Applications of Perturbative NRQCD

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We review the theoretical ideas and tools required to arrive at a next-to-next-to-leading order (NNLO) description of the heavy quark-antiquark production cross section in $e^+e^-$ annihilation for the case that the center of mass kinetic energy of the quarks is larger than $\Lambda_{QCD}$. In this case the NNLO cross section can be calculated with purely perturbative methods. We present details of the calculation and discuss two applications, the determination of the bottom quark mass from $\Upsilon$ sum rules and the top-antitop production cross section close to threshold in $e^+e^-$ annihilation.

1 Introduction

Within the last year significant progress has been achieved in the perturbative description of heavy quark-antiquark ($Q\bar{Q}$) pairs in the kinematic regime close to threshold using the concepts of effective field theories. In this talk we report on some of these developments from the point of view of practical applications. There have also been many interesting new results concerning the systematics of the effective field theory description of $Q\bar{Q}$ pairs at threshold which would deserve to be presented in detail but can only be mentioned peripherally due to lack of space.

To be definite, we consider the total cross section $\sigma^{thr} = \sigma(e^+e^- \to Q\bar{Q} + \text{anything})$ for the c.m. energies $\sqrt{s} \approx 2M_Q$, $M_Q$ being the heavy quark mass. In this kinematic regime the $Q\bar{Q}$ dynamics is characterized by the fact that the heavy quarks have small c.m. velocities. Considering the strong dependence of the velocity ($v = \sqrt{1 - 4M_Q^2/s}$) in the threshold regime on $M_Q$ it is obvious that, at least in principle, accurate calculations of the cross section may lead to

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precise determinations of the quark mass. The situation, however, is not simple due to the fact that the $Q\bar{Q}$ system in the nonrelativistic regime is governed by (at least) three scales. This makes this system particularly complicated. The three scales are $M_Q$, the quark mass, $M_Qv$, the relative momentum, and $M_Qv^2$, the kinetic energy in the c.m. frame. This has three important consequences which are symptomatic for all nonrelativistic $Q\bar{Q}$ systems:

1. Because $v \ll 1$, the three scales are widely separated ($M_Q \gg M_Qv \gg M_Qv^2$). The theoretical tools required to describe the $Q\bar{Q}$ pair strongly depend on whether the scales each are smaller or larger then $\Lambda_{\text{QCD}}$. The most transparent situation arises if $M_Qv^2 > \Lambda_{\text{QCD}}$ because in this case the theoretical tools resemble most closely those of QED bound state calculations. In fact, for the two applications mentioned in this talk it is ensured that this condition is satisfied.

2. Because ratios of the three scales arise, the description of the nonrelativistic $Q\bar{Q}$ pair involves a double expansion in $\alpha_s$ and $v$. This means that the standard multi-loop expansion in $\alpha_s$ breaks down. The most prominent indication of this fact is the so called Coulomb singularity, originating from the ratio $M_Q/M_Qv$, which corresponds to a singular $(\alpha_s/v)^n$-behavior in the $n$-loop correction to the amplitude $\gamma \to Q\bar{Q}$ for $v \to 0$. The latter singularity is caused by the 00-component of the gluon propagator and, in Coulomb gauge, is directly related to the exchange of longitudinally polarized gluons. This singularity has to be treated by resumming diagrams involving longitudinal gluon exchange to all orders in $\alpha_s$. The Coulomb singularity exists in the nonrelativistic limit, but there are also power-like and logarithmic divergences in $v$ which are suppressed by powers of the velocity. At this point we would like to specify more clearly what “resummation” means in this context. Strictly speaking it would mean a resummation of the perturbation series in $\alpha_s$ to all orders, where the respective coefficients are expanded up to a certain power in $v$. This means that the resummation would be carried out in the (formal) limit $\alpha_s \ll v \ll 1$. The resulting series would then (uniquely) define analytic functions which could be continued to the region of interest $|v| \sim \alpha_s$. Typical structures like $Q\bar{Q}$ bound states can only be observed after this continuation. Of course, this would be a highly cumbersome and inefficient method. It is therefore mandatory to reformulate the problem in terms of wave equations. The solutions of these wave equations are equivalent to the results of the resummation method. As a matter of convenience we will also call the wave equation method “resummation” for the rest of this talk.

3. From point 1 we can conclude that the c.m. velocity of the heavy quarks should satisfy the condition $v > (\Lambda_{\text{QCD}}/M_Q)^{1/2}$ to ensure that the scale $M_Qv^2$ is perturbative. In addition, relativistic corrections are not suppressed by fac-
tors of $\pi$ like multi-loop corrections. This means that relativistic corrections can be quite sizeable. The corrections of $\mathcal{O}(v^2)$, called NNLO from now on, can be estimated to be of order $20-30\%$ for $b\bar{b}$ and $5\%$ for $t\bar{t}$. Thus, the calculation of higher order relativistic corrections is mandatory in order to achieve sufficient theoretical accuracy and to test the reliability of the perturbative description itself. Obviously the perturbative treatment works better if $M_Q$ is large.

In order to arrive at a reliable theoretical description of $\sigma^{thr}$ we have to go through two steps: first, we have to address the question how to organize the calculation systematically keeping in mind points 1-3, and, second, we have to actually carry out the calculation itself. In Section 2 we will briefly address the systematics and in Section 3 we will present the calculation of the photon mediated cross section at NNLO in the nonrelativistic expansion. Section 4 is devoted to two applications, the determination of the bottom quark mass and the $t\bar{t}$ cross section at threshold in $e^+e^-$ annihilation.

2 Systematics and NRQCD

A very economical approach to systematically deal with the problems described previously is to take advantage of the separation of the scales $M_Q$, $M_Qv$ and $M_Qv^2$ using the concepts of effective field theories. In the following we outline the conceptual steps to arrive at a NNLO description of a nonrelativistic $Q\bar{Q}$ pair for $\Lambda_{QCD} < M_Qv^2$ without going very far into formal considerations. The basic idea of the effective field theory approach is to integrate out momenta above the scales relevant for the nonrelativistic dynamics of the $Q\bar{Q}$ pair. Doing this, one always has to keep in mind the relation of each of the scales to $\Lambda_{QCD}$. (If $M_Q$ were of order as or even smaller than $\Lambda_{QCD}$ a nonrelativistic expansion would be meaningless in the first place.)

Suppose that $M_Q$ is larger than $\Lambda_{QCD}$. In this case we can integrate out momenta of order $M_Q$ because they are not responsible for the nonrelativistic dynamics of the $Q\bar{Q}$ pair. Starting from QCD we then arrive at an effective field theory in which the heavy quarks and the gluons interacting with them only carry momenta below $M_Q$. This forces us to introduce different fields for heavy quark and antiquark. The resulting theory is called nonrelativistic QCD (NRQCD) and its Lagrangian reads

$$\mathcal{L}_{NRQCD} = -\frac{1}{2} \text{Tr} G^{\mu\nu} G_{\mu\nu} + \sum_{\text{light quarks}} \bar{q} i\gamma_\mu q + \psi^\dagger \left[ iD_\mu + a_1 \frac{D^2}{2M_Q} \right] \psi + \ldots$$
\[ + \psi^\dagger \left[ + a_2 \frac{D^4}{8 M_Q^2} + \frac{a_3 g}{2 M_Q} \sigma \cdot B + \frac{a_4 g}{8 M_Q} (D \cdot E - E \cdot D) \right] \psi + \]

\[ + \psi^\dagger \left[ \frac{a_5 g}{8 M_Q^2} i \sigma (D \times E - E \times D) \right] \psi + \ldots + \chi^\dagger \chi \text{ bilinears.} \quad (1) \]

The gluonic and light quark degrees of freedom are described by the conventional relativistic Lagrangian, whereas the heavy quark and antiquark are described by the Pauli spinors \( \psi \) and \( \chi \), respectively. For convenience all color indices are suppressed. Only those terms relevant for the NNLO cross section are displayed, where we have omitted the straightforward antitop bilinears. The latter can be obtained through charge conjugation symmetry. The effects coming from momenta or order \( M_Q \) are encoded in the short-distance coefficients \( a_1, \ldots, a_5 \). They can be determined as a perturbative series in \( \alpha_s \) at the scale \( \mu_{\text{hard}} = M_Q \) through the matching procedure. If \( \Lambda_{\text{QCD}} \) were of order of or even larger than \( M_Q \) (which is essentially the situation for \( c \bar{c} \)) this would be all we could do using perturbation theory. For \( \Lambda_{\text{QCD}} \) smaller than \( M_Q \), however, one can go further and also integrate out gluonic (and light quark) momenta of order \( M_Q \). The resulting theory has been called “potential NRQCD” (PNRQCD) in Ref. 2 and is characterized by the fact that its Lagrangian contains spatially non-local four-fermion interactions which are nothing else than instantaneous (static) \( Q \bar{Q} \) potentials. The “short-distance” coefficients of the corresponding operators describe the gluonic (and light quark) effects from momenta of order \( M_Q \) and can be calculated perturbatively at the scale \( \mu_{\text{soft}} = M_Q \). To NNLO (i.e. including potentials suppressed by at most \( \alpha_s^2 \), \( \alpha_s/M_Q \) or \( 1/M_Q^2 \) relative to the Coulomb potential) the relevant \( Q \bar{Q} \) potentials read (\( a_s \equiv \alpha_s(\mu_{\text{soft}}) \), \( C_A = 3 \), \( C_F = 4/3 \), \( T = 1/2 \), \( \bar{\mu} \equiv \epsilon^2 \mu_{\text{soft}}, r \equiv |\vec{r}| \))

\[ V_c(\vec{r}) = - \frac{C_F a_s}{r} \left( \frac{a_s}{4 \pi} \left[ 2 \beta_0 \ln(\bar{\mu} r) + a_1 \right] + \left( \frac{a_s}{4 \pi} \right)^2 \left[ \beta_0^2 \left( 4 \ln^2(\bar{\mu} r) + \frac{\pi^2}{3} \right) + 2 \left( 2 \beta_0 a_1 + \beta_1 \right) \ln(\bar{\mu} r) + a_2 \right] \right) \quad (2) \]

\[ V_{6F}(\vec{r}) = \frac{C_F a_s \pi}{M_Q^2} \left[ 1 + \frac{8}{3} \vec{S}_t \vec{S}_\bar{t} \right] \delta^{(3)}(\vec{r}) + \frac{C_F a_s}{2 M_Q^2 r} \left[ \nabla^2 + \frac{1}{r^2} r (\vec{r} \nabla) \nabla \right] \]

\[ - \frac{3 C_F a_s}{M_Q^2 \vec{r}^2} \left[ \frac{1}{3} \vec{S}_t \vec{S}_\bar{t} - \frac{1}{r^2} \left( \vec{S}_t \vec{r} \right) \left( \vec{S}_\bar{t} \vec{r} \right) \right] + \frac{3 C_F a_s}{2 M_Q^2 \vec{r}^3} \vec{L} (\vec{S}_t + \vec{S}_\bar{t}) \right) \quad (3) \]
\[ V_{\text{NA}}(\vec{r}) = -\frac{C_A C_F a_s^2}{2 M_Q r^2}, \]  

(4)

where \( \vec{S}_t \) and \( \vec{S}_{\bar{t}} \) are the top and antitop spin operators, \( \vec{L} \) is the angular momentum operator and \( \beta_{0,1} \) are the one- and two-loop beta-functions. The constants \( a_{1,2} \) have been calculated in Refs. [3, 4]. \( V_c \) is the Coulomb (static) potential and \( V_{\text{BF}} \) the Breit-Fermi potential known from positronium. \( V_{\text{NA}} \) is a purely non-Abelian potential generated through non-analytic terms in the one-loop vertex corrections to the Coulomb potential involving the triple gluon vertex. The remaining dynamical gluon (light quark) fields can only carry momenta of order \( M_Q v^2 \) and describe radiation and retardation effects. If \( \Lambda_{\text{QCD}} < M_Q v^2 \) one can show that those retardation effects are of NNNLO in \( \sigma^{\text{thr}} \) using arguments known from QED and taking into account how the gluon self-coupling scales with \( v \) for gluonic momenta of order \( M_Q v^2 \). [This only works because the \( Q\bar{Q} \) pair is produced in a color singlet state!] This means that retardation effects (and the scale \( M_Q v^2 \)) can be ignored at NNLO and that the \( Q\bar{Q} \) dynamics can be described by a two-body positronium-like Schrödinger equation equation of the form 

\[
\left( -\frac{\vec{\nabla}^2}{M_Q} - \frac{\vec{\nabla}^4}{4 M_Q^3} + \left[ V_c(\vec{r}) + V_{\text{BF}}(\vec{r}) + V_{\text{NA}}(\vec{r}) \right] - E \right) G(\vec{r}, \vec{r}^\prime, E) = \delta^{(3)}(\vec{r} - \vec{r}^\prime),
\]  

(5)

containing the heavy quark kinetic energy up to NNLO and the instantaneous potentials [(2)–(4)]. In Eq. (5) \( M_Q \) is defined as the pole mass. If \( \Lambda_{\text{QCD}} \) were of order or even larger than \( M_Q v^2 \), on the other hand, the coupling of the radiation gluon with the heavy quark would become of order one and retardation effects would be NNLO. In this case a perturbative calculation of \( \sigma^{\text{thr}} \) would be impossible at NNLO because retardation effects would be non-perturbative. For this reason we have to make sure that the condition \( M_Q v^2 > \Lambda_{\text{QCD}} \) is satisfied, if the NNLO expression for \( \sigma^{\text{thr}} \) derived in the following section shall be trusted.

### 3 Calculation of the Total Cross Section

For simplicity we only consider the photon mediated total cross section. The inclusion of the \( Z \) exchange is trivial for the vector current contributions. The contributions from the axial-vector current can be easily implemented at NNLO because the axial-vector current produces the \( Q\bar{Q} \) pair in a P-wave state which leads to a suppression \( \propto v^2 \) relative to the vector current contribution.
We start from the fully covariant expression for the normalized total cross section \( R_{Q\bar{Q}} \equiv \sigma^{thr} / \sigma(e^+ e^- \to \mu^+ \mu^-) \)

\[
R_{Q\bar{Q}}(q^2) = \frac{4 \pi Q^2}{q^2} \text{Im} \left[ -i \int dx e^{i q \cdot x} \langle 0 | T j^b_{\mu}(x) j^b_{\mu}(0) | 0 \rangle \right]
\]

\[
eq \frac{4 \pi Q^2}{q^2} \text{Im} \left[ -i \langle 0 | T \bar{j}^b_{\mu}(q) \bar{j}^b_{\mu}(-q) | 0 \rangle \right], \tag{6}
\]

and expand the electromagnetic current which produces/annihilates the \( Q\bar{Q} \) pair with c.m. energy \( \sqrt{q^2} \) in terms of \( ^3S_1 \) NRQCD currents up to dimension eight \( (i = 1, 2, 3) \)

\[
\tilde{j}_i(q) = b_1 \left( \tilde{\psi}^\dagger \sigma_i \tilde{\chi} \right)(q) - \frac{b_2}{6M_Q^2} \left( \tilde{\psi}^\dagger \sigma_i \left( -\frac{i}{2} \bar{D} \right)^2 \tilde{\chi} \right)(q) + \ldots ,
\]

\[
\tilde{j}_i(-q) = b_1 \left( \tilde{\chi}^\dagger \sigma_i \tilde{\psi} \right)(-q) - \frac{b_2}{6M_Q^2} \left( \tilde{\chi}^\dagger \sigma_i \left( -\frac{i}{2} \bar{D} \right)^2 \tilde{\psi} \right)(-q) + \ldots , \tag{7}
\]

where the constants \( b_1 \) and \( b_2 \) are short-distance coefficients normalized to one at the Born level. Only the spatial components of the currents contribute at the NNLO level. Inserting expansion (7) back into Eq. (6) leads to the nonrelativistic expansion of the NNLO cross section

\[
R_{\text{NNLO}}^{thr}(E) = \frac{\pi Q^2}{M_Q^2} C_1(\mu_{\text{hard}}, \mu_{\text{fac}}) \text{Im} \left[ A_1(E, \mu_{\text{soft}}, \mu_{\text{fac}}) \right] - \frac{4 \pi Q^2}{3M_Q^4} C_2(\mu_{\text{hard}}, \mu_{\text{fac}}) \text{Im} \left[ A_2(E, \mu_{\text{soft}}, \mu_{\text{fac}}) \right] + \ldots , \tag{8}
\]

where

\[
A_1 = i \langle 0 \left| \left( \tilde{\psi}^\dagger \bar{\sigma} \tilde{\chi} \right) \left( \tilde{\chi}^\dagger \bar{\sigma} \tilde{\psi} \right) \right| 0 \rangle , \tag{9}
\]

\[
A_2 = \frac{1}{2} i \langle 0 \left| \left( \tilde{\psi}^\dagger \bar{\sigma} \tilde{\chi} \right) \left( \tilde{\chi}^\dagger \bar{\sigma} \left( -\frac{i}{2} \bar{D} \right)^2 \tilde{\psi} \right) \right| 0 \rangle + \text{h.c.} \tag{10}
\]

The cross section is expanded in terms of a sum of absorptive parts of nonrelativistic current correlators, each of them multiplied by a short-distance coefficient. In fact, the right-hand side (RHS) of Eq. (8) just represents an application of the factorization formalism proposed in \cite{5}. The second term on the RHS of Eq. (8) is suppressed by \( v^2 \), i.e. of NNLO. This can be seen explicitly
by using the equations of motion from the NRQCD Lagrangian, which relates the correlator $A_2$ directly to $A_1$,

$$A_2 = M_Q E A_1. \quad (11)$$

Relation (11) has also been used to obtain the coefficient $-4/3$ in front of the second term on the RHS of Eq. (8). The nonrelativistic current correlators $A_{1,2}$ contain the resummation of the singular terms mentioned previously. They depend on the renormalization scale $\mu_{\text{soft}}$ through the potentials (2)–(4). The constants $C_1$ and $C_2$ (which are also normalized to one at the Born level), on the other hand, describe short-distance effects and, therefore, depend on the hard scale $\mu_{\text{hard}}$. They only represent a simple power series in $\alpha_s$ (where the coefficients contain numbers and logarithms of $M_Q$, $\mu_{\text{fac}}$ and $\mu_{\text{hard}}$) and do not contain any resummations in $\alpha_s$. At NNLO they have to be calculated up to order $\alpha_s^2$ because we count $\alpha_s/v$ of order one for a perturbative nonrelativistic $Q\bar{Q}$ system.

The nonrelativistic correlators $A_{1,2}$ are calculated by determining the Green function of the Schrödinger equation (5) where $V_{\text{BF}}$ is evaluated for the $3S_1$ configuration. The NNLO relation between the correlator $A_1$ and Green function reads

$$A_1 = 6 N_c \left[ \lim_{|\vec{r}|,|\vec{r}'| \to 0} G(\vec{r},\vec{r}',E) \right]. \quad (12)$$

Eq. (12) can be quickly derived from the facts that $G(\vec{r},\vec{r}',\tilde{E})$ describes the propagation of a quark-antiquark pair which is produced and annihilated at relative distances $|\vec{r}|$ and $|\vec{r}'|$, respectively, and that the $Q\bar{Q}$ pair is produced and annihilated through the electromagnetic current at zero distances. Therefore $A_1$ must be proportional to $\lim_{|\vec{r}|,|\vec{r}'| \to 0} G(\vec{r},\vec{r}',E)$. The correct proportionality constant can then be determined by considering production of a free (i.e. $\alpha_s = 0$) $Q\bar{Q}$ pair in the nonrelativistic limit. (In this case the Born cross section in full QCD can be easily compared to the imaginary part of the Green function of the free nonrelativistic Schrödinger equation.) The correlator $A_2$ is determined from $A_1$ via relation (11). We would like to emphasize that the zero-distance Green function on the RHS of Eqs. (12) contains UV divergences from the higher dimensional NNLO effects which have to be regularized. In the actual calculations carried out in Refs. [6,7] we have imposed the explicit short-distance cutoff $\mu_{\text{fac}}$, called factorization scale. This is the reason why the correlators also depend on $\mu_{\text{fac}}$. One way to solve Eq. (12) is to start from the well known Green function $G_0^{(0)}$ of the nonrelativistic Coulomb problem and to incorporate all the higher order terms via first and second order Rayleigh-Schrödinger time-independent perturbation theory.
To determine the short-distance constant $C_1$ up to $O(\alpha_s^2)$ we can expand expression (8) in the (formal) limit $\alpha_s \ll v \ll 1$ (for $\mu_{\text{soft}} = \mu_{\text{hard}}$) up to $O(\alpha_s^2)$ and demand equality (i.e. match) to the total cross section obtained at the two-loop level in full QCD keeping terms up to NNLO in an expansion in $v$. In this limit fixed multi-loop perturbation theory (i.e. an expansion in $\alpha_s$) as well as the nonrelativistic approximation (i.e. a subsequent expansion in $v$) are feasible. Because the full QCD cross section is independent of $\mu_{\text{fac}}$, $C_1$ also depends on $\mu_{\text{fac}}$. We call this kind of matching calculation at the level of the final result “direct matching”. The consistency of the effective field theory approach ensures that $C_1$ only contains contributions from momenta of order $M_Q$ and does not have any terms singular in $v$. Details of this calculation and references regarding the important calculations of the two-loop cross section in full QCD at NNLO in the velocity expansion can be found in Ref.\[8]. A very economical method to calculate Feynman diagrams in full QCD in an expansion in $v$ using dimensional regularization, the “Threshold Expansion”, has been developed in Ref.\[8].

4 Applications

4.1 Bottom Quark Mass from $\Upsilon$ Mesons

Due to causality, derivatives of the electromagnetic current-current correlator with respect to $q^2$ at $q^2 = 0$ are directly related to the total photon mediated cross section of bottom quark-antiquark production in $e^+e^-$ annihilation,

$$P_n = \frac{4\pi^2 Q_b^2}{n! q^2} \left( \frac{d}{dq^2} \right)^n \Pi_{\mu}(q) \bigg|_{q^2=0} = \int \frac{ds}{s^{n+1}} R_{bb}(s). \quad (13)$$

Assuming global duality, $P_n$ can be either calculated from experimental data for the total cross section in $e^+e^-$ annihilation\[7] or theoretically using quantum chromodynamics (QCD). It is the basic idea of this sum rule to set the moments calculated from experimental data, $P_n^{\text{exp}}$, equal to those determined theoretically from QCD, $P_n^{\text{th}}$, and to use this relation to determine the bottom quark mass (and the strong coupling) by fitting theoretical and experimental moments for various values of $n$. At this point it is important to set the range of allowed values of $n$ for which the moments $P_n^{\text{th}}$ can be trusted using the NNLO cross section calculated in the preceding section. As mentioned, we have to make sure that $M_b\langle v \rangle^2 > \Lambda_{\text{QCD}}$ where $\langle v \rangle$ is the effective c.m. bottom quark velocity in a particular moment. One can show that $\langle v \rangle \sim 1/\sqrt{n}$ for large $n$.

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\[b\] At the level of precision in this work the $Z$ mediated cross section can be safely neglected.
Figure 1: (a) Allowed region in the $M_b^{\text{Pole}}$-$\alpha_s$ plane for the unconstrained fit using the moments at NNLO. The gray shaded region represents the allowed region. The dots are points of minimal $\chi^2$ for a large number of models. Experimental errors are included at the 95% CL level and have been drawn as narrow gray ellipses centered at the dots of minimal $\chi^2$. (b) Allowed $M_b^{\text{Pole}}$ values for a given value of $\alpha_s$. The gray shaded region corresponds to the allowed ranges for the NNLO analysis and the striped region for the NLO analysis. Experimental errors are included at the 95% CL level. It is illustrated how the allowed range for $M_b^{\text{Pole}}$ at NNLO is obtained if $0.114 \leq \alpha_s(M_Z) \leq 0.122$ is taken as an input.

This means that $n$ should be sufficiently smaller than $15 - 20$. It is interesting that the same conclusion can be drawn from the Poggio-Quinn-Weinberg argument that the effective energy smearing range contained in the moments should be larger than $\Lambda_{\text{QCD}}$. On the other hand, $n$ has to be large enough that the use of the cross section at threshold is justified because the energy regime close to threshold dominates $P_n$ only in this case. In our analysis we have taken the range $4 \leq n \leq 10$. Larger values of $n$ increase the danger of large systematic errors. To determine the allowed range for the bottom quark mass we have fitted $P_n^{\text{th}}$ to $P_n^{\text{exp}}$ for various sets of $n$'s. It turns out that the theoretical errors are much larger than the experimental ones. The dominant theoretical errors come from the dependence of $P_n^{\text{th}}$ on the scale $\mu_{\text{soft}}$. We have combined experimental and theoretical errors by using the “scanning” method in which a large number of statistical fits is carried out for various
Table 1: Recent determinations of bottom quark masses using QCD sum rules for the Υ mesons. \(m_b(m_b)\) refers to the \(\overline{\text{MS}}\) mass. NLO refers to analyses including corrections of order \(\alpha_s\) to the nonrelativistic limit and NNLO to analyses including corrections of order \(\alpha_s^2\), \(\alpha_s v\) and \(v^2\). No order is indicated for Ref. 10 because the bound state poles have not been taken into account for \(P_{\text{th}}\) in that analysis. For entries where also values of \(\alpha_s\) are given a simultaneous fit for mass and QCD coupling has been carried out. In our analysis (Ref. 6) the errors are not written as Gaussian errors but as allowed ranges.

| authors | order | \(n\) | \(\mathcal{M}_P^{\text{pole}}\) [GeV] | \(m_b(m_b)\) [GeV] |
|---------|-------|------|--------------------------------|------------------|
| Ref. 4  | NLO   | 4 – 10 | 4.64 – 4.92 | \(\alpha_s(M_Z) = .086 - 1.32\) |
|         | NNLO  | 4 – 10 | 4.14 – 4.37 | \(\alpha_s(M_Z) = .096 - 1.24\) |
|         | NNLO  | 4 – 10 | 4.18 – 4.35 | \(\alpha_s(M_Z) = .096 - 1.24\) |
| Ref. 4  | NLO   | 8 – 20 | 4.82 \pm .001 | \(\alpha_s(M_Z) = .109 \pm .001\) |
|         | NNLO  | 8 – 20 | 4.604 \pm .014 | \(\alpha_s(M_Z) = .118^{+ .007}_{-.006}\) |
| Ref. 11 | NLO   | 10 – 20 | 4.75 \pm .04 | \(\alpha_s(M_Z) = .120^{+ .010}_{-.008}\) |
| Ref. 12 | NNLO  | 10 – 20 | 4.78 \pm .04 | |
| Ref. 13 | NNLO  | 14 – 18 | 4.20 \pm .10 | |
| Ref. 14 | NNLO  | 8 – 12 | 4.80 \pm .06 | |

"reasonable" choices for the scales, each called "a model". It is believed that this method represents a conservative way to combine experimental and large theoretical errors. Our result for a simultaneous fit for the bottom pole mass and \(\alpha_s\) is displayed in Fig. 1a. In Fig. 1b the result for the pole mass is displayed if \(\alpha_s\) is taken as an input. It is conspicuous that the extracted values for the pole mass are quite different for both methods. This variation could be explained from the fact that the pole mass is not defined beyond an accuracy of \(\Lambda_{\text{QCD}}\) due to its strong infrared sensitivity. It therefore seems to be more advantageous to extract a short-distance mass like the \(\overline{\text{MS}}\) mass. Taking into account the strong correlations between pole mass and \(\alpha_s\) and using the two-loop conversion formula between pole and \(\overline{\text{MS}}\) mass we find very good agreement in the mass determination for both methods (see Tab. 1). Using the moments at NLO we also found errors which were much more conservative than the ones given in the NLO analysis of Ref. 9. In Tab. 1 we give a compilation of all recent sum rule determinations of bottom quark masses based on experimental data from the Υ mesons.
4.2 Top Quark Pair Production Cross Section at Threshold

The measurement of the $t\bar{t}$ production lineshape at threshold ($\sqrt{s} \approx 2M_t$) is among the first tasks of the Next Linear Collider (NLC). Due to the large top quark width ($\Gamma_t(t \rightarrow Wb) \approx (G_F/\sqrt{2})M_t^3/8\pi \approx 1.5$ GeV) which serves as a natural infrared cutoff and smearing mechanism one can in fact calculate the lineshape locally without imposing any additional smearing. To be more specific, the effective c.m. velocity $v_{\text{eff}}$ of the top quarks is of order $\sqrt{E^2 + \Gamma_t^2}/M_t$ which means that $M_t v_{\text{eff}}^2$ is larger than $\Lambda_{\text{QCD}}$ for any nonrelativistic energy. Physically this cutoff mechanism arises because the top quarks decay weakly before hadronization effects can set in. This also leads to the phenomenon that there are no individual narrow toponium resonances because the latter are so broad that they are almost completely smeared out. In other words, there will be no toponium spectroscopy as we know it from charmonia or bottomonia. However, the large decay width allows for remarkably accurate measurements of top quark properties without the diluting effects of hadronization. It is believed that measurements at the $t\bar{t}$ threshold at the NLC will provide the most precise determination of the top quark mass. In view of the potentially sizeable NNLO relativistic corrections it is mandatory to calculate and control the NNLO effects. Our calculation of the NNLO cross section has been designed for stable quarks and can, strictly speaking, not be used as the NNLO cross section for $t\bar{t}$. Nevertheless we can illustrate the impact of the NNLO corrections by using the naive replacement $E \rightarrow E + i\Gamma_t$ in Eq. (8) where $\Gamma_t$ is the free top quark width. This prescription has been proven to be correct in the nonrelativistic limit and should be sufficient to give us the bulk of the NNLO relativistic corrections to the $t\bar{t}$ cross section at threshold. In Fig. 2 the LO (dotted lines), NLO (dashed lines) and NNLO (solid lines) normalized cross sections are plotted versus $E = \sqrt{q^2 - 2M_t^{\text{pole}}}$ in the range $-5$ GeV $< E < 5$ GeV for $M_t^{\text{pole}} = 175$ GeV, $\alpha_s(M_z) = 0.118$ and $\Gamma_t = 1.43$ GeV. For the scales the choices $\mu_{\text{soft}} = 50$ (upper lines), 75 (middle lines) and 100 GeV (lower lines) and $\mu_{\text{hard}} = \mu_{\text{fac}} = M_t$ have been made and two-loop running of the strong coupling has been used. It is evident that the NNLO corrections are large. The behavior of the NNLO corrections clearly indicates that the convergence of the perturbative series for the $t\bar{t}$ cross section is much worse than expected from the general arguments given by Fadin and Khoze. Further examinations of the NNLO cross section including a study of the impact of the use of a mass other than the pole one and a proper treatment of the top width are mandatory.

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As already mentioned, we also have not yet included the Z exchange contributions.
5 Conclusions

During the last year there has been significant progress in the understanding of perturbative heavy quark-antiquark systems in the kinematic regime close to threshold. Using the concepts of effective field theories our knowledge has increased at the conceptual level and a number of previously unknown NNLO corrections have been determined. In this talk I have reviewed the ideas involved to perturbatively calculate and then apply the NNLO corrections of the total $Q\bar{Q}$ production cross section at threshold.

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