POLYHEDRAL DIVISORS OF AFFINE TRINOMIAL HYPERSURFACES

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Abstract: We find the general form of the polyhedral divisors corresponding to the natural torus action of complexity 1 on affine trinomial hypersurfaces. Some explicit computations of the divisors for the particular classes of the hypersurfaces are given.

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§ 1. Introduction

One of the main problems of modern algebraic geometry is to find an effective description for the particular classes of algebraic varieties and study the geometric properties of varieties in terms of the description. The classical example of this approach is the theory of toric varieties. Therein the normal algebraic variety with an action of an algebraic torus with open orbit is determined by a fan of rational polyhedral cones (see, for instance, [1, 2]). Over the past decades, many results have been obtained that relate the geometry of a variety with the combinatorial properties of the corresponding fans.

The natural generalization of this theory is the study of the arbitrary actions of an algebraic torus. The complexity of the action, defined in 1986 by Vinberg in [3], is an important invariant characterizing the action. In the case of an algebraic torus action, this is the codimension of a generic orbit of the action. In 2006 in [4], Altmann and Hausen proposed to describe the action of a torus of the arbitrary complexity on an affine variety using its own polyhedral divisors. This approach has become widespread. Details can be found in the survey [5]. Thus, of great importance becomes the problem of explicitly computing the polyhedral divisors for particular affine varieties. This description is most effective in the case of actions of complexity 1.

The theory of Cox rings relates an arbitrary action of complexity 1 with the affine varieties defined by trinomial equations. The case of one trinomial equation leads to affine trinomial hypersurfaces. These varieties are of unflagging interest from the standpoint of affine algebraic geometry. The question of calculating the polyhedral divisors that correspond to the action of a torus on the trinomial hypersurface was also previously studied in [6]. Namely, Arzhantsev obtained the polyhedral divisors corresponding to the action of a torus on factorial trinomial hypersurfaces.

In this paper, we find the general form of the polyhedral divisor corresponding to the canonical torus action of complexity 1 on an affine trinomial hypersurface. In Section 5, we explicitly compute the polyhedral divisor for rational trinomial hypersurfaces. Some classification of the hypersurfaces is given and the general form of the divisor for each of the classes is calculated. The two-dimensional case is considered separately: The corresponding affine surfaces are widely known as the Pham–Brieskorn surfaces. In the final example, we consider the case of an irrational hypersurface; in this example there arises a curve of genus 1.

§ 2. Polyhedral Divisors

Let \( \mathbb{K} \) be an algebraically closed field of characteristic 0 and let \( Y \) be a normal algebraic variety. We will also assume that \( Y \) is semiprojective, i.e. projective over some affine variety. A prime divisor \( D \) on the variety \( Y \) is an irreducible closed subvariety of codimension 1 in \( Y \). A Weil divisor on \( Y \) is a finite
sum
\[ D = \sum_i n_i \cdot D_i, \]
where \( D_i \subseteq Y \) are prime divisors and \( n_i \in \mathbb{Z} \). For each divisor \( D \), the support of \( D \) is defined as
\[ \text{supp}(D) := \bigcup_{n_i \neq 0} D_i. \]

The Weil divisors on a variety \( Y \) constitute the group that is denoted by \( \text{Div}(Y) \). We can also define the group of rational Weil divisors \( \text{Div}_Q(Y) := \mathbb{Q} \otimes \mathbb{Z} \text{Div}(Y) \). A Weil divisor \( D \) is called effective if \( n_i \geq 0 \) for each \( i \) (denoted by \( D \geq 0 \)). Let \( f \) be a nonzero rational function on a normal variety \( Y \). We can associate a Weil divisor with \( f \) by the rule: \( D_i \) are prime divisors corresponding to the zeros and poles of \( f \), \( n_i < 0 \) if \( D_i \) corresponds to a pole of \( f \), and \( n_i > 0 \) if \( D_i \) corresponds to a zero of \( f \); in turn, \( |n_i| \) is equal to the order of the corresponding zero or pole. The divisor \( D \) obtained from \( f \) as above is called principal and denoted by \( (f) \). A Cartier divisor on \( Y \) is a locally principal divisor, i.e., a divisor \( D \) such that there is an open cover of \( Y \) by affine patches \( U_i \) such that for each \( i \) the restriction of \( D \) to \( U_i \) coincides with the restriction to \( U_i \) of the principal divisor of some function \( f_i \in \mathbb{K}(Y) \). The restriction of a divisor \( D \) to an open subset \( U \subseteq Y \) is defined as
\[ D|_U := \sum_i n_i \cdot (D_i \cap U). \]

The Cartier divisors on \( Y \) constitute the subgroup in \( \text{Div}(Y) \) which is denoted by \( \text{CaDiv}(Y) \), and \( \text{CaDiv}_Q(Y) := \mathbb{Q} \otimes \mathbb{Z} \text{CaDiv}(Y) \) respectively. With \( D \in \text{Div}(Y) \) there is associated the subspace of \( \mathbb{K}(Y) \) defined as
\[ \Gamma(Y, \mathcal{O}(D)) := \{ f \in \mathbb{K}(Y)^\times \mid (f) + D \geq 0 \} \cup \{0\}. \]

Given \( f \in \Gamma(Y, \mathcal{O}(D)) \), define the zero set \( Z(f) := \text{supp}((f) + D) \). A divisor \( D \in \text{Div}(Y) \) is called semiample if there exists \( n \in \mathbb{N} \) such that
\[ Y = \bigcup_{f \in \Gamma(Y, \mathcal{O}(nD))} Y_f, \]
where \( Y_f := Y \setminus Z(f) \). A divisor \( D \) is called big if there exists \( n \in \mathbb{N} \), such that \( Y_f \) is affine for some \( f \in \Gamma(Y, \mathcal{O}(nD)) \).

Recall some notions from convex geometry. The cone generated by vectors \( v_1, \ldots, v_n \) in a rational vector space \( V \) is the subset of this space which is defined as follows:
\[ \text{cone}(v_1, \ldots, v_n) := \left\{ \sum_{i=1}^{n} \lambda_i v_i \in V \mid \lambda_i \in \mathbb{Q}_{\geq 0} \right\}. \]

A polyhedron in a rational affine space is an intersection of finitely many affine subspaces. Given a polyhedron \( \Delta \), define the set of relatively interior points relint(\( \Delta \)); i.e., the points of \( \Delta \) that are interior if we regard \( \Delta \) as a polyhedron in the minimal affine space containing \( \Delta \).

An algebraic torus \( T \) is a direct product of several copies of the multiplicative group of the field. The two lattices are associated with a torus: the lattice of one-parameter subgroups \( N \) and the dual character lattice \( M := \text{Hom}(N, \mathbb{Z}) \) (see [1, 2]). Henceforth, we consider polyhedra in the rational vector spaces \( N_Q := \mathbb{Q} \otimes \mathbb{Z} N \) and \( M_Q \) that is defined similarly.

To each cone \( \sigma \subseteq N_Q \) there corresponds the cone in \( M_Q \) defined by
\[ \sigma' := \left\{ u \in M_Q \mid \forall v \in \sigma \langle u, v \rangle \geq 0 \right\} \]
which is called the dual cone to \( \sigma \). The recession cone \( \sigma \) of a polyhedron \( \Delta \subseteq N_Q \) is the cone
\[ \sigma := \{ v \in N_Q \mid \forall v' \in \Delta, \forall t \in \mathbb{Q}_{\geq 0} v' + tv \in \Delta \}. \]
The addition of polyhedra can be introduced as the Minkowski summation. With this operation, the set of polyhedra with recession cone $\sigma$ constitutes the commutative monoid $\text{Pol}_\sigma^+(N_Q)$. Denote the Grothendieck group of this monoid by $\text{Pol}_\sigma(N_Q)$.

**Definition 2.1.** A polyhedral Weil divisor associated with a cone $\sigma$ is an element of the group

$$\text{Div}_Q(Y, \sigma) := \text{Pol}_\sigma(N_Q) \otimes \mathbb{Z} \text{Div}(Y).$$

For every $u \in \sigma^\vee$, we have the linear functional

$$\text{Div}_Q(Y, \sigma) \to \text{Div}_Q(Y), \quad \mathcal{D} = \sum_i \Delta_i \cdot D_i \mapsto \mathcal{D}(u) := \sum_i \min_{v \in \Delta_i} \langle u, v \rangle \cdot D_i.$$

**Definition 2.2.** A polyhedral divisor $\mathcal{D} \in \text{Div}_Q(Y, \sigma)$ is called proper or a pp-divisor (proper polyhedral divisor) if $D$ is representable as

$$\mathcal{D} = \sum_i \Delta_i \cdot D_i,$$

where $\Delta_i \in \text{Pol}_\sigma^+(N_Q)$, while $D_i \in \text{Div}(Y)$ are prime divisors, $\mathcal{D}(u) \in \text{CaDiv}_Q(Y)$, and $\mathcal{D}(u)$ is semiample for every $u \in \sigma^\vee$ and big for all $u \in \text{relint}(\sigma^\vee)$.

The proper divisors constitute the subgroup $\text{PDPDiv}_Q(Y, \sigma) \subseteq \text{Div}_Q(Y, \sigma)$. Let us formulate the main theorem of [4]:

**Theorem 2.1.** Let $\mathcal{D} \in \text{PDPDiv}_Q(Y, \sigma)$, where $Y$ is a normal semiprojective variety, and $\sigma$ is a cone in $N_Q$. Then to $\mathcal{D}$ there corresponds the normal affine variety $X = \text{Spec}(A[Y, \mathcal{D}])$ of dimension $\text{rank}(N) + \text{dim}(Y)$ with an effective action of a torus $T$ with the lattice of one-parameter subgroups $N$, where

$$A[Y, \mathcal{D}] = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))).$$

A torus action is defined by an $M$-grading on $A$. Conversely, for each normal affine variety $X$ with a given effective torus action, there exist a normal semiprojective variety $Y$ and a pp-divisor $\mathcal{D}$ on it such that $X$ is isomorphic to $\text{Spec}(A[Y, \mathcal{D}])$.

Thus, for defining an effective torus action of complexity 1 on a trinomial hypersurface, it suffices to find a semiprojective straight line $Y$, a cone $\sigma$, and a polyhedral divisor $\mathcal{D} \in \text{PDPDiv}_Q(Y, \sigma)$.

§ 3. Statement of the Problem and Formulation of Results

Fix naturals $n_0$, $n_1$, $n_2$ and put $n := n_0 + n_1 + n_2$. Given $n_i$, consider $l_i \in \mathbb{Z}_{\geq 0}^{n_i}$ and define the monomial

$$T_{l_i} := T_{l_{i0}}^{l_{i0}} \ldots T_{l_{in_i}}^{l_{in_i}} \in K[T_{ij}; \ i = 0, 1, 2, j = 0, \ldots, n_i].$$

A hypersurface $X$ is called trinomial if $X$ is defined in $K^n$ by the equation $T_{0i}^{l_{0i}} + T_{1i}^{l_{1i}} + T_{2i}^{l_{2i}} = 0$. We will assume that $n_il_i > 1$ for $i = 0, 1, 2$; otherwise, $X$ is isomorphic to an affine space. Introduce the notations that we will need below:

$$d_i := \gcd(l_{i0}, \ldots, l_{in_i}), \quad d := \gcd(d_0, d_1, d_2),$$

$$d_{ij} := \gcd\left(\frac{d_i}{d_j}, \frac{d_j}{d_i}\right), \quad \overline{d} := dd_0d_1d_2.$$
Proposition 3.1 [9, Proposition 5.5]. A trinomial hypersurface $X$ is rational if and only if $X$ belongs to one of the two types:

Type I. $d = 1$, $d_{01} = d_{02} = 1$, and $d_{12} = s$ (up to renumbering).

Type II. $d = 2$ and $d_{01} = d_{02} = d_{12} = 1$.

A trinomial hypersurface is endowed with a natural torus action for which the codimension of a generic orbit of this action is equal to 1. In this case the action is said to have complexity 1. The action is defined as follows: Consider the standard action of the $n$-dimensional torus $T$ on $\mathbb{K}^n$. Denote by $N$ the lattice of one-parameter subgroups of this kind. Let $e_1, \ldots, e_n$ be the basis of the lattice. The torus $T \subseteq \mathbb{T}$, which is the stabilizer of the ideal $I(X) \subseteq \mathbb{K}[T]$, has dimension $n - 2$. It is convenient to define $T$ by using the integer $(2 \times n)$-matrix

$$L = \begin{pmatrix} -l_0 & l_1 & 0 \\ -l_0 & 0 & l_2 \end{pmatrix}$$

which defines a linear mapping $L : \mathbb{N} \to \mathbb{Z}^2$. It is not hard to see that $T$ is a torus with the lattice of one-parameter subgroups $N = \text{Ker}(L)$ which acts on $X$.

Let $F : N \to \mathbb{N}$ be the embedding. Define some mapping $S : \mathbb{N} \to N$ so as $S \circ F = \text{id}_N$. Note that $S$ is defined non-uniquely.

The algebra $\mathbb{K}[X]$ admits an $M$-grading corresponding to the above-described action, where $M = \text{Hom}(N, \mathbb{Z})$ is the character lattice of $T$. We will illustrate the construction by an example.

Example 3.1. Consider the trinomial hypersurface $X \subseteq \mathbb{K}^4$ defined by the equation $T_{01}^3 + T_{11}^5 + T_{21}T_{22} = 0$. In this case $L$ has the form

$$L = \begin{pmatrix} -3 & 5 & 0 & 0 \\ -3 & 0 & 1 & 1 \end{pmatrix}.$$ 

The kernel $N$ of the corresponding linear mapping is generated by the vectors $(5, 3, 0, 15)$ and $(0, 0, 1, -1)$. Thus, the action of the 2-dimensional torus $T$ on $X$ has the form

$$(t_1, t_2) \cdot (t_{01}, t_{11}, t_{21}, t_{22}) = (t_1^3t_{01}, t_1^5t_{11}, t_2t_{21}, t_1^{15}t_2^{-1}t_{22}),$$

and the $M$-grading on $\mathbb{K}[X]$ corresponding to this action is defined by

$$\deg(T_{01}) = (5, 0), \quad \deg(T_{02}) = (3, 0), \quad \deg(T_{11}) = (0, 1), \quad \deg(T_{21}) = (15, -1).$$

The main result of the article is as follows:

Theorem 3.1. Suppose that a variety $X \subseteq \mathbb{K}^n$ is defined by the equation $T_{00}^d + T_{1i}^d + T_{2j}^d = 0$, and the $(n - 2)$-dimensional torus $T$ with the lattice of one-parameter subgroups $N \subseteq \mathbb{N}$ acts on $X$ as described above. Then, for the polyhedral divisor

$$\mathfrak{D} = D_0 \cdot \Delta_0 + D_1 \cdot \Delta_1 + D_2 \cdot \Delta_2 \in \text{PPDiv}_Q(Y, \sigma),$$

corresponding to this action, the curve $Y$ lies in the weighted projective plane $\mathbb{P}(d_{12}, d_{02}, d_{01})$ and is defined by the equation

$$w_0^{d_{12}/d_{12}} + w_1^{d_{02}/d_{02}} + w_2^{d_{01}/d_{01}} = 0,$$

the divisor $D_i$ on $Y$ is defined by the equation $w_i = 0, i = 0, 1, 2$, and the polyhedra $\Delta_i$ are defined by the recession cone $\sigma = S(\mathbb{Q}_{\geq 0}^n \cap N_Q)$ and the corresponding vertex sets

$$V(\Delta_0) = S\left(\frac{d}{d_{12}l_{0j}} e_k\right), \quad j = k, \ k = 1, \ldots, n_0,$$

$$V(\Delta_1) = S\left(\frac{d}{d_{02}l_{1j}} e_k\right), \quad j = k - n_0, \ k = n_0 + 1, \ldots, n_0 + n_1,$$

$$V(\Delta_2) = S\left(\frac{d}{d_{01}l_{2j}} e_k\right), \quad j = k - n_0 - n_1, \ k = n_0 + n_1 + 1, \ldots, n.$$
§ 4. Proof of the Main Result

We use the method for computing polyhedral divisors of [4, Section 11]. Namely, consider the action $T \times X \to X$ as the restriction of the diagonal action of $T$ on $\mathbb{K}^n$. Thus, $T$ lies in the torus $\mathbb{T}$ of all invertible diagonal matrices. The embedding $T \subseteq \mathbb{T}$ corresponds to the embedding of the one-parameter subgroups $F : N \subseteq \overline{N}$. Describe the divisor $\mathcal{D}_{\text{toric}}$ corresponding to the action of $T$ on $\mathbb{K}^n$. Consider the exact sequence

$$0 \to N \xrightarrow{F} \mathbb{N} \xrightarrow{P} N' \to 0,$$

where $N' = \mathbb{N}/N$ and $P$ is the canonical projection. Let $\Sigma$ the minimal fan in $N'$ containing all cones $P(\sigma)$, where $\sigma$ ranges over all faces of the cone $\mathbb{Q}_{\geq 0}^n \subseteq \mathbb{N}_Q$. The toric variety $W$ corresponding to $\Sigma$ is the variety on which we will construct the divisor $\mathcal{D}_{\text{toric}}$ for the action of $T$ on $\mathbb{K}^n$. Let $R$ be the set of one-dimensional cones in $\Sigma$. Given $\rho \in R$, consider a generating vector $v_\rho$ of the subgroup $\rho \cap N'$. With each such vector, associate the polyhedron

$$\Delta_\rho := S\left(\mathbb{Q}_{\geq 0}^n \cap P^{-1}(v_\rho)\right) \subseteq \mathbb{N}_Q.$$

In these notations, the divisor $\mathcal{D}_{\text{toric}}$ has the form

$$\mathcal{D}_{\text{toric}} = \sum_{\rho \in R} \Delta_\rho \cdot D_\rho,$$

where $D_\rho \subseteq W$ is the $T$-invariant prime divisor corresponding to the edge $\rho$. Turn to the variety $X$ and the torus action on $X$. In this case the desired divisor $\mathcal{D}$ is obtained by restricting $\mathcal{D}_{\text{toric}}$ to the normalization of the closure of the image of $X \cap O$ under $P$, where $O$ is the open orbit of the action $\mathbb{T} \times \mathbb{K}^n \to \mathbb{K}^n$.

Apply this construction to the case under consideration. Take the mapping $L : \mathbb{Z}^n \to \mathbb{Z}^2$ as $P$. Since $L$ can be nonsurjective, we must assume that $N' = \text{Im}(L) \subseteq \mathbb{Z}^2$. Find the variety $W$. Obviously, $\text{Im}(L)$ is generated by the images of the vectors $e_i$, where $\{e_1, \ldots, e_n\}$ is a standard basis for $\mathbb{Z}^n$. This implies that $N' \subseteq \mathbb{Z}^2$ is generated by the vectors

$$v_0 = (-d_0, -d_0), \quad v_1 = (d_1, 0), \quad v_2 = (0, d_2).$$

The fan $\Sigma$ is also generated by $v_0, v_1, v_2$ in the sense that each proper subset of the set $\{v_0, v_1, v_2\}$ generates a cone $\sigma \in \Sigma, \sigma \subseteq N'_Q$.

**Proposition 4.1.** Suppose that a lattice $N$ is generated by vectors $\{v_0, \ldots, v_n\}$ such that

$$\sum_{i=0}^n q_i v_i = 0, \quad q_i \in \mathbb{Z}_{>0}, \quad \gcd(q_0, \ldots, q_n) = 1.$$

Then $v_i = a_i u_i, i = 0, \ldots, n$, where $u_i$ is a generator of the semigroup $\mathbb{Q}_{\geq 0} v_i \cap N$ and

$$a_i := \gcd(q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n).$$

**Proof.** This fact is demonstrated in [10, Lemma 1(c)]. By Proposition 4.1, the lattice $N'$ is generated by the vectors

$$u_0 = dd_{01}d_{02} \cdot (-1, -1), \quad u_1 = dd_{01}d_{12} \cdot (1, 0), \quad u_2 = dd_{02}d_{12} \cdot (0, 1).$$

For understanding which variety the given vector corresponds to, we use the assertion whose proof can be found in [1, Section 2.3]:

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Proposition 4.2. Let $N$ be a lattice and let $\{e_1, \ldots, e_n\} \subseteq N$ be a basis for $N$. Consider the vectors
\[ v_0 = -\frac{1}{q_0} \sum_{i=1}^{n} e_i, \quad v_i = \frac{1}{q_i} e_i, \quad i = 1, \ldots, n, \]
where $q_i \in \mathbb{Z}_{>0}$, $i = 0, \ldots, n$. Assume that the lattice $N'$ is generated by $v_1$. Then the fan $\Sigma$ generated by $\{v_0, \ldots, v_n\}$ corresponds to the weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$.

Taking the lattice with basis $\{(\tilde{d},0), (0,\tilde{d})\}$ as the initial lattice and using Proposition 4.1, we see that the variety $W$ is $\mathbb{P}(d_{12}, d_{02}, d_{01})$. We infer that $\mathcal{D}_{\text{toric}}$ has the form
\[ \mathcal{D}_{\text{toric}} = \Delta_0 \cdot D_0 + \Delta_1 \cdot D_1 + \Delta_2 \cdot D_2, \]
where $D_i$ is the divisor on $W$ corresponding to the ray $\mathbb{Q}_{\geq 0} u_i$, i.e., $D_i$ is defined in the homogeneous coordinates $[w_0 : w_1 : w_2]$ by the equation $w_i = 0$. Describe the polyhedral coefficients of $\mathcal{D}_{\text{toric}}$. In view of the above, we must calculate
\[ \Delta_i = S(\mathbb{Q}_{\geq 0} \cap L^{-1}(u_i)), \quad i = 0, 1, 2. \]

Since $\text{Ker}(L) = N_Q$, it is clear that the recession cone of $\Delta_i$ is $\sigma = S(\mathbb{Q}_{\geq 0} \cap N_Q)$. Consider the images of the vectors of the standard basis of $N_Q$:
\[ L(e_k) = \frac{l_{0j}}{dd_{01}d_{02}} u_0, \quad j = k, \ k = 1, \ldots, n_0; \]
\[ L(e_k) = \frac{l_{1j}}{dd_{01}d_{12}} u_1, \quad j = k - n_0, \ k = n_0 + 1, \ldots, n_0 + n_1; \]
\[ L(e_k) = \frac{l_{2j}}{dd_{02}d_{12}} u_2, \quad j = k - n_0 - n_1, \ k = n_0 + n_1 + 1, \ldots, n. \]

This easily implies that
\[ \mathbb{Q}_{\geq 0} \cap L^{-1}(u_0) = \left\{ \sum_{k=1}^{n_0} \lambda_k \frac{dd_{01}d_{02}}{l_{0j}} e_k + v \mid \lambda_k \geq 0, \sum_{k=1}^{n_0} \lambda_k = 1, v \in N_Q \right\} \]
\[ \mathbb{Q}_{\geq 0} \cap L^{-1}(u_1) = \left\{ \sum_{k=n_0+1}^{n_0+n_1} \lambda_k \frac{dd_{01}d_{12}}{l_{1j}} e_k + v \mid \lambda_k \geq 0, \sum_{k=n_0+1}^{n_0+n_1} \lambda_k = 1, v \in N_Q \right\} \]
\[ \mathbb{Q}_{\geq 0} \cap L^{-1}(u_2) = \left\{ \sum_{k=n_0+n_1+1}^{n} \lambda_k \frac{dd_{02}d_{12}}{l_{2j}} e_k + v \mid \lambda_k \geq 0, \sum_{k=n_0+n_1+1}^{n} \lambda_k = 1, v \in N_Q \right\}. \]

Thus, the vertex sets of $\Delta_i$, $i = 0, 1, 2$, have the form
\[ V(\Delta_0) = \left\{ S(\frac{dd_{01}d_{02}}{l_{0j}} e_k) \mid j = k, \ k = 1, \ldots, n_0 \right\}, \]
\[ V(\Delta_1) = \left\{ S(\frac{dd_{01}d_{12}}{l_{1j}} e_k) \mid j = k - n_0, \ k = n_0 + 1, \ldots, n_0 + n_1 \right\}, \]
\[ V(\Delta_2) = \left\{ S(\frac{dd_{02}d_{12}}{l_{2j}} e_k) \mid j = k - n_0 - n_1, \ k = n_0 + n_1 + 1, \ldots, n \right\}, \]
and the polyhedra themselves are respectively equal to
\[ \Delta_i = S(\text{conv}(V(\Delta_i) + \mathbb{Q}_{\geq 0} \cap N_Q)), \quad i = 0, 1, 2. \]
So, $\mathcal{D}_{\text{toric}}$ is constructed. Let us now consider the mapping $L$ on $T \subseteq \mathbb{K}^n$ and $(\mathbb{K}^*)^2 \subseteq \mathbb{P}(d_{12}, d_{02}, d_{01})$. Represent $L$ as the composition of the mappings corresponding to the mappings of the lattices

$$N \xrightarrow{L} N_1 \longrightarrow N_2 \longrightarrow N',$$

where $N_1 = \mathbb{Z}^2$ is the lattice containing $N'$ and generated by the vectors $\{(1, 0), (0, 1)\}$. It is not hard to see that the maximal cones of the fan $\Sigma$ in this lattice are

$$\text{cone}((1, 0), (0, 1)), \quad \text{cone}((1, 0), (1, -1)), \quad \text{cone}((0, 1), (-1, -1)),$$

which corresponds to the projective space $\mathbb{P}^2$. The mapping of the corresponding tori looks as

$$(t_0, t_1, t_2) \mapsto [1 : t_0^{-l_0} t_1^l : t_0^{-l_0} t_2^l] = [t_0^l : t_1^l : t_2^l],$$

where $t_i = (t_{i1}, \ldots, t_{in})$, $t_i^l = (t_{i1}^l, \ldots, t_{in}^l)$. The image of the closure of $X \cap O$ under this mapping is defined in the homogeneous coordinates $[x_0 : x_1 : x_2]$ on $\mathbb{P}^2$ by the equation $x_0 + x_1 + x_2 = 0$. The lattice $N_2$ is generated by $\{(d, 0), (0, d)\}$; passage to $N_2$ corresponds to the mapping of $\mathbb{P}^2$ onto itself $[x_0 : x_1 : x_2] \mapsto [x_0^d : x_1^d : x_2^d]$, and, respectively, the closure of the image of $X \cap O$ under this mapping goes to the curve $x_0^d + x_1^d + x_2^d = 0$. Finally, passage to $N'$ corresponds to the factorization of $\mathbb{P}^2$ by the action of the group $\mathbb{N}^*/N_2 \cong \mu_{d_{12}} \oplus \mu_{d_{02}} \oplus \mu_{d_{01}}$, where $\mu_k$ is the group of $k$th roots of unity. The action is defined as

$$(\zeta_0, \zeta_1, \zeta_2) \cdot [w_0 : w_1 : w_2] = [\zeta_0 w_0 : \zeta_1 w_1 : \zeta_2 w_2],$$

where $(\zeta_0, \zeta_1, \zeta_2) \in \mu_{d_{12}} \oplus \mu_{d_{02}} \oplus \mu_{d_{01}}$. Note that the isomorphism corresponding to the factorization mapping has the form

$$\phi : \mathbb{P}^2/(\mu_{d_{12}} \oplus \mu_{d_{02}} \oplus \mu_{d_{01}}) \rightarrow \mathbb{P}(d_{12}, d_{02}, d_{01}), \quad x_0^{d_{12}} \mapsto w_0, \ x_1^{d_{02}} \mapsto w_1, \ x_2^{d_{01}} \mapsto w_2.$$

Note that we used the following assertion of [2, Proposition 1.3.18].

**Proposition 4.3.** Suppose that $N$ is a lattice, $N' \subseteq N$ is a sublattice of finite index in $N$; $M'$ is the dual lattice to $M$; $G = N/N'$; and $\sigma \subseteq \mathbb{N}_Q = N'_Q$ is a strictly convex rational cone, while $T$ and $T'$ are the tori corresponding to $N$ and $N'$. Then

1. $G \cong \text{Hom}_G(M'/M, \mathbb{K}^*) = \text{Ker}(T' \rightarrow T)$;
2. $G$ acts on the algebra $\mathbb{K}[\sigma^\vee \cap M']$, and the algebra of invariant functions $\mathbb{K}[\sigma^\vee \cap M']^G$ coincides with $\mathbb{K}[\sigma^\vee \cap M]$;
3. $G$ acts on $U_{\sigma, N'} := \text{Spec}(\mathbb{K}[\sigma^\vee \cap M'])$ and there exists a morphism $\phi : U_{\sigma, N'} \rightarrow U_{\sigma, N}$ constant on the orbits of $G$; moreover, a bijection $U_{\sigma, N'}/G \cong U_{\sigma, N}$ is defined.

As a result we find that, in the homogeneous coordinates $[w_0 : w_1 : w_2]$ on the variety $W = \mathbb{P}(d_{12}, d_{02}, d_{01})$, the closure of the image of $X \cap O$ under $L$ is defined by the equation $w_0^{d_{01}d_{02}} + w_1^{d_{01}d_{12}} + w_2^{d_{02}d_{12}} = 0$. For finishing the proof of Theorem 3.1, it remains to show that this curve is normal.

Consider the $\mathbb{Z}$-graded algebra $\mathbb{K}[w_0, w_1, w_2]$ whose grading is defined by the degrees of the generators, namely,

$$\deg w_0 = d_{12}, \quad \deg w_1 = d_{02}, \quad \deg w_2 = d_{01},$$

and the mapping $\pi$ from this algebra into the algebra of three variables with the standard $\mathbb{Z}$-grading:

$$\pi : \mathbb{K}[w_0, w_1, w_2] \rightarrow \mathbb{K}[x_0, x_1, x_2], \quad w_0 \mapsto x_0^{d_{12}}, \ w_1 \mapsto x_1^{d_{02}}, \ w_2 \mapsto x_2^{d_{01}}.$$  

Observe that $\pi$ sends the homogeneous polynomials in the algebra $\mathbb{K}[w_0, w_1, w_2]$ to homogeneous polynomials in $\mathbb{K}[x_0, x_1, x_2]$, which makes it possible to give

**Definition 4.1.** The standard cover of a projective curve $Y \subseteq \mathbb{P}(d_{12}, d_{02}, d_{01})$ defined by the equation $f = 0$, $f \in \mathbb{K}[w_0, w_1, w_2]$, is the curve $\tilde{Y} \in \mathbb{P}^2$ defined by the equation $\pi(f) = 0$.  

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The mapping \( \pi \) is the quotient mapping by a finite group. Hence, if \( \tilde{Y} \) is a normal variety then so is \( Y \). For curves, smoothness is equivalent to normality, i.e., we must check the smoothness of the projective curve

\[
\tilde{Y} = \{ [x_0 : x_1 : x_2] \in \mathbb{P}^2 : x_0^d + x_1^d + x_2^d = 0 \}.
\]

In the standard affine patch \( U_0 : x_0 = 1 \), in the coordinates \( y_1 = \frac{x_1}{x_0}, \ y_2 = \frac{x_2}{x_0} \), the curve \( \tilde{Y} \) is defined by the equality \( 1 + y_1^d + y_2^d = 0 \). It is not hard to see that \( \tilde{Y} \) is smooth for all \( d \). Similarly, we can consider the patches \( U_1 \) and \( U_2 \). Thus, \( \tilde{Y} \) is smooth and hence normal. Theorem 3.1 is proved.

Remark 4.1. Describe the structure of the divisors \( D_i \) in more detail. To this end, consider once again the standard cover of the curve \( Y \). The full preimage of \( D_i \) under \( \pi \) looks as follows:

\[
\pi^{-1}(D_i) = \{ [x_0 : x_1 : x_2] \in \tilde{Y} \mid x_i = 0 \}.
\]

Consider, for instance, \( D_0 \). It is not hard to see that the points of the preimage of \( D_0 \) in \( \mathbb{P}^2 \) are defined by the equations \( x_0 = 0 \) and \( x_1^d + x_2^d = 0 \), i.e., these are exactly the points \( \{ [0 : 1 : \zeta^k], \ k = 0, \ldots, d - 1 \} \), with \( \zeta^d = -1 \), where \( \zeta \) is a primitive root of unity of degree \( 2d \). The mapping \( \pi \) takes these points to \( \{ [0 : 1 : \zeta_{k_0}^k], \ k = 0, \ldots, dd_2 - 1 \} \), where \( \eta_0 \) is a primitive root of unity of degree \( dd_2 \). Similarly, the divisors \( D_1 \) and \( D_2 \) are representable as

\[
D_1 = \{ [1 : 0 : \zeta_{k_1}^k], \ k = 0, \ldots, dd_2 - 1 \}, \quad D_2 = \{ [0 : 1 : \zeta_{k_2}^k : 1], \ k = 0, \ldots, dd_1 - 1 \},
\]

where \( \eta_1 \) and \( \eta_2 \) are primitive roots of unity of degrees \( dd_2 \) and \( dd_1 \) respectively.

Remark 4.2. Observe that the planar projective curve obtained in Theorem 3.1 has genus

\[
g = \frac{d}{2}(d - (d_0 + d_0 + d_1)) + 1,
\]

which follows from [9, Theorem 5.3].

§ 5. Examples of Calculations of Polyhedral Divisors

In this section, we explicitly compute the polyhedral divisors corresponding to the several classes of affine trinomial hypersurfaces. Let us start from rational hypersurfaces. We apply Theorem 3.1 to each of the two classes of rational curves described in Proposition 3.1 considering the factorial case separately.

5.1. Type I. The factorial case. As was observed, the factoriality of \( X \) is equivalent to the fact that \( d_0, d_1, \) and \( d_2 \) are pairwise coprime. Using Theorem 3.1, we see that, in this case, the curve \( Y \) on which the divisor defining the torus action on \( X \) is located lies in \( \mathbb{P}^2 \) and is defined in the homogeneous coordinates by the equation \( w_0 + w_1 + w_2 = 0 \). Thus, for the factorial case, we obtain

**Proposition 5.1.** Let \( X \) be a factorial trinomial hypersurface. The divisor \( \mathcal{D} \) corresponding to the natural torus action of complexity 1 on \( X \) looks as

\[
\mathcal{D} = \Delta_0 \cdot \{0\} + \Delta_1 \cdot \{1\} + \Delta_2 \cdot \{\infty\} \in \text{PPDiv}_Q(\mathbb{P}^1, \sigma),
\]

\[
\Delta_0 = \text{conv} \left( \left\{ S \left( \frac{1}{l_0} e_k \right) \mid j = k, \ k = 1, \ldots, n_0 \right\} \right) + \sigma,
\]

\[
\Delta_1 = \text{conv} \left( \left\{ S \left( \frac{1}{l_1} e_k \right) \mid j = k - n_0, \ k = n_0 + 1, \ldots, n_0 + n_1 \right\} \right) + \sigma,
\]

\[
\Delta_2 = \text{conv} \left( \left\{ S \left( \frac{1}{l_2} e_k \right) \mid j = k - n_0 - n_1, \ k = n_0 + n_1 + 1, \ldots, n \right\} \right) + \sigma,
\]

where the cone \( \sigma \) is defined as in Theorem 3.1.
We conclude that the desired divisor is equal to
\[(t_1, t_2) \cdot (T_{01}, T_{11}, T_{21}, T_{22}) = (t_1^5 T_{01}, t_1^3 T_{11}, t_2 T_{21}, t_1^2 t_2^{-1} T_{22}).\]
Consequently, the matrices \(F\) and \(S\) have the form
\[
F = \begin{pmatrix} 5 & 3 & 0 & 15 \\ 0 & 0 & 1 & -1 \end{pmatrix}^T, \quad S = \begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]
As a result, we infer that the desired divisor is equal to
\[
\mathcal{D} = \left( \left( \frac{2}{3}, 0 \right) + \sigma \right) \cdot \{0\} + \left( \left( -\frac{3}{5}, 0 \right) + \sigma \right) \cdot \{1\} + (\{0\} \times [0, 1] + \sigma) \cdot \{\infty\} \in \text{PPDiv}_Q(\mathbb{P}^1, \sigma),
\]
where \(\sigma = S((Q_{d_0}^4 \cap F(Q^2)) = \text{cone}((1, 0), (1, 15))\).

**5.2. Type II. The nonfactorial case.** Without loss of generality, we may assume that \(d = 1, d_{01} = d_{02} = 1, d_{12} = s \geq 2\). We see from Theorem 3.1 that the divisor corresponding to the torus action on this hypersurface is given on the curve \(Y \subseteq \mathbb{P}(s, 1, 1)\) defined by the equation \(w_0 + w_1^s + w_2^s = 0\). We can identify this curve with \(\mathbb{P}^1\) by the isomorphism
\[
\phi_T : Y \rightarrow \mathbb{P}^1, \quad [w_0 : w_1 : w_2] \mapsto [w_1 : w_2],
\]
\[
\phi_T^{-1} : \mathbb{P}^1 \rightarrow Y, \quad [z_0 : z_1] \mapsto [-z_0^s - z_1^s : z_0 : z_1].
\]
In this case Theorem 3.1 takes the following form:

**Proposition 5.2.** Let \(X\) be a rational trinomial hypersurface of type I. Then the divisor \(\mathcal{D}\) corresponding to the natural torus action of complexity 1 on \(X\) looks as
\[
\mathcal{D} = \Delta_0 \cdot \sum_{k=0}^{s-1} \{\zeta^k\} + \Delta_1 \cdot \{0\} + \Delta_2 \cdot \{\infty\} \in \text{PPDiv}_Q(\mathbb{P}^1, \sigma),
\]
\[
\Delta_0 = \text{conv}\left( \left\{ S\left( \frac{1}{l_{0j}} e_k \right) \mid j = k, k = 1, \ldots, n_0 \right\} \right) + \sigma,
\]
\[
\Delta_1 = \text{conv}\left( \left\{ S\left( \frac{s}{l_{1j}} e_k \right) \mid j = k - n_0, k = n_0 + 1, \ldots, n_0 + n_1 \right\} \right) + \sigma,
\]
\[
\Delta_2 = \text{conv}\left( \left\{ S\left( \frac{s}{l_{2j}} e_k \right) \mid j = k - n_0 - n_1, k = n_0 + n_1 + 1, \ldots, n \right\} \right) + \sigma,
\]
where \(\zeta\) is a primitive \(s\)th root of unity and the cone \(\sigma\) is defined as in Theorem 3.1.

**Example 5.2.** Let \(X\) be defined by the equation \(T_{01}^2 + T_{11}^3 + T_{21}^3 T_{22}^3 = 0\). In this case,
\[
L = \begin{pmatrix} -2 & 3 & 0 & 0 \\ -2 & 0 & 3 & 3 \end{pmatrix}, \quad F = \begin{pmatrix} 3 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}^T, \quad S = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]
We conclude that the desired divisor is equal to
\[
\mathcal{D} = \left( \left( \frac{1}{2}, 0 \right) + \sigma \right) \cdot \{1\} + \{\zeta\} + \{\zeta^2\} + \left( \left( -\frac{1}{3}, 0 \right) + \sigma \right) \cdot \{0\} + \{0\} \times \left[ 0, \frac{1}{5} \right] + \sigma \cdot \{\infty\} \in \text{PPDiv}_Q(\mathbb{P}^1, \sigma),
\]
where \(\sigma = \text{cone}((1, 0), (1, 2))\) and \(\zeta^3 = 1, \zeta \neq 1\).
5.3. Rational hypersurfaces of type II. Consider the remaining class of rational hypersurfaces. In this case $d_1 = d_2 = d_1 = 1$, and $d = 2$. By Theorem 3.1, the divisor corresponding to the torus action on $X$ is on the curve $Y \subseteq \mathbb{P}^2$ defined by the equation $w_0^2 + w_1^2 + w_2^2 = 0$. This projective curve has genus 0 and hence is isomorphic to $\mathbb{P}^1$ like in the previous examples. An isomorphism can be constructed, for instance, as follows:

$$
\phi_{II} : Y \to \mathbb{P}^1, \quad [w_0 : w_1 : w_2] \mapsto \begin{cases}
[w_0 + iw_1 : w_2], & w_0 \neq -iw_1, \\
[w_2 : iw_1 - w_0], & w_0 \neq iw_1,
\end{cases}
$$

$$
\phi_{II}^{-1} : \mathbb{P}^1 \to Y, \quad [z_0 : z_1] \mapsto [z_0^2 - z_1^2 : -i(z_0^2 + z_1^2) : 2z_0z_1],
$$

where $i^2 = -1$. We infer that, for rational hypersurfaces of type II, Theorem 3.1 looks like

**Proposition 5.3.** Let $X$ be a rational trinomial hypersurface of type II. Then the divisor $\mathcal{D}$ corresponding to the natural torus action of complexity 1 on $X$ has the form

$$
\mathcal{D} = \Delta_0 \cdot (\{1\} + \{-1\}) + \Delta_1 \cdot (\{i\} + \{-i\}) + \Delta_2 \cdot (\{0\} + \{\infty\}) \in \text{PPDiv}(\mathbb{P}^1, \sigma),
$$

$$
\Delta_0 = \text{conv}\left(\left\{ S\left(\frac{2}{l_{0j}} e_k \right) \mid j = k, \ k = 1, \ldots, n_0 \right\}\right) + \sigma,
$$

$$
\Delta_1 = \text{conv}\left(\left\{ S\left(\frac{2}{l_{1j}} e_k \right) \mid j = k - n_0, \ k = n_0 + 1, \ldots, n_0 + n_1 \right\}\right) + \sigma,
$$

$$
\Delta_2 = \text{conv}\left(\left\{ S\left(\frac{2}{l_{2j}} e_k \right) \mid j = k - n_0 - n_1, \ k = n_0 + n_1 + 1, \ldots, n \right\}\right) + \sigma,
$$

where $i^2 = -1$ and the cone $\sigma$ is defined as in Theorem 3.1.

**Example 5.3.** Let $X$ be defined by the equation $T_{01}^2 + T_{11}^4 + T_{21}^2 T_{22}^4 = 0$. In this case,

$$
L = \begin{pmatrix} -2 & 4 & 0 & 0 \\ -2 & 0 & 2 & 4 \end{pmatrix}, \quad F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}^T, \quad S = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.
$$

We conclude that the desired divisor is equal to

$$
\mathcal{D} = \left( (1, 0) + \sigma \right) \cdot (\{1\} + \{-1\}) + \left( \left( -\frac{1}{2}, -\frac{1}{2} \right) + \sigma \right) \cdot (\{i\} + \{-i\})
$$

$$
+ \left( \{0\} \times \left[ 0, \frac{1}{2} \right] + \sigma \right) \cdot (\{0\} + \{\infty\}) \in \text{PPDiv}(\mathbb{P}^1, \sigma),
$$

where $\sigma = \text{cone}(\{1, 0\}, \{1, -1\})$ and $i^2 = -1$.

5.4. Pham–Brieskorn surfaces. Let $n_0 = n_1 = n_2 = 1$. Then $X$ is defined in $\mathbb{A}^3$ by the equation $x_0^d + x_1^d + x_2^d = 0$. These surfaces are called Pham–Brieskorn surfaces. On $X$ there exists an action of the one-dimensional torus of complexity 1. It is easy to see that for these surfaces we can always choose matrices $F$ and $S$ so that $\sigma = \mathbb{Q}_{\geq 0}$. For this we must choose $F$ so that its elements be nonnegative. The mapping $F$ can be defined by the vector $f = (d/d_0, d/d_1, d/d_2)$, where $d := \text{lcm}(d_0, d_1, d_2)$. Its elements are coprime in their totality, and so there exists a vector $s = (s_0, s_1, s_2)$ defining the mapping $S$ such that $(f, s) = 1$.

**Proposition 5.4.** Let $X$ be a Pham–Brieskorn surface defined by the equation $x_0^d + x_1^d + x_2^d = 0$. Then the divisor $\mathcal{D}$ corresponding to the action of the one-dimensional torus on $X$ looks as

$$
\mathcal{D} = \Delta_0 \cdot D_0 + \Delta_1 \cdot D_1 + \Delta_2 \cdot D_2 \in \text{PPDiv}(Y, \mathbb{Q}_{\geq 0}),
$$

$$
\Delta_0 = \left( s_0 \frac{dd_0d_2}{d_0}, +\infty \right), \quad \Delta_1 = \left( s_1 \frac{dd_0d_2}{d_1}, +\infty \right), \quad \Delta_2 = \left( s_2 \frac{dd_0d_2}{d_2}, +\infty \right),
$$

where the curve $Y$ and the divisors $D_i$ on it are defined as in Theorem 3.1.
Example 5.4. Let $X$ be defined in $\mathbb{K}^3$ by the equation $x_0^2 + x_1^3 + x_2^6 = 0$. The vectors $f$ and $s$ defining the corresponding mappings have the form $f = (3, 2, 1), s = (1, -1, 0)$. The curve $Y$ lies in $\mathbb{P}(3, 2, 1)$ and is defined by the equation $w_0^2 + w_1^3 + w_2^6 = 0$. By Remark 4.2, this is a curve of genus 1. We infer that the divisor corresponding to the torus action on $X$ is equal to
\[ \mathcal{D} = [1, +\infty) \cdot D_0 + [-1, +\infty) \cdot D_1, \]
\[ D_0 = \sum_{k=1}^{3} \{ [0 : \zeta^k : 1] \}, \quad D_1 = \{ [\tau : 0 : 1] \} + \{ [-\tau : 0 : 1] \} \in \text{PPDiv}_Q(Y, \mathbb{Q}_{\geq 0}), \]
where $\zeta^3 = 1$, $\zeta \neq 1$.

Remark 5.1. Example 5.4 demonstrates that we can choose a mapping $S$ so that some of the polyhedra $\Delta_i$ are equal to the cone $\sigma$.

5.5. The irrational case. In conclusion, construct the divisor for the torus action of complexity 1 on an irrational three-dimensional hypersurface.

Example 5.5. Let $X$ be defined in $\mathbb{K}^4$ by the equation $T_0^2 + T_1^3 + T_2^6 = 0$,
\[ L = \begin{pmatrix} -2 & 3 & 0 & 0 \\ -2 & 0 & 6 & 6 \end{pmatrix}, \quad F = \begin{pmatrix} 3 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}^T, \quad S = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]
The curve $Y$ coincides with the curve obtained in the previous example. As was pointed out, this is a curve of rank 1. We conclude that the divisor corresponding to the action of $X$ is equal to
\[ \mathcal{D} = ((1, 0) + \sigma) \cdot D_0 + ((-1, 0) + \sigma) \cdot D_1 + ([0] \times [0, 1] + \sigma) \cdot D_2, \]
\[ D_0 = \sum_{k=1}^{3} \{ [0 : \zeta^k : 1] \}, \quad D_1 = \{ [\tau : 0 : 1] \} + \{ [-\tau : 0 : 1] \}, \quad D_2 = \{ [1 : -1 : 0] \}, \]
where $\sigma = \text{cone}((1, 0), (1, 1)), \ i^2 = -1, \ zeta^3 = 1, \ zeta \neq 1$.

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