BALANCED HERMITIAN METRICS FROM SU(2)-STRUCTURES

M. FERNÁNDEZ, A. TOMASSINI, L. UGARTE AND R. VILLACAMPÀ

ABSTRACT. We study the intrinsic geometrical structure of hypersurfaces in 6-manifolds carrying a balanced Hermitian SU(3)-structure, which we call balanced SU(2)-structures. We provide conditions which imply that such a 5-manifold can be isometrically embedded as a hypersurface in a manifold with a balanced SU(3)-structure. We show that any 5-dimensional compact nilmanifold has an invariant balanced SU(2)-structure as well as new examples of balanced Hermitian SU(3)-metrics constructed from balanced SU(2)-structures. Moreover, for \( n = 3, 4 \), we present examples of compact manifolds, endowed with a balanced SU(\( n \))-structure, such that the corresponding Bismut connection has holonomy equal to SU(\( n \)).

1. Introduction

Let \((J, g)\) be a Hermitian structure on a manifold \(M\), with Kähler form \(F\) and Lie form \(\theta\). The 3-form \(JdF\) is the torsion of the Bismut connection of \((J, g)\), that is, the unique Hermitian connection with totally skew-symmetric torsion \([2]\). If \(JdF\) is closed and non-zero (so \((M, J, g)\) is not a Kähler manifold) the Hermitian structure is called strong Kähler with torsion.

Hermitian structures for which the Lie form \(\theta\) vanishes identically or, equivalently, \(F^2 = F \wedge F\) is closed, are called balanced. If in addition, there is an SU(3)-structure with Kähler form \(F\) and a complex volume (3, 0)-form \(\Psi = \Psi_+ + i\Psi_-\) on \(M\) such that \(dF^2 = d\Psi_+ = d\Psi_- = 0\), we say that \((F, \Psi_+, \Psi_-)\) is a balanced SU(3)-structure. Such structures are a generalization of integrable SU(3)-structures, which are defined by a triplet \((F, \Psi_+, \Psi_-)\) satisfying \(dF = d\Psi_+ = d\Psi_- = 0\). Balanced SU(3)-structures have been very useful in physics for the construction of explicit compact supersymmetric valid solutions to the heterotic equations of motion in dimension six \([7]\).

Recently, Conti and Salamon \([5]\) introduced the notion of hypo structures on 5-manifolds as those structures corresponding to the restriction of an integrable SU(3)-structure on a 6-manifold \(M\) to a hypersurface \(N\) of \(M\). They are a generalization in dimension 5 of Sasakian-Einstein metrics. In fact, Sasakian-Einstein metrics correspond to Killing spinors and hypo structures are induced by generalized Killing spinors. In terms of differential forms, a hypo structure on a 5-manifold \(N\) is determined by a quadruplet \((\eta, \omega_i, 1 \leq i \leq 3)\) of differential forms, where \(\eta\) is a nowhere vanishing 1-form and \(\omega_i\) are 2-forms on \(N\) satisfying certain relations (see \([5]\) in Section 2). If the forms \(\eta\) and \(\omega_i\) satisfy

\[
d\eta = -2\omega_3, \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d\omega_2 = -3\eta \wedge \omega_1,
\]

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then $N$ is a Sasakian-Einstein manifold, that is, a Riemannian manifold such that $N \times \mathbb{R}$ with the cone metric is Kähler and Ricci flat [3], so $N \times \mathbb{R}$ has holonomy contained in SU(3) or, equivalently, it has an integrable SU(3)-structure.

In the general case of a hypo structure, in [5] it is proved that a real analytic hypo structure on $N$ can be lifted to an integrable SU(3)-structure on $N \times I$, for some open interval $I$, if $(\eta, \omega_i, 1 \leq i \leq 3)$ belongs to a one-parameter family of hypo structures $(\eta(t), \omega_i(t), 1 \leq i \leq 3)$ satisfying the evolution equations (10) given in Section 3. Moreover, any oriented hypersurface of a 6-manifold with an integrable SU(3)-structure is naturally endowed with a hypo structure (see Section 2 for details).

Our purpose in this paper is to study the geometrical structure of an oriented hypersurface in a 6-manifold equipped with a balanced SU(3)-structure, which we call balanced SU(2)-structure. These are SU(2)-structures defined by a quadruplet $(\eta, \omega_i, 1 \leq i \leq 3)$ of differential forms satisfying the relations given by (4), and they can be considered as a generalization to dimension five of the holomorphic symplectic structures in dimension four. We prove that any compact nilmanifold of dimension 5 has a balanced SU(2)-structure although three of the 5-nilmanifolds do not admit an invariant hypo structure [3]. Furthermore, we provide conditions under which a balanced SU(2)-structure on $N$ can be lifted to a balanced SU(3)-structure on $N \times \mathbb{R}$. This allows us to exhibit examples of (non-invariant) balanced Hermitian metrics and, in particular of complex structures, on nilmanifolds not admitting invariant balanced Hermitian metrics.

To this end, in Section 2 we show that there exists a balanced SU(2)-structure on any oriented hypersurface $f: N \to M$ of a 6-manifold $M$ with a balanced SU(3)-structure. Furthermore, we prove that any balanced SU(2)-structure on $N$ can be lifted to a balanced Hermitian SU(3)-structures on $N \times \mathbb{R}$ if and only if it satisfies the evolution balanced SU(2) equations (9) established in Theorem 3.1 of Section 3.

On the other hand, in Proposition 2.6 we characterize the circle bundles $S^1 \subset N \to X$ with a balanced SU(2)-structure induced by a holomorphic symplectic structure on $X$. In particular, we get that the product $X \times \mathbb{R}$ has a balanced SU(2)-structure. Then, in Section 3 we solve the evolution balanced SU(2) equations (9) for the compact 5-nilmanifolds not having a hypo structure as well as for the balanced SU(2)-structures on manifolds which are the total space of a circle bundle over Kodaira-Thurston manifold with a certain holomorphic symplectic structure. In this way, we get new examples of balanced Hermitian SU(3)-metrics.

Finally, in Section 4 we describe in detail several examples of compact manifolds obtained as quotient of solvable Lie groups, and endowed with a balanced SU($n$)-structure, for $n = 3, 4$. In particular, we show that the holonomy of the Bismut connection is equal to SU($n$). Moreover, many of such examples are holomorphic parallelizable.

2. Balanced SU(2)-structures

In this section we introduce a special type of SU(2)-structures on 5-manifolds, namely balanced SU(2)-structures, which permit construct new Hermitian balanced metrics by solving suitable evolution equations. First we recall some facts about SU(2)-structure on a 5-dimensional manifold. (For more details, we refer e.g. to [4].) Let $N$ be a 5-dimensional manifold and let $L(N)$ be the principal bundle of
linear frames on $N$. An SU(2)-structure on $N$ is an SU(2)-reduction of $L(N)$. We have the following (see [5 Prop.1])

**Proposition 2.1.** SU(2)-structures on a 5-manifold $N$ are in $1:1$ correspondence with quadruplets $(\eta, \omega_1, \omega_2, \omega_3)$, where $\eta$ is a 1-form and $\omega_i$ are 2-forms on $N$ satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \eta \neq 0,$$

for some 4-form $v$, and

$$i_X \omega_3 = i_Y \omega_1 \Rightarrow \omega_2(X, Y) \geq 0,$$

where $i_X$ denotes the contraction by $X$.

Equivalently, an SU(2)-structure on $N$ can be viewed as the datum of $(\eta, \omega_3, \Phi)$, where $\eta$ is a 1-form, $\omega_3$ is a 2-form and $\Phi = \omega_1 + i \omega_2$ is a complex 2-form such that

$$\eta \wedge \omega_3^2 \neq 0, \quad \Phi^2 = 0$$

$$\omega_3 \wedge \Phi = 0, \quad \Phi \wedge \overline{\Phi} = 2\omega_3^2$$

and $\Phi$ is of type $(2,0)$ with respect to $\omega_3$.

As a corollary of the last Proposition, we obtain the useful local characterization of SU(2)-structures (see [5]):

**Corollary 2.2.** If $(\eta, \omega_1, \omega_2, \omega_3)$ is an SU(2)-structure on a 5-dimensional manifold $N$, then locally, there exists a basis of 1-forms $\{e^1, \ldots, e^5\}$ such that

$$\eta = e^1, \quad \omega_1 = e^{24} + e^{53}, \quad \omega_2 = e^{25} + e^{34}, \quad \omega_3 = e^{23} + e^{45}.$$

As a consequence, SU(2)-structures naturally arise on hypersurfaces of 6-manifolds with an SU(3)-structure. Indeed, let $f : N \to M$ be an oriented hypersurface of a 6-manifold $M$ endowed with an SU(3)-structure $(F, \Psi_+, \Psi_-)$ and denote by $U$ the unit normal vector field. Then $N$ inherits an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ given by

$$\eta = -i_U F, \quad \omega_1 = i_U \Psi_-, \quad \omega_2 = -i_U \Psi_+, \quad \omega_3 = f^* F.$$ 

Conversely, an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on $N$ induces an SU(3)-structure $(F, \Psi_+, \Psi_-)$ on $N \times \mathbb{R}$ given by

$$F = \omega_3 + \eta \wedge dt, \quad \Psi = \Psi_+ + i \Psi_- = (\omega_1 + i \omega_2) \wedge (\eta + idt),$$

where $t$ is a coordinate on $\mathbb{R}$.

**Definition 2.3.** An SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on a 5-dimensional manifold $N$ is called balanced if it satisfies the following equations

$$d(\omega_1 \wedge \eta) = 0, \quad d(\omega_2 \wedge \eta) = 0, \quad d(\omega_3 \wedge \omega_3) = 0.$$

In [5], an SU(2)-structure is said to be hypo if

$$d(\omega_1 \wedge \eta) = d(\omega_2 \wedge \eta) = d\omega_3 = 0.$$ 

Hence, it is obvious that any hypo structure is balanced. However, there are nilmanifolds admitting no invariant hypo structure, but having balanced SU(2)-structures. In fact, if the Lie algebra underlying a 5-dimensional compact nilmanifold $N$ is isomorphic to $(0, 0, 0, 12, 14), (0, 0, 12, 13, 23)$ or $(0, 0, 12, 13, 14 + 23)$, then
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There is no invariant hypo structure on \( N \) (see [5]). It is easy to check that the SU(2)-structure given by
\[
\eta = e^1, \quad \omega_1 = e^{24} + e^{53}, \quad \omega_2 = e^{25} + e^{34}, \quad \omega_3 = e^{23} + e^{45},
\]
satisfies (4) on each one of these three Lie algebras. Therefore, we get the following

**Proposition 2.4.** Any 5-dimensional compact nilmanifold has an invariant balanced SU(2)-structure.

There exist also 5-dimensional solvable non-nilpotent Lie algebras with no invariant hypo structure, but having a balanced SU(2)-structure. For example, let us consider the Lie algebra \( g \) whose dual is spanned by \((e_1, \ldots, e_5)\) such that
\[
de_1 = 0, \quad de_2 = 0, \quad de_3 = e^{13}, \quad de_4 = -e^{14}, \quad de_5 = e^{34}.
\]
Then \( g \) is a 5-dimensional solvable non-nilpotent Lie algebra. The simply-connected Lie group \( G \) associated with \( g \) has a uniform discrete subgroup \( \Gamma \), so that \( N = \Gamma \backslash G \) is a compact solvmanifold. In fact, the manifold \( N \) is the topological product of the unit circle by the compact solvable 4-dimensional manifold, studied in [1], which is a circle bundle over the compact solvmanifold \( Sol(3) \). A straightforward calculation shows that the following forms
\[
\eta = e^1, \quad \omega_1 = e^{24} + e^{53}, \quad \omega_2 = e^{25} + e^{34}, \quad \omega_3 = e^{23} + e^{45},
\]
satisfy
\[
d(\omega_1 \wedge \eta) = d(\omega_3 \wedge \eta) = d(\omega_2) = 0,
\]
and thus they define a balanced SU(2)-structure on \( N \). However, \( N \) has not invariant hypo SU(2)-structures. First, using Hattori’s theorem [11], we have that the real cohomology groups of \( N \) of degree \( \leq 2 \) are
\[
H^0(N) = \langle 1 \rangle, \quad H^1(N) = \langle [e^1], [e^2] \rangle, \quad H^2(N) = \langle [e^{12}] \rangle.
\]
Now, let us suppose that \( N \) has an invariant hypo SU(2)-structure \((\eta, \omega_1, \omega_2, \omega_3)\). Then
\[
\omega_3 = ae^{12} + be^{13} + ce^{14} + fe^{34},
\]
for some real numbers \( a, b, c \) and \( f \). Therefore, \( \omega_3^2 = 2afe^{1234} \), and so \( \eta = e^5 + \sum_{i=1}^4 \lambda_i e^i \) since \( \omega_3 \wedge \eta \) is a volume form. On the other hand,
\[
\omega_1 = \sum_{i,j=1}^4 a_{ij} e^{ij},
\]
and
\[
\omega_2 = \sum_{i,j=1}^4 b_{ij} e^{ij},
\]
for some real numbers \( a_{ij} \) and \( b_{ij} \). Now, the conditions \( d(\omega_1 \wedge \eta) = d(\omega_2 \wedge \eta) = 0 \) imply that
\[
\omega_1 = a_{13} e^{13} + a_{14} e^{14} + a_{34} e^{34},
\]
and
\[
\omega_2 = b_{13} e^{13} + b_{14} e^{14} + b_{34} e^{34},
\]
which implies that \( \omega_1^2 = \omega_2^2 = 0 \). This is not possible for an SU(2)-structure on \( N \).

Balanced SU(2)-structures on 5-manifolds are related to Hermitian balanced structures in six dimensions as the next proposition shows. First, recall that a
balanced SU(3)-structure on a 6-manifold $M$ is an SU(3)-structure $(F, \Psi = \Psi_+ + i\Psi_-)$ (where $F$ is the Kähler form of an almost Hermitian structure and $\Psi = \Psi_+ + i\Psi_-$ is a complex volume form) such that $F^2$ and $\Psi$ are closed. The latter condition implies that the underlying almost complex structure is integrable. Notice that any balanced SU(3)-structure is in particular half-flat \cite{4, 12, 6}, that is, it satisfies $dF^2 = d\Psi_+ = 0$.

**Proposition 2.5.** Let $f : N \rightarrow M$ be an immersion of an oriented 5-manifold into a 6-manifold with an SU(3)-structure. If the SU(3)-structure is balanced then the SU(2)-structure on $N$ given by (2) is balanced.

**Proof.** From (2) it follows that $\omega_1 \wedge \eta = f^* \Psi_+, \omega_2 \wedge \eta = f^* \Psi_-$ and $\omega_3 \wedge \omega_3 = f^* F^2$. Now, if $F^2$ and $\Psi$ are closed then the induced structure is half-balanced. $\square$

Let $(X, J)$ be a complex surface. By definition, a holomorphic symplectic structure on $X$ is the datum of a $d$-closed and non-degenerate $(2, 0)$-form $\omega$ on $X$. Let $g$ be a $J$-Hermitian metric on $X$ and $\omega_3$ be the fundamental form of $(g, J)$. Then, up to a conformal change, we may assume that $\omega_3^2 = \omega_1^2 = \omega_2^2$.

Then we have the following

**Proposition 2.6.** Let $(X, J)$ be a complex surface equipped with a holomorphic symplectic structure $\omega = \omega_1 + i\omega_2$, and let $\omega_3$ be the Kähler form of a $J$-Hermitian metric. Then, for any integral closed 2-form $\Omega$ on $X$ annihilating $\cos \theta \omega_1 + \sin \theta \omega_2$ and $\sin \theta \omega_1 - \cos \theta \omega_2$ for some $\theta$, there is a principal circle bundle $\pi : N \rightarrow X$ with connection form $\rho$ such that $\Omega$ is the curvature of $\rho$ and such that the SU(2)-structure $(\eta, \omega_1^\theta, \omega_2^\theta, \omega_3^\theta)$ on $N$ given by

\[
\begin{align*}
\eta &= \rho, \\
\omega_1^\theta &= \pi^*(\cos \theta \omega_1 + \sin \theta \omega_2), \\
\omega_2^\theta &= \pi^*(- \sin \theta \omega_1 + \cos \theta \omega_2), \\
\omega_3^\theta &= \pi^*(\omega_3)
\end{align*}
\]

is a balanced SU(2)-structure.

**Proof.** As previously remarked, we may assume that $\omega_1^2 = \omega_2^2 = \omega_3^2$. Since $d\omega_1 = d\omega_2 = 0$ and $d\eta = \pi^*(\Omega)$, a simple calculation shows that

\[
d(\omega_1^\theta \wedge \eta) = \omega_1^\theta \wedge d\eta = \pi^*((\cos \theta \omega_1 + \sin \theta \omega_2) \wedge \Omega) = 0,
\]

and

\[
d(\omega_2^\theta \wedge \eta) = \omega_2^\theta \wedge d\eta = \pi^*((- \sin \theta \omega_1 + \cos \theta \omega_2) \wedge \Omega) = 0.
\]

The existence of a principal circle bundle in the conditions above follows from a well known result by Kobayashi \cite{13}. $\square$

**Remark 2.7.** Notice that $\Omega = 0$ satisfies the hypothesis in the previous proposition for each $\theta$ and one gets the trivial circle bundle $N = X \times \mathbb{R}$ with the balanced SU(2)-structure which is the natural extension to $N$ of the holomorphic symplectic structure on $X$. 

Remark 2.8. Following [9, Def.1.1], a symplectic couple on an oriented 4-manifold \( X \) is a pair of symplectic forms \( (\omega_1, \omega_2) \) such that \( \omega_1 \wedge \omega_2 = 0 \) and \( \omega_1^2, \omega_2^2 \) are volume forms defining the positive orientation. A symplectic couple is called conformal if \( \omega_1^2 = \omega_2^2 \). By [9, Thm.1.3] it follows that \( X \) admits a conformal symplectic couple if and only if \( X \) is diffeomorphic to (a) a complex torus, (b) a K3 surface or (c) a primary Kodaira surface. In the hypothesis of prop. 2.8 \( (\omega_1, \omega_2) \) defines a conformal symplectic couple on the 4-manifold \( X \) and \( \omega_3 \) is a non-degenerate 2-form on \( X \) such that \( \omega_i \wedge \omega_3 = 0 \), \( i = 1, 2 \) and \( \omega_1^2 = \omega_2^2 = \omega_3^2 \).

Next we illustrate the construction given in Proposition 2.6 by showing principal circle bundles over holomorphic symplectic manifolds in the cases (a) and (c) of Remark 2.8. Let us consider the closed 4-manifold \( X = \Gamma \setminus G \), where the Lie algebra \( g \) of \( G \) has the following structure equations

\[
de^1 = 0, \quad de^2 = 0, \quad de^3 = 0, \quad de^4 = -\epsilon e^{23} \quad (\epsilon = 0, 1).
\]

Clearly \( X \) is the Kodaira-Thurston manifold for \( \epsilon = 1 \) and a 4-torus for \( \epsilon = 0 \). Consider the complex structure \( J \) on \( X \) defined by the complex \((1,0)\)-forms

\[
\varphi^1 = e^1 + ie^4, \quad \varphi^2 = e^2 + ie^3,
\]

so that

\[
\omega = \varphi^1 \wedge \varphi^2 = (e^{12} + e^{14}) + i(e^{13} - e^{24}) = \omega_1 + i\omega_2
\]

is a holomorphic symplectic structure on \( X \). The metric \( g \) on \( X \) given by \( g = \sum_{i=1}^{4} e^i \otimes e^i \) is \( J \)-Hermitian and the fundamental form of \((g, J)\) is precisely \( \omega_3 = e^{14} + e^{23} \), so we are in the conditions of Proposition 2.6. Observe that \( \omega_3 \) is closed only when \( X \) is a 4-torus, i.e. \( \epsilon = 0 \).

Now let \( \Omega \) be a closed 2-form on \( X \) such that

\[
\Omega \wedge (\cos \theta \omega_1 + \sin \theta \omega_2) = 0 = \Omega \wedge (\sin \theta \omega_1 - \cos \theta \omega_2),
\]

for some \( \theta \). A direct calculation shows that

\[
\Omega = a(e^{12} - e^{34}) + b(e^{13} + e^{24}) + (\epsilon - 1)c_1 e^{14} + c_2 e^{23}
\]

satisfies (7) for all \( \theta \), where \( a, b, c_1, c_2 \) are constant. Applying Proposition 2.6 we have a balanced \( SU(2) \)-structure on the total space \( N \), which is non-hypo if \( \epsilon = 1 \). In this case it is easy to see that \( N \) is a compact 5-nilmanifold with underlying Lie algebra isomorphic to either \((0, 0, 0, 0, 12)\) or \((0, 0, 0, 12, 13 + 24)\).

3. Evolution equations and Hermitian balanced metrics

Next we establish evolution equations that allow the construction of new balanced Hermitian metrics in dimension six from balanced \( SU(2) \)-structures in dimension five.

Theorem 3.1. Let \((\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))\) be a family of \( SU(2) \)-structures on a 5-manifold \( N \), for \( t \in I = (a, b) \). Then, the \( SU(3) \)-structure on \( M = N \times I \) given by

\[
F = \omega_3(t) + \eta(t) \wedge dt, \quad \Psi = (\omega_1(t) + i\omega_2(t)) \wedge (\eta(t) + idt),
\]

is a balanced \( SU(3) \)-structure on \( M \).
is balanced Hermitian if and only if \((\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))\) is a balanced SU(2)-structure for any \(t\) in the open interval \(I\), and the following evolution equations

\[
\begin{align*}
\partial_t(\omega_1 \wedge \eta) &= -d\omega_2, \\
\partial_t(\omega_2 \wedge \eta) &= d\omega_1, \\
\partial_t(\omega_3 \wedge \omega_3) &= -2 d(\omega_3 \wedge \eta),
\end{align*}
\]

(9)

are satisfied.

**Proof.** A direct calculation shows that the SU(3)-structure given by (8) satisfies

\[
dF^2 = d(\omega_3 \wedge \omega_3) + (\partial_t(\omega_3 \wedge \omega_3) + 2 d(\omega_3 \wedge \eta)) \wedge dt,
\]

and

\[
d\Psi = d(\omega_1 \wedge \eta) - (\partial_t(\omega_1 \wedge \eta) + d\omega_2) \wedge dt + i d(\omega_2 \wedge \eta) - i (\partial_t(\omega_2 \wedge \eta) - d\omega_1) \wedge dt.
\]

The forms \(F^2\) and \(\Psi\) are both closed if and only if \((\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))\) is a balanced SU(2)-structure for any \(t \in I\), and satisfies the equations (9).

**Remark 3.2.** Notice that in the special case when the balanced SU(2)-structures in the family are hypo, the last equation in (9) reduces to

\[
\omega_3 \wedge (\partial_t \omega_3 + d\eta) = 0.
\]

Hence, any solution of the hypo evolution equations introduced in [5], namely

\[
\begin{align*}
\partial_t(\omega_1 \wedge \eta) &= -d\omega_2, \\
\partial_t(\omega_2 \wedge \eta) &= d\omega_1, \\
\partial_t \omega_3 &= -d\eta,
\end{align*}
\]

(10)

is trivially a solution of (9), and the resulting SU(3)-structure is integrable because \(F\) in (8) is closed.

Next we give some explicit solutions to the balanced evolution equations (9). Our first example is the 5-manifold \(N = X \times \mathbb{R}\), where \(X\) is the Kodaira-Thurston manifold. Let us consider the non-hypo balanced SU(2)-structure given by (6) for \(\epsilon = 1\) and with zero curvature, namely

\[
\eta = e^5, \quad \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}.
\]

The family of non-hypo balanced SU(2)-structures on \(N\) given by \(\eta(t) = e^5\) and \(\omega_1(t) = e^{12} + e^3(e^4 - t e^5), \quad \omega_2(t) = e^{13} + (e^4 - t e^5)e^2, \quad \omega_3(t) = e^1(e^4 - t e^5) + e^{23}\), coincides with the previous one for \(t = 0\) and satisfies the evolution equations (9) for any \(t \in \mathbb{R}\).

Denote by \(G\) the simply-connected nilpotent Lie group with Lie algebra \((0,0,0,12)\), so that \(N = \Gamma \setminus G\) for some lattice \(\Gamma\) of maximal rank in \(G\). It follows from Proposition 3.1 that the SU(3)-structure on \(G \times \mathbb{R}\) given by

\[
\begin{align*}
F &= e^{14} + e^{23} - t e^{15} + e^5 \wedge dt, \\
\Psi_+ &= e^{125} + e^{345} - (e^{13} - e^{24} + t e^{25}) \wedge dt, \\
\Psi_- &= e^{135} - e^{245} + (e^{12} + e^{34} - t e^{35}) \wedge dt,
\end{align*}
\]

(11)

is balanced Hermitian.

Next we find explicit solutions of (9) for the non-hypo nilpotent Lie algebras \((0,0,12,14)\) and \((0,0,12,13,23)\).
Lie algebra \((0,0,12,14)\): It is straightforward to check that the family of balanced SU(2)-structures given by
\[
\eta(t) = \sqrt{\frac{2-3t}{2}} e^1, \\
\omega_1(t) = \frac{1}{2} \left( \sqrt{\frac{2}{2-3t}} - \frac{2-3t}{2} \right) e^{23} + \sqrt{\frac{2-3t}{2}} e^{24} - \sqrt{\frac{2}{2-3t}} e^{35}, \\
\omega_2(t) = \sqrt{\frac{2-3t}{2}} e^{25} + \sqrt{\frac{2-3t}{2}} e^{34}, \\
\omega_3(t) = e^{23} - \frac{1}{2} \left( 1 - \frac{2-3t}{2} \sqrt{\frac{2-3t}{2}} \right) e^{24} + e^{45},
\]
satisfies the evolution equations (9) for \(t \in \mathbb{R} - \{2/3\}\). Observe that the volume form of the associated Riemannian metric along the family is given by
\[
\omega_1(t) \wedge \omega_1(t) \wedge \eta(t) = 2 \sqrt{\frac{2-3t}{2}} e^{12345},
\]
so the orientation for \(t \in (-\infty, 2/3)\) is opposite to the orientation for \(t \in (2/3, \infty)\).

Let \(I = (-\infty, 2/3)\) and denote by \(G\) the simply-connected nilpotent Lie group with Lie algebra \((0,0,12,14)\). The basis of 1-forms on the product manifold \(G \times I\) given by
\[
\alpha^1 = e^2, \quad \alpha^2 = e^3, \quad \alpha^3 = \sqrt{\frac{2-3t}{2}} e^4, \quad \alpha^4 = \frac{1}{2} \sqrt{\frac{2-3t}{2}} (e^2 + 2e^5) - \frac{2-3t}{2} e^2, \\
\alpha^5 = \sqrt{\frac{2-3t}{2}} e^1, \quad \alpha^6 = dt,
\]
is orthonormal with respect to the Riemannian metric associated to the balanced SU(3)-structure on \(G \times I\). The Hermitian balanced structure on \(G \times I\) is given by
\[
F = e^{23} - \frac{1}{2} e^{24} + e^{45} + \frac{2-3t}{4} \sqrt{\frac{2-3t}{2}} e^{24} + \sqrt{\frac{2-3t}{2}} e^1 \wedge dt, \\
\Psi_+ = \frac{1}{2} e^{123} - e^{135} - \frac{2-3t}{4} \sqrt{\frac{2-3t}{2}} e^{123} + \sqrt{\frac{(2-3t)^2}{4}} e^{124} \\
- \left( \sqrt{\frac{2}{2-3t}} e^{25} + \sqrt{\frac{2-3t}{2}} e^{34} \right) \wedge dt, \\
\Psi_- = e^{125} + \sqrt{\frac{2}{2-3t}} e^{134} \\
+ \left( \sqrt{\frac{2}{2-3t}} e^{23} - \frac{2-3t}{4} e^{23} + \sqrt{\frac{2-3t}{2}} e^{24} - \sqrt{\frac{2}{2-3t}} e^{45} \right) \wedge dt.
\]

Lie algebra \((0,0,12,13,23)\): A direct calculation shows that the family of balanced SU(2)-structures given by
\[
\eta(t) = \frac{2}{e^4} e^3, \\
\omega_1(t) = \frac{2-t}{4} (e^{15} + e^{42}), \\
\omega_2(t) = \frac{t(2-t)(t-4)}{4} e^{12} + \frac{2-t}{4} (e^{14} + e^{25}), \\
\omega_3(t) = e^{12} - \frac{(2-t)^2(t-4)}{8} e^{25} - \frac{(2-t)^2}{4} e^{45},
\]
satisfies the evolution equations (9) for \(t \in \mathbb{R} - \{2\}\). Observe that the volume form of the associated Riemannian metric along the family is given by
\[
\omega_1(t) \wedge \omega_1(t) \wedge \eta(t) = -2 e^{12345},
\]
so it remains constant.

Let $I = (-\infty, 2)$ and denote by $G$ the simply-connected nilpotent Lie group with Lie algebra $(0, 0, 12, 13, 23)$. The basis of 1-forms on the product manifold $G \times I$ given by

\[
\begin{align*}
\alpha^1 &= e^3, \\
\alpha^2 &= e^2, \\
\alpha^3 &= \frac{2-t}{2} e^5, \\
\alpha^4 &= \frac{t(2-t)(t-4)}{4} e^2 + \frac{2-t}{2} e^4, \\
\alpha^5 &= \frac{2}{2-t} e^3, \\
\alpha^6 &= dt,
\end{align*}
\]

is orthonormal with respect to the Riemannian metric associated to the balanced SU(3)-structure on $G \times I$. The Hermitian balanced structure on $G \times I$ is given by

\[
\begin{align*}
F &= e^{12} - \frac{(t(2-t)^2(t-4))}{8} e^{25} - \frac{(2-t)^2}{4} e^{45} + \frac{2}{2-t} \epsilon^3 \wedge dt, \\
\Psi_+ &= -\epsilon^{135} + \epsilon^{234} - \frac{2-t}{2} \left(\frac{(t-4)}{2} e^{12} + \epsilon^{14} + \epsilon^{25}\right) \wedge dt, \\
\Psi_- &= -\epsilon^{134} - \epsilon^{235} + \frac{(t-4)}{2} \epsilon^{123} + \frac{2-t}{2} (\epsilon^{15} - \epsilon^{24}) \wedge dt.
\end{align*}
\]

As a consequence of our previous examples we conclude:

**Corollary 3.3.** The 6-dimensional simply-connected nilpotent Lie groups $H_8$, $H_{16}$ and $H_{17}$ corresponding to the Lie algebras $\mathfrak{h}_8 = \langle 0, 0, 0, 0, 12 \rangle$, $\mathfrak{h}_{16} = \langle 0, 0, 0, 12, 14, 24 \rangle$ and $\mathfrak{h}_{17} = \langle 0, 0, 0, 0, 12, 15 \rangle$ have balanced Hermitian SU(3)-structures.

It is worthy to remark that $H_8$ and $H_{16}$ have invariant complex structures, but none of them admit invariant compatible balanced metric [15]. On the other hand, $H_{17}$ has no invariant complex structures.

Finally, recall that according to [12] if $M$ is a 6-manifold with a family $(F(s), \Psi_+(s), \Psi_-(s))$ of half-flat structures, $s \in I = (a, b)$, satisfying the evolution equations

\[
\partial_s \Psi_+ = dF, \quad F \wedge \partial_s F = -d\Psi_-,
\]

then the product manifold $M \times I$ has a Riemannian metric whose holonomy is contained in $G_2$; in fact, the form $\varphi = F(s) \wedge ds + \Psi_+(s)$ defining the $G_2$-structure is parallel. Therefore, the balanced Hermitian SU(3)-structures [11], [12] and [13] can be lifted to a metric with holonomy in $G_2$.

4. Holonomy of Bismut connection of balanced Hermitian metrics on solvmanifolds

In this section we provide some examples of compact Hermitian manifolds, endowed with an SU($n$)-structure whose associated metric is balanced and such that the corresponding Bismut connection has holonomy equal to SU($n$), for $n = 3$ and 4.

Let $(M, J, g)$ be a Hermitian manifold and $F$ be the Kähler form of the Hermitian structure $(g, J)$. Denote by $\nabla^{LC}$ the Levi Civita connection of the metric $g$. Then the Bismut connection $\nabla^B$ of $(M, J, F, g)$ is characterized by the following formula

\[
g(\nabla_X^B Y, Z) = g(\nabla_X^{LC} Y, Z) + \frac{1}{2} T(X, Y, Z), \quad \forall X, Y, Z \in \Gamma(M, TM),
\]

where the torsion form $T$ is given by

\[
T(X, Y, Z) = JdF(X, Y, Z).
\]
We will need to compute the curvature of $\nabla^B$. In order to do this, we will use the Cartan structure equations,

\[
\begin{cases}
    d\omega^i + \sum_{j=1}^{2n} \omega^i_j \wedge e^j = \tau^i, & i = 1, \ldots, 2n \\
    \omega^i_j + \omega^i_{2n} = 0, & i, j = 1, \ldots, 2n,
\end{cases}
\]

\[
\begin{cases}
    d\omega^i_j + \sum_{r=1}^{2n} \omega^i_r \wedge \omega^j_r = \Omega^i_j, & i, j = 1, \ldots, 2n \\
    \Omega^i_j + \Omega^j_i = 0, & i, j = 1, \ldots, 2n,
\end{cases}
\]

where $\{e^1, \ldots, e^{2n}\}$ is an orthonormal coframe, $\omega^i_j$ are the connection 1-forms, $\tau^i$ are the torsion 2-forms and $\Omega^i_j$ are the curvature 2-forms.

If $\{f_1, \ldots, f_{2n}\}$ denotes the dual frame of $\{e^1, \ldots, e^{2n}\}$, then

\[
\tau^i = \sum_{j<k=1}^{2n} T_{ijk} e^j \wedge e^k, \quad i = 1, \ldots, 2n,
\]

where $T_{ijk} = T(e_i, e_j, e_k)$.

4.1. Dimension six. There are two 3-dimensional complex-parallelizable (non-Abelian) solvable Lie groups [14]. The complex structure equations are given by:

(I) $d\varphi^1 = d\varphi^2 = 0$, $d\varphi^3 = \varphi^{12}$;

(II) $d\varphi^1 = 0$, $d\varphi^2 = \varphi^{12}$, $d\varphi^3 = -\varphi^{13}$.

Here (I) is the Lie group underlying the Iwasawa manifold. In the following we show that they can be endowed with a balanced Hermitian SU(3)-structure and determine the holonomy of the corresponding Bismut connection.

**Theorem 4.1.** Any 3-dimensional complex-parallelizable (non-Abelian) solvable Lie group has a Hermitian metric such that the holonomy of its Bismut connection is equal to SU(3).

**Proof.** First we show that for the standard Hermitian balanced structure $(J, g)$ on the Lie group (I) the holonomy of its Bismut connection equal to SU(3).

Let us consider the real basis $(e^1, \ldots, e^6)$ given by

\[
\varphi^1 = e^1 + i e^2, \quad \varphi^2 = e^3 + i e^4, \quad \varphi^3 = e^5 + i e^6.
\]

In this case the structure equations are

\[
de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{13} - e^{24}, \quad de^6 = e^{14} + e^{23}.
\]

Then, the complex structure $J$ is given by

\[
Je^1 = -e^2, \quad Je^2 = e^1, \quad Je^3 = -e^4, \quad Je^4 = e^3, \quad Je^5 = -e^6, \quad Je^6 = e^5.
\]

The fundamental form $F$ associated with the $J$-Hermitian metric $g = \sum_{i=1}^6 e^i \otimes e^i$ is given by

\[
F = e^{12} + e^{34} + e^{56}.
\]

Since $dF = e^{136} - e^{145} - e^{235} - e^{246}$, we get that $g$ is balanced and the torsion $T = JdF$ is given by

\[
T = -e^{135} - e^{146} - e^{236} + e^{245}.
\]
The non-zero connection 1-forms of the Bismut connection $\nabla^B$ are

$$\omega^1_3 = -e^3, \quad \omega^1_6 = -e^4, \quad \omega^2_2 = e^4, \quad \omega^2_6 = -e^3,$$
$$\omega^3_3 = e^1, \quad \omega^3_6 = e^2, \quad \omega^4_5 = -e^2, \quad \omega^4_6 = e^1.$$  

The (linearly independent) curvature forms of the Bismut connection are

$$\Omega^1_2 = 2e^{34}, \quad \Omega^1_3 = -e^{13} - e^{24}, \quad \Omega^2_3 = e^{14} - e^{23}, \quad \Omega^3_4 = 2e^{12},$$

and the (linearly independent) curvature forms of the Bismut connection are

$$\nabla^B_{\mathfrak{E}_1}(\Omega^1_2) = -2(e^{36} - e^{45}), \quad \nabla^B_{\mathfrak{E}_2}(\Omega^1_2) = 2(e^{35} + e^{46}),$$
$$\nabla^B_{\mathfrak{E}_3}(\Omega^2_3) = 2(e^{16} - e^{25}), \quad \nabla^B_{\mathfrak{E}_4}(\Omega^3_4) = -2(e^{15} + e^{26}).$$

Therefore, $\text{Hol}(\nabla^B) = \text{SU}(3)$.

For case (II) we consider the real basis $(e^1, \ldots, e^6)$ given by

$$\varphi^1 = e^1 + i e^2, \quad \varphi^2 = e^3 + i e^4, \quad \varphi^3 = e^5 + i e^6.$$

In terms of this basis, the structure equations are

$$de^1 = de^2 = 0, \quad de^3 = e^{13} - e^{24}, \quad de^4 = e^{14} + e^{23},$$
$$de^5 = -e^{15} + e^{26}, \quad de^6 = -e^{16} - e^{25}.$$  

Then, the complex structure $J$ is given by

$$Je^1 = -e^2, \quad Je^2 = e^1, \quad Je^3 = -e^4, \quad Je^4 = e^3, \quad Je^5 = -e^6, \quad Je^6 = e^5.$$  

Then the fundamental form $F$ associated with the $J$-Hermitian metric $g = \sum_{i=1}^6 e^i \otimes e^i$ is given by

$$F = e^{12} + e^{34} + e^{56}.$$  

Since $dF = 2(e^{134} - e^{156})$, we get that $g$ is balanced and the torsion $T = JdF$ is given by

$$T = -2(e^{234} - e^{256}).$$

The non-zero connection 1-forms of the Bismut connection $\nabla^B$ are

$$\omega^1_3 = -e^3, \quad \omega^1_6 = -e^4, \quad \omega^2_2 = e^4, \quad \omega^2_6 = -e^3,$$
$$\omega^3_3 = e^1, \quad \omega^3_6 = e^2, \quad \omega^4_5 = -e^2, \quad \omega^4_6 = e^1.$$  

The (linearly independent) curvature forms of the Bismut connection are

$$\Omega^1_3 = -e^{13} - e^{24}, \quad \Omega^2_4 = e^{14} + e^{23}, \quad \Omega^3_5 = -e^{15} - e^{26}, \quad \Omega^4_6 = -e^{16} + e^{25},$$
$$\Omega^3_4 = -2e^{34}, \quad \Omega^2_3 = -e^{35} - e^{46}, \quad \Omega^3_6 = e^{36} - e^{45}, \quad \Omega^4_5 = -2e^{56}.$$  

Therefore, the holonomy of $\nabla^B$ is again equal to $\text{SU}(3)$.

Next, we study the holonomy of the Bismut connection for Hermitian balanced metrics on 6-nilmanifolds. First of all, we recall [15] that if a nilmanifold $M = \Gamma \backslash G$ admits a Hermitian balanced metric (not necessarily invariant), then the Lie algebra $\mathfrak{g}$ of $G$ is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}$, where $\mathfrak{h}_1 = (0, 0, 0, 0, 0, 0)$ is the Abelian Lie algebra and

$$\mathfrak{h}_2 = (0, 0, 0, 12, 34), \quad \mathfrak{h}_5 = (0, 0, 0, 13 + 42, 14 + 23),$$
$$\mathfrak{h}_3 = (0, 0, 0, 0, 12 + 34), \quad \mathfrak{h}_6 = (0, 0, 0, 0, 12, 13),$$
$$\mathfrak{h}_4 = (0, 0, 0, 0, 12, 14 + 23), \quad \mathfrak{h}_{19}^{-} = (0, 0, 0, 12, 23, 14 - 35).$$
where \( \mathfrak{h}_5 \) is the Lie algebra underlying the Iwasawa manifold considered in Theorem 4.1. Therefore, the Lie algebra \( \mathfrak{h}_{19} \) is the unique 6-dimensional 3-step nilpotent Lie algebra admitting balanced structures.

On the other hand, in [S][Corollary 6.2, Example 6.3] an explicit balanced Hermitian metric on \( \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6 \) is given such that the holonomy of the corresponding Bismut connection is equal to \( \text{SU}(3) \) for \( \mathfrak{h}_4, \mathfrak{h}_5 \) and \( \mathfrak{h}_6 \). For \( \mathfrak{h}_3 \) they show that there is a reduction of the holonomy group, whereas it remains to study the Lie algebra \( \mathfrak{h}_2 \) because the particular coefficients given at the end of the proof of [S][Corollary 6.2] correspond to \( \mathfrak{h}_6 \).

Next we prove that \( \mathfrak{h}_2 \) and \( \mathfrak{h}_{19} \) also admit a balanced Hermitian \( \text{SU}(3) \)-structure such that the holonomy of its Bismut connection also equals \( \text{SU}(3) \). That is, we prove the following

**Theorem 4.2.** Let \( M \) be a 6-dimensional nilmanifold admitting invariant balanced Hermitian structures \((g, J)\). If the first Betti number of \( M \) is \( \leq 4 \), then there is \((g, J)\) such that the holonomy of its Bismut connection is equal to \( \text{SU}(3) \).

**Proof.** A 6-dimensional nilmanifold with \( b_1(M) \leq 4 \) admitting invariant balanced Hermitian structures \((g, J)\) has underlying Lie algebra isomorphic to \( \mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6 \) or \( \mathfrak{h}_{19} \).

In order to define a balanced Hermitian \( \text{SU}(3) \)-structure on \( \mathfrak{h}_2 \) such that the holonomy of its Bismut connection is equal to \( \text{SU}(3) \), let us consider the structure equations

\[
\begin{align*}
d e^1 &= d e^2 = d e^3 = d e^4 = 0, & d e^5 &= e^{13} - e^{24}, & d e^6 &= -2 e^{12} + e^{14} + e^{23} + 2 e^{34},
\end{align*}
\]

and the complex structure \( J \) given by

\[
J e^1 = -e^2, \quad J e^2 = e^1, \quad J e^3 = -e^4, \quad J e^4 = e^3, \quad J e^5 = -e^6, \quad J e^6 = e^5.
\]

Firstly, we notice that the structure equations (16) correspond to the Lie algebra \( \mathfrak{h}_2 \). To see this it is sufficient to express the equations with respect to the new basis

\[
\begin{align*}
f^1 &= -2 e^2 + \sqrt{3} e^3 + e^4, & f^2 &= e^1 - \sqrt{3} e^2 + 2 e^3, & f^3 &= 2 e^2 + \sqrt{3} e^3 - e^4, \\
f^4 &= e^1 + \sqrt{3} e^2 + 2 e^3, & f^5 &= -\sqrt{3} e^5 - e^6, & f^6 &= -\sqrt{3} e^5 + e^6.
\end{align*}
\]

Now, the fundamental form \( F \) associated to the \( J \)-Hermitian metric \( g = \sum_{i=1}^6 e^i \otimes e^i \) is given by \( F = e^{12} + e^{34} + e^{56} \). Since

\[
d F = e^{136} - e^{246} + 2 e^{125} - e^{145} - e^{235} - 2 e^{345},
\]

we get that \( g \) is balanced and the torsion \( T \) is given by

\[
T = -2 e^{126} - e^{135} - e^{146} - e^{236} + 2 e^{245} + 2 e^{346}.
\]

The (linearly independent) curvature forms for the Bismut connection are

\[
\begin{align*}
\Omega_2^1 &= -2(2 e^{12} - e^{14} - e^{23} - 3 e^{34}), & \Omega_3^1 &= -(e^{13} + e^{24}), \\
\Omega_4^1 &= -(e^{14} - e^{23}), & \Omega_5^1 &= -2 e^{46}, \\
\Omega_6^1 &= 2 e^{36}, & \Omega_7^1 &= 2(3 e^{12} - e^{14} - e^{23} - 2 e^{34}), \\
\Omega_8^1 &= -2 e^{26}, & \Omega_9^1 &= 2 e^{16}.
\end{align*}
\]

Therefore, \( \text{Hol}(\nabla_B) = \text{SU}(3) \).

We have given a balanced metric with \( \text{Hol}(\nabla_B) = \text{SU}(3) \) for each case. \( \square \)
To complete the proof, we show that there is a balanced Hermitian SU(3)-
structure \((J, g)\) on \(\mathfrak{h}_{19}\) with holonomy of its Bismut connection equal to SU(3).

We know that the structure equations of \(\mathfrak{h}_{19}\) are

\[
d e^1 = d e^2 = d e^3 = 0, \quad d e^4 = e^{12}, \quad d e^5 = e^{23}, \quad d e^6 = e^{14} - e^{35}.
\]

We define the complex structure \(J\) given by

\[
J e^1 = e^3, \quad J e^2 = 4 e^6, \quad J e^3 = -e^1, \quad J e^4 = -e^5, \quad J e^5 = e^4, \quad J e^6 = \frac{1}{4} e^2.
\]

Then, the fundamental form \(F\) associated to the \(J\)-Hermitian metric \(g = \sum_{i=1}^{6} e^i \otimes e^i\) is given by \(F = -e^{13} - e^{26} + e^{45}\) and

\[
d F = -e^{124} + e^{125} - e^{234} - e^{235}.
\]

Therefore, we get that \(g\) is balanced and the torsion \(T\) is given by

\[
T = e^{146} - e^{156} - e^{346} - e^{356}.
\]

The non-zero connection 1-forms of the Bismut connection \(\nabla^B\) are

\[
\omega_1 = -\frac{1}{2} e^4, \quad \omega_2 = -\frac{1}{2} e^2 - e^6, \quad \omega_3 = \frac{1}{2} e^6, \quad \omega_4 = \frac{1}{2} e^5, \quad \omega_5 = -\frac{1}{2} e^3, \quad \omega_6 = \frac{1}{2} e^3.
\]

and the (linearly independent) curvature forms for the Bismut connection are

\[
\Omega_1 = -\frac{1}{4}(3e^{12} + 2e^{16} + 2e^{36}), \quad \Omega_2 = \frac{1}{4}(e^{26} + e^{45}),
\]

\[
\Omega_4 = -\frac{1}{4}(e^{14} - e^{35}), \quad \Omega_5 = \frac{1}{4}(2e^{14} - e^{15} - e^{34} - e^{35}),
\]

\[
\Omega_6 = -\frac{1}{4}(e^{16} + 3e^{23} + 2e^{36}), \quad \Omega_7 = \frac{1}{4}(e^{24} - 2e^{46} - e^{56}),
\]

\[
\Omega_8 = \frac{1}{4}(e^{25} + e^{46} - 2e^{56}), \quad \Omega_9 = \frac{1}{2}(e^{13} - e^{45}).
\]

Therefore, \(\text{Hol}(\nabla^B) = \text{SU(3)}\).

To finish this section, we show an example of a six-dimensional compact solv-
manifold with a balanced metric such that the holonomy of its Bismut connection
is equal to SU(3).

Let \(g\) be the solvable Lie algebra whose structure equations are

\[
de e^1 = 0, \quad de^2 = 0, \quad de^3 = e^1 \wedge e^3, \quad de^4 = -e^1 \wedge e^4, \quad de^5 = e^1 \wedge e^5, \quad de^6 = -e^1 \wedge e^6.
\]

Let \(G\) be the simply-connected Lie group whose Lie algebra is \(g\). It can be easily
checked that, for any \(X \in g\), \(\text{ad}_X\) has real eigenvalues, i.e. \(g\) is completely solvable.

Thus \(G\) has a uniform discrete subgroup \(\Gamma\), such that \(M^6 = \Gamma \backslash G\) is a 6-dimensional
compact solvmanifold.

Define an (integrable) complex structure \(J\) on \(M^6\) by setting

\[
J e^1 = -e^2, \quad J e^2 = e^1, \quad J e^3 = -e^5, \quad J e^4 = -e^6, \quad J e^5 = e^3, \quad J e^6 = e^4,
\]

and a \(J\)-Hermitian structure \(g\) on \(M\) by \(g = \sum_{i=1}^{6} e^i \otimes e^i\). Then \(F = \sum_{i=1}^{3} e^{2i-1} \wedge e^{2i}\).

Hence

\[
d F = 2(e^{135} - e^{146})
\]

and consequently, the torsion form of the Bismut connection is given by

\[
T = 2(e^{246} - e^{235}).
\]
By the above expression, the only non-zero components of the torsion are
\[ T_{235} = -2, \quad T_{246} = 2. \]

By solving (14), we obtain that the non-zero connection forms are given by
\[
\begin{align*}
\omega_1^1 &= -e^3, & \omega_1^4 &= e^4, & \omega_1^5 &= -e^5, & \omega_1^6 &= e^6, \\
\omega_2^3 &= e^5, & \omega_2^4 &= -e^6, & \omega_2^5 &= -e^3, & \omega_2^6 &= e^4, \\
\omega_3^3 &= e^5, & \omega_3^4 &= e^2, & \omega_3^5 &= 0, & \omega_3^6 &= -e^2
\end{align*}
\]

Therefore, we get
\[
\begin{align*}
\Omega_2^1 &= 2 \left( e^{35} + e^{46} \right), & \Omega_2^4 &= -e^{13} - e^{25}, & \Omega_2^5 &= -e^{14} - e^{26}, & \Omega_2^6 &= -e^{15} + e^{23}, \\
\Omega_3^1 &= -e^{16} + e^{24}, & \Omega_3^2 &= e^{15} - e^{23}, & \Omega_3^3 &= e^{16} - e^{24}, & \Omega_3^4 &= -e^{13} - e^{25}, \\
\Omega_3^5 &= -e^{14} - e^{26}, & \Omega_3^6 &= e^{34} + e^{56}, & \Omega_5^3 &= -2e^{35}, & \Omega_5^6 &= e^{36} + e^{45}, \\
\Omega_5^4 &= e^{36} + e^{45}, & \Omega_5^4 &= -2e^{46}, & \Omega_5^6 &= e^{34} + e^{56}.
\end{align*}
\]

By the above expression for the curvature forms, we obtain that \( \text{Hol} (\nabla^B) = \text{SU}(3) \).

On the other hand, the manifold \( M \) has no Kähler structures. Indeed, in view of the Main Theorem in [10], a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus. Finally, note that
\[ dF^2 = 2 \left( e^{135} - e^{146} \right) \wedge \left( e^{12} + e^{35} + e^{46} \right) = 0, \]
i.e. \( g \) is a balanced metric.

4.2. **Balanced Hermitian metrics on compact 8-solvmanifolds.** Here we present examples of compact balanced Hermitian 8-manifolds with an SU(4) structure such that the holonomy of its Bismut connection is all the Lie group SU(4). Furthermore, according to [14], all the examples are holomorphic parallelizable.

**Example 4.3.** Consider the Lie algebra defined by the complex structure equations
\[ d\varphi^1 = d\varphi^2 = 0, \quad d\varphi^3 = -\varphi^{12}, \quad d\varphi^4 = -2\varphi^{13}. \]

Let \( e^1, \ldots, e^8 \) be the basis given by \( e^{2j-1} + ie^{2j} = \varphi^j \), for \( j = 1, \ldots, 4 \).

The corresponding real structure equations are
\[
\begin{align*}
de^1 &= de^2 = de^3 = de^4 = 0, & de^5 &= -e^{13} + e^{24}, \\
&= -e^{14} - e^{23}, & de^7 &= -2(e^{15} - e^{26}), & de^8 &= -2(e^{16} + e^{25}).
\end{align*}
\]

Let \( J \) be the complex structure given by
\[
\begin{align*}
Je^1 &= -e^2, & Je^2 &= e^1, & Je^3 &= -e^4, & Je^4 &= e^3, \\
Je^5 &= -e^6, & Je^6 &= e^5, & Je^7 &= -e^8, & Je^8 &= e^7.
\end{align*}
\]

Then fundamental form \( F \) associated with the \( J \)-Hermitian metric
\[ g = \sum_{i=1}^8 e^i \otimes e^i \] is given by \( F = \sum_{j=1}^4 e^{2j-1} \wedge e^{2j} \). Since
\[ dF = -e^{136} + e^{145} - 2e^{158} + 2e^{167} + e^{235} + e^{246} + 2e^{257} + 2e^{268}, \]
we get that \( g \) is balanced and the torsion \( T \) is given by
\[ T = JdF = e^{135} + e^{146} + 2e^{157} + 2e^{168} + e^{236} - 2e^{245} + 2e^{258} - 2e^{267}. \]
The non-zero connection 1-forms of the Bismut connection $\nabla^B$ are
\[
\omega_3 = e^3, \quad \omega_4 = e^4, \quad \omega_5 = 2e^5, \quad \omega_6 = 2e^6, \quad \omega_7 = -e^4, \quad \omega_8 = e^3, \quad \omega_9 = -2e^1.
\]
\[
\omega_2 = -2e^6, \quad \omega_2 = 2e^5, \quad \omega_3 = -e^1, \quad \omega_3 = -e^2, \quad \omega_4 = e^2, \quad \omega_5 = -e^1, \quad \omega_6 = -e^1.
\]
The following curvature forms for the Bismut connection are linearly independent:
\[
\Omega_1 = -e^{13} - e^{24}, \quad \Omega_4 = -e^{14} + e^{23}, \quad \Omega_5 = -4(e^{15} + e^{26}), \quad \\
\Omega_6 = -4(e^{16} - e^{25}), \quad \Omega_8 = 2(e^{12} - e^{34}), \quad \Omega_7 = -2(e^{35} + e^{46}), \quad \Omega_9 = -8(e^{12} + e^{56}).
\]
This gives a 9-dimensional space. Moreover, the following 6 covariant derivatives of the curvature forms
\[
\nabla^B_{E_2}(\Omega_1) = -16(e^{57} + e^{68}), \quad \nabla^B_{E_3}(\Omega_1) = -2(e^{37} + e^{48}), \quad \nabla^B_{E_4}(\Omega_5) = -2(e^{38} - e^{47}),
\]
\[
\nabla^B_{E_5}(\Omega_5) = -8(e^{58} - e^{67}), \quad \nabla^B_{E_6}(\Omega_7) = -4(e^{18} - e^{27}), \quad \nabla^B_{E_7}(\Omega_1) = 4(e^{17} + e^{28}),
\]
are linearly independent, therefore $\text{Hol}(\nabla^B) = SU(4)$.

**Example 4.4.** Consider the complex structure equations
\[
d\varphi^1 = 0, \quad d\varphi^2 = \varphi^{12}, \quad d\varphi^3 = c\varphi^{13}, \quad d\varphi^4 = -(1+c)\varphi^{14},
\]
where $c(1+c) \neq 0$. Let $e^1, \ldots, e^8$ be the basis given by $\varphi^j = e^{2j-1} + ie^{2j}$, for $j = 1, \ldots, 4$.

The corresponding real structure equations are
\[
de^1 = de^2 = 0, \quad de^3 = e^{13} - e^{24}, \quad de^4 = e^{14} + e^{23}, \quad \\
de^5 = c(e^{15} - e^{26}), \quad de^6 = c(e^{16} + e^{25}), \quad de^7 = -(1+c)(e^{17} - e^{28}), \quad de^8 = -(1+c)(e^{18} + e^{27}).
\]
Let $J$ be the complex structure given by
\[
Je^1 = -e^2, \quad Je^2 = e^1, \quad Je^3 = -e^4, \quad Je^4 = e^3, \quad \\
Je^5 = -e^6, \quad Je^6 = e^5, \quad Je^7 = -e^8, \quad Je^8 = e^7.
\]
Then $g = \sum_{i=1}^8 e^i \otimes e^i$ is a $J$-Hermitian metric and the fundamental form is given by $F = \sum_{j=1}^4 e^{2j-1} \wedge e^{2j}$. Hence
\[
dF = 2(e^{134} + ce^{156} - (1+c)e^{178})
\]
and consequently we get that $g$ is balanced and the torsion $T$ is expressed by
\[
T = JdF = -2(e^{234} + ce^{256} - (1+c)e^{278}).
\]
A direct computation shows that the non-zero connection 1-forms of the Bismut connection $\nabla^B$ are
\[
\omega_3 = -e^3, \quad \omega_4 = -e^4, \quad \omega_5 = -ce^5, \quad \omega_6 = -ce^6, \quad \omega_7 = (1+c)e^7, \\
\omega_8 = (1+c)e^8, \quad \omega_9 = e^1, \quad \omega_2 = -e^3, \quad \omega_2 = e^6, \quad \omega_6 = -ce^5, \quad \\
\omega_2 = -(1+c)e^8, \quad \omega_8 = (1+c)e^7, \quad \omega_3 = 2e^2, \quad \omega_5 = 2ce^2, \quad \omega_5 = -2(1+c)e^2.
\]
The following curvature forms for the Bismut connection are linearly independent:

\[ \Omega_1 = -(\alpha^{13} + \alpha^{24}), \quad \Omega_4 = -\alpha^{14} + \alpha^{23}, \quad \Omega_2 = -c^2(\alpha^{15} + \alpha^{26}), \]
\[ \Omega_6 = -c^2(\alpha^{16} - \alpha^{25}), \quad \Omega_7 = -(1 + c)^2(\alpha^{17} + \alpha^{28}), \]
\[ \Omega_8 = -(1 + c)^2(\alpha^{18} - \alpha^{27}), \quad \Omega_3 = -2\alpha^{34}, \quad \Omega_5 = -c(\alpha^{35} + \alpha^{46}), \quad \Omega_9 = -c(\alpha^{36} - \alpha^{45}), \]
\[ \Omega_7 = (1 + c)(\alpha^{37} + \alpha^{48}), \quad \Omega_8 = (1 + c)(\alpha^{38} - \alpha^{47}), \quad \Omega_{10} = -2c^2\alpha^{56}, \]
\[ \Omega_9 = c(1 + c)(\alpha^{57} + \alpha^{68}), \quad \Omega_{11} = c(1 + c)(\alpha^{58} - \alpha^{67}), \quad \Omega_{12} = -2(1 + c)^2\alpha^{78}. \]

Therefore, \( \text{Hol}(\nabla^B) = \text{SU}(4) \) for any value of the parameter \( c \).

**Example 4.5.** Consider the complex structure equations

\[ d\varphi^1 = 0, \quad d\varphi^2 = \varphi^{12}, \quad d\varphi^3 = -2\varphi^{13}, \quad d\varphi^4 = \varphi^{14} - \varphi^{12}. \]

Let \( e^1, \ldots, e^8 \) be the basis given by \( \varphi^j = e^{2j-1} + ie^{2j} \), for \( j = 1, \ldots, 4 \).

The corresponding real structure equations are

\[ de^1 = de^2 = 0, \quad de^3 = e^{13} - e^{24}, \quad de^4 = e^{14} + e^{23}, \quad de^5 = -2(e^{15} - e^{26}), \]
\[ de^6 = -2(e^{16} + e^{25}), \quad de^7 = -e^{13} + e^{17} + e^{24} - e^{28}, \]
\[ de^8 = -e^{14} + e^{18} - e^{23} + e^{27}. \]

Let \( J \) be the complex structure given by

\[ Je^1 = -e^2, \quad Je^2 = e^1, \quad Je^3 = -e^4, \quad Je^4 = e^3, \quad Je^5 = -e^6, \quad Je^6 = e^5, \quad Je^7 = -e^8, \quad Je^8 = e^7. \]

The fundamental form \( F \) associated to the \( J \)-Hermitian metric \( g = \sum_{i=1}^{8} e^i \otimes e^i \) is given by \( F = \sum_{j=1}^{4} e^{2j-1} \wedge e^{2j} \). In such a case,

\[ dF = 2e^{134} - e^{138} + e^{147} - 4e^{156} + 2e^{178} + e^{237} + e^{248} \]

and the torsion \( T \) is given by

\[ T = dJF = e^{137} + e^{148} - 2e^{234} + e^{238} - e^{247} + 4e^{256} - 2e^{278}. \]

Again, \( g \) is balanced. The non-zero connection 1-forms of the Bismut connection \( \nabla^B \) are

\[ \omega^1_3 = -e^3, \quad \omega^1_4 = -e^4, \quad \omega^1_5 = 2e^5, \quad \omega^1_6 = 2e^6, \quad \omega^1_7 = e^3 - e^7, \]
\[ \omega^3_8 = e^4 - e^8, \quad \omega^3_9 = e^4, \quad \omega^3_5 = -e^3, \quad \omega^3_6 = -2e^6, \quad \omega^3_7 = 2e^5, \]
\[ \omega^7_8 = -e^4 + e^8, \quad \omega^7_9 = e^3 - e^7, \quad \omega^7_5 = 2e^2, \quad \omega^7_6 = -e^1, \]
\[ \omega^8_3 = -e^2, \quad \omega^8_4 = e^2, \quad \omega^8_5 = -e^1, \quad \omega^8_6 = -4e^2, \quad \omega^8_7 = 2e^2. \]
The following 15 curvature forms for the Bismut connection are linearly independent:

\[
\begin{align*}
\Omega_1 &= -2e^{13} + e^{17} - 2e^{24} + e^{28}, & \Omega_4 &= -2e^{14} + e^{18} + 2e^{23} - e^{27}, \\
\Omega_2 &= -4(e^{15} + e^{26}), & \Omega_5 &= -4(e^{16} - e^{25}) \\
\Omega_3 &= e^{13} - e^{17} + e^{24} - e^{28}, & \Omega_6 &= e^{14} - e^{18} - e^{23} + e^{27}, \\
\Omega_7 &= 2(e^{12} - e^{34}), & \Omega_8 &= 2(e^{35} + e^{46}), \\
\Omega_9 &= 2(e^{36} - e^{45}), & \Omega_9 &= -(e^{37} + e^{48}) \\
\Omega_{10} &= 2e^{34} - e^{38} - 2e^{43} + e^{47}, & \Omega_{11} &= -8e^{56}, \\
\Omega_{12} &= 2(e^{35} + e^{46} + e^{57} + e^{68}), & \Omega_{13} &= 2(-e^{36} + e^{45} + e^{58} - e^{67}), \\
\Omega_{14} &= -2(e^{12} + e^{34} - e^{38} + e^{47} + e^{78}).
\end{align*}
\]

Therefore, \(\text{Hol}(\nabla^B) = SU(4)\).

**Example 4.6.** Let \(\varphi^j = e^{2j-1} + i e^{2j}, j = 1, \ldots, 4\) be complex \((1,0)\)-forms satisfying

\[
d\varphi^1 = 0, \quad d\varphi^2 = \varphi^{12}, \quad d\varphi^3 = -\varphi^{13}, \quad d\varphi^4 = -\varphi^{23}.
\]

Hence

\[
\begin{align*}
de^1 &= 0, & de^2 &= 0, & de^3 &= e^{13} - e^{24}, & de^4 &= e^{14} + e^{23}, \\
de^5 &= -e^{15} + e^{26}, & de^6 &= -e^{16} - e^{25}, & de^7 &= -e^{35} + e^{46}, & de^8 &= -e^{36} - e^{45}.
\end{align*}
\]

Let \(g\) be the real 8-dimensional Lie algebra whose dual is spanned by \(\{e^1, \ldots, e^8\}\), and let \(G\) be the simply-connected Lie group whose Lie algebra is \(g\). It is immediate to see that \(g\) is a 3-step solvable but not completely solvable Lie algebra. In fact, \(g' = \langle e_3, e_4, e_5, e_6, e_7, e_8 \rangle, g'' = \langle e_7, e_8 \rangle, g''' = \{0\}\). However, by [14] it turns out that \(G\) has a uniform discrete subgroup \(\Gamma\), such that \(M^8 = \Gamma \backslash G\) is a compact solvmanifold of dimension 8.

Define an (integrable) complex structure \(J\) on \(M^8\) by setting

\[
Je^1 = -e^2, \quad Je^2 = e^1, \quad Je^3 = -e^4, \quad Je^4 = -e^3, \\
Je^5 = -e^6, \quad Je^6 = e^5, \quad Je^7 = -e^8, \quad Je^8 = e^7,
\]

and a \(J\)-Hermitian structure \(g\) on \(M\) by setting \(g = \sum_{i=1}^8 e^i \otimes e^i\). Then \(F = \sum_{j=1}^4 e^{2j-1} \wedge e^{2j}\). Hence

\[
dF = 2(e^{134} - e^{156}) - e^{358} + e^{468} + e^{367} + e^{457}
\]

and consequently

\[
T = 2(e^{256} - e^{234}) - e^{467} + e^{357} + e^{458} + e^{368}.
\]
Therefore we obtain that the non zero connection and curvature forms of the Bismut connection are given respectively by

\( \omega_1 = -e^3, \quad \omega_2 = -e^4, \quad \omega_3 = e^5, \quad \omega_4 = e^6, \quad \omega_5 = e^4, \quad \omega_6 = -e^3 \) \tag{18}

and by

\[ \begin{align*}
\Omega_1 &= 2(e^{34} + e^{56}), \quad \Omega_2 = -e^{24} - e^{13}, \quad \Omega_3 = e^{23} - e^{14}, \quad \Omega_4 = -e^{26} - e^{15} \\
\Omega_5 &= e^{25} - e^{16}, \quad \Omega_6 = -e^{23} + e^{14}, \quad \Omega_7 = -e^{13} - e^{24}, \quad \Omega_8 = e^{16} - e^{25} \\
\Omega_9 &= -e^{15} - e^{26}, \quad \Omega_{10} = 2(-e^{34} + e^{56}), \quad \Omega_{11} = -e^{26} - e^{15}, \quad \Omega_{12} = e^{25} - e^{16} \\
\Omega_{13} &= -e^{25} + e^{16}, \quad \Omega_{14} = -e^{26} - e^{15}, \quad \Omega_{15} = 2(-e^{56} + e^{34}), \quad \Omega_{16} = -e^{24} - e^{13} \\
\Omega_{17} &= e^{23} - e^{14}, \quad \Omega_{18} = -e^{23} + e^{14}, \quad \Omega_{19} = -e^{24} - e^{13}, \\
\Omega_{20} &= -2(e^{56} + e^{34}).
\end{align*} \]

Hence the six 2-forms

\[ \begin{align*}
\Omega_1 &= 2(e^{34} + e^{56}), \quad \Omega_2 = -e^{24} - e^{13}, \quad \Omega_3 = e^{23} - e^{14}, \\
\Omega_4 &= e^{25} - e^{16}, \quad \Omega_5 = -e^{26} - e^{15}, \quad \Omega_6 = e^{25} - e^{16}, \quad \Omega_7 = 2(-e^{34} + e^{56})
\end{align*} \]

are linearly independent. Therefore, \( \dim(\mathfrak{hol}(\nabla^B)) \geq 6 \).

We need to compute the covariant derivative of the curvature. By the above expression for the connection forms, we get:

\[ \begin{align*}
\nabla^B e^1 &= e^3 \otimes e^3 + e^4 \otimes e^4 - e^5 \otimes e^5 - e^6 \otimes e^6 \\
\nabla^B e^2 &= -e^4 \otimes e^4 + e^3 \otimes e^3 + e^6 \otimes e^6 - e^5 \otimes e^5 \\
\nabla^B e^3 &= -e^3 \otimes e^1 + e^4 \otimes e^2 - 2 e^2 \otimes e^4 - e^5 \otimes e^7 - e^6 \otimes e^8 \\
\nabla^B e^4 &= -e^4 \otimes e^1 - e^3 \otimes e^2 + 2 e^2 \otimes e^3 + e^6 \otimes e^7 - e^5 \otimes e^8 \\
\nabla^B e^5 &= e^5 \otimes e^1 - e^6 \otimes e^2 + 2 e^2 \otimes e^6 + e^3 \otimes e^7 + e^4 \otimes e^8 \\
\nabla^B e^6 &= e^6 \otimes e^1 + e^5 \otimes e^2 - 2 e^2 \otimes e^5 - e^4 \otimes e^7 + e^3 \otimes e^8 \\
\nabla^B e^7 &= e^7 \otimes e^1 - e^6 \otimes e^4 - e^3 \otimes e^5 + e^4 \otimes e^6 \\
\nabla^B e^8 &= e^8 \otimes e^1 + e^5 \otimes e^4 - e^4 \otimes e^5 - e^3 \otimes e^6.
\end{align*} \]
By the above formulas, a straightforward computation, taking into account (19), gives

\[ \nabla^B e_3 \Omega^1_2 = 2 \left( -e^{14} + e^{23} - e^{67} + e^{58} \right) \]

\[ \nabla^B e_4 \Omega^1_2 = 2 \left( e^{13} + e^{24} - e^{68} - e^{57} \right) \]

\[ \nabla^B e_5 \Omega^1_2 = 2 \left( e^{47} - e^{38} + e^{16} - e^{25} \right) \]

\[ \nabla^B e_6 \Omega^1_2 = 2 \left( e^{48} + e^{37} - e^{26} - e^{15} \right) \]

\[ \nabla^B e_3 \Omega^1_4 = -e^{35} - e^{28} - e^{46} - e^{17} \]

\[ \nabla^B e_4 \Omega^1_4 = -e^{18} + e^{27} + e^{36} - e^{45} \]

\[ \nabla^B e_5 \Omega^1_4 = e^{45} + e^{27} - e^{36} - e^{18} \]

\[ \nabla^B e_6 \Omega^1_4 = -e^{35} + e^{28} - e^{46} + e^{17} \]

\[ \nabla^B e_6 \Omega^1_5 = 2(e^{36} - e^{12}) \]

Therefore, we have proved the following

**Proposition 4.7.** The holomorphic parallelizable solvmanifold \( M = \Gamma \backslash G \), where \( \{ \varphi^1, \ldots, \varphi^4 \} \) are the complex \((1,0)\)-forms satisfying (17), it is endowed with an Hermitian metric with Kähler form

\[ F = \frac{i}{2} \sum_{j=1}^{4} \varphi^j \wedge \varphi^j \]

satisfying \( dF^3 = 0 \) and the infinitesimal holonomy of the Bismut connection is \( su(4) \).

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M. Fernández: Departamento de Matemáticas, Facultad de Ciencia Tecnología, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
E-mail address: marisa.fernandez@ehu.es

A. Tomassini: Dipartimento di Matematica, Università di Parma, Viale G.P. Usberti 53/A, 43100 Parma, Italy
E-mail address: adriano.tomassini@unipr.it

L. Ugarte and R. Villacampa: Departamento de Matemáticas - I.U.M.A., Universidad de Zaragoza, Campus Plaza San Francisco, 50009 Zaragoza, Spain
E-mail address: ugarte@unizar.es, raquelvg@unizar.es