The 1-Yamabe equation on graph

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Abstract

We study the following 1-Yamabe equation on a connected finite graph

$$\Delta_1 u + g \text{Sgn}(u) = h |u|^{\alpha - 1} \text{Sgn}(u),$$

where $\Delta_1$ is the discrete 1-Laplacian, $\alpha > 1$ and $g, h > 0$ are known. We show that the above 1-Yamabe equation always has a nontrivial solution $u \geq 0$, $u \neq 0$.

0 Introduction

Let $(M^n, g)$ be a smooth $n \geq 3$ dimensional Riemannian manifold with scalar curvature $R$. The well known smooth Yamabe problem asks if there exists a smooth Riemannian metric $\tilde{g}$ that conformal to $g$ and has constant scalar curvature $\tilde{R}$. This leads to the consideration of the following smooth Yamabe equation

$$\Delta u - \frac{n-2}{4(n-1)} R u + \frac{n-2}{4(n-1)} \tilde{R} u^{\frac{n+2}{n-2}} = 0, \quad u > 0.$$ 

If $u \in C^\infty(M)$ is a solution, then $\tilde{g} = u^{4/(n-2)} g$ has scalar curvature $\tilde{R}$. Today the Yamabe problem is completely understood thanks to basic contributions by Yamabe, Trudinger, Aubin and Schoen. We refer to [1], [21]-[24], the survey [20] and the references therein.

In case $n = 2$, the Yamabe problem reduces to the famous uniformization theorem, which asks for the consideration of the Kazdan-Warner equation

$$\Delta u = K - \tilde{K} e^{2u}$$

If $u \in C^\infty(M)$ is a solution, then $\tilde{g} = e^{2u} g$ has Gaussian curvature $\tilde{K}$. We refer to [14]-[16] for the solvability of the smooth Kazdan-Warner equation.

In the series work [11]-[13], Grigor’yan-Lin-Yang observed that one can establish similar results on graphs. In [11], they studied the Kazdan-Warner equation $\Delta u = c - he^n$ on a finite graph $G$, and gave various conditions such that the equation has a solution. They
characterize the solvability of the equation completely except for the critical case which was finally settled down by the first author Ge [6] of this paper. Ge [8], Zhang-Chang [25] then generalized their results to the $p$-th Kazdan-Warner equation $\Delta_p u = c - he^u$, where $\Delta_p$ is a type of discrete $p$-Laplace operator, see (1.4) in this paper for a definition. On infinite graphs which are analogues of noncompact manifolds, the absence of compactness indicates the difficulty to give a complete characterization of the solvability of the corresponding equation. Under some additional assumptions on the graphs and functions, Ge-Jiang [10], Keller-Schwarz [17] get some results with totally different techniques and assumptions. In [12], Grigor’yan-Lin-Yang studied the Yamabe type equation $-\Delta u - \alpha u = |u|^{p-2}u$ on a finite domain $\Omega$ of an infinite graph, with $u = 0$ outside $\Omega$. Ge [7] studied the $p$-th Yamabe equation $\Delta_p u + hu^{p-1} = \lambda f u^{\alpha-1}$ on a finite graph under the assumption $\alpha \geq p > 1$, which was generalized to $\alpha < p$ by Zhang-Lin [26]. Ge-Jiang [9] further generalized Ge’s results to get a global positive solution on an infinite graph under suitable assumptions.

In case $p > 1$, the $p$-Laplace operator $\Delta_p$ exhibits more or less similar properties with the standard Laplace operator $\Delta$, both on the smooth manifolds and graphs. However, the 1-Laplace operator $\Delta_1$ looks much different. The solutions for equations involving $\Delta$ and $\Delta_p$ are smooth, while those for $\Delta_1$ may be discontinuous. Since solutions in many interesting problems, e.g., in the signal processing and in the image processing, etc., may be discontinuous, the 1-Laplace operator $\Delta_1$ has been received much attention in recent years. In this paper, we study the 1-th Yamabe equation on a finite graph. Different with the study of the $p$-th equations, variational methods can not be used here directly. The main idea of the proof is to take $\Delta_1$ as in some sense the limit of $\Delta_p$ as $p \to 1$.

1 Settings and main results

Let $G = (V, E)$ be a finite graph, where $V$ denotes the vertex set and $E$ denotes the edge set. Fix a vertex measure $\mu : V \to (0, +\infty)$ and an edge measure $w : E \to (0, +\infty)$ on $G$. The edge measure $w$ is assumed to be symmetric, that is, $w_{xy} = w_{yx}$ for each edge $x \sim y$. Let $C(V)$ be the set of all real functions defined on $V$, then $C(V)$ is a finite dimensional linear space with the usual function additions and scalar multiplications. Professor Chang Kung-Ching (see [3], or [4, 5]) introduced a 1-Laplace operator on graphs as the following

$$\Delta_1 f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \text{Sgn}(f(y) - f(x)) \quad (1.1)$$
for any $f \in C(V)$ and $x \in V$, in which

$$\text{Sgn}(t) = \begin{cases} 
1 & \text{if } t > 0 \\
-1 & \text{if } t < 0 \\
[-1, 1] & \text{if } t = 0
\end{cases}$$

(1.2)
is a set valued function.

**Remark 1.** The addition of two subsets $A, B \subset \mathbb{R}^n$ is the set $\{x + y|x \in A, y \in B\}$, and for a scalar $\alpha$, the scalar multiplication $\alpha A$ is the set $\{\alpha x|x \in A\}$.

In this paper, our main task is to study the following 1-th Yamabe equation

$$\Delta_1 u + g \text{Sgn}(u) = h|u|^\alpha \text{Sgn}(u),$$

(1.3)
with $\alpha > 1$, and $g, h \in C(V)$ be positive.

Note at each vertex $x \in V$, $\Delta_1 f(x)$ is a subset of $\mathbb{R}$. For each function $u \in C(V)$, set

$$A^+(x) = -\Delta_1 u(x) + g(x)\text{Sgn}(u(x))$$

and

$$A^-(x) = h(x)|u(x)|^{\alpha-1}\text{Sgn}(u(x)).$$

We say $u$ is a solution of the 1-th Yamabe equation (1.3) if at each vertex $x \in V$,

$$A^+(x) \cap A^-(x) \neq \emptyset.$$  

Obviously, if $u = 0$ everywhere on $V$, both $A^+(x)$ and $A^-(x)$ contains 0 as an element. Hence $u = 0$ is always a trivial solution of the 1-th Yamabe equation (1.3). We want to know if there is a nontrivial solution to (1.3). Our main result reads as

**Theorem 1.1.** Let $\alpha > 1$, $g, h \in C(V)$ be positive. The 1-th Yamabe equation (1.3) has a nontrivial solution $u \geq 0$, $u \neq 0$.

For any $p > 1$, the $p$-th discrete graph Laplace operator $\Delta_p : C(V) \to C(V)$ is

$$\Delta_p f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}|f(y) - f(x)|^{p-2}(f(y) - f(x))$$

(1.4)
for any $f \in C(V)$ and $x \in V$. If $f(y) = f(x)$, we require $|f(y) - f(x)|^{p-2}(f(y) - f(x)) = 0$. $\Delta_p$ is a nonlinear operator when $p \neq 2$ (see [8] for more properties about $\Delta_p$).

The main idea of the proof is to take $\Delta_1$ as in some sense the limit of $\Delta_p$ as $p \to 1$. Hence we shall first establish an existence result of the positive solution $u_p$ to the following $p$-th Yamabe equation

$$-\Delta_p u_p + g u_p^{p-1} = h u_p^{\alpha-1}$$

(1.5)
for each $p > 1$. By proving a uniform bound for $u_p$, and taking limit $p \to 1$, we get a nontrivial solution to the 1-th Yamabe equation (1.3).

It is remarkable that one can see other interesting phenomenons when $p \to 1$. There are many such works on smooth domains (see Kawohl-Fridman [18] and Kawohl-Schuricht [19]) and on graphs (see Chang [34, 41, 5]).

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2  Sobolev embedding

For any $f \in C(V)$, define the integral of $f$ over $V$ with respect to the vertex weight $\mu$ by

$$\int_V f \, d\mu = \sum_{x \in V} \mu(x) f(x).$$

Set $\text{Vol}(G) = \int_V d\mu$. Similarly, for any function $\psi$ defined on the edge set $E$, we define the integral of $\psi$ over $E$ with respect to the edge weight $w$ by

$$\int_E \psi \, dw = \sum_{x \sim y} w_{xy} \psi_{xy}.$$  

Specially, for any $f \in C(V)$,

$$\int_E |\nabla f|^p \, dw = \sum_{x \sim y} w_{xy} |f(y) - f(x)|^p,$$

where $|\nabla f|$ is defined on the edge set $E$, and $|\nabla f|_{xy} = |f(y) - f(x)|$ for each edge $x \sim y$.

Next we consider the Sobolev space $W^{1,p}$ on the graph $G$. Define

$$W^{1,p}(G) = \left\{ f \in C(V) : \int_E |\nabla f|^p \, dw + \int_V |f|^p \, d\mu < +\infty \right\},$$

and

$$\|f\|_{W^{1,p}(G)} = \left( \int_E |\nabla f|^p \, dw + \int_V |f|^p \, d\mu \right)^{\frac{1}{p}}.$$  

Since $G$ is a finite graph, then $W^{1,p}(G)$ is exactly $C(V)$, a finite dimensional linear space. This implies the following Sobolev embedding:

**Lemma 2.1.** (Sobolev embedding) Let $G = (V,E)$ be a finite graph. The Sobolev space $W^{1,p}(G)$ is pre-compact. Namely, if $\{\varphi_n\}$ is bounded in $W^{1,p}(G)$, then there exists some $\varphi \in W^{1,p}(G)$ such that up to a subsequence, $\varphi_n \to \varphi$ in $W^{1,p}(G)$.

**Remark 2.** The convergence in $W^{1,p}(G)$ is in fact pointwise convergence.
3 The $p$-th equation has a positive solution $u_p$

Our first main technical observation is the following.

**Theorem 3.1.** Let $G$ be a finite connected graph. Assume $\alpha \geq p > 1$, $g, h > 0$. Then the following $p$-th Yamabe equation

$$-\Delta_p u + g|u|^{p-1} = \lambda h|u|^\alpha - 1 \quad (3.1)$$

on $G$ always has a positive solution $u$ for some constant $\lambda > 0$. Moreover, $u$ satisfies

$$\int_V h\varphi^\alpha d\mu = 1. \quad (3.2)$$

The first part of the theorem was established by Ge [7]. The second part of the theorem is a consequence of the proof of the main theorem of [7]. For completeness, we give a direct and great simplified proof here. It is remarkable that the condition $g > 0$ is not needed so as (3.1) to have a positive solution. $g > 0$ is used here (in fact, $g \geq 0$ and $g \neq 0$ is enough) to guarantee that $\lambda > 0$.

**Proof.** We minimize the nonnegative functional (note $g \geq 0$)

$$I(\varphi) = \int_E |\nabla \varphi|^p d\nu + \int_V g\varphi^p d\mu$$

in the non-empty set

$$\Gamma = \left\{ \varphi \in W^{1,p}(G) : \int_V h\varphi^\alpha d\mu = 1, \varphi \geq 0 \right\}.$$

Let

$$\beta = \inf_{\varphi \in \Gamma} I(\varphi).$$

Take a sequence of functions $u_n \in \Gamma$ such that $I(u_n) \to \beta$. Obviously, $\{u_n\}$ is bounded in $W^{1,p}(G)$. Therefore by the Sobolev embedding Lemma 2.1, there exists some $u \in C(V)$ such that up to a subsequence, $u_n \to u$ in $W^{1,p}(G)$. We may well denote this subsequence as $u_n$. Because the set $\Gamma$ is closed, we see $u \in \Gamma$, that is, $u \geq 0$, and

$$\int_V h\varphi^\alpha d\mu = 1. \quad (3.2)$$

Based on the method of Lagrange multipliers, one can consider the nonrestraint minimization of the following functional

$$J(\varphi) = I(\varphi) + \gamma \left( \int_V h\varphi^\alpha d\mu - 1 \right)$$
and calculate the Euler-Lagrange equation of $u$ as follows:

$$- \Delta p u + gu^{p-1} + \frac{\gamma \alpha}{p} hu^{\alpha-1} \geq 0$$

(3.3)

where $\gamma$ is a constant. (3.3) implies $u > 0$. In fact, note the graph $G$ is connected, if $u > 0$ is not satisfied, since $u \geq 0$ and not identically zero (this can be seen from (3.2)), there is an edge $x \sim y$, such that $u(x) = 0$, but $u(y) > 0$. Now look at $\Delta p u(x)$,

$$\Delta p u(x) = \frac{1}{\mu(x)} \sum_{z \sim x} w_{xz} |u(z) - u(x)|^{p-2}(u(z) - u(x)) > 0,$$

which contradicts (3.3). Hence $u > 0$ is in the interior of the space $\{ \varphi : \varphi \geq 0 \}$ and hence then (3.3) becomes an equality

$$- \Delta p u + gu^{p-1} = \lambda hu^{\alpha-1},$$

(3.4)

where $\lambda = -\frac{\gamma \alpha}{p}$. Multiplying $u$ at the two sides of (3.4) and integrating, we have

$$\int_E |\nabla u|^p dw + \int_V gu^p d\mu = \int_V (-u \Delta p u d\mu + gu^p) d\mu = \lambda \int_V hu^\alpha d\mu.$$

This leads to

$$\lambda = \frac{\int_E |\nabla u|^p dw + \int_V gu^p d\mu}{\int_V hu^\alpha d\mu},$$

from which we see $\lambda > 0$ and hence the conclusion.

4 The solutions $u_p$ are uniformly bounded

For each $p \in (1, \alpha)$, let $u_p > 0$ be a solution to the $p$-th Yamabe equation (3.1). Thus

$$- \Delta p u_p + gu_p^{p-1} = \lambda_p hu_p^{\alpha-1},$$

(4.1)

where

$$\lambda_p = \frac{\int_E |\nabla u_p|^p dw + \int_V gu_p^p d\mu}{\int_V hu_p^{\alpha} d\mu} > 0.$$  

(4.2)

Moreover,

$$\int_V hu_p^{\alpha} d\mu = 1.$$  

(4.3)

In the following, we use $c(\alpha, h, G)$ as a constant depending only on the information of $\alpha$, $h$ and $G$, use $c(\alpha, g, h, G)$ as a constant depending only on the information of $\alpha, g, h$ and $G$. Note that the information of $G$ contains $V, E$, the vertex measure $\mu$ and the edge weight $w$. For any function $f \in C(V)$, we denote $f_m = \min_{x \in V} f(x)$ and $f_M = \max_{x \in V} f(x)$. 


Lemma 4.1. There are positive constants $c_1(\alpha, h, G) \geq 1$ and $c_2(\alpha, h, G) \leq 1$ so that
\[
c_2(\alpha, h, G) \leq \max_{x \in V} u_p(x) \leq c_1(\alpha, h, G). \tag{4.4}
\]

Proof. The above estimates come from (4.3). For all $p \in (1, \alpha)$ and $x \in V$, by
\[
h(x)u_p^\alpha(x)\mu(x) \leq \int_V h u_p^\alpha d\mu = 1,
\]
we see $u_p(x) \leq (h(x)\mu(x))^{-1/\alpha} \leq (h\mu)^{-1/\alpha} \lor 1 = c_1(\alpha, h, G)$. Let $|V|$ be the number of all vertices, then from
\[
1 = \int_V h u_p^\alpha d\mu \leq (h\mu)_M |V| \max_{x \in V} u_p^\alpha(x)
\]
we see $\max_{x \in V} u_p(x) \geq (h\mu)_M |V|^{-1/\alpha} \lor 1 = c_2(\alpha, h, G). \quad \square$

Lemma 4.2. There are positive constants $c_1(\alpha, g, h, G) \geq 1$ and $c_2(\alpha, g, h, G) \leq 1$ so that
\[
c_2(\alpha, g, h, G) \leq \lambda_p \leq c_1(\alpha, g, h, G). \tag{4.5}
\]

Proof. Assume $u_p$ attains its maximum at $x_0 \in V$, then $\Delta_p u_p(x_0) \leq 0$ by the definition of $\Delta_p$. From (4.11), we have $-\Delta_p u_p(x_0) + g(x_0)u_p^{p-1}(x_0) = \lambda_p h(x_0)u_p^{\alpha-1}(x_0)$. Hence
\[
\lambda_p = \frac{-\Delta_p u_p(x_0) + g(x_0)u_p^{p-1}(x_0)}{h(x_0)u_p^{\alpha-1}(x_0)}
\geq g(x_0)h(x_0)^{\alpha-1}u_p^{\alpha-\alpha}(x_0)
\geq (gh^{-1})m c_1(\alpha, h, G)^{1-\alpha} \lor 1
= c_2(\alpha, g, h, G),
\]
where we have used $u_p^{\alpha-\alpha}(x_0) \geq c_1(\alpha, h, G)^{p-\alpha} \geq c_1(\alpha, h, G)^{1-\alpha}$ in the last inequality.

Similarly, from
\[
|\Delta_p u_p(x_0)| \leq \frac{1}{\mu(x_0)} \sum_{y \sim x_0} w_{x_0 y} |u_p(y) - u_p(x_0)|^{p-1}
\leq \frac{1}{\mu(x_0)} \sum_{y \sim x_0} w_{x_0 y} (2u_p(x_0))^{p-1}
\leq c(\alpha, G)u_p^{p-1}(x_0)
\]
we obtain
\[
\lambda_p = \frac{-\Delta_p u_p(x_0) + g(x_0)u_p^{p-1}(x_0)}{h(x_0)u_p^{\alpha-1}(x_0)}
\leq c(\alpha, g, h, G)u_p^{\alpha-\alpha}(x_0)
\leq c(\alpha, g, h, G)c_2(\alpha, h, G)^{1-\alpha} \lor 1
= c_1(\alpha, g, h, G)
\]
\[7\]
where we have used \( u_p^{p-\alpha}(x_0) \leq c_2(\alpha, h, G)^{p-\alpha} \leq c_2(\alpha, h, G)^{1-\alpha} \) in the last inequality. \( \square \)

**Lemma 4.3.** For every \( p \in (1, \frac{\alpha+1}{2}) \), the following equation

\[
- \Delta_p \hat{u}_p + g\hat{u}_p^{p-1} = h\hat{u}_p^{\alpha-1}
\]  

(4.6)

has a positive solution \( \hat{u}_p \) with \( c_4(\alpha, g, h, G) \leq \hat{u}_p \leq c_3(\alpha, g, h, G) \).

**Proof.** Set \( \hat{u}_p = u_p \lambda_p^{\alpha-1} \), then it is easy to see \( \hat{u}_p \) is a positive solution of (4.6). From the estimates (4.5), we get

\[
c_2^{\alpha+1/2} \leq c_2(\alpha, g, h, G)^{1/p} \leq \lambda_p^{\alpha-1} \leq c_1(\alpha, g, h, G)^{1/p} \leq c_1^{\alpha-1}.
\]

Combining with the estimate (4.4), we get the conclusion. \( \square \)

## 5 Proof of Theorem 1.1

Since \( G \) is finite graph, we can choose a function \( u \in C(V) \) and a sequence \( p_n \downarrow 1 \), so that \( \hat{u}_{p_n} \to u \). Since \( \max \hat{u}_p \) is uniformly bounded by Lemma 4.1 and Lemma 4.3, we see \( u \geq 0 \) and \( u \neq 0 \). We can always choose a subsequence of \( p_n \), which is still denoted as \( \{p_n\} \) itself, a function \( \xi \in C(V) \) and an edge weight \( \eta \) defined on \( E \), so that at each vertex \( x \in V \),

1. \( \hat{u}_{p_n}(x)^{\alpha-1} \to \hat{u}(x)^{\alpha-1} \).
2. \( \hat{u}_{p_n}(x)^{p_n-1} \to \xi(x) \in [0, 1] \).
3. \( |\hat{u}_{p_n}(y) - \hat{u}_{p_n}(x)|^{p_n-2}(\hat{u}_{p_n}(y) - \hat{u}_{p_n}(x)) \to \eta(x, y) \in [-1, 1] \), for \( y \sim x \).

It is easy to see

\[
- \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \eta(x, y) + g(x)\xi(x) \in A^+(x) \quad \text{and} \quad h(x)u(x)^{\alpha-1} \in A^-(x).
\]

Observe the equality

\[
- \Delta_p \hat{u}_p(x) + g(x)\hat{u}_p(x)^{p-1} = h(x)\hat{u}_p(x)^{\alpha-1},
\]

and let \( p_n \to 1 \), we obtain

\[
- \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \eta(x, y) + g(x)\xi(x) = h(x)u(x)^{\alpha-1}.
\]

This shows \( A^+(x) \cap A^-(x) \neq \emptyset \) for every vertex \( x \in V \). That is Theorem 1.1
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