A GENERALIZATION OF THE WEAK AMENABILITY OF BANACH ALGEBRAS

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Abstract. Let $A$ be a Banach algebra and let $\varphi$ and $\psi$ be continuous homomorphisms on $A$. We consider the following module actions on $A$,
\[ a \cdot x = \varphi(a)x, \quad x \cdot a = x\psi(a) \quad (a, x \in A). \]
We denote by $A_{\langle \varphi, \psi \rangle}$ the above $A$-module. We call the Banach algebra $A$, $(\varphi, \psi)$-weakly amenable if every derivation from $A$ into $(A_{\langle \varphi, \psi \rangle})^*$ is inner. In this paper among many other things we investigate the relations between weak amenability and $(\varphi, \psi)$-weak amenability of $A$. Some conditions can be imposed on $A$ such that the $(\varphi'', \psi'')$-weak amenability of $A^{**}$ implies the $(\varphi, \psi)$-weak amenability of $A$.

1. Introduction and preliminaries

Let $A$ be a Banach algebra and let $X$ be a Banach $A$-module. Then a derivation from $A$ into $X$ is a (bounded) linear map $D: A \longrightarrow X$ such that for every $a, b \in A$, $D(ab) = D(a) \cdot b + a \cdot D(b)$. If $x \in X$, the map $a \mapsto a \cdot x - x \cdot a$, $(a \in A)$ is a derivation. A derivation of this form is called an inner derivation. The set of all bounded linear operators from $A$ into $X$ is denoted by $\mathcal{B}(A, X)$. The set of all bounded linear operators from $A$ into $X$ is denoted by $\mathcal{Z}^1(A, X)$, and the set of all inner derivations from $A$ into $X$ is denoted by $\mathcal{B}^1(A, X)$. Then $H^1(A, X) = \frac{\mathcal{Z}^1(A, X)}{\mathcal{B}^1(A, X)}$ is the first Hochschild cohomology group of $A$ with coefficients in $X$.

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Let $A$ be a Banach algebra and $X$ be a Banach $A$-module. Then $X^*$ is the dual of Banach $A$-module $X$, and is also a Banach $A$-module as well, if for each $a \in A$, $x \in X$ and $x^* \in X^*$ we define

$$\langle a \cdot x^*, x \rangle = \langle x^*, x \cdot a \rangle, \quad \langle x^* \cdot a, x \rangle = \langle x^*, a \cdot x \rangle.$$ 

A Banach algebra $A$ is amenable if every derivation from $A$ into every dual Banach $A$-module is inner, equivalently if $H^1(A, X^*) = \{0\}$ for every Banach $A$-module $X$, this definition was introduced by Johnson in [12]. A is weakly amenable if $H^1(A, A^*) = \{0\}$; this definition generalizes that introduced by Bade, Curtis and Dales in [1]. We introduce the following new definition of amenability which is related to homomorphisms of Banach algebras.

Let $A$ be a Banach algebra and let $\varphi$ and $\psi$ be continuous homomorphisms on $A$. We consider the following module actions on $A$,

$$a \cdot x := \varphi(a)x, \quad x \cdot a := x\psi(a) \quad (a, x \in A).$$

We denote the above $A$-module by $A(\varphi, \psi)$.

Let $X$ be an $A$-module. A bounded linear mapping $d : A \to X$ is called a $(\varphi, \psi)$-derivation if

$$d(ab) = d(a) \cdot \varphi(b) + \psi(a) \cdot d(b) \quad (a, b \in A).$$

A bounded linear mapping $d : A \to X$ is called a $(\varphi, \psi)$-inner derivation if there exists $x \in X$ such that

$$d(a) = x \cdot \varphi(a) - \psi(a) \cdot x \quad (a \in A).$$

Derivations of this form are studied in [14, 15, 16].

**Definition 1.1.** Let $A$ be a Banach algebra and let $\varphi$ and $\psi$ be continuous homomorphisms on $A$. Then $A$ is called $(\varphi, \psi)$-weakly amenable if $H^1(A, (A(\varphi, \psi))^*) = \{0\}$.

Let $A$ and $B$ be Banach algebras. We denote by $\text{Hom}(A, B)$ the metric space of all bounded homomorphisms from $A$ into $B$, with the metric derived from the usual linear operator norm $\| \cdot \|$ on $B(A, B)$ and denote $\text{Hom}(A, A)$ by $\text{Hom}(A)$. The following assertions hold for any Banach algebra $A$.

(a) If $A$ is amenable then $A$ is an $(\varphi, \psi)$-weakly amenable for each $\varphi$ and $\psi$ in $\text{Hom}(A)$.

(b) $A$ is weakly amenable if and only if $A$ is an $(\text{id}, \text{id})$-weakly amenable ($\text{id}$=identity homomorphism).

(c) Let $A$ be a commutative weakly amenable Banach algebra. Then $Z^1(A, X) = \{0\}$ for each Banach $A$-module $X$ [3, Theorem 2.8.63]. Therefore $A$ is $(\varphi, \psi)$-weakly amenable for all $\varphi, \psi \in \text{Hom}(A)$ if and only if $A$ is commutative and weakly amenable.

**Definition 1.2.** Let $A$ be a Banach algebra, $X$ be a Banach $A$-module and let $\varphi, \psi \in \text{Hom}(A)$. A derivation $D : A \to X$ is called approximately $(\varphi, \psi)$-inner if there exists a net $(x_\alpha)$ in $X$ such that, for all $a \in A$, $D(a) = \lim_\alpha x_\alpha \cdot \varphi(a) - \psi(a) \cdot x_\alpha$ in norm.
Definition 1.3. A Banach algebra $A$ is approximately $(\varphi, \psi)$-weakly amenable if every derivation $D : A \to (A_{(\varphi, \psi)})^*$ is approximately $(\varphi, \psi)$-inner.

Whenever $\varphi = \psi = \text{id}$, this is just the definition of approximate weak amenability developed by Ghahramani and Loy in [9].

Definition 1.4. Let $A$ be an algebra, and let $\varphi \in \Phi_A \cup \{0\}$ ($\Phi_A$ be the character space of $A$). A linear functional $d$ on $A$ is a point derivation at $\varphi$ if

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in A).$$

Throughout this paper $A$ denotes a Banach algebra and $A^{**}$ is the second dual of $A$ equipped with the first Arens product. This product can be characterized as the extension to $A^{**} \times A^{**}$ of the bilinear map $A \times A \to A : (a, b) \to ab$ with the following properties:

i) for fixed $b'' \in A^{**}$, $a'' \mapsto a''b''$ is $w^*$-continuous on $A^{**}$.

ii) for fixed $b \in A$, $a'' \mapsto ba''$ is $w^*$-continuous on $A^{**}$.

The image of $A$ in $A^{**}$ under the canonical embedding is denoted by $\hat{A}$.

In section 2 we prove the main results for this new concept of amenability. In section 3, we develop the relation between the $(\varphi, \psi)$-weak amenability of a Banach algebra $A$ and $A^{**}$. Finally in section 4 we give some examples to show that the new concept of amenability is different from amenability and weak amenability.

2. $(\varphi, \psi)$-WEAK AMENABILITY

Let $A$ be a Banach algebra, and let $A^2 = \text{span}\{ab : a, b \in A\}$.

Proposition 2.1. Let $A$ be Banach algebra and let $\varphi, \psi \in \text{Hom}(A)$ such that $\varphi(a)b = a\psi(b)$ for all $a, b \in A$. If $A$ is $(\varphi, \psi)$-weakly amenable, then $\overline{A^2} = A$, where $\overline{A^2}$ is the closure of $A^2$ in $A$.

Proof. Suppose $\overline{A^2} \neq A$. Take $a_0 \in A \setminus \overline{A^2}$ and $f \in A^*$ such that $f|_{A^2} = 0$ and $\langle f, a_0 \rangle = 1$. Define $d : A \to (A_{(\varphi, \psi)})^*$ by $d(a) = \langle f, a \rangle f$. It is easy check that $d$ is a $(\varphi, \psi)$-derivation. Since $A$ is $(\varphi, \psi)$-weakly amenable, $d$ is $(\varphi, \psi)$-inner, so that there is a $g \in (A_{(\varphi, \psi)})^*$ such that $d(a) = g \cdot \varphi(a) - \psi(a) \cdot g$, for all $a \in A$. So we have $\langle da_0, a_0 \rangle = 1$. On the other hand

$$\langle d(a_0), a_0 \rangle = \langle g, \varphi(a_0)a_0 \rangle - \langle g, a_0\psi(a_0) \rangle = 0.$$

This is a contradiction. \hfill $\Box$

Corollary 2.2. Let $A$ be a Banach algebra. Then $A$ is $(0, 0)$-weakly amenable if and only if $\overline{A^2} = A$.

Proof. Let $A$ be $(0, 0)$-weakly amenable. Then by the above theorem, $\overline{A^2} = A$. For the converse let $d : A \to (A_{(0, 0)})^*$ be a $(0, 0)$-derivation. Then we have $d(A^2) = \{0\}$. Since $d$ is continuous, we have $d = 0$. So $d$ is $(0, 0)$-inner. \hfill $\Box$

Let $A$ be a weakly amenable Banach algebra or $A$ be a Banach algebra with a bounded left (right) approximate identity. Then $A^2$ is dense in $A$. Thus $A$ is $(0, 0)$-weakly amenable.
Proposition 2.3. Let $A$ be a Banach algebra and $\varphi, \psi$ and $\lambda$ are continuous homomorphisms from $A$ into $A$. If $\varphi$ is an epimorphism and $A$ is $(\psi \circ \varphi, \lambda \circ \varphi)$-weakly amenable, then $A$ is $(\psi, \lambda)$-weakly amenable.

Proof. Let $d : A \rightarrow (A(\psi, \lambda))^*$ be a continuous $(\psi, \lambda)$-derivation, and $D = d \circ \varphi$. We see that $D$ is a $(\psi \circ \varphi, \lambda \circ \varphi)$-derivation. So there exists a $f \in (A(\psi, \lambda))^*$ such that for each $a \in A$, $D(a) = f \cdot (\psi \circ \varphi)(a) - (\lambda \circ \varphi)(a) \cdot f$. Let $b \in A$. Then there exists a $a \in A$ such that $\varphi(a) = b$ and so

$$d(b) = d(\varphi(a)) = D(a) = f \cdot \psi(\varphi(a)) - \lambda(\varphi(a)) \cdot f = f \cdot \psi(b) - \lambda(b) \cdot f.$$  

Thus $d$ is an $(\psi, \lambda)$-inner. \qed

Corollary 2.4. Let $A$ be a Banach algebra and let $\varphi \in \text{Hom}(A)$. If $\varphi$ is an epimorphism and $A$ is $(\varphi^n, \varphi^n)$-weakly amenable for some $n \in \mathbb{N}$. Then $A$ is weakly amenable.

There are Banach algebras which are $(\varphi, \varphi)$-weakly amenable where $\varphi$ is not an epimorphism, and $A$ is weakly amenable. This will be presented in Examples 4.3 and 4.4. The converse of the Corollary 2.4 is true when $\varphi^2 = 1_A$ or $\varphi$ is an epimorphism such that $\varphi^2|_{[A]} = 1_A$ where $[A] = \{ab - ba | a, b \in A\}$. In the following theorems and corollaries we prove the above claims.

Theorem 2.5. Let $A$ be a Banach algebra and let $\psi, \lambda, \varphi \in \text{Hom}(A)$ and $\varphi^2 = 1_A$. If $A$ is $(\psi, \lambda)$-weakly amenable, then $A$ is $(\psi \circ \varphi, \lambda \circ \varphi)$-weakly amenable.

Proof. Let $D : A \rightarrow (A(\psi, \lambda))^*$ be a $(\psi \circ \varphi, \lambda \circ \varphi)$-derivation and let $d = D \circ \varphi^{-1}$. It can be shown that $d$ is a $(\psi, \lambda)$-derivation. Thus there exist a $f \in (A(\psi, \lambda))^*$ such that for all $a \in A$, $d(a) = f \cdot \psi(\varphi^{-1}(a)) - \lambda(\varphi^{-1}(a)) \cdot f$ and so we have $D(a) = D(\varphi^{-1}(\varphi(a))) = f \cdot \psi(\varphi(a)) - \lambda(\varphi(a)) \cdot f$, i.e., $D$ is an $(\psi \circ \varphi, \lambda \circ \varphi)$-inner derivation. \qed

Corollary 2.6. If $A$ is weakly amenable and $\varphi \in \text{Hom}(A)$ such that $\varphi^2 = 1_A$, then $A$ is $(\varphi^n, \varphi^n)$-weakly amenable for all $n \in \mathbb{N}$.

Theorem 2.7. Let $\varphi, \psi \in \text{Hom}(A)$ and let $A$ be $(\psi, \psi)$-weakly amenable. If $\varphi|_{[A]} = \text{id}$ and $\varphi$ is an epimorphism, then $A$ is $(\varphi, \varphi, \varphi \circ \psi)$- weakly amenable.

Proof. Suppose that $D : A \rightarrow (A(\varphi, \psi))^*$ is an $(\varphi, \psi, \varphi \circ \psi)$-derivation. Set $d : A \rightarrow (A(\psi, \psi))^*$ as follows

$$\langle d(a), b \rangle := \langle D(a), \varphi(b) \rangle.$$  

Then $d$ is a $(\psi, \psi)$-derivation. Thus there exists a $f \in (A(\psi, \psi))^*$ such that for every $a \in A$, $d(a) = f \cdot \psi(a) - \psi(a) \cdot f$. Let $b \in A$. Since $\varphi$ is onto, there exists $b_1 \in A$ such that $b = \varphi(b_1)$. So

$$\langle D(a), b \rangle = \langle d(a), b_1 \rangle = \langle f \cdot \psi(a) - \psi(a) \cdot f, b_1 \rangle = \langle f, \varphi(\psi(a)b_1 - b_1\psi(a)) \rangle = \langle f \cdot \varphi \circ \psi(a) - \varphi \circ \psi(a) \cdot f, b \rangle.$$  

Therefore $D$ is an $(\varphi \circ \psi, \varphi \circ \psi)$-inner. \qed
Corollary 2.8. Let $A$ be a Banach algebra and let $\varphi \in \text{Hom}(A)$. Suppose that $A$ is weakly amenable and $\varphi$ is an epimorphism such that $\varphi|_A = \text{id}$. Then $A$ is $(\varphi^n, \varphi^n)$-weakly amenable for all $n \in \mathbb{N}$.

Proposition 2.9. Let $A$ be a Banach algebra and $\varphi \in \text{Hom}(A)$. Suppose that $A$ is $(\varphi^n, \varphi^n)$-weakly amenable for all $n \in \mathbb{N}$, and $\varphi^n \to 1_A$ in norm. Then $A$ is approximately weakly amenable.

Proof. Let $D : A \to A^*$ be a derivation. For every $n \in \mathbb{N}$ set $D_n : A \to (A(\varphi^n, \varphi^n))^*$, $D_n(a) = D(\varphi^n(a))$. It is clear that $D_n$ is an $(\varphi^n, \varphi^n)$-derivation. So there exists a sequence $(f_n)$ in $A^*$ such that $D_n(a) = f_n \cdot \varphi^n(a) - \varphi^n(a) \cdot f_n$. Since $\varphi^n(a) \to a$, $D_n(a) \to D(a)$. Therefore $D(a) = \lim_n (f_n \cdot a - a \cdot f_n).$ \hfill $\Box$

In the proof of the Proposition 2.9, if the sequence $(f_n)$ has an accumulation point then $A$ is weakly amenable.

Theorem 2.10. Let $A$ be a Banach algebra, $\varphi \in \text{Hom}(A)$ and $0 \neq \psi \in \Phi_A$. Let $A$ be $(\varphi, \varphi)$-weakly amenable and $\text{Im}\varphi \not\subset \ker\psi$. Then there are no non-zero continuous point derivations at $\psi \circ \varphi$.

Proof. Let $\psi \in \Phi_A$ and let $\varphi \in \text{Hom}(A)$. Then $\psi \circ \varphi \in \Phi_A$. Suppose that $d = d_{\psi \circ \varphi} : A \to \mathbb{C}$ is a point derivation at $\psi \circ \varphi$. We define $D : A \to (A(\varphi, \varphi))^*$ by $D(a) := d(a)\psi$. Then clearly $D$ is a $(\varphi, \varphi)$-derivation. Since $A$ is $(\varphi, \varphi)$-weakly amenable, there exists a $f \in (A(\varphi, \varphi))^*$ such that $D(a) = f \cdot \varphi(a) - \varphi(a) \cdot f$. On the other hand since $\text{Im}\varphi \not\subset \ker\psi$, there exist $a_1 \in A$ such that $\psi(\varphi(a_1)) = 1$. If $d_{\psi \circ \varphi}$ is a non-zero point derivation, then $\ker(\psi \circ \varphi) \subset \ker d_{\psi \circ \varphi}$. In fact if $\ker(\psi \circ \varphi) \subset \ker d_{\psi \circ \varphi}$, then there is an $\alpha \in \mathbb{C}$ such that $d_{\psi \circ \varphi} = \alpha(\psi \circ \varphi)$. So

$$2\alpha = 2\alpha((\psi \circ \varphi)(a_1)) = 2d_{\psi \circ \varphi}(a_1)$$

$$= 2d_{\psi \circ \varphi}(a_1)\psi \circ \varphi(a_1)$$

$$= d_{\psi \circ \varphi}(a_1)\psi \circ \varphi(a_1) + \psi \circ \varphi(a_1)d_{\psi \circ \varphi}(a_1)$$

$$= d_{\psi \circ \varphi}(a_1^2) = \alpha(\psi \circ \varphi)(a_1^2) = \alpha.$$

Thus $\alpha = 0$, i.e. $d = 0$ which is a contradiction.

Therefore there exist $a_2 \in \ker(\psi \circ \varphi)$ such that $d(a_2) = 1$. Put $a_0 = a_1 + (1 - d(a_1))a_2$, then

$$\psi \circ \varphi(a_0) = \psi \circ \varphi(a_1) + (1 - d(a_1))\psi \circ \varphi(a_2) = 1,$$

and

$$d(a_0) = d(a_1) + (1 - d(a_1))d(a_2) = d(a_1) + 1 - d(a_1) = 1.$$

Therefore

$$1 = d(a_0)\psi(\varphi(a_0))$$

$$= \langle D(a_0), \varphi(a_0) \rangle = \langle f \cdot \varphi(a_0) - \varphi(a_0) \cdot f, \varphi(a_0) \rangle$$

$$= \langle f, (\varphi(a_0))^2 \rangle - \langle f, (\varphi(a_0))^2 \rangle = 0.$$

Which is a contradiction. \hfill $\Box$
By using the Theorem 2.10, if $A$ is a weakly amenable Banach algebra, then there is no non-zero continuous point derivation on $A$. Therefore Theorem 2.10 could be considered as a generalization of [3, Theorem 2.8.63]. If $A$ is approximately $(\varphi, \varphi)$-weakly amenable, then the Theorem 2.10 is also true.

**Theorem 2.11.** Let $\varphi \in \text{Hom}(A)$ and $\varphi^2 = \varphi$. Suppose that $A$ and $\ker \varphi$ are weakly amenable, $\text{Im} \varphi$ is an ideal of $A$. Then $A$ is $(\varphi, \varphi)$-weakly amenable.

**Proof.** Let $D : A \to (A_{\varphi, \varphi})^*$ be a $(\varphi, \varphi)$-derivation. Take $d : A \to A^*$ with $\langle d(a), b \rangle := \langle D(a), \varphi(b) \rangle$, and so $d$ is a derivation. Then there exists a $f \in A^*$ such that $d(a) = f \cdot a - a \cdot f$, $(a \in A)$.

Since $\varphi : A \to \text{Im} \varphi$ is a projection, $A = \text{Im} \varphi \oplus \ker \varphi$ where $\text{Im} \varphi$ and $\ker \varphi$ are closed ideals of $A$. Let $a \in A$. Then there exist $a_1, a_2 \in A$ such that $a = a_1 + a_2$ where $a_1 \in \text{Im} \varphi$ and $a_2 \in \ker \varphi$. Since $\ker \varphi$ is weakly amenable, $(\ker \varphi)^2 = \ker \varphi$. So, there is a net $(t_\alpha s_\alpha)_\alpha \subset (\ker \varphi)^2$ such that $t_\alpha s_\alpha \to a_2$, and $D(a_2) = \lim_\alpha D(t_\alpha s_\alpha) = \lim_\alpha (D(t_\alpha) \cdot \varphi(s_\alpha) - \varphi(t_\alpha) \cdot D(s_\alpha)) = 0$.

Therefore

$$D(a) = D(a_1) = D(\varphi(a_1)) = D(\varphi(a)),$$

so we have

$$\langle D(a), \varphi(b) \rangle = \langle D(\varphi(a)), \varphi(b) \rangle = \langle d(\varphi(a)), b \rangle = \langle f \cdot \varphi(a) - \varphi(a) \cdot f, b \rangle = \langle f, \varphi(a) b - b \varphi(a) \rangle = \langle d(-b), \varphi(a) \rangle = \langle D(-\varphi(b)), \varphi^2(a) \rangle = \langle d(-\varphi(b)), \varphi(a) \rangle = \langle f \cdot \varphi(a) - \varphi(a) \cdot f, b \varphi(b) \rangle.$$

On the other hand $b = b_1 + b_2$ such that $b_1 \in \text{Im} \varphi$, $b_2 \in \ker \varphi$ we have $\varphi(b) = \varphi(\varphi^2(b)) = \varphi(b_1) = b_1$, hence

$$\langle D(a), b_2 \rangle = \lim_\alpha \lim_\beta \langle D(s_{1\beta} s_{2\beta}), t_{1\alpha} t_{2\alpha} \rangle = \lim_\alpha \lim_\beta \langle D(s_{1\beta}) \cdot \varphi(s_{2\beta}) + \varphi(s_{1\beta}) \cdot D(s_{2\beta}), t_{1\alpha} t_{2\alpha} \rangle = \lim_\alpha \lim_\beta \langle D(s_{1\beta}), \varphi(s_{2\beta}) t_{1\alpha} t_{2\alpha} \rangle + \lim_\alpha \lim_\beta \langle D(s_{2\beta}), t_{1\alpha} t_{2\alpha} \varphi(s_{1\beta}) \rangle = 0,$$

since $s_{1\beta}, s_{2\beta} \in \ker \varphi$. $\varphi(s_{2\beta}) t_{1\alpha} t_{2\alpha}, t_{1\alpha} t_{2\alpha} \varphi(s_{1\beta}) \in \text{Im} \varphi \cap \ker \varphi = \{0\}$. Therefore $A$ is $(\varphi, \varphi)$-weakly amenable.

□

3. **$(\varphi, \psi)$-Weak Amenability of the Second Dual**

Let $A$ be a Banach algebra. We consider $A^{**}$ the second dual of $A$. It is known that the Banach algebra $A$ inherits amenability from $A^{**}$ [11]. No example is yet known whether this fails if one considers the weak amenability instead, but the property is known to hold for the Banach algebras $A$ which are left ideals in $A^{**}$ [10], the dual Banach algebras [8], the Banach algebras $A$ which are Arens regular and every derivation from $A$ into $A^*$ is weakly compact [5], Banach algebras for which the second adjoint of each derivation $D : A \to A^*$ satisfies $D''(A^{**}) \subseteq \text{WAP}(A)$, and the Banach algebras $A$ which are right ideals
Let $A$ be a Banach algebra and let $\varphi, \psi \in \text{Hom}(A)$ such that $\varphi(a)b = a\psi(b)$ for all $a, b \in A$. If $A^{**}$ is $(\varphi'', \psi'')$-weakly amenable, then by Proposition 2.1, $A^{**2} = A^{**}$. Thus we can show that $A^2 = A$ [8, Proposition 2.1]. So by Corollary 2.2, $A$ is $(0, 0)$-weakly amenable. A question remains whether the $(\varphi'', \psi'')$-weak amenability of $A^{**}$ implies the $(\varphi, \psi)$-weak amenability of $A$ and vice versa.

$L^1(\mathbb{R})$ is (id, id)-weakly amenable but $L^1(\mathbb{R})^{**}$ is not (id, id)-weakly amenable. In general if $G$ is a nondiscrete locally compact group then $L^1(G)^{**}$ is not (id, id)-weakly amenable [4], but $L^1(G)$ is (id, id)-weakly amenable [13].

For an infinite compact metric space $X$, $\text{lip}_0(X)$ is (id, id)-weakly amenable, for $0 < \alpha < 1/2$, but $\text{lip}_0(X)^{**}$ is not (id, id)-weakly amenable [1].

**Proposition 3.1.** Let $A$ be Banach algebra and $\varphi \in \text{Hom}(A)$, if $A^{**}$ is $(\varphi'', 0)$-weakly amenable, then $A$ is $(\varphi, 0)$-weakly amenable.

**Proof.** Suppose that $D : A \to (A(\varphi, 0))^*$ is a continuous $(\varphi, 0)$-derivation. Take $a'', b'' \in A^{**}$ and take bounded nets $(a_\alpha)$ and $(b_\beta)$ in $A$ with $a_\alpha \to a''$, $b_\beta \to b''$ in the $w^*$-topology of $A^{**}$. Then $D'' : A^{**} \to (A(\varphi'', 0))^*$ is an $(\varphi'', 0)$-derivation because

$$D''(a''b'') = w^* - \lim_{\alpha, \beta} D''(\hat{a_\alpha} \hat{b_\beta}) = w^* - \lim_{\alpha, \beta} (D(a_\alpha) \cdot \varphi(b_\beta)) = D''(a'') \cdot \varphi''(b'').$$

Therefore there exists $a''_0 \in A^{***}$ such that $D''(a'') = a''_0 \varphi''(a'')$ for all $a'' \in A^{**}$. We obtain $D(a) = a_0 \varphi(a)$ for all $a \in A$, where $a_0$ is the restriction of $a''_0$ to $A$. Thus $A$ is an $(\varphi, 0)$-weakly amenable.

If $A^{**}$ is $(0, \psi'')$-weakly amenable, then $A$ is $(0, \psi)$-weakly amenable if and only if $D''(a''b'') = \psi''(a'') \cdot D''(b'')$ if and only if $\psi''(a'') \cdot D''(b'') = w^* - \lim_\alpha \psi''(\hat{a_\alpha}) \cdot D''(b'')$ [9]. The last equality is true if $A^{**}$ is a Banach algebra under the second Arens product. Let $A$ be a Banach algebra with a bounded approximate identity, then $A$ is $(\varphi, 0)$ and $(0, \psi)$-weakly amenable (see Example 4.2).

For a Banach algebra $A$, let $A^{op}$ be the Banach algebra with underlying Banach space $A$ and with product $\circ$ given by $a \circ b = ba$. We have the following simple observation.

**Proposition 3.2.** Let $A$ be Banach algebra and $\varphi, \psi \in \text{Hom}(A)$. Then $A$ is $(\varphi, \psi)$-weakly amenable if and only if $A^{op}$ is $(\psi, \varphi)$-weakly amenable.
Let $A$ be a Banach algebra and $\varphi, \psi \in \text{Hom}(A)$. Let $A^{**}$ be $(\varphi'', \psi'')$-weakly amenable, and suppose $A^\prime$ is a left ideal in $A^{**}$. Then $A$ is $(\varphi, \psi)$-weakly amenable.

**Proof.** It is known that $\varphi''(\widehat{a}) = \varphi(\widehat{a})$ and $\psi''(\widehat{a}) = \psi(\widehat{a})$, for all $a \in A$. The proof of the Theorem is similar to [10, Theorem 2.3].

A Banach algebra $A$ is said to be dual if there is a closed submodule $A_*$ of $A^*$ such that $A = A_*^\prime$. Let $i : A_* \to A^*$ be the canonical embedding and $i'$ be the adjoint of $i$. If $a \in A$, we have $i'(\widehat{a}) = \widehat{a}$. Obviously $i$ is norm-continuous, hence $i'$ is $w^*$-continuous. Let $a'' \in A^{**}$ and take nets $(a_\alpha)$ and $(b_\beta)$ in $A$ such that $A(a_\alpha) \to a''$ and $b_\beta \to b''$. Then

$$i'(a''b'') = i'(w^* - \lim_{\alpha} \widehat{a}_\alpha \widehat{b}_\beta) = w^* - \lim_{\alpha} i'(\widehat{a}_\alpha \widehat{b}_\beta)$$

$$= w^* - \lim_{\alpha} (\widehat{a}_\alpha \widehat{b}_\beta) = (w^* - \lim_{\alpha} \widehat{a}_\alpha)(w^* - \lim_{\beta} \widehat{b}_\beta)$$

$$= i'(w^* - \lim_{\alpha} \widehat{a}_\alpha)i'(w^* - \lim_{\beta} \widehat{b}_\beta) = i'(a'')i'(b'').$$

Hence $i'$ is an algebra homomorphism from $A^{**}$ onto $A$. Let $\varphi : A \to A$ be a continuous homomorphism. Then the second conjugate $\varphi''$ is $w^*$-continuous and

$$\langle i'(\varphi''(\widehat{a})), b \rangle = \langle \varphi''(\widehat{a}), i(b) \rangle = \langle \varphi''(i'(\widehat{a})), b \rangle = \langle \varphi''(i'(\widehat{a})), b \rangle.$$

Hence $i'/(\varphi''(\widehat{a})) = \varphi''(i'(\widehat{a}))$. We know that $\varphi''|_A = \varphi$, if $a'' \in A^{**}$, $(a_\alpha) \subset A$, $A(a_\alpha) \to a''$, then

$$\varphi(i'(a'')) = \varphi''(i'(\widehat{a})) = \varphi''(i'(w^* - \lim_{\alpha} \widehat{a}_\alpha)) = w^* - \lim_{\alpha} \varphi''(i'(\widehat{a}_\alpha))$$

$$= w^* - \lim_{\alpha} i'(\varphi''(\widehat{a}_\alpha)) = i'(w^* - \lim_{\alpha} \widehat{a}_\alpha) = i'(\varphi''(a'')).$$

**Theorem 3.4.** Let $A$ be a dual Banach algebra and let $\varphi, \psi \in \text{Hom}(A)$. If $A^{**}$ is $(\varphi'', \psi'')$-weakly amenable, then $A$ is $(\varphi, \psi)$-weakly amenable.

**Proof.** Let $i$ be as above. Suppose that $d : A \to (A(\varphi, \psi))^*$ is an $(\varphi, \psi)$-derivation. Set $D = i'' \circ d \circ i' : A^{**} \to (A^{**}(\varphi'', \psi''))^*$, then for every $a'', b'', c'' \in A^{**}$ we have

$$\langle D(a''b''), c'' \rangle = \langle d(i'(a''))i'(b''), c'' \rangle$$

$$= \langle d(i'(a'')) \cdot \varphi(i'(b'')) + \psi(i'(a'')) \cdot d(i'(b''))i'(c''), d(i'(a''))i'(b'') \rangle$$

$$= \langle d(i'(a'')) \cdot \varphi''(i'(b'')) + \psi''(i'(a'')) \cdot d(i'(a''))i'(c'') \rangle + \langle d(i'(a''))i'(b'') \cdot i''(c''), i'' \cdot c'' \cdot \psi''(a'') \rangle$$

$$= \langle (i'' \circ d \circ i'(a'')) \cdot \varphi''(b'')c'' + (i'' \circ d \circ i'(b'')) \cdot c'' \psi''(a''), d(a'') \cdot b'' \cdot c'' \rangle$$

$$= \langle D(a'') \cdot \varphi''(b'') + \psi''(a'') \cdot D(b''), c'' \rangle.$$
Now let \( R : A^{***} \rightarrow A^* \) be the restriction map. Set \( a'_0 = R(a''_0) \). For every \( a, b \in A \) we have
\[
\langle d(a), b \rangle = \langle d(i'(\hat{a}), i'(\hat{b})) \rangle = \langle i'' \circ d \circ i'(\hat{a}), \hat{b} \rangle
\]
\[
= \langle D(\hat{a}), \hat{b} \rangle = \langle a''_0 \cdot \varphi''(\hat{a}), \hat{b} \rangle - \langle \psi''(\hat{a}) \cdot a'_0, \hat{b} \rangle
\]
\[
= \langle a''_0, \varphi''(\hat{a}) \rangle - \langle a''_0, b \psi''(\hat{a}) \rangle = \langle a''_0, \varphi(a)b \rangle - \langle a''_0, b \psi(a) \rangle
\]
\[
= \langle R(a''_0), \varphi(a)b \rangle - \langle R(a''_0), b \psi(a) \rangle = \langle a'_0, \varphi(a) - \psi(a) \rangle \cdot a'_0, b \rangle.
\]
So \( d(a) = a'_0 \cdot \varphi(a) - \psi(a) \cdot a'_0 \). Therefore \( d \) is an \((\varphi, \psi)\)-inner. \( \square \)

4. Examples

**Example 4.1.** Let \( A \) be a commutative weakly amenable Banach algebra. A Banach \( A \)-module \( X \) is called symmetric if \( a.x = x.a \), for \( a \in A \) and \( x \in X \). Then for every symmetric Banach \( A \)-module \( X \) we have \( H^1(A, X) = \{0\} \) \([1]\). On the other hand for every \( \varphi \in Hom(A), (A(\varphi, \varphi))^* \) is a symmetric Banach \( A \)-module. Thus \( A \) is \((\varphi, \varphi)\)-weakly amenable.

**Example 4.2.** Let \( A \) be a Banach algebra with a bounded right approximate identity \((e_a)\). Let \( D : A \rightarrow (A(0,0))^* \) be a derivation. Then for every \( a, b \in A \), we have \( D(ab) = \varphi(a) \cdot D(b) \). Since \( D \) is bounded, \((D(e_a))\) is a bounded net in \((A(0,0))^*\). Let \( f \in (A(0,0))^* \) be a cluster point of \((D(e_a))\). We can suppose that \( w^* - \lim_\alpha D(e_\alpha) = f \) in \((A(0,0))^*\). Then for every \( a \in A \), we have \( w^* - \lim_\alpha aD(e_\alpha) = af \) in \((A(0,0))^*\). Thus we have
\[
D(a) = \lim_\alpha D(ae_\alpha) = \lim_\alpha \psi(a) \cdot D(e_\alpha) = \psi(a) \cdot f.
\]
This means that \( D \) is \((0, \psi)\)-inner. So \( A \) is \((0, \psi)\)-weakly amenable. Similarly every Banach algebra with a bounded left approximate identity is \((\varphi, 0)\)-weakly amenable. So every group algebra and \( C^*\)-algebra are \((\varphi, 0)\) and \((0, \psi)\)-weakly amenable.

**Example 4.3.** Let \( A = l^1(\mathbb{N}) \) with the product \( ab := a(1)b \ (a, b \in l^1(\mathbb{N})) \). For every \( \varphi, \psi \in Hom(A), A \) is \((\varphi, \psi)\)-weakly amenable \([16]\). It is easy to check that \( A \) does not have a bounded right approximate identity, thus \( A \) is not amenable.

**Example 4.4.** Let \( S = \{x_1, x_2, x_3, x_4, x_5\} \) be a semigroup with \( x_1^2 = x_1, x_1x_2 = x_2, x_3x_1 = x_3, x_3x_2 = x_4 \) and all other products equal to \( x_5 \). We identify the elements of \( S \) with the point masses on \( S(\delta_x := x) \). We know that, for any semigroup \( S \),
\[
l^1(S) = \left\{ \sum_{s \in S} \alpha_s \delta_s : \sum_{s \in S} | \alpha_s | < \infty, s \in S, \alpha_s \in \mathbb{C} \right\}
\]
is a Banach algebra with the norm \( \| \sum_{s \in S} \alpha_s \delta_s \| = \sum_{s \in S} | \alpha_s | \) and the convolution reduced to \( \delta_s \ast \delta_t = \delta_{st} \), for \( s, t \in S \) (see citeDal for details). In our case
\[
l^1(S) = \left\{ \lambda = \sum_{n=1}^5 \alpha_n x_n : \{\alpha_n\}_{n=1}^5 \subset \mathbb{C}, \{x_n\}_{n=1}^5 \subset S, \|\lambda\| = \sum_{n=1}^5 |\alpha_n| \right\},
\]
and $l^1(S)$ is weakly amenable \cite{2}. Since $S$ is not regular semigroup, $l^1(S)$ is not amenable \cite{6}. Let $\varphi, \psi : l^1(S) \rightarrow l^1(S)$ be continuous homomorphisms and $D : l^1(S) \rightarrow (l^1(S)_{(\varphi, \psi)})^*$ be a $(\varphi, \psi)$-derivation we show that $D = 0$. Therefore $D$ is an $(\varphi, \psi)$-inner derivation for each $\varphi, \psi \in Hom(l^1(S))$. If $x, y \in S$ we show that $\langle Dx, y \rangle = 0$.

Suppose that $\varphi(x) = \sum_{k=1}^{5} \alpha_{jk} x_k$, $\psi(x) = \sum_{k=1}^{5} \beta_{jk} x_k$, $(1 \leq j \leq 5, \alpha_{jk}, \beta_{jk} \in \mathbb{C})$. Since $\varphi(x_1^2) = \varphi(x)$, $\alpha_{11} = \alpha_{11}, \alpha_{11}\alpha_{12} = \alpha_{12}, \alpha_{13}\alpha_{11} = \alpha_{13}, \alpha_{13}\alpha_{12} = \alpha_{14}$.

I) If $\alpha_{11} = 0$, then $\alpha_{12} = \alpha_{13} = \alpha_{14} = 0, \alpha_{15}^2 = \alpha_{15}$. It is easy to show that above $(\varphi, \psi)$-derivation is zero.

II) If $\alpha_{11} = \beta_{11} = 1$ then

\[ \varphi(x_j) = x_1 + \sum_{k=2}^{5} \alpha_{jk} x_k, \quad \psi(x_j) = x_1 + \sum_{k=2}^{5} \beta_{jk} x_k \quad (2 \leq j \leq 5). \quad (4.1) \]

We put $\langle Dx_i, x_j \rangle = t_{ij}, (i, j \in \{1, 2, 3, 4, 5\}, x_i, x_j \in S, t_{ij} \in \mathbb{C})$. Also \[ \langle Dx_i, x_j \rangle = \langle Dx_i, \varphi(x_k) x_j \rangle + \langle Dx_k, x_j \psi(x_i) \rangle \quad (4.2) \]

for all $i, j, k \in \{1, 2, 3, 4, 5\}$. Since $x_1^2 = x_1, t_{1j} = \langle Dx_1, \varphi(x_1) x_j \rangle + \langle Dx_1, x_j \psi(x_1) \rangle$. Therefore \[ t_{14} = t_{15} = (2 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15})t_{15}. \]

If

\[ \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15} = 1 \quad (4.3) \]

since $\varphi(x_1^2) = \varphi(x_1)$ and $\psi(x_1^2) = \psi(x_1)$, we have $\alpha_{14} = \alpha_{12}\alpha_{13}, \beta_{14} = \beta_{12}\beta_{13}$ and

\[ \alpha_{12}^2 + \alpha_{13}^2 + \alpha_{12}\alpha_{13}^2 + \alpha_{15}^2 + 2\alpha_{12}\alpha_{13} + 2\alpha_{12}\alpha_{13}\alpha_{15} + 3\alpha_{12}\alpha_{13} + 2\alpha_{12}\alpha_{15} + 2\alpha_{12}\alpha_{13} + \alpha_{12} + \alpha_{13} + \alpha_{15} = 0, \quad (4.4) \]

\[ \beta_{12}^2 + \beta_{13}^2 + \beta_{12}\beta_{13}^2 + \beta_{15}^2 + 2\beta_{12}\beta_{13} + 2\beta_{12}\beta_{13}\beta_{15} + 3\beta_{12}\beta_{13} + 2\beta_{12}\beta_{15} + 2\beta_{13}\beta_{15} + \beta_{12} + \beta_{13} + \beta_{15} = 0. \quad (4.5) \]

From (4.4) and (4.5) the following relation is obtained

\[ \begin{align*}
\alpha_{15} &= -\alpha_{12}\alpha_{13} - \alpha_{12} - \alpha_{13} + 1 \\
\text{or} \\
\alpha_{15} &= -\alpha_{12}\alpha_{13} - \alpha_{12} - \alpha_{13}
\end{align*} \quad (4.6) \]

and

\[ \begin{align*}
\beta_{15} &= -\beta_{12}\beta_{13} - \beta_{12} - \beta_{13} + 1 \\
\text{or} \\
\beta_{15} &= -\beta_{12}\beta_{13} - \beta_{12} - \beta_{13}
\end{align*} \quad (4.7) \]

by inserting these solutions in (4.3), we get the contradictions: $3 = 0, 2 = 0, 1 = 0$. Therefore

\[ t_{14} = t_{15} = 0. \quad (4.8) \]

Since $\varphi(x_5^2) = \varphi(x_5)$ and $\psi(x_5^2) = \psi(x_5)$, similar to the above

\[ t_{54} = t_{55} = 0. \quad (4.9) \]
From (4.1), (4.2) and (4.8), we deduce that
\[ t_{24} = t_{25} = (1 + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15})t_{25}. \]

Since \( Dx_5 = Dx_2x_1 \),
\[ t_{5j} = \langle Dx_2, \varphi(x_1)x_j \rangle + \langle Dx_1, x_j \psi(x_2) \rangle. \] (4.10)

From (4.8), (4.9) and (4.10), we have
\[ (1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15})t_{25} = 0. \]

If \( \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15} = 0 \) and \( 1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} = 0 \), from the relations \( \alpha_{14} = \alpha_{12}\alpha_{13}, \beta_{14} = \beta_{12}\beta_{13} \), (4.6) and (4.7), we conclude the contradictions: \( 1 = -1 \) and \( 1 = 0 \). Therefore
\[ t_{24} = t_{25} = 0. \] (4.11)

Since \( Dx_4 = Dx_3x_2 \),
\[ t_{4j} = \langle Dx_3, \varphi(x_2)x_j \rangle + \langle Dx_2, x_j \psi(x_3) \rangle. \] (4.13)

From (4.1) and (4.13), we have
\[ t_{44} = t_{45} = (1 + \sum_{k=2}^{5} \alpha_{2k})t_{25} + (1 + \sum_{k=2}^{5} \beta_{3k})t_{35}. \] (4.14)

From (4.11), (4.12) and (4.14), we conclude that
\[ t_{44} = t_{45} = 0. \]

If \( t_{11} \neq 0 \), we can conclude that \( \alpha_{43} = 0 \) and \( \alpha_{43} = 2 \), which is a contradiction. Hence \( t_{11} = 0 \). By using of the above relations \( t_{ij} = 0 \) for every \( i \) and \( j \). Therefore \( D = 0 \).

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