GAC, SAVINGS, AND UNBOUNDED INPUTS

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ABSTRACT. Let a control system and a target be given on an open subset of an Euclidean space. The existence of a Control Lyapunov Function – namely a positive definite, semi-concave, solution of the Hamilton-Jacobi inequality corresponding to the control vector field – guarantees Global Asymptotic Controllability (GAC). In this case, however, minimization is not an issue. Instead, if a Lagrangean with non-negative values is considered as well, an optimal control problem can be defined in relation to the corresponding integral functional.

In the first part of the present paper we show that the existence of a Minimum Restraint Function – a solution of a strict Hamilton-Jacobi inequality involving the Lagrangian and a non-negative savings multiplier – provides not only global asymptotic controllability but also savings, namely a state-dependent upper bound for the infima. This extends a former result, where the control set was assumed to be compact. Here we allow unbounded controls and replace inputs' values' compactness with a quite mild hypothesis concerning the dependence of the data on inputs: such condition is met, for instance, by control vector fields that are compositions of Lipschitz maps with polynomials and exponentials of the control variable.

In the second part of the paper we focus on the case when the dynamics is a polynomial in the control variable. Through some analysis of convexity properties of vector-valued polynomials’ ranges, we prove some simplified versions of the main result, in terms of either affine representability or reduction to weak subsystems for the original dynamics.

1. Introduction

1.1. The general case.

Let us consider an optimal control problem of the form

$\dot{x} = F(x, u); \quad x(0) = z \in \Omega \setminus C; \quad (1)$

$W(z) := \inf_{(x,u)} \mathcal{I}(x, u); \quad \mathcal{I}(x, u) := \int_{[0,T_{z,u}]} l(x(t), u(t)) dt; \quad (2)$

where: (i) the state $x$ ranges over $\Omega \setminus C$, $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) being a open subset and $C \subset \Omega$ being a closed target with compact boundary; (ii) $x(\cdot)$ is a trajectory corresponding to the control $u(\cdot)$ and such that $x(0) = z$; (iii) $T_{z,u}$ (possibly equal to $+\infty$) is the least time for
x to approach the target C; and (iv) the Lagrangean l verifies
\[ l(x, u) \geq 0 \]
for all \((x, u) \in (\Omega \setminus C) \times U\).

We will first focus on the unboundedness of control set U. Secondly, we shall specialize dynamics that are on polynomial in the control \(u\). Let us point out that, in connection with the investigation of uniqueness and regularity of solutions for Hamilton-Jacobi equations, dynamics and Lagrangeans with unbounded controls and polynomial growth have been addressed in [23], by embedding the problem in a space-time problem through techniques of graph’s reparameterization – see e.g. [4, 5, 9, 13, 20, 27, 26, 31]. With similar arguments (see also [19]) necessary conditions for the existence of (possibly impulsive) minima of input-polynomial optimal control problems have been studied in [22].

Our main aim consists in establishing sufficient conditions that guarantee, simultaneously, Global Asymptotic Controllability (GAC)\(^1\) and a continuous bound for the infimum value \(W(z)\). Under the additional assumption that the control ranges over a compact subset, the problem has been investigated in [22], where the existence of a special kind of Lyapunov Function, the Minimum Restraint Function, has been introduced as a sufficient condition for both GAC and value upper estimate, even in the case when classical transversality conditions for the corresponding boundary value problem are not met, as in the so-called cheap control problems – see e.g. [3, 6, 14, 15, 16, 21, 29]. In Section 3 we shall extend the results of [22] to unbounded Lagrangean-dynamics pairs. To compensate the lack of compactness generated by the inputs’ unboundedness, here we shall introduce a state-dependent rescaling, which in fact allows us to treat a class of problems which is much wider then those with polynomial growth. More precisely, we shall assume the following hypothesis:

**Hypothesis A:** For every compact subset \(K \subset \Omega \setminus C\) the function
\[ (\mathcal{L}, \mathcal{F})(x, u) := \frac{(l, \mathcal{F})}{1 + |(l, \mathcal{F})(x, u)|}(x, u) \]
is uniformly continuous\(^2\) on \(K \times U\).

Let us immediately observe that Hypothesis A is quite weak: in particular it allows for a vast class of control fields \((l, \mathcal{F})(x, u)\), which contains not only the \((x\)-dependent\) polynomials in \(u_1, \cdots, u_m, |u_1|, \cdots, |u_m|, |u|\) but also compositions of polynomials with exponential and Lipschitz continuous functions.

In particular, it is easy to check that Hypothesis A’ below is sufficient for Hypothesis A to hold true:

**Hypothesis A’:** \((l, \mathcal{F})\) is continuous with respect to the state variable \(x\) an locally Lipschitz with respect to the control variable \(u\) and there exists a continuous function

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\(^1\)Namely, the task of approaching a target C, possibly in infinite time, with trajectories ranging on suitable sublevels determined by the initial positions \(z \in \Omega \setminus C\), see Definition 3.5.

\(^2\)Notice, incidentally, that we make no assumptions implying uniqueness of a solution of \(\dot{x} = \mathcal{F}(x, u)\) \(x(0) = z\) for a given control \(u\).
\[ \eta : \Omega \setminus C \to [0, +\infty] \text{ such that} \]

\[
\left| \frac{D_u(l, F)}{(1 + |l(F)|)^2} \right| (x, u) \leq \eta(x) \quad \text{for almost every } (x, u) \in (\Omega \setminus C) \times U. \tag{4}
\]

Let us neglect several technical details, including the definition of GAC (see Definition 3.5), and let us bring forward the statement of Theorem 3.1 (here labeled Theorem 1.1). For this purpose we need to introduce the Hamiltonian

\[ H_{l,F}(x, p_0, p) := \inf_{u \in U} \left\langle (p_0, p), (l(x, u), F(x, u)) \right\rangle \tag{5} \]

and to give the notion of Minimum Restraint Function:

**Definition 1.1.** \(^3\) Let \( V : \Omega \setminus C \to [0, +\infty) \) be a locally semiconcave, positive definite, proper function. We say that \( V \) is a Minimum Restraint Function (in short, MRF) for \((l, F, C)\) if there exists a savings multiplier \( \bar{p}_I \in [0, +\infty) \) such that

\[
\max_{p \in D^* V(x)} H_{l,F}(x, \bar{p}_I, p) < 0 \quad \forall x \in \Omega \setminus C. \tag{6}
\]

**Theorem 1.1.** Let \( V \) be a Minimum Restraint Function for the problem \((l, F, C)\) and assume Hypothesis A. Then:

(i) the system \( F \) is globally asymptotically controllable to \( C \);

(ii) if \( V \) has savings multiplier \( \bar{p}_I > 0 \), then

\[ W(x) \leq \frac{V(x)}{\bar{p}_I} \quad \forall x \in \Omega \setminus C. \tag{7} \]

**Remark 1.1.** Let us notice, incidentally, that by taking \( l \equiv 0 \), one obtains a Lyapunov-Function-type result for GAC with unbounded dynamics. Let us also point out that, both in the bounded and in the unbounded case, an approach based on Minimum Restraint Functions likely proves useful for a generalization of feedback stabilization results (see e.g. [2, 8, 17] and references therein).

### 1.2. The case of control-polynomial dynamics.

In Sections 4 and 5 we will specialize our investigation to dynamics which are polynomial in the control \( u \in U \subseteq \mathbb{R}^m \), where \( U \) is allowed to be unbounded, in particular equal to \( \mathbb{R}^m \). Namely

\[ F(x, u) := f(x) + \sum_{\alpha=1}^{m} u_{\alpha_1} g_{\alpha_1}(x) + \cdots + \sum_{\alpha_1 \leq \cdots \leq \alpha_d} u_{\alpha_1} \cdots u_{\alpha_d} g_{\alpha_1 \cdots \alpha_d}(x). \]

Actually, a careful investigation of elementary algebraic properties of the convex hull \( \text{co} F(x, \mathbb{R}^m) \) proves essential for the application of the general result to the polynomial case.

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\(^3\)Angle brackets are more commonly used for inner products, while here they denote the standard duality on \((\mathbb{R}^{1+n})^* \times \mathbb{R}^{1+n}\). We avoid the use of the more standard square brackets to rule out any confusion with Lie bracketing.

\(^4\)See the precise concept in Definition 3.4 where, as soon as \( \Omega \subseteq \mathbb{R}^n \), one also posits \( V_0 \in \mathbb{R} \cup \{+\infty\} \) such that \( V(\Omega \setminus C) \leq V_0 \) and \( \lim_{n \to \infty} V(x_n) = V_0 \), for every sequence \((x_n)\) in \( \Omega \) converging to a point of \( \partial \Omega \).
Let us point out that an analogous study, focusing upon the interplay between convexity and polynomial dependence of both the dynamics and the Lagrangean, has been pursued, notably in [25], in order to establish existence of optimal solutions. Before developing the theory, we will begin, in Section 2, with a fully worked out toy problem where the dynamics depends quadratically on the control: rather than showing a sophisticated application of the theory, with this example we aim to acquaint the reader, in a trivialized setting, with most basic dynamical features due to the input-polynomial dependence.

**Balanced systems.** As a natural instance where the algebraic structure may be exploited to recover a cleaner picture of the dynamics, in Section 4 we will consider a class of control-polynomial systems which can be “represented” (in a sense to be made precise) by control-affine systems with controls ranging in a neighborhood of the origin. It is sufficient to consider the system
\[ \dot{x} = f(x) + g_1(x)w_1 + g_2(x)w_2 + g_3(x)w_3 \]
\[ (w_1, w_2, w_3) \in \mathbb{R}^3 \]
(see Theorem 4.1). If a system is representable as an affine system, via closure-of-trajectories arguments one can import various controllability results valid for control-affine systems. In particular, we shall prove that balanced systems (see Section 4) can be actually represented as control-affine systems.

**Weak subsystems.** Another direction in the sense of simplifying the search of a MRF will be pursued in Section 5, where one relies on weak subsystems, which are in fact parameterized selections of the set-valued function \( x \mapsto \text{co} F(x, \mathbb{R}^m) \). Specifically, we single out the maximal degree subsystem and, for any \( \lambda \) in the \( m \)-dimensional unit simplex, the \( \lambda \)-diagonal subsystems (see Definition 5.1 and Section 5.1, respectively). The idea of utilizing subsystems might sound contradictory: in principle, the smaller is the quantity of available velocity directions, the more unlikely the discovery of a MRF will result. On the other hand, the diminished complexity of the dynamics might ease the guess of a MRF, which would automatically be a MRF for the original problem. To give a hint of the results which can be obtained through parameterized, set-valued selections of \( x \mapsto \text{co} F(x, \mathbb{R}^m) \), let us just anticipate the result (see Theorem 5.1) concerning the maximal degree subsystem \( F_{\text{max}} \), which is defined as
\[ F_{\text{max}}(x, u) := f(x) + \sum_{\alpha_1 \leq \cdots \leq \alpha_d} u_{\alpha_1} \cdots u_{\alpha_d} g_{\alpha_1 \cdots \alpha_d}(x). \]

**Theorem 1.2.** Let us assume Hypothesis \( A_{\text{max}} \) (see Section 5.1) on the Lagrangean \( l \) and let us posit the existence of a Minimum Restraint Function \( V \) for the problem \((l, F_{\text{max}}, C)\) with savings multiplier \( \bar{p}_I \). Then \( V \) is a Minimum Restraint Function for the original system \((l, F, C)\) as well, with the same savings multiplier \( \bar{p}_I \).

\(^5\)The adjective weak refers to the fact that in general they fail to be actual selection of \( F(x, \mathbb{R}^m) \).
Perhaps it is worthwhile pointing out a significative similarity between Theorem 1.2 and some results in [5], which provide an explanation of certain counterintuitive phenomena in mechanics, like the stabilizability of equilibria (and even of some non equilibrium states) of a pendulum with oscillating pivot.

2. A WORKED OUT TOY EXAMPLE

To see how some of the theoretical material below works, let us examine a very simple, fully computable, example. Obviously, many non trivial difficulties of the general case are left aside – e.g. for here the control is scalar-valued. Actually, this section can be read with almost no knowledge of the next ones. More complex examples will be illustrated amid the statements of the major results.

Consider the control system
\[ \dot{x} = F(x, u) := f(x) + g(x)u + h(x)u^2 \]
where \( x = (x_1, x_2)^T \in \mathbb{R}^2 \), \( f(x) = -h(x) = x \), \( g(x) = (x_2, -x_1)^T \), and \( u \in \mathbb{R} \).

The solution issuing from a point \( z \) corresponding to a \( L^2 \) control \( u : [0, +\infty[ \to \mathbb{R} \) is given by
\[
x(t) = e^{\int_{[0,t]}(1-u^2(s))ds} \begin{pmatrix} z_1 \cos(v(t)) - z_2 \sin(v(t)) \\ z_1 \sin(v(t)) + z_2 \cos(v(t)) \end{pmatrix},
\]
where we have set
\[
v(t) := \int_{[0,t]} u(s)ds \quad \forall t \in [0, +\infty[.
\]
Notice that
\[ |x(t)| = |z|e^{\int_{[0,t]}(1-u^2(s))ds}. \]

In points (i)-(v) below we associate some integral functionals and targets with equation [8].

(i) Let us choose a real number \( \rho > 0 \) and the unit circle
\[ C := \{ x \in \mathbb{R}^2 \mid |x| = \rho \} \]
be a target. Consider the Lagrangean \( l(x, u) = u^2 \), so that the corresponding cost functional reads
\[ I(x, u) = \int_{[0,T_{z,u}]} u^2(s)ds. \]
Let us regard the system as defined on \( \Omega := \mathbb{R}^2 \setminus \{0\} \). Let us observe that, because of radial symmetry, the guess of a MRF function for the system governed by the maximal degree subsystem
\[ \dot{x} = F^{max}(x, u) = x(1 - u^2) \]
might be easier than the search of a MRF for the original system. In fact, the Hamiltonian corresponding to the dynamics $F_{\text{max}}$ is

$$H_{l,F_{\text{max}}}(x,p_I,p) = \inf_{u \in \mathbb{R}} \{ \bar{p}_I(1 + u^2) + \langle p_I, x \rangle(1 - u^2) \}.$$  

For instance, it is natural to look for a $\bar{p}_I > 0$ such that the map

$$V(x) = |x| - \rho \quad x \in \mathbb{R} \backslash \{0\}$$

is a MRF for the problem $(l,F_{\text{max}},C)$ with savings multiplier $\bar{p}_I$ (notice in particular that $V$ tends to $V_0 = \rho$ along any sequence approaching $\partial \Omega = \{0\}$). Actually, it is trivial to see that $V$ is a MRF with savings multiplier $\bar{p}_I$ provided

$$\bar{p}_I < \rho. \quad (10)$$

Furthermore, observe that the Lagrangean $l(x,u)$ satisfies Hypothesis $A_{\text{max}}$ with $L = 1$. Therefore, in view of Theorem 5.1 the system $F$ is globally asymptotically controllable to $C$, and $W(x) \leq \frac{V(x)}{\bar{p}_I} \bar{p}_I$ for all $x \in \Omega \setminus C$, where $W$ is the value function, namely $W(z) := \inf_{(x,u)} I(x,u)$.

(ii) Let us consider the same control system $[8]$, this time defined on the whole $\mathbb{R}^2$, with the same cost functional, but let us modify the target by setting $C := \{0\}$.

Making the radius $\rho$ of the target in the previous example going to zero one would be tempted to try the function $V(x) = |x|$ as a MRF function. However a MRF with a positive savings multiplier fails to exist in this case, for this would provide a bound for the infimum value of the functional: instead, in view of (9) for every control such that the trajectory approach the origin at $T_{z,u}$, the value of the functional diverges to $+\infty$.

(iii) Let us try to understand how minimizing sequences of controls are made.

In case (i), if $z$ is inside the disc, namely $|z| < \rho$, one simply implements the control $u = 0$ and let the state evolve according to the radial dynamics $\dot{x} = x$ until it meets the circle.

Instead, if $z$ is outside the circle, namely $|z| > \rho$, one will try to push the state along the radial direction so that to diminish its norm to $\rho$. Intuitively this can be better and better done by implementing controls $(u_n)_{n \in \mathbb{N}}$ defined on a interval $[0,T]$ which switch faster and faster between values $a$ and $-a$ ($a > 0$), so that $\int_{[0,T]} u_n dt = 0$ and

$$|u_n|^2 = Ta^2 = T - \log(\rho/|z|). \quad (11)$$

The limit behavior of the corresponding solutions is a trajectory that goes along the radial direction towards the origin and reaches the circle at time $T$. Notice that, while the time $T$ can be arbitrarily small, in order that (11) makes sense one must have $a > 1$ (and the smaller is the time $T$ the larger is the constant $a$). This is consistent with intuition: while $\int_{[0,T]} u_n dt$ must be negligible in order to annihilate the transverse component of the velocity, $a$ must be greater than 1 in order to overcome the radial component of the velocity,
Figure 1. This figure represents the parabola $F(x, [-1.5, 1.5])$ and its convex hull when $x = (1, 1)^T$. Notice in particular that, for every $x \in \mathbb{R}^2 \setminus \{0\}$, $co F(x, [-1.5, 1.5])$ is a neighborhood of the origin.

namely $f(x) - u_n^2 g(x)$ needs to be directed towards the origin. This also matches with the consideration that a sequence of solutions tends to a solution of the convexified dynamics: indeed the condition $a > 1$ characterizes the circumstance that the set $co F(x, [-a, a])$ contains a vector of the form $-\eta x$, $\eta > 0$ — see Figure 1.

The fact that in case (ii) the functional blows up when reaching the origin is saying that one can still implement fast switching controls to go towards the origin, but the cost $I$ one has to pay to do it is infinite.

(iv) In both cases (i) and (ii), if one agrees to spend a non-zero time $T$ to reach the target from outside, then controls can be taken uniformly bounded in $L^\infty$.

Instead, if we replace the functional $I$ with

$$\tilde{I}(x, u) := \int_{0, T_{z, u}} (1 + u^2) dt = T_{z, u} + \int_{0, T_{z, u}} u^2 dt$$

we can make the time $T_{z, u}$ to tend to zero at the condition of taking larger and larger controls. In fact it can be easily checked that (in both cases (i) and (ii)) if $z \neq 0$ then any minimizing sequence $(u_n)$ is such that

$$|u_n|_\infty \to +\infty.$$

Furthermore, in case (ii), when the target is the origin, one also has

$$|u_n|_2 \to \infty.$$

(v) It may even happen that Theorem 5.1 determines the value function: if $C := \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ and $\bar{T} := \int_{0, T_{z, u}} dt = T_{z, u}$ is simply the time to reach the target, it is easy to verify that $V(x) := |x| - 1$ for all $x \in \mathbb{R} \setminus \{0\}$ is a MRF with a savings multiplier $\bar{p}_T$ arbitrarily large. Therefore, by Theorem 5.1 it follows, in particular, that the value function, i.e. the minimum time, is bounded by $\tilde{V}(x)/\bar{p}_T$ for any positive $\bar{p}_T$. Therefore the minimum time is identically equal to zero: from any initial state there exist trajectories reaching the target with arbitrarily small time, which matches with the argument in (iii).

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This is due to invariance of the pair dynamics-functional up to reparameterization.
(iv) It is possible that one looses some important information when exploiting only the subsystem \( \dot{x} = F_{\text{max}}(x,u) \). Indeed the state might need to move along circles, as in the case when the initial position is \( z = (1,0)^{tr} \) and the target is, say,
\[
C := \{(x_1,x_2)^{tr} \mid (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1/2\}.
\]
In this case the function \( V(x) = (d(x,C))^2 \) is a MRF function for the original system, but it is not a MRF for the subsystem \( \dot{x} = F_{\text{max}}(x,u) \). Notice that in order to move along circles it is sufficient to implement controls \( u \) with values in \( \{-1,1\} \).7

3. GAC and Cost Estimate

3.1. Preliminary concepts and notation. Let us recall some basic notions, which will be needed to express the results of the paper.

**Definition 3.1** (Positive definiteness and proper functions). Let \( \Omega, C \subset \mathbb{R}^n \) respectively be an open and a closed set. Let \( V : \Omega \setminus C \to \mathbb{R} \) (where \( \Omega \setminus C \) denotes the closure of \( \Omega \setminus C \) in the relative topology of \( \Omega \)) be a continuous function. Then \( V \) is positive definite on \( \Omega \setminus C \) if \( V(x) > 0 \) for all \( x \in \Omega \setminus C \) and \( V(x) = 0 \) for all \( x \in \partial C \).

Moreover the function \( V \) is called proper on \( \Omega \setminus C \) if the pre-image \( V^{-1}(K) \) of any compact set \( K \subset [0, +\infty) \) is compact.

**Definition 3.2** (Semiconcavity). Let \( \Omega \subset \mathbb{R}^n \) be an open set, and let \( V : \Omega \to \mathbb{R} \) be a continuous function. \( V \) is said to be locally semiconcave on \( \Omega \) if for any point \( x \in \Omega \) there exist \( R > 0 \) and \( \rho > 0 \) such that

\[
V(z_1) + V(z_2) - 2V\left(\frac{z_1 + z_2}{2}\right) \leq R|z_1 - z_2|^2 \quad \forall z_1, z_2 \in B_n(x, \rho).
\]

Let us remind that locally semiconcave functions are locally Lipschitz. Actually, they are twice differentiable almost everywhere (see e.g. [7]).

**Definition 3.3** (Limiting gradient). Let \( \Omega \subset \mathbb{R}^n \) be an open set, and let \( V : \Omega \to \mathbb{R} \) be a locally Lipschitz function. For every \( x \in \Omega \) let us set

\[
D^*V(x) := \left\{ w \in \mathbb{R}^n \mid w = \lim_{k \to \infty} \nabla V(x_k), \quad x_k \in \text{DIFF}(V) \setminus \{x\}, \lim_{k \to \infty} x_k = x \right\}
\]

where \( \nabla \) denotes the classical gradient operator and \( \text{DIFF}(V) \) is the set of differentiability points of \( V \). \( D^*V(x) \) is called the set of limiting gradients of \( V \) at \( x \).

**Remark 3.1.** For every \( x \in \Omega \), \( D^*V(x) \) is a nonempty, compact subset of \( \mathbb{R}^n \) (more precisely, of the cotangent space \( T^*_x \Omega \)). Notice that, in general, \( D^*V(x) \) is not convex. Actually, the convex hull \( \text{co} D^*V(x) \) coincides with Clarke’s generalized gradient.

Let us extend the definition of Minimum Restraint Function given in [22] to the case of unbounded control sets.

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7We point out that the goal of the whole example is stressing, in a simple situation, the role of the dynamics’ polynomial dependence on control \( u \). Actually, from the view point of general theory the example is trivial, since at every point \( x \neq 0 \) the system is locally controllable at the first-order: \( 0 \in \text{int}(\text{co} \mathcal{F}(x,U)) \) (see e.g. [20]).
Definition 3.4. Let $V : \Omega \setminus C \to [0, +\infty)$ be a locally semiconcave, positive definite, proper function. We say that $V$ is a Minimum Restraint Function for $(l, F, C)$ if $\overline{p_I} \in [0, +\infty]$ if
\begin{equation}
\max_{p \in D^*V(x)} H_{l,F}(x, \overline{p_I}, p) < 0 \quad \forall x \in \Omega \setminus C
\end{equation}
and, moreover, there is $V_0$, possibly equal to $+\infty$, such that
\begin{equation}
V(\Omega \setminus C) \leq V_0 \quad \text{and} \quad \lim_{n \to \infty} V(x_n) = V_0
\end{equation}
for every sequence $(x_n)$ in $\Omega$ converging to a point of $\partial \Omega$.

As customary, we shall use $KL$ to denote the set of all continuous functions
\begin{equation}
\beta : [0, +\infty] \times [0, +\infty] \to [0, +\infty]
\end{equation}
such that: (1) $\beta(0, t) = 0$ and $\beta(\cdot, t)$ is strictly increasing and unbounded for each $t \geq 0$; (2) $\beta(r, \cdot)$ is decreasing for each $r \geq 0$; (3) $\beta(r, t) \to 0$ as $t \to +\infty$ for each $r \geq 0$. For brevity, let us use the notation $d(x)$ in place of $d(x, C)$.

Definition 3.5. The system $(\overline{l}, \overline{F})$ is globally asymptotically controllable to $C$—shortly, $(\overline{l}, \overline{F})$ is GAC to $C$—provided there is a function $\beta \in KL$ such that, for each initial state $z \in \Omega \setminus C$, there exists an admissible trajectory-control pair $(x, u) : [0, +\infty] \to \mathbb{R}^n \times U$ that verifies
\begin{equation}
d(x(t)) \leq \beta(d(z), t) \quad \forall t \in [0, +\infty[. \tag{8}
\end{equation}

3.2. The main result. Let us recall our main hypothesis

Hypothesis A: For every compact subset $K \subset \Omega \setminus C$ the function
\begin{equation}
(\overline{l}, \overline{F})(x, u) := \frac{(l, F)}{1 + |(l, F)(x, u)|}(x, u)
\end{equation}
is uniformly continuous on $K \times U$.

This assumption, which allows for a wide class of problems (see Introduction) will be standing throughout the whole paper. The following result generalizes the MRF method (see Theorem 1.1 in [22]) to the case of unbounded controls.

Theorem 3.1. Let $V$ be a Minimum Restraint Function for the problem $(l, F, C)$. Then:

(i) the system $F$ is globally asymptotically controllable to $C$;

(ii) if $V$ has savings multiplier $\overline{p_I} > 0$, then
\begin{equation}
W(x) \leq \frac{V(x)}{\overline{p_I}} \quad \forall x \in \Omega \setminus C
\end{equation}
where $W$ is the value function, namely
\begin{equation}
W(z) := \inf_{(x,u)} \mathcal{I}(x, u).
\end{equation}

\begin{footnote}
By convention we establish that, if $T_{z,u} < +\infty$, the trajectory $x[z; u](t)$ is prolonged to $[0, +\infty[$, with $x(t) = \bar{z}$ for all $t \geq T_{z,u}$, where $\bar{z} := \lim_{t \to T_{z,u}^{-}} x[z; u](t)$.
\end{footnote}
Proof. Let us consider the $1 + n$-dimensional control vector field $(\bar{l}, \bar{F})$ defined in (3) and the rescaled optimal control problem $(\bar{l}, \bar{F}, C)$ defined as

$$y'(s) = \bar{F}(y, v) \quad y(0) = z;$$

(16)

$$\bar{W}(z) := \inf_{(y,v)} \mathcal{I}_{(\bar{l}, \bar{F})}(y, v) \quad \mathcal{I}_{(\bar{l}, \bar{F})}(y, v) := \int_{[0, S_{z,v}]} \bar{l}(y(s), v(s))ds$$

where the apex $'$ denotes differentiation with respect to the parameter $s$, and $S_{z,v} \in [0, +\infty]$ is the exit time (in $s$) from $\Omega \setminus C$.

The Hamiltonian to the problem $(\bar{l}, \bar{F})$ is given by

$$H_{\bar{l}, \bar{F}}(x, p_0, p) = \inf_{u \in U} \left\langle \left(p_0, p\right), \left(\bar{l}(x, u), \bar{F}(x, u)\right) \right\rangle.$$

Clearly, $H_{\bar{l}, \bar{F}}$ is continuous and verifies

$$|H_{\bar{l}, \bar{F}}(x, p_0, p)| \leq |(p_0, p)|$$

for all $(x, p_0, p) \in (\Omega \setminus C) \times \mathbb{R}^{1+n}$.

It is also trivial to check that, for every $(x, p_0, p) \in (\Omega \setminus C) \times \mathbb{R}^{1+n}$,

(17) $$H_{\bar{l}, \bar{F}}(x, p_0, p) < 0 \iff H_{\bar{l}, \bar{F}}(x, p_0, p) < 0.$$

In particular, the hypothesis that $V$ is a MRF for $(l, F, C)$ with savings multiplier $\bar{p}_T$ is equivalent to the fact that $V$ is a MRF for $(\bar{l}, \bar{F}, C)$ with savings multiplier $\bar{p}_T$. Moreover, because of Hypothesis A, the problem $(\bar{l}, \bar{F}, C)$ meets the hypotheses of Theorem 3.2 below. Therefore:

(i) the system $\bar{F}$ is globally asymptotically controllable to $C$, i.e. there exists a function $\beta \in KL$ such that, for every $z \in \Omega \setminus C$, there exists an admissible trajectory-control pair $(y, v) : [0, +\infty[ \to \Omega \times U$ for $F$ that verifies

(18) $$y(0) = z \quad d(y(s)) \leq \beta(d(z), s) \quad \forall s \in [0, +\infty[;$$

(ii) moreover, if $\bar{p}_T > 0$, then

(19) $$\bar{W}(z) \leq \frac{V(z)}{\bar{p}_T} \quad \forall z \in \Omega \setminus C.$$

For any $z \in \Omega \setminus C$ consider the pair $(y, v)$ whose existence is stated in (i), and set

$$t(s) := \int_{[0,s]} (1 + \|l(F)(y(\eta), v(\eta))\|^{-1}d\eta \quad \forall s \in [0, S_{z,v}]$$

$$x(t) := y \circ s(t) \quad u(t) := v \circ s(t) \quad \forall t \in [0, t(S_{z,v})]$$

where $s(\cdot)$ is the inverse of $t(\cdot)$ and $S_{z,v}$ is the $s$-exit time of the trajectory $y(\cdot)$. In view of Proposition 3.1 $(x, u)(\cdot)$ is a trajectory-control pair for $\dot{x} = F(x, u)$. Moreover, by (18)-(19) and the identity

(20) $$W = \bar{W}$$

one obtains

(21) $$x(0) = z \quad d(x(t)) \leq \beta(d(z), s(t)) \quad \forall t \in [0, +\infty[$$

(20) is a straightforward consequence of Proposition 3.1.
and, if $\bar{p}_I > 0$,

\begin{equation}
W(z) \leq \frac{V(z)}{\bar{p}_I} \quad \forall z \in \Omega \setminus C.
\end{equation}

Notice that $t(s) \leq s$ for all $s$, consequently $t \leq s(t)$ for all $t$. Since for every $z$ the map $\beta(z, \cdot)$ is decreasing, one gets

$$
\beta(z, s(t)) \leq \beta(z, t)
$$

for all $t$. Then, it follows by (18)

\begin{equation}
d(x(t)) \leq \beta(d(z), t) \quad \forall t \in [0, +\infty[
\end{equation}

so the theorem is proved. $\square$

The main step of the proof of Theorem 3.1 is based on Theorem 3.2 below, which concerns GAC and optimization for a Lagrangean-Dynamics pair $(l, F)$ defined on $(\Omega \setminus C) \times U$, the control set $U$ being possibly unbounded. More precisely we will assume the following boundedness uniform continuity hypothesis:

**Hypothesis $A_{UC}$** The vector field $(l, F)$ is continuous on $(\Omega \setminus C) \times U$ and, for every compact subset $K \subset \Omega \setminus C$, it is bounded and uniformly continuous on $K \times U$.

**Theorem 3.2.** Let us consider the exit time optimal control problem

$$
\dot{x} = F(x, u) \quad x(0) = z \quad W(z) := \inf_{(x, u)} \int_0^{T_{z, u}} l(x(t), u(t)) dt
$$

where we assume that $(l, F)$ satisfies Hypothesis $A_{UC}$. For a given, closed target $C$ with compact boundary, $T_{z, u}$ is the infimum time (possibly equal to $+\infty$) such that $\lim_{t \to T_{z, u}} d(x(t), C) = 0$.

Let $V$ be a Minimum Restraint Function for the problem $(l, F)$. Then:

(i) the system $F$ is globally asymptotically controllable to $C$;

(ii) moreover, if $V$ has savings multiplier $\bar{p}_I > 0$,

\begin{equation}
W(z) \leq \frac{V(z)}{\bar{p}_I} \quad \forall z \in \Omega \setminus C.
\end{equation}

We choose to omit here an explicit (and unavoidably technical) proof for the following reason: under the additional hypothesis that the control set $U$ is compact, Theorem 3.2 has been proved in [22, Theorem 1.1]; and, as it is trivial to verify, the utilization of the compactness of $U$ consists in the fact that hypothesis $A_{UC}$ turns out to be trivially verified; so, by simply replacing the compactness of $U$ with hypothesis $A_{UC}$ and repeating verbatim the arguments in [22], one gets a proof of Theorem 3.2.

We conclude this section by stating and briefly justifying a reparameterization result which has been used in the proof of Theorem 3.1.

Consider the rescaled optimal control problem

\begin{equation}
y'(s) = \bar{F}(y, v) \quad y(0) = z; \quad \bar{W}(z) = \inf_{(y, v)} \int_{[0, S_{z, v}]} \bar{l}(y(s), v(s)) ds
\end{equation}
where the apex ′ denotes differentiation with respect to the parameter \( s \), and \( S_{x,v} \in [0, +\infty] \) is the exit time (in \( s \)) from \( \Omega \setminus \mathcal{C} \).

Let \((y,v) : [0,S[ \mapsto (\Omega \setminus \mathcal{C}) \times U\) be a path, with \( y \) absolutely continuous and \( v \) Borel measurable. Set

\[
t(s) := \int_0^s w_0(y(\eta), v(\eta))d\eta \quad \forall s \in [0,S[
\]

\((x(t), u(t)) := (y, v) \circ s(t) \quad \forall t \in [0,T[\)

where \( T := t(S) \) and \( s : [0, S[ \mapsto [0, T[ \) denotes the inverse of \( t(\cdot) \).

**Proposition 3.1.** The path \((y,v)\) is a trajectory-control pair for \( \bar{F} \) if and only if \((x,u)\) is a trajectory-control pair for \( F \). Furthermore, in this case,

\[
\int_{[0,S]} \bar{l}(y(s), v(s))ds = \int_{[0,T]} l(x(t), u(t))dt.
\]

It follows, in particular, that

\[
W(z) = \bar{W}(z)
\]

for all \( z \in \Omega \setminus \mathcal{C} \).

**Proof.** Since \( t = t(s) \) is absolutely continuous and \( t'(s) > 0 \) almost everywhere, by M. A. Zareckii’s Theorem (see e.g. [24, Theorem 4, page 253] or, for a more general statement, [11, Theorem 2.10.13, page 177]) the inverse map \( s(\cdot) = t^{-1}(\cdot) \) is absolutely continuous. In particular, \( x = y \circ s \) is absolutely continuous, and \( u = v \circ s \) turns out to be Borel measurable as well. Hence the claim follows by some standard applications of the chain rule. \( \square \)

### 4. Control-polynomial systems: affine representability

In this section and in the next one we will assume the dynamics \( F \) to be a polynomial of degree \( d \geq 0 \) in the control variable \( u \):

\[
F(x, u) = f(x) + \sum_{\alpha=1}^{m} u_\alpha g_\alpha(x) + \cdots + \sum_{\alpha_1 \leq \cdots \leq \alpha_d} u_{\alpha_1} \cdots u_{\alpha_d} g_{\alpha_1 \cdots \alpha_d}(x)
\]

the functions \( f, g_\alpha, \ldots, g_{\alpha_1 \cdots \alpha_d} \) being continuous. We will develop the idea of getting advantage from a careful study of the vectogram’s convex hull, a task which, though not trivial, is natural because of the polynomial structure.

We address here the possibility of representing a control-polynomial system – actually, its convexification – by means of the control-affine dynamics

\[
F_{aff}(x, w) := f(x) + \sum_{\alpha_1} w_{\alpha_1} g_{\alpha_1}(x) + \cdots + \sum_{\alpha_1 < \cdots < \alpha_d} w_{\alpha_1 \cdots \alpha_d} g_{\alpha_1 \cdots \alpha_d}(x),
\]

\[\text{Notice that, coherently with the fact that we just assume continuity for both } F \text{ and } \bar{F}, \text{ no implication of solutions’ uniqueness for } \dot{x} = F \text{ or } \dot{y} = \bar{F} \text{ can be inferred from Proposition 3.1.}\]

\[\text{In some classical literature, as well as in some recent papers, objects akin to the convex hull of the image of the vector valued function that maps } u \in \mathbb{R}^m \text{ into the (suitably ordered) sequence of all monomials of } u \text{ up to the degree } d, \text{ are refereed as spaces of moments, see e.g. [11,10,13,25,26].}\]
where all inputs $w_{\alpha_1,\ldots,\alpha_k}$, with $k = 1, \ldots, d$, range in $\mathbb{R}$ and are mutually independent.

Obviously, such a representation in general is not valid, as shown for example by the trivial case $F(x, u) = g(x)u + h(x)u^2$.

We will show that this affine representation is valid for balanced systems (see Definition 4.1), where the only non-zero terms are those corresponding to control monomial such that each component $u_\alpha$ appears with degree equal either to 0 or to a fixed odd positive number $K_\alpha$. The advantage of a control-affine of the dynamics is obvious, given the great amount of results on controllability issues concerning control-affine systems (see also Remark 4.2).

To state precisely the main result, let us give some definitions.

Definition 4.1 (Balanced systems). We say that the control-polynomial dynamics (26) is balanced if there exist an $m$-tuple $K = (K_1, \ldots, K_m)$ of positive odd numbers and a positive integer number $\bar{d} \leq d$ such that

$$F(x, u) = f(x) + \sum_{\alpha_1} u^{K_{\alpha_1}} g_{\alpha_1}(x) + \cdots + \sum_{\alpha_1 < \cdots < \alpha_{\bar{d}}} u^{K_{\alpha_1} \cdots K_{\alpha_{\bar{d}}}} g_{\alpha_1 \cdots \alpha_{\bar{d}}}(x),$$

where we have set $u^{K_{\alpha_1} \cdots K_{\alpha_k}} := u^{K_{\alpha_1} \cdots K_{\alpha_k}}$.

In other words, we are assuming that the only nonzero monomials are those in which every control $u_\alpha$, $\alpha = 1, \ldots, m$, has exponent either 0 or $K_\alpha$.

Remark 4.1. If a control-polynomial dynamics (26) of degree $d$ is balanced, then the relation between $\bar{d}$ and $d$ is given by $d = \max \left\{ \sum_{j=1}^{\bar{d}} K_{\alpha_j} \mid 1 \leq \alpha_1 < \cdots < \alpha_{\bar{d}} \leq m \right\}$.

Moreover, if $m \geq \bar{d}$ the number of non-drift terms of a balanced system $F$ is equal to $M = \sum_{k=1}^{m} \binom{m}{k} = 2^m - 1$. Indeed for every $k \leq \bar{d}$, the number of the terms of the form $u^{K_{\alpha_1} \cdots K_{\alpha_k}} g_{\alpha_1 \cdots \alpha_k}$ is equal to $\binom{m}{k}$.

We shall adopt the following notations:

$$U_r := [-r, r]^m$$

for $r \in [0, \infty]$;

$$W_r^K := [-\bar{r}, \bar{r}]^M$$

$\bar{r} := \frac{1}{M} \min\{r^j K_\alpha \mid \alpha = 1, \ldots, m; j = 1, \bar{d}\}$

for $r \in [0, \infty]$;

and

$$U_{+\infty} := \mathbb{R}^m; \quad W_{+\infty} := \mathbb{R}^M.$$

Theorem 4.1 below establishes that balanced systems can be regarded as control-affine systems with independent control variables.

We shall make the following assumption on the pair $(l, U)$:

Theorem 4.1: The control set $U$ and the Lagrangean $l$ are such that, for every $x \in \Omega \setminus C$, the map $l(x, \cdot) : U \to \mathbb{R}$ is bounded.
Hypothesis $A_{bal}$ standing, we define the (non-negative, continuous) function
\[ \ell(x) := \sup_{u \in U} l(x,u). \]

We shall also mean that a $m$-tuple of positive odd numbers $K = (K_1, \ldots, K_m)$ and a non-negative $r$ (possibly equal to $+\infty$) are chosen and that the control set for the minimum problems $(\ell, F_{aff}, C)$ and $(l, F, C)$ coincide with $W^K_r$ and $U_r$, respectively.

**Theorem 4.1.** Let us assume Hypothesis $A_{bal}$ and let $V$ be a Minimum Restraint Function with savings multiplier $\bar{p}_I \geq 0$ for the affine problem $(\ell, F_{aff}, C)$. Then the map $V$ is a Minimum Restraint Function for the original (non-affine) problem $(l, F, C)$ as well, with the same savings multiplier $\bar{p}_I$. In particular, the control system $\dot{x} = F(x,u)$ is GAC to $C$ and, if $\bar{p}_I > 0$,
\[ W(z) := \inf_{(x,u)} \mathcal{I}(x,u) \leq \frac{V(z)}{\bar{p}_I} \quad \forall z \in \Omega \setminus C. \]

**Proof.** Let $x \in \Omega \setminus C$. By assumption one has
\[ \min_{w \in W^K_r} \left( \langle \bar{p}_I, p \rangle, \left( \ell(x), F_{aff}(x,w) \right) \right) < 0 \]
for all $p \in D^*V(x)$. By Lemma 4.1 below, $F_{aff}(x, W^K_r) \subseteq co F(x, U_r)$, so
\[ \min_{u \in U_r} \left( \langle p_I, p \rangle, \left( \ell(x), F(x,u) \right) \right) < 0 \]
for all $p \in D^*V(x)$, which by the definition of $\ell$, yields
\[ \min_{u \in U_r} \left( \langle p_I, p \rangle, \left( l(x,u), F(x,u) \right) \right) < 0, \]
for all $p \in D^*V(x)$. This concludes the proof. \hfill \Box

**Lemma 4.1.** Let $m \geq \bar{d}$. Then for every $r \in [0, +\infty)$
\[ F_{aff}(x, W^K_r) \subseteq co F(x, U_r) \quad \forall x \in \Omega \setminus C. \tag{27} \]

**Remark 4.2.** Besides implying Theorem 4.1, Lemma 4.1 gives access to classical results on control-affine systems for the study of local controllability of balanced systems. For instance consider the driftless $K$-balanced system (with $d = 8$, $K = (1, 3, 5)$ and $\bar{d} = 2$)
\[ \dot{x} = F(x,u) = g_1(x)u_1u_2^3 + g_2(x)u_1u_2^5 + g_2(x)u_2^3u_3^5, \]
with $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and
\[ g_1(x) = (1, 0, x_2, 0)^{tr}; \quad g_2(x) = (0, 1, -x_1, 0)^{tr}; \quad g_3(x) = (0, 0, 0, 1)^{tr}. \]
Notice that $\{(u_1u_2^3, u_1u_2^5, u_3^3u_3^5) \mid (u_1, u_2, u_3) \in \mathbb{R}^3\} \subseteq \mathbb{R}^3$: for instance
\[ (0, 1, 1) \notin \{(u_1u_3^3, u_1u_3^5, u_3^5u_3^5) \mid (u_1, u_2, u_3) \in \mathbb{R}^3\}, \]
so $F$ cannot parameterized as control-linear vector field with controls in $\mathbb{R}^3$. However, by Lemma 4.1 the control-linear vector field
\[ F_{aff}(x,w) = \sum_{j=1}^3 g_j(x)w_j \quad w = (w_1, w_2, w_3) \in \mathbb{R}^3 \]
satisfies

\[ \mathcal{F}_{aff}(x, W^K_r) \subset \text{co}(\mathcal{F}(x, U_r)) \quad \forall x \in \mathbb{R}^4; \quad \forall r > 0. \]

For example, we have that

\[ g_2(x) + g_3(x) \notin \mathcal{F}(x, U_r), \quad g_2(x) + g_3(x) = \frac{1}{2} \mathcal{F}(x, (1, 0, 1)) + \frac{1}{2} \mathcal{F}(x, (0, 1, 1)). \]

**Remark 4.3.** With reference to system (28), let us see a simple utilization of the affine representability of \( \mathcal{F}_{aff} \). Let us notice that system (28) verifies the so-called Lie algebra rank condition:

\[ \text{Lie}_x \{g_1, g_2, g_3\} = \mathbb{R}^4 \quad \forall x \in \mathbb{R}^4. \]

Therefore, when the control set coincides with \( W^K_r \), for some \( r > 0 \), by Chow-Rashevsky’s Theorem the system \( \dot{x} = \mathcal{F}_{aff}(x, w) \) turns out to be small time locally controllable. Now, by Lemma 4.1

\[ \mathcal{F}_{aff}(x, W^K_r) \subset \text{co}(\mathcal{F}(x, U_r)) \quad \forall x \in \Omega \setminus \mathcal{C}. \]

Consequently, we can deduce by a relaxation argument that the system \( \dot{x} = \mathcal{F}(x, u) \), equipped with control set \( U_r \), is small time locally controllable as well.

### 4.1. Proof of Lemma 4.1

We prove the Lemma in the case all components of the \( m \)-tuple \( K \) are equal to 1, i.e., \( K = (1, \ldots, 1) \) (this assumption implies \( \bar{d} = d \), see Remark 4.1). Indeed, to prove the theorem when \( K \) is general \( m \)-tuple of odd numbers it is sufficient to apply the result to the rescaled control-polynomial vector field

\[ \hat{\mathcal{F}}(x, u) := \mathcal{F}(x, u^K_1, \ldots, u^K_m). \]

Fix \( k \in \mathbb{N} \) and denote by \( \{1, -1\}^k \) the set of \( k \)-tuples \((s_1, \ldots, s_k)\) with \( s_j \in \{-1, 1\} \). Denote by \( P(S) \) the power set of a set \( S \) and consider the set-valued map \( S_k : \{1, -1\} \to P(\{-1, 1\}^k) \) defined by

\[ S_k(s) = \left\{ (s_1, \ldots, s_k) \in \{-1, 1\}^k \mid s_1 \cdots s_k = s \right\}. \]

Let us begin with a combinatorial statement:

**Claim A:** Let \( k, d \in \mathbb{N}, k \leq d \). For every \( 1 \leq \alpha_1 < \cdots < \alpha_k \leq d \) and for every \( s \in \{-1, 1\} \)

\[ \sum_{(s_1, \ldots, s_d) \in S_d(s)} s_{\alpha_1} \cdots s_{\alpha_k} = 0. \tag{29} \]

To prove **Claim A** let us fix \( k, d \in \mathbb{N}, k < d \) and notice that

\[ \sum_{(s_1, \ldots, s_k) \in \{-1,1\}^k} s_1 s_2 \cdots s_k = 0. \tag{30} \]

\(^{13}\) Chow-Rashevsky’s Theorem requires the dynamics to be symmetric, so that the fact \( \mathcal{F} \) is balanced is crucial in the above arguments. For instance, the unbalanced control system \( \dot{x} = g_1(x)u_1^2 + g_2(x)u_1 u_3 + g_2(x)u_2 u_3 \), though constructed with the same vector field as (28), fails to be controllable at \( x = 0 \), for \( \dot{x}_1 \geq 0 \).
This concludes the proof of

We continue the proof of Lemma 4.1 by proving a statement on the convex hull

In view of (30) and of (31), for every $s \in \{−1, 1\}$

Now, fix an index sequence $1 \leq \alpha_1 < \cdots < \alpha_k \leq d$ and an auxiliary $k$-ple $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_k) \in \{-1, 1\}^k$. One has

$$\# \left\{ (s_1, \ldots, s_k) \in \{-1, 1\}^k \mid s_{\alpha_h} = \bar{s}_h; \ h = 1, \ldots, k \right\} = 2^{d-k}.$$  

Therefore, by a symmetry argument,

$$\# \{ (s_1, \ldots, s_d) \in S_d(s) \mid s_{\alpha_h} = \bar{s}_h; \ h = 1, \ldots, k \} = 2^{d-k-1} \ \forall s \in \{-1, 1\}.  \tag{31}$$

In view of (30) and of (31), for every $s \in \{-1, 1\}$

$$\sum_{(s_1, \ldots, s_d) \in S_d(s)} s_{\alpha_1} \cdots s_{\alpha_k} = 2^{d-k-1} \sum_{(s_{\alpha_1}, \ldots, s_{\alpha_k}) \in \{-1, 1\}^k} s_{\alpha_1} \cdots s_{\alpha_k} = 0.$$  

This concludes the proof of Claim A.

We continue the proof of Lemma 4.1 by proving a statement on the convex hull $co \ F(x, U_r)$.

**Claim B:** Let $d \leq m$. For every $k \leq d$, and for every index sequence $1 \leq \alpha_1 < \cdots < \alpha_k \leq d$ and for every $w \in [-r^k, r^k]$

$$f(x) + wg_{\alpha_1 \cdots \alpha_k}(x) \in co \ F(x, U_r).  \tag{32}$$

To prove Claim B, let us fix $k \leq d$, $1 \leq \alpha_1 < \cdots < \alpha_k \leq d$ and $w \in [-r^k, r^k]$. Denote by $s(w)$ the sign of $w$ and select from $[-r, r]^k$ a vector $(u_1, \ldots, u_k)$ satisfying $u_1 \cdots u_k = w$. Introduce an ordering on the set $\bar{S}_k(s(w))$ so that we can index its elements and write $S_k(s(w)) = \{(s_1^{(h)}, \ldots, s_k^{(h)})\}_{h=1}^{2^{k-1}}$. Define

$$u^{(h)} := \sum_{j=1}^{k} s_j^{(h)} |u_j| e_{\alpha_j} \ \forall h \in \{1, \ldots, 2^{k-1}\},$$

where $e_{\alpha}$ is the $\alpha$-th element of the canonical base of $\mathbb{R}^m$. Notice that $2^{k-1}$ is the cardinality of $S_d(s(w))$. By construction one has $u^{(h)} \in [-r, r]^m = \bar{U}_r$ and

$$u^{(h)}_{\alpha_1} \cdots u^{(h)}_{\alpha_k} = w.$$  

Let $A_j$ be the set of subsequences of $(\alpha_1, \ldots, \alpha_k)$ of length $j$, namely

$$A_j = \{(\alpha_{k_1}, \ldots, \alpha_{k_j}) \mid \alpha_1 \leq \alpha_{k_1} < \cdots < \alpha_{k_j} \leq \alpha_k\}.$$  

By Claim A, for every $j < k$ and every subsequence $(\alpha_{k_1}, \ldots, \alpha_{k_j}) \in A_j$ one has

$$\sum_{h=1}^{2^{k-1}} u^{(h)}_{\alpha_{k_1}} \cdots u^{(h)}_{\alpha_{k_j}} |u_{k_1}| \cdots |u_{k_j}| \sum_{(s_1, \ldots, s_k) \in S_k(s)} s_{k_1} \cdots s_{k_j} = 0.$$
Then
\[
\sum_{h=1}^{2k-1} \frac{1}{2k-1} F(x, u(h)) = f(x) + \sum_{(\alpha_k) \in A_1} \frac{1}{2k-1} \left( \sum_{h=1}^{2k-1} u_{\alpha_1}^{(h)} \right) g_{\alpha_1}(x)
\]

\[
+ \sum_{(\alpha_k, \alpha_k_2) \in A_2} \frac{1}{2k-1} \left( \sum_{h=1}^{2k-1} u_{\alpha_k_1}^{(h)} u_{\alpha_k_2}^{(h)} \right) g_{\alpha_{k_1} \alpha_{k_2}}(x) + \cdots + w g_{\alpha_1 \cdots \alpha_k}(x)
\]

\[
= f(x) + w g_{\alpha_1 \cdots \alpha_k}(x),
\]
which concludes the proof of Claim B.

To end the proof of Lemma 4.1 in case \( K = (1, \ldots, 1) \), it suffices to remark that for every \( k = 1, \ldots, d \), \([-\bar{r}, \bar{r}] \subseteq M[-r^k, r^k] \). Therefore Claim B implies that for every \( w \in [-\bar{r}, \bar{r}]^M = W^{(1,1,\ldots,1)} \)

\[
F_{aff}(x, w) = f(x) + \sum_{\alpha_1} w_{\alpha_1} g_{\alpha_1}(x) + \cdots + \sum_{\alpha_1 < \cdots < \alpha_d} w_{\alpha_1 \cdots \alpha_d} g_{\alpha_1 \cdots \alpha_d}(x) =
\]

\[
\sum_{k=1}^p \sum_{\alpha_1 < \cdots < \alpha_k} \frac{1}{M} (f(x) + M w_{\alpha_1 \cdots \alpha_d} g_{\alpha_1 \cdots \alpha_d}(x)) \in \text{co}\ F(x, U_r).
\]

5. Control-polynomial systems: weak subsystems

Instead of assuming any structure in the polynomial-control affine dynamics, we focus here on the search of special, algebraically “simple”, control subsystems which, from the viewpoint of set-valued analysis, are (set-valued) selections of the convex-valued multifunction \( x \mapsto \text{co}\ F(x, \mathbb{R}^m) \). We will examine two significative cases: the maximal degree subsystem and the \( \lambda \)-diagonal subsystems.

5.1. Maximal degree weak subsystems.

Theorem 5.1 below extends in several directions a result contained in [5] and valid for the case \( d = 2 \). It states that in order to test if a function \( V \) is a MRF function it is sufficient to test it on the (simpler) maximal degree problem

\[
(33)\quad \dot{x} = F^{\max}(x, u), \quad \inf_{u} \int_{0,T_x,u} l(x(t), u(t))dt,
\]

where the maximal degree control-polynomial vector field \( F^{\max} \) is defined by

\[
F^{\max}(x, u) := f(x) + \sum_{\alpha_1 \leq \cdots \leq \alpha_d} u_{\alpha_1 \cdots \alpha_d} g_{\alpha_1 \cdots \alpha_d}(x).
\]

We shall assume the following additional hypothesis on the Lagrangean:
**Hypothesis A**\(_{\text{max}}\): There exist non negative continuous functions \(M_1 = M_1(x)\), \(M_2 = M_2(x, u)\) such that

\[
I(x, u) = M_0(x) + M_1(x, u),
\]

and with \(M_1\) satisfying \(M_1(x, 0) = 0\) and

\[
M_1(x, ku) \leq k^d M_1(x, u) \quad \forall k \geq 1, \quad x \in \Omega \setminus C, \quad u \in \mathbb{R}^m.
\]

Notice that Lagrangeans of the form

\[
I(x, u) = l_0(x) + l_1(x)|u| + \cdots + l_p(x)|u|^d,
\]

where the maps \(l_i(\cdot)\) are continuous and non-negative, verify Hypothesis \(A_{\text{max}}\).

**Theorem 5.1.** Let us assume Hypothesis \(A_{\text{max}}\), and let \(V\) be a Minimum Restraint Function for the maximal degree problem \((l, \mathcal{F}^\text{max}_\lambda, C)\), with savings multiplier \(\bar{p}_I\). Then the map \(V\) is a Minimum Restraint Function for the original problem \((l, \mathcal{F}, C)\), with the same savings multiplier. In particular, the control system \(\dot{x} = F(x, u)\) is GAC to \(C\) and, if \(\bar{p}_I > 0\),

\[
W(z) := \inf_{(x, u)} I(x, u) \leq \frac{V(z)}{-\bar{p}_I} \quad \forall z \in \Omega \setminus C.
\]

**Proof.** Now, assume by contradiction that there exists \(x \in \Omega \setminus C\) and \(p \in D^*V(x)\) such that

\[
\bar{p}_I l(x, u) + \langle p, f(x) \rangle + \sum_{\alpha=1}^m u_\alpha \langle p, g_\alpha(x) \rangle + \cdots + \sum_{\alpha_1 \leq \cdots \leq \alpha_d} u_{\alpha_1} \cdots u_{\alpha_d} \langle p, g_{\alpha_1 \cdots \alpha_d}(x) \rangle \geq 0
\]

for all \(u \in \mathbb{R}^m\). By taking \(u = 0\) we obtain

\[
\bar{p}_I M_0(x) + \langle p, f(x) \rangle \geq 0.
\]

By assumption, there exists \(\tilde{u} \in \mathbb{R}^m\) and \(\eta > 0\) such that

\[
\bar{p}_I l(x, \tilde{u}) + \langle p, f(x) \rangle + \sum_{\alpha_1 \leq \cdots \leq \alpha_d} \tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_d} \langle p, g_{\alpha_1 \cdots \alpha_d}(x) \rangle = -\eta.
\]

Moreover, \((36) + (37)\) imply

\[
\bar{p}_I k^d M_1(x, \tilde{u}) + k^d \sum_{\alpha_1 \leq \cdots \leq \alpha_d} \tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_d} \langle p, g_{\alpha_1 \cdots \alpha_d}(x) \rangle \leq -\eta k^d
\]
for any $k \geq 0$. Hence, for every $k \geq 1$

$$\tilde{p}_T l(x, k\tilde{u}) + \langle p, f(x) \rangle + k \sum_{\alpha=1}^{m} \tilde{u}_\alpha \langle p, g_\alpha(x) \rangle + \ldots$$

$$+ k^{d-1} \sum_{\alpha_1 \leq \cdots \leq \alpha_{d-1}} \tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_{d-1}} \langle p, g_{\alpha_1 \ldots \alpha_{d-1}}(x) \rangle + k^{d} \sum_{\alpha_1 \leq \cdots \leq \alpha_{d}} \tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_{d}} \langle p, g_{\alpha_1 \ldots \alpha_{d}}(x) \rangle \leq$$

$$\tilde{p}_T k^d l(x, \tilde{u}) + \tilde{p}_T M_0(x) + \langle p, f(x) \rangle + k \sum_{\alpha=1}^{m} \tilde{u}_\alpha \langle p, g_\alpha(x) \rangle + \ldots$$

$$+ k^{d-1} \sum_{\alpha_1 \leq \cdots \leq \alpha_{d}} \tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_{d}} \langle p, g_{\alpha_1 \ldots \alpha_{d}}(x) \rangle + k^{d} \sum_{\alpha_1 \leq \cdots \leq \alpha_{d}} \tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_{d}} \langle p, g_{\alpha_1 \ldots \alpha_{d}}(x) \rangle \leq$$

$$k \sum_{\alpha=1}^{m} \tilde{u}_\alpha \langle p, g_\alpha(x) \rangle + \ldots + k^{d-1} \sum_{\alpha_1 \leq \cdots \leq \alpha_{d-1}} \tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_{d-1}} \langle p, g_{\alpha_1 \ldots \alpha_{d-1}}(x) \rangle - \eta k^d.$$

If $k$ is sufficiently large the last term is negative, which contradicts $[35]$. □

**Remark 5.1.** The choice of a different subsystem may be more effective for the guessing of a Minimum Restraint Function; see Example 5.1 in Section 5.2, where a new class of subsystems (the $\lambda$-diagonal subsystems) is introduced and we prove a result which is analogous to Theorem 5.1.

**Remark 5.2.** On the one hand, the thesis of Theorem 5.1 cannot be extended the case of bounded control sets. For instance, if $d = 3$, $n = m = 1$, $U = [-1, 1]$, $C = \{0\}$, $l \equiv 0$, and $F(x, u) = (u^2 + u^3)x$, for any initial datum $x \in \mathbb{R}$ one has $\dot{x}[z, u] \geq 0$, so the system is not GAC to $C$ and no Lyapunov Function can exist. However, $V(x) = x^2$ is a Lyapunov Function for $(l, F^{max})$ and consequently, the system $\dot{x} = F^{max}(x, u)$ is globally asymptotic controllable to $C$.

On the other hand, it may happen that some symmetry argument may allow the extension of Theorem 5.1 for a special class of polynomial control systems with bounded control sets. Consider, for instance, a problem where $d = 2$, $U$ is a (compact) symmetric control set (i.e. $u \in U$ implies $-u \in U$) and, for all $x \in \Omega \setminus C$, $l(x, \cdot)$ is an even function. For example:

$$\inf \int_0^{T_x,u} [(|u| + x^2 u^2)] dt$$

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 + h_{1,1}(x)u_1^2 + h_{2,2}(x)u_2^2 + h_{1,2}(x)u_1u_2, \quad U = [-1, 1]^2.$$
Notice that
\[(l, \mathcal{F}^{\text{max}})(x, u) = \frac{1}{2}(l, \mathcal{F})(x, u) + \frac{1}{2}(l, \mathcal{F})(x, -u) \in \text{co}(l, \mathcal{F})(x, U) \quad \forall x \in \Omega \setminus C, \ u \in U.\]
Therefore, for every \((x, (p_0, p)) \in (\Omega \setminus C) \times \mathbb{R}^{1+n},\) one has
\[H_{l, \mathcal{F}^{\text{max}}}(x, p_0, p) < 0 \Rightarrow H_{l, \mathcal{F}}(x, p_0, p) < 0.\]
Consequently a map \(V\) is Minimum Restraint Function for \((l, \mathcal{F}^{\text{max}}, C)\) with savings multiplier \(\bar{p}_T\) if and only if \(V\) is a Minimum Restraint Function for \((l, \mathcal{F}, C)\) with the same savings multiplier \(\bar{p}_T.\) Then Theorem 3.1 applies and, consequently, Theorem 5.1 turns out to be extended to this case.

5.2. Diagonal weak subsystems.

Definition 5.1. For every \(\lambda\) belonging to the simplex \(\Lambda := \{\lambda \in \mathbb{R}^m \mid \sum_{\alpha=1}^{m} \lambda_m \leq 1; \ \lambda_\alpha \geq 0\},\)
\[\mathcal{F}^{\text{diag}}_{\lambda}(x, u) := f(x) + \sum_{i=1}^{d} \left( \sum_{\alpha=1}^{m} \lambda_{d-i}^\alpha g_{\alpha\alpha\cdots\alpha}(x) u_{\alpha}^i \right),\]
will be called the \(\lambda\)-diagonal control vector field corresponding to \(\mathcal{F}\) and \(\lambda.\)

For instance, when \(d = 2\) and \(d = 3,\) one has
\[\mathcal{F}^{\text{diag}}_{\lambda}(x, u) = f(x) + \sum_{\alpha=1}^{m} \lambda_{2}^\alpha g_{\alpha}(x) u_{\alpha} + \sum_{\alpha=1}^{m} g_{\alpha\alpha}(x) u_{\alpha}^2.\]
and
\[\mathcal{F}^{\text{diag}}_{\lambda}(x, u) = f(x) + \sum_{\alpha=1}^{m} \lambda_{3}^\alpha g_{\alpha}(x) u_{\alpha} + \sum_{\alpha=1}^{m} g_{\alpha\alpha}(x) u_{\alpha}^2 + \sum_{\alpha=1}^{m} g_{\alpha\alpha\alpha}(x) u_{\alpha}^3,\]
respectively.

Remark 5.3. Notice that \(\mathcal{F}(x, 0) = f(x),\) so
\[(39) \quad \mathcal{F}^{\text{diag}}_{\lambda}(x, u) = \sum_{\alpha=0}^{m} \lambda_{\alpha} \mathcal{F}(x, \lambda_{\alpha}^{-\frac{1}{d}} u_{\alpha} e_{\alpha}),\]
where \(e_1, \cdots, e_m\) denotes the basis of \(\mathbb{R}^m\) and we set \(e_0 := 0, \ \lambda_0 := 1 - \sum_{\alpha=1}^{m} \lambda_\alpha.\) Since \(\sum_{\alpha=0}^{m} \lambda_\alpha = 1,\) this implies that
\[(40) \quad \mathcal{F}^{\text{diag}}_{\lambda}(x, \mathbb{R}^m) \subseteq \text{co} \mathcal{F}(x, \mathbb{R}^m).\]

We shall assume the following additional hypothesis on the Lagrangean:

Hypothesis A_{diag}: There exists a real number \(M_0 > 0\) such that, for every \(\lambda \in \Lambda\) verifying \(\lambda_\alpha > 0, \ \alpha = 1, \cdots, m,\) one has

\[^{15}\text{In the particular case } i = d \text{ and } \lambda_\alpha = 0, \text{ the formal expression } \lambda_{d-i}^\alpha (\neq 0^0) \text{ is meant equal to } 1.\]
this, together with (44), implies

\[
\lambda \alpha l(x, \frac{u_\alpha}{\sqrt{\lambda}} e_\alpha) \leq M_0 l(x, u) \quad \forall u \in \mathbb{R}^m.
\]

**Remark 5.4.** Notice that for every \( q \geq 1 \)

\[
l(x, u) := l_0(x) + l_1(x)|u| + \cdots + l_q(x)|u|^q,
\]
does verify Hypothesis \( A_{\text{diag}} \) (with \( M_0 = \sqrt{2} \)). As a model, simple case, one could consider \( l(x, u) = |u|^q, q \geq d \), so that the functional to be minimized would be nothing but the \( L^q \)-norm of \( u \) (at the power \( q \)).

**Theorem 5.2.** Assume Hypothesis \( A_{\text{diag}} \) holds true for a suitable \( M_0 \geq 0 \), and let \( V \) be a Minimum Restraint Function for the \( \lambda \)-diagonal problem \((l, F_\lambda^{\text{diag}}, C)\), with savings multiplier \( \bar{p}_T \). Then the map \( V \) is a Minimum Restraint Function for the original problem \((l, F, C)\), with savings multiplier \( \frac{\bar{p}_T}{M_0} \) if \( M_0 > 0 \) and with arbitrarily large savings multiplier \( \bar{p}_0 \) if \( M_0 = 0 \).

In particular, the control system \( \dot{x} = F \) is GAC and, if the savings multiplier \( \bar{p}_T \) of \( V \) is positive,

\[
W(z) := \inf_{(x, u)} I(x, u) \leq \frac{M_0 V(z)}{\bar{p}_T} \quad \forall z \in \Omega \backslash C.
\]

**Proof.** Set \( \lambda_0 = 1 - \sum_{\alpha=1}^m \lambda_\alpha \) and \( e_\alpha = 0 \). First assume \( M_0 > 0 \). Then for every \( \alpha = 0, \ldots, m \), every \( (x, u) \in (\Omega \backslash C) \times \mathbb{R}^m \) and every \( p \in D^*V(x) \), one has

\[
\lambda_\alpha H_{l, F}(x, \frac{\bar{p}_T}{K}, p) \leq \lambda_\alpha \left( \left( \frac{\bar{p}_T}{K}, p \right), (l, F)(x, \lambda_\alpha^{-\frac{1}{2}} u_\alpha e_\alpha) \right)
\]

that, summing up for \( \alpha = 0, \ldots, m \), yields

\[
H_{l, F}(x, \frac{\bar{p}_T}{K}, p) \leq \sum_{\alpha=0}^m \lambda_\alpha \left( \left( \frac{\bar{p}_T}{K}, p \right), (l, F)(x, \lambda_\alpha^{-\frac{1}{2}} u_\alpha e_\alpha) \right) \leq \frac{\bar{p}_T}{M_0} M_0 l(x, u) + \left( p, F_\lambda^{\text{diag}}(x, u) \right) \]

\[
= \bar{p}_T l(x, u) + \left( p, F_\lambda^{\text{diag}}(x, u) \right).
\]

Since by hypothesis \( \max_{p \in D^*V(x)} H_{l, F_\lambda^{\text{diag}}}(x, \bar{p}_T, p) < 0 \), then there exists \( \tilde{u} \) such that

\[
\bar{p}_T l(x, \tilde{u}) + \left( p, F_\lambda^{\text{diag}}(x, \tilde{u}) \right) < 0 \quad \forall p \in D^*V(x),
\]

this, together with by (44), implies

\[
H_{l, F}(x, \frac{\bar{p}_T}{M_0}, p) < 0 \quad \forall p \in D^*V(x)
\]

This is due to the elementary inequalities

\[
|u_1| + \cdots + |u_m| \leq \sqrt{2} |u| \quad (|u_1|^q + \cdots + |u_m|^q)^\frac{1}{q} \leq |u| \quad \forall q > 1.
\]
which indeed is the thesis of the theorem. Assume otherwise $M_0 = 0$. Then $l \equiv 0$, consequently $W(z) \equiv 0$ and \([43]\) is trivially verified. Since $V$ is a Minimum Restraint Function for $(l, F^\text{diag}_\lambda, C)$ and since $l \equiv 0$, for every $x \in \Omega \setminus C$ there exists $\bar{u} \in \mathbb{R}^m$ such that $\langle p, F^\text{diag}_\lambda(x, \bar{u}) \rangle < 0$ for all $p \in D^*V(x)$. Consequently, for every $\bar{p}_0 \in \mathbb{R}$ and for every $p \in D^*V(x)$

$$H_{l,F}(x, \bar{p}_0, p) = \inf_{u \in \mathbb{R}^m} \langle p, F(x, u) \rangle \leq \sum_{\alpha=0}^{m} \lambda_{\alpha} \langle p, F(x, \lambda^{-\frac{1}{2}}u_{\alpha}e_{\alpha}) \rangle$$

$$= \langle p, F^\text{diag}(x, u) \rangle < 0.$$  

This gives the thesis in the case $M_0 = 0$ and completes the proof. \hfill \Box 

**Example 5.1.** Consider the exit time problem in $\mathbb{R}^2 \setminus \{0\}$

\[(45)\]

\[\dot{x} = F(x, u) := x + \left(\frac{1}{|r|} \right) u_1 u_2 - \left(\frac{1}{1} \right) u_1^2 - \left(\frac{0}{1} \right) u_2^2 + 3xu_1^2u_2 \quad x(0) = z; \quad u \in \mathbb{R}^2\]

\[W(z) := \inf_{(x,u)} \int_{[0,T\in\mathbb{R}]} l(x(t), u(t)) dt \quad \text{with} \quad l(x, u) := x^2 |u|^2\]

referred to the target $C := \{0\}$.

Let $\Phi : [0, +\infty[ \to \mathbb{R}$ be a smooth convex function such that $\Phi(0) = 0$, $\Phi'(0) \geq 1$, and suppose that we aim to verify that a function of the form

$$V(x) = \Phi(|x|^2)$$

is a MRF function for our problem, possibly with a savings multiplier $\bar{p}_T > 0$.

We begin with observing that the maximal degree subsystem

$$\dot{x} = F^{\text{max}}(x, u) = x + 3xu_1^2u_2$$

do not give any useful information. Indeed

$$H_{l,F^{\text{max}}}(x, \bar{p}_T, \nabla(V)(x)) = \inf_u \left\{ \langle \nabla V(x) , F^{\text{max}}(x, u) \rangle + \bar{p}_T x^2 |u|^2 \right\}$$

$$= \inf_u \left\{ 2\Phi'(|x|^2) |x|^2(1 + 3u_1^2u_2^2) + p_T x^2 |u|^2 \right\} \geq 0$$

for all $x \in \mathbb{R}^2 \setminus \{0\}$ and $\bar{p}_T \geq 0$.

On the other hand, by considering the diagonal subsystem

$$\dot{x} = F^{\text{diag}}(\frac{1}{2}, \frac{1}{2}) = x - \left(\frac{\sqrt{1/2}}{0} \right) u_1^2 - \left(\frac{0}{\sqrt{1/2}} \right) u_2^2,$$

if $\bar{p}_T < 1(\leq \Phi'(|x|^2)$ for all $x \in \mathbb{R}^2$), we get

$$H_{l,F^{\text{diag}}(\frac{1}{2}, \frac{1}{2})}(x, \bar{p}_T, \nabla(V)(x)) \leq \inf_u \left\{ |x|^2 \left(\Phi'(|x|^2)(2 - u^2) + \bar{p}_T u^2 \right) \right\} = -\infty,$$

i.e., $V$ is a MRF with savings multiplier $\bar{p}_T$ for the problem $(l, F^{\text{diag}}(\frac{1}{2}, \frac{1}{2})$. Therefore, in view of Theorem \[5.2\], $V$ is a MRF with savings multiplier $\bar{p}_T$ for the problem \(45\) as well.
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