FINITE GROUPS OF BIMEROMORPHIC SELFMAPS OF NON-UNIRULED KÄHLER THREEFOLDS

YU. G. PROKHOROV AND C. A. SHRAMOV

Abstract. We prove the Jordan property for groups of bimeromorphic selfmaps of three-dimensional compact Kähler varieties of non-negative Kodaira dimension and positive irregularity.

Contents

1. Introduction 1
2. Preliminaries 3
3. Equivariant fibrations 5
4. Pseudoautomorphisms 9
5. Albanese map 10
6. Indeterminacy loci 12
7. More on compact complex surfaces 14
8. Proof of the main theorem 15
References 17

1. INTRODUCTION

In the theory of automorphism groups of complex varieties, the following notion plays an important role.

Definition 1.1 ([Pop11, Definition 2.1]). A group \( \Gamma \) is called Jordan (alternatively, one says that \( \Gamma \) has Jordan property), if there exists a constant \( J = J(\Gamma) \) such that any finite subgroup \( G \subset \Gamma \) contains a normal abelian subgroup \( A \subset G \) of index at most \( J \).

It appears that Jordan property holds for many groups of geometric origin, including automorphism groups and groups of bimeromorphic selfmaps of many compact complex varieties. For instance, if \( X \) is a compact complex surface, then the automorphism group of \( X \) is always Jordan (see [PS21, Theorem 1.6]), while its group of bimeromorphic selfmaps is Jordan if and only if \( X \) is not bimeromorphic to a product of \( \mathbb{P}^1 \) and an elliptic curve (see [PS21, Theorem 1.7]). As for higher dimensions, much less is understood. For instance, it is unclear whether the automorphism group of an arbitrary compact complex variety is Jordan or not; a positive answer is known only for the neutral component of such a group, see [Pop18, Theorem 5]. However, there are several important results in the case of Kähler varieties. To start with, the following theorem was proved in [Kim18].

Theorem 1.2. Let \( X \) be a compact Kähler variety. Then the automorphism group of \( X \) is Jordan.

This work is supported by the Russian Science Foundation under grant №18-11-00121.
Theorem 1.2 was recently generalized to the case of automorphism groups of compact complex spaces of Fujiki’s class $C$ in [MPZ20] (cf. [PS19]). In [PS20b], groups of bimeromorphic selfmaps of uniruled compact Kähler varieties of dimension 3 were studied from the point of view of Jordan property. In higher dimensions, partial results for $\mathbb{P}^1$-bundles over appropriate bases were obtained in [BZ21a] and [BZ21b] (cf. [BZ21c]). In particular, [Zar19] provides plenty of examples of uniruled compact Kähler varieties with non-Jordan groups of bimeromorphic selfmaps.

Denote by $\kappa(X)$ the Kodaira dimension of a compact complex variety $X$, and by $q(X)$ its irregularity; we refer the reader to Section 2 for details. Recall that a three-dimensional compact Kähler variety is uniruled if and only if its Kodaira dimension is negative, see [HP16, Corollary 1.4]: this is a very deep fact which follows from the results of [Bru06], [DP03], and [HP16]. The purpose of this paper is to prove the following theorem concerning the groups of bimeromorphic selfmaps of non-uniruled three-dimensional compact Kähler varieties; we refer the reader to Section 2 for notation and conventions.

**Theorem 1.3.** Let $X$ be a three-dimensional compact Kähler variety with $\kappa(X) \geq 0$ and $q(X) > 0$. Then the group of bimeromorphic selfmaps of $X$ is Jordan.

The plan of the paper is as follows. In Section 2 we collect basic definitions and auxiliary facts concerning compact complex varieties and manifolds. In Section 3 we study groups of bimeromorphic selfmaps that preserve fibrations on compact complex varieties. In Section 4 we consider groups of pseudoautomorphisms of compact Kähler manifolds. In particular, we show in Proposition 4.5 that the group of pseudoautomorphisms of a compact Kähler manifold is always Jordan; a projective analog of this result is well known to experts. In Section 5 we collect information about Albanese maps of compact complex varieties. In Section 6 we discuss some technical facts on the indeterminacy loci of pseudautomorphisms. In Section 7 we make a couple of additional observations on compact complex surfaces and their automorphism groups. Finally, in Section 8 we complete the proof of Theorem 1.3.

For a non-uniruled algebraic variety $X$ of arbitrary dimension a much stronger assertion than Theorem 1.3 is valid: the group of birational selfmaps of $X$ is Jordan without any restriction on the irregularity, see [PS14, Theorem 1.8(ii)]. The proof of this result in [PS14] uses the equivariant Minimal Model Program (see [Pro21] for a brief introduction). The proofs in this paper also use the MMP for three-dimensional compact Kähler varieties which is available due to [HP16]. We expect that an analog of Theorem 1.3 is valid in arbitrary dimension and can be proved independently of the MMP.

Furthermore, our proof of Jordan property for pseudoautomorphism groups is based on Fujiki’s result from [Fuj81] (see Theorem 4.4 below) which is known only for smooth Kähler varieties. It is desirable to generalize it for Kähler varieties with terminal singularities. This would allow to largely simplify our arguments, and to remove the assumption that $q(X) > 0$ in Theorem 1.3.

We are grateful to Ch. Hacon who pointed out a gap in the first draft of the paper. We also thank the anonymous referee for helpful suggestions.

---

1After this paper was written, A. Golota in [Gol21] generalized the result of Fujiki to the case of singular Kähler varieties and proved a stronger version of Theorem 1.3.
2. Preliminaries

In this section we collect some auxiliary facts about compact complex varieties and manifolds. We refer the reader to [Uen75] for the most basic facts and definitions. In particular, by a *complex variety* we mean an irreducible reduced complex space. A *morphism* of complex varieties is a holomorphic map between them. A smooth complex variety is called a *complex manifold*. A *complex surface* is a complex manifold of dimension 2.

Given a compact complex variety $X$, by $a(X)$ we denote the algebraic dimension of $X$, i.e. the transcendence degree of the field of meromorphic functions on $X$. By $q(X)$ we denote the irregularity of $X$, i.e. the dimension of $H^1(X', \mathcal{O}_{X'})$, where $X'$ is an arbitrary compact complex manifold bimeromorphic to $X$. Similarly, the Kodaira dimension $\kappa(X)$ is defined as the Kodaira dimension of its smooth compact model $X'$, see [Uen75, Definition 6.5]. By $\text{Bim}(X)$ we denote the group of biholomorphic selfmaps of $X$. A *Zariski open subset* of $X$ is a subset of the form $X \setminus \Sigma$, where $\Sigma$ is a closed (analytic) subset in $X$. A typical point of $X$ is a point of some non-empty Zariski open subset of $X$; a typical fiber of a (meromorphic) map $\phi: X \to Y$ is a fiber over a typical point of $Y$. For a meromorphic map $\chi: X \to Y$, we denote by $\text{Ind}(\chi)$ the indeterminacy locus of $\chi$, i.e. the minimal closed analytic subset $V \subset X$ such that the restriction of $\chi$ to $X \setminus V$ is holomorphic.

Recall that the canonical class of a normal complex variety $X$ is the reflexive rank one sheaf

$$\omega_X = j_* \mathcal{O}_{X_0}^{\dim(X)}$$

where $X_0 \subset X$ is the smooth locus and $j: X_0 \hookrightarrow X$ is the embedding, see [Rei80] §1A. Note that in contrast with the projective case the canonical class $\omega_X$ need not be represented by a Weil divisor. However, $\omega_X$ is always represented by a Weil divisor locally, in a small analytic neighborhood of any point. Thus, sometimes we will abuse notation and write $K_X$ instead of $\omega_X$. Note also that throughout this paper we consider varieties of non-negative Kodaira dimension; for such a variety $X$ the reflexive sheaf

$$\omega^{[n]}_X = (\omega_X^\otimes n)^{\vee\vee}$$

is represented by a Weil divisor $nK_X$ for some positive integer $n$. Indeed, for such a divisor one can take $nK_X = f_* D$, where $f: \tilde{X} \to X$ is a resolution of singularities, and $D$ is an effective divisor from non-empty linear system $|nK_{\tilde{X}}|$.

For the basic terminology, definitions, and facts concerning the Kähler Minimal Model Program we refer to [HP16]. We emphasize only the following distinctions. Since the canonical class $\omega_X$ need not be represented by a Weil divisor, in the case of non-algebraic complex varieties the definition of a normal $\mathbb{Q}$-factorial singularity includes the requirement that the sheaf $\omega^{[n]}_X$ is invertible for some $n$. We remind the reader that a terminal singularity is a normal $\mathbb{Q}$-Gorenstein singularity of a complex variety such that all of its discrepancies are positive. In particular, if a variety $X$ has only terminal (or just normal $\mathbb{Q}$-Gorenstein) singularities, the intersection numbers of $\omega_X$ with all the curves on $X$ are well defined. A canonical singularity is a normal $\mathbb{Q}$-Gorenstein singularity of a complex variety such that all of its discrepancies are non-negative. All canonical (and in particular terminal) singularities are rational, see e.g. [KM98, Theorem 5.22].

One says that a group $\Gamma$ has *bounded finite subgroups* if there exists a constant $B = B(\Gamma)$ such that every finite subgroup of $\Gamma$ has order at most $B$. 

3
Lemma 2.1. Let
\[ 1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma'' \]
be an exact sequence of groups. Suppose that the group $\Gamma''$ has bounded finite subgroups. Then the group $\Gamma$ is Jordan if and only if the group $\Gamma'$ is Jordan.

Proof. Obvious. \qed

A group $\Gamma$ is called strongly Jordan if it is Jordan and there exists a constant $r = r(\Gamma)$ such that every finite subgroup of $\Gamma$ is generated by at most $r$ elements.

Lemma 2.2 ([PS14, Lemma 2.8]). Let
\[ 1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma'' \]
be an exact sequence of groups. Suppose that the group $\Gamma''$ is strongly Jordan and the group $\Gamma'$ has bounded finite subgroups. Then the group $\Gamma$ is Jordan.

The following classical theorem was proved by H. Minkowski.

Theorem 2.3 (see e.g. [Ser07, Theorem 1]). For every positive integer $n$ the group $GL_n(\mathbb{Z})$ has bounded finite subgroups.

Corollary 2.4. Let $\Lambda$ be a finitely generated abelian group. Then the group $\text{Aut}(\Lambda)$ has bounded finite subgroups.

The following assertion is well known, see e.g. [BL04, Proposition 1.2.1] or [PS21, Theorem 8.4].

Theorem 2.5. Let $T$ be a complex torus of dimension $n$. Then
\[ \text{Aut}(T) \cong T \rtimes \Gamma, \]
where $\Gamma$ is a subgroup of $GL_{2n}(\mathbb{Z})$.

The next result follows from [Uen75, Lemma 9.11] and [Gra18, Lemma 3.5].

Proposition 2.6. Let $X$ be a compact complex variety with rational singularities, and let $T$ be a complex torus. Let $\zeta: X \longrightarrow T$ be a meromorphic map. Then $\zeta$ is holomorphic.

Corollary 2.7. Let $T$ be a complex torus of dimension $n$. Then the group $\text{Bim}(T)$ is Jordan. Moreover, there exists a positive integer $r = r(n)$ which depends only on $n$ but not on $T$ such that every finite subgroup of $\text{Bim}(T)$ is generated by at most $r$ elements. In particular, $\text{Bim}(T)$ is strongly Jordan.

Proof. One has $\text{Bim}(T) = \text{Aut}(T)$ by Proposition 2.6. Thus the first assertion follows from Theorems 2.3 and 2.5 together with Lemma 2.1 (alternatively, it can be obtained from Theorem 1.2). The second assertion follows directly from Theorems 2.5 and 2.3. \qed

Theorem 2.8 (see [Uen75, Corollary 14.3]). Let $X$ be a compact complex variety with $\dim(X) = \kappa(X)$. Then the group $\text{Bim}(X)$ is finite.

Lemma 2.9. Let $X$ and $Y$ be compact complex varieties, and let $\phi: X \longrightarrow Y$ be a dominant meromorphic map. Suppose that $\kappa(X) \geq 0$. Let $F$ be a typical fiber of $\phi$, and let $F'$ be an irreducible component of $F$. Then $\kappa(F') \geq 0$. 

4
Proof. Applying the resolution of singularities and indeterminacies, we may assume that $X$ and $Y$ are smooth, and the map $\phi$ is holomorphic. In particular, this means that $F$ is a compact complex manifold, and $F'$ is a connected component of $F$. By adjunction one has

$$\omega_{F'} \cong \omega_X|_{F'}.$$  

Therefore, since $\omega_X^n$ is represented by an effective divisor for some positive integer $n$, the same holds for $\omega_{F'}$. Thus, we have $\kappa(F') \geq 0$. \hfill \Box

For most of the compact complex surfaces, their groups of bimeromorphic selfmaps are Jordan. More precisely, the following is known.

**Theorem 2.10** (see [PS21, Theorem 1.7]). Let $X$ be a compact complex surface with $\kappa(X) \geq 0$. Then the group $\text{Bim}(X)$ is strongly Jordan.

**Remark 2.11.** Let $X$ be a (smooth connected) compact complex curve. It is easy to see that the group $\text{Bim}(X) = \text{Aut}(X)$ is strongly Jordan.

If a group acts on a compact Kähler manifold of non-negative Kodaira dimension with a fixed point, then it has bounded finite subgroups.

**Theorem 2.12** ([PS20a, Theorem 1.5]). Let $X$ be a compact Kähler manifold of non-negative Kodaira dimension, and let $P$ be a point on $X$. Then the stabilizer of $P$ in $\text{Aut}(X)$ has bounded finite subgroups.

### 3. Equivariant fibrations

In this section we make several observations about groups of bimeromorphic selfmaps that preserve fibrations on complex varieties.

Given a dominant meromorphic map $\alpha: X \rightarrow Y$ of compact complex varieties, we denote by $\text{Bim}(X;\alpha)$ the subgroup in $\text{Bim}(X)$ that consists of all bimeromorphic selfmaps of $X$ which map the fibers of $\alpha$ again to the fibers of $\alpha$. In other words, if $\chi \in \text{Bim}(X;\alpha)$, then for every two points $P, Q \in X \setminus \text{Ind}(\chi)$ such that $\alpha(P) = \alpha(Q)$ we have $\alpha(\chi(P)) = \alpha(\chi(Q))$. There is a natural homomorphism

$$h_{\alpha}: \text{Bim}(X;\alpha) \longrightarrow \text{Bim}(Y),$$

and the map $\alpha$ is equivariant with respect to $\text{Bim}(X;\alpha)$. Denote by $\text{Bim}(X)_{\alpha}$ the kernel of the homomorphism $h_{\alpha}$.

The proof of the following lemma is similar to that of [PS20b, Lemma 4.1].

**Lemma 3.1.** Let $X$ and $Y$ be compact complex varieties, and let $\alpha: X \rightarrow Y$ be a dominant meromorphic map. Then there is a constant $I = I(\alpha)$ with the following property. Let $G_i, i \in \mathbb{N}$, be a countable family of finite subgroups in $\text{Bim}(X)_{\alpha}$. Then there exists a reduced fiber $F$ of the map $\alpha$, and its irreducible component $F'$ of dimension $\dim(X) - \dim(Y)$, such that in every group $G_i$ there is a subgroup of index at most $I$ which is isomorphic to a subgroup of $\text{Bim}(F')$. Moreover, if $\dim(Y) > 0$, and we are given a countable union $\Xi$ of proper closed analytic subsets in $Y$, then the fiber $F$ can be chosen so that the point $\alpha(F)$ does not lie in $\Xi$.
Proof. Let $\Delta \subset Y$ be the minimal closed subset of $Y$ such that every point $P$ of $Y \setminus \Delta$ is smooth, the fiber $\alpha^{-1}(P)$ is reduced, and every irreducible component of $\alpha^{-1}(P)$ has dimension $\dim(X) - \dim(Y)$. Then $Y \setminus \Delta$ is a dense open subset of $Y$. Enlarging $\Delta$ if necessary, we may assume that the fibers over all the points of $Y \setminus \Delta$ have the same number $N$ of irreducible components. Denote

$$G = \bigcup_i G_i;$$

thus, $G$ is a countable set of elements in $\text{Bim}(X)_{\alpha}$. Let $\gamma$ be an element of the group $\text{Bim}(X)_{\alpha}$. Consider the set $\nabla_{\gamma} \subset Y$ consisting of all the points $P$ for which $\text{Ind}(\gamma)$ contains an irreducible component of the fiber $\alpha^{-1}(P)$. Thus the map $\gamma$ is defined in a typical point of every irreducible component of the fiber $\alpha^{-1}(P)$ over every point $P \in Y \setminus \nabla_{\gamma}$. Moreover, for any point

$$P \in Y \setminus (\Delta \cup \nabla_{\gamma} \cup \nabla_{\gamma^{-1}})$$

the restriction $\gamma|_{F'}$ of the map $\gamma$ to every irreducible component $F'$ of the fiber $\alpha^{-1}(P)$ is a bimeromorphic map of $F'$ to its image $\gamma(F')$, and the image $\gamma(F')$ does not coincide with an image of any other irreducible component of $\alpha^{-1}(P)$.

Consider the subset $D_{\gamma} \subset Y \setminus (\Delta \cup \nabla_{\gamma})$ consisting of all the points $P$ such that for a typical point $Q$ of some irreducible component of the fiber $\alpha^{-1}(P)$ one has $\gamma(Q) = Q$. Let $\overline{D}_{\gamma}$ be the closure of $D_{\gamma}$ in $Y$. Then for any point

$$P \in Y \setminus (\Delta \cup \nabla_{\gamma} \cup \nabla_{\gamma^{-1}} \cup \overline{D}_{\gamma})$$

the restriction $\gamma|_{F'}$ of the map $\gamma$ to every irreducible component $F'$ of the fiber $\alpha^{-1}(P)$ is not the identity map of $F'$, provided that $\gamma$ itself is not the identity map of $X$.

The sets $\Delta$, $\nabla_{\gamma}$, and $\overline{D}_{\gamma}$ are proper closed analytic subsets in $Y$. If $\dim(Y) > 0$, fix also a subset $\Xi$ which is a countable union of proper closed analytic subsets in $Y$. Since the field $\mathbb{C}$ is uncountable, $Y$ cannot be represented as a countable union of proper closed subsets. Hence the complement

$$U = Y \setminus \left( \Xi \cup \Delta \cup \bigcup_{\gamma \in G \setminus \{\text{id} \}} (\nabla_{\gamma} \cup \overline{D}_{\gamma}) \right)$$

is non-empty.

Let $P$ be a point of $U$, and let $F$ be the fiber of $\alpha$ over $P$. Every element $\gamma \in G$ defines a permutation of the set of $N$ irreducible components of $F$. Thus, for every $i$ we have a homomorphism $G_i \to \mathfrak{S}_N$ to the symmetric group of degree $N$. Denote by $K_i \subset G_i$ the kernel of this homomorphism. Then the index of $K_i$ in $G_i$ is at most $N! = |\mathfrak{S}_N|$. Moreover, every element of $K_i$ maps every irreducible component $F'$ of $F$ to itself, and every non-trivial element of $K_i$ restricts to a non-trivial bimeromorphic selfmap of $F'$. Therefore, all the groups $K_i$ are embedded into the group $\text{Bim}(F')$.

Lemma 3.2. Let $X$ and $Y$ be compact complex varieties, and let $\alpha: X \rightarrow Y$ be a dominant meromorphic map. Let $F$ be a typical fiber of $\alpha$. Suppose that for any irreducible component $F'$ of $F$ the group $\text{Bim}(F')$ is Jordan. Suppose also that the image $h_{\alpha}(\text{Bim}(X; \alpha)) \subset \text{Bim}(Y)$ has bounded finite subgroups. Then the group $\text{Bim}(X; \alpha)$ is Jordan. In particular, if under these
assumptions the map $\alpha$ is equivariant with respect to the whole group $\text{Bim}(X)$, then $\text{Bim}(X)$ is Jordan.

Proof. Suppose that the group $\text{Bim}(X; \alpha)$ is not Jordan. By Lemma 2.1, this means that the group $\text{Bim}(X)_\alpha$ is also not Jordan. Hence the group $\text{Bim}(X)_\alpha$ contains a countable family of subgroups $G_i$, $i \in \mathbb{N}$, such that the minimal indices $J_i$ of normal abelian subgroups of $G_i$ form an unbounded sequence. On the other hand, by Lemma 3.1, there exists a constant $I$, a typical fiber $F$ of the map $\alpha$, and an irreducible component $F'$ of $F$, such that every $G_i$ contains a subgroup $K_i$ of index at most $I$ which can be embedded into the group $\text{Bim}(F')$. Since $\text{Bim}(F')$ is a Jordan group, we conclude that the minimal index of an abelian subgroup of $K_i$ is bounded by a constant $J$ independent of $i$. Therefore, the minimal index of an abelian subgroup of $G_i$ is bounded by the constant $IJ$, and hence the minimal index of a normal abelian subgroup of $G_i$ is also bounded. This contradicts the unboundedness of the indices $J_i$. □

Corollary 3.3. Let $X$ be a three-dimensional compact complex variety, let $Z$ be a compact complex surface, and let $\alpha: X \dasharrow Z$ be a dominant meromorphic map. Suppose that the image $h_\alpha(\text{Bim}(X; \alpha)) \subset \text{Aut}(Z)$ has bounded finite subgroups. Then the group $\text{Bim}(X; \alpha)$ is Jordan. In particular, if under these assumptions the map $\alpha$ is equivariant with respect to the whole group $\text{Bim}(X)$, then $\text{Bim}(X)$ is Jordan.

Proof. Let $F$ be a typical fiber of $\alpha$, and let $F'$ be an irreducible component of $F$. Then $F'$ is a smooth projective curve. Hence $\text{Bim}(F') = \text{Aut}(F')$ is Jordan, see Remark 2.1. Therefore, the required assertion follows from Lemma 3.2. □

Corollary 3.4. Let $X$ be a three-dimensional compact complex variety with $\kappa(X) \geq 0$, let $B$ be a curve, and let $\alpha: X \dasharrow B$ be a dominant meromorphic map. Suppose that the image $h_\alpha(\text{Bim}(X; \alpha)) \subset \text{Aut}(B)$ has bounded finite subgroups. Then the group $\text{Bim}(X; \alpha)$ is Jordan. In particular, if under these assumptions the map $\alpha$ is equivariant with respect to the whole group $\text{Bim}(X)$, then $\text{Bim}(X)$ is Jordan.

Proof. Let $F$ be a typical fiber of $\alpha$, and let $F'$ be an irreducible component of $F$. Then $\kappa(F') \geq 0$ by Lemma 2.9. Hence $\text{Bim}(F')$ is Jordan by Theorem 2.10. Therefore, the required assertion follows from Lemma 3.2. □

Corollary 3.5. Let $X$ be a three-dimensional compact complex variety with $\kappa(X) \geq 0$, let $A$ be a smooth projective curve, and let $\alpha: X \dasharrow A$ be a dominant meromorphic map. Suppose that either the genus of $A$ is at least 2, or $A$ is elliptic and the image $h_\alpha(\text{Bim}(X; \alpha)) \subset \text{Aut}(A)$ preserves a non-empty finite subset of $A$. Then the group $\text{Bim}(X; \alpha)$ is Jordan. In particular, if under these assumptions the map $\alpha$ is equivariant with respect to the whole group $\text{Bim}(X)$, then $\text{Bim}(X)$ is Jordan.

Proof. We see from the assumptions that the group $h_\alpha(\text{Bim}(X; \alpha))$ is finite. Therefore, the assertion follows from Corollary 3.3. □

Corollary 3.6. Let $X$ be a three-dimensional compact Kähler variety with $\kappa(X) \geq 0$, and let $\alpha: X \dasharrow T$ be a dominant meromorphic map to a two-dimensional complex torus $T$. Suppose that there exists a subvariety $V \subseteq T$ invariant with respect to $h_\alpha(\text{Bim}(X; \alpha))$. Then the group $\text{Bim}(X; \alpha)$ is Jordan. In particular, if under these assumptions the map $\alpha$ is equivariant with respect to the whole group $\text{Bim}(X)$, then $\text{Bim}(X)$ is Jordan.
Proof. Recall that the action of $h_\alpha(Bim(X;\alpha))$ on $T$ is regular by Proposition 2.6. If the group $h_\alpha(Bim(X;\alpha))$ has bounded finite subgroups, then $Bim(X;\alpha)$ is Jordan by Corollary 3.3.

Let $V_1$ be an irreducible component of $V$, and let $\Gamma_1 \subseteq h_\alpha(Bim(X;\alpha))$ be the stabilizer of $V_1$. Then $\Gamma_1$ is a finite index subgroup in $h_\alpha(Bim(X;\alpha))$. If $\Gamma_1$ acts on $V_1$ with a fixed point, then it has bounded finite subgroups by Theorem 2.12 (or by Theorems 2.5 and 2.3). Hence $h_\alpha(Bim(X;\alpha))$ has bounded finite subgroups as well. In particular, this applies to the case when $V_1$ is a point itself.

Suppose that $V_1$ is a curve; then $V_1$ is not rational. If $V_1$ is singular, then a subgroup of finite index in $\Gamma_1$ acts on $T$ with a fixed point, and so $\Gamma_1$ and $h_\alpha(Bim(X;\alpha))$ have bounded finite subgroups. Thus, we assume that $V_1$ is smooth. If $g(V_1) > 1$, then the group $\text{Aut}(V_1)$ is finite. This means that a subgroup of finite index in $\Gamma_1$ acts on $V_1$ trivially, and in particular has a fixed point. As before, we conclude that $\Gamma_1$ and $h_\alpha(Bim(X;\alpha))$ have bounded finite subgroups in this case.

Therefore, we may assume that $V_1$ is an elliptic curve. Consider the quotient torus $T_1 = T/V_1$, where the group structure on $T$ is chosen in such a way that $V_1$ is a subgroup of $T$ (i.e., the neutral element of $T$ is contained in $V_1$). Note that $T_1$ and the quotient map $T \to T_1$ do not depend on this choice, and hence the map $T \to T_1$ is $\Gamma_1$-equivariant. Thus the composition $X \to T \to T_1$ is equivariant with respect to the preimage $\tilde{\Gamma}_1$ of $\Gamma_1$ in $Bim(X;\alpha)$. Moreover, the image of $\tilde{\Gamma}_1$ (or of $\Gamma_1$) in $\text{Aut}(T_1)$ preserves a point on the elliptic curve $T_1$. Therefore, the group $\tilde{\Gamma}_1$ is Jordan by Corollary 3.3. Since $\tilde{\Gamma}_1$ has finite index in $Bim(X;\alpha)$, the latter group is Jordan as well. \qed

Lemma 3.7. Let $X$ and $Y$ be compact complex varieties, and let $\alpha: X \dashrightarrow Y$ be a dominant meromorphic map. Let $F$ be a typical fiber of $\alpha$. Suppose that for any irreducible component $F'$ of $F$ the group $Bim(F')$ has bounded finite subgroups. Suppose also that the image $h_\alpha(Bim(X;\alpha)) \subset Bim(Y)$ is strongly Jordan. Then the group $Bim(X;\alpha)$ is Jordan. In particular, if under these assumptions the map $\alpha$ is equivariant with respect to the whole group $Bim(X)$, then $Bim(X)$ is Jordan.

Proof. As in the proof of Lemma 3.2 suppose that the group $Bim(X;\alpha)$ is not Jordan. By Lemma 2.2, this means that the group $Bim(X;\alpha)$ has unbounded finite subgroups. In other words, the group $Bim(X;\alpha)$ contains a countable family of subgroups $G_i$, $i \in \mathbb{N}$, such that the orders of $G_i$ are unbounded. On the other hand, by Lemma 2.1 there exists a constant $I$, a typical fiber $F$ of the map $\alpha$, and an irreducible component $F'$ of $F$, such that every $G_i$ contains a subgroup $K_i$ of index at most $I$ which can be embedded into the group $Bim(F')$. Since $Bim(F')$ has bounded finite subgroups, we see that the orders of $K_i$ are bounded. This contradicts the unboundedness of the orders of the groups $G_i$. \qed

Lemma 3.7 immediately implies

Corollary 3.8. Let $X$ and $Y$ be compact complex varieties, and let $\alpha: X \dashrightarrow Y$ be a dominant meromorphic map such that a typical fiber of $\alpha$ is finite. Suppose that the group $Bim(Y)$ is strongly Jordan. Then the group $Bim(X;\alpha)$ is Jordan. In particular, if under these assumptions the map $\alpha$ is equivariant with respect to the whole group $Bim(X)$, then $Bim(X)$ is Jordan.
In this section we consider groups of pseudoautomorphisms of compact Kähler manifolds, and establish the Jordan property for groups of bimeromorphic selfmaps of three-dimensional compact Kähler varieties of Kodaira dimension zero.

Recall that a biholomorphic selfmap \( f: X \rightarrow X \) of a compact complex variety \( X \) is called a pseudoautomorphism if there are non-empty Zariski open subsets \( U_1, U_2 \subset X \) such that \( \text{codim}_X(X \setminus U_i) \geq 2 \), and \( f \) restricts to an isomorphism
\[
f|_{U_1}: U_1 \xrightarrow{\sim} U_2.
\]
The pseudoautomorphisms of \( X \) form a subgroup in \( \text{Bim}(X) \) which we denote by \( \text{PAut}(X) \).

**Lemma 4.1** (Negativity Lemma [Sho93, 1.1], [Wan21, Lemma 1.3]). Let \( f: \tilde{V} \to V \) be a proper bimeromorphic morphism between normal complex varieties. Let \( D \) be a Cartier divisor on \( \tilde{V} \) such that \(-D\) is \( f\)-nef. Then \( D \) is effective if and only if \( f_* D \) is effective.

The following assertion is well known to experts (see e.g. [Kol89, Lemma 4.3]). We provide its proof for the convenience of the reader.

**Lemma 4.2.** Let \( \chi: X \to X' \) be a bimeromorphic map of compact complex varieties with terminal singularities such that \( \omega_{X'} \) is nef. Then \( \chi^{-1} \) does not contract any divisors.

**Proof.** By definition of terminal singularities, for a sufficiently divisible positive integer \( m \) the sheaves \( \omega^m_{X'} \) and \( \omega^m_{X'} \) are invertible. Let
\[
\begin{array}{c}
X \xrightarrow{\chi} Y \\
\downarrow p \quad \downarrow q \\
\end{array}
\]
be a common resolution, i.e. a commutative diagram where \( Y \) is a compact complex manifold, and \( p \) and \( q \) are proper bimeromorphic morphisms. By making further blowups if necessary, we may assume that the exceptional sets \( \text{Exc}(p) \) and \( \text{Exc}(q) \) are of pure codimension one. Set \( Y_p = p(\text{Exc}(p)) \) and \( Y_q = q(\text{Exc}(q)) \). Then \( p \) induces an isomorphism of open subsets \( Y \setminus \text{Exc}(p) \) and \( X \setminus \text{Y}_p \); similarly, \( q \) induces an isomorphism of \( Y \setminus \text{Exc}(q) \) and \( X' \setminus \text{Y}_q \). This implies that one can write
\[
\omega_{Y}^m \cong p^*(\omega^m_{X'}) \otimes \mathcal{O}_Y \left( m \left( \sum a_i E_i + \sum b_j F_j \right) \right) \cong q^*(\omega^m_{X'}) \otimes \mathcal{O}_Y \left( m \left( \sum c_i E_i + \sum d_k G_k \right) \right),
\]
where \( E_i \) (respectively, \( F_j \), respectively, \( G_k \)) are exceptional divisors with respect to both \( p \) and \( q \) (respectively, \( p \)-exceptional but not \( q \)-exceptional, respectively, \( q \)-exceptional but not \( p \)-exceptional). Thus, none of the three divisors \( \sum E_i \), \( \sum F_j \), and \( \sum G_k \) has a common irreducible component with any of the other two. Since \( X \) and \( X' \) have only terminal singularities, the rational numbers \( a_i \), \( b_j \), \( c_i \), and \( d_k \) are strictly positive.

Consider the Cartier divisor
\[
D = m \left( \sum (c_i - a_i) E_i + \sum d_k G_k - \sum b_j F_j \right)
\]
Clearly, the push-forward \( p_* D \) is effective. Since
\[
\mathcal{O}_Y(-D) \cong q^* \omega^m_{X'} \otimes p^* \omega^{-m}_{X},
\]
we see that the divisor \(-D\) is \(p\)-nef. Thus \(D\) is effective by Lemma 4.1. Hence, one has \(\sum F_j = 0\), i.e. the map \(\chi^{-1}\) does not contract any divisors.

**Corollary 4.3.** Let \(X\) be a compact complex variety with terminal singularities and \(\omega_X\) nef. Then \(\text{Bim}(X)\) acts on \(X\) by pseudo-automorphisms.

The following result is due to A. Fujiki.

**Theorem 4.4 ([Fuj81 Corollary 3.3]).** Let \(X\) be a compact Kähler manifold, and let \(g\) be its pseudoautomorphism. Suppose that for a Kähler class \(\alpha\) on \(X\) its push-forward \(g_\ast \alpha\) is again a Kähler class. Then \(g^{-1}\) is a morphism.

Theorem 4.4 allows to study the group of pseudoautomorphisms of compact Kähler manifolds.

**Proposition 4.5.** Let \(X\) be a compact Kähler manifold. Then the group of its pseudoautomorphisms is Jordan.

*Proof.* The group \(\text{PAut}(X)\) naturally acts on \(H^2(X, \mathbb{Z})\). Let \(\text{PAut}'(X)\) be the kernel of this action. Then \(\text{PAut}'(X)\) preserves the Kähler class. By Theorem 4.4 the group \(\text{PAut}'(X)\) consists of biholomorphic automorphisms. Hence the group \(\text{PAut}'(X)\) is Jordan by Theorem 1.2. On the other hand, the quotient \(\text{PAut}(X)/\text{PAut}'(X)\) acts faithfully on the finitely generated abelian group \(H^2(X, \mathbb{Z})\), and thus has bounded finite subgroups by Corollary 2.4. Therefore, the group \(\text{PAut}(X)\) is Jordan by Lemma 2.1.

**Remark 4.6.** In [Fuj81], Theorem 4.4 is proved for manifolds (smooth varieties). We do not know if this result, and thus Proposition 4.5, can be generalized to the case of singular Kähler varieties.

Applying Proposition 4.5 together with Corollary 4.3, we obtain

**Corollary 4.7.** Let \(X\) be a compact Kähler manifold such that \(\omega_X\) is nef. Then the group \(\text{Bim}(X)\) is Jordan.

### 5. Albanese map

In this section we collect information about Albanese maps of compact complex manifolds. Let \(X\) be a compact complex manifold. We denote by

\[
\alpha: X \longrightarrow \text{Alb}(X)
\]

the Albanese morphism of \(X\), see [Uen75 Definition 9.6]. The map \(\alpha\) has the following universal property. If \(\zeta: X \rightarrow T\) is a morphism to an arbitrary complex torus \(T\), then there exists a unique homomorphism of complex tori \(\xi: \text{Alb}(X) \rightarrow T\) that fits into the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\zeta} & T \\
\downarrow \alpha & & \xi \\
\text{Alb}(X) & & \\
\end{array}
\]

**Remark 5.1.** More generally, the Albanese morphism is well defined (and has the above universal property) for compact complex varieties with rational singularities, see [Gra18 Theorem 3.3] and [Gra18 Remark 3.4].
The next assertion is well known to experts, but we provide its proof for the reader’s convenience. We will use it several times without further reference in the sequel.

**Proposition 5.2.** Let $X$ be a compact complex manifold. There is a natural biregular action of the group $\text{Bim}(X)$ on $\text{Alb}(X)$ such that the morphism $\alpha$ is equivariant.

**Proof.** Let $\varphi: X \to X$ be an arbitrary bimeromorphic map. The composition $\alpha \circ \varphi$ is holomorphic by Proposition 2.6. By the universal property of $\alpha$ there exist a unique translation

$$t_a: \text{Alb}(X) \longrightarrow \text{Alb}(X)$$

by an element $a \in \text{Alb}(X)$ and a unique homomorphism of complex tori

$$\psi: \text{Alb}(X) \longrightarrow \text{Alb}(X)$$

such that

$$\alpha \circ \varphi = t_a \circ \psi \circ \alpha.$$

In other words, there exists a unique morphism of abstract varieties

$$\theta_\varphi = t_a \circ \psi: \text{Alb}(X) \longrightarrow \text{Alb}(X)$$

that fits to the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
\text{Alb}(X) & \xrightarrow{\theta_\varphi} & \text{Alb}(X)
\end{array}
$$

It is easy to see that the correspondence is functorial: for any two bimeromorphic maps $\varphi': X \to X$ and $\varphi'': X \to X$ one has

$$\theta_{\varphi'} \circ \theta_{\varphi''} = \theta_{\varphi' \circ \varphi''}.$$

In particular, the correspondence $\varphi \mapsto \theta_\varphi$ defines a group homomorphism

$$\text{Bim}(X) \longrightarrow \text{Aut}(\text{Alb}(X)).$$

\[ \square \]

**Remark 5.3.** If $X$ is a normal compact complex variety with rational singularities and $q(X) = 1$, then the fibers of the Albanese map $\alpha: X \to A = \text{Alb}(X)$ are connected. Indeed, otherwise $\alpha$ admits a non-trivial Stein factorisation

$$\alpha: X \longrightarrow A'.

Let $\beta$ be the composition of $\alpha'$ with the embedding of $A'$ into its Jacobian $J(A')$. Then the map $\beta: X \to J(A')$ must factor through $\alpha$ by the universal property of the Albanese map, which is clearly impossible.

Recall that a compact complex subvariety $Y$ of a complex torus $T$ is said to *generate* $T$ if for some positive integer $n$ every point of $T$ can be represented as a sum of $n$ points of $Y$; see [Uen75, Definition 9.13]. For instance, a proper subtorus of $T$ (containing the neutral element of the group $T$) does not generate $T$.  

11
Proposition 5.4. Let $T$ be a complex torus, and let $Y \subset T$ be a compact complex subvariety. Then there exists a canonically defined fibration $Y \to Z$ whose typical fiber is a subtorus $T_1 \subset T$ and the base $Z$ is a (possibly singular) compact complex variety with $\dim(Z) = \kappa(Z)$. Moreover, $Z$ is a point if and only if $Y$ is a translation of a subtorus $T_1 \subset T$.

Proof. Take the largest subtorus of $T_1 \subset T$ such that $Y$ is invariant under translations by all elements of $T_1$, and then put $Z = Y/T_1$. See [Uen75, Theorem 10.9] and its proof for details. □

Lemma 5.5. Let $X$ be a three-dimensional compact complex manifold with $\kappa(X) \geq 0$. Suppose that $\alpha$ is not surjective. Then the group $\text{Bim}(X)$ is Jordan.

Proof. Let $Y = \alpha(X) \subseteq \text{Alb}(X)$. By Proposition 5.4 there exists a canonically defined fibration $\gamma: Y \to Z$, where $\dim(Z) = \kappa(Z)$. The group $\text{Bim}(Z)$ is finite by Theorem 2.8. Recall that $Y$ generates $\text{Alb}(X)$, see [Uen75, Lemma 9.14]. Since $Y$ contains the neutral element of the group $\text{Alb}(X)$, this implies that it is not contained in a proper subtorus of $\text{Alb}(X)$. Hence $Z$ is not a point by Proposition 5.4.

Now we have a morphism $\lambda = \gamma \circ \alpha: X \to Z$ that is equivariant with respect to the group $\text{Bim}(X)$. Let $F$ be a typical fiber of $\lambda$, and let $F'$ be its connected component. Then $F'$ is a compact complex manifold of dimension at most 2, and $\kappa(F') \geq 0$ by Lemma 2.9. Thus the group $\text{Bim}(F')$ is Jordan by Theorem 2.10 and Remark 2.11. Therefore, the group $\text{Bim}(X)$ is Jordan by Lemma 3.2. □

Corollary 5.6. Let $X$ be a three-dimensional compact complex manifold with $\kappa(X) \geq 0$. Suppose that $\text{Bim}(X)$ is not Jordan. Then $\dim(\text{Alb}(X)) < \dim(X)$. In particular, if $X$ is Kähler, then $q(X) < \dim(X)$.

Proof. We know from Lemma 5.5 that $\alpha$ is surjective. In particular, this implies that $\dim(\text{Alb}(X)) \leq \dim(X)$. Suppose that $\dim(\text{Alb}(X)) = \dim(X)$. Then a typical fiber of $\alpha$ is finite. Applying Corollaries 3.8 and 2.7 we see that the group $\text{Bim}(X)$ is Jordan, which is not the case by assumption.

If $X$ is a compact Kähler manifold, then $\dim(\text{Alb}(X)) = q(X)$. □

6. Indeterminacy loci

In this section we make some observations concerning the indeterminacy loci of bimeromorphic maps of three-dimensional compact complex varieties.

Lemma 6.1. Let $X$ be a normal three-dimensional compact complex variety, and let $\chi: X \to X$ be a pseudoautomorphism. Let $\beta: X \to Z$ be a morphism to a curve $Z$ such that $\chi$ maps the fibers of $\beta$ again to the fibers of $\beta$. Then the indeterminacy locus $\text{Ind}(\chi)$ is contained in a finite set of fibers of $\beta$. 
Proof. Assume the contrary, i.e. there is an irreducible curve \( C \subset \text{Ind}(\chi) \) that dominates \( Z \). Let \( \tilde{Y} \) be the normalization of the graph of \( \chi \). Thus, there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\chi} & X' \\
\downarrow{f} & & \downarrow{f'} \\
S & \xrightarrow{\zeta} & S \\
\end{array}
\]

Since \( \chi \) is not defined at a typical point of \( C \), there exists a two-dimensional component \( E \) of the \( p \)-exceptional set that dominates \( C \). Since the map \( \tilde{Y} \to X \times X \) is finite onto its image, no curves on \( \tilde{Y} \) are contracted by both maps \( p \) and \( q \). Hence \( C' = q(E) \) is a curve, and a typical fiber \( \Gamma \) of \( q_E : E \to C' \) dominates \( C \). Thus \( \Gamma \) meets the proper transform \( \tilde{F} \) of any fiber \( F = \beta^{-1}(z) \), and so \( q(\tilde{F}) \) contains \( C' \). In other words, \( C' \) is contained in every fiber of \( \beta \), which gives a contradiction. \( \square \)

The next proposition is a relative analog of the well-known assertion about decomposition of certain maps into flops, see for instance [Kol89, Theorem 4.9]. The proof in our case follows the same scheme. We outline it for convenience of the reader.

**Proposition 6.2.** Let \( f : X \to S \) and \( f' : X' \to S \) be proper morphisms with one-dimensional fibers, where \( X \) and \( X' \) are three-dimensional Kähler varieties with terminal \( \mathbb{Q} \)-factorial singularities and \( S \) is a smooth surface. Suppose that both \( K_X \) and \( K_{X'} \) are nef over \( S \). Let \( \chi : X \dashrightarrow X' \) be a bimeromorphic map such that it is an isomorphism in codimension one, and the indeterminacy loci \( \text{Ind}(\chi) \) and \( \text{Ind}(\chi^{-1}) \) are contained in (a finite set of) fibers of \( f \) and \( f' \), respectively. Suppose that there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\chi} & X' \\
\downarrow{f} & & \downarrow{f'} \\
S & \xrightarrow{\zeta} & S \\
\end{array}
\]

where \( \zeta \) is a bimeromorphic map of \( S \). Then \( \zeta \) is an isomorphism, and \( \chi \) is a composition of flops in the curves that are contained in (a finite set of) fibers over \( S \).

**Outline of the proof.** By assumption, \( \chi \) is an automorphism outside a finite union of fibers of \( f \). This means that \( \zeta \) induces an automorphism of a smooth open subset \( S^0 \subset S \) that is a complement to a finite subset of \( S \). Therefore, by Hartogs’s extension theorem \( \zeta \) is an isomorphism on the whole \( S \).

To prove that \( \chi \) is a composition of flops in the curves contracted by \( f \), we may assume that \( S \) is a small analytic neighborhood of a point \( s \in S \). Thus, we may assume that \( f' \) is a projective morphism, because it has one-dimensional fibers. In other words, there exists a divisor \( D' \) on \( X' \) that is very ample over \( S \): one can construct \( D' \) as a union of discs meeting the fiber \( f'^{-1}(s) \) transversely. Similarly, we see that the morphism \( f \) is projective.

Let \( D \subset X \) be the proper transform of \( D' \). For small positive \( \varepsilon \), run a \( (K_X + \varepsilon D) \)-MMP over \( S \). This is possible because \( f \) is projective, see [Nak87, §4]. Every step of this MMP is a flop, and we end up with \( X' \). \( \square \)

**Corollary 6.3.** In the notation of Proposition 6.2, set \( \Theta = f(\text{Sing}(X)) \). Then \( \zeta(\Theta) = \Theta \).
Proof. A flop preserves the analytic types of singularities of the ambient variety, see [Kol89, Theorem 2.4]. Thus, the required assertion follows from Proposition 6.2. □

The following fact is well-known.

**Lemma 6.4** (see e.g. [Kul83, Lemma 1.8]). Let $X$ be a three-dimensional compact complex variety with at worst terminal singularities, and let $\chi: X \to X$ be a pseudoautomorphism. Then any irreducible component of the indeterminacy locus $\text{Ind}(\chi)$ is a rational curve.

### 7. More on compact complex surfaces

In this section we make some additional observations on compact complex surfaces and their automorphism groups.

**Lemma 7.1.** Let $S$ be a smooth compact Kähler surface with $0 \leq \kappa(S) \leq 1$ and $q(S) > 0$. Then $\chi_{\text{top}}(S) = 0$ if and only if $S$ contains no rational curves.

**Proof.** We may assume that $S$ is minimal. Indeed, according to the Enriques–Kodaira classification (see e.g. [BHPVdV04, §VI.1]), every minimal surface of non-negative Kodaira dimension has a non-negative topological Euler characteristic. Thus, any non-minimal compact complex surface of non-negative Kodaira dimension has a positive topological Euler characteristic (and contains rational curves).

Suppose that $\kappa(S) = 0$. Since $q(S) > 0$, we conclude that $S$ is either a complex torus, or a bielliptic surface. In both of these cases $\chi_{\text{top}}(S) = 0$ and $S$ contains no rational curves.

Suppose that $\kappa(S) = 1$. Let $\pi: S \to B$ be the pluricanonical map; thus, $\pi$ is an elliptic fibration over a curve $B$. If the genus $g(B) > 0$, then all rational curves on $S$ are contained in a finite set of fibers of $\pi$, and $\chi_{\text{top}}(S) = \sum \chi_{\text{top}}(F_i)$, where $F_i$ are degenerate fibers. Clearly, $\chi_{\text{top}}(S) > 0$ if and only if $\pi$ has a fiber that is not an elliptic curve. Such a fiber is a union of rational curves.

Assume that $B \cong \mathbb{P}^1$. We are going to show that in this case $\chi_{\text{top}}(S) = 0$ and $S$ does not contain rational curves. Indeed, the Albanese map $\alpha: S \to A$ cannot be constant on the fibers of $\pi$, because otherwise it would factor through the Albanese map of $\mathbb{P}^1$, which is impossible since $q(S) > 0$. Therefore, any fiber $S_b$, $b \in B$, of the map $\pi$ does not contain rational curves; in other words, it is of type $m_10$ in the Kodaira classification of degenerate fibers of elliptic fibrations (see for instance [BHPVdV04 §V.7]). Hence, one has

$$\chi_{\text{top}}(S) = \sum_{b \in B} \chi_{\text{top}}(S_b) = 0.$$  

We claim that $q(S) = 1$. Indeed, for a typical fiber $S_o$ of $\pi$ the image $\alpha(S_o)$ is an elliptic curve on $A$. Consider the quotient map $A \to A' = A/\alpha(S_o)$. The image of $\alpha(S_o)$ is a point on $A'$, and so for any fiber $S_b$ of $\pi$ the image of $\alpha(S_b)$ on $A'$ is also a point. Thus, the image of $\alpha(S)$ on $A'$ is a curve dominated by the (rational) base of the fibration $\pi$. Since $A'$ contains no rational curves, we conclude that the image of $\alpha(S)$ on $A'$ is a point. Hence $\alpha(S) = \alpha(S_o) = A$.

Therefore, $A$ is an elliptic curve and $\alpha$ has connected fibers, see Remark 5.3. Since $S$ is a Kähler surface, we know that $b_1(S) = b_3(S) = 2q(S)$ and

$$b_2(S) = \chi_{\text{top}}(S) - 2 + 4q(S) = 2.$$  

Hence all the fibers of $\alpha$ are irreducible.
Let $F_1, \ldots, F_r$ be all the singular fibers of $\alpha$, let $m_1, \ldots, m_r$ be their multiplicities, and let $F$ be a typical fiber of $\alpha$. Thus, $F$ is numerically equivalent to $m_i F_i$, and $\chi_{\text{top}}(F_i) = 2 - 2 p_a(F_i) = -K_S \cdot F_i$. Similarly, one has

$$\chi_{\text{top}}(F_i) \geq 2 - 2 p_a(F_i) = -K_S \cdot F_i = -\frac{1}{m_i} K_S \cdot F.$$ 

Thus we have

$$0 = \chi_{\text{top}}(S) = \sum_{i} \left( \chi_{\text{top}}(F_i) - \chi_{\text{top}}(F) \right) \geq K_S \cdot F \cdot \sum_{i} \left( 1 - \frac{1}{m_i} \right).$$

Since $S$ is not covered by rational curves, we know that $K_S \cdot F > 0$. This shows that $m_i = 1$ and $\chi_{\text{top}}(F_i) = 2 - 2 p_a(F_i)$ for all $i$. Therefore, $\alpha$ is a smooth morphism. Now if $S$ contains a rational curve $C$, then $C$ must be a (smooth) fiber of $\alpha$. But then

$$-2 = 2 g(C) - 2 = K_S \cdot C = K_S \cdot F \geq 0,$$

which is a contradiction. \qed

The next result was proved in \cite{PS20a}.

**Proposition 7.2** (\cite{PS20a}, Corollary 4.1). Let $S$ be a compact Kähler surface with $\kappa(S) \geq 0$. Suppose that the group $\text{Bim}(S)$ has unbounded finite subgroups. Then either $\kappa(S) = 1$, or $S$ is bimeromorphic to a complex torus or a bielliptic surface.

The following fact is a version of Proposition 7.2 for automorphism groups.

**Proposition 7.3.** Let $S$ be a minimal compact Kähler surface with $\kappa(S) \geq 0$. Suppose that the group $\text{Aut}(S)$ has unbounded finite subgroups. Then either $\kappa(S) = 1$, or $S$ is either a complex torus, or a bielliptic surface, or a surface with $\kappa(S) = 1$.

**Proof.** One has $\kappa(S) < 2$ by Theorem 2.8. If $\kappa(S) = 0$, then by Proposition 7.2 the surface $S$ is either a complex torus, or a bielliptic surface. In both of these cases the required assertions clearly hold.

Suppose that $\kappa(S) = 1$. Let $\pi: S \to B$ be the pluricanonical fibration. If $\pi$ has a fiber $S_b = \pi^{-1}(b)$ over some point $b \in B$ which is not of type $m I_0$, then a finite index subgroup $\Gamma \subset \text{Aut}(S)$ fixes a singular point of $F_{\text{red}}$. Since $S$ is Kähler, this implies that $\text{Aut}(S)$ has bounded finite subgroups by Theorem 2.12. Thus we may assume that all the fibers of $\pi$ are of type $m I_0$. Hence, one has $\chi_{\text{top}}(S) = 0$. By Noether formula this gives $\chi(\Theta_S) = 0$, so that $q(S) = 1 + p_g(S) > 0$. \qed

### 8. Proof of the main theorem

In this section we complete the proof of Theorem 1.3.

**Lemma 8.1.** Let $X$ be a three-dimensional compact Kähler variety with $\kappa(X) \geq 0$. Suppose that there exists a dominant $\text{Bim}(X)$-equivariant meromorphic map $f: X \dashrightarrow Z$ to a smooth projective curve $Z$ of positive genus. Then $\text{Bim}(X)$ is Jordan.

**Proof.** Running the MMP on $X$, we may assume that $X$ has at worst terminal $\mathbb{Q}$-factorial singularities and $\omega_X$ is nef. Since $Z$ is not a rational curve, $f$ is holomorphic: otherwise its composition with the embedding of $Z$ into its Jacobian is a non-holomorphic map to a complex torus, which
is impossible by Proposition 2.6. Applying the Stein factorization, we may assume that the fibers of \( f \) are connected. Thus, any smooth fiber \( F \) of \( f \) is a minimal surface of non-negative Kodaira dimension; in particular, one has \( \text{Bim}(F) = \text{Aut}(F) \). Note also that \( \text{Bim}(X) \) acts on \( Z \) by automorphisms.

Suppose that \( \text{Bim}(X) \) is not Jordan. Then it follows from Lemma 3.7 and Remark 2.11 that for a typical fiber \( F \) of \( f \) the group \( \text{Aut}(F) \) has unbounded finite subgroups. Therefore, by Proposition 7.3 one has \( 0 \leq \kappa(F) \leq 1 \), \( \chi_{\text{top}}(F) = 0 \), and \( q(F) > 0 \). Recall that the topological Euler characteristic is constant in smooth families of compact manifolds. Hence for any smooth fiber \( F \) of \( f \) one has \( \chi_{\text{top}}(F) = 0 \). Furthermore, since the irregularity of a compact complex surface is uniquely determined by the first Betti number (see e.g. [BHPVdV04, Theorem IV.2.7]), it is also constant in smooth families, so that \( q(F) > 0 \). Also, we see that \( \kappa(F) \geq 0 \) by the adjunction formula, and \( K_F^2 = 0 \), so that \( \kappa(F) \leq 1 \). This means that any smooth fiber of \( f \) contains no rational curves by Lemma 7.4.

Let \( F \) be a smooth fiber of \( f \), and let \( \gamma \in \text{Bim}(X) \) be an arbitrary element. By Lemma 6.4 any irreducible component \( C \) of the indeterminacy locus \( \text{Ind}(\gamma) \) is a rational curve. Since \( g(Z) > 0 \), the curve \( C \) cannot dominate \( Z \). On the other hand, we already know that \( C \not\subseteq F \). Hence \( C \) is disjoint from \( F \), i.e. \( \gamma \) is holomorphic near \( F \). Then \( \gamma(F) \) also does not contain rational curves and so \( \gamma^{-1} \) is holomorphic near \( \gamma(F) \). This implies that \( \gamma \) is an isomorphism near \( F \). Hence \( \gamma(F) \) is a smooth fiber, i.e. any element \( \gamma \in \text{Bim}(X) \) maps smooth fibers to smooth fibers (and vice versa, maps singular fibers to singular fibers).

Now let \( F_1, \ldots, F_r, r \geq 0 \), be all the singular fibers of \( f \). Then \( \text{Bim}(X) \) acts biholomorphically on the complement \( X \setminus \bigcup F_i \). If \( r = 0 \), then \( \text{Bim}(X) \) acts on the whole \( X \) by automorphisms. In this case \( \text{Bim}(X) = \text{Aut}(X) \) is Jordan by Theorem 1.2. If \( r > 0 \), then \( \text{Bim}(X) \) permutes the points \( f(F_1), \ldots, f(F_r) \). Therefore, \( \text{Bim}(X) \) is Jordan by Corollary 3.5.

Lemma 8.2. Let \( X \) be a three-dimensional compact Kähler variety with \( \kappa(X) \geq 0 \). Suppose that there exists a dominant \( \text{Bim}(X) \)-equivariant meromorphic map \( f : X \to Z \) to a compact complex surface \( Z \) with \( \kappa(Z) \geq 0 \). Then \( \text{Bim}(X) \) is Jordan.

Proof. Running the MMP, we may assume that the singularities of \( X \) are at worst terminal \( \mathbb{Q} \)-factorial and \( \omega_X \) is nef. Then \( \text{Bim}(X) \) acts on \( X \) by pseudo-automorphisms, see Corollary 4.3. Furthermore, we may assume that \( Z \) is a minimal surface. So, the induced action of \( \text{Bim}(X) \) on \( Z \) is biholomorphic. We may also assume that the group \( \text{Aut}(Z) \) has unbounded finite subgroups, because otherwise \( \text{Bim}(X) \) is Jordan by Corollary 3.3. Note that the surface \( Z \) is Kähler by [Var89, Theorem 5]. Thus, we know from Proposition 7.3 that \( Z \) is ether a complex torus, or a bielliptic surface, or a surface with \( \kappa(Z) = 1 \).

Assume that \( \kappa(Z) = 1 \). Let \( \psi : X \to B \) be the composition of \( f \) with the pluricanonical fibration \( Z \to B \). The image of \( \text{Bim}(X) \) in \( \text{Aut}(B) \) is finite by [PS20a, Proposition 1.2]. Hence \( \text{Bim}(X) \) is Jordan by Corollary 3.3. The same argument works if \( Z \) is a bielliptic surface: in this case there is an \( \text{Aut}(Z) \)-equivariant elliptic fibration \( Z \to \mathbb{P}^1 \) (see [BHPVdV04, §V.5]), and by Kodaira’s canonical bundle formula (see [BHPVdV04, Theorem V.12.1]) it has 3 or 4 multiple fibers. Hence the image of \( \text{Bim}(X) \) in \( \text{Aut}(\mathbb{P}^1) \) is finite and \( \text{Bim}(X) \) is again Jordan.

Finally, assume that \( Z \) is a complex torus and \( \text{Bim}(X) \) is not Jordan. The map \( f \) is holomorphic in this case, see Proposition 2.6.
Suppose that $f$ has a two-dimensional fiber. Denote by $\Sigma$ the image under $f$ of the union of all two-dimensional fibers of $f$. Then $\Sigma$ is a finite non-empty subset of $\mathbb{Z}$. Since a pseudoautomorphism cannot contract a two-dimensional fiber, we conclude that $\Sigma$ is invariant under the action of the image of Bim($X$) on $\mathbb{Z}$. Thus, the group Bim($X$) is Jordan by Corollary 3.6.

Now suppose that all fibers of $f$ are one-dimensional. By Lemma 6.1, the indeterminacy locus of every bimeromorphic selfmap of $X$ is contained in a finite set of fibers of $f$. Therefore, it follows from Corollary 6.3 that the group Bim($X$) preserves the image of the singular locus of $X$ on $\mathbb{Z}$, which is a finite non-empty subset of $\mathbb{Z}$ by assumption. Again by Corollary 3.6 this implies that the group Bim($X$) is Jordan. \hfill $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\alpha: X \to A$ be the Albanese map. By Corollary 5.6 we may assume that $q(X) \leq 2$, and by Lemma 5.5 we may assume that $\alpha$ is surjective. Now the assertion of the theorem is given by Lemma 8.1 if $q(X) = 1$, and by Lemma 8.2 if $q(X) = 2$. \hfill $\square$

References

[BHPVdV04] W. Barth, K. Hulek, Ch. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 2004.

[BL04] Ch. Birkenhake, H. Lange. *Complex abelian varieties*. Second edition. Grundlehren der mathematischen Wissenschaften, 302. Springer-Verlag, Berlin, 2004.

[Bru06] M. Brunella. A positivity property for foliations on compact Kähler manifolds. *Internat. J. Math.*, 17(1):35–43, 2006.

[BZ21a] T. Bandman and Yu. Zarhin. Bimeromorphic automorphism groups of certain $\mathbb{P}^1$-bundles. *European J. Math.*, 7:641–670, 2021.

[BZ21b] T. Bandman and Yu. Zarhin. Automorphism groups of $\mathbb{P}^1$-bundles over a non-uniruled base. \texttt{arXiv:2103.07015}, 2021.

[BZ21c] T. Bandman and Yu. Zarhin. Simple complex tori of algebraic dimension 0. \texttt{arXiv:2106.10308}, 2021.

[DP03] J.-P. Demailly and Th. Peternell. A Kawamata–Viehweg vanishing theorem on compact Kähler manifolds. *J. Differ. Geom.*, 63(2):231–277, 2003.

[Fuj81] A. Fujiki. A theorem on bimeromorphic maps of Kähler manifolds and its applications. *Publ. Res. Inst. Math. Sci.*, 17:735–754, 1981.

[Go12] A. Golota. Jordan property for groups of bimeromorphic automorphisms of compact Kähler threefolds. \texttt{arXiv:2112.02679}, 2021.

[Gra18] P. Graf. Algebraic approximation of Kähler threefolds of Kodaira dimension zero. *Math. Ann.*, 371(1–2):487–516, 2018.

[HP16] A. Höring and Th. Peternell. Minimal models for Kähler threefolds. *Invent. Math.*, 203(1):217–264, 2016.

[Kim18] J. H. Kim. Jordan property and automorphism groups of normal compact Kähler varieties. *Commun. Contemp. Math.*, 20(3):1750024, 9, 2018.

[KM98] J. Kollár and Sh. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol89] J. Kollár. Flips. *Nagoya Math. J.*, 113:15–36, 1989.

[Kul83] Vik. S. Kulikov. Decomposition of birational mappings of three-dimensional varieties outside of codimension 2. *Math. USSR-Izvestiya*, 21(1):187–200, 1983.

[MPZ20] Sh. Meng, F. Perroni, and D.-Q. Zhang. Jordan property for automorphism groups of compact spaces in Fujiki’s class $C$. \texttt{arXiv:2011.09381}, 2020.
[Nak87] N. Nakayama. The lower semicontinuity of the plurigenera of complex varieties. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 551–590. North-Holland, Amsterdam, 1987.

[Pop11] V. Popov. On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties. In Peter Russell’s Festschrift, *Proceedings of the conference on Affine Algebraic Geometry held in Professor Russell’s honour, 1–5 June 2009*, McGill Univ., Montreal, volume 54 of *Centre de Recherches Mathématiques CRM Proc. and Lect. Notes*, pages 289–311, 2011.

[Pop18] V. L. Popov. The Jordan property for Lie groups and automorphism groups of complex spaces. *Math. Notes*, 103(5):811–819, 2018.

[Pro21] Yu. Prokhorov. Equivariant minimal model program. *Russian Math. Surv.*, 76(3):461–542, 2021.

[PS14] Yu. Prokhorov and C. Shramov. Jordan property for groups of birational selfmaps. *Compositio Mathematica*, 150(12):2054–2072, 2014.

[PS19] Yu. Prokhorov and C. Shramov. Automorphism groups of Moishezon threefolds. *Math. Notes*, 106(4):651–655, 2019.

[PS20a] Yu. Prokhorov and C. Shramov. Bounded automorphism groups of compact complex surfaces. *Mat. Sb.*, 211(9):1310–1322, 2020.

[PS20b] Yu. Prokhorov and C. Shramov. Finite groups of bimeromorphic selfmaps of uniruled Kähler threefolds. *Izv. Ross. Akad. Nauk Ser. Mat.*, 84(5):978–1001, 2020.

[PS21] Yu. Prokhorov and C. Shramov. Automorphism groups of compact complex surfaces. *Int. Math. Res. Notices*, 2021(14):10490–10520, 2021.

[Rei80] M. Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[Ser07] J.-P. Serre. Bounds for the orders of the finite subgroups of $G(k)$. In *Group representation theory*, pages 405–450. EPFL Press, Lausanne, 2007.

[Sho93] V. Shokurov. 3-fold log flips. *Russ. Acad. Sci. Izv. Math.*, 40(1):95–202, 1993.

[Uen75] K. Ueno. *Classification theory of algebraic varieties and compact complex spaces*. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975. Notes written in collaboration with P. Cherenack.

[Var89] J. Varouchas. Kähler spaces and proper open morphisms. *Math. Ann.* 283(1): 13–52, 1989.

[Wan21] J. Wang. On the Iitaka conjecture $C_{n,m}$ for Kähler fibre spaces. *Ann. Fac. Sci. Toulouse Math. (6)*, 30(4):813–897, 2021.

[Zar19] Yu. Zarhin. Complex tori, theta groups and their Jordan properties. *Tr. Mat. Inst. Steklova*, 307:22–50, 2019.

*Yuri Prokhorov*
Steklov Mathematical Institute of RAS, 8 Gubkina street, Moscow 119991, Russia.
HSE University, Russian Federation, Laboratory of Algebraic Geometry, 6 Usacheva str., Moscow, 119048, Russia.
prokhoro@mi-ras.ru

*Constantin Shramov*
Steklov Mathematical Institute of RAS, 8 Gubkina street, Moscow 119991, Russia.
HSE University, Russian Federation, Laboratory of Algebraic Geometry, 6 Usacheva str., Moscow, 119048, Russia.
costya.shramov@gmail.com