Optimal control synthesis for affine nonlinear dynamic systems

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Abstract. The problem of optimal control synthesis of complex nonlinear and multidimensional systems oriented on the building of integrated control systems with the adaptation and self-organization properties is of much current interest. The development of optimization methods employed in control problems demonstrated their effectiveness and has led to the construction of algorithms of approximate solution of optimization problems that significantly extends the class of control problems that can be set. In this work we consider a synthesis of optimal control for affine nonlinear system, in which a special emphasis is placed on the degenerate problems of optimal control. Our approach is based on the extension of the method of minimization of generalized work function by properly transforming the initial nondegenerate problem of optimal control into degenerate problem of optimal control synthesis. The search for the solution is based on the suboptimal strategy that includes the construction of minimizing sequences that converge interval-wise to the optimal solution of the initial problem. We obtain some optimality conditions as well as the necessary and sufficient conditions of existence of optimal solution.

1. Introduction
Optimal control theory offers modern methods regarding the control of systems, and provide a lot of optimization problems, which can be described by optimal control formulations. It plays a significant role in the analysis of control problems arising in different areas such as engineering, automatics, mathematical physics, and physical sciences in general. The optimal control theory provides us with powerful results like dynamic programming [1] or the Pontryagin maximum principle. However, solutions are mainly numerical. Nowadays, the real-life optimal control problems are fully nonlinear. Recently, piecewise (control) affine models has become a relevant and powerful tool in the approximation of general smooth nonlinear systems that usually manage to capture many features of general physical systems, and enable a tractable mathematical analysis. The use of optimal control in the class of linear systems permits a substantial reduction of the computations determining the laws of optimal control. It is also an efficient method for solving nonlinear optimal control problems. Geometric methods and in particular the Lie brackets generated by the fields of vectors defining the nonlinear system represent a remarkable mathematical tool for the control of affine systems [2, 3, 4, 5, 6, 7, 8, 9].

Generally, optimal control theory deals with optimization problems involving a controlled dynamical system. A controlled dynamical system is a dynamical system in which the trajectory can be altered continuously in time by choosing a control parameter \( u(t) \) continuously in time. A (deterministic) controlled dynamical system is usually governed by an ordinary differential
equation. By choosing the value of \( u(t) \), the state trajectory \( x(t) \) can be controlled. The objective of controlling the state trajectory is to minimize a certain cost functional.

In this work we consider a synthesis of optimal control for affine nonlinear system, in which a special emphasis is placed on the degenerate problems of optimal control [10, 11, 12]. Our approach is based on the extension of the method of minimization of generalized work function by properly transforming the initial nondegenerate problem of optimal control into degenerate problem of optimal control synthesis. We obtain some optimality conditions as well as the necessary and sufficient conditions of existence of optimal solution.

2. The formulation of optimal control problem

We consider the class of controlled nonlinear dynamic systems

\[
\dot{x}(t) = f(t, x(t), u(t)), \quad t > 0
\]

where \( x \in \mathbb{R}^n \), \( f \) is a smooth function, and the control \( u = (u_1, u_2, \ldots, u_m) \) is a piecewise continuous function that takes values in an open subset \( U \) of \( \mathbb{R}^m \), \( u \in U \subset \mathbb{R}^m \), and \( u_i \in \mathbb{R}^1 \), \( i = 1, 2, \ldots, m \). The system (1) is defined on an interval \( \tau = [t_0, \tau] \) with the initial condition \( x(t_0) = x_0 \in \mathbb{R}^n \), and the \((x, u)\) represents a trajectory-control pair defined on the interval \( \tau \). For \( x_0 \in \mathbb{R}^n \) and \( \tau > 0 \), a control \( u \in U \) is said to be admissible if the trajectory \( x(\cdot, x_0, u) \) of (1) associated to \( u \) and starting at \( x_0 \) is well-defined on \([t_0, \tau]\). We denote by \( \mathcal{U} \) the class of admissible controls.

The optimal control problem is to find the control function \( u(t) \) that minimize the functional

\[
J(t, x(t), u(t)) = \Phi(x(t_\tau)) + \int_{t_0}^{t_\tau} \varphi(\theta, x(\theta), u(\theta)) \, d\theta,
\]

where \( \Phi \) is a scalar differentiable endpoint function – a function of state of (1) at the endpoint time \( t_\tau \) of the time interval \( \tau \); \( \varphi \) is a scalar function differentiable with respect to all arguments.

The set of pairs \((x(t), u(t))\) that satisfy the above conditions is referred to as the set of the admissible trajectory-control pairs and is denoted by \( \mathcal{P} \).

The functional (2) attains its infimum

\[
\inf_{u} J(t, x_{\text{opt}}(t), u(t)) = J(t, x_{\text{opt}}(t), u_{\text{opt}}(t)) = B(t, x_{\text{opt}}(t))
\]

on the optimal trajectory \( x_{\text{opt}}(t) \) and under the optimal control \( u_{\text{opt}}(t) \) if the solution of the optimal control problem (1)-(2) exists. The functional (3) is called the supporting functional and coincides with the Bellman function \( B \).

The main property of the Bellman function is that at the endpoint time \( t_\tau \) it numerically equals the endpoint function \( \Phi \) of the functional (2)

\[
B(t_\tau, x_{\text{opt}}(t_\tau)) = \Phi(x_{\text{opt}}(t_\tau)),
\]

and satisfies the Bellman equation

\[
\frac{\partial B(t, x_{\text{opt}})}{\partial t} + \frac{\partial B(t, x_{\text{opt}})}{\partial x_{\text{opt}}} f(t, x_{\text{opt}}, u_{\text{opt}}) + \varphi(t, x_{\text{opt}}, u_{\text{opt}}) = 0
\]

otherwise.

The reduction of the above optimal control problem to the degenerate case is obtained by redefining the functional (3) such that the conditions (4) and (5) would hold for any arbitrary variation \( \delta u \) of control \( u(t) \). In other words, we impose auxiliary conditions under which the
pair \( (x_{\text{opt}}(t), u_{\text{opt}}(t) + \delta u(t)) \), where \( x_{\text{opt}}(t) \) is the optimal trajectory and \( u_{\text{opt}}(t) + \delta u(t) \) is any control, provides the optimal solution to the control problem.

By taking the total differential with respect to \( t \) in (3) and adding the term \( \varphi \left( t, x_{\text{opt}}, u_{\text{opt}} \right) \), we obtain

\[
\inf_u \left( \dot{J}(t, x_{\text{opt}}(t), u(t)) + \varphi \left( t, x_{\text{opt}}, u_{\text{opt}} \right) = \dot{B}(t, x_{\text{opt}}(t)) + \varphi \left( t, x_{\text{opt}}, u_{\text{opt}} \right). \right. \]

(6)

Since

\[
\dot{B}(t, x_{\text{opt}}(t)) = \frac{\partial B(t, x_{\text{opt}}(t))}{\partial t} + \frac{\partial B(t, x_{\text{opt}}(t))}{\partial x_{\text{opt}}} f(t, x_{\text{opt}}(t), u_{\text{opt}}(t)),
\]

(7)

it follows that the right-hand side of (6) contains in fact the left-hand side of (7).

The requirement that the Bellman equation is satisfied, leads from (7) to

\[
\inf_u \left( \dot{J}(t, x_{\text{opt}}(t), u(t)) + \varphi \left( t, x_{\text{opt}}, u_{\text{opt}} \right) = 0. \right. \]

(8)

From the total differential of the functional \( J(t, x_{\text{opt}}(t), u(t)) \) in (8), we get

\[
\inf_u \left( \frac{\partial J(t, x_{\text{opt}}, u)}{\partial t} + \frac{\partial J(t, x_{\text{opt}}, u)}{\partial x_{\text{opt}}} f(t, x_{\text{opt}}, u) + \varphi \left( t, x_{\text{opt}}, u_{\text{opt}} \right) = 0. \right. \]

(9)

Expanding the functions \( f \) and \( \varphi \) with respect to \( \delta u \) into Taylor series in the neighborhood of optimal trajectory \( x_{\text{opt}}(t) \) and considering only linear terms, we find from (9)

\[
\inf_u \left( \left( \frac{\partial \varphi(t, x_{\text{opt}}, u)}{\partial u} + \frac{\partial J(t, x_{\text{opt}}, u)}{\partial x_{\text{opt}}} \frac{\partial f(t, x_{\text{opt}}, u)}{\partial u} \right) \delta u + \mathcal{O}(\delta^2 u) \right) = 0, \]

(10)

where \( \mathcal{O}(\delta^2 u) \) is the sum of higher order terms in the expansion of \( f \) and \( \varphi \) into Taylor series.

The equation (10) leads to the following two cases:

(i) the variation \( \delta u \) around \( x_{\text{opt}} \) is equal to zero, which means that the control that corresponds to the extremum of the functional is used;

(ii) the coefficients of expansion are equal to zero, which results in the independence of the value of the functional along the optimal trajectory on the control chosen.

The former case is not of much interest, while the latter one corresponds to the degenerate control problem and requires certain extension of control problem statement.

Assume that in (10) the relation

\[
\frac{\partial \varphi(t, x_{\text{opt}}, u)}{\partial u} + \frac{\partial J(t, x_{\text{opt}}, u)}{\partial x_{\text{opt}}} \frac{\partial f(t, x_{\text{opt}}, u)}{\partial u} = 0
\]

(11)

holds true for any control \( u \in \mathcal{U} \).

For \( u = u_{\text{opt}} \) the equation (11), in view of (3), can be written as

\[
\frac{\partial \varphi(t, x_{\text{opt}}, u)}{\partial u} + \frac{\partial B(t, x_{\text{opt}}(t))}{\partial x_{\text{opt}}} \frac{\partial f(t, x_{\text{opt}}, u)}{\partial u} = 0.
\]

(12)

The equation (12) provides the necessary condition of local optimality of system (1) and determines the optimal control of the given control problem. The condition given by the equation (12) means that the equation (9) minimized with respect to the control vector \( u \) is equivalent to the Bellman equation (5) for every \( u \in \mathcal{U} \).
Hence, from (3), (5) and (12) we get the following representation for the functional $J$, which holds for any admissible control $u \in \mathcal{U}$

$$J(t, x_{\text{opt}}(t), u(t)) = \Phi(x(t_\tau)) + \int_{t_0}^{t_\tau} R(\theta, x_{\text{opt}}(\theta), u(\theta)) d\theta, \quad (13)$$

where

$$R(t, x_{\text{opt}}, \delta u) = \varphi(t, x_{\text{opt}}, u_{\text{opt}}) + \frac{\partial \varphi(t, x_{\text{opt}}, u)}{\partial u} \delta u$$

is a scalar function that redefines the integrand of supporting functional up to the integrand of the functional of the given control problem.

It should be noted that sufficient optimality conditions of approximate control synthesis are stated in terms of an auxiliary function possessing the property of Lyapunov function [13]. However, our approach is based on the introduction of supporting functional having the property of the Bellman function – the Cauchy problem (4)-(5), as well as on the linearization of system (1) with respect to the \textit{apriori} unknown vector-function of optimal control. At the same time, the functional (13) is turned out to be semidefinite and is considered to be nonclassical functional as it contains the vector $u_{\text{opt}}$. The resolution of uncertainty in the system and in the functional consists of two stages:

(i) the initial nondegenerate problem of control synthesis is redefined to become singular in order to include the limiting control functions to the set of admissible controls $\mathcal{U}$; it should be done in such a way that the extended control problem would contain the optimal solution; for $u = u_{\text{opt}}$ the control problem formulations coincide.

(ii) search for the proper control function, appropriate for the initial control problem conditions, with respect to the vector-function $u_{\text{opt}}$ that resolves the uncertainty as the system (1) is functioning.

The existence of supporting functional and uniqueness of solution of the Cauchy problem (4)-(5), as well as of the initial control synthesis problem (1)-(2) is not uniquely determined. Moreover, special control modes, $\mathcal{U} \subset \mathbb{R}^m$, and impulsive control modes, $\mathcal{U} = \mathbb{R}^m$, arise in the degenerate control problem. For these modes, the regular optimality conditions, such as Jacobi condition and Legendre-Clebsch condition are turned out to be trivially satisfied and thus are noneffective. Therefore, the transition from the initial control synthesis problem to the degenerate control problem makes it necessary to introduce additional optimality conditions for the extended set of admissible controls. The investigation of degenerate optimal control problem leads to more complicated and deep necessary optimality conditions, which does not remove the problem of existence and uniqueness of control synthesis solution.

Nevertheless, for the whole class of optimal control problems, the existence and uniqueness problem can be solved by means of replacing the Cauchy problem (4)-(5) by a simpler sufficient optimality condition in the form of Lyapunov equation with the boundary condition given by the endpoint function $\Phi$ of the functional (2). Formally, this replacement is achieved by a certain reformulation of the initial control synthesis problem by introducing the additional isoperimetric condition [14] into the functional (2).

3. Analytic solution of optimal control problem

Consider the optimal control synthesis for control affine nonlinear system

$$\dot{x}(t) = \psi(t, x(t)) + \mu(t, x(t))u(t) \quad (14)$$
with the integrand in the functional (2) to be
\[ \varphi(t, x, u) = Q(t, x) + L(t, u, u_{opt}) \]  

(15)

where \( x \) is an \( n \)-dimensional state vector, \( x \in \mathbb{R}^n \); \( u \) is an \( m \)-dimensional control vector, \( u \in \mathbb{R}^m \); \( t \) is time in the given time interval; \( \psi \) and \( \mu \) are differentiable and continuous with respect to \( x \) vector function and matrix function respectively; \( Q \) is a scalar function that determines conditions imposed on trajectory \( x(t) \) on the interval \( \tau = [t_0, t_\tau] \); \( L(t, u, u_{opt}) \) is a scalar payoff function; \( u_{opt} \) is a locally optimal \textit{apriori} unknown control vector.

The degenerate control problem is to define such a structure of the function \( L(t, u, u_{opt}) \) for which, from one side, instead of the Cauchy problem (4)-(5) one can use the sufficient optimality condition in the form of the Lyapunov equation
\[ \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x_{opt}} \psi + Q = 0 \]  

(16)

with the boundary condition
\[ \Phi(t_\tau, x_{opt}(t_\tau)) = \Phi(x_{opt}(t_\tau)), \]  

(17)

and, from the other side, there exists the possibility of choosing the appropriate control strategy with respect to vector \( u_{opt} \), which may include also iterative algorithms.

The latter condition means that the solution of equation (16) should coincide with the Bellman function \( \Phi(t, x_{opt}) = B(t, x_{opt}) \), and supporting functional should be the optimal Lyapunov function on the given time interval for any admissible control vector. For this reason, the direct usage of the case \( u = u_{opt} \) should be excluded from the search of optimal control that make use of the solution of the Cauchy problem (16)-(17).

For the problem (14)-(15), which slightly simplifies the initial control synthesis problem, the necessary condition of local optimality (12) would be
\[ \frac{\partial L(t, u, u_{opt})}{\partial u} + \frac{\partial B(t, x_{opt})}{\partial x_{opt}} \mu(t, x_{opt}) = 0. \]  

(18)

In this case, the equation (9) minimized with respect to control vector \( u \) can be replaced by the Lyapunov equation (16) if:
(i) the condition of local optimality (18) is satisfied;
(ii) the additional condition
\[ L(t, u_{opt}, u_{opt}) = \frac{\partial L(t, u, u_{opt})}{\partial u} u_{opt} \]  

(19)

is provided.
(iii) for \( u = u_{opt} \) it holds that
\[ \frac{\partial B(t, x_{opt})}{\partial x_{opt}} \mu(t, x_{opt}) u_{opt} + L(t, u_{opt}, u_{opt}) = 0. \]  

(20)

The equations (18)-(20) when \( \Phi(t, x_{opt}) = B(t, x_{opt}) \) define the structure of the function \( L(t, u, u_{opt}) \) for the sufficient optimality condition in the form of Lyapunov equation.

The scalar function \( L(t, u, u_{opt}) \) can normally be given as
\[ L(t, u, u_{opt}) = L_1(t, u) + L_2(t, u_{opt}), \]  

(21)

where \( L_1 \) and \( L_2 \) are certain quadratic forms of vector functions \( u \) and \( u_{opt} \).

The choice of scalar functions \( L_1 \) and \( L_2 \) determines different optimal control problem statements, which we study below.

Let us consider the case \( L_1(t, u) = 0.5u^T M^{-1} u \) and \( L_2(t, u_{opt}) = 0 \). Then we have
**Theorem 3.1.** Let \( L_1(t, u) = 0.5u^T M^{-1} u \) and \( L_2(t, u_{\text{opt}}) = 0 \), where \( M \) is a positively definite matrix of constant coefficients. Then the equation (18) gives the condition for the infimum of the functional (2)

\[
u = u_{\text{opt}} = -M \mu^T \frac{\partial B^T}{\partial x_{\text{opt}}}
\]

and the equations (19) and (20) do not hold.

**Proof.** The statement of the theorem can be directly verified by substitution of scalar function \( L(t, u_{\text{opt}}, u_{\text{opt}}) = L_1(t, u) \) into equations (19) and (20).

We prove the theorem by contradiction. Suppose that equation (20) holds for some control vector function \( u = u^* \) that satisfies the solution (22) and for which the functional (2) attains locally its infimum. Then, equation (9) minimized with respect to control vector \( u \) has the form

\[
\frac{\partial B(t, x_{\text{opt}})}{\partial t} + \frac{\partial B(t, x_{\text{opt}})}{\partial x_{\text{opt}}} \psi(t, x_{\text{opt}}) + Q(t, x_{\text{opt}}) + 0.5(u^*)^T M^{-1} u^* - u_{\text{opt}}^T M^{-1} u_{\text{opt}} = 0.
\]

Whence it follows that Lyapunov equation (16) holds for \( u^* = 0 \) or \( u^* = 2u_{\text{opt}} \). This contradicts the condition \( u = u_{\text{opt}} \) in the formula (22). \( \square \)

Therefore, the sufficient optimality condition in the form of Lyapunov equation in the above formulation of optimal control problem does not hold. This means that the control functions, under which the functional (2) attains its infimum at some arbitrary time moment from the given interval, do not coincide with the optimal control functions obtained using the necessary condition of local optimality (18). Hence, the functional of the initial control problem is required to be redefined up to nonclassical functional.

Now we put \( L_1(t, u) = 0.5u^T M^{-1} u \) and \( L_2(t, u_{\text{opt}}) = 0.5u_{\text{opt}}^T M^{-1} u_{\text{opt}} \). The term \( L_2(t, u_{\text{opt}}) = 0.5u_{\text{opt}}^T M^{-1} u_{\text{opt}} \) serves as the additional isoperimetric condition

\[
\int_{t_0}^{t_f} L_2(\theta, u_{\text{opt}}) d\theta = C
\]

introduced to the functional (2), which thereafter takes the form of generalized work function

\[
J = \Phi(x(t^T)) + \int_{t_0}^{t_f} \left( Q(\theta, x) + 0.5u^T M^{-1} u + 0.5u_{\text{opt}}^T M^{-1} u_{\text{opt}} \right) d\theta.
\]

The solution to the above control problem is given by the formula (22), but it also holds when \( \Phi(t, x_{\text{opt}}) = B(t, x_{\text{opt}}) \).

We have the following

**Theorem 3.2.** The conditions (18), (19), and (20) hold true if and only if the controls for which the generalized work function (24) attains locally its infimum are the optimal controls.

**Proof.** The proof is similar to that of previous theorem. Suppose that equation (20) holds for some control vector function \( u = u^* \) that satisfies the solution (22) and that differs from the optimal control vector. Then, for the equation (9) minimized with respect to control vector \( u \) we get

\[
\frac{\partial B(t, x_{\text{opt}})}{\partial t} + \frac{\partial B(t, x_{\text{opt}})}{\partial x_{\text{opt}}} \psi(t, x_{\text{opt}}) + Q(t, x_{\text{opt}}) + 0.5(u^* - u_{\text{opt}})^T M^{-1} (u^* - u_{\text{opt}}) = 0.
\]

Comparing the conditions on control in the formulas (22) and (25), we obtain that the sufficient optimality condition in the form of Lyapunov equation (16) holds only for \( u = u_{\text{opt}} \). \( \square \)
The condition \( u = u_{\text{opt}} \) is a very strong requirement imposed on the generalized work function. To weaken this requirement and introduce the limiting control functions we use nonquadratic conditions that we impose on scalar functions \( L_1 \) and \( L_2 \) by means of regularizing parameters, the powers of control vectors \( u \) and \( u_{\text{opt}} \) or their quadratic forms [14]. To this end, we redefine the functional (24) and consider the degenerate case as follows. We define a function \( L(t,u,u_{\text{opt}}) \) such that the Lyapunov equation (16) is satisfied for any admissible control vector \( u \). To do this, we introduce the additional term \(-0.5(u^* - u_{\text{opt}})^T M^{-1}(u^* - u_{\text{opt}})\) to the left-hand side of equation (25). Since the vector function \( u = u^* \) is assumed to be arbitrary, the function \( L(t,u,u_{\text{opt}}) \) is redefined by adding the term \( L_3(t,u,u_{\text{opt}}) = -0.5(u - u_{\text{opt}})^T M^{-1}(u - u_{\text{opt}}) \) in the control problem with generalized work function (24). Then the function \( L(t,u,u_{\text{opt}}) \) has the form

\[
L(t,u,u_{\text{opt}}) = L_1(t,u) + L_2(t,u_{\text{opt}}) + L_3(t,u,u_{\text{opt}}) = 0.5u^T M^{-1}u + 0.5u_{\text{opt}}^T M^{-1}u_{\text{opt}} - 0.5(u - u_{\text{opt}})^T M^{-1}(u - u_{\text{opt}})
\]

**Theorem 3.3.** Let \( L(t,u,u_{\text{opt}}) = u^T M^{-1}u_{\text{opt}} \). Then the equation (18) gives the condition for the infimum of the functional (2)

\[
u_{\text{opt}} = -M\mu^T \frac{\partial \Phi^T}{\partial x_{\text{opt}}}
\]

for all admissible control vector \( u \), and the formulas (19) and (20) hold true.

**Proof.** The proof of the statement is analogous to those of previous theorems.

Hence, the functional for which the Lyapunov equation (16) is satisfied for any admissible control \( u \), is given by

\[
J = \Phi(x(t^T)) + \int_{t_0}^{t^T} (Q(\theta,x) + u^T M^{-1}u_{\text{opt}}) d\theta.
\]

The functional (27) is referred to as weighted generalized work criterion and has the similar to generalized work function (24) meaning. The term \( u^T M^{-1}u_{\text{opt}} = (u_{\text{opt}} + \delta u)^T M^{-1}u_{\text{opt}} \) characterizes the dissipative kinetic energy of optimal process (11) – \( L_3 \) is a kinetic energy of dissipation. The integral of this term determines the generalized work of dissipative system.

4. **Necessary and sufficient optimality conditions**

In the optimal control problem we have formulated the degeneracy of optimality conditions is due to the linearity of controls in equation (11) and in the functional (27). The meaning of this degeneracy is that it reveals itself as the result of transition from nondegenerate optimal control problem with generalized work function (24) to singular problem, for which the regular optimality conditions (Jacobi, Legendre-Clebsch) still remain valid. So, unlike the ordinary formulation of degenerate control problem, in which the singular curve is required to be found, in our formulation it is known – it is the optimal trajectory that can be found using (26). The necessary and sufficient conditions of existence of optimal solution of control problem with the functional (27) to be minimized are given in the following proposition.
Theorem 4.1. For the existence of optimal control vector function (26) that delivers the infimum to the functional (27) of degenerate optimal control problem for all controls $u \in U \subset \mathbb{R}^m$, it is necessary and sufficient that the following conditions satisfy

$(i)$

$$\dot{\Phi}_{xx} + \Phi_{xx}\psi_x + \psi^T_x \Phi_{xx} + Q_{xx} = 0 \quad (28)$$

with the boundary conditions $\Phi_{xx}(t^T) = \Phi_x$ if $x(t^T) \neq 0$ and $\Phi_{xx}(t_0) = 0$ if $x(t^T) = 0$.

$(ii)$

$$Q_{xx} = Q^T_{xx} \geq 0. \quad (29)$$

$(iii)$

$$\Phi_{xx}(\psi_x \mu - \dot{\mu}) \geq 0. \quad (30)$$

$(iv)$

$$\mu^T(Q_{xx} \mu + 2\Phi_{xx}(\psi_x \mu - \dot{\mu})) > 0. \quad (31)$$

Proof. The theorem is proved using integral transformations and substitution technique [15], with the assumption that the functions $\Phi, Q, \psi, \mu$ are differentiable with respect to $x$. \qed

5. Conclusion

The proposed degenerate optimal control problem formulation allows us to study multiconnected, multicomponent control objects, and involves the multivaricance of control goal attainability. The optimal control problem with the functional (27) allows one to include limiting control functions, $U \subset \mathbb{R}^m$, using the quadratic conditions and choose the appropriate control strategy with respect to the optimal solution (26) of control problem. Another advantage is that one can develop iterative algorithms of approximate optimal control synthesis.

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