Effective Random Matrix Theory description of chaotic Andreev billiards

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An effective random matrix theory description is developed for the universal gap fluctuations and the ensemble averaged density of states of chaotic Andreev billiards for finite Ehrenfest time. It yields a very good agreement with the numerical calculation for Sinai-Andreev billiards. A systematic linear decrease of the mean field gap with increasing Ehrenfest time $\tau_E$ is observed but its derivative with respect to $\tau_E$ is in between two competing theoretical predictions and close to that of the recent numerical calculations for Andreev map. The exponential tail of the density of states is interpreted semi-classically.

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Recently, mesoscopic ballistic two dimensional normal (N) dots in contact with a superconductor (S) have been extensively studied. Such hybrid systems are commonly called Andreev billiards. In the most recent works, interest has shifted to mesoscopic fluctuations of the excitation spectrum of these systems. Since the sub-gap spectrum determines the tunneling conductance of the superconductor, interest has shifted to mesoscopic fluctuations of the sub-gap spectrum.

The first-dimensional Andreev billiards (N) dots in contact with a superconductor (S) have been experimentally and theoretically. It was shown by Melsen et al. that integrable Andreev billiards are gapless, whereas systems with classically chaotic dots possess an energy gap on the scale of the Thouless energy $E_T = h/(2\pi\tau_D)$, where $\tau_D = \pi A/(Wv_F)$ is the mean dwell time in the normal dot (here $A$ is the area of the normal dot, $W$ is the width of the superconducting region, and $v_F$ is the Fermi velocity). For such systems, it is assumed that $\delta_N << E_T << \Delta$, where $\delta_N = 2\pi\hbar^2/(mA)$ is the mean level spacing of the isolated normal dot with effective mass $m$ of the electrons and $\Delta$ is the bulk order parameter of the superconductor. In further studies, it was concluded that in chaotic cases, the lowest energy level $E_1$ of the system varies from sample to sample with a universal probability distribution $P(x)$ given in Ref. if the energy levels are rescaled as $x = (E_1 - E_g)/\Delta_g$, where the mean-field level of the gap $E_g$ and the width of the distribution $\Delta_g$ are given by

$$E_g = E_g^{\text{RMT}} = 2\gamma^{5/2}E_T,$$

$$\Delta_g = \Delta_g^{\text{RMT}} = c'M^{1/3}\delta_N.$$  

Here $\gamma = \frac{1}{2}(\sqrt{5} - 1)$ is the golden ratio, $c' = [(15 - 6\sqrt{5})/20]^{1/3}/2\pi$, $M = \text{Int}[kWF/\pi]$ is the number of open channels in the S region and $k_F$ is the Fermi wave number (\text{Int}[\cdot] stands for the integer part).

Equations are strictly valid only in the RMT limit, i.e., when the Ehrenfest time $\tau_E = (1/\lambda)\ln(L/k_F)$ tends to zero ($\tau_E$ is the time needed for a wave packet of minimal size $L = 2\pi/k_F$ to spread to the characteristic size $L$ of the classically chaotic normal dot with Lyapunov exponent $\lambda$). For finite but small enough Ehrenfest time, Silvestrov et al. and Vavilov and Larkin predicted that to lowest order in $\tau_E/\tau_D$ the mean-field gap $E_g$ decreases linearly by increasing the ratio $\tau_E/\tau_D$. The first numerical evidence for the distribution $P(x)$ in the RMT limit and the dependence of $E_g$ on the ratio $\tau_E/\tau_D$ was presented by Jacquod et al. modeling the hybrid system with the one dimensional Andreev map.

From an experimental point of view, more realistic candidates for studying quantum chaos in hybrid systems would be two dimensional Andreev billiards with classically chaotic normal region. However, to date no numeri-

![FIG. 1: A normal dot (N) of Sinai billiard in contact with a superconductor (S).](image-url)
DOS. For clarity, we would like to mention that the recently developed theory\cite{10} for the mean field gap based on the concept of a reduced phase space can also be considered as an effective RMT description. However, our effective RMT description deals with the fluctuation of the gap in Andreev billiards.

In our numerical work we calculated the exact energy levels of the so-called Sinai-Andreev (SA) billiard in which the normal dot is a Sinai billiard (see Fig. 1). The energy levels of the Andreev billiards are the positive eigenvalues $E$ (measured from the Fermi energy) of the Bogoliubov-de Gennes equation\cite{12}. To obtain the exact energy levels of an Andreev billiard we used the recently derived general and quantum mechanically exact secular equation expressed in terms of the scattering matrix $S_0(E)$ of the normal region\cite{13}. The scattering matrix $S_0(E)$ was calculated by expanding the wave function in the N region in terms of Bessel functions.

To ensure that the long classical trajectories (compared to the characteristic length of the system) starting and ending at the N-S interface are truly chaotic the following geometrical constrains should be applied: $h = a - W$ (the superconductor is placed at the top of the vertical border of the Sinai billiard), $R + W \geq a$ and $R \geq a/\sqrt{2}$. Otherwise, there may exist arbitrary long trajectories without bouncing on the circular part of the Sinai billiard. For such intermittent trajectories the return probability decays as $P_r(s) \sim 1/s^3$ ($s$ is the length of the trajectory) and this results in a gapless energy spectrum of the Andreev billiard\cite{2, 4}.

For realistic systems the Ehrenfest time is always finite and an important question arises, namely how the predictions of the RMT for the gap distribution and the scaling parameters $E_g$ and $\Delta_g$ are affected. Provided that for small enough $\tau_E/\tau_P$ the distribution of the gap in the rescaled variable $x$ remains the same as that in the RMT limit, the following equations hold

$$E_g = \langle E_1 \rangle - \langle x \rangle \Delta_g, \quad \Delta_g = \delta E_1 / \delta x,$$

where $\langle E_1 \rangle$ is the mean value, $\delta E_1 = \sqrt{\langle E_1^2 \rangle - \langle E_1 \rangle^2}$ is the standard deviation of $E_1$, while $\langle x \rangle \approx 1.21$ and $\delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \approx 1.27$ are calculated from the gap distribution $P(x)$\cite{6}. Although the distribution of $E_1$ is unknown, its mean and standard deviation can be numerically estimated from the data of an ensemble of the SA billiard, hence $E_g$ and $\Delta_g$ follow.

Figure 2 shows our numerical results for the integrated distribution $F(x) = \int_{0}^{x} P(x') dx'$ together with the theoretical prediction (the distribution $P(x)$ is shown in the inset). In our numerics we used 5000 slightly different realizations of the SA billiard by varying the geometrical parameters $R$, $W$ and the Fermi wave numbers $k_F$ (for the parameters of the SA billiard see\cite{14}). From Eq. (2) we found that $E_g \approx 0.5 E_F$ and $\Delta_g \approx 0.118 E_F$ and they are different from those given in the RMT limit, $E_g^{\text{RMT}} = 0.6 E_T$ and $\Delta_g^{\text{RMT}} = 0.097 E_T$. It is clear from the figure that the numerical result for $F(x)$ (using Eq. (2)) is also different from that of the theoretical prediction in the RMT limit. However, the agreement between our numerical results obtained using Eq. (2) and the universal distribution function $F(x)$ is excellent, without any adjustable parameters. Similar numerical results were found for other ensembles of the SA billiard. These results imply that for systems with non-zero Ehrenfest time the gap distribution is still given by the universal function $P(x)$ provided the ‘renormalized’ parameters $E_g$ and $\Delta_g$ obtained from Eq. (2) are used. One can regard the distribution $P(x)$ (and the integrated distribution $F(x)$) with the renormalized parameters as a result of an effective RMT description.

The necessity of the renormalization of $E_g$ and $\Delta_g$ can be interpreted as their dependence on the Ehrenfest time. Since the characteristic length of the system is uncertain, two definitions of the Ehrenfest time, proposed in Ref.\cite{11} and used in numerical simulations of Ref.\cite{8}, were here adapted for numerical calculations:

$$\tau_E^{(1)} = \frac{1}{2\lambda} \ln \frac{W^2}{\lambda_F L_c},$$

$$\tau_E^{(2)} = \frac{1}{2\lambda} \ln \frac{L_{av}}{\lambda_F},$$

where $L_c$ is the average length of the part of the trajectory lying between two consecutive bounces at the curved boundary segment of the Sinai billiard and $L_{av} = \pi A / K$ with perimeter $K$ of the billiard is the mean chord length in the normal region. They are parametrically different, but their numerical values are of the same magnitude.
For the numerical results shown in Fig. 2, the ratio of the Ehrenfest time and the dwell time is about $\tau_E^{(1)}/\tau_D \approx 0.1$ and $\tau_E^{(2)}/\tau_D \approx 0.26$ (for calculation of the Lyapunov exponent see [13]).

It was predicted theoretically [10, 11] and demonstrated numerically using the Andreev map [8] that to lowest order in $\tau_E/\tau_D$ the gap $E_g$ decreases linearly by increasing the ratio of the Ehrenfest time and the dwell time for ratio much less than one. From our numerical study a systematic decrease of $E_g$ as a function of ratio $\tau_E^{(1)}/\tau_D$ is found as shown in Fig. 3 (for details of the system parameters see [13]). Similar result has been obtained using definition [8] of the Ehrenfest time. The error bars represent the standard deviations which are calculated by taking into account the correlation between the first energy levels $E_1$ for systems with slightly different parameters (see [10] for details).

Assuming that

$$\frac{E_g}{E_g^{RMT}} = \beta - \alpha \frac{\tau_E^{(1)}}{\tau_D}, \quad (4)$$

we found that $\alpha = 0.7 \pm 0.2$ and $\beta = 0.95 \pm 0.02$. Similar results were obtained by using [8]: $\alpha = 0.9 \pm 0.3$ and $\beta = 1.10 \pm 0.07$. According to the two competing theories the universal values of $\alpha = 0.23$ and $\beta = 1$ (Ref. [11]) and $\alpha = 2$ and $\beta = 1$ (Ref. [10]) were predicted, while from the numerics for Andreev map [8] only $\alpha = 0.59 \pm 0.08$ is universal value. From our numerics the value $\alpha$ is in between the two theoretical predictions and within the numerical errors it agrees with the result for the Andreev map. We found that the values of $\Delta_g$ obtained from [2] are slightly greater than those predicted from [10]. However, no obvious functional form can be deduced from our data for the dependence of $\Delta_g$ on $\tau_E/\tau_D$.

For further support of the effective RMT description of the universal gap fluctuation we compare the numerically obtained ensemble averaged DOS $\langle \rho(E) \rangle$ with the effective RMT prediction. The ensemble averaged DOS $\langle \rho_{eff}(E) \rangle$ in the effective RMT description can be calculated from $\langle \rho_{RMT}(x) \rangle$ given in the RMT limit using the renormalized parameters $E_g$ and $\Delta_g$ when ‘scaling back’ the variable $x = (E - E_g)/\Delta_g$ into the energy variable $E$. In the RMT limit $\langle \rho_{RMT}(x) \rangle = -x\text{Ai}^2(x) + [\text{Ai}'(x)]^2 + \frac{x}{2}\text{Ai}(x)\left[1 - \int_x^\infty \text{Ai}(y) \, dy\right]$ is again a universal function of $x$ (see note 20 in [6]). It can be seen from Fig. 4 that the agreement between our numerically obtained ensemble averaged DOS $\langle \rho(E) \rangle$ and $\langle \rho_{eff}(E) \rangle$ is excellent at the edge of the spectrum (where the theory is valid) for ratio $\tau_E^{(1)}/\tau_D \approx 0.063$ (the same holds for other ratios not shown). For larger energies the DOS is around $2/\delta N$ as it is expected. For convenience the mean field DOS $\rho_{eff}$ in the RMT limit is also plotted in the figure. In case of two dimensional Andreev billiards the ensemble averaged DOS $\langle \rho(E) \rangle$ shown in Fig. 4 is the first numerical evidence for the prediction $\langle \rho_{eff}(E) \rangle$ obtained from the theoretical result $\langle \rho_{RMT}(x) \rangle$ (see Fig. 2 of Ref. [6]).

From Fig. 4 one can also see that the agreement between the Bohr-Sommerfeld approximation (BS) [2] and our numerics is quite good at the bottom of the spectrum. This suggests a semi-classical explanation. In a work by Schomerus and Beenakker [4] close correspondence has been found between the morphology of the phase space and the density of low energy excitations. Ref. [2] finds that the semi-classical prediction of the DOS for systems with fully chaotic phase space has no definite gap but it becomes exponentially small below the energy $\approx 0.5E_T$. Furthermore in case of systems with mixed phase space and strong coupling of the regular islands to the superconductor, the above defined ‘gap’ is substantially reduced, namely by a factor $\tau_D/t^*$, where $t^*$ is the mean dwell time of trajectories in the chaotic part of the
phase space.

The gap of the phase reduction can also be applied to our SA billiard, since this is a system with mixed phase space (for geometries studied in this work approximately 10% of the phase space was regular). To calculate the mean dwell time $t^*$ in the chaotic part of the phase space, only those trajectories are taken into account, for which the size of the bunch of trajectories started in its $\lambda_F$ neighborhood at the N-S interface is increased to the characteristic length scale of the billiard before returning to the superconducting lead. For the parameters of the SA billiard used in our calculations this means that one has to exclude trajectories which either do not bounce on the circular part or bounce on it only once. From our numerics we found (for the ensemble corresponding to Fig. 4) that the effective dwell time of these trajectories is $t^* \approx 1.66 t_0$ hence the semi-classically obtained gap is reduced to the value $\approx 0.3 E_T$. This result is just about the energy value where the numerically found DOS becomes exponentially small and thus it confirms the semi-classical picture developed in Ref. [4]. Note also that below the value $\approx 0.3 E_T$ (i.e., for $x \lesssim -2$) the RMT DOS $\langle \rho_{\text{RMT}}(x) \rangle$ is also exponentially small.

While the BS approximation is quite successful in predicting the density of low energy excitations, it is clear from Fig. 6 that the edge of the spectrum can be better predicted using our effective RMT description. This may also reveal the limits of the BS approximation. (We observed similar deviations for other ensembles not shown).

In summary we have numerically shown that for finite but small enough Ehrenfest time the distribution of the rescaled first energy level $E_1$ of an ensemble of chaotic Andreev billiards can be treated by an effective RMT description. In this model the scaling parameters $E_g$ and $\Delta_q$, extracted from the data of an ensemble rescale the distribution of $E_1$ such that it agrees with $P(x)$ given in the RMT limit. Our numerical results also show that to lowest order in $t_0/\tau_D$ the mean field gap $E_g$ decreases linearly with the Ehrenfest time but the slope is between the two competing theories and close to that of the recent numerical calculations for the Andreev map. Calculation of the ensemble averaged DOS gives further confirmation of our effective RMT description. Our numerics suggest that the exponential tail of the DOS can be well interpreted semi-classically.

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