Digital fundamental groups and edge groups of clique complexes

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Abstract
In previous work, we have defined—intrinsically, entirely within the digital setting—a fundamental group for digital images. Here, we show that this group is isomorphic to the edge group of the clique complex of the digital image considered as a graph. The clique complex is a simplicial complex and its edge group is well-known to be isomorphic to the ordinary (topological) fundamental group of its geometric realization. This identification of our intrinsic digital fundamental group with a topological fundamental group—extrinsic to the digital setting—means that many familiar facts about the ordinary fundamental group may be translated into their counterparts for the digital fundamental group: The digital fundamental group of any digital simple closed curve is \( \mathbb{Z} \); a version of the Seifert–van Kampen Theorem holds for our digital fundamental group; every finitely presented group occurs as the (digital) fundamental group of some digital image. We also show that the (digital) fundamental group of every 2D digital image is a free group.

Keywords Digital topology · Digital image · Fundamental group · Edge group · Simplicial complex · Clique complex · Tolerance space · Seifert–van Kampen theorem · Finitely presented group · Free group

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1 Introduction

A digital image $X$ is a finite subset $X \subseteq \mathbb{Z}^n$ of the integral lattice in some $n$-dimensional Euclidean space, together with a particular adjacency relation on the set of points. This is an abstraction of an actual digital image which consists of pixels (in the plane, or higher dimensional analogues of such). Digital topology refers to the use of notions and methods from (algebraic) topology to study digital images. The idea in doing so is that such notions can provide useful theoretical background for certain steps of image processing, such as contour filling, border and boundary following, thinning, and feature extraction or recognition (e.g., see p. 273 of Kong and Rosenfeld 1989). There is an extensive literature on digital topology (see e.g., Rosenfeld 1986; Kong and Rosenfeld 1989; Boxer 1999; Evako 2006 and the references therein). As a contribution to this literature, in Lupton et al. (2021, 2022a, b) we have started to build a general “digital homotopy theory” that brings many of the strengths of homotopy theory to the digital setting. In Lupton et al. (2021) we focussed on the fundamental group. Our definition of the digital fundamental group in Lupton et al. (2021)—see below for a résumé—is intrinsic, in the sense that it is defined directly in terms of a digital image, using ingredients such as homotopy of based loops defined within the digital setting. Indeed, a crucial component of our development in Lupton et al. (2021) involves the notion of subdivision of a digital image—a construction that relies on the “cubical” setting of the integer lattice and which does not translate out of the digital setting in any obvious way. One of the main results of Lupton et al. (2021) shows that this process of subdivision preserves the fundamental group of a digital image (Th.3.23 of Lupton et al. 2021).

In this paper, we make significant advances on the development of Lupton et al. (2021). The main result is the following.

**Theorem** (Theorem 3.6) Let $X$ be a digital image and $\text{cl}(X)$ its clique complex. The digital fundamental group of $X$, as defined in Lupton et al. (2021), is isomorphic to the edge group of $\text{cl}(X)$.

See Sect. 3 for descriptions of the clique complex of a digital image and the edge group of a simplicial complex. Now it is known that the edge group of a simplicial complex is isomorphic to the fundamental group—in the ordinary, topological sense—of the geometric realization of the simplicial complex (see Maunder 1996, Th.3.3.9, repeated as Theorem 3.4 here). It follows that the relatively unfamiliar digital fundamental group may be identified with the much more familiar topological fundamental group of a space that is associated to the digital image in a fairly transparent way. With this identification we may, with care over one or two technical points, translate many known results about the topological fundamental group into their counterparts for the digital fundamental group. Doing so adds greatly to our understanding of the digital fundamental group.

An overview of the organization of the paper and our results follows. Section 2 summarizes some basics of digital topology and our definition of the digital fundamental group from Lupton et al. (2021), and gives one technical result from Lupton et al. (2021) that we rely on in the sequel. We have tried to keep this material to the minimum necessary for understanding our results here, and refer to Lupton et al. (2021)
for fuller details. Section 3 contains our main result. We review clique complexes and edge groups, and prove the isomorphism asserted in the Theorem above. In Sect. 4 we begin to draw consequences of this isomorphism. In Theorem 4.2 we show that the digital fundamental group of any digital simple closed curve (Definition 4.1) is $\mathbb{Z}$. In Theorem 4.6 we deduce a version of the Seifert–van Kampen theorem for the digital fundamental group. The conclusion is the same as the topological theorem, but we require an extra (mild) hypothesis in addition to the usual connectivity hypotheses. We use this result to give concrete examples of digital images with interesting fundamental groups. Example 4.13 shows that a one-point union of two digital simple closed curves has non-abelian digital fundamental group (a free group on two generators, in fact). Example 4.18 shows that a certain digital image—which we construct as a “digital projective plane”—has torsion in its digital fundamental group (which is $\mathbb{Z}_2$, in fact). These examples are deduced from special cases of our digital Seifert–van Kampen theorem (Corollary 4.12 and Corollary 4.15). To the best of our knowledge, these are the first examples given of digital images with fundamental group—in any sense—that is not free abelian. More generally, we are able to realize any finitely presented group as the digital fundamental group of a digital simple closed curve in Theorem 4.19. In the final Sect. 5, we show that the digital fundamental group of every 2D digital image is a free group. This result does not follow automatically from the isomorphism of Theorem 3.6. Rather, we establish it after some preliminary results in Sect. 5 about shortening of paths that are of interest in their own right.

The fundamental group is not new in digital topology (see Kong 1989; 1999, for example). But our approach and development in Lupton et al. (2021) and here differs from versions previously used in digital topology. We give some discussion of these differences now. As we pointed out in Lupton et al. (2021), our fundamental group differs from that of Boxer (1999) for basic examples of digital images. This difference derives from differences in the notion of homotopy, and is explained in some detail in Lupton et al. (2021). Ayala et al. (2003) work in a setting in which digital images have extra structure that our notion of digital image does not have a priori. By making different choices of their “weak lighting function,” for example, one can arrive at different notions of a fundamental group that on a digital simple closed curve take $\mathbb{Z}$ or the trivial group. Furthermore, Ayala et al. (2003) does not actually define a fundamental group in the digital setting. Rather, their “digital” fundamental group is defined extrinsically to be the edge group of an auxiliary complex; they do not work in terms of loops and homotopies in the actual digital image itself, as we do. A digital image in our sense only conforms to one of the general “device models” considered in Ayala et al. (2003), namely, the standard cubical decomposition of Euclidean n-space $\mathbb{R}^n$. Working within that device model, and using $(3^n - 1)$-adjacency in $\mathbb{Z}^n$, as we do consistently, we do not know whether it is possible to make a uniform, once and for all, choice of extra structure for which the corresponding fundamental group of Ayala et al. (2003) determined by such a choice agrees with our fundamental group. If not, then our notions of fundamental group are basically different. But even if it were, it is unlikely that such a matching would extend to any other aspects of our more general digital homotopy theory. For example, maps of digital images and homotopies of them are not discussed in the body of work surrounding (Ayala et al. 2003).
We end this introduction by mentioning a more general notion than that of a digital image to which many of our results apply. A tolerance space is a set with a symmetric, reflexive binary relation (which we interpret as an adjacency relation on the points of the set). Poston (1971) referred to the use of notions from (algebraic) topology in a tolerance space setting as fuzzy geometry, and used “fuzzy” terminology throughout. Sossinsky, however, makes a sharp distinction between tolerance spaces and more general “fuzzy mathematics” (see §5, ‘Tolerance is Crisp, Not Fuzzy,’ of Sossinsky 1986). For a recent, detailed history of tolerance spaces together with further examples of applications of tolerance spaces, see Peters and Wasilewski (2012). Every digital image is a tolerance space. Conversely, every finite tolerance space may be embedded in some $\mathbb{Z}^n$ as a digital image, preserving the adjacencies (we explain how in Proposition 4.16 below). But there may be many ways to “realize” a given tolerance space as a digital image. Thus, a digital image may be thought of as a tolerance space together with a particular choice of embedding into some $\mathbb{Z}^n$. Our focus is on developing homotopy theory in the context of digital images. However, many of our results apply just as well to tolerance spaces. The main difference between the two concepts, as regards our general “digital homotopy theory” program, concerns subdivision. Whereas a digital image has canonical subdivisions (that are defined in terms of the ambient $\mathbb{Z}^n$), a tolerance space does not. Generally speaking, then, results that we prove about a digital image $X$ may be interpreted equally well as results about a general tolerance space $X$, so long as the proofs do not involve subdividing $X$. Examples of this include the results of Lupton et al. (2021) through Theorem 3.22 there—including the definition of the fundamental group, its independence of the choice of basepoint, and its behaviour with respect to products. Also, Theorem 3.6 of this paper and its consequences in Sect. 4 apply equally well to tolerance spaces as to digital images (the proofs involve subdivisions of intervals—the domains of paths and loops, but do not involve subdivisions of the digital image/tolerance space). In contrast to this fluidity between digital images and tolerance spaces, Proposition 4.16 essentially concerns digital images. In Remarks 4.14 and 5.7 we emphasize other aspects of our work that are likewise particular to digital images.

## 2 Digital topology and a digital fundamental group

We review some notation and terminology from digital topology, and give a brief summary of our fundamental group from Lupton et al. (2021). Because we are dealing with the fundamental group, our basic object of interest is a based digital image; maps and homotopies will preserve basepoints.

### 2.1 Adjacency and continuity

A based digital image $X$ means a finite subset $X \subseteq \mathbb{Z}^n$ of the integral lattice in some $n$-dimensional Euclidean space together with a choice of a distinguished point $x_0 \in X$, which we refer to as the basepoint of $X$, and the following reflexive, symmetric binary relation on $X$ that we refer to as adjacency: two (not necessarily distinct) points
For a positive integer $N$ we use the notation $I_N$ or $[0, N]$ for the digital interval of length $N$. Namely, $I_N \subseteq \mathbb{Z}$ consists of the integers from 0 to $N$ (inclusive) with consecutive integers adjacent. Thus, we have $I_1 = [0, 1] = \{0, 1\}$, $I_2 = [0, 2] = \{0, 1, 2\}$, and so-on. Occasionally, we may use $I_0$ to denote the singleton point $\{0\} \subseteq \mathbb{Z}$. We will consistently choose $0 \in I_N$ as the basepoint of an interval.

For based digital images $X \subseteq \mathbb{Z}^n$ and $Y \subseteq \mathbb{Z}^m$, a function $f : X \rightarrow Y$ is continuous if $f(x) \sim_Y f(y)$ whenever $x \sim_X y$, and it is based if $f(x_0) = y_0$. By a based map of based digital images we mean a continuous based function.

### 2.2 Paths, loops and homotopies

Let $(Y, y_0)$ be a based digital image with $Y \subseteq \mathbb{Z}^n$. For any $N \geq 1$, a based path of length $N$ in $Y$ is a based path $\alpha : I_N \rightarrow Y$ (with $\alpha(0) = y_0$). Unlike in the topological setting, where any path may be taken with the fixed domain $[0, 1]$, in the digital setting we must allow paths of different lengths to have different domains. A based loop of length $N$ in $Y$ is a based path $\gamma : I_N \rightarrow Y$ that satisfies $\gamma(0) = \gamma(N) = y_0$.

A based digital image $(X, x_0)$ is connected if, for any $x \in X$ there is some based path $\alpha : I_N \rightarrow X$ (for some $N \geq 0$) with $\alpha(N) = x$.

The product of based digital images $(X, x_0)$ with $X \subseteq \mathbb{Z}^m$ and $(Y, y_0)$ with $Y \subseteq \mathbb{Z}^n$ is the based digital image $(X \times Y, (x_0, y_0))$. Here, the Cartesian product $X \times Y \subseteq \mathbb{Z}^m \times \mathbb{Z}^n \cong \mathbb{Z}^{m+n}$ has the adjacency relation $(x, y) \sim_{X \times Y} (x', y')$ when $x \sim_X x'$ and $y \sim_Y y'$.

Two based maps of based digital images $f, g : X \rightarrow Y$ are based homotopic if, for some $N \geq 1$, there is a (continuous) based map

$$H : X \times I_N \rightarrow Y,$$

with $H(x, 0) = f(x)$ and $H(x, N) = g(x)$, and $H(x_0, t) = y_0$ for all $t = 0, \ldots, N$. Then $H$ is a based homotopy from $f$ to $g$, and we write $f \approx g$.

We specialize this to the context of based loops as follows. Based loops $\alpha, \beta : I_M \rightarrow Y$ (of the same length) are based homotopic as based loops if there is a based homotopy $H : I_M \times I_N \rightarrow Y$ with $H(0, t) = H(M, t) = y_0$ for all $t \in I_N$. We refer to such a homotopy as a based homotopy of based loops and we write $\alpha \approx \beta$, even though the homotopy is more restrictive here than in the general based sense. The context should make it clear exactly what we intend our homotopies to preserve.

### Remarks 2.1

We point briefly to some differences between our conventions here and those in much of the digital topology literature. For fuller details on these differences see Remarks 2.1 and 2.5 of Lupton et al. (2021).

In the digital topology literature adjacency in $X$ is commonly defined as a relation between distinct points only. Namely, adjacency is commonly defined as a symmetric...
(but not reflexive) binary relation over $X$. Adjusting adjacency to be a reflexive and symmetric binary relation over $X$, as we choose to do, does not appear to cause any harm and doing so allows us to avoid frequent repetition of clauses like “or $x = x'$” in our arguments and definitions. Also, our convention on adjacency is consistent with that used in the tolerance space literature. We also depart from most of the digital topology literature in choosing and fixing a particular adjacency relation in $\mathbb{Z}^n$. In the literature, it is usual to allow for various choices of adjacency: in the plane $\mathbb{Z}^2$, for instance, “4-adjacency” is admitted as well as the “8-adjacency” that we choose and fix (see, e.g. Section 2 of Boxer 1999).

(Based) homotopy of digital maps has been studied by Boxer and others (see, e.g. Boxer 1999, 2005). Our definition of homotopy is visually the same as that used by these authors and in most of the literature. There is a technical difference, however, in that other authors usually take the “graph product” adjacency relation in the product $X \times I_N$, and not the adjacency relation we use (cf. remarks after Definition 2.5 of Boxer 2006). The adjacency relation we use on the product is often called the strong product, or the normal product in a graph theory setting. The fact that our homotopies must preserve more adjacencies has important consequences.

2.3 Trivial extensions and subdivision-based homotopy of based paths or loops

In our broader digital homotopy theory program, subdivision of digital images plays a prominent role. For the purposes of this paper, however, we only use a relaxed version of subdivision of an interval in the notion of a trivial extension of a path or loop. For a treatment of subdivision, see Lupton et al. (2022b) in which we discuss subdivision of maps as well as of general digital images.

Let $\alpha : I_M \to X$ be a path. A trivial extension of $\alpha$ is any path $\alpha' : I_{M'} \to X$ of the following form. For each $i$ with $0 \leq i \leq M$, choose $t_i \in \mathbb{Z}$ with $t_i \geq 0$. Then define $\alpha'$ by

$$
\alpha'(s) := \begin{cases} 
\alpha(0) & 0 \leq s \leq t_0 \\
\alpha(1) & t_0 + 1 \leq s \leq t_0 + 1 + t_1 \\
\alpha(2) & t_0 + t_1 + 2 \leq s \leq t_0 + t_1 + 2 + t_2 \\
\vdots & \\
\alpha(M) & \sum_{i=0}^{M-1} t_i + M \leq s \leq \sum_{i=0}^{M} t_i + M.
\end{cases}
$$

If we choose each $t_i = 0$, then we retrieve the original path $\alpha$. Generally, a trivial extension of $\alpha$ is a prolonged version (a re-parametrization) of $\alpha$ that repeats the value $\alpha(i)$ an extra $t_i$ times, to produce a path $\alpha' : I_{M'} \to X$ with the same image in $X$ as that of $\alpha$, but of (longer) length

$$
M' = \sum_{i=0}^{M} t_i + M.
$$
This device allows us to compare loops of different lengths and also provides flexibility in deforming loops by (based) homotopies.

Two based loops \( \alpha : I_M \to Y \) and \( \beta : I_N \to Y \) (generally of different lengths) are subdivision-based homotopic as based loops if there are trivial extensions to the same length, \( \alpha' : I_{M'} \to Y \) and \( \beta' : I_{N'} \to Y \) of \( \alpha \) and \( \beta \), respectively, with \( M' = N' \), and \( \alpha' \) and \( \beta' \) based homotopic as based loops.

In Lupton et al. (2021) (see particularly Props. B.3.2 and 3.12), we show that subdivision-based homotopy of based loops is an equivalence relation on the set of all based loops (of all lengths) in \( Y \). Denote by \( [\alpha] \) the (subdivision-based homotopy) equivalence class of based loops represented by a based loop \( \alpha : I_M \to Y \). Thus, for instance, we have \( [\alpha] = [\alpha'] \) for any trivial extension \( \alpha' : I_{M'} \to Y \) of \( \alpha \).

### 2.4 Concatenation of paths and loops

Suppose \( \alpha : I_M \to Y \) and \( \beta : I_N \to Y \) are paths—not necessarily based paths—in \( Y \) that satisfy \( \alpha(M) \sim_Y \beta(0) \). Their concatenation is the path \( \alpha \cdot \beta : I_{M+N+1} \to Y \) of length \( M + N + 1 \) in \( Y \) defined by

\[
\alpha \cdot \beta(t) := \begin{cases} 
\alpha(t) & 0 \leq t \leq M \\
\beta(t - (M + 1)) & M + 1 \leq t \leq M + N + 1.
\end{cases}
\]

If \( \alpha(M) = \beta(0) \), then our definition means that we pause for a unit interval when attaching the end of \( \alpha \) to the start of \( \beta \).

Given two based loops \( \alpha : I_M \to Y \) and \( \beta : I_N \to Y \), we form their product by concatenation:

\[
\alpha \cdot \beta : I_{M+N+1} \to Y
\]

is the based loop of length \( M + N + 1 \) defined by (1). We pause at the basepoint for a unit interval when attaching the end of \( \alpha \) to the start of \( \beta \). This product of based loops is strictly associative, as is easily checked.

### 2.5 The fundamental group

For \( Y \subseteq \mathbb{Z}^n \) a based digital image, denote the set of subdivision-based homotopy equivalence classes of based loops in \( Y \) by \( \pi_1(Y; y_0) \). As we show in Lupton et al. (2021), setting \( [\alpha] \cdot [\beta] = [\alpha \cdot \beta] \) for based loops \( \alpha : I_M \to Y \) and \( \beta : I_N \to Y \) gives a well-defined product on the set \( \pi_1(Y; y_0) \). This product is associative, since concatenation of based loops itself is associative. Now for any path \( \gamma : I_M \to Y \), let \( \overline{\gamma} : I_M \to Y \) denote the reverse path, defined as \( \overline{\gamma}(t) := \gamma(M - t) \). If \( \alpha \) is a based loop in \( Y \), then so too is its reverse \( \overline{\alpha} \). For any \( N \geq 0 \), write \( C_N : I_N \to Y \) for the constant loop defined by \( C_N(t) = y_0 \) for \( 0 \leq t \leq N \). Since any trivial extension of a constant loop is again a (longer) constant loop, it follows that all the constant loops \( C_N \) represent the same subdivision-based homotopy equivalence class of based loops, which we denote by \( e \in \pi_1(Y; y_0) \). Then \( \pi_1(Y; y_0) \) is a group, with \( e \) a two-sided
identity element and \([\overline{a}]\) a two-sided inverse element of \([a]\), for each \([a] \in \pi_1(Y, y_0)\). Full details of the topics summarized in this subsection are developed in Section 3 of Lupton et al. (2021).

### 3 Edge groups and clique complexes

The edge group of a simplicial complex is a group defined, like the digital fundamental group of a digital image or the fundamental group of a topological space, in terms of equivalence classes of edge loops (namely, loops consisting of edge paths). The equivalence relation is given by a combinatorial notion of homotopy. We repeat some of equivalence classes of edge loops (namely, loops consisting of edge paths). The group of a digital image or the fundamental group of a topological space, in terms of equivalence classes of edge loops (namely, loops consisting of edge paths). The equivalence relation is given by a combinatorial notion of homotopy. We repeat some of the definitions from (§3.3 Maunder 1996).

Suppose that \(K\) is a simplicial complex with 1-skeleton consisting of vertices \(V\) and edges \(E\). An edge path is a finite sequence \([v_0, v_1, \ldots, v_n]\) of vertices in \(V\) such that, for each \(i\) with \(0 \leq i \leq n - 1\), we have \(v_i = v_{i+1}\) or \([v_i, v_{i+1}] \in E\) an edge of \(K\). An edge path is an edge loop if we have, in addition, \(v_0 = v_n\).

**Definition 3.1** By an elementary edge-homotopy (relative the endpoints) we mean one of the following operations on edge paths:

(a) If \(v_i = v_{i+1}\), for some \(i\) with \(0 \leq i \leq n - 1\), then replace an edge path \([v_0, \ldots, v_i, v_{i+1}, v_{i+2}, \ldots v_n]\) with \([v_0, \ldots, v_i, v_{i+2}, \ldots v_n]\). Namely, delete a repeated vertex. Or, conversely, for any \(i\) with \(0 \leq i \leq n\), replace an edge path \([v_0, \ldots, v_i, v_{i+1}, \ldots v_n]\) with \([v_0, \ldots, v_i, v_{i+1}, \ldots v_n]\). Namely, insert a repeat of a vertex.

(b) If \([v_{i-1}, v_i, v_{i+1}]\) form a simplex of \(K\), for some \(i\) with \(1 \leq i \leq n - 1\), replace an edge path \([v_0, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots v_n]\) with \([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots v_n]\). Or, conversely, for any \(i\) with \(0 \leq i \leq n - 1\), replace an edge path \([v_0, \ldots, v_i, v_{i+1}, \ldots v_n]\) with \([v_0, \ldots, v_i, v, v_{i+1}, \ldots v_n]\) for any \(v \in V\) for which \([v_i, v, v_{i+1}]\) form a simplex of \(K\).

We say that two edge paths are edge-homotopic (relative their endpoints) if one can apply a finite sequence of elementary edge homotopies, of types (a) and (b) in any order or combination, so as to start with one of the edge paths and arrive at the other. We refer to the sequence of elementary edge homotopies as an edge homotopy from one edge path to the other. If two edge paths \(\alpha = [v_0, v_1, \ldots, v_n]\) and \(\beta = [w_0, w_1, \ldots, w_m]\) with \(v_0 = w_0\) and \(v_n = w_m\) are edge-homotopic (relative their endpoints), then we write \(\alpha \approx_e \beta\).

If \(K\) is a based simplicial complex with basepoint \(v_0 \in V\), and if the two edge paths in question are edge loops, each of which starts and finishes at \(v_0\), then we will refer to an edge homotopy of based loops. Two edge paths \(\alpha = [v_0, v_1, \ldots, v_n]\) and \(\beta = [w_0, w_1, \ldots, w_m]\) with \(v_n = w_0\) may be concatenated to form the edge path

\[
\alpha \cdot \beta = [v_0, v_1, \ldots, v_n, w_1, w_2, \ldots, w_m].
\]

**Remark 3.2** This concatenation differs from the way in which we concatenate suitable digital paths. In fact, we could just as well concatenate edge paths in the way in which
we do our digital paths, requiring only that \( \{v_n, w_0\} \) be an edge in \( K \). However, since we want to cite results from the literature, we use the standard way of concatenating edge paths. Doing so causes no problems for us in the development.

Suppose that \( K \) is a based simplicial complex with basepoint \( v_0 \in V \). Edge homotopy of based loops is an equivalence relation on the set of edge loops based at \( v_0 \). However, since we want to cite results from the literature, we use the standard way of concatenating edge paths. Doing so causes no problems for us in the development.

Denote the equivalence class of an edge loop \( \alpha \) by \([\alpha]\), and the set of all equivalence classes by \( E(K; v_0) \). Just as for the fundamental group, defining 
\[
[\alpha] \cdot [\beta] = [\alpha \cdot \beta] \in E(K; v_0)
\]
gives a well-defined product of equivalence classes. Concatenation of edge loops is associative, and so this product is associative. Each edge path \( \alpha = \{v_0, v_1, \ldots, v_n\} \) has a reverse, which is the edge path \( \overline{\alpha} = \{v_n, v_{n-1}, \ldots, v_0\} \). One confirms that \( \alpha \cdot \overline{\alpha} \) and \( \overline{\alpha} \cdot \alpha \) are both edge-homotopic, as based loops, to a constant loop at \( v_0 \).

**Remark 3.3** Although intuitively we may think of type (b) elementary edge homotopies as collapsing or expanding a 2-simplex, in fact there is no requirement that \( \{v_{i-1}, v_i, v_{i+1}\} \) be a 2-simplex. Indeed, to reduce a concatenation of the form \( \alpha \cdot \overline{\alpha} \) to the trivial loop \( \{v_0\} \) requires collapsing terms such as \( \ldots, v_i, v_{i+1}, v_i, \ldots \), in which \( \{v_i, v_{i+1}\} \) is an edge, to \( \ldots, v_i, v_i, \ldots \).

Then the equivalence class of the trivial loop \([\{v_0\}]\) plays the role of a two-sided identity element, and \( [\alpha]^{-1} = [\overline{\alpha}] \) defines inverses, making \( E(K; v_0) \) into a group, called the edge group of \( K \) (based at \( v_0 \)).

Now let \(|K|\) denote the geometric realization of \( K \). We have the following result.

**Theorem 3.4** (Maunder 1996, Th.3.3.9) There is an isomorphism of groups
\[
E(K; v_0) \cong \pi_1(|K|; v_0),
\]
where the right-hand side denotes the ordinary fundamental group of \(|K|\) as a topological space.

This result is often given as a means of computing the fundamental group of a topological space or, at least, arriving at a presentation of it. We refer to Maunder (1996) for details of the material just reviewed.

Now suppose that \( X \) is a digital image. We may associate to \( X \) its clique complex, which we denote by \( \text{cl}(X) \) and which is a simplicial complex whose simplices are determined by the cliques of \( X \). Namely, the vertices of \( \text{cl}(X) \) are the vertices of \( X \). The 2-cliques of \( X \), namely pairs of adjacent points, are the 1-simplices of \( X \), and so-on. In general, the \((n+1)\)-cliques of \( X \) are the \( n \)-simplices of \( \text{cl}(X) \). Now observe that the set of simplices \( \text{cl}(X) \) satisfies the requirements to be an (abstract) simplicial complex (a subset of a clique is again a clique).

We will show that the (digital) fundamental group of a digital image \( X \) is isomorphic to the edge group of the clique complex \( \text{cl}(X) \). The basic idea is to associate to each based loop \( \alpha: I_M \to X \), in an obvious way, its corresponding edge loop
\[
e(\alpha) = \{\alpha(0), \alpha(1), \ldots, \alpha(M)\}
\]
of vertices in $\text{cl}(X)$. Note that (digital) continuity of $\alpha$ ensures that $e(\alpha)$ is an edge path in $\text{cl}(X)$. Furthermore, it is an edge loop because we also have $\alpha(0) = \alpha(M) = x_0$. Then we wish to define a homomorphism

$$\phi: \pi_1(X; x_0) \to E(\text{cl}(X); x_0),$$

by setting $\phi([\alpha]) = [e(\alpha)]$. From the next lemma, it will follow that this $\phi$ is well-defined; we will complete the proof that $\phi$ gives an isomorphism of groups following that.

**Lemma 3.5** Let $\alpha, \beta: I_M \to X$ and $\gamma: I_R \to X$ be based loops in a digital image $X$.

(a) We have $e(\alpha') \approx_c e(\alpha)$ for $\alpha'$ any trivial extension of $\alpha$.
(b) If $\alpha \approx \beta: I_M \to X$ as based loops in $X$, then we have an edge homotopy of based edge loops $e(\alpha) \approx_c e(\beta)$ in $\text{cl}(X)$.
(c) If $\alpha$ and $\gamma$ are subdivision-based homotopic as based loops in $X$, then we have an edge homotopy of based edge loops $e(\alpha) \approx_c e(\gamma)$ in $\text{cl}(X)$.

**Proof** (a) In particular, we may consider elementary trivial extensions $\alpha_i: I_{M+1} \to X$ of $\alpha$, defined as

$$\alpha_i(s) := \begin{cases} \alpha(s) & 0 \leq s \leq i \\ \alpha(s-1) & i + 1 \leq s \leq M + 1 \end{cases}$$

for each $i \in I_M$. Then we have $e(\alpha) \approx_c e(\alpha_i)$, for each $i$ with $0 \leq i \leq M$, using elementary edge homotopies of type (a) from Definition 3.1. Since any trivial extension of $\alpha$ may be achieved as a finite sequence of elementary trivial extensions starting from $\alpha$, so too the edge loop $e(\alpha \circ \rho_k)$ may be achieved as the corresponding finite sequence of elementary edge homotopies of type (a) of the edge loop $e(\alpha)$.

(b) Suppose we have a based homotopy of based loops $H: I_M \times I_N \to X$ from $\alpha$ to $\beta$, for some $N$. For each $t$ with $0 \leq t \leq N$, define a loop $H_t: I_M \to X$ by $H_t(s) = H(s, t)$ for $s \in I_M$. Continuity of $H$ means that, for each $t$ with $0 \leq t \leq N - 1$ and for each $s \sim s'$ in $I_M$, we have $H_t(s) \sim_X H_{t+1}(s')$. Now define, for each $q$ with $0 \leq q \leq N - 1$, a homotopy $G_q: I_M \times I_M \to X$ by setting

$$G_q(s, t) := \begin{cases} H_{q+1}(s) & 0 \leq s \leq t \\ H_q(s) & t + 1 \leq s \leq M. \end{cases}$$

If $(s, t) \sim (s', t')$ in $I_M \times I_M$, then we have $s \sim s'$ in $I_M$. Because both $H_{q+1}(s)$ and $H_q(s)$ are adjacent to both $H_{q+1}(s')$ and $H_q(s')$, we have $G_q(s, t) \sim G_q(s', t')$ in $X$, and so $G_q$ is continuous. Then $G_q$ is a homotopy from $H_q$ to $H_{q+1}$ for each $q$ with $0 \leq q \leq N - 1$, and we have $G_q(s, M) = G_{q+1}(s, 0) = H_{q+1}(s)$ for each $s \in I_M$, each $q$ with $0 \leq q \leq N - 2$. 

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Now we may assemble the $G_t$ together into a homotopy

$$G : I_M \times I_{MN} \to X$$

by setting

$$G(s, t) := G_q(s, r)$$

if $t = Mq + r$ for $0 \leq q \leq N - 1$ and $0 \leq r \leq M - 1$

and then $G(s, MN) = G_{N-1}(s, M) = H_N(s)$. We check that this $G$ is continuous. We have $G = G_q$ when restricted to the rectangle $I_M \times [Mq, M(q + 1)]$, and so $G$ is continuous when restricted to each such rectangle. But if we have $(s, t) \sim (s', t')$ in $I_M \times I_{MN}$, then in particular we have $|t' - t| \leq 1$ and hence both $(s, t)$ and $(s', t')$ must lie in at least one such rectangle. Then $G(s, t) = G_q(s, t) \sim_X G_q(s', t') = G(s, t)$ for some $q$, and it follows that $G$ is continuous on $I_M \times I_{MN}$.

**Claim.** For each $t$ with $0 \leq t \leq MN - 1$, define the two based loops $\eta, \eta' : I_M \to X$ by $\eta(s) := G(s, t)$ and $\eta'(s) := G(s, t + 1)$. Then we have an edge homotopy of based edge loops $e(\eta) \approx_e e(\eta')$ in $\text{cl}(X)$.

**Proof of Claim.** The loops $\eta$ and $\eta'$ differ in at most one value, which means that, for some $S$ and $1 \leq S \leq M - 1$, we have $\eta(s) = \eta'(s)$ for $s \neq S$, and $\eta(S) \sim_X \eta'(S)$ (these may agree also, in which case we have $\eta = \eta'$). This follows from the way in which we have constructed the homotopy $G$. Then an edge homotopy from $e(\eta)$ to $e(\eta')$ is given by the sequence of elementary edge homotopies

$$\ldots, \eta(S - 1), \eta(S), \eta(S + 1), \ldots \approx_e \ldots, \eta(S - 1), \eta'(S), \eta(S), \eta(S + 1), \ldots \approx_e \ldots, \eta(S - 1), \eta'(S), \eta(S + 1), \ldots,$$

in which the first edge path is $e(\eta)$ and the last is $e(\eta')$. The first elementary edge homotopy inserts the vertex $\eta'(S)$ between $\eta(S - 1)$ and $\eta(S)$. This is permissible since $G(S - 1, t), G(S, t)$ and $G(S, t + 1)$ form a 3-clique in $X$, from continuity of $G$. The second elementary edge homotopy deletes the vertex $\eta(S)$ from between $\eta'(S)$ and $\eta(S + 1)$. Again, this is permissible since continuity of $G$ implies that $G(S, t + 1), G(S, t)$ and $G(S + 1, t)$ form a 3-clique in $X$. **End of Proof of Claim.**

Since $e(\eta) \approx_e e(\eta')$ in $\text{cl}(X)$ for each $t$, transitivity of edge homotopies now gives $e(\alpha) \approx_e e(\beta)$ in $\text{cl}(X)$.

(c) Now suppose that $\alpha$ and $\gamma$ are subdivision-based homotopic as based loops in $X$ so, for some $\alpha'$ and $\gamma'$ trivial extensions to the same length of $\alpha$ and $\gamma$, we have a based homotopy of based loops $\alpha' \approx \gamma'$. But then we have edge homotopies of based edge loops

$$e(\alpha) \approx_e e(\alpha') \approx_e e(\gamma') \approx_e e(\gamma)$$

in $\text{cl}(X)$, with the first and third edge homotopies coming from part (a), and the middle edge homotopy from part (b). Then part (c) follows from transitivity of edge homotopies.
Theorem 3.6 Let $X$ be a digital image and $\text{cl}(X)$ its clique complex. The map

$$\phi : \pi_1(X; x_0) \to E(\text{cl}(X); x_0),$$

defined by setting $\phi([\alpha]) = [e(\alpha)]$ is an isomorphism of groups.

Proof The map $\phi$ is well-defined by Lemma 3.5. The concatenation of two based loops $\alpha \cdot \beta$ has a repeat of the basepoint at times $M$ and $M + 1$ (if $\alpha$ is of length $M$). But then we may use an elementary edge homotopy of type (a) to delete this repetition, so that we have

$$e(\alpha \cdot \beta) \approx e e(\alpha) \cdot e(\beta),$$

where the right-hand side refers to (the standard) concatenation of edge loops in $\text{cl}(X)$. It follows that $\phi$ is a homomorphism. Any edge loop $\{v_0, \ldots, v_M\}$ in $\text{cl}(X)$ may be viewed as $e(\alpha)$, where $\alpha : I_M \to X$ is the path $\alpha(i) = v_i$ for $0 \leq i \leq M$. Continuity of $\alpha$ follows because $v_i$ and $v_{i+1}$ must be adjacent in $X$ for there to be an edge joining them in $\text{cl}(X)$. So $\phi$ is evidently onto.

It remains to show that $\phi$ is also injective. For this it is sufficient to show that if two edge loops, which—as we just observed—we may assume are of the form $e(\alpha)$ and $e(\beta)$ for loops $\alpha$ and $\beta$ in $X$, are homotopic via an elementary edge homotopy, then the loops $\alpha$ and $\beta$ are subdivision-based homotopic. So first suppose that $e(\beta)$ is edge homotopic to $e(\alpha)$ by an elementary edge homotopy of type (a). Then the paths $\alpha$ and $\beta$ differ by a repeated vertex and so one is a trivial extension of the other: we have $[\beta] = [\alpha]$ in $\pi_1(X; x_0)$. Next, suppose that $e(\beta)$ is edge homotopic to $e(\alpha)$ by an elementary edge homotopy of type (b)—addition of a vertex $v$ between two vertices $\alpha(j)$ and $\alpha(j+1)$ with $[\alpha(j), v, \alpha(j+1)]$ a simplex of $\text{cl}(X)$. But if $\{v_j, v, v_{j+1}\}$ is a simplex of $\text{cl}(X)$, then we have $\alpha(j) \sim \alpha(j + 1)$ and $v$ is adjacent to both of these in $X$. Let $\alpha_j$ denote the elementary trivial extension of $\alpha$ obtained by repeating the value $\alpha(j)$, as in the proof of (a) of Lemma 3.5. We may define a homotopy

$$H : I_{M+1} \times I_1 \to X,$$

assuming $\alpha$ is of length $M$, by setting

$$H(s, t) := \begin{cases} 
\alpha(s) & 0 \leq s \leq j \\
\alpha(j) & (s, t) = (j + 1, 0) \\
v & (s, t) = (j + 1, 1) \\
\alpha(s - 1) & j + 2 \leq s \leq M + 1.
\end{cases}$$

It is easy to confirm that $H$ is continuous, and that it is a based homotopy of based loops $\alpha_j \approx \beta$. In $\pi_1(X; x_0)$, then, we have

$$[\alpha] = [\alpha_j] = [\beta],$$
where the first re-write follows because $\alpha_j$ is a trivial extension of $\alpha$ and the second follows from the homotopy $H$ above. Thus, for each type of elementary edge homotopy, we have established that $e(\alpha) \approx e(\beta)$ implies $[\alpha] = [\beta] \in \pi_1(X; x_0)$. Injectivity of $\phi$ follows, and this completes the proof.  

4 Direct consequences for the digital fundamental group

Because so much is known about edge groups of simplicial complexes and the fundamental groups of topological spaces, it is now easy to compile many basic results about the digital fundamental group. We simply translate known facts and results from the topological setting to the digital setting, wherever feasible. We begin by considering digital images of the kind known in the digital topology literature as a “digital simple closed curve” (see Section 3 of Boxer 1999). Conceptually, these are digital images that correspond in the topological setting to a circle up to homeomorphism.

**Definition 4.1** Consider a set $C = \{x_0, x_1, \ldots, x_{N-1}\}$ of $N$ (distinct) points in $\mathbb{Z}^n$, with $N \geq 4$ and for any $n \geq 2$. We say that $C$ is a digital simple closed curve of length $N$ if we have adjacencies $x_i \sim_C x_{i+1}$ for each $0 \leq i \leq N - 2$, and $x_{N-1} \sim_C x_0$, and no other adjacencies amongst the elements of $C$.

We may parametrize a digital simple closed curve as a loop $\alpha: I_N \to C$ (in various ways).

**Theorem 4.2** $\pi_1(C; x_0) \cong \mathbb{Z}$ for every digital simple closed curve $C$.

**Proof** The clique complex of a digital simple closed curve is a cycle graph, with geometric realization an actual circle $S^1$ up to homeomorphism. The result follows from Theorems 3.6, 3.4 and the well-known, basic calculation of $\pi_1(S^1; x_0) \cong \mathbb{Z}$ (e.g. (Massey 1991, Th.II.5.1)).

**Remark 4.3** We have shown in Lupton et al. (2021) that the particular 4-point digital simple closed curve $D := \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ in $\mathbb{Z}^2$, which we called the diamond and which may be viewed as an actual digital circle, has fundamental group $\pi_1(D; x_0) \cong \mathbb{Z}$. This computation was done staying within digital topology, using some results we developed in Lupton et al. (2022a). This gives a computation of $\pi_1(S^1; x_0) \cong \mathbb{Z}$ independently of the usual topological argument, through the identifications

$$\pi_1(S^1; x_0) \cong \pi_1(|\text{cl}(D)|; x_0) \cong \pi_1(D; x_0)$$

of Theorems 3.6 and 3.4. Furthermore, these theorems allow us to lever the single computation $\pi_1(D; x_0) \cong \mathbb{Z}$ into a computation of the fundamental group of any digital simple closed curve $C$, because the geometric realizations $|\text{cl}(D)|$ and $|\text{cl}(C)|$ are both homeomorphic to the circle $S^1$. Note that digital simple closed curves of different lengths are not (digitally) based-homotopy equivalent. We suspect that any two digital simple closed curves are subdivision-based homotopy equivalent (as we...
defined this notion in Lupton et al. 2021). However, we are as yet unable to establish this because the arguments become bogged down in lengthy expositional details. The digital fundamental group is preserved by this notion of subdivision-based homotopy equivalence of digital images. But the isomorphism $\pi_1(D; x_0) \cong \pi_1(C; x_0)$, for any digital simple closed curve $C$, is available to us without having to establish $D$ and $C$ as subdivision-based homotopy equivalent. These comments indicate that, speaking generally, enlarged or reduced versions of a digital image should have the same fundamental group as the original, even though they will not be digitally homotopy equivalent, and even though we may not be able to show them subdivision-based homotopy equivalent. This is because we may—at the fundamental group level—pass into the topological setting, enlarge or reduce there, and then pass back into the digital setting.

We now deduce a general result that enables calculation of many examples. In the topological setting, the Seifert–van Kampen theorem describes the fundamental group of a union $\pi_1(U \cup V; x_0)$ as a pushout—in the category of groups and group homomorphisms (a.k.a. the free product with amalgamation, see, e.g. Rotman 1995)—involving the fundamental groups $\pi_1(U; x_0)$, $\pi_1(V; x_0)$ and $\pi_1(U \cap V; x_0)$ and the obvious (induced) homomorphisms.

We will need to place certain mild constraints on the union.

**Definition 4.4** Suppose $U$ and $V$ are digital images in some $\mathbb{Z}^n$. Denote by $U' = \{ v \in V \mid v \notin V \cap U \}$ the complement of $U$ in $U \cup V$ and by $V' = \{ u \in U \mid u \notin U \cap V \}$ the complement of $V$ in $U \cup V$. We say that $U$ and $V$ have disconnected complements (in $U \cup V$) if $U'$ and $V'$ are disconnected from each other. That is, $U$ and $V$ have disconnected complements when the set of pairs $\{u, v\}$ with $u \in V'$, $v \in U'$ and $u \sim_{U \cup V} v$ is empty.

**Lemma 4.5** Suppose $U$ and $V$ are digital images in some $\mathbb{Z}^n$.

(a) We have $\text{cl}(U) \cap \text{cl}(V) = \text{cl}(U \cap V)$.

(b) In general we have $\text{cl}(U) \cup \text{cl}(V) \subseteq \text{cl}(U \cap V)$ and if we assume that $U$ and $V$ have disconnected complements, then we also have $\text{cl}(U) \cup \text{cl}(V) = \text{cl}(U \cup V)$.

**Proof**

(a) Suppose $\sigma$ is an $n$-simplex in $\text{cl}(U) \cap \text{cl}(V)$ with vertices $\{x_0, x_1, \ldots, x_n\} \subseteq U \cup V$. Then the vertices form an $(n+1)$-clique in $U$ and an $(n+1)$-clique in $V$, hence they form an $(n+1)$-clique in $U \cap V$. Thus $\sigma$ is also in $\text{cl}(U \cap V)$. The containment in the other direction is just as easy to see.

(b) Suppose $\sigma$ is an $n$-simplex in $\text{cl}(U) \cup \text{cl}(V)$. Then its vertices $\{x_0, x_1, \ldots, x_n\}$ form an $(n+1)$-clique in $U$ or in $V$. It does no harm to view these vertices as lying in $U \cup V$, and hence $\sigma \in \text{cl}(U \cap V)$. This shows the general inclusion. Conversely, suppose we have $\sigma$ an $n$-simplex in $\text{cl}(U \cup V)$. If $n = 0$, this simply means that $x_0$ is a vertex in either $U$ or $V$, and thus $\sigma \in \text{cl}(U) \cup \text{cl}(V)$. If $n \geq 1$, however, we need the disconnected complements hypothesis to obtain this inclusion. The vertices $\{x_0, x_1, \ldots, x_n\}$ form an $(n+1)$-clique in $U \cup V$, and the disconnected complements hypothesis entails that this clique must lie entirely in $U$ or entirely in $V$: we cannot have some points of a clique in $V'$ and other points of the same clique in $U'$, in the notation of Definition 4.4. That is, we have $\text{cl}(U \cup V) \subseteq \text{cl}(U) \cup \text{cl}(V)$ and hence the two are equal. \qed
Theorem 4.6 (Digital Seifert–van Kampen) Let $U$ and $V$ be connected digital images in some $\mathbb{Z}^n$ with connected intersection $U \cap V$. Choose $x_0 \in U \cap V$ for the basepoint of $U \cap V$, $U$, $V$, and $U \cup V$. If $U$ and $V$ have disconnected complements, then

\[
\begin{align*}
\pi_1(U \cap V; x_0) & \xrightarrow{i_1} \pi_1(U; x_0) \\
& \Downarrow i_2 \Downarrow \psi_1 \\
\pi_1(V; x_0) & \xrightarrow{\psi_2} \pi_1(U \cup V; x_0)
\end{align*}
\]

is a pushout diagram of groups and homomorphisms, with $i_1$, $i_2$, $\psi_1$ and $\psi_2$ the homomorphisms of fundamental groups induced by the inclusions $U \cap V \to U$, $U \cap V \to V$, $U \to U \cup V$ and $V \to U \cup V$ respectively.

Proof We have isomorphisms $\pi_1(U \cap V; x_0) \cong E(\text{cl}(U) \cap \text{cl}(V); x_0)$ and $\pi_1(U \cup V; x_0) \cong E(\text{cl}(U) \cup \text{cl}(V); x_0)$, from Lemma 4.5 and Theorem 3.6. Now we may apply the ordinary Seifert–van Kampen theorem from the topological setting in the form for simplicial complexes (see, e.g. Rotman 1995, Th.11.60) to the inclusions of connected simplicial (sub-) complexes

\[
\begin{align*}
\text{cl}(U) \cap \text{cl}(V) & \longrightarrow \text{cl}(U) \\
& \Downarrow \Downarrow \\
\text{cl}(V) & \longrightarrow \text{cl}(U) \cup \text{cl}(V)
\end{align*}
\]

and conclude the result via Theorems 3.6 and 3.4. \qed

Remark 4.7 We make no assumptions about any of the induced homomorphisms $i_1$, $i_2$, $\psi_1$ and $\psi_2$ being injective. Depending on the circumstances, some or all of them, in various combinations, may be injective. But none of them need be injective.

Remark 4.8 Theorem 4.6 identifies $\pi_1(U \cup V; x_0)$ up to isomorphism, although it does so indirectly in terms of a universal property. For $U$ and $V$ that satisfy the hypotheses, a more concrete description of $\pi_1(U \cup V; x_0)$ may be given as follows (see Th.11.58 of Rotman (1995) for example). We have an isomorphism

\[
\pi_1(U \cup V; x_0) \cong \frac{\pi_1(U; x_0) * \pi_1(V; x_0)}{N},
\]

where $\pi_1(U; x_0) * \pi_1(V; x_0)$ denotes the free product and $N$ the normal subgroup generated by $\{i_1(g)i_2(g^{-1}) \mid g \in \pi_1(U \cap V; x_0)\}$. Or, in terms of presentations, if $\pi_1(U; x_0) = \langle G_1 \mid R_1 \rangle$ and $\pi_1(V; x_0) = \langle G_2 \mid R_2 \rangle$, where the $G_i$ and $R_i$ are sets of generators and relations, then we have a presentation

\[
\pi_1(U \cup V; x_0) = \langle G_1 \cup G_2 \mid R_1 \cup R_2 \cup \{i_1(g)i_2(g^{-1}) \mid g \in \pi_1(U \cap V; x_0)\} \rangle.
\]
Remark 4.9 The conclusion of Theorem 4.6 need not hold if $U$ and $V$ do not have disconnected complements. For example, take $U, V \subseteq \mathbb{Z}^2$ as follows.

$$U = \{(1, 0), (0, 1)\}, \quad V = \{(1, 0), (0, -1), (-1, 0)\},$$

so that $U \cup V = D$, the diamond, and $U \cap V = \{(1, 0)\}$. Then we have $(-1, 0) \sim (0, 1)$ with $(-1, 0) \in U'$ and $(0, 1) \in V'$, and so $U$ and $V$ do not have disconnected complements. Furthermore, we know from Lupton et al. (2021) or Theorem 4.2 above that $\pi_1(D; (1, 0)) \cong \mathbb{Z}$, whereas here we have $U$ and $V$ both contractible with trivial fundamental group. Evidently, the conclusion of the theorem does not hold. Specifically, here, the issue is that—concomitant with $U'$ and $V'$ not being disconnected—we have $\text{cl}(U) \cup \text{cl}(V)$ strictly contained in (not equal to) $\text{cl}(U \cup V)$.

Remark 4.10 It is possible to prove Theorem 4.6 entirely within the digital setting (without relying on Theorems 3.6 and 3.4). Surprisingly, perhaps, we are able to prove Theorem 4.6 by adapting the argument that is used in Massey (1977) to prove the topological Seifert–van Kampen theorem there. That argument uses the Lebesgue covering lemma, from the theory of compact metric spaces. In our digital setting, we find that it is possible to follow the same argument without really having to develop a substitute for this ingredient. It turns out that dividing a rectangle $I_M \times I_N$ into unit squares achieves the same purpose as does dividing the rectangle $I \times I$ into subrectangles of diameter less than the Lebesgue number of a certain covering of $I \times I$ in the topological setting.

Remark 4.11 Ayala et al. (2003) have a Seifert–van Kampen theorem for the digital fundamental groups they consider. However, as we mentioned in the introduction, their approach is effectively to define the fundamental group as that of an associated simplicial complex, so it a priori will obey the Seifert–van Kampen theorem and possess any other properties of the topological fundamental group. The difference between that approach and ours is that we have an intrinsic, self-contained construction of the fundamental group in the digital setting, and we need to establish Theorems 3.6 and 3.4 in order to make use of the properties of the topological fundamental group.

There are some special cases of Theorem 4.6 that are especially useful. First, consider the case in which the intersection has trivial fundamental group (cf. Massey 1991, Th.IV.3.1).

**Corollary 4.12** Suppose $U$ and $V$ satisfy the hypotheses of Theorem 4.6 (including disconnected complements) and, in addition, we have $\pi_1(U \cap V; x_0) = \{e\}$. Then we have

$$\pi_1(U \cup V; x_0) \cong \pi_1(U; x_0) \ast \pi_1(V; x_0),$$

where the right-hand side denotes the free product of groups.

**Proof** Direct from Theorem 4.6. □
In particular, if we have \( U \cap V = \{x_0\} \), so that \( U \cup V \) is a one-point union of \( U \) and \( V \), and if \( U \) and \( V \) have disconnected complements in \( U \cup V \), then we have \( \pi_1(U \cup V; x_0) \cong \pi_1(U; x_0) * \pi_1(V; x_0) \).

**Example 4.13** Let \( D = \{(1, 0), (0, 1), (-1, 0), (0, -1)\} \) be the diamond in \( \mathbb{Z}^2 \), with basepoint \((1, 0)\). The “double diamond” in \( \mathbb{Z}^2 \), with basepoint \((0, 0)\) pictured in Fig. 1 may be viewed as a one-point union \( D \vee D \) of two isomorphic copies of \( D \). With \( U \) and \( V \) the right-hand and the left-hand copies of \( D \), respectively, we have \( U \cap V = \{(0, 0)\} \), a single point. Since \( \pi_1(D; x_0) \cong \mathbb{Z} \), it follows from Corollary 4.12 that we have \( \pi_1(D \vee D; x_0) \cong \mathbb{Z} * \mathbb{Z} \). Alternatively, we could just as well deduce the same conclusion by observing that \( \text{cl}(D \vee D) \) has geometric realization homeomorphic to \( S^1 \vee S^1 \), the one-point union of two circles, and using the well-known result that \( \pi_1(S^1 \vee S^1; x_0) \cong \mathbb{Z} * \mathbb{Z} \) (e.g., Massey 1991, Ex.IV.3.1) together with Theorems 3.6 and 3.4. This example illustrates that a digital image may have non-abelian fundamental group.

**Remark 4.14** The notation \( D \vee D \) used in Example 4.13 follows that used for the topological one-point union (the coproduct in the pointed setting). But this construction requires some care in the digital setting. For instance, in the topological setting we may take \( I \vee I \) to be the subset \( I \times \{0\} \cup \{0\} \times I \) of the product \( I \times I \). In the digital setting, however, with the unit digital interval \( I_1 \) in the role of \( I \), taking \( U = \{(0, 0), (1, 0)\} \) and \( V = \{(0, 0), (0, 1)\} \) would not produce the desired \( I_1 \vee I_1 \) as \( U \cup V \) because this \( U \) and \( V \) do not have disconnected complements. The one-point union of tolerance spaces, by contrast, does not present this issue: one identifies the two tolerance spaces at their basepoints in the abstract, and no additional, undesired adjacencies are forced. We discussed a similar distinction between digital and tolerance settings, in the context of pushouts, in Remark 4.7 of Lupton et al. (2022a) and the results surrounding it there.

Another special case of Theorem 4.6 that is often useful is the case in which one of \( U \) or \( V \) is contractible or, at least, has trivial fundamental group (cf. Massey 1991, Th.IV.4.1).

**Corollary 4.15** Suppose \( U \) and \( V \) satisfy the hypotheses of Theorem 4.6 (including disconnected complements) and, in addition, we have \( \pi_1(V; x_0) = \{e\} \). Then \( \psi_1 : \pi_1(U) \to \pi_1(U \cup V) \) is an epimorphism, and its kernel is the smallest normal subgroup of \( \pi_1(U) \) containing the image \( \phi_1[\pi_1(U \cap V)] \).
Our next example will display a digital image with fundamental group isomorphic to $\mathbb{Z}_2$. Our approach here is to “reverse-engineer” a digital image $X$ so that the geometric realization of $\text{cl}(X)$ is homeomorphic to the real projective plane $\mathbb{R}P^2$. The approach depends in part on being able to realize a graph as a digital image. We now describe a general procedure for doing this.

Recall our discussion of tolerance spaces from the introduction. A tolerance space may be viewed as a graph (no double edges or loops at a vertex), and vice versa, by interpreting “adjacent vertices” in the tolerance space as “vertices connected by an edge” in the graph (ignoring self-adjacencies). In the following, and in the sequel, by an “isomorphism” across the structures of digital images, on the one hand, and simple graphs/tolerance spaces, on the other, we mean an adjacency-preserving bijection of the vertices with an adjacency-preserving inverse (again, ignoring self-adjacencies).

**Proposition 4.16** If $G$ is a finite graph (a finite tolerance space), then $G$ may be isomorphically embedded as a digital image with vertices in the hypercube $[-1, 1]^{n-1} \subseteq \mathbb{Z}^{n-1}$, where $n = |G|$, the number of vertices.

**Proof** Work by induction on $n$. Induction starts with $n = 1$ (or $n = 2$), where there is nothing to show.

Inductively assume that, if $|G| \leq n$, then we may embed $G$ as a digital image in $[-1, 1]^{n-1}$. Suppose we have a graph $G'$ with $n + 1$ vertices. Choose any vertex $x \in G'$ and write $G' = G \cup \{x\}$ with $|G'| = n$. Embed $G$ as a digital image in $[-1, 1]^{n-1} \subseteq \mathbb{Z}^{n-1} \subseteq \mathbb{Z}^{n-1} \times \mathbb{Z} = \mathbb{Z}^n$. Then each vertex $y \in G$ has coordinates $y = (y_1, \ldots, y_{n-1}, 0) \in \mathbb{Z}^n$, and we have $y_i \in \{\pm 1, 0\}$ for $i = 1, \ldots, n - 1$. Denote by $\text{lk}(v)$ the (vertices of the) link of a vertex $v$ in a graph, namely, the set of vertices (other than $v$) connected by an edge to $v$. Now separate the vertices of $G$ into the disjoint union $G = \text{lk}(x) \cup \text{lk}(x)^C$. For each $y \in \text{lk}(x)^C$, move it down to the plane $y_n = -1$. In other words, adjust the embedding of $G$ in $\mathbb{Z}^n$ using the isomorphism of digital images $\phi: G \to \overline{G}$ given by

$$\phi(y_1, \ldots, y_{n-1}, 0) = \begin{cases} (y_1, \ldots, y_{n-1}, 0) & \text{if } y \in \text{lk}(x) \\ (y_1, \ldots, y_{n-1}, -1) & \text{if } y \in \text{lk}(x)^C \end{cases}$$

This is an isomorphism, since we have—for $y, y' \in \mathbb{Z}^{n-1} \times \{0\} \subseteq \mathbb{Z}^n$

$$y \sim_{\mathbb{Z}^n} y' \iff (y_1, \ldots, y_{n-1}) \sim_{\mathbb{Z}^{n-1}} (y'_1, \ldots, y'_{n-1}) \iff \phi(y) \sim_{\mathbb{Z}^n} \phi(y').$$

So we now have $G$ embedded in $\mathbb{Z}^n$ as a digital image with $\text{lk}(x) \subseteq [-1, 1]^{n-1} \times \{0\} \subseteq \mathbb{Z}^n$ and $\text{lk}(x)^C \subseteq [-1, 1]^{n-1} \times \{-1\} \subseteq \mathbb{Z}^n$. Add $x$ as the point $x = e_n = (0, \ldots, 0, 1)$. This point is adjacent to every point in $[-1, 1]^{n-1} \times \{0\} \subseteq \mathbb{Z}^n$, and hence to every point of $\text{lk}(x)$ as we have embedded it. Furthermore, $x = e_n$ is not adjacent to any point of $[-1, 1]^{n-1} \times \{-1\} \subseteq \mathbb{Z}^n$, and so this produces exactly the adjacencies of $x$ from $G'$. This completes the induction; the result follows. □
Example 4.17  Consider the cycle graph of length 4 (complete bipartite graph $K_{2,2}$) illustrated in Fig. 2. Start by embedding the edge $\{v_0, v_1\}$ in $\mathbb{Z}$ as $\{0, 1\}$ and follow the inductive step in the proof of Proposition 4.16 adding vertex $v_2$ and then vertex $v_3$. This leads to an embedding of the graph as the digital image in $\mathbb{Z}^3$ with vertices

$$v_0 = (0, -1, 0), \quad v_1 = (1, 0, -1), \quad v_2 = (0, 1, 0), \quad v_3 = (0, 0, 1).$$

Example 4.18  As announced above, we now construct a digital image $X$ that may be viewed as a digital version of the real projective plane $\mathbb{R}P^2$. Start with a suitable triangulation of $\mathbb{R}P^2$. Notice that some care must be taken here. For example, the triangulation of $\mathbb{R}P^2$ given in (Massey 1991, Ex.I.6.2) (see Figure 1.13 on p. 15 of Massey 1991) is not suitable. This is because the clique complex of that triangulation, considered as a graph, contains simplices that are not part of the triangulation (the triangulation has “empty” simplices, and so is not a clique, or flag complex). For example, with reference to the notation of (Massey 1991, Ex.I.6.2), the 3-clique 123 does not correspond to a 2-simplex of the triangulation. Indeed, the triangulation of (Massey 1991, Ex.I.6.2), considered as a graph, is actually a complete graph, and so its clique complex would be a 5-simplex, with contractible geometric realization.

Instead, we may use the triangulation of $\mathbb{R}P^2$ (represented as the disc with antipodal points of the boundary circle identified) illustrated in Fig. 3.

Observe that this triangulation, considered as a graph (after making the identifications indicated), contains 3-cliques, each of which corresponds to a 2-simplex of the triangulation, and does not contain any 4-cliques. Therefore, if $G$ is the (abstract) graph, or tolerance space illustrated, its clique complex will give $\text{cl}(G) = K$, where $K$ is the (abstract) simplicial complex indicated, and thus $|\text{cl}(G)|$ will be homeomorphic to $\mathbb{R}P^2$.

It remains to display the abstract graph/tolerance space $G$ as a digital image, up to isomorphism. Proposition 4.16 provides a recipe for doing so which, starting from an edge embedded in $\mathbb{Z}$, results in a digital image $X$ in $\mathbb{Z}^{12}$. For this digital image $X$, by construction, we have $\text{cl}(X)$ isomorphic to the complex represented by $G$, as a simplicial complex, and thus the geometric realization $|\text{cl}(X)|$ is homeomorphic to $\mathbb{R}P^2$. As is well-known, we have $\pi_1(\mathbb{R}P^2; x_0) \cong \mathbb{Z}_2$ (see Massey 1991, Ex.V.5.2, for example). From Theorems 3.6 and 3.4, it follows that we have

$$\pi_1(X; x_0) \cong \mathbb{Z}_2.$$
Notice that it would also be possible to calculate $\pi_1(X; x_0) \cong \mathbb{Z}_2$ using Corollary 4.15, mimicking the steps in the argument used for (Massey 1991, Ex.V.5.2). This example illustrates that a digital image may have torsion in its fundamental group.

Finally, for this section, we use the approach of Example 4.18 to show the following general realization result.

**Theorem 4.19** Every finitely presented group occurs as the (digital) fundamental group of some digital image.

**Proof** Suppose $G$ is a finitely presented group with finite presentation

$$G = \langle g_1, \ldots, g_n \mid R_1, \ldots, R_m \rangle.$$

Here, each $R_j$ is a word in the $g_i$ and their inverses $g_i^{-1}$. We may suppose these words are in reduced form (no occurrences of a generator juxtaposed with its own inverse). First we build in the usual way, but taking care to avoid empty simplices, a two-dimensional simplicial complex with this $G$ as edge group. For the one-skeleton, take an $n$-fold one-point union of length-4 cycle graphs with vertices

$$V = \{v_0\} \cup \bigcup_{i=1}^{n} \{v_i, v_{i,1}, v_{i,2}, v_{i,3}\}$$

and edges

$$E = \bigcup_{i=1}^{n} \{\{v_0, v_{i,1}\}, \{v_{i,1}, v_{i,2}\}, \{v_{i,2}, v_{i,3}\}, \{v_{i,3}, v_0\}\}.$$
The case in which \( n = 2 \) is illustrated in Fig. 4 below. The edge group of this graph is the free group on \( n \) generators, which we may identify with the free group \( \langle g_1, \ldots, g_n \rangle \) in an obvious way. Namely, each generator \( g_i \) corresponds to the edge loop \( \{v_0, v_{i,1}, v_{i,2}, v_{i,3}, v_0\} \) of length 4. The inverse of a generator corresponds to the reverse path: \( g_i^{-1} \) corresponds to the edge loop \( \{v_0, v_{i,3}, v_{i,2}, v_{i,1}, v_0\} \). Because each of the generating cycle graphs is of length four, there are no 3-cliques in this graph, hence no empty 2-simplices.

Next, for each relator \( R_j \), we wish to attach a (triangulated) disk so as to introduce this relation into the edge group. Here, again, we just have to be careful not to introduce any empty 3-simplices. We may achieve this as follows. Consider a single relator \( R \). Suppose \( R \) is a word

\[
R = g_{j_1}^{\epsilon_1} \cdots g_{j_k}^{\epsilon_k}
\]

of length \( k \) in the letters \( \{g_i, g_i^{-1}\} \), with each \( \epsilon_r \) either 1 or \(-1\). Define a cycle graph \( C \) of length \( 4k \) whose vertices we list in order as

\[
V_C = \{w_1, w_{1,1}, w_{1,2}, w_{1,3}, w_2, w_{2,1}, w_{2,2}, w_{2,3}, w_3, \ldots, w_k, w_{k,1}, w_{k,2}, w_{k,3}\},
\]

with adjacent vertices of this list joined by an edge of \( C \), as well as the last vertex \( w_{k,3} \) and the first vertex \( w_1 \) joined by an edge. Take a copy of this cycle graph \( C' \) with vertices

\[
V_{C'} = \{w'_1, w'_{1,1}, w'_{1,2}, w'_{1,3}, w'_2, w'_{2,1}, w'_{2,2}, w'_{2,3}, w'_3, \ldots, w'_k, w'_{k,1}, w'_{k,2}, w'_{k,3}\}.
\]

Now join the \( i \)th listed vertex of \( C \) to the \( i \)th and \((i + 1)\)st listed vertices of \( C' \) (treating the \((4k + 1)\)st as the first). This creates a “triangulated annulus,” with \( C \) as outer boundary and \( C' \) as inner boundary. Finally, add another vertex \( w \), and join this vertex to every vertex of \( C' \). A case in which \( k = 3 \) is illustrated in Example 4.20 below (see Fig. 5).

So far, we have built a triangulated disk that has no 4-cliques. Now attach this disk to the one-point union of length-4 cycle graphs, according as the letters of the relator \( R \). Namely, identify for each \( i \) the edge loops (vertex-for-vertex and edge-for-edge)

\[
w_i, w_{i,1}, w_{i,2}, w_{i,3}, w_{i+1} \text{ with } \begin{cases} v_0, v_{i,1}, v_{i,2}, v_{i,3}, v_0 & \text{if } \epsilon_i = 1 \\ v_0, v_{i,3}, v_{i,2}, v_{i,1}, v_0 & \text{if } \epsilon_i = -1. \end{cases}
\]

Now it is standard that attaching this disk in this way introduces the relation \( R \) into the edge group (and no other relations). The main point here, though, is that we have introduced the desired relation by building a 2-dimensional simplicial complex that has no empty 2-simplices, and no 4-cliques (hence no empty 3- or higher simplices). Considering the 1-skeleton of the complex after attaching the disk as a graph, its clique complex is the 2-dimensional complex we have constructed.

It is clear that we may apply this last step to each of the relators \( R_j \). Doing so constructs a 2-dimensional simplicial complex \( K \), that is the original one-point union...
of $n$ length-4 cycle graphs, with $m$ triangulated disks attached as in the step above. The edge group of $K$, by construction, is $G$. As a (finite, simple) graph, we may embed the one-skeleton of $K$ into some $\mathbb{Z}^n$ (possibly a high-dimensional such) as a digital image, following the scheme of Proposition 4.16. Furthermore, from the way in which we have constructed and attached the triangulated disks, the clique complex of this digital image, considered as the graph we started from, is exactly $K$. Then the (digital) fundamental group of this digital image is $G$, as follows from Theorem 3.6. \hfill \Box

Example 4.20 We illustrate the above result with an example. Take

$$G = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} \rangle.$$  

Following the recipe of the proof of Theorem 4.19, we start with a graph that is a one-point union of two cycle graphs of length 4:

Next, we construct a triangulated disk whose boundary corresponds to the relation we wish to introduce. Once again following the recipe of the proof of Theorem 4.19, this will consist of: a cycle graph of length 12; an intermediate cycle graph of the same length; an evident triangulation of the “annulus” with these cycle graphs as boundary; a cone-point added to “cone-off” the inner cycle graph. In Fig. 5, we have illustrated the result, and also indicated the identifications we make along the boundary, with vertices and edges identified with their counterparts in the one-point union illustrated above.

Identifying the boundary of this triangulated disk, in the way indicated, to the one-point union of length-4 cycle graphs illustrated in Fig. 4 results in a 2-dimensional simplicial complex whose edge group is $G$. This simplicial complex has $7 + 12 + 1 = 20$ vertices. Following our general scheme for embedding a graph as a digital image, we may realize the one-skeleton of this complex as a digital image in some $\mathbb{Z}^n$ with $n \leq 19$ (considerably less should be possible). Furthermore, the clique complex of this digital image, considered as the graph that we realized, has clique complex exactly this simplicial complex, with edge group $G$. This digital image realizes the group $G$.

Remark 4.21 Theorem 4.19, Proposition 4.16, and Examples 4.17, 4.18 and 4.20 taken together raise interesting questions. First, is every homotopy type the geometric realization of a simplicial complex that is a clique complex? As we saw in Example 4.18, triangulations commonly used to represent a space as a simplicial complex need not be clique complexes. Second, suppose we have a homotopy type represented as the geometric realization of some clique complex $\text{cl}(G)$. Then the recipe of Proposition 4.16
Fig. 5 Triangulated disk, plus attachments

will display $G$ as a digital image, but the embedding dimension that results may be unnecessarily high. For instance, following the recipe in Example 4.17 leads to an embedding of the cycle graph in $\mathbb{Z}^3$. But this cycle graph may also be embedded in $\mathbb{Z}^2$ as the diamond $D$ of Remark 4.3. In this case, it is evident that the cycle graph cannot be embedded in $\mathbb{Z}$. Generally, however, when we have a graph $G$ whose $E(G, v_0)$ gives some group of interest, it does not seem easy to determine the minimal embedding dimension of $G$ as a digital image. For example, it is easy to see that the graph of Example 4.18 may be embedded as a digital image in $\mathbb{Z}^n$ for various $n$ somewhat lower than 12. Is it possible to have a digital image in $\mathbb{Z}^4$ whose clique complex has geometric realization homeomorphic to $\mathbb{R}P^2$? As another question along these lines, we ask: which groups might be obtained as the fundamental groups of 3D digital images?

5 Path shortening and 2D digital images

Whilst Theorems 3.6 and 3.4 allow us to use many results from the topological setting in the digital setting, they do not automatically resolve all questions about the digital fundamental group. For example, as just remarked, it is not immediately clear which groups might be obtained as the fundamental groups of 3D digital images. In this
last section we will show (Theorem 5.6 below) that the fundamental group of every 2D digital image is a free group. Now, the clique complex of a 2D image, generally speaking, is a simplicial complex with simplices of dimension up to 3. There is no general reason why such a simplicial complex should have fundamental group that is a free group. So some argument is required, either in the digital setting or, using Theorems 3.6 and 3.4, in the simplicial complex setting or in the topological setting. We argue in the digital setting. To prepare for this result, we establish some basic results about paths and digital simple closed curves.

Definition 5.1 Let \( X \subseteq \mathbb{Z}^r \) be any digital image. Suppose we have two points \( a, b \in X \) that are non-adjacent. We say that a set of \( n+2 \) (distinct) points \( P = \{a, x_1, \ldots, x_n, b\} \subseteq X \) with \( n \geq 1 \) is a contractible path in \( X \) from \( a \) to \( b \) of length \( n+1 \) if we have adjacencies \( a \sim_X x_1, x_i \sim_X x_{i+1} \) for each \( 1 \leq i \leq n - 1 \), and \( x_n \sim_X b \), and no other adjacencies amongst the elements of \( P \).

The relationship on pairs of points of having a contractible path from one to the other is clearly symmetric: a contractible path from \( a \) to \( b \) will serve as a contractible path from \( b \) to \( a \). The nomenclature is justified by the following observations.

Lemma 5.2 Suppose we have a set of points \( P = \{a, x_1, \ldots, x_n, b\} \subseteq X \) that is a contractible path in \( X \).

(a) Setting \( \alpha(0) = a, \alpha(n+1) = b, \) and \( \alpha(i) = x_i \) for \( 1 \leq i \leq n \) defines a a path \( \alpha: I_{n+1} \to X \).

(b) This path gives an isomorphism of digital images \( I_{n+1} \cong P \).

(c) With \( a \in P \) as basepoint, \( P \) is a based-contractible subset of \( X \) (contractible in itself, not just in \( X \)).

Proof (a) The given data about \( P \) ensure continuity of \( \alpha \).

(b) The path \( \alpha \) has continuous inverse \( g: P \to I_{n+1} \) given by \( g(a) = 0, g(n+1) = b, \) and \( g(x_i) = i \) for \( 1 \leq i \leq n \).

(c) The homotopy \( H: I_{n+1} \times I_{n+1} \to I_{n+1} \) defined by

\[
H(i, t) = \begin{cases} 
  i & 0 \leq i \leq n + 1 - t \\
  n + 1 - t & n + 2 - t \leq i \leq n + 1 
\end{cases}
\]

is a contracting homotopy of the interval (cf. Example 3.20 of Lupton et al. 2021). Now the isomorphisms \( \alpha \) and \( g \) of part (b) define a homotopy

\[
G = \alpha \circ H \circ (g \times \text{id}_{I_{n+1}}): P \times I_{n+1} \to P,
\]

that satisfies \( G(p, 0) = \alpha \circ g(p) = p \) and \( G(p, n+1) = \alpha(0) = a \) for each \( p \in P \). The homotopy \( G \) also satisfies \( G(a, t) = \alpha \circ H(0, t) = \alpha(0) = a \) for each \( t \in I_{n+1} \), so it is a based contraction of \( P \) in the sense asserted.

If we remove a point from a digital simple closed curve, we obtain a contractible path. Whilst there are many contractible paths that may be “completed” to a digital
simple closed curve by the addition of a suitable point, there are examples of con-
tractible paths that may not be completed to a digital simple closed curve, even when
we have \( a \) and \( b \) adjacent to a common point of \( X \).

**Example 5.3** Take \( X \subseteq \mathbb{Z}^2 \) by \( X = \{(-1, 0), (0, 0), (1, 0)\} \). Then \( X \) is a contractible
path that cannot be completed to a digital simple closed curve by a single point. This
is because the only points not in \( X \) adjacent to \( a = (-1, 0) \) and \( b = (1, 0) \) are \((0, 1)\)
and \((0, -1)\), which are adjacent to \((0, 0)\).

We have the following “shortening lemma.”

**Lemma 5.4** Let \( X \subseteq \mathbb{Z}^n \) be any digital image. For non-adjacent points \( a, b \in X \), if
there is path in \( X \) from \( a \) and \( b \), then there is a contractible path in \( X \) from \( a \) to \( b \).

**Proof** Let \( \gamma \) be a shortest path in \( X \) from \( a \) to \( b \). If \( \gamma(i) \) and \( \gamma(i + k) \) are adjacent,
for some \( k > 1 \), then \( \gamma' \) obtained from \( \gamma \) by skipping \( \gamma(i + 1), \ldots, \gamma(i + k - 1) \) is
shorter: a contradiction. Thus, \( \gamma \) is contractible. \( \square \)

We have one more ingredient to prepare for our main result.

**Definition 5.5** (Based Homotopy Equivalence) Let \( f : X \to Y \) be a based map of
based digital images. If there is a based map \( g : Y \to X \) such that \( g \circ f \approx \text{id}_X \) and
\( f \circ g \approx \text{id}_Y \) (based homotopies), then \( f \) is a based-homotopy equivalence, and \( X \)
and \( Y \) are said to be based-homotopy equivalent, or to have the same based-homotopy type.

In this definition, the notation “\( \approx \)” denotes based homotopy of based maps, as
we recalled in Sect. 2. As we remarked in Lupton et al. (2021), the notion of based
homotopy equivalence of digital images is often too rigid to be of much use as a notion
of “same-ness” for digital images. However, in the following result, we do find a use
for it. It follows easily from Lemma 3.18 of Lupton et al. (2021) that if \( X \) and \( Y \) are
based-homotopy equivalent digital images, then their digital fundamental groups are
isomorphic.

We are now ready to prove the main result of this section.

**Theorem 5.6** Let \( X \in \mathbb{Z}^2 \) be a connected 2D digital image. Then \( \pi_1(X; x_0) \) is a free
group.

**Proof** We argue by induction on the (finite) number of points \( n \) in the digital image.
Induction starts with \( n = 1, 2, \) or \( 3, \) where there is nothing to prove (\( X \) is contractible
to a point in these cases, so has \( \pi_1(X; x_0) \cong \{e\} \)).

So assume inductively that, for any 2D digital image with \( n \) or fewer points, the
fundamental group is free. Now suppose \( X \) is a digital image with \( n + 1 \) points.

We may totally order the points of \( X \) by lexicographic order. That is, \((x_1, y_1) > (x_2, y_2)\)
if \( x_1 > x_2 \), and \((x, y_1) > (x, y_2)\) if \( y_1 > y_2 \). Suppose that \( x \in X \) is the
maximum in this ordering. The possible neighbours of \( x \) in \( X \) are illustrated as follows
(there are at most 4 of them):

\[
\begin{array}{c|cc}
\text{a} & & \\
\text{c} & \text{x} & \\
\text{b}_1 & \text{b}_2 & \\
\end{array}
\]

The link of \(x\), denoted by \(\text{lk}(x)\), is \(\{a, c, b_1, b_2\} \cap X\). First note that in the exceptional case in which \(X = \{x\} \cup \text{lk}(x)\), which would entail \(n\) being relatively small, and \(X\) consisting of at most the 5 points illustrated, then \(X\) itself will be contractible, with trivial fundamental group. So from now on, assume that we have points in \(X\) in addition to those of \(\{x\} \cup \text{lk}(x)\). Furthermore, \(\text{lk}(x)\) must be non-empty, otherwise \(X\) would be disconnected; we assume a choice of basepoint in \(\text{lk}(x)\). We divide and conquer, based on the form of this link.

**Case 1:** \(c \in \text{lk}(x)\). In this case, we claim that \(X\) is based-homotopy equivalent to \(X - \{x\}\). In fact we show that \(X - \{x\}\) is a deformation retract of \(X\). Define a retraction \(r: X \to X - \{x\}\) on each \(y \in X\) by

\[
r(y) = \begin{cases} 
  y & y \neq x \\
  c & y = x.
\end{cases}
\]

Let \(i: X - \{x\} \to X\) denote the inclusion. We have \(r \circ i = \text{id}: X - \{x\} \to X - \{x\}\). We claim that

\[
H(y, t) = \begin{cases} 
  y & t = 0 \\
  i \circ r(y) & t = 1
\end{cases}
\]

defines a (continuous) homotopy \(H: X \times I_1 \to X\). To confirm continuity, suppose we have \((y, t) \sim (y', t')\) in \(X \times I_1\). If neither \(y\) nor \(y'\) are \(x\), then we have \(H(y, t) = y\) and \(H(y', t') = y'\), which are adjacent because \((y, t) \sim (y', t')\) implies \(y \sim y'\) in \(X\). If both \(y\) and \(y'\) are \(x\), then we have \(H(x, 0) = x\) and \(H(x, 1) = c\), which are adjacent. So the only remaining adjacencies we need check are for \(H(x, 0)\) and \(H(y', t')\) and for \(H(x, 1)\) and \(H(y', t')\) with \(y' \neq x\) but \(y' \sim_x x\). But then we have \(H(x, 0) = x \sim y' = H(y', t')\), and \(H(x, 1) = c \sim y' = H(y', t')\). This latter follows because the only possibilities for \(y' \sim_x x\) are from \(\text{lk}(x)\), and each of these is also adjacent to \(c\). This completes the check of the continuity of \(H\), so it is a homotopy \(\text{id}_X \approx i \circ r: X \to X\). With a choice of basepoint in \(\text{lk}(x)\), \(H\) is evidently a based homotopy \(\text{id}_X \approx i \circ r\). Then, as claimed, \(X\) is based-homotopy equivalent to \(X - \{x\}\) and thus these two digital images have isomorphic fundamental groups. Since \(X - \{x\}\) has \(n\) vertices, its fundamental group is a free group, by our induction hypothesis, and so this establishes the induction step in this case.

For the remainder of the argument, we suppose that \(c\) is absent, so that \(\text{lk}(x) \subseteq \{a, b_1, b_2\}\).

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**Case 2:** \( \{b_1, b_2\} \subseteq \text{lk}(x) \). In this case, we claim that \( X \) is based-homotopy equivalent to \( X - \{b_2\} \). Again, we show that \( X - \{b_2\} \) is a deformation retract of \( X \). This is similar to the previous case. Define a retraction \( r: X \to X - \{b_2\} \) on each \( y \in X \) by

\[
 r(y) = \begin{cases} 
 y & y \neq b_2 \\
 b_1 & y = b_2.
\end{cases}
\]

Let \( i: X - \{b_2\} \to X \) denote the inclusion. We have \( r \circ i = \text{id}: X - \{b_2\} \to X - \{b_2\} \). We claim that

\[
 H(y, t) = \begin{cases} 
 y & t = 0 \\
 i \circ r(y) & t = 1
\end{cases}
\]

defines a (continuous) homotopy \( H: X \times I_1 \to X \). To confirm continuity, suppose we have \((y, t) \sim (y', t')\) in \( X \times I_1 \). If neither \( y \) nor \( y' \) are \( b_2 \), then we have \( H(y, t) = y \) and \( H(y', t') = y' \), which are adjacent because \((y, t) \sim (y', t')\) implies \( y \sim y' \) in \( X \). If both \( y \) and \( y' \) are \( b_2 \), then we have \( H(b_2, 0) = b_2 \) and \( H(b_2, 1) = b_1 \), which are adjacent. So the only remaining adjacencies we need check are for \( H(b_2, 0) \) and \( H(y', t') \) and for \( H(b_2, 1) \) and \( H(y', t') \) with \( y' \neq b_2 \) but \( y' \sim_X b_2 \). But then we have \( H(b_2, 0) = b_2 \sim y' = H(y', t') \), and \( H(b_2, 1) = b_1 \sim y' = H(y', t') \). This latter follows because the only possibilities for \( y \sim b_2 \) are from \([x, b_1, z_1, z_2] \) (see the figure below, and recall that there are no points in \( X \) with first coordinate greater than that of \( x \)—also, we are supposing \( c \) is absent from \( X \), but it does not affect the argument here even if we include it), and each of these is also adjacent to \( b_1 \).

\[
\begin{array}{c}
  a \\
  \hline
  c & x \\
  \hline
  b_1 & b_2 \\
  \hline
  z_1 & z_2
\end{array}
\]

This completes the check of the continuity of \( H \), so it is a homotopy \( \text{id}_X \approx i \circ r: X \to X \). If we choose, say, basepoint \( b_1 \), then \( H \) is evidently a based homotopy \( \text{id}_X \approx i \circ r \).

Then the induction step goes through in this case just as in the previous case.

**Case 3:** \( \text{lk}(x) = \{a\}, \text{lk}(x) = \{b_1\}, \) or \( \text{lk}(x) = \{b_2\} \). In this case, choose whichever point is in \( \text{lk}(x) \) as basepoint, and set \( U = \{x\} \cup \text{lk}(x) \) and \( V = X - \{x\} \). Then \( X = U \cup V \) and \( U \cap V = \text{lk}(x) \), a single point (the basepoint). Furthermore, the complements \( U' = X - (\{x\} \cup \text{lk}(x)) \) and \( V' = \{x\} \) are disconnected. It follows from Corollary 4.12 that we have

\[
\pi_1(X; x_0) \cong \pi_1(U; x_0) * \pi_1(V; x_0),
\]

the free product of groups. Furthermore, \( U \) is isomorphic to the unit interval \( I_1 \), which is (based) contractible and so we have \( \pi_1(U; x_0) \cong \{e\} \). Thus \( \pi_1(X; x_0) \cong \pi_1(V; x_0) \),
and our induction hypothesis gives that \( \pi_1(V; x_0) \) is a free group, as \( V \) has fewer points than \( X \). The induction step is complete in this case also.

The only remaining possibilities for the link of \( x \), now are \( \text{lk}(x) = \{a, b_1\} \) and \( \text{lk}(x) = \{a, b_2\} \). Notice that, here, we cannot use the same sets \( U \) and \( V \) to decompose \( X \) in the previous case, since Theorem 4.6 requires a connected intersection \( U \cap V \).

Instead, we take up each of these cases with an argument that uses the contractible path material earlier in the section.

**Case 4:** \( \text{lk}(x) = \{a, b_1\} \). Recall that we assume \( X \neq \{x\} \cup \text{lk}(x) \) and so there are points in \( X \) other than those adjacent to \( x \), and there exists a path from \( x \) to those points but only by passing through either \( a \) or \( b_1 \). There are two sub-cases: (1) in which \( a \) and \( b_1 \) are not connected by a path in \( X - \{x\} \); and (2) in which \( a \) and \( b_1 \) are connected by a path in \( X - \{x\} \). Take sub-case (1) first. Here, \( X - \{x\} \) must fall into the two components defined by

\[
X_a = \{ p \in X - \{x\} \mid p \text{ is connected to } a \text{ by a path in } X - \{x\} \}
\]

and

\[
X_{b_1} = \{ p \in X - \{x\} \mid p \text{ is connected to } b_1 \text{ by a path in } X - \{x\} \},
\]

the first of which contains \( a \) and the second of which contains \( b_1 \). We explain this assertion as follows. Take \( y \in X - \{x\} \) and suppose that \( y \notin X_a \). Since \( X \) is connected, there is some path from \( y \) to \( b_1 \) in \( X \). If this path does not contain \( x \), then it is a path in \( X - \{x\} \) from \( y \) to \( b_1 \) and so \( y \in X_{b_1} \). Otherwise, consider the first occurrence of \( x \) in the path from \( y \) to \( b_1 \). The part of the path from \( y \) to the point preceding this first occurrence of \( x \) is a path in \( X - \{x\} \) from \( y \) to a point of \( \text{lk}(x) \). But if this point is \( a \), we would have \( y \in X_a \), which we said was not the case. Therefore, it is \( b_1 \) and we have \( y \in X_{b_1} \). Furthermore, not only do we have \( X - \{x\} = X_a \cup X_{b_1} \) but these components must be disconnected, in the sense that no point of \( X_a \) is adjacent to any point of \( X_{b_1} \). For if we were to have \( p \in X_a \) and \( q \in X_{b_1} \) with \( p \sim q \), then we could concatenate a path in \( X - \{x\} \) from \( a \) to \( p \) with a path in \( X - \{x\} \) from \( q \) to \( b_1 \), to obtain a path in \( X - \{x\} \) from \( a \) to \( b_1 \). But we are currently assuming there is no such path. From all this, it follows that if we set \( U = X_a \cup \{x\} \) and \( V = X_{b_1} \cup \{x\} \), then \( U \cap V = \{x\} \). So take the basepoint as \( x_0 = x \), and notice that the complements \( U' = X_{b_1} \) and \( V' = X_a \) are disconnected, by the preceding discussion. In effect, we have identified \( X \) as a one-point union \( U \vee V \). It follows from Corollary 4.12 that we have

\[
\pi_1(X; x_0) \cong \pi_1(U; x_0) \ast \pi_1(V; x_0).
\]

Since \( U \) and \( V \) both contain fewer points than \( X \), our induction hypothesis gives that their fundamental groups are free groups. The free product of free groups is again a free group, and the induction step is complete in this sub-case (1).

Now go back to sub-case (2), in which \( a \) and \( b_1 \) are connected by a path in \( X - \{x\} \). By Lemma 5.4 we may shorten this path to a contractible path in \( X - \{x\} \). So without loss of generality, suppose we have a contractible path \( P \) in \( X - \{x\} \) from \( a \) to \( b_1 \).
Now set $U = P \cup \{x\}$ and $V = X - \{x\}$. Then $U \cap V = P$, which is connected and contractible. If $X \neq P \cup \{x\}$, choose $a = x_0$ and observe that the complements $U' = X - (P \cup \{x\})$ and $V' = \{x\}$ are disconnected. We may apply Corollary 4.12 again to obtain

$$\pi_1(X; x_0) \cong \pi_1(U; x_0) \ast \pi_1(V; x_0).$$

Then both $U$ and $V$ contain fewer points than $X$, and it follows as in the previous sub-case that $\pi_1(X; x_0)$ is a free group. However, it is possible that we have $X = P \cup \{x\}$, in which case $V$ will have (one) fewer points than $X$, but $U$ will have the same number, and we are not able to apply our inductive hypotheses to $U$. But in this situation, notice that $U = P \cup \{x\}$ is a digital simple closed curve, with $\pi_1(U; a) \cong \mathbb{Z}$ by Theorem 4.2. Now $\mathbb{Z}$ is a free group, and our inductive hypothesis applied to $V$ yields $\pi_1(X; x_0) \cong \mathbb{Z} \ast \pi_1(V; x_0)$: a free product of free groups, and hence a free group. So we have closed the induction in sub-case (2). This completes the induction in Case 4.

**Case 5:** $\text{lk}(x) = \{a, b_2\}$. This case may be handled with an argument identical to that just used for Case 4, only replacing $b_1$ with $b_2$. We omit the details.

This exhausts all cases for the link of $x$, and completes the induction. The result follows.

**Remark 5.7** We return briefly to the themes of Remark 4.21. A 2D digital image cannot contain cliques of more than 4 vertices (in general a unit $n$-cube of $2^n$ vertices is the maximal clique possible in a digital image in $\mathbb{Z}^n$). Therefore, a tolerance space (or graph) that contains a 5-clique cannot be realized as a 2D digital image. For tolerance spaces that contain only 4-cliques, Theorem 5.6 provides a necessary condition for their being realized as a 2D digital image. Namely, if a tolerance space may be realized as a 2D digital image, then its fundamental group—in the sense of this paper—must be free. Likewise, Theorem 5.6 provides a necessary condition for a clique complex (of dimension no more than 3) to be the clique complex of a 2D digital image.

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**References**

Ayala, R., Domínguez, E., Francés, A.R., Quintero, A.: Homotopy in digital spaces. Discrete Appl. Math. 125(1), 3–24 (2003)

Boxer, L.: A classical construction for the digital fundamental group. J. Math. Imaging Vis. 10(1), 51–62 (1999)

Boxer, L.: Properties of digital homotopy. J. Math. Imaging Vis. 22(1), 19–26 (2005)

Boxer, L.: Digital products, wedges, and covering spaces. J. Math. Imaging Vis. 25(2), 159–171 (2006)

Evako, A.V.: Topological properties of closed digital spaces: one method of constructing digital models of closed continuous surfaces by using covers. Comput. Vis. Image Underst. 102, 134–144 (2006)

Kong, T.Y.: A digital fundamental group. Comput. Graph. 13, 159–166 (1989)

Kong, T.Y., Rosenfeld, A.: Digital topology: a comparison of the graph-based and topological approaches. In: Reed, G.M., Roscoe, A.W., Wachter, R.F. (eds.) Topology and Category Theory in Computer Science, vol. 1991, pp. 273–289. Oxford Science Publication, Oxford (1989)
Lupton, G., Oprea, J., Scoville, N.: A fundamental group for digital images. J. Appl. Comput. Topol. 5(2), 249–311 (2021)
Lupton, G., Oprea, J., Scoville, N.: Homotopy theory in digital topology. Discrete Comput. Geom. 67(1), 112–165 (2022a)
Lupton, G., Oprea, J., Scoville, N.: Subdivision of maps in digital topology. Discrete Comput. Geom. 67(3), 698–742 (2022b)
Massey, W.S.: Algebraic Topology: An Introduction. Springer, New York (1977). Reprint of the 1967 edition, Graduate Texts in Mathematics, vol. 56
Massey, W.S.: A Basic Course in Algebraic Topology. Graduate Texts in Mathematics, vol. 127. Springer, New York (1991)
Maunder, C.R.F.: Algebraic Topology. Dover Publications, Inc., Mineola (1996). Reprint of the 1980 edition
Peters, J.F., Wasilewski, P.: Tolerance spaces: origins, theoretical aspects and applications. Inf. Sci. 195, 211–225 (2012)
Poston, T.: Fuzzy Geometry. Thesis , University of Warwick (1971)
Rosenfeld, A.: ‘Continuous’ functions on digital pictures. Pattern Recognit. Lett. 4, 177–184 (1986)
Rotman, J.J.: An Introduction to the Theory of Groups. Graduate Texts in Mathematics, vol. 148, 4th edn. Springer, New York (1995)
Sossinsky, A.B.: Tolerance space theory and some applications. Acta Appl. Math. 5(2), 137–167 (1986)

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