Energy backflow in unidirectional spatiotemporally localized wavepackets

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I. INTRODUCTION

In general terms, the phenomenon of backflow takes place when some quantity (probability or energy density flow, local momentum, etc.) in some spatio-temporal region of a wavefield is directed backward with respect to the directions of all plane-wave constituents of the wavefield \cite{1,2}. Position probability backflow specific to quantum particles, such as electrons, has been termed ‘quantum backflow,’ and this subject is actively studied (see the newest review \cite{3} and references therein). A dispute arose recently over the distinction between the quantum backflow and backflow phenomena known in classical field theories \cite{2,4}. In our opinion, the contradistinction here is largely of a terminological nature. At least nobody doubts that backflow is a wave phenomenon that may occur in all kinds of wavefields, particularly in those describable by the Schrödinger or the Maxwell or the wave equations in free space.

Indeed, already a simple field of four appropriately polarized and directed electromagnetic plane waves exhibit prominent energy backflow described by the Poynting vector whose direction is reversed with respect to the direction of propagation of the resultant wave \cite{3,7}. In the physical optics community the energy backflow in sharply focused light has been known more than an half a century and has been thoroughly studied theoretically recently \cite{8-11}. In the context of quantum backflow, monochromatic optical fields have been used for recent experimental verifications of the effect \cite{11,12}. Energy backflow in electromagnetic Bessel beams has been analytically demonstrated in \cite{13,14}. In the context of our present subject, important is the theoretical study \cite{15} of backflow in pulsed electromagnetic X waves, which belong to the class of the so-called localized waves.

Localized waves (LWs)—also known as space-time wavepackets (STWP)—have been studied intensively during the past thirty years (see \cite{16,30} for pertinent literature). They constitute spatiotemporally localized solutions to various hyperbolic equations governing acoustical, electromagnetic and quantum wave phenomena and can be classified according to their group velocity as luminal LWs, or focus wave modes (FWM), superluminal LWs, or X waves (XWs), and subluminal LWs. Further details can be found in two edited monographs on the subject \cite{31,52} and in the recent thorough review article \cite{33}. In general, both linear and nonlinear LW pulses exhibit distinct advantages in comparison to conventional quasi-monochromatic signals. Their spatiotemporal confinement and extended field depths render them especially useful in diverse physical applications. Experimental demonstrations have been performed in the acoustical and optical regimes \cite{21,22,25,34,38}. Work, however, has been carried out at microwave frequencies recently \cite{39,41}.

An important question—widely discussed especially in the early stages of the theoretical study of LWs at the end of the last century—has been the physical realizability of localized waves. For example, two- or three-dimensional electromagnetic luminal localized waves in free space involve the characteristic variables $\zeta = z - ct$ and $\eta = z + ct$ of the one-dimensional scalar wave equation. Consequently, they contain both forward and backward propagating components. Tweaking free parameters appearing in the wavepackets can significantly reduce the backward components. However, it is principally crucial whether the plane-wave constituents of the wavepacket propagate only in the positive $z$-direction or also backward. In the first case not only the wave can be launched from an aperture as a freely propagating beam but, also, the very question of energy backflow is meaningful.

In the literature, the LWs with forward-propagating plane-wave constituents have been called somewhat mis-
leadingly ‘causal’. In the following discussion we shall use the term ‘unidirectional’. It must be pointed out, however, that, in general, a wavepacket as a whole can propagate in the positive $z$-direction despite the fact that its plane-wave constituents are omnidirectional; and *vice versa*: the group velocity of a packet may have a negative $z$-component despite the fact that the $z$-components of the wavevectors of all its plane-wave constituents are positive. Also, for LWs the group velocity typically differs from the energy velocity, see [42, 43].

Our aim in this article is to theoretically study several representatives of a class of causal, purely unidirectional finite-energy localized waves with particular emphasis on their energy backflow characteristics. Specifically, we shall try to ascertain the role of the vector nature and polarization properties of a light field in the emergence of the backflow effect and its strength. The paper is organized as follows. In the next section we consider several finite-energy unidirectional localized waves known from the literature. Extended unidirectional LWs will be introduced in Sec. 3. Sec. 4 is devoted to a detailed analysis of the backflow characteristics of a unidirectional vectorial LW derived from the so-called splash mode solution of the scalar wave equation. This LW is a generalization of the unidirectional solution used by Bialynicki-Birula et al. [2] to demonstrate the existence of the backflow in electromagnetic fields. Section 5 is devoted to the derivation of a finite-energy Hopfion-like spatiotemporally localized wave that is devoid of energy backflow. Concluding remarks are made in Sec. 6.

### II. CAUSAL, SCALAR UNIDIRECTIONAL LOCALIZED WAVES

The feasibility of a finite-energy, causal, unidirectional localized wave was first addressed by Lekner [1] using the Fourier synthesis

$$\psi_+ (\rho, z, t) = \int_0^\infty dk e^{-ikz} \int_0^\infty dk e^{-ikt} \int_0^\infty d\kappa \kappa J_0 (\kappa \rho) \delta \left( -\kappa^2 - k_x^2 + k_z^2 \right) F(k, k_z, k). \quad (1)$$

Choosing the spectrum

$$F_1 (k_z, k) = \frac{k_z e^{-ka}}{\sqrt{k^2 - k_z^2}}, \ a > 0, \quad (2)$$

one obtains

$$\psi_+ (\rho, z, t) = \int_0^\infty dk e^{-k(a + it)} \int_0^k dk_z k_z e^{ik_z z} J_0 \left( \rho \sqrt{k^2 - k_z^2} \right). \quad (3)$$

Carrying out the integrations, Lekner has derived the causal unidirectional solution

$$\psi_+ (\rho, z, t) = \frac{a^4 \tilde{a} \left( \tilde{a}^2 + \rho^2 \right)^2 - 3 \tilde{a}^2 \left( \tilde{a}^2 + \rho^2 \right) - 16iz \left( \tilde{a}^2 + \rho^2 \right)^{3/2}}{3 \left( \tilde{a}^2 + \rho^2 \right)^{3/2} \left( \tilde{a}^2 + \rho^2 + z^2 \right)^3}, \quad (4)$$

with $\tilde{a} = a + it$. By interchanging the order of integrations in Eq. (3) and using table integrals from [42], p. 191, No. 10 and p. 133, No. 3, we obtained a more compact expression for the same unidirectional solution:

$$\psi (\rho, z, t) = \frac{a^3 \tilde{a}}{3 \rho^2 + \tilde{a}^2} \left( 3 \sqrt{\rho^2 + \tilde{a}^2} - 3iz \right) \left( \sqrt{\rho^2 + \tilde{a}^2} - iz \right)^3. \quad (5)$$

A formal study of a causal, unidirectional localized wave was undertaken by So, Plachenov and Kiselev [6] starting from the finite-energy luminal splash mode [18, 19]

$$\psi (\rho, z, t) = \frac{1}{a_1 + i (z - ct)} \frac{1}{a_2 - i (z + ct) + \rho^2}, \quad (6)$$

where $a_{1,2}$ are positive free parameters. As mentioned in the introduction, this is a bidirectional solution. However, for $a_2 > a_1$, the backward components are significantly reduced. So et al. [6] decomposed the denominator in Eq. (6) as follows:

$$\psi (\rho, z, t) = \frac{1}{z^2 - S^2} \frac{1}{S \left( \frac{1}{z_s - S} - \frac{1}{z_s + S} \right)} = \frac{1}{2} (u_+ - u_-) ; \quad S = \sqrt{ct_x^2 - \rho^2}; \quad t_s = t + \frac{i (a_2 + a_1)}{2c}, \ z_s = z + \frac{i (a_2 - a_1)}{2}. \quad (7)$$

They proved that the expression $u_+ = S^{-1} (z_s - S)^{-1}$ satisfies the wave equation and is solution propagating in the positive $z$-direction. On the other hand, $u_-$ is a solution propagating purely in the negative $z$-direction. They did not analyze whether $u_+$ is unidirectional (causal) in the sense we use here, that all its plane-wave constituents propagate solely into the hemisphere with $k_z > 0$. 


Note that the wavefunctions of a seemingly different type of solution was derived
where \( a > 0 \), which they derived by a Fourier synthesis and limiting the integration in \( k \)-space to one of
the hemispheres (with \( k_z > 0 \) or \( k_z < 0 \), respectively). Here, we carry out the synthesis in a slightly modified
version that incorporates complex values \( z_s = z + iz_\omega, \) \( \omega \equiv (a_2 - a_1)/2 \) of the \( z \)-coordinate and, thus, a two-
parameter solution like in Eq. (7). Since the solution is axially symmetric, we base the synthesis on the zeroth-order Bessel beams; specifically,

\[
h_{\pm}(\rho, z, t) = \frac{1}{2} \int_{-\infty}^{\infty} dk_z H(\pm k_z) e^{ik_zz} G(\kappa, k_z),
\]

where \( H(\cdot) \) designates the Heaviside unit step function which ensures the integration in one hemisphere only. Choosing the spectrum

\[
G(\kappa, k_z) = \frac{1}{\sqrt{\kappa^2 + k_z^2}} e^{-ac\sqrt{\kappa^2 + k_z^2}}, \quad a > 0
\]

and introducing the new variable \( \lambda = \sqrt{\kappa^2 + k_z^2} \), one obtains

\[
h_{\pm}(\rho, z, t) = \frac{1}{2} \int_{-\infty}^{\infty} dk_z H(\pm k_z) e^{ik_zz} \int_{k_z}^{\infty} d\lambda J_0 \left( \rho \sqrt{\lambda^2 - k_z^2} \right) e^{-a\lambda}.
\]

The integration over \( \lambda \) is carried out \[45\], p. 191, No. 9, yielding

\[
h_{\pm}(\rho, z, t) = \frac{1}{2\sqrt{c^2(\alpha + it)^2 + \rho^2}} \times \int_{-\infty}^{\infty} dk_z H(\pm k_z) e^{ik_zz} \left| k_z \right| \sqrt{c^2(\alpha + it)^2 + \rho^2}.
\]

Finally, the integration over \( k_z \) results in the unidirectional solutions given in Eq. (8), where \( z \) is replaced by \( z_s \), while the restriction \( |\text{Im } z_s| < a \) avoids singularity. Note that the wavefunctions \( h_{\pm} \) given in Eq. (8), but modified in this manner, are equivalent, respectively, to \( u_+ / 2 \) and \( u_- / 2 \) in Eq. (7), provided that, in addition, \( a = (a_1 + a_2) / 2 \).

Without detailed discussion of its unidirectional features, a seemingly different type of solution was derived by Wong and Kaminer in 2017 \[47\]; specifically,

\[
\psi(\rho, z, t) = -\frac{a + it}{k_0 R^2} \left( \frac{1}{k_0 R} f^{s-1} + \frac{s + 1}{s} f^{s-2} \right);
\]

\[
f = 1 - k_0 \left( iz + a - \tilde{R} \right) / s, \quad \tilde{R} = \sqrt{(a + it)^2 + \rho^2},
\]

where \( k_0 = \omega_0 / c = 2\pi / \lambda_0 \). This solution can be derived from the Fourier synthesis in Eq. (1). Assuming, first,

\[
\psi(\rho, z, t) = \frac{1}{2 \sqrt{c^2(\alpha + it)^2 + \rho^2}} \times \int_{-\infty}^{\infty} dk_z H(\pm k_z) e^{ik_zz} \left| k_z \right| \sqrt{c^2(\alpha + it)^2 + \rho^2}.
\]

The solution given in Eq. (13) results from differentiation of \( \psi_{1+}(\rho, z, t) / k_0 \) with respect to time.

All the unidirectional wavepackets discussed in this section are finite-energy solutions of the three-dimensional scalar wave equation in free space. An important question is how do they propagate in the positive \( z \)-direction? We shall answer this question in connection to the Bialynicki-Birula et al. \[2\] wavepacket which seems to be the simplest. Consider the part of the solution involving \( z \) and \( t \); specifically,

\[
z = -i \sqrt{c^2(\alpha + it)^2 + \rho^2}.
\]

Then, the real group speed is given by

\[
v_g(\rho, t) = \text{Re} \left\{ \frac{\partial}{\partial t} z(\rho, t) \right\} = c \text{Re} \left\{ \frac{c (a + it)}{\sqrt{c^2(\alpha + it)^2 + \rho^2}} \right\}.
\]
very large values of time. A similar behavior is exhibited for larger values of $\rho$.

This behavior is different from a finite-energy unidirectional scalar wavepacket moving at a fixed speed. An example is provided in Appendix A for a solution to the equation of acoustic pressure under conditions of uniform flow.

III. EXTENDED SCALAR UNIDIRECTIONAL LOCALIZED WAVES

Courant and Hilbert [48] have pointed out that a "relatively undistorted" progressive solution to the homogeneous three-dimensional (3D) scalar wave equation in vacuum assumes the form

$$\psi(\vec{r}, t) = \frac{1}{g(\vec{r}, t)} f[\theta(\vec{r}, t)],$$

(19)

where $f(\cdot)$ is essentially an arbitrary function, $\theta(\vec{r}, t)$, referred to as the "phase" function, is a solution to the nonlinear characteristic equation

$$\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 - \frac{1}{c^2} \left(\frac{\partial \theta}{\partial t}\right)^2 = 0,$$

(20)

and $g(\vec{r}, t)$ is an "attenuation" function; the latter depends on the choice of $\theta(\vec{r}, t)$, but not in a unique manner. Along this vein, a very general class of solutions to the homogeneous scalar wave equation in free space is given as

$$\psi_+(\vec{r}, t) = \frac{1}{g(\vec{r}, t)} f[\theta(\alpha, \beta)];$$

$$g(\vec{r}, t) \equiv \sqrt{\rho^2 - c^2(t - it_s)^2};$$

$$\alpha(\vec{r}, t) \equiv \sqrt{\rho^2 - c^2(t - it_s)^2 - i(z + iz_s)};$$

$$\beta(\vec{r}, t) \equiv \frac{\rho e^{-i\phi}}{ic(t - it_s) + g(\vec{r}, t)},$$

(21)
in polar coordinates. Here, $z_s$ and $t_s$ are free positive parameters. In the sequel, we shall discuss in detail the specific azimuthally asymmetric solution

$$\psi_+(\rho, \phi, z, t) = \frac{1}{2g(\vec{r}, t)} \frac{1}{\alpha^2(\vec{r}, t)} e^{-p\alpha(\vec{r}, t) \beta^m(\vec{r}, t)},$$

(22)

where $p$ is a positive free parameter. It should be noted that for $m = 0$, $p = 0$ and $q = 1$, the solution $\psi_+(\vec{r}, t)$ is identical to the Bialynicki-Birula azimuthally symmetric expression $h_+$ given in Eq. (8) if $z_s = 0$ and $t_s = \alpha$. Also, for $m = 0$, $p = 0$ and $q = s + 1$, $\psi_+(\vec{r}, t)$ is a slight variation of the expression $\psi_+^{(1)}(\rho, z, t)$ in Eq. (16).

IV. ENERGY BACKFLOW IN UNIDIRECTIONAL SPATIOTEMPORAL LOCALIZED WAVES

A. Scalar-valued wave theory

The energy transport equation corresponding to the (3+1)D homogeneous scalar wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \psi(\vec{r}, t) = 0$$

(23)

governing the real-valued wave function $\psi(\vec{r}, t)$ in free space is given as

$$\nabla \cdot \vec{S} + \frac{\partial}{\partial t} U = 0,$$

(24)

where

$$U = \frac{1}{2c^2} \left(\frac{\partial}{\partial t} \psi\right)^2 + \frac{1}{2} \nabla \psi \cdot \nabla \psi$$

(25)

is the energy density ($J/m^3$) and

$$\vec{S} = -\frac{\partial}{\partial t} \psi \nabla \psi$$

(26)

is the energy flow density vector ($W/m^2$).

An examination of energy backflow is accomplished by examining the properties of the $z$--component of the energy flow vector corresponding to the real part of the extended unidirectional scalar complex wave function $\psi_+(\rho, \phi, z, t)$ given in Eq. (22). Plots are shown in Fig. 2 of $S_z$ versus $\rho$ for four axial positions, first for $m = 0$, $p = 0$, $q = 1$ and then, in Fig. 2(b) for $m = 3$, $p = 0$, $q = 1$. For both plots the parameter values are $t_s = 0.3$ and $z_s = 0.1$. The time is fixed at $t = 0.5$, with the speed of light of vacuum normalized to unity. Thus, the parameters and time are given in dimensionless length units and therefore the results are applicable not only in optical but also in microwave, etc. regions. For example, if the chosen values of the parameters $t_s$ and $z_s$, as well as the time $t$ were in micrometers, then $t = 0.5$ would correspond to 1.67 femtoseconds which would also
be approximately equal to the pulsewidth. The value $t = 0.5$ has been chosen since at $t = 0$ the backflow is absent but at higher values of $t$ the pulse would spread out too much and the energy flow would become negligible.

We studied the simplest case with $m = 0$, $p = 0$, $q = 1$ in more detail by plotting the local energy transport velocity vector field, $\vec{V} = \vec{S}/U$, see Fig. 3. The plots demonstrate that at the instant $t = 0.5$ energy flows backward in the region ($\rho < 0.3, z > 0.5$). In a sense the situation resembles the sea drawback phenomenon, where the water recedes ahead of the tsunami wave peak.

A complex-valued version of the wavefunction of Eq. (22) exhibits practically no backflow effect because for its imaginary part the backflow regions have different locations as compared to those of the real part. Therefore, negative values of $S_z$ from the real part are compensated by much stronger positive values of $S_z$ from the imaginary part and vice versa since the energy flow vectors from both parts sum up additively. The same holds for the wavefunction of Eq. (5) or Eq. (4).

An interesting question is whether the power flux, that is the integral of the $z$-component of the energy flow vector over a transverse plane at fixed values of $z$ and $t$ can be negative, at least for a small circular disk. Indeed, this is the case. From Fig. 2(a), we use the dimensionless values $z = 0.9$ and $t = 0.5$. Then, the integration results in the plot are shown in Fig. 4.

We see that the negative flux increases in absolute value up to the radius $\simeq 0.26$ of the disk, in full accordance with Fig. 3.

To conclude, it is somewhat surprising that a scalar solution to the wave equation exhibits the backflow effect because earlier studies have instilled an opinion that the effect appears in electromagnetic fields of specific polarization. Our results indicate that backflow is possible not only in optical fields describable by scalar approximation but also in acoustical fields.
Implicit in the article by Bialynicki-Birula et al. \[2\] is that the examination of the energy backflow characteristics of a vector-valued unidirectional localized wave is based on a complex Riemann-Silberstein vector \[\vec{H}\] derived from the vector Hertz potential \[\vec{H} = 2(\vec{a}_t + i\vec{a}_s) \psi\] in Cartesian coordinates, or \[\vec{H} = \exp(\vec{a}_r + i\vec{a}_\phi) \psi\] in cylindrical coordinates, where the complex-valued wave function \[\psi(r,t)\] is a solution of the \((3+1)D\) scalar wave equation in free space; specifically, the function \[h_+\] in Eq. (8). The complex-valued Riemann-Silberstein vector is defined as

\[
\vec{F} = \nabla \times \nabla \times \vec{H} + \frac{i}{c} \frac{\partial}{\partial t} \nabla \times \vec{H}.
\]  

(27)

It obeys the equations

\[
\nabla \times \vec{F} - \frac{i}{c} \frac{\partial}{\partial t} \vec{F} = 0, \quad \nabla \cdot \vec{F} = 0,
\]  

(28)

that are exactly equivalent to the homogeneous Maxwell equations for the free-space real electric and magnetic fields \[\vec{E}\] and \[\vec{B}\], defined in terms of \[\vec{F}\] as follows:

\[
\vec{F} = \sqrt{\frac{\varepsilon_0}{2}} \left( \vec{E} + ic\vec{B} \right).
\]  

(29)

The importance of the specific choice for the vector Hertz potential is that the corresponding Riemann-Silberstein vector, with the scalar wave function \[\psi(r,t)\] being any luminal spatiotemporally localized wave [e.g., the splash mode in Eq. (6)], is null, that is, it has the property \[\vec{F} \cdot \vec{F} = 0\]. Equivalently, the two Lorentz-invariant quantities \[I_1 = \vec{E} \cdot \vec{B}\] and \[I_2 = |\vec{E}|^2 - c^2|\vec{B}|^2\] are both equal to zero. Under certain restrictions, the resultant field can be a pure Hopfion \[53\] exhibiting interesting topological properties, such as linked and knotted field lines, or a Hopfion-like structure, such as the one established by Bialynicki-Birula et al. \[2\].

In the discussion below, the formalism described above will be followed, however based on the simpler vector Hertz potential \[\vec{H} = \psi \vec{a}_z\]. The Poynting vector, defined in terms of the real fields as \[\vec{S} = \vec{E} \times \vec{H}\], can be written in terms of the Riemann-Silberstein vector and its complex conjugate as follows: \[\vec{S} = -i\vec{F}^* \times \vec{F}\]. An examination of energy backflow will be accomplished by examining the properties of the \(z\)-component of the Poynting vector corresponding to the extended unidirectional scalar complex wave function \[\psi_+ (\rho, \phi, z, t)\] given in Eq. (22).

Plots are shown in Fig. 5 of \(S_z\) versus \(\rho\) for three axial positions, first for \(m = 0, p = 0, q = 1\) and then for \(m = 3, p = 0, q = 1\). Note that the values of \(t\) and \(z\), as well as of the parameters \(t_s\) and \(z_s\), differ from those of Fig. 2. We see that the vector-valued versions of both waves in a certain spatiotemporal region exhibit the backflow effect which is weak but comparable to that of the scalar-valued waves.
V. UNIDIRECTIONAL HOPFION-LIKE SPATIOTEMPORALLY LOCALIZED WAVE WITHOUT ENERGY BACKFLOW

The basic (or pure) Hopfion is a finite-energy luminal spatiotemporally localized solution to Maxwell’s equations in free space with unique topological properties. Specifically, all electric and magnetic field lines are closed loops, and any two electric (or magnetic) field lines are linked once with one another. However, the basic Hopfion has equally distributed forward (along the positive z-direction) and backward components. By construction, then, it exhibits energy backflow. The unidirectional Hopfion-like wave structure in shares some of the topological characteristics with the pure Hopfion but exhibits energy backflow. A unidirectional Hopfion-like wave packet devoid of energy backflow will be constructed in this section. Toward this goal, a technique due originally to Whittaker [54] and Bateman [55] (see, also, [56, 57] for modern applications) will be used. First, the following two quantities, \( \vec{\alpha} (\vec{r}, t) \) and \( \vec{\beta} (\vec{r}, t) \), known as Bateman conjugate functions, will be defined in terms of the functions \( \alpha (\vec{r}, t) \) and \( \beta (\vec{r}, t) \) in Eq. (21):

\[
\vec{\alpha} (\vec{r}, t) = \frac{1}{\alpha^2 (\vec{r}, t)}, \quad \vec{\beta} (\vec{r}, t) = \beta^* (\vec{r}, t).
\]

Any functional of these two functions obeys the nonlinear characteristic Eq. (20). Furthermore, these two functions obey the Bateman constraint

\[
\nabla \vec{\alpha} \times \nabla \vec{\beta} - \frac{i}{c} \left( \frac{\partial \vec{\alpha}}{\partial t} \nabla \vec{\beta} - \frac{\partial \vec{\beta}}{\partial t} \nabla \vec{\alpha} \right) = 0.
\]

Under these assumptions, the complex vector

\[
\vec{F} = \nabla \vec{\alpha} \times \nabla \vec{\beta}
\]

is a null Riemann-Silberstein vector governed by the expressions in Eq. (28) and related to the real electric and magnetic fields as given in Eq. (29). These fields have some of the topological characteristics of those associated with a pure Hopfion. Fig. 6 shows the linkages of the electric and magnetic field lines. Similar linkages characterize any two electric (or magnetic) field lines. The basic Hopfion is characterized by linked single closed field line loops. In our case, we have linked bundles of field lines instead.

The \( z \)-component of the Poynting vector \( \vec{S} = -i \vec{F}^* \times \vec{F} \) is plotted in Fig. 7. No energy backflow is present in this case.

![FIG. 6. Linkages of electric and magnetic field lines at a fixed value of time, with the speed of light in vacuum normalized to unity. The parameter values are \( t_s = 0.3 \) and \( z_s = 0.1 \).]

![FIG. 7. Plot of the \( z \)-component of the Poynting vector versus \( z \) and \( \rho \) for three values of time, \( t \). The parameter values are \( t_s = 0.3 \) and \( z_s = 0.1 \).]

The ratio of the Poynting vector and the electromagnetic volume density is the local energy transport velocity

\[
\vec{V} (\vec{r}, t) = \vec{S}/U, \text{ with the energy density given in terms of the Riemann-Silberstein vector as } U (\vec{r}, t) = \vec{F}^* \cdot \vec{F}.
\]
Due to the nullity of $\vec{F}$, the modulus of the energy transport velocity is equal to the speed of light $c$, although $\vec{V}(\rho, t)$ may vary in space and time. In the case of the basic Hopfion, the local energy velocity depends on the $z$ and $t$ through the combination $z - ct$; that is, it evolves along the $z$-direction without any deformation. Such a structure is known as a Robinson congruence. In the case under consideration in this section the local energy transport velocity is altogether independent of the coordinate $z$. The plot in Fig. 8 shows the $z$-component of the local energy transport velocity versus $\rho$ for three values of time. The absence of energy backflow is clearly evident.

![Plot](image)

**FIG. 8.** Plot of the $z$-component of the local energy transport velocity versus $\rho$ for three values of time, $t = 0.1, 1$ and $t = 3$.

**VI. CONCLUDING REMARKS**

As already has been mentioned, it is crucial whether the plane-wave constituents of a localized wavepacket propagate only in the positive $z$-direction or, also, backward. In the first case, not only the wave can be launched from an aperture as a freely propagating beam but, also, the very question of energy backflow is meaningful. Since most of the analytically constructed finite-energy spatiotemporally localized waves (luminal, subluminal, and superluminal) are acausal in the sense that they include both forward and backward propagating components, several attempts have been made to create close replicas of such waves that can be causally launched as forward beams from apertures (see, e.g., [21] and [58]), or even derive exact causal unidirectional wavepackets [59, 60].

Based usually on the Huygens principle, dealing with the former is computationally intensive. On the other hand, the latter are quite complicated analytically. The study of energy backflow in this article has been confined to relatively simple causal unidirectional finite-energy localized waves arising from a factorization of the basic splash mode [18, 19]. Specific results are given for the energy backflow exhibited in known azimuthally symmetric unidirectional wavepackets, as well as in novel azimuthally asymmetric extensions. Using the Bateman-Whittaker technique, a novel finite-energy unidirectional null localized wave has been constructed that is devoid of energy backflow and has some of the topological properties of the basic Hopfion.

The study of energy backflow of the vector-valued unidirectional localized study in Sec. 4 was based on the Riemann-Silberstein complex vector that results in electromagnetic and magnetic fields that both have nonzero $z$ components (non-TE and non-TM). Although specific results have not been incorporated in this article, an examination of the Poynting vectors associated individually with pure TE and TM fields associated to the scalar unidirectional localized wave $\psi_+ (\rho, \phi, z, t)$ given in Eq. (22) shows the presence of energy backflow. This is altogether different from the cases of the superposition of four plane waves [5–7], Bessel beams [13–14], and a pulsed electromagnetic X wave [15], all of which require a superposition of TE and TM fields for the appearance of energy backflow.

Finally, some remarks about possible experimental studies and relation of our results to quantum optics.

As was said already in Section III, the results (incl. numerical plots) are applicable irrespectively of the frequency range of the EM field. At low frequencies up to the microwave region, the measurements for studying the non-stationary behavior of the vector-valued Poynting vector in pulsed fields considered here require time resolution up to nanosecond range. For that purpose, known sensor-based techniques developed for monochromatic fields (see, e.g., [61]) would be applicable. However, to the best of our knowledge, no experiments have been carried out for electromagnetic or acoustic energy backflow associated with pulsed spatiotemporally localized wavefunctions.

As far as experiments on the electromagnetic energy backflow effect in the optical range is concerned, to date a few studies with monochromatic fields can be found: in addition to Refs. 11 and 12, studies of the effect in nanoscale focuses [62, 63] are appropriate to mention here. Distribution of the Poynting vector in optical fields can be measured via the motion of probe Rayleigh particles or via investigation of polarization in passage through an anisotropic crystal [64]. However, optical experiments on the fields considered in our paper are hardly feasible today because they would need near-single-cycle light pulses and—consequently—sub-femtosecond temporal resolution.

The expressions derived in this paper also apply to quantum optics: as it is known, the spatio-temporal dependence of the (quantum mechanical) wave function of a single photon, treated as a particle-like object, is given by the Riemann-Silberstein vector of the corresponding classical EM field. Moreover, as shown in [64], in terms of quantum weak measurements, observation of averaged trajectories of single photons can be considered as measurement of the distribution of the Poynting vector in the corresponding classical optical field, incl. backflow effects.
VII. APPENDIX A

The equation of acoustic pressure under conditions of uniform flow is given as follows:

\[
\left[ \nabla^2 - \frac{1}{u_0^2} \left( \frac{\partial}{\partial t} + \bar{u} \cdot \nabla \right)^2 \right] p(r, t) = 0. \tag{A1}
\]

Here, \(u_0\) is the speed of sound in the rest frame of the medium and \(\bar{u}\) is the uniform velocity of the background flow. In the special case where \(\bar{u} = u\bar{a}_z\) and \(u = u_0\) the resulting equation for the acoustic pressure simplifies as follows:

\[
\left( \nabla^2_t - \frac{2}{u_0} \frac{\partial^2}{\partial \partial z} - \frac{1}{u_0^2} \frac{\partial^2}{\partial t^2} \right) p(r, t) = 0. \tag{A2}
\]

Under this assumption, several exact analytical infinite-energy nondiffracting and finite-energy slowly non-diffracting spatiotemporally localized wave solutions are supported. One such solution is the finite-energy unidirectional splash-like mode

\[
p(\rho, z, t) = \frac{1}{a_1 + i(z - 2u_0 t)} \left( a_2 - iz + \frac{\rho^2}{a_1 + i(z - 2u_0 t)} \right)^{-q}, \tag{A3}
\]

with \(a_{1,2}\) positive parameters. This is a finite-energy unidirectional wavepacket moving along the \(z\)-direction at the fixed speed \(v = 2u_0\).

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