Canonical quantization of spin 1 matter fields.
Preliminary results.

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Abstract. In this work we present preliminary results on the canonical quantization of spin 1 matter fields, i.e. massive fields transforming in the \((1,0) \oplus (0,1)\) representation of the Homogeneous Lorentz Group. The classical field theory is based on the projection onto subspaces of well defined parity in this representation.

1. Introduction
The Standard Model (SM) is constructed using only a few of the representations of the Homogeneous Lorentz Group (HLG): the \((0,0)\) for the Higgs, \((1/2,0)\) and \((0,1/2)\) for quarks and leptons and \((1/2,1/2)\) for the gauge bosons. The possible inclusion of higher spin matter fields in extensions of the SM requires to have consistent quantum field theory for \(j > 1\) which is a long standing problem. Recently, the covariant structure of operators acting on the \((j,0) \oplus (0,j)\) representation was elucidated based on the covariant properties of parity [1] and a proposal for the description of fields transforming in this representation was put forth based on the projection onto subspaces of well defined parity [2]. Here we present preliminary results on the canonical quantization of the simplest case, \(j = 1\).

2. Parity and covariant basis for \((1,0) \oplus (0,1)\)
The Homogeneous Lorentz Group (HLG) generated by rotations and boosts \(\{J, K\}\) is isomorphic to the group \(SU(2)_A \otimes SU(2)_B\) generated by
\[
A = \frac{1}{2}(J + iK), \quad B = \frac{1}{2}(J - iK).
\]
(1)
The quantum states with well defined transformation properties under the HLG can be labeled as \(|a, m_a; b, m_b\rangle\) corresponding to the eigenvalues of \(\{A^2, A_3, B^2, B_3\}\) and the irreducible representations are labeled by the corresponding two \(SU(2)\) good quantum numbers \((a, b)\).
Under parity \(J \rightarrow J, K \rightarrow -K\) and \(A \rightarrow B, B \rightarrow A\). Parity maps \((a, b) \rightarrow (b, a)\), i.e. for \(a \neq b\), the simple representations \((a, b)\) are not invariant subspaces under this transformation. The construction of theories invariant under parity requires to use fields transforming in the \((a, b) \oplus (b, a)\). For the special case \(b = 0\), we get \(A = J\), i.e. the representation is \((j,0) \oplus (0,j)\). In this case, using the basis \(|j, m)_{R}\) and \(|j, m)_{L}\) for the ”right” \((j,0)\) and ”left” \((0,j)\) representations, the generators of the HLG are given by
\[
M_{ij} = \epsilon_{ijk} \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix}, \quad M^{0i} = \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}
\]
(2)
where $\tau_i$ are the conventional rotation matrices for spin $j$. In general for the $(a, b)$ representations we can define the chirality operator

$$\chi = \frac{i}{4a(a+1) - 4b(b+1)} M_{\mu\nu} \tilde{M}^{\mu\nu}. \quad (3)$$

Since $\chi$ is proportional to a Casimir operator of the HLG, we have the commutation rule

$$[\chi, M_{\mu\nu}] = 0. \quad (4)$$

In the rest frame, the parity and chirality operators for the specific $(j, 0) \oplus (0, j)$ representations are given by

$$\chi = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \Pi = \left( \begin{array}{cc} 0 & \mathbf{1}_{2j+1} \\ \mathbf{1}_{2j+1} & 0 \end{array} \right). \quad (5)$$

Using the explicit form of the boost generators in Eq. (2), $K_i = \tau_i$ and the Jordan algebra satisfied by these matrices it is possible to explicitly construct the boost operator. Using it on the rest frame states $u(0, \lambda)$ we obtain the states of arbitrary momentum $u(p, \lambda)$.

It was shown in [1] that the set \{$1, \chi, S^{\mu\nu}, \chi S^{\mu\nu}, M_{\mu\nu}, C^{\mu\nu\alpha\beta}$\} form a covariant basis for operators acting on the $(1, 0) \oplus (0, 1)$ representation where

$$S^{\mu\nu} = \Pi \left[ \eta^{\mu\nu} - i \left( \eta^{0\mu} M^{0\nu} + \eta^{0\nu} M^{0\mu} \right) - \left\{ M^{0\mu}, M^{0\nu} \right\} \right], \quad (6)$$

is a symmetric traceless tensor ($S^{\mu\mu} = 0$). The $C$ tensor is defined as

$$C^{\mu\nu\alpha\beta} = 4 \left\{ M^{\mu\nu}, M^{\alpha\beta} \right\} + 2 \left\{ M^{\mu\alpha}, M^{\nu\beta} \right\} - 2 \left\{ M^{\mu\beta}, M^{\nu\alpha} \right\} - 8 \left( \eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha} \right). \quad (7)$$

It has the symmetries of the Weyl tensor

$$C^{\mu\nu\alpha\beta} = -C^{\nu\mu\alpha\beta} = -C^{\alpha\mu\nu\beta}, \quad C^{\mu\nu\alpha\beta} = C^{\alpha\beta\mu\nu}. \quad (8)$$

The contraction of any index pair vanishes and it satisfies the Bianchi identity

$$C^{\mu\nu\alpha\beta} + C^{\mu\alpha\beta\nu} + C^{\mu\beta\alpha\nu} = 0. \quad (9)$$

The $S$ tensor satisfy the following commutation relations

$$i \left[ S^{\mu\nu}, S^{\alpha\beta} \right] = \eta^{\mu\alpha} M^{\nu\beta} + \eta^{\mu\beta} M^{\nu\alpha} + \eta^{\nu\alpha} M^{\mu\beta} + \eta^{\nu\beta} M^{\mu\alpha},$$

$$\left\{ S^{\mu\nu}, S^{\alpha\beta} \right\} = \frac{4}{3} \left( \eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\nu\alpha} \eta^{\mu\beta} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta} \right) - \frac{1}{6} \left( C^{\alpha\nu\mu\beta} + C^{\mu\beta\alpha\nu} \right), \quad (10)$$

and anticommutes with the chirality operator

$$\{ \chi, S^{\mu\nu} \} = 0. \quad (11)$$

States of well defined parity satisfy the corresponding projection condition in the rest frame

$$\frac{1}{2} (1 + \Pi) u(0, \lambda) = u(0, \lambda). \quad (12)$$

Boosting this equation we get

$$(S^{\mu\nu} p_{\mu} p_{\nu} - m^2) u(p, \lambda) = 0. \quad (13)$$
Defining the corresponding wave function as usual

$$\psi(x) = u(p, \lambda)e^{-ip \cdot x}$$  \(14\)

we get

$$(S^{\mu \nu} \partial_\mu \partial_\nu + m^2)\psi(x) = 0.$$  \(15\)

This equation was considered by Weinberg and Joos long ago [3]. It can be derived from the following Lagrangian

$$\mathcal{L} = \partial_\mu \bar{\psi}S^{\mu \nu} \partial_\nu \psi - m^2 \bar{\psi} \psi$$  \(16\)

where $\bar{\psi} = \psi^\dagger \Pi$. This Lagrangian has an obvious global $U(1)$ symmetry whose conserved current is

$$J^\alpha = i q \left[ (\partial_\mu \bar{\psi}) S^\mu \psi - \bar{\psi} S^\alpha \partial_\nu \psi \right].$$  \(17\)

The free Lagrangian has a chiral decomposition

$$\mathcal{L} = \partial_\mu \bar{\psi} R S^{\mu \nu} \partial_\nu \psi R + \partial_\mu \bar{\psi} L S^{\mu \nu} \partial_\nu \psi L - m^2 (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$$  \(18\)

where $\psi_R = \frac{1}{2} (1 + \chi) \psi$ and $\psi_L = \frac{1}{2} (1 - \chi) \psi$, such that in the massless case it is symmetric under independent rotations of the left and right components, i.e., it is chiral symmetric.

3. Canonical quantization

First we expand the corresponding fields

$$\Psi(x) = \sum_{p, r} \frac{1}{\sqrt{2V_0 p_0}} \left[ c_r(p) u_r(p) e^{-ip \cdot x} + d^+_r(p) u^*_r(p) e^{ip \cdot x} \right],$$  \(19\)

$$\bar{\Psi}(x) = \sum_{p, r} \frac{1}{\sqrt{2V_0 p_0}} \left[ c^+_r(p) \bar{u}_r(p) e^{ip \cdot x} + d_r(p) \bar{u}^*_r(p) e^{-ip \cdot x} \right].$$  \(20\)

Here, $r$ stands for the polarization and we use the shorthand notation $u_r(p) = u(p, r)$. The charged conjugated spinor is given by [2]

$$u^c(p, r) = \Gamma u^*(p, r)$$  \(21\)

with the matrix

$$\Gamma = \begin{pmatrix} 0 & U \\ U^{-1} & 0 \end{pmatrix}$$  \(22\)

where $U = e^{-i\pi J_3}$. The coefficients $c_r(p), d_r(p), c^+_r(p), d^+_r(p)$ satisfy the usual commutation relations. The spinors $u, u^c$ satisfy

$$\left( S(p) - m^2 \right) u_r(p) = 0, \quad \left( S(p) - m^2 \right) u^*_r(p) = 0,$$  \(23\)

where $S(p) \equiv S^{\alpha \mu} p_\mu p_\alpha$. Notice that both particle and antiparticle satisfy the same free equation, this occurs because charge conjugation and parity commute [2]. In spite of this, it can be shown that the interacting equations satisfied by the field and its charge conjugated differ in the sign of the coupling.

The momentum densities of the field are

$$\Pi_d = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}_{d,0}} = S^{\alpha \mu}_{da} (\partial_\mu \Psi)_a,$$  \(24\)
Now, we rewrite the term $S$ in order to reduce this equation we use Eq. (10) to show that

$$
\Pi_d = \frac{\partial \mathcal{L}}{\partial \Psi_{d,0}} = (\partial_\mu \bar{\Psi})_a S_{ad}^{\partial \mu}.
$$

(25)

The commutators of these momenta with their respective fields are given by

$$
[\Pi_d, \Psi_b] = (\partial_\mu \bar{\Psi})_a S_{ad}^{\partial \mu} \Psi_b - \Psi_b (\partial_\mu \bar{\Psi})_a S_{ad}^{\partial \mu},
$$

(26)

and

$$
[\bar{\Pi}_d, \bar{\Psi}_b] = S_{da}^{\partial \mu} (\partial_\mu \Psi)_a \bar{\Psi}_b - \bar{\Psi}_b S_{da}^{\partial \mu} (\partial_\mu \Psi)_a.
$$

(27)

In the following we give some details of the calculation of Eq. (26). The calculation for the other commutator is quite similar. After substituting the Fourier expansion for the field and momentum density we get

$$
[\Pi_d(x_1), \Psi_b(x_2)] = -i \sum_r \frac{p_\mu}{2Vp_0} \left[ \bar{u}_{ra}(\mathbf{p}) u_{rb}(\mathbf{p}) S_{ad}^{\partial \mu} e^{ip(x_1-x_2)} + \bar{u}_{ra}(\mathbf{p}) u_{rb}^c(\mathbf{p}) S_{ad}^{\partial \mu} e^{ip(x_2-x_1)} \right].
$$

(28)

The states of well defined parity satisfy

$$
\sum_r u_{ra}(\mathbf{p}) \bar{u}_{rb}(\mathbf{p}) = \left( \frac{S(p) + m^2}{2m^2} \right)_{ab}.
$$

(29)

Since the charge conjugated states have the same parity, they satisfy the same relation

$$
\sum_r u_{ra}^c(\mathbf{p}) \bar{u}_{rb}^c(\mathbf{p}) = \left( \frac{S(p) + m^2}{2m^2} \right)_{ab}.
$$

(30)

Substituting this expression in Eq. (28) and evaluating at equal times $x^0_1 = x^0_2 = 0$ we obtain

$$
[\Pi_d(x_1), \Psi_b(x_2)]_{x^0,1,2=0} = -i \sum_r \frac{p_\mu}{2Vp_0} \left( \frac{S(p) + m^2}{2m^2} \right)_{ba} \left[ S_{ad}^{\partial \mu} e^{ip(x_1-x_2)} + S_{ad}^{\partial \mu} e^{-ip(x_1-x_2)} \right].
$$

(31)

changing $p_i \to -p_i$ in the second term after some algebra we get

$$
[\Pi_d(x_1), \Psi_b(x_2)]_{x^0,1,2=0} = -i \sum_r e^{ip(x_1-x_2)} \frac{1}{2Vp_0} \left[ \left( S^{ij} p_i p_j + S^{0i} p_0 p_i + \frac{m^2}{2m^2} \right)_{ba} S_{ad}^{\partial \mu} \right] + p_i \left( \frac{2S^{0i} p_0 p_i}{m^2} \right)_{ba} S_{ad}^{\partial \mu}.
$$

(32)

In order to reduce this equation we use Eq. (10) to show that

$$
S^2(p) = p^4.
$$

(33)

Now, we rewrite the term $S^2(p)$ as

$$
S^2(p) = S^{ij} S^{km} p_i p_m p_j + 2 \left\{ S^{ij}, S^{0l} \right\} p_i p_0 p_j + \left\{ S^{ij}, S^{00} \right\} p_0^2 p_i p_j + 4 S^{0i} S^{0l} p_0^2 p_i + 2 \left\{ S^{0i}, S^{00} \right\} p_0^3 p_i + p_0^4.
$$

(34)

Using Eq. (33) and comparing terms with equal powers of $p_0$ we obtain

$$
\left\{ S^{ij}, S^{00} \right\} p_i p_j + 4 S^{0i} S^{0l} p_0 p_i = -2p^2.
$$

(35)
This result can be ever reduced by taking advantage of Eq. (10) with \( \mu = i, \nu = j, \alpha = \beta = 0 \) which yields \([S^{ij}, S^{00}] = 0\). The final result is

\[
[\Pi_d(x_1), \Psi_b(x_2)]_{x_1^0 = 0} = -i\delta(x_1 - x_2) \left( \frac{\delta_{bd} + S^{00}_{b0}}{2} \right) \Lambda^+_{bd}(0). \tag{36}
\]

where \( \Lambda^+(0) \) is the positive parity projector in the rest frame. This is an unusual but not an unexpected result since our particle and antiparticle live in the same parity subspace of \((1, 0) \oplus (0, 1)\) and the projector \( \Lambda^+(0) \) is like the identity in this subspace.

A similar calculation for the remaining commutator yields

\[
[\Pi_d(x), \Psi_b(y)]_{x^0 = 0} = -i\delta(x - y) \left( \frac{\delta_{db} + S^{00}_{db}}{2} \right) \Lambda^+_{db}(0). \tag{37}
\]

The stress tensor is given by

\[
T^{\mu \nu} = \partial^\nu \bar{\Psi} S^{\mu \alpha} \partial_\alpha \Psi + \partial_\alpha \bar{\Psi} S^{\alpha \nu} \partial^\mu \Psi - \eta^{\mu \nu} \mathcal{L}. \tag{38}
\]

Integrating \( T^{00} \) we get the total energy as

\[
H = \int N \left\{ T^{00} \right\} d^3x = \int N \left\{ \partial_\mu \bar{\Psi} S^{00} \partial_\mu \Psi - \partial_\mu \bar{\Psi} S^{ij} \partial_\mu \Psi + m^2 \bar{\Psi} \Psi \right\} d^3x. \tag{39}
\]

Substituting the Fourier expansion in this equation some factors like

\[
\bar{u}_r(p) \left( S^{0\alpha} p_0 p_\alpha \right) u_s(p) \tag{40}
\]

appear. By taking advantage of the property:

\[
\bar{u}_r(p) = \bar{u}_r(p) \frac{S^{0\nu} p_\nu p_\nu}{m^2}, \quad u_s(p) = \frac{S^{0\mu} p_\mu p_\mu}{m^2} u_s(p) \tag{41}
\]

these terms can be reduced. First we insert Eq.(41) in Eq.(40), then we commute the \( S \)'s and use again Eq.(41) to obtain

\[
\bar{u}_r(p) \left( S^{0\alpha} p_0 p_\alpha \right) u_s(p) = \frac{1}{2m^2} \bar{u}_r(p) \left( S^{0\nu} S^{0\alpha} \right) p_0 p_\alpha u_s(p) p_\mu p_\nu. \tag{42}
\]

Using Eq.(10) the result for the commutators inside Eq.(42) is

\[
\left\{ S^{0\nu}, S^{0\alpha} \right\} = \frac{4}{3} \left( \eta^{\mu 0} \eta^{\nu \alpha} + \eta^{\mu 0} \eta^{\nu \alpha} - \frac{1}{2} \eta^{\mu \nu} \eta^{0 \alpha} \right) - \frac{1}{6} \left( C^{\mu 0\nu \alpha} + C^{\mu \alpha 0 \nu} \right). \tag{43}
\]

Using this result we get

\[
\bar{u}_r(p) \left( S^{0\alpha} p_0 p_\alpha \right) u_s(p) = \frac{1}{2m^2} \bar{u}_r(p) \left( 2m^2 \left( p^0 \right)^2 - \frac{1}{6} \left( C^{\mu 0\nu \alpha} + C^{\mu \alpha 0 \nu} \right) p_\mu p_\nu p_0 p_\alpha \right) u_s(p), \tag{44}
\]

and using now symmetry properties in Eq.(8) we obtain

\[
\bar{u}_r(p) \left( S^{0\alpha} p_0 p_\alpha \right) u_s(p) = \left( p^0 \right)^2 \bar{u}_r(p) u_s(p) = \left( p^0 \right)^2 \delta_{rs}. \tag{45}
\]

Finally, inserting this result in Eq. (39) it is straightforward to show that

\[
H = \sum_r \int d^3p \left[ c_r^+ (p) c_r (p) + d_r^+ (p) d_r (p) \right] p_0. \tag{46}
\]
Notice that the algebraic properties of covariant basis of operators are crucial in order to get this result.

A similar calculation for \( T^{\mu\nu} \) yield the momentum of the field as

\[
P_i = \sum_r \int d^3p p_i \left[ c_r^\dagger(p)c_r(p) + d_r^\dagger(p)d_r(p) \right].
\]  

(47)

Also, the total charge associated to the current in Eq. (17) is

\[
Q = \int d^3x N \left\{ J^0 \right\} = q \sum_r \int d^3p \left( d_r^\dagger(p)d_r(p) - c_r^\dagger(p)c_r(p) \right).
\]  

(48)

4. Propagator

The Feynman propagator is defined as

\[
i \Gamma_F(x - y)_{ab} \equiv \langle 0 | T \{ \phi_a(x)\bar{\phi}_b(y) \} | 0 \rangle
\]  

(49)

Using the expansion of the fields and \( c_r(p) | 0 \rangle = d_r(p) | 0 \rangle = 0 \), we get

\[
i \Gamma_F(x - y)_{ab} = \begin{cases} 
\sum_{p, r} \frac{1}{2Vp_0} u_{ra}(p)\bar{u}_{rb}(p) e^{-ip(x-y)} & x_0 > y_0 \\
\sum_{p, r} \frac{1}{2Vp_0} u_{ra}^c(p)\bar{u}_{rb}^c(p) e^{ip(x-y)} & y_0 > x_0
\end{cases}
\]  

(50)

Using Eq.(29) and Eq.(30) in the propagator we obtain

\[
i \Gamma_F(x - y)_{ab} = \begin{cases} 
\sum_{p} \frac{1}{2Vp_0} \left( \frac{S(p) + m^2}{2m^2} \right)_{ab} e^{-ip(x-y)} & x_0 > y_0 \\
\sum_{p} \frac{1}{2Vp_0} \left( \frac{S(p) + m^2}{2m^2} \right)_{ab} e^{ip(x-y)} & y_0 > x_0.
\end{cases}
\]  

(51)

A four-dimensional integral representation can be obtained following the conventional \( i\epsilon \) prescription to handle the poles in the energy plane. We obtain

\[
i \Gamma_F(x - y) = \frac{i}{(2\pi)^4} \int d^4p \frac{\left( S(p) + m^2 \right) e^{-ip(x-y)}}{2m^2 (p^2 - m^2 + i\epsilon)}.
\]  

(52)

However, using Eq.(33) it can be shown that this is not the appropriate integral representation for the two-point Green function for our theory since

\[
\left( S(i\partial_x) - m^2 \right) \Gamma_F(x) = \frac{1}{(2\pi)^4} \int \frac{(S^2(p) - m^4) e^{-ip(x)}d^4p}{2m^2 (p^2 - m^2 + i\epsilon)}
\]  

\[
= \frac{1}{(2\pi)^4} \int \frac{(p^2 + m^2) e^{-ip(x)}d^4p}{2m^2} \neq \delta(x).
\]  

(53)

Clearly, due to the algebraic properties of the operator \( S(p) \), an appropriate integral representation must contain the \( p^4 - m^4 \) denominator but in such a way that the tachyonic pole at \( p^2 = -m^2 \) do not contribute to the Green function. This can be formally
done by an appropriate choice of the contour integral. Choosing the following contour

we get the integral representation

$$\Gamma_F(x-y) = \frac{1}{(2\pi)^4} \int_C \frac{(S(p) + m^2) e^{-ip(x-y)} d^4p}{(p^4 - m^4)}.$$  \hspace{1cm} (54)

We can trace this problem to the fact that Eq.(12) is a projector only on the mass-shell. For arbitrary values of $p^2$ the covariant parity projectors is

$$P(p) = \frac{1}{2} \left( 1 + \frac{S(p)}{p^2} \right),$$  \hspace{1cm} (55)

thus a possibility is to use the mass-projector combined with the parity projector, i.e. to consider

$$\frac{p^2}{m^2} \frac{1}{2} \left( 1 + \frac{S(p)}{p^2} \right) \psi(p, \lambda) = \psi(p, \lambda).$$  \hspace{1cm} (56)

This amounts to change

$$S^{\mu\nu} \to \Sigma^{\mu\nu} = \frac{1}{2} (S^{\mu\nu} + \eta^{\mu\nu})$$  \hspace{1cm} (57)

in our starting equation of motion. In this case the calculations go along the ones outlined here and we obtain a two-point Green function with the conventional $p^2 - m^2$ denominator. This theory is actually equivalent to the formalism used in R\chi PT for the effective description of $j = 1$ hadrons \[4\]. The eventual disadvantage of this formulation is that in this case we loose the chiral decomposition in Eq.(18) and we do not get a chiral theory in the massless limit.

5. Conclusions

We present preliminary results on the canonical quantization of spin 1 fields with well definite (positive) parity transforming in the $(1,0) \oplus (0,1)$ representation of the HLG. Conventional results are obtained for the energy, momentum and vector charge. The algebraic structure of the covariant basis of operators acting on this space is crucial in getting these results. In the simplest formulation we get a chiral structure in the mass-less limit and an unconventional propagator whose properties are worth to explore further. A conventional propagator is obtained for a formulation using the true (off-mass-shell) projector onto positive parity subspace combined with a mass-projector. In this case, however we loose chiral symmetry in the massless case.
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