On the center of mass of isolated systems with general asymptotics

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Abstract
We propose a definition of center of mass for asymptotically flat manifolds satisfying the Regge–Teitelboim condition at infinity. This definition has a coordinate-free expression and natural properties. Furthermore, we prove that our definition is consistent both with the one proposed by Corvino and Schoen and another by Huisken and Yau. The main tool is a new density theorem for data satisfying the Regge–Teitelboim condition.

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1. Introduction

A 3-manifold $M$ with a Riemannian metric $g$ and a 2-tensor $K$ is called a vacuum initial data set $(M, g, K)$ if $g$ and $K$ satisfy the constraint equations

\[ R_g - |K|^2 + H^2 = 0, \quad \text{div}_g(K) - dH = 0, \quad (1.1) \]

where $R_g$ is the scalar curvature of $M$ and $H = \text{Tr}_g(K) = g^{ij}K_{ij}$. We say $(M, g, K)$ is asymptotically flat (AF) if it is a vacuum initial data set and there exists a coordinate $\{x\}$ outside a compact set, say $B_{R_0}$, in $M$ such that, for some $\delta \in (1/2, 1]$,

\[ g_{ij}(x) = \delta_{ij} + h_{ij}(x), \quad h_{ij} = O(|x|^{-\delta}), \quad K_{ij}(x) = O(|x|^{-1-\delta}), \quad (1.2) \]

and similarly for higher derivatives up to the third derivatives on $g$ and up to the second derivatives on $K$. For AF manifolds, the ADM mass $m$ is defined by

\[ m = \frac{1}{16\pi} \lim_{r \to \infty} \int_{|x| = r} \sum_{i,j} (g_{ij,i} - g_{ii,j})v^j g \, d\sigma_g, \]

where $v^j$ is the outward normal.

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where \( |x| = r \) is the Euclidean sphere, \( v_g \) is the unit outward normal vector field with respect to the metric \( g \) and \( \sigma_g \) is the volume form induced from \((M, g)\). We remark that if one replaces \( v_g \) by the unit outward normal vector field \( v_0 \) with respect to the Euclidean metric and \( \sigma_g \) by the volume form \( d\sigma_0 \) with respect to the induced metric from the Euclidean space, the limit is the same. Similarly, the remark holds for (1.4), (1.5) and (1.8).

For the case in which we are interested, we require asymptotic symmetry on \((M, g, K)\). We say \((M, g, K)\) is asymptotically flat satisfying the Regge–Teitelboim condition (AF-RT) if \((M, g, K)\) is AF, and \( g, K \) satisfy these asymptotically even/odd conditions

\[
\begin{align*}
G_{ij}(x) &= O(|x|^{-1-\delta}), \\
K_{ij}(x) &= O(|x|^{-2-\delta}),
\end{align*}
\]

and on higher derivatives, where \( f^{\text{odd}}(x) = f(x) - f(-x) \) and \( f^{\text{even}}(x) = f(x) + f(-x) \), [RT74]. Note that \( f^{\text{odd}} \) and \( f^{\text{even}} \) are only defined outside \( B_{R_0} \), in which the coordinates \( x \) are defined. It is proved by Corvino and Schoen in [CS06] that AF-RT manifolds form a dense subset of AF manifolds in some suitable weighted Sobolev space.

For \((M, g, K)\) satisfying AF-RT, we propose an intrinsic definition of the center of mass suggested by Richard Schoen,

\[
C^\alpha_i = \frac{1}{16\pi m} \lim_{r \to \infty} \int_{|x| = r} \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) Y(\alpha)_i Y(\alpha)_j \sigma_g, \quad \alpha = 1, 2, 3, \tag{1.4}
\]

where \( R_{ij} \) is the Ricci curvature of \( M \) and \( Y(\alpha) = (|x|^2 - 2x^a x^b \frac{\partial}{\partial x^b})^{\frac{\alpha_0}{2}} \) is a Euclidean conformal Killing vector field. Einstein summation convention is used throughout the paper. The definition is intrinsic because it is a flux integral at infinity of the three-dimensional Einstein tensor contracted with two vector fields \( Y(\alpha)_i \) and \( v_g \), and \( Y(\alpha) \) has a geometric meaning. Moreover, the surface of integration can be defined geometrically (proposition 3.1). If \((M, g, K)\) is AF-RT, the limit converges. Note that the above expression is not defined when the ADM mass \( m \) is zero. However, the center of mass cannot be well defined when \( m \) is zero because a basic version of positive mass theorem shows that \( M \) is actually the Euclidean space [SY79, SY81].

This intrinsic definition is motivated by a similar expression of the ADM mass when \( Y(\alpha) \) in definition (1.4) is replaced by \(-2x^i \frac{\partial}{\partial x^i}\), the radial direction Euclidean conformal Killing vector field,

\[
m = \frac{1}{16\pi} \lim_{r \to \infty} \int_{|x| = r} \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) (-2x^i) v^j \sigma_g. \tag{1.5}
\]

The Euclidean conformal Killing vector fields \(-2x^i \frac{\partial}{\partial x^i}\) and \( Y(\alpha) \) generate dilation and translation at infinity. A detailed discussion about these two vector fields can be found in [CW08]. Another motivation comes from the spatial Schwarzschild metric

\[
g^S(x) = \left( 1 + \frac{m}{2|x-p|} \right)^4, \quad \delta = 1 + \frac{2m}{|x|} + \frac{2mx \cdot p}{|x|^3} + \frac{3m^2}{|x|^2} + O(|x|^{-3}).
\]

If we replace the metric in (1.4) by \( g^S \), \( C_f \) is precisely the vector \( p \) which indicates the center of the manifold. It is worth mentioning that although in the Schwarzschild case \( p \) is not a point in the manifold, one can develop polar coordinates using concentric spheres centered at \( p \).

\[1\] The intrinsic definition can be generalized to \( n \)-dimensional manifolds if we replace the factor \( 1/(16\pi m) \) by a suitable constant depending on \( n \) and \( \alpha_0 \), where \( \alpha_0 \) is the volume of the unit ball in \( \mathbb{R}^n \). Hence the following arguments work more generally for \( n \)-dimensional manifolds, but for simplicity, we assume \( n = 3 \).

\[2\]
Other notions of the center of mass have been proposed. Huisken and Yau [HY96] define the center of mass for \((M, g, K)\) which is spherically asymptotically flat (SAF), i.e. \((M, g, K)\) is AF and
\[
g_{ij}(x) = \left(1 + \frac{2m}{|x|}\right) \delta_{ij} + p_{ij}
\]
\[
p_{ij}(x) = O(|x|^{-2}), \quad \partial^a p_{ij}(x) = O(|x|^{-2-|a|}), \quad 1 \leq |a| \leq 4.
\]
They prove the existence and uniqueness of the constant mean curvature foliation \(\{M_r\}\) for SAF manifolds, where the mean curvature of \(M_r\) is \((2/r) - (4m/r^2) + O(r^{-3})\). They also show that the geometric center defined as follows converges:
\[
C_{\alpha HY} = \lim_{r \to \infty} \frac{\int_{M_r} z^\alpha \, d\sigma_0}{\int_{M_r} d\sigma_0}, \quad \alpha = 1, 2, 3,
\]
where \(z\) is the position vector of \(M_r\) in \(M\). This definition is also motivated by the spatial Schwarzschild manifold in which the constant mean curvature foliation is \(\{|x - p| = r\}\).

Another definition in [CS06] is defined for AF-RT manifolds by
\[
C_{\alpha CS} = \frac{1}{16\pi m} \lim_{r \to \infty} \left( \int_{|x| = r} \sum_{i,j} x^\alpha (g_{ij,i} - g_{ii,j}) v_i^g \, d\sigma_g - \int_{|x| = r} \sum_{i,j} (h_{ia} v_i^g - h_{ij} v_j^g) \, d\sigma_g \right),
\]
where we recall that \(h_{ij} = g_{ij} - \delta_{ij}\), see also [BO87, BLP03]. One application of this definition in [CS06] is for AF-RT manifolds by
\[
C_{I} = C_{CS}.
\]

The main purpose of this paper is to prove that the intrinsic definition (1.4) is equivalent to the Corvino–Schoen definition (1.8). Moreover, for SAF manifolds in which the unique foliation of constant mean curvature surfaces exists, our intrinsic definition is equal to the Huisken–Yau definition (1.8). In other words, the intrinsic definition is a coordinate-free expression of the Corvino–Schoen definition and it generalizes the Huisken–Yau definition.

**Theorem 1.** Assume \((M, g, K)\) is AF-RT (1.3). Then
\[
C_I = C_{CS}.
\]

**Theorem 2.** Assume \((M, g, K)\) is SAF (1.6). Then
\[
C_I = C_{HY}.
\]

We would like to make a note that Corvino and Wu [CW08] recently have a result about the equivalence of these definitions under the assumption that the metric \(g\) is conformally flat at infinity with vanishing scalar curvature. In that special case, they are able to derive some explicit estimates. However, it seems that their approach cannot be generalized to AF-RT metrics.

This paper is organized as follows. In section 3, we prove a density theorem (theorem 2.2) for AF-RT manifolds. The theorem is crucial for most of the arguments in the paper and may be of independent interest. In section 4, we discuss natural properties of the intrinsic definition. Theorems 1 and 2 are proved in sections 5 and 6, respectively.
2. The density theorem

Let \((M, g, K)\) be a vacuum initial data set. We introduce the momentum tensor
\[
\pi^{ij} = K^{ij} - \text{Tr}_g(K) g^{ij}.
\]
The constraint equations (1.1) then take the form
\[
R_g + \frac{1}{2} (\text{Tr}_g \pi)^2 - |\pi|^2 g = 0 \quad \text{div}_g(\pi) = 0. \tag{2.1}
\]
We define
\[
\Phi(g, \pi) = \left( R_g + \frac{1}{2} (\text{Tr}_g \pi)^2 - |\pi|^2 g, \text{div}_g(\pi) \right).
\]
In the case that \((M, g, K)\) is AF-RT, Corvino and Schoen [CS06] prove that AF manifolds can be approximated by AF-RT manifolds in some weighted Sobolev space. Moreover, instead of requiring smooth solutions to the constraint equations (2.1), their theorem works for solutions with weak regularity. Before we state the theorem, we need the following definitions.

**Definition 2.1** (Linear and angular momentum). The linear momentum \((P_1, P_2, P_3)\) and the angular momentum \((J_1, J_2, J_3)\) are defined as follows:
\[
P_i = \frac{1}{8\pi} \lim_{r \to \infty} \int_{|x| = r} \pi_{ij} v^j_g \, dg,
\]
\[
J_{\alpha} = \frac{1}{8\pi} \lim_{r \to \infty} \int_{|x| = r} \pi_{jk} Z(\alpha)^k_j v^k_g \, dg,
\]
where \(Z(\alpha)\) give the rotation fields in \(\mathbb{R}^3\), for example \(Z(1) = x^2 \partial_x^3 - x^3 \partial_x^2\).

**Remark.** The linear momentum is well defined for AF manifolds and the angular momentum is well defined for AF-RT manifolds.

**Definition 2.2** (Weighted Sobolev spaces). For a non-negative integer \(k\), a non-negative real number \(p\) and a real number \(\delta\), we say \(f \in W^{k,p-\delta}_{\text{loc}}(M)\) if
\[
\|f\|_{W^{k,p-\delta}_{\text{loc}}(M)} \equiv \left( \int_M \sum_{|\alpha| \leq k} \left( \left| D^\alpha f \right| \rho^{\rho(\alpha + \delta) |x|^{-3}} \right)^p \, ds_g \right)^{\frac{1}{p}} < \infty,
\]
where \(\alpha\) is a multi-index and \(\rho\) is a continuous function with \(\rho = |x|\) on \(M \setminus B_0\).

When \(p = \infty\),
\[
\|f\|_{W^{k,\infty}_{\text{loc}}(M)} = \sum_{|\alpha| \leq k} \text{ess sup}_M |D^\alpha f| \rho^{\rho(\alpha + \delta)}.
\]

**Definition 2.3** (Harmonic asymptotics). \((M, g, \pi)\) is said to have harmonic asymptotics if \((M, g, \pi)\) is AF and
\[
g = u^4 \delta, \quad \pi = u^2 (L_\delta X) \tag{2.2}
\]
outside a compact set for some \(u\), \(X\) tending to 1, 0, respectively, where, for any metric \(g\), \(L_\delta X\) is the operator associated with the Lie derivative \(L_X g\) defined by
\[
L_X g = L_X g - \text{div}_g(X) g.
\]

By the constraint equations (2.1), \(u\) and \(X\) in definition 2.3 satisfy the following equations outside the compact set:
\[
8 \Delta_\delta u = \left( -|L_\delta X|^2 + \frac{1}{2} (\text{Tr}_\delta (L_\delta X))^2 \right) u,
\]
\[
\Delta_\delta X^i + 4 u^{-1} u_j (L_\delta X)_j^i - 2 u^{-1} u^i \text{Tr}_\delta (L_\delta X) = 0,
\]

where \( u_t = u' = \frac{\partial u}{\partial x_0} \). Asymptotic flatness requires \( u \) and \( X \) tend to 1 and 0, respectively, at some decay rate. Using the decay conditions on \( u \) and \( X \), we have \( \Delta u = O(|x|^{-2-2\delta}) \) and \( \Delta X = O(|x|^{-2-2\delta}) \). As shown in [B86], the asymptotic behavior implies that outside a compact set
\[
\begin{align*}
u &= 1 + \frac{a}{|x|} + O(|x|^{-1-\delta}), \\
X' &= \frac{b'}{|x|} + O(|x|^{-1-\delta})
\end{align*}
\] (2.3)
for some constants \( a \) and \( b' \). Note that this clearly implies that \((M, g, K)\) is AF-RT. Furthermore, if \((M, g, K)\) is AF-RT, \( u^{\text{odd}}(x) \) and \((X')^{\text{odd}}\) satisfy equations with better decay: \( \Delta u^{\text{odd}} = O(|x|^{-4-2\delta}) \) and \( \Delta (X')^{\text{odd}} = O(|x|^{-4-2\delta}) \). Hence
\[
\begin{align*}
u^{\text{odd}}(x) &= \frac{c \cdot x}{|x|^3} + O(|x|^{-2-\delta}), \\
(X')^{\text{odd}}(x) &= \frac{d(i) \cdot x}{|x|^3} + O(|x|^{-2-\delta}), \\
\left(\nu^{\text{odd}}(x) - \frac{c \cdot x}{|x|^3}\right)_k &= O(|x|^{-3-\delta}), \\
\left((X')^{\text{odd}}(x) - \frac{d(i) \cdot x}{|x|^3}\right)_k &= O(|x|^{-3-\delta}),
\end{align*}
\] (2.4)
for some vectors \( c, d(i) \) which are quantities corresponding to the center of mass and angular momentum of \((g, \pi)\). Since we assume \( g, K \) with the pointwise regularity, the above identities hold pointwisely. However, we can generalize the above discussions to the setting of weighted Sobolev spaces and have more general results as follows.

**Theorem 2.1** (CS, theorem 1). Let \((g_{ij} - \delta_{ij}, \pi_{ij}) \in W_{-\frac{3}{2}}^{3,p}(M) \times W_{-\frac{3}{2}}^{3,p}(M)\) be a vacuum initial data set, where \( \delta \in (1/2, 1) \) and \( p > 3/2 \). There is a sequence of solutions \((\tilde{g}_k, \tilde{\pi}_k)\) with harmonic asymptotics satisfying (2.2) and (2.3). Given any \( \epsilon > 0 \), there exist \( k_0 > 0 \) so that
\[
\|g - \tilde{g}_k\|_{W_{-\frac{3}{2}}^{3,p}(M)} \leq \epsilon, \quad \|\pi - \tilde{\pi}_k\|_{W_{-\frac{3}{2}}^{3,p}(M)} \leq \epsilon, \quad \text{for } k \geq k_0.
\]
Moreover, the mass and linear momentum of \((\tilde{g}_k, \tilde{\pi}_k)\) are within \( \epsilon \) of those of \((g, \pi)\).

The theorem says that the solutions with harmonic asymptotics (2.2) are dense among general solutions. More remarkably, the mass and linear momentum which can be explicitly expressed for solutions with harmonic asymptotics converge to the original initial data set in these weighted Sobolev spaces. However, in the above theorem the center of mass does not seem to converge, neither is the center of mass \( C_{ij} \) generally defined for AF manifolds. Therefore, we would like to modify their theorem and prove, in some weighted Sobolev space, solutions with harmonic asymptotics form a dense subset inside AF-RT solutions so that the center of mass and angular momentum converge. The precise statement is as follows.

**Theorem 2.2** (Density theorem). Let \((g - \delta, \pi) \in W_{-\frac{3}{2}}^{3,p}(M) \times W_{-\frac{3}{2}}^{3,p}(M)\) be a vacuum initial data set and \((g^{\text{odd}}, \pi^{\text{even}}) \in W_{-\frac{3}{2}}^{3,p}(M \setminus B_{\delta_0}) \times W_{-\frac{3}{2}}^{3,p}(M \setminus B_{\delta_0})\), where \( \delta \in (1/2, 1) \) and \( p > 3 \). There is a sequence of solutions \((\tilde{g}_k, \tilde{\pi}_k)\) with harmonic asymptotics satisfying (2.2), (2.3) and (2.4). Given any \( \epsilon > 0 \) and \( \delta_0 \in (0, \delta) \), there exist \( R \) and \( k_0 = k_0(R) \) so that \((\tilde{g}_k, \tilde{\pi}_k)\) is within an \( \epsilon \)-neighborhood of \((g, \pi)\) in the \( W_{-\frac{3}{2}}^{3,p}(M) \times W_{-\frac{3}{2}}^{3,p}(M)\) norm and
\[
\|g_k^{\text{odd}}\|_{W_{-\frac{3}{2}}^{3,p}(M | B_{\delta_0})} \leq \epsilon, \quad \|\pi_k^{\text{even}}\|_{W_{-\frac{3}{2}}^{3,p}(M | B_{\delta_0})} \leq \epsilon, \quad \text{for } k \geq k_0.
\]
Moreover, the mass, linear momentum, center of mass and angular momentum of \((\tilde{g}_k, \tilde{\pi}_k)\) are within \( \epsilon \) of those of \((g, \pi)\).

First, we briefly describe Corvino and Schoen’s construction of the approximating solutions \((\tilde{g}_k, \tilde{\pi}_k)\).
Sketch of the proof of theorem 2.1. Let \((\hat{g}_k, \hat{\pi}_k)\) be 2-tensors cut off from the original solutions \((g, \pi), \hat{g}_k = \xi_k g + (1 - \xi_k)\delta = \hat{\xi}_k h, \hat{\pi}_k = \xi_k \pi, \) where \(\xi_k\) is a smooth cut-off function

\[
\xi_k(x) = \begin{cases} 
1 & \text{between 0 and 1} \\
0 & \text{when } |x| \leq 2k
\end{cases}
\]

\(\xi_k\) is chosen so that \(|D\xi_k| \leq c/k\) and \(|D^2\xi_k| \leq c/k^2\) for some constant \(c\) independent of \(k\). Then we let

\[
\hat{g}_k = u_1^2 \hat{g}_k, \quad \hat{\pi}_k = u_1^2 (\hat{\pi}_k + \mathcal{L}_{\hat{g}_k} X_k).
\]

To simplify notations, we denote \(\mathcal{L}_{\hat{g}_k}\) by \(\mathcal{L}\), and we also drop the subindex \(k\) when it is clear from context. In order to find \(u\) near 1 and \(X\) near 0 at infinity so that \((\hat{g}, \hat{\pi})\) is a vacuum initial data set, we need to solve the following systems for \(u\) and \(X\) from the constraint equations (2.1):

\[
\mu = u^{-3} (-8\Delta_u u + (R(\hat{g}) - |\hat{\pi} + \mathcal{L}X|^2 + \frac{1}{4}(\text{div}(\hat{\pi} + \mathcal{L}X)^2)u) = 0
\]

\[
(\text{div}(\hat{\pi}))(i) = u^{-2} ((\text{div}(\hat{\pi} + \mathcal{L}X))_i + 4u^{-1} \hat{g}^{jk} u_j (\hat{\pi} + \mathcal{L}X)_{ik} - 2u^{-1} u_i \text{Tr}(\hat{\pi} + \mathcal{L}X) = 0,
\]

\[
i = 1, 2, 3. \quad (2.5)
\]

Consider the map \(T : (1, 0) + W^{1,p}_{\text{g, p}}(M) \times W^{2,p}_{\text{g, p}}(M) \rightarrow W^{1,2}_{\text{g, p}}(M) \times W^{0,2}_{\text{g, p}}(M)\) defined by

\[
T(u, X) = (\mu, \text{div}(\hat{\pi})).\]

It is known that \(DT_{(1, 0)}\) is a Fredholm operator of index 0. For the Fredholm operator of index 0, the operator is injective if and only if it is surjective. However, it is not clear whether \(DT_{(1, 0)}\) is surjective. Corvino and Schoen enlarge the domain and utilize initial data sets of the form \((u^2 \hat{g} + h, u^2 (\hat{\pi} + \mathcal{L}X) + q)\) with

\[
\Phi(u^2 \hat{g} + h, u^2 (\hat{\pi} + \mathcal{L}X) + q) = (0, 0),
\]

where \(h\) and \(q\) are symmetric (0, 2)-tensors with compact supports. Then they prove that the operator \(D\Phi_{(1, 0)}\) maps surjectively onto \(W^{1,2}_{\text{g, p}}(M) \times W^{0,2}_{\text{g, p}}(M)\) for \(p > 1\) and \(\delta \in (0, 1)\).

Since \(DT_{(1, 0)}\) is Fredholm of index 0, we have

\[
W^{1,2}_{\text{g, p}}(M) \times W^{0,2}_{\text{g, p}}(M) = \text{Ker}(DT_{(1, 0)}) \oplus W_1,
\]

\[
W^{1,2}_{\text{g, p}}(M) \times W^{0,2}_{\text{g, p}}(M) = \text{Range}(DT_{(1, 0)}) \oplus \text{span}\{V_1, \ldots, V_N\},
\]

where \(W_1\) is an \(n\)-dimensional linear subspace and \(\{V_1, \ldots, V_N\}\) is a basis for the cokernel of \(DT_{(1, 0)}\). Because \(D\Phi_{(1, 0)}\) is surjective, we can choose \((\hat{h}_1, q_1), \ldots, (\hat{h}_N, q_N)\) so that \(D\Phi_{(1, 0)}(\hat{h}_i, q_i) = V_i\). Supports of those \((\hat{h}_i, q_i)\) may not be compact, but there exist \((\tilde{h}_i, \tilde{q}_i)\) with compact supports close enough to \((\hat{h}_i, q_i)\) so that \(V_i = D\Phi_{(1, 0)}(\tilde{h}_i, \tilde{q}_i)\) still span a complementing subspace for \(\text{Range}(DT_{(1, 0)})\).

Let \(W_2 = \text{span}\{(\tilde{h}_1, \tilde{q}_1), \ldots, (\tilde{h}_N, \tilde{q}_N)\}\). \(W = W_1 \times W_2\) is a Banach space inside \(W^{1,2}_{\text{g, p}}(M) \times W^{0,2}_{\text{g, p}}(M)\). Define the map \(\overline{T}\) from \(((1, 0), (0, 0)) + W_1 \times W_2 \rightarrow W^{1,2}_{\text{g, p}}(M) \times W^{0,2}_{\text{g, p}}(M)\) by

\[
\overline{T}(u, X, (h, q)) = \Phi(u^2 \hat{g} + h, u^2 (\hat{\pi} + \mathcal{L}X) + q).
\]

\(\overline{T}\) is an isomorphism by construction. Hence the inverse function theorem asserts that \(\overline{T}\) is an isomorphism from a fixed (independent of \(k\)) neighborhood of \(((1, 0), (0, 0))\) to a fixed neighborhood of \(\Phi(\hat{g}, \hat{\pi})\). Because \((0, 0)\) is contained in the image when \(k\) is large, there exists a unique \(((u, X), (h, q))\) within that fixed neighborhood of \(((1, 0), (0, 0))\) such that \(\Phi(u^2 \hat{g} + h, u^2 (\hat{\pi} + \mathcal{L}X) + q) = (0, 0)\) for \(k\) large.

Note that the supports of \(h, q\) in the proof may not be uniformly bounded in \(k\), but it is important in the proof of theorem 2.2 that \(h, q\) have compact supports uniformly bounded in
Therefore, we need to carefully choose the cokernel of $DT_{(1,0)}$ for $k$ large. This choice is described by the following lemma.

**Lemma 2.4.** $V$ and $W$ are Banach spaces. Assume $S_k : V \rightarrow W$ is a sequence of Fredholm operators and $S_k$ converges (in the operator norm) to some Fredholm operator $S'$. If $W = \text{Range } S' \oplus W'$ for some finitely dimensional closed subspace $W'$, we can choose the cokernel of $S_k$ inside the cokernel $W'$ of $S'$, i.e. $W = \text{Range } S_k \oplus W_k$ for some $W_k \subset W'$, for $k$ large.

**Proof.** Since $S'$ is Fredholm, there exists $V'$ such that $V = \text{Ker } S' \oplus V'$. Consider for any bounded linear operator $S : V \rightarrow W$, the map

$$\tau_S : V' \oplus W' \rightarrow W$$

given by $\tau_S(v, w) = Sv + w$. Then $\tau_S$ is an isomorphism for $k$ large since $\tau_S'$ is an isomorphism by construction and isomorphism is an open condition in the space of linear operators. We then have

$$W = S_k(V') \oplus W'.$$

Therefore, for any $v \in V$, $S_k(v)$ can be decomposed uniquely into $S_k(v') + S_k(v) - S_k(v')$ for some $v' \in V'$ and $S_k(v) - S_k(v') \in W'$. Hence $v$ can be decomposed uniquely into $v = v' + (v - v')$ as well, where $S_k(v - v') \in W'$. Let $U_k = \{u \in V : S_k(u) \in W'\}$. It is easy to see that $U_k$ is a closed space in $V$ and

$$V = V' \oplus U_k.$$

Note that $\text{Ker } S_k$ is a finite dimensional subspace in $U_k$, so we can write

$$V = V' \oplus \text{Ker } S_k \oplus Z_k$$

for some closed subspace $Z_k \subset U_k$. $S_k(Z_k)$ is closed in $W'$, so there is $W_k \subset W'$ such that $S_k(Z_k) \oplus W_k = W'$ and hence

$$W = S_k(V') \oplus S_k(Z_k) \oplus W_k = \text{Range } S_k \oplus W_k. \quad \square$$

**Corollary 2.5.** The supports of $h$ and $q$ in the proof of theorem 2.1 can be chosen to be uniformly bounded for $k$ large.

Next, the key lemma (lemma 2.8) used to prove theorem 2.2 is an a priori type estimate for second-order elliptic equations $Pv = f$ which have a symmetric property at infinity. Roughly speaking, if we know $P$ and $f$ are even (or odd, respectively) then we hope solutions $v$ will be even (or odd, respectively) as well. However, this is not true generally because boundary values can affect solutions dramatically. For example, consider two harmonic functions $u_1, u_2$ in $\mathbb{R}^3 \setminus B_{R_0}$,

$$u_1 = \frac{1}{|x|} \quad \text{and} \quad u_2 = \frac{e \cdot x}{|x|^3}.$$

Both $|u_1|$ and $|u_2|$ tend to zero at infinity. However, $u_1$ is even and $u_2$ is odd. They are solutions for different boundary values on the inner boundary. Nevertheless, in the case that the boundary value is very small, we will show the symmetry of the solutions is not affected much in the region away from the boundary. Before we state the lemma, we will give some definitions of the operators we consider. This class of operators is discussed in detail in [B86].

**Definition 2.6.** Let $P$ defined as $Pu = a^{ij}(x)\partial^2_{ij}u + b^i(x)\partial_i u + c(x)u$ be an elliptic operator.
(1) \( P \) is said to be asymptotic (at rate \( \tau \)) to an elliptic operator \( \tilde{P} \), \( \tilde{P}u = \tilde{a}^{ij}(x)\partial_{ij}u + \tilde{b}^i(x)\partial_iu + \tilde{c}(x)u \), if there exist \( q \in (3, \infty) \), \( \tau \geq 0 \) and a constant \( C \) such that over the region \( M \setminus B_R \),
\[
\|a^{ij} - \tilde{a}^{ij}\|_{W^{q,r}_x} + \|b^i - \tilde{b}^i\|_{W^{q,r}_x} + \|c - \tilde{c}\|_{W^{q,2}_x} \leq C.
\]

(2) \( P^{\text{odd}} \) is the odd part of the operator, defined on \( M \setminus B_R \) by
\[
P^{\text{odd}} = (a^{ij})^{\text{odd}} \partial_{ij} + (b^i)^{\text{even}} \partial_i + c^{\text{odd}}.
\]

Remark. We assume \( P : W^{s,p}_{-a} \rightarrow W^{s-2,p}_{-a} \) for \( s = 2 \) or \( 3 \), \( 1 < p < q \) and a non-integer \( a > 0 \), and define
\[
\|P\|_{\text{op}, M', B_R} = \sup \{ \|Pw\|_{W^{s-2,p}_{-a}(M)} : \|w\|_{W^{s,p}_{-a}(M)} = 1, \supp w \subset M' \setminus B_R \}.
\]

From Definition 2.6, we have
\[
\|P - \tilde{P}\|_{W^{s-2,p}_{-a}(M), \text{loc}} \leq \|a^{ij} - \tilde{a}^{ij}\|_{W^{q,r}_x} + \|b^i - \tilde{b}^i\|_{W^{q,r}_x} + \|c - \tilde{c}\|_{W^{2,q}_x},
\]

as \( R \to \infty \). Therefore, \( \|P - \tilde{P}\|_{\text{op}, M', B_R} \to 0 \) as \( R \to \infty \).

Definition 2.7. We say a sequence of elliptic operators \( P_k \), \( P_ku = a_{(k)}^{ij}(x)\partial_{ij}u + b_{(k)}^i(x)\partial_iu + c_{(k)}(x)u \), is asymptotic to \( P \) uniformly if, given \( \epsilon \), there exist \( R \) and \( k_0 \) such that
\[
\|a_{(k)}^{ij} - a^{ij}\|_{W^{q,r}_x} + \|b_{(k)}^i - b^i\|_{W^{q,r}_x} + \|c_{(k)} - c\|_{W^{2,q}_x} \leq \epsilon \quad \text{for all} \quad k > k_0.
\]

Lemma 2.8. Let \( \Delta_3 \) be the Euclidean Laplacian in \( \mathbb{R}^3 \). Let \( \{P_k\} \) be a sequence of second-order elliptic operators asymptotic to \( \Delta_3 \) uniformly, and for \( s = 2 \) or \( 3 \), \( 1 < p < q \) and a non-integer \( a > 0 \),
\[
P_k : W^{s,p}_{-a} (M) \rightarrow W^{s-2,p}_{-a} (M).
\]

Assume \( \{u_k\} \subset W^{s,p}_{-a} (M) \), \( \{f_k\} \subset W^{s-2,p}_{-a} (M) \) are sequences of functions. We also assume \( \{u_k\}^{\text{odd}} \in W^{s,p}_{-a}(M \setminus B_{k_0}) \), \( \|u_k\|_{W^{s,p}_{-a}(M \setminus B_{k_0})} \to 0 \) as \( k \to \infty \) and \( \{v_k\}^{\text{odd}} \) satisfy
\[
P_k(v_k) = f_k. \]

Then there exist \( R \) large and \( k_0 \) large such that
\[
\|v_k\|_{W^{s,p}_{-a}(M \setminus B_{k_0})} \leq c \|f_k\|_{W^{s-2,p}_{-a}(M \setminus B_{k_0})} + c \quad \text{for all} \quad k > k_0,
\]

where \( c = c(s, p, a, \|P_k - \Delta_3\|_{\text{op}, M', B_R}), R = R(s, p, a) \), \( \|P_k - \Delta_3\|_{\text{op}, M', B_R} \) and \( k_0 = k_0(R) \).

Proof. Because \( \{v_k\}^{\text{odd}} \) is not a global function, we multiply it by a smooth function \( \phi \),
\[
\phi(x) = \begin{cases} 
0 & \text{when } |x| \leq R \\
|\phi| - 1 & \text{when } R \leq |x| \leq 2R \\
1 & \text{when } |x| \geq 2R,
\end{cases}
\]

where \( \phi \) satisfies \( |D\phi| \leq c/R, |D^2\phi| \leq c/R \) for some constant \( c \) independent of \( R \), where the real number \( R \) will be chosen later. We have \( \phi(v_k) \in W^{s,p}_{-a}(M) \) and
\[
P_k(\phi(v_k)) = \phi f_k + a_{(k)}^{ij}(x) \frac{\partial^2 \phi}{\partial x^i \partial x^j} (v_k)^{\text{odd}} + 2a_{(k)}^{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial (v_k)^{\text{odd}}}{\partial x^j} + b_{(k)}^i \frac{\partial \phi}{\partial x^i} (v_k)^{\text{odd}}.
\]

By [B86, theorem 1.7], there exists a constant \( c_1 = c_1(s, p, a) \) such that
\[
\|\phi(v_k)\|_{W^{s,p}_{-a}(M)} \leq c_1 \|\Delta_3(\phi(v_k))^{\text{odd}}\|_{W^{s-2,p}_{-a}(M)}.
\]
We then consider the difference between $\Delta$ and $P_k$ by viewing them as the operators from $W^{s,2}_{1,p}(M)$ to $W^{-2,2}_{1,s}(M)$. By the remark after definition 2.6, the operator norm $\|P_k - \Delta\|_{op;M, \theta}$ tends to zero as $R \to \infty$ uniformly in $k$ because $P_k$ is asymptotic to $\Delta$ uniformly. Therefore,

$$\|\phi(v_k)^{odd}\|_{W^{s,2}_{1,p}(M)} \leq c_1 \|(P_k - \Delta)\)(\phi(v_k)^{odd})\|_{W^{-2,2}_{1,s}(M)} + c_1 \|P_k(\phi(v_k)^{odd})\|_{W^{-2,2}_{1,s}(M)}$$

Choose $R$ such that $c_1 \|P_k - \Delta\|_{op;M, \theta} \leq 1/2$ and move that term to the left-hand side. Then

$$\|\phi(v_k)^{odd}\|_{W^{s,2}_{1,p}(M)} \leq 2 c_1 \|P_k(\phi(v_k)^{odd})\|_{W^{-2,2}_{1,s}(M)}$$

where $A_R = \{x : R \leq |x| \leq 2R\}$. Because $\|(v_k)^{odd}\|_{W^{-1,1}_{s,p}(M\setminus B_R)} \to 0$ as $k \to \infty$, for $\epsilon = \frac{1}{2R}$, there exists $k_0$ so that for all $k > k_0$

$$\|\phi(v_k)^{odd}\|_{W^{s,2}_{1,p}(M)} \leq 2 R \|(v_k)^{odd}\|_{W^{-1,1}_{s,p}(A_R)} \leq 2 R \epsilon = 1.$$

As a result, we have

$$\|\phi(v_k)^{odd}\|_{W^{s,2}_{1,p}(M\setminus B_{2R})} \leq c_2 \|f_k\|_{W^{-2,2}_{1,s}(M\setminus B_{2R})} + c_3 \quad \text{for all} \quad k > k_0.$$

**Proof of theorem 2.2.**

(1) Estimates on $u^{odd}$ and $(X_i)^{odd}$. We construct $(\hat{g}_k, \hat{\pi}_k)$ as in theorem 2.1 in the form $\hat{g}_k = u_k^2 \hat{g}_k + h_k, \hat{\pi}_k = u_k^2(\hat{\pi}_k + \zeta_k X_k) + g_k$. Recall that $k$ is the radius of which we cut off the original data. Again, we drop the subindex $k$ when it is clear from context. By theorem 2.1, $u$ and $X$ exist and satisfy the system of the constraint equations (2.5). From the constraint equation (2.5) for $u$ in $M\setminus B_R$, we have

$$0 = \Delta \hat{g} u - \frac{1}{8} \left( R(\hat{g}) - |\hat{\pi} + \mathcal{L}X|_{\hat{g}}^2 + \frac{1}{2} (\text{Tr}_{\hat{g}}(\hat{\pi} + \mathcal{L}X))^2 \right) u$$

$$\hat{g}^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sqrt{\hat{g}} \left( \frac{\partial}{\partial x^i} \hat{g}^{ij} \right) \frac{\partial u}{\partial x^j} - \frac{1}{8} \left( R(\hat{g}) - |\hat{\pi} + \mathcal{L}X|_{\hat{g}}^2 + \frac{1}{2} (\text{Tr}_{\hat{g}}(\hat{\pi} + \mathcal{L}X))^2 \right) u$$

$$= P_i u.$$

On $M\setminus B_R$, $u^{odd}$ satisfies the following equation:

$$P_i u^{odd} = (P_i u)(x) - (P_i u)(-x) - p_1^{odd}(u(-x))$$

$$= \frac{1}{8} \left( R(\hat{g}) - |\hat{\pi} + \mathcal{L}X|_{\hat{g}}^2 + \frac{1}{2} (\text{Tr}_{\hat{g}}(\hat{\pi} + \mathcal{L}X))^2 \right) u(-x)$$

$$- (\hat{g}^{ij}(x) - \hat{g}^{ij}(-x)) \frac{\partial^2}{\partial x^i \partial x^j} u(-x)$$

$$- (\sqrt{\hat{g}}^{-1}(x) \left( \frac{\partial}{\partial x^i} \hat{g}^{ij}(x) \sqrt{\hat{g}}(x) \right)$$
where

\[ f_1 = \frac{1}{16} (\hat{\alpha} + LX)^2 + \frac{1}{2} (\text{Tr}_\text{g}(\hat{\alpha} + LX))^2 )^{\text{odd}} u(-x), \quad f_2 = P_2 u^{\text{odd}} - f_1. \]

\( f_1 \) contains the terms involving \( X \). We will use a bootstrap argument to improve its decay rate. \( f_2 \) contains the terms which have expected a good decay rate already, such as \( \hat{g}^{\text{odd}}, \hat{\alpha}^{\text{even}} \). A direct calculation tells us

\[ \| f_1 \|_{W^{1,\infty}((M, \mathcal{B}_0))} \leq c, \quad \| f_2 \|_{W^{1,\infty}((M, \mathcal{B}_0))} \leq c. \]

We emphasize that throughout this proof \( c \) is a constant independent of \( k \).

From the constraint equations (2.5) for \( X \), we have

\[ (\text{div}_\text{g}(LX))_i + \text{div}_\text{g}(\hat{\alpha})_i + 4u^{-1} \hat{g}^{ij} u_i (\hat{\alpha} + LX)_{ik} - 2u^{-1} u_k \text{Tr}_\text{g}(\hat{\alpha} + LX) = 0. \]

If we compute the first term in local coordinates, we have

\[ \text{div}_\text{g}(LX)_i = (\Delta X)_i + \text{Ric}_{ik} X^k. \]

Then we define

\[ P_2 X_i \equiv \hat{g}^{kl} \frac{\partial}{\partial x^k} (X^p \hat{\Gamma}^q_{pl} \hat{g}_{pq}) - \hat{g}^{kl} X^p \hat{\Gamma}^q_{lp} \hat{g}_{qr} + \text{Ric}_{ik} X^k. \]

We define \( P_2 X_i = F_i \), where \( F_i \) contains the remainder terms from the above identities,

\[ F_i = \hat{g}^{kl} \frac{\partial}{\partial x^k} (X^p \hat{\Gamma}^q_{pl} \hat{g}_{pq}) + \hat{g}^{kl} X^p \hat{\Gamma}^q_{lp} \hat{g}_{qr} - \text{Ric}_{ik} X^k - \text{div}_\text{g}(\hat{\alpha})_i - 4u^{-1} \hat{g}^{ij} u_i (\hat{\alpha} - LX)_{ik} + 2u^{-1} u_k \text{Tr}_\text{g}(\hat{\alpha} + LX). \]

Then we define

\[ P_2 (X_i)^{\text{odd}} = (F_i)^{\text{odd}} - \left( \hat{g}^{kl} \frac{\partial^2}{\partial x^k \partial x^l} - \hat{g}^{kl} (-x) \frac{\partial^2}{\partial x^k \partial x^l} \right) (X_i(-x)) \]

where

\[ a_i = - (\text{div}_\text{g}(\hat{\alpha})_i)^{\text{odd}}, \quad b_i = P_2 (X_i)^{\text{odd}} - a_i, \]

where \( b_i \) contains \( u^{\text{odd}} \) and \( X^{\text{odd}}_i \) and we will bootstrap to improve its decay rate. A straightforward calculation gives us

\[ \| a_i \|_{W^{0,\infty}((M, \mathcal{B}_0))} \leq c, \quad \| b_i \|_{W^{0,\infty}((M, \mathcal{B}_0))} \leq c. \]

From the above, we derive the system

\[ P_1 u^{\text{odd}} = f_1 + f_2, \quad P_2 (X_i)^{\text{odd}} = a_i + b_i, \quad i = 1, 2, 3. \]

We are at the stage that we can apply lemma 2.8 (with \( \alpha = 2\delta - 1 > \delta \)) for each equation because \( P_1 \) and \( P_2 \) are obviously asymptotic to \( \Delta_3 \) and

\[ \| (a_k)^{\text{odd}} \|_{W^{2,\infty}((M, \mathcal{B}_0))} \leq \| (a_k)^{\text{odd}} \|_{W^{2,\infty}((M, \mathcal{B}_0))} \leq 2 \| u_k - 1 \|_{W^{2,\infty}((M, \mathcal{B}_0))} \rightarrow 0, \]

\[ \| (X_k)^{\text{odd}} \|_{W^{2,\infty}((M, \mathcal{B}_0))} \leq \| (X_k)^{\text{odd}} \|_{W^{2,\infty}((M, \mathcal{B}_0))} \leq 2 \| (X_k)^{\text{odd}} \|_{W^{2,\infty}((M, \mathcal{B}_0))} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \]

Hence, there exist \( R_1 \) and \( k_1 \) such that for all \( k > k_1 \)

\[ \| (a_k)^{\text{odd}} \|_{W^{2,\infty}((M, \mathcal{B}_0))} \leq c, \quad \| (X_k)^{\text{odd}} \|_{W^{2,\infty}((M, \mathcal{B}_0))} \leq c. \]
Once we derive these estimates, the decay rates for $f_1, b_1$ are improved. The bootstrap argument allows us to conclude that for some $R_2 \geq R_1, k_2 \geq k_1$,
\[
\| (u_k)_{\text{odd}} \|_{W^{3,6}_{1-\delta}(M', B_{R_2})} \leq c, \quad \| (X_j)_{k} \|_{W^{2,6}_{1-\delta}(M', B_{R_2})} \leq c, \quad \text{for all } k > k_2.
\]

Therefore, for any given $\epsilon$ and $\delta_0 \in (0, \delta)$, there exist $R$ and $k_0$ so that for all $k > k_0$
\[
\| (u_k)_{\text{odd}} \|_{W^{3,6}_{1-\delta}(M', B_{R})} \leq C R^{\delta_0-\delta} \leq \epsilon, \quad \| (X_j)_{k} \|_{W^{2,6}_{1-\delta}(M', B_{R})} \leq \epsilon.
\]

Furthermore, by corollary 2.5, the supports of $(h_k, q_k)$ are uniformly bounded in $k$. Hence we have
\[
\| \tilde{g}_k \|_{W^{3,6}_{1-\delta}(M', B_{R})} \leq \epsilon.
\]

Similarly, we have the estimate for $\tilde{\alpha}$.

(2) **Convergence of the center of mass and the angular momentum.** To prove the center of mass and angular momentum of $(\tilde{g}, \tilde{\alpha})$ converge to those of $(g, \pi)$, the same idea of proving the convergence of mass and linear momentum in [CS06] is employed.

\[
C^g_\ell(g) - C^g_\ell(\tilde{g}) \leq \left| C^g_\ell(g) - \int_{|x| \geq r} \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) Y^i (g) Y^j (g) \sigma_g \right|
\]
\[
+ \left| \int_{|x| \geq r} \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) Y^i (g) Y^j (g) \sigma_g - \int_{|x| = r} \left( \tilde{R}_{ij} - \frac{1}{2} \tilde{R}_g \tilde{g}_{ij} \right) \tilde{Y}^i (\tilde{g}) \tilde{Y}^j (\tilde{g}) \tilde{\sigma}_{\tilde{g}} \right|
\]
\[
+ \left| C^\tilde{g}_\ell(\tilde{g}) - \int_{|x| \geq r} \left( \tilde{R}_{ij} - \frac{1}{2} \tilde{R}_g \tilde{g}_{ij} \right) \tilde{Y}^i (\tilde{g}) \tilde{Y}^j (\tilde{g}) \tilde{\sigma}_{\tilde{g}} \right|
\]

The first and the third terms can be written as integrals over $\{|x| \geq r\}$ by the divergence theorem. We will also use the fact that $\{|x| \geq r\}$ is centrically symmetric, i.e. $x, -x \in \{|x| \geq r\}$, to estimate those integrals over $\{|x| \geq r\}$. For the first integral

\[
C^g_\ell(g) - \int_{|x| \geq r} \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) Y^i (g) Y^j (g) \sigma_g
\]
\[
= \int_{|x| \geq r} x^a R_g + 2 x^a (g^{ij} - \delta^{ij}) \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right)
\]
\[
+ \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) Y^k (g) \Gamma^i_{kj} d\sigma_g.
\]

Let $H = \{x \in M' \setminus B_r : x^1 \geq 0\}$ be the half space. Then we can rewrite the above integral as follows:

\[
\int_H x^a (R_g \sqrt{g} + R_g (-x) \sqrt{g} \text{odd}) dx
\]
\[
- \int_H 2 x^a (g^{ij}) \text{odd} \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) \sqrt{g} dx
\]
\[
- \int_H 2 x^a (g^{ij} - \delta^{ij}) \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) \sqrt{g} dx
\]
\[
+ \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) \sqrt{g} \text{odd} dx
\]
\[
+ \int_H Y^k (g) \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) \sqrt{g} \text{odd} d\sigma_g
\]
\[
+ \int_H Y^k (g) \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) \left( \Gamma^i_{kj} \sqrt{g} dx
\]
\[
+ \int_H Y^k (g) \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) \left( (\Gamma^i_{kj}) \text{even} \sqrt{g} + \Gamma^i_{kj} (-x) \sqrt{g} \text{odd} \right) dx.
\]
We substitute $R_\gamma$ in the first integral by using the constraint equation $R_\gamma = -\frac{1}{2}(\text{Tr}_g \pi)^2 + |\pi|^2_{g'}$. We then bound above integrals symbolically using the Hölder inequality,

\[
c_1 \int_H |x|(\|\pi_{\text{even}}\|_{|x|} + |\pi|^2_{g'}g'_{\text{odd}} + |g'_{\text{odd}}|D^2g) + |g - \delta||D^2(g_{\text{odd}}))| \, dx
\]

\[
+ c_1 \int_H |x|(D^2(g_{\text{odd}})||Dg| + |D^2g||D(g_{\text{odd}})| + |D^2g||Dg||g_{\text{odd}})) \, dx
\]

\[
\leq c_2 \left(\|\pi_{\text{even}}\|^2_{W^{1, \infty}_{\gamma}} \|\pi\|^2_{W^{1, \infty}_{\gamma}} + |g|^2_{W^{1, \infty}_{\gamma}} ||g_{\text{odd}}||_{W^{1, \infty}_{\gamma}}
\right)
\]

\[
+ \|g_{\text{odd}}\|^2_{W^{1, \infty}_{\gamma}} \|g - \delta\|_{W^{1, \infty}_{\gamma}} \right)^{1-2\delta},
\]

(2.7)

where $c_1$ and $c_2$ are constants independent of $g$, $\pi$ and $r$. The weighted Sobolev norms above are over the region $\{|x| \geq r\}$. Similarly,

\[
\left| C^{\nu}_{\gamma}(\tilde{g}) - \int_{|x| = r} \left( R_{ij} - \frac{1}{2} R_{ij} \right) (Y^\nu)_{ij} v^\nu_g \, d\sigma_g \right|
\]

\[
\leq c_2 \left(\|\pi_{\text{even}}\|^2_{W^{1, \infty}_{\gamma}} \|\pi\|^2_{W^{1, \infty}_{\gamma}} + |g|^2_{W^{1, \infty}_{\gamma}} ||g_{\text{odd}}||_{W^{1, \infty}_{\gamma}}
\right)
\]

\[
+ \|g_{\text{odd}}\|^2_{W^{1, \infty}_{\gamma}} \|g - \delta\|_{W^{1, \infty}_{\gamma}} \right)^{1-2\delta}.
\]

(2.8)

For the surface integral, we can assume $k \geq r$ (recall $k$ is the radius of which we cut off the original data) and then $\tilde{g} = u^2g$ on $\{|x| < 2r\}$. Hence we have

\[
\left| \int_{|x| = r} \left( R_{ij} - \frac{1}{2} R_{ij} \right) (Y^\nu)_{ij} v^\nu_g \, d\sigma_g \right|
\]

\[
= \int_{|x| = r} \left( (R_{ij} - u^2 R_{ij}) - \frac{1}{2} (R_{ij} - u^6 R_{ij}) \right) (Y^\nu)_{ij} v^\nu_g \, d\sigma_g
\]

\[
= \int_{|x| = r} \left( (1 - u^2) R_{ij} - \frac{1}{2} (1 - u^6) R_{ij} \right) (Y^\nu)_{ij} v^\nu_g \, d\sigma_g
\]

\[
+ \int_{|x| = r} \left( u^2 (R_{ij} - R_{ij}) - \frac{1}{2} u^6 (R_{ij} - R_{ij}) \right) (Y^\nu)_{ij} v^\nu_g \, d\sigma_g
\]

\[
\leq c_6 \max_{|x| = r} (|1 - u| |x|^2) \max_{|x| = r} (|D^2(g)| |x||x|^2) \right)^{1-2\delta}
\]

\[
+ c_6 \max_{|x| = r} |D^2(g - \tilde{g})| |x|^{1-2\delta}.r^{2-\delta}.
\]

Recall the Sobolev inequality for weighted Sobolev spaces (see [B86]). For $n - sp < 0$ where $n = \dim M = 3$, we have $\|u\|_{W^{s, p}_\gamma(M)} \leq c \|u\|_{W^{s, p}_\gamma(M)}$. Therefore, for $p > 3$,

\[
\max_{|x| = r} |D^2(g - \tilde{g})| |x|^{2+\delta} \leq \max_{M, \gamma} |D^2(g - \tilde{g})| |x|^{2+\delta}
\]

\[
= \|D^2(g - \tilde{g})\|_{W^{s, p}_\gamma(M, \gamma)} \leq c \|D^2(g - \tilde{g})\|_{W^{s, p}_\gamma(M, \gamma)}
\]

\[
\leq c \|g - \tilde{g}\|_{W^{s, p}_\gamma(M, \gamma)} r^{2-\delta}.
\]

(2.9)

To conclude the argument, for any given $\epsilon$, let $R_1$ be a constant so that the right-hand side of (2.7) is bounded by $\epsilon/16$. Using estimates (2.7), (2.8) and (2.9) and choosing
\[ k \geq k_0 \text{ so that } \| 1 - u_k \|_{W^{1,p}_s(M)}, \| g - \tilde{g}_k \|_{W^{1,p}_s(M)}, \| \pi - \tilde{\pi}_k \|_{W^{1,p}_s(M)}, \| \tilde{\pi}_k \|_{W^{1,p}_s(M) \setminus B_{R_k}} \]
and \[ \tilde{g}_k \text{ are small, we can conclude for all } k \geq k_0, \]
\[
\left| C^\alpha_f (g) - C^\alpha_f (\tilde{g}_k) \right| < \epsilon.
\]

To prove that angular momentum of \( \tilde{g} \) is close to that of \( g \), we compute
\[
J_a = \int_{|x| = r} \pi_{jk} (Z_{(a)})^j v^k_x \, d\sigma_{\tilde{g}} = \int_{|x| = r} \text{div}_g \left( \pi_{jk} Z_{(a)}^j \right) \, d\text{vol}_{\tilde{g}}
\]
\[
= \int_{|x| = r} \pi_{jk} g^{kl} \left( \frac{\partial}{\partial x^l} (Z^a)^j + Z^m_{(a) \Gamma^j}_{ml} \right) \, d\text{vol}_{\tilde{g}}
\]
\[
= \int_{|x| \geq r} \left( \pi_{jk} \frac{\partial}{\partial x^l} (Z^a)^j + \pi_{jk} (g^{kl} - \delta^{kl}) \frac{\partial}{\partial x^l} Z_{(a)}^j + \pi_{jk} g^{kl} Z_{(a) \Gamma^j}_{ml} \right) \, d\text{vol}_{\tilde{g}}.
\]

The first term is zero since \( \pi_{jk} \) is a symmetric tensor and \( Z^a \) is a Euclidean Killing vector field. The other terms after integration are bounded by \( C_\epsilon^{1-\delta} \). Then the rest of the argument works the same as the case of center of mass. \( \square \)

In the proof of convergence of the center of mass and the angular momentum, we showed that the limits at infinity can be approximated by surface integrals at the finite radius and the difference for surface integrals at the finite radius is arbitrarily small. Since we will use this argument several times through this paper, we formulate the argument into the following lemma.

**Lemma 2.9.** Let \( F(g) \) be a vector field depending on \( x, g_{ij}, Dg_{ij}, D^2g_{ij} \) smoothly defined in \( M \setminus B_{R_k} \). Let \( a, b > 0 \), and let \( r \) be the radius. Assume the following estimates hold for \( k > r \), and for \( g, \tilde{g}_k \) satisfying the assumptions in theorem 2.2 (density theorem).

(1)
\[
\left| \int_{|x| = r} \text{div}_{g} F(g) \, d\text{vol}_{g} \right| \leq c \left( \\| \pi^{\text{even}} \|_{W^{1,p}_s(M) \setminus B_{R_k}} \\| \pi \|_{W^{1,p}_s(M) \setminus B_{R_k}} \right. \\
+ \left. \| \pi \|_{W^{1,p}_s(M) \setminus B_{R_k}} \| g \|_{W^{1,p}_s(M) \setminus B_{R_k}} \right) \| g - \delta \|_{W^{1,p}_s(M) \setminus B_{R_k}} \right)^r \cdot r^{-a}. \tag{2.10}
\]

(2)
\[
\left| \int_{|x| = r} F(g) \cdot v_x \, d\sigma_{g} - \int_{|x| = r} F(\tilde{g}_k) \cdot v_{\tilde{g}_k} \, d\sigma_{\tilde{g}_k} \right| \leq c \left( \left( \| 1 - u \|_{W^{1,p}_s(M) \setminus B_{R_k}} + \| g - \tilde{g}_k \|_{W^{1,p}_s(M) \setminus B_{R_k}} \right) \right)^r \cdot r^b. \tag{2.11}
\]

Then given \( \epsilon > 0 \), there exists \( k_0 \) such that
\[
\lim_{r \to \infty} \int_{|x| = r} F(g) \cdot v_x \, d\sigma_g = \lim_{r \to \infty} \int_{|x| = r} F(\tilde{g}_k) \cdot v_{\tilde{g}_k} \, d\sigma_{\tilde{g}_k} + \epsilon, \quad \text{for all } k > k_0.
\]

**Remark.** If we replace the normal vectors and the volume forms of the integrals in the above assumptions by those with respect to the induced metric in the Euclidean space, the analogous result is the following: assume the following estimates hold:

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Then given $\epsilon > 0$, there exists $k_0$ such that
\[
\lim_{r \to \infty} \int_{|x| = r} F(g) \cdot \nu_0 \, d\sigma_0 = \lim_{r \to \infty} \int_{|x| = r} F(\bar{g}_k) \cdot \nu_0 \, d\sigma_0 + \epsilon, \quad \text{for all } k > k_0.
\]

3. Properties of the intrinsic definition

Bartnik [B86, section 4] proves that the ADM mass is a geometric invariant and satisfies natural properties. Along the same lines, we will show that the intrinsic definition of the center of mass (1.4) is well defined and has the corresponding change of coordinate under the transformation at infinity. We will first show that the intrinsic definition is robust in the sense that we can integrate over a general class of surfaces, and those integrals converge to the same vector.

Proposition 3.1. Suppose $(M, g, K)$ is AF-RT (1.3). Let $\{D_k\}_{k=1}^{\infty} \subset M$ be closed sets such that the sets $S_k = \partial D_k$ are connected two-dimensional $C^1$-submanifolds without boundary which satisfy
\[
\begin{align*}
& r_k = \inf\{|x| : x \in S_k\} \to \infty \quad \text{as } k \to \infty \\
& r_k^{-2} \text{area}(S_k) \text{ is bounded} \quad \text{as } k \to \infty \\
& \text{vol}(D_k \setminus D_k^-) = O(r_k^3), \quad \text{where } D_k^- = D_k \cap \{-D_k\} \quad \text{and } 3^- \text{ is a number less than } 3.
\end{align*}
\]

Then the center of mass defined by
\[
C_\alpha^\omega = \frac{1}{16\pi m} \lim_{k \to \infty} \int_{S_k} \left( R_{ij} - \frac{1}{2} R g_{ij} \right) Y_i^{\alpha} Y_j^\alpha \, d\sigma_g, \quad \alpha = 1, 2, 3,
\]
is independent of the sequence $\{S_k\}$.

Remark. The first two conditions (3.1) and (3.2) on $S_k$ are the conditions considered by Bartnik [B86] to ensure the ADM mass is well defined. The volume growth condition (3.3) allows us to consider a general class of surfaces which are roughly symmetric; that is, the non-symmetric region $D_k \setminus D_k^-$ of $D_k$ has the volume growth slightly less than the volume growth of arbitrary regions in $M$. 

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Proof. As in (2.6), the divergence theorem gives us
\[
\int_{S_k} \left( R_{ij} - \frac{1}{2} R g_{ij} \right) Y^i_x v^j_x \, d\sigma_g = \int_{S_k} \left( R_{ij} - \frac{1}{2} R g_{ij} \right) Y^i_x v^j_x \, d\sigma_g \\
+ \int_{D_x \setminus D_t} x'^a R_g + 2 x'^a (g^{ij} - \delta^{ij}) \left( R_{ij} - \frac{1}{2} R g_{ij} \right) + \left( R_{ij} - \frac{1}{2} R g_{ij} \right) Y^i_x v^j_x \, d\, vol_g.
\]
We can decompose the integral over \( D_k \setminus D_t = \{ D_k \setminus D_t \} \cup \{ D_t \setminus D_k \} \cup \{ D_k \setminus D_t \} \) into three integrals, where \( D_k \setminus D_t \) converges because of the volume growth condition (3.3). The second integral over \( D_k \setminus D_t \) converges because the region is centrically symmetric and the initial data set is AF-RT. After taking limits, the right-hand side has a limit independent of the sequence \( \{S_k\} \). \( \square \)

Assume \( \{x\} \) and \( \{y\} \) are two AF coordinates on \((M \setminus B_{R_0}, g)\). Assume \( F \) is the transition function between these two coordinates and \( y = F(x) \). It is shown that the only possible coordinates changes for AF manifolds at infinity are rotation and translation [B86, corollary 3.2]. More precisely, there is a rigid motion of \( \mathbb{R}^3 \), \((O^i, a) \in O(3, \mathbb{R}) \times \mathbb{R} \) so that
\[
|F(x) - (Ox + a)| \in W_{0, \infty}^2 (\mathbb{R}^3 \setminus B_{R_0}).
\]
Similarly, if we use the same argument in that corollary for AF-RT manifolds, we derive
\[
F^{odd}(x) \in W_{-1, \infty}^2 (\mathbb{R}^3 \setminus B_{R_0}).
\]
Because the center of mass is a quantity depending on the coordinates, we now show that centers of mass in \( \{x\} \) and \( \{y\} \) coordinates have the corresponding translation and rotation. An interesting phenomenon is that compared to rotation, translation is a more subtle rigid motion. If the translation \( a \) is not zero, the density theorem (theorem 2.2) is involved in the proof.

**Theorem 3.1.** Let \( \{x\} \) and \( \{y\} \) be two distinct AF-RT (1.3) coordinates for \((M, g, K)\) satisfying the change of coordinates as we described above. Assume \( C_{I,x} \) and \( C_{I,y} \) are the centers of mass defined by the intrinsic definition (1.4) in these two coordinates, then
\[
C_{I,y} = OC_{I,x} + a.
\]

**Proof.** The metric \( g \) in the cotangent spaces induced from \( \{x\} \) and \( \{y\} \) can be written locally as
\[
dx^2 = g_{ij}(x) \, dx^i \, dx^j = g_{ij}(y) \, dy^i \, dy^j = \tilde{g}_{ij}(F(x)) \, d(\mathcal{O}_x^i x^k + a) \, d(\mathcal{O}_y^j x^l + a) + O(|x|^{-1}).
\]
Expanding the above terms, we get
\[
g_{ij}(x) = \tilde{g}_{ij}(F(x)) \mathcal{O}_x^i \mathcal{O}_y^j + e_1(x),
\]
where the error term \( e_1 \in W_{-1, \infty}^2 \) and \( e_1^{odd} \in W_{-1, \infty}^2 \). Then a straightforward calculation gives us
\[
\frac{\partial^2 g_{ij}}{\partial x^m \partial x^n} = \frac{\partial^2 \tilde{g}_{ij}}{\partial y^p \partial y^q} \mathcal{O}_x^p \mathcal{O}_x^m \mathcal{O}_y^n \mathcal{O}_y^q + e_2,
\]
and formulae for Ricci and scalar curvatures,
\[
R_{ij}(x) = \tilde{R}_{ij}(F(x)) \mathcal{O}_x^i \mathcal{O}_y^j + e_3(x), \quad R(x) = \tilde{R}(F(x)) + e_4(x),
\]
where \( e_q(x) \in O(|x|^{-q}) \) and \( e_q^{odd}(x) = O(|x|^{-q}) \) for \( q = 2, 3, 4 \). To calculate \( C_{I,y} \), proposition 3.1 allows us to integrate over \(|y - a| = r\),
\[
C_{I,y} = \frac{1}{16\pi m} \lim_{r \to \infty} \int_{|y - a| = r} \left( \tilde{R}_{ij}(y) - \frac{1}{2} \tilde{R}(y) \tilde{g}_{ij}(y) \right) |y|^2 \delta^{ai} - 2y^a y^j \frac{y^j - a^j}{|y - a|} \, d\sigma_0.
\]
We would like to replace $y = F(x)$ in the above identity. First, we have

$$|y|^2 \delta^{ai} - 2 y^a y^i = (|x|^2 - 2 \sigma^a \sigma^i) \delta^{ai} + 2 \sigma x \cdot a \delta^{ai} - 2 (a^a \sigma^i x^k + \sigma^a \delta^{ai}) + O(1),$$

where $e_5 \in W^{2,\infty}$ and $e_5^{dd} \in W^{2,\infty}$. Using the facts that $\tilde{R}_{ij}(F), \tilde{R}(F)$ and $g(F)$ are asymptotically even

$$\int \delta^{ai} \frac{y^j - a^j}{|y|} \, d\sigma_0$$

By the fact that $\tilde{R}_{ij}(F), \tilde{R}(F)$ and $g(F)$ are asymptotically even

$$\int \frac{\tilde{R}_{ij}(F(x)) - \frac{1}{2} \tilde{R}(F(x)) \tilde{g}_{ij}(F(x))}{|x|^2} \bigg(\frac{2 \sigma^a \sigma^i x^k - 2 \sigma^a x^k}{|x|} \bigg) \, d\sigma_0$$

$$+ \frac{1}{2} \tilde{R}(F(x)) g_{ij}(F(x)) \bigg(\frac{2 \sigma^a \sigma^i x^k - 2 \sigma^a x^k}{|x|} \bigg) \, d\sigma_0$$

$$+ \int \frac{\tilde{R}_{ij}(F(x)) - \frac{1}{2} \tilde{R}(F(x)) \tilde{g}_{ij}(F(x))}{|x|^2} \bigg(\frac{2 \sigma^a \sigma^i x^k - 2 \sigma^a x^k}{|x|} \bigg) \, d\sigma_0$$

We claim that the first integral $I_1$ converges to $16\pi m \sigma^a \sigma^i$, the second integral $I_2$ converges to $16\pi ma^a$, and the third integral $I_3$ converges to 0.

$$I_1 = \int \frac{\tilde{R}_{ij}(F(x)) - \frac{1}{2} \tilde{R}(F(x)) \tilde{g}_{ij}(F(x))}{|x|^2} \bigg(\frac{2 \sigma^a \sigma^i x^k - 2 \sigma^a x^k}{|x|} \bigg) \, d\sigma_0$$

$$= \sigma^a \int \frac{R_{ij}(x) - \frac{1}{2} R_g g_{ij}(x)}{|x|^2} \bigg(\frac{2 \sigma^a \sigma^i x^k - 2 \sigma^a x^k}{|x|} \bigg) \, d\sigma_0$$

$$= 16\pi m \sigma^a \sigma^i + O(r^{-1}).$$

By using the formula for the ADM mass (1.5),

$$I_2 = a^a \int \frac{R_{ij}(x) - \frac{1}{2} R_g g_{ij}(x)}{|x|^2} \bigg(\frac{2 \sigma^a \sigma^i x^k - 2 \sigma^a x^k}{|x|} \bigg) \, d\sigma_0 + O(r^{-1})$$

To see that $I_3$ converges to 0, we simplify the expression by letting $b = \sigma^a a^i$, and then

$$I_3 = \sigma^a \int \frac{R_{ij}(x) - \frac{1}{2} R_g g_{ij}(x)}{|x|^2} \bigg(\frac{2 \sigma^a \sigma^i x^k - 2 \sigma^a x^k}{|x|} \bigg) \, d\sigma_0.$$

By the fact that $(2x \cdot b \delta^{jk} - 2 \delta^{jk} b^j) \cdot \frac{1}{|x|} = 0$, the scalar curvature term vanishes after taking a limit, so we only need to prove

$$\sigma^a \int \frac{R_{ij}(x)(2x \cdot b \delta^{jk} - 2 \delta^{jk} b^j)}{|x|} \, d\sigma_0 = O(r^{-1}).$$

It is sufficient to prove

$$\int \frac{R_{ij}(x)(2x \cdot b \delta^{jk} - 2 \delta^{jk} b^j)}{|x|} \, d\sigma = O(r^{-1}).$$

A straightforward calculation shows this identity holds if the metric is conformally flat outside a compact set, so we would like to approximate $g$ by $\tilde{g}$ which have harmonic asymptotics.
and then apply lemma 2.9. We need to check if conditions (2.10) and (2.11) hold. Using the divergence theorem, we have

\[ \int_{|x| \geq r} \text{div}_g(R_{ij}(x)(2x \cdot b \delta^{jk} - 2x^\beta b^k)) \, d\text{vol}_g \]

\[ = \int_{|x| \geq r} (\text{div}_g R_{ij})(2x \cdot b \delta^{jk} - 2x^\beta b^k) + R_{ij}(2b^j \delta^{jk} - 2\delta^{jk} b^k) \, d\text{vol}_g \]

\[ + \int_{|x| \geq r} R_{ij}(2x \cdot b \delta^{jk} - 2x^\beta b^k) g^{ln} \Gamma^m_{mn} \, d\text{vol}_g. \]

Using the divergence theorem, we have

\[ \int_{|x| \geq r} |x| \geq r \, \text{div}_g(R_{kl}(x)(2x \cdot b \delta^{lk} - 2x^\beta b^k)) \, d\text{vol}_g \]

\[ \leq c \left( \| \pi \|_{W^{1,p}_{-\frac{1}{2},1}(M \setminus \Omega)} + \| g - \bar{g} \|_{W^{1,p}_{-\frac{1}{2},1}(M \setminus \Omega)} \right)^{r^{-1-\delta}}. \]

Hence condition (2.10) holds. Condition (2.11) holds because the Sobolev inequality implies

\[ \int_{|x| = r} R_{ij}(x)(2x \cdot b \delta^{jk} - 2x^\beta b^k) v_i^0 \, d\sigma_g = - \int_{|x| = r} \bar{R}_{ij}(x)(2x \cdot b \delta^{jk} - 2x^\beta b^k) v_i^0 \, d\sigma_{\bar{g}} \]

\[ \leq c \left( \| 1 - u \|_{W^{1,p}_{-\frac{1}{2},1}(M \setminus \Omega)} + \| g - \bar{g} \|_{W^{1,p}_{-\frac{1}{2},1}(M \setminus \Omega)} \right)^{r^{-1-\delta}}. \]

By lemma 2.9, we conclude

\[ \lim_{r \to \infty} \int_{|x| = r} R_{ij}(x)(2x \cdot b \delta^{jk} - 2x^\beta b^k) v_i^0 \, d\sigma_g \]

\[ = \lim_{r \to \infty} \int_{|x| = r} \bar{R}_{ij}(x)(2x \cdot b \delta^{jk} - 2x^\beta b^k) v_i^0 \, d\sigma_{\bar{g}} + \epsilon = \epsilon. \]

Since \( \epsilon \) can be chosen arbitrarily small, \( I_3 \) converges to 0. Therefore,

\[ C_{\alpha \beta}^{I,\gamma} = \frac{1}{16\pi m} \lim_{r \to \infty} I_1 + I_2 + I_3 = O^\rho p C_{\alpha \beta}^{I,\gamma} + a^\alpha. \]

\[ \Box \]

4. The Corvino–Schoen center of mass

Corvino and Schoen [CS06] define center of mass (1.8) for AF-RT manifolds. In this section, we will show that the intrinsic definition we propose is actually equal to the Corvino–Schoen definition (theorem 1). In other words, the intrinsic definition (1.4) is a coordinate-free expression of the Corvino–Schoen definition.

Assume \( \bar{g} \) is the approximating solution in theorem 2.2 (density theorem). A straightforward calculation shows

\[ C_{\alpha \beta}^{I,\gamma} = \frac{1}{16\pi m} \lim_{r \to \infty} I_1 + I_2 + I_3 = O^\rho p C_{\alpha \beta}^{I,\gamma} + a^\alpha. \]

Proof of theorem 1. First note that the Corvino–Schoen definition is equal to the following expression where the normal vector and the volume form are with respect to the induced metric in the Euclidean space

\[ C_{\alpha \beta}^{I,\gamma} = \frac{1}{16\pi m} \lim_{r \to \infty} \int_{|x| = r} x^\alpha (g_{ij,j} - g_{ii,j}) v_i^0 \, d\sigma_0 - \int_{|x| = r} (h_{ij} v_i^0 - h_{ij} v_i^0) \, d\sigma_0. \]
A direct calculation shows
\[
\left( \int_{|x| = r_1} x^a (g_{ij}, i - g_{ii}, j) v_0^i \, d\sigma_0 - \int_{|x| = r_1} \left( h_{ia} v_0^i - h_{ii} v_0^a \right) \, d\sigma_0 \right)
= \left( - \int_{|x| = r_1} x^a (g_{ij}, i - g_{ii}, j) v_0^i \, d\sigma_0 - \int_{|x| = r_1} \left( h_{ia} v_0^i - h_{ii} v_0^a \right) \, d\sigma_0 \right)
= \int_{r_1 \leq |x| \leq r_2} x^a (g_{ii}, i - g_{ii}, j) \, d\sigma_0 - \int_{|x| = r_1} \left( h_{ia} v_0^i - h_{ii} v_0^a \right) \, d\sigma_0.
\]

It is easy to see that conditions (2.12) and (2.13) in lemma 2.9 (and the remark afterward) hold because \( g_{ii, i} - g_{ii, j} \) is the leading order term of \( R_g \), and by the constraint equation (2.1),
\[
\left| \int_{|x| \geq r} x_a (g_{ij}, i - g_{ii}, j) \, d\sigma_0 \right| \leq c \left( \| \pi \|_W^{2,2} (M, \mathcal{B}_R) \| \pi^\text{even} \|_W^{1,2} (M, \mathcal{B}_R) \right) + \| \delta \|_W^{2,2} (M, \mathcal{B}_R)^{p/2}.
\]
and
\[
\left| \left( \int_{|x| = r} x_a (g_{ij}, i - g_{ii}, j) v_0^i \, d\sigma_0 - \int_{|x| = r} \left( h_{ia} v_0^i - h_{ii} v_0^a \right) \, d\sigma_0 \right) \right| \leq c \| g - \bar{g} \|_W^{2,2} (M, \mathcal{B}_R)^{2p/2}.
\]

Then we can apply lemma 2.9 and have
\[
C_{CS}^g = \frac{1}{16\pi m} \lim_{r \to \infty} \left( \int_{|x| = r} x_a (g_{ij}, i - g_{ii}, j) v_0^i \, d\sigma_0 - \int_{|x| = r} \left( h_{ia} v_0^i - h_{ii} v_0^a \right) \, d\sigma_0 \right)
= \frac{1}{16\pi m} \lim_{r \to \infty} \left( \int_{|x| = r} x_a (\bar{g}_{ij, i} - \bar{g}_{ii, j}) v_0^i \, d\sigma_0 - \int_{|x| = r} \left( \bar{h}_{ia} v_0^i - \bar{h}_{ii} v_0^a \right) \, d\sigma_0 \right) + \epsilon
= C_{CS}^g (\bar{g}) + \epsilon.
\]

5. The Huisken–Yau center of mass

In the case that \( (M, g, K) \) is SAF (1.6), Huisken–Yau [HY96] and Ye [Y96] prove the existence and uniqueness of the constant mean curvature foliation in the exterior region of \( M \), if \( m > 0 \). Huisken and Yau use the volume-preserving mean curvature flow to evolve each Euclidean sphere centered at the origin with a large radius and they show that the Euclidean sphere converges to a surface with constant mean curvature. Furthermore, they prove that those surfaces with constant mean curvature are approximately round and their approximate centers converge as per equation (1.7):
\[
C_{HY}^\alpha = \lim_{r \to \infty} \frac{\int_{M_r} z^a \, d\sigma_0}{\int_{M_r} \, d\sigma_0}, \quad \alpha = 1, 2, 3.
\]
we will show that \( p \) converges to \( C_{CS} \) and use this fact to prove \( C_{HY} = C_{CS} \). Then \( C_1 = C_{HY} \) follows because the Corvino–Schoen center of mass \( C_{CS} \) is equal to the intrinsic definition \( C_1 \) by theorem 1. Before we prove theorem 2, we will need a technical lemma (lemma 5.1) which suggests that the center of the surface \( p \) converges to \( C_{CS} \).

Let \( v \) be the outward unit normal vector field on \( S_R(p) \) with respect to the metric \( g \). As \( R \) varies, \( v \) is well defined in a tubular neighborhood of \( S_R(p) \). Therefore, the mean curvature at \( y \in S_R(p) \) is \( \text{div}_g \mathbf{v} = \text{div}_g v \), the divergence operator of the ambient manifold \( M \). Recall that \( g = (1 + (2m/|y|)) \delta_{ij} + p_{ij} \). At \( y \),

\[
\nu = \frac{\nabla |y - p|}{|\nabla |y - p||} = \left( 1 - \frac{m}{|y|} + \frac{3m^2}{2|y|^2} + \frac{1}{2} \frac{p_{kl}}{y} \frac{(y^q - p^q)(y^r - p^r)}{|y - p|^2} \right) \frac{y^i - p^i}{|y - p|} \frac{\partial}{\partial y^i} - p_{kl} \frac{y^k - p^k}{|y - p|} \frac{\partial}{\partial y^l} + O(R^{-3}).
\]

A straightforward calculation gives us the mean curvature on \( S_R(p) \) equal to

\[
\text{div}_g v = \frac{4m}{|y - p|^2} - \frac{6m(y - p) \cdot p}{|y - p|^4} + \frac{9m^2}{|y - p|^3}
+ \frac{1}{2} \sum_{i,j,k} p_{ij,k} \frac{(y^q - p^q)(y^r - p^r)(y^s - p^s)}{|y - p|^3} + 2 \sum_{i,j,k} p_{ij} \frac{(y^q - p^q)(y^r - p^r)}{|y - p|^3} \frac{y^r - p^r}{|y - p|} + E_0,
\]

where \( |E_0| \leq \frac{c}{R} (1 + |p|) \) for some constant \( c \) depending only on the metric \( g \).

**Lemma 5.1.** For \( R \) large,

\[
\int_{|y - p| = R} (y^\alpha - p^\alpha) \left( \frac{1}{2} \sum_{i,j,k} p_{ij,k} \frac{(y^q - p^q)(y^r - p^r)(y^s - p^s)}{|y - p|^3} \right) \, d\sigma_0
+ \int_{|y - p| = R} (y^\alpha - p^\alpha) \left( \frac{2}{2} \sum_{i,j} p_{ij} \frac{(y^q - p^q)(y^r - p^r)}{|y - p|^3} \right) \, d\sigma_0
+ \int_{|y - p| = R} (y^\alpha - p^\alpha) \left( \frac{1}{2} \sum_{i,j,k} p_{ij,k} \frac{y^q - p^q}{|y - p|} \right) \, d\sigma_0
= -8m\pi C_{CS}^2 + O(1/R^3), \quad \alpha = 1, 2, 3.
\]

**Proof of lemma 5.1.** We denote the first integral by

\[
\mathcal{I}(R) = \int_{|y - p| = R} (y^\alpha - p^\alpha) \left( \frac{1}{2} \sum_{i,j,k} p_{ij,k} \frac{(y^q - p^q)(y^r - p^r)(y^s - p^s)}{|y - p|^3} \right) \, d\sigma_0.
\]

In the proof, we will rewrite \( \mathcal{I}(R) \), and then some cancellation allows us to rearrange the left-hand side of (5.2) so that it has an expression corresponding to \( C_{CS}^2 \).

Since the coordinate is only defined outside a compact set of \( M \), we can use the divergence theorem only in the annular region \( A = \{ R \leq |y - p| \leq R_1 \} \),

\[
\mathcal{I}(R_1) = \mathcal{I}(R) + \frac{1}{2} \int_A \left( p_{ij,k} \frac{(y^l - p^l)(y^s - p^s)(y^r - p^r)}{|y - p|^2} \right) \, d\text{vol}_0
= \mathcal{I}(R_1) + \frac{1}{2} \int_A \left( p_{ij,k} \frac{(y^l - p^l)(y^s - p^s)(y^r - p^r)}{|y - p|^2} \right) \, d\text{vol}_0.
\]
\[- \frac{1}{2} \int_A p_{ij,i} \left( \frac{(y^j - p^j)(y^k - p^k)(y^\alpha - p^\alpha)}{|y - p|^2} \right) \, d\nu 0 + \frac{1}{2} \int_A p_{ij,k} \left( \frac{(y^j - p^j)(y^k - p^k)(y^\alpha - p^\alpha)}{|y - p|^2} \right) \, d\nu 0.\]

Using the divergence theorem and simplifying the expression, we get an identity containing purely boundary terms

\[ I(R_1) = I(R) + J(R_1) - J(R), \quad \text{for all } R_1 > R, \]

where

\[ J(R) = \frac{1}{2} \int_{|y - p| = R} (y^\alpha - p^\alpha) p_{ij,j} \frac{y^j - p^j}{|y - p|} \, d\nu 0 \]

\[- \frac{1}{2} \int_{|y - p| = R} (y^\alpha - p^\alpha) p_{ii,j} \frac{y^j - p^j}{|y - p|} \, d\nu 0 + \frac{1}{2} \int_{|y - p| = R} p_{ii} \frac{y^\alpha - p^\alpha}{|y - p|} \, d\nu 0. \]

To prove that \( I(R) = J(R) \), we would like to apply lemma 2.9. It is easy to check that conditions (2.12) and (2.13) hold, so we get

\[ I(R) - J(R) = \lim_{R_1 \to \infty} (I(R_1) - J(R_1)) = \lim_{R_1 \to \infty} (I_{p}(R_1) - J_{p}(R_1)) + \epsilon, \]

where \( I_{p} \) and \( J_{p} \) denote the integrals that we obtain by replacing \( p_{ij} \) by \( \tilde{p}_{ij} \) in \( I \) and \( J \), where \( \tilde{p}_{ij} \) are \( O(|y|^{-2}) \) terms in the approximating solutions \( \tilde{g}_{ij} \) in theorem 2.2 (density theorem).

More precisely, \( \tilde{g}_{ij} \) has this expansion

\[ \tilde{g}_{ij} = \left( 1 + \frac{a}{|y|} \right) \delta_{ij} + \text{O}(|y|^{-2}), \]

and \( \tilde{p}_{ij} \) is defined by

\[ \tilde{p}_{ij} = \tilde{g}_{ij} - \left( 1 + \frac{a}{|y|} \right) \delta_{ij}. \]

Because \( \tilde{p}_{ij} = \frac{c}{|y|} + \text{O}(|y|^{-3}) \) as in (2.4), it is easy to see that

\[ \lim_{R_1 \to \infty} \frac{I_{p}(R_1)}{J_{p}(R_1)} = 0. \]

Therefore, we conclude \( I(R) = J(R) \). We then replace \( I(R) \) by \( J(R) \) in identity (5.2) and derive that the left-hand side is equal to

\[- \frac{1}{2} \left( \int_{|y - p| = R} (y^\alpha - p^\alpha) (p_{ij,j} - p_{ii,j}) \frac{y^j - p^j}{|y - p|} \, d\nu 0 \right) \]

\[ - \frac{1}{2} \left( \int_{|y - p| = R} p_{ii} \frac{y^\alpha - p^\alpha}{|y - p|} \, d\nu 0 \right). \]

We rewrite the above integrals into the summation of an expression of the center of mass (1.8) and remainder terms, and the explicit calculations give

\[ - \frac{1}{2} \left( \int_{|y - p| = R} (g_{ij,j} - g_{ii,j}) \frac{y^j - p^j}{|y - p|} \, d\nu 0 \right) \]

\[ - \frac{1}{2} \left( \int_{|y - p| = R} g_{ii} \frac{y^\alpha - p^\alpha}{|y - p|} \, d\nu 0 \right). \]
In case, $d = H(p, R, \lambda \phi) \in \Sigma_1$.

Proof of theorem 2. Let $F_{p, R} : S_1(0) \to M$ be an embedding defined by $y = F_{p, R}(x) = Rx + p$, that is, $F_{p, R}(S_1(0)) = S_R(p)$, the Euclidean sphere centered at $p$ with the radius $R$ in $M$. We consider the perturbation along the normal direction on $S_R(p)$ defined by $\Sigma = \{ y + \lambda \phi \nu : y \in S_R(p) \}$ for a parameter $\lambda > 0$, and for $\phi \in C^{2, \alpha}(S_R(p))$ with $\| \phi \|_{C^{2, \alpha}} \leq 1$. We denote the mean curvature on $\Sigma$ by $H(p, R, \lambda \phi)$. Using this notation, $H(p, R, 0) = \text{div}_v \nu$, the mean curvature of $S_R(p)$. By Taylor's theorem for mappings between two Banach spaces, we have the following expansion in the $\phi$-component at $0$:

$$
H(p, R, \lambda \phi) = H(p, R, 0) + dH(p, R, 0)(\lambda \phi) + \int_0^1 (1 - s)(d^2H(p, R, s(\lambda \phi))(\lambda \phi, \lambda \phi)) \, ds,
$$

where $dH$ and $d^2H$ are the first and second Fréchet derivatives in the $\phi$-component. In our case, $dH(p, R, 0)$ is the linearized mean curvature operator on $S_R(p)$, i.e.

$$
dH(p, R, 0) = \Delta_{S_1(p)} + |A|_g^2 + \text{Ric}(v, v),
$$

where $\Delta_{S_1(p)}$ is the Laplacian operator on $S_R(p)$ with respect to the induced metric from $g$, $A$ is the second fundamental form on $S_R(p)$ and $\text{Ric}(\cdot, \cdot)$ is the Ricci curvature of $M$. In [HY96], the estimates on the eigenvalues $\lambda_1, \lambda_2$ of $A$ are derived and

$$
|A|_g^2 = \lambda_1^2 + \lambda_2^2 = \frac{2}{R^2} + O(R^{-3}), \quad \text{Ric}(v, v) = O(R^{-3}).
$$

For the second Fréchet derivative in the Taylor expansion, we have

$$
d^2H(p, R, s(\lambda \phi))(\lambda \phi, \lambda \phi) = \frac{\partial^2}{\partial t^2}H(p, R, t(\lambda \phi)) \bigg|_{t=0}.
$$

The right-hand side is the second derivative of the mean curvature of the surface $\{ y + s(\lambda \phi) \nu : y \in S_R(p) \}$. For $R$ large, the unit outward normal vector field on $\{ y + s(\lambda \phi) \nu : y \in S_R(p) \}$ is close to $\nu$, and a straightforward calculation gives us

$$
\left| \frac{\partial^2}{\partial t^2}H(p, R, t(\lambda \phi)) \right| \leq C \lambda^2(|R_{ijkl}| |A||\phi|^2 + |A||\phi||\partial^2\phi| + |A|^2|\phi|^2),
$$

where the constant $C$ is independent of $p$, $R$ and $\phi$. Therefore,

$$
H(p, R, \lambda \phi) = H(p, R, 0) + \Delta_{S_1(p)}(\lambda \phi) + (|A|_g^2 + \text{Ric}(v, v))(\lambda \phi) + \int_0^1 (1 - s)(d^2H(p, R, s(\lambda \phi))(\lambda \phi, \lambda \phi)) \, ds. \tag{5.3}
$$

Let $G$ and $E_1$ be defined as follows where $G$ is the lower order terms of the mean curvature of $S_R(p)$ from (5.1),

$$
G(y) = \frac{1}{2} p_{ij,k} \frac{(y^i - p^i)(y^j - p^j)(y^k - p^k)}{|y - p|^3} + 2 p_{ij} \frac{(y^i - p^i)(y^j - p^j)}{|y - p|^3} - p_{ij,k} \frac{y^i - p^i}{|y - p|} - \frac{p_{ij}}{|y - p|} + \frac{1}{2} p_{ij} \frac{y^i - p^j}{|y - p|},
$$

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and

\[ E_1(y) = E_0 + \left( |A|^2 - \frac{2}{R^2} \right)(\lambda \phi) + \text{Ric}(\nu, \nu)(\lambda \phi)
\]
\[ + \int_0^1 (1 - s)(\partial^2 H(p, R, s \lambda \phi))(\lambda \phi, \lambda \phi) \, ds. \]

\( G \) is the good term that gives us \( C_{CS} \) as indicated in lemma 5.1, and \( E_1(y) \) is an error term bounded as follows for some constant \( c \) independent of \( p, R \) and \( \phi \):

\[ |E_1| \leq c \left( 1 + |p| \right) + c \frac{1}{R^3} (\lambda |\phi| + \lambda^2 |\phi|^2 + R^2 \lambda^2 |\phi||\partial^2 \phi|). \]

From identities (5.1) and (5.3),

\[ H(p, R, \lambda \phi) = \frac{2}{R} - \frac{4m}{R^2} + \frac{6m(y - p) \cdot p}{R^4} + \frac{9m^2}{R^3} + G(y) + \lambda \Delta_s(p) \phi + \frac{2}{R^2} \lambda \phi + E_1(y). \]

To find the surface \( S_1 \) with constant mean curvature, we need to find \( p, R, \phi \) such that

\[ H(p, R, \lambda \phi) = \frac{2}{R} - \frac{4m}{R^2}. \]

It is equivalent to solving

\[ 0 = \frac{6m(y - p) \cdot p}{R^3} + \frac{9m^2}{R^3} + G(y) + \lambda \Delta_s(p) \phi + \frac{2}{R^2} \lambda \phi + E_1(y). \quad (5.4) \]

We pull back equation (5.4) via the map \( F_{p,R} \), and we get the following equation on \( S_1(0) \):

\[ 0 = \frac{6m \cdot p}{R^3} + \frac{9m^2}{R^3} + G \circ F_{p,R}(x) + \lambda \Delta_{S_1(0)} \psi(x) + \frac{2}{R^2} \lambda \psi(x) + E_1 \circ F_{p,R}(x) \quad (5.5) \]

for \( p, R, \psi(x) \), where \( \Delta_{S_1(0)} \) is the Laplacian on \( S_1(0) \) with respect to the pull-back metric and \( \psi = \phi \circ F_{p,R}(x) = \phi(Rx + p) \) is the pull back of \( \phi \). Define the operator \( L \) by

\[ L \equiv -\Delta_0 - 2 : C^{2,\alpha}(S_1(0)) \rightarrow C^{0,\alpha}(S_1(0)), \]

where \( \Delta_0 \) is Laplacian on \( S_1(0) \) with respect to the round metric induced by the Euclidean metric on \( \mathbb{R}^3 \). Because the metric \( \text{g} \) is asymptotically flat, the difference between \( \Delta_0 \) and \( \Delta_{S_1(0)} \) is small and can be treated as the error term. Therefore, identity (5.5) is equal to

\[ 0 = \frac{6m \cdot p}{R^3} + \frac{9m^2}{R^3} + G \circ F_{p,R}(x) - \frac{1}{R^2} \lambda L \psi(x) + \tilde{E}_1 \circ F_{p,R}(x), \]

where \( \tilde{E}_1 \) has the same bound as \( E_1 \). Let \( \lambda = R^{-a} \) for some fixed \( a \in (0, 1) \) and multiply \( R^{2a} \) on both sides of the above equation, then

\[ L \psi(x) = \frac{6m \cdot p}{R^{1-a}} + \frac{9m^2}{R^{1-a}} + R^{2a} G \circ F_{p,R}(x) + R^{2a} \tilde{E}_1 \circ F_{p,R}(x). \quad (5.6) \]

Furthermore, since \( \partial_x \psi = (\partial_x \phi) R \),

\[ |\tilde{E}_1 \circ F_{p,R}(x)| \leq C \frac{1 + |p|}{R^3} \left( |\psi| + |\psi|^2 + \frac{1}{R^2} |\partial^2 \psi| |\psi| \right). \]

In order to find \( p, R \) and \( \psi \) to solve (5.6), we first perturb \( p = p(R, \psi) \) so that the right-hand side of (5.6) is inside Range \( L \) for any \( R \) and \( \psi \). We will also show that \( p = C_{CS} + \varepsilon \), where \( \varepsilon \) is the error term containing lower order terms in \( R \) and \( \|\psi\| \). Second, using an iteration process and the Schauder estimate, we can find a solution \( \psi \) for \( R \) large.


(1) **Perturb the center p.** \( L \) has a kernel equal to \( \text{span}\{x^1, x^2, x^3\} \) because translation preserves the mean curvature. Since \( L \) is self-adjoint, \( C^{0,\alpha}(S_1(0)) \) has the \( L^2 \)-orthogonal decomposition \( C^{0,\alpha}(S_1(0)) = \text{Range}L \oplus \text{span}\{x^1, x^2, x^3\} \). We would like to find \( p \) so that the right-hand side of (5.6) is orthogonal to \( \text{span}\{x^1, x^2, x^3\} \). That is, we want to find \( p \) so that for \( \alpha = 1, 2, 3 \),

\[
\int_{S_1(0)} x^\alpha \left( \frac{6mx \cdot p}{R^{1-a}} + \frac{9m^2}{R^{1-a}} \right) \, d\sigma_0 \\
+ \int_{S_1(0)} x^\alpha (R^{2a}G \circ F_{p,R}(x) + R^{2a}E_1 \circ F_{p,R}(x)) \, d\sigma_0 = 0.
\]  

(5.7)

We calculate each term above separately. A direct calculation gives the first integral

\[
\int_{S_1(0)} x^\alpha \left( \frac{6mx \cdot p}{R^{1-a}} + \frac{9m^2}{R^{1-a}} \right) \, d\sigma_0 = \frac{8m\pi p^\alpha}{R^{1-a}}.
\]

From the area formula and lemma 5.1, we have

\[
\int_{S_1(0)} x^\alpha R^{2a}G \circ F_{p,R}(x) \, d\sigma_0 = \int_{S_1(0)} \frac{y^\alpha - p^\alpha}{R} R^{2a}G(y) R^{-2} \, d\sigma_0 \\
= \frac{1}{R^{1-a}} \int_{S_1(0)} (y^\alpha - p^\alpha) G(y) \, d\sigma_0 = \frac{-8m\pi C^{\alpha}_L}{R^{1-a}} + O(R^{-2a}).
\]

Moreover, the error term can be bounded by

\[
\left| \int_{S_1(0)} x^\alpha R^{2a}E_1 \circ F_{p,R}(x) \, d\sigma_0 \right| \leq \frac{c}{R^{2-a}} (1 + |p|) \\
+ \frac{c}{R} (||p|| + |Dp||p| + |Dp|^2 + |p||Dp||D^2p|).
\]

Since the ADM mass \( m > 0 \), we can choose \( p \)

\[
p(R, \psi) = C_L + e(R, \psi)
\]

so that identity (5.7) holds, where

\[
|e(R, \psi)| \leq \frac{c}{R} (1 + |p|) + \frac{c}{R^a} (||p|| + |Dp||p| + |Dp|^2 + |p||Dp||D^2p|).
\]

(2) **Find the solution \( \psi \) by iteration.** We consider the isomorphism \( L : (\text{Ker} \, L)^\perp \to \text{Range}(L) \) for \( (\text{Ker} \, L)^\perp \subset C^{2,\alpha}(S_1(0)) \) and \( \text{Range}(L) \subset C^{0,\alpha}(S_1(0)) \). If we denote the right-hand side of (5.6) by \( f(p, R, \psi) \), i.e.

\[
f(p, R, \psi) = \frac{6mx \cdot p}{R^{1-a}} + \frac{9m^2}{R^{1-a}} + R^{2a}G \circ F_{p,R}(x) + R^{2a}E_1 \circ F_{p,R}(x),
\]

we know \( f(p(R, \psi), R, \psi) \in \text{Range}(L) \) for any \( R, \psi \) with \( ||\psi||_{C^{2\alpha}} \leq 1 \). Therefore, any \( \psi_0 \) with \( ||\psi_0||_{C^{2\alpha}} \leq 1 \) is \( \psi_1 \in (\text{Ker} \, L)^\perp \) such that

\[
L\psi_1 = f(p(R, \psi_0), R, \psi_0).
\]

Moreover, we use the Schauder estimate and the fact that \( \psi_1 \in (\text{Ker} \, L)^\perp \),

\[
||\psi_1||_{C^{2\alpha}} \leq c \left| f(p(R, \psi_0), R, \psi_0) \right|_{C^{2\alpha}} \\
\leq \frac{c}{R^{1-a}} (|C_L| + 1) + \frac{c}{R^a} (||\psi_0||_{C^{2\alpha}} + ||\psi_0||_{C^{2\alpha}}^2) \\
\leq \frac{c}{R^{1-a}} (|C_L| + 1) + \frac{2c}{R^a}.
\]
For any $R$ large enough (independent of $\psi_0$), we have $\|\psi_1\|_{C^{2,\alpha}} \leq 1$. We continue the iteration process and get a sequence of functions $\{\psi_k\}_{k=0}^\infty$ satisfying
\[
L\psi_{k+1} = f(p(R, \psi_k), R, \psi_k) \quad \text{and} \quad \|\psi_{k+1}\|_{C^{2,\alpha}} \leq 1.
\]
The Arzela–Ascoli theorem says that there exists $\psi_\infty \in C^{2,\mu}(S_1(0))$ such that a subsequence of $\{\psi_k\}$ converges to $\psi_\infty$ in $C^{2,\mu}$ for $0 < \mu < \alpha$. Moreover, $\psi_\infty$ is a solution to (5.6),
\[
L\psi_\infty = f(p(R, \psi_\infty), R, \psi_\infty).
\]
Let $\phi_\infty(y) = \psi_\infty \circ F_{R^{-1}}(y)$, then the surface $M_R = \{z : z = y + R^{-a}\phi_\infty v, y \in S_R(p)\}$ has constant mean curvature equal to $(2/R) - (4m/R^2)$.

To complete the proof of theorem 2, we need to compute
\[
\lim_{R \to \infty} \int_{M_R} \frac{z^a}{\int_{M_R} \delta \sigma_0} d\sigma_0.
\]
By the uniqueness of the constant mean curvature foliation, the $\{M_R\}$ are equal to those constructed in [HY96], and therefore (5.9) converges to the Huisken–Yau center of mass $C_{HY}$. We now prove that (5.9) also converges to the Corvino–Schoen center of mass $C_{CS}$. Let $F$ be the diffeomorphism defined by $F(y) = y + R^{-a}\phi_\infty v$, then
\[
\int_{M_R} \frac{z^a}{\int_{M_R} \delta \sigma_0} d\sigma_0 = \int_{S_\alpha(p)} (y^a + R^{-a}\phi_\infty v^a) JF d\sigma_0 \int_{S_\alpha(p)} JF d\sigma_0,
\]
where $JF$ is the Jacobian from the area formula, $JF = 1 + O(R^{-1-a})$. Now we can use the fact that the area of the Euclidean sphere is $O(R^2)$ and the estimate for the center $p$ in (5.8) to conclude
\[
\int_{M_R} \frac{z^a}{\int_{M_R} \delta \sigma_0} d\sigma_0 = p^a + \int_{S_\alpha(p)} (y^a - p^a)(1 + O(R^{-1-a})) d\sigma_0 + \int_{S_\alpha(p)} R^{-a}\phi_\infty v^a JF d\sigma_0
\]
\[
= C_{CS}^a + e(R, \phi_\infty) + O(R^{-a}).
\]
Therefore, after taking limits, the Huisken–Yau center of mass $C_{HY}^a$ is equal to the Corvino–Schoen center of mass $C_{CS}^a$ and, therefore, is equal to $C_{I}^a$. □

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References

[B86] Bartnik R 1986 Mass of an asymptotically flat manifold Commun. Pure Appl. Math. 39 661–93
[BLP03] Baskaran D, Lau S R and Petrov A N 2003 Center of mass integral in canonical general relativity Ann. Phys. 307 90–131
[BO87] Beig R and ‘O Murchadha N 1987 The Poincaré group as the symmetry group of canonical general relativity Ann. Phys. 174 463–98
[CS06] Corvino J and Schoen R 2006 On the asymptotics for the vacuum Einstein constraint equations J. Diff. Geom. 73 185–217

[CW08] Corvino J and Wu H 2008 On the center of mass of isolated systems Class. Quantum Grav. 25 085008

[HY96] Huisken G and Yau S-T 1996 Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature Invent. Math. 124 281–331

[RT74] Regge T and Teitelboim C 1974 Role of surface integrals in the Hamiltonian formulation of general relativity Ann. Phys. 88 286–318

[SY79] Schoen R and Yau S T 1979 On the proof of the positive mass conjecture in general relativity Commun. Math. Phys. 65 45–76

[SY81] Schoen R and Yau S T 1981 Proof of the positive mass theorem: II Commun. Math. Phys. 79 231–60

[Y96] Ye R 1996 Foliation by constant mean curvature spheres on asymptotically at manifolds Geometric Analysis and the Calculus of Variations (Cambridge, MA: Int. Press) pp 369–83