\textbf{p-Biharmonic Hypersurfaces in Einstein Space and Conformally Flat Space}

\begin{abstract}
In this paper, we present some new properties for \( p \)-biharmonic hypersurfaces in Riemannian manifold. We also characterize the \( p \)-biharmonic submanifolds in an Einstein space. We construct a new example of proper \( p \)-biharmonic hypersurfaces. We present some open problems.

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\end{abstract}

1 Introduction

Let \( \varphi : (M^m, g) \longrightarrow (N^n, h) \) be a smooth map between Riemannian manifolds. The \( p \)-energy functional of \( \varphi \) is defined by

\[ E_p(\varphi; D) = \frac{1}{p} \int_D |d\varphi|^p v_g, \]

where \( D \) is a compact domain in \( M \), \( |d\varphi| \) the Hilbert-Schmidt norm of the differential \( d\varphi \), \( v_g \) the volume element on \( (M^m, g) \), and \( p \geq 2 \).

A smooth map is called \( p \)-harmonic if it is a critical point of the \( p \)-energy functional (1). We have

\[ \frac{d}{dt} E_p(\varphi_t; D) \bigg|_{t=0} = -\int_D h(\tau_p(\varphi), v) v_g, \]

where \( \{\varphi_t\}_{t \in (-\epsilon, \epsilon)} \) is a smooth variation of \( \varphi \) supported in \( D \), \( v = \frac{\partial \varphi_t}{\partial t} \bigg|_{t=0} \) the variation vector field of \( \varphi \), and \( \tau_p(\varphi) = \text{div}^M (|d\varphi|^{p-2} d\varphi) \) the \( p \)-tension field of \( \varphi \).

Let \( \nabla^M \) the Levi-Civita connection of \( (M^m, g) \), and \( \nabla^\varphi \) the pull-back connection on \( \varphi^{-1}TN \), the map \( \varphi \) is \( p \)-harmonic if and only if (see [1][3][7])

\[ |d\varphi|^{p-2} \tau(\varphi) + (p-2)|d\varphi|^{p-3} d\varphi (\text{grad}^M |d\varphi|) = 0, \]

where \( \tau(\varphi) = \text{trace}_g \nabla d\varphi \) is the tension field of \( \varphi \) (see [2][6]). The \( p \)-bienergy functional of \( \varphi \) is defined by

\[ E_{2,p}(\varphi; D) = \frac{1}{2} \int_D |\tau_p(\varphi)|^2 v_g. \]

We say that \( \varphi \) is a \( p \)-biharmonic map if it is a critical point of the \( p \)-bienergy functional (4), the Euler-Lagrange equation of the \( p \)-bienergy functional is given by (see [1][11])

\[ \tau_{2,p}(\varphi) = -|d\varphi|^{p-2} \text{trace}_g R^N (\tau_p(\varphi), d\varphi) d\varphi - \text{trace}_g \nabla^\varphi |d\varphi|^{p-2} \nabla^\varphi \tau_p(\varphi) \]

\[ -(p-2) \text{trace}_g \nabla < \nabla^\varphi \tau_p(\varphi), d\varphi > |d\varphi|^{p-4} d\varphi = 0, \]
where $R^N$ is the curvature tensor of $(N^n, h)$ defined by

$$R^N(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z, \quad \forall X, Y, Z \in \Gamma(TN),$$

and $\nabla^N$ the Levi-Civita connection of $(N^n, h)$. The $p$-energy functional (resp. $p$-bienergy functional) includes as a special case ($p = 2$) the energy functional (resp. bienergy functional), whose critical points are the usual harmonic maps (resp. biharmonic maps) [9].

A submanifold in a Riemannian manifold is called a $p$-harmonic submanifold (resp. $p$-biharmonic submanifold) if the isometric immersion defining the submanifold is a $p$-harmonic map (resp. $p$-biharmonic map). Will call proper $p$-biharmonic submanifolds a $p$-biharmonic submanifolds which is non $p$-harmonic.

## 2 Main Results

Let $(M^m, g)$ be a hypersurface of $(N^{m+1}, \langle \cdot, \cdot \rangle)$, and $i : (M^m, g) \hookrightarrow (N^{m+1}, \langle \cdot, \cdot \rangle)$ the canonical inclusion. We denote by $\nabla^M$ (resp. $\nabla^N$) the Levi-Civita connection of $(M^m, g)$ (resp. of $(N^{m+1}, \langle \cdot, \cdot \rangle)$), $\text{grad}^M$ (resp. $\text{grad}^N$) the gradient operator in $(M^m, g)$ (resp. in $(N^{m+1}, \langle \cdot, \cdot \rangle)$), $B$ the second fundamental form of the hypersurface $(M^m, g)$, $A$ the shape operator with respect to the unit normal vector field $\eta$, $H$ the mean curvature of $(M^m, g)$, $\nabla^\perp$ the normal connection of $(M^m, g)$, and by $\Delta$ (resp. $\Delta^\perp$) the Laplacian on $(M^m, g)$ (resp. on the normal bundle of $(M^m, g)$ in $(N^{m+1}, \langle \cdot, \cdot \rangle)$) (see [2, 13, 15]). Under the notation above we have the following results.

**Theorem 1.** The hypersurface $(M^m, g)$ with the mean curvature vector $H = f \eta$ is $p$-biharmonic if and only if

$$\begin{cases}
-\Delta^M(f) + f|A|^2 - f \text{Ric}^N(\eta, \eta) + m(p-2) f^3 = 0; \\
2A(\text{grad}^M f) - 2f(\text{Ricci}^N \eta)^\top + (p-2 + \frac{m}{2}) \text{grad}^M f^2 = 0,
\end{cases}
$$

where $\text{Ric}^N$ (resp. $\text{Ricci}^N$) is the Ricci curvature (resp. Ricci tensor) of $(N^{m+1}, \langle \cdot, \cdot \rangle)$.

**Proof.** Choose a normal orthonormal frame $\{e_i\}_{i=1, \ldots, m}$ on $(M^m, g)$ at $x$, so that $\{e_i, \eta\}_{i=1, \ldots, m}$ is an orthonormal frame on the ambient space $(N^{m+1}, \langle \cdot, \cdot \rangle)$. Note that, $\text{d}(X) = X$, $\nabla^\perp Y = \nabla^N_{\otimes} Y$, and the $p$-tension field of $i$ is given by $\tau_p(i) = m^2 f \eta$. We compute the $p$-tension field of $i$

$$\tau_{2,p}(i) = -|\text{d}|^{p-2} \text{trace}_g R^N(\tau_p(i), \text{d}) \text{d}\i
\begin{align*}
(p-2) \text{trace}_g \nabla \langle \nabla^\perp \tau_p(i), \text{d}\i \rangle |\text{d}|^{p-4} \text{d}\i
\text{trace}_g |\text{d}|^{p-2} \nabla^\perp \tau_p(i).
\end{align*}
$$

The first term of (7) is given by

$$-|\text{d}|^{p-2} \text{trace}_g R^N(\tau_p(i), \text{d}) \text{d}\i = -|\text{d}|^{p-2} \sum_{i=1}^m R^N(\tau_p(i), \text{d} e_i) \text{d}(e_i) = -m^{p-1} f \sum_{i=1}^m R^N(\eta, e_i) e_i = -m^{p-1} f \text{Ricci}^N \eta = -m^{p-1} f \left[(\text{Ricci}^N \eta)^\perp + (\text{Ricci}^N \eta)^\top\right].
$$

We compute the second term of (7)

$$-(p-2) \text{trace}_g \nabla \langle \nabla^\perp \tau_p(i), \text{d}\i \rangle |\text{d}|^{p-4} \text{d}\i = -(p-2)m^{p-2} \sum_{i,j=1}^m \nabla^N_{e_i} \langle \nabla^N_{e_j} f \eta, e_i \rangle e_j,$n

$$\sum_{i=1}^m \langle \nabla^N_{e_i} f \eta, e_i \rangle = \sum_{i=1}^m \langle e_i(f) \eta, e_i \rangle + f \langle \nabla^N_{e_i} \eta, e_i \rangle
$$

$$= -f \sum_{i=1}^m \langle e_i, B(e_i, e_i) \rangle
$$

$$= -m f^2.$$
Thus, at \( x \), we have the following

\[
- (p - 2) \text{trace}_g \nabla (\nabla^1 \tau_p (i), d\bar{i}) |d\bar{i}|^{p-4} d\bar{i} = m^{p-1} (p - 2) \left( \text{grad}^M f^2 + m f^3 \eta \right).
\]

The third term of (7) is given by

\[
- \text{trace}_g \nabla^1 |d\bar{i}|^{p-2} \nabla^1 \tau_p (i) = -m^{p-1} \sum_{i=1}^m \nabla^N \nabla^N e_i \eta
\]

\[
= -m^{p-1} \left[ \nabla^N \left( e_i (f) \eta + f \nabla^N \eta \right) \right]
\]

\[
= -m^{p-1} \left[ \Delta^M (f) \eta + 2 \nabla^N \text{grad}^M f \eta + m \sum_{i=1}^m \nabla^N \nabla^N e_i \eta \right].
\]

Thus, at \( x \), we obtain

\[
\sum_{i=1}^m \nabla^N \nabla^N e_i \eta = \sum_{i=1}^m \nabla^N \left[ \left( \nabla^N e_i \eta \right)^T + \left( \nabla^N \eta \right)^T \right]
\]

\[
= - \sum_{i=1}^m \nabla^N A(e_i)
\]

\[
= - \sum_{i=1}^m \nabla^N A(e_i) - \sum_{i=1}^m B(e_i, A(e_i)).
\]

Since \( \langle A(X), Y \rangle = \langle B(X, Y), \eta \rangle \) for all \( X, Y \in \Gamma(TM) \), we get

\[
\sum_{i=1}^m \nabla^M A(e_i) \eta = \sum_{i,j=1}^m \langle \nabla^M A(e_i), e_j \rangle e_j
\]

\[
= \sum_{i,j=1}^m \left[ e_i \langle A(e_i), e_j \rangle e_j - \langle A(e_i), \nabla^M e_j \rangle e_j \right]
\]

\[
= \sum_{i,j=1}^m e_i \langle B(e_i, e_j), \eta \rangle e_j
\]

\[
= \sum_{i,j=1}^m e_i \langle \nabla^N e_i, \eta \rangle e_j
\]

\[
= \sum_{i,j=1}^m \langle \nabla^N \nabla^N e_i, \eta \rangle e_j.
\]

By using the definition of curvature tensor of \( \left( N^{m+1}, \langle \cdot, \cdot \rangle \right) \), we conclude

\[
\sum_{i=1}^m \nabla^M A(e_i) \eta = \sum_{i,j=1}^m \left[ \langle R^N (e_i, e_j) e_i, \eta \rangle e_j + \langle \nabla^N \nabla^N e_i, \eta \rangle e_j \right]
\]

\[
= \sum_{i,j=1}^m \left[ -\langle R^N (\eta, e_i) e_i, e_j \rangle e_j + \langle \nabla^N e_i, \eta \rangle e_j \right]
\]

\[
= - \sum_{j=1}^m \langle \text{Ricci}^N \eta, e_j \rangle e_j + \sum_{i,j=1}^m e_j \langle \nabla^N e_i, \eta \rangle e_j - \sum_{i,j=1}^m \langle \nabla^N e_i, \nabla^N \eta \rangle e_j
\]

\[
= -\langle \text{Ricci}^N \eta \rangle^T + m \text{grad}^M f.
\]
On the other hand, we have
\[
\sum_{i=1}^{m} B(e_i, A(e_i)) = \sum_{i=1}^{m} (B(e_i, A(e_i)), \eta) \eta = \sum_{i=1}^{m} (A(e_i), A(e_i)) \eta = |A|^2 \eta. \tag{14}
\]
Substituting (11), (13) and (14) in (10), we obtain
\[
- \text{trace}_g \nabla^i |d_i|^{p-2} \nabla^i \tau_p(i) = -m^{p-1} [\Delta^M(f) \eta - 2A(\text{grad}^M f) + f(\text{Ricci}^N \eta)^T \\
- \frac{m}{2} \text{grad}^M f^2 - f|A|^2 \eta], \tag{15}
\]
The Theorem follows by (7)-(9), and (15).

The Corollary follows by Theorem 1.

Corollary 2. A hypersurface \((M^m, g)\) in an Einstein space \((N^{m+1}, \langle , \rangle)\) is \(p\)-biharmonic if and only if its mean curvature function \(f\) is a solution of the following PDEs
\[
\begin{aligned}
\Delta^M(f) + f|A|^2 + m(p - 2)f^3 - \frac{S}{m+1}f &= 0; \\
2A(\text{grad}^M f) + (p - 2 + \frac{m}{2}) \text{grad}^M f^2 &= 0,
\end{aligned} \tag{16}
\]
where \(S\) is the scalar curvature of the ambient space.

Proof. It is well known that if \((N^{m+1}, \langle , \rangle)\) is an Einstein manifold then \(\text{Ric}^N(X, Y) = \lambda \langle X, Y \rangle\) for some constant \(\lambda\), for any \(X, Y \in \Gamma(TN)\). So that
\[
S = \text{trace}_{\langle , \rangle} \text{Ric}^N
= \sum_{i=1}^{m} \text{Ric}^N(e_i, e_i) + \text{Ric}^N(\eta, \eta)
= \lambda(m + 1),
\]
where \(\{e_i\}_{i=1,...,m}\) is a normal orthonormal frame on \((M^m, g)\) at \(x\). Since \(\text{Ric}^N(\eta, \eta) = \lambda\), on conclude that
\[
\text{Ric}^N(\eta, \eta) = \frac{S}{m+1}.
\]
On the other hand, we have
\[
(\text{Ricci}^N \eta)^T = \sum_{i=1}^{m} (\text{Ricci}^N \eta, e_i) e_i = \sum_{i=1}^{m} \text{Ric}^N(\eta, e_i) e_i = \sum_{i=1}^{m} \lambda(\eta, e_i) e_i = 0.
\]
The Corollary follows by Theorem 1.

Theorem 3. A totally umbilical hypersurface \((M^m, g)\) in an Einstein space \((N^{m+1}, \langle , \rangle)\) with non-positive scalar curvature is \(p\)-biharmonic if and only if it is minimal.
Proof. Take an orthonormal frame \( \{e_i, \eta\}_{i=1}^{m} \) on the ambient space \((N^{m+1}, (,))\) such that \( \{e_i\}_{i=1}^{m} \) is an orthonormal frame on \((M^m, g)\). We have
\[
f = \langle H, \eta \rangle = \frac{1}{m} \sum_{i=1}^{m} \langle B(e_i, e_i), \eta \rangle = \frac{1}{m} \sum_{i=1}^{m} \langle g(e_i, e_i)\beta \eta, \eta \rangle = \beta,
\]
where \( \beta \in C^\infty(M) \). The \( p \)-biharmonic hypersurface equation (16) becomes
\[
\begin{cases}
-\Delta^M(\beta) + m(p-1)\beta^3 - \frac{s}{m+1} \beta = 0; \\
(p - 1 + \frac{m}{2}) \beta \text{ grad}^M \beta = 0,
\end{cases}
\]
Solving the last system, we have \( \beta = 0 \) and hence \( f = 0 \), or
\[
\beta = \pm \sqrt{\frac{s}{m(m+1)(p-1)}},
\]
it’s constant and this happens only if \( s \geq 0 \). The proof is complete. \( \square \)

3 \( p \)-biharmonic hypersurfaces in conformally flat space

Let \( i : M^m \hookrightarrow \mathbb{R}^{m+1} \) be a minimal hypersurface with the unit normal vector field \( \eta \). \( \tilde{\iota} : (M^m, \tilde{g}) \hookrightarrow (\mathbb{R}^{m+1}, \tilde{h} = e^{2\gamma}h) \), \( x \mapsto \tilde{\iota}(x) = i(x) = x \), where \( \gamma \in C^\infty(\mathbb{R}^{m+1}) \), \( h = (,)_{\mathbb{R}^{m+1}} \), and \( \tilde{g} \) is the induced metric by \( \tilde{h} \), that is
\[
\tilde{g}(X,Y) = e^{2\gamma}g(X,Y) = e^{2\gamma}(X,Y)_{\mathbb{R}^{m+1}},
\]
where \( g \) is the induced metric by \( h \). Let \( \{e_i, \eta\}_{i=1}^{m} \) be an orthonormal frame adapted to the \( p \)-harmonic hypersurface on \((\mathbb{R}^{m+1}, h)\), thus \( \{\tilde{e}_i, \tilde{\eta}\}_{i=1}^{m} \) becomes an orthonormal frame on \((\mathbb{R}^{m+1}, \tilde{h})\), where \( \tilde{e}_i = e^{-\gamma}e_i \) for all \( i = 1, \ldots, m \), and \( \tilde{\eta} = e^{-\gamma}\eta \).

Theorem 4. The hypersurface \((M^m, \tilde{g})\) in the conformally flat space \((\mathbb{R}^{m+1}, \tilde{h})\) is \( p \)-biharmonic if and only if
\[
\begin{cases}
\eta(\gamma)e^{-\gamma} \left[ -\Delta^M(\gamma) - m \text{ Hess}_{\gamma}^R(\eta, \eta) + (1 - m) \right| \text{ grad}^M \gamma |^2 \\
-|A|^2 + m(1-p)\eta(\gamma)^2 + \Delta^M(\eta(\gamma)e^{-\gamma}) + (m - 2)(\text{ grad}^M \gamma)(\eta(\gamma)e^{-\gamma}) = 0; \\
-2A(\text{ grad}^M(\eta(\gamma)e^{-\gamma})) + 2(1-m)\eta(\gamma)e^{-\gamma}A(\text{ grad}^M \gamma) + (2p - m)\eta(\gamma) \text{ grad}^M(\eta(\gamma)e^{-\gamma}) = 0,
\end{cases}
\]
where \( \text{ Hess}_{\gamma}^R \) is the Hessian of the smooth function \( \gamma \) in \((\mathbb{R}^{m+1}, h)\).

Proof. By using the Kozul’s formula, we have
\[
\begin{cases}
\bar{\nabla}^M_X Y = \nabla^M_X Y + X(\gamma) Y + Y(\gamma) X - g(X,Y) \text{ grad}^M \gamma; \\
\bar{\nabla}^R_{U} V = \nabla^R_{U} V + U(\gamma) V + V(\gamma) U - h(U,V) \text{ grad}^R \gamma,
\end{cases}
\]
for all \( X, Y \in \Gamma(TM) \), and \( U, V \in \Gamma(T\mathbb{R}^{m+1}) \). Consequently
\[
\bar{\nabla}^L_X d\tilde{\iota}(Y) = \bar{\nabla}^L_X Y = \bar{\nabla}^R_{d\tilde{\iota}(X)} Y = \bar{\nabla}^R_{d\tilde{\iota}(X)^{-1}} Y = \bar{\nabla}^R_{d\tilde{\iota}(X)} Y + X(\gamma) Y + Y(\gamma) X - h(X,Y) \text{ grad}^R \gamma,
\]
and the following
\[ d\tilde{\Omega}(\nabla^M Y) = d\Omega(\nabla^M Y) + X(\gamma)\Omega(Y) + \gamma(X, Y)\Omega - g(X, Y)\Omega(\text{grad}^M Y). \]

From equations (19) and (20), we get
\[ (\nabla d\tilde{\Omega})(X, Y) = \nabla^T_d d\tilde{\Omega}(Y) - d\tilde{\Omega}(\nabla^M Y) = \partial (\nabla d\tilde{\Omega})(X, Y) + g(X, Y)[\text{grad}^M Y - \text{grad}^{m+1} Y] = B(X, Y) - g(X, Y)\eta(\gamma)\eta. \]

So that, the mean curvature function \( \tilde{f} \) of \( (M^m, \tilde{g}) \) (in \( \mathbb{R}^{m+1} \)) is given by \( \tilde{f} = -\eta(\gamma)e^{-\gamma} \). Indeed, by taking traces in (21), we obtain
\[ e^{2\gamma}\tilde{H} = H - \eta(\gamma)\eta. \]

Since \( (M^m, g) \) is minimal in \( (\mathbb{R}^{m+1}, \tilde{g}) \), we find that \( \tilde{H} = -e^{-2\gamma}\eta(\gamma)\eta \), that is \( \tilde{H} = -e^{-\gamma}\eta(\gamma)\tilde{g} \).

With the new notations the equation (6) for \( p \)-biharmonic hypersurface in the conformally flat space becomes
\[ \begin{cases} 
-\Delta(\tilde{f}) + \tilde{f}A_{\eta}^2 - f\tilde{R} - m(p - 2)\tilde{f}^3 & = 0; \\
2A(\text{grad} \tilde{f}) - 2f(\tilde{R} - m(p - 2 + \frac{m}{2})\text{grad} \tilde{f}^2 & = 0, 
\end{cases} \]

A straightforward computation yields
\[ \tilde{\Omega}^{\text{Ricci}}_{\eta} = e^{-2\gamma}\left[ \text{Ricci}^{\text{Ricci}}_{\eta} - \Delta^{\text{Ricci}}_{\eta} - \text{grad}^{\text{Ricci}}_{\eta}\gamma \right] + (1 - m)|\text{grad}^{\text{Ricci}}_{\eta}\gamma|^2 \eta - (1 - m)\eta(\gamma)\text{grad}^{\text{Ricci}}_{\eta}\gamma; \]
\[ \tilde{\Omega}^{\text{Ricci}}_{\eta}(\tilde{\eta}, \tilde{\eta}) = \tilde{h}(\text{Ricci}^{\text{Ricci}}_{\eta}, \tilde{\eta}, \tilde{\eta}) \]
\[ = e^{-2\gamma}h(\text{Ricci}^{\text{Ricci}}_{\eta}, \eta, \eta) \]
\[ = e^{-2\gamma}h(\text{Ricci}^{\text{Ricci}}_{\eta} - \Delta^{\text{Ricci}}_{\eta} - \text{grad}^{\text{Ricci}}_{\eta}\gamma) + (1 - m)|\text{grad}^{\text{Ricci}}_{\eta}\gamma|^2 \eta - (1 - m)\eta(\gamma)\text{grad}^{\text{Ricci}}_{\eta}\gamma; \]
\[ = e^{-2\gamma}\left[ -\Delta^{\text{Ricci}}_{\eta} - \text{grad}^{\text{Ricci}}_{\eta}\gamma + (1 - m)\text{Hess}^{\text{Ricci}}_{\eta}(\eta, \eta) + (1 - m)|\text{grad}^{\text{Ricci}}_{\eta}\gamma|^2 \right] - (1 - m)\eta(\gamma)^2; \]
\[ \tilde{\Omega}^{\text{Ricci}}_{\eta}(\tilde{\eta}, \tilde{\eta}) = \sum_{i=1}^{m} h(\text{Ricci}^{\text{Ricci}}_{\eta}, \tilde{\eta}, e_i)e_i \]
\[ = (1 - m)e^{-3\gamma}\sum_{i=1}^{m} \left[ h(\text{grad}^{\text{Ricci}}_{\eta}\gamma, e_i)e_i - \eta(\gamma)h(\text{grad}^{\text{Ricci}}_{\eta}\gamma, e_i)e_i \right] \]
\[ = (1 - m)e^{-3\gamma}\left[ \sum_{i=1}^{m} h(\text{grad}^{\text{Ricci}}_{\eta}\gamma, e_i)e_i - \eta(\gamma)\text{grad}^{\text{Ricci}}_{\eta}\gamma \right] \]
\[ = (1 - m)e^{-3\gamma}\left[ \sum_{i=1}^{m} h(\text{grad}^{\text{Ricci}}_{\eta}\gamma, e_i) + \sum_{i=1}^{m} h(\text{grad}^{\text{Ricci}}_{\eta}\gamma, e_i - \eta(\gamma) \text{grad}^{\text{Ricci}}_{\eta}\gamma \right] \]
\[ = (1 - m)e^{-3\gamma}\left[ \text{grad}^{\text{Ricci}}_{\eta}\gamma + \eta(\gamma)\text{grad}^{\text{Ricci}}_{\eta}\gamma \right] \]
\[ = (1 - m)e^{-3\gamma}\left[ \text{grad}^{\text{Ricci}}_{\eta}\gamma + A(\text{grad}^{\text{Ricci}}_{\eta}\gamma - \eta(\gamma)\text{grad}^{\text{Ricci}}_{\eta}\gamma) \right]; \]
\( \Delta(f) = e^{-2\gamma}[\Delta(f) + (m - 2)d\bar{f}(\text{grad}^M \gamma)] \)
\[ = e^{-2\gamma}[-\Delta(\eta(\gamma)e^{-\gamma}) - (m - 2)(\text{grad}^M \gamma)(\eta(\gamma)e^{-\gamma})]; \quad (25) \]

\[ |\tilde{A}|^2_g = \sum_{i=1}^{m} g(\tilde{A}e_i, \tilde{A}e_i) \]
\[ = \sum_{i=1}^{m} g(\tilde{A}e_i, \tilde{A}e_i) \]
\[ = \sum_{i=1}^{m} h(\nabla_{\tilde{e}_i}^m \tilde{\eta}, \nabla_{\tilde{e}_i}^m \tilde{\eta}) \]
\[ = \sum_{i=1}^{m} h(\nabla_{\tilde{e}_i}^m \tilde{\eta} + e_i(\gamma)\tilde{\eta} + \tilde{\eta}(\gamma)e_i, \nabla_{\tilde{e}_i}^m \tilde{\eta} + e_i(\gamma)\tilde{\eta} + \tilde{\eta}(\gamma)e_i) \]
\[ = \sum_{i=1}^{m} [h(\nabla_{\tilde{e}_i}^m \tilde{\eta}, \nabla_{\tilde{e}_i}^m \tilde{\eta}) + 2\tilde{\eta}(\gamma)h(\nabla_{\tilde{e}_i}^m \tilde{\eta}, e_i) + e_i(\gamma)^2e^{-2\gamma} + 2e_i(\gamma)h(\nabla_{\tilde{e}_i}^m \tilde{\eta}, \tilde{\eta})] + m\tilde{\eta}(\gamma)^2. \quad (26) \]

The first term of (26) is given by
\[ \sum_{i=1}^{m} h(\nabla_{\tilde{e}_i}^m \tilde{\eta}, \nabla_{\tilde{e}_i}^m \tilde{\eta}) = \sum_{i=1}^{m} h(-e^{-\gamma}e_i(\gamma)\eta + e^{-\gamma}\nabla_{\tilde{e}_i}^m \tilde{\eta}, -e^{-\gamma}e_i(\gamma)\eta + e^{-\gamma}\nabla_{\tilde{e}_i}^m \tilde{\eta}) \]
\[ = \sum_{i=1}^{m} [e^{-2\gamma}e_i(\gamma)^2 + e^{-2\gamma}h(\nabla_{\tilde{e}_i}^m \tilde{\eta}, \nabla_{\tilde{e}_i}^m \tilde{\eta})] \]
\[ = e^{-2\gamma}|\text{grad}^M \gamma|^2 + e^{-2\gamma}|A|^2. \]

The second term of (26) is given by
\[ 2\tilde{\eta}(\gamma) \sum_{i=1}^{m} h(\nabla_{\tilde{e}_i}^m \tilde{\eta}, e_i) = -2e^{-\gamma}\eta(\gamma) \sum_{i=1}^{m} h(e^{-\gamma}\eta, \nabla_{\tilde{e}_i}^m e_i) \]
\[ = -2me^{-2\gamma}\eta(\gamma)h(\eta, H) \]
\[ = 0. \]

Here \( H = 0 \). We have also
\[ 2 \sum_{i=1}^{m} e_i(\gamma)h(\nabla_{\tilde{e}_i}^m \tilde{\eta}, \tilde{\eta}) = \sum_{i=1}^{m} e_i(\gamma)e_ih(\tilde{\eta}, \tilde{\eta}) \]
\[ = \sum_{i=1}^{m} e_i(\gamma)e_i(e^{-2\gamma}) \]
\[ = -2e^{-2\gamma} \sum_{i=1}^{m} e_i(\gamma)^2 \]
\[ = -2e^{-2\gamma}|\text{grad}^M \gamma|^2. \]

Thus
\[ |\tilde{A}|^2_g = e^{-2\gamma}|A|^2 + m e^{-2\gamma}\eta(\gamma)^2. \quad (27) \]

We compute
\[ \text{grad}^M f = e^{-2\gamma} \sum_{i=1}^{m} e_i(\bar{f})e_i \]
\[ = -e^{-2\gamma}\text{grad}^M(\eta(\gamma)e^{-\gamma}); \quad (28) \]
and the following
\[
\tilde{A}(\tilde{\nabla} f) = -\tilde{\nabla} f \tilde{\eta} =
\]
\[
-\tilde{\nabla} f e^{-\gamma} \eta
\]
\[
e^{-\gamma}(\nabla f)(\gamma) \eta - e^{-\gamma} \tilde{\nabla} f \eta
\]
\[
e^{-3\gamma} \nabla M (\eta e^{-\gamma})(\gamma) \eta + e^{-3\gamma} \tilde{\nabla} M (\eta e^{-\gamma}) \eta
\]
\[
e^{-3\gamma} \nabla M (\eta e^{-\gamma})(\gamma) \eta + e^{-3\gamma} \tilde{\nabla} M (\eta e^{-\gamma}) \eta
\]
\[
e^{-3\gamma} \eta (\gamma) \nabla M (\eta e^{-\gamma}) - e^{-3\gamma} \tilde{A}(\nabla M \eta e^{-\gamma}).
\]  
(29)

Substituting (23) – (29) in (22), and by simplifying the resulting equation we obtain the system (18).

Remark 5.

1. Using Theorem 4 we can construct many examples for proper p-biharmonic hypersurfaces in the conformally flat space.

2. If the functions \( \gamma \) and \( \eta(\gamma) \) are non-zero constants on \( M \), then according to Theorem 4 the hypersurface \( (M^m, \tilde{g}) \) is p-biharmonic in \( (\mathbb{R}^{m+1}, \tilde{h}) \) if and only if

\[ |A|^2 = m(1-p)\eta(\gamma)^2 - m\eta(\gamma) \]

Example 6. The hyperplane \( i : \mathbb{R}^m \leftrightarrow (\mathbb{R}^{m+1}, e^{2\gamma(x)}h), x \in (x, c), \) where \( \gamma \in C^\infty(\mathbb{R}), h = \sum_{i=1}^m dx_i^2 + dz^2 \), and \( c \in \mathbb{R} \), is proper p-biharmonic if and only if \( (1-p)\gamma'(c)^2 - \gamma''(c) = 0 \). Note that, the smooth function

\[ \gamma(z) = \ln(c_1(p-1)z + c_2(p-1)) \]

\[ p-1 \]

\[ c_1, c_2 \in \mathbb{R}, \]

is a solution of the previous differential equation (for all c).

Example 7. Let \( M \) be a surface of revolution in \( \{(x, y, z) : \mathbb{R}^3 | z > 0 \} \). If \( M \) is part of a plane orthogonal to the axis of revolution, so that \( M \) is pararametrized by

\[ (x_1, x_2) \mapsto (f(x_2) \cos(x_1), f(x_2) \sin(x_1), c), \]

for some constant \( c > 0 \). Here \( f(x_2) > 0 \). Then, \( M \) is minimal, and according to Theorem 4 the surface \( M \) is proper p-biharmonic in 3-dimensional hyperbolic space \( (\mathbb{H}^3, z^\frac{2}{p-2}h), \) where \( h = dx^2 + dy^2 + dz^2 \).

Open Problems.

1. If \( M \) is a minimal surface of revolution contained in a catenoid, that is \( M \) is parametrized by

\[ (x_1, x_2) \mapsto \left(a \cosh \left(\frac{x_2}{a} + b \right) \cos(x_1), a \cosh \left(\frac{x_2}{a} + b \right) \sin(x_1), x_2 \right), \]

where \( a \neq 0 \) and \( b \) are constants. Is there \( p \geq 2 \) and \( \gamma \in C^\infty(\mathbb{R}^3) \) such that \( M \) is proper p-biharmonic in \( (\mathbb{R}^3, e^{2\gamma} (dx^2 + dy^2 + dz^2)) \)?

2. Is there a proper p-biharmonic submanifolds in Euclidean space \( (\mathbb{R}^n, dx_1^2 + ... + dx_n^2) \)?

References

[1] P. Baird, S. Gudmundsson, p-Harmonic maps and minimal submanifolds, Math. Ann. 294 (1992), 611-624.
[2] P. Baird, J. C. Wood, Harmonic morphisms between Riemannain manifolds, Clarendon Press Oxford 2003.
[3] B. Bojarski and T. Iwaniec, p-Harmonic equation and quasiregular mappings, Partial differential equations (Warsaw, 1984), 25-38, Banach Center Publ., vol. 19, PWN, Warsaw, 1987.
[4] B-Y. Chen, Total Mean Curvature and Submanifolds of Finite Type, Series in Pure Mathematics, 1. World Scientific Publishing Co., Singapore, 1984.
M. Djaa and A. M. Cherif, *On Generalized f-biharmonic Maps and Stress f-bienergy Tensor*. Journal of Geometry and Symmetry in Physics JGSP 29 (2013), pp. 65-81.

J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109-160.

A. Fardoun, *On equivariant p-harmonic maps*, Ann.Inst. Henri. Poincare, 15 (1998), 25-72.

Y. Han and W. Zhang, *Some results of p-biharmonic maps into a non-positively curved manifold*, J. Korean Math. Soc. 52 (2015), No. 5, pp. 1097-1108.

G. Y. Jiang, *2-Harmonic maps between Riemannian manifolds*, Annals of Math., China, 7A(4) (1986), 389-402.

E. Loubeau, S. Montaldo, And C. Oniciuc, *the stress-energy tensor for biharmonic maps*, arXiv:math/0602021v1 [math.DG] 1 Feb 2006.

A. Mohammed Cherif, *On the p-harmonic and p-biharmonic maps*, J. Geom. (2018) 109:41

C. Oniciuc, *Biharmonic maps between Riemannian manifolds*, An. Stiint. Univ. Al.I. Cuza Iasi Mat (N.S.) 48 (2002), 237-248.

O’Neil, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

Ye-Lin Ou, *Biharmonic hypersurfaces in Riemannian manifolds*, Pacific Journal of Mathematics, Vol. 248, No. 1, 2010.

Y. Xin, *Geometry of harmonic maps*, Fudan University, 1996.