Decay of Gaussian correlations in local thermal reservoirs

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Abstract
In this paper we examine the decay of quantum correlations for the radiation field in a two-mode squeezed thermal state in contact with local thermal reservoirs. Two measures of the evolving quantum correlations between the modes are compared: the entanglement of formation and the quantum discord. We derive analytic expressions of the entanglement-death time in two special cases: when the reservoirs for each mode are identical, as well as when a single reservoir acts on the first mode only. In the latter configuration, we show that all the pure Gaussian states lose their entanglement at the same time, determined solely by the field-reservoir coupling. Also investigated is the evolution of the Gaussian quantum discord for the same choices of thermal baths. We notice that the discord can increase in time above its initial value in a special situation, namely, when the input state is mixed, and local measurements are performed on the attenuated mode. This enhancement of discord is stronger for zero-temperature reservoirs and increases with the input degree of mixing.

Keywords: entanglement, discord, dissipation, Gaussian states

1. Introduction

The extent to which the non-classical properties of one-mode field states survive in the presence of noise and losses has been investigated since the early years of quantum optics [1, 2]. A routine operation like the transmission of light beams through an optical fibre could produce a substantial degradation of their non-classical properties. As an example, it was found that squeezing properties are altered by an admixture with thermal noise and disappear completely for values of thermal mean photon occupancy exceeding the threshold 1/2 [3]. From a more recent quantum-information perspective, a lot of work is concentrated on correlations such as entanglement and discord in multi-partite systems [4–6]. While correlations associated with entanglement [4] are defined in connection with global transformations of bipartite quantum states, the concept of quantum discord arises from local (marginal) actions and measurements performed on one subsystem [8, 9]. Its definition contains an optimization over the set of all one-party measurements, which in the case of mixed states could be a challenging problem. Note that in the pure-state case, entanglement and discord coincide, and therefore they measure the total amount of correlations between parties. In the mixed-state case, quantum discord is a measure of quantumness whose relation to entanglement is not a simple one. A survey of recent progress and applications of classical and quantum correlations quantified by quantum discord and other measures can be found in references [5, 6].

When states of a two-party quantum system need to interact with a noisy channel, a drastic modification of their quantum correlations is expected to occur [11, 12]. For instance, it was found that quantum and classical correlations for a system of two qubits evolving in Markovian dephasing channels can display different dynamics [13]. Quite recently, the effect of local noisy channels on quantum correlations in finite-dimensional quantum systems was investigated [14, 15]. It was found that while entanglement does not increase under local channels, other correlations can become larger when the input state is not pure. In continuous-variable settings, a similar behaviour was recently noticed for the...
Gaussian discord of two-mode mixed states under single-mode Gaussian dissipative channels [16]. Local Gaussian thermal and phase-sensitive reservoirs modify the entanglement properties of two-mode Gaussian states, as is interestingly pointed out in references [17–19]. Evolutions of more general Gaussian correlations were also investigated in reference [20–23].

In this work we analyze the decay of quantum correlations of the field in a two-mode Gaussian state due to the interaction of the modes with separate thermal baths. We focus on two measures of quantum correlations, namely, the entanglement of formation (EF) and the quantum discord, and evaluate them for a damped two-mode squeezed thermal state (STS) [24, 25]. Our choice of this important particular class of Gaussian states is motivated by recent result [26] that for a two-mode STS, the exact discord according to its original definition [8, 9] is achieved with an optimal measurement which is Gaussian. Consequently, what was called the Gaussian discord and was derived in reference [29, 30] is actually the exact discord. On the other hand, in an interesting paper [31] it is shown that for a two-mode STS, a Gaussian character of the discord implies that the EF is also Gaussian. Accordingly, the Gaussian EF written explicitly in our paper [32] is equally an exact result. As such, when dealing with a dissipative evolution that preserves the STSs, we have the rare privilege to fully describe the decay of two types of correlations by analytic means.

Our paper is structured as follows. In section 2 we recapitulate several properties of a two-mode STS: the covariance matrix (CM), Simon’s separability criterion [27], the EF [32] as a measure of inseparability, and the quantum discord as derived in references [29, 30]. In section 3 we present our main results. In particular, we evaluate them for a damped two-mode squeezed thermal state (2.6) with the action of a two-mode squeeze operator,

$$\hat{S}_{12}(r, \phi) = \exp \left[ i \sqrt{\frac{\pi}{2}} \left( e^{i\phi} \hat{a}_1 \hat{a}_2 - e^{-i\phi} \hat{a}_1 \hat{a}_2^+ \right) \right]$$

on a two-mode thermal state with the mean photon occupancies \(\bar{n}_1\) and \(\bar{n}_2\):

$$\hat{\rho}_{ST} = \hat{S}_{12}(r, \phi) \hat{\rho}_T (\bar{n}_1, \bar{n}_2) \hat{S}_{12}^+(r, \phi).$$

Its CM has the following block structure [24, 25]:

$$\mathcal{C} = \begin{pmatrix} b_1 I_2 & C \\ C & b_2 I_2 \end{pmatrix},$$

where

$$b_1 > \frac{1}{2}, \quad b_2 > \frac{1}{2}.$$

In equation (2.2), \(I_2\) denotes the 2 \(\times\) 2 identity matrix, and \(C\) is the symmetric 2 \(\times\) 2 matrix

$$C = \begin{pmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{pmatrix},$$

(\(c > 0\)).

Recall that the CM of a two-mode STS, equations (2.2) and (2.3), has the standard-form parameters [24, 25]:

$$b_1 = \left( \bar{n}_1 + \frac{1}{2} \right) \cosh(r)^2 + \left( \bar{n}_2 + \frac{1}{2} \right) \sinh(r)^2,$$

$$b_2 = \left( \bar{n}_1 + \frac{1}{2} \right) \sinh(r)^2 + \left( \bar{n}_2 + \frac{1}{2} \right) \cosh(r)^2,$$

$$c = (\bar{n}_1 + \bar{n}_2 + 1) \sinh(r) \cosh(r).$$

We emphasize that in many applications one can take advantage of a formal definition of a two-mode STS as being an undisplaced and unscaled two-mode Gaussian state described by three standard-form parameters: \(b_1 > \frac{1}{2}, \ b_2 > \frac{1}{2}, \ c > 0\). If \(b_1 \geq b_2\), these parameters must satisfy the uncertainty inequality

$$\left( b_1 + \frac{1}{2} \right) \left( b_2 - \frac{1}{2} \right) - c^2 \geq 0.$$

If \(b_1 < b_2\), one has to interchange the parameters \(b_1\) and \(b_2\) in equation (2.5) [25]. The standard form of the CM is given by equation (2.2) with the 2 \(\times\) 2 matrix \(C\) written for \(\phi = 0\), i.e., becoming proportional to the Pauli matrix \(\sigma_z\): \(c = c \sigma_z\). Within this formal treatment, equation (2.1) and its companions, equations (2.2)–(2.4), represent a parametrization of a two-mode STS with a clear experimental relevance.

It is known that, according to Williamson’s theorem [33], the CM of a two-mode Gaussian state can be diagonalized by a symplectic transformation. We get thus an important ingredient in describing the state, namely, the symplectic eigenvalues of the CM. For a two-mode STS they are [25]:

$$\kappa = \frac{1}{2} \left( b_1 + b_2 \right)^2 - 4c^2 \pm (b_1 - b_2).$$

In the parametrization (2.4), we get \(\kappa = \bar{n}_{1,2} + \frac{1}{2}\).

It is worth mentioning Simon’s separability criterion for two-mode Gaussian states [27]. Simon has proven that preservation of the non-negativity of the density matrix under partial transposition is not only a necessary [28] but also a
sufficient condition for the separability of two-mode Gaussian states [27]. Accordingly, a two-mode Gaussian state is separable when the condition \( \kappa^-_{\text{pt}} \geq \frac{1}{2} \) is met. We have denoted by \( \kappa^\pm_{\text{pt}} \) the symplectic eigenvalues of the CM corresponding to the partial transpose of the density matrix. For a two-mode STS one finds:

\[
\kappa^\pm_{\text{pt}} = \frac{1}{2} \left[ b_1 + b_2 \pm \sqrt{(b_1 - b_2)^2 + 4c^2} \right]. \tag{2.7}
\]

A simplified form of the separability condition for a two-mode STS that we shall use in what follows reads [24, 25]:

\[
\left( b_1 - \frac{1}{2} \right) \left( b_2 - \frac{1}{2} \right) - c^2 \geq 0. \tag{2.8}
\]

In the usual parametrization (2.4), the separability condition (2.8) for a two-mode STS reduces to an inequality that the squeeze parameter \( r \) has to fulfill [24, 25]:

\[
r \leq r_s, \quad r_s := \ln \left[ \frac{\sqrt{n_1 + 1} (n_2 + 1) + \sqrt{n_1 n_2} \sqrt{n_2 + 1}}{\sqrt{n_1 + n_2 + 1}} \right]. \tag{2.9}
\]

Recall the specific property that any separable two-mode STS is also a classical state in the quantum-optical meaning, i.e., it has a well-behaved Glauber–Sudarshan \( P \) representation. Before proceeding, let us note that, apart from the vacuum state, the only undisplaced and unscaled pure two-mode Gaussian states are the squeezed vacuum ones:

\[
\hat{\rho}_{SV} = |\psi_{SV}\rangle \langle \psi_{SV}|, \quad |\psi_{SV}\rangle := \hat{S}_{12}(r, \phi) |0, 0\rangle. \tag{2.10}
\]

The pure states (2.10) belong to the set of entangled two-mode STSs and are characterized by the identities: \( b_1 = b_2 = b, \ c > 0, \) and \( b^2 - c^2 = \frac{1}{4} \). Their standard-form parameters (2.4) read \( b = \frac{1}{\sqrt{3}} \cosh(2r), \ c = \frac{1}{\sqrt{3}} \sinh(2r) \) and, while the symplectic eigenvalues (2.6) and (2.7) become \( \kappa^\pm = \frac{1}{2} \) and, respectively, \( \kappa^-_{\text{pt}} = b \pm c \).

In what follows, we provide an overview of two measures of quantum cross-correlations for a two-mode STS, namely, the EF and the quantum discord. The EF is defined as an optimization over all pure state-decompositions of the given state [35]:

\[
E_F(\hat{\rho}) := \inf \left\{ \sum_k p_k E\left( |\psi_k\rangle \langle \psi_k| \right) \right\},
\]

\[
\hat{\rho} = \sum_k p_k |\psi_k\rangle \langle \psi_k|. \tag{2.11}
\]

In the expression above, we have denoted by \( E\left( |\psi_k\rangle \langle \psi_k| \right) \) the relative entropy of entanglement of the pure bipartite state \( |\psi_k\rangle \langle \psi_k| \). We here focus on the case of a two-mode STS \( \hat{\rho}_{ST} \) and recall that an expression for its EF could be obtained when restricting the optimization in equation (2.11) to Gaussian pure-state decompositions only. For further convenience, we introduce the entropic function

\[
h(x) := \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) - \left( x - \frac{1}{2} \right) \ln \left( x - \frac{1}{2} \right), \quad x \geq \frac{1}{2}. \tag{2.12}
\]

It was proven that the Gaussian EF can be expressed in terms of the function \( h(x) \) as:

\[
E_F(\hat{\rho}_{ST}) = h(x_m). \tag{2.13}
\]

In equation (2.13), the parameter \( x_m \) is given in terms of the entries of the CM [32]:

\[
x_m = \frac{(b_1 + b_2) (b_1 b_2 - c^2 + \frac{1}{4}) - 2c \sqrt{D}}{(b_1 + b_2)^2 - 4c^2}. \tag{2.14}
\]

Here \( D := (b_1 b_2 - c^2)^2 - \frac{1}{4} \left( b_1^2 + b_2^2 - 2c^2 \right) + \frac{1}{16} \geq 0 \) is the main symplectic invariant of a two-mode STS: its non-negativity results from the Robertson–Schrödinger uncertainty relation. In particular, for any two-mode squeezed vacuum state, \( D = 0 \), so that \( x_m = b \), and thus its EF is equal to the common von Neumann entropy of its reduced one-mode thermal states, i.e., \( E_F(\hat{\rho}_{SV}) = h(b) \).

The difference between two classically equivalent definitions of the mutual information provides another measure of the total amount of quantum correlations in a two-party quantum state, called discord [7–9]. For an arbitrary composite quantum system, we start to examine a bipartite state \( \hat{\rho}_{AB} \) by writing down the quantum mutual information between its subsystems, \( A \) and \( B \):

\[
I(\hat{\rho}_{AB}) := S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}_{AB}). \tag{2.15}
\]

In equation (2.15), \( \hat{\rho}_{AB} = \text{Tr}_{B}(\hat{\rho}_{AB}) \) denote the reduced states of the two parties, and \( S(\hat{\rho}) \) stands for the von Neumann entropy of the state \( \hat{\rho} \): \( S(\hat{\rho}) := -\text{Tr}[\hat{\rho} \ln(\hat{\rho})] \). Another quantum analogue of the classical mutual information depends on the effects of the quantum measurements made on a single subsystem, say, \( B \), felt by the other one, \( A \). Let us consider a general measurement performed on subsystem \( B \) and described by a \( b \) positive-operator-valued measure \( \text{POVM} \) : \( \{ E_k^B \}, \ E_k^B = \langle \hat{M}_k^B \rangle \hat{M}_k^B \geq 0, \ \sum_k \hat{E}_k^B = \hat{1}_B \). Here the symbols \( \hat{M}_k^B \) and \( \hat{E}_k^B \) designate the corresponding measurement operators and, respectively, the POVM elements [10]. Any selective measurement made on \( B \) induces a disturbance on subsystem \( A \), modifying the state \( \hat{\rho}_A \) if one obtains the outcome \( j \) of such a marginal measurement, then the initial state \( \hat{\rho}_A \) collapses into the post-measurement reduced state:

\[
\hat{\rho}_{A}\hat{E}_j^B = \frac{1}{p_j^B} \text{Tr}_B \left[ \hat{\rho}_{AB} \left( \hat{1}_A \otimes \hat{E}_j^B \right) \right]. \tag{2.16}
\]

In equation (2.16), \( p_j^B \) is the probability of the outcome \( j \); \( p_j^B = \text{Tr}_B \left[ \hat{\rho}_{AB} \left( \hat{1}_A \otimes \hat{E}_j^B \right) \right] \). The quantum conditional
entropy, given the non-selective measurement \( \{ \hat{E}_j \} \), is defined as a convex sum of the von Neumann entropies of the post-measurement reduced states (2.16), which is taken over all the possible outcomes:

\[
S(\hat{\rho}_{A|\{\hat{E}_j^*\}}) = \sum_j p_j^B S(\hat{\rho}_{A|\hat{E}_j^*}).
\]

(2.17)

Since \( \hat{\rho}_{A} = \sum_j p_j^B \hat{\rho}_{A|\hat{E}_j^*} \), the difference \( S(\hat{\rho}_{A}) - S(\hat{\rho}_{A|\hat{E}_j^*}) \) is non-negative, owing to the concavity of the von Neumann entropy [10]. This difference between the entropy of a mixture of post-measurement reduced states (2.16) and the average entropy of such a state is mainly due to a loss of ignorance by creation of classical correlations [5, 9]. Accordingly, the maximal information gained about subsystem \( A \) from all the possible one-party measurements performed on subsystem \( B \) has the expression [7–9]:

\[
J(\hat{\rho}_{AB})|_{\hat{E}_j^*} = S(\hat{\rho}_{A}) - \min\{\hat{E}_j^*\} S(\hat{\rho}_{A|\hat{E}_j^*}) \geq 0.
\]

(2.18)

The above maximization is necessary due to what is called the quantumness of measurements, i.e., the non-commutativity of the measurement operators, as well as of their residual states [36]. The quantum \( A \)-discord is then defined as the difference between the von Neumann mutual information (2.15) and the measurement-induced quantum mutual information (2.18) [7, 8]:

\[
D_A(\hat{\rho}_{AB}) := I(\hat{\rho}_{AB}) - J(\hat{\rho}_{AB})|_{\hat{E}_j^*} \geq 0.
\]

(2.19)

It is therefore a one-end measure of the total amount of quantum correlations of the two-party state \( \hat{\rho}_{AB} \). Similarly, the quantum \( B \)-discord, which takes account of all the local quantum measurements performed on subsystem \( A \), is

\[
D_B(\hat{\rho}_{AB}) := I(\hat{\rho}_{AB}) - J(\hat{\rho}_{AB})|_{\hat{E}_j^*} \geq 0.
\]

(2.20)

Quite recently, the above-defined discord [7–9] has been calculated for two-mode Gaussian states under the approach of restricting the set of all one-party quantum measurements to the Gaussian ones [29, 30]. We were thus provided with an analytic formula for what is called the Gaussian discord. Moreover, according to reference [26], at least for the states analyzed here, namely, the two-mode STSs, the Gaussian discord is the exact discord. Thus the quantum discords (2.19) and (2.20) turn out to have very simple expressions in terms of one-mode von Neumann entropies:

\[
D_1(\hat{\rho}_{ST}) = h(b_2) - h(\kappa_+ - h(\kappa_-) + h(y)
\]

\[
D_2(\hat{\rho}_{ST}) = h(b_1) - h(\kappa_+) - h(\kappa_-) + h(z).
\]

(2.21)

Here \( h \) is the entropic function (2.12), and \( \kappa_+, \kappa_- \) are the symplectic eigenvalues given in equation (2.6). Besides, we have introduced the notations:

\[
y := b_1 - \frac{c^2}{b_2 + \frac{1}{2}},
\]

\[
z := b_2 - \frac{c^2}{b_1 + \frac{1}{2}}.
\]

(2.22)

Remark that, for symmetric two-mode STSs \((b_1 = b_2 =: b)\), the identity \( y = z \) holds, and, therefore, \( D_1(\hat{\rho}_{ST}) = D_2(\hat{\rho}_{ST}) \). Furthermore, for pure two-mode Gaussian states, we get \( y = z = \frac{c}{2} \) and \( D_1(\hat{\rho}_{ST}) = D_2(\hat{\rho}_{ST}) = h(b) \), i.e., the discord and the entanglement coincide, as expected [5].

### 3. Evolution of a two-mode state with two local thermal reservoirs

We consider an arbitrary two-mode field state having the annihilation operators \( \hat{a}_1, \hat{a}_2 \), and the density operator \( \hat{\rho} \). Each mode is in contact with a local thermal bath. We denote the mean photon occupancies of the two thermal reservoirs by \( n_j \) \((j = 1, 2)\), respectively, and the corresponding damping rates by \( \gamma_j \) \((j = 1, 2)\). In the interaction picture, the quantum optical master equation that describes this type of coupling is

\[
\frac{\partial \hat{\rho}}{\partial t} = \frac{\gamma_1}{2} \left( 2\hat{a}_1 \hat{\rho} \hat{a}_1 - \hat{a}_1 \hat{a}_1 \hat{\rho} - \hat{a}_1 \hat{a}_1 \hat{\rho} \right) + \frac{n_1}{2} \left( 2\hat{a}_1 \hat{\rho} \hat{a}_1 - \hat{a}_1 \hat{a}_1 \hat{\rho} - \hat{a}_1 \hat{a}_1 \hat{\rho} \right) + \frac{n_1}{2} \left( 2\hat{a}_2 \hat{\rho} \hat{a}_2 - \hat{a}_2 \hat{a}_2 \hat{\rho} - \hat{a}_2 \hat{a}_2 \hat{\rho} \right) + \frac{n_1}{2} \left( 2\hat{a}_2 \hat{\rho} \hat{a}_2 - \hat{a}_2 \hat{a}_2 \hat{\rho} - \hat{a}_2 \hat{a}_2 \hat{\rho} \right).
\]

(3.1)

As in our recent work [34] for the one-mode case, instead of the master equation (3.1), we employ the equivalent differential equation for the two-mode characteristic function \( \chi(\lambda_1, \lambda_2, t) \) \(=\) \(\text{Tr} [(\hat{D}_1(\lambda_1) \otimes \hat{D}_2(\lambda_2)) \hat{\rho}(t)]\). Here \( \hat{D}_1(\lambda_1) \) and \( \hat{D}_2(\lambda_2) \) are the Weyl displacement operators of the modes: \( \hat{D}_j(\lambda_j) := \exp(\hat{\lambda}_j \hat{a}_j - \hat{\lambda}_j^* \hat{a}_j^\dagger) \), \((j = 1, 2)\). We finally find the solution:

\[
\chi(\lambda_1, \lambda_2, t) = \chi(\lambda_1 e^{-\frac{\gamma_1 t}{2}}, \lambda_2 e^{-\frac{\gamma_2 t}{2}}, 0) \\
\times \exp \left[ - \left( \frac{n_1}{2} \right) (1 - e^{-\gamma_1 t}) |\lambda_1^f |^2 \right] \\
\times \exp \left[ - \left( \frac{n_2}{2} \right) (1 - e^{-\gamma_2 t}) |\lambda_2^f |^2 \right].
\]

(3.2)

Let us inspect the asymptotic behaviour of the solution (3.2) of the master equation (3.1). When we take \( t \to \infty \) in equation (3.2), we get the characteristic function of the two-
mode thermal state imposed by the two reservoirs:

\[ \lim_{t \to \infty} \chi(\lambda_1, \lambda_2, t) = \exp \left[ -\left( \hat{n}_{R1} + \frac{1}{2} \right) |\lambda_1|^2 - \left( \hat{n}_{R2} + \frac{1}{2} \right) |\lambda_2|^2 \right]. \] (3.3)

Note that this two-mode steady state, which is independent of the input state, is a product state without any correlations between modes.

Given the structure of the time-dependent characteristic function (3.2), any input Gaussian state preserves its Gaussian form at any time during the mode damping. In particular, an initial two-mode STS remains a two-mode STS at any subsequent time. Its evolving CM has the following standard-form entries:

\[ \begin{align*}
\frac{b_1(t)}{c(t)} &= b_1 e^{-\gamma t} + \left( \hat{n}_{R1} + \frac{1}{2} \right) \left( 1 - e^{-\gamma t} \right), \\
\frac{b_2(t)}{c(t)} &= b_2 e^{-\gamma t} + \left( \hat{n}_{R2} + \frac{1}{2} \right) \left( 1 - e^{-\gamma t} \right), \\
\frac{c(t)}{c(t)} &= c \exp \left[ -\frac{1}{2} \left( \hat{n}_1 + \hat{n}_2 \right) t \right].
\end{align*} \] (3.4)

In view of equations (3.4), the CM of the damped two-mode STS becomes asymptotically diagonal:

\[ \lim_{t \to \infty} \mathcal{V}(t) = \left( \hat{n}_{R1} + \frac{1}{2} \right) I_2 \oplus \left( \hat{n}_{R2} + \frac{1}{2} \right) I_2. \] (3.5)

Therefore, the two-mode steady state is a product one whose factors are precisely the single-mode thermal states conditioned by the corresponding reservoirs. We thus recover the previous general conclusion in the special case of an an initial two-mode STS.

To sum up, any measure of quantum bipartite correlations in the Gaussian approach available for a two-mode STS, such as the EF [32] or the quantum discord [29, 30], can readily be applied for a decaying two-mode STS on account of equations (3.4).

4. Evolution of a two-mode squeezed thermal state with two identical local thermal reservoirs

What are we expecting to occur when a two-mode quantum state is subjected to a dissipative interaction, such as that described by the master equation (3.1)? In general, we expect a substantial reduction of the non-classical properties of the state, which entails a decrease of its quantum correlations such as entanglement and discord. More specifically, in the important particular case of two-mode Gaussian states, we can notice from the very beginning an important difference between the ways in which these two measures of quantum cross-correlations actually decay. Indeed, on the one hand, according to condition (2.8), the entanglement of the input state is expected to vanish at a finite time. This process has been called the entanglement sudden death in the case of qubits [11, 12]. On the other hand, it is known that the only zero-discord two-mode Gaussian states are the product ones [29, 30]. Taking account of the time-dependent two-mode characteristic function (3.2), as well as of its steady-state form (3.3), we infer that only the latter describes a product state without any correlations between the modes. Therefore, it is reasonable to believe that only asymptotically a damped two-mode Gaussian state could lose all its quantum cross-correlations measured by the Gaussian discord.

For the sake of simplicity and in order to get a versatile analytic results, we consider here the particular case when the two local reservoirs are identical: \( \gamma_1 = \gamma_2 = \gamma \) and \( \hat{n}_{R1} = \hat{n}_{R2} = \hat{n}_R \). In this case, the CM of an arbitrary damped two-mode Gaussian state reads:

\[ \mathcal{V}(t) = e^{-\gamma t} \mathcal{V}(0) + \left( \hat{n}_R + \frac{1}{2} \right) \left( 1 - e^{-\gamma t} \right) I_4. \] (4.1)

Here \( \mathcal{V}(0) \) is the input CM, and \( I_4 \) denotes the \( 4 \times 4 \) identity matrix. Equation (4.1) tells us that an input state with no local squeezing does not change its character during damping: for instance, a symmetric state remains symmetric, and a two-mode STS evolves, maintaining its form of an STS. We restrict ourselves now to this latter case. When employing the entries of the time-dependent CM (4.1) in the separability condition (2.8), one finds a simple expression of the time required by a damped two-mode STS to reach the separability threshold:

\[ t_s = \frac{1}{\gamma} \ln \left( 1 + \frac{1}{2} - \frac{\kappa_{PT}^-}{\hat{n}_R} \right), \] (4.2)

Here \( \kappa_{PT}^- \) is the smallest symplectic eigenvalue of the CM of the partially transposed input density matrix, which is given by equation (2.7). We see that in the special case of zero-temperature baths, the entanglement disappears only asymptotically. In all other cases, the entanglement vanishes at finite times. It is worth stressing that for any damping two-mode STS this sudden death of entanglement [11] has an additional significance: it happens to be a \textit{quantum-classical transition} in the sense of quantum optics. This means that beyond the time \( t_s \), equation (4.2), the damped two-mode STS possesses a Glauber–Sudarshan \( P \) distribution which is a genuine probability density.

In figures 1(a) and (b) we plot the evolution of the entanglement of formation, equation (2.13), as well as that of the quantum discords \( D_1 \) and \( D_2 \), equation (2.21), for an asymmetrical mixed STS. The aspect of the plots follows closely our above remarks on the robustness of discord against noise in comparison with the fragility of entanglement. Note that the discords \( D_1 \) and \( D_2 \) are very close and can be distinguished only for very distant values of the thermal mean photon occupancies \( \bar{n}_1 \) and \( \bar{n}_2 \). When the reservoirs are noisy, \( \langle \hat{n}_R \rangle > 0 \), both the EF and the discords \( D_1, D_2 \) are strongly diminished. Contact with zero-temperature baths, as in figure 1(b), produces a slower decay of all correlations, which in this case disappear only asymptotically.
Figure 1. Evolution of the EF (dot-dashed blue line) and of the discords $D_1$ (black line) and $D_2$ (dashed red line) for an input STS in interaction with two local identical thermal reservoirs. We have employed the following parameters. (a) The two-mode STS is characterized by the parameters $\tilde{n}_1 = 10$, $\tilde{n}_2 = 0.1$, $r = 2$ and the reservoir by $\bar{n}_R = 0.5$. (b) For the same input state we use $\bar{n}_R = 0$. (c) We plot the EF and the discord $D_1 = D_2$ (dashed red line) for an input pure state having the squeeze parameter $r = 2$. The reservoir is noisy with $\bar{n}_R = 0.5$.

Figure 1(c) we consider an input pure state, namely, a two-mode squeezed vacuum state in contact with a noisy bath. At the time $t = 0$ the entanglement and discord coincide, but their time developments look very different. Notice that, in view of equation (4.1), a thermalized two-mode squeezed vacuum state evolves into a symmetric two-mode STS having $D_1 = D_2$.

5. Evolution of a two-mode squeezed thermal state with a single local thermal reservoir

For finite-dimensional quantum systems, it was recently found that while entanglement does not increase under local channels, other correlations such as discord can become larger when the input state is not pure [14, 15]. In continuous-variable settings, a similar behaviour was noticed for the Gaussian discord of mixed two-mode states under one-mode Gaussian dissipative channels [16, 23]. To investigate here such an interaction with analytic means and results, we consider an input STS having only the mode 1 in contact with a thermal reservoir. When specializing the master equation (3.1) to the values $\gamma_1 = \gamma$, $\tilde{n}_{R1} = \tilde{n}_R$, $\gamma_2 = 0$, $\bar{n}_{R2} = 0$, the standard-form entries (3.4) of the damped CM become:

$$b_1(t) = b_1 e^{-\gamma t} + \left(\tilde{n}_R + \frac{1}{2}\right)(1 - e^{-\gamma t}),$$

$$b_2(t) = b_2,$$

$$c(t) = c \exp\left(-\frac{1}{2}\gamma t\right).$$

(5.1)

By insertion of the time-dependent parameters (5.1) into the separability condition (2.8) one finds the time at which the EF of a damped STS vanishes:

$$t_c = \frac{1}{\gamma} \ln \left[1 - \frac{1}{\tilde{n}_R \left(b_2 - \frac{1}{2}\right) - c^2}\right].$$

(5.2)

Note that in the case of a pure Gaussian input $\left(b_1 = b_2 = b, c > 0, b^2 - c^2 = \frac{1}{2}\right)$, the time of the entanglement death (5.2), denoted below by $t_c$, is independent of the input two-mode squeezed vacuum state, being determined only by the field-reservoir coupling:

$$t_c = \frac{1}{\gamma} \ln \left(1 + \frac{1}{\bar{n}_R}\right).$$

(5.3)

We have also checked that the time of the entanglement death has the same expression (5.3) for an input squeezed vacuum state with additional local squeezings on both modes.

Besides, in our recent paper [34], we found that, for some classes of one-mode states displaying initially certain negativities of their Glauber–Sudarshan $P$ representation, $t_c$ is the ultimate time at which the $P$ function becomes positive, due to the field interaction with a thermal reservoir. Otherwise stated, for some types of one-mode states, a sudden quantum-classical transition (in the quantum-optical sense) occurs at times not exceeding the time $t_c$, equation (5.3). This happens to be precisely the time of the entanglement sudden death for any two-mode squeezed vacuum state.

Regarding the evolution of the Gaussian discord, we expect it to decay eventually very slowly and to vanish only asymptotically. Indeed, according to equation (5.1), the CM of the damped STS has an asymptotically diagonal form:

$$\lim_{t \to \infty} V(t) = \left(\tilde{n}_R + \frac{1}{2}\right) I_2 \bigoplus b_2 I_2.$$  

(5.4)

The steady state of the field is therefore the product of two single-mode thermal states: the state of the damped mode 1, which is imposed by the thermal reservoir owing to their interaction, and that of the freely-evolving mode 2, which is its reduced state remaining constant in time and thus equal to its input at $t = 0$.

The damping of a pure-state input deserves additional remarks. According to equation (5.1), although at the moment $t = 0$ the three measures of quantum correlations EF, $D_1$, and $D_2$ coincide, they behave subsequently quite differently, because an input two-mode squeezed vacuum state evolves into an asymmetric STS. Figure 2(c) displays the evolution of the EF as well as those of both discords $D_1$ and $D_2$, which are all monotonic, as predicted in [14]. However, the discord $D_2$,
corresponding to local measurements performed on the damped mode 1, survives much longer than both the EF and the discord $D_1$.

The case of an initial mixed state can be tackled by using equations (2.13), (2.21), and (5.1) for obtaining the expressions of the EF and the discords $D_1$ and $D_2$. We plot in figures 2(a) and (b) their time evolution for the same input state, but with a noisy bath (a) and a zero-temperature reservoir (b). An enhancement of $D_2$ is noticed in both panels (a) and (b). The discord $D_2$ presents a clear maximum in the latter situation and is much enhanced with respect to its value at the moment $t = 0$. This can be interpreted as a creation of quantum correlations similar to those first explored for finite-dimensional systems [14, 15]. Furthermore, in reference [16] it was found that an enhancement of the discord $D_2$ can be noticed even when the input Gaussian state is separable.

6. Concluding remarks

In order to draw some conclusions on the effects produced by local dissipation on quantum correlations of a two-mode STS, we compare now the decay of entanglement and discord for the two situations studied above. Figure 3 displays our results for the EF (lower (blue) curves) and $D_2$ (upper (black) plots) in the cases of both one and two local identical reservoirs for a mixed STS (panels (a) and (b)) and for a pure Gaussian state (c).

We represent here the case of zero-temperature reservoirs (panels (a) and (c)) to show a better preservation of all correlations in comparison with the noisy bath considered in panel (b). We can see that in all cases both Gaussian discords $D_1$ and $D_2$ survive longer than the EF. This is expected, because only asymptotically the damped Gaussian state becomes a product one. Thus the Gaussian discord, which is a measure of quantum bipartite correlations, proves to be quite robust against dissipation in all the above-mentioned situations. However, only for a configuration with one local thermal bath, there is an enhancement of the discord $D_2$. Since this can be larger than the discord of the input state, it means that the field-reservoir interaction generates quantum correlations of the discord type. A final conclusion arising from figure 3 is quite interesting. In all the analyzed situations (mixed or pure input states, noisy or zero-temperature reservoirs), the configuration with one local thermal bath performs better than that with two local identical baths. This is valid when analyzing both the magnitude of correlations and their preservation in time.

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References

[1] Vourdas A and Weiner R M 1987 Phys. Rev. A 36 5866
[2] Milburn G J and Walls D F 1988 Phys. Rev. A 38 1087
[3] Marian P and Marian T A 1996 J. Phys. A: Math. Gen. 29 6233
[4] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865
[5] Modi K, Brodutch A, Cable H, Paterek T and Vedral V 2012 Rev. Mod. Phys. 84 1655
[6] Adesso G, Ragy S and Lee A R 2014 Open Syst. Inf. Dyn. 21 1440001
[7] Zurek W H 2000 Ann. Phys. (Leipzig) 9 855
[8] Ollivier H and Zurek W H 2001 Phys. Rev. Lett. 88 017901
[9] Henderson L and Vedral V 2001 J. Phys. A 34 6899
[10] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[11] Yu T and Eberly J H 2002 Phys. Rev. B 66 193306
[12] Mazzola L, Pilo J and Maniscalco S 2010 Phys. Rev. Lett. 104 200401
[13] Streltsov A, Kampermann H and Bruss D 2011 Phys. Rev. Lett. 107 170502
[14] Ciccarello F and Giovannetti V 2012 Phys. Rev. A 85 010102(R)
[15] Ciccarello F and Giovannetti V 2012 Phys. Rev. A 85 022308
[16] Serafini A, Illuminati F, Paris M G A and De Siena S 2004 Phys. Rev. A 69 022318
[17] Goyal S K and Ghosh S 2010 Phys. Rev. A 82 042337
[18] Souza L A M, Drumond R C, Nemes M C and Fonseca Romero K M 2012 Opt. Commun. 285 4453
[19] Buono D, Nocerino G, Porzio A and Solimeno S 2012 Phys. Rev. A 86 042308
[20] Barbosa F A S, de Faria A J, Coelho A S, Cassemiro K N, Villar A S, Nussenzveig P and Martinelli M 2011 Phys. Rev. A 84 052330
[21] Issar A 2012 Phys. Scr. T147 14015
[22] Madsen L S, Berni A, Lassen M and Andersen U L 2012 Phys. Rev. Lett. 109 030402
[23] Marian P, Marian T A and Scutaru H 2001 J. Phys. A: Math. Gen. 34 6969
[24] Marian P, Marian T A and Scutaru H 2003 Phys. Rev. A 68 062309
[25] Pirandola S, Spedalieri G, Braunstein S L, Cerf N J and Lloyd S 2014 Phys. Rev. Lett. 113 140405
[26] Simon R 2000 Phys. Rev. Lett. 84 2726
[27] Peres A 1996 Phys. Rev. Lett. 77 1413
[28] Giorda P and Paris M G A 2010 Phys. Rev. Lett. 105 020503
[29] Adesso G and Datta A 2010 Phys. Rev. Lett. 105 030501
[30] Olivares S and Paris M G A 2012 Int. J. Mod. Phys. B 27 134504
[31] Marian P and Marian T A 2008 Phys. Rev. Lett. 101 220403
[32] Williamson J 1936 Amer. J. Math. 58 141
[33] Simon R 2000 Phys. Rev. Lett. 84 2726