Geodesic rays of the $N$-body problem

J. M. Burgos† and E. Maderna*

†CINVESTAV–CONACYT, Ciudad de México, México.
*IMERL, Universidad de la República, Uruguay.

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Abstract

For the Newtonian $N$-body problem, we study the Jacobi-Maupertuis metric of the nonnegative energy levels. We show that the geodesic rays are expansive, that is to say, all the distances between the bodies must be divergent functions. More precisely, we prove that the evolution of such motions asymptotically decomposes into free particles and subsystems in completely parabolic expansion. The theorem applies in particular to the maximal characteristic curves of any given global viscosity solution of the stationary Hamilton-Jacobi equation $H(x, d_x u) = h$.

Keywords: $N$-body problem, geodesic ray, Jacobi-Maupertuis metric.

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1 Introduction

The classification of the final evolutions for solutions of the classical $N$-body problem not experiencing singularities in finite time (either collisions or pseudo-collisions) were developed during the 20th century, starting with the pioneering works of Chazy \[3, 4\]. The classification, in terms of the asymptotic behaviour of the mutual distances between the bodies, ends up being subordinated to a first description that takes into account the order of growth of these distances. More precisely, we are referring to the Marchal-Saari primary description for the possible final evolution of a Newtonian universe, which applies to all motions, except those that may eventually present an extremely complex behaviour.

Let us start by remembering this basic result. For this we need to fix some notation. Let $E$ be an Euclidean space, in which the $N$ bodies (punctual masses) evolve according to the Newton’s law of universal gravitation. At every moment during the motion, the different positions of the bodies form a configuration $x(t) = (r_1(t), \ldots, r_N(t)) \in \Omega$, where $\Omega \subset E^N$ denotes the open and dense set of configurations without collisions, that is, such that $i \neq j$ implies $r_i \neq r_j$. We will also denote by $R(t) = \max \{\|r_i(t) - r_j(t)\| \mid i < j\}$ the greatest distance between two bodies at time $t$, or equivalently, the diameter of the subset of the Euclidean space $E$ defined by the configuration $x(t)$. Then the Marchal-Saari theorem can be formulated as follows:
Theorem (Marchal-Saari [16]). Let \( x : [t_0, +\infty) \to \mathbb{E}^N \) be a solution of the \( N \)-body problem. Either
\[
\limsup R(t)/t = +\infty,
\]
or there is a configuration \( a \in \mathbb{E}^N \) such that
\[
x(t) = ta + O(t^{2/3}).
\]

Motions in the first case of this alternative are called \textit{superhyperbolic}. To date, the existence of superhyperbolic motions has not been proven, and it is known that their occurrence require the presence of at least four bodies. However, there is evidence that they could exist and probably the elucidation of this dilemma will require an effort of the same kind as that which recently concluded with the resolution of the Painlevé conjecture by Xue [20].

In this paper, we address the study of solutions to the \( N \)-body problem that, in addition to being defined for all future time \( t > t_0 \), also satisfy a strong global variational property, namely of being geodesic rays. As we will see, this variational property excludes the possibility of superhyperbolicity, and leads us naturally to the study of the \( O(t^{2/3}) \) term established by the Marchal-Saari description. To be more explicit, we will now say what this variational property consists of.

As it is well known, negative energy levels are not projected over the entire configuration space. Indeed, the Newtonian potential is the function
\[
U(x) = \sum_{i<j} m_i m_j r_{ij}^{-1}
\]
where \( m_i > 0 \) is the value of the mass located at position \( r_i \in \mathbb{E} \), \( r_{ij} = \| r_i - r_j \| \), and the kinetic energy of a vector \( v = (v_1, \ldots, v_N) \in \mathbb{E}^N \) is given by the quadratic form
\[
K(v) = \frac{1}{2} \sum_{i=1}^N m_i \| v_i \|^2.
\]
Thus, the energy constant of a given motion \( x(t) \) is \( h = K(\dot{x}(t)) - U(x(t)) \), which implies that the motion must be confined to the \textit{Hill’s region}
\[
\Omega_h = \{ x \in \Omega \mid U(x) \geq -h \}.
\]
Clearly \( \Omega_h = \Omega \) if and only if \( h \geq 0 \). We also recall that trajectories of a fixed energy level \( h \) are nothing other than the geodesics of the Jacobi-Maupertuis Riemannian metric within the corresponding Hill’s region. More specifically, this metric can be written as the conformal scaling
\[
j_h = 2(h + U) g_m
\]
of the flat metric \( g_m \) induced on \( \mathbb{E}^N \) by the mass scalar product. The reader will find a full development of this geometric point of view in the second author’s recent work with Venturelli [13]. In this article, the existence of geodesic rays is established for arbitrary initial positions of the bodies and for any choice of limit shape \( a \in \Omega \), thus giving rise to motions of the hyperbolic type. Let us recall the notion of geodesic ray. A \textit{geodesic ray} of a Riemannian manifold \((M, g)\) is an isometric embedding of the half-line \([0, +\infty)\) within \( M \), or in other words,
it is an arclength parameterized geodesic $\gamma : [0, +\infty) \to M$ such that all of its restrictions to compact subintervals are minimizing geodesics.

Geodesic rays are essential elements for the study of non-compact spaces, and it is through them that Gromov introduced the geometric notion of boundary at infinity. However, taking into account that our motivation is the study of dynamics, we will give this notion a broader meaning.

**Definition 1.** We will say that a curve $x : [0, +\infty) \to \Omega$ is a geodesic ray for the metric $j_h$ whenever its arclength parameterization is an isometric embedding of the half-line $[0, +\infty)$ into $(\Omega, j_h)$.

In fact, we will see that this does not change things much, since each geodesic ray has a unique arclength parameterization, as well as it has a unique parameterization with constant energy $h$, the latter being also defined over an unbounded time interval. Moreover, this constant energy parameterization is indeed a true motion of the $N$-body problem, because it is a minimizer of the Lagrangian action functional (see corollaries 3 and 4 below).

We are now in position to summarize the state of the art knowledge on geodesic rays of the classical $N$-body problem, as well as the contribution we will make in this paper. We start by discussing their existence. In what follows we assume, for all considered motions, that the center of mass is at rest.

(i) It is unknown whether there are geodesic rays of negative energy levels. It is not difficult to deduce that in every negative energy motion, defined over an unbounded time interval, at least one relative position $r_i(t) - r_j(t)$ between two bodies must have a limit vector for a sequence of times $t_n \to +\infty$. It seems reasonable to think that this form of recurrence could contradict a global minimization property as strong as that of being a geodesic ray. However, we will not address this issue in this article.

(ii) For $h = 0$, the existence of geodesic rays for arbitrary initial positions of the bodies was first established in [11], as calibrating curves of any global viscosity solution of the Hamilton-Jacobi equation. In [19], Percino and Sánchez Morgado proved the existence of geodesic rays with prescribed minimal central configuration as its limit shape. On the other hand, it was proved in [7] that the geodesic rays of the zero energy level are completely parabolic motions. This means that $\dot{x}(t) \to 0$ as $t \to +\infty$, or equivalently, that $r_{ij}(t) \approx t^{2/3}$ for all $i < j$. As Chazy pointed out in [3], the shape of the configuration of a completely parabolic motion (as well as of a total collision motion) must approximate the set of central configurations. In fact, he introduced the notion of central configuration for the general $N$-body problem as the set of these limit shapes. In [12], the second author observes that the set of central configurations being limit shape of minimizing parabolic motions also agree with the set of limit shapes of minimizing total collision motions, giving thus rise to the notion of minimizing central configuration. It is not difficult to see that any minimal configuration is minimizing. However, it is unknown whether there are minimizing configurations that are not minimal. Barutello and Secchi [1] have established a sufficient condition to ensure that certain configurations are not minimizing. This result allows to prove that for the three body problem with equal masses, only the equilateral Lagrange
configuration is minimizing, and this holds for homogeneous potentials in a certain range including the Newtonian case. Moreover, the geodesic rays of the zero energy level of the planar three body problem were studied by Moeckel, Montgomery and Sánchez Morgado in [18], where it is showed for an open set of values of the masses, that the only minimizing configuration is the Lagrange equilateral one, and that any completely parabolic motion with equilateral limit shape admits a restriction which is a geodesic ray (that is, the restriction to $t \geq t_0$ for some time $t_0$).

(iii) For $h > 0$ now it is known that there are, as we have said, at least geodesic rays for arbitrary limit shape without collisions $a \in \Omega$ and starting from any initial positions of the bodies [13]. These are therefore (completely) hyperbolic motions. The existence of partially hyperbolic motions is part of a current research by the first author, see [2]. These motions are those of the form $x(t) = ta + O(t^{2/3})$ such that their limit shapes have collisions, $a \in E^N \setminus \Omega$, but $a \neq 0$. For this kind of motion there is associated a natural cluster partition, defined as follows. If $x(t) = (r_1(t), \ldots, r_N(t))$ and $a = (a_1, \ldots, a_N)$, then $a_i = a_j$ if and only if $r_{ij}(t) = O(t^{2/3})$. The partition of the set of bodies is then defined by this equivalence relation. Therefore, the motion consists of a family of clusters expanding between them with asymptotically linear rate, while inside them the distances grow at most with a rate of order $t^{2/3}$. For hyperbolic motions there are recent results by Duignan, Moeckel, Montgomery and Yu [8], in particular that the limit shape $a$ of a hyperbolic motion is a real analytic function of the initial conditions of the motion.

In this paper we prove that any geodesic ray of a positive energy level is expansive, meaning that $r_{ij}(t) \to +\infty$ for all $i < j$. Clearly, we can also define these motions as those for which $U(x(t)) \to 0$. Our main theorem will be formulated for free time minimizers of the action functional

$$\mathcal{A}_h(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t)) \, dt + hT$$

acting on the set of absolutely continuous curves $\gamma : [0, T] \to E^N$. We recall that as usual, we say that a curve defined on a noncompact interval is minimizing when all of its restrictions to compact subintervals are so. Then, as a corollary, we will obtain that the same holds for geodesic rays of the metric $j_h$.

**Theorem 1.** Let $h > 0$, and let $x : [0, +\infty) \to \Omega$ be a free time minimizer of the action functional $\mathcal{A}_h$. Then, writing $x(t) = (r_1(t), \ldots, r_n(t))$ we have

$$\lim_{t \to +\infty} \| r_i(t) - r_j(t) \| = +\infty$$

for all $1 \leq i < j \leq N$.

As we recall with more precision in the next sections, the dynamics of the clusters, or subsystems, was also analysed by Marchal and Saari in [10]. They proved that each subsystem behaves asymptotically like a pure $P$-body problem $1 < P < N$, in the sense that energy and an angular momentum relationships are asymptotically satisfied.
For geodesic rays of a positive energy level, we will prove that each of the subsystems must be of zero asymptotic energy. In particular, we will also be able to deduce, using again the known results by Marchal and Saari, that if one of the subsystem consists of \( P \) bodies, then the configuration which defines must converge to the set of central configurations of the \( P \)-body problem with the same value of the masses.

Actually, Theorem 1 is nothing but a trivial corollary of the following one, which gives a more accurate description of the final evolution of geodesic rays of the nonnegative energy levels. In particular, it generalizes the main theorem in [7]. The proof will be obtained essentially by combining the decomposition of the Lagrangian action in terms of the partial actions of the subsystems, with the aforementioned results of Marchal and Saari that we recall in section 4.1.

Notation. In what follows we write \( f \approx g \) whenever \( f \) and \( g \) are positive functions such that their quotient is eventually bounded between two positive constants.

\[ \text{Theorem 2. Let } h \geq 0 \text{ and } x = (r_1, \ldots, r_N) : [0, +\infty) \rightarrow \Omega \text{ be a free time minimizer of the action functional } A_h. \text{ Then for all } 1 \leq i < j \leq N \text{ we have} \]
\[ r_{ij}(t) \approx t^{2/3} \text{ or } r_{ij}(t) \approx t. \]
If all the distances grow like \( t^{2/3} \) then \( h = 0 \) and the motion is completely parabolic. Otherwise \( h > 0 \), and the motion is completely hyperbolic whenever all the distances grow like \( t \).

Roughly speaking, the theorem says that every free time minimizer of \( A_h \), defined over an unbounded time interval, can asymptotically be seen as a family of subsystems which mutually separate each other with a speed of higher order than that of the separation of the bodies within them. Moreover, it says that asymptotically, each non trivial subsystem (i.e. including at least two bodies) can itself be considered as a completely parabolic expanding system in uniform translation.

Now, concerning the geometric interpretation let us say that, for a curve defined over a compact interval, it is well known the equivalence between the variational property of being a free time minimizer of \( A_h \), and the property of being a minimizing geodesic for the Jacobi-Maupertuis metric \( j_h \). However, we will see that the equivalence between, on the one hand being a free time minimizer of \( A_h \), defined over \([0, +\infty)\), and on the other being a geodesic ray of the metric \( j_h \) (parameterized with constant energy) is in fact a consequence of Theorem 1. In this sense, we prove below (in Section 3) the following two corollaries.

\[ \text{Corollary 3. Let } h \geq 0, \text{ and let } \gamma : [0, +\infty) \rightarrow \Omega \text{ be an arclength parameterized geodesic ray of the Jacobi-Maupertuis metric } j_h. \text{ Let } \gamma(s) = (q_1(s), \ldots, q_N(s)), \text{ and } q_{ij}(s) = \| q_i(s) - q_j(s) \|. \text{ Then we have:} \]
\[ (i) \text{ The parameterization of } \gamma \text{ with constant energy } h \text{ is a free time minimizer of } A_h \text{ and moreover, it is well defined for all } t \in [0, +\infty). \]
\[ (ii) \text{ If } h > 0, \text{ then for all } 1 \leq i < j \leq N \text{ either} \]
\[ q_{ij}(s) \approx s^{2/3} \text{ or } q_{ij}(s) \approx s, \]
and \( q_{ij}(s) \approx s^2 \) for all \( 1 \leq i < j \leq N \) in the case \( h = 0 \).

\[ \text{Corollary 4. Let } h \geq 0. \text{ If } x : [0, +\infty) \rightarrow \Omega \text{ is a free time minimizer of the action functional } A_h, \text{ then } x \text{ is a geodesic ray of the metric } j_h. \]
2 The variational setting and preliminaries

The preliminary aspects that follow will be approached in a synthetic way. For more precise details, or proofs of the statements, we recommend that the reader consult mainly the references [7, 13], as well as those we will indicate.

As usual we define the Lagrangian as the function

\[ L(x,v) = K(v) + U(x) = \frac{1}{2} \| v \|^2 + \sum_{i<j} \frac{m_i m_j}{r_{ij}} \]

where \( \| v \|^2 = m_1 \| v_1 \|^2 + \cdots + m_N \| v_N \|^2 \) for any \( v = (v_1, \ldots, v_N) \) and \( r_{ij} \) is the distance \( \| r_i - r_j \| \) between bodies \( i \) and \( j \) of the configuration \( x = (r_1, \ldots, r_N) \).

Of course \( L < +\infty \) only for \( (x,v) \in T\Omega = \Omega \times E^N \). It should be noted that the potential energy of the system is the opposite of the Newtonian potential, that is to say, \( V = -U \).

The Lagrangian action

\[ A(\gamma) = \int_0^\tau L(\gamma(t), \dot{\gamma}(t)) \, dt \]

is well defined for absolutely continuous curves in \( E^N \) and it is clear that for some curves \( \gamma \) passing through collisions, its value is \( A(\gamma) = +\infty \).

It is well known, by experts in the field, that this does not pose a problem for the application of variational methods, but the fact that curves passing through collisions can have finite action. This was already noticed by Poincaré, claiming that the minimizers of the action could a priori not be genuine solutions, since they could present collision singularities. Today there is no doubt that the resolution of this difficulty, thanks to the celebrated Marchal’s Theorem [15, 5, 10], which assures that Poincaré’s objection is not feasible, has made it possible to fruitfully exploit variational methods.

To be explicit, we define for \( x, y \in E^N \) and \( \tau > 0 \) the sets of curves

\[ C(x,y,\tau) = \{ \gamma : [0,\tau] \to E^N \text{ absolutely continuous} \mid \gamma(0) = x, \gamma(\tau) = y \} \]

and

\[ C(x,y) = \bigcup_{\tau > 0} C(x,y,\tau) \, . \]

Usually in the literature, the actions are considered on curves in Sobolev spaces. It is not difficult to see that in our context, this amounts to restricting our space of absolutely continuous curves to that formed by those with finite action. On the other hand, we prefer to keep the widest space, since this is where Tonelli’s theory naturally applies. We also need to define, for each \( h \in \mathbb{R} \), the following action potentials:

\[ \phi_h : E^N \times E^N \to \{ -\infty \} \cup \mathbb{R} \]

\[ \phi_h(x,y) = \inf \{ A_h(\gamma) \mid \gamma \in C(x,y) \} \, , \]

where \( A_h(\gamma) \) is defined for a curve \( \gamma \in C(x,y,\tau) \) as the action of \( L + h \) on \( \gamma \), that is to say,

\[ A_h(\gamma) = \int_0^\tau L(\gamma(t), \dot{\gamma}(t)) \, dt + h\tau \, . \]
For $h < 0$ it can easily be seen that $\phi_h = -\infty$. For $h \geq 0$, it is well known that $(E^N, \phi_h)$ is a metric space, and indeed a length space. Moreover, the restriction of $\phi_h$ to $\Omega$, the space of configurations without collisions, is precisely the Riemannian distance induced by the Jacobi-Maupertuis metric $j_h$.

### 2.1 Semistatic curves

Before continuing, let us digress briefly to show the relationship between the motions we study here, and those that naturally arise in the Aubry-Mather theory for Tonelli Lagrangians (see for instance [6, 9, 14]). What is now clear, is that for the $N$-body problem the semistatic curves must escape, hence both the Aubry and the Mather sets turn out to be empty. However, we find it is more suggestive to think that these sets live in fact on the boundary at infinity for a convenient compactification, and that eventually the inclusion of one in the other could be strict for some particular values of the masses.

Following Mañé, we define the critical value $c = c(L)$ of the Lagrangian as

$$c = \inf \{ h \in \mathbb{R} \mid \phi_h(x, y) > -\infty \text{ for all } x, y \in E^N \},$$

It follows that for $h < c$ we have $\phi_h = -\infty$ everywhere, and also that for any value of $h \geq c$, the action potential $\phi_h$ defines a distance in $E^N$. Moreover, for $h \geq c$ and for any pair of configurations $x, y \in E^N$ there is always a curve $\gamma \in \mathcal{C}(x, y, \tau)$ for some $\tau > 0$ realizing the $\phi_h$ distance between $x$ and $y$, that is to say, such that $\mathcal{A}_h(\gamma) = \phi_h(x, y)$. As a consequence of Marchal’s Theorem these curves avoid collisions at any interior time and are true solutions.

A curve $\gamma$ defined on an interval $[t_0, +\infty)$ is called semistatic whenever

$$\mathcal{A}_c(\gamma|_{[a, b]}) = \phi_c(\gamma(a), \gamma(b))$$

for $c = c(L)$ the critical value, and for any compact subinterval $[a, b] \subset [t_0, +\infty)$. Observing that for any natural Lagrangian of the form $L = K - V$, the critical value is nothing but $c = \sup V$, we get in particular for the $N$-body problem that $c(L) = 0$. For this reason, here the critical action potential is $\phi_0$, and the semistatic curves are the free time minimizers of the Lagrangian action. Since these free time minimizers are completely parabolic motions [7] and this type of motion have configurations whose shapes approximate central configuration shapes, their limit shapes define a very special class of central configurations. Indeed, the set of minimizing configurations is defined in [12] precisely in this way. On the other hand, we have the set of minimal configurations, composed by those configurations which minimize $U$ among all configurations with the same size. Clearly any minimal configuration is a minimizing one. This is because a minimal configuration is obviously a central configuration, and the corresponding parabolic homothetic motion is a free time minimizer with fixed shape. This suggest that probably these two sets of configurations will play an interesting role, respectively as the Aubry and the Mather sets at infinity.

However, it should be said that for the positive energy levels this analogy does not have such a clear interpretation. Indeed for $h > 0$, every configuration without collisions $\alpha \in \Omega$ can be obtained as the limit shape of hyperbolic motions which are free time minimizers of $A_h$. 
2.2 Upper bounds for the action potentials

The following are estimates for minimal actions between configurations. All of them are deduced from the first one which gives an upper bound for \( \phi(x, y, \tau) \), where \( x \) and \( y \) are given configurations, \( \tau \) is a positive real number, and

\[
\phi(x, y, \tau) = \min \{ A_0(\gamma) = A(\gamma) \mid \gamma \in \mathcal{C}(x, y, \tau) \}.
\]

We will say that the configuration \( x = (r_1, \ldots, r_N) \) is contained in the ball \( B(r, R) \subset E \) whenever for all \( i = 1, \ldots, N \) we have \( \| r_i - r \| < R \).

**Theorem** (see [11], Prop. 4, p. 1023). There are positive constants \( \alpha_0 \) and \( \beta_0 \) such that, for any two configurations \( x \) and \( y \) contained in a given ball \( B \subset E \) of radius \( R > 0 \), and for any \( \tau > 0 \), there is a curve \( \gamma \in \mathcal{C}(x, y, \tau) \) such that

(i) for all \( t \in [0, \tau] \) the configuration \( \gamma(t) \) is contained in the ball with the same center as \( B \) but with radius \( 6NR \), and

(ii)

\[
A(\gamma) \leq \alpha_0 MN^N \frac{R^2}{\tau} + \beta_0 M^2 N^4 \frac{\tau}{R}
\]

where \( M = m_1 + \cdots + m_N \) is the total mass of the system.

From this construction of canonical paths between configurations, and using well chosen cluster partitions, the following upper bounds in terms of \( \| x - y \| \) can be deduced.

**Theorem** (see [11], Prop. 9, p. 1031). There are positive constants \( \alpha_1 \) and \( \beta_1 \) which only depend on the number of bodies \( N \) and the total mass \( M \), such that for all \( x, y \in E^N \) and any \( \tau > 0 \) we have

\[
\phi(x, y, \tau) \leq \alpha_1 \frac{\rho^2}{\tau} + \beta_1 \frac{\tau}{\rho}
\]

whenever \( \rho > \| x - y \| \). In particular, in the case where \( x \neq y \) the upper bound also applies for \( \rho = \| x - y \| \).

Finally, from this last result we can obtain the following upper bound for the action potentials (see also Theorem 2 in [11] for the critical case \( h = 0 \)).

**Theorem** (see [13], Theorem 2.11 p. 517). There are positive constants \( \alpha \) and \( \beta \) only depending on the number of bodies \( N \) and the total mass \( M \), such that for all \( x, y \in E^N \) we have

\[
\phi_h(x, y) \leq \left( \alpha \| x - y \| + \beta h \| x - y \|^2 \right)^{1/2}
\]

for any value of \( h \geq 0 \).

3 Geodesic rays as free time minimizers

So far we have assumed a total equivalence up to reparameterizations of, on the one hand the geodesic rays of the metric \( j_h \) of a given energy level \( h \geq 0 \), and
on the other the free time minimizers of the action functional $A_h$. As we will see now, this assumption deserves a non trivial justification.

For this, we will denote by $L_h(x)$ the length of an absolutely continuous curve $x : [a, b] \rightarrow \Omega$ with respect to the $j_h$ metric, that is,

$$L_h(x) = \int_a^b \| \dot{x}(t) \|_h \, dt = \int_a^b [2(h + U(x(t)))]^{\frac{1}{2}} \| \dot{x}(t) \| \, dt.$$  

Moreover, we recall that for any of these curves we have that $A_h(x) \geq L_h(x)$, with equality if and only if the curve $x$ has constant energy $h$.

### 3.1 Proofs of corollaries \(3\) and \(4\)

**Proof of Corollary \(3\)**: Let $\gamma : [0, +\infty) \rightarrow \Omega$ be a geodesic ray of $j_h$, for a given $h \geq 0$. The reparameterization of $\gamma$ as a curve of constant energy $h$ is the curve $x : [0, t^*) \rightarrow \Omega$, with $t^* \in (0, +\infty]$, defined by $x(t) = \gamma(\sigma(t))$ where

$$\sigma : [0, t^*) \rightarrow [0, +\infty)$$

is a diffeomorphism such that $\sigma(0) = 0$, and such that

$$\frac{1}{2} \left\| \frac{d\gamma}{ds}(\sigma(t)) \dot{\sigma}(t) \right\|^2 - U(\gamma(\sigma(t))) = h$$

for all $t \in [0, t^*)$. On the other hand, since we are assuming that $\gamma$ is arclength parameterized, for any $s \geq 0$ we have

$$1 = \left\| \frac{d\gamma}{ds}(s) \right\|_h^2 = j_h \left( \frac{d\gamma}{ds}(s), \frac{d\gamma}{ds}(s) \right) = 2(h + U(\gamma(s))) \left\| \frac{d\gamma}{ds}(s) \right\|^2.$$  

Thus we deduce that $\sigma$ is well defined as the maximal solution of the Cauchy problem

$$\dot{\sigma}(t) = 2h + 2U(\gamma(\sigma(t)))$$

with $\sigma(0) = 0$, which clearly satisfies $\lim_{t \rightarrow t^*} \sigma(t) = +\infty$.

In order to prove the first statement, we observe that for any $[a, b] \subset [0, t^*)$, we have

$$A_h(x |_{[a,b]}) = L_h(x|_{[a,b]}) = L_h(\gamma |_{[\sigma(a), \sigma(b)]}) = \sigma(b) - \sigma(a)$$

because $x$ has constant energy $h$ and the length of a curve does not depend on the chosen parameterization. Therefore, since $\gamma$ is a minimizing geodesic, we conclude that $x |_{[a,b]}$ is a free time minimizer of $A_h$ for all $[a, b] \subset [0, t^*)$, and it only remains to prove that $t^* = +\infty$.

Suppose $t^* < +\infty$. Either $x$ has a limit configuration as $t \rightarrow t^*$, or $x$ presents a noncollision singularity. In both cases, a contradiction can be deduced with the upper bound for the action potential $\phi_h$. Since $\sigma(t)$ is precisely the $j_h$-length of $x |_{[0,t]}$, if $x(t) \rightarrow x^*$ as $t \rightarrow t^*$, then for all $t \in [0, t^*)$ we have

$$\sigma(t) = \phi_h(x_0, x(t)) < \phi_h(x_0, x^*)$$

where $x_0 = x(0)$, contradicting the fact that $\sigma(t) \rightarrow +\infty$. On the other hand, if we assume that $x$ presents a noncollision singularity at $t = t^*$, then the Von
Zeipel’s Theorem \cite{17, 21} implies that \( \| x(t) \| \) is unbounded on \([0, t^*]\). Therefore \( r_n = \| x(t_n) - x_0 \| \to +\infty \) for some sequence \( t_n \to t^* \). Let us denote by \( A_n \) the Lagrangian action of the restriction of \( x \) to \([0, t_n]\), that is

\[
A_n = A(x |_{[0, t_n]}) = \frac{1}{2} \int_0^{t_n} \| \dot{x}(t) \|^2 \, dt + \int_0^{t_n} U(x(t)) \, dt .
\]

By neglecting the potential term, and applying the Cauchy-Schwarz inequality with the kinetic term, we get the lower bound

\[
r_n^2 \leq \left( \int_0^{t_n} \| \dot{x}(t) \| \, dt \right)^2 \leq t_n \int_0^{t_n} \| \dot{x}(t) \|^2 \, dt < 2t_n A_n ,
\]

and therefore

\[
\frac{r_n^2}{2t_n} + h t_n < A_n + h t_n = \phi_h(x_0, x(t_n))
\]

for all \( n \in \mathbb{N} \). But the upper bound for the action potential cited in the previous section (Theorem 2.11 in \cite{13}) gives

\[
\phi_h(x_0, x(t_n)) \leq (\alpha r_n + \beta h r_n^2)^{1/2}
\]

and this inequality is incompatible with the previous one for \( r_n \) big enough. Hence we have proved that \( t^* = +\infty \) as we wanted.

We prove now the second statement of the corollary. We begin by considering the particular case \( h = 0 \). In this case, we have that \( x(t) \) is a free time minimizer of \( A_0 \), therefore, according to Theorem 2 all the mutual distances satisfy

\[
\rho_{ij}(t) \approx t^{2/3} .
\]

In turn, this implies that \( \dot{\sigma}(t) = U(x(t)) \approx t^{-2/3} \), hence that \( \sigma(t) \approx t^{1/3} \). Now we define \( \tau \) as the inverse function of \( \sigma \), and then we get that \( \tau(s) \approx s^3 \). Since \( \rho_{ij}(s) = r_{ij}(\tau(s)) \), we conclude that \( \rho_{ij}(s) \approx s^2 \) for all \( i < j \). On the other hand, if \( h > 0 \) then we have

\[
\frac{\mu}{t} \leq U(x(t)) \leq \frac{\nu}{t^{2/3}}
\]

for some positive constants \( \mu \) and \( \nu \), and for \( t > 0 \) large enough. Thus we get

\[
2ht + \mu \log(t) + \eta \leq \sigma(t) \leq 2ht + 3\nu t^{1/3} + \xi
\]

for \( t \) large enough. We conclude that \( \sigma(t) \approx t \) and therefore, its inverse function satisfies \( \tau(s) \approx s \). The proof is then achieved as before, by using Theorem 2 and the fact that \( \rho_{ij}(s) = r_{ij}(\tau(s)) \).

Proof of Corollary 4. Let \( x : [0, +\infty) \to \Omega \) be a free time minimizer of \( A_h \) and let \( \gamma : [0, s^*) \to \Omega \) be the arclength parameterization of the curve, where \( s^* \in (0, +\infty) \). Thus we have \( \gamma(\sigma(t)) = x(t) \) where \( \sigma : [0, +\infty) \to [0, s^*) \) is the length of \( x |_{[0, t]} \). Since \( x \) is a free time minimizer we know that, for each \( t > 0 \)

\[
\phi_h(x_0, x(t)) = A_h(x |_{[0, t]}) = L_h(x |_{[0, t]}) = \sigma(t) ,
\]

which says that \( \gamma \) is a minimizing geodesic of the metric \( j_h \). To see that \( \gamma \) is a geodesic ray we only have to prove that \( \gamma \) has infinite length, that is \( s^* = +\infty \). Since a free time minimizer can not be superhyperbolic (Lemma 7 below), we
have that all mutual distances in the motion \( x(t) \) satisfy \( r_{ij}(t) = O(t) \). Then we must have

\[
\frac{\lambda}{t} \leq U(x(t))
\]

for some constant \( \lambda > 0 \) and \( t \) large enough. This suffices for \( \sigma \) to be divergent, since

\[
\sigma(t) \geq \int_0^t U(x(\zeta)) \, d\zeta.
\]

In particular \( s^* = \lim_{t \to +\infty} \sigma(t) = +\infty \). \( \square \)

4 Systems and subsystems

4.1 The cluster’s asymptotic energy

In the same quoted paper by Marchal and Saari, the research continues with a finer study on the behaviour of the subsystems that naturally appears in any non superhyperbolic motion. In this section we recall some of these results in that direction that we will use later in the proof of the main theorem.

Let us define first, for a motion \( x(t) = ta + O(t^{2/3}) \) as \( t \to +\infty \), its associated natural cluster partition, as well as the geometry of that decomposition.

**Definition 2.** For a motion \( x(t) = ta + O(t^{2/3}) \) as \( t \to +\infty \) its corresponding natural partition of the index set \( \mathcal{N} = \{1, \ldots, N\} \) is the one for which \( i, j \in \mathcal{N} \) belong to the same class if and only if the mutual distance \( r_{ij}(t) \) grow as \( O(t^{2/3}) \). Equivalently, if \( a = (a_1, \ldots, a_N) \), then the natural partition is defined by the relation \( i \sim j \) if and only if \( a_i = a_j \). The partition classes will be called clusters.

**Definition 3.** Let \( \mathcal{P} \) be a given partition of \( \mathcal{N} \), and let \( x = (r_1, \ldots, r_N) \in E^N \) be a given configuration. For each cluster \( A \in \mathcal{P} \) we define the mass of the cluster

\[
m_A = \sum_{i \in A} m_i,
\]

the center of mass of the cluster

\[
y_A = \frac{1}{m_A} \sum_{i \in A} m_i r_i,
\]

and the relative positions \( s_i = r_i - y_A \), being \( A \) the cluster containing \( i \in \mathcal{N} \).

Notice that for \( i, j \in A \) we have \( s_{ij} = \|s_i - s_j\| = r_{ij} \).

**Definition 4.** Let \( \mathcal{P} \) be any given partition of \( \mathcal{N} \). Then for any given curve \( \gamma(t) = (r_1(t), \ldots, r_N(t)) \) in \( E^N \) we define, for each cluster \( A \in \mathcal{P} \), the functions

\[
K_A(t) = \frac{1}{2} \sum_{i \in A} m_i \|\dot{s}_i(t)\|^2 \quad \text{and} \quad U_A(t) = \sum_{i,j \in A, i<j} \frac{m_i m_j}{s_{ij}(t)},
\]

as well as the function \( H_A(t) = K_A(t) - U_A(t) \), that we will call the energy of the cluster \( A \) of the curve \( \gamma \).
Theorem (Marchal-Saari [16], Theorem 2 p.165). Let \( x(t) \) be a motion defined for all \( t \geq t_0 \) which is not superhyperbolic. If \( P \) is its associated natural partition of \( N \), then for every cluster \( A \in P \), there is a constant \( h_A \leq 0 \) such that

\[
H_A(t) = h_A + O(t^{-5/3}) .
\]

Theorem (Marchal-Saari [16], Corollary 4 p.166). Under the same hypothesis, if \( A \in P \) is such that \( h_A = 0 \), then \( r_{ij}(t) \approx t^{2/3} \) for all \( i, j \in A \) with \( i \neq j \).

Furthermore, we also have the following.

Theorem (Marchal-Saari [16], Corollary 5 p.166). Under the same hypothesis, if \( h_A < 0 \) then the function

\[
r_A(t) = \min \{ r_{ij}(t) \mid i, j \in A, i \neq j \}
\]

is bounded.

4.2 Geometry of the cluster partition

Before proceeding with the proofs of the main results, let us better describe the geometry of the cluster decomposition of the configuration space associated with a given partition of the system. We believe that a convenient notation is necessary to be able to express this description synthetically. For this purpose, we start with the following definitions dependent on a given partition \( P \) of the set \( N = \{1, \ldots, N\} \).

The space of the cluster centers will be the space \( E_P = \{ (u_A)_{A \in P} \} \) endowed with the inner product which weights each center with the total mass of the cluster. More precisely, for any pair \( u, u' \in E_P \),

\[
\langle u, u' \rangle = \sum_{A \in P} m_A \langle u_A, u'_A \rangle_E
\]

where as before, \( m_A = \sum_{i \in A} m_i \). Second, for each cluster \( A \in P \) we define the space of centered configurations of the cluster

\[
E^A_0 = \{ z = (s_i)_{i \in A} \in E^A \mid \sum_{i \in A} m_i s_i = 0 \},
\]

also endowed with its corresponding mass inner product

\[
\langle z, z' \rangle = \sum_{i \in A} m_i \langle s_i, s'_i \rangle_E .
\]

Notice that for this product, \( E^A_0 \) is precisely the orthogonal complement of the diagonal \( \Delta_A \subset E^A \). Also notice that \( \dim E^P = \#P \dim E \), and that for each \( A \in P \) we have \( \dim E^A_0 = (\#A - 1) \dim E \).

Finally, we define the product of all these inner product spaces, that is,

\[
D_P = E^P \times \prod_{A \in P} E^A_0
\]

and actually, it is well known that this space is isometric to the usual mass inner product space defined on the original space of configurations \( E^N \).
Proposition 5. For any partition $\mathcal{P}$ of $\mathcal{N}$, the linear map

$$d : E^N \to D_\mathcal{P}$$

given by

$$x = (r_1, \ldots, r_N) \mapsto d(x) = (y(x), (z_A)_{A \in \mathcal{P}}),$$

where $y(x) = (y_A)_{A \in \mathcal{P}}$ is the configuration produced by the centers of mass of the clusters, and $z_A = (s_i)_{i \in A}$ are the configurations of the clusters given by the relative positions $s_i = r_i - y_A$ is an isometry.

Proof. It is trivial to see that the linear map $d$ takes values in the space $D_\mathcal{P}$. This follows from the fact that the center of mass of each configuration $z_A$ is precisely

$$\sum_{i \in A} m_i s_i = \sum_{i \in A} m_i (r_i - y_A) = \sum_{i \in A} m_i r_i - (\sum_{i \in A} m_i) y_A = m_A y_A - m_A y_A = 0.$$

Let us prove now that $d$ is an isometric immersion. Let $x, x' \in E^N$ be any two configurations. If $d(x) = (y, (z_A)_{A \in \mathcal{P}})$ and $d(x') = (y', (z'_A)_{A \in \mathcal{P}})$ are the corresponding cluster decompositions, then we have

$$\langle d(x), d(x') \rangle = \langle y, y' \rangle + \sum_{A \in \mathcal{P}} \langle z_A, z'_A \rangle$$

$$= \sum_{A \in \mathcal{P}} m_A \langle y_A, y'_A \rangle + \sum_{A \in \mathcal{P}} \sum_{i \in A} m_i \langle s_i, s'_i \rangle$$

$$= \sum_{A \in \mathcal{P}} \sum_{i \in A} m_i \langle y_A, y'_A \rangle + \sum_{A \in \mathcal{P}} \sum_{i \in A} m_i \langle r_i - y_A, r'_i - y'_A \rangle$$

$$= \sum_{i=1}^{N} m_i \langle r_i, r'_i \rangle = \langle x, x' \rangle.$$

Finally, we conclude that $d$ is an isometry since

$$\dim D_\mathcal{P} = \#\mathcal{P} \dim E + \sum_{A \in \mathcal{P}} (\#A - 1) \dim E = N \dim E = \dim E^N.$$

In particular we deduce the following useful corollary.

Corollary 6. For any partition $\mathcal{P}$ of $\mathcal{N}$, the decomposition $d(x) = (y, (z_A)_{A \in \mathcal{P}})$ of a given configuration $x \in E^N$ with respect to the cluster partition $\mathcal{P}$ satisfies

$$\|x\|^2 = \|y\|^2 + \sum_{A \in \mathcal{P}} \|z_A\|^2.$$

4.3 Decomposition of the Lagrangian action

The cluster decomposition of the configurations will be used to decompose the Lagrangian action of a curve. For this, let us first observe that for any curve $\gamma(t)$ in $E^N$ with decomposition $d(\gamma(t)) = (\xi(t), (\sigma_A(t))_{A \in \mathcal{P}})$ we have $\dot{\sigma}_A(t) \in E^A_0$ at
any point of differentiability of $\gamma$. In particular, for absolutely continuous curves the previous lemma gives the almost everywhere identity
\[
\|\dot{\gamma}\|^2 = \|\dot{\xi}\|^2 + \sum_{A \in \mathcal{P}} \|\dot{\sigma}_A\|^2.
\]

In order to decompose the Lagrangian action of a curve we also need to decompose the Newtonian potential. For that purpose, given a partition $\mathcal{P}$ of $\mathcal{N}$, for any configuration $z \in E^A_0$, say $z = (s_i)_{i \in A}$, we write
\[
U_A(z) = \sum_{i,j \in A, i < j} \frac{m_i m_j}{s_{ij}},
\]
where $s_{ij} = \|s_i - s_j\|$. Moreover, if $x = (r_1, \ldots, r_N) \in E^N$, for any pair of different clusters $A, B \in \mathcal{P}$, we define the interaction potential $W_{AB}$ by
\[
W_{AB}(x) = \sum_{i \in A, j \in B} \frac{m_i m_j}{r_{ij}}.
\]
Now, if the cluster decomposition of $x$ is $(y_c, (z_A)_{A \in \mathcal{P}})$, where $z_A = (s_i)_{i \in A}$, then clearly we have $r_{ij} = s_{ij}$ for all $i, j \in A$ and any cluster $A \in \mathcal{P}$. Therefore we can write
\[
U(x) = \sum_{A \in \mathcal{P}} U_A(z_A) + W_P(x)
\]
where
\[
W_P(x) = \sum_{A, B \in \mathcal{P}, A \neq B} \frac{1}{2} W_{AB}(x).
\]

Finally we conclude that for any absolutely continuous curve $\gamma : [a, b] \to E^N$, the Lagrangian action decomposes as
\[
A(\gamma) = \int_a^b \frac{1}{2} \|\dot{\xi}\|^2 + \sum_{A \in \mathcal{P}} \left( \int_a^b \frac{1}{2} \|\dot{\sigma}_A\|^2 + U_A(\sigma_A) \right) + \int_a^b W_P(\gamma)
\]
where $\xi(t)$ is the configuration of the cluster centers of $\gamma(t)$, and $\sigma_A(t)$ is the centered configuration defined by the cluster $A$ of $\gamma(t)$, for each $A \in \mathcal{P}$.

## 5 Proof of the main theorem

We start by showing that curves in our hypothesis are not superhyperbolic. This result was first established in the previous work by the second author with Venturelli for the study of calibrating curves of some viscosity solutions of the Hamilton-Jacobi equation, and we transcribe here the proof for the sake of completeness.

**Lemma 7** ([13], p.534). If $x : [0, + \infty) \to E^N$ is a free time minimizers of $A_h$ with $h > 0$, then $\gamma$ is not superhyperbolic.

**Proof.** Suppose we have a sequence $t_n \to + \infty$ such that $\|x(t_n)\| t_n^{-1} \to + \infty$. We will prove that this is in contradiction with the upper bound for $\phi_h$ given
Therefore we will conclude that \( \| x(t) \| < \lambda t \) for some constant \( \lambda \) and \( t \) big enough, which says that \( x(t) \) is not superhyperbolic.

Let us call \( r_n = \| x(t_n) - x(0) \| \) and \( A_n = A(x|_{[0,t_n]}) \). Then we have, as in the proof of Corollary 3, on the one hand, that

\[
\frac{r_n^2}{2t_n} + ht_n \leq A_n + ht_n = \phi_h(x(0), x(t_n)),
\]

and on the other hand, because of the estimate given in section 2.2, that

\[
\phi_h(x(0), x(t_n)) \leq (\alpha r_n + \beta h r_n^2)^{1/2}.
\]

These inequalities are incompatible for \( n \) large enough, since \( r_n t_n^{-1} \to +\infty \).

\[ \square \]

5.1 A closer look to the Lagrangian action estimate

In the proof of our theorem we will make use of a slight modification of the fundamental result stated in section 2.2. More precisely, we refer to the existence of a curve joining two given configurations in a given time, for which an upper bound of its Lagrangian action is obtained.

The curve provided by the theorem, a curve \( \gamma \in \mathcal{C}(x, y, \tau) \) is built in [11] in canonical way from the data, namely the configurations \( x, y \) in \( E^N \) and time \( \tau > 0 \). Moreover, for configurations \( x = (r_1, \ldots, r_N) \) and \( y = (s_1, \ldots, s_N) \) contained in a ball of radius \( R \) of \( E \), that is such that

\[
\| r_i - c \| < R \quad \text{and} \quad \| s_i - c \| < R
\]

for some \( c \in E \) and for all \( i \in \mathcal{N} \), the construction is such that the configuration \( \gamma(t) = (\gamma_1(t), \ldots, \gamma_N(t)) \) is contained in the concentric ball of radius \( 6NR \) for all \( t \in [0, \tau] \).

As we will see, the canonical path \( \gamma \in \mathcal{C}(x, y, \tau) \) can be constructed with the following additional condition: if \( x \) and \( y \) are configurations with the same center of mass, then the center of mass of the curve \( \gamma(t) \) is constant and equals the common center of the configurations \( x \) and \( y \). However, to achieve this we must allow the configuration \( \gamma(t) \) to be contained in a ball of radius twice as large, that is, being able to assure that \( \| \gamma_i(t) - c \| < 12NR \) for all \( i \in \mathcal{N} \) and for all \( t \in [0, \tau] \).

One possible way to prove this, which is not the one we will address, consists of adapting the proof offered in [11]. Indeed, the construction of the curve \( \gamma \) is obtained there by reparameterizing the concatenation of two linear paths with a common vertex in an intermediate configuration \( z \). To achieve the estimation of the action, that intermediate configuration \( z = (z_1, \ldots, z_N) \) must be required to have all mutual distances greater than or equal to \( 6R \). In order for the argument to be valid even in the case \( \dim E = 1 \), a collinear \( z \) configuration is used, in which the body \( z_1 \) is in the center of the ball, and the distance between \( z_i \) and \( z_{i+1} \) is exactly \( 6R \). This explains the upper bound \( 6NR \) for the radius of the ball containing the curve \( \gamma \). It is not difficult to verify that the additional condition we are looking for, is trivially obtained if we translate the intermediate configuration \( z \) in such a way that its center of mass coincides with the midpoint of the segment joining the center of \( x \) and the center of \( y \). It is also observed
that the path obtained with this modification could not be contained in the ball of radius \(6NR\), but is necessarily contained in the ball of radius \(12NR\).

Instead, to avoid these verifications for the reader, we propose the use of the following simple lemma.

**Lemma 8.** Let \(x, y \in E^N\), \(\tau > 0\) and \(\gamma \in C(x, y, \tau)\). If \(\xi(t) \in E\) denotes the center of mass of \(\gamma(t) = (\gamma_1(t), \ldots, \gamma_N(t))\) for each \(t \in [0, \tau]\) then the curve \(\gamma' \in C(x, y, \tau)\) defined by

\[
\gamma'_i(t) = \frac{(\tau - t)}{\tau} \xi(0) + \frac{t}{\tau} \xi(\tau) + \gamma_i(t) - \xi(t)
\]

satisfies \(A(\gamma') \leq A(\gamma)\) and moreover, if \(B \subseteq E\) is a ball containing \(\gamma(t)\) for all \(t \in [0, \tau]\) then \(\gamma'(t)\) is all the time contained in the ball concentric with \(B\) and twice the radius.

**Proof.** Consider the trivial partition of \(E^N\) with only one cluster \(A = N\). In this case, the orthogonal decomposition of \(E^N\) is simply reduced to identifying each configuration \(x = (r_1, \ldots, r_N)\) with the pair \((y, (r_1 - y, \ldots, r_N - y))\), being \(y \in E\) the center of mass of the configuration \(x\).

Using the corresponding orthogonal decomposition of \(E^N\), that is to say \(E^N = E \times E^N\), and writing \(\sigma(t) = (\gamma_1(t) - \xi(t), \ldots, \gamma_N(t) - \xi(t))\) for \(t \in [0, \tau]\), we have that \(\gamma(t) = (\xi(t), \sigma(t))\) and that \(\gamma'(t) = (\xi'(t), \sigma(t))\), where

\[
\xi'(t) = \frac{(\tau - t)}{\tau} \xi(0) + \frac{t}{\tau} \xi(\tau)
\]

is the linear path from the center of \(x\) to the center of \(y\).

Therefore, using now the decomposition for the action, we get

\[
A(\gamma) - A(\gamma') = \int_0^\tau \frac{1}{2} M \|\dot{\xi}(t)\|^2 dt - \int_0^\tau \frac{1}{2} M \|\dot{\xi}'(t)\|^2 dt
\]

where \(M\) is the total mass of the system. We conclude that \(A(\gamma) \geq A(\gamma')\) since the Bunyakowsky’s inequality implies that

\[
\int_0^\tau \|\dot{\xi}(t)\|^2 dt \geq \frac{1}{\tau} \left(\int_0^\tau \|\dot{\xi}(t)\| dt\right)^2 \geq \frac{1}{\tau} \|\xi(\tau) - \xi(0)\|^2
\]

with equality holding if and only if \(\xi(t) = \xi'(t)\) for all \(t \in [0, \tau]\). Concerning the second assertion, its proof is immediate since if \(\gamma(t)\) is contained in some ball \(B \subseteq E\) for all \(t \in [0, \tau]\), then both \(\xi(t)\) and \(\xi'(t)\) are in \(B\) all the time. \(\square\)

### 5.2 Proof of Theorem 2

**Proof.** Let \(x : [0, +\infty) \to \Omega\) be a free time minimizer of \(A_h\), with \(h \geq 0\), say \(x(t) = (r_{11}(t), \ldots, r_{NN}(t))\). By Lemma 7 we know that \(x(t)\) is not superhyperbolic, and that \(x(t) = ta + O(t^{2/3})\).

In the case \(h = 0\), we already know that \(x(t)\) is a completely parabolic motion, see 7. Some times this is expressed by the condition \(\dot{x}(t) = 0\), but equivalently we must have \(r_{ij}(t) \approx t^{2/3}\) for all \(i \neq j\). It remains then to consider
the case $h > 0$, in which $a \neq 0$. However, the following proof works in all cases, that is, even when $h = 0$.

Let us write $a = (a_1, \ldots, a_N)$, and call $\mathcal{P}$ the natural partition associated to the motion $x$, that is $i \sim j$ if and only if $a_i = a_j$. According to the description of the asymptotic behaviour of the clusters also given by Marchal and Saari and recalled in section 2.4 for each cluster $A \in \mathcal{P}$ either the asymptotic energy is $h_A = 0$ and $r_{ij}(t) \approx t^{2/3}$ for any $i, j \in A$ with $i \neq j$, or $h_A < 0$ and the minimal distance $r_A(t) = \min \{ r_{ij}(t) \mid i \neq j, \ i, j \in A \}$ is bounded. Therefore, it suffices to prove that $h_A = 0$ for all $A \in \mathcal{P}$ or equivalently, that $r_A(t)$ is unbounded for all $A \in \mathcal{P}$.

The cluster decomposition associated to our partition will be denoted by $d(x(t)) = (y(t), (z_A(t)))_{A \in \mathcal{P}}$. Recall that $y(t) = (y_A(t))_{A \in \mathcal{P}}$ is the configuration in $E^P$ composed by the cluster centers, and that $z_A(t) = (s_i(t))_{i \in A} \in E^A_0$ is given for each cluster $A \in \mathcal{P}$ by the relative positions $s_i(t) = r_i(t) - y_A(t)$.

Now we use the fact that $x(t) = ta + O(t^{2/3})$. This implies that there is $R > 0$ such that $\| r_i(t) - ta_i \| < Rt^{2/3}$ for all $t > t_0$ for some $t_0 > 0$. Thus for each cluster $A$ we also have $\| y_A(t) - ta_i \| < Rt^{2/3}$ whenever $i \in A$, hence $\| s_i(t) \| < 2Rt^{2/3}$ for all $i \in A$ and every $t > t_0$. This means that the configuration $z_A(t) \in E_0^A$ is contained in the ball with center at the origin of $E$ and radius $2Rt^{2/3}$ for $t > t_0$.

Suppose that there is a cluster $A \in \mathcal{P}$ and $\rho > 0$ such that $r_A(t) < \rho$ for all $t \geq 0$. Then we deduce that

$$U_A(x(t)) \geq k \geq \frac{m_0^2}{\rho} > 0$$

for all $t \geq 0$, where $m_0 = \min \{ m_1, \ldots, m_N \}$. In particular we can state that,

$$\int_t^{t+2t} L_A(z_A(s), \dot{z}_A(s)) \, ds \geq kt$$

for any $t > 0$, where $L_A : TE_0^A \to \mathbb{R}$ is the Lagrangian given by

$$L_A(z, w) = \frac{1}{2} \| w \|^2 + U_A(z).$$

We will show that for $t > t_0$ big enough, this lower bound is incompatible with the minimizing property of the motion $x(t)$. To do this, for each $t > t_0$ we consider a new curve $\gamma_t \in C(x(t), x(2t), t)$ as follows. For the cluster $A \in \mathcal{P}$ that is supposed to be non expanding, we choose a curve $\sigma_A(s)$ in $E_0^A$, joining $z_A(t)$ with $z_A(2t)$ and defined for $s \in [t, 2t]$, in such a way that we have

$$\int_t^{t+2t} L_A(\sigma_A(s), \dot{\sigma}_A(s)) \, ds \leq \alpha_0 n_A^3 \frac{R_0^2}{t} + \beta_0 m_A^2 \frac{n_A^3}{R_0^2} \frac{t}{R_{2t}},$$

where $m_A$ is the total mass of the cluster, $n_A = \#A$ is the number of bodies in the cluster, $R_0 = 2Rt^{2/3}$ is the radius of the ball containing both configurations $z_A(t)$ and $z_A(2t)$ in $E_0^A$, and $\alpha_0$ and $\beta_0$ are the positive constants given by the fundamental theorem stated in section 2.2. Note that here we are applying this theorem to the Lagrangian $L_A$ in $E^A$. Moreover, according to Lemma 8 we can assume that $\sigma_A(s) \in E_0^A$ for all $s \in [t, 2t]$, and that it is contained in the ball with center at the origin and radius $12 n_A R_{2t}$. Thus we have

$$\int_t^{t+2t} L_A(\sigma_A(s), \dot{\sigma}_A(s)) \, ds \leq k' t^{1/3}$$
for some constant $k' > 0$, and this for all $t > t_0$. Then we define the curve 
$\gamma_t \in C(x(t), x(2t), t)$ by replacing in the cluster decomposition of the original 
curve the component $z_A(s)$ by the curve $\sigma_A(s)$.

In order to get a contradiction, finally we compare de Lagrangian action of 
the curve $x \mid_{[t,2t]}$ with that of the new curve $\gamma_t$. Using the decomposition of 
the action we see, after cancellation of the common terms, that

$$\mathcal{A}(\gamma_t) - \mathcal{A}(x \mid_{[t,2t]}) \leq k't^{1/3} - k t + \int_t^{2t} W_P(\gamma_t(s)) - W_P(x(s)) \, ds.$$  

Let us fix for a while the value of $t > t_0$. If we write 
$\gamma_t(s) = (r'_1(s), \ldots, r'_N(s))$ then we have that

$$W_P(\gamma_t(s)) - W_P(x(s)) = \sum_{i \in A} \sum_{j \not\in A} m_i m_j \left( \frac{1}{\|r_i(s) - r_j(s)\|} - \frac{1}{\|r'_i(s) - r'_j(s)\|} \right).$$

By the previous considerations, for any $i \in A$ and $j \not\in A$ we also have

$$\|r_i(s) - r_j(s)\| \geq \|sa_i - sa_j\| - \|r_i(s) - sa_i\| - \|r_j(s) - sa_j\|$$

as well as

$$\|r'_i(s) - r'_j(s)\| \geq \|sa_i - sa_j\| - \|r'_i(s) - y_A(s)\| - \|y_A(s) - sa_i\| - \|r_j(s) - sa_j\|$$

$$\geq s \|a_{ij}\| - 12n_A R s^{2/3} - 2R s^{2/3}.$$  

Therefore we deduce that there is a constant $k'' > 0$ and $t_1 > t_0$ such that

$$|W_P(\gamma_t(s)) - W_P(x(s))| \leq \frac{k''}{s}$$

for all $s \in [t,2t]$, whenever $t \geq t_1$, which in turn implies that

$$\mathcal{A}(\gamma_t) - \mathcal{A}(x \mid_{[t,2t]}) \leq k't^{1/3} - k t + k'' \log(2).$$

For $t > t_1$ big enough we have that this upper bound becomes negative, meaning 
that $\mathcal{A}(x \mid_{[t,2t]}) > \phi(x(t), x(2t), t)$. But clearly this is in contradiction with 
the hypothesis that $x$ is a free time minimizer of $\mathcal{A}_h$.

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\[\square\]
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