Exhaustion of the curve graph via rigid expansions

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Abstract

For an orientable surface $S$ of finite topological type with genus $g \geq 3$, we construct a finite set of curves whose union of iterated rigid expansions is the curve graph $C(S)$. The set constructed, and the method of rigid expansion, are closely related to Aramayona and Leiniger’s finite rigid set in [1] and [2], and in fact a consequence of our proof is that Aramayona and Leininger’s set also exhausts the curve graph via rigid expansions.

Introduction

In this article we consider an orientable surface $S_{g,n}$ of finite topological type with genus $g \geq 3$ and $n \geq 0$ punctures. The mapping class group of $S_{g,n}$, denoted by $\text{Mod}(S_{g,n})$ is the group of orientation preserving self-homeomorphisms of $S_{g,n}$. The extended mapping class group of $S_{g,n}$, denoted by $\text{Mod}^*(S_{g,n})$ is the group of isotopy classes of self-homeomorphisms of $S_{g,n}$.

In order to study these groups, Harvey in 1979 (see [7]) introduced the curve complex of a surface as the simplicial complex whose vertices are isotopy classes of essential curves, and simplices are defined by disjointness (see Section 1 for details). We call the 1-skeleton of the curve complex the curve graph, which we denote by $C(S_{g,n})$.

There is a natural link between the curve complex and $\text{Mod}(S_{g,n})$ and $\text{Mod}^*(S_{g,n})$. Ivanov (in [14]) linked the curve complex to $\text{Mod}^*(S_{g,n})$ via simplicial automorphisms, while Harer (in [6]) linked the curve complex with $\text{Mod}(S_{g,n})$ by their (co-)homology.

On one hand, in [14], [15] and [17] it was proved that for most surfaces every automorphism of the curve graph is induced by a homeomorphism of $S_{g,n}$, with the well-known exception of $S_1, 2$. Later on, there were generalizations of this result for larger classes of simplicial maps (see [11], [12], [13], [3]), until Shackleton (see [18]) proved that any locally injective self-map of the curve graph is induced by a homeomorphism (for surfaces of high-enough complexity).

Thereafter, Aramayona and Leininger introduced in [1] the concept of a rigid set of the curve graph, which is a full subgraph $Y$ such that any locally injective map from $Y$ to $C(S_{g,n})$ is the restriction to $Y$ of an automorphism, unique up to the pointwise stabilizer of $Y$ in $\text{Aut}(C(S_{g,n}))$. By Shackleton’s result, the curve graph itself is a rigid set. In [1] they also construct a finite rigid set for any orientable surface of finite topological type. See Section 3 below.

On the other hand, it is a well-known result by Harer [6] that the curve complex is homotopically equivalent to a bouquet of spheres, which is used to determine the virtual cohomological dimension of the mapping class group.

Later on, Birman, Broaddus and Menasco in [4] proved that Aramayona and Leininger’s finite rigid set either is (for $g = 0$ and $n \geq 5$) or contains (for $g \geq 1$ and $n \leq 1$) a $\text{Mod}(S_{g,n})$-module generator of the reduced homology of the curve complex. Thus, they link the (co-)homological and simplicial sides of the study of the mapping class group and curve complexes.

Afterwards, Aramayona and Leininger proved in [2] that for almost all surfaces of finite topological type, there exists an increasing sequence of finite rigid sets that exhaust the curve graph, each of which has trivial pointwise stabilizer in $\text{Mod}^*(S_{g,n})$. Note that this is not trivial, given that there exist examples of supersets of a rigid set that are not rigid themselves.
While their proof is effective for the result, it does not lend itself to improving other results concerning simplicial maps. In this work, we prove a similar result to theirs; however, we use a method developed in \cite{2} for expanding subgraphs. This method can be used to obtain new results concerning edge-preserving maps; the details of these results are given in \cite{8} and will appear in a second paper \cite{9}. We call this method rigid expansion.

We define the first rigid expansion of a subgraph $Y$, denoted as $Y^{1}$, as the union of $Y$ with all the curves uniquely determined by subsets of $Y$, where a curve $\beta$ is uniquely determined by a subset $B$ of $C(S_{g,n})$ if it is the unique curve disjoint from every element in $B$. We also define 

$$Y^{0} = Y \text{ and, inductively, } Y^{k} = (Y^{k-1})^{1}.$$

Note in particular that if $\beta$ is uniquely determined by $B$, then for every $h \in \text{Mod}^{*}(S_{g,n})$ we have that $h(\beta)$ is uniquely determined by $h(B)$.

Now we can state the main result of this work.

**Theorem A.** Let $S_{g,n}$ be an orientable surface of finite topological type with genus $g \geq 3$, $n \geq 0$ punctures, and empty boundary. There exists a finite subgraph of $C(S_{g,n})$ whose union of iterated rigid expansions is equal to $C(S_{g,n})$.

The proof of Theorem A is divided into two cases: the closed surface case (see Theorem 2.1 in Section 2) and the punctured surface case (see Theorem 3.3 in Section 3). We begin by defining a particular set of curves (based on the rigid set introduced in \cite{1}) that we call the principal set, and use a Humphries-Lickorish generating set of $\text{Mod}(S_{g,n})$ to see that the positive and negative translations of the principal set are contained in some rigid expansion of it (the principal set); afterwards, the iterated use of this result allows us to see that most topological types of curves are in some rigid expansion, while the rest of the topological types are uniquely determined by finite sets of curves from the previous cases.

Afterwards, in Section 4 we reintroduce the rigid set of \cite{1}, denoted by $\mathcal{X}(S_{g,n})$. Note that while Birman, Broaddus and Menasco’s homological spheres in \cite{4} (which is a subset of $\mathcal{X}(S_{g,n})$ for $g \geq 1$ and $n \leq 1$) are not contained in the principal set of a closed surface, they are contained in their first rigid expansion. Then, we use Theorem A to obtain an analogous result for Aramayona and Leininger’s finite rigid set.

**Theorem B.** Let $S_{g,n}$ be an orientable surface of genus $g \geq 3$, $n \geq 0$ punctures and empty boundary. Then $\bigcup_{i \in \mathbb{N}} \mathcal{X}(S_{g,n})^{i} = C(S_{g,n})$.

We must remark that this work is the published version of the first two chapters of the author’s Ph.D. thesis, and as was mentioned before these results are used to obtain new results on simplicial maps of different graphs. In particular we use these results in \cite{9} to prove that under certain conditions on the surfaces, all edge-preserving maps between a priori different curve graphs are actually induced by homeomorphisms between the underlying surfaces.

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1 Preliminaries

We suppose $S_{g,n}$ is an orientable surface of finite topological type with empty boundary, genus $g \geq 3$ and $n$ punctures. The mapping class group of $S_{g,n}$, denoted by $\text{Mod}(S_{g,n})$, is the group of isotopy classes of orientation preserving self-homeomorphisms of $S_{g,n}$; the extended mapping class of $S_{g,n}$, denoted by $\text{Mod}^{*}(S_{g,n})$, is the group of isotopy classes of all self-homeomorphisms of $S_{g,n}$. Note that $\text{Mod}(S_{g,n})$ is an index 2 subgroup of $\text{Mod}^{*}(S_{g,n})$. 

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A curve $\alpha$ is the topological embedding of the unit circle into the surface. We often abuse notation and call “curve” the embedding, its image on $S_{g,n}$ or its isotopy class. The context makes clear which use we mean.

A curve is essential if it is neither null-homotopic nor homotopic to the boundary curve of a neighbourhood of a puncture.

The (geometric) intersection number of two (isotopy classes of) curves $\alpha$ and $\beta$ is defined as follows:

$$i(\alpha, \beta) := \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$  

Let $\alpha$ and $\beta$ be two curves on $S_{g,n}$. As a convention for this work, we say $\alpha$ and $\beta$ are disjoint if $i(\alpha, \beta) = 0$ and $\alpha \neq \beta$.

Under the conditions on $S_{g,n}$ imposed above, we define the curve graph of $S_{g,n}$, denoted by $\mathcal{C}(S_{g,n})$, as the simplicial graph whose vertices are the isotopy classes of essential curves on $S_{g,n}$, and two vertices span an edge if the corresponding curves are disjoint.

Let $\beta$ be an essential curve on $S_{g,n}$ and $B$ a set of curves on $S_{g,n}$. We say $\beta$ is uniquely determined by $B$, denoted $\beta = \langle B \rangle$, if $\beta$ is the unique essential curve on $S_{g,n}$ that is disjoint from every element in $B$, i.e.

$$\{\beta\} = \bigcap_{\gamma \in B} \mathcal{A}(\gamma),$$

where $\mathcal{A}(\gamma)$ denotes the link of $\gamma$ in $\mathcal{C}(S_{g,n})$.

Let $Y \subset \mathcal{C}(S_{g,n})$; the first rigid expansion of $Y$ is defined as

$$Y^1 := Y \cup \{\beta : \beta = \langle B \rangle, B \subset Y\};$$

we also define $Y^0 = Y$ and, inductively, $Y^k = (Y^{k-1})^1$.

## 2 Closed surface case

In this section, we suppose that $S$ is a closed surface of genus $g \geq 3$. This section is divided as follows: Subsection 2.1 gives some definitions, fixes the principal set, states the main result of the section, and gives the proof of said result pending the proof of a technical lemma; Subsections 2.2, 2.3, and 2.4 give the proofs of the claims for the technical lemma.

### 2.1 Statement and proof of Theorem 2.1

Let $k \in \mathbb{Z}^+$ and $C = \{\gamma_0, \ldots, \gamma_k\}$ be an ordered set of $k + 1$ curves in $S$. It is called a chain of length $k + 1$ if $i(\gamma_i, \gamma_{i+1}) = 1$ for $0 \leq i \leq k - 1$, and $\gamma_i$ is disjoint from $\gamma_j$ for $|i - j| > 1$. On the other hand, $C$ is called a closed chain of length $k + 1$ if $i(\gamma_i, \gamma_{i+1}) = 1$ for $0 \leq i \leq k$ modulo $k + 1$, and $\gamma_i$ is disjoint from $\gamma_j$ for $|i - j| > 1$ (modulo $k + 1$); a closed chain is maximal if it has length $2g + 2$. A subchain is an ordered subset of either a chain or a closed chain which is itself a chain, and its length is its cardinality.

Recalling that $k \geq 1$, note that if $C$ is a chain (or a subchain), then every element of $C$ is a nonseparating curve. Also, if $C$ has odd length, a closed regular neighbourhood $N(C)$ has two boundary components; we can call these curves the bounding pair associated to $C$.

Let $\mathcal{C} = \{\alpha_0, \ldots, \alpha_{2g+1}\}$ be the closed chain in $S$ depicted in Figure 1. Observe it is a maximal closed chain, and given any other maximal closed chain $C$ there exists an element of $\text{Mod}(S)$ that maps $C$ to $\mathcal{C}$ (see [5]).

We define the set $\mathcal{B}$ as the union of the bounding pairs associated to the subchains of odd length of $\mathcal{C}$.

Now we are able to state the main result for the closed surface case.
Figure 1: The set $\mathcal{C} = \{\alpha_0, \ldots, \alpha_{2g+1}\}$ and the curve $\zeta$ – one of the curves of the bounding pair associated to $\{\alpha_0, \alpha_1, \alpha_2\}$.

![Diagram of curves and notation](image)

Figure 2: The curves $\alpha$ and $\beta$ in black, with the curve $\tau_\alpha(\beta)$ in blue.

Theorem 2.1. Let $S$ be an orientable closed surface with genus $g \geq 3$, and let $\mathcal{C}$ and $\mathcal{B}$ be defined as above. Then $\bigcup_{i \in \mathbb{N}} (\mathcal{C} \cup \mathcal{B}) = \mathcal{C}(S)$.

The idea of the proof is as follows. Let $\zeta$ be the curve depicted in Figure 1, we define the set $\mathcal{G} = \{\alpha_0, \ldots, \alpha_{2g-1}, \zeta\}$. Note that Humphries and Lickorish proved that the Dehn twists along the elements of $\mathcal{G}$ generate $\text{Mod}(S)$ (see [10]). Also recall that an essential curve $\alpha$ on $S$ is separating if $S \setminus \{\alpha\}$ is disconnected, and it is called nonseparating otherwise.

First we prove that the image of $\mathcal{C} \cup \mathcal{B}$ under the Dehn twist along any element of $\mathcal{G}$ is contained in $(\mathcal{C} \cup \mathcal{B})$. Afterwards we note that any nonseparating curve in $\mathcal{C}(S)$ is the image of an element in $\mathcal{G}$ under an orientation preserving mapping class, and thus is contained in $(\mathcal{C} \cup \mathcal{B})^k$ for some $k$. Finally, we show that every separating curve in $\mathcal{C}(S)$ is uniquely determined by some finite subset of nonseparating curves, and thus also lies in $(\mathcal{C} \cup \mathcal{B})^k$ for some $k$.

Before passing to the proof of Theorem 2.1 we give the necessary notation and state a technical lemma.

Let $\alpha, \beta \in \mathcal{C}(S)$ and $A, B \subset \mathcal{C}(S)$. We denote by $\tau_\alpha(\beta)$ the right Dehn twist of $\beta$ along $\alpha$, $\tau_\alpha(B) = \bigcup_{\gamma \in B} \{\tau_\alpha(\gamma)\}$ and $\tau_A(B) = \bigcup_{\gamma \in A} \tau_\gamma(B)$. Observe that if $\alpha$ and $\beta$ are such that $i(\alpha, \beta) = 1$, we have:

$$\tau_\alpha(\beta) = \tau_{\beta}^{-1}(\alpha) \quad \tau_\alpha^{-1}(\beta) = \tau_\beta(\alpha); \quad (1)$$

See Proposition 3.9 in [2] or Figure 2 for a proof.

The key technical lemma for the proof of Theorem 2.1 is the following.

Lemma 2.2. $\tau_{\mathcal{G}}^{-1}((\mathcal{C} \cup \mathcal{B}) \subset (\mathcal{C} \cup \mathcal{B})^4$.

Note that, as was mentioned in the Introduction, if $\beta = \langle B \rangle$ we have that for any $h \in \text{Mod}^\ast(S)$, $h(\beta) = \langle h(B) \rangle$. This allows the iterated use of the lemma.

Assuming this lemma (which we prove in the following subsections) we embark on the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\gamma$ be a nonseparating curve and $\alpha \in \mathcal{G}$. There exists an orientation preserving mapping class $h \in \text{Mod}(S)$ such that $\gamma = h(\alpha)$. As was mentioned above,
the Dehn twists along the elements of $\mathcal{G}$ generate Mod($S$). Thus, for some $\gamma_1, \ldots, \gamma_m \in \mathcal{G}$ and some $n_1, \ldots, n_m \in \mathbb{Z}$ we have that $\gamma = \tau_{\gamma_1}^{n_1} \circ \cdots \circ \tau_{\gamma_m}^{n_m}(\alpha)$. By an inductive use of Lemma 2.2 we have that $\gamma \in (\mathcal{C} \cup \mathcal{B})^4|n_1|+\cdots+|n_m|)$. Hence, every nonseparating curve is an element of $\bigcup_{i \in \mathbb{N}} (\mathcal{C} \cup \mathcal{B})^i$.

Let $\gamma$ be a separating curve. Note that up to homeomorphism there exist only a finite number of separating curves. Moreover, as can be seen in Figure 3, every such curve can be uniquely determined by a pair of chains of cardinalities $2g'$ and $2g''$, where $g'$ and $g''$ are the genera of the connected components of $S \setminus \{\gamma\}$. Then, there exist chains $C_1$ and $C_2$ such that $\gamma = \langle C_1 \cup C_2 \rangle$. By the previous case, $C_1 \cup C_2 \subset (\mathcal{C} \cup \mathcal{B})^k$ for some $k \in \mathbb{N}$; thus $\gamma \in (\mathcal{C} \cup \mathcal{B})^{k+1}$. Therefore $C(S) = \bigcup_{i \in \mathbb{N}} (\mathcal{C} \cup \mathcal{B})^i$. □

As stated before, the rest of this section is dedicated to the proof of Lemma 2.2 which (using that $\zeta \in \mathcal{B}$) is divided as follows:

**Claim 1:** $\tau_{\zeta}^{\pm 1}(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B})^2$.

**Claim 2:** $\tau_{\zeta}^{\pm 1}(\mathcal{B}) \cup \tau_{\zeta}^{\pm 1}(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B})^3$.

**Claim 3:** $\tau_{\zeta}^{\pm 1}(\mathcal{B}) \subset (\mathcal{C} \cup \mathcal{B})^4$.

Note that since we only need to prove the lemma for Dehn twists along elements of $\mathcal{G}$, we only need to prove Claim 3 for $\zeta$.

Before going further, we introduce the notation used in the proofs of said claims.

Let $\mathcal{C}' = \{\gamma_0, \ldots, \gamma_{2g+1}\}$ be a maximal closed chain in $S$. The sets $\mathcal{C}'_o = \{\gamma_i \in \mathcal{C}' : i \text{ is odd}\}$ and $\mathcal{C}'_e = \{\gamma_i \in \mathcal{C}' : i \text{ is even}\}$, satisfy that $S \setminus \mathcal{C}'_o$ and $S \setminus \mathcal{C}'_e$ have two connected components, each homeomorphic to $S_0, g+1$. We denote by $S^+_e$ and $S^-_e$ the connected components of $S \setminus \mathcal{C}'_e$, and by $S^+_o$ and $S^-_o$ the connected components of $S \setminus \mathcal{C}'_o$. See Figure 4 for an example. Let $1 \leq k \leq g - 1$, and $\{\gamma_i, \ldots, \gamma_{i+2k}\}$ (with the indices modulo $2g + 2$) be a subchain of $\mathcal{C}'$. We denote by $[\gamma_i, \ldots, \gamma_{i+2k}]^+$ the curve in the associated bounding pair that is contained in either $S^+_o$ or $S^+_e$. Analogously, we denote by $[\gamma_i, \ldots, \gamma_{i+2k}]^-$ the curve in the associated bounding pair contained in either $S^-_o$ or $S^-_e$.

![Figure 4: The set $\mathcal{C}'_e$ and the corresponding $S^+_e$ and $S^-_e$.](image-url)
Remark 2.3. Note that according to this notation, \( \zeta = [\alpha_0, \alpha_1, \alpha_2]^- \).

We partition the set \( \mathcal{B} \) into \( \mathcal{B}^+, \mathcal{B}^-, \mathcal{B}^e_+ \) and \( \mathcal{B}^e_- \), depending on whether \( \beta \in \mathcal{B} \) is contained in \( S_0^+, S_0^-, S_e^+ \) or \( S_e^- \) respectively. We write \( \mathcal{B}^+ = \mathcal{B}^e_+ \cup \mathcal{B}^e_- \) and \( \mathcal{B}^- = \mathcal{B}^e_- \cup \mathcal{B}^e_+ \).

2.2 Proof of Claim 1: \( \tau_{\alpha_2}^{\pm 1}(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B})^2 \)

To prove the claim, we start with a pair of particular curves and we show that is enough to prove the claim via the action of a particular subgroup of \( \text{Mod}(S) \).

The following lemma is heavily based on Lemma 5.3 in [2]. However, its proof has been modified to emphasize the arguments that are used to obtain a more general result which is repeatedly used in the following subsections.

**Lemma 2.4.** \( \tau_{\alpha_2}^{\pm 1}(\alpha_2 g^-) \in (\mathcal{C} \cup \mathcal{B})^2 \).

**Proof.** Using the set

\[
C_+ = \{\alpha_{2g+1}, \alpha_1, \alpha_2, \ldots, \alpha_{2g-2}, \alpha_{2g-1}, [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}]^+, [\alpha_2, \ldots, \alpha_{2g-2}]^+\},
\]

we obtain the curve \( \gamma_+ \in (\mathcal{C} \cup \mathcal{B})^1 \) as the curve uniquely determined by \( C_+ \), see Figure 5. Then, letting

\[
C'_+ = \{\alpha_0, \ldots, \alpha_{2g-3}, [\alpha_{2g-2}, \alpha_{2g-1}, \alpha_{2g}]^+, [\alpha_{2g-2}, \alpha_{2g-1}, \alpha_{2g}]^-, \gamma_+\},
\]

we have that \( \tau_{\alpha_2}(\alpha_2 g^-) = (C'_+) \in (\mathcal{C} \cup \mathcal{B})^2 \).

Analogously we have that \( \tau_{\alpha_2}^{-1}(\alpha_2 g^-) = (C'_-) \in (\mathcal{C} \cup \mathcal{B})^2 \). See [3] for more details. Therefore \( \tau_{\alpha_2}^{\pm 1}(\alpha_2 g^-) \in (\mathcal{C} \cup \mathcal{B})^2 \). \( \square \)

Figure 5: Above, the sets \( C_+ \) (left) and \( C'_+ \) (right). Below, the curves \( \gamma_+ \) (left) and \( \tau_{\alpha_2}(\alpha_2 g^-) \) (right), uniquely determined by the sets \( C_+ \) and \( C'_+ \) respectively.

Let \( H_{\mathcal{C}} \subset \text{Mod}^+ (S) \) be the setwise stabilizer of \( \mathcal{C} \).

**Remark 2.5.** Observe that \( H_{\mathcal{C}}(\mathcal{B}) = \mathcal{B} \), and for all \( h \in H_{\mathcal{C}} \) we have that \( h(\mathcal{B}^+), h(\mathcal{B}^-) \in \{\mathcal{B}^+, \mathcal{B}^e\} \). Moreover, \( H_{\mathcal{C}} \) can be partitioned as \( H_{\mathcal{C}} = H_{\mathcal{C}}^+ \cup \iota H_{\mathcal{C}}^- \), where \( H_{\mathcal{C}}^+ \) is the subgroup such that \( H_{\mathcal{C}}^+(\mathcal{B}^+) = \mathcal{B}^+ \), and \( \iota \) the hyperelliptic involution (which exchanges \( S_0^+ \) (resp. \( S_e^+ \)) and \( S_0^- \) (resp. \( S_e^- \))). Also note that \( H_{\mathcal{C}}^+ \) acts transitively on \( \mathcal{C} \).

**Lemma 2.6.** Let \( h \in H_{\mathcal{C}}, k \in \mathbb{N} \) and \( \gamma \in (\mathcal{C} \cup \mathcal{B})^k \). Then \( h(\gamma) \in (\mathcal{C} \cup \mathcal{B})^k \).

**Proof.** We proceed by induction; if \( k = 0 \), we obtain the result by construction. If \( k \geq 1 \) let \( \gamma \in (\mathcal{C} \cup \mathcal{B})^{k}(\mathcal{C} \cup \mathcal{B})^{k-1} \), as such \( \gamma = (C_0) \) with \( C_0 \subset (\mathcal{C} \cup \mathcal{B})^{k-1} \); then \( h(\gamma) = \langle h(C_0) \rangle \), but by induction \( h(C_0) \subset (\mathcal{C} \cup \mathcal{B})^{k-1} \), thus \( h(\gamma) \in (\mathcal{C} \cup \mathcal{B})^k \). \( \square \)
Armed with Lemma 2.6, we are ready to prove Claim 1.

**Proof of Claim 1.** Let \( \alpha_i, \alpha_j \in \mathcal{C} \) with \( i \neq j \). We want to prove that \( \tau_{\alpha_i}^{\pm 1}(\alpha_j) \in (\mathcal{C} \cup \mathcal{B})^2 \). If \( |i - j| > 1 \) (modulo \( 2g + 2 \)), then the curves are disjoint and we have that \( \tau_{\alpha_i}^{\pm 1}(\alpha_j) = \alpha_j \in \mathcal{C} \). Suppose then that \( |i - j| = 1 \). There exists an element \( h \in H^{\pm}_{\mathcal{C}} \) such that either \( h(\alpha_{2g}) = \alpha_i \) and \( h(\alpha_{2g-1}) = \alpha_j \) if \( i = j + 1 \), or \( h(\alpha_{2g}) = \alpha_j \) and \( h(\alpha_{2g-1}) = \alpha_i \) if \( j = i + 1 \). Repeating the procedure of the proof of Lemma 2.4 precomposing by \( h \) and using Lemma 2.6 we obtain that \( \tau_{\alpha_i}^{\pm 1}(\alpha_j) \in (\mathcal{C} \cup \mathcal{B})^2 \). \( \square \)

This finishes the proof of Claim 1. However, the proofs of Lemma 2.4 and Claim 1 give us a slightly more general result, which is often used in the rest of this section. Its objective is to reduce the problems posed in the following claims when finding convenient maximal closed chains and showing that particular curves dependent on said chains are uniquely determined by elements in the expansions of \( \mathcal{C} \cup \mathcal{B} \).

**Lemma 2.7.** Let \( \{\gamma_0, \ldots, \gamma_{2g+1}\} \) be a maximal closed chain in \( S \) that is contained in \( (\mathcal{C} \cup \mathcal{B})^k \) for some \( k \in \mathbb{N} \). If \( \{\gamma_{2g-3}, \gamma_{2g-2}, \gamma_{2g-1}\}, \{\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}\} \in (\mathcal{C} \cup \mathcal{B})^{k+1} \) and \( \{\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}\} \in (\mathcal{C} \cup \mathcal{B})^{k+2} \), then \( \tau_{\gamma_{2g}}^{\pm 1}(\gamma_{2g-1}) \in (\mathcal{C} \cup \mathcal{B})^{k+2} \). Moreover, \( \tau_{f(\gamma_{2g})}^{\pm 1}(f(\gamma_{2g-1})) \in (\mathcal{C} \cup \mathcal{B})^{k+3} \) for any \( f \in H^{\pm}_{\mathcal{C}} \).

**Proof.** Given that up to the action of \( \text{Mod}(S) \), \( \mathcal{C} \) is the only maximal closed chain, there exists \( h \in \text{Mod}(S) \) such that \( h(\alpha_i) = \gamma_i \) for \( i \in \{0, \ldots, 2g+1\} \). Then, we repeat the procedure of the proof of Lemma 2.4 precomposing by \( h \) and using Lemma 2.6 getting \( \tau_{\gamma_{2g}}^{\pm 1}(\gamma_{2g-1}) = \tau_{h(\alpha_{2g-1})}^{\pm 1}(h(\alpha_{2g-1})) \in (\mathcal{C} \cup \mathcal{B})^{k+3} \).

Let \( f \in H^{\pm}_{\mathcal{C}} \). Using Lemma 2.6 we can apply the result above to \( fh(\mathcal{C}) \) and we get that \( \tau_{f(\gamma_{2g})}^{\pm 1}(f(\gamma_{2g-1})) \in (\mathcal{C} \cup \mathcal{B})^{k+3} \). \( \square \)

**2.3 Proof of Claim 2:** \( \tau_{\mathcal{C}}^{\pm 1}(\mathcal{B}) \cup \tau_{\mathcal{B}}^{\pm 1}(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B})^3 \)

As in Subsection 2.2, we first note that letting \( \alpha \in \mathcal{C} \) and \( \beta \in \mathcal{B} \), if \( \alpha \) and \( \beta \) are disjoint, there is nothing to prove and so we assume \( i(\alpha, \beta) \neq 0 \). By construction we then have that \( i(\alpha, \beta) = 1 \). In part 1, we first establish the claim for a particular family of maximal closed chains that verify the conditions of Lemma 2.7, proving that \( \tau_{[\alpha_0, \ldots, \alpha_{2l}]}^{\pm 1}(\alpha_{2l+1}) \in (\mathcal{C} \cup \mathcal{B})^3 \) for all \( 1 \leq l \leq g - 2 \); then, via the action of \( H^{\pm}_{\mathcal{C}} \), we prove that \( \tau_{\mathcal{C}}^{\pm 1}(\mathcal{C} \cup \mathcal{B}^{-}) \subset (\mathcal{C} \cup \mathcal{B})^3 \). In part 2, we finish the proof via the action of the hyperelliptic involution \( \iota : S \to S \) mentioned in Remark 2.5.

**Part 1:** Let \( 1 \leq l \leq g - 2 \). We define the following maximal closed chain:

\[ \mathcal{C}_l = \{\alpha_1, \alpha_2, \ldots, \alpha_{2l}, \alpha_{2l+1}, [\alpha_0, \ldots, \alpha_{2l}]^-, \alpha_{2g+1}, \alpha_{2g}, \alpha_{2g-1}, \ldots, \alpha_{2l+3}, [\alpha_2, \ldots, \alpha_{2l+2}]^+\}. \]

We refer the reader to Figure 6 for an example of such a maximal closed chain.

We now prove, using Lemma 2.7, that \( \tau_{\mathcal{C}}^{\pm 1}(\mathcal{C} \cup \mathcal{B}^-) \subset (\mathcal{C} \cup \mathcal{B})^3 \).

In order to facilitate the use of Lemma 2.7, we cyclically reorder the elements of \( \mathcal{C}_l \) as follows:

\[ \gamma_0 = \alpha_{2g}, \quad \gamma_1 = \alpha_{2g-1}, \quad \ldots, \quad \gamma_{2g-1} = \alpha_{2l+1}, \quad \gamma_{2g} = [\alpha_0, \ldots, \alpha_{2l}]^-, \quad \text{and} \quad \gamma_{2g+1} = \alpha_{2g+1}. \]

Again, see Figure 6 for an example.

By inspection we can verify that \( \mathcal{C}_l \) satisfies the conditions of Lemma 2.7 and thus \( \tau_{\gamma_{2g}}^{\pm 1}(\gamma_{2g-1}) = \tau_{[\alpha_0, \ldots, \alpha_{2l}]}^{\pm 1}(\alpha_{2l+1}) \in (\mathcal{C} \cup \mathcal{B})^3 \), however for the sake of completeness we give a detailed account of which set of curves uniquely determines the needed curves.

For \( 1 \leq l \leq g - 2 \) we have:
In the case of \( l = 1 \) we have:

\[
\begin{align*}
[\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^+ &= [\alpha_0, \alpha_1, \alpha_2]^-
s
[\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^- &= \langle \alpha_2, \alpha_3, [\alpha_0, \alpha_1, \alpha_2]^- \rangle,
\end{align*}
\]

In the cases with \( l > 1 \) we have:

\[
\begin{align*}
[\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^+ &= [\alpha_0, \alpha_{2l-1}, \alpha_{2l}]^+
[\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^- &= \langle \alpha_2, \alpha_{2l+1}, \alpha_0, \alpha_{2l}^- \rangle,
\end{align*}
\]

Letting \( l \) vary from 1 to \( g-2 \), and applying Lemma 2.7, we have that \( \tau_{g-1}^{\pm 1}(\gamma_{2g-1}) = \tau_{[\alpha_0, ..., \alpha_{2l}]}^{\pm 1}(\alpha_{2l+1}) \in (\mathcal{C} \cup \mathcal{B})^3 \) for all \( 1 \leq l \leq g-2 \).

Now, using the fact that \( H_\mathcal{E}^+ \subset \text{Mod}^*(S) \) acts transitively on \( \mathcal{C} \), we have as a consequence that it also acts transitively on each of the sets \( \{[\alpha_i, ..., \alpha_{i+2}]^- : 0 \leq i \leq 2g + 1 \} \) for \( 1 \leq l \leq g-2 \). This implies that given \( [\alpha_i, ..., \alpha_{i+2}]^- \in \mathcal{B}^- \), there exists \( h \in H_\mathcal{E}^+ \) such that \( h([\alpha_i, ..., \alpha_{i+2}]^-) = [\alpha_{i+1}, \alpha_{i+2}]^- \). Thus, by Lemmas 2.6 and 2.7, we have then that \( \tau_{\mathcal{B}}^{\pm 1}(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B})^3 \), and by Equation 1 we obtain that \( \tau_{\mathcal{B}}^{\pm 1}(\mathcal{C}) \cup \tau_{\mathcal{B}}^{\pm 1}(\mathcal{B}^-) \subset (\mathcal{C} \cup \mathcal{B})^3 \).

### Part 2

To prove the rest of the cases, recall that the hyperelliptic involution \( i \) is an element of \( \text{stabilizer}(\mathcal{C}) \) and \( i([\alpha_i, ..., \alpha_{i+2k}]^+) = [\alpha_i, ..., \alpha_{i+2k}]^- \) for all \( \{\alpha_i, ..., \alpha_{i+2k}\} \subset \mathcal{C} \). Given that (as was shown in part 1) for all \( 1 \leq l \leq g-2 \), the families of maximal closed chains \( H_\mathcal{E}^+(\mathcal{C}) \) satisfy the conditions of Lemma 2.7, we have that Lemma 2.6 yields the same is true for the maximal closed chains \( iH_\mathcal{E}^+(\mathcal{C}) \). Therefore

\[
\tau_{\mathcal{C}}^{\pm 1}(iH_\mathcal{E}^+(\mathcal{C})) \cup \tau_{\mathcal{B}^-}^{\pm 1}(iH_\mathcal{E}^+(\mathcal{C})) = \tau_{\mathcal{C}}^{\pm 1}(\mathcal{C}) \cup \tau_{\mathcal{B}^-}^{\pm 1}(\mathcal{B}^-) \subset (\mathcal{C} \cup \mathcal{B})^3,
\]

as desired.

### 2.4 Proof of Claim 3: \( \tau_{\mathcal{C}}^{\pm 1}(\mathcal{B}) \subset (\mathcal{C} \cup \mathcal{B})^4 \)

Recall \( \zeta = [\alpha_0, \alpha_1, \alpha_2]^- \) and let \( \gamma \in \mathcal{B} \). In the cases where \( \zeta \) is disjoint from \( \gamma \), we have \( \tau_{\mathcal{C}}^{\pm 1}(\gamma) = \gamma \in \mathcal{C} \cup \mathcal{B} \). So we assume that \( i(\gamma, \zeta) \neq 0 \), which by construction implies

\[
\gamma \in \{[\alpha_1, ..., \alpha_{2k+1}]^\pm, [\alpha_3, ..., \alpha_{2k+1}]^\pm, [\alpha_2, ..., \alpha_{2l}]^- : 1 \leq k \leq g-1, 2 \leq l \leq g-1 \}.
\]
In these cases there exist subsets of $C_0 \subset \mathcal{C}$ and $\{\beta_0\} \subset \mathcal{B}$, such that $\gamma = \langle C_0 \cup \{\beta_0\} \rangle$ and $\beta_0$ is disjoint from $\zeta$. Note that $\tau^{\pm 1}_\zeta(C_0) \subset \langle \mathcal{C} \cup \mathcal{B} \rangle^3$ by claim 2, and $\tau^{\pm 1}_\zeta(\beta_0) = \beta_0 \in \mathcal{C} \cup \mathcal{B}$ by construction. Therefore

$$\tau^{\pm 1}_\zeta(\gamma) = \tau^{\pm 1}_\zeta(\langle C_0 \cup \beta_0 \rangle) = (\tau^{\pm 1}_\zeta(C_0) \cup \tau^{\pm 1}_\zeta(\beta_0)) \in \langle \mathcal{C} \cup \mathcal{B} \rangle^4.$$ 

For a more detailed account on $C_0$ and $\beta_0$ see [8]. For some examples see Figures 7 and 8.

![Figure 7](image1.png)  
Figure 7: An example of $\gamma = [\alpha_1, \ldots, \alpha_{2k+1}]^+$, and the corresponding $C_0$ and $\beta_0$.

![Figure 8](image2.png)  
Figure 8: An example of $\gamma = [\alpha_2, \ldots, \alpha_{2k}]^-$, and the corresponding $C_0$ and $\beta_0$.

### 3 Exhaustion of $\mathcal{C}(S)$ for punctured surfaces

In this section, we suppose that $S = S_{g,n}$ with genus $g \geq 3$ and $n \geq 1$ punctures. In Subsection 3.1 we fix the principal set and give notation; in addition to a principal set of curves analogous to $\mathcal{C}$ and $\mathcal{B}$, we introduce some auxiliary curves to aid the exposition in Subsection 3.2, we also prove they are in specific expansions of the principal set, and state and prove several technical propositions; in Subsection 3.3 we prove the main theorem, pending the proof of a technical lemma; Subsections 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9 give the proofs of the claims for the technical lemma.
3.1 Statement of Theorem 3.3

The idea of the proof of the analogous result to Theorem 2.1 is the same as in the closed surface case. Using arguments similar to those of Theorem 2.1 we show that every nonseparating curve is in some expansion and then use that to prove the same for the separating curves.

However, the presence of punctures induce several small but important changes, both in the principal set of curves that is used (which while analogous to the closed case, is not as symmetric and thus induces changes in the proofs), and in the manner auxiliary curves are used. For this reason, we first introduce the sets $\mathcal{C}$ and $\mathcal{B}_0$ whose union is the principal set and then we state in detail the theorem to prove.

Let $\mathcal{C}_0 = \{\alpha_0, \ldots, \alpha_{2g+1}\}$ be the chain depicted in Figure 9 and $\mathcal{C}_f = \{\alpha_0^1, \alpha_1, \ldots, \alpha_0^n\}$ be the multicurve also depicted in Figure 9 and $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_f$.

Figure 9: $\mathcal{C}_0$ in blue and $\mathcal{C}_f$ in red for $S_{0,4}$.

\begin{center}
\begin{tikzpicture}
\end{tikzpicture}
\end{center}

**Remark 3.1.** Note that for $i \in \{0, \ldots, n - 1\}$, we have that $S \setminus \{\alpha_0^i, \alpha_0^{i+1}\}$ has a connected component homeomorphic to a thrice punctured sphere. Also note that for each $j \in \{0, \ldots, n\}$, the set $C_j = \{\alpha_0^0, \alpha_1, \ldots, \alpha_{2g+1}\}$ is a maximal closed chain.

Adapting the notation used in the previous section, for $i \in \{0, \ldots, n\}$, let us consider the closed chain $C_i \subset \mathcal{C}$. Then we denote the curve $\alpha_0^i$ by $\alpha_0$ to simplify notation when it is understood that $\alpha_0 \in C_i$. As such $C_i$ has the subsets: $\mathcal{C}_{i(o)} = \{\alpha_j \in C_i : j \text{ is odd}\}$ and $\mathcal{C}_{i(e)} = \{\alpha_j \in C_i : j \text{ is even}\}$. These subsets are such that:

- $S \setminus \mathcal{C}_{i(e)}$ has two connected components, $S^+_{i(e)} = S_{0,i+g+1}$ and $S^-_{i(e)} = S_{0,n-i+g+1}$.
- $S \setminus \mathcal{C}_{i(o)}$ has two connected components, $S^+_{i(o)} = S_{0,n+g+1}$ and $S^-_{i(o)} = S_{0,g+1}$.

Recalling that the subindices are modulo $2g + 2$, we denote by $[\alpha_j, \ldots, \alpha_{j+2k}]^+$ for $0 < k < g - 1$, the boundary component of a closed regular neighbourhood $N(\{\alpha_j, \ldots, \alpha_{j+2k}\})$, that is contained in either $S^+_{i(o)}$ or in $S^+_{i(e)}$. Analogously, $[\alpha_j, \ldots, \alpha_{j+2k}]^-$ denotes the boundary component of a closed regular neighbourhood $N(\{\alpha_j, \ldots, \alpha_{j+2k}\})$, that is contained in either $S^-_{i(o)}$ or in $S^-_{i(e)}$.

**Remark 3.2.** Note that this notation is the same as in the closed surface case for the set $\mathcal{B}$; however, when $0 \in \{j, \ldots, j+2k\}$ (modulo $2g + 2$), the curves $[\alpha_j, \ldots, \alpha_{j+2k}]^\pm$ depend on the choice of $i \in \{0, \ldots, n\}$ (recall $\alpha_0$ stands for $\alpha_0^0$).

Let $J = \{2l, \ldots, 2(l+k)\}$, for some $1 \leq k \leq g - 1$, be a proper interval in the cyclic order modulo $2g + 2$. Let also $\beta_j^\pm = [\alpha_{2l}, \ldots, \alpha_{2l+k}]^\pm$ (with $\alpha_0 = \alpha_1^0$ when necessary). See Figure 10 for examples. We define

$\mathcal{B}_0 := \{\beta_j^\pm : J = \{2l, \ldots, 2(l+k)\}, 1 \leq k \leq g - 1\}$.
Now that we have the principal set of curves, we are able to state the punctured case version of Theorem A.

**Theorem 3.3.** Let $S$ be an orientable surface with genus $g \geq 3$ and $n \geq 1$ punctures; let also $\mathcal{C}$ and $\mathcal{B}_0$ be defined as above. Then $\bigcup_{i \in \mathbb{N}} (\mathcal{C} \cup \mathcal{B}_0)^i = \mathcal{C}(S)$.

### 3.2 Auxiliary curves

We need for the proof some auxiliary curves and some technical results.

For $0 \leq i \leq n - 2$, we define

$$
\epsilon^{i,i+2} := \langle \mathcal{C}_0 \cup (\mathcal{C}_j \setminus \{\alpha_0^{i+1}\}) \rangle;
$$

note that $\epsilon^{i,i+2} \in \mathcal{C}^1$; this can be seen using Figure 9 and removing $\alpha_0^{i+1}$ for the chosen $i$.

For $0 \leq i < j \leq n$ with $j - i > 2$, we define the curve:

$$
\epsilon^{i,j} := \langle \mathcal{C}_0 \cup (\mathcal{C}_j \setminus \{\alpha_0^k : i < k < j\}) \cup \{\epsilon^{k,k+2} : i \leq k \leq j - 2\} \rangle;
$$

note that $\epsilon^{i,j} \in \mathcal{C}^2$.

Then, we define the set:

$$
D := \{\epsilon^{i,j} : j - i > 1\} \subset \mathcal{C}^2.
$$

We now expand this set. Let $0 \leq i \leq j \leq n$. We define

$$
C^{i,j} := \{\alpha_0^0, \ldots, \alpha_0^0, \alpha_1, \ldots, \alpha_{2g+1}\}, \quad \text{and} \quad E^{i,j} := \{\epsilon^{k,l} \in D : 0 \leq k < l \leq i, \quad \text{or} \quad j \leq k < l \leq n\}.
$$

Note that $C^{i,j} = C_i$, and $E^{i,j} = D \setminus \{\epsilon^{k,l} : (\exists \gamma \in C^{i,j}) \quad i(\epsilon^{k,l}, \gamma) \neq 0\}$.

Let $k \in \{1, \ldots, g\}$, and define (see Figure 11)

$$
\epsilon^{(i,j)}_k := \langle (C^{i,j} \setminus \{\alpha_0^k\}) \cup E^{i,j} \rangle;
$$

for $k = 0$ we take $\epsilon^{(i,j)}_0 := \epsilon^{i,j}$, and for $k = -1$ we take $\epsilon^{(i,j)}_{-1} = \epsilon^{(i,j)}$. Finally we define

$$
\mathcal{E} := D \cup \{\epsilon^{(i,j)}_k : 0 \leq i \leq j \leq n, \ k \in \{1, \ldots, g\}\}
$$

Note that $\mathcal{E} \subset \mathcal{C}^3$ by construction. See Figure 12 for examples of curves in $\mathcal{E}$.

**Remark 3.4.** Note that the sets $D$ and $\mathcal{E}$ are only defined when $n \geq 2$. For this reason, if $n = 1$ we define $D = \mathcal{E} = \emptyset$. 

![Figure 10: Examples of curves $\beta^+_{(2,3,4)}$ and $\beta^-_{(0,1,2,3,4)}$.](image.png)
Figure 11: The curve $\epsilon_k^{(i,j)} := \langle (C_i^j \{\alpha_k\}) \cup E_i^j \rangle$.

Figure 12: Examples of curves in $\mathcal{E}$.

Now, in the case where $1 \leq j \leq n - 1$. We define the following curves (see Figure 13 for examples):

$$\beta_{\{0,1,2\}}^{j,+} := \langle (\mathcal{C}_f \{\alpha_k : k < j\}) \cup (\mathcal{C}_0 \{\alpha_3, \alpha_{2g+1}\}) \cup E_{j,n} \cup \{\epsilon_k^{(1,n-1) : 2 \leq k \leq g}\};$$

$$\beta_{\{0,1,2\}}^{j,-} := \langle (\mathcal{C}_f \{\alpha_k : k > j\}) \cup (\mathcal{C}_0 \{\alpha_3, \alpha_{2g+1}\}) \cup E_{n,j} \cup \{\epsilon_k^{(1,n-1) : 2 \leq k \leq g}\};$$

in the case where $j = n$, we define (see Figure 14):

$$\beta_{\{0,1,2\}}^{n,+} := \langle (\mathcal{C}_0 \{\alpha_3, \alpha_{2g+1}\}) \cup \{\alpha_0^n\} \cup \emptyset \cup \{\beta^{+}_{\{4,\ldots,2g\}}\};$$

$$\beta_{\{0,1,2\}}^{n,-} := \beta_{\{4,\ldots,2g\}}^{-} \quad (\in \mathcal{B}_0).$$

Similarly, we define the following curves: in the case where $1 \leq j \leq n - 1$, we define (see Figure 15 for examples):

$$\beta_{\{2g,2g+1,0\}}^{j,+} := \langle (\mathcal{C}_0 \{\alpha_1, \alpha_{2g-1}\}) \cup (\mathcal{C}_f \{\alpha_k^0 : k < j\}) \cup E_{j,n} \cup \{\epsilon_k^{(1,n-1) : 1 \leq k \leq g-1}\};$$

$$\beta_{\{2g,2g+1,0\}}^{j,-} := \langle (\mathcal{C}_0 \{\alpha_1, \alpha_{2g-1}\}) \cup (\mathcal{C}_f \{\alpha_k^0 : k > j\}) \cup E_{n,j} \cup \{\epsilon_k^{(1,n-1) : 1 \leq k \leq g-1}\};$$

in the case where $j = n$, we define (see Figure 16 for examples):

$$\beta_{\{2g,2g+1,0\}}^{n,+} := \langle (\mathcal{C}_0 \{\alpha_1, \alpha_{2g-1}\}) \cup \{\alpha_0^g\} \cup \emptyset \cup \{\beta^{+}_{\{2,\ldots,2g-2\}}\}).$$
proof into two cases according to the parity of

**Proof.** Suppose $i = 1, 2, \ldots, n$.

Let $\beta^\pm_{\{0,1,2\}}$, $\beta^\pm_{\{0,1,2\}}$, and $\beta^\pm_{\{0,1,2\}}$ be as in the previous section. However, some of these missing curves are not needed for the proof of Theorem 3.3, some particular curves are needed for an idea of “translation” as in the previous section.

\[
\beta^\pm_{\{2g,2g+1,0\}} := \beta^\pm_{\{2g,2g+1,0\}}.
\]

Note that $\beta^\pm_{\{0,1,2\}}, \beta^\pm_{\{2g,2g+1,0\}} \in \mathcal{C}^4$ for all $1 \leq j \leq n-1$, and $\beta^\pm_{\{0,1,2\}}, \beta^\pm_{\{2g,2g+1,0\}} \in (\mathcal{C} \cup \mathcal{B})^4$. Then, we define the set

\[
\mathcal{B}_T := \{\beta^+_j, \beta^-_j : J \in \{\{0, 1, 2\}, \{2g, 2g + 1, 0\}\}, 1 \leq i \leq n\} \subset (\mathcal{C} \cup \mathcal{B})^4.
\]

The set $\mathcal{B}_0 \cup \mathcal{B}_T$ and the set $\mathcal{B}$ of the previous section are quite similar. However, $\mathcal{B}_0 \cup \mathcal{B}_T$ is not as symmetric as $\mathcal{B}$ and it has a sense of incompleteness, for example that we are not including the boundary components of regular neighbourhoods of chains of odd length whose first curve has odd index. While some of these missing curves are not needed for the proof of Theorem 3.3, some particular curves are needed for an idea of “translation” as in the previous section.

**Proposition 3.5.** Let $k \in \mathbb{Z}$. Then $\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)+} \in (\mathcal{C} \cup \mathcal{B}_0)^4$ for any choice of $i \in \{0, \ldots, n\}$ (with $a_0 = a_0^0$ when necessary).

**Proof.** Suppose $n \geq 2$. If $k = 0$, $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)+}] \in (\mathcal{C} \cup \mathcal{B}_0)^4$. We split the rest of the proof into two cases according to the parity of $k$.

If $k \neq 0$ is even, then $k = 2(t + 2)$ for $t \in \{-1, \ldots, g - 2\}$, and so $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)+}] = [\alpha_{2t+5}, \ldots, \alpha_{2t+1}]^+$ (recall the subindices are modulo $2g + 2$). Thus, if $i \in \{1, \ldots, n-1\}$ we get,

\[
[a_{2t+5}, \ldots, a_{2t+1}]^+ = ([a_{2t+5}, \ldots, a_{2t+1}] \cup \{a_{2t+3}\} \cup E^{k,i} \cup \{4(1,n-1), 4(1,n-1)} \in (\mathcal{C} \cup \mathcal{B}_0)^4,
\]
Figure 15: Examples of curves $\beta_{2g,2g+1,0}^1$ and $\beta_{2g,2g+1,0}^3$ in $S_{4,4}$.

Figure 16: The curve $\beta_{2g,2g+1,0}^4 := \langle (\mathcal{C}_0 \setminus \{\alpha_1, \alpha_{2g-1}\}) \cup \{\alpha_0^n\} \cup \mathcal{D} \cup \{\beta_{2g,2g-2}^\pm\} \rangle$.

see Figure 17; if $i \in \{0, n\}$ we get,

$$[\alpha_{2l+5}, \ldots, \alpha_{2l+1}]^+ = \langle \{\alpha_{2l+5}, \ldots, \alpha_{2l+1}\} \cup \{\alpha_{2l+3}\} \cup \mathcal{D} \rangle \in (\mathcal{C} \cup \mathcal{B}_0)^3.$$

If $k$ is odd, then $k = 2l - 1$ for some $0 \leq l \leq g$, and so $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+ = [\alpha_{2l}, \ldots, \alpha_{2l-4}]^+$ (recall the subindices are modulo $2g + 2$). Thus, if $i = 0$ we have that $[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ = \alpha_{2l-2}$; if $i = 1$ we have that $[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ \in \mathcal{B}_0$; if $i \in \{2, \ldots, n-1\}$ we have (see Figure 18):

$$[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ = \langle \{\alpha_{2l}, \ldots, \alpha_{2l-4}\} \cup \{\alpha_{2l-2}\} \cup \{\alpha_j : j > i\} \cup \{e_{l-1}^{(1-n-1)}\} \rangle \in (\mathcal{C} \cup \mathcal{B}_0)^4;$$

and finally, if $i = n$ we have

$$[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ = \langle \{\alpha_{2l}, \ldots, \alpha_{2l-4}\} \cup \{\alpha_{2l-2}\} \cup \mathcal{D} \rangle \in (\mathcal{C} \cup \mathcal{B}_0)^3.$$

Now, if $n = 1$, we can uniquely determine the curves $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+$ in the same way as above, recalling that in this instance $\mathcal{D} = \mathcal{C} = \emptyset$ and taking the cases $i = 0$ and $i = n$.

Therefore, for $i \in \{0, \ldots, n\}$, $k \in \mathbb{Z}$, $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+ \in (\mathcal{C} \cup \mathcal{B}_0)^4$.

Note that for $n = 1$, the curves $[\alpha_{k+1}, \ldots, \alpha_{k+(2g+1)}]^+$ are elements of $(\mathcal{C} \cup \mathcal{B}_0)^4$ for any $k \in \mathbb{Z}$ and any choice of $i \in \{0, 1\}$.

Now we have the following proposition.
Figure 17: \([\alpha_{2l+5}, \ldots, \alpha_{2l+1}]^+ = \langle \{\alpha_{2l+5}, \ldots, \alpha_{2l+1}\} \cup \{\alpha_{2l+3}\} \cup E^{l,1} \cup \{\ell^{(1,n-1)}, \ell'_{l+2}\} \rangle\) for \(l = 0\) and \(i = 2\), in \(S_{1,4}\).

**Proposition 3.6.** Let \(k \in \mathbb{Z}\). Then \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm \in (\mathcal{C} \cup \mathcal{B}_0)^6\) for any choice of \(i \in \{0, \ldots, n\}\) (with \(\alpha_0 = \alpha_0^i\) when necessary).

**Proof.** We start by proving that for \(k \in \mathbb{Z}\) even, \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm \in (\mathcal{C} \cup \mathcal{B}_0)^4\) (part 1); afterwards, we prove that for \(k\) odd, \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^- \in (\mathcal{C} \cup \mathcal{B}_0)^4\) (part 2); finally we prove that for \(k\) odd, \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^+ \in (\mathcal{C} \cup \mathcal{B}_0)^6\) (part 3).

**Part 1:** If \(k\) is even, \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm \in (\mathcal{C} \cup \mathcal{B}_0)^4\) with the exception of \([\alpha_{2g}, \alpha_{2g+1}, \alpha_0^0]^-\), and \([\alpha_0^0, \alpha_1, \alpha_2]^-\) (we can verify that \([\alpha_{2g}, \alpha_{2g+1}, \alpha_0^0]^+ = \beta^+_\{2, \ldots, 2g-2\}\) and \([\alpha_0^0, \alpha_1, \alpha_2]^+ = \beta^+_\{4, \ldots, 2g\}\). This happens since \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm\) is an element of \(\mathcal{B}_0\) or \(\mathcal{B}_T\), for \(k\) even with the aforementioned exceptions. So, for the first exception we have the following (see Figure 19):

\([\alpha_{2g}, \alpha_{2g+1}, \alpha_0^0]^- = \langle (C_0 \{\alpha_1, \alpha_{2g-1}\}) \cup \mathcal{D} \cup \{\beta^\pm_{\{2, \ldots, 2g-2\}}\} \rangle\).

And for the second exception we have (see Figure 20),

\([\alpha_0^0, \alpha_1, \alpha_2]^- = \langle (C_0 \{\alpha_3, \alpha_{2g+1}\}) \cup \mathcal{D} \cup \{\beta^\pm_{\{4, \ldots, 2g\}}\} \rangle\).

Therefore, for \(k\) even (with \(\alpha_0 = \alpha_0^i\) when necessary), \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm \in (\mathcal{C} \cup \mathcal{B}_0)^4\).

**Part 2:** Here we prove the case \(n \geq 2\), leaving the analogous details of the case \(n = 1\) to the reader (see \[8\]).

Let \(i \in \{0, \ldots, n\}\). For \(k\) odd, we have to prove that each of the curves \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^-\) is in \((\mathcal{C} \cup \mathcal{B}_0)^4\). Let \(k \in \{3, 5, \ldots, 2g + 1\} \{2g - 1\}\), then (see Figure 21):

\([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^- = \langle (\mathcal{C} \{\alpha_{k-1}, \alpha_{k+3}\}) \cup \mathcal{D} \rangle\).

We also have

\([\alpha_1, \alpha_2, \alpha_3]^- = \langle (\mathcal{C}_0 \{\alpha_4\}) \cup \mathcal{D} \rangle,\)

\([\alpha_{2g-1}, \alpha_{2g}, \alpha_{2g+1}]^- = \langle (\mathcal{C}_0 \{\alpha_{2g-2}\}) \cup \mathcal{D} \rangle\).

Therefore, for \(k\) odd (with \(\alpha_0 = \alpha_0^i\) when necessary), \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^- \in (\mathcal{C} \cup \mathcal{B}_0)^4\).
Figure 18: \( [\alpha_{2l},\ldots,\alpha_{2l-4}]^+ = (\{\alpha_{2l},\ldots,\alpha_{2l-4}\} \cup \{\alpha_{2l-2}\} \cup \{\alpha_j : j > i\} \cup E_{i,i} \cup \{\epsilon_{l-1}^{(1,n-1)}\}) \) for \( l = 3 \) and \( i = 2 \) in \( S_{4,4} \).

Figure 19: An illustration of \( [\alpha_{2g},\alpha_{2g+1},\alpha_0^0]^- = ((C_0 \setminus \{\alpha_1,\alpha_{2g-1}\}) \cup D \cup \{\beta_{2,\ldots,2g-2}\} \cup \{\epsilon_{l}^{(1,n-1)}\}) \).

**Part 3:** As above, we only prove the case \( n \geq 2 \); for the details of the case \( n = 1 \) see [8].

Let \( i \in \{0,\ldots,n\} \), \( k \) be odd. Similarly to the previous part, we have to prove that \( [\alpha_k,\alpha_{k+1},\alpha_{k+2}]^+ \in (C \cup \mathcal{B}_0)^6 \). Let \( k \in \{3,5,\ldots,2g-3\} \). Thus (see Figure 22),

\[
[\alpha_k,\alpha_{k+1},\alpha_{k+2}]^+ = ((C_0 \setminus \{\alpha_{k-1},\alpha_{k+3}\}) \cup \{\alpha_k,\alpha_{k+1},\alpha_{k+2}\})^-
\]

We then have (see Figure 23),

\[
[\alpha_{2g-1},\alpha_{2g},\alpha_{2g+1}]^+ = (\mathcal{C}_0 \setminus \{\alpha_{2g-2}\}) \cup \mathcal{D} \cup \left( \bigcup_{l \in \{1,\ldots,g-1\}} \epsilon_l^{(1,n-1)} \right) \cup \{[\alpha_{2g-1},\alpha_{2g},\alpha_{2g+1}]^-\}
\]

\[
[\alpha_1,\alpha_2,\alpha_3]^+ = (\mathcal{C}_0 \setminus \{\alpha_4\}) \cup \mathcal{D} \cup \left( \bigcup_{l \in \{2,\ldots,g\}} \epsilon_l^{(1,n-1)} \right) \cup \{[\alpha_1,\alpha_2,\alpha_3]^+\}
\]
Figure 20: An illustration of \([\alpha_0, \alpha_1, \alpha_2]^- = \langle (C_0 \setminus \{\alpha_3, \alpha_{2g+1}\}) \cup \mathcal{D} \cup \{\beta_{4\ldots 2g}\} \rangle\).

Figure 21: \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^- = \langle (C_0 \setminus \{\alpha_{k-1}, \alpha_{k+3}\}) \cup \mathcal{D} \rangle \) for \(k = 3\) in \(S_{4,4}\).

For \(i \in \{1, \ldots, n - 1\}\), we get (see Figure 24),

\[ [\alpha_{2g+1, \alpha_0^i, \alpha_1}]^+ = \langle (C_i \setminus \{\alpha_2, \alpha_{2g}\}) \cup E^{i,i} \cup \left( \bigcup_{l \in \{1, \ldots, g\}} \{\epsilon^{(1,n)}_{i,l}\} \right) \cup \{[\alpha_{2g+1, \alpha_0, \alpha_1}]^-\} \rangle; \]

for \(i \in \{0, n\}\), to prove the result for \([\alpha_{2g+1, \alpha_0^i, \alpha_1}]^+\), we need the auxiliary curve (see Figure 25):

\[ [\alpha_3, \ldots, \alpha_{2g-1}]^+ = \langle (C \setminus \{\alpha_2, \alpha_{2g}\}) \cup \{[\alpha_{2g+1, \alpha_0, \alpha_1}]^-\} \rangle \in (C \cup \mathcal{D}_0)^5; \]

and so (see Figure 26),

\[ [\alpha_{2g+1, \alpha_0^i, \alpha_1}]^+ = \langle (C \setminus \{\alpha_2, \alpha_{2g}\}) \cup \mathcal{D} \cup \{[\alpha_{2g+1, \alpha_0, \alpha_1}]^-\cup\{[\alpha_3, \ldots, \alpha_{2g-1}]^+\} \rangle. \]

Therefore, for \(k \in \mathbb{Z}\) (with \(\alpha_0 = \alpha_0^1\) when necessary), \([\alpha_k, \alpha_{k+1, \alpha_{k+2}]}^\pm \in (C \cup \mathcal{D}_0)^6\).
Finally, we define the set of auxiliary curves $\mathcal{B}$. Note that by construction, $\mathcal{B} \subset (\mathcal{C} \cup \mathcal{B}_0)^6$.

$$
\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_T \cup \bigcup_{i \in \{0, \ldots, m\}} \{[\alpha_{k+1}, \ldots, \alpha_{k+2g-1}]^+, [\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm\}.
$$

### 3.3 Proof of Theorem 3.3

Having defined the principal set $\mathcal{C} \cup \mathcal{B}_0$ and constructed the auxiliary sets $\mathcal{D} \subset \mathcal{C}$ and $\mathcal{B}_T \subset \mathcal{B}$, we state some results to ease the proofs of the following section, as well as give necessary notation and the proof of Theorem 3.3.

**Proposition 3.7.** Let $h \in \text{Mod}^*(S)$ and $Y \subset \mathcal{C}(S)$. If $h(Y) \subset (\mathcal{C} \cup \mathcal{B}_0)^k$ for some $k \in \mathbb{Z}$, then $h(Y^m) \subset (\mathcal{C} \cup \mathcal{B}_0)^{k+m}$.

**Proof.** Let $\gamma \in Y^1$. If $\gamma \in Y$, then $h(\gamma) \in (\mathcal{C} \cup \mathcal{B}_0)^k \subset (\mathcal{C} \cup \mathcal{B}_0)^{k+1}$ by hypothesis. Otherwise, there exists a set $A \subset Y$ such that $\gamma = \langle A \rangle$. Given that $h$ is a mapping class we have that $h(\gamma) = \langle h(A) \rangle$, and since by hypothesis $h(A) \subset h(Y) \subset (\mathcal{C} \cup \mathcal{B}_0)^k$ we get that $h(\gamma) \in (\mathcal{C} \cup \mathcal{B}_0)^{k+1}$. This implies that $h(Y^1) \subset (\mathcal{C} \cup \mathcal{B}_0)^{k+1}$. By induction, we obtain that $h(Y^m) \subset (\mathcal{C} \cup \mathcal{B}_0)^{k+m}$.

As a consequence of this proposition, since $\mathcal{C} \cup \mathcal{E} \cup \mathcal{B} \subset (\mathcal{C} \cup \mathcal{B}_0)^6$, we have the following corollary.
Figure 24: The curve \([\alpha_{2g+1}, \alpha_0, \alpha_1]\) = \(\left( \bigcup_{i \in \{1, \ldots, g\}} \{ \epsilon_i^{(1,n-1)} \} \right) \cup \{ \alpha_{2g+1}, \alpha_0, \alpha_1 \}\).

Figure 25: \([\alpha_3, \ldots, \alpha_{2g-1}] = (\{ \alpha_2, \alpha_{2g} \}) \cup \{ \alpha_{2g+1}, \alpha_0, \alpha_1 \} \) for \(j = 3\) in \(S_{4,4}\).

Corollary 3.8. Let \(h \in \text{Mod}^*(S)\). If \(h(C \cup B_0) \subset (C \cup B_0)^k\) for some \(k \in \mathbb{Z}\), then \(h(C \cup E \cup B) \subset (C \cup B_0)^{k+6}\).

An outer curve \(\alpha\) is a separating curve such that cutting along \(\alpha\) one of the resulting connected components is homeomorphic to a thrice-punctured sphere. Let \(\alpha, \beta \in C(S)\) and \(A, B \subset C(S)\). We denote by \(\eta_\alpha(\beta)\) the half-twist of \(\beta\) along \(\alpha\) and \(\eta_A(B) = \bigcup_{\gamma \in A} \eta_\gamma(B)\).

We must recall that the half-twist \(\eta_\alpha\) is defined if and only if \(\alpha\) is an outer curve, and there is exactly one half-twist along \(\alpha\) if \(S \not\cong S_{0,4}\). Let \(H := \{ \epsilon_i^{1,2j} \in \mathcal{D} : 2 \leq i \leq n \}\), then we can state the following lemma.

Lemma 3.9. Let \(\zeta = \beta_+^{2g-2,2g-1,2g}\), and \(H\) as above. Then \(\tau_{H}(C \cup B_0)^{18}\).\)

Assuming this lemma (for which we give a proof in the following subsections) we can proceed to prove Theorem 3.3 as follows.

Proof of Theorem 3.3 Let \(G = (C \setminus \{\alpha_{2g+1}\}) \cup \{\zeta\}\). Recalling Lickorish-Humphries Theorem (see [10], [10] and Section 4 of [5]), we have that \(\text{Mod}(S)\) is generated by the Dehn twists along \(G\) if \(n \leq 1\) and by the Dehn twists along \(G\) and the half twists along \(H\) if \(2 \leq n\). By Lemma 3.9, we know that \(\tau_{G}^{\pm1}(C \cup B_0)^{18}\).

Let us denote by \(f_\gamma\) the Dehn twist along \(\gamma\) if \(\gamma \in G\) or the half-twist along \(\gamma\) if \(\gamma \in H\).

Now let \(\gamma\) be either a nonseparating curve or an outer curve, and \(\alpha\) an element in \(G\) or \(H\).
\[
\alpha_{2g+1}, \alpha_0, \alpha_1^+ = (\{C_1 \setminus \{\alpha_2, \alpha_2g\}\} \cup \mathcal{D} \cup \{[\alpha_{2g+1}, \alpha_0^-, \alpha_1], [\alpha_3, \ldots, \alpha_{2g-1}]^+\})
\]
for \(j = 3\) in \(S_{4,4}\).

respectively. There exists \(h \in \text{Mod}(S)\) such that \(\gamma = h(\alpha)\). Thus, by an iterated use of Lemma 3.9 there are some curves \(\gamma_1, \ldots, \gamma_i \in \mathcal{G} \cup \mathcal{H}\) and some exponents \(n_1, \ldots, n_i \in \mathbb{Z}\), such that:

\[
\gamma = f_{\gamma_1}^{n_1} \circ \cdots \circ f_{\gamma_i}^{n_i}(\alpha) \in (\mathcal{G} \cup \mathcal{B}_0)^{18(|n_1| + \ldots + |n_i|)}.
\]

So, every nonseparating curve and every outer curve is an element of \(\bigcup_{i \in \mathbb{N}} (\mathcal{G} \cup \mathcal{B}_0)^i\).

Figure 27: Above, a separating curve \(\gamma\) with every connected component of \(S \setminus \{\gamma\}\) of positive genus; below, a separating curve \(\gamma\) with a connected component of \(S \setminus \{\gamma\}\) of genus zero, the set \(F_1\) in blue, the set \(F_2\) in black, and \(\gamma\) in red.

Let \(\gamma\) be a nonouter separating curve. We can always find sets \(F_1\) and \(F_2\) containing only nonseparating and outer curves, such that \(\gamma = \langle F_1 \cup F_2 \rangle\), see Figure 27. By the previous case, \(F_1 \cup F_2 \subset (\mathcal{G} \cup \mathcal{B}_0)^k\) for some \(k \in \mathbb{N}\); thus \(\gamma \in (\mathcal{G} \cup \mathcal{B}_0)^{k+1}\). Therefore \(C(S) = \bigcup_{i \in \mathbb{N}} (\mathcal{G} \cup \mathcal{B}_0)^i\). □

Now, to prove Lemma 3.9, due to Corollary 3.8 we only need to prove that

\[
\tau_{\mathcal{G} \cup \mathcal{B}_0}^{\pm 1}(\mathcal{G} \cup \mathcal{B}_0) \cup \eta_{\mathcal{G}}^{\pm 1}(\mathcal{G} \cup \mathcal{B}_0) \subset (\mathcal{G} \cup \mathcal{B}_0)^{12}.
\]

For this, we divide the proof into the following claims:

**Claim 1:** \(\tau_{\mathcal{G}}^{\pm 1}(\mathcal{G}) \subset (\mathcal{G} \cup \mathcal{B}_0)^8\)

**Claim 2:** \(\tau_{\mathcal{G}}^{\pm 1}(\mathcal{G} \cup \mathcal{B}_0) \subset (\mathcal{G} \cup \mathcal{B}_0)^{12}\)

**Claim 3:** \(\tau_{\mathcal{G}}^{\pm 1}(\mathcal{G}) \subset (\mathcal{G} \cup \mathcal{B}_0)^8\)

**Claim 4:** \(\tau_{\mathcal{G}}^{\pm 1}(\mathcal{G} \cup \mathcal{B}_0) \subset (\mathcal{G} \cup \mathcal{B}_0)^{10}\)

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3.4 Proof of Claim 1: $\tau_{\mathcal{G}}^{\pm 1}(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B}_0)^7$

Let $\alpha_j, \alpha_k \in \mathcal{C}$ (taking $\alpha_0 = \alpha_0'$ when necessary). If $|k - j| > 1$, $i(\alpha_j, \alpha_k) = 0$ and so we have that $\tau_{\mathcal{G}}^{\pm 1}(\alpha_j) = \alpha_j \in \mathcal{C}$. We then only need to prove for the case when $|k - j| = 1$.

In contrast to the closed surface case $\text{Mod}^*(S)$ does not act transitively on $\mathcal{C}$ (it has two orbits), so here we first prove that $\tau_{\mathcal{G}}^{\pm 1}(\alpha_{g-1})$ and $\tau_{\mathcal{G}}^{\pm 1}(\alpha_{g-2})$ are elements of $(\mathcal{C} \cup \mathcal{B}_0)^8$, then we use the action of a subgroup of $\text{Mod}^*(S)$ to prove Claim 1.

Let $A \subset \mathcal{C}(S)$. We define $E(A) := \{\epsilon \in \mathcal{G} : i(\epsilon, \delta) = 0 \text{ for all } \delta \in A\}$.

Following [2], as in Subsection 2.2, we prove first that $\tau_{\mathcal{G}}^{\pm 1}(\alpha_{g-1}) \in (\mathcal{C} \cup \mathcal{B}_0)^8$.

Lemma 3.10. $\tau_{\mathcal{G}}^{\pm 1}(\alpha_{g-1}) \in (\mathcal{C} \cup \mathcal{B}_0)^8$.

Proof. Taking the set

$C_1 = \{\alpha_{g+1}, \alpha_1, \alpha_2, \ldots, \alpha_{2g-4}, \alpha_{2g-2}, [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}], [\alpha_{2g-4}, \alpha_{2g-3}, \alpha_{2g-2}]^+, [\alpha_1, \ldots, \alpha_{2g-1}]^+, \},$

then, by Proposition 3.6, we get that $\gamma_+ := (C_1 \cup E(C_1)) \in (\mathcal{C} \cup \mathcal{B}_0)^7$. See Figure 28.

Figure 28: Examples of $C_1$ and $E(C_1)$ above, and the corresponding curve $\gamma_+$ below.

Letting

$C_1' = \{\alpha_1, \ldots, \alpha_{2g-3}, [\alpha_{2g-2}, \alpha_{2g-1}, \alpha_{2g}]^+, [\alpha_{2g-2}, \alpha_{2g-1}, \alpha_{2g}]^-, \gamma_+\},$

we then have that $\tau_{\mathcal{G}}(\alpha_{g-1}) = (C_1' \cup E(C_1')) \in (\mathcal{C} \cup \mathcal{B}_0)^7$ (see Figure 29).

As in Subsection 2.2, let $C_-$ be the set obtained by substituting $[\alpha_{g-4}, \alpha_{g-3}, \alpha_{g-2}]^+ \in C_+$ for $[\alpha_{g-4}, \alpha_{g-3}, \alpha_{g-2}]^-$, $\gamma_- = (C_- \cup E(C_-))$, and $C'$ be the set obtained by substituting $\gamma_+$ in $C_+$ for $\gamma_-$. As before, we have that $\tau_{\mathcal{G}}^{-1}(\alpha_{g-1}) = (C' \cup E(C')) \in (\mathcal{C} \cup \mathcal{B}_0)^8$.

To prove that $\tau_{\mathcal{G}}^{\pm 1}(\alpha_{g-2}) \in (\mathcal{C} \cup \mathcal{B}_0)^8$, we cannot proceed as in the closed case. This is due to the fact that for any choice of $i$, there are nonhomeomorphic connected components of $S \setminus C_i$. This implies it is possible that there is no homeomorphism that leaves $C_i$ invariant but sends $\alpha_{g-2}$ into $\alpha_{g-1}$. So, we first prove a proposition for an auxiliary curve used only in this proof, and then follow a method similar to Lemma 3.10.
Figure 29: Examples of $C'_1$ and $E(C'_1)$ above, and $\tau_{2g}(\alpha_{2g-1})$ below.

**Proposition 3.11.** Let $k \in \mathbb{Z}$. Then we have that $[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^{-} \in (\mathcal{C} \cup \mathcal{R}_0)^2$ for any choice of $i \in \{0, \ldots, n\}$ (with $\alpha_0 = \alpha'_0$ when necessary).

**Proof.** If $2k \neq 2 \pmod{2g+2}$ and $i \neq n$, then we have (see Figure 30)

$[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^{-} = \langle \{\alpha_0^j : 0 \leq j \leq i\} \cup \{\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}\} \cup \{\alpha_{2k+2g}\} \cup \{e^{j,k} : i \leq j < k \leq n\} \rangle$.

Figure 30: The curve $[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^{-} = \langle \{\alpha_0^j : 0 \leq j \leq i\} \cup \{\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}\} \cup \{\alpha_{2k+2g}\} \cup \{e^{j,k} : i \leq j < k \leq n\} \rangle$.

If $2k \neq 2 \pmod{2g+2}$ and $i = n$, then we have $[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^{-} = \alpha_{2k+2g}$.

If $2k = 2 \pmod{2g+2}$ then $[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^{-} = \beta_{2g+2}^{-2} = \alpha_0^{n}$.

Now, we prove:

**Lemma 3.12.** $\tau_{2g-1}^{\pm 1}(\alpha_{2g-2}) \in (\mathcal{C} \cup \mathcal{R}_0)^8$. 

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Proof. Let \( i \in \{0, \ldots, n\} \). Taking the set
\[
C_{2+} = \{\alpha_{2g}, \alpha_0, \alpha_1, \ldots, \alpha_{2g-5}, [\alpha_{2g-4}, \alpha_{2g-3}, \alpha_{2g-2}]^+, [\alpha_{2g-5}, \alpha_{2g-4}, \alpha_{2g-3}]^+, [\alpha_0, \ldots, \alpha_{2g-2}]^\pm, \}
\]
then, by Lemma 3.6 we get that \( \gamma_+ := (C_{2+} \cup E(C_{2+})) \in (\mathcal{C} \cup \mathcal{B}_0)^7 \) (see Figure 31).

\[\text{Figure 31: The curve } \gamma_+ := (C_{2+} \cup E(C_{2+})).\]

Letting
\[
C'_{2+} = \{\alpha_0^i, \ldots, \alpha_{2g-4}, [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}]^+, [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}]^-, \gamma_+\},
\]
we then have that \( \tau_{\alpha_{2g-1}}(\alpha_{2g-2}) = (C'_{2+} \cup E(C'_{2+})) \in (\mathcal{C} \cup \mathcal{B}_0)^8 \) (see Figure 32).

\[\text{Figure 32: The curve } \tau_{\alpha_{2g-1}}(\alpha_{2g-2}) = (C'_{2+} \cup E(C'_{2+})) \in (\mathcal{C} \cup \mathcal{B}_0)^8.\]

As in the case for \( \tau_{\alpha_{2g}}^{-1}(\alpha_{2g-1}) \) of Lemma 3.10 substituting the analogous curves, we also have that \( \tau_{\alpha_{2g-1}}^{-1}(\alpha_{2g-2}) \in (\mathcal{C} \cup \mathcal{B}_0)^8. \)

Now, let \( h_i \) be the mapping class obtained by cutting \( S \) along \( C_i \) and rotating the resulting (sometimes punctured) discs so that \( h_i(\alpha_i) = \alpha_{i+2} \) (with \( \alpha_0 = \alpha_0^i \)). See Figure 33.
Figure 33: An example of $h_i$ in $S_{4,3}$, for which $h_i(S_{i(o)}^+) = S_{i(o)}^+$, $h_i(S_{i(o)}^-) = S_{i(o)}^-$, $h_i(S_{i(e)}^+) = S_{i(e)}^+$ and $h_i(S_{i(e)}^-) = S_{i(e)}^-$. In particular $h_i(\alpha_j) = \alpha_{j+2}$. Also, $h_i \in \text{stab}_\mu(E^{n,i})$ and if $k \in \{1, \ldots, g-1\}$, $h_i(\epsilon_k^{(1,n-1)}) = \epsilon_{k+1}^{(1,n-1)}$. 
Proof of Claim 1: We can precompose by appropriate elements of the group \( \langle h_i \rangle \) on \( C_i \) to translate the elements of \( C_j^+, C_j^- \) and the corresponding sets for the negative exponents of the Dehn twists, and following the procedure for \( \tau_{\alpha_2g-1}^{\pm 1}(1g), \tau_{\alpha_{3g-1}}^{\pm 1}(1g) \in (C \cup B_0)^8 \) we have that \( \tau_{\alpha_j}^{\pm 1}(1g) \in (C \cup B_0)^8 \). Therefore \( \tau_{\alpha_j}^{\pm 1}(C) \subset (C \cup B_0)^8 \). □

3.5 Proof of Claim 2: \( \tau_{\alpha_j}^{\pm 1}(C) \subset (C \cup B_0)^8 \)

Let \( \alpha \in \mathcal{C} \) and \( \beta \in \mathcal{B} \); if \( i(\alpha, \beta) = 0 \), we have that \( \tau_{\alpha}^{\pm 1}(\beta) = \beta \in \mathcal{C} \cup \mathcal{B} \). So, we assume this is not the case. Now, to prove the claim we first see that every curve in \( \mathcal{B}_0 \) can be taken to be a curve uniquely determined by a set \( C \cup E \cup B \) such that \( C \in \mathcal{C}, E \in \mathcal{E}, B \in \mathcal{B} \), and with every element in \( B \) disjoint from \( \alpha \). Since \( \tau_{\alpha}^{\pm 1} \) are mapping classes, we get that \( \tau_{\alpha}^{\pm 1}(1g) = \langle \tau_{\alpha}^{\pm 1}(C \cup E \cup B) \rangle = \langle \tau_{\alpha}^{\pm 1}(C) \cup \tau_{\alpha}^{\pm 1}(E) \cup B \rangle \). Using the result in Claim 1 we get that \( \tau_{\alpha}^{\pm 1}(1g) \subset (C \cup B_0)^8 \), and by Proposition 3.7 \( \tau_{\alpha}^{\pm 1}(1g) \subset (C \cup B_0)^12 \). Therefore \( \tau_{\alpha}^{\pm 1}(1g) \subset (C \cup B_0)^8 \).

For a detailed account of the sets \( C, E \) and \( B \), see [8].

3.6 Proof of Claim 3: \( \tau_{\alpha_j}^{\pm 1}(1g) \subset (C \cup B_0)^8 \)

Recall \( \zeta = \beta_{g-2,2g-1,2g}^+ \), which is disjoint from every element in \( \mathcal{C} \setminus \{\alpha_{2g-3,2g+1}\} \). This implies we only need to prove that \( \tau_{\alpha_j}^{\pm 1}(1g) \subset (C \cup B_0)^8 \).

We proceed as in Claim 1, using the following ordered maximal closed chain

\[
\gamma_0 = \alpha_1^1, \gamma_1 = \alpha_2, \gamma_2 = \alpha_2g-5, \gamma_3 = \alpha_2g-4 = \beta_{2g-4,2g-3,2g-2}^+; \\
\gamma_4 = \alpha_2g-3, \gamma_5 = \alpha_2g-2, \gamma_6 = \alpha_2g-1, \gamma_7 = \alpha_2g, \gamma_8 = \zeta, \gamma_9 = \alpha_2g+1, \\
(see Figure 34) \text{ and proving first that } \tau_{\alpha_j}(1g) \subset (C \cup B_0)^8.
\]

Figure 34: The ordered maximal closed chain for \( S = S_{5,4} \), which is used for the case of \( \tau_{\alpha_j}^{\pm 1}(\alpha_{2g-3}) \).

Using the set

\[
C_{1g} = \{\alpha_{2g+1}, \alpha_1, \alpha_2, \ldots, \alpha_{2g-5}, \beta_{2g-4,2g-3,2g-2}^+; \alpha_{2g-2}, \alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}^+; \beta_{2g-2,2g-4}^+, [\alpha_1, \ldots, \alpha_2g-1]_+ \} \\
\text{for genus } g \geq 4,
\]

\[
= \{\alpha_1, \alpha_2, \beta_{2,2,4}^+, \alpha_4, [\alpha_3, \alpha_4, \alpha_5]_+, [\alpha_1, \ldots, \alpha_5]_+ \} \\
\text{for genus } g = 3,
\]
and Propositions 3.6 and 3.5 we obtain the curve \( \gamma_+ = (C_1 \cup \mathcal{D}) \in (\mathcal{C} \cup \mathcal{B}_0)^7 \). Then, using the set

\[
C'_1 = \{ \alpha_1, \ldots, \alpha_{2g-5}, \beta_{2g-4,2g-3,2g-2}, \alpha_{2g-2}, \alpha_{2g-3}, \zeta, \alpha_{2g} \},
\]

we obtain the curve \( \gamma'_+ = (\mathcal{C}_f \cup \mathcal{C}'_1) \in (\mathcal{C} \cup \mathcal{B}_0)^8 \). Finally, using the set

\[
C''_1 = \{ \alpha_1, \ldots, \alpha_{2g-5}, \beta_{2g-4,2g-3,2g-2}, \alpha_{2g-1}, \gamma_+', \alpha_{2g}, \gamma_+ \},
\]

we obtain that \( \tau_-(\alpha_{2g-3}) = (C''_1 \cup \mathcal{C}_f) \in (\mathcal{C} \cup \mathcal{B}_0)^8 \).

For \( \tau_-(\alpha_{2g-3}) \) we proceed analogously, substituting the appropriate curves, getting that

\[
\tau_-(\alpha_{2g-3}) \in (\mathcal{C} \cup \mathcal{B}_0)^8.
\]

To prove that \( \tau^\pm_+(\alpha_{2g+1}) \in (\mathcal{C} \cup \mathcal{B}_0)^8 \) we proceed analogously, using the ordered maximal closed chain

\[
\gamma_0 = \alpha_{2g-2}; \gamma_1 = \alpha_{2g-1}; \gamma_2 = \beta_{2g-4,2g-3,2g-2}; \gamma_3 = \alpha_{2g-5}.
\]

\[
\gamma_4 = \alpha_{2g-6}, \ldots; \gamma_{2g-4} = \alpha_2; \gamma_{2g-3} = \alpha_1; \gamma_{2g-2} = \alpha_0; \gamma_{2g-1} = \alpha_{2g+1}; \gamma_{2g} = \zeta; \gamma_{2g+1} = \alpha_{2g-3}
\]

(see Figure 35). Note that this closed chain as a set, is the same closed chain as the previous case but with the order reversed.

Figure 35: The ordered maximal closed chain for \( S = S_{5,4} \), which is used for the case of \( \tau^\pm_+(\alpha_{2g+1}) \).

We have then \( \tau^\pm_+(\alpha_{2g+1}) \in (\mathcal{C} \cup \mathcal{B}_0)^8 \).

### 3.7 Proof of Claim 4: \( \tau^\pm_+(\mathcal{C} \cup \mathcal{B}_0) \subset (\mathcal{C} \cup \mathcal{B}_0)^{10} \)

Given that \( \zeta \) is disjoint from every curve \( \beta \in \mathcal{B}_0 \) of the form \( \beta_{2l-2(l+k)} \) for some \( l \in \mathbb{N} \) and \( k \in \mathbb{Z}^+ \), it follows that \( \tau^\pm_+(\beta_{2l-2(l+k)}) = \beta_{2l-2(l+k)} \in \mathcal{B}_0 \). So, we need to prove the result for \( \beta = \beta_{2l-2(l+k)} \); we do so dividing into several cases in the following way (See Figure 36 for examples):

1. \( \beta \) is of the form \( \beta_{2l-2(l+k)} \) for \( l < l + k \).
2. \( \beta \) is of the form \( \beta_{2l-2(l+k)} \) for \( l > l + k \).

To prove the claim, as in Claim 2 in Subsection 3.5, we just have to remember that every curve in \( \mathcal{B}_0 \) can be taken to be a curve uniquely determined by a set \( C \cup E \cup B \) such that \( C \subset \mathcal{C} \), \( E \subset \mathcal{E} \), \( B \subset \mathcal{B} \), and with every element in \( B \) disjoint from \( \zeta \). For a detailed account of these sets see [8].
3.8 Proof of Claim 5: $\eta_+^1(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B}_0)^7$

Recall $\mathcal{H} := \{\epsilon^{2i} \in \mathcal{D} : 2 \leq i \leq n\}$. Given that $i(\alpha, \epsilon^{2i-1}) = 0$ for all $\alpha \in \mathcal{C}\{\alpha_0^{-1}\}$, to prove the claim we just need to prove that $\eta_+^1(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B}_0)^7$. We do so following [2].

Let $i \in \{1, \ldots, n\}$; we define (see Figure 37)

$$\beta^i = \langle\{\alpha_1\} \cup \{\alpha_3, \ldots, \alpha_{2g+1}\} \cup \{\beta^\pm_{4i,2g}\} \cup \{\beta^\pm_{0,1,2} : j < i\} \cup \{\beta^\pm_{0,1,2} : i \leq k\} \rangle \in (\mathcal{C} \cup \mathcal{B}_0)^5.$$ 

Then, for $i \in \{1, \ldots, n\}$, we define (see Figure 38)

$$\gamma_+^i = \langle\{\alpha_0, \alpha_1, \alpha_2\} \cup \{\alpha_4, \ldots, \alpha_{2g}\} \cup \{\beta^\pm_{4i,2g}\} \cup \{\beta^j : i \neq j\} \rangle \in (\mathcal{C} \cup \mathcal{B}_0)^6,$$

$$\gamma_-^i = \langle\{\alpha_0, \alpha_1, \alpha_2\} \cup \{\alpha_4, \ldots, \alpha_{2g}\} \cup \{\beta^\pm_{4i,2g}\} \cup \{\beta^j : i \neq j\} \rangle \in (\mathcal{C} \cup \mathcal{B}_0)^6.$$

Finally, for $i \in \{2, \ldots, n\}$, we get that $\eta_{i-2,i}(\alpha_0^{-1}) = \langle\{\mathcal{C}\{\alpha_{2g+1}, \alpha_0^{-1}, \alpha_1\}\} \cup \{\gamma_+^i\} \rangle$ and $\eta_{i-2,i}(\alpha_0^{-1}) = \langle\{\mathcal{C}\{\alpha_2g+1, \alpha_0^{-1}, \alpha_1\}\} \cup \{\gamma_-^i\} \rangle$ (see Figure 39). Therefore $\eta_+^1(\mathcal{C}) \subset (\mathcal{C} \cup \mathcal{B}_0)^7$.

3.9 Proof of Claim 6: $\eta_+^1(\mathcal{C} \cup \mathcal{B}_0) \subset (\mathcal{C} \cup \mathcal{B}_0)^{11}$

Let $\beta \in \mathcal{B}_0$, as such it is of the form $\beta^\pm_{2l,2(l+k)}$ for some $l \in \mathbb{N}$ and some $k \in \mathbb{Z}^+$. If $0 < l < l + k$, then $\beta$ and $\epsilon^{i(i+2)}$ are disjoint for $i \in \{0, \ldots, n - 2\}$. This implies that
\( \eta_{l+1, l+2}(\beta) = \beta. \)

If \( l = 0 \) and \( k \in \{2, \ldots, g-1\} \), either we have that (see Figure 40)

\[
\beta = \left( c \setminus \{\alpha_0^0, \alpha_{2g+1}, \alpha_{2k+1}\} \right) \cup \left( \bigcup_{l \in \{k+1, \ldots, g\}} \epsilon^{(1,n-1)}_l \right),
\]

or we have that (see Figure 41)

\[
\beta = \left( C_1 \setminus \{\alpha_{2k+1}, \alpha_{2g+1}\} \right) \cup \{\alpha_0^0\} \cup E^{i,1} \cup \left( \bigcup_{l \in \{k+1, \ldots, g\}} \epsilon^{(1,n-1)}_l \right).
\]

Using Claim 5 and Proposition 3.7, we obtain that \( \eta_{l+1, i+2}(\beta) \in (c \cup B_0)^{11} \) for \( i \in \{0, \ldots, n-2\} \).

If \( \beta \) is of the form \( \beta_{2l, \ldots, 2l+k}^{\pm} \) for \( l > l+k \), following the proof of Claim 2, we have that \( \beta = (C \cup E \cup B) \) with \( C \subset c, E \subset \mathcal{E} \) and \( B \) a singleton of a curve disjoint from \( \epsilon^{i, i+2} \). Using the result from Claim 5 we get that \( \eta_{l+1, i+2}(C) \subset (c \cup B_0)^7 \), and applying Proposition 3.7 we
Figure 40: Examples of $\beta = \beta^+_{\{2l,\ldots,2(l+k)\}}$ with $l = 0$ and $k = 3$. The elements in $E$ are coloured red.

Figure 41: Examples of $\beta = \beta^-_{\{2l,\ldots,2(l+k)\}}$ with $l = 0$ and $k = 3$. The elements in $E$ are coloured red.

obtain that $\eta^{\pm 1}_{\alpha_{i+1}}(E) \subset (E \cup B_0)^{10}$. Finally, this implies that $\eta^{\pm 1}_{\alpha_{i+1}}(\beta) \in (E \cup B_0)^{11}$, for $i \in \{0,\ldots,n-2\}$.

4 Rigid sets

In this section we suppose $S = S_{g,n}$ with genus $g \geq 3$, and $n \geq 0$ punctures. Here we reintroduce the finite rigid set from [1] (see Subsections 4.1 for the closed surface case and 4.2 for the punctured surface case, below) and prove Theorem [B].

4.1 $X(S)$ for closed surfaces

Let $E$ and $B$ be as in Section [2] and $J$ be a subinterval (modulo $2g+2$) of $\{0,\ldots,2g+1\}$ such that $|J| < 2g-1$.

If $J = \{j,\ldots,j+2k-1\}$ for some $j \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, we get the following curve:

$$\sigma_J := \langle \{\alpha_j,\ldots,\alpha_{j+2k-1}\} \cup \{\alpha_{j+2k+1},\ldots,\alpha_{j-2}\} \rangle.$$
For examples see Figure 42.

\[ \sigma_{\{0,1\}} \text{ in red, and } \sigma_{\{4,5,6,7\}} \text{ in blue.} \]

This way, we define the following set:

\[ \mathcal{S} := \{ \sigma_J : |J| = 2k \text{ for some } k \in \mathbb{Z}^+ \}. \]

If \( J = \{j, \ldots, j+2k\} \) for some \( j \in \mathbb{N} \) and \( k \in \mathbb{Z}^+ \), let we get the following curves:

\[
\mu^+_j,J := \langle \{\beta_j^+, \alpha_{j+2k+1}\} \cup \{\alpha_j, \ldots, \alpha_{j+k-1}\} \cup \{\beta^-_{(j+2, \ldots, j+2k+2)}\} \cup \{\alpha_{j+2k+3}, \ldots, \alpha_{j-2}\} \rangle,
\]

\[
\mu^-_j,J := \langle \{\beta^-_j, \alpha_{j+2k+1}\} \cup \{\alpha_j, \ldots, \alpha_{j+k-1}\} \cup \{\beta^+_{(j+2, \ldots, j+2k+2)}\} \cup \{\alpha_{j+2k+3}, \ldots, \alpha_{j-2}\} \rangle.
\]

For examples see Figure 43.

\[ \mu_{3,\{0,1,2\}} \text{ in red, and } \sigma_{7,\{4,5,6\}} \text{ in blue.} \]

This way, we define the following set:

\[ \mathcal{A} := \{ \mu^+_i,J : J = \{j, \ldots, j+2k\} \text{ for some } k \in \mathbb{Z}^+, i = j + 2k + 1 \} \]

Finally, we have the set

\[ \mathcal{X}(S) := \mathcal{C} \cup \mathcal{B} \cup \mathcal{S} \cup \mathcal{A} \]

Recall that, as was mentioned in the Introduction, this set was proved to be rigid in \( \mathbb{P} \), and by construction has trivial pointwise stabilizer in \( \text{Mod}^*(S) \).

### 4.2 \( \mathcal{X}(S) \) for punctured surfaces

Let \( \mathcal{C}, \mathcal{B}_0, \mathcal{D} \) and \( \mathcal{B}_T \) be as in Section 3.

For \( 0 \leq i \leq j \leq n \), we denote by \( N_{i,j}^1 \) and \( N_{i,j}^{2g+1} \) closed regular neighbourhoods of the chains \( \{\alpha^i_0, \alpha^j_0, \alpha_1\} \) and \( \{\alpha^i_0, \alpha^j_0, \alpha_{2g+1}\} \) respectively.

Note that \( N_{i,j}^1 \) is a two-holed torus if \( j - i \geq 1 \) (one of the boundary components will not be essential if \( j - i = 1 \)). Also, \( S \setminus N_{i,j}^1 \) is the disjoint union of a subsurface homeomorphic to
an at least once-punctured open disc, and a subsurface homeomorphic to \( S_{g-1,n-(j-1)+1} \).

If \( j-i > 1 \), one of the boundary components of \( N_{2(i)}^{i} \) is the curve \( c^{i,j} \). On the other hand, for \( 0 \leq i \leq j \leq n \), we denote by \( \sigma_{i,j}^{1} \) the boundary component of \( N_{1}^{i,j} \) such that one of the connected components of \( S \setminus \{ \sigma_{i,j}^{1} \} \) is homeomorphic to \( S_{1,j-i+1} \).

We denote by \( \sigma_{2g+1}^{i,j} \) the analogous boundary curves of \( N_{2g+1}^{i,j} \) (whenever they are essential).

Then, we define

\[
\mathcal{J}_{T} := \{ \sigma_{i,j}^{1} : l \in \{1, 2g+1\}, 0 \leq i \leq j \leq n \}
\]

Now, let \( J \) be a subinterval of \( \{0, \ldots, 2g+1\} \) (modulo \( 2g+2 \)) such that \( |J| \leq 2g \).

If \( J = \{i, \ldots, i+2k-1\} \) for some \( k \in \mathbb{Z}^{+} \), let \( \sigma_{J} = [\alpha_{i}, \ldots, \alpha_{i+2k-1}] \) (with \( \alpha_{0} = \alpha_{0}^{0} \) if necessary). We define

\[
\mathcal{J}_{0} := \{ \sigma_{J} : J = \{i, \ldots, i+2k-1\}, k \in \mathbb{Z}^{+} \}.
\]

If \( J = \{2l, \ldots, 2(l+k)\} \), for some \( k \in \mathbb{Z}^{+} \), and \( j = 2(l+k)+1 \), then \( i(\alpha_{j}, \beta_{j}^{+}) = i(\alpha_{j}, \beta_{j}^{-}) = 1 \).

Let \( \mu_{i,j}^{\pm} \) be the boundary curve of a regular neighbourhood of \( \{\alpha_{j}, \beta_{j}^{\pm}\} \). Analogously, let \( \mu_{i,j}^{-} \) be the boundary curve of a regular neighbourhood of \( \{\alpha_{j}, \beta_{j}^{+}\} \). We define

\[
\mathcal{M} : = \{ \mu_{i,j}^{\pm} : J = \{2l, \ldots, 2(l+k)\}, k \in \mathbb{Z}^{+}, j = 2(l+k)+1 \}.
\]

Therefore, we define:

\[
\mathcal{X} : = \mathcal{C} \cup \mathcal{B} \cup \mathcal{J}_{T} \cup \mathcal{J}_{0} \cup \mathcal{M}_{T} \cup \mathcal{B}_{0} \cup \mathcal{M}.
\]

Recall that, as was mentioned in the Introduction, this set was proved to be rigid in [1], and by construction has trivial pointwise stabilizer in \( \text{Mod}^{\ast}(S) \).

### 4.3 Proof of Theorem [3]

The set \( \mathcal{X}(S) \) is studied in [1] and [2], and it is proven to be a finite rigid set of \( \mathcal{C}(S) \) (Theorems 5.1 and 6.1 in [1]). Also, by construction, we have that the principal sets used in Sections 2 and 3 (\( \mathcal{C} \cup \mathcal{B} \) for closed surfaces, \( \mathcal{C} \cup \mathcal{B}_{0} \) for punctured surfaces) are contained in their respective \( \mathcal{X}(S) \), which gives us the proof of Theorem [3].

**Proof of Theorem [2]** Since \( \mathcal{C} \cup \mathcal{B} \subset \mathcal{X}(S) \) for \( S \) closed (and \( \mathcal{C} \cup \mathcal{B}_{0} \subset \mathcal{X}(S) \) for \( S \) a punctured surface), we have that \( (\mathcal{C} \cup \mathcal{B})^{k} \subset \mathcal{X}(S)^{k} \) for any \( k \in \mathbb{N} \) (analogously \( (\mathcal{C} \cup \mathcal{B}_{0})^{k} \subset \mathcal{X}(S)^{k} \) for any \( k \in \mathbb{N} \)). This implies that \( \bigcup_{i \in \mathbb{N}}(\mathcal{C} \cup \mathcal{B})^{i} = \bigcup_{i \in \mathbb{N}}\mathcal{X}(S)^{i} \) (analogously \( \bigcup_{i \in \mathbb{N}}(\mathcal{C} \cup \mathcal{B}_{0})^{i} = \bigcup_{i \in \mathbb{N}}\mathcal{X}(S)^{i} \)).

This coupled with Theorem 2.1 and 3.3 gives us the desired result. \qed

Recalling that in Proposition 3.5 in [2], Aramayona and Leininger prove that the rigid expansions of rigid set are themselves rigid, note that Theorem [3] gives and alternative proof of Theorem 1.1 in [2].

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