MEASURE OF COMPACTNESS FOR FILTERS IN PRODUCT SPACES: KURATOWSKI-MRÓWKA IN CAP REVISITED

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Abstract. The first author introduced a measure of compactness for families of sets, relative to a class of filters, in the context of convergence approach spaces. We characterize a variety of maps (types of quotient maps, closed maps, and variants of perfect maps) as those respecting this measure of compactness under one form or another. We establish a product theorem for measure of compactness that yields as instances new product theorems for spaces and maps, and new product characterizations of spaces and maps, thus extending existing results from the category of convergence spaces to that of convergence approach spaces. In particular, results of the Mrówka-Kuratowski type are obtained, shedding new light on existing results for approach spaces.

1. Introduction

R. Lowen introduced [11] Approach Spaces as a powerful tool bridging the gap between metric, topological and uniform spaces. In that setting, many concepts have been unified via the introduction of measures (e.g., measures of connectedness, measure of compactness and of its variants [2, 1, 10]). Consequently, as the category Ap of approach spaces became a natural object of study, so did its quasitopos, extensional [7, 8], and cartesian closed [20, 6] hulls. In particular, the category Cap of convergence approach spaces, which contains both Ap and the category Conv of convergence spaces both reflectively and coreflectively emerged as a convenient setting for integrating metric-like and topological-like studies.

In [19], the first author introduced a general measure of compactness in Cap relative to a class of filters that applied to filters to the effect that all known measures of compactness-like properties for approach spaces (as in [2, 11, 10]), as well as limit functions for various important reflections, were recovered as instances. The purpose of the present paper is to study its preservation under maps and finite products. As a consequence, new results on measures of compactness and its variants for product of spaces are obtained, and characterizations “à la Mrówka-Kuratowski” are obtained for a range of spaces and maps, including perfect maps and its variants, and various kinds of quotient maps. Maybe more importantly, all these results appear as instances of a single unifying principle. In particular, this viewpoint sheds new light on the results of [5], which are among those revisited here.

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1.1. Terminology: convergence approach spaces and its subcategories. Let $(FX, ≤)$ denote the set of filters on $X$, ordered by inclusion (inverse to the monad order). Let $UX$ be the subset of $FX$ formed by ultrafilters on $X$ and, given $G ∈ FX$, let $U(G)$ be the set of ultrafilters that are finer than $G$. If $A ⊂ X$, then $A^+ := \{ B ⊂ X : A ⊆ B \}$ and if $A ⊂ 2^X$ then $A^+ := ∪_{A ∈ A} A^+$.

Following [7] and [8], we call convergence-approach limit on $X$ a map $λ : FX → [0, ∞]^X$ which fulfills the properties:

(CAL1) \[ ∀x ∈ X, \ λ(\{x\}^+) = 0; \]

(CAL2) \[ G ≥ F \implies λ(F) ≥ λ(G); \]

(CAL3) \[ ∀F, G ∈ FX, λ(F ∨ G) = λ(F) \lor λ(G), \]

where of course (CAL2) follows from (CAL3) and is therefore redundant.

$(X, λ)$, shortly $X$, is called a convergence-approach space. A map $f : X → Y$ between two convergence-approach spaces is a contraction if

\[ λ_Y(f(\mathcal{F})) (f(\cdot)) ≤ λ_X(\mathcal{F})(\cdot), \]

for every $\mathcal{F} ∈ FX$. The category with convergence-approach spaces as objects and contractions as morphisms is a cartesian-closed topological category denoted Cap [7]. Each convergence space $X$ can be considered as a convergence-approach space by stating

\[ λ_X(\mathcal{F})(x) = \begin{cases} 0 & \text{if } x ∈ \lim_X \mathcal{F} \\ \infty & \text{otherwise.} \end{cases} \]

Moreover, the category Conv of convergence spaces (and continuous maps) is included both reflectively and coreflectively in Cap. Indeed, if $λ$ is a convergence-approach, then its Conv-coreflection is $c(λ)$ defined by $x ∈ \lim_c(λ) \mathcal{F}$ if and only if $λ(\mathcal{F})(x) = 0$, while its Conv-reflection is $r(λ)$ defined by $x ∈ \lim_r(λ) \mathcal{F}$ if and only if $λ(\mathcal{F})(x) < ∞$.

A convergence-approach $λ$ is a pseudo-approach space [8] if

(PSAP) \[ ∀\mathcal{F} ∈ FX, \ λ(\mathcal{F}) = ∨_{U ∈ U(\mathcal{F})} λ(U); \]

and it is a pre-approach space [7] if (CAL3) is strengthened to

(PRAP) \[ λ(\bigwedge_{j ∈ J} F_j) = ∨_{j ∈ J} λ(F_j), \] for any family $(F_j)_{j ∈ J}$ of filters.

The category Psap of pseudo-approach spaces (and contractions) contains the category Pstop of pseudotopological spaces (and continuous maps) and the category Prtop of pre-approach spaces contains the category Prtop of pretopological spaces both reflectively and coreflectively (via the restrictions of $c$ and $r$). Both Psap and Prap are reflective subcategories of Cap.

We say that two families $A$ and $B$ of subsets of a set $X$ mesh, in symbol $A \# B$, if $A ∩ B ≠ ∅$ whenever $A ∈ A$ and $B ∈ B$. We write $A \# B$ for $\{A\} \# B$. The grill of a family $A$ of subsets of $X$ is $A^# = \{H ⊂ X : H \# A\}$.

An approach space is a pre-approach space fulfilling

(AP) \[ λ(\bigcup_{F ∈ FX} \bigcap_{x ∈ F} G(x))(\cdot) ≤ λ(\mathcal{F})(\cdot) + ∨_{x ∈ X} λ(G(x))(x), \]
for any $F \in FX$ and any $G(\cdot) : X \to FX$.

The category $\text{Top}$ of topological spaces (with continuous maps) is a reflective and coreflective (via the restrictions of $r$ and $c$) subcategory of the category $\text{Ap}$ of approach spaces \cite{11}. There are several other equivalent descriptions of $\text{Ap}$ and $\text{Prap}$ (see \cite{12} and \cite{9} for details).

1.2. Measures of compactness. The adherence function of a filter $H$ in a convergence approach space $(X, \lambda)$ is

$$\text{adh}_\lambda H(\cdot) = \bigwedge_{G \in H} \lambda(G)(\cdot) = \bigwedge_{U \in \mathcal{U}(H)} \lambda(U)(\cdot).$$

We are now in a position to recall the main definitions and results of \cite{19}.

Let $(X, \lambda)$ be a $\text{Cap}$-object, and let $D$ be a class of filters. The measure of $D$-compactness of a filter $F$ at $A \subset [0, \infty]^X$ is

$$c^A_D(F) = \bigvee_{D \in D, \phi \in A} \bigwedge_{x \in X} \text{adh}_\lambda D + \phi(x).$$

This definition is motivated by the special case where $\lambda = r(\lambda)$ and $A \subset 2^X$ via the identification of $A \subset X$ with the indicator function $\theta_A$ of $A$ taking the value 0 on $A$ and $\infty$ on $A^c$. In this case, a filter $F$ is $D$-compact at $A$ (in the sense of \cite{4}) if and only if $c^A_D(F) = 0$. By a convenient abuse of notation, we will write $c^A_D(F)$ for $c_{\{\theta_A\}}^D(F)$ whenever $A \subset X$.

Notice that

$$c^A_D(F) = \bigvee_{D \in D, a \in A} \text{adh}_\lambda D(a).$$

In this paper, we are primarily concerned with measures of compactness at a set, like in (3).

When particularized to principal filters in approach spaces, the measures of compactness \cite{10} and relative compactness (for $D = F$ the class of all filters), relative countable compactness (for $D = F_1$ the class of countably based filters), and relative sequential compactness \cite{2}, as well as the measure of Lindelöf \cite{1} (for $D = F_{\Lambda_1}$ the class of countably deep filters, that is, those closed under countable intersections), are all instances, as shown in \cite{19, Examples 4-8}.

A subset $A$ of a $\text{Cap}$-space $X$ is $D$-compact if $c^A_D A = 0$ and relatively $D$-compact if $c^X_D A = 0$. In particular, if $D = F$ is the class of all filters, we call $A$ compact if $c^A_F A = 0$, in contrast to the terminology of R. Lowen and his collaborators who normally call such a set 0-compact (e.g., \cite{5}), and reserve the term compact for the smaller class of spaces whose topological coreflection is compact in the topological sense.

1.3. Endoreflectors of Conv and Cap. S. Dolecki presented in \cite{3} a unified treatment of several important concrete endoreflectors and endocoreflectors of $\text{Conv}$. In particular, given a class $\mathcal{J}$ of filters, he defined the modifications $\text{Adh}_\mathcal{J} \xi$ and $\text{Base}_\mathcal{J} \xi$ of a convergence $\xi$ on $X$ as follows:

$$\lim_{\text{Adh}_\mathcal{J} \xi} F = \bigcap_{J \in \mathcal{J} \# F} \text{adh}_\xi J,$$
where $\text{adh}_\xi \mathcal{J} := \bigcup_{U \in \mathcal{U} (\mathcal{J})} \lim_\xi U$, and

$$\lim_{\text{Base}_\xi \mathcal{J}} \mathcal{F} = \bigcup_{J \supseteq \mathcal{J} \leq \mathcal{F}} \lim_\xi J.$$ 

If the class $\mathcal{J}$ is independent of the convergence, stable by finite infimum and stable by relation (4), then $\text{Adh}_\mathcal{J}$ is (the restriction to objects of) a reflector and $\text{Base}_\mathcal{J}$ is (the restriction to objects of) a coreflector. In particular, when $\mathcal{J}$ is respectively the class $\mathbb{F}$ of all filters, the class $\mathbb{F}_1$ of countably based filters and the class $\mathbb{F}_0$ of principal filters, then $\text{Adh}_\mathcal{J}$ is the reflector from $\text{Conv}$ onto the category of pseudotopological, paratopological and pretopological spaces respectively; and $\text{Base}_\mathcal{J}$ is the identity functor of $\text{Conv}$, the coreflector from $\text{Conv}$ onto first-countable convergence spaces and the coreflector from $\text{Conv}$ onto finitely generated convergence spaces, respectively.

As observed in [16], the definitions of the reflectors $\text{Adh}_\mathcal{J}$ and of the coreflectors $\text{Base}_\mathcal{J}$ extend from $\text{Conv}$ to $\text{Cap}$ via

$$\text{(Adh}_\mathcal{J} \lambda \text{)}(\mathcal{F})(x) = \bigvee_{J \ni H \subseteq \mathcal{F}} \text{adh}_\lambda H(x),$$

and

$$\text{(Base}_\mathcal{J} \lambda \text{)}(\mathcal{F})(\cdot) = \bigwedge_{J \ni G \leq \mathcal{F}} \lambda(G)(\cdot).$$

When $\mathcal{J}$ is respectively the class of all filters and of principal filters, $\text{Adh}_\mathcal{J}$ is respectively the reflector on $\text{Psap}$ and on $\text{Prap}$. Moreover, the category $\text{Parap}$ of para-approach spaces is introduced as the category of fixed points for $\text{Adh}_\mathcal{J}$ with the class $\mathbb{J}$ of countably based filters. Notice that (4) gives an explicit description of the reflection of a $\text{Cap}$-object on $\text{Psap}$, $\text{Parap}$ or $\text{Prap}$, but not on $\text{Ap}$.

A convergence approach space $(X, \lambda)$ is called $\mathcal{J}$-based if $\lambda = \text{Base}_\mathcal{J} \lambda$ (equivalently, $\lambda \geq \text{Base}_\mathcal{J} \lambda$).

Measures of $\mathbb{D}$-compactness for filters generalize both usual measure of compactness for sets and approach limits. It is this very fact that allows to derive a variety of corollaries from any result on the measure of $\mathbb{D}$-compactness of filters. With our definitions, it is immediate that:

**Theorem 1.** [19, Theorem 9]

$$(\text{Adh}_\mathcal{J} \lambda \text{)}(\mathcal{F})(x) = c^\mathcal{J}_x(\mathcal{F}).$$

The $\text{Ap}$-reflection of a convergence-approach space can also be characterized in similar terms [19, Theorem 10].

1.4. **Calculus of relations.** Recall that $R \subseteq X \times Y$ can be seen as a multivalued map $R : X \Rightarrow Y$ with $y \in R(x)$ whenever $(x, y) \in R$. We denote $R^- : Y \Rightarrow X$ the inverse relation. If $R : X \Rightarrow Y$ is a relation and $\mathcal{F} \in \mathcal{F} X$ then

$$R[\mathcal{F}] := \left\{ R(F) := \bigcup_{x \in F} R(x) : F \in \mathcal{F} \right\}.$$  

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1. For more general conditions, see [3].
Comparing the extremes of these inequalities, we have that
\[ J \subset \text{cap} \]
and we can define
\[ J[F] := \{ J(F) : F \in F, J \in J \} \]
which is a (possibly degenerate) filter on \( X \). \( J^{-}[G] \) is defined similarly and is a (possibly degenerate) filter on \( X \). With these notations, (6) immediately extends to
\[ (F \times G)\# J \iff J[F]\# G \iff F\# J^{-}[G]. \]
As a general convention, all classes of filters contain the degenerate filter of each set.

Let \( D \) and \( J \) be two classes of filters. Then \( D \) is a \( J \)-composable class of filters if for every pair of sets \( X \) and \( Y \), when \( D \in D \times X \) and \( J \in \mathcal{J}(X \times Y) \), \( J[D] \in D \times Y \). In particular, if \( D \) is \( D \)-composable, we say that \( D \) is \emph{composable}. For instance the classes \( F_0 \) of principal filters, \( F_1 \) of countably based filters, and \( F_{\lambda,1} \) of countably deep filters are all composable classes containing \( F_0 \), so that they are in particular \( F_0 \)-composable. In contrast, the class \( E \) of sequential filters is not \( F_0 \)-composable.

2. Compact relations in \text{Cap}

As we set out to extend some of the results of [17] from \text{Conv} to \text{Cap}, a necessary first step is to extend to \text{Cap} the characterizations of various types of quotient maps and of perfect-like maps in terms of preservation of compactness established in [18]. This is the purpose of this section.

A relation \( R : X \rightarrow Y \) is \( D \)-compact if for every \( F \in \mathcal{F}X \) and \( A \subset X \),
\[ c_D^{R[A]}(R[F]) \leq c_D^A(F). \]

**Lemma 2.** Let \( R : X \rightarrow Y \) be a \( D \)-compact relation and \( J \subset D \) be classes of filters with \( J \) an \( F_0 \)-composable class. Then \( R \) is also \( J \)-compact.

**Proof.** Let \( F \in \mathcal{F}X \) and \( G \in J \), with \( G \# R[F] \). By (6), \( R^{-}[G] \# F \) and \( R^{-}[G] \in J \) since \( J \) is an \( F_0 \)-composable class. Therefore \( \bigwedge_{x \in A} \text{adh} R^{-}[G](x) \leq c_J^A(F) \), so that for every \( \epsilon > 0 \), there is an \( x_\epsilon \in A \) and a \( U_\epsilon \subset \text{U}(R^{-}[G]) \) such that
\[ c_J^A(F) + \epsilon \geq \lambda_X(U_\epsilon)(x_\epsilon). \]

By the \( D \)-compactness of the relation \( R \), we see that
\[ \lambda_X(U_\epsilon)(x_\epsilon) \geq c_D^A(U_\epsilon) \geq c_D^{R[A]}(R[U_\epsilon]). \]

Since \( J \subset D \), \( c_D^B H \geq c_D^B H \) for any filter \( H \) and set \( B \). Moreover, \( R[U_\epsilon] \# G \), so that, in view of (8),
\[ \lambda_X(U_\epsilon)(x_\epsilon) \geq c_J^{R[A]}(R[U_\epsilon]) \geq \bigwedge_{y \in R[A]} \text{adh} G(y). \]

Comparing the extremes of these inequalities, we have that
\[ \bigwedge_{y \in R[A]} \text{adh} G(y) \leq c_J^A F + \epsilon, \]
for every \( \epsilon \), so that \( c_J^{R[A]} R[F] \leq c_J^A F. \)  \( \square \)
Lemma 3. Let $\mathcal{D}$ be an $\mathbb{F}_0$-composable class of filters. Then $R : X \Rightarrow Y$ is a $\mathcal{D}$-compact relation if and only if

\begin{equation}
\lambda_X(\mathcal{F})(x) \geq c_D^{R(x)} R[\mathcal{F}]
\end{equation}

for every $\mathcal{F} \in \mathbb{F}X$ and every $x \in X$.

Proof. Assume that $R$ is a $\mathcal{D}$-compact relation. For every $x \in X$ and $\mathcal{F} \in \mathbb{F}X$,

\[
c_D^{R(x)} R[\mathcal{F}] \leq c_D^{\{x\}}(\mathcal{F}) = \bigvee_{\mathcal{D} \ni \mathcal{F} \ni \mathcal{G} \ni \mathcal{D}} \lambda_X(\mathcal{G})(x) \leq \lambda_X(\mathcal{F})(x),
\]

because $\mathcal{F} \# \mathcal{D}$.

Conversely, assume (9) for every $\mathcal{F} \in \mathbb{F}X$ and every $x \in X$, and given a filter $\mathcal{F} \in \mathbb{F}X$, consider a $\mathcal{D}$-filter $\mathcal{D}$ that meshes with $R[\mathcal{F}]$. By (6), $R^{-}[\mathcal{D}] \# \mathcal{F}$, and $R^{-}[\mathcal{D}] \in \mathcal{D}$ because $\mathcal{D}$ is $\mathbb{F}_0$-composable, so that for every $A \subset X$,

\[
\bigwedge_{x \in A} \text{adh}_X R^{-}[\mathcal{D}](x) \leq c_D^A(\mathcal{F}).
\]

Thus, for any $\epsilon > 0$, there is an $x_\epsilon \in A$ and $\mathcal{U}_\epsilon \in U(R^{-}[\mathcal{D}])$ such that

\[
\lambda(\mathcal{U}_\epsilon)(x_\epsilon) \leq c_D^A(\mathcal{F}) + \epsilon.
\]

By (9) applied to $\mathcal{U}_\epsilon$ and $x_\epsilon$,

\[
c_D^{R(x_\epsilon)} R[\mathcal{U}_\epsilon] \leq c_D^A(\mathcal{F}) + \epsilon.
\]

Moreover, $R[\mathcal{U}_\epsilon] \# \mathcal{D}$ because $\mathcal{U}_\epsilon \# R^{-}[\mathcal{D}]$, so that

\[
\bigwedge_{y \in R(A)} \text{adh}_Y \mathcal{D}(y) \leq \bigwedge_{y \in R(x_\epsilon)} \text{adh}_Y \mathcal{D}(y) \leq c_D^{R(x_\epsilon)} R[\mathcal{U}_\epsilon].
\]

We conclude that for every $\epsilon > 0$,

\[
\bigwedge_{\mathcal{D} \ni \mathcal{F} \ni R[\mathcal{F}]} \bigwedge_{y \in R(A)} \text{adh}_Y \mathcal{D}(y) = c_D^{R(A)} R[\mathcal{F}] \leq c_D^A(\mathcal{F}) + \epsilon,
\]

which yields the desired property that $c_D^{R(A)} R[\mathcal{F}] \leq c_D^A(\mathcal{F})$. \hfill \square

Corollary 4. Let $\mathcal{D}$ be an $\mathbb{F}_0$-composable class of filters and let $f : (X, \lambda_X) \rightarrow (Y, \lambda_Y)$ with $Y = \text{Adh}_\mathcal{D}Y$. The following are equivalent:

1. $f$ is a contraction;
2. $f$ is a compact relation;
3. $f$ is a $\mathcal{D}$-compact relation.

Proof. (1 $\Rightarrow$ 2). If $f$ is a contraction then

\[
\lambda_X(\mathcal{F})(x) \geq \lambda_Y(f[\mathcal{F}]) f(x) \geq c_F^{\{f(x)\}} f[\mathcal{F}]
\]

and Lemma 3 applies to the effect that $f$ is a compact relation. (2 $\Rightarrow$ 3) is obvious, and (3 $\Rightarrow$ 1) follows from Theorem 1 and $Y = \text{Adh}_\mathcal{D}Y$. \hfill \square

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In particular, \( F_0 \)-compact (equivalently compact) maps between pre-approach spaces are exactly the contractive ones.

Since \( c_D^{\{x\}} \mathcal{F} \leq \lambda(\mathcal{F})(x) \), Lemma 3 immediately gives:

**Corollary 5.** Let \( \mathbb{D} \) be an \( \mathbb{F}_0 \)-composable class, then \( R : X \rightarrow Y \) is a \( \mathbb{D} \)-compact relation if and only if for every \( \mathcal{F} \in \mathbb{F} \mathcal{X} \), and \( x \in X \),

\[
c_D^{\{x\}}(\mathcal{F}) \geq c_D^{R(x)}[\mathcal{F}] = c_D^{R(x)} R(\mathcal{F}).
\]

If \( \mathbb{D} \) is a class of filters that contains \( \mathbb{F}_0 \), then, in view of Lemma 2, a \( \mathbb{D} \)-compact relation \( R \) is also \( \mathbb{F}_0 \)-compact, and for each \( x \in X \),

\[
c_D^{R(x)} R(x) \leq c_D^{\{x\}}(x) = 0,
\]

so that \( R(x) \) is a \( \mathbb{D} \)-compact subset of \( Y \). When \( Y \) is an approach space, the converse is true:

**Theorem 6.** Let \( X \) be a convergence approach space, let \( Y \) be an approach space, let \( \mathbb{D} \) be an \( \mathbb{F}_0 \)-composable class of filters, and let \( R : X \rightarrow Y \) be an \( \mathbb{F}_0 \)-compact relation. If for every \( x \in X \), \( R(x) \) is a \( \mathbb{D} \)-compact subset of \( Y \), then \( R \) is a \( \mathbb{D} \)-compact relation.

**Proof.** In view of 5, it suffices to show that \( c_D^{\{x\}}(\mathcal{F}) \geq c_D^{R(x)}[\mathcal{F}] \) for every \( x \in X \) and \( \mathcal{F} \in \mathbb{F} \mathcal{X} \), equivalently, that given \( x \in X \) and \( \mathcal{F} \in \mathbb{F} \mathcal{X} \), for every \( \mathcal{D} \in \mathbb{D} \) with \( \mathcal{D} \# R(\mathcal{F}) \),

\[
\bigwedge_{y \in R(x)} \text{adh} \mathcal{D}(y) \leq c_D^{\{x\}}(\mathcal{F}).
\]

By 5, \( R^{-}[\mathcal{D}]\# \mathcal{F} \), and \( R^{-}[\mathcal{D}] \in \mathbb{D} \) because \( \mathbb{D} \) is an \( \mathbb{F}_0 \)-composable class. Thus \( \text{adh}_Y R^{-}[\mathcal{D}](x) \leq c_D^{\{x\}}(\mathcal{F}) \).

For every \( \epsilon > 0 \), there is a \( \mathcal{U}_\epsilon \# R^{-}[\mathcal{D}] \) such that \( \lambda_X(\mathcal{U}_\epsilon)(x) \leq c_D^{\{x\}}(\mathcal{F}) + \epsilon \). Since \( R \) is an \( \mathbb{F}_0 \)-compact relation,

\[
c_D^{R(x)} R[\mathcal{U}_\epsilon] \leq c_D^{\{x\}}(\mathcal{U}_\epsilon) \leq \lambda(\mathcal{U}_\epsilon)(x) \leq c_D^{\{x\}}(\mathcal{F}) + \epsilon.
\]

Since \( \mathcal{D} \# R[\mathcal{U}_\epsilon] \), then for every \( \mathcal{D} \in \mathbb{D} \), we have that \( \bigwedge_{y \in R(x)} \text{adh} \mathcal{D}(y) \leq c_D^{R(x)} R[\mathcal{U}_\epsilon] \). Setting \( \alpha = c_D^{\{x\}}(\mathcal{F}) + \epsilon \) and \( D^{(\alpha)} = \{ D^{(\alpha)} : D \in \mathbb{D} \} \in \mathbb{D} \), we have that \( \bigwedge_{y \in R(x)} \text{adh} \mathcal{D}^{(\alpha)}(y) = 0 \) because \( D^{(\alpha)} \# R(x) \) and \( R(x) \) is a \( \mathbb{D} \)-compact subset of \( Y \). Since \( Y \) is an approach space,

\[
\text{adh} \mathcal{D}(y) \leq \text{adh} \mathcal{D}^{(\alpha)}(y) + \alpha
\]

so that, given that \( \bigwedge_{y \in R(x)} \text{adh} \mathcal{D}^{(\alpha)}(y) = 0 \),

\[
\alpha = \left( \bigwedge_{y \in R(x)} \text{adh} \mathcal{D}^{(\alpha)}(y) \right) + \alpha
\]

\[
= \bigwedge_{y \in R(x)} \left( \text{adh} \mathcal{D}(y) + \alpha \right)
\]

\[
c_D^{\{x\}}(\mathcal{F}) + \epsilon = \alpha \geq \bigwedge_{y \in R(x)} \text{adh} \mathcal{D}(y).
\]
Since this inequality is true for every $\epsilon > 0$ we obtain (10) and the conclusion follows. □

**Corollary 7.** Let $X$ be a convergence approach space, let $Y$ be an approach space, let $\mathbb{D}$ be an $\mathbb{F}_0$-composable class of filters. Then a relation $R : X \Rightarrow Y$ is $\mathbb{D}$-compact if and only if it is $\mathbb{F}_0$-compact and for every $x \in X$, $R(x)$ is a $\mathbb{D}$-compact subset of $Y$.

**Proof.** If $\mathbb{D}$ is $\mathbb{F}_0$-composable, then in particular $\mathbb{F}_0 \subseteq \mathbb{D}$. Indeed, Let $A \in \mathbb{F}_0(X)$ and let $\mathbb{D}$ be a non-degenerate $\mathbb{D}$-filters on $X$. Let $R := X \times A$. Then $R[\mathbb{D}] = A$. Thus, Lemma 2 applies to the effect that if $R$ is a $\mathbb{D}$-compact relation, it is also $\mathbb{F}_0$-compact, and as observed before each $R(x)$ is $\mathbb{D}$-compact. The converse is Theorem 6. □

2.1. Closed and perfect maps. Lowen et al. introduced in [5] a notion of closed maps in $\mathbb{A}p$ and checked that this class of morphisms satisfies the conditions to be a categorically well-behaved class of closed morphisms in the sense of [14]. Namely, a map $f : (X, \lambda_X) \rightarrow (Y, \lambda_Y)$ between two approach spaces is closed-expansive, which we will abridge as closed, if for every $y \in Y$ and $A \subseteq X$

$$\bigwedge_{x \in f^{-1}(y)} \text{adh}_X A(x) \leq \text{adh}_Y f(A)(y).$$

We extend this definition to convergence approach spaces.

**Proposition 8.** A map $f : (X, \lambda_X) \rightarrow (Y, \lambda_Y)$ between convergence approach spaces is closed (in the sense of [17]) if and only if $f^{-1} : Y \Rightarrow X$ is an $\mathbb{F}_0$-compact relation.

**Proof.** Assume that $f : X \rightarrow Y$ is closed. According to Lemma 3 we need to show that for every $G \in FY$ and $y \in Y$, $c_{\mathbb{F}_0} f^{-1}(y) f^{-}[G] \leq \lambda_Y (G)(y)$. If $A \# f^{-}[G]$, then $f(A) \# G$ so that $\text{adh}_Y f(A)(y) \leq \lambda_Y (G)(y)$ and, in view of (11),

$$\bigwedge_{x \in f^{-1}(y)} \text{adh}_X A(x) \leq \lambda_Y (G)(y).$$

Since this is true for every $A \# f^{-}[G]$, $c_{\mathbb{F}_0} f^{-1}(y) f^{-}[G] \leq \lambda_Y (G)(y)$.

Conversely, assume that $f : Y \Rightarrow X$ is an $\mathbb{F}_0$-compact relation. To show (11), note that if $G \# f(A)$ then $f^{-1}[G] \# A$ so that, by $\mathbb{F}_0$-compact of $f^{-}$ and Lemma 3,

$$\bigwedge_{x \in f^{-1}(y)} \text{adh}_X A(x) \leq \bigwedge_{G \# f(A)} \lambda_Y (G)(y) = \text{adh}_Y f(A)(y).$$

Let us call a map $f : X \rightarrow Y$ between convergence approach spaces $\mathbb{D}$-perfect if $f^{-} : Y \Rightarrow X$ is a $\mathbb{D}$-compact relation. In view of Proposition 8, $\mathbb{F}_0$-perfect means closed. Moreover, Corollary 7 applies to $f^{-}$ to the effect that:

**Theorem 9.** Let $\mathbb{D}$ be an $\mathbb{F}_0$-composable class of filters, $X$ be an approach space and $Y$ be a convergence approach space. A map $f : X \rightarrow Y$ is $\mathbb{D}$-perfect if and only if $f$ is closed and for every $y \in Y$, $f^{-} y$ is $\mathbb{D}$-compact.
2.2. Quotient maps. S. Dolecki observed in [3] that the notions of quotient, hereditarily quotient, countably biquotient, biquotient, almost open maps (in the sense of [13]) can be extended from the category Top of topological spaces to the category Conv of convergence spaces by noting that a map between topological spaces is hereditarily quotient, countably biquotient, biquotient, almost open respectively, if it is quotient when regarded in the category of pretopological spaces, paratopological spaces, pseudotopological spaces, and convergence spaces respectively. To fully extend the notions to Conv, he further observed that if \( f : X \to Y \) is onto between topological spaces, seen as convergence spaces, the map is quotient in a reflective subcategory if the convergence of \( Y \) is finer than the reflection of the final convergence for \( f \) and \( X \).

We can proceed exactly the same way in Cap: Given a map \( f : (X, \lambda_X) \to Y \), there is the finest limit function \( \lambda_{fX} \) on \( Y \) making \( f \) a contraction, that is, the quotient structure in Cap. Given an \( F_0 \)-composable class \( D \), \( \Adh_D \) (given by (4)) defines a reflector, and the inequality
\[
\lambda_Y \geq \Adh_D \lambda_{fX}
\]
characterizes the fact that a surjective map \( f : (X, \lambda_X) \to (Y, \lambda_Y) \) is quotient in the full reflective category of Cap of objects fixed by \( \Adh_D \). When \( D = F_0 \), (12) defines hereditarily quotient maps between Cap spaces. Of course, a map between two topological spaces is hereditarily quotient if and only if it is hereditarily quotient when domain and codomain are seen as convergence approach spaces. Similarly, (12) for \( D = F_1 \) defines countably biquotient maps, and (12) for \( D = F \) defines biquotient maps. Naturally, we call a surjective map \( f : (X, \lambda_X) \to (Y, \lambda_Y) \) \( D \)-quotient if (12) holds.

\( D \)-quotient maps are also instances of \( D \)-compact relations. To see that, we need to use both the final Cap structure \( \lambda_{fX} \) but also the initial Cap structure \( \lambda_{f-Y} \), that is, the coarsest Cap structure on \( X \) making the map \( f : X \to (Y, \lambda_Y) \) a contraction. Initial and final structures in Cap are described in [8, Proposition 2.3] to the effect that
\[
\lambda_{f-Y}(F)(x) = \lambda_Y(f[F])(f(x))
\]
and
\[
\lambda_{fX}(G)(y) = \begin{cases} 
0 & \text{if } G = \{y\}^\uparrow \\
\bigwedge_{x \in f^{-Y}} \bigwedge_{F \in F_X, f[F] \leq G} \lambda_X(F)(x) & \text{otherwise.}
\end{cases}
\]

**Lemma 10.** If \( f : (X, \lambda_X) \to (Y, \lambda_Y) \) is onto and \( D \in FY \)
\[
\operatorname{adh}_{fX} D(y) = \bigwedge_{x \in f^{-Y}} \operatorname{adh}_X f^{-[D]}(x).
\]

**Proof.** Since \( f \) is onto, for each \( U \in U(D) \) there is \( W \in U(f^{-[D]}) \) with \( f[W] = U \) so that
\[
\lambda_{fX} U(y) = \bigwedge_{x \in f^{-Y}} \lambda_X W(x) \geq \bigwedge_{x \in f^{-Y}} \operatorname{adh}_X f^{-[D]}(x).
\]

On the other hand, if \( W \in U(f^{-[D]}) \) then \( f[W] \in U(D) \) and \( f : (X, \lambda_X) \to (Y, \lambda_{fX}) \) is a contraction, so that, for every \( x \in f^{-Y} \),
\[
\lambda_X W(x) \geq \lambda_{fX} f[W](y) \geq \operatorname{adh}_{fX} D(y)
\]
and we conclude that
\[ \bigwedge_{x \in f^{-}y} \text{adh}_X f^{-}[\mathcal{D}](x) \geq \text{adh}_X f^{-}[\mathcal{D}](y). \]

\[ \square \]

**Theorem 11.** Let \( \mathcal{D} \) be an \( F_0 \)-composable class of filters. Let \( f : (X, \lambda_X) \to (Y, \lambda_Y) \) be onto. Then \( f \) is \( \mathcal{D} \)-quotient if and only if \( f : (X, \lambda_{f^{-}Y}) \to (Y, \lambda_{fX}) \) is a \( \mathcal{D} \)-compact relation.

**Proof.** Assume \( f \) is \( \mathcal{D} \)-quotient. Then given \( \mathcal{F} \in F_X \) and \( x \in X \),
\[ \lambda_{f^{-}Y}(\mathcal{F})(x) = \lambda_Y(f[\mathcal{F}](f(x)) \geq \text{Adh}_{\mathcal{D}} F_X(f[\mathcal{F}](f(x)) = c_{\mathcal{D}}^{f(x)} f[\mathcal{F}] \]
where the measure of \( \mathcal{D} \)-compactness is in \((Y, \lambda_{fX})\). In view of Lemma, \( f : (X, \lambda_{f^{-}Y}) \to (Y, \lambda_{fX}) \) is a \( \mathcal{D} \)-compact relation.

Conversely, assume that \( f : (X, \lambda_{f^{-}Y}) \to (Y, \lambda_{fX}) \) is a \( \mathcal{D} \)-compact relation. Let \( \mathcal{G} \in F_Y \) and \( y \in Y \). Since \( f \) is onto, \( \mathcal{G} = f[f^{-}[\mathcal{G}]] \), and there is \( x \in f^{-}y \), so that
\[ \lambda_Y(\mathcal{G})(y) = \lambda_{f^{-}Y}(f^{-}[\mathcal{G}](x) \geq c_{\mathcal{D}}^{f^{-}[\mathcal{G}]} \]
where the right hand side is measured in \((Y, \lambda_{fX})\). In view of Theorem, \( \lambda_Y \geq \text{Adh}_{\mathcal{D}} \lambda_{fX} \).

\[ \square \]

Note that Corollary does not apply to \( \mathcal{D} \)-quotient maps in general, for even if \( X \) is an approach space, \((Y, \lambda_{fX})\) generally fails to be.

The table below gathers the terminology we use for various instances of \( F_0 \)-composable classes of filters:

| Class \( \mathcal{D} \) of filters | Adh\( \mathcal{D} \)-fixed spaces | \( \mathcal{D} \)-compact filter | \( \mathcal{D} \)-quotient map | \( \mathcal{D} \)-perfect map |
|--------------------------------------|----------------------------------|-------------------------------|-------------------------------|-----------------------------|
| \( F \) (All filters)               | \textbf{Psap}                     | Compact                       | biquotient                   | Perfect                     |
| \( F \land_1 \) (countably deep)   | \textbf{Hypoap}                  | Lindelöf                      | weakly biquotient            | inversely Lindelöf          |
| \( F_1 \) (Countably Based)        | \textbf{Parap}                   | Countably compact             | countably biquotient         | Countably perfect           |
| \( F_0 \) (Principal)              | \textbf{Prap}                    | Finitely compact              | hereditarily quotient        | Closed                      |

**Proposition 12.** Let \( \mathcal{D} \) be an \( F_0 \)-composable class of filters. A \( \mathcal{D} \)-perfect surjective map is \( \mathcal{D} \)-quotient. In particular, in \( \text{Cap} \), surjective perfect maps are biquotient and surjective closed maps are hereditarily quotient, hence quotient in \( \text{Ap} \).

**Proof.** Let \( \mathcal{F} \in F_X \) and \( x \in X \), and let \( y := f(x) \), \( \mathcal{G} := f[\mathcal{F}] \). Then, in view of (13),
\[ \lambda_{f^{-}Y} f^{-}(\mathcal{F})(x) = \lambda_Y(\mathcal{G})(y) \geq c_{\mathcal{D}}^{f^{-}y} f^{-}[\mathcal{G}] \]
because \( f^{-} : Y \ni X \) is \( \mathcal{D} \)-compact. Moreover,
\[ c_{\mathcal{D}}^{f^{-}y} f^{-}[\mathcal{G}] = \bigvee_{\mathcal{D} \ni f^{-}y} \bigwedge_{x \in f^{-}y} \text{adh}_X f^{-}[\mathcal{D}(x)] \]

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so that, if $\mathcal{D} \in \mathbb{D}(Y)$ with $\mathcal{D} \# \mathcal{G}$ then $f^-[\mathcal{D}] \# f^-[\mathcal{G}]$ and $f^-[\mathcal{D}] \in \mathbb{D}(X)$ by $F_0$-composability, so that
\[
\bigwedge_{x \in f^-y} \operatorname{adh}_X f^-[\mathcal{D}](x) \leq c_{\mathcal{D}^{-}}^{-} f^-[\mathcal{G}] \leq \lambda_Y(\mathcal{G})(y).
\]
By Lemma 10, $\operatorname{adh}_X \mathcal{D}(y) = \bigwedge_{x \in f^-y} \operatorname{adh}_X f^-[\mathcal{D}](x)$, and we conclude that $\operatorname{Adh}_\mathcal{D} \lambda_X \leq \lambda_Y$. □

3. PRODUCT OF MEASURES OF COMPACTNESS

The main result to be applied in the next section is the following extension from $\textbf{Conv}$ to $\textbf{Cap}$ of [17, Theorem 1]:

**Theorem 13.** Let $(X, \lambda_X)$ be a convergence approach space, $A \subset X$, $\alpha \in [0, \infty)$, and let $\mathcal{F} \in \mathbb{F}X$. Let $\mathbb{D}$ be a composable class of filters that contains principal filters. The following are equivalent:

1. $c_0^A(\mathcal{F}) \leq \alpha$;
2. For every convergence approach space $(Y, \lambda_Y)$, every $B \subset Y$, and every $\mathcal{G} \in \mathbb{D}(Y)$,
   \[
   c_0^{A \times B}(\mathcal{F} \times \mathcal{G}) \leq \alpha \lor c_0^B(\mathcal{G});
   \]
3. For every $\mathbb{D}$-based atomic topological approach space $Y$, with non-isolated point $\infty$ and neighborhood filter $\mathcal{N}(\infty)$,
   \[
   c_0^{A \times (\infty)}(\mathcal{F} \times \mathcal{N}(\infty)) \leq \alpha.
   \]

Note that the case where $\alpha = \infty$ is trivially true.

**Proof.**

(1) $\implies$ (2) Assume that $c_0^A(\mathcal{F}) \leq \alpha$. Let $\mathcal{D} \in \mathbb{D}(X \times Y)$ with $\mathcal{D} \# (\mathcal{F} \times \mathcal{G})$. By [17], $\mathbb{D}^{-}[\mathcal{G}] \# \mathcal{F}$ and moreover, $\mathbb{D}^{-}[\mathcal{G}] \in \mathbb{D}X$ because $\mathcal{G} \in \mathbb{D}Y$ and $\mathbb{D}$ is composable. Since $c_0^A(\mathcal{F}) \leq \alpha$, for every $\epsilon > 0$ there is a $\mathcal{U}_c \# \mathcal{D}^{-}[\mathcal{G}]$ and $u_c \in A$ such that $\lambda_X(\mathcal{U}_c)(u_c) \leq \alpha + \epsilon$. By [17], so that there is $\mathcal{W}_c \in \mathbb{U}(\mathbb{D}[\mathcal{U}_c])$ and $b_c \in B$ such that $\lambda_Y(\mathcal{W}_c)(b_c) \leq c_0^B(\mathcal{G}) + \epsilon$. Moreover, $\mathcal{W}_c \# \mathcal{D}[\mathcal{U}_c]$ so that $\mathcal{D} \# (\mathcal{W}_c \times \mathcal{U}_c)$, and thus
\[
\bigwedge_{(a, b) \in A \times B} \operatorname{adh}_{A \times B} \mathcal{D}(a, b) \leq (\alpha + \epsilon) \lor (c_0^B(\mathcal{G}) + \epsilon).
\]
Since this is true for every $\epsilon > 0$, we obtain the result.

(2) $\implies$ (3) is obvious because $F_0 \subseteq \mathbb{D}$.

(3) $\implies$ (1) Assume that $c_0^A(\mathcal{F}) > \alpha$, and let $\beta$ be such that $c_0^A(\mathcal{F}) > \beta > \alpha$. Then there is a $\mathcal{D} \in \mathbb{D}$ with $\mathcal{D} \# \mathcal{F}$ such that
\[
\bigwedge_{a \in A} \operatorname{adh}_X \mathcal{D}(a) \geq \beta.
\]
We construct a $\mathbb{D}$-based topological approach space with underlying set $Y = X \cup \{\infty\}$ where $\infty \notin X$. Set every point of $X$ to be isolated in $Y$, that is, for every $x \in X$, $\lambda_Y(\mathcal{F})(x) = \infty$ for every $\mathcal{F} \neq \{x\}^\uparrow$ and $\lambda_Y(\{x\}^\uparrow)(x) = 0$. Let $\mathcal{N}_Y(\infty) := \mathcal{D} \wedge \{x\}^\uparrow \in \mathbb{D}$, that is, $\lambda_Y(\mathcal{F})(\infty) = 0$ if $\mathcal{F} \geq \mathcal{N}_Y(\infty)$ and $\lambda_Y(\mathcal{F}) = \infty$ otherwise.
Let
\[
\Delta := \{(x, x) : x \in X\}^\uparrow \in \mathbb{F}_0(X \times Y).
\]
Note that $\Delta^\#((F \times N_Y(\infty)))$ because $D^\#F$ in $X$. We claim that

$$\bigwedge_{a \in A} \text{adh}_{X,Y} \Delta(a,\infty) > \alpha \lor \lambda_Y(N_Y(\infty))(\infty) = \alpha,$$

which yields $c_{p_D}^{A \times (\infty)}(F \times N_Y(\infty)) > \alpha$.

To verify (15), note that for every $a \in A$,

$$\text{adh}_{X,Y} \Delta(a,\infty) = \bigwedge_{\Delta \leq H} \lambda_{X,Y}(H)(a,\infty) = \bigwedge_{\Delta \leq H} \lambda_X(p_X[H])(a) \lor \lambda(p_Y[H])(\infty).$$

If $p_Y[H] \nmid D$, then $\lambda_Y(p_Y[H])(\infty) = \infty$ so $\text{adh}_{X,Y} \Delta(a,\infty) > \beta$. Otherwise, $D^\#p_Y[H]$ and $\Delta \leq H$ so that $D^\#p_X[H]$, and thus, $\lambda_X(p_X[H])(a) \geq \beta$. Either way, $\text{adh}_{X,Y} \Delta(a,\infty) \geq \beta$, proving our claim. \qed

In order to apply this result to product of maps, we need the following extension from Conv to Cap of [17, Corollary 12]:

**Theorem 14.** Let $D$ be a composable class of filters containing principal filters, and let $X$ and $Y$ be two convergence approach spaces. The following are equivalent:

1. $R : X \Rightarrow Y$ is a $D$-compact relation.
2. For every $D$-based convergence approach space $Z$, $R \times \text{Id}_Z : X \times Z \Rightarrow Y \times Z$ is a $D$-compact relation.
3. For every atomic topological $D$-based approach space $Z$, $R \times \text{Id}_Z : X \times Z \Rightarrow Y \times Z$ is an $D_0$-compact relation.

**Proof.** (1 $\Rightarrow$ 2) Let $R : X \Rightarrow Y$ be a $D$-compact relation. We show that

$$\lambda_X(F)(x) \lor \lambda_Z(G)(z) \geq c_D^{R(x) \times \{z\}}(R[F] \times G)$$

for every $F \in F_X$, $G \in F_Z$, $x \in X$, and $z \in Z$. To this end, note that for every $D \in D$ such that $D \leq G$,

$$c_D^{R(x) \times \{z\}}(R[F] \times G) \leq c_D^{R(x) \times \{z\}}(R[F] \times D) \leq c_D^{R(x)}(R[F]) \lor c_D^{\{z\}}(D) \text{ by Theorem 13} \leq \lambda_X(F)(x) \lor \lambda_Z(D)(z) \text{ so that}$$

$$c_D^{R(x) \times \{z\}}(R[F] \times G) \leq \bigwedge_{D \in D \leq G} (\lambda_X(F)(x) \lor \lambda_Z(D)(z)) \leq \lambda_X(F)(x) \lor \lambda_Z(G)(z),$$

because $Z$ is $D_0$-based.

(2 $\Rightarrow$ 3) Is trivial.

(3 $\Rightarrow$ 1) Let $F \in F_X$ and $x \in X$. We want to show that

$$c_D^{R(x)} R[F] \leq \alpha := \lambda_X(F)(x).$$
Since $R \times Id_Z$ is $\mathbb{F}_0$-compact for every topological $\mathbb{D}$-based atomic approach space $Z$, then for every such $(Z, \lambda_Z)$ with non-isolated point $\infty$, 
\[ c_{\mathbb{F}_0}^{R(x) \times \{\infty\}}(R[\mathcal{F}] \times \mathcal{N}(\infty)) \leq \alpha \lor 0 = \alpha, \]
and Theorem 13 applies to the effect that that $c_{\mathbb{D}}^{R(x)} R[\mathcal{F}] \leq \alpha$. \hfill \Box

**Remark 15 (on infinite products).** [19, Theorem 14] provides a Tychonoff Theorem for the general measure $c_{\mathbb{D}}^{A}(\mathcal{F})$ as defined in [2]. However, there is an obvious error in the proof tantamount to writing that 
\[ \bigvee_{i \in I} a_i + \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a_i + b_i) \]
which is obviously false. This problem disappears when $A$ is restricted to a family of subsets, rather than functions. In this case, $\bigvee_{i \in I} A_i$ becomes $\prod_{i \in I} A_i$. Thus [19, Theorem 14] should read:

**Theorem 16.** Let $(X_i, \lambda_i)_{i \in I}$ be a family of convergence approach spaces, let $A_i \subset 2^{X_i}$, and let $\mathcal{F}$ be a filter on $\prod_{i \in I} X_i$. Then 
\[ c_{\prod_{i \in I} A_i}(\mathcal{F}) = \bigvee_{i \in I} c_{A_i}(p_i[\mathcal{F}]), \]
where $p_i : \prod_{i \in I} X_i \to X_i$ is the $i^{th}$-projection.

4. **Applications**

Taking $\mathcal{F} = \{X\}$, $A = X$, $\mathcal{G} = \{Y\}$, $B = Y$ in Theorem 13, we obtain an extension to Cap in terms of measure of compactness of the classical topological fact that a product of a compact space with a space that is respectively compact, countably compact, or Lindelöf is also compact, countably compact, or Lindelöf, respectively:

**Corollary 17.** Let $\mathbb{D}$ be a composable class of filters containing principal filters, and let $X$ and $Y$ be two convergence approach spaces.
\[ c_{\mathbb{D}}(X \times Y) \leq c_{\mathbb{D}}X \lor c_{\mathbb{D}}Y. \]

In particular, (for $\mathbb{D} = \mathbb{F}_1$) the measure of countable compactness (in the sense of [2]) of a product of two spaces is not larger than the supremum of the measure of countable compactness and measure of compactness of the factors. Similarly, for $\mathbb{D} = \mathbb{F}_{\Lambda_1}$, the measure of Lindelöf (in the sense of $\Pi$) of a product of two spaces is not larger than the supremum of the measure of Lindelöf and measure of compactness of the factors.

Both instances appear to be new, even if they are probably part of the folklore on approach spaces.

On the other hand, applying Theorem 13 with $\mathcal{F} = \{X\}$, $\alpha = 0$, and $A = X$, yields the following generalization of the Kuratowski-Mrówka characterization of compactness:

**Corollary 18.** Let $\mathbb{D}$ be a composable class of filters containing principal filters, and let $X$ be a convergence approach space. Then the following are equivalent:
(1) $X$ is $\mathcal{D}$-compact;
(2) For every $\mathcal{D}$-based convergence approach space $(Y, \lambda_Y)$, $p_Y : X \times Y \to Y$ is $\mathcal{D}$-perfect;
(3) For every atomic $\mathcal{D}$-based atomic topological (approach) space $Y$, $p_Y : X \times Y \to Y$ is closed.

Proof. (1 $\Rightarrow$ 2) : To see that $p_Y : X \Rightarrow X \times Y$ is $\mathcal{D}$-compact, we need to show that $c_{\mathcal{D}}^{X \times \{y\}}(X \times G) \leq \lambda_Y(G)(y)$ for any $G \in \mathcal{F}Y$ and $y \in Y$. For each $D \in \mathcal{D}$ with $D \leq G$, Theorem 13 applies for $\mathcal{F} = \{X\}$, $A = \{X\}$, $G = D$, $B = \{y\}$ and $\alpha = c_{\mathcal{F}}^{X}(X) = 0$ to the effect that $c_{\mathcal{D}}^{X \times \{y\}}(X \times G) \leq c_{\mathcal{D}}^{X \times \{y\}}(X \times D) \leq c_{\mathcal{F}}^{\{y\}}(D) \leq \lambda_Y(D)(y)$ so that

$c_{\mathcal{F}}^X(D) = \bigwedge_{D \supseteq \mathcal{D} \subseteq G} \lambda_Y(D)(y) = \lambda_Y(G)(y).

(2 $\Rightarrow$ 3) is clear, and (3 $\Rightarrow$ 1) because (3) means that for every $\mathcal{D}$-based atomic topological approach space $Y$, with non-isolated point $\infty$, $c_{\mathcal{F}_0}^{X \times \{\infty\}}(X \times \mathcal{N}(\infty)) \leq c_{\mathcal{F}}^{\{\infty\}}(\mathcal{N}(\infty)) = 0$ so that Theorem 13 applies to the effect that $c_{\mathcal{D}}^{X}X = 0$. □

In particular, when $\mathcal{D}$ ranges over the classes $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_{\Lambda}$ respectively, we obtain the instances below. By analogy with the case of topological spaces, we call first-countable an $\mathcal{F}_1$-based convergence approach space, and a $P$-convergence approach space one that is $\mathcal{F}_{\Lambda}$-based. On the other hand, we call an $\mathcal{F}_0$-based convergence approach space finitely generated, because a pre-approach space is finitely generated in the sense of [9] if and only if it is finitely generated in this sense.

Corollary 19. Let $(X, \lambda_X)$ be a convergence approach space. Then the following are equivalent:

1. $X$ is compact;
2. For every convergence approach space $(Y, \lambda_Y)$, $p_Y : X \times Y \to Y$ is perfect;
3. For every atomic topological (approach) space $Y$, $p_Y : X \times Y \to Y$ is closed.

Corollary 20. Let $(X, \lambda_X)$ be a convergence approach space. The following are equivalent:

1. $X$ is countably compact;
2. For every first-countable $\mathcal{F}$ convergence approach space $(Y, \lambda_Y), p_Y : X \times Y \to Y$ is countably perfect;
3. For every first-countable atomic topological (approach) space $Y$, $p_Y : X \times Y \to Y$ is closed.

Corollary 21. Let $(X, \lambda_X)$ be a convergence approach space. The following are equivalent:

2 As noted in [17] in the context of Conv, we could extend this to spaces based in contours of countably based filters, but this is of little importance here. In the case of topological spaces, they turn out to be exactly subsequential spaces, that is, subspaces of sequential spaces. See [17] for details.
(1) $X$ is Lindelöf;
(2) For every $P$-convergence approach space $(Y, \lambda_Y), p_Y : X \times Y \to Y$ is inversely Lindelöf;
(3) For every atomic topological $P$-space $Y$ (seen as an approach space) $p_Y : X \times Y \to Y$ is closed.

On the other hand, Theorem 14 combined with Proposition 8 readily gives:

**Corollary 22.** Let $\mathbb{D}$ be a composable class of filters containing principal filters. Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \to Y$. Then the following are equivalent:

1. $f$ is $\mathbb{D}$-perfect;
2. For every $\mathbb{D}$-based convergence approach space $Z$, $f \times \text{Id}_Z : X \times Z \to Y \times Z$ is $\mathbb{D}$-perfect;
3. For every atomic topological $\mathbb{D}$-based approach space $Z, f \times \text{Id}_Z : X \times Z \to Y \times Z$ is closed.

Similarly, Theorem 14 combines with Theorem 11 to the effect that:

**Corollary 23.** Let $\mathbb{D}$ be a composable class of filters containing principal filters. Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \to Y$ be a surjective map. Then the following are equivalent:

1. $f$ is $\mathbb{D}$-quotient;
2. For every $\mathbb{D}$-based convergence approach space $Z$, $f \times \text{Id}_Z : X \times Z \to Y \times Z$ is $\mathbb{D}$-quotient;
3. For every atomic topological $\mathbb{D}$-based approach space $Z, f \times \text{Id}_Z : X \times Z \to Y \times Z$ is hereditarily quotient.

In particular, when $\mathbb{D} = \mathbb{F}$ is the class of all filters, we obtain:

**Corollary 24.** Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \to Y$ be a surjective map. Then the following are equivalent:

1. $f$ is biquotient;
2. For every convergence approach space $Z, f \times \text{Id}_Z : X \times Z \to Y \times Z$ is biquotient;
3. For every atomic topological convergence approach space $Z, f \times \text{Id}_Z : X \times Z \to Y \times Z$ is hereditarily quotient.

**Corollary 25.** Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \to Y$. Then the following are equivalent:

1. $f$ is perfect;
2. For every convergence approach space $Z, f \times \text{Id}_Z : X \times Z \to Y \times Z$ is perfect;
3. For every atomic topological approach space $Z, f \times \text{Id}_Z : X \times Z \to Y \times Z$ is closed.

In [5], Lowen and al. call a map $f : X \to Y$ between two approach spaces proper if $f \times \text{Id}_Z$ is closed for every approach space $Z$. In view of Corollary 24, our perfect maps extend to Cap the concept of proper maps of [5]. Additionally, the equivalence between (1) and (2) in [5, Proposition 3.3] states that a map between two approach spaces is proper if and only if it is closed and has compact fibers (0-compact in the terminology of
Theorem 9 for $D = \mathbb{F}$ and Corollary 26 recover this equivalence, and delineate the conditions of an extension of this result to $\text{Cap}$ (namely, $X$ needs to remain an approach space, but $Y$ can be an arbitrary convergence approach space). At any rate, the proper (no pun intended) notion yielding a characterization of maps whose product with every identity map is closed (in $\text{Cap}$ and not only $\text{Ap}$) appears to be that of perfect maps, which ultimately depends on that of compact relation. That the condition reduces to the closedness of the map and compactness of the fibers is specific to $\text{Ap}$, as shows Theorem 9.

Maybe more importantly, the viewpoint in terms of $D$-compact relations unveils the relationships between similar characterizations in terms of products of variants of perfect maps on one hand (Corollary 22) and variants of quotient maps on the other hand (Corollary 23) as two instances of the same result. While this was already observed in [17] in the concept of $\text{Conv}$, it is remarkable that this turns out to extend fully to $\text{Cap}$.

On the other hand, letting $D$ range over other classes ($\mathbb{F}_1$, $\mathbb{F}_{\Lambda 1}$, $\mathbb{F}_0$) yields other variants of Corollaries 24 and 25:

**Corollary 26.** Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \rightarrow Y$ be a surjective map. Then the following are equivalent:

1. $f$ is countably biquotient;
2. For every first-countable convergence approach space $Z$, $f \times \text{Id}_Z : X \times Z \rightarrow Y \times Z$ is countably biquotient;
3. For every atomic first-countable topological approach space $Z$, $f \times \text{Id}_Z : X \times Z \rightarrow Y \times Z$ is hereditarily quotient.

**Corollary 27.** Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \rightarrow Y$. Then the following are equivalent:

1. $f$ is countably perfect;
2. For every first-countable convergence approach space $Z$, $f \times \text{Id}_Z : X \times Z \rightarrow Y \times Z$ is countably perfect;
3. For every atomic first-countable topological approach space $Z$, $f \times \text{Id}_Z : X \times Z \rightarrow Y \times Z$ is closed.

**Corollary 28.** Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \rightarrow Y$. Then the following are equivalent:

1. $f$ is weakly biquotient;
2. For every $P$-convergence approach space $Z$, $f \times \text{Id}_Z : X \times Z \rightarrow Y \times Z$ is weakly biquotient;
3. For every atomic topological approach $P$-space $Z$, $f \times \text{Id}_Z : X \times Z \rightarrow Y \times Z$ is hereditarily quotient.

**Corollary 29.** Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \rightarrow Y$. Then the following are equivalent:

1. $f$ is inversely Lindelöf;
2. For every $P$-convergence approach space $Z$, $f \times \text{Id}_Z : X \times Z \rightarrow Y \times Z$ is inversely Lindelöf;
(3) For every atomic topological approach $P$-space $Z$, $f \times Id_Z : X \times Z \to Y \times Z$ is closed.

**Corollary 30.** Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \to Y$ be a surjective map. Then the following are equivalent:

1. $f$ is hereditarily biquotient;
2. For every finitely generated convergence approach space $Z$, $f \times Id_Z : X \times Z \to Y \times Z$ is hereditarily quotient;
3. For every atomic topological finitely generated approach space $Z$, $f \times Id_Z : X \times Z \to Y \times Z$ is hereditarily quotient.

**Corollary 31.** Let $(X, \lambda_X)$ and $(Y, \lambda_Y)$ be two convergence approach spaces, and let $f : X \to Y$. Then the following are equivalent:

1. $f$ is closed;
2. For every finitely generated convergence approach space $Z$, $f \times Id_Z : X \times Z \to Y \times Z$ is closed;
3. For every atomic finitely generated topological approach space $Z$, $f \times Id_Z : X \times Z \to Y \times Z$ is closed.

Finally, let us note that applying Theorem 13 for $A = \{x\}$ and $B = \{y\}$, yields, via Theorem 1, the following extension to $\text{Cap}$ of [17, Theorem 8]:

**Corollary 32.** Let $\mathbb{D}$ be a composable class of filters containing principal filters. Let $(X, \lambda_X)$ be a convergence approach space and let $\lambda_2$ be another convergence approach structure on $X$. The following are equivalent:

1. $\lambda_2 \geq \text{Adh}_\mathbb{D}\lambda_X$;
2. For every $\mathbb{D}$-based convergence approach space $(Y, \lambda_Y)$,
   \[ \text{Adh}_\mathbb{D}(\lambda_X \times \lambda_Y) \leq \lambda_2 \times \text{Adh}_\mathbb{F}\lambda_Y \]
3. For every $\mathbb{D}$-based atomic topological approach space $(Y, \lambda_Y)$,
   \[ \text{Adh}_{\mathbb{F}_0}(\lambda_X \times \lambda_Y) \leq \lambda_2 \times \lambda_Y. \]

The significance of this type of results appears fully in the context of modified duality as developed in [15, 16]. For instance, when $\mathbb{D}$ is the class of all filters, ($1 \implies 2$) simply shows that the reflector on pseudo-approach spaces $\text{Adh}_\mathbb{F}$ commutes with (finite) products. As a result $\text{Prap}$ is cartesian-closed. More importantly, since $\text{Adh}_{\mathbb{F}_0}$ is the projector on $\text{Prap}$, ($3 \implies 1$) shows (see [15, 16] for details) that $\text{Psap}$ is the cartesian-closed hull of $\text{Prap}$, which is [8, Theorem 5.9]. See the aforementioned references for details and other applications of results akin to Corollary 32.

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