ON THE CO-ORDINATED CONVEX FUNCTIONS

M. EMIN ÖZDEMIR, ÇETIN YILDIZ, AND AHMET OCAK AKDEMIR

Abstract. In this paper we established new integral inequalities which are more general results for co-ordinated convex functions on the co-ordinates by using some classical inequalities.

1. INTRODUCTION

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a < b \). The following double inequality

\[
\frac{f\left(\frac{a+b}{2}\right)}{b-a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

is well known in the literature as Hadamard’s inequality. Both inequalities hold in the reversed direction if \( f \) is concave.

In [4], Dragomir defined convex functions on the co-ordinates as following;

**Definition 1.** Let us consider the bidimensional interval \( \Delta = [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \), \( c < d \). A function \( f : \Delta \rightarrow \mathbb{R} \) will be called convex on the co-ordinates if the partial mappings \( f_y : [a,b] \rightarrow \mathbb{R}, f_y(u) = f(u,y) \) and \( f_x : [c,d] \rightarrow \mathbb{R}, f_x(v) = f(x,v) \) are convex where defined for all \( y \in [c,d] \) and \( x \in [a,b] \). Recall that the mapping \( f : \Delta \rightarrow \mathbb{R} \) is convex on \( \Delta \) if the following inequality holds,

\[
f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)
\]

for all \((x,y), (z,w) \in \Delta \) and \( \lambda \in [0,1] \).

In [4], Dragomir established the following inequalities of Hadamard’s type for co-ordinated convex functions on a rectangle from the plane \( \mathbb{R}^2 \).
Theorem 1. Suppose that \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). Then one has the inequalities;

\[
\begin{align*}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f(x, \frac{c+d}{2}) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\
&\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy \\
&\leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \\
&+ \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\
&\leq f(a, c) + f(a, d) + f(b, c) + f(b, d).
\end{align*}
\]

The above inequalities are sharp.

In [1], Bakula and Pečarić established several Jensen type inequalities for co-ordinated convex functions and in [5], Hwang et al. gave a mapping \( F \), discussed some properties of this mapping and proved some Hadamard-type inequalities for Lipschitzian mapping in two variables. In [2], Özdemir et al. established new Hadamard-type inequalities for co-ordinated \( m \)-convex and \((\alpha, m)\)-convex functions. On all of these, in [3], the authors proved some Hadamard-type inequalities for co-ordinated convex functions as followings;

Theorem 2. Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial t \partial s} \) is a convex function on the co-ordinates on \( \Delta \), then one has the inequalities:

\[
\begin{align*}
&\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \\
&\leq \frac{1}{(b-a)(d-c)} \left[ \int_a^b \int_c^d f(x, y) dx dy - A \right] \\
&\leq \frac{1}{16} \left( \frac{\partial^2 f}{\partial t \partial s}(a,c) + \frac{\partial^2 f}{\partial t \partial s}(a,d) + \frac{\partial^2 f}{\partial t \partial s}(b,c) + \frac{\partial^2 f}{\partial t \partial s}(b,d) \right)
\end{align*}
\]

where

\[
A = \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{d-c} \int_c^d [f(a, y) + f(b, y)] dy \right].
\]

Theorem 3. Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right| \), \( q > 1 \), is a convex function on
the co-ordinates on $\Delta$, then one has the inequalities:

\[
(1.3) \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right| - \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dxdy - A \leq \frac{(b - a)(d - c)}{4(p + 1)^q} \left( \frac{\partial^2 f}{\partial t \partial s}^q (a, c) + \frac{\partial^2 f}{\partial t \partial s}^q (a, d) + \frac{\partial^2 f}{\partial t \partial s}^q (b, c) + \frac{\partial^2 f}{\partial t \partial s}^q (b, d) \right)^\frac{1}{q}
\]

where

\[A = \frac{1}{2} \left\{ \frac{1}{(b - a)} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{(d - c)} \int_c^d [f(a, y) + f(b, y)] dy \right\} \]

and $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 4.** Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$, $q \geq 1$, is a convex function on the co-ordinates on $\Delta$, then one has the inequalities:

\[
(1.4) \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right| - \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dxdy - A \leq \frac{(b - a)(d - c)}{16} \left( \frac{\partial^2 f}{\partial t \partial s}^q (a, c) + \frac{\partial^2 f}{\partial t \partial s}^q (a, d) + \frac{\partial^2 f}{\partial t \partial s}^q (b, c) + \frac{\partial^2 f}{\partial t \partial s}^q (b, d) \right)^\frac{1}{q}
\]

where

\[A = \frac{1}{2} \left\{ \frac{1}{(b - a)} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{(d - c)} \int_c^d [f(a, y) + f(b, y)] dy \right\} \]

In [\$], authors proved following inequalities for co-ordinated convex functions.

**Theorem 5.** Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$ is a convex function on the co-ordinates on $\Delta$, then the following inequality holds:

\[
(1.5) \left| f \left( \frac{a + b}{2}, c + \frac{d}{2} \right) \left[ -1 \left( \frac{a + b}{2}, y \right) + \frac{1}{(b - a)} \int_a^b f \left( x, c + \frac{d}{2} \right) dx \right. \right.
\]

\[
+ \left. \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx \right| \leq \frac{(b - a)(d - c)}{64} \left[ \left| \frac{\partial^2 f}{\partial s \partial s} (a, c) \right| + \left| \frac{\partial^2 f}{\partial s \partial s} (b, c) \right| + \left| \frac{\partial^2 f}{\partial s \partial s} (a, d) \right| + \left| \frac{\partial^2 f}{\partial s \partial s} (b, d) \right| \right].
\]
Theorem 6. Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$, $q > 1$, is a convex function on the co-ordinates on $\Delta$, then the following inequality holds:

$$
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \frac{1}{(b-a)(d-c)} \int_c^d f(x, y) \, dy \right|
$$

$$
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy - \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx
$$

$$
\leq \frac{4}{(b-a)(d-c)}
$$

$$
\times \left( \frac{\left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q}{4} \right)^{\frac{1}{q}}.
$$

Theorem 7. Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$, $q > 1$, is a convex function on the co-ordinates on $\Delta$, then the following inequality holds:

$$
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \frac{1}{(b-a)(d-c)} \int_c^d f(x, y) \, dy \right|
$$

$$
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy - \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx
$$

$$
\leq \frac{16}{(b-a)(d-c)}
$$

$$
\times \left( \frac{\left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q}{4} \right)^{\frac{1}{q}}.
$$

In [7], Özdemir et al. proved the following Theorem which is involving an inequality of Simpson’s type;

Theorem 8. Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$. If $\frac{\partial^2 f}{\partial x \partial y}$ is a convex function on the co-ordinates on $\Delta$, then the following inequality holds:

$$
\left| f \left( a, \frac{c+d}{2} \right) + f \left( b, \frac{c+d}{2} \right) + 4f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right|
$$

$$
+ f \left( a, c \right) + f \left( b, d \right) + f \left( a, d \right) + f \left( b, c \right)
$$

$$
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A
$$

$$
\leq \frac{25(b-a)(d-c)}{72}
$$

$$
\times \left( \frac{\left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right| + \left| \frac{\partial^2 f}{\partial x \partial y} (a, d) \right| + \left| \frac{\partial^2 f}{\partial x \partial y} (b, c) \right| + \left| \frac{\partial^2 f}{\partial x \partial y} (b, d) \right|}{72} \right).\]
where

\[
A = \frac{1}{6(b-a)} \int_a^b \left[ f(x, c) + 4f \left( x, \frac{c+d}{2} \right) + f(x, d) \right] dx \\
+ \frac{1}{6(a-c)} \int_c^d \left[ f(a, y) + 4f \left( \frac{a+b}{2}, y \right) + f(b, y) \right] dy.
\]

The main purpose of this paper is to establish a new lemma which gives more general results and different type inequalities for special values of \( \lambda \) and to prove several inequalities.

2. MAIN RESULTS

In order to prove our main theorems we need the following lemma:

**Lemma 1.** Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a differentiable function on \( \Delta \) where \( a < b, c < d \) and \( \lambda \in [0,1] \), if \( \frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta) \), then the following equality holds:

\[
(1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \lambda^2 f(a,c) + f(a,d) + f(b,c) + f(b,d)
\]
\[
+ \frac{\lambda(1-\lambda)}{2} \left[ f \left( \frac{a+b}{2}, c \right) + f \left( \frac{a+b}{2}, d \right) + f \left( a, \frac{c+d}{2} \right) + f \left( b, \frac{c+d}{2} \right) \right]
\]
\[
- (1-\lambda) \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy - (1-\lambda) \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx
\]
\[
- \lambda \frac{1}{2(b-a)} \int_a^b \left[ f(x,d) + f(x,c) \right] dx - \lambda \frac{1}{2(d-c)} \int_c^d [f(a,y) + f(b,y)] dy
\]
\[
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x)M(y) \frac{\partial^2 f}{\partial x \partial y}(x,y) dxdy
\]

where

\[
K(x) = \begin{cases} 
  x - (a + \lambda \frac{b-a}{2}), & x \in [a, a+b] \\
  x - (b - \lambda \frac{b-a}{2}), & x \in [a+b, b] 
\end{cases}
\]

and

\[
M(y) = \begin{cases} 
  y - (c + \lambda \frac{c-d}{2}), & y \in [c, c+d] \\
  y - (d - \lambda \frac{c-d}{2}), & y \in [c+d, d] 
\end{cases}
\]
Proof. Integration by parts, we get

\[
\begin{align*}
\int_a^b \int_c^d K(x)M(y) \frac{\partial^2 f}{\partial x \partial y}(x, y) dy dx &= \int_a^b K(x) \left[ \int_c^d \left( y - \left( c + \frac{d - c}{2} \right) \right) \frac{\partial^2 f}{\partial x \partial y}(x, y) dy \right. \\
&\quad + \int_c^d \left( y - \left( d - \frac{d - c}{2} \right) \right) \frac{\partial^2 f}{\partial x \partial y}(x, y) dy \left. \right] dx \\
&= \int_a^b K(x) \left[ \left( y - \left( c + \frac{d - c}{2} \right) \right) \frac{\partial f}{\partial x}(x, y) \right|_c^d + \left. \int_c^d \frac{\partial f}{\partial x}(x, y) dy \right] dx \\
&\quad + \left( y - \left( d - \frac{d - c}{2} \right) \right) \frac{\partial f}{\partial x}(x, y) \left. \right|_c^d - \int_c^d \left. \frac{\partial f}{\partial x}(x, y) dy \right] dx \\
&= \int_a^b K(x) \left[ (1 - \lambda) (d - c) \frac{\partial f}{\partial x} \left( x, \frac{c + d}{2} \right) \\
&\quad + \left( \frac{d - c}{2} \right) \left( \frac{\partial f}{\partial x}(x, c) + \frac{\partial f}{\partial x}(x, d) \right) - \int_c^d \left. \frac{\partial f}{\partial x}(x, y) dy \right] dx.
\end{align*}
\]

Integrating by parts again, we obtain

\[
\begin{align*}
\int_a^b \int_c^d K(x)M(y) \frac{\partial^2 f}{\partial x \partial y}(x, y) dy dx &= (1 - \lambda)^2 (b - a)(d - c) \left[ f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right] \\
&\quad + \lambda^2 (b - a)(d - c) \left[ f \left( a, c \right) + f \left( a, d \right) + f \left( b, c \right) + f \left( b, d \right) \right] \\
&\quad + \frac{\lambda(1 - \lambda)(b - a)(d - c)}{2} \left[ f \left( \frac{a + b}{2}, c \right) + f \left( \frac{a + b}{2}, d \right) \\
&\quad + f \left( \frac{a + d}{2} \right) + f \left( \frac{b + d}{2} \right) \right] - (1 - \lambda)(b - a) \int_c^d f \left( \frac{a + b}{2}, y \right) dy \\
&\quad - (1 - \lambda)(d - c) \int_a^b f \left( x, \frac{c + d}{2} \right) dx - \lambda^2 \left( d - c \right) \int_a^b \left[ f(x, d) + f(x, c) \right] dx \\
&\quad - \lambda \frac{(b - a)}{2} \int_c^d \left[ f(a, y) + f(b, y) \right] dy + \int_a^b \int_c^d f(x, y) dy dx.
\end{align*}
\]

Dividing both sides of the above equality by \((b - a)(d - c)\), we have the required result. \qed
Theorem 9. Let \( f : \Delta = [a,b] \times [c,d] \to \mathbb{R} \) be a differentiable function on \( \Delta \). If \( \left| \frac{\partial^2 f}{\partial x \partial y} \right| \) is convex function on the co-ordinates on \( \Delta \), then one has the inequality:

\[
\begin{align*}
| (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \lambda^2 f (a, c) + f (a, d) + f (b, c) + f (b, d) & + \frac{\lambda(1 - \lambda)}{2} \left[ f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, c \right) + f \left( \frac{a + b}{2}, d \right) + f \left( \frac{a}{2}, \frac{c + d}{2} \right) + f \left( \frac{b}{2}, \frac{c + d}{2} \right) \right] \\
& - (1 - \lambda) \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy - (1 - \lambda) \frac{1}{b - a} \int_a^b f (x, \frac{c + d}{2}) dx \\
& - \frac{1}{b - a} \int_a^b \int_c^d f (x, y) dx dy & \leq \frac{(b - a)(d - c)}{16} \left[ 2\lambda^2 - 2\lambda + 1 \right]^2 \\
& \times \left( \frac{\left| \frac{\partial^2 f}{\partial x \partial y} \right| (a, c) + \left| \frac{\partial^2 f}{\partial x \partial y} \right| (a, d) + \left| \frac{\partial^2 f}{\partial x \partial y} \right| (b, c) + \left| \frac{\partial^2 f}{\partial x \partial y} \right| (b, d)}{4} \right).
\end{align*}
\]

Proof. From Lemma 1 and property of the modulus, we can write

\[
\begin{align*}
| (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \lambda^2 f (a, c) + f (a, d) + f (b, c) + f (b, d) & + \frac{\lambda(1 - \lambda)}{2} \left[ f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, c \right) + f \left( \frac{a + b}{2}, d \right) + f \left( \frac{a}{2}, \frac{c + d}{2} \right) + f \left( \frac{b}{2}, \frac{c + d}{2} \right) \right] \\
& - (1 - \lambda) \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy - (1 - \lambda) \frac{1}{b - a} \int_a^b f (x, \frac{c + d}{2}) dx \\
& - \frac{1}{b - a} \int_a^b \int_c^d f (x, y) dx dy & \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d |K(x) M(y)| \left| \frac{\partial^2 f}{\partial x \partial y} \right| (x, y) dy dx.
\end{align*}
\]
By using the change of variables $y = sd + (1 - s) c$, $(d - c) ds = dy$, we obtain

\[
\left| (1 - \lambda)^2 f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) + \lambda^2 f(a, c) + f(a, d) + f(b, c) + f(b, d) \right|
\]

\[
+ \lambda \left(1 - \lambda\right)\left[ f\left(\frac{a + b}{2}, c\right) + f\left(\frac{a + b}{2}, d\right) + f\left(a, \frac{c + d}{2}\right) + f\left(b, \frac{c + d}{2}\right) \right]
\]

\[
- (1 - \lambda) \frac{1}{b - a} \int_c^b f\left(\frac{a + b}{2}, y\right) dy - (1 - \lambda) \frac{1}{b - a} \int_a^b f\left(x, \frac{c + d}{2}\right) dx
\]

\[
- \lambda \frac{1}{2(d - c)} \int_a^b f(x, d) + f(x, c) dx - \lambda \frac{1}{2(d - c)} \int_c^d [f(a, y) + f(b, y)] dy
\]

\[
+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx
\]

\[
\leq \frac{d - c}{b - a} \int_a^b |K(x)| \left\{ \int_0^{\frac{1}{2}} \left( \frac{\lambda}{2} - s \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, sd + (1 - s) c) \right| ds \right.
\]

\[
+ \int_{\frac{1}{2}}^1 \left( s - \frac{\lambda}{2} \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, sd + (1 - s) c) \right| ds
\]

\[
+ \int_{\frac{1}{2}}^{1 - \frac{1}{2}} \left( 1 - \frac{\lambda}{2} - s \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, sd + (1 - s) c) \right| ds
\]

\[
+ \int_{1 - \frac{1}{2}}^1 \left( s - 1 + \frac{\lambda}{2} \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, sd + (1 - s) c) \right| ds \}
\]

Since $\left| \frac{\partial^2 f}{\partial x \partial y} \right|$ is convex function on the co-ordinates on $\Delta$, we have

\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d |K(x)M(y)| \left| \frac{\partial^2 f}{\partial x \partial y} (x, y) \right| dy dx
\]

\[
\leq \frac{d - c}{b - a} \int_a^b |K(x)| \left\{ \int_0^{\frac{1}{2}} s \left( \frac{\lambda}{2} - s \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, d) \right| ds + \int_{\frac{1}{2}}^1 \left( s - \frac{\lambda}{2} \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, c) \right| ds
\]

\[
+ \int_{\frac{1}{2}}^1 s \left( 1 - \frac{\lambda}{2} - s \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, d) \right| ds + \int_{\frac{1}{2}}^{1 - \frac{1}{2}} \left( 1 - s \right) \left( 1 - \frac{\lambda}{2} - s \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, c) \right| ds
\]

\[
+ \int_{1 - \frac{1}{2}}^1 s \left( s - 1 + \frac{\lambda}{2} \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, d) \right| ds + \int_{1 - \frac{1}{2}}^1 \left( s - 1 + \frac{\lambda}{2} \right) \left| \frac{\partial^2 f}{\partial x \partial y} (x, c) \right| ds \}\right\} dx.
By calculating the above integrals, we obtain

\[
\begin{align*}
&\left| (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \lambda^2 f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \\
&+ \frac{\lambda(1-\lambda)}{2} \left[ f \left( \frac{a+b}{2}, c \right) + f \left( \frac{a+b}{2}, d \right) + f \left( a, \frac{c+d}{2} \right) + f \left( b, \frac{c+d}{2} \right) \right] \\
&- (1-\lambda) \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy - (1-\lambda) \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx \\
&- \lambda \frac{1}{2(b-a)} \int_a^b \left[ f(x,d) + f(x,c) \right] dx - \lambda \frac{1}{2(d-c)} \int_c^d \left[ f(a,y) + f(b,y) \right] dy \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx \\
&\leq \frac{b-c}{b-a} \int_a^b \left| K(x) \right| \left\{ \left| \frac{\partial^2 f}{\partial x \partial y} (x,c) \right| + \left| \frac{\partial^2 f}{\partial x \partial y} (x,d) \right| \right\}.
\end{align*}
\]

By a similar argument for other integrals, by using the change of variable \( x = \lambda b + (1-\lambda)a, \ (b-a)dt = dx \) and convexity of \( \left| \frac{\partial^2 f}{\partial x \partial y} (x,y) \right| \) on the co-ordinates on \( \Delta \), we deduce the result. Which completes the proof. \( \square \)

**Remark 1.** If we choose \( \lambda = 1 \) in Theorem 5, we have the inequality (1.3).

**Remark 2.** If we choose \( \lambda = 0 \) in Theorem 5, we have the inequality (1.3).

**Remark 3.** If we choose \( \lambda = \frac{1}{4} \) in Theorem 5, we have the inequality (1.8).

**Theorem 10.** Let \( f : \Delta = [a,b] \times [c,d] \to \mathbb{R} \) be a differentiable function on \( \Delta \). If \( \left| \frac{\partial^2 f}{\partial x \partial y} \right| \) is convex function on the co-ordinates on \( \Delta \), then one has the inequality:

\[
\begin{align*}
&\left| (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \lambda^2 f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \\
&+ \frac{\lambda(1-\lambda)}{2} \left[ f \left( \frac{a+b}{2}, c \right) + f \left( \frac{a+b}{2}, d \right) + f \left( a, \frac{c+d}{2} \right) + f \left( b, \frac{c+d}{2} \right) \right] \\
&- (1-\lambda) \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy - (1-\lambda) \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx \\
&- \lambda \frac{1}{2(b-a)} \int_a^b \left[ f(x,d) + f(x,c) \right] dx - \lambda \frac{1}{2(d-c)} \int_c^d \left[ f(a,y) + f(b,y) \right] dy \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx \\
&\leq \frac{(b-a)(d-c)}{4(p+1)^2} \left[ 2\lambda^2 - 2\lambda + 1 \right]^2 \\
&\times \left( \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a,c) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a,d) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b,c) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b,d) \right)^{\frac{1}{q}}
\end{align*}
\]

for \( q > 1 \), where \( q = \frac{p}{p-1} \).
Proof. Let $p > 1$. From Lemma 1 and using the H"{o}lder inequality for double integrals, we can write

\[
\begin{align*}
&\left| (1-\lambda)^2 f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \lambda^2 f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \\
&\quad + \frac{\lambda(1-\lambda)}{2} \left[ f\left(\frac{a+b}{2},c\right) + f\left(\frac{a+b}{2},d\right) + f\left(a,\frac{c+d}{2}\right) + f\left(\frac{b+c}{2}\right) \right] \\
&\quad - (1-\lambda) \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2},y\right) dy - (1-\lambda) \frac{1}{b-a} \int_a^b f(x,\frac{c+d}{2}) dx \\
&\quad - \lambda \frac{1}{2(b-a)} \int_a^b \int_c^d |K(x)M(y)|^p dy dx - \lambda \frac{1}{2(d-c)} \int_c^d \int_d^c |K(x)M(y)|^p dy dx \\
&\quad + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dxdy \\
\leq &\frac{1}{(b-a)(d-c)} \left( \int_a^b \int_c^d |K(x)M(y)|^p dy dx \right)^{\frac{1}{q}} \left( \int_a^b \int_c^d \left| \frac{\partial^2 f}{\partial x \partial y}(x,y) \right|^q dy dx \right)^{\frac{1}{q}}.
\end{align*}
\]

Since $\left| \frac{\partial^2 f}{\partial x \partial y} \right|$ is convex function on the co-ordinates on $\Delta$, by taking into account the change of variable $x = tb + (1-t)a$, $(b-a)dt = dt$ and $y = sd + (1-s)c$, $(d-c)ds = dy$, we have

\[
\left| \frac{\partial^2 f}{\partial x \partial y}(tb + (1-t)a, y) \right|^q \leq t \left| \frac{\partial^2 f}{\partial x \partial y}(b, y) \right|^q + (1-t) \left| \frac{\partial^2 f}{\partial x \partial y}(a, y) \right|^q
\]

and

\[
\begin{align*}
&\left| \frac{\partial^2 f}{\partial x \partial y}(tb + (1-t)a, sd + (1-s)c) \right|^q \\
\leq &\ t s \left| \frac{\partial^2 f}{\partial x \partial y}(b, d) \right|^q + t (1-s) \left| \frac{\partial^2 f}{\partial x \partial y}(b, c) \right|^q \\
&\quad + s (1-t) \left| \frac{\partial^2 f}{\partial x \partial y}(a, d) \right|^q + (1-t) (1-s) \left| \frac{\partial^2 f}{\partial x \partial y}(a, c) \right|^q
\end{align*}
\]
Thus, we obtain
\[
\left| (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \lambda^2 f(a, c) + f(a, d) + f(b, c) + f(b, d) \right|
\]
\[\begin{align*}
+ & \frac{\lambda(1 - \lambda)}{2} \left[ f \left( \frac{a + b}{2}, c \right) + f \left( \frac{a + b}{2}, d \right) + f \left( \frac{a}{2}, \frac{c + d}{2} \right) + f \left( \frac{b}{2}, \frac{c + d}{2} \right) \right] \\
- & (1 - \lambda) \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy - (1 - \lambda) \frac{1}{b - a} \int_a^b f(x, \frac{c + d}{2}) dx \\
- & \lambda \frac{1}{2(6 - a)} \int_a^b f(x, d) + f(x, c) dx - \lambda \frac{1}{2(d - c)} \int_c^d [f(a, y) + f(b, y)] dy \\
+ & \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx
\end{align*}\]
\[
\leq \frac{(b - a)(d - c)}{4(p + 1)^2} [2\lambda^2 - 2\lambda + 1]^2
\]
\[
\frac{1}{4} \left( \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^q (a, c) + \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^q (a, d) + \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^q (b, c) + \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^q (b, d) \right) ^{\frac{1}{q}}.
\]
Which completes the proof. \(\square\)

**Remark 4.** Under the assumptions of Theorem 6, if we choose \(\lambda = 1\), we have the inequality \((1.3)\).

**Remark 5.** Under the assumptions of Theorem 6, if we choose \(\lambda = 0\), we have the inequality \((1.0)\).

**Corollary 1.** Under the assumptions of Theorem 6, if we choose \(\lambda = \frac{1}{4}\), we have the inequality
\[
\left| f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + 4f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + 4f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right|
\]
\[\begin{align*}
+ & \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{36} + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx - \lambda
\end{align*}\]
\[
\leq \frac{25(b - a)(d - c)}{324(p + 1)^2}
\]
\[
\times \left( \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^q (a, c) + \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^q (a, d) + \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^q (b, c) + \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^q (b, d) \right) ^{\frac{1}{q}}.
\]
where
\[
A = \frac{1}{6(b - a)} \int_a^b \left[ f(x, c) + 4f \left( \frac{x + d}{2} \right) + f(x, d) \right] dx
\]
\[\begin{align*}
+ & \frac{1}{6(d - c)} \int_c^d \left[ f(a, y) + 4f \left( \frac{a + b}{2}, y \right) + f(b, y) \right] dy.
\end{align*}\]
Corollary 2. In Corollary 1, since $\frac{1}{4} < \frac{1}{(p+1)^p} < 1$, for $p > 1$, we have the following inequality:

$$
\left| f\left(\frac{a+c}{2}\right) + f\left(\frac{b,c+d}{2}\right) + 4f\left(\frac{a+b+c}{2}\right) + f\left(\frac{a+b}{2},c\right) + f\left(\frac{a+b}{2},d\right) \right|
$$

$$\leq \frac{25 (b-a) (d-c)}{324} \times \left( \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a,c) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a,d) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b,c) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b,d) \right)^{\frac{1}{q}}.
$$

Theorem 11. Let $f: \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a differentiable function on $\Delta$. If $\left| \frac{\partial^2 f}{\partial x \partial y} \right|^q$ is convex function on the co-ordinates on $\Delta$ and $q \geq 1$, then one has the inequality:

$$
\left| (1-\lambda)^2 f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + \lambda^2 f\left(\frac{a+b}{2},c\right) + f\left(a,d\right) + f\left(b,c\right) + f\left(b,d\right) \right|
$$

$$\leq \frac{b-a)(d-c)}{16} \left[ 2\lambda^2 - 2\lambda + 1 \right] \times \left( \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a,c) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a,d) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b,c) + \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b,d) \right)^{\frac{1}{q}}.
$$
Proof: From Lemma 1 and using the well-known Power-mean inequality, we can write

\[
\begin{align*}
(1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \lambda^2 & \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \\
+ \lambda(1 - \lambda) & \left[ f\left( \frac{a + b}{2}, c \right) + f\left( \frac{a + b}{2}, d \right) + f\left( \frac{a + c}{2}, d \right) + f\left( \frac{b + c}{2}, d \right) \right] \\
& - (1 - \lambda) \frac{1}{d - c} \int_c^d f\left( \frac{a + b}{2}, y \right) dy - (1 - \lambda) \frac{1}{b - a} \int_a^b f\left( x, \frac{c + d}{2} \right) dx \\
& - \lambda \frac{1}{2(b - a)} \int_a^b \int_c^d \left| f(x, d) + f(x, c) \right| dx - \lambda \frac{1}{2(d - c)} \int_c^d \left[ f(a, y) + f(b, y) \right] dy \\
& + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx \\
\leq \frac{1}{(b - a)(d - c)} & \left( \int_a^b \int_c^d |K(x)M(y)| dy dx \right)^{1 - \frac{1}{q}} \\
& \times \left( \int_a^b \int_c^d |K(x)M(y)| \left| \frac{\partial^2 f}{\partial x \partial y} (x, y) \right|^q dy dx \right)^{\frac{1}{q}}.
\end{align*}
\]

Since \( \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q \) is convex function on the co-ordinates on \( \Delta \), by taking into account the change of variable \( x = tb + (1 - t)a, (b - a) dt = dt \) and \( y = sd + (1 - s)c, (d - c) ds = dy \), we have

\[
\left| \frac{\partial^2 f}{\partial x \partial y} (tb + (1 - t)a, y) \right|^q \leq t \left| \frac{\partial^2 f}{\partial x \partial y} (b, y) \right|^q + (1 - t) \left| \frac{\partial^2 f}{\partial x \partial y} (a, y) \right|^q
\]

and

\[
\left| \frac{\partial^2 f}{\partial x \partial y} (tb + (1 - t)a, sd + (1 - s)c) \right|^q \leq ts \left| \frac{\partial^2 f}{\partial x \partial y} (b, d) \right|^q + t(1 - s) \left| \frac{\partial^2 f}{\partial x \partial y} (b, c) \right|^q \\
+ s(1 - t) \left| \frac{\partial^2 f}{\partial x \partial y} (a, d) \right|^q + (1 - t)(1 - s) \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q.
\]
Corollary 3. Under the assumptions of Theorem 7, if we choose \( \lambda = 1 \), we have

\[
\left| (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \lambda^2 f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| + \frac{1}{2} (b - a) \int_c^d f(x, y) dy - \lambda \frac{1}{2(d - c)} \int_c^d [f(x, y) + f(b, y)] dy
\]

\[
- \frac{1}{16} [2\lambda^2 - 2\lambda + 1]^2 \times \left( \frac{\partial^2 f}{\partial x \partial y} \right)^q (a, c) + \frac{\partial^2 f}{\partial x \partial y} \left( a, d \right) \right)^q (a, d) + \frac{\partial^2 f}{\partial x \partial y} \left( b, c \right) + \frac{\partial^2 f}{\partial x \partial y} \left( b, d \right) \right)^q (b, d).
\]

Which completes the proof. \( \square \)

Remak 6. Under the assumptions of Theorem 7, if we choose \( \lambda = 1 \), we have the inequality \((1.3)\).

Remark 7. Under the assumptions of Theorem 7, if we choose \( \lambda = 0 \), we have the inequality \((1.7)\).

Corollary 3. Under the assumptions of Theorem 7, if we choose \( \lambda = \frac{1}{4} \), we have

\[
\left| f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f\left( b, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, c \right) + f \left( \frac{a + b}{2}, d \right) \right|
\]

\[
+ \frac{1}{9} \int_c^d f(x, y) dy - \frac{1}{36} \int_a^d f(x, y) dy dx - A
\]

\[
\leq \frac{25(b - a)(d - c)}{36^2}
\]

\[
\times \left( \frac{\partial^2 f}{\partial x \partial y} \right)^q (a, c) + \frac{\partial^2 f}{\partial x \partial y} \left( a, d \right) + \frac{\partial^2 f}{\partial x \partial y} \left( b, c \right) + \frac{\partial^2 f}{\partial x \partial y} \left( b, d \right) \right)^q (b, d).
\]

where

\[
A = \frac{1}{6(b - a)} \int_a^b \left[ f(x, c) + 4f \left( x, \frac{c + d}{2} \right) + f(x, d) \right] dx
\]

\[
+ \frac{1}{6(d - c)} \int_c^d \left[ f(a, y) + 4f \left( \frac{a + b}{2}, y \right) + f(b, y) \right] dy.
\]

References

[1] M.K. Bakula and J. Pečarić, On the Jensen's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Math.*, 5, 2006, 1271-1292.
[2] M.E. Özdemir, E. Set, M.Z. Sarıkaya, Some new Hadamard’s type inequalities for co-ordinated $m$–convex and $(\alpha, m)$–convex functions, Accepted.

[3] M.Z. Sarıkaya, E. Set, M. Emin Özdemir and S.S. Dragomir, New some Hadamard’s type inequalities for co-ordinated convex functions, Accepted.

[4] S.S. Dragomir, On Hadamard’s inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Math.*, 5, 2001, 775-788.

[5] D. Y. Hwang, K. L. Tseng and G. S. Yang, Some Hadamard’s inequalities for co-ordinated convex functions in a rectangle from the plane, *Taiwanese Journal of Mathematics*, 11 (2007), 63-73.

[6] M.E. Özdemir, H. Kavurmacı, A.O. Akdemir and M. Avcı, Inequalities for convex and $s$–convex functions on $\Delta = [a, b] \times [c, d]$, Submitted.

[7] M.E. Özdemir, A.O. Akdemir and H. Kavurmacı, On the Simpson’s inequality for co-ordinated convex functions, Submitted.

Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey

E-mail address: emos@atauni.edu.tr

E-mail address: yildizc@atauni.edu.tr

★ Ağrı İbrahim Çeçen University, Faculty of Science and Letters, Department of Mathematics, 04100, AĞRI, Turkey

E-mail address: ahmetakdemir@agri.edu.tr