Langmuir waves in semi-relativistic spinless quantum plasmas

A. Yu. Ivanov, P. A. Andreev, and L. S. Kuzmenkov
Physics Faculty, Moscow State University, Moscow, Russian Federation.
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Many particle quantum hydrodynamics based on the Darwin Hamiltonian (the Hamiltonian corresponding to the Darwin Lagrangian) is considered. A force field appearing in corresponding Euler equation is considered in details. Contributions from different terms of the Darwin Hamiltonian in the Euler equation are traced. For example, the relativistic correction to the kinetic energy of particles leads to several terms in the Euler equation, these terms have different form. One of them has a form similar to a term appearing from the Darwin term. Hence, the two different mechanisms give analogous contributions in wave dispersion. Microscopic analog of the Biot-Savart law, called the current-current interaction and describing an interaction of moving charges via the magnetic field, is also included in our description. The semi-relativistic generalization of the quantum Bohm potential is obtained. Contribution of the relativistic effects in the spectrum of plasma collective excitations is considered.

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I. INTRODUCTION

There is fast growing interest to the theory of the relativistic [1]-[8] and the semi-relativistic (weakly-relativistic) [9] quantum plasmas. In this paper we develop the many-particle quantum hydrodynamics (QHD) [10]-[16] in the semi-relativistic approximation. In this way we are going to discuss relations between the quantum, thermal, and semi-relativistic effects in the system of many charged particles. As the result, we present complete theory including effects mentioned above for the spinless charged particles.

Spin leads to effects appearing in the semi-relativistic approximation. However, it plays significant role in non-relativistic physical systems, for example, in ferromagnetic materials. Spin dynamics is also very important in physics of the quantum plasma, where electrons and positrons are most widespread objects, and their spin is an inherent dynamical property. Over the last decade a lot of papers have been dedicated to studies of spin dynamics in quantum plasmas, especially by means of the quantum hydrodynamics and Vlasov-like kinetic equations. However, it is very interesting and important to understand the quantum many-particle physics appearing from consideration of the Darwin Hamiltonian the Hamiltonian corresponding to the Darwin Lagrangian, which is the spinless analog of the Breit Hamiltonian [17]. The Darwin Hamiltonian contains both the non-relativistic terms, which describe kinetic energy of particles and the Coulomb interaction, and semi-relativistic terms. They describe the relativistic correction to the kinetic energy of particle (RCKE), the interaction energy of moving charges (which is also called the current-current interaction), the Darwin term proportional to \( \nabla E \) and the term, describing interparticle interaction, proportional to the Dirac delta function, corresponding to the Darwin term, we call it the Darwin interaction. The current-current interaction presents the Biot-Savart-Laplace law. The RCKE and the current-current interaction should be important when studying the relativistic beams in the plasmas.

Suggestion was made in Ref. [9] that in some cases contribution of the RCKE much smaller than the Darwin term. However, our studies of the semi-relativistic effects in the quantum plasmas based on the quantum hydrodynamics method show that the RCKE leads to existence of terms in the semi-relativistic Euler equation. One of these terms has form close to the only term brought by the Darwin term and Darwin interaction. Thus the Darwin term and the RCKE must be considered together.

Contribution of the RCKE and the current-current interaction in the plasma wave dispersion have been considered recently [18] in terms of the many-particle quantum hydrodynamics developed in Refs. [10]-[16], but the Darwin term was not considered there. Another derivation of the QHD equations for systems of charged spinning particles suggested later can be found in Refs. [19], [20]. Some aspects of the quantum plasma physics were reviewed in Ref. [21].

The Darwin term is the semi-relativistic trace of the Zitterbewegung contribution in the Langmuir wave dispersion. The Zitterbewegung effect has been actively studied [22]-[30]. It has been considered for electrons in semiconductors [23], [24], ions [22], [24] and the quantum gases of neutral atoms [28], [31]. Consequently it is worthwhile to point out that the RCKE gives the contribution in the equations of collective motion counteractive to the Darwin term and Darwin interaction contributions.

This paper is dedicated to comparison of the RCKE, the Darwin interaction and the current-current interac-
tion contributions in the Euler equation, obtaining of the explicit form of the semi-relativistic pressure tensor and its influence on the dispersion properties of the longitudinal waves.

Our paper is organized as follows. In Sec. II we discuss basic Hamiltonian and compare contributions of different terms. In Sec. III a set of QHD equations is presented in semi-relativistic approximation. Different contributions in the Euler equation are discussed. In Sec. IV the method of dispersion equation obtaining is described, linearized set of the semi-relativistic Euler equations is presented. In Sec. IV dispersion relation for the quantum semi-relativistic Langmuir waves is calculated and discussed. In Sec. V brief summary of obtained results is presented.

II. THE MODEL DESCRIPTION

The equations of quantum hydrodynamics are derived from the non-stationary Schrodinger equation for system of N particles:

$$\iota \hbar \partial_t \psi(R,t) = \hat{H} \psi(R,t)$$  \hspace{1cm} (1)

with Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_{Rel} + \hat{H}_D,$$  \hspace{1cm} (2)

where

$$\hat{H}_0 = \sum_i \left( \frac{1}{2m_i} \hat{D}_i^2 + e_i \varphi_{i,ext} \right) + \frac{1}{2} \sum_{i,j \neq i} e_i e_j G_{ij},$$  \hspace{1cm} (3)

$$\hat{H}_{Rel} = - \sum_i \frac{1}{8m_i^2 c^2} \hat{D}_i^4 - \frac{1}{2} \sum_{i,j \neq i} \frac{e_i e_j}{2m_i m_j c^2} G_{ij} \alpha^\alpha \beta^\beta \hat{D}_i^\alpha \hat{D}_j^\beta,$$  \hspace{1cm} (4)

and

$$\hat{H}_D = - \sum_i \frac{e_i \hbar^2}{8m_i^2 c^2} \nabla_i E_{i,ext}$$

$$- \frac{1}{2} \sum_{i,j \neq i} \frac{\pi e_i e_j \hbar^2}{2 c^2} \left( \frac{1}{m_i^2} + \frac{1}{m_j^2} \right) \delta(\hat{r}_i - \hat{r}_j).$$  \hspace{1cm} (5)

This Hamiltonian corresponds to the spin independent part of the Breit Hamiltonian (see [17 sections 33 and 83, and [31 formula (4.74b)]). All terms except the fourth and sixth terms correspond to the classic Hamiltonian derived from the Darwin Lagrangian (see [32 section 65]). The following designations are used in the equation [22-24]: $e_i, m_i$ are the charge and the mass of particle, $\hbar$ is the Planck constant and $c$ is the speed of light, $D_i^\alpha = -i\hbar \partial_i^\alpha - e_i \varphi_{i,ext}^\alpha / c$ is the covariant derivative, $\varphi_{i,ext}^\alpha, A_{i,ext}^\alpha$ the potentials of an external electromagnetic field, $\partial_i^\alpha = \nabla_i^\alpha$ is the spatial derivatives, $G_{ij} = 1/r_{ij}$ is the Green functions of the Coulomb interaction, $r_{ij} = r_i - r_j$,$ G_{ij}^\alpha\beta = \frac{\delta_{ij}^\alpha\beta + \frac{r_{ij}^\alpha}{r_{ij}^\beta}}{r_{ij}^2}$  \hspace{1cm} (6)

is the Green functions of the current-current interaction, $\psi(R,t)$ is psi-function of N particle system, $R = (r_1, ..., r_N)$. Let us consider physical meaning of the terms in the Hamiltonian [22]

We consider the Hamiltonian as the sum of three parts: the non-relativistic part $H_0$, the relativistic part $H_{Rel}$, and the quantum-relativistic terms $H_D$. The first term in the non-relativistic part of the Hamiltonian $H_0$ is the kinetic energy, the first term in $H_{Rel}$ is the RCCE, the second term in $H_0$ is the potential energy of the classic charge in the external electric field, the first term in $H_D$ is the quantum contribution in the energy of the charge being in the external electric field, which is called the Darwin term. All these terms are valid for each particle, as they describe kinematic properties and interaction with the external field. They present the first groups of terms in Hamiltonians [3]. The second term in $H_0, H_{Rel}, H_D$ describe inter-particle interactions. First of all, the Coulomb interaction is presented by the third term in $H_0$. The second term in $H_D$ describes a quantum contribution to the interaction of charges. It is the Darwin interaction. The second term in $H_{Rel}$ describes the current-current interaction, which is the microscopic analog of the Biot-Savart law.

The first term in $H_D$ shows a semi-relativistic contribution to the force acting from the external electric field on charged particle (the Darwin term). The second term presents the interaction between two particles, which can be considered as a semi-relativistic addition to the Coulomb interaction (the Darwin interaction). If we have deal with interaction of two electrons, the Darwin interaction is

$$H_D = -\pi \left( \frac{\epsilon \hbar}{mc} \right)^2 \delta(\hat{r}_i - \hat{r}_j),$$  \hspace{1cm} (7)

where $r_i$ and $r_j$ are the coordinates of the two electrons. The explicit form of the Darwin interaction was derived from the scattering amplitude in the quantum electrodynamics. Now we have to compare the Darwin term describing interaction with the external field, the first term in formula [14], which appearing in the semi-relativistic limit of the Dirac equation [17], and the Darwin interaction presented by the second term in $H_D$ [17]. Admitting that $\Delta_i(1/|\hat{r}_i - \hat{r}_j|) = -4\pi \delta(\hat{r}_i - \hat{r}_j)$ and introducing the microscopic electric field caused by particle $j$ acting on particle $i$ as $E_{ij} = -\nabla_i (e_j/r_{ij})$, we see that the second term in Hamiltonian [5] can be represented as

$$H_D = -\frac{e_i \hbar^2}{8c^2} \left( \frac{1}{m_i^2} + \frac{1}{m_j^2} \right) \nabla_i E_{ij}.$$  \hspace{1cm} (8)

In formula [3] we used general dependence of masses for interacting particles obtained in Ref. [17]. In formula
we have assumed $m_i = m_j = m$. Comparing the first term in the Hamiltonian and formula we get that these terms coincide if $m_j \to \infty$ that corresponds to the Dirac equation. The Dirac equation describes motion of an electron in an external field, so motion of the electron gives no influence on the external field. Consequently, a mass of source of external field can be considered as equal to infinity. However, if we consider interaction of two electrons, we have $m_i = m_j$ and from \[ H_D = -\frac{e^2}{4m^2c^2} \nabla_i E_{ij}, \] what differs in two times from the first term in \[. \] It was expected that discussed terms should coincide due to the superposition principle, so we have to put additional factor two in the first term in the Hamiltonian, but we keep in mind that we can make another choice and accept the consequence of the Dirac equation. At discussion of wave dispersion we consider consequences of the both choices.

We have deal with the methods based on a certain equation, in our case it is the Schrödinger equation \ref{eq:Schrodinger}, describing system evolution in terms of the Hamiltonian for particles. In the relativistic case the Dirac equation is the corresponding equation. However the Dirac equation describes the quantum motion of one relativistic electron in an external electromagnetic field. There is no proper equation describing the quantum or classical motion of many relativistic electrons in terms of a Hamiltonian, since the Hamiltonian of electromagnetic field has to be included and the field should be considered as independent variable, as it is in the quantum electrodynamics. Thus there is no proper many-particle generalization of the Dirac equation. Consequently the Dirac equation do not allow to derive many-particle relativistic hydrodynamic directly. Even semi-relativistic hydrodynamics could not be derived by means the Dirac equation. However, the Breit Hamiltonian obtained from the quantum electrodynamic scattering amplitude of two charged spinning particles describes the semi-relativistic system of two particles (see Ref. \ref{breit} section 83). It is easy to generalize the Breit Hamiltonian on system of $N$ particles, where $N > 2$. Including the fact that we consider spinless particles, we see that the many particle Breit Hamiltonian corresponds to the classic Hamiltonian obtained from the Darwin Lagrangian (see Ref. \ref{environmental} section 65). But the Breit Hamiltonian contains the Darwin term and the Darwin interaction having quantum semi-relativistic nature. So it does not appear in the classic semi-relativistic theory.

For short references below we introduce new function $G_{ij}$, which is defined as $G_{ij} = G_{ij} - \frac{\hbar^2}{4m^2c^2} \delta(r_i - r_j)$.

$G_{ij}$ leads to existence of two force field terms in the Euler equation. Let us consider how they emerge during derivation of the semi-relativistic Euler equation. We differentiate the current $j$ appearing in the continuity equation with respect to time and use the Schrödinger equation. One of these terms appears due to commutation of $\tilde{G}_{ij}$ with the momentum operator $\hat{\rho}^a_i$ in the current $j$. Let us point out that the operator $\hat{\rho}^a_i$ exists in the current $j$ due to presence of the kinetic energy operator in the Hamiltonian \ref{eq:Schrodinger}. In the self-consistent field approximation this term has following form:

$$\mathbf{F}_C = -e^2 n \nabla \int d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left( -\frac{\pi \hbar^2}{m^2c^2} \delta(\mathbf{r} - \mathbf{r}') \right) n(\mathbf{r}', t).$$

The self-consistent field approximation allows to introduce the electric field $\mathbf{E}$ caused by the charges. It has following explicit form

$$\mathbf{E} = -e \nabla \int d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} n(\mathbf{r}', t),$$

where $n$ is the particle concentration, and field $\mathbf{E}$ satisfy to the quasi-electrostatic Maxwell equations:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = 0,$$

and

$$\nabla \mathbf{E}(\mathbf{r}, t) = 4\pi \sum_a c_a n_a(\mathbf{r}, t),$$

where subindex ”$a$” describes species of particles. We are interested in dispersion of the Langmuir waves, which are the high frequency oscillations. Consequently electrons give main contribution there. Thus we can neglect motion of ions and consider ions as motionless. Having mixture of electrons and ions we work with a stable system. Presence of motionless ions reveals in the Poisson equation \ref{eq:poisson}, where ions cancel equilibrium charge density of the electrons. So perturbations of electric field are caused by the perturbation of the electron density

$$\nabla \mathbf{E}(\mathbf{r}, t) = 4\pi \delta n(\mathbf{r}, t),$$

where $\delta n$ is the perturbation of electron concentration.

Now force field $\mathbf{F}_C$ \ref{eq:F_C} takes form

$$\mathbf{F}_C = e n \mathbf{E} + \frac{\pi \hbar^2}{m^2c^2} \nabla n,$$

where the concentration under the space derivative represents source of the field. So, using equation \ref{eq:poisson} for the mentioned concentration we have come to

$$\mathbf{F}_C = e n \mathbf{E} + \frac{\hbar^2}{4m^2c^2} n \nabla (\nabla \mathbf{E}).$$

Presented here form of the second term corresponds to the semi-relativistic contribution in the force acting on the charged particle from external electric field obtained from the Dirac equation \ref{eq:Schrodinger}. The second term in formula \ref{eq:F_C} gives general form of the Darwin interaction force field. In some cases it can be rewritten in terms of the self-consistent electric field. We have done this representation for electron-ion plasmas with motionless ions. Formula \ref{eq:F_C} is useful for comparison of the Darwin interaction with the RCCE.
The second term associated with $G_{ij}$ appears due to the RCKE. Or, more precisely, it exists due to simultaneous account of the RCKE and interaction of charges with the electric field, which is the sum of the external field and interparticle electric field. Hence let us call it the RCKE-electric field (RCKE-EF) interaction. In the self-consistent field approximation it appears as

$$
F_{st}^\alpha = \frac{e\hbar^2}{4m^2c^2} \partial_\beta (\partial_\alpha E_\beta \cdot n). \tag{17}
$$

As the RCKE-EF interaction has semi-relativistic origin, we can write $G_{ij}$ instead of $G_{ij}$ in this term. The RCKE also gives other terms in the force field, all of them are presented below in the Euler equation.

The force caused by the Darwin term was accounted in Ref. [9], when the force field presented by formula (17) was not considered in Ref. [9] at derivation of the Langmuir wave dispersion. The interparticle Darwin interaction was not considered in Ref. [9] either. Thus, we are going to generalize result of Ref. [9] calculating contribution of the force field (17) in the Langmuir wave dispersion. We also present equations of semi-relativistic collective motion of plasma. But we consider hydrodynamic equations, when the kinetic equation was considered in Ref. [9].

Equations (16) and (17) are very similar, but they have two differences. The first difference is distinction in tensor structure and the second one is the fact that equation (17) contains concentration under the spatial derivative, whereas in formula (16) concentration contains as an external multiplier.

### III. EQUATIONS OF QUANTUM HYDRODYNAMICS

In the previous section we have shown similarity of the RCKE-EF and the Darwin interaction. One of the aims of the paper is to compare contribution of these terms in the QHD equations and the Langmuir wave dispersion. We want to trace separate contribution of the each term. Thus we need to mark them.

The first equation of the QHD set is the continuity equation

$$
\partial_t n + \nabla j = 0. \tag{18}
$$

In that equation a function of current $j(r, t) = n(r, t)v(r, t)$ is arisen, where $v(r, t)$ is the velocity field.

The second equation of the QHD set is the Euler equation, but in the semi-relativistic approximation function $j(r, t)$ appeared in the continuity equation is the particle current. However, in contrast to non-relativistic case we can not call it momentum density, thus the Euler equation is the equation of particle current evolution [33].

This equation has form

$$
mn(\partial_t + v^\beta \nabla^\beta) v^\alpha + \partial_\beta P_{\alpha\beta} = enE^\alpha + \frac{e}{c}\varepsilon^{\alpha\beta\gamma}nv^\beta B^\gamma + \frac{e\hbar^2}{8m^2c^2} n\partial^\alpha (\partial^\beta E^\beta_{\text{ext}}) + \frac{e\hbar^2}{4m^2c^2} n\partial^\alpha (\partial^\beta E^\beta_{\text{int}})
$$

$$
+ \frac{e\hbar^2}{4m^2c^2} \partial_\beta (\partial_\alpha E_\beta \cdot n)
$$

$$
- \frac{e^2\hbar^2}{8m^2c^2} \partial_\beta n \int dr' \partial_\gamma G_{\beta\gamma}(r - r') \partial_\alpha n(r', t)
$$

$$
- \frac{e^3}{2mc^2} \int dr' G_{\alpha\beta}(r - r') E_\beta(r', t)n(r', t)
$$

$$
+ \frac{e^2}{2c^2} \int dr'[\partial_\alpha G_{\beta\gamma}(r - r') - \partial_\beta G_{\alpha\gamma}(r - r')]\pi_{\beta\gamma}(r, r', t)
$$

$$
+ \frac{e^2}{2mc^2} \int dr' \partial_\gamma G_{\alpha\beta}(r - r') \times
$$

$$
x [mn(r', t)v_\beta(r', t)v_\gamma(r', t) + P_{\beta\gamma}(r', t)], \tag{19}
$$

where $E = E_{\text{ext}} + E_{\text{int}}$ and $B$ are the electric and magnetic fields, $n\varepsilon$ is the density of thermal energy including quantum part (which is an analog of the quantum Bohm potential), $e^{\alpha\beta}$ is the antisymmetric symbol (the Levi-Civita symbol), $P_{\alpha\beta}$ is the pressure tensor, which is the semi-relativistic generalization of the sum of non-relativistic thermal pressure $p^{\alpha\beta}$ and the quantum Bohm potential $T^{\alpha\beta}$. In right-hand side of equation (19) a force field locates. The force field consists of the Lorentz force and specific quantum semi-relativistic terms, which are discussed below. $\pi_{\alpha\beta}(r, r', t)$ is presented explicitly and considered below after analysis of $P_{\alpha\beta}$ structure. The vector potential appears in the form

$$
A_{\alpha\beta}^{\text{int}}(r, t) = \frac{e}{2c} \int dr' G_{\alpha\beta}(r - r')n(r', t)v_\beta(r', t), \tag{20}
$$

which gives contribution in the Lorentz force, the second term in right-hand side of equation (19) along with external magnetic field. Magnetic field $B = \nabla \times A_{\text{int}}$ satisfies to the quasi-magnetostatic Maxwell equation:

$$
\nabla \times B = \frac{4\pi}{c} j, \tag{21}
$$

and

$$
\nabla B = 0. \tag{22}
$$

We believe that it is worthwhile to admit that we have not neglected time derivatives in the Maxwell equations (22) and (21). We do not present these terms, because...
they do not appear in the semi-relativistic approximation. The Hamiltonian \[2\] contains the Coulomb interaction and the current-current interaction (the Biot–Savart law). Obtained Maxwell equations correspond to the Hamiltonian. We can put these well-known time derivatives back in the Maxwell equations. However, this step breaks logic of semi-relativistic description. So, we keep it as it describes an electron-electron interaction.

Before discussion of pressure tensor \(P^{\alpha\beta}\) we explain the physical meaning of the force field terms presented in the right-hand side of the Euler equation \([10]\). It is especially important as some of these terms are presented for the first time.

The first two terms present the density of the Lorentz force. The self-consistent part of the Coulomb interaction gives contribution in the first term. The second term contains contribution of the current-current interaction in the self-consistent field approximation. We should admit that all terms in the Euler equation are presented in the self-consistent field approximation. Actually, only a part of the whole contribution from the current-current interaction came in the Lorentz force, it also leads to several other terms. They are seventh–tenth terms of the Euler equation. In fact, the terms eight–ten already appear in the classic semi-relativistic hydrodynamics, but in the quantum theory these terms have more rich structure. First of all, they contain contribution of the quantum Bohm potential. In one particle case we lose contribution of quantum correlations, which is not considered in this paper, but they naturally appear in the many-particle QHD. We neglect them here considering the self-consistent field approximation only. However they contain contribution of the quantum Bohm potential along with the thermal pressure.

The third term corresponds to the Darwin term. The fourth term corresponds to the Darwin interaction. Terms three and four have same nature, but they have different coefficients. The Darwin term appears from the Dirac equation describing motion of an electron in an external field. Then the Darwin interaction comes from quantum electrodynamic scattering amplitude of two particles. The third and fourth terms contain electric field. The third (fourth) term includes the external (interparticle Coulomb) electric field \(E_{ext}\) (\(E_{int}\)). Due to different coefficients before these terms we can not combine them together having full electric field \(E = E_{ext} + E_{int}\). We should not expect additivity of electric fields in these terms, since formula \(10\) used in the fourth term is asymptotic. Original force field in presented by formula \(15\). The original formula does not contain electromagnetic field. It is presented in terms of the particle concentration of interacting species \(F_D = \frac{\pi e^2 n}{m_e c^2} \nabla n\). In our case we have one species, so it describes an electron-electron interaction. In this paper we consider plasmas without external electric field. Consequently the third term equals to zero. In Ref. \([9]\) authors have deal with the Darwin term appearing from the Dirac equation, which corresponds to the interaction of electrons with the external field. As a consequence they get coefficient in two times less then we get from the interparticle interaction.

The terms five and six present contribution of the RCKE. The fifth term has simple structure, it contains divergence \(\nabla \beta\) of the tensor which is product of particle concentration on \(\nabla \alpha E^\beta\) and presents the RCKE-EF interaction. In the sixth term, which contains a number of terms in square brackets, the first set of them is the convolution of \(E^\beta\) with the tensor which is current of the particle current \(j\), and as a part of this current we have the pressure tensor \(P^{\alpha\beta}\). As the sixth term of the Euler equation has semi-relativistic nature, we should consider only non-relativistic part of \(P^{\alpha\beta}\). The second set of terms in the sixth term is the product of electric field \(E^\alpha\) on the energy density. The energy density was separated on two parts there. First of them is the kinetic energy density of a local ordered motion. We need to say that \(n \varepsilon\) is the energy density which consists of two parts: thermal energy and quantum contribution – an analog of the quantum Bohm potential. In one particle case we lose contribution of thermal motion and quantum-thermal terms, and get quantum terms arising for non-interacting particles. \(\varepsilon\) gives no contribution in considered below problem, therefore we do not present its explicit form.

Explicit form of the tensor \(P^{\alpha\beta}\) is

\[
P_{\alpha\beta}(r, t) = \int dR \sum_{i=1}^{N} \delta(r - r_i)a^2 \times
\]

\[
\times \left[ m u_{i\alpha} u_{i\beta} - \frac{\hbar^2}{2m} \left( 1 - \frac{v_i^2}{c^2} \right) \partial_{\alpha\beta} \ln a \right.
\]

\[
+ \frac{\hbar^2}{2mc^2} \left( \partial_{\alpha\beta} v_{i\gamma} \partial_{\alpha\beta} v_{i\gamma} + v_{i\gamma} \partial_{\alpha\beta} v_{i\gamma} \right)
\]

\[
+ \frac{\hbar^2}{4mc^2} \left( v_{i\alpha} \partial_{\alpha\beta} + v_{i\beta} \partial_{\alpha\beta} \right) \left( \partial_{i\alpha} v_{i\gamma} + 2v_{i\gamma} \partial_{i\alpha} \ln a \right)
\]

\[
- \frac{\hbar^4}{4m^3c^2a^2} \left( a \partial_{\alpha\beta} \partial_{\alpha\beta} \Delta_a + \partial_{\alpha\beta} a \Delta_a \right.
\]

\[
- \left. \partial_{\alpha\beta} a \partial_{\alpha\beta} \Delta_a - \partial_{\alpha\beta} a A_{\alpha\beta} \Delta_a \right) \right]
\]

\[
+ \int dR \sum_{i=1, j=1, i \neq j}^{N} \delta(r - r_i)a^2 \frac{\hbar^2 c^2}{4m^3c^2} \times
\]

\[
\times \left( G_{ij}^{\alpha\gamma} \partial_{\alpha\beta} \partial_{i\gamma} \ln a + G_{ij}^{\alpha\gamma} \partial_{\beta\gamma} \partial_{i\gamma} \ln a \right),
\]

where \(v_{i\alpha}\) is the velocity of i-th particle, and it is the sum of the velocity field \(v^\alpha(r, t)\) and thermal velocity
\( u^\alpha_i \), \( a \) is the amplitude of the wave function \( \psi(R,t) = a \exp(iS/\hbar) \), velocity of \( i \)-th particle \( v^\alpha_i \) connects with the phase of the wave function as

\[
v^\alpha_i = \frac{s^\alpha_i}{m_i} - \frac{s^\beta_i s^\gamma_i}{2m_i^3 c^2} + \frac{\hbar^2}{2m_i^3 c^2} \left[s^\alpha_i a^{-1} \Delta_i a + \partial_\alpha^i (s^\beta_i \partial_\beta \ln a) + \frac{1}{2} \partial_\alpha^i \partial_\beta^i s^\beta_i \right] - \sum_{j=1, j \neq i}^{N} \frac{e_i e_j}{2m_i m_j c^2} G_{ij} \delta^{\alpha\beta}, \tag{24}
\]

where \( s^\alpha_i = \partial_\alpha^i S - \frac{\hbar}{8m_i^3 c^2} A^\alpha_i \). The first term in formula (23) is the non-relativistic pressure tensor, which is the semi-relativistic generalization of the pressure of ideal gas. The second term in this formula consist of two parts, the first of them is the non-relativistic part and consider non-relativistic one only.

The first two terms have non-relativistic nature. The other terms are semi-relativistic, most of them are proportional to \( v^2/c^2 \), except of the four last terms. Thermal pressure \( p^{\alpha\beta} \) does not depend on interaction, so we can use equation of state for ideal gas, and we write

\( p^{\alpha\beta} = n k_B T \delta^{\alpha\beta} \), where \( k_B \) is the Boltzmann constant, \( T \) is the temperature, \( \delta^{\alpha\beta} \) is the Kronecker symbol. When \( p^{\alpha\beta} \) stays in a semi-relativistic term we should neglect semi-relativistic part and consider non-relativistic one only.

Let us repeat a part of the semi-relativistic quantum Bohm potential existing in linear approximation, assuming that an equilibrium condition is described by non-zero uniform concentration \( n_0 \neq 0 \) and zero velocity field \( \mathbf{v}_0 = 0 \). Hence we obtain

\[
\partial_\beta T^{\alpha\beta} = -\frac{\hbar^2}{4m} \partial^\alpha \Delta n - \frac{\hbar^4}{8m^3 c^2} \partial^\alpha \Delta \delta n. \tag{27}
\]

We have two terms. The first of them appears from the first term in the formula (25). The second term in formula (27) is the linear part of the first term in the last group of terms, which does not contain the velocity field, in formula (26).

Here we present explicit form of \( \pi_{\alpha\beta}(r, r', t) \), which is the part of the seventh term in the force field

\[
\pi_{\alpha\beta}(r, r', t) = \int \prod_{j=1}^{N} dr_j \sum_{i,j=1, i \neq j}^{N} \delta^2(u^\alpha_i u_{j\beta} - \frac{\hbar^2}{2m^2} \partial_\alpha \partial_\beta \ln a). \tag{28}
\]

To close the QHD set of equations we should find approximate connection between \( \pi_{\alpha\beta}(r, r', t) \) and other hydrodynamic quantities. Calculating \( \pi_{\alpha\beta}(r, r', t) \) for the system of independent particles we get \( \pi_{\alpha\beta}(r, r', t) = 0 \). Thus, in the first approximation we do not need to account contribution of \( \pi_{\alpha\beta}(r, r', t) \) in the QHD equations.

IV. DISPERSION EQUATION FOR QUANTUM SEMI-RELATIVISTIC LANGMUIR WAVES

To get semi-relativistic effects in the form of analytic simple formulas we consider quantum motion of electrons on the background of motionless ions. We consider the small perturbations of equilibrium state like

\[
n = n_0 + \delta n, \quad \mathbf{v} = 0 + \delta \mathbf{v}, \quad \mathbf{E} = 0 + \delta \mathbf{E}, \quad \mathbf{B} = 0 + \delta \mathbf{B}, \quad \delta \rho = 3m v_F^2 \delta n, \tag{29}
\]

where \( m \) is the mass of the electron. In equations (18), (19) and the Maxwell equations (12), (13), (21) and (22), \( v_F \) is the average thermal velocity (for the case of degenerate electrons we should write \( \frac{v_F}{\sqrt{3}} (1 - \frac{1}{2} \frac{\delta n}{n}) \), instead of \( 3v_F \), see Appendix, where \( v_{Fe} = \sqrt{3n_0 e^2 / m} \) is the Fermi velocity). Substituting these relations into the set of equations and neglecting by nonlinear terms, we obtain a system of linear homogeneous equations in partial
derivatives with constant coefficients. Carrying the following representation for small perturbations \( \delta f \)
\[
\delta f = f(\omega, k) \exp(-\omega t + ik\mathbf{r})
\]
yields a homogeneous system of algebraic equations.

The Euler equation (19) is very complicated, thus we allow ourselves to present the algebraic form of linearized Euler equation
\[
-\omega mn_0 \delta v^\alpha + ik^\alpha \left( 3mv^2_{se} + \frac{h^2k^2}{4m} - \frac{h^4k^4}{8m^3c^2} \right) \delta n
\]
\[
= en_0 E^\alpha - k^\alpha k^\beta \frac{en_0 h^2}{4m^2c^2} \delta E^\beta - \frac{e^3 n_0^2}{2mc^2} G^{\alpha\beta}(k) \delta E^\beta, \quad (28)
\]
where \( G^{\alpha\beta}(k) \) is the Fourier image of the current-current interaction Green function, its explicit form is
\[
G^{\alpha\beta}(k) = \frac{8\pi}{k^2} \left( \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right).
\]
The last term in the Euler equation (19) gives a linear term due to the linear part of the quantum Bohm potential, which is a part of \( P^\gamma \), but it is equal to zero because of the structure of \( G^{\alpha\beta}(k) \). The last term in equation (28) gives no contribution in the dispersion of the Langmuir waves. We also admit that we do not consider the temperature-relativistic effects \( \sim T/mc^2 \).

The electric field is assumed to have a nonzero value. Expressing all quantities through the electric field, we come to the equation
\[
\omega^2 = \omega^2_{Le} \left( 1 - \frac{h^2k^2}{2m^2c^2} \right)
\]
\[
+ \left( \frac{3v^2_{se}}{V^2_{Fe}(1 - \frac{v^2}{10c^2})} \right) + \frac{h^2k^2}{4mc^2} - \frac{h^4k^4}{8m^3c^2} k^2, \quad (29)
\]
where \( \omega_{Le} \) is the Langmuir frequency, \( \omega^2_{Le} = 4\pi e^2 n_0/m. \) The first group of terms in the right-hand side of (29) consists of three parts: the Langmuir frequency, contribution of the RCKE-EF and Darwin interactions, where the RCKE-EF interaction leads to \(-\frac{h^2k^2}{2mc^2}\), and the Darwin interaction leads to the same structure \(-\frac{h^2k^2}{4m^2c^2}\). Together they give \(-\frac{5h^2k^2}{2mc^2}\). The second group in equation (29) consists of three parts: contribution of the pressure (thermal motion or Fermi pressure), the next part is the well-known quantum Bohm potential, the last part is the contribution of the RCKE via the semi-relativistic part of the pressure tensor, or, speaking in other terms, it is the semi-relativistic part of the quantum Bohm potential.

Comparing formula (29) with the results of Ref. [9] we should mention several differences. In Ref. [9] authors do not have the quantum Bohm potential contribution \( \sim h^2k^4 \). They also do not obtain the semi-relativistic part of the quantum Bohm potential. Most dramatic difference arises at consideration of the second term in the first group of terms in formula (29) with the corresponding result of Ref. [9]. Here we have \(-\frac{1}{8} \frac{h^2 k^2}{m^2 c^2} \). In Ref. [9] it was found as \( \frac{h^2 k^2}{m^2 c^2} \). We have difference in signs and magnitude of coefficients. It looks like they choose another sign before the Darwin term (see Ref. [9] the fifth term in formula (5) and formula (10) of our paper). As we mention they neglected the RCKE \( D^4 \), because they did not want to consider the semi-relativistic part of the quantum Bohm potential. As we have shown the RCKE gives several different contributions. Consequently the RCKE-EF interaction was also lost in Ref. [9], which gives half of the term under consideration. Moreover they applied, for the electron-electron interaction, the Darwin term describing interaction of the charges with external electric field. Which is two times smaller than the electron-electron Darwin interaction (see terms three and four in the right-hand side of formula (10) of our paper). Altogether we see why the coefficient obtained in Ref. [9] in four times smaller than our results.

A. Estimations

In this subsection we consider system of degenerate electrons. Our aim is to find parameters of system when semi-relativistic effects are noticeable. To this end we represent spectrum of the semi-relativistic Langmuir waves in terms of the Bohm velocity \( v_B \) defined as \( v_B^2 = \frac{h^2k^2}{4m^2} \). The Bohm velocity is not a constant since it depends on the wave vector \( v_B = v_B(k) \). The spectrum reappears in the following form
\[
\omega^2 = \omega^2_{Le} \left( 1 - 2 \frac{v_B^2}{c^2} \right)
\]
\[
+ \frac{1}{3} \frac{v^2_{Fe}}{V^2_{Fe}(1 - \frac{v^2}{10c^2})} \right) k^2 + v^2_B k^2 \left( 1 - 2 \frac{v_B^2}{c^2} \right). \quad (30)
\]

Contribution of the semi-relativistic effects is noticeable at large Bohm velocity. The Bohm velocity \( v_B \) increased with increasing of the wave vector, which is bounded above. This limitation is related to average interparticle distance \( \bar{a} \) giving minimal wavelength, and, consequently, maximal wave vector \( k_{max} \). At small wavelength limit, we find \( \omega^2_{Le} \sim n_0, \) \( v^2_B \sim n_0^{2/3}, \) \( k^2 \sim \frac{1}{\bar{a}^2} \sim n_0^{2/3}, \) \( v_B \sim k^2 \sim n_0^{2/3}. \) We are interested in a regime when the Bohm velocity \( v_B \) is comparable with the speed of light \( c \), but \( v_B \ll c. \) Otherwise we do not get the semi-relativistic regime. Let us estimate parameters of system at \( \frac{v_B}{c} \approx 0.1. \) In this case the semi-relativistic effects RCKE-EF+Darwin interactions and the semi-relativistic quantum Bohm potential decrease the corresponding terms on two percent. We obtain \( k_{max} = 1.2 \)
V. CONCLUSION

We gave derivation of the many-particle QHD equations for the semi-relativistic system of spinless charged particles. Contribution of the RCKE, Coulomb, Darwin and current-current interactions in the Euler equation is obtained. Contributions from different terms are compared. It is shown that simultaneous account of the RCKE and Darwin interaction is necessary, because the RCKE gives a number of terms having different structure, and one of them has structure of the term connected with the Darwin interaction in the Hamiltonian. The RCKE leads to the complex structure of the pressure tensor. The semi-relativistic part of the pressure tensor contains terms proportional to the ratio of the velocity field to the square of light speed. It also includes the several terms proportional to $\hbar^4/c^2$ and contains more higher spatial derivatives than in the non-relativistic quantum Bohm pressure.

Using developed approximation of the QHD equations we studied dispersion dependence $\omega(k)$ of the semi-relativistic Langmuir waves. We got contribution of the RCKE, which gives two terms in $\omega(k)$, they are the RCKE-EF interaction and the semi-relativistic part of the quantum Bohm potential contributions, and the Darwin giving one term. We have obtained that the RCKE-EF and the Darwin interactions give an equal contribution in the dispersion dependence of the Langmuir waves.

We have developed the QHD method with all semi-relativistic effects in spinless plasmas for further research of linear and nonlinear effects in the semi-relativistic quantum plasmas.

The spinless semi-relativistic effects play important role in plasmas of spinless particles and in plasmas of spinning particles. Spin-dependent semi-relativistic interactions were considered in literature earlier: the spin-spin interaction was studied in Refs. [11, 12, 14], the spin-current interaction is considered in [12], the spin-orbit interaction was included in the QHD equations in Refs. [14] and [37].

We conclude that this paper fulfills the program of development of the semi-relativistic hydrodynamics based on the Breit Hamiltonian.

VI. APPENDIX: EQUATION OF STATE FOR SEMI-RElativistic Degenerate Fermi Gas

For relativistic fermions, assuming relativistic relation between energy and momentum of each particle $\epsilon_i = \sqrt{p_i^2 c^2 + m^2 c^4}$ and applying usual technics (see [38], see sections 56, 58), one can find equation of state

$$p = \frac{\pi}{3} \frac{m^4 c^5}{(2\pi\hbar)^3} \Xi \left( \frac{p_F}{mc} \right),$$

where $p_F = (3\pi^2 n_0)^{1/3} \hbar$ is the Fermi momentum, and

$$\Xi(x) = 8 \int_0^x \frac{\xi^4}{\sqrt{1 + \xi^2}} d\xi.$$

We consider the semi-relativistic limit. Consequently we have $\frac{\hbar}{mc} \ll 1$, and $x \ll 1, \xi \ll 1$.

In the semi-relativistic approximation we can write equation of state in the following form

$$p = p_{NR} \left( 1 - \frac{1}{14} \frac{v_{Fe}^2}{c^2} \right),$$

where we have used $p_{NR}$ for the non-relativistic Fermi pressure $p_{NR} = \frac{h^2}{2m} (3\pi^2)^\frac{3}{2} \bar{n}_0^\frac{5}{2}$.

For linear perturbation of the pressure [38] we obtain

$$\delta p = \frac{\partial p}{\partial n} \delta n = \frac{1}{3} m v_{Fe}^2 \left( 1 - \frac{1}{10} \frac{v_{Fe}^2}{c^2} \right)$$

[1] F. Haas, B. Eliasson, P. K. Shukla, Phys. Rev. E 86, 036406 (2012).
[2] F. Haas, B. Eliasson, P. K. Shukla, Phys. Rev. E 85, 056411 (2012).
[3] F. Haas, A. Bret, P. K. Shukla, Phys. Rev. E 80, 066407 (2009).
[4] F. Haas and A. Bret, EPL 97, 26001 (2012).
[5] M. Akbari-Moghanjoughi, Phys. Plasmas 20, 042706 (2013).
[6] J. Zhu and P. Ji, Phys. Rev. E 81, 036406 (2010).
[7] H. A. Shah, W. Musood, M. N. S. Qureshi, and N. L. Tsintsadze, Phys. Plasmas 18, 102306 (2011).
[8] F. A. Asenjo, V. Munoz, J. A. Valdivia, and S. M. Mahajan, Phys. Plasmas 18, 012107 (2011).
[9] F. A. Asenjo, J. Zamanian, M. Marklund, G. Brodin, and P. Johansson, New J. Phys. 14, 073042 (2012).
[10] L. S. Kuz'menkov and S. G. Maksimov, Teor. i Mat. Fiz., 118, 287 (1999) [Theoretical and Mathematical Physics 118, 227 (1999)].
[11] L. S. Kuz'menkov, S. G. Maksimov, and V. V. Fedoseev, Teor. i Mat. Fiz. 126, 136 (2001) [Theoretical and Mathematical Physics 126, 110 (2001)].
[12] P. A. Andreev and L. S. Kuz'menkov, Russian Phys. Jour. 50, 1251 (2007).
[13] P. A. Andreev, L. S. Kuz’menkov, Moscow University Physics Bulletin 62, N.5, 271 (2007).
[14] P. A. Andreev and L. S. Kuz’menkov, Int. J. Mod. Phys. B 26, 1250186 (2012).
[15] P. A. Andreev, L. S. Kuzmenkov, Phys. Rev. A 78, 053624 (2008).
[16] P. A. Andreev, L. S. Kuzmenkov, M. I. Trukhanova, Phys. Rev. B 84, 245401 (2011).
[17] V. B. Berestetskii, E. M. Lifshitz, L. P. Pitaevskii, Quantum Electrodynamics, Vol. 4, 2nd ed. (Butterworth-Heinemann, 1982).
[18] A. Yu. Ivanov and P. A. Andreev, Russ. Phys. J. 56, 325 (2013).
[19] M. Marklund and G. Brodin, Phys. Rev. Lett. 98, 025001 (2007).
[20] G. Brodin, M. Marklund, New J. Phys. 9, 277 (2007).
[21] P. K. Shukla, B. Eliasson, Rev. Mod. Phys. 83, 885 (2011).
[22] L. Lamata, J. Casanova, R. Gerritsma, C. F. Roos, J. J. Garca-Ripoll, and E. Solano, New Journal of Physics 13, 095003 (2011).
[23] L. Lamata, J. Leon, T. Schatz, and E. Solano, Phys. Rev. Lett. 98, 253005 (2007).
[24] C. Schneider, D. Porras, and T. Schaeftz, Rep. Prog. Phys. 75, 024401 (2012).
[25] Wlodek Zawadzki and Tomasz M. Rusin, J. Phys.: Condens. Matter 23, 143201 (2011).
[26] Tutul Biswas and Tarun Kanti Ghosh, J. Phys.: Condens. Matter 24, 185304 (2012).
[27] Q. Zhang, J. B. Gong, and C. H. Oh, Eur. Phys. Lett. 96, 10004 (2011).
[28] Dan-wei Zhang, Zi-dan Wang, Shi-liang Zhu, Front. Phys. 7, 31 (2012).
[29] D. Witthaut, T. Salger, S. Kling, C. Grossert, and M. Weitz, Phys. Rev. A 84, 033601 (2011).
[30] O. Boada, A. Celi, J. I. Latorre, and M. Lewenstein, New Journal of Physics 13, 035002 (2011).
[31] P. Strange, (1998) Relativistic Quantum Mechanics with Applications in Condensed Matter and Atomic Physics (Cambridge: Cambridge University Press).
[32] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Butterworth-Heinemann, 1975).
[33] P. A. Andreev, arXiv:1208.0988.
[34] J. Zhu and P. Ji, Plasma Physics and Controlled Fusion, 54, 065004 (2012).
[35] J. T. Mendonca, Phys. Plasmas 18, 062101 (2011).
[36] D. A. Uzdensky and S. Rightley, arXiv:1401.5110.
[37] P. A. Andreev and L. S. Kuzmenkov, PIERS Proceedings, Marrakesh, Morocco, March 20-23, p. 1047 (2011).
[38] L. D. Landau, E. M. Lifshitz (1980). Statistical Physics. Vol. 5 (3rd ed.). Butterworth-Heinemann.