HOMOLOGICAL PROPERTIES OF GRAPH MANIFOLDS

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Abstract. We consider the following properties of compact oriented irreducible graph-
manifolds: to contain a $\pi_1$-injective surface (immersed, virtually embedded or embedded),
be (virtually) fibered over $S^1$, and to carry a metric of nonpositive sectional curvature. It
turns out that all these properties can be described from a unified point of view.

§0 introduction

We consider the class $\mathcal{M}$ of graph-manifolds (cf.[BK1]). We say that a 3-manifold $M$
belongs to $\mathcal{M}$ if $M$ is a compact closed orientable Haken 3-manifold and its canonical
decomposition surface (JSJ-surface) $T$ splits $M$ into Seifert pieces with orientable base
orbifolds of negative Euler characteristic. These Seifert pieces are called maximal blocks.

We say that a manifold $M \in \mathcal{M}$ contains an immersed (resp. embedded) surface $S$ if there is an immersion (resp. embedding) $g: S \to M$. Such a surface is called $\pi_1$-injective, if the induced homomorphism of fundamental groups $g_*: \pi_1(S) \to \pi_1(M)$ is injective. Such a surface is called horizontal if $g: S \to M$ is transversal to the Seifert fibers in each maximal block of $M$. (Horizontality implies $\pi_1$-injectivity [RW]).

We say that an immersed $\pi_1$-injective surface $g: S \to M \in \mathcal{M}$ is virtually embedded if there exists a finite covering $p: \tilde{M} \to M$ and a $\pi_1$-injective embedding $\tilde{g}: S \to \tilde{M}$ such that $g = p \circ \tilde{g}$. Finally we say that a manifold $M \in \mathcal{M}$ is virtually fibered over $S^1$ if $M$ has a finite cover which is fibered over the circle. Note, that the fiber of such a bundle has a negative Euler characteristic.

Consider the following properties of $M \in \mathcal{M}$:

(Im) $M$ contains an immersed $\pi_1$-injective surface $S$ with $\chi(S) < 0$;
(HI) $M$ contains an immersed $\pi_1$-injective horizontal surface $S$ with $\chi(S) < 0$;
(F) $M$ is fibered over $S^1$;
(E) $M$ contains an embedded $\pi_1$-injective surface $S$ with $\chi(S) < 0$;
(VF) $M$ is virtually fibered over $S^1$;
(VE) $M$ contains a virtually embedded $\pi_1$-injective surface $S$ with $\chi(S) < 0$;
(NPC) $M$ carries a metric of nonpositive sectional curvature (NPC-metric),
where \( \chi(S) \) is the Euler characteristic of \( S \).

These properties were intensively studied in the last decade by different authors. In the paper [LW], a simple obstruction for (virtual) fibration of graph manifolds was described. The question about existence of NPC-metrics on graph manifolds was treated in the paper [L]. In the papers [BK1,2], an explicit criterion which permits to solve whether or not a given graph manifold of the class \( \mathfrak{M} \) carries a NPC-metric was obtained. In the papers [N1,2], explicit criteria which permit to solve whether or not a given graph manifold satisfies one of these properties \( \mathbf{F}, \mathbf{VE} \), and \( \mathbf{Im} \) were proved; an implicit criterion for the property \( \mathbf{VF} \) was obtained in [N1] as well. In the papers [RW,M], the question whether or not a given \( \pi_1 \)-injective immersion of a surface in a graph manifold is a virtual embemdent was analyzed. If a graph manifold \( M \) of the class \( \mathfrak{M} \) carries a NPC-metric then it is virtually fibered over the circle, as it was proved in the author work [S]. So, we can present relationships between these properties by the following diagram:

\[
E \quad \Rightarrow \quad VE \quad \Rightarrow \quad Im
\]
\[
\uparrow \quad \uparrow \quad \downarrow
\]
\[
F \quad \Rightarrow \quad VF \quad \Rightarrow \quad HI
\]
\[
\uparrow
\]
\[
NPC
\]

(the implication \( \mathbf{Im} \leftrightarrow \mathbf{HI} \) follows from [N2]).

As it follows from S. Buyalo and V. Kobel’skii paper [BK1], a graph manifold \( M \in \mathfrak{M} \) admits a NPC-metric iff a difference equation over a finite graph is solvable. From the other side, the same equation was independently obtained by W. Neumann in the paper [N2], where \( \pi_1 \)-injective surfaces in graph manifolds were investigated. It turns out that the equation describes all the features of graph manifolds we have considered above (theorem II below). The equation generalizes the Laplace equation on graphs (it contains long “covariant” differences, which depends on topological invariants of \( M \)) [B]. We give the equation in 1.4 and call it the Buyalo-Kobel’skii-Neumann equation (BKN-equation).

To learn the properties \( \mathbf{Im-NPC} \) we also propose an equivalent approach which is based on their homological nature (theorem I below). This approach calls on no topological invariants of manifolds except their JSJ-splitting so it is more common than BKN-equation. However, BKN-equation is more suitable for computations.

§1. **Main definitions and results**

1.1 **Notations.** Let \( M \) be a graph-manifold of the class \( \mathfrak{M} \) and \( \mathcal{T} \) be its JSJ-surface. The surface splits \( M \) into a finite set of Seifert fibered pieces: \( M|\mathcal{T} = \{M_v| v \in V\} \). So \( M \) can be obtained by pasting together Seifert bundles \( M_v, v \in V \) along their boundary tori. Define the index set \( W \) by \( \{T_w| w \in W\} = \{\partial M_v| v \in V\} \) and by definition put \( \partial v = \{w \in W|T_w \subset \partial M_v\} \), so that we have \( \partial M_v = \{T_w| w \in \partial v\} \). Denote by \( -w \) the
element of $W$ such that the tori $T_w$ and $T_{-w}$ are glued together in $M$ providing JSJ-torus $T_{|w|}$. These collections $V$ and $W$ form a graph $\Gamma_M(V, W)$, which is conjugated to the JSJ-splitting of $M$. Namely, $V$ is the vertices set and $W$ is the edges one. A vertex $v \in V$ is the initial vertex for an edge $w \in \partial v$.

From now on we fix an orientation of $M$ and also we fix some orientations of the Seifert fibers in the maximal blocks. Let $f_v \in H_1(M_v)$ be the class of the Seifert fiber in $M_v$ and let $f_w \in H_1(T_{|w|})$, $w \in \partial v$ be the homological class which is defined by $(t_w)_* f_w = f_v$, where $(t_w)_* : H_1(T_{|w|}) \rightarrow H_1(M_v)$ is the map induced by inclusion.

Let $l \in H^1(M_v)$ be a cohomological class in a block $M_v$. If $T_w$ is a boundary torus of $M_v$ and $T_{-w} \subset \partial M_{v'}$ then we define

$$\langle l, f_{v'} \rangle_{|w|} = \iota_w^*(l_{-w}).$$

The aim of this paper is to present all properties we have introduced as modifications of the following one:

**1.2 Definition.** Let $M$ be a graph-manifold of the class $\mathfrak{M}$ and $\Gamma_M(V, W)$ be its graph. We say that a nontrivial set $\{l_v \in H^1(M_v) | v \in V\}$ of (rational) cohomological classes satisfies the compatibility property (or is a CP-system on $M$), if

- CP1 $|\langle l_{v'}, f_v \rangle_{|w|}| \leq l_v(f_w)$ for each triple $v, w, v'$ such that $w \in \partial v$ and $-w \in \partial v'$;
- CP2 if $|\langle l_{v'}, f_v \rangle_{|w|}| = l_v(f_w)$ and $|\langle l_v, f_{v'} \rangle_{|w|}| = l_{v'}(f_{v'})$ then $\iota_{-w}^* l_{v'} = \pm \iota_w^* l_v$;
- CP3 if $l_v(f_w) = 0$ then $l_v = 0$.

Not all graph manifolds have CP-systems.

**Theorem I.** A graph manifold $M$ of the class $\mathfrak{M}$ satisfies the property $\text{Im}$ (resp. HI, F, E, VF, VE, NPC) if and only if there exists a CP-system on $M$ (resp. there exists a CP-system $\{l_v | v \in V\}$ on $M$ such that

- (HI) $l_v(f_w) > 0$;
- (F) $\langle l_{v'}, f_v \rangle_{e} = \epsilon_v \cdot \langle l_v, f_{v'} \rangle_{e} \neq 0$, where $\epsilon : V \rightarrow \{\pm 1\}$ is a function;
- (E) $\langle l_{v'}, f_v \rangle_{e} = l_v(f_{v'})$;
- (VF) $l_v(f_w) > 0$ and $\langle l_{v'}, f_v \rangle_{e} \cdot l_{v'}(f_{v'}) = \langle l_v, f_{v'} \rangle_{e} \cdot l_v(f_{v'})$;
- (VE) $\langle l_{v'}, f_v \rangle_{e} \cdot l_{v'}(f_{v'}) = \langle l_v, f_{v'} \rangle_{e} \cdot l_v(f_{v'})$;
- (NPC) $|\langle l_{v'}, f_v \rangle_{e}| < l_v(f_w)$ and $\langle l_{v'}, f_v \rangle_{e} \cdot l_{v'}(f_{v'}) = \langle l_v, f_{v'} \rangle_{e} \cdot l_v(f_{v'})$,

for each triple $v, w, v'$ ($e = |w|$) as in CP1.

This theorem follows from theorems 2.1, 2.2, 3.1, 3.2, 4.2, 4.3, 5.1. □

**Remark.** S. Buyalo gave similar conditions for $\text{Im}$, HI, E, and NPC (unpublished). He analyzed collections of homological classes on JSJ-tori and used the Waldhausen basises which were the basic construction in [BK1,2].

**1.3 Invariants of graph manifolds.** Let $M$ be a manifold of the class $\mathfrak{M}$. The choosing orientation of $M$ induces orientations on maximal blocks $\{M_v | v \in V\}$ and
hence on each torus $T_w$, $w \in W$. So, we have an “area” isomorphism $is_w : \Lambda^2 H_1(T_1) \to \mathbb{Q}$. By definition, put $a \wedge_w b = is_w(a \wedge b)$. Note that $a \wedge_w b = -a \wedge_w b$ isomorphically choosing orientations on $T_w$ and $T_{-w}$ are opposite.

Now we define the intersection index $b_w \in \mathbb{Z} \setminus \{0\}$, $w \in W$ by

$$b_w = f_w \wedge_w f_{-w}.$$ 

It is clear that $b_w = b_{-w}$.

Consider a maximal block $M_v$ in $M \in \mathfrak{M}$. An element $z_v \in H_1(\partial M_v)$ is called an adjoint element for the Seifert fibration of $M_v$, if $f_w \wedge_w p_w(z_v) = 1$ for each $w \in \partial v$, where $p_w : H_1(\partial M_v) \to H_1(T_v)$ is the canonical projection ($\partial M_v = \bigcup_{w \in \partial v} T_w$). For each adjoint element $z_v \in H_1(\partial M_v)$ we have $\iota_w(z_v) = -e_{M_v}(z_v) \cdot f_v$, where $e_{M_v}(z_v)$ is a rational number [N2]. The ratio $e_{M_v}(z_v)$ is called the Euler number of the Seifert fibration of $M_v$ with respect to the linear foliation of $\partial M_v$ induced by $z_v \in H_1(\partial M_v)$. This number was introduced in [NR].

1.3.1 Lemma [N2]. Let $M_v$ be a maximal Seifert block in a graph manifold $M$ and let $p : S \to M_v$ be a proper $\pi_1$-injective immersion of an oriented surface with negative Euler characteristic. Consider the (oriented) boundary of $S$ and let $\{C_{wi}, i = 1, \cdots, k_w\}$ be its components on $T_w$. Then for each adjoint element $z_v \in H^1(\partial M_v)$ we have

$$\sum_{i=1}^{k_w} f_w \wedge_w [C_{wi}] = a \neq 0 \quad \text{for each } T_w \subset \partial M_v.$$ 

$$\sum_{w, i} p_w(z_v) \wedge_w [C_{wi}] = -ae_{M_v}(z_v).$$

Conversely, suppose that for each boundary component $T_w$, $w \in \partial v$ a family $C_{w1}, \cdots, C_{wk}$ of immersed curves transverse to the Seifert fibres of $M_v$ is given satisfying homology relations (1), (2). Then there exist integers $d_0 > 0, n_0 > 0$ so that for any positive integer multiple $d$ of $d_0$ and $n$ of $n_0$ the family of curves $d(C_{w1})^n$, $w \in \partial v, i = 1, \cdots, k_w$, obtained as follows, bounds a connected immersed horizontal surface. We take $d(C_{wi})^n$ as $d$ copies of the immersed curve obtained by going $n$ times around the curve $C_{wi}$.

Lemma 1.3.2. Let $M_v$ be a maximal Seifert block in a graph manifold $M$ and for each $w \in \partial v$ a disjoint family of simple oriented curves $\{C_{wi}, i = 1, \cdots, k_w\} \subset T_w$ is given. If the conditions (1) and (2) are satisfied then the set of curves bounds an embedded horizontal surface in $M_v$. If the number $a$ in (1) and (2) is zero then there is a set of incompressible, boundary incompressible disjoint embedded vertical annuli in $M_v$ whose boundary is homotopic to the family of curves.

Proof. Consider the classes $c_w = \sum_{i=1}^{k_w} [C_{wi}] \in H_1(T_w)$. For each adjoint element $z_v \in H_1(\partial M_v)$ we can write $c_w = ap_w(z_v) - (p_w(z_v) \wedge_w c_w)f_w$. Let $c \in H_1(\partial M_v)$ be the
class such that \( p_w(c) = c_w \). Then for any \( l \in H^1(M_v) \) we have \( l(\iota_* c) = 0 \):

\[
\begin{align*}
  l(\iota_* c) &= \sum_{w \in \partial v} \iota_w^* l(c_w) = \sum_{w \in \partial v} \iota_w^* l(a p_w(z_v) - (p_w(z_v) \wedge c_w) f_w) = \\
  a \sum_{w \in \partial v} \iota_w^* l(p_w(z_v)) - \sum_{w \in \partial v} (p_w(z_v) \wedge c_w) \iota_w^* l(f_w) = a \sum_{w \in \partial v} l(\iota_w^* p_w(z_v)) - \\
  l(f_v) \sum_{w \in \partial v} (p_w(z_v) \wedge c_w) = -a e_{M_v}(z_v) l(f_v) + a e_{M_v}(z_v) l(f_v) = 0.
\end{align*}
\]

It means that \( \iota_* c = 0 \), so there exists an integer class \( s \in H_2(M_v; \partial M_v) \) such that \( s \mapsto c \) under the canonical homomorphism \( H_2(M_v; \partial M_v) \to H_1(\partial M_v) \). The class \( s \) can be realized as an embedded surface \( S \subset M_v \).

If \( a = 0 \) then all curves are homotopic to the Seifert fibers (by (1)) and there are even number of curves (by (2)). It is obvious that there are required annuli. \( \square \)

Now we define the charge \( k_v \) of a maximal block \( M_v \) in a graph manifold \( M \in \mathfrak{M} \) by \( k_v = -e_{M_v}(\phi_v) \), where \( \phi_v \) is the element of \( H_1(\partial M_v) \) such that \( p_w(\phi_v) = f_w/b_w \), \( w \in \partial v \).

The charge \( k_v \) of a block \( M_v \subset M \) is independent of the orientations of the Seifert fibers in blocks of \( M \). It depends only on orientation of \( M \). See, also [BK1].

The graph \( \Gamma_M(V, W) \) providing with the numbers \( \{ b_w, k_v \mid w \in W, v \in V \} \) is called the labelled graph of \( M \) and is denoted by \( X_M \) (here and further we omit “arguments” of \( X_M \)).

1.4 The BKN-equation. Let \( M \in \mathfrak{M} \) be a graph manifold and let \( X_M \) be its labelled graph. The equation

\[
\sum_{w \in \partial v} \frac{\gamma_w a_w(v)}{b_w} = k_v \cdot a_v,
\]

(the symbol \( w(v) \) denotes the terminal vertex for \( w \)) with \( \{ a, \gamma \mid v \in V, w \in W \} \) as unknown rational numbers is called the BKN-equation over \( X_M \). The BKN-equation is said to be solvable if there exists a nontrivial (\( a \neq 0 \)) rational solution \( \{ a, \gamma \} \) such that \( a_v \geq 0, |\gamma_w| \leq 1 \) and \( \gamma_w \gamma_{-w} \neq -1 \).

The following lemma describes the correspondence between CP-systems and solutions of the BKN-equation.

1.5 Lemma. Let \( M \) be a graph manifold of the class \( \mathfrak{M} \). The BKN-equation over \( X_M \) is solvable if and only if there exists a CP-system \( \{ l_v \mid v \in V \} \) on \( M \).

Proof. Let \( \{ a, \gamma \} \) be a solution of the BKN-equation, and let

\[
W_0 = \{ w \in W \mid \text{if } w \in \partial v \text{ then } a_w a_{w(v)} = 0 \}.
\]
Now we define a new solution \( \{ a, \gamma' \} \) of the BKN-equation by

\[
\gamma'_w = \begin{cases} 
\gamma_w & w \notin W_0 \\
0 & w \in W_0 
\end{cases}
\]

Consider an element \( l_w \in H^1(T|w|) \) such that

\[
l_w(f_w) = a_v, \quad l_w(f_{-w}) = \gamma'_w a_{w(v)},
\]

and the element \( l_v \in H^1(\partial M_v) \) such that \( p_w(l_v) = l_w \) for each \( w \in \partial v \). Using lemma 1.3.1 it is not difficult to verify that the element \( l_v \) lies in the image of \( H^1(M_v) \) under the canonical map \( i^* \) which is induced by inclusion \( \iota : \partial M_v \to M_v \). Now we get the required \( l_v \) as an (arbitrary) element of \( (i^*)^{-1} l_v \). If \( l_v = 0 \) we choose \( l_v = 0 \).

Conversely, let \( \{ l_v \mid v \in V \} \) be a CP-system on \( M \). To find a solution of the BKN-equation we can use the above two formulas:

\[
a_v = l_v(f_v) \quad \text{and} \quad \gamma_w = \begin{cases} 
\frac{(l_v f'_w)|w|}{a_{w(v)}} & \text{if } a_{w(v)} \neq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

So we get

\[
\sum_{w \in \partial v} \gamma_w a_{w(v)} b_w = \sum_{w \in \partial v} \frac{(l_v f'_w)|w|}{b_w} = \sum_{w \in \partial v} i^*_w l_v \left( \frac{f_{-w}}{b_w} \right)
\]

\[
= l_v \left( \sum_{w \in \partial v} (t_w) \frac{f_{-w}}{b_w} \right) = -e_{M_v}(\phi_v) l_v(f_v) = k v a_v. \quad \square
\]

It is easy to translate theorem I from CP-language to BKN-language.

**Theorem II.** A manifold \( M \in \mathfrak{M} \) satisfies the property \( \text{Im} \) (resp. \( \text{HI}, \text{F}, \text{E}, \text{VF}, \text{VE}, \text{NPC} \)) if and only if the BKN-equation over \( \Gamma_M(V,W) \) has a solution (resp. has a solution \( \{ a, \gamma \} \) such that

- (HI) \( a_v > 0 \), and if \( |\gamma_w| = 1 \) then \( \gamma_{-w} = \gamma_w \);
- (F) \( a_v > 0 \) and \( \gamma_w = \gamma_{-w} = \epsilon_v \epsilon_w \) for a function \( \epsilon : V \to \{ \pm 1 \} \);
- (E) either 1) \( \gamma_w = \gamma_{-w} = 1 \) or 2) \( a_v = 0 \) and \( \gamma_w = \gamma_{-w} = 0 \) for any \( w \in \partial v \);
- (VF) \( a_v > 0 \) and \( \gamma_w = \gamma_{-w} \);
- (VE) \( \gamma_w = \gamma_{-w} \);
- (NPC) \( a_v > 0 \) and \( \gamma_w = \gamma_{-w} \in (-1; 1) \),

for each \( v \in V, w \in W \).

This theorem follows from theorem I and lemma 1.5. \( \square \)
Remark. Items \( \text{Im} \) and \( \text{HI} \) of theorem II was proved by W. Neumann in [N2]. The item \( \text{NPC} \) was proved by S. Buyalo and V. Kobel’skii in [BK1].

1.6 Matrices and reduced criteria. Consider the following square matrices \( A^\epsilon_M = (a^\epsilon_{vv'}) \), \( A^+_M = (a^+_{vv'}) \), and \( H_M = (h_{vv'}) \) over the graph \( \Gamma_M \):

\[
a^\epsilon_{vv'} = \begin{cases} 
  k_v - \sum_{w(v)=v'} \frac{a_{vw}}{b_w}, & \text{if } v = v' \\
  - \sum_{w(v)=v'} \frac{a_{vw}}{b_w}, & \text{if } v \neq v'
\end{cases}, \quad a^+_{vv'} = \begin{cases} 
  |k_v| - \sum_{w(v)=v'} \frac{1}{|b_w|}, & \text{if } v = v' \\
  - \sum_{w(v)=v'} \frac{1}{|b_w|}, & \text{if } v \neq v'
\end{cases}
\]

\[
h_{vv'} = \begin{cases} 
  s(v)k_v - \sum_{w(v)=v'} \frac{1}{|b_w|}, & \text{if } v = v' \\
  - \sum_{w(v)=v'} \frac{1}{|b_w|}, & \text{if } v \neq v' \text{ and } k_v \cdot k_{v'} > 0 \\
  0, & \text{otherwise}
\end{cases}
\]

Here \( \epsilon : W \to \{\pm 1\} \) is a function. The matrix \( -A^\epsilon_M \) for \( \epsilon \equiv 1 \) is the “reduced plumbing matrix” from [N1]. The second matrix is equal to \( -A_- \) from [N2]. The function \( s : V \to \{0, \pm 1\} \) is constructed as follows. Vertices \( v, v' \in V \) of the graph \( \Gamma_M \) are called equivalent \( v \sim v' \) if there exists a path \( v_0 = v, v_1, v_2, \ldots, v_n = v' \) in the graph \( \Gamma_M \) such that \( k_{v_i} \cdot k_{v_{i+1}} > 0 \), for each \( i = 0, \ldots, n-1 \). An edge \( w \in W \) and a vertex \( v \in V \) are called equivalent \( w \sim v \) if \( k_v \neq 0 \) and \( v \) is equivalent to both the ends of \( w \). Edges \( w, w' \in W \) are called equivalent if either \( w' = -w \) or there exists a vertex \( v \in V \) such that \( w \sim v \sim w' \). Now we define the graph of signed components \( G(U, E_0) \) of the triple \( \{\Gamma_M(V,W), B, K\} \) as the factor graph \( \Gamma_M/\sim \). This graph \( G(U, E_0) \) was firstly defined in [BK2].

If the graph \( G(U, E_0) \) is not bipartite one then we put \( s(v) = 0 \) for each vertex \( v \in V \). Let \( G(U, E_0) \) be a bipartite graph and \( U = P \cup N \) be a splitting to parts such that \( P \cap \{v \in V \mid k_v > 0\} \neq \emptyset \). In this case we put \( s(v) = 1 \) if \( v \in P \) and \( s(v) = -1 \) if \( v \in N \).

If the graph \( G(U, E_0) \) is bipartite then the matrix \( H_M \) coincides with a matrix from [BK2]. If a graph manifold \( M \) has no block with zero charge then \( G(U, E_0) \) is bipartite and \( s(v) = \text{sgn} \ k_v \), if in addition the graph \( \Gamma_M \) has no loops then \( H_M \) can be represented as the matrix \( -(P_- \oplus N) \) from the paper [N1].

A matrix \( A \) is called semipositive defined (resp. supersingular) if \( x^tAx \geq 0 \) (resp. \( Ax = 0 \)) for each tuple \( x \) (resp. for some tuple \( x \) with no zero entry). A square matrix \( A' \) is called a principal submatrix of a square matrix \( A \) if it can be obtained from \( A \) by deleting some rows and the corresponding columns.

Theorem III. A manifold \( M \in \mathfrak{M} \) satisfies the property \( \text{Im}, \text{HI}, \text{F}, \text{E}, \text{VF}, \text{VE}, \text{and NPC} \) if and only if the following conditions (respectively) hold:

(Im, HI) either 1) diagonal elements of \( A^+ \) has the same sign and the matrix \( A^+_M \) is semipositive defined and singular or 2) the matrix \( A^+_M \) has a negative eigenvalue;
(F) the matrix $A_M$ is supersingular;

(E) there exists $\epsilon: W \to \{\pm 1\}$ such that the matrix $A_M^\epsilon$ has a supersingular principal submatrix;

(VF) either 1) the matrix $H_M$ is semipositive defined and supersingular or 2) the matrix $H_M$ has a negative eigenvalue;

(VE) the matrix $H_M$ has a principal submatrix which satisfies the requirement of the previous item

(NPC) either 1) $H_M \equiv 0$ or 2) the matrix $H_M$ has a negative eigenvalue.

The item F of this theorem was proved in [N1], the items Im and VE were proved in [N1,2] for manifolds whose graphs has no loops, the item NPC was proved in [BK2], and the item VF was proved by the author in [S]. We give no proof of the remain item E in this draft version. □

1.7 Conventions. Let $g : S \to M$ be a $\pi_1$-injective immersion. Rubinstein and Wang [RW] proved that each surface $S$ with negative Euler characteristic in a graph-manifold $M$ is $\pi_1$-injective if and only if it can be properly homotoped so that any connected component of $S \cap M_v$ is either vertical or horizontal for each maximal block $M_v$. Further we assume that all immersed surfaces already put in this position.

Let $g : S \to M$ be a $\pi_1$-injective not horizontal immersion and let $g' : S' \to M$ be the immersion such that $g'(S')$ is the boundary of a collar for $g(S)$ in $M$. If $g : S \to M$ is a $\pi_1$-injective immersion (embedding) then $g' : S' \to M$ has the same property. If $g : S \to M$ is a $\pi_1$-injective virtual embedding then $g' : S' \to M$ has the same property. Converse is also true [RW].

Further we assume that each vertical annulus of $g(S) \setminus T$ has a parallel copy.

2. Immersed and horizontal immersed surfaces in graph manifolds

2.1. The implication i$\iff$iii of the following theorem was proved by W. Neumann in [N2].

Theorem (cf. [N3]). Let $M$ be a manifold of the class $\mathcal{M}$. The following three properties are equivalent.

i $M$ contains an immersed $\pi_1$-injective surface of negative Euler characteristic.

ii there exists a CP-system on $M$

iii BKN-equation over $\Gamma_M$ is solvable.

Proof. Let $g : S \to M$ be a $\pi_1$-injective immersion. and let $C = g^{-1}(T) \subset S$. The set $S \setminus C$ is a disjoint union $\bigcup_{\alpha \in A} S_\alpha$ of connected components. Now we orient the surfaces $S_\alpha$, $\alpha \in A$ in the following way. Choosing orientation of $M$ and orientations of the Seifert fibers in maximal blocks induce an orientation on each surface $S_\alpha$ which is not annulus. By conventions 1.7, vertical part of $g(S) \cap M_v$ is a set of parallel pairs of annuli. We orient the annuli so that parallel annuli have opposite orientations.
Let $T_{|w|} \subset \partial M_v \cap \partial M_{v'}$ be a JSJ-torus and $C$ be a connected component of $g^{-1}(T_{|w|})$. The curve $C$ is oriented as a boundary component of $g^{-1}(M_v)$ and at the same time it is oriented as a boundary component of $g^{-1}(M_{v'})$. The curve is said to be consistent (resp. nonconsistent) if these two orientations coincide (resp. opposite). The sum of homological classes of consistent (resp. nonconsistent) curves on $T_{|w|}$ which are oriented as boundary components of $g(S) \cap M_v$ we denote by $c^+_w$ (resp. $c^-_w$), where $w \in \partial v$.

By Poincare duality $H_2(M_v; \partial M_v) \simeq H^1(M_v)$ we get a class $l_v \in H^1(M_v)$ which is dual for $[g(S) \cap M_v] \in H_2(M_v; \partial M_v)$. It is clear that

$$l^*_w l_v(x) = x \wedge (c^+_w + c^-_w), \quad l^*_w l^*_{v'}(x) = x \wedge (c^+_w - c^-_w)$$

for each $x \in H_1(T_{|w|})$, where $w \in \partial v, -w \in \partial v'$. Due to orientations on components of $g(S) \cap M_v$ we have $a^+_w = f_w \wedge c^+_w \geq 0$ and $a^-_w = f_w \wedge c^-_w \geq 0$.

Since

$$l_v(f_v) = a^+_w + a^-_w, \quad \langle l_v', f_v \rangle_{|w|} = a^+_w - a^-_w$$

we have $|\langle l_v', f_v \rangle_{|w|}| \leq l_v(f_v)$ (CP1).

If either $a^+_w = 0$ or $a^-_w = 0$ then (by convention 1.7) we conclude that no one of curves $g(S) \cap T_{|w|}$ is parallel to Seifert fiber of $M_v$ as well as of $M_{v'}$. Therefore either $a^-_w = a^-_{-w} = 0$ or $a^+_w = a^+_{-w} = 0$. So we have $\langle l_v', \cdot \rangle_{|w|} = \pm \langle l_v', \cdot \rangle_{|w|}$ and it is equal to CP2.

Finally, if $l_v(f_v) = 0$ then $g(S) \cap M_v$ is a set of pairs of oppositely oriented parallel annuli. So we have $l_v = 0$ (CP3).

ii$\Leftrightarrow$iii It is the assertion of lemma 1.5.

ii$\Rightarrow$i Multiplying all $\{l_v\}_{v \in V}$ by suitable integer, we can assume that $l_v$ is an integer class for each $v \in V$.

Let $c^+_w \in H_1(T_{|w|})$ (resp. $c^-_w \in H_1(T_{|w|})$) be the homological class which is dual by Poincare for $l^*_w l_v + c^+_w l^*_{v'} \in H^1(T_{|w|})$ (resp. for $l^*_w l_v - c^-_w l^*_{v'} \in H^1(T_{|w|})$), where $w \in \partial v, -w \in \partial v'$. It is obvious that $c^+_w = \pm c^-_w$. Choose a pair of nonoriented curves $C^+_w$ and $C^-_w$ on each JSJ-torus $T_{|w|}$ in $M$ so that these curves represent (with some orientation) the classes $c^+_w$ and $c^-_w$ respectively. If one of the classes (or both) is zero then we do not consider the corresponding curve.

It is easy to see that

$$f_w \wedge (c^+_w + c^-_w) = 2l_v ((l_w)_* f_w) = 2l_v(f_v),$$

$$\sum_{w \in \partial v} p_w(z_v) \wedge (c^+_w + c^-_w) = 2l_v \left( \sum_{w \in \partial v} (l_w)_* p_w(z_v) \right) = -2e_{M_v}(z_v) \cdot l_v(f_v).$$

for each adjacent element $z_v \in H_1(\partial M_v), v \in V$. If $p_w(z_v) = f_{-w}/b_w$ for each $w \in \partial v$ then the right side of $(2')$ is $k_v l_v(f_v)$. The equalities $(1')$ and $(2')$ coincide with the corresponded equalities from lemma 1.3.1 if $l_v(f_v) > 0$. 

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As above, by \( d(C)^n \) we denote \( d \) copies of a curve obtained by going \( n \) times around the curve \( c \). Using lemma 1.3.1 we can find integers \( d_v, n_v \) such that for any integer pair \( d, n \) with \( d_v \) dividing \( d \) and \( n_v \) dividing \( n \) the curves \( d(C^\pm_{|w|})^n, w \in \partial v \) span a surface (horizontal if \( l_v(f_v) \neq 0 \)) in \( M_v \). Taking appropriate integers \( D \) and \( N \) we obtain a surface by fitting together the parts \( S_v \) spanning on \( D(C^\pm_{|w|})^N, w \in \partial v \) for each \( v \in V \) such that \( l_v(f_v) > 0 \). If \( l_v(f_v) > 0 \) for each \( v \in V \) we obtain the desired surface.

Consider a block \( M_v \subseteq M \) such that \( l_v = 0 \). For each \( T_w \subseteq \partial M_v \) we either have two set of parallel curves \( D(C^+_{|w|})^N \) and \( D(C^-_{|w|})^N \) on \( T_{|w|} \) or have no curves. In the first case the curves are parallel to the linear foliation of \( T_w \) which is induced from \( M_v \). So we can find \( D \) parallel immersed vertical annuli in \( M_v \) which are not boundary parallel and their boundary is \( D(C^+_{|w|})^N \cup D(C^-_{|w|})^N \). The horizontal parts \( S_v \) (in blocks \( M_v \) with \( l_v(f_v) \neq 0 \)) as above and the annuli fit together providing the desired surface. □

2.2 The HI property. In this section we prove

**Theorem (cf. [N2]).** Let \( M \) be a manifold of the class \( \mathfrak{M} \). The following three properties are equivalent.

i. \( M \) contains a horizontal immersed surface of negative Euler characteristic.

ii. There is a CP-system \( \{l_v \mid v \in V\} \) on \( M \) such that \( l_v(f_v) > 0 \) for each \( v \in V \).

iii. BKN-equation over \( \Gamma_M \) has a nontrivial solution \( \{a, \gamma\} \) such that \( a_v > 0 \) for each \( v \in V \).

**Proof.** The equivalence ii⇔iii follows from lemma 1.5.

\( [i \Rightarrow ii] \) Using previous theorem it sufficient to prove the condition \( l_v(f_v) > 0 \). It is true, due to horizontality.

\( [ii \Rightarrow i] \) The surface fitting as in the last part of the previous proof is horizontal in each block \( M_v \) inasmuch \( l_v(f_v) > 0 \) for each \( v \in V \) and the surface is connected in each maximal block. □

3. Fibering and embedded surfaces

The map \( p : M \to S^1 \) is homotopic to a fibration iff \( p_*(f_v) \neq 0 \) for each \( v \in V \) [NR]. So, if \( p \) is a fibration then any fiber \( p^{-1}(pt) \) (surface) is transversal to Seifert fibers (circles) in each block so such a surface is horizontal.

**3.1 Theorem (cf. [N1]).** Let \( M \) be a manifold of the class \( \mathfrak{M} \) and \( \Gamma_M(V, W) \) be its graph. The following conditions are equivalent.

i. \( M \) is fibered over \( S^1 \) with oriented surface of negative Euler characteristic as a fiber.

ii. There exists a CP-system \( \{l_v\}_{v \in V} \) on \( M \) such that \( l_v(f_v) > 0 \) for each \( v \in V \) and in addition for each triple \( v, w, v' \) as in CP1 (definition 1.2) we have \( \langle l_{v'}, f_v \rangle_{|w|} = \epsilon_{v'}\epsilon_v l_v(f_v) \), where \( \epsilon : V \to \{\pm 1\} \) is a function.
iii There exists a solution \( \{ a, \gamma \} \) of BKN-equation such that \( a_v > 0 \) for each \( v \in V \) and \( \gamma_w = \gamma_{-w} = \epsilon_{w'} \epsilon_v \) for each triple \( v, w, v' \) as in CP1 (definition 1.2), where \( \epsilon : V \to \{ \pm 1 \} \) is a function.

**Proof.** The equivalence ii\( \Rightarrow \)iii follows from lemma 1.5.

i\( \Rightarrow \)ii Let \( p : M \to S^1 \) be a fibering map and \( p^{-1}(pt) \) is a fiber \( S, \chi(S) < 0 \). For each \( v \in V \) by \( p_v : M_v \to S^1 \) we denote the restriction of \( p \) to the maximal block \( M_v \).

Put by definition \( \epsilon_v = \text{sgn} \langle \alpha, p_v f_v \rangle \) where \( \alpha \) is a generator of \( H_1(S^1; \mathbb{Z}) \). The desired classes \( l_v \in H^1(M_v) \), \( v \in V \) we define as follows

\[
l_v = \epsilon_v \cdot p_v^* \alpha,
\]

It is clear that \( l_v(f_v) = |\langle \alpha, p_v f_v \rangle| > 0 \) and is equal to the geometric intersection number between \( S \cap M_v \) and a Seifert fiber in \( M_v \) which are transversal each other. To conclude the proof of \( i \Rightarrow ii \) it remains to note that

\[
\langle l_v, f_v' \rangle_{w'} = \epsilon_v \langle p_v\alpha, f_v' \rangle_{w'} = \epsilon_v \langle \iota_w^* p_v\alpha, f_{-w} \rangle = \epsilon_v \langle \alpha, p_v f_v' \rangle = \epsilon_v \epsilon_{w'} l_{v'}(f_v').
\]

[\( ii \Rightarrow i \)] Due to \( \epsilon_v l_v(f_v) \neq 0 \) we can find a fibering map \( p_v : M_v \to S^1 \) using the canonical isomorphism \( H^1(M_v) \cong [M_v, S^1] \) so that \( p_v^* (\alpha) = \epsilon_v l_v \) [N1]. Since \( \iota_w^* \epsilon_v l_v = \iota_{-w}^* \epsilon_v \iota_{w'} \iota_{v'} \) for each triple \( v, w, v' \) we can choose fibering maps \( p_v \) and \( p_{v'} \) so that \( p_{v'} |_{T_{v'} [w]} = p_{v} |_{T_v [w]} \) for each \( T_{v'} [w] \subset M_v \cap M_{v'} \). By [NR] there is a fibering map \( p : M \to S^1 \) such that \( p|_{M_v} = p_v \). \( \square \)

**3.2 Theorem (cf. [N1]).** Let \( M \) be a manifold of the class \( \mathcal{M} \) and \( \Gamma_M(V, W) \) be its graph. The following conditions are equivalent.

i \( M \) contains an embedded surface of negative Euler characteristic.

ii There exists a CP-system \( \{ l_v \}_{v \in V} \) on \( M \) such that for each triple \( v, w, v' \) as in CP1 (definition 1.2) we have \( |\langle l_v, f_v' \rangle_{w'}| = l_v(f_v) \).

iii There exists a solution \( \{ a, \gamma \} \) of the BKN-equation over \( \Gamma_M(V, W) \) such that either \( \gamma_w = \gamma_{-w} = \pm 1 \) or \( a_v = 0 \) and \( \gamma_w = \gamma_{-w} = 0 \) for each \( w \in \partial v \).

**Proof.** The equivalence ii\( \Leftrightarrow \)iii follows from lemma 1.5.

[i\( \Rightarrow \)ii] Let \( g : S \to M \) be a \( \pi_1 \)-injective embedding. Consider the boundary \( S' \) of \( N(g(S)) \), where \( N \) denotes a normal neighborhood. We orient the connected components of \( S' \setminus T \) as in the proof of theorem 2.1. Let \( l_v \in H^1(M_v) \) be the class, which is dual for \( [S' \cap M_v] \in H_2(M_v, \partial M_v) \). Since the surface \( S' \) has negative Euler characteristic, we have \( l_v(f_v) \neq 0 \) at least for one block \( M_{v'} \subset M \). Consider a JSJ-torus \( T_{v'} [w] \) separating blocks \( M_v \) and \( M_{v'} \). Since \( g \) is an embedding, each intersection \( S' \cap M_v \) either vertical or horizontal. If \( T_{v'} [w] \) does not intersect the embedded surface then \( \iota_v^* l_v = \pm \iota_{-w}^* l_{v'} \) and we have nothing to do. Assume that \( l_v(f_v) > 0 \) then either \( l_{v'}(f_{v'}) > 0 \) or \( l_{v'}(f_{v'}) = 0 \). In the first case we have \( \iota_v^* l_v = \pm \iota_{-w}^* l_{v'} \) indeed \( M_v \) and \( M_{v'} \) contain only horizontal parts of
the embedded surface so we get $\langle l_{v'}, f_v \rangle |_{w'} = \pm l_v(f_v)$. In the second case pairs of opposite oriented annuli are occurred in $M_{v'}$ and corresponded pairs of parallel horizontal surfaces are occurred in $M_v$. It gives $\langle l_{v'}, f_v \rangle |_{w'} = \langle l_v, f_{v'} \rangle |_{w'} = 0$ therefore $\iota_w^*l_{v'} = 0$ for each $-w \in \partial v'$. In any case we have $\langle |(l_w, f_{v'})_e | = l_v(f_v)$ and $\langle |(l_w, f_{v'})_e | = l_v(f_{v'})$. So $\{ l_v | v \in V \}$ is the desired CP-system.

[ii$\Rightarrow$i] Let $\{ l_v | v \in V \}$ be a CP-system which is satisfied ii and $T_{|w|}$ be a JSJ-torus in $M$ such that $w \in \partial v$, and $-w \in \partial v'$. As in the theorem’s 2.1 proof we can consider the pair of homological classes $c^+_w$ and $c^-_w$. There are three cases:

1. both $l_v$ and $l_{v'}$ are not zero,
2. either $l_v = 0$ or $l_{v'} = 0$ but not both,
3. both $l_v$ and $l_{v'}$ are zero.

In the first case ii and CP2 imply $\iota_w^*l_{v'} = \pm \iota_w^*l_v$ and there is only one non-zero homological class (either $c^+_w$ or $c^-_w$). In the second case either $\iota_w^*l_{v'}(f_w) = 0$ or $\iota_w^*l_v(f_{-w}) = 0$ respectively and we have two parallel curves which are parallel to the linear foliation on $T_{|w|}$ which is induced either from $M_v$ or from $M_{v'}$. In the third case there are no curves on $T_{|w|}$.

By lemma 1.3.2 there exists an embedded $\pi_1$-injective surface $S \subset M$ such that the curves in $S \cap T$ realize the classes $c^+_w$, $c^-_w$ for all $w \in W$. □

4. Virtual fibering and virtually embedded surfaces

Let $M$ be a graph manifold of the class $\mathcal{M}$. A covering $p : \tilde{M} \to M$ is called s-characteristic (or characteristic), if its restriction on each JSJ-torus $T \subset \tilde{M}$ is organized as follows: there exists a basis $(\tilde{a}, \tilde{b})$ of the group $\pi_1(T)$ and a basis $(a, b)$ of the group $\pi_1(p(T))$ such that $(p|_T)_*\tilde{a} = sa$ and $(p|_T)_*\tilde{b} = sb$ for a positive integer $s$ [LW]. Let $q : \Gamma_{\tilde{M}}(\tilde{V}, \tilde{W}) \to \Gamma_M(V, W)$ be the map between the graphs which is induced by $p$, and let $d_u s^2$ be the degree of the restriction $p|_{M_u} : M_{u} \to M_{q(u)}$, $u \in \tilde{V}$. Note, that the number $s^2 \sum_{q(u) = v} d_u$ is independent of $v \in V$ and is equal to the degree of $p$. In the following lemma we collect some useful results from [LW].

4.1 Lemma (Luecke and Wu). Let $p : \tilde{M} \to M$ be a finite covering of graph manifold $M$.

1. There exists a covering $\tilde{p} : \tilde{M} \to \tilde{M}$ such that $\tilde{p} = p \circ \tilde{p}$ is a characteristic covering [LW, prop.4.4]. In this case
2. the charges of vertices and the intersection indexes for edges of $\Gamma_{\tilde{M}}$ are connect by the following relations: $k_u = k_{\tilde{q}(u)}d_u$, $b_l = b_{\tilde{q}(l)}$, where $u \in \tilde{V}$, $l \in \tilde{W}$ and $\tilde{q} : \Gamma_{\tilde{M}} \to \Gamma_M$ is the map induced by $\tilde{p}$ [LW, prop.2.3].

4.2 Proposition. Let $M$ be a graph manifold of the class $\mathcal{M}$ and let $\Gamma_M(V, W)$ be its graph. If $M$ contains a virtually embedded $\pi_1$-injective surface of negative Euler
characteristic then BKN-equation has a solution \{a, \gamma\} such that \(\gamma_w = \gamma_{-w}\) for each \(w \in W\). Moreover, if the surface is horizontal then \(a_v > 0\) for each \(v \in V\).

**Proof.** Let \(M\) satisfies \(\text{VE}\). By item 1 of the previous lemma we have an \(s\)-characteristic covering \(p: \hat{M} \to M\) and a \(\pi_1\)-injective embedding \(g: S \to \hat{M}\). By theorem 3.2 there exists a solution \(\{x, \mu\}\) of the BKN-equation over \(\Gamma_{\hat{M}}\) such that \(\mu_l = \mu_{-l} \in \{0, \pm 1\}\) for each \(l \in \hat{W}\):

\[
k_u x_u = \sum_{l \in \partial u} \frac{\mu_l}{b_l} x_{l^+},
\]

where \(l^+\) is the terminal edge for \(l\). Using the item 2 of lemma 4.1 we have

\[
(*) \quad k_{\hat{q}(u)} d_u x_u = \sum_{w \in \partial \hat{q}(u)} \frac{1}{b_w} \sum_{l \in \partial u \cap \hat{q}^{-1}(w)} \mu_l x_{l^+}.
\]

Let \(M_w, w \in W\) be a rectangular matrix given by

\[
(M_w)_{uu'} = \sum_{\{l \in \hat{q}^{-1}(w) \mid u' = l(u)\}} \mu_l
\]

The columns of \(M_w\) are corresponded to vertices of \(\hat{q}^{-1}(w^+)\) as well as its rows are corresponded to vertices of \(\hat{q}^{-1}(w^-)\). Since \(\mu_l = \mu_{-l}\) we have \(M_w = M^t_w\), where \(t\) is the transposition. Let \(X_v, v \in V\) be a tuple consisting of entries \(x_u, u \in \hat{q}^{-1}(v)\) and let \(D_v\) be a diagonal matrix \(\text{diag}\{d_u, u \in \hat{q}^{-1}(v)\}\). In these notations the equality (\(*)\) can be rewrited as a vector one:

\[
(**) \quad k_v D_v X_v = \sum_{w \in \partial v} \frac{1}{b_w} M_w X_{w^+}.
\]

Put \(t_v = +\sqrt{X^t_v D_v X_v}\) and

\[
\lambda_w = \frac{X^t_w M_w X_{w^+}}{t_{w^-} \cdot t_{w^+}} \quad \text{if} \quad t_{w^-} \cdot t_{w^+} \neq 0
\]

and \(\lambda_w = 0\), if \(t_{w^-} \cdot t_{w^+} = 0\). Multiplying both sides of (\(**)\) by \(X^t_w\) from the left, we get

\[
k_v t^2_v = \sum_{w \in \partial v} \frac{\lambda_w}{b_w} t_v t_{w^+}.
\]

Since \(\lambda_w = 0\) for each \(w \in \partial v\) if \(t_v = 0\), we have

\[
k_v t_v = \sum_{w \in \partial v} \frac{\lambda_w}{b_w} t_{w^+}.
\]
Claim. \(|\lambda_w| \leq 1\).

If \(A = (a_{ik})\) is \(m \times n\)-matrix, \(x = (x_i) - m\)-tuple, \(y = (y_k) - n\)-tuple, and all entries \(a_{ik}, x_i, y_k\) are nonnegative then

\[ x^t Ay = \sum_{i,k} x_i a_{ik} y_k = \sum_i x_i \left( \sum_k \sqrt{a_{ik} \cdot a_{ik}} y_k \right) \leq \]

\[ \sum_i \left( \sum_k a_{ik} \right) \cdot \sqrt{\sum_k y_k^2} \leq \]

\[ \sqrt{\sum_i \left( \sum_k a_{ik} \right)^2} \cdot \sqrt{\sum_k y_k^2} \]

by the Cauchy inequality acting twice.

Let \(M'_w\) be the matrix which is obtained from \(M_w\) by taking absolute values of all its entries. It is clear that \(|X'_v M_w X_w(v)| \leq X'_v M'_w X_w(v)|.\) The previous inequality with the matrix \(M'_w\) and the tuples \(X_v, X_w(v),\) gives

\[ X'_v M'_w X_w(v) \leq \sqrt{X'_v D_v|w| X_v \cdot X'_w D_w|w| X_w(v),} \]

where \(D_v|w|\) is the diagonal matrix with diagonal entries which is defined by

\[ d_v|w| = \sum_{l \in \partial u \cap \hat{q}^{-1}(w)} |\mu_l|, \quad u \in \hat{q}^{-1}(v). \]

Since \(|\mu_2| \leq 1\) we have \(d_v|w| \leq \#(\partial u \cap \hat{q}^{-1}(w)) = d_u\) hence

\[ |X'_v M_w X_w(v)| \leq X'_v M'_w X_w(v) \leq \sqrt{X'_v D_v|w| X_v \cdot X'_w D_w|w| X_w(v)} \leq \]

\[ \sqrt{X'_v D_v X_v \cdot X'_w D_w|w| X_w(v)} = t_v t(w(v)). \]

The claim is proved.

Note, that \(\lambda_{-w} = \lambda_w\) since \(M'_w = M_{-w}\). If the surface \(g(S)\) is horizontal then \(t_v > 0\) for each \(v \in V\). So we have a solution \(\{t, \lambda\}\) of the BKN-equation over \(\Gamma(V, W),\) with prescribed properties. \(\square\)

4.3 Theorem. Let \(M\) be a graph manifold of the class \(\mathcal{M}\). The following three properties are equivalent.

i. \(M\) is virtually fibered over a circle.

ii. There exists a CP-system of cohomological classes \(\{l_v \mid v \in V\}\) on \(M\) such that \(l_v(f_v) > 0\) for each \(v \in V,\) and \(\langle l_w, f_v|w| \rangle \cdot l_w(f_v') = \langle l_v, f_v'|w| \rangle \cdot l_v(f_v)\) for each triple \(v, w, v'\) as in \(CP1\) (definition 1.2).

iii. The BKN-equation over \(\Gamma_M\) has a nontrivial solution \(\{a, \gamma\}\) such that \(a_v > 0\) and \(\gamma_w = \gamma_{-w}\) for each \(v \in V, w \in W.\)
Proof. The equivalence ii $\iff$ iii follows from lemma 1.5. The implication i $\Rightarrow$ ii is in proposition 4.2.

The implication ii $\Rightarrow$ i is proved in our previous work [S]. The proof uses the main result of [RW]. □

Now we need some technique which is useful in what follows. Let $M$ be a manifold of the class $\mathcal{M}$, let $\Gamma_M(V, W)$ be its graph and let $V_0 \subset V$ be a subset of its maximal blocks. Remove the blocks $\{M_v | v \in V_0\}$ from $M$ and let

$$M \setminus V_0 := M \setminus \bigcup_{v \in V_0} M_v$$

be a (possible not connected) manifold with boundary. The manifold $M \setminus V_0$ is canonically framed as follows. Consider $T_{\partial V_0}$, a boundary torus of $M \setminus V_0$ (here the initial point of $w$ is in $V_0$ and its terminal point is in $V \setminus V_0$), and choose $r_w$ as a curve on $T_w$ which corresponds to the Seifert fibration of the removing block $M_v$, $w \in \partial v$. Let $(M \setminus V_0)^D$ be the closed manifolds by performing Dehn filling along all $r_w$, $T_{|w|} \subset \partial(M \setminus V_0)$. This manifold is defined up to homeomorphism. The charges of maximal blocks of $(M \setminus V_0)^D$ are coincides with the charges of corresponding blocks of $M$ [N1]. The graph of $(M \setminus V_0)^D$ is a subgraph of $\Gamma_M(V, W)$.

4.4 Theorem (cf. [N1]). Let $M$ be a manifold of the class $\mathcal{M}$ and $\Gamma_M(V, W)$ be its graph. The following conditions are equivalent.

i $M$ contains a virtually embedded $\pi_1$-injective surface of negative Euler characteristic.

ii There exists a CP-system of cohomological classes $\{l_v | v \in V\}$ on $M$ such that $\langle l_v, f_w \rangle_{|w|} = l_w(f_v) = l_v(f_w)$ for each triple $v, w, v'$ as in CP1 (definition 1.2).

iii The BKN-equation over $\Gamma_M$ has a solution $\{a, \gamma\}$ such that $\gamma_w = \gamma_{-w}$ for each $v \in V, w \in W$.

Proof. The equivalence ii $\iff$ iii follows from lemma 1.5.

The implication i $\Rightarrow$ ii is in proposition 4.2.

iii $\Rightarrow$ i Let $\{a, \gamma\}$ be a solution of the BKN-equation which satisfies iii and $V_0 = \{v \in V | a_v = 0\}$. Consider a manifold $M_1$ with boundary which is a connected part of $M \setminus V_0$, and let $M_1^D$ be the corresponding closed graph manifold (may be Seifert fibered). It is easy to see that the collection $\{a_v, \gamma_w | v \in V_1, w \in W_1\}$ is a solution of the BKN-equation over $\Gamma_M(V_1, W_1) \subset \Gamma_M(V, W)$ and it satisfies to iii of theorem 4.3. By the theorem, there exists a finite covering $p_1^D : \tilde{M}_1^D \to M_1^D$ and a horizontal embedding $\tilde{g}_1^D : S_1^D \to \tilde{M}_1^D$, where $S_1^D$ is a closed oriented surface. Let $p_1 : \tilde{M}_1 \to M_1$ be the covering which is induced by the inclusion $\iota_1 : M_1 \to M_1^D$, i.e. $p_1 = \iota_1^* p_1^D$. We also have a natural proper horizontal embedding $\tilde{g}_1 : S_1 \to \tilde{M}_1$, where $S_1$ is the surface $S_1^D$ with some number of holes and $\chi(S_1) < 0$. 
Now we claim that there exists a covering \( p : \tilde{M} \to M \) such that \( \nu^*(p) = p_1 \) for the natural inclusion \( \nu : M_1 \to M \). Really, this inclusion induces the injective map \( \nu_* : \pi_1(M_1) \to \pi_1(M) \). Such a way we have an embedding \((S_1, \partial S_1) \to (\tilde{M}, \tilde{T})\) where \( \tilde{T} \) is the JSJ-surface for \( \tilde{M} \). The surface \( S_1 \) is contained in \( M_1 \subset M \). Let \( C \) be a connected component of \( \partial S_1 \cap T \) where \( T \in \tilde{T} \) and \( \partial S \cap T \neq \emptyset \). The torus \( T \) must separates two blocks. One of the blocks say \( M_u \) must be preimage (under \( p \)) of one of removing blocks \( M_v, v \in V_0 \). As it easy to see the curve \( C \) is homotopic to the linear foliation of \( T \) which is induced from the Seifert space \( M_u \). Let \((S_2, \partial S_2) \subset (\tilde{M}, \tilde{T})\) be a surface obtained as two parallel copies of \( S_1 \). The intersection \( M_u \cap S_2 \) consists on even number of curves on \( \partial M_u \). Each curve parallel to the linear foliation of the boundary component it lies. It is not difficult to see that there are a set of incompressible, boundary-incompressible annuli in \( M_u \) whose boundary is \( M_u \cap S_2 \). So we have found a \( \pi_1 \)-injective surface in some finite cover of \( M \). \( \Box \)

5. NPC-metrics on graph manifolds

5.1 Theorem (cf. [BK2]). Let \( M \) be a manifold of the class \( \mathcal{M} \) and \( \Gamma_M(V, W) \) be its graph. The following conditions are equivalent.

i \( M \) carries an NPC-metric.

ii there exists a CP-system \( \{l_v\}_{v \in V} \) on \( M \) such that \((-\langle l_v, f_v \rangle_e) < l_v(f_v) \) and \( \langle l_v', f_e \rangle_c \cdot l_{v'}(f_e) = \langle l_v, f_e \rangle_{|w} \cdot l_v(f_e) \) for each triple \( v, e, v' \) as in CP1 (definition 1.2);

iii there exists a solution \( \{a, \gamma\} \) of CE such that \( a_v > 0 \) for each \( v \in V \) and \( \gamma_w = \gamma_{-w} \in (-1, 1) \) for each \( w \in W \).

Proof. The equivalence ii\(\iff\)iii follows from lemma 1.5.

iii\(\iff\)i If each Seifert block of \( M \) is a product surface \( \times S^1 \) then the equivalence follows from [BK1, Proposition 8.1]. The general case is proved by I. Andreeva (unpublished). \( \Box \)

References

[B] S. Buyalo, Metrics of nonpositive curvature on graph manifolds and electromagnetic fields on graphs, Nauchnye Zapiski POMI 280 (2001), 3–45.

[BK1] S. Buyalo, V. Kobel’skii, Geometrization of graph-manifolds. II. Conformal geometrization, St. Petersburg Math. J. 7 (1996), no. 2, 185–216.

[BK2] S. Buyalo, V. Kobel’skii, Geometrization of graph-manifolds. II. Isometric geometrization, St. Petersburg Math. J. 7 (1996), no. 3, 387–404.

[L] Leeb B., Manifolds with(out) metrics of non-positive curvature, Invention. Math. 122 (1995), 277–289.

[M] S. Matsumoto, Separability criterion for graph-manifold groups, Topology Appl. 93 (1999), 17–33.

[N1] W.D. Neumann, Commensurability and virtual fibration for graph manifolds, Topology 39 (1996), 355–378.
[N2] W. D. Neumann, *Immersed and virtually embedded surfaces in graph manifolds*, Algebraic and Geometric Topology 1 (2001), 411-426.

[NR] Neumann W. D., Raymond F., *Seifert manifolds, plumbing, μ-invariant and orientation reversing maps*, Springer, Berlin, Lect. Notes in Math. 664 (1978), 162–195.

[RW] Rubinstein J. H., Wang S., *π₁-injective surfaces in graph-manifolds*, Comment. Math. Helv. 73 (1998), 499–515.

[S] P. Svetlov, *Non-positively curved graph manifolds are virtually fibered over the circle*, preprint, arXiv math.GT/0108010 (2001), 10 p.

[Th] Thurston W., *Hyperbolic structures on 3-manifolds, II: surface groups and 3-manifolds which fiber over the circle*, preprint, arXiv math.GT/9801045 (1998), 32 p.

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