Diminishing inverse transfer and non-cascading dynamics in surface quasi-geostrophic turbulence

Chuong V. Tran

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

Abstract

The inverse transfer in two-dimensional turbulence governed by the surface quasi-geostrophic (SQG) equation is studied. The nonlinear transfer of this system conserves the two quadratic quantities $\Psi_1 = \langle |(-\Delta)^{1/4}\psi|^2 \rangle / 2$ and $\Psi_2 = \langle |(-\Delta)^{1/2}\psi|^2 \rangle / 2$ (kinetic energy), where $\psi$ is the streamfunction and $\langle \cdot \rangle$ denotes a spatial average. In the limit of infinite domain, the kinetic energy density $\Psi_2$ remains bounded. For power-law inverse-transfer region, the inverse flux of $\Psi_1$ diminishes as it proceeds toward sufficiently low wavenumbers, implying that no persistent inverse cascade of $\Psi_1$ is sustainable. The unrealizability of an inverse cascade of $\Psi_1$ implies that there is no direct cascade of $\Psi_2$. Hence, the dual-cascade picture which is widely believed to be realizable in two-dimensional Navier–Stokes turbulence does not apply to SQG turbulence. Numerical results supporting the theoretical predictions are presented.

Key words:
Surface quasi-geostrophic turbulence, Inverse transfer, Diminishing inverse flux
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1 Introduction

The motion of a three-dimensional stratified rotating fluid is characterized by the geostrophic balance between the Coriolis force and pressure gradient. The dynamics governed by the first order departure from this linear balance is known as quasi-geostrophic dynamics (see for example [4,5,20,22]), which can be described in terms of the (three-dimensional) geostrophic streamfunction $\psi(\mathbf{x}, t)$. The vertical dimension $z$ is usually taken to be semi-infinite and
doubly periodic conditions are usually imposed on the horizontal flow. Normally decay conditions are required as \( z \to \infty \). At the flat surface boundary \( z = 0 \), the vertical gradient of \( \psi(x, t) \) matches the temperature field \( T(x, t) \), i.e. \( T(x, t) |_{z=0} = \partial_z \psi(x, t) |_{z=0} \). For flows with zero potential vorticity, this surface temperature field can be identified with \((-\Delta)^{1/2} \psi\), where \( \Delta \) is the (horizontal) two-dimensional Laplacian. The conservation equation governing the advection of the active temperature field \((-\Delta)^{1/2} \psi\) by the surface flow \( u = (-\partial_y \psi, \partial_x \psi) \) is [2,13,20,21]

\[
\partial_t (-\Delta)^{1/2} \psi + J(\psi, (-\Delta)^{1/2} \psi) = 0, \tag{1}
\]

where \( J(\vartheta, \theta) = \partial_x \vartheta \partial_y \theta - \partial_x \theta \partial_y \vartheta \). Eq. (1) is known as the SQG equation.

In this paper, a forced-dissipative version of (1) is studied. A dissipative term of the form \( \mu \Delta \psi \), where \( \mu > 0 \), which results from Ekman pumping at the surface, is considered (cf. [6,24]). Since \((-\Delta)^{1/2} \psi\) is the advected quantity, this physical dissipation mechanism corresponds to the hypoviscous dissipation operator \( \mu (-\Delta)^{1/2} \). The dissipation coefficient \( \mu \) has the dimensions of velocity and is not vanishingly small in the atmospheric context. The system is assumed to be driven by a forcing \( f \). Thus, the forced-dissipative SQG equation can be written as

\[
\partial_t (-\Delta)^{1/2} \psi + J(\psi, (-\Delta)^{1/2} \psi) = \mu \Delta \psi + f. \tag{2}
\]

The Jacobian operator \( J(\cdot, \cdot) \) admits the identities

\[
\langle \phi J(\vartheta, \theta) \rangle = -\langle \vartheta J(\phi, \theta) \rangle = -\langle \theta J(\vartheta, \phi) \rangle, \tag{3}
\]

where \( \langle \cdot \rangle \) denotes the spatial average. As a consequence, the nonlinear term in (2) obeys the conservation laws

\[
\langle \psi J(\psi, (-\Delta)^{1/2} \psi) \rangle = \langle (-\Delta)^{1/2} \psi J(\psi, (-\Delta)^{1/2} \psi) \rangle = 0. \tag{4}
\]

It follows that the two quadratic quantities \( \Psi_1 = \langle |(-\Delta)^{1/4} \psi|^2 \rangle / 2 \) and \( \Psi_2 = \langle |(-\Delta)^{1/2} \psi|^2 \rangle / 2 \) (kinetic energy) are conserved by nonlinear transfer.

The simultaneous conservation of \( \Psi_1 \) and \( \Psi_2 \) by advective nonlinearities imposes strict constraints on the transfer (redistribution) of these quantities in wavenumber space. Let us consider the transfer of an amount \( \Psi_1 = \epsilon \), initially distributed around a given wavenumber \( s \), which corresponds to an initial \( \Psi_2 = s \epsilon \). Let \( \Psi_1(k) \) and \( \Psi_2(k) \) be the resulting redistributions of \( \Psi_1 = \epsilon \) and of \( \Psi_2 = s \epsilon \). Given arbitrary wavenumbers \( p < s \) and \( q > s \), one has
\[
\frac{1}{\epsilon} \int_{q}^{\infty} \Psi_1(k) \, dk \leq \frac{1}{q\epsilon} \int_{q}^{\infty} \Psi_2(k) \, dk \leq \frac{s}{q}, \tag{5}
\]
\[
\frac{1}{s\epsilon} \int_{0}^{p} \Psi_2(k) \, dk \leq \frac{p}{s\epsilon} \int_{0}^{p} \Psi_1(k) \, dk \leq \frac{p}{s}, \tag{6}
\]
where the two conservation laws and straightforward inequalities have been used. The left-hand side of (5) [(6)] is the fraction of \(\epsilon [s\epsilon] \) that gets transferred to wavenumbers \(k \geq q [k \leq p] \). This fraction is bounded from above by \(s/q [p/s] \), implying that no significant fraction of \(\epsilon [s\epsilon] \) can be transferred to wavenumbers \(k \gg s [k \ll s] \). This type of constraint on the nonlinear transfer of the invariants is a common feature in incompressible fluid systems in two dimensions. (Some familiar systems in this category are the Charney–Hasegawa–Mima equation [11,12] and the class of \(\alpha \) turbulence equations [21], which includes both the Navier–Stokes and the SQG equations.) The implication is that when the said initial sources spread out in wavenumber space, \(\Psi_1 [\Psi_2] \) is preferentially transferred toward lower [higher] wavenumbers. According to the classical theory of two-dimensional turbulence [1,14,15,17], which was originally formulated for high-Reynolds number Navier–Stokes fluids and subsequently thought to apply to two-dimensional incompressible fluids in general, this preferential transfer achieves the extreme limit by transferring virtually all \(\epsilon \) to \(k \ll s \) (inverse cascade) and virtually all \(s\epsilon \) to \(k \gg s \) (direct cascade). The transfer of the invariants in this manner is known as the dual cascade.\(^1\)

However, it is shown in this work that the preferential transfer of \(\Psi_1 \) and of \(\Psi_2 \) is not as dramatic as predicted but rather has a limited extent. More accurately, it is shown that an upper bound on the inverse flux of \(\Psi_1 \) across a low wavenumber \(\ell \) vanishes (uniformly in time) as \(\ell/s \to 0 \), where \(s \) is the characteristic forcing wavenumber, thereby ruling out the existence of a persistent inverse cascade of \(\Psi_1 \) toward the low wavenumbers. The unrealizability of an inverse cascade implies that there is no direct cascade of \(\Psi_2 \) [24]. The physical reasons behind this behaviour are that the kinetic energy density of SQG dynamics remains bounded in the limit of infinite domain, a consequence of the hypoviscous dissipation operator \(\mu(-\Delta)^{1/2} \), and that the inverse flux of \(\Psi_1 \) across \(\ell \) is proportional to the energy content of the wavenumber region \(k \leq \ell \). These two properties of SQG turbulence, when combined, imply that the inverse flux of \(\Psi_1 \) becomes smaller for progressively lower \(\ell \), thus ruling out the existence of a persistent inverse cascade.

In the next section some preliminary estimates, which are employed in the

\(^1\) For some recent discussion on the possibility of a dual cascade in various two-dimensional systems, including the Navier–Stokes and SQG equations, see [24,25,27,29].
derivations of the main results in Section 3, are presented. Section 4 examines the implications of the results in Section 3 for the long-time dynamics and spectral distribution of kinetic energy. Section 5 presents some numerical results in support of the theoretical prediction of no cascades in SQG dynamics. The paper ends with some discussion in the final section.

2 Preliminary estimates

This section presents a simple analytic inequality and reviews the boundedness of the energy density [24]. The former applies to bounded domains only, while the latter is valid for both bounded and unbounded cases.

For a doubly periodic domain $L \times L$, the (complex) Fourier representation of $\psi$ is

$$\psi(x) = \sum_k \exp\{i k \cdot x\} \hat{\psi}(k). \quad (7)$$

Here $k = k_0(n, m)$, where $k_0 = 2\pi/L$ is the lowest wavenumber and $n$ and $m$ integers not simultaneously zero. For a given wavenumber $\ell$, let $\psi^<$ and $\psi^>$ denote, respectively, the components of $\psi$ spectrally supported by the disk $d = \{k|k < \ell\}$ and its complement $D = \{k|k \geq \ell\}$, i.e.

$$\psi^< = \sum_{k \in d} \exp\{i k \cdot x\} \hat{\psi}(k), \quad \psi^> = \sum_{k \in D} \exp\{i k \cdot x\} \hat{\psi}(k). \quad (8)$$

For the lower-wavenumber component $\psi^<$, the following inequality holds:

$$\sup_x |\nabla \psi^<| \leq \sum_{k \in d} k|\hat{\psi}(k)| \leq \left( \sum_{k \in d} 1 \right)^{1/2} \left( \sum_{k \in d} k^2|\hat{\psi}(k)|^2 \right)^{1/2} = c\frac{\ell}{k_0^2} \Psi_1^{1/2} \quad (9)$$

where $c$ is an absolute constant of order unity and $\Psi_1^{1/2} = \langle|\nabla \psi^<|^2\rangle/2$ the large-scale energy density associated with the wavenumbers $k < \ell$. In (9) the Cauchy–Schwarz inequality is used in the second step, and the sum $\sum_{k \in d} 1 \approx \ell^2/k_0^2$ represents the number of wavevectors in $d$.

On multiplying (2) by $\psi$ and $(-\Delta)^{1/2}\psi$ and taking the spatial averages of the resulting equations over the domain, noting from the conservation laws that the nonlinear terms identically vanish, one obtains evolution equations for $\Psi_1$ and $\Psi_2$. 

4
\[
\frac{d}{dt} \Psi_1 = -2\mu \Psi_2 + \langle f \psi \rangle, \\
\frac{d}{dt} \Psi_2 = -2\mu \Psi_3 + \langle f(-\Delta)^{1/2} \psi \rangle,
\]

(10)  

(11)

where \( \Psi_3 = \langle |(-\Delta)^{3/4} \psi|^2 \rangle / 2 \). Using the Cauchy–Schwarz and Young inequalities, one obtains the upper bounds on the injection terms in (10) and (11):

\[
\langle f \psi \rangle \leq \langle |(-\Delta)^{1/2} \psi|^2 \rangle^{1/2} \langle |(-\Delta)^{-1/2} f|^2 \rangle^{1/2} \leq \mu \Psi_2 + \mu^{-1} F_{-2}, \\
\langle f(-\Delta)^{1/2} \psi \rangle \leq \langle |(-\Delta)^{3/4} \psi|^2 \rangle^{1/2} \langle |(-\Delta)^{-1/4} f|^2 \rangle^{1/2} \leq \mu \Psi_3 + \mu^{-1} F_{-1}, 
\]

(12)

where the norms of \( f \) are defined by \( F_{-1} = \langle |(-\Delta)^{-1/4} f|^2 \rangle / 2 \) and \( F_{-2} = \langle |(-\Delta)^{-1/2} f|^2 \rangle / 2 \). These norms are assumed to be finite. Substituting (12) in (10) and (11) yields

\[
\frac{d}{dt} \Psi_1 \leq -\mu \Psi_2 + \mu^{-1} F_{-2}, \\
\frac{d}{dt} \Psi_2 \leq -\mu \Psi_3 + \mu^{-1} F_{-1}.
\]

(13)  

(14)

Let the overline denote the asymptotic average (existence is assumed). One can deduce the following upper bounds for \( \overline{\Psi}_2 \) and \( \overline{\Psi}_3 \):

\[
\overline{\Psi}_2 \leq \mu^{-2} F_{-2}, \\
\overline{\Psi}_3 \leq \mu^{-2} F_{-1}.
\]

(15)  

(16)

Ineqs. (15) and (16) apply to both the unbounded and bounded domains. In the study of turbulent transfer, the spectral support of \( f \) is required to be in the intermediate wavenumber region, so as to render inverse- and direct-transfer ranges free of sources. The limit of infinite domain \( (k_0 \to 0) \), for which the classical theory is formulated, is taken in a straightforward manner: in the limit \( L \to \infty \), the dissipation coefficient \( \mu \), the injection densities, and the forcing characteristic length scale (cf. [30]) are held fixed. In this limit both \( F_{-1} \) and \( F_{-2} \) are bounded if \( F = \langle |f|^2 \rangle / 2 \) is bounded since \( F_{-2} \leq k_m^{-1} F_{-1} \leq k_m^{-2} F \), where \( k_m \) is the minimum wavenumber of the spectral support of \( f \). A persistent inverse cascade of \( \Psi_1 \) toward ever-lower wavenumbers necessarily requires \( d\Psi_1/dt > 0 \), which, by (13), implies \( \Psi_2 < \mu^{-2} F_{-2} \). Therefore, since the main concern of the subsequent analyses is the realizability of an inverse cascade, for the rest of this paper the (instantaneous) energy \( \Psi_2 \) is assumed to be bounded in the limit \( k_0 \to 0 \).
3 Diminishing inverse transfer

This section reports the main result of this paper. In the subsequent analyses, rigorous estimates are supplemented by the usual assumption of power-law scaling for the energy spectrum. An inverse-transfer range $\Psi_2(k) = ak^{-\alpha}$, for $k < s$, is assumed. Here $s$ may be taken to be the minimum wavenumber of the spectral support of the forcing. An upper bound for the total transfer of $\Psi_1$ into the low-wavenumber region $[k_0, \ell \ll s]$, i.e. the inverse flux of $\Psi_1$ across $\ell$, is derived in terms of the kinetic energy $\Psi_2$ and of the large-scale kinetic energy $\Psi_2^\geq$. It is shown that this flux diminishes in the limit $\ell/s \to 0$. The implication is that no persistent inverse cascade of $\Psi_1$ is realizable.

The evolution of $\Psi_1^{\geq} = \langle \left| (-\Delta)^{1/4} \psi^{\leq} \right|^2 \rangle / 2$ is governed by

$$\frac{d}{dt} \Psi_1^{\geq} = -\langle \psi<J(\psi, (-\Delta)^{1/2} \psi) \rangle - 2\mu \Psi_2^{\leq}$$

$$= -\langle \psi<J(\psi^>, (-\Delta)^{1/2} \psi) \rangle - 2\mu \Psi_2^{\leq}$$

$$= \langle (-\Delta)^{1/2} \psi J(\psi^>, \psi^<) \rangle - 2\mu \Psi_2^{\leq}$$

$$\leq \langle (-\Delta)^{1/2} \psi ||\nabla \psi^>|| \nabla \psi^<|| \rangle - 2\mu \Psi_2^{\leq}, \quad (17)$$

where (3) has been used in both the second and third steps. The final triple-product term represents an upper bound for the inverse flux of $\Psi_1$ across the wavenumber $\ell$ that drives the large-scale dynamics.

In the limit of infinite domain, no finite rigorous estimates of the nonlinear term in (17) are available. Nevertheless, since the norm $|\nabla \psi^<|$ represents a measure of the large-scale fluid velocity (associated with $k < \ell$), it could heuristically be identified with $\Psi_2^{1/2}$. Hence, a rough estimate of $|\nabla \psi^<|$ would be $|\nabla \psi^<| \approx 2^{-1/2} c \langle |\nabla \psi^<|^2 \rangle^{1/2} = c' \Psi_2^{1/2}$, where $c'$ is a constant. Substituting this estimate into (17) yields

$$\frac{d}{dt} \Psi_1^{\leq} \leq 2c' \Psi_2^{1/2} \Psi_2^{1/2} \Psi_2^{1/2} - 2\mu \Psi_2^{\leq} \leq 2c' \Psi_2^{1/2} \Psi_2^{1/2} - 2\mu \Psi_2^{\leq}, \quad (18)$$

where $\Psi_2^{\leq} = \langle |\nabla \psi^<|^2 \rangle / 2$ is the small-scale energy, associated with $k \geq \ell$. Since $\Psi_2$ is bounded and since its spectrum $\Psi_2(k)$ is assumed to obey power-law scaling in the inverse-transfer region, $\Psi_2^{\leq}$ necessarily diminishes as $\ell \to 0$. Therefore, the quantity $2c' \Psi_2^{1/2} \Psi_2$, which bounds the inverse flux of $\Psi_1$ across $\ell$, can become arbitrarily small for sufficiently low $\ell$. Thus, no persistent inverse cascade of $\Psi_1$ is realizable provided that the foregoing heuristic estimate of $|\nabla \psi^<|$ can be assumed.

A rigorous version of the above calculation can be deduced for the bounded
case. By applying (9) to (17) one obtains

$$\frac{d}{dt} \Psi_1^\leq \leq \sup_x |\nabla \psi^\leq| \langle|(-\Delta)^{1/2}\psi||\nabla \psi^\leq\rangle - 2\mu \Psi_2^\leq$$

(19)

$$\leq 2c \frac{\ell}{k_0} \Psi_2^{1/2}\Psi_2^{1/2}\Psi_2^{1/2} - 2\mu \Psi_2^\leq \leq 2c \frac{\ell}{k_0} \left(\frac{\Psi_2^\leq}{\Psi_2}\right)^{1/2} \Psi_2^{3/2} - 2\mu \Psi_2^\leq.$$

The difference between (18) and (19) is the presence of the ratio $\ell/k_0$ in the latter. Hence, the arguments in the preceding paragraph go through without change if the ratio $\ell/k_0$ is either held fixed or allowed to grow at some certain rate as $\ell \to 0$. This condition can be stated more explicitly by making use of the assumed spectrum $\Psi_2(k) = a k^{-\alpha}$ for $k < s$. In the limit $k_0 \to 0$, the large-scale energy $\Psi_2^\leq$ and the total energy $\Psi_2$ can be estimated as

$$\Psi_2^\leq = a \int_0^\ell k^{-\alpha} dk = \frac{a}{1-\alpha} \ell^{1-\alpha},$$

(20)

$$\Psi_2 \geq a \int_0^s k^{-\alpha} dk = \frac{a}{1-\alpha} s^{1-\alpha},$$

(21)

where $\alpha < 1$, in accord with the boundedness of energy. It follows that

$$2c \frac{\ell}{k_0} \left(\frac{\Psi_2^\leq}{\Psi_2}\right)^{1/2} \Psi_2^{3/2} \leq 2c \frac{\ell}{k_0} \left(\frac{\ell}{s}\right)^{(1-\alpha)/2} \Psi_2^{3/2}.$$

(22)

Given a bounded $\Psi_2$ and a ratio $\ell/k_0$ that does not grow as rapidly as $(s/\ell)^{(1-\alpha)/2}$ as $\ell$ becomes small, this upper bound on the inverse flux of $\Psi_1$ clearly diminishes as $\ell/s \to 0$. Hence, for a sufficiently wide inverse-transfer range $[\ell, s]$, the advective nonlinearities of SQG turbulence are incapable of transferring a significant amount of the injection of $\Psi_1$ to the low-wavenumber region $[k_0, \ell]$.

It is interesting to generalize the above calculations to other models of incompressible fluid turbulence in two dimensions. Pierrehumbert et al. [21] propose to consider the so-called $\alpha$-turbulence models, for which the unforced and inviscid dynamics are governed by

$$\partial_t (-\Delta)^{\alpha/2}\psi + J(\psi, (-\Delta)^{\alpha/2}\psi) = 0,$$

(23)

where $\alpha$ is a positive number. The two invariants of this system are $\Psi_\alpha = \langle|(-\Delta)^{\alpha/4}\psi|^2\rangle/2$ and $\Psi_{2\alpha} = \langle|(-\Delta)^{\alpha/2}\psi|^2\rangle/2$. The inverse transfer of $\Psi_\alpha$ across $\ell$ can be estimated as
\[
\langle \psi^< J(\psi, (-\Delta)^{\alpha/2} \psi) \rangle = \langle ( -\Delta)^{\alpha/2} \psi^< J(\psi^>, \psi^<) \rangle \\
\leq \langle ( -\Delta)^{\alpha/2} \psi^< |\nabla \psi^<| \rangle \\
\leq \sup_x |\nabla \psi^<| \langle ( -\Delta)^{\alpha/2} \psi^< |\nabla \psi^<| \rangle \\
\leq 2c \frac{\ell}{k_0} \Psi_2^{1/2} \Psi_{2\alpha}^{1/2} \Psi_2^{1/2}.
\]

(24)

There are cases for which \( \Psi_2^< \to 0 \) in the limit \( \ell \to 0 \), leading to the diminishing of the inverse flux across \( \ell \). For example, if the direct-cascading candidate \( \Psi_{2\alpha} \) remains bounded \(^{2}\) in the limit \( k_0 \to 0 \) and if \( \alpha < 1 \), then \( \Psi_2^< \leq \ell^{2-2\alpha} \Psi_{2\alpha}^< = \ell^{2-2\alpha} \langle ( -\Delta)^{\alpha/2} \psi^< \rangle^2 / 2 \), which certainly vanishes as \( \ell \to 0 \) because \( \Psi_{2\alpha}^< (\leq \Psi_{2\alpha}) \) is bounded. On the other hand, \( \Psi_2^> \) converges toward the low wavenumbers for the same reason. There remains the modest assumption that \( \Psi_2^> \) also converges toward the high wavenumbers. Given all these, the preceding arguments of no inverse cascade go through without change.

**Remark 1.** For technical reasons, it is difficult to generalize the analyses in this section to the truly unbounded case (i.e. the case \( \ell/k_0 \to \infty \), regardless of the ratio \( \ell/s \)). The main difficulty lies in the intrinsic domain-size dependence of the nonlinear term, thereby making its rigorous estimates, such as the one in (19) and the subsequent estimate (22), diverge in the limit \( k_0 \to 0 \).

**Remark 2.** If the viscous operator \( \mu(-\Delta)^{1/2} \) is replaced by one of higher degree, as is often done in numerical simulations, then the boundedness of energy in the limit \( k_0 \to 0 \) cannot be guaranteed. Nevertheless, if boundedness of energy is assumed, then the above calculations go through without change since they do not refer at all to the dissipation mechanism.

**Remark 3.** The above procedure, when applied to the nonlinear term of the two-dimensional Navier–Stokes equations, yields an estimate of the inverse energy flux that is energy-dependent. This behaviour makes it challenging to estimate the inverse energy transfer since the energy is supposed to grow with progressively wider inverse-transfer range. Nevertheless, it can be shown that for the Kolmogorov–Kraichnan \( k^{-5/3} \) energy spectrum the energy that gets transferred onto \( k_0 \) can be bounded from above by a constant independent of the width of the inverse-cascading range, i.e. independent of \( k_0 \). This result, together with related issues such as the Kolmogorov constant and the Kraichnan conjecture of energy condensation at \( k_0 \), is the subject of a separate study.

**Remark 4.** Recently, Eyink [10] raised the possibility of energy condensation in bounded two-dimensional Navier–Stokes turbulence, a conjecture by Kraichnan [14], and argued that this possibility cannot be ruled out in some cases.

\(^2\) This is certainly the case for SQG (\( \alpha = 1 \)) and Navier–Stokes (\( \alpha = 2 \)) turbulence, for which the energy and enstrophy, respectively, remain bounded in the limit of infinite domain.
recent theoretical results. Kraichnan [14], in an attempt to apply his dual-
cascade hypothesis (initially formulated for unbounded fluids) to bounded tur-
bulence, predicts that the inverse energy cascade, upon reaching $k_0$, deposits
energy to this wavenumber, and that this process continues until growth of
energy at $k_0$ is limited by its own dissipation, resulting in what may be termed
an “energy condensate”. The energy dissipation by this condensate alone is
supposed to account for virtually all the energy injection, so that the energy
condensate is also an enstrophy condensate, although the latter is of a lesser
degree. The realization of such a “singular” energy and enstrophy concentra-
tion at $k_0$ (or around $k_0$) is required to maintain the proposed dual cascade
in the bounded case. Recent numerical results seemed to suggest otherwise:
as the turbulence approaches a steady state, a $k^{-3}$ energy spectrum forms at
the large scales [3,27]. Nevertheless, the a priori exclusion of the Kraichnan
scenario by some recent theoretical studies, such as Constantin et al. [7], Tran
and Shepherd [29], and Kuksin [16], does not seem to be fully justified. The
SQG dynamics allows for no possibility of such a condensate.

4 Approach to steady dynamics and spectral distribution of energy

This section features some physical interpretations of the results derived in the
preceding section. The non-cascading dynamics of SQG turbulence is discussed
together with a review of the constraint on the spectral distribution of energy
derived by Tran [24].

It is customary in the study of 2D turbulence to consider the scenario in which
the fluid is driven around a wavenumber $s$ by steady injections $\langle f\psi \rangle = \epsilon$ and
$\langle f(-\Delta)^{1/2}\psi \rangle = s\epsilon$. The result in the preceding section implies that an inverse
transfer of a nonzero fraction of $\epsilon$ to sufficiently low wavenumbers requires that
the energy spectrum in the inverse-transfer region be no shallower than $a k^{-1}$.
But then the energy would grow at least as rapidly as $a \ln(s/\ell)$ as $s/\ell \to \infty$,
eventually leading to a balance between the injection $\epsilon$ and the dissipation
$2\mu \Psi_2$. This result suggests two plausible routes to the long-time high-Reynolds
number dynamics (for some suitably defined Reynolds number). First, if a $k^{-1}$
inverse-cascading range is realized, the inverse flux decreases as it proceeds
toward lower wavenumbers since the dissipation of $\Psi_1$, given by $2\mu \Psi_2$, grows
logarithmically. $\Psi_1$ eventually becomes steady, as described above. Second,
suppose that an inverse-cascading range with energy spectrum shallower than
$k^{-1}$ is realized. The inverse flux decreases for the reason discussed in the
preceding section as it proceeds toward sufficiently low wavenumbers. As a
result, growth of $\Psi_1$ (and of $\Psi_2$) occurs alongside the existing inverse-transfer
range. Eventually the dissipation $2\mu \Psi_2$ reaches the injection $\epsilon$ and $\Psi_1$ becomes

\footnote{Incidentally, dimensional analyses predict a $k^{-1}$ inverse-cascading range.}
steady. If power-law scaling is maintained, the slope of the energy spectrum in the inverse-transfer range approaches $-1$. The low-wavenumber end of this range $\ell$ can be calculated from $2\mu a \ln(s/\ell) = \epsilon$.

Near steady dynamics can be achieved after the inverse flux of $\Psi_1$ across $\ell$ becomes sufficiently less than its dissipation $2\mu \Psi_2$, which should then be comparable to $\epsilon$. Hence, by replacing $\Psi_2$ in the expression on the right-hand side of (22) by $\epsilon/2\mu$ and requiring that the resulting expression be no larger than $\epsilon$, one obtains a condition for this near steady picture:

$$\left( \frac{\ell}{k_0} \right)^2 \left( \frac{\ell}{s} \right)^{1-\alpha} \leq \frac{\mu^3}{\epsilon},$$

(25)

where a constant factor of order unity has been dropped. For fixed (per-unit-area) injection rate $\epsilon$, forcing wavenumber $s$, and dissipation coefficient $\mu$, no significant fraction of $\epsilon$ can be transferred to wavenumbers $k \leq \ell$, where $\ell$ satisfies (25). $\Psi_1$ necessarily becomes near steady, with the low-wavenumber region $[k_0, \ell]$ at best weakly excited.

For steady dynamics the balances $2\mu \Psi_2 = \epsilon$ and $2\mu \Psi_3 = s \epsilon$ are achieved. It follows that $s \Psi_2 = \Psi_3$, or in terms of the energy spectrum $\Psi_2(k)$,

$$\int_{k_0}^{s} (s-k) \Psi_2(k) \, dk = 0.$$  

(26)

This equation can be used to estimate the slopes of the energy spectrum if power-law scaling is assumed for both the inverse- and direct-transfer ranges. For this purpose, let us consider the following spectrum

$$\Psi_2(k) = \begin{cases} ak^{-\alpha} & \text{if } \ell < k < s, \\ bk^{-\beta} & \text{if } s < k < k_\nu, \end{cases} \quad as^{-\alpha} = bs^{-\beta},$$

(27)

where $a$, $b$, $\alpha$, $\beta$ are constants, and $k_\nu$ is the highest wavenumber in the range $k^{-\beta}$, beyond which the spectrum is supposed to be steeper than $k^{-\beta}$. By substituting this spectrum into (26) and making the respective substitutions $\kappa = k/s$ for $k < s$, and $\kappa = s/k$ for $k > s$, one obtains [24]

$$\int_{\ell/s}^{1} (1-\kappa)^{-\alpha} \, d\kappa \approx \int_{s/k_\nu}^{1} (1-\kappa)^{\beta-3} \, d\kappa,$$

(28)

where the contribution from both $k \leq \ell$ and $k \geq k_\nu$ has been dropped. It follows that if $\ell/s \geq s/k_\nu$, then $-\alpha \leq \beta - 3$. Hence, the constraint
\[ \alpha + \beta \geq 3 \]  

(29)

holds. Since \( \alpha \leq 1 \), \( \beta \) satisfies \( \beta \geq 2 \), meaning that for \( k > s \), the spectrum \( \Psi_3(k) \) of \( \Psi_3 \) is no shallower than \( k^{-1} \). Hence, the energy dissipation cannot occur mainly at \( k \gg s \). Thus there is no direct cascade, a dynamical behaviour consistent with no inverse cascade of \( \Psi_1 \).

**Remark 5.** Although no persistent inverse flux is possible, in the limit \( k_0 \to 0 \), \( \Psi_1 \) could become unbounded, growing at a rate that fluctuates about zero and has a vanishingly small positive average. The unboundedness of \( \Psi_1 \) requires that the energy spectrum of the inverse-transfer range have a non-positive slope, so that the spectrum \( \Psi_1(k) \) is at least as steep as \( k^{-1} \). For a given set of physical parameters, the issue of whether or not \( \Psi_1 \) becomes divergent (in the limits \( k_0 \to 0 \) and \( t \to \infty \)) is interesting but is beyond the scope of the present work.

**Remark 6.** In some sense SQG turbulence is relatively “simple”. Given spectrally localized steady injections about the forcing wavenumber \( s \), the dynamics should eventually become non-cascading. For a \( k^{-1} \) transient inverse-cascading range, the approach to steady dynamics is rather slow for two reasons. First, because \( \Psi_1(k) \propto k^{-2} \), the low-wavenumber end of the inverse-cascading range \( \ell \) proceeds relatively slowly toward lower wavenumbers, even during the early stages for which the inverse cascade is relatively strong \( (d\Psi_1/dt \approx \epsilon) \). Second, growth of energy is only logarithmic in \( \ell^{-1} \), giving rise to rather slow growth of the dissipation of \( \Psi_1 \) toward the lower wavenumbers.

**Remark 7.** Dimensional analyses, without references to any particular dissipation mechanisms, predict a \( k^{-1} \) inverse-transfer range and a \( k^{-5/3} \) direct-transfer range. The former is consistent with an inverse cascade of \( \Psi_1 \), but the latter allows for virtually no energy to get transferred to the small scales.\(^4\) Before a viscous dissipation mechanism is taken into consideration, a \( k^{-5/3} \) spectrum means that virtually no energy gets transferred to the high wavenumbers. In the presence of a viscous dissipation operator of the form \( \propto (-\Delta)^{\delta} \), for \( 0 \leq \delta \leq 1/3 \), instead of the natural dissipation operator \( \mu(-\Delta)^{1/2} \), a \( k^{-5/3} \) direct-transfer range means that the spectral energy dissipation scales as \( k^{-5/3+2\delta} \), which is no shallower than \( k^{-1} \), thereby allowing for virtually no energy to be dissipated at its high-wavenumber end.

**Remark 8.** The simultaneous conservation of \( \Psi_1 \) and \( \Psi_2 \) leads to an increase

\(^4\) These predictions seem to be mutually inconsistent since the inverse-transfer of \( \Psi_1 \) via a \( k^{-1} \) spectrum is incompatible with the “frozen-in” of energy due to the \( k^{-5/3} \) direct-transfer range. This picture is quite contrary to that of two-dimensional Navier–Stokes turbulence, for which a \( k^{-3} \) enstrophy-transfer range means that virtually all enstrophy gets transferred away from the forcing region, even before dissipative effects are considered.
in $\Psi_4$ (enstrophy) when an initial spectral peak spreads out in wavenumber space [24]. This explains the observed formation of strong “fronts” in numerical simulations of SQG turbulence [8,9,18], even in the absence of a direct energy cascade as discussed above.

5 Numerical results

This section reports results from numerical simulations that illustrate the diminishing inverse transfer and no cascades of SQG dynamics. Numerical studies in the literature have thus far failed to recognize these properties of SQG turbulence (see for example [19,21,23,26,28]). Tran and Bowman [28], however, notice that $\Psi_1$ is “reluctant” to cascade to the large scales, even when the natural dissipation operator $\mu(-\Delta)^{1/2}$ is replaced by ones with higher degrees, allowing for relatively weaker dissipation at the large scales.

Equation (2) is simulated in a doubly periodic square of side $2\pi$, where the modal forcing $\hat{f}(k)$ is nonzero only for those wavevectors $k$ having magnitudes lying in the interval $K = [9.5, 10.5]$:

$$
\hat{f}(k) = s\epsilon \frac{\hat{\psi}(k)}{N^2 \sum_{|p|=k} |\hat{\psi}(p)|^2},
$$

(30)

where $s\epsilon = 1$ is the constant energy injection rate and $N$ the number of discrete wavenumbers in $K$. The wavenumber $s \approx 10$ is defined such that $s^{-1}$ is the mean of $k^{-1}$ over $K$. The (constant) injection rate of $\Psi_1$ is $\epsilon \approx 0.1$. Dealiased $683^2$ and $1365^2$ pseudospectral simulations ($1024^2$ and $2048^2$ total modes) were performed. For Navier–Stokes turbulence, these resolutions are sufficient to simulate an inverse cascade that carries about a quarter of the energy injection to the large scales via a discernible $k^{-5/3}$ range [24]. For SQG turbulence, even the higher resolution turns out to be insufficient to simulate a noticeable transient inverse cascade. Two dissipation coefficients were used: $\mu = 0.05$ and $\mu = 0.025$. The lower and higher resolutions were used for the stronger and weaker dissipation, respectively. Both simulations were initialized with the spectrum $\Psi_2(k) = 10^{-2} \pi k/(100 + k^2)$.

Figure 1 shows the time-averaged (between $t = 19.7$ and $t = 20.3$)\(^5\) kinetic energy spectrum for the case $\mu = 0.05$. The average energy is $\Psi_2 = 0.99$. The dissipation of $\Psi_1$, averaged for the same period, is $2\mu\Psi_2 = 0.099$, which

\(^5\) The dissipation time at the forcing wavenumber $s \approx 10$ is $(2s\mu)^{-1} \approx 1$, so that the time $t \approx 20$ is sufficiently long for the evolution of $\Psi_2$. In fact, it was observed that both $\Psi_1$ and $\Psi_2$ became steady at $t \approx 12$. 

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Fig. 1. The time-averaged steady-state energy spectrum $\Psi_2(k)$ vs. $k$ for the dissipation coefficient $\mu = 0.05$. The average energy is $\Psi_2 = 0.99$, implying that the dissipation of $\Psi_1$, averaged in the same period, is 0.099. This amounts to virtually all of the injection rate $\epsilon \approx 0.1$. Hence, no inverse cascade of $\Psi_1$ exists and both $\Psi_1$ and $\Psi_2$ are steady.

amounts to virtually all the injection rate $\approx 0.1$. The energy was observed to increase monotonically from $t = 0$ to $t \approx 12$, by which time $d\Psi_1/dt \approx 0$; no significant inverse transfer was observed throughout the period. Instead, growth of energy occurs mainly around the forcing region. The steady spectrum is relatively shallow in the low-wavenumber range (see Fig. 1), meaning that the lowest wavenumbers are virtually unexcited. The small-scale energy spectrum scales as $k^{-3.6}$, so that the spectrum $\Psi_3(k)$ of the energy dissipation agent $\Psi_3$ scales as $k^{-2.6}$. This scaling means that the energy dissipation occurs mainly around the forcing region, consistent with the weak inverse transfer.

A somewhat stronger transient inverse transfer was observed for the case $\mu = 0.025$. Fig. 2 shows a near steady kinetic energy spectrum averaged between $t = 15.1$ and $t = 15.6$. The average energy is $\Psi_2 = 1.96$. The dissipation of $\Psi_1$ averaged for the same period is $2\mu\Psi_2 = 0.098$, which amounts to virtually all of the injection rate $\approx 0.1$. The large-scale spectrum is better “filled up” than that of the previous case, due to a stronger inverse transfer during the transient phase. Nevertheless, no significant fraction of $\epsilon$ reaches the lowest wavenumbers. The energy dissipation occurs around the forcing region, as is evident from the steep small-scale spectrum (see Fig. 2).

It is expected that for higher resolutions (so that simulations with smaller $\mu$ are possible), transient inverse fluxes can become more noticeable and steeper inverse-transfer ranges can be realized. Given limited resolutions, Tran and Bowman [28] use high-degree dissipation operators $\propto \Delta$ and $\propto (-\Delta)^{3/2}$. In
both cases, weak inverse cascades are observed, but the transient energy spectra of the inverse-transfer region are considerably shallower than $k^{-1}$. These spectra cannot support a persistent inverse cascade, as argued in Section 3. However, an interesting possibility arises. Due to both the inability of the nonlinear transfer to sustain a constant (wavenumber-independent) inverse flux and $d\Psi_1/dt > 0$, growth of $\Psi_1$ ought to occur within the existing inverse-transfer range, thereby causing this range to become steeper. This process may continue until the inverse-transfer range eventually exceeds the $k^{-1}$ threshold. Because of the high degrees of viscosity, such a slope steepening causes insignificant increase in the dissipation of $\Psi_1$. As a result, a positive growth rate $d\Psi_1/dt$ could be maintained for a steeper-than-$k^{-1}$ inverse-transfer range, which might then be able to support a persistent inverse cascade.

6 Conclusion

In this paper, the advective transfer of SQG turbulence is studied. The main result obtained is an upper bound for the inverse flux of the inverse-cascading candidate $\Psi_1$. This upper bound diminishes as the flux proceeds toward sufficiently low wavenumbers, thereby ruling out the existence of a persistent inverse cascade. The unrealizability of an inverse cascade entails that there is no direct cascade. This is the first rigorous example of the dynamics of incompressible fluids in two dimensions that exhibits no cascades.
There are two essential features of SQG turbulence that facilitate the proof of non-cascading dynamics. First, the inverse flux of $\Psi_1$ across a low wavenumber $\ell$ can be uniformly (in time) estimated in terms of the kinetic energy $\Psi_2$ and of $\Psi_2^\varphi$, the large-scale energy associated with the low-wavenumber region $k < \ell$. Second, the energy $\Psi_2$ remains bounded in the limit of infinite domain. The former is an intrinsic property of the advective nonlinear term, and the latter is due to the hypoviscous nature of the dissipation of SQG dynamics. The boundedness of energy means that if the turbulence is driven at some fixed energy density rate around some fixed wavenumber $s$, virtually no energy can be acquired by wavenumbers $k \ll s$. This means that in the limit $\ell/s \to 0$, $\Psi^\varphi_2$ can become arbitrarily small, leading to an arbitrarily small inverse flux of $\Psi_1$ across $\ell$. Hence, for sufficiently low wavenumber $\ell$ no significant inverse transfer of $\Psi_1$ across $\ell$ is possible.

Numerical simulations of SQG turbulence were performed. The results show no significant inverse transfer of $\Psi_1$ to the large scales, thereby lending strong support to the prediction of no cascades. The large-scale energy spectra are shallower than $k^{-1}$, and the small-scale spectra are steeper than $k^{-2}$, consistent with the theoretical prediction.

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