript{14} and Fig. 2 in Ref.\textsuperscript{15}), all in the context of VPIC simulations, there is a good reason to believe that others must have encountered the same mystery in the first-principles kinetic simulations where the physics of $T_e\perp$ degrees of freedom are retained (plausible examples include Fig. 9 in Ref.\textsuperscript{10} where kinetic simulations with continuum discretization as opposed to particle-in-cell method is deployed, and Fig. 13 in Ref.\textsuperscript{16} where electrostatic 2D particle-in-cell simulations was performed for $E \times B$ plasmas).

This paper aims to elucidate the underlying physics that would resolve the mystery of $T_e\perp$ spike in the plasma sheath by considering a normal $B$ to the walls. We will show that such behavior is associated with a negative gradient of the parallel electron heat flux of the perpendicular degrees of freedom, $q_{es}$. As defined in the original formalism by Chew et al.\textsuperscript{17},

$$q_{es} \equiv (1/2) \int m_e w_\perp^2 w_\parallel f_e d^3v$$ \hspace{1cm}(1)

with $v$ the particle velocity, $w$ the electron peculiar velocity, $w_\parallel \equiv w \cdot b$, $w_\perp \equiv w - w_\parallel b$, and $b \equiv B/B$. This can be compared with the parallel electron heat flux of parallel degrees of freedom, which takes the form

$$q_{en} \equiv \int m_e w_\parallel^2 w_\parallel f_e d^3v.$$ \hspace{1cm}(2)

Parallel streaming loss in the neighborhood of the plasma sheath leads to a truncation of the electron distribution function in the direction that is opposite to the sheath flow, the asymmetry of
which yields a wall-bound $q_{en}$ even in the absence of collisions and wave-particle interaction. In sharp contrast, finite pitch-angle scattering is required to both isotropize the parallel and perpendicular electron temperature, and produce a finite $q_{es}$. There are two mechanisms for the pitch-angle scattering of electrons in the neighborhood of the plasma sheath: one is Coulomb collisions, which is the reason for $T_{e\perp}$ spike in the sheath of collisional plasma that has $Kn^{*h} > 1$ but not $Kn^{*h} \gg 1^{13,15}$, and the other is resonant wave-electron interaction in a nearly collisionless plasma with $Kn^{*h} \gg 1$. The self-excited wave instability could be whistler waves in a magnetized plasma$^{18,19}$ or Weibel instability in an unmagnetized plasma$^5$. It should be noted that, although the effects of whistler waves/turbulence on the electron particle flux$^{18,19}$ and heat flux of the parallel degree of freedom$^{20}$, $q_{en}$, have been well documented, its role in $q_{es}$ modulation and thus a spike of $T_{e\perp}$ in the sheath of a nearly collisionless plasma have not been reported yet.

The subtle physics of $T_{e\perp}$ spike in the sheath region can be demonstrated in the archetypal example of a one-dimensional (1D) three-velocity (3V) plasma in a slab geometry with a strong magnetic field normal to the absorbing boundaries. To compensate for the particle loss to the walls, we introduce an upstream source that draws from a local Maxwellian of fixed source temperature. As a result, a steady state can be sustained. We must emphasize that the rise of $T_{e\perp}$ towards the wall appears not only in the steady state but also in its early time evolution process. However, the steady state will be employed to illustrate the role of $q_{es}$ in the $T_{e\perp}$ spike. Moreover, it is worth noting that the aforementioned two mechanisms for the pitch-angle scattering of electrons have different electrostatic/electromagnetic natures in that the whistler waves can be excited only in electromagnetic simulations while the collisions work equally in the electromagnetic and electrostatic simulations. Indeed, the first-principles kinetic simulations using VPIC$^{21}$ confirm no differences, even in the sheath region, of strongly collisional plasmas between the electromagnetic and electrostatic simulations. Before elucidating the underlying physics for the $T_{e\perp}$ spike in the sheath, we briefly discuss the VPIC simulation setup, which is similar to that in Refs. $^{14}$ and $^{15}$. Specifically, a uniform proton-electron plasma with density $n_0$ and temperature $T_0$ is initially filled in the simulation domain, and a strong magnetic field is introduced so that the plasma has a low-$\beta$ with $\beta \approx 1.4\%$. The plasma source with temperature $T_0$ is in the middle of the domain $x \in [3L/8, 5L/8]$ with two absorbing walls at $x = 0$ and $x = L \equiv 256\lambda_D$. The resolution of the simulation is $\Delta x = 0.4\lambda_D$ with 2500 macro-particles per cell (note that same results are obtained for simulations with higher resolution, $\Delta x = 0.1\lambda_D$, and number of macro-particles, 10000). For collisional plasmas, we will use an artificial Coulomb logarithm $\ln \Lambda$ to obtain different collisional
regimes characterized by the nominal Knudsen number $Kn = \lambda_{mf}/\lambda_D$, where Takizuka and Abe’s method\cite{22} is employed as the collisional model in VPIC.

The rest of the paper is organized as follows. Sections II and III will consider, respectively, a collisional and collisionless plasma with a normal $B$ to the walls, where $T_{e\perp}$ spikes near the wall. In section IV we will discuss a collisional plasma in an oblique magnetic field with a small angle to the walls, which would provide additional decompressional cooling to $T_{e\perp}$ (with respect to the plasma flow direction) so that $T_{e\perp}$ spike would disappear. Section V will conclude.

II. COLLISIONAL PLASMA WITH A NORMAL MAGNETIC FIELD TO THE WALLS

In a strongly collisional plasma (but $Kn^{th} > 1$ is still satisfied), the kinetic instabilities like the whistler instability are suppressed\cite{23}. As a result, the magnetic field is unperturbed such that the parallel direction is still along $x$ and $\nabla \cdot \hat{b} = 0$. This indicates that the $T_{e\perp}$ spike in the sheath has an electrostatic characteristic in a collisional plasma (e.g., see Fig. I). As a result, the anisotropic energy equations for electrons in the steady state sheath region read\cite{17,24}

$$n_e u_{ex} \frac{\partial T_{e\parallel}}{\partial x} + 2n_e T_{e\parallel} \frac{\partial u_{ex}}{\partial x} + \frac{\partial q_{en}}{\partial x} = Q_{ee} + Q_{ei}, \quad (3)$$

$$n_e u_{ex} \frac{\partial T_{e\perp}}{\partial x} + \frac{\partial q_{es}}{\partial x} = -(Q_{ee} + Q_{ei})/2, \quad (4)$$

where $n_e$ is the electron density and $u_{ex}$ is the parallel electron flow. Notice that we use $\parallel$ in the subscript only for the parallel electron temperature (to distinguish from $T_{e\perp}$) while using $x$ for the parallel direction in all the other quantities. For the energy transfer due to collisions, we only keep the dominant temperature isotropization terms $Q_{ee,ei}$ for illustration purpose\cite{14}, where $Q_{ee,ei}$ denote electrons colliding with electrons and ions, respectively.

In the regime of $Kn^{th} > 1$, the collisions in the sheath region are sufficiently weak that $Q_{ee,ei}$ are subdominant in Eqs. (3), (4). Ignoring them, one finds the remaining difference between the two equations is the presence of the decompressional cooling term $\partial u_{ex}/\partial x$ in Eq. (3), which is absent in Eq. (4). For $T_{e\parallel}$, the decompressional cooling term overwhelms the conduction flux contribution, so

$$n_e u_{ex} \frac{\partial T_{e\parallel}}{\partial x} \approx -2n_e T_{e\parallel} \frac{\partial u_{ex}}{\partial x}, \quad (5)$$
FIG. 1: The electron temperature and density (corresponding to the right y-axis) and heat flux (corresponding to the left y-axis) in a steady state from the first-principles simulation using VPIC\textsuperscript{21}. The nominal Knudsen number $Kn$, defined as the ratio of the initial electron mean free path to the Debye length, is $Kn = 20$. $n_0$, $T_0$ and $v_{th,0}$ are initial plasma density, temperature and electron thermal speed. The time-averaging (but not spatial-averaging) employed in Ref.\textsuperscript{14} over a long period in the steady state is utilized to overcome the PIC noise. We note that the electrostatic and electromagnetic simulations provide the same results so only the former are plotted here.

which says that decompressional cooling due to an accelerating sheath flow produces a decreasing $T_{e\parallel}$ as the plasma approaches the wall\textsuperscript{12}. It is important to note that this happens despite the wall-bound $q_{en}$ heat flux drops in magnitude towards the wall, which contributes a heating mechanism for $T_{e\parallel}$, except that it is simply too weak compared with decompressional cooling. For $T_{e\perp}$, the gradient of heat flux $q_{es}$ is the only term that drives $T_{e\perp}$ variation,

$$n_e u_{ex} \frac{\partial T_{e\perp}}{\partial x} \approx -\frac{\partial q_{es}}{\partial x}. \quad (6)$$

Because of this, as the non-neutral sheath reduces the wall-bound heat flux $q_{es}$, like what it does to $q_{en}$, the heating effect that $\partial q_{es}/\partial x$ brings, would heat up the $T_{e\perp}$ in the sheath.

In the mostly collisionless sheath of a collisional bulk plasma, we have previously shown\textsuperscript{12}, with the help of a truncated bi-Maxwellian (TBM) model for sheath electron distribution, that

$$q_{en} \approx -\Gamma_{e\parallel}^{se} e\Delta\Phi + \Gamma_{e\parallel}^{se} \left( T_{e\parallel} - \frac{3}{2} T_{e\parallel} \right) \quad (7)$$
where the electron particle flux through the sheath $\Gamma_{se}$ and a nominal temperature $T^0_e$ are both constants, the potential drop is defined as $\Delta \Phi(x) = \Phi^0 - \Phi(x)$ with $\Phi^0$ the wall potential, and $T_e(x)$ is the local electron parallel temperature. For large ion-electron mass ratio, $m_i/m_e \gg 1$, the sheath potential variation is a few times greater than $T^0_e$, so the heat flux $q_{en}$ is in the particle flow direction for $\Delta \Phi < 0$ in the sheath. Furthermore, since $-\Delta \Phi$ drops in magnitude as the wall is approached, the $q_{en}$ would decrease in magnitude towards the wall as well. Numerical results in Ref. 12 confirm that $q_{en} \approx -\Gamma_{se} e \Delta \Phi$ is closely followed in first-principles kinetic simulations, so the spatial gradient of $q_{en}$ is set by the gradient length scale of $\Phi(x)$,

$$\frac{\partial q_{en}}{\partial x} \approx -\Gamma_{se} e \Delta \Phi \frac{\partial}{\partial x} \ln(-\Delta \Phi) \approx q_{en} \frac{\partial}{\partial x} \ln(-\Delta \Phi) < 0. \quad (8)$$

Since the electron density closely follows the Boltzmann distribution in the TBM model, which agrees well with the first-principles kinetic simulation results, one has

$$\frac{\partial}{\partial x} \ln(-\Delta \Phi) \approx -\frac{T^0_e}{e \Delta \Phi} \frac{\partial}{\partial x} \ln n_e. \quad (9)$$

For $-e \Delta \Phi/T^0_e \sim 1$, we have

$$\frac{\partial q_{en}}{\partial x} \sim q_{en} \frac{\partial \ln n_e}{\partial x}. \quad (10)$$

The TBM model completely misses the $q_{es}$ physics as it assumes Maxwellian distribution in $T_{e\perp}$ so it enforces $q_{es} = 0$. It turns out that $q_{es} \neq 0$ in the cases of both $Kn_{sh} > 1$ and $Kn_{sh} \gg 1$, but for different physics considerations. For a collisional bulk plasma with $Kn_{sh} > 1$, one can see from the simulations that outside the sheath region, the collisions are so strong that the total parallel heat flux $q_x = (q_{en} + 2q_{es})/2$ nearly follows the Braginskii’s closure $q_x = -\kappa \partial T_e/\partial x$. The plasma is also nearly isotropic outside the sheath as shown in Fig. 1 where $T_{e\parallel} \approx T_{e\perp}$. As a result, $q_{es}$ and $q_{en}$ have the same trend as $q_x$ outside the sheath region, which guarantees that $q_{es}$ has the same sign with $q_{en}$ even in the sheath region despite the temperature anisotropy there. By the heat flux definition of both $q_{en}$ and $q_{es}$, their spatial gradient is related to the spatial gradient of the distribution function, which can be written in terms of thermodynamic variables. One insight from the TBM model is that the dominant terms are

$$\frac{\partial f_e}{\partial x} = n_e \frac{\partial f_e}{\partial n_e} \frac{\partial \ln n_e}{\partial x} + T_{e\parallel} \frac{\partial f_e}{\partial T_{e\parallel}} \frac{\partial \ln T_{e\parallel}}{\partial x}. \quad (11)$$

The plasma potential dependence primarily enters through $n_e$, but there is also a contribution through $T_{e\parallel}$ via the parallel velocity cutoff. In Fig. 2 first-principles kinetic simulations confirm
the TBM prediction of
\[ \left| \frac{\partial \ln n_e}{\partial x} \right| \gg \left| \frac{\partial \ln T_{e\parallel}}{\partial x} \right|, \] (12)
and further establishes
\[ \left| \frac{\partial \ln n_e}{\partial x} \right| \gg \left| \frac{\partial \ln T_{e\perp}}{\partial x} \right|. \] (13)

Combining Eqs. (11) and from Eq. (1), one finds
\[ \frac{\partial q_{es}}{\partial x} \sim \frac{1}{2} \int m_e w_{\perp}^2 w_{\parallel} n_e \frac{\partial f_e}{\partial n_e} d^3 v \frac{\partial \ln n_e}{\partial x} \sim q_{es} \frac{\partial \ln n_e}{\partial x}. \] (14)

Recalling the fact that \( q_{es} \) has the same sign as \( q_{en} \) and \( \Gamma_{se} \parallel \), we come to the interesting conclusion that the gradient of \( q_{es} \) has the same sign as that of \( q_{en} \), which is negative in the sheath region. As a result, \( T_{e\perp} \) will arise towards the wall as predicted by Eq. (6) and confirmed by simulation data in Fig. 1.

![Graph](image1.png)

**FIG. 2:** Logarithms of \( n_e, T_{e\parallel}, T_{e\perp}, \) and \( q_{en,s} \) near the left boundary for the same simulation in Fig. 1. Here \( n_e \) is normalized by \( n_0 \), \( T_{e\parallel}, T_{e\perp} \) by \( T_0 \), and \( q_{en,s} \) by the maximum value of \( q_{en,s} \).

### III. COLLISIONLESS PLASMA WITH A NORMAL MAGNETIC FIELD TO THE WALLS

The limiting case of the \( Kn^{sh} \gg 1 \) regime is a collisionless plasma. In the absence of Coulomb collisions, the pitch-angle scattering of electrons can be facilitated by the electron interaction with electromagnetic waves. The most obvious candidate of the electromagnetic waves in the sheath
problem of $Kn^s \gg 1$ is the parallel-propagating whistler waves that are robustly excited by the electrostatically trapped electrons\textsuperscript{19}, which arise naturally due to the ambipolar potential in the sheath region. The Fourier analysis of the perturbed perpendicular magnetic field in the VPIC simulation shows that the most unstable mode of whistler instability has wavelength $k_x \lambda_D \approx 0.7$ and growth rate $\gamma = 7 \times 10^{-3} \omega_{pe}$ (detailed analyses of the dispersion relation and growth rate of whistler instability driven by the trapped electrons can be found in Ref.\textsuperscript{19}). Note that the interaction of electrons with the self-excited whistler waves causes temperature isotropization by reducing $T_{e\perp}$ from $T_0$, which is accompanied by a non-zero $q_{es}$. It is worth noting that the role of whistler instability in reducing $T_{e\perp}$ and causing its spike in the sheath is further highlighted by the comparison of the electromagnetic simulation against the electrostatic simulation, where $T_{e\perp}$ remains unchanged ($T_{e\perp} = T_0$) and $q_{es} = 0$ in the latter case, where $T_{e\parallel}$ is similarly reduced toward the boundary due to decompressional cooling. Such a sharp contrast in $T_{e\perp}$ and $q_{es}$ between the electromagnetic and electrostatic simulations also reveals that the PIC noise does not induce effective pitch-angle scattering in the collisionless limit. As in the collisional case, we focus on a steady state in which the time-averaged plasma state variables remain nearly constant.

In contrast to the high-$\beta$ plasma (e.g., due to a weak equilibrium magnetic field as in Ref.\textsuperscript{20}), the amplitude of saturated whistler waves is small $\delta B/B_0 \approx 0.1$ in our case for low-$\beta$ fusion
plasma with $\beta = 1.4\%$. In our 1D problem, instead of forming whistler turbulence due to the whistler instability, standing structures, including those of $q_{en,s}$, are observed in VPIC simulations as shown in Fig. [3] which reinforces the fact that wave-particle interaction can produce the heat flux $q_{es}$. However, in contrast to the electron temperature where the fluctuations are smaller than the time-averaged values, $\bar{T}_{e\|,\perp} < \langle T_{e\|,\perp} \rangle_T$, the electron heat flux is dominated by the fluctuations, $\bar{q}_{en,s}/ \langle q_{en,s} \rangle_T \sim 10$, where the time average $\langle \rangle_T$ is taken in a period much larger than the whistler period $\Delta T \gg \omega_{ce}^{-1}$. This indicates that the heat flux is largely independent of the temperature but determined by the whistler waves. It is worth noting that the interaction of electrons with self-excited whistler waves cannot completely remove the temperature anisotropy, especially for low-$\beta$ plasma as shown in Fig. [3]. As a result, $T_{e\|}$ is small but $T_{e\perp}$ is large in the collisionless case compared to those in the collisional case in Fig. [1].

To get rid of the fast oscillations due to wave-particle interaction, we can focus on the time-averaged quantities as shown in Fig. [4]. For the sake of simplicity, we ignore the time-averaged symbol $\langle \rangle_T$ in the following. One of the most important findings is that the $T_{e\perp}$ spike near the wall still prevails (although the spike width is slightly smaller), which is still associated with the negative gradient of $q_{es}$. The underlying physics is similar to that in the collisional case, where the time-averaged energy equations for electron temperature are similar to Eq. (3, 4):

$$n_e u_{ex} \frac{\partial T_{e\|}}{\partial x} + 2n_e T_{e\|} \frac{\partial u_{ex}}{\partial x} + \frac{\partial q_{en}}{\partial x} = Q,$$

(15)

$$n_e u_{ex} \frac{\partial T_{e\perp}}{\partial x} + \frac{\partial q_{es}}{\partial x} = -Q/2,$$

(16)

Here we assumed that the variation of $B$ is so small that $B$ is still normal to the wall. In addition, the whistler mode is rather coherent in the case considered and thus the fluctuation driven fluxes are tiny compared with the turbulent case, which are ignored compared with their mean values. $Q$ stands for the energy exchange between the perpendicular and parallel directions due to the wave-particle interaction in which we assume that there is no energy exchange between the whistler waves and electrons in the steady state.

Just like the collisional case where the plasma in the sheath region is nearly collisionless so that $Q_{ee,ei}$ is negligibly small, $Q$ term should also be ignored in the sheath region due to two reasons: 1) the deep drop of $T_{e\|}$ (via the reduction of the trap-passing boundary in the electron distribution function) makes the resonance $\omega - kv_{\|} = \omega_{ce}$ less efficient; 2) the large spatial gradient of $T_{e\|,\perp}$ and $q_{en,s}$ in the sheath makes $Q$ less important. Then we come to the same striking realization that
the spike of $T_{e\perp}$ near the wall is associated with a negative gradient of $q_{es}$ as in the collisional case. Considering the similar role of collisions and wave-particle interaction in the pitch-angle scattering and temperature isotropization, $q_{es}$ should have similar behaviors as $q_{en}$ outside the sheath region just like in the collisional case, which flows from the source region to the walls (e.g., see Fig. 4). So $q_{es}$ would have the same sign as $q_{en}$ in the sheath. While in the sheath region, the gradient of the electron density still dominates so that the gradient of $q_{es}$ has the same sign as that of $q_{en}$, e.g., see Eqs. (10, 14), which is negative. As a result, $T_{e\perp}$ will spike up near the wall.

FIG. 4: Line plots of time-averaged electron temperature and density (right y-axis) and heat flux (left y-axis) in a steady state near the left boundary at $x = 0$, where the period for averaging is $\Delta T = 6.5 \times 10^4 \omega_{ce}^{-1}$. They are from the same simulation as shown in Fig. 3.

IV. THE CASE OF AN OBLIQUE MAGNETIC FIELD INTERCEPTING THE WALL

There are additional complications in the case of a collisional plasma in an oblique magnetic field that intercepts the wall at a small angle. The subtlety can be appreciated by projecting the plasma energy equations with respect to the magnetic field:

\[ n_e \mathbf{u} \cdot \nabla T_{e\parallel} + 2n_e T_{e\parallel} \nabla \parallel \mathbf{u}\parallel + \nabla \parallel q_{en} = 0, \]  \hspace{1cm} (17)

\[ n_e \mathbf{u} \cdot \nabla T_{e\perp} + n_e T_{e\perp} \nabla \perp \mathbf{u}\perp + \nabla \perp q_{es} = 0, \]  \hspace{1cm} (18)

where $\nabla \parallel = \hat{b} \cdot \nabla$, $\nabla \perp \mathbf{u}\perp = \nabla \cdot \mathbf{u} - \nabla \parallel u\parallel$. One can verify this derivation by ignoring the collisional contribution in the sheath and combining the steady-state equations for $n_e, P_{e\parallel} \equiv n_e T_{e\parallel}$, and $P_{e\perp} \equiv$
\( n_e T_{e\perp} \), which are given in Eqs. (3,7,8) of Ref. [28]. A further approximation is that the magnetic field is uniform in the sheath, so \( \nabla \cdot \hat{b} = 0 \). The electron flow field is allowed to have components both parallel and perpendicular to the magnetic field, \( \mathbf{u} = u_{\parallel} \hat{b} + u_{\perp} \). Here we focus on small-\( \beta \) plasmas.

If \( \mathbf{B} \) is normal to the walls, the electron flow \( \mathbf{u} \) is aligned with the magnetic field \( \mathbf{B} \) so that \( u_{\perp} = 0 \). As a result, we recover Eqs. (2,4). Whereas, for an oblique magnetic field with a small angle to the walls, the sheath field acceleration is bending the ion flow in the Chodura layer from the direction aligned with the magnetic field towards the direction normal to the wall. In the analogous electron Chodura layer, which is next to the wall and of electron gyroradius in width, the electron flow \( \mathbf{u} \) also tilts away from the magnetic field line and toward the wall, so that

\[
\nabla_{\perp} \cdot u_{\perp} \neq 0. 
\]

This non-zero divergence of the perpendicular electron flow, as we will show below, provides an additional cooling mechanism that can overcome the heating effect of \( \nabla_{\parallel} q_{es} \).

**FIG. 5:** The projections of the parallel and perpendicular plasma flow onto the x-direction. The simulation setup is the same as Fig. 1 but with an oblique magnetic field with \( \theta = 10^\circ \).

Let the wall-intercepting magnetic field \( \mathbf{B} \) lie in the x-y plane, and take the form \( \mathbf{B} = B \sin(\theta) \mathbf{e}_x + B \cos(\theta) \mathbf{e}_y \) with a small angle \( \theta \), where the wall is in the y-z plane. Assuming that the plasma is uniform in the y-z plane, all the spatial derivatives can be projected into the x-direction: \( \mathbf{u} \cdot \nabla = u_\parallel \nabla_x, \nabla_\parallel = \sin(\theta) \nabla_x \), and \( \nabla_{\perp} = \mathbf{e}_x \cos(\theta) \nabla_x \). As a result, \( \nabla_{\parallel} u_{\parallel} = \nabla_x u_{\parallel x} \) and \( \nabla_{\perp} \cdot u_{\perp} = \nabla_x u_{\perp x} \) with \( u_{\parallel x}, u_{\perp x} \) being the projection of the parallel plasma flow and perpendicular flow in the x-y plane onto the x-direction. Notice that for \( \theta = \pi/2 \) we readily recover the normal magnetic field case. For an oblique magnetic field with a small angle \( \theta \), \( u_{\perp x} \) will increase approaching the wall due to the electron pressure drive (e.g., see Fig. 5) so that \( \nabla_{\perp} \cdot u_{\perp} > 0 \) in Eq. (18), which provides decompressional cooling. In contrast to ions, where the acceleration of the projected flow
in the sheath is mainly via \( u_\perp \) due to the sheath electric field, both the parallel and perpendicular electron flows contribute nearly equally to the plasma flow at x-direction as shown in Fig. 5 since their acceleration drive is the electron pressure. As a result, \( \nabla_x u_\parallel \approx \nabla_x u_\perp \) for electrons so that the decompressional cooling for \( T_e\parallel \) and \( T_e\perp \) would be similar, which overwhelms the heating due to \( \nabla || q_{en,s} \). Therefore, there is no spike in the parallel and perpendicular electron temperature and thus any other projected temperature (e.g., see Fig. 6).

\[ \nabla_x u_\perp \approx \nabla_x u_\parallel \]

\[ \nabla || q_{en,s} \]

\[ \nabla_x u_\parallel \approx \nabla_x u_\perp \]

\[ \nabla || q_{en,s} \]

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\[ \nabla || q_{en,s} \]

\[ \nabla_x u_\parallel \approx \nabla_x u_\perp \]

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the large ambipolar potential. However, for an oblique magnetic field intercepting the wall with a small angle, the situation is different in that the decompressional cooling will overwhelm the heating due to the electron heat flux in both the parallel and perpendicular directions. As a result, there will be no spike in the electron temperature in any direction.

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