On Quasinormal Modes for Gravitational Perturbations of Bardeen Black Hole

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In this paper we investigate gravitational perturbations of a regular black hole, particularly the Bardeen solution. Such system is solution of Einstein equations that do not have a singularity at the origin of the radial symmetry. However it still have events horizons depending on the values of the characteristic parameters of the solution. When a black hole is perturbed, it oscillates. It gives rise to damped vibrating modes which are known as quasinormal modes. It is calculated the quasinormal frequencies of a regular black hole using the third order WKB approximation for gravitational perturbations. The results are presented in tables I, II and III.

I. INTRODUCTION

In 1957 Regge and Wheeler explored the stability of Schwarzschild solution over gravitational perturbations [1]. They discovered that it leads to a Schrödinger-like equation with a specific effective potential and a frequency associated to the temporal dependence. Since then others perturbations for different systems have been analyzed and lead to similar equations [2]. The frequencies in those equations can be calculated by means semi-analytic or numerical methods. Such frequencies are discrete and complex numbers which can be obtained by the imposition of specific boundary conditions such as the existence of outgoing waves at the spatial infinity and ingoing waves at the event horizon. This is known as quasinormal modes. These modes are damped oscillations of the spacetime geometry that can be used to investigate fundamental features of the gravitational field. The real part is related to the oscillation itself while the imaginary part refers to the rate at each mode is damped. Therefore quasinormal modes may be important to gravitational waves physics and it could become an environment to study the basic effects of some sort of quantized gravity, since the quasinormal frequencies are discrete quantities.

Defining a singularity is an intricate task, in what concerns black holes the presence of a fundamental singularity can cause serious difficulties when one tries to predict the future from the past, which led Einstein himself to doubt about the physical realization of the first solutions of the field equations of general relativity [3]. It’s believed that the singularities appearing in some solutions, such as Schwarzschild’s solution, is due to highly symmetric assumptions used to get them. However the singularity is not a privilege of general relativity since the collapse of a spherical dust in newtonian theory also yields a singular point. Over the years a great effort has been directed to deal with such problem by means the development of the so called cosmic censorship hypotheses [4, 5]. The singularity would be inaccessible to an observer outside the event horizon and always covered by it which allows the avoidance of a naked singularity. This is a route to contour the problem, not to solve it, since the singularity, which is a place where no physics can be done, is still there. Since there is neither a physical significance nor a experimental support for the presence of a singularity, one could seek for non-singular solutions of field equations. In fact, the ideas of gravastars and regular black holes arose from this kind of feeling. The last objects are interesting because they do not have a fundamental singularity at the origin of the coordinate system. Event horizons can be present but they could be removed under some change of variables.

The first regular solution of Einstein equations that describes a black hole was due to Bardeen who got his solution only approximatively [6]. Later it was discovered that such a solution could be viewed as the gravitational collapse of a magnetic monopole to which a nonlinear electromagnetic energy-momentum tensor works as the source of field equations [7]. This has lead the Bardeen metric to the category of an exact solution of Einstein equations. Others regular solutions was obtained by the coupling of some matter field, such as scalar field, to gravitation in a cosmological context [8, 9]. Because of their singularity-free feature regular black holes could be great candidates to represent realistic final stages of collapsing regular configurations such as a common star. Although the success of the computation of quasinormal modes of black holes or neutron stars, it has not been done for regular black holes for gravitational perturbations.

In this paper the quasinormal modes of Bardeen black hole, for gravitational perturbations, are calculated. Such modes are the solution of Einstein equations in the presence of a non-linear electromagnetic field which is an entirely new paradigm when compared to what is done in [1] for Schwarzschild black hole, since it is a vacuum solution. However the first step is to construct axial perturbations once they are simpler than polar perturbations. It will be chosen a gauge in which there is no dependence.

The paper is organized as follows. In section II the
general theory of black holes stability of a spherical symmetric background metric for gravitational perturbations is summarized. The master equation and the effective potential are obtained for the Bardeen solution which was the first one describing a regular black hole. In section III the complex frequencies for such a solution has been calculated. It was done using WKB approximation of third order, the results are then organized in tables I, II and III. Finally in the last section the concluding remarks are presented.

II. GRAVITATIONAL PERTURBATIONS OF A REGULAR BLACK HOLE

In this section we carry out a treatment to deal with a gravitational perturbation of regular black holes. A generic spherically symmetric background metric $\eta_{\mu\nu}$, is defined by the line element

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) , \quad (1)$$

thus the metric tensor can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The perturbations $h_{\mu\nu}$ can be decomposed as

$$\delta G_{13} = \frac{1}{2} \left\{ \frac{1}{f(r)} \left[ \frac{\partial^2 h_1}{\partial t^2} - \frac{\partial^2 h_0}{\partial t \partial r} + 2 \frac{\partial h_0}{\partial t} \right] - \left( \frac{2}{r^2} - \frac{2}{r} \frac{df}{dr} - \frac{d^2 f}{dr^2} \right) \right\} h(\theta) - \frac{h_1}{2r^2} \left( \frac{d^2 h(\theta)}{d\theta^2} - \frac{\cos \theta \, dh(\theta)}{d\theta} \right) .$$

Therefore given a background metric which is a solution of the unperturbed Einstein equation, it’s possible to find the gravitational perturbations by the above equation once the perturbed energy-momentum tensor is settled.

A. Master Equation for Bardeen Solution

There are many works in the literature concerning Bardeen black hole. Geodesic structure of test particles was studied in [10], while the features of Bardeen solution as a gravitational lens was given in [11]. The quasinormal modes of such a spacetime, for scalar field perturbations, was calculated in [12] and a good revision can be find in [13].

Let us consider a non-linear electromagnetic energy-momentum tensor given by [7]

$$\delta G_{\mu\nu} = k \delta T_{\mu\nu} ,$$

where $k = 8\pi$ in units such that $G = c = 1$ and

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} (h_{\mu\nu} R + \eta_{\mu\nu} \delta R) ,$$

$$\delta R = \eta^{\mu\alpha} \eta^{\nu\beta} \delta \Gamma_{\alpha\beta} ,$$

$$\delta R_{\mu\nu} = -\nabla_\alpha \delta \Gamma^\alpha_{\mu\nu} + \nabla_\nu \delta \Gamma^\alpha_{\mu\alpha} ,$$

$$\delta \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \eta^\alpha_{\nu\gamma} (\partial_\nu h_{\beta\nu} + \partial_\nu h_{\gamma\nu} - \partial_\nu h_{\beta\gamma}) .$$

Thus, from eq. (2), the above equations yield

$$\delta G_{23} = \frac{1}{2} \left[ -\frac{1}{f(r)} \frac{\partial h_0}{\partial t} + \frac{\partial f(r)h_1}{\partial r} \right] \left[ \frac{dh(\theta)}{d\theta} - \frac{2}{\sin \theta} \frac{dh(\theta)}{d\theta} \right] .$$

and

$$T^\nu_\mu = 2 (L_F F^\mu_\lambda F^{\nu\lambda} - \delta^\nu_\mu L) ,$$

where $L_F = \delta L/\delta F$, with $F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and

$$L = \frac{3}{4} \frac{m}{|a|^3} \left( \frac{\sqrt{2\alpha^2 F}}{1 + \sqrt{2\alpha^2 F}} \right)^{5/2} .$$

The parameter $\alpha$ represents a magnetic monopole and $m$ its mass. If we consider

$$F_{\mu\nu} = 2\delta^\mu_\lambda \delta^\nu_\alpha \alpha \sin \theta ,$$

then we find $F = \alpha^2 / 2r^4$. Therefore coupling this model of nonlinear electrodynamics to the Einstein equations, it is possible to find an exact solution [6, 7]. This solution
is known as the Bardeen black hole and it can be put in the form (1), with
\[ f(r) = 1 - \frac{2mr^2}{(r^2 + \alpha^2)^{3/2}}. \]

Such a solution was first obtained only approximately and it was the first regular black hole ever proposed. We can picture this system as a self-gravitating magnetic monopole of charge \( \alpha \) and mass \( m \). An interesting feature of the Bardeen solution is that it can have zero, one or two horizons of events depending on the choice of the magnetic charge \( \alpha \). Although it is always a regular black hole at \( r = 0 \) for \( \alpha \neq 0 \), it describes a regular spacetime only when the following inequality holds:
\[ \alpha^2 \leq \frac{16}{27} m^2. \]

Clearly for \( \alpha = 0 \) the solution reduces to the well known Schwarzschild metric which do not represent a regular black hole.

Here we intend to obtain the master equation for gravitational perturbations of Bardeen spacetime. Thus the non-vanishing components of the perturbed energy-momentum tensor are
\[ \delta T_{03} = -2kLh_0(r,t)h(\theta), \]
\[ \delta T_{13} = -2kLh_1(r,t)h(\theta). \]

Therefore the perturbed Einstein equations lead to
\[ \frac{1}{2} \left[ -\frac{1}{f(r)} \frac{\partial f(r) h_1}{\partial t} + \frac{\partial [f(r)h_1]}{\partial r} \right] = 0, \quad (3) \]
\[ \frac{1}{f(r)} \left[ \frac{\partial^2 h_1}{\partial t^2} - \frac{\partial^2 h_0}{\partial \theta \partial r} + 2 \frac{\partial h_0}{r \partial t} \right] + \left[ \frac{\gamma - 2}{r^2} + \frac{2 df}{dr} + \frac{d^2 f}{dr^2} + 2kL \right] h_1 = 0, \quad (4) \]
\[ \frac{d^2 h}{d\theta^2} - \cos \theta \frac{dh}{d\theta} + \gamma h = 0. \quad (5) \]

for the functions \( h(\theta), h_0(r,t) \) and \( h_1(r,t) \). From the eq. (5), it follows that \( \gamma = l(l+1) \) and \( h(\theta) = P_l(\cos \theta) \) which are the Legendre Polynomials.

If we use the definition \( \psi = \left( \frac{1}{f} \right) f(r)h_1(r,t) \), then eqs. (3) and (4) can be combined into a single one which reads
\[ \frac{\partial^2 \psi}{\partial t^2} - f^2 \frac{\partial^2 \psi}{\partial r^2} - f \frac{df}{dr} \frac{\partial \psi}{\partial r} + f \left[ \frac{l(l+1) + 2(f-1)}{r^2} + \frac{1}{f} \frac{df}{dr} + \frac{d^2 f}{dr^2} + 2kL \right] \psi = 0. \quad (6) \]

In order to put such equation in a more familiar form, we change the variable \( r \) to the "tortoise" coordinate, defined by \( dx = \frac{dr}{f(r)} \), that leads to
\[ \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi = 0, \quad (7) \]

with the effective potential given by
\[ V = f \left[ \frac{l(l+1) + 2(f-1)}{r^2} + \frac{1}{r} \frac{df}{dr} + \frac{d^2 f}{dr^2} + 2kL \right]. \]

The temporal dependence is assumed to be given by \( \psi = e^{-i \omega t} \phi \), thus we have
\[ \left[ \frac{\partial^2}{\partial x^2} + \omega^2 - V(x) \right] \phi(x) = 0, \quad (8) \]

which is the master equation for gravitational perturbations of the Bardeen solution as the background metric. With the solution of this equation it would be possible to construct the perturbed metric, then all the features of spacetime would be known. The frequencies \( \omega \) represent dissipative modes in time, thus after the perturbation the whole black hole will oscillate and then go to a stable configuration. Such modes oscillate with the so called quasinormal frequencies.

### III. QUASINORMAL MODES IN WKB APPROXIMATION

The WKB method is a semi-analytic technique used to solve Schrödinger-type equations like eq. (8). Once the solution \( \phi(x) \) is known it is possible to use two conditions to properly establish it: it is requested that there are only ongoing waves at the infinity and only ingoing waves at the horizon of events. Usually the frequency in WKB approximation is written as
\[ \omega = \frac{\sqrt{\omega^2 - V_0}}{\sqrt{\omega^2 - V_0}} - \sum_{i=2}^{k} \Lambda_i = n + \frac{1}{2}, \]

where \( V_0 \) and \( V_0'' \) are the effective potential and its second derivative respectively, taken at the point of the maximum of \( V \). The second term in the left-hand side of the above equation represents the higher order WKB corrections (from 2nd until the k-th order [13]). In general a great feature of the WKB method is that it gets better and better as one take the higher orders of approximation (for low values of \( n \) and \( l \)). As a matter of fact it is a very accurate procedure even when compared to numeric methods [14].

In this paper we will work with WKB approximation of third order [10] which is given by
\[ \omega^2_{n,l} = \left[ V_0 + (-2V_0'')^{1/2} \Lambda \right] - i \left( n + \frac{1}{2} \right) \left(-2V_0''\right)^{1/2}(1+\Omega), \quad (9) \]
and is a point of maximum of the effective potential. It should be noted that $x$ is the well known tortoise coordinate.

Using expression (5), it is possible to calculate the quasinormal frequencies which are presented in tables II and III for Bardeen space-time. All results refer to quasinormal modes for gravitational perturbations. They were calculated in units of m which means $\omega_{nl}$ is dimensionless.

Firstly we note that the Bardeen spacetime for $\alpha = 0$ is exactly the Schwarzschild spacetime. It’s well known that the ground state of the quasinormal modes for Schwarzschild metric is given by $\{l, n\} = \{2, 0\}$, thus

\begin{table}[h]
\centering
\caption{Quasinormal modes of Bardeen space-time for $\alpha = 0$.}
\begin{tabular}{lll}
\hline
$l$ & $n$ & $\omega_{nl}$ \\
\hline
1 & 0 & 0.1171192993 - 0.08879106527i \\
1 & 1 & 0.55501430875 - 0.28730577745i \\
2 & 0 & 0.3731620888 - 0.08921749033i \\
2 & 1 & 0.3460174754 - 0.2749155289i \\
2 & 2 & 0.3029353684 - 0.471064944i \\
3 & 0 & 0.5992651163 - 0.09272839457i \\
1 & 1 & 0.5823546522 - 0.2814060077i \\
2 & 1 & 0.5531995934 - 0.4766840022i \\
3 & 1 & 0.5157471801 - 0.6774290938i \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Quasinormal modes of Bardeen space-time for $\alpha = 0.3$.}
\begin{tabular}{lll}
\hline
$l$ & $n$ & $\omega_{nl}$ \\
\hline
0 & 0 & 0.0598299242 - 0.3650702720i \\
1 & 0 & 0.1884164237 - 0.2186679687i \\
1 & 1 & 0.6363972564 - 0.7536992941i \\
2 & 0 & 1.3294036135 - 1.4939390929i \\
2 & 1 & 0.4066060007 - 0.08797961013i \\
2 & 2 & 0.3918842187 - 0.2699024702i \\
3 & 0 & 0.3720487816 - 0.4598638795i \\
3 & 1 & 0.3501499693 - 0.6529380130i \\
\hline
\end{tabular}
\end{table}
TABLE III. Quasinormal modes of Bardeen space-time for $\alpha = \frac{4}{\pi}$.

| $l$ | $n$ | $\omega_{nl}$ |
|-----|-----|----------------|
| 0   | 0   | $0.6218228066 - 0.07896506498i$ |
|     |     | $0.9315004445 - 0.07399275203i$ |
| 1   | 0   | $0.6767694809 - 0.07815597820i$ |
|     |     | $0.8943294277 - 0.3736175009i$ |
| 1   | 1   | $0.6619780155 - 0.2360238599i$ |
|     |     | $0.8584153696 - 0.5278493353i$ |
| 2   | 0   | $0.6340480798 - 0.3978834282i$ |
|     |     | $1.109517971 - 0.07301443485i$ |
| 2   | 1   | $0.7828924561 - 0.07622286290i$ |
|     |     | $1.099525117 - 0.2193579532i$ |
| 2   | 2   | $0.7688856410 - 0.2299687824i$ |
|     |     | $1.079605666 - 0.3666520402i$ |
| 3   | 0   | $0.7420488856 - 0.3872459558i$ |
|     |     | $1.049912024 - 0.515543322i$ |
| 3   | 1   | $0.7042736801 - 0.5493249063i$ |
|     |     | $1.010709699 - 0.6666705011i$ |

IV. CONCLUSION

In this article we have analyzed gravitational perturbations of a regular black hole, by calculating the quasinormal frequencies. Such results were obtained for the Bardeen solution of Einstein equations that represent a regular black hole. To obtain the quasinormal modes it has been used third order WKB approximation which yields good results when compared to accurate numerical techniques, for low values of the numbers $n$ and $l$. It is such a surprise to find that the function $h(\theta)$ is equal to the Legendre Polynomials since that in the Regge-Wheeler gauge the "$\theta$" dependence is given by the function $h(\theta) = \sin \theta \partial_{\theta} P_l(\cos \theta)$. However in the Regge-Wheeler gauge it appears the value $l(l+1)$ in the radial equation which is the same quantity found here due to the eigenvalue of Legendre equation. The results are summarized in tables II, III and IV. The quasinormal modes for Schwarzschild were already known, for instance see table III of reference [10], however they are charted here to serve as a comparison guide.

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