ON CERTAIN SUMS OF NUMBER THEORY

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Abstract. We study sums of the shape \( \sum_{n \leq x} f(\lfloor x/n \rfloor) \) where \( f \) is either the von Mangoldt function or the Dirichlet-Piltz divisor functions. We improve previous estimates when \( f = \Lambda \) and \( f = \sigma \), and provide new results when \( f = \sigma_r \) with \( r \geq 3 \), breaking the \( \frac{1}{2} \)-barrier in each case. The functions \( f = \mu^2 \) and \( f = 2^\omega \) are also investigated.

1. Introduction and results

Recently, there has been a great deal of interest in estimating sums of the form

\[
\sum_{n \leq x} f(\lfloor x/n \rfloor)
\]

where \( \lfloor x \rfloor \) is the integer part of \( x \in \mathbb{R} \), and \( f \) is an arithmetic function. Historically, the first one goes back to Dirichlet in the middle of the 19th century when he proved that

\[
\sum_{n \leq x} \lfloor x/n \rfloor = x \log x + x(2\gamma - 1) + O(\sqrt{x}).
\]

Subsequently, the exponent in the error term has been improved, the best result to date being \( x^{517/1648 + \varepsilon} \), which is due to Bourgain & Watt [4]. In [2], the authors established a quite general result involving arithmetic functions \( f \) which are not too large. More precisely, if \( f \) satisfies

\[
\sum_{n \leq x} |f(n)|^2 \ll x^\alpha
\]

for some \( \alpha \in (0, 2) \), then it is proved that

\[
\sum_{n \leq x} f(\lfloor x/n \rfloor) = x \sum_{n=1}^\infty \frac{f(n)}{n(n+1)} + O \left( x^{\frac{1}{2}(\alpha+1)}(\log x)^{\frac{1}{2}(\alpha+1)+o(1)} \right).
\]

This estimate was then improved independently by Wu [13, Theorem 1.2] and Zhai [14, Theorem 1] who proved that

\[
\sum_{n \leq x} f(\lfloor x/n \rfloor) = x \sum_{n=1}^\infty \frac{f(n)}{n(n+1)} + O \left( x^{\frac{1}{2}(\alpha+1)}(\log x)^{\theta} \right)
\]

provided that \( f(n) \ll n^\alpha (\log n)^\theta \) for some \( \alpha \in [0, 1) \) and \( \theta > 0 \). For arithmetic functions \( f \) satisfying the Ramanujan hypothesis \( f(n) \ll n^\varepsilon \), this implies

\[
\sum_{n \leq x} f(\lfloor x/n \rfloor) = x \sum_{n=1}^\infty \frac{f(n)}{n(n+1)} + O \left( x^\frac{1}{2} + x^\varepsilon \right).
\]

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The question of breaking the \( \frac{1}{2} \)-barrier for specific arithmetic functions \( f \) then arises naturally. Using Vaughan’s identity and the exponent pair \( (\frac{1}{6}, \frac{2}{3}) \), Ma and Wu \[9\] showed that

\[
\sum_{n \leq x} \Lambda \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n(n+1)} + O \left( x^{\frac{35}{71} + \varepsilon} \right).
\]

In a similar but simpler way, Ma and Sun \[8\] proved that

\[
\sum_{n \leq x} \tau \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\tau(n)}{n(n+1)} + O \left( x^{\frac{11}{23} + \varepsilon} \right).
\]

Note that \( \frac{35}{71} \approx 0.4929\ldots \) and \( \frac{11}{23} \approx 0.4782\ldots \). The aim of this work is to improve these results when \( f = \Lambda \) and \( f = \tau \), to extend them to the case \( f = \tau_r \) for some fixed integer \( r \geq 2 \), and also to study the cases \( f = \mu^2 \) and \( f = 2^\omega \).

**Theorem 1.1.** Let \((k, \ell)\) be an exponent pair satisfying \( k \leq \frac{1}{6}, \) \( 3k + 4\ell \geq 1 \) and \( \ell^2 + \ell + 3 - k(5 - \ell) - 9k^2 > 0 \). For any \( \varepsilon > 0 \) and \( x \) sufficiently large, we have

\[
\sum_{n \leq x} \Lambda \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n(n+1)} + O_{\varepsilon} \left( x^{\frac{14(k+1)}{29k - r + 30} + \varepsilon} \right).
\]

**Theorem 1.2.** Let \( r \geq 2 \) be any fixed integer and \((k, \ell)\) be an exponent pair satisfying

\[
1 - \ell > k(r - 1).
\]

For any \( \varepsilon > 0 \) and \( x \) sufficiently large, we have

\[
\sum_{n \leq x} \tau_r \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\tau_r(n)}{n(n+1)} + O_{\varepsilon,r} \left( x^{\frac{k(r-1) + (\ell + r - 1)}{k(r-1) + (\ell + r - 1) - 1} + \varepsilon} \right).
\]

It is proved in \[3\], Theorem 6] that \( (\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon) \) is an exponent pair. We use this result in the cases \( f = \Lambda \) and \( f = \tau \), and the exponent pair \( A \left( \frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon \right) = (\frac{13}{104} + \varepsilon, \frac{76}{97} + \varepsilon) \) when \( f = \tau_3 \). For \( r \geq 4 \), the condition \[4\] requires having \( k \) very small. Recently, some improvements in exponential sums have appeared in the literature. As an application, Heath-Brown \[7\], Theorem 2] proved that, for all \( m \in \mathbb{Z}_{\geq 3} \)

\[
(k, \ell) = \left( \frac{2}{(m - 1)^2(m + 2)}, 1 - \frac{3m - 2}{m(m - 1)(m + 2)} + \varepsilon \right)
\]

is an exponent pair. For the function \( \tau_r \) with \( r \geq 4 \), we use this result with \( m = 2r - 1 \). Putting altogether, we derive the next estimates.
Corollary 1.3. Let $r \geq 4$ be any fixed integer. For any $\varepsilon > 0$ and $x \geq e$ sufficiently large, we have

$$
\sum_{n \leq x} \Lambda \left(\left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n(n+1)} + O\left( x^{\frac{97}{202} + \varepsilon} \right);
$$

$$
\sum_{n \leq x} \tau \left(\left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\tau(n)}{n(n+1)} + O\left( x^{\frac{19}{42} + \varepsilon} \right);
$$

$$
\sum_{n \leq x} \tau_3 \left(\left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\tau_3(n)}{n(n+1)} + O\left( x^{\frac{97}{202} + \varepsilon} \right);
$$

$$
\sum_{n \leq x} \tau_r \left(\left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\tau_r(n)}{n(n+1)} + O\left( x^{\frac{19}{42} + \varepsilon} \right).
$$

Note that $\frac{97}{202} \approx 0.4778$, $\frac{19}{42} = 0.475$, $\frac{283}{574} \approx 0.493$ and

| $r$ | 4   | 5   | 6   |
|-----|-----|-----|-----|
| $\frac{1}{2} - \frac{1}{2(4r^3-1)}$ | 0.498 | 0.499 | 0.4994 |

For the functions $\mu_2 = \mu^2$ and $2^\omega$, we have the following estimates.

Theorem 1.4. For any $\varepsilon > 0$ and $x \geq e$ sufficiently large, we have

$$
\sum_{n \leq x} \mu_2 \left(\left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n(n+1)} + O\left( x^{\frac{1919}{2028} + \varepsilon} \right).
$$

Theorem 1.5. Let $(k, \ell)$ be an exponent pair such that $k + \ell < 1$. For any $\varepsilon > 0$ and $x \geq e$ sufficiently large, we have

$$
\sum_{n \leq x} 2^\omega \left(\left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{2^\omega(n)}{n(n+1)} + O\left( x^{\frac{2(k+1)}{2k+1} + \varepsilon} \right).
$$

In particular

$$
\sum_{n \leq x} 2^\omega \left(\left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^{\infty} \frac{2^\omega(n)}{n(n+1)} + O\left( x^{\frac{97}{202} + \varepsilon} \right).
$$

Note that $\frac{1919}{2028} \approx 0.4496$ and $\frac{97}{202} \approx 0.4802$.

2. Notation

If $f$ and $g$ are any arithmetic functions, $f \ast g$ is the Dirichlet convolution product defined by

$$(f \ast g)(n) = \sum_{d \mid n} f(d)g(n/d).$$

Let $\mu$ be the Möbius function, $\Lambda = \mu \ast \log$ is the von Mangoldt function, and $\mu_2 = \mu^2$ is the characteristic function of the set of squarefree numbers. As usual, $\omega(n)$ is the number of distinct prime factors of $n$ with the convention $\omega(1) = 0$, so that $2^\omega(n)$ counts the number of unitary divisors of $n$. If $r \geq 1$ is any fixed positive integer, the Dirichlet-Piltz divisor function $\tau_r$ is inductively defined by $\tau_1 = 1$ and, for $r \geq 2$, $\tau_r = \tau_{r-1} \ast 1$, and it is customary to set $\tau = \tau_2$. Finally, for any $x \in \mathbb{R}$, $e(x) = e^{2\pi i x}$ and $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$ is the 1st Bernoulli function.
The next result relates our problem to estimating certain exponential sums.

**Proposition 3.1.** Let $x \geq e$ large, $f : \mathbb{Z}_{\geq 1} \rightarrow \mathbb{C}$ satisfying $f(n) \ll n^\varepsilon$ and let $x^{1/3} \leq N < x^{1/2}$ be a parameter. Then, for all $H \in \mathbb{Z}_{\geq 1}$

$$
\sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^\infty \frac{f(n)}{n(n+1)}
$$

$$
\quad + O \left\{ N x^\varepsilon + x^\varepsilon \max_{N \leq D \leq x/N} \left( \frac{D}{H} + \sum_{h \leq H} \frac{1}{h} \sum_{a=0}^1 \left| \sum_{D < d \leq 2D} f(d) e \left( \frac{hx}{d+a} \right) \right| \right) \right\}.
$$

**Proof.** Note first that the series in the main term above converges absolutely. Following [8, 9], we split the sum into two subsums

$$
\sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \left( \sum_{n \leq N} + \sum_{N < n \leq x} \right) f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) := S_1 + S_2.
$$

where $x^{1/3} \leq N < x^{1/2}$ is a parameter at our disposal. Trivially

$$
S_1 \ll x^\varepsilon \sum_{n \leq N} \frac{1}{n^\varepsilon} \ll N x^\varepsilon.
$$

Next

$$
S_2 = \sum_{d \leq x/N} f(d) \left( \left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{x}{d+1} \right\rfloor \right) + O \left\{ x^\varepsilon \left( 1 + x N^{-2} \right) \right\}
$$

$$
= \sum_{d \leq x/N} f(d) \left( \frac{x}{d(d+1)} - \psi \left( \frac{x}{d} \right) + \psi \left( \frac{x}{d+1} \right) \right) + O \left( x^{1+\varepsilon} N^{-2} \right)
$$

$$
= x \sum_{d=1}^\infty \frac{f(d)}{d(d+1)} - x \sum_{d>x/N} \frac{f(d)}{d(d+1)} + \sum_{d \leq N} f(d) \left( \psi \left( \frac{x}{d+1} \right) - \psi \left( \frac{x}{d} \right) \right)
$$

$$
+ \sum_{N < d \leq x/N} f(d) \left( \psi \left( \frac{x}{d} \right) - \psi \left( \frac{x}{d+1} \right) \right) + O \left( x^{1+\varepsilon} N^{-2} \right).
$$

Now the condition $f(n) \ll n^\varepsilon$ entails that

$$
\left| \sum_{d \leq N} f(d) \left( \psi \left( \frac{x}{d+1} \right) - \psi \left( \frac{x}{d} \right) \right) \right| \ll \sum_{d \leq N} |f(d)| \ll N^{1+\varepsilon}
$$

and, by partial summation

$$
\sum_{d>x/N} \frac{f(d)}{d(d+1)} \ll \left( \frac{x}{N} \right)^{\varepsilon-1}.
$$

Therefore

$$
S_2 = x \sum_{d=1}^\infty \frac{f(d)}{d(d+1)} + \sum_{N < d \leq x/N} f(d) \left( \psi \left( \frac{x}{d+1} \right) - \psi \left( \frac{x}{d} \right) \right) + O \left( N x^\varepsilon + x^{1+\varepsilon} N^{-2} \right)
$$
and note that \( xN^{-2} \leq N \) since \( N \geq x^{1/3} \). Hence

\[
\sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{d=1}^{\infty} \frac{f(d)}{d(d+1)} + \sum_{N < d \leq x/N} f(d) \left( \psi \left( \frac{x}{d} \right) - \psi \left( \frac{x}{d+1} \right) \right) + O(Nx^\varepsilon).
\]

We complete the proof with the usual Vaaler’s approximation of the function \( \psi \) by trigonometric polynomials \([11]\), implying the asserted result. \( \square \)

4. **Useful decompositions**

The next result is Vaughan’s identity \([12]\) or \([5, \text{Chapter 24}]\). We use the functions

\[
1_U^-(n) = \begin{cases} 1, & \text{if } n \leq U; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and} \quad 1_U^+(n) = \begin{cases} 1, & \text{if } n > U; \\ 0, & \text{otherwise}. \end{cases}
\]

**Proposition 4.1.** Let \( F : [1, \infty) \to [0, \infty) \) be any map. For all \( 1 \leq U \leq R^{1/2} \)

\[
\sum_{R < n \leq 2R} \Lambda(n) e(F(n)) = \sum_{n \leq U} \mu(n) \sum_{\frac{R}{n} < m \leq \frac{2R}{n}} e(F(mn)) \log m - \sum_{n \leq U^2} a_n \sum_{\frac{R}{n} < m \leq \frac{2R}{n}} e(F(mn)) - \sum_{U < n \leq \frac{2R}{n}} \Lambda(n) \sum_{\frac{R}{n} < m \leq \frac{2R}{n}} b_m e(F(mn))
\]

with

\[ a_n := (\mu 1_U^- \ast \Lambda 1_U^-)(n) \quad \text{and} \quad b_n := (\mu 1_U^- \ast 1)(m). \]

A similar result holds for the Möbius function.

**Proposition 4.2.** Let \( 1 < R < R_1 \leq 2R \) and let \( F : [1, \infty) \to [0, \infty) \) be any map. For all \( 1 \leq U \leq R^{1/2} \)

\[
\sum_{R < n \leq R_1} \mu(n) e(F(n)) = -\sum_{n \leq U} a_n \sum_{\frac{R}{n} < m \leq \frac{R_1}{n}} e(F(mn)) \log m - \sum_{U < n \leq \frac{R_1}{n}} \sum_{\frac{max(U, R)}{n} < m \leq \frac{R_1}{n}} \mu(m) e(F(mn))
\]

with

\[ a_n := (\mu 1_U^- \ast \mu 1_U^-)(n) \quad \text{and} \quad b_n := (\mu 1_U^- \ast 1)(n). \]

The usual Dirichlet hyperbola principle, a proof of which can be found for instance in \([10, \text{Theorem 2.4.1}]\), can be slightly extended to the following form. The proof is well-known.

**Lemma 4.3** (Dirichlet hyperbola principle). Let \( f, g : \mathbb{Z}_{\geq 1} \to \mathbb{C} \) be two arithmetic functions and \( h : [1, \infty) \to \mathbb{C} \) be any map. For all \( 1 \leq U \leq x \)

\[
\sum_{n \leq x} (f \ast g)(n)h(n) = \sum_{n \leq U} f(n) \sum_{m \leq x/n} g(m)h(mn) + \sum_{n \leq U} g(n) \sum_{m \leq x/n} f(m)h(mn) - \sum_{n \leq U} \sum_{m \leq x/U} f(n)g(m)h(mn).
\]

Specifying \( h(n) = e(F(n)) \) where \( F : [1, \infty) \to [0, \infty) \) is any function, we immediately derive the next tool.
Corollary 4.4. Let $f, g : \mathbb{Z}_{\geq 1} \to \mathbb{C}$ be two arithmetic functions and $F : [1, \infty) \to [0, \infty)$ be any map. For all $R < R_1 \in \mathbb{Z}_{\geq 1}$ and $1 \leq U \leq R$

$$\sum_{R < n \leq R_1} (f \ast g)(n) e(F(n)) = \sum_{n \leq UR_1 \over R} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn))$$

$$+ \sum_{n \leq U} g(n) \sum_{R n < m \leq R_1 n} f(m) e(F(mn)) - \sum_{U < n \leq UR_1 \over R} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn)).$$

Proof. By Lemma [3] we first derive

$$\sum_{R < n \leq R_1} (f \ast g)(n) e(F(n)) = \sum_{n \leq U} f(n) \sum_{R n < m \leq R_1 n} g(m) e(F(mn)) + \sum_{n \leq U} g(n) \sum_{R n \leq m \leq R_1 n} f(m) e(F(mn))$$

$$+ \sum_{R n < m \leq R_1 n} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn))$$

$$= S_1 + S_2 + S_3 - S_4.$$

For $S_3$, interchanging the sums and then the indices yields

$$S_3 = \sum_{n \leq U} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn))$$

$$= \sum_{n \leq U} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn)) + \sum_{U \leq n \leq UR_1 \over R} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn))$$

$$= S_4 + \sum_{U \leq n \leq UR_1 \over R} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn))$$

and, in the 2nd sum, since $U < n \leq UR_1 \over R$, we have $R_n < R \leq R_1 n$, so that

$$S_3 - S_4 = \sum_{U \leq n \leq UR_1 \over R} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn)) - \sum_{U \leq n \leq UR_1 \over R} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn))$$

$$= \sum_{n \leq UR_1 \over R} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn)) - S_1 - \sum_{U \leq n \leq UR_1 \over R} f(n) \sum_{R n \leq m \leq R_1 n} g(m) e(F(mn))$$

implying the asserted result. 

5. The von Mangoldt function

The following result relates certain exponential sums of primes with the sum of Theorem [11].

Proposition 5.1. Assume there exist real numbers $\alpha, \beta > 0$, $0 \leq \gamma < 1$ such that $2\alpha + \beta < 1$, $\alpha(\gamma - 3) \leq \beta - \gamma$, $\alpha(\gamma + 1) + \gamma(\beta - 2) + 1 \geq 0$, and, for all $z \geq 1$ and all integers $R \leq z^{2/3}$, we have for all $\varepsilon \in \left(0, \frac{1}{2}\right)$

$$z^{-\varepsilon} \sum_{R \leq n \leq 2R} \Lambda(n) e\left(\frac{\overline{z}}{n}\right) \ll z^\alpha R^\beta + R^\gamma.$$

Then, for $x \geq e$ large

$$\sum_{n \leq x} \Lambda\left(\left\lfloor \frac{x}{n}\right\rfloor\right) = x \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n(n+1)} + O\left(x^{3/5 + \varepsilon}\right).$$
Proof. By Proposition 3.1 it suffices to estimate
\[ \sum_{D < d \leq 2D} \Lambda(d) e \left( \frac{hx}{d + a} \right) \]
where \( a \in \{0, 1\} \) and for all \( x^{1/3} \leq N < x^{1/2} \), \( N < D \leq xN^{-1} \), \( H \in \mathbb{Z}_{\geq 1} \) and \( 1 \leq h \leq H \). Note that \( \frac{x}{d+1} = \frac{x}{d} - \frac{x}{d(d+1)} \), so that, by Abel summation, we get
\[ \sum_{D < d \leq 2D} \Lambda(d) e \left( \frac{hx}{d + 1} \right) = \sum_{D < d \leq 2D} \Lambda(d) e \left( \frac{hx}{d} \right) e \left( -\frac{hx}{d(d+1)} \right) \]
\[ \ll \left( 1 + \frac{hx}{D^2} \right) \max_{D \leq D_i \leq 2D} \left| \sum_{D < d \leq D_i} \Lambda(d) e \left( \frac{hx}{d} \right) \right| \]
\[ \ll (hx)^\varepsilon \left\{ (hx)^{1+\alpha} D^{\beta-2} + (hx)^\alpha D^\beta + hxD^{\gamma-2} + D^\gamma \right\} \]
where we used (6) assuming also \( D \leq x^{2/3} \), and therefore
\[ (Hx)^{-\varepsilon} \left\{ \frac{D}{H} + \sum_{h \leq H} \frac{1}{h} \sum_{a=0}^{1} \left| \sum_{D < d \leq 2D} \Lambda(d) e \left( \frac{hx}{d + a} \right) \right| \right\} \]
\[ \ll \frac{D}{H} + (Hx)^{1+\alpha} D^{\beta-2} + (Hx)^\alpha D^\beta + hxD^{\gamma-2} + D^\gamma \]
provided that \( H \geq 1 \) and \( N < D \leq \min \left( xN^{-1}, x^{2/3} \right) = xN^{-1} \), since \( N \geq x^{1/3} \). Using Srinivasan optimization lemma on the parameter \( H \), we derive
\[ x^{-\varepsilon} \left( \frac{D}{H} + \sum_{h \leq H} \frac{1}{h} \sum_{a=0}^{1} \left| \sum_{D < d \leq 2D} \Lambda(d) e \left( \frac{hx}{d + a} \right) \right| \right) \ll (x^{1+\alpha} D^{\alpha+\beta-1})^{\frac{1}{\alpha+\beta}} + x^{1+\alpha} D^{\beta-2} \]
\[ + (x^\alpha D^{\alpha+\beta})^{\frac{1}{\alpha+\beta}} + x^\alpha D^\beta + x^{1/2} D^{\gamma+1} + xD^{\gamma-2} + D^\gamma. \]
Hence the error term of Proposition 3.1 is, for all \( x^{1/3} \leq N < x^{1/2} \) and up to \( x^\varepsilon \)
\[ \ll N + (x^{1+\alpha} N^{\alpha+\beta-1})^{\frac{1}{\alpha+\beta}} + x^{1+\alpha} N^{\beta-2} \]
\[ + (x^{2\alpha+\beta} N^{-\alpha-\beta})^{\frac{1}{\alpha+\beta}} + x^{\alpha+\beta} N^{-\beta} + x^{1/2} N^{\gamma+1} + xN^{\gamma-2} + \left( \frac{x}{N} \right)^\gamma. \]
Now choose \( N = \frac{x^{1+\alpha}}{2^{\alpha+\beta}} \). Note that the condition \( 2\alpha + \beta < 1 \) entails that \( \frac{1+\alpha}{\alpha+\beta} < \frac{1}{2} \), and clearly \( \frac{1+\alpha}{2-\beta} > \frac{1}{3} \). We obtain that the error term is, up to \( x^\varepsilon \)
\[ \ll \frac{1+\alpha}{3-\beta} + x^{\frac{\alpha^2+\beta(3\beta-2)(3\beta-4)}{(1-\alpha)(3-\beta)}} + x^{\frac{(\alpha+1)(1-\beta)(3\beta-2)}{4-\beta}} + x^{\frac{2(\alpha+2)(3-\beta)}{2(\alpha+2)(3-\beta)}} + x^{\frac{2(\alpha+2)(3-\beta)}{2(\alpha+2)(3-\beta)}} + x^{\frac{\alpha(1+\beta-\gamma)}{1+\beta-\gamma}} \]
and note that the conditions \( 2\alpha + \beta < 1 \), \( \alpha(\gamma-3) \leq \beta - \gamma \) and \( \alpha(\gamma+1) + \gamma(\beta-2) + 1 \geq 0 \) imply that the 1st term dominates the other terms, completing the proof.

Bounds for the sum (6) do already exist in the literature. For instance, in [6, Theorem 9], the authors proved that
\[ \sum_{R < n \leq 2R} \Lambda(n) e \left( \frac{z}{n} \right) \ll z^{1/12} R^{19/24} (\log R)^{11/4} \]
provided that $1 \leq R \leq \frac{1}{5} x^{3/5}$, so that Proposition 5.1 used with $(\alpha, \beta, \gamma) = \left(\frac{1}{12} : \frac{19}{24}, 0\right)$ yields

\[
\sum_{n \leq x} \Lambda \left(\frac{x}{n}\right) = x \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n(n+1)} + O_{\varepsilon} \left(x^{26/53 + \varepsilon}\right)
\]

which is slightly better than (2). The next result is a consequence of Proposition 4.1

**Proposition 5.2.** Let $R \geq 1$ be a large integer and $F : [1, \infty) \to [0, \infty)$ be any map. Assume there exist $A, B > 0$ such that, for all integers $M, N \geq 1$

\[
\sum_{N < n \leq 2N} \max_{\frac{D}{n} < M \leq \frac{2R}{n}} \left| \sum_{\frac{R}{m} < m \leq M} e(F(mn)) \right| \leq A \quad \text{for } N \leq R^{1/3}
\]

\[
\max_{M \geq 1} \max_{N \geq R} \left| \sum_{N < n \leq 2N} \alpha_n \sum_{M < m \leq 2M} \beta_m e(F(mn)) \right| \leq B \quad \text{for } R^{1/2} \leq N \leq 2R^{2/3}
\]

uniformly for all complex-valued sequences $(\alpha_n)$ and $(\beta_m)$ satisfying $|\alpha_n| \leq 1$ and $|\beta_m| \leq 1$. Then

\[
\sum_{R < n \leq 2R} \Lambda(n) e(F(n)) \ll (A + B) R^\varepsilon.
\]

**Proof.** Set $S_1$, $S_2$, and $S_3$ the three sums of Proposition 4.1, where we choose $U = R^{1/3}$ and we used the normalized coefficients $\alpha_n := \alpha_n / \log R$ and $\beta_m := b_m 2^{-\omega(m)}$, so that $|\alpha_n| \leq 1$ and $|\beta_m| \leq 1$. Using partial summation and (7), we derive

\[
S_1 \ll \max_{N \leq R^{1/3}} \sum_{N < n \leq 2N} \max_{\frac{D}{n} < M \leq \frac{2R}{n}} \left| \sum_{\frac{R}{m} < m \leq M} e(F(mn)) \right| (\log R)^2 \ll A(\log R)^2.
\]

We split $S_2$ into three subsums, namely

\[
(\log R)^{-1} S_2 = \left(\sum_{n \leq R^{1/3}} + \sum_{R^{1/3} < n \leq R^{1/2}} + \sum_{R^{1/2} < n \leq R^{2/3}}\right) \alpha_n \sum_{\frac{R}{m} < m \leq \frac{2R}{n}} e(F(mn)) := S_{21} + S_{22} + S_{23}.
\]

As for $S_1$, we immediately derive $S_{21} \ll A(\log R)^2$. Inverting the summations in $S_{22}$ and using (8) yields

\[
S_{22} = \sum_{R^{1/2} < n \leq 2R^{2/3}} \max_{\frac{R}{m} < m \leq \min\left(\frac{R^{1/2} 2R}{n}\right)} \alpha_m e(F(mn)) \log R
\]

\[
\ll \max_{R^{1/2} < N \leq 2R^{2/3}} \max_{M \geq 1} \max_{M N \geq R} \left| \sum_{N < n \leq 2N} \sum_{M < m \leq 2M} \alpha_m e(F(mn)) \right| (\log R)^3
\]

\[
\ll \max_{R^{1/2} < N \leq 2R^{2/3}} \max_{M \geq 1} \max_{M N \geq R} \left| \sum_{N < n \leq 2N} \sum_{M < m \leq 2M} \alpha_m e(F(mn)) \right| (\log R)^4 \ll B(\log R)^4
\]
where $I_n$ is any subinterval of $(\frac{R}{n}, \frac{2R}{n}]$. Also,

$$S_{23} \ll \max_{R^{1/3} < N \leq R^{2/3}} \max_{M \geq 1 \atop MN > R} \left| \sum_{N < n \leq 2N} \sum_{M < m \leq 2M} e(F(mn)) \right|(\log R)^3$$

$$\leq \max_{R^{1/3} < N \leq R^{2/3}} \max_{M \geq 1 \atop MN > R} \left| \sum_{N < n \leq 2N} \sum_{M < m \leq 2M} e(F(mn)) \right|(\log R)^4 \ll B(\log R)^4.$$  

Finally, the sum $S_3$ is split into two subsums, namely

$$S_3 = \left( \sum_{R^{1/3} < n \leq R^{1/2}} + \sum_{R^{1/2} < n \leq R^{2/3}} \right) \Lambda(n) \sum_{\frac{n}{R} < m \leq \frac{4R}{n}} b_m e(F(mn))$$

and each of these subsums are treated similarly as for $S_{22}$ and $S_{23}$. The proof is complete. \(\square\)

**Proposition 5.3** (Baker). Let $X > 0$, $1 < R < R_1 \leq 2R$, $M, N \geq 1$ such that $MN \asymp R$, $M \ll N$ and $M \ll X$, $(a_n), (b_m) \in \mathbb{C}$ such that $|a_n|, |b_m| \leq 1$, $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq 1$, $\beta < 0$ and $\alpha + \beta < 2$. Set $L := \log(RX + 2)$. If $(k, \ell)$ is an exponent pair, then

$$L^{-2} \sum_{N < n \leq 2N} a_n \sum_{M < m \leq 2M} b_m e\left( X \left( \frac{m}{M} \right)^{\alpha} \left( \frac{n}{N} \right)^{\beta} \right) \ll X^{1/6} \left( R^{5k+4}M^{\ell-k} \right)^{\frac{1}{12k+1}} + R \left( M^{-1/2} + N^{-1/4} \right).$$

Taking an integer $r \geq 1$, and applying Proposition 5.3 with $\alpha = \beta = -1$ and $X = z(MN)^{-r}$, we derive the next estimate.

**Corollary 5.4.** Let $z \geq 1$, $r \in \mathbb{Z}_{\geq 1}$, $1 < R < R_1 \leq 2R$ such that $R \leq z^\frac{1}{2r}$, $M, N \geq 1$ such that $MN \asymp R$ and $R^{1/2} \ll N \ll R^{2/3}$, and let $(a_n), (b_m) \in \mathbb{C}$ such that $|a_n|, |b_m| \leq 1$. If $(k, \ell)$ is an exponent pair and $L := \log(z + 2)$, then

$$L^{-2} \sum_{N < n \leq 2N} a_n \sum_{M < m \leq 2M} b_m e\left( \frac{z}{(mn)^r} \right) \ll z^{1/6} R \left( R^{(4r-1)(k(0-2r)+4r)} + R^{7/8}. \right.$$

**Proof.** Let $S_{II}$ be the sum of the left-hand side. First note that the condition $N \gg R^{1/2}$ entails that $N^2 \gg R \asymp MN$, and hence $M \ll N$. Furthermore, since $R \leq z^\frac{1}{2r}$ and $N \gg R^{1/2}$, we get

$$M^{r+1}N^{r} \asymp R^{r+1}N^{-1} \ll R^{r+1/2} \ll z$$

and therefore $M \ll z(MN)^{-r}$. Proposition 5.3 may be applied with $X = z(MN)^{-r}$, yielding

$$L^{-2} S_{II} \ll z^{1/6} \left( R^{(k(5-r)+4r)} M^{(-\ell-k)} \right)^{\frac{1}{12k+1}} + RM^{-1/2} + RN^{-1/4}$$

with $R^{k(5-r)+4r}M^{\ell-k} \ll R^{\ell+(k+1)(4r)}N^{k-\ell} \ll R^\frac{1}{2} (2(4r-1)+k(9-2r)+\ell)$, $RM^{-1/2} \ll (RN)^{1/2} \ll R^{5/6}$ and $RN^{-1/4} \ll R^{7/8}$, and hence

$$L^{-2} S_{II} \ll z^{1/6} R^{\frac{(2(4r-1)+k(9-2r)+\ell)}{12(k+1)}} + R^{7/8} \quad \text{Note that Baker's results in [1] are stated with an extra multiplicative condition $R < mn \leq R_1$, but the author removes it at the start of the proof at the cost of a factor log $R$.}$$
completing the proof. □

We are now in a position to establish the main estimate of this section.

**Proposition 5.5.** Let \( R, z \geq 8 \) large such that \( R \leq z^{2/3} \). If \((k, \ell)\) is an exponent pair satisfying \( k \leq \frac{1}{6} \) and \( 20k^2 + k(23 - 8\ell) + 2 - 7\ell > 0 \), then, for all \( \varepsilon \in (0, \frac{1}{2}) \)

\[
z^{-\varepsilon} \sum_{R < n \leq 2R} \Lambda(n) e \left( \frac{z}{n} \right) \ll z^{1/6} R^{2k + \ell + 6} + R^{7/8}.
\]

**Proof.** The sums (8) are treated with Proposition 5.4 with \( r = 1 \). It remains to estimate the sums (7), for which we apply the exponent pair (8), yielding

\[
\max_{N \in R^{1/3}} \left\| \sum_{N < n \leq 2N} \max_{\frac{R}{n} < m \leq 2R} \sum_{\frac{R}{m} < \ell < M} e \left( \frac{z}{mn} \right) \right\| \ll \max_{N \in R^{1/3}} \left\| \sum_{N < n \leq 2N} \left\{ \left( \frac{z}{R} \right)^k \left( \frac{R}{n} \right)^{t-k} + \frac{R^2}{n z} \right\} \right\|
\]

\[
\ll \max_{N \in R^{1/3}} \left( z^k R^{\ell - 2k} N^{1-\ell + k} + R^2 z^{-1} \right)
\]

\[
\ll z^k R^{1 + 2k - 3\beta} + R^2 z^{-1}.
\]

and note that \( R^2 z^{-1} \leq R^{7/8} \) and \( z^k R^{1 + 2k - 3\beta} \leq z^{1/6} R^{2k + 6} \) since \( k \leq \frac{1}{6} \) and \( 20k^2 + k(23 - 8\ell) + 2 - 7\ell > 0 \). The proof is complete. □

**Proof of Theorem 1.1.** The proof consists of a simple verification of the hypotheses of Proposition 5.1 with

\[
(\alpha, \beta, \gamma) = \left( \frac{1}{6}, \frac{7k + \ell + 6}{12(k+1)}, \frac{7}{8} \right).
\]

The condition \( 3k + 4\ell \geq 1 \) ensures that \( \alpha(\gamma - 3) \leq \beta - \gamma \).

□

6. The Dirichlet-Piltz divisor functions

We first derive the analog of Proposition 5.1 for the function \( \tau_r \).

**Proposition 6.1.** Let \( r \in \mathbb{Z}_{\geq 1} \) fixed, and assume there exist real numbers \( \alpha, \beta > 0 \) such that \( 2\alpha + \beta < 1 \) and \( 4\alpha + 2\beta > 1 \) and, for all \( z \geq 1 \) and all integers \( 1 \leq R \leq z \), we have for all \( \varepsilon \in (0, \frac{1}{2}] \)

\[
z^{-\varepsilon} \left\{ \left| \sum_{R < n \leq 2R} \tau_r(n) e \left( \frac{z}{n} \right) \right| + \left| \sum_{R < n \leq 2R} \tau_r(n) e \left( \frac{z}{n + 1} \right) \right| \right\} \ll z^\alpha R^\beta + R^2 z^{-1}.
\]

Then, for \( x \geq e \) large

\[
\sum_{n \leq x} \tau_r \left( \left[ \frac{x}{n} \right] \right) = x \sum_{n=1}^\infty \tau_r(n) \frac{n}{n(n+1)} + O_{\varepsilon} \left( x^{2\alpha + \beta + 1 + \varepsilon} \right).
\]

**Proof.** Let \( x^{1/3} \leq N < x^{1/2} \). Using (8), we derive

\[
(Hx)^{-\varepsilon} \left\{ \frac{D}{H} + \sum_{h \leq H} \frac{1}{h} \sum_{a=0}^{1} \sum_{D < d \leq 2D} \tau_r(d) e \left( \frac{hx}{d + a} \right) \right\} \ll \frac{D}{H} + (Hx)^\alpha D^\beta + D^2 x^{-1}
\]

for all \( H \in \mathbb{Z}_{\geq 1} \) and all \( N < D \leq L \leq x/N \). Using Srinivasan optimization lemma on \( H \), we get

\[
x^{-\varepsilon} \left( \frac{D}{H} + \sum_{h \leq H} \frac{1}{H} \sum_{a=0}^{1} \sum_{D < d \leq 2D} \tau_r(d) e \left( \frac{hx}{d + a} \right) \right) \ll (x^\alpha D^{\alpha + \beta})^{1/\alpha + 1} + x^\alpha D^\beta + D^2 x^{-1}
\]
and hence the error term does of Proposition 3.1 not exceed, up to a factor $x^\varepsilon$

$$\ll N + \left(x^{2\alpha+\beta}N^{-\alpha-\beta}\right)^{\frac{1}{2n+\beta+1}} + x^{a+\beta}N^{-\beta} + xN^{-2}.$$ 

Now choosing $N = \frac{x^{2\alpha+\beta}}{2n+\beta+1}$ yields the asserted result plus the extra terms

$$\frac{x^{2\alpha^2+\alpha(\beta+1)+\beta}}{2n+\beta+1} + \frac{1}{x^{2n+\beta+1}}$$

which are absorbed by the term $x^{2n+\beta+1}$ whenever $2\alpha + \beta < 1$ and $4\alpha + 2\beta > 1$. Also note that these two conditions ensure that $x^{1/3} \leq N < x^{1/2}$, as required. The proof is complete.

The treatment of the sum (9) rests on the next result which can be seen as an extension of the definition of the exponent pairs.

**Proposition 6.2.** Let $R, r \in \mathbb{Z}_{\geq 1}$ and $F \in C^\infty [R, 2R]$, positive-valued, and such that there exists $T > 0$ such that, for all $j \in \mathbb{Z}_{\geq 0}$ and all $x \in [R, 2R]$, we have $|F^{(j)}(x)| \asymp TR^{-j}$. If $(k, \ell)$ is an exponent pair, then

$$\sum_{R < n \leq 2R} \tau_r(n) e(F(n)) \ll T^{k}R^{\frac{\ell-k}{r}+1+\frac{1}{r}}(\log R)^r + RT^{-1}(\log R)^{r+1}.$$ 

**Proof.** We use induction on $r$, the case $r = 1$ being the definition of the exponent pairs. Assume the result is true for some $r \geq 1$. Using Corollary 4.4 with $f = \tau_r$, $g = 1$ and $U = R^{\frac{1}{r+1}}$, we derive

$$\sum_{R < n \leq 2R} \tau_{r+1}(n) e(F(n)) \ll \sum_{n \leq 2R^{\frac{1}{r+1}}} \tau_r(n) \left| \sum_{\frac{R}{n} < m \leq 2R} e\left(\frac{F(mn)}{m}\right) \right|$$

$$+ \sum_{n \leq R^{\frac{1}{r+1}}} \left| \sum_{\frac{R}{n} < m \leq 2R} \tau_r(m) e\left(\frac{F(mn)}{m}\right) \right| + \sum_{R^{\frac{1}{r+1}} < n \leq 2R^{\frac{1}{r+1}}} \tau_r(n) \left| \sum_{\frac{R}{n} < m \leq R^{\frac{1}{r+1}}} e\left(\frac{F(mn)}{m}\right) \right|$$

and the induction hypothesis entails that

$$\sum_{R < n \leq 2R} \tau_{r+1}(n) e(F(n)) \ll \sum_{n \leq 2R^{\frac{1}{r+1}}} \tau_r(n) \left( T^k R^{\ell-k} n^{k-\ell} + \frac{R}{nT} \right)$$

$$+ \sum_{n \leq R^{\frac{1}{r+1}}} \left( T^k R^{\ell-k} + \frac{1}{r} n \left( \frac{2R^{\frac{1}{r+1}}}{R^{\frac{1}{r+1}}} - 1 \right) (\log R)^r + \frac{R}{nT} (\log R)^{r+1} \right)$$

$$+ \sum_{R^{\frac{1}{r+1}} < n \leq 2R^{\frac{1}{r+1}}} \tau_r(n) \left( T^k R^{\ell-k} n^{k-\ell} + \frac{R}{nT} \right)$$

where we used in the 3rd sum the fact that $\left( \frac{R}{n}, R^{\frac{1}{r+1}} \right) \subset \left( \frac{R}{n}, \frac{2R}{n} \right)$. The bound

$$\sum_{n \leq z} \frac{\tau_r(n)}{n^\alpha} \ll z^{1-\alpha} (\log z)^r \quad (0 \leq \alpha \leq 1)$$

enables us to derive

$$\sum_{R < n \leq 2R} \tau_{r+1}(n) e(F(n)) \ll T^k R^{\ell-k} + \frac{1}{r} (\log R)^{r+1} + RT^{-1}(\log R)^{r+2}$$

completing the proof. □
Proof of Theorem 1.2. By Proposition 6.2, we obtain
\[ z^{-\varepsilon} \left\{ \sum_{R < n \leq 2R} \tau_1(n) e \left( \frac{z}{n^2} \right) + \sum_{R < n \leq 2R} \tau_2(n) e \left( \frac{z}{n + 1} \right) \right\} \ll z^k R^{\ell - k - \frac{1}{r} - k} + R^2 z^{-1}. \]
and the condition \( 1 - \ell > k(r - 1) \) ensures that \( 2\alpha + \beta < 1 \) and \( 4\alpha + 2\beta > 1 \) when \( \alpha = k \) and \( \beta = \frac{\ell - k}{r} + 1 - \frac{1}{r} - k. \) Now the result follows by applying Proposition 6.1 with these values of \( \alpha \) and \( \beta. \)

\[ \Box \]

7. The functions \( \mu_2 \) and \( 2^\varepsilon \)

In this section, we will make use of the function \( \chi_2 \) defined by
\[ \chi_2(n) := \begin{cases} \mu(m), & \text{if } n = m^2 \\ 0, & \text{otherwise.} \end{cases} \]
We first need the following lemma.

Lemma 7.1. Let \( 1 < R < R_1 \leq 2R, \ z \geq 1 \) and assume \( R \leq z^{2/5} \). If \( (k, \ell) \) is an exponent pair, then, for all \( \varepsilon > 0 \)
\[ R^{-\varepsilon} \sum_{R < n \leq R_1} \mu(n) e \left( \frac{z}{n^2} \right) \ll z^{1/6} R^{5k/2(\ell+3)} + z^k R^{2k + 1 - 8k} + R^{7/8}. \]
In particular
\[ R^{-\varepsilon} \sum_{R < n \leq R_1} \mu(n) e \left( \frac{z}{n^2} \right) \ll z^{1/6} R^{5k/97} + R^{7/8}. \]

Proof. The proof is similar to that of Proposition 5.2 so that we only sketch the main details. Let \( S_1, S_2 \) and \( S_3 \) be the sums of Proposition 4.2 used with \( U = R^{1/3} \). By partial summation and using the exponent pair \((k, \ell)\), we get
\[ R^{-\varepsilon} S_1 \ll z^k R^{k - 3k} \sum_{n \in R_1} \frac{1}{n^{k - 1}} + R^3 z^{-1} \ll z^k R^{2k + 1 - 4k} + R^3 z^{-1}. \]
Splitting \( S_2 \) into two subsums
\[ S_2 = \left( \sum_{R^{1/3} < n \leq R^{1/2}} + \sum_{R^{1/2} < n \leq R^{2/3}} \right) a_n \sum_{\frac{R}{n} < m \leq R_1} \log m e \left( \frac{z}{(mn)^2} \right) := S_{21} + S_{22} \]
we interchange the summations in \( S_{21} \), so that
\[ S_{21} = \sum_{R^{1/2} < n \leq R_1} \log n \sum_{\max \left( R^{1/3}, \frac{R}{n} \right) < m \leq \min \left( R^{1/2}, \frac{R_1}{n} \right)} a_m e \left( \frac{z}{(mn)^2} \right) \]
so that Corollary 5.3 with \( r = 2 \) yields
\[ R^{-\varepsilon} S_{21} \ll \max_{R^{1/2} < N \leq R^{2/3}} \max_{M \geq 1} \left| \sum_{N < n \leq 2N} \sum_{M < m \leq 2M} a_m e \left( \frac{z}{(mn)^2} \right) \right| \ll z^{1/6} R^{5k/2(\ell+3)} + R^{7/8}. \]
where $\alpha_n := a_n 2^{-\omega(n)}$ and $l_m = \log m / \log R$, and similarly

$$R^{-\varepsilon} S_{22} \ll \max_{R^{1/2} < N < R^{2/3}} \max_{M \geq 1} \sum_{M \leq n \leq R^{4/3}} \sum_{M \leq m \leq n} \alpha_n l_m e \left( \frac{z}{(mn)^2} \right) \ll z^{1/6} R^{\frac{55}{194} + 40/355} + R^{7/8}.$$

The argument is similar for $S_3$, splitting the sum in two

$$S_3 = \left( \sum_{R^{1/3} < n < R^{1/2}} + \sum_{R^{1/2} < n < R^{1-1/3}} \right) b_n \sum_{\frac{n}{m} < \frac{R}{n}} \sum_{\frac{R}{n} < m \leq \frac{R}{n}} \mu(m) \left( \frac{z}{(mn)^2} \right) := S_{31} + S_{32}$$

and estimating $S_{31}$ and $S_{32}$ as $S_{21}$ and $S_{22}$. Also note that $R^3 z^{-1} \leq R^{7/8}$. The last part of the proposition follows with the use of Bourgain’s exponent pair $(\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon)$, yielding the asserted estimate with an extra term $z^{1/84} R^{5/14}$, which is absorbed by the first one. □

**Proposition 7.2.** Let $z, R \geq 1$ such that $R \leq z^{7/10}$. Then, for all $\varepsilon > 0$

$$R^{-\varepsilon} \sum_{R < n \leq 2R} \mu_2(n) e \left( \frac{z}{n} \right) \ll z^{\frac{497}{14337}} R^{\frac{15}{16}}.$$

**Proof.** If $R < z^{\frac{68}{488}}$, then trivially

$$\sum_{R < n \leq 2R} \mu_2(n) e \left( \frac{z}{n} \right) \ll R \ll z^{\frac{68}{488}} \ll z^{\frac{497}{14337}} R^{\frac{15}{16}},$$

so that we may assume $R \geq z^{\frac{68}{488}}$. Using Corollary 4.4 with $f = \chi_2$ and $g = 1$, we derive for all $1 \leq U \leq R$

$$\sum_{R < n \leq 2R} \mu_2(n) e \left( \frac{z}{n} \right) = \sum_{n \leq \sqrt{U}} \mu(n) \sum_{\frac{R}{n} < m \leq \frac{R}{n}} e \left( \frac{z}{mn^2} \right) + \sum_{n \geq \sqrt{U}} \sum_{\sqrt{U} < m \leq \sqrt{2U}} \mu(m) e \left( \frac{z}{m^2 n} \right) - \sum_{\sqrt{U} < n \leq \sqrt{2U}} \mu(n) \sum_{\frac{R}{n} < m \leq \frac{R}{n}} e \left( \frac{z}{mn^2} \right).$$

For $S_1$ and $S_3$, we apply the exponent pair $(k, \ell)$ yielding

$$|S_1| \leq \sum_{n \leq \sqrt{2U}} \left| \sum_{\frac{R}{n} < m \leq \frac{R}{n}} e \left( \frac{z}{mn^2} \right) \right| \ll \sum_{n \leq \sqrt{2U}} \left\{ \left( \frac{z}{R} \right) \left( \frac{R}{n^2} \right)^{\ell-k} + \frac{R^2}{n^2 z} \right\} \ll z^k R^{\ell-2k} \sum_{n \leq \sqrt{2U}} \frac{1}{n^{2(\ell-k)}} + R^2 z^{-1}$$

and similarly for $S_3$. Choosing $(k, \ell) = BA \left( \frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon \right) = \left( \frac{55}{194} + \varepsilon, \frac{55}{97} + \varepsilon \right)$, we derive

$$R^{-\varepsilon} (S_1 + S_3) \ll z^{\frac{55}{194}} \sum_{n \leq \sqrt{2U}} \frac{1}{n^{55/97}} + R^2 z^{-1} \ll z^{\frac{55}{194} U^{21/97}} + R^2 z^{-1}.$$
Now noticing that the condition $R \leq z^{7/10}$ entails that $\sqrt{R/n} \leq (z/n)^{2/5}$ for all $n \in \mathbb{Z}_{\geq 1}$, we use (10) for the sum $S_2$, which gives

$$R^{-\varepsilon} |S_2| \leq \sum_{n \leq x} \sqrt{\frac{x}{n}} \sum_{m \leq \sqrt{\frac{x}{n}}} \mu(m) e \left( \frac{z}{m^2 n} \right) \ll \sum_{n \leq x} \left\{ \left( \frac{z}{n} \right)^{1/6} \left( \frac{R}{n} \right)^{19/97} + \left( \frac{R}{n} \right)^{7/16} \right\} \ll z^{1/6} R^{5/6} U^{-371/582} + RU^{-9/16}.$$  

We choose $U = (z^{-68} R^{485})^{1/497}$. Note that the condition $R \geq z^{68/485}$ ensures that $1 \leq U \leq R$, and this choice of $U$ yields the asserted result with the extra terms $z^{153/774} R^{487} + R^{2} z^{-1}$, which are both easily seen to be dominated by the term $z^{1/497/13774} R^{15/71}$ via the hypothesis $R \leq z^{7/10}$.  

**Proof of Theorem 1.4.** By Proposition 3.1 we derive

$$\sum_{n \leq x} \mu_2 \left( \left| \frac{x}{n} \right| \right) = x \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n(n+1)} + R(x)$$

with, for all $x^{1/3} \leq N < x^{1/2}$ and all $H \geq 1$

$$x^{-\varepsilon} R(x) \ll N + \max_{N < D \leq x / N} \left\{ \frac{D}{H} + \sum_{h \leq H} \frac{1}{h} \left( \sum_{D < d \leq 2D} \mu_2(d) e \left( \frac{hx}{d} \right) \right) \right\} \ll N + \max_{N < D \leq x / N} \left\{ \frac{D}{H} + \sum_{h \leq H} \left( 1 + \frac{hx}{D^2} \right) \left( Hx \right)^{3497/1144} D^{15/1144} \right\} \ll N + \max_{N < D \leq x / N} \left\{ (x^{17271} D^{-7.367})^{1/31345} + x^{17271} D^{-1.47} + (x^{1.3497} D^6 407)^{1/17271} + x^{3497/1144} D^{15/1144} \right\}$$

where we used Srinivasan optimization lemma in the last line, and the fact that $\min \left( \frac{x}{N}, x^{7/10} \right) = \frac{x}{N}$. Therefore

$$x^{-\varepsilon} R(x) \ll N + (x^{17271} N^{-7.367})^{1/31345} + x^{17271} N^{-1.47} + (x^{9.904} N^{-6.407})^{1/17271} + x^{6.407/1144} N^{-1.47}$$

and choosing $N = x^{1.479/2368}$ yields the asserted bound.  

The next result is the analog of Proposition 7.2 for the function $2^\omega$. The proof is much simpler.

**Proposition 7.3.** Let $z, R \geq 1$, $(k, \ell)$ be an exponent pair and assume $R \leq z^{2(k+1)/3(k+1) - \varepsilon}$. Then, for all $\varepsilon > 0$

$$R^{-\varepsilon} \sum_{R < n \leq 2R} 2^{\omega(n)} e \left( \frac{z x}{n} \right) \ll z^k R^{1 + \varepsilon/2}.$$
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PROOF. Using Corollary 4.4 with \( f = \chi_2, \ g = \tau \) and \( U = R \), we derive

\[
\sum_{R < n \leq 2R} 2^{\omega(n)} e \left( \frac{z}{n} \right) = \sum_{n \leq \sqrt{2R}} \mu(n) \sum_{\frac{n}{R} < m \leq \frac{2R}{n}} \tau(m) e \left( \frac{z}{mn^2} \right)
\]

and Proposition 6.2 with \( r = 2 \) yields

\[
R^{-\varepsilon} \sum_{R < n \leq 2R} 2^{\omega(n)} e \left( \frac{z}{n} \right) \ll \sum_{n \leq \sqrt{2R}} \left\{ \left( \frac{z}{R} \right)^k \left( \frac{R}{n^2} \right)^{\frac{1+\ell-k}{2}} + \frac{R^2}{n^2 z} \right\}
\]

\[
\ll z^k R^{1+\ell-k} \sum_{n \leq \sqrt{2R}} \frac{1}{n^{1+\ell-k}} + R^2 z^{-1} \ll z^k R^{1+\ell-k}
\]

since \( 1 + \ell - k \geq 1 \) and the term \( R^2 z^{-1} \) is absorbed by the term \( z^k R^{1+\ell-k} \) with the help of the hypothesis \( R \leq z^{2(k+1)\varepsilon} \). \( \square \)

**Proof of Theorem 1.5.** First note that, if \( x^{1/3} \leq N < x^{1/2} \) and if \( (k, \ell) \) is an exponent pair, then

\[
x^{-\frac{2(k+1)\varepsilon}{2}} \geq x^{2/3} \geq \frac{x}{N}.
\]

Now we proceed as in the proof of Theorem 1.4 above, using Propositions 3.1 and 7.3 to derive

\[
\sum_{n \leq x} 2^{\omega([x/n])} = x \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n(n+1)} + R(x)
\]

with, for all \( x^{1/3} \leq N < x^{1/2} \) and all \( H \geq 1 \)

\[
x^{-\varepsilon} R(x) \ll N + \max_{N < D \leq x/N} \left\{ \frac{D}{H} + \sum_{h \in H} \frac{1}{h} \left( \left| \sum_{D < d \leq 2D} 2^{\omega(d)} e \left( \frac{hx}{d} \right) \right| + \left| \sum_{D < d \leq 2D} 2^{\omega(d)} e \left( \frac{hx}{d+1} \right) \right| \right) \right\}
\]

\[
\ll N + \max_{N < D \leq \min \left\{ x/N, x^{3(2k+1)-(k+1)\varepsilon} \right\}} \left\{ \frac{D}{H} + \sum_{h \in H} \frac{1}{h} \left( 1 + \frac{Hx}{D^2} \right) (hx)^k D^{1+\ell-3k} \right\}
\]

\[
\ll N + \max_{N < D \leq \min \left\{ x/N, x^{3(2k+1)-(k+1)\varepsilon} \right\}} \left\{ \frac{D}{H} + (Hx)^{1+k} D^{-\frac{3+3k-\ell}{2}} + (Hx)^k D^{1+\ell-3k} \right\}
\]

\[
\ll N + \max_{N < D \leq \min \left\{ x/N, x^{3(2k+1)-(k+1)\varepsilon} \right\}} \left( x^{\frac{1+k}{2}} D^{-\frac{1+k-\ell}{2(k+1)}} + x^{1+k} D^{-\frac{3+3k-\ell}{2}} + x^{1+k} D^{1+\ell-3k} \right)
\]

where we used Srinivasan optimization lemma again, and hence

\[
x^{-\varepsilon} R(x) \ll N + x^{\frac{1+k}{2}} N^{-\frac{1+k-\ell}{2(k+1)}} + x^{1+k} N^{-\frac{3+3k-\ell}{2}} + x^{1+k} N^{-\frac{1+k}{2(k+1)}} + x^{1+k} N^{-\frac{1+k-3k}{2}}
\]

Choose \( N = x^{\frac{2(k+1)}{2k+1}} \). Note that the condition \( k + \ell < 1 \) ensures that \( x^{1/3} \leq N < x^{1/2} \). We then get the asserted result with the following two extra terms

\[
x^{\frac{5k^2+3k+3\ell-(1+\ell)}{2(k+1)(3k-\ell)+6}} + x^{\frac{3k^2+(3+2k-\ell)(1+\ell)}{2(3k-\ell)+6}}
\]

which are absorbed by the first term via the condition \( k + \ell < 1 \). The last part of the Theorem is derived with the exponent pair \( (k, \ell) = (\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon) \). \( \square \)
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