Enumerations for Permutations by Circular Peak Sets

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Abstract

The circular peak set of a permutation $\sigma$ is the set $\{\sigma(i) \mid \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$. In this paper, we focus on the enumeration problems for permutations by circular peak sets. Let $cp_n(S)$ denote the number of the permutations of order $n$ which have the circular peak set $S$. For the case with $|S| = 0, 1, 2$, we derive the explicit formulas for $cp_n(S)$. We also obtain some recurrence relations for the sequence $cp_n(S)$ and give the formula for $cp_n(S)$ in the general case.

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1 Introduction

Throughout this paper, let \( [m, n] := \{m, m + 1, \ldots, n\} \), \([n] := [1, n]\) and \([m, n] = \emptyset\) if \(m > n\). Let \(S_n\) be the set of all the permutations in the set \([n]\). We will write permutations of \(S_n\) in the form \(\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n))\). We say that a permutation \(\sigma\) has a circular descent of value \(\sigma(i)\) if \(\sigma(i) > \sigma(i + 1)\) for any \(i \in [n - 1]\). The circular descent set of a permutation \(\sigma\), denoted \(CDES(\sigma)\), is the set \(\{\sigma(i) \mid \sigma(i) > \sigma(i + 1)\}\). For any \(S \subseteq [n]\), define a set \(CDES_n(S)\) as \(CDES_n(S) = \{\sigma \in S_n \mid CDES(\sigma) = S\}\) and use \(cdes_n(S)\) to denote the number of the permutations in the set \(CDES_n(S)\), i.e., \(cdes_n(S) = |CDES_n(S)|\).

In a join work [2], Hungyung Zhang et al. derive the explicit formula for \(cdes_n(S)\). As an application of the main results in [2], they also give the enumeration of permutation tableaux according to their shape and generalizes the results in [4]. Moreover, Robert J.Clarke et al. [3] gave the conceptions of linear peak and cyclic peak and studied some new Mahonian permutation statistics. In this paper, we say that a permutation \(\sigma\) has a circular peak of value \(\sigma(i)\) if \(\sigma(i - 1) < \sigma(i) > \sigma(i + 1)\) for any \(i \in [2, n - 1]\). The circular peak set of a permutation \(\sigma\), denoted \(CP(\sigma)\), is the set \(\{\sigma(i) \mid \sigma(i - 1) < \sigma(i) > \sigma(i + 1)\}\). For example, the circular peak set of \(\sigma = (48362517)\) is \(\{5, 6, 8\}\). Since \(\sigma\) has no circular peaks when \(n \leq 2\), we always suppose that \(n \geq 3\). For any \(S \subseteq [n]\), we define a set \(CP_n(S)\) as \(CP_n(S) = \{\sigma \in S_n \mid CP(\sigma) = S\}\). Obviously, if \(\{1, 2\} \subseteq S\), then \(CP_n(S) = \emptyset\).

Example 1.1

\[CP_5(\{4, 5\}) = \{14253, 14352, 24153, 34152, 24351, 34251, 15243, 15342, 25143, 35142, 25341, 35241\} \]
Suppose that $S = \{i_1, i_2, \ldots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$. Pierre Bouchard et al. [1] proved that the necessary and sufficient conditions for $CP_n(S) \neq \emptyset$ are $i_j \geq 2j + 1$ for all $j \in [k]$.

Let $\mathcal{P}_n = \{S \mid CP_n(S) \neq \emptyset\}$. We can make the set $\mathcal{P}_n$ into a poset $\mathcal{P}_n$ by defining $S \leq T$ if $S \subseteq T$ as sets. See [1] for the properties of the poset $\mathcal{P}_n$.

In this paper, we focus on the enumerations for permutations by circular peak sets. Let $cp_n(S)$ denote the number of the permutations in the sets $CP_n(S)$, i.e., $cp_n(S) = |CP_n(S)|$. For the case with $|S| = 0, 1, 2$, we derive the explicit formulas for $cp_n(S)$. For the general case, we consider the recurrence relations for $cp_n(S)$. We find that if $n \geq 3$ and $S \subseteq [n - 1]$ with $CP_n(S) \neq \emptyset$, then $cp_n(S)$ satisfies the following recurrence relation

$$cp_n(S \cup \{n\}) = [n - 2 - 2|S|]cp_{n-1}(S) + \sum_{j \not\in S} 2cp_{n-1}(S \cup \{j\}).$$

But we are more interested in another recurrence relation as follows: suppose $n \geq 3$, $k \geq 0$ and $S \subseteq [n - k - 1]$ with $CP_n(S) \neq \emptyset$, then

$$cp_n(S \cup [n - k + 1, n]) = 2(k + 1)cp_{n-1}(S \cup [n - k, n - 1]) + k(k + 1)cp_{n-2}(S \cup [n - k, n - 2]).$$

First, let $k \geq 1$, $n \geq \max\{3, 2k\}$ and $CP_n([n - k + 1, n]) \neq \emptyset$, by this recurrence relation, we conclude that

$$cp_n([n - k + 1, n]) = k(k + 1) \sum_{i=1}^{k} (-1)^{i+1}b_{k,i} \left[(2k + 2)^{n-2k} - (2k + 2 - 2i)^{n-2k}\right],$$

where the coefficients $b_{k,i}$ satisfy the following recurrence relation:

$$b_{k+1,i} = \begin{cases} 
  k(k + 1)^2 \sum_{j=1}^{k} (-1)^{j+1}b_{k,j} & \text{if } i = 1 \\
  k(k + 1)^{k+2-i}b_{k,i-1} & \text{if } 2 \leq i \leq k + 1
\end{cases}$$
with initial condition $b_{1,1} = \frac{1}{2}$. And then we study the connections between this recurrence relation and the circular-peak path with weight. A circular-peak path is a lattice path in the first quadrant starting at $(r, 0)$ and ending at $(n, k)$ with only two kinds of steps—horizon step $H = (1, 0)$ and rise step $R = (2, 1)$. We consider a circular-peak path $P$ from $(r, 0)$ to $(n, k)$ as a word of $n-r-k$ letters using only $H$ and $R$. Let $P_{r,n,k}$ be the set of all the circular-peak paths from the vertices $(r, 0)$ to $(n, k)$. Given an integer $i$ and $P = e_1e_2 \cdots e_{n-k-r} \in P_{r,n,k}$, if the step $e_j$ connects the vertices $(x, y)$ with $(x + 1, y)$, then the weight of $e_j$, denoted $w_i(e_j)$, is $2i + 2(y + 1)$; if the step $e_j$ connects the vertices $(x, y)$ with $(x + 2, y + 1)$, then the weight of $e_j$, denoted $w_i(e_j)$, is $(y + i + 1)(y + i + 2)$; at last, let $w_i(P) = \prod_{j=1}^{n-k-r} w_i(e_j)$ be the weight of the circular-peak path $P$ and $w(i, r, n, k) = \sum_{P \in P_{r,n,k}} w_i(P)$. It is proved that if $k \geq 0$, $n \geq k + 4$, $S \subseteq [3, n - k - 1]$ with $CP_n(S) \neq \emptyset$ and $r = \max S$, then

$$\text{cp}_n(S \cup [n - k + 1, n]) = \sum_{i=0}^{k} w(i, r, n-i, k-i)\text{cp}_{r+i}(S \cup [r+1, r+i]).$$

We must give the formula for $w(i, r, n, k)$ as follows:

$$w(i, r, n, k) = 2^{n-r-2k} \prod_{m=0}^{k-1} (m + i + 1)(m + i + 2) \sum_{i=0}^{k} \prod_{m=0}^{k} [i + m + 1]^{t_m},$$

where the sum is over all $(k+1)$-tuples $(t_0, t_1, \cdots, t_k)$ such that $\sum_{m=0}^{k} t_m = n-r-2k$ and $t_m \geq 0$.

For any $S \subseteq [3, n]$, define the type of the set $S$ as $(r_1^{k_1}, r_2^{k_2}, \cdots, r_m^{k_m})$ if $S = \bigcup_{i=1}^{m} [r_i - k_i + 1, r_i]$ such that $r_i \leq r_{i+1} - k_{i+1} - 1$ for all $i \in [1, m-1]$. We may state one of the main results of the paper as follows:

$$\text{cp}_n(S) = 2^{n-r_m} \sum_{i_1=0}^{k_m} \sum_{i_2=0}^{k_{m-1}+i_1} \cdots \sum_{i_{m-1}=0}^{k_2+i_{m-2}-1} \prod_{j=1}^{i_{m-1}} w(i_j, r_{m-j}, r_{m-j+1} + i_{j-1} - i_j, k_{m-j+1} + i_{j-1} - i_j)\text{cp}_{r_1+i_{m-1}}([r_1 - k_1 + 1, r_1 + i_{m-1}]),$$
where \( i_0 = 0 \), when \( m \geq 2 \) and \( CP_n(S) \neq \emptyset \).

The paper is organized as follows. In Section 2, we will consider the enumerations of the permutations in the sets \( CP_n(S) \) with \( |S| = 0, 1, 2 \). In Section 3, we will derive some recurrence relations for the sequences \( cp_n(S) \) and give the formula for \( cp_n(S) \) in the general case. In the Appendix, we list all the values of \( cp_n(S) > 0 \) for \( 3 \leq n \leq 8 \).

# 2 The Enumerations for The Permutations in The Set \( CP_n(S) \) with \( |S| = 0, 1, 2 \)

In this section, we will consider the enumeration problems of the permutations in the sets \( CP_n(S) \) with \( |S| = 0, 1, 2 \).

Let \( cp_n(S) \) denote the number of the elements in the sets \( CP_n(S) \), i.e., \( cp_n(S) = |CP_n(S)| \). First, we need the following lemma.

**Lemma 2.1** Suppose \( n \geq 3 \) and \( S \subseteq [n] \) with \( CP_n(S) \neq \emptyset \). Then

1. \( cp_{n+1}(S) = 2cp_n(S) \), and
2. let \( m = \max S \), then \( cp_n(S) = 2^{n-m}cp_m(S) \) for any \( n \geq m \).

**Proof.** (1) It is easy to check that \( ((n+1)\sigma(1)\cdots\sigma(n)) \in CP_{n+1}(S) \) and \( (\sigma(1)\cdots\sigma(n)(n+1)) \in CP_{n+1}(S) \) for any \( \sigma = (\sigma(1)\cdots\sigma(n)) \in CP_n(S) \). Conversely, for any \( \sigma \in CP_{n+1}(S) \), the position of the letter \( n+1 \) is 1 or \( n+1 \), i.e., \( \sigma^{-1}(n+1) = 1 \) or \( n+1 \), since \( n+1 \notin S \). Hence, \( cp_{n+1}(S) = 2cp_n(S) \).

(2) Iterating the identity in (1), we immediately obtain that \( cp_n(S) = 2^{n-m}cp_m(S) \). \( \blacksquare \)
For any $\sigma \in \mathfrak{S}_n$, let $\tau$ be a subsequence $(\sigma(j_1)\sigma(j_2)\cdots\sigma(j_k))$ of $(\sigma(1)\cdots\sigma(n))$, where $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Define $\text{red}_{\sigma,\tau}$ as an increasing bijection of $\{\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_k)\}$ onto $[k]$. Let $\text{red}_\sigma(\tau) = (\text{red}_{\sigma,\tau}(\sigma(j_1))\text{red}_{\sigma,\tau}(\sigma(j_2))\cdots\text{red}_{\sigma,\tau}(\sigma(j_k)))$.

**Theorem 2.1** Let $n \geq 3$. Then

1. $\text{cp}_n(\emptyset) = 2^{n-1}$,
2. $\text{cp}_n(\{i\}) = 2^{n-2}(2^{i-2} - 1)$ for any $i \in [3, n]$, and
3. $\text{cp}_n(\{i, j\}) = 2^{n-3}(2^{i-2} - 1)(2^{j-i-1} - 1) + 2^{n+j-i-5}3(3^{i-2} - 2^{i-1} + 1)$ for any $i, j \in [3, n]$ and $i < j$.

**Proof.** (1) For any $\sigma \in \mathfrak{S}_n$, suppose that the position of the letter 1 is $i + 1$, i.e., $\sigma^{-1}(1) = i + 1$. Then $\sigma \in CP_n(\emptyset)$ if and only if $\sigma$ has the form $\sigma(1) > \cdots > \sigma(i+1) < \cdots < \sigma(n)$. For each letter $j \neq 1$, the position of $j$ has two possibilities at the left or right of 1. Hence, $\text{cp}_n(\emptyset) = 2^{n-1}$.

(2) By Lemma 2.1(2), we first consider the number of the permutations in the set $CP_i(\{i\})$, where $i \geq 3$. For any $\sigma \in CP_i(\{i\})$, suppose that the position of the letter $i$ is $k + 1$, i.e., $\sigma^{-1}(i) = k + 1$, then $1 \leq k \leq i - 2$, $\text{red}_\sigma(\sigma(1)\cdots\sigma(k)) \in CP_k(\emptyset)$ and $\text{red}_\sigma(\sigma(k+2)\cdots\sigma(i)) \in CP_{i-k-1}(\emptyset)$. There are $\binom{i-1}{k}$ ways to form the set $\{\sigma(1), \ldots, \sigma(k)\}$. So, $\text{cp}_i(\{i\}) = \sum_{k=1}^{i-2} \binom{i-1}{k}2^{k-1}2^{i-k-2} = 2^{i-2}(2^{i-2} - 1)$. Hence, $\text{cp}_n(\{i\}) = 2^{n-2}(2^{i-2} - 1)$.

(3) It is easy to check that the identity holds when $i = 3$ and $j = 4$. By Lemma 2.1(2), we first consider the number of the permutations in the set $CP_j(\{i, j\})$, where $3 \leq i < j$. 

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We begin from the case $\sigma \in CP_j(\{i, j\})$ with $\sigma^{-1}(i) < \sigma^{-1}(j)$. Let

$$T_1(\sigma) = \{\sigma(k) \mid \sigma(k) < i \text{ and } k < \sigma^{-1}(i)\},$$

$$T_2(\sigma) = \{\sigma(k) \mid \sigma(k) < i \text{ and } \sigma^{-1}(i) < k < \sigma^{-1}(j)\},$$

$$T_3(\sigma) = \{\sigma(k) \mid \sigma(k) < i \text{ and } k > \sigma^{-1}(j)\}.$$

Note that $T_k(\sigma) \neq \emptyset$ for $k = 1, 2$ since $\sigma$ must have a circular peak $i$ and $\bigcup_{k=1}^3 T_k(\sigma) = [i - 1]$. Let

$$T_4(\sigma) = \{\sigma(k) \mid i < \sigma(k) < j \text{ and } k < \sigma^{-1}(i)\}$$

$$T_5(\sigma) = \{\sigma(k) \mid i < \sigma(k) < j \text{ and } \sigma^{-1}(i) < k < \sigma^{-1}(j)\}.$$

We discuss the following two subcases.

**Subcase 1.** $T_3(\sigma) = \emptyset$

Let $T_6(\sigma) = \{\sigma(k) \mid i < \sigma(k) < j, k > \sigma^{-1}(j)\}$. Then $T_6(\sigma) \neq \emptyset$ since $\sigma$ must have a circular peak $j$ and $\bigcup_{k=4}^6 T_k(\sigma) = [i + 1, j - 1]$. For $k = 1, 2, 6$, the subsequences of $\sigma$, which is determined by the elements in $T_k(\sigma)$, corresponds to a permutation in $CP_{|T_k(\sigma)|}(\emptyset)$. The subsequences of $\sigma$, which is determined by the elements in $T_4(\sigma)$ and $T_5(\sigma)$, are decreasing and increasing, respectively. So, the number of the permutations under this subcase is

$$\sum_{(T_1, T_2)} \binom{i - 1}{|T_1|, |T_2|} 2^{|T_1| - 1} 2^{|T_2| - 1} \sum_{(T_4, T_5, T_6)} \binom{j - i - 1}{|T_4|, |T_5|, |T_6|} 2^{|T_6| - 1} = 2^{j-4}(2^{i-2} - 1)(2^{j-1} - 1),$$

where the first sum is over all pairs $(T_1, T_2)$ such that $T_i \neq \emptyset$ for $i = 1, 2$ and $T_1 \cup T_2 = [i - 1]$; the second sum is over all triples $(T_4, T_5, T_6)$ such that $T_6 \neq \emptyset$ and $T_4 \cup T_5 \cup T_6 = [i + 1, j - 1]$.

**Subcase 2.** $T_3(\sigma) \neq \emptyset$
Suppose that \( \min T_3(\sigma) = s \), let

\[
T_6(\sigma) = \{ \sigma(k) \mid i < \sigma(k) < j \text{ and } \sigma^{-1}(j) < k < \sigma^{-1}(s) \}
\]

\[
T_7(\sigma) = \{ \sigma(k) \mid i < \sigma(k) < j \text{ and } k > \sigma^{-1}(s) \}.
\]

Then, for \( k = 1, 2, 3 \), the subsequences of \( \sigma \), which is determined by the elements in \( T_k(\sigma) \), corresponds to a permutation in \( CP | T_k(\sigma) | (\emptyset) \). The subsequences of \( \sigma \), which is determined by the elements in \( T_4(\sigma) \) and \( T_6(\sigma) \), are decreasing. The subsequences of \( \sigma \), which is determined by the elements in \( T_5(\sigma) \) and \( T_7(\sigma) \), are increasing. So, the number of the permutations under this subcase is

\[
\sum_{(T_1, T_2, T_3)} \left( i - 1 \right) 2^{\lvert T_1 \rvert - 1} 2^{\lvert T_2 \rvert - 1} 2^{\lvert T_3 \rvert - 1} = 2^{2j-i-6} \cdot 3(3^{i-2} - 2^{i-1} + 1),
\]

where the sum is over all triples \( (T_1, T_2, T_3) \) such that \( T_i \neq \emptyset \) for \( i = 1, 2, 3 \) and \( T_1 \cup T_2 \cup T_3 = [i - 1] \).

Similarly, we may consider the case \( \sigma \in CP_j(\{i, j\}) \) with \( \sigma^{-1}(i) > \sigma^{-1}(j) \). Therefore, \( cp_j(\{i, j\}) = 2[2^{j-4}(2^{i-2} - 1)(2^{j-i-1} - 1) + 2^{2j-i-6} \cdot 3(3^{i-2} - 2^{i-1} + 1)] \). In general, for any \( n \geq 3 \) and \( 3 \leq i < j \leq n \),

\[
cp_n(\{i, j\}) = 2^{n-3}(2^{i-2} - 1)(2^{j-i-1} - 1) + 2^{n+j-i-5} \cdot 3(3^{i-2} - 2^{i-1} + 1).
\]

### 3 The recurrence relations for the sequence \( cp_n(S) \)

In this section, we will derive some recurrence relations for the sequence \( cp_n(S) \).
Lemma 3.1 Let $n \geq 3$ and $S \subseteq [n-1]$. Then
\[
 cp_n(S \cup \{n\}) = [n - 2 - 2|S||cp_{n-1}(S) + \sum_{j \notin S, j < n} 2cp_{n-1}(S \cup \{j\}).
\]

Proof. Suppose $\sigma \in CP_{n-1}(S)$. We want to form a new permutation $\tau \in CP_n(S \cup \{n\})$ by inserting the letter $n$ into $\sigma$. For any $j \in S$, since the letter $j$ still must be a circular peak in the new permutation, we can’t insert $n$ into $\sigma$ beside $j$. But the letter $n$ must be a circular peak. So, there are $(n - 2 - 2|S|)$ ways to form a new permutation $\tau$ from $\sigma$ such that $\tau \in CP_n(S \cup \{n\})$.

For any $j \notin S$ with $j < n$ and $\sigma \in CP_{n-1}(S \cup \{j\})$, we must insert $n$ into $\sigma$ beside $j$ such that $n$ becomes a circular peak. So, there are 2 ways to form a new permutation $\tau$ from $\sigma$ such that $\tau \in CP_n(S \cup \{n\})$.

Hence,
\[
 cp_n(S \cup \{n\}) = [n - 2 - 2|S||cp_{n-1}(S) + \sum_{j \notin S, j < n} 2 \cdot cp_{n-1}(S \cup \{j\}).
\]

For any $S \in [n]$, suppose $S = \{i_1, i_2, \ldots, i_k\}$, let $x_S$ stand for the monomial $x_{i_1}x_{i_2}\cdots x_{i_k}$; In particular, let $x_\emptyset = 1$. Given $n \geq 3$, we define a generating function
\[
g_n(x_1, x_2, \ldots, x_n; y) = \sum_{\sigma \in \mathfrak{S}_n} x_{CP(\sigma)}y^{|CP(\sigma)|}.
\]
We also write $g_n(x_1, x_2, \ldots, x_n; y)$ as $g_n$ for short.

Corollary 3.1 Let $n$ be a positive integer with $n \geq 3$ and $g_n = \sum_{\sigma \in \mathfrak{S}_n} x_{CP(\sigma)}y^{|CP(\sigma)|}$. Then $g_n$ satisfies the following recursion:
\[
g_{n+1} = [2 + (n - 1)x_{n+1}y]g_n + 2x_{n+1} \sum_{i=1}^{n} \frac{\partial g_n}{\partial x_i} - 2x_{n+1}y^2 \frac{\partial g_n}{\partial y}.
\]
for all $n \geq 3$ with initial condition $g_3 = 4 + 2x_3y$, where the notation $\frac{\partial g_n}{\partial y}$ denote partial differentiation of $g_n$ with respect to $y$.

**Proof.** Obviously, $g_3 = 4 + 2x_3y$ and $\sum_{\sigma \in S_n} x_{CP(\sigma)} y_{\lvert CP(\sigma) \rvert} = \sum_{S \subseteq [2,n]} c_p(S) x_S y_{\lvert S \rvert}$. Hence,

$$g_{n+1} = \sum_{S \subseteq [n+1]} c_p(S) x_S y_{\lvert S \rvert}$$

$$= \sum_{S \subseteq [n+1], n+1 \in S} c_p(S) x_S y_{\lvert S \rvert} + \sum_{S \subseteq [n+1], n+1 \notin S} c_p(S) x_S y_{\lvert S \rvert}$$

$$= \sum_{S \subseteq [n]} \left[ (n-1 - 2|S|)c_p(S) + \sum_{i \in [n] \setminus S} 2c_p(S \cup \{i\}) \right] x_S x_{n+1} y_{\lvert S \rvert+1} + 2g_n$$

$$= 2 \sum_{S \subseteq [n]} \sum_{i \in [n] \setminus S} c_p(S \cup \{i\}) x_S x_{n+1} y_{\lvert S \rvert+1} - 2 \sum_{S \subseteq [n]} |S| c_p(S) x_S x_{n+1} y_{\lvert S \rvert+1}$$

$$+ [2 + (n-1)x_{n+1}y]g_n.$$

Note that

$$\frac{\partial g_n}{\partial y} = \sum_{S \subseteq [n]} |S| c_p(S) x_S y_{\lvert S \rvert-1}$$

and

$$\sum_{S \subseteq [n]} \sum_{i \in [n] \setminus S} c_p(S \cup \{i\}) x_S x_{n+1} y_{\lvert S \rvert+1}$$

$$= \sum_{S \subseteq [n], S \neq \emptyset} c_p(S) x_{n+1} y_{\lvert S \rvert} \sum_{i \in S} x_i$$

$$= x_{n+1} \sum_{i=1}^n \frac{\partial g_n}{\partial x_i}.$$

Therefore,

$$g_{n+1} = [2 + (n-1)x_{n+1}y]g_n + 2x_{n+1} \sum_{i=1}^n \frac{\partial g_n}{\partial x_i} - 2x_{n+1}y^2 \frac{\partial g_n}{\partial y}.$$
Lemma 3.2 Suppose that $k \geq 0$ and $n \geq k + 4$. Let $S \subseteq [3, n - k - 1]$ with $CP_n(S) \neq \emptyset$. Then

$$cp_n(S \cup [n - k + 1, n]) = 2(k + 1)cp_{n-1}(S \cup [n - k, n - 1]) + k(k + 1)cp_{n-2}(S \cup [n - k, n - 2]).$$

Proof. For any $\sigma \in CP_n(S \cup [n - k + 1, n])$, we consider the following four cases.

Case 1. There is no letters $i \in [n - k + 1, n]$ such that the position of $i$ is beside $n - k$ in $\sigma$, i.e., $|\sigma^{-1}(i) - \sigma^{-1}(n - k)| = 1$. Then $\sigma^{-1}(n - k) = 1$ or $n$ since the permutation $\sigma$ hasn’t a circular peak $n - k$. We can obtain a new permutation $\tau$ by exchanging the positions of $n - k$ and $n$ in $\sigma$. Clearly, $\tau \in CP_n(S \cup [n - k, n - 1])$. Lemma 2.1 (1) tells us that $cp_n(S \cup [n - k, n - 1]) = 2cp_{n-1}(S \cup [n - k, n - 1])$). Hence, the number of permutations under this case is $2 \cdot cp_{n-1}(S \cup [n - k, n - 1])$.

Case 2. There are exact two letters $j, m \in [n - k + 1, n]$ such that $|\sigma^{-1}(j) - \sigma^{-1}(n - k)| = 1$ and $|\sigma^{-1}(m) - \sigma^{-1}(n - k)| = 1$. Deleting $j$ and $m$, we obtain a subsequence $\tau$ of $\sigma$. Then $red_{\sigma}(\tau) \in CP_{n-2}(S \cup [n - k, n - 2])$. Note that there are $k(k - 1)$ ways to form the pairs $(j, m)$. Hence, the number of permutations under this case is $k(k - 1)cp_{n-2}(S \cup [n - k, n - 2])$.

Case 3. There is exact one letter $j \in [n - k + 1, n]$ such that $|\sigma^{-1}(j) - \sigma^{-1}(n - k)| = 1$. Then there are $k$ ways to form the set $\{j\}$. Let $\tau$ be the subsequence of $\sigma$ obtained by deleting $j$. There are the following two subcases.

Subcase 3.1. $\sigma^{-1}(n - k) \neq 1$ and $n$. Then $red_{\sigma}(\tau) \in CP_{n-1}(S \cup [n - k, n - 1])$. Hence, the number of permutations under this subcase is $k \cdot cp_{n-1}(S \cup [n - k, n - 1])$.

Subcase 3.2. $\sigma^{-1}(n - k) = 1$ or $n$. Then $red_{\sigma}(\tau) \in CP_{n-2}(S \cup [n - k, n - 2])$. Hence, the number of permutations under this subcase is $k \cdot cp_{n-2}(S \cup [n - k, n - 2])$.

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So,

\[ cp_n(S \cup [n - k + 1, n]) \]

\[ = 2cp_{n-1}(S \cup [n - k, n - 1]) + k(k - 1)cp_{n-2}(S \cup [n - k, n - 2]) \]

\[ + 2k \cdot cp_{n-1}(S \cup [n - k, n - 1]) + 2k \cdot cp_{n-2}(S \cup [n - k, n - 2]) \]

\[ = 2(k + 1)cp_{n-1}(S \cup [n - k, n - 1]) + k + 1)cp_{n-2}(S \cup [n - k, n - 2]). \]

Setting \( S = \emptyset \) in Lemma 3.2, we derive the following results.

**Theorem 3.1** Let \( k \geq 1, \ n \geq \max\{3, 2k\} \) and \( CP_n([n - k + 1, n]) \neq \emptyset \). Suppose that

\[ cp_n([n - k + 1, n]) = k(k + 1) \sum_{i=1}^{k} (-1)^{i+1}b_{k,i} \left[(2k + 2)^n - (2k + 2 - 2i)^n \right]. \]

Then the coefficients \( b_{k,i} \) satisfy the recurrence relation as follows:

\[ b_{k+1,i} = \begin{cases} 
  k(k + 1)^2 \sum_{j=1}^{k} (-1)^{j+1}b_{k,j} & \text{if } i = 1 \\
  k(k + 1)^{k+2-i}b_{k,i-1} & \text{if } 2 \leq i \leq k + 1 
\end{cases} \]

**Proof.** Lemma 2.1 tells us that \( cp_n([n, n]) = cp_n([n]) = 4^{n-2} - 2^{n-2} \). Hence, \( b_{1,1} = \frac{1}{2} \). Lemma 3.2 implies that

\[ cp_n([n - k + 1, n]) = 2(k + 1)cp_{n-1}([n - k, n - 1]) + k + 1)cp_{n-2}([n - k, n - 2]). \]

By comparing the coefficients, we obtain the desired results.
By Theorem 3.1, we compute the values of $b_{k,i}$ for all $1 \leq k \leq 4$ as follows.

| $i$  | 1   | 2   | 3   | 4   |
|------|-----|-----|-----|-----|
| $k = 1$ | $\frac{1}{2}$ |   |   |   |
| 2    | 2   | $\frac{1}{2}$ |   |   |
| 3    | 27  | 12  | 1   |   |
| 4    | 768 | 486 | 96  | 3  |

Table 1. The values of $b_{k,i}$ for $1 \leq k \leq 4$

**Corollary 3.2** Let $k \geq 0$, $n \geq \max\{3, 2k\}$ and $CP_n([n - k + 1, n]) \neq \emptyset$. Suppose that

$$cp_n([n - k + 1, n]) = \sum_{i=0}^{k} (-1)^i a_{k,i} (2k + 2 - 2i)^{n-2k}.$$  

Then the coefficients $a_{k,i}$ satisfy the following recurrence relation:

$$a_{k+1,i} = \begin{cases} 
\sum_{j=1}^{k+1} (-1)^{j+1} a_{k+1,j} & \text{if } i = 0 \\
(k + 1)(k + 2)\frac{k+2-i}{i} a_{k,i-1} & \text{if } 1 \leq i \leq k + 1 
\end{cases}$$

with initial condition $a_{0,0} = \frac{1}{2}$. Let $f_k(x) = \sum_{i=0}^{k} a_{k,i} x^i$ for any $k \geq 0$, then $f_0(x) = \frac{1}{2}$, $f_{k+1}(-1) = 0$ and $f_k(x)$ satisfies the differential equation $f'_{k+1}(x) = (k + 1)^2 (k + 2) f_k(x) - (k + 1)(k + 2)x f'_k(x)$ for any $k \geq 0$, where the notation “′” denotes the differentiation of functions.

**Proof.** When $k = 0$, by Lemma 2.1 we have $cp_n([n + 1, n]) = cp_n(\emptyset) = 2^{n-1}$. Hence, $a_{0,0} = \frac{1}{2}$. When $k \geq 1$, it is easy to check that

$$a_{k,i} = \begin{cases} 
k(k + 1) \sum_{j=1}^{k} (-1)^{j+1} b_{k,j} & \text{if } i = 0 \\
k(k + 1) b_{k,i} & \text{if } 1 \leq i \leq k 
\end{cases}$$
Hence, by Theorem 3.1, we have

\[
a_{k+1,i} = \begin{cases} 
\sum_{j=1}^{k+1} (-1)^{j+1}a_{k+1,j} & \text{if } i = 0 \\
(k+1)(k+2)^{k+2-i}a_{k,i-1} & \text{if } 1 \leq i \leq k+1
\end{cases}
\]

with initial condition \(a_{0,0} = \frac{1}{2}\). Let \(f_k(x) = \sum_{i=0}^{k} a_{k,i}x^i\) for any \(k \geq 0\). Clearly, \(f_0(x) = \frac{1}{2}\).

Note that \(a_{k,i} = \sum_{j=1}^{k} (-1)^{j+1}a_{k,j}\) for any \(k \geq 1\). Hence, \(f_{k+1}(-1) = 0\) if \(k \geq 0\). Since \(a_{k+1,i} = (k+1)(k+2)^{k+2-i}a_{k,i-1}\) for any \(i \in [k+1]\), we have

\[
\sum_{i=1}^{k+1} a_{k+1,i}x^i = \sum_{i=1}^{k+1} (k+1)(k+2)^{k+2-i}a_{k,i-1}x^i
\]

Simple computations tell us that \(f'_{k+1}(x) = (k+1)^2(k+2)f_k(x) - (k+1)(k+2)xf_k'(x)\). ■

By Corollary 3.2 we compute the values of \(a_{k,i}\) for all \(0 \leq k \leq 3\) as follows.

|        | \(i=0\) | 1   | 2   | 3   |
|--------|---------|-----|-----|-----|
| \(k=0\)| \(\frac{1}{2}\) |     |     |     |
|        | 1       | 1   | 1   |     |
|        | 2       | 9   | 12  | 3   |
|        | 3       | 192 | 324 | 144 | 12  |

Table 2. The values of \(a_{k,i}\) for \(0 \leq i \leq 3\)

Recall that \(w(i, r, n, k) = \sum_{P \in P_{r,n,k}} w_i(P)\), where \(P_{r,n,k}\) is the set of all the circular-peak path from the vertices \((r, 0)\) to \((n, k)\) and \(w_i(P)\) is the weight of path \(P \in P_{r,n,k}\).

**Lemma 3.3**

\[
w(i, r, n, k) = 2^{n-r-2k} \prod_{m=0}^{r-1} (m+i+1)(m+i+2) \sum_{m=0}^{k} (i+m+1)^{t^m}
\]
where the sum is over all \((k + 1)\)-tuples \((t_0, t_1, \cdots, t_k)\) such that \(\sum_{m=0}^{k} t_m = n - r - 2k\) and \(t_m \geq 0\).

**Proof.** Suppose that \(P = e_1 e_2 \cdots e_{n-k-r} \in P_{r,n,k}\). Let \(R = \{j \mid e_j = R\}\), then \(|R| = k\).

Furthermore, we may suppose that \(R = \{e_{j_1}, \cdots, e_{j_k}\}\), where \(0 = j_0 < j_1 < j_2 < \cdots < j_k \leq n - k - r = j_{k+1}\). Hence,

\[
w_i(P) = \prod_{m=0}^{k} [2i + 2m + 2]^{j_{m+1}-j_m-1} \prod_{m=0}^{k-1} (m + i + 1)(m + i + 2).
\]

Let \(t_m = j_{m+1} - j_m - 1\) for any \(0 \leq m \leq k\), then \(t_m \geq 0\) and \(\sum_{m=0}^{k} t_m = n - r - 2k\). So,

\[
w(i, r, n, k) = \sum_{m=0}^{k} \prod_{m=0}^{k} [2i + 2m + 2]^{t_m} \prod_{m=0}^{k-1} (m + i + 1)(m + i + 2)
= 2^{n-r-2k} \prod_{m=0}^{k-1} (m + i + 1)(m + i + 2) \sum_{m=0}^{k} [i + m + 1]^{t_m}
\]

where the sum is over all \((k + 1)\)-tuples \((t_0, t_1, \cdots, t_k)\) such that \(\sum_{m=0}^{k} t_m = n - r - 2k\) and \(t_m \geq 0\).

\[
\]

**Lemma 3.4** Suppose \(n \geq 3\) and \(k \geq 0\). Let \(S \subseteq [3, n - k - 1]\) with \(CP_n(S) \neq \emptyset\) and \(r = \max S\). Then

\[
\text{cp}_n(S \cup [n - k + 1, n]) = \sum_{i=0}^{k} w(i, r, n - i, k - i) \text{cp}_{r+i}(S \cup [r + 1, r + i])
\]

**Proof.** We view the set \(S \cup [n - k + 1, n]\) as a vertex \((n, k)\). Connect the vertices \((n, k)\) with \((n-1,k)\) (resp.\((n-2,k-1)\)) and give this edge a weight \(2(k+1)\) (resp.\(k(k+1)\)). We draw the obtained graph as follows:
Conversely, from the graph, we also can derive the recurrence relation in Lemma 3.2. Repeating this processes, we will obtain the following graph with weight.

Hence, \( cp_n(S \cup [n-k+1, n]) \) can be expressed as a linear combination of \( cp_{r+i}(S \cup [r+1, r]) \), \( cp_{r+i}(S \cup [r+1, r+1]) \), \( \cdots \), and \( cp_{r+i}(S \cup [r+1, r+k]) \). It is easy to check that the coefficient of \( cp_{r+i}(S \cup [r+1, r+i]) \) is \( w(i, r, n-i, k-i) \) which is the sum of the weights of all the circular peak path from \( (r+i, i) \) to \( (n, k) \). So, \( cp_n(S \cup [n-k+1, n]) = \sum_{i=0}^{k} w(i, r, n-i, k-i) cp_{r+i}(S \cup [r+1, r+i]) \).

For any \( S \subseteq [3, n] \), suppose that the type of the set \( S \) is \( (r_1^{k_1}, r_2^{k_2}, \cdots, r_m^{k_m}) \). We have studied the case with \( m = 1 \) in Theorem 3.1. We will consider the case with \( m \geq 2 \) in the following theorem.

**Theorem 3.2** Let \( n \geq 3 \), \( k \geq 0 \) and \( S \subseteq [3, n] \). Suppose that \( CP_n(S) \neq \emptyset \) and the type of
the set $S$ is $(r_1^{k_1}, r_2^{k_2}, \ldots, r_m^{k_m})$ with $m \geq 2$, where $m \geq 2$. Then
\[
\text{cp}_n(S) = 2^{n-r_m} \sum_{i_1=0}^{k_m} \sum_{i_2=0}^{k_{m-1}+i_1} \cdots \sum_{i_{m-1}=0}^{k_m-i_{m-1}} \prod_{j=1}^{m-1} w(i_j, r_{m-j}, r_{m-j+1} + i_{j-1} - i_j, k_{m-j+1} + i_{j-1} - i_j) \text{cp}_{r_1+i_{m-1}([r_1 - k_1 + 1, r_1 + i_{m-1}])}
\]
where $i_0 = 0$.

**Proof.** Iterating the identity in Lemma 3.4, we obtain the desired results. 

**Example 3.1** Let $S = \{i, j\}$ such that $3 \leq i < j - 1$, $j \leq n$ and $CP_n(S) \neq \emptyset$. The type of $S$ is $(i^1, j^1)$. By Theorem 3.2, we have $\text{cp}_n(\{i, j\}) = 2^{n-j}[w(0, i, j, 1)\text{cp}_i(\{i\}) + w(1, i, j - 1, 0)\text{cp}_{i+1}(\{i, i+1\})]$. Lemma 3.3 implies that $w(0, i, j, 1) = 2^{i-1}(2^{j-i-1} - 1)$ and $w(1, i, j - 1, 0) = 2^{i-2}$. Theorem 2.1 tells us that $\text{cp}_i(\{i\}) = 2^{i-1}(2^{i-1} - 1)$. From Corollary 3.2 it follows that $\text{cp}_{i+1}(\{i, i+1\}) = \sum_{m=0}^{2} (-1)^m a_{2,m}(6-2m)^{i-3} = 9 \cdot 6^{i-3} - 12 \cdot 4^{i-3} + 3 \cdot 2^{i-3}$. Hence, we obtain the formula $\text{cp}_n(\{i, j\}) = 2^{n-3}(2^{i-2} - 1)(2^{j-i-1} - 1) + 2^{n+j-i-5} \cdot 3(3^{i-2} - 2^{i-1} + 1)$ again.

### 4 Appendix

For convenience to check the identities given in the previous sections, by the computer search, for $3 \leq n \leq 8$, we obtain the number $\text{cp}_n(S)$ of the permutations in the sets $CP_n(S) \neq \emptyset$ and list them in Table 3.
\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$n = 3$ & $S = \emptyset$ & (3) & & & & \\
\hline
 4   & 2   & & & & & \\
\hline
$n = 4$ & $S = \emptyset$ & (3) & (4) & & & \\
\hline
 8   & 4   & 12 & & & & \\
\hline
$n = 5$ & $S = \emptyset$ & (3) & (4) & (5) & (3, 5) & (4, 5) \\
\hline
16  & 8   & 24 & 56 & 4 & 12 & \\
\hline
$n = 6$ & $S = \emptyset$ & (3) & (4) & (5) & (6) & (3, 5) & (3, 6) & (4, 5) & (4, 6) \\
\hline
 32  & 16  & 48 & 112 & 240 & 8 & 24 & 24 & 72 & \\
\hline
$S = (5, 6)$ & & & & & & & & & \\
\hline
144 & & & & & & & & & \\
\hline
$n = 7$ & $S = \emptyset$ & (3) & (4) & (5) & (6) & (7) & (3, 5) & (3, 6) & (3, 7) \\
\hline
 64  & 32  & 96 & 224 & 480 & 992 & 16 & 48 & 112 & \\
\hline
$S = (4, 5)$ & (4, 6) & (4, 7) & (5, 6) & (5, 7) & (6, 7) & (3, 5, 7) & (3, 6, 7) & (4, 5, 7) & \\
\hline
 48  & 144 & 336 & 288 & 688 & 1200 & 8 & 24 & 24 & \\
\hline
$S = (4, 6, 7)$ & (5, 6, 7) & & & & & & & & \\
\hline
 72  & 144 & & & & & & & & \\
\hline
$n = 8$ & $S = \emptyset$ & (3) & (4) & (5) & (6) & (7) & (8) & (3, 5) & (3, 6) \\
\hline
 128 & 64  & 192 & 448 & 960 & 1984 & 4032 & 32 & 96 & \\
\hline
$S = (3, 7)$ & (3, 8) & (4, 5) & (4, 6) & (4, 7) & (4, 8) & (5, 6) & (5, 7) & (5, 8) & \\
\hline
 224 & 480 & 96 & 288 & 672 & 1440 & 576 & 1376 & 2976 & \\
\hline
$S = (6, 7)$ & (6, 8) & (7, 8) & (3, 5, 7) & (3, 5, 8) & (3, 6, 7) & (3, 6, 8) & (3, 7, 8) & (4, 5, 7) & \\
\hline
 2400 & 5280 & 8640 & 16 & 48 & 48 & 144 & 288 & 48 & \\
\hline
$S = (4, 5, 8)$ & (4, 6, 7) & (4, 6, 8) & (4, 7, 8) & (5, 6, 7) & (5, 6, 8) & (5, 7, 8) & (6, 7, 8) & & \\
\hline
 144 & 144 & 432 & 864 & 288 & 864 & 1728 & 2880 & & \\
\hline
\end{tabular}
\end{center}
\caption{\(cp_n(S)\) for \(3 \leq n \leq 8\) with \(CP_n(S) \neq \emptyset\)}
\end{table}

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