Three aspects of bosonized supersymmetry and linear differential field equation with reflection

Jorge Gamboa\textsuperscript{a}*, Mikhail Plyushchay\textsuperscript{a,b}†, Jorge Zanelli\textsuperscript{a,c}‡

\textsuperscript{a}Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile
\textsuperscript{b}Institute for High Energy Physics, Protvino, Russia
\textsuperscript{c}Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago, Chile

Abstract

Recently it was observed by one of the authors that supersymmetric quantum mechanics (SUSYQM) admits a formulation in terms of only one bosonic degree of freedom. Such a construction, called the minimally bosonized SUSYQM, appeared in the context of integrable systems and dynamical symmetries. We show that the minimally bosonized SUSYQM can be obtained from Witten’s SUSYQM by applying to it a non-local unitary transformation with a subsequent reduction to one of the eigenspaces of the total reflection operator. The transformation depends on the parity operator, and the deformed Heisenberg algebra with reflection, intimately related to parabosons and parafermions, emerges here in a natural way. It is shown that the minimally bosonized SUSYQM can also be understood as supersymmetric two-fermion system. With this interpretation, the bosonization construction is generalized to the case of $N = 1$ supersymmetry in 2 dimensions. The same special unitary transformation diagonalises the Hamiltonian operator of the 2D massive free Dirac theory. The resulting Hamiltonian is not a square root like in the Foldy-Wouthuysen case, but is linear in spatial derivative. Subsequent reduction to ‘up’ or ‘down’ field component gives rise to a linear differential equation with reflection whose ‘square’ is the massive Klein-Gordon equation. In the massless limit this becomes the self-dual Weyl equation. The linear differential equation with reflection admits generalizations to higher dimensions and can be consistently coupled to gauge fields. The bosonized SUSYQM can also be generated applying the nonlocal unitary transformation to the Dirac field in the background of a nonlinear scalar field in a kink configuration.

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\*E-mail: jgamboa@lauca.usach.cl
\†E-mail: mplyushc@lauca.usach.cl
\‡E-Mail: jz@cecs.cl
1 Introduction

Supersymmetric quantum mechanics (SUSYQM) was introduced by Witten \cite{Witten} as a toy model for studying supersymmetry breaking mechanism that would solve the hierarchy problem. Since then, SUSYQM has found many important applications. Their exhaustive list alongside with corresponding references can be found, e.g., in the review papers \cite{Fromhof, Nepomechie}. A new aspect of SUSYQM, recently discussed in the context of integrable systems and dynamical symmetries, is its minimally bosonized form \cite{Bosonized}. The name of the construction stems from its formulation in terms of only one bosonic degree of freedom. The supercharge and Hamiltonian operators of the minimally bosonized supersymmetry are given by

\[
Q_1 = -\frac{i}{\sqrt{2}} \left( \frac{d}{dx} + W_-(x)R \right), \quad Q_2 = iRQ_1, \quad H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + W^2 - W'R \right),
\]

where \( W_-(x) = -W_-(x) \) is the odd superpotential, and \( R \) is the reflection operator, \( R^2 = 1, \quad Rx = -xR \). It was shown in \cite{Bosonized} that the construction can in principle be extended to the case of \( OSp(2|2) \) supersymmetry and that it is related to the Witten’s SUSYQM in a nontrivial way, but the exact form of the relationship was not established.

In this paper we investigate the relationship between conventional and bosonized forms of SUSYQM and discuss two other aspects of bosonized supersymmetry: its interpretation as supersymmetry of a system of identical fermions and as a symmetry in a system of a Dirac field in the background of a nonlinear scalar field. The bosonization construction applied to the 2D massive Dirac field allows us also to obtain a linear differential field equation with reflection whose ‘square’ is the Klein-Gordon equation.

The paper is organized as follows. In Section 2 we find the nonlocal unitary transformation and subsequent reduction procedure relating the conventional form of Witten’s SUSYQM with the bosonized form. Here we discuss general properties of the bosonized supersymmetry and arrive naturally at the deformed Heisenberg algebra with reflection emerging in some integrable systems \cite{Fromhof, Nepomechie} and closely related to parabosons and parafermions \cite{Parabosons, Parafermions}. In Section 3 we show that bosonized supersymmetry can be understood as a supersymmetry of the system of two identical fermions. This interpretation allows us to generalize the bosonization construction to the case of \( N = 1 \) supersymmetry in two spatial dimensions. In Section 4 we find that the same nonlocal unitary transformation, which relates the conventional supersymmetry with the bosonized one, diagonalises the Hamiltonian of the (1 + 1)-dimensional massive Dirac equation. Subsequent reduction of the transformed Dirac equation supplies us with the linear differential equation with reflection whose ‘squared form’ is the Klein-Gordon equation. In the massless limit this becomes the self-dual Weyl equation. We generalize the obtained linear equation to higher (arbitrary) dimensions and show that it admits switching on gauge interactions of special form. Finally, we observe how the bosonized SUSYQM appears under application of the same nonlocal unitary transformation to the 2D system of Dirac field in the background of a soliton. Here the corresponding phases of the bosonized supersymmetry signal the presence or absence of fermionic zero modes. We conclude in Section 5 giving the list of open problems for further investigation.
2 Special unitary transformation and bosonized SUSY

In this Section we show how the minimally bosonized supersymmetric quantum mechanics [4] can be related to Witten’s SUSYQM [1] by applying to the latter the special nonlocal unitary transformation with a subsequent reduction to one of the eigenspaces of the total reflection operator. The form of such bosonization procedure is found by analyzing the structure of the simplest supersymmetric quantum mechanical system (superoscillator) and then is employed for the general case of \( N = 1 \) supersymmetry.

2.1 Simplest bosonized supersymmetric system

Let us consider the superoscillator [9] given by the Hamiltonian

\[
H = \frac{1}{2} \{b^+, b^-\} + \frac{1}{2} [f^+, f^-],
\]

(2.1)

where \( b^\pm \) and \( f^\pm \) are the bosonic and fermionic creation-annihilation operators,

\[
[b^-, b^+] = 1, \quad \{f^-, f^+\} = 1, \quad f^{+2} = f^{-2} = 0, \quad [b^\pm, f^\pm] = 0.
\]

The supercharge operators

\[
Q_\pm = b^\mp f^\pm
\]

(2.2)

are dynamical integrals of motion forming together with the Hamiltonian (2.1) the superalgebra of \( N = 1 \) supersymmetry:

\[
\{Q_+, Q_-\} = H, \quad Q_+^2 = Q_-^2 = 0, \quad [H, Q_\pm] = 0.
\]

(2.3)

The Hamiltonian is the sum of the bosonic and fermionic number operators, \( H = N_b + N_f \), \( N_b = b^+ b^- \), \( [N_b, b^\pm] = \pm b^\pm \), \( N_f = f^+ f^- \), \( [N_f, f^\pm] = \pm f^\pm \). Thus, in the basis \( |n_b, n_f\rangle \), \( N_b|n_b, n_f\rangle = n_b|n_b, n_f\rangle \), \( N_f|n_b, n_f\rangle = n_f|n_b, n_f\rangle \), \( n_b = 0, 1, \ldots, n_f = 0, 1 \), \( H \) is diagonal and its spectrum is

\[
E_n = \left\lfloor \frac{n + 1}{2} \right\rfloor,
\]

(2.4)

where \( n = 2n_b + n_f = 0, 1, \ldots \), and \( \lfloor \cdot \rfloor \) means the integer part. The state \( |0, 0\rangle \) is the SUSY singlet corresponding to \( E_0 = 0 \), whereas all other states are SUSY doublets with \( E_{2k-1} = E_{2k} > 0, k = 1, \ldots \). The operator

\[
F = [f^+, f^-] = -(-1)^N_f,
\]

satisfies the relations \( \{F, f^\pm\} = 0, F^2 = 1 \). \( F \) is the grading operator, which may also be interpreted as fermionic reflection operator. Since \( N_f = \frac{1}{2} (1 + F) \), instead of \( n_f \), one can characterize the complete basis by the eigenvalues of the operator \( F: F|n_b, \varphi\rangle = \varphi|n_b, \varphi\rangle \), \( \varphi = \pm 1 \). The operators \( f^\pm \) and \( F \) may be represented with Pauli matrices, \( f^\pm = \frac{1}{2}(\sigma_1 \pm i \sigma_2), F = \sigma_3 \).

Let us rearrange the entire spectrum:

\[
E_n = E_n^+ \oplus E_n^- , \quad E_n^+ = 2 \left\lfloor \frac{n+1}{2} \right\rfloor, \quad E_n^- = 2 \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad n = 0, 1, \ldots
\]

(2.5)
For \( n \neq 0 \), \( E_n^+ \) are doubly degenerate, but \( E_n^- = 0 \) is nondegenerate. On the other hand, the spectrum \( E_n^- \) is doubly degenerate for all \( n \). Thus, \( E_n^+ \) is the spectrum of a system with unbroken SUSY, whereas \( E_n^- \) corresponds to some another system in a phase of spontaneously broken SUSY. The corresponding eigenstates are

\[
|n, \epsilon\rangle \equiv (|n_b = 2n, \varphi = -\epsilon \cdot 1\rangle, |n_b = 2n + 1, \varphi = \epsilon \cdot 1\rangle), \quad H|n, \epsilon\rangle = E_n^\epsilon|n, \epsilon\rangle.
\]

The subspaces with \( \epsilon = + \) and \( \epsilon = - \) are separated by the operator \( \mathcal{R} = -RF, \mathcal{R}|n, \epsilon\rangle = \epsilon|n, \epsilon\rangle \), where \( R = (-1)^{N_0} \) is the reflection (parity) bosonic operator, \( \{R, b^\pm\} = 0, R^2 = 1 \).

\( \mathcal{R} \) represents the total reflection operator, \( \{\mathcal{R}, b^\pm\} = 0, \{\mathcal{R}, f^\pm\} = 0, \mathcal{R}^2 = 1 \), and there is a unitary transformation relating \( \mathcal{R} \) to the fermionic reflection operator \( F = \sigma_3, U\sigma_3U^\dagger = \mathcal{R} \).

The corresponding unitary operator may be written in the form

\[
U = \exp(i\pi S_- \Pi_+) = S_+ - R S_-,
\]

where \( S_\pm = \frac{1}{2}(1 \pm \sigma_1) \) and \( \Pi_\pm = \frac{1}{2}(1 \pm R) \) are the projector operators, \( S_\pm^2 = S_\pm, S_\pm S_- = 0 \), \( S_+ + S_- = 1, \Pi_\pm^2 = \Pi_\pm, \Pi_+ \Pi_- = 0, \Pi_+ + \Pi_- = 1 \). From (2.6) one can see that \( U^\dagger = U^{-1} = U \), and calling \( A' = UAU^{-1} \), then

\[
\sigma_1' = \sigma_1, \quad \sigma_2' = -R\sigma_2, \quad \sigma_3' = -R\sigma_3 = \mathcal{R}, \quad b^{\pm'} = b^\pm\sigma_1 \equiv a^\pm.
\]

\( U \) transforms eigenstates of \( F \) into eigenstates of \( \mathcal{R} \): \( |n, \epsilon\rangle = U|n, \varphi\rangle \). The initial (untransformed) supercharges and Hamiltonian can be written in terms of the transformed operators \( a^\pm, \mathcal{R} \) as

\[
Q_\pm = a^\pm \frac{1}{2}(1 \pm RR), \quad H = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}RR, \quad R = (-1)^{N_a},
\]

where \( N_a \equiv a^+a^- \), and we have taken into account that \( N_b = N_a \).

Let us consider now the restriction of these supersymmetry generators to the eigenspaces of the operator \( \mathcal{R} \). We have \( Q_\pm|n, \epsilon\rangle = Q_\pm^\epsilon|n, \epsilon\rangle, H|n, \epsilon\rangle = H^\epsilon|n, \epsilon\rangle \), where

\[
Q_\pm^\epsilon = a^\pm \Pi_\pm \epsilon, \quad H^\epsilon = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}R\epsilon.
\]

Since the operators \( Q_\pm \) and \( H \) commute with \( \mathcal{R} \), their restrictions (2.8) satisfy the same \( N = 1 \) superalgebra (2.3). Thus, reducing the system to the subspaces with \( \epsilon = + \) or \( \epsilon = - \) does not affect the supersymmetry algebra. As a result, we find two systems described only by the bosonic operators \( a^\pm \) and given by the Hamiltonians \( H^\epsilon, \epsilon = +, - \). In the case of \( \epsilon = + \) the system exhibits the spectrum \( E_n^+ \) corresponding to the exact supersymmetry, whereas the choice \( \epsilon = - \) gives the system in the phase of spontaneously broken supersymmetry with the spectrum \( E_n^- \). The operators \( Q_\pm^\epsilon \) given in terms of only bosonic operators \( a^\pm \) are the corresponding supercharge operators. Thus, bosonized supersymmetry in the exact or spontaneously broken phases can be produced via the unitary transformation (2.6) with a subsequent reduction of the system to one of the eigenspaces of the total reflection operator.

The same result may be obtained applying the unitary transformation \( U \) to the supercharges and Hamiltonian (since \( U = U^{-1} \)) with the subsequent reduction of the system to the corresponding eigenspaces of the operator \( F \). Indeed, the transformed operators \( Q_\pm^\epsilon \) and \( H^\epsilon \) take the form

\[
Q_\pm^\epsilon = b^\pm \frac{1}{2}(1 \pm R\sigma_3), \quad H^\epsilon = \frac{1}{2}\{b^+, b^-\} - \frac{1}{2}R\sigma_3,
\]
and commute with the untransformed operator $F = \sigma_3$. Restricting these operators on the eigenspaces of $F$ with $\varphi = +1$ or $-1$, and then changing the notation $\varphi \to \epsilon$, $b^\pm \to a^\pm$, we arrive at the bosonized supercharge and Hamiltonian operators (2.8). This alternative way is more convenient for generalizations.

In the coordinate representation, where $b^\pm = \frac{1}{\sqrt{2}}(x \mp \frac{d}{dx})$, the operator $R$ acts as $R\psi(x) = \psi(-x)$, and we see that the operator (2.6) generates the unitary transformation

$$
\Psi(x) \to \Psi'(x) = U\Psi(x) = S_+\Psi(x) - S_-\Psi(-x),
$$

which is nonlocal: the transformed state at point $x$ is a linear combination of initial states taken at points $x$ and $-x$.

### 2.2 Bosonized SUSYQM: general case

We now generalize the bosonization construction starting with an arbitrary supersymmetric quantum mechanical system. Consider the SUSYQM system characterized by the supercharges $Q_\pm$ and the Hamiltonian $H$ defined by the superpotential $W(x)$ [1]:

$$
Q_\pm = \frac{1}{\sqrt{2}} \left( \pm \frac{d}{dx} + W(x) \right) \cdot \frac{1}{2}(\sigma_1 \pm i\sigma_2), \quad H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + W^2(x) + W'(x)\sigma_3 \right). \quad (2.10)
$$

Let us decompose $W(x)$ into even and odd parts, $W(x) = W_+(x) + W_-(x)$, $W_\pm(-x) = \pm W_\pm(x)$, and realize the unitary transformation with the operator (2.6). Under it, the operators (2.10) are transformed into

$$
Q'_\pm = \frac{1}{\sqrt{2}} \left( \pm \frac{d}{dx} + W_+(x)\sigma_1 + W_-(x) \right) \frac{1}{2}(1 \pm \sigma_3 R),
$$

$$
H' = \frac{1}{2} \left( -\frac{d^2}{dx^2} + (W_+(x)\sigma_1 + W_-(x))^2 - (W'_+(x)\sigma_1 + W'_-(x))\sigma_3 R \right).
$$

If the even part of the superpotential vanishes, $W_+ = 0$, (as in the case of the superoscillator considered above, $W(x) = W_-(x) = x$), the operators $Q'_\pm$ and $H'$ commute with $\sigma_3$ and do not mix ‘up’ and ‘down’ states. Restricting these operators on the ‘up’ ($\epsilon = +$) and ‘down’ ($\epsilon = -$) eigenspaces of $\sigma_3$, we get

$$
Q^\epsilon_\pm = \frac{1}{\sqrt{2}} \left( \pm \frac{d}{dx} + W_-(x) \right) \cdot \Pi_{\pm \epsilon}, \quad H^\epsilon = \frac{1}{2} \left( -\frac{d^2}{dx^2} + W_-^2(x) - \epsilon W'_-(x)R \right),
$$

that generalizes the simplest bosonized supersymmetric system (2.8). The restricted operators $Q^\epsilon_\pm$ and $H^\epsilon$ form the $N = 1$ superalgebra (2.3) both for $\epsilon = +$ and $\epsilon = -$. These two cases are related in a simple way: $Q^\epsilon_\pm(W_-) = -Q_{-\epsilon}^\epsilon(W_-)$, $H^\epsilon(W_-) = H^{-\epsilon}(W_-)$. Therefore, the general case of the bosonized SUSYQM can be given by the hermitian supercharge and Hamiltonian operators

$$
Q_1 = -\frac{i}{\sqrt{2}} \left( \frac{d}{dx} + W_- R \right), \quad Q_2 = iRQ_1, \quad H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + W_-^2 - W'_- R \right), \quad (2.11)
$$
containing arbitrary odd function (superpotential) \( W_-(x) \) and satisfying the superalgebra 
\[ \{ Q_i, Q_j \} = 2\delta_{ij}H, \ [H, Q_i] = 0, \ i, j = 1, 2. \]  
The hermitian supercharges are related to \( Q^\pm \) as \( Q_1 = i(Q^+_1 - Q^-_1), \ Q_2 = Q^+_1 + Q^-_1. \)

The generators of bosonized supersymmetry act on the space of wave functions \( \Psi(x) \), 
where the action of reflection operator \( R \) is defined by \( R\Psi(x) = \Psi(-x) \). It should be noted 
that the operator \( i\sqrt{2}Q_1 \) is related to \( \frac{d}{dx} \) through 
\[ i\sqrt{2}Q_1 \exp(-\omega_+ R) = \exp(\omega_+ R) \frac{d}{dx} \]
involving the nonunitary operator \( \exp(\omega_+ R) \), \( \omega_+(x) = \int^x W_-(y)dy \). The argument of the 
exponent is operator-valued, and it can be verified that 
\[ -2Q_1^2 \exp(-\omega_+ R)\Psi(x) = \exp(-\omega_+ R)(\frac{d}{dx} + 2W_-(x))\frac{d}{dx}\Psi(x). \]

This means that the action of \( i\sqrt{2}Q_1 \) is not reducible to the simple derivative.

Let us also note that for the particular case, \( W_-(x) = -\frac{\nu}{2x} \), the operator \( i\sqrt{2}Q_1 \) coincides 
with the Yang-Dunkl operator [10] \( D_\nu = \frac{d}{dx} - \frac{\nu}{2x}R \) which occurs in the Calogero model [5,6], 
where \( R \) is an exchange operator. With the extended differential operator \( D_\nu \), we arrive at 
the deformed Heisenberg algebra with reflection 
\[ [a^-, a^+] = 1 - R, \quad R^2 = 1, \quad \{ a^\pm, R \} = 0, \quad (2.12) \]
with \( a^\pm = \frac{1}{\sqrt{2}}(x \mp iD_\nu) \). This algebra is intimately related to parabosons [4] and parafermions [8], 
and to the \( osp(1|2) \) and \( osp(2|2) \) superalgebras [4,8].

Let us now decompose the wave function into even and odd parts, \( \Psi(x) = \Psi_+(x) + \Psi_-(x), \) 
\( \Psi_+(x) \) and \( \Psi_-(x) \) involving the nonunitary operator \( \exp(\omega_+ R) \), \( \omega_+(x) = \int^x W_-(y)dy \). The argument of the 
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even and odd wave functions in the bosonized version of SUSYQM. As we shall see below, these parity restrictions on the superpotential and wave functions imply that the bosonized SUSYQM can be understood as a supersymmetric system of two identical fermions.

In correspondence with Eqs. (2.14), (2.15), if the bosonized supersymmetric system is in the phase of spontaneously unbroken supersymmetry, its vacuum state has to satisfy either $A^-\Psi_+ = 0$ or $A^+\Psi_- = 0$. Solutions to these equations are $\Psi_+^{(0)} = N_+ \exp(-\int x W_-(y) dy)$, $\Psi_-^{(0)} = N_- \exp(+\int x W_-(y) dy)$. The second solution would be odd only for $N_- = 0$. Therefore, if the superpotential $W_-$ is such that $\Psi_+^{(0)}$ is normalizable, it is the case of exact bosonized SUSY, i.e. the supersymmetric vacuum state (if it exists) is always described by an even function. In the bosonized version, like the case of conventional SUSYQM, the SUSY partner eigenfunctions belonging to the same energy eigenvalue $E > 0$ are related by the operators (2.13), $\Psi_- \propto A^-\Psi_+$, $\Psi_+ \propto A^+\Psi_-$. Examples of the systems with unbroken bosonized supersymmetry are provided by the following superpotentials:

$$W_- = \epsilon x^{2k+1}, \quad k = 0, 1, \ldots, \quad W_- = \epsilon \frac{x}{|x|}, \quad W_- = \epsilon \sinh x, \quad W_- = \epsilon \tanh x, \quad \epsilon = \pm.$$ 

A superpotential of the form $W_- = \epsilon x - \frac{\nu}{2x}$ gives rise to exact SUSY when $\epsilon = +$ and $\nu > -1$. The supercharge and Hamiltonian operators for this system can be represented in the form (2.8) in terms of creation-annihilation operators of the deformed Heisenberg algebra with reflection (2.12). This system turns out to be isospectral to the bosonized superoscillator in the phase of exact supersymmetry (with $E_n^+ = 2[(n + 1)/2]$) discussed in the previous subsection.

On the other hand, for $\epsilon = -$ the system with superpotential $W_- = \epsilon x - \frac{\nu}{2x}$ is in the phase of the spontaneously broken supersymmetry with the spectrum $E_n^- = 2[n/2] + 1 + \nu$. In this case the deformation parameter defines the scale of supersymmetry breaking (for details, see ref. [4]).

### 3 SUSY of two-fermion system and bosonized SUSY

As we saw, the bosonized form of SUSYQM can be formally obtained by imposing the corresponding parity restrictions on the superpotential and ‘up’ and ‘down’ states of the conventional SUSYQM. But analogous parity restrictions on the physical operators and wave functions emerge when a system of identical fermions is described. Having in mind this observation, here we show that the minimally bosonized supersymmetric quantum mechanics can be understood as a supersymmetric two-fermion system. With this interpretation, the bosonization construction is generalized to the case of $N = 1$ supersymmetry in two spatial dimensions.

Let us consider a system of two identical fermions on the line. It can be described by the wave function $\Psi_{s_1,s_2}(x_1,x_2) = -\Psi_{s_2,s_1}(x_2,x_1)$, where indices $s_1$, $s_2$ correspond to spin degrees of freedom of the particles, and we assume that unlike the 1-dimensional coordinate space, the spin operator space is 3-dimensional. It is convenient to pass over to the center

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1Let us indicate the difference of the present case of fermions with spin from the systems of spinless fermions on the line, where the problem of self-adjointness of physical operators is essential [11, 12, 13].
of mass and relative coordinates, \( X = \frac{1}{2}(x_1 + x_2), \) \( x = x_1 - x_2, \) as well as to the basis of the total spin, \( J_i = \frac{1}{2}(\sigma_i \otimes 1 + 1 \otimes \sigma_i), \) \( i = 1, 2, 3. \) Then, omitting the dependence on center of mass coordinate, we describe the two-fermion system by the wave functions of the form

\[
\Psi_f(x) = \chi^{j_3}_a \psi^{j_3}(x) + \chi_a \psi_+(x),
\]

(3.1)

where \( j_3 = +1, 0, -1. \) Here \( \chi^{j_1}_s = |+\rangle |+\rangle, \chi^{-1}_s = |−\rangle |−\rangle \) and \( \chi^0_s = \sqrt{\frac{1}{2}}(|+\rangle |−\rangle + |−\rangle |+\rangle) \) are symmetric spin states forming a vector triplet, \( J_i J_s \chi^{j_3}_s = 2 \chi^{j_3}_s, \) \( J_3 \chi^{j_3}_s = j_3 \chi^{j_3}_s, \) and \( \chi_a = \sqrt{\frac{1}{2}}(|+\rangle |−\rangle - |−\rangle |+\rangle) \) is antisymmetric spin-0 singlet state, \( J_i J_a \chi_a = J_3 \chi_a = 0; \psi^{j_3} \) are odd functions, \( \psi^{j_3}(-x) = -\psi^{j_3}(x), \) whereas \( \psi_+ \) is an even function, \( \psi_+(-x) = \psi_+(x). \) To simplify the notation, below we denote \( \chi^{±1}_s, \chi^0_s \) and \( \psi^0 \) by \( \chi^{±}, \chi_a \) and \( \psi_-, \) respectively.

We want to realize \( N = 1 \) supersymmetry on this system of two identical fermions. This means that our task is to write the supercharge and Hamiltonian operators in a form similar to that in Witten’s SUSYQM:

\[
Q_1 = \frac{1}{\sqrt{2}} \left( -i \frac{d}{dx} \sigma_1 - W(x) \sigma_2 \right), \quad Q_2 = i \sigma_3 Q_1,
\]

(3.2)

\[
H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + W^2(x) - W'(x) \sigma_3 \right).
\]

(3.3)

Let us try to do this in a way that respects the rotational \( J_3^2 \)-symmetry of the spin space. The list of nontrivial independent operators respecting \( J_3^2 \) symmetry is given by the operators commuting or anticommuting with \( J_3, \)

\[
\Sigma_1 = \frac{1}{2}(\sigma_3 \otimes 1 - 1 \otimes \sigma_3), \quad \Sigma_2 = \frac{1}{2}(\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1), \quad \Sigma_3 = \frac{1}{2}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2),
\]

\[
\Xi_1 = \frac{1}{2}(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2), \quad \Xi_2 = \frac{1}{2}(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1),
\]

and by \( J_3 \equiv \Xi_3 \) itself. Operators \( \Sigma_i \) annihilate the states \( \chi^{±}, \chi_a, \) i.e. they are proportional effectively to \( 1 - J_3^2 \) which projects on the \( j_3 = 0 \) subspace, whereas operators \( \Xi_i, \) being proportional to \( J_3^2, \) annihilate the states \( \chi_a, \chi_a. \) Rearranging 4-dimensional spin space into the direct sum of \( j_3 = 0 \) and \( j_3^2 = 1 \) subspaces, we find that in the basis \( (\chi_s, \chi_a) \oplus (\chi^{±}, \chi_a) \) the action of the listed operators can be represented as

\[
\Sigma_i = \sigma_i (1 - J_3^2), \quad \Xi_i = J_3^2 \sigma_i,
\]

(3.4)

i.e. \( \Sigma_1 \chi_{a(a)} = \chi_{a(a)}, \Sigma_3 \chi_{a(a)} = +(-) \chi_{a(a)}, \Sigma_1 \chi^{±} = 0, \Sigma_2 = i \Sigma_1 \Sigma_3, \Xi_1 \chi^{±} = \chi^{±}, \Xi_3 \chi^{±} = ± \chi^{±}, \Xi_1 \chi_{a(a)} = 0, \Xi_2 = i \Xi_1 \Xi_3. \)

The ‘physical’ operators are those transforming the states of the form \( (\Sigma, \Xi) \) into the states of the same form (i.e. one should remember that we are dealing with the system of two identical fermions). The algebra of physical operators is generated by

\[
\mathcal{A}_+ = f_+(x, \frac{d}{dx}) \mathcal{O}_+, \quad \mathcal{A}_- = f_-(x, \frac{d}{dx}) \mathcal{O}_-,
\]

(3.5)

where \( f_+(x, -\frac{d}{dx}) = ± f_+(x, \frac{d}{dx}), \mathcal{O}_+ = 1, \Sigma_3, \Xi_i, \) \( i = 1, 2, 3, \) and \( \mathcal{O}_- = \Sigma_1, \Sigma_2. \)
Taking into account the explicit form of the operators (3.4), we see that the supercharge operators cannot take the form (3.2) on the subspace with \( j^2_3 = 1 \) since the differential operator \( \frac{d}{dx} \) is odd. On the other hand, defining the operators

\[
Q_1 = \frac{1}{\sqrt{2}} \left( -i \frac{d}{dx} \Sigma_1 - W_-(x) \Sigma_2 \right), \quad Q_2 = i \Sigma_3 Q_1, \tag{3.6}
\]

\[
\mathcal{H} = \frac{1}{2} \left( -\frac{d^2}{dx^2} + W^2_-(x) \right) \mathcal{I} - W'_-(x) \Sigma_3 \tag{3.7}
\]

where \( W_-(x) \) is an odd function, and \( \mathcal{I} = 1 \oplus (1 - J^2_3) \), we find that they satisfy \( N = 1 \) superalgebra, \( \{ Q_a, Q_b \} = 2 \delta_{ab} \mathcal{H}, \ [ \mathcal{H}, Q_a ] = 0 \). On the subspace \( j_3 = 0 \), these operators take the form of operators (3.2), (3.3). Moreover, since these operators annihilate \( j^2_3 = 1 \) states, we conclude that the Hamiltonian (3.7) and the supercharges (3.6) provide an \( N = 1 \) supersymmetry realized on the system of two identical fermions. Spin states with \( j_3 = 1 \) and \( j_3 = -1 \) are supersymmetric vacuum states with zero energy regardless of whether supersymmetry in \( j_3 = 0 \) sector is exact or spontaneously broken.

The only essential difference of the supersymmetry realized in the sector \( j_3 = 0 \) from Witten’s SUSYQM is that in the two-fermion system the superpotential \( W_-(x) \) must be an odd function. But we have exactly the same restriction in the case of minimally bosonized supersymmetric quantum mechanics and one can establish the one-to-one correspondence of supersymmetric \( j_3 = 0 \) subsystem with the bosonized supersymmetric system. To this end, let us consider another system described by the scalar wave function \( \Psi(x) = \Psi_+(x) + \Psi_-(x), \) \( \Psi_+(x) = \pm \Psi_+(x) \). The subspaces formed by even and odd functions are mutually orthogonal, they can be distinguished by the parity operator \( R, \) \( R \Psi(x) = \Psi(\pm x); \) \( R \Psi_\pm(x) = \pm \Psi_\pm(x) \). Even operators \( f_+(x, \frac{d}{dx}) \) map these two subspaces into themselves, whereas odd operators \( f_-(x, \frac{d}{dx}) \) interchanged these two subspaces. Taking into account that \( \Sigma_2 = i \Sigma_1 \Sigma_3 \), we arrive at the one-to-one correspondence between the \( N = 1 \) supersymmetry realized on the system of two identical fermions in the \( j_3 = 0 \) sector and the bosonized supersymmetry through the following identifications:

| States | Fermion system \((j_3 = 0)\) | Bosonized SUSYQM |
|--------|-----------------------------|-----------------|
| \( \chi_+ \psi_- \) | \( \Psi_- \) | \( \Psi_+ \) |
| \( \chi_0 \psi_+ \) |               |               |
| Operators | \( f_+ \mathcal{I} \) | \( f_+ \) |
|         | \( f_+ \Sigma_3 \) | \( f_+ R \) |
|         | \( f_- \Sigma_1 \) | \( f_- \) |
|         | \( f_- \Sigma_2 \) | \( i f_- R \) |

The \( N = 1 \) supersymmetry can also be realized in the two-fermion system in 2 dimensions. In this case the supercharges and the Hamiltonian are

\[
Q_1 = \frac{1}{\sqrt{2}} \pi_a \Sigma_a, \quad Q_2 = i \Sigma_3 Q_1, \quad \mathcal{H} = \frac{1}{2} (\pi_a^2 \mathcal{I} - \Sigma_3 B_+), \tag{3.8}
\]

where \( \pi_a = -i \partial^a - A^a(x), \) \( \partial^a = \partial_a = \partial/\partial x^a, \) \( a = 1, 2, \) \( A_-(x) \) is a two-dimensional antisymmetric vector potential, \( A_-(x) = -A_-(x), \) and the magnetic field \( B_+(x) = \)
\[ \partial_1 A^2_\| \| - \partial_2 A^1_\| \| \] is symmetric under inversion \( x \to -x \). In this case the states \( j_3 = \pm 1 \) are also supersymmetric vacuum states with zero energy. The \( N = 1 \) supersymmetry realized in the \( j_3 = 0 \) subspace of two-fermion system corresponds formally to SUSYQM of the two-dimensional (plane) spin-1/2 particle with gyromagnetic ratio \( g = 2 \) interacting with a magnetic field given by a generic vector potential \( A(x) \) \[2, 3\] (there is no parity restriction on the vector gauge potential):

\[ Q_1 = \frac{1}{\sqrt{2}} \pi^a \sigma_a, \quad Q_2 = i \sigma_3 Q_1, \quad H = \frac{1}{2} (\pi^2 - \sigma_3 B). \] (3.9)

The \( N = 1 \) supersymmetry of the \( j_3 = 0 \) subspace of the system (3.8) gives rise to the bosonized 2-dimensional supersymmetric system in the same way as we indicated in the one dimensional case with the only difference that the reflection (parity) operator \( R \) is now given by \( R = R_1 R_2 \), where \( R_1, R_2 \) are reflection operators with respect to \( x^1 \) and \( x^2 \): \( \{ R_1, x^1 \} = \{ R_2, x^2 \} = 0 \), \( [ R_1, x^2 ] = [ R_2, x^1 ] = 0 \), \( R_1^2 = R_2^2 = 1 \), \( [ R_1, R_2 ] = 0 \). The operator \( R \) is also the operator of space rotation by the angle \( \pi \), \( R = \exp(-i\pi L) \), where \( L = i(x^2 \partial_1 - x^1 \partial_2) \) is the operator of orbital angular momentum.

It is known that the conventional \( N = 1 \) SUSYQM can be constructed in 3-dimensional coordinate space in the form analogous to that of 2-dimensional space (3.9), but this time the vector gauge potential must be an antisymmetric function, \( A(x) = -A(-x) \equiv A_-(x) \).

In this case, the \( N = 1 \) supercharge and Hamiltonian operators have the form \[2\]

\[ Q_1 = \frac{1}{\sqrt{2}} \pi_-^j \sigma_j, \quad Q_2 = i P Q_1, \quad H = \frac{1}{2} (\pi_-^2 - B_+^j \sigma_j), \] (3.10)

where \( B_+^j = \epsilon_{jkl} \partial_k A^-_l \) is a supersymmetric pseudovector of magnetic field, and \( P \) is the parity operator, \( \{ x^i, P \} = 0, i = 1, 2, 3 \), \( P^2 = 1 \). Since in the supercharge \( Q_1 \) all \( \sigma \)-matrix factors are multiplied by odd operators \( \pi_-^j \), it is clear that this 3-dimensional supersymmetry cannot be reproduced in the two-fermion system: the reason is that the form of the physical operators (3.3) requires that \( \Sigma_3 \) should be multiplied only by even space operator. As a consequence, it is not possible to construct the bosonized analog of the supersymmetry (3.10) in the way described in Section 2.

### 4 Reflection-dependent unitary transformation applied to Dirac field theory

It is well known that Witten’s SUSYQM is related in some aspects to the Dirac field theory \[2, 3, 14\] (see, e.g., Section 4.3 below). With this motivation, here we apply the special unitary transformation (2.6) to the 2D Dirac field theory. First, we find that the transformation diagonalises the Hamiltonian operator of the 2D free massive Dirac field. Unlike the Foldy-Wouthuysen case, the resulting Hamiltonian is not of a square root form but is linear in space derivative and contains a space reflection (parity) operator. Subsequent reduction to ‘up’ or ‘down’ field component breaks the Poincaré invariance of the theory, but supplies a linear differential equation with reflection whose ‘squared form’ is the massive Klein-Gordon equation. In the massless limit this linear equation becomes the self-dual Weyl equation.
Then we show that the linear differential equation with reflection admits the generalization to higher dimensions and also allows gauge interactions. Finally, we show that the bosonized \( N = 1 \) SUSYQM emerges if the nonlocal unitary transformation is applied to the Dirac field theory in a kink background.

### 4.1 Linear differential equation with reflection

Let us consider free massive Dirac equation in \( 1 + 1 \) dimensions,

\[
[i(\gamma^0 \partial_t + \gamma^1 \partial_x) - m] \Psi(t, x) = 0,
\]

and apply to it the unitary transformation

\[
\Psi(t, x) \rightarrow \Psi'(t, x) = U \Psi(t, x), \quad U = \exp [i\pi S_- \Pi_+] = S_+ - RS_-.
\] (4.1)

Here \( S_\pm = \frac{1}{2}(1 \pm \gamma_5) \), \( \gamma_5 = \gamma^0 \gamma^1 \), \( \Pi_+ = \frac{1}{2}(1 + R) \), and \( R \) is the space reflection (parity) operator, \( R t = t R, Rx = -x R, R^2 = 1 \). Transformation (4.1) is generated by the operator which is the product of the projector on the subspace of even functions, \( \Pi_+ \Psi(t, x) = \Psi_+(t, x) \), \( \Psi_+(t, x) = \Psi_+(t, -x) \), and of the chiral projector \( S_- \). In representation \( \gamma^0 = \sigma_3, \gamma^1 = i\sigma_2 \), operator \( U \) is reduced to the unitary operator introduced in Section 2. With this unitary transformation, \( A \rightarrow A' = UAU^{-1} \). For \( A = t, x, \sigma_i, i = 1, 2, 3 \), this means \( t \rightarrow t' = t, x \rightarrow x' = \sigma_1 x, \sigma_1 \rightarrow \sigma'_1 = \sigma_1, \sigma_2 \rightarrow \sigma'_2 = -R\sigma_2, \sigma_3 \rightarrow \sigma'_3 = -R\sigma_3 \). Multiplying the transformed Dirac equation by \( \sigma_3 \), we arrive at

\[
[iR(\partial_t + \partial_x) + m \sigma_3] \Psi'(t, x) = 0.
\] (4.2)

Thus, the transformed Dirac field \( \Psi' \) satisfies the equation \( i\partial_t \Psi' = H' \Psi' \) with the Hamiltonian

\[
H' = -i\sigma_2 - R\sigma_3 m.
\] (4.3)

The Hamiltonian (4.3) has diagonal matrix form and therefore (4.1) is analogous to the Foldy-Wouthuysen (FW) transformation. However, the Hamiltonian (4.3) is not of the FW square root form, but the relation

\[
H'^2 = -\partial_x^2 + m^2
\]

is satisfied here due to the dependence of the Hamiltonian on the reflection operator \( R \). Like the FW case, the transformation (4.1) is nonlocal: the transformed field at point \( x \) is a linear combination of the positive chirality component of the field at point \( x \) and of the negative chirality component taken at point \( -x \):

\[
\Psi'(t, x) = S_+ \Psi(t, x) - S_- \Psi(t, -x).
\]

Since the transformed Dirac equation (4.2) has a diagonal form, one could reduce the theory to that of either up, \( \Psi'_1 \equiv \psi^+ \), or down, \( \Psi'_2 \equiv \psi^- \), field component, each of which satisfies the corresponding linear differential equation,

\[
[iR(\partial_t + \partial_x) + \epsilon m] \psi^\epsilon = 0, \quad \epsilon = +, -.
\] (4.4)
In the light-cone coordinates \( x_\pm = t \pm x \), Eq. (4.4) is

\[
i\partial_\pm \psi^\epsilon(x_+, x_-) + \epsilon m \psi^\epsilon(x_-, x_+) = 0.
\]

Both fields \( \psi^+ \) and \( \psi^- \) satisfy the massive Klein-Gordon equation as a consequence of equation (4.4), which we call the linear differential equation with reflection. However, the reduction of the Dirac field theory given by Eq. (4.2) to the one-component field theory corresponding to Eq. (4.4) destroys the Poincaré invariance. This is clear from the form of the space translation and Lorentz transformations,

\[
\delta_\kappa \Psi' = \kappa \cdot \sigma_1 \partial_x \Psi', \quad \delta_\lambda \Psi' = \lambda \cdot \sigma_1 \left( t \partial_x + x \partial_t - \frac{1}{2} \right) \Psi',
\]

which mixes the components \( \Psi'_1 = \psi^+ \) and \( \Psi'_2 = \psi^- \). Here \( \kappa \) and \( \lambda \) are the corresponding infinitesimal transformation parameters. It is interesting to note that in spite of this breaking of Poincaré invariance, the theory given by Eq. (4.4) has a formal analogy with the initial, non-transformed Dirac equation. Indeed, decomposing the field \( \psi^\epsilon \) in even and odd parts, \( \psi^\epsilon = \psi^+ + \psi^- \), \( \psi^\epsilon(t, -x) = \pm \psi^\epsilon(t, x) \), we represent one equation (4.4) with \( \epsilon = + \) or \( - \) in the form of two equations:

\[
i(\partial_t \psi^\epsilon_+ + \partial_x \psi^\epsilon_-) + \epsilon m \psi^\epsilon_+ = 0, \quad -i(\partial_t \psi^\epsilon_- + \partial_x \psi^\epsilon_+) + \epsilon m \psi^\epsilon_- = 0.
\]

These two equations resemble the Dirac equation \( i(\gamma^0 \partial_t + \gamma^1 \partial_x) + \epsilon m \psi^\epsilon = 0 \) written in terms of ‘up’, \( \Psi^1_1 \), and ‘down’, \( \Psi^1_2 \), components, i.e. formally \( \psi^+ \) and \( \psi^- \) play the role analogous to \( \Psi^1_1 \) and \( \Psi^1_2 \), respectively.

The broken Poincaré invariance can be ‘restored’ in the \( m = 0 \) limit. To do this, first one can note that in this limit Eq. (4.4) is the massless self-dual equation for a one-component Weyl fermion: \( (\partial_t + \partial_x) \psi = 0 \) (13). For this field, the space-time translation and Lorentz transformations have the infinitesimal form

\[
\delta_\tau \psi = \tau \partial_t \psi, \quad \delta_\kappa \psi = \kappa \partial_x \psi, \quad \delta_\lambda \psi = \lambda \left( t \partial_x + x \partial_t - \frac{1}{2} \right) \psi,
\]

and the corresponding theory is Poincaré invariant. So, if one ignores the connection of the theory given by Eq. (4.4) with the initial massive Dirac theory and postulates for the field \( \psi^\epsilon \) the same form of Poincaré transformations (4.6), then in the corresponding field action \( S = \int L d^2 x \) with

\[
L^\epsilon = \bar{\psi}^\epsilon \left[ iR(\partial_t + \partial_x) + \epsilon m \right] \psi^\epsilon, \quad \bar{\psi} = \psi^\dagger R,
\]

it is the mass term that breaks space translation and Lorentz invariance (but does not break time translation invariance). In this way, the field satisfying the linear differential equation with reflection (4.4) can be viewed as a massive Poincaré non-invariant generalization of the Poincaré invariant theory of massless Weyl field.

### 4.2 Higher-dimensional generalization and switching on interactions

The Lagrangian (4.7) can be generalized to the case of \((2 + 1)\) dimensions:

\[
L^\epsilon = \bar{\psi}^\epsilon L^\epsilon \psi^\epsilon, \quad L^\epsilon = \Delta + \epsilon m, \quad \Delta = iR_2[R_1(\partial_0 + \partial_1) + \partial_2],
\]

where \( \Delta \) is the mass term.
where \( \bar{\psi}^\epsilon = \psi'^\dagger R_1 R_2, \epsilon = +, - \), \( R_1^2 = R_2^2 = 1 \), \( R_1 R_2 = R_2 R_1 \), \( R_1 x_1 = -x_1 R_1 \), \( R_2 x_2 = -x_2 R_2 \), \( R_1 x_2 = x_2 R_1 \), \( R_2 x_1 = x_1 R_2 \), \( R_i x_0 = x_0 R_i \), \( i = 1, 2 \). The equation of motion \( L^\epsilon \psi^\epsilon(x_0, x_i) = 0 \) is a linear differential equation whose ‘square’ is the \( D = 2 + 1 \) Klein-Gordon equation: multiplying \( L^\epsilon \) by the operator \( \Delta - \epsilon m \), and using the relation \( \Delta^2 = -\partial_0^2 + \partial_1^2 + \partial_2^2 \), we get the latter equation. The generalization to the arbitrary case of \( D = d + 1 \) is achieved via the following formal substitutions in the corresponding \((d + 1)\)-dimensional Dirac equation:

\[
\gamma^d \partial_d \to R_d \partial_d, \quad \gamma^{d-1} \partial_{d-1} \to R_d R_{d-1} \partial_{d-1}, \ldots, \quad \gamma^2 \partial_2 \to R_d R_{d-1} \ldots R_2 \partial_2,
\]

\[
\gamma^1 \partial_1 \to R_d R_{d-1} \ldots R_1 \partial_1, \quad \gamma^0 \partial_0 \to R_d R_{d-1} \ldots R_1 \partial_0.
\]

The quadratic equation following from Eq. (4.11) is

\[
[i R(\mathcal{D}_0 + \mathcal{D}_1) + \sigma_3 m + i e \sigma_2 (A_{0+} + A_{1+})] \Psi' = 0,
\]

where \( \mathcal{D}_0 = \partial_t - i e A_{0+}, \mathcal{D}_1 = \partial_x - i e A_{1-} \), and \( A_\mu = A_{\mu+} + A_{\mu-}, A_{\mu\pm}(t, -x) = \pm A_{\mu\pm}(t, x) \). In the case when the electric field \( F = \partial_0 A_1 - \partial_1 A_0 \) is such that its even part is zero, \( F_+(t, x) = 0 \), the non-diagonal term can be removed from Eq. (4.10) by the gauge transformation \( A_\mu \to A_\mu + \partial_\mu \Lambda \) with \( \Lambda(t, x) = -\int_0^t A_{1+}(t, x') dx' + \lambda(t) \). As a result the equation takes the diagonal form

\[
[i R(\mathcal{D}_0 + \mathcal{D}_1) + \sigma_3 m] \Psi = 0.
\]

The quadratic equation following from Eq. (4.11) is

\[
[\mathcal{D}_\mu \mathcal{D}^\mu + i e \mathcal{F} - m^2] \Psi' = 0,
\]

where \( \mathcal{F} = F_- \). In this case the transformed gauge potential components satisfy the following parity restriction conditions: \( A_0(t, -x) = A_0(t, x), A_1(t, -x) = -A_1(t, x) \), i.e. \( A_0 \) and \( A_1 \) obey the same (anti)commutation relations with space reflection operator \( R \) as the time, \( t \), and space, \( x \). Reducing Eq. (4.11), we find the generalization of Eq. (4.4) for the case of \( U(1) \) interaction:

\[
[i R(\mathcal{D}_0 + \mathcal{D}_1) + \epsilon m] \psi^\epsilon = 0, \quad \epsilon = +, -,
\]

where \( \psi^+ = \Psi'_1, \psi^- = \Psi'_2, \mathcal{D}_0 = \partial_t - i e A_0, \mathcal{D}_1 = \partial_x - i e A_1 \), with \( A_0 \) and \( A_1 \) being even and odd functions of \( x \), respectively. Eq. (4.13) leads to the second order equation for the field \( \psi^\epsilon \) of the form (4.12). Note that as in a free case, equation (4.13) and the corresponding second order equation decomposed in terms of even and odd parts of \( \psi^\epsilon \), \( \psi^\epsilon = \psi^+_\epsilon + \psi^-_\epsilon \), are formally analogous to Dirac equation \( (i \gamma^\mu D_\mu + m) \psi^\epsilon = 0 \) and associated Klein-Gordon equation \( (D_\mu \bar{D}^\mu + i e \mathcal{F} - m^2) \Psi = 0 \), decomposed in ‘up’ and ‘down’ components.

Introducing the corresponding restrictions on the gauge potential, one can generalize the free equation (4.4) to higher-dimensions with a \( U(1) \) interaction. For \( D = 2 + 1 \), this is achieved by imposing the following parity restrictions:

\[
A_0(x_0, x_1, x_2) = A_0(x_0, -x_1, x_2) = A_0(x_0, x_1, -x_2), \quad A_1(x_0, x_1, x_2) = -A_1(x_0, -x_1, x_2) = A_1(x_0, x_1, -x_2), \quad A_2(x_0, x_1, x_2) =
\]

\[
-\partial_0 A_1(t, x) dx' + \lambda(t).
\]

As a result the equation takes the diagonal form

\[
[i R(\mathcal{D}_0 + \mathcal{D}_1) + \sigma_3 m] \Psi = 0.
\]
\[ A_2(x_0, -x_1, x_2) = -A_2(x_0, x_1, -x_2). \] In this case the associated quadratic equation is given by
\[ [D_\mu D^\mu + ie(F_{01} + (F_{02} - F_{12})R_1) - m^2]\psi = 0, \tag{4.14} \]
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). In order to include the U(1) gauge interaction in arbitrary \( D = d + 1 \), we have to introduce the obvious parity restrictions for the components of the vector gauge potential \( A_\mu \) generalizing the \( D = 2 + 1 \) case: the components have to satisfy the same (anti)commutation relations with the set of \( d \) reflection operators \( R_i, i = 1, \ldots, d \), as the coordinates \( x_\mu \) do.

The described construction also works for non-Abelian gauge interaction. It is sufficient to take the field \( \psi^x(t, x) \) in the corresponding representation of the internal gauge group, and impose on the algebra-valued vector gauge potential the same parity restrictions, as in the case of the U(1) interaction.

### 4.3 Dirac field in a kink background and bosonized SUSYQM

Conventional SUSYQM underlies the dynamics of Dirac field propagating in a background of a stationary scalar field soliton [3]. Here we show that the bosonized SUSYQM can also be revealed for a Dirac field in a kink background. Analogously to the case of Witten’s super-symmetry, the exact or broken phases of minimally bosonized SUSYQM may be associated with the presence or absence of fermionic zero modes in a system.

To see this, let us consider the Lagrangian \([14]\) to the Dirac field \( \Psi \), the fermion part becomes
\[ \mathcal{L}_f = \bar{\Psi}[i\gamma^\mu \partial_\mu + gW(\phi)]\Psi, \tag{4.15} \]
with \( \mathcal{L}_s = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \) describing nonlinear scalar field \( \phi \) interacting with Dirac field \( \Psi \). The Lagrangian \([14]\) was used in the context of fermion number fractionalization [16] observed in certain polymers like polyacetylene, and also in supersymmetric field theories in \( 1 + 1 \) [17] (for \( V = \frac{1}{2}v^2 \) and \( gW = v' \) with \( v = v(\phi) \) being a superpotential).

Applying the unitary transformation \([14]\) to the Dirac field \( \Psi \), the fermion part becomes
\[ \mathcal{L}_f = \bar{\Psi}[iR(\partial_t + \partial_x) - g\sigma_3W_+(\phi) - ig\sigma_2W_-(\phi)]\Psi', \quad \Psi' = \Psi'^t R, \]
where we have used the decomposition \( W(\phi) = W_+(\phi) + W_-(\phi), \quad W_\pm(\phi(t, x)) = \pm W_\pm(\phi(t, -x)) \). With an additional unitary transformation \( \Psi' \rightarrow \psi = U'\Psi' \) generated by \( U' = \exp(i\frac{\pi}{4}\sigma_1) \), we get finally for the Lagrangian \([14]\)
\[ \mathcal{L} \rightarrow \mathcal{L} = \mathcal{L}_s + \bar{\psi}[iR(\partial_t + \partial_x) + g\sigma_3W_+(\phi) - ig\sigma_2W_-(\phi)]\psi, \quad \bar{\psi} = \psi^t R. \tag{4.16} \]

Suppose now that in the free case (\( g = 0 \)), the nonlinear scalar field theory has a static kink solution \( \phi_c \), satisfying the equation \( d^2\phi_c(x)/dx^2 = dV(\phi)/d\phi|_{\phi=\phi_c} \), and require that \( W(\phi) \) is such that \( W_+(\phi_c) = 0 \). To see that this requirement for \( W \) is not too tough, consider, as an example, \( \phi^4 \) model with \( V(\phi) = \lambda^2(\phi^2 - a^2)^2, \quad \lambda > 0, \quad a > 0 \), and sine-Gordon model with \( V(\phi) = \cos \phi \). These models have kink solutions of the form
\[ \phi^4: \ \phi_c(x) = a \tanh \left( \frac{a \lambda}{\sqrt{2}}(x - X_1) \right), \quad sG: \ \phi_c(x) = 4 \tan^{-1} \exp(x - X_2), \]
where $X_1$, $X_2$ are some constants (see, e.g., [18]). For $\phi$ close to corresponding kink solution, the choice $W(\phi) = W_-(\Phi(\phi))$, with arbitrary odd function $W_-$ and

$$\phi^4 : \Phi(\phi) = \frac{\sqrt{2}}{a\lambda} \tanh^{-1}(a^{-1}\phi) + X_1, \quad sG : \Phi(\phi) = \ln \tan(\phi/4) + X_2$$

satisfies the imposed requirement for $\phi^4$ and sine-Gordon models. In the case of $\phi^4$ model, kink solution with $X_1 = 0$ admits, obviously, a simple choice $W(\phi) = \phi$.

Then, for field $\psi$ propagating in this static background the equations of motion take the diagonal form for up, $\psi^+, \phi$, and down, $\psi^-, \phi$, components:

$$[iR(\partial_t + \partial_x) - \epsilon igW_-(\phi_c)]\psi^\epsilon = 0, \quad \epsilon = +, -.$$ 

Since $\phi_c$ is static, one can factorize $\psi^\epsilon(t, x)$ as $\psi^\epsilon(t, x) = \exp(-i\omega t)\psi^\epsilon(x)$, $\omega = \text{const}$. As a result, we arrive at the equation $H^\epsilon\psi^\epsilon(x) = \omega\psi^\epsilon(x)$ with $H^\epsilon = -i(\partial_x + \epsilon gW_-(\phi_c(x))R)$. The Hamiltonian $H^\epsilon$ has the form of the supercharge of the bosonized supersymmetric quantum mechanical system with the superpotential $W^\epsilon(x) = \epsilon gW_-(\phi_c(x))$:

$$\hat{H}^\epsilon = Q_1^\epsilon = -i\left(\frac{d}{dx} + W^\epsilon(x)R\right).$$

The bosonized supercharges $Q_1^\epsilon$ and $Q_2^\epsilon = iRQ_1^\epsilon$ and the operator $H^\epsilon = [-\frac{d^2}{dx^2} + W_-^2 - \frac{i}{dx}W_-R]$, form an $N = 1$ SUSY algebra: $\{Q_1^\epsilon, Q_2^\epsilon\} = 2\delta_{ij}H^\epsilon$, $[H^\epsilon, Q_1^\epsilon] = [H^\epsilon, Q_2^\epsilon] = 0$. Only one of the two equations $\frac{d}{dx}\psi^\epsilon(x) = -W^\epsilon(x)\psi^\epsilon(x)$, $\epsilon = +, -$, can have a normalized solution, and if so, $\psi(x)$ is an even function. In this case, the corresponding associated bosonized quantum mechanical system is in the phase of exact SUSY, and there is a fermionic zero mode solution (corresponding to $\omega = 0$) in the theory given by Lagrangian (1.16). On the other hand, if the equations $Q_1^\epsilon\psi^\epsilon = 0$, $\epsilon = +, -$, have no normalizable solutions, the corresponding quantum mechanical bosonized supersymmetric systems are in a phase of spontaneously broken supersymmetry, and the theory (1.16) has no fermionic zero modes.

5 Discussion and outlook

To conclude, let us indicate some problems that deserve further attention.

Our construction results in even and odd supersymmetry generators (2.11) given in terms of only bosonic operators, i.e. supersymmetry algebra is preserved here. This is in contrast to the hidden supersymmetry observed by Gozzi [19] at the level of the generating functional of Witten’s SUSYQM, where the information on the supersymmetry algebra disappears after integrating away the anticommuting variables. On the other hand, at the moment for us it is not clear how supersymmetry transformations should be understood in the bosonized SUSYQM. This is not clear either in the case of conventional SUSYQM if the fermionic operators are realized in matrix form, without turning to the holomorphic (Grassmann) representation. Some light on this problem could be shed by the observed possibility of interpreting the minimally bosonized SUSYQM as supersymmetry of identical fermions.

The minimally bosonized SUSYQM has a nonlocal nature brought about by the reflection operator in the unitary transformation (2.3) and in the bosonized supersymmetry generators.
In this sense, the construction is analogous to the bosonization of fermionic theories in $1 + 1$ dimensions \cite{20}, which, in turn, is a generalization of the one-dimensional Jordan-Wigner transformation for spin systems \cite{21} and its higher-dimensional extensions \cite{22}. An open problem we will address elsewhere is constructing the $(1 + 1)$-dimensional field analog of the minimally bosonized SUSYQM which could be realized in terms of only one scalar field in the simplest case.

According to the constructions realized in Section 2, two different bosonized supersymmetric systems given by the superpotentials $W_-(x)$ and $-W(x)$ together form a system, unitary equivalent to one conventional SUSYQM system with the superpotential $W_-(x)$ (or $-W_-(x)$). On the other hand, it is an open question what is the conventional supersymmetric system being equivalent to one bosonized system with superpotential $W_-(x)$. In the simplest case, the bosonized supersymmetric system with $W_-(x) = x$ is equivalent to the superoscillator with $W(x) = \pm \sqrt{2}x$, where the coefficient $\sqrt{2}$ accounts for the difference in the energy spectra $E_n$ and $E^+_n$ given by Eqs. (2.4) and (2.5). However, it is not obvious which is the conventional supersymmetric system in the phase of spontaneously broken supersymmetry possessing the same spectrum $E_n$ in (2.3), as the bosonized supersymmetric system with $W_-(x) = -x$.

We saw that the superpotential $W_-(x) = x - \frac{\nu}{2}$ gives the isospectral family of the bosonized supersymmetric systems specified by the parameter $\nu > -1$ and containing the bosonized superoscillator system in the phase of exact supersymmetry ($\nu = 0$). It could be expected that bosonized SUSYQM should provide a universal recipe for constructing isospectral supersymmetric families. That is, given a bosonized supersymmetric system with the superpotential $W_-(x)$, one could produce other bosonized supersymmetric systems isospectral to the first one.

The deformed Heisenberg algebra with reflection (2.12) is closely related to parabosons and parafermions \cite{7,8}, and we saw that it emerges naturally in bosonized SUSYQM. We also showed that bosonized SUSYQM can be understood as the supersymmetry realized in a system of two identical fermions. Therefore, it would be interesting to investigate the relationship between the parabosonic and parafermionic systems on one hand, and the supersymmetric two-fermion systems on the other. Besides, the results of Section 3 seem to indicate that they could be generalized to relate $n$-particle spinless integrable models involving exchange operators \cite{5} with $n$-particle integrable systems of identical fermions.

Conventional supersymmetry plays an important role in the theory of $(1 + 1)$-dimensional integrable systems \cite{3,23}. It would be interesting to investigate the applications of bosonized SUSYQM to the theory of $(1 + 1)$-dimensional integrable systems on the half-line \cite{24}. One notes also that the $(1 + 1)$-dimensional linear differential equation with reflection has some formal analogies with the theory of a massless boson field on the half-line related, in turn, via bosonization to the massless fermion field on the half-line \cite{25}. Indeed, in both theories spatial translation and Lorentz invariance are broken, whereas the theories are invariant under time translations. Mixed boundary conditions for the boson field on the half line contain a parameter of dimension of mass and they are introduced adding to the action functional the mass boundary term. It is such a term, like mass term in our Lagrangian (4.7), that breaks Lorentz and translation invariance. An idea analogous to the introduction of a mass parameter to produce mixed boundary conditions was used earlier in ref. \cite{13} to transform $(1 + 1)$-dimensional massive scalar field theory into relativistic anyon field...
theory. Thus, a natural and attractive problem would be to quantize the theory given by Lagrangian (4.7) and investigate its relation to the field theories on the half-line, especially to the relativistic model of massive anyons [13]. It would be also interesting to answer the question of whether the higher dimensional generalizations of the (1 + 1)-dimensional linear differential equation with reflection can be obtained from the corresponding higher dimensional Dirac equation through a unitary transformation with reduction analogous to our construction of Section 4.

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