Kinetic energy dissipation and fluctuations in strongly-damped heavy-ion collisions within the stochastic mean-field approach

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\textbf{Background:} Microscopic mean-field approaches have been successful in describing the most probable reaction outcomes in low-energy heavy-ion reactions. However, those approaches are known to severely underestimate dispersions of observables around the average values that has limited their applicability. Recently it has been shown that a quantal transport approach based on the stochastic mean-field (SMF) theory significantly improves the description, while its application has been limited so far to fragment mass and charge dispersions.

\textbf{Purpose:} In this work, we extend the quantal transport approach based on the SMF theory for relative kinetic energy dissipation and angular momentum transfer in low-energy heavy-ion reactions.

\textbf{Methods:} Based on the SMF concept, analytical expressions are derived for the radial and tangential friction and associated diffusion coefficients. Those quantal transport coefficients are calculated microscopically in terms of single-particle orbitals within the time-dependent Hartree-Fock (TDHF) approach.

\textbf{Results:} As the first application of the proposed formalism, we consider the radial linear momentum dispersion, neglecting the coupling between radial and angular momenta. We analyze the total kinetic energy (TKE) distribution of binary reaction products in the $^{136}\text{Xe}^{+}\text{Pb}$ reaction at $E_{cm} = 526$ MeV and compare with experimental data. From time evolution of single-particle orbitals in TDHF, the radial diffusion coefficient is computed on a microscopic basis, while a phenomenological treatment is introduced for the radial friction coefficient. By solving the quantal diffusion equation for the radial linear momentum, the dispersion of the radial linear momentum is obtained, from which one can construct the TKE distribution. We find that the calculations provide a good description of the TKE distribution for large values of energy losses, $\text{TKE} \gtrsim 150$ MeV. However, the calculations underestimate the TKE distribution for smaller energy losses. Further studies are needed to improve the technical details of calculations.

\textbf{Conclusions:} It has been shown that the quantal transport approach based on the SMF theory provides a promising basis for the microscopic description of the TKE distribution as well as the isotopic distributions in damped collisions of heavy ions at around the Coulomb barrier.

\section{I. INTRODUCTION}

The nuclear dissipation plays a major role in nuclear dynamics such as heavy-ion collisions as well as nuclear fission. In order to understand the nuclear dissipation mechanism, a large amount of investigations have been carried out both experimentally and theoretically over many years \cite{1,4}. In low-energy heavy-ion collisions at around the Coulomb barrier, the one-body dissipation-fluctuation mechanism originating from nucleon exchange is essential. The time-dependent Hartree-Fock (TDHF) approach provides a microscopic basis for describing dissipative collisions at low energies. It incorporates with the one-body dissipation mechanism and successfully describes the most probable dynamical path of reaction dynamics \cite{5-10}. However, it is well known that the mean-field treatment of the TDHF approach severely underestimates dynamical fluctuations around the most probable path. Recent applications of the so-called time-dependent random phase approximation (TDRPA), which is based on the generalized variational principle of Balian and Vénéroni \cite{11,13}, provides a possible prescription for calculating dispersions of one-body observables in low-energy heavy-ion reactions. The latter approach has been applied to calculate mass and charge dispersions in heavy-ion collisions \cite{14-17}. Although there was an attempt to quantify kinetic energy fluctuations in dissipative collisions in the past \cite{18}, its practical applications are still scarce. This work is the first step toward the fully microscopic description of dissipation and fluctuations of the relative motion of colliding nuclei based on an alternative approach, the stochastic mean-field (SMF) theory \cite{19,20}.

It is crucially important to develop a microscopic basis for describing fluctuations in the kinetic energy dissipation for providing a reliable prediction for producing unknown unstable nuclei. In recent years, deep-inelastic collisions such as multinucleon transfer and quasifission processes have engaged substantial interests, regarding the possibility of efficient production of unknown neutron-rich heavy nuclei. Production of transactinide nuclei in the superheavy region in deep-inelastic or quasifission type processes in damped collisions of two heavy nuclei has been explored. Besides, multinucleon transfer reactions at energies around the Coulomb barrier are expected to be useful to produce neutron-rich heavy nuclei along the neutron magic number $N = 126$. (See, e.g., Refs. \cite{21,22}, for recent reviews.) To provide a reliable prediction for production of yet-unknown unstable nuclei, it is of paramount importance to properly describe not only dispersions of mass and charge of reaction products, as was greatly improved by the recent developments of the SMF approach \cite{23-32}, but also the distribution of dissipated relative kinetic

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energy during the collision. The latter is directly connected with excitation energies of reaction products, which should not be too large to maximize the production yield. Regarding the ongoing worldwide experimental effort aiming at producing unknown neutron-rich heavy nuclei [22–40], it is an imperative task to develop a fully microscopic framework for dissipation and fluctuations of the relative motion of colliding nuclei associated with nucleon exchange.

In this work, we develop a quantal transport formalism for dissipation and fluctuations of the relative kinetic energy and the relative angular momentum transfer based on the SMF approach. Analytical expressions for the radial and tangential friction and associated diffusion coefficients are derived on the microscopic basis. As a first step toward the fully microscopic description of energy and angular momentum dissipation in low-energy heavy-ion reactions, in the present work, we consider dissipation of the relative radial linear momentum, neglecting its coupling with the angular momentum transfer. The kinetic energy dissipation in the collision of $^{136}$Xe+$^{208}$Pb at $E_{c.m.} = 526$ MeV is analyzed with the newly developed approach and the total kinetic energy (TKE) distribution is compared with the available experimental data [23].

The article is organized as follows. In Sec. II, we present derivation of the Langevin equations for the relative radial momentum and the orbital angular momentum. In Sec. III, quantal expressions of diffusion coefficients for the radial and angular momenta, and the joint probability distribution function for these quantities are given. In Sec. IV, the numerical results of the TKE distribution for the $^{136}$Xe+$^{208}$Pb reaction at $E_{c.m.} = 526$ MeV are presented and compared with the experimental data. A summary and conclusions are given in Sec. V.

II. FLUCTUATION OF THE RELATIVE MOMENTA WITHIN THE SMF APPROACH

A. Remarks on the SMF approach

The SMF approach goes beyond the standard TDHF description and provides a microscopic basis for describing the fluctuations around the most probable path [19, 20]. In the SMF approach, instead of a single deterministic event in TDHF, an ensemble of mean-field events is considered, which is associated with a distribution law. The single-particle density matrix of an event $\lambda$ is given by

$$\rho^\lambda(r, r', t) = \sum_{ij} \phi^*_j(r, t; \lambda)\rho^\lambda_{ji}(r', t; \lambda),$$

(1)

where the wave functions in each event $\lambda$ obey the TDHF equation under own self-consistent mean field of the event. According to the basic postulate of the SMF approach, elements of the density matrix $\rho^\lambda_{ji}$ at the initial state have uncorrelated Gaussian distribution with the average values $\bar{\rho}^\lambda_{ji} = n_j\delta_{ji}$ and the variances determined according to

$$\delta\rho^\lambda_{ji} = n_j - n_j\delta_{ji}$$

and $n_j$ denotes the average occupation numbers of the single particle states. Here and hereafter, the bar over quantities represents the ensemble average over the stochastically generated events. At zero temperature the occupation numbers are zero or one, while at finite temperatures they are specified according to the Fermi-Dirac distribution. The distribution row (2) ensures that an ensemble average of observables recovers the quantal expressions for the mean and the variance at the initial state.

In the special case, where colliding nuclei maintain a dinuclear structure (cf. Fig. 1 showing a typical density distribution in the $^{136}$Xe+$^{208}$Pb reaction to be analyzed in Sec. IV), it is possible to analyze reaction dynamics in terms of a few macroscopic variables, such as relative linear and angular momenta, and mass and charge asymmetries of the dinuclear system. In this case, the SMF approach gives rise to a set of coupled Langevin equations for the macroscopic variables, which provides a quantal diffusion description of complex reaction dynamics in terms of a few relevant macroscopic variables. With the quantal diffusion equations, one can calculate not only the mean values of observables, which coincide with the TDHF results, but also distributions of the observables. For details of the SMF approach we refer readers to Refs. [19, 20, 24, 26]. We also refer to recent applications of the SMF approach for the multinucleon transfer mechanism in the dissipation heavy-ion collisions in Refs. [23, 32].

![FIG. 1. Density profile in the reaction plane at a certain instant in the $^{136}$Xe+$^{208}$Pb reaction at $E_{c.m.} = 526$ MeV with initial orbital angular momentum of $l = 200h$. The beam direction is parallel to the $-x$ direction and the impact parameter vector is parallel to $+y$ direction. The orientation angle of the dinuclear system is indicated by $\theta (= 52.2^\circ$ at this instant). The red dot represents the center of mass position of the system. The position vectors of projectile-like and target-like fragments in the center-of-mass frame are indicated by $R^+$ and $R^-$, respectively, where the relative distance at this instant is $R = |R^+ - R^-| = 13.7$ fm. The dashed line indicates the position of the window plane placed at the minimum density location.](image-url)
B. Rate of change of the relative linear momentum

In this section, let us recall basic equations that characterize the relative motion of colliding nuclei. We define the relative distance, \( \mathbf{R}(t) \), the reduced mass, \( \mu(t) \), and the relative linear momentum, \( \mathbf{P}(t) \), in terms of the TDHF solutions with the help of the window dynamics, see Fig. 1. Figure 1 illustrates the elongation axis (the solid line) and the window plane (the dashed line) at a certain instant in the \(^{138}\)Xe+\(^{208}\)Pb reaction at \( E_{\text{c.m.}} = 526 \text{ MeV} \) with the initial angular momentum \( l = 200h \). The orientation angle is indicated by \( \theta \) in the reaction plane. The elongation axis of the dinuclear system can be determined by diagonalizing the mass quadruple tensor at any instant. The window plane is perpendicular to the elongation axis and passes through the minimum density location on the elongation axis. For description of the details of the window dynamics we refer to Appendix A in Ref. [27].

In Fig. 1 the position vectors pointing the mean center-of-mass position of projectile- and target-like fragments in the center-of-mass frame are indicated by \( \mathbf{R}^+ \) and \( \mathbf{R}^- \), respectively. In terms of the local density \( \rho_\lambda \) and the current density \( j_\lambda \), the relative motion of colliding nuclei. We define the relative mass position of projectile- and target-like fragments in an event \( \lambda \), the masses, the center-of-mass positions, and the relative linear momenta of the projectile- and target-like fragments are, respectively, given by

\[
M_\lambda^\pm(t) = m \int \, d\mathbf{r} \, \Theta(\pm x') \rho_\lambda(\mathbf{r}, t),
\]

\[
\mathbf{R}_\lambda^\pm(t) = m \int \, d\mathbf{r} \, \Theta(\pm x') \mathbf{r} \rho_\lambda(\mathbf{r}, t)/M_\lambda^\pm(t),
\]

\[
\mathbf{P}_\lambda(t) = m \int \, d\mathbf{r} \, \Theta(\pm x') j_\lambda(\mathbf{r}, t),
\]

where \( j_\lambda(\mathbf{r}, t) \) denotes the current density in the event \( \lambda \),

\[
j_\lambda(\mathbf{r}, t) = \frac{\hbar}{2m_\lambda} \sum_{ij} \left[ \phi^*_i(\mathbf{r}, t ; \lambda) \nabla \phi_i(\mathbf{r}, t ; \lambda) - \phi_i(\mathbf{r}, t ; \lambda) \nabla \phi^*_i(\mathbf{r}, t ; \lambda) \right] \rho_{ji}^\lambda.
\]

In Eqs. (3)–(5) we neglect the fluctuations in the window geometry and specify the mean window position by a theta function \( \Theta(\pm x') \), where \( x'(t) = [x - x_0(t)] \cos \theta(t) + [y - y_0(t)] \sin \theta(t) \) measures distance from the window, \( \theta(t) \) is the initially smaller angle between the elongation axis and the beam direction, and \( (x_0(t), y_0(t)) \) is the position of the center of the window.

With the quantities introduced above, we can define the relative coordinate, \( \mathbf{R}_\lambda = \mathbf{R}^+_\lambda - \mathbf{R}^-_\lambda \), the reduced mass, \( \mu_\lambda = M^+_\lambda M^-_\lambda / (M^+_\lambda + M^-_\lambda) \), and the relative linear momentum,

\[
\mathbf{P}_\lambda = \mu_\lambda \dot{\mathbf{R}}_\lambda = \frac{M^+_\lambda \mathbf{P}^+_\lambda - M^-_\lambda \mathbf{P}^-_\lambda}{M^+_\lambda + M^-_\lambda} = \mu_\lambda [\dot{\mathbf{R}}_\lambda \hat{e}_R + \mathbf{R}_\lambda \dot{\theta}_\lambda \hat{e}_\theta].
\]

Here, \( \dot{\mathbf{R}}_\lambda = \dot{\mathbf{R}}^+_\lambda - \dot{\mathbf{R}}^-_\lambda \) denotes the relative velocity vector, where the velocities of the projectile- and target-like fragments can be defined by \( \dot{\mathbf{R}}^+_\lambda = \mathbf{P}^+_\lambda / M^+_\lambda \) and \( \dot{\mathbf{R}}^-_\lambda = \mathbf{P}^-_\lambda / M^-_\lambda \). In the second line of Eq. (7), the relative velocity is decomposed into the radial and tangential components with the unit vectors in respective directions,

\[
\hat{e}_R = \cos \theta \hat{e}_x + \sin \theta \hat{e}_y,
\]

\[
\hat{e}_\theta = -\sin \theta \hat{e}_x + \cos \theta \hat{e}_y.
\]

Neglecting the rate of change of the reduced mass, one finds the following expression for the rate of change of the relative momentum:

\[
\frac{d\mathbf{P}_\lambda}{dt} = \mu_\lambda \left[ \left( \mathbf{R}_\lambda - \mathbf{R}_\lambda \right) \dot{\theta}_\lambda \right] \dot{e}_R + \left( \mathbf{R}_\lambda \dot{\theta}_\lambda \right) \dot{e}_\theta + \frac{1}{\mu_\lambda} \frac{dL_\lambda}{dt} \dot{\theta}_\lambda,
\]

where we have introduced the radial component of the relative linear momentum \( K_\lambda \) and the relative orbital angular momentum \( L_\lambda \) as defined as

\[
K_\lambda \equiv \dot{e}_R \cdot \mathbf{P}_\lambda,
\]

\[
L_\lambda \equiv \mu_\lambda R_\lambda^2 \dot{\theta}_\lambda.
\]

The first and the second terms of Eq. (10) denote the rate of changes of the radial and the tangential components, respectively.

C. Stochastic equations for the relative momenta

In the SMF approach, we can express the rate of change of the projectile- and target-like fragments in an event \( \lambda \) as

\[
\frac{d\mathbf{P}_\lambda}{dt} = \pm m \int \, d\mathbf{r} \, \delta(\mathbf{x}) \dot{\mathbf{P}}_\lambda(\mathbf{x}, t) + m \int \, d\mathbf{r} \nabla \left( \dot{\mathbf{x}} \cdot \mathbf{P}_\lambda(\mathbf{x}, t) \right),
\]

Employing the TDHF equation for the single-particle orbitals in the event \( \lambda \), it is possible to write down the rate of change of the radial and the tangential components of the linear momentum of the fragments in the following form:

\[
\hat{e}_\alpha \cdot \frac{d\mathbf{P}_\lambda^\pm}{dt} = \pm \int \, d\mathbf{r} \, \delta(\mathbf{x}) \dot{\mathbf{P}}_\lambda^\pm(\mathbf{x}, t) = \int \, d\mathbf{r} \, \Theta(\pm x') \hat{e}_\alpha \cdot \dot{\mathbf{j}}_\lambda(\mathbf{r}, t)
\]

\[
\left\{ \text{[Potential terms]} \right\}
\]

with

\[
A_{ji}^\alpha = \frac{\hbar^2}{4m} \left[ \phi_i(\mathbf{r}, t ; \lambda) \nabla (\hat{e}_\alpha \cdot \nabla \phi^*_j(\mathbf{r}, t ; \lambda)) + \text{c.c.} \right],
\]

\[
B_{ji}^\alpha = \frac{\hbar^2}{4m} \left[ (\nabla \phi_j(\mathbf{r}, t ; \lambda)) \hat{e}_\alpha \cdot \nabla \phi^*_i(\mathbf{r}, t ; \lambda) + \text{c.c.} \right],
\]

where \( \hat{e}_\alpha \) indicates the unit vector in the radial (\( \alpha = R \)) or the tangential (\( \alpha = \theta \)) direction. The ‘[Potential terms]’ in Eq. (14) represents terms associated with mean-field potentials other than the kinetic term in the single-particle Hamiltonian in the event \( \lambda \).
From Eq. (14), we can derive a Langevin equation for the rate of change of the relative linear momentum,

\[
\frac{dP^\lambda}{dt} = \int d\mathbf{r} \ g(x') \ \hat{z}' m j_\lambda(r, t) + \text{[Potential terms]} + f^\lambda(t),
\]

where the first and the second terms represent the forces arising from the motion of the window plane and the potential terms, respectively. The quantity \( f^\lambda(t) \) is the fluctuating dynamical force due to nucleon exchange between projectile- and target-like fragments. Its radial (\( \alpha = R \)) and tangential (\( \alpha = \theta \)) components are given by

\[
f^\lambda_\alpha(t) = \sum_{ji} Y_{ji}(t) \rho^\lambda_{ji},
\]

where we have introduced a shorthand notation,

\[
Y^\alpha_{ji}(t) \equiv \int d\mathbf{r} \ g(x') \ \hat{e}_\alpha \cdot \left( A^\alpha_{ji}(t) - B^\alpha_{ji}(t) \right).
\]

In obtaining this result we employed a partial integration in Eq. (14) and used the following relations:

\[
\begin{align*}
\frac{\partial}{\partial x} \Theta(x') &= \delta(x') \cos \theta, \\
\frac{\partial}{\partial y} \Theta(x') &= \delta(x') \sin \theta.
\end{align*}
\]

In Eq. (17) the delta function has been replaced with a smoothing function \( \delta(x') \to g(x') \) expressed as a Gaussian,

\[
g(x') = \frac{1}{\sqrt{2\pi\kappa}} \exp \left[ -\frac{1}{2} \left( \frac{x'}{\kappa} \right)^2 \right],
\]

with a dispersion \( \kappa = 1.0 \) fm which is on the same order as the lattice spacing in the numerical calculations. By projecting Eq. (17) along the radial and the tangential directions, together with Eq. (10), we obtain two coupled Langevin equations for the radial and angular momenta [41, 42]:

\[
\begin{align*}
\frac{dK^\lambda}{dt} - \frac{L^2}{\mu^2 R^4} &= \int d\mathbf{r} \ g(x') \ \hat{z}' m \ \hat{e}_R \cdot \mathbf{j}_\lambda(r, t) + f^\lambda_R(t) \\
&\quad + \text{[Potential terms]}, \\
\frac{1}{R^2} \frac{dL^\lambda}{dt} &= \int d\mathbf{r} \ g(x') \ \hat{z}' m \ \hat{e}_\theta \cdot \mathbf{j}_\lambda(r, t) + f^\lambda_\theta(t) \\
&\quad + \text{[Potential terms]}.
\end{align*}
\]

In the right-hand side of these expressions, the first and the third terms represent the force due to the motion of the window plane and the conservative force due to nuclear and electrical potential energies, respectively. The fluctuating forces, \( f^\lambda_R(t) \) and \( f^\lambda_\theta(t) \), represent the dynamical forces arising from nucleon exchange between projectile- and target-like fragments. These dynamical forces provide the dominant mechanism for the dissipation and fluctuations of the relative momentum in damped collisions of heavy ions, such as deep-inelastic and quasifission processes.

The ensemble average of these equations of motion are equivalent to the TDHF description for the radial and angular components of the relative linear momentum. Consequently, we use the mean values of TKE and the orbital angular momentum obtained from the TDHF approach. We employ the Langevin equations, Eqs. (23) and (24), for describing fluctuations around their mean values. There are two different sources for fluctuations of the dynamical forces \( f^\lambda_R(t) \) and \( f^\lambda_\theta(t) \) induced by nucleon exchange: (i) fluctuations due to different set of wave functions in each event \( \lambda \), and (ii) fluctuations introduced by the stochastic part \( \delta \rho^\lambda_{ji} \) of the density matrix at the initial state. The former part of fluctuations can be approximately described in terms of the fluctuating components of the radial and angular momentum as \( f^\lambda_R(t) \to f_R^{\lambda\text{dis}}(K_\lambda) \) and \( f^\lambda_\theta(t) \to f_\theta^{\lambda\text{dis}}(L_\lambda) \). Here, \( f_R^{\lambda\text{dis}}(K_\lambda) \) and \( f_\theta^{\lambda\text{dis}}(L_\lambda) \) are the mean values of the radial and tangential components of the dissipative part of the dynamical forces expressed in terms of fluctuating radial and angular momenta, respectively. We assume that the amplitudes of the fluctuations are sufficiently small, so that we can linearize the Langevin equations, Eq. (23) and (24), around the mean values to give

\[
\begin{align*}
\frac{\partial}{\partial t} \delta K_\lambda &= \frac{2L}{\mu R^3} \delta L_\lambda + \left( \frac{\partial f_R^{\lambda\text{dis}}}{\partial K_\lambda} \right) \delta K_\lambda + \delta f^\lambda_R, \\
\frac{\partial}{\partial t} \delta L_\lambda &= \left( \frac{\partial f_\theta^{\lambda\text{dis}}}{\partial L_\lambda} \right) \delta L_\lambda + R \delta f^\lambda_\theta,
\end{align*}
\]

where \( \delta K_\lambda = K_\lambda - \langle K_\lambda \rangle \) and \( \delta L_\lambda = L_\lambda - \langle L_\lambda \rangle \) are the fluctuating components of the radial and angular momenta, respectively. The fluctuating forces originating from the potential energy terms are expected to have a small effect on the fluctuations of the relative momentum. In these expressions, we neglect these forces as well as the force due to the motion of the window given by the first terms in the right-hand side of Eqs. (23) and (24). Also, we neglect the fluctuations in the reduced mass and the relative distance between the centers of the fragments. The quantities, \( \mu(t), R(t), \) and \( L(t) \), are, respectively, the mean values of the reduced mass, the relative distance, and the relative orbital angular momentum of the colliding system, which are determined by the TDHF equation. The derivatives of dissipative forces on the right-hand side of Eqs. (25) and (26) are related to the reduced radial and tangential friction coefficients:

\[
\begin{align*}
\frac{\partial f_R^{\lambda\text{dis}}}{\partial K} &= -\gamma_R(t), \\
\frac{\partial f_\theta^{\lambda\text{dis}}}{\partial L} &= -\gamma_\theta(t).
\end{align*}
\]

An analysis of the radial friction force and the reduced friction coefficients are presented in Appendix B.

Multiplying both sides of Eqs. (25) and (26) by \( \delta K_\lambda \) and \( \delta L_\lambda \), respectively, and taking the ensemble average, we obtain a set of coupled differential equations for the variances [43]:
In this expression, the summations in the first term run over the complete set of states originating from the projectile. We introduce a similar subtraction in the second term of Eq. (34). As shown in Appendix A using the closure relation in a diabatic approximation of the TDHF orbitals, it is possible to eliminate the complete set of the projectile (target) states in the first (second) term. As a result, the radial, the tangential and the mixed diffusion coefficients are given by the following compact expression:

\[
D_{\alpha\beta}(t) = \int_0^t dt' \int \! dr \, \bar{g}(x') \left[ G_T(\tau) J_{\alpha\beta}^T(r, \ell) + G_P(\tau) J_{\alpha\beta}^P(r, \ell) \right]
- \int_0^t dt \left[ \sum_{h \in T, h' \in P} Y_{hh'}^\alpha(t) Y_{h'h}^{\alpha*}(t - \tau) + \sum_{h \in P, h' \in T} Y_{hh'}^\alpha(t) Y_{h'h}^{\alpha*}(t - \tau) \right].
\]

In the first term, the quantity \( J_{\alpha\beta}^T(r, \ell) \) is given by

\[
J_{\alpha\beta}^T(r, \ell) = \frac{\hbar}{m} \sum_{h \in P} \left[ \mu_{h\alpha}^R(r, \ell) \right] \left[ \mu_{h\beta}^R(r, \ell) \right] \times \left| \text{Im} \left[ \phi_h^\alpha(r, \ell) \hat{e}_n \cdot \nabla \phi_h^\alpha(r, \ell) \right] \right|.
\]

where \( \ell = (t + t')/2 = t - \tau/2 \). This expression represents the magnitude of the nucleon flux that carries the product of the momentum components \( \mu_{h\alpha}^R(r, \ell) \) and \( \mu_{h\beta}^R(r, \ell) \) from the target-like fragment in the perpendicular (\( \alpha, \beta = R \)) and tangential (\( \alpha, \beta = \theta \)) directions to the window plane. The quantity \( J_{\alpha\beta}^P(r, t - \tau/2) \) is given by a similar expression and it represents the magnitude of the nucleon flux from the projectile-like fragment. The radial and tangential components of the nucleon flow velocities are determined by

\[
u_{\alpha}^b(r, \ell) = \frac{\hbar}{m} \sum_{h \in P} \left[ \phi_h^\alpha(r, \ell) \hat{e}_n \cdot \nabla \phi_h^\alpha(r, \ell) \right] \left| \phi_h(r, \ell) \right|^2.
\]

We observe that there is a close analogy between the quantal expression of the diffusion coefficients and the classical ones in a random walk problem. The first term in the quantal expression gives the sum of the nucleon flux across the window from the target-like to the projectile-like fragments and vice versa, which is integrated over the memory. Each nucleon transfer across the window in both directions carries the product of the momentum components which increases the rate of change of the momentum dispersion. This is analogous to the random walk problem, in which the diffusion coefficient is given by the sum of the rate for forward and backward steps. The second term in the quantal expression stands for the Pauli blocking effects in the nucleon transfer mechanism, which does not have a classical counterpart. The quantities in the Pauli blocking factors are determined by hole-hole elements of the matrices \( Y_{hh'}^\alpha(t) \) and \( Y_{h'h}^{\alpha*}(t) \) which are defined in Eq. (19) with Eqs. (15) and (16).

\[\delta K \quad \delta K \quad \delta L\]

III. TOTAL KINETIC ENERGY DISTRIBUTION WITHIN THE SMF APPROACH

A. Momentum diffusion coefficients

The momentum diffusion coefficients for the radial and angular momenta are defined as the time integral over the history of the autocorrelation functions of the stochastic forces,

\[
D_{\alpha\beta}(t) = \int_0^t dt' \, \delta f_{\alpha}(t) \delta f_{\beta}(t').
\]

The stochastic parts of the radial and tangential forces are given by,

\[
\delta f_{\alpha}(t) = \sum_{ij} Y_{ji}^\alpha(t) \delta \rho_{ji}^\alpha.
\]

Using the basic postulate of the SMF approach, we can analytically take the ensemble average, and the correlation functions of the random force on radial and tangential directions read

\[
\delta f_{\alpha}(t) \delta f_{\beta}(t') = \Re \left[ \sum_{p \in P, h \in T} Y_{hp}^\alpha(t) Y_{hp}^{\beta*}(t') + \sum_{p \in T, h \in P} Y_{hp}^\alpha(t) Y_{hp}^{\beta*}(t') \right].
\]

In this expression, the summations in the first term run over the particle states originating from the projectile \( p \in P \) and the hole states originating from the target \( h \in T \), while in the second term the summations run in the opposite way. By adding and subtracting the hole-hole terms, the first term in this expression can be written as,

\[
\sum_{p \in P, h \in T} Y_{hp}^\alpha(t) Y_{hp}^{\beta*}(t') = \sum_{h \in T, a \in P} Y_{ha}^\alpha(t) Y_{ha}^{\beta*}(t') - \sum_{h \in T, h' \in P} Y_{hh'}^\alpha(t) Y_{hh'}^{\beta*}(t').
\]

\[\delta K \quad \delta K \quad \delta L\]
B. Total kinetic energy distribution

It is possible to determine the joint probability distribution function of the radial linear momentum \( K \) and the orbital angular momentum \( L \) for each initial orbital angular momentum \( l \), \( P_l(K, L) \), employing the coupled Langevin equations, Eqs. (25) and (26). It is well known that these coupled Langevin equations are equivalent to the Fokker-Planck description for the joint probability distribution \( P_l(K, L) \) [40].

When the radial and tangential friction forces have linear dependence on the radial and the angular momenta, the solution of the joint probability distribution can be expressed as a correlated Gaussian function:

\[
P_l(K, L) = \frac{\exp[-C_l(K, L)]}{2\pi\sigma_{Kl}(l)\sigma_{Ll}(l) \sqrt{1 - \eta_l^2}},
\]

where

\[
C_l(K, L) = \frac{1}{2(1 - \eta_l^2)} \left[ \left( \frac{K - K_t}{\sigma_{Kl}(l)} \right)^2 + \left( \frac{L - L_t}{\sigma_{Ll}(l)} \right)^2 - 2\eta_l \frac{K - K_t}{\sigma_{Kl}(l)} \frac{L - L_t}{\sigma_{Ll}(l)} \right].
\]

Here, the correlation factor is defined as \( \eta_l = \sigma_{Ll}^2(l)/\sigma_{Kl}(l)\sigma_{Ll}(l) \). \( K_t \equiv K(l) \) and \( L_t \equiv L(l) \) denote the mean values of the radial and the angular momenta for each value of the initial orbital angular momentum \( l \), respectively, which are determined by solving the TDHF equation.

The mean values of the radial and angular momenta, \( K_t \) and \( L_t \), are obtained by solving the TDHF equation. In practice, we follow the reaction dynamics up to a certain instant, say \( t = t_f \), at which binary products are well separated spatially. Denoting the relative distance at this instant as \( R_t = R(t_f) \), the asymptotic value of TKE of the outgoing fragments is given by \( E_{\text{kin}}^\infty(K, L) = K^2/2\mu + L^2/2\mu R_t^2 + Z_1Z_2e^2/R_t \). For a given initial angular momentum \( l \), we define the TKE distribution \( G_l(E) \) as

\[
G_l(E) = \int dK dL \delta(E - E_{\text{kin}}^\infty(K, L)) P_l(K, L).
\]

Note that \( K \) and \( L \) in the above expression correspond to the radial and the angular momenta at the instant \( t = t_f \), respectively, and \( E \) stands here for the asymptotic TKE. It is to mention that \( \mu \) and \( Z_{1,2} \) are, in general, \( l \) dependent quantities, and the fluctuations in the mass and charge asymmetries may affect the TKE fluctuations. However, we neglect the effects of mass and charge fluctuations on the TKE distribution and retain the mean values of the mass and charge asymmetry for each angular momentum.

In practice, the mixed diffusion coefficients, \( D_{KL}(t) \) and \( D_{LK}(t) \), are expected to be much smaller than the radial and the angular momentum diffusion coefficients, \( D_{KK}(t) \) and \( D_{LL}(t) \). Hence, in the present work, we neglect the mixed dispersion term \( \sigma_{KL}(t) \) in Eq. (29) and the coupling between the radial and angular momenta. In such a case, the expression can be greatly simplified by the asymptotic limit, \( R \to \infty \), leading to

\[
G_l(E) = \int dK dL \delta(E - E_{\text{kin}}^\infty(K, L)) P_l(K, L),
\]

where \( E_{\text{kin}}^\infty(K) = K^2/2\mu \) and \( P_l(K) \) is the probability distribution of the radial momentum. Notice that by taking the limit \( R \to \infty \) the centrifugal part of the kinetic energy and the Coulomb energy entirely transformed into the radial TKE, and \( K \to \infty \) in Eq. (42) corresponds to the asymptotic value of the radial momentum for \( R \to \infty \). After taking the integral over the angular momentum variable, the asymptotic radial momentum distribution becomes a simple Gaussian,

\[
P_l(K) = \frac{1}{\sqrt{2\pi\sigma_{Kl}(l)}} \exp\left[ -\frac{1}{2} \frac{(K - K_t^\infty)^2}{\sigma_{Kl}(l)} \right],
\]

where the mean value of the asymptotic radial momentum is related to the mean asymptotic TKE from TDHF, \( E_{\text{kin}}^\infty(l) \), by \( K_t^\infty = (2\mu E_{\text{kin}}^\infty(l))^{1/2} \). After a trivial integration, we obtain the asymptotic TKE distribution,

\[
G_l(E) = \frac{1}{\sqrt{8\pi E \sigma_{Kl}(l)}} \exp\left[ -\frac{1}{2} \frac{(E - \sqrt{E_{\text{kin}}^\infty(l)})^2}{\sigma_{Kl}(l)} \right],
\]

where \( \sigma_{Kl}(l) \equiv \sigma_{KL}/\sqrt{2\mu} \). To obtain the radial dispersion \( \sigma_{KL}(l) \), we solve the quantal diffusion equation for the radial component,

\[
\frac{d\sigma_{KL}^2}{dt} = -2\gamma R \sigma_{KL}^2 + 2D_{KK}.
\]

We note that the unit of the TKE distribution \( G_l \) is MeV\(^{-1} \), hence the fraction of events with final TKE in the energy range \( \Delta E \) in MeV is given by \( G_l\Delta E \).

IV. RESULTS FOR XE+PB COLLISIONS

In this section, as the first application of the proposed formalism given in the preceding sections, we present calculations of the TKE distribution for the \( ^{136}\text{Xe}^{+}\text{Pb} \) reaction at \( E_{c.m.} = 526 \text{ MeV} \), for which extensive experimental data reported by Kozulin et al. [53] are available. TDHF calculations were carried out for a range of initial orbital angular momenta \( l \). The results of TDHF calculations for a set of observables in the \( ^{136}\text{Xe}^{+}\text{Pb} \) reaction at \( E_{c.m.} = 526 \text{ MeV} \) are presented in Table 1. We mention here that for the \( ^{136}\text{Xe}^{+}\text{Pb} \) system the average numbers of transferred nucleons are small, reflecting a small charge asymmetry and possible shell effects in the reactants. Nucleons are, however, actively exchanged during the collision, which is the source of dissipation and fluctuations of observables, such as mass, charge, TKE, and scattering angles, in low-energy heavy-ion reactions. For this reaction, mean TKEL reaches around 175 MeV for small angular momenta, while contact time is rather short (< 2 zs). We note that fragments for \( l < 100h \) are outside of the experimental angular coverage.
TABLE I. A list of numerical results of the TDHF calculations for the $^{136}$Xe$^{208}$Pb reaction at $E_{c.m.} = 526$ MeV. From left to right columns, it shows: the initial orbital angular momentum, $l$, in $\hbar$, the final average relative orbital angular momentum, $L_t$, in $\hbar$, neutron and proton numbers of projectile-like (target-like) fragment, $N_1$ and $Z_1$ ($N_2$ and $Z_2$), mean total kinetic energy loss (TKEL) in MeV, contact time, $t_{\text{contact}}$, in fm/c, scattering angles in center-of-mass frame, $\theta_{\text{c.m.}}$, and those in laboratory frame for projectile-like (target-like) fragment, $\phi_1^{\text{lab}}$ ($\phi_2^{\text{lab}}$), in degrees. The contact time is defined as duration in which the minimum density between two colliding nuclei exceeds half the saturation density, $\rho_{\text{sat}}/2 = 0.08$ fm$^{-3}$.

| $l$ | $t_{\text{contact}}$ | TKEL | $\theta_{\text{c.m.}}$ | $\phi_1^{\text{lab}}$ | $\phi_2^{\text{lab}}$ |
|-----|---------------------|------|-------------------|-----------------|-----------------|
| 0   | 0                   | 0.0  | 823.51            | 55.1            | 125.0            |
| 50  | 39                  | 0.0  | 82.4              | 55.1            | 124.9            |
| 100 | 78                  | 0.0  | 82.0              | 54.5            | 125.4            |
| 110 | 85                  | 0.0  | 81.9              | 54.3            | 125.5            |
| 120 | 95                  | 0.0  | 81.8              | 54.3            | 125.6            |
| 130 | 104                 | 0.0  | 81.7              | 54.3            | 125.7            |
| 140 | 114                 | 0.0  | 81.9              | 54.4            | 125.5            |
| 150 | 124                 | 0.0  | 82.4              | 54.7            | 125.0            |
| 160 | 134                 | 0.0  | 83.1              | 55.2            | 124.3            |
| 170 | 142                 | 0.0  | 83.7              | 55.6            | 123.7            |
| 180 | 149                 | 0.0  | 83.9              | 55.8            | 123.5            |
| 190 | 158                 | 0.0  | 83.6              | 55.6            | 123.9            |
| 200 | 166                 | 0.0  | 83.0              | 55.4            | 124.5            |
| 210 | 173                 | 0.0  | 82.7              | 55.2            | 124.9            |
| 220 | 179                 | 0.0  | 82.3              | 55.0            | 125.3            |
| 230 | 185                 | 0.0  | 81.8              | 54.8            | 125.8            |
| 240 | 194                 | 0.0  | 81.5              | 54.6            | 126.2            |
| 250 | 203                 | 0.0  | 81.3              | 54.3            | 126.4            |
| 260 | 214                 | 0.0  | 81.2              | 54.4            | 126.6            |
| 270 | 225                 | 0.0  | 81.2              | 54.4            | 126.6            |
| 280 | 240                 | 0.0  | 81.4              | 54.5            | 126.5            |
| 290 | 258                 | 0.0  | 81.5              | 54.5            | 126.4            |
| 300 | 277                 | 0.0  | 81.5              | 54.4            | 126.4            |
| 310 | 295                 | 0.0  | 81.6              | 54.3            | 126.4            |
| 320 | 311                 | 0.0  | 81.7              | 54.2            | 126.2            |
| 330 | 325                 | 0.0  | 81.9              | 54.1            | 126.1            |
| 340 | 337                 | 0.0  | 81.9              | 54.0            | 126.1            |
| 350 | 347                 | 0.0  | 81.9              | 54.0            | 126.1            |

FIG. 2. Reduced radial friction coefficients $\gamma_{R}$ in the $^{136}$Xe$^{208}$Pb reaction at $E_{c.m.} = 526$ MeV with initial orbital angular momenta of $l = 100$, 200, 250, and 300 (in units of $\hbar$) are shown as functions of time. Nevertheless, using the analogy to the random walk problem, we have extracted from TDHF an approximate expression for the radial friction force and the radial friction coefficients. Details of this analysis are given in Appendix B.

In Figs. 2–4, we show examples of the computational results for the collisions of $^{136}$Xe$^{208}$Pb at $E_{c.m.} = 526$ MeV for four typical initial angular momenta, $l$ = 100 (solid line), 200 (dash-dotted line), 250 (dashed line), and 300 (dotted line) in units of $\hbar$, as functions of time. Figure 2 shows the reduced radial friction coefficients $\gamma_{R}(t, l)$ given by Eq. (36), which were extracted from TDHF employing the method explained in detail in Appendix B. We observe that the radial friction coefficient develops when two nuclei collide at around $t = 200$–400 fm/c. The magnitude of the friction coefficient increases with decreasing the initial orbital angular momentum $l$, for which contact times are longer, indicating that larger amount of the relative kinetic energy is converted into internal excitations at smaller orbital angular momenta, as expected. In Fig. 3 we show the quantal momentum diffusion coefficient $D_{KK}(t)$ given by Eq. (36), which is calculated microscopically based on occupied single-particle orbitals within the

FIG. 3. Radial-momentum diffusion coefficients $D_{KK}$ in the $^{136}$Xe$^{208}$Pb reaction at $E_{c.m.} = 526$ MeV with initial orbital angular momenta of $l = 100$, 200, 250, and 300 (in units of $\hbar$) are shown as functions of time.
TDHF approach. Again, the magnitude of the diffusion coefficient increases with decreasing the initial orbital angular momentum \( l \). From the results, we find that the diffusion coefficient has a relatively long tail as compared to the friction coefficient shown in Fig. 2. It is related to the fact that the quantal diffusion coefficient is governed by nucleon exchange which lasts even after the turning point through a neck structure of the dinuclear system (cf. contact times shown in Table II).

Having the radial friction and momentum diffusion coefficients, \( \gamma_R(t) \) and \( D_{KK}(t) \), at hand, we solve the differential equation (45) and the results are shown in Fig. 4. From the figure, we see that the variances of the radial momentum \( \sigma_{KK} \) show somewhat complicated behavior as a function of time. The variance grows in time and saturates when two nuclei reseparate. We notice that the asymptotic value of \( \sigma_{KK} \) is smallest for \( l = 100 \) and decreases with increasing \( l \) at values from 100 to 200, then increases for \( l = 250 \), and then decreases again for \( l = 300 \).

Table II illustrates the variance of the radial momentum dispersion \( \sigma_{KK} \), in MeV/c, the modified radial momentum dispersion, \( \tilde{\sigma}_{KK} = \sigma_{KK}/\sqrt{2\mu} \), in MeV/c, and the dispersion of total kinetic energy (TKE), \( \sigma_{TKE} \approx 2\tilde{\sigma}_{KK}\sqrt{E_{\text{kin}}^\infty} \), in MeV.

From Table II, we see that the asymptotic value of the largest dispersion occurs for \( l = 0 \). In this case, we find \( \tilde{\sigma}_{KK}^2 = \sigma_{KK}^2/2\mu \approx 1.98 \text{ MeV} \), which is much smaller than \( E_{\text{kin}}^\infty(l) \approx 353 \text{ MeV} \), confirming the correspondence with the mean TKE from TDHF. We can also calculate the variance of TKE for each value of angular momentum as

\[
\sigma_{TKE}^2(l) = \int dE (E - E_{\text{kin}}^\infty(l))^2 G_l(E) \\
\approx 4\tilde{\sigma}_{KK}^2(l) E_{\text{kin}}^\infty(l). \tag{47}
\]

Dispersion of TKE grows linearly with the square root of the mean value, \( \sigma_{TKE} \approx 2\tilde{\sigma}_{KK}\sqrt{E_{\text{kin}}^\infty} \). For example for the initial angular momentum \( l = 0 \), dispersion is about \( \sigma_{TKE} \approx 37 \text{ MeV} \). This indicates the total excitation energy of the primary fragments have quite large dispersion values. Large values of dispersions of the excitation energies may

| \( l \) (fm) | \( \sigma_{KK} \) (MeV/c) | \( \tilde{\sigma}_{KK} \) (MeV/c\(^{1/2} \)) | \( \sigma_{TKE} \) (MeV) |
|---|---|---|---|
| 0 | 553.7 | 1.406 | 37.00 |
| 50 | 497.2 | 1.263 | 33.60 |
| 100 | 392.7 | 0.999 | 26.47 |
| 110 | 367.3 | 0.934 | 24.73 |
| 120 | 342.4 | 0.871 | 23.05 |
| 130 | 319.4 | 0.813 | 21.52 |
| 140 | 298.3 | 0.759 | 20.07 |
| 150 | 277.9 | 0.706 | 18.64 |
| 160 | 268.7 | 0.681 | 17.87 |
| 170 | 266.0 | 0.674 | 17.54 |
| 180 | 261.0 | 0.661 | 17.16 |
| 190 | 265.4 | 0.672 | 17.40 |
| 200 | 282.9 | 0.718 | 18.49 |
| 210 | 302.8 | 0.769 | 19.51 |
| 220 | 317.0 | 0.805 | 20.05 |
| 230 | 329.7 | 0.838 | 20.47 |
| 240 | 344.2 | 0.876 | 20.77 |
| 250 | 357.5 | 0.910 | 20.73 |
| 260 | 366.6 | 0.933 | 20.20 |
| 270 | 365.5 | 0.930 | 18.93 |
| 280 | 352.0 | 0.896 | 16.62 |
| 290 | 330.3 | 0.840 | 13.74 |
| 300 | 295.4 | 0.752 | 10.52 |
| 310 | 233.6 | 0.595 | 6.80 |
| 320 | 152.0 | 0.387 | 3.18 |
| 330 | 85.8 | 0.218 | 1.24 |
| 340 | 34.5 | 0.088 | 0.38 |
| 350 | 36.2 | 0.092 | 0.34 |
FIG. 5. The total kinetic energy (TKE) distribution, $G_l$ defined by Eq. \( (44) \), is shown in the $l$-TKE plane, where $l$ represents the initial orbital angular momentum of the reaction, calculated for the \(^{136}\text{Xe} + ^{208}\text{Pb} \) reaction at $E_{\text{c.m.}} = 526$ MeV.

have an important effect on deexcitation processes of the primary fragments.

Finally, to make a comparison with the experimental data [33], we evaluate the yield of the reaction outcomes as a function of total kinetic energy loss (TKEL, i.e., $E_{\text{c.m.}} - E_{\text{kin}}^\infty$) by summing up contributions from each initial orbital angular momentum,

$$Y(E_{\text{c.m.}} - E_{\text{kin}}^\infty) = Y_0 \sum_{l=100}^{350} (2l + 1)G_l(E_{\text{kin}}^\infty). \quad (48)$$

Figure 6 shows a comparison of the calculations with experimental data for the collisions of \(^{136}\text{Xe} + ^{208}\text{Pb} \) at $E_{\text{c.m.}} = 526$ MeV. The normalization constant $Y_0$ is adjusted to the data at a suitable point. From the figure we find that the calculations provide good description for the large energy-loss segment of the data, TKEL $\gtrsim 150$ MeV. However, it underestimates the count curve over the lower energy-loss segment. This behavior is a result of apparent large dispersions of the TKE distribution over the range $l = 200h$–$300h$, which may be due to the over prediction of the radial momentum diffusion coefficients and/or the approximate description of the radial friction for the large angular momentum region. Although further improvements of the formalism are mandatory, we consider that the quantal diffusion approach based on the SMF theory provides a promising microscopic basis for quantifying kinetic energy dissipation and fluctuations in low-energy heavy-ion reactions.

V. SUMMARY

The stochastic mean-field (SMF) approach goes beyond the standard mean-field approximation, describing dynamical fluctuations of the collective motion in heavy-ion collisions at low energies. In the time-dependent Hartree-Fock (TDHF) approach, dynamical evolution of the colliding system is described by a single Slater determinant which is determined by a given set of initial conditions. In the SMF approach, on the other hand, an ensemble of the mean-field (TDHF) events for stochastically-generated initial conditions is considered. The initial conditions for each event are specified by the quantal and thermal fluctuations, and each event evolves according to the self-consistent mean-field Hamiltonian of that event. As a result, the SMF approach provides not only the mean values, but the entire distribution functions of the observables.

For low-energy heavy-ion reactions in which the colliding system maintains a dinuclear structure, instead of generating an ensemble of stochastic mean-field events, the evolution of the system can be described in terms of a few representative macroscopic variables, such as the relative linear momentum, the orbital angular momentum, the mass and charge asymmetries of the colliding system. In such a case, we can deduce effective equations of motion for the macroscopic variables by adiabatic or geometric projection of the stochastically-generated mean-field events on a macroscopic subspace, in a manner similar to the Mori formalism [34]. For deep-inelastic or quasifission processes in which dinuclear structure is maintained, the geometric projection with the help of the window dynamics is more suitable to deduce the effective equations for the macroscopic variables. Being consistent with the Mori formalism, the effective equations for the macroscopic variables take the form of generalized Langevin equations which offer quantal diffusion description of the dynamical evolution of the colliding system. The Langevin equations are characterized by transport coefficients, i.e., diffusion and drift coefficients. It is possible to deduce analytical expressions for transport coefficients by carrying out suitable averages over the ensemble generated by the SMF approach. Employing the closure relation in the diabatic approximations of the TDHF wave functions, we can express the diffusion coefficients in
terms of the occupied single-particle orbitals in TDHF. Therefore, it provides a very practical and powerful framework for the microscopic description of fluctuations of collective variables. This result is consistent with the quantal fluctuation-dissipation theorem of the non-equilibrium statistical mechanics \cite{41,42}. The theorem states that the diffusion coefficients which provide the source of fluctuations can be expressed in terms of the mean-field properties.

In previous studies we employed the quantal diffusion approach to investigate multinucleon transfer mechanism in low-energy heavy-ion collisions. In the present work, we have developed a formalism for describing the total kinetic energy (TKE) distributions of binary reaction products. We have deduced an effective transport equation for the relative linear momentum based on the SMF approach by the projection technique with the help of window dynamics. The radial and the tangential components of this equation provide quantal description of the random walk problem. We find that the dispersion of the TKE distribution reach rather large values which may have important effects in the deexcitation mechanism of heavy-ion reactions. In the present work, we have developed a formalism for describing the total kinetic energy loss (TKEL) distribution for large values of the orbital angular momentum (i.e., reactions at large initial TKEL). The underestimation of the TKEL data for lower values of TKEL (i.e., reactions at small TKEL values (TKEL < 150 MeV), but underestimate the data for lower values of TKEL (i.e., reactions at large initial orbital angular momenta). The underestimation of the TKEL distribution for large values of the orbital angular momentum may be partly due to the approximate description of the radial friction coefficient and/or the neglected coupling between the radial and tangential components of the linear momentum. This work put an important step forward for the microscopic description of low-energy heavy-ion reactions, including distributions of various observables, and further improvements of the proposed formalism are in order.

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**Appendix A: Momentum diffusion coefficient**

In this Appendix, we derive the quantal expression of the momentum diffusion coefficient given by Eq. (36). Employing a partial integration in the expression \( Y_{ha}^{\alpha}(t) \) in Eq. (19), we have

\[
Y_{ha}^{\alpha}(t) = \frac{\hbar^2}{m} \int d\mathbf{r}_1 \left[ g(x'_1) \left( \hat{\epsilon}_a \cdot \nabla_1 \hat{e}_R \cdot \nabla_1 \phi_h(\mathbf{r}_1, t) \right) + \frac{1}{2} \hat{\epsilon}_a \cdot \nabla_1 g(x'_1) \hat{e}_R \cdot \nabla_1 \phi_h(\mathbf{r}_1, t) + \frac{1}{2} \hat{e}_R \cdot \nabla_1 g(x'_1) \hat{\epsilon}_a \cdot \nabla_1 \phi_h(\mathbf{r}, t) + \frac{1}{4} \left[ \hat{\epsilon}_a \cdot \nabla_1 \hat{e}_R \cdot \nabla_1 g(x'_1) \right] \phi_h(\mathbf{r}_1, t) \right] \phi_0(\mathbf{r}, t), \tag{A1}
\]

and its complex conjugation,

\[
Y_{ha}^{\alpha^*}(t) = \frac{\hbar^2}{m} \int d\mathbf{r}_1 \left[ g(x'_1) \left( \hat{\epsilon}_a \cdot \nabla_1 \hat{e}_R \cdot \nabla_1 \phi_h(\mathbf{r}_1, t) \right) + \frac{1}{2} \hat{\epsilon}_a \cdot \nabla_1 g(x'_1) \hat{e}_R \cdot \nabla_1 \phi_h(\mathbf{r}_1, t) + \frac{1}{2} \hat{e}_R \cdot \nabla_1 g(x'_1) \hat{\epsilon}_a \cdot \nabla_1 \phi_h(\mathbf{r}, t) + \frac{1}{4} \left[ \hat{\epsilon}_a \cdot \nabla_1 \hat{e}_R \cdot \nabla_1 g(x'_1) \right] \phi_h^*(\mathbf{r}_1, t) \right] \phi_0^*(\mathbf{r}, t). \tag{A2}
\]

With the diabatic property of the TDHF wave functions, we can shift the wave functions back and forth during short time intervals \( \tau = t - t' \) to have an approximate relation,

\[
\phi_\alpha(\mathbf{r}, t') \approx \phi_\alpha(\mathbf{r} - u_\tau, t), \tag{A3}
\]

where \( u_\tau \) denotes a small displacement during the short time interval with flow velocity \( u \). Using the closure relation,

\[
\sum_\alpha \phi_\alpha^*(\mathbf{r}_1, t) \phi_\alpha(\mathbf{r}_2 - u_\tau, t) = \delta(\mathbf{r}_1 - \mathbf{r}_2 + u_\tau), \tag{A4}
\]

we obtain:

\[
\sum_{h \in T, \alpha \in P} Y_{ha}^{\alpha} Y_{ha}^{\beta^*}(t') = \sum_{h \in T} \int d\mathbf{r}_1 d\mathbf{r}_2 \delta(\mathbf{r}_1 - \mathbf{r}_2 + u_\tau) W_h^{\alpha}(\mathbf{r}_1, t) W_h^{\beta^*}(\mathbf{r}_2, t). \tag{A5}
\]
First, we consider the case for $\alpha = \beta = R$. The radial part $W^R_h(r_1, t)$ reads

$$W^R_h(r_1, t) = \frac{\hbar^2}{m} \left[ \hat{g}(x')[(\hat{e}_R \cdot \nabla_1)\hat{e}_R \cdot \nabla_1)\phi_h(r_1, t)] + \frac{1}{2}(\hat{e}_R \cdot \nabla_1 g(x')\hat{e}_R \cdot \nabla_1 \phi_h(r_1, t) + \frac{1}{2}(\hat{e}_R \cdot \nabla_1 g(x')\hat{e}_R \cdot \nabla_1 \phi_h(r_1, t) + \frac{1}{4}(\hat{\alpha}_1 \cdot \nabla_1 (\hat{e}_R \cdot \nabla_1)g(x')\phi_h(r_1, t)] \right].$$

(A6)

Using the expression for $g(x')$ given by Eq. (22), we find

$$W^K_h(r_1, t) = \frac{\hbar^2}{m} g(x') \left[ (\hat{e}_R \cdot \nabla_1)^2 \phi_h(r_1, t)] - \frac{x'}{\kappa^2}(\hat{e}_R \cdot \nabla_1 \phi_h(r_1, t)] + \frac{1}{4\kappa^2} [x'^2 - \kappa^2] \phi_h(r_1, t)].$$

(A7)

and its complex conjugation,

$$W^K_h(r_2, t) = \frac{\hbar^2}{m} g(x') \left[ (\hat{e}_R \cdot \nabla_2)^2 \phi_h^*(r_2, t)] - \frac{x'}{\kappa^2}(\hat{e}_R \cdot \nabla_2 \phi_h^*(r_2, t)] + \frac{1}{4\kappa^2} [x'^2 - \kappa^2] \phi_h^*(r_2, t)].$$

(A8)

Let us introduce the following coordinate transformations,

$$\mathbf{R} = (r_1 + r_2)/2, \quad \mathbf{r} = r_1 - r_2,$$

(A9)

and its inverse,

$$r_1 = \mathbf{R} + r/2, \quad r = \mathbf{R} - r/2.$$  

(A10)

Because of the delta function, we can immediately carry out the integration over $\mathbf{r}$ in Eq. (A3) and make the substitution for $\mathbf{r} = -u_h \tau$, and introduce diabatic shifts in the wave functions,

$$\phi_h(r_1, t) = \phi_h(\mathbf{R} + r/2, t) = \phi_h(\mathbf{R} - u_h \tau/2, t) \approx \phi_h(\mathbf{R}, \tilde{t}),$$

(A11)

$$\phi_h(r_2, t) = \phi_h(\mathbf{R} - r/2, t) = \phi_h(\mathbf{R} + u_h \tau/2, t) \approx \phi_h(\mathbf{R}, \tilde{t}),$$

(A12)

with $\tilde{t} \equiv (t + t')/2$. We can express product of the Gaussian factors as $g(x_1')g(x_2') = \tilde{g}(X')\tilde{G}(x')$, where

$$\tilde{g}(X') = \frac{1}{\sqrt{4\pi \kappa}} \exp \left[ - \frac{X'^2}{\kappa} \right],$$

(A13)

$$\tilde{G}(X') = \frac{1}{\sqrt{4\pi \kappa}} \exp \left[ - \frac{X'^2}{2\kappa} \right].$$

(A14)

with $x' = \hat{e}_R \cdot \mathbf{r} = -\hat{e}_R \cdot u_h \tau = u^h_R \tau$ and $X' = \hat{e}_R \cdot \mathbf{R}$, where $u^h_R \mathbf{R}$, denotes the component of the flow velocity of the hole states perpendicular to the window, which may, in general, depend on the mean position $\mathbf{R} = (r_1 + r_2)/2$ and the mean time $t = (t + t')/2$. In the product $W^R_h(r_1, t)W^K_h(r_2, t')$, there are linear, second, third, and fourth order terms in $x_1$ and $x_2$ in the integrand of Eq. (A5). The integrand contains a product of two sharp Gaussians, $\tilde{g}(X')$ and $\tilde{G}(x')$, which provides the memory kernel in the integrand. Taking the averages over the memory kernel and over the sharp Gaussian $\tilde{g}(X')$, all terms in the integrand of Eq. (A5) which are proportional to the powers of $x_1$ and $x_2$ vanish. We obtain the similar results for other components of Eq. (A5) with $\alpha, \beta = R, \theta$, and we find

$$\sum_{h \in T, \alpha \in P} Y^{\alpha \beta}_h(t)Y^{\alpha \beta}_h(t') = \left( \frac{\hbar}{m} \right)^2 \sum_{h \in T} \int d\mathbf{R} \tilde{g}(X') \frac{G^h_T(\tau)}{[u^h_T(\mathbf{R}, \tilde{t})]} \times [(\hat{\alpha}_\alpha \cdot \nabla)(\hat{e}_R \cdot \nabla) \phi_h(\mathbf{R}, \tilde{t})]$$

$$\times [(\hat{\alpha}_\beta \cdot \nabla)(\hat{e}_R \cdot \nabla) \phi_h(\mathbf{R}, \tilde{t})]^*,$$

(A15)

where the memory kernel is defined as

$$G^h_T(\tau) = \frac{1}{\sqrt{4\pi \tau T}} \exp \left[ - \left( \frac{\tau}{2\tau T} \right)^2 \right],$$

(A16)

with the memory time, $\tau^h = \kappa/|u^h_R|$. We can write the wave functions as $\phi_h(\mathbf{r}, t) = |\phi_h(\mathbf{r}, t)| \exp(i\mathbf{q}_h \cdot \mathbf{r})$. Since the phase factor behaves like the velocity potential, neglecting derivative of the amplitude of the wave function, we have the approximation result:

$$\hat{e}_R \cdot \nabla) \phi_h \approx i \hat{e}_R \cdot \nabla Q_h$$

$$= i \phi_h(\mathbf{R}, \tilde{t}) \frac{m}{\hbar} u^h_R(\mathbf{R}, \tilde{t}).$$

(A17)

In a similar manner, we can express the second derivative of the wave function as,

$$\hat{e}_R \cdot \nabla) \phi_h \approx i(\hat{e}_R \cdot \nabla) \phi_h(\mathbf{R}, \tilde{t}) u^h_R(\mathbf{R}, \tilde{t}) \approx -\phi_h(\mathbf{R}, \tilde{t}) m^2 u^h_R(\mathbf{R}, \tilde{t}) u^h_R(\mathbf{R}, \tilde{t}).$$

(A18)

We can write the radial ($\alpha = R$) and the tangential ($\alpha = \theta$) flow velocities in the flowing form,

$$u^h_R(\mathbf{R}, \tilde{t}) = \frac{m}{\hbar} \text{Im}[\phi_h^*(\mathbf{R}, \tilde{t}) \hat{e}_R \cdot \nabla \phi_h(\mathbf{R}, \tilde{t})]/|\phi_h(\mathbf{R}, \tilde{t})|^2.$$  

(A19)

Incorporating this expression, Eq. (A15) becomes,

$$\sum_{h \in T, \alpha \in P} Y^{\alpha \beta}_h(t)Y^{\alpha \beta}_h(t') = \sum_{h \in T} \int d\mathbf{R} \tilde{g}(X') G_T(\tau) J^T_{\alpha \beta}(\mathbf{R}, \tilde{t}).$$

(A20)

Here, $J^T_{\alpha \beta}(\mathbf{R}, \tilde{t})$ is given in Eq. (37), and $G_T(\tau)$ denotes the average memory kernel given by Eq. (A16), which is evaluated with the average value of the flow velocity of the hole states originating from the target. The second term on the right side of Eq. (34) is evaluated in a similar manner, and we obtain the expression given by Eq. (36) for the momentum diffusion coefficients.
Appendix B: Radial friction coefficient

In this Appendix, we discuss an analysis of the radial friction coefficient based on the mean-field solution of TDHF. The TDHF description contains the one-body dissipation of relative kinetic energy and the transfer of the relative angular momentum into the intrinsic degrees of freedom. The dominant mechanism for the one-body dissipation is nucleon exchange between projectile-like and target-like fragments. However, a microscopic derivation of the so-called window formula for the reduced friction coefficients from the TDHF approach is not trivial. Here, we consider the analogy with the random walk problem to deduce the reduced friction coefficients from the mean description of TDHF. By taking the ensemble average in Eq. (12), the mean evolution of the rate of change of the relative momentum is given by

$$\frac{\partial}{\partial t} P = \int dr \left[ g(x') \dot{x}' m j(r, t) + [\text{Potential terms}] + f(t) \right],$$

where $j(r, t) = \frac{\hbar}{2m} \sum_h \text{Im} \left[ \phi_h^* (r, t) \nabla \phi_h (r, t) \right]$. This equation is equivalent to the TDHF description of the relative momentum. The first and second terms on the right hand side are the conservative forces on the relative motion due the motion of the window and the potential terms. In the last term $f(t)$ represents the dynamical force due to nucleon exchange between the projectile- and target-like fragments with the radial and the tangential components,

$$f_\alpha (t) = \int dr \left[ g(x') \dot{\epsilon}_\alpha \cdot \sum_h \left( A^h_{\alpha h} - B^h_{\alpha h} \right) \right],$$

(B2)

Using the approximate result of Eq. (A17) in Appendix A we can express the component of the dynamical force due to nucleon exchange as

$$f_\alpha (t) = -\frac{\hbar}{m} \int dr \left[ g(x') \sum_h m u^h (r, t) \nabla \phi_h (r, t) \right] \right],$$

(B3)

where the summation runs over the hole states originating from both projectile and target nuclei. The dynamical force involves both the conservative and dissipative forces. In order to infer the dissipative part of the dynamical force, we use the analogy to the Langevin description of the random walk problem. As seen in Eq. (50), the direct terms of the momentum diffusion coefficients are determined by the sum of nucleon fluxes that carry the product of momentum components from projectile to target and vice versa. In analogy to the description of the random walk, the components of the dissipative force are determined by the net momentum flux across the window as follows:

$$f^\text{diss}_\alpha (t) = -\frac{\hbar}{m} \int dr \left[ g(x') \sum_{h \in \mathcal{P}} m u^h (r, t) \right] \right],$$

where $K_l = \dot{\epsilon} R \cdot P_l$ is the radial component of the relative linear momentum. From this relation, in principle, it should be possible to deduce the radial friction coefficient for each value of the initial angular momentum $l$. In Fig. (B1) we show the radial friction force in (a) and the radial momentum in (b) for the central collisions ($l = 0$) as a function of time. The radial momentum vanishes at the turning point which occurs at $t = 313$ fm/c (see Fig. (B1) b), solid line). We expect that the radial friction force also vanishes at the turning point. However, we notice that the friction force vanishes at a slightly later time, $t = 360$ fm/c. The time shift may originate from the
approximate expression of the friction force, Eq. (B4), which overestimates the net momentum flux across the window, and the shift becomes larger for increasing the orbital angular momentum.

In order to extract the radial friction coefficients for all values of the initial angular momentum $l$, we employ an approximate method as described below. In Fig. B1(b), we introduce a smoothing of the radial momentum so that the friction force and the radial momentum vanish at the same instant. The smoothed (averaged over a time interval of 220 fm/c) radial momentum, say $\bar{K}_{t,l}(t)$, is shown by the dashed line in Fig. B1(b), and the vertical line indicates the instant at which both the friction force and the radial momentum become vanishingly small. Then, we define the reduced friction coefficient for $l = 0$ as

$$\gamma_R(t, l = 0) = -\frac{f^{\text{diss}}_{R,t,l}(t, l = 0)}{K_{t,l}(t)}.$$  

Note that the phenomenological relation (B5) has been used.

In Fig. B2 a), we show the extracted friction coefficient according to Eq. (B6) in the central collision ($l = 0$) as a function of time (solid line). We find that dissipation occurs mainly during the incoming phase until the turning point at around $t = 360$ fm/c (indicated by a vertical line), and only a small fraction of dissipation takes place during the outgoing phase after the turning point until the separation of the fragments. We should thus ignore an unphysical negative tail after $t = 460$ fm/c. Figure B2(b) shows the friction coefficient $\gamma_R$ in the central collision ($l = 0$) as a function of the relative distance $R(t)$. In this work, we parametrize the friction coefficient during the incoming phase by an exponential function as

$$\gamma_R^{\text{inc}}[R(t)] = c_1 \exp\left[-c_2 \left(R(t) - c_3\right)\right],$$

with $c_1 = 9.97$ fm/c, $c_2 = 1.04$ fm$^{-1}$, and $c_3 = 6.86$ fm.

The friction force reaches the maximum value at the minimum distance $R_{\text{min}}$. In the outgoing phase, from the minimum distance $R_{\text{min}}$ to reseparation, we adopt another form with an exponential damping factor as

$$\gamma_R^{\text{out}}[R(t)] = \gamma_R^{\text{inc}}[R(t)] c_4 \exp\left[-c_5 \left(R(t) - R_{\text{min}}\right)\right],$$

where $c_4 = 1.96$ and $c_5 = 0.95$. Note that $c_4 > 1$ has been used, because the minimum distance is reached at $t = 313$ fm/c, while the obtained friction coefficient has a peak at slightly later time. We joint the two expressions for the incoming and outgoing phases smoothly around the turning point. Assuming that the friction coefficients scale with the relative distance for all initial angular momentum in a similar manner as for the central collision, we express the reduced radial friction coefficient for non-zero $l$ values as

$$\gamma_R(t, l) = N_l \gamma_R[R_l(t)].$$

Here, $\gamma_R(R)$ is the friction coefficients given by Eqs. (B7) and (B8) extracted from the $l = 0$ case as a function of the relative distance. The normalization factor $N_l$ is determined by matching the dissipated energy with the mean TKEL calculated by TDHF for each initial angular momentum, i.e.,

$$E_l^{\text{diss}} = \int dt \frac{\gamma_R(t, l) K^2_l(t)}{\mu(t)} = E_{\text{kin}}^\infty(l).$$

In Fig. B3, we show the magnitude of the normalization constant $C_l$ as a function of the initial orbital angular momentum.
The dissipative force $F^{\text{diss}}_l$ is shown as a function of the radial linear momentum $K_l$ for the $^{136}$Xe+$^{208}$Pb reaction at $E_{\text{cm.}} = 526$ MeV with a range of $l$ values. Colors represent the value of the initial orbital angular momenta. (a) The dissipative force obtained with Eq. (B5) based on the single-particle orbitals in TDHF. (b) The reconstructed dissipative force according to the phenomenological expression, $F^{\text{diss}}_l = -\gamma_R K_l$, where $\gamma_R$ here is expressed as the fitted function given by Eqs. (B7) and (B8).

In Fig. B4, we compare the original radial friction force as given by Eq. (B4) in panel (a) and the reconstructed radial friction force using the approximate treatment of Eq. (B9) in panel (b) as functions of the radial momentum for a range of initial angular momenta $l$. It is visible that the original radial friction forces shown in (a) does not vanish at the turning point at which the radial momentum changes its sign. On the other hand, the reconstructed friction forces as functions of the radial momentum shown in (b) give rise to the expected behavior and provide a support for the reduced friction coefficients that we obtained using the approximate procedure.
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