VANISHING GEODESIC DISTANCE FOR THE RIEMANNIAN METRIC WITH GEODESIC EQUATION THE KDV-EQUATION

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Abstract. The Virasoro-Bott group endowed with the right-invariant $L^2$-metric (which is a weak Riemannian metric) has the KdV-equation as geodesic equation. We prove that this metric space has vanishing geodesic distance.

1. Introduction

It was found in [11] that a curve in the Virasoro-Bott group is a geodesic for the right invariant $L^2$-metric if and only if its right logarithmic derivative is a solution of the Korteweg-de Vries equation, see [12]. Vanishing geodesic distance for weak Riemannian metrics on infinite dimensional manifolds was first noticed on shape space Imm$(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$ for the $L^2$-metric in [7, 3.10]. In [8] this result was shown to hold for the general shape space Imm$(M, N)/\text{Diff}(M)$ for any compact manifold $M$ and Riemannian manifold $N$, and also for the right invariant $L^2$-metric on each full diffeomorphism group with compact support Diff$_c(N)$. In particular, Burgers’ equation is related to the geodesic equation of the right invariant $L^2$-metric on Diff$(S^1)$ or Diff$_c(\mathbb{R})$ and it thus also has vanishing geodesic distance. We even have

Result. [8] The weak Riemannian $L^2$-metric on each connected component of the total space Imm$(M, N)$ for a compact manifold $M$ and a Riemannian manifold $(N, g)$ has vanishing geodesic distance.

This result is not spelled out in [8] but it follows from there: Given two immersions $f_0, f_1$ in the same connected component, we first connect their shapes $f_0(M)$ and $f_1(M)$ by a curve of length $< \varepsilon$ in the shape space Imm$(M, N)/\text{Diff}(M)$ and take the horizontal lift to get a curve of length $< \varepsilon$ from $f_0$ to an immersion $f_1 \circ \varphi$ in the connected component of the orbit through $f_1$. Now we use the induced metric $f_1^*g$ on $M$ and the right invariant $L^2$-metric induced on Diff$(M)$ to get a curve in Diff$(M)$ of length $< \varepsilon$ connecting $\varphi$ with $\text{Id}_M$. Evaluating at $f_1$ we get curve in Imm$(M, N)$ of length $< \varepsilon$ connecting $f_1 \circ \varphi$ with $f_1$.

In this article we show that the right invariant $L^2$-metric on the Virasoro-Bott groups (see [24]) has vanishing geodesic distance. This might be related to the fact that the Riemannian exponential mapping is not a diffeomorphism near 0, see [2] for Diff$(S^1)$ and [3] for the Virasoro group over $S^1$. See [10] for information on conjugate points along geodesics.

Date: January 11, 2013.

2000 Mathematics Subject Classification. Primary 35Q53, 58B20, 58D05, 58D15, 58E12.

Key words and phrases. diffeomorphism group, Virasoro group, geodesic distance.

All authors were supported by ‘Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 21030’.
2. The Virasoro-Bott groups

2.1. The Virasoro-Bott groups. Let $\text{Diff}_S(\mathbb{R})$ be the group of diffeomorphisms of $\mathbb{R}$ which rapidly fall to the identity. This is a regular Lie group, see [6, 6.4]. The mapping

$$c : \text{Diff}_S(\mathbb{R}) \times \text{Diff}_S(\mathbb{R}) \to \mathbb{R}$$

$$c(\varphi, \psi) := \frac{1}{2} \int \log(\varphi \circ \psi) d \log \psi' = \frac{1}{2} \int \log(\varphi' \circ \psi) d \log \psi'$$

satisfies $c(\varphi, \varphi^{-1}) = 0$, $c(\text{Id}, \psi) = 0$, $c(\varphi, \text{Id}) = 0$ and is a smooth group cocycle, called the Bott cocycle:

$$c(\varphi_2, \varphi_3) - c(\varphi_1 \circ \varphi_2, \varphi_3) + c(\varphi_1, \varphi_2 \circ \varphi_3) - c(\varphi_1, \varphi_2) = 0.$$

The corresponding central extension group $\text{Vir} := \mathbb{R} \times_c \text{Diff}_S(\mathbb{R})$, called the Virasoro-Bott group, is a trivial $\mathbb{R}$-bundle $\mathbb{R} \times \text{Diff}_S(\mathbb{R})$ that becomes a regular Lie group relative to the operations

$$\left( \begin{array}{c} \varphi \\ \psi \end{array} \right) \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = \left( \begin{array}{c} \varphi \circ \psi \\ \alpha + \beta + c(\varphi, \psi) \end{array} \right),$$

$$\left( \begin{array}{c} \varphi \\ \psi \end{array} \right)^{-1} = \left( \begin{array}{c} \varphi^{-1} \\ -\alpha \end{array} \right) \quad \varphi, \psi \in \text{Diff}_S(\mathbb{R}), \alpha, \beta \in \mathbb{R}.$$

Other versions of the Virasoro-Bott group are the following: $\mathbb{R} \times_c \text{Diff}_+(\mathbb{S}^1)$, $\mathbb{R} \times_c \text{Diff}_c(\mathbb{R})$. One can also apply the homomorphism $\exp(\alpha)$ to the center and replace it by $\mathbb{S}^1$. To be specific, we shall treat the most difficult case $\text{Diff}_S(\mathbb{R})$ in this article. All other cases require only obvious minor changes in the proofs.

2.2. The Virasoro Lie algebra. The Lie algebra of the Virasoro-Bott group $\mathbb{R} \times_c \text{Diff}_S(\mathbb{R})$ is $\mathbb{R} \times \mathcal{X}_S(\mathbb{R})$ (where $\mathcal{X}_S(\mathbb{R}) = S(\mathbb{R})\partial_x$) with the Lie bracket

$$\left[ \left( \begin{array}{c} X \\ a \end{array} \right), \left( \begin{array}{c} Y \\ b \end{array} \right) \right] = \left( \begin{array}{c} -[X, Y] \\ \omega(X, Y) \end{array} \right) = \left( \begin{array}{c} X'Y - XY' \\ \omega(X, Y) \end{array} \right)$$

where

$$\omega(X, Y) = \omega(X)Y = \int X'dY' = \int X'Y''dx = \frac{1}{2} \int \det \left( \begin{array}{cc} X' & Y' \\ X'' & Y'' \end{array} \right) dx,$$

is the Gelfand-Fuks Lie algebra cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, which is a bounded skew-symmetric bilinear mapping satisfying the cocycle condition

$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0.$$

It is a generator of the 1-dimensional bounded Chevalley cohomology $H^2(\mathfrak{g}, \mathbb{R})$ for any of the Lie algebras $\mathfrak{g} = \mathcal{X}(\mathbb{R})$, $\mathcal{X}_c(\mathbb{R})$, or $\mathcal{X}_S(\mathbb{R}) = S(\mathbb{R})\partial_x$. The Lie algebra of the Virasoro-Bott Lie group is thus the central extension $\mathbb{R} \times_\omega \mathcal{X}_S(\mathbb{R})$ induced by this cocycle. We have $H^2(\mathcal{X}_c(\mathbb{M}), \mathbb{R}) = 0$ for each finite dimensional manifold of dimension $\geq 2$ (see [4]), which blocks the way to find a higher dimensional analog of the Korteweg-de Vries equation in a way similar to that sketched below.

To complete the description, we add the adjoint action:

$$\text{Ad} \left( \begin{array}{c} \varphi \\ \psi \end{array} \right) \left( \begin{array}{c} X \\ Y \\ a \end{array} \right) = \left( \begin{array}{c} \text{Ad}(\varphi)Y = \varphi \circ Y \circ \varphi^{-1} \\ b + \int S(\varphi)Y dx \end{array} \right)$$

where the Schwartzian derivative $S$ is given by

$$S(\varphi) = \left( \frac{\varphi'''}{\varphi''} \right)' - \frac{1}{2} \left( \frac{\varphi'''}{\varphi''} \right)^2 = \frac{\varphi'''}{\varphi''} - \frac{3}{2} \left( \frac{\varphi''}{\varphi'} \right)^2 = \log(\varphi')' - \frac{1}{2}(\log(\varphi'))^2$$
which measures the deviation of $\varphi$ from being a Möbius transformation:

$$S(\varphi) = 0 \iff \varphi(x) = \frac{ax + b}{cx + d} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

The Schwartzian derivative of a composition and an inverse follow from the action of the de Vries equation with its natural companions adjoint action: Virasoro algebra (via the weak Riemannian metric). We need the transpose of the $\varphi$ which measures the deviation of $\varphi$ from being a Möbius transformation:

$$S(\varphi \circ \psi) = (S(\varphi) \circ \psi)(\psi')^2 + S(\psi), \quad S(\varphi^{-1}) = -\frac{S(\varphi)}{(\varphi')^2} \circ \varphi^{-1}$$

2.3. The right invariant $L^2$-metric and the KdV-equation. We shall use the $L^2$-inner product on $\mathbb{R} \times \mathbb{R}$:

$$\left\langle \begin{pmatrix} X \\ a \end{pmatrix}, \begin{pmatrix} Y \\ b \end{pmatrix} \right\rangle := \int XY \, dx + ab.$$  

We use the induced right invariant weak Riemannian metric on the Virasoro group. According to [1], see [9] for a proof in the notation and setup used here, a curve $t \mapsto (\varphi_t, \alpha_t)$ in the Virasoro-Bott group is a geodesic if and only if

$$\begin{pmatrix} u_t \\ a_t \end{pmatrix} = -\text{ad} \begin{pmatrix} u \\ a \end{pmatrix}^T \begin{pmatrix} u \\ a \end{pmatrix} = \begin{pmatrix} -3u_x u - au_{xxx} \\ 0 \end{pmatrix} \quad \text{where}$$

$$\begin{pmatrix} u(t) \\ a(t) \end{pmatrix} = \partial_t \begin{pmatrix} \varphi(s) \\ \alpha(s) \end{pmatrix} \bigg|_{s=t} = \partial_t \left( \begin{pmatrix} \varphi(s) \circ \varphi^{-1} \circ \varphi(t) \end{pmatrix} \bigg|_{s=t} \right),$$

$$\begin{pmatrix} u \\ a \end{pmatrix} = \left( \alpha_t + \frac{\varphi_t \circ \varphi^{-1}}{2\varphi_x^2} \right) dx.$$  

since we have

$$2\partial_t c(\varphi(s), \varphi(t) \circ \varphi^{-1})|_{s=t} = \partial_t \int \log((\varphi(s) \circ \varphi(t) \circ \varphi^{-1}) d \log((\varphi(t) \circ \varphi^{-1})^t) dx$$

$$= \int \frac{\varphi_t(t) \circ \varphi^{-1}(t) - \varphi_t(t) \circ \varphi^{-1}(t)}{\varphi(t) \circ \varphi^{-1}(t)} \left( \varphi(t) \circ \varphi^{-1}(t) \right)^t dx$$

$$= -\int \frac{\varphi_t(t)^2}{\varphi(t)^2} dx = -\int \frac{\varphi_{xx} \varphi^2}{\varphi_x^2} (t) dx.$$  

Thus $a$ is a constant in time and the geodesic equation is hence the Korteweg-de Vries equation

$$u_t + 3u_x u + au_{xxx} = 0$$

with its natural companions

$$\varphi_t = u \circ \varphi, \quad \alpha_t = a + \int \frac{\varphi_{xx} \varphi^2}{2\varphi_x^2} dx.$$  

To be complete, we add the invariant momentum mapping $J$ with values in the Virasoro algebra (via the weak Riemannian metric). We need the transpose of the adjoint action:

$$\left( \text{Ad} \begin{pmatrix} \varphi^T_x \\ c \end{pmatrix}, \begin{pmatrix} Y \\ b \end{pmatrix}, \begin{pmatrix} Z \\ c \end{pmatrix} \right) = \left( \begin{pmatrix} Y \\ b \end{pmatrix}, \text{Ad} \begin{pmatrix} \varphi_x \\ c + S(\varphi) \end{pmatrix} \right)$$

$$= \left( \begin{pmatrix} Y \\ b \end{pmatrix}, \begin{pmatrix} \varphi x Z \\ c \end{pmatrix} + \left( c + \int S(\varphi) \, dx \right) \right)$$
= \int Y((\phi' \circ \phi^{-1})(Z \circ \phi^{-1}) \, dx + bc + \int bS(\phi)Z \, dx
\]

Thus, the invariant momentum mapping is given by

\[
J\left(\left(\phi_\alpha, \left(\begin{array}{c} Y \\ b \end{array}\right)\right)\right) = \text{Ad} \left(\phi_\alpha^T \right) \left(\begin{array}{c} Y \\ b \end{array}\right) = \left(\begin{array}{c} (Y \circ \phi)(\phi')^2 + bS(\phi) \\ b \end{array}\right).
\]

Along a geodesic \( t \mapsto g(t, \cdot) = (\phi(t), a(t)) \), the momentum

\[
J\left(\left(\phi_\alpha, \left(\begin{array}{c} u = \phi_t \circ \phi^{-1} \\ a \end{array}\right)\right)\right) = \left(\begin{array}{c} (u \circ \phi)^2_2 + aS(\phi) \\ a \end{array}\right) = \left(\begin{array}{c} \phi_t \phi^2_2 + aS(\phi) \\ a \end{array}\right)
\]
is constant in \( t \).

2.4. Lifting curves to the Virasoro-Bott group. We consider the extension

\( \mathbb{R} \to \mathbb{R} \times_{c} \text{Diff}_S(\mathbb{R}) \to \text{Diff}_S(\mathbb{R}) \).

Then \( p \) is a Riemannian submersion for the right invariant \( L^2 \)-metric on \( \text{Diff}_S(\mathbb{R}) \), i.e., \( Tp \) is an isometry on the orthogonal complements of the fibers. These complements are not integrable; in fact, the curvature of the corresponding principal connection is given by the Gelfand-Fuks cocycle. For any curve \( \phi(t) \) in \( \text{Diff}_S(\mathbb{R}) \) its horizontal lift is given by

\[
\left( a(t) = a(0) - \int_0^t \frac{\phi_x \phi'_{xx}}{\phi^2_x} \, dx \, dt \right)
\]
since the right translation to \((\text{Id}, 0)\) of its velocity should have zero vertical component, see [8]. The horizontal lift has the same length and energy as \( \phi \).

3. Vanishing of the geodesic distance

3.1. Theorem. **On all Virasoro-Bott groups mentioned in 2.1 geodesic distance for the right invariant \( L^2 \)-metric vanishes.**

The rest of this section is devoted to the proof of theorem 3.1 for the most difficult case \( \mathbb{R} \times_{c} \text{Diff}_S(\mathbb{R}) \).

3.2. Proposition. **Any two diffeomorphisms in \( \text{Diff}_S(\mathbb{R}) \) can be connected by a path with arbitrarily short length for the right invariant \( L^2 \)-metric.**

In [8] for \( \text{Diff}_c(\mathbb{R}) \) it was first shown that there exists one non-trivial diffeomorphism which can be connected to \( \text{Id} \) with arbitrarily small length. Then, it was shown that the diffeomorphisms with this property form a normal subgroup. Since \( \text{Diff}_c(\mathbb{R}) \) is a simple group this concluded the proof. But \( \text{Diff}_S(\mathbb{R}) \) is not a simple group since \( \text{Diff}_c(\mathbb{R}) \) is a normal subgroup. So, we have to elaborate on the proof of [8] as follows.

**Proof.** We show that any rapidly decreasing diffeomorphism can be connected to the identity by an arbitrarily short path. We will write this diffeomorphism as \( \text{Id} + g \), where \( g \in \mathcal{S}(\mathbb{R}) \) is a rapidly decreasing function with \( g' > -1 \). For \( \lambda = 1 - \varepsilon < 1 \) we define

\[
\phi(t, x) = x + \max(0, \min(t - \lambda x, g(x))) - \max(0, \min(t + \lambda x, -g(x))).
\]
This is a (non-smooth) path defined for \( t \in (-\infty, \infty) \) connecting the identity in Diff_{S}(\mathbb{R}) with the diffeomorphism \((\text{Id} + g)\). We define \( \psi(t, x) = \varphi(\tan(t), x) \ast G_\varepsilon(t, x) \), where \( G_\varepsilon(t, x) = \frac{1}{\pi} G_1\left(\frac{t}{\varepsilon}, x\right) \) is a smoothing kernel with \( \text{supp}(G_\varepsilon) \subseteq B_2(0) \) and \( \int G_\varepsilon \, dx \, dt = 1 \). Thus \( \psi \) is a smooth path defined on the finite interval \(-\frac{\pi}{2} < t < \frac{\pi}{2}\) connecting the identity in Diff_{S}(\mathbb{R}) with a diffeomorphism arbitrarily close to \((\text{Id} + g)\) for \( \varepsilon \) small. (Compare figure [1](#) for an illustration.)

The \( L^2 \)-energy of \( \psi \) is

\[
E(\psi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} (\psi_t \circ \psi^{-1})^2 \, dx \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \psi_t^2 \psi_x \, dx \, dt
\]

where \( \psi^{-1}(t, x) \) stands for \( \psi(t, \cdot)^{-1}(x) \). We have

\[
\partial_n \max(0, \min(a, b)) = I_{0 \leq a \leq b}, \quad \partial_n \max(0, \min(a, b)) = I_{0 \leq b \leq a},
\]

and therefore

\[
\psi_x(t, x) = \varphi_x(\tan(t), x) \ast G_\varepsilon
\]

\[
= (1 - \lambda I_{0 \leq \tan(t) - \lambda x \leq g(x)} + g'(x) I_{0 \leq g(x) \leq \tan(t) - \lambda x}
- \lambda I_{0 \leq \tan(t) + \lambda x \leq -g(x)} + g'(x) I_{0 \leq -g(x) \leq \tan(t) + \lambda x}) \ast G_\varepsilon,
\]

\[
= (1 + \tan(t)^2) \varphi_t(\tan(t), x) \ast G_\varepsilon
\]

\[
= (1 + \tan(t)^2)(I_{0 \leq \tan(t) - \lambda x \leq g(x)} - I_{0 \leq \tan(t) + \lambda x \leq -g(x)}) \ast G_\varepsilon.
\]

Note that these functions have disjoint support when \( \varepsilon = 0, \lambda = 1 - \varepsilon = 1 \).

**Claim.** The mappings \( \varepsilon \mapsto \psi_t \) and \( \varepsilon \mapsto (\psi_x - 1) \) are continuous into each \( L^p \) with \( p \) even. (The proofs are simpler when \( p \) is even because there are no absolute values to be taken care of.) To prove the claim, we calculate

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} (1 + \tan(t)^2) \varphi_t(\tan(t), x)^p \, dx \, dt = \int_{\mathbb{R}^2} \varphi_t(t, x)^p (1 + t^2)^{p-1} \, dx \, dt
\]

\[
= \int_{\mathbb{R}^2} (I_{0 \leq -\lambda x \leq g(x)} + I_{0 \leq t + \lambda x \leq -g(x)}) (1 + t^2)^{p-1} \, dx \, dt
\]

\[
= \int_{g(x) \geq 0} \int_{-\lambda x}^{\lambda x + g(x)} (1 + t^2)^{p-1} \, dt \, dx + \int_{g(x) < 0} \int_{-\lambda x}^{\lambda x} (1 + t^2)^{p-1} \, dt \, dx
\]

\[
= \int_{\mathbb{R}} \left| F(t) \right|_{t = \lambda x + g(x)} \right|_{t = -\lambda x} \, dx = \int_{\mathbb{R}} |F(\lambda x + g(x)) - F(\lambda x)| \, dx,
\]

where \( F(\lambda x + g(x)) - F(\lambda x) \) is a polynomial without constant term in \( g(x) \) with coefficients also powers of \( \lambda \). Integrals of the form \( \int_{\mathbb{R}} |(\lambda x)^k g(x)|^k \, dx \) with \( k_1 \geq 0, k_2 > 0 \) are finite and continuous in \( \lambda = 1 - \varepsilon \) since \( g \) is rapidly decreasing. This shows that \( \|(1 + \tan(t)^2) \varphi_t(\tan(t), x)\|_p \) depends continuously on \( \varepsilon \). Furthermore the sequence \( (1 + \tan(t)^2) \varphi_t(\tan(t), x) \) converges almost everywhere for \( \varepsilon \to 0 \), thus it also converges in measure. By the theorem of Vitali, this implies convergence in \( L^p \), see for example [12], theorem 16.6]. Convolution with \( G_\varepsilon \) acts as approximate unit in each \( L^p \), which proves the claim for \( \psi_t \). For \( \psi_x - 1 \) it follows similarly.

The above claim implies that

\[
E(\psi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \psi_t^2 \psi_x \, dx \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \psi_t^2 (\psi_x - 1) \, dx \, dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \psi_t^2 \, dx \, dt
\]
viewed as a mapping on $L^4 \times L^4 \times L^2$ (first summand) and on $L^2 \times L^2$ (second summand) is continuous in $\varepsilon$. It also vanishes at $\varepsilon = 0$ since then $\psi_x$ and $\psi_t$ have disjoint support. The Cauchy-Schwarz inequality $L(\psi)^2 < \pi E(\psi)$ implies that $L(\psi)$ goes to zero as well. Ultimately, $\psi(\varpi) = (\text{Id} + g) \star G_\varepsilon$ is arbitrarily close to $\text{Id} + g$. \hfill \Box

3.3. Lemma. For any $a \in \mathbb{R}$ there exists an arbitrarily short path connecting $(\text{Id})_0$ and $(\text{Id})_a$, i.e., $\text{dist}_{\text{Vir}}^2 \left( (\text{Id})_0, (\text{Id})_a \right) = 0$.

Proof. The aim of the following argument is to construct a family of paths in the diffeomorphism group, parametrized by $\varepsilon$, with the following properties: all paths in the family start and end at the identity and their length in the diffeomorphism group with respect to the $L^2$ metric tends to 0 as $\varepsilon \to 0$. By letting $\varepsilon$ be time-dependent, we are able to control the endpoint $a(T)$ of the horizontal lift for certain diffeomorphisms.

We consider the function

$$f(z, a, \varepsilon) = \max(0, \min(z, a)) \star G_\varepsilon(z)G_\varepsilon(a)$$

$$= \int \int \max(0, \min(z - \varpi, a - \varpi))G_\varepsilon(\varpi)G_\varepsilon(\varpi) \, d\varpi \, d\varpi$$

$$= \int \int \max(0, \min(z - \varepsilon\varpi, a - \varepsilon\varpi))G_{\varepsilon}(\varpi)G_{\varepsilon}(\varpi) \, d\varpi \, d\varpi$$

$$= \varepsilon f\left(\frac{z}{\varepsilon}, \frac{a}{\varepsilon}, 1\right)$$

(1)

where $G_\varepsilon(z) = \frac{1}{\varepsilon}G_1(\frac{z}{\varepsilon})$ is a function with $\text{supp}(G_\varepsilon) \subseteq [-\varepsilon, \varepsilon]$ and $\int G_\varepsilon \, dx = 1$. Furthermore, let $g : \mathbb{R} \to [0, 1]$ be a function with compact support contained in $\mathbb{R}_{>0}$ and $g' > -1$, so that $x + g(x)$ is a diffeomorphism. For $0 < \lambda < 1$ and $t \in [0, T]$ let

$$\varphi(t, x) = x + f(t - \lambda x, g(x), \varepsilon(t))$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{path.png}
\caption{The path $\varphi(t,)$ defined in 3.3 connecting Id to Id+$g$, plotted at $t - \Delta < t < t + \Delta$. Between the dashed lines, $g \equiv 1$ is constant.}
\end{figure}
be the path going away from the identity (since \( \text{supp}(g) \subset \mathbb{R}_{>0} \), see also figure [1]). For given \( \varepsilon_0 > 0 \), let

\[
\psi(t, x) = x + f(T - t - \lambda x, g(x), \varepsilon_0)
\]

the path leading back again. The only difference to \([8]\) is that the parameter \( \varepsilon \) may vary along the path.

We shall need some derivatives of \( \varphi \) and \( f \):

\[
\varphi_t(t, x) = f_z(t - \lambda x, g(x), \varepsilon(t)) + \dot{\varepsilon}(t)f_z(t - \lambda x, g(x), \varepsilon(t))
\]

\[
\varphi_x(t, x) = 1 - \lambda f_z(t - \lambda x, g(x), \varepsilon(t)) + f_a(t - \lambda x, g(x), \varepsilon(t))g'(x)
\]

\[
f_z(z, a, \varepsilon) = \int_{-\infty}^{z} \int_{-\infty}^{a-w} G_\varepsilon(w)G_\varepsilon(w + b) \, dw \, db
\]

\[
f_a(z, a, \varepsilon) = \int_{-\infty}^{a} \int_{-\infty}^{\varepsilon} G_\varepsilon(w)G_\varepsilon(w + b) \, dw \, db
\]

\[
f_\varepsilon(z, a, \varepsilon) = \frac{1}{\varepsilon} \left( f(z, a, \varepsilon) - z f_z(z, a, \varepsilon) - a f_a(z, a, \varepsilon) \right)
\]

\[
f_{z\varepsilon}(z, a, \varepsilon) = G_\varepsilon(z) \int_{-\infty}^{a} G_\varepsilon(b) \, db - \int_{-\infty}^{z} G_\varepsilon(w)G_\varepsilon(w - (z - a)) \, dw
\]

**Claim 1.** The path \( \varphi \) followed by \( \psi \) still has arbitrarily small length for the \( L^2 \)-metric.

We are working with a fixed time interval \([0, 2T]\). Thus arbitrarily small length is equivalent to arbitrarily small energy. The energy is given by

\[
(2) \quad \iint \varphi_t^2 \varphi_x \, dx \, dt = \iint (f_z + \dot{\varepsilon} f_z)^2 (1 - \lambda f_z + f_a g') \, dx \, dt
\]

Looking at the formula for \( f_z \) we see that \( \varepsilon f_z \) is bounded on a domain with bounded \( a \). Thus \( \| \varepsilon f_z \|_\infty \to 0 \) can be achieved by choosing \( \varepsilon \), such that \( |\varepsilon| \leq C \varepsilon^{3/2} \). We will see later that this is possible. Inspecting \( \varphi_t(t, x) \) and looking at the formulas for \( f_z \) and \( f \) we see that for \( t - \lambda x < -\varepsilon(t) \) and for \( t - \lambda x - g(x) > 2\varepsilon(t) \) we have \( \varphi_t(t, x) = 0 \). Thus the domain of integration is contained in the compact set

\[
[0, T] \times \left[ \frac{T + \| g \|_\infty + 2\| \varepsilon \|_\infty}{\lambda}, \frac{T + \| \varepsilon \|_\infty}{\lambda} \right].
\]

Therefore, it is enough to show that the \( L^\infty \)-norm of the integrand in \((2)\) goes to zero as \( \| \varepsilon \|_\infty \) goes to zero. For all terms involving \( \dot{\varepsilon} f_z \) this is true by the above assumption since \( (1 - \lambda f_z + f_a g') \) and \( \varepsilon f_z \) are bounded. For the remaining parts \( f_z^2(1 - \lambda f_z) \) and \( f_z f_a g' \) we follow the argumentation of \([8]\). For \( t \) fixed and \( \lambda \) close to 1, the function \( 1 - \lambda f_z \), when restricted to the support of \( f_z \), is bigger than \( \varepsilon(t) \) only on an interval of length \( O(\varepsilon(t)) \). Hence we have

\[
\int_0^T \int_{\mathbb{R}} f_z^2(1 - \lambda f_z) \, dx \, dt \leq \| f_z \|_\infty^2 \int_0^T (1 - \lambda f_z) \, dx \, dt = O(\| \varepsilon \|_\infty).
\]

For the last part, we note that the support of \( f_z f_a \) is contained in the set \( |g(x) - (t - \lambda x)| \leq 2\varepsilon \). Now we define \( x_0 < x_1 \) by \( g(x_0) + \lambda x_0 = T - 2\| \varepsilon \|_\infty \) and \( g(x_1) + \lambda x_1 = T + 2\| \varepsilon \|_\infty \). Then

\[
\int_0^T \int_{\mathbb{R}} f_z^2 f_a g' \, dx \, dt \leq T \| f_z \|_\infty^2 \| f_a \|_\infty \int_{\text{supp}(f_z^2 f_a)} g' \, dx
\]
\[ = T(g(x_1) - g(x_0)) \leq 4T\|\varepsilon\|_\infty. \]

The estimate for \( \psi \) is similar and easier. This proves claim 1.

Claim 2. For every \( a \in \mathbb{R} \) and \( \delta > 0 \) we may choose \( \varepsilon(t) \) with \( \|\varepsilon\|_\infty < \delta \) such that

\[
\int_0^T \int_{\mathbb{R}} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2} \, dx \, dt + \int_0^T \int_{\mathbb{R}} \frac{\psi_{tx}\psi_{xx}}{\psi_x^2} \, dx \, dt = a.
\]

We will subject \( \varepsilon \) and \( g \) to several assumptions. First, we partition the interval \( [0, T] \) equidistantly into \( 0 < T_A < T_E < T \) and the \((t,x)\)-domain into two parts, namely \( A_1 = (0, T_A] \cup [T_E, T) \times \mathbb{R} \) and \( A_2 = [T_A, T_E] \times \mathbb{R} \). We want \( g(x) \equiv 1 \) on a neighborhood of the interval \( \frac{1}{\lambda}(T_A - 1, \frac{1}{\lambda}T_E) \). We choose \( \varepsilon(t) \) to be constant \( \varepsilon \equiv \varepsilon_0 \) on \([0, T_A] \cup [T_E, T]\) and to be symmetric in the sense, that \( \varepsilon(t) = \varepsilon(T - t) \). In addition, we want \( \varepsilon(t) \) small enough, such that \( g(x) \equiv 1 \) on \( \frac{1}{\lambda}(T_A - 1 - 2\varepsilon(t)), \frac{1}{\lambda}(T_E + \varepsilon(t)) \).

On \( A_1 \) we have \( \varepsilon(t) \equiv \varepsilon_0 \). This implies \( \psi_{tx}(t, x) = -\varphi_{tx}(T - t, x), \psi_x(t, x) = \varphi_x(T - t, x) \) and \( \psi_{xx}(t, x) = \varphi_{xx}(T - t, x) \). Hence

\[
\int \int_{A_1} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2} \, dx \, dt + \int \int_{A_1} \frac{\psi_{tx}\psi_{xx}}{\psi_x^2} \, dx \, dt = 0.
\]

Let \( A_2 = [T_A, T_E] \times \mathbb{R} \) be the region, where \( \varepsilon(t) \) is not constant. In the interior, where

\[
-\varepsilon(t) < t - \lambda x < g(x) + 2\varepsilon(t)
\]

we have by assumption \( g(x) \equiv 1 \). Therefore, one has in this region:

\[
\varphi_x(t, x) = -\lambda f_x(t - \lambda x, 1, \varepsilon(t)),
\]

\[
\varphi_{xx}(t, x) = \lambda^2 f_{xx}(t - \lambda x, 1, \varepsilon(t)),
\]

\[
\varphi_{tx}(t, x) = -\lambda f_{tx}(t - \lambda x, 1, \varepsilon(t)) - \lambda f_{xx}(t - \lambda x, 1, \varepsilon(t)) \varepsilon(t)
\]

We divide the integral over \( A_2 \) into two symmetric parts

\[
\int_{T_A}^{T/2} \int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t+\varepsilon(t))} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2} \, dx \, dt + \int_{T/2}^{T_E} \int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t+\varepsilon(t))} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2} \, dx \, dt
\]

and apply the following variable substitution to the second integral

\[
\tilde{t} = T - t, \quad \tilde{x} = x + \frac{1}{\lambda}(\tilde{t} - t).
\]

Thus \( \tilde{t} - \lambda \tilde{x} = t - \lambda x \). Together with \( \varepsilon(t) = \varepsilon(\tilde{t}) \) this implies

\[
\varphi_x(t, x) = \varphi_x(\tilde{t}, \tilde{x}), \quad \varphi_{xx}(t, x) = \varphi_{xx}(\tilde{t}, \tilde{x}).
\]

Since \( \varepsilon(t) = -\varepsilon(\tilde{t}) \) changes sign, the term containing \( \varepsilon(t) \) cancels out and leaves only

\[
\varphi_{tx}(t, x) + \varphi_{tx}(\tilde{t}, \tilde{x}) = -2\lambda f_{xx}(t - \lambda x, 1, \varepsilon(t)).
\]

A simple calculation shows that the integration limits transform

\[
\int_{T/2}^{T_E} \int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t+\varepsilon(t))} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2} \, dx \, dt = \int_{T/2}^{T/2} \int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t-1-2\varepsilon(t))} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2} \, d\tilde{x} \, d\tilde{t}
\]

to those of the first integral. Therefore, the sum of the integrals gives

\[
\int \int_{A_2} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2} \, dx \, dt = -2\lambda^3 \int_{T_A}^{T/2} \int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t+\varepsilon(t))} \frac{f_{xx}(t - \lambda x, 1, \varepsilon(t))^2}{(1 - \lambda f_{xx}(t - \lambda x, 1, \varepsilon(t)))^2} \, dx \, dt.
\]
From formula (1) we see:

\[ f_2(z, \alpha, \varepsilon) = f_2\left(\frac{z}{\varepsilon}, \frac{\alpha}{\varepsilon}, 1\right), \quad f_{zz}(z, \alpha, \varepsilon) = \frac{1}{\varepsilon} f_{zz}\left(\frac{z}{\varepsilon}, \frac{\alpha}{\varepsilon}, 1\right). \]

We can use this to rewrite the above integral:

\[
\int \int_{A_2} \frac{\varphi_{tx} \varphi_{xx}}{\varphi_x^2} \, dx \, dt = -2\lambda^2 \int_{T_A}^{T/2} f_{zz}(t, \lambda, \varepsilon(t)) \left(1 - \lambda f_z(t, \lambda, z, \varepsilon(t))\right)^2 \, dx \, dt
\]

Looking at the formula for \( f_{zz} \)

\[ f_{zz}(z, \frac{1}{\varepsilon}, 1) = G_1(z) - \int_{-\infty}^{z} G_1(w) G_1(w - (\frac{1}{\varepsilon})) \, dw \]

we see that \( f_{zz}(z, \frac{1}{\varepsilon}, 1) \) is non-zero only on the intervals \( |z| < 1 \) and \( |z - \frac{1}{\varepsilon}| < 2 \).

For small \( \varepsilon \), these are two disjoint regions. Therefore, the above integral equals

\[
\int \int_{A_2} \frac{\varphi_{tx} \varphi_{xx}}{\varphi_x^2} \, dx \, dt = -2\lambda^2 \int_{T_A}^{T/2} \frac{1}{\varepsilon(t)} \left( f_{zz}(z, \frac{1}{\varepsilon(t)}, 1) \right) \, dz \, dt
\]

For \( z \) bounded and sufficiently small \( \varepsilon(t) \), the functions under the integral do not depend on \( \varepsilon(t) \) any more as can be seen from the definitions of \( f_z \) and \( f_{zz} \). Thus

\[ I = \lambda^2 \int_{-1}^{1} \frac{f_{zz}(z, \frac{1}{\varepsilon(t)}, 1)}{\left(1 - \lambda f_z(z, \frac{1}{\varepsilon(t)}, 1)\right)^2} \, dz + \lambda^2 \int_{-2}^{2} \frac{f_{zz}(z, \frac{1}{\varepsilon(t)}, 1)}{\left(1 - \lambda f_z(z, \frac{1}{\varepsilon(t)}, 1)\right)^2} \, dz, \]

is independent of \( t \) and we have

\[ \int \int_{A_2} \frac{\varphi_{tx} \varphi_{xx}}{\varphi_x^2} \, dx \, dt = -I \int_{T_A}^{T} \frac{1}{\varepsilon(t)} \, dt. \]

The same calculations can be repeated for the return path \( \psi \), where \( \varepsilon = \varepsilon_0 \) is constant in time:

\[ \int \int_{A_2} \psi_{tx} \psi_{xx} \, dx \, dt = I \int_{T_A}^{T} \frac{1}{\varepsilon_0} \, dt. \]

Note that the sign is positive now, which comes from the \( t \)-derivative. Putting everything together gives us

\[ a = \int \int \frac{\varphi_{tx} \varphi_{xx}}{\varphi_x^2} + \psi_{tx} \psi_{xx} \, dx \, dt = I \int_{T_A}^{T} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon(t)}\right) \, dt \]
Let \( \varepsilon(t) = \varepsilon_0 + \varepsilon_1^{3/2}b(t) \) where \( b(t) \) is a bump function with height 1 and \( \varepsilon_1 \) is a small constant. Note that \( \varepsilon(t) \) satisfies \( |\varepsilon| \leq \|b\|_{\infty} \varepsilon_1^{3/2} \). Choosing \( \varepsilon_0 \) and \( \varepsilon_1 \) small independently we may produce any \( a \in \mathbb{R} \).

**Proof of Theorem 3.1.** Let \( (\varphi, a) \in \mathbb{R} \times \text{Diff}_S(\mathbb{R}) \). By proposition 3.2 we get a smooth family \( \varphi(\delta, t, x) \) for \( \delta > 0 \) and \( t \in [0, 1] \) such that \( \varphi(\delta, t, a) \in \text{Diff}_S(\mathbb{R}) \), \( \varphi(\delta, 0) = \text{Id}_S \), \( \varphi(\delta, 1, a) = \varphi \), and such that the length of \( t \mapsto \varphi(\delta, t, a) \) is \( < \delta \).

Using Proposition 2.4 consider the horizontal lift \( (\varphi(\delta, t, a), \dot{a}(\delta, t)) \in \text{Diff}_S(\mathbb{R}) \) of this family which connects \( (\text{Id}, 0) \) with \( (a(\delta, 1)) \) for each \( \delta > 0 \) and has length \( < \delta \). But one can see from the proof of lemma 3.3 that \( a(\delta, 1) \) becomes unbounded for \( \delta \to 0 \).

Using Lemma 3.3 we can find a horizontal path \( t \mapsto (\varphi(\delta, t, a(\delta, t))^{-1}) \) for \( t \in [0, 1] \) in the Virasoro group of length \( < \delta \) connecting \( (\text{Id}, 0) \) with \( (a(-a(\delta, 1))) \). Then the curve \( t \mapsto (\varphi(\delta, t, a(\delta, t)))^{-1} \cdot (\varphi(\delta, t, a(\delta, 1))) \cdot (\varphi(\delta, t, a(\delta, 1)))^{-1} \cdot (\varphi(\delta, t, a(\delta, 1)))^{-1} \) connects \( (\varphi(\delta, 1)) = (\text{Id}) \cdot (a(\delta, 1)) \) with \( (\varphi(\delta, 1)) = (\text{Id}) \cdot (a(-a(\delta, 1))) \) and it has length \( < \delta \).

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