Introducing a hybrid radiative transfer method for smoothed particle hydrodynamics

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ABSTRACT
A new means of incorporating radiative transfer into smoothed particle hydrodynamics (SPH) is introduced, which builds on the success of two previous methods – the polytropic cooling approximation as devised by Stamatellos et al. and flux-limited diffusion. This hybrid method preserves the strengths of its individual components, while removing the need for atmosphere matching or other boundary conditions to marry optically thick and optically thin regions. The code uses a non-trivial equation of state to calculate temperatures and opacities of SPH particles, which captures the effects of H2 dissociation, Hβ ionization, He0 and He+ ionization, ice evaporation, dust sublimation, molecular absorption, bound-free and free–free transitions and electron scattering. The method is tested in several scenarios, including (i) the evolution of a 0.07 M⊙ protoplanetary disc surrounding a 0.5 M⊙ star; (ii) the collapse of a 1 M⊙ protostellar cloud and (iii) the thermal relaxation of temperature fluctuations in a static homogeneous sphere.

Key words: accretion, accretion discs – hydrodynamics – radiative transfer – methods: numerical – stars: formation.

1 INTRODUCTION
Smoothed particle hydrodynamics (SPH) (Gingold & Monaghan 1977; Lucy 1977; Monaghan 1992) is a Lagrangian method which represents a fluid by a distribution of particles. Each particle is assigned a mass, position, internal energy and velocity: state variables such as density and pressure can then be calculated by interpolation – see reviews by Monaghan (1992, 2005). The effects of gravitation are also included as standard in most SPH codes. Recently, many authors have constructed versions of SPH with effects of radiative transfer (Whitehouse & Bate 2004; Stamatellos & Whitworth 2005; Stamatellos et al. 2005; Mayer et al. 2007). A full description of polychromatic three-dimensional radiative transfer is currently not possible within SPH (at least while current computational limitations prevent it). Even describing a snapshot from a simulation using full polychromatic radiative transfer is quite expensive (Stamatellos & Whitworth 2005; Stamatellos et al. 2005). In the past, approximations to individual features of radiative transfer were used – for example, the cooling time formalism: U = c_s t_{cool} (Rice et al. 2003), which only describes energy loss from the system, and does not model the transport of energy between neighbouring particles.

An example of where radiative transfer plays a fundamental role is in the physics of gravitational instabilities (GIs) in protoplanetary discs. The simple parametrization of the cooling time method allowed these GIs to be probed and characterized effectively. The gravitational fragmentation of protoplanetary discs is key to the disc instability theory of giant planet formation (Boss 1997). However, it is disputed whether fragmentation can indeed occur, with a strong debate between different groups using different methods of simulation and different formalisms for radiative transfer. At the time of writing, there is no strong consensus as to whether GI and fragmentation in protoplanetary discs can be a realistic mechanism for giant planet formation.

For fragmentation to occur, the disc must become gravitationally unstable, so that gravity can overcome the pressure and rotational support. The disc can become gravitationally unstable to axisymmetric instabilities if (Toomre 1964)

\[ Q = \frac{c_s \kappa}{\pi G \Sigma} < 1, \]

(1)

where c_s is the local sound speed, κ is the epicyclic frequency and Σ is the surface density of the disc. In a Keplerian disc, the epicyclic frequency is replaced by the angular frequency, Ω. If the perturbation is non-axisymmetric, the condition becomes Q < 1.5 – 1.7 (Durisen et al. 2007). Further to this condition, the cooling of the gas must be efficient enough to radiate away energy gained by the compression during the contraction. Use of the cooling time formalism allowed a quantitative statement of this second condition: t_{cool} ≤ 3Ω^{-1} (Gammie 2001; Rice et al. 2003; Mejía, Durisen & Pickett 2005).
More recent efforts have used more sophisticated approximations to capture the equation of state (EoS) of the gas, and model realistic radiative cooling and radiation transport, using both SPH (Whitehouse & Bate 2006; Mayer et al. 2007) and grid codes (Cai et al. 2008; Boley et al. 2007a), but these are becoming increasingly complex, with some methods requiring mapping of the photosphere (which is often of non-trivial geometric shape), and extra conditions to be applied there [matching atmospheres as in Cai et al. (2008), or specifying cooling at the photosphere as in Mayer et al. (2007)]. Also, identifying the photosphere often requires extra free parameters, the changing of which will affect the final results. The latest radiative transfer approximations [such as Boley et al. (2007a), which solves the full radiative transfer equation explicitly in the vertical direction] are now attempting to remove this parametrization.

This paper presents a new radiative transfer approximation, which relies on two separate methods working in tandem. The first method is the polytropic approximation devised by Stamatellos et al. (2007), which models the cooling of particles over a range of optical depths \((0 < \tau \lesssim 10^3)\). Its formulation ensures that cooling will be at its most efficient where the optical depth is around unity, in accordance with the definition of the photosphere. This avoids the necessity of explicitly computing the location of the photosphere, or imposing boundary conditions upon it. The second method is the flux-limited diffusion approximation, used by many authors (Bodenheimer et al. 1990; Cleary & Monaghan 1999; Whitehouse & Bate 2004, 2006; Boley et al. 2007a; Mayer et al. 2007; Boss 2008; Cai et al. 2008) to simulate radiation transport in optically thick regimes. Although these methods are combined to create the new algorithm. In Section 2, the constituents of the new hybrid algorithm are presented; it is then shown how these methods are combined to create the new algorithm. In Section 3, the results from the following test scenarios are given: the evolution of a 0.07 M⊙ protoplanetary disc used as an example by Pickett et al. (2003), Mejia et al. (2005), Boley et al. (2006) and Cai et al. (2008); the collapse of a 1 M⊙ molecular cloud (Masunaga & Inutsuka 2000) and the thermal relaxation of a static sphere with seeded temperature fluctuations (Spiegel 1957; Masunaga, Miyama & Inutsuka 1998). Finally, in Section 4 the method is summarized, and some indications of future work are given.

## 2 Method

### 2.1 Polytropic cooling

The polytropic approximation uses an SPH particle’s density \(\rho_i\), temperature \(T_i\), and gravitational potential \(\psi_i\) to estimate a mean optical depth for the particle (Stamatellos et al. 2007). The approximation is achieved as follows. Assume that the particle is embedded in a spherically symmetric polytropic ‘pseudo-cloud’ (which need not be in hydrostatic equilibrium). The properties of the cloud are calculable analytically (using the Lane–Emden equation), given the particle’s (dimensionless) radius \(\xi\) from the centre: \(R = \xi R_0\). Therefore, by appropriate selection of the central values of density and temperature \(\rho_c\), \(T_c\), and the scalelength \(R_0\), the particle’s own values can be recovered:

\[
\rho_i = \rho_c \theta^n(\xi).
\]

\[
T_i = T_c \theta(\xi),
\]

\[
\psi_i = -4\pi G \rho_i R_0^2 \phi(\xi).
\]

Here, \(\theta\) is the solution to the Lane–Emden equation for a polytrope of index \(n\), and

\[
\phi(\xi) = -\frac{d\psi}{d\xi}(\xi) + \psi(\xi)
\]

(5)

(where \(\psi(\xi)\) is the boundary of the polytrope and \(R_0\) satisfies

\[
R_0 = \left[ \frac{-\psi(\theta(\xi))}{4\pi G \rho \phi(\xi)} \right]^{1/2}.
\]

This provides tools to calculate a column density from any given (dimensionless) radius to the boundary of the cloud:

\[
\Sigma_i(\xi) = \int_{\xi}^{\xi=\xi'_B} \rho_i \theta^n(\xi') R_0 \, d\xi'.
\]

(7)

However, it is assumed that the value of \(\xi\) for the particle is unknown. Instead, a value for the column density is arrived at by performing a mass weighted average over all possible values of \(\xi\) out to the polytrope’s boundary:

\[
\Sigma_i = \int_{\xi=\xi_B}^{\xi=\xi_B} \rho_i \theta^n(\xi') R_0 \, d\xi'.
\]

(9)

The total (dimensionless) mass of the polytrope is \([-\xi_B^2 \frac{d\phi}{d\xi}(\xi_B)]\) and \(\theta^n(\xi) \xi^2 d\xi\) is the dimensionless mass element between \([\xi, \xi + d\xi]\). In real terms, \(\Sigma\) becomes a simple algebraic quantity:

\[
\Sigma_i = \xi_0 \left[ \frac{-\psi(\xi)}{4\pi G} \right]^{1/2}
\]

(10)

with the integral folded into the constant \(\xi_0\), which is dependent only on the polytropic index \(n\):

\[
\xi_0 = \left[ \frac{-\xi_B^2 \frac{d\phi}{d\xi}(\xi_B)}{\psi(\xi)} \right]^{-1}
\]

\[
\int_{\xi=\xi_B}^{\xi=\xi_B} \int_{\xi=B}^{\xi=\xi_B} \theta^n(\xi') \, d\xi' \sin^2 \frac{\theta^n(\xi)}{\phi(\xi)} \, d\xi'.
\]

(11)

Stamatellos et al. (2007) show that this constant is insensitive to the value of \(n\). They select \(n = 2\) for their work, as this would give a polytropic exponent of \(3/2\), in keeping with polytropic exponents of protostars in quasi-static equilibrium. When using the polytropic formalism alone, the results of this paper will assume \(n = 2\) (and hence \(\xi_0 = 0.368\)) except where otherwise stated. The simple expression for the column density illustrates its ability to capture the effects of the local environment (through the presence of \(\rho\) and the effects of the system’s global geometry (through the gravitational potential \(\psi\)).

In the same vein, a mass weighted optical depth can be calculated. The optical depth from any radius to the edge of the pseudocloud is

\[
\tau_i(\xi) = \int_{\xi=\xi}^{\xi=\xi_B} \kappa_i(\rho_i \theta^n(\xi'), T_i(\theta(\xi'))) \rho_i \theta^n(\xi') R_0 \, d\xi'.
\]

(12)

Substituting for \(\rho_i\), \(T_c\) and \(R_0\) gives

\[
\tau_i(\xi) = \left[ \frac{-\psi(\xi)}{4\pi G \rho \theta^2(\xi)} \right]^{1/2}
\]

\[
\int_{\xi=\xi_B}^{\xi=\xi_B} \kappa_i \left( \rho_i \left( \frac{\theta^n(\xi)}{\theta(\xi)} \right)^n, T_i \left( \frac{\theta(\xi)}{\theta(\xi)} \right) \right) \left( \frac{\theta(\xi)}{\theta^n(\xi)} \right)^n \, d\xi'.
\]

(13)
Taking a mass weighted average then gives a rather messy:

$$\tau_i = \left[ -\xi_i^2 \frac{d\theta}{d\xi}(\xi) \right]^{-1} \left[ \frac{-\psi_i \rho_i}{4\pi G} \right]^{1/2} \int_{\xi = \xi_i}^{\xi = \xi_i + \lambda_i} \int_{\xi = \xi_i}^{\xi = \xi_i + \lambda_i} \kappa \left( \frac{\frac{\partial(\xi)}{\partial(\xi)}}{\phi(\xi)} \right) T_i \left[ \frac{\theta(\xi)}{\theta(\xi)} \right] \theta(\xi) d\xi' \left[ \frac{\theta(\xi)}{\phi(\xi)} \right]^{1/2} \xi^2 d\xi. \quad (14)$$

This is a complicated function to calculate during a simulation. However, using the previous result for $\Sigma$, a mass weighted opacity can be defined,

$$\kappa = \frac{\tau}{\Sigma}, \quad (15)$$

and this can be evaluated in advance, and stored for later interpolation. Hence, for a given $(\rho, T)$

$$\kappa(\rho, T) = \left[ -\xi_i^2 \frac{d\theta}{d\xi}(\xi) \right]^{-1} \left[ \frac{-\psi_i \rho_i}{4\pi G} \right]^{1/2} \int_{\xi = \xi_i}^{\xi = \xi_i + \lambda_i} \int_{\xi = \xi_i}^{\xi = \xi_i + \lambda_i} \kappa \left( \frac{\frac{\partial(\xi)}{\partial(\xi)}}{\phi(\xi)} \right) T_i \left[ \frac{\theta(\xi)}{\theta(\xi)} \right] \theta(\xi) d\xi' \left[ \frac{\theta(\xi)}{\phi(\xi)} \right]^{1/2} \xi^2 d\xi. \quad (16)$$

The interpretation of this result is important: embedding the particle at some position in the polytrope ensures that the environment immediately surrounding the particle has a strong effect on its optical depth, and hence its emissivity. This allows, for example, insolation of hot particles by cooler surroundings. It is vital at this juncture to appreciate the meaning of this: the formalism is attempting to compensate for absorption of escaping radiation by modifying the net radiative losses of the particles using the polytrope approximation.

If the net cooling term for SPH particle $i$ is $\dot{u}_{i,\text{cool}}$, then this then becomes

$$\dot{u}_{i,\text{cool}} = \frac{4\sigma}{\Sigma_k(\rho_i, T_i) + \kappa_i^{-1}(\rho_i, T_i)} \sum_{b} \frac{4m_b}{\rho_b \kappa_b} k_i k_b (T_i - T_b) \frac{r_{ib} \cdot \nabla W}{|r_{ib}|^2}. \quad (18)$$

The summation index $b$ describes the nearest neighbours of the particle (which is tracked by SPH to evaluate density fields and other requisite variables), $W$ is the smoothing kernel, $r_{ib}$ is the separation vector between particles $i$ and $b$ and $k_i$ describes the thermal conductivity of the particle. The gradient of the kernel is everywhere negative, so if $T_i > T_b$, the summation will be negative (i.e. energy will flow from particle $i$ to $b$, in accordance with the laws of thermodynamics). If the system’s energy budget is defined entirely by diffusion, the particles will exchange energy amongst themselves in order to reduce temperature gradients; the long-term evolution of the system will be towards a single equilibrium temperature. This ‘washing out’ of temperature gradients is of critical importance: when simulating protoplanetary discs, the temperature profile (both radially and vertically) can define the regions of the disc where possible fragmentation can occur, and hence the regions where giant planets may form (Boss 1997). Any process which affects these profiles will influence where these regions are located.

It should be noted at this point that all energy changes due to these diffusion terms are pairwise, i.e. any energy loss by one particle will be matched by the gain in its counterpart. This means that the total energy change over the entire system due to diffusion must be zero:

$$\sum_i \dot{u}_{i,\text{diff}} = 0. \quad (19)$$

This is an important feature, which allows it to be used in the hybrid method, as will be shown later. The thermal conductivity is

$$k_i = \frac{16\sigma}{\rho_b \kappa_i} \lambda_i T_i^4. \quad (20)$$

where $\kappa_i$ is the opacity, $\sigma$ is the Stefan–Boltzmann constant and $\lambda_i$ is the flux limiter. Bodenheimer et al. (1990) describe an expression for $\lambda_i$ which is calculated from the local radiation field:

$$\lambda_i(R_i) = \frac{2 + R_i}{6 + 3R_i + R_i^2}. \quad (21)$$

Here, $R_i$ is a function of the radiation energy density at the particle’s position, $u_r(r_i)$:

$$R_i = \frac{\left| \sum u_r(r_i) \right|}{u_r(r_i) \rho_b \kappa_i}. \quad (22)$$

Studying the expression for $R_i$, there are two limiting cases.

(i) When the region is very optically thick, $\rho$ and $\kappa$ become large (and the radiation field becomes uniform), and hence $R_i \to 0$. In this limit, the flux limiter $\lambda_i \to 1/3$, in accordance with the diffusion approximation.
(ii) In the very optically thin limit, $R$, becomes very large, and $\lambda_i \to 0$, ending energy transport by diffusion.

This approximation is valid in the optically thick regime (and to lower optical depths with the use of the flux limiter, which prevents energy exchange as the mean free path of the radiation becomes prohibitively large). Limitations of this method are

(i) It does not model radiation well at very low optical depths (where energy exchange disappears)

(ii) It does not allow the system to lose energy (i.e. it does not model radiative cooling). Instead, this cooling must be added using a prescription which assumes prior knowledge of the geometry of the system being studied, and invokes resolution-dependent free parameters (e.g. Mayer et al. 2007).

### 2.3 The hybrid method

Comparing the limitations of the above two methods, it is clear that a union of these two procedures should be complementary: polytropic cooling handles the important energy loss from the system (which flux-limited diffusion cannot) and flux-limited diffusion handles the detailed exchange of heat between neighbouring fluid elements (which polytropic cooling cannot). Indeed, polytropic cooling’s inability to model the detailed exchange of heat between neighbouring fluid elements – and flux-limited diffusion’s inability to model energy loss – allows the two methods to work together correctly, modelling all aspects of the system’s energy budget without encroaching on each other. The energy equation simply becomes

$$\dot{u}_{i,\text{total}} = \dot{u}_{i,\text{hydro}} + \dot{u}_{i,\text{cool}} + \dot{u}_{i,\text{diff}},$$

where $\dot{u}_{i,\text{hydro}}$ describes the energy change due to the hydrodynamics of the system, e.g. compressive $PdV$ heating. The true advantage in using this hybrid method is in its simplicity.

(i) By construction, the hybrid method is fully three-dimensional, and capable of handling arbitrary particle geometries.

(ii) There is no requirement to grid the system.

(iii) The algorithm is continuous over a wide range of optical depths, so there are no requirements to match separate atmospheres at some boundary.

(iv) As no extra boundary conditions are required, there are no extra parameters to be specified, so the simulation’s results are only dependent on the traditional SPH parameters (particle number, smoothing length, etc.).

However, the hybrid method still suffers from some disadvantages.

(i) This method is still unable to model frequency-dependent radiative transfer.

(ii) As with polytropic cooling, the hybrid method is better suited to modelling the cooling of spherical geometries.

### 2.4 Updating energy: a semi-implicit scheme

The use of an explicit scheme to update energy can result in very short time-steps. To avoid this, a modified version of the implicit scheme adopted by Stamatellos et al. (2007) is used. This models each particle’s approach to its equilibrium temperature $T_{eq,i}$, which satisfies

$$\dot{u}_{i,\text{hydro}} + \frac{4\sigma}{\sum_i \kappa_i (\rho_i, T_{eq,i}) + \kappa_i^{-1} (\rho_i, T_{eq,i})} \left[ T_{eq,i}^4 - T_i^4 \right] + \dot{u}_{i,\text{diff}} = 0.$$  

From this, the equilibrium internal energy $u_{eq,i} = u(\rho_i, T_{eq,i})$ can be calculated, and hence the thermalization time-scale

$$t_{\text{therm}} = \frac{u_{eq,i} - u_i}{\dot{u}_{i,\text{total}}}.$$  

With the knowledge of how quickly each particle can be thermalized, the particle’s energy can be updated thus

$$u_i(t + \Delta t) = u_i(t) \exp \left( \frac{-\Delta t}{t_{\text{therm}}} \right) + u_{eq,i} \left[ 1 - \exp \left( -\frac{-\Delta t}{t_{\text{therm}}} \right) \right].$$

For particles that will thermalize very quickly ($t_{\text{therm}} \ll \Delta t$), which would result in very short time-steps, this equation reduces to

$$u_i(t + \Delta t) \approx u_{eq,i},$$

i.e. the particle rapidly reaches equilibrium. If thermalization happens on a long time-scale ($t_{\text{therm}} \gg \Delta t$), then the equation becomes

$$u_i(t + \Delta t) \approx u_i(t) + \left[ u_{eq,i} - u_i(t) \right] \frac{\Delta t}{t_{\text{therm}}}.$$  

### 2.5 Properties of the dust and gas

Vital to any radiative transfer method is how the variables it uses (temperature, opacity, etc.) are evaluated. SPH evolves only the density and internal energy of the particles; hence, some kind of prescription is required in order to obtain the requisite data. Essentially, what is required is $T(\rho, \kappa, \kappa)$, (the opacity law) and tabulate these values, which can then be interpolated to achieve the correct results. The EoS and the opacity law used in this work are similar to that of Stamatellos et al. (2007), where a full description is available (see also Black & Bodenheimer 1975; Boley et al. 2007b). This work assumes hydrogen and helium mass fractions of $X = 0.7$, $Y = 0.3$ and a fixed ortho- to para-hydrogen ratio of 3:1. The dependence of the various variables on temperature can be seen in Figs 1–4. Figs 1 and 2 show the activation of the various energy states of hydrogen and helium gas as temperature increases: Fig. 3 shows the opacity law as calculated according to the prescription of Bell & Lin (1994). Fig. 4 shows the mass weighted opacity as discussed in Stamatellos et al. (2007). The opacity law captures many different opacity regimes of the gas.
including ice and dust opacities, as well as molecular absorption, bound-free and free–free interactions and electron scattering.

3 TESTS

The code used to perform these tests is based on the SPH code developed by Bate, Bonnell & Price (1995). It uses variable individual smoothing lengths \( h_i \) in order that the number of nearest neighbours for any particle is \( 50 \pm 20 \). It uses individual particle time-steps to allow dense regions to be simulated with greater time resolution while preventing oversimulation of less dense regions. A binary tree is employed to calculate neighbour lists and gravity forces. The standard artificial viscosity is also used. All simulations are sufficiently populated to satisfy the Jeans resolution condition of Bate & Burkert (1997) for Jeans masses of \( 30M_\odot \) or less. These conditions are sufficient for the cloud simulations performed.

For the disc simulations performed, the Toomre length becomes important in the regions that are unstable, and places stricter resolution conditions. As the disc simulation is Keplerian, the following relation can be used between the Jeans length and the Toomre length (Nelson 2006)

\[
\lambda_T = \sqrt{2Qf\lambda_J},
\]

where \( f \sim 1 \) represents the conversion factor between surface and volume densities. As the disc is marginally unstable \( (Q \sim 1) \), the Toomre length can be simply calculated. Converting this (assuming a homogeneous sphere) into a Toomre Mass, it is calculated that the disc simulation can resolve Toomre masses of \( \sim 85M_\odot \) or more.

3.1 The evolution of a protoplanetary disc

As a means of comparison with previous results, the conditions used for this test are those proposed by Pickett et al. (2003), and used in a series of papers describing radiative transfer in protoplanetary discs (Pickett et al. 2003; Mejía et al. 2005; Boley et al. 2006; Cai et al. 2008). The model is a 0.07 M_\odot Keplerian disc which extends to 40 au, orbiting a star of 0.5 M_\odot. Initially, the surface density profile is \( \Sigma \sim r^{-1/2} \), with a temperature profile of \( T \sim r^{-1} \). The disc is modelled using \( 2.5 \times 10^5 \) SPH particles, with one sink particle representing the star. The disc is immersed in a radiation field of \( T_0(r) = 3K \); the effects of disc irradiation by the central star are not included.

The properties of the evolved disc using both the hybrid method and the polytropic cooling approximation alone are shown in Figs 5 and 6. For both methods, several key phases are identified: the initial settling phase, during which the disc adjusts its outer radius by axisymmetric evolution and ring formation and contraction occur; the ‘burst’ phase where non-axisymmetric instabilities, in the form of spiral waves, begin to grow and the later asymptotic phase, where the disc’s radial extent is more firmly established and the GI is regulated (cf. Boley et al. 2006). As the evolution of this asymptotic phase continues, the low-\( m \) modes begin to dominate. In terms of time-scale, the settling phase lasts until \( t \sim 500 \) yr, the burst phase until \( t \sim 1200 \) yr, where the asymptotic phase then begins, resulting in a quasi-equilibrium state.

Comparing the hybrid method against the results of using the polytropic cooling approximation alone, there are significant differences: the hybrid method transports more mass radially outward, which can be seen in the surface density of the disc (Fig. 6, top left panel). This has several important consequences: it allows the
optical depth to be reduced in the region $r \sim 20-40$ au, which allows an increase in radiative cooling; this in turn allows the outer disc to be cooler, and for the outer regions of the disc ($r > 20$ au) to become less stable (as can be seen in the other panels of Fig. 6). Snapshots of the disc under both methods can be seen in Fig. 7. Note the stronger spiral structure in the disc under the hybrid method, with instabilities extending to larger radii. All these differences are critical if the formation of giant planets by GI is to be effectively tested by simulation.

Comparing the hybrid method to the results of Boley et al. (2006, 2007a), the two are qualitatively consistent. Each has a burst phase and an asymptotic phase; each has a two-component surface density profile (approximately flat at lower radii, with a cut off at larger radii). There are also some quantitative consistencies: the optical depth from the mid-plane to the surface in the hybrid method reaches unity at $R \sim 27$ au, which is coincident with the region of the disc that is most unstable (i.e. the Toomre Q parameter is at a global minimum) – which is in keeping with the work of Boley et al. It can also be seen (by comparing the surface density profiles of the hybrid method and polytropic cooling) that there appears to be a surplus of matter within $R \sim 20-27$ au, and a slight deficit at $R \sim 27-40$ au, indicating that $R \sim 27$ au may be the location where mass transport switches from inward to outward, again consistent with the results of Boley et al. However, there are some important differences to be considered. The burst phase of the hybrid method is notably weaker, and the disc undergoes less radial spreading. This also means that in the asymptotic phase, the unstable region is much narrower in radius. The first component of the surface density profile also appears to be flatter at lower radii for the hybrid method.

It should be noted at this point that there are mitigating factors at work: the EoS and opacity law used in this work are different from that of Boley et al; also, they fix the star at the centre of their grid: the star used in these results is allowed to move. The differences in the EoS and the opacity law will have a stronger effect in the hotter inner regions of the disc, perhaps explaining the differences in surface density profile, and the lack of radial spreading. It should also be noted that the inner disc stays somewhat hotter than expected (for both polytropic cooling alone and for the hybrid method). This may be due to SPH viscosity: as the distance to the centre decreases, the magnitude of the SPH viscosity increases, and may become significant (relative to the effective gravitational viscosity). This possibility will be investigated in more detail at a later date.

Although exciting spiral waves in all three cases (polytropic cooling alone, the hybrid method and the work of Boley et al.), the instability in the disc does not lead to fragmentation. Also, the disc is only Toomre unstable at larger radii, which does not bode well for in situ formation of Jovian objects at $R \leq 20$ au (at least in these conditions).

### 3.2 The collapse of a 1M⊙ cloud

The collapse of a non-rotating molecular cloud was then simulated. The spherical, uniform-density cloud contains 1 M⊙ of material (populated by $5 \times 10^5$ SPH particles), and has a radius of 10^4 au (which gives a density of $\rho_0 = 1.41 \times 10^{-19}$ g cm$^{-3}$), and is immersed in a background radiation field of $T_0(r) = 5$ K. These conditions were initially investigated by Masunaga & Inutsuka (2000) by solving the full radiative transfer in three-dimensions (with
the hydrodynamics solved in one-dimension), and were revisited by Stamatellos et al. (2007). These conditions therefore represent not only a solid test of the code’s ability to match Masunaga & Inutsuka’s data, but also allow us to compare with the results of Stamatellos et al. to identify the effects of adding flux-limited diffusion.

In the initial phase, the collapse is isothermal: the temperature remains at approximately 5 K through 7 orders of magnitude in density (see Fig. 8, left-hand panel), until the central density reaches $\rho \sim 10^{-12}$ g cm$^{-3}$. The cloud then becomes optically thick, and the temperature starts to rise. As the temperature reaches $T \sim 100$ K, the rotational degrees of freedom of molecular hydrogen are activated, slowing the temperature increase slightly (this can be seen in the small bump in the left-hand panel of Fig. 8). The increased heating in the centre eventually decelerates the contraction at around $\rho = 10^{-9}$ g cm$^{-3}$, and the first core is formed. The contraction and heating of this core proceed until the central temperature is around $T \sim 2000$ K. The H$_2$ present begins to dissociate, using some of the available compressive energy due to the contraction. This allows a second collapse, which can continue until most of the H$_2$ is dissociated. After this, the contraction decelerates again at around $\rho = 10^{-3}$ g cm$^{-3}$, and the second core forms.

The dotted line in the left-hand panel of Fig. 8 shows the evolution of the Masunaga cloud using polytropic cooling alone. Both methods approximate the data of Masunaga & Inutsuka (2000) well (diamonds in Fig. 8). However, there are two key differences: the
The key analytical result is the dispersion relation \( \omega(k) \), which is shown as the solid lines in Fig. 9. The points in each panel are obtained by calculating \( \omega/\gamma \) individually for all \( 2 \times 10^5 \) SPH particles using equations (31) and (33), and by calculating the mean. This is done for five separate instants in the simulation, corresponding to maximum temperatures of 10.14, 10.13, 10.1, 10.05 and 10.02 K, respectively, and is plotted for several runs with different cloud opacities (i.e. different values of \( \kappa_0/k \)). Error bars for these points indicate the sample standard deviation. The left-hand panel shows the results using polytropic cooling only; the right-hand panel shows the results using the hybrid method. In the optically thin regime (low \( \kappa_0/k \)), both methods deliver the same results. As the optical depth increases, the hybrid method approximates the curve better, as it can model the local radiation transport that occurs in the optically thick limit. However, both methods underestimate the analytical value of \( \omega \), reflecting their approximate nature.

For extra comparison, the temperature profiles of the cloud for polytropic cooling and the hybrid method are shown in Figs 10 and 11. In the optically thin case (Fig. 10), the two panels are basically identical, since flux-limited diffusion is not active in this limit; both illustrate the decaying sinusoidal function described in equation (31). In the optically thick case (Fig. 11), the curve for polytropic cooling begins to spread, filling the regions between the troughs/peaks and 10 K. The same panel for the hybrid method shows less spreading, retaining a more robust sinusoidal pattern.

### 3.3 The spiegel test

As a final test, the thermal relaxation of a static, spherical cloud with a well-defined temperature perturbation allows comparison of the hybrid method with analytic results. The cloud is uniform in density, with \( \rho = 10^{-19} \) g cm\(^{-3} \), and a radius of \( R = 10^4 \) au. The equilibrium temperature is taken to be \( T_0 = 10 \) K, and an initial temperature perturbation which satisfies

\[
T(r) = T_0 + \Delta T_0 \frac{\sin kr}{kr},
\]

where \( \Delta T_0 = 0.15 \) K is the amplitude and \( k = \frac{\pi}{2500 \text{au}} \) is the characteristic wavenumber (Spiegel 1957; Masunaga et al. 1998). At a later time \( t \), this perturbation evolves according to Masunaga et al. (1998):

\[
T(r, t) = T_0 + \Delta T_0 \frac{\sin kr}{kr} e^{-\omega(k)v}.
\] (31)

In equation (31),

\[
\omega(k) = \gamma \left[ 1 - \frac{\kappa_0}{k} \cot^{-1} \left( \frac{\kappa_0}{k} \right) \right]
\]

and

\[
\gamma = \frac{16\sigma T_0^4}{\rho c_v}.
\] (33)

Here, \( \kappa_0 \) is the opacity at equilibrium and \( c_v \) is the heat capacity of the material. This test was also performed by Stamatellos et al. (2007), and hence provides an extra means of comparing polytropic cooling and the hybrid method.

### 4 CONCLUSIONS

This paper has presented a new means of modelling radiative transfer in SPH by fusing two well-tested methods, polytropic cooling and flux-limited diffusion, in order that they may complement each other, and perform the functions that the other cannot. By this fusion, the physics of three-dimensional frequency-averaged radiative transfer is captured without the need for complex boundary conditions, photosphere mapping or extra parameters. Temperatures and opacities are obtained using a non-trivial EoS which captures the effects of H\(_2\) dissociation, H\(^+\) ionization, He\(^0\) and He\(^+\) ionization, ice evaporation, dust sublimation, molecular absorption, bound-free

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**Figure 8.** Evolution of the central density of the Masunaga cloud – the left-hand panel shows the evolution of central temperature with increasing central density, the right-hand panel shows the time evolution of the central density. The solid lines represent the hybrid method, the dotted lines represent polytropic cooling only and the diamonds represent the data of Masunaga & Inutsuka (2000).
The algorithm is fast: only a 6 per cent increase in CPU time is incurred in comparison to standard SPH simulations performed with a barotropic EoS. It has shown itself to be accurate in the tests outlined in the previous section: the evolution of a protoplanetary disc (with parameters proposed by Pickett et al. 2003; Mejía et al. 2005) from a uniform state through ring formation and contraction to instability; the complex thermal history of a collapsing molecular cloud (as studied by Masunaga & Inutsuka 2000) and the smoothing of temperature fluctuations in a homogeneous, static sphere (Spiegel 1957; Masunaga et al. 1998). However, the scheme is still approximate, and can only partially describe radiative effects that occur over mid-range distances (unlike the scheme proposed by Boley et al. 2007a, albeit in the vertical direction only).
Comparisons with simulations using polytropic cooling alone have shown that the hybrid method is, in effect, only a small correction to the polytropic cooling method, which however can become important in some problems where temperature gradients and system geometries become complex (e.g. the protoplanetary disc simulations described in this paper).

Future work will see this algorithm applied to a variety of protostellar and protoplanetary environments, primarily a study of initial conditions for disc formation and evolution, as well as the effects of interactions between discs and binary companions.

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