On superintegrable systems separable in Cartesian coordinates

Yu. A. Grigoriev, A.V. Tsiganov
yury.grigoryev@gmail.com, andrey.tsiganov@gmail.com
Saint Petersburg State University

Abstract

We continue the study of superintegrable systems of Thompson’s type separable in Cartesian coordinates. An additional integral of motion for these systems is the polynomial in momenta of $N$-th order which is a linear function of angle variables and the polynomial in action variables. Existence of such superintegrable systems is naturally related to the famous Chebyshev theorem on binomial differentials.

Keywords: Superintegrable systems, higher order integrals of motion, Fokas-Lagersrom system

1 Introduction

In 1984 Thompson proved superintegrability of the Hamiltonian

$$H = p_x^2 + p_y^2 + a(x - y) \frac{2}{2n-1}, \quad n \in \mathbb{Z}_+,$$

where $n$ is an arbitrary positive integer [18]. To simplify the notation it is best to make a 45 degree rotation $q_1 = x + y$ and $q_2 = x - y$ as in [11]. Such superintegrable systems are still being studied up till now, see [1, 9, 10, 16] and references within.

In this note we prove that dynamical system with Hamiltonian

$$H = p_1^2 + p_2^2 + a q_1^{M_1} + b q_2^{M_2}, \quad a, b \in \mathbb{R}, \quad (1.1)$$

is superintegrable, if exponents $M_1$ and $M_2$ belong to the following sequence of positive rational numbers

$$M = 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \quad n \in \mathbb{Z}_+, \quad (1.2)$$

or sequence of negative rational numbers

$$M = 0, -2, -\frac{2}{3}, -\frac{2}{5}, -\frac{2}{7}, \cdots, -\frac{2}{2n-1}. \quad (1.3)$$

These two sequences of exponents are distinguished according to the Chebyshev theorem on binomial differentials [4]. The corresponding additional first integral is a polynomial with respect to momenta $p_1$ and $p_2$.

We also discuss nonseparable systems with Hamiltonians

$$H = p_1^2 + p_2^2 + (a q_1^{M_1} + b) q_2^{M_2}, \quad (1.4)$$

where $M_{1,2}$ belong to [1, 2, 3] and present a new integrable deformation of the Fokas-Lagerstrom system [5, 11]. The corresponding integral of motion is a polynomial in the momenta of the sixth degree.
2 Thompson’s type systems

There are many integrable and superintegrable systems with algebraic potentials, see [3, 5, 8, 10, 11, 12, 13, 15, 17, 18]. For arbitrary rational $M_1, M_2$ Hamiltonian $H$ is also an algebraic function well-defined in some part of the plane. In the same domain of definition we introduce variables

\[ I_1 = p_1^2 + aq_1^{M_1}, \quad I_2 = p_2^2 + bq_2^{M_2}, \]

\[ \omega_1 = -\int^{q_1} \frac{dx}{\sqrt{p_1^2 + aq_1^{M_1} - ax^{M_1}}}, \quad \omega_2 = -\int^{q_2} \frac{dx}{\sqrt{p_2^2 + bq_2^{M_2} - bx^{M_2}}}, \]

with canonical Poisson brackets

\[ \{\omega_j, I_k\} = \delta_{jk}, \quad \{I_j, I_k\} = \{\omega_j, \omega_k\} = 0, \quad j, k = 1, 2, \]

and equations of motion

\[ \dot{I}_{1,2} = 0, \quad \dot{\omega}_{1,2} = \frac{\partial H}{\partial I_{1,2}} = 1, \quad \text{with } H = I_1 + I_2. \]

For the completely integrable system the Liouville-Arnold theorem implies that almost all points of the phase space are covered by a system of open toroidal domains with the action-angle coordinates $I_1, \ldots, I_n; \omega_1, \ldots, \omega_n$. In these coordinates the completely integrable system has the form

\[ \dot{I}_k = 0, \quad \dot{\omega}_k = \frac{\partial H}{\partial I_k}, \quad k = 1, \ldots, n, \]

and symplectic structure is canonical $\Omega = \sum dI_k \wedge d\omega_k$. The variables $I_{1,2}$ and $\omega_{1,2}$ satisfy standard equations of motion and have canonical Poisson structure $P = \Omega^{-1}$. So, we will call them the formal action-angle variables which are well-defined functions on the original Cartesian variables only in some part of the cotangent bundle to plane.

By definition Hamiltonian $H$ is in the involution with action variables $I_{1,2}$ and with any function on the difference of the angle variables

\[ X = F(I_1, I_2, \omega_1 - \omega_2), \]

see discussion in [19, 20, 21, 22]. Below we prove that $X$ is the polynomial in momenta $p_{1,2}$ if $M_{1,2}$ belong to (1.2) or (1.3) because in this case $\omega_1, \omega_2$ are given by elementary functions. More general case when some function on difference $\omega_1 - \omega_2$ are elementary functions on original variables we do not consider here, see discussion and examples in [7, 19, 20, 21, 22].

Let us recall that expressions of the form

\[ x^m(\alpha + \beta x^n)^p dx, \]

where $\alpha, \beta$ are arbitrary coefficients and $m, n, p$ are rational numbers, are called differential binomials. According to the Chebyshev theorem integrals on differential binomials

\[ \int x^m(\alpha + \beta x^n)^p dx, \]

can be evaluated in terms of elementary functions if and only if:

1. $p$ is an integer, then we expand $(\alpha + \beta x^n)^p$ by the binomial formula in order to rewrite the integrand as a rational function of simple radicals $x^{j/k}$. Then we make a substitution $x = t^r$, where $r$ is the largest of all denominators $k$, remove the radicals entirely and obtain integral on rational function.
2. \( \frac{m+1}{n} \) is an integer, then we set \( t = \alpha + \beta x^n \) to obtain integral
\[
\int x^m (\alpha + \beta x^n)^p \, dx = \frac{1}{2} \beta^{-\frac{m+1}{n}} \int t^p (t - \alpha)^{\frac{m+1}{n} - 1} \, dt
\]
which belongs to Case 1.

3. \( \frac{m+1}{n} + p \) is an integer, then we transform the integral by factoring out \( x^n \)
\[
\int x^m (\alpha + \beta x^n)^p \, dx = \int x^{m+n} (\alpha x^{-n} + \beta)^p \, dx.
\]
The result is a new integral of the differential binomial which belongs to Case 2.

In our case (2.5) we have
\[
\alpha = I_{1,2}, \quad \beta = 1, \quad m = 0, \quad n = M, \quad p = -1/2.
\]
Hence action variables \( \omega_1 \) and \( \omega_2 \) is expressed via elementary functions only if
\[
\frac{1}{M} \text{ is integer} \quad \text{or} \quad \frac{1}{M} - \frac{1}{2} \text{ is integer.}
\]

In order to avoid logarithmic term \( \ln(t) = \int t^{-1} \, dt \) in (2.5), which is also an elementary function, we have to consider only zero, positive and negative values of \( M \), respectively.

For \( M_k \) from (1.2) action variables (2.5) are
\[
M_k = 0, \quad \omega = \frac{2q_k}{p_k}, \quad M_k = \frac{1}{n_k} > 0, \quad \omega_k = \text{polynomial of order } 2n_k - 1.
\]

For \( M_k \) from (1.3) action variables (2.5) are
\[
M_k = 0, \quad \omega = \frac{2q_k}{p_k}, \quad M_k = -\frac{2}{2n - 1} < 0, \quad \omega = \text{polynomial of order } 2n_k - 1,
\]
where \( I_k, k = 1, 2, \) is the corresponding action variable. Let us show a few explicit formulae for positive exponents
\[
M_2 = 1, \quad \omega_2 = \frac{p_2}{b}, \quad M_2 = \frac{1}{3}, \quad \omega_2 = \frac{p_2 (3b q_2^{2/3} + 4b q_2^{1/3} q_2^{1/3} p_2^2 + 8/5 p_2^4)}{b^3},
\]
and negative exponents
\[
M_2 = \frac{2}{3}, \quad \omega_2 = \frac{p_2 (3b q_2^{1/3} + 2q_2^{2/3})}{2 \left( p_2^2 + b q_2^{2/3} \right)^{1/2}}, \quad M_2 = -\frac{2}{5}, \quad \omega_2 = -\frac{p_2 (5b q_2^{1/5} + 10/3b q_2^{3/5} p_2^2 + q_2 p_2^4)}{2 \left( p_2^2 + b q_2^{2/5} \right)^{3/2}}.
\]

Other partial or generic expressions for integrals may be found in textbooks, tables of integrals or any computer algebra system.

**Proposition 1** A Hamiltonian system defined by \( H \) (1.1) has a polynomial first integral \( X_N \) of order \( N \), if \( M_1 \) and \( M_2 \) belong to (1.2) or (1.3):

1. if \( M_1 = 1/n_1 \) and \( M_2 = 1/n_2 \), then
\[
X_{2n-1} = \omega_1 - \omega_2, \quad \text{where} \quad n = \max(n_1, n_2);
\]
2. if \( M_1 = -2/(2n_1 - 1) \) and \( M_2 = -2/(2n_2 - 1) \), then
\[
X_{2n-1} = (\omega_1 - \omega_2) I_1^{n_1} I_2^{n_2}, \quad \text{where} \quad n = n_1 + n_2;
\]

3. **Proposition 1** A Hamiltonian system defined by \( H \) (1.1) has a polynomial first integral \( X_N \) of order \( N \), if \( M_1 \) and \( M_2 \) belong to (1.2) or (1.3):

1. if \( M_1 = 1/n_1 \) and \( M_2 = 1/n_2 \), then
\[
X_{2n-1} = \omega_1 - \omega_2, \quad \text{where} \quad n = \max(n_1, n_2);
\]
2. if \( M_1 = -2/(2n_1 - 1) \) and \( M_2 = -2/(2n_2 - 1) \), then
\[
X_{2n-1} = (\omega_1 - \omega_2) I_1^{n_1} I_2^{n_2}, \quad \text{where} \quad n = n_1 + n_2;
\]
3. if \( M_1 = 1/n_1 \) and \( M_2 = -2/(2n_2 - 1) \), then
\[
X_{2n-1} = (\omega_1 - \omega_2)I_2^{n^2}, \quad \text{where} \quad n = n_1 + n_2;
\]

4. if \( M_1 = 0 \) and \( M_2 = 1/n \), then
\[
X_{2n} = p_1(\omega_1 - \omega_2), \quad \text{where} \quad p_1 = \sqrt{I_1};
\]

5. if \( M_1 = 0 \) and \( M_2 = -2/(2n - 1) \), then
\[
X_{2n} = p_1(\omega_1 - \omega_2)I_2^n, \quad \text{where} \quad p_1 = \sqrt{I_1}.
\]

This integral of motion \( X_N \) is functionally independent from \( I_{1,2} \) \((\exists,?)\).

Cases 1 and 5 were studied in \([10]\) and \([18]\), respectively.

Let us show some “compact” examples of polynomial integrals \( X_N \) with \( N = 8 \):
\[
V = a q_2^{1/4}, \quad X_8 = p_1 \left[ p_2^7 + \frac{7}{2} p_2^3 V + \frac{35}{8} p_2^3 V^2 + \frac{35}{16} p_2^3 V^3 \right] + \frac{35 a^4 q_1}{128},
\]
and for \( V = a q_2^{-2/7} \)
\[
X_8 = p_2^7 (p_1 q_2 - p_2 q_1) + p_2^3 \left( \frac{21}{5} q_2 p_1 - 4 q_1 p_2 \right) + V + p_2^3 (7 q_1 p_1 - 6 q_2 p_2) V^2 + p_2 (7 q_2 p_1 - 4 q_1 p_2) V^3 - q_1 V^4.
\]

Here we multiply the expressions from Proposition 1 by a constant in order to bring the principal part of these polynomials to standard form used in \([1,9,16]\).

Of course, any polynomial combinations of \( I_{1,2} \) and \( X_N \) are also integrals of motion. For instance, there are other integrals of motion that are functions on \((\omega_1 - \omega_2)^2\). It is interesting that for \( M_{1,2} \neq 0 \) they are polynomials in momenta of less degree \( N - 2 \). For negative \( M \) such integrals have the following form
\[
Y_{N-1} = \frac{X_N^2}{I_1^{n_1} I_2^{n_2}} + \frac{\alpha I_1^{n_1}}{I_1^{n_2}} + \frac{\beta I_2^{n_1}}{I_2^{n_2}} = (\omega_1 - \omega_2)^2 I_1^{n_1} I_2^{n_2} + \frac{\alpha I_1^{n_1}}{I_1^{n_2}} + \frac{\beta I_2^{n_1}}{I_2^{n_2}},
\]
where \( \alpha, \beta \) are polynomials in \( a, b \) and binomial coefficients. For instance, Hamiltonian
\[
H = p_1^2 + p_2^2 + \frac{a}{q_1^2} + \frac{b}{q_2^{2/3}}
\]
is in the involution with polynomial in momenta \( X_7 = (\omega_1 - \omega_2)I_1 I_2^3 \) of the seventh degree and with the following polynomial of the sixth degree
\[
Y_6 = 4(\omega_1 - \omega_2)^2 I_1 I_2^3 + \frac{4 a b^5 I_1}{9 I_2^3} + \frac{a I_1^3}{I_2^3}
\]
\[
= p_2^3 (p_1 q_2 - p_2 q_1)^2 + \frac{a}{q_1^2} \left[ \frac{q_1^2 p_1^3}{q_1^4 (q_1^2 - 27 p_1^2 q_2^2 q_1^2)} + \frac{11 b q_2^{6/5} p_2^2}{3} + \frac{64 b^5 q_2^{6/5}}{9} \right] + \frac{b q_2^{3/2}}{q_1^2}
\]
\[
+ \frac{b^5 (64 b^5 q_2^{6/5} - 90 q_2 q_1 p_1 (p_1 q_2 - p_2 q_1))}{9 q_2^2 q_1} + \frac{b^5 (64 b^5 q_2^{6/5} - 90 q_2 q_1 (p_1 q_2 - p_2 q_1))}{3 q_2^2 q_1}.
\]

More symmetric Hamiltonian
\[
H = p_1^2 + p_2^2 + \frac{a}{q_1^{2/3}} + \frac{b}{q_2^{2/3}}
\]
Proposition 2

Let us start with the following theorem from [15].

3 Nonseparable systems

For instance, let us take

\[ H = \frac{p_1^2}{I_1} + \frac{p_2^2}{I_2} + aq_1^{M_1} + aq_2^{M_2} \]

is integrable in the Liouville sense, then either

\[ Y_6 = 4I_1^2I_2^2(\omega_1 - \omega_2)^2 + \frac{4a^3I_2^2}{I_1^2} + \frac{4b^3I_1^2}{I_2^2} \]

or

\[ Y_6 = \frac{p_1^2p_2^2(p_1q_2 - p_2q_1)^2}{q_1^2} + 2(p_1q_2 - p_2q_2) \left( \frac{ap_2^2(q_2p_1 - 2q_1p_2)}{q_1^{2/3}} + \frac{bq_1^2(2q_2p_1 - q_1p_2)}{q_2^{2/3}} \right) \]

In a similar manner we can construct integrals of motion \( Y_{N-1} \) for positive exponents \( M \) and for composition of the positive and negative exponents.

3 Nonseparable systems

Let us start with the following theorem from [15].

Proposition 2 If a Hamiltonian system defined by

\[ H = \frac{p_1^2}{I_1} + \frac{p_2^2}{I_2} + aq_1^{M_1} + aq_2^{M_2} \]

is integrable in the Liouville sense, then either

\[ M_1 + M_2 = \frac{2}{2p + 1} \quad \text{or} \quad M_1 + M_2 = \frac{2(p + 1)}{p(p - 1)}, \quad p \in \mathbb{Z} \quad (3.7) \]

for a certain integer \( p \).

The conditions (3.7) are only necessary for the integrability. Only some of the potentials satisfying these conditions are integrable.

We can obtain the known list of these integrable systems considering deformations of the Thompson integrals of motion

\[ \tilde{H} = H + U(q_1, q_2), \quad \tilde{Z}_N = Z_N + \Delta Z_{N-2}, \quad Z_N = F(I_1, I_2, \omega_1 - \omega_2). \quad (3.8) \]

Here \( H \) is given by (1.1) at \( M_1 = 0 \), \( Z_N \) is some fixed polynomial in momenta of degree \( N \), whereas potential \( U(q_1, q_2) \) and polynomial \( \Delta Z_{N-2} \) of degree \( N - 2 \) have to be obtained by solving equation

\[ \{ \tilde{H}, \tilde{Z}_N \} = \sum_{i,j=1}^{n} \left( \frac{\partial \tilde{H}}{\partial q_i} \frac{\partial \tilde{Z}_N}{\partial p_j} - \frac{\partial \tilde{H}}{\partial p_j} \frac{\partial \tilde{Z}_N}{\partial q_i} \right) = 0. \]

For instance, let us take

\[ H = p_1^2 + p_2^2 + bq_2^{-2/3}, \quad I_1 = p_1^2, \quad I_2 = p_2^2 + bq_2^{-2/3}, \]

\[ X_4 = p_1^2(q_2p_1 - q_1q_2) + p_2 (3q_2p_1 - 2q_1p_2) bq_2^{-2/3} - b_2q_1q_2^{-4/3} \]

\[ Y_4 = p_2^2(q_2p_1 - q_1q_2)^2 + (q_2p_1 - q_1p_2)(2q_2p_1 - 4q_1p_2) bq_2^{-2/3} + b^2q_1^2q_2^{-4/3}, \]

where \( X_4 \) and \( Y_4 \) are integrals of motion considered in previous Section.

It is easy to find deformation of this Hamiltonian

\[ \tilde{H} = p_1^2 + p_2^2 + (aq_1 + b) q_2^{-2/3}. \]
which is in involution with two functionally independent integrals of motion

\[ \tilde{Z}_3 = \sqrt{I_1(3H - I_1)} + \Delta Z_2 = p_1(2p_1^2 + 3p_2^2) + \frac{3a(2p_1q_1 + 3p_2q_2)}{2q_2^{2/3}} + \frac{3bp_1}{q_2^{2/3}} \]

and

\[ \tilde{Z}_4 = I_1(2H - I_1) + \Delta Z_3 = p_1^2(p_1^2 + 2p_2^2) + \frac{2ap_1(p_1q_2 - 3p_2q_1)}{q_1^{2/3}} + \frac{2bp_2}{q_2^{2/3}} + \frac{9q_2^{2/3}}{2}. \]

Properties of this superintegrable system are discussed in [3, 17]. Similar deformation

\[ \tilde{H} = p_1^2 + p_2^2 + \left( aq_1^{-2/3} + b \right) q_2 \] (3.11)
is integrable with first integral

\[ \tilde{Z}_4 = I_1^4 + \Delta Z_3 = p_1^4 + \frac{2ap_1(p_1q_2 - 3p_2q_1)}{q_1^{2/3}} + \frac{9aq_2^{4/3}}{4} - \frac{a^2(9q_1^2 - 2q_2^2)}{2q_1^{4/3}}. \]

On the one hand, both Hamiltonians (3.10) and (3.11) can be obtained from the Hamiltonians of various Holt systems [3,11,12] using shift \( q_1 \to q_1 + \alpha \). On the other hand, all the Holt systems can be considered as deformations (3.8) of the Thompson superintegrable system (3.9), see [3].

Next integrable deformation (3.8) of the same system (3.9) can be obtained using sixth order polynomial in momenta

\[ Z_6 = I_1^2p_2^2(q_2q_2 - q_1p_2)^2 + \cdots \]

which now depends on the angle variables. In this case solving equation \( \{ \tilde{H}, \tilde{Z}_N \} = 0 \) one gets a new integrable deformation of the Fokas-Lagerstrom system [5].

**Proposition 3** Hamiltonian

\[ \tilde{H} = p_1^2 + p_2^2 + \left( aq_1^{-2/3} + b \right) q_2^{-2/3} \] (3.12)
is in involution with the following integral of motion

\[ \tilde{Z}_6 = Z_6 + \Delta Z_4 = \frac{(p_1p_2(2p_1^2 - 2q_2q_1))^2}{q_1^{2/3}q_2^{2/3}} - 2a \left( \frac{p_1p_2(2p_1^2 - 2q_2q_1)}{q_1^{2/3}q_2^{2/3}} - \frac{6p_1(p_1q_2^2 + 4p_1q_2^2 - 2p_2q_1^2)}{q_1^{2/3}q_2^{2/3}} \right) \]

\[ + \quad a^2 \left( \frac{(p_1q_2 - 2p_2q_1)^2}{q_1^{2/3}q_2^{2/3}} + \frac{4b}{q_1^{2/3}} \right) + \frac{2bp_1^2(q_2q_2 - p_2q_1)(2p_1q_2 - 2p_2q_1)}{q_2^{2/3}} + \frac{b^2q_1^2q_2^2}{q_2^{2/3}}. \]

which is polynomial in momenta of the sixth degree.

For \( b = 0 \) potential in \( \tilde{H} \) (3.12) coincides with the so-called Fokas-Lagerstrom potential [5,11]

\[ U = \frac{a}{(x^2 - y^2)^{2/3}}, \]

after a 45 degree rotation

\[ q_1 = x - y, \quad q_2 = x + y. \]

It is easy to directly prove that for other pairs of exponents \( (M_1, M_2) \) in (1.4), which satisfy conditions (3.7), the first integral has to be a degree more than five in the momenta.

**4 Conclusion**

In this note we have carried out a systematic study of superintegrable Hamiltonian systems separable in Cartesian coordinates using action-angle variables, which play a fundamental role in classical and quantum mechanics. It is enough to say that they are the key points in the Kolmogorov–Arnold–Moser theory, in the geometric and semi-classical quantization.
Previously in [19, 20, 21, 22], we have already constructed polynomial integrals of motion using addition theorems for the action-angle variables. For instance, by adding action variables

\[ I_1 = p_1^2 + m^2 q_1^2 + a q_1, \quad I_2 = p_2^2 + n^2 q_2^2 + \frac{b}{q_2}, \quad m, n, a, b \in \mathbb{R}, \]

one gets Hamiltonian

\[ H = I_1 + I_2 = p_1^2 + p_2^2 + m^2 q_1^2 + n^2 q_2^2 + a q_1 + \frac{b}{q_2}, \]

which is in involution with the following integral of motion

\[ X = F(I_1, I_2, \omega_2 - \omega_1), \quad \{H, X\} = 0, \]

which is functionally independent from \( I_{1,2} \). Here

\[ \omega_1 = -\frac{1}{m} \arctan \left( \frac{2m^2 q_1 + a}{2mp_1} \right), \quad \omega_2 = \frac{1}{4n} \arctan \left( \frac{p_2^2 - n^2 q_2^2 + b q_2^{-2}}{2nq_2 p_2} \right), \]

are the corresponding action variables. For integer \( m \) and half-integer \( n \) this integral could be polynomial in momenta

\[ X_{2n+m} = (-a^2 - 4m^2 I_1)^n (4n^2 b - I_2^2)^{m/2} e^{i m (\omega_1 - \omega_2)}, \]

which is obtained using an addition theorem for logarithmic (inverse trigonometric) functions.

In this note we use the simplest addition theorems for polynomials (rational functions). Because we know how to add polynomials in quantum variables we could try to study quantum counterpart of the Hamiltonian (1.1) using quantum analogs of the action-angle variables [6, 13]. The main problem here is that classical action-angle variables are defined only in some domain of the cotangent bundle of plane. In [1, 10, 16] authors de facto found quantum action-angle variables in the framework of the Bohr-Sommerfeld quantization in the Cartesian coordinates. It will be interesting to obtain these quantum action-angle variable in geometric or semi-classical quantization.

We are very grateful to the referees for thorough analysis of the manuscript, constructive suggestions and proposed corrections, which certainly lead to a more profound discussion of the results. The work was supported by the Russian Science Foundation (project 18-11-00032).

References

[1] Abouamal I., Winternitz P., Fifth-order superintegrable quantum system separating in Cartesian coordinates. Doubly exotic potentials, Journal of Mathematical Physics, (2018), v.59, 022104.

[2] Arnold V.I., Mathematical methods of classical mechanics, Berlin, Heidelberg, New York: Springer, 1978.

[3] Campoamor-Stursberg R., Cariñena J.F., Rañada M.F., Higher-order superintegrability of a Holt related potential, J. Phys. A: Math. Theor., (2014), v.46, n.43, 435202.

[4] Chebyshev P.L., Sur l’intégration des différentielles irrationnelles, J. Math. Pures Appl., (1853), v.18, pp. 87-111; Oeuvres vol. 1, pp. 147-168.

[5] Fokas S., Lagerström P.A., Quadratic and cubic invariants in classical mechanics, J. Math. Anal. Appl., (1980), v.74, pp. 325-341.

[6] Giachetta G., Mangiarotti L., Sardanashvily G., Geometric and algebraic topological methods in quantum mechanics, Singapore; Hackensack, N.J.: World Scientific, (2005).
[7] Gonera C., *On the superintegrability of TTW model*, Physics Letters A, (2012), v.376, pp.2341-2343.

[8] Grammaticos B., Dorizzi B., Ramani A., *Hamiltonians with high-order integrals and the "weak-Painlevé" concept*, J. Math. Phys., (1984), v.25, pp. 3470-3473.

[9] Gravel S., *Hamiltonians separable in cartesian coordinates and third-order integrals of motion*, J. Math. Phys., (2004), v. 45, pp. 1003–1019.

[10] Güngör, Kuru S., Negro J., Nieto L.M., *Heisenberg-type higher order symmetries of superintegrable systems separable in cartesian coordinates*, Nonlinearity, (2017), v.30, pp.1788–1808.

[11] Hietarinta J., *Direct methods for the search of the second invariant*, Phys.Rept., (1987) , v.147, n.2, pp. 87-154.

[12] Holt C.R., *Construction of new integrable Hamiltonians in two degrees of freedom*, J. Math. Phys., (1982), v.23, pp. 1037-1046.

[13] Lewis R.H., Lawrence W.E., Harris, J.D., *Quantum action-angle variables for the harmonic oscillator*, Physical Review Letters, (1996), v.77, n.26, pp.5157-5159.

[14] Maciejewski A.J., Przybylska M., Tsiganov A.V., *On algebraic construction of certain integrable and super-integrable systems*, Physica D, (2011), v. 240, p.1426-1448.

[15] Maciejewski A.J., Przybylska M., *Integrability of Hamiltonian systems with algebraic potentials*, Phys. Lett. A, (2016), v.380, n.1, pp.76-82.

[16] Marquette I., Sajedi M., Winternitz P., *Fourth order superintegrable systems separating in cartesian coordinates I. Exotic quantum potentials*, Journal of Physics A: Mathematical and Theoretical, (2017), v. 50, 315201.

[17] Post S., Winternitz P., *A nonseparable quantum superintegrable system in 2D real Euclidean space*, J. Phys. A: Math. Theor., (2011), v. 44, n. 16, 162001.

[18] Thompson G., *Polynomial constants of motion in flat space* J. Math. Phys., (1984), v.25, pp. 3474-3478.

[19] Tsiganov A.V., *On maximally superintegrable systems*, Reg. Chaot. Dyn., (2008), v.13, n.3, pp.178-190.

[20] Tsiganov A.V., *Addition theorems and the Drach superintegrable systems*, J. Phys. A: Math. Theor., (2008), v. 41, 335204.

[21] Tsiganov A.V., *Leonard Euler: addition theorems and superintegrable systems*, Reg. Chaot. Dyn., (2009), v.14, n.3, pp.389-406.

[22] Tsiganov A.V., *Superintegrable Stäckel systems on the plane: elliptic and parabolic coordinates*, SIGMA, (2012), v.8, 031.