A 5-ENGEL ASSOCIATIVE ALGEBRA WHOSE GROUP OF UNITS IS NOT 5-ENGEL

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Abstract. Let $R$ be an associative ring with unity and let $[R]$ and $U(R)$ denote the associated Lie ring (with $[a,b] = ab - ba$) and the group of units of $R$, respectively. In 1983 Gupta and Levin proved that if $[R]$ is a nilpotent Lie ring of class $c$ then $U(R)$ is a nilpotent group of class at most $c$. The aim of the present note is to show that, in general, a similar statement does not hold if $[R]$ is $n$-Engel. We construct an algebra $R$ over a field of characteristic $\neq 2,3$ such that the Lie algebra $[R]$ is 5-Engel but the group $U(R)$ is not.

1. Introduction

Let $R$ be an associative ring with unity and let $[R]$ and $U(R)$ denote the associated Lie ring (with $[a,b] = ab - ba$) and the group of units of $R$, respectively. It is known that if $[R]$ is a nilpotent Lie ring of class $c$ then $U(R)$ is a nilpotent group of class at most $c$ (Gupta and Levin [7]). Also, if $[R]$ is metabelian then $U(R)$ is metabelian as well (Krasilnikov [9] and Sharma and Srivastava [17]).

If $R$ is an associative ring such that $[R]$ is centre-by-metabelian then $U(R)$, in general, is not centre-by-metabelian. For instance, if $F$ is an infinite field of characteristic 2 and $R = M_2(F)$ is the algebra of all $2 \times 2$ matrices over $F$ then it is well-known that $[R]$ is centre-by-metabelian but $U(R)$ does not satisfy any non-trivial identity; in particular, $U(R)$ is not centre-by-metabelian.

However, suppose that $R$ is a unital associative algebra over a field of characteristic 0 generated (as a unital algebra) by its nilpotent elements. Then if $[R]$ is centre-by-metabelian then $U(R)$ is also centre-by-metabelian (Krasilnikov and Riley [11]). Moreover, for such an algebra $R$, if $[R]$ satisfies an arbitrary multilinear Lie commutator identity then $U(R)$ satisfies the corresponding group commutator identity (see [11] for precise definitions); for example, if $[R]$ is solvable of length $n$ then $U(R)$ is also solvable of length at most $n$.

It is natural to ask whether a similar result holds for Lie commutator identities that are not multilinear; in particular, whether it holds for the Engel identity. We show that this is, in general, not the case.

More precisely, let $[x,y] = [x,(1)y] = xy - yx$ and let $[x,(k+1)y] = [[[x,(k)y],y],y]$ for $k \geq 1$. Recall that a Lie ring $L$ is $n$-Engel if $[u,(n)v] = 0$ for all
for all \( u, v \in L \). Similarly, a group \( G \) is \( n \)-Engel if \( (u,(n) v) = 1 \) for all \( u, v \in G \), where \( (x, y) = (x,1) y = x^{-1} y^{-1} xy \) and \( (x, (k+1) y) = ((x, k) y, y) \) for \( k \geq 1 \). We are concerned with the following question.

**Question.** Let \( F \) be a field of characteristic 0 and let \( R \) be a unital associative \( F \)-algebra. Suppose that the Lie algebra \([R]\) is \( n \)-Engel. Is the group \( U(R) \) also \( n \)-Engel?

If \( n = 2 \) then the answer to this question is “yes”. Indeed, it is well-known (see, for instance, [18, Theorem 3.1.1]) that if \([R]\) is 2-Engel and \( \text{char} F \neq 3 \) then \([R]\) is nilpotent of class 2. Hence, by [7], \( U(R) \) is nilpotent of class at most 2 and, therefore, is 2-Engel. If \( n = 3 \) and the algebra \( R \) is generated (as a unital algebra) by its nilpotent elements then the answer is “yes” as well; this can be deduced from the results of [4] and [5] (see also [1]).

However, in general the answer to the question above is “no”. Our result is as follows.

**Theorem 1.1.** Let \( F \) be a field of characteristic \( \neq 2, 3 \). Then there is a unital associative \( F \)-algebra \( R \) such that \([R]\) is a 5-Engel Lie algebra but \( U(R) \) is not a 5-Engel group. This algebra \( R \) is generated (as a unital \( F \)-algebra) by 2 nilpotent elements.

Note that if \( R \) is an associative unital ring and \([R]\) is \( n \)-Engel then \( U(R) \) is \( m \)-Engel for some \( m = m(n) \) (Riley and Wilson [16] and independently Amberg and Sysak [1]). If \( R \) is an algebra over a field of characteristic 0 then the existence of such \( m = m(n) \) follows also from the results of Zelmanov [19] and Gupta and Levin [7]. One can check that in our example below the group \( U(R) \) is 6-Engel (and nilpotent of class 7).

We obtain Theorem 1.1 as a corollary of the result below about the adjoint groups of associative algebras.

Let \( R \) be an associative ring with or without unity. It can be easily checked that \( R \) is a monoid with respect to adjoint multiplication defined by \( u \circ v = u + v + uv \) \((u, v \in R)\). The group of units of this monoid is called the adjoint group \( R^o \) of \( R \). It is well-known that if \( R \) is nilpotent, that is, if \( R^n = \{0\} \) for some positive integer \( n \), then \( R^o = R \). On the other hand, if \( R \) is a ring with unity 1 then \( R^o \) is isomorphic to the group of units \( U(R) \) (the mapping \( R^o \to R \) such that \( a \to 1 + a \) is an isomorphism of \( R^o \) onto \( U(R) \)).

Note that one can easily deduce from the results of [7, 9, 17] that, for an associative ring \( R \), if the Lie ring \([R]\) is nilpotent of class \( c \) or metabelian then the adjoint group \( R^o \) is also nilpotent of class at most \( c \) or metabelian, respectively. Furthermore, if \( R^o = R \) then the converse also holds: if \( R^o \) is nilpotent of class \( c \) then \([R]\) is nilpotent of class \( c \) (Du [6]) and if \( R^o \) is metabelian then \([R]\) is metabelian (Amberg and Sysak [3]).

Let \( F \) be a field and let \( A \) be the free associative \( F \)-algebra without 1 on free generators \( x, y \). Let \( m(x, y), n(x, y) \in A \) be monic monomials
in $x,y$. If $n(x,y) = m_1(x,y)m(x,y)m_2(x,y)$ for some monic monomials $m_1(x,y), m_2(x,y) \in A \cup \{1\}$ we say that $m(x,y)$ divides $n(x,y)$ and $n(x,y)$ is a multiple of $m(x,y)$.

Let $I$ be the ideal in $A$ generated by the following elements:

i) all monomials of degree 8;

ii) all monomials of degree greater than 2 in $x$;

iii) all monic monomials of degree 7 except $yxy^3xy$ and $y^2xyxy^2$;

iv) all monic monomials of degree less than 7 which do not divide the monomials $yxy^3xy$ and $y^2xyxy^2$;

v) the polynomial $2xy^3xy - 5yxyxy^2 - 2yxy^3x + 5y^2xyxy$;

vi) the polynomial $2yxy^3xy - 5y^2xyxy^2$.

Let $B = A/I$. It can be easily seen that $B^8 = 0$. Thus, the associative algebra $B$ is nilpotent and, therefore, $B^6 = B$.

**Theorem 1.2.** Let $F$ be a field of characteristic $\neq 2,3$ and let $B = A/I$. Then $[B]$ is a 5-Engel Lie algebra but the adjoint group $B^0$ is not 5-Engel.

To deduce Theorem 1.1 from Theorem 1.2 we embed a non-unital $F$-algebra $B$ into its unital hull. Let $B_1 = F \oplus B$ be a direct sum of $F$-vector spaces $F$ and $B$. Then $B_1$ has a natural associative algebra structure in which the elements of $F$ act on $B$ by scalar multiplication. Further, the element 1 of $F$ is unity for $B_1$ and the set $1 + B$ forms a group under multiplication isomorphic to the adjoint group $B^0$ (the mapping $B^0 \to B_1$ such that $u \to 1 + u$ is an isomorphism of $B^0$ onto $1 + B$).

It can be easily checked that, since $[B]$ is 5-Engel, so is the Lie ring $[B_1]$. On the other hand, since $B^0$ is not 5-Engel, so are the subgroup $(1 + B) \simeq B^0$ of $U(B_1)$ and the group $U(B_1)$. Thus, we have

**Corollary 1.3.** Let $F$ be a field of characteristic $\neq 2,3$ and let $B_1$ be the unital hull of $B$. Then the Lie algebra $[B_1]$ is 5-Engel but the group $U(B_1)$ is not 5-Engel.

Theorem 1.1 follows immediately from Corollary 1.3 (with $R = B_1$).

Recall that if $L$ is a nilpotent Lie algebra over a field of characteristic 0 then $L$ is a group with the multiplication $*$ defined by the Baker-Campbell-Hausdorff formula: $x * y = \log(e^x e^y) = x + y + \frac{1}{2}[x,y] + \ldots$ (for details see, for example, [8, §9.2]). We denote this group by $L^*$. If $L^*$ is an $n$-Engel group for some $n \geq 1$ then $L$ is an $n$-Engel Lie algebra; this is well-known and can be deduced, for instance, from [8 Lemma 10.12 (d)]). However, the converse statement is false.

Indeed, if $B$ is a nilpotent associative algebra over a field of characteristic 0 then it is well-known that $[B]^* \simeq B^0$ (see, for instance, [11]). Thus, by Theorem 1.2 we have

**Corollary 1.4.** Let $F$ be a field of characteristic 0 and let $B = A/I$ be the associative $F$-algebra defined above. Let $L = [B]$. Then $L$ is a 5-Engel Lie algebra such that the group $L^*$ is not 5-Engel.
Remarks. 1. Theorems [11] [12] Corollary [1.3] and their proofs remain valid for algebras over a unital associative and commutative ring $F$ such that $6 \neq 0$ in $F$.

2. One can check that if $R$ is a nilpotent associative algebra over an infinite field and the group $R^\circ$ is $n$-Engel then the Lie algebra $[R]$ is also $n$-Engel.

3. For an associative ring $R$, the Lie ring $[R]$ is nilpotent of class $c$ if and only if the adjoint semigroup $(R, \circ)$ is nilpotent of class $c$ in the sense of Mal’cev [12] or Neumann-Taylor [13]. The “only if” part of this statement has been proved independently by Krasilnikov [10] and Riley and Tasic [15] and the “if” part by Amberg and Sysak [2]. Note that if a group $G$, viewed as a semigroup, is Mal’cev or Neumann-Taylor nilpotent of class $c$ then $G$ is a nilpotent group of class $c$ in the usual sense [12] [13].

In [14] Riley posed the following problem:

Given any positive integer $n$, does there exist a semigroup variety $P_n$ with the property that, for every associative ring $R$, the Lie ring $[R]$ is $n$-Engel if and only if the adjoint semigroup $(R, \circ)$ lies in $P_n$?

This problem is not yet solved. Theorem 1.2 shows that, if such a variety of semigroups $P_n$ exists, the groups that belong to $P_n$ are not necessarily $n$-Engel.

2. Proof of Theorem 1.2

Let $I_0$ be the two-sided ideal in $A$ generated by the polynomials i)–iv) above. Let $C = A/I_0$. It is clear that $C = \bigoplus_{k=7}^{k=7} C(k)$ where $C(k)$ is the linear span of the monomials of degree $k$ in $x + I_0$, $y + I_0$.

It follows easily from the item iv) that

$$x^2, xy^2 x, y^4, y^2 xy^3, x y y x^2, y^3 xy x \in I_0.$$

On the other hand, it can be easily checked that all monic monomials in $x, y$ satisfying iv) are multiples of the monomials above. Note that every monomial in $x, y$ of degree 1 in $x$ and of degree 5 or 6 in $y$ belongs to $I_0$ because such a monomial is a multiple of either $y^4$ or $y^2 xy^2$ and the latter monomials belong to $I_0$. It is straightforward to check that an $F$-basis of $C(6)$ is formed by the images of

$$xy^3 xy, y xy x^2, x y x^2, x^2 xy y$$

and an $F$-basis of $C(7)$ is formed by the images of

$$y xy^3 x y, y^2 xy x^2.$$

By the definition of $I$, $I/I_0$ is the ideal of $C$ generated by the images of the polynomials

$$h_1 = 2 x y^3 x y - 5 y x x y y^2 - 2 y x y x^3 x + 5 y^2 x y x y$$

and

$$h_2 = 2 x y x^3 x y - 5 y^2 x y x y^2.$$
Note that \(xh_1 \equiv h_1 x \equiv 0 \pmod{I_0},\) \(yh_1 \equiv -h_1 y \equiv h_2 \pmod{I_0}\) and \(xh_2 \equiv h_2 x \equiv yh_2 \equiv y2y \equiv 0 \pmod{I_0}\). Hence, \(h_1 + I_0\) and \(h_2 + I_0\) form an \(F\)-basis of the ideal \(I/I_0\). In particular, \(C(7) \cap I/I_0\) is a one-dimensional vector subspace in \(C(7)\) generated by the image of \(h_2\) and \(C(7)/(C(7) \cap I/I_0)\) is a one-dimensional vector space generated by the image of \(y^2xyxy^2\).

Let \(a = x + I, b = y + I\). Then, by (1), we have

\[
a^2 = ab+b^2 = b^2ab = abab = ba = 0.
\]

It is clear that \(B = \bigoplus_{k=1}^{7} B(k)\) where \(B(k)\) is the linear span of monomials of degree \(k\) in \(a, b\). Note that \(B(7) \simeq C(7)/(C(7) \cap I/I_0)\) is a one-dimensional vector subspace in \(B\) generated by \(y^2abab^2\).

To prove that \([B]\) is 5-Engel it suffices to check that \([u, (5)v] = 0\) for all \(u, v \in B\). Let

\[
u = \alpha_1a + \beta_1b + \gamma_1ab + \delta_1 ba + \mu_1 b^2 + u', \quad v = \alpha_2a + \beta_2b + \gamma_2 ab + \delta_2 ba + \mu_2 b^2 + v',
\]

where \(u'\) and \(v'\) are linear combinations of monomials in \(a, b\) of degree at least 3. Then it is straightforward to check that

\[
[u, (5)v] = \alpha_1[a_{(5)}v] + \beta_1[b_{(5)}v] + \gamma_1[ab_{(5)}v] + \delta_1[ba_{(5)}v] + \mu_1[b^2_{(5)}v]
= \alpha_1\beta_2 f_0 + \alpha_1\alpha_2\beta_2 f_1 + \alpha_1\beta_1^3 f_2 + \alpha_2\beta_3 f_3 + \alpha_1\beta_2 \delta_2 f_4 + \alpha_1\alpha_2 \beta_2^2 \mu_2 f_5
+ \beta_1\alpha_2 \beta_1^2 f_6 + \beta_1\alpha_2 \beta_2 f_7 + \beta_1\beta_2^2 \gamma_2 f_8 + \beta_1\alpha_2 \beta_3 \gamma_2 f_9 + \beta_1\beta_2 \delta_2 f_{10}
+ \beta_1\alpha_2 \beta_2 \delta_2 f_{11} + \beta_1\alpha_2 \beta_2^2 \mu_2 f_{12} + \beta_1\alpha_2 \beta_2 \gamma_2 f_{13} + \gamma_1 \beta_2 f_{14}
+ \gamma_1 \alpha_2 \beta_2^3 f_{15} + \delta_1 \beta_2 f_{16} + \delta_1 \alpha_2 \beta_2 f_{17} + \mu_1 \alpha_2 \beta_2 f_{18} + \mu_1 \alpha_2 \beta_2^3 f_{19},
\]

where \(f_i\) are multihomogeneous polynomials in \(a, b\) of (total) degree 6 or 7.

Recall that every monomial in \(x, y\) of degree 1 in \(x\) and of degree 5 or 6 in \(y\) belongs to \(I_0\). Hence, every monomial in \(a, b\) of degree 1 in \(a\) and of degree 5 or 6 in \(b\) is equal to 0. It follows immediately that \(f_0 = f_4 = f_6 = f_8 = f_{10} = f_{12} = f_{14} = f_{16} = f_{18} = 0\).

It is straightforward to check that

\[
f_1 = [a, b, a, b, b, b] + [a, b, b, a, b, b] + [a, b, b, a, b, a] + [a, b, b, b, b, a],
\]

\[
f_2 = [a, ab, b, b, b, b] + [a, ab, b, b, b] + [a, b, ab, b, b]
+ [a, b, b, ab, b] + [a, b, b, b, ab],
\]

\[
f_3 = [a, ba, b, b, b, b] + [a, ba, b, b, b] + [a, b, ba, b, b]
+ [a, b, b, ba, b] + [a, b, b, b, ba],
\]

\[
f_4 = [a, b^2, a, b, b, b] + [a, b^2, a, b, b] + [a, b^2, b, a, b] + [a, b^2, b, b, a]
+ [a, b, b^2, a, b] + [a, b, b^2, b, a] + [a, b, b^2, b, b] + [a, b, a, b^2, b, b]
+ [a, b, b, b^2, a] + [a, b, b, b^2] + [a, b, a, b, b^2] + [a, b, b, a, b^2],
\]

\[
f_5 = [a, b^2, a, b, b, b] + [a, b^2, a, b, b] + [a, b^2, b, a, b] + [a, b^2, b, b, a]
+ [a, b, b^2, a, b] + [a, b, b^2, b, a] + [a, b, a, b^2, b, b] + [a, b, a, b, b^2, b]
+ [a, b, b, b^2, a] + [a, b, b, b^2] + [a, b, a, b, b^2] + [a, b, b, a, b^2],
\]

\[
f_6 = [a, b^3, a, b, b, b] + [a, b^3, a, b, b] + [a, b^3, b, a, b] + [a, b^3, b, b, a]
+ [a, b, b^3, a, b] + [a, b, b^3, b, a] + [a, b, a, b^3, b, b] + [a, b, a, b, b^3, b]
+ [a, b, b, b^3, a] + [a, b, b, b^3] + [a, b, a, b, b^3] + [a, b, b, a, b^3],
\]
\[ f_7 = [b, a, a, b, b, b] + [b, a, b, a, b, b] + [b, a, b, b, a, b] + [b, a, b, b, b, a] = -f_1, \]

\[ f_9 = [b, a, ab, b, b, b] + [b, a, b, ab, b, b] + [b, a, b, b, ab, b] + [b, a, b, b, b, ab] + [b, ab, a, b, b, b] + [b, ab, b, a, b, b] + [b, ab, b, b, a, b] + [b, ab, b, b, b, a] \]

\[ f_{11} = [b, a, ba, b, b, b] + [b, a, b, ba, b, b] + [b, a, b, b, ba, b] + [b, a, b, b, b, ba] + [b, ba, a, b, b, b] + [b, ba, b, a, b, b] + [b, ba, b, b, a, b] + [b, ba, b, b, b, a] \]

\[ f_{13} = [b, a, b^2, a, b, b] + [b, a, b^2, b, a, b] + [b, a, a, a, b^2, b, b] + [b, a, a, b, b^2, b] + [b, a, a, b, a, b^2] + [b, a, a, b, b, a^2] + [b, a, a, b, b, b] + [b, a, b, a, b^2] \]

\[ f_{15} = [ab, a, b, b, b, b] + [ab, a, b, b, b, b] + [ab, a, b, b, b, b] + [ab, b, a, b, b, a] + [ab, b, b, a, b, a] + [ab, b, b, b, a, b] + [ab, b, b, b, b, a] \]

\[ f_{17} = [ba, a, b, b, b, b] + [ba, a, b, b, b, b] + [ba, a, b, b, b, b] + [ba, b, a, b, b, b] + [ba, b, b, a, b, b] + [ba, b, b, b, a, b] \]

\[ f_{19} = [b^2, a, a, b, b, b] + [b^2, a, b, a, b, b] + [b^2, a, b, b, a, b] + [b^2, a, b, b, b, a] \]

To proceed further we need the following lemma which is well-known and can be easily proved by induction.

**Lemma 2.1.** \[ x, (k) y = \sum_{i=0}^{k} (k^i)y^i x y^{k-i} \].

Now we will check that \( f_1 = 0 \). We have

\[ [a, b, a] = [ab - ba, a] = aba - ba^2 - a^2 b + aba = 2aba \]

because, by (2), \( a^2 = 0 \). Therefore,

\[ [a, b, a, b, b, b] = 2[aba, b, b, b]. \]

By Lemma 2.1,

\[ [aba, b, b, b] = abab^3 - 3babab^2 + 3b^2 abab - b^3 aba, \]

where, by (2), \( abab^3 = b^3 aba = 0 \). Hence,

\[ [aba, b, b, b] = -3babab^2 + 3b^2 abab \]

and

\[ [a, b, a, b, b, b] = -6babab^2 + 6b^2 abab. \]

It is straightforward to check that \( [a, b, b, a] = [a, b, a, b] \) so

\[ [a, b, a, b, b, b] = [a, b, a, b, b, b]. \]

Further, by Lemma 2.1,

\[ [a, b, b, b] = ab^3 - 3bab^2 + 3b^2 ab - b^3 a \]
Therefore,

By (2), \( \text{bab}^2a = b^3a^2 = a^2b^3 = ab^2ab = 0 \), we have

\[ [a, b, b, b, a] = 2ab^3a + 3b^2aba + 3abab^2. \]

By (2), \( \text{bab}^3 = b^3aba = 0 \) so

\[ [a, b, b, b, a, b] = 2ab^3ab + 3b^2abab + 3abab^3 - 2bab^3a - 3b^3aba - 3babab^2. \]

By Lemma 2.1 and (2),

\[ [a, b, b, b, b, a] = ab^4 - 4bab^3 + 6b^2ab^2 - 4b^3ab + b^4a = -4ab^3 - 4b^3ab \]

so, again by (2), we have

\[ [a, b, b, b, b, a, b] = -4ab^3a - 4b^3aba + 4abab^3 + 4ab^3ab = -4ab^3a + 4ab^3ab. \]

Thus, by (3), (4), (5) and (6), we have

\[ f_1 = -12babab^2 + 12b^2abab + 2ab^3ab + 3b^2abab - 2bab^3a - 3babab^2 - 4bab^3a + 4ab^3ab \]

\[ = -15babab^2 + 15b^2abab - 6bab^3a + 6ab^3ab. \]

By the item v) of the definition of the ideal I, \( f_1 = 0 \). Since \( f_7 = -f_1 \), we have \( f_7 = 0 \) as well.

One can check in a similar way using Lemma 2.1 (2) and the item vi) of the definition of the ideal I that \( f_2 = f_3 = f_5 = f_9 = f_{11} = f_{13} = f_{15} = f_{17} = f_{19} = 0 \).

More precisely, one can check using the relations (2) that

\[ [b^2, a, b, b, b] = [b^2, a, b, a, b, b] = [b^2, a, b, b, a, b] = [b^2, a, b, b, b, a] = 0. \]

It follows that \( f_{19} = 0 \). Similarly, \( f_5 = f_{13} = 0 \) because \( f_5 \) and \( f_{13} \) are sums of certain commutators and one can check using (2) that all these commutators are equal to 0.

Further, it can be checked using (2) that \( f_2 = f_3 = 0 \) although the commutator summands of \( f_2 \) and \( f_3 \) are not, in general, equal to 0. Finally, one needs the relations (2) as well as the item vi) of the definition of the ideal I to check that \( f_9 = f_{11} = f_{15} = f_{17} = 0 \).

Thus, \([B] \) is a 5-Engel Lie ring, as required.

Now we prove that the group \( B^{0} \) is not 5-Engel. We will check that in \( U(B_1) \)

\[ ((1 + a), (1 + b)) = 1 + 6b^2abab^2 \ne 1. \]

Hence, the subgroup \( 1 + B \) of \( U(B_1) \) is not 5-Engel. Since \( B^{0} \) is isomorphic to \( 1 + B \), the group \( B^{0} \) is not 5-Engel as well.

Note that if \( u \in B \) then \( u^8 = 0 \) so \((1 + u)^{-1} = 1 - u + u^2 - \cdots - u^7 \). It is straightforward to check that, for all \( u, v \in B \),

\[ ((1 + u), (1 + v)) = 1 + [u, v] - u^2v + uvu + v^2u - vuv + w \]
where \( w \) is a linear combination of monomials of degree at least 4 in \( u, v \).

It follows that if \( u = u(a, b) \) is a linear combination of some monomials of degree \( k \geq 2 \) in \( a, b \) and, possibly, some monomials of degree \( > k \) then

\[
(8) \quad ((1 + u), (1 + b)) = 1 + [u, b] + b^2u - bab + w' = 1 + [u, b] - b[u, b] + w'
\]

where \( w' \in B^{k+3} \).

By (7), we have

\[
((1 + a), (1 + b)) = 1 + [a, b] + aba - a^2b + b^2a - bab + w_1
\]

\[
= 1 + [a, b] + aba + b^2a - bab + w_1
\]

where \( w_1 \in B^4 \).

Let \( u_1 = [a, b] + aba + b^2a - bab + w_1 \); then \( ((1 + a), (1 + b)) = 1 + u_1 \). By (8), we have

\[
((1 + a), (2) (1 + b)) = ((1 + u_1), (1 + b)) = 1 + [u_1, b] - b[u_1, b] + w'_2
\]

\[
= 1 + [a, b, b] + [(aba + b^2a - bab), b] - b[a, b, b] + w_2
\]

\[
= 1 + [a, b, b] + abab - baba - 2b[a, b, b] + w_2
\]

where \( w'_2, w_2 \in B^5 \).

Similarly, one can check that

\[
((1 + a), (3) (1 + b)) = 1 + [a, b, b, b] + abab^2 - 2babab
\]

\[
+ b^2aba - 3b[a, b, b, b] + w_3
\]

where \( w_3 \in B^6 \),

\[
((1 + a), (4) (1 + b)) = 1 + [a, b, b, b, b] - 3babab^2 + 3b^2abab - 4b[a, b, b, b] + w_4
\]

where \( w_4 \in B^7 \) and

\[
((1 + a), (5) (1 + b)) = 1 + [a, b, b, b, b, b] + 6b^2abab^2 - 5b[a, b, b, b, b, b] + w_5
\]

where \( w_5 = 0 \) because \( w_5 \in B^8 \) and \( B^8 = 0 \) and \( [a, b, b, b, b] = 0 \) because the Lie algebra \([B]\) is 5-Engel.

Thus, \((1 + a), (5) (1 + b)) = 1 + 6b^2abab^2\). Since \( B_{(7)} \) is a one-dimensional vector subspace in \( B \) generated by \( b^2abab^2 \), we have \( b^2abab^2 \neq 0 \) and \((1 + a), (5) (1 + b)) \neq 1 \). It follows that \( B^0 \) is not 5-Engel group, as required.

The proof of Theorem 1.2 is completed.

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