RANDOM-LIKE PROPERTIES OF CHAOTIC FORCING

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ABSTRACT. We prove that skew systems with a sufficiently expanding base have approximate exponential decay of correlations, meaning that the exponential rate is observed modulo an error. The fiber maps are only assumed to be Lipschitz regular and to depend on the base in a way that guarantees diffusive behaviour on the vertical component. The assumptions do not imply an hyperbolic picture and one cannot rely on the spectral properties of the transfer operators involved. The approximate nature of the result is the inevitable price one pays for having so mild assumptions on the dynamics on the vertical component. However, the error in the approximation goes to zero when the expansion of the base tends to infinity. The result can be applied beyond the original setup when combined with acceleration or conjugation arguments, as our examples show.

1. INTRODUCTION

One of the main questions of modern dynamical systems theory is: to which extent a deterministic chaotic system resembles a random process? This question has been addressed in various contexts from different point of views (see [You19] for a review). Here we study it in relation to forcing, and in particular we investigate the similarities between random and (sufficiently chaotic) deterministic forcing focusing on the statistical properties of the forced system.

A forced system is a system whose intrinsic dynamics is affected by an external influence typically coming from the interaction with another system or the surrounding environment. The forcing can be modelled to be random, e.g. obtained by adding to the dynamics a noise term independent in time, or deterministic, i.e. dependent on a variable that evolves in time following a deterministic law.

In the random case, classical results from the theory of Markov chains show that if there is enough diffusion, e.g. if the forcing adds smooth unbounded noise to the dynamics, then the forced system has a stationary measure that describes its asymptotic statistical behaviour, and exhibits memory loss and annealed exponential decay of correlations (among others [DMT95, BY93]). In contrast, if the forcing is deterministic, it is well known that even just to prove existence of a physically relevant invariant measure one needs to impose strong assumptions both on the intrinsic dynamic and on the forcing, often leading to some degree of hyperbolicity of the system and/or a good spectral picture of the operators involved (see literature below).

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¹For precise definitions and a comparison between deterministic and random forcing see Section A.3 in the Appendix.
In this paper we prove that, if the forcing has a “diffusive effect” and is generated by a uniformly expanding map with high expansion, then the deterministic system has an approximate stationary measure and exhibits approximate decay of correlations. We postpone rigorous definitions to later sections. Loosely speaking, an approximate stationary measure describes the asymptotic statistical properties of the system modulo a controlled error, and by an approximate exponential decay of correlations we mean that measurements of observables along orbits exhibit exponential decay of correlations also modulo an error. Most importantly, these errors go to zero when the expansion of the map generating the forcing goes to infinity. In other words we could say that, when the expansion of the map generating the forcing goes to infinity, the deterministic forcing becomes indistinguishable from random forcing with respect to the statistical properties we analyze.

It’s important to remark that our requirements do not ensure global hyperbolic properties or a good spectral picture, and even the existence of a physically relevant invariant measure cannot be deduced from the assumptions. The price that we pay is the approximate nature of the result. Its relevance, however, is clear when having an eye to applications; here decorrelation estimates come from observations of real-world systems and are intrinsically affected by a measurement error: if this error is larger than the approximation error in the decorrelation estimate, exact and approximate decay of correlations are indistinguishable.

Our approach is quite flexible and we expect it to be adaptable to a variety of situations beyond the current working assumptions, for example in situations with lower regularity, or in combination with various conjugation arguments (see Section 5 for some generalizations).

1.1. Literature review. In mathematical terms, a forced system in discrete time can be described by a skew-product transformation which is a map \( F : \Omega \times X \to \Omega \times X \) such that

\[
F(\omega, x) = (g(\omega), f(\omega, x))
\]

where \( g : \Omega \to \Omega \) and \( f : \Omega \times X \to X \). The set \( \Omega \) is called the base of the skew-product, while \( X \) is referred to as the vertical fiber. The main characteristic of a skew-product is that the evolution on the vertical fiber \( X \) depends on the state of the base \( \Omega \), but not vice versa.

The literature on skew-products is vast to the extent that there are entire research trends studying particular aspects of these systems (e.g. iterated function systems, random dynamical systems, smoothness of invariant graphs over skew-products, etc.). Here we focus on those works dealing with statistical properties of skew products that have a “deterministic” base, such as \([\text{Gou07, Ste11, SW13, GRS15, BE17, Bje18, DFGTV18, NTV18, WW18, Haf19, Klo20, DFGTV20}]\) and references therein. These works usually only require \( g \) to be a measure preserving ergodic transformation or, at most, to exhibit some uniform hyperbolicity. However, they restrict the fiber map \( f \) to one of some particular classes to ensure contraction or hyperbolic properties (exact or averaged) of the vertical fiber. In contrast, our results make only mild regularity assumptions on \( f \), but require that \( g \) is uniformly expanding with large minimal expansion.
As a consequence of our requirements, the map $F$ is likely to have a dominated splitting of the tangent space and be partially hyperbolic (see e.g. [HP06, Sam16]) with an expanding direction roughly aligned with the base dominating the other invariant directions. To put our work under this perspective, let us remind that available results on existence of physical measures and decay of correlations for partially hyperbolic systems often assume low dimensional geometry either of the phase space or of some invariant directions, and/or nonvanishing Lyapunov exponents ([CM00, ABV00, Dol04a, Dol04b, Tsu05, ADLP17, TY20]) which, in general, are not granted in our setup. More recent results give sufficient conditions for partially hyperbolic systems to have exponential decay of correlations by turning qualitative topological conditions such as accessibility ([BW10]), into quantitative properties of the operators involved ([CL20, PRH20]). The systems we consider do not fit in these results due to lack of smoothness, but it is unclear if the assumptions can be verified even for those systems in our setup which have the required regularity.

As the base map is much more chaotic than the vertical fibers, our setup is reminiscent of fast-slow systems (see [DL18, CL20, CFKM20, KKM20] among many others). However, the dynamic of our skew-products does not present separation of time-scales since at each time step it can produce displacements of the same order in both the base system and the vertical fibers.

1.2. Organization of the paper. In Section 2 we present the setting, the results, some examples, a sketch of the proof. In Section 3 we prove our result in the simpler situation where the map in the base has no distortion and the phase space is 2D. In Section 4 we prove our main theorem in full generality. In Section 5 we discuss some generalizations. In the appendices we gather some background material and results on Markov chains (in Appendix A), disintegration of measures (in Appendix B), and some computations involving the Kantorovich-Wasserstein distance that are used throughout the proofs (in Appendix C).

2. Setting and Results

2.1. Setting. Let’s consider a map $F$ as in (1) where we set $\Omega = \mathbb{T}^{m_1}$ and $X = \mathbb{T}^{m_2}$, here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the 1D torus and $m_1, m_2$ two positive integers. In the following we will denote by $|p_1 - p_2|$ the distance between $p_1, p_2 \in \mathbb{T}^N$ regardless of the specific $N \in \mathbb{N}$. For $I \subseteq \mathbb{T}^m$ be a set we denote by $\text{Op}(I)$ its open part.

2.1.1. The base map $g$. Consider $g : \mathbb{T}^{m_1} \rightarrow \mathbb{T}^{m_1}$ a $C^2$ local diffeomorphism. In particular, there is $d \in \mathbb{N}$ and $I = \{I_i\}_{i=1}^d$ a partition of $\mathbb{T}^{m_1}$ such that: $\text{Op}(I_i) = I_i \mod 0$, $\{g_i := g|_{I_i}\}_{i=1}^d$ with $g_i : I_i \rightarrow \mathbb{T}^{m_1}$ are invertible branches of $g$, and $g_i|\text{Op}(I_i)$ is $C^2$. Call $\{h_i := g_i^{-1}\}_{i=1}^d$ the corresponding inverses.

We assume that $g$ satisfies the following assumptions:

(H0.1) $\exists \sigma > 1$ s.t. $\|Dg_\omega v\| \geq \sigma \|v\|$ \quad $\forall \omega \in \mathbb{T}^{m_1}$, $v \in \mathbb{R}^{m_1}$,

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{m_1}$, and

(H0.2) $\exists D > 0$ s.t. $\frac{|Dg_{h_i}(\omega_1)|}{|Dg_{h_i}(\omega_2)|} \leq e^{D|\omega_1 - \omega_2|}$ \quad $\forall \omega_1, \omega_2 \in \mathbb{T}^{m_1}$ and $\forall i$. 
where $|Dg_{h_{i}(\omega)}|$ denotes the determinant of $Dg_{h_{i}(\omega)}$. Condition [H0.1] states that the differential of $g$ expands vectors in tangent space of a factor at least $\sigma > 1$, which [H0.2] imposes a uniform bound on the distortion. It is well known that $g$ has a unique absolutely continuous invariant probability (a.c.i.p.) measure (see [BG97], [Via97] and references therein). We call this measure and $\rho_g := \frac{d\nu_g}{d\text{Leb}_{T^{m_1}}}$ its density, where $\text{Leb}_{T^{m_1}}$ is the Lebesgue measure on $T^{m_1}$.

2.1.2. The vertical fiber maps $f$. We assume $f : T^{m_1} \times T^{m_2} \rightarrow T^{m_2}$ to be at least Lipschitz, and denote by $L \geq 0$ the Lipschitz constant, namely

\[
L := \inf_{(\omega,x_1) \neq (\omega,x_2)} \frac{|f(\omega,x_1) - f(\omega,x_2)|}{|(\omega_1,x_1) - (\omega_2,x_2)|}.
\]

Let $\{f_\omega\}_{\omega \in T^{m_1}}$ be the collection of maps $f_\omega : T^{m_2} \rightarrow T^{m_2}$ i.e. $f_\omega(\cdot) := f(\omega,\cdot)$. We write $f(\cdot, x)$ for the maps $f(\cdot, x) : T^{m_1} \rightarrow T^{m_2}$ obtained by fixing $x \in T^{m_2}$ and letting $\omega \in T^{m_1}$ vary. We let $\pi_1 : T^{m_1} \times T^{m_2} \rightarrow T^{m_1}$ be the projection onto the horizontal $T^{m_1}$-coordinate and, given a measure $\mu$ on $T^{m_1} \times T^{m_2}$, we refer to $\pi_{1*}\mu$ as the horizontal marginal of $\mu$. We also denote by $\pi_2 : T^{m_1} \times T^{m_2} \rightarrow T^{m_2}$ the projection onto the vertical $T^{m_2}$-coordinate and refer to $\Pi \mu := \pi_{2*}\mu$ as the vertical marginal of the measure $\mu$.

2.1.3. $\mathcal{P}$, the random counterpart of $F$. In the following, $\mathcal{M}_1(Y)$ denotes the space of Borel probability measures on the compact metric space $Y$.

For $\mu \in \mathcal{M}_1(T^{m_2})$, define the push-forward $f_\omega_*\mu(A) = \mu(f_\omega^{-1}(A))$ for any measurable $A \subseteq T^{m_2}$, and define the operator $\mathcal{P} : \mathcal{M}_1(T^{m_2}) \rightarrow \mathcal{M}_1(T^{m_2})$

\[
\mathcal{P}\mu := \int_{T^{m_1}} d\nu_g(\omega) f_\omega_*\mu = \int_{T^{m_1}} d\omega \rho_g(\omega) f_\omega_*\mu.
\]

Notice that $\mathcal{P}$ is the generator for a discrete time stationary Markov process with transition kernel

\[
P(x, A) := \int_{T^{m_1}} \delta_{f_\omega(x)}(A) \rho_g(\omega) d\omega.
\]

where $\delta_{f_\omega(x)}$ denotes the Dirac mass at $f_\omega(x)$. These operators are well studied in the literature and sufficient conditions under which $\mathcal{P}$ has a spectral gap in various functional spaces are known (see e.g. [HM08], [SB93], [Str14] and Appendix A).

It is important to notice that if at each time step one was to apply a map $\{f_\omega\}_{\omega \in T^{m_1}}$ sampled independently with respect to $\nu_g$, then the operator $\mathcal{P}$ would describe the evolution of the vertical marginal. In other terms, one can think of the Markov chain generated by $\mathcal{P}$ as the “random counterpart” of the deterministic evolution given by $F$ which instead selects the map $f_\omega$ at each time-step according to the deterministic process $\omega$, $g(\omega), g^2(\omega), ...$ generated by $g$.

2.2. Main Assumption. Assumption [H] below requires that the Markov chain generated by $\mathcal{P}$ is geometrically ergodic with respect to the Total Variation (TV) distance (see Appendix A for definitions).

Assumption H. There are $C > 0, \lambda \in (0,1)$ such that

\[
d_{TV}(\mathcal{P}^n \mu, \mathcal{P}^n \nu) \leq C \lambda^n d_{TV}(\mu, \nu),
\]

for all $\mu, \nu \in \mathcal{M}_1(T^{m_2})$. 
Notice that by a Krylov-Bogolyubov argument, it follows that there is a unique \( \eta_0 \in \mathcal{M}_1(T^{m_2}) \) invariant under \( P \), i.e. such that \( P\eta_0 = \eta_0 \), which is called a stationary measure for the Markov process generated by \( P \). Also notice that Assumption [H] is a condition on \( P \), and therefore it depends on \( f : T^{m_1} \times T^{m_2} \to T^{m_2} \) and \( \nu_g \) only.

### 2.3. Main Result

When describing the statistical properties of a skew-product such as \( F \), we adopt the following point of view. We assume to have access to observations of measurable functions \( \varphi : T^{m_2} \to \mathbb{R} \) along the orbits of the system. Picking as reference measure on \( T^{m_1} \times T^{m_2} \) the Lebesgue measure \( \text{Leb}_{T^{m_1} \times T^{m_2}} \) gives rise to the sequence of dependent random variables

\[
\{ \varphi \circ \pi_2 \circ F^n \}_{n=1}^{+\infty}
\]

on \( (T^{m_1} \times T^{m_2}, \text{Leb}_{T^{m_1} \times T^{m_2}}) \).

For \( \varphi, \psi : T^{m_2} \to \mathbb{R} \) in suitable functional spaces, we ask if there are constants \( A \in \mathbb{R} \) and \( \tilde{\lambda} \in (0, 1) \) such that

\[
\left| \int_{T^{m_1} \times T^{m_2}} \varphi(\pi_2 F^n(\omega, x))\psi(x) \, d\omega dx - A \right| = O(\tilde{\lambda}^n)
\]

When (3) is satisfied, the system is said to have exponential annealed decay of correlations. The term annealed refers to the fact that the observables \( \varphi, \psi \) depend on the vertical \( T^{m_2} \)-coordinate only, and therefore the correlations are averaged with respect to the horizontal \( T^{m_1} \)-coordinate.

As already argued in the introduction, our systems have little hope to satisfy (3), but the following theorem shows that \( F \) exhibits exponential annealed decay of correlations, up to a given precision that depends on the expansion of the base system.

**Theorem 2.1.** Let \( F \) satisfy assumptions [H0.1], [H0.3] and Assumption [H] with datum \( m_1, m_2 \in \mathbb{N}, D, L, C > 0, \sigma > 1, \tilde{\lambda} \in (0, 1) \). For every \( \varepsilon > 0 \) there is \( \sigma_0 > \max\{1, L\} \) (depending on \( \varepsilon \) and all the datum but \( \sigma \)) such that if \( \sigma > \sigma_0 \), then there are \( \tilde{\lambda} \in (0, 1), \tilde{C} > 0 \) and a probability measure \( \eta \in \mathcal{M}_1(T^{m_2}) \) such that

\[
\left| \int_{T^{m_1} \times T^{m_2}} \varphi(\pi_2 F^n(\omega, x))\psi(x) \, d\omega dx - \int_{T^{m_2}} \varphi(x) d\eta(x) \int_{T^{m_2}} \psi(x) dx \right| \leq C_{\varphi, \psi}(\tilde{C}\tilde{\lambda}^n + \varepsilon)
\]

for all \( \psi \in L^1(T^{m_2}; \mathbb{R}) \) and \( \varphi \in \text{Lip}(T^{m_2}; \mathbb{R}) \) where \( C_{\varphi, \psi} > 0 \) depends on \( \varphi, \psi \) but not from \( n, \varepsilon \).

In fact, we will prove something stronger. Loosely speaking, we show that under the assumptions of Theorem 2.1, for any \( \mu \in \mathcal{M}_1(T^{m_1} \times T^{m_2}) \) which is sufficiently regular, in a sense that will be specified below, the distance between the vertical marginal of \( F^n \mu \) and \( \eta \) can be upper bounded by \( O(\tilde{\lambda}^n + \varepsilon) \) (see Proposition 4.5 for a rigorous statement).

We call this phenomenon approximate memory loss. For a definition and an example of (exact) memory loss see e.g. [OSY09].

The measure \( \eta \) above plays the role of an approximate stationary measure for the forced system. In the case with no distortion, e.g. \( g(\omega) = \sigma \omega \mod 1 \) with \( \sigma \geq 2 \), \( \eta \) equals \( \eta_0 \), the stationary measure of \( P \). As shown in Section 4.3 when there is distortion, \( \eta \)

\[\text{Here the distance is with respect to the Wasserstein-Kantorovich metric defined in equation (4) below.}\]
can be different from \( \eta_0 \), and is related to the fixed point of another operator, called \( L \), introduced in Section 4.2.

**Remark 2.1.**

- Given \( D \) and \( L \), one might need a large minimal expansion \( \sigma_0 \) to ensure that \( \varepsilon > 0 \) is small. Examples of base maps \( g \) with given distortion, and arbitrarily large minimal expansions \( \sigma_0 \) can be constructed easily by fixing any map \( g_0 : T^m \to T^m \) satisfying (H0.1)-(H0.2), and considering \( g := g_0^n \) with high \( n \in \mathbb{N} \). With this definition, \( g \) has minimal expansion equal to the minimal expansion of \( g \) raised to the power \( n \in \mathbb{N} \), and distortion uniformly bounded with respect to \( n \).
- Existence of an invariant measure which is physical or with some smoothness such as an SRB measure (see [You02] for definitions) has little hope in general. One reason is the low regularity of \( F \) which is only Lipschitz. However, imposing higher regularity, e.g. \( F \) globally \( C^{1+\alpha} \), would not be enough as the domination that (possibly) results from the high expansion in the base, even if it can lead to existence of positive Lyapunov exponents, cannot ensure existence of an SRB or physical measure by itself, and all the more reasons not to expect exact exponential decay of correlations.
- We can give an explicit bound for the constant \( C_{\varphi, \psi} \). Letting \( \psi - \int_{T^m} \psi = \psi_1 - \psi_2 \) with \( \psi_1, \psi_2 \geq 0 \) being the positive and negative components of \( \psi - \int_{T^m} \psi \),
  \[
  C_{\varphi, \psi} \leq 2 \|\psi\|_{L^1}(\text{Lip}(\varphi) + 1).
  \]
- As mentioned in the introduction, whenever one has additional information on the fiber maps \( \{f_\omega\}_{\omega \in T} \), other approaches could lead to more precise statements.

2.4. **Examples.** One way to ensure that Assumption \( \mathcal{H} \) holds is by imposing two main regularity requirements on \( f \) with respect to the horizontal variable \( \omega \), i.e. with respect to the forcing: 1) Regularity condition: \( f \) is \( C^k \) in the variable \( \omega \) for a sufficiently large \( k \).
- Non-degeneracy condition: the differential of \( f \) with respect to \( \omega \) is invertible which, for every \( x \in T^{m_2} \), makes the function \( f(\cdot, x) : T^{m_1} \to T^{m_2} \) a local diffeomorphism on its range (notice that for this requirement to hold \( m_1 \) has to be equal to \( m_2 \)).

**Example 2.1.** Let’s consider \( m_1 = m_2 = m \), and assume that for any \( x \in T^m \) \( f(\cdot, x) : T^m \to T^m \) is a \( C^2 \) local diffeomorphism or, equivalently, \( \{f_\omega\}_{\omega \in T^m} \) is a family of maps with \( C^2 \) dependence on the parameter \( \omega \) such that the differential \( (Df(\cdot, x))_\omega \) is bijective for every \( x, \omega \in T^m \).

From Eq. \( \mathcal{H} \) one can deduce that

\[
\mathcal{P} \delta_x = f(\cdot, x)_* \nu_g
\]

and since \( f(\cdot, x) \) is a non-singular transformation, the expression of its Perron-Frobenius operator gives

\[
\frac{d\mathcal{P} \delta_x}{d\text{Leb}_{T^m}}(y) = \sum_k \frac{\rho_g(y_k)}{\|Df(\cdot, x)\|_{y_k}^m}
\]

where the sum is over all the preimages \( y_k \) of \( y \) under the map \( f(\cdot, x) \). \( \frac{d\mathcal{P} \delta_x}{d\text{Leb}_{T^m}} \) is in \( C^1 \) since \( |Df(\cdot, x)| \) and \( \rho_g \) are \( C^1 \) functions. It is also uniformly bounded away from zero, as there is \( c_1 > 0 \) such that \( \rho_g > c_1 \) (see e.g. [Via97]), and there is \( K_1 > 0 \).
such that \(|(Df(\cdot, x))_\omega| \leq K_1\) for every \(\omega, x \in \mathbb{T}^m\). This implies that for every \(x \in \mathbb{T}^m\),
\[
d_{\text{Leb}_{\mathbb{T}^m}}(y) > cK_1,
\]
i.e. the densities of the transition probabilities are all uniformly bounded away from zero. It is well known that the Markov chain generated by \(P\) is geometrically ergodic, i.e. satisfies Assumption \([H]\) (see Theorem A.1 in the Appendix).

The following example is a subcase of the example above and shows one of the simplest possible nontrivial setups.

**Example 2.2** (System with additive deterministic noise). \(f(\omega, x) = T(x) + h(\omega)\), where \(T : \mathbb{T}^m \to \mathbb{T}^m\) is any Lipschitz map, and \(h : \mathbb{T}^m \to \mathbb{T}^m\) is a local \(C^2\) diffeomorphism.

Let us stress that these sufficient conditions for Assumption \([H]\): 1) are by no means necessary; 2) give no control on a single fiber map \(f_\omega\), beyond the requirement that it is Lipschitz regular; 3) do not imply good spectral properties for \(F_\omega\).

2.5. **Sketch of the proof**. Given two probability measures \(\mu_1, \mu_2\) on \(\mathbb{T}^{m_2}\), the Kantorovich-Wasserstein distance between them is defined as
\[
d_W(\mu_1, \mu_2) := \inf_{\gamma \in \mathcal{C}(\mu_1, \mu_2)} \int_{\mathbb{T}^{m_2} \times \mathbb{T}^{m_2}} |x - y| d\gamma(x, y)
\]
where \(\mathcal{C}(\mu_1, \mu_2)\) is the set of couplings of \(\mu_1\) and \(\mu_2\), i.e. the set of all probability measures on \(\mathbb{T}^{m_2} \times \mathbb{T}^{m_2}\) with marginals \(\mu_1\) and \(\mu_2\) respectively on the first and second factor.

The class of measures defined below plays a central role in the proof of Theorem 2.1.

**Definition 2.1.** Given a probability measure \(\mu\) on \(\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}\), we say that \(\mu\) has **Lipschitz disintegration along vertical fibres**, or simply **Lipschitz disintegration**, if there is a disintegration \(\{\mu_\omega\}_{\omega \in \mathbb{T}^{m_1}}\) of \(\mu\), with respect to the measurable partition \(\{\omega\} \times \mathbb{T}^{m_2}\)\(\omega \in \mathbb{T}^{m_1}\), such that the map \(\omega \mapsto \mu_\omega\) from \((\mathbb{T}^{m_1}, |\cdot|)\) to \((\mathcal{M}_1(\mathbb{T}^{m_2}), d_W)\) is Lipschitz.

Let
\[
\text{Lip}(\mu) := \inf_{\omega_1 \neq \omega_2} \frac{d_W(\mu_{\omega_1}, \mu_{\omega_2})}{|\omega_1 - \omega_2|}
\]
the Lipschitz constant of \(\omega \mapsto \mu_\omega\).

Since \(\mathcal{M}_1(\mathbb{T})\) with the metric \(d_W\) is complete, if \(\mu\) admits a Lipschitz disintegration, this disintegration is unique and \(\text{Lip}(\mu)\) is well defined.\(^3\)

To prove Theorem 2.1 we are going to study the evolution of probability measures on \(\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}\) that have a Lipschitz disintegration with a focus on the evolution of their vertical marginals. To do so we follow the steps below.

1) First of all we show that under the assumptions of the main theorem, if \(\mu \in \mathcal{M}_1(\mathbb{T}^{m_1} \times \mathbb{T}^{m_2})\) has Lipschitz disintegration, so does \(F_n^g \mu\) for any \(n \in \mathbb{N}\), and \(\text{Lip}(F_n^g \mu)\) is bounded uniformly in \(n \in \mathbb{N}\) (see Proposition 4.2). This is a consequence of the uniform (high) expansion of the map \(g\). This result shows the existence of an invariant class of measures whose disintegration is smooth along the \(\mathbb{T}^{m_1}\)-coordinate, and is proved by using an explicit expression for a disintegration of \(F_n \mu\) in terms of a disintegration of \(\mu\) (see Proposition 4.1).

\(^3\)In Appendix B we gather statements, such as the above, about disintegration of measures that will be used throughout the paper.
2) Next, we use the above fact to show that the vertical marginal of $F^n \mu$ can be approximated by looking at the action of an auxiliary operator, $L$, that acts on a suitable decomposition of $\mu$, and that, unlike $F$, has contraction properties that can be exploited (see [18] for the definition of $L$, Proposition 4.3, and Proposition 4.4).

3) The above approximation allows to show that under application of $F^*$, the system exhibits approximate exponential memory loss on its vertical marginal. By this we mean that given any two probability measures $\mu_1, \mu_2 \in M_1(T^{m_1} \times T^{m_2})$ with Lipschitz disintegration, the Kantorovich-Wasserstein distance between vertical marginals $\Pi F^n \mu_1$ and $\Pi F^n \mu_2$ shrinks exponentially fast modulo an approximation error (see Proposition 4.5).

4) Finally, we use the above approximate memory loss to prove the approximate decay of correlations.

In Section 3, we give a proof of Theorem 2.1 in the simpler case where: $m_1 = m_2 = 1$; the dynamics in the base is smooth and has no distortion, i.e. $D = 0$. Under these assumptions $g : T \to T$ can be written as

\[(A0+) \quad g(\omega) = \sigma \omega \mod 1, \quad \sigma \in \mathbb{N}\setminus\{1\}.\]

This is for the sake of presentation since in this case the treatment of points 2) and 3) does not require the introduction of the auxiliary operator $L$, whose role is played by $P$. This makes the proof much easier than in the fully general case.

Remark 2.2. Picking a metric on $M_1(T^{m_2})$ as weak as the Wasserstein metric $d_{W}$ is crucial to our arguments. Without further assumptions on $\{f_\omega\}_{\omega \in \Omega}$, measures with Lipschitz disintegration with respect to the $d_{TV}$ distance, for example, may not constitute an invariant class with respect to the action of $F^*$.

3. Case without distortion

In this section we work under Assumption [A0+]. Namely we consider $g : T \to T$ defined as $g(\omega) = \sigma \omega \mod 1$, where $\sigma \in \mathbb{N}\setminus\{1\}$. Recall that under these assumptions $\nu_g = \text{Leb}_T$ and $\rho_g$ is constant equal to one. Take $\mu \in M_1(T \times T)$ having horizontal marginal Leb$_T$. We study the evolution of $\mu$ under applications of $F^*$. First of all notice that as a consequence of the product structure of $F$ and invariance of Leb$_T$ under $g$, also $F^* \mu$ has horizontal marginal equal to Leb$_T$. The evolution of the disintegration along vertical fibres, instead is described in the following proposition.

Proposition 3.1. Let $\mu$ be a probability measure on $T \times T$ with horizontal marginal equal to Leb$_T$. Let $\{\mu_\omega\}_{\omega \in T}$ be a disintegration of $\mu$ along vertical fibres, then a disintegration of $F^* \mu$ along vertical fibres is given by $\{(F^* \mu)_\omega\}_{\omega \in T}$ with

\[(5) \quad (F^* \mu)_\omega = \frac{1}{\sigma} \sum_{i=0}^{\sigma-1} f_{\omega + i\sigma} \mu_{\omega + i\sigma}.\]

The proof is a particular case of Proposition [4.1] therefore we omit the details. Sufficient to say that, in this setting one has, for an interval $I$

\[(F^* \mu)_\omega(I) = \lim_{\delta \to 0} \frac{(F^* \mu)([\omega - \delta, \omega + \delta] \times I)}{2\delta} \]
where the numerator can be easily controlled. Next, recall Definition 2.1. In the proposition below we use [5] to deduce that if $\mu$ has Lipschitz disintegration, then so do all its iterates $F^n_\sigma \mu$, and if $\sigma$ is sufficiently large, then their Lipschitz constants are all uniformly bounded.

Before moving to the next proposition we recall for the reader’s convenience a property of Wasserstein spaces (see e.g. [Vil09] for details). Given a Borel signed measure $\xi$ on $T$ with $\xi(T) = 0$, consider the Wasserstein norm

$$\|\xi\|_W := \sup_{\varphi \in \text{Lip}(T)} \int_T \varphi \, d\xi.$$ 

Recall that we denoted by Lip$(1)(T) := \{\varphi : T \to \mathbb{R} : \text{Lip}(\varphi) \leq 1\}$ the Lipschitz functions on $(T, | \cdot |)$ with Lipschitz constant less or equal to one (we write Lip$(1)$ when there is no risk of confusion). The Kantorovich-Wasserstein distance defined in [4] can be rewritten as

$$d_W(\mu, \nu) = \|\mu - \nu\|_W = \sup_{\varphi \in \text{Lip}(1)} \int_T \varphi(x) \, d(\mu - \nu)(x)$$

which will simplify the notation later in the proofs.

**Proposition 3.2.** Let $\mu$ be a probability measure on $T \times T$ with horizontal marginal Leb$_\sigma$ and Lipschitz disintegration $\{\mu_\omega\}_{\omega \in T}$. Then the disintegration of $F_\sigma \mu$ defined in Eq. [5] is also Lipschitz and

$$\text{Lip}(F_\sigma \mu) \leq L\sigma^{-1} \text{Lip}(\mu) + L\sigma^{-1}.$$

**Proof.**

$$d_W((F_\sigma \mu)_\omega, (F_\sigma \mu)_{\omega'}) = \sup_{\varphi \in \text{Lip}(1)} \int_T \varphi \, d \left( \frac{1}{\sigma} \sum_{i=0}^{\sigma-1} f_{\omega + i \frac{\omega}{\sigma}} \mu_{\omega + i \frac{\omega}{\sigma}} - \frac{1}{\sigma} \sum_{i=0}^{\sigma-1} f_{\omega + i \frac{\omega}{\sigma}} \mu_{\omega + i \frac{\omega}{\sigma}} \right)$$

$$\leq \frac{1}{\sigma} \sum_{i=0}^{\sigma-1} \sup_{\varphi \in \text{Lip}(1)} \int_T \varphi \, d \left( f_{\omega + i \frac{\omega}{\sigma}} \mu_{\omega + i \frac{\omega}{\sigma}} - f_{\omega + i \frac{\omega}{\sigma}} \mu_{\omega + i \frac{\omega}{\sigma}} \right)$$

Calling $\omega_i := \frac{\omega + i \frac{\omega}{\sigma}}{\sigma}$, and $\omega_i' := \frac{\omega + i \frac{\omega}{\sigma}}{\sigma}$ for brevity, we have

$$\sup_{\varphi \in \text{Lip}(1)} \int_T \varphi \, d \left( f_{\omega_i \mu_{\omega_i}} - f_{\omega_i' \mu_{\omega_i'}} \right) = d_W(f_{\omega_i \mu_{\omega_i}}, f_{\omega_i' \mu_{\omega_i'}})$$

$$\leq d_W(f_{\omega_i \mu_{\omega_i}}, f_{\omega_i' \mu_{\omega_i}}) + d_W(f_{\omega_i' \mu_{\omega_i}}, f_{\omega_i' \mu_{\omega_i}}).$$

We bound the first term from above. Notice that for any $\xi \in M_1(T)$ and $\varphi \in \text{Lip}(1)$

$$\int_T \varphi \, d(f_{\omega_i \mu_{\omega_i}} \xi - f_{\omega_i' \mu_{\omega_i}} \xi) = \int_T (\varphi \circ f_{\omega_i}(x) - \varphi \circ f_{\omega_i'}(x)) \, d\xi(x)$$

$$\leq \text{Lip}(\varphi) \int_T |f_{\omega_i}(x) - f_{\omega_i'}(x)| \, d\xi(x)$$

$$\leq L |\omega_i - \omega_i'| \leq L\sigma^{-1} |\omega - \omega'|$$

where $L$ is the Lipschitz constant of $f$. The above implies

$$d_W(f_{\omega_i \mu_{\omega_i}}, f_{\omega_i' \mu_{\omega_i}}) \leq L\sigma^{-1} |\omega - \omega'|.$$
The second term can be bounded using an analogous computation
\[ d_W(f_\omega^* \mu_\omega, f_\omega'^* \mu'_\omega) \leq L d_W(\mu_\omega, \mu'_\omega) \leq L \text{Lip}(\mu) \sigma^{-1} |\omega - \omega'|. \]
where we used that the Lipschitz constant of \( f_\omega \) is equal to the Lipschitz constant of \( f_\omega^* \) (see Lemma C.1 in the Appendix) which is upper bounded by \( L \) as in \( \text{H0.3} \).

Putting everything together we obtain
\[ d_W((F_\omega^* \mu_\omega), (F_\omega' \mu_\omega')) \leq L \sigma^{-1} \left[ 1 + \text{Lip}(\mu) \right] |\omega - \omega'|. \]

□

As a corollary to the previous proposition, for \( \sigma \) sufficiently large, we obtain the existence of an invariant class of measures whose disintegration has Lipschitz dependence on the variable \( \omega \in T \), and whose Lipschitz constant goes to zero as \( \sigma \to \infty \). More precisely, let’s define the set \( \mathcal{M}_{1,\text{Leb}}(T \times T) \) of probability measures on \( T \times T \) with horizontal marginal \( \text{Leb}_T \). Let’s call \( \Gamma_\ell \subset \mathcal{M}_{1,\text{Leb}} \) the set of those probability measures that have Lipschitz disintegration with Lipschitz constant at most \( \ell \):
\[ \Gamma_\ell := \{ \mu \in \mathcal{M}_{1,\text{Leb}} : \text{Lip}(\mu) \leq \ell \}. \]

**Corollary 3.1.** If \( \sigma > L \), then the set \( \Gamma_\ell \) is invariant under the push-forward \( F_\omega \) for every \( \ell \geq \ell_0 \) with
\[ \ell_0 := \frac{L \sigma^{-1}}{1 - L \sigma^{-1}}. \]

The following proposition controls the evolution of vertical marginals for two probability measures in \( \Gamma_{\ell_0} \) under application of \( F_\omega \). In the statements below, the constants \( C \) and \( \lambda \) are the same as those in Assumption [H].

**Proposition 3.3 (Approximate Memory Loss).** For every \( \varepsilon > 0 \) there is \( \sigma_0(\varepsilon) > L \) such that if \( \sigma > \sigma_0(\varepsilon) \) then
i) \( d_W(\Pi F_\omega^n \mu_1, \Pi F_\omega^n \mu_2) \leq C \lambda^n + \varepsilon, \quad \forall \mu_1, \mu_2 \in \Gamma_{\ell_0}; \)
ii) \( d_W(\Pi F_\omega^n \mu, \eta_0) \leq C \lambda^n + \varepsilon, \quad \forall \mu \in \Gamma_{\ell_0}; \)
where \( \eta_0 \) is the stationary measure for \( \mathcal{P} \).

**Proof.** Let \( \mu := \mu_1 - \mu_2 \) and recall that \( \Pi \mu = \int_T \mu_\omega \, d\omega \) is the vertical marginal of \( \mu \). Since
\[ d_W(\mu_\omega, \mu'_\omega) \leq \ell_0 |\omega - \omega'| \leq \ell_0 \]
then \( d_W(\mu_\omega, \Pi \mu) \leq \ell_0 \) (see Lemma C.2 in the Appendix). Therefore,
\[ \Pi F_\omega \mu = \int_T f_\omega^* \mu_\omega \, d\omega = \int_T f_\omega^* (\mu_\omega - \Pi \mu) \, d\omega + \int_T f_\omega^* (\Pi \mu) \, d\omega \]
\[ = \int_T f_\omega^* (\mu_\omega - \Pi \mu) \, d\omega + \mathcal{P}(\Pi \mu), \]
where \( \mathcal{P} \) is defined in Eq. (2) and, by Lemma C.1
\[ \left\| \int_T f_\omega^* (\mu_\omega - \Pi \mu) \, d\omega \right\|_W \leq L \ell_0. \]
For higher iterates, one gets the telescopic sum
\[
\Pi F_n^\ast \mu = \int_T d\omega_{n-1} f_{\omega_{n-1}}((F_n^1)_{\omega_{n-1}} - \Pi F_n^1 \mu) + \int_T d\omega_{n-1} f_{\omega_{n-1}}(\Pi F_n^{n-1} \mu)
\]
and by triangular inequality
\[
\|\Pi F_n^\ast \mu\|_W \leq \|\Pi^n \Pi \mu\|_W + \left|\sum_{i=0}^{n-1} \int_T d\omega_{n-1} f_{\omega_{n-1}}((F_n^i)_{\omega_{n-1}} - \Pi F_n^i \mu)\right|_W.
\]
For the first term in the above inequality,
\[
\|\Pi^n \Pi \mu\|_W = d_W(\Pi^n \Pi \mu_1, \Pi^n \Pi \mu_2) \leq d_{TV}(\Pi^n \Pi \mu_1, \Pi^n \Pi \mu_2) \leq C\lambda^n
\]
where the first inequality follows by \(d_W \leq d_{TV}\) (see Lemma C.3), and the last inequality is Assumption H. For the second term, each summand can be treated as follows
\[
\left|\sum_{i=0}^{n-1} \int_T d\omega_{n-1} f_{\omega_{n-1}}((F_n^i)_{\omega_{n-1}} - \Pi F_n^i \mu)\right|_W \leq \sup_{\omega} \text{Lip}(f_{\omega})^{n-1-i} \sup_{\omega} \left|\Pi F_n^i \mu_1 - \Pi F_n^i \mu_2\right|_W
\]
\[
\leq 2L^{n-1-i} \ell_0.
\]
Now, one can pick \(n_0 \in \mathbb{N}\) such that \(C\lambda^{n_0} \leq \varepsilon/2\), and \(\sigma_0 > 0\) so that
\[
\frac{L\sigma_0^{-1}}{1 - L\sigma_0^{-1}} \sum_{i=0}^{n_0-1} L^{n_0-1-i} = \ell_0 \sum_{i=0}^{n_0-1} L^{n_0-1-i} \leq \varepsilon/2.
\]
This way, if \(n \leq n_0\)
\[
\|\Pi F_n^\ast \mu\|_W \leq C\lambda^n + 2\ell_0 \sum_{i=0}^{n-1} L^i
\]
\[
\leq C\lambda^n + 2\ell_0 \sum_{i=0}^{n_0-1} L^i
\]
\[
\leq C\lambda^n + \varepsilon/2
\]
and if \(n \geq n_0\),
\[
d_W(\Pi F_n^\ast \mu_1, \Pi F_n^\ast \mu_2) = d_W(\Pi F_n^{n_0} F_n^{n-n_0} \mu_1, \Pi F_n^{n_0} F_n^{n-n_0} \mu_2),
\]
and since \(F_n^{n-n_0} \mu_1\) and \(F_n^{n-n_0} \mu_2\) both belong to \(\Gamma_{\ell_0}\)
\[
d_W(\Pi F_n^{n_0}(F_n^{n-n_0} \mu_1), \Pi F_n^{n_0}(F_n^{n-n_0} \mu_2)) \leq C\lambda^{n_0} + \varepsilon/2 \leq C\lambda^n + \varepsilon
\]
which proves point i).
For point ii), going back to (6) and picking \(n_0\) and \(\sigma_0\) as above, for any \(\mu \in \Gamma_{\ell_0}\) and \(n \leq n_0\)
\[
d_W(\Pi F_n^\ast \mu, \eta_0) \leq d_W(\Pi F_n^\ast \mu, \Pi \nu) + d_W(\Pi \nu, \eta_0)
\]
\[
\leq C\lambda^n + \varepsilon/2
\]
while for $n \geq n_0$ we use an analogous computation and get
\[
d_W(\Pi F^n_*\mu, \eta_0) \leq d_W(\Pi F^n_* F^{n-n_0}_*\mu, \mathcal{P}^{n_0}\Pi F^{n-n_0}_*\mu) + d_W(\mathcal{P}^{n_0}\Pi F^{n-n_0}_*\mu, \eta_0)
\leq C\lambda^{n_0} + \varepsilon/2
\leq C\lambda^n + \varepsilon
\]

We can now proceed with the proof of the main theorem in the case without distortion.

Proof of Theorem 2.1 under condition (A0+). Assume that $\int \psi(x)dx = 0$. Then $\psi = \psi_1 - \psi_2$ where $\psi_1, \psi_2 \geq 0$ are the positive and negative parts of $\psi$ and $\int \psi_1 = \int \psi_2 =: M$. Take $\mu$ the measure on $\mathbb{T} \times \mathbb{T}$ defined as
\[
d_\mu(\omega, x) = M^{-1}(\psi_1(\omega) - \psi_2(\omega))d\omega dx.
\]
It follows that $\mu = \mu_1 - \mu_2$ where $\mu_1, \mu_2$ are probability measures having constant disintegrations $\mu_{1,\omega} = M^{-1}\psi_1(\omega)dx$ and $\mu_{2,\omega} = M^{-1}\psi_2(\omega)dx$. In particular, $\mu_1, \mu_2 \in \Gamma_{\ell_0}$.

Now, picking $\sigma_0$ as in Proposition 3.3 if $\sigma > \sigma_0$
\[
\int_{\mathbb{T} \times \mathbb{T}} \varphi \circ F^n(\omega, x)\psi(x)d\omega dx = M \int_{\mathbb{T} \times \mathbb{T}} \varphi(x)d(F^n_*\mu)(\omega, x)
\leq M\text{Lip}(\varphi)(C\lambda^n + \varepsilon)
\]
where for (10) we used the duality property of the push-forward and the definition of $\mu$, and in (12) we used that $\varphi$ does not depend on $\omega \in \mathbb{T}$ and Proposition 3.3.

If $\int \psi \neq 0$ consider $\tilde{\psi} := \psi - \int \psi$.
\[
\int_{\mathbb{T} \times \mathbb{T}} \varphi \circ F^n(\omega, x)\psi(x)d\omega dx = \int_{\mathbb{T} \times \mathbb{T}} \varphi \circ F^n(\omega, x)\tilde{\psi}(x)d\omega dx + \int_{\mathbb{T} \times \mathbb{T}} \varphi \circ F^n(\omega, x)d\tilde{\psi}(x)d\omega dx
\]
and from point ii) of Proposition 3.3 the above is less than $\text{Lip}(\varphi)[C\lambda^n + \varepsilon]$. By triangular inequality
\[
\left| \int_{\mathbb{T} \times \mathbb{T}} \varphi \circ F^n(\omega, x)\psi(x)d\omega dx - \int_{\mathbb{T}} \varphi(x)d\eta_0(x) \right| \leq C_{\varphi, \psi}(C\lambda^n + \varepsilon)
\]
where $C_{\varphi, \psi} \leq \frac{3}{2}||\psi||_{L_1}(\text{Lip}(\varphi) + 1)$. □
Remark 3.1. As a remark, note that if one tries to estimate the quantifier $\varepsilon$ in Theorem 2.1 for a given datum, by inspecting the proof of this simpler case, one realizes that a bound for $\varepsilon$ is proportional to the smallest number one gets from the sequence $\{\max\{C\lambda^n, 2\sigma^{-1}L^n\}\}_{n \in \mathbb{N}}$. Since the first sequence is decreasing, while the second is increasing, the optimal trade off is achieved when they are of about the same size. Thus imposing that the two numbers are equal, one obtains the estimate $\varepsilon \lesssim \sigma^{-\gamma}$ for some $\gamma > 0$ which depends on $C$, $\lambda$, and $L$.

4. General case: proof of Theorem 2.1

4.1. Control on the disintegration along vertical fibres. Take a measure $\mu_0$ on $\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$ with horizontal marginal equal to $\nu_0 \in \mathcal{M}_1(\mathbb{T}^{m_1})$ which is absolutely continuous with respect to Lebesgue, and let $\mu_1 := F_\ast \mu_0$. It follows from the skew-product structure of $F$ that the horizontal marginal of $\mu_1$ equals $\nu_1 := g_\ast \nu_0$. We will denote by $\rho_1$ the density of $\nu_1$.

Recall from Section 2 that $g$ is a local diffeomorphism, $g_i$ are its invertible branches, and $h_i$ their inverses. Then an explicit expression of $\rho_1$ in terms of $\rho_0$ is given by

$$\rho_1(\omega) = \frac{d}{d \sum_{i=1}^d \rho_0(\omega_i)} \forall \omega \in \mathbb{T}^{m_1}$$

where we denote by $\omega_i = h_i(\omega)$ the preimages of $\omega$ and $|Dg_\omega|$ for $|Dg(\omega)|$.

For $k \in \{0, 1\}$, let $\{\mu_{k, \omega}\}_{\omega \in \mathbb{T}^{m_1}}$ be a disintegration of $\mu_k$ w.r.t. the measurable partition $\{\omega\} \times \mathbb{T}^{m_2}\}_{\omega \in \mathbb{T}^{m_1}}$. For a definition and some results on disintegrations see Appendix B.

Proposition 4.1. A disintegration $\{\mu_{1, \omega}\}_{\omega \in \mathbb{T}^{m_1}}$ of $\mu_1$ is given by

$$\mu_{1, \omega} = \frac{1}{\rho_1(\omega)} \sum_{i=1}^d \rho_0(\omega_i) \left| \frac{1}{Dg_\omega} \right| \int_{h_i(B_\delta(\omega))} d\mu_0(s) \mu_0(\omega_i).$$

Proof. Let $B_\delta(\omega) \subset \mathbb{T}^{m_1}$ be the Euclidean ball centered at $\omega$ of radius $\delta$. By Theorem B.1 in Appendix B for Leb$_{\mathbb{T}^{m_1}}$ a.e. $\omega$

$$\mu_{1, \omega} = \lim_{\delta \to 0} \frac{\int_{B_\delta(\omega)} d\rho_1(s) \mu_{1, s}}{\int_{B_\delta(\omega)} d\rho_1(s)}$$

where the limit is with respect to the weak* topology. Using the definition of disintegration and that $\mu_1(B_\delta(\omega) \times I) = \mu_0(F^{-1}(B_\delta(\omega) \times I))$, for every measurable set $I$ on $\mathbb{T}^{m_2}$ one gets, for $\delta > 0$ sufficiently small,

$$\int_{B_\delta(\omega)} d\rho_1(s) \mu_{1, s} = \sum_{i=1}^d \int_{h_i(B_\delta(\omega))} d\rho_0(s) f_{s, \ast} \mu_0(s).$$

\footnote{Since $g$ is a local diffeomorphism is in particular nonsingular and its push-forward sends absolutely continuous measures to absolutely continuous measures.}
By changing variables, \( s = h_i(s') \), and multiplying and dividing by \( \rho_1(s) \), the above equals

\[
\int_{B_\delta(\omega)} ds' \rho_1(s') \left[ \frac{1}{\rho_1(s')} \sum_{i=1}^d \frac{\rho_0(s'_i)}{|Dg_{s'_i}|} f_{s'_i} \mu_{0,s'_i} \right],
\]

where we denoted \( s'_i = h_i(s') \). Applying Lebesgue’s differentiation theorem, Eq. \([14]\) becomes

\[
\mu_{1,\omega} = \frac{1}{\rho_1(\omega)} \sum_{i=1}^d \frac{\rho_0(\omega)}{|Dg_{\omega}|} f_{\omega} \mu_{0,\omega}.
\]

\( \square \)

The formula for the evolution of disintegrations in \([13]\) depends on \( \nu_0 \) and \( \nu_1 \), the horizontal marginals of the measures \( \mu_0 \) and \( \mu_1 \). Thanks to assumptions on \( g \), the evolution of the horizontal can be controlled (see Lemma 4.1 below).

Consider for a \( a \geq 0 \), the cone of log-Lipschitz functions,

\[
\mathcal{V}_a := \left\{ \varphi : T^{m1} \rightarrow \mathbb{R}^+ : \frac{\varphi(\omega)}{\varphi(\omega')} \leq e^{a|\omega-\omega'|} \right\}.
\]

The following lemma gathers some standard facts about uniformly expanding maps with bounded distortion, such as \( g \).

**Lemma 4.1.** Let \( g : T^{m_1} \rightarrow T^{m_1} \) be a \( C^2 \) local diffeomorphism satisfying \([H0.1]-[H0.2]\), and let \( \rho_0 \) and \( \rho_1 \) be as above.

i) If \( \rho_0 \in \mathcal{V}_a \), then \( \rho_1 \in \mathcal{V}_{\sigma^{-1}a + D} \). In particular, if \( a \geq a_0 := \frac{D}{e^{-\sigma} - 1} \), then \( \rho_1 \in \mathcal{V}_a \);  

ii) If \( \rho_0 \in \mathcal{V}_{a_0} \), calling \( \rho_n \) the density of \( \nu_n := g^n \nu_0 \), there are \( C_g > 0 \) and \( \lambda_g \in (0,1) \) such that

\[
\|\rho_n - \rho_g\|_\infty := \sup_{\omega \in T^{m1}} |\rho_n(\omega) - \rho_g(\omega)| \leq C_g \lambda_g^n.
\]

**Proof.** See e.g. \([Liv95]\), \([Via97]\). \( \square \)

**Remark 4.1.** In point ii) of Lemma 4.1, one can choose \( C_g := C_g(D, \sigma) \) and \( \lambda_g := \lambda_g(D, \sigma) \). Moreover, fixed \( D \), the two functions can be chosen to be decreasing with respect to \( \sigma \). This implies that fixed \( D > 0 \) and \( \sigma_0 > 1 \), there are constants \( C \) and \( \lambda \in (0,1) \) such that \( C_g < C \) and \( \lambda_g < \lambda \) for any \( g \) satisfying \([H0.1]-[H0.2]\) with \( \sigma \geq \sigma_0 \).

From now on we will restrict our analysis to probability measures on \( T^{m_1} \times T^{m_2} \) whose horizontal marginals belong to \( \mathcal{V}_a \) for some \( a > 0 \).

**Proposition 4.2.** Assume \( \rho_0 \in \mathcal{V}_a \) for some \( a \geq a_0 \) and that \( \mu_0 \) has Lipschitz disintegration. Then the disintegration of \( \mu_1 \) given in \([13]\) is Lipschitz and

\[
\text{Lip} (\mu_1) \leq \sigma^{-1} L \text{Lip} (\mu_0) + \left[ C_a + \sigma^{-1} L \right]
\]

where \( C_a := e^{(a+\sigma^{-1}a + D) C_1} \) and \( C_1 \) is the diameter of \( T^{m_1} \).
Proof. The proof is analogous to that of Proposition 3.2, although one has to work with \([13]\), rather than the simpler formula \([5]\).

\[
d_W(\mu_{1,\omega}, \mu_{1,\omega'}) =
\]

\[
= \sup_{\varphi \in \text{Lip}^1} \int \varphi \left( \sum_{i=1}^{d} \frac{1}{\rho_1(\omega)} \frac{\rho_0(\omega_i)}{|Dg_{\omega_i}|} f_{\omega_i, \mu_{0,\omega_i}} - \frac{1}{\rho_1(\omega')} \frac{\rho_0(\omega'_i)}{|Dg_{\omega'_i}|} f_{\omega_i, \mu_{0,\omega'_i}} \right)
\]

\[
\leq \sup_{\varphi \in \text{Lip}^1} \int \varphi \left( \sum_{i=1}^{d} \frac{1}{\rho_1(\omega)} \frac{\rho_0(\omega_i)}{|Dg_{\omega_i}|} f_{\omega_i, \mu_{0,\omega_i}} - \frac{1}{\rho_1(\omega')} \frac{\rho_0(\omega'_i)}{|Dg_{\omega'_i}|} f_{\omega_i, \mu_{0,\omega'_i}} \right) + \sum_{i=1}^{d} \int \varphi \left( \frac{1}{\rho_1(\omega')} \frac{\rho_0(\omega'_i)}{|Dg_{\omega'_i}|} f_{\omega_i, \mu_{0,\omega_i}} - \frac{1}{\rho_1(\omega')} \frac{\rho_0(\omega'_i)}{|Dg_{\omega'_i}|} f_{\omega_i, \mu_{0,\omega'_i}} \right)
\]

\[
=: A + B
\]

where to get the inequality we added and subtracted the same quantity and distributed the sup.

**Upper bound for** \(A\). To bound the first term

\[
A = \sup_{\varphi \in \text{Lip}^1} \sum_{i=1}^{d} \frac{1}{\rho_1(\omega)} \frac{\rho_0(\omega_i)}{|Dg_{\omega_i}|} \int \left( 1 - \frac{1}{\rho_1(\omega')} \frac{\rho_0(\omega'_i)}{|Dg_{\omega'_i}|} \right) \varphi d(f_{\omega_i, \mu_{0,\omega_i}})
\]

\[
\leq \frac{1}{\rho_1(\omega)} \sum_{i=1}^{d} \frac{\rho_0(\omega_i)}{|Dg_{\omega_i}|} \left| 1 - e^{(a + \sigma^{-1}a + D)|\omega - \omega'|} \right|
\]

where \(C_1 > 0\) is the diameter of \(T_m^1\). To estimate the ratio in parenthesis, we used that \(\rho_0 \in \mathcal{V}_a\) with \(a > a_0\) implies \(\rho_1 \in \mathcal{V}_a\), the fact that \(|\varphi| \leq 1\), and \([H0.2]\).

**Upper bound for** \(B\). The second term can be bounded by

\[
d_W \left( \sum_{i=1}^{d} \frac{1}{\rho_1(\omega')} \frac{\rho_0(\omega'_i)}{|Dg_{\omega'_i}|} f_{\omega_i, \mu_{0,\omega_i}}, \sum_{i=1}^{d} \frac{1}{\rho_1(\omega')} \frac{\rho_0(\omega'_i)}{|Dg_{\omega'_i}|} f_{\omega_i, \mu_{0,\omega'_i}} \right) \leq 
\]

\[
= \max_i d_W \left( f_{\omega_i, \mu_{0,\omega_i}}, f_{\omega'_i, \mu_{0,\omega'_i}} \right),
\]

where we used that \(\sum_{i=1}^{d} \frac{\rho_0(\omega'_i)}{|Dg_{\omega'_i}|} = 1\) and Lemma C.4 about the Wasserstein distance between convex combinations of measures.

By triangular inequality,

\[
d_W \left( f_{\omega_i, \mu_{0,\omega_i}}, f_{\omega'_i, \mu_{0,\omega'_i}} \right) \leq d_W \left( f_{\omega_i, \mu_{0,\omega_i}}, f_{\omega'_i, \mu_{0,\omega'_i}} \right) + d_W \left( f_{\omega'_i, \mu_{0,\omega_i}}, f_{\omega'_i, \mu_{0,\omega'_i}} \right).
\]

For the first term, pick any \(\varphi \in \text{Lip}^1\) and any probability measure \(\xi\)

\[
\int \varphi d(f_{\omega_i, \xi} - f_{\omega'_i, \xi}) = \int (\varphi \circ f_{\omega_i} - \varphi \circ f_{\omega'_i}) d\xi
\]

\[
\leq L|\omega_i - \omega'_i| \leq L\sigma^{-1}|\omega - \omega'|.
\]
So
\[ d_W(f\omega, \mu_0, f\omega', \mu_0) \leq L \sigma^{-1} |\omega - \omega'|. \]

For the second term, using the fact that for every \( \omega \in T^{m_1} \), \( \text{Lip}(f\omega) = L \),
\[ d_W(f\omega', \mu_0, f\omega', \mu_0') \leq L \text{Lip}(\mu_0) |\omega_i - \omega_i'| \]
\[ \leq L \text{Lip}(\mu_0) \sigma^{-1} |\omega - \omega'|, \]
which implies
\[ B \leq \sigma^{-1} L [1 + \text{Lip}(\mu_0)] |\omega - \omega'|. \]

Putting together the estimates for \( A \) and \( B \)
\[ \text{Lip}(\mu_1) \leq \sigma^{-1} L \text{Lip}(\mu_0) + [C_a + \sigma^{-1} L] \]
\[ \square \]

As a corollary to the previous proposition we obtain the existence of an invariant class of measures with Lipschitz disintegration and with uniformly bounded Lipschitz constant. More precisely, let’s define \( \Gamma_{\ell,a} \) the set of probability measures that have a Lipschitz disintegration with constant at most \( \ell \) and horizontal marginal with density in \( \mathcal{V}_a \):
\[ \Gamma_{\ell,a} := \left\{ \mu \in \mathcal{M}_1(T^{m_1} \times T^{m_2}) : \text{Lip}(\mu) \leq \ell \text{ and } d_{\pi_1, \mu} \text{Leb}_{T^{m_1}} \in \mathcal{V}_a \right\}. \]

Corollary 4.1.

i) If \( \rho \in \mathcal{V}_a \) with \( a \leq a_0 := \frac{D}{1 - \sigma^{-1}}, \sigma > L \), then \( F^*_\rho(\Gamma_{\ell,a}) \subset \Gamma_{\ell,a_0} \) for every \( \ell \geq \ell_0 \)
\[ (15) \]
\[ \ell_0 := \frac{\sigma^{-1} L + C_{a_0}(1 + D)}{(1 - \sigma^{-1} L)}. \]

ii) If there are \( a > 0 \) and \( \ell > 0 \) such that \( \mu \in \Gamma_{\ell,a} \), then for every \( \delta > 0 \) there is \( N \in \mathbb{N} \) such that
\[ \text{Lip}(F^n_\mu) \leq \ell_0 + \delta \]
for all \( n > N \).

Proof. To prove i), recall that the horizontal marginal of \( F^*_\mu \) is the push-forward under \( g \) of the horizontal marginal of \( \mu \). Since the horizontal marginal of \( \mu \) has density in \( \mathcal{V}_{a_0} \), by the inclusion \( \mathcal{V}_a \subset \mathcal{V}_{a_0} \) and Lemma 4.1, the horizontal marginal of \( F^*_\mu \) belongs to \( \mathcal{V}_{a_0} \). By Proposition 4.2 and the choice of \( \ell_0 \), it follows that if \( \text{Lip}(\mu) \leq \ell_0 \) then also \( \text{Lip}(F^*_\mu) \leq \ell_0 \).

For point ii) notice that the horizontal marginal of \( F^n_\mu \) belongs to \( \Gamma_{\ell_n,a_n} \) for some \( a_n \) and \( \ell_n \) such that \( a_n \to a_0 \) as \( n \to \infty \) by Lemma 4.1 and \( \ell_n \to \ell_0 \) as \( n \to \infty \) by Proposition 4.2. The claim follows easily. \( \square \)
4.2. Tracking the evolution of the vertical marginal. Let’s consider $\mathcal{M}_{1,\nu_g}(\mathbb{T}^{m_1} \times \mathbb{T}^{m_2})$ the set of Borel probability measures on $\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$ having horizontal marginal equal to $\nu_g$, the invariant measure for $g$, and recall that for $\mu \in \mathcal{M}_{1,\nu_g}(\mathbb{T}^{m_1} \times \mathbb{T}^{m_2})$, the vertical marginal is given by

$$
\Pi\mu = \int_{\mathbb{T}^{m_2}} d\omega \rho_g(\omega) \mu_\omega.
$$

For every $i = 1, \ldots, d$, call

$$
\overline{\rho}_i := \nu_g(I_i) = \int_{I_i} d\omega \rho_g(\omega)
$$

the measure of $I_i$ with respect to the invariant measure of $g$. Define the map $\Delta : \mathcal{M}_{1,\nu_g}(\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}) \rightarrow (\mathcal{M}_1(\mathbb{T}^{m_2}))^d$ in the following way

$$(\Delta \mu)_i := \overline{\rho}_i^{-1} \int_{I_i} d\omega \rho_g(\omega) \mu_\omega,
$$

i.e. $(\Delta \mu)_i$ is the average of the disintegration $\{\mu_\omega\}_{\omega \in \mathbb{T}^{m_1}}$ on $I_i$ with respect to the invariant measure of $g$. The map $\Delta$ gives a decomposition of $\mu$ which can be viewed as a coarse-graining of the disintegration of $\{\mu_\omega\}_{\omega \in \mathbb{T}^{m_1}}$. Moreover, for any $\mu \in \mathcal{M}_{1,\nu_g}(\mathbb{T}^{m_1} \times \mathbb{T}^{m_2})$

$$
\Pi\mu = \sum_{i=1}^d \overline{\rho}_i (\Delta \mu)_i,
$$

therefore, by keeping track of $\Delta(F^*_n \mu)$, we can keep track of $\Pi F^*_n \mu$.

Consider also

$$
F_i := \overline{\rho}_i^{-1} \int_{I_i} d\omega \rho_g(\omega) f_{\omega^*},
$$

which is the average of the operators $\{f_{\omega^*}\}_{\omega \in \mathbb{T}^{m_1}}$ on $I_i$ w.r.t. $\nu_g$ restricted to $I_i$ and normalized. A lemma below shows that $F_i$ is an approximation of $f_{\omega^*}$ for $\omega \in I_i$. The smaller is the size of $I_i$, i.e. the larger is $\sigma > 0$, the better is the approximation.

For every $1 \leq i, j \leq d$, define the operators

$$
L_{ij} := \overline{\rho}_i^{-1} \left( \int_{I_i} d\omega \rho_g(\omega_j) \right) F_j;
$$

and consider the operator $\mathcal{L} : (\mathcal{M}_1(\mathbb{T}^{m_2}))^d \rightarrow (\mathcal{M}_1(\mathbb{T}^{m_2}))^d$

$$
(\mathcal{L} \mu)_i = \sum_{j=1}^d L_{ij}(\mu)_j.
$$

Remark 4.2. Before moving on, let us stress why the above mappings $\Delta$ and $\mathcal{L}$ are important: $\Delta(F^*_n \mu)$ and $\mathcal{L}(\Delta \mu)$ are very close when the expansion of $g$ is very large. This will let us prove that for fixed $n$, we can approximate $\Delta(F^*_n \mu)$ with $\mathcal{L}^n(\Delta \mu)$ when the expansion of $g$ is sufficiently large, with the advantage that $\mathcal{L}$ has good contraction properties.
The remark above is formalised in the following propositions. For $\mu_1, \mu_2 \in (\mathcal{M}_1(\mathbb{T}^n))^d$ we define 
\[
d_W(\mu_1, \mu_2) = \max_{i=1, \ldots, d} d_W((\mu_1)_i, (\mu_2)_i).
\]

**Proposition 4.3.** If $\mu \in \Gamma_{\ell_0, a_0}$, with $\ell_0$ and $a_0$ as in Corollary 4.1 then there is a constant $K_# > 0$ uniform in $\sigma$, and $C_3 : (1, +\infty) \to \mathbb{R}^+$ decreasing such that 
\[
d_W(\Delta(F^m \mu), \mathcal{L}^n(\Delta \mu)) < K_# L^{n+1} \left( \sigma^{-1} + C_3(\sigma) \|\rho_0 - \rho_g\|_\infty \right)
\]
where $\rho_0$ is the density of the horizontal marginal of $\mu$.

**Proof.** Let’s call $\nu_0$ the horizontal marginal of $\mu$, and $\rho_0 \in \mathcal{V}_{a_0}$ its density. Let’s denote by $\nu_n := F^n \nu_0$ and by $\rho_n := \frac{d\nu_n}{d\nu_0}.$ By Lemma 4.1 $\rho_n \in \mathcal{V}_a$ for every $n \in \mathbb{N}_0$.

First, let’s prove (19) for $n = 1$. Recalling the disintegration (13),
\[
(\Delta(F, \mu))_i = \bar{p}_i^{-1} \int_{I_i} d\omega \rho_g(\omega)(F, \mu)\omega
\]
\[= \bar{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_g(\omega)}{\rho_1(\omega)} \sum_{j=1}^{d} \frac{\rho_0(\omega_j)}{|Dg_{\omega_j}|} f_{\omega_j} \mu_{\omega_j} + \]
\[= \bar{p}_i^{-1} \int_{I_i} d\omega \left( \frac{\rho_g(\omega)}{\rho_1(\omega)} - 1 \right) \sum_{j=1}^{d} \frac{\rho_0(\omega_j)}{|Dg_{\omega_j}|} f_{\omega_j} \mu_{\omega_j} + \]
\[+ \bar{p}_i^{-1} \int_{I_i} d\omega \sum_{j=1}^{d} \frac{\rho_0(\omega_j) - \rho_g(\omega_j)}{|Dg_{\omega_j}|} f_{\omega_j} \mu_{\omega_j} + \]
\[+ \bar{p}_i^{-1} \int_{I_i} d\omega \sum_{j=1}^{d} \frac{\rho_g(\omega_j)}{|Dg_{\omega_j}|} f_{\omega_j} \mu_{\omega_j}
\]
where in the last equality we added and subtracted the same terms. We denote by $A$ the term in (20) and by $B$ the term in (21). With this notation,
\[
d_W((\Delta(F, \mu))_i, (\mathcal{L}\Delta \mu)_i) = \left\| A + B + \sum_{j=1}^{d} \bar{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_g(\omega_j)}{|Dg_{\omega_j}|} (f_{\omega_j} - F_j) \mu_{\omega_j} + \right\|_W
\]
\[+ \sum_{j=1}^{d} \bar{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_g(\omega_j)}{|Dg_{\omega_j}|} F_j(\mu_{\omega_j} - (\Delta \mu)_j) \right\|_W.
\]
Call $\text{Lip}_b^1(\mathbb{T}^n; \mathbb{R})$, $\text{Lip}_b^1$ for brevity, the set of Lipschitz functions from $\mathbb{T}^n$ to $\mathbb{R}$ with zero integral. When compute the above $\| \cdot \|_W$, taking the supremum over $\text{Lip}_b^1$ or $\text{Lip}_b^1$, doesn’t matter, as the integrals of $\varphi$ and that of $\varphi - f \varphi$ are the same. Notice that for $\varphi \in \text{Lip}_b^1$, $|\varphi| \leq C_2$, where $C_2$ is the diameter of $\mathbb{T}^n$.
\[
d_W((\Delta(F, \mu))_i, (\mathcal{L}\Delta \mu)_i) = \sup_{\varphi \in \text{Lip}_b^1} \int I_i \varphi d \left( A + B + \sum_{j=1}^{d} \bar{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_g(\omega_j)}{|Dg_{\omega_j}|} (f_{\omega_j} - F_j) \mu_{\omega_j} \right) + \]
\[+ \sum_{j=1}^{d} \bar{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_g(\omega_j)}{|Dg_{\omega_j}|} F_j(\mu_{\omega_j} - (\Delta \mu)_j) \right\|_W.
\]
(24) 
\[ + \sum_{j=1}^{d} \bar{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_{\varphi}(\omega_j)}{D\omega_j} F_j(\mu_{\omega_j} - (\Delta \mu)_j) \].

Let’s call
\[ \delta_n := \|\rho_n - \rho_g\|_{\infty} = \sup_{\omega \in T^m \cap I_i} |\rho_n(\omega) - \rho_g(\omega)|. \]

Since \( \rho_1 \in \mathcal{V}_a, |\rho_1| \geq e^{-C_1D} \) where \( C_1 > 0 \) is the diameter of \( T^m \). Therefore
\[ \left| 1 - \frac{\rho_g(\omega)}{\rho_1(\omega)} \right| \leq \frac{1}{\rho_1(\omega)} |\rho_g(\omega) - \rho_1(\omega)| \leq e^{C_1D} \delta_1. \]

Now we distribute the sup among the four terms on the RHS of (23), and estimate each of them separately.

\[
\sup_{\varphi \in \text{Lip}^0} \int_{\varphi \in \text{Lip}^0} \varphi d\mu \leq \bar{p}_i^{-1} \int_{I_i} d\omega \sum_{j=1}^{d} \rho_0(\omega_j) \frac{\varphi(\omega_j)}{D\omega_j} \sup_{\varphi \in \text{Lip}^0} \int_{T^m} \varphi(x) d \left[ \frac{\rho_g(\omega)}{\rho_1(\omega)} - 1 \right] f_{\omega_j, \mu_{\omega_j}}(x) \]
\[
\leq \bar{p}_i^{-1} \int_{I_i} d\omega \sum_{j=1}^{d} \rho_0(\omega_j) \frac{\varphi(\omega_j)}{D\omega_j} C_2 e^{C_1D} \delta_1 \]
\[
= C_2 e^{C_1D} \delta_1 \bar{p}_i^{-1} \int_{I_i} d\omega \rho_1(\omega) \]
\[
= C_2 e^{C_1D} \delta_1 \nu_1(I_i) \]
\[
\leq C_2 e^{3C_1D} \delta_1. \]

Then for \( B \)
\[
\sup_{\varphi \in \text{Lip}^0} \int_{T^m} \varphi d\mu = \sup_{\varphi \in \text{Lip}^0} \int_{T^m} \varphi(x) d \left[ \bar{p}_i^{-1} \int_{I_i} d\omega \sum_{j=1}^{d} \rho_0(\omega_j) - \rho_g(\omega_j) \right] f_{\omega_j, \mu_{\omega_j}} d\omega \]
\[
\leq \bar{p}_i^{-1} \int_{I_i} d\omega \sum_{j=1}^{d} \frac{1}{D\omega_j} \sup_{\varphi \in \text{Lip}^0} \int_{T^m} \varphi(x)(\rho_0(\omega_j) - \rho_g(\omega_j)) d\mu_{\omega_j}(x) \]
\[
\leq \bar{p}_i^{-1} \int_{I_i} d\omega \sum_{j=1}^{d} \frac{1}{D\omega_j} C_2 \beta_0 \]
\[
= \frac{\nu_g(I_i)}{\nu_1(I_i)} C_2 \beta_0 \]
\[
\leq e^{2C_1D} C_2 \beta_0. \]

For the third term in the big parenthesis of Eq. (23), using the definition of \( F_j \)
\[
\left| \int_{I_i} d\omega \sum_{j=1}^{d} \bar{p}_i^{-1} \int_{I_i} d\omega \rho_0(\omega_j) d(\mu_{\omega_j} - F_j) \right| =
\]
\[ \sum_{j=1}^{d} \int_{I_j} \overline{p}_j^{-1} d\omega \overline{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} \int \varphi(x) d(f_{\omega_j \ast} - f_{\omega \ast})\mu_{\omega_j}(x) \]

\[ = \sum_{j=1}^{d} \int_{I_j} \overline{p}_j^{-1} d\omega \overline{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} \int (\varphi \circ f_{\omega_j}(x) - \varphi \circ f_{\omega}(x)) d\mu_{\omega_j}(x) \]

\[ \leq \sum_{j=1}^{d} \int_{I_j} \overline{p}_j^{-1} d\omega \overline{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} \int |\varphi \circ f_{\omega_j}(x) - \varphi \circ f_{\omega}(x)| d\mu_{\omega_j}(x) \]

\[ \leq \sum_{j=1}^{d} \int_{I_j} \overline{p}_j^{-1} d\omega \overline{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} L \text{diam}(I_j) \]

\[ \leq L\sigma^{-1}C_1 \sum_{j=1}^{d} \overline{p}_j^{-1} |I_j| \overline{p}_i^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} \]

\[ \leq L\sigma^{-1}C_1 \sum_{j=1}^{d} \overline{p}_j^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} \]

\[ \leq L\sigma^{-1}C_1 \]

where we used that

\[ |\varphi \circ f_{\omega_j}(x) - \varphi \circ f_{\omega}(x)| \leq |f_{\omega_j}(x) - f_{\omega}(x)| \leq L|\omega_j - \omega| \leq L \text{diam}(I_j) \leq L\sigma^{-1}C_1, \]

recall that \( L \) is the Lipschitz constant of \( f \) and \( C_1 = \text{diam}(\mathbb{T}^m) \); and that

\[ \overline{p}_i^{-1} \int_{I_i} d\omega \sum_{j=1}^{d} \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} = \overline{p}_i^{-1} \int_{I_i} d\omega \rho_{\theta}(\omega) = 1. \]

For the last term, on [24],

\[ \int \varphi(x) d \left( \sum_{j=1}^{d} \overline{p}_j^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} \mathcal{F}_j(\mu_{\omega_j} - (\Delta \mu)_j) \right) (x) = \]

\[ \sum_{j=1}^{d} \overline{p}_j^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} \int \varphi(x) d\mathcal{F}_j(\mu_{\omega_j} - (\Delta \mu)_j)(x) \]

\[ \leq \sum_{j=1}^{d} \overline{p}_j^{-1} \int_{I_i} d\omega \frac{\rho_{\theta}(\omega_j)}{|Dg_{\omega_j}|} \text{Lip}(\mathcal{F}_j) dW(\mu_{\omega_j}, (\Delta \mu)_j) \]

\[ \leq L\ell_0 C_1 \sigma^{-1} \]

where in the last step we used that \( \text{Lip}(\mathcal{F}_j) \leq \sup_{\omega} \text{Lip}(f_{\omega \ast}) \leq L \) and that

\[ dW(\mu_{\omega_j}, (\Delta \mu)_j) \leq \ell_0 |\text{diam}(I_j)| \leq \ell_0 C_1 \sigma^{-1}. \]
Putting all of the above together we conclude that there is $K_\# > 0$ (independent of $\sigma > 0$) such that

$$d_W((\Delta(F^*_\sigma \mu)), (\mathcal{L}\Delta \mu)_i) \leq K_\#(\sigma^{-1} + \delta_0 + \delta_1).$$

Now, since for every $k \in \mathbb{N}$, $F^k_\sigma \mu \in \mathcal{M}^{1,\nu}(T^m \times T^m) \cap \Gamma_{\ell_0}$, by repeated applications of the triangular inequality

$$d_W(\Delta F^* \mu, \mathcal{L}^n \Delta \mu) \leq \sum_{k=0}^{n-1} d_W(\mathcal{L}^{n-k-1} \Delta F^*_\sigma \mu, \mathcal{L}^{n-k} \Delta (F^k_\sigma \mu)) \leq K_\# \sum_{k=0}^{n-1} L^{n-k} [\ell_0 \sigma^{-1} + \delta_k + \delta_{k+1}]$$

$$\leq K_\# (\ell_0 \sigma^{-1} + \sum_{k=0}^{n-1} \delta_k + \delta_{k+1}) \sum_{k=0}^{n-1} L^{n-k} \leq K_\# (\ell_0 \sigma^{-1} + C_3 \delta_0) \sum_{k=0}^{n-1} L^{n-k}.$$

where we used that by Lemma 4.1, $\delta_k \leq C g_\lambda^k \delta_0$, and $C_3 := \frac{2C g_\lambda}{1-\lambda}$.

The operator $\mathcal{L}$ has good spectral properties. To prove it, we are going to need the following lemma

**Lemma 4.2.** The following inequality holds

$$\bar{p}_i^{-1} \bar{p}_j^{-1} \int_{I_i} d\omega \rho_g(\omega_j) |Dg_{\omega_j}| > p,$$

with

$$p := e^{-C_1 D(\frac{3}{1-\sigma^{-1}}+1) \in (0, 1)}$$

where $D$ is the bound on the distortion of the map $g$, and $C_1$ is the diameter of $T^m$.

**Proof.** Recall that $\omega_j$ is shorthand notation for $h_j(\omega)$. Since

$$\int_{T^m} d\omega \frac{1}{|Dg_{\omega_j}|} = \text{Leb}_{T^m}(I_j) =: |I_j|$$

and $|Dg \circ h_j|$ is continuous, there is $\omega_0$ such that

$$\frac{1}{|Dg(h_j(\omega_0))|} = |I_j|$$

Recalling the notation $\omega_j = h_j(\omega)$, the bound on the distortion ([H0.2]) gives

$$|I_j|^{-1} |Dg_{\omega_j}|^{-1} = \left|\frac{|Dg(h_j(\omega_0))|}{|Dg(h_j(\omega))|}\right| \geq e^{-D|\omega_0-\omega|} \geq e^{-DC_1},$$

where we used that by Lemma 4.1, $\delta_k \leq C g_\lambda^k \delta_0$, and $C_3 := \frac{2C g_\lambda}{1-\lambda}$.\]
where $C_1$ equals the diameter of $\mathbb{T}^{m_1}$ w.r.t. the Euclidean distance.

Also, recall that $\rho_g \in \mathcal{V}_{a_0}$ with $a_0 = \frac{D}{\sigma}$, therefore $e^{-C_1a_0} \leq \rho_g \leq e^{C_1a_0}$ and $\overline{p}_i \leq e^{C_1a_0}|I_i|$. Putting the above considerations together

$$
\overline{p}_i^{-1}
\rho_j
\int_{I_i} d\omega \frac{\rho_g(\omega_j)}{|Dg_{\omega_j}|} \geq e^{-C_1a_0}|I_j|^{-1}e^{-C_1a_0}|I_i|^{-1}
\int_{I_i} d\omega \frac{\rho_g}{|Dg_{\omega_j}|} \\
\geq e^{-3C_1a_0}|I_i|^{-1}
\int_{I_i} e^{-DC_1} \\
\geq e^{-3C_1a_0-DC_1}
$$

$\square$

**Remark 4.3.** Notice that $p$ depends on $\sigma$, but for $D$ fixed, $p$ increases with $\sigma > 1$. In particular, assuming that $\sigma \geq \sigma_0 > 1$, we get

$$
p \geq e^{-C_1D\left[1+\frac{2}{1-\sigma_0}\right]}.
$$

In a proposition below we show that the operator $\mathcal{L}$ has good contracting properties with respect to $d_{TV}$. First we state a couple of lemmas and definitions.

**Lemma 4.3.** For every $\mu_1, \mu_2 \in (\mathcal{M}_1(\mathbb{T}))^d$

$$
d_{TV}(\mathcal{L}\mu_1, \mathcal{L}\mu_2) \leq d_{TV}(\mu_1, \mu_2).
$$

**Proof.** By definition of Total Variation distance, transfer operators are weak contractions with respect to $d_{TV}$; in particular, for any $\eta_1, \eta_2 \in \mathcal{M}_1(\mathbb{T})$ and any $\omega \in \mathbb{T}^{m_1}$

$$
d_{TV}(f_\omega \ast \eta_1, f_\omega \ast \eta_2) \leq d_{TV}(\eta_1, \eta_2),
$$

and therefore

$$
d_{TV}(F_j \eta_1, F_j \eta_2) \leq d_{TV}(\eta_1, \eta_2)
$$

for any $j$.

By formula\[29\] one gets that for every $i$

$$
d_{TV}(\mathcal{L}\mu_1)_{(i)}, (\mathcal{L}\mu_2)_{(i)} =
$$

$$
= d_{TV}
\left(\sum_j \overline{p}_i^{-1}
\left(\int_{I_i} d\omega \rho_g(\omega_j) \right) F_j(\mu_1)_{(j)}, \sum_j \overline{p}_i^{-1}
\left(\int_{I_i} d\omega \rho_g(\omega_j) \right) F_j(\mu_2)_{(j)}\right)
$$

$$
\leq \sum_j \overline{p}_i^{-1}
\left(\int_{I_i} d\omega \rho_g(\omega_j) \right) d_{TV}(F_j(\mu_1)_{(j)}, F_j(\mu_2)_{(j)})
$$

$$
\leq \sum_j \overline{p}_i^{-1}
\left(\int_{I_i} d\omega \rho_g(\omega_j) \right) d_{TV}(\mu_1_{(j)}, \mu_2_{(j)})
$$

$$
\leq d_{TV}(\mu_1, \mu_2).
$$

$\square$
Lemma 4.2 implies that $\mathcal{L}$ can be decomposed in the following way: there are $p_{ij} \geq 0$ such that

$$\mathcal{L}_{ij} = p \int_{I_j} d\omega \rho_g(\omega) f_{\omega^j} + p_{ij} \mathcal{F}_j. \tag{30}$$

Define $\mathcal{L}_1 : (\mathcal{M}_1(\mathbb{T}^m_2))^d \to (\mathcal{M}_1(\mathbb{T}^m_2))^d$ as

$$(\mathcal{L}_1)_{ij} = \int_{I_j} d\omega \rho_g(\omega) f_{\omega^j}$$

and $\mathcal{L}_2 : (\mathcal{M}_1(\mathbb{T}^m_2))^d \to (\mathcal{M}_1(\mathbb{T}^m_2))^d$ as

$$(\mathcal{L}_2)_{ij} := p_{ij} \mathcal{F}_j$$

so that $\mathcal{L} = p\mathcal{L}_1 + \mathcal{L}_2$.

**Lemma 4.4.** For every $n \in \mathbb{N}$ the following decomposition holds

$$\mathcal{L}^n \mu := p^n \mathcal{L}_1^n \mu + (1 - p^n) \mathcal{R}_n \mu,$$

where $\mathcal{R}_n : (\mathcal{M}_1(\mathbb{T}^m_2))^d \to (\mathcal{M}_1(\mathbb{T}^m_2))^d$ is such that

$$d_{TV}(\mathcal{R}_n \mu_1, \mathcal{R}_n \mu_2) \leq d_{TV}(\mu_1, \mu_2)$$

for all $\mu_1, \mu_2 \in (\mathcal{M}_1(\mathbb{T}^m_2))^d$.

**Proof.** Let’s start noticing that, by definition, $\sum_j p_{ij} = (1 - p)$ for every $i$, in fact comparing equations (30) and (17) follows that

$$p \bar{p}_i + p_{ij} = \bar{p}_j^{-1} \left( \int_{I_i} d\omega \rho_g(\omega_j) \right)$$

and

$$\sum_j (p \bar{p}_j + p_{ij}) = p + \sum_j p_{ij} = \sum_j \bar{p}_i^{-1} \left( \int_{I_i} d\omega \rho_g(\omega_j) \right) = 1.$$

Now we prove the statement of the lemma by induction on $n \in \mathbb{N}$. For $n = 1$, $\mathcal{R}_1 = (1 - p)^{-1} \mathcal{L}_2$ and recalling (29)

$$d_{TV}(p \mathcal{L}_2 \mu_1, (1 - p)^{-1} \mathcal{L}_2 \mu_2) =$$

$$= \max_l \left( \sum_j (1 - p)^{-1} p_{ij} \mathcal{F}_j(\mu_1)_j, \sum_j (1 - p)^{-1} p_{ij} \mathcal{F}_j(\mu_2)_j \right)$$

$$\leq \max_j \sum_j (1 - p)^{-1} p_{ij} d_{TV}(\mu_1)_j, (\mu_2)_j$$

$$\leq d_{TV}(\mu_1, \mu_2).$$

Now assume that the statement is true for $n - 1$.

$$\mathcal{L}^n = \mathcal{L} \mathcal{L}^{n-1} = p^n \mathcal{L}_1^n + p^{n-1}(1 - p)\mathcal{R}_1 \mathcal{L}_1^{n-1} + (1 - p^{n-1}) \mathcal{L} \mathcal{R}_{n-1}.$$

Define

$$\mathcal{R}_n := \frac{(1 - p)p^{n-1} \mathcal{R}_1 \mathcal{L}_1^{n-1} + (1 - p^{n-1}) \mathcal{L} \mathcal{R}_{n-1}}{1 - p^n}$$
and by Lemma \ref{lemma:4.3} applied to \( \mathcal{L} \) and \( \mathcal{L}_1 \), the inductive step, \((1 - p)p^{n-1} + (1 - p^{n-1}) = 1 - p^n\), and Lemma \ref{lemma:C.4},
\[
d_{TV}(\mu_1, \mu_2) \leq d_{TV}(\mu_1, \mu_2).
\]

We are now ready to show that \( \mathcal{L} \) has good contraction properties with respect to the Total Variation distance. The proof uses a coupling argument.

**Proposition 4.4.** There are \( C_{\mathcal{L}} > 0 \) and \( \lambda_{\mathcal{L}} \in (0, 1) \) such that for any \( \mu_1, \mu_2 \in (\mathcal{M}_1(\mathbb{T}^{m_2}))^d \)
\[
d_{TV}(\mathcal{L}^n_{\mathcal{L}} \mu_1, \mathcal{L}^n_{\mathcal{L}} \mu_2) \leq C_{\mathcal{L}} \lambda_{\mathcal{L}}^n d_{TV}(\mu_1, \mu_2).
\]

**Proof.** Notice that all the rows of the operator \( \mathcal{L}_1 \) are equal, therefore, for any \( \mu \in (\mathcal{M}_1(\mathbb{T}^{m_2}))^d \), also all the components of \( \mathcal{L}_1 \mu \) are equal, i.e. there is \( \mu' \in \mathcal{M}_1(\mathbb{T}^{m_2}) \) such that \( (\mathcal{L}_1 \mu)_i = \mu'_i \). By definition of \( \mathcal{L}_1 \) follows that
\[
(\mathcal{L}_1^n \mu)_i = \sum_{j=1}^d (\mathcal{L}_1)_ij \mu' = \sum_{j=1}^d \int_{I_j} d\omega f_{ij}(\omega) \mu' = \mathcal{P} \mu'
\]
and by induction
\[
(\mathcal{L}_1^n \mu)_i = \mathcal{P}^{n-1} \mu'
\]
for every \( i \) and \( n > 1 \).

Pick \( n_0 > 1 \) such that \( C_{\mathcal{L}} \lambda_{\mathcal{L}}^{n_0-1} \leq \frac{1}{2} \). Then it follows from \ref{eq:32} and Assumption \ref{H} that for all \( n > 1 \)
\[
d_{TV}(\mathcal{L}_1^n \mu_1, \mathcal{L}_1^n \mu_2) \leq \frac{1}{2} d_{TV}(\mu_1, \mu_2).
\]
which implies
\[
d_{TV}(\mathcal{L}^n_{\mathcal{L}} \mu_1, \mathcal{L}^n_{\mathcal{L}} \mu_2) \leq p^n d_{TV}(\mathcal{L}_1^n \mu_1, \mathcal{L}_1^n \mu_2) + (1 - p^n) d_{TV}(\mu_1, \mu_2)
\]
\[
\leq \left( 1 - \frac{1}{2} p^n \right) d_{TV}(\mu_1, \mu_2).
\]

Define \( \lambda_{\mathcal{L}} := (1 - \frac{1}{2} p^n)^{\frac{1}{n_0}} \) and \( C_{\mathcal{L}} := \lambda_{\mathcal{L}}^{n_0} \). For every \( n \in \mathbb{N} \) there are \( k \in \mathbb{N} \) and \( 0 \leq r < n_0 \) such that \( n = kn_0 + r \), and by Lemma \ref{lemma:4.3}
\[
d_{TV}(\mathcal{L}_1^n \mu_1, \mathcal{L}_1^n \mu_2) = d_{TV}(\mathcal{L}_1^r \mathcal{L}_{\mathcal{L}}^{kn_0} \mu_1, \mathcal{L}_1^r \mathcal{L}_{\mathcal{L}}^{kn_0} \mu_2)
\]
\[
\leq d_{TV}(\mathcal{L}_1^{kn_0} \mu_1, \mathcal{L}_1^{kn_0} \mu_2)
\]
\[
\leq \lambda_{\mathcal{L}}^{nk_0} d_{TV}(\mu_1, \mu_2)
\]
\[
\leq C_{\mathcal{L}} \lambda_{\mathcal{L}}^n d_{TV}(\mu_1, \mu_2).
\]

The contraction properties of \( \mathcal{L} \), \ref{eq:31}, and the weak*-compactness of \( (\mathcal{M}_1(\mathbb{T}^{m_2}))^d \) imply the existence of \( \eta_0 \in (\mathcal{M}_1(\mathbb{T}^{m_2}))^\ast \) such that
\[
\mathcal{L} \eta_0 = \eta_0.
\]
The following proposition is the analogous of Proposition 3.3 in the case without distortion and proves approximated memory loss for the vertical marginals under application of $F_*$. In this case, there is an extra difficulty as, in order to prove Theorem 2.1, the class of probability measures we start from should include those having horizontal marginal equal to $\text{Leb}_{\mathbb{T}^m}$, which in general can be different from the invariant measure $\nu_g$.

**Proposition 4.5** (Approximate Memory Loss). Fix $D, L > 0$. Given any $\varepsilon > 0$, there is $C_L' > 0$ and $\sigma_0 > L$ such that for any $\sigma > \sigma_0$, $F$ satisfying (H0.1)-(H0.3)

\[ d_W(\Pi F_*^n, \Pi F_*^{n+1}) \leq C_L' \lambda_L + \varepsilon, \quad \forall t \in \mathbb{N} \]

for any $n, m \in \mathbb{N}$

\[ d_W(\Delta F_*^n(F_*^m \mu_1), \Delta F_*^{n+1}(F_*^{m+1} \mu_2)) \]

\[ \leq d_W(\mathcal{L}^n \Delta F_*^n \mu_1, \mathcal{L}^{n+1} \Delta F_*^{n+1} \mu_2) + d_W(\mathcal{L}^{n+1} \Delta F_*^{n+1} \mu_1, \mathcal{L}^{n+1} \Delta F_*^{n+1} \mu_2) \]

\[ \leq C_2d_{TV}(\mathcal{L}^n \Delta F_*^m \mu_1, \mathcal{L}^{n+1} \Delta F_*^{m+1} \mu_2) + 2K_{\#}L^{n+1}(\sigma^{-1} + C_3(\sigma))\|\rho_m - \rho_g\|_{\infty} \]

where we used triangular inequality, Lemma C.3 (recall that $C_2$ is the diameter of $\mathbb{T}^{m^2}$), and Proposition 4.3.

For every $\varepsilon > 0$, pick $n_0, m_0 \in \mathbb{N}$, and $\sigma_0$ large enough so that $C_2C_L\lambda_L^{n_0} \leq \varepsilon/2$, and

\[ 2K_{\#}L^{n+1}(\sigma^{-1} + C_3(\sigma))\|\rho_m - \rho_g\|_{\infty} \leq \frac{\varepsilon}{2} \]

Notice that $m_0$ is a transient one waits for the horizontal marginal to get sufficiently close to $\nu_g$ while $n_0$ is the time one waits for $\mathcal{L}$ to contract by the desired amount.

Calling $C_L := C_2C_L\lambda_L^{n_0}$

\[ d_W(\Delta F_*^{n+m_0} \mu_1, \Delta F_*^{n+m_0} \mu_2) \leq C_L\lambda_L^{n_0} + \varepsilon/2 \leq C_L'\lambda_L^{n+m_0} + \varepsilon, \]

for every $n$, in fact if $n \leq n_0$

\[ d_W(\Delta F_*^{n+m_0} \mu_1, \Delta F_*^{n+m_0} \mu_2) \leq C_L'\lambda_L^{n_0} + \varepsilon/2 \]

and if $n \geq n_0$

\[ d_W(\Delta F_*^{n+m_0} \mu_1, \Delta F_*^{n+m_0} \mu_2) \leq C_L'\lambda_L^{n_0} + \varepsilon/2 \leq C_L'\lambda_L^{n} + \varepsilon. \]

Recall that

\[ \Pi F_*^{n+m_0} \mu_j = \sum_{i=1}^{d} \overline{p}_i(\Delta F_*^{n+m_0} \mu_j)_i \]
and since the above is a convex combination, using Lemma C.4

\[ d_W(\Pi F_x^{n+m} \mu_1, \Pi F_x^{n+m} \mu_2) \leq d_W(\Delta F_x^{n+m} \mu_1, \Delta F_x^{n+m} \mu_2) \leq C' \lambda_x^{n+m} + \varepsilon \]

which proves point i) with \( t \geq m_0 \). If \( t \leq m_0 \), by the definition of \( d_W \)

\[ d_W(\Pi F_x^{n+m} \mu_1, \Pi F_x^{n+m} \mu_2) = C_2 \]

the diameter of \( T^{m_2} \). Therefore, picking \( C' = \max \{ C' \lambda_x^{n+m}, C_2 \lambda_x^{m} \} \)

we get

\[ d_W(\Pi F_x^{n+m} \mu_1, \Pi F_x^{n+m} \mu_2) \leq C' \lambda_x^{n+m} + \varepsilon \]

for all \( t \in \mathbb{N} \) which concludes the proof of point i).

To prove point ii), recall that \( \eta_0 \in (\mathcal{M}_1(\mathbb{T}^{m_2}))^d \) is fixed by \( \mathcal{L} \). Now

\[
d_W(\Delta F^{n+m} \mu \eta_0) \leq d_W(\Delta F^{n+m} \mu \eta_0) + d_W(\mathcal{L}^n \Delta F^{n+m} \mu \eta_0) + C_2 d_{TV} (\mathcal{L}^n \Delta F^{n+m} \mu \eta_0) \leq K \# L^{n+1}(\sigma - 1) + C_3(\sigma) \| \rho_m - \rho_g \|_{\infty} + C_2 C \lambda_x^{n}.
\]

In a way completely analogous to the proof of point i) one can show that for every \( \varepsilon > 0 \) there are \( \sigma_0 \) sufficiently large and \( C'' \lambda_x^{n} > 0 \) such that for \( \sigma > \sigma_0 \)

\[ d_W(\Delta F^{n+m} \mu \eta_0) \leq C'' \lambda_x^{n} + \varepsilon. \]

By definition of \( d_W \), the above means that \( d_W(\Delta F^{n+m} \mu \eta_0,i) \leq C'' \lambda_x^{n} + \varepsilon \) for all \( i \), which implies that

\[
 d_W \left( \sum_i \pi_i (\Delta F^{n+m} \mu \eta_0,i), \sum_i \pi_i (\eta_0,i) \right) \leq C'' \lambda_x^{n} + \varepsilon.
\]

Since \( \Pi F^{n+m}_x = \sum_i \pi_i (\Delta F^{n+m} \mu \eta_0,i) \), the statement follows. \( \square \)

**Proof of Theorem 2.2**. With all the work above done, the proof of the theorem is almost identical to the case without distortion. The only difference is that instead of \( \eta_0 \) in the proof of the case without distortion, one has to substitute \( \sum_i \pi_i (\eta_0,i) \), and apply Proposition 4.5 in place of Proposition 3.3. \( \square \)

### 4.3. Fixed point for \( \mathcal{L} \) and fixed point for \( \mathcal{P} \)

Point ii) of Proposition 4.5 shows that if \( \sigma \) is sufficiently large, then the vertical marginal of \( F^{n} \mu \) becomes close to \( \overline{\eta} = \sum_i \pi_i (\eta_0,i) \). The purpose of this section is to remark that, in general, \( \overline{\eta} \) is different (and possibly quite far) from \( \eta_0 \), the stationary measure of \( \mathcal{P} \). We prove this fact in an indirect way by showing that the unique fixed point of \( \mathcal{P} := \int_T d\omega \rho_g(\omega)f_{\omega}, \eta_0 \), and the unique fixed point of \( \mathcal{P}' := \int_T d\omega \rho_g(\omega)(f_{g^{k-1}(\omega)} \circ ... \circ f_{\omega}), \eta_0 \), that we will call \( \eta_0' \), can be in general very different for some \( k > 1 \). If this is the case, \( \Pi F^{nk}_x \text{Leb}_{\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}} \) cannot become close to both \( \eta_0 \) and \( \eta_0' \), and since \( \mathcal{P}' \) is the random counterpart of \( F^k \), it implies that \( \sum_i \pi_i (\eta_0,i) \) can be far from the fixed points of \( \mathcal{P} \) or/and \( \mathcal{P}' \). At the end of the section we also give numerical evidence that \( \overline{\eta} \) can be different from \( \eta_0 \) when the map \( g \) has nonzero distortion.
For simplicity of exposition, we are going to present an example that does not satisfy the smoothness requirements of Theorem 2.1. However, with a small modification on a set of arbitrarily small measure, the system can be made as smooth as one likes and all the considerations below carry over to the smoothed version.

First of all, we define the map $g = g_{M,\kappa} : \mathbb{T} \to \mathbb{T}$ where $M \in \mathbb{N}$ and $\kappa \in (0, 1)$ are parameters. We identify $\mathbb{T}$ with $[0, 1]$ in the usual way and divide $[0, 1]$ into $2^M$ intervals of equal length $I_j := \left[\frac{j-1}{2^M}, \frac{j}{2^M}\right]$. Let $\kappa' = 1 - \kappa$. Define for $0 \leq j \leq M - 1$

$g_{M,\kappa}(\omega) := \begin{cases} \frac{M}{\kappa}\omega - \frac{j}{2^M} & \omega \in [j/2M, (j + \kappa)/2M] \\ \frac{M}{\kappa'}\omega - \frac{j + \kappa - \kappa'}{2^M} & \omega \in [(j + \kappa)/2M, (j + 1)/2M] \end{cases}$

and for $M \leq j \leq 2M - 1$

$g_{M,\kappa}(\omega) := \begin{cases} \frac{M}{\kappa}\omega - \frac{j}{2^M} & \omega \in [j/2M, (j + \kappa)/2M] \\ \frac{M}{\kappa'}\omega - \frac{j + \kappa' - \kappa}{2^M} & \omega \in [(j + \kappa)/2M, (j + 1)/2M] \end{cases}$

The graph of $g_{5,0.99}$ is presented in Figure 1.

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and for $M \leq j \leq 2M - 1$

$g_{M,\kappa}(\omega) := \begin{cases} \frac{M}{\kappa}\omega - \frac{j}{2^M} & \omega \in [j/2M, (j + \kappa)/2M] \\ \frac{M}{\kappa'}\omega - \frac{j + \kappa' - \kappa}{2^M} & \omega \in [(j + \kappa)/2M, (j + 1)/2M] \end{cases}$

The graph of $g_{5,0.99}$ is presented in Figure 1.

It is easy to verify that $g_{M,\kappa}$ is piecewise affine, uniformly expanding, and keeps the Lebesgue measure invariant. Also, the minimal expansion of $g_{M,\kappa}$ can be made arbitrarily large by letting $M \to \infty$.

Notice that for $1 \leq j \leq M$,

$g_{M,\kappa}([j/2M, (j + \kappa)/2M]) = [0, 1/2]$ and $g_{M,\kappa}([(j + \kappa)/2M, (j + 1)/2M]) = [1/2, 1]$ while for $M + 1 \leq j \leq 2M$

$g_{M,\kappa}([j/2M, (j + \kappa')/2M]) = [0, 1/2]$ and $g_{M,\kappa}([(j + \kappa')/2M, (j + 1)/2M]) = [1/2, 1]$.

Picking $\kappa \approx 1$, most of the points in the interval $[0, 1/2]$ are mapped back to $[0, 1/2]$, and also most of the points of $[1/2, 1]$ are mapped back to $[1/2, 1]$. More precisely, defining
\[ V_1 := [0, 1/2], \quad V_2 := [1/2, 1] \]

\[ V_{i,n} := \{ \omega \in V_i : g^{k}_{M,n}(\omega) \in V_i \text{ for } 0 \leq k \leq n - 1 \}; \]

\[ V_{i,n} \subset V_i \text{ is such that, for any } n \in \mathbb{N}, \]

\[ |V_{i,n}| \to 1/2 \text{ as } \kappa \to 1. \]

Fix \( \varepsilon > 0 \) a small number. Pick \( \varphi : \mathbb{T} \to \mathbb{T} \) a \( N - S \) diffeomorphism such that

\[ |\varphi(x) - x| \leq \varepsilon \]

and define

\[ f_\omega(x) = \begin{cases} 2\omega & \omega \in I_1 \\ \varphi + a\omega & \omega \in I_2. \end{cases} \]

One can check that \( P := \int_{\mathbb{T}} d\omega f_\omega \) maps a small closed ball around \( \text{Leb}_\mathbb{T} \) into itself, with the diameter of the ball going to zero (in Total Variation distance) when \( a \to 0 \). This implies that the unique fixed point of \( P \) is close to \( \text{Leb}_\mathbb{T} \).

To ease the notation, from now on we write \( \varphi_n := g^{n-1}_\omega \circ \cdots \circ g_\omega \) and study \( P' := \int_{\mathbb{T}} d\omega (\varphi_n)_* \). Fix \( \Delta > 0 \) small. For any \( \varpi \in \mathbb{T} \) and \( (\varpi_k)_{k=0}^{n-1} \) with \( \varpi_k \in V_2 \), consider \( (\varpi_k')_{k=0}^{n-1} \) with \( \varpi_k' = \varphi(\varpi_k) + a\omega_k \). Pick \( n \in \mathbb{N} \) large and \( a > 0 \) small so that for any \( \varpi_k \in [N - \Delta, N + \Delta] \) and \( (\varpi_k')_{k=0}^{n-1} \) as above, \( \varpi_{n-1} \in [S - \Delta, S + \Delta] \). One can find \( \kappa \) close enough to one so that \( |V_{1,n}| = |V_{2,n}| = 0.49 \), which implies

\[ P'\eta = \int_{V_{1,n}} d\omega (\varphi_n)_* \eta + \int_{V_{2,n}} d\omega (\varphi_n)_* \eta + \int_{(V_{1,n} \cup V_{2,n})^c} d\omega (\varphi_n)_* \eta \]

\[ = 0.49 \text{Leb}_\mathbb{T} + \int_{V_{2,n}} d\omega (\varphi_n)_* \eta + \int_{(V_{1,n} \cup V_{2,n})^c} d\omega (\varphi_n)_* \eta. \]

Given the expression of \( P' \), if \( \eta'_0 \) is such that \( P'\eta'_0 = \eta'_0 \) then, \( \eta'_0 = 0.49 \text{Leb} + 0.51\eta_1 \), where \( \eta_1 \) is some probability measure. This implies that

\[ P'\eta_0([S - \Delta, S + \Delta]) = (0.49P' \text{Leb} + 0.51P'\eta_1)([S - \Delta, S + \Delta]) \]

\[ = 0.49 \int_{V_{2,n}} d\omega (\varphi_n)_* \text{Leb}([S - \Delta, S + \Delta]) + \]

\[ + (1 - 0.49^2)\eta_2([S - \Delta, S + \Delta]) \]

\[ > 0.49^2(1 - 2\Delta) \]

where \( \eta_2 \) above is some probability measure. Since \( \Delta > 0 \) is arbitrary, \( \eta'_0([S - \Delta, S + \Delta]) \approx 1/4 \) while \( \eta_0([S - \Delta, S + \Delta]) \approx 2\Delta \) which makes \( \eta_0 \) and \( \eta'_0 \) two very far apart measures with respect to most metrics (e.g. \( d_{TV}, d_W, \ldots \)).

In Figure 3 below we compare numerical simulations of the distribution of mass on the vertical marginal after several iterations of skew-products \( F \) with different base maps \( g \). For each such map, we consider several initial conditions sampled randomly and uniformly on \([0, 1] \times [0, 1] \), let \( F \) act for a while on these points, then take their vertical coordinates, and plot them on a histogram. When the expansion in the base is large, we

\[ A \text{ North-South (NS) diffeomorphism is a diffeomorphism with exactly two fixed points: one attracting, the South Pole (S), and one repelling, the North Pole (N), such that for any } x \neq N, \varphi^n(x) \to S. \]

Furthermore, \( \varphi'(N) > 1 \) and \( \varphi'(S) < 1 \) so that the two fixed points are hyperbolic.
Figure 2. For different base maps $g$, we consider $10^4$ initial conditions $\{(\omega_k, x_k)\}_{k=1}^{10^4}$ sampled randomly and uniformly on $[0, 1] \times [0, 1]$, let $F(\omega, x) = (g(\omega), f(\omega, x))$ act for 100 time steps to obtain $\{F^{100}(\omega_k, x_k)\}_{k=1}^{10^4}$, take the vertical $x$-coordinates of these points, and plot them on a histogram. The different $g$ maps used are indicated above the histograms. The fiber maps are the same throughout and as in (36) with $\varphi(x) = x - 0.01 \sin(2\pi x)$ and $a = 0.001$. The last panel shows a numerical approximation for $\eta_0$ obtained as in the deterministic case by applying $F$ to $\{(\omega_k, x_k)\}_{k=1}^{10^4}$ but where, instead of having $g$ in the base, we sampled the $\omega$-coordinate at random independently (both w.r.t. time and initial conditions) and uniformly on $[0, 1]$ using the random number generator built in the programming language.
expect the distribution given by the histogram to be close to \( \bar{\eta} \). We compare the case of base maps with no distortion, \( g(\omega) = \sigma \omega \mod 1 \), against base maps \( g_{M,K} \) defined above. We also simulate numerically \( \eta_0 \), the stationary measure for \( \mathcal{P} \) (as given by the random number generator of the programme). The fiber maps \( f_\omega \) are of the kind described in \([30]\).

In the case without distortion, when the minimal expansion in the base increases, we can see that the simulated \( \bar{\eta} \) becomes very close to \( \eta_0 \) (as per Proposition 3.3 point ii)), while in the case with distortion, \( \bar{\eta} \) and \( \eta_0 \) are different.

5. Generalizations and limitations

In this section we discuss a few generalizations of the results and techniques presented above, and also some of the limitations. Before proceeding with the generalizations, we would like to stress that the goal of this paper was not to give a result in its greatest generality possible, but rather to present some techniques that we believe can be applied (with different levels of additional effort) to various setups.

5.1. Regularity assumptions on \( g \). The regularity assumptions on the map \( g \) can be revised to fit other situations. For example, \( \Omega \) could be a compact manifold with border such as \( \Omega = [0, 1]^{m_1} \) with \( g \) piecewise \( C^2 \) with onto branches. By this we mean that there are open sets \( \{I_i\}_{i=1}^d \) partitioning \( \Omega \) modulo sets of measure zero, and such that \( g|_{I_i} : I_i \to (0, 1)^{m_1} \) is a \( C^2 \) uniformly expanding diffeomorphism with bounded distortion.

For the system in Section 3 i.e. when \( m_1 = 1 \) and no distortion, this corresponds to considering maps \( g : [0, 1] \to [0, 1] \) for which there are \( n \in \mathbb{N} \) and \( 0 =: a_0 < a_1 < \ldots < a_n < a_{n+1} := 1 \) such that \( g|_{(a_i,a_{i+1})} \) is \( C^2 \) and onto \((0,1)\). It is easy to check that all the proof of statements in Section 3 hold, \textit{mutatis mutandis}, for maps \( g \) satisfying these assumptions.

Also the assumption that \( g \) must be \( C^2 \) (or piecewise \( C^2 \)) is not necessary, and can be substituted by \( g \) being \( C^{1+\alpha} \) (or piecewise \( C^{1+\alpha} \)), meaning that \( g \) is once differentiable and with \( \alpha \)-Hölder differential (or same property, but piecewise).

5.2. Robustness under conjugacy. Consider a map \( \hat{g} : \Omega \to \Omega \) and assume that there is an invertible map \( h : \Omega \to \mathbb{T}^{m_1} \) which is measurable and with measurable inverse, such that \( \hat{g} := h^{-1} \circ g \circ h \) for a map \( g : \mathbb{T}^{m_1} \to \mathbb{T}^{m_1} \). Consider \( \hat{f} : \Omega \times \mathbb{T}^{m_2} \to \Omega \times \mathbb{T}^{m_2} \) and the skew-product system \( \hat{F} : \Omega \times \mathbb{T}^{m_2} \to \Omega \times \mathbb{T}^{m_2} \)

\[
\hat{F}(\omega, x) = (\hat{g}(\omega), \hat{f}(\omega, x)).
\]

Then, if one can show that the skew-system \( F : \mathbb{T}^{m_1} \times \mathbb{T}^{m_2} \to \mathbb{T}^{m_1} \times \mathbb{T}^{m_2} \)

\[
F(\omega, x) = (g(\omega), f(\omega, x))
\]

with \( f(\omega, x) := \hat{f}(h^{-1}\omega, x) \), satisfies an approximate decay of correlations (as in Theorem 2.1), then so does \( \hat{F} \). This is made precise in the following proposition.

**Proposition 5.1.** Suppose \( \hat{F} : \Omega \times \mathbb{T}^{m_2} \to \Omega \times \mathbb{T}^{m_2} \) and \( F : \mathbb{T}^{m_1} \times \mathbb{T}^{m_2} \to \mathbb{T}^{m_1} \times \mathbb{T}^{m_2} \) are as above, and assume that for some \( \varepsilon > 0 \), \( \eta \) a probability measure, \( \hat{C} > 0 \) and \( \tilde{\lambda} \in (0,1) \)
the conclusion of Theorem 2.1 holds for \( F \). Then, defining \( \nu := (h^{-1})_* \text{Leb}_{T^m} \)

\[
\left| \int_{\Omega \times T^m} \varphi(\pi_2 \hat{F}^m(\omega, x)) \psi(x) d\nu(\omega) - \int_{T^m} \varphi(x) d\eta(x) \int_{T^m} \psi(x) dx \right| \leq C_{\varphi, \psi}(\tilde{C} \lambda^n + \varepsilon)
\]

for all \( \psi \in L^1(\mathbb{T}^m; \mathbb{R}) \) and \( \varphi \in \text{Lip}(\mathbb{T}^m; \mathbb{R}) \).

**Proof.** Take \( \psi \in L^1(\mathbb{T}^m; \mathbb{R}) \) and \( \varphi \in \text{Lip}(\mathbb{T}^m; \mathbb{R}) \). Define \( \nu = (h^{-1})_* \text{Leb}_{T^m} \) a probability measure on \( \Omega \). Let’s call \( H := h \times \text{id} \) which is invertible with inverse \( H^{-1} = h^{-1} \times \text{id} \).

\[
\int_{\Omega \times T^m} \psi \circ \pi_2 \circ \hat{F}^n d\nu \otimes \text{Leb} = \int_{\Omega \times T^m} \psi \circ \pi_2 \circ H^{-1} \varphi \circ \pi_2 \circ \hat{F}^n \circ H^{-1} dH_\nu(\nu \otimes \text{Leb})
\]

\[
= \int_{\Omega \times T^m} \psi \circ \pi_2 H \circ \hat{F}^n \circ H^{-1} d\text{Leb}
\]

\[
= \int_{\Omega \times T^m} \psi \circ \pi_2 \circ F^n d\text{Leb}.
\]

Therefore, from the assumptions, there is \( C_{\varphi, \psi} > 0 \) such that

\[
\left| \int_{\Omega \times T^m} \psi \circ \pi_2 \circ \hat{F}^n d\nu \otimes \text{Leb} - \int_{T^m} \varphi(x) d\eta(x) \int_{T^m} \psi(x) dx \right| \leq C_{\varphi, \psi}(\tilde{C} \lambda^n + \varepsilon)
\]

\( \square \)

As an example, one can use Theorem 2.1 to prove approximate decay of correlation in case the forcing is driven by a power of the logistic map \( \hat{g} := 4x(1-x) \). In fact, it is well known \( \hat{g} \) is conjugate to the tent map via \( g \) = \( (h^{-1})_* \text{Leb}_{T^1} \) defined by

\[
g(x) = \begin{cases} 
2x & x \in [0,1/2) \\
1 - 2x & x \in [1/2,1]
\end{cases}
\]

via a \( C^1 \) map \( h : [0,1] \to [0,1] \). Analogously, for any \( n \in \mathbb{N} \), also \( \hat{g}^n \) is conjugate to \( g := g^n \) via \( h \), and \( g \) is in the class of maps admitted by the generalization in Section 5.1 for which one can apply Theorem 2.1.

5.3. **More or less regular disintegrations.** In Definition 2.1 we have given the definition of Lipschitz disintegration \( \{\mu_\omega\}_{\omega \in \Omega} \) and later we have shown how, under the hypotheses of Theorem 2.1, certain classes of measures with Lipschitz disintegration were kept invariant by the dynamics. Measures having Hölder disintegration can be defined in a completely analogous way, and they can be used to define classes of invariant measures for example in the case where \( f : \Omega \times X \to X \) is only Hölder and not Lipschitz.

Analogously, one could think of defining measures having disintegrations of higher regularity, e.g. differentiable for a suitable notion of differentiability for curves in \( \mathcal{M}_1(X) \), and exploit these classes.

5.4. **Limitations of the approach.** A more substantial and also natural step forward from Theorem 2.1 would be considering \( g \) an invertible uniformly hyperbolic map, like an Anosov diffeomorphism or a map with an Axiom A attractor. Unfortunately it seems hard to extend the techniques in this paper to this case. The main reason is that we need the contraction properties of the inverse branches of \( g \): for invertible uniformly hyperbolic systems, some directions are contracted when taking preimages, but others are expanded and this spoils the arguments.
For the same reason our approach is evidently ill-suited to treat skew-products with quasi-periodic base (see e.g. [DF15]).

Appendix A. Markov chains and random dynamics

In this section we report some classical results about geometric ergodicity of Markov chains ([DMT95, MT93, HM11, Clo15]), and we relate this to random dynamical systems in discrete time.

A.1. Markov chains and geometric ergodicity.

Definition A.1. Given a Polish space $S$, the state space, endowed with a countably generated $\sigma-$field $\mathcal{B}(S)$, a discrete time Markov process is a sequence of random variables $\{X_t\}_{t \in \mathbb{N}_0}$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}[X_n | X_{n-1}, ..., X_0] = \mathbb{E}[X_n | X_{n-1}].$$

The Markov process is called stationary if $\mathbb{E}[X_n | X_{n-1}]$ does not depend on $n$, and $\mathbb{P}: S \times \mathcal{B}(S) \to \mathbb{R}^+$ is the associated transition kernel satisfying

$$\mathbb{P}(X_{n+1} \in A | X_n = x) = P(x, A).$$

For every $x \in S$, $P(x, \cdot)$ defines a probability measure with the following meaning: $P(x, A)$ is the probability that $X_{n+1} \in A$ given that at time $n$ one has observed $X_n = x$.

Given a stationary Markov process and $n \in \mathbb{N}$, one can extend the notion of kernel to higher iterates: Define $P^n: S \times \mathcal{B}(S) \to \mathbb{R}^+$

$$P^n(x, A) = \mathbb{P}(X_{n+m} \in A | X_n = x).$$

For any $n \in \mathbb{N}$, $P^n$ generates an action on the set of measures on $(S, \mathcal{B}(S))$ in the following way. Given $\mu$ a measure on $S$, define

$$\mathcal{P}\mu(A) := \int_S P(x, A) d\mu(x).$$

and

$$\mathcal{P}^n\mu(A) := \int_S P^n(x, A) d\mu(x).$$

Using the properties of transition kernels one can prove that, $\{\mathcal{P}^n\}$ satisfies the semi-group property

$$\mathcal{P}^n \circ \mathcal{P}^m = \mathcal{P}^{n+m}$$

making $\mathcal{P}$ the generator of a semi-group action on probability measures on $(S, \mathcal{B}(S))$.

Definition A.2. A stationary Markov chain is said to be geometrically ergodic if there are $C > 0$ and $\lambda \in (0, 1)$ such that

$$d_{TV}(P^n(x_1, \cdot), P^n(x_2, \cdot)) \leq C\lambda^n, \quad \forall x_1, x_2 \in S.$$

For a definition of the Total Variation distance $d_{TV}$ see the beginning of Sec. C. From Definition [A.2] follows that if a Markov chain is geometrically ergodic, then there is a probability measure $\eta_0$ such that, for every probability measure $\mu$ on $(S, \mathcal{B}(S))$,

$$d_{TV}(\mathcal{P}^n\mu, \eta_0) \leq C\lambda^n.$$

The measure $\eta_0$ satisfies $\mathcal{P}(\eta_0) = \eta_0$ and is also called a stationary distribution or stationary measure.
A.2. **Sufficient conditions for geometric ergodicity.** In this subsection we give a sufficient condition that ensures geometric ergodicity of a stationary Markov chain. Weaker conditions working in more general setups are available and involve petite sets [DMT95] or Lyapunov functions [HMT1].

**Theorem A.1** ([MT93]). Let \( \{X_n\}_{n \in \mathbb{N}_0} \) be a stationary Markov chain on \((S, \mathcal{B}(S))\) with transition kernel \( P : S \times \mathcal{B}(S) \to \mathbb{R}^+ \). Assume there is \( \nu \) a probability measure, \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that

\[
P^{n_0}(x, \cdot) \geq \varepsilon \nu(\cdot), \quad \forall x \in S.
\]

Then the Markov chain is geometrically ergodic.

A.3. **Randomly forced systems and Markov chains.** In this section we discuss the difference, in terms of mathematical definitions, between random and deterministic forcing.

By **random forcing**, we mean that given a probability space \((\Omega, \nu)\) and \( f : \Omega \times X \to X \), at the \( n \)-th iteration we apply the map \( f_n := f(\zeta_n, \cdot) : X \to X \), where \( \{\zeta_n\}_{n \in \mathbb{N}} \) is an i.i.d sequence of random variables defined on some probability space \((\Xi, P)\) with values in \( \Omega \) and distributed according to \( \nu \). Fixed \( w \in \Xi \), the forward orbits of the system are given by

\[
O(x) := \{ f_{n_0}(w) \circ \ldots \circ f_1(w) \circ f_0(w)(x) : n \in \mathbb{N}_0 \}, \quad \forall x \in X.
\]

An important example of random forcing is given by additive i.i.d. noise: Consider \( X = \mathbb{T}^m \), or any other set with an additive structure, a map \( T : X \to X \) and \( \{\zeta_n\}_{n \in \mathbb{N}_0} \) an i.i.d. sequence of random variables with values in \( X \) and distributed according to \( \nu \), then taking \( \Omega = X \) define \( f : \Omega \times X \to X \) as

\[
f(\omega, x) := T(x) + \omega.
\]

Composing at time \( n \in \mathbb{N} \) by \( f_n \) corresponds to considering the recursive equation

\[
\mathcal{X}_{n+1} = T(\mathcal{X}_n) + \zeta_n, \quad \forall n \in \mathbb{N}_0.
\]

where \( \mathcal{X}_n \) is the state of the system at time \( n \). What the above means is that, calling \((\Xi, P)\) the underlying probability space where \( \{\zeta_n\}_{n \in \mathbb{N}_0} \) are defined, \( \{\mathcal{X}_n\}_{n \in \mathbb{N}_0} \) are random variables satisfying

\[
P(\mathcal{X}_{n+1} \in A | \mathcal{X}_n = x_n) = P(\zeta_n \in (A - T(x_n))),
\]

and thus \( \{\mathcal{X}_n\}_{n \in \mathbb{N}_0} \) is a Markov chain. In the above, \( T \) denotes the intrinsic dynamics, i.e. the dynamics the system would have if it did not receive any forcing, while \( \zeta_n \) is the random forcing noise term.

**Deterministic forcing** is also represented as application at time \( n \in \mathbb{N} \) of the map \( f(\zeta_n, \cdot) : X \to X \). However, in this case the sequence \( \{\zeta_n\}_{n \in \mathbb{N}} \) is not required to be independent, but it should satisfy \( \zeta_{n+1} = g(\zeta_n) \) for some transformation \( g : \Omega \to \Omega \) that preserves the measure \( \nu \). This corresponds also to the general definition of **random dynamical system** usually given in the literature (see [Arn98]).

The difference between random and deterministic forcing is not a stark one. In fact one can show that random forcing is a particular case of deterministic forcing where \( g \) is an appropriate shift map. In fact given \( f : \Omega \times X \to X \) and \( \{\zeta_n\}_{n \in \mathbb{N}_0} \) an i.i.d sequence with values in \( \Omega \) distributed as \( \nu \), we can construct the probability space \((\Omega', \nu')\) with

\[
(\xi_n) = (\zeta_n - f(\omega, (\zeta_0, \ldots, \zeta_{n-1})))_{n \in \mathbb{N}_0}
\]
Ω′ := Ω′0 and ν′ := ν ⊗ Ω′0. Now define the sequence of identically distributed random variables \( \{ζ_k\}_{k ∈ N_0} \) in Ω′ with \( ζ_k := \{ζ_{n+k}\}_{n ∈ N_0} \), and \( f′ : Ω′ × X → X \)
\[ f′(ζ, x) := f((ζ)_0, x) \]
where \((ζ)_0\) denotes the first term of the sequence \( ζ \in Ω′ \). With this definition we also have
\[ ζ_{k+1} = σ_{k+1}(\{ζ_n\}_{n ∈ N_0}) = σ(ζ_{k-1}) \]
where \( σ : Ω′ → Ω′ \) is the left shift which is easy to check that keeps the measure \( ν′ \) invariant.

Appendix B. Disintegration of measure and Rohlin’s theorem

The following definitions and results are taken from [Sim12], adapted to the level of generality needed in this paper.

**Definition B.1.** Let \((X, µ)\) be a topological probability space, \( Y \) a metric space and \( π : X → Y \) a measurable function. Call \( ˆµ := π∗µ \). A system of conditional measures of \( µ \) with respect to \((X, π, Y)\) is a collection of measures \( \{µ_y\}_{y ∈ Y} \) such that

1) For \( ˆµ \)–almost every \( y ∈ Y \), \( µ_y \) is a probability measure on \( π^{-1}(y) \).

2) For every measurable subset \( B ⊂ X \), \( y ↦→ µ_y(B) \) is measurable and
\[ µ(B) = \int µ_{π^{-1}(y)}(B)d ˆµ(y). \]

When \( Y \) in the above definition is a measurable partition of \( X \) and \( π(x) \) is the unique element of the partition to which \( x \) belongs, then we also call \( \{µ_y\}_{y ∈ Y} \) a disintegration of \( µ \).

**Definition B.2.** In the same setup of Definition B.1 the topological conditional measure of \( µ \) with respect to \((X, π, y, Y)\) is the weak∗ limit (if it exists)
\[ µ_y := \lim_{ε → 0^+} µ_{π^{-1}(B(y, ε))} \]
where \( B(y, ε) \) is the ball centered at \( y \) with radius \( ε \) with respect to the metric on \( Y \) and
\[ µ_{π^{-1}(B(y, ε))}(I) = \frac{µ(π^{-1}(B(y, ε)) ∩ I)}{µ(π^{-1}(B(y, ε)))}. \]

**Theorem B.1** (Theorem 2.2 [Sim12]). Let \((X, µ)\) be a compact metric probability space, let \( Y \) be a separable Riemannian manifold. Let \( π : X → Y \) be measurable. Then for \( ˆµ \)–almost every \( y ∈ Y \), the topological conditional measure of \( µ \) with respect to \((X, π, y, Y)\) exists as in Definition B.1. Furthermore the collection of measures \( \{µ_y\}_{y ∈ Y} \) is a system of conditional measures as in Definition B.1 (if \( µ_y \) does not exist, set \( µ_y = 0 \)).

Appendix C. Wasserstein distance: some computations

Consider a compact metric space \((Y, d)\). Then the Kantorovich-Wasserstein between \( µ_1, µ_2 ∈ M_1(Y) \) is defined as
\[ d_W(µ_1, µ_2) := \sup_{γ ∈ C(µ_1, µ_2)} \int_{Y × Y} d(s, s′)dγ(s, s′) \]
where $C(\mu_1, \mu_2)$ is the set of all couplings between $\mu_1$ and $\mu_2$. If we consider instead of the metric $d$ the discrete metric $d_{\text{dis}}$ defined as

$$d_{\text{dis}}(s, s') = \begin{cases} 1 & s = s' \\ 0 & s \neq s' \end{cases}$$

We have that

$$d_{\text{TV}}(\mu_1, \mu_2) := \sup_{\gamma \in C(\mu_1, \mu_2)} \int_{Y \times Y} d_{\text{dis}}(s, s') d\gamma(s, s').$$

**Lemma C.1.** Let $(Y, d)$ be a metric space, $T : Y \to Y$ a Lipschitz transformation with Lipschitz constant $\text{Lip}(T)$. Then for any $\xi_1, \xi_2 \in M_1(Y)$

$$d_W(T_*\xi_1, T_*\xi_2) \leq \text{Lip}(T) d_W(\xi_1, \xi_2).$$

**Proof.**

$$d_W(T_*\xi_1, T_*\xi_2) = \sup_{\varphi \in \text{Lip}^1(Y)} \int_Y \varphi(y) d(T_*\xi_1 - T_*\xi_2)(y)$$

$$= \sup_{\varphi \in \text{Lip}^1(Y)} \int_Y \varphi \circ T d(\xi_1 - \xi_2)(y)$$

$$\leq \sup_{\varphi \in \text{Lip}^1(Y)} \text{Lip}(\varphi \circ T) d_W(\xi_1, \xi_2)$$

and $\text{Lip}(\varphi \circ T) = \text{Lip}(\varphi) \text{Lip}(T)$. \qed

**Remark C.1.** The above lemma can be read in the following way: If $T : (Y, d) \to (Y, d)$ is Lipschitz, then $T_* : (M_1(Y), d_W) \to (M_1(Y), d_W)$ is Lipschitz with $\text{Lip}(T_*) = \text{Lip}(T)$.

**Lemma C.2.** Consider $(S, \nu)$ a measurable space with $\nu$ a probability measure, and $Y$ a compact metric space. Assume that $\{\mu_s\}_{s \in S}$ is a family of measures belonging to $M_1(Y)$ and that $\exists \ell > 0$ s.t. $d_W(\mu_s, \mu_{s'}) \leq \ell$ for for every $s, s' \in S$. Then the measure $\bar{\mu} \in M_1(Y)$ defined as

$$\bar{\mu}(A) := \int_S d\nu(s) \mu_s(A)$$

is such that $d_W(\bar{\mu}, \mu_s) \leq \ell$ for all $s \in S$.

**Proof.** Pick $s \in S$

$$d_W(\bar{\mu}, \mu_s) = \sup_{\varphi \in \text{Lip}^1(Y)} \int_Y \varphi(y) d(\bar{\mu} - \mu_s)(y)$$

$$= \sup_{\varphi \in \text{Lip}^1(Y)} \int_Y \int_S d\nu(s') \varphi(y) d(\mu_{s'} - \mu_s)(y)$$

$$\leq \int_S d\nu(s') \sup_{\varphi \in \text{Lip}^1(Y)} \int_Y \varphi(y) d(\mu_{s'} - \mu_s)(y)$$

$$\leq \int_S d\nu(s') d_W(\mu_s, \mu_{s'})$$

$$\leq \ell.$$ \qed
Lemma C.3. Let $(Y,d)$ be a bounded metric space and call $\text{diam}(Y)$ its diameter. Then
\[ d_{W}(\mu_{1}, \mu_{2}) \leq \text{diam}(Y) d_{TV}(\mu_{1}, \mu_{2}). \]

Proof.
\[
d_{W}(\mu_{1}, \mu_{2}) = \sup_{\gamma \in \mathcal{C}(\mu_{1}, \mu_{2})} \int_{Y \times Y} d(s, s') d\gamma(s, s') \\
\leq \sup_{\gamma \in \mathcal{C}(\mu_{1}, \mu_{2})} \int_{Y \times Y} \text{diam}(Y) d_{dis}(s, s') d\gamma(s, s') \\
= \text{diam}(Y) d_{TV}(\mu_{1}, \mu_{2}).
\]
\[\square\]

Lemma C.4. Assume $\{\mu_{i}\}_{i=1}^{n}$ and $\{\mu'_{i}\}_{i=1}^{n}$ are probability measures in $\mathcal{M}_{1}(Y)$ and $\{b_{i}\}_{i=1}^{n}$, $b_{i} > 0$, are weights with $\sum_{i=1}^{n} b_{i} = 1$. Then
\[ d_{W}\left(\sum_{i=1}^{n} b_{i} \mu_{i}, \sum_{i=1}^{n} b_{i} \mu'_{i}\right) \leq \max_{i} d_{W}(\mu_{i}, \mu'_{i}). \]

Proof.
\[
d_{W}\left(\sum_{i=1}^{n} b_{i} \mu_{i}, \sum_{i=1}^{n} b_{i} \mu'_{i}\right) \leq \sup_{\varphi \in \text{Lip}^{1}} \int_{Y} \varphi d\left(\sum_{i=1}^{n} b_{i} \mu_{i} - \sum_{i=1}^{n} b_{i} \mu'_{i}\right) \\
\leq \sum_{i=1}^{n} b_{i} \sup_{\varphi \in \text{Lip}^{1}} \int_{Y} \varphi d(\mu_{i} - \mu'_{i}) \\
\leq \sum_{i=1}^{n} b_{i} d_{W}(\mu_{i}, \mu'_{i}) \\
\leq \max_{i} d_{W}(\mu_{i}, \mu'_{i}).
\]
\[\square\]

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