A CONDITIONAL QUASI-GREEDY BASIS OF $l_1$

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Abstract. We show that the Lindenstrauss basic sequence in $l_1$ may be used to construct a conditional quasi-greedy basis of $l_1$, thus answering a question of Wojtaszczyk. We further show that the sequence of coefficient functionals for this basis is not quasi-greedy.

1. Introduction

In what follows $\{e_i\}_{i=1}^\infty$ denotes the standard unit vector basis of $l_1$. In [3], the following concept was studied:

Definition. Let $X$ be a Banach space with dual $X^*$ and let $(x_i, x_i^*)_{i \in F}$ be a fundamental biorthogonal system in $X \times X^*$ with $\inf_{i \in F} \|x_i\| > 0$ and $\sup_{i \in F} \|x_i^*\| < \infty$. For $m \in \mathbb{N}$, define the operator $G_m$ by

$$G_m(x) = \sum_{i \in A} x_i^*(x)x_i,$$

where $A \subset F$ is a set of cardinality $m$ such that $|x_i^*(x)| \geq |x_k^*(x)|$ whenever $i \in A$ and $k \notin A$ (note that the set $A$ depends on $x$ and may not be unique; nevertheless, $G_m$ is well-defined). Then $(x_i, x_i^*)_{i \in F}$ is a quasi-greedy system provided that the operators $G_m$ satisfy:

$$\lim_m G_m(x) = x, \quad \text{for each } x \in X.$$

Equivalently, by [3, Theorem 1], $(x_i, x_i^*)_{i \in F}$ is a quasi-greedy system if there exists a constant $C$ so that for every $x \in X$ and for every $m \in \mathbb{N}$, we have

$$\|G_m(x)\| \leq C\|x\|.$$

If $\{x_i\}_{i=1}^\infty$ is, in addition, basic, it will be said to be a quasi-greedy basis. A sequence $\{z_i\}_{i=1}^\infty$ is unconditional for constant coefficients if
there exist positive numbers $c$ and $C$ such that
\[
  c \sum_{i=1}^{m} z_i \leq \sum_{i=1}^{m} \epsilon_i z_i \leq C \sum_{i=1}^{m} z_i
\]
for any positive integer $m$ and for any sequence of signs $\{\epsilon_i\}_{i=1}^{m}$.

Remark. By \[5, Proposition 2\] a quasi-greedy basis is necessarily unconditional for constant coefficients.

In \[5\], Wojtaszczyk shows that, for $1 < p < \infty$, the space $l_p$ possesses a conditional quasi-greedy basis. Since, by the Remark, a quasi-greedy basis of $l_1$ is quite close to being equivalent to $\{e_i\}_{i=1}^{\infty}$, and since $\{e_i\}_{i=1}^{\infty}$ is the unique normalized unconditional basis of $l_1$ up to equivalence \[3\], it is of interest to find a conditional quasi-greedy basis of $l_1$. In particular, the existence of such a basis would show that $l_1$ does not have a unique (up to equivalence) normalized basis that is unconditional for constant coefficients. In this note, we show that indeed such a basis exists.

Our example is derived from the basic sequence constructed by Lindenstrauss in \[2\]. This is a monotone, conditional basic sequence in $l_1$ whose closed linear span is a $\mathcal{L}_1$-space possessing no unconditional basis. We denote this sequence by $\{x_i\}_{i=1}^{\infty}$ and its associated sequence of coefficient functionals by $\{x_i^*\}_{i=1}^{\infty}$. They are defined as follows: For $i \in \mathbb{N}$,
\[
x_i = e_i - \frac{1}{2}(e_{2i+1} + e_{2i+2}).
\]
Considering $x_i^*$ as an element of $l_\infty/[x_i]^{\perp}$, we write $x_i^* = y_i^* + [x_i]^{\perp}$ where the $y_i^* \in l_\infty$ are as defined by Holub and Retherford (see \[4\]):
For each $i \in \mathbb{N}$, let $\alpha_i$ be the finite sequence of positive integers defined by the following conditions:
1) $\alpha_i(1) = i$.
2) $\alpha_i(j) = \alpha_i(j-1) - (\lfloor \alpha_i(j-1)/2 \rfloor + 1)$ for admissible $j < i$ (that is, such that $\alpha_i(j) > 0$), where $\lfloor k \rfloor$ denotes the greatest integer less than or equal to $k$.

Then $y_i^*$ is defined by
\[
y_i^* = \sum_{j=1}^{\lfloor \alpha_i \rfloor} \left(\frac{1}{2}\right)^{j-1} e_{\alpha_i(j)}
\]
Thus, for example,
\[
\begin{align*}
  y_1^* &= (1, 0, \ldots) \\
  y_2^* &= (0, 1, 0, \ldots) \\
  y_3^* &= \left(\frac{1}{2}, 0, 1, 0, \ldots\right) \\
  y_4^* &= \left(\frac{1}{2}, 0, 0, 1, 0, \ldots\right) \\
  y_5^* &= (0, \frac{1}{2}, 0, 0, 1, 0, \ldots) \\
  y_6^* &= (0, \frac{1}{2}, 0, 0, 0, 1, 0, \ldots) \\
  y_7^* &= \left(\frac{1}{4}, 0, \frac{1}{2}, 0, 0, 1, 0, \ldots\right) \\
  y_8^* &= \left(\frac{1}{4}, 0, \frac{1}{2}, 0, 0, 0, 1, 0, 0, \ldots\right).
\end{align*}
\]

The properties of \(\{x_i\}_{i=1}^\infty\) which we shall use in the sequel are summarized in the following fact (see [2], [3], and, also, [4]).

**Fact.** The sequence \(\{x_i\}_{i=1}^\infty\) satisfies:

1) \(\{x_i\}_{i=1}^\infty\) is a monotone basic sequence ([4, p. 455]).
2) \([x_i]\) has no unconditional basis ([4, p. 455]).
3) For \(n \in \mathbb{N}\), there exists an isomorphism \(T_n\) from \([x_i : 1 \leq i \leq n]\) onto \(l_1^n\) satisfying \(\|T_n\|\|T_n^{-1}\| \leq 2\) ([3, Ex. 8.1]).

2. Results

We now construct the basis heralded in the Introduction. To do this, it suffices by [3, Proposition 3] to construct such a basis in a space isomorphic to \(l_1\). Towards this end, define, for each \(n \in \mathbb{N}\),
\[
F_n = [x_i : 1 \leq i \leq n].
\]
Let
\[
\mathfrak{X} = \left(\bigoplus_{i=1}^\infty F_i\right)_1.
\]
We claim that the natural basis \(\{\tilde{x}_i\}_{i=1}^\infty\) of \(\mathfrak{X}\) obtained from the \(x_i\)'s is the desired sequence. Indeed, it follows from the Fact that \(\{\tilde{x}_i\}_{i=1}^\infty\) is a monotone, conditional basis of \(\mathfrak{X}\), and that \(\mathfrak{X}\) is isomorphic to \(l_1\). Moreover, assuming for the moment that \(\{x_i\}_{i=1}^\infty\) is quasi-greedy, for \(m \in \mathbb{N}\) and \(y = \sum y_i \in \mathfrak{X}\), we have
\[
\|G_m(y)\| \leq \sup_{k \in K_m} \sum_{i=1}^\infty \|G_{k(i)}(y_i)\| \leq C \sum_{i=1}^\infty \|y_i\| = C\|y\|,
\]
where $C$ is as in the Definition, $K_m = \{ k : \mathbb{N} \to \mathbb{N} \cup \{0\} : \sum_{i=1}^{\infty} k(i) = m \}$, and $G_0(x) = 0$ for each $x \in \mathcal{X}$ (note that the operators $G_j$ appearing in the above inequality are defined with respect to two different sequences). Thus, the fact that $\{\tilde{x}_i\}_{i=1}^{\infty}$ is quasi-greedy will be a consequence of the following theorem.

**Theorem.** The sequence $\{x_i\}_{i=1}^{\infty}$ satisfies

\[
3 \left\| \sum_{i \in S_1 \cup S_2} \alpha_i x_i \right\| \geq \left\| \sum_{i \in S_1} \alpha_i x_i \right\|,
\]

whenever $S_1$ and $S_2$ are disjoint finite subsets of $\mathbb{N}$ with

\[
\min_{i \in S_1} |\alpha_i| \geq \max_{i \in S_2} |\alpha_i|.
\]

**Proof.** For $A \subseteq \mathbb{N}$ and $x \in l_1$, we denote by $P_A x$ the vector in $l_1$ whose $j^{\text{th}}$-coordinate is $x(j)$ if $j \in A$ and zero otherwise. Let $S_1$, $S_2$, and $\{\alpha_i\}_{i \in S_1 \cup S_2}$ be as above. Set

\[
x = \sum_{i \in S_1} \alpha_i x_i \quad \text{and} \quad y = \sum_{i \in S_2} \alpha_i x_i,
\]

and define the sets:

\[
A_0 = \left\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = 1 \right\},
\]

\[
B_0 = \left\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = -1/2 \right\},
\]

and

\[
C_0 = \left\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = 1/2 \right\}.
\]

First, we concentrate our attention on the set $B_0$. We define the sets:

\[
W_1 = \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_0 \},
\]

\[
A_1 = \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_1 \},
\]

and

\[
B_1 = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_1 \}.
\]
Finally, set
\[ y_0 = x \text{ and } y_1 = \sum_{i \in W_1} \alpha_i x_i. \]

Note that \( A_1, B_1, B_0, \) and \( C_0 \) are mutually disjoint. We also have from the triangle inequality that
\[ \| P_{B_0} y_0 \| \leq \| P_{B_0} (y_0 + y_1) \| + \| P_{B_0} y_1 \|. \]

But, since \( \| P_{B_0} y_1 \| = \| P_{A_1} y_1 \| + \| P_{B_1} y_1 \| \), we obtain from \( 3 \) that
\[ (* \# \#) \| P_{B_0} (y_0 + y_1) \| + \| P_{B_1} y_1 \| \geq \| P_{B_0} y_0 \| - \| P_{A_1} y_1 \|. \]

Concentrating our attention now on the set \( B_1 \), we let
\[ W_2 = \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_1 \}, \]
\[ A_2 = \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_2 \}, \]
and
\[ B_2 = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_2 \}. \]

Set
\[ y_2 = \sum_{i \in W_2} \alpha_i x_i. \]

Then \( A_1, A_2, B_1, B_2, B_0, \) and \( C_0 \) are mutually disjoint and, as above, we have
\[ (*) \| P_{B_1} (y_1 + y_2) \| + \| P_{B_2} y_2 \| \geq \| P_{B_1} y_1 \| - \| P_{A_2} y_2 \|. \]

In general, at the \( l^{th} \) step of the induction, we set
\[ W_l = \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_{l-1} \}, \]
\[ A_l = \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_l \}, \]
and
\[ B_l = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_l \}. \]

Set
\[ y_l = \sum_{i \in W_l} \alpha_i x_i. \]
Then the sets $A_j, B_j$ (1 ≤ $j$ ≤ $l$), $B_0$, and $C_0$ are mutually disjoint, and we have

(*) \[ \|P_{B_{i-1}}(y_{i-1} + y_i)\| + \|P_{B_i}y_i\| \geq \|P_{B_{i-1}}y_{i-1}\| - \|P_{A_i}y_i\|. \]

This process must end at some stage $k$ with $W_{k+1} = \emptyset$. Summing the inequalities (*) so obtained and simplifying, we have:

(4) \[ \sum_{i=1}^{k} \|P_{B_{i-1}}(y_{i-1} + y_i)\| + \|P_{B_i}y_k\| \geq \|P_{B_0}y_0\| - \sum_{i=1}^{k} \|P_{A_i}y_i\|. \]

But, for each $i = 0, 1, \ldots, k$, we have $B_i \cap \text{supp} \ y_j = \emptyset$ for $j \notin \{i, i+1\}$. Moreover, the mutually disjoint sets $A_i$ ($i = 1, 2, \ldots, k$) are contained in $A_0$. Recalling that $y_0 = x$, from these two observations the inequality (4) reduces to

(5) \[ \sum_{i=0}^{k} \|P_{B_i}(x + y)\| \geq \|P_{B_0}x\| - \|P_{A_0}y\|. \]

Since the sets $C_0, A_0, B_0, B_1, B_2, \ldots, B_k$ are mutually disjoint, we have

\[ \|x + y\| \geq \|P_{A_0}(x + y)\| + \sum_{i=0}^{k} \|P_{B_i}(x + y)\| + \|P_{C_0}(x + y)\|; \]

and so, using (4):

(6) \[ \|x + y\| \geq \|P_{A_0}(x + y)\| - \|P_{A_0}y\| + \|P_{B_0}x\| + \|P_{C_0}x\|. \]

Using (2), we have $\|P_{A_0}(x + y)\| \geq \|P_{A_0}y\|$, and thus from (2) we have

(7) \[ \|x + y\| \geq \|P_{B_0}x\| + \|P_{C_0}x\|. \]

Also, by (2), we have $2\|P_{A_0}(x + y)\| \geq \|P_{A_0}x\|$; thus,

(8) \[ 2\|x + y\| \geq \|P_{A_0}x\|. \]

Since $\|x\| = \|P_{A_0}x\| + \|P_{B_0}x\| + \|P_{C_0}x\|$, we may now obtain (3) by adding the inequalities (7) and (8). \(\square\)

**Remark.** We note that the sequence of coefficient functionals for $\{\tilde{x}_i\}_{i=1}^{\infty}$ is not quasi-greedy. To see this, it is enough to show that $\{x^*_i\}_{i=1}^{\infty}$ is not quasi-greedy. Towards this end, note that

(9) \[ \left\| \sum_{i=1}^{2^n+1-2} (-1)^i x^*_i \right\| \leq \left\| \sum_{i=1}^{2^n+1-2} (-1)^i y^*_i \right\| = 1. \]
However, defining
\[ \{\alpha_n\}_{n=1}^{\infty} = \{1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots \} \]
and
\[ z_n = \sum_{i=1}^{2^{n+1}-2} \alpha_i x_i = e_1 + e_2 + \sum_{i=1}^{2^{n+1}-2} \frac{1}{2^n+1} e_2 \cdot (2^{n+1} - 1 + i), \]
we obtain
\[ \left( \sum_{i=1}^{2^{n+1}-2} x_i^* \right) \geq \frac{1}{\|z_n\|} \sum_{i=1}^{2^{n+1}-2} x_i^* (z_n) = \frac{1}{4} \left( \sum_{i=1}^{2^{n+1}-2} x_i^* , e_1 + e_2 \right) = \frac{n}{2}. \]

It follows from (9), (10), and the Remark of the Introduction that \( \{x_i^*\}_{i=1}^{\infty} \) is not quasi-greedy.

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