Influence of a center of mass position on beam natural frequencies of vibration

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Abstract. The paper discusses the effect of beam center of mass displacements on the values of their first natural vibration frequencies. Three schemes of beam restraint with a displacement of the center of mass along all axes of a coordinate system and around them are considered. It is established that if the center of mass is in the center of the beam, then any of its small linear and angular displacements slightly affect the first natural frequencies of vibration. If the center of mass was near the support, its longitudinal displacements along the beam will significantly affect the first natural frequency of bending vibrations. The transverse displacements of the center of mass will change only the natural frequencies of the torsional vibrations of the beam, which are usually quite small. Small angular displacements of the beam center of mass do not change the first natural frequencies of vibration at any location relative to the beam.

1. Introduction
The theory of the beam vibration [1-10] models beams a design with a continuously distributed mass that is the correct approach and gives exact decisions. When calculating frame designs it is usually supposed that the center of mass of each beam is located in the middle of its length. It is right for beams of constant cross-section which have no additional elements along the length.
At the same time, many beam structures are made prefabricated, that is, consisting of several different elements. As a result of a random change in the relative arrangement of elements during assembly, the position of the center of mass of the entire beam changes. Displacement of the center of mass leads to a change in the stiffness characteristics of the structure and its dynamic properties, such as natural frequencies, vibration modes, effective modal masses, etc.

Simplifying the problem of vibration of a beam with a continuously distributed mass to the problem of vibration of its center of mass will certainly lead to a deterioration in the accuracy of the resulting solutions [11-15]. However, the purpose of this paper is only to obtain analytical constraints in a general form that will show the main consequences of shifting the center of mass of the beam.

In this paper, consider how the linear and angular displacements of the center of mass of a beam affect its first natural vibration frequencies. All displacements of the center of mass will be considered relative to its initial position on the beam. Using beam vibration theory will allow obtaining simple analytical equations that will explicitly help to establish the desired dependencies and identify the most dangerous displacement directions of the center of mass.
2. Beam restraints

The study examines several common methods of anchoring a beam, as shown in figure 2. The concentrated mass $m$ is the center of mass of the entire beam, including the connections of additional elements: flanges, couplings, etc. With this approach, the rest of the beam is weightless.

![Beam supports](image1)

Figure 1. Beam supports.

After the beam is assembled, the position of its center of mass can shift in any direction. To specify offsets of the position of the center of mass $m$ in the local coordinate system (figure 2).

![Beam center of mass offsets](image2)

Figure 2. Beam center of mass offsets.

In the modal analysis, it is usually necessary to determine the first 6 natural frequencies and forms of beam vibrations. In practice, however, bending vibrations in the plane of least stiffness are the most dangerous. In this case, the vibration frequency value will be the smallest of all and is a critical parameter. Torsional vibrations of the beam, which must be taken into account in the calculation, can also be dangerous. Thus, we will limit in this work the possible vibrations to only two types: bending and torsional.

3. Assessing the influence of center of mass offsets

Consider the effect of linear and angular offsets of the center of mass of the beam on its first frequency of transverse and torsional vibrations separately. The equations for evaluation of the first natural frequency of bending vibrations of the beam at hinged fixation (figure 2, a) is determined by the equation [16-21]:

$$
\omega_1 = \frac{1}{m(a \cdot b)^2 l} \frac{3EJ}{l},
$$

where $J_{min}$ is the minimum moment of inertia; $E$ is the Young module.

Excluding parameter $b$ from (1), we obtain an expression for determining the first natural frequency of bending vibrations in the direction of minimum stiffness:

$$
\omega_1 = \frac{1}{\frac{a(l-a)}{m}} \frac{3EJ}{l}.
$$

(2)
In the first approximation, the minimum value of the first natural frequency of the torsional vibrations of the beam can be estimated by dependence [21]:

\[ \omega_1 = \sqrt{C_\varphi / J_\varphi}. \]  

Equation (3) is obtained for the condition of cantilever support of the beam in question but has a similar appearance for other types of supports.

3.1. The \( \alpha \) - offset

After making the beam, let the actual position of the center of mass \( m \) (figure 2) be shifted in the direction of the axis \( \alpha \) by the value \( \delta_{\alpha m} \). Then the expression (2) takes the form:

\[ \omega_1(\delta_{\alpha m}) = \frac{1}{(a \pm \delta_{\alpha m})^2 - (a \pm \delta_{\alpha m})^2} \sqrt{\frac{3EJl}{m}} \]  

The analysis of the dependency (4) shows that it is quadratic, and the result of the calculation will be greatly influenced by the distance \( a \), which determines the position of the center of mass \( m \) relative to the supports (figure 1, a). For example, figure 3 shows the change in the first natural bending frequency depending on the deviation \( \delta_{\alpha m} \) for two options of mass arrangement \( m \): at \( a = l/2 \) (figure 1, a) and \( a = l/10 \) (figure 1, b).

![Figure 3. Plot \( \omega_1(\delta) \).](image)

In the first case \( a=l/2 \), the mass is displaced by \( \delta_{\alpha m} \) from the center of the section to one of the supports, which increases the rigidity of the structure. Since \( \delta_{\alpha m} \ll a \), this displacement leads to a slight increase in the value of the vibration frequency (figure 1, a).

The location of the center of mass in the middle of the supports \( (a=l/2) \) leads to the minimum values of the first natural frequency of beam vibration and the largest amplitudes of forced vibrations. In order to increase the rigidity of the structure, a support is installed near the center of mass \( m \), while the distance \( a \) becomes small and comparable to the installation error: \( \delta_{\alpha m} \sim a \). As a result, even small error values \( \delta_{\alpha m} \) lead to significant changes in the first natural frequency of beam vibrations, as shown in figure 1, b for the case \( a = l/10 \).

The displacement \( \delta_{\alpha m} \) of the center of mass will not affect the frequency of torsional vibrations, since it does not lead to a change in the terms of the corresponding expression (4).
3.2. The \( \beta \) - offset

The displacement of the center of mass in the transverse direction by the value \( \delta_{\beta m} \) (figure 2) will not cause a change in the value of the first natural frequency of bending vibrations of the beam (2), since the distances to the supports \( a \) and \( b \) during such an offset will remain unchanged, and local change of the moment of inertia at the place of setting mass \( m \) can be neglected, taking into account the accepted assumptions about the small and linear vibrations [4]. The transverse displacement will cause a change in the polar moment of inertia of the mass \( m \) and, therefore, the frequency of torsional vibrations of the beam. The polar moment of inertia \( J_p \) can be represented consisting of two parts:

\[
J_p = J_{cm} + J_m
\]  

(5)

Where \( J_{beam} \) is the polar moment of inertia of a beam; \( J_m \) is the moment of inertia of \( m \), which can be estimated approximately by the equation:

\[
J_m = m \cdot R_{equ}^2,
\]

(6)

where \( R_{equ} \) is the equivalent radius of structure with mass \( m \).

In case mass \( m \) is structurally a disk with radius \( R \), then \( R_{equ} = R \). For a beam having a non-symmetrical cross-section, it is common practice to replace the radius with an equivalent value, which is calculated according to various empirical formulas [4], the specific form of which in our case does not play a significant role.

Taking into account the displacement \( \delta_{\beta m} \) and expression (6), equation (5) takes the form:

\[
J_m(\delta_{\beta m}) = m \cdot (R_{equ} + \delta_{\beta m})^2.
\]

(7)

The minimum value of the first natural frequency taking into account (5) can be estimated by the equation:

\[
\omega_1 = \frac{GJ_{cm}}{\sqrt{(J_{cm} + J_m(\delta_{\beta m})) l}} = \frac{\sqrt{GJ_{cm}}}{\sqrt{(J_{cm} + m \cdot (R_{equ} + \delta_{\beta m})^2) l}},
\]

(8)

where \( G \) is the shear modulus.

In practice, the concentrated mass \( m \) is a flange or clutch connection, the inertial parameters of which are usually significantly higher than that of the beam. For example, for waveguides, the polar moments of inertia of the flange connection and the beam will be related as \( 12:2 \), that is, differ by more than an order.

Therefore, when examining the dependence (8) in the first approximation, it can be assumed:

\[
\omega_1 = \frac{GJ_{cm}}{\sqrt{J_m(\delta_{\beta m}) l}} = \frac{\sqrt{GJ_{cm}}}{\sqrt{m \cdot (R_{equ} + \delta_{\beta m})^2 l}} = \sqrt{\frac{C_p}{(R_{equ} + \delta_{\beta m}) l}}.
\]

(9)

The obtained dependence (9) determines the values of the first natural frequency of torsional oscillations from the error value \( \delta_{\beta m} \) of mass setting \( m \).

Similarly, it is possible to obtain a dependence of the value of the first natural frequency of the torsional oscillations on the value of the error \( \delta_{cm} \) in the transverse direction along the \( z \) coordinate, which will have the form (9).

3.3. The angular offsets

Studies on the influence of angular displacements (figure 2) of the center of mass showed that they cause minor changes in moments of inertia. In the first approximation, angular offsets can be neglected when estimating the natural frequencies of the beam.
4. Discussion
The obtained research results showed that bending vibrations, the frequency of which depends on the longitudinal offsets of the center of mass of the beam, are the most dangerous. This dependence is strongly manifested for short beams and with the initial location of the center of mass of the beam near the support. Other types of displacements have little effect on bending vibrations.

The torsional vibrations of the beam are sensitive to transverse displacements of the center of mass, but usually, the corresponding value of the frequency of these vibrations is not the smallest. Therefore, in each case, it is necessary to make an appropriate frequency calculation for all types of vibrations.

The study found that small angular offsets of the center of mass of the beam practically do not affect its dynamic behavior and frequency parameters. This statement is valid only for beam linear vibrations and requires clarification when any nonlinearities appear in the problem.

5. Conclusion
The paper studies the effect of beam center of mass displacements on its first natural vibration frequency. Each direction of center of mass offsets is considered and the most dangerous offset is revealed and analytical equations are obtained. The results can be used when designing beams, manufacturing, specifying the degree of accuracy of their geometry and element connections.

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