Convergence problem of the Kawahara equation on the real line

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Abstract. In this paper, we consider the convergence problem of the Kawahara equation
\begin{equation*}
    u_t + \alpha \partial_x^5 u + \beta \partial_x^3 u + \partial_x(u^2) = 0
\end{equation*}
on the real line with rough data. Firstly, by using Strichartz estimates as well as high-low frequency idea, we establish two crucial bilinear estimates, which are just Lemmas 3.1-3.2 in this paper; we also present the proof of Lemma 3.3 which shows that $s > -\frac{1}{2}$ is necessary for Lemma 3.2. Secondly, by using frequency truncated technique and high-low frequency technique, we show the pointwise convergence of the Kawahara equation with rough data in $H^s(\mathbb{R})(s \geq \frac{1}{4})$; more precisely, we prove
\begin{equation*}
    \lim_{t \to 0} u(x, t) = u(x, 0), \quad a.e. x \in \mathbb{R},
\end{equation*}
where $u(x, t)$ is the solution to the Kawahara equation with initial data $u(x, 0)$. Lastly, we show
\begin{equation*}
    \lim_{t \to 0} \sup_{x \in \mathbb{R}} |u(x, t) - U(t)u_0| = 0
\end{equation*}
with rough data in $H^s(\mathbb{R})(s > -\frac{1}{2})$.

Keywords: Kawahara equation; Strichartz estimates; Pointwise convergence; Uniform convergence

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1. Introduction

In this paper, we investigate the Cauchy problem for the Kawahara equation

\[ u_t + \alpha \partial_x^5 u + \beta \partial_x^3 u + \partial_x (u^2) = 0, \quad x \in \mathbb{R}, \]  

\[ u(x, 0) = u_0(x), \]  

where \( \alpha \neq 0, \beta \) are real numbers.

The Kawahara equation arises in the study of the water waves with surface tension, in which the Bond number takes on the critical value, where the Bond number represents a dimensionless magnitude of surface tension in the shallow water regime [2, 33, 37]. Some authors have studied the Cauchy problem for the Kawahara equation on the real line [7, 9, 17, 18, 28, 29, 34, 36, 53–55] and [35] on the torus and [9] on the half-line. By using the Fourier restriction norm method introduced in [6] and developed in [30, 31], Chen et al. [10] and Jia and Huo [29] independently proved that the Cauchy problem for (1.1) is locally well-posed in \( H^s(\mathbb{R})(s > -\frac{7}{4}) \). Chen and Guo [11] proved that the Cauchy problem for (1.1) is globally well-posed in \( H^s(\mathbb{R})(s \geq -\frac{7}{4}) \) with the aid of I-method introduced in [13]. Kato [34] proved that the Cauchy problem for the Kawahara equation is locally well-posed in \( H^s(\mathbb{R})(s \geq -2) \) with the aid of some resolution spaces and ill-posed in \( H^s(\mathbb{R})(s < -2) \) by using the argument of [1]. Kato [36] proved that the Cauchy problem for the Kawahara equation is globally well-posed in \( H^s(\mathbb{R})(s > -\frac{38}{21}) \) with the aid of I-method.

Now we recall the research history of pointwise convergence problem of some linear dispersive equations. Carleson [8] firstly investigated the pointwise convergence problem of one dimensional Schrödinger equation in \( H^s(\mathbb{R})(s \geq 1/4) \). Some authors have investigated the pointwise convergence problem of the Schrödinger equation in dimensions \( n \geq 2 \) [3–5, 12, 16, 19–21, 23, 24, 26, 39, 42–44, 47–52] and established an improved maximal inequality for 2D fractional order Schrödinger operators [45] and established the maximal estimates for Schrödinger equation with inverse-square potential [46]. Dahlberg and Kenig [19] showed that the pointwise convergence problem of the Schrödinger equation is invalid in \( H^s(\mathbb{R}^n)(s < \frac{1}{4}) \). Bourgain [5] presented counterexamples showing that when \( s < \frac{n}{2(n+1)}(n \geq 2) \), the pointwise convergence problem of \( n \) dimensional Schrödinger equation does not hold in \( H^s(\mathbb{R}^n) \). Recently, Du et al. [22] proved that the pointwise convergence problem of two dimensional Schrödinger equation is valid in \( H^s(\mathbb{R}^2) \) with \( s > \frac{1}{3} \). Du and Zhang [24] showed that the pointwise convergence problem of \( n \) dimensional Schrödinger equation is valid for data in \( H^s(\mathbb{R}^n)(s > \frac{n}{2(n+1)}, n \geq 3) \).

Compaan [14] studied the smooth property and dispersive blow-up of semilinear Schrödinger equation. Compaan et al. [15] showed the convergence problem of the
nonlinear Schrödinger flows with rough data and random data. Linares and Ramos [40, 41] showed the pointwise convergence results for the flow of the generalized Zakharov-Kuznetsov equation.

In this paper, motivated by [14, 15, 40, 41], we show \( \lim_{t \to 0} u(x, t) = u(x, 0), \ a.e. x \in \mathbb{R} \), where \( u(x, t) \) is the solution to the Kawahara equation with rough data in \( H^s(\mathbb{R}) (s \geq \frac{1}{4}) \); we also show

\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}} |u(x, t) - U(t)u_0| = 0
\]

with rough data in \( H^s(\mathbb{R}) (s > -\frac{1}{2}) \).

We present some notations before stating the main results. \(|E|\) denotes by the Lebesgue measure of set \( E \). \( A \sim B \) means that there exists \( C > 0 \) such that \( \frac{1}{C} |A| \leq |B| \leq C |A| \). We define a smooth jump function \( \eta(\xi) \) such that \( \eta(\xi) = 1 \) for \( |\xi| \leq 1 \) and \( \eta(\xi) = 0 \) for \( |\xi| > 2 \) and \( \phi(\xi) = \alpha \xi^5 - \beta \xi^3 \). We define

\[
a = \max \left\{ 1, \left( \frac{3\beta}{5\alpha} \right)^{\frac{1}{2}} \right\},
\]

\[
\mathcal{F}_x f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx,
\]

\[
\mathcal{F}_x^{-1} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(x) dx,
\]

\[
\mathcal{F} f(\xi, \tau) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\xi - it\tau} f(x, t) dx dt,
\]

\[
\mathcal{F}^{-1} f(\xi, \tau) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\xi + it\tau} f(x, t) dx dt,
\]

\[
U(t)f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi - it\phi(\xi)} \mathcal{F}_x u_0(\xi) d\xi,
\]

\[
P^a f = \frac{1}{2\pi} \int_{|\xi| \geq a} e^{ix\xi} \mathcal{F}_x f(\xi) d\xi,
\]

\[
P_a f = \frac{1}{2\pi} \int_{|\xi| \leq a} e^{ix\xi} \mathcal{F}_x f(\xi) d\xi.
\]

The space \( H^s(\mathbb{R}) \) is the completion of the Schwartz function space on \( \mathbb{R} \) with respect to the norm \( \| f \|_{H^s(\mathbb{R})} = \| \langle \xi \rangle^s \mathcal{F}_x f \|_{L^2_\xi(\mathbb{R})} \), where \( \langle \xi \rangle^s = (1 + |\xi|^2)^{\frac{s}{2}} \). The space \( X_{s, b}(\mathbb{R}^2) \) is defined to be the completion of the Schwartz function space on \( \mathbb{R}^2 \) with respect to the norm

\[
\| u \|_{X_{s, b}(\mathbb{R}^2)} = \| \langle \xi \rangle^s \langle \sigma \rangle^b \mathcal{F} u(\xi, \tau) \|_{L^2_{\xi, \tau}}.
\]

Here, \( \sigma = \tau + \phi(\xi) \).

The main results are as follows:
Theorem 1.1. (Bilinear estimate related to Kawahara equation) For $s \geq -\frac{7}{4} + 4\epsilon$, $b' = -\frac{1}{2} + 2\epsilon$ and $b = \frac{1}{2} + \epsilon$. Then, we have

$$\|\partial_x (u_1 u_2)\|_{X_{s,b'}} \leq C \|u_1\|_{X_{s,b}} \|u_2\|_{X_{s,b}}. \quad (1.3)$$

Remark 1. Theorem 1 is the new proof of Corollary 3.3 of [29]. Jia and Huo used the Cauchy-Schwarz inequality and Strichartz estimates to establish Corollary 3.3 of [29]. In this paper, we only use the Strichartz estimates to establish Theorem 1.1. Moreover, Lemma 2.6 plays an important role in establishing Theorem 1.1.

Theorem 1.2. (Bilinear estimate related to Kawahara equation). Let $s_1 = \frac{1}{2} + 2\epsilon$, $b' = -\frac{1}{2} + 2\epsilon$, $b = \frac{1}{2} + \epsilon$ and $s_2 \geq -\frac{1}{2} + \epsilon$. Then, we have

$$\|\partial_x (u_1 u_2)\|_{X_{s_1,b'}} \leq C \|u_1\|_{X_{s_2,b}} \|u_2\|_{X_{s_2,b}}. \quad (1.4)$$

Remark 2. From the proof process of Theorem 1.5, Theorem 1.2 plays an import part in establishing Theorem 1.5.

Theorem 1.3. (Necessity of $s > -\frac{1}{2}$ in Theorem 2) Let $s \leq -\frac{1}{2}$, $s_1 = \frac{1}{2} + 2\epsilon$, $b' = -\frac{1}{2} + 2\epsilon$, $b = \frac{1}{2} + \frac{1}{2}$. Then, the following bilinear estimate

$$\|\partial_x (uv)\|_{X_{s_1,b'}} \leq C \|u\|_{X_{s,b}} \|v\|_{X_{s,b}} \quad (1.5)$$

fails.

Remark 3. From Theorem 1.3, we know that $s_2 > -\frac{1}{2}$ is necessary in proving Theorem 1.2. From the proof process of Theorem 1.5, we know that $X_{s,b} \hookrightarrow C(\mathbb{R}; H^s)(b > \frac{1}{2}, s \in \mathbb{R})$, which is just Lemma 2.7 in this paper and Theorem 1.2 plays the key role in proving Theorem 5. From the proof process of Theorem 1.5, we know that $b > \frac{1}{2}$ is necessary in proving Theorem 1.2. Thus, we require $b > \frac{1}{2}$ in Theorem 1.3.

Theorem 1.4. (Pointwise convergence of Kawahara equation) Let $u_0 \in H^s(\mathbb{R})(s \geq \frac{1}{4})$ and $u$ be the solution to (1.1). Then, we have

$$\lim_{t \to 0} u(x, t) = u_0(x), \quad (1.6)$$

for almost everywhere $x \in \mathbb{R}$.

Remark 4: Follow the idea of Proposition 4.3 of [15], we present the outline of Theorem 1.4. From Lemma 2.5 established in this paper, we have

$$\|u\|_{L^4_t L^\infty_x} \leq C \|u\|_{X_{s,b}}(s \geq \frac{1}{4}). \quad (1.7)$$
We consider the frequency truncated Kawahara equation

\[ \partial_t u_N + \alpha \partial_x^5 u_N + \beta \partial_x^3 u_N + \partial_x P_N ((u_N)^2) = 0, \]

\[ u_N(x, 0) = P_N u_0. \]

(1.8) (1.9)

Firstly, we prove

\[ \lim_{N \to \infty} \| u - u_N \|_{L^4_x L^\infty_t} = 0, \]

(1.10)

which is just Lemma 4.1 in this paper. Since \( u_N \) is smooth, for all \( x \in \mathbb{R} \), we have

\[ \lim_{t \to 0} u_N(x, t) = P_N u_0(x). \]

(1.11)

Since

\[ |u - u_0| \leq |u - u_N| + |u_N - P_N u_0| + |P_N u_0|, \]

(1.12)

we have

\[ \limsup_{t \to 0} |u - u_0| \leq \limsup_{t \to 0} |u - u_N| + |P_N u_0|. \]

(1.13)

For arbitrary \( \lambda > 0 \), by using the Chebyshev inequality, (1.13) and Sobolev embedding, we have

\[ |\{ x \in \mathbb{R} : \limsup_{t \to 0} |u - u_0| > \lambda \}| \leq |\{ x \in \mathbb{R} : \limsup_{t \to 0} |u - u_N| > \frac{\lambda}{2} \}| \]

\[ + |\{ x \in \mathbb{R} : |P_N u_0| > \frac{\lambda}{2} \}| \leq C \lambda^{-4} \| u - u_N \|_{L^4_x L^\infty_t}^4 + C \lambda^{-2} \| P_N u_0 \|_{L^2} \]

\[ \leq C \lambda^{-4} \| u - u_N \|_{L^4_x L^\infty_t}^4 + C \lambda^{-2} \| P_N u_0 \|_{H^s}. \]

(1.14)

Since \( u_0 \in H^s(\mathbb{R})(s \geq \frac{1}{2}) \), we have

\[ \| P_N u_0 \|_{H^s} \to 0 \]

(1.15)

as \( N \to \infty \). By using (1.10) and (1.15), we have

\[ |\{ x \in \mathbb{R} : \limsup_{t \to 0} |u - u_0| > \lambda \}| = 0. \]

(1.16)

**Theorem 1.5.**(Uniform convergence of Kawahara equation) Let \( u_0 \in H^s(\mathbb{R})(s > -\frac{1}{2}) \) and \( u \) be the solution to (1.1). Then, we have

\[ \limsup_{t \to 0} x \in \mathbb{R} \| u(x, t) - U(t) u_0 \| = 0. \]

(1.17)
Remark 5: Inspired by the idea of [14], we present the outline of Theorem 1.5. Firstly, from [29] and the proof process of Lemma 4.1 in this paper, we have that the Cauchy problem for the Kawahara equation possesses a unique solution with data in $H^s(R)(s > -\frac{7}{4})$. From the proof process of Lemma 4.1, we have

$$u - U(t)u_0 = \eta \left( \frac{t}{T} \right) \int_0^t U(t-t')\partial_x(u^2)dt'. \quad (1.18)$$

From (1.18) and Theorem 1.2, we have

$$\|u - U(t)u_0\|_{X_{s_1,b}} = \|\eta \left( \frac{t}{T} \right) \int_0^t U(t-t')\partial_x(u^2)dt'\|_{X_{s_1,b}} \leq C\|\partial_x(u^2)\|_{X_{s_1,b}} \leq C\|u\|_{X_{s_2,b}}^2 \leq 2C^3\|u_0\|_{H^{s_2}(R)} < \infty. \quad (1.19)$$

Here $s_1 = \frac{1}{2} + 2\epsilon, s_2 = -\frac{1}{2} + \epsilon$. Since $X_{s_1,b} \hookrightarrow C([-T, T]; H^{s_1}(R)) \hookrightarrow C([-T, T]; C(R))$, from (1.19), we have

$$\limsup_{t \to 0} x \in R |u(x, t) - U(t)u_0| = \limsup_{t \to 0} x \in R \left| \eta \left( \frac{t}{T} \right) \int_0^t U(t-t')\partial_x(u^2)dt' \right| = 0. \quad (1.20)$$

Here, we use Lemma 2.7 and $H^{s_1}(R) \hookrightarrow C(R)(s_1 = \frac{1}{2} + 2\epsilon)$.

Remark 6: We can use Lemma 2.5 established in this paper and Theorem 1.5 to present an alternative proof of Theorem 1.4. Combining Lemma 2.5 established in this paper with the proof of Lemma 2.3 of [21], we immediately obtain

$$U(t)u_0 \longrightarrow u_0 \quad a.e. \quad (1.21)$$

as $t \to 0$ for data in $H^s(R)(s \geq \frac{1}{4})$. From (1.17), we know

$$u \longrightarrow U(t)u_0 \quad a.e. \quad (1.22)$$

as $t \to 0$ for data in $H^s(R)(s > -\frac{1}{2})$. By using the triangle inequality, we have

$$|u - u_0| \leq |u - U(t)u_0| + |u_0 - U(t)u_0| \longrightarrow 0 \quad a.e. \quad (1.23)$$

as $t \to 0$ for data in $H^s(R)(s \geq \frac{1}{4})$. Thus, we have

$$u \longrightarrow u_0 \quad a.e. \quad (1.24)$$

as $t \to 0$ for data in $H^s(R)(s \geq \frac{1}{4})$. Thus, we give an alternative proof of Theorem 1.4.

Remark 7: Compaan et al. [15] studied the pointwise convergence and uniform convergence of the semilinear Schrödinger equation with rough data and random data. In this paper, we investigate the pointwise convergence and uniform convergence of the Kawahara equation with rough data. Kawahara equation is a quasilinear evolution equation,
thus, the structure of its is much more complicated than the structure of the semilinear Schrödinger equation.

**Remark 8:** The proof of Theorem 1.5 mainly depends on the Theorem 1.2, which is optimal due to Theorem 1.3. Thus, the result of Theorem 1.5 is optimal in the sense of Theorems 1.2, 1.3.

The rest of the paper is arranged as follows. In Section 2, we give some preliminaries. In Section 3, we prove three bilinear estimates, which are just Theorems 1.1-1.3. In Section 4, we give the proof of the Theorem 1.4. In Section 5, we give the proof of the Theorem 1.5.

2. Preliminaries

In this section, we present some preliminaries.

**Lemma 2.1.** Let $T \in (0, 1)$, $s \in \mathbb{R}$, $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$ and $f \in H^s(\mathbb{R}), g \in X_{s,b}(\mathbb{R}^2)$. Then, we have

$$
\|\eta(t)U(t)f\|_{X_{s,\frac{1}{2}+s}} \leq C\|f\|_{H^s(\mathbb{R})}, \tag{2.1}
$$

$$
\left\|\eta\left(\frac{t}{T}\right)\int_0^t U(t - \tau)g(\tau)d\tau\right\|_{X_{s,b}(\mathbb{R}^2)} \leq CT^{1+b' - b}\|g\|_{X_{s,b}(\mathbb{R}^2)}. \tag{2.2}
$$

For the proof of Lemma 2.1, we refer the readers to [6, 27, 30].

**Lemma 2.2.** Let

$$
\phi(\xi) = \alpha \xi^5 - \beta \xi^3, \sigma = \tau + \phi(\xi), \sigma_j = \tau_j + \phi(\xi_j)(1 \leq j \leq 2).
$$

Then, we have

$$
|\sigma - \sigma_1 - \sigma_2| = 5|\alpha||\xi||\xi_1||\xi_2|\left|\xi^2 + \xi_1^2 - \xi_1 - \frac{3\beta}{5\alpha}\right|. \tag{2.3}
$$

Moreover, when $|\xi| \geq 2a$ or $|\xi_1| \geq 2a$, where a is defined as in [29], then, (2.3) implies that one of the following cases always occurs:

$$
\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} = |\sigma| \geq C|\xi||\xi_1||\xi_2|\max\{|\xi|^2, |\xi_1|^2\}, \tag{2.4}
$$

$$
\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} = |\sigma_1| \geq C|\xi||\xi_1||\xi_2|\max\{|\xi|^2, |\xi_1|^2\}, \tag{2.5}
$$

$$
\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} = |\sigma_2| \geq C|\xi||\xi_1||\xi_2|\max\{|\xi|^2, |\xi_1|^2\}. \tag{2.6}
$$

Lemma 2.2 can be seen [29].
Lemma 2.3. Let $b > \frac{1}{2}$ and $D \geq 4a$, $a$ is defined as in [29]. Then, we have

\[
\|P^D u\|_{L_t^4 L_x^3} \leq C\|u\|_{X_{0,\frac{1}{2}b}}, \tag{2.7}
\]
\[
\|P^D u\|_{X_{0,-\frac{1}{2}}} \leq C\|u\|_{L_t^4 L_x^3}, \tag{2.8}
\]
\[
\|D_x^3 P^D u\|_{L_t^4 L_x^\infty} \leq C\|u\|_{X_{0,b}}, \tag{2.9}
\]
\[
\|u\|_{L_t^{12}} \leq C\|u\|_{X_{0,b}}, \tag{2.10}
\]
\[
\|u\|_{L_t^4} \leq C\|u\|_{X_{0,\frac{3b}{2}}}, \tag{2.11}
\]
\[
\|P^D U(t)u_0\|_{L_t^4 L_x^\infty} \leq C\|u_0\|_{H^\frac{1}{2}(\mathbb{R})}, \tag{2.12}
\]
\[
\|P^D D_x^3 u\|_{L_t^4} \leq C\|u\|_{X_{0,b}}. \tag{2.13}
\]

Proof. For the proof of (2.7)-(2.9), we refer the readers to Lemma 2.6 of [54]. For the proof of (2.10), we refer the readers to (2.21) of [29]. Interpolating (2.10) with

\[
\|u\|_{L_t^2} = C\|u\|_{L_t^{\frac{8}{3}}},
\]
yields (2.11). For the proof of (2.12), (2.13), we refer the readers to (2.10) and (2.13) of [29], respectively.

We have completed the proof of Lemma 2.3.

Lemma 2.4. Let $s \geq \frac{1}{4}$. Then we have

\[
\|U(t)u_0\|_{L_t^4 L_x^\infty} \leq C\|u_0\|_{H^s(\mathbb{R})}. \tag{2.14}
\]

Proof. By using (2.12) and the Sobolev embeddings Theorem $W^{\frac{1}{4}+\varepsilon,4}(\mathbb{R})$, we have

\[
\|U(t)u_0\|_{L_t^4 L_x^\infty} \leq C \|P^D U(t)u_0\|_{L_t^4 L_x^\infty} + C\|P^D U(t)u_0\|_{L_t^4 L_x^\infty}
\]
\[
\leq C\|u_0\|_{H^{\frac{1}{2}}(\mathbb{R})} + C\|D_x^{\frac{1}{4}+\varepsilon} P^D U(t)u_0\|_{L_t^4}
\]
\[
\leq C\|u_0\|_{H^{\frac{1}{2}}(\mathbb{R})} + C\|\mathcal{F}_x^{-1}\left(|\alpha\xi^5 + \beta\xi^3|^{\frac{1}{4}+\varepsilon} \chi_{|\xi|\leq D(\xi)}\mathcal{F}_x u_0(\xi)\right)\|_{L_t^4}
\]
\[
\leq C\|u_0\|_{H^{\frac{1}{2}}(\mathbb{R})} + C\|\mathcal{F}_x^{-1}\left(|\alpha\xi^5 + \beta\xi^3|^{\frac{1}{4}+\varepsilon} \chi_{|\xi|\leq D(\xi)}\mathcal{F}_x u_0(\xi)\right)\|_{L_t^2}
\]
\[
\leq C\|u_0\|_{H^{\frac{1}{2}}(\mathbb{R})} + C\|\alpha\xi^5 + \beta\xi^3|^{\frac{1}{4}+\varepsilon} \chi_{|\xi|\leq D(\xi)}\mathcal{F}_x u_0(\xi)\|_{L_t^2}
\]
\[
\leq C\|u_0\|_{H^{\frac{1}{2}}(\mathbb{R})} \leq C\|u_0\|_{H^s(\mathbb{R})}. \tag{2.15}
\]

Here, $a$ is defined as in [29] and $D \geq 4a$.

We have completed the proof of Lemma 2.4.

Lemma 2.5. Let $s \geq \frac{1}{4}$ and $b > \frac{1}{2}$. Then, we have

\[
\|u\|_{L_t^4 L_x^\infty} \leq C\|u\|_{X_{s,b}}. \tag{2.16}
\]
Proof. By changing variable \( \tau = \lambda - \phi(\xi) \), we derive

\[
\begin{align*}
    u(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\xi + it\tau} \mathcal{F} u(\xi, \tau) d\xi d\tau \\
    &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\xi + it(\lambda - \phi(\xi))} \mathcal{F} u(\xi, \lambda - \phi(\xi)) d\xi d\lambda \\
    &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\lambda} \left( \int_{\mathbb{R}} e^{ix\xi - it\phi(\xi)} \mathcal{F} u(\xi, \lambda - \phi(\xi)) d\xi \right) d\lambda.
\end{align*}
\]

By using (2.14), (2.17) and Minkowski’s inequality, for \( b > \frac{1}{2} \), we derive

\[
\|u\|_{L^4_t L^\infty_x} \leq C \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} e^{ix\xi - it\phi(\xi)} \mathcal{F} u(\xi, \lambda - \phi(\xi)) d\xi \right\|_{L^4_t L^\infty_x} d\lambda
\]

\[
\leq C \int_{\mathbb{R}} \left\| \mathcal{F} u(\xi, \lambda - \phi(\xi)) \right\|_{H^s} d\lambda
\]

\[
\leq C \left[ \int_{\mathbb{R}} (1 + |\lambda|)^{2b} \left\| \mathcal{F} u(\xi, \lambda - \phi(\xi)) \right\|^2_{H^s} d\lambda \right]^{\frac{1}{2}} \left[ \int_{\mathbb{R}} (1 + |\lambda|)^{-2b} d\lambda \right]^{\frac{1}{2}}
\]

\[
\leq C \left[ \int_{\mathbb{R}} (1 + |\tau + \phi(\xi)|)^{2b} \left\| \mathcal{F} u(\xi, \tau) \right\|^2_{H^s} d\tau \right]^{\frac{1}{2}} = \|u\|_{X_{s,b}}.
\]

This completes the proof of Lemma 2.5.

**Lemma 2.6.** Let \( b = \frac{1}{2} + \epsilon \). Then, we have

\[
\|I(u_1, u_2)\|_{L^2_{xt}} \leq C \prod_{j=1}^{2} \|u_j\|_{X_{0,b}},
\]

\[
\|I_k(u_1, u_2)\|_{L^2_{xt}} \leq C \prod_{j=1}^{2} \|u_j\|_{X_{0,b}} \quad (1 \leq k \leq 2),
\]

where

\[
\mathcal{F} I(u_1, u_2)(\xi, \tau) = \int_{\xi = \xi_1 + \xi_2, |\xi| \geq 4a, \tau = \tau_1 + \tau_2} |\xi_1^4 - \xi_2^4|^{\frac{1}{2}} \mathcal{F} u_1(\xi_1, \tau_1) \mathcal{F} u_2(\xi_2, \tau_2) d\xi_1 d\tau_1,
\]

\[
\mathcal{F} I_k(u_1, u_2)(\xi, \tau) = \int_{\xi = \xi_1 + \xi_2, |\xi| \geq 4a, \tau = \tau_1 + \tau_2} |\xi_1^4 - \xi_2^4|^{\frac{1}{2}} \prod_{j=1}^{2} \mathcal{F} u_j(\xi_j, \tau_j) d\xi_1 d\tau_1 \quad (1 \leq k \leq 2).
\]

For the proof of Lemma 2.6, we refer the readers to Theorem 3.1 of [54].

**Lemma 2.7.** Let \( b > \frac{1}{2} \). Then, we have \( X_{s,b}(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}; H^s(\mathbb{R})) \).

For the proof of Lemma 2.7, we refer the readers to Lemma 4 of [25].

3. Bilinear estimates

In this section, we prove Theorems 1.1-1.3.

To prove Theorem 1.1, it suffices to prove Lemma 3.1.
Lemma 3.1. Let $s \geq -\frac{7}{4} + 4\epsilon$, $b' = -\frac{1}{2} + 2\epsilon$ and $b = \frac{1}{2} + \frac{\epsilon}{2}$. Then, we have

$$
\|\partial_x (u_1 u_2)\|_{X_{s, b'}} \leq C \prod_{j=1}^{2} \|u_j\|_{X_{s, b'}}. \quad (3.1)
$$

**Proof.** To prove (3.1), by duality, it suffices to prove

$$
\left| \int_{\mathbb{R}^2} \partial_x (u_1 u_2) h dx dt \right| \leq C \|h\|_{X_{-s, -b'}} \prod_{j=1}^{2} \|u_j\|_{X_{s, b'}}. \quad (3.2)
$$

We define

$$
\int^* = \int_{\xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2},
$$

$$
f_j(\xi, \tau) = \langle \xi, \tau \rangle^b \mathcal{F} u_j(\xi, \tau)(j = 1, 2),
$$

$$
g(\xi, \tau) = \langle \xi \rangle^{-s} \langle \tau \rangle^{-b'} \mathcal{F} h(\xi, \tau).
$$

To prove (3.2), it suffices to prove

$$
\int_{\mathbb{R}^2} \int^* \frac{|\xi| |\xi|}{\langle \sigma \rangle} \prod_{j=1}^{2} \frac{\prod_{j=1}^{2} \langle \xi_j \rangle}{\langle \sigma_j \rangle} f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) g(\xi, \tau) d\xi_1 d\tau_1 d\xi d\tau
$$

$$
\leq C \|g\|_{L^2_{\xi r}} \|f_1\|_{L^2_{\xi r}} \|f_2\|_{L^2_{\xi r}} \quad (3.3)
$$

with the aid of the Plancherel identity. We define

$$
K_1(\xi_1, \tau_1, \xi, \tau) = \frac{|\xi| |\xi|}{\langle \sigma \rangle} \prod_{j=1}^{2} \frac{\prod_{j=1}^{2} \langle \xi_j \rangle}{{\langle \sigma_j \rangle}},
$$

$$
\mathcal{F} F_j(\xi, \tau) = \frac{f_j(\xi, \tau)}{\langle \sigma_j \rangle^b}(j = 1, 2), \mathcal{F} G(\xi, \tau) = \frac{g(\xi, \tau)}{\langle \sigma \rangle^{-b'}},
$$

$$
I_1 = \int_{\mathbb{R}^2} \int^* K_1(\xi_1, \tau_1, \xi, \tau) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) g(\xi, \tau) d\xi_1 d\tau_1 d\xi d\tau.
$$

Without loss of generality, we can assume that $|\xi_1| \geq |\xi_2|$. Obviously,

$$
\Omega = \{ (\xi_1, \xi, \tau) \in \mathbb{R}^4 : \xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2, |\xi_1| \geq |\xi_2| \} \subset \bigcup_{j=1}^{6} \Omega_j,
$$

and

$$
\Omega_1 = \{ (\xi_1, \tau_1, \xi, \tau) \in \Omega : |\xi_1| \leq 4a \},
$$

$$
\Omega_2 = \{ (\xi_1, \tau_1, \xi, \tau) \in \Omega : |\xi_1| > 4a, |\xi_1| > 4|\xi_2|, |\xi_2| \leq a \},
$$

$$
\Omega_3 = \{ (\xi_1, \tau_1, \xi, \tau) \in \Omega : |\xi_1| > 4a, |\xi_1| > 4|\xi_2|, |\xi_2| > a \},
$$

$$
\Omega_4 = \{ (\xi_1, \tau_1, \xi, \tau) \in \Omega : |\xi_1| > 4a, |\xi_2| \leq |\xi_1| \leq 4|\xi_2|, \xi_1 \xi_2 \geq 0 \},
$$

$$
\Omega_5 = \{ (\xi_1, \tau_1, \xi, \tau) \in \Omega : |\xi_1| > 4a, |\xi_2| \leq |\xi_1| \leq 4|\xi_2|, \xi_1 \xi_2 < 0, 4|\xi| \geq |\xi_2| \},
$$

$$
\Omega_6 = \{ (\xi_1, \tau_1, \xi, \tau) \in \Omega : |\xi_1| > 4a, |\xi_2| \leq |\xi_1| \leq 4|\xi_2|, \xi_1 \xi_2 < 0, 4|\xi| < |\xi_2| \}.
$$
(1) When \((\xi_1, \xi_1, \tau, \tau) \in \Omega_1\), which yield \(|\xi| \leq |\xi_1| + |\xi_2| \leq 8a\), therefore, we have
\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq \frac{C}{\langle \sigma \rangle^{-b} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq \frac{C}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}.
\] (3.4)

By using (3.4), the Cauchy-Schwarz inequality, the Plancherel identity and the Hölder inequality as well as (2.11), we have
\[
I_1 \leq C \int_{\mathbb{R}^2} \int_{*} \frac{f_1 f_2 g}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} d\xi_1 d\tau_1 d\xi d\tau
\]
\[
\leq C \left\| \int_{*} \frac{f_1 f_2}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} d\xi_1 d\tau_1 \right\| \|g\|_{L_2^g}^2
\]
\[
\leq C \|F_1 F_2\|_{L_2^g} \|g\|_{L_2^g}
\]
\[
\leq C \|F_1\|_{L_2^g} \|F_2\|_{L_2^g} \|g\|_{L_2^g}
\]
\[
\leq C \|F_1\|_{L_2^g} \|F_2\|_{L_2^g} \|g\|_{L_2^g}.
\] (3.5)

(2) When \((\xi_1, \xi_1, \tau_1, \tau) \in \Omega_2\), which yield \(|\xi_1| \sim |\xi|\) and \(|\xi_2| \leq a\), therefore, we have
\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi|}{\langle \sigma \rangle^{-b} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1^4 - \xi_2^4|^{\frac{1}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}.
\] (3.6)

By using (3.6), the Cauchy-Schwarz inequality and the Plancherel identity as well as Lemma 2.6, we have
\[
I_1 \leq C \int_{\mathbb{R}^2} \int_{*} \frac{|\xi_1^4 - \xi_2^4|^{\frac{1}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} f_1 f_2 g d\xi_1 d\tau_1 d\xi d\tau
\]
\[
\leq C \left\| \int_{*} \frac{|\xi_1^4 - \xi_2^4|^{\frac{1}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} f_1 f_2 d\xi_1 d\tau_1 \right\| \|g\|_{L_2^g}^2
\]
\[
\leq C \|F_1\|_{L_2^g} \|F_2\|_{L_2^g} \|g\|_{L_2^g} \leq C \|F_1\|_{L_2^g} \|F_2\|_{L_2^g} \|g\|_{L_2^g}.
\] (3.7)

(3) When \((\xi_1, \xi_1, \tau_1, \tau) \in \Omega_3\), which yield \(|\xi_1| \sim |\xi|, |\xi_1| \geq 4a > 2a, |\xi_2| > a\), then, we consider (2.4)-(2.6), respectively.

When (2.4) is valid, since \(s \geq -\frac{7}{4} + 4\epsilon\), then, we have
\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq \frac{|\xi_1^{1+4b} |\xi_2^{1+4b} - s}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1^{1+4b} |\xi_2^{1+4b} - s}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1^4 - \xi_2^4|^{\frac{1}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}.
\] (3.8)
This case can be proved similarly to Case (2).

When (2.5) is valid, which yield $\langle \sigma_1 \rangle^{-b} \langle \sigma \rangle^b \leq \langle \sigma_1 \rangle^b \langle \sigma \rangle^{-b}$, since $s \geq -\frac{7}{4} + 4\epsilon$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq \frac{\langle \xi_1 \rangle^{1+b-s} \langle \xi_2 \rangle^{-s+b}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \leq C \frac{\langle \xi_1 \rangle^{1+b+6\epsilon}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \leq C \frac{\langle \xi_1 \rangle^{\frac{3}{2}} - \langle \xi_2 \rangle^{\frac{1}{2}}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b}. \quad (3.9)$$

This case can be proved similarly to Case (2).

When (2.6) is valid, which yield $\langle \sigma_2 \rangle^{-b} \langle \sigma \rangle^b \leq \langle \sigma_2 \rangle^b \langle \sigma \rangle^{-b}$, since $s \geq -\frac{7}{4} + 4\epsilon$, then we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq \frac{\langle \xi_1 \rangle^{1+b} \langle \xi_2 \rangle^{-s+b}}{\langle \sigma_1 \rangle^b \langle \sigma \rangle^b} \leq C \frac{\langle \xi_1 \rangle^{\frac{3}{2}+6\epsilon}}{\langle \sigma_1 \rangle^b \langle \sigma \rangle^b} \leq C \frac{\langle \xi_1 \rangle^{\frac{3}{2}}}{\langle \sigma_1 \rangle^b \langle \sigma \rangle^b}. \quad (3.10)$$

By using (3.10), the Cauchy-Schwarz inequality, (2.8)-(2.9) and the Hölder inequality, we have

$$I \leq C \int_{\mathbb{R}^2} \int_{\sigma} \frac{\langle \xi_1 \rangle^{\frac{3}{2}}}{\langle \sigma \rangle^b \langle \sigma \rangle^b} f_1 f_2 g d\xi_1 d\xi_2 d\tau_1 d\xi_3 d\tau$$

$$\leq C \left\| \langle \sigma \rangle^{-b} \int_{\sigma} \frac{\langle \xi_1 \rangle^{\frac{3}{2}}}{\langle \sigma \rangle^b} f_1 f_2 d\xi_1 d\tau_1 \right\|_{L^2_{\xi_1 \tau_1}} \|h\|_{L^2_{\xi_2 \tau_2}} \leq C \left\| P^4a((D^{\frac{3}{2}}_s P^D F_1) \mathcal{F}^{-1} f_2) \right\|_{X_{0,-b}} \|g\|_{L^2_{\xi_2 \tau_2}}$$

$$\leq C \|D^{\frac{3}{2}}_s P^D F_1 \mathcal{F}^{-1} f_2\|_{L^4_{\xi_2 \tau_2}} \|g\|_{L^2_{\xi_2 \tau_2}} \leq C \left\| D^{\frac{3}{2}}_s P^D F_1 \right\|_{L^4_{\xi_2 \tau_2}} \|g\|_{L^2_{\xi_2 \tau_2}}$$

$$\leq C \|f_1\|_{L^2_{\xi_2 \tau_2}} \|f_2\|_{L^2_{\xi_2 \tau_2}} \|g\|_{L^2_{\xi_2 \tau_2}}. \quad (3.11)$$

(4) When $(\xi_1, \xi, \tau_1, \tau) \in \Omega_4$, which yield $|\xi_1| \sim |\xi_2| \sim |\xi|, |\xi_1| \geq 4a$, then we consider (2.4)-(2.6), respectively.

When (2.4) is valid, since $s \geq -\frac{7}{4} + 4\epsilon$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{\langle \xi_1 \rangle^{1+5b-s} \langle \xi_2 \rangle^{-s+b}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{\langle \xi_1 \rangle^{\frac{3}{2}+6\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{\langle \xi_1 \rangle^{\frac{3}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}. \quad (3.12)$$

By using (3.12), the Cauchy-Schwarz inequality, and the Plancherel identity as well as (2.7), (2.8), we have

$$I_1 \leq C \int_{\mathbb{R}^2} \int_{\sigma} \frac{\langle \xi_1 \rangle^{\frac{3}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} f_1 f_2 g d\xi_1 d\xi_2 d\tau_1 d\xi_3 d\tau$$

$$\leq C \left\| \int_{\mathbb{R}^2} \frac{\langle \xi_1 \rangle^{\frac{3}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \int_{\sigma} f_1 f_2 d\xi_1 d\tau_1 \right\|_{L^2_{\xi_1 \tau_1}} \|g\|_{L^2_{\xi_2 \tau_2}} \leq C \left\| f_1 \right\|_{L^2_{\xi_2 \tau_2}} \|f_2\|_{L^2_{\xi_2 \tau_2}} \|g\|_{L^2_{\xi_2 \tau_2}}. \quad (3.13)$$

When (2.5) is valid, which yield $\langle \sigma_1 \rangle^{-b} \langle \sigma \rangle^b \leq \langle \sigma_1 \rangle^b \langle \sigma \rangle^{-b}$, since $s \geq -\frac{7}{4} + 4\epsilon$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{\langle \xi_1 \rangle^{1+5b-s} \langle \xi_2 \rangle^{-s+b}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \leq C \frac{\langle \xi_1 \rangle^{\frac{3}{2}+6\epsilon}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \leq C \frac{\langle \xi_2 \rangle^{\frac{3}{2}}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \langle \sigma \rangle^b. \quad (3.14)$$
This case can be proved similarly to (3.13).

When (2.6) is valid, which yield \( \langle \sigma_2 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_2 \rangle^{b} \langle \sigma \rangle^{-b} \), since \( s \geq -\frac{7}{4} + 4\varepsilon \), we have

\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{\xi_1^{1+5b'-s}}{(\sigma_1)^{b}(\sigma)^{-b}} \leq C \frac{\xi_1^{1+6\varepsilon}}{(\sigma_1)^{b}(\sigma)^{-b}} \leq C \frac{\xi_1^{3}|\xi|^{\frac{3}{2}}}{(\sigma_1)^{b}(\sigma)^{-b}}. \tag{3.15}
\]

This case can be proved similarly to (3.13).

(5) When \((\xi_1, \xi, \tau_1, \tau) \in \Omega_5\), which yield \( |\xi_1| \sim |\xi_2| \sim |\xi|, |\xi_1| \geq 4a \).

This case can be proved similarly to Case (4).

(6) When \((\xi_1, \xi, \tau_1, \tau) \in \Omega_6\), which yield \( |\xi_1| \sim |\xi_2|, |\xi_1| \geq 4a, 4|\xi| < |\xi_2| \). Then, we consider (2.4)-(2.6), respectively.

When (2.4) is valid, since \( s \geq -\frac{7}{4} + 4\varepsilon \), we have

\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{\xi_1^{1+b'}|\xi_1|^{4b'-2s}}{\prod_{j=1}^{2} (\sigma_j)^{b}} \leq C \frac{\xi_1^{b} |\xi_1|^2}{\prod_{j=1}^{2} (\sigma_j)^{b}} \leq C \frac{\xi_1^{2} - \xi_2^{2}}{\prod_{j=1}^{2} (\sigma_j)^{b}}. \tag{3.16}
\]

This case can be proved similarly to (3.7).

When (2.5) is valid, which yield \( \langle \sigma_1 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_1 \rangle^{-b} \langle \sigma \rangle^{b} \), since \( s \geq -\frac{7}{4} + 4\varepsilon \), we have

\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{\xi_1^{1+b'}|\xi_1|^{4b'-2s}}{\langle \sigma_1 \rangle^{b}(\sigma)^{-b}} \leq C \frac{\xi_1^{b} |\xi_1|^2}{\langle \sigma_1 \rangle^{b}(\sigma)^{-b}} \leq C \frac{\xi_1^{2} - \xi_2^{2}}{\langle \sigma_1 \rangle^{b}(\sigma)^{-b}}. \tag{3.17}
\]

This case can be proved similarly to (3.7).

When (2.6) is valid, which yield \( \langle \sigma_2 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_2 \rangle^{b} \langle \sigma \rangle^{-b} \), since \( s \geq -\frac{7}{4} + 4\varepsilon \), we have

\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{\xi_1^{1+b'}|\xi_1|^{4b'-2s}}{\langle \sigma_2 \rangle^{b}(\sigma)^{-b}} \leq C \frac{\xi_1^{b} |\xi_1|^2}{\langle \sigma_2 \rangle^{b}(\sigma)^{-b}} \leq C \frac{\xi_2^{2} - \xi_1^{2}}{\langle \sigma_2 \rangle^{b}(\sigma)^{-b}}. \tag{3.18}
\]

This case can be proved similarly to (3.7).

This completes the proof of Lemma 3.1.

To prove Theorem 1.2, it suffices to prove Lemma 3.2.

**Lemma 3.2.** Let \( s_1 = \frac{1}{2} + 2\varepsilon \), \( b = \frac{1}{2} + \varepsilon \), \( b' = -\frac{1}{2} + 2\varepsilon \), \( s_2 \geq -\frac{1}{2} + \varepsilon \). Then, we have

\[
\| \partial_x (u_1 u_2) \|_{X_{s_1,b'}} \leq C \| u_1 \|_{X_{s_2,b}} \| u_2 \|_{X_{s_2,b}}. \tag{3.19}
\]

**Proof.** To prove (3.1), by duality, it suffices to prove

\[
\left| \int_{\mathbb{R}^2} \partial_x (u_1 u_2) h dx \right| \leq C \| h \|_{X_{-s_1,-b'}} \| u_1 \|_{X_{s_2,b}} \| u_2 \|_{X_{s_2,b}}. \tag{3.20}
\]
We define
\[
\int_\ast = \int_{\xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2},
\]
\[
f_j(\xi_j, \tau_j) = \langle \xi_j \rangle^{s_2} \langle \sigma_j \rangle^b \mathcal{F} u_j(\xi_j, \tau_j) (j = 1, 2),
\]
\[
g(\xi, \tau) = \langle \xi \rangle^{-s_1} \langle \sigma \rangle^{-b'} \mathcal{F} h(\xi, \tau).
\]

To prove (3.20), it suffices to prove
\[
\int_{\mathbb{R}^2} \int_\ast \frac{|\xi| \langle \xi \rangle^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^2 \langle \xi_j \rangle^{s_2} \langle \sigma_j \rangle^b} f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) g(\xi, \tau) d\xi_1 d\tau_1 d\xi_2 d\tau_2
\]
\[
\leq C \|g\|_{L^2_{\xi_2}} \|f_1\|_{L^\infty_{\xi_2}} \|f_2\|_{L^2_{\xi_2}}.
\]

with the aid of the Plancherel identity. We define
\[
K_2(\xi_1, \tau_1, \xi, \tau) = \frac{|\xi| \langle \xi \rangle^{s_1}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^2 \langle \xi_j \rangle^{s_2} \langle \sigma_j \rangle^b},
\]
\[
\mathcal{F} F_j(\xi_j, \tau_j) = \frac{f_j(\xi_j, \tau_j)}{\langle \sigma_j \rangle^b} (j = 1, 2), \mathcal{F} G(\xi, \tau) = \frac{g(\xi, \tau)}{\langle \sigma \rangle^{-b'}},
\]
\[
I = \int_{\mathbb{R}^2} \int_\ast K_2(\xi_1, \tau_1, \xi_2, \tau_2) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) g(\xi, \tau) d\xi_1 d\tau_1 d\xi_2 d\tau_2.
\]

Without loss of generality, we can assume that $|\xi_1| \geq |\xi_2|$, we have
\[
A = \{ (\xi_1, \xi, \tau_1, \tau) \in \mathbb{R}^4 : \xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2, |\xi_1| \geq |\xi_2| \} \subset \bigcup_{j=1}^6 A_j,
\]
where $A_j (1 \leq j \leq 6, j \in \mathbb{N})$ are defined as in Lemma 3.1.
(1) When $(\xi_1, \xi, \tau_1, \tau) \in A_1$, which yield $|\xi| \leq |\xi_1| + |\xi_2| \leq 8a$, therefore, we have
\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq \frac{C}{\langle \sigma \rangle^{-b'} \prod_{j=1}^2 \langle \sigma_j \rangle^b}. \tag{3.22}
\]
This case can be proved similarly to Case (1) of Lemma 3.1.
(2) When $(\xi_1, \xi, \tau_1, \tau) \in A_2$, which yield $|\xi_1| \sim |\xi|$ and $|\xi_2| \leq a$, therefore, we have
\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi|^2}{\langle \sigma \rangle^{-b'} \prod_{j=1}^2 \langle \sigma_j \rangle^b} \leq C \frac{|\xi|^4 - |\xi_2|^{\frac{1}{2}}}{\prod_{j=1}^2 \langle \sigma_j \rangle^b}. \tag{3.23}
\]
This case can be proved similarly to Case (2) of Lemma 3.1.
(3) When $(\xi_1, \xi, \tau_1, \tau) \in A_3$, which yield $|\xi_1| \sim |\xi|, |\xi_1| \geq 4a, |\xi_2| > a$, then, we consider (2.4)-(2.6), respectively.
When (2.4) is valid, then, we have
\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq \frac{|\xi_1|^{1+s_1-s_2+4b'}|\xi_2|^{-s_2+b'}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \\
\leq C \frac{|\xi_1|^{\frac{3}{4}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1^{\frac{3}{4}} - \xi_2^{\frac{3}{4}}|^{\frac{1}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}.
\] (3.24)

This case can be proved similarly to Case (2) of Lemma 3.1.

When (2.5) is valid, which yield \( \langle \sigma_1 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_2 \rangle^{b'} \langle \sigma \rangle^{-b} \), then, we have
\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq \frac{|\xi_1|^{1+s_1-s_2+4b'}|\xi_2|^{-s_2+b'}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \\
\leq C \frac{|\xi_1|^{\frac{9}{8}}}{\langle \sigma_1 \rangle^b \langle \sigma \rangle^b} \leq C \frac{|\xi_1|^{\frac{3}{4}} |\xi_2|^{\frac{3}{4}}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b}.
\] (3.25)

This case can be proved similarly to Case (2) of Lemma 3.1.

When (2.6) is valid, which yield \( \langle \sigma_2 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_2 \rangle^{b'} \langle \sigma \rangle^{-b} \), then we have
\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq \frac{|\xi_1|^{1+s_1-s_2+4b'}|\xi_2|^{-s_2+b'}}{\langle \sigma_1 \rangle^b \langle \sigma \rangle^b} \\
\leq C \frac{|\xi_1|^{\frac{9}{8}}}{\langle \sigma_1 \rangle^b \langle \sigma \rangle^b} \leq C \frac{|\xi_1|^{\frac{3}{4}} |\xi_2|^{\frac{3}{4}}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b}.
\] (3.26)

This case can be proved similarly to (2.6) of Case (3) in Lemma 3.1.

When \((\xi_1, \xi, \tau_1, \tau) \in A_4\), which yield \( |\xi_1| \sim |\xi_2| \sim |\xi|, |\xi_1| \geq 4a > 2a \), then, we consider (2.4)-(2.6), respectively.

When (2.4) is valid, we have
\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi_1|^{1+s_1-2s_2+5b'}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \\
\leq C \frac{2}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leq C \frac{|\xi_1|^{\frac{3}{4}} |\xi_2|^{\frac{3}{4}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^b}.
\] (3.27)

This case can be proved similarly to (2.6) of Case 4.

When (2.5) is valid, which yield \( \langle \sigma_1 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_2 \rangle^{b'} \langle \sigma \rangle^{-b} \), then, we have
\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi_1|^{1+s_1-2s_2+5b'}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \\
\leq C \frac{2}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \leq C \frac{|\xi_1|^{\frac{3}{4}}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b} \leq C \frac{|\xi_2|^{\frac{3}{4}} |\xi_1|^{\frac{3}{4}}}{\langle \sigma_2 \rangle^b \langle \sigma \rangle^b}.
\] (3.28)
This case can be proved similarly to (2.6) of Case 4.

When (2.6) is valid, which yield \( \langle \sigma_2 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_2 \rangle^{b'} \langle \sigma \rangle^{-b} \), we have

\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi_1|^{1+s} \langle \xi \rangle^{s_1} |\xi_1|^{-2s_2+4b'}}{\langle \sigma \rangle^{b} \langle \sigma \rangle^{-b}} \leq C \frac{|\xi_1|^{1+s_1} |\xi_1|^{2s_2+5b'}}{\langle \sigma \rangle^{b} \langle \sigma \rangle^{-b}} \leq C \frac{|\xi_1|^3 |\xi|^3}{\langle \sigma \rangle^{b} \langle \sigma \rangle^{-b}}. \tag{3.29}
\]

This case can be proved similarly to (2.6) of Case 4.

(5) When \( (\xi_1, \xi, \tau_1, \tau) \in A_5 \), which yield \( |\xi_1| \sim |\xi_2| \sim |\xi_1| \geq 4a \). This case can be proved similarly to Case (4).

(6) When \( (\xi_1, \xi, \tau_1, \tau) \in A_6 \), which yield \( |\xi_1| \sim |\xi_2|, |\xi_1| \geq 4a, 4|\xi| < |\xi_2| \), then, we consider (2.4)-(2.6), respectively.

When (2.4) is valid, we have

\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi_1|^{1+b} \langle \xi \rangle^{s_1} |\xi_1|^{-2s_2+4b'}}{\prod_{j=1}^{2} \langle \sigma \rangle^{b_j}} \leq C \frac{|\xi_1|^4 - |\xi_2|^4}{\prod_{j=1}^{2} \langle \sigma \rangle^{b_j}}. \tag{3.30}
\]

This case can be proved similarly to Case 2 of Lemma 3.1.

When (2.5) is valid, which yield \( \langle \sigma_1 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_2 \rangle^{b'} \langle \sigma \rangle^{-b} \), then, we have

\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi_1|^{1+b} \langle \xi \rangle^{s_1} |\xi_1|^{-2s_2+4b'}}{\langle \sigma \rangle^{b} \langle \sigma \rangle^{-b}} \leq C \frac{|\xi_1|^4 - |\xi_2|^4}{\langle \sigma \rangle^{b} \langle \sigma \rangle^{-b}}. \tag{3.31}
\]

This case can be proved similarly to Case 2 of Lemma 3.1.

When (2.6) is valid, which yield \( \langle \sigma_2 \rangle^{-b} \langle \sigma \rangle^{b'} \leq \langle \sigma_2 \rangle^{b'} \langle \sigma \rangle^{-b} \), then, we have

\[
K_2(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi_1|^{1+b} \langle \xi \rangle^{s_1} |\xi_1|^{-2s_2+4b'}}{\langle \sigma \rangle^{b} \langle \sigma \rangle^{-b}} \leq C \frac{|\xi_1|^4 - |\xi_2|^4}{\langle \sigma \rangle^{b} \langle \sigma \rangle^{-b}}. \tag{3.32}
\]

This case can be proved similarly to Case 2 of Lemma 3.1.

This completes the proof of Lemma 3.2.

To prove Theorem 1.3, it suffices to prove Lemma 3.3.

**Lemma 3.3.** For \( s \leq -\frac{1}{2} \), \( b' = -\frac{1}{2} + 2\epsilon \) and \( b = \frac{1}{2} + \xi \). Then, we have that

\[
\| \partial_x (u_1u_2) \|_{X^{\frac{1}{2}+2\epsilon, b'}} \leq C \| u_1 \|_{X^{s, b}} \| u_2 \|_{X^{s, b}} \tag{3.33}
\]

fails.
Proof. We define

\[ A = \left\{ (\xi, \tau) \in \mathbb{R}^2 : |\xi - N| \leq N^{-\frac{3}{4}}, |\tau - (5\alpha N^4 + 3\beta N^2)\xi + 4\alpha N^5 + 2\beta N^3| \leq \frac{1}{2} \right\}, \]

\[ B = \left\{ (\xi, \tau) \in \mathbb{R}^2 : |\xi - 2N^{-\frac{3}{4}}| \leq N^{-\frac{3}{4}}, |\tau - (5\alpha N^4 + 3\beta N^2)\xi| \leq 1 \right\}, \]

\[ R = \left\{ (\xi, \tau) \in \mathbb{R}^2 : |\xi - N| \leq \frac{N}{4}, |\tau - (5\alpha N^4 + 3\beta N^2)\xi + 4\alpha N^5 + 2\beta N^3| \leq \frac{1}{2} \right\}, \]

\[ f(\xi, \tau) = \chi_A(\xi, \tau), g(\xi, \tau) = \chi_B(\xi, \tau). \]

Here, \( A, B, R \) are defined as in example 2 of [34]. Then, we have

\[ f * g(\xi, \tau) \geq CN^{-\frac{3}{2}} \chi_R(\xi, \tau). \] (3.34)

Combining (3.33) with (3.34), we have

\[ N^{\frac{3}{4} + 2\varepsilon} N^{-\frac{3}{4}} \leq CN^{\frac{5}{2}} N^{-\frac{3}{4}} N^{-\frac{3}{4}}, \] (3.35)

which is equivalent to \( s \geq -\frac{1}{2} + \frac{3\varepsilon}{4} \). This contradicts with \( s \leq -\frac{1}{2} \).

This ends the proof of Lemma 3.3.

4. Proof of Theorem 1.4

Lemma 4.1. Let \( u_0 \in H^s(\mathbb{R})(s \geq \frac{1}{4}) \). Then, we have

\[ \lim_{N \to \infty} \| u - u_N \|_{L^1_x L^\infty_t} = 0. \] (4.1)

Proof. Inspired by the idea of the proof process of Theorem 1.1 of [15]. We firstly prove that the Cauchy problem for (1.1) is locally well-posed in \( H^s(\mathbb{R})(s > -\frac{1}{2}) \). We define

\[ \Phi(u) = \eta(t)U(t)f + \eta \left( \frac{t}{T} \right) \int_0^t U(t - \tau)\partial_x(u^2)d\tau, \] (4.2)

\[ B = \left\{ u \in X_{s,b} : \| u \|_{X_{s,b}} \leq 2C\| u_0 \|_{H^s} \right\}. \] (4.3)

By using (4.1)-(4.2) and Lemma 3.1, for \( T \leq \left( \frac{1}{4C^2\| f \|_{H^s}} \right)^{\frac{2}{7}} \), we have

\[ \| \Phi(u) \|_{X_{s,b}} \leq \| \eta(t)U(t)f \|_{X_{s,b}} + \left\| \eta \left( \frac{t}{T} \right) \int_0^t U(t - \tau)\partial_x(u^2)d\tau \right\|_{X_{s,b}} \]

\[ \leq C\| f \|_{H^s} + CT^{\frac{3}{7}} \| \partial_x(u^2) \|_{X_{s,b1}} \]

\[ \leq C\| f \|_{H^s} + CT^{\frac{3}{7}} \| u \|_{X_{s,b}}^2 \]

\[ \leq C\| f \|_{H^s} + CT^{\frac{3}{7}}(2C\| f \|_{H^s})^2 \leq 2C\| f \|_{H^s}. \] (4.4)
and
\[ \| \Phi(u) - \Phi(v) \|_{X_{s,b}} \leq \| \eta \left( \frac{t}{T} \right) \int_0^t U(t - \tau) \partial_x (u^2 - v^2) d\tau \|_{X_{s,b}} \]
\[ \leq CT^{\frac{3}{2}} \| u - v \|_{X_{s,b}} \left[ \| u \|_{X_{s,b}} + \| v \|_{X_{s,b}} \right] \leq 2C^2 T^{\frac{3}{2}} \| f \|_{H^s} \| u - v \|_{X_{s,b}} \]
\[ \leq \frac{1}{2} \| u - v \|_{X_{s,b}}. \] (4.5)

Thus, \( \Phi \) is a contraction mapping from \( B \) to \( B \). Consequently, \( \Phi \) has a fixed point. That is \( \Phi(u) = u \). From (4.5), we have
\[ \| u - v \|_{X_{s,b}} \leq \frac{1}{2} \| u - v \|_{X_{s,b}}. \] (4.6)

We can assume that \( u_N \) is the solution to the truncated Kawahara equation
\[ \partial_t u_N + \alpha \partial_x^5 u_N + \beta \partial_x^3 u_N + \partial_x ((u_N)^2) = 0, \] (4.7)
with the initial data \( P_N u_0 \). We can see that \( u_N \) is smooth, as the initial data \( P_N u_0 \) is smooth with \( u_0 \in H^s(\mathbb{R})(s \geq \frac{1}{4}) \). Let \( u := u_\infty \) be the solution to the Kawahara equation with initial data \( u_0 = P_\infty u_0 \). We define
\[ \Phi(u_N) := \eta(t) U(t) P_N u_0 - \eta \left( \frac{t}{T} \right) \int_0^t U(t - t') P_N \partial_x ((u_N)^2) dt'. \] (4.8)

Obviously, by using a proof similar to above, \( \Phi \) is a contraction mapping on the ball \( \left\{ u_N : \| u_N \|_{X_{s,\frac{1}{4}+\epsilon}} \leq 2C \| u_0 \|_{H^s(\mathbb{R})} \right\} \). By using Lemma 2.5, we have
\[ \left\| \sup_{t \in [0,T]} |u - u_N| \right\|_{L^2_t} \leq C \| u - u_N \|_{X_{s,\frac{1}{4}+\epsilon}} (s \geq \frac{1}{4}). \] (4.9)

For \( t \in [-T, T] \), we have
\[ u - u_N = \eta(t) U(t) P_N u_0(x) - \eta \left( \frac{t}{T} \right) \int_0^t U(t - t') (\partial_x (u^2)) - P_N \partial_x ((u_N)^2) dt'. \] (4.10)

Then, by using Lemma 2.1, we have
\[ \| u - u_N \|_{X_{s,\frac{1}{4}+\epsilon}} \leq C \| P_N u_0 \|_{H^s(\mathbb{R})} + CT^{\frac{3}{2}} \| \partial_x (u^2) - P_N \partial_x (u_N^2) \|_{X_{s,\frac{1}{4}+2\epsilon}}. \] (4.11)

Since
\[ \partial_x (u^2) - P_N \partial_x (u_N^2) = P_N (\partial_x (u^2) - \partial_x (u_N^2)) + P_N (\partial_x (u_N^2)), \] (4.12)

we have
\[ \| u - u_N \|_{X_{s,\frac{1}{4}+\epsilon}} \leq C \| P_N u_0 \|_{H^s(\mathbb{R})} + CT^{\frac{3}{2}} \| P_N (\partial_x (u^2) - \partial_x (u_N^2)) \|_{X_{s,\frac{1}{4}+2\epsilon}} + CT \| P_N (\partial_x (u_N^2)) \|_{X_{s,\frac{1}{4}+2\epsilon}}. \] (4.13)
By using Lemmas 2.1, 3.2 and (4.7), for $T \leq \left( \frac{1}{4C^2\|f\|_{H^s}} \right)^{\frac{1}{2}}$, we have

\[ T^{\frac{3}{2}} \| P_N (\partial_x (u^2) - \partial_x (u_N^2)) \|_{X_{s,-\frac{1}{2}+2\epsilon}} \]
\[ \leq CT^{\frac{3}{2}} \| P_N (\partial_x [(u + u_N)(u - u_N)]) \|_{X_{s,-\frac{1}{2}+2\epsilon}} \]
\[ \leq CT^{\frac{3}{2}} \| u + u_N \|_{X_{s,\frac{1}{2}+\epsilon}} \| u - u_N \|_{X_{s,\frac{1}{2}+\epsilon}} \leq CT^{\frac{3}{2}} (\| u \|_{X_{s,\frac{1}{2}+\epsilon}} + \| u_N \|_{X_{s,\frac{1}{2}+\epsilon}}) \| u - u_N \|_{X_{s,\frac{1}{2}+\epsilon}} \]
\[ \leq 2CT^{\frac{3}{2}} \| u_0 \|_{H^s(R)} \| u - u_N \|_{X_{s,\frac{1}{2}+\epsilon}} \leq \frac{1}{2} \| u - u_N \|_{X_{s,\frac{1}{2}+\epsilon}}. \]  

(4.14)

inserting (4.14) into (4.13), we have

\[ \| u - u_N \|_{X_{s,\frac{1}{2}+\epsilon}} \]
\[ \leq C \| P_N u_0 \|_{H^s(R)} + \| P_N (\partial_x (u^2)) \|_{X_{s,-\frac{1}{2}+2\epsilon}} + \frac{1}{2} \| u - u_N \|_{X_{s,\frac{1}{2}+\epsilon}}. \]  

(4.15)

From (4.15), we have

\[ \| u - u_N \|_{X_{s,\frac{1}{2}+\epsilon}} \leq 2C \| P_N u_0 \|_{H^s(R)} + \| P_N (\partial_x (u^2)) \|_{X_{s,-\frac{1}{2}+2\epsilon}}. \]  

(4.16)

From Lemma 3.1, we have

\[ \| \partial_x (u^2) \|_{X_{s,-\frac{1}{2}+2\epsilon}} \leq C \| u \|_{X_{s,\frac{1}{2}+\epsilon}}^2 \leq 4C^3 \| u_0 \|_{H^s(R)}^2 < \infty. \]  

(4.17)

From (4.17), we have

\[ \| P_N (\partial_x (u^2)) \|_{X_{s,-\frac{1}{2}+2\epsilon}} \to 0 \]  

(4.18)

as $N \to \infty$. Since $u_0 \in H^s(R)(s \geq \frac{1}{4})$, we have

\[ \| P_N u_0 \|_{H^s(R)} \to 0 \]  

(4.19)

as $N \to \infty$.

Inserting (4.18), (4.19) into (4.16) yields (4.1).

This completes the proof of Lemma 4.1.

To obtain Theorem 1.4, it suffices to prove Lemma 4.2.

**Lemma 4.2.** Let $u_0 \in H^s(R)(s \geq \frac{1}{4})$. Then, we have $u(x, t) \to u_0(x)$ as $t \to 0$ for almost everywhere $x \in R$.

**Proof.** Inspired by the idea of the proof of Proposition 3.3 of [15], we present the proof of Lemma 4.2. Since $u_N$ is smooth, for all $x \in R$, we have

\[ \lim_{t \to 0} u_N(x, t) = P_N u_0(x). \]  

(4.20)
By using the triangle inequality, we have
\[ |u - u_0| \leq |u - u_N| + |u_N - P_Nu_0| + |u_0 - P_Nu_0|. \]  
(4.21)

By using (4.21), we have
\[ \lim_{t \to 0} \sup |u - u_0| \leq \lim_{t \to 0} \sup |u - u_N| + |P_Nu_0|. \]  
(4.22)

For \( \lambda > 0 \), by using the Chebyshev inequality and Sobolev embedding as well as (4.22), we have
\[ \left| \{ x \in \mathbb{R} : \lim_{t \to 0} \sup |u - u_0| > \lambda \} \right| \leq \left| \{ x \in \mathbb{R} : \lim_{t \to 0} \sup |u - u_N| > \frac{\lambda}{2} \} \right| + \left| \{ x \in \mathbb{R} : |P_Nu_0| > \frac{\lambda}{2} \} \right| \leq C \lambda^{-4} \| u - u_N \|_{L^4 L^\infty}^4 + \lambda^{-2} \| P_Nu_0 \|_{L^2} \]  
(4.23)

Since \( u_0 \in H^s(\mathbb{R})(s \geq \frac{1}{4}) \), we have
\[ \| P_Nu_0 \|_{H^s} \to 0 \]  
(4.24)
as \( N \to \infty \). By using Lemma 4.1 and (4.24), we have
\[ \left| \{ x \in \mathbb{R} : \lim_{t \to 0} \sup |u - u_0| > \lambda \} \right| = 0. \]

This completes the proof of Lemma 4.2.

5. Proof of Theorem 1.5

In this section, we use Lemmas 2.1, 3.1 to prove Theorem 1.5.

**Proof of Theorem 1.5.** Inspired by page 7 of [14], we present the proof of Theorem 1.5. Obviously,
\[ u(x, t) - U(t)u_0(x) = \eta \left( \frac{t}{T} \right) \int_0^t U(t - t')\partial_x(u^2)dt'. \]  
(5.1)

Let \( s_1 = \frac{1}{2} + 2\epsilon, b = \frac{1}{2} + \epsilon, s_2 \geq -\frac{1}{2} + \epsilon \). By using Lemmas 2.1, 3.1 and (5.1), we have
\[ \| u - U(t)u_0 \|_{X_{s_1,b}} = \left\| \eta \left( \frac{t}{T} \right) \int_0^t U(t - t')(\partial_x(u^2))dt' \right\|_{X_{s_1,b}} \leq C T^{\frac{3}{2}} \| \partial_x(u^2) \|_{X_{s_1,-\frac{1}{2} + 2\epsilon}} \leq C \| u \|_{X_{s_2,b}}^2 \leq 2 C^3 \| u_0 \|_{H^{s_2}(\mathbb{R})}^2 < \infty. \]  
(5.2)

Thus, from Lemma 2.7 and \( H^{s_1}(\mathbb{R}) \hookrightarrow C(\mathbb{R})(s_1 = \frac{1}{2} + 2\epsilon) \), we have \( u - U(t)u_0 \in X_{s_1,b} \hookrightarrow C([-T, T], H^{s_1}(\mathbb{R})) \hookrightarrow C([-T, T]; C(\mathbb{R})) \). Thus, we have
\[ \lim_{t \to 0} \sup_{x \in \mathbb{R}} |u(x, t) - U(t)u_0| = 0. \]  
(5.3)
This ends the proof of Theorem 1.5.

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**References**

**References**

[1] I. Bejenaru, T. Tao, Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation, *J. Funct. Anal.* 233(2006), 228-259.

[2] J. L. Bona, R. Smith, A model for the two-way propagation of water waves in a channel, *Math. Proc. Cambridge Philos. Soc.* 79(1976), 167-182.

[3] J. Bourgain, A remark on Schrödinger operators, *Israel J. Math.* 77(1992), 1-16.

[4] J. Bourgain, On the Schrödinger maximal function in higher dimension, *Proc. Steklov Inst. Math.* 280(2013), 46-60.

[5] J. Bourgain, A note on the Schrödinger maximal function, *J. Anal. Math.* 130(2016), 393-396.

[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, I, II, *Geom. Funct. Anal.* 3(1993), 107-156, 209-262.

[7] A. Boukarou, K. Guerbati, K. Zennir, Local well-posedness and time regularity for a fifth-order shallow water equations in analytic Gevrey-Bourgain spaces, *Monatsh. Math.* 193(2020), 763-782.

[8] L. Carleson, Some analytical problems related to statistical mechanics. Euclidean Harmonic Analysisi. Lecture Notes in Mathematics, vol. 779, pp. 5.45, Springer, Berlin, (1979).

[9] M. Cavalcante, C. Kwak, The initial-boundary value problem for the Kawahara equation on the half-line. *Nonli. Diff. Eqns. Appl.* 27(2020), Paper No. 45, 50 pp.

[10] W. G. Chen, J. F. Li, C. X. Miao, J. X. Wu, Low regularity solutions of two fifth-order KdV type equations, *J. Anal. Math.* 107 (2009), 221C238.

[11] W. G. Chen, Z. H. Guo, Global well-posedness and I method for the fifth order Korteweg-de Vries equation, *J. Anal. Math.* 114(2011), 121-156.
[12] C. Cho, S. Lee and A. Vargas, Problems on pointwise convergence of solutions to the Schrödinger equation, *J. Fourier Anal. Appl.* 18(2012), 972-994.

[13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, *J. Amer. Math. Soc.* 16 (2003), 705-749.

[14] E. Compaan, A smoothing estimate for the nonlinear Schrödinger equation, UIUC Research Experiences for Graduate Students report, 2013.

[15] E. Compaan, R. Lucà, G. Staffilani, Pointwise convergence of the Schrödinger flow, *Int. Math. Res. Not.* 2021, 599-650.

[16] M. Cowling, Pointwise behavior of solutions to Schrödinger equations. In: Harmonic Analysis (Cortona, 1982). Lecture Notes in Mathematics, vol. 992, pp. 83-90. Springer, Berlin, (1983).

[17] S. B. Cui, D. G. Deng, S. P. Tao, Global existence of solutions for the Cauchy problem of the Kawahara equation with $L^2$ initial data, *Acta Math. Sin. (Engl. Ser.)* 22(2006), 1457-1466.

[18] S. B. Cui, S. P. Tao, Strichartz estimates for dispersive equations and solvability of the Kawahara equation, *J. Math. Anal. Appl.* 304(2005), 683-702.

[19] B. E. Dahlberg, C. E. Kenig, A note on the almost everywhere behavior of solutions to the Schrödinger equation. In: Proceedings of Italo-American Symposium in Harmonic Analysis, University of Minnesota. Lecture Notes in Mathematics, vol. 908, pp. 205-208. Springer, Berlin, (1982).

[20] C. Demeter, S. Guo, Schrödinger maximal function estimates via the pseudoconformal transformation, arXiv: 1608.07640.

[21] X. Du, A sharp Schrödinger maximal estimate in $\mathbb{R}^2$, Dissertation, 2017.

[22] X. Du, L. Guth, X. Li, A sharp Schrödinger maximal estimate in $\mathbb{R}^2$, *Ann. Math.* 188(2017), 607-640.

[23] X. Du, L. Guth, X. Li and R. Zhang, Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates, *Forum Math. Sigma*, 6(2018).

[24] X. Du and R. Zhang, Sharp $L^2$ estimates of the Schrödinger maximal function in higher dimensions, *Ann. Math.* 189(2019), 837-861.

[25] M. B. Erdogan, N. Tzirakis, The Initial Value Problem for KdV, University of Illinois, Urbana-Champaign, IL, 2013. Lecture Notes.
[26] G. Gigante and F. Soria, On the the boundedness in $H^{1/4}$ of the maximal square function associated with the Schrödinger equation, *J. Lond. Math. Soc.* 77(2008), 51-68.

[27] A. Grünrock, New applications of the Fourier restriction norm method to well-posedness problems for nonlinear Evolution Equations, Ph.D. Universität Wuppertal, 2002, Germany, Dissertation.

[28] Z. H. Huo, The Cauchy problem for the fifth order shallow water equation, *Acta Math. Appl. Sin. Engl. Ser.* 21(2005), 441-454.

[29] Y. L. Jia, Z. H. Huo, Well-posedness for the fifth-order shallow water equations, *J. Diff. Eqns.* 246(2009), 2448-2467.

[30] C. E. Kenig, G. Ponce, L. Vega, The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, *Duke Math. J.* 71(1993), 1-21.

[31] C. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, *J. Amer. Math. Soc.* 9(1996), 573-603.

[32] R. Killip, J. Murphy, M. Visan, Almost sure scattering for the energy-critical NLS with radial data below $H^1(\mathbb{R}^4)$, *Commun. Partial Diff. Eqns.* 44(2019), 51-71.

[33] T. Kawahara, Oscillatory solitary waves in dispersive media, *J. Phys. Soc. Japan*, 33(1972), 260-264.

[34] T. Kato, Local well-posedness for Kawahara equation, *Adv. Diff. Eqns.* 16(2011), 257-287.

[35] T. Kato, Low regularity well-posedness for the periodic Kawahara equation, *Diff. Int. Eqns.* 25(2012), 1011-1036.

[36] T. Kato, Global well-posedness for the Kawahara equation with low regularity, *Commun. Pure Appl. Anal.* 12(2013), 1321-1339.

[37] S. Kichenassamy, P. J. Olver, Existence and nonexistence of solitary wave solutions to higher-order model evolution equations. *SIAM J. Math. Anal.* 23(1992), 1141-1166.

[38] S. Klainerman, M. Machedon, Smoothing estimates for null forms and applications, *Int. Math. Res. Not.* 1994, no. 9, 383ff., approx. 7 pp.

[39] S. Lee, On pointwise convergence of the solutions to Schrödinger equation in $\mathbb{R}^2$. *Int. Math. Res. Not.* 2006, 32597.
[40] F. Linares, J. P. G. Ramos, Maximal function estimates and local well-posedness for the generalized Zakharov-Kuznetsov equation, *SIAM J. Math. Anal.* 53(2021), 914-936.

[41] L. Linares, J. G. Ramos, The Cauchy problem for the $L^2$-critical generalized Zakharov-Kuznetsov equation in dimension 3, *Comm. Partial Diff. Eqns.* 46(2021), 1601-1627.

[42] R. Lucà and K. M. Rogers, An improved neccessary condition for Schrödinger maximal estimate, arXiv: 1506.05325.

[43] R. Lucà and K. M. Rogers, Coherence on fractals versus pointwise convergence for the Schrödinger equation, *Commun. Math. Phys.* 351(2017), 341-359.

[44] R. Lucà, K. M. Rogers, A note on pointwise convergence for the Schrödinger equation, *Math. Proc. Cambridge Philos. Soc.* 166(2019), 209-218.

[45] C. Miao, J. Yang and J. Zheng, An improved maximal inequality for 2D fractional order Schrödinger operators, *Stud. Math.* 230(2015), 121-165.

[46] C. Miao, J. Zhang and J. Zheng, Maximal estimates for Schrödinger equation with inverse-square potential, *Pac. J. Math.* 273(2015), 1-19.

[47] A. Moyua, A. Vargas and L. Vega, Schrödinger maximal function and restriction properties of the Fourier transform, *Int. Math. Res. Not.* 1996(1996), 793-815.

[48] K. M. Rogers, A. Vargas and L. Vega, Pointwise convergence of solutions to the nonelliptic Schrödinger equation, *Indiana Univ. Math. J.* 55(2006), 1893-1906.

[49] K. M. Rogers and P. Villarroella, Sharp estimates for maximal operators associated to the wave equation, *Ark. Mat.* 46(2008), 143-151.

[50] S. Shao, On localization of the Schrödinger maximal operator, arXiv: 1006.2787v1.

[51] P. Sjölin, Regularity of solutions to the Schrödinger equation, *Duke Math. J.* 55(1987), 699-715.

[52] L. Vega, Schrödinger equations: pointwise convergence to the initial data, *Proc. Am. Math. Soc.* 102(1988), 874-878.

[53] H. Wang, S. B. Cui, D. G. Deng, Global existence of solutions for the Kawa-hara equation in Sobolev spaces of negative indices, *Acta Math. Sin. (Engl. Ser.)* 23(2007), 1435-1446.

[54] W. Yan, Y. S. Li, The Cauchy problem for Kawahara equation in Sobolev spaces with low regularity, *Math. Methods Appl. Sci.* 33(2010), 1647-1660.
[55] W. Yan, Y. S. Li, Ill-posedness of Kawahara equation and Kaup-Kupershmidt equation, *J. Math. Anal. Appl.* 380(2011), 486-492.