Correspondence between 2 Calabi-Yau Categories and Quivers

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In Section 8 of [2], M. Kontsevich and Y. Soibelman proved that the equivalence classes of a certain type of 3-dimensional Calabi-Yau categories are in one-to-one correspondence with the gauge equivalence classes of quiver with minimal potential \((Q, W)\). This note gives an analogue in 2-dimensional Calabi-Yau case. We assume that \(k\) is a field of characteristic zero.

**Theorem.** Let \(\mathcal{C}\) be a 2-dimensional \(k\)-linear Calabi-Yau category generated by a finite collection \(\mathcal{E} = \{E_i\}_{i \in I}\) of generators satisfying

\[
\begin{align*}
\text{• } & \text{Ext}^0(E_i, E_i) = k \cdot \text{id}_{E_i}, \\
\text{• } & \text{Ext}^0(E_i, E_j) = 0, \forall i \neq j, \\
\text{• } & \text{Ext}^{<0}(E_i, E_j) = 0, \forall i, j.
\end{align*}
\]

The equivalence classes of such categories with respect to \(A_\infty\)-transformations preserving the Calabi-Yau structure and \(\mathcal{E}\), are in one-to-one correspondence with finite symmetric quivers with even number of loops at each vertex.

**Proof.** Let’s denoted by \(\mathcal{A}\) the set of equivalence classes of such 2 Calabi-Yau categories, and \(\mathcal{B}\) the set of finite symmetric quivers with even number of loops at each vertex.

Given such a category \(\mathcal{C}\), we associate a quiver \(Q\) whose vertices \(\{i\}_{i \in I}\) are in one-to-one correspondence with \(\mathcal{E} = \{E_i\}_{i \in I}\), and the number of arrows from \(i\) to \(j\) is equal to \(\dim \text{Ext}^1(E_i, E_j)\). Since \(\mathcal{C}\) is 2 Calabi-Yau, we have \(\dim \text{Ext}^1(E_i, E_j) = \dim \text{Ext}^1(E_j, E_i)\), so \(Q\) is symmetric. The supersymmetric non-degenerate pairing on \(\text{Ext}^\bullet(E_i, E_i)\) leads
to a symplectic pairing on $\text{Ext}^1(E_i, E_i)$, thus $\dim \text{Ext}^1(E_i, E_i)$ is even, which means that the number of loops at each vertex is even. This construction defines a map $\Phi : \mathcal{A} \to \mathcal{B}$.

To prove that $\Phi$ is a bijection, we consider $\mathcal{C}$ with single generator $E$, and quiver $Q$ with single vertex for simplicity. The general case can be proved in a similar way.

Let $Q$ be a quiver with one vertex and $|J| = 2n$ loops, where $J$ is the set of loops. We will construct a 2 Calabi-Yau category with one generator $E$, such that $2n = \dim \text{Ext}^1(E, E)$. Assuming that such a category exists, we will find an explicit formula for the potential on $A = \text{Hom}^\bullet(E, E)$. Let’s consider the graded vector space

$$\text{Ext}^\bullet(E, E)[1] = \text{Ext}^0(E, E)[1] \oplus \text{Ext}^1(E, E) \oplus \text{Ext}^2(E, E)[-1] = k[1] \oplus k^{2n} \oplus k[-1].$$

We introduce graded coordinates on $\text{Ext}^\bullet(E, E)[1]$:

a) the coordinate $\alpha$ of degree 1 on $\text{Ext}^0(E, E)[1]$,

b) the coordinate $\beta$ of degree $-1$ on $\text{Ext}^2(E, E)[-1]$,

c) the coordinates $x_i, \xi_i, i = 1, ..., n$ of degree 0 on $\text{Ext}^1(E, E) = \text{Ext}^1(E, E)^\vee$.

The Calabi-Yau structure gives rise to the minimal potential $W = W(\alpha, x_i, \xi_i, \beta)$, which is a series of cyclic words on the space $\text{Ext}^\bullet(E, E)[1]$. Furthermore, $A$ defines a non-commutative formal pointed graded manifold endowed with a symplectic structure (c.f. [3]). The potential $W$ satisfies the equation $\{W, W\} = 0$, where $\{\cdot, \cdot\}$ is the corresponding Poisson bracket.

We need to construct the formal series $W$ of degree 1 in cyclic words on the graded vector space $k[1] \oplus k^{2n} \oplus k[-1]$, satisfying $\{W, W\} = 0$ with respect to the Poisson bracket

$$\{f, g\} = \sum_{i=1}^n \left[ \frac{\partial}{\partial x_i} ; \frac{\partial}{\partial \xi_i} \right] (f, g) + \left[ \frac{\partial}{\partial \alpha} ; \frac{\partial}{\partial \beta} \right] (f, g).$$

Let $W_{can} = \alpha^2 \beta + \sum_{i=1}^n (\alpha x_i \xi_i - \alpha \xi_i x_i)$. This potential makes $\text{Ext}^\bullet(E, E)$ into a 2 Calabi-Yau algebra with associative product and the unit. The multiplications are as follows: the multiplication of $\text{Ext}^0(E, E)$ and the other components is scalar product, and is a non-degenerate bilinear form on the components $\text{Ext}^1(E, E) \otimes \text{Ext}^1(E, E) \to \text{Ext}^2(E, E) \simeq k$. 

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In addition, 

\[ \{W_{can}, W_{can}\} = \sum_{i=1}^{n} [\frac{\partial W_{can}}{\partial x_i}, \frac{\partial W_{can}}{\partial \xi_i}] + [\frac{\partial W_{can}}{\partial \alpha}, \frac{\partial W_{can}}{\partial \beta}] \]

\[ = \sum_{i=1}^{n} (\xi_i \alpha - \alpha \xi_i)(\alpha x_i - x_i \alpha) - (\alpha x_i - x_i \alpha)(\xi_i \alpha - \alpha \xi_i) \]

\[ + (\alpha \beta + \beta \alpha + \sum_{j=1}^{n} (\xi_j x_j - x_j \xi_j)\alpha^2 - \alpha^2(\alpha \beta + \beta \alpha + \sum_{k=1}^{n} (x_k \xi_k - \xi_k x_k)) \]

\[ = 0 \]

The above construction from \( Q \) to \( \mathcal{C} \) shows that \( \Phi \) is a surjection.

Finally, we need to check that \( \Phi \) is an injection. The 2 Calabi-Yau algebras we are considering can be thought of as deformations of the 2 Calabi-Yau algebra \( A_{can} = Ext^*(E, E) \) corresponding to the potential \( W_{can} \). The deformation theory of \( A_{can} \) is controlled by a differential graded Lie algebra (DGLA) \( g_{can} = \bigoplus_{n \in \mathbb{Z}} g^n_{can} \), which is a DG Lie subalgebra of the DGLA \( \widehat{g} = \prod_{k \geq 1} Cycl^k(A_{can}[1])^\vee = \bigoplus_{n \in \mathbb{Z}} \widehat{g}^n \). Here we write \( \widehat{g}^n = \{ W | coh.deg W = n \} \), and \( g^n_{can} = \{ W \in \widehat{g}^n | cyc.deg W \geq n + 2 \} \), where coh.deg means the cohomological degree of \( W \), and cyc.deg means the number of letters \( \alpha, x_i, \xi_i, \beta, i = 1, \ldots, n \) that each term of \( W \) contains. In these DGLAs, the Lie bracket is given by the Poisson bracket and the differential is given by \( d = \{ W_{can} \cdot \cdot \} \).

The DGLA \( g_{can} \) is a DG Lie subalgebra of \( \widehat{g} \) because of the following reason: \( d \) preserves \( g_{can} \) since it increases both coh.deg and cyc.deg by 1, and the Poisson bracket restricts to \( g_{can} \) since for any \( W_1 \in g^m_{can} \) and \( W_2 \in g^n_{can} \) with cyc.deg \( W_1 = l_1 \) and cyc.deg \( W_2 = l_2 \), we have cyc.deg \( \{ W_1, W_2 \} = l_1 + l_2 - 2 = m + n + 2 \). As vector spaces, \( \widehat{g} = g_{can} \bigoplus \mathfrak{g} \), where \( \mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}^n \), and \( \mathfrak{g}^n = \{ W \in \widehat{g}^n | cyc.deg W < n + 2 \} \). For the same reason as \( g_{can} \), we have that \( \mathfrak{g} \) is also a DG Lie subalgebra of \( \widehat{g} \). It follows that \( g_{can} \) is a direct summand of the complex \( \widehat{g} \). The latter is quasi isomorphic to the cyclic complex \( CC_*(A_{can})^\vee \). Let \( A_{can}^+ \subset A_{can} \) be the non-unital \( A_\infty \)-subalgebra consisting of terms of positive cohomological degree. Then for the cyclic homology, \( HC_*(A_{can}) \simeq HC_*(A_{can}^+ \bigoplus HC_*(k) \). Thus to compute the cohomology of the dual complex \( \widehat{g} \), we only need to consider the space of cyclic series in variables \( x_i, \xi_i, \beta, i = 1, \ldots, n \) (corresponds to \( HC_*(A_{can}^+)^\vee \)), and the one in variable \( \alpha \) (corresponds to \( HC_*(k)^\vee \)). We have that the series in \( \alpha \) don’t contribute to the cohomology of \( g_{can} \) since they do not belong to \( g_{can} \). Moreover, the cohomological degree of series in \( x_i, \xi_i, \beta, i = 1, \ldots, n \) is non-positive. Hence \( H^{\geq 1}(g_{can}) = 0 \), which means that the deformation of \( A_{can} \) is trivial. Thus, \( \Phi \) is an injection. \( \square \)
Remark. Suppose that the 2 Calabi-Yao category over an algebraically closed field is endowed with a stability condition, then a polystable object has formal endomorphism algebra. A special case of the coherent sheaves on a projective K3 surface is proven in [1].

References

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