Integrability in the dynamics of axially symmetric membranes

Jens Hoppe

Abstract. Bäcklund-type transformations in four-dimensional space-time and an intriguing reduced zero-curvature formulation for axially symmetric membranes, with diffeomorphism- resp. Lorentz-symmetries reappearing after orthonormal gauge-fixing, are found.

Axially symmetric membranes in $\mathbb{R}^{1,3}$ are the simplest case in a notoriously difficult class of problems that have long withstood to be solved. In [1], an orthonormally gauge fixed light-cone formulation was given (in terms of only one field, with polynomial Hamiltonian and no explicit gage-fixing constraint remaining), in [2] an important relation with hydrodynamics was found and in [3] the first non-trivial explicit solutions (after the ‘obvious’ spherically symmetric one, 40 years ago [5]). The existence of transformations similar to those of Bäcklund and Bianchi [6], resp. Thybaut [7], was conjectured in [8] [4], and several non-trivial observations concerning axially symmetric membranes were mentioned in part IV of [9]. Recently [10], in the context of new explicit (infinite-energy) solutions [11], an astonishingly simple transformation between different orthonormal parametrizations was observed. These transformations (discussed thoroughly in the first part of the present paper), mapping solutions (of different PDE’s) into each other (at first describing the same zero mean curvature world-volume $\mathcal{M}_3 \subset \mathbb{R}^{1,3}$) are similar to the above mentioned Bäcklund transformations, that originally were used in the context of generating 2-dimensional constant negative curvature surfaces (in $\mathbb{R}^3$).

One way (see e.g. [8]) to describe axially symmetric membrane motions in 4-dimensional Minkowski-space is via solutions $(r(t, \varphi), z(t, \varphi))$ of

$$
\dot{r} r' + \dot{z} z' = 0, \quad \dot{r}^2 + \dot{z}^2 + \frac{r^2(r'^2 + z'^2)}{\rho^2} = 1
$$

(1)

(where $'$ and $'$ denote differentiation with respect to time $t$ and ‘angle’ $\varphi$), $\rho(\varphi)$ may be chosen to be constant, $\rho = \rho_0 = r_0^2$, (by reparametrizing $\varphi \to \tilde{\varphi}(\varphi), \frac{d\tilde{\varphi}}{d\varphi} = \rho(\varphi)$; in some cases, cp. [10],paying the price of having non-compact ranges), which (by scaling $t$, $r$ and $z$) could then allow one to put $\rho = 1$; hence describing membrane solutions, resp. the
3-dimensional world-volume swept out in space-time, by

\[
x^\mu(t, \varphi, \psi) = \begin{pmatrix} t \\ r(t, \varphi) \cos \psi \\ r(t, \varphi) \sin \psi \\ z(t, \varphi) \end{pmatrix} = \begin{pmatrix} t \\ \vec{x}(t, \varphi) \end{pmatrix}
\]

\[(\dot{r} \pm \frac{r'}{r_0})^2 + (\dot{z} \pm \frac{z'}{r_0})^2 = 1\]

with \(r_0 = 1\).

As long as \((\dot{r})\) and \((\dot{z}')\), velocity- and tangent-vector of a time dependent curve in \(\mathbb{R}^2\), are linearly independent, (1) implies the second-order ‘membrane’ equations

\[\mathbf{r}_{\alpha \beta} = \begin{pmatrix} 1 - \dot{r}^2 - z^2 & 0 & 0 \\ 0 & -(r'^2 + z'^2) & 0 \\ 0 & 0 & -r^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x^\mu}{\partial \varphi^\alpha} \frac{\partial x^\nu}{\partial \varphi^\beta} \eta_{\mu\nu} \end{pmatrix}_{\alpha\beta=0,1,2} \]

\[
\Delta x^\mu = \frac{1}{\sqrt{G}} \partial_\alpha \sqrt{G} G^{\alpha\beta} \partial_\beta x^\mu = \frac{1}{\sqrt{G}} \left( \partial^2_t - \partial_\varphi \frac{r'^2}{r_0} \partial_\varphi - \left( \frac{r'^2 + z'^2}{r_0} \right) \partial_\psi \right) x^\mu = 0.
\]

The (at first astonishing) fact that the second-order equations (3) follow from the first-order ‘gauge-conditions’ (1) has to do with \(\Delta x^\mu\) always being purely normal to the world-volume \(M_3\) (\(\partial_\alpha x_\mu, \Delta x^\mu \equiv 0\)), i.e. the 4 equations \(\Delta x^\mu = 0\) having only one independent component - which can be taken to be \(\frac{1}{3} n_\mu \Delta x^\mu = \frac{1}{3} G^{\alpha\beta} (H_{\alpha\beta} := n_\mu \partial_\alpha^2 x^\mu)\), the mean-curvature of \(M_3\), but (generically equivalently) also (only/simply) as one of the 4 equations; due to the choice of \(\varphi^0\) and the orthogonal parametrization provided by the first part of (1) (as well as the \(U(1)\)-symmetry) \(\Delta x^0 = 0\) becomes a (first order) conservation law, \(\partial_t (\sqrt{G} G^{00}) = 0\), which is the second part of (1)-the local conserved quantity being \(\rho\) (which is the energy-density \(\mathcal{H}\) in a Hamiltonian O(ortho)N(ormal) ‘3+1’ description of membranes [12] [14] [21] [13]).

While ‘integrability’ of the above equations has been suspected for quite some time (cp. [4], [8], [15], [9]), the present paper reveals more concrete, substantial, signs of integrability.
1. (Bäcklund-) Transformations between Different Orthonormal Parametrizations

Instead of choosing $\varphi^0 = x^0$ (= the time $t$ of a Lorentz-observer), choose $\varphi^0$ to be $x^3$, i.e.

$$\tilde{x}^\mu(u,v,w) = \begin{pmatrix} T(u,v) \\ R(u,v) \cos w \\ u \sin w \end{pmatrix} = \begin{pmatrix} \tilde{x}^\alpha(u,v,w) \\ u \end{pmatrix},$$

and again an orthogonal parametrization (i.e. $v = \varphi^1$ such that $\tilde{G}_{01} = 0; \tilde{G}_{02} = 0 = \tilde{G}_{12}$ because of the $u(1)$-symmetry; implying $\partial_u(\tilde{\rho} := \sqrt{\tilde{G}_{00}}) = 0$ if (5) describes a stationary point of the volume-functional). In [10] the particular (membrane) solution (stationary point)

$$T(u,v) = H(u) \cdot v, \quad R(u,v) = H(u)\sqrt{v^2 - 1}$$

$$H'^2 = H^4 + 1, \quad \tilde{x}^\alpha = H(u)\left(\frac{u}{\sqrt{v^2 - \frac{1}{2} \cos w}}\right)$$

(in order to have $\tilde{\rho} = 1 = \rho$, $v$ here equals $\cosh v$ in [10], and $\varphi$ here corresponds to $\frac{1}{2} \varphi^2$, there) was found; note that (6) is not a reparametrization of Dirac’s $SO(3)$ invariant solution, $\tilde{x}(t,\varphi,\psi) = D(t)\left(\sqrt{1-\frac{1}{2} \varphi^2 \cos \psi} \right)_{\varphi \in [-1,1]}$, $D^2 + D^4 = 1$, but its $SO(2,1)$ invariant ‘cousin’. The general orthonormality conditions and corresponding equations of motion for (3), in complete analogy to (1) and (3), are

$$T_u T_v - R_u R_v = 0, \quad T_u^2 - R_u^2 - 1 = \frac{R^2(R^2 - T^2)}{R^4}$$

$$R^4_0 T_{uu} = (R^2 T_v)_v, \quad R^4_0 R_{uu} = (R^2 R_v)_v - R(R^2 - T^2).$$

While it is obvious that (5) and (2), for one and the same world-volume, must give

$$R(u,v) = v(t,\varphi), \quad t = T(u,v), \quad u = z(t,\varphi),$$

with some concrete relation $\varphi = \phi(u,v)$ resp. $v = v(t,\varphi)$, the important observation, in the example (6) e.g. allowing to conclude

$$\phi(u,v) = \sqrt{H^4 + 1} \left(\frac{1}{2} (v^2 - 1)\right) =: \varphi$$

(corresponding to $\frac{1}{2} \varphi^2$ in [10]) is that the (‘Bäcklund’) transformations between the two triply orthogonal parametrizations of $\mathfrak{M}_3$, are given
by first order linear PDE’s, following from
\[
\tilde{G}_{\alpha\beta} = \begin{pmatrix}
T_u^2 - R_u^2 & 0 & 0 \\
0 & -(R_v^2 - T_v^2) & 0 \\
0 & 0 & -R^2
\end{pmatrix}
\]
(10)
\[
= \begin{pmatrix}
1 - r^2 - z^2 & 0 & 0 \\
0 & -r^2 + z^2 & 0 \\
0 & 0 & -r^2
\end{pmatrix} J
\]
\[
J = \frac{\partial t, \varphi, \psi}{\partial u, v, w} = \begin{pmatrix}
T_u & T_v & 0 \\
\phi_u & \phi_v & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

implying in particular \(G_{00}T_u T_v + G_{11} \phi_u \phi_v = 0\), \(\tilde{G}_{00} \dot{u} u' + \tilde{G}_{11} \dot{v} v' = 0\).

Due to
\[
1 = \frac{\rho^2}{r_0^2} = \frac{r^2(r^2 + z^2)}{1 - r^2 - z^2} = \frac{G_{22} G_{11}}{r_0^4 G_{00}} = 1 = \frac{\tilde{G}_{22} \tilde{G}_{11}}{R_0^4 G_{00}}
\]
(11)
\[
= \frac{R^2 (R_v^2 - T_v^2)}{T_u^2 - R_v^2 - 1} = \frac{\rho^2}{R_0^4} = 1
\]

(hence \(r_0^4 \frac{G_{11}}{G_{00}} = G_{22} = -r^2(t, \varphi) = -R^2(u, v) = \tilde{G}_{22} = \frac{\tilde{G}_{00} R_0^4}{G_{11}}, \)

implies
\[
r_0^4 \phi_u \phi_v = r^2 T_u T_v = R^2 R_u R_v
\]
\[
R_0^4 \dot{v} v' = R^2 \dot{u} u' = r^2 \dot{z} z' = -r^2 \dot{r} r'
\]
i.e.
\[
R_0^4 T_u \phi_u T_v \phi_v = R_0^4 \dot{v} v' = R^2 = r^2 = r_0^4 \phi_u \phi_v = \frac{\dot{u} v}{u' v'} = \frac{\dot{u}}{u} \cdot \frac{r_0^4}{R_0^4},
\]

hence
\[
\dot{u}^2 = v^2 \frac{R_0^4}{r_0^4}, \quad T_u^2 = \phi_v^2 \cdot \frac{r_0^4}{R_0^4}
\]
(14)

when using that, given the transformation
\[
(z(t, \varphi) = u, v) \leftrightarrow (T(u, v) = t, \phi(u, v) = \varphi)
\]

(15)
\[
\begin{pmatrix}
T_u & T_v \\
\phi_u & \phi_v
\end{pmatrix} = \begin{pmatrix}
\dot{u} & u' \\
\dot{v} & v'
\end{pmatrix}^{-1} = \frac{1}{(\delta := \dot{u} v' - u' \dot{v})} \begin{pmatrix}
v' & -u' \\
-\dot{v} & \dot{u}
\end{pmatrix},
\]

(15)
simplifying (12) even further (while making some sign choice/s for $\phi$ and $v$),
\begin{equation}
\frac{r^2}{R^2_0}\phi_v = T_u, \quad r_0^2\phi_u = \frac{R^2_0}{R_0^2}T_v; \\
\dot{u} = v'R_0^2, \quad R_0^2\dot{v} = \frac{r^2}{r_0^2}u'
\end{equation}
(which itself implies $T_{uu} = \left(\frac{R^2_0}{R_0^2}T_v\right)_u$, resp. $\ddot{z} = \left(\frac{r^2}{r_0^2}z\right)'$; note that every appearance of $v$ with a factor $R_0^2$). Solving the linear first order PDE’s for $\phi(u, v)$ (resp. $v(t, \varphi)$) then provides a (‘Bäcklund’) map between solutions $(T(u, v), R(u, v))$ satisfying (17)/(18) and $(r(t, \varphi), z(t, \varphi))$ satisfying (11)/(13) - which one can easily prove as follows: suppose e.g. $T$ and $R$ satisfy (17), and $\phi$ is a solution of (16); define $r(t, \varphi) := R(u(t, \varphi), v(t, \varphi))$, where $(u, v) \leftrightarrow (t = T(u, v), \varphi = \phi(u, v))$. Then one first straightforwardly verifies that
\begin{equation}
\ddot{r}' + \ddot{z}' = (\ddot{u}R_u + \ddot{v}R_v)(u'R_u + v'R_v) + \dddot{u}u' \\
\quad = \delta((\phi_v R_u - \phi_u R_v)\left(-T_v R_u + T_u R_v - T_v \phi_v\right) \\
\quad = \frac{R_0^2}{r_0^2}\delta((T_u R_u - \frac{R^2_0}{R_0^2}R_v T_v)(-T_v R_u + T_u R_v) - T_u T_v) = \ldots = 0.
\end{equation}
On the other hand implies
\begin{equation}
\frac{\dot{r}^2 + z'^2}{r_0^4} = \frac{R^2_v - T^2_v}{R_0^4 T^2_u - R^2 T^2_v}\left( = \frac{1}{r_0^2} \frac{R^2_v - T^2_v}{|J|}\right) \\
1 - \dot{r}'^2 - z'^2 = \frac{T^2_u - \frac{R^2_v}{r_0^2} - 1}{T^2_u - \frac{R^2_v}{r_0^2} T^2_v}\left( = \frac{r_0^2}{R_0^2} \frac{T^2_u - \frac{R^2}{r_0^2} - 1}{|J|}\right).
\end{equation}

While the above relations/transformations between the two orthonormal parametrizations are very nice, they do not a priori help to find new solutions. One should look for symmetries that one could use (on either of the two sides) which can help to generate from one (original/simple) solution ‘most’ of the other solutions by a mixture of solving the linear first-order PDE(s) (16), and applying symmetry transformations. As it is rather tedious to carry along the factors of $r_0$ and $R_0$, both will now be put = 1 (until specifically needed) as was already done.

1It is important to note that there are two very different kinds of transformations, which effect $r_0(R_0)$, hence could be used to make them = 1: scaling of all space-time coordinate, $x^\mu \rightarrow \lambda x^\mu$ (resulting in $r_0 \rightarrow \lambda r_0$ resp. $R_0 \rightarrow \lambda R_0$; explaining why the conserved quantity $\rho_0 = \int \rho(\varphi)d\varphi$ was chosen to be denoted by the square of the constants $r_0$ and $R_0$- for any given solution, such a scaling (changing the actual/physical/geometric solution) may be used to put the conserved quantity
in the explicit \((u, v)\) example [6] (cp. [11]/[10]), \(T(u, v) = H(u) \cdot v, R(u, v) = H(u)\sqrt{v^2 - 1}\), where \(\phi(u, v) = \sqrt{H^4 + \frac{1}{2}(v^2 - 1)}\), cp.[9], resp. \(u = (t, \varphi)\) and \(v\) related to \(t\) and \(\varphi = \phi(u, v)\) by

\[
\frac{2\varphi}{\sqrt{1 + H^4}} = \frac{t^2}{H^2} - 1, \quad \frac{4\varphi^2}{(v^2 - 1)^2} - \frac{t^4}{v^4} = 1,
\]

and \(r(t, \varphi) = R(u(t, \varphi), v(t, \varphi))\).

2. Reduced, Single-field, Parametric description

As the second order equations for \(T(u, v)\) resp. \(z(t, \varphi)\) arise as compatibility/integrability equations of first-order equations, one may wonder whether this is also true for \(r(t, \varphi) = R(u, v)\). As (7) implies

\[
(\phi_v =)T_u = F(R, R_u, R_v) = F[R] = Y_{uv}
\]

(19)

\[
(\frac{1}{R^2}\varphi_u - Y_{vv} =)T_v = \frac{R_u R_v}{F}
\]

with

\[
F_{\pm}^2 = \frac{1}{2}(R_u^2 + R^2 R_v^2 + 1) \pm \sqrt{\frac{1}{4}(R_u^2 + R^2 R_v^2 + 1)^2 - R^2 R_u R_v^2}
\]

\[
= : P \pm \sqrt{P^2 - Q^2} \geq 0
\]

cross-differentiation,

\[
(F)_v = \left(\frac{R_u R_v}{F}\right)_u, \quad (F)_u = \left(\frac{R^2 R_u R_v}{F}\right)_v
\]

will result in a second-order PDE for \(R\). The consequences of (21) (allowing for four different choices of \(F\)) require a careful discussion, which will now follow for the corresponding equations,

\[
\dot{z} = f[r]
\]

(23)

\[
\frac{\dot{z}' = -i r r'}{f[r]}
\]

\[
f_{\pm}^2 = \frac{1}{2}(1 - \dot{r}^2 - r^2 r'^2) \pm \sqrt{\frac{1}{2}(1 - \dot{r}^2 - r^2 r'^2)^2 - r^2 r'^2} = : p \pm \sqrt{p^2 - q^2}.
\]

= 1; or: changing the scale of \(v\) or \(\varphi\) (which both are reparametrizations, i.e. not changing the geometric/physical solution; hence \(\rho = 1 = \tilde{\rho}\) is no real loss of generality, but for compact membranes one, in some cases, has to be careful to not generate multicovering)
A straightforward, but astonishingly lengthy/tedious calculation of $2f(f' + \frac{r'f}{f})$ gives
\[ \pm \frac{r'}{\sqrt{p^2 - q^2}} \times \text{LHS of } \text{(25)} \] times the LHS of
\[ (\ddot{r} - r^2 \dddot{r} - rr'^2)(p \pm \sqrt{p^2 - q^2}) + \frac{i^2}{r}(p \mp \sqrt{p^2 - q^2}) + i^2(\ddot{r} - r^2 \dddot{r}) = 0. \]
Denoting $(\ddot{r} - r^2 \dddot{r} - rr'^2)$ by $D_0[r]$, and using
\[ f_{\pm} f_{\mp} = rr' = q \] (25) can be written as
\[ (\dot{r}^2 + (p \pm \sqrt{p^2 - q^2}))(D_0[r] + \frac{1}{r}(p \mp \sqrt{p^2 - q^2})) = 0, \]
which implies
\[ D_{\pm}[r] := \ddot{r} - r^2 \dddot{r} - rr'^2 + \frac{1}{r}(p \mp \sqrt{p^2 - q^2}) = 0, \]
which is the correct second-order PDE for $r(t, \varphi)$, as the last term equals $rz'^2$. An analogous (similarly tedious, equally intricate) calculation,
\[ f^2 = p \pm \sqrt{p^2 - q^2}, \] gives
\[ \frac{2f}{r} \sqrt{p^2 - q^2}(\dot{f} + (r \dot{f})) = -(D_0[r] + \frac{1}{r}(p \mp \sqrt{p^2 - q^2}))(r^2 \dot{r}'^2 + (p \pm \sqrt{p^2 - q^2})). \]
For definiteness take
\[ f_{\pm} = \frac{1}{2} \left( \sqrt{1 - (\dot{r} - rr')^2} \mp \sqrt{1 - (\dot{r} + rr')^2} \right) = \frac{1}{2}(\sqrt{-} \pm \sqrt{+}); \]
multiplying (29) by an overall minus-sign would not change (26), nor (24), but $(\dot{r}' r_{\pm} = \frac{-f_{\pm}}{r}, \text{ cp. (26)}\)$ simply correspond to $z \rightarrow -z$, consistent with
\[ z + rz' = f_{\pm} - f_{\mp} = \pm \sqrt{1 - (\dot{r} + rr')^2} \]
\[ z - rz' = f_{\pm} + f_{\mp} = \sqrt{1 - (\dot{r} - rr')^2}. \]
Note that if $r = r_+$ is a solution of $D_+[r_+] = 0$ (with $z = z_+$ then obtained from $\dot{z}_+ = f_+[r_+]$, $z'_+ = -\frac{1}{r}f_+[r_+]$) $r_+$ will generically not satisfy $D_-[r_+] = 0$; so the somewhat curious situation occurs that the solutions of (23) resp. (cp.(3)) $D_z[r] := \ddot{r} - r^2 \dddot{r} - rr'^2 + rz'^2 = 0$ fall into two (more or less disjoint) sets, $(r_+, z_+)$ and $(r_-, z_-)$.

3. Characteristic Coordinates

In [9] many different aspects of ‘characteristic coordinates’ were discussed. Here, it is perhaps simplest to say that one can e.g. use the
differmorphism invariance of the Volume-functional to choose coordinates \(\theta_+\), \(\theta_-\), \(\theta\) on the \(U(1)\) invariant world-volume \(\mathcal{M}_3\) to choose the 2 tangent vectors \(x_\pm : = \frac{\partial x}{\partial \theta_\pm}\) to be null, i.e.

\[
(31) \quad t_\pm^2 - r_\pm^2 - z_\pm^2 = 0.
\]

Then any of the 3 conditions

\[
(32) \quad \partial_+(rt_-) + \partial_-(rt_+) = 0,
\]

\[
(33) \quad \partial_+(rz_-) + \partial_-(rz_+) = 0,
\]

\[
(34) \quad \partial_+(rr_-) + \partial_-(rr_+) + x_+x_- = 0,
\]

will guarantee that \(\mathcal{M}_3\) has zero mean curvature. The metric tensor, its determinant, and its inverse in these coordinates are

\[
G_{\alpha\beta} = \begin{pmatrix} 0 & x_+x_- & 0 \\ x_+x_- & 0 & 0 \\ 0 & 0 & -r^2 \end{pmatrix}
\]

\[
G^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & x_+x_- & 0 \\ 0 & 0 & -\frac{1}{r^2} \end{pmatrix}
\]

\[
\sqrt{G} = r|x_+x_-|.
\]

\[
(36) \quad \partial_\alpha \sqrt{G} G^{\alpha\beta} \partial_\beta = \pm \left( \partial_+ r \partial_- + \partial_- r \partial_+ - \frac{x_+x_-}{r} \partial_\theta^2 \right),
\]

applied to \(x^\mu\), gives \((32)-(34)\) (\(\pm\) corresponding to the sign of \(x_+x_-\)). Note the slight, but crucial, difference to the case of a string world-sheet, where \((31)\) implies \(\sqrt{G} G^{\alpha\beta} = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), hence minimal surfaces in 3-dimensional Minkowski-space corresponding to sums of 2 Null-curves \((x = \psi(\theta_+) + \phi(\theta_-), \psi'^2 = 0 = \phi'^2)\). For axially symmetric membranes \((35)\) results in equations whose solutions due to the extra factor(s) of \(r\), are ‘not exactly’ sums of two Null-curves, but ‘rather close’. Given the crucial conditions \((31)\) each of the second order equations will imply the other ones. Note that this formulation is manifestly invariant with respect to boosts in the \(z\)-direction, i.e. ‘hyperbolic rotations’

\[
(37) \quad \dot{t} = c_\gamma t + s_\gamma z
\]

\[
\dot{z} = s_\gamma t + c_\gamma z.
\]

Consider now what happens to \((31)-(32)\) when transforming to the \((u, v)\) (or \((t, \varphi)\)) coordinates that have been discussed above. For
$(\theta_+ \theta_-) \leftrightarrow (u(= z(\theta_+, \theta_-)), v)$ e.g.

\[
\left( \frac{\partial uv}{\partial \theta_+ \partial \theta_-} \right) = \begin{pmatrix} u_+ & u_- \\ v_+ & v_- \end{pmatrix} = \begin{pmatrix} \partial_+ \theta_+ & \partial_+ \theta_- \\ \partial_- \theta_+ & \partial_- \theta_- \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} \partial_- \theta_- & -\partial_- \theta_+ \\ -\partial_+ \theta_- & \partial_+ \theta_+ \end{pmatrix}
\]

(38)

one gets

\[0 = r^2_+ - r^2_2 - z^2_\pm = (T_u u_+ + T_v v_\pm)^2 - (R_u u_\pm + R_v v_\pm)^2 - u^2_\pm
\]

(39)

\[= u^2_\pm (T^2_u - R^2_u - 1) - v^2_\pm (R^2_v - T^2_v) + 2 u_\pm v_\pm (T_u T_v - R_u R_v)
\]

\[= (R^2_v - T^2_v) (u^2_\pm R^2 - v^2_\pm)
\]

\[\equiv 0,
\]

i.e.

(40)

\[u_\pm R = \mp v_\pm,
\]

corresponding to $\Delta > 0$; equal signs in (40), allowed for by (39), must be excluded as otherwise the transformation between $\theta_+, \theta_-$ and $u, v$ could not be invertible. Note that ‘again’ the second order $z$-equation (now (39)) is automatic, as the compatibility condition following from (40). Using (38),

(41)

\[\partial_+ \theta_\pm \pm R(u, v) \partial_+ \theta_\pm = 0
\]

which from another point of view could also be taken as the definition of characteristic coordinates.

\[0 = (T_u + RT_v)^2 - (R_u + RR_v)^2 - 1
\]

(42)

\[= (t_+ (\partial_+ + R \partial_\v) \theta_+ + t_- (\partial_+ + R \partial_\v) \theta_-)^2 - (r_+ (\partial_+ + R \partial_\v) \theta_+ + (r_- \partial_+ + R \partial_\v) \theta_-)^2 - 1
\]

\[\equiv 4 (\partial_+ \theta_-)^2 (t_+^2 - r_+^2) - 1
\]

\[= 4 (\partial_+ \theta_-)^2 (t_-^2 - r_-^2 - z_-^2)
\]

(i.e (7) implying (30) when using (38)/(41)).
4. Symmetries

Although (1) and (7) at first sight appear to clearly not be invariant under the Lorentz-transformations (37), they are if extending (37) to also involve the additional parameters, i.e. when involving explicit gauge-compensating transformations on $\varphi$, resp. $v$; namely when defining the following field-dependent transformations:

\begin{equation}
\tag{43}
t = T(u, v) \rightarrow \tilde{T}(u, v) = c_\gamma T(u, v) + s_\gamma u = c_\gamma t + s_\gamma z(t, \varphi)
\end{equation}

\begin{equation}
\tag{44}
z(t, \varphi) = u \rightarrow \tilde{u} = c_\gamma u + s_\gamma T(u, v) = c_\gamma z(t, \varphi) + s_\gamma t
\end{equation}

\begin{equation}
\tag{45}
R_0^2 v(t, \varphi) \rightarrow \tilde{R}_0^2 \tilde{v} = c_\gamma R_0^2 v + s_\gamma r_0^2 \tilde{\phi}(u, v) = c_\gamma R_0 v(t, \varphi) + s_\gamma r_0^2(t, \varphi)
\end{equation}

\begin{equation}
\tag{46}
r = R \rightarrow \tilde{R}(u, v) = R(u, v) = r(t, \varphi) = \tilde{r}(t, \varphi)
\end{equation}

(from now on again $r_0 = 1 = R_0$) where $\phi(u, v)$ (analogously for $v(t, \varphi)$), defined by (16), in characteristic coordinates ($\theta_+, \theta_-$), $\phi(u, v) = \phi(\theta_+, \theta_-)$ satisfies (in analogy to (40))

\begin{equation}
\tag{47}
r t_{\pm} = \mp \phi_{\pm}
\end{equation}

(making (32) a compatibility consequence, resp. a consistent definition of $\varphi$ if (32) is satisfied) corresponding to

\begin{equation}
\tag{48}
\tilde{\theta}_{\pm}(t, \varphi) = \theta_{\pm}(u, v),
\end{equation}

and one can easily check the consistency of (43) with (41):

\begin{equation}
\tag{49}
\tilde{T}_a \tilde{T}_b = \tilde{R}_a \tilde{R}_b
\end{equation}

One way to prove that (43) leaves (7) invariant is to first verify, by a slightly tedious (but straight-forward) calculation, using

\begin{equation}
\tag{47}
\left( \frac{\partial uv}{\partial \tilde{u} \tilde{v}} \right) = \left( \frac{\partial \tilde{u} \tilde{v}}{\partial uv} \right)^{-1} = \frac{1}{\Delta} \begin{pmatrix} \tilde{v}_u & -\tilde{u}_v \\ -\tilde{v}_u & \tilde{u}_v \end{pmatrix}
\end{equation}

\begin{equation}
\tag{48}
\partial_a = \frac{1}{\Delta} \left( \tilde{v}_u \partial_u - \tilde{\nu}_a \partial_e \right)
\end{equation}

that (43) implies

\begin{equation}
\tag{49}
\tilde{T}_a \tilde{T}_b = \tilde{R}_a \tilde{R}_b
\end{equation}
(both sides, multiplied by $\Delta^2$, turn out to be equal to $T_v(T_u(c^2 + s^2) + s_c c(1 + T_u^2 - R^2 T_v^2))$, when using (7) and (16)), and then note that (using (14)) (33) implies
\[ \tilde{u}_\pm R = \mp \tilde{v}_\pm, \]
which (using an argument analogous to (33)) implies
\[ (50) \tilde{T}_u^2 - \tilde{R}_u^2 - 1 = \tilde{R}_v^2 (\tilde{R}_v^2 - \tilde{T}_v^2). \]
(33) applied to any given solution (of (7)) will produce a (other/new) solution; in the example $T = H v, R = H \sqrt{v^2 - 1}, \phi = H^1_2 (v^2 - 1)$ e.g. one would get $\tilde{T}(u, v) = c H v + s u, \tilde{R} = R = H \sqrt{v^2 - 1}, \tilde{u} = s H v + c u, \tilde{v} = cv + s H^1_2 (v^2 - 1)$.

A second useful observation is to note that $\tilde{T}(\tilde{u}, \tilde{v} = v) = \lambda T\left(\frac{v}{\lambda}, v\right), \tilde{R}(\tilde{u}, \tilde{v} = v) = \lambda R\left(\frac{v}{\lambda}, v\right), \tilde{\phi}(\tilde{u}, \tilde{v} = v) = \phi(\frac{v}{\lambda}, \tilde{v} = v), \tilde{R}_0 = \lambda R_0$, corresponding to $\lambda \cdot \mathfrak{M}_3$, will solve the equations, if $T, R, \phi, R_0$ do. At this point some additional comments about $\rho_0 = R_0^2$ (the power 2 having been chosen such that $R_0$ scales linearly with $\lambda$, s.a, resp. that with the Dirac-solution, $r^2(t) + \frac{r'(t)}{r_0} = 1, r_0$ corresponds to the maximal radius of the pulsating sphere; mathematically, the significance of $r_0$ (resp. $R_0$) for compact membranes/ranges of the parameters is given by Moser’s lemma [16]) are perhaps useful: first of all note that the above $\lambda$-scaling is a ‘physical/geometric’ scaling, meaning: would in the description 2 similarly describe $\lambda \mathfrak{M}_3$, and in particular require $\tilde{r}_0 = \lambda r_0$. Secondly, two reasons for not always having put $\tilde{\rho}_0$ (or/and $\rho_0$) = 1 (which would have saved one with carrying factors of $R_0$ and $r_0$ around): if using the above scaling symmetry to reach the value 1, which on either one side one could certainly do, it would a priori not be clear (nor, most likely, be the case) whether that particular scaling would make the conserved constant $= 1$ on the other side; if on the other hand, using $v \to \kappa v$ and or scaling the ‘angle’ $\varphi$ (on each side those scalings could be used independently) this could for compact membranes cause multi-coverings; in examples with non-compact range, like the example [10] it would be possible and convenient to put $\rho = 1$.

Both symmetries (boosts, and scaling) however, though on each side differently implemented (in the $(u, v)$ resp. $(t, \varphi)$ parametrization) have the same geometric action/effect on the two sides (in case of the scaling symmetry e.g. describing ‘$\lambda \cdot \mathfrak{M}^3$’ instead of $\mathfrak{M}_3$). Still, the existence, (and simplicity) of the (‘Bäcklund’) transformation (16) that maps solutions in one parametrization to solutions in the other is a sign of ‘integrability’.

Note that, curiously, each of the equations in (22), alone, leads to the
equation
\[ \tilde{D}_\pm[R] := R_{uu} - R^2 R_{vv} - R R_v^2 - R T_v^2 \]
(52)
\[ ( = \tilde{D}_0[R] - \frac{1}{R} F_\pm^2 ) = 0, \]
if \( F \) in (22) is chosen to be \( F_+ \) or \( F_- \),
(53)
\[ F_\pm = \frac{1}{2} \left( \sqrt{1 + (R_u + R R_v)^2} \pm \sqrt{1 + (R_u - R R_v)^2} \right), \]
while the fact that (due to the relation of \( T \) to \( \phi \), resp. the PDE satisfied by \( T \)) both equations in (22) hold can be used to derive that
(54)
\[ \left( \begin{array}{c} F_u \\ F_v \end{array} \right) = \frac{1}{1 - \frac{Q^2}{F^2}} \left( \begin{array}{c} \frac{Q}{R R_v^2} - \frac{R Q}{F^2} \\ -1 \end{array} \right) \frac{1}{F} \left( \begin{array}{c} (R Q)_v \\ (\frac{R}{R_u})_v \end{array} \right), \]
where \( Q := R R_u R_v \).

5. Light-Cone Formulation

Yet another triply orthogonal parametrization (given already in [1], but rarely used – see however p.52-55 of [9]) should be mentioned, namely
(55)
\[ \tilde{x}_\mu = \left( \begin{array}{c} \tau + \frac{\zeta}{2} \\ R \cos v \\ \tau - \frac{\zeta}{2} \end{array} \right), \quad \partial_\tau \tilde{x}_\mu = \left( \begin{array}{c} 1 + \frac{\zeta}{2} \\ R \cos v \\ 1 - \frac{\zeta}{2} \end{array} \right), \]
\[ \tilde{x}^{\mu} = \left( \begin{array}{c} \frac{\zeta}{2} \\ R \cos v \\ -\frac{\zeta}{2} \end{array} \right), \quad \tilde{G}_{\alpha \beta} = \left( \begin{array}{ccc} 2 \dot{\zeta} - \dot{R}^2 & 0 & 0 \\ 0 & -R^2 & 0 \\ 0 & 0 & -R^2 \end{array} \right), \]
with
(56)
\[ (\dot{G}_{01} = ) \zeta' - \dot{R} R' = 0, \quad 2 \dot{\zeta} = \dot{R}^2 + \frac{R^2 R''}{\eta^2}, \]
i.e. \( \dot{G}_{01} = 0 (\equiv \dot{G}_{10}) \) and \( \dot{G}_{11} \dot{G}_{22} = \eta^2 \dot{G}_{00} \) (\( R \) and \( \zeta \) are functions of \( \tau \) and a spatial parameter, \( \mu \)). (56) implies both
(57)
\[ \eta^2 \ddot{R} = R^2 R'' + R R^2 \left( = R (R R')' = (R^2 R')' - R' R^2 \right), \]
and
(58)
\[ \eta^2 \ddot{\zeta} = (R^2 \zeta')', \]
which are the correct equations,
(59)
\[ \partial_\alpha \sqrt{\tilde{G}} \tilde{G}^{\alpha \beta} \partial_\beta \tilde{x}^\mu = (\partial_\tau^2 - \frac{1}{\eta^2} \partial_\mu R^2 \partial_\mu - \frac{R^2}{\eta^2} \partial_v^2) \tilde{x}^\mu = 0. \]
While the fact that (57) are Hamiltonian equations, with respect to

\[ H[R, P; \eta, \zeta_0] = \frac{1}{2\eta} \int (P^2 + R^2 R'^2) d\mu \]  

(with the weight, \( \eta = \int \rho (\mu) d\mu \), of the conserved light-cone density \( \rho \) acquiring yet another meaning/importance) is well-noted, it is in the context of integrability most likely crucial to not ‘forget’ \( \zeta \) (the traditional point of view has been to view (56) solely as determining \( \zeta \), which—when considering (60) – has ‘dropped out’, except for the zero-mode \( \zeta_0 \), canonically conjugate to \( \eta \); it should be worthwhile studying in detail the ‘reconstruction algebra’, introduced in [17], for this axially symmetric case). That (56) equally implies (58) could be used to introduce \( Y(\tau, \mu) \) via

\[ (\eta^2 \dot{R}^2 + R'^2 R^2) \frac{1}{2} = Y' (= \dot{\zeta}) \]

\[ R^2 \dot{R} R' = Y (= R^2 \zeta'), \]

implying

\[ (\frac{1}{R^2} \dot{Y}) = Y'' \frac{1}{\eta^2} \]

and then, with

\[ w' := \zeta - \zeta_0, \quad \dot{w} = \frac{1}{\eta^2} Y \]

get

\[ \eta^2 \ddot{w} = R^2 w''. \]

For one particular (‘starting’) solution \( R \), could one take \( w \) to be a linear-combination of specific solutions of (64), which then (calculating \( \zeta \) from (63), and \( R \) from (60)) would /ad infinitum/ give new solutions?). Note also the following puzzle (and its resolution): under boosts in the \( z \)-direction, \( \eta \) and \( \tau \) are known to be multiplied by \( e^{\gamma} \in \mathbb{R} \). How does that fit into the above mentioned relations with the \((t, \varphi)\) resp. \((u, v)\) representation? (in particular: how can one justify \( \mu \) in (60) to be invariant under such boosts?)

\[ \begin{pmatrix} \sim_{G_{00}} & 0 & 0 \\ 0 & \sim_{G_{11}} & 0 \\ 0 & 0 & \sim_{G_{22}} \end{pmatrix} = \begin{pmatrix} \dot{\tau} & \dot{\mu} & 0 \\ \tau' & \mu' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\dot{\zeta} - \dot{\bar{R}}^2 & 0 & 0 \\ 0 & -R'^2 & 0 \\ 0 & 0 & -R^2 \end{pmatrix} \begin{pmatrix} \dot{\tau} & \tau' & 0 \\ \dot{\mu} & \mu' & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
where \(\cdot\) and \(\prime\) denote differentiations with respect to \(t\) and \(\varphi\) (or \(u\) and \(v\), depending on what one takes on the left) always gives

\[
\hat{G}_{00}\ddot{\tau}^\prime + \hat{G}_{11}\ddot{\mu}^\prime = 0, \quad \hat{\rho}^2 = \frac{\hat{G}_{11}\hat{G}_{22}}{\hat{G}_{00}} = \eta^2 = \text{Const.,}
\]

(66)

\[
\frac{\ddot{\mu}^\prime}{\ddot{\tau}^\prime} = \frac{R^2}{\eta^2} = \frac{\dot{r}^2}{\eta^2} = \frac{1}{\eta^2} \frac{\mu}{\mu^\prime}\tau^\prime
\]

hence

(67)

\[
\eta\ddot{\mu} = \dot{r}^2\tau^\prime, \quad \eta\mu^\prime = \dot{\tau}.
\]

So

(68)

\[
2\eta\ddot{\mu} = r^2\dot{z}^\prime, \quad 2\eta\mu^\prime = 1 + \dot{z},
\]

resp.

(69)

\[
2\eta\ddot{\mu}_u = R^2\ddot{\tau}(u, v) = R^2T_v = \phi_u
\]

\[
2\eta\ddot{\mu}_v = 1 + T_u = 1 + \phi_v,
\]

hence

(70)

\[
\ddot{\mu}(u, v) = \left(v + \phi(u, v)\right)\frac{1}{2\eta}, \quad \ddot{\tau}(u, v) = \frac{1}{2}(u + T(u, v))
\]

\[
2\eta\mu(t, \varphi) = \varphi + v(t, \varphi), \quad 2\tau(t, \varphi) = t + z(t, \varphi)
\]

which indeed is invariant under (43), as

(71)

\[
v + \phi(u, v) \rightarrow (c_\gamma + s_\gamma)(v + \phi(u, v)) = e^{\gamma}(v + \phi(u, v))
\]

as well as making \(\eta\frac{\partial}{\partial \tau}\) invariant too. Note that (60) and (70)/(16) imply (i.e. again, a conformal equivalence in the upper \(2 \times 2\) part of the metric)

(72)

\[
\frac{T_u^2 - R_u^2 - 1}{[(1 + T_u)^2 - R^2T_v^2]} = \frac{s^2s^2}{4\eta^2}
\]

\[
\frac{R_v^2 - T_v^2}{[(1 + T_u)^2 - R^2T_v^2]} = \frac{s^2}{4\eta^2},
\]

which is consistent, as the ratio is \(R^2\), and the second equation also follows by using

(73)

\[
s' = \partial_\eta s = (u_\eta\partial_u + v_\eta\partial_v)R
\]

\[
= \frac{2\eta}{[(1 + T_u)^2 - R^2T_v^2]}[(1 + T_u)R_v - T_vR_u].
\]

Similarly, also using

(74)

\[
\zeta' = (u_\eta\partial_u + v_\eta\partial_v)(T(u, v) - u) = T_v\frac{4\eta}{\Delta}
\]

\[
\equiv \dot{s}s'.
\]
one finds
\[
\dot{s} = \frac{2}{\Delta} \left( (1 + T_u) \partial_u - R^2 T_v \partial_v \right) R
\]
(75)
\[
= \frac{2}{\Delta} \left( R_u (1 + T_u) - R^2 T_v R_u \right)
\]
\[
= \frac{T_v}{\sqrt{\Delta} \sqrt{R^2_v - T^2_v}}
\]
and/hence
(76)
\[
\left( (1 + T_u) \partial_v - T_v \partial_u \right) \frac{T_v}{\sqrt{\Delta} \sqrt{R^2_v - T^2_v}} = \left( (1 + T_u) \partial_u - R^2 T_v \partial_v \right) \frac{\sqrt{R^2_v - T^2_v}}{\sqrt{\Delta}}.
\]

6. More Bäcklund Transformations

Due to (16), (23) implies two compatibility equations, namely (for \( r_0 = 1 = R_0 \))

(77) \[ [1] := f' + \left( \frac{1}{r} \tilde{f} \right) = 0, \quad [2] := \dot{f} + (r \tilde{f})' = 0, \]
where \( f \) is one of the 4 solutions
\[
f_{++} = +\sqrt{p + \sqrt{p^2 - q^2}} = +\sqrt{f^2_+};
\]
\[
f_{+-} = -\sqrt{p + \sqrt{p^2 - q^2}} = -\sqrt{f^2_+};
\]
\[
f_{-+} = +\sqrt{f^2} = +\sqrt{p - \sqrt{p^2 - q^2}};
\]
\[
f_{--} = -\sqrt{p - \sqrt{p^2 - q^2}}
\]
and \( \tilde{f} \) another one, such that
(79) \[ f \tilde{f} = rr' = q; \]

each of the 4 solutions in (78) satisfies the ‘linear’ PDE
(80) \[ \left( \frac{1}{r} \tilde{f} \right)' = (r \tilde{f})'' \]

and the set given in (78) coincides with the set (cp.(29))
(81) \[ g_{\delta\delta'} = \frac{\delta}{2} \sqrt{1 - (\dot{r} - rr')^2} + \frac{\delta'}{2} \sqrt{1 - (\dot{r} + rr')^2} =: \frac{\delta}{2} \sqrt{-} + \frac{\delta'}{2} \sqrt{+}. \]
Using both parts of (77) somewhat simplifies the tedious derivation of (28); e.g. for \( f = g_{++} = f_+ \), \( \tilde{f} = g_{+-} = f_- \):

\[
\begin{align*}
\dot{r}(\sqrt{-} + \sqrt{+}) + (\sqrt{-} - \sqrt{+}) - \frac{\dot{r}}{r}(\sqrt{-} - \sqrt{+}) &= 0 \\
\dot{r}(\sqrt{-} - \sqrt{+}) + (\sqrt{-} + \sqrt{+}) + r'(\sqrt{-} - \sqrt{+}) &= 0,
\end{align*}
\]

\([1]+[2] \quad r_{-rr'}\) giving

\[
\begin{align*}
(\partial_t + r \partial_\varphi)(\partial_t - r \partial_\varphi)r + \frac{1}{r}(p + q - \sqrt{p^2 - q^2}) &= 0, \\
(\partial_t - r \partial_\varphi)(\partial_t + r \partial_\varphi)r + \frac{1}{r}(p - q - \sqrt{p^2 - q^2}) &= 0,
\end{align*}
\]

implying \( \dot{r}[82]_1 = -rr'[82]_2 \) – note also (the overall sign in front of \( \sqrt{-} \) being positive if \( f = f_+ \))

\[
(\partial_t \pm r \partial_\varphi)z = f \mp \tilde{f} = \sqrt{\pm} = \sqrt{1 - (\dot{r} \pm rr')^2}
\]

and that (82) implies

\[
(\partial_t \pm r \partial_\varphi)\sqrt{\mp} = \pm \frac{1}{2r}(\dot{r} \mp rr')(\sqrt{-} - \sqrt{+});
\]

and that the two equations in (82) (resp. (77)) in characteristic coordinates read (consistent with \([9]\); note that \( 2\dot{r} = \frac{r_+}{t_+} + \frac{r_-}{t_-} \))

\[
\begin{align*}
\left(\frac{t_{++}}{t_+ t_-} + \frac{\dot{r}}{r} \right) \cdot \left(\frac{z_+}{t_+} - \frac{z_-}{t_-} \right) &= 0 \\
2z_{++,} &= \frac{z_-}{t_-}(t_{++} - r't_+ t_-) + \frac{z_+}{t_+}(t_{++} + r't_+ t_-),
\end{align*}
\]

but it is important to stress that, as done when first deriving (28) (= (84) = (83)), each of the two equations in (77), alone, is sufficient to give (28), – whose solutions, via (23) and (2), give (not counting \( z \rightarrow -z \) ) ‘half’ of the (time-like) axially minimal 3-manifolds in \( \mathbb{R}^{1,3} \). This may lead one to speculate that the 2 (equivalent) options in (77) reflect a bi-Hamiltonian nature of the problem (which is one of the formal routes to get infinitely many conserved quantities); but where (what exactly) are the Hamiltonians?

First of all, what about a Lagrangian description of (28)? One could...
try to ‘parametrize’ the unparametric radial action for \( r = r(t, z) \) (cp. e.g. [8])

\[
S_r[r(t, z)] = \int r \sqrt{1 - \dot{r}^2 + r'^2} \, dt \, dz,
\]

i.e. considering \((t = x^0, x^3 = z(t, \varphi)) \leftrightarrow (t = \varphi^0, \varphi = \varphi^1(x^0, x^3))\),

\[
\left( \begin{array}{cc} \frac{\partial x^0}{\partial \varphi} & 0 \\ \frac{\partial x^3}{\partial \varphi} & \partial \varphi \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ \dot{z} & z' \end{array} \right)^{-1} = \left( \begin{array}{cc} 1 & 0 \\ -\frac{\dot{z}}{z'} & \frac{1}{z'} \end{array} \right)
\]

\[
(\dot{r} =) \partial_{x^0} r = (\partial_t + \frac{\partial x^0}{\partial \varphi} \partial_{\varphi}) r = (\partial_t - \frac{\dot{z}}{z'} \partial_{\varphi}) r \rightarrow \dot{r} + \frac{\ddot{r}}{\dot{r}} \dot{r}'
\]

\[
(\dot{r}' =) \partial_{x^3} r = \frac{\partial x^3}{\partial \varphi} \partial_{\varphi} r = \frac{1}{z'} \partial_{\varphi} r \rightarrow -\frac{rr'}{\dot{r}}
\]

which (using \( dt \, dz = dt \, d\varphi \mid \frac{\partial x^0}{\partial \varphi} \mid = |z'| \, dt \, d\varphi \)) gives

\[
S_{rz} = S[r(t, \varphi), z(t, \varphi)] = \int \frac{r \sqrt{r'^2 + z'^2 - (\dot{r} z' - r' \dot{z})^2}}{\sqrt{r'^2 + \ddot{f}^2 - (\dot{r} \ddot{f} + rr' \dot{f})^2}} \, dt \, d\varphi
\]

(91)

with \( f, \ddot{f} \) given by (29).

While \( S_{rz} \) is certainly correct (and coincides with \( \int \sqrt{G} \, d\varphi^0 \, d\varphi \, d\psi \) when \( \varphi^0 = t \), and integrating out the axial symmetry variable \( \psi \), \( S_r[r(t, \varphi)] \) must be taken with great care, as inserting (23) simply into the Lagrangian, rather than into the equations of motion could easily give wrong conclusions). As \( S_{rz} \) is unconstrained, one may wonder how it relates to (1), whose first part is easily motivated/‘achieved’ by noting that (91) is invariant under time-dependent reparametrizations \( \varphi \rightarrow \tilde{\varphi}(\varphi, t) \), which clearly allows to choose \( \tilde{r} \tilde{r}' + \dot{z} \tilde{z}' = 0 \). Note that defining \( \Pi_r := \frac{\dot{r}}{\dot{r}'} \) and \( \Pi_z := \frac{\dot{z}}{\dot{r}} \) does give nice expression in terms of the Jacobian \( \dot{r} \tilde{z}' - r' \dot{z} \), and in particular \( \Pi_r \ddot{r} + \Pi_z \dot{z} = 0 \), but – no surprise – does not allow to express \( \dot{r} \) and \( \dot{z} \) in terms of \( \Pi_r \) and \( \Pi_z \).

More importantly: what is the ‘symmetry’ allowing for the second part of (1)? That one can indeed choose \( \varphi(r, z) \) such that both parts in (1) hold is specific to the minimal-surface problem, namely \( t(r, z) \) (the time at which the surface reaches the point \( (r, z) \) in space) satisfying (cp. [8])

\[
\vec{\nabla} \left( \frac{r \cdot \vec{\nabla} t}{\sqrt{(\nabla t)^2 - 1}} \right) = 0.
\]

(92)
After the hodograph transformation $t, \varphi \leftrightarrow r, z$ (cp. [8], [9]), (93) reads

\[
\left( \begin{array}{c}
\dot{r} \\
\dot{z}
\end{array} \right) = \left( \begin{array}{cc}
t_r & t_z \\
\varphi_r & \varphi_z
\end{array} \right)^{-1} \frac{1}{\det} \left( \begin{array}{cc}
\varphi_z & -t_z \\
-\varphi_r & t_r
\end{array} \right)
\]

(94) reads

\[
\vec{\nabla} \varphi \cdot \vec{\nabla} t = 0, \quad ((\nabla t)^2 - 1)((\nabla \varphi)^2 - r^2) = r^2,
\]

which is solvable, resp. solved, by

(95) \[
\varphi_r = \frac{\pm rt_z}{\sqrt{(\nabla t)^2 - 1}}, \quad \varphi_z = \frac{\mp rt_r}{\sqrt{(\nabla t)^2 - 1}},
\]

(the consistency being precisely (92)); as on the other hand

(96) \[
t_r = \frac{\mp \varphi_z}{\sqrt{(\nabla \varphi)^2 - r^2}}, \quad t_z = \frac{\pm \varphi_r}{\sqrt{(\nabla \varphi)^2 - r^2}},
\]

we have derived yet another sign of integrability, namely (95)/(96) being classical Bäcklund-transformations between solutions of (92) and solutions of (97)

\[
\left( \frac{\varphi_r}{\sqrt{(\nabla \varphi)^2 - r^2}} \right)_r + \left( \frac{\varphi_z}{\sqrt{(\nabla \varphi)^2 - r^2}} \right)_z = \vec{\nabla} \left( \frac{\vec{\nabla} \varphi}{\sqrt{(\nabla \varphi)^2 - r^2}} \right) = 0.
\]

In analogy with (91) one could of course equally well, via \((t = T, z = u) \leftrightarrow (u, v = V(u, T(u, v)))\), derive

(98) \[
S_{TR} = \int \sqrt{R_u T_v - R_v T_u)^2 - (R_v^2 - T_v^2) R} \, dudv
\]

from (89) (or (5)) and then, by choosing \(v = V(u, T(u, v))\) such that (7) (implying (1)) holds, and again interchanging independent and dependent variables \((u, v) \leftrightarrow T, R, u = z(T, R), v = v(T, R), \)

(99) \[
\left( \begin{array}{cc}
T_u & T_v \\
R_u & R_v
\end{array} \right) = \left( \begin{array}{cc}
u_T & u_R \\
v_R & v_T
\end{array} \right)^{-1} \frac{1}{\det} \left( \begin{array}{cc}
v_R & -u_R \\
-v_T & u_T
\end{array} \right),
\]

obtaining

\[
\begin{align*}
& u_T v_T = u_R v_R \quad (i.e. \partial_\alpha u \partial^\alpha v = 0) \\
& v_R^2 - v_T^2 - (u_T v_R - u_R v_T)^2 = R^2(u_T^2 - u_R^2) \quad (= R^2 u^\alpha u_\alpha) \\
& (1 - u^\alpha u_\alpha)(R^2 - v^\beta v_\beta) = R^2,
\end{align*}
\]
which is solved by

\begin{align}
\tilde{v}_T &= \frac{R u_R}{\sqrt{1 - u^\alpha u_\alpha}}, \quad \tilde{v}_R &= \frac{R u_T}{\sqrt{1 - u^\alpha u_\alpha}}, \\
\tilde{u}_R &= \frac{v_T}{\sqrt{R^2 - v_\beta v_\beta}}, \quad \tilde{u}_T &= \frac{v_R}{\sqrt{R^2 - v_\beta v_\beta}},
\end{align}

implying

\begin{align}
\partial_\alpha \left( \frac{r z^\alpha}{\sqrt{1 - z^\beta z_\beta}} \right) = 0, \quad \partial_\alpha \left( \frac{v^\alpha}{\sqrt{R^2 - v_\beta v_\beta}} \right) = 0
\end{align}

(which are the Euler-Lagrange equations corresponding to \(- \int r \sqrt{1 - \dot{z}^2 + z'^2} \, dr \, dt\), resp. \(- \int \sqrt{R^2 - \dot{v}^2 + v'^2} \, dR \, dt\)).

7. Zero (Gauss) Curvature Condition(s) for Unconstrained Motion(s)

The (semi-) final, calculationally (together with (28)) most difficult/tedious (though beautiful, as giving the simplest possible of all zero-curvature conditions), aspect of integrability in the extremality-properties of axially symmetric membranes, reported here, has to do with the reduced description following from (23), namely considering simply the planar motion of the curves

\begin{align}
\tilde{u}(t, \varphi) &= \begin{pmatrix} r(t, \varphi) \\ z(t, \varphi) \end{pmatrix} \\
\tilde{v}(t, \varphi) &= \begin{pmatrix} \dot{r} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ f[r] \end{pmatrix} \\
\tilde{u}'(t, \varphi) &= \begin{pmatrix} r' \\ z' \end{pmatrix} = \begin{pmatrix} r' \\ -f[r] r' \end{pmatrix}
\end{align}

resp.

\begin{align}
\tilde{v}(t, \varphi) &= \begin{pmatrix} r(t, \varphi) \\ v(t, \varphi) \end{pmatrix} \\
\tilde{v}(t, \varphi) &= \begin{pmatrix} \dot{r} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ -r \tilde{f}[r] = r^2 z' \end{pmatrix} \\
\tilde{v}'(t, \varphi) &= \begin{pmatrix} r' \\ v' \end{pmatrix} = \begin{pmatrix} r' \\ f[r] = \dot{z} \end{pmatrix},
\end{align}
which can be used to define 2-dimensional metrics

\[
(u_{ab}) = \begin{pmatrix}
\dot{r}^2 + \dot{z}^2 = g_1 = H_1^2 = \dot{r}^2 + f^2 \\
0 \\
r^2 + z^2 = g_2 = H_2^2 = r^2 + \dot{f}^2
\end{pmatrix}
\]

\[
(\tilde{g}_{ab}) = (v_{ab}) = \begin{pmatrix}
\dot{r}^2 + v^2 = g_{11} = \dot{r}^2 + r^2 \tilde{f}^2 \\
\ddot{r}r' + \dot{v}v' = g_{12} = \ddot{r}r' - r\dot{f} \dot{f} \\
r^2 + v^2 = g_{22} = r^2 + f^2
\end{pmatrix}
\]

(note that \((\tilde{g}_{ab}) = (v_{ab})\) is not diagonal, and its determinant is the square of \((\dot{r}\dot{z} - r^2r'z') = (\dot{r}f + rr' \dot{f})\)). While calculating the Gauss-curvature \(K\) from the LHS expressions in \((105)\) one trivially gets zero (as the expressions for the entries of the metric then explicitly come from planar curves, sweeping out part of \(\mathbb{R}^2\)), this is not the case for the RHS expressions. For the simpler, diagonal, case Liouville’s formula

\[
K = \frac{1}{\sqrt{g}} \left[ (\sqrt{g} g_{11}^2) - (\sqrt{g} g_{11}^2) \right]
\]

simplifies to the standard consistency condition for diagonal metrics (see e.g. the first line of equation(3) in [18], where, quoting Lamé, the conditions for triply orthogonal coordinate systems in \(\mathbb{R}^3\) (!) are given),

\[
\left( \frac{\dddot{H}_2}{H_1} \right) + \left( \frac{\dddot{H}_1}{H_2} \right) = 0 \quad (\equiv -H_1 H_2 K),
\]

i.e.

\[
\left( \frac{r'\dot{r}' + (\frac{f}{r})(\frac{\dot{f}}{r})}{\dot{r}'r'(\frac{\dot{f}}{r} + \frac{f}{r})} \right) + \left( \frac{\dddot{r}'r' + f f'}{\dot{r}'r'(\frac{\dot{f}}{r} + \frac{f}{r})} \right) = 0
\]

(note that they heavily involve 3\(^{rd}\)-derivatives of \(r\)). Due to \((23)\) one knows that, using the second-order equation for \(r\), \(D[r] = 0\) (cp. \((28)\)), the above must hold, but it is important to find out ‘how exactly’. Using the first PDE in \((77)\) (the one that was shown to reduce to \((27)\)) one can write the two numerators above as

\[
\begin{align*}
r'\dot{r}' + \left( \frac{\dot{f}}{r} \right) \left[ \left( \frac{\dot{f}}{r} \right) + f' \right] - \dot{r}' f' \frac{\dot{f}}{f} &= r' (\dot{r}' f - \dot{r}f') + \frac{\ddot{f}}{r} \quad \left[ f[1] \right] \\
\ddot{r}'f' + f f' &= \dddot{r}'f' + f \left[ f' + \left( \frac{\dot{f}}{r} \right) \right] - f \left( \frac{\dddot{r}'f'}{f} \right) \\
&= -\dddot{r}' + \dot{r}' \dot{f} \frac{\dot{f}}{f} + f[1] \\
&= \frac{f'}{f} (-\ddot{r}f + \ddot{r}f) + f[1].
\end{align*}
\]
As the (common) denominator contains only first derivatives, it is then easy to see that all third-order terms not involving \([1]\) cancel, \((\ddot{r}f - \dot{r}\hat{f} + \dot{r}\hat{f}' - \ddot{r}f) = 0\), and the terms involving \([1]\) give, with the help of (27)

\[
(aD)' + (bD)' = \left(\frac{\dot{r}' - \dot{f}\hat{r}}{f^2 + \hat{f}^2}\right)' + \left(\frac{\dot{f} - \dot{r}\hat{f}}{f^2 + \hat{f}^2}\right)' = 0,
\]

as the terms not involving \([1]\) cancel

\[
\begin{align*}
(f\dot{r}' - f'\hat{r})' + (\dot{f} - \dot{r}\hat{f})' = 0,
\end{align*}
\]

(one way to see the cancellations in \((111)\) is to look at the coefficients of \(f, f^2\hat{f}, f'\hat{f}, \hat{f}, f', f\hat{f}\), i.e. without calculating the derivatives of \(f\), or using its form; \((111)\) holds for \(any f\). Note that while \(D[r] = 0\) implies \(K = 0\) which was clear from the beginning, the two conditions are not//yet// equivalent, as \((110)\) could in principle hold by a less trivial vanishing -mechanism. Where could additional ‘help’ come from? To use \((77)\) for \((108)\) seemed to not easily lead to conclusions. The \(v\)-curves, \((104)\), however (which by the central ‘Bäcklund’ relations \((16)\) also describe planar motion in terms of \(f\) and \(\hat{f}\)), though more complicated (s.b.), can be shown to give an equation of the form

\[
(cD)' + (dD)' = eD,
\]

hence together with \((110)\) presumably giving equivalence of \(D[r] = 0\) and the vanishing of the Gauss curvature(s) (for motions satisfying \((23)\)).

\[(106)\], together with

\[
\tilde{g}\Gamma^2_{11} = -\frac{1}{2}g_{12}\dot{g}_{11} + \frac{1}{2}g_{11}(2\dot{g}_{12} - g'_{11})
\]

\[
\tilde{g}\Gamma^2_{12} = -\frac{1}{2}g_{12}\dot{g}'_{11} + \frac{1}{2}g_{11}\dot{g}_{22}
\]

gives, with \(\sqrt{\tilde{g}} = (f\dot{r} + \hat{f}rr')\),

\[
(f\dot{r} + \hat{f}rr')\tilde{K} = \left[\frac{(qr - \dot{r}r')(\ddot{r}r + r\dot{f}(\dot{r}\hat{f}))}{(\dot{r}' + f'\hat{r})^2} + \frac{(\ddot{r}r' - qr) - (\dot{r}\hat{f}' + \hat{f}r(\dot{f}\hat{f}'))'}{\sqrt{\tilde{g}}}\right]'
\]

\[
- \left[\frac{r'\dot{r}' + f\dot{f}}{\sqrt{\tilde{g}}} + \frac{(qr - \dot{r}r')(\ddot{r}r' + (\dot{f}r)(\dot{f}\hat{f}'))'}{\sqrt{\tilde{g}(\dot{r}' + f'\hat{r})^2}}\right].
\]

Using \((77)\) one may try to convince oneself that again all third-order terms not involving \([1]\) or \([2]\) cancel. As \(D[r] = 0\) (which is of second-order) implies \(\tilde{K} = 0\), \((113)\) must then reduce to an equation of the
form \((cD) + (dD)' = eD\), and it seems reasonable to assume that \(D = 0\) is indeed equivalent to \(K = 0 = \tilde{K}\).

8. (Multi-) Hamiltonian Structures

The two equations in (102) are Hamiltonian with respect to

\[
H_z = \int dr \sqrt{\pi^2 + r^2} \sqrt{1 + \dot{z}^2} = H_z[z, \pi]
\]

\[
H_V = \int dR \sqrt{P^2 + 1} \sqrt{R^2 + v'^2} = H_V[V, P],
\]

as can be easily checked, using canonical Poisson structures:

\[
\dot{z} = \frac{\delta H_z}{\delta \pi} = \pi \frac{\sqrt{1 + \dot{z}^2}}{\sqrt{\pi^2 + r^2}}, \quad \dot{V} = \frac{\delta H_V}{\delta P} = P \frac{\sqrt{R^2 + V'^2}}{\sqrt{P^2 + 1}},
\]

\[
\dot{\pi} = -\frac{\delta H_z}{\delta z} = z' \frac{\sqrt{\pi^2 + r^2}}{\sqrt{1 + \dot{z}^2}}, \quad \dot{P} = -\frac{\delta H_V}{\delta V} = V' \frac{\sqrt{R^2 + V'^2}}{\sqrt{R^2 + V'^2}}
\]

(115)

(the 2+1-dimensional version of \(H_z\) was discussed in \([19]\), and its 1+1-dimensional version goes back to \([20]\)). Note also the relations

\[
\dot{v} = \frac{rz'}{\sqrt{1 - \dot{z}^2 + z'^2}} = \frac{\sqrt{r^2 + v'^2}}{\sqrt{1 + \dot{z}^2}},
\]

\[
\dot{z} = \frac{v' \sqrt{1 + \dot{z}^2}}{\sqrt{r^2 + v'^2}}, \quad v' = \frac{r \dot{z}}{\sqrt{1 - \dot{z}^2 + z'^2}}
\]

and

\[
\left( \begin{array}{c} z' \\ v' \end{array} \right) = \left( \begin{array}{cc} 0 & \partial_r \\ \partial_r & 0 \end{array} \right) \left( \begin{array}{c} \frac{\delta H}{\delta z'} \\ \frac{\delta H}{\delta v'} \end{array} \right), \quad H = \int \sqrt{1 + \dot{z}^2} \sqrt{r^2 + v'^2} \, dr.
\]

As \((q, p) \leftrightarrow (P = \mp q, Q = \mp p)\), i.e. interchanging coordinates and momenta (with one − sign), are canonical transformations, from a Hamiltonian point of view the Bäcklund-transformations (101), resp. (96)/(95), could be considered as auto-Bäcklund-transformations (as the Hamiltonians in (114) are ‘self-dual’). Also note that in the case of compact membranes, boundary conditions (e.g. the range of \(r\)) are tacitly assumed to work out; e.g. in (117), for the spherically symmetric solution,

\[
z(t, r) = D^2(t) - r^2, \quad \sqrt{1 - \dot{z}^2 + z'^2} = \frac{D^3}{\sqrt{D^2 - r^2}}
\]

\[
\dot{v} = -\frac{r^2}{D^2}, \quad v' = r \frac{D}{D^2}, \quad v(t, r) = \pm \frac{r^2}{2} \frac{\sqrt{1 - D^2}}{D^2},
\]

(118)
\[ \frac{d}{dt} H = \frac{d}{dt} \int_0^{D(t)} dr \sqrt{1 + z'^2 \sqrt{r^2 + v'^2}} \]

\[ = \frac{d}{dt} \lim_{\varepsilon \to 0} \int_0^{D(t) - \varepsilon} \frac{r \ dr}{D\sqrt{D^2 - r^2}} \]

\[ = \frac{d}{dt} \lim_{\varepsilon \to 0} \left( \frac{-\sqrt{D^2 - r^2}}{D} \right)_{0}^{D(t) - \varepsilon} = 0 \]

resp. (as one would do for other conserved quantities)

\[ \frac{d}{dt} H = \lim_{\varepsilon \to 0} \left( \frac{\dot{D}}{\sqrt{2\varepsilon D}} - \frac{\dot{D}}{D^2} \int_0^{D(t) - \varepsilon} \frac{r \ dr}{\sqrt{2\varepsilon D^2 - r^2}} \right) = 0 \]

9. \( D[r] \), Again

Consider a conservation law

\[ g' + \dot{h} = 0 \]

where \( g = g(\dot{r}, r', r) \) and \( h(\dot{r}, r', r) \) satisfy

\[ g_u \equiv \frac{\partial g}{\partial \dot{r}} = -\frac{\partial h}{\partial r'} \equiv -h_w, \quad g_w \equiv \frac{\partial g}{\partial r'} = -r^2 \frac{\partial h}{\partial \dot{r}} \equiv r^2 h_u \]

(122)

\[ \text{(which the 2 choices } g = f_+ = \dot{z}_+^{(+)}, \quad h = \frac{1}{r} f_- = -z'_+^{(+)}) \text{ resp. } g = f_- = \dot{z}_-^{(-)}, \quad h = -\frac{1}{r} f_+ = -z'_-^{(-)} \text{ both do) (121)} \]

then reads

\[ h_\dot{r}(\dot{r} - r^2 \dot{r}') + \dot{r} h_r + r' g_r = 0; \]

note that the terms proportional to \( \dot{r}' \) have cancelled because of the \(-\) sign in (122) \(_1\) (in the more common, Lagrangian, origin of (121), where \( g = \frac{\partial C}{\partial r'} \) and \( h = \frac{\partial C}{\partial r} \) one would have a + sign).

Assuming the 2 basic relations

\[ g^2 + r^2 h^2 = 1 - \dot{r}^2 - r^2 \dot{r}'^2 \]

\[ gh = r \dot{r}' \left( = \frac{q}{r} \right), \]

(124)

(which actually imply (122)) one trivially derives

\[ g_r h + gh_r = 0, \quad g_\dot{r} h + gh_\dot{r} = \dot{r}', \quad g_\dot{r}' h + gh_\dot{r}' = \dot{r}, \]

(125)

\[ g g_\dot{r} + hh_\dot{r} = -r(r^2 + h^2), \quad g_\dot{r} + r^2 h h_\dot{r} = -\dot{r}, \quad g_\dot{r}' + r^2 h h_\dot{r}' = -r^2 \dot{r}' \]
which implies the ‘kinematical’ relations

\[ \vec{m}_u = \left( \frac{g_r}{h_r} \right) = \frac{1}{r^2 h^2 - g^2} \begin{pmatrix} \dot{r} & g r \cr -r' & -\dot{r} \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} =: K \vec{m} \]

(126) \[ \vec{m}_w = \left( \frac{g_r}{h_r} \right) = \frac{1}{r^2 h^2 - g^2} \begin{pmatrix} r^2 r' & r^2 \dot{r} \\ -r & -r^2 r' \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} =: N \vec{m} \]

\[ \vec{m}_r = \left( \frac{g_r}{h_r} \right) = \frac{r^2 (r^2 + h^2)}{r^2 h^2 - g^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} =: S \vec{m} \]

\[(r^2 h^2 - g^2) \cdot (123) \] then becomes

(127) \[ (g r' + h \dot{r})(\dddot{r} - r^2 r'') = -r(r^2 + h^2)(\dot{r} r' - r^2 \dot{g}). \]

While for a moment one may wonder about the signs in front of \( \dot{r} h \) and \( g r' \) (as it would be so easy to simply divide by their sum resp. difference, if the relative signs on the 2 sides were the same), one again has to argue ‘with hindsight’ in order to derive the desired conclusion

(128) \[ D[r] := \dddot{r} - r^2 r''' - r r'' + r h^2 = 0, \]

by writing (127) as

(129) \[ (g r' + h \dot{r})(D[r] + r r'' - r h^2) + r(r^2 + h^2)(\dot{r} h - r' \dot{g}) = 0 \]

and then showing that all terms not containing \( D \) identically cancel (as well as noting that, as assumed, \( g r' + h \dot{r} = \dot{z} r' - z' \dot{r} \neq 0 \)).

Analogously for

(130) \[ \vec{g}' \dot{h} = 0; \]

assuming

(131) \[ \vec{h}_w = \frac{\partial \vec{h}}{\partial r'} = -\frac{\partial \vec{g}}{\partial r} = -\vec{g}_u, \]

\[ \vec{g}_w = \frac{\partial \vec{g}}{\partial r'} = -r^2 \frac{\partial \vec{h}}{\partial r} = -r^2 \vec{h}_u, \]

which \( \vec{h} = f_\pm (= \dot{z}_\pm = v'_\pm (t, \varphi)) \) and \( \vec{g} = r^2 \dot{h} = -r^2 z'_\pm = -\dot{v}_\pm \) do, (130) becomes (cp. 123)

(132) \[ \vec{h}_r (\dddot{r} - r^2 r'') + \vec{h}_r \ddot{r} + \vec{g}_r r' = 0. \]

Now, however, assuming (cp. 124)

(133) \[ \vec{g}^2 + r^2 \vec{h}^2 = r^2 (1 - r^2) - r^2 r' = r^2 [1 - r^2 - r^2 r'^2] \]

\[ \vec{g} \dot{h} = r^2 (\dot{r} r') \]
one gets
\[
\begin{aligned}
\tilde{g}_r \dot{h} + \tilde{g} \ddot{h}_r &= 2 r \dot{r} \dot{r}', \\
\tilde{g}_r \dot{h} + \tilde{g} \ddot{h}_r &= r^2 \dot{r}', \\
\tilde{g}_r \dot{h} + \tilde{g} \ddot{h}_r &= r^2 \dot{r}'
\end{aligned}
\]
(134)
\[
\begin{aligned}
\tilde{g} \ddot{g}_r + r^2 \ddot{h} \ddot{h}_r &= r (1 - \dot{r}^2 - 2 r^2 \dot{r}'^2 - \tilde{h}^2), \\
\tilde{g} \ddot{g}_r + r^2 \ddot{h} \ddot{h}_r &= - \dot{r} r^2,
\end{aligned}
\]
which implies the (again, ‘kinematical’) relations
\[
\begin{aligned}
\left(\begin{array}{ccc}
\tilde{g}_r & \tilde{g}_r' & \tilde{g}_r \\
\dot{h}_r & h_r & - \dot{h}_r \\
h_r & \dot{h}_r & \ddot{h}_r
\end{array}\right) = \frac{1}{r^2 \ddot{h}^2 - \ddot{g}} \left(\begin{array}{ccc}
2 \tilde{r} \dot{r}' & - \tilde{g} & r^2 \dot{r}' \\
- \tilde{g} & - r^2 \dot{r}' & - r^2 \dot{r}' \\
r^2 \dot{r}' & - r^2 \dot{r}' & r (1 - \dot{r}^2 - 2 r^2 \dot{r}'^2 - \tilde{h}^2)
\end{array}\right)
\end{aligned}
\]
i.e.
\[
\begin{aligned}
\tilde{n}_u &= \left(\begin{array}{c}
\tilde{g}_r \\
\ddot{h}_r \\
h_r
\end{array}\right) = \frac{1}{r^2 \ddot{h}^2 - \ddot{g}} \left(\begin{array}{ccc}
2 \tilde{r} \dot{r}' & - \tilde{g} & r^2 \dot{r}' \\
- \tilde{g} & - r^2 \dot{r}' & - r^2 \dot{r}' \\
r^2 \dot{r}' & - r^2 \dot{r}' & r (1 - \dot{r}^2 - 2 r^2 \dot{r}'^2 - \tilde{h}^2)
\end{array}\right) \left(\begin{array}{c}
\tilde{g} \\
\dot{h} \\
\ddot{h}
\end{array}\right) =: \tilde{K} \tilde{n} \\
\tilde{n}_w &= \left(\begin{array}{c}
\tilde{g}_r' \\
\ddot{h}_r' \\
h_r'
\end{array}\right) = \frac{1}{r^2 \ddot{h}^2 - \ddot{g}} \left(\begin{array}{ccc}
r^4 \dot{r}' & - \tilde{g} & r^4 \dot{r}' \\
- \tilde{g} & - r^4 \dot{r}' & - r^4 \dot{r}' \\
r^4 \dot{r}' & - r^4 \dot{r}' & r (1 - \dot{r}^2 - 2 r^2 \dot{r}'^2 - \tilde{h}^2)
\end{array}\right) \left(\begin{array}{c}
\tilde{g} \\
\dot{h} \\
\ddot{h}
\end{array}\right) =: \tilde{N} \tilde{n} \\
\tilde{n}_r &= \left(\begin{array}{c}
\tilde{g}_r \\
\dot{h}_r \\
- \dot{h}_r
\end{array}\right) = \frac{1}{r^2 \ddot{h}^2 - \ddot{g}} \left(\begin{array}{ccc}
r (1 - \dot{r}^2 - 2 r^2 \dot{r}'^2 - \tilde{h}^2) & - 2 r \ddot{r} & r (1 - \dot{r}^2 - 2 r^2 \dot{r}'^2 - \tilde{h}^2)
\end{array}\right) \left(\begin{array}{c}
\tilde{g} \\
\dot{h} \\
\ddot{h}
\end{array}\right) =: \tilde{S} \tilde{n}
\end{aligned}
\]
so that, in analogy with (127),
\[
\begin{aligned}
\cdot
\cdot
\cdot
\end{aligned}
\]
(137)
\[
\begin{aligned}
r^2 (\dot{r}' \tilde{g} + \ddot{h}) (\dddot{r} - r^2 \dddot{r}'') &= \dddot{r} \left[ - 2 \ddot{r} \dddot{r}' + (1 - \dot{r}^2 - 2 r^2 \dot{r}'^2 - \tilde{h}^2) \dddot{h} \right] \\
&+ \dddot{r}' \left[ - (1 - \dot{r}^2 - 2 r^2 \dot{r}'^2 - \tilde{h}^2) \tilde{g} + 2 r \dddot{r} \dddot{r}' \tilde{h} \right]
\end{aligned}
\]
is obtained; writing \(\dddot{r} - r^2 \dddot{r}'' = D[r] + r r^2 - \frac{1}{r^2} \ddot{g}^2\), \(r^2 (\dddot{r}' \tilde{g} + \dddot{h}) D[r] = 0\) is derived, as all terms not containing \(D\) again identically cancel (note that the condition \(r^2 \dddot{h}^2 - \ddot{g}^2 = - r^2 (r^2 \dddot{h}^2 - \tilde{g}^2) \neq 0\), needed to obtain (126) / (136) from (125) / (134) was implicitly assumed from the start, as \(\dddot{z}^2 = r^2 z^2\) implies that one of \(\sqrt{\dddot{z}^2} := \sqrt{1 - (\dddot{r} \pm \dddot{r}' r)^2}\) vanishes, making derivatives of \(f\), resp. \(gh \tilde{h}\) diverge there; apart from those critical points, however, (121) and (130) present 2, genuinely different, ways to obtain the (same) equation of motion, \(D[r] = 0\).

In forthcoming work the progress made in understanding the dynamics of axially symmetric membranes will be applied to general minimal hypersurfaces, Lorentzian and Euclidean.
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Email address: jens.r.hoppe@gmail.com

Braunschweig University, Germany