SCALING LIMIT OF SEMIFlexible POLYMERS: A PHASE TRANSITION

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ABSTRACT. We consider a semiflexible polymer in $\mathbb{Z}^d$ which is a random interface model with a mixed gradient and Laplacian interaction. The strength of the two operators is governed by two parameters called lateral tension and bending rigidity, which might depend on the size of the graph. In this article we show a phase transition in the scaling limit according to the strength of these parameters: we prove that the scaling limit is, respectively, the Gaussian free field, a “mixed” random distribution and the continuum membrane model in three different regimes.

1. Introduction

In this article we study a model which is a special instance of a more general class of random interfaces. Random interfaces are fields $\phi = (\phi_x)_{x \in \mathbb{Z}^d}$, whose distribution is specified by a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$, $d \geq 1$. The density is given in terms of an energy function $H$ called Hamiltonian and has the form

$$P_{\Lambda}(d\phi) := \frac{e^{-H(\phi)}}{Z_{\Lambda}} \prod_{x \in \Lambda} d\phi_x \prod_{x \in \mathbb{Z}^d \setminus \Lambda} \delta_0(d\phi_x),$$

(1.1)

where $\Lambda \subset \mathbb{Z}^d$ is a finite subset, $d\phi_x$ is the Lebesgue measure on $\mathbb{R}$, $\delta_0$ is the Dirac measure at 0, and $Z_{\Lambda}$ is a normalizing constant. We are imposing zero boundary conditions: almost surely $\phi_x = 0$ for all $x \in \mathbb{Z}^d \setminus \Lambda$, but the definition holds for more general boundary conditions. A special case is when the Hamiltonian is given by

$$H(\phi) = \sum_{x \in \mathbb{Z}^d} \left( \kappa_1 \| \nabla \phi_x \|^2 + \kappa_2 (\Delta \phi_x)^2 \right)$$

(1.2)

where $\nabla$ is the discrete gradient and $\Delta$ is the discrete Laplacian defined by

$$\nabla f(x) = (f(x + e_i) - f(x))_{i=1}^d$$

$$\Delta f(x) = \frac{1}{2d} \sum_{i=1}^d (f(x + e_i) + f(x - e_i) - 2f(x))$$

for any $x \in \mathbb{Z}^d$, $f : \mathbb{Z}^d \to \mathbb{R}$, and $\kappa_1$, $\kappa_2$ are two parameters. In the physics literature, the above Hamiltonian is considered to be the energy of a semiflexible membrane (or semiflexible polymer if $d = 1$) where the parameters $\kappa_1$ and $\kappa_2$ are the lateral tension and the bending rigidity, respectively (Leibler (2004), Lipowsky (1995), Ruiz-Lorenzo et al. (2005)). It is sometimes natural to have the parameters $\kappa_1$ and $\kappa_2$ depending on the length of the polymer chain. When $\kappa_2 = 0$, the model is the purely gradient model and it is known as the discrete Gaussian free field. In this case the Hamiltonian is governed by the surface area of the interface. When $\kappa_1 = 0$, the model is called the membrane, or Bilaplacian, model. In this case the Hamiltonian is
governed by the curvature of the interface. More generally the Hamiltonian is governed by an interplay the surface area and the curvature, hence one considers the model dependent on both gradient and Laplacian interaction. The main aim of this article is to show how the dependency on the size of the set $\Lambda$ of $\kappa_1$ and $\kappa_2$ affects the scaling limit of $P_\Lambda$.

When $\kappa_1 = 0$ or $\kappa_2 = 0$, the scaling limit of the model is well-understood. The literature on the discrete Gaussian free field is huge due to its connection to various other probabilistic objects and we refer the interested reader to the lecture notes and survey articles Berestycki (2015), Biskup (2017), Sheffield (2007). We refer to Caravenna and Deuschel (2009), Cipriani et al. (2018b), Hryniv and Velenik (2009) for the scaling limit of the membrane model in $d \geq 1$. The literature on the case when $\kappa_1 > 0, \kappa_2 > 0$ is limited and has been considered in the works of Borecki (2010), Borecki and Caravenna (2010), Cipriani et al. (2018a), Sakagawa (2018). Borecki (2010) and Borecki and Caravenna (2010) introduced this model as the $(\nabla + \Delta)$-model (we will also refer to it as “mixed model”) with constant $\kappa_1, \kappa_2$. They studied in $d = 1$ the influence of pinning in order to understand the localization behavior of the polymer. The results were extended to higher dimensions, together with further properties of the free energy, in Sakagawa (2018). In Cipriani et al. (2018a) the scaling limit of the $(\nabla + \Delta)$-model is studied. There it is shown that if one takes the lattice size to go to zero, under a suitable scaling the Laplacian term is dominated by the gradient and the limit becomes the Gaussian free field. A very natural question, which we aim at investigating in this paper, is what happens if we increase the strength of the Laplacian part. To the best of our knowledge, the influence of the polymer length through $\kappa_1, \kappa_2$ has not been systematically addressed in the literature.

We now briefly describe the phase transition picture which appears in the scaling limit. We restrict our focus to $d = 1$ for heuristic explanations. Let us consider the Hamiltonian described in (1.2). We take $\Lambda = \{1, \ldots, N - 1\}$ for $N \in \mathbb{N}, \kappa_1 = 1/4$ and $\kappa_2 = \kappa(N)/2$. In $d = 1$ in the DGFF case ($\kappa_2 = 0$) it is well-known that the finite volume measure can be given by a random walk bridge and in the membrane case ($\kappa_1 = 0$) by an integrated random walk bridge (Caravenna and Deuschel (2008)). Therefore the scaling limit for the DGFF and membrane turns out to be Brownian bridge and the integrated Brownian bridge, respectively. In $d = 1$, a representation for the $(\nabla + \Delta)$—model using random walks was obtained in Borecki (2010). The details of the representation are recalled in Appendix C.

Let $\gamma$ and $\sigma$ be as in (C.1) and (C.2), respectively. Let $(\tilde{\varepsilon}_i)_{i \in \mathbb{Z}^+}$ be i.i.d. normal random variables with mean zero and variance $\sigma^2/(1 - \gamma)^2$. For $n \geq 1$, let $W_n = S_n - U_n$, where $S_n = \sum_{k=1}^n \tilde{\varepsilon}_k$ and $U_n = \gamma^n \tilde{\varepsilon}_1 + \gamma^{n-1} \tilde{\varepsilon}_2 + \cdots + \gamma \tilde{\varepsilon}_n$. From Borecki (2010, Proposition 1.10) it is known that the finite volume measure of the model is given by the joint distribution of $(W_n)_{1 \leq n \leq N - 1}$ conditioned on $W_N = W_{N+1} = 0$. We look at the unconditional process and see how the parameter $\kappa(N)$ changes the variance. It follows from (C.1) and (C.2) that

$$\sigma^2 \approx \frac{1}{\kappa(N)} \quad \text{and} \quad (1 - \gamma) \approx \frac{1}{\sqrt{\kappa(N)}}.$$

So for the case when $\kappa(N) \ll N^2$ we have

$$\text{Var}(S_{N-1}) \approx N, \quad \text{Var}(U_{N-1}) \approx \sqrt{\kappa(N)} \quad \text{and} \quad \text{Cov}(S_{N-1}, U_{N-1}) \approx \sqrt{\kappa(N)}$$

which together imply that $\text{Var}(W_{N-1}) \approx N$, thus the random walk dominates with its scaling $\sqrt{N}$.

When $\kappa(N) \gg N^2$ the situation is a bit more complicated and one can compute that (see Appendix C)

$$\text{Var}(W_{N-1}) \approx \frac{N^3}{\kappa(N)}.$$

It turns out that the Laplacian part dominates under this scaling. When $\kappa(N) \sim N^2$ then the contribution from $S_{N-1}$ and $U_{N-1}$ is similar and hence both the gradient and Laplacian interaction come into picture. The reader can see a simulation of the free boundary case that is, the trajectories of $(W_n)_{1 \leq n \leq N}$, in Figure 1
and Figure 2. We plotted the two cases $\kappa \ll N^2$ and $\kappa \gg N^2$ in different pictures as the height scalings are different.

Figure 1. This is a simulation of some trajectories of $(W_n)_{1 \leq n \leq N}$ with $N = 10^4$ and $\kappa = 0$, $\kappa = 2 \times 10^2$, $\kappa = 2 \times 10^4$, $\kappa = 2 \times 10^6$.

Figure 2. This is a simulation of some trajectories of $(W_n)_{1 \leq n \leq N}$ with $N = 10^3$ and $\kappa = 2 \times 10^6.5$, $\kappa = 2 \times 10^7$, $\kappa = 2 \times 10^8$.

We stress that in the above description we did not consider boundary effects which can cause considerable difficulty in understanding these processes explicitly. In Appendix C we have pointed out the conditional representation of $W_{N-1}$. One can see that it is not easy to determine whether the above transition can be pushed to the conditional processes and hence the finite volume measure. The aim of this article is to
go beyond such representations and show the above transition holds true in general dimensions and get the explicit limits in each of the cases. In this respect, we also record that the integrated random walk representations of \( d = 1 \) cannot be extended to \( d > 1 \). In a recent work, the authors of the present article introduced a finite difference method to approximate solutions of PDEs to successfully obtain the scaling limit of the membrane model and the \((\nabla + \Delta)\)-model with fixed coefficients (see Cipriani et al. (2018a,b)). The idea was inspired by the work Thomée (1964). Finite difference methods was also employed in the works Müller and Schweiger (2019), Schweiger (2019) to obtain important estimates on the discrete Green’s function of the membrane model.

The main results of the article are as follows. We consider the model on \( \Lambda_N \subseteq \mathbb{Z}^d \) for a suitable \( \Lambda_N \) defined later in Section 2. Also, we assume \( \kappa_1 = 1/(4d) \), \( \kappa_2 = \kappa(N)/2 \) and distinguish three regimes for \( \kappa \).

(a) Let \( \kappa \gg N^2 \). In \( d \geq 1 \), we show that the appropriately rescaled field converges to the continuum membrane model. The continuum membrane model is roughly a centered Gaussian process whose covariance is given by the Green’s function of the Bilaplacian Dirichlet problem. For \( d \geq 4 \), in Theorem 2.8 we show the convergence takes place in a distributional space (more precisely a negatively-indexed Sobolev space). In \( d = 1, 2 \) and \( 3 \) we show in Theorem 2.1 that the limiting Gaussian process has continuous paths.

(b) Let \( \kappa \sim 2dN^2 \). In \( d \geq 2 \) we show (Theorem 2.8) that the rescaled field converges to a random distribution in an appropriate Sobolev space and the covariance of the limiting Gaussian field is given by the Dirichlet problem involving the elliptic operator \(-\Delta_c + \Delta^2_c\). In \( d = 1 \), again we show (in Theorem 2.1) the convergence takes place in the space of continuous functions.

(c) Let \( \kappa \ll N^{1/2} \). In \( d \geq 2 \) we show (in Theorem 2.8) that the rescaled field converges in distribution to the Gaussian free field. Again, since the Gaussian free field is a random distribution the convergence takes place in a negatively-indexed Sobolev space. In \( d = 1 \), we show (in Theorem 2.1) that the limiting process is the Brownian bridge, confirming the heuristics presented above. We believe the threshold \( \kappa \ll N^{1/2} \) can be extended to \( \kappa \ll N^2 \) but we could not reach this boundary through our methods.

To derive the above results, the main technique we use is the approximation of the solution of a continuum Dirichlet problem with its discrete counterpart. Using Sobolev estimates it can be shown that the closeness of the solutions is related to the approximation of the discrete elliptic operator to the continuum one. This idea has been already employed in Cipriani et al. (2018b) and Cipriani et al. (2018a). But in the present scenario, the discrete elliptic operators have coefficients which depend on \( N \) and hence the estimates of Thomée (1964) are not applicable directly. In addition, the rough behaviour around the boundary in the case of constant coefficients was dealt with by considering a truncation of the discrete elliptic operator. The operators were rescaled around the boundary and this helped in controlling their behavior. The same technique becomes a bit more involved in the present case and the truncation requires more care. We deal with these technical issues in Section 2.3. In passing, we believe that the result in Section 2.3 is of independent interest and can be applied to discrete elliptic operators where coefficients depend on the approximation size.

Structure of the article. In Section 2 we state our main results precisely. Furthermore, in its Subsection 2.3 we discuss the approximation technique and the norm estimates in detail, while in Subsection 2.4 we mention some open problems. In Section 3 we derive the proof of Theorem 2.8 and in Section 4 we deal with the lower dimensional case (Theorem 2.1). In Section 5 we provide a proof of the approximation results stated in Subsection 2.3. These are mainly improvements of the results of Thomée (1964).

Notation. For real-valued functions \( f(\cdot), g(\cdot) \) we write \( f \gg g, f \sim g, f \approx g, f \ll g \) when \( \lim_{n \to \infty} f/g \) equals \( \infty, 1, c \) and \( 0 \), respectively, where \( c \) is a non zero constant which may be \( 1 \) also. Also we write \( f \asymp g \).
if there exist two positive constants \( c_\ell, c_r \) such that \( c_\ell g(n) \leq f(n) \leq c_r g(n) \) for all \( n \). We denote by \( C \) a universal constant that may change from line to line within the same equation.

2. Set-up and main results

Let \( \Lambda \) be a finite subset of \( \mathbb{Z}^d \), \( d \geq 1 \), and \( \mathbf{P}_\Lambda \) and \( H(\varphi) \) be as in (1.1) and (1.2) respectively. It follows from Lemma 1.2.2 of Kurt (2008) that the Gibbs measure (1.1) on \( \mathbb{R}^{|\Lambda|} \) with Hamiltonian (1.2) exists. Note that (1.2) can be written as

\[
H(\varphi) = \frac{1}{2} \langle \varphi, (-4d\kappa_1\Delta + 2\kappa_2\Delta^2)\varphi \rangle_{\mathcal{E}(\mathbb{Z}^d)}. \tag{2.1}
\]

Let \( d \geq 1 \). Let \( D \) be a bounded domain in \( \mathbb{R}^d \). For \( N \in \mathbb{N} \), let \( D_N = N\overline{D} \cap \mathbb{Z}^d \). Let us denote by \( \Lambda_N \) the set of points \( x \) in \( D_N \) such that, for every direction \( i, j \), also the points \( x \pm e_i, x \pm (e_i \pm e_j) \) are all in \( D_N \). In other words, \( \Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d \) is the largest set satisfying \( \partial_2 \Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d \) where \( \partial_2 \Lambda_N := \{ y \in \mathbb{Z}^d \setminus \Lambda_N : \text{dist}(y, \Lambda_N) \leq 2 \} \) is the double (outer) boundary of \( \Lambda_N \) of points at \( \ell^1 \) distance at most 2 from it. We consider the model with \( \Lambda = \Lambda_N, \kappa_1 = 1/4d, \kappa_2 = \kappa(N)/2 \) and want to study what happens when we tune suitably the parameter \( \kappa(N) \) as \( N \) tends to infinity. We assume \( \kappa_1 \) to be constant as it is easy to state the results in this format. Also for simplicity we write \( \kappa \) for \( \kappa(N) \). We just note here that if we write \( G_{\Lambda_N}(x, y) := \mathbf{E}_{\Lambda_N}(\varphi_x \varphi_y) \), it follows from Lemma 1.2.2 of Kurt (2008) that \( G_{\Lambda_N} \) solves the following discrete boundary value problem: for \( x \in \Lambda_N \)

\[
\begin{cases}
(-\Delta + \kappa\Delta^2)G_{\Lambda_N}(x, y) = \delta_x(y) & y \in \Lambda_N \\
G_{\Lambda_N}(x, y) = 0 & y \notin \Lambda_N.
\end{cases} \tag{2.2}
\]

To describe the main results we need some elliptic operators. We first introduce them and the corresponding Dirichlet problem. Let \( L \) denote one of the following three elliptic operators:

\[
L = \begin{cases}
-\Delta_c, \\
\Delta_c^2, \\
-\Delta_c + \Delta_c^2,
\end{cases} \tag{2.3}
\]

where \( \Delta_c \) is the Laplace operator defined by \( \Delta_c = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \). We consider the following continuum Dirichlet problem:

\[
\begin{cases}
Lu(x) = f(x) & x \in D \\
D^\alpha u(x) = 0 & |\alpha| \leq m - 1, \ x \in \partial D.
\end{cases} \tag{2.4}
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_d) \) is a multi-index with \( \alpha_i \)'s being non-negative integers, \( |\alpha| := \sum_{i=1}^d \alpha_i, m = 1 \) if \( L = -\Delta_c \) and \( m = 2 \) in the other cases.

2.1. Lower dimensional results. We first present the results in lower dimensions where we show that convergence takes place in the space of continuous functions. In this case we consider \( D = (0, 1)^d \). Also here, according to the behaviour of \( \kappa \) as \( N \to \infty \) we have three different limits. To verify the convergence in the space of continuous functions we shall need to continuously interpolate the discrete model. In \( d = 1 \) the linear interpolation gives a continuous process but for higher dimensions there might be many ways. We stick to the following natural way. We will need this interpolation in \( d = 2 \) and 3 only when \( \kappa \gg N^2 \). We define the continuous interpolation \( \{ \Psi_N \}_{N \in \mathbb{N}} \) in the following fashion:

- For \( d = 1 \) and \( t \in \overline{D} \)
  \[
  \Psi_N(t) = c_N(1) \left[ \varphi_{[Nt]} + (Nt - [Nt])(\varphi_{[Nt]+1} - \varphi_{[Nt]}) \right]. \tag{2.5}
  \]
For $d = 2$ and $t = (t_1, t_2) \in \overline{D}$
\[
\Psi_N(t) = c_N(2) \left[ \varphi_{[Nt]} + \{Nt_i\} \left( \varphi_{[Nt]} - \varphi_{[Nt]} e_i \right) + \{Nt_j\} \left( \varphi_{[Nt]} + e_i + e_j - \varphi_{[Nt]} e_i \right) \right], \quad \text{if } \{Nt_i\} \geq \{Nt_j\}
\]
where $i, j \in \{1, 2\}, i \neq j$.

For $d = 3$ and $t = (t_1, t_2, t_3) \in \overline{D}$
\[
\Psi_N(t) = c_N(3) \left[ \varphi_{[Nt]} + \{Nt_i\} \left( \varphi_{[Nt]} - \varphi_{[Nt]} e_i \right) + \{Nt_j\} \left( \varphi_{[Nt]} + e_i + e_j - \varphi_{[Nt]} e_i \right) + \{Nt_k\} \left( \varphi_{[Nt]} + e_i + e_j + e_k - \varphi_{[Nt]} e_i \right) \right], \quad \text{if } \{Nt_i\} \geq \{Nt_j\} \geq \{Nt_k\}
\]
where $i, j, k \in \{1, 2, 3\}$ and pairwise different. Here $(e_i)_{i=1}^d$ denotes the standard basis for $\mathbb{R}^d$ and $c_N(d)$, $d = 1, 2, 3$, are scaling factors which are specified in the following result.

**Theorem 2.1.** We have the following convergence results.

1. $\kappa \gg N^2$. Let $1 \leq d \leq 3$. Define a continuously interpolated field $\Psi_N$ as in (2.5), (2.6) and (2.7) with
   \[
   c_N(d) = (2d)^{-1} \sqrt{\kappa N^{d+1}}.
   \]
   Then we have, as $N \to \infty$, that the field $\Psi_N$ converges in distribution to $\Psi^{\Delta^2}$ in the space of continuous functions on $\overline{D}$, where $\Psi^{\Delta^2}$ is defined to be the centered continuous Gaussian process on $\overline{D}$ with covariance $G_D(\cdot, \cdot)$, the Green’s function for the following biharmonic Dirichlet problem:
   \[
   \begin{cases}
   \Delta^2 u(x) = f(x), & x \in D \\
   \partial^\alpha u(x) = 0, & \forall |\alpha| \leq 1, x \in \partial D.
   \end{cases}
   \]

2. $\kappa \sim 2dN^2$. Let $d = 1$. Define the continuously interpolated field $\Psi_N$ as in (2.5) with
   \[
   c_N(1) = (2d)^{-\frac{1}{2}} N^{-\frac{3}{2}}.
   \]
   Define $\Psi^{-\Delta+\Delta^2}$ to be the continuous Gaussian process in $\overline{D}$ with covariance $G_D(\cdot, \cdot)$, where $G_D$ is the Green’s function for the problem
   \[
   \begin{cases}
   -\frac{d^2 u}{dx^2}(x) + \frac{d^4 u}{dx^4}(x) = f(x), & x \in D \\
   u(x) = \frac{d u}{dx}(x) = 0 & x \in \partial D.
   \end{cases}
   \]
   Then $\Psi_N$ converges in distribution to the field $\Psi^{-\Delta+\Delta^2}$ in the space of continuous functions on $\overline{D}$.

3. $\kappa \ll N^{\frac{3}{2}}$. Let $d = 1$. Define the continuously interpolated field $\Psi_N$ as in (2.5) with
   \[
   c_N(1) = (2d)^{-\frac{1}{2}} N^{-\frac{3}{2}}.
   \]
   Then as $N \to \infty$, $\Psi_N$ converges in distribution to the Brownian bridge, $\Psi^{-\Delta}$, in the space of continuous functions on $\overline{D}$.

**Remark 2.2.** When $\kappa_1 = 0$ and $\kappa_2 = 1$ in (1.2) the $d = 1$ case was first studied in Caravenna and Deuschel (2009), where they showed that the limiting distribution is given by an integrated Brownian bridge (for a more precise definition see Theorem 1.2 of Caravenna and Deuschel (2009)). The higher dimensional case was studied in Cipriani et al. (2018b). It was shown in Cipriani et al. (2018b) that for $d = 2, 3$ the discrete membrane model converges to a Gaussian process with continuous paths and the methods in that article can be seen to be valid in $d = 1$ also. By uniqueness of the limit in $C[0, 1]$ it follows that the limiting Gaussian
process in $d = 1$ for the case $\kappa \gg N^2$ (Theorem 2.1 (1)) can be described using the integrated Brownian bridge, the limit matching that of Caravenna and Deuschel (2009).

2.2. Higher dimensional results. We present now the results in higher dimensions where we show convergence in the space of distributions. In order to make our statements precise, we need to introduce three (negative ordered) Sobolev spaces denoted respectively as $H^s_{\Delta^2}(D)$, $H^s_{-\Delta+\Delta^2}(D)$ and $H^{-\Delta}(D)$. We are going to recall some basic notations on Sobolev spaces and also some facts about the eigenvalues of the elliptic operators involved in our problem.

2.2.1. Basics of Sobolev spaces. Let us first describe the standard Sobolev space. Let $C^\infty_c(D)$ denote the space of infinitely differentiable functions $u : D \to \mathbb{R}$ with compact support inside $D$. For $\alpha = (\alpha_1, \ldots, \alpha_d)$ a multi-index define

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u.$$ 

Suppose $f, g \in L^1_{loc}(D)$. We say that $g$ is the $\alpha$-th weak partial derivative of $f$ (written $D^\alpha f = g$) if

$$\int_D f D^\alpha u \, dx = (-1)^{|\alpha|} \int_D g u \, dx \quad \forall u \in C^\infty_c(D).$$

The Sobolev space $W^{k,p}$ is defined in the usual way as

$$W^{k,p} = \{ f \in L^1_{loc}(D) : D^\alpha f \in L^p(D), |\alpha| \leq k \}.$$

Denote by $H^k(D) := W^{k,2}(D), k = 0, 1, \ldots$, which is a Hilbert space with norm

$$\|f\|_{H^k(D)} = \left( \sum_{|\alpha| \leq k} \int_D |D^\alpha f|^2 \, dx \right)^{1/2}.$$

It is true that if $a > b$ then $H^a(D) \subset H^b(D)$. Let us define another Hilbert space,

$$H^k_0(D) := \overline{C^\infty_c(D)}^{\| \cdot \|_{H^k(D)}}$$

and let $H^{-k}(D) = [H^k_0(D)]^*$ be its dual.

2.2.2. Continuum membrane model. We briefly give the definition of the Sobolev space $H^s_{\Delta^2}(D)$ and the continuum membrane model. For a more detailed discussion see Cipriani et al. (2018b). By the spectral theorem for compact self-adjoint operators and elliptic regularity one can show that there exist smooth eigenfunctions $\{u_j\}_{j \in \mathbb{N}}$ of $\Delta^2$ corresponding to the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ such that $\{u_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for $L^2(D)$. Now for any $s > 0$ we define the following inner product on $C^\infty_c(D)$:

$$\langle f, g \rangle_{s, \Delta^2} := \sum_{j \in \mathbb{N}} \lambda_j^{s/2} \langle f, u_j \rangle_{L^2} \langle u_j, g \rangle_{L^2}.$$ 

Then $H^s_0(D)$ is defined to be the Hilbert space completion of $C^\infty_c(D)$ with respect to this inner product. We define $H^{-s}(D)$ to be its dual and the dual norm is denoted by $\| \cdot \|_{-s, \Delta^2}$. The following definition is from Cipriani et al. (2018b, Proposition 3.9) and provides a description of the continuum membrane model $\Psi \Delta^2$.

**Definition 2.3.** Let $(\xi_j)_{j \in \mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set

$$\Psi \Delta^2 := \sum_{j \in \mathbb{N}} \lambda_j^{-1/2} \xi_j u_j.$$ 

Then $\Psi \Delta^2 \in H^{-s}_{\Delta^2}(D)$ a.s. for all $s > (d - 4)/2$ and is called the continuum membrane model.
2.2.3. Continuum mixed model. We define the space $\mathcal{H}_{-\Delta + \Delta^2}^s$ analogously to $\mathcal{H}_{\Delta}^{-s}$. One can find smooth eigenfunctions $\{v_j\}_{j \in \mathbb{N}}$ of $-\Delta_c + \Delta_c^2$ corresponding to eigenvalues $0 < \mu_1 \leq \mu_2 \leq \cdots \to \infty$ such that $\{v_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of $L^2(D)$. One can define, for $s > 0$, the following inner product for functions from $C_c^\infty(D)$:

$$\langle f, g \rangle_{s, \text{mixed}} := \sum_{j \in \mathbb{N}} \mu_j^{-s/2} \langle f, v_j \rangle_{L^2} \langle v_j, g \rangle_{L^2}.$$ 

Let $\mathcal{H}_0(D)$ be the completion of $C_c^\infty(D)$ with the above inner product and $\mathcal{H}_{-\Delta + \Delta^2}^s$ be its dual. The dual norm is denoted by $\| \cdot \|_{s, \text{mixed}}$. We describe the details on this space in Appendix C. The following definition is proved as Proposition B.5 in Appendix C.

**Definition 2.4.** Let $(\xi_j)_{j \in \mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set

$$\Psi_{-\Delta + \Delta^2}^{-s} := \sum_{j \in \mathbb{N}} \mu_j^{-1/2} \xi_j v_j.$$ 

Then $\Psi_{-\Delta + \Delta^2}^{-s} \in \mathcal{H}_{-\Delta + \Delta^2}^s(D)$ a.s. for all $s > (d - 4)/2$ and is called the continuum mixed model.

2.2.4. Gaussian free field. Here also we briefly give the definition of the Sobolev space $H_{-\Delta}^{-s}(D)$ and the Gaussian free field. For a detail discussion see Cipriani et al. (2018a). By the spectral theorem for compact self-adjoint operators and elliptic regularity we know that there exist smooth eigenfunctions $(w_j)_{j \in \mathbb{N}}$ of $-\Delta_c$ corresponding to the eigenvalues $0 < \nu_1 \leq \nu_2 \leq \cdots \to \infty$ such that $(w_j)_{j \geq 1}$ is an orthonormal basis of $L^2(D)$. Now for any $s > 0$ we define the following inner product on $C_c^\infty(D)$:

$$\langle f, g \rangle_{s, \Delta_c} := \sum_{j \in \mathbb{N}} \nu_j^{-s} \langle f, w_j \rangle_{L^2} \langle w_j, g \rangle_{L^2}.$$ 

Then $H_{0}^s(D)$ can be defined as the completion of $C_c^\infty(D)$ with respect to this inner product. We define $H_{-\Delta}^{-s}(D)$ to be its dual and the dual norm is denoted by $\| \cdot \|_{-s, \Delta_c}$. We give the definition of the Gaussian free field in the next Proposition.

**Definition 2.5** (Cipriani et al. (2018a, Proposition 10)). Let $(\xi_j)_{j \in \mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set

$$\Psi_{-\Delta}^{-s} := \sum_{j \in \mathbb{N}} \nu_j^{-1/2} \xi_j w_j.$$ 

Then $\Psi_{-\Delta}^{-s} \in H_{-\Delta}^{-s}(D)$ a.s. for all $s > d/2 - 1$ and is called the Gaussian free field.

**Remark 2.6.** We define different spaces with respect to different eigenfunctions of the operators. It is not clear to us if these spaces coincide for a general domain. One might note that $\mathcal{H}_{0}^2(D) = H_{0}^2(D) = \mathcal{H}_0^2(D)$, with the latter space endowed with the equivalent norm $\| \cdot \|_{\text{mixed}}$. We are not aware of a result which gives the norm equivalence between the spaces $\mathcal{H}_0^s(D)$, $\mathcal{H}_0^s(D)$ and $H_{0}^s(D)$. In this article we are not pursuing this line of research; what is important for us are the specific norms that determine the limiting variance of the discrete fields.

**Remark 2.7.** Note that we have used the same notation for the fields both in higher as well as in lower dimensions, although they do not live in the same spaces. The relation of the fields comes through the Dirichlet problem. For $f \in C_c^\infty(D)$, one can easily show that

$$\mathbb{E}[(\Psi^L, f)^2] = \int_{D^2} G_L(x, y) f(x) f(y) \, dx \, dy$$

where $\Psi^L$ is one of the three fields associated to the elliptic operator $L$ as in (2.3) and $G_L$ is the Green’s function of the Dirichlet problem (2.4).
We are now ready to state our main results in the higher dimensional case.

**Theorem 2.8.** Assume that $D$ has smooth boundary. Depending on the behaviour of $\kappa$ as $N \to \infty$ we have the following three convergence results.

1. $\kappa \gg N^2$. Let $d \geq 4$. Define $\Psi_N$ by

$$\langle \Psi_N, f \rangle := (2d)^{-1} \sqrt{\kappa} N^{-\frac{d+4}{2}} \sum_{x \in \Lambda_N} \varphi_{N,x} f(x), \quad f \in H^{s_d}_0(D). \quad (2.9)$$

Then we have, as $N \to \infty$, that the field $\Psi_N$ converges in distribution to the continuum membrane model $\Psi_\Delta^2$ in the topology of $H^{-s}_\Delta(D)$ for $s > s_d$, where

$$s_d := \frac{d}{2} + 2 \left( \left\lfloor \frac{d}{2} \right\rfloor + 1 \right) + \left\lfloor \left\lfloor \frac{d}{2} \right\rfloor + 6 \right\rfloor - 1. \quad (2.10)$$

2. $\kappa \sim 2dN^2$. Let $d \geq 2$. Define $\Psi_N$ by

$$\langle \Psi_N, f \rangle := (2d)^{-\frac{1}{2}} N^{-\frac{d+2}{2}} \sum_{x \in \Lambda_N} \varphi_{N,x} f(x), \quad f \in H^{s_d}_0(D). \quad (2.11)$$

Then, as $N \to \infty$, the field $\Psi_N$ converges in distribution to $\Psi^\Delta_{-\Delta + \Delta^2}$ in the topology of $H^{-s}_\Delta(D)$ for $s > s_d$ where $s_d$ is as in (2.10).

3. $\kappa \ll N^2$. Let $d \geq 2$. Define $\Psi_N$ by

$$\langle \Psi_N, f \rangle := (2d)^{-\frac{1}{2}} N^{-\frac{d+2}{2}} \sum_{x \in \Lambda_N} \varphi_{N,x} f(x), \quad f \in H^{s_d}_0(D). \quad (2.12)$$

Then, as $N \to \infty$, the field $\Psi_N$ converges in distribution to the zero boundary Gaussian free field $\Psi^\Delta$ in the topology of $H^{-s}_\Delta(D)$ for $s > d/2 + \lfloor d/2 \rfloor + 3/2$.

2.3. **Main ingredients in the proofs.** We prove both Theorem 2.1 and Theorem 2.8 by first showing finite dimensional convergence and secondly tightness. As the measures are Gaussian with mean zero, the finite dimensional convergence follows from the convergence of the covariance. However the behaviour of the covariance of the model is not known explicitly. Therefore we use the expedient of boundary value problems to achieve both goals. The key fact which allows us to employ PDE techniques is that the covariance satisfies the discrete boundary value problem (2.2). For the proof of our main theorems we will compute in Theorem 2.9 the magnitude of the error one commits in approximating the solution of the Dirichlet problem (2.4) by its discrete counterpart. We follow the idea given in Thomée (1964) to deal with a general class of elliptic operators, but we restrict our discussion to three particular operators of our interest. In the present section we only state the error estimate leaving the proof for Section 5. Let $V$ be any bounded domain in $\mathbb{R}^d$ satisfying the uniform exterior ball condition (UEBC), which states that there exists $\delta > 0$ such that for any $z \in \partial V$ there is a ball $B_\delta(c)$ of radius $\delta$ with center at some point $c$ satisfying $B_\delta(c) \cap V = \{z\}$. We mention here that any domain with $C^2$ boundary satisfies the UEBC.

Let $h > 0$. We will call the points in $h\mathbb{Z}^d$ the grid points in $\mathbb{R}^d$. We consider $L_h$ to be a discrete approximation of $L$ given by

$$L_hu = \begin{cases} 
(-\Delta_h + \rho_1(h)\Delta^2_h)u & \text{if } L = -\Delta_c \\
(-\rho_2(h)\Delta_h + \Delta^2_h)u & \text{if } L = \Delta^2_c \\
(-\Delta_h + \rho_3(h)\Delta^2_h)u & \text{if } L = -\Delta_c + \Delta^2_c 
\end{cases} \quad (2.13)$$
where $\Delta_h$ is defined by
\[
\Delta_h u(x) := \frac{1}{h^2} \sum_{i=1}^{d} (u(x + he_i) + u(x - he_i) - 2u(x)),
\]
u is any function on $h\mathbb{Z}^d$ (called a grid function) and $\rho_i(h)$ are functions of $h$ taking values in the positive real line such that
\[
\lim_{h \to 0} \rho_i(h) = \begin{cases} 
0 & i = 1, 2; \\
1 & i = 3.
\end{cases}
\]
Let $V_h$ be the set of grid points in $\overline{V}$ i.e. $V_h = \overline{V} \cap h\mathbb{Z}^d$. For any grid point $x$ we define the points $x \pm he_i, \, x \pm h(e_i \pm e_j)$ with $1 \leq i, \, j \leq d$ to be its neighbors. We say that $x$ is an interior grid point in $V_h$ if all its neighbors are in $V_h$. Let $R_h$ be the set of interior grid points in $V_h$ and $B_h := V_h \setminus R_h$ be the set of grid points near the boundary. We divide $R_h$ further into $R_h^*$ and $B_h^*$, where $R_h^*$ is the set of $x$ in $R_h$ such that all its neighbors are in $R_h$ and $B_h^*$ is the set of remaining points in $R_h$. Thus we have
\[
V_h = B_h \cup R_h = B_h \cup B_h^* \cup R_h^*.
\]
Denote by $D_h$ the set of grid functions vanishing outside $R_h$. For a grid function $f$ we define $R_h f \in D_h$ by
\[
R_h f(x) = \begin{cases} f(x) & x \in R_h \\
0 & x \notin R_h.
\end{cases}
\]
Define for grid-functions vanishing outside a finite set
\[
\langle u, \, v \rangle_{h, \text{grid}} := h^d \sum_{x \in h\mathbb{Z}^d} u(x)v(x),
\]
\[
\|u\|_{h, \text{grid}} := \langle u, \, u \rangle_{h, \text{grid}}^{1/2}.
\]
We now define the finite difference analogue of the Dirichlet problem (2.4). For given $h$, we look for a function $u_h(\cdot)$ defined on $V_h$ such that
\[
L_h u_h(x) = f(x), \quad x \in R_h
\]
and
\[
u_h(x) = 0, \quad x \in B_h.
\]
The uniqueness of the solution of (2.15) and (2.16) is shown in Lemma 5.5. We are now ready to state the error estimate result which forms the core result of this article.

**Theorem 2.9.** Depending on $L$ we have the following error bounds.

1. $L = \Delta_h^2$. Let $u \in C^5(\overline{V})$ be the solution of the Dirichlet problem (2.4). If $e_h := u - u_h$ then we have for all sufficiently small $h$
\[
\|R_h e_h\|^2_{h, \text{grid}} \leq C \left[ M_0^2 h^2 + M_2^2 (\rho_2(h))^2 + M_2^2 h \right].
\]

2. $L = -\Delta_c + \Delta_h^2$. Let $u \in C^5(\overline{V})$ be the solution of the Dirichlet problem (2.4). If $e_h := u - u_h$ then we have for all sufficiently small $h$
\[
\|R_h e_h\|^2_{h, \text{grid}} \leq C \left[ M_0^2 h^2 + M_2^2 (\rho_3(h) - 1)^2 + M_2^2 h + M_2^2 h^4 \right].
\]

3. $L = -\Delta_c$. Suppose $\rho_1(h) \ll h^{3/2}$. Let $u \in C^4(\overline{V})$ be a solution of the Dirichlet problem (2.4). If $e_h := u - u_h$ then for sufficiently small $h$ we have
\[
\|R_h e_h\|^2_{h, \text{grid}} \leq C \left[ M_0^2 h^3 + M_2^2 h + M_1^2 h^{-3}(\rho_1(h))^2 \right].
\]
In all the cases $M_k := \sum_{|\alpha| \leq k} \sup_{x \in V} |D^\alpha u(x)|$.

**Remark 2.10.** Observe that there is an extra condition on $\rho_1(h)$ in Theorem 2.9 (3). When proving this result in our setting we shall take $\rho_1(h) = \kappa h^2/(2d)$ with $h = 1/N$; this will give the restriction $\kappa \ll N^{1/2}$, instead of the conjectured $\kappa \ll N^2$. As remarked at the beginning of Section 5, the operator $-\Delta_c$ is defined through a Dirichlet problem with one boundary condition, whereas in the discrete Dirichlet problem involving $L_h$ two boundary conditions are needed. The contribution of $\Delta^2_h$ is negligible in the limit but in the finite setting it is not. The truncated discrete operator $L_{h,1}$ which we also introduce (see (5.11)) is rescaled around the boundary points adding to this technical issue. Our present method does not circumvent this obstacle, still we expect the results to be true for $\kappa \ll N^2$.

### 2.4. Open problems and discussions.

In this subsection we list some open problems.

1. As it is clear from Theorem 2.1 and Theorem 2.8, we do not cover the whole range of behaviours for $\kappa(N)$, that is, the case $N^{1/2} \ll \kappa \ll N^2$ remains open. We conjecture that then the limit should be the Gaussian free field. With free boundary conditions, in $d = 1$, we have briefly argued in the Introduction that the transition point should be $N^2$. However in our approach boundary issues cannot be circumvented. The main obstacle comes from the fact that the discrete operator

$$-\Delta^1_N + \frac{\kappa}{2dN^2} \Delta^2_N$$

has interactions up to distance two from the boundary, but the limiting operator is $-\Delta_c$ which is defined up to the boundary. This discrepancy comes up when we consider the truncation of the operator on a thin layer surrounding the boundary. Improvements in Theorem (3) will be helpful. In $d = 1$ we performed some simulations (a histogram is presented in Figure 3) in which the pointwise variance of our process matches with the variance of the Brownian bridge.

![Figure 3](image_url)

**Figure 3.** This is an histogram of the interpolated field in $d = 1$ at the point $1/2$ with $N = 100$, $\kappa = 2 \times 100^{3/2}$ and 1000 iterations. The variance is approximately 0.2485 (vs. theoretical value of 0.25, the variance of the Brownian bridge at the point 1/2).

2. Let $\varepsilon \geq 0$ and consider the following pinned measure on $\mathbb{R}^{V_N}$, with $V_N$ being a box of side length $N$:

$$P_{\varepsilon,N} = \frac{1}{Z_{\varepsilon,N}} e^{-H(\phi)} \prod_{x \in V_N} (\varepsilon \delta_0(d\phi_x) + d\phi_x) \prod_{x \in \mathbb{Z}^d \setminus V_N} \delta_0(d\phi_x)$$
Here $H(\phi)$ is as in (2.1). Let $F(\varepsilon)$ be the free energy of the above system, namely,

$$F(\varepsilon) = \lim_{N \to \infty} \frac{1}{N} \log \frac{Z_{\varepsilon,N}}{Z_{0,N}}.$$ 

If $F(\varepsilon) > 0$ then the above pinned measure is said to be localized, otherwise it is delocalized. We call $\varepsilon_c$ the supremum of all delocalized $\varepsilon$. It would be interesting to see if the above model with $\kappa_1$ and $\kappa_2$ depending on $N$ shows a phase transition with respect to localization. The case when $\kappa_1$ and $\kappa_2$ do not depend on $N$ was studied in Borecki and Caravenna (2010). The case of $\kappa_1 = 0$ and $d = 1$ was extensively studied in the literature, see Caravenna and Deuschel (2008, 2009).

(3) Extremes of interface models are also to be investigated. From Theorem 2.1 it follows that the maximum of the $(\nabla + \Delta)$–model with varying coefficients converges after appropriate rescaling to the supremum of a Gaussian process. We summarise the cases in which we are able to identify the limiting rescaled maximum:

- $\kappa \ll N^{1/2}$ and $d = 1$;
- $\kappa \sim 2dN^2$ and $d = 1$;
- $\kappa \gg N^2$ and $d = 1, 2, 3$;

All the remaining cases are not known yet and it would be interesting to see if the existing methods can be pushed to cover other dimensions. The challenge in this problem arises because the behaviour of the Green’s function is hard to determine. A similar situation was recently handled by Schweiger (2019) to determine the extremes of the two-dimensional membrane model. He found out estimates for the Green’s function and applied the methods of Ding et al. (2017) to show that the limit of the maximum is a shifted Gumbel distribution.

### 3. Proof of Theorem 2.8

We now give the proof of each of the three parts of Theorem 2.8.

#### 3.1. Proof of finite dimensional convergence.

We first show that for $f \in C_c^\infty(D)$

$$\Phi_N(x,y) \overset{d}{\to} \begin{cases} (\Phi^2, f) & \kappa \gg N^2 \\ (\Phi^{\Delta}, f) & \kappa \sim 2dN^2 \\ (\Phi, f) & \kappa \ll N^{1/2} \end{cases} \tag{3.1}$$

We begin by noting that $(\Phi_N, f)$ is a centered Gaussian random variable. Hence to show the above convergence it is enough to show that $\text{Var}(\Phi_N, f)$ converges to the variance of the Gaussian on the right hand side of (3.1). We denote $G_{\frac{1}{N}}(x,y) := E_{\Lambda_N} [\varphi_{nx} \varphi_{ny}]$. Note that by (2.2), we have for all $x \in \frac{1}{N} \Lambda_N$,

$$\begin{align*}
\kappa \gg N^2 : & \quad \left\{ \begin{array}{ll}
\left( -\frac{2dN^2}{\kappa} \Delta \frac{1}{N} + \Delta^2 \frac{1}{N} \right) G_{\frac{1}{N}}(x,y) = \frac{4d^2 N^4}{\kappa} \delta_x(y), & y \in \frac{1}{N} \Lambda_N \\
G_{\frac{1}{N}}(x,y) = 0 & y \notin \frac{1}{N} \Lambda_N.
\end{array} \right. \\
\kappa \sim 2dN^2 : & \quad \left\{ \begin{array}{ll}
\left( -\Delta \frac{1}{N} + \frac{\kappa}{2dN^2} \Delta^2 \frac{1}{N} \right) G_{\frac{1}{N}}(x,y) = 2dN^2 \delta_x(y), & y \in \frac{1}{N} \Lambda_N \\
G_{\frac{1}{N}}(x,y) = 0 & y \notin \frac{1}{N} \Lambda_N.
\end{array} \right. \\
\kappa \ll N^{1/2} : & \quad \left\{ \begin{array}{ll}
\left( -\Delta \frac{1}{N} + \frac{\kappa}{2dN^2} \Delta^2 \frac{1}{N} \right) G_{\frac{1}{N}}(x,y) = 2dN^2 \delta_x(y), & y \in \frac{1}{N} \Lambda_N \\
G_{\frac{1}{N}}(x,y) = 0 & y \notin \frac{1}{N} \Lambda_N.
\end{array} \right.
\end{align*} \tag{3.2-3.4}$$
Now considering all the three cases we can rewrite the variance as

\[ \text{Var}[(\Psi_N, f)] = N^{-d} \sum_{x \in \frac{1}{N} \Lambda_N} H_N(x)f(x) \]

where for \( x \in \frac{1}{N} D_N \),

\[ H_N(x) = \begin{cases} (2d)^{-2} \kappa N^{-4} \sum_{y \in \frac{1}{N} \Lambda_N} G_{\frac{1}{N}}(x, y)f(y) & \kappa \gg N^2 \\ (2d)^{-1} N^{-2} \sum_{y \in \frac{1}{N} \Lambda_N} G_{\frac{1}{N}}(x, y)f(y) & \kappa \sim 2dN^2 \\ (2d)^{-1} N^{-2} \sum_{y \in \frac{1}{N} \Lambda_N} G_{\frac{1}{N}}(x, y)f(y) & \kappa \ll N^{1/2} \end{cases} \]

It is immediate from (3.2), (3.3), (3.4) that \( H_N \) is the solution of the following Dirichlet problem:

\[ \begin{cases} \left( \frac{-2dN^2}{\kappa} \Delta_{\frac{1}{N}} + \Delta_{\frac{2}{N}} \right) H_N(x) = f(x), & x \in \frac{1}{N} \Lambda_N \\ H_N(x) = 0, & x \notin \frac{1}{N} \Lambda_N. \end{cases} \]  

(3.5)

\[ \begin{cases} \left( -\frac{\Delta}{N} + \frac{\kappa}{2dN^2} \Delta_{\frac{2}{N}} \right) H_N(x) = f(x), & x \in \frac{1}{N} \Lambda_N \\ H_N(x) = 0, & x \notin \frac{1}{N} \Lambda_N. \end{cases} \]  

(3.6)

\[ \begin{cases} \left( -\frac{\Delta}{N} + \frac{\kappa}{2dN^2} \Delta_{\frac{2}{N}} \right) H_N(x) = f(x), & x \in \frac{1}{N} \Lambda_N \\ H_N(x) = 0, & x \notin \frac{1}{N} \Lambda_N. \end{cases} \]  

(3.7)

Observe that we get the discrete Dirichlet problem involving the operator \( L_h \) defined in (2.13) with \( h = 1/N \) and

\[ \rho_1(h) := \kappa h^2/2d, \quad \rho_2(h) := 2d/\kappa h^2, \quad \rho_3(h) := \kappa h^2/2d. \]

We now consider the following continuum Dirichlet problem with the elliptic operator \( L \) as in (2.3):

\[ \begin{cases} Lu(x) = f(x) & x \in D \\ D^\alpha u(x) = 0 & |\alpha| \leq m - 1, x \in \partial D. \end{cases} \]  

(3.8)

where \( m = 1 \) if \( L = -\Delta \) and \( m = 2 \) in the other two cases. We set \( L := \Delta^2 \) when \( \kappa \gg N^2 \), \( L := -\Delta \) when \( \kappa \ll N^{1/2} \) and \( L := -\Delta + \Delta^2_\kappa \) when \( \kappa \sim 2dN^2 \). Define \( e_N(x) = H_N(x) - u(x) \) for \( x \in \frac{1}{N} D_N \).

Then from Theorem 2.9 we have

\[ N^{-d} \sum_{x \in \frac{1}{N} \Lambda_N} e_N(x)^2 \leq \begin{cases} C \left( \frac{1}{N^2} + \frac{4d^2N^4}{\kappa^2} + \frac{1}{N} \right) & \kappa \gg N^2 \\ C \left( \frac{1}{N^2} + \left( \frac{2dN^2}{\kappa} - 1 \right)^2 \right) & \kappa \sim 2dN^2. \\ C \left( \frac{1}{N^2} + \frac{\kappa}{N} + \frac{\kappa^2}{N^2} \right) & \kappa \ll N^{1/2} \end{cases} \]  

(3.9)

Hence we get that

\[ \text{Var}[(\Psi_N, f)] = \sum_{x \in \frac{1}{N} \Lambda_N} e_N(x)f(x)N^{-d} + \sum_{x \in \frac{1}{N} \Lambda_N} u(x)f(x)N^{-d}. \]  

(3.10)

Note that by Cauchy-Schwarz inequality and (3.9) the first term goes to zero as \( N \to \infty \). The second term converges to

\[ \sum_{x \in \frac{1}{N} \Lambda_N} u(x)f(x)N^{-d} \to_{N \to \infty} \int_D u(x)f(x) \, d\, x. \]  

(3.11)
Notice that by integration by parts we have
\[
\int_D u(x)f(x)\,dx = \begin{cases}
\|u\|_{2,\Delta}^2 = \|f\|_{2,\Delta}^2 & L = \Delta^2 \\
\|u\|_{2,\text{mixed}}^2 = \|f\|_{2,\text{mixed}}^2 & L = -\Delta_c + \Delta^2 \\
\|u\|_{1,\Delta_c}^2 = \|f\|_{1,\Delta_c}^2 & L = -\Delta_c 
\end{cases}
\]

On the other hand from the definition it follows that
\[
\Var[(\Psi^2, f)] = \sum_{j \in \mathbb{N}} \lambda_j^{-1} \langle u_j , f \rangle_{L^2}^2 = \|f\|_{-2,\Delta}^2
\]
\[
\Var[(\Psi^2, f)] = \sum_{j \in \mathbb{N}} \mu_j^{-1} \langle v_j , f \rangle_{L^2}^2 = \|f\|_{-2,\text{mixed}}^2
\]
\[
\Var[(\Psi^2, f)] = \sum_{j \in \mathbb{N}} \nu_j^{-1} \langle w_j , f \rangle_{L^2}^2 = \|f\|_{-1,\Delta_c}^2
\]

Consequently we obtain (3.1).

3.2. Tightness. To show tightness we shall need the following bounds on the eigenfunctions \((u_j)_{j \in \mathbb{N}}\), \((v_j)_{j \in \mathbb{N}}\) and \((w_j)_{j \in \mathbb{N}}\) of \(\Delta^2, -\Delta_c + \Delta^2\) and \(-\Delta_c\) respectively. They can obtained from Evans (2002, Chapter 5, Theorem 6 (ii)) and a repeated application of Gazzola et al. (2010, Corollary 2.21).

**Lemma 3.1.**

1. For the eigenfunctions \((u_j)_{j \in \mathbb{N}}\) of \(\Delta^2\) in Problem (2.4) there exists a constant \(C > 0\) (which may change among equations) such that
\[
\sup_{x \in D} |u_j(x)| \leq C \lambda_j^{l_0}, \quad \sum_{|\alpha| \leq 2} \sup_{x \in D} |D^\alpha u_j(x)| \leq C \lambda_j^{l_2}, \quad \sum_{|\alpha| \leq 5} \sup_{x \in D} |D^\alpha u_j(x)| \leq C \lambda_j^{l_5}. \tag{3.12}
\]

2. For the eigenfunctions \((v_j)_{j \in \mathbb{N}}\) of \(-\Delta_c + \Delta^2\) in Problem (2.4), there exists a constant \(C > 0\) (which may change among equations) such that
\[
\sup_{x \in D} |v_j(x)| \leq C \mu_j^{l_0}, \quad \sum_{|\alpha| \leq 2} \sup_{x \in D} |D^\alpha v_j(x)| \leq C \mu_j^{l_2}, \quad \sum_{|\alpha| \leq 5} \sup_{x \in D} |D^\alpha v_j(x)| \leq C \mu_j^{l_5}. \tag{3.13}
\]

3. For the eigenfunctions \((w_j)_{j \in \mathbb{N}}\) of \(-\Delta_c\) in Problem (2.4), there exists a constant \(C > 0\) (which may change among equations) such that
\[
\sup_{x \in D} |w_j(x)| \leq C \nu_j^{2l_0}, \quad \sum_{|\alpha| \leq 1} \sup_{x \in D} |D^\alpha w_j(x)| \leq C \nu_j^{2l_1}, \quad \sum_{|\alpha| \leq 3} \sup_{x \in D} |D^\alpha w_j(x)| \leq C \nu_j^{2l_3}. \tag{3.14}
\]

where
\[
l_k := \left[ \frac{1}{4} \left( \left\lfloor \frac{d}{2} \right\rfloor + k + 1 \right) \right], \quad k = 0, \ldots, 5.
\]

We can now begin to show tightness.

**Case 1: \(\kappa \gg N^2\).** Our target is to show that the sequence \((\Psi_N)_{N \in \mathbb{N}}\) is tight in \(H^{-\delta}(D)\) for all \(s > s_d\). It is enough to show that
\[
\limsup_{N \to \infty} \mathbf{E}_{\Lambda_N} \|\Psi_N\|_{-s,\Delta}^2 < \infty \quad \forall \ s > s_d. \tag{3.15}
\]
The tightness of \((\Psi_N)_{N \in \mathbb{N}}\) would then follow immediately from (3.15) and the fact that, for \(0 \leq s_1 < s_2\), \(\mathcal{H}^{s_1}_{\Delta} (D)\) is compactly embedded in \(\mathcal{H}^{s_2}_{\Delta} (D)\) (for a proof of this fact see Cipriani et al. (2018b, Theorem 3.15)).

From the definition of dual norm it is immediate that we have
\[
E_{\Lambda_N} \left[ \|\Psi_N\|_{-s, \Delta_2}^2 \right] \leq \sum_{j \in \mathbb{N}} \lambda_j^{-s/2} E_{\Lambda_N} \left[ (\Psi_N, u_j)^2 \right].
\]

Note that \(u = \lambda_j^{-1} u_j\) is the unique solution of (2.4) with \(L = \Delta_2^2\) for \(f := u_j\). Define \(e_{N,j}\) to be the error between the solution of the discrete Dirichlet problem (3.5) and the continuum one (3.8) with input datum \(f := u_j\). Now as in (3.10) we have
\[
E_{\Lambda_N} \left[ (\Psi_N, u_j)^2 \right] = \sum_{x \in \frac{1}{N} \Lambda_N} e_{N,j}(x) u_j(x) N^{-d} + \sum_{x \in \frac{1}{N} \Lambda_N} \lambda_j^{-1} u_j(x) u_j(x) N^{-d} \leq C \sup_{x \in D} |u_j(x)| \left( N^{-d} \sum_{x \in \frac{1}{N} \Lambda_N} e_{N,j}(x)^2 \right)^{1/2} + C \lambda_j^{-1} \left( \sup_{x \in D} |u_j(x)| \right)^2. \tag{3.16}
\]

Using Theorem 2.9 (1) along with the bounds (3.12) we obtain
\[
E_{\Lambda_N} \left[ (\Psi_N, u_j)^2 \right] \leq C \lambda_j^{l_0} \left[ \lambda_j^{2l_2-2} N^{-2} + \lambda_j^{2l_2-2} d^2 N^4 \kappa^{-2} + \lambda_j^{2l_2-2} N^{-1} \right]^{1/2} + C \lambda_j^{2l_0-1} \leq C \lambda_j^{l_0+l_5-1}.
\]

Therefore we have
\[
E_{\Lambda_N} \left[ \|\Psi_N\|_{-s, \Delta_2}^2 \right] \leq C \sum_{j \in \mathbb{N}} \lambda_j^{-\frac{s}{2}} \lambda_j^{l_0+l_5-1}.
\]

Thus
\[
\limsup_{N \to \infty} E_{\Lambda_N} \left[ \|\Psi_N\|_{-s, \Delta_2}^2 \right] < \infty \quad \text{if} \quad \sum_{j \in \mathbb{N}} \lambda_j^{-\frac{s}{2}+l_0+l_5-1} < \infty.
\]

Now using \(\lambda_j \sim c(d) j^{4/d}\) (see Proposition 3.8 of Cipriani et al. (2018b)) we obtain that \(\sum_{j \in \mathbb{N}} \lambda_j^{-\frac{s}{2}+l_0+l_5-1}\) is finite whenever \(s > s_d\). Thus we have proved (3.15).

**Case 2:** \(\kappa \sim 2 d N^2\). Due to the compact embedding of the spaces \(\mathcal{H}^{s}_{-\Delta_+ \Delta_2} (D)\), to show that the sequence \((\Psi_N)_{N \in \mathbb{N}}\) is tight in \(\mathcal{H}^{s}_{-\Delta_+ \Delta_2} (D)\) for all \(s > s_d\), it is enough to show that
\[
\limsup_{N \to \infty} E_{\Lambda_N} \left[ \|\Psi_N\|_{-s, \text{mixed}}^2 \right] < \infty \quad \forall \ s > s_d. \tag{3.17}
\]

As in the previous case, by definition of dual norm we have
\[
E_{\Lambda_N} \left[ \|\psi_N\|_{-s, \text{mixed}}^2 \right] \leq \sum_{j \in \mathbb{N}} \mu_j^{-s/2} E_{\Lambda_N} \left[ (\psi_N, v_j)^2 \right].
\]

Note that \(u = \mu_j^{-1} v_j\) is the unique solution of (2.4) with \(L = -\Delta_c + \Delta_c^2\) for \(f := u_j\). Define \(e_{N,j}\) to be the error between the solution of the discrete Dirichlet problem (3.6) and the continuum one (3.8) with \(f := v_j\). Now as in (3.16) we have
\[
E_{\Lambda_N} \left[ (\Psi_N, v_j)^2 \right] \leq C \sup_{x \in D} |v_j(x)| \left( N^{-d} \sum_{x \in \frac{1}{N} \Lambda_N} e_{N,j}(x)^2 \right)^{1/2} + C \mu_j^{-1} \left( \sup_{x \in D} |v_j(x)| \right)^2.
\]
Using Theorem 2.9 (2) along with the bounds (3.13) we obtain
\[ E_{\Lambda_N}[(\Psi_N, v_j)^2] \leq C \mu_j^{l_0} \left[ \mu_j^{2l_5-2} N^{-2} + \mu_j^{2l_5-2} \left( \frac{\kappa}{2dN^2} - 1 \right)^2 + \mu_j^{2l_5-2} N^{-1} \right]^{1/2} + C \mu_j^{2l_0-1}. \]

Therefore we have
\[ E_{\Lambda_N} \left[ \|\Psi_N\|^2_{s,\text{mixed}} \right] \leq C \sum_{j \in \mathbb{N}} \mu_j^{-\frac{s}{2}} \mu_j^{l_0+l_5-1}. \]

Thus
\[ \limsup_{N \to \infty} E_{\Lambda_N} \left[ \|\Psi_N\|^2_{s,\text{mixed}} \right] < \infty \quad \text{if} \quad \sum_{j \in \mathbb{N}} \mu_j^{-\frac{s}{2}+l_0+l_5-1} < \infty. \]

From Proposition B.4 we obtain that \( \sum_{j \in \mathbb{N}} \mu_j^{-\frac{s}{2}+l_0+l_5-1} < \infty \) whenever \( s > s_d \). Thus we have proved (3.17).

**Case 3:** \( \kappa \ll N \frac{1}{2} \). The arguments are similar to the previous two cases and hence we just indicate the required bounds. To show tightness in \( H^{-s}_\Delta (D) \) it is enough to show
\[ \limsup_{N \to \infty} E_{\Lambda_N} \left[ \|\Psi_N\|^2_{-s,\Delta_c} \right] \leq \sum_{j \in \mathbb{N}} \nu_j^{-s} E_{\Lambda_N} \left[ (\Psi_N, w_j)^2 \right] < \infty \quad \forall s > d/2 + \lfloor d/2 \rfloor + 3/2. \] (3.18)

Setting \( e_{N,j} \) to be the error between the solution of the discrete Dirichlet problem (3.7) and the continuum one (3.8) with \( f := w_j \) we obtain
\[ E_{\Lambda_N} \left[ (\Psi_N, w_j)^2 \right] \leq C \sup_{x \in D} |w_j(x)| \left( N^{-d} \sum_{x \in \Lambda_N} e_{N,j}(x)^2 \right)^{1/2} + C \nu_j^{-1} \left( \sup_{x \in D} |w_j(x)| \right)^2. \]

Using Theorem 2.9 (3) along with the bounds (3.14) we can conclude the following upper bound for \( E_{\Lambda_N} \left[ (\Psi_N, w_j)^2 \right] \):
\[ C \sup_{x \in D} |w_j(x)| \left[ \left( \nu_j^{-1} \nu_j^{\frac{d}{2} + d} \right)^2 + \left( \nu_j^{-1} \nu_j^{\frac{d}{2} + 2} \right)^2 \right] \frac{1}{2} + C \nu_j^{-1} \left( \sup_{x \in D} |w_j(x)| \right)^2. \]

Now a consequence of the above and (3.14) is that
\[ E_{\Lambda_N} \left[ (\Psi_N, w_j)^2 \right] \leq C \nu_j^{\frac{d}{2} + \frac{3}{2}}. \] (3.19)

Therefore we have
\[ E_{\Lambda_N} \left[ \|\Psi_N\|^2_{-s,\Delta_c} \right] \leq C \sum_{j \in \mathbb{N}} \nu_j^{-s} \nu_j^{\frac{d}{2} + \frac{3}{2}}. \]

Thus
\[ \limsup_{N \to \infty} E_{\Lambda_N} \left[ \|\Psi_N\|^2_{-s,\Delta_c} \right] < \infty \quad \text{if} \quad \sum_{j \in \mathbb{N}} \nu_j^{-s+\frac{d}{2}+\frac{3}{2}} < \infty. \]

But \( \nu_j \sim C j^\frac{2}{3} \) and \( \sum_{j \in \mathbb{N}} j^{\frac{2}{3}(-s+\frac{d}{2}+\frac{3}{2})} < \infty \) whenever \( s > d/2 + \lfloor d/2 \rfloor + 3/2 \). Thus we have proved (3.18).

For all the cases we now have the tightness and the convergence of \( (\Psi_N, f) \) for all \( f \in C_c^\infty (D) \). A standard uniqueness argument completes the proof of Theorem 2.8, using the fact that \( C_c^\infty (D) \) is dense in \( H^s_\Delta (D), 3H^s_{-\Delta,\Delta_c} (D) \) and \( H^s_{-\Delta} (D) \) respectively.
4. Proof of Theorem 2.1

In this section we prove Theorem 2.1 by showing finite dimensional convergence and tightness. The proof is similar to the proofs of the lower dimensional results in Cipriani et al. (2018b) and Cipriani et al. (2018a). First we will show tightness and the finite dimensional convergence is similar to the proof of Theorem 2.8. To show tightness we use the following Theorem, whose proof follows from that of Theorem 14.9 of Kallenberg (2006).

**Theorem 4.1.** Let \( X^1, X^2, \ldots \) be continuous processes on \( \mathbb{V} \) with values in a complete separable metric space \( (S, \rho) \). Assume that \( (X^N_0) \) is tight in \( S \) and that for constants \( \alpha, \beta > 0 \)
\[
E[\rho(X^n_s, X^n_t)^\alpha] \leq C \|s - t\|^{\alpha + \beta}, \quad s, t \in \mathbb{V}
\] (4.1)
uniformly in \( n \). Then \( (X^n) \) is tight in \( C(\mathbb{V}, S) \) and for every \( c \in (0, \beta/\alpha) \) the limiting processes are almost surely H"older continuous with exponent \( c \).

We shall elaborate on the membrane case, that is when \( \kappa \gg N^2 \) and the other two cases will be similar and hence we shall indicate only the crucial bounds which are required in the proof of the cases \( \kappa \sim 2dN^2 \) and \( \kappa \ll N^{1/2} \).

4.1. **Case 1:** \( \kappa \gg N^2 \): First we want to show that the sequence \( \{\Psi_N\}_{N \in \mathbb{N}} \) is tight in \( C(D) \). We need the following bounds.

**Lemma 4.2.**

1. For any \( x, y \in \mathbb{Z}^d \)
   \[ |G_{\Lambda_N}(x, y)| \leq C \kappa^{-1} N^{4-d}. \]
2. For \( z \in \mathbb{Z}^d \)
   \[
   E_{\Lambda_N} \left[ (\varphi_{z+e_i} - \varphi_z)^2 \right] \leq \begin{cases} 
   C \kappa^{-1} N & d = 1 \\
   C \kappa^{-1} \log N & d = 2 \\
   C \kappa^{-1} & d = 3
   \end{cases}.
   \]

**Proof.** To show the first inequality we bound \( G_{\Lambda_N}(z, z) \). One can show using Theorem 5.1 of Brascamp and Lieb (1976) that

\[
G_{\Lambda_N}(z, z) \leq \kappa^{-1} E_{\Lambda_N}^{MM}(\varphi_z^2)
\]

where \( P_{\Lambda_N}^{MM} \) denotes the law of the membrane model on \( \Lambda_N \) with zero boundary conditions outside \( \Lambda_N \). The bound for the \( d = 1 \) case can now be obtained using Lemma A.1. For \( d = 2, 3 \) we obtain the bound from Theorem 1.1 of Müller and Schweiger (2019).

For the second part the Brascamp-Lieb inequality yields

\[
E_{\Lambda_N}[(\varphi_{z+e_i} - \varphi_z)^2] \leq \kappa^{-1} E_{\Lambda_N}^{MM}[(\varphi_{z+e_i} - \varphi_z)^2].
\]

The bound now follows from Lemma A.1 (for \( d = 1 \)) and Theorem 1.1 of Müller and Schweiger (2019) (for \( d = 2, 3 \)).

Observe that the process \( \{\Psi_N(t)\}_{t \in \mathbb{V}} \) is Gaussian. Using Lemma 4.2 (1) it is easy to see that \( \{\Psi_N(0)\} \) is tight. Again, using the properties of Gaussian laws, to show (4.1) it is enough to prove the following Lemma.

**Lemma 4.3.** There exists \( C > 0 \) such that
\[
E_{\Lambda_N} \left[ |\Psi_N(t) - \Psi_N(s)|^2 \right] \leq C \|t - s\|^{1+b}
\] (4.2)
for all \( t, s \in \mathbb{V} \), uniformly in \( N \), where \( b = 1 \) in \( d = 1 \), \( b \in (0, 1) \) in \( d = 2 \) and \( b = 0 \) in \( d = 3 \).

**Proof of Lemma 4.3.** First we consider \( d = 1 \). To show (4.2) we consider the following two cases.
(I) Suppose \( t, s \in [x, x + 1/N] \) for some \( x \in N^{-1}D_N \). Then we have
\[
\Psi_N(t) - \Psi_N(s) = (2d)^{-1} \sqrt{K} N^{-3/2} \left[(Nt - Ns)(\varphi_{N\{x+1\}} - \varphi_{N\{x\}})\right].
\]

Now using Lemma 4.2 (2) we get (4.2).

(II) Next suppose \( s \in [x, x + 1/N] \) and \( t \in [y, y + 1/N] \) for some \( x, y \in N^{-1}D_N \) and \( t > x + 1/N \). In this case if \( |t - s| \leq 1/N \) then one can obtain (4.2) using (I) and a suitable point in between. So we assume \( |t - s| > 1/N \). We first note that
\[
E_{\Lambda_N} \left[|\Psi_N(y) - \Psi_N(x)|^2\right] = (2d)^{-2} K N^{-3} E_{\Lambda_N} [(\varphi_{Ny} - \varphi_{Nx})^2] \\
\leq C K N^{-3} |\varphi_{Ny} - \varphi_{Nx}|^2 \\
\leq C (y - x)^2,
\]

where we have used Lemma A.1 to get the last inequality. Now using (I) we obtain
\[
E_{\Lambda_N} \left[|\Psi_N(t) - \Psi_N(s)|^2\right] \leq C \left(E_{\Lambda_N} \left[|\Psi_N(t) - \Psi_N(y)|^2\right] + E_{\Lambda_N} \left[|\Psi_N(y) - \Psi_N(x)|^2\right] + E_{\Lambda_N} \left[|\Psi_N(x) - \Psi_N(s)|^2\right]\right) \leq C|t - s|^2.
\]

Next we consider \( d = 2 \). We fix \( b \in (0, 1) \) and let \( t, s \in T \). We split the proof into a few cases.

Case 1: Suppose \( t, s \) belong to the same smallest square box in the lattice \( \frac{1}{N} \mathbb{Z}^2 \). First assume \( |Nt| = |Ns| \), that is, the points are in the interior and not touching the top and right boundaries. In this case if we have \( \{Nt_1\} \geq \{Nt_2\} \) and \( \{Ns_1\} \geq \{Ns_2\} \). Then by definition of the interpolation we have
\[
\Psi_N(t) - \Psi_N(s) = (2d)^{-1} \sqrt{K} \left[(t_1 - s_1) (\varphi_{N\{t_1\}+e_1} - \varphi_{N\{t_1\}}) + (t_2 - s_2) (\varphi_{N\{t_2\}+e_2} - \varphi_{N\{t_2\}+e_2})\right].
\]

So from the above expression we have
\[
E_{\Lambda_N} \left[|\Psi_N(t) - \Psi_N(s)|^2\right] \leq 2(2d)^{-2} K \left[(t_1 - s_1)^2 E_{\Lambda_N} [(\varphi_{N\{t_1\}+e_1} - \varphi_{N\{t_1\}})^2] + (t_2 - s_2)^2 E_{\Lambda_N} [(\varphi_{N\{t_2\}+e_2} - \varphi_{N\{t_2\}+e_2})^2]\right].
\]

Now from Lemma 4.2 (2) and \( |t_1 - s_1|, |t_2 - s_2| < N^{-1} \) we obtain (4.2). The argument is similar if one has \( \{Nt_1\} \leq \{Nt_2\} \) and \( \{Ns_1\} \leq \{Ns_2\} \).

Again if \( \{Nt_1\} \geq \{Nt_2\} \) and \( \{Ns_1\} < \{Ns_2\} \), or if \( \{Nt_1\} < \{Nt_2\} \) and \( \{Ns_1\} \geq \{Ns_2\} \) then we consider the point \( u \) on the line segment joining \( t \) and \( s \) such that \( Nu \) is the point of intersection of the line segment joining \( Nt, Ns \) and the diagonal joining \( \lfloor Nt \rfloor, \lfloor Nt \rfloor + e_1 + e_2 \). Then we have using the above computations
\[
E_{\Lambda_N} \left[|\Psi_N(t) - \Psi_N(s)|^2\right] \leq 2E_{\Lambda_N} \left[|\Psi_N(t) - \Psi_N(u)|^2\right] + 2E_{\Lambda_N} \left[|\Psi_N(u) - \Psi_N(s)|^2\right] \\
\leq C \left[\|t - u\|^{1+b} + \|u - s\|^{1+b}\right] \leq C\|t - s\|^{1+b}.
\]

Now the other case, that is, when \( \lfloor Nt \rfloor \neq \lfloor Ns \rfloor \), follows from above by continuity.

Case 2: Suppose \( t, s \) do not belong to the same smallest square box in the lattice \( \frac{1}{N} \mathbb{Z}^2 \). In this case if \( \|t - s\| \leq 1/N \) then one can obtain (4.2) by Case 1 and a suitable point in between. So we assume \( \|t - s\| > 1/N \). Depending on whether \( Nt \) and \( Ns \) belong to the discrete lattice we split the proof in two broad cases.
Suppose \( x, y \in \frac{1}{N} \mathbb{Z}^2 \). Then using Brascamp-Lieb inequality we obtain

\[
E_{\Lambda_N} \left[ |\Psi_N(t) - \Psi_N(s)|^2 \right] \leq \kappa^{-1} E_{\Lambda_N}^{MM} \left[ |\Psi_N(t) - \Psi_N(s)|^2 \right].
\]

Now similarly as in the proof of Lemma 2.6 of Cipriani et al. (2018b) we obtain

\[
E_{\Lambda_N} \left[ |\Psi_N(t) - \Psi_N(s)|^2 \right] \leq C \|t - s\|^{1+b}.
\]

Sub-case 2 (b) Suppose at least one between \( t, s \) does not belong to \( \frac{1}{N} \mathbb{Z}^2 \). Then

\[
E_{\Lambda_N} \left[ |\Psi_N(t) - \Psi_N(s)|^2 \right] \leq 3E_{\Lambda_N} \left[ |\Psi_N(t) - \Psi_N\left( \frac{|Nt|}{N} \right)|^2 \right]
+ 3E_{\Lambda_N} \left[ |\Psi_N\left( \frac{|Nt|}{N} \right) - \Psi_N\left( \frac{|Ns|}{N} \right)|^2 \right]
+ 3E_{\Lambda_N} \left[ |\Psi_N\left( \frac{|Ns|}{N} \right) - \Psi_N(s)|^2 \right]
\]

\[
\leq C \left[ \|t - \frac{|Nt|}{N}\|^{1+b} + \left\| \frac{|Nt|}{N} - \frac{|Ns|}{N} \right\|^{1+b} + \left\| \frac{|Ns|}{N} - s \right\|^{1+b} \right] \leq C \|t - s\|^{1+b}.
\]

Note that for the last inequality we have used our assumption \( \|t - s\| > 1/N \).

Finally we consider \( d = 3 \). Let \( t, s \in \mathcal{D} \). We split the proof into cases similar to those of \( d = 2 \). We give a brief description. For Case 1, suppose \( t, s \) belong to the same smallest cube in the lattice \( \frac{1}{N} \mathbb{Z}^3 \). First assume \([Nt] = [Ns]\). In this case if \(|Nt_1| \geq |Nt_2| \geq |Nt_3|\) and \(|Ns_1| \geq |Ns_2| \geq |Ns_3|\) then it follows from the definition of interpolation

\[
E_{\Lambda_N} \left[ (\Psi_N(t) - \Psi_N(s))^2 \right] \leq 3(2d)^{-2} \kappa N[(t_1 - s_1)^2 E_{\Lambda_N} \left[ (\varphi_{\lfloor Nt_1 \rfloor}^2 - \varphi_{\lfloor Nt_1 \rfloor})^2 \right]
+ (t_2 - s_2)^2 E_{\Lambda_N} \left[ (\varphi_{\lfloor Nt_2 \rfloor}^2 + e_1 - \varphi_{\lfloor Nt_2 \rfloor})^2 \right]
+ (t_3 - s_3)^2 E_{\Lambda_N} \left[ (\varphi_{\lfloor Nt_3 \rfloor}^2 + e_1 + e_2 - \varphi_{\lfloor Nt_3 \rfloor})^2 \right]].
\]

Now from Lemma 4.2 (2) and the fact that \(|t_1 - s_1|, |t_2 - s_2|, |t_3 - s_3| < 1/N\) we have (4.2). Note that this is a particular case of \( t, s \) lying in the same tetrahedral portion of the cube. Hence if \( t, s \) lie in the same tetrahedral portion of the cube then by similar arguments (4.2) holds. If \( t, s \) do not lie in the same tetrahedral part then we consider points (at most 3) on the line segment joining them such that two consecutive between \( t \), the selected points and \( s \) lie in the same tetrahedral part. Then applying the previous argument we can obtain (4.2). The case when \([Nt] \neq [Ns]\) follows by continuity. For Case 2, we describe Sub-case 2(a) which turns out to be simpler in \( d = 3 \). The rest of the argument is similar to that in \( d = 2 \). Suppose \( t, s \in \frac{1}{N} \mathbb{Z}^3 \) with \( \|t - s\| > 1/N \). Then using Brascamp-Lieb inequality we obtain

\[
E_{\Lambda_N} \left[ |\Psi_N(t) - \Psi_N(s)|^2 \right] \leq \kappa^{-1} E_{\Lambda_N}^{MM} \left[ |\Psi_N(t) - \Psi_N(s)|^2 \right].
\]

Now similarly as in the proof of Lemma 2.6 of Cipriani et al. (2018b) we obtain

\[
E_{\Lambda_N} \left[ |\Psi_N(t) - \Psi_N(s)|^2 \right] \leq C \|t - s\|.
\]

To conclude the finite dimensional convergence we first show the convergence of the covariance matrix. For \( x, y \in \mathcal{D} \cap N^{-1} \mathbb{Z}^d \) we define

\[
G_{\frac{1}{N}}(x, y) := (2d)^{-2} \kappa N^{d-4} G_{\Lambda_N}(Nx, Ny).
\]
We now interpolate $G^I_N$ in a piece-wise constant fashion on small squares of $\mathcal{D} \times \mathcal{D}$ to get a new function $G^I_N$. We show that $G^I_N$ converges uniformly to $G_D$ on $\mathcal{D} \times \mathcal{D}$. Indeed, let $F_N := G^I_N - G_D$. Similarly as in the proof of the finite dimensional convergence in Theorem 2.8 (1) it follows that, for any $f, g \in C^\infty_c(D)$,

$$\lim_{N \to \infty} \sum_{x, y \in \frac{1}{N} \mathcal{D}_N} N^{-2d} G^I_N(x, y) f(x) g(y) = \iint_{D \times D} G_D(x, y) f(x) g(y) \, dx \, dy.$$ 

Again from Riemann sum convergence we have

$$\lim_{N \to \infty} \sum_{x, y \in \frac{1}{N} \mathcal{D}_N} N^{-2d} G_D(x, y) f(x) g(y) = \iint_{D \times D} G_D(x, y) f(x) g(y) \, dx \, dy.$$ 

Thus we get

$$\lim_{N \to \infty} \sum_{x, y \in \frac{1}{N} \mathcal{D}_N} N^{-2d} F_N(x, y) f(x) g(y) = 0. \quad (4.3)$$

Note that $G_D$ is bounded and

$$\sup_{x, y \in \frac{1}{N} \mathcal{D}_N} |G_N(x, N_y)| \leq C \kappa^{-1} N^{4-d}.$$ 

These imply that

$$\sup_{x, y \in \mathcal{D}} |F_N(x, y)| \leq C.$$ 

Thus $F_N$ has a subsequence converging uniformly to some function $F$ which is bounded by $C$. With abuse of notation we denote this subsequence by $F_N$. We then have

$$\lim_{N \to \infty} \sum_{x, y \in \frac{1}{N} \mathcal{D}_N} N^{-2d} F_N(x, y) f(x) g(y) = \iint_{D \times D} F(x, y) f(x) g(y) \, dx \, dy.$$ 

Uniqueness of the limit gives

$$\iint_{D \times D} F(x, y) f(x) g(y) \, dx \, dy = 0$$

by (4.3). From this we obtain that $F(x, y) = 0$ for almost every $x$ and almost every $y$. The definition by interpolation of $G^I_N$ ensures that $F$ is pointwise equal to zero. Finally, the fact that the original sequence $F_N$ converges uniformly to zero follows using the subsequence argument.

We now show the finite dimensional convergence. First let $t \in \mathcal{D}$. We write

$$\Psi_N(t) = \Psi_{N,1}(t) + \Psi_{N,2}(t)$$

where $\Psi_{N,1}(t) := (2d)^{-1} \sqrt{\kappa} N^{-d+4} \phi_N(t)$ and $\Psi_{N,2}(t) := \psi_N(t) - \psi_{N,1}(t)$. From Lemma 4.2 (2) it follows that $\mathbb{E}_N[\Psi_{N,2}(t)^2]$ goes to zero as $N$ tends to infinity. Therefore to show that $\Psi_N(t) \xrightarrow{d} \Psi(t)$ it is enough to show that $\mathbb{V} \text{ar}[\Psi_{N,1}(t)] \to G_D(t, t)$. But we have

$$\mathbb{V} \text{ar}[\Psi_{N,1}(t)] = (2d)^{-2} \kappa N^{d-4} G_{\Psi}([Nt], [Nt]) = G^I_N(t, t) \to G_D(t, t)$$

since the sequence $F_N$ converges to zero uniformly. Since the variables under consideration are Gaussian, one can show the finite dimensional convergence using the convergence of the Green’s functions.
4.2. Case 2-3: $\kappa \sim 2dN^2$, $\kappa \ll N^{1/2}$: In this case also we use Theorem 4.1 to show tightness. Using the Brascamp-Lieb inequality and an argument similar to the proof of Lemma 13 of Cipriani et al. (2018a) we obtain the following bounds in both cases.

**Lemma 4.4.** We have

$$G_{\Lambda_N}(x, x) \leq E_{\Lambda_N}^{GFF}(\varphi^2_x) \leq CN \quad \text{for all } x \in \mathbb{Z}. \quad (4.4)$$

Also there exists $C > 0$ such that for all $x, y \in \mathbb{Z}$

$$E_{\Lambda_N}[(\varphi_x - \varphi_y)^2] \leq E_{\Lambda_N}^{GFF}[(\varphi_x - \varphi_y)^2] \leq C|y - x|. \quad (4.5)$$

where $P_{\Lambda_N}^{GFF}$ denote the law of the discrete Gaussian free field on $\Lambda_N$ with zero boundary conditions outside $\Lambda_N$.

Once we have these bounds, the rest of the proof is similar to that of the one-dimensional result in the $\kappa \gg N^2$ case. In the case of $\kappa \ll N^{1/2}$ we need the following additional information for the identification of the limit. The Green’s function $G_D$ for the problem

$$\begin{cases}
-d^2 u(x) = f(x) & x \in D \\
u(x) = 0 & x \in \partial D
\end{cases}$$

is given by

$$G_D(x, y) = x \land y - xy, \quad x, y \in \overline{D}$$

which also turns out to be the covariance function of the Brownian bridge. To avoid repetitions of the arguments, we skip the details of these cases.

5. Proof of Theorem 2.9

This section is devoted to proof of the error estimation result in Theorem 2.9. To estimate the error we need to develop some Sobolev inequalities in the general setting which involves consistency between discrete and continuous operator. The content of this section can be of independent interest and can possibly be applied to general interface models. We would like to stress that although we follow the ideas involved in Thomée (1964), we cannot quote the results from there verbatim as the coefficients of the discrete operators do not depend on the scaling of the lattice. Also another important remark is that the discrete Dirichlet problem involving the operators $L_h$ introduced in (2.13) requires two boundary conditions but the definition of the limiting operator $-\Delta_c$ involves only one boundary conditions. This in turn will affect the definition of the truncated operator $L_{h,1}$ (see (5.11)) around the boundary points.

5.1. Sobolev-type norm inequalities. The main aim of this Subsection is to have an estimate on the $\ell^2$ norm of a function on the grid in terms of the operator $L_h$ (and its truncated version). Later this turns out to be useful as we use the convergence of $L_h$ to $L$. We continue with all the definitions and notations from Section 2.3.

The notion of discrete forward and backward derivatives will be essential in the following arguments.

$$\partial_j u(x) := \frac{1}{h}(u(x + he_j) - u(x)),$$
$$\bar{\partial}_j u(x) := \frac{1}{h}(u(x) - u(x - he_j)),$$
$$\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d},$$
$$\bar{\partial}^\alpha := \bar{\partial}_1^{\alpha_1} \cdots \bar{\partial}_d^{\alpha_d},$$
where \( \alpha = (\alpha_1, \ldots, \alpha_d) \) is a multi-index. It is easy to see that
\[
\langle \partial_j u, v \rangle_{h, \text{grid}} = \langle u, \bar{\partial}_j v \rangle_{h, \text{grid}}
\]
for grid-functions vanishing outside a finite set. We now define
\[
\|u\|_{h, m} := \left( \sum_{|\alpha| \leq m} \|\partial^\alpha u\|^2_{h, \text{grid}} \right)^{\frac{1}{2}}
\]
and obtain the following Lemma.

**Lemma 5.1.** There are constants \( C = C_j \) independent of \( u \) and \( h \) such that
\[
\|u\|_{h, \text{grid}} \leq C \|\partial_j u\|_{h, \text{grid}}, \quad u \in \mathcal{D}_h, \quad j = 1, \ldots, d,
\]
and for fixed \( m \geq 1 \),
\[
\|u\|_{h, \text{grid}} \leq C \|u\|_{h, m}, \quad u \in \mathcal{D}_h.
\]

**Proof.** Since \( u \equiv 0 \) outside \( R_h \), we have for \( x \in R_h \)
\[
u(x) = -h \sum_{l=0}^{\infty} \partial_j u(x + lhe_j).
\]
As \( V \) is bounded, the number of non-zero terms is \( O(h^{-1}) \). Hence by Cauchy-Schwarz inequality we have
\[
u(x)^2 \leq Ch \sum_{z \in \{x + lhe_j : -\infty < l < \infty\}} (\partial_j u(z))^2.
\]
Summing over \( \{x + lhe_j : -\infty < l < \infty\} \) and using the fact that the number of non-zero terms is \( O(h^{-1}) \) we obtain
\[
\sum_{z \in \{x + lhe_j : -\infty < l < \infty\}} u(z)^2 \leq C \sum_{z \in \{x + lhe_j : -\infty < l < \infty\}} (\partial_j u(z))^2.
\]
Now we obtain (5.1) by summing over the remaining components of \( x \) and multiplying by \( h^d \). Then (5.2) follows by a repeated application of (5.1). \( \square \)

We will need the following norm which rescales the function near the boundary:
\[
|||u|||_{h, m} := \left( h^d \left( \sum_{x \in R_h^*} u(x)^2 + \sum_{x \in B_h^*} (h^{-m}u(x))^2 \right) \right)^{\frac{1}{2}}, \quad u \in \mathcal{D}_h.
\]
We can relate the weighted Sobolev norm \(||| \cdot |||_{h, m}|| \) to \( \| \cdot \|_{h, m} \) with this bound:

**Lemma 5.2.** There is a constant \( C \) independent of \( u \) and \( h \) such that
\[
|||u|||_{h, m} \leq C \|u\|_{h, m}, \quad u \in \mathcal{D}_h.
\]

**Proof.** For this lemma we use the following fact. There is a natural number \( K \) such that for all sufficiently small \( h \), the following is valid: consider for any \( x \in B_h^* \) all half-rays through \( x \). At least one of them contains \( m \) consecutive grid-points outside \( R_h \) within distance \( Kh \) from \( x \). Note that we are interested only in the cases \( m = 1 \) or \( m = 2 \). This fact is easy to observe when \( m = 1 \). For \( m = 2 \) see Cipriani et al. (2018b, Proposition A.2). Let \( x \in B_h^* \). We first consider the case when the half-ray in the \( x_1 \)-direction
contains, within distance $K h$ from $x$, $m$ consecutive grid-points outside $R_h$. Let $x - (K_0 + 1) h e_1$, where $K_0 + m \leq K$, be the first of the $m$ consecutive points. It is then easy to see that

$$h^{-m} u(x) = \sum_{j=0}^{K_0} \binom{m+j-1}{j} \partial_1^m u(x - j h e_1).$$

So

$$(h^{-m} u(x))^2 \leq C (K_0 + 1) \sum_{j=0}^{K_0} (\partial_1^m u(x - j h e_1))^2 = C \sum_{j=0}^{K_0} (\partial_1^m u(x - (j+2) h e_1))^2.$$

Similar inequalities hold in the cases of the other half-rays where in the above $\partial_1$ has to be replaced by the derivative in the direction of the corresponding half-ray. With this observation we obtain

$$h^d \sum_{x \in \mathcal{B}_h^*} (h^{-m} u(x))^2 \leq C \|u\|^2_{h,m}.$$ 

And by definition

$$h^d \sum_{x \in \mathcal{R}_h^*} u(x)^2 \leq \|u\|^2_{h,m}.$$

This completes the proof. □

We rewrite $L_h$ in (2.13) as

$$L_h u(x) = h^{-2m} \sum_{\eta} c_\eta u(x + \eta h),$$

where $\eta = (\eta_1, \ldots, \eta_d)$ with the $\eta_j$’s being integers and the $c_\eta$’s being real numbers which may depend on $h$. We now define the characteristic polynomial of $L_h$ by

$$p(\theta) := \sum_{\eta} c_\eta e^{i \langle \eta, \theta \rangle},$$

where $\theta = (\theta_1, \ldots, \theta_d)$ and $\langle \eta, \theta \rangle = \sum_{j=1}^d \eta_j \theta_j$. We have the following Lemma:

**Lemma 5.3.**

$$\langle L_h u, u \rangle_{h, grid} = h^{d-2m} (2\pi)^{-d} \int_S p(\theta)|\hat{u}(\theta)|^2 d\theta, \quad u \in \mathcal{D}_h.$$ 

where

$$\hat{u}(\theta) = \sum_{\xi \in \mathbb{Z}^d} u(\xi h) e^{-i \langle \xi, \theta \rangle}$$

and $S = \{\theta : |\theta_j| \leq \pi, \ j = 1, \ldots, d\}$.

**Proof.** We expand

$$\langle L_h u, u \rangle_{h, grid} = h^d \sum_{x \in \mathbb{Z}^d} L_h u(x) u(x)$$

$$= h^{d-2m} \sum_{x \in \mathbb{Z}^d} \sum_{\eta \in \mathbb{Z}^d} c_\eta u(x + \eta h) u(x)$$

$$= h^{d-2m} \sum_{x, \xi \in \mathbb{Z}^d} c_{\xi - x \ h} u(\xi) u(x).$$

By inverting (5.4) we have

$$c_\eta = (2\pi)^{-d} \int_S p(\theta) e^{-i \langle \eta, \theta \rangle} d\theta.$$

We rewrite $L_h$ in (2.13) as

$$L_h u(x) = h^{-2m} \sum_{\eta} c_\eta u(x + \eta h),$$

where $\eta = (\eta_1, \ldots, \eta_d)$ with the $\eta_j$’s being integers and the $c_\eta$’s being real numbers which may depend on $h$. We now define the characteristic polynomial of $L_h$ by

$$p(\theta) := \sum_{\eta} c_\eta e^{i \langle \eta, \theta \rangle},$$

where $\theta = (\theta_1, \ldots, \theta_d)$ and $\langle \eta, \theta \rangle = \sum_{j=1}^d \eta_j \theta_j$. We have the following Lemma:

**Lemma 5.3.**

$$\langle L_h u, u \rangle_{h, grid} = h^{d-2m} (2\pi)^{-d} \int_S p(\theta)|\hat{u}(\theta)|^2 d\theta, \quad u \in \mathcal{D}_h.$$ 

where

$$\hat{u}(\theta) = \sum_{\xi \in \mathbb{Z}^d} u(\xi h) e^{-i \langle \xi, \theta \rangle}$$

and $S = \{\theta : |\theta_j| \leq \pi, \ j = 1, \ldots, d\}$.

**Proof.** We expand

$$\langle L_h u, u \rangle_{h, grid} = h^d \sum_{x \in \mathbb{Z}^d} L_h u(x) u(x)$$

$$= h^{d-2m} \sum_{x \in \mathbb{Z}^d} \sum_{\eta \in \mathbb{Z}^d} c_\eta u(x + \eta h) u(x)$$

$$= h^{d-2m} \sum_{x, \xi \in \mathbb{Z}^d} c_{\xi - x \ h} u(\xi) u(x).$$

By inverting (5.4) we have

$$c_\eta = (2\pi)^{-d} \int_S p(\theta) e^{-i \langle \eta, \theta \rangle} d\theta.$$
Thus
\[
\langle L_h u, u \rangle_{h, \text{grid}} = h^{d-2m} \sum_{x, \xi \in h \mathbb{Z}^d} (2\pi)^{-d} \int_{S} p(\theta) e^{-i \langle \frac{\xi}{h} + \theta \rangle} d\theta u(\xi) u(x)
\]
\[= h^{d-2m}(2\pi)^{-d} \int_{S} p(\theta) |\hat{u}(\theta)|^2 d\theta. \]

We will also need

Lemma 5.4. There is a constant $C$ independent of $u$ and $h$ such that
\[
\|u\|_{h,m}^2 \leq C \sum_{j=1}^{d} \|\partial_j^m u\|_{h, \text{grid}}^2, \quad u \in \mathcal{D}_h.
\]

Proof. We first prove that if $\alpha$ is a multi-index with $|\alpha| = m$ then
\[
\langle \bar{\partial}^\alpha \partial^\alpha u, u \rangle_{h, \text{grid}} \leq \langle Q_h u, u \rangle_{h, \text{grid}}, \quad u \in \mathcal{D}_h, \tag{5.5}
\]
where $Q_h$ is the difference operator
\[
Q_h u := \sum_{j=1}^{d} \bar{\partial}_j^m \partial_j^m u. \tag{5.6}
\]

Similar to (5.4) we can show the characteristic polynomial of $\bar{\partial}^\alpha \partial^\alpha$ and $Q_h$ are respectively
\[
q_1(\theta) = 2^m \prod_{j=1}^{d} (1 - \cos \theta_j)^{\alpha_j}
\]
and
\[
q_2(\theta) = 2^m \sum_{j=1}^{d} (1 - \cos \theta_j)^m.
\]

Now by the inequality between arithmetic and geometric mean we have
\[
q_1(\theta) \leq 2^m \sum_{j=1}^{d} m^{-1} \alpha_j (1 - \cos \theta_j)^m \leq q_2(\theta).
\]

Using Lemma 5.3 we obtain (5.5), which implies
\[
\|\partial^\alpha u\|_{h, \text{grid}}^2 \leq \sum_{j=1}^{d} \|\partial_j^m u\|_{h, \text{grid}}^2, \quad u \in \mathcal{D}_h.
\]

For $|\alpha| < m$, one can show using Lemma 5.1
\[
\|\partial^\alpha u\|_{h, \text{grid}}^2 \leq C \sum_{j=1}^{d} \|\partial_j^m u\|_{h, \text{grid}}^2, \quad u \in \mathcal{D}_h.
\]

Hence the proof is complete.
5.2. Errors in the Dirichlet problem. We have shown some discrete Sobolev inequalities till now. We now relate these directly to our discrete operators. We start dealing with each of the operators separately. Before we do so let us show here the existence and uniqueness of the solution of the discrete boundary value problem (2.15)-(2.16).

Lemma 5.5. The finite difference Dirichlet problem (2.15)-(2.16) has exactly one solution for arbitrary $f$.

Proof. Since $u \equiv 0$ in $B_h$, Equation (2.15) can be considered as a linear system of equations with the same number of equations as of unknowns (the number of points in $R_h$). Therefore it is sufficient to prove that the corresponding homogeneous system has only the trivial solution i.e. $u \equiv 0$ in $R_h$. This follows from Lemmas 5.6, 5.7 and 5.8 depending on the operator $L$.

5.2.1. Bilaplacian case: proof of Theorem 2.9 (1). In this subsection we consider $L := \Delta^2$. Recall $\rho^2(h) \to 0$ and we have for $x \in h\mathbb{Z}^d$.

$$ L_h u(x) = \frac{1}{h^4} \left[ -h^2 \rho^2(h) \sum_{i=1}^{d} (u(x + he_i) + u(x - he_i) - 2u(x)) 
 + \sum_{i,j=1}^{d} \{ u(x + h(e_i + e_j)) + u(x - h(e_i + e_j)) + u(x + h(e_i - e_j)) + u(x - h(e_i - e_j))
 - 2(u(x + he_i) - 2u(x - he_i) - 2u(x + he_j) - 2u(x - he_j) + 4u(x)) \right]. $$

We define the operator $L_{h,2}$ as follows:

$$ L_{h,2} f(x) = \begin{cases} L_h f(x) & x \in R_h^1 \\ h^2 L_h f(x) & x \in B_h^* \\ 0 & x \notin R_h. \end{cases} \tag{5.7} $$

Then we have the following Lemma involving $L_{h,2}$.

Lemma 5.6. There exists a constant $C > 0$ independent of $u$ and $h$ such that

$$ \| u \|_{h,2} \leq C \| L_{h,2} u \|_{h,\text{grid}}, \quad u \in \mathcal{D}_h. $$

Proof. We consider the characteristic polynomial of $L_h$ and observe that

$$ p(\theta) = -h^2 \rho^2(h) \sum_{i=1}^{d} (2 \cos \theta_i - 2) 
 + \sum_{i,j=1}^{d} [2 \cos (\theta_i + \theta_j) + 2 \cos (\theta_i - \theta_j) - 4 \cos \theta_i - 4 \cos \theta_j + 4] 
 = h^2 \rho^2(h) \sum_{i=1}^{d} (2 - 2 \cos \theta_i) + \sum_{i,j=1}^{d} [4(1 - \cos \theta_i)(1 - \cos \theta_j)] 
 \geq 4 \sum_{i=1}^{d} (1 - \cos \theta_i)^2. $$

Hence by Lemmas 5.4 and 5.3 we obtain for $u \in \mathcal{D}_h$

$$ \| u \|_{h,2}^2 \leq C \sum_{j=1}^{d} \| \partial_j^2 u \|^2_{h,\text{grid}} = C \langle Q_h u, u \rangle_{h,\text{grid}} \leq C \langle L_h u, u \rangle_{h,\text{grid}}, $$
where $Q_h$ is the difference operator defined in (5.6) with $m = 2$. Again we have

$$
\langle L_h u, u \rangle_{h, \text{grid}} = h^d \left[ \sum_{x \in B_h^*} L_{h, 2} u(x) \left( h^{-2} u(x) \right) + \sum_{x \in R_h^*} L_{h, 2} u(x) u(x) \right]
$$

Therefore by Cauchy-Schwarz inequality we have

$$
\| \langle L_h u, u \rangle_{h, \text{grid}} \| \leq C \| L_{h, 2} u \|_{h, \text{grid}} \| u \|_{h, 2}.
$$

Thus from Lemma 5.2 we have

$$
\| u \|_{h, 2}^2 \leq C \| L_{h, 2} u \|_{h, \text{grid}} \| u \|_{h, 2} \leq C \| L_{h, 2} u \|_{h, \text{grid}} \| u \|_{h, 2}
$$

This completes the proof. $\square$

We have now all the ingredients to show Theorem 2.9 (1).

**Proof of Theorem 2.9 (1).** We denote all constants by $C$ and they do not depend on $u, f$. Using Taylor expansion we have for all $x \in R_h$ and for small $h$

$$
L_{h, 2} u(x) = h^{-2} \rho_2(h) R_2(x) + L u(x) + h^{-4} R_5(x)
$$

where $|R_2(x)| \leq C M_2 h^2$ and $|R_5(x)| \leq C M_5 h^5$. We thus obtain, for $x \in R_h$,

$$
L_{h, 2} e_h(x) = L_{h, 2} u(x) - L_{h, 2} u_h(x) = h^{-2} \rho_2(h) R_2(x) + h^{-4} R_5(x).
$$

(5.8)

For $x \in R_h^*$ we have

$$
L_{h, 2} R_h e_h(x) = L_{h, 2} R_h e_h(x) = L_{h, 2} e_h(x) = h^{-2} \rho_2(h) R_2(x) + h^{-4} R_5(x).
$$

For $x \in B_h^*$ at least one among $x \pm h(e_i \pm e_j)$, $x \pm h e_i$ is in $B_h \cap B_h$. For any $y \in B_h \setminus \partial V$ we consider a point $b(y)$ on $\partial V$ of minimal distance to $y$. Note that this distance is at most $2h$. Now using Taylor expansion and the fact that the value of $u$ and all its first order derivatives are zero at $b(y)$ one sees that

$$
u(y) = u_h(y) + R_2(y)
$$

where $|R_2'(y)| \leq C M_2 h^2$. For $x \in B_h^*$ denote by $S(x)$ the neighbors of $x$ which are in $B_h \setminus \partial V$ i.e. $S(x) = \{ y : y \in B_h \setminus \partial V \cap \{ x \pm h e_i, x \pm h(e_i \pm e_j) : 1 \leq i, j \leq d \} \}$.

Therefore, for $x \in B_h^*$,

$$
L_{h, 2} R_h e_h(x) = h^2 L_h R_h e_h(x)
$$

$$
= h^2 \left\{ L_h e_h(x) - h^{-4} \sum_{y \in S(x)} \left( h^2 \rho_2(h) C(y) e_h(y) + C'(y) e_h(y) \right) \right\}
$$

$$
= h^2 \left\{ h^{-2} \rho_2(h) R_2(x) + h^{-4} R_5(x) \right\} + (C \rho_2(h) + C' h^{-2}) R_2''(x)
$$

(5.8)
We define the operator $L_{h,2}$. Lemma 5.7.

\section{Laplacian + Bilaplacian case: proof of Theorem 2.9 (2)}

In this subsection we consider $L = -\Delta_e + \Delta_e^2$. Recall $\rho_3(h) \to 1$ and we have for $x \in h\mathbb{Z}^d$,

\begin{align*}
L_h u(x) &= \frac{1}{h^4} \left[ -h^2 \sum_{i=1}^d (u(x + he_i) + u(x - he_i) - 2u(x)) \\
&\quad + \rho_3(h) \sum_{i,j=1}^d \{u(x + (e_i + e_j)) + u(x - (e_i + e_j)) + u(x + (e_i - e_j)) + u(x - (e_i - e_j)) \\
&\quad - 2(u(x + he_i) - 2u(x - he_i) - 2(u(x + he_j) - 2u(x - he_j) + 4u(x)) \} \right].
\end{align*}

We define the operator $L_{h,2}$ as in (5.7) and obtain

**Lemma 5.7.** There exists a constant $C > 0$ independent of $u$ and $h$ such that

$$
\|u\|_{h,2} \leq C \|L_{h,2} u\|_{h,\text{grid}}, \quad u \in \mathcal{D}_h.
$$
Proof. We observe that
\[
p(\theta) = -h^2 \sum_{i=1}^{d} (2 \cos \theta_i - 2) + \rho_3(h) \sum_{i,j=1}^{d} \left[ 2 \cos (\theta_i + \theta_j) + 2 \cos (\theta_i - \theta_j) - 4 \cos \theta_i - 4 \cos \theta_j + 4 \right]
\]
\[
= h^2 \sum_{i=1}^{d} (2 - 2 \cos \theta_i) + \rho_3(h) \sum_{i,j=1}^{d} \left[ 4(1 - \cos \theta_i)(1 - \cos \theta_j) \right]
\geq 4\rho_3(h) \sum_{i=1}^{d} (1 - \cos \theta_i)^2.
\]
Hence by Lemma 5.4 and 5.3 we obtain for \(u \in \mathcal{D}_h\)
\[
\|u\|_{h,2}^2 \leq C \sum_{j=1}^{d} \|\partial^2_j u\|_{h,\text{grid}}^2 = C \langle Q_h u, u \rangle_{h,\text{grid}} \leq C(\rho_3(h))^{-1} \langle L_h u, u \rangle_{h,\text{grid}} \leq C \langle L_h u, u \rangle_{h,\text{grid}},
\]
where \(Q_h\) is the difference operator defined in (5.6) with \(m = 2\). The rest of the proof is similar to Lemma 5.6 and hence omitted. \(\square\)

We now prove the approximation result in this case.

Proof of Theorem 2.9 (2). As before the constant \(C\) does not depend on \(u\) and \(f\). Using Taylor expansion we have for all \(x \in R_h\) and for small \(h\)
\[
L_h u(x) = Lu(x) + (\rho_3(h) - 1)\Delta^2_e u(x) + h^{-2}R_4(x) + \rho_3(h)h^{-4}R_5(x)
\]
where \(|R_4(x)| \leq CM_4 h^4, |R_5(x)| \leq CM_5 h^5\). We obtain for \(x \in R_h\)
\[
L_h e_h(x) = L_h u(x) - L_h u_h(x)
= Lu(x) + (\rho_3(h) - 1)\Delta^2_e u(x) + h^{-2}R_4(x) + \rho_3(h)h^{-4}R_5(x) - L_h u_h(x)
= (\rho_3(h) - 1)\Delta^2_e u(x) + h^{-2}R_4(x) + \rho_3(h)h^{-4}R_5(x).
\]
For \(x \in R^*_h\) we have
\[
L_{h,2} R_h e_h(x) = L_h R_h e_h(x) = L_h e_h(x) = (\rho_3(h) - 1)\Delta^2_e u(x) + h^{-2}R_4(x) + \rho_3(h)h^{-4}R_5(x). \quad (5.9)
\]
As in the case of \(\Delta^2_e\) we have for any \(y \in B_h \setminus \partial V\)
\[
u(y) = u_h(y) + R_2(y)
\]
where \(|R_2(y)| \leq CM_2 h^2\). Therefore, for \(x \in B^*_h\),
\[
L_{h,2} R_h e_h(x) = h^2 L_h R_h e_h(x)
= h^2 \left\{ L_h e_h(x) - h^{-d} \sum_{y \in S(x)} \left( h^2 C(y) e_h(y) + \rho_3(h) C'(y) e_h(y) \right) \right\}
= h^2 (\rho_3(h) - 1)\Delta^2_e u(x) + h^{-2}R_4(x) + \rho_3(h)h^{-4}R_5(x)
+ C R_2''(x) + Ch^{-2} \rho_3(h) R_2'(x) \quad (5.10)
\]
where \( S(x) \) is defined similarly as in \( \Delta^2 \) case, \( C(y), C'(y) \) are constants depending on \( y \) and \( |R'_2(x)| \leq CM_2h^2 \), \( |R''_2(x)| \leq CM_2h^2 \). We have

\[
\|L_{h,2}R_he_h\|_{h,\text{grid}}^2 = h^d \sum_{x \in R_h} (L_{h,2}R_he_h(x))^2
\]

which, using the bounds (5.9)-(5.10), turns into

\[
\|L_{h,2}R_he_h\|_{h,\text{grid}}^2 \leq Ch^d \sum_{x \in R_h^*} ((\rho_3(h) - 1)^2M_4^2 + M_5^2h^4 + (\rho_3(h))^2M_6^2h^2)
\]

\[
+ Ch^d \sum_{x \in B_h^*} (h^4(\rho_3(h) - 1)^2M_4^2 + M_5^2h^6 + (\rho_3(h))^2M_6^2h^6 + M_5^2M_6^2h^8 + C((\rho_3(h) - 1)^2M_4^2
\]

\[
+ M_5^2h^9 + (\rho_3(h))^2M_6^2h^7 + M_5^2M_6^2h^5) \leq C[(\rho_3(h) - 1)^2M_4^2 + M_5^2h^4 + (\rho_3(h))^2M_6^2h^4 + h^5(\rho_3(h) - 1)^2M_4^2
\]

\[
+ M_5^2h^5 + (\rho_3(h))^2M_6^2h^4 + M_5^2M_6^2h^5] \leq C \left[ M_5^2h^2 + M_5^2(\rho_3(h) - 1)^2 + M_4^2h^4 + M_4^2h \right].
\]

5.2.3. Laplacian case: proof of Theorem 2.9 (3). In this subsection we consider \( L = -\Delta_c \). In this case we have \( \rho_1(h) \to 0 \) and for \( x \in h\mathbb{Z}^d \),

\[
L_h u(x) = \frac{1}{h^2} \left[ - \sum_{i=1}^d (u(x + he_i) + u(x - he_i) - 2u(x))
\]

\[
+ \frac{\rho_1(h)}{h^2} \sum_{i,j=1}^d (u(x + h(e_i + e_j)) + u(x - h(e_i + e_j)) + u(x + h(e_i - e_j))
\]

\[
+ u(x - h(e_i - e_j)) - 2u(x + he_i) - 2u(x - he_i) - 2u(x + he_j) - 2u(x - he_j) + 4u(x)) \right].
\]

We define the operator \( L_{h,1} \) as follows:

\[
L_{h,1}f(x) = \begin{cases} L_h f(x) & x \in R_h^* \\
h L_h f(x) & x \in B_h^* \\ 0 & x \notin R_h. \end{cases}
\] (5.11)

Then we have

**Lemma 5.8.** There exists a constant \( C > 0 \) independent of \( u \) and \( h \) such that

\[
\|u\|_{h,1} \leq C \|L_{h,1}u\|_{h,\text{grid}}, \quad u \in \mathcal{D}_h.
\]
Proof. The proof is similar to Lemma 5.6 once we observe the following lower bound of the characteristic polynomial of $L_h$:

\[
p(\theta) = -\sum_{i=1}^{d} (2 \cos \theta_i - 2) + \frac{\rho_1(h)}{h^2} \sum_{i, j=1}^{d} [2 \cos (\theta_i + \theta_j) + 2 \cos (\theta_i - \theta_j) - 4 \cos \theta_i - 4 \cos \theta_j]
\]

\[
= \sum_{i=1}^{d} (2 - 2 \cos \theta_i) + \frac{\rho_1(h)}{h^2} \sum_{i, j=1}^{d} [4(1 - \cos \theta_i)(1 - \cos \theta_j)]
\]

\[
\geq 2 \sum_{i=1}^{d} (1 - \cos \theta_i).
\]

We are now ready to prove Theorem 2.9 (3).

Proof of Theorem 2.9 (3). As before the constant $C$ does not depend on $u, f$. Using Taylor expansion we have for all $x \in R_h$ and for small $h$

\[
L_h u(x) = Lu(x) + h^{-2} R_4(x) + h^{-4} \rho_1(h) R_4'(x)
\]

where $|R_4(x)| \leq CM_4 h^4$ and $|R_4'(x)| \leq CM_4 h^4$. We thus obtain, for $x \in R_h$,

\[
L_h e_h(x) = L_h u(x) - L_h u_h(x) = h^{-2} R_4(x) + h^{-4} \rho_1(h) R_4'(x).
\]  

(5.12)

For $x \in R_h^*$ we have

\[
L_{h,1} R_h e_h(x) = L_h R_h e_h(x) = L_h e_h(x) = h^{-2} R_4(x) + h^{-4} \rho_1(h) R_4'(x).
\]

As in the case of $\Delta^2$ we have for any $y \in B_h \setminus \partial V$

\[
u(y) = u_h(y) + R_1(y)
\]

where $|R_1(y)| \leq CM_1 h$. Therefore for $x \in B_h^*$,

\[
L_{h,1} R_h e_h(x) = h L_h R_h e_h(x)
\]

\[
= h \left\{ L_h e_h(x) - \sum_{y \in S(x)} \left( h^{-2} C(y) e_h(y) + h^{-4} \rho_1(h) C'(y) e_h(y) \right) \right\}
\]

\[
= h^{-1} R_4(x) + h^{-3} \rho_1(h) R_4'(x) + h^{-1} C R_1'(x) + C h^{-3} \rho_1(h) R_1''(x).
\]  

(5.13)

where $S(x)$ is defined similarly as in the $\Delta^2$ case and $|R_1'(x)|, |R_1''(x)| \leq CM_1 h$. So we write

\[
\|L_{h,1} R_h e_h\|_{h, \text{grid}}^2 = h^d \sum_{x \in R_h} (L_{h,1} R_h e_h(x))^2
\]

\[
= h^d \left[ \sum_{x \in R_h^*} (L_{h,1} R_h e_h(x))^2 + \sum_{x \in B_h^*} (L_{h,1} R_h e_h(x))^2 \right]
\]
The first and second terms are given respectively by (5.12) and (5.13). Using the bounds for \( \mathcal{R}_1, \mathcal{R}'_1, \mathcal{R}''_4, \mathcal{R}_4 \) and \( \mathcal{R}'_4 \) we have

\[
\|L_{h,1} R_h e_h\|^2_{h,\text{grid}} \leq C h^d \left[ \sum_{x \in B^*_h} (M^2_h h^4 + M^2_h (\rho_1(h))^2) \right. \\
+ \sum_{x \in B^*_h} \left. \left( M^2_h h^6 + M^2_h h^2 (\rho_1(h))^2 + M^2_h + M^2_h h^{-4}(\rho_1(h))^2 \right) \right] \\
\leq C \left[ M^2_h h^4 + M^2_h (\rho_1(h))^2 + M^2_h h^7 + M^2_h h^3 (\rho_1(h))^2 + M^2_h + M^2_h h^{-3}(\rho_1(h))^2 \right] \\
\leq C \left[ M^2_h h^3 + M^2_h h + M^2_h h^{-3}(\rho_1(h))^2 \right]
\]

where the second last inequality holds as the number of points in \( B^*_h \) is \( O(h^{-(d-1)}) \) and in the last inequality \( (\rho_1(h))^2 \ll C h^3 \). Now using Lemma 5.1 and Lemma 5.8, the bound on \( \|R_h e_h\|^2_{h,\text{grid}} \) follows.

\[\square\]

**APPENDIX A. COVARIANCE BOUND FOR MM IN \( d = 1 \)**

In this section we consider \( d = 1 \) and the Membrane model \( (\varphi_x)_{x \in V_N} \) on \( V_N = \{1, \ldots, N - 1\} \) with zero boundary conditions outside \( V_N \). We want to show the following bound:

**Lemma A.1.** There exists a constant \( C > 0 \) such that

\[
E_{V_N}[(\varphi_x - \varphi_{x+1})^2] \leq C N, \quad x \in \mathbb{Z}.
\]

**Proof.** Let \( \{X_i\}_{i \in \mathbb{N}} \) be a sequence of i.i.d. standard Gaussian random variables. We define \( \{Y_i\}_{i \in \mathbb{Z}^+} \) to be the associated random walk starting at 0, that is,

\[
Y_0 = 0, \quad Y_n = \sum_{i=1}^n X_i, \quad n \in \mathbb{N},
\]

and \( \{Z_i\}_{i \in \mathbb{Z}^+} \) to be the integrated random walk starting at 0, that is, \( Z_0 = 0 \) and for \( n \in \mathbb{N} \)

\[
Z_n = \sum_{i=1}^n Y_i.
\]

Then one can show that \( P_{V_N} \) is the law of the vector \( (Z_1, \ldots, Z_{N-1}) \) conditionally on \( Z_N = Z_{N+1} = 0 \) (Caravenna and Deuschel, 2008, Proposition 2.2). So we have that

\[
E_{V_N}[(\varphi_{i+1} - \varphi_i)^2] = E[(Z_{i+1} - Z_i)^2 | Z_N = Z_{N+1} = 0] = E[Y_{i+1}^2 | Z_N = Z_{N+1} = 0].
\]

Hence it is enough to find a bound for \( E[Y_i^2 | Z_N = Z_{N+1} = 0] \) for \( i = 1, \ldots, N - 1 \). The covariance matrix \( \Sigma \) for \( (Y_1, \ldots, Y_{N-1}, Z_N, Z_{N+1}) \) can be partitioned as

\[
\Sigma = \begin{bmatrix} A & B \\ B & D \end{bmatrix}
\]

where \( A \) is a \( (N-1) \times (N-1) \) matrix with entries

\[
A(i, j) = \text{Cov}(Y_i, Y_j) = i \wedge j.
\]

\( B(i, j) \) and \( C(i, j) \) are \( (N-1) \times 2 \) and \( 2 \times (N-1) \) matrices respectively, with \( C = B^T \) and

\[
B(i, j) = \text{Cov}(Y_i, Z_{j+N-1}) = \sum_{l=1}^{j+N-1} i \wedge l.
\]
Finally, $D$ is a $2 \times 2$ matrix with

$$D(i, j) = \text{Cov}(Z_{i+N-1}, Z_{j+N-1}).$$

It easily follows that

$$D = \frac{1}{6} \begin{bmatrix} N(N+1)(2N+1) & N(N+1)(2N+4) \\ N(N+1)(2N+4) & (N+1)(N+2)(2N+3) \end{bmatrix}.$$  \hspace{1cm} (A.1)

It is well known that $(Y_1, \ldots, Y_{N-1})|Z_N = Z_{N+1} = 0)$ is a Gaussian vector with mean zero and covariance matrix given by $A - BD^{-1}C$. The inverse of $D$ is as follows. Observe

$$\gamma_N := \det(D) = \frac{1}{36}N(N+1)^2(8N^2 + 3N + 6)$$

and

$$D^{-1} = \frac{1}{\gamma_N} \begin{bmatrix} D(2, 2) - D(1, 2) \\ -D(2, 1) & D(1, 1) \end{bmatrix}.$$  

Now the diagonal element of $BD^{-1}C$ can be determined:

$$(BD^{-1}C)(i, i) = \frac{1}{\gamma_N} \left[ \left( \sum_{l=1}^{N} i \wedge l \right)^2 D(2, 2) - \left( \sum_{l=1}^{N} i \wedge l \right) \left( \sum_{l=1}^{N+1} i \wedge l \right) D(1, 2) \right.$$

$$- \left( \sum_{l=1}^{N} i \wedge l \right) \left( \sum_{l=1}^{N+1} i \wedge l \right) D(1, 2) + \left( \sum_{l=1}^{N+1} i \wedge l \right)^2 D(1, 1) \right].$$

Plugging in the entries $D(i, j)$ from (A.1) and simplifying we get

$$(BD^{-1}C)(i, i) = \frac{i^2(N+1)}{24\gamma_N} \left[ 6N^2 - 12Ni + 6i^2 + 4N \right] > 0$$

This shows that for $i = 1, 2, \ldots, N - 1$,

$$\mathbb{E}[Y_i^2|Z_N = Z_{N+1} = 0] = A(i, i) - (BD^{-1}C)(i, i) < i.$$  

Similar bound can be obtained for $\mathbb{E}[Y_i^2|Z_N = Z_{N+1} = 0]$ and this completes the proof. \hfill \Box

**APPENDIX B. DETAILS ON THE SPACE $\mathcal{H}^{-\frac{s}{2}}_{-\Delta + \Delta^2}$**

In this section we briefly describe few of the details regarding the space $\mathcal{H}^{-\frac{s}{2}}_{-\Delta + \Delta^2}$ and also about the spectral theory of $-\Delta_c + \Delta_c^2$. This is an elliptic operator, and the spectral theory is similar to that of either $-\Delta_c$ or $\Delta_c^2$. First recall the standard Sobolev inner products on $H^s_0(D)$ and $H^s_0(D)$. They are

$$\langle u, v \rangle_1 = \int_D \nabla u \cdot \nabla v \, d x, \quad u, v \in H^s_0(D)$$

and

$$\langle u, v \rangle_2 = \int_D \Delta u \Delta v \, d x, \quad u, v \in H^s_0(D)$$

and they induce norms on $H^s_0(D)$ and $H^s_0(D)$ respectively which are equivalent to the standard Sobolev norms (Gazzola et al., 2010, Corollary 2.29). We now consider the following inner product on $H^s_0(D)$:

$$\langle u, v \rangle_{\text{mixed}} := \int_D \nabla u \cdot \nabla v \, d x + \int_D \Delta u \Delta v \, d x, \quad u, v \in H^s_0(D).$$

Clearly the norm induced by this inner product is equivalent to the norm $\| - H^s_0$ (by integration by parts). We consider $H^{-2}(D)$ to be the dual of $(H^2_0(D), \| \cdot \|_{\text{mixed}})$.

We now give some results whose proofs are similar to Theorem 3.2 and 3.3 of Cipriani et al. (2018b).
Remark B.1. For each $j \in \mathbb{N}$ one has $v_j \in C^\infty_c(D)$. Moreover $v_j$ is an eigenfunction of $-\Delta_c + \Delta_c^2$ with eigenvalue $\mu_j$. Indeed, we have for all $v \in H^2_0(D)$

$$\langle (-\Delta_c + \Delta_c^2)v_j, v \rangle_{L^2} = \langle (-\Delta_c)v_j, v \rangle_{L^2} + \langle (\Delta_c^2)v_j, v \rangle_{L^2} = \langle v_j, v \rangle_{\text{mixed}} = \mu_j \langle v_j, v \rangle_{L^2}$$

where “GI” stands for Green’s first identity

$$\int_D u\Delta v \, dV = -\int_D \nabla u \cdot \nabla v \, dV + \int_{\partial D} u\nabla v \cdot \mathbf{n} \, dS.$$

Thus $v_j$ is an eigenfunction of $-\Delta_c + \Delta_c^2$ with eigenvalue $\mu_j$ in the weak sense. The smoothness of $v_j$ follows from the fact that $-\Delta_c + \Delta_c^2$ is an elliptic operator with smooth coefficients and the elliptic regularity theorem (Folland, 1999, Theorem 9.26). Hence $v_j$ is an eigenfunction of $-\Delta_c + \Delta_c^2$ with eigenvalue $\mu_j$.

Remark B.2. As a consequence of the above, one easily has that

$$\|f\|_{\text{mixed}}^2 = \sum_{j \geq 1} \mu_j \langle f, v_j \rangle_{L^2}^2$$

for any $f \in H^2_0(D)$.

For any $v \in C^\infty_c(D)$ and for any $s > 0$ we define

$$\|v\|_{s,\text{mixed}}^2 := \sum_{j \in \mathbb{N}} \mu_j^{s/2} \langle v, v_j \rangle_{L^2}^2.$$

We define $\mathcal{H}^s_0(D)$ to be the Hilbert space completion of $C^\infty_c(D)$ with respect to the norm $\| \cdot \|_{s,\text{mixed}}$. Then $(\mathcal{H}^s_0(D), \| \cdot \|_{s,\text{mixed}})$ is a Hilbert space for all $s > 0$.

Remark B.3.

- Note that for $s = 2$ we have $\mathcal{H}^2_0(D) = (H^2_0(D), \| \cdot \|_{\text{mixed}})$ by Remark B.2.

- $i : \mathcal{H}^2_0(D) \hookrightarrow L^2(D)$ is a continuous embedding.

For $s > 0$ we define $\mathcal{H}_{-\Delta + \Delta^2}^{-s} = (\mathcal{H}^s_0(D))^*$, the dual space of $\mathcal{H}^s_0(D)$. Then we have

$$\mathcal{H}^s_0(D) \subseteq L^2(D) \subseteq \mathcal{H}_{-\Delta + \Delta^2}^{-s}(D).$$

One can show using the Riesz representation theorem that for $s > 0$, and $v \in L^2(D)$ the norm of $\mathcal{H}_{-\Delta + \Delta^2}^{-s}(D)$ is given by

$$\|v\|_{-s,\text{mixed}}^2 := \sum_{j \in \mathbb{N}} \mu_j^{-s/2} \langle v, v_j \rangle_{L^2}^2.$$
Before we show the definition of the continuum mixed model, we need an analog of Weyl’s law for the eigenvalues of the operator \(-\Delta_c + \Delta_c^2\).

**Proposition B.4** (Beals (1967, Theorem 5.1), Pleijel (1950)). There exists an explicit constant \(c\) such that, as \(j \uparrow +\infty\),

\[
\mu_j \sim c^{-d/4} j^{4/d}.
\]

**Proof.** We want to apply Theorem 5.1 of Beals (1967) for \(A := -\Delta_c + \Delta_c^2\). First note that \(A\) is an elliptic operator of order \(m = 4\) defined on \(D\) having smooth coefficients. Let us consider \(A_1 := (-\Delta_c + \Delta_c^2)|_{H^4(D) \cap H^2_0(D)}\). Clearly, \(A_1 : H^4(D) \cap H^2_0(D) \to L^2(D)\) and also \(C_c^\infty(D) \subset D(A_1) \subset H^4(D)\), where \(D(A_1)\) is the domain of \(A_1\). By elliptic regularity we have \(D(A_1^p) \subset H^{4p}, p = 1, 2, \ldots\) We first show that \(A_1\) is self-adjoint. Note that as \(C_c^\infty(D) \subset D(A_1)\) and \(C_c^\infty(D)\) is dense in \(L^2(D)\), \(A_1\) is densely defined. Again, by Green’s identity we have for all \(u, v \in H^4(D) \cap H^2_0(D)\)

\[
\langle (-\Delta_c + \Delta_c^2)u, v \rangle_{L^2} = \langle \nabla u, \nabla v \rangle_{L^2} + \langle \Delta_c u, \Delta_c v \rangle_{L^2} = \langle u, (-\Delta_c + \Delta_c^2)v \rangle_{L^2}.
\]

Thus \(A_1\) is symmetric. Also by Corollary 2.21 of Gazzola et al. (2010) we observe that image of \(A_1\) is self-adjoint. Note that as \(C_c^\infty(D) \subset D(A_1)\) and \(C_c^\infty(D)\) is dense in \(L^2(D)\), \(A_1\) is densely defined. Again, by Green’s identity we have for all \(u, v \in H^4(D) \cap H^2_0(D)\)

The result we will prove now shows the well-posedness of the series expansion for \(\Psi^{-\Delta+\Delta^2}\).

**Proposition B.5.** Let \((\xi_j)_{j \in \mathbb{N}}\) be a collection of i.i.d. standard Gaussian random variables. Set

\[
\Psi^{-\Delta+\Delta^2} := \sum_{j \in \mathbb{N}} \mu_j^{-1/2} \xi_j v_j.
\]

Then \(\Psi^{-\Delta+\Delta^2} \in \mathcal{F}_c^{-s} -\Delta+\Delta^2(D)\) a.s. for all \(s > (d-4)/2\).

**Proof.** Fix \(s > (d-4)/2\). Clearly \(v_j \in L^2(D) \subseteq \mathcal{F}_c^{-s}(D)\). We need to show that \(\|\psi\|_{-s,mixed} < +\infty\) almost surely. Now this boils down to showing the finiteness of the random series

\[
\|\psi\|_{-s,mixed}^2 = \sum_{j \geq 1} \mu_j^{-s/2} \left( \sum_{k \geq 1} \mu_k^{-1/2} u_k \xi_k, v_j \right)^2 = \sum_{j \geq 1} \mu_j^{-\frac{s}{2} - 1} \xi_j^2
\]

where the last equality is true since \((v_j)_{j \geq 1}\) form an orthonormal basis of \(L^2(D)\). Observe that the assumptions of Kolmogorov’s two-series theorem are satisfied: indeed using Proposition B.4 one has

\[
\sum_{j \geq 1} \mathbb{E} \left( \mu_j^{-\frac{d}{2} - 1} \xi_j^2 \right) < c \sum_{j \geq 1} j^{-\frac{d}{2}(d+1)} < +\infty
\]

for \(s > (d-4)/2\) and

\[
\sum_{j \geq 1} \text{Var} \left( \mu_j^{-\frac{d}{2} - 1} \xi_j^2 \right) < c \sum_{j \geq 1} j^{-\frac{d}{2}(d+2)} < +\infty
\]

for \(s > (d-8)/4\). The result then follows. \(\square\)
Appendix C. Random Walk Representation of the \((\nabla + \Delta)\)-Model in \(d = 1\) and Estimates

In this Appendix we recall some of the notations about the \(d = 1\) case which were used in the heuristic explanations of the Introduction. We take advantage of the representation of the mixed model given in Borecki (2010, Subsection 3.3.1) in our setting. To do that we set \(\beta_N := 16\kappa_N\).

Let

\[
\gamma = \left(\frac{1 + \beta_N - \sqrt{1 + 2\beta_N}}{1 + \beta_N + \sqrt{1 + 2\beta_N}}\right)^{1/2}
\]  

(C.1)

and let \((\varepsilon_i)_{i \in \mathbb{Z}^+}\) be i.i.d. \(N(0, \sigma^2)\) with

\[
\sigma^2 = \frac{4}{(1 + \beta_N + \sqrt{1 + 2\beta_N})}.
\]  

(C.2)

Define

\[
Y_n = \gamma^{n-1} \varepsilon_1 + \ldots + \gamma^n \varepsilon_n = \sum_{i=1}^{n} \gamma^{n-i} \varepsilon_i.
\]

Let the integrated walk be denoted by

\[
W_n = \sum_{i=1}^{n} Y_i = r_{n-1} \varepsilon_1 + \ldots + r_0 \varepsilon_n = \sum_{i=1}^{n} r_{n-i} \varepsilon_i
\]

where \(r_{n-i} = \sum_{i=0}^{n-i} \gamma^i\).

We consider the case when \(\kappa_N \to \infty\) and note that then \(\gamma = \gamma_N \to 1\) and \(\sigma^2_N = \sigma^2 \to 0\). The following representation will give an idea on how the phase transition occurs in the mixed model:

\[
W_n = \frac{1}{1 - \gamma}(\varepsilon_1 + \ldots + \varepsilon_n) - \frac{1}{1 - \gamma}(\gamma^n \varepsilon_1 + \gamma^{n-1} \varepsilon_2 + \ldots + \gamma \varepsilon_n).
\]

We recall the following Proposition from Borecki (2010, Proposition 1.10).

**Proposition C.1.** Let \(P_N(\cdot)\) be the mixed model with 0 boundary conditions. Then

\[
P_N(\cdot) = P((W_1, \ldots, W_{N-1}) \in \cdot | W_N = W_{N+1} = 0)
\]

Let \((\tilde{\varepsilon}_i)_{i \in \mathbb{Z}}\) be i.i.d. \(N(0, \sigma^2 / (1 - \gamma)^2)\). Then \(W_n\) can be written as

\[
W_n = S_n - U_n
\]

where \(S_n = \sum_{k=1}^{n} \tilde{\varepsilon}_k\) and \(U_n = \gamma^n \tilde{\varepsilon}_1 + \gamma^{n-1} \tilde{\varepsilon}_2 + \ldots + \gamma \tilde{\varepsilon}_n\). The conditional integrated random walk process has a representation, stated in Proposition 3.7 of Borecki (2010). Let

\[
P\left((\tilde{W}_1, \ldots, \tilde{W}_{N-1}) \in \cdot \right) = P((W_1, \ldots, W_{N-1}) \in \cdot | W_N = W_{N+1} = 0)
\]

Then

\[
\tilde{W}_k = W_k - W_{N} r_1(k) - W_{N+1} r_2(k)
\]

where \(r_1(k) = s_1(k)/r(k)\) and \(r_2(k) = s_2(k)/r(k)\). The definitions of \(r(k)\) and \(s_i(k)\) for \(i = 1, 2\) are as follows:

\[
r(k) = (-1 + \gamma)(-1 + \gamma^{N+1}) \left(-N + \gamma(2 + N + \gamma^{N}(-2 + (-1 + \gamma)N))\right),
\]

\[
s_1(k) = (-k + \gamma(1 - \gamma k + k)) + \gamma^{3+2N+k}(1 + \gamma^k(-1 + (-1 + \gamma)k))
+ \gamma^{N-k}(-\gamma + \gamma^3)(1 - k + N) + \gamma^{2+2k}(2 + N - \gamma(1 + N)) + \gamma(1 + N - \gamma(2 + N)),
\]
and

\[ s_2(k) = \gamma(\gamma^{1+k} + k - \gamma(1+k)) + \gamma^{2+2N-k}(-1 + \gamma^k(1+k - \gamma k)) \\
+ \gamma^{1+N-k}(\gamma + \gamma^k(-1 + \gamma^2)(k-N) - N + \gamma N + \gamma^{1+2k}(-1 + (-1 + \gamma N))) \]

Let us consider the unconditional process \( W_n \). Note that

\[
\text{Var}(S_n) = \frac{n\sigma^2}{(1-\gamma)^2}, \quad \text{Var}(U_n) = \frac{\sigma^2 \gamma^2 (1 - \gamma^{2n})}{(1-\gamma)^2(1-\gamma^2)},
\]

and

\[
\text{Cov}(S_n, U_n) = \frac{\gamma \sigma^2 (1 - \gamma^n)}{(1-\gamma)^2(1-\gamma^2)}.
\]

So from here we have

\[
\text{Var}(W_n) = \frac{n\sigma^2}{(1-\gamma)^2} - \frac{\sigma^2 \gamma^2 (1 - \gamma^n)^2}{(1-\gamma)^2(1+\gamma)} - \frac{2 \sigma^2 \gamma (1 - \gamma^n)}{(1-\gamma)^2(1+\gamma)}.
\]

(C.3)

From the above expressions one can show that \( \text{Var}(W_{N-1}) \sim N \) when \( \kappa = \kappa_N \ll N^2 \). We now derive the variance estimate when \( \kappa \gg N^2 \). For ease of writing, denote

\[
\zeta = \frac{1}{\beta_N} + \sqrt{\frac{1}{\beta_N}} \sqrt{\frac{1}{\beta_N}} + 2 \to 0.
\]

Furthermore \( \gamma = 1/(1 + \zeta) \) and \( \sigma^2 = 2/\beta_N(1 + \zeta) \). Rewriting (C.3) in terms of \( \zeta \) we have

\[
\text{Var}(W_{N-1}) = \frac{2(N-1)(1 + \zeta)^2}{\zeta^2 \beta_N(1 + \zeta)} - \frac{2(1 + \zeta)(1 - (1 + \zeta)^{-N-1})^2}{\beta_N \zeta^3(2 + \zeta)} - \frac{4(1 + \zeta)^2(1 - (1 + \zeta)^{-N-1})}{\beta_N \zeta^3(2 + \zeta)}
\]

\[
= \frac{2(1 + \zeta)}{\beta_N(2 + \zeta) \zeta^3} \left[ (N-1)(2 + \zeta) \zeta - (1 - (1 + \zeta)^{-N-1})^2 - 2(1 + \zeta)(1 - (1 + \zeta)^{-N-1}) \right].
\]

(C.4)

Using a Taylor series expansion of the fourth order for the second and third summands in (C.4) (since coefficients up to \( \zeta^2 \) get cancelled) we obtain that

\[
\text{Var}(W_{N-1}) \approx \frac{(1 + \zeta)N(N-1)^2}{\beta_N(2 + \zeta)} \approx \frac{N^3}{\beta_N} \approx \frac{N^3}{\kappa_N}.
\]

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