Volume growth and topological entropy of certain partially hyperbolic systems

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Abstract

Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$ admitting a partially hyperbolic splitting $TM = E^s \oplus E^1 \oplus \cdots \oplus E^l \oplus E^u$ where $E^s$ is uniformly contracting, $E^u$ is uniformly expanding and $\dim E^i = 1$, $1 \leq i \leq l$. We prove an entropy formula w.r.t. the volume growth rate of subspaces in the tangent bundle:

$$h_{\text{top}}(f) = \lim_{n \to +\infty} \frac{1}{n} \log \int \max_{V \subset T_x M} |\det Df^n|_V |dx.$$  

1 Introduction

There are many invariants (e.g., various entropies, volume growth) that can measure the complexity of a dynamical system. The relationships between them were studied. One interesting inequality is an upper bound of the topological entropy:

$$h_{\text{top}}(f) \leq \limsup_{n \to +\infty} \frac{1}{n} \log \int ||(Df^n)^{\wedge}|| dx$$  

(1)

where $f$ is a smooth diffeomorphism on a compact Riemannian manifold $M$ and $(Df^n)^{\wedge}$ is the induced (by $Df^n$) map between exterior algebras of the tangent spaces $T_x M$ and $T_{f^n x} M$. Here $|| \cdot ||$ is the norm on operators, induced from the Riemannian metric.

This was obtained by Przytychi [17] for $C^{1+\alpha}$ diffeomorphisms. Later on, it has been proved by Kozlovski [14] that for $C^\infty$ diffeomorphisms, it is in fact an equality. In this work, we would like to extend Przytychi-Kazlovski’s results in a different setting: some $C^1$ partially hyperbolic diffeomorphisms.

Hereafter, we always assume that $V$ denotes a linear space (or subspace). Our main result is the following:

**Theorem A.** Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$. Assume there is a partially hyperbolic splitting $TM = E^s \oplus E^1 \oplus \cdots \oplus E^l \oplus E^u$ where $E^s$ is uniformly contracting, $E^u$ is uniformly expanding and $\dim E^i = 1$, $1 \leq i \leq l$. Then

$$h_{\text{top}}(f) = \liminf_{n \to +\infty} \frac{1}{n} \log \int \max_{V \subset T_x M} |\det Df^n|_V |dx = \limsup_{n \to +\infty} \frac{1}{n} \log \int \max_{V \subset T_x M} |\det Df^n|_V |dx.$$ 

Note that $||(Df^n)^{\wedge}||$ has a geometrical explanation:

$$||(Df^n)^{\wedge}|| = \max_{V \subset T_x M} |\det Df^n|_V |.$$

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In this paper, we will use that latter notation as it provides more direct computation.

The dynamics of hyperbolic systems was understood very well. Beyond uniform hyperbolicity, partially hyperbolic diffeomorphisms as in Theorem A inherit some strong hyperbolicity. But the dynamics of these systems is not as clear as the hyperbolic case. The splitting type in Theorem A has been shown to be abundant among diffeomorphisms away from homolnic tangencies by Crovisier-Sambarino-Yang [9]. For some other related work about the partial hyperbolicity with multi 1-D centers, one can see [11], [7].

There are several other works which establish the relationship between entropy and the growth rates of volumes. Here we give a partial list.

- In \(C^\infty\) setting, Yomdin [20] showed a formula between the topological entropy and other form of volume growth on sub-manifolds (in contrast with our volume growth which is on the tangent space).

- Also in \(C^\infty\) setting, Burguet [4] has shown that for Lebesgue almost every point \(x \in M\), there is some invariant probability measure in the limit set of \(\{\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}\}_{n \in \mathbb{N}}\) with entropy larger than (or equal to) the volume growth rate on the direction with positive upper Lyapunov exponents. He also gave some counterexample to show that it is not true if the system only has finite regularity.

- Cogswell [6] showed in \(C^2\) setting that one can use the volume growth rate of one single local unstable manifold to bound above the metric entropy. Recently, a preprint [21] extends Cogswell’s result which bounds the metric entropy also by a mixture between volume growth rate (or many other invariants) and positive Lyapunov exponents.

- In the \(C^{1+\alpha}\) setting, by using Pesin theory, Newhouse [15] showed that the metric entropy is bounded above by the volume growth rate of sub-manifolds which are transverse to the stable manifolds.

- In some \(C^1\) partially hyperbolic setting (or dominated splitting), Saghin [18], Guo-Liao-Sun-Yang [12] proved that the metric entropy can be bounded above by a mixture between the positive Lyapunov exponents and the volume growth of some sub-manifold. In [8] and [10], for Lebesgue almost every point \(x \in M\), a lower bound of the metric entropy of the measures in the limit set of \(\{\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}\}_{n \in \mathbb{N}}\) is established w.r.t. the sum of the Lyapunov exponents on the stronger sub-bundle.

Most of the known results (e.g. [18], [12], [17]) are concentrated on establishing inequalities for entropies (upper bound) when the system are not smooth enough. Now we can get a more precise relationship between the topological entropy and the volume growth in the \(C^1\) partially hyperbolic setting. So we have to do more work on the lower bound of the topological entropy.

Our strategy is to estimate the volume growth on dynamical balls from above and below. Roughly speaking, we prove that for sufficiently many points \(x \in M\) and sufficiently small number \(\delta > 0\),

\[
\int_{B(x,n,\delta)} \max_{V \subset T_x M} |\det Df^n|_V |dz| \approx \text{sub-exponentially small}.
\]

Usually, the above estimate of the volume growth on dynamical balls is established only for some Lyapunov regular points. But the corresponding estimation on Lyapunov regular points is not sufficient (the Lyapunov regular points might have zero Lebesgue measure) to get an entropy formula (equality). Therefore, we have to get a uniform bound on the volume growth on the dynamical balls at all points in \(M\) (see Lemma 2.7). Unfortunately, Lemma 2.7 only gives estimation of the volume growth on the stronger direction (w.r.t. to a dominated splitting). We come up with a general result in Section 3 to verify that the maximal volume growth can be
achieved by the stronger direction not only for Lyapunov regular points but also for all points in \( M \). This eventually leads us to the entropy formula.

## 2 Volume growth of dynamical balls: estimation on topological entropy

Let \( f \) be a \( C^1 \) diffeomorphism on a compact manifold \( M \). We first introduce the concept of partially hyperbolic splitting.

We say \( f \) admits a dominated splitting \( TM = E \oplus_F F \) if \( E, F \) are both \( Df \)-invariant and there are two constants \( C > 0 \) and \( \lambda \in (0, 1) \) such that for any integer \( k \in \mathbb{N} \), any \( x \in M \) and any non-zero vectors \( v_E \in E(x), v_F \in F(x) \), we have

\[
\frac{\|Df^k_x(v_E)\|}{\|v_E\|} \leq C \lambda^k \cdot \frac{\|Df^k_x(v_F)\|}{\|v_F\|}.
\]

We say \( f \) admits a partially hyperbolic splitting \( TM = E^s \oplus \cdots \oplus E^l \oplus E^u \), if

- For each \( i = 0, 1, \ldots, l \), \( TM = (E^s \oplus E^1 \oplus \cdots \oplus E^i) \oplus (E^{i+1} \oplus \cdots \oplus E^l \oplus E^u) \) is a dominated splitting.

- \( E^s \) is uniformly contracting and \( E^u \) is uniformly expanding: there are two constants \( C > 0 \) and \( \lambda \in (0, 1) \) such that for any integer \( k \in \mathbb{N} \), any \( x \in M \) and any non-zero vectors \( v^s \in E^s(x), v^u \in E^u(x) \), we have

\[
\frac{\|Df^k_x(v^s)\|}{\|v^s\|} \leq C \lambda^k, \quad \frac{\|Df^{-k}_x(v^u)\|}{\|v^u\|} \leq C \lambda^k.
\]

### 2.1 Upper bound of topological entropy

In this subsection, our goal is to bound above the topological entropy by the volume growth rate.

**Proposition 2.1.** Let \( f \) be a \( C^1 \) diffeomorphism on a compact manifold \( M \). Assume there is a partially hyperbolic splitting \( TM = E^s \oplus E^1 \oplus \cdots \oplus E^l \oplus E^u \) with \( \dim E^i = 1, 1 \leq i \leq l \). Then

\[
\hat{h}(f) \leq \liminf_{n \to \infty} \frac{1}{n} \log \int_{\mathbb{S}^n} \max_{v \in T_x M} |\det Df^n_x| v | dx.
\]

Let \( h(f, \mu) \) denote the metric entropy of an invariant measure \( \mu \). Proposition 2.1 above is a direct application of the following result.

**Lemma 2.2.** Let \( f \) be a \( C^1 \) diffeomorphism on a compact manifold \( M \). Assume there is a partially hyperbolic splitting \( TM = E^s \oplus E^1 \oplus \cdots \oplus E^l \oplus E^u \) with \( \dim E^i = 1, 1 \leq i \leq l \). Then for any ergodic measure \( \mu \),

\[
\hat{h}(f, \mu) \leq \liminf_{n \to \infty} \frac{1}{n} \log \int |\det Df^n_x|_{F^k} | dx
\]

where \( F^k = E^k \oplus E^{k+1} \oplus \cdots \oplus E^l \oplus E^u \) and \( k \) is the smallest integer such that the Lyapunov exponent on \( E^k \) is non-negative.
Theorem 1.2 in [12] gives a similar upper bound where the volume growth was established with \( \text{lim sup} \). Here we improve their result by using \( \text{lim inf} \) for partially hyperbolic systems with multi 1-D centers. Similar ideas can also be found in [14] for \( C^{1+\alpha} \) \( (\alpha > 0) \) systems (see Section 3.3 in [14]).

We postpone the proof of Lemma 2.2. As a consequence of Lemma 2.2, we first prove Proposition 2.1.

Proof of Proposition 2.1. First we note that for any \( 1 \leq k \leq l \) and \( x \in M \),

\[
| \det Df^m_x |_{E^k} \leq \max_{V \subset T_x M} | \det Df^n_x |_V |.
\]

Recall the variational principal (see [19, Theorem 8.6]):

\[
h_{\text{top}}(f) = \sup_{\mu} h(f, \mu)
\]

where the supreme is taken over all ergodic measures. Then the conclusion is a direct consequence of Lemma 2.2.

Now the rest of this subsection is devoted to the proof of Lemma 2.2.

Let \( B(x, \delta) \) be the ball in \( M \) centered at \( x \) with radius \( \delta \). We define the dynamical ball

\[
B(x, n, \delta) = \bigcap_{i=0}^{n-1} f^{-i}(B(f^i(x), \delta)).
\]

Assume there is a dominated splitting \( TM = E \oplus\subset F \). Given \( \alpha > 0 \), we define the cone along \( F \) with width \( \alpha \):

\[
C^\alpha_F = \{ v_E + v_F | x \in M, v_E \in E_x, v_F \in F_x, |v_E| \leq \alpha \cdot |v_F| \}.
\]

A sub-manifold \( \gamma \) with \( \dim \gamma = \dim F \) is called tangent to the cone \( C^\alpha_F \) if for any \( x \in \gamma \), \( T_x \gamma \subset C^\alpha_F \).

We next give a key estimation on the lower bound of the volume growth of dynamical balls.

Lemma 2.3. Let \( f \) be a \( C^1 \) diffeomorphism on a compact manifold \( M \). Assume there is a dominated splitting \( TM = E \oplus\subset E^{\text{cu}} \). Let \( \mu \) be an ergodic measure such that the Lyapunov exponents on \( E \) are negative and the Lyapunov exponents on \( E^{\text{cu}} \) are non-negative. Then for any \( \epsilon, \delta > 0 \), there are a subset \( \Omega \) with \( \mu(\Omega) > \frac{1}{2} \) and an integer \( N_0 \) such that for any \( x \in \Omega \),

\[
\int_{B(x, n, \delta)} | \det Df^m_x |_{E^{\text{cu}}} dz \geq e^{-n\epsilon}, \quad n \geq N_0.
\]

Proof. Given a small number \( \epsilon > 0 \) (much smaller than the gaps between the Lyapunov exponents of \( \mu \)), define

\[
\Omega^N_\epsilon = \{ x \in M | m(Df^n_x |_{E^{\text{cu}}}) \geq e^{-\frac{n\epsilon}{2}}, \forall n \geq N \}
\]

where \( m(A) \) denotes the minimum norm of the linear map \( A \), i.e., \( m(A) = \inf_{v} \frac{\| A(v) \|}{\| v \|} \).

We note that \( \mu(\Omega^N_\epsilon) \rightarrow 1 \) as \( N \rightarrow +\infty \).

For \( \mu \) almost every point \( x \in \Omega^N_\epsilon \), we define two increasing sequence of integers \( n_0 < n_1 < n_2 < \cdots \) and \( r_0 \leq r_1 \leq r_2 \leq \cdots \) inductively:

- Define \( r_0 = n_0 = 0 \),
Suppose that \( r_i \) and \( n_i \) have been defined. If \( f^{n_i}(x) \in \Omega^N_x \), we then define \( n_{i+1} = n_i + N \) and \( r_{i+1} = r_i \). If \( f^{n_i}(x) \notin \Omega^N_x \), we define \( n_{i+1} = n_i + r \) and \( r_{i+1} = r_i + r \) where \( r \) is the smallest positive integer such that \( f^{n_{i+r}}(x) \in \Omega^N_x \).

Let
\[
j(n) = \max\{n_i \mid n_i \leq n\}, \quad r_x(n) = N + n - j(n) + \max\{r_i \mid n_i \leq n\}.
\]
We note by definition,
\[
r_x(n) \leq N + \#\{i \leq n \mid f^i(x) \notin \Omega^N_x\}.
\]

Let \( \delta_0 \) be a small number such that for any \( z \in \Delta \), the exponential map \( \exp_z \) is a local diffeomorphism.

For each \( n \in \mathbb{N} \) and \( \tau \leq \delta \), we use \( \mathcal{W}(f^n(x), \tau) \) to denote the connected component in the ball \( B(f^n(x), \tau) \) of
\[
\mathcal{W}(\exp_x(E_x^u(\delta))^n)(f^n(x))
\]
which contains \( f^n(x) \). Let \( C = \max_{x \in M} ||Df_x|| \). We define
\[
D_n = f^{-n}(\mathcal{W}(f^n(x), C^{-r_x(n)}. C^{-N} e^{-(n-r_x(n))\varepsilon})).
\]

By domination, there are some \( T \in \mathbb{N} \) and some small number \( \alpha > 0 \) (independent of \( x \)) such for any embedded sub-manifold \( \gamma \) tangent to the cone \( C^\alpha_{\mathcal{E}^{cu}} \) and any \( k \geq T \), \( f^k(\gamma) \) is also tangent to \( C^\alpha_{\mathcal{E}^{cu}} \). By shrinking \( \delta_0 \) if necessary (independent of \( x \)), we can assume for any \( x \) and its corresponding \( D_n, f^k(D_n) \) is tangent to the small cone \( C^\alpha_{\mathcal{E}^{cu}} \) for \( k \in \mathbb{N} \). Moreover we also assume that for any \( \delta \leq \delta_0 \), any \( \gamma \) tangent to \( C^\alpha_{\mathcal{E}^{cu}} \), any \( k \in \mathbb{N} \) and any \( y, z \in \gamma \), if for any \( 0 \leq i \leq k - 1 \), \( d(f^i(y), f^i(z)) \leq \delta \), then
\[
\frac{m(Df^k|T_{xy})}{m(Df^k|E_x)} \geq e^{-\frac{4\varepsilon}{\delta}}.
\]

**Claim.** For each \( x \in \Omega^N_x \) and each \( n \), \( D_n \subset B(x, n, \delta) \cap \exp_x(E_x^{cu}(\delta)) \).

**Proof.** We first show that \( D_j \subset D_{n_i} \) for each \( i \) and each \( j \geq n_i \). We prove it with two steps:

1. \( D_{n_{i+1}} \subset D_{n_i} \) for each \( i \),
2. \( D_j \subset D_{n_i} \) for each \( n_i \leq j \leq n_{i+1} \).

For the first property, it is sufficient to show that \( f^{n_{i+1} - n_i}(f^{n_i}(D_{n_i})) \) covers \( f^{n_{i+1}}(D_{n_{i+1}}) \), i.e., \( f^{n_{i+1}}(D_{n_{i+1}}) \subset f^{n_{i+1} - n_i}(f^{n_i}(D_{n_i})) \). By definition of \( r_x(n) \), the radius of \( f^{n_i}(D_{n_i}) \) is
\[
C^{-r_x(n_i)} \cdot C^{-N} e^{-(n_i - r_x(n_i))\varepsilon}
\]
and the radius of \( f^{n_{i+1}}(D_{n_{i+1}}) \) is
\[
C^{-r_x(n_{i+1})} \cdot C^{-N} e^{-(n_{i+1} - r_x(n_{i+1}) - N)\varepsilon}.
\]
If \( f^{n_i}(x) \in \Omega^N_x \), then by definition,
\[
\text{Proof.}
\]

As a consequence, the radius of \(f^{n_i+1-n_i}(f^{n_i}(D_{n_i}))\) is at least larger than the radius of \(f^{n_i}(D_{n_i})\) times the factor \(e^{-N\delta}\) (which is exactly the radius of \(f^{n_i+1}(D_{n_i+1})\)). By comparing the radii, we conclude that \(f^{n_i+1-n_i}(f^{n_i}(D_{n_i}))\) covers \(f^{n_i+1}(D_{n_i+1})\).

Now for the case that \(f^{n_i}(x) \notin \Omega^n_\varepsilon\), by definition, we have

\[
\bullet n_{i+1} - n_i \leq N, r_{i+1} - r_i = 0 \quad \text{and consequently,} \quad r_x(n_i) = r_x(n_{i+1}),
\]

\[
\bullet \text{the expansion rate of } f^{n_i+1-n_i} \quad \text{(which is } f^N) \quad \text{on } f^{n_i}(D_{n_i}) \quad \text{is at least } e^{-N\delta}. \quad \text{Indeed, we note by definition that for any point } y \quad \text{in } f^{n_i}(D_{n_i}) \quad \text{and any } 0 \leq j \leq N - 1, d(f^j(f^{n_i}(x)), f^j(y)) \leq \delta. \quad \text{We then conclude by Equation (2) and the fact } f^{n_i}(x) \in \Omega^n_\varepsilon.
\]

As a consequence, for every point \(x \in f^{n_i}(D_{n_i})\), any \(0 \leq j \leq N - 1\), \(d(f^j(x), f^j(y)) \leq \delta\). By the inequality (3), it remains to check the property for each \(i \leq j < n_{i+1}\) and consequently, for \(f^j(x) \notin \Omega^n_\varepsilon\).

To see the second property, for any \(n_i \leq j < n_{i+1}\), by definition, we note that the difference between the two radii of \(f^n(D_{n_i})\) and \(f^j(D_j)\) is a factor \(C^{-\gamma}\) which is the maximal contraction for \(f^n\). Then get that \(f^{n_i}(D_{n_i})\) covers \(f^{n_i+1}(D_{n_i+1})\). This implies \(D_j \subset D_{n_i}\).

Now we have shown that \(D_j \subset D_{n_i}\) for each \(i\) and each \(j \geq n_i\). As a direct consequence, it gives that for any \(n, y \in D_n\), any \(n_i \leq n\) and any \(n_i \leq j \leq \min\{n, n_i + r_{i+1} - r_i\},\)

\[
d(f^j(x), f^j(y)) \leq C^{-N\delta}. \tag{3}
\]

To show \(D_n \subset B(x, n, \delta) \cap \exp_x(E^\text{cu}_x(\delta))\), it is sufficient to verify that for any \(y \in D_n\) and any \(0 \leq j \leq n\), \(d(f^j(x), f^j(y)) \leq \delta\). By the inequality (3), it remains to check the property for each \(n_i < j < n_{i+1}\) in the case of \(n_{i+1} - n_i = N\) with \(f^{n_i}(x) \in \Omega^n_\varepsilon\). Indeed, this is guaranteed by the fact that \(d(f^{n_i}(x), f^{n_i}(y)) \leq C^{-N\delta}\). The proof of the claim is complete.

Given a number \(\delta > 0\), we write \(\gamma_x = B(x, n, \delta) \cap \exp_x(E^\text{cu}_x(\delta))\) for short. Write \(d = \dim M\).

**Claim.** For any \(0 < \delta \leq \delta_0\), there are \(N_0, N_1 \in \mathbb{N}\) and a subset \(\Omega_\varepsilon \subset \Omega^{N_1}_\varepsilon\) with \(\mu(\Omega_\varepsilon) > \frac{1}{2}\) such that for every point \(x \in \Omega^{N}_\varepsilon\),

\[
\int_{\gamma_x} |\det Df^n|_{T_\gamma_x} dt \geq e^{-3d\varepsilon} \quad n \geq N_0.
\]

**Proof.** For a recurrent point \(x \in \Omega^{N}_\varepsilon\), by the previous claim, we have

\[
\int_{\gamma_x} |\det Df^n|_{T_\gamma_x} dt \geq \int_{D_0} |\det Df^n|_{T_\gamma_x} dt
\]

\[
= \text{Vol}(\mathcal{W}(f^n(x), C^{-r_x(n)} \cdot C^{-N\delta} \cdot e^{-(n-r_x(n))\varepsilon}))
\]

\[
\geq \left(C^{-r_x(n)} \cdot C^{-N\delta} \cdot e^{-(n-r_x(n))\varepsilon}\right)^{\dim E^\text{cu}}.
\]

Recall that \(r_x(n) \leq N + \#\{i \leq n \mid f^i(x) \notin \Omega^n_\varepsilon\}\) and \(\mu(\Omega^n_\varepsilon) \to 1\) as \(N \to +\infty\). Also note that by recurrence (ergodicity), for \(\mu\) almost every \(x \in \Omega^n_\varepsilon\),

\[
\lim_{n \to +\infty} \frac{\#\{i \leq n \mid f^i(x) \in \Omega^n_\varepsilon\}}{n} = \mu(\Omega^n_\varepsilon).
\]

As a consequence, for \(\mu\) almost every \(x \in \Omega^n_\varepsilon\),

\[
\limsup_{n \to +\infty} \frac{r_x(n)}{n} \leq 1 - \mu(\Omega^n_\varepsilon).
\]
Hence we can choose $N_1$ large enough such that the set
\[
\tilde{\Omega}_\varepsilon^{N_1} = \left\{ x \in \Omega_\varepsilon^{N_1} \mid \lim_{n \to +\infty} \left( C^{-r_x(n)} \cdot e^{-(n-r_x(n))\varepsilon} \right)^{\dim E^{cu}} \geq e^{-2dn\varepsilon} \right\}
\]
carries $\mu$-measure larger than $\frac{2}{3}$. We then choose $N_0$ large enough and a subset $\Omega_\varepsilon \subset \tilde{\Omega}_\varepsilon^{N_1}$ with $\mu(\Omega_\varepsilon) > \frac{1}{2}$ such that for any $x \in \Omega_\varepsilon$,
\[
\left( C^{-r_x(n)} \cdot C^{-N_1\delta} \cdot e^{-(n-r_x(n))\varepsilon} \right)^{\dim E^{cu}} \geq e^{-3dn\varepsilon}, \quad \forall n \geq N_0.
\]
We complete the proof. 

Recall that we assume the Lyapunov exponents of $\mu$ on $E$ are negative. For a $C^1$ diffeomorphism with dominated splitting, F. Abdenur, C. Bonatti and S. Crovisier gives a non-uniform version of stable manifold theorem (see Proposition 8.9 in [1]) which states that for $\mu$ almost every point $x$, there is a local stable manifold $W^s_{\loc}(x)$ whose dimension is $\dim E$. Let $W^s_\tau(x)$ denote the local stable manifold with radius $\tau > 0$. We note that the radii of these local stable manifolds might not be uniformly bounded from below (like Pesin theory in $C^{1+\alpha}$ setting). We choose a number $0 < \tau \ll \delta$ small enough and a subset $\Omega \subset \Omega_\varepsilon$ with $\mu$-measure larger than $\frac{1}{2}$ such that for each $x \in \Omega$,

- the local stable manifold $W^s_{\loc}(x)$ has radius at least $\tau$, i.e., $W^s_\tau(x) \subset W^s_{\loc}(x)$.
- For any $n \in \mathbb{N}$, $W^s_\tau(x) \subset B(x, n, \delta)$.

For $x \in \Omega$, we now consider the local foliation
\[
\mathcal{F}_x = \{ \exp_x \left( \exp_x^{-1}(z) + E^{cu}_x(\delta) \right) \mid z \in W^s_\tau(\tau) \}.
\]

For simplicity, we write $\gamma_x(z) = B(x, n, \delta) \cap \exp_x \left( \exp_x^{-1}(z) + E^{cu}_x(\delta) \right)$. We note that by shrinking $\delta_0$ if necessary, we can assume that for any $\delta \leq \delta_0$, any $z \in B(x, n, \delta)$ and any $t \in \gamma_x$,

- there is a smooth diffeomorphism $h_z : \gamma_x \to \gamma_x(z)$ with $|\det(Dh_z)| \geq \frac{1}{2}$,
- since $\gamma_x \subset B(x, n, \delta)$,
\[
\frac{|\det Df^n_z|_{E^{cu}}}{|\det Df^n_t|_{T^1, \gamma_x}} \geq e^{-n\varepsilon}.
\]

Applying Fubini’s theorem to the foliation $\mathcal{F}_x$, by the above claim, we get that for any $n \geq N_0$,
\[
\int_{B(x, n, \delta)} |Df^n_z|_{E^{cu}} dz \geq \int_{W^s_\tau(x)} \left( \int_{\gamma_x(s)} |\det Df^n_t|_{E^{cu}}|dt| \right) ds \\
\geq \frac{1}{2} e^{-n\varepsilon} \int_{W^s_\tau(x)} \left( \int_{\gamma_x} |\det Df^n_t|_{T^1, \gamma_x}|dt| \right) ds \\
\geq \frac{1}{2} \tau^{\dim E} e^{-4dn\varepsilon}.
\]
Replacing $\varepsilon$ by $\frac{\varepsilon}{4n}$ and increasing $N_0$ if necessary, we can assume that for any $x \in \Omega$,
\[
\int_{B(x, n, \delta)} |Df^n_z|_{E^{cu}} dz \geq e^{-n\varepsilon}, \quad \forall n \geq N_0.
\]

We get the desired estimation for all $\delta$ small enough. Since here we are considering a lower bound, this is true for all $\delta > 0$. We complete the proof. 

\qed
A direct application of Lemma 2.3 on our situation is the following:

**Corollary 2.4.** Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$. Assume there is a partially hyperbolic splitting $TM = E^s \oplus E^u$ with $\dim E^i = 1$, $1 \leq i \leq l$. Then for any $\varepsilon, \delta > 0$ and any ergodic measure $\mu$, there are a subset $\Omega$ with $\mu(\Omega) > \frac{1}{2}$ and an integer $N_0$ such that for any $x \in \Omega$,

$$
\int_{B(x,n,\delta)} |\det Df^n_x|_{F^k} \, dz \geq e^{-n\varepsilon}, \quad n \geq N_0
$$

where $F^k = E^k \oplus E^{k+1} \oplus \cdots \oplus E^l \oplus E^u$ and $k$ is the smallest integer such that the Lyapunov exponent of $\mu$ on $E^k$ is non-negative (write $k = u$ if the Lyapunov exponent on $E^l$ is still negative).

Now we prove Lemma 2.2. We first recall a result of Katok (Theorem 1.1 in [13]).

Given $\lambda > 0$, let $S_\lambda(n, \tau)$ be the minimum number of dynamical balls $\{B(x,n,\tau)\}$ whose union has $\mu$-measure larger than or equal to $\lambda$. Recall that a subset $S \subset M$ is called a $(n, \tau)$ spanning set if $\{B(x,n,\tau)\}_{x \in S}$ covers $M$. Moreover, the spanning set $S$ is called minimal if its cardinality is smaller than or equal to the cardinality of any other spanning set. Given a subset $\Omega$, let $S(n, \tau, \Omega)$ denote a minimal $(n, \tau)$ spanning set of $\Omega$.

**Lemma 2.5** (Katok, [13]). Let $f$ be a homeomorphism on a compact metric space $X$ and let $\mu$ be an ergodic measure. Then for any $\lambda \in (0, 1)$,

$$
h(f, \mu) = \lim \liminf_{n \to \infty} \frac{1}{n} \log \#S_\lambda(n, \tau) = \lim \limsup_{n \to \infty} \frac{1}{n} \log \#S_\lambda(n, \tau).
$$

**Proof of Lemma 2.5.** Given $\varepsilon, \delta > 0$ and an ergodic measure $\mu$, let $\Omega$ be the set with $\mu(\Omega) > \frac{1}{2}$ in Corollary 2.4 and let $N_0$ be the corresponding integer. Let $S(n, 2\delta, \Omega)$ be a minimal $(n, 2\delta)$ spanning set of $\Omega$. By definition, for any $y_1, y_2 \in S(n, 2\delta, \Omega)$,

$$
B(y_1, n, \delta) \cap B(y_2, n, \delta) = \emptyset.
$$

For any $n \geq N_0$, by Corollary 2.4

$$
\int_M |\det Df^n_x|_{F_k} \, dx \geq \#S(n, 2\delta, \Omega) \cdot \min_{x \in S(n, 2\delta, \Omega)} \int_{B(x,n,\delta)} |\det Df^n_y|_{F_k} \, dy \geq \#S(n, 2\delta, \Omega) \cdot e^{-n\varepsilon}.
$$

Then

$$
\lim \inf_{n \to \infty} \frac{1}{n} \log \int_M |\det Df^n_x|_{F_k} \, dx + \varepsilon \geq \lim \inf_{n \to \infty} \frac{1}{n} \log \#S(n, 2\delta, \Omega).
$$

Since $\mu(\Omega) > \frac{1}{2}$, we have $S(n, 2\delta, \Omega) \geq S_{\frac{1}{2}}(n, 2\delta)$. By Lemma 2.5 and the arbitrariness of $\varepsilon$ and $\delta$, we get the result.

## 2.2 Lower bound of topological entropy

In this subsection, we bound from below the topological entropy by the volume growth rate along the dominated sub-bundles.

**Proposition 2.6.** Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$. Assume there is a partially hyperbolic splitting $TM = E^s \oplus E^1 \oplus E^2 \oplus \cdots \oplus E^l \oplus E^u$ with $\dim E^i = 1$, $1 \leq i \leq l$. Then

$$
h_{\text{top}}(f) \geq \limsup_{n \to \infty} \frac{1}{n} \log \int_{0 \leq i \leq l} \max_{0 \leq i \leq l} |\det Df^n_x|_{F_i} \, dx
$$

where the bundle $F^i = E^{i+1} \oplus E^{i+2} \oplus \cdots \oplus E^l \oplus E^u$. 
Combining Proposition 2.6 above with Proposition 3.1 in the next section, we will get the
desired lower bound of the topological entropy.

The key tool to prove Proposition 2.6 is to estimate from above the volume growth of the
dynamical balls which is stated in the following. We remark that in Corollary 2.4 we estimate
from below the volume growth of the dynamical balls for the partially hyperbolic systems with
multi 1-D center. To estimate from above, we do not have to know on which sub-bundle the
negative and non-negative Lyapunov exponents are separated. This brings two advantages: a
general domination (not have to be partially hyperbolic) is enough and a probabilistic argument
can be avoided so that we can consider the dynamical behavior of all points.

**Lemma 2.7.** Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$. Assume there is a
dominated splitting $TM = E \oplus \gamma$. Then there is a constant $C$ such that for any $\varepsilon > 0$, there
is a constant $\delta > 0$ such that for any $x \in M$ and $n \in \mathbb{N},$

$$\int_{B(x, \varepsilon, \delta)} |\det Df^n|_{\gamma}| \, dy \leq Ce^{n\varepsilon}. \quad (4)$$

**Proof.** Indeed, to prove the lemma, it is sufficient to find some $\delta_0 > 0$ such that Equation (4)
above holds for all $\delta \leq \delta_0$ with some constant $C$ only depending on $\delta_0$.

For $\delta > 0$, write

$$\tilde{B}(x, n, \delta) \triangleq \exp_{x}^{-1}(B(x, n, \delta)) \subset T_x M.$$

First note that there is $\delta_0 \ll 1$ small enough such that for any $\delta \leq \delta_0$, any $x \in M$ and any
$n \in \mathbb{N},$

$$\int_{B(x, \varepsilon, \delta)} |\det Df^n|_{\gamma}| \, dy \leq 2 \int_{B(x, \varepsilon, \delta)} |\det Df^n|_{\gamma} \circ \exp_x(w) \, dw. \quad (5)$$

Applying Fubini Theorem to the foliation $\mathcal{F}_x \triangleq \{s + F_x \mid s \in E_x\}$, we get that

$$\int_{\tilde{B}(x, n, \delta)} |\det Df^n|_{\gamma} \circ \exp_x(w) \, dw = \int_{\tilde{B}^E(x, n, \delta)} \left( \int_{\tilde{B}^F(x, n, \delta)} |\det Df^n|_{\gamma} \circ \exp_x((t, s)) \, dt \right) ds \quad (6)$$

where $\tilde{B}^E(x, n, \delta) = \tilde{B}(x, n, \delta) \cap E_x$ and $\tilde{B}^F(x, n, \delta) = \tilde{B}(x, n, \delta) \cap (s + F_x)$.

Write for simplicity

$$\gamma_x(s) \triangleq \exp_x(\tilde{B}^F(x, n, \delta)).$$

We fix a cone $C^\tau_F$ for some number $\tau > 0$ such that for each $\delta \leq \delta_0$, all the corresponding
$\{\gamma_x(s)\}$ are tangent to $C^\tau_F$. We also fix a constant $\hat{C}$ such that for any embedded sub-manifold
$\gamma$ tangent to $C^\tau_F$ with radius less than $\delta_0$, the induced Lebesgue volume is bounded above:

$$\text{Leb}(\gamma) \leq \hat{C}.$$

We note that the domination gives that there is some uniform integer $N \in \mathbb{N}$ such for any
embedded sub-manifold $\gamma$ tangent to $C^\tau_F$ and any $n \geq N$, $f^n(\gamma)$ is also tangent to $C^\tau_F$.

As a consequence, for any $\delta \leq \delta_0$ and any $x \in M$,

$$\text{Leb}(f^n(\gamma_x(0))) \leq \hat{C}, \quad n \geq N. \quad (7)$$

Moreover, given any $\varepsilon > 0$, by shrinking $\delta_0$ if necessary, we can assume that for any $\delta \leq \delta_0$,
any $x \in M$, $n \in \mathbb{N}$, any $s \in \tilde{B}^E(x, n, \delta)$ and $y \in \gamma_x(0)$, $z \in \gamma_x(s),$

$$\frac{|\det Df^n|_{\gamma|x(0)}|}{|\det Df^n|_{\gamma|x(0)}} \leq e^{n\varepsilon}. \quad (8)$$
Also note that we can identity \( \gamma_x(0) \) with any \( \gamma_x(s) \) through a smooth diffeomorphism \( h_s : \gamma_x(0) \to \gamma_x(s) \) with \( |\det (Dh_s)| \leq 2 \). Combining the formulas (7) and (8), we have for \( n \geq N \) and \( s \in \overline{B}^E(x, n, \delta) \),

\[
\int_{B^E(x, n, \delta)} |\det D f^n|_{F^i} \circ \exp_x(t, s) \, dt \leq 2e^{ne} \int_{\gamma_x(0)} |\det D f^n|_{T^y \gamma_x(0)} \, dy = e^{ne} \text{Leb}(f^n(\gamma_x(0))) \leq \tilde{C} e^{ne}.
\]

As a consequence, by letting \( C_0 = 4\tilde{C}, (5) \) and (6) immediately give that

\[
\int_{B(x, n, \delta)} \max_{0 \leq i \leq l} |\det D f^n|_{F^i} \, dy \leq C e^{ne}, \quad n \geq N.
\]

Replacing \( C_0 \) by a larger number \( C \) if necessary, we can assume that the above inequality holds w.r.t. \( C \) for all \( n \in \mathbb{N} \) and \( \delta \leq \delta_0 \).

For partially hyperbolic systems with multi one dimensional centers, we have the following application of Lemma 2.7.

**Corollary 2.8.** Let \( f \) be a \( C^1 \) diffeomorphism on a compact manifold \( M \). Assume there is a partially hyperbolic splitting \( TM = E^s \oplus \ltimes E^l \oplus \ltimes E^u \) where \( \dim E^i = 1, 1 \leq i \leq l \). Then there is a constant \( C \) such that for any \( \varepsilon > 0 \), there is \( \delta > 0 \) such that for any \( x \in M \) and any \( n \in \mathbb{N} \),

\[
\int_{B(x, n, \delta)} \max_{0 \leq i \leq l} |\det D f^n|_{F^i} \, dy \leq C e^{ne}
\]

where the bundle \( F^i = E^{i+1} \oplus E^{i+2} \oplus \cdots \oplus E^l \oplus E^u \).

**Proof.** By applying Lemma 2.7 for \( l \) times, we can get a constant \( C \) such that for any \( \varepsilon > 0 \), there is \( \delta > 0 \) such that for any \( x \in M \) and \( n \in \mathbb{N} \),

\[
\max_{0 \leq i \leq l} \int_{B(x, n, \delta)} |\det D f^n|_{F^i} \, dy \leq C e^{\frac{n\varepsilon}{l}}.
\]

On the other hand, by shrinking \( \delta \), we can assume for any \( p, q \in M \), any \( 0 \leq i \leq l \) and any \( n \in \mathbb{N} \), if \( p \in B(q, n, \delta) \), then

\[
\frac{|\det D f^n|_{F^i}}{|\det D f^n|_{F^i}} \leq e^{\frac{n\varepsilon}{l}}.
\]

Hence

\[
\int_{B(x, n, \delta)} \max_{0 \leq i \leq l} |\det D f^n|_{F^i} \, dy \leq e^{\frac{n\varepsilon}{l}} \int_{B(x, n, \delta)} \max_{0 \leq i \leq l} |\det D f^n|_{F^i} \, dy = e^{\frac{n\varepsilon}{l}} \int_{B(x, n, \delta)} |\det D f^n|_{F^i} \, dy \leq e^{\frac{n\varepsilon}{l}} \max_{0 \leq i \leq l} \int_{B(x, n, \delta)} |\det D f^n|_{F^i} \, dy \leq C e^{ne}.
\]

Now we are ready to give the lower bound of the topological entropy.

**Proof of Proposition 2.6.** For \( \delta > 0 \) and \( n \in \mathbb{N} \), let \( S(n, \delta, M) \) be a minimal spanning set of \( M \). By Corollary 2.8 there is a constant \( C \) such that for any \( \varepsilon > 0 \), there are some \( \delta > 0 \) and \( n \in \mathbb{N} \),
\[
\int \max_{0 \leq i \leq l} |\det Df^n_x|_{F^i} \, dx \leq \#S(n, \delta, M) \cdot \max_{x \in S(n, \delta, M)} \int_{B(x, n, \delta)} \max_{0 \leq i \leq l} |\det Df^n_y|_{F^i} \, dy \\
\leq \#S(n, \delta, M) \cdot Ce^{n\varepsilon}.
\]

Then
\[
\limsup_{n \to \infty} \frac{1}{n} \log \int \max_{0 \leq i \leq l} |\det Df^n_x|_{F^i} \, dx - \varepsilon \leq \limsup_{n \to \infty} \frac{1}{n} \log \#S(n, \delta, M).
\]

Note that by the definition of topological entropy, for any \( \delta > 0 \), the right side above is always less than or equal to the topological entropy. Hence by the arbitrariness of \( \varepsilon \), we get the result.

\[\tag{9}\]

Remark 2.9. By a similar argument in Proposition 2.6, for the case of dominated splitting \( TM = E \oplus F \), Lemma 2.7 gives an interesting result:

\[\]

3 The volume growth rate along subspaces

Our goal in this section is to relate the volume growth rate along these subspaces \( F^k \) in Proposition 2.6 to volume growth rate along all subspaces.

Proposition 3.1. Let \( f \) be a \( C^1 \) diffeomorphism on a compact manifold \( M \). Assume there is a partially hyperbolic splitting \( TM = E^s \oplus E^1 \oplus E^2 \oplus \cdots \oplus E^l \oplus E^u \) with \( \dim E^i = 1 \), \( 1 \leq i \leq l \). Then

\[
\limsup_{n \to \infty} \frac{1}{n} \log \int \max_{V \subset T_x M} |\det Df^n_x|_V \, dx = \limsup_{n \to \infty} \frac{1}{n} \log \int \max_{0 \leq i \leq l} |\det Df^n_x|_{F^i} \, dx
\]

where the bundle \( F^i = E^{i+1} \oplus E^{i+2} \oplus \cdots \oplus E^l \oplus E^u \).

In order to prove the above result, we first show a general result that for a dominated splitting \( TM = E \oplus F \), the maximal growth rate over subspaces with dimension \( \dim F \) is uniformly bounded above by the growth rate of \( F \).

Proposition 3.2. Let \( f \) be a \( C^1 \) diffeomorphism on a compact manifold \( M \). Assume there is a dominated splitting \( TM = E \oplus F \). Then

\[
\limsup_{n \to \infty} \frac{1}{n} \max_{n \leq i \leq l} \left( \log \max_{V \subset T_x M} |\det Df^n_x|_V - \log |\det Df^n_x|_F \right) = 0.
\]

We postpone the proof of Proposition 3.2. We next use Proposition 3.2 to prove Proposition 3.1 in this section.

Proof of Proposition 3.1. First we note that in formula (9), the left side is always larger than or equal to the right side. Hence we next show the reverse.

For simplicity, we write \( s = \dim E^s \) and \( u = \dim E^u \). Given \( \varepsilon > 0 \), applying Proposition 3.2 for \( l \) times, there is some \( T_1 \in \mathbb{N} \) such that for any \( x \in M \) and any \( n \geq T_1 \),

\[
\max_{0 \leq i \leq l} \max_{V \subset T_x M} |\det Df^n_x|_V \leq e^{n\varepsilon} \cdot \max_{0 \leq i \leq l} |\det Df^n_x|_{F^i}.
\]

\[\tag{10}\]
Claim. There exists some $T_2 \in \mathbb{N}$ such that for any $1 \leq i_s \leq s$, $1 \leq i_u \leq u$, any $x \in M$ and any $n \geq T_2$,

$$\max_{\dim V = i_s + l + u} \max_{V \subset T_x M} |\det Df^n_x|_V \leq \max_{\dim V = i_s + l} \max_{V \subset T_x M} |\det Df^n_x|_V,$$

and

$$\max_{\dim V = i_u} \max_{V \subset T_x M} |\det Df^n_x|_V \leq \max_{\dim V = i_u} \max_{V \subset T_x M} |\det Df^n_x|_V.$$

Together with formula (10), we then get that for any $x \in M$ and any $n \geq \max\{T_1, T_2\}$,

$$\max_{V \subset T_x M} |\det Df^n_x|_V \leq e^{\varepsilon n} \cdot \max_{0 \leq i \leq l} |\det Df^n_x|_{F_1}.$$

This implies

$$\limsup_{n \to \infty} \frac{1}{n} \log \max_{V \subset T_x M} |\det Df^n_x|_V | dx \leq \varepsilon + \limsup_{n \to \infty} \frac{1}{n} \log \max_{0 \leq i \leq l} |\det Df^n_x|_{F_1} | dx.$$

By the arbitrariness of $\varepsilon$, we then get formula (9). It remains to prove the claim.

Proof of Claim. Let $T_2$ be a positive integer such that for any $x \in M$ and any $v, w \in E^n_x$, $w \in E^n_x$ with $||v|| = 1, ||w|| = 1$,

$$||Df^n_x(v)|| < 1, \quad ||Df^{-n}_x(w)|| < 1, \quad \forall n \geq T_2.$$

Given $x \in M$, consider a subspace $V \subset T_x M$ with $\dim V = i_s + l + u$. Choose any $i_s$ dimensional subspace $V_s \subset E^n_x \cap V$. Let $V_s^\perp$ be the orthogonal subspace of $V_s$ in $V$. We note that $V_s^\perp$ has dimension $l + u$. Since $V_s \subset E^n_x$, for any $n \geq T_2$,

$$|\det Df^n_x|_V \leq |\det Df^n_x|_{V_s^\perp} \times |\det Df^n_x|_{V_s} \leq |\det Df^n_x|_{V_s^\perp}.$$

This proves inequality (11).

Next we consider a subspace $V \subset T_x M$ with $\dim V = i_u$. For any $n \geq T_2$, let $(Df^n_x(V))^\perp$ be the orthogonal subspace of $Df^n_x(V)$ in $T^n_{f^n,x} M$. We note that $(Df^n_x(V))^\perp$ has dimension $M - i_u$. Hence we can find a $u - i_u$ dimensional subspace $V_u^\perp \subset Df^n_{f^n,x}(Df^n_x(V))^\perp \cap E^n_{f^n,x}$.

By definition, since $Df^n_x(V_u^\perp)$ is orthogonal to $Df^n_x(V)$ and $Df^n_x(V_u^\perp) \subset E^n_{f^n,x}$, $V \oplus V_u^\perp$ has $u$ dimension and

$$|\det Df^n_x|_{V \oplus V_u^\perp} = \left( |\det Df^{-n}_{f^n,x}|_{Df^n_x(V)^\perp \oplus Df^n_{f^n,x}(V_u^\perp)} \right)^{-1} \geq \left( |\det Df^{-n}_{f^n,x}|_{Df^n_x(V)^\perp} \times |\det Df^{-n}_{f^n,x}|_{Df^n_{f^n,x}(V_u^\perp)} \right)^{-1} \geq \left( |\det Df^{-n}_{f^n,x}|_{Df^n_x(V)} \right)^{-1} = |\det Df^n_x|_V.$$

This proves the inequality (12).
Now it remains to prove Proposition \ref{prop:lyapunov}. We prepare some lemmas.

We next introduce the classical Oseledets’ Theorem \cite{os2}. For more detailed discussions, see
the book \cite{wu} and Appendix C.1 in \cite{ku}. Let $\Phi : \mathcal{E} \to \mathcal{E}$ be a continuous vector bundle automorphism. Write $\mathcal{E}_x$ the fiber of $\mathcal{E}$ at the point $x$ and $\Phi_x : \mathcal{E}_x \to \mathcal{E}_{\Phi(x)}$ the action of $\Phi$ at the fiber $\mathcal{E}_x$. Define the linear cocycle induced by $\Phi$ as

$$\Phi^n_x \triangleq \begin{cases} 
\Phi^{n-1}_x \circ \Phi^{n-2}_x \circ \cdots \circ \Phi_x, & n > 0; \\
\Phi^{-1}_{f^{-n}(x)} \circ \Phi^{-1}_{f^{-n+1}(x)} \circ \cdots \circ \Phi^{-1}_{f^{-1}(x)}, & n \leq -1.
\end{cases}$$

The Oseledet’s Theorem \cite{os2} states that

**Theorem 3.3.** Consider the system $(M, f, \mathcal{E}, \Phi)$ as above. There are an $f$-invariant subset $R$ with total measure $\mu(R) = 1$ for any invariant measure $\mu$, an $\Phi$-invariant measurable decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_l$ and finitely many measurable functions $\lambda_1(\cdot) < \lambda_2(\cdot) < \cdots < \lambda_\rho(x)(\cdot)$ such that for any $x \in R$ and any nonzero vector $v \in \mathcal{E}_x$, we have

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \| \Phi^n_x(v) \| = \lambda_j(x),$$

Moreover, for any $j \leq \rho(x)$,

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \det Df^n_x|_{\oplus_{i=1}^l \mathcal{E}_i} = \sum_{i=1}^j \lambda_i(x) \cdot \dim \mathcal{E}_i.$$ 

The set $R$ in Theorem 3.3 is called the Lyapunov regular set and $x \in R$ is called a Lyapunov regular point. These numbers $\lambda_1(\cdot) < \lambda_2(\cdot) < \cdots < \lambda_\rho(x)(\cdot)$ are called the Lyapunov exponents of $x$ and the splitting $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_l$ is called the Oseledets’ splitting. Usually the set $R$ is not the whole manifold $M$.

Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$. Now we apply the abstract Oseledets’ Theorem (Theorem 3.3) to our situation with $\mathcal{E}$ being the tangent bundle and $\Phi$ being the derivative $DF$.

**Lemma 3.4.** Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$. Assume there is a dominated splitting $TM = E \oplus F$. There is an $f$-invariant subset $R$ (the Lyapunov regular set) such that for any $x \in R$ and any $V_x \subset T_x M$ with $\dim V_x = \dim F$,

$$\lambda(x, V_x) \leq \lambda(x, F(x)).$$

**Proof.** Write $d = \dim M$ and $k = \dim F$. Let $R$ be the Lyapunov regular set in Theorem 3.3. We write $\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_l$ the Oseledets’ splitting (See Theorem 3.3). It is sufficient to prove that for any $x \in R$,

1. there is a number $j_F \leq \rho(x)$ ($\rho(x)$ is from Theorem 3.3) such that $\dim F = \sum_{i=j_F}^{\rho(x)} \dim \mathcal{E}_i$.
2. $\lambda(x, F(x)) = \sum_{i=j_F}^{\rho(x)} \lambda_i(x) \dim \mathcal{E}_i$. 

\[ \square \]
3. for any $V_x \subset T_x M$ with $\dim V_x = \dim F$,

$$\lambda(x, V_x) \leq \sum_{i=j_F}^{\rho(x)} \lambda_i(x) \dim \mathcal{E}_i.$$ 

Next we prove the three properties above.

**Property 1.** It is a consequence of domination by noting that there is some $j_F$ such that

$$E(x) = \bigoplus_{i=1}^{j_F-1} \mathcal{E}_i.$$ 

(*)

**Property 2.** By domination, the angles between $E$ and $F$ is uniformly bounded below. As a consequence,

$$\lim_{n \to +\infty} \frac{1}{n} \log |\det Df^n_x| \leq \lambda(x, E(x)) + \lambda(x, F(x)).$$

For $x \in R$, by Theorem 3.3 we have

$$\lim_{n \to +\infty} \frac{1}{n} \log |\det Df^n_x| = \sum_{i=1}^{\rho(x)} \lambda_i(x) \cdot \dim \mathcal{E}_i$$

and together with the formula (*) above, we also have

$$\lambda(x, E(x)) = \sum_{i=1}^{j_F-1} \lambda_i(x) \cdot \dim \mathcal{E}_i.$$ 

We then get the second property.

**Property 3.** Given $x \in R$, write $d_i = \dim \mathcal{E}_i$. Let $e_1, e_2, \cdots, e_d$ be a basis of $T_x M$ such that $e_1, e_2, \cdots, e_{d_1} \in \mathcal{E}_1, e_{d_1+1}, e_{d_1+2}, \cdots, e_{d_1+d_2} \in \mathcal{E}_2, e_{d_1+d_2+1}, e_{d_1+d_2+2}, \cdots, e_{d_1+d_2+d_3} \in \mathcal{E}_3$ and so on. Let

$$\{v^i = \sum_{j=1}^{d} \delta^i_j \cdot e_j \mid \delta^i_j \in \mathbb{R}\}_{1 \leq i \leq k}$$

be a basis of the subspace $V_x$. Write the coordinate matrix $A$ of $\{v^i\}$ w.r.t. $\{e_j\}$ as

$$A = [\mathbb{B} | \mathbb{C}] = \begin{bmatrix} \delta^1_1 & \delta^1_2 & \delta^1_3 & \cdots & \delta^1_{d-k+1} & \cdots & \delta^1_{d-1} & \delta^1_d \\ \delta^2_1 & \delta^2_2 & \delta^2_3 & \cdots & \delta^2_{d-k+1} & \cdots & \delta^2_{d-1} & \delta^2_d \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta^k_1 & \delta^k_2 & \delta^k_3 & \cdots & \delta^k_{d-k+1} & \cdots & \delta^k_{d-1} & \delta^k_d \end{bmatrix}$$

where $\mathbb{B}$ is a $k \times (d-k)$ matrix and $\mathbb{C}$ is a $k \times k$ matrix. Up to a change of the basis of $V_x$, we may assume the basis $\{v^i\}$ is such that the $k \times k$ matrix $\mathbb{C}$ above is lower triangular:

$$\mathbb{C} = \begin{bmatrix} * & 0 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & 0 & \cdots & 0 \\ * & * & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & 0 \end{bmatrix}$$
Define
\[ \lambda(x, v^i) \triangleq \limsup_{n \to +\infty} \frac{1}{n} \log \| Df^n_x(v^i) \|. \]

The way we choose the basis (s.t. \( C \) is lower triangular) implies that \( \lambda(x, v^i) = \lambda(x, e_{t_i}) \) where
\[ t_i = \max\{ j \mid \delta_{ij}^i \neq 0 \}. \]

Hence
\[ \lambda(x, V_x) \leq k \sum_{i=1}^{k} \lambda(x, e_{t_i}) = \sum_{i=1}^{k} \lambda(x, e_{t_i}). \]

Since different \( v^i \) corresponds to different \( e_{t_i} \),
\[ \sum_{i=1}^{k} \lambda(x, e_{t_i}) \leq \sum_{i=j, \rho} \lambda_i(x) \dim e^i. \]

We get the third property.

The following abstract result is from Lemma 2 in [5].

**Lemma 3.5.** Let \( T \) be a continuous map on a compact metric space \( X \) and let \( \phi : X \to \mathbb{R} \) be a continuous function. Assume that for any invariant measure \( \mu \), \( \int \phi d\mu \leq 0 \). Then
\[ \limsup_{n \to +\infty} x \to \infty \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(x)) \leq 0. \]

**Proof of Proposition 3.2.** By Lemma 3.4, there is an \( f \)-invariant subset \( R \) with total measure such that for any \( x \in R \) and any \( V_x \subset T_x M \) with \( \dim V_x = \dim F \),
\[ \lambda(x, V_x) \leq \lambda(x, F(x)). \]

Write \( k = \dim F \). Let \( Gr(k, M) \) be the \( k \)-th Grassmannian manifold of \( M \), i.e.,
\[ Gr(k, M) = \{ V \mid V \subset T_x M \text{ is a linear space with } \dim V = k, x \in M \}. \]

It is a compact metric space. Consider the induced map \( \tilde{f} : Gr(k, M) \to Gr(k, M) \) defined by
\[ \tilde{f}(V_x) = Df_x(V_x), \quad V_x \in Gr(k, M). \]

Let \( \mathcal{E} \) be the continuous vector bundle on \( Gr(k, M) \) with \( \mathcal{E}_{V_x} = T_x M, V_x \in Gr(k, M) \). Let \( \Phi : \mathcal{E} \to \mathcal{E} \) be the bundle automorphism with \( \Phi_{V_x} = Df_x, V_x \in Gr(k, M) \). Let \( \pi \) be the projection from \( Gr(k, M) \) to \( M \) defined by \( \pi(V_x) = x \). Let \( \tilde{R} \) be the Lyapunov regular set (see Lemma 3.3) w.r.t. the system \( (Gr(k, M), \tilde{f}, \mathcal{E}, \Phi) \). Note that by definition, \( \pi(\tilde{R}) = R \). Define \( \phi : Gr(k, M) \to \mathbb{R} \) as
\[ \phi(V_x) = \log |\det Df_x|_{V_x}| - \log |\det Df_x|_{F(x)}|. \]

By definition, if \( V_y \) is close to \( V_x \), then \( y \) is close to \( x \). As a consequence, \( \log |\det Df_y|_{V_y}| \) is close to \( \log |\det Df_x|_{V_x}| \) and \( \log |\det Df_y|_{F(y)}| \) is close to \( \log |\det Df_x|_{F(x)}| \). Hence \( \phi \) is a continuous function on \( Gr(k, M) \). Since \( \pi(\tilde{R}) = R \), given any \( \tilde{f} \)-invariant measure \( \tilde{\mu} \) on \( Gr(k, M) \), for
\( \tilde{\mu} \) almost every point \( V_x \in Gr(k, m) \), we have that \( x \in R \). By Birkhoff ergodic theorem, 
\[ \frac{1}{n} \sum_{j=0}^{n-1} \phi(\tilde{f}^j(V_x)) \] converges. Then by the definition of \( \phi \) and Lemma 3.4, we have
\[ \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(\tilde{f}^j(V_x)) = \lambda(x, V_x) - \lambda(x, F(x)) \leq 0. \]
Hence
\[ \int \phi \, d\tilde{\mu} = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(\tilde{f}^j(V_x)) \leq 0. \]
Then by applying Lemma 3.5 to \( \tilde{f} \) and \( Gr(k, M) \), we get that
\[ \limsup_{n \to +\infty} \max_{V_x \in Gr(k, m)} \frac{1}{n} \left( \log | \det Df^n_{x | V_x} | - \log | \det Df^n_{x | F} | \right) \leq 0. \]
This gives the statement in Proposition 3.2.

4 Proof of Theorem A

With the preparations above, we now prove Theorem A. The part
\[ h_{top}(f) \leq \liminf_{n \to \infty} \frac{1}{n} \log \int \max_{V \subset T_x M} | \det Df^n_{x | V} | \, dx. \]
is proved in Proposition 2.1.
The other part is directly proved with the following two steps:

- Proposition 2.6
  \[ h_{top}(f) \geq \limsup_{n \to \infty} \frac{1}{n} \log \int \max_{1 \leq i \leq l} | \det Df^n_{x | F^i} | \, dx \]
  where the bundle \( F^i = E^i \oplus E^{i+1} \oplus \cdots \oplus E^l \oplus E^u \).

- Proposition 3.1
  \[ \limsup_{n \to \infty} \frac{1}{n} \log \int \max_{V \subset T_x M} | \det Df^n_{x | V} | \, dx = \limsup_{n \to \infty} \frac{1}{n} \log \int \max_{0 \leq i \leq l} | \det Df^n_{x | F^i} | \, dx. \]
  The proof is now complete.

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