Double field formulation of Yang-Mills theory

IMTAK JEON,† KANGHOON LEE‡ AND JEONG-HYUCK PARK†

‡Center for Quantum Spacetime, Sogang University, Shinsu-dong, Mapo-gu, Seoul 121-742, Korea

†Department of Physics, Sogang University, Shinsu-dong, Mapo-gu, Seoul 121-742, Korea

E-mail: imtak@sogang.ac.kr kanghoon@sogang.ac.kr park@sogang.ac.kr

Abstract

Based on our previous work on the differential geometry for the closed string double field theory, we construct a Yang-Mills action which is covariant under $O(D, D)$ T-duality rotation and invariant under three types of gauge transformations: non-Abelian Yang-Mills, diffeomorphism and one-form gauge symmetries. In double field formulation, in a manifestly covariant manner our action couples a single $O(D, D)$ vector potential to the closed string double field theory. In terms of undoubled component fields, it couples a usual Yang-Mills gauge field to an additional one-form field and also to the closed string background fields which consist of a dilaton, graviton and a two-form gauge field. Our resulting action resembles a twisted Yang-Mills action.

PACS: 11.25.-w

Keywords: Double field theory, T-duality, Differential geometry.
1 Introduction

The low energy effective action for a closed string massless sector takes the following well-known form:

\[ S_{\text{eff.}} = \int d^Dx \sqrt{-g} e^{-2\phi} \left[ R_g + 4(\partial \phi)^2 - \frac{1}{12} H^2 \right], \tag{1.1} \]

where \( g_{\mu\nu} \) is the \( D \)-dimensional spacetime metric with its scalar curvature, \( R_g \); \( \phi \) is the string theory dilaton; and \( H \) is the three form field strength of a two form gauge field, \( B_{\mu\nu} \). In a double field theory (DFT) formalism developed by Hull et al., in \([1-4]\), the above action was reformulated as

\[ S_{\text{DFT}} = \int dg^{2D} e^{-2d} \left[ \mathcal{H}^{AB} (4\partial_A \partial_B d - 4\partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD}) \right. \]

\[ \left. + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} \right]. \tag{1.2} \]

Herein the spacetime dimension is formally doubled from \( D \) to \( 2D \) with coordinates \( x^\mu \to y^A = (\tilde{x}_\mu, x^\nu) \); \( d \) denotes the double field theory ‘dilaton’ given by \( e^{-2d} = \sqrt{-g} e^{-2\phi} \); and \( \mathcal{H}_{AB} \) is a \( 2D \times 2D \) matrix of the form,

\[ \mathcal{H}_{AB} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\nu} B_{\nu\sigma} \\ B_{\rho\mu} g^{\kappa\nu} & g_{\rho\sigma} - B_{\rho\mu} g^{\kappa\lambda} B_{\lambda\sigma} \end{pmatrix}. \tag{1.3} \]
All the spacetime indices, $A, B, C, \cdots$, are 2D-dimensional vector indices which can be raised or lowered by the $O(D, D)$ invariant constant metric, $\eta$,

$$
\eta := \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
$$

(1.4)

As a field theory counterpart of the level matching condition in closed string theory, it is required that all the fields in double field theory as well as all of their possible products should be annihilated by the $O(D, D)$ d’Alembert operator, $\partial^2 = \partial_A \partial^A$,

$$
\partial^2 \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0.
$$

(1.5)

This constraint, which one may call ‘the level matching constraint’, actually means that the theory is not truly doubled: there is a choice of coordinates $(\tilde{x}', x')$, related to the original coordinates $(\tilde{x}, x)$, by an $O(D, D)$ rotation, in which all the fields do not depend on the $\tilde{x}'$ coordinates [3]. Remarkably, while the double field theory action, $S_{DFT}$ (1.2), reduces to the effective action, $S_{\text{eff}}$ (1.1), upon the level matching constraint, the double field theory formulation manifests the $O(D, D)$ covariance of the action and hence the T-duality first noted by Buscher [5–7] and further studied in [9–16].

However, what is not obvious about the above DFT action (1.2) is that it possesses gauge symmetry, which must be the case [4,17], since restricted on the $x$-hyperplane the action (1.2) is nothing but a rewriting of the effective action (1.1) while the latter surely enjoys both the $D$-dimensional diffeomorphism, $x^\mu \to x^\mu + \delta x^\mu$, and the gauge symmetry of the two form field, $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$. That is to say, in contrast to the effective action (1.1) where the gauge symmetry is manifest yet T-duality is not, in the DFT action given in the form (1.2) it is quite the opposite.

In order to manifest both the $O(D, D)$ structure and the gauge symmetry, in our previous work [18], we conceived a differential geometry characterized by a projection satisfying the following defining properties,

$$
P_A^B P_B^C = P_A^C, \quad P_{AB} = P_{BA}.
$$

(1.6)

Further demanding that the upper left $D \times D$ block of $2^D - 1$ is non-degenerate, the projection is related to the matrix, $\mathcal{H}_{AB}$ (1.3), by

$$
P_A^B = \frac{1}{2}(\delta_A^B + \mathcal{H}_A^B).
$$

(1.7)

1Note that throughout our paper, the equivalence symbol, ‘$\equiv$’, denotes the equality up to the level matching constraint (1.5).

2Without imposing the level matching constraint, the $O(D, D)$ transformation surely corresponds to a Noether symmetry of the $2D$-dimensional field theory. After imposing the constraint, the double field theory is, by nature, $D$-dimensional: it lives on a $D$-dimensional hyperplane. As the $O(D, D)$ transformation then rotates the entire hyperplane, the $O(D, D)$ rotation acts a priori as a duality rather than a Noether symmetry of the $D$-dimensional theory. After further dimensional reductions, it becomes a Noether symmetry of the reduced action, as verified by Buscher [5–7] (c.f. [8]).
In terms of a certain differential operator compatible with the projection – which we review later – we were able to identify the underlying differential geometry of the double field theory and, in particular, to rewrite the original DFT action (1.2) in a compact manner:

\[ S_{\text{DFT}} = \int \, dy^2 \, d^2e \, e^{-2d} \, \mathcal{H}^{AB} \left( 4 \nabla_A d \nabla_B d + S_{AB} \right) . \] (1.8)

In this paper, we apply our differential geometric tools in [18] to Yang-Mills theory with an arbitrary gauge group, G. We construct a Yang-Mills action which is covariant under the O(D, D) rotation and invariant under three-types of gauge transformations: non-Abelian Yang-Mills, diffeomorphism and one-form gauge symmetries. The latter two amount to the DFT gauge symmetry, as summarized below:

- **O(D, D) T-duality**
- **Gauge symmetry**
  - Yang-Mills gauge symmetry
  - DFT gauge symmetry
  - Diffeomorphism
  - One-form gauge symmetry for \( B_{\mu\nu} \)

In double field formulation, our action couples a single O(D, D) vector potential to the closed string double field theory (1.3), keeping the O(D, D) T-duality and all the gauge symmetries manifest. In terms of undoubled component fields, the T-duality works in a nontrivial way and the action couples a usual Yang-Mills gauge field, \( A_{\mu} \), to an additional one-form field, \( \phi_{\mu} \), and also to the closed string background fields which consist of the dilaton, graviton and the two-form gauge field, \( \phi, g_{\mu\nu}, B_{\mu\nu} \).

In section 2, we review our previous work [18] on the differential geometry for the closed string double field theory, and set up our notations. In section 3, we present our O(D, D) covariant Yang-Mills theory, both in the double field formulation (subsection 3.1) and also in terms of undoubled component fields (subsection 3.2). We conclude with some comments in section 4.

---

\(^3\)Shortly after our work [18], an alternative approach to the underlying differential geometry of the double field theory was proposed by Hohm and Kwak [19] based on earlier works by Siegel [12, 13]. It differs from our approach, as it postulates a covariant derivative whose connection is not \textit{a priori} a physical variable of the double field theory.
2 Differential geometry compatible with a projection: review

In double field theory, the usual definition of Lie derivative is generalized to \[^4\][16][18]

\[
\tilde{L}_X T_{A_1 A_2 \ldots A_n} := X^B \partial_B T_{A_1 A_2 \ldots A_n} + \omega \partial_B X^B T_{A_1 A_2 \ldots A_n} + \sum_{i=1}^{n} 2 \partial_{[A_i} X_{B]} T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n} , \tag{2.1}
\]

where \( \omega \) is the weight of each field, \( T_{A_1 A_2 \ldots A_n} \), and \( X^A \) is a local gauge parameter, of which half corresponds to the \( D \)-dimensional diffeomorphism parameter, \( \delta x^\mu \), and the other half matches the one-form gauge symmetry parameter, \( \Lambda_\nu \). Up to the level matching constraint (1.5), the commutator of them is closed by the \( c \)-bracket introduced by Siegel \[^5\].

\[
\tilde{L}_X, \tilde{L}_Y \equiv \tilde{L}_{[X,Y]} = X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B . \tag{2.2}
\]

By definition in double field theory, covariant tensors (\( \omega = 0 \)) or tensor densities follow the gauge transformation rule dictated by the generalized Lie derivative,

\[
\delta_X T_{A_1 A_2 \ldots A_n} = \tilde{L}_X T_{A_1 A_2 \ldots A_n} . \tag{2.3}
\]

Examples include for a tensor, \( \mathcal{H}_{AB} \), and for a scalar density with weight one, \( e^{-2d} \), such that

\[
\delta_X \mathcal{H}_{AB} = \tilde{L}_X \mathcal{H}_{AB} = X^C \partial_C \mathcal{H}_{AB} + (\partial_A X_C - \partial_C X_A) \mathcal{H}_{CB} + (\partial_B X_C - \partial_C X_B) \mathcal{H}_{AC} ,
\]

\[
\delta_X (e^{-2d}) = \tilde{L}_X (e^{-2d}) = \partial_A (X^A e^{-2d}) . \tag{2.4}
\]

The latter suggests, with \( \tilde{L}_X (e^{-2d}) = -2(\tilde{L}_X d) e^{-2d} \),

\[
\delta_X d = \tilde{L}_X d := X^A \partial_A d - \frac{1}{2} \partial_B X^B . \tag{2.5}
\]

The DFT action (1.2) is indeed invariant under the above gauge transformation (2.4), as first shown in \[^4\].

In our previous work \[^18\], we introduced the following projection-compatible derivative, \( \nabla_C \), which acts on tensors, tensor densities as well as their derivative-descendants as

\[
\nabla_C T_{A_1 A_2 \ldots A_n} = \partial_C T_{A_1 A_2 \ldots A_n} - \omega \Gamma_B^{BC} T_{A_1 A_2 \ldots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}^{B} T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n} , \tag{2.6}
\]

where the connection is, with the projection, (1.6), (1.7), and its complementary projection, \( \bar{P} := 1 - P \), given by

\[
\Gamma_{CAB} := 2 P_{[A} \bar{P}_{B]} E \partial_C P_{DE} + 2 ( \bar{P}_{[A} \bar{P}_{B]} E - P_{[A} P_{B]} E ) \partial_D P_{EC} . \tag{2.7}
\]

\(^4\)Upon the level matching constraints the \( c \)-bracket itself reduces to the Courant bracket \[^20\], as recognized in \[^2\].

\(^5\)Another example of a covariant tensor is the \( c \)-bracket of two covariant vectors, \( \delta_X ([X,Y]_C) \equiv \tilde{L}_X ([X,Y]_C) \tag{2.11} \).
This connection was uniquely determined in terms of the projections and their derivatives\(^6\) by requiring

\[
\nabla_A \eta_{BC} = 0, \quad \nabla_A P_{BC} = 0, \quad (2.8)
\]

and

\[
\Gamma_{CAB} + \Gamma_{CBA} = 0, \quad \Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0. \quad (2.9)
\]

The unique feature of the projection-compatible derivative is that, acting on a covariant tensor, although it does not lead to a covariant quantity, the resulting derivative satisfies (2.8), (2.9), (2.12) and further that

\[
\nabla_{A} \partial_{B} \partial_{C} = \partial_{A} \partial_{B} \partial_{C}, \quad \nabla_{A} \partial_{B} \partial_{C} = \partial_{A} \partial_{B} \partial_{C} + \frac{1}{2} \Gamma^{B}_{BA} \nabla_{C} d. \quad (2.14)
\]

\[
\delta_{X} - \bar{\partial}_{X} \nabla_{C} T_{A_{1}A_{2} \ldots A_{n}} = \sum_{i=1}^{n} \left( P_{A_{i}} D P_{B} E P_{C} F + \bar{P}_{A_{i}} D \bar{P}_{B} E \bar{P}_{C} F \right) \partial_{F} \partial_{D} X_{E} T_{A_{1} \ldots A_{i-1} B A_{i+1} \ldots A_{n}}, \quad (2.10)
\]

after being contracted properly with the projections, it can be covariantized as

\[
\left( \delta_{X} - \bar{\partial}_{X} \right) \left( P_{C} D P_{A_{1}} B_{1} \bar{P}_{A_{2}} B_{2} \ldots \bar{P}_{A_{n}} B_{n} \nabla_{D} T_{B_{1} B_{2} \ldots B_{n}} \right) = 0, \quad (2.11)
\]

\[
\left( \delta_{X} - \bar{\partial}_{X} \right) \left( \bar{P}_{C} D P_{A_{1}} B_{1} \bar{P}_{A_{2}} B_{2} \ldots P_{A_{n}} B_{n} \nabla_{D} T_{B_{1} B_{2} \ldots B_{n}} \right) = 0. \quad (2.11)
\]

Thanks to the symmetric properties (2.9), all the ordinary derivatives in the definitions of the generalized Lie derivative (2.1) and the e-bracket (2.2) can be replaced by our projection-compatible derivatives\(^7\)

\[
\bar{\partial}_{X} T_{A_{1} \ldots A_{n}} = X^{B} \nabla_{B} T_{A_{1} \ldots A_{n}} + \omega \nabla_{B} X^{B} T_{A_{1} \ldots A_{n}} + \sum_{i=1}^{n} 2 \nabla_{[A_{i} X]} T_{A_{1} \ldots A_{i-1} B A_{i+1} \ldots A_{n}}, \quad (2.12)
\]

| X, Y | B |
|---|---|
| X \nabla_{B} Y^{A} - Y^{B} \nabla_{B} X^{A} + \frac{1}{2} Y^{B} \nabla^{A} X_{B} - \frac{1}{2} X^{B} \nabla^{A} Y_{B}. |

Postulating this property to hold also for the gauge transformation of the dilaton (2.5), and writing

\[
\nabla_{A} (e^{-2d}) = (-2 \nabla_{A} d) e^{-2d}, \quad \nabla_{A} \nabla_{B} (e^{-2d}) = (-2 \nabla_{A} \nabla_{B} d + 4 \nabla_{A} d \nabla_{B} d) e^{-2d}, \quad (2.13)
\]

it is natural further to set, as if \( \nabla_{A} d \) has trivial weight,

\[
\nabla_{A} d := \partial_{A} d + \frac{1}{2} \Gamma^{B}_{BA}, \quad \nabla_{A} \nabla_{B} d := \partial_{A} \nabla_{B} d + \Gamma_{AB}^{C} \nabla_{C} d. \quad (2.14)
\]

\(^6\)One possible generalization of (2.7) which we have not taken seriously is to include the dilaton and its derivative in the connection,

\[
\Gamma_{CAB} \rightarrow \Gamma'_{CAB} := \Gamma_{CAB} - \frac{1}{D+1} \left( P_{CAB} P_{BD} - P_{CAB} P_{AD} + P_{CAB} P_{BD} - P_{CAB} P_{AD} \right) \nabla^{B} d.
\]

The resulting derivative satisfies (2.8), (2.9), (2.12) and further that \( \nabla' d = \partial_{A} d + \frac{1}{2} \Gamma'_{BA} = 0 \), whilst it does not affect the covariant quantities in (2.11). However, it becomes singular in the case of \( D = 1 \).

\(^7\)The weight of a gauge symmetry parameter is taken to be zero, such that \( \nabla_{A} X^{B} = \partial_{A} X^{B} + \Gamma_{A}^{B} C X^{C} \).
Now, with the curvature defined in standard way,

\[ R_{CDAB} := \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC} E^E \Gamma_{BED} - \Gamma_{BC} E^E \Gamma_{AED}, \quad (2.15) \]

if we set

\[ S_{ABCD} := \frac{1}{2} (R_{ABCD} + R_{CDAB} - \Gamma^E_{AB} \Gamma_{ECD}), \quad S_{AB} := S^C_{ACB}, \quad (2.16) \]

the following quantities are all gauge covariant \[18\].

\[ \mathcal{R}_{AB} := P_A^C \tilde{P}_B^D \left( S_{CD} + 2 \nabla (C \nabla D) d \right), \quad (2.17) \]

\[ \mathcal{R} := \mathcal{H}^{AB} \left( 4 \nabla_A \nabla_B d - 4 \nabla_A d \nabla_B d + S_{AB} \right), \quad (2.18) \]

\[ P^{AB} (\nabla_A - 2 \nabla_A d) V_B, \quad (2.19) \]

\[ \tilde{P}^{AB} (\nabla_A - 2 \nabla_A d) V_B, \quad (2.20) \]

\[ P^{AB} P_{C_1}^{D_1} \ldots P_{C_n}^{D_n} \left[ \nabla_A \nabla_B T_{D_1 \ldots D_n} - 2 (\nabla_A d) \nabla_B T_{D_1 \ldots D_n} \right], \quad (2.21) \]

\[ \tilde{P}^{AB} P_{C_1}^{D_1} \ldots P_{C_n}^{D_n} \left[ \nabla_A \nabla_B T_{D_1 \ldots D_n} - 2 (\nabla_A d) \nabla_B T_{D_1 \ldots D_n} \right], \quad (2.22) \]

in addition to the ones in \ref{2.11}.

As a matter of fact, up to a surface term, the double field theory Lagrangian in \ref{1.8} is equivalent to \( e^{-2d} \mathcal{R} \), while its equations of motion for the dilaton and the projection are \( \mathcal{R} = 0 \) and \( \mathcal{R}_{(AB)} = 0 \) respectively.

Some useful identities to note are

\[ S_{ABCD} = S_{[AB][CD]} \] \[ S_{ABCD} = S_{CDAB} \] \[ S_{A[BCD]} = 0 \] \[ P_{C}^D P_{A_1}^{B_1} P_{A_2}^{B_2} \ldots P_{A_n}^{B_n} \nabla_D T_{B_1 B_2 \ldots B_n}, \quad (2.23) \]

\[ \tilde{P}_{C}^D P_{A_1}^{B_1} P_{A_2}^{B_2} \ldots P_{A_n}^{B_n} \nabla_D T_{B_1 B_2 \ldots B_n}. \]

\[ 4 \nabla_A \nabla^A d - 4 \nabla_A d \nabla^A d + S \equiv 0. \quad (2.26) \]

\[ \text{Successive application of } (2.23) \text{ with more than one covariant vectors also leads to the following gauge covariant higher order derivatives:} \]

\[ (\prod_{i=1}^m V_i^B P_B^C \nabla_C) P_{A_1}^{B_1} P_{A_2}^{B_2} \ldots P_{A_n}^{B_n} T_{B_1 B_2 \ldots B_n}, \]

\[ (\prod_{i=1}^m V_i^B \tilde{P}_B^C \nabla_C) P_{A_1}^{B_1} P_{A_2}^{B_2} \ldots P_{A_n}^{B_n} T_{B_1 B_2 \ldots B_n}. \]
Under an arbitrary infinitesimal transformation of the projection satisfying
\[ \delta P = P \delta P \bar{P} + \bar{P} \delta PP, \] (2.27)
the connection and \( S_{ABCD} \) transform as
\begin{align*}
\delta \Gamma_{CAB} &= 2 P [A \bar{P}]^E \nabla C \delta P_{DE} + 2 (\bar{P} [A \bar{P}]^E - P [A \bar{P}]^E) \nabla D \delta P_{EC} \\
&- \Gamma_{FDE} \delta (P_C F P_A D P_B E + \bar{P}_C F \bar{P}_A D \bar{P}_B E), \quad (2.28) \\
\delta S_{ABCD} &= \nabla [A \delta \Gamma_{B} CD + \nabla [C \delta \Gamma_{D}] AB].
\end{align*}

3 \( O(D, D) \) covariant Yang-Mills theory

3.1 Double field formulation

Our main result in the present paper comes from generalizing the previous analysis on the covariant quantities, especially (2.23), to Yang-Mills theory with a generic non-Abelian gauge group, \( G \). We postulate a DFT vector potential, \( V_A \), which is in the adjoint representation of the Lie algebra of the gauge group, \( G \). For a DFT tensor, \( T_{A_1 A_2 \cdots A_n} \), which is in the fundamental representation of \( G \), we define with the projection-compatible derivative (2.6),
\[ D_C T_{A_1 A_2 \cdots A_n} := \nabla_C T_{A_1 A_2 \cdots A_n} - i V_C T_{A_1 A_2 \cdots A_n}, \] (3.1)
This derivative is covariant with respect to the usual Yang-Mills gauge symmetry: with \( g \in G \), under
\begin{align*}
T_{A_1 A_2 \cdots A_n} &\rightarrow g T_{A_1 A_2 \cdots A_n}, \\
V_A &\rightarrow g V_A g^{-1} - i (\partial_A g) g^{-1}, \quad (3.2)
\end{align*}
the derivative transforms in a covariant fashion,
\[ D_C T_{A_1 A_2 \cdots A_n} \rightarrow g D_C T_{A_1 A_2 \cdots A_n}. \] (3.3)
Note that the projection and the dilaton are all Yang-Mills gauge singlets such that the projection-compatible derivative (2.6) does not change under the Yang-Mills gauge transformation.
The commutator of the above derivatives reads
\[ [D_A, D_B] T_{C_1 C_2 \cdots C_n} = -i F_{AB} T_{C_1 C_2 \cdots C_n} - \Gamma^{D}_{AB} D_D T_{C_1 C_2 \cdots C_n} + \sum_{i=1}^{n} R_{C_1 DAB} T_{C_1 \cdots C_{i-1} D C_{i+1} \cdots C_n}, \] (3.4)
where $R_{CDAB}$ is the curvature given in (2.15), and $F_{AB}$ is the field strength of the vector potential,

$$F_{AB} = \partial_A V_B - \partial_B V_A - i [V_A, V_B],$$

which surely transforms covariantly under the Yang-Mills gauge transformation,

$$F_{AB} \rightarrow g F_{AB} g^{-1}.$$  \hspace{1cm} (3.6)

However, this field strength is not DFT gauge covariant,

$$\delta_X F_{AB} \neq \tilde{L}_X F_{AB}.$$  \hspace{1cm} (3.7)

It is necessary to utilize the projection compatible derivative as in (2.23). Hence, instead of (3.5) we consider

$$F_{AB} := \nabla_A V_B - \nabla_B V_A - i [V_A, V_B] = F_{AB} - \Gamma^C_{AB} V_C.$$  \hspace{1cm} (3.8)

Although it is not covariant under the Yang-Mills gauge symmetry,

$$F_{AB} \rightarrow g F_{AB} g^{-1} + i \Gamma^C_{AB} (\partial C g) g^{-1},$$  \hspace{1cm} (3.9)

when its two $O(D, D)$ vector indices are projected into opposite chiralities,

$$P^A B C \bar{P}^D C F_{CD},$$  \hspace{1cm} (3.10)

it becomes covariant with respect to both the Yang-Mills and the DFT gauge symmetries, thanks to the level matching constraint (1.5) imposed on the explicit expression of the connection (2.7),

$$P^A C \bar{P}^B D F_{CD} \rightarrow P^A C \bar{P}^B D F_{CD} g^{-1},$$  \hspace{1cm} (3.11)

$$\delta_X (P^A C \bar{P}^B D F_{CD}) \equiv \tilde{L}_X (P^A C \bar{P}^B D F_{CD}).$$

Therefore, our double field formulation of a Yang-Mills action is

$$S_{YM} = g_{YM}^{-2} \int d^2 y e^{-2d} \text{Tr}(P^{AB} \bar{P}^{CD} F_{AC} F_{BD}),$$  \hspace{1cm} (3.12)

which can be coupled to the closed string DFT (1.8) as

$$S_{DFT} + S_{YM} = \int d^2 y e^{-2d} \left[ h^{AB} (4 \nabla_A d \nabla_B d + S_{AB}) + g_{YM}^{-2} \text{Tr}(P^{AB} \bar{P}^{CD} F_{AC} F_{BD}) \right].$$  \hspace{1cm} (3.13)

These actions are manifestly $O(D, D)$ covariant, and invariant under both the Yang-Mills and the DFT gauge transformations.
3.2 Component field formulation

Here we rewrite the above double field formulation of a Yang-Mills action (3.12) in terms of ordinary undoubled $D$-dimensional component fields, in a similar fashion that the closed string DFT action, $S_{\text{DFT}}$ (1.8), reduces to the more familiar looking effective action, $S_{\text{eff}}$ (1.1), upon the level matching constraint.

We first decompose the DFT vector potential into a chiral and an anti-chiral vectors,

$$ V_A = V_A^+ + V_A^- $$

such that $H_A^B V_B^\pm = \pm V_A^\pm$. The chiral and anti-chiral vectors assume the following generic forms,

$$ V_A^+ = \frac{1}{2} \begin{pmatrix} A^+ \lambda \\ (g+B)_{\mu \nu} A^+ \nu \end{pmatrix}, \quad V_A^- = \frac{1}{2} \begin{pmatrix} -A^- \lambda \\ (g-B)_{\mu \nu} A^- \nu \end{pmatrix}. $$

With the field redefinition,

$$ A_\mu := \frac{1}{2} (A_\mu^+ + A_\mu^-), \quad \phi_\mu := \frac{1}{2} (A_\mu^+ - A_\mu^-), $$

which is equivalent to $A_\mu^\pm = A_\mu \pm \phi_\mu$, the DFT vector potential can be parametrized by

$$ V_A = \begin{pmatrix} \phi^\lambda \\ A_\mu + B_{\mu \nu} \phi^\nu \end{pmatrix}. $$

Note that the $D$-dimensional vector indices, $\mu, \nu$, are here and henceforth freely raised or lowered by the $D$-dimensional metric, $g_{\mu \nu}$, in the usual manner.

Direct computation shows, turning off the $\tilde{x}$-dependence,

$$ P_A^C \hat{P}_B^D F_{CD} \equiv \frac{1}{4} \begin{pmatrix} -\hat{f}^\lambda \mu & \hat{f}^\lambda \tau (g+B)_{\tau \nu} \\ -(g+B)_{\rho \sigma} \hat{f}^\sigma \mu & (g+B)_{\rho \sigma} \hat{f}^\sigma \tau (g+B)_{\tau \nu} \end{pmatrix}, $$

where we set

$$ \hat{f}_{\mu \nu} := f_{\mu \nu} - D_\mu \phi_\nu - D_\nu \phi_\mu + i [\phi_\mu, \phi_\nu] + H_{\mu \nu \lambda} \phi^\lambda, $$

$$ D_\mu \phi_\nu := \nabla_\mu \phi_\nu - i [ A_\mu, \phi_\nu ] = \partial_\mu \phi_\nu - \phi_\lambda \gamma_{\mu \nu}^\lambda - i [ A_\mu, \phi_\nu ], $$

$$ f_{\mu \nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - i [ A_\mu, A_\nu ], $$

$$ H_{\lambda \mu \nu} := \partial_\lambda B_{\mu \nu} + \partial_\mu B_{\nu \lambda} + \partial_\nu B_{\lambda \mu}. $$

9
Unlike (2.6) and (3.1), in our $D$-dimensional notation, $\nabla_\mu$ denotes the usual diffeomorphism covariant derivative involving the Christoffel symbol, $\gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$, and $D_\mu$ is the diffeomorphism plus Yang-Mills gauge covariant derivative.

It is worth while to note

$$\hat{f}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + i [A_\mu, A_\nu] + H_{\mu\nu\lambda} \phi^\lambda,$$

$$\hat{f}_{[\mu\nu]} = f_{\mu\nu} + i [\phi_\mu, \phi_\nu] + H_{\mu\nu\lambda} \phi^\lambda,$$

$$\hat{f}_{(\mu\nu)} = -(D_\mu \phi_\nu + D_\nu \phi_\mu),$$

and for (3.18)

$$P_A^C \tilde{P}_B^D \mathcal{F}_{CD} \equiv P_A^C \tilde{P}_B^D \begin{pmatrix} 0 & 0 \\ 0 & \hat{f}_{\mu\nu} \end{pmatrix}_{CD}.$$  (3.21)

Now, from (3.18), it is straightforward to show that the Yang-Mills action in the double field formulation (3.12) reduces to

$$S_{YM} \equiv g_{YM}^{-2} \int d^Dx \sqrt{-g} e^{-2\phi} \text{Tr} \left( -\frac{1}{4} \hat{f}_{\mu\nu} \hat{f}^{\mu\nu} \right),$$  (3.22)

and hence,

$$S_{DFT} + S_{YM} \equiv \int d^Dx \sqrt{-g} e^{-2\phi} \left[ R_g + 4(\partial \phi)^2 - \frac{1}{12} H^2 - \frac{1}{4} g_{YM}^{-2} \text{Tr} \left( \hat{f}^2 \right) \right].$$  (3.23)

Explicitly, we have for $S_{YM}$ (3.22),

$$\text{Tr} \left( \hat{f}_{\mu\nu} \hat{f}^{\mu\nu} \right) = \text{Tr} \left( f_{\mu\nu} f^{\mu\nu} + 2D_\mu \phi_\nu D^\mu \phi^\nu + 2D_\mu \phi_\nu D^\nu \phi^\mu - [\phi_\mu, \phi_\nu] [\phi^\mu, \phi^\nu] \\ + 2i f_{\mu\nu} [\phi^\mu, \phi^\nu] + 2 (f^{\mu\nu} + i [\phi^\mu, \phi^\nu]) H_{\mu\nu\sigma} \phi^\sigma + H_{\mu\sigma\rho} H^{\mu\nu\rho} \phi^\sigma \phi^\tau \right).$$  (3.24)

The above actions (3.22), (3.23) are clearly invariant under both the Yang-Mills and the DFT gauge symmetries. Moreover, though not manifest, by construction it enjoys T-duality.
4 Comments

We recall the DFT tensor (3.10) which is fully covariant under the $O(D,D)$ T-duality as well as all the gauge symmetries, to set

$$\hat{F}_{AB} := P_A^C \hat{P}_B^D \mathcal{F}_{CD}. \quad (4.1)$$

Apart from $\text{Tr}(\hat{F}^{AB} \hat{F}_{AB})$ which essentially leads to our DFT formulation of the Yang-Mills action (3.12), the following quantity of even power in the field strength is also fully covariant,

$$\text{Tr} \left( \hat{F}^{A_1 B_1} \hat{F}^{A_2 B_2} \hat{F}^{A_3 B_3} \cdots \hat{F}^{A_n B_n} \right). \quad (4.2)$$

Due to the chirality of $\hat{F}_{AB}$, there is no covariant scalar with odd power. Especially, for the Abelian group, $G = U(1)$, we obtain another covariant quantity,

$$\det \left( \eta_{AB} + \kappa \hat{F}_{AC} \hat{F}^C_B \right) = \det \left( \eta_{AB} + \kappa \hat{F}_{CA} \hat{F}^C_B \right), \quad (4.3)$$

where $\kappa$ is a constant and the determinant is taken over the $O(D,D)$ vector indices, $A, B$. Since this is a scalar rather than a scalar density, there appears no compulsory reason to take a square root of the determinant constructing a Born-Infeld type action.

In the presence of a curved $D$-brane, string theory can force a topological twisting on a usual Yang-Mills theory, converting scalars into one-form [28]. Especially, when a pure Yang-Mills theory in $(D+D)$-dimensions is reduced to $D$-dimensions, the Lorentz symmetry group coincides with the $R$-symmetry group. If we diagonalize these two, as in topological twisting theories [29–33], we may obtain the following maximally twisted action,

$$S_{\text{twisted}} \equiv -g_{\text{YM}}^{-2} \int d\tau^D \sqrt{-g} \text{Tr} \left( \frac{1}{4} f_{\mu \nu} f^{\mu \nu} + \frac{1}{2} D_\mu \phi_\nu D^\mu \phi^\nu - \frac{1}{4} [\phi_\mu, \phi_\nu][\phi^\mu, \phi^\nu] + \frac{1}{2} R_{\mu \nu} \phi^\mu \phi^\nu \right). \quad (4.4)$$

Intriguingly this twisted action resembles our Yang-Mills action (3.22), although they differ in some details. More precise string theory interpretation of our double field formulation of Yang-Mills theory is desirable (for some related works we refer [34–36]). Doubled sigma-model formalism [37–40] may provide useful insights.

---

9 Generalization to non-Abelian Born-Infeld action is also doable following various prescriptions, e.g. [22][27].
10 On the other hand, due to the chirality of $\hat{F}_{AB}$, $\det(\eta_{AB} + \kappa \hat{F}_{AB})$ is trivial.
11 To confirm the difference, it is necessary to use the identity,

$$[D_\mu, D_\nu] \phi^\nu + R_{\mu \nu} \phi^\nu + i [f_{\mu \nu}, \phi^\nu] = 0.$$
Note added: After submitting the first version of this manuscript to arXiv, a related work by Hohm and Kwak appeared \cite{41}. Their paper attempts the double field theory formulation of the heterotic string effective action, and hence the inclusion of Yang-Mills theories. It is based on an enlarged, yet broken, $O(D, D + n)$ T-duality, which differs from ours, \emph{i.e.} unbroken $O(D, D)$.

Acknowledgements
We are grateful to Seung Ki Kwak for useful discussion which motivated our work. The work was supported by the National Research Foundation of Korea(NRF) grants funded by the Korea government(MEST) with the grant numbers 2005-0049409 (CQUeST) and 2010-0002980.
References

[1] C. Hull and B. Zwiebach, JHEP 0909, 099 (2009) [arXiv:0904.4664 [hep-th]].
[2] C. Hull and B. Zwiebach, JHEP 0909, 090 (2009) [arXiv:0908.1792 [hep-th]].
[3] O. Hohm, C. Hull and B. Zwiebach, JHEP 1007, 016 (2010) [arXiv:1003.5027 [hep-th]].
[4] O. Hohm, C. Hull and B. Zwiebach, JHEP 1008, 008 (2010) [arXiv:1006.4823 [hep-th]].
[5] T. H. Buscher, Phys. Lett. B 159 (1985) 127.
[6] T. H. Buscher, Phys. Lett. B 194 (1987) 59.
[7] T. H. Buscher, Phys. Lett. B 201 (1988) 466.
[8] O. Hohm, arXiv:1101.3484 [hep-th].
[9] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B 322 (1989) 167.
[10] A. A. Tseytlin, Phys. Lett. B 242, 163 (1990).
[11] A. A. Tseytlin, Nucl. Phys. B 350, 395 (1991).
[12] W. Siegel, Phys. Rev. D 48, 2826 (1993) [arXiv:hep-th/9305073].
[13] W. Siegel, Phys. Rev. D 47, 5453 (1993) [arXiv:hep-th/9302036].
[14] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, Phys. Lett. B 336 (1994) 183 [arXiv:hep-th/9406206].
[15] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rept. 244, 77 (1994) [arXiv:hep-th/9401139].
[16] M. Grana, R. Minasian, M. Petrini and D. Waldram, JHEP 0904 (2009) 075 [arXiv:0807.4527 [hep-th]].
[17] S. K. Kwak, JHEP 1010 (2010) 047 [arXiv:1008.2746 [hep-th]].
[18] I. Jeon, K. Lee and J.-H. Park, arXiv:1011.1324 [hep-th].
[19] O. Hohm and S. K. Kwak, arXiv:1011.4101 [hep-th].
[20] T. Courant, Dirac Manifolds, Trans. Amer. Math. Soc. 319: 631-661, 1990.
[21] M. Gualtieri, Ph.D. Thesis “Generalized complex geometry,” arXiv:math/0401221.
[22] T. Hagiwara, J. Phys. A 14, 3059 (1981).
[23] P. C. Argyres and C. R. Nappi, Nucl. Phys. B 330, 151 (1990).
[24] A. A. Tseytlin, Nucl. Phys. B 501, 41 (1997) [arXiv:hep-th/9701125].

[25] J.-H. Park, Phys. Lett. B 458 (1999) 471 [arXiv:hep-th/9902081].

[26] E. Serie, T. Masson and R. Kerner, Phys. Rev. D 68 (2003) 125003 [arXiv:hep-th/0307105].

[27] E. Serie, T. Masson and R. Kerner, Phys. Rev. D 70 (2004) 067701 [arXiv:hep-th/0408012].

[28] M. Bershadsky, C. Vafa and V. Sadov, Nucl. Phys. B 463, 420 (1996) [arXiv:hep-th/9511222].

[29] E. Witten, Commun. Math. Phys. 117, 353 (1988).

[30] C. Vafa and E. Witten, Nucl. Phys. B 431, 3 (1994) [arXiv:hep-th/9408074].

[31] J. P. Yamron, Phys. Lett. B 213, 325 (1988).

[32] N. Marcus, Nucl. Phys. B 452, 331 (1995) [arXiv:hep-th/9506002].

[33] J.-H. Park and D. Tsimpis, Nucl. Phys. B 776 (2007) 405 [arXiv:hep-th/0610159].

[34] E. Bergshoeff, I. Entrop and R. Kallosh, Phys. Rev. D 49 (1994) 6663 [arXiv:hep-th/9401025].

[35] E. Bergshoeff, B. Janssen and T. Ortin, Class. Quant. Grav. 13 (1996) 321 [arXiv:hep-th/9506156].

[36] G. Chalmers and W. Siegel, arXiv:hep-th/9712191.

[37] C. M. Hull, JHEP 0510, 065 (2005) [arXiv:hep-th/0406102].

[38] C. M. Hull, JHEP 0707, 080 (2007) [arXiv:hep-th/0605149].

[39] D. S. Berman, N. B. Copland and D. C. Thompson, Nucl. Phys. B 791 (2008) 175 [arXiv:0708.2267 [hep-th]].

[40] D. S. Berman and D. C. Thompson, Phys. Lett. B 662 (2008) 279 [arXiv:0712.1121 [hep-th]].

[41] O. Hohm and S. K. Kwak, arXiv:1103.2136 [hep-th].

14