Log-concavity of Lucas sequences of first kind

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Abstract

In these notes we address the study of the log-concave operator acting on Lucas sequences of first kind. We will find for which initial values a generic Lucas sequence is log-concave, and using this we show when the same sequence is infinite log-concave. The main result will help to fix the log-concavity of some well known recurrent sequences like Fibonacci and Mersenne numbers. Some possible generalization for a complete classification of the log-concave operator applied to general linear recurrent sequences is proposed.

1 Introduction

Log-concave sequences arise in many areas of algebra, combinatorics, and geometry as detailed by the survey article of Brenti [1]. During the years there have been some studies on the log-operator \( \mathcal{L} \) acting on recurrent sequences such as the work of Asai [2] on Bell numbers, Bóna [3] on sequences counting permutations, Liu [4] on combinatorial sequences and McNamara [5] with his work on Pascal’s triangle. Lucas sequences where first introduced in 1874 by the French mathematician Edouard Lucas, an extensive reference is the book of Koshy [6]. By definition let \( P, Q \) two integer numbers such that \( P^2 - 4Q \geq 0 \), then the Lucas sequence of first kind \( U_n(P, Q) \) is the recurrent sequence defined by \( U_0 = 0, U_1 = 1, U_2 = p, U_n = PU_{n-1} - QU_{n-2} \). As special case for some \( P, Q \) the Lucas sequence associated becomes a well known sequence, for example \( L(1, -1, n) = F_n \) where \( F_n \) is the Fibonacci sequence. In these notes we study the log-operator on these sequence to address the general problem to find which \( P, Q \) integer the corresponding Lucas sequence \( U_n(P, Q) \) is log-concave or \( \infty \)-log concave. In section one we will introduce some basic definition and some basic results on log-operator acting on recurrent sequences. Section two will show a general result on how to solve the log-concavity problem on a generic Lucas sequence of first kind. Last section will propose a generalization of the methods used on Lucas sequence to generic linear recurrent sequences.

2 Basic definition

We now remark some definitions of the log-operator. We refer to the notation to McNamara [5]. Let us start with
**Definition 1.** Let \((a_n)_{n \in \mathbb{N}}\) a real sequence we define the log-operator as a function \(L : \mathbb{R} \rightarrow \mathbb{R}\) such that \(b_n = L(a_n) = a_n^2 - a_{n-1}a_{n+1}\). If \(b_n \geq 0\) for all \(n \in \mathbb{N}\) then the sequence \((a_n)_{n \in \mathbb{N}}\) is said to be log-concave.

Considering that log-concavity can deal with negative indexes, by convention we will extend a sequence \((a_n)_{n \in \mathbb{N}}\) to a sequence \((a_n)_{n \in \mathbb{Z}}\) where by definition \(a_n = 0\) if \(n < 0\). In the same way if the sequence is finite so \(n \leq m, m \in \mathbb{N}\) then all other indexes \(n > m\) will be zero.

**Definition 2.** A real sequence \((a_n)_{n \in \mathbb{N}}\) is said to be \(i\)-fold log-concave for \(i \in \mathbb{N}, i \geq 1\) if \(L^i(a_n)\) is a nonnegative sequence. Where \(L^i(a_n)\) is the log-operator applied to a sequence \((a_n)_{n \in \mathbb{N}}\) \(i\)-times so \(L^i = L \circ L \circ \cdots \circ L\).

Using McNamara [5] notation:

**Definition 3.** A real sequence \((a_n)_{n \in \mathbb{N}}\) is said to be \(\infty\)-log concave if \(L^i(a_n)\) is a nonnegative sequence for all \(i \in \mathbb{N}, i \geq 1\).

So log-concavity in the ordinary sense is 1-fold log-concavity. To study log-concavity on Lucas sequences, we need some preliminary results, like the following:

**Lemma 4.** Let \((a_n)_{n \in \mathbb{N}}\) a sequence where \(a_n = k\) for all \(n \in \mathbb{N}\) and \(k\) is a real number, then \((a_n)_{n \in \mathbb{N}}\) is \(\infty\)-log concave.

**Proof.** It is easy to check that
\[
b_n = L(a_n) = L(a_n) = k^2 - (k \cdot k) = 0
\]
for all \(n \in \mathbb{N}\). It is also clear that the all zeros sequence \(b_n\) is invariant by the log-operator that is \(L(b_n) = b_n\). Being \(b_n \geq 0\) that means that also \(L(b_n) \geq 0\) so the all zeros sequence is \(\infty\)-log concave. \(\square\)

In the same way it is also easy to check that

**Lemma 5.** Let \((a_n)_{n \in \mathbb{N}}\) a sequence where for all \(n \in \mathbb{N}\) \(a_n = kb^n\) where \(k, b \in \mathbb{R}, k \neq 0, b \neq 0\) then \((a_n)_{n \in \mathbb{N}}\) is \(\infty\)-log concave.

**Proof.** By direct check
\[
L(a_n) = (a_n)^2 - a_{n-1}a_{n+1} = k^2b^{2n} - k^2b^{n-1+n+1} = k^2b^{2n} - k^2b^{2n} = 0
\]
for all \(n \in \mathbb{N}\). So \(a_n\) is 1-fold log-concave and the result sequence is the all zeros sequence than considering lemma 4 then the sequence \(a_n\) is also \(\infty\)-log concave. \(\square\)

In next section we will detail our analysis of the log-operator to the Lucas sequence.
3 Log-operator and Lucas sequences

In these section we address the study of log-concavity, of a Lucas sequence of first kind Let start with the Lucas sequence definition:

**Definition 6.** Let \((P, Q) \in \mathbb{Z} \times \mathbb{Z}\) two non-zero integer such that \(P^2 - 4Q \geq 0\) and let \(n \in \mathbb{N}\) an index. A Lucas sequence \(U_n(P, Q)\) of first kind is a recurrent sequence defined as follows:

\[
\begin{align*}
U_0 &= 0 \\
U_1 &= 1 \\
U_n &= PU_{n-1} - QU_{n-2}.
\end{align*}
\]

Choosing the correct \(P, Q\) it is possible to obtain some well known sequences for example:

- If \(P = 1, Q = -1\) then the Lucas sequence \(U_n(1, -1) = F_n\) where \(F_n\) is the Fibonacci sequence.
- If \(P = 2, Q = -1\) then the Lucas sequence \(U_n(2, -1)\) is the sequence of Pell numbers.
- If \(P = 1, Q = -2\) then the Lucas sequence \(U_n(1, 2)\) is the sequence of Jacobsthal numbers.
- If \(P = 3, Q = 2\) then the Lucas sequence \(U_n(3, 2)\) is the sequence of Mersenne numbers.

The main result of this section will prove for which initial \(P, Q\) the resulting Lucas sequence is \(\infty\)-log concave. Let us start by showing that in general if we choose a generic couple \(P, Q\) the Lucas sequence \(U_n(P, Q)\) is not 1-fold log-concave.

We use the following proposition

**Proposition 7.** The Fibonacci sequence \(F_n\) is not 1-fold log-concave.

**Proof.** Considering the log-operator applied to \(F_n\) we have

\[
b_n = \mathcal{L}(F_n) = F_n^2 - F_{n-1}F_{n+1};
\]

now by the Cassini’s identity

\[
F_{n-1}F_{n+1} - F_n^2 = (-1)^n
\]

we obtain

\[
F_n^2 - F_{n-1}F_{n+1} = (-1) \cdot (-1)^n = (-1)^{n+1}.
\]

So

\[
\mathcal{L}(F_n) = (-1)^{n+1}.
\]

thus \(F(n)\) is not 1-fold log-concave. If we applied the \(\mathcal{L}\) operator to the sequence \(b_n\) and we calculate \(\mathcal{L}^2(F(n)) = \mathcal{L}(\mathcal{L}(b_n))\) we obtain

\[
\mathcal{L}^2(F(n)) = ((-1)^{n+1})^2 - (-1)^{n+2} \cdot (-1)^n = ((-1)^{n+1})^2 - (-1)^{2n+2} = 1 - 1 = 0
\]

so after applying the log-operator more than once we obtain a sequence that is log-concave.

\(\square\)
We will now fix for what initial parameter $P, Q$ the generate Lucas sequence $U_n(P, Q)$ is a 1-fold log-concave Lucas sequence, and in these cases where for what $P, Q$ the Lucas sequence becomes $\infty$-log concave. Instead of trying to apply directly the log-operator to the generic expression of the Lucas sequence $U_n(P, Q)$, we will use a more treatable expression for $U_n(P, Q)$. To do this, we first need to recall [7] that:

**Remark 8.** Let $U_n(P, Q)$ a Lucas sequence of first kind, than the characteristic equation of the recurrence relation is

$$x^2 - Px + Q = 0$$

that has discriminant $D = P^2 - 4Q$. If the discriminant is positive so $D \geq 0$ then the roots of the characteristic equation are

$$a = \frac{P + \sqrt{D}}{2}, \quad b = \frac{P - \sqrt{D}}{2}$$

and so if $D \geq 0$ it is possible to rewrite $U_n(P, Q)$ in the following way

$$U_n(P, Q) = \frac{a^n - b^n}{a - b} = \frac{a^n - b^n}{\sqrt{D}}.$$  \hspace{1cm} (4)

Armed with this expression for Lucas sequence, we will divide our study in two main cases let us start with the simpler one.

**Proposition 9.** Let $U_n(P, Q)$ a Lucas sequence where $P, Q$ are two integer and the discriminant $D$ of the characteristic equation associated with $U_n(P, Q)$ is zero then the Lucas sequence associated is 1-fold log-concave.

**Proof.** It is easy to see that if $D = 0$ then $P^2 - 4Q = 0$ and so there exists and integer $S$ such that $P = 2S$ and $Q = S^2$. Using this fact the Lucas sequence associated can be rewritten in the form

$$U_n = nS^{n-1}. \hspace{1cm} (5)$$

So now, applying the $\mathcal{L}$ operator, we see that

$$\mathcal{L}(U_n) = nS^{n-1}$$

and so $\mathcal{L}(U_n) \geq 0$ for all $S \in \mathbb{Z}$. This prove that $U_n$ is 1-fold log-concave.

From proposition 9 we have also the following corollary

**Corollary 10.** Let $U_n(P, Q)$ a Lucas sequence where $P, Q$ are two integer and there exist an $S \in \mathbb{Z}$ such that $P = 2S$ and $Q = S^2$ then the Lucas sequence associated is $\infty$-log concave.
Proof. We have seen that under the hypothesis $L(U_n) = (S^{n-1})^2 = (S^2)^{n-1}$. By changing the index we have that the original sequence become a sequence of the form $b_k = S^k$ where $k \in \mathbb{Z}, k = 2n - 2, k \geq -2$. Considering that for negative indexes $b_k = 0$ we have that by lemma 5 the sequence $b_k$ is $\infty$-log concave and so $U_n$.

Let now consider the general case

If $D = P^2 - 4Q > 0$ by remark 8 it is possible to rewrite $U_n(P, Q)$ in the following way

$$U_n(P, Q) = \frac{a^n - b^n}{a - b} = \frac{a^n - b^n}{\sqrt{D}}$$

where

$$a = \frac{P + \sqrt{D}}{2}, \quad b = \frac{P - \sqrt{D}}{2}$$

we notice that, using direct calculation we have

$$L(U_n) = \frac{(a^n - b^n)^2}{\sqrt{D}^2} - \left[ \frac{a^{n-1} - b^{n-1}}{\sqrt{D}} \cdot \frac{a^{n+1} - b^{n+1}}{\sqrt{D}} \right]$$

$$= \frac{a^{2n} - 2a^n b^n + b^{2n}}{D} - \frac{a^{n-1} - b^{n-1} - a^{n+1} b^{n+1} + b^{n+1 + n-1}}{D}$$

$$= \frac{a^{n+1} b^{n-1} - 2a^n b^n + a^{n-1} b^{n+1}}{D}$$

$$= \frac{a^{n-1} b^{n-1} (a^2 - 2ab + b^2)}{D}$$

$$= \frac{(ab)^{n-1} (a - b)^2}{D}$$

now then by definition

$$ab = \frac{P + \sqrt{D}}{2} \cdot \frac{P - \sqrt{D}}{2} = \frac{1}{4}(P^2 - D) = \frac{1}{4}(P^2 - P^2 + 4Q) = Q$$

and

$$a - b = \frac{P + \sqrt{D}}{2} - \frac{P - \sqrt{D}}{2} = \frac{2P}{2} = P.$$ 

So finally we have

$$L(U_n) = \frac{Q^{n-1} P^2}{D}$$

(10)
So $\mathcal{L}(U_n) \geq 0$ if $Q \geq 0$. Combining this with the assumption that $P^2 - 4Q \geq 0$ we have that $U_n(P, Q)$ is 1-fold log-concave if

$$\begin{cases} Q \geq 0 \\ P^2 - 4Q > 0 \end{cases}$$

that gives the following set of solutions $Q \geq 0 \land P > 2\sqrt{Q}$ or $Q \geq 0 \land P < -2\sqrt{Q}$.

We can summarize the result in the following

**Theorem 11.** Let $P, Q$ two integer such that $Q \geq 0 \land P > 2\sqrt{Q}$ or $Q \geq 0 \land P < -2\sqrt{Q}$, then the associated Lucas sequence $U_n(P, Q)$ is 1-fold log-concave.

Using theorem 11 and the lemma 5, it is easy to check that

**Corollary 12.** Let $P, Q$ two integer such that $Q \geq 0 \land P > 2\sqrt{Q}$ or $Q \geq 0 \land P < -2\sqrt{Q}$. Then the Lucas sequence $U_n(P, Q)$ is $\infty$-log concave.

**Proof.** Under the hypothesis we have that

$$b_n = \mathcal{L}(U_n) = \frac{Q^{n-1}P^2}{D}$$

that is a sequence of the form $k_na^n$ and by lemma 5 $U_n(P, Q)$ is $\infty$-log concave. \qed

At the end using the corollary 12 we can check that:

- $U_n(1, -1)$ is the Fibonacci sequence that is not 1-fold log-concave and so neither $\infty$-log concave.
- $U_n(2, -1)$ is the sequence of Pell numbers that is not 1-fold log-concave and so neither $\infty$-log concave.
- $U_n(1, -2)$ is the sequence of Jacobsthal numbers that is not 1-fold log-concave and so neither $\infty$-log concave.
- $U_n(3, 2)$ is the sequence of Mersenne numbers that is $\infty$-log concave.

## 4 Conclusion

In these notes we have studied the log-operator applied to a generic Lucas sequence of first kind $U_n$. We have shown that for initial parameter $Q \geq 0$, $P \geq 2Q$ or $Q \geq 0$, $P \leq 2Q$, the associate Lucas sequence of first kind is $\infty$-log concave. As result we find that Fibonacci, Pell and Jacobsthal sequences are not $\infty$-log concave but the Mersenne numbers sequence is $\infty$-log concave. There is a natural question that arise from these results. As shown the key fact, that a sequence is recurrent, allow the sequence to be expressed in a more treatable way before applying the log-operator. It would be interesting giving a generic linear recurrent sequence that satisfy a generic characteristics equation of order $k$, to find sufficient condition on the coefficient of the equation to be sure that the sequence is 1-fold log-concave and after this fix which conditions leads to a $\infty$-log concave sequence. Formalizing a little, giving a
recurrent sequence define as \( a_n = k_1a_{n-1} + k_2a_{n-2} + \ldots + k_ma_{n-m} \) that has a characteristic equation \( a_n - k_1a_{n-1} - k_2a_{n-2} - \cdots - k_ma_{n-m} = 0 \) is there is a sufficient condition on the \( k_1, k_2, \ldots, k_m \) integer coefficient such that \( a_n \) is 1-fold log-concave and \( \infty \)-log concave. This question would be subject of further study.

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