On spectral spread of generalized distance matrix of a graph

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ABSTRACT
For a simple connected graph G, let $D(G)$, $Tr(G)$, $D^1(G)$ and $D^2(G)$, respectively, are the distance matrix, the diagonal matrix of the vertex transmissions, distance Laplacian matrix and the distance signless Laplacian matrix. The generalized distance matrix $D_{\alpha}(G)$ of G is the convex linear combinations of $Tr(G)$ and $D(G)$ and is defined as $D_{\alpha}(G) = \alpha Tr(G) + (1 - \alpha)D(G)$, for $0 \leq \alpha \leq 1$. As $D_0(G) = D(G)$, $D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha - \beta)D^1(G)$, this matrix reduces to merging the distance spectral and distance signless Laplacian spectral theories. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the eigenvalues of $D_{\alpha}(G)$ and let $SD_{\alpha}(G) = \lambda_1(G) - \lambda_n(G)$ be the generalized distance spectral spread of the graph G. In this paper, we obtain bounds for the generalized distance spectral spread $SD_{\alpha}(G)$. We also obtain a relation between the generalized distance spectral spread $SD_{\alpha}(G)$ and the distance spectral spread $SD(G)$. Further, we obtain lower bounds for $SD_{\alpha}(G)$ of bipartite graphs involving different graph parameters and we characterize the extremal graphs for some cases. We also obtain lower bounds for $SD_{\alpha}(G)$ in terms of clique number and independence number of the graph G and characterize the extremal graphs for some cases.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$, denoted by $G(V(G), E(G))$. We use standard terminology; for concepts not defined here, we refer the reader to any standard graph theory monograph, such as [1] or [2].

In a graph G, the distance between two vertices $u, v \in V(G)$, denoted by $d_{uv}$, is defined as the length of a shortest path between $u$ and $v$. The diameter of G is the maximum distance between any two vertices of G. The distance matrix of G, denoted by $D(G)$, is defined as $D(G) = (d_{uv})_{u, v \in V(G)}$. Till now, the distance spectrum of a connected graph has been investigated extensively; for spectral properties of $D(G)$ see the survey [3]. The transmission $Tr_G(v)$ of a vertex $v$ is defined to be the sum of the distances from $v$ to all other vertices in G, i.e. $Tr_G(v) = \sum_{u \in V(G)} d_{uv}$. A graph G is said to be k-transmission...
regular if $Tr_G(v) = k$, for each $v \in V(G)$. The Wiener index of a graph $G$, denoted by $W(G)$, is the sum of the distances between all unordered pairs of vertices in $G$. Clearly, $W(G) = \frac{1}{2} \sum_{v \in V(G)} Tr_G(v)$. For a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $Tr_G(v_i)$ has also been referred as the transmission degree $Tr_i$ and hence the transmission degree sequence is given by $\{Tr_1, Tr_2, \ldots, Tr_n\}$. The second transmission degree of $v_i$, denoted by $T_i$, is given by $T_i = \sum_{j=1}^{n} d_{ij}Tr_j$. Let $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of $G$. The works [4–6] investigated the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = Tr(G) - D(G)$ is referred to as the distance Laplacian matrix of $G$, while the matrix $D^Q(G) = Tr(G) + D(G)$ is the distance signless Laplacian matrix of $G$. For some recent results on distance (signless) Laplacian matrix, we refer to [4–15] and the references therein.

The concept of generalized distance matrix of graphs was recently put forward in [16, 17]. The generalized distance matrix of a graph $G$, denoted by $D_\alpha(G)$, is a convex combination of $Tr(G)$ and $D(G)$, and is defined as $D_\alpha(G) = \alpha Tr(G) + (1 - \alpha)D(G)$, where $0 \leq \alpha \leq 1$. Since $D_0(G) = D(G)$, $2D_{\frac{1}{2}}(G) = D^Q(G)$, $D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha - \beta)D^L(G)$, any result regarding the spectral properties of generalized distance matrix, has its counterpart for each of these particular graph matrices, and these counterparts follow immediately from a single proof. In fact, this matrix reduces to merging the distance and distance signless Laplacian spectral theories.

Since the matrix $D_\alpha(G)$ is a real symmetric matrix, therefore its eigenvalues can be arranged as $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$, where the largest eigenvalue $\partial_1$ is called the generalized distance spectral radius of $G$. (From now onwards, we will denote $\partial_1(G)$ by $\partial_1$). As $D_\alpha(G)$ is non-negative and irreducible, by the Perron-Frobenius theorem, $\partial_1$ is a simple (with multiplicity 1) eigenvalue and there is a unique positive unit eigenvector $X$ corresponding to $\partial_1$, which is called the generalized distance Perron vector of $G$. For some recent results on spectral properties of the generalized distance matrix, see [16–19] and the references therein.

Let $M$ be a symmetric matrix of order $n$ having eigenvalues $w_i, i = 1, 2, \ldots, n$. If $w_1$ and $w_n$ are respectively, the largest and smallest eigenvalues of the symmetric matrix $M$, the spread of the matrix $M$ is defined as $S(M) = w_1 - w_n$. In the literature, several papers can be seen regarding the parameter $S(M)$ for any symmetric matrix $M$. The spread of a matrix is an attractive topic and as such the investigation of the spread of some matrices of a graph become interesting. When $M$ is restricted to a particular graph matrix, the parameter $S(M)$ has attracted much attention of the researchers as it is clear from the fact that various papers can be found in the literature in this direction. For a particular graph matrix (like adjacency, Laplacian, signless Laplacian, distance, etc.), the much studied problem about the parameter $S(M)$ is to obtain bounds in terms of various graph parameters. Another problem worth to mention is to characterize the extremal graphs for the parameter $S(M)$ for a graph matrix, in some class of graphs.

Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the distance eigenvalues of the graph $G$. The distance spread of a connected graph $G$ was considered in [20] and is defined as $S_D(G) = \rho_1 - \rho_n$. Various lower and upper bounds for the parameter $S_D(G)$ can be found in [20].

Motivated by the results obtained for the distance spread, You et al. [15] put forward the concept of the distance signless Laplacian spread of a connected graph $G$ as $S_D^Q(G) =$
where $q_1^D - q_n^D$ and obtained various lower and upper bounds for the parameter $S_{D^0}(G)$ in terms of different graph parameters.

Here we define the generalized distance spectral spread of a graph $G$ as the difference between the largest and smallest eigenvalues of $D_{\alpha}(G)$, that is,

$$S_{D_{\alpha}}(G) = \partial_1(G) - \partial_n(G).$$

It is clear from the above discussion that $S_{D_0}(G) = S_D(G)$ and $2S_{D_{\frac{1}{2}}}(G) = S_{D^0}(G)$. Therefore, the parameter $S_{D_{\alpha}}(G)$ is a generalization of the already studied parameters $S_D(G)$ and $S_{D^0}(G)$. The motive of this paper is to obtain bounds for the parameter $S_{D_{\alpha}}(G)$, in terms of different graph parameters and to characterize the extremal graphs.

In a graph $G$, a clique is a maximal complete subgraph and the order of the maximal clique is called the clique number of the graph $G$ and is denoted by $\omega(G)$ or $\omega$. A subset $S$ of the vertex $V(G)$ of a graph $G$ is said to be an independent set if the induced subgraph $\langle S \rangle$ is an empty graph. The cardinality of the maximal possible independent set in a graph $G$ is called its independence number.

The rest of the paper is structured as follows. In Section 2, we obtain some bounds for the generalized distance spectral spread $S_{D_{\alpha}}(G)$ of graphs. We also obtain a relation between the generalized distance spectral spread $S_{D_{\alpha}}(G)$ and the distance spectral spread $S_D(G)$. In Section 3, we obtain lower bounds for the generalized distance spectral spread $S_{D_{\alpha}}(G)$ of bipartite graphs involving different graph parameters and characterize the extremal graphs for some cases. In Section 4, we obtain lower bounds for the generalized distance spectral spread $S_{D_{\alpha}}(G)$ in terms of clique number and independence number of the graph $G$ and characterize the extremal graphs for some cases.

2. Bounds for $S_{D_{\alpha}}(G)$ for any graph

In this section, we obtain some bounds for $S_{D_{\alpha}}(G)$, in terms of various graph parameters. We also establish a relation between the generalized distance spectral spread and the distance spectral spread of a graph $G$. We start with known results which will be used in the proofs of our main results in the sequel.

**Lemma 2.1 ([16]):** Let $G$ be a connected graph of order $n$. Then, $\partial_1(G) \geq \frac{2W(G)}{n}$, with equality if and only if $G$ is a transmission regular graph.

The following is the well known Weyl’s inequality and can be found in [21]. Note that the equality case was discussed in [22].

**Lemma 2.2:** Let $X$ and $Y$ be Hermitian matrices of order $n$ such that $Z = X + Y$. Then

$$\lambda_k(Z) \leq \lambda_j(X) + \lambda_{k-j+1}(Y), \quad n \geq k \geq j \geq 1,$$

$$\lambda_k(Z) \geq \lambda_j(X) + \lambda_{k-j+n}(Y), \quad n \geq j \geq k \geq 1,$$

where $\lambda_i(M)$ is the $i^{th}$ largest eigenvalue of the matrix $M$. In either of these inequalities, equality holds if and only if there exists a unit vector that is an eigenvector to each of the three eigenvalues involved.
The proof of the following lemma is similar to that of [23, Lemma 2], so is omitted here.

**Lemma 2.3:** A connected graph $G$ has two distinct $D_\alpha(G)$ eigenvalues if and only if $G$ is a complete graph.

If $G$ is a $k$-transmission regular graph, then $Tr(G) = kI_n$ (where $I_n$ is the identity matrix of order $n$) and so

$$D_\alpha(G) = \alpha Tr(G) + (1 - \alpha)D(G) = k\alpha I_n + (1 - \alpha)D(G).$$

Therefore, if $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ are the generalized distance eigenvalues of the $k$-transmission regular graph $G$ having distance eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, then we have $\partial_i = k\alpha + (1 - \alpha)\rho_i$, $1 \leq i \leq n$. Thus, for a $k$-transmission regular graph $G$, we have

$$SD_{\alpha}(G) = \partial_1 - \partial_n = (1 - \alpha)(\rho_1 - \rho_n) = (1 - \alpha)SD(G).$$

Hence, for a $k$-transmission regular graph $G$, the study of generalized distance spectral spread $SD_{\alpha}(G)$ is same as that of distance spectral spread $SD(G)$.

For $\frac{1}{2} \leq \alpha \leq 1$, it has been shown [16] that the generalized distance matrix $D_\alpha(G)$ of the connected graph $G$ is a positive semi-definite matrix. Therefore, for $\frac{1}{2} \leq \alpha \leq 1$, clearly $\partial_n \geq 0$ implies that $SD_{\alpha}(G) = \partial_1 - \partial_n \leq \partial_1$, with equality if and only if $\partial_n = 0$. From this, for $\frac{1}{2} \leq \alpha \leq 1$, it is clear that any upper bound for generalized distance spectral radius gives an upper bound for the generalized distance spectral spread $SD_{\alpha}(G)$.

Now, we obtain a relation between the generalized distance spectral spread $SD_{\alpha}(G)$ and the distance spectral spread $SD(G)$ of a connected graph $G$.

**Theorem 2.4:** Let $G$ be a connected graph of order $n$ having transmission degree sequence \{Tr$_1$, Tr$_2$, \ldots , Tr$_n$\}. If $Tr_{\text{max}} = \max_{1 \leq i \leq n} Tr_i$ and $Tr_{\text{min}} = \min_{1 \leq i \leq n} Tr_i$, then

$$(1 - \alpha)SD(G) - \alpha \left(Tr_{\text{max}} - Tr_{\text{min}}\right) \leq SD_{\alpha}(G) \leq \alpha \left(Tr_{\text{max}} - Tr_{\text{min}}\right) + (1 - \alpha)SD(G).$$

Equality occurs on both sides if and only if $G$ is a transmission regular graph.

**Proof:** Let $G$ be a connected graph of order $n$ having generalized distance matrix $D_\alpha(G)$. Since $D_\alpha(G) = \alpha Tr(G) + (1 - \alpha)D(G)$, by taking $k = 1, j = 1$ in first inequality and $k = 1, j = n$ in second inequality of Lemma 2.2, it follows that

$$\lambda_n(\alpha Tr(G)) + \lambda_1((1 - \alpha)D(G)) \leq \lambda_1(D_\alpha(G)) \leq \lambda_1(\alpha Tr(G)) + \lambda_1((1 - \alpha)D(G)).$$

(2)

Again taking $k = n, j = 1$ in first inequality and $k = n, j = n$ in second inequality of Lemma 2.2, we get

$$\lambda_n(\alpha Tr(G)) + \lambda_n((1 - \alpha)D(G)) \leq \lambda_1(D_\alpha(G)) \leq \lambda_1(\alpha Tr(G)) + \lambda_n((1 - \alpha)D(G)).$$

(3)

From (2) and (3), we have

$$\lambda_1(D_\alpha(G)) - \lambda_n(D_\alpha(G)) \leq \alpha \left(\lambda_1(Tr(G)) - \lambda_n(Tr(G))\right) + (1 - \alpha)\left(\lambda_1(D(G)) - \lambda_n(D(G))\right).$$
and
\[
\lambda_1(D_\alpha(G)) - \lambda_n(D_\alpha(G)) \\
\geq (1 - \alpha)\left(\lambda_1(D(G)) - \lambda_n(D(G))\right) - \alpha \left(\lambda_1(\text{Tr}(G)) - \lambda_n(\text{Tr}(G))\right).
\]

The result now follows from these inequalities. Equality occurs in (1) if and only if equality occurs in (2) and (3). Suppose that equality occurs on the right of (2). Then by Lemma 2.2, the eigenvalues \(\lambda_1(D_\alpha(G)), \lambda_1(\text{Tr}(G))\) and \(\lambda_1(D(G))\) of the matrices \(D_\alpha(G), \text{Tr}(G)\) and \(D(G)\) have the same unit eigenvector \(x\). Similarly, if equality occurs on the left of (2.2), then again by Lemma 2.2, the eigenvalues \(\lambda_1(D_\alpha(G)), \lambda_n(\text{Tr}(G))\) and \(\lambda_1(D(G))\) of the matrices \(D_\alpha(G), \text{Tr}(G)\) and \(D(G)\) have the same unit eigenvector \(x\). From this it follows that the eigenvalues \(\lambda_1(\text{Tr}(G))\) and \(\lambda_n(\text{Tr}(G))\) of the matrix \(\text{Tr}(G)\) have the same eigenvector, which is only possible if \(\lambda_1(\text{Tr}(G)) = \lambda_n(\text{Tr}(G))\). This implies that \(\text{Tr}(G) = \lambda_1(\text{Tr}(G))I_n\), that is, \(G\) is a \(\lambda_1(\text{Tr}(G))\) transmission regular graph. In a similar way we can discuss the equality in (2.3).

Conversely, if \(G\) is a \(k\)-transmission regular graph, then \(D_\alpha(G) = k\alpha I_n + (1 - \alpha)D(G)\) and so equality holds in (2) and (3).

From inequality (2), we obtain the following bounds for the generalized distance spectral radius \(\partial_1\)
\[
\alpha \text{Tr}_{\min} + (1 - \alpha)\rho_1 \leq \partial_1 \leq \alpha \text{Tr}_{\max} + (1 - \alpha)\rho_1,
\]
with equality on both sides if and only if \(G\) is a transmission regular graph. In the literature, we can find various bounds for the distance spectral radius \(\rho_1\). All those bounds together with inequality (4) give bounds for the generalized distance spectral radius \(\partial_1\).

Similarly, from inequality (3), we get the following bounds for the smallest generalized distance eigenvalue \(\partial_n\)
\[
\alpha \text{Tr}_{\min} + (1 - \alpha)\rho_n \leq \partial_n \leq \alpha \text{Tr}_{\max} + (1 - \alpha)\rho_n,
\]
with equality on both sides if and only if \(G\) is a transmission regular graph.

The following result gives a lower bound for \(S_{D_\alpha}(G)\) in terms of the order \(n\), the Wiener index \(W(G)\) and the parameter \(\alpha\).

**Theorem 2.5:** If \(G\) is a connected graph of order \(n \geq 3\) having Wiener index \(W(G)\) and let \(\alpha \in [0, 1)\). Then
\[
S_{D_\alpha}(G) \geq \left(1 - \alpha\right)\frac{2\alpha W(G)}{n - 1},
\]
with equality if and only if \(G \cong K_n\).

**Proof:** Let \(\partial_1, \partial_2, \ldots, \partial_n\) be the generalized distance eigenvalues of \(G\). Note that \(\sum_{i=1}^n \partial_i = 2\alpha W(G)\). So \(2\alpha W(G) \geq \partial_1 + (n - 1)\partial_n\) and therefore \(\partial_n \leq \frac{(2\alpha W - \partial_1)}{n - 1}\). Thus, using
Lemma 2.1, we have
\[ SD_\alpha (G) \geq \partial_1 - \frac{(2\alpha W(G) - \partial_1)}{n-1} = \frac{n}{n-1} \partial_1 - \frac{2\alpha W(G)}{n-1} \geq \left(1 - \alpha\right) \frac{2\alpha W(G)}{n-1}. \]

Equality in (6) holds if and only if \( \partial_2 = \cdots = \partial_n \) and \( \partial_1 = \frac{2W(G)}{n} \). That is, if and only if \( G \) is a transmission regular graph with exactly two distinct eigenvalues. Using Lemma 2.3, it follows that \( G \) is the complete graph \( K_n \).

Conversely, if \( G \cong K_n \), it can be seen by direct computations that equality occurs in (6).

Now, we obtain another lower bound for \( SD_\alpha (G) \) in terms of the order \( n \), the Wiener index \( W(G) \) and the parameter \( \alpha \).

**Theorem 2.6:** Let \( G \) be a connected graph of order \( n \geq 3 \) having Wiener index \( W(G) \) and let \( \alpha \in [0,1) \). Then
\[
SD_\alpha (G) \geq \frac{1}{n} \left( 2W(G) - \sqrt{n^2(2(1-\alpha)^2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2) - 4W^2(G)} \right),
\]
with equality if and only if \( G \cong K_n \).

**Proof:** Clearly
\[
2(1-\alpha)^2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 = \sum_{i=1}^{n} \partial_i^2 \geq \partial_1^2 + (n-1) \partial_2^2. \tag{7}
\]
Therefore,
\[
\partial_n \leq \sqrt{\frac{2(1-\alpha)^2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \partial_1^2}{n-1}}.
\]
Thus
\[
SD_\alpha (G) \geq \partial_1 - \sqrt{\frac{2(1-\alpha)^2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \partial_1^2}{n-1}}.
\]
Now, using Lemma 2.1, the result follows. Equality in (7) holds if and only if \( \partial_2 = \cdots = \partial_n \) and \( \partial_1 = \frac{2W(G)}{n} \). That is, if and only if \( G \) is a transmission regular graph with exactly two distinct eigenvalues. Using Lemma 2.3, it follows that \( G \) is the complete graph \( K_n \).

Conversely, if \( G \cong K_n \), it can be seen by direct computations that equality occurs in (7).}

Next, we obtain a lower bound for the generalized distance spectral spread \( SD_\alpha (G) \) in terms of the Wiener index \( W(G) \) and the order \( n \) of the connected graph \( G \).
**Theorem 2.7:** Let $G$ be a connected graph of order $n \geq 3$ having Wiener index $W(G)$ and let $\alpha \in [0, 1)$. Then

$$S_{D_{\alpha}}(G) \geq \frac{2}{n} \sqrt{n \left(2(1-\alpha)^2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 \right) - 4\alpha^2 W^2(G)}.$$  

**Proof:** Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ be the generalized distance eigenvalues of $G$. We assume that the non-negative real numbers $a_2, a_3, \ldots, a_{n-1}, b_2, b_3, \ldots, b_{n-1}$ are defined by the equations

$$a_i \partial_1 = a_i^2 + b_i \partial_1^2, a_i + b_i = 1, \quad i = 2, 3, \ldots, n - 1.$$  

Putting $a = 1 + \sum_{i=2}^{n-1} a_i$ and $b = 1 + \sum_{i=2}^{n-1} b_i$, then $a + b = n$ and $\sum_{i=1}^{n} \partial_i^2 = a \partial_1^2 + b \partial_n^2$. On the other hand, we have $\partial_i = (a_i \partial_1^2 + b_i \partial_n^2)(a_i + b_i)$, so that $\partial_i \geq a_i \partial_1 + b_i \partial_n, i = 2, 3, \ldots, n - 1$, and therefore $\sum_{i=1}^{n} \partial_i \geq a \partial_1 + b \partial_n$. Now, observing that $ab \leq \frac{(a+b)^2}{4} = \frac{n^2}{4}$, we obtain

$$n \sum_{i=1}^{n} \partial_i^2 - \left(\sum_{i=1}^{n} \partial_i\right)^2 \leq (a \partial_1^2 + b \partial_n^2)(a + b) - (a \partial_1 + b \partial_n)^2$$

$$= ab(\partial_1 - \partial_n)^2 \leq \frac{n^2}{4}(\partial_1 - \partial_n)^2.$$ 

Consequently, from (8), we get

$$S_{D_{\alpha}}(G) \geq \frac{2}{n} \sqrt{n \left(2(1-\alpha)^2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 \right) - 4\alpha^2 W^2(G)},$$

as desired.  

For a real rectangular matrix $M = (w_{ij})_{m \times n}$, let $\|M\|_{F} = (\sum_{i=1}^{m} \sum_{j=1}^{n} |w_{ij}|^2)^{\frac{1}{2}}$ be the Frobenius norm of $M$. If $M$ is a square matrix, its trace is denoted by $tr(M)$. If $w_1$ and $w_n$ are, respectively, the largest and smallest eigenvalues of the symmetric matrix $M$, the spread of the matrix $M$ is defined as $S(M) = w_1 - w_n$. The following observation is due to Mirsky [24].

**Lemma 2.8:** Let $M$ be an $n$-square normal matrix. Then

$$S(M) \leq \left(2\|M\|_{F}^2 - \frac{2}{n}(tr(M))^2\right)^{\frac{1}{2}},$$ 

with equality if and only if the eigenvalues $w_1, w_2, \ldots, w_n$ of $M$ satisfy the identity $w_2 = w_3 = \cdots = w_{n-1} = \frac{w_1 + w_n}{2}$.

An upper bound for $S_{D_{\alpha}}(G)$ in terms of the order $n$, the Wiener index $W(G)$ and the transmission degrees of the connected graph $G$ is as follows.
Theorem 2.9: Let $G$ be a connected graph of order $n \geq 3$ having Wiener index $W(G)$ and let $\alpha \in [0, 1]$. Then

$$S_{D_\alpha}(G) \leq \left(2(1-\alpha)^2 \sum_{i \neq j} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} \text{Tr}_i^2 - \frac{8}{n} \alpha^2 W^2(G) \right)^{\frac{1}{2}},$$

with equality if and only if $\partial_2 = \partial_3 = \cdots = \partial_{n-1} = \frac{\partial_1 + \partial_n}{2}$.

Proof: Since the matrix $D_\alpha(G)$ is normal with $\|D_\alpha(G)\|_F^2 = (1-\alpha)^2 \sum_{i \neq j} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} \text{Tr}_i^2$ and $\text{tr}(D_\alpha(G)) = 2\alpha W(G)$, the result follows from Lemma 2.8, by taking $M = D_\alpha(G)$.

For $\alpha = 1$, it can be seen that equality occurs in Theorem 2.9 if and only if $G$ is a transmission regular graph. In general, equality occurs in Theorem 2.9 if and only if $\partial_2 = \partial_3 = \cdots = \partial_{n-1} = \frac{\partial_1 + \partial_n}{2}$. This gives $\text{tr}(D_\alpha(G)) = n\partial_2$, which implies that $2\alpha W(G) = n\partial_2$, that is, $\partial_2 = \frac{2\alpha W(G)}{n}$. Since

$$\frac{2\alpha W(G)}{n} = \alpha \frac{e^T D_\alpha(G)e}{e^Te} \leq \alpha \partial_1,$$

where $e$ is the all one $n$-vector, it follows that $\partial_2 \leq \alpha \partial_1$. So, we have $\partial_1 + \partial_n = 2\partial_2 \leq 2\alpha \partial_1$, implying that $\partial_n \leq (2\alpha - 1)\partial_1$, if $\alpha > \frac{1}{2}$ and $(1 - 2\alpha)\partial_1 + \partial_n \leq 0$, if $\alpha \leq \frac{1}{2}$. Since equality occurs in (2.9) if and only if $G$ is a transmission regular graph, it follows that equality occurs in Theorem 2.9 if and only if $G$ is a transmission regular graph with $\partial_2 = \partial_3 = \cdots = \partial_{n-1} = \alpha \partial_1$ and $\partial_n \leq (2\alpha - 1)\partial_1$, for $\alpha > \frac{1}{2}$ and $(1 - 2\alpha)\partial_1 + \partial_n \leq 0$, for $\alpha \leq \frac{1}{2}$. In particular, if $\alpha = 0$, then equality occurs if and only if $\partial_1 = -\partial_n$ and $\partial_2 = \cdots = \partial_{n-1} = 0$.

We observe that the characterization of the connected graphs which attain equality in Theorem 2.9 seems not to be easy. Therefore, we have the following problem which will be of interest for the future research.

Problem 2.10: Characterize all connected graphs $G$ of order $n \geq 3$ with $\partial_2 = \partial_3 = \cdots = \partial_{n-1} = \frac{\partial_1 + \partial_n}{2}$, where $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ are the generalized distance eigenvalues of $G$.

3. Bounds for $S_{D_\alpha}(G)$ for bipartite graphs

In this section, we obtain lower bounds for $S_{D_\alpha}(G)$ in terms of the maximum degree $\Delta$, the order $n$, the transmission degrees, the average distance degrees and the Wiener index of a bipartite graph $G$.

Let $M$ be a real matrix of order $n$ described in the following block form

$$M = \begin{pmatrix}
M_{11} & M_{12} & \cdots & M_{1t} \\
: & : & \cdots & : \\
M_{t1} & M_{t2} & \cdots & M_{tt}
\end{pmatrix}, \quad (M*)$$

where the diagonal blocks $M_{ii}$ are $n_i \times n_i$ matrices for any $i \in \{1, 2, \ldots, t\}$ and $n = n_1 + \cdots + n_t$. For any $i, j \in \{1, 2, \ldots, t\}$, let $b_{ij}$ denote the average row sum of $M_{ij}$, that is, $b_{ij}$ is
the sum of all entries in $M_{ij}$ divided by the number of rows. Then $B(M) = (b_{ij})$ is called the quotient matrix of $M$. If in addition, for each pair $i, j$, $M_{ij}$ has constant row sum, then $B(M)$ is called the equitable quotient matrix of $M$. The following lemma can be found in [1].

**Lemma 3.1:** Let $M$ be a symmetric matrix which has the block form as $(M^*)$ and $B$ be the quotient matrix of $M$. Then the eigenvalues of $B$ interlace the eigenvalues of $M$. That is, if $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_r$ are the eigenvalues of the matrices $M$ and $B$, respectively, then $a_i \geq b_1 \geq a_{n-r+i}$, for all $i = 1, 2, \ldots, r$. Moreover, if $B$ is the equitable quotient matrix of $M$, then each eigenvalue of $B$ is an eigenvalue of $M$.

The following observation can be found in [25].

**Lemma 3.2:** Let $M$ be a matrix of order $n$ having eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_n$ and let $B$ be a principal submatrix of $M$ of order $r$ having eigenvalues $b_1 \geq b_2 \geq \cdots \geq b_r$. Then $a_i \geq b_i \geq a_{n-r+i}$, for all $i = 1, 2, \ldots, r$.

The following observation can be found in [16].

**Lemma 3.3:** Let $G$ be a connected graph of order $n$ and $S$ be a subset of the vertex set $V(G)$, such that $N(x) = N(y)$ for any $x, y \in S$, where $|S| = t$.

(i) If $S$ is an independent set, then $Tr(x)$ is a constant for each $x \in S$ and $D_\alpha(G)$ has $\alpha(Tr(x) + 2) − 2$ as an eigenvalue with multiplicity at least $t−1$.

(ii) If $S$ is a clique, then $Tr(x)$ is a constant for each $x \in S$ and $D_\alpha(G)$ has $\alpha(Tr(x) + 1) − 1$ as an eigenvalue with multiplicity at least $t−1$.

The following result gives the generalized distance spectrum of the complete bipartite graph $K_{r,s}$.

**Lemma 3.4:** The generalized distance spectrum of the complete bipartite graph $K_{r,s}$ consists of the eigenvalue $\alpha(2r + s) − 2$ with multiplicity $r-1$, the eigenvalue $\alpha(2s + r) − 2$ with multiplicity $s-1$ and the remaining two eigenvalues as $x_1 \geq x_2$, where $x_1, x_2 = \frac{\alpha(s+r)+2(s+r)-4\pm\sqrt{(r^2+s^2)(\alpha-2)^2+2r\alpha^2}}{2}$.

**Proof:** Let $V_1$ and $V_2$ be the partite sets of $K_{r,s}$ such that the degree of each vertex in $V_1$ is $s$ and the degree of each vertex in $V_2$ is $r$. It is clear that $Tr(v_i) = 2r + s − 2$, for all $v_i \in V_1$ and $Tr(u_j) = 2s + r − 2$, for all $u_j \in V_2$. Now, using Lemma 3.3 with $S = V_1$ and then with $S = V_2$, we get the eigenvalue $\alpha(2r + s) − 2$ with multiplicity $r-1$ and the eigenvalue $\alpha(2s + r) − 2$ with multiplicity $s-1$. The remaining two eigenvalues (as the quotient matrix for the graph $K_{r,s}$ is an equitable quotient matrix) can be obtained from the quotient matrix of the matrix $D_\alpha(K_{r,s})$ given by $B = \left(\begin{array}{cc}
\alpha_1 + 2r - 2 & s(1-\alpha) \\
(1-\alpha) & r(1-\alpha)
\end{array}\right)$. ■

Let $t_{vi} = \frac{1}{deg(v_i)\sum_{v_j,v_j \in E(G)} Tr(v_j)}$ be the average distance degree of the vertex $v_i$. A lower bound for $S_{D_\alpha}(G)$ in terms of the maximum degree $\Delta$, the order $n$, the average distance degrees and the Wiener index $W(G)$ of a bipartite graph $G$ is as follows.
**Theorem 3.5:** Let $G$ be a connected bipartite graph on $n \geq 3$ vertices with maximum degree $\Delta$ and Wiener index $W$. Let $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_k) = \Delta$, where $v_i \in V(G)$ for $1 \leq i \leq k$.

(i) If $\Delta \leq n - 2$, then

$$S_{D_\alpha}(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{\alpha_i^2 - 4\beta_i(\Delta + 1)(n - \Delta - 1)}}{\Delta + 1(n - \Delta - 1)},$$

where $\alpha_i = \alpha n(t_{vi} \Delta + Tr(v_i) - 2\Delta^2) + 2n\Delta^2 + (\Delta + 1)(2W - 2t_{vi}\Delta - 2Tr(v_i))$ and $\beta_i = 2\alpha W(t_{vi}\Delta + Tr(v_i) - 2\Delta^2) + 4W\Delta^2 - (t_{vi}\Delta + Tr(v_i))^2$.

(ii) If $\Delta = n - 1$, then $S_{D_\alpha}(G) = \begin{cases} n + \sqrt{n^2 - 3n + 3}, & \text{if } \alpha = 0, \\ \frac{\sqrt{\alpha^2 - 2(n - 2n + 2) + 2(n - 1)(\alpha^2 - 2)}}{\Delta + 1(n - \Delta - 1)}, & \text{if } \alpha \neq 0. \end{cases}$

**Proof:** Let $G$ be a connected bipartite graph on $n \geq 3$ vertices with maximum degree $\Delta$, Wiener index $W$ and let $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_k) = \Delta$, where $v_i \in V(G)$ for $1 \leq i \leq k$. We consider the following two cases:

(i) Let $\Delta \leq n - 2$. For $1 \leq i \leq k$, let $N(v_i) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_\Delta}\}$ be the neighbour set and $N[v_i] = N(v_i) \cup \{v_i\}$. Let $V(G) = \{v_j, u_{i_1}, u_{i_2}, \ldots, u_{i_\Delta}, w_1, w_2, \ldots, w_{n-\Delta-1}\}$ and let $V(G) = V_1 \cup V_2$ be the bipartite partition of $V(G)$. Since $G$ is bipartite, if $v_i \in V_1$, then $u_{ij} \in V_2$ and if $v_i \in V_2$, then $u_{ij} \in V_1$, for $j = 1, 2, \ldots, \Delta$. By suitably labelling the vertices of $G$, it can be seen that the generalized distance matrix of the graph $G$ can be written as $D_{\alpha}(G) = \begin{pmatrix} X & (1-\alpha)Y \\ \alpha Tr(Y) & -Z \end{pmatrix}$, where

$$X = \begin{pmatrix} \alpha Tr(v_1) & 1-\alpha & 1-\alpha & \cdots & 1-\alpha \\ 1-\alpha & \alpha Tr(u_{i_1}) & 2(1-\alpha) & \cdots & 2(1-\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-\alpha & 2(1-\alpha) & 2(1-\alpha) & \cdots & \alpha Tr(u_{i_\Delta}) \end{pmatrix},$$

$$Y = \begin{pmatrix} d(v_1, w_1) & \cdots & d(v_1, w_{n-\Delta-1}) \\ d(u_{i_1}, w_1) & \cdots & d(u_{i_1}, w_{n-\Delta-1}) \\ \vdots & \cdots & \vdots \\ d(u_{i_\Delta}, w_1) & \cdots & d(u_{i_\Delta}, w_{n-\Delta-1}) \end{pmatrix},$$

and

$$Z = \begin{pmatrix} \alpha Tr(w_1) & (1-\alpha)d(w_1, w_2) & \cdots & (1-\alpha)d(w_1, w_{n-\Delta-1}) \\ (1-\alpha)d(w_2, w_1) & \alpha Tr(w_2) & \cdots & (1-\alpha)d(w_2, w_{n-\Delta-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)d(w_{n-\Delta-1}, w_1) & (1-\alpha)d(w_{n-\Delta-1}, w_2) & \cdots & \alpha Tr(w_{n-\Delta-1}). \end{pmatrix}$$
It can be seen that the quotient matrix $B$ of the matrix $D_\alpha(G)$ is

$$B = \begin{pmatrix}
\frac{2\Delta^2(1 - \alpha) + \alpha(t_v \Delta + \text{Tr}(v_i))}{\Delta + 1} & \frac{(1 - \alpha)(t_v \Delta + \text{Tr}(v_i) - 2\Delta^2)}{\Delta + 1} \\
\frac{(1 - \alpha)(t_v \Delta + \text{Tr}(v_i) - 2\Delta^2)}{n - \Delta - 1} & \frac{\alpha(2W - t_v \Delta - \text{Tr}(v_i)) + (1 - \alpha)}{(2W - 2t_v \Delta - 2\text{Tr}(v_i) + 2\Delta^2)}
\end{pmatrix}. $$

The eigenvalues of the matrix $B$ are

$$x_1, x_2 = \frac{\alpha_i \pm \sqrt{\alpha_i^2 - 4\beta_i(\Delta + 1)(n - \Delta - 1)}}{2(\Delta + 1)(n - \Delta - 1)},$$

where $x_1 \geq x_2$, $\alpha_i = \alpha(n(t_v \Delta + \text{Tr}(v_i) - 2\Delta^2) + 2n\Delta^2 + (\Delta + 1)(2W - 2t_v \Delta - 2\text{Tr}(v_i))$ and $\beta_i = 2\alpha W(t_v \Delta + \text{Tr}(v_i) - 2\Delta^2) + 4W\Delta^2 - (t_v \Delta + \text{Tr}(v_i))^2$. Now, using Lemma 3.1, the result follows in this case.

(ii) If $\Delta = n - 1$, then $G$ being a bipartite graph implies that $G \cong K_{1,n-1}$. The result now follows from Lemma 3.3, by taking $r = 1$ and $s = n - 1$.

For $\Delta \leq n - 2$, it will be interesting to characterize the graphs attaining the lower bound in Theorem 3.5. Therefore, we have the following problem.

**Problem 3.6:** For $\Delta \leq n - 2$, characterize the connected graphs $G$ of order $n \geq 3$, which attains the lower bound in Theorem 3.5.

Taking $r = a$ and $s = n - a$ with $s \geq r$ in Lemma 3.4, it follows that the generalized distance spectrum of $K_{a,n-a}$ is

$$\left\{ \alpha(n + a) - 2^{[a-1]}, \alpha(2n - a) - 2^{[s-1]}, \frac{(\alpha + 2)n - 4 \pm \sqrt{\sigma}}{2} \right\},$$

where $\sigma = n^2\alpha^2 - (n^2 + 2\alpha^2 - 2an)4\alpha + 4(n^2 + 3an + 3a^2)$. We have $\frac{(\alpha + 2)n - 4 \pm \sqrt{\sigma}}{2} \geq \alpha(2n - a) - 2$ implying that $n(2 - 3\alpha) + 2a\alpha + \sqrt{\sigma} \geq 0$. Consider the function $f(\alpha) = n^2\alpha^2 - (n^2 + 2\alpha^2 - 2an)4\alpha + 4(n^2 + 3an + 3a^2)$, $\alpha \in [0, 1]$. It is easy to see that $f(\alpha)$ is a decreasing function for $\alpha < \frac{2(n^2 - 2an + 2a^2)}{n^2}$. Since $n \geq 2a$, so $\frac{2(n^2 - 2an + 2a^2)}{n^2} \geq 1$, and thus it follows that $f(\alpha)$ is a decreasing function for $\alpha \leq 1$. Therefore, $f(\alpha) \geq f(1) = (n - 2a)^2$ implies that $\sqrt{\sigma} \geq n - 2a$, for all $\alpha$. Now, $n(2 - 3\alpha) + 2a\alpha + \sqrt{\sigma} \geq n(2 - 3\alpha) + 2a\alpha + n - 2a = 3n - 2a + \alpha(2a - 3n) = (3n - 2a)(1 - \alpha) \geq 0$, for all $\alpha$, so it follows that $\partial_1(K_{a,n-a}) = \frac{(\alpha + 2)n - 4 \pm \sqrt{\sigma}}{2}$. Proceeding, similarly it can be seen that $\frac{(\alpha + 2)n - 4 \pm \sqrt{\sigma}}{2} \geq \alpha(n + a) - 2$ for all $\alpha \in [0, 1]$ and $a \neq 1$. Therefore, for $a \neq 1$, we have $\partial_n(K_{a,n-a}) = \alpha(n + a) - 2$. 


If $\alpha \neq 0$ and $a \neq 1$, then $\partial_n(K_{a,n-a}) = \alpha(n + a) - 2$ and so
\[
S_{D_{\alpha}}(K_{a,n-a}) = \frac{n(2 - \alpha) - 2a\alpha + \sqrt{\sigma}}{2}.
\]
It is easy to see that $S_{D_{\alpha}}(K_{a,n-a})$ is a decreasing function of $a$ for all $a \in [1, \frac{n}{2}]$. Therefore, it follows that
\[
S_{D_{\alpha}}(K_{2,n-2}) \geq S_{D_{\alpha}}(K_{3,n-3}) \geq \cdots \geq S_{D_{\alpha}}(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}).
\]
If $\alpha \neq 0$ and $a = 1$, then $\partial_n(K_{1,n-1}) = (\alpha + 2)n - 4 - \sqrt{\sigma}$ and so
\[
S_{D_{\alpha}}(K_{1,n-1}) = \sqrt{n^2\alpha^2 - (n^2 + 2 - 2n)4\alpha + 4(n^2 - 3n + 3)}.
\]
By direct computation it can be seen that
\[
\frac{n(2 - \alpha) - 4\alpha + \sqrt{\sigma}}{2} \leq \sqrt{n^2\alpha^2 - (n^2 + 2 - 2n)4\alpha + 4(n^2 - 3n + 3)},
\]
which implies that $S_{D_{\alpha}}(K_{1,n-1}) \geq S_{D_{\alpha}}(K_{2,n-2})$ Thus, for $\alpha \neq 0$, it follows that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the complete bipartite graph of order $n$ having minimum generalized distance spectral spread and $K_{1,n-1}$ has the maximum generalized distance spectral spread.

If $\alpha = 0$, proceeding similarly as above it can be seen that the same conclusion holds. Thus we have proved the following.

**Theorem 3.7:** Among all complete bipartite graphs $K_{a,n-a}$, with $a \leq n-a$, of order $n$, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ has the minimum and $K_{1,n-1}$ has the maximum generalized distance spectral spread.

The next observation follows from Lemma 2.2 and can be found in [16].

**Lemma 3.8:** Let $G$ be a connected graph of order $n$ and let $\frac{1}{2} \leq \alpha \leq 1$. If $G' = G - e$ is a connected graph obtained from $G$ by deleting an edge $e$, then $\partial_i(G') \geq \partial_i(G)$, for all $1 \leq i \leq n$.

Now, we obtain a lower bound for $S_{D_{\alpha}}(G)$ in terms of the order $n$ of the bipartite graph $G$.

**Theorem 3.9:** Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then, for $\alpha = 0$,
\[
S_{D_{\alpha}}(G) \geq n + \sqrt{\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil},
\]
with equality if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. For $\frac{1}{2} \leq \alpha \leq 1$, let $x$ be the vertex with minimum transmission degree and $y, z$ be neighbours of $x$ with minimum, second minimum transmission degree. Then
\[
S_{D_{\alpha}}(G) \geq \frac{\alpha n + 2n - 4 + \sqrt{\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lceil \frac{n}{2} \right\rceil^2(\alpha - 2)^2 + 2\left\lfloor \frac{n}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil(\alpha^2 - 2)}}{2} - \beta,
\]
where $\beta$ is the smallest eigenvalue of the matrix $M = \begin{pmatrix} \alpha & 1 - \alpha & 1 - \alpha \\ 1 - \alpha & \alpha & 1 - \alpha \\ 1 - \alpha & 2(1 - \alpha) & \alpha & 2(1 - \alpha) \end{pmatrix}$. Equality occurs if and only if $G \cong K_{1,2}$. 
Proof: If $\alpha = 0$, the proof follows from [20, Theorem 3.4]. Suppose that $\alpha \neq 0$. Let $V(G)$ be the vertex set of the bipartite graph $G$ and let $V(G) = V_1 \cup V_2$ be the bipartite partition of $V(G)$. Let $|V_1| = r$ and $|V_2| = s$, with $r + s = n$. Clearly, $G$ can be obtained by deleting some edges in $K_{rs}$. Therefore, for $\frac{1}{2} \leq \alpha \leq 1$, from Lemmas 3.8 and 3.4, it follows that

$$\partial_1(G) \geq \partial_1(K_{rs}) = \frac{n+2n-4+\sqrt{(r^2+s^2)(\alpha-2)^2+2rs(\alpha^2-2)}}{2}. $$

As seen above, the quantity $\partial_1(K_{rs}) = \frac{n+2n-4+\sqrt{(r^2+s^2)(\alpha-2)^2+2rs(\alpha^2-2)}}{2}$ is minimal for $r = [\frac{n}{2}]$ and $s = [\frac{n}{2}]$. Thus, it follows that $\partial_1(G) \geq \frac{n+2n-4+\sqrt{(\frac{n^2}{4}+\frac{n^2}{4})(\alpha-2)^2+2\sqrt{\frac{n^2}{2}}(\alpha^2-2)}}{2}$. Let $x \in V(G)$ be the vertex with minimum transmission degree in $G$. Then it is clear that $x$ is a non-pendent vertex of $G$. Let $y$ and $z$ be the vertices in the neighbourhood of $x$ having the minimum and the second minimum transmission degree. Since the vertices $x$, $y$, and $z$ forms the subgraph $K_{1,2}$ of $G$, it follows that the matrix $M$ (given in the statement) is a principal submatrix of $D_\alpha(G)$. Therefore, using Lemma 3.2, we have $\partial_n(G) \leq \lambda_3(M)$. The first part of the result now follows. From the discussion done so far it is clear that equality occurs if and only if $G \cong K_{1,2}$.

From Theorem 3.9, for $\alpha = 0$, it follows that the graph $K_{[\frac{n}{2}],[\frac{n}{2}]}$ has the minimum generalized distance spectral spread. For $\alpha \neq 0$, Theorem 3.7 gives an insight that this may be true, in general. Therefore, we state the following conjecture.

Conjecture 3.1: Among all connected bipartite graphs of order $n$, the graph $K_{[\frac{n}{2}],[\frac{n}{2}]}$ has the minimum generalized distance spectral spread.

4. Bounds for $S_{D_{\alpha}}(G)$ in terms of clique number and independence number

In this section, we obtain lower bounds for generalized distance spectral spread $S_{D_{\alpha}}(G)$ in terms of the clique number and the independence number of the graph $G$.

First we obtain a lower bound for $S_{D_{\alpha}}(G)$ in terms of clique number $\omega$ and the Wiener index $W(G)$ of the graph $G$.

Theorem 4.1: Let $G$ be a connected graph on $n \geq 3$ vertices having clique number $\omega \geq 2$ and Wiener index $W$. Let $G_1, G_2, \ldots, G_k$ be all cliques of order $\omega$ and $s_i = \sum_{v_j \in V(G_i)} \text{Tr}(v_j)$.

(i) If $\omega \leq n - 1$, then

$$S_{D_{\alpha}}(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{\alpha_i^2 - 4\beta_i\omega(n - \omega)}}{(\omega)(n - \omega)},$$

where $\alpha_i = s_i(\alpha n - 2\omega) + \omega(1 - \alpha)(2W + n(\omega - 1))$ and $\beta_i = 2W\omega(\omega - 1) - s_i^2 + 2W\alpha(s_i - \omega(\omega - 1))$.

(ii) If $\omega = n$, then $S_{D_{\alpha}}(G) = (1 - \alpha)n$.

Proof: Let $G$ be a connected graph on $n \geq 3$ vertices having clique number $\omega$ and Wiener index $W$. Let $G_i$, $1 \leq i \leq k$, be the cliques of $G$ having clique number $\omega$. We consider the following two cases:

(i) Let $\omega \leq n - 1$. For $1 \leq i \leq k$, let $V(G_i) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_\omega}\}$ be the vertex set of $G_i$. Let $V(G) \setminus V(G_i) = \{w_1, w_2, \ldots, w_{n-\omega}\}$. By suitably labelling the vertices of $G$, it can
be seen that the generalized distance matrix of the graph $G$ can be written as $D_\alpha(G) = \begin{pmatrix} X & (1-\alpha)Y \\ (1-\alpha)Y^T & Z \end{pmatrix}$, where

$$X = \begin{pmatrix} \alpha Tr(u_1) & 1-\alpha & 1-\alpha & \cdots & 1-\alpha \\ 1-\alpha & \alpha Tr(u_2) & (1-\alpha) & \cdots & (1-\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-\alpha & (1-\alpha) & (1-\alpha) & \cdots & \alpha Tr(u_\omega) \end{pmatrix},$$

$$Y = \begin{pmatrix} d(u_1,w_1) & \cdots & d(u_1,w_{n-\omega}) \\ d(u_2,w_1) & \cdots & d(u_2,w_{n-\omega}) \\ \vdots & \vdots & \vdots \\ d(u_\omega,w_1) & \cdots & d(u_\omega,w_{n-\omega}) \end{pmatrix},$$

and

$$Z = \begin{pmatrix} \alpha Tr(w_1) & (1-\alpha)d(w_1,w_2) & \cdots & (1-\alpha)d(w_1,w_{n-\omega}) \\ (1-\alpha)d(w_2,w_1) & \alpha Tr(w_2) & \cdots & (1-\alpha)d(w_2,w_{n-\omega}) \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)d(w_{n-\omega},w_1) & (1-\alpha)d(w_{n-\omega},w_2) & \cdots & \alpha Tr(w_{n-\omega}) \end{pmatrix}.$$

It can be seen that the quotient matrix $B$ of the matrix $D_\alpha(G)$ is

$$B = \begin{pmatrix} (1-\alpha)\omega(\omega-1) + \alpha s_i & (1-\alpha)(s_i - \omega(\omega-1)) \\ (1-\alpha)(s_i - \omega(\omega-1)) & 2W + s_i(\alpha - 2) + \omega(1-\alpha)(\omega-1) \end{pmatrix}.$$ 

The eigenvalues of the matrix $B$ are

$$x_1, x_2 = \frac{\alpha_i \pm \sqrt{\alpha_i^2 - 4\beta_i \omega(n-\omega)}}{2\omega(n-\omega)},$$

where $x_1 \geq x_2, \alpha_i = s_i(\alpha n - 2\omega) + \omega(1-\alpha)(2W + n(\omega-1))$ and $\beta_i = 2W\omega(\omega-1) - s_i^2 + 2W\alpha(s_i - \omega(\omega-1))$. Now, using Lemma 3.1, the result follows in this case.

(ii) If $\omega = n$, then $G$ is $K_n$. Since the generalized distance spectrum of the complete graph $K_n$ is $\{n-1, n\alpha - 1^{[n-1]}\}$, the result follows in this case also.

For $\omega \leq n - 1$, it will be interesting to characterize the graphs attaining the lower bound in Theorem 4.1. Therefore, we have the following problem.

**Problem 4.2:** For $\omega \leq n - 1$, characterize the connected graphs $G$ of order $n \geq 3$, which attain the lower bound in Theorem 4.1.

A complete split graph, denoted by $CS_{t,k}$, $t + k = n$, is the graph consisting of a clique on $t$ vertices and an independent set on the remaining $k = n-t$ vertices, such that each vertex of the clique is adjacent to every vertex of the independent set. The next observation follows from Lemma 3.3 and can be found in [16].
Lemma 4.3: The generalized distance spectrum of the complete split graph $CS_{t,n-t}$ is

$$\{\alpha n - 1^{t-1}, \alpha(2n - t) - 2^{t-1}, x_1, x_2\},$$

where $x_1 \geq x_2$, $x_1, x_2 = \frac{2n-t+\alpha n-3+\sqrt{\theta}}{2}$ and $\theta = (5 - 4\alpha)t^2 + (6\alpha n - 8n - 4\alpha + 6)t + n^2(\alpha - 2)^2 + 2n\alpha - 4n + 1$.

If $G$ is a connected graph with independence number $k = 1$, then clearly $G \cong K_n$ and so $SD_{\alpha}(G) = (1 - \alpha)n$. Therefore, we need to consider $k \geq 2$. The next theorem gives a lower bound for $SD_{\alpha}(G)$ in terms of independence number $k$ of the graph $G$.

Theorem 4.4: Let $G$ be a connected graph with $n \geq 3$ vertices having independence number $k \geq 2$. Then for $\alpha = 0$,

$$SD_{\alpha}(G) \geq \frac{n + t + 1}{2} + \frac{1}{2}(n - t + 1)^2 + 4t^2 - 4t,$$

with equality if and only if $G \cong CS_{t,k}$, $k = n - t$. For $\frac{1}{2} \leq \alpha \leq 1$, let $x$ be the vertex with minimum transmission degree and $y, z$ be neighbours of $x$ with minimum, second minimum transmission degree. Then

$$SD_{\alpha}(G) \geq \frac{2n - t + \alpha n - 3 + \sqrt{\theta}}{2} - \beta,$$

where $\theta = (5 - 4\alpha)t^2 + (6\alpha n - 8n - 4\alpha + 6)t + n^2(\alpha - 2)^2 + 2n\alpha - 4n + 1$ and $\beta$ is the smallest eigenvalue of the matrix $M = \begin{pmatrix}
\alpha Tr(x) & (1-\alpha) & (1-\alpha) \\
(1-\alpha) & \alpha Tr(y) & 2(1-\alpha) \\
(1-\alpha) & 2(1-\alpha) & \alpha Tr(z)
\end{pmatrix}$. Equality occurs if and only if $G \cong K_{1,2}$.

Proof: If $\alpha = 0$, the proof follows from [20, Theorem 3.6]. Suppose that $\alpha \neq 0$. Let $V(G)$ be the vertex set of the graph $G$ and let $V(G) = V_1 \cup V_2$ be a partition of $V(G)$ such that the subgraph induced by $V_2$ is an empty graph and $V_1 = V(G) - V_2$. Let $|V_1| = t$ and $|V_2| = k = n - t$. Clearly, $G$ can be obtained by deleting some edges in the complete split graph $CS_{t,k}$. Therefore, for $\frac{1}{2} \leq \alpha \leq 1$, from Lemmas 3.8 and 3.4, it follows that

$$d_1(G) \geq d_1(CS_{t,n-t}) = \frac{2n-t+\alpha n-3+\sqrt{\theta}}{2},$$

where $\theta$ is defined above. Let $x \in V(G)$ be the vertex with minimum transmission degree in $G$. Then it is clear that $x$ is a non-pendent vertex of $G$. Let $y$ and $z$ be the vertices in the neighbourhood of $x$ having the minimum and the second minimum transmission degree. Since the vertices $x, y, z$ form the subgraph $K_{1,2}$ of $G$, it follows that the matrix $M$ (given in the statement) is a principal submatrix of $D_{\alpha}(G)$. Therefore, using Lemma 3.2, we have $d_n(G) \leq \lambda_3(M)$. The first part of the result now follows. From the discussion done so far it is clear that equality occurs if and only if $G \cong K_{1,2}$.

5. Concluding remarks

We conclude the paper with the following remark.

Remark 5.1: As mentioned in the introduction, for $\alpha = 0$, the generalized distance matrix $D_{\alpha}(G)$ is same as the distance matrix $D(G)$ and for $\alpha = \frac{1}{2}$, twice the generalized distance
matrix $D_{\alpha}(G)$ is same as the distance signless Laplacian matrix $D^{Q}(G)$. Therefore, if in particular, we put $\alpha = 0$ and $\alpha = \frac{1}{2}$, in all the results obtained in Sections 2–4, we obtain the corresponding bounds for the distance spectral spread $S_{D}(G)$ and the distance signless Laplacian spectral spread $S_{D^{Q}}(G)$.

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