From Lieb–Thirring Inequalities to Spectral Enclosures for the Damped Wave Equation

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Abstract. Using a correspondence between the spectrum of the damped wave equation and non-self-adjoint Schrödinger operators, we derive various bounds on complex eigenvalues of the former. In particular, we establish a sharp result that the one-dimensional damped wave operator is similar to the undamped one provided that the $L^1$ norm of the (possibly complex-valued) damping is less than 2. It follows that these small dampings are spectrally undetectable.

1. Introduction

Consider a classical or a quantum system described by the damped wave equation

$$u_{tt} + a(x)u_t - \Delta_x u = 0 \quad (1)$$

in the space–time variables $(x,t) \in \mathbb{R}^d \times (0,\infty)$, where the ‘damping’ $a$ is a complex-valued function. The positive (respectively, negative) part of the real part of $a$ corresponds to the dissipation (respectively, excitation) of mechanical or electromagnetic waves, while the imaginary part of $a$ can be interpreted as a conservative perturbation of a Dirac (quasi-)particle.

Writing $U := (u, u_t)^T$, it is customary to replace the scalar second-order equation (1) by the first-order evolution system $U_t = A_a U$ with the matrix-valued damped wave operator

$$A_a := \begin{pmatrix} 0 & 1 \\ \Delta & -a \end{pmatrix}, \quad \text{dom } A_a := H^2(\mathbb{R}^d) \times \dot{H}^1(\mathbb{R}^d), \quad (2)$$

acting in the Hilbert space $\mathcal{H} := \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. A careful analysis of the stationary problem $A_a \Psi = \mu \Psi$ in $\mathcal{H}$, where $\mu \in \mathbb{C}$ is a spectral parameter, provides information on the behaviour of the time-dependent solutions $u$ of (1). It is easy to see that the spectral problem is related to the operator pencil (see Lemma 1 below)

$$S_{\mu a} \psi := -\Delta \psi + \mu a \psi = -\mu^2 \psi \quad \text{in } L^2(\mathbb{R}^d). \quad (3)$$
As an example of usefulness of the spectral data, let us recall the collaboration of the first author with Freitas [20], where it is shown that the damped wave equation (1) becomes unstable whenever the real-valued damping $a$ achieves a sufficiently negative minimum. The strategy of [20] is based on establishing spectral asymptotics of the family of Schrödinger operators $S_{\mu a}$ as $\mu \to +\infty$. Consequently, $A_a$ possesses a real positive point $\mu$ in the spectrum, which is responsible for a global instability of (1).

The present paper is partially motivated by a remark of Weidl [31] that Lieb–Thirring inequalities known for $S_{\mu a}$ could provide more quantitative information on the location of the real spectrum of $A_a$. Moreover, in view of the unprecedented interest in non-self-adjoint Schrödinger operators in the near past, new complex extensions of the Lieb–Thirring inequalities have been derived in recent years. Consequently, the T. Weidl’s observation has become interesting for the complex spectrum of $A_a$ as well. In this paper we go beyond the usual setting by considering even complex-valued dampings.

The literature on damped wave systems is rather extensive and we restrict ourselves on quoting the following recent works and refer the interested reader for further references therein. The relationship between the damped wave and Dirac equations is discussed on an abstract level in [23] and related one-dimensional Lieb–Thirring-type inequalities can be found in [8, 9]. An extensive spectral analysis of the wave operator with possibly unbounded damping is performed in [21]. Basic eigenvalue bounds for weakly damped wave systems can be deduced from [19]. The perturbation of eigenvalues of abstract damped wave operators has been recently studied in the framework of Krein spaces in [28]. Resolvent estimates for an abstract dissipative operator are derived in [4, 30].

The structure of this paper is as follows. In Sect. 2 we comment on basic properties of the damped wave operator $A_a$ and state a crucial correspondence between its eigenvalue problem and the operator pencil (3). Self-adjoint and non-self-adjoint Lieb–Thirring-type inequalities are applied in Sects. 3 and 4, respectively. In particular, in Theorem 4 we prove that the point spectrum of the one-dimensional damped wave operator $A_a$ is empty provided that $\|a\|_{L^1(\mathbb{R})} < 2$. We interpret this result as that small dampings cannot be detected by measuring eigenfrequencies of the wave system. In Sect. 5 we strengthen this observation to a complete lack of spectral detectability by showing that the non-self-adjoint operator $A_a$ is actually similar to the skew-adjoint undamped operator $A_0$ under the same smallness condition.

2. Preliminaries

Although the damped wave equation can be made meaningful for certain unbounded dampings (cf. [21]), our standing hypothesis is that the damping $a : \mathbb{R}^d \to \mathbb{C}$ is bounded, i.e.,

$$a \in L^\infty(\mathbb{R}^d).$$

Then it is easy to see that there exists a constant $c$ such that $A_a + c$ is a generator of a $C^0$-semigroup of contractions (cf. [20, App. B]), so (1) is well
posed. Equivalently, $A_a$ is a quasi-$m$-accretive operator. In particular, the operator $A_a$ is closed, so its spectrum is well defined.

Notice that $A_a$ is skew-adjoint (i.e., $iA_a$ is self-adjoint) if, and only if, $a$ is purely imaginary. In particular, the undamped operator $A_0$ is skew-adjoint. To have this symmetry result, it is important that we consider the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^d)$ (defined as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $\psi \mapsto \|\nabla \psi\|$) in the definition of the Hilbert space $\mathcal{H}$, see (2). On the other hand, it is important to keep in mind that $\dot{H}^1(\mathbb{R}^d)$ is not a subset of $L^2(\mathbb{R}^d)$.

With an abuse of notation, we denote by $-\Delta$ the distributional Laplacian as well as its self-adjoint realisation in $L^2(\mathbb{R}^d)$ with domain $\text{dom}(-\Delta) := H^2(\mathbb{R}^d)$. For any bounded potential $V : \mathbb{R}^d \to \mathbb{C}$, the Schrödinger operator $S_V = -\Delta + V$ with domain $S_V = H^2(\mathbb{R}^d)$ is a well defined $m$-sectorial operator.

We say that $V$ is *vanishing at infinity* and write $V \xrightarrow{\infty} 0$, provided that
$$\lim_{R \to +\infty} \|V\|_{L^\infty(\mathbb{R}^d \setminus B_R(0))} = 0,$$
where $B_R(0)$ denotes the ball of radius $R > 0$ centred at the origin. Then $V$ is a relatively compact perturbation of $-\Delta$. Consequently, $\sigma_e(S_V) = \sigma_e(S_0) = [0, +\infty)$ for any choice of the essential spectrum (namely, $\sigma_{e1}, \ldots, \sigma_{e5}$ in the notation of [11, Chapt. IX]). More specifically, it is clear for the components $\sigma_{e1}, \ldots, \sigma_{e4}$, which are preserved by relatively compact perturbations, and the result extends to the widest choice $\sigma_{e5}$ because $S_V$ is $m$-sectorial (cf. [25, Prop. 5.4.4]).

It is easy to see that the spectrum of the undamped operator $A_0$ is purely continuous and equal to the imaginary axis; in particular, $\sigma(A_0) = \sigma_e(A_0) = i\mathbb{R}$. If $a$ is vanishing at infinity, then the damping represents a relatively compact perturbation of $A_0$. Consequently, $\sigma_{ek}(A_a) = \sigma_{ek}(A_0) = i\mathbb{R}$ for $k = 1, \ldots, 4$. To see that it is true also for $\sigma_{e5}$, it is enough to notice that each connected component of $\mathbb{C} \setminus i\mathbb{R}$ intersects the resolvent set of $A_a$ due to (4). In summary,
$$a \xrightarrow{\infty} 0 \implies \sigma_e(A_a) = \sigma_e(A_0) = i\mathbb{R}.$$  

Since the essential spectrum is independent of $a$, the present paper focuses on the point spectrum of $A_a$. The following lemma specifies the equivalence between the spectral problem for (2) and the operator pencil (3) in the case of eigenvalues (not necessarily discrete).

**Lemma 1.** Assume (4). For every $\mu \in \mathbb{C}$,
$$-\mu^2 \in \sigma_p(S_{\mu a}) \iff \mu \in \sigma_p(A_a).$$

**Proof.** Assuming $-\mu^2 \in \sigma_p(S_{\mu a})$, there exists a non-trivial function $\psi_1 \in H^2(\mathbb{R}^d)$ such that
$$S_{\mu a} \psi_1 = -\mu^2 \psi_1.$$  
Then $\Psi := (\psi_1, \mu \psi_1)^T \in \text{dom} A_a$ and
$$(A_a \Psi)^T = (\mu \psi_1, \Delta \psi_1 - a \mu \psi_1) = (\mu \psi_1, -S_{\mu a} \psi_1) = \mu (\psi_1, \mu \psi_1) = \mu \Psi^T.$$  
Conversely, assuming $\mu \in \sigma_p(A_a)$, there exists a non-trivial $\Psi = (\psi_1, \psi_2)^T \in \text{dom} A_a$ such that $A_a \Psi = \mu \Psi$. In other words, $\psi_1 \in H^2(\mathbb{R}^d)$, $\psi_2 \in \dot{H}^1(\mathbb{R}^d)$ and
$$\psi_2 = \mu \psi_1, \Delta \psi_1 - a \psi_2 = \mu \psi_2.$$  
Combining the latter two equations, we get $S_{\mu a} \psi_1 = -\mu^2 \psi_1$ with $\psi_1 \neq 0$. \qed
3. Real Spectrum and Real-Valued Damping

In this section, we assume that the damping $a$ satisfying (4) is \textit{real-valued} and focus on \textit{real} eigenvalues $\mu \in \mathbb{R}$. Recalling the correspondence of Lemma 1, it is enough to consider the auxiliary Schrödinger operators $S_V$ with real-valued potentials $V$. Then $S_V$ is self-adjoint, so its spectrum is purely real. Let us denote by $\{\lambda_n(V)\}_{n=1}^N$ the non-decreasing sequence of negative discrete eigenvalues of $S_V$, where each eigenvalue is repeated according to its multiplicity. The set can be either empty ($N = 0$) or finite ($1 \leq N < \infty$) or infinite ($N = \infty$).

Our starting point are the self-adjoint Lieb–Thirring inequalities (cf. [26,27])

$$\sum_{n=1}^N |\lambda_n(V)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma + \frac{d}{2}},$$

where $L_{\gamma,d}$ is a positive constant independent of $V$ and $V_\pm := \frac{1}{2}(|V| \pm V)$. More specifically, it is known that such a bound holds with a \textit{finite} constant $L_{\gamma,d}$ if, and only if,

$$\gamma \geq \frac{1}{2} \quad \text{if} \quad d = 1,$$
$$\gamma > 0 \quad \text{if} \quad d = 2,$$
$$\gamma \geq 0 \quad \text{if} \quad d \geq 3. \tag{6}$$

The sharp values of the constants $L_{\gamma,d}$ are not known explicitly for all the admissible values of $\gamma$. In special cases, however, one knows that $L_{\frac{1}{2},1} = \frac{1}{2}$ and

$$L_{\gamma,d} = L_{\gamma,d}^{cl} := \frac{\Gamma(\gamma + 1)}{2^d \pi^{\frac{d}{2}} \Gamma(\gamma + \frac{d}{2} + 1)}$$

for $d \geq 1, \gamma \geq \frac{3}{2}$ are the best possible values.

A direct combination of the Lieb–Thirring inequalities with Lemma 1 yields the following basic results.

\textbf{Theorem 1.} Suppose that $a$ is real-valued and assume (4). Let $\gamma$ be any number satisfying (6). If $\mu$ is a positive (respectively, negative) eigenvalue of $A_a$ and $a_- \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d)$ (respectively, $a_+ \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d)$), then

$$\mu^\gamma - \frac{d}{2} \leq L_{\gamma,d} \int_{\mathbb{R}^d} a_-^{\gamma + \frac{d}{2}} \quad \text{(respectively,} \quad (-\mu)^\gamma - \frac{d}{2} \leq L_{\gamma,d} \int_{\mathbb{R}^d} a_+^{\gamma + \frac{d}{2}}) \right).$$

On the other hand, if $a_- \in L^d(\mathbb{R}^d)$ (respectively, $a_+ \in L^d(\mathbb{R}^d)$) and

$$\int_{\mathbb{R}^d} a_-^d < \frac{1}{L_{\frac{d}{2},d}^d}, \quad \text{(respectively,} \quad \int_{\mathbb{R}^d} a_+^d < \frac{1}{L_{\frac{d}{2},d}^d}),$$

then $A_a$ has no positive (respectively, negative) eigenvalues.
Proof. From Lemma 1 we get that real \( \mu \in \sigma_p(A_a) \) if, and only if, there exists \( n \in \mathbb{N} \) such that \( \lambda_n(\mu a) = -\mu^2 \). Hence, (5) implies
\[
\begin{align*}
\mu^{2\gamma} &= |\lambda_n(\mu a)|^\gamma \leq L_{\gamma,d} \mu^{\gamma + \frac{d}{2}} \int_{\mathbb{R}^d} a_{\gamma + \frac{d}{2}} \quad \text{for} \quad \mu \in \sigma_p(A_a) \cap (0, +\infty), \\
|\mu|^{2\gamma} &= |\lambda_n(\mu a)|^\gamma \leq L_{\gamma,d} |\mu|^{\gamma + \frac{d}{2}} \int_{\mathbb{R}^d} a_{\gamma + \frac{d}{2}} \quad \text{for} \quad \mu \in \sigma_p(A_a) \cap (-\infty, 0).
\end{align*}
\]
(7)

This proves the first part of the theorem. The case \( 0 \in \sigma_p(A_a) \) cannot occur because the spectrum of \( S_0 \) is purely continuous. Dividing (7) by \( |\mu|^{\gamma + \frac{d}{2}} \), and eventually putting \( \gamma = \frac{d}{2} \), which can be done for all \( d \geq 1 \), we conclude with the second part of the theorem. \( \square \)

To continue, we restrict ourselves to \( d = 1 \). In this case the Buslaev–Faddeev–Zakharov trace formulae (cf. [12]) provides us with a lower bound for the sum of square roots of the eigenvalues of \( S_{\mu a} \), namely
\[
\sum_{n=1}^{N} |\lambda_n(\mu a)|^{\frac{1}{2}} \geq -\frac{\mu}{4} \int_{\mathbb{R}} a.
\]
(8)

Of course, the inequality is non-trivial only if \( \mu \int_{\mathbb{R}} a < 0 \). The latter is known to be a sufficient condition which guarantees that inf \( \sigma(S_{\mu a}) < 0 \). Assuming in addition \( a \xrightarrow{\infty} 0 \), it follows that \( S_{\mu a} \) possesses at least one negative eigenvalue. The number of eigenvalues \( N \) of \( S_{\mu a} \) is controlled from above by the Bargmann bound (cf. [29, Prob. 22])
\[
N \leq 1 + |\mu| \int_{\mathbb{R}} |a(x)||x| \, dx.
\]
Consequently, for \( \mu \) satisfying the inequality
\[
|\mu| < \left( \int_{\mathbb{R}} |a(x)||x| \, dx \right)^{-1}
\]
the operator \( S_{\mu a} \) has exactly one negative eigenvalue \( \lambda_1(\mu a) \). Recalling the correspondence of Lemma 1, it follows from (8) that \( \mu \in \sigma_p(A_a) \) satisfies the estimate
\[
|\mu| = |\lambda_1(\mu a)|^{\frac{1}{2}} \geq -\frac{\mu}{4} \int_{\mathbb{R}} a.
\]
(10)

This implies \( \int_{\mathbb{R}} a \geq -4 \) for \( \mu > 0 \) and \( \int_{\mathbb{R}} a \leq 4 \) for \( \mu < 0 \). In summary, we have established the following result.

Theorem 2. Let \( d = 1 \). Suppose that \( a \) is real-valued and in addition to (4) assume \( a \in L^1(\mathbb{R}, |x| \, dx) \) and \( a \xrightarrow{\infty} 0 \). Let \( \mu \) be a real eigenvalue of \( A_a \). If \( \mu > 0 \) and \( \int_{\mathbb{R}} a < -4 \) (or \( \mu < 0 \) and \( \int_{\mathbb{R}} a > 4 \)), then
\[
|\mu| \geq \left( \int_{\mathbb{R}} |a(x)||x| \, dx \right)^{-1}.
\]

Finally, combining the Lieb–Thirring inequalities with the Buslaev–Faddeev–Zakharov trace formulae, we obtain the following quantitative estimates on the location of real eigenvalues of the one-dimensional damped wave...
operator. The presence of the coupling parameter $\alpha$ is useful for studying the stability of solutions of (1) in the spirit of [18, 20, 22].

**Theorem 3.** Let $d = 1$. Suppose that $a$ is real-valued and in addition to (4) assume $a \in L^1(\mathbb{R}, |x| \, dx)$ and $a \to 0$. Let $\int_{\mathbb{R}} a < 0$ (respectively, $\int_{\mathbb{R}} a > 0$). For any $\mu > 0$ (respectively, $\mu < 0$) satisfying (9), there exists exactly one $\alpha > 0$ satisfying

$$2 \left( \int_{\mathbb{R}} a^- \right)^{-1} \leq \alpha \leq -4 \left( \int_{\mathbb{R}} a^+ \right)^{-1}$$

(respectively, $2 \left( \int_{\mathbb{R}} a^- \right)^{-1} \leq \alpha \leq 4 \left( \int_{\mathbb{R}} a^+ \right)^{-1}$)

such that $\frac{\mu}{\alpha}$ is an eigenvalue of $A_{\alpha a}$.

**Proof.** Assuming $\mu > 0$ and $\int_{\mathbb{R}} a < 0$ together with $a \to 0$ and (9), the operator $S_{\mu a}$ possesses exactly one negative eigenvalue. Applying (7) with $\gamma = \frac{1}{2}$ and (10), we get

$$-\frac{\mu}{4} \int_{\mathbb{R}} a \leq |\lambda_1(\mu a)|^{\frac{1}{2}} \leq \frac{\mu}{2} \int_{\mathbb{R}} a^-.$$

Similarly, for $\mu < 0$ and $\int_{\mathbb{R}} a > 0$, we get

$$-\frac{\mu}{4} \int_{\mathbb{R}} a \leq |\lambda_1(\mu a)|^{\frac{1}{2}} \leq \frac{|\mu|}{2} \int_{\mathbb{R}} a^+.$$

These estimates together with Lemma 1 prove the theorem. \hfill $\square$

### 4. Complex Spectrum and Complex-Valued Damping

In this section, we use the recent progress in spectral theory of non-self-adjoint Schrödinger operators and state results about complex eigenvalues of the damped wave operator. There is no obstacle to consider complex-valued dampings at the same time.

Let us start with dimension $d = 1$. The celebrated result of Davies et al. (see [1, Thm. 4] and [10, Corol. 2.16]) states that every eigenvalue $\lambda(V)$ of $S_V$ with $V \in L^1(\mathbb{R})$ satisfies the bound

$$|\lambda(V)|^{\frac{1}{2}} \leq \frac{1}{2} \int_{\mathbb{R}} |V(x)| \, dx.$$  \hfill (11)

Moreover, the bound is known to be optimal for step-like potentials approximating the Dirac potential. Now, assuming in addition to (4) that $a \in L^1(\mathbb{R})$, if $\mu$ is any eigenvalue of $A_a$ (necessarily it must be non-zero, cf. the proof of Theorem 1), Lemma 1 ensures that there exists $\lambda(\mu a) \in \sigma_p(S_{\mu a})$ such that

$$|\mu| = |\lambda_j(\mu a)|^{\frac{1}{2}} \leq \frac{1}{2} |\mu| \int_{\mathbb{R}} |a(x)| \, dx,$$

where the inequality is due to (11). Dividing by $|\mu|$, it follows that the point spectrum of $A_a$ is empty provided that the $L^1$-norm of $a$ is small, namely $\|a\|_{L^1(\mathbb{R})} < 2$. Let us summarise the observation into the following theorem.
Theorem 4. Let $d = 1$. In addition to (4) assume $a \in L^1(\mathbb{R})$. If $\|a\|_{L^1(\mathbb{R})} < 2$, then $\sigma_p(A_a) = \emptyset$. Moreover, the constant 2 is optimal.

Proof. It remains to argue about the optimality. Our strategy is to show that for any number slightly greater than 2 there exists a damping $W$ with the $L^1$-norm equal to this number such that $A_W$ has an eigenvalue. For this we choose the analytically computable case, where the damping is a step-like potential

$$W(x) := \begin{cases} 0 & \text{if } x < -b, \\ a & \text{if } -b < x < b, \\ 0 & \text{if } x > b, \end{cases} \quad \text{with } a < 0 < b.$$ 

The eigenvalue equation $A_W \Psi = \mu \Psi$ reduces to $\Delta \psi - \mu W \psi - \mu^2 \psi = 0$, where $\psi$ is the first component of $\Psi$. It is enough to analyse the situation of real $\mu$. For $\mu = 0$ or $\mu = -a$ we get just a trivial solution to the eigenvalue equation. Also, using the spectral correspondence of Lemma 1, we know that all the real eigenvalues of $A_W$ must be positive, provided that $W \leq 0$ and $\mu \leq \|a\|_{L^\infty(\mathbb{R})} = -a$. Thus the only interval where we can find a real eigenvalue is $(0, -a)$.

Now, for $\psi$ to lie in $H^2(\mathbb{R}) \subset C^1(\mathbb{R})$ and be non-trivial, it can be easily verified that the secular equation

$$F(\mu) := 2\sqrt{- (\mu a + \mu^2)} \cos \left( 2b \sqrt{- (\mu a + \mu^2)} \right) + (a + 2\mu) \sin \left( 2b \sqrt{- (\mu a + \mu^2)} \right) = 0$$

must be satisfied. For $\mu \in (0, -a)$ this is equivalent to

$$G(\mu) := \sqrt{- (\mu a + \mu^2)} F(\mu) = 0.$$ 

We compute

$$\lim_{\mu \to 0^+} G(\mu) = 0 = \lim_{\mu \to -a^-} G(\mu),$$

$$\lim_{\mu \to 0^+} G'(\mu) = 2a(-1 + c),$$

$$\lim_{\mu \to -a^-} G'(\mu) = 2a(1 + c),$$

where $c := -ba = \frac{1}{2} \|W\|_{L^1(\mathbb{R})}$. We observe that for $c > 1$ both the derivatives have the same sign which, together with the fact that the limit points of this continuous function $G(\mu)$ are the same, implies that there exists $\mu^* \in (0, -a)$ such that $G(\mu^*) = 0$. Hence, $\mu^* \in \sigma_p(A_W)$ which proves the desired optimality. □

Remark 1. Taking $b := (2|a|)^{-1}$ and $\alpha \in \mathbb{C}$, the complexified step-like potential $\alpha W$ converges in the sense of distributions to $\alpha \delta$ as $a \to -\infty$, where $\delta$ is the Dirac delta function. Replacing (formally) $a$ by $\alpha \delta$ in (3), one arrives at
the pencil problem
\begin{equation}
\begin{cases}
-\psi'' = -\mu^2 \psi & \text{in } \mathbb{R} \setminus \{0\}, \\
\psi(0^+) - \psi(0^-) = 0, \\
\psi'(0^+) - \psi'(0^-) = \mu \alpha \psi(0).
\end{cases}
\end{equation}
(12)

There exists no admissible solution \( \psi \in H^2(\mathbb{R} \setminus \{0\}) \), unless \( \alpha = -2 \) (respectively, \( \alpha = 2 \)) in which case every \( \mu \in \mathbb{C} \) with \( \Re \mu > 0 \) (respectively, with \( \Re \mu < 0 \)) is an 'eigenvalue! Since \( \|aW\|_{L^2(\mathbb{R})} = |\alpha| \), this is another support for the optimality of the constant 2 in Theorem 4. However, we are not aware of any result about a spectral approximation of the operator pencil (3) by bounded potentials.

The damped wave equation on a finite interval with the damping being a Dirac delta function was previously studied in [3], [2, Sect. 4.1.1] and [7]. In Sect. 5, we argue that the absence of eigenvalues follows as a consequence of the similarity of \( A_a \) to the undamped wave operator \( A_0 \), provided that \( \|a\|_{L^1(\mathbb{R})} < 2 \).

In higher dimensions \( d \geq 2 \), we use the robust result of R. Frank et al. (see [15, Thm. 1] and [17, Thm. 3.2]) stating that every eigenvalue \( \lambda(V) \) of \( S_V \) with \( V \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d) \), \( 0 < \gamma \leq \frac{1}{2} \), satisfies the bound
\begin{equation}
|\lambda(V)|^\gamma \leq D_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^\gamma \frac{d}{2} \, dx,
\end{equation}
(13)
where \( D_{\gamma,d} \) is a constant independent of \( V \). Using Lemma 1, it follows that any eigenvalue \( \mu \) of \( A_a \) satisfies
\begin{equation}
|\mu|^{\gamma} = |\lambda_j(\mu a)|^{\gamma} \leq D_{\gamma,d} |\mu|^{\gamma + \frac{d}{2}} \int_{\mathbb{R}^d} |a(x)|^{\gamma + \frac{d}{2}} \, dx.
\end{equation}
(14)

We therefore conclude with the following theorem.

**Theorem 5.** Let \( d \geq 2 \). In addition to (4) assume \( a \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d) \) with \( 0 < \gamma \leq \frac{1}{2} \). There exists a constant \( D_{\gamma,d} \) such that, for any eigenvalue \( \mu \in \sigma_p(A_a) \), one has
\begin{equation}
|\mu|^{\gamma} \leq D_{\gamma,d} \int_{\mathbb{R}^d} |a(x)|^{\gamma + \frac{d}{2}} \, dx.
\end{equation}

The formal analogue of (13) for \( \gamma = 0 \) and \( d \geq 3 \) states that there exists a dimensional positive constant \( D_{0,d} \) such that if
\begin{equation}
D_{0,d} \int_{\mathbb{R}^d} |V(x)|^{\frac{d}{2}} \, dx < 1,
\end{equation}
then \( S_V \) has no eigenvalues. The case of discrete eigenvalues is due to Frank [15, Thm. 2], while possibly embedded eigenvalues were covered by [17, Thm. 3.2]. Independently, the total absence of eigenvalues under a weaker hypothesis for \( d = 3 \) and alternative conditions in higher dimensions was obtained by Fanelli et al. in [13] (see also [5,6,14]). In fact, the spectral stability of \( S_V \) for potentials \( V \) which are small in some sense goes back to the abstract result of Kato’s [24] that we shall recall for other purposes in the following section. Here we just mention that a straightforward combination
of Lemma 1 and (15) implies that (14) holds also for $\gamma = 0$ and $d \geq 3$. Consequently, we get the following formal analogue of Theorem 5.

**Theorem 6.** Let $d \geq 3$. In addition to (4) assume $a \in L^{\frac{d}{2}}(\mathbb{R}^d)$. There exists a positive constant $D_{0,d}$ such that, for any eigenvalue $\mu \in \sigma_p(A_a)$, one has

$$|\mu|^{-\frac{d}{2}} \leq D_{0,d} \int_{\mathbb{R}^d} |a|^\frac{d}{2}.$$ 

5. **Similarity**

In this section, we come back to the one-dimensional setting of Theorem 4. We find the result appealing because it implies that small dampings cannot be distinguished from the undamped system just by measuring eigenfrequencies. In fact, the following result shows a much stronger result that small dampings are indeed spectrally undetectable.

**Theorem 7.** Let $d = 1$. In addition to (4) assume $a \in L^1(\mathbb{R})$. If $\|a\|_{L^1(\mathbb{R})} < 2$, then the operators $A_a$ and $A_0$ are similar to each other. Moreover, the constant 2 is optimal.

Here the similarity means that there exists an operator $W \in \mathcal{B}(\mathcal{H})$ such that $W^{-1} \in \mathcal{B}(\mathcal{H})$ and $A_a = WA_0W^{-1}$. In other words, $iA_a$ is quasi-self-adjoint (cf. [25]), because $iA_0$ is self-adjoint. Then the spectra of $A_a$ and $A_0$ coincide. In particular, $A_a$ must have the same eigenvalues as $A_0$. Since the spectrum of $A_0$ is purely continuous, Theorem 4 follows as a direct consequence of Theorem 7.

**Proof.** The optimality of the constant 2 follows from the proof of Theorem 4. Indeed, for any number strictly larger than 2, there exists a damping $a$ whose $L^1$-norm equals that number, while $A_a$ possesses eigenvalues. This would violate the similarity.

To prove the similarity, we use the abstract result of Kato [24, Thm. 1.5]. Writing

$$iA_a = iA_0 + (-i \text{sgn } \bar{a} B)^* B \quad \text{with} \quad B := \begin{pmatrix} 0 & 0 \\ 0 & |a|^\frac{1}{2} \end{pmatrix},$$

where sgn is the complex sign (defined by $\text{sgn } f := f/|f|$ if $f \neq 0$ and $\text{sgn } f := 0$ if $f = 0$), it is enough to show that the bounded operators $B$ and $-i \text{sgn } \bar{a} B$ are relatively smooth with respect to $iA_0$ (cf. [24, Def. 1.2]) and

$$\sup_{\xi \in \mathbb{C} \setminus \mathbb{R}} \|K_\xi\| < 1,$$

where $K_\xi := BR(\xi, iA_0)Bi \text{sgn } a$ 

is the Birman–Schwinger operator. Here we use the notation $R(\xi, T) := (T - \xi)^{-1}$ for the resolvent of an operator $T$ at point $\xi \in \mathbb{C} \setminus \sigma(T)$. The relative smoothness is a rather complicated condition in general, but it reduces to reasonable criteria when $iA_0$ is self-adjoint (cf. [24, Thm. 5.1]). Using in addition the simple structure of the intertwining operators $B$ and $-i \text{sgn } \bar{a} B$ in our case, everything reduces to verifying the unique condition

$$\sup_{\xi \in \mathbb{C} \setminus \mathbb{R}} \|\tilde{K}_\xi\| < 1,$$

where $\tilde{K}_\xi := BR(\xi, iA_0)B$. 

(16)
It can be easily verified that

$$R(ξ, iA_0) = R(ξ^2, -Δ) \left( \begin{array}{c} ξ \\ iΔ ξ \end{array} \right).$$

Consequently,

$$\tilde{K}_ξ = |a|^\frac{1}{2} ξ R(ξ^2, -Δ) |a|^\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and therefore

$$\|\tilde{K}_ξ\| = |ξ| \left\| |a|^\frac{1}{2} R(ξ^2, -Δ) |a|^\frac{1}{2} \right\|,$$

where $\|\cdot\|$ denotes the operator norm both in $H$ and $L^2(\mathbb{R})$ on the left- and right-hand side, respectively. The latter will be estimated by the Hilbert–Schmidt norm $\|\cdot\|_{HS}$. To this purpose, we recall the explicit formula for the integral kernel of $R(z, -Δ)$:

$$G_z(x, y) = \frac{e^{-\sqrt{-z} |x-y|}}{2\sqrt{-z}},$$

where $x, y \in \mathbb{R}, z \in \mathbb{C} \setminus [0, +∞)$ and the principal branch of the square root is used. Using the elementary estimate

$$|G_z(x, y)| \leq \frac{1}{2\sqrt{|z|}},$$

we therefore get

$$\|\tilde{K}_ξ\|^2 \leq |ξ|^2 \left\| |a|^\frac{1}{2} R(ξ^2, -Δ) |a|^\frac{1}{2} \right\|^2_{HS}$$

$$= |ξ|^2 \int_{\mathbb{R} \times \mathbb{R}} |a(x)||G_{ξ^2}(x, y)|^2 |a(y)| \, dx \, dy$$

$$\leq \frac{\|a\|^2_{L^1(\mathbb{R})}}{4},$$

where the last bound is independent of $ξ$. Recalling (17), Kato’s similarity condition [24, Thm. 1.5] holds provided that $\|a\|_{L^1(\mathbb{R})} < 2$. □

**Remark 2.** Theorem 7 does not seem to admit direct higher-dimensional analogues. Technically (leaving aside the incompatibility with the higher-dimensional bounds on the eigenvalues of the damped wave operator established in the present paper), uniform (in $ξ$) estimates on the Birman–Schwinger operator $k_ξ := |a|^\frac{1}{2} R(ξ^2, -Δ) |a|^\frac{1}{2}$ can be deduced from the deep results of Frank and Sabin [16], it is true, but uniform estimates on the operator $ξk_ξ$ relevant for the damped wave equation (cf. (16)) seem to hold in dimension 1 only, due to the very special homogeneity of $R(ξ^2, -Δ)$ with respect to $ξ$. 
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