Temperature Correlations in a Finite Universe

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ABSTRACT

We study the effect of a finite topology on the temperature correlations of the cosmic microwave background in a flat universe. Analytic expressions for the angular power spectrum are given for all possible finite flat models. We examine the angular correlation function itself, pointing out visible and discrete features that arise from topology. While observations of the power spectrum on large angular scales can be used to place bounds on the minimum topology length, cosmic variance generally restricts us from differentiating one flat topology from another. Schemes that acknowledge anisotropic structures, such as searches for ghosts, circles or geometric patterns, will be needed to further probe topology.

Key words: cosmic microwave background—large-scale structure of Universe

1 INTRODUCTION

There is no reason to believe that the universe is infinite. While general relativity specifies the local curvature of spacetime, the global geometry of the universe remains unspecified. From this point of view, an infinite universe is assumed only to simplify theoretical calculations and is subject to observational verification. Although inflation would push topology scales far out of view, recent observations (Spinrad et al. 1997; Garnavich et al. 1998) suggest that the curvature of the universe may deviate from flat sufficiently to be measurable. If curvature is observable, then how can we assume topology is not?

As it represents the largest volume observable, the Cosmic Microwave Background (CMB) is uniquely sensitive to the global geometry of the universe. Already the Cosmic Background Explorer (COBE) results have been used to place constraints on flat topologies and limited open topologies that are orders of magnitude better than other approaches (Gott 1980; Sokolov 1993; Stevens, Scott, & Silk 1993; Starobinsky 1993; Lachieze-Rey & Luminet 1993; Lehoucq, Lachieze-Rey, & Luminet 1996; He, Oliveria-Costa, Smoot, & Starobinsky 1997; Levin, Barrow, Bunn, & Silk 1997; Levin, Scannapieco, & Silk 1998, hereafter LSS). The increased sensitivity of the next generation of CMB experiments has inspired renewed interest in the search for topology (Cornish, Spergel, & Starkman 1996; Cornish, Spergel, & Starkman 1997; Cornish, Spergel, & Starkman 1998; Cornish, Spergel, & Starkman 1999; Cornish, Spergel, & Starkman 2000). In LSS, we were able to quantify the effects of topology on the CMB by considering all possible compactifications of flat space. We solved for the spectrum of fluctuations explicitly which allowed us to create realizations of finite universes and compare typical angular power spectra to the COBE data. Here we extend those results by computing the ensemble-averaged angular power spectrum, as opposed to just obtaining realizations. Since we know the modes explicitly from LSS, our task here is to reduce the angular power spectrum to a simple analytic expression for each of the six orientable topologies. Generic features in the spectrum can then be identified without ambiguity.

2 TEMPERATURE FLUCTUATIONS

The primary cause of CMB temperature fluctuations is lumps in the geometry of spacetime at the time of decoupling. The fluctuations can be decomposed into eigenmodes and written in any compact, flat spacetime as

$$\frac{\delta T}{T}(\hat{n}) \propto \sum_k \hat{\Phi}_k \exp \left( i \Delta \eta \hat{k} \cdot \hat{n} \right),$$

with $\Delta \eta$ the conformal time between today and decoupling. On a compact manifold, the usual continuous spectrum of eigenvalues becomes discretized, hence the sum in the equation. The $\hat{\Phi}_k$ are primordially seeded Gaussian amplitudes that obey the reality condition $\hat{\Phi}_k = \hat{\Phi}^*_k$ and a set of relations that depend on the topology (LSS).

With this decomposition we can construct the correlation function between any two points on the sky as

$$C(\hat{n}, \hat{n}’) = \left\langle \frac{\delta T}{T}(\hat{n}) \frac{\delta T}{T}(\hat{n}’) \right\rangle$$
As the fundamental domain has a particular orientation on the sky, the correlation is not simply a function of the angular separation between \( \hat{n} \) and \( \hat{n}' \) as it is in the infinite case.

From this expression, \( C_\ell \) can be determined using the orthogonality relations of the Legendre polynomials:

\[
C_\ell = \frac{1}{4\pi} \int d\Omega \int d\Omega' C(\hat{n}, \hat{n}') P_\ell(\mu),
\]

where \( \mu = \hat{n} \cdot \hat{n}' \). Expanding the exponential and Legendre polynomials in terms of spherical harmonics, this becomes

\[
C_\ell \propto \sum \left\langle \hat{\Phi}_\ell^a \hat{\Phi}_\ell^{a'} \right\rangle \exp \left( i \Delta \eta(\vec{k} \cdot \hat{n} - \vec{k}' \cdot \hat{n}') \right) (2\ell + 1)^{-1}
\]

\[
\times \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\vec{k}) Y_{\ell,m}(\vec{k}').
\]

As \( \left\langle \hat{\Phi}_\ell^a \hat{\Phi}_\ell^{a'} \right\rangle \) and the spectrum of eigenvalues are known for all six possible flat topologies, we can use this expression to compute \( C_\ell \) for each of the possible cases.

### 3 COMPACT SPACES

The simplest topology is the hypertorus, which is built out of a parallelepiped by identifying \((x, y, z) \rightarrow (x + h, y + b, z + c)\). The identification leads to a restriction of the eigenvalue spectrum, \( \ell = 2\pi(j/h, w/b, n/c) \) with the \( j, w, n \) running over all integers. With this restriction, the \( C_\ell \)'s become

\[
C_\ell \propto \sum_{jwn} \frac{P(k)}{k^3} j_\ell(\Delta \eta k)^2,
\]

where \( P(k) \propto 1 \) for a flat power spectrum. This is in agreement with Stevens, Scott, & Silk (1993). In Fig. 1 we plot this expression for three different topology scales for a flat power spectrum normalized by what we would expect for a universe with no topology; that is \( C_\ell \propto V^{-1} \pi^2 \ell(\ell + 1) \), where \( V \) is the volume of the fundamental domain. The normalization is absolute, such that an infinite universe would be represented by normalization \( \ell(\ell + 1)C_\ell = 1 \). Cosmic variance is estimated as \( C_\ell \sqrt{2/(2\ell + 1)} \), as for an infinite universe, although the true variances for any given topology would be slightly different.

There are a number things to note here. In the upper panel we see that the low \( \ell \) modes are damped, with the suppression becoming more severe as the topology scale decreases. From the discretization of the wave vector, \( \vec{k} \), it is clear that there is a minimum eigenvalue corresponding to the longest wavelength that can fit inside the fundamental domain. This maximum wavelength can be associated with an angular scale above which we do not expect to find fluctuations. As the association between real space and angular perturbations causes some averaging over \( k \) modes, the damping is smeared over a range of \( \ell \) values.

In addition to the damping at low \( \ell \), a finite topology also causes jaggs at higher \( \ell \) values, extending to values above \( \ell = 60 \) for the torus of size \( 0.66\Delta \eta \). The jaggy features not only suppress many of the higher \( C_\ell \)'s but actually cause enhancement at selected \( \ell \) values. This ringing in the \( C_\ell \)'s can be understood as caused by the presence of a discrete set of harmonics of the fundamental domain in the matter power spectrum. The discretization not only draws power away from values that are disallowed, but enhances power at certain typical angular scales. Another way to understand this effect is to consider the presence of multiple copies of the same point. One can imagine that given a topology scale, there are certain angles at which multiple images tend to fall, while at other angles such correlations are disallowed by the geometry of the fundamental domain. This effect will become more apparent when we consider the angular correlation function itself.

In Fig. 2 we plot the \( C_\ell \)'s for three different tori with the same volume but different aspect ratios. Here we see that the suppression at low \( \ell \) is much more severe in the elongated configurations than in the model with equal sides. Thus the damping is much more dependent on the minimum dimension of the parallelepiped than the overall volume of the fundamental domain.

Returning to Eq. (3) we are able to calculate the effect of a finite volume on the \( C_\ell \)'s for more complicated flat topologies. Three other spacetimes are constructed from a parallelepiped. The first twisted parallelepiped we consider has opposite faces identified with one pair rotated through the angle \( \pi \). The eigenmodes are \( \vec{k} = 2\pi(j/h, w/b, n/2c) \), with the additional relation \( \hat{\Phi}_{jwn} = \hat{\Phi}_{-j,-wn} e^{i\pi/2} \). In this case the \( C_\ell \)'s become

\[
C_\ell \propto \sum_{jwn} \frac{P(k)}{k^3} j_\ell(\Delta \eta k)^2 \sum_{m=-\ell}^{\ell} |Y_{\ell,m}(\vec{k})|^2 (1 + e^{i\pi(a+m)}).
\]

Another possible compact space identifies opposite faces of the parallelepiped with one face rotated by \( \pi/2 \). The discrete eigenmodes are \( \vec{k} = 2\pi(j/h, w/b, n/4c) \), with the additional relations \( \hat{\Phi}_{jwn} = \hat{\Phi}_{-j,-wn} e^{i\pi/2} = \hat{\Phi}_{-j,-wn} e^{i\pi n} = \hat{\Phi}_{-j,-wn} e^{i\pi n/2} \). The \( C_\ell \)'s are given by

\[
C_\ell \propto \sum_{jwn} \frac{P(k)}{k^3} j_\ell(\Delta \eta k)^2 \sum_{m=-\ell}^{\ell} |Y_{\ell,m}(\vec{k})|^2 \sum_{m=-\ell}^{\ell} |Y_{\ell,m}(\vec{k})|^2 (1 + e^{i\pi(a+m)\pi/2} + e^{i\pi(a+m)\pi/2}).
\]
The last parallelepiped is unique among the flat topologies in that it has a fundamental domain of volume $2hbc$ and is thus a “double” parallelepiped. It is described by the following identifications (Wolf 1967): Translate along $x$ and then rotate around $x$ by $\pi$ so that $(x, y, z) \rightarrow (x + h, -y, -z)$. Next, translate along $y$ and $z$, then rotate around $y$ by $\pi$ so that $(x, y, z) \rightarrow (-x, y + b, -z + c)$. Finally, translate along $x$, $y$, and $z$, then rotate around $z$ by $\pi$ so that $(x, y, z) \rightarrow (-x + h, -y + b, z + c)$. The discrete spectrum for this space is $\vec{k} = \pi(j/h, w/b, n/4c)$. With relations as given in LSS we find

$$C_\ell \propto \sum_{jwn} \left( \frac{2\pi}{h} \right)^4 \frac{(2\pi)^2}{k^2} \sum_{m=-\ell}^\ell \left( Y_{\ell,m}(\hat{k}) (1 + e^{i(m+j+w+n)\pi}) + Y_{\ell,-m}(\hat{k}) e^{i\pi(e^{i(m+j)\pi} + e^{i(w+n)\pi})} \right).$$

In Fig. 3 we plot the normalized $C_\ell$s for each of these spaces, taking the topology scale equal to the horizon size. In the upper panel we see that both the $\pi$-twisted and $\pi/2$ twisted tori have $C_\ell$s that are almost identical to that of the torus. It may seem that the maximum wavelength would be bigger for twisted spaces since a wave can wrap more than once across the fundamental domain before completing a full circuit. However, a closer inspection shows that the relations between eigenmodes places a cutoff in these spectra at the same place as in the torus (LSS). While it is true that in some sense, there is power at smaller eigenvalues in the twisted direction, the angular average discards this asymmetric information. Here we see that not only is the cutoff the same for square parallelepipeds, but the $C_\ell$s are damped in almost the same manner. The harmonics appear to be the distinguishing feature between topologies, but unfortunately fall well within cosmic variances.

The multiply twisted case is somewhat less damped than the other cases, due to the “double” nature of the fundamental domain. Halving the volume lessens these differences, although the relations in this topology still allow slightly larger modes than the other cases. This is illustrated in the lower panel of Fig. 3.

The last two possible compact flat spaces are based on a hexagonal tiling. In the first of these cases, the opposite sides of the hexagon are identified while in the $z$ direction the faces are rotated relative to each other by $2\pi/3$. The potential can be written as

$$\Phi = \sum_{n_2n_3n_z} \Phi_{n_2n_3n_z} e^{i\Omega x z} \times \exp \left[ \frac{2\pi}{h} \left( n_2 \left( x - \frac{1}{\sqrt{3}} y \right) + n_3 \left( x + \frac{1}{\sqrt{3}} y \right) \right) \right]$$

with the eigenmodes $\vec{k} = 2\pi((n_2 + n_3)/h, -(n_2 + n_3)/b, n_3/3c)$. The relations on this space (LSS) result in $C_\ell$s given by

$$C_\ell \propto \sum_{n_2n_3n_z} \left( \frac{2\pi}{h} \right)^4 \frac{(2\pi)^2}{k^2} \sum_{m=-\ell}^\ell \left| Y_{\ell,m}(\hat{k}) \right|^2$$

$$\left( 1 + e^{2i(n_2 + m)\pi/3} + e^{4i(n_2 + m)\pi/3} \right).$$

The last possibility identifies the $z$-faces after rotation by $\pi/3$. The potential can still be written as (11) with $\vec{k} = 2\pi((n_2 + n_3)/h, -(n_2 + n_3)/b, n_3/6c)$ and a set of relations among the $\Phi_{\ell}$, the $C_\ell$s are given by

$$C_\ell \propto \sum_{n_2n_3n_z} \left( \frac{2\pi}{h} \right)^4 \frac{(2\pi)^2}{k^2} \sum_{m=-\ell}^\ell \left| Y_{\ell,m}(\hat{k}) \right|^2$$

$$\left( 1 + e^{i(n_3 + m)\pi/3} + e^{2i(n_3 + m)\pi/3} + e^{4i(n_3 + m)\pi/3} + e^{5i(n_3 + m)\pi/3} \right).$$

The volume of both of these topologies is $h^2\pi \Delta y^3$. In Fig. 3 we plot normalized $C_\ell$s for both hexagonal spaces with unit volumes and with $h = c = \Delta y$. Again, the damping is quite similar to the torus, despite the fact that modes in the $\pi/3$ torus must wrap around a full six times before being associated with the same point. The $C_\ell$s of unit volume are a somewhat better match.

4 ANGULAR CORRELATION FUNCTION

We now turn our attention to the angular correlation function itself. While $C(\theta)$ can in principle be obtained numerically from the $C_\ell$s through a Legendre transform, an analytical expression can be obtained for the special case of the
torus by carrying out an angular average over the sky with the angle between $\hat{n}$ and $\hat{n}'$ fixed. This gives us

$$C(\theta) \propto \sum_{jwn} \frac{P(k) \sin(2\Delta\eta k \sin(\theta/2))}{k^3} \frac{1}{2\Delta\eta k \sin(\theta/2)}.$$  \hspace{1cm} (11)

In the upper panel of Fig. 4 we plot $C(\theta)$ for a flat power spectrum. Note the secondary peaks at angular separations of 60 and 90 degrees. These occur because the relationship between the topology and the horizon size is such that for certain orientations, multiple copies of the same 2π/3-twisted space appear in the upper panel and the π/3-twisted space in the lower panel.

Figure 4. Normalized $C_l$s for the hexagonal cases. The short dashed lines correspond to $h = c = \Delta\eta$, the long dashed lines to $V = \Delta\eta^3$, and the solid to the unit torus. The 2π/3-twisted space appears in the upper panel and the π/3-twisted space in the lower panel.

While each pairing is completely correlated, the averaging of $C(\hat{n}, \hat{n}')$ over all orientations causes the secondary peaks in $C(\theta)$ to be dwarfed by the small angle correlations. This suggests that the most accurate determinations of topology will employ an approach that retains orientation information. Such methods are a topic of current research (de Oliveria-Costa, Smoot, & Starobinsky 1996; Cornish, Spergel, & Starkman 1997; Cornish, Spergel, & Starkman 1997; Bond, Pogosyan, & Souradeep 1997; Ferreira & Magueijo 1997; Levin et al. 1998).

5 CONCLUSIONS

As there are an infinite number of compact open spaces, all of which support chaotic flows, many theoretical advances must be made before these cases can be approached. While we have only considered flat topologies, at least some of the features we observe, such as discrete harmonics of the fundamental domain, enhancement due to ringing, and multiple peaks in the correlation function, can be expected to play a role in any finite universe. While the absence of large-scale damping places a weak lower bound on the minimum topology length, the angular power spectrum is in general a poor measure of topology. Not only does it discard vital anisotropic information, but cosmic variance prevents it from discriminating a hexagonal prism from a hypertorus.

More inventive methods have been suggested such as a search for circles in the sky (Cornish, Spergel, & Starkman 1997; Cornish, Spergel, & Starkman 1997), pattern formation (Levin, Barrow, Bunn, & Silk 1997; Levin et al. 1998), or a method of images (Bond, Pogosyan, & Souradeep 1997). However we look for these features, topology is as important as the curvature or the primordial power spectrum in determining the structure of the microwave background. Just as curvature may be observable with a new generation of experiments, so too may be the global structure of the universe.

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