EQUIVARIANT MAPS BETWEEN CALOGERO-MOSER SPACES

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Abstract. We add a last refinement to the results of [BW1] and [BW2] relating ideal classes of the Weyl algebra to the Calogero-Moser varieties: we show that the bijection constructed in those papers is uniquely determined by its equivariance with respect to the automorphism group of the Weyl algebra.

1. Introduction and statement of results

Let \( A \) be the Weyl algebra \( \mathbb{C} \langle x, y \rangle / (xy - yx - 1) \), and let \( \mathcal{R} \) be the space of noncyclic right ideal classes of \( A \) (that is, isomorphism classes of noncyclic finitely generated rank 1 torsion-free right \( A \)-modules). Let \( \mathcal{C} \) be the disjoint union of the Calogero-Moser spaces \( \mathcal{C}_n \) \((n \geq 1)\): we recall that \( \mathcal{C}_n \) is the space of all simultaneous conjugacy classes of pairs of \( n \times n \) matrices \((X, Y)\) such that \([X, Y] + I\) has rank 1. It is a smooth irreducible affine variety of dimension \( 2n \) (see [W]). For simplicity, in what follows we shall use the same notation \((X, Y)\) for a pair of matrices and for the corresponding point of \( \mathcal{C}_n \). Let \( G \) be the group of \( \mathbb{C} \)-automorphisms of \( A \), and let \( \Gamma \) and \( \Gamma' \) be the isotropy groups of the generators \( y \) and \( x \) of \( A \). Thus \( \Gamma \) consists of all automorphisms of the form

\[
\Phi_p(x) = x - p(y), \quad \Phi_p(y) = y
\]

where \( p \) is a polynomial; and similarly \( \Gamma' \) consists of all automorphisms of the form

\[
\Psi_q(x) = x, \quad \Psi_q(y) = y - q(x)
\]

where \( q \) is a polynomial. According to Dixmier (see [D]), \( G \) is generated by the subgroups \( \Gamma \) and \( \Gamma' \). There is an obvious action of \( G \) on \( \mathcal{R} \); we let \( G \) act on \( \mathcal{C} \) by the formulae

\[
(1.1) \quad \Phi_p(X, Y) = (X + p(Y), Y), \quad \Psi_q(X, Y) = (X, Y + q(X)).
\]

According to [BW1] this \( G \)-action is transitive on each space \( \mathcal{C}_n \). The main result of [BW1] was the following.

**Theorem 1.1.** There is a bijection between the spaces \( \mathcal{R} \) and \( \mathcal{C} \) which is equivariant with respect to the above actions of \( G \).

This bijection constructed in [BW1] was obtained in a quite different way in [BW2]. The proof in [BW2] that the two constructions agree used the fact that equivariance was known in both cases; thus to prove that the bijections coincide, it was enough to check one point in each \( G \)-orbit, that is, in each space \( \mathcal{C}_n \). The result to be proved in the present note is that even this (not difficult) check was unnecessary.

**Theorem 1.2.** There is only one \( G \)-equivariant bijection between the spaces \( \mathcal{R} \) and \( \mathcal{C} \).
Clearly, it is equivalent to show that there is no nontrivial $G$-equivariant bijection from $\mathcal{C}$ to itself. We shall show a little more, namely, that (apart from the identity) there is no $G$-equivariant map (for short: $G$-map) at all from $\mathcal{C}$ to itself. Since a $G$-map must take each orbit onto another orbit, that amounts to the following assertion.

**Theorem 1.3.**

(i) For any $n \geq 1$, let $f : \mathcal{C}_n \to \mathcal{C}_n$ be a $G$-map. Then $f$ is the identity.

(ii) For $n \neq m$ there is no $G$-map from $\mathcal{C}_n$ to $\mathcal{C}_m$.

Since $\mathcal{C}_n$ and the action of $G$ on it are defined by simple formulae involving matrices, the proof of Theorem 1.3 is just an exercise in linear algebra. Quite possibly there is a simpler solution to the exercise than the one given below.

The first part of Theorem 1.3 is equivalent to the statement that the isotropy group of any point of $\mathcal{C}$ (or $\mathcal{R}$) coincides with its normalizer in $G$ (see section 6 below); in particular, these isotropy groups are not normal in $G$, confirming a suspicion of Stafford (see [St], p. 636). Stafford’s conjecture seems to have been the motivation for Kouakou’s work [K], which contains a result equivalent to ours. The proof in [K] looks quite different from the present one, because Kouakou does not use the spaces $\mathcal{C}_n$, but rather the alternative description of $\mathcal{R}$ (due to Cannings and Holland, see [CH]) as the adelic Grassmannian of $\mathcal{W}$. I have not entirely succeeded in following the details of [K]; in any case, it seems worthwhile to make available the independent verification of the result offered here.

**Remark.** We have excluded from $\mathcal{R}$ the cyclic ideal class, corresponding to the Calogero-Moser space $\mathcal{C}_0$ (which is a point). The reason is very trivial: since there is always a map from any space to a point, part (ii) of Theorem 1.3 would be false if we included $\mathcal{C}_0$. However, Theorem 1.2 would still be true.

### 2. Proof of Theorem 1.3 in the case $n < m$

If we accept (cf. [BW1], section 11) that the $\mathcal{C}_n$ are homogeneous spaces for the (infinite-dimensional) algebraic group $G$, then Theorem 1.3 becomes obvious in the case $n < m$. Indeed, any $G$-map from $\mathcal{C}_n$ to $\mathcal{C}_m$ would have to be a surjective map of algebraic varieties, which is clearly impossible if $n < m$, because then $\mathcal{C}_m$ has greater dimension ($2m$) than $\mathcal{C}_n$. For readers who are not convinced by this argument, we offer a more elementary one, based on the following lemma.

**Lemma 2.1.** Let $f : \mathcal{C}_n \to \mathcal{C}_m$ be a $G$-map. Suppose that $f(X, Y) = (P, Q)$, and that $P$ is diagonalizable. Then every eigenvalue of $P$ is an eigenvalue of $X$.

**Proof.** Let $\chi$ be the minimum polynomial of $X$; then in $\mathcal{C}_m$ we have

$$(P, Q) = f(X, Y) = f(X, Y + \chi(X)) = (P, Q + \chi(P))$$

(where the last step used the fact that $f$ has to commute with the action of $\Psi_\chi \in G$). That means that there is an invertible matrix $A$ such that

$$APA^{-1} = P \text{ and } AQA^{-1} = Q + \chi(P).$$

We may assume that $P = \text{diag}(p_1, \ldots, p_m)$ is diagonal. Then since the $p_i$ are distinct (see [W], Proposition 1.10), $A$ is diagonal too, so taking the diagonal entries in the last equation gives $q_i = q_i + \chi(p_i)$, whence $\chi(p_i) = 0$ for all $i$. Thus $\chi(P) = 0$, so the minimum polynomial of $P$ divides $\chi$. The lemma follows. \hfill $\square$
Corollary 2.2. If \( n < m \) there is no \( G \)-map \( f : \mathcal{C}_n \to \mathcal{C}_m \).

Proof. Choose \((P, Q) \in \mathcal{C}_m\) with \( P \) diagonalizable. Since \( \mathcal{C}_m \) is just one \( G \)-orbit, \( f \) is surjective, so we can choose \((X, Y) \in \mathcal{C}_n\) with \( f(X, Y) = (P, Q) \). But then Lemma 2.1 says that \( X \) is an \( n \times n \) matrix with more than \( n \) distinct eigenvalues, which is impossible.

3. THE BASE-POINT

A useful subgroup of \( G \) is the group \( R \) of scaling transformations, defined by

\[
R_\lambda(x) = \lambda x, \quad R_\lambda(y) = \lambda^{-1} y \quad (\lambda \in \mathbb{C}^\times).
\]

It acts on \( \mathcal{C}_n \) in a similar way:

\[
(3.1) \quad R_\lambda(X, Y) = (\lambda^{-1} X, \lambda Y).
\]

Lemma 3.1. Suppose that the conjugacy class \((X, Y) \in \mathcal{C}_n \) is fixed by the group \( R \). Then \( X \) and \( Y \) are both nilpotent.

Proof. Let \( \mu \) be an eigenvalue of (say) \( Y \). Then for any \( \lambda \in \mathbb{C}^\times \), \( \lambda \mu \) is an eigenvalue of \( \lambda Y \), which is (by hypothesis) conjugate to \( Y \). Thus \( \lambda \mu \) is an eigenvalue of \( Y \) for every \( \lambda \in \mathbb{C}^\times \), which is impossible unless \( \mu = 0 \). Hence all eigenvalues of \( Y \) must be 0, that is, \( Y \) must be nilpotent. The same argument applies to \( X \).

The converse to Lemma 3.1 is also true, but we shall use that fact only for the pair \((X_0, Y_0)\) given by

\[
(3.2) \quad X_0 = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 2 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & n-1 & 0 \\
\end{pmatrix}, \quad Y_0 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}.
\]

We shall regard \((X_0, Y_0)\) as the base-point in \( \mathcal{C}_n \). In the rather trivial case \( n = 1 \), we have \( \mathcal{C}_1 = \mathbb{C}^2 \), and we interpret \((X_0, Y_0)\) as \((0, 0)\).

Lemma 3.2. The (conjugacy class of) the pair \((X_0, Y_0) \in \mathcal{C}_n \) is fixed by the group \( R \).

Proof. For \( \lambda \in \mathbb{C}^\times \), let \( d(\lambda) \) be the diagonal matrix

\[
d(\lambda) := \text{diag}(\lambda, \lambda^2, \ldots, \lambda^n).
\]

Then \( d(\lambda)^{-1} X d(\lambda) = \lambda^{-1} X \) and \( d(\lambda)^{-1} Y d(\lambda) = \lambda Y \).

Corollary 3.3. Let \( f : \mathcal{C}_n \to \mathcal{C}_m \) be a \( G \)-map, and let \( f(X_0, Y_0) = (P, Q) \). Then \( P \) and \( Q \) are nilpotent.

Proof. This follows at once from Lemmas 3.1 and 3.2, since a \( G \)-map must respect the fixed point set of any subgroup of \( G \).
4. Proof of Theorem 1.3 in the case $n > m$

The remaining parts of the proof use the following trivial fact.

**Lemma 4.1.** Let $(X, Y) \in C_n$, let $p$ be any polynomial, and let $\chi$ be divisible by the minimum polynomial of $X + p(Y)$. Then the automorphism $\Phi_p \Psi \Phi_p$ fixes $(X, Y)$.

**Proof.** Since $\chi(X + p(Y)) = 0$ we have

$$\Phi_p \Psi \Phi_p(X, Y) = \Phi_p \Psi(X + p(Y), Y) = \Phi_p(X + p(Y), Y) = (X, Y),$$

as claimed. □

**Proposition 4.2.** If $n > m > 0$ there is no $G$-map $f : C_n \to C_m$.

**Proof.** We apply Lemma 4.1 to the base-point $(X_0, Y_0) \in C_n$, with $p(t) = t^{n-1}$. The minimum (= characteristic) polynomial of $X_0 + Y_0^{n-1}$ is

$$\chi(t) := \det(tI - X_0 - Y_0^{n-1}) = t^n - (n-1)!.$$ (4.1)

Now suppose that $f : C_n \to C_m$ is a $G$-map, and let $f(X_0, Y_0) = (P, Q)$: according to Corollary 3.3 $P$ and $Q$ are nilpotent. They are of size less than $n$, so we have $P^{n-1} = Q^{n-1} = 0$. Thus

$$\Phi_p \Psi \Phi_p(P, Q) = \Phi_p \Psi(P + Q^{n-1}, Q) = \Phi_p \Psi(P, Q) = \Phi_p(P, Q + P^{n-1} - (n-1)!I) = \Phi_p(P, Q - (n-1)!I) = (\text{something}, Q - (n-1)!I).$$

Now, $Q - (n-1)!I$ is not conjugate to $Q$ (because their eigenvalues are different), hence $\Phi_p \Psi \Phi_p$ does not fix $(P, Q)$. So by Lemma 4.1 the isotropy group of $(X_0, Y_0)$ is not contained in the isotropy group of $f(X_0, Y_0)$. This contradiction shows that $f$ does not exist. □

5. Proof of Theorem 1.3 in the case $n = m$

It remains to show that there is no nontrivial $G$-map from $C_n$ to itself. Note that because $C_n$ is a single orbit, any such map must be bijective, and must map each point of $C_n$ to a point with the same isotropy group. In the case $n = 1$ the result follows (for example) from Lemma 2.1 so from now on we shall assume that $n \geq 2$. Let $f : C_n \to C_n$ be a $G$-map, and let $f(X_0, Y_0) = (P, Q)$. Again, Corollary 3.3 says that $P$ and $Q$ are nilpotent. We aim to show that $(P, Q)$ can only be $(X_0, Y_0)$, whence $f$ is the identity. We remark first that if $Q^{n-1} = 0$, then the calculation in the proof of Proposition 4.2 still gives a contradiction; thus the Jordan form of $Q$ consists of just one block, so we may assume that $Q = Y_0$. Now, it is not hard to classify all the points $(X, Y_0) \in C_n$ with $X$ nilpotent (see [W], p.26 for the elementary argument): there are exactly $n$ of them, and they
all have the form \((X(a), Y_0)\), where \(a := (a_1, \ldots, a_{n-1})\) and \(X(a)\) denotes the subdiagonal matrix

\[
X(a) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & a_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & a_{n-1} \\
\end{pmatrix}.
\]

(5.1)

The possible vectors \(a\) that give points of \(C_n\) are

\[
a = (1, 2, \ldots, r-1; -(n-r), \ldots, -2, -1) \quad \text{for} \quad 1 \leq r \leq n
\]

(5.2)

so \(r = n\) gives \(X_0\). Thus so far we have shown that \(f(X_0, Y_0)\) must be one of these points \((X(a), Y_0)\). To finish the argument, we need the following easy calculations of characteristic polynomials (the first generalizes (4.1)):

\[
det(tI - X(a) - Y_0^{n-1}) = t^n - \prod_{1}^{n-1} a_i;
\]

(5.3)

\[
det(tI - X(a) - Y_0^{n-2}) = t^n - \left(\prod_{1}^{n-2} a_i + \prod_{1}^{n-1} a_i\right) t,
\]

(5.4)

where the last formula holds only for \(n \geq 3\). If \(a\) is one of the vectors \((5.2)\) with \(1 < r < n\), then the right hand side of \((5.3)\) is just \(t^n\); that is, \(X(a) + Y_0^{n-2}\) is nilpotent. In fact it is easy to check that the pair \((X(a) + Y_0^{n-2}, Y_0)\) is conjugate to \((X(a), Y_0)\); that is, the map \((X, Y) \mapsto (X + Y_0^{n-2}, Y)\) fixes \((X(a), Y_0)\). It does not fix \((X_0, Y_0)\), so \(f(X_0, Y_0)\) cannot be any of these points \((X(a), Y_0)\).

It remains only to see that \(f\) cannot map \((X_0, Y_0)\) to the pair corresponding to \(r = 1\) in \((5.2)\); let us call it \((X_1, Y_0)\).

If \(n\) is even we use \((5.3)\): the characteristic polynomial of \(X_0 + Y_0^{n-1}\) is \(\chi(t) := t^n - (n-1)!\) while the characteristic polynomial of \(X_1 + Y_0^{n-1}\) is \(t^n + (n-1)!\), so that \(\chi(X_1 + Y_0^{n-1}) = -2(n-1)!I\). We now apply Lemma 4.1 with \(p(t) = t^{n-1}\). According to that lemma, the map \(\Phi_{-p} \psi \Phi_p\) fixes \((X_0, Y_0)\); on the other hand

\[
\Phi_{-p} \psi \Phi_p(X_1, Y_0) = \Phi_{-p} \psi(X_1 + Y_0^{n-1}, Y_0) = \Phi_{-p}(X_1 + Y_0^{n-1}, Y_0 - 2(n-1)!I) = (\text{something}, Y_0 - 2(n-1)!I).
\]

Since \(Y_0 - 2(n-1)!I\) is not conjugate to \(Y_0\), this shows that \(\Phi_{-p} \psi \Phi_p\) does not fix \((X_1, Y_0)\). Thus in this case \(f(X_0, Y_0)\) cannot be equal to \((X_1, Y_0)\).

Finally, if \(n\) is odd, we have a similar calculation using \((5.3)\). Setting \(\alpha := (n-1)! + (n-2)!\), the characteristic polynomial of \(X_0 + Y_0^{n-2}\) is \(\chi(t) := t^n - \alpha t\) while the characteristic polynomial of \(X_1 + Y_0^{n-2}\) is \(t^n + \alpha t\), so that \(\chi(X_1 + Y_0^{n-2}) = -2\alpha(X_1 + Y_0^{n-2})\). We now apply Lemma 4.1 with \(p(t) = t^{n-2}\). The map \(\Phi_{-p} \psi \Phi_p\) fixes \((X_0, Y_0)\); on the other hand

\[
\Phi_{-p} \psi \Phi_p(X_1, Y_0) = \Phi_{-p} \psi(X_1 + Y_0^{n-2}, Y_0) = \Phi_{-p}(X_1 + Y_0^{n-2}, Y_0 - 2\alpha(X_1 + Y_0^{n-2}) = (\text{something}, Y_0 - 2\alpha(X_1 + Y_0^{n-2})).
\]
The matrix $Y_0 - 2\alpha(X_1 + Y_0^{n-2})$ is not nilpotent, for example because its square does not have trace zero. Hence $\Phi_p \Psi_\chi \Phi_p$ does not fix $(X_1, Y_0)$, and the proof is finished.

6. Other formulations of Theorem 1.3

The remarks in this section are at the level of “groups acting on sets”: that is, we may as well suppose that $\mathcal{R}$ denotes any set acted on by a group $G$. We are interested in the condition

\[(6.1) \text{ there is no nontrivial } G\text{-map } f: \mathcal{R} \to \mathcal{R}\]

(“nontrivial” means “not the identity map”). As we observed above, that is equivalent to the two conditions

\[(6.2a) \text{ each } G\text{-orbit in } \mathcal{R} \text{ satisfies } (6.1);\]
\[(6.2b) \text{ if } O_1 \text{ and } O_2 \text{ are distinct orbits, there is no } G\text{-map from } O_1 \text{ to } O_2.\]

Let us reformulate these conditions in terms of the isotropy groups $G_M$ of the points $M \in \mathcal{R}$. If $H$ and $K$ are subgroups of $G$, then any $G$-map from $G/H$ to $G/K$ to must have the form $\varphi(gH) = g(xK)$ for some $x \in G$. This is well-defined if and only if we have

\[x^{-1}Hx \subseteq K.\]

In the case $H = K$, that says that $x \in N_G(H)$, where $N_G$ denotes the normalizer in $G$: it follows that the $G$-maps from $G/H$ to itself correspond 1–1 to the points of $N_G(H)/H$. Thus the conditions (6.2) are equivalent to

\[(6.3a) \text{ for any } M \in \mathcal{R}, \text{ we have } G_M = N_G(G_M);\]
\[(6.3b) \text{ if } M \text{ and } N \text{ are on different orbits, no conjugate of } G_M \text{ is in } G_N.\]

Finally, we note that the conditions (6.3) are equivalent to the single assertion

\[(6.4) \text{ if } G_M \subseteq G_N, \text{ then } M = N.\]

Indeed, suppose (6.4) holds, and let $x \in N_G(G_M)$, that is, $xG_Mx^{-1} \subseteq G_M$, or $G_{xM} \subseteq G_M$. By (6.4), we then have $xM = M$, that is, $x \in G_M$. Thus (6.4) $\Rightarrow$ (6.3a). Now, if (6.3b) is false, we have $xG_Mx^{-1} \subseteq G_N$, that is, $G_{xM} \subseteq G_N$, for some $x \in G$ and some $M, N$ on different orbits. But since they are on different orbits, $xM \neq N$, so (6.4) is false. Thus (6.4) $\Rightarrow$ (6.3a).

Conversely, suppose (6.3a) holds, and let $M, N$ be such that $G_M \subseteq G_N$. By (6.3a), $M$ and $N$ are on the same orbit, so $M = xN$ for some $x \in G$; hence $G_M = xG_Nx^{-1} \subseteq G_N$. Thus $x \in N_G(G_N)$, so by (6.3a), $x \in G_N$; hence $M = N$, as desired.

It is in the form (6.4) that our result is stated in [K].

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