A VARIATIONAL APPROXIMATION SCHEME FOR ELASTODYNAMIC PROBLEMS USING A NEW CLASS OF ADMISSIBLE MAPPINGS

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Abstract. We consider a variational approximation scheme for the 3D elastodynamics problem. Our approach uses a new class of admissible mappings that are closed with respect to the space of mappings with finite distortion.

Key words and phrases: elastodynamics, mapping with finite distortion, polyconvexity, variational approximation scheme.

The motion of a deformable solid body can be written:
\[ \frac{\partial^2 y}{\partial t^2} = \nabla \cdot S(\nabla y), \]
where \( y: \Omega \times [t_0, t_1] \rightarrow \mathbb{R}^3 \) is the displacement and \( S \) is the first Piola–Kirchhoff stress tensor. This equation can also be written as a system of conservation laws for the deformation gradient \( F = D_x y \) and the velocity \( v = \partial_t y \)

\[
\begin{align*}
\frac{\partial}{\partial t} F_{i\alpha} &= \frac{\partial}{\partial x_\alpha} v_i, \\
\frac{\partial}{\partial t} v_i &= \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} S_{i\alpha}(F),
\end{align*}
\]

In the case of hyperelastic materials, the tensor \( S \) can be expressed as the gradient of a scalar function \( W(F) \). This function is called the stored energy function, \( W: \mathbb{R}^{3 \times 3} \rightarrow [0, \infty) \), where \( \mathbb{R}^{3 \times 3} \) stands for \((3 \times 3)\)-matrices, i.e. \( S(F) = \left[ \frac{\partial W}{\partial F_{ij}}(F) \right] \). In order to prove the existence of solutions to the system of equations in (1) it would normally be necessary that \( W \) be convex. However, this is incompatible with the known physics of elastic materials such as the requirement of frame-indifference, i.e. the principle that certain properties of the system are invariant under arbitrary coordinate transformations [4].

This suggests the replacement of the condition of convexity with a weaker condition such as polyconvexity (for further details see, e.g. [1,3]). More precisely, we may assume that
\[ W(F) = G(F, \text{cof} F, \text{det} F) \]

2010 Mathematics Subject Classification. 46E30 and 46E35.

This work was partially supported by a Grant of the Russian Foundation of the Russian Science Foundation (Agreement No. 16-41-02004).
holds for some convex function $G(F, Z, w)$, where $\text{cof } F$ and $\det F$ are the cofactor matrix (i.e. transposed adjunctive matrix $\text{cof } F = \text{adj } F^T$) and determinant of the matrix $F$ respectively.

At present, the question of the existence of a solution to the elastostatic problem has been thoroughly studied. A review of basic works and open problems can be found, for example, in [2]. Furthermore, the reader is referred to [5] for local existence of the classical solution of the elastodynamical system with rank-one convex and polyconvex stored energy functions. The existence of global weak solutions, excluding some particular cases [8], is still an open problem. Nevertheless, the existence of a global measure-valued solution was proven in [7] using a variational approximation scheme.

The variational approximation method or, in the nomenclature of E. De Giorgi [6], the minimizing movements method, is a method by which the limit of a minimizing sequence of iterations to the variational problem for an appropriate functional is found. The technique developed in [7] uses the variational approximation scheme to establish a link between elastostatics and elastodynamics. The method is based on the observation that the solution of the system (1) meets the additional conservation laws (the idea was suggested independently by P. G. Le Floch and T. Qin [12])

\[
\frac{\partial}{\partial t} \det F = \sum_{i, \alpha = 1}^{3} \frac{\partial}{\partial x_{\alpha}} ((\text{cof } F)_{i\alpha} v_{i}),
\]

\[
\frac{\partial}{\partial t} (\text{cof } F)_{k\gamma} = \sum_{i, j, \alpha, \beta = 1}^{3} \frac{\partial}{\partial x_{\alpha}} (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta} v_{i}),
\]

where $\epsilon_{ijk}$ is the permutation symbol. Then, introducing new variables $Z = \text{cof } F$, $w = \det F$ and setting $\Xi = (F, Z, w)$, we can use (1) together with (2) to derive the enlarged system

\[
\partial_{t} v_{i} = \sum_{\alpha, A} \partial_{\alpha} \left( \frac{\partial G}{\partial \Xi^{A}} \frac{\partial \Phi^{A}}{\partial F_{i\alpha}(F)} \right) = \sum_{\alpha} \partial_{\alpha} (g_{\alpha \gamma}(\Xi; F)),
\]

\[
\partial_{t} \Xi^{A} = \sum_{i, \alpha} \partial_{\alpha} \left( \frac{\partial \Phi^{A}}{\partial F_{i\alpha}} (F) v_{i} \right),
\]

where

\[
g_{\alpha \gamma}(F, Z, w; F^0) = \frac{\partial G}{\partial F_{i\alpha}} + \sum_{j, k, \beta, \gamma} \frac{\partial G}{\partial Z_{k\gamma}} \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta}^{0} + \frac{\partial G}{\partial w} (\text{cof } F^{0})_{i\alpha}.
\]

and $\Phi(F) = (F, \text{cof } F, \det F)$. Further, time discretization gives the variational problem

\[
\min_{C} \int_{\Omega} \frac{1}{2} (v - v^{0})^{2} + G(F, Z, w) \, dx,
\]
with initial data $v^0(x)$, $F^0(x)$, $Z^0(x)$, $w^0(x)$, and the admissible set involves the “additional” constraints

$$
C = \{(v, F, Z, w) \in L_2(\Omega) \times L_p(\Omega) \times L_q(\Omega) \times L_r(\Omega), p > 4, q, r \geq 2 : \}
$$

$$
\begin{align*}
\frac{1}{h} (F_{i\alpha} - F_{i0}^0) &= \partial_{\alpha} v_i, \\
\frac{1}{h} (Z_{k\gamma} - Z_{k0}^0) &= \sum_{i,j,\alpha,\beta} \epsilon_{ijk} \partial_{\alpha} (\epsilon_{\alpha\beta\gamma} F_{j\beta}^0 v_i), \\
\frac{1}{h} (w - w^0) &= \sum_{i,\alpha} \partial_{\alpha} ((\text{cof} F^0)_{i\alpha} v_i) 
\end{align*}
$$

In this article, we consider the variational approximation scheme in a new class of admissible mappings, in function classes stemming from quasi-conformal analysis, and derive the Euler–Lagrange equations in the cases of smaller regularity ($p \geq 3$), finite distortion ($|F|_3 \leq Mw$) and nonnegative Jacobian ($w \geq 0$) requirements. Recall that a mapping $f: \Omega \to \mathbb{R}^n$ is called the mapping with finite distortion, $f \in \text{FD}(\Omega)$, if $f \in W^{1,1}_{\text{loc}}(\Omega)$, $J(x, f) \geq 0$ almost everywhere (henceforth abbreviated as a.e.) in $\Omega$ and

$$
|Df(x)|^p \leq K(x) J(x, f) \quad \text{a.e. in } \Omega,
$$

where $1 \leq K(x) < \infty$ a.e. in $\Omega$ (see for example [9]). We also note that the problem of the approximation preserving the constraint $\det F > 0$ is still open, except for the very special case of radial elastodynamics [10]. This condition on the deformation gradient is necessary to ensure that the mappings representing motion are orientation-preserving i.e. that the deformations are physical.

For the sake of simplicity we will work with periodic boundary conditions, i.e. the domain $\Omega$ is taken to be a three dimensional torus. Consider the stored energy function $W: \Omega \times M^3 \to \mathbb{R}$ with the following properties:

**(H1) Polyconvexity:** there exists a convex $C^2$-function $G: M^3 \times M^3 \times \mathbb{R}_+ \to \mathbb{R}$ such that for all $F \in M^3$, $\det F \geq 0$, the equality

$$
G(F, \text{cof} F, \det F) = W(F)
$$

holds.

**(H2) Coercivity:** there are constants $C_1 > 0, C_2 \in \mathbb{R}, p \geq 3, q, r \geq 2$ such that

$$
G(F, Z, w) \geq C_1 \left( |F|^p + |Z|^q + w^r \right) + C_2.
$$

**(H3)** There is a constant $c > 0$ such that

$$
G(F, Z, w) \leq c(|F|^p + |Z|^q + w^r + 1).
$$
(H4) There is a constant $C > 0$ such that the inequality

$$|\partial F G|^p + |\partial Z G|^p + |\partial w G|^p \leq C(|F|^p + |Z|^q + w^r + 1)$$

holds for $p' = \frac{p}{p-1}$ if $p > 3$ and $p' = \frac{3}{2}$ if $p = 3$.

Then the iteration scheme is constructed by solving

$$\frac{v_i^j - v_i^{j-1}}{h} = \sum_{\alpha, A} \partial_\alpha \left( \frac{\partial G}{\partial \Xi_A} (\Xi^{J-1}) \frac{\partial \Phi_A}{\partial F_{\alpha}} (F^{J-1}) \right),$$

$$\frac{(\Xi^j - \Xi^{j-1})A}{h} = \sum_{i, \alpha} \partial_\alpha \left( \frac{\partial \Phi_A}{\partial F_{\alpha}} (F^{J-1}) v_i^j \right).$$

The $J$-th iterates are given by

$$(v^J, \Xi^J) = (v^J, F^J, Z^J, w^J) = (S_h)^J(v^0, F^0, Z^0, w^0),$$

where a solution operator $S_h$ is defined by

(4a) $$\frac{1}{h} (v_i - v_i^0) = \sum_{\alpha} \partial_\alpha g_{i\alpha} (F, Z, w; F^0),$$

(4b) $$\frac{1}{h} (F_{i\alpha} - F_{i\alpha}^0) = \partial_\alpha v_i,$$

(4c) $$\frac{1}{h} (Z_{k\gamma} - Z_{k\gamma}^0) = \sum_{i,j,\alpha,\beta} \partial_\alpha (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta}^0 v_i),$$

(4d) $$\frac{1}{h} (w - w^0) = \sum_{i,\alpha} \partial_\alpha ((\text{cof } F^0)_{i\alpha} v_i).$$

Given $M \in L^s(\Omega), s > 2$, consider the space $X = L_2(\Omega) \times L_p(\Omega) \times L_q(\Omega) \times L_r(\Omega)$ and the set of admissible mappings

$$\mathcal{A} = \{ (v, F, Z, w) \in X, I(v, F, Z, w) < \infty, |F(x)|^3 \leq M(x)^w(x) \text{ a.e. in } \Omega, w(x) \geq 0 \text{ a.e. in } \Omega, \text{ and for every } \theta \in C_0^\infty(\Omega, \mathbb{R}^3) \}$$

\begin{align*}
\int_{\Omega} \frac{1}{h} (F_{i\alpha} - F_{i\alpha}^0) dx &= - \int_{\Omega} v_i \partial_\alpha \theta dx, \\
\int_{\Omega} \frac{1}{h} (Z_{k\gamma} - Z_{k\gamma}^0) dx &= - \int_{\Omega} \sum_{i,j,\alpha,\beta} \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta}^0 v_i \partial_\alpha \theta dx, \\
\int_{\Omega} \frac{1}{h} (w - w^0) dx &= - \int_{\Omega} \sum_{i,\alpha} (\text{cof } F^0)_{i\alpha} v_i \partial_\alpha \theta dx.
\end{align*}

Let the initial data satisfy $y^0 = y(0) \in W_1^1(\Omega) \cap FD(\Omega), v^0 = \partial_t y(0) \in L_2(\Omega), F^0 = Dy^0 \in L_p(\Omega), Z^0 = \text{cof } Dy^0 \in L_q(\Omega), w^0 = \det Dy^0 \in L_r(\Omega), |F^0(x)|^3 \leq M(x)^w(x),$
\( \nu^0(x) \geq 0 \) a.e. in \( \Omega \) and
\[
\int_{\Omega} \frac{1}{2}(\nu^0)^2 + G(F^0, Z^0, w^0) \, dx < \infty.
\]

It is easy to see that the following assertions hold.

**Lemma 1.** The admissible set \( A \) is nonempty.

**Lemma 2.** The admissible set \( A \) is invariant with respect to the relations
\[
\sum_{\alpha} \partial_{\alpha} Z_{i\alpha} = 0,
\]
\[
\partial_{\beta} F_{i\alpha} - \partial_{\alpha} F_{i\beta} = 0.
\]

In particular, if \( F^0 \) is a differential then so is \( F \), and, thus, there exists the mapping
\( y \in W^{1}_p(\Omega) \) such that \( \partial_{\alpha} y_i = F_{i\alpha} \).

Consider the minimization problem for the functional
\[
I(v, F, Z, w) = \int_{\Omega} \frac{1}{2}|v - v^0|^2 + G(F, Z, w) \, dx.
\]

**Theorem 1.** There exists \( (v, F, Z, w) \in A \) satisfying
\[
I(v, F, Z, w) = \inf_A I(v', F', Z', w').
\]

Furthermore, if \( G \) is a strictly convex function then the minimizer \( (v, F, Z, w) \in A \) is unique.

The proof of Theorem 1 is based on the next theorem.

**Theorem 2.** Let \( \{(v_n, F_n, Z_n, w_n)\}_{n \in \mathbb{N}} \subset A \) and \( S = \sup \{ I(v_n, F_n, Z_n, w_n) : n \in \mathbb{N} \} < \infty \). Then there exist \( (v, F, Z, w) \in X \) and a subsequence \( (v_{\mu}, F_{\mu}, Z_{\mu}, w_{\mu}) \) such that
\[
\begin{align*}
  v_{\mu} &\rightharpoonup v & \text{in } L_2(\Omega), \\
  F_{\mu} &\rightharpoonup F & \text{in } L_p(\Omega), \\
  Z_{\mu} &\rightharpoonup Z & \text{in } L_q(\Omega), \\
  w_{\mu} &\rightharpoonup w & \text{in } L_r(\Omega),
\end{align*}
\]

Moreover, \( (v, F, Z, w) \in A \) and
\[
I(v, F, Z, w) \leq \liminf_{n \to \infty} I(v_n, F_n, Z_n, w_n) = s < \infty.
\]

This statement can be proven by applying the techniques and methods of papers [7, 11, 13].

We will now show that the minimizer of (6) over the admissible set \( A \) satisfies the weak form of the system of equations (4). To derive the Euler–Lagrange equations, we assume that the minimizer \( (v, F, Z, w) \) meets \( w(x) \geq \gamma > 0 \) a.e. in \( \Omega \).
We fix “direction” \( \theta = (\theta_1, \theta_2, \theta_3) \in C_0^\infty(\Omega, \mathbb{R}^3) \) such that
\[
|F + hD\theta|^n \leq M\left(w + \sum_{i,\alpha} h \cof F_{i\alpha}^0 \partial_\alpha \theta_i\right) \quad \text{and} \quad \sum_{i,\alpha} \cof F_{i\alpha}^0 \partial_\alpha \theta_i \in L_\infty(\Omega).
\]
For
\[
|\varepsilon| \leq \varepsilon^0 = \frac{\gamma}{\| \sum_{i,\alpha} h \cof F_{i\alpha}^0 \partial_\alpha \theta_i \|_{L_\infty} + 1}
\]
we set
\[
\varepsilon(\delta v_i, \delta F_{i\alpha}, \delta Z_{k\gamma}, \delta w) = \varepsilon\left(\theta_i, h\partial_\alpha \theta_i, \sum_{i,j,\alpha,\beta} h\epsilon_{ijk}\epsilon_{\alpha\beta\gamma} F_{j\beta}^0 \partial_\alpha \theta_i, \sum_{i,\alpha} h\partial_\alpha \cof F_{i\alpha}^0 \partial_\alpha \theta_i\right)
\]
and a variation
\[
\Xi^\varepsilon = (v^\varepsilon, F^\varepsilon, Z^\varepsilon, w^\varepsilon) = (v, F, Z, w) + \varepsilon(\delta v_i, \delta F_{i\alpha}, \delta Z_{k\gamma}, \delta w).
\]

One can readily see that such variations fulfill the conditions (5) since columns of the matrix \( \cof F^0 \) are divergence-free vector fields. Additional requirements on \( \theta \) allow us to conclude that \( \Xi^\varepsilon = (v^\varepsilon, F^\varepsilon, Z^\varepsilon, w^\varepsilon) \) belongs to the admissible set \( \mathcal{A} \).

Furthermore, using the mean value theorem and Lebesgue’s dominated convergence theorem we find that
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(I(v_i + \varepsilon \delta v_i, F_{i\alpha} + \varepsilon \delta F_{i\alpha}, Z_{k\gamma} + \varepsilon \delta Z_{k\gamma}, w + \varepsilon \delta w) - I(v_i, F_{i\alpha}, Z_{k\gamma}, w))
\]
\[
= \int_{\Omega} \sum_i \theta_i(v_i - v_i^0) + \sum_{i,\alpha} h\partial_\alpha \theta_i g_{i\alpha}(F, Z, w; F^0) \, dx
\]
for \( \varepsilon^* \in [0, \varepsilon] \). The last equality is the weak form of (4a). To apply the dominated convergence theorem we use the integrability properties of \( g_{i\alpha} \) derived from hypothesis (H4) and Young’s inequality [7].

In the case \( p > 4 \), following to [7], we find the mapping \( y : (0, \infty) \times \Omega \to \mathbb{R}^3, y \in W_1^\infty([0, \infty]; L_2) \cap L_\infty([0, \infty]; W_p^1) \) such that the conditions \( \partial_t y = v, D_x y = F, \cof D_x y = Z, \det D_x y = w \) are fulfilled. Moreover, they satisfy the weak form of the additional conservation laws [2] and \( y \) is the measure-valued solution of the system [3].

Acknowledgment. The author is grateful to professor Sergey Vodop’yanov for suggesting the problem and useful discussions.

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