Comment on the paper of Leonard Parker and Yang Zhang

“Cosmological perturbations of a relativistic condensate”

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Abstract

The “standard” inflationary formula for density perturbations is often being used in the literature and, in particular, it has been used in the paper of Parker and Zhang. Among other things, this formula suggests that the contribution of density perturbations to the microwave background anisotropies is much larger than the contribution of gravitational waves in the limit of the de Sitter inflation. It is shown that this formula is incorrect.

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In a recent paper, Parker and Zhang [1] consider cosmological perturbations that can be possibly produced in the early Universe. This is an interesting problem having important observational implications. The paper of Parker and Zhang is transparent and honest, in the sense that the authors clearly identify what they derive and what they use from the previous literature. The final numerical estimates of the paper [1] rely entirely on a formula which the authors take from the inflationary literature. This formula, see Eqs. (22), (43), relates the amplitude of perturbations today with the amplitude of perturbations during the inflationary stage. The formula suggests an enormous increase of the amplitude, if the equation of state at the inflationary phase has happened to be sufficiently close to the de Sitter one. The point of my comment is that this formula is incorrect, as we will see below, and the results based on this formula cannot be trusted.

It appears that the use of this formula is connected to a certain disorientation as for the mechanism responsible for the production of cosmological perturbations. In the introductory part of their paper, Parker and Zhang associate the perturbations with the existence of a particle horizon in de Sitter space. If it is the particle horizon that is responsible for the generation of perturbations, then the question arises why the perturbations cannot be generated by the particle horizon of the radiation-dominated Universe. Later, Parker and Zhang associate the perturbations with the “horizon Hawking temperature” (apparently, they refer to the paper of Gibbons and Hawking [2] where the event horizon of the exact de Sitter solution was considered). The notion of the event horizon is global, not local. In realistic cosmological models that are usually discussed, there is no event horizon at all, despite the possible presence of an intermediate stage of the quasi-de Sitter expansion. If it is the event horizon that is responsible for the generation of cosmological perturbations, then — there is no event horizon, there is no horizon temperature, there is nothing to discuss. As a result, the authors of Ref. [1] were not surprised by the formula which essentially states that one can produce an arbitrarily large amount of density perturbations by practically doing nothing.

The formula used in [1] has been derived as a continuation of the previous studies on
this subject. Parker and Zhang refer to the papers [3-8]. We will start from the paper of Hawking [7] which seems to be clearer than others in expressing the basic idea and intentions. The papers [5,6,7] are similar in many respects.

Hawking considers a scalar field \( \phi \) running slowly down an effective scalar field potential. He discusses the inhomogeneous fluctuations \( \phi_1(t, x) \) in the field \( \phi = \phi_0(t) + \phi_1(t, x) \) which mean that on a surface of constant time there will be some regions where the \( \phi \) field has run further down the hill than in other regions. He introduces a new time coordinate \( \bar{t} = t + \delta t(t, x) \) in such a way that the variations of the field are removed and the surfaces of constant time are surfaces of constant \( \phi \). Since the scalar field transforms as \( \phi_0 + \phi_1 \rightarrow \phi_0 + \phi_1 - \dot{\phi}_0 \delta t \), the required condition is achieved by the time coordinate shift \( \delta t = \phi_1/\dot{\phi}_0 \). Note that for a given \( \phi_1 \) the time shift is larger, the smaller is \( \dot{\phi}_0 \). Then Hawking says that the change of time coordinate will introduce inhomogeneous fluctuations in the rate of expansion \( H \). He and other authors take \( \delta H \sim H^2 \delta t \). From here they come, implicitly or explicitly, to the dimensionless amplitude of density perturbations

\[
\frac{\delta \rho}{\rho} \sim \frac{\delta H}{H} \sim H \delta t \sim \frac{H \phi_1}{\dot{\phi}_0}.
\]  

(1)

Some authors write explicitly \( \phi_1 \sim H \) and \( \delta \rho/\rho \sim H^2/\dot{\phi}_0 \).

The analysis has been done at the inflationary stage. To obtain the today's amplitude of density perturbations in wavelengths, say, of the order of the today's Hubble radius, it is recommended to calculate the right hand side of Eq. (1) at the moments of time when the scales of our interest were “crossing horizon” during inflationary epoch. In one or another version this formula appears in the most of inflationary literature and because of numerous repetitions it has grown to the ”standard” one. According to this formula, the amplitude of density perturbations becomes larger if one takes the \( \dot{\phi}_0 \) smaller.

The authors of [5,6,7] work with a specific scalar field potential, so the numerical value of \( \dot{\phi}_0 \) and the numerical value of \( \delta \rho/\rho \) following from Eq. (1) turn out to be dependent on the self-coupling constant in the potential. These authors are concerned about the unacceptably large amplitude of density perturbations that they have produced. But it is not a concern
about the fact that the Einstein equations play no role in this argumentation. It is a tricky
detail in the scalar field potential that the authors of [5,6,7] do not like.

Now let us show what is wrong with the argumentation of [5,6,7]. Let us consider a
scalar field $\phi$ with arbitrary potential. Write the field as $\phi = \phi_0(t) + \phi_1(t)Q$ where $Q$ is the
$n$-th spatial harmonic, $Q^i_\;^i + n^2Q = 0$. Write the perturbed metric in the form

$$ds^2 = -dt^2 + a^2(t)[(1 + h(t)Q)\delta_{ij} + h(t)n^{-2}Q_{;i,j}]dx^i dx^j .$$

The de Sitter solution corresponds to $\dot{\phi}_0 = 0$, $a(t) \sim e^{Ht}$, and $H(t) = \dot{a}/a = \text{const}$. It follows
from the Einstein equations that the (linear) contribution $\epsilon_\phi$ of the scalar field perturbations
to the total energy density $\epsilon = \epsilon_0 + \epsilon_\phi$ can be written as

$$\epsilon_\phi = \phi_0 \left\{ \dot{\phi}_1 - \phi_1 \left[ \ln(a^3\dot{\phi}_0) \right] \right\} Q .$$

The contribution $\epsilon_\phi$, as well as other components of the perturbed energy-momentum tensor,
vanish in the de Sitter limit $\dot{\phi}_0 \to 0$. Thus, the first conclusion we have to make is that in the
de Sitter limit there is no linear density perturbations at all. The scalar field perturbations
are uncoupled from gravity, they are not accompanied by linear perturbations of the energy-
momentum tensor and they are not accompanied by linear perturbations of the gravitational
field. The general solution to the perturbed Einstein equations is a set of purely coordinate
solutions totally removable by appropriate coordinate transformations. The scalar field
perturbations reduce to a test field whose role is to identify events in the spacetime. One
can still ask about a coordinate system such that the surfaces of constant time $\tau$, $\tau = \phi_1(t, x)$
are surfaces of constant $\phi$. But the perturbation of the expansion rate of this new coordinate
system will have nothing to do with the energy density perturbations. [An attempt of cutting
the de Sitter space-time along a surface of the new time and simply joining it to the radiation-
dominated stage would have shown that perturbations at the radiation-dominated stage are
completely determined by the coordinate solutions at the de Sitter stage. This result would
have only signaled about a mistake that has been made. Indeed, the perturbed equations
become singular at the matching surface and dealing with the solutions requires special care,
see Ref. [12].]
Now let us assume that $\dot{\phi}_0$ is not zero. Transformation of time $\bar{t} = t + \chi(t)Q$ generates a Lie transformation of the scalar field:

$$\phi_0(t) + \phi_1(t)Q \rightarrow \phi_0(t) + [\phi_1(t) - \dot{\phi}_0(t)\chi(t)]Q .$$

If one wants the transformed field to be homogeneous one takes $\chi(t) = \phi_1(t)/\dot{\phi}_0(t)$. The same transformation of time generates Lie transformations of the metric. The transformed $g_{\alpha\alpha}$ component is $\bar{g}_{\alpha\alpha} = -1 + 2\chi Q$, the transformed $g_{ik}$ components are described by $\ddot{h} = h - 2(\dot{a}/a)\chi$. There appear also the $g_{oi}$ components but they will not participate in our linear analysis. The expansion rate of a given frame of reference is determined by the trace of the deformation tensor [9]:

$$D = \frac{1}{2\sqrt{-g_{\alpha\alpha}}} \frac{\partial(g_{ik} - g_{\alpha\alpha}g_{\alpha k}/g_{\alpha\alpha})}{\partial t} g^{ik} .$$

In the linear approximation and before the transformation,

$$D \approx 3H + \frac{1}{2}(3\dot{h} - \ddot{h})Q .$$

After the transformation, $\bar{D} = D - 3\dot{H}\chi Q$. So, the introduced inhomogeneous fluctuation in the rate of expansion is $\delta H = -\dot{H}\chi Q = \ddot{h} \delta t$, not $\delta H = H^2\delta t$ assumed in Refs. [5,6,7].

The Einstein equation for energy density $D^2/3 = \kappa\epsilon$ is satisfied before and after the transformation, since the variation of $H$ is balanced by the variation of the energy density. The transformed energy density is

$$\bar{\epsilon} = \epsilon_0 + \epsilon_\phi - \dot{\epsilon}_0\chi Q = \epsilon_0 + \dot{\phi}_0^2 \left( \frac{\phi_1}{\phi_0} \right) Q .$$

Thus, if one makes the $\dot{\phi}_0$ smaller, the energy density perturbation decreases according to the Einstein equations, and it increases according to the conjectures of Refs. [5,6,7].

The situation becomes even more disturbing if one recalls that the formula (1) has been seemingly confirmed by more detailed studies. People did really write the perturbed Einstein equations. Moreover, it was done in the framework of the so-called gauge-invariant
formalism, the whole purpose of which is to eliminate coordinate solutions and to work exclusively with something “physical”. Parker and Zhang quote the paper [4]. It is useful to consider also the paper [10] which summarizes the previous work and gives a clearer exposition.

In terms of the gauge-invariant potential \( \Phi \), the basic equation of [10] is

\[
\Phi'' + 2\frac{(a/\phi_0)'}{(a/\phi_0)} \Phi' - \nabla^2 \Phi + 2\phi_0' \left( \frac{\mathcal{H}}{\phi_0} \right)' \Phi = 0 \tag{2}
\]

where \( ' = \frac{d}{d\eta} \), \( dt = a d\eta \), \( \mathcal{H} = a'/a \), \( \nabla^2 \Phi = -n^2 \Phi \). Equation (2) is exactly the same equation as the basic equation (2.23) of Ref. [4]. These equations were derived from the original perturbed Einstein equations with the help of manipulations aimed at expressing the equations in terms of the gauge-invariant potentials. Parker and Zhang refer to a conservation law found in Ref. [4]. Indeed, in terms of the quantity \( \zeta \) defined as

\[
\zeta = \frac{2}{3} H^{-1} \dot{\Phi} + \Phi \left( 1 + \frac{w}{1 + \omega} \right) \Phi
\]

where \( w = p/\epsilon \), Eq. (2) takes on the form

\[
\frac{3}{2} \dot{\zeta} H (1 + w) = -\frac{n^2}{a^2} \Phi
\]

In the long-wavelength limit \( n^2 \to 0 \), the authors of Refs. [10] and [4] neglect the right-hand side of this equation and arrive at the “conservation law”:

\[
\zeta \approx \text{const} \tag{3}
\]

They use the constancy of \( \zeta \) all the way from the first “horizon crossing” at \( t_i \) to the second “horizon crossing” at \( t_f \). Returning to the definition of \( \zeta \) and remembering that \( 1 + w(t_f) \) is of the order of 1 while \( 1 + w(t_i) \) is much smaller than 1, one can derive

\[
\Phi(t_f) \sim \Phi(t_i)[1 + w(t_i)]^{-1} \tag{4}
\]

The authors of Refs. [4,10] emphasize that this formula is in agreement with Eq. (1). It is essentially this formula that has been used by Parker and Zhang. Equation (4) suggests
an arbitrarily large production of density perturbations for no other reason but simply because the $1 + w(t_i)$ was very close to zero. This formula cannot be correct. I realize perfectly well that what I qualify here as obviously incorrect is definitely considered by others as obviously correct. Otherwise somebody would raise a voice of protest against the ease with which inflationists generate tremendous amounts of various substances (some of them are even claiming that they can “overclose” our Universe). However, judging from the literature, it is not only that there are no voices of protest but there is rather an element of competition as for who was the first to proclaim the “standard” inflationary results. For instance, the authors of [11] address the inflationary claims about density perturbations as “first quantitatively calculated in [7,5,6] [and which] have been successfully quantitatively confirmed by the COBE discovery”.]

Now let us show what is wrong with the derivation of Eqs. (3) and (4). Use the background equations in order to express the coefficients of Eq. (2) in terms of the scale factor $a(\eta)$ and its derivatives. Introduce a new variable $\mu$ according to the definition

$$
\Phi = \frac{1}{2n^2} \frac{a'}{a} \gamma \left( \frac{\mu}{a\sqrt{\gamma}} \right)'
$$

where $\gamma = -\dot{H}/H^2 = 1 + (a/a')'$. So far, the variable $\mu$ is simply a new variable replacing $\Phi$, but the importance of $\mu$ is in that the original perturbed Einstein equations require this variable to satisfy the equation

$$
\mu'' + \mu \left[ n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] = 0
$$

Equation (6) is satisfied, Eq. (7) is satisfied too, but not vice versa. Equation (7) is equivalent to

$$
\frac{1}{a^2\gamma} \left[ a^2 \gamma \left( \frac{\mu}{a\sqrt{\gamma}} \right)' \right]' + n^2 \frac{\mu}{a\sqrt{\gamma}} = X
$$

7
where $X$ is arbitrary constant. Use the definition of $\zeta$ and Eq. (5) to show that

$$\zeta = \frac{1}{2n^2} \frac{1}{a^2 \gamma} \left[ \frac{a^2 \gamma}{a^2 \gamma} \right]'' .$$

In the lowest approximation of $n^2 \to 0$ the second term in Eq. (8) can be neglected. This gives $\zeta \approx X/2n^2 = \text{const}$ and explains the origin of Eq. (3).

Thus, the “conservation law” (3) can only be used for the derivation of Eq. (4) if one is willing to make a mistake, that is to forget that the constant $X$ must be equal to zero.

The possible relative contributions of density perturbations and gravitational waves to the observed microwave background anisotropies is a subject of active study. In the center of discussion are usually the ”consistency relations” which say that the gravity wave contribution goes to zero if the spectrum of perturbations approaches the Harrison-Zeldovich form. In reality, these ”consistency relations” are simply a manifestation of inconsistency of the ”standard” inflationary theory from which they are derived.

In conclusion, if the “standard” inflationary results are incorrect and cannot be trusted, what is the amount of density perturbations that can be generated in the early Universe? My part of answer is formulated in Ref. [12].
REFERENCES

[1] L. Parker and Y. Zhang, Phys. Rev. D 51, 2703 (1995).

[2] G. Gibbons and S. Hawking, Phys. Rev. D 15, 541 (1977).

[3] R. Brandenberger, R. Kahn and W. H. Press, Phys. Rev. D 28, 1809 (1983); R. Brandenberger and R. Kahn, ibid. 29, 2172 (1984).

[4] J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D 28, 679 (1983).

[5] A. A. Starobinsky, Phys. Lett. 117B, 175 (1982).

[6] A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. 49, 1110 (1982).

[7] S. W. Hawking, Phys. Lett. 115B, 295 (1982).

[8] A. Vilenkin and L. Ford, Phys. Rev. D 26, 1231 (1982).

[9] A. L. Zel’manov, Doklady Acad. Nauk USSR. 107, 815 (1956) [Sov. Phys. Doklady. 1, 227 (1956)].

[10] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Reports 215, 6 (1992).

[11] D. Polarski and A. A. Starobinsky. Semiclassicality and Decoherence of Cosmological Perturbations, Report LMPM/95-4, gr-qc 9504030.

[12] L. P. Grishchuk, Phys. Rev. D 50, 7154 (1994); Cosmological Perturbations of Quantum Mechanical Origin and Anisotropy of the Microwave Background Radiation, Report WUGRAV-94-11, gr-qc 9410025; Statistics of the Microwave Background Anisotropies Caused by the Squeezed Cosmological Perturbations, Report WUGRAV-95-6, gr-qc 9504045.