Arithmetic-Geometric Mean: Evaluation of Parameter from Observed Data Containing Itself and Random Error

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Abstract

Recently some methods have been developed for determining the value of parameter from observed data containing a single parameter and random error since the existing statistical methods of estimation in such situation fail in finding out the appropriate value of the parameter. The methods, so developed, involve huge computational tasks. Moreover, a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. For these two limitations, one method for the same has been developed here which involves lesser computational tasks than those involved in the methods developed so far. Moreover, the method described here can be applicable in the case of finite set of data. This paper describes the derivation of the method and one numerical application of the method in determining the central tendency of each of annual maximum and annual minimum of surface air temperature at Guwahati.

Key words: 1; AGM 2; parameter 3; random error 4; evaluation of parameter 5; surface air temperature 6; central tendency

1. Introduction:

There are many situations where the observations are composed of some parameter and random errors i.e. the observations

\[ x_1, x_2, \ldots, x_n \]

are composed of some parameter \( \mu \) and random errors \( \varepsilon_i \) which implies

\[ \text{observed data} = \mu + \varepsilon_i \]
\[ x_i = \mu + \varepsilon_i \quad (i = 1, 2, \ldots, n) \quad \text{---------------------- (1.1)} \]

[Chakrabarty (2014a, 2014b, 2014c, 2015a, 2015b, 2015c, 2015d, 2015e)]. The existing methods of estimation of \( \mu \) namely least squares method, maximum likelihood method, minimum variance unbiased method, method of moment and method of minimum chi-square, [Aldrich (2000), Anders (1999), Barnard (1949), Birnbaum (1962), Ivory (1825), Kendall & Stuart (1977), Lehmann & Casella George (1998), Lucien (1990), Walker & Lev (1965)] provides \( T \) as estimator of the parameter \( \mu \) where \( T \) is given by

\[ T = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{---------------------- (1.2)} \]

It has been shown that this estimator \( T \) of the parameter \( \mu \) suffers from an error \( e \) given by

\[ e = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \quad \text{---------------------- (1.3)} \]

which is not zero usually. In other words, none of these methods can provide appropriate value of the parameter \( \mu \) [Chakrabarty (2014a, 2014b, 2014c)].

In some recent studies, some methods have been developed for determining the appropriate value of the parameter \( \mu \) involved in the model described by equation (1.1) [Chakrabarty (2014a, 2014b, 2014c, 2015a, 2015b, 2015c, 2015d, 2015e, 2015f, 2016a, 2016b, 2019a, 2019c)]. In these studies some methods have been developed for determining the appropriate value of the parameter \( \mu \) when \( \varepsilon_i \) occurs due to random cause.

The first method, developed for the same is based on computing sequence of interval value of \( \mu \) with decreasing length of interval and then to find out the shortest interval value of \( \mu \) [Chakrabarty (2014a, 2014b, 2014c, 2015d)] while the second one is based on stable mid range and median (Chakrabarty, 2015b) and the third one on the convergence of statistic i.e. some function of the available numerical data (Chakrabarty, 2017). The fourth one (Chakrabarty, 2018a) has been developed on the basis of Pythagorean means [Kolmogorov (1930), O'Meara (1989), Riedweg (2005), Cornelli, McKirahan & Meris (2013), de Carvalho (2016), Chakrabarty (2018b, 2018c, 2018d, 2018e, 2019d, 2019e)] while the fifth one (Chakrabarty, 2019b) for the same is based on the probabilistic convergence of Pythagorean means.
The methods, developed so far for determining the appropriate value of the parameter from observed data containing the parameter itself and random error, involve huge computational tasks. Moreover, a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. For these two probable drawbacks, one method for the same purpose has been developed here which involves lesser computational tasks than those involved in the methods developed so far. Moreover, the method described here can be applicable in the case of finite set of data. This paper describes the derivation of the method and one numerical application of the method in determining the central tendency of each of annual maximum and annual minimum of surface air temperature at Guwahati.

2. Arithmetic-Geometric Mean (AGM):
In mathematics, the arithmetic–geometric mean (AGM) [David (2004), Hazewinkel (2001)], of two positive real numbers \(x\) and \(y\) is defined as follows:

Let us denote \(x\) & \(y\) by \(a_0\) & \(g_0\) respectively.

Then define the two interdependent sequences \(\{a_n\}\) and \(\{g_n\}\) by

\[
a_{n+1} = \frac{1}{2} (a_n + g_n) \quad \& \quad g_{n+1} = (a_n g_n)^{1/2}
\]

where the square root takes the principal value.

These two sequences converge to the same number, called the Arithmetic-Geometric Mean (abbreviated by AGM) of \(x\) & \(y\); which is denoted by \(M(x, y)\) or sometimes by \(agm(x, y)\).

**Proof of Existence of \(M(x, y)\)**

The three Pythagorean means namely Arithmetic Mean (AM), Geometric Mean (GM) & Harmonic Mean (HM) satisfy the inequality

\[
AM \geq GM \geq HM
\]

From the inequality one can conclude that

\[
g_n \leq a_n
\]

and thus

\[
g_{n+1} = (a_n g_n)^{1/2} \geq (g_n g_n)^{1/2} = g_n
\]

i.e. \(g_{n+1} \geq g_n\) since \((g_n g_n)^{1/2} = g_n\)

This means that the sequence \(\{g_n\}\) is nondecreasing.
Moreover, the sequence \( \{g_n\} \) is bounded above by the larger of \( x \) and \( y \) (which follows from the fact that both the arithmetic and geometric means of two numbers lie between them).

In the mathematical field of real analysis, the monotone convergence theorem [Weir (1973), Yeh (2006)] states that if a sequence is increasing and bounded above by a supremum, then the sequence will converge to the supremum; in the same way, if a sequence is decreasing and is bounded below by an infimum, it will converge to the infimum.

Thus, by the monotone convergence theorem, the sequence is convergent. Therefore, there exists a finite number \( g \) such that

\[
g_n \text{ converges to } g \text{ as } n \text{ approaches infinity.}
\]

Again, \( a_n \) can be expressed as

\[
a_n = \frac{g_{n+1}^2}{g_n}
\]

This implies that the limiting value of \( a_n \) as \( n \) approaches infinity is \( g \).

**Therefore**, \( a_n \) converges to \( g \) as \( n \) approaches infinity.

**Thus**, both of \( a_n \) & \( g_n \) converges to \( g \) as \( n \) approaches infinity.

### 3. Evaluation of \( \mu \)

If the observations

\[
x_1, x_2, \ldots, x_N
\]

are composed of some parameter \( \mu \) and random errors then the observations can be expressed as

\[
x_i = \mu + \varepsilon_i \quad (i = 1, 2, \ldots, N)
\]

where

(i) \( x_1, x_2, \ldots, x_N \) are observed data,

(ii) \( \mu \) is the parameter

& (iii) \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \) are the random errors associated to

\[
x_1, x_2, \ldots, x_N
\]

respectively which assume positive and negative values in random order.
In this situation,
\[ A(x_1, x_2, \ldots, x_N) \rightarrow \mu \quad \text{as} \quad N \rightarrow \infty \]
where
\[ A(x_1, x_2, \ldots, x_N) = \frac{1}{N} \sum_{i=1}^{N} x_i \]
and
\[ G(x_1, x_2, \ldots, x_N) \rightarrow \mu \quad \text{as} \quad N \rightarrow \infty \]
where
\[ G(x_1, x_2, \ldots, x_N) = \left( \prod_{i=1}^{N} e_i \right)^{1/N} \]
This implies that the common converging value of \(A(x_1, x_2, \ldots, x_N)\) and \(G(x_1, x_2, \ldots, x_N)\) is the value of \(\mu\).
It is to be noted that the converging value may not be possible to be obtained for a finite set of observed values namely
\[ x_1, x_2, \ldots, x_N \]
In order to obtain the value of \(\mu\), in this case, let us write
\[ A(x_1, x_2, \ldots, x_N) = A_0 \]
\& \[ G(x_1, x_2, \ldots, x_N) = G_0 \]
and then define the two interdependent sequences \(\{A_n\}\) and \(\{G_n\}\) as
\[ A_{n+1} = \frac{1}{2} (A_n + G_n) \]
\& \[ G_{n+1} = (A_n G_n)^{1/2} \]
Then, both of \(A_n\) \& \(G_n\) converges to \(C\) as \(n\) approaches infinity.
Now, it is required to verify whether this \(C\) is equal to \(\mu\).
From the model it is obtained that
\[ A_0 = \mu + \epsilon_0 \quad \text{&} \quad G_0 = \mu + e_0 \]
The inequality of Pythagorean means namely
\[ AM \geq GM \]
implies that
\[ A_0 \geq G_0 \quad \text{i.e.} \quad \epsilon_0 \geq e_0 \]
Thus \[ A_1 = \mu + \epsilon_1 \quad \text{where} \quad \epsilon_1 = \frac{1}{2} (\epsilon_0 + e_0) \leq \epsilon_0 \]
In general, corresponding to \(A_{n+1}\), it holds that
\[ \varepsilon_{n+1} = \frac{1}{2} (\varepsilon_n + e_n) \leq \varepsilon_n \]

This implies \( \varepsilon_n \) converges to 0 i.e. \( A_n \) converges to \( \mu \).

By the existence of AGM, \( G_n \) also converges to \( \mu \).

3. Application to Numerical Data:

Observed data considered here are the data on each of annual maximum & annual minimum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013. The objective here is to evaluate the central tendency of each of annual maximum & annual minimum of surface air temperature at Guwahati

3.1. Annual Maximum of Surface Air Temperature at Guwahati:

The following table (Table – 3.1.1) shows the observed data on annual maximum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013:

| TPR No (i) | Observed Value \( (x_i) \) | TPR No (i) | Observed Value \( (x_i) \) | TPR No (i) | Observed Value \( (x_i) \) | TPR No (i) | Observed Value \( (x_i) \) |
|-----------|--------------------------|-----------|--------------------------|-----------|--------------------------|-----------|--------------------------|
| 1         | 37.1                     | 12        | 35.1                     | 23        | 37.4                     | 34        | 38.0                     |
| 2         | 36.6                     | 13        | 35.8                     | 24        | 39.4                     | 35        | 36.6                     |
| 3         | 36.0                     | 14        | 36.5                     | 25        | 36.4                     | 36        | 38.0                     |
| 4         | 35.7                     | 15        | 36.7                     | 26        | 38.1                     | 37        | 37.3                     |
| 5         | 39.0                     | 16        | 37.2                     | 27        | 36.3                     | 38        | 37.3                     |
| 6         | 36.1                     | 17        | 36.5                     | 28        | 39.9                     | 39        | 38.0                     |
| 7         | 39.2                     | 18        | 38.4                     | 29        | 37.4                     | 40        | 37.2                     |
| 8         | 39.0                     | 19        | 37.2                     | 30        | 37.5                     | 41        | 37.3                     |

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Here the observed values $x_i$ ($i = 1, 2, 3, \ldots, 43$) can be assumed to be composed of a parameter $\mu$ (representing the central tendency of annual maximum) and random errors so that the observed values follow the model described by equation (1.1).

**Evaluation of Value of $\mu$ (the central tendency of annual maximum)**

The computed values of arithmetic mean and the geometric mean of the observed values, shown in Table – 3.1.1, are found to be

\[
37.2093023255814 \quad \text{and} \quad 37.192287148576076781925812747586
\]

respectively.

Let us write

\[
A_0 = 37.2093023255814 \quad \text{and} \quad G_0 = 37.192287148576076781925812747586
\]

In this case the iterations give the values which are given in the following table (Table – 3.1.2):

| $n$ | $A_n$ | $G_n$ |
|-----|-------|-------|
| 0   | 37.2093023255814 | 37.192287148576076781925812747586 |
| 1   | 37.200794737078738390642906373793 | 37.200794250670695306539564311732 |
| 2   | 37.200794250670695306539564311732 | 37.200794250670695306539564311732 |
| 3   | 37.2079425067069371656824015813 | 37.2079425067069371656824015813 |

The digits in $A_n$ and $G_n$, which are agreed, have been underlined in the above table.

The AGM of

\[
37.2093023255814 \quad \text{and} \quad 37.192287148576076781925812747586
\]

is the common limit of these two sequences which is
Thus the value of $\mu$, the central tendency of annual maximum of surface air temperature at Guwahati, is 37.20079425067069371656824015813 Degree Celsius.

### 3.2. Annual Minimum of Surface Air Temperature at Guwahati:

The following table (Table – 3.2.1) shows the observed data on annual maximum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013.

As earlier, the observed values $x_i$ ($i = 1, 2, 3, \ldots, 43$) can in this case also be assumed to be composed of a parameter $\mu$ (representing the central tendency of annual minimum) and random errors so that the observed values follow the model described by equation (1.1).

#### Table – 3.2.1

| TPR No ($i$) | Observed Value ($x_i$) | TPR No ($i$) | Observed Value ($x_i$) | TPR No ($i$) | Observed Value ($x_i$) | TPR No ($i$) | Observed Value ($x_i$) |
|-------------|------------------------|-------------|------------------------|-------------|------------------------|-------------|------------------------|
| 1           | 6.6                    | 12          | 6.4                    | 23          | 7.4                    | 34          | 8.0                    |
| 2           | 6.6                    | 13          | 7.5                    | 24          | 5.9                    | 35          | 7.9                    |
| 3           | 5.9                    | 14          | 8.3                    | 25          | 8.4                    | 36          | 6.7                    |
| 4           | 8.2                    | 15          | 4.9                    | 26          | 7.8                    | 37          | 9.6                    |
| 5           | 5.0                    | 16          | 6.1                    | 27          | 7.5                    | 38          | 6.4                    |
| 6           | 6.3                    | 17          | 7.8                    | 28          | 9.4                    | 39          | 7.8                    |
| 7           | 7.4                    | 18          | 8.6                    | 29          | NA                     | 40          | 9.9                    |
| 8           | 6.6                    | 19          | 7.7                    | 30          | NA                     | 41          | 8.6                    |
| 9           | 6.2                    | 20          | 9.2                    | 31          | NA                     | 42          | 7.0                    |
| 10          | 7.3                    | 21          | 6.7                    | 32          | 8.9                    | 43          | 6.4                    |
Determination of Value of µ (the central tendency of annual minimum)

The computed values of arithmetic mean and the geometric mean of the observed values, shown in Table – 3.1.1, are found to be

7.032 and 6.9882108302873798619833480810597

Let us write

\[ A_0 = 7.032 \]
\[ G_0 = 6.9882108302873798619833480810597 \]

In this case the iterations give the values which are given in the following table (Table – 3.2.2):

| \( n \) | \( A_n \) | \( G_n \) |
|-------|---------|---------|
| 0     | 7.032   | 6.9882108302873798619833480810597 |
| 1     | 7.0101054151436899309916740405299 | 7.0100712235027152532408857295979 |
| 2     | 7.0100883193232025921162798850639 | 7.0100883193023564155188951988585 |
| 3     | 7.0100883193127795038175875419612 | 7.0100883193127795038175797930738 |
| 4     | 7.0100883193127795038175836675175 | 7.0100883193127795038175836675175 |

The digits in \( A_n \) and \( G_n \), which are agreed, have been underlined in the above table.

The AGM of

7.032 and 6.9882108302873798619833480810597

is the common limit of these two sequences which is

7.0100883193127795038175836675175

Thus the value of \( \mu \), the central tendency of annual minimum of surface air temperature at Guwahati, is 7.0100883193127795038175836675175 Degree Celsius.
4. Conclusion
The methods developed so far, for determining the appropriate value of the parameter from observed data containing the parameter itself and random error involve huge computational tasks. The method described here involves lesser computational tasks than those involved in the methods developed so far.

Moreover, a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. The method described here can be applicable in the case of finite set of data.

Regarding the findings obtained on annual maximum and annual minimum of surface air temperature at Guwahati, the following conclusion can be drawn:

4.1. The value of central tendency of annual maximum of surface air temperature at Guwahati has here been obtained as 37.20079425067069371656824015813 Degree Celsius. However, the value of the same was found to be 37.2 Degree Celsius by the earlier methods of the same [2014c, 2015a, 2015b, 2015c, 2015d, 2015e, 2017, 2018a, 2019b]. This means, the method developed here can yield more accurate value.

4.2. The central tendency of annual maximum of surface air temperature at Guwahati has here been obtained as 7.0100883193127795038175175 Degree Celsius. However, it was found not possible to obtain the value of central tendency of the same by the earlier methods [2014c, 2015a, 2015b, 2015c, 2015d, 2015e, 2017, 2018a, 2019b].

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