Plaquette Invariants and the Flavour Symmetric Description of Quark and Neutrino Mixings

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Abstract

We present a complete set of new flavour-permutation-symmetric mixing observables. We give expressions for these “plaquette invariants”, both in terms of the mixing matrix elements alone, and in terms of manifestly Jarlskog-invariant functions of fermion mass matrices. While these quantities are unconstrained in the Standard Model, we point out that remarkably, in the case of leptonic mixing, the values of most of them are consistent with zero, corresponding to certain phenomenological symmetries. We give examples of their application to the flavour-symmetric description of both lepton and quark mixings, showing for the first time how to construct explicitly weak-basis invariant constraints on the mass matrices, for a number of phenomenologically valid mixing ansatze.

To be published in Physics Letters B

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1 The Jarlskogian and Plaquette Invariance

Jarlskog’s CP-violating invariant, $J$ \[^1\], is important in the phenomenology of both quarks and leptons. As well as parameterising the violation of a specific symmetry, it has two other properties which set it apart from most other mixing observables. First, its value (up to its sign) is independent of any flavour labels\[^1\]. Mixing observables are in general dependent on flavour labels, eg. the moduli-squared of mixing matrix elements, $|U_{\alpha i}|^2$, certainly depend on $\alpha$ and $i$. Indeed, $J$ itself is often calculated in terms of a subset of four mixing matrix elements, namely those forming a given plaquette\[^2\], or box \[^3\]:

$$J = \text{Im}(\Pi_{\gamma k}) = \text{Im}(U_{\alpha i}U_{\alpha j}^*U_{\beta i}^*U_{\beta j}). \quad (1)$$

However, it is well-known \[^1\] that the value of $J$ does not depend on the choice of plaquette (ie. on its flavour labels, $\gamma$ and $k$ above) - it is “plaquette-invariant”. This special feature originates in the fact that $J$ is *flavour-symmetric*, carrying information sampled evenly across the whole mixing matrix. We point-out that in fact, *any* observable function of the mixing matrix elements, flavour-symmetrised (eg. by summing over both rows and columns), and written in terms of the elements of a single plaquette (eg. using unitarity constraints), will be similarly plaquette-invariant. Both its expression in terms of mixing matrix elements, as well as its value, will be independent of the particular choice of plaquette.

The second exceptional property of $J$ is that it may be particularly simply related to the fermion mass-(or Yukawa) matrices:

$$J = -i \frac{\text{Det}[L, N]}{2L_\Delta N_\Delta} \quad (2)$$

where for leptons, $L$ and $N$ are the charged-lepton and neutrino mass matrices respectively\[^4\] (in an arbitrary weak basis) and $L_\Delta = (m_e - m_\mu)(m_\mu - m_\tau)(m_\tau - m_e)$ (with an analogous definition for $N_\Delta$ in terms of neutrino masses and likewise for the quarks).

In this paper, we introduce and classify several new plaquette-invariant (ie. flavour-symmetric mixing) observables, which, in common with $J$, are independent of flavour labels and may be simply related to the mass matrices. Again, in common with $J$, our

\[^1\]We focus on the leptons, although many of our considerations may be applied equally well to the quarks. In the leptonic case, neutrino mass eigenstate labels $i = 1...3$ take the analogous role to the charge $-\frac{1}{3}$ quark flavour labels in the quark case. In this sense, we will often use the term “flavour” to include neutrino mass eigenstate labels, as well as charged lepton flavour labels.

\[^2\]We use a cyclic labelling convention such that $\beta = \alpha + 1$, $\gamma = \beta + 1$, $j = i + 1$, $k = j + 1$, all indices evaluated mod 3.

\[^3\]Throughout this paper, $L$ and $N$ are taken to be Hermitian, either by appropriate choice of the flavour basis for the right-handed fields, or as the Hermitian squares, $MM^\dagger$, of the relevant mass or Yukawa coupling matrices. The variables $m_\alpha$, $m_i$ generically refer to their eigenvalues in either case.
new observables parameterise the violation of certain phenomenological symmetries which have already been considered significant \[4, 5, 6\] in leptonic mixing. In the next section, we define more precisely what we mean by flavour symmetry.

2 The $S_{3\ell} \times S_{3\nu}$ Flavour Group

Our starting point is the matrix of moduli-squared of the mixing matrix elements:

$$P = \begin{pmatrix} |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu1}|^2 & |U_{\mu2}|^2 & |U_{\mu3}|^2 \\ |U_{\tau1}|^2 & |U_{\tau2}|^2 & |U_{\tau3}|^2 \end{pmatrix}. \quad (3)$$

The $P$-matrix can be simply related to weak-basis invariant functions of the fermion mass matrices \[7\], a feature which we develop later. Under permutations of the charged lepton flavour labels (ie. rows), $P$ transforms as the 3-dimensional (natural) representation of $S_3$ of lepton flavour, $S_{3\ell}$. Similarly, for independent permutations of the neutrino mass eigenstate labels (ie. columns, or neutrino “flavour” labels, see Footnote 1), $P$ transforms as another copy of the natural representation of $S_3$, denoted $S_{3\nu}$ here. Hence, we have a $3 \times 3$ natural representation of the group $S_{3\ell} \times S_{3\nu}$. It is well-known that the natural representation of $S_3$ is reducible.

We introduce here a convenient parameterisation of the $P$-matrix:

$$P = D + \tilde{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} + \begin{pmatrix} -w - x & w & x \\ -y - z & y & z \\ w + x + y + z & -w - y & -x - z \end{pmatrix}. \quad (4)$$

The four parameters $w, x, y$ and $z$ appearing in the reduced $P$-matrix, $\tilde{P} = P - D$ above, completely specify the mixing, up to the sign of the $CP$ violation parameter $J$ \[7\]. For example, tribimaximal mixing \[8\] corresponds to $w = 0, x = -1/3, y = 0$ and $z = 1/6$, values which are consistent with current neutrino data \[9\]. We define $\tilde{p}$ as the $2 \times 2$ plaquette in the top right-hand corner of $\tilde{P}$:

$$\tilde{p} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \quad (5)$$

and note that it transforms under flavour permutations as a $2 \times 2$ (real) irreducible representation of the $S_{3\ell} \times S_{3\nu}$ group.

Our prototype flavour-symmetric mixing observable, $J$, is invariant under even permutations of the charged lepton and neutrino flavour labels, and flips sign under odd permutations, ie. it transforms as a $T \times T$ representation under $S_{3\ell} \times S_{3\nu}$ (where $T$ means the “odd”, or alternating, representation of $S3$). By analogy, we denote as

\[4\] The top-right (ie. “$\tau_1$”) plaquette is chosen simply on the grounds that its elements correspond to the most directly measured elements of the mixing matrix in both the lepton and quark cases.
“flavour-symmetric”, all observables which transform as (pseudo-)scalars\(^5\) under the \(S_{3\ell} \times S_{3\nu}\) group. We focus here on functions of the mixing matrix elements alone (see Footnote 8) which are homogeneous in the \(\bar{p}\)-matrix, classifying them as quadratic, cubic etc. (there are no non-trivial linear plaquette invariants).

The set of polynomials in \(w, x, y\) and \(z\) at any given order comprise a representation of \(S_{3\ell} \times S_{3\nu}\) which may be decomposed into irreducible representations. The set of quadratic polynomials, \((2 \times 2) \otimes (2 \times 2)\), contains exactly one each of \(1 \times 1\), \(1 \times \bar{1}\) and \(\bar{1} \times 1\), as does the cubic \((2 \times 2) \otimes (2 \times 2) \otimes (2 \times 2)\) (for orders \(\geq 4\), there are multiple 1-dimensional representations of each symmetry). Hence, for orders up to cubic, such (pseudo-)scalar quantities are uniquely defined (up to an arbitrary normalisation) by their order in \(\bar{p}\) and their symmetry under \(S_{3\ell} \times S_{3\nu}\). While standard techniques exist\(^\[10\]\) for performing these reductions, for orders \(\leq 3\) the required polynomial forms are anyway easily obtained, symmetrising appropriately, eg. as indicated in Section 1.

3 New Plaquette Invariant Mixing Observables
In Table I, we introduce our new flavour-symmetric mixing observables (ie. plaquette invariants) for order \(\leq 3\), and summarise the experimental information on each for both leptons and quarks. We postpone giving the explicit expressions for them until Section 4. We normalise the quantities listed in the first column of the table so that their maximum value is unity. For comparison, we also include \(J\).

As is well-known, for quarks and for Dirac neutrinos, four parameters are sufficient to completely determine the mixing matrix (also in the Majorana case, as far as flavour oscillations are concerned). We therefore expect that fixing the values of any four independent plaquette invariants must completely determine the mixing matrix, up to discrete ambiguities inherent to the built-in flavour symmetry. A natural set would be \(F, G, C\) and \(A\), being the lowest-order set possible treating leptons and neutrinos (or up- and down-like quarks) symmetrically. For example, the phenomenologically successful tribimaximal [8] mixing scheme corresponds to the set of constraints \(F = C = A = 0, G = \frac{1}{6}\), constituting the first flavour-symmetric description of exact tribimaximal mixing (one could of course substitute the constraint \(J = 0\) for the condition on \(G\)). These constraints determine tribimaximal mixing only up to \((6 \times 6 = 36)\) \(S_{3\ell} \times S_{3\nu}\) permutations, with the observed mixing breaking the flavour symmetry spontaneously [12].

We consider also two extremes of mixing. The now-excluded, highly symmetric case of trimaximal mixing [13], in which all elements of the mixing matrix have magnitude

\(^5\)We adopt the term (pseudo-)scalars to denote any of the \(1 \times 1, \bar{1} \times \bar{1}, 1 \times \bar{1}\) and \(\bar{1} \times 1\) representations of \(S_{3\ell} \times S_{3\nu}\).
Table 1: Properties and values of plaquette-invariant observables. The experimentally allowed ranges were estimated (90% CL) from compilations of experimental results [9, 11], neglecting any correlations between the input quantities.

$1/\sqrt{3}$, is given by: $\mathcal{F} = \mathcal{G} = \mathcal{C} = \mathcal{A} = 0$ (our variables parameterise deviations from this unique form). It is somewhat remarkable that the experimental values of three of these quantities (Table 1) are consistent with zero for the leptons. This near-vanishing of the leptonic observables is a flavour-symmetric expression of the presence of large mixing angles in the lepton sector. By contrast, the case of no mixing corresponds to the constraints $\mathcal{F} = \mathcal{G} = \mathcal{C} = \mathcal{A} = 1$, and it is notable that for the quarks, the experimental values of our variables are all quite close to unity, a flavour-symmetric expression of the observed smallness of the quark mixing angles.

4 Expression and Interpretation of the New Observables

We give explicit expressions for our new plaquette-invariant observables in terms of the mixing parameters $w$, $x$, $y$ and $z$ (as defined in Eq. (4)). We also give them in terms of the fermion mass matrices, to emphasise the analogy with $J$ (cf. Eq. (2)). We use a relationship between $\tilde{p}$ and the mass matrices, derivable from the relations of Jarlskog and Kleppe [14]:

$$\tilde{p} = \tilde{M}_\ell^T \cdot \tilde{T} \cdot \tilde{M}_\nu$$

where $\tilde{T}_{mn} := \text{Tr}(\tilde{L}^m \tilde{N}^n)$,

and $\tilde{L}^m := L^m - \frac{1}{3} \text{Tr}(L^m)$ is the reduced (ie. traceless) $m$th power of the $L$ mass matrix (and similarly for $\tilde{N}^n$). The $2 \times 2$ matrix $\tilde{T}$ is closely related to the $T$-matrix introduced in [7], and contains complete information about the mixing, assuming the
\[ L, N\text{-eigenvalues are known. The transformation matrices are given by:} \]

\[ \tilde{M}_L = \frac{1}{L_\Delta} \begin{pmatrix} m_\mu^2 - m_\tau^2 & m_\mu^2 - m_e^2 \\ m_\mu^2 - m_\tau^2 & m_\tau^2 - m_e^2 \end{pmatrix}, \quad \tilde{M}_\nu = \frac{1}{N_\Delta} \begin{pmatrix} m_3^2 - m_1^2 & m_1^2 - m_2^2 \\ m_3^2 - m_1^2 & m_1^2 - m_2^2 \end{pmatrix}, \]

\[ (7) \]

with \( \text{Det} \tilde{M}_L = L_\Delta^{-1} \) etc. We emphasise that although we chose \( \tilde{p} \) (ie. a particular plaquette of \( \tilde{P} \)) as our \( 2 \times 2 \) representation of the flavour group, all the following formulae are completely independent of this choice.

4.1 \( \mathcal{F} \) (Quadratic \( 1 \times 1 \))

Despite the fact that \( \mathcal{F} \) is quadratic in the \( P \)-matrix, it turns out (perhaps surprisingly) to be expressible as the determinant of \( P \) (clearly odd under both \( S3_\ell \) and \( S3_\nu \)):

\[ \mathcal{F} = \text{Det} P = 3 (wz - xy) = 3 \text{Det} \tilde{p} = 3 \frac{\text{Det} \tilde{T}}{L_\Delta N_\Delta}. \]

\[ (8) \]

We note the striking similarity between the form of \( \mathcal{F} \) in terms of mass matrices given by the last equality here, and the RHS of Eq. (2).

From Table 1, we see that the data are compatible with \( \mathcal{F} = 0 \). We have met this condition before: for non-trivial mass spectrum, it is equivalent to the “determinant condition” \( N_\Delta \mathcal{F} = 0 \), derived in [12] (which ensures that the mass-constraining Lagrange multipliers can be determined in that case). As long as there are no degeneracies in either \( L \) or \( N \), we see from Eq. (8) that the determinant condition may be simply recast as a condition on the mass matrices \( L_\Delta N_\Delta \).

The condition \( \mathcal{F} = 0 \) is satisfied by any mixing matrix having a trimaximally mixed row or column, such as, eg. the experimentally viable \( \nu_2 = \frac{1}{\sqrt{3}}(1, 1, 1)^T \), mass eigenstate [4, 5], as is readily verified by direct substitution, eg. \( w = y = 0 \) in Eq. (8). The condition is also satisfied by any mixing matrix exhibiting \( \mu - \tau \) reflection symmetry \( (P_{\mu i} = P_{\tau i}, \forall i [6]) \), or indeed by any mixing matrix having two rows (resp. columns) each of whose corresponding elements have equal moduli. In particular, the tribimaximal mixing matrix [8] satisfies \( \mathcal{F} = 0 \), on two counts [4], as this mixing ansatz has both a trimaximally mixed \( \nu_2 \) and \( \mu - \tau \) reflection symmetry.

More generally, \( \mathcal{F} \) measures the “acoplanarity” of the “\( P \)-vectors”, ie. when \( \mathcal{F} = 0 \), the plane defined by any pair of \( P \)-vectors is independent of the choice of pair (in the \( \nu_1, \nu_2, \nu_3 \) basis, the \( e - \mu, \mu - \tau \) and \( \tau - e \) planes coincide iff \( \mathcal{F} = 0 \)). Indeed, the \( \mathcal{F} = 0 \) symmetry is sufficient to protect the flavour composition of any remote source against analysis, since the asymptotic \( (L/E\text{-averaged}) \) matrix of oscillation probabilities, \( <P>_{\infty} = PP^T \) [7], cannot be inverted in that case.

\[ ^{6}\text{The columns of } \tilde{M}_L \text{ and } \tilde{M}_\nu \text{ are labelled by charged lepton labels (} e \text{ and } \mu \text{ here) and neutrino flavour labels (} 2 \text{ and } 3 \text{ here) respectively, in accordance with our choice of plaquette, } \tilde{p}, \text{ Eq. (5).} \]

\[ ^{7}\text{The determinant condition may equivalently be expressed in terms of the original } 3 \times 3 \text{ } T\text{-matrix} \]

\[ [7] \text{ as } \text{Det } T = 0, \text{ or in terms of the determinant of a matrix of cubic commutators, see [15].} \]
4.2 $G$ (Quadratic $1 \times 1$)

$G$ is quadratic in $\tilde{p}$ (and equivalently in $\tilde{T}$):

$$G = (w + x + y + z)^2 + (w^2 + x^2 + y^2 + z^2) - (wz + xy)$$

(9)

$$= 2 \text{Tr} [\tilde{p}^T \phi \tilde{p} \phi] = \frac{2 \text{Tr} (\tilde{T}^T L_G \tilde{T} N_G)}{(L_{\Delta} N_{\Delta})^2},$$

(10)

where $\phi \equiv \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$ is defined by its invariance under transformations with all six $2 \times 2$ (real) $S^3$ permutation matrices: $S_i \phi S_i^T = \phi$. $L_G$ is a (symmetric) matrix of weak-basis-invariant functions of $L$ depending only on masses, given by:

$$L_G = \frac{3}{2} \begin{pmatrix} \text{Tr} [(L^2)^2] & -\text{Tr} [L \cdot L^2] \\ -\text{Tr} [L \cdot L^2] & \text{Tr} [L^2] \end{pmatrix},$$

(11)

where we denote the reduced mass matrix, $\tilde{L}^1$, as $\tilde{L}$ for simplicity. A similar definition applies for $N_G$ in terms of the $\tilde{N}^n$.

Traces (and determinants) of functions of the mass (or Yukawa) matrices (eg. $\text{Tr} (L^m N^n)$) such as enter the numerators of our expansions, eg. in Eqs. (8) and (10) (also Eqs. (13) and (16) below), are themselves always flavour-invariant ($1 \times 1$ under $S^3_\ell \times S^3_\nu$). In the general case, they depend in a non-trivial way on the mixing matrix elements and the relevant (mass) eigenvalues. By contrast, the particular combinations appearing here always factorise into powers of $L_{\Delta}$ and $N_{\Delta}$, and the relevant plaquette-invariant (which has no dependence on the mass eigenvalues) $8$ eg. $(L_{\Delta} N_{\Delta})^2 \times G$ in Eq. (10). Turning to the denominators (eg. Eq. (10)), the discriminant-like factors, $L_{\Delta}^m$ and $N_{\Delta}^m$ (which cancel the mass-dependence), arise from the structure of the transformation matrices, Eq. (7). $L_{\Delta}$ and $N_{\Delta}$ transform as $\bar{T}$ under $S^3_\ell$ and $S^3_\nu$ respectively, so that the denominator always carries the symmetry of the mixing observable.

We note that $G$ is unique among our set of four $L \leftrightarrow N$ symmetric plaquette-invariants, $F$, $G$, $C$, $A$, in having an experimentally allowed range of values for leptons (Table 1) which is not consistent with zero, being instead consistent with the tri-bimaximal value, $G = \frac{1}{6} \simeq 0.17$. Furthermore, it can be shown that $G$ is in fact, the only one of the four which can be non-zero if each of the other three is zero.

The quadratic $1 \times 1$ and $\mathbf{T} \times 1$ of $S^3_\ell \times S^3_\nu$ are both identically zero, so that in search of additional non-trivial plaquette invariants, we must now move to higher order.

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8In this sense, we distinguish our flavour-symmetric mixing observables (ie. plaquette-invariants), having no dependence on the mass eigenvalues, from the more general class of flavour-symmetric functions of the mass matrices.
4.3 $\mathcal{C}$ (Cubic $1 \times 1$)
$\mathcal{C}$ is cubic in $\tilde{p}$ (and equivalently in $\tilde{T}$):

$$
\mathcal{C} = 9(xyz + wyz + wxz + wxy) + \frac{9}{2}[xy(x + y) + wz(w + z)] \\
= \frac{3}{2} \frac{\tilde{T}_{mn} \tilde{T}_{pq} \tilde{T}_{rs} \mathcal{L}_c^{(mpq)} \mathcal{N}_c^{(nqs)}}{(L_\Delta N_\Delta)^2}
$$

where the charged lepton mass tensor, $\mathcal{L}_c^{(mpq)}$, is constructed from flavour-symmetric observables of the lepton mass matrix ($L_m := \text{Tr} L^m$ etc.). $\mathcal{L}_c$ is symmetric in all its indices so that it has only four independent elements:

$$
\mathcal{L}_c^{222} = 3, \quad \mathcal{L}_c^{122} = -2L_1, \quad \mathcal{L}_c^{112} = \frac{1}{2}(3L_1^2 - L_2), \quad \mathcal{L}_c^{111} = L_3 - L_1^3
$$

with analogous expressions for $\mathcal{N}_c$. We note that the pattern remarked-on in the previous section indeed continues, the denominator in Eq. (13) having even powers of both $L_\Delta$ and $N_\Delta$, ensuring the overall $1 \times 1$ symmetry under the flavour group.

For lepton data \cite{9} we see from Table \textbf{1} that $\mathcal{C}$ is consistent with zero. We note further that any mixing scheme having a trimaximally-mixed column (or row) satisfies the condition $\mathcal{C} = 0$ (in addition to the constraint $\mathcal{F} = 0$ discussed already in Section 4.1). Hence, $\mathcal{C}$ and $\mathcal{F}$ parameterise deviations from democracy (also called “magic-square”) symmetry \cite{5}, the symmetry which ensures one trimaximally-mixed column.

4.4 $\mathcal{A}$ (Cubic $\bar{T} \times \bar{T}$)
$\mathcal{A}$ is cubic in $\tilde{p}$ (and equivalently in $\tilde{T}$):

$$
\mathcal{A} = 2(w^3 - x^3 - y^3 + z^3) + 3[wx(w - x) + wy(w - y) + wz(z - y)] \\
+ xz(z - x) + xy(w + z) - wz(x + y) + \frac{3}{2}[wz(w + z) - xy(x + y)]
$$

$$
= \frac{81}{2} \frac{\tilde{T}_{mn} \tilde{T}_{pq} \tilde{T}_{rs} \mathcal{L}_A^{(mpq)} \mathcal{N}_A^{(nqs)}}{(L_\Delta N_\Delta)^3}
$$

where the charged lepton mass tensor, $\mathcal{L}_A^{(mpq)}$, is again constructed from flavour-symmetric observables of the lepton mass matrix. $\mathcal{L}_A$ is again also symmetric in all its indices so that it is completely determined by four elements as follows:

$$
\mathcal{L}_A^{222} = -\text{Tr} [\bar{L}^3], \quad \mathcal{L}_A^{122} = \text{Tr} [\bar{L}^2 \cdot \bar{L^2}], \\
\mathcal{L}_A^{112} = -\text{Tr} [\bar{L} \cdot (\bar{L}^2)^2], \quad \mathcal{L}_A^{111} = \text{Tr} [(\bar{L}^2)^3]
$$

with analogous expressions for $\mathcal{N}_A$ (the $\bar{L}^m$ were defined just below Eq. (13)). The odd powers of $L_\Delta$ and $N_\Delta$ in the denominator of Eq. (16) ensure the required transformation property under the flavour group, as expected.
For lepton data \cite{9} we see from Table 1 that $A$ is consistent with zero. We note further that any $P$-matrix with two rows (or two columns) equal (e.g., with $\mu$-$\tau$-symmetry), satisfies the condition $A = 0$ (in addition to the constraint $F = 0$ discussed already in Section 4.1). Hence, $A$ and $F$ parameterise deviations from $\mu$-$\tau$-symmetry \cite{6} and/or any of its permutations. A kind of duality is now apparent between this $\mu$-$\tau$-symmetric mixing scheme and the S3 Group mixing (democracy/magic-square symmetry) \cite{5} scheme, with each requiring $F = 0$, and additionally $A = 0$ and $C = 0$ respectively (we encounter a generalisation of this duality later, Eqs. (23)-(24)).

4.5 $B$, $D$ (Cubic $T \times 1$, $1 \times T$) and Higher-order Plaquette Invariants

The set of cubic plaquette invariants is completed by $B$ and $D$ where:

$$
B = 3\sqrt{3} \left[ (w^2 x + w x^2 - y^2 z - z^2 y + wxy + wxz - xyz - wyz) \\
+ \frac{1}{2} (w^2 z - wz^2 + yx^2 - y^2 x) \right]
$$

and $D$ is given by exchanging the roles of $L$ and $N$ throughout ($x \leftrightarrow y$, etc.), and making an overall (conventional) sign flip. The tensors $L_A, N_C$ etc. are those already defined above. The denominator, $L_\Delta^3 N_\Delta^2$ in Eq. (19), carries the transformation properties as usual. Unlike $F$, $G$, $C$ and $A$, the observables $B$ and $D$ are not $L \leftrightarrow N$ symmetric (or not up $\leftrightarrow$ down symmetric in the quark case). They are also not independent of those defined earlier, satisfying the identities:

$$
\begin{align*}
A^2 + B^2 + C^2 + D^2 &= G(3F^2 + G^2)/2 \\
AC + BD &= F(F^2 + 3G^2)/4.
\end{align*}
$$

Plaquette invariants of order $\geq 4$ are not uniquely defined by their symmetry under the flavour group. All are however expressible in terms of those we have already encountered. They may be flavour even-even (e.g., $F^2$, $G^2$, $GC$ etc.), odd-odd (e.g., $FG$, $FC$, $AC$, $BD$ etc.) or odd-even (e.g., $BG$, $BC$, $FD$) etc. Of particular interest is the square of the $CP$ violation parameter, $J^2$, which is even-even, but is not homogeneous in $\bar{p}$. It may be written in terms of the invariants already discussed:

$$
18J^2 = 1/6 - G + (4/3) C - (1/2) F^2.
$$

Its physical range, $0 < 18J^2 < \frac{1}{6}$, implies non-trivial boundaries for the space of $G$, $C$ and $F^2$. Clearly, $J^2$ can be expressed in terms of mass matrices, either via Eq. (23), or by squaring Eq. (2). While plaquette-invariants may be constructed at any order, and may be homogeneous or not, we consider those introduced here to be elemental.
5 Application to Flavour Symmetric Descriptions of Mixing

Our plaquette invariants may be used to describe fermion mixing in terms which are independent of flavour labels, and we have given in Sections 3 and 4, some examples for particular lepton mixing schemes. In this section, we expand the list of mixing schemes considered, see Table 2, where we summarise the correspondence between these schemes, constraints on our flavour-symmetric mixing observables, and the phenomenological symmetries to which they correspond.

| Mixing Ansatz               | F | G  | C  | A  | Corresponding Symmetries          | 18J²  | B   | D   |
|-----------------------------|---|-----|----|----|-----------------------------------|-------|-----|-----|
| No Mixing                   | 1 | 1   | 1  | 1  | -                                 | 0     | 0   | 0   |
| Tribimaximal Mixing* [8]   | 0 | 1/6 | 0  | 0  | Dem., μ-τ, CP                      | 0     | 0   | 1/12√3 |
| Trimaximal Mixing [13]      | 0 | 0   | 0  | 0  | Dem., μ-τ                          | 1/6   | 0   | 0   |
| S3 Group Mixing* [4, 5]    | 0 | –   | 0  | –  | Democracy                          | –     | 0   | –   |
| Two Equal P-Rows* [6]      | 0 | –   | –  | 0  | e.g. μ-τ                           | –     | 0   | –   |
| Two Equal P-Columns        | 0 | –   | –  | 0  | e.g. 1-2                           | –     | 0   | 0   |
| Altarelli-Feruglio* [10]   | 0 | –   | 62/8 | 0 | μ-τ, CP                            | 0     | 0   | –   |
| Tri-χmaximal Mixing* [4]   | 0 | –   | 0  | 0  | Dem., μ-τ                          | –     | 0   | –   |
| Tri-φmaximal Mixing* [4]   | 0 | 1/6 | 0  | –  | Dem., CP                           | 0     | 0   | –   |
| Bi-maximal Mixing [17]     | 0 | 1/8 | –  | 1/32 | CP, μ-τ, 1-2 | 0     | 0   | 0   |

Table 2: Particular mixing schemes and their corresponding descriptions in terms of constraints on plaquette invariants, and symmetries. Those marked with an asterisk (*) are currently phenomenologically viable. Although the four \( L \leftrightarrow N \) symmetric variables, \( F, G, C \) and \( A \), are sufficient, we include \( B \) and \( D \) (and \( J^2 \)) for completeness.

Setting the values of \( F, C, A \) and \( G \) or \( J^2 \) equal to those given in Table 2 gives for the first time, flavour-symmetric statements of the respective mixing schemes. For example, the constraints \( F = C = A = 0 \) correspond to the \( μ-τ \)-symmetric and democratic “tri-χmaximal” mixing ansatz [4]. Such constraints are of course readily cast in manifestly weak-basis invariant form [18], using our expressions for our mixing observables in terms of mass matrices.

As well as the conditions summarised in Table 2 one can construct single flavour-symmetric constraints corresponding to less restrictive conditions on the mixing matrix, which are nevertheless interesting, and phenomenologically viable, eg.:

\[
8C^3 - 27F^2(CG - AF) = 0 \quad \Rightarrow \quad |U_{ai}|^2 = \frac{1}{3} \quad \text{(for any particular } \alpha, i) \quad (23)
\]

\[
8B^3 - 27F^2(BG - DF) = 0 \quad \Rightarrow \quad |U_{ai}|^2 = |U_{bi}|^2 \quad \text{(for any particular } \alpha \neq \beta, i) \quad (24)
\]
The two constraints, $F = 0$, $C = 0$, highlighted earlier, corresponding to democracy symmetry, are clearly a special case of Eq. (23). Each of the constraints, Eqs. (23)-(24), is a ninth-order equation in $\tilde{p}$, having one solution for each of the nine possible locations of the constrained element of $U$.

Finally, we give the (two) simultaneous flavour-symmetric constraints which correspond to “any element of $U$ equal to zero” (eg. $P_{e3} \equiv |U_{e3}|^2 = 0$, consistent with the CHOOZ bound [19]):

$$\text{Det} K = 0; \quad J = 0;$$  \hspace{1cm} (25)

where 54 $\text{Det} K = 2A + F(F^2 - 2C - 1)$ and $K$ is the matrix of real parts of plaquette-products of $U$ [20] [7], ie. the $CP$-conserving analogue of $J^\ell$. We point-out that the first of the two conditions in Eq. (25), is in fact consistent with mixing data for both leptons and quarks, this being manifestly so for leptons (see eg. Table 1). In the quark case, for the known values [11] of the Wolfenstein parameters, $\lambda, A$ and $\rho^2 + \eta^2$, $\text{Det} K = 0$ corresponds to $\alpha = (88 \pm 1)^\circ$, consistent with the latest fits [11] which give $\alpha = (90^{+7}_{-3})^\circ$. Indeed, the two constraints “$\text{Det} K = 0$, $J$ small”, together provide a unified and flavour-symmetric, partial description of both lepton and quark mixing matrices, being associated with the existence of at least one small element in each mixing matrix, $U_{e3}$ and $V_{ub}$ respectively.

6 Summary
We have introduced several mixing observables which, like $J$, are (pseudo-)scalars under the flavour-symmetry group $S_3^\ell \times S_3^\nu$ (or the analogous group for the quarks, $S_3^u \times S_3^D$). We have shown how, like $J$, they can also be expressed quite simply in terms of weak-basis invariant functions of the mass matrices divided by powers of the mass matrix discriminants. Our new observables “measure” the violations of certain symmetries (again in analogy to $J$) associated with the phenomenologically successful tribimaximal [8] scheme. It is remarkable that in the case of the leptons, most of these observables are consistent with zero, corresponding to the previously identified democracy and $\mu-\tau$ symmetries. Such plaquette-invariant observables may be applied to construct explicitly flavour-symmetric constraints on the quark and lepton mixing matrices, even though the observed mixing matrices are not themselves flavour-symmetric, the flavour-symmetry being spontaneously broken. The main result of this paper is the set of such constraints, summarised in Table 2, and the corresponding weak-basis invariant constraints in terms of the fermion mass matrices.

$^9$The $\text{Det} K = 0$ condition, Eq. (25), may also be written in terms of mass matrices using the $Q$-matrix ($Q$ is the matrix of quadratic mass matrix commutators related to $K$ by a simple moment transform [12] [7]). The condition becomes simply $\text{Det} Q = 0$. 

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Acknowledgments

This work was supported by the UK Science and Technology Facilities Council (STFC). PFH acknowledges the hospitality of the Centre for Fundamental Physics (CfFP) at the Rutherford Appleton Laboratory, and the Particle Physics Group at the University of Hawaii, Manoa, where some of this work was carried out. We acknowledge useful comments made by Probir Roy and Sandip Pakvasa and are grateful to R. Krishnan for independently cross-checking many of the formulae presented herein.

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