Dynamo action due to $\alpha$ fluctuations in a shear flow: mean–field theory

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ABSTRACT

We present an analytical theory of the growth of a large–scale mean magnetic field in a linear shear flow with fluctuations in time of the $\alpha$ parameter (equivalently, kinetic helicity). Using shearing coordinates and Fourier variables we derive a set of coupled integro–differential equations, governing the dynamics of the mean magnetic field, that are non perturbative in the rate of shear. When the $\alpha$ fluctuations are of white–noise form, the mean electromotive force (EMF) is identical to the negative diffusive form derived by Kraichnan for the case of no shear; the physical reason is that shear takes time to act, and white–noise fluctuations have zero correlation time. We demonstrate that the white–noise case does not allow for large–scale dynamo action. We then allow for a small but non zero correlation time and show that, for a slowly varying mean magnetic field, the mean EMF has additional terms that depend on a combination of shear and $\alpha$ fluctuations; the mean–field equations now reduce to a set of coupled partial differential equations. A dispersion relation for modes is derived and studied in detail for growing solutions. Our salient results are: (i) a necessary condition for dynamo action giving the minimum value of shear required; (ii) two types of dynamos depending on the different forms taken by the growth rate as a function of wavenumber; (iii) explicit expressions for the growth rate and wavenumber of the fastest growing mode; these are not only consistent with the scalings with shear seen in numerical simulations, but also provide an estimate of the strength of $\alpha$ fluctuations.

Key words: magnetic fields — MHD — dynamo — galaxies: magnetic fields — turbulence
1 INTRODUCTION

The magnetic fields observed in many astrophysical bodies — such as the Sun, stars, galaxies and clusters of galaxies — are thought to be generated by electric currents in the turbulent plasmas flowing in these objects (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Ruzmaikin, Shukurov & Sokoloff 1998; Kulsrud 2004; Brandenburg & Subramanian 2005). The field shows structure on a wide range of scales, both smaller and larger than the outer scale of the turbulence. Of particular interest to the present investigation is the large–scale ordered structure of the field. When the turbulent motions are helical (i.e when mirror–symmetry is broken), the well–known $\alpha$–effect can amplify seed magnetic fields and maintain them in the face of turbulent dissipation (Moffatt 1978; Parker 1979; Krause & Rädler 1980). However, it is not clear whether astrophysical turbulence has a mean helicity that is large enough to sustain such a large–scale turbulent dynamo. Kraichnan (1976) considered dynamo action when $\alpha$ is a stochastic quantity with zero mean, and demonstrated that the $\alpha$ fluctuations would contribute to a decrease in the turbulent diffusivity. When the fluctuations are large enough, there is negative diffusion, and the magnetic field would grow on all spatial scales, with the fastest rates of growth on the smallest scales. Thus $\alpha$ (or helicity) fluctuations by themselves can result only in a small–scale dynamo. A feature common to astrophysical flows is velocity shear, and it is only relatively recently that the question of large–scale dynamo action in shear flows with $\alpha$ fluctuations began receiving attention (Vishniac & Brandenburg 1997; Sokolov 1997; Silant’ev 2000). The interest in this question has grown significantly, and a number of investigations have pursued this problem both analytically and numerically (Proctor 2007; Brandenburg et al 2008; Yousef et al 2008; Kleedorf & Rogachevskii 2008; Rogachevskii & Kleedorf 2008; Sur & Subramanian 2009; Heinemann, McWilliams & Schekochihin 2011; McWilliams 2012; Mitra & Brandenburg 2012; Proctor 2012; Richardson & Proctor 2012; Tobias & Cattaneo 2013).

The numerical simulations of Yousef et al (2008) and Brandenburg et al (2008) are clear demonstrations of large–scale dynamo action in a shear flow with turbulence that is, on average non–helical. The evidence for the growth of a large–scale mean magnetic field is particularly compelling, as can be seen from the ‘butterfly diagrams’ of Figure 3 in Yousef et al (2008) and Figure 8 in Brandenburg et al (2008). It is also significant, as shown in Figure 10

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of Brandenburg et al (2008), that the $\alpha$ parameter fluctuates about a mean value close to zero. Yousef et al (2008) also derive scalings of important quantities as a function of $S$, the rate of shear. Their Figures 2 and 4 demonstrate that the wavenumber of the fastest growing mode is $K_m \propto |S|^{1/2}$, and its growth rate is $\gamma_m \propto |S|$. The goal of the present paper is to construct a mean–field theory of large–scale dynamo action in a shear flow with zero–mean $\alpha$ fluctuations, and use this to understand the scaling relations derived by Yousef et al (2008) in their numerical simulations.

In Section 2 we derive the equations governing the space–time evolution of the mean magnetic field: we begin with the standard $\alpha^2\Omega$ equations, allow $\alpha$ to fluctuate in time stochastically with zero mean, and use the method of shearing coordinates and Fourier variables presented earlier in Sridhar & Subramanian (2009a,b); Sridhar & Singh (2010); Singh & Sridhar (2011). A notable feature is that the equations are non–perturbative in the shear, and the expression for the mean electromotive force (EMF) involves the time integral of the mean magnetic field and the two–point time correlation function of the $\alpha$ fluctuations. With no further approximations, this is applied in Section 3 to the case of white–noise $\alpha$ fluctuations (which has zero correlation time), and general conclusions are derived regarding the limited nature of dynamo action. In Section 4, we consider the theory for the case of a small and non zero correlation time and when the mean magnetic field is slowly varying. A concise and physically appealing expression for the mean EMF is derived, and the partial differential equations (PDEs) governing mean–field evolution is exhibited. Dynamo solutions of these PDEs are investigated in Section 5, and the dispersion relation for modes is studied in detail giving the necessary condition for large–scale dynamo action, as well as two types of behaviour of the growth rate as a function of wavenumber. In Section 6 we show that our theory predicts scaling relations for the wavenumber of the fastest growing mode and its wavenumber, that are consistent with the numerical simulations of Yousef et al (2008). We also estimate that the level of $\alpha$ fluctuations in their simulations should be close to the threshold value predicted by our theory; this is consistent with the fact that the simulations of Yousef et al (2008) have modest values of fluid and magnetic Reynolds numbers. Conclusions follow in Section 7.
2 DERIVATION OF THE MEAN–FIELD EQUATIONS

Let \( (e_1, e_2, e_3) \) be the unit vectors of Cartesian coordinate system that is locally comoving with the mean velocity of a turbulent shear flow. The position vector and time are denoted by \( X = (X_1, X_2, X_3) \) and \( \tau \). Let \( V(X) \) be the mean velocity field, and \( v(X, \tau) \) be the fluctuating velocity of the small–scale turbulence that has an ‘eddy’ outer scale \( \ell_t \) and turnover time \( \tau_t \). We denote by \( B(X, \tau) \) the magnetic field that has been averaged over these spatial and temporal scales. The time evolution of this magnetic field is governed by the well–known \( \alpha^2 \Omega \) equation,

\[
\frac{\partial B}{\partial \tau} = \nabla \times [V \times B + \alpha B - (\eta + \eta_t) \nabla \times B] ; \quad \nabla \cdot B = 0 .
\]  

(1)

Here \( \alpha \) is a pseudoscalar contributed by the kinetic helicity of the turbulence; \( \eta \) and \( \eta_t \) are the microscopic and turbulent magnetic diffusivities, respectively. Both \( \alpha \) and \( \eta_t \) are quantities that have been averaged over scales \( \ell_t \) and \( \tau_t \).

The problem studied here concerns stochastic fluctuations of \( \alpha \) and its consequences for magnetic field evolution, over spatial scales much larger than \( \ell_t \) and \( \tau_t \). We assume that (i) the \( \alpha \) fluctuations have zero mean, \( \alpha = 0 \), and such slow spatial variations that \( \alpha = \alpha(\tau) \); (ii) the shear flow \( V = SX_1 e_2 \), where \( S \) is the constant rate of shear parameter; (iii) \( \eta_T = (\eta + \eta_t) \) is constant. The once–averaged magnetic field can be written as \( B = \mathcal{B} + b \), as the sum of a mean (over \( \alpha \) fluctuations) magnetic field \( \mathcal{B} \), and a fluctuating component \( b \) where \( \mathcal{B} = 0 \). Applying Reynolds averaging to (1), we obtain the following equation governing the dynamics of the mean magnetic field:

\[
\left( \frac{\partial}{\partial \tau} + SX_1 \frac{\partial}{\partial X_2} \right) \mathcal{B} - S\mathcal{B}_1 e_2 = \nabla \times \mathcal{E} + \eta_T \nabla^2 \mathcal{B} , \quad \nabla \cdot \mathcal{B} = 0 ,
\]  

(2)

where \( \mathcal{E} = \alpha \mathcal{B} \) is the mean electromotive force (EMF) due to \( \alpha \) fluctuations in a shearing background. To obtain a closed equation for \( \mathcal{B} \), it is necessary to determine the mean EMF as a functional of \( \mathcal{B} \), for which it is necessary to solve for \( b \) as a functional of \( \mathcal{B} \). The fluctuating field, \( b \), obeys

\[
\left( \frac{\partial}{\partial \tau} + SX_1 \frac{\partial}{\partial X_2} \right) b - Sb_1 e_2 = \nabla \times (\alpha \mathcal{B}) , \quad \nabla \cdot b = 0 , \quad b(X,0) = 0 ,
\]  

(3)

in the standard first–order–smoothing approximation. Equation (3) is homogeneous in time but inhomogeneous in space. Following the approach of Sridhar & Subramanian (2009a,b);

1 To order of magnitude, \( \alpha \sim -\frac{1}{3} \tau_t \langle v \cdot (\nabla \times v) \rangle \) and \( \eta_t \sim \frac{1}{3} \tau_t \langle v \cdot v \rangle \).
Sridhar & Singh (2010); Singh & Sridhar (2011), we rewrite it in sheared coordinates \( x = (x_1, x_2, x_3) \) and new time \( t \), defined by \( x_1 = X_1 \), \( x_2 = X_2 - S \tau X_1 \), \( x_3 = X_3 \), \( t = \tau \). We also define new magnetic field variables, which are component-wise equal to the old variables: \( \overline{H}(x, t) = \overline{B}(X, \tau) \) and \( h(x, t) = b(X, \tau) \). Then (3) gives:

\[
\frac{\partial h}{\partial t} - Sh_1 e_2 = \alpha \nabla \times \overline{H}, \quad \nabla \cdot h = 0, \quad h(X, 0) = 0,
\]

(4)

where \((\nabla)_i = (\partial/\partial x_i - S \delta_{ij} \partial/\partial x_j)\) depends on both \( x \) and \( t \), and \( \nabla \cdot \overline{H} = 0 \). Equation (4) is inhomogeneous in time but homogeneous in sheared coordinate \( s \), which allows us to solve it using Fourier transforms, while being fully nonperturbative in the shear parameter \( S \). Define Fourier–transformed magnetic fields: \( \tilde{H}(k, t) = \int d^3x \exp(-i k \cdot x) \overline{H}(x, t) \) and \( \tilde{h}(k, t) = \int d^3x \exp(-i k \cdot x) h(x, t) \), where \( k \) is a sheared wavevector. Using this in (4), we see that \( \tilde{h} \) satisfies,

\[
\frac{\partial \tilde{h}}{\partial t} - S \tilde{h}_1 e_2 = i \alpha K \times \overline{H}, \quad K \cdot \tilde{h} = 0, \quad \tilde{h}(k, 0) = 0,
\]

(5)

where \( K(k, t) = (k_1 - Stk_2, k_2, k_3) \) is a time–dependent wavevector, and \( K \cdot \overline{H} = 0 \). It may be verified that the solution to (5) is:

\[
\tilde{h}(k, t) = \int_0^t dt' \alpha(t') \left[ iK' \times \overline{H'} \right] + e_2 S \int_0^t dt' (t - t') \alpha(t') \left[ iK' \times \overline{H'} \right],
\]

(6)

where the primes in \( K' \) and \( \overline{H'} \) mean that they are to be evaluated at time \( t' \). Using (6), the mean EMF can be written as:

\[
\mathcal{E} = \overline{\alpha(\tau) b(X, \tau)} = \overline{\alpha(t) h(x, t)}
\]

\[
= \int \frac{d^3k}{(2\pi)^3} \exp(i k \cdot x) \int_0^t dt' \alpha(t) \alpha(t') \left[ iK' \times \overline{H'} \right] + e_2 S \int \frac{d^3k}{(2\pi)^3} \exp(i k \cdot x) \int_0^t dt' (t - t') \alpha(t) \alpha(t') \left[ iK' \times \overline{H'} \right]
\]

(7)

This is the most general expression for the mean EMF in our theory, where \( \mathcal{E} \) has been expressed as a functional of the mean field \( \overline{B}(X, \tau) \), or equivalently its Fourier–transform.

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2 Just like the old variables, the new variables are expanded in the fixed Cartesian basis of the lab frame. i.e. \( \overline{H} = \Pi_1 e_1 + \Pi_2 e_2 + \Pi_3 e_3 \), where \( \Pi_i(x, t) = \overline{B}_i(X, \tau) \), and similarly for \( h \).

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in sheared coordinates, \( \tilde{H}(k,t) \). Of particular interest is the case when the \( \alpha \) fluctuations are stationary in time. Define

\[
\alpha(t)\alpha(t') = 2\eta_\alpha D(t-t'),
\]

where \( \eta_\alpha \) is a positive constant characterizing the strength of the fluctuations, and the (normalized) correlation function \( D(t) \) is an even function of \( t \) which is normalized as,

\[
\int_{-\infty}^{\infty} dt \ D(t) = 2 \int_{0}^{\infty} dt \ D(t) = 1.
\]

The \( \alpha \) correlation time is defined by,

\[
\tau_\alpha = 2 \int_{0}^{\infty} dt \ t D(t).
\]

Using (8) in (7), the mean EMF for stationary \( \alpha \) fluctuations is:

\[
\mathcal{E} = 2\eta_\alpha \int \frac{d^3 k}{(2\pi)^3} \exp(i k \cdot x) \int_{0}^{t} ds D(s) \left[ iK(k,t-s) \times \tilde{H}(k,t-s) \right] + 2e_2\eta_\alpha S \int \frac{d^3 k}{(2\pi)^3} \exp(i k \cdot x) \int_{0}^{t} ds s D(s) \left[ iK(k,t-s) \times \tilde{H}(k,t-s) \right].
\]

Equation (2), together with either (7) or (10) for the mean EMF, is a closed set of integro–differential equations governing the dynamics of the mean magnetic field \( \overline{B} \). These are, of course, difficult to solve in general; but there is one important case for which they can be solved exactly; this is the case of white–noise \( \alpha \) fluctuations, as discussed below.

### 3 WHITE–NOISE \( \alpha \) FLUCTUATIONS

For white–noise \( \alpha \) fluctuations the normalized correlation function, \( D(t) = \delta(t) \), is equal to the Dirac delta–function. From (2) we see that correlation time vanishes, \( \tau_\alpha = 0 \), as should be the case. From (10), the corresponding mean EMF is:

\[
\mathcal{E}_{WN} = \eta_\alpha \int \frac{d^3 k}{(2\pi)^3} \exp(i k \cdot x) \left[ iK \times \tilde{H} \right] = \eta_\alpha \int \frac{d^3 K}{(2\pi)^3} \exp(i K \cdot X) \left[ iK \times \overline{B} \right] = \eta_\alpha \nabla \times \overline{B},
\]

because \( d^3 k = d^3 K \), \( k \cdot x = K \cdot X \) and \( \tilde{H} = \overline{B} \). Thus the white–noise mean EMF, \( \mathcal{E}_{WN} \), is identical to the one derived by Kraichnan (1976) for \( \alpha \) fluctuations in the absence of shear. The physical reason why shear does not contribute to \( \mathcal{E}_{WN} \) is that white–noise has
zero correlation time, so shear has no time to act; this statement is true for any finite value of \(S\), the rate of shear parameter.

Substituting (11) in (2), we get the following partial differential equation for the evolution of the mean magnetic field:

\[
\left( \frac{\partial}{\partial \tau} + SX_1 \frac{\partial}{\partial X_2} \right) \overline{B} - S\overline{E}_1e_2 = \eta_K \nabla^2 \overline{B} \tag{12}
\]

This is just the induction equation for the mean magnetic field, in a background shear flow, with a Kraichnan–type diffusivity, \(\eta_K = (\eta_T - \eta_\alpha)\), to which \(\alpha\) fluctuations contribute a decrement. The general solution to (12) can be written as a superposition of shearing waves:

\[
\overline{B}(X, \tau) = \int \frac{d^3k}{(2\pi)^3} \exp (i\mathbf{K}(k, t) \cdot \mathbf{X}) \widetilde{G}(k, t) \left[ \widetilde{M}(k) + e_2St\widetilde{M}_1(k) \right], \tag{13}
\]

where \(\widetilde{M}(k)\) are any initially specified vector wave amplitudes, which satisfy the transversality condition \(\mathbf{k} \cdot \widetilde{M}(k) = 0\). The quantity \(\widetilde{G}(k, t)\) is a shear–diffusive Green’s function,

\[
\widetilde{G}(k, t) = \exp \left[ -\eta_K \left( k^2t - Sk_1k_2t^2 + (S^2/3)k_2^2t^3 \right) \right], \tag{14}
\]

which determines growth or decay of the shearing waves.

1. For weak \(\alpha\) fluctuations, \(\eta_\alpha < \eta_T\) so that \(\eta_K > 0\), and waves of all wavenumbers \(\mathbf{k}\) eventually decay. Waves with \(k_2 = 0\) (also sometimes called axisymmetric waves) decay as \(\exp [-\eta_Kk^2t]\), in a manner that is independent of shear. The asymptotic decay of non–axisymmetric waves \((k_2 \neq 0)\) is more rapid, like \(\exp [-\eta_K(S^2/3)k_2^2t^3]\), where \(\alpha\) fluctuations and shear contribute together.

2. For strong \(\alpha\) fluctuations, \(\eta_\alpha > \eta_T\) so that \(\eta_K < 0\), and waves of all wavenumbers \(\mathbf{k}\) eventually grow. Axisymmetric waves grow like \(\exp [\eta_Kk^2t]\), due to negative diffusion, in a shear–independent manner, just like in [Kraichnan (1976)]. Non–axisymmetric waves grow even more rapidly, like \(\exp [\eta_K(S^2/3)k_2^2t^3]\), where shear acts in conjunction with the negative diffusion due to \(\alpha\) fluctuations. Since the growth rate (for either kind of wave) is proportional to the square of the wavenumber, the fastest growing modes are those with the smallest spatial scales which, in our theory, should be of order a few eddy outer scales \(\ell_t\). This is a small–scale dynamo.

3. In either case, weak or strong \(\alpha\) fluctuations, the necessary condition for dynamo action is that \(\eta_K < 0\), which is identical to [Kraichnan (1976)] for the case of no shear; however, the
growth rates (or damping rates when \( \eta_K > 0 \)) of non–axisymmetric modes depend both on \( \eta_K \) and \( S \), and is a generalization of Kraichnan (1976) to the case of a shearing background.

Thus we conclude that white noise \( \alpha \) fluctuations in a linear shear flow can give rise only to small–scale dynamo action, for any finite value of \( S \), the rate of shear parameter. Our interest is in dynamo action on larger spatial scales, to explore which we must consider the case of a nonzero \( \tau_\alpha \), as studied in the next section.

4 MEAN–FIELD EQUATIONS FOR SMALL \( \tau_\alpha \) AND SLOWLY VARYING MEAN MAGNETIC FIELD

Our goal here is to derive the mean–field equations when \( \tau_\alpha \), the \( \alpha \) correlation time, is not zero but small: i.e. when \(|S\tau_\alpha| < 1\). Then general expression for the mean EMF in (10) can be simplified when the mean field is a slowly varying function of space and time. This means that \( \widetilde{H}(k, t-s) \) is appreciable only for small enough wavenumbers, and changes over time scales larger than \( \tau_\alpha \). Noting that \( \mathcal{D}(s) \) should become small when \( s > \tau_\alpha \), we see that we need to solve for the mean field behaviour only for short times. Using the general solution given in (13), we can read off:

\[
\widetilde{H}(k, t-s) \simeq \exp \left[ \eta_K K^2(k, t) s \right] \left[ \widetilde{H}(k, t) - e_2 sS \widetilde{H}_1(k) \right],
\]

when \( 0 \leq s < \tau_\alpha \) and \( |S\tau_\alpha| < 1 \). (15)

By definition, \( K(k, t-s) = K(k, t) + e_1 sS k_2 \). It is straightforward to work out the product, \( K(k, t-s) \times \widetilde{H}(k, t-s) \) and substitute this in (10). The exponential pre–factor in (15) can be dropped because this is appreciable for \( K^2 \) large enough to matter for a small–scale dynamo. The upper limit of the integrals can be extended to \(+\infty\), because the integrand is strongly convergent. Then the mean EMF is:
\[
E(X, \tau) = \eta \int \frac{d^3k}{(2\pi)^3} \exp(ik \cdot x) \left[ iK(k, t) \times \widetilde{H}(k, t) \right] 
+ \eta S\tau \int \frac{d^3k}{(2\pi)^3} \exp(ik \cdot x) \left[ i(k_3e_1 - K_1e_3)\widetilde{H}_1(k, t) \right.
\left. + i(k_2e_3 - k_3e_2)\widetilde{H}_2(k, t) \right] 
- 2\eta w(\tau) \int \frac{d^3k}{(2\pi)^3} \exp(ik \cdot x) i(k_2e_3 - k_3e_2)\widetilde{H}_1(k, t) ,
\]
\hspace*{1cm} (16)

where \( w(\tau) \) is a dimensionless measure of the width of \( \mathcal{D}(t) \), defined as:
\[
w(\tau) = \frac{1}{\tau^2} \int_0^\infty dt \ t^2 \mathcal{D}(t) > 0 .
\]
\hspace*{1cm} (17)

The right side of (16) can be simplified considerably\(^3\) and the mean EMF cast in a physically appealing form:
\[
E(X, \tau) = \eta \nabla \times \overline{B} + \beta_1 e_2 \times \nabla \overline{B}_1 + \beta_1 e_1 \times \nabla \overline{B}_2 - 2\beta_2 e_1 \times \nabla \overline{B}_1 ,
\]
\hspace*{1cm} (18)

where the new diffusivity coefficients, \( \beta_1 = \eta \alpha S\tau \) and \( \beta_2 = \eta \alpha w(\tau) \). Of the four terms on the right side, the first term is just \( E^{\text{WN}} \), whereas the others are new terms that depend on both shear and \( \alpha \) fluctuations. Substituting (18) in (2), we obtain the mean-field equation:
\[
\left( \frac{\partial}{\partial \tau} + SX_1 \frac{\partial}{\partial X_2} \right) \overline{B} - S\overline{B}_1 e_2 = \nabla^2(\psi_1 e_1 + \psi_2 e_2) - \nabla \left( \frac{\partial \psi_1}{\partial X_1} + \frac{\partial \psi_2}{\partial X_2} \right) + \eta K \nabla^2 \overline{B} ,
\]
\[
\nabla \cdot \overline{B} = 0 ;
\]

where \( \psi_1 = \beta_1 B_2 - 2\beta_2 B_1, \psi_2 = \beta_1 B_1, \) and \( \beta_1 = \eta \alpha S\tau, \beta_2 = \eta \alpha w(\tau) \). \hspace*{1cm} (19)

\hspace*{0.5cm} \text{Thus is our main general result: equations (19) are a set of coupled, linear partial differential equations, which govern the dynamics of the mean magnetic field, } \overline{B}(X, \tau) \text{, when shear acts}

\(^3\) Use \((k_2e_3 - k_3e_2) = e_1 \times K, (k_3e_1 - K_1e_3) = e_2 \times K, d^3k = d^3K, k \cdot x = K \cdot X \) and \( \widetilde{H} = \overline{H} \), and do the Fourier integrals.
in conjunction with $\alpha$ fluctuations. They are valid when the $\alpha$ correlation time is small, and the mean-field varies slowly in space and time, as discussed earlier:

$$ |S\tau_\alpha| < 1, \quad w_2 |S\tau_\alpha|^2 < 1, \quad |\eta_K| K^2 \tau_\alpha < 1, $$

where $K^{-1}$ is a typical scale of spatial variations (in lab coordinates) of the mean field.

We see that, when shear and $\alpha$ fluctuations act together, there is a mean EMF that gives, besides the usual Kraichnan–type diffusivity $\eta_K$, additional terms that depend on the ‘horizontal’ components $\overline{B}_1$ and $\overline{B}_2$ but not $\overline{B}_3$. This falls within the framework of the usual mean-field dynamics: shearing of the cross-shear component $\overline{B}_1$ to produce the shear-wise component $\overline{B}_2$, and a dynamo action — due to the $\beta$ terms — that depends on the shearing rate $S$, as well as the strength $\eta_\alpha$, correlation time $\tau_\alpha$ and width $w_\alpha$ of the $\alpha$ fluctuations.

5 DYNAMO SOLUTIONS OF THE MEAN–FIELD EQUATIONS

Here we investigate the simplest of the ‘dynamo’ solutions of (19), and make predictions which are then compared with published numerical simulations. Since the dynamo action mainly involves the coupling of cross-shear ($\overline{B}_1$) and shear-wise ($\overline{B}_2$) components, we can study the subset of solutions with $\overline{B}_3 = 0$, where $\overline{B}_1$ and $\overline{B}_2$ depend only on $X_3$ and $\tau$; then $\nabla \cdot \overline{B} = 0$ is automatically satisfied. This case also happens to be the one studied most extensively in numerical simulations, where averaging over the coordinates $X_1$ and $X_2$ is useful in reducing noise. Then the mean-field equations reduce to:

\[
\begin{align*}
\frac{\partial \overline{B}_1}{\partial \tau} &= \beta_1 \frac{\partial^2 \overline{B}_2}{\partial X_3^2} + (\eta_K - 2\beta_2) \frac{\partial^2 \overline{B}_1}{\partial X_3^2}, \\
\frac{\partial \overline{B}_2}{\partial \tau} &= \beta_1 \frac{\partial^2 \overline{B}_1}{\partial X_3^2} + S \overline{B}_1 + \eta_K \frac{\partial^2 \overline{B}_2}{\partial X_3^2}.
\end{align*}
\]

(21)

We seek modal solutions of the form, $\overline{B} = \text{Re} \{ \overline{B}_0 \exp [\lambda \tau + i K X_3] \}$, where $\overline{B}_0$ is any constant complex vector in the horizontal $(e_1, e_2)$ plane, and $K$ is a constant (real) wavenumber. Using this in (21), we get the dispersion relation:

\[
\lambda_{\pm}(K) = - (\eta_K - \beta_2) K^2 \pm \sqrt{(\beta_1^2 + \beta_2^2) K^4 - S \beta_1 K^2},
\]

(22)

Since $\lambda_{\pm}(K)$ is an even function of $K$, we will henceforth assume that $K$ is positive. The real part of $\lambda_{\pm}(K)$ determines whether the mode grows or damps. It is always true that
Re $\{\lambda_+(K)\} \geq \text{Re} \{\lambda_-(K)\}$, so we attention focus on the growth rate, $\gamma(K) = \text{Re} \{\lambda_+(K)\}$, because we are interested in the fastest growing modes. To study this, it is convenient to define two positive dimensionless (control) parameters:

$$\sigma = |S\tau_\alpha| \quad \text{and} \quad \mu = \frac{\eta_\alpha}{\eta_T}. \quad (23)$$

Then $\eta_K = (\eta_T - \eta_\alpha) = \eta_T(1 - \mu)$ is negative when $\mu > 1$, and positive when $0 < \mu < 1$. In the former case, $\alpha$ fluctuations are strong and exceed turbulent diffusion; even in the white–noise limit ($\tau_\alpha = 0$) we always have a small–scale dynamo, as noted in the previous section. The latter, $0 < \mu < 1$, is the more interesting case in which all modes are damped for white–noise $\alpha$ fluctuations. The question of interest is: can shear, together with $\alpha$ fluctuations, lead to large–scale dynamo action when the fluctuations are weak (i.e. when $0 < \mu < 1$)?

To address this question we include the conditions of (20) and limit consideration to the following ranges of parameters:

$$0 \leq \mu < 1, \quad 0 \leq \sigma < 1, \quad 0 \leq w_\alpha \sigma^2 < 1, \quad 0 \leq \eta_T K^2 (1 - \mu) \sigma < |S|. \quad (24)$$

The parameters $\mu$ and $\sigma$ are restricted to the unit square in the $\mu$-$\sigma$ plane. The width $w_\alpha$ can take any positive value and, when it exceeds unity, we need only consider a smaller part of the unit square where $0 \leq \sigma < w_\alpha^{-1/2}$. The last of the constraints implies that the largest wavenumber that is allowed in our theory is:

$$K_m = \left(\frac{|S|}{\eta_T}\right)^{1/2} \frac{1}{\sqrt{(1 - \mu)\sigma}}. \quad (25)$$

The dispersion relation (22) also provides us with a characteristic wavenumber:

$$K_c = \left(\frac{|S|}{\eta_T}\right)^{1/2} \frac{1}{\sqrt{\mu\sigma(1 + w_\alpha^2\sigma^2)}}. \quad (26)$$

The square–root term in (22) is imaginary when $K < K_c$, so these low wavenumber modes are oscillatory with frequencies given by,

$$\omega_\pm(K) = \text{Im} \{\lambda_\pm(K)\} = \pm \eta_T \mu \sigma[1 + w_\alpha^2\sigma^2]^{1/2} K \sqrt{K_c^2 - K^2} \quad \text{for} \quad K < K_c. \quad (27)$$

The waves can be either growing or damped, and it is this property that is of the greatest interest to us. Below we study the properties of the growth rate, $\gamma(K)$, as a function of
various parameters; it is convenient to think of $\mu$ and $w_\alpha$ as given, while $\sigma$ is varied. It is useful to define two functions,

$$
\sigma_c(\mu, w_\alpha) = \frac{(1 - \mu)}{\sqrt{\mu [2 \mu - 1 + 2 w_\alpha (1 - \mu)]}} \quad \text{and} \quad \sigma_0(\mu, w_\alpha) = \sqrt{\frac{(1 - \mu)}{w_\alpha \mu}}, \quad (28)
$$

It is straightforward to prove that when $\sigma_c$ and $\sigma_0$ both are less than unity, we always have $\sigma_c \leq \sigma_0$. The significance of these two quantities is explained below. From [22] the growth rate can be written as:

$$
\gamma(K) = \begin{cases} 
  w_\alpha \eta T \mu [\sigma^2 - \sigma_0^2] K^2 & \text{if } 0 \leq K < K_c \\
  w_\alpha \eta T \mu [\sigma^2 - \sigma_0^2] K^2 + \eta T \mu \sigma [1 + w_\alpha^2 \sigma_0^2]^{1/2} K \sqrt{K^2 - K_c^2} & \text{if } K \geq K_c.
\end{cases} \quad (29)
$$

Two types of dynamo action are allowed by (29):

**Type I dynamo** (1 > $\sigma_0$ > $\sigma$ > $\sigma_c$): 
When $\sigma < \sigma_0$, the growth rate $\gamma(K)$ is negative for all wavenumbers $0 < K < K_c$, so these modes are damped and oscillatory, with frequency given by (27). The curve has a cusp at $K = K_c$ and begins increasing for larger $K$. This increase is monotonic when $\sigma$ exceeds $\sigma_{\text{min}} = \left( w_\alpha \sigma_0^2 / \sqrt{1 + 2 w_\alpha^2 \sigma_0^2} \right)$ and $\gamma(K_0) = 0$, as shown in Figure 1a, where

$$
K_0 = \left( \frac{|S|}{\eta T} \right)^{1/2} \sqrt{\frac{\mu \sigma}{\mu [\mu + 2 w_\alpha (1 - \mu)] \sigma^2 - (1 - \mu)^2}}. \quad (30)
$$

When $\sigma$ is close to and just a bit larger than $\sigma_{\text{min}}$, the zero–crossing point $K_0$ is larger than $K_m$, which is the maximum wavenumber allowed in our theory. For larger values of $\sigma$ the

---

**Figure 1.** Growth rate $\gamma(K)$ plotted as a function of wavenumber $K$. The left panel is for a Type I dynamo, where purely growing modes exist for $0 < K < K_m$. The right panel is for a type II dynamo, where all $0 < K < K_m$ grow; wavenumbers $K < K_c$ are growing and oscillatory, whereas $K_c \leq K < K_m$ are purely growing.
Figure 2. Phase diagram in the $\mu$-$\sigma$ unit square. The dark and light gray regions correspond to Type I and Type II dynamos. All modes are damped in the region left in white. The solid and dashed lines plot the functions $\sigma_c(\mu, 1)$ and $\sigma_0(\mu, 1)$, respectively.

gap between $K_m$ and $K_0$ lessens and, when $\sigma$ exceeds $\sigma_c$, $K_0$ becomes less than $K_m$. This condition which gives us the following necessary condition for dynamo action:

$$1 > \sigma > \sigma_c(\mu, w_\alpha).$$

Thus there are purely growing modes for $K_0 < K < K_m$.

**Type II dynamo ($1 > \sigma > \sigma_0$):**

From (29) we can see that the growth rate $\gamma(K)$ is positive, and a monotonically increasing function of $K$, as shown in Figure 1b; as earlier the curve has a cusp at $K = K_c$. Modes with $K < K_c$ are growing and oscillatory with frequency given by (27), whereas modes with $K_c < K < K_m$ are purely growing.

In Figure 2 we show the regions of dynamo action, in the unit square of the $\mu$-$\sigma$ plane (for

---

4 For a Type I dynamo we always have $K_c < K_0 < K_m$. It is straightforward to prove that $K_c < K_m$ for Type II dynamos too; to do this we note that $(K_m/K_c) = (1 + w_\alpha^2\sigma^2)/(1 - \mu)$, is an increasing function of $\sigma$, for given $\mu$ and $w_\alpha$. We know that this ratio is greater than unity for the Type I case. So it must be even larger for the Type II case where $\sigma$ takes larger values. Thus, for both types it must be that $K_m \geq K_c$.  

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$w_\alpha = 1$), as the areas shaded gray. The solid and dashed lines are the functions $\sigma_c(\mu)$ and $\sigma_0(\mu)$, respectively, given in (28). Type I and Type II dynamos operate in the dark and light gray regions, respectively; there is no dynamo action in the region left in white. Our theory predicts that there is a substantial region of the unit square in the $\mu$-$\sigma$ plane in which there are exponentially growing modes. This is in contrast to the case of white–noise $\alpha$ fluctuations where there is no region of the unit square which admits growing solutions; the square would be all white for white–noise $\alpha$ fluctuations.

For both types of dynamo action, I and II, the fastest growing modes have wavenumbers close to $K_m$, and (29) gives the (maximum) growth rate as:

$$\gamma_m = w_\alpha \eta \mu \left[ \sigma^2 - \sigma_0^2 \right] K_m^2 + \eta \mu \sigma [1 + w_\alpha \sigma^2]^{1/2} K_m \sqrt{K_m^2 - K_c^2} > 0. \quad (32)$$

## 6 COMPARISON WITH NUMERICAL SIMULATIONS

The shear dynamo problem was examined numerically by Yousef et al (2008) and Brandenburg et al (2008). These simulations clearly demonstrated the growth of large–scale magnetic field due to non–helical, white–noise, random forcing at small scales, in a background shear flow. As pointed out in Brandenburg et al (2008), and shown explicitly in Singh & Sridhar (2011) in the limit of small fluid Reynolds number (Re), that the white–noise random stirring gives rise to a fluctuating velocity field with a non zero correlation times. The emergence of large–scale features in the magnetic field has been shown in Figures (7) and (8) of Brandenburg et al (2008), which are called butterfly diagrams. As the simulations reported in Yousef et al (2008) were performed at relatively low Reynolds numbers (Re $= R_m = 5$), the small scale dynamo action was hence impossible, and thus the growth of magnetic energy in such simulations could be attributed to large–scale dynamo action. Brandenburg et al (2008) also measured fluctuations in the mean–field transport coefficients, $\alpha$ and $\eta$–tensors, by test–field method, and showed that each component of $\alpha$–tensor fluctuates with zero mean in a shearing background. As for high reynolds number flows the only long relevant time scale in the problem is the shearing time, so we set $\tau_\alpha = |S|^{-1}$ to compare our results with direct numerical simulations (DNS) of Yousef et al (2008) and Brandenburg et al (2008).

We need to choose a value for the width $w_\alpha$ of the normalized correlation function, $D(t)$. This function itself has not yet been measured in DNS. We note that, for an exponential form $D(t) = (2\tau_\alpha)^{-1} \exp \left[ -|t|/\tau_\alpha \right]$, the width is unity. In this work we set $w_\alpha = 1$, noting
that it may be necessary to use other values in the future. Then the quantities defined in equation (33) are:

\[
K_c = \left( \frac{|S|}{\eta T} \right)^{1/2} \frac{1}{\sqrt{2\mu}}, \quad K_0 = \left( \frac{|S|}{\eta T} \right)^{1/2} \frac{\mu}{\sqrt{2\mu(2-\mu)-1}},
\]
\[
K_m = \left( \frac{|S|}{\eta T} \right)^{1/2} \frac{1}{\sqrt{1-\mu}}, \quad \gamma_m = |S| \left[ \frac{2\mu - 1 + \sqrt{\mu(3\mu-1)}}{1-\mu} \right].
\]

Figure 3 shows plots of \(K_c\), \(K_0\) and \(K_m\) as functions of \(\mu\). Some points of interest are:

1. The necessary condition for dynamo action is \(\mu > \mu_c = (3 - \sqrt{5})/2 = 0.382\ldots\)
2. For \(\mu\) between \(\mu_c\) and \(\mu_0 = 1/2\), there is a Type I type dynamo, with purely growing modes of wavenumbers \(K_0 < K < K_m\) (shown as the region shaded dark gray).
3. For \(1/2 < \mu < 1\), a Type II type dynamo operates in the region shaded light gray. The dotted line separates the overstable solutions from the purely growing ones: there are purely growing solutions for \(K_c < K < K_m\), and growing oscillatory solutions for \(K < K_c\).

The fastest growing modes are purely growing ones, with wavenumber \(K_m\) and maximum growth rate \(\gamma_m\) as given in equation (33). We first note that the scalings, \(K_m \propto |S|^{1/2}\) and \(\gamma_m \propto |S|\),
are consistent with the DNS of Yousef et al. (2008). Then, using our expression for $\gamma_m$ from (33) and Figure 2 of Yousef et al. (2008), we estimate that $\mu = \eta_\alpha/\eta_T \simeq 0.383$ in the DNS. This value is just above but close to the threshold value of $\mu_c = 0.382 \ldots$ for the level of $\alpha$ fluctuations needed for a dynamo to operate in our theory. Since the DNS of Yousef et al. (2008) has modest values of fluid and magnetic Reynolds numbers ($\text{Re} = \text{Rm} = 5$), the fluctuations must be modest and close to the threshold for dynamo action. This is consistent with our estimate.

7 CONCLUSIONS

We have presented a mean-field theory of large-scale dynamo action in a linear shear flow that has stochastic, zero-mean fluctuations in time of the $\alpha$ parameter. The equations that govern the space-time evolution of the mean magnetic field are non-perturbative in $S$, the rate of shear; they are derived using the methods of shearing coordinates and Fourier variables presented earlier in Sridhar & Subramanian (2009a,b); Sridhar & Singh (2010); Singh & Sridhar (2011). The expression for the mean EMF is a time integral, with integrand that has both the mean magnetic field and $\mathcal{D}(t)$, the normalized two-point time correlation function of the $\alpha$ fluctuations. Thus the most general form of the mean-field theory is given by a set of coupled set of linear integro-differential equations.

An immediate and instructive application of this theory is to the case of white-noise $\alpha$ fluctuations which has zero correlation time ($\tau_\alpha = 0$). The expression for the mean EMF is identical to the case of $\alpha$ fluctuations in the absence of shear, derived earlier by Kraichnan (1976); in other words, shear has no effect at all on the mean EMF when the $\alpha$ fluctuations are of white-noise form. The physical reason why shear plays such a limited role is that shear needs time to act, whereas white-noise fluctuations have zero correlation time. The corresponding mean-field equation is a PDE where shear enters only in the background flow, so the detailed behaviour of the solutions depends on shear and is, indeed, different from the case when shear is absent, as discussed in Section 3. Here we recall just the most important point: (i) The necessary condition for dynamo action is identical to the case without shear, treated by Kraichnan (1976); namely, $\eta_\alpha > \eta_T$, that the $\alpha$-diffusivity must be sufficiently strong; (ii) when $\eta_\alpha > \eta_T$, modes of all wavenumbers grow by (shear modified) negative diffusion with the highest wavenumbers growing fastest, so white noise $\alpha$ fluctuations in a linear shear flow cannot lead to large-scale dynamo action.
Thus a large-scale shear dynamo of this form must necessarily have non zero $\alpha$ correlation time ($\tau_\alpha \neq 0$). The integral expression for the mean EMF can be significantly simplified for the physically important case when the mean magnetic field is a slowly varying function of space and time. We obtain a concise and physically appealing expression for the mean EMF which has, in addition to a Kraichnan-type diffusion term (which is the only term present for white-noise $\alpha$ fluctuations), additional terms that depend on the first-order spatial gradients of the ‘horizontal’ components, $\overline{B}_1$ and $\overline{B}_2$, of the mean magnetic field. These terms are proportional to new diffusivity coefficients, $\beta_1 = \eta_\alpha S \tau_\alpha$ and $\beta_2 = \eta_\alpha w_\alpha (S \tau_\alpha)^2$, which depend on shear, as well as the strength, correlation time and width of the $\alpha$ fluctuations. Whereas $\beta_1$ can be of either sign, $\beta_2$ is always positive. The evolution of the mean magnetic field is now governed by a set of coupled PDEs, which are more tractable than the integro-differential equations of the general theory.

In order to investigate dynamo solutions and derive consequences that can be compared with numerical solutions, we consider the case when the mean-field is a function only of the spatial coordinate $X_3$ and time $\tau$. Then the diffusivity $\beta_2$ contributes negatively to the diffusion of the cross-shear component $\overline{B}_1$, reducing the coefficient to $(\eta_K - 2 \beta_2)$, which is below the Kraichnan value $\eta_K$; the shear-wise component $\overline{B}_2$ has a diffusion coefficient equal to $\eta_K$. The diffusivity $\beta_1$ plays a more important role, resulting in a mutual coupling of $\overline{B}_1$ and $\overline{B}_2$. We then derive a dispersion relation for Fourier modes, and study this in some detail, as a function of three dimensionless parameters, which are $\sigma = |S \tau_\alpha|$, $\mu = (\eta_\alpha / \eta_T)$ and $w_\alpha$.

We give below our salient results, where the quantities $(\sigma_c, \sigma_0, K_c, K_0, K_m, \gamma_m)$ are defined in Section 5:

1. The necessary condition for large-scale dynamo action is $1 > \sigma > \sigma_c(\mu, w_\alpha)$.

2. There are two types of dynamos allowed in our theory:

   - **Type I dynamos** occur when $1 > \sigma_0(\mu, w_\alpha) > \sigma > \sigma_c(\mu, w_\alpha)$. The modes are damped and oscillatory for wavenumbers $K < K_c$, and purely growing for wavenumbers in the range $K_c \leq K < K_m$.

   - **Type II dynamos** occur when $1 > \sigma > \sigma_0(\mu, w_\alpha)$. The modes are growing and oscillatory for wavenumbers $K < K_c$, and purely growing for $K_c < K < K_m$.

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5 It is necessary that $\sigma < 1$ for the PDEs — rather than the integro-differential equations — to be a valid description of mean-field dynamics. Although it is not necessary, we also consider $\mu < 1$; otherwise the $\alpha$ fluctuations are so strong that one would anyway have a Kraichnan-type negative diffusivity. In all the figures, and formulae comparing with numerical simulations we have chosen $w_\alpha = 1$, corresponding to an exponential-type $\alpha$ correlation function.
3. For both Type I and Type II dynamos, the fastest growing modes are purely growing ones which have wavenumbers close to $K_m$, and growth rate $\gamma_m$.

4. Our theory predicts that there is a substantial region of the unit square in the $\mu$-$\sigma$ plane, shown in phase diagram of Figure 2, in which there are exponentially growing modes. This is in contrast to the case of white–noise $\alpha$ fluctuations where there is no region of the unit square which admits growing solutions.

For comparison with numerical simulations, we obtain expressions for the wavenumber $K_m \propto |S|^{1/2}$ and growth rate $\gamma_m \propto |S|$, of the fastest growing dynamo mode, which are in agreement with the numerical simulations of Yousef et al (2008); Brandenburg et al (2008). We also estimate that the level of $\alpha$ fluctuations in the simulations of Yousef et al (2008) should be close to the threshold value predicted by our theory; this is consistent with the fact that these simulations have modest values of fluid and magnetic Reynolds numbers. Our analysis of growing modes is valid for $|S\tau_\alpha| < 1$, but this is not a necessary restriction. The most general formulation, given by (2) together with (10), is valid for arbitrary values of $|S\tau_\alpha|$ because the development until this point is non perturbative in the shear. This is a closed set of integro–differential equations governing the dynamics of the mean magnetic field $\mathcal{B}$, which must be addressed if we are to understand mean–field theory for large shear and non zero $\alpha$ correlation times.

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