Spanning trees and spanning closed walks with small degrees

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Abstract

Let $G$ be a graph and let $f$ be a positive integer-valued function on $V(G)$. In this paper, we show that if for all $S \subseteq V(G)$, $\omega(G \setminus S) < \sum_{v \in S}(f(v) - 2) + 2 + \omega(G[S])$, then $G$ has a spanning tree $T$ containing an arbitrary given matching such that for each vertex $v$, $dT(v) \leq f(v)$, where $\omega(G \setminus S)$ denotes the number of components of $G \setminus S$ and $\omega(G[S])$ denotes the number of components of the induced subgraph $G[S]$ with the vertex set $S$. This is an improvement of several results. Next, we prove that if for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \sum_{v \in S}(f(v) - 1) + 1$, then $G$ admits a spanning closed walk passing through the edges of an arbitrary given matching meeting each vertex $v$ at most $f(v)$ times. This result solves a long-standing conjecture due to Jackson and Wormald (1990).

Keywords:
Spanning tree; spanning closed walk; toughness; connected factor; matching.

1 Introduction

In this article, all graphs have no loop, but multiple edges are allowed and a simple graph is a graph without multiple edges. Let $G$ be a graph. The vertex set, the edge set, the maximum degree, and the number of components of $G$ are denoted by $V(G)$, $E(G)$, $\Delta(G)$, and $\omega(G)$, respectively. The degree $d_G(v)$ of a vertex $v$ is the number of edges of $G$ incident to $v$. Let $F$ be a subgraph of $G$. For an edge set $E$, we denote by $F - E$ the graph obtained from $F$ by removing the edges of $E$ from $F$. Likewise, we denote by $F + E$ the graph obtained from $F$ by inserting the edges of $E$ into $F$. For convenience, we use $e$ instead of $E$ when $E = \{e\}$. For two edge sets $E_1$ and $E_2$, we also use the notation $E_1 + E_2$ for the union of them. For a vertex $v$, we denote by $d_G(v, F)$ the number of edges $uv$ of $G$ such that $u$ and $v$ are not in the same component of $F$. The graph $F$ is said to be trivial, if it has no edge. The graph obtained from $G$ by contracting any component of $F$ is denoted by $G/F$. Let $S \subseteq V(G)$. We denote by $G[S]$ the induced subgraph of $G$ with the vertex set $S$ containing precisely those edges of $G$ whose ends lie in $S$. The graph obtained from $G$ by removing all vertices of $S$ is denoted by $G \setminus S$. We denote by $G \setminus [S, F]$ the graph obtained from $G$ by removing all edges incident to the vertices of $S$ except the edges of $F$. Note that while the vertices of $S$
are deleted in $G \setminus S$, no vertices are removed in $G \setminus [S, F]$. The set $S$ is called independent, if there is no edge of $G$ connecting vertices in $S$. We denote by $e_G(S)$ the number of edges of $G$ with both ends in $S$. Moreover, the number of edges of $G$ with both ends in $S$ joining different components of $F$ is denoted by $e_G(S, F)$. Let $g$ and $f$ be two nonnegative integer-valued functions on $V(G)$. A $(g, f)$-factor of $G$ is a spanning subgraph $H$ such that for each vertex $v$, $g(v) \leq d_H(v) \leq f(v)$. For a set $A$ of integers, an $A$-factor is a spanning subgraph with vertex degrees in $A$. A tree (forest) $T$ is said to be an $f$-tree ($f$-forest), if for each vertex $v$, $d_T(v) \leq f(v)$. Likewise, an $f$-walk ($f$-trail) in a graph refers to a walk (trail) meeting each vertex $v$ at most $f(v)$ times. A graph is called $K_{1,n}$-free, if it has no induced subgraph isomorphic to the complete bipartite graph $K_{1,n}$. For a positive real number $t$, a graph $G$ is said to be $t$-tough, if $\omega(G \setminus S) \leq \max\{1, \frac{1}{t}|S|\}$ for all $S \subseteq V(G)$. A graph is called $k$-tree-connected, if it has $k$ edge-disjoint spanning trees. Throughout this article, all variables $k$ are positive integers.

In 1976 Frank and Gyárfás investigated orientations of graphs with bounded out-degrees on certain connectivity properties. A special case of their result can conclude the following theorem.

**Theorem 1.1.** ([5, 14]) Let $G$ be a graph with an independent set $X \subseteq V(G)$. If for all $S \subseteq X$, $\omega(G \setminus S) \leq \sum_{v \in S}(f(v) - 1) + 1$, then $G$ has a spanning tree $T$ such that for each $v \in X$, $d_T(v) \leq f(v)$, where $f$ is a positive integer-valued function on $X$.

In 1989 Win [16] established a result related to spanning trees and toughness of graphs, and Ellingham, Nam, and Voss (2002) generalized it as the following. Former, Ellingham and Zha (2000) [3] found the following fact for constant function form. A consequence, every $1$-tough graph must have a spanning $3$-tree containing an arbitrary given perfect matching.

**Theorem 1.2.** ([2]) Let $G$ be a connected graph with a spanning forest $F$ of which every component contains at least $c$ vertices. Let $f$ be a positive integer-valued function on $V(G)$. If for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \sum_{v \in S}(f(v) - 2) + 2$, then $G$ has a spanning tree $T$ containing $F$ such that for each vertex $v$,

$$d_T(v) \leq \begin{cases} f(v) + d_F(v), & \text{if } c = 1; \\ f(v) + d_F(v) - 1, & \text{if } c \geq 2. \end{cases}$$

Liu and Xu (1998) and Ellingham, Nam, and Voss (2002) independently investigated spanning trees with small degrees in highly edge-connected graphs and found the following theorem.

**Theorem 1.3.** ([2, 12]) Every $k$-edge-connected simple graph $G$ has a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq \left\lceil \frac{d_G(v)}{k} \right\rceil + 2$.

Recently, the present author (2015) refined Theorem 1.3 and concluded the next theorems.

**Theorem 1.4.** ([6]) Every $k$-edge-connected graph $G$ has a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq \left\lceil \frac{d_G(v) - 2}{k} \right\rceil + 2$. 

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Theorem 1.5. ([6]) Every k-tree-connected graph $G$ has a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq \lceil \frac{\omega(v) - 1}{k} \rceil + 1$.

In this paper, we improve Theorems 1.2 to the following stronger version which can conclude Theorems 1.1, 1.2, 1.4, and 1.5 (not necessarily directly). In particular, it can conclude that every 1-tough graph must have a spanning 3-tree containing an arbitrary given (not necessarily perfect) matching.

Theorem 1.6. Let $G$ be a graph with a spanning forest $F$. Let $f$ be a positive integer-valued function on $V(G)$. If for all $S \subseteq V(G)$, $\omega(G \setminus S) < \sum_{v \in S} (f(v) - 2) + 2 + \omega(G[S])$, then $G$ has a spanning tree $T$ containing $F$ such that for each vertex $v$,

$$d_T(v) \leq \begin{cases} f(v) + d_F(v), & \text{if } d_F(v) = 0; \\ f(v) + d_F(v) - 1, & \text{if } d_F(v) \geq 1. \end{cases}$$

Ellingham and Zha (2000) established a sufficient toughness condition for extending a spanning forest with non-trivial components to a spanning tree only by inserting a matching. Later, Ellingham, Nam, and Voss (2002) developed their result to the following theorem.

Theorem 1.7. ([2]) Let $G$ be a connected graph with a spanning forest $F$ of which every component contains at least $c$ vertices with $c \geq 2$. Let $h$ be a nonnegative integer-valued function on $V(G)$. If for every $S \subseteq V(G)$, at least one of the following conditions holds:

- $\omega(G \setminus S) < \sum_{v \in S} \left( \frac{1}{2}h(v) - \frac{1}{2} \right) + 2$.
- $\omega(G \setminus S) < \sum_{v \in S} \frac{c-2}{2c-2} h(v) + 2 + \frac{1}{\rho - 1} \min\{h(v) : v \in S\}$ and $\min\{h(v) : v \in S\} > 0$, where $\rho = c \min\{h(v) : v \in S\}$.

then $G$ has a spanning tree $T$ containing $F$ such that for each vertex $v$, $d_T(v) \leq h(v) + d_F(v)$.

In this paper, we provide a common improvement for both items of Theorem 1.7 as the following stronger version. More generally, we will introduce a combined version for this result and Theorem 1.6. Owing to its complicated form, we postpone it until Section 5.

Theorem 1.8. Let $G$ be a graph with a spanning forest $F$ of which every component contains at least $c$ vertices with $c \geq 2$. Let $h$ be a nonnegative integer-valued function on $V(G)$. If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) < \sum_{v \in S} \left( \frac{c}{2c-2} h(v) - \frac{1}{c-1} \right) + 2 + \frac{1}{c-1} \omega(G[S]),$$

then $G$ has a spanning tree $T$ containing $F$ such that for each vertex $v$, $d_T(v) \leq h(v) + d_F(v)$.

In 1990 Jackson and Wormald [7] conjectured that every $\frac{1}{n-1}$-tough graph with $n \geq 2$ admits a spanning closed $n$-walk. They also observed that this conjecture is true for $\frac{1}{n-2}$-tough graphs, when $n \geq 3$. Later, Ellingham and Zha (2000) proved the remaining case $n = 2$ for 4-tough graphs by making the following result. In Section 6, we solve this conjecture completely as mentioned in the abstract.
Theorem 1.9.([3]) Every 4-tough simple graph of order at least three admits a connected \([2,3]\)-factor containing an arbitrary given 2-factor.

2 Preliminary result

Here, we state the following fundamental theorem which was studied in [2, Theorem 1] for the case that \(M\) is the trivial matching.

Theorem 2.1. Let \(G\) be a graph with a spanning forest \(F\), and let \(M\) be a matching of \(G\) whose non-trivial components are vertex-disjoint from non-trivial components of \(F\). Let \(h\) be a nonnegative integer-valued function on \(V(G)\). If \(T\) is a spanning \((h+d_F)\)-forest of \(G\) containing \(F\cup M\) with the minimum \(\omega(T)\), then there exists a subset \(S\) of \(V(G)\) with the following properties:

1. \(\omega(G \setminus [S,F]) = \omega(T \setminus [S,F])\).
2. For each vertex \(u\) of \(S\), \(d_T(u) = h(u) + d_F(u)\).

Proof. Define \(V_0 = \emptyset\). For any \(S \subseteq V(G)\) and \(u \in V(G) \setminus S\), let \(A(S,u)\) be the set of all spanning \((h+d_F)\)-forests \(T'\) of \(G\) containing \(F \cup M\) such that \(\omega(T') = \omega(T)\), and also \(T'\) and \(T\) have the same edges, except for some of the edges of \(G\) whose ends are in \(V(C) \setminus S\), where \(C\) is the component of \(T \setminus [S,F]\) containing \(u\). Now, for each positive integer \(n\), recursively define \(V_n\) as follows:

\[
V_n = V_{n-1} \cup \{ v \in V(G) \setminus V_{n-1} : d_{T'}(v) = h(v) + d_F(v) \text{ for all } T' \in A(V_{n-1}, v) \}.
\]

Now, we prove the following claim.

Claim. Let \(x\) and \(y\) be two vertices in different components of \(T \setminus [V_{n-1}, F]\). If \(xy \in E(G) \setminus E(T)\), then \(x \in V_n\) or \(y \in V_n\).

Proof of Claim. By induction on \(n\). Suppose, to the contrary, that \(x\) and \(y\) are in different components of \(T \setminus [V_{n-1}, F]\), \(xy \in E(G) \setminus E(T)\), and \(x,y \notin V_n\). Let \(X\) and \(Y\) be the vertex sets of the components of \(T \setminus [V_{n-1}, F]\) containing \(x\) and \(y\), respectively. Since \(x, y \notin V_n\), there exist \(T_x \in A(V_{n-1}, x)\) and \(T_y \in A(V_{n-1}, y)\) with \(d_{T_x}(x) < h(x) + d_F(x)\) and \(d_{T_y}(y) < h(y) + d_F(y)\). For \(n = 1\), define \(T'\) to be the spanning forest of \(G\) containing \(F \cup M\) with

\[
E(T') = E(T) + xy - E(T[X]) + E(T_x[X]) - E(T[Y]) + E(T_y[Y])
\]

Since \(T'\) is a spanning \((h+d_F)\)-forest and \(\omega(T') < \omega(T)\), we arrive at a contradiction. Now, suppose \(n \geq 2\). By the induction hypothesis, \(x\) and \(y\) are in the same component of \(T \setminus [V_{n-2}, F]\). Let \(P\) be the unique path connecting \(x\) and \(y\) in \(T\). Notice that the vertices of \(P\) lie in the same component of \(T \setminus [V_{n-2}, F]\). Pick \(e \in E(P) \setminus E(F)\) such that \(e\) is incident to a vertex \(z \in V_{n-1} \setminus V_{n-2}\). According to the assumption on \(M\)
and $F$, if $e \in E(M)$ then for the other edge $e' \in E(P \setminus \{e\}$ incident to $z$, we must have $e' \notin E(F) \cup E(M)$. We may therefore assume that $e \notin E(M)$. Now, let $T'$ be the spanning forest of $G$ containing $F \cup M$ with

$$E(T') = E(T) - e + xy - E(T[X]) + E(T_z[X]) - E(T[Y]) + E(T_y[Y]).$$

It is not hard to check that $d_{T'}(z) < d_T(z) \leq h(z) + d_F(z)$ and $T'$ lies in $A(V_{n-2}, z)$. Since $z \in V_{n-1}$, we arrive at a contradiction. Hence the claim holds.

When $F$ is the trivial spanning forest, Theorem 2.1 can be reformulated to the following simpler version. For our purposes in Section 4, this special case would be sufficient.

**Corollary 2.2.** Let $G$ be a graph with a matching $M$ and let $f$ be a positive integer-valued function on $V(G)$. If $T$ is a spanning $f$-forest of $G$ containing $M$ with the minimum $\omega(T)$, then there exists a subset $S$ of $V(G)$ with the following properties:

1. $\omega(G \setminus S) = \omega(T \setminus S)$.
2. For each vertex $v$ of $S$, $d_T(v) = f(v)$.

## 3 Connected $(d_F, f + d_F - 1)$-factors

The following lemma establishes a simple but important property of forests.

**Lemma 3.1.** Let $T$ be a forest with a spanning forest $F$. If $S \subseteq V(T)$ and $\mathcal{F} = T \setminus E(F)$, then

$$\sum_{v \in S} d_F(v) = \omega(T \setminus [S, F]) - \omega(T) + e_F(S).$$

**Proof.** By induction on the number of edges of $\mathcal{F}$ which are incident to the vertices in $S$. If there is no edge of $\mathcal{F}$ incident to a vertex in $S$, then the proof is clear. Now, suppose that there exists an edge $e = uu' \in E(\mathcal{F})$ with $|S \cap \{u, u'\}| \geq 1$. Hence

1. $\omega(T) = \omega(T \setminus e) - 1$,
2. $\omega(T \setminus [S, F]) = \omega((T \setminus e) \setminus [S, F])$,
3. $e_F(S) = e_{\mathcal{F} \setminus e}(S) + |S \cap \{u, u'\}| - 1$. 

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4. \( \sum_{v \in S} d_F(v) = \sum_{v \in S} d_{F\setminus e}(v) + |S \cap \{u, u'\}|. \)

Therefore, by the induction hypothesis on \( T \setminus e \) with the spanning forest \( F \) the lemma holds.

The following theorem is essential in this section.

**Theorem 3.2.** Let \( G \) be a connected graph with \( X \subseteq V(G) \) and with a factor \( F \). Let \( \lambda \in [0,1] \) be a real number and let \( \eta : X \to (\lambda, \infty) \) be a real function. If for all \( S \subseteq X \),

\[
\omega(G \setminus [S, F]) < \sum_{v \in S} (\eta(v) - 2) + 2 - \lambda e_G(S, F) + 1 - \lambda,
\]

then \( G \) has a connected factor \( H \) containing \( F \) such that for each \( v \in X \), \( d_H(v) \leq [\eta(v) - \lambda] + d_F(v) - 1 \).

**Proof.** For each vertex \( v \), define

\[
h(v) = \begin{cases} d_G(v) + 1, & \text{if } v \notin X; \\ \lceil \eta(v) - \lambda \rceil - 1, & \text{if } v \in X. \end{cases}
\]

First, suppose that \( F \) is a forest. Let \( T \) be a spanning \((h + d_F)\)-forest of \( G \) containing \( F \) with the minimum \( \omega(T) \). Define \( S \) to be a subset of \( V(G) \) with the properties described in Theorem 2.1. If \( S \) is empty, then \( \omega(T) = \omega(G) = 1 \) and the theorem clearly holds. So, suppose \( S \) is nonempty. If \( v \in V(G) \setminus X \), then \( d_T(v) \leq d_G(v) < h(v) + d_F(v) \). This implies that \( S \subseteq X \). Put \( \mathcal{F} = T \setminus E(F) \). By Lemma 3.1 and Theorem 2.1,

\[
\sum_{v \in S} h(v) = \sum_{v \in S} d_F(v) = \omega(T \setminus [S, F]) - \omega(T) + e_{\mathcal{F}}(S),
\]

and so

\[
\omega(T) = \omega(G \setminus [S, F]) - \sum_{v \in S} h(v) + (1 - \lambda)e_{\mathcal{F}}(S) + \lambda e_{\mathcal{F}}(S).
\]

Since \( e_{\mathcal{F}}(S) \leq |S| - 1 \) and \( e_{\mathcal{F}}(S) \leq e_G(S, F) \), by the assumption, we therefore have

\[
\omega(T) \leq \omega(G \setminus [S, F]) - \sum_{v \in S} (\eta(v) - \lambda - 1) + (1 - \lambda)(|S| - 1) + \lambda e_G(S, F) < 2.
\]

Hence \( \omega(T) = 1 \) and the theorem holds. Now, suppose that \( F \) is not a forest. Remove some of the edges of the components of \( F \) until the resulting graph \( F' \) becomes a forest such that their components have the same vertices. It is enough, now, to apply the theorem on \( F' \) and finally add the edges of \( E(F) \setminus E(F') \) to that explored tree.

The following corollary provides a necessary and sufficient condition for the existence of a spanning tree with the described properties.

**Corollary 3.3.** Let \( G \) be a graph with a spanning forest \( F \) and let \( X \subseteq V(G) \) with \( e_G(X, F) = 0 \). Then \( G \) has a spanning tree \( T \) containing \( F \) such that for each \( v \in X \), \( d_T(v) \leq h(v) + d_F(v) \), if and only if for all \( S \subseteq X \), \( \omega(G \setminus [S, F]) \leq \sum_{v \in S} h(v) + 1 \), where \( h \) is a nonnegative integer-valued function on \( X \).
Proof. Assume that $G$ has a spanning tree $T$ containing $F$ such that for each $v \in X$, $d_T(v) \leq h(v) + d_F(v)$. Put $\mathcal{F} = T \setminus E(F)$ and let $S \subseteq X$. According to the assumption on $X$, one can conclude that $|\mathcal{F}(S)| = 0$. Since for each $v \in S$, $d_{\mathcal{F}}(v) \leq h(v)$, and $\omega(T) = 1$, with respect to Lemma 3.1, $\omega(G \setminus [S, F]) \leq \omega(T \setminus [S, F]) = \sum_{v \in S} d_{\mathcal{F}}(v) + 1 \leq \sum_{v \in S} h(v) + 1$. To prove the converse, one can apply Theorem 3.2 with $\lambda = 1$ and $\eta(v) = h(v) + 2$. Note that $G$ is connected, because $\omega(G \setminus [\emptyset, F]) \leq 1$. □

This corollary shows an application of Corollary 3.3. It can also be deduced from Corollary 4.3.

Corollary 3.4. ([5], see Page 5 in [14]) Let $G$ be a graph with an independent set $X \subseteq V(G)$. Then $G$ has a spanning tree $T$ such that for each $v \in X$, $d_T(v) \leq f(v)$, if and only if $\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1$ for all $S \subseteq X$, where $f$ is a positive integer-valued function on $X$.

Proof. Apply Corollary 3.3 and the fact that $\omega(G \setminus [S, F]) = \omega(G \setminus S) + |S|$ when $F$ is the trivial spanning forest. □

Corollary 3.5. Let $G$ be a connected graph and let $f$ be a positive integer-valued function on $V(G)$. If for all $S \subseteq V(G)$,

$$\omega(G \setminus [S, F]) \leq \sum_{v \in S} (f(v) - 2) + 2,$$

then $G$ has a connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq f(v) + d_F(v) - 1$.

Proof. Apply Theorem 3.2 with $\eta = f$ and $\lambda = 0$. □

3.1 Graphs with high essential edge-connectivity

The following lemma provides two upper bounds on $\omega(G \setminus [S, F])$ depending on two parameters of connectivity of $G/F$ and $d_G(v, F)$ of the vertices $v$ in $S$.

Lemma 3.6. Let $G$ be a graph with a factor $F$ and let $S \subseteq V(G)$. Then

$$\omega(G \setminus [S, F]) \leq \begin{cases} \sum_{v \in S} \left(\frac{d_G(v, F)}{k} + 1\right) - \frac{2}{k} e_G(S, F), & \text{if } G/F \text{ is } k\text{-edge-connected and } S \neq \emptyset; \\
\sum_{v \in S} \frac{d_G(v, F)}{k} + 1 - \frac{1}{k} e_G(S, F), & \text{if } G/F \text{ is } k\text{-tree-connected.} \end{cases}$$

Proof. First, assume that $G/F$ is $k$-edge-connected and $S$ is nonempty. Thus there are at least $k(\omega(G \setminus [S, F]) - |S|)$ edges of $G$ with exactly one end in $S$ joining different components of $G \setminus [S, F]$, because $S$ is nonempty and there are at least $\omega(G \setminus [S, F]) - |S|$ components of $G \setminus [S, F]$ without any vertex of $S$. Note that we might have $\omega(G \setminus [S, F]) < |S|$. On the other hand, there are $\sum_{v \in S} d_G(v, F) - 2 e_G(S, F)$ edges of $G$ with exactly one end in $S$ joining different components of $F$. Hence we have

$$k(\omega(G \setminus [S, F]) - |S|) \leq \sum_{v \in S} d_G(v, F) - 2 e_G(S, F).$$
Next, assume that $G/F$ is $k$-tree-connected. Thus there are at least $k(\omega(G \setminus [S,F]) - 1)$ edges of $G$ with at least one end in $S$ joining different components of $G \setminus [S,F]$. On the other hand, there are $\sum_{v \in S} d_G(v,F) - e_G(S,F)$ edges of $G$ with at least one end in $S$ joining different components of $F$. Hence we have

\[
k(\omega(G \setminus [S,F]) - 1) \leq \sum_{v \in S} d_G(v,F) - e_G(S,F).
\]

These inequalities complete the proof. \qed

The following theorem generalizes Theorems 1.4 and 1.5.

**Theorem 3.7.** Let $G$ be a graph with a factor $F$. Then $G$ has a connected factor $H$ containing $F$ such that for each vertex $v$,

\[
d_H(v) \leq \begin{cases} \left\lfloor \frac{d_G(v) - d_F(v) - 2}{k} \right\rfloor + d_F(v) + 2, & \text{if } G/F \text{ is } k\text{-edge-connected;} \\ \left\lfloor \frac{d_G(v) - d_F(v) - 1}{k} \right\rfloor + d_F(v) + 1, & \text{if } G/F \text{ is } k\text{-tree-connected.} \end{cases}
\]

Furthermore, for an arbitrary given vertex $u$, the upper bound can be reduced to $\lceil (d_G(u) - d_F(u))/k \rceil + d_F(u)$.

**Proof.** We may assume that $k \geq 2$, as the assertions trivially hold when $k = 1$. Since $G$ is connected, it is obvious that $\omega(G \setminus \emptyset, F) = 1$. Let $S$ be a nonempty subset of $V(G)$. If $G/F$ is $k$-edge-connected, then by Lemma 3.6, we have

\[
\omega(G \setminus [S,F]) \leq \sum_{v \in S} \left( \frac{d_G(v,F)}{k} + 1 - \frac{2}{k} e_G(S,F) \right) \leq \begin{cases} \sum_{v \in S} (\eta(v) - 2) + \frac{2}{k} e_G(S,F) + 1 - \frac{3}{k}, & \text{if } u \in S; \\ \sum_{v \in S} (\eta(v) - 2) - \frac{2}{k} e_G(S,F), & \text{if } u \notin S, \end{cases}
\]

where $\eta(u) = \frac{d_G(u) - d_F(u) + 3}{k}$ and $\eta(v) = \frac{d_G(v) - d_F(v) + 3}{k}$ for all $v \in V(G) \setminus u$. If $G$ is $k$-tree-connected, then by Lemma 3.6, we also have

\[
\omega(G \setminus [S,F]) \leq \sum_{v \in S} \frac{d_G(v,F)}{k} + 1 - \frac{1}{k} e_G(S,F) \leq \begin{cases} \sum_{v \in S} (\eta(v) - 2) + \frac{2}{k} e_G(S,F) + 1 - \frac{2}{k}, & \text{if } u \in S; \\ \sum_{v \in S} (\eta(v) - 2) + 1 - \frac{1}{k} e_G(S,F), & \text{if } u \notin S, \end{cases}
\]

where $\eta(u) = \frac{d_G(u) - d_F(u) + 2}{k}$ and $\eta(v) = \frac{d_G(v) - d_F(v) + 2}{k}$ for all $v \in V(G) \setminus \{u\}$. Hence the assertions follow from Theorem 3.2 with $\lambda \in \{2/k, 1/k\}$. \qed

## 4 Connected $(d_F, f + \max\{0, d_F - 1\})$-factors

Our aim in this section is to prove Theorem 1.6 and give several applications of it on connected factors. We begin with the following lemma that allows us to make the proof simpler. This lemma can also develop a result due to Rivera-Campo [15], who gave a sufficient condition for the existence of a spanning tree with bounded maximum degree containing an arbitrary given matching.
Lemma 4.1. Let $G$ be a graph with a factor $F$. If a maximal matching of $F$ can be extended to a spanning tree $T$, then $F$ itself can be extended to a connected factor $H$ such that for each vertex $v$,

$$d_H(v) \leq d_T(v) + \max\{0, d_F(v) - 1\}.$$ 

Proof. Choose a maximal matching $M$ of $F$ which can be extended to a spanning tree $T$. Define $G_0 = T \cup F$. Let $T_0$ be a spanning tree of $G_0$ containing $M$ such that $d_{T_0}(u) \leq d_T(u)$ for all $u \in A = \{v \in V(G) : d_M(v) = 0\}$. According to the maximality of $M$, the vertex set $A$ must be an independent set of $F$. Otherwise, we can insert a new edge of $F$ into $M$ to expand it to a large matching which is a contradiction. Note that $T$ is a natural candidate for $T_0$. Consider $T_0$ with the maximum $|E(F) \cap E(T_0)|$. Define $H = T_0 \cup F$. We claim that $H$ is the desired factor we are looking for. Let $v \in V(H)$. If $d_M(v) = 1$ or $d_F(v) = 0$, then

$$d_H(v) \leq d_{G_0}(v) \leq d_T(v) + d_F(v) - d_M(v) = d_T(v) + \max\{0, d_F(v) - 1\}.$$ 

So, suppose that $v \in A$ and $d_F(v) > 0$. Define $F_0$ to be the factor of $F$ with $E(F_0) = E(F) \cap E(T_0)$. To complete the proof, we are going to show that $d_{F_0}(v) > 0$, which can imply that $d_H(v) \leq d_{T_0}(v) + d_F(v) - d_{F_0}(v) \leq d_T(v) + d_F(v) - 1$. Suppose, to the contrary, that $d_{F_0}(v) = 0$. Pick $vx \in E(F)$ so that $vx \notin E(T_0)$. Thus there exists an edge $vy \in E(T_0)$ such that the graph $T_0' = T_0 - vy + vx$ is still connected (we might have $x = y$). Since $d_{F_0}(v) = 0$, we must have $vy \notin E(F)$ and so $vy \notin E(M)$. Thus the spanning tree $T_0'$ must contain the edges of $M$. According to this construction, $d_{T_0}(u) = d_{T_0}(u)$ for all $u \in V(G) \setminus \{x, y\}$.

![Figure 1: An example for which the vertex $x$ is the center of a star component of $F$.](image)

Moreover, $d_{T_0}(x) = d_{T_0}(x) + 1$ and $d_{T_0}(y) = d_{T_0}(y) - 1$ when $x \neq y$, and $d_{T_0}(x) = d_{T_0}(x)$ when $x = y$. Since $vx \in E(F)$ and $v \in A$, we must have $x \notin A$. Therefore, $d_{T_0}(u) \leq d_{T_0}(u) \leq d_T(u)$ for all $u \in A$. Since $|E(F) \cap E(T_0)| = |E(F) \cap E(T_0)| + 1$, we derive a contradiction to the maximality of $T_0$, as desired. 

The following theorem is essential in this section.

Theorem 4.2. Let $G$ be a graph with $X \subseteq V(G)$ and with a factor $F$. Let $f$ be a positive integer-valued function on $X$. If for all $S \subseteq X$,

$$\omega(G \setminus S) < \sum_{v \in S} (f(v) - 2) + 2 + \omega(G[S]),$$

then $G$ has a connected factor $H$ containing $F$ such that for each $v \in X$, $d_H(v) \leq f(v) + \max\{0, d_F(v) - 1\}$. 

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**Proof.** For each \( v \in V(G) \setminus X \), define \( f(v) = d_G(v) + 1 \). Choose a matching \( M \) of \( F \) and let \( T \) be a spanning \( f \)-forest of \( G \) containing \( M \) with the minimum \( \omega(T) \). Define \( S \) to be a subset of \( V(G) \) with the properties described in Corollary 2.2. If \( v \in V(G) \setminus X \), then \( d_T(v) \leq d_G(v) < f(v) \). This implies that \( S \subseteq X \). By Lemma 3.1 and Corollary 2.2,

\[
\sum_{v \in S} f(v) = \sum_{v \in S} d_T(v) = \omega(T \setminus S) + |S| - \omega(T) + \epsilon_T(S),
\]

and hence

\[
\omega(T) = \omega(G \setminus S) - \sum_{v \in S} (f(v) - 1) + e_T(S).
\]

Since \( e_T(S) \leq |S| - \omega(G[S]) \), by the assumption, we therefore have

\[
\omega(T) \leq \omega(G \setminus S) - \sum_{v \in S} (f(v) - 2) - \omega(G[S]) < 2.
\]

Thus \( T \) is a spanning \( f \)-tree of \( G \) containing \( M \). Since \( M \) is an arbitrary matching, by Lemma 4.1, one can conclude that the factor \( F \) itself can be extended to a connected factor \( H \) such that for each vertex \( v \), \( d_H(v) \leq f(v) + \max\{0, d_F(v) - 1\} \). Hence the theorem is proved. \( \square \)

**Corollary 4.3.** Let \( G \) be a graph with an independent set \( X \subseteq V(G) \). Let \( f \) be a positive integer-valued function on \( X \). If for all \( S \subseteq X \),

\[
\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1,
\]

then every factor \( F \) can be extended to a connected factor \( H \) such that for each \( v \in X \), \( d_H(v) \leq f(v) + \max\{0, d_F(v) - 1\} \).

**Proof.** Let \( S \) be a subset of \( X \). Since \( X \) is an independent set, we must have \( \omega(G[S]) = |S| \) which implies that \( \omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1 < \sum_{v \in S} (f(v) - 2) + 2 + \omega(G[S]) \). Now, it is enough to apply Theorem 4.2. \( \square \)

Ellingham, Nam, and Voss [2] discovered the following result, when \( g' \) is a positive function.

**Corollary 4.4.** Let \( G \) be a connected graph. If for all \( S \subseteq V(G) \), \( \omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 2) + 2 \), then every \( (g', f') \)-factor can be extended to a connected \( (g', f' + f - 1) \)-factor, where \( g' \) is a nonnegative integer-valued function on \( V(G) \), and \( f' \) and \( f \) are positive integer-valued functions on \( V(G) \).

**Proof.** Since \( G \) is connected, it is obvious that \( \omega(G \setminus 0) < 2 + \omega(G[0]) \). Let \( S \) be a nonempty subset of \( V(G) \). Since \( \omega(G[S]) \geq 1 \), we must have \( \omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 2) + 2 < \sum_{v \in S} (f(v) - 2) + 2 + \omega(G[S]) \). Thus by Theorem 4.2, we can extend an arbitrary given \( (g', f') \)-factor \( F \) to a connected factor \( H \) such that for each vertex \( v \), \( g'(v) \leq d_F(v) \leq d_H(v) \leq f(v) + \max\{0, d_F(v) - 1\} \). Since \( f'(v) \) is positive, we have \( d_H(v) \leq f(v) + f'(v) - 1 \), regardless of \( d_F(v) = 0 \) or not. Thus \( H \) is the desired connected factor. \( \square \)
When we consider the special case \((g', f') = (0, 1)\), Corollary 4.4 becomes simpler as the following result.

**Corollary 4.5.** Let \(G\) be a connected graph. If for all \(S \subseteq V(G)\), \(\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 2) + 2\), then \(G\) has a spanning \(f\)-tree containing an arbitrary given matching, where \(f\) is a positive integer-valued function on \(V(G)\).

**Remark 4.6.** Note that if every matching of a graph \(G\) can be extended to a spanning \(f\)-tree, then for all \(S \subseteq V(G)\), we must have \(\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1\), where \(\alpha'(G[S])\) denotes the number of edges in a maximum matching of the induced graph \(G[S]\). In fact, if a graph \(G\) has a spanning \(f\)-tree \(T\) containing a given forest \(M\), then by Lemma 3.1, we clearly have \(\omega(G \setminus S) \leq \omega(T \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1 - e_M(S)\) for all \(S \subseteq V(G)\).

The following lemma allows us to establish the next result, and also will be employed in the last section.

**Lemma 4.7.** (Ellingham, Nam, and Voss [2]) If \(G\) is a connected \(K_{1,n}\)-free simple graph with \(n \geq 2\), then \(\omega(G \setminus S) \leq (n - 2)|S| + 1\) for all \(S \subseteq V(G)\).

Xu, Liu, and Tokuda [17] discovered the following result, when \(g'\) is a positive function.

**Corollary 4.8.** If \(G\) is a connected \(K_{1,n}\)-free simple graph with \(n \geq 2\), then every \((g', f')\)-factor can be extended to a connected \((g', f' + n - 1)\)-factor, where \(g'\) is a nonnegative integer-valued function on \(V(G)\) and \(f'\) is a positive integer-valued function on \(V(G)\).

**Proof.** Apply Lemma 4.7 and Corollary 4.4 with \(f(v) = n\). \(\Box\)

### 4.1 Graphs with high edge-connectivity

A special case of Lemma 3.6 can easily conclude the following lemma, because \(e_G(S, F) = e_G(S) \geq |S| - \omega(G[S])\) and \(\omega(G \setminus [S, F]) = \omega(G \setminus S) + |S|\) when \(F\) is the trivial spanning forest.

**Lemma 4.9.** Let \(G\) be a graph with \(S \subseteq V(G)\). Then

\[
\omega(G \setminus S) \leq \begin{cases} 
\sum_{v \in S} \frac{d_G(v) - 2}{k} + \frac{2}{k} \omega(G[S]), & \text{if } G \text{ is } k\text{-edge-connected and } S \neq \emptyset; \\
\sum_{v \in S} (\frac{d_G(v) - 1}{k} - 1) + 1 + \frac{1}{k} \omega(G[S]), & \text{if } G \text{ is } k\text{-tree-connected}.
\end{cases}
\]

Another generalization of Theorems 1.4 and 1.5 is given in the next theorem.

**Theorem 4.10.** Let \(G\) be a graph with \(X \subseteq V(G)\). Then every factor \(F\) can be extended to a connected

\[
\omega(G \setminus X) \leq \begin{cases} 
\sum_{v \in X} \frac{d_G(v) - 2}{k} + \frac{2}{k} \omega(G[X]), & \text{if } G \text{ is } k\text{-edge-connected and } X \neq \emptyset; \\
\sum_{v \in X} (\frac{d_G(v) - 1}{k} - 1) + 1 + \frac{1}{k} \omega(G[X]), & \text{if } G \text{ is } k\text{-tree-connected}.
\end{cases}
\]
factor $H$ such that for each $v \in X$,

$$d_H(v) \leq \max\{0, d_F(v) - 1\} + \begin{cases} \left\lceil \frac{d_G(v) - 2}{k} \right\rceil + 2, & \text{if } G \text{ is } k \text{-edge-connected;} \\ \left\lceil \frac{d_G(v) - 1}{k} \right\rceil + 1, & \text{if } G \text{ is } k \text{-tree-connected;} \\ \left\lceil \frac{d_G(v)}{k} \right\rceil + 1, & \text{if } G \text{ is } k \text{-edge-connected and } X \text{ is independent;} \\ \left\lceil \frac{d_G(v)}{k} \right\rceil, & \text{if } G \text{ is } k \text{-tree-connected and } X \text{ is independent.} \end{cases}$$

Furthermore, for an arbitrary given vertex $u$, the upper bound can be reduced to $\lceil d_G(u)/k \rceil + \max\{0, d_F(u) - 1\}$.

**Proof.** We may assume that $k \geq 2$, as the assertions trivially hold when $k = 1$. Since $G$ is connected, it is obvious that $\omega(G \setminus \emptyset) < 2 + \omega(G[\emptyset])$. Let $S$ be a nonempty subset of $X$ so that $\omega(G[S]) \geq 1$. If $G$ is $k$-edge-connected, then by Lemma 4.9, we have

$$\omega(G \setminus S) \leq \sum_{v \in S} \left( \frac{d_G(v) - 2}{k} + \frac{2}{k} \right) \omega(G[S]) < \sum_{v \in S} (f(v) - 2) + 2 + \omega(G[S]),$$

where $f(u) = \left\lceil \frac{d_G(u) + 1}{k} \right\rceil - 1$ and $f(v) = \left\lceil \frac{d_G(v) - 2}{k} \right\rceil + 2$ for all $v \in V(G) \setminus \{u\}$. Note that $f(u) = \left\lceil \frac{d_G(u)}{k} \right\rceil \geq 1$ because of $d_G(u) \geq k$. If $G$ is $k$-tree-connected, then by Lemma 4.9, we also have

$$\omega(G \setminus S) \leq \sum_{v \in S} \left( \frac{d_G(v) - 1}{k} - 1 \right) + 1 + \frac{1}{k} \omega(G[S]) < \sum_{v \in S} (f(v) - 2) + 2 + \omega(G[S]),$$

where $f(u) = \left\lceil \frac{d_G(u) + 1}{k} \right\rceil - 1$ and $f(v) = \left\lceil \frac{d_G(v) - 1}{k} \right\rceil + 1$ for all $v \in V(G) \setminus \{u\}$. Thus the first two assertions follow from Theorem 4.2. Now, suppose that $X$ is an independent set. If $G$ is $k$-edge-connected, then by Lemma 4.9, we have

$$\omega(G \setminus S) \leq \sum_{v \in S} \left( \frac{d_G(v)}{k} - 1 \right) < \sum_{v \in S} (f(v) - 1) + 2,$$

where $f(u) = \left\lceil \frac{d_G(u) + 1}{k} \right\rceil - 1$ and $f(v) = \left\lceil \frac{d_G(v)}{k} \right\rceil + 1$ for all $v \in X \setminus \{u\}$. If $G$ is $k$-tree-connected, then by Lemma 4.9, we also have

$$\omega(G \setminus S) \leq \sum_{v \in S} \left( \frac{d_G(v)}{k} - 1 \right) + 1 < \sum_{v \in S} (f(v) - 1) + 2,$$

where $f(u) = \left\lceil \frac{d_G(u) + 1}{k} \right\rceil - 1$ and $f(v) = \left\lceil \frac{d_G(v)}{k} \right\rceil$ for all $v \in X \setminus \{u\}$. Thus the second two assertions follow from Corollary 4.3. □

**Corollary 4.11.** If $G$ is a $k$-edge-connected graph with $\Delta(G) \leq k(n-2) + 2$ and $n \geq 2$, then every $(g', f')$-factor can be extended to a connected $(g', f' + n - 1)$-factor, where $g'$ is a nonnegative integer-valued function on $V(G)$ and $f'$ is a positive integer-valued function on $V(G)$.

**Proof.** Let $F$ be a $(g', f')$-factor of $G$. By Theorem 4.10, we can extend $F$ to a connected factor $H$ such that for each vertex $v$, $g'(v) \leq d_F(v) \leq d_H(v) \leq n + \max\{0, d_F(v) - 1\}$. Since $f'(v)$ is positive, we have $d_H(v) \leq n + f'(v) - 1$, regardless of $d_F(v) = 0$ or not. Hence $H$ is the desired connected factor. □
The following corollary makes a strengthened version for Lemma 2.2 (ii) in [7].

**Corollary 4.12.** Let $G$ be a graph with a matching $M$ and let $f$ be a positive integer-valued function on $V(G)$. If $G$ admits a spanning closed $f$-walk passing through the edges of $M$, then it has a spanning $(f+1)$-tree $T$ containing the edges of $M$. Furthermore, for an arbitrary given vertex $u$, we can have $d_T(u) \leq f(u)$.

**Proof.** Let $H$ be the Eulerian graph with $V(H) = V(G)$ obtained from a spanning closed $f$-walk of $G$ passing through the edges of $M$, by inserting $t$ copies of every edge $e$ of $G$ into $H$ in which $e$ is used $t$ times in the desired walk. Since $H$ is Eulerian, it is 2-edge-connected. Thus by Theorem 4.10, one can conclude that the graph $H$ has a spanning $(f+1)$-tree $T$ containing the edges of $M$ with $d_T(u) \leq f(u)$, and so does $G$. □

Another strengthened version of Lemma 2.2 (ii) in [7] is given in the following theorem. This result allows one to deduce Corollary 4.5 form Theorem 6.5. In Section 6, we will conversely show that Corollary 4.5 implies Theorem 6.5, and the two are therefore equivalent.

**Theorem 4.13.** Let $G$ be a graph with a matching $M$ and let $f$ be a positive integer-valued function on $V(G)$. If $G$ admits a spanning $f$-walk (not necessarily closed) passing through the edges of $M$, then it has a spanning $(f+1)$-tree containing the edges of $M$.

**Proof.** Let $H$ be the connected graph with $V(H) = V(G)$ obtained from a spanning $f$-walk of $G$ passing through the edges of $M$, by inserting $t$ copies of every edge $e$ of $G$ into $H$ in which $e$ is used $t$ times in the desired walk. Note that all vertices of $H$, except possibly two vertices, have even degrees. This shows that the graph $H$ can be made 2-edge-connected by adding at most one edge. Let $S \subseteq V(G)$. Obviously, for all components $C$ of $H \setminus S$, there is at least one edge with exactly one edge in $V(C)$. Moreover, all components of $C$ of $H \setminus S$, except possibly two components, have at least two edges of $H$ with exactly one end in $V(C)$. Therefore, there are at least $2\omega(H \setminus S) - 2$ edges of $H$ with exactly on edge in $S$. This implies that $2\omega(H \setminus S) - 2 \leq \sum_{v \in S} d_H(v) - 2e_H(S)$ and hence $\omega(H \setminus S) \leq \sum_{v \in S} (d_H(v)/2 - 1) + \omega(H[S])$. Thus by Theorem 4.2, one can conclude that the graph $H$ has a spanning $(f+1)$-tree $T$ containing the edges of $M$, and so does $G$. □

5  Toughness and the existence of connected $\{r, r+1\}$-factors

In this section, we shall introduce a combined stronger version for Theorems 1.6 and 1.8. For this purpose, we need to establish the following lemma that provides a relationship between $\omega(G \setminus S)$ and $\omega(G \setminus [S,F])$.

**Lemma 5.1.** Let $G$ be a graph with a spanning forest $F$. Let $c \in [2, \infty)$ be a real number and let $\xi : V(G) \to [0,1]$ be a real function in which for every component $C$ of $F$, $\sum_{v \in V(C)} \xi(v) \geq 1 - \frac{1}{c-1}(|V(C)| - 1)$. 

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If $S \subseteq V(G)$, then
\[ \omega(G \setminus [S, F]) \leq \omega(G \setminus S) + \frac{1}{c-1}e_F(S) + \sum_{v \in S} \xi(v). \]

**Proof.** Since every component $C$ of $F$ whose vertices entirely lie in the set $S$ has exactly $|V(C)| - 1$ edges with both ends in $S$, we must have $1 \leq \sum_{v \in V(C)} \xi(v) + \frac{1}{c-1}e_C(S)$. Thus $t_s \leq \sum_{v \in S'} \xi(v) + \frac{1}{c-1}e_F(S') \leq \sum_{v \in S} \xi(v) + \frac{1}{c-1}e_F(S)$, where $S'$ is the set of all vertices belonging to the components of $F$ whose vertices entirely lie in the set $S$, and $t_s$ is the number of those components. Therefore, $\omega(G \setminus [S, F]) \leq \omega(G \setminus S) + t_s \leq \sum_{v \in S} \xi(v) + \frac{1}{c-1}e_F(S)$. Hence the lemma holds. \(\square\)

The following theorem is essential in this section.

**Theorem 5.2.** Let $G$ be a graph with a factor $F$ and let $h$ be a nonnegative integer-valued function on $V(G)$. Let $c \in [2, \infty)$ be a real number and let $\xi : V(G) \to [0, 1]$ be a real function in which for every component $C$ of $F$, $\sum_{v \in V(C)} \xi(v) \geq 1 - \frac{1}{c-1}(|V(C)| - 1)$. If for every $S \subseteq V(G)$,
\[ \omega(G \setminus S) < \sum_{v \in S} (h(v) - \frac{1}{c-1} - \xi(v)) + 2 + \frac{1}{c-1} \omega(G[S]) - \frac{c-2}{c-1} \min\{\sum_{v \in S} h(v)\}, |S| - \omega(G[S])\],
then $G$ has a connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq h(v) + d_F(v)$.

**Proof.** First, suppose that $F$ is a forest. Let $T$ be a spanning $(h + d_F)$-forest of $G$ containing $F$ with the minimum $\omega(T)$. Define $S$ to be a subset of $V(G)$ with the properties described in Theorem 2.1. Put $F = T \setminus E(F)$. By Lemma 3.1 and Theorem 2.1,
\[ \sum_{v \in S} h(v) = \sum_{v \in S} d_F(v) = \omega(T \setminus [S, F]) - \omega(T) + e_F(S), \]
and so
\[ \omega(T) = \omega(G \setminus [S, F]) - \sum_{v \in S} h(v) + e_F(S). \] (1)

Also, by Lemma 5.1,
\[ \omega(G \setminus [S, F]) \leq \omega(G \setminus S) + \frac{1}{c-1}e_F(S) + \sum_{v \in S} \xi(v). \] (2)

Since $e_F(S) + e_F(S) = e_T(S) \leq |S| - \omega(G[S])$,
\[ e_F(S) + \frac{1}{c-1}e_F(S) \leq \frac{c-2}{c-1}e_F(S) + \frac{1}{c-1}(|S| - \omega(G[S])). \]

In addition, since $e_F(S) \leq \frac{1}{2} \sum_{v \in S} d_F(v) = \frac{1}{2} \sum_{v \in S} h(v)$, we must have
\[ e_F(S) + \frac{1}{c-1}e_F(S) \leq \frac{c-2}{c-1} \min\{\frac{1}{2} \sum_{v \in S} h(v), |S| - \omega(G[S])\} + \frac{1}{c-1}(|S| - \omega(G[S])). \] (3)

Therefore, Relations (1), (2), and (3) can conclude that
\[ \omega(T) \leq \omega(G \setminus S) - \sum_{v \in S} (h(v) - \frac{1}{c-1} - \xi(v)) - \frac{1}{c-1} \omega(G[S]) + \frac{c-2}{c-1} \min\{\sum_{v \in S} \frac{h(v)}{2}, |S| - \omega(G[S])\} < 2. \]
Hence \( \omega(T) = 1 \) and the theorem holds. Now, suppose that \( F \) is not a forest. Remove some of the edges of the components of \( F \) until the resulting graph \( F' \) becomes a forest such that their components have the same vertices. It is enough, now, to apply the theorem on \( F' \) and finally add the edges of \( E(F) \setminus E(F') \) to that explored tree.

\[ \square \]

**Corollary 5.3.** Let \( G \) be a simple graph and let \( F \) be a factor of \( G \) with \( \Delta(F) \leq 2 \). Then \( F \) can be extended to connected factor \( H \) with \( \Delta(H) \leq 3 \), if for all \( S \subseteq V(G) \), \( \omega(G \setminus S) \leq \frac{1}{4} |S| + 1 \).

**Proof.** For each vertex \( v \), define \( h(v) = 3 - d_F(v) \). Also, define \( \xi(v) = 1 \) when \( d_F(v) = 0 \), and define \( \xi(v) = 0 \) otherwise. According to these definitions, \( 1/4 \leq \frac{3}{2}h(v) - \frac{1}{c-1} - \xi(v) \). Thus if \( S \subseteq V(G) \), then \( \omega(G \setminus S) \leq \frac{1}{4} |S| + 1 < \sum_{v \in S} \left( \frac{c}{2c-2} h(v) - \frac{1}{c-1} - \xi(v) \right) + 2 \), where \( c = 3 \). Let \( C \) be a component of \( F \). Since \( F \) has no multiple edges, if \( C \) has a vertex with degree two, then it must contain at least three vertices, which implies that \( \sum_{v \in V(C)} \xi(v) \geq 0 \geq 1 - (|V(C)| - 1)/2 \). Otherwise, we again have \( \sum_{v \in V(C)} \xi(v) \geq 1 - (|V(C)| - 1)/2 \) regardless of \( C \) contains two vertices with degree one or not. Hence it is enough to apply Theorem 5.2 to complete the proof.

The following corollary improves Theorem 1.8 and implies Theorem 1.6.

**Corollary 5.4.** Let \( G \) be a graph with a factor \( F \) of which every non-trivial component contains at least \( c \) vertices with \( c \geq 2 \). Let \( h \) be a nonnegative integer-valued function on \( V(G) \). If for all \( S \subseteq V(G) \),

\[
\omega(G \setminus S) < \sum_{v \in S} \left( \frac{c}{2c-2} h(v) - \frac{1}{c-1} \right) + 2 + \frac{1}{c-1} \omega(G[S]) - |\{ v \in S : d_F(v) = 0 \}|
\]

then \( G \) has a connected factor \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \leq h(v) + d_F(v) \).

**Proof.** For each vertex \( v \), define \( \xi(v) = 1 \) when \( d_F(v) = 0 \), and define \( \xi(v) = 0 \) otherwise. Let \( C \) be a component of \( F \). If \( C \) is a non-trivial component, then by the assumption, we must have \( \sum_{v \in V(C)} \xi(v) \geq 0 \geq 1 - (|V(C)| - 1)/(c-1) \). Thus \( \sum_{v \in V(C)} \xi(v) \geq 1 - (|V(C)| - 1)/2 \) regardless of \( C \) is a trivial component or not. Hence it is enough to apply Theorem 5.2 to complete the proof.

**Corollary 5.5.** Let \( G \) be a graph with a factor \( F \). Let \( f \) be a positive integer-valued function on \( V(G) \). If for all \( S \subseteq V(G) \),

\[
\omega(G \setminus S) < \sum_{v \in S} \left( f(v) - 2 \right) + 2 + \omega(G[S]),
\]

then \( G \) has a connected factor \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \leq f(v) + \max\{0, d_F(v) - 1\} \).

**Proof.** Apply Corollary 5.4 with \( c = 2 \) and \( h(v) = f(v) - 1 + \max\{0, 1 - d_F(v)\} \).

We here derive the following corollary from Corollary 5.4 by comparing their conditions.
Corollary 5.6. ([2]) Let $G$ be a connected graph with a factor $F$ of which every component contains at least $c$ vertices with $c \geq 2$. Let $h$ be a nonnegative integer-valued function on $V(G)$. If for every $S \subseteq V(G)$, at least one of the following conditions holds:

- $\omega(G \setminus S) < \sum_{v \in S} \left( \frac{1}{2} h(v) - \frac{1}{c} \right) + 2$.
- $\omega(G \setminus S) < \sum_{v \in S} \frac{c^2 - 2}{2c - 2} h(v) + 2 + \frac{1}{\rho - 1}$ and $\min\{h(v) : v \in S\} > 0$, where $\rho = c \min\{h(v) : v \in S\}$.

then $G$ has a connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq h(v) + d_F(v)$.

Proof. Since $G$ is connected, it is obvious that $\omega(G \setminus \emptyset) < 2 + \frac{1}{c-1} \omega(G[\emptyset])$. Let $S$ be a nonempty subset of $V(G)$ and set $S_0 = \{v \in S : h(v) = 0\}$. Take $S_0'$ to be a subset of $S_0$ with $|S_0| - 1 \leq |S_0'| < |V(G)|$. If $|S_0| \geq 1$, then the first condition of the theorem must hold for the vertex set $S_0'$. Thus $1 \leq \omega(G \setminus S_0') < -|S_0'|/c + 2$. This implies that $|S_0'| < c$. Hence $|S_0| \leq c$ regardless of $S_0$ is empty or not. Thus

$$\sum_{v \in S} \left( \frac{1}{2} h(v) - \frac{1}{c} \right) \leq \sum_{v \in S} \left( \frac{c}{2c - 2} h(v) - \frac{1}{c - 1} \right) + \frac{|S_0|}{c(c - 1)} \leq \sum_{v \in S} \left( \frac{c}{2c - 2} h(v) - \frac{1}{c - 1} \right) + \frac{1}{c - 1}.$$  

In addition, if $|S_0| = 0$ then

$$\sum_{v \in S} \frac{\rho - 2}{2\rho - 2} h(v) = \sum_{v \in S} \left( \frac{c}{2c - 2} - \frac{1}{2c - 2} - \frac{1}{2\rho - 2} \right) h(v) \leq \sum_{v \in S} \left( \frac{c}{2c - 2} h(v) - \frac{1}{c - 1} \right).$$

More precisely, $\frac{1}{2c - 2} + \frac{1}{2\rho - 2} = \frac{1}{c - 1}$ when $\rho = c$, and $\frac{1}{2c - 2} h(v) \geq \frac{1}{c - 1}$ when $\rho \geq 2c$. Therefore, by the assumption, one can conclude that $\omega(G \setminus S) < \sum_{v \in S} \left( \frac{c}{2c - 2} h(v) - \frac{1}{c - 1} \right) + 2 + \frac{1}{c - 1} \omega(G[S])$. Hence it is enough to apply Corollary 5.4 to complete the proof. \hfill \Box

When we consider the special case $h = 1$, Corollary 5.6 becomes simpler as the following version.

Corollary 5.7. ([3]) Let $G$ be a connected graph with a factor $F$ of which every component contains at least $c$ vertices with $c \geq 2$. If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \frac{c - 2}{2c - 2} |S| + 2 + \frac{1}{2c - 2},$$

then $G$ has a connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \in \{d_F(v), d_F(v) + 1\}$.

Enomoto, Jackson, Katerinis, and Saito (1985) [4] showed that every $r$-tough graph $G$ of order at least $r + 1$ with $r |V(G)|$ even admits an $r$-factor. For the case that $r |V(G)|$ is odd, the same arguments can imply that the graph $G$ admits a factor such that whose degrees are $r$, except for a vertex with degree $r + 1$. A combination of Corollary 5.4 and this result can conclude the following corollary.

Corollary 5.8. ([2, 3]) Every $r$-tough graph of order at least $r + 1$ with $r \geq 3$ admits a connected $\{r, r + 1\}$-factor.
Proof. We may assume that $G$ is an $r$-tough simple graph, by deleting multiple edges from $G$ (if necessary). Let $F$ be an \{r, r + 1\}-factor of $G$ such that each of whose vertices has degree $r$, except for at most one vertex $u$ with degree $r + 1$ [4]. Note that every component of $F$ must contain at least $r + 1$ vertices. Let $S$ be a subset of $V(G)$. Since $G$ is $r$-tough,

\[
\omega(G \setminus S) \leq \frac{1}{r} |S| + 1 \leq \frac{r - 1}{2r} |S| + 1 + \sum_{v \in S} \left( \frac{c}{2c - 2} h(v) - \frac{1}{c - 1} \right) + 2 + \frac{1}{c - 1} \omega(G[S]),
\]

where $c = r + 1$, $h(u) = 0$, and $h(v) = 1$ for each vertex $v$ with $v \neq u$. Thus by applying Corollary 5.4, the graph $G$ has a connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq h(v) + d_F(v)$, and also $d_H(u) = d_F(u)$. This implies that $H$ is a connected \{r, r + 1\}-factor. \hfill \Box

Remark 5.9. Note that Corollary 5.8 can be proved by Corollary 5.7, except for the case that $r |V(G)|$ is odd, and it can be proved by Corollary 5.6, except for the case that $r = 3$ and $|V(G)|$ is odd.

6 Applications to spanning closed walks

Our aim in this section is to prove a long-standing conjecture due to Jackson and Wormald [7] with a stronger version. Before doing so, we state some results on spanning parity forests.

6.1 Spanning parity $f$-forests

In 1985 Amahashi [1] introduced a criterion for the existence of a spanning forest with bounded maximum degree in which all vertices have odd degree. Later, Yuting and Kano (1988) generalized it by establishing the following theorem. We denote below by $\text{odd}(G)$ the number of components of $G$ with odd order.

**Theorem 6.1.** ([18]) Let $G$ be a graph and let $f$ be an odd positive integer-valued function on $V(G)$. Then $G$ has a spanning $f$-forest with odd degree vertices if and only if for all $S \subseteq V(G)$,

\[
\text{odd}(G \setminus S) \leq \sum_{v \in S} f(v).
\]

Kano, Katona, and Szabó (2009) studied a more general version for Theorem 6.1 which gives a criterion for the existence of parity $f$-forests. We denote below by $\text{odd}_h(G)$ the number of components of $G$ with odd number of vertices $v$ with $h(v)$ odd.

**Theorem 6.2.** ([9]) Let $G$ be a graph and let $h$ be a nonnegative integer-valued function on $V(G)$. Then $G$ has a spanning $h$-forest $F$ such that for each vertex $v$, $d_F(v)$ and $h(v)$ have the same parity, if and only if for all $S \subseteq V(G)$,

\[
\text{odd}_h(G \setminus S) \leq \sum_{v \in S} h(v).
\]
Proof. The proof presented here is introduced in [9, Section 1] implicitly. Obviously, if $G$ has a spanning $h$-forest $F$ such that for each vertex $v$, $d_F(v)$ and $h(v)$ have the same parity, then for every $C$ component of $G \setminus S$ with $\sum_{v \in V(C)} h(v)$ odd, there must be an edge of $F$ with one end in $V(C)$ and the other one in $S$, where $S \subseteq V(G)$. Thus $\text{odd}_h(G \setminus S) \leq \sum_{v \in S} d_F(v) \leq \sum_{v \in S} h(v)$. Now, it remains to prove the sufficiency. Define $G_0$ to be the graph obtained from $G$ by adding a new vertex $u'$ and a pendant edge $u'u$ corresponding to each $u \in U = \{v \in V(G) : h(v) \text{ is even}\}$. Let us define $f(u) = h(u) + 1$ and $f(u') = 1$ for all $u \in U$, and $f(v) = h(v)$ for all $v \in V(G) \setminus U$. Note that $f$ is an odd positive integer-valued function on $V(G_0)$. It is not hard to check that for every $S \subseteq V(G_0)$,

$$\text{odd}(G_0 \setminus S) \leq \text{odd}_h(G \setminus S) + |S \cap (U \cup U')|,$$

where $U' = \{u' : u \in U\}$. This implies that $\text{odd}(G_0 \setminus S) \leq \sum_{v \in S \setminus U'} h(v) + |S \cap (U \cup U')| = \sum_{v \in S} f(v)$. Thus by Theorem 6.1, the graph $G_0$ has a spanning $f$-forest $F_0$ in which all vertices have odd degree. Obviously, this forest contains all inserted pendant edges and so by removing them we can find a forest $F$ of $G$ with the desired properties. \hfill \Box

We shall below derive a conclusion of Theorem 6.2, which will be used in the subsequent subsection. This result is proved in [10] when $f$ is an odd positive integer-valued function.

Corollary 6.3. Let $G$ be a graph and let $f$ be a positive integer-valued function on $V(G)$. Then for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1,$$

if and only if for every $Q \subseteq V(G)$ with even size, the graph $G$ has a spanning $f$-forest $F$ such that $Q = \{v \in V(G) : d_F(v) \text{ is odd}\}$.

Proof. We first prove the necessity. Let $Q$ be a subset of $V(G)$ with even size. For each vertex $v$, define $h(v)$ to be either $f(v)$ or $f(v) - 1$ such that $\{v \in V(G) : h(v) \text{ is odd}\} = Q$. Clearly, $\sum_{v \in V(G)} h(v)$ is even. Let $S \subseteq V(G)$. By the assumption, $\text{odd}_h(G \setminus S) \leq \omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1 \leq \sum_{v \in S} h(v) + 1$. It is easy to check that $\text{odd}_h(G \setminus S) + \sum_{v \in S} h(v)$ and $\sum_{v \in V(G)} h(v)$ have the same parity and so $\text{odd}_h(G \setminus S)$ and $\sum_{v \in S} h(v)$ have the same parity. Thus $\text{odd}_h(G \setminus S) \leq \sum_{v \in S} h(v)$. Therefore, by Theorem 6.2, the graph $G$ has a spanning $h$-forest $F$ such that for each vertex $v$, $d_F(v)$ and $h(v)$ have the same parity. Hence the necessity is proved.

Now, we shall prove the sufficiency. Let $S \subseteq V(G)$. We may assume that $G \setminus S$ contains a component $C'$. For each $v \in S$, define $h(v) = f(v) - 1$, and for each $v \in V(G) \setminus S$, define $h(v)$ to be either $f(v)$ or $f(v) + 1$ such that for every other component $C$ of $G \setminus S$, $\sum_{v \in V(C)} h(v)$ is odd, and also $\sum_{v \in V(C')} h(v)$ and $\sum_{v \in V(G) \setminus V(C)} h(v)$ have the same parity. Since $\sum_{v \in V(G)} h(v)$ is even, by the assumption, $G$ has a spanning $f$-forest $F$ such that for each vertex $v$, $d_F(v)$ and $h(v)$ have the same parity. Thus $F$ is also a spanning $h$-forest of $G$ and $\omega(G \setminus S) = \text{odd}_h(G \setminus S) + 1 \leq \sum_{v \in S} h(v) + 1 = \sum_{v \in S} (f(v) - 1) + 1$. Hence the proof is completed. \hfill \Box
The following result improves the upper bounds in Theorems 1.4 and 1.5 when the existence of parity forests are considered.

**Corollary 6.4.** Let $G$ be a connected graph with $Q \subseteq V(G)$, where $|Q|$ is even. Then $G$ has a spanning forest $F$ such that $Q = \{v \in V(G) : d_F(v) \text{ is odd}\}$, and for each vertex $v$,

$$d_F(v) \leq \begin{cases} \left\lceil \frac{d_G(v)}{k} \right\rceil + 1, & \text{if } G \text{ is } k\text{-edge-connected;} \\ \left\lceil \frac{d_G(v)}{k} \right\rceil, & \text{if } G \text{ is } k\text{-tree-connected.} \end{cases}$$

Furthermore, for an arbitrary given vertex $u$, the upper bound can be reduced to $\lfloor \frac{d_G(u)}{k} \rfloor$.

**Proof.** Since $G$ is connected, it is obvious that $\omega(G \setminus 0) = 1$. Let $S$ be a nonempty subset of $V(G)$. If $G$ is $k$-edge-connected, then by Lemma 4.9, we have

$$\omega(G \setminus S) \leq \sum_{v \in S} \frac{d_G(v)}{k} < \sum_{v \in S} (f(v) - 1) + 2,$$

where $f(u) = \left\lceil \frac{d_G(u)+1}{k} \right\rceil - 1$ and $f(v) = \left\lceil \frac{d_G(v)}{k} \right\rceil + 1$ for all $v \in V(G) \setminus \{u\}$. Note that $f(u) = \left\lfloor \frac{d_G(u)}{k} \right\rfloor$. If $G$ is $k$-tree-connected, then by Lemma 4.9, we also have

$$\omega(G \setminus S) \leq \sum_{v \in S} \left( \frac{d_G(v)}{k} - 1 \right) + 1 < \sum_{v \in S} (f(v) - 1) + 2,$$

where $f(u) = \left\lceil \frac{d_G(u)+1}{k} \right\rceil - 1$ and $f(v) = \left\lfloor \frac{d_G(v)}{k} \right\rfloor$ for all $v \in V(G) \setminus \{u\}$. Hence the assertions follow from Corollary 6.3. \qed

6.2 Jackson-Wormald Conjecture is true

The following theorem gives a sufficient condition for the existence of spanning $f$-walks. Note that under this condition, the desired spanning $f$-walk is not necessarily closed. For example, consider two copies of the complete graph of odd order $n$ with $n \geq 3$ and add a perfect matching $M$ between them. The resulting connected graph $G$ does not have a spanning closed 1-walk passing through the edges $M$, while satisfies $\omega(G \setminus S) \leq 2$ for all vertex sets $S$.

**Theorem 6.5.** Let $G$ be a connected graph and let $f$ be a positive integer-valued function on $V(G)$. If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 2,$$

then $G$ admits a spanning $f$-walk passing through the edges of an arbitrary given matching (and a spanning $f$-walk passing through the edges of an arbitrary given connected $(1,f+1)$-factor).

**Proof.** By Corollary 4.5, the graph $G$ has a spanning $(f+1)$-tree $T$ containing an arbitrary given matching (to prove the second assertion, $T$ can play the role of an arbitrary given connected $(1,f+1)$-factor). Let
$G_0$ be the graph obtained from $G$ by adding a new vertex $v_0$ and joining it to all other vertices. Define $f(v_0) = 2$. It is easy to check that for every $S \subseteq V(G_0)$, $\omega(G_0 \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1$, regardless of $v_0 \in S$ or not. Thus by Corollary 6.3, the graph $G_0$ contains a spanning $f$-forest $F_0$ such that $d_{F_0}(v_0) \in \{0, 2\}$ and for each $v \in V(G)$, $d_{F_0}(v)$ and $d_{T}(v)$ have the same parity. Let $F$ be the factor of $G$ obtained from $F_0$ by removing the vertex $v_0$. Insert a new copy of $F$ into $T$ and call the resulting connected graph $H$. Obviously, for each $v \in V(G) \setminus A$, $d_H(v)$ must be even and $d_H(v) = d_F(v) + d_T(v) \leq 2f(v) + 1$, where $A$ is the set of neighbours of $v_0$ in $F_0$. Note that $|A| \in \{0, 2\}$. Moreover, for each $v \in A$, $d_F(v) \leq f(v) - 1$ and so $d_H(v) = d_F(v) + d_T(v) \leq 2f(v)$. Therefore, the graph $H$ admits a spanning $f$-trail and so $G$ admits a spanning $f$-walk. □

To guarantee the existence of spanning closed $f$-walks, we need to push down the upper bound in Theorem 6.5 only by one as the next theorem.

**Theorem 6.6.** Let $G$ be a graph and let $f$ be a positive integer-valued function on $V(G)$. If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1,$$

then $G$ admits a spanning closed $f$-walk passing through the edges of an arbitrary given matching (and a spanning closed $f$-walk passing through the edges of an arbitrary given connected $(1, f+1)$-factor).

**Proof.** The graph $G$ must automatically be connected, because $\omega(G \setminus \emptyset) = 1$. Thus by Corollary 4.5, the graph $G$ has a spanning $(f + 1)$-tree $T$ containing an arbitrary given matching (to prove the second assertion, $T$ can play the role of an arbitrary given connected $(1, f+1)$-factor). By Corollary 6.3, the graph $G$ contains a spanning $f$-forest $F$ such that for each vertex $v$, $d_F(v)$ and $d_T(v)$ have the same parity. Insert a new copy of $F$ into $T$ and call the resulting connected graph $H$. For each vertex $v$, $d_H(v)$ must be even and $d_H(v) = d_F(v) + d_T(v) \leq 2f(v) + 1$. Therefore, the graph $H$ admits a spanning closed $f$-trail and so $G$ admits a spanning closed $f$-walk. □

**Corollary 6.7.** A simple graph $G$ admits a spanning closed $2$-walk passing through the edges of an arbitrary given factor $F$ with maximum degree at most 2, if for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \frac{1}{2}|S| + 1$.

**Proof.** By applying Corollary 5.3, $F$ can be extended to connected factor $H$ with $\Delta(H) \leq 3$. Thus by Theorem 6.6, the graph $G$ admits a spanning closed $2$-walk passing through the edges of $H$ and so does $F$. □

The following corollary is an immediate consequence of Lemma 4.7 and Theorem 6.6.

**Corollary 6.8.** ([7]) Every connected $K_{1,n}$-free simple graph with $n \geq 2$ has a spanning closed $(n-1)$-walk.
The next corollary improves Theorem 4.2 in [7] and implies Corollary 3.1 in [11]. Note that there are infinitely many $k$-connected $K_{1,n}$-free simple graphs with $k \geq 2$ and $n \geq 3$ having no spanning closed $\lfloor \frac{n-1}{k} \rfloor$-walks, which were constructed by Jin and Li [8].

**Corollary 6.9.** Every $k$-connected $K_{1,n}$-free simple graph with $n \geq 2$ has a spanning closed $\lfloor \frac{n-1}{k} \rfloor + 1$-walk.

**Proof.** We repeat the proof of Theorem 4.2 in [7]. Let $S$ be a cutset of $G$. Since $G$ is $k$-connected, every component of $G \setminus S$ is joined to at least $k$ vertices in $S$. Since $G$ is $K_{1,n}$-free, every vertex of $S$ is joined to at most $n-1$ components of $G \setminus S$. Hence $\omega(G \setminus S)k \leq (n-1)|S|$. Therefore, for all vertex sets $S$, $\omega(G \setminus S) \leq (n-1)|S|/k+1$, regardless of $S$ is a cutset or not. Hence the assertion follows from Theorem 6.6 with replacing $\lceil (n-1)/k \rceil + 1$ instead of $f(v)$. □

The following result confirms Conjecture 2.1 in [7]. Note that there are infinitely many graphs with toughness approaching $\frac{1}{n-\frac{5}{8}}$ having no spanning closed $n$-walks, which were constructed by Ellingham and Zha [3].

**Corollary 6.10.** Every $\frac{1}{n-1}$-tough graph with $n \geq 2$ admits a spanning closed $n$-walk.

**Proof.** If $S \subseteq V(G)$, then by the assumption, we have $\omega(G \setminus S) \leq \max\{1, (n-1)|S|\} \leq (n-1)|S| + 1$. Thus the assertions follows from Theorem 6.6 with setting $f(v) = n$. □

The next result confirms Conjecture 23 in [2]. Note that there are infinitely many $r$-edge-connected $r$-regular simple graphs with $r \geq 3$ having no spanning closed 1-walks, which were constructed by Meredith [13].

**Corollary 6.11.** Every $r$-edge-connected $r$-regular graph admits a spanning closed 2-walk.

**Proof.** Apply Lemma 4.9 and Corollary 6.10. □

Finally, we propose the following conjecture to make a stronger version for Theorem 6.6.

**Conjecture 6.12.** Let $G$ be a graph and let $f$ be a positive integer-valued function on $V(G)$. If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1 + \frac{1}{2} \omega(G[S]),$$

then $G$ admits a spanning closed $f$-walk passing through the edges of an arbitrary given matching.

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