Gorenstein Dimensions
under Base Change

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Abstract. The so-called ‘change-of-ring’ results are well-known expressions which present several connections between projective, injective and flat dimensions over the various base rings. In this note we extend these results to the Gorenstein dimensions over Cohen-Macaulay local rings.

Introduction

In 1967 Auslander \cite{1} introduced the Gorenstein dimension of a finitely generated module over a commutative noetherian ring. This provides a characterization of Gorenstein local rings analogous to the well–known Auslander-Buchsbaum-Serre characterization of regular local rings. But since the Gorenstein dimension is only defined for finitely generated modules, the analogy is not complete. In the 1990s

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Enochs and Jenda in [10] introduced extensions of Auslander’s Gorenstein dimension, the so-called *Gorenstein projective* and *Gorenstein flat dimensions*, and the dual notion, the *Gorenstein injective dimension* and they get good results when the base ring is Gorenstein.

Using Foxby equivalence, a nice theory for Gorenstein projective, flat and injective dimensions over Cohen–Macaulay local rings was given in [13] and [15].

In this note we establish connections between Gorenstein projective, Gorenstein flat and Gorenstein injective dimensions of complexes over the various base rings. The note is based on the paper ” L.Khatami and S.Yassemi, *Gorenstein injective and Gorenstein flat dimensions under base change*, To appear in Comm. Algebra”. Section 5 and also statements about Gorenstein projective dimension have been added to the main paper later.

In section 1 we give the fundamental definitions and results of hyperhomological algebra. Details can be found in [14] and this is the main reference of section 1.

In section 2 the Auslander and Bass classes over a Cohen–Macaulay local ring with a dualizing complex are introduced. It is well-known (cf. [4]) that over such a ring a homologically bounded complex has finite Gorenstein projective/flat (res. Gorenstein injective) dimension if and only if it is in the Auslander (res. Bass) class. Then in this section we prove some change of ring results about Auslander and Bass classes which will be used in our main results. For example we prove that:

Let $R$ and $S$ be $Q$–algebras such that $R$ is a Cohen-Macaulay local ring with a dualizing complex. If $X$ and $Y$ are homologically bounded $(R,S)$- bicomplexes and $F$ and $I$ two $S$-complexes of finite flat and injective dimension, respectively, then the following hold:
(i) If $X$ belongs to the Auslander class of $R$ then $\mathbf{R}\text{Hom}_S(X, I)$ belongs to the Bass class of $R$ and $X \otimes_S^L F$ belongs to the Auslander class of $R$.

(ii) If $Y$ belongs to the Bass class of $R$ then $\mathbf{R}\text{Hom}_S(Y, I)$ belongs to the Auslander class of $R$ and $Y \otimes_S^L F$ belongs to the Bass class of $R$.

The main results of this note are proved in sections 3, 4 and 5.

In section 3 we prove the following result that is a generalization of [5; 6.4.13] and [17; (1.4)-(1.5)].

Let $R$ and $S$ be $Q$–algebras such that $R$ is a Cohen–Macaulay local ring with a dualizing module. If $X \in \mathcal{C}(R, S)$ ($X$ is a homologically bounded complexes of $(R, S)$–bimodules) and $Y \in \mathcal{C}(\square)(S)$, then

(i) $\text{Gpd}_R(X \otimes_S^L Y) \leq \text{pd}_SY + \text{Gpd}_RX$.

(ii) $\text{Gfd}_R(X \otimes_S^L Y) \leq \text{fd}_SY + \text{Gfd}_RX$.

(iii) $\text{Gid}_R(\mathbf{R}\text{Hom}_S(X, Y)) \leq \text{id}_SY + \text{Gfd}_RX$.

(iv) If $Y \in \mathcal{T}(S)$, then
$\text{Gfd}_R(\mathbf{R}\text{Hom}_S(X, Y)) \leq \text{Gid}_RX + \sup Y$

In section 4 we work on finite local ring homomorphisms of finite flat dimension. We prove the next result which can be viewed as a generalization of the classical results for flat and injective dimension.

Let $\varphi : (R, m) \to (S, n)$ be a finite local ring homomorphism of Cohen–Macaulay local rings with finite flat dimension, (that is $R$ and $S$ are both Cohen–Macaulay rings with unique maximal ideals $m$ and $n$ respectively such that $\varphi(m) \subseteq n$ and
where $S$ is a finite $R$–module with finite flat dimension over $R$.) Then the following hold for a homologically bounded $R$–complex $X$.

(i) $\text{Gpd}_S(S \otimes_R^L X) \leq \text{Gpd}_RX$.

(ii) $\text{Gfd}_S(S \otimes_R^L X) \leq \text{Gfd}_RX$.

(iii) $\text{Gid}_S(\text{RHom}_R(S, X)) \leq \text{Gid}_RX$.

In section 5 we study the connections between Gorenstein dimensions of an $S$–complex over $R$ and $S$, when $\phi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a quasi Gorenstein local ring homomorphism and $R$ a Cohen-Macaulay ring which admits a dualizing module.

Convention. Throughout this paper by a ring we mean a commutative noetherian ring with non-zero identity.

1. Homological Algebra

This section fixes the notation and sums up a few basic results. The main reference is [12] but also one can consult [4].

An $R$–complex $X$ is a sequence of $R$–modules $X_\ell$ and $R$–linear maps $\partial^X_\ell$, $\ell \in \mathbb{Z}$,

$$X = \cdots \to X_{\ell+1} \xrightarrow{\partial^X_{\ell+1}} X_\ell \xrightarrow{\partial^X_\ell} X_{\ell-1} \to \cdots$$

such that $\partial^X_\ell \partial^X_{\ell+1} = 0$ for all $\ell \in \mathbb{Z}$. $X_\ell$ and $\partial^X_\ell$ are called the module in degree $\ell$ and the $\ell$th differential of $X$, respectively.

The supremum and infimum of $X$ are defined as

$$\text{sup } X = \sup \{ \ell \in \mathbb{Z} | H_\ell(X) \neq 0 \}, \text{ and}$$

$$\text{inf } X = \inf \{ \ell \in \mathbb{Z} | H_\ell(X) \neq 0 \}.$$
The symbol $\mathcal{C}(R)$ denotes the category of $R$–complexes and morphisms of $R$–complexes.

The full subcategories $\mathcal{C}_{\ll}(R), \mathcal{C}_{\gg}(R), \mathcal{C}_{\square}(R)$ and $\mathcal{C}_0(R)$ of $\mathcal{C}(R)$ consist of complexes $X$ with $X_\ell = 0$, for respectively $\ell \gg 0$, $\ell \ll 0$, $|\ell| \gg 0$, and $\ell \neq 0$. The full subcategories $\mathcal{C}_{\ll}(R), \mathcal{C}_{\gg}(R)$ and $\mathcal{C}_{\square}(R)$ of $\mathcal{C}(R)$ consist of those $X$ with $\text{H}(X)$ belonging to $\mathcal{C}_{\ll}(R)$, $\mathcal{C}_{\gg}(R)$, and $\mathcal{C}_{\square}(R)$, respectively.

The right derived functor of the homomorphism functor of $R$–complexes and the left derived functor of the tensor product of $R$–complexes are denoted by $\mathbf{R}\text{Hom}_R(-,-)$ and $- \otimes_R L$, respectively.

The following inequalities hold for $X, Z \in \mathcal{C}_{\gg}(R)$ and $Y \in \mathcal{C}_{\ll}(R).$(cf. [12])

$$\sup(\mathbf{R}\text{Hom}_R(X,Y)) \leq \sup Y - \inf X; \text{ and}$$

$$\inf(X \otimes_R Z) \geq \inf X + \inf Z.$$ 

A complex $X \in \mathcal{C}_{\square}(R)$ is said to be of finite projective (respectively, injective or flat) dimension if $X \simeq U$, where $U$ is a complex of projective (respectively, injective or flat) modules and $U_\ell = 0$ for $|\ell| \gg 0$.

The full subcategories of $\mathcal{C}_{\square}(R)$ consisting of complexes of finite projective, injective and flat dimension are denoted by $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$, respectively.

If $X$ belongs to $\mathcal{C}_{\square}(R)$, then the following inequalities hold when $P \in \mathcal{P}(R)$, $I \in \mathcal{I}(R)$ and $F \in \mathcal{F}(R).$(cf. [12])
\[
\inf(\text{RHom}_R(P, X)) \geq \inf X - \text{pd}_R P;
\]
\[
\inf(\text{RHom}_R(X, I)) \geq -\sup X - \text{id}_R I; \text{ and}
\]
\[
\sup(F \otimes_R^L X) \leq \text{fd}_R F + \sup X.
\]

Let \( R \) and \( S \) be commutative \( Q \)-algebras. Then there are the following identities of equivalence of \( Q \)-complexes (cf. section 9 of \([14]\))

**Commutativity.** If \( X \in \mathcal{C}(\subset)(R) \) and \( Y \in \mathcal{C}(R) \), then
\[
X \otimes_R^L Y = Y \otimes_R^L X.
\]

**Associativity.** If \( X \in \mathcal{C}(\subset)(R) \), \( Y \in \mathcal{C}(R, S) \) and \( Z \in \mathcal{C}(\subset)(S) \), then
\[
(X \otimes_R^L Y) \otimes_S^L Z = X \otimes_R^L (Y \otimes_S^L Z).
\]

**Adjointness.** If \( X \in \mathcal{C}(\subset)(R) \), \( Y \in \mathcal{C}(R, S) \) and \( Z \in \mathcal{C}(\subset)(S) \), then
\[
\text{RHom}_R(X, \text{RHom}_S(Y, Z)) = \text{RHom}_S(\text{RHom}_R(X, Y), Z).
\]

**Tensor Evaluation.** If \( R \) is noetherian and \( X \in \mathcal{C}(f)(R) \), \( Y \in \mathcal{C}(\sqcap)(R, S) \) and \( Z \in \mathcal{C}(\sqcup)(S) \), then
\[
\text{RHom}_R(X, Y \otimes_S^L Z) = \text{RHom}_R(X, Y) \otimes_S^L Z,
\]
provided \( X \in \mathcal{P}(f)(R) \) or \( Z \in \mathcal{F}(S) \).

**Hom Evaluation.** If \( R \) is noetherian and \( X \in \mathcal{C}(f)(R) \), \( Y \in \mathcal{C}(\sqcap)(R, S) \) and \( Z \in \mathcal{C}(\sqcup)(S) \), then
\[
X \otimes_R^L \text{RHom}_S(Y, Z) = \text{RHom}_S(\text{RHom}_R(X, Y), Z),
\]
provided \( X \in \mathcal{P}(f)(R) \) or \( Z \in \mathcal{I}(S) \).

(These results actually hold under less restrictive boundedness conditions mentioned above (and in \([14]\)) but we never use this wider generality.)
2. The Auslander and Bass classes

First recall that a complex $D$ is a dualizing complex for a local ring $R$ when $D \in \mathcal{T}^{(f)}(R)$ and $R = \mathbf{R}\text{Hom}_R(D, D)$. An $R$–module $K$ which is a dualizing complex for $R$ is said to be a dualizing module.

It is known that if $R$ admits a dualizing module, it is a Cohen–Macaulay ring, and if $R$ is a Cohen–Macaulay ring then every dualizing complex of $R$ has only one nonzero homology module which can be considered as a dualizing module for $R$.

Convention. In the rest of this section $(R, \mathfrak{m}, k)$ will be a local ring with unique maximal ideal $\mathfrak{m}$, residue field $k$, and dualizing complex $D$.

The Auslander class, $\mathcal{A}(R)$, of $R$ is the full subcategory of $\mathcal{C}(R)$ consisting of all $R$–complexes $X$ satisfying

1. $X \in \mathcal{C}(\square)(R)$;
2. $D \otimes_R^L X \in \mathcal{C}(\square)(R)$; and
3. The canonical morphism $X \to \mathbf{R}\text{Hom}_R(D, D \otimes_R^L X)$ is an isomorphism in $\mathcal{C}(R)$.

The Bass class, $\mathcal{B}(R)$, of $R$ is the full subcategory of $\mathcal{C}(R)$ consists of all $R$–complexes $Y$ satisfying

1. $Y \in \mathcal{C}(\square)(R)$;
2. $\mathbf{R}\text{Hom}_R(D, Y) \in \mathcal{C}(\square)(R)$; and
3. The canonical morphism $Y \leftarrow D \otimes_R^L \mathbf{R}\text{Hom}_R(D, Y)$ is an isomorphism in $\mathcal{C}(R)$.
The following propositions will be used to prove the main results of this note.

2.1 Proposition Let $R$ and $S$ be two $Q$–algebras. Let $X, Y \in C(□)(R, S)$, $F \in \mathcal{F}(S)$, and $I \in \mathcal{I}(S)$. Then the following hold:

(i) If $X \in \mathcal{A}(R)$ then $\text{RHom}_S(X, I) \in \mathcal{B}(R)$ and $(X \otimes_S^L F) \in \mathcal{A}(R)$.

(ii) If $Y \in \mathcal{B}(R)$ then $\text{RHom}_S(Y, I) \in \mathcal{A}(R)$ and $(Y \otimes_S^L F) \in \mathcal{B}(R)$.

Proof. (i) Suppose that $X \in \mathcal{A}(R)$. Since $X \in C(□)(R, S)$ and $I \in \mathcal{I}(S)$ and $F \in \mathcal{F}(S)$, we have that $\text{RHom}_S(X, I) \in C(□)(R)$ and $X \otimes_S^L F \in C(□)(R)$. From the equalities

$$\text{RHom}_R(D, \text{RHom}_S(X, I)) = \text{RHom}_S(D \otimes_R^L X, I);$$

and

$$D \otimes_R^L (X \otimes_S^L F) = (D \otimes_R^L X) \otimes_S^L F$$

we get that $\text{RHom}_R(D, \text{RHom}_S(X, I))$ and $D \otimes_R^L (X \otimes_S^L F)$ are both homologically bounded since so is $D \otimes_R^L X$.

Finally note that

$$D \otimes_R^L (\text{RHom}_R(D, \text{RHom}_S(X, I))) = D \otimes_R^L (\text{RHom}_S(D \otimes_R^L X, I)) = \text{RHom}_S(\text{RHom}_R(D, D \otimes_R^L X), I)$$

which is canonically isomorphic to $\text{RHom}_S(X, I)$; and

$$\text{RHom}_R(D, D \otimes_R^L (X \otimes_S^L F)) = \text{RHom}_R(D, (D \otimes_R^L X) \otimes_S^L F) = \text{RHom}_R(D, D \otimes_R^L X) \otimes_S^L F$$

which is canonically isomorphism with $X \otimes_S^L F$.

Therefore $\text{RHom}_S(X, I) \in \mathcal{B}(R)$ and $X \otimes_S^L F \in \mathcal{A}(R)$.

The proof of part (ii) is analogous to the proof of part (i). □
Recall that an injective $S$–module $J$ is called \textit{faithfully injective} if for all non-zero $S$–modules $M$ we have $\text{Hom}_S(M, J) \neq 0$.

A flat $S$–module $P$ is called \textit{faithfully flat} if for all nonzero $S$–modules $M$ we have $M \otimes_S P \neq 0$.

Since in this note all rings are assumed to be noetherian it is clear that the module $J = \bigoplus_{n \in \text{max} S} E_S(S/n)$, the sum of injective envelopes of the $S/n$, for all maximal ideals of $S$, is a faithfully injective $S$–module and $S$ itself is a faithfully flat $S$–module.

\textbf{2.2 Proposition} With the same conditions as those of the proposition (2.1), if $J$ is a faithfully injective $S$–module and $P$ is a faithfully flat $S$–module, then the following hold:

(i) $X \in \mathcal{A}(R)$ if and only if $\text{Hom}_S(X, J) \in \mathcal{B}(R)$

(ii) $X \in \mathcal{A}(R)$ if and only if $X \otimes_S P \in \mathcal{A}(R)$

(iii) $X \in \mathcal{B}(R)$ if and only if $\text{Hom}_S(Y, J) \in \mathcal{A}(R)$

(iv) $X \in \mathcal{A}(R)$ if and only if $Y \otimes_S P \in \mathcal{B}(R)$.

\textit{Proof}. The “only if” parts are clear from Proposition (2.1) since $\mathbf{R}\text{Hom}_S(Z, J)$ and \text{L}$Z \otimes_S P$ are isomorphic to $\text{Hom}_S(Z, J)$ and $Z \otimes_S P$ respectively, for all $Z \in \mathcal{C}(S)$.

Now we prove the “if” part of (i). The other claims can be proved with similar techniques.

Since $J$ is a faithfully injective $S$–module, if $\text{Hom}_S(X, J)$ is homologically bounded
then so is $X$. We also have
\[ \text{Hom}_S(D \otimes^L_R X, J) = \text{RHom}_S(D \otimes^L_R X, J) \]
\[ = \text{RHom}_R(D, \text{Hom}_S(X, J)) \in \mathcal{C}(\square)(R) \]
then $D \otimes^L_R X \in \mathcal{C}(\square)(R)$. Finally
\[ \text{Hom}_S(\text{RHom}_R(D, D \otimes^L_R X), J) = \text{RHom}_S(\text{RHom}_R(D, D \otimes^L_R X), J) \]
\[ = D \otimes^L_R \text{RHom}_S(D \otimes^L_R X, J) \]
\[ = D \otimes^L_R \text{RHom}_R(D, \text{RHom}_S(X, J)) \]
\[ = D \otimes^L_R (\text{RHom}_R(D, \text{Hom}_S(X, J))). \]
Then $\text{Hom}_S(\text{RHom}_R(D, D \otimes^L_R X), J)$ is canonically isomorphism with $\text{Hom}_S(X, J)$ and hence $\text{RHom}_R(D, D \otimes^L_R X)$ is canonically isomorphism with $X$. \qed

2.3 Proposition Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a finite local homomorphism of local rings such that $S$ has finite flat dimension over $R$. If $R$ has a dualizing complex $D$, then for $X \in \mathcal{C}(\square)(R)$ the following hold:

(i) If $X \in \mathcal{A}(R)$ then $S \otimes^L_R X \in \mathcal{A}(S)$

(ii) If $X \in \mathcal{B}(R)$ then $\text{RHom}_R(S, X) \in \mathcal{B}(S)$.

Proof. Note that the $S$–complex $\tilde{D} = \text{RHom}_R(S, D)$ is a dualizing complex for $S$, cf. [14; 15.28]. Also recall that since $R$ is noetherian and $S$ has finite flat dimension over $R$ it has finite projective dimension over $R$ too.

(i): See [6; 6.6].

(ii): It is clear that $\text{RHom}_R(S, X) \in \mathcal{C}(\square)(S)$. We also have the following equalities:
\[ \text{RHom}_S(\tilde{D}, \text{RHom}_R(S, X)) = \text{RHom}_R(\tilde{D} \otimes^L_S S, X) \]
\[ = \text{RHom}_R(\tilde{D}, X) \]
\[ = S \otimes^L_R \text{RHom}_R(D, X) \]
The latter is homologically bounded since $\operatorname{RHom}_R(D, X) \in \mathcal{C}(\square)(R)$ and $\operatorname{fd}_R S < \infty$, hence $\operatorname{RHom}_S(\tilde{D}, \operatorname{RHom}_R(S, X)) \in \mathcal{C}(\square)(S)$. Finally note that

$$\tilde{D} \otimes^L_S \operatorname{RHom}_S(\tilde{D}, \operatorname{RHom}_R(S, X)) = \tilde{D} \otimes^L_S (\operatorname{RHom}_R(\tilde{D}, X))$$

$$= \tilde{D} \otimes^L_R (\operatorname{RHom}_R(D, X))$$

$$= \operatorname{RHom}_R(S, D \otimes^L_R (\operatorname{RHom}_R(D, X))).$$

The complex $X$ represents $D \otimes^L_R \operatorname{RHom}_R(D, X)$ canonically, then $\operatorname{RHom}_R(S, X)$ canonically represents $\tilde{D} \otimes^L_S \operatorname{RHom}_S(\tilde{D}, \operatorname{RHom}_R(S, X))$. □

3. Gorenstein dimensions

In this section we study the Gorenstein dimensions of complexes of modules over a Cohen–Macaulay local ring which admits a dualizing module.

Let $P \in \mathcal{C}^P(R)$ be homologically trivial. $P$ is called a complete projective resolution if the complex $\operatorname{Hom}_R(P, Q)$ is homologically trivial for every projective $R$–module $Q$.

A module $M$ is said to be Gorenstein projective if there exists a complete projective resolution $P$ with $C^P_0 \cong M$. Observe that every projective module is obviously Gorenstein projective.

The notation $\mathcal{C}^{GP}(R)$ is used for the full subcategory (of $\mathcal{C}(R)$) of complexes of Gorenstein projective modules.

The Gorenstein projective dimension of $X \in \mathcal{C}(\square)(R)$, $\operatorname{Gpd}_R X$, is defined as

$$\operatorname{Gpd}_R X = \inf \{ \sup \{ l \in \mathbb{Z} | A_l \neq 0 \} | X \cong A \in \mathcal{C}^{GP}(R) \}.$$  

(The set over which infimum is taken is non–empty since any complex $X \in \mathcal{C}(\square)(R)$ has a projective resolution that belongs to $\mathcal{C}^{GP}(R)$. )

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In particular, if $M$ is a non–zero $R$–module then $\text{Gpd}_R M$ is the smallest integer $n \geq 0$, such that there is an exact sequence

$$0 \to T_n \to \cdots \to T_1 \to T_0 \to M \to 0$$

where each $T_i$ is a non–zero Gorenstein projective module.

**3.1 Theorem** ([5], 4.4.5–4.4.16) Let $R$ be a Cohen–Macaulay local ring which admits a dualizing module. For a complex $X \in \mathcal{C}_{(\square)}(R)$ the next three conditions are equivalent.

(i) $X \in \mathcal{A}(R)$

(ii) $\text{Gpd}_R X < \infty$

(iii) $X \in \mathcal{C}_{(\square)}(R)$ and $\text{Gpd}_R X \leq \sup X + \dim R$.

Furthermore, if $X \in \mathcal{A}(R)$, then

$$\text{Gpd}_R X = \sup \{\inf U - \inf (R\text{Hom}_R(X, U))|U \in \mathcal{F}(R) \land U \neq 0\}$$

$$= \sup \{-\inf (R\text{Hom}_R(X, Q))|Q \in \mathcal{C}_0^P(R)\}$$

$$= \sup \{\inf U - \inf (R\text{Hom}_R(X, U))|U \in \mathcal{I}(R) \land U \neq 0\}$$

$$= \sup \{-\inf (R\text{Hom}_R(X, T))|T \in \mathcal{I}_0(R)\}.$$
The notation \( \mathcal{C}^{GF}(R) \) is used for the full subcategory (of \( \mathcal{C}(R) \)) of complexes of Gorenstein flat modules.

The *Gorenstein flat dimension* of \( X \in \mathcal{C}(R) \), \( \text{Gfd}_R X \), is defined as

\[
\text{Gfd}_R X = \inf \{ \sup \{ l \in \mathbb{Z} | A_l \neq 0 \} \mid X \simeq A \in \mathcal{C}^{GF}(R) \}.
\]

(The set over which infimum is taken is non–empty since any complex \( X \in \mathcal{C}(R) \) has a projective resolution that belongs to \( \mathcal{C}^{GF}(R) \).)

In particular, if \( M \) is a non–zero \( R \)–module then \( \text{Gfd}_R M \) is the smallest integer \( n \geq 0 \), such that there is an exact sequence

\[
0 \to T_n \to \cdots \to T_1 \to T_0 \to M \to 0
\]

where each \( T_i \) is a non–zero Gorenstein flat module.

**3.2 Theorem** ([5], 5.2.6–5.4.6) Let \( R \) be a Cohen–Macaulay local ring which admits a dualizing module. For a complex \( X \in \mathcal{C}(R) \) the next three conditions are equivalent.

(i) \( X \in \mathcal{A}(R) \)

(ii) \( \text{Gfd}_R X < \infty \)

(iii) \( X \in \mathcal{C}(R) \) and \( \text{Gfd}_R X \leq \sup X + \dim R \).

Furthermore, if \( X \in \mathcal{A}(R) \), then

\[
\text{Gfd}_R X = \sup \{ \sup (U \otimes_R^L X) - \sup U \mid U \in \mathcal{I}(R) \land U \neq 0 \}
\]

\[
= \sup \{ \sup (U \otimes_R^L X) - \sup U \mid U \in \mathcal{F}(R) \land U \neq 0 \}
\]

\[
= \sup \{ \sup (J \otimes_R^L X) \mid J \in \mathcal{C}_0^l(R) \}
\]

\[
= \sup \{ \sup (T \otimes_R^L X) \mid T \in \mathcal{I}_0^l(R) \}.
\]
Note that it is clear from the definition of the Gorenstein flat dimension that the inequality \( \text{Gfd} \ X \leq \text{fd}_R X \) holds. It is known that if \( \text{fd}_R X < \infty \) then equality holds. (cf. [5; (5.2.9)].)

Let \( I \in \mathcal{C}^I(R) \) be homologically trivial. \( I \) is called a complete injective resolution if the complex \( \text{Hom}_R(J, I) \) is homologically trivial for every injective \( R \)-module \( J \).

A module \( N \) is said to be Gorenstein injective if there exists a complete injective resolution \( I \) with \( Z_0^I \cong N \). Observe that every injective module is obviously Gorenstein injective.

The notation \( \mathcal{C}^{GI}(R) \) is used for the full subcategory (of \( \mathcal{C}(R) \)) of complexes of Gorenstein injective modules.

The Gorenstein injective dimension of \( Y \in \mathcal{C}_(\subseteq)(R) \), \( \text{Gid}_R Y \), is defined as

\[
\text{Gid}_R Y = \inf \{ \sup \{ \ell \in \mathbb{Z} | B_{-\ell} \neq 0 \} | Y \cong B \in \mathcal{C}^{GI}_\subseteq(R) \}
\]

(The set over which infimum is taken is non-empty since any complex \( Y \in \mathcal{C}_(\subseteq)(R) \) has an injective resolution that belongs to \( \mathcal{C}^{GI}_\subseteq(R) \).)

So in particular, if \( M \) is a non-zero \( R \)-module then \( \text{Gid}_R M \) is the smallest integer \( n \geq 0 \), such that there is an exact sequence

\[
0 \to M \to H^0 \to H^1 \to \cdots \to H^n \to 0
\]

where each \( H^i \) is a non-zero Gorenstein injective module.

3.3 Theorem ([5], 6.2.5) Let \( R \) be a Cohen–Macaulay local ring which admits a dualizing module. For a complex \( Y \in \mathcal{C}_(\subseteq)(R) \) the following conditions are equivalent.

(i) \( Y \in B(R) \)
(ii) $\text{Gid}_R Y < \infty$

(iii) $Y \in C(\square)(R)$ and $\text{Gid}_R Y \leq -\inf Y + \dim R$.

Furthermore if $Y \in B(R)$, then

$$
\text{Gid}_R Y = \sup \{-\sup U - \inf (\text{RHom}_R(U, Y)) | U \in \mathcal{I}(R) \land U \neq 0\}
$$

$$
= \sup \{-\inf (\text{RHom}_R(J, Y)) | J \in \mathcal{C}_0^I(R)\}.
$$

Note that from the definition of the Gorenstein injective dimension that the inequality $\text{Gid}_R Y \leq \text{id}_R Y$ holds. It is known that over a local ring with dualizing module if $\text{id}_R Y < \infty$ then equality holds. (cf. [5; (6.2.6)].)

In the next theorem $R$ and $S$ are assumed to be $Q$-algebras.

**3.4 Theorem** Let $R$ be a Cohen–Macaulay local ring with a dualizing module $D$.

If $X \in \mathcal{C}(R, S)$ and $Y \in \mathcal{C}(\square)(S)$, then

(i) $\text{Gpd}_R(X \otimes_S^L Y) \leq \text{pd}_S Y + \text{Gpd}_R X$.

(ii) $\text{Gfd}_R(X \otimes_S^L Y) \leq \text{fd}_S Y + \text{Gfd}_R X$.

(iii) $\text{Gid}_R(\text{RHom}_S(X, Y)) \leq \text{id}_S Y + \text{Gfd}_R X$.

(iv) If $Y \in \mathcal{I}(S)$, then

$$
\text{Gfd}_R(\text{RHom}_S(X, Y)) \leq \text{Gid}_R X + \sup Y.
$$

**Proof.** To prove each inequality we assume that the right hand side terms are finite, the inequality is clear if one of the terms is infinite.

By (3.1, 3.2, 3.3) and (2.1), finiteness of the right hand side terms of each inequality implies the finiteness of the left hand term side of it.
(i) We have
\[ \text{Gpd}_R(X \otimes_S Y) = \sup \{-\inf(\text{RHom}_R(X \otimes_S Y, T) | T \in \mathcal{C}_0^p(S))\} \]
\[ = \sup \{-\inf(\text{RHom}_S(Y, \text{RHom}_R(X, T)) | T \in \mathcal{C}_0^p(R))\} \]
\[ \leq \sup \{-\inf(\text{RHom}_R(X, T)) + \text{pd}_S Y | T \in \mathcal{C}_0^p(R)\} \]
\[ = \text{Gpd}_R X + \text{pd}_S Y. \]

(ii) We have
\[ \text{Gfd}_R(X \otimes_S Y) = \sup \{\sup((J \otimes_R (X \otimes_S Y)) | J \in \mathcal{C}_0^f(R))\} \]
\[ = \sup \{\sup((J \otimes_R X) \otimes_S Y)) | J \in \mathcal{C}_0^f(R)\} \]
\[ \leq \sup \{\sup(J \otimes_R X) + \text{fd}_S Y | J \in \mathcal{C}_0^f(R)\} \]
\[ = \text{Gfd}_R X + \text{fd}_S Y. \]

(iii) We have
\[ \text{Gid}_R(\text{RHom}_S(X, Y)) = \sup \{-\inf((\text{RHom}_R(J, \text{RHom}_S(X, Y))) | J \in \mathcal{C}_0^f(R))\} \]
\[ = \sup \{-\inf(\text{RHom}_S(J \otimes_R X, Y)) | J \in \mathcal{C}_0^f(R)\} \]
\[ \leq \sup \{\sup(J \otimes_R X) + \text{id}_S Y | J \in \mathcal{C}_0^f(R)\} \]
\[ = \text{Gfd}_R X + \text{id}_S Y. \]

(iv) We have
\[ \text{Gfd}_R(\text{RHom}_S(X, Y)) = \sup \{\sup((J \otimes_R \text{RHom}_S(X, Y)) | J \in \mathcal{I}_0^f(R))\} \]
\[ = \sup \{\sup(\text{RHom}_S(\text{RHom}_R(J, X), Y)) | J \in \mathcal{I}_0^f(R))\} \]
\[ \leq \sup \{-\inf(\text{RHom}_R(J, X) + \text{sup}Y | J \in \mathcal{I}_0^f(R))\} \]
\[ \leq \text{Gid}_R X + \text{sup}Y. \]

\[ \square \]

3.5 Corollary Let \( R \) be a Cohen-Macaulay local ring with a dualizing module. If \( \phi : R \to S \) is a ring homomorphism and \( Y \in \mathcal{C}_S(S) \), then
(i) \( \text{Gpd}_R Y \leq \text{pd}_S Y + \text{Gpd}_R S \).

(ii) \( \text{Gfd}_R Y \leq \text{fd}_S Y + \text{Gfd}_R S \).

(iii) \( \text{Gid}_R Y \leq \text{id}_S Y + \text{Gfd}_R S \).

(iv) If \( \text{id}_R Y < \infty \), then \( \text{Gfd}_R Y \leq \text{Gid}_R S + \sup Y \). \qed

3.6 Corollary With the same conditions as (3.5), if \( S \) is a Gorenstein local ring then

\[ \text{Gfd}_R S \leq \text{Gid}_R S \leq \text{Gfd}_R S + \text{dim} S. \]

In particular if \( S \) is self-injective, then \( \text{Gfd}_R S = \text{Gid}_R S \). \qed

3.7 Proposition Let \( R \) and \( S \) be \( Q \)-algebras such that \( R \) is a Cohen–Macaulay local ring with a dualizing module. If \( J \) is a faithfully injective \( S \)-module and \( P \) is a faithfully flat \( S \)-module, then the following hold for \( X \in \mathcal{C}(□)(R, S) \).

(i) \( \text{Gfd}_R (X \otimes_S P) = \text{Gfd}_R X \)

(ii) \( \text{Gid}_R (\text{Hom}_S(X, J)) = \text{Gfd}_R X \)

(iii) \( \text{Gfd}_R (\text{Hom}_S(X, J)) < \infty \) then \( \text{Gid}_R X < \infty \).

Proof. (i), (ii) By (2.2(i),(ii)), (3.2) and (3.3) we have that \( \text{Gid}_R \text{Hom}_S(X, J) < \infty \) if and only if \( \text{Gfd}_R X < \infty \) if and only if \( \text{Gfd}_R (X \otimes_S P) < \infty \).

The equalities are clear from the proof of (3.4), since the inequalities there become equalities for faithfully injective \( J \) and faithfully flat \( P \).

(iii) Use (3.2), (3.3) and (2.2 (iii)). \qed
Remark. It is natural to ask about the equality of part (c). We were unable to prove that equality always holds. We do not even know whether an $R$–module $M$ is Gorenstein injective if and only if $\text{Hom}_R(M, E)$ is Gorenstein flat, for every injective $R$–module $E$.

3.8 Corollary Let $R$ be a Cohen–Macaulay local ring with a dualizing module and $\varphi : R \to S$ a ring homomorphism. If $X \in C(\square)(S)$, and $p$ is a prime ideal of $S$

(i) If $\text{Gfd}_R X_p < \infty$, then

$$\text{Gid}_R(\text{Hom}_S(X, E_S(S/p))) = \text{Gfd}_R X_p.$$ 

(ii) If $\text{Gid}_R X_p < \infty$, then

$$\text{Gfd}_R(\text{Hom}_S(X, E_S(S/p))) \leq \text{Gid}_R X_p.$$ 

Proof. Note that

$$\text{Hom}_S(X, E_S(S/p)) = \text{Hom}_S(X, \text{Hom}_{S_p}(S_p, E_S(S/p))) = \text{Hom}_{S_p}(X_p, E_S(S/p)).$$

Now use (3.7). 

4. Finite local ring homomorphism

In this section $\varphi : (R, m) \to (S, n)$ is a finite local ring homomorphism of Cohen–Macaulay local rings. We also assume that $R$ has a dualizing module $D$, and consequently $(R\text{Hom}_R(S, D))$ is a dualizing module for $S$.

4.1 Theorem If $\text{fd}_R S < \infty$, then the following hold for $X \in C(\square)(R)$. 

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(i) \( \text{Gpd}_S(S \otimes_R^L X) \leq \text{Gpd}_R X. \)

(ii) \( \text{Gfd}_S(S \otimes_R^L X) \leq \text{Gfd}_R X. \)

(iii) \( \text{Gid}_S(\text{RHom}_R(S, X)) \leq \text{Gid}_R X. \)

**Proof.** We prove each inequality when the right hand side term of it is finite, otherwise there is nothing to prove.

(i) We assume that \( \text{Gpd}_R X < \infty \). Then by (2.3(i)) \( \text{Gpd}_S(S \otimes_R^L X) < \infty \).

Therefore

\[
\text{Gpd}_S(S \otimes_R^L X) = \sup \{-\inf(\text{RHom}_S(S \otimes_R^L X, Q)) | Q \in \mathcal{C}_0^P(S)\}
\]

\[
= \sup \{-\inf(\text{RHom}_R(X, \text{RHom}_S(S, Q))) | Q \in \mathcal{C}_0^P(S)\}
\]

\[
= \sup \{-\inf(\text{RHom}_R(X, Q)) | Q \in \mathcal{C}_0^P(S)\}
\]

By [2; (4.2(b))] \( Q \) has finite projective dimension over \( R \) if it has finite projective dimension over \( S \). Then

\[
\text{Gpd}_S(S \otimes_R^L X) \leq \text{Gpd}_R X.
\]

(ii) We assume that \( \text{Gfd}_R X < \infty \). Then by (2.3(i)) \( \text{Gfd}_S(S \otimes_R^L X) < \infty \).

Therefore

\[
\text{Gfd}_S(S \otimes_R^L X) = \sup \{\sup(J \otimes_S(S \otimes_R^L X)) | J \in \mathcal{C}_0^I(S)\}
\]

\[
= \sup \{\sup(J \otimes_R^L X) | J \in \mathcal{C}_0^I(S)\}
\]

By [2; (4.2(b))] \( J \) has finite injective dimension over \( R \) if it has finite injective dimension over \( S \). Then

\[
\text{Gfd}_S(S \otimes_R^L X) \leq \text{Gfd}_R X.
\]
(iii) We assume that $\text{Gid}_R X < \infty$. Then by (2.3(ii)) $\text{Gid}_S(\mathcal{R}\text{Hom}_R(S, X)) < \infty$. And

$$
\text{Gid}_S(\mathcal{R}\text{Hom}_R(S, X)) = \sup \{-\inf(\mathcal{R}\text{Hom}_S(J, \mathcal{R}\text{Hom}_R(S, X))) | J \in \mathcal{C}_0^I(S)\}
$$

$$
= \sup \{-\inf(\mathcal{R}\text{Hom}_R(J, X)) | J \in \mathcal{C}_0^I(S)\}
$$

$$
\leq \text{Gid}_R X.
$$

☐

Recall that for $x \in \mathfrak{m}$ the complex $0 \to R \xrightarrow{\Delta} R \to 0$ concentrated in degrees one and zero is called the Koszul complex of $x$ and denoted $K(x)$. For $x = x_1, \ldots, x_n \in \mathfrak{m}$ the Koszul complex $K(x)$ of $x_1, \ldots, x_n$ is the complex $K(x_1) \otimes_R \cdots \otimes_R K(x_n)$. Note that $K(x)$ is a homologically bounded $R$–complex of finite free $R$–modules.

The sequence $x = x_1, \ldots, x_n$ of elements of $\mathfrak{m}$ are $R$–regular if and only if $K(x)$ has zero homology modules except at $K(x)_0$, and when this is the case the homology module in degree zero is $R/\langle x \rangle$. (cf. [4; 1.6.19]).

4.2 Proposition Let $x = x_1, \ldots, x_n \in R$ be an $R$–regular sequence and let $X \in \mathcal{C}_0(H)(R)$, then the following hold

(i) $\text{Gpd}_{R/\langle x \rangle} (K(x) \otimes_R X) \leq \text{Gpd}_R X$

(ii) $\text{Gfd}_{R/\langle x \rangle} (K(x) \otimes_R X) \leq \text{Gfd}_R X$

(iii) $\text{Gid}_{R/\langle x \rangle} (\text{Hom}_R(K(x), X)) \leq \text{Gid}_R X$.

Proof. It is clear from remark (4.2) that $K(x)$ is a free resolution for $R/\langle x \rangle$, then the above inequalities are consequences of (4.1). ☐

Remark. Note that the converses inequalities of (4.2(i) and (ii)) hold when $X$ is a
finite $R$-module and $x$ a $X$–regular sequence (cf. [5]). However we could not prove the converse inequality in the general case.

5. Quasi-Gorenstein ring homomorphisms

In [3], Avramov and Foxby have defined quasi-Gorenstein ring homomorphisms.

5.1 Definition  Let $\phi : (R, m) \to (S, n)$ be a local ring homomorphism. Let $D$ be the dualizing complex of $R$, then $\phi$ is said to have finite Gorenstein dimension, if and only if $S$ belongs to $\mathcal{A}(R)$; and $\phi$ is said to be quasi-Gorenstein at $n$, if and only if it has finite Gorenstein dimension and $D \otimes^L_R S$ is a dualizing complex for $S$.

They have also proved ([3], 7.9) that if $\phi$ is quasi-Gorenstein at $n$, then an $S$-complex $X$ is in $\mathcal{A}(S)$, respectively, $\mathcal{B}(S)$, if and only if it is in $\mathcal{A}(R)$, respectively, $\mathcal{B}(R)$.

In this appendix, using this fact, we present connections between Gorenstein dimensions of an $S$-complex over $R$ and $S$, when $\phi : R \to S$ is a quasi-Gorenstein ring homomorphism.

5.2 Theorem  Assume that $(R, m)$ is a Cohen-Macaulay local ring with a dualizing module $D$. If $\phi : (R, m) \to (S, n)$ is quasi-Gorenstein at $n$, then the following inequalities hold for $X \in \mathcal{C}(S)$.

(i) $\text{Gpd}_R X \leq \text{Gpd}_S X + \text{Gpd}_R S$.

(ii) $\text{Gfd}_R X \leq \text{Gfd}_S X + \text{Gfd}_R S$.

Proof. Since $S \in \mathcal{A}(R)$, the module $D \otimes_R S$ represents $D \otimes^L_R S$ (cf. [5 ,3.4.6]). Then $S$ is a Cohen-Macaulay local ring with a dualizing module and hence $\text{Gpd}_S X$
and \( \text{Gfd}_S X \) are finite if and only if \( \text{Gpd}_R X \) and \( \text{Gfd}_R X \) are finite.

Now let \( X \in \mathcal{A}(S) \), then using fundamental equalities we have

\[
\text{Gpd}_R X = \sup \{- \inf(\text{RHom}_R(X, T)) | T \in \mathcal{I}_0(R)\}
\]

\[
= \sup \{- \inf(\text{RHom}_R(X \otimes_S^L S, T)) | T \in \mathcal{I}_0(R)\}
\]

\[
= \sup \{- \inf(\text{RHom}_S(X, \text{RHom}_R(S, T)) | T \in \mathcal{I}_0(R)\}
\]

\[
\leq \sup \{\inf(\text{RHom}_R(S, T)) - \inf(\text{RHom}_S(X, \text{RHom}_R(S, T)) | T \in \mathcal{I}_0(R)\}
\]

\[
+ \sup \{- \inf(\text{RHom}_R(S, T)) | T \in \mathcal{I}_0(R)\}
\]

\[
\leq \text{Gpd}_S X + \text{Gpd}_R S.
\]

And

\[
\text{Gfd}_R X = \sup \{\sup(U \otimes_R^L X) | U \in \mathcal{F}_0(R)\}
\]

\[
= \sup \{\sup(U \otimes_R^L (S \otimes_S^L X)) | U \in \mathcal{F}_0(R)\}
\]

\[
= \sup \{\sup((U \otimes_R^L S) \otimes_S^L X) | U \in \mathcal{F}_0(R)\}
\]

\[
\leq \sup \{\sup((U \otimes_R^L S) \otimes_S^L X) - \sup(U \otimes_R^L S) | U \in \mathcal{F}_0(R)\}
\]

\[
+ \sup \{\sup(U \otimes_R^L S) | U \in \mathcal{F}_0(R)\}
\]

\[
\leq \text{Gfd}_S X + \text{Gfd}_R S.
\]

\( \square \)

Avramov and Foxby have proved that if \( \phi : R \to S \) is quasi-Gorenstein at \( n \), the inequalities of the above theorem become equalities provided that \( S \) is a finite \( R \)-module and \( X \) a finite \( S \)-module.\(^3\) Note that when this is the case, the Gorenstein projective and flat dimensions are equal to the Auslander’s G-dimension.

To prove the dual of the theorem, we provide some preliminaries.

For an complex \( X \in \mathcal{C}(\square)(R) \), Christensen, Foxby and Frankild defined the small restricted covariant Ext-dimension of \( X \) as follows.

\(^3\) L. Avramov informed us that this result is a consequence of a result due to Golod.
\[ \text{rid}_R X = \sup \{- \inf (R \text{Hom}_R(T, X)) | T \in \mathcal{P}_0^{(f)} \} = \sup \{- \sup U - \inf (R \text{Hom}_R(U, X)) | U \in \mathcal{F}^{(f)}(R) \}. \]

They have proved that the inequality \( \text{rid}_R X \leq \text{id}_R X \) always holds, and it becomes equality if \( \text{cmd} R \leq 1 \) and \( \text{id}_R X < \infty \). (cf. [7])

### 5.3 Theorem

Let \( M \) be an \( R \)-module. The following inequality always holds.

\[ \text{rid}_R M \leq \text{Gid}_R M. \]

The equality holds if \( \text{cmd} R \leq 1 \) and \( \text{Gid}_R M < \infty \).

**Proof.** If \( \text{Gid}_R M \) is not finite, then the claim is obvious. Now let \( n = \text{Gid}_R M \) be finite. We prove the theorem by induction on \( n \).

If \( n = 0 \), then \( M \) is a Gorenstein injective module and then there exists an exact complex

\[ I = \cdots \to I_1 \to I_0 \to M \to 0 \]

with \( I_j \) s injective modules.

If \( T \) is a finite \( R \)-module of finite projective dimension \( t \), then since \( \text{Ext}_R^i(T, M) = \text{Ext}_R^{i+t}(T, Z_t) \), we have \( \text{Ext}_R^i(T, M) = 0 \) for all \( i > 0 \). Therefore \( \text{rid}_R M = 0 \).

If \( n > 0 \), then by [17, 2.45], there exists a Gorenstein injective \( R \)-module \( G \) and an \( R \)-module \( C \) with \( \text{id}_R C = \text{Gid}_R C = n - 1 \), such that the following sequence is exact.

\[ 0 \to M \to G \to C \to 0 \]

For any finite \( R \)-module \( T \) of finite projective dimension, we have the exact sequence

\[ 0 = \text{Ext}_R^{i-1}(T, C) \to \text{Ext}_R^i(T, M) \to \text{Ext}_R^i(T, G) = 0 \]
for $i > n$.

Therefore $\text{rid}_R M \leq \text{Gid}_R M$.

Now let $\text{cmd} R \leq 1$. To prove the inverse inequality, note that since $G$ is a Gorenstein injective module, there exists an injective $R$-module $E$ and an exact sequence

$$0 \to K \to E \to G \to 0$$

with $K$ a Gorenstein injective $R$-module, too.

On the other hand we have the isomorphisms $C \cong G/M$ and $G \cong E/K$. Then there exists a submodule $L$ of $E$ such that $K \subseteq L$ and $M \cong L/K$, and then $C \cong E/L$.

From the following exact sequence, we get $\text{id}_R L \leq \text{id}_R C = n$.

$$0 \to L \to E \to C \to 0$$

Now consider the exact sequence

$$0 \to K \to L \to M \to 0.$$ 

Since $\text{Gid}_R M = n$, there exists an injective $R$-module $J$ with $\text{Ext}_R^n(J, M) \neq 0$. Hence from the exact sequence

$$\text{Ext}_R^n(J, L) \to \text{Ext}_R^n(J, M) \to \text{Ext}_R^{n+1}(J, K) = 0,$$

we get $\text{Ext}_R^n(J, L) \neq 0$ and then $\text{id}_R L \geq n$.

Therefore $\text{rid}_R L = \text{id}_R L = n$, and then there exists a finite $R$-module $Q$ of finite projective dimension such that $\text{Ext}_R^n(Q, L) \neq 0$.

Finally the exactness of

$$0 = \text{Ext}_R^n(Q, K) \to \text{Ext}_R^n(Q, L) \to \text{Ext}_R^n(Q, M)$$

implies that $\text{Ext}_R^n(Q, M) \neq 0$ and then $\text{rid}_R M \geq n$. \qed
5.4 Theorem Assume that \((R, \mathfrak{m})\) is a Cohen-Macaulay local ring with a dualizing module \(D\). If \(\phi : (R, \mathfrak{m}) \to (S, \mathfrak{n})\) is quasi-Gorenstein at \(\mathfrak{n}\), then the following inequalities hold for an \(S\)-module \(M\).

\[
\text{Gid}_R M \leq \text{Gid}_S M + \text{Gfd}_R S.
\]

Furthermore, the equality holds if \(S\) is a finite \(R\)-module and \(M\) a finite \(S\)-module.

Proof. Since \(\phi\) is quasi-Gorenstein, \(\text{Gid}_R M\) is finite, if and only if \(\text{Gid}_S M\) is finite. Let \(\text{Gid}_S M < \infty\), then

\[
\text{Gid}_R M = \sup \{-\inf(\text{RHom}_R(T, M)| T \in \mathcal{P}_0^{(f)}(R)) \}
\]

\[
= \sup \{-\inf(\text{RHom}_R(T, \text{RHom}_S(S, M))| T \in \mathcal{P}_0^{(f)}(R)) \}
\]

\[
= \sup \{-\inf(\text{RHom}_S(T \otimes_R S, M))| T \in \mathcal{P}_0^{(f)}(R)) \}
\]

\[
\leq \sup \{-\sup(T \otimes_R S) - \inf(\text{RHom}_S(T \otimes_R S, M))| T \in \mathcal{P}_0^{(f)}(R)) \}
\]

\[
+ \sup \{\sup(T \otimes_R S)| T \in \mathcal{P}_0^{(f)}(R)) \}
\]

\[
\leq \text{Gid}_S M + \text{Gfd}_R S.
\]

Now let \(S\) be a finite \(R\)-module and \(M\) a finite \(S\)-module. By [5, 6.2.15], the following equalities hold.

\[
\text{Gid}_R M = \text{depth}R \quad \text{and} \quad \text{Gid}_S M = \text{depth}S.
\]

And then the requested equality follows by Auslander-Bridger formula. \(\square\)

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