Thinness for Scalar-Negative Singular Yamabe Metrics

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ABSTRACT. This paper deals with the conformal deformation of the standard metric in a domain on the sphere to a complete metric with the constant scalar curvature. The problem of description of domains allowing such deformation originates in the works of Loewner and Nirenberg, and Schoen and Yau concerned with the locally conformally flat manifolds. The goal of this work is to apply ideas from the nonlinear potential theory to the problem. They allow, in particular, to solve the problem in the case of the constant negative scalar curvature.

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1 Introduction

1.1 Singular Yamabe problem

Yamabe problem [56] was to prove that for any compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\) we can find a metric conformal to \(g\) with a constant scalar curvature by solving a certain variational problem. This was proved in three subsequent contributions by Trudinger [53], Aubin [2], and Schoen [17], see also [28], [51] for an exposition. Later other proofs were given, see [3] for a survey. To a large extent, it is the Yamabe problem that stimulated the development of the modern geometric analysis. An intensive work was done on Yamabe problem for manifolds with boundary. In this case seeks a conformal metric with constant curvatures in the interior and on the boundary, cf. e.g. [12], [13], [14]. There were also generalisations to non-Riemannian settings [21], [19].

In 1988 Schoen and Yau arrived at a different (non-variational) problem [50], [48]. Namely, is it possible to characterise domains on the unit sphere admitting a conformal deformation of the standard metric to a complete metric with a constant scalar curvature? Schoen and Yau were led to this problem by their research on geometry and topology of locally conformally flat manifolds and were mainly interested in the case of non-negative curvature. The case of negative scalar curvature goes back to an early paper by Loewner and Nirenberg [31]. Further motivations for the problem can be found in #36 from [57], [48].

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The goal of this paper is to introduce methods of nonlinear potential theory to this problem. They allow, in particular, to solve the problem in the negative curvature case. Our Theorem 1.1 states that the conformal deformation to a complete scalar-negative metric is possible in \( \Omega \) if and only if its complement is not thin. Thinness, see sec.1.2, is a basic concept in potential theory first introduced by Wiener in his works on the classical Dirichlet problem. Developments in nonlinear potential theory easily allow to relate thinness with geometric properties. Let us now describe the previous work and our results on the problem in more details.

Under the conformal change of metric \( g = u^{4/(n-2)} \hat{g} \), \( n \geq 3 \), the scalar curvature changes according to the formula

\[
R(g) = u^{-(n+2)/(n-2)} \left( -\frac{4(n-1)}{n-2} \Delta u + R(\hat{g})u \right).
\]

Here \( \Delta u = \text{div} (\text{grad} u) \) is the Laplace-Beltrami operator on \((S^n, \hat{g})\) and \( R(\hat{g}) = n(n-1) \) is the scalar curvature of the standard metric \( \hat{g} \) induced by the embedding \( S^n \hookrightarrow R^{n+1} \). Thus analytically for given \( \Omega \subset S^n \) and \( R \in \{-1, 0, 1\} \) one seeks a smooth solution to the following problem:

\[
\frac{4(n-1)}{n-2} \Delta u - R(\hat{g})u + Ru^{(n+2)/(n-2)} = 0 \quad \text{in} \quad \Omega \\
u > 0 \quad \text{in} \quad \Omega \\
u^{4/(n-2)} \hat{g} \text{ is complete metric} \quad \text{in} \quad \Omega.
\]

The case \( R = 1 \) is regarded as the hardest among the three. The present work focuses on the negative curvature case

\[
(1.3) \quad R = -1.
\]

Survey [42] by McOwen describes the progress on the problem and open problems. Let us explain how our results fit in the general picture. We set \( K = S^n \setminus \Omega \).

Investigations of the negative curvature case were started in 1974 by Loewner and Nirenberg [31]. They proved that if the problem (1.2), (1.3) admits a solution then the complement of \( \Omega \) must satisfy

\[
\mathcal{H}^{(n-2)/2}(K) = \infty.
\]

Here \( \mathcal{H}^\alpha \) denotes the Hausdorff \( \alpha \)-measure. Their work together with Aviles [4] and Veron [54] showed that if \( K \) is a smooth submanifold of \( S^n \) of the corresponding dimension \( k > (n-2)/2 \) then problem (1.2), (1.3) has a solution. Mazzeo [35] showed, in particular, that for such \( K \) the solution is unique. Finn [15], [16], [17] established the solvability under weaker conditions on \( K \). Namely he required that it has a structure similar to (actually, more general than) Lipschitz submanifold of the corresponding dimension, see also [18]. The gap between such requirements and the sufficient condition of Loewner and Nirenberg still remained broad. In section 1.3 we show how all these results follow from Theorem 1.1.

This paper studies (1.2), (1.3) in dimensions \( n \geq 3 \). In the case of \( S^2 \) the complete metric conformal to \( \hat{g} \) and having the constant negative curvature is called the Poincare metric. The equation for Poincare metric is slightly different from (1.2). Mazzeo and Taylor [41] proved that the Poincare metric in \( \Omega \subset S^2 \) always exists provided the complement \( K \) has at least two distinct points.

In 1988 Schoen and Yau [50] were led by their research on locally conformally flat manifolds to the case \( R \geq 0 \). They found that a necessary condition for solvability of (1.2) with \( R \geq 0 \), as opposed to the case (1.3), is smallness of \( K \). For example, they proved that if the solution exists then the Newtonian capacity of \( K \) must vanish. They also established that the solvability in this case implies that

\[
\mathcal{H}^{-+(n-2)/2}(K) = 0 \quad \text{for all} \quad \varepsilon > 0.
\]
Thus the Hausdorff dimension \((n - 2)/2\) separates the cases of the negative and non-negative curvature. Similarly to the negative curvature case, the existence of a solution is known at the moment only in cases when \(K\) has much more structure that vanishing Hausdorff measure or capacity. Despite the similarity in statements, the results in the case \(R = 1\) are much more difficult to prove. In a seminal paper \([49]\) Schoen established the existence of \((1.2)\) with \(R = 1\) when \(K\) is a finite number (at least two) of points. Mazzeo and Pacard \([36]\) generalising earlier results \([40], [46], [37]\) extended Schoen’s result to the case when \(K\) is a finite number of disjoint smooth submanifolds of the dimension \(k \leq (n - 2)/2\). There is also a construction of a solution using Kleinian groups in the case when \(K\) is a certain Cantor-type set \([50]\).

The case \(R = 0\) is easier because equation \((1.2)\) becomes linear. In this case it is known that the solution exists provided that \(K\) is essentially a finite union of Lipschitz submanifolds of dimension \(k \leq (n - 2)/2\) \([8], [32], [22]\).

In the paper we are interested only in the basic problem of existence for \((1.2)\). However, other questions about solutions of \((1.2)\) can be asked as well. For example problems of uniqueness, asymptotic behaviour of \(u\) near \(\partial\Omega\), structure of moduli space of solutions, gluing different solutions, are investigated in \([24], [38], [39], [42]\). Some of the results mentioned above hold for more general manifolds than \(S^n\). The result directly related to the present paper was proved by Aviles and McOwen \([5], [6], [7]\). They established that an open subset \(\Omega\) of any closed Riemannian manifold \((M, g)\) admits a complete metric with the constant negative scalar curvature conformal to \(g\) provided \(K\) is a finite union of closed smooth submanifolds of dimensions \(k > (n - 2)/2\). In a future publication we introduce a suitable capacity and extend our Theorem 1.1 to more general manifolds.

1.2 Main theorem

We investigate the solvability of \((1.2)\) by attracting ideas from the nonlinear potential theory. Let us recall a fundamental result form the classical potential theory for the Laplace equation. This is the Wiener test for the classical Dirichlet problem for harmonic functions \([55]\). Wiener theorem states that the Dirichlet problem

\[
\begin{align*}
\Delta w &= 0 \quad \text{in } D \\
 w &= f \quad \text{on } \partial D
\end{align*}
\]

in a bounded domain \(D \subset \mathbb{R}^n\), \(n \geq 3\), is solvable for all boundary data \(f \in C(\partial D)\) if and only if \(\mathbb{R}^n \setminus D\) is not thin. Explicitly the latter means that

\[
\int_0^1 \frac{\text{cap}(B(x, r) \setminus D)}{\text{cap}(B(x, r))} \frac{dr}{r} = +\infty \quad \text{for any } x \in \partial D.
\]

Here 1 can be replaced by any \(\delta > 0\), and \(\text{cap}\) is the classical (electrostatic) capacity. Our main Theorem 1.1 states that problem \((1.2), (1.3)\) admits a solution if and only if a Wiener-type test with a certain capacity holds.

Let us sketch the definition of the capacity appropriate for problem \((1.2)\), see section 2.2 for more details. Take a compact set \(E \subset S^n\), \(n \geq 3\), with

\[
\text{diam}_{\hat{g}}(E) \leq \pi/3.
\]

After a rotation we can assume that such \(E\) lies in the southern hemisphere. We set

\[
\mathcal{C}(E) = \inf \left\{ \int_{S^n} |\nabla^2 \varphi|^{(n+2)/4} d\text{vol}_{\hat{g}} \right\},
\]

\[
\frac{1}{(n+2)/(n-2)} + \frac{1}{(n+2)/4} = 1.
\]
Here symbols $\text{dvol}_{\tilde{g}}$, $\nabla$, and $|\cdot|$, stand respectively for the volume element, connection, and norm with respect to the metric $\tilde{g}$. The infimum is taken over all $\varphi \in C^\infty(S^n)$ such that $\varphi|_E \geq 1$ and $\varphi \equiv 0$ on the northern hemisphere. Essentially, $C$ is the Bessel capacity for the Sobolev space $W^{2,(n+2)/4}(\mathbb{R}^n)$. Bessel capacities have been intensively investigated in the nonlinear potential theory. Nonlinear potential theory originates in early works of Maz’ya and Serrin in the 1960s and was extensively developed later in 1970s and 1980s by many authors. Our paper heavily relies on it. The main references will be monographs by Adams and Hedberg [1], Maz’ya [34], and Ziemer [58]. There the reader can also find a rich bibliography and historical notes. Now we state the main theorem.

**Theorem 1.1** Let $\Omega \subset S^n$, $n \geq 3$, be an open set and $K = S^n \setminus \Omega$. Then the following properties are equivalent:

(i) In $\Omega$ there exists a complete metric with constant negative scalar curvature conformal to $\tilde{g}$.

(ii) The compactum $K$ is not thin, that is for any $p \in K$

$$
\int_0^{1/2} \left( \frac{C(B(p,r) \cap K)}{C(B(p,r))} \right)^{2/(n-2)} \frac{dr}{r} = +\infty.
$$

_iWiener test (1.4)_ is a capacitary condition on $K$. Geometric properties of the capacity $C$ are well understood due to investigations in nonlinear potential theory. Using the information available there, we show in section 1.3 that more transparent geometric results easily follow from Theorem 1.1.

In view of Theorem 1.1 it would be interesting to clarify how the condition

$$C(S^n \setminus \Omega) = 0$$

relates to the conformal deformation to nonnegative scalar curvature $R \geq 0$. In [27] we apply potential theory ideas to the scalar flat case $R = 0$. In this situation as opposed to Theorem 1.1 the set $K$ should be small.

We mention that ideas from potential theory have been used in conformal geometry before. For example Schoen and Yau [50], [51] used capacity related to Sobolev space $W^{1,q}(\mathbb{R}^n)$. Capacity $C$ was implicitly used at some stage in [24] to prove distribution removability of isolated singularities for the equation in (1.2) with $R = 1$.

**Remark 1.2** Intuitively, the completeness condition forces solutions of (1.2) to blow up in some sense near $\partial \Omega$. Dhersin and LeGall [9] considered the problem

$$
\begin{align*}
\Delta u - u^2 &= 0 \quad \text{in } D \\
\quad &\quad \text{when } x \to \partial D
\end{align*}
$$

in general domains $D \subset \mathbb{R}^n$. They proved that a Wiener test characterises domains $D$ for which (1.3) is solvable. Their paper provided an important inspiration for our work, although the consideration in [27] is based on probabilistic methods. In fact, there is a strong connection between $u$ in (1.3) and a certain branching random process (so-called Brownian snake) [27], [10]. To find an adequate probabilistic interpretation for $p > 2$ is an important open problem in the area [30], [24], [10]. However, in [26] we established a solvability criterion for

$$\Delta u - u^p = 0$$

blowing up at the boundary in the full range of $p > 1$. Relying there on entirely analytic ideas we establish estimates for solutions in terms of the capacity associated with the variational integral

$$
\int_{\mathbb{R}^n} |D^2 \varphi|^{p'}, \quad \varphi \in C_0^\infty(\mathbb{R}^n), \quad \frac{1}{p} + \frac{1}{p'} = 1.
$$
Comparison of Theorem 1.1 with the condition from [26] implies that any conformal factor in (1.2), (1.3) must blow up pointwisely near $K$. The interplay between Brownian motion on a Riemannian manifold and geometric properties of the manifold is an important subject. We refer to Grigor’yan’s survey [20] for an exposition of the classical and new results. It would be interesting to understand the relation between super-Brownian motion, Brownian snakes and the geometric problems from this paper (or properties of Riemannian manifolds in general). Some results in this direction can be found in [11].

1.3 Examples

We illustrate how Theorem 1.1 allows to establish the existence for (1.2), (1.3) in concrete situations. In particular, apparently all necessary or sufficient conditions from previous papers can be easily derived from (1.4). The reason for this is that capacity $C$ had appeared before in different problems related to the interaction between nonlinear potentials and the Littlewood-Paley theory. As a result, it was intensively studied in the 1970s-1980s and its geometric properties are well known. They are carefully documented e.g. in [1], [34], [58].

Example 1.3 The necessity of Loewner-Nirenberg condition [31]

$\mathcal{H}^{(n-2)/2}(K) = +\infty$

for solvability of (1.2), (1.3) follows immediately from Theorem 1.1 and the implication

$\mathcal{H}^{(n-2)/2}(K) < +\infty \implies C(K) = 0,$

valid for our capacity $C$, see [1] and (2.15) in section 2.2 below.

Example 1.4 Assume that $K$ is a smooth immersed submanifold in $S^n$ of dimension $k$, (1.6)

\[ k > (n-2)/2. \]

Then (1.2), (1.3) has a solution. In fact, according to Theorem 1.1 we need to show that (1.4) holds. Fix any $p \in K$. By the definition of immersion there exists an open smooth submanifold $E$ of dimension $k$ embedded in $S^n$ such that $p \in E$ and $E \subset K$. Therefore

$C(K \cap B(p, r)) \geq C(E \cap B(p, r)).$

The exponential map $\exp_p$ is a diffeomorphism of a neighbourhood of the origin 0 in $T_pS^n$. In the sufficiently small neighbourhood of $p$ the smooth submanifold $E$ is well approximated by the image under $\exp_p$ of a neighbourhood of 0 in $T_pE$, $T_pE \subset T_pS^n$. Capacities of a set and its image under a diffeomorphism are equivalent [1]. Hence utilising the scaling property (2.13) we find a small number $r_0 > 0$, such that

$C(E \cap B(p, r)) \geq C(E \cap B(p, r_0)) \left( \frac{r}{r_0} \right)^{(n-2)/2} C(E \cap B(p, r_0))$ for $r \in (0, r_0).

Here the constant $C(E) > 0$ depends on the smoothness of $E$. Capacity of the ball can be estimated by $C(B(p, r)) \asymp r^{(n-2)/2}$ for $r \in (0, r_0)$, see (2.10). Now (1.4) follows.
Example 1.5 Let \( d = \varepsilon + (n-2)/2 \), \( \varepsilon > 0 \). Assume that for any \( x \in K \) there is a positive constant \( C \) such that
\[
\mathcal{H}^d_\infty(K \cap B(x,r)) \geq C r^d
\]
for all \( r \) near 0. That is, the Hausdorff \( d \)-content of \( K \) is big at all small scales. Then the conformal metric from Theorem 1.4 exists. This follows at once from (2.14), (2.16). Density condition (1.7) allows to recover the results of Finn [15], [16], [17] about sets \( K \) with stratified cone-type tangent structure. Also (1.7) allows to establish existence of the complete metrics in the cases when \( K \) satisfies different geometric conditions invariant with respect to quasiconformal maps, see e.g. [25].

Example 1.6 Let \( K \) be the Lebesgue cusp. That is for a fixed \( \rho > 0 \) and for a continuous positive nondecreasing function \( h \) on the real line, \( h(r) = O(r) \), \( r \to 0 \), we set
\[
T_h = \{ x \in \mathbb{R}^n : 0 \leq x_n \leq \rho, \ (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leq h(x_n) \}.
\]
Then define \( K \) to be the preimage of \( T_h \) under the stereographic projection,
\[
K = \sigma^{-1}(T_h).
\]
The existence of the singular conformal metric \( g \) from Theorem 1.4 in \( \mathbb{S}^n \setminus K \) depends on the dimension \( n \). If \( n = 3 \) then \( g \) always exists. For higher dimensions \( g \) exists if and only if
\[
\int_0^1 \frac{dr}{r \log (r/h(r))} = +\infty \quad \text{for} \quad n = 4
\]
\[
\int_0^1 \left( \frac{h(r)}{r} \right)^{(n-4)/(n-2)} \frac{dr}{r} = +\infty \quad \text{for} \quad n > 4.
\]
Indeed, we only need to check that (1.2) holds for the South pole \( S = \sigma^{-1}(0) \). To verify (1.4), first recall that for the cylinder
\[
\Pi = \sigma^{-1}(\{(-\delta,\delta) \times \cdots \times (-\delta,\delta) \times (r/2,r)\}), \quad 4\delta < r,
\]
with \( r > 0 \) small enough, the capacity is given by the following formulae [37], Ch. 9:
\[
\mathcal{C}(\Pi) \asymp \mathcal{C}(B(S,r)) \quad \text{for} \quad n = 3,
\]
\[
\mathcal{C}(\Pi) \asymp \frac{1}{\log (r/\delta)} \mathcal{C}(B(S,r)) \quad \text{for} \quad n = 4,
\]
\[
\mathcal{C}(\Pi) \asymp \left( \frac{\delta}{r} \right)^{(n-4)/2} \mathcal{C}(B(S,r)) \quad \text{for} \quad n > 4.
\]
Now just apply elementary estimate (2.28).

1.4 Organisation of the paper

In section 2 we introduce the capacity and use it to prove some preliminary estimates for solutions of the equation. We also describe there the unique feature of equation (1.2), (1.3). Namely, the existence of a \( u_\Omega \) dominating all other solutions pointwisely.

In section 3 we prove the crucial estimates for solutions of (1.2), (1.3) in terms of the capacity. The principal difficulty in the proof of the main Theorem 1.1 is the analysis of the completeness condition in (1.2). This involves understanding the behaviour of the conformal factor \( u \) near \( \partial \Omega \) under no assumptions (say, when proving (i) \( \Rightarrow \) (ii) in Theorem 1.1 on the structure of \( \Omega \). The first estimate
in section 3, Theorem 3.1 controls the solution pointwisely away from \( \partial \Omega \). The second estimate, Theorem 3.2, provides the integral control when we stay arbitrarily close to \( \partial \Omega \).

With all this background in place we proceed to prove our main result, Theorem 1.1, in section 4. The sufficiency of the Wiener test (1.4) will follow rather straightforwardly from the pointwise estimate from section 3. To prove the necessity we suppose that the negation of (1.4) holds. In other words, suppose that the complement of \( \Omega \) is thin at some point. We will find a curve in \( \Omega \) approaching this point, such that its length with respect to \( u_\Omega \) (and hence with respect to any other solution of (1.2), (1.3)) is finite. How to construct such a curve without any assumptions on \( \partial \Omega \)? The key idea here is to reduce this issue to an integral estimate. To achieve this we bring in the estimate from Theorem 3.2.

Throughout this paper, we will use the notation

\[
q = \frac{n + 2}{n - 2}, \quad q' = \frac{n + 2}{4}, \quad \frac{1}{q} + \frac{1}{q'} = 1.
\]

By \( g_E \) we denote the Euclidean metric in \( \mathbb{R}^n \) and by \( \hat{g} \) the standard metric on the sphere induced by \( g_E \). By \( B(p,r) \) we denote the ball of radius \( r \) centered at \( p \) for \( \hat{g} \) or \( g_E \). It will be clear from the context which metric is taken. For an integer \( j \) we put \( r_j = 2^{-j} \). By \( B_j \) we denote the dyadic ball in \( \mathbb{R}^n \), \( B_j = B(0, r_j) \). We denote the Green’s function for the Laplacian in \( B(0, R) \subset \mathbb{R}^n \) by \( G_R \).

By \( C, \tilde{C}, C_1, \ldots \), we denote positive constants depending only on the dimension. The value of \( C, \tilde{C}, C_1, \ldots \), may vary even within the same line. We write

\[ A \lesssim B \quad (A \gtrsim B) \]

if

\[ A \leq CB \quad (A \geq CB) \]

for some \( C \). We write

\[ A \asymp B \]

if

\[ A \lesssim B \lesssim A. \]

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2 Preliminary on the equation and capacity

2.1 Equation on sphere and in space

Let \( g \) be a metric on a manifold \( M \) of dimension \( n \), \( n \geq 3 \). The operator

\[
\mathcal{L}_g = -4 \frac{n-1}{n-2} \Delta_g + R(g)
\]

from (1.1) is called conformal Laplacian [51]. If we change the metric conformally

\[ \hat{g} = \varphi^{4/(n-2)} g, \]
then

\[ R(\hat{g}) = \phi^{-(n+2)/(n-2)} \mathcal{L}_g \phi, \]
\[ \mathcal{L}_{\hat{g}} v = \phi^{-(n+2)/(n-2)} \mathcal{L}_g (\phi v). \]

More generally, let \( \tilde{M} \) be another manifold with the metric \( \tilde{g} \), and let \( f: M \to \tilde{M} \) be a diffeomorphism. Assume that \( f \) changes the metric conformally

\[ f^* \tilde{g} = \phi^{4/(n-2)} g. \]

Then

\[ f^* (R(\tilde{g})) = \phi^{-(n+2)/(n-2)} \mathcal{L}_{\tilde{g}} \phi, \]
\[ f^* (\mathcal{L}_{\tilde{g}} v) = \phi^{-(n+2)/(n-2)} \mathcal{L}_g (\phi f^* v). \]

Now, the stereographic projection \( \sigma: S^n \setminus \{N\} \to \mathbb{R}^n \) is a conformal diffeomorphism between \( (S^n \setminus \{N\}, \hat{g}) \) and \( (\mathbb{R}^n, g_E) \) because

\[ (\sigma^{-1})^* \hat{g} = \left( \frac{2}{1+|x|^2} \right)^2 g_E = \Upsilon^{4/(n-2)} g_E \]

with

\[ \Upsilon(x) = \left( \frac{2}{1+|x|^2} \right)^{(n-2)/2} x \in \mathbb{R}^n. \]

According to the above formulae for the conformal changes we have the following correspondence.

Let \( \Omega \in S^n \), \( N \subset \Omega \), and let the function \( v \) satisfy

\[ v^{-(n+2)/(n-2)} \mathcal{L}_{\hat{g}} v = -1, \quad v > 0 \quad \text{in} \quad \Omega. \]

Then the function

\[ u(x) = \Upsilon(x) (\sigma^{-1})^* v(x) \]
\[ = \Upsilon(x)v(\sigma^{-1}x), \quad x \in \mathbb{R}^n, \]

satisfies

\[ u^{-(n+2)/(n-2)} \mathcal{L}_{g_E} u = -1, \quad u > 0 \quad \text{in} \quad \sigma(\Omega). \]

Thus after multiplication by a constant, \( u \) satisfies

\[ u > 0, \quad \Delta u - u^{(n+2)/(n-2)} = 0. \]

Conversely, for any solution \( u \) of \( 2.3 \) defined in \( \sigma(\Omega) \) set

\[ v = \sigma^* (u/\Upsilon). \]

Then after multiplication by a constant, \( v \) satisfies \( 2.1 \) in \( \Omega \setminus \{N\} \). Moreover we know \( 3.1 \) that for any \( u \) solving \( 2.3 \) in a neighbourhood of infinity in \( \mathbb{R}^n \), there exists a constant \( A > 0 \) such that

\[ u(x) = \frac{A}{|x|^{n-2}} + o \left( \frac{1}{|x|^{n-2}} \right), \quad x \to \infty. \]

Hence we can extend \( v \) to \( N \) by continuity, remove the isolated singularity, and conclude that \( 2.1 \) holds.
Clearly the metric $v^{4/(n-2)}g$ is complete in $\Omega$ for $v$ from (2.1) if and only if $u^{4/(n-2)}g_E$ is complete in $\sigma(\Omega) \cup \{\infty\}$ for the corresponding $u$ from (2.2), (2.3).

The previous discussion shows that the existence of the singular Yamabe metric in a domain on the sphere is equivalent to finding a complete solution of (2.3) in the exterior domain in $\mathbb{R}^n$. Let us describe the main features of equation (2.3) in $\mathbb{R}^n$. Omitted proofs can be found for example in [31].

The crucial fact about solutions of (2.3) that will be used constantly in this paper is the elliptic comparison principle. As a consequence of this principle, local regularity estimates hold for $u$. In particular, if $u \in L^q_{\text{loc}}$ is a distributional solution of (2.3) then, in fact, $u \in C^\infty_{\text{loc}}$ and $u$ is the classical solution. Moreover, let $u$ be any solution of (2.3) in an open set $O \subset \mathbb{R}^n$. Then

$$u(x) \lesssim \frac{1}{\text{dist}(x, \partial O)^{(n-2)/2}} \quad \text{for all} \quad x \in O.$$  

This is an estimate uniform in $u$. It was first discovered by Keller [23] and Osserman [15], and also follows from the comparison principle.

Estimate (2.4) combined with the elliptic Perron argument implies the existence of the finite solution $u_O$ which is maximal in $O$. It means that the inequality

$$u \leq u_O \quad \text{in} \quad O$$

holds for any other $u$ solving (2.3) in $O$. Clearly

$$u_{O_1} \leq u_{O_2} \quad \text{in} \quad O_2 \quad \text{when} \quad O_1 \supset O_2.$$  

Let $K_1, \ldots, K_m$ be compact sets in $\mathbb{R}^n$, let

$$K = K_1 \cup \cdots \cup K_m,$$

and let $u, u_1, \ldots, u_m$ be the maximal solutions of (2.3) in $K^c$, $K_1^c$, $\ldots$, $K_m^c$ respectively. Then the H"older inequality and the comparison ensure that

$$m^{-(n-2)/(n+2)} \sum_{i=1}^m u_i \leq u \leq \sum_{i=1}^m u_i \quad \text{in} \quad K^c.$$  

If $x_0 \in \partial O$ and $(\partial O) \cap B(x_0, r)$ is a smooth hypersurface for some $r > 0$, then

$$u_O(x) \text{dist}(x, \partial O)^{(n-2)/2} \to \left(\frac{n(n-2)}{4}\right)^{(n-2)/4},$$

when $x \to (\partial O) \cap B(x_0, r)$.

Asymptotic behaviour (2.6) holds in fact for any solution of (2.3) blowing up at $(\partial O) \cap B(x_0, r)$.

Solutions to (2.3) exhibit the following dilation invariance: for all $a > 0$ and $r > 0$,

$$u \text{ solves (2.3) in } B(0, r) \quad \iff \quad a^{(n-2)/2}u(a \cdot) \text{ solves (2.3)} \quad \text{in } B(0, r/a).$$  

Finally consider equation (2.1) on the sphere. As a direct consequence of the properties of the stereographic projection which we discussed above, there exists the maximal solution of (2.1) and the estimates analogous to (2.4)–(2.6) hold.
2.2 Capacity

In this paragraph we define the capacity $C$ for subsets of the unit sphere $S^n$. Essentially it is a particular Bessel capacity $C$ in $\mathbb{R}^n$. Omitted proofs of the statements about $C$ can be found in monographs [1], [34], and [58]. By $S^n_S$ and $S^n_N$ we denote southern and northern hemispheres.

Fix the spherical cup around the south pole $S$ by writing
\begin{equation}
U = \left\{ x \in S^n : d_{\hat{g}}(S, x) \leq \pi/3 \right\}, \quad U \subset S^n_S.
\end{equation}

Take any compact set $K \subset S^n$ with $\text{diam}_{\hat{g}}(K) \leq \pi/3$.

The rotation group $SO(n+1)$ acts transitively on $S^n \hookrightarrow \mathbb{R}^{n+1}$. Map $K$ by a rotation $\Phi \in SO(n+1)$ in a way that $\Phi(K) \subset U$. Define
\begin{equation}
C(K) = \inf \left\{ \int_{S^n} |\nabla^2 \varphi|^{(n+2)/4} \, dvol_{\hat{g}} : \varphi \in C_\infty(S^n), \varphi|_{S^n_N} = 0, \varphi|_{\Phi(K)} \geq 1 \right\}.
\end{equation}

We will prove that different choices of $\Phi \in SO(n+1)$ lead to equivalent capacities. First we give an alternative description of the capacity. Stereographic projection $\sigma$ is a smooth quasiisometry between $S^n_S$ and $B(0,1) \subset \mathbb{R}^n$. Hence
\begin{equation}
C(K) \asymp \inf \left\{ \int_{\mathbb{R}^n} |D^2 \psi|^{(n+2)/4} \, dx : \psi \in C_\infty(\mathbb{R}^n), \psi|_{\sigma(K)} \geq 1 \right\}.
\end{equation}

Let us introduce the corresponding capacity for sets in $\mathbb{R}^n$. For a compact set $E \subset B((0,1)) \subset \mathbb{R}^n$ its Bessel capacity is defined as
\begin{equation}
C(E) = \inf \left\{ \int_{\mathbb{R}^n} |D^2 \psi|^{(n+2)/4} \, dx : \psi \in C_\infty(\mathbb{R}^n), \psi|_{E} \geq 1 \right\}.
\end{equation}

Notice that the set $\sigma \circ \Phi(K)$ stays away from the boundary of the unit ball: $\sigma \circ \Phi(K) \subset B(0,99/100)$.

Hence properties of Bessel capacities imply that for $E = \sigma \circ \Phi(K)$ the right hand sides of (2.11) and (2.10) are equivalent. Thus
\begin{equation}
C(K) \asymp C(\sigma \circ \Phi(K)).
\end{equation}

Now take another $\tilde{\Phi} \in SO(n+1)$, $\tilde{\Phi}(K) \subset U$. The same variational procedure as (2.9) gives the new capacity $\tilde{C}(K)$. Bessel capacity (2.11) of a compactum and of its image under a bi-Lipschitz homeomorphism are equivalent. Apply this to the locally bi-Lipschitz map
\[ \sigma \circ \hat{\Phi} \circ \Phi^{-1} \circ \sigma^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n \]
which sends $\sigma \circ \Phi(K)$ to $\sigma \circ \tilde{\Phi}(K)$, and utilise (2.12) to derive that
\[ C(K) \asymp \tilde{C}(K). \]
Clearly property (1.4) of the set to be not thin does not change when we pass to an equivalent capacity. Set functions $C$ and $\tilde{C}$ enjoy subadditivity and monotonicity properties. Standard scheme of axiomatic potential theory extends them to arbitrary sets as the outer measure.

Now we list some well-known metric estimates for the capacity. The following important scaling holds:

\begin{equation}
C(tE) \asymp t^{(n-2)/2}C(E), \quad t \in (0,1), \quad E \subset B(0,1).
\end{equation}

Next, for $\alpha > 0$ the Hausdorff $\alpha$-content of $E \subset \mathbb{R}^n$ (or $E \subset S^n$) is defined as

$$H_\alpha^\infty(E) = \inf \sum_j r_j^\alpha,$$

where the infimum is taken over all coverings of $E$ by countable unions of euclidean balls $\{B(x_j, r_j)\}$ in $\mathbb{R}^n$ (or $d_g$-balls in $S^n$). The set function $H_\alpha^\infty$ is subadditive and monotone. For the Hausdorff measure we have

$$H^\alpha(E) = 0 \iff H_\alpha^\infty(E) = 0.$$

There is a strong connection between the capacity and the Hausdorff content and measure. For any $\alpha > (n-2)/2$ there is a constant $C(n,\alpha) > 0$ such that

\begin{equation}
(H_\alpha^\infty(E))^{(n-2)/2} \leq C(n,\alpha)C(E)^\alpha, \quad E \subset B(0,1).
\end{equation}

Hence sets of the capacity 0 have the Hausdorff dimension at most $(n-2)/2$. In the converse direction the following implication holds for the Hausdorff measure:

\begin{equation}
H^{(n-2)/2}(E) < +\infty \implies C(E) = 0
\end{equation}

for $E \subset B(0,1)$. According to (2.12) statements (2.14) and (2.15) also hold for $\tilde{C}$. From (2.12), (2.13), and (2.14) we also derive that

\begin{equation}
C(B(p,r)) \asymp r^{(n-2)/2}, \quad p \in S^n, \quad 0 \leq r \leq \pi/6.
\end{equation}

### 2.3 An estimate

In this paragraph we provide an integral estimate for any solution $u$ of (2.3) outside a compact set $K$. It will be frequently used in the sequel. More precisely, the following lemma produces a cut-off function $\eta$ which vanishes in a neighbourhood of $K$, equals 1 away from $K$, and bounds the rate of a possible blow-up of $u$ via estimates (2.18) and (2.19).

**Lemma 2.1** Let $K \subset B(0,1)$ be a compact set in $\mathbb{R}^n$, $n \geq 3$, and

$$m \geq \frac{n+2}{2}.$$

Let $u$ solve (2.3) in $K^c$. Then there exists a function $\varphi \in C_0^\infty(B(0,2))$ such that $0 \leq \varphi \leq 1$ in $B(0,2)$, $\varphi = 1$ in an open neighbourhood of $K$, and

\begin{equation}
\int_{B(0,2)} |D^2 \varphi|^{(n+2)/4} \lesssim C(K),
\end{equation}

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and such that for \( \eta = (1 - \varphi)^m \) the inequalities

\[
\int_{\mathbb{R}^n} u(|D\eta| + |\Delta\eta|) \leq C(m, n)C(K),
\]

(2.18)

\[
\int_{\mathbb{R}^n} u^{(n+2)/(n-2)}\eta \leq C(m, n)C(K)
\]

(2.19)

hold.

**Proof.** 1. The open set \( K^c \) can be approximated from the interior by domains with smooth boundaries. Consequently, by standard continuity properties of capacity, we can assume in the proof that \( K \) is a disjoint union of a finite number of closed domains with smooth boundaries. We set \( \hat{B} = B(0, 2) \).

We claim that there exists a function \( \varphi \in C^\infty_0(\hat{B}) \) with \( 0 \leq \varphi \leq 1 \) in \( \hat{B} \) and \( \varphi = 1 \) in an open neighbourhood of \( K \) such that (2.17) holds. To prove this, we first recall a well-known result in nonlinear potential theory \[1\] Chapter 2, \[34\] Chapter 9, that states that there exists a function \( \tilde{\varphi} \in C^\infty_0(\hat{B}) \) such that

\[\tilde{\varphi}|_K \geq 1, \quad \int_{\hat{B}} |D^2\tilde{\varphi}|^q = C(K), \quad \|\tilde{\varphi}\|_{L^\infty(\hat{B})} \lesssim 1.\]

Next, take a function \( H \in C^\infty(\mathbb{R}^1) \) such that \( H(t) = 0 \) for \( t < 1/3 \), \( H(t) = 1 \) for \( t > 1/2 \).

Now we take \( \varphi \) to be the smooth truncation of \( \tilde{\varphi} \), \( \varphi = H(\tilde{\varphi}) \). Then

\[
\int_{\hat{B}} |D^2\varphi|^q \lesssim \int_{\hat{B}} |H''(\varphi)'|D\varphi|^q + \int_{\hat{B}} |H'(\varphi)'||D^2\varphi|^q.
\]

To obtain (2.17), we just apply the Gagliardo-Nirenberg interpolation inequality \[34\], Chapter 9, to the first term: if \( 1 < r < \infty \), then for any \( f \in C^\infty(\hat{B}) \)

\[
\|Df\|_{L^{2r}(\hat{B})} \lesssim \|D^2f\|_{L^r(\hat{B})}^{1/2} \|f\|_{L^{r}(\hat{B})}^{1/2}.
\]

(2.20)

We remark that arguments of this type are well known, cf. \[34\] Chapter 9, \[1\] Chapter 3.

2. Let \( u \) be a solution of (2.8). Take any \( \varepsilon > 0 \). Appealing to decay (2.4), we choose \( R = R(\varepsilon) \), \( R > 4 \), such that

\[
u \leq \varepsilon \quad \text{on} \quad \partial B(0, R).
\]

Set \( B = B(0, R) \), \( \hat{B} \subset B \). Let \( v \) solve the problem

\[
\begin{cases}
\Delta v - v^q = 0 & \text{in} \quad B \setminus K \\
v(x) \to +\infty & \text{when} \quad x \to K \\
v = 0 & \text{on} \quad \partial B.
\end{cases}
\]

Then

\[
\Delta (v + \varepsilon) - (v + \varepsilon)^q \leq 0 \quad \text{in} \quad B \setminus K.
\]

Hence by asymptotic condition (2.6) and the comparison principle

(2.21)

\[
u \leq v + \varepsilon \quad \text{in} \quad B \setminus K.
\]

In what follows we first prove (2.18) (2.19) for \( v \) and then let \( \varepsilon \) vanish.
3. Let $\psi = 1 - \varphi$. We claim that

\[
(2.22) \quad \int_B v^q \psi^m \leq C(m, n) C(K) \quad \text{for } m \geq 2q'.
\]

In fact, by Green’s formula

\[
\int_B v^q \psi^m = \int_B (\Delta v) \psi^m = \int_B v \Delta (\psi^m) + \int_{\partial B} (\psi^m \frac{\partial v}{\partial \nu} - v \frac{\partial \psi^m}{\partial \nu}),
\]

where $\nu$ is the outer normal on $\partial B$. Since $\psi|_{|x|\geq 2} = 1$ we conclude that

\[
\frac{\partial \psi^m}{\partial \nu} = 0 \quad \text{on } \partial B.
\]

By the comparison principle, $v|_{B\setminus K} > 0$. Hence

\[
\int_{\partial B} \psi^m \frac{\partial v}{\partial \nu} \leq 0.
\]

Using the Hölder inequality, we compute:

\[
(2.23) \quad \int_B v^q \psi^m \leq \int_B v \Delta (\psi^m) \leq \int_B v |\Delta (\psi^m)| \leq m \int_B v^q |\Delta \psi| + m(m - 1) \int_B (v^q |D\psi|^2)
\]

\[
\leq m \left( \int_B v^q \psi^m \right)^{1/q} \left( \int_{\hat{B}} |\Delta \psi|^{q'} \right)^{1/q'}
\]

\[
\leq m \left( \int_B v^q \psi^m \right)^{1/q} \left( \int_{\hat{B}} D\psi|^{2q'} \right)^{1/q'}
\]

where

\[
X = m - q', \quad Y = m - 2q'.
\]

We can assume that the left-hand side in (2.23) is positive. From (2.23) it then follows that

\[
\int_B v^q \psi^m \leq m^{2q'} \int_{\hat{B}} (|\Delta \varphi|^{q'} + |D\varphi|^{2q'}).
\]

Applying inequality (2.24), we obtain

\[
\int_B v^q \psi^m \leq C(m, n) \int_{\hat{B}} |D^2 \varphi|^{q'}
\]

and (2.22) follows from (2.17).

4. We claim that

\[
(2.25) \quad \int_B v(|\Delta \eta| + |D \eta|) \leq C(m, n) C(K).
\]
In fact, we have by the same calculations as in (2.23):

\[
\int_A |v\Delta \eta| \leq m \left( \int_A |vq^{(m-1)q}| \right)^{1/q} \left( \int_A |\Delta \varphi'|^{q'} \right)^{1/q'} + m(m-1) \left( \int_A |vq^{(m-2)q}| \right)^{1/q} \cdot \left( \int_A |D\varphi'|^{2q'} \right)^{1/q'}.
\]

(2.26)

\[
\int_A |v|D\eta| \leq m \left( \int_A |vq^{(m-1)q}| \right)^{1/q} \left( \int_A |D\varphi'|^{q'} \right)^{1/q'}.
\]

(2.27)

For \( m \geq 2q' \) we have

\[
(m-2)q \geq 2q', \quad (m-1)q \geq 2q' + q.
\]

Thus we can use (2.22) to estimate the integrals containing \( v^q \) in (2.26) and (2.27). Applying interpolation inequality (2.20) to the last term in (2.26), we conclude on the basis of (2.17) that

\[
\int_A |v\Delta \eta| \leq C(m,n)C(K)^{1/q} \left( \int_A |D^2 \varphi'|^{q'} \right)^{1/q'} \leq C(m,n)C(K).
\]

Similarly, applying the Poincaré inequality to the last integral in (2.27) gives

\[
\int_A |v|D\eta| \leq C(m,n)C(K).
\]

We conclude that (2.25) indeed holds.

5. From (2.21) and (2.25) we obtain

\[
\int_{\mathbb{R}^n} u(|D\eta| + |\Delta \eta|) = \int_B u(|D\eta| + |\Delta \eta|) \leq C(m,n) \left[ C(K) + \varepsilon \int_B (|D\eta| + |\Delta \eta|) \right].
\]

To establish (2.18) we let \( \varepsilon \to 0 \) both in (2.21) and in the last inequality. A similar limit argument applied to (2.22) gives us (2.19). 

Finally, we record a useful elementary inequality, (see for example [1] or [3]). Let \( J \in \mathbb{Z} \), and let the function \( \zeta: (0, r_J) \to \mathbb{R}^1 \) be either nondecreasing or nonincreasing. Then for any \( \kappa \in \mathbb{R} \)

\[
\sum_{j=J+1}^{\infty} \zeta(r_j)r_j^{\kappa} \leq \int_{0}^{r_J} \zeta(r)r^{\kappa} \frac{dr}{r} \leq \sum_{j=J}^{\infty} \zeta(r_j)r_j^{\kappa}.
\]

(2.28)

3 Capacitary estimates

In this section we prove first estimates on \( u \) near \( \partial \Omega \). We will work in \( \mathbb{R}^n \) instead of \( S^n \). According to sections \( 2.1, 2.2 \) transition to the sphere is immediate.

Theorems from this section will play the following role in the proof of the main result. Let \( u \) solve

\[
(3.1) \quad u > 0, \quad \Delta u - u^q = 0
\]
outside a compact set $K \subset \mathbb{R}^n$. When estimating the length of a curve $\gamma$ in the metric $u^{4/(n-2)}g_E$ we will distinguish two regions. In the first region $\gamma$ is far enough from $K$. Then pointwise estimate (3.2) from Theorem 3.1 will be applied. In the second region $\gamma$ is arbitrarily close to $K$. Then we will use integral estimate (3.16) from Theorem 3.2.

**Theorem 3.1** Let $K \subset B(0,r)$ be a compact set in $\mathbb{R}^n$, $0 < r < 1$, $n \geq 3$, Let $u$ be the maximal solution of (3.1) in $K^c$. Then

$$u(x) \asymp \frac{C(K)}{|x|^{n-2}}, \quad |x| \geq 2r.$$ (3.2)

**Proof.** [of the upper estimate in (3.2)]

1. According to the scalings (2.7), (2.13) we need to prove that for a compact set $K$, $K \subset B(0,1)$, the following estimate holds:

$$u(x) \lesssim C(K) \text{ for all } x \text{ such that } 2 \leq |x| \leq 3.$$ (3.3)

Fix any such $x$. Let $\eta$ be the function for our set $K$ from Lemma 2.1 with some fixed $m$.

2. Utilising decay (2.4) we can choose $R > 0$ so big that we have

$$u(x) = (u\eta)(x) \lesssim \int_{B(0,R)} G_R(x,y)\Delta(u\eta)(y) \, dy.$$ (3.4)

Denote further $B = B(0,R)$, $G = G_R$. Equation (3.1) gives us

$$\Delta(u\eta) = (\Delta u)\eta + 2DuD\eta + u(\Delta \eta) \geq 2DuD\eta + u\Delta \eta.$$ Substituting this into (3.4) and integrating by parts to deduce that

$$u(x) \lesssim -2\int_B D_yG(x,y)D\eta(y)u(y) \, dy$$

$$-\int_B G(x,y)u(y)\Delta \eta(y) \, dy.$$ Next, the choice of $x$ and elementary bounds for $G$ give that

$$u(x) \lesssim \int_{\mathbb{R}^n} u(|D\eta| + |\Delta \eta|).$$ Now estimate (3.5) from Lemma 2.1 leads us to (3.3). ■

**Proof.** [of the lower estimate in (3.2)]

1. According to the scalings (2.7), (2.13) we need to prove that for a compact set $K$, $K \subset B(0,1)$, the following estimate holds:

$$u(x) \gtrsim C(K) \text{ for all } x \text{ such that } 2 \leq |x| \leq 3.$$ (3.5)

Taking a suitable approximation we can assume that $K$ in (3.5) is the closure of a finite number of domains with smooth boundaries.

Now we recall the fundamental result in potential theory, [I] Ch. 2. The Bessel kernel $J_2 \in C^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ is defined via the formula

$$(1 - \Delta)^{-1}f = J_2 * f \quad \text{for all } f \in \mathcal{S}.$$
It satisfies the estimates (see, for instance, Chapter 1):

\( J_2(x) \approx |x|^{-n+2} \quad \text{for} \quad x \in B(0,1), \)

\( J_2(x) \approx e^{-|x| |x|^{-n+1}/2} \quad \text{for} \quad x \in B(0,1)^c. \)

The theorem from nonlinear potential theory states that there exists a Radon measure \( \mu^K, \mu^K \geq 0, \)

such that

\[ \text{supp}(\mu^K) \subset K, \]

and

\[ C(K) \approx \mu^K (K) \approx \int_{\mathbb{R}^n} (J_2 * \mu^K)^q. \]

Hence, after the regularisation of \( \mu^K \) and a possible additional smooth approximation of \( K \) we obtain a function \( g \in C_0(\mathbb{R}^n), \ g \geq 0, \) such that

\[ \text{supp}(g) \subset K, \]

and

\[ C(K) \approx \int_{\mathbb{R}^n} g \approx \int_{\mathbb{R}^n} (J_2 * g)^q. \]

2. Set \( R = 10 \) and \( B = B(0,R) \). For a fixed \( \varepsilon > 0 \) consider the Dirichlet problem

\[
\begin{cases}
\Delta v = v^q - \varepsilon g & \text{in} \ B \\
v = 0 & \text{on} \ \partial B.
\end{cases}
\]

As a simple consequence of the comparison principle, it has the unique smooth solution \( v = v_\varepsilon, \ v \geq 0 \) in \( B \). Our goal will be to show that there exists \( \varepsilon > 0, \ \varepsilon = \varepsilon(n), \) such that

\( v(x) \geq C(K) \quad \text{for all} \quad x \text{ such that} \quad 2 \leq |x| \leq 3. \)

To prove this we set \( G(x,y) = G_R(x,y), \) and note that by comparison principle

\[ v(x) \leq -\varepsilon \int_B G(x,y)g(y) \, dy \quad \text{for all} \quad x \in B. \]

Consequently

\[ v(x) = -\varepsilon \int_B G(x,y)g(y) \, dy + \int_B G(x,y)v(y)^q \, dy \]

\[ \geq \varepsilon \int_B |G(x,y)| g(y) \, dy \]

\[ -\varepsilon^q \int_B |G(x,y)| \left( \int_B |G(y,z)| g(z) \, dz \right)^q \, dy \]

\[ = \varepsilon I(x) - \varepsilon^q II(x) \quad \text{for all} \quad x \in B. \]

Hence, to obtain we need to estimate \( I \) from below and \( II \) from above.

3. Define

\[ S = \{ x \in \mathbb{R}^n : 2 \leq |x| \leq 3 \}. \]

The sets \( S, \ \text{supp}(g), \) and \( \partial B \) are located at a distance at least 1 from each other. Consequently applying and invoking the elementary properties of \( G, \) we derive

\[ I(x) = \int_{\text{supp}(g)} |G(x,y)| g(y) \, dy \]

\[ \geq \int_{\mathbb{R}^n} g \]

\[ \geq C(K) \quad \text{for all} \quad x \in S. \]
4. We claim that
\begin{equation}
II(x) \lesssim C(K) \quad \text{for all } x \in S.
\end{equation}
Indeed, fix $x_0 \in S$. Introduce the shell
\[ \tilde{S} = \{ x \in \mathbb{R}^n : 2 - 1/100 \leq x \leq 3 + 1/100 \}, \]
and utilise estimate (3.6) for $J_2$ to write
\begin{equation}
II(x_0) \lesssim \int_{\tilde{S}} \frac{1}{|x_0 - y|^{n-2}} \left( - \int_B G(y, z) g(z) dz \right)^q dy 
+ \int_{B \setminus \tilde{S}} \frac{1}{|x_0 - y|^{n-2}} (J_2 * g)^q (y) dy 
= X + Y.
\end{equation}

We estimate $X$ and $Y$ separately.

To estimate $X$ define the function $H: B \to \mathbb{R}^1$ by writing
\[ H(y) = -\int_B G(y, z) g(z) dz, \quad y \in B. \]
Notice that according to (3.7), $H$ is positive in $B$ and harmonic in $B \setminus K$. Consequently $H^q$ is subharmonic in $B \setminus K$. Hence by the mean value property
\begin{align*}
X & \lesssim \left( \max_{\tilde{S}} H \right)^q \int_{\tilde{S}} \frac{dy}{|x_0 - y|^{n-2}} \\
& \lesssim \max_{\tilde{S}} H^q \\
& \lesssim \int_B H(y)^q dy.
\end{align*}

Now (3.6) and (3.8) allow us to conclude that
\begin{equation}
X \lesssim \int_{\mathbb{R}^n} (J_2 * g)^q (y) dy 
\lesssim C(K).
\end{equation}

To estimate $Y$ notice that
\[ |x_0 - y| \geq 1/100 \quad \text{for all } y \in B \setminus \tilde{S}. \]

Therefore utilising (3.8) we derive
\begin{align*}
Y & \lesssim \int_{B \setminus \tilde{S}} (J_2 * g)^q (y) dy \\
& \lesssim \int_{\mathbb{R}^n} (J_2 * g)^q \\
& \lesssim C(K).
\end{align*}

Substituting (3.14) and (3.15) into (3.13) we deduce (3.12).

5. Now we conclude the proof of the theorem. First we establish (3.9). Substitute (3.11) and (3.12) into (3.10):
\[ v(x) \geq (\varepsilon C_1(n) - \varepsilon^q C_2(n)) C(K) \quad \text{for all } x \in S. \]
Choosing the suitable ε > 0 derive (3.9).

Finally, the regularity of K implies that our maximal solution u blows up near K as in (2.6). Therefore

\[ u \geq v \quad \text{on} \quad \partial(B \setminus K). \]

Owing to (3.7) and the comparison principle,

\[ u \geq v \quad \text{in} \quad B \setminus K. \]

This inequality and (3.9) complete the proof of (3.5).

Next we establish an integral estimate for any solution of (3.1). This is Theorem 3.2 below. It has a particularly simple proof when \( n \geq 4 \) and hence

\[ \frac{2}{n-2} \leq 1. \]

In this case it follows more or less directly from the representation formula for solution of the linear Poisson equation. However, such approach does not work for \( n = 3 \) because the singularity of the Green function is too strong then. Proof of Theorem 3.2 given below does not use representation formula. Instead we rely on techniques common in quasilinear elliptic regularity theory. Such arguments were first used by Moser [43, 44] for linear equations, and by Trudinger [52] for nonlinear equations.

**Theorem 3.2** Let \( K \subset B(0,1) \) be a compact set, \( \varphi \) be the function from Lemma 2.1, and let

\[ m = \frac{n+2}{2} + 100n. \]

Then for any \( u \) solving (3.1) in \( K^c \) the estimate

\[ \int_{B(0,10)} u^{2/(n-2)} (1 - \varphi)^m \lesssim C(K)^{2/(n-2)} \]

holds.

**Proof.** 1. We claim that for any number \( \varepsilon \), \( 0 < \varepsilon < 1 \), the inequality

\[ \left( \int_{\mathbb{R}^n} u^{(1-\varepsilon)n/(n-2)} |\zeta|^{2n/(n-2)} \right)^{(n-2)/n} \leq C(\varepsilon) \left( \int_{\mathbb{R}^n} u^{1-\varepsilon} |D\zeta|^2 + \int_{\mathbb{R}^n} u^{q-\varepsilon} |\zeta|^2 \right) \]

(3.17)

holds for all functions \( \zeta \in C_0^\infty(\mathbb{R}^n) \), such that \( \zeta = 0 \) in an open neighbourhood of \( K \). In fact, multiplying the equation

\[ \Delta u - u^q = 0 \quad \text{in} \quad \mathbb{R}^n \setminus K \]

by \( u^{-\varepsilon} \zeta^2 \), integrating by parts, and invoking the formula

\[ D(u^{-\varepsilon} \zeta^2) = -\varepsilon u^{-\varepsilon-1} \zeta^2 Du + 2\zeta u^{-\varepsilon} D\zeta, \]

we deduce that

\[ \varepsilon \int_{\mathbb{R}^n} |Du|^2 u^{-\varepsilon-1} \zeta^2 \leq 2 \left( \int_{\mathbb{R}^n} |Du||D\zeta| u^{-\varepsilon} |\zeta| \right) + \int_{\mathbb{R}^n} u^{q-\varepsilon} \zeta^2. \]
Since
\[ |Du| |Dζ| u^{-\varepsilon} |ζ| \leq \delta |Du|^2 u^{-\varepsilon-1} ζ^2 + \frac{1}{4\delta} |Dζ|^2 u^{-\varepsilon+1} \]
for each \( \delta > 0 \), we derive that
\[ \int_{\mathbb{R}^n} |Du|^2 u^{-\varepsilon-1} ζ^2 \leq C(\varepsilon) \left( \int_{\mathbb{R}^n} u^{1-\varepsilon} |Dζ|^2 + \int_{\mathbb{R}^n} u^{q-\varepsilon} ζ^2 \right). \]

After some calculations we find
\[ \int_{\mathbb{R}^n} \left| D \left( \frac{u^{(1-\varepsilon)/2}}{2} \right) \right|^2 \leq C(\varepsilon) \left( \int_{\mathbb{R}^n} u^{1-\varepsilon} |Dζ|^2 + \int_{\mathbb{R}^n} u^{q-\varepsilon} ζ^2 \right). \]

Now (3.17) follows from the Sobolev inequality applied to the left hand side.

2. Set \( B = B(0, 10) \) and \( \hat{B} = B(0, 20) \). In (3.17), choose \( \varepsilon \in (0, 1) \) such that
\[ (1 - \varepsilon) \frac{n}{n-2} = \frac{2}{n-2}. \]

Then select a smooth cutoff function \( \theta \in C_0^\infty(\hat{B}) \), such that \( \theta = 1 \) on \( B \). Take the function \( \eta = (1 - \varphi)^m \) from Lemma 2.1. Now set \( \zeta = \eta \theta \), in estimate (3.17) to discover that
\[ \left( \int_B u^{2/(n-2)} (1 - \varphi)^m \right)^{(n-2)/2} \lesssim \left( \int_{\mathbb{R}^n} u^{1-\varepsilon} |D\eta|^2 \right) \left( \int_{\mathbb{R}^n} u^{1-\varepsilon} |D\theta|^2 \right) \left( \int_{\mathbb{R}^n} u^{q-\varepsilon} (\eta \theta)^2 \right)^{1/(1-\varepsilon)}. \]

We estimate three integrals in the right hand side of (3.18) as follows. Applying Holder inequality and estimates (2.17), (2.19) from Lemma 2.1 we deduce that
\[ \int_{\mathbb{R}^n} u^{1-\varepsilon} |D\eta|^2 \lesssim \left( \int_B u^{q(1-\varepsilon)} (1 - \varphi)^{(n+2)/2} \right)^{1/q} \left( \int_B |D\varphi|^{2q'} \right)^{1/q'} \lesssim C(K)^{(q-\varepsilon)/q}. \]

Estimate (5.22) from Theorem 3.1 implies
\[ \int_{\mathbb{R}^n} u^{1-\varepsilon} |D\theta|^2 \lesssim \|u\|_{L^\infty(B \setminus \hat{B})}^{1-\varepsilon} \lesssim C(K)^{1-\varepsilon}. \]

Finally, owing to Holder inequality and (2.19) we have
\[ \int_{\mathbb{R}^n} u^{q-\varepsilon} (\eta \theta)^2 \lesssim C(K)^{(q-\varepsilon)/q}. \]

Substituting these estimates in (3.18) we arrive at
\[ \left( \int_B u^{2/(n-2)} (1 - \varphi)^m \right)^{(n-2)/2} \lesssim C(K) + C(K)^{(q-\varepsilon)/(q-\varepsilon q)} \lesssim C(K). \]

This is assertion (3.16).  

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4 Proof of the Wiener test for conformal metrics

4.1 Sufficiency

We prove the implication (ii) ⇒ (i) in Theorem 1.1.

1. In the proof we will work on $\mathbb{S}^n$. A curve $\gamma: [0, +\infty) \to \Omega$ is said to converge to infinity if for every compact set $M \subset \Omega$, there is a time $T$, $0 < T < +\infty$, such that $\gamma(t) \notin M$ for all $t > T$. By a version of the Hopf-Rinow theorem, a metric is complete in $\Omega$ if and only if every smooth curve converging to infinity has the infinite length.

Let $u_\Omega$ be the maximal solution of the conformal scalar curvature equation (2.1) in $\Omega$. We set $\mathcal{C} = u_\Omega^{4/(n-2)}$.

Fix any smooth curve $\gamma: [0, +\infty) \to \Omega$ converging to infinity. To prove statement (i) in Theorem 1.1 we need to show that

$$L_g(\gamma) = \int_0^\infty u_\Omega(\gamma)^{2/(n-2)} \mathcal{C}(\gamma, \gamma)^{1/2} \, dt = +\infty.$$ 

In the rest of the proof we establish (4.1).

2. Compactness of $\mathbb{S}^n$ and convergence of $\gamma$ to infinity imply the existence of a point $p \in K$ such that

$$d_{\mathcal{C}}(\gamma(T_k), p) \to 0 \quad \text{for a sequence } \{T_k\}, \quad T_k \to +\infty.$$ 

For $j = 1, 2, \ldots$ we define $\Gamma_j$ to be that part of $\gamma$ whose image is contained in the shell $S_j$,

$$S_j = \left\{ x \in \mathbb{S}^n : r_j < d_{\mathcal{C}}(x, p) < r_{j-1} \right\}.$$

The smoothness of $\gamma$ implies that for any $j \geq j_0$ the set $\Gamma_j$ is at most a countable union of open smooth curves. Utilising condition (4.2) we deduce that $\Gamma_j \neq \emptyset$, and moreover

$$L_{\mathcal{C}}(\Gamma_j) \geq \frac{r_j}{100} \quad \text{for all } j \geq j_0.$$

After a rotation we can assume that

$$K \cap \overline{B}(p, r_j) \subset U \quad \text{for all } j \geq j_0 - 10,$$

where $U$ is cup (2.8) around the south pole from the definition of $C$. We claim that for all $j \geq j_0$ the inequality

$$L_{\mathcal{C}}(\Gamma_j) \geq \frac{r_j}{100} \quad \text{for all } j \geq j_0.$$

holds. In fact, define the open set $\Omega_j \subset \Omega$, $\Omega_j \supset \Omega$, by writing

$$\Omega_j = \mathbb{S}^n \setminus (K \cap \overline{B}(p, r_{j+2})).$$

Let $u_j$ be the maximal solution to our equation (2.1) in $\Omega_j$. Pull estimate (3.2) from Theorem 3.1 back to the sphere via the stereographic projection, keeping in mind that the conformal factor in (2.2) satisfies

$$\Upsilon(x) \approx 1 \quad \text{for all } x, \quad |x| \leq 10.$$
We discover that
\[ u_{\Omega}(x) \geq u_j(x) \geq C \left( \frac{K \cap B(p, r_j+2)}{r_j^{n-2}} \right) \]
for all \( x \in S_j \cap \Omega \).

Let \( I_j, I_j \subset (0, +\infty) \), be the open set such that
\[ \Gamma_j : I_j \rightarrow \Omega \cap S_j. \]

Then we derive that
\[ L_g(\Gamma_j) = \int_{I_j} u_{\Omega}(\gamma)^{2/(n-2)} \hat{g}(\dot{\gamma}, \gamma)^{1/2} dt \]
\[ \geq \left( \inf_{S_j} u_{\Omega} \right)^{2/(n-2)} L_{\hat{g}}(\Gamma_j) \]
\[ \geq \left( C \left( K \cap B(p, r_j+2) \right) \right)^{2/(n-2)} r_j, \]
thereby obtaining (4.3).

3. We claim that (4.1) holds. Indeed, the sets \( S_j \) are disjoint, and thus
\[ L_g(\gamma) \geq \sum_{j \geq 1} L_g(\Gamma_j). \]

To each term with sufficiently large number in this sum we apply estimate (4.3) and recall (2.16) to derive that
\[ L_g(\gamma) \geq \sum_{j \geq j_0} \left( C \left( K \cap B(p, r_j+2) \right) \right)^{2/(n-2)} r_j + 100 \]
\[ \geq \sum_{j \geq j_0 + 100} \left( \frac{C(B(p, r_j) \cap K)}{C(B(p, r_j))} \right)^{2/(n-2)}. \]

Finally utilise (2.28) and (1.4) to establish (4.1). This completes the proof of implication \((ii) \Rightarrow (i)\) in Theorem 1.1.

4.2 Necessity

Now we prove the implication \((i) \Rightarrow (ii)\) in Theorem 1.1.

1. Seeking a contradiction assume that \((ii)\) does not hold. Hence
\[ \int_0^{1/2} \left( \frac{C(B(P, r) \cap K)}{C(B(P, r))} \right)^{2/(n-2)} \frac{dr}{r} < +\infty \]
for some \( P \in \partial \Omega \). The desired contradiction will follow if the maximal solution of (2.11) does not give the metric complete in \( \Omega \). Let \( U \) be this maximal solution, and let
\[ g = U^{2/(n-2)} \hat{g}. \]
According to the Hopf-Rinow theorem, to prove the non-completeness of \( g \) we must show that there exists a smooth curve \( c \),
\[
c: [0, 1) \to \Omega, \quad \text{such that } d_g(c(t), P) \to 0 \text{ as } t \to 1, \quad \text{and } L_g(c) < +\infty.
\]
(4.5)

2. First we reformulate claim (4.5). Fix a parameter \( \rho > 0 \), which we will later choose small. Set
\[
\tilde{K} = K \cap \overline{B}(P, \rho).
\]
Let \( U_1 \) be the maximal solution of (2.1) in \( \tilde{\Omega} \),
\[
\tilde{\Omega} = S^n \setminus \tilde{K},
\]
and let \( U_2 \) be the maximal solution of (2.1) in \( \Omega \cup B(P, \rho) \). From (2.11) we deduce that
\[
U^{2/(n-2)} \leq U_1^{2/(n-2)} + U_2^{2/(n-2)} \text{ in } \Omega.
\]
At the same time (2.4) implies
\[
U_2(x) \leq C(\rho) \text{ for all } x \in B(P, \rho/2).
\]
Therefore (4.5) is equivalent to the same statement with \( \Omega \) replaced by \( \tilde{\Omega} \), and \( g \) replaced by
\[
\tilde{g} = U_1^{2/(n-2)} g.
\]
To prove this statement it will be convenient to transform the problem to \( \mathbb{R}^n \).

Applying a suitable rotation and stereographic projection we can achieve that \( P \) is mapped to 0. We denote the image of \( K \) under such map by the same letter \( K \). In \( \mathbb{R}^n \) we set
\[
K_J = K \cap \overline{B}_j \quad \text{and } B = B(0, 1).
\]
By \( u \) we denote the conformal pullback (2.2) of \( U_1 \),
\[
u(x) = \Upsilon(x) U_1(\sigma^{-1} x), \quad x \in \sigma(\tilde{\Omega}).
\]
As it is shown in section 2.1, \( u \) is the maximal solution of (2.2) in \( \mathbb{R}^n \setminus K_J \) for some \( J \), \( J = J(\rho) \). From (4.4) and (2.12) we deduce that
\[
\int_0^1 \left( \frac{C(B(0, r) \cap K_J)}{C(B(0, r))} \right)^{2/(n-2)} \frac{dr}{r} < +\infty.
\]
(4.6)

Finally, to establish (4.5) we must prove that \( u^{2/(n-2)} g_E \) is not complete, that is
\[
\int_\gamma u^{2/(n-2)} ds < +\infty \quad \text{for a smooth curve } \gamma: [0, 1) \to B \setminus K_J,
\]
(4.7)

such that \( \gamma(t) \to 0 \) as \( t \to 1 \).

3. We intend to establish (4.7). The construction of \( \gamma \) in (4.7) will be indirect. More precisely, let us first reduce the proof of (4.7) to an integral estimate for our maximal solution \( u \).

We assert that it is possible to choose large enough \( J \) in (4.7) (equivalently, to choose small enough \( \rho > 0 \)) such that there exists a compact set \( \Sigma \),
\[
K_J \subset \Sigma \subset B,
\]
with the following two properties:

\[(4.8) \quad \int_{B \setminus \Sigma} u(x)^{2/(n-2)} \frac{1}{|x|^{n-1}} \, dx < +\infty, \]

and

\[(4.9) \quad \mathcal{H}^{n-1}(\pi(\Sigma \setminus \{0\})) < \mathcal{H}^{n-1}(\partial B), \]

where \( \pi \) is the radial projection on \( \partial B \),

\[\pi: B \setminus \{0\} \to \partial B, \quad x \mapsto \frac{x}{|x|}.\]

This assertion is the core of the proof. Before passing to its verification we conclude the current step by showing that (4.8), (4.9) immediately imply (4.7) and hence the theorem.

Indeed, for \( \omega \in \partial B \) we define the interval \( \ell(\omega) \) by writing

\[\ell(\omega) = \{x \in \mathbb{R}^n : x = s\omega, \ 0 < s \leq 1\}.\]

Set

\[\Xi = \partial B \setminus \pi(\Sigma \setminus \{0\}).\]

First notice that

\[\pi^{-1}(\Xi) \subset B \setminus \Sigma.\]

Hence, using the polar coordinates \((r, \omega), \ r > 0, \ \omega \in \partial B\), we deduce at once from (4.8) that

\[+\infty > \int_{\pi^{-1}(\Xi)} u(x)^{2/(n-2)} \frac{1}{|x|^{n-1}} \, dx = \int_{\Xi} \int_{0}^{1} u(x(r, \omega))^{2/(n-2)} \frac{1}{r^{n-1}} r^{n-1} \, dr \, d\mathcal{H}^{n-1}(\omega) = \int_{\Xi} \left( \int_{\ell(\omega)} u^{2/(n-2)} \, ds \right) \, d\mathcal{H}^{n-1}(\omega).\]

Next, apply (4.9) to discover that

\[\mathcal{H}^{n-1}(\Xi) = \mathcal{H}^{n-1}(\partial B) - \mathcal{H}^{n-1}(\pi(\Sigma \setminus \{0\})) > 0.\]

Consequently

\[\int_{\ell(\omega_0)} u^{2/(n-2)} \, ds < +\infty \quad \text{for some} \quad \omega_0 \in \Xi.\]

By our definitions

\[\ell(\omega_0) \cap K_j = 0,\]

and we conclude that (4.7) holds for the curve \( \gamma = \ell(\omega_0) \).

Thus, to establish the theorem it is left to construct \( \Sigma \) satisfying (4.8) and (4.9). The rest of the proof is devoted entirely to this construction.

4. For any \( j \geq J \) let

\[v = v_j\]

be the maximal solution for \( K_{j-2} \). We now apply Theorem 3.2 and scalings (2.7), (2.12) to bound \( v \).

Applying the scaling

\[x \mapsto r_{j-3} x, \quad x \in \mathbb{R}^n,\]
to estimate (4.10) we deduce that there exists a function

$$\varphi = \varphi_j,$$

such that:

$$\varphi \in C_0^\infty(B(0,2)), \quad 0 \leq \varphi \leq 1 \quad \text{in} \quad B(0,2),$$

$$\varphi = 1 \quad \text{in a neighbourhood of} \quad K_{j-2},$$

(4.10)

$$\int_{B(0,2)} |D^2 \varphi|^{(n+2)/4} \lesssim C(K_{j-2}),$$

and

$$\frac{1}{r_{j-2}^{n-1}} \int_{B_{j-2}} v^{2/(n-2)} (1 - \varphi)^m \lesssim \left( \frac{C(K_{j-2}/r_{j-3})}{C(B_{j-2})} \right)^{2/(n-2)}$$

(4.11)

5. Now we construct the compactum $\Sigma$ for (4.8), (4.9). Set

$$S_j = \{ x : r_j \leq |x| \leq r_{j-1} \}.$$

First for a fixed $j \geq J$, define the compact set $E_j$ by writing

$$E_j = \left\{ x \in S_j : \varphi_j(x) \geq \frac{99}{100} \right\},$$

where the function $\varphi_j$ is taken from (4.10), (4.11). Then define

$$\Sigma = \bigcup_{j=J}^\infty E_j \bigcup \{0\}.$$

According to the construction, the set $\Sigma$ is compact and

$$K_j \subset \Sigma \subset B.$$

We claim that (4.13) holds. Indeed, the definition of the capacity and (4.10) imply that

$$C(E_j) \leq \int_{B(0,2)} \left| D^2 \left( \frac{100}{99} \varphi_j \right) \right|^{(n+2)/4} \lesssim C(K_{j-2}).$$

The metric estimate (2.24) therefore ensures

$$\mathcal{H}_{n-1}^{n-1}(E_j)^{(n-1)/2} \lesssim C(K_{j-2})^{n-1}.$$

The projection $\pi$ restricted to $S_j$ distorts the distances at most $1/r_j$ times. Consequently

$$\mathcal{H}_{n-1}^{n-1}(\pi(S_j \setminus \{0\})) \leq \sum_{j=J}^\infty \mathcal{H}_{n-1}^{n-1}(\pi(E_j)) \leq \sum_{j=J}^\infty \frac{1}{r_j^{n-1}} \mathcal{H}_{n-1}^{n-1}(E_j) \lesssim \sum_{j=J}^\infty \frac{1}{r_j^{n-1}} C(K_{j-2})^{(n-1)/2/(n-2)} \lesssim \left( \sum_{j=J-2}^\infty \left( \frac{C(K_j)}{C(B_j)} \right)^{2/(n-2)} \right)^{n-1}. $$
According to (2.28) and (4.6) we can make the last series as small as we wish by choosing $J$ large enough. Thus for any $\varepsilon > 0$ we may fix $J$ in (4.6) so that
\[ \mathcal{H}_{\infty}^{n-1}(\pi(\Sigma \setminus \{0\})) < \varepsilon. \]

For sets lying on an $s$-dimensional smooth submanifold, the Hausdorff $s$-measure is equivalent to the Lebesgue $s$-measure. Consequently
\[ \mathcal{H}_{\infty}^{n-1}(F) \gtrsim \mathcal{H}^{n-1}(F) \quad \text{for any} \quad F \subset \partial B. \]

This gives (4.9).

6. It is left to prove (4.8). Splitting the integral there we find that
\[
\int_{B \setminus \Sigma} u(x) \frac{1}{|x|^{n-1}} \, dx \lesssim \sum_{j=J}^{\infty} \int_{S_j \setminus \Sigma} u^{2/(n-2)}.
\]

Thus our task is to estimate $u$ in $S_j \setminus \Sigma$. Fix $j \geq J$. Let as before $v$ be the maximal solution for $K_{j-2}$. For $l = 1, 2, \ldots, j - 2$ let $w_l$ be the maximal solution for $K \cap S_l$. From (2.5) we deduce that
\[ u \leq v + \sum_{l=1}^{j-2} w_l \quad \text{in} \quad S_j. \]

Next observe that
\[ |x - y| \approx r_l \quad \text{for all} \quad x \in S_j, \quad y \in S_l, \quad l \leq j - 2. \]

Hence applying the scaled estimate (3.2) from Theorem 3.1 to $w_l$, we derive that
\[
u(x) \lesssim v(x) + \sum_{l=1}^{j-2} \frac{C(K \cap S_l)}{r_l^{n-2}} \lesssim v(x) + \sum_{l=1}^{j-2} \frac{C(K_l)}{r_l^{n-2}} \quad \text{for all} \quad x \in S_j.
\]

To estimate $v$ in $S_j$ notice that (4.11) and the definition of $\Sigma$ ensure that
\[
\frac{1}{r_j^{n-1}} \int_{S_j \setminus \Sigma} v^{2/(n-2)} \lesssim \left( \frac{C(K_{j-2})}{C(B_{j-2})} \right)^{2/(n-2)}.
\]

Utilising (4.13) we thereupon conclude that
\[
\frac{1}{r_j^{n-1}} \int_{S_j \setminus \Sigma} u^{2/(n-2)} \lesssim \frac{1}{r_j^{n-1}} \int_{S_j \setminus \Sigma} v^{2/(n-2)} + \frac{1}{r_j^{n-1}} \left( \sum_{l=1}^{j-2} \frac{C(K_l)}{r_l^{n-2}} \right)^{2/(n-2)} |S_j|
\leq \left( \frac{C(K_{j-2})}{C(B_{j-2})} \right)^{2/(n-2)} + r_j \left( \sum_{l=1}^{j-2} \frac{C(K_l)}{r_l^{n-2}} \right)^{2/(n-2)}.
\]
Now continue (4.12) to find
\[
\int_{B \setminus \Sigma} u(x) \frac{1}{|x|^{n-1}} \, dx \lesssim \sum_{j=1}^{\infty} \left( \frac{C(K_j)}{C(B_j)} \right)^{2/(n-2)} + \sum_{j=1}^{\infty} r_j \left( \sum_{l=1}^{j} \frac{C(K_l)}{r_l^{n-2}} \right)^{2/(n-2)}
\]
(4.14)

Thus to prove (4.8) we need to bound I and II.

7. Utilising (2.28) we deduce at once that
\[
I \lesssim \int_{0}^{1} \left( \frac{C(K \cap B(0, r))}{C(B(0, r))} \right)^{2/(n-2)} \frac{dr}{r}.
\]

To estimate II we define the function \( \Phi: (0, 1) \to \mathbb{R}^1 \) by writing
\[
\Phi(r) = C(K \cap B(0, r)), \quad 0 < r < 1.
\]

First assume that \( n \geq 4 \) and hence
\[
\frac{2}{n-2} \leq 1.
\]

In this case by the simple change of the summation order we discover that
\[
II \leq \sum_{j=1}^{\infty} r_j \sum_{l=1}^{j} \left( \frac{C(K_l)}{r_l^{n-2}} \right)^{2/(n-2)}
\]
\[
\leq \sum_{l=1}^{\infty} \left( \frac{C(K_l)}{r_l^{n-2}} \right)^{2/(n-2)} r_l
\]
\[
\leq \int_{0}^{1} \left( \frac{\Phi(r)}{r^{n-2}} \right)^{2/(n-2)} \, dr.
\]

Assume next that \( n = 3 \), and hence
\[
\frac{2}{n-2} = 2
\]

Then Hardy’s inequality implies that
\[
II \lesssim \int_{0}^{1} \left( \frac{\Phi(r)}{r} \right)^{2/(n-2)} \, dr
\]
\[
\lesssim \int_{0}^{1} \left( \frac{\Phi(r)}{r} \right)^2 \, dr.
\]

Thus for any \( n \geq 3 \) we have
\[
II \lesssim \int_{0}^{1} \left( \frac{\Phi(r)}{r^{(n-2)/2}} \right)^{2/(n-2)} \frac{dr}{r}.
\]

Returning to (4.14) and recalling (2.16) we derive
\[
\int_{B \setminus \Sigma} u(x) \frac{1}{|x|^{n-1}} \, dx \lesssim \int_{0}^{1} \left( \frac{C(K \cap B(0, r))}{C(B(0, r))} \right)^{2/(n-2)} \frac{dr}{r}.
\]

Employing (4.10) we establish (4.8). This completes the proof of the implication \((i) \Rightarrow (ii)\) in Theorem 1.1.
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