SIMPLY CONNECTED INDEFINITE HOMOGENEOUS SPACES
OF FINITE VOLUME

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Abstract. Let $M$ be a simply connected pseudo-Riemannian homogeneous space of finite volume with isometry group $G$. We show that $M$ is compact and that the solvable radical of $G$ is abelian and the Levi factor is a compact semisimple Lie group acting transitively on $M$. For metric index less than three, we find that the isometry group of $M$ is compact itself. Examples demonstrate that $G$ is not necessarily compact for higher indices. To prepare these results, we study Lie algebras with abelian solvable radical and a nil-invariant symmetric bilinear form. For these, we derive an orthogonal decomposition into three distinct types of metric Lie algebras.

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1. Introduction and main results

In this article we are interested in the isometry groups of simply connected homogeneous pseudo-Riemannian manifolds of finite volume. D’Ambra [3, Theorem 1.1] showed that a simply connected compact analytic Lorentzian manifold (not necessarily homogeneous) has compact isometry group, and she also gave an example of a simply connected compact analytic manifold of metric signature $(7, 2)$ that has a non-compact isometry group.

Here we study homogeneous spaces for arbitrary metric signature. Our main tool is the structure theory of the isometry Lie algebras developed by the authors in [2]. The metric on the homogeneous space induces a symmetric bilinear form on the isometry Lie algebra, and as shown in [1, 2], the existence of a finite invariant measure then implies that this bilinear form is nil-invariant. The first main result is the following theorem:

Theorem A. Let $M$ be a connected and simply connected pseudo-Riemannian homogeneous space of finite volume, $G = \text{Iso}(M)^0$, and let $H$ be the stabilizer subgroup in $G$ of a point in $M$. Let $G = KR$ be a Levi decomposition, where $R$ is the solvable radical of $G$. Then:

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(1) \( M \) is compact.
(2) \( K \) is compact and acts transitively on \( M \).
(3) \( R \) is abelian. Let \( A \) be the maximal compact subgroup of \( R \). Then \( A = \mathbb{Z}(G)^\circ \). More explicitly, \( R = A \times V \) where \( V \cong \mathbb{R}^n \) and \( V^K = 0 \).
(4) \( H \) is connected. If \( \dim R > 0 \), then \( H = (H \cap K)E \), where \( E \) and \( H \cap K \) are normal subgroups in \( H \), \( (H \cap K) \cap E \) is finite, and \( E \) is the graph of a non-trivial homomorphism \( \varphi : R \to K \), where the restriction \( \varphi|_A \) is injective.

In Section 4 we give examples of isometry groups of compact simply connected homogeneous \( M \) with non-compact radical. However, for metric index 1 or 2 the isometry group of a simply connected \( M \) is always compact:

**Theorem B.** The isometry group of any simply connected pseudo-Riemannian homogeneous manifold of finite volume with metric index \( \ell \leq 2 \) is compact.

As follows from Theorem A the isometry Lie algebra of a simply connected pseudo-Riemannian homogeneous space of finite volume has abelian radical. This motivates a closer investigation of Lie algebras with abelian radical that admit nil-invariant symmetric bilinear forms in Section 3. Our main result is the following algebraic theorem:

**Theorem C.** Let \( \mathcal{G} \) be a Lie algebra whose solvable radical \( \mathcal{R} \) is abelian. Suppose \( \mathcal{G} \) is equipped with a nil-invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \) such that the kernel \( \mathcal{G}^1 \) of \( \langle \cdot, \cdot \rangle \) does not contain a non-trivial ideal of \( \mathcal{G} \). Let \( \mathcal{K} \times S \) be a Levi subalgebra of \( \mathcal{G} \), where \( \mathcal{K} \) is of compact type and \( S \) has no simple factors of compact type. Then \( \mathcal{G} \) is an orthogonal direct product of ideals

\[
\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3,
\]

with

\[
\mathcal{G}_1 = \mathcal{K} \times \mathcal{A}, \quad \mathcal{G}_2 = S_0, \quad \mathcal{G}_3 = S_1 \times S_1^*,
\]

where \( \mathcal{R} = \mathcal{A} \times S_1 \) and \( S = S_0 \times S_1 \) are orthogonal direct products, and \( \mathcal{G}_3 \) is a metric cotangent algebra. The restrictions of \( \langle \cdot, \cdot \rangle \) to \( \mathcal{G}_2 \) and \( \mathcal{G}_3 \) are invariant and non-degenerate. In particular, \( \mathcal{G}^1 \subseteq \mathcal{G}_1 \).

For the definition of metric cotangent algebra, see Section 2. We call an algebra \( \mathcal{G}_1 = \mathcal{K} \times \mathcal{A} \) with \( \mathcal{K} \) semisimple of compact type and \( \mathcal{A} \) abelian a Lie algebra of **Euclidean type**. By Theorem A isometry Lie algebras of compact simply connected pseudo-Riemannian homogeneous spaces are of Euclidean type. However, not every Lie algebra of Euclidean type appears as the isometry Lie algebra of a compact pseudo-Riemannian homogeneous space. In fact, this is the case for the Euclidean Lie algebras \( \mathcal{E}_n = \mathfrak{so}_n \times \mathbb{R}^n \) with \( n \neq 3 \).

**Theorem D.** The Euclidean group \( \mathcal{E}_n = \mathfrak{O}_n \times \mathbb{R}^n, \ n \neq 1, 3, \) does not have compact quotients with a pseudo-Riemannian metric such that \( \mathcal{E}_n \) acts isometrically and almost effectively.

Note that \( \mathcal{E}_n \) acts transitively and effectively on compact manifolds with finite fundamental group, as we remark at the end of Section 3.

**Notations and conventions.** For a Lie group \( G \), we let \( G^\circ \) denote the connected component of the identity. For a subgroup \( H \) of \( G \), we write \( \text{Ad}_G(H) \) for the adjoint representation of \( H \) on the Lie algebra \( \mathfrak{g} \) of \( G \), to distinguish it from the adjoint representation \( \text{Ad}(H) \) on its own Lie algebra \( \mathcal{H} \).
The solvable radical $R$ of $G$ is the maximal connected solvable normal subgroup of $G$. The solvable radical $R$ of $G$ is the maximal solvable ideal of $G$. The semisimple Lie algebra $\mathfrak{g} = G/\langle R \rangle$ is a direct product $\mathfrak{k} \times S$, where $\mathfrak{k}$ is a semisimple Lie algebra of compact type, meaning its Killing form is definite, and $S$ is semisimple without factors of compact type.

The center of a group $G$, or a Lie algebra $G$, is denoted by $Z(G)$, or $Z(\mathfrak{g})$, respectively. Similarly, the centralizer of a subgroup $H$ in $G$ (or a subalgebra $\mathfrak{h}$ in $\mathfrak{g}$) is denoted by $Z^G_H$ or $Z^G_\mathfrak{h}$.

The action of a Lie group $G$ on a homogeneous space $M$ is (almost) effective if the stabilizer of any point in $M$ does not contain a non-trivial (connected) normal subgroup of $G$.

If $V$ is a $G$-module, then we write $V^G = \{ v \in V \mid \forall g \in G : gv = v \}$ for the module of $G$-invariants. Similarly, $V^G = \{ v \in V \mid \forall x \in G : xv = 0 \}$ for a $G$-module.

For direct products of Lie algebras $G_1, G_2$ we write $G_1 \times G_2$, whereas $G_1 + G_2$ or $G_1 \oplus G_2$ refers to sums as vector spaces.

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2. Nil-invariant bilinear forms

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra, let $\text{Inn}(\mathfrak{g})$ denote the inner automorphism group of $\mathfrak{g}$ and $\overline{\text{Inn}(\mathfrak{g})}$ its Zariski closure in $\text{Aut}(\mathfrak{g})$. A symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is called nil-invariant if for all $x_1, x_2 \in \mathfrak{g}$,

$$\langle \varphi x_1, x_2 \rangle = -\langle x_1, \varphi x_2 \rangle$$

for all nilpotent elements $\varphi$ of the Lie algebra of $\overline{\text{Inn}(\mathfrak{g})}$. For a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we say $\langle \cdot, \cdot \rangle$ is $\mathfrak{h}$-invariant if for all $x \in \mathfrak{h}$, $\text{ad}_\mathfrak{g}(x)$ is skew-symmetric for $\langle \cdot, \cdot \rangle$.

The kernel of $\langle \cdot, \cdot \rangle$ is the subspace

$$\mathfrak{g}^\perp = \{ x \in \mathfrak{g} \mid \langle x, y \rangle = 0 \text{ for all } y \in \mathfrak{g} \}.$$

We use a Levi decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = (\mathfrak{k} \times S) \ltimes \mathfrak{r},$$

where $\mathfrak{k}$ is semisimple of compact type, $S$ is semisimple without factors of compact type, and $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$. Let further $\mathfrak{g}_s = S \ltimes \mathfrak{r}$.

Theorem 2.1 ([2 Theorem A]). Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with nil-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}_s}$ denote the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{g}_s$. Then:

(1) $\langle \cdot, \cdot \rangle_{\mathfrak{g}_s}$ is invariant by the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}_s$.
(2) $\langle \cdot, \cdot \rangle$ is invariant by the adjoint action of $\mathfrak{g}_s$.

This implies some orthogonality relations that will be useful later on:

$$\mathfrak{s} \perp [\mathfrak{k}, \mathfrak{g}], \quad \mathfrak{k} \perp [\mathfrak{s}, \mathfrak{g}].$$
Theorem 2.2 ([2 Corollary C]). Let $\mathcal{G}$ be a finite-dimensional real Lie algebra with nil-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$, where we further assume that $\mathcal{G}^1$ does not contain any non-zero ideal of $\mathcal{G}$. Let $\mathbb{Z}(\mathcal{G}_s)$ denote the center of $\mathcal{G}_s$. Then

$$\mathcal{G}^1 \subseteq \mathfrak{K} \times \mathbb{Z}(\mathcal{G}_s) \quad \text{and} \quad [\mathcal{G}^1, \mathcal{G}_s] \subseteq \mathbb{Z}(\mathcal{G}_s) \cap \mathcal{G}^1.$$ 

We say that $\langle \cdot, \cdot \rangle$ has relative index $\ell$ if the induced scalar product on $\mathcal{G}/\mathcal{G}^1$ has index $\ell$. For relative index $\ell \leq 2$, we have a general structure theorem for $\mathcal{G}$.

Theorem 2.3 ([2 Theorem D]). Let $\mathcal{G}$ be a finite-dimensional real Lie algebra with nil-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ of relative index $\ell \leq 2$, and assume that $\mathcal{G}^1$ does not contain any non-zero ideal of $\mathcal{G}$. Then:

1. The Levi decomposition of $\mathcal{G}$ is a direct sum of ideals $\mathcal{G} = \mathfrak{K} \times \mathfrak{S} \times \mathfrak{R}$.
2. $\mathcal{G}^1$ is contained in $\mathfrak{K} \times \mathbb{Z}(\mathfrak{K})$ and $\mathcal{G}^1 \cap \mathfrak{K} = 0$.
3. $\mathfrak{S} \perp (\mathfrak{K} \times \mathfrak{R})$ and $\mathfrak{K} \perp [\mathfrak{R}, \mathfrak{R}]$.

2.1. Cotangent algebras. Let $\mathcal{L}$ be a Lie algebra. A cotangent algebra constructed from $\mathcal{L}$ is a Lie algebra $\mathcal{G} = \mathcal{L} \times \mathcal{L}^*$ where $\mathcal{L}$ acts on its dual space $\mathcal{L}^*$ by its coadjoint representation. We call $\mathcal{G}$ a metric cotangent algebra if it has a non-degenerate invariant scalar product $\langle \cdot, \cdot \rangle$ such that $\mathcal{L}^*$ is totally isotropic.

2.2. Invariance by $\mathcal{G}^1$. We are mainly interested in nil-invariant bilinear forms $\langle \cdot, \cdot \rangle$ on $\mathcal{G}$ induced by pseudo-Riemannian metrics on homogeneous spaces. In this case, $\langle \cdot, \cdot \rangle$ is invariant by the stabilizer subalgebra $\mathcal{G}^1$. We can then further sharpen the statement of Theorem 2.2.

Proposition 2.4. Let $\mathcal{G}$ and $\langle \cdot, \cdot \rangle$ be as in Theorem 2.2. If in addition $\langle \cdot, \cdot \rangle$ is $\mathcal{G}^1$-invariant, then

$$[\mathcal{G}^1, \mathcal{G}_s] = 0.$$

The proof is based on the following immediate observations:

Lemma 2.5. Suppose $\langle \cdot, \cdot \rangle$ is $\mathcal{G}^1$-invariant. Then $[[\mathfrak{K}, \mathcal{G}^1], \mathcal{G}_s] \subseteq \mathcal{G}^1 \cap \mathcal{G}_s$.

and

Lemma 2.6. Let $\mathcal{H}$ be any Lie algebra and $\mathcal{V}$ a module for $\mathcal{H}$. Suppose that the subalgebra $\mathcal{Q}$ of $\mathcal{H}$ is generated by the subspace $\mathcal{M}$ of $\mathcal{H}$. Then $\mathcal{Q} \cdot \mathcal{V} = \mathcal{M} \cdot \mathcal{V}$.

Together with

Lemma 2.7. Let $\mathfrak{K}$ be semisimple of compact type and $\mathfrak{K}_0$ a subalgebra of $\mathfrak{K}$. Then the subalgebra $\mathcal{Q}$ generated $\mathcal{M} = \mathfrak{K}_0 + [\mathfrak{K}, \mathfrak{K}_0]$ is an ideal of $\mathfrak{K}$.

Proof. Put $\mathcal{Z} = \mathbb{Z}(\mathfrak{K}_0)$. Then $[\mathcal{Z}, \mathcal{M}] \subseteq \mathcal{M}$ and $[[\mathfrak{K}, \mathfrak{K}_0], \mathcal{M}] \subseteq \mathcal{M} + [\mathcal{M}, \mathcal{M}]$. Since $\mathfrak{K} = [\mathfrak{K}, \mathfrak{K}_0] + \mathcal{Z}$, this shows $[\mathfrak{K}, \mathcal{M}] \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is linearly spanned by the iterated commutators of elements of $\mathcal{M}$, $[\mathfrak{K}, \mathfrak{K}] \subseteq \mathcal{Q}$. 

Proof of Proposition 2.4. Let $\mathfrak{K}_0$ be the image of $\mathcal{G}^1$ under the projection homomorphism $\mathcal{G} \to \mathfrak{K}$. Note that by Theorem 2.2 above, $[\mathcal{G}^1, \mathcal{G}_s] = [\mathfrak{K}_0, \mathcal{G}_s]$. Let $\mathcal{Q} \subseteq \mathfrak{K}$ be the subalgebra generated by $\mathcal{M} = \mathfrak{K}_0 + [\mathfrak{K}, \mathfrak{K}_0]$ and consider $\mathcal{V} = \mathcal{G}_s$ as a module for $\mathcal{Q}$. Since $\mathcal{Q}$ is an ideal of $\mathfrak{K}$, $[\mathcal{Q}, \mathcal{V}]$ is a submodule for $\mathfrak{K}$, that is, $[\mathfrak{K}, [\mathcal{Q}, \mathcal{V}]] \subseteq [\mathcal{Q}, \mathcal{V}]$. By Lemmas 2.5 and Theorem 2.2 we have $[\mathcal{Q}, \mathcal{V}] = [\mathcal{M}, \mathcal{V}] \subseteq \mathcal{G}^1 \cap \mathbb{Z}(\mathcal{G}_s)$. Hence, $\mathfrak{J} = [\mathcal{M}, \mathcal{V}] \subseteq \mathcal{G}^1$ is an ideal in $\mathcal{G}$, with $\mathfrak{J} \supseteq [\mathcal{G}^1, \mathcal{G}_s] = [\mathfrak{K}_0, \mathcal{G}_s]$. Since $\mathcal{G}^1$ contains no non-trivial ideals of $\mathcal{G}$ by assumption, we conclude that $\mathfrak{J} = 0$. 

□
3. Metric Lie algebras with abelian radical

In this section we study finite-dimensional real Lie algebras \( \mathcal{G} \) whose solvable radical \( \mathcal{R} \) is abelian and which are equipped with a nil-invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \).

3.1. An algebraic theorem. The Lie algebras with abelian radical and a nil-invariant symmetric bilinear form decompose into three distinct types of metric Lie algebras.

**Theorem** Let \( \mathcal{G} \) be a Lie algebra whose solvable radical \( \mathcal{R} \) is abelian. Suppose \( \mathcal{G} \) is equipped with a nil-invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \) such that the kernel \( \mathcal{G}^1 \) of \( \langle \cdot, \cdot \rangle \) does not contain a non-trivial ideal of \( \mathcal{G} \). Let \( \mathcal{K} \times \mathcal{S} \) be a Levi subalgebra of \( \mathcal{G} \), where \( \mathcal{K} \) is of compact type and \( \mathcal{S} \) has no simple factors of compact type. Then \( \mathcal{G} \) is an orthogonal direct product of ideals

\[
\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3,
\]

with

\[
\mathcal{G}_1 = \mathcal{K} \times \mathcal{A}, \quad \mathcal{G}_2 = \mathcal{S}_0, \quad \mathcal{G}_3 = \mathcal{S}_1 \times \mathcal{S}_1^*,
\]

where \( \mathcal{K} = \mathcal{A} \times \mathcal{S}_1^* \) and \( \mathcal{S} = \mathcal{S}_0 \times \mathcal{S}_1 \) are orthogonal direct products, and \( \mathcal{G}_3 \) is a metric cotangent algebra. The restrictions of \( \langle \cdot, \cdot \rangle \) to \( \mathcal{G}_2 \) and \( \mathcal{G}_3 \) are invariant and non-degenerate. In particular, \( \mathcal{G}^1 \subseteq \mathcal{G}_1 \).

We split the proof into several lemmas. Consider the submodules of invariants \( \mathcal{R}^\mathcal{S}, \mathcal{R}^\mathcal{K} \subseteq \mathcal{R} \). Since \( \mathcal{S}, \mathcal{K} \) act reductively, we have

\[
[\mathcal{S}, \mathcal{R}] \oplus \mathcal{R}^\mathcal{S} = \mathcal{R} = [\mathcal{K}, \mathcal{R}] \oplus \mathcal{R}^\mathcal{K}.
\]

Then \( \mathcal{A} = \mathcal{R}^\mathcal{S}, \mathcal{B} = [\mathcal{S}, \mathcal{R}^\mathcal{K}] \) and \( \mathcal{C} = [\mathcal{S}, \mathcal{R}] \cap [\mathcal{K}, \mathcal{R}] \) are ideals in \( \mathcal{G} \) and \( \mathcal{R} = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C} \). Recall from Theorem 2.1 that \( \langle \cdot, \cdot \rangle \) is in particular \( \mathcal{S} \)- and \( \mathcal{R} \)-invariant.

**Lemma 3.1.** \( \mathcal{C} = 0 \) and \( \mathcal{R} \) is an orthogonal direct sum of ideals in \( \mathcal{G} \)

\[
\mathcal{R} = \mathcal{A} \oplus \mathcal{B}
\]

where \( [\mathcal{K}, \mathcal{R}] \subseteq \mathcal{A} \) and \( [\mathcal{S}, \mathcal{R}] = \mathcal{B} \).

**Proof.** The \( \mathcal{S} \)-invariance of \( \langle \cdot, \cdot \rangle \) immediately implies \( \mathcal{A} \perp \mathcal{B} \). Since \( \mathcal{R} \) is abelian, \( \mathcal{R} \)-invariance implies \( \mathcal{C} \perp \mathcal{R} \). Since \( \mathcal{C} \perp (\mathcal{S} \times \mathcal{K}) \) by (2.2), this shows \( \mathcal{C} \) is an ideal contained in \( \mathcal{G}^1 \), hence \( \mathcal{C} = 0 \). Now \([\mathcal{K}, \mathcal{R}] \subseteq \mathcal{A} \) and \([\mathcal{S}, \mathcal{R}] = \mathcal{B} \) by definition of \( \mathcal{A} \) and \( \mathcal{B} \). \( \square \)

**Lemma 3.2.** \( \mathcal{G} \) is a direct product of ideals

\[
\mathcal{G} = (\mathcal{K} \times \mathcal{A}) \times (\mathcal{S} \times \mathcal{B}),
\]

where \( (\mathcal{K} \times \mathcal{A}) \perp (\mathcal{S} \times \mathcal{B}) \).

**Proof.** The splitting as a direct product of ideals follows from Lemma 3.1. The orthogonality follows together with (2.2) and the fact that the \( \mathcal{S} \)-invariance of \( \langle \cdot, \cdot \rangle \) implies \( \mathcal{S} \perp \mathcal{A} \) and \( \mathcal{K} \perp \mathcal{B} \). \( \square \)

**Lemma 3.3.** \( \mathcal{G}^1 \subseteq \mathcal{K} \times \mathcal{A} \) and \( \mathcal{S} \times \mathcal{B} \) is a non-degenerate ideal of \( \mathcal{G} \).

**Proof.** \( \mathcal{Z}(\mathcal{G}^1) = \mathcal{A} \), therefore \( \mathcal{G}^1 \subseteq \mathcal{K} \times \mathcal{A} \) by Theorem 2.2. Since also \( (\mathcal{S} \times \mathcal{B}) \perp (\mathcal{K} \times \mathcal{A}) \), we have \( (\mathcal{S} \times \mathcal{B}) \cap (\mathcal{S} \times \mathcal{B})^1 \subseteq \mathcal{G}^1 \subseteq \mathcal{K} \times \mathcal{A} \). It follows that \( (\mathcal{S} \times \mathcal{B}) \cap (\mathcal{S} \times \mathcal{B})^1 = 0 \). \( \square \)
To complete the proof of Theorem C it remains to understand the structure of the ideal $S \times B$, which by Theorem 2.1 and the preceding lemmas is a Lie algebra with an invariant non-degenerate scalar product given by the restriction of $\langle \cdot, \cdot \rangle$.

**Lemma 3.4.** $B$ is totally isotropic. Let $S_0$ be the kernel of the $S$-action on $B$. Then $S_0 = B^1 \cap S$.

*Proof.* Since $\langle \cdot, \cdot \rangle$ is $\mathcal{K}$-invariant and $\mathcal{K}$ is abelian, $B$ is totally isotropic. For the second claim, use $B \cap S^1 = 0$ and the invariance of $\langle \cdot, \cdot \rangle$. □

**Lemma 3.5.** $S$ is an orthogonal direct product of ideals $S = S_0 \times S_1$ with the following properties:

1. $S_1 \times B$ is a metric cotangent algebra.
2. $[S_0, B] = 0$ and $S_0 = B^1 \cap S$.

*Proof.* The kernel $S_0$ of the $S$-action on $B$ is an ideal in $S$, and by Lemma 3.4 orthogonal to $B$. Let $S_1$ be the ideal in $S$ such that $S = S_0 \times S_1$. Then $S_0 \perp S_1$ by invariance of $\langle \cdot, \cdot \rangle$.

$S_1$ acts faithfully on $B$ and so $S_1 \cap B^1 = 0$ by Lemma 3.4. Moreover, $S_1 \times B$ is non-degenerate since $S \times B$ is. But $B$ is totally isotropic by Lemma 3.4, so non-degeneracy implies $\dim S_1 = \dim B$. Therefore $B$ and $S_1$ are dually paired by $\langle \cdot, \cdot \rangle$.

Now identify $B$ with $S_1^\ast$ and write $\xi(s) = \langle \xi, s \rangle$ for $\xi \in S_1^\ast$, $s \in S_1$. Then, once more by invariance of $\langle \cdot, \cdot \rangle$,

$$[s, \xi](s') = \langle [s, \xi], s' \rangle = \langle \xi, -[s, s'] \rangle = \xi(-\text{ad}(s) s') = (\text{ad}^*(s) \xi)(s')$$

for all $s, s' \in S_1$. So the action of $S_1$ on $S_1^\ast$ is the coadjoint action. This means $S \times B$ is a metric cotangent algebra (cf. Subsection 2.1). □

*Proof of Theorem C.* The decomposition into the desired orthogonal ideals follows from Lemmas 3.2 to 3.5. The structure of the ideals $G_2$ and $G_3$ is Lemma 3.6. □

The algebra $G_1$ in Theorem C is of Euclidean type. Let $G = \mathcal{K} \times V$, with $V \cong \mathbb{R}^n$, be an algebra of Euclidean type and let $\mathcal{K}_0$ be the kernel of the $\mathcal{K}$-action on $V$. Proposition 2.4 and the fact that the solvable radical $V$ is abelian immediately give the following:

**Proposition 3.6.** Let $G = \mathcal{K} \times V$ be a Lie algebra of Euclidean type, and suppose $G$ is equipped with a symmetric bilinear form that is nil-invariant and $G^1$-invariant, such that $G^1$ does not contain a non-trivial ideal of $G$. Then

$$G^1 \subseteq \mathcal{K}_0 \times V.$$

The following Examples 3.7 and 3.8 show that in general a metric Lie algebra of Euclidean type cannot be further decomposed into orthogonal direct sums of metric Lie algebras. Both examples will play a role in Section 4.

**Example 3.7.** Let $\mathcal{K}_1 = SO_3$, $V_1 = \mathbb{R}^3$, $V_0 = \mathbb{R}^3$ and $G = (SO_3 \times V_1) \times V_0$ with the natural action of $SO_3$ on $V_1$. Let $\varphi : V_1 \to V_0$ be an isomorphism and put

$$\mathcal{H} = \{(0, v, \varphi(v)) \mid v \in V_0\} \subset (\mathcal{K}_0 \times V_1) \times V_0.$$ 

We can define a nil-invariant symmetric bilinear form on $G$ by identifying $V_1 \cong SO_3^\ast$ and requiring for $k \in \mathcal{K}_1$, $v_0 \in V_0$, $v_1 \in V_1$,

$$\langle k, v_0 + v_1 \rangle = v_1(k) - \varphi^{-1}(v_0)(k),$$
and further \( K_1 \perp K_1 \), \((V_0 \oplus V_1) \perp (V_0 \oplus V_1)\). Then \(\langle \cdot, \cdot \rangle\) has signature \((3, 3, 3)\) and kernel \(\mathcal{H} = G^4\), which is not an ideal in \(G\). Note that \(\langle \cdot, \cdot \rangle\) is not invariant. Moreover, \(K_1 \times V_1\) is not orthogonal to \(V_0\). A direct factor \(K_0\) can be added to this example at liberty.

**Example 3.8.** We can modify the Lie algebra \(G\) from Example 3.7 by embedding the direct summand \(V_0 \cong \mathbb{R}^3\) in a torus subalgebra in a semisimple Lie algebra \(K_0\) of compact type, say \(K_0 = SO_6\), to obtain a Lie algebra \(G = (K_1 \times V_1) \times K_0\). We obtain a nil-invariant symmetric bilinear form of signature \((15, 3, 3)\) on \(G\) by extending \(\langle \cdot, \cdot \rangle\) by a definite form on a vector space complement of \(V_0\) in \(K_0\). The kernel of the new form is still \(G^4 = \mathcal{H}\).

### 3.2. Nil-invariant bilinear forms on Euclidean algebras

A *Euclidean algebra* is a Lie algebra \(\mathfrak{e}_n = SO_n \ltimes \mathbb{R}^n\), where \(SO_n\) acts on \(\mathbb{R}^n\) by the natural action.

By a *skew pairing* of a Lie algebra \(L\) and an \(L\)-module \(V\) we mean a bilinear map \(\langle \cdot, \cdot \rangle : L \times V \to \mathbb{R}\) such that \(\langle x, yv \rangle = -\langle y, xv \rangle\) for all \(x, y \in L\), \(v \in V\). Note that any nil-invariant symmetric bilinear form on \(\mathfrak{g} = \mathfrak{k} \ltimes \mathbb{R}^n\) yields a skew pairing of \(\mathfrak{k}\) and \(\mathbb{R}^n\).

**Proposition 3.9 (\([2] \text{ Proposition A.5}\)).** Let \(\langle \cdot, \cdot \rangle : SO_3 \times V \to \mathbb{R}\) be a skew pairing for the (non-trivial) irreducible module \(V\). If the skew pairing is non-zero, then \(V\) is isomorphic to the adjoint representation of \(SO_3\) and \(\langle \cdot, \cdot \rangle\) is proportional to the Killing form.

**Example 3.10.** Consider \(G = SO_3 \ltimes \mathbb{R}^n\) with a nil-invariant symmetric bilinear form \(\langle \cdot, \cdot \rangle\), and assume that the action of \(SO_3\) is irreducible. By Proposition 3.9, either \(SO_3 \perp \mathbb{R}^n\), or \(n = 3\) and \(SO_3\) acts by its coadjoint representation on \(\mathbb{R}^3 \cong \mathfrak{so}_3^*\), and \(\langle \cdot, \cdot \rangle\) is the dual pairing. In the first case, \(\mathbb{R}^n\) is an ideal in \(G^4\), and in the second case, \(\langle \cdot, \cdot \rangle\) is invariant and non-degenerate.

**Example 3.11.** Let \(G\) be the Euclidean Lie algebra \(SO_4 \ltimes \mathbb{R}^4\) with a nil-invariant symmetric bilinear form \(\langle \cdot, \cdot \rangle\). Since \(SO_4 \cong SO_3 \times SO_3\), and here both factors \(SO_3\) act irreducibly on \(\mathbb{R}^4\), it follows from Example 3.10 that in \(G\), \(\mathbb{R}^4\) is orthogonal to both factors \(SO_3\), hence to all of \(SO_4\). In particular, \(\mathbb{R}^4\) is an ideal contained in \(G^4\).

**Theorem 3.12.** Let \(\langle \cdot, \cdot \rangle\) be a nil-invariant symmetric bilinear form on the Euclidean Lie algebra \(SO_n \ltimes \mathbb{R}^n\) for \(n \geq 4\). Then the ideal \(\mathbb{R}^n\) is contained in \(G^4\).

**Proof.** For \(n = 4\), this is Example 3.11. So assume \(n > 4\). Consider an embedding of \(SO_4\) in \(SO_n\) such that \(\mathbb{R}^n = \mathbb{R}^4 \oplus \mathbb{R}^{n-4}\), where \(SO_4\) acts on \(\mathbb{R}^4\) in the standard way and trivially on \(\mathbb{R}^{n-4}\). By Example 3.11, \(SO_4 \perp \mathbb{R}^4\). Since \(\mathbb{R}^{n-4} \subseteq [SO_n, \mathbb{R}^n]\), the nil-invariance of \(\langle \cdot, \cdot \rangle\) implies \(SO_4 \perp \mathbb{R}^{n-4}\). Hence \(\mathbb{R}^n \perp SO_4\).

The same reasoning shows that \(\text{Ad}(g)SO_4 \perp \mathbb{R}^n\), where \(g \in SO_n\). Then \(\mathcal{B} = \sum_{g \in SO_n} \text{Ad}(g)SO_4\) is orthogonal to \(\mathbb{R}^n\). But \(\mathcal{B}\) is clearly an ideal in \(SO_n\), so \(\mathcal{B} = SO_n\) by simplicity of \(SO_n\) for \(n > 4\).

**Theorem D.** The Euclidean group \(E_n = O_n \ltimes \mathbb{R}^n\), \(n \neq 1, 3\), does not have compact quotients with a pseudo-Riemannian metric such that \(E_n\) acts isometrically and almost effectively.

**Proof.** For \(n > 3\), such an action of \(E_n\) would induce a nil-invariant symmetric bilinear form on the Lie algebra \(SO_n \ltimes \mathbb{R}^n\) without non-trivial ideals in its kernel, contradicting Theorem 3.12.
For \( n = 2 \), the Lie algebra \( \mathfrak{e}_2 \) is solvable, and hence by Baues and Globke \[1\], any nil-invariant symmetric bilinear form must be invariant. For such a form, the ideal \( \mathbb{R}^2 \) of \( \mathfrak{e}_2 \) must be contained in \( \mathfrak{e}_2^1 \), and therefore action cannot be effective.

Note that \( \mathfrak{e}_3 \) is an exception, as it is the metric cotangent algebra of \( \text{SO}_3 \). □

Remark. Clearly the Lie group \( \mathfrak{e}_n \) admits compact quotient manifolds on which \( \mathfrak{e}_n \) acts almost effectively. For example take the quotient by a subgroup \( F \rtimes \mathbb{Z}^n \), where \( F \subset \text{O}_n \) is a finite subgroup preserving \( \mathbb{Z}^n \). Compact quotients with finite fundamental group also exist. For example, for any non-trivial homomorphism \( \varphi : \mathbb{R}^n \to \text{O}_n \), the graph \( H \) of \( \varphi \) is a closed subgroup of \( \mathfrak{e}_n \) isomorphic to \( \mathbb{Z}^n \), and the quotient \( M = \mathfrak{e}_n / H \) is compact (and diffeomorphic to \( \text{O}_n \)). Since \( H \) contains no non-trivial normal subgroup of \( \mathfrak{e}_n \), the \( \mathfrak{e}_n \)-action on \( M \) is effective. Theorem \[D\] tells us that none of these quotients admit \( \mathfrak{e}_n \)-invariant pseudo-Riemannian metrics.

4. Simply connected compact homogeneous spaces with indefinite metric

Let \( M \) be a connected and simply connected pseudo-Riemannian homogeneous space of finite volume. Then we can write

\[
M = G / H
\]

for a connected Lie group \( G \) acting almost effectively and by isometries on \( M \), and \( H \) is a closed subgroup of \( G \) that contains no non-trivial connected normal subgroup of \( G \) (for example, \( G = \text{Iso}(\mathbb{R}^n) \)). Note that \( H \) is connected since \( M \) is simply connected.

Let \( \mathfrak{g}, \mathfrak{h} \) denote the Lie algebras of \( G, H \), respectively. Recall that the pseudo-Riemannian metric on \( M \) induces an \( \mathfrak{h} \)-invariant and nil-invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \), and the kernel of \( \langle \cdot, \cdot \rangle \) is precisely \( \mathfrak{g}^1 = \mathfrak{h} \) and contains no non-trivial ideal of \( \mathfrak{g} \).

We decompose \( G = KSR \), where \( K \) is a compact semisimple subgroup, \( S \) is a semisimple subgroup without compact factors, \( R \) the solvable radical of \( G \).

**Proposition 4.1.** The subgroup \( S \) is trivial and \( M \) is compact.

**Proof.** As \( M \) is simply connected, \( H = H^0 \). Now \( H \leq KR \) by Theorem 2.2. On the other hand, since \( M \) has finite invariant volume, the Zariski closure of \( \text{Ad}_{\mathfrak{g}}(H) \) contains \( \text{Ad}_{\mathfrak{g}}(S) \), see Mostow \[7\] Lemma 3.1. Therefore \( S \) must be trivial. It follows from Mostow’s result \[6\] Theorem 6.2 on quotients of solvable Lie groups that \( M = (KR)/H \) is compact. □

We can therefore restrict ourselves in (4.1) to groups \( G = KR \) and connected uniform subgroups \( H \) of \( G \).

The structure of a general compact homogeneous manifold with finite fundamental group is surveyed in Onishchik and Vinberg \[8\] II.5.§2. In our context it follows that

\[
G = L \rtimes V
\]

where \( V \) is a normal subgroup isomorphic to \( \mathbb{R}^n \) and \( L = KA \) is a maximal compact subgroup of \( G \). The solvable radical is \( R = A \rtimes V \). Moreover, \( V^L = 0 \). By a theorem of Montgomery \[5\] (also \[8\] p. 137), \( K \) acts transitively on \( M \).

The existence of a \( G \)-invariant metric on \( M \) further restricts the structure of \( G \).
Proposition 4.2. The solvable radical $R$ of $G$ is abelian. In particular, $R = A \times V$, $V^K = 0$ and $A = Z(G)^\circ$.

Proof. Let $Z(R)$ denote the center of $R$ and $N$ its nilradical. Since $H$ is connected, $H \subseteq KZ(R)^\circ$ by Theorem 2.2. Hence there is a surjection $G/H \to G/(KZ(R)^\circ) = R/Z(R)^\circ$. It follows that $Z(R)^\circ$ is a connected uniform subgroup. Therefore the nilradical $N$ of $R$ is $N = T Z(R)^\circ$ for some compact torus $T$. But a compact subgroup of $N$ must be central in $R$, so $T \subseteq Z(R)$. Hence $N \subseteq Z(R)$, which means $R = N$ is abelian. □

Combined with (4.2), we obtain

\[(4.3) \quad G = KR = (K_0 A) \times (K_1 \ltimes V),\]

with $K = K_0 \times K_1$, $R = A \times V$, where $K_0$ is the kernel of the $K$-action on $V$.

For any subgroup $Q$ of $G$ we write $H_Q = H \cap Q$.

Lemma 4.3. $[H, H] \subseteq H_K$. In particular, $H_K$ is a normal subgroup of $H$.

Proof. By Proposition 3.6 and the connectedness of $H$, we have $H \subseteq K_0 R$. Since $R$ is abelian, $[H, H] \subseteq H_{K_0}$. □

If $G$ is simply connected, we have $K \cap R = \{e\}$. Then let $p_K$, $p_R$ denote the projection maps from $G$ to $K$, $R$.

Lemma 4.4. Suppose $G$ is simply connected. Then $p_R(H) = R$.

Proof. Since $K$ acts transitively on $M$, we have $G = KH$. Then $R = p_R(G) = p_R(H)$. □

Proposition 4.5. Suppose $G$ is simply connected. Then the stabilizer $H$ is a semidirect product $H = H_K \rtimes E$, where $E$ is the graph of a homomorphism $\varphi : R \to K$ that is non-trivial if $\dim R > 0$. Moreover, $\varphi(R \cap H) = \{e\}$.

Proof. The subgroup $H_K$ is a maximal compact subgroup of the stabilizer $H$. By Lemma 4.3, $H = H_K \times E$ for some normal subgroup $E$ diffeomorphic to a vector space. By Lemma 4.4, $H$ projects onto $R$ with kernel $H_K$, so that $E \cong R$. Then $E$ is necessarily the graph of a homomorphism $\varphi : R \to K$. If $\dim R > 0$, then $\varphi$ is non-trivial, for otherwise $R \subseteq H$, in contradiction to the almost effectivity of the action. Observe that $R \cap H \subseteq E$. Therefore $\varphi(R \cap H) \subseteq H_K \cap E = \{e\}$. □

Now we can state our main result:

Theorem A. Let $M$ be a connected and simply connected pseudo-Riemannian homogeneous space of finite volume, $G = \text{Iso}(M)^\circ$, and let $H$ be the stabilizer subgroup in $G$ of a point in $M$. Let $G = KR$ be a Levi decomposition, where $R$ is the solvable radical of $G$. Then:

1. $M$ is compact.
2. $K$ is compact and acts transitively on $M$.
3. $R$ is abelian. Let $A$ be the maximal compact subgroup of $R$. Then $A = Z(G)^\circ$. More explicitly, $R = A \times V$ where $V \cong \mathbb{R}^n$ and $V^K = 0$.
4. $H$ is connected. If $\dim R > 0$, then $H = (H \cap K)E$, where $E$ and $H \cap K$ are normal subgroups in $H$, $(H \cap K) \cap E$ is finite, and $E$ is the graph of a non-trivial homomorphism $\varphi : R \to K$, where the restriction $\varphi|_A$ is injective.
Example 4.7. Start with the action of $K$ (Gromov [4, Theorem 3.5.C]).

Connected pseudo-Riemannian manifold has finitely many connected components below in which we construct some $M$.

Proof. Suppose there is a compact Lie group $G$ that contains $K$ as a subgroup. As the action of $K$ on $V$ is non-trivial, there exists an element $v \in V \subset C$ such that $\text{Ad}_{C}(v)$ has non-trivial unipotent Jordan part. But by compactness of $C$, every $\text{Ad}_{C}(g), g \in C$, is semisimple, a contradiction. □

Example 4.7. Start with $G_1 = (\tilde{\text{SO}}_3 \rtimes \mathbb{R}^3) \rtimes \mathbb{T}^3$, where $\tilde{\text{SO}}_3$ acts on $\mathbb{R}^3$ by the coadjoint action, and let $\varphi : \mathbb{R}^3 \to \mathbb{T}^3$ be a homomorphism with discrete kernel. Put

$$H = \{ (I_3, v, \varphi(v)) \mid v \in \mathbb{R}^3 \}.$$ 

The Lie algebras $\mathfrak{g}_1$ of $G_1$ and $\mathfrak{h}$ of $H$ are the corresponding Lie algebras from Example 3.7. We can extend the nil-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_1$ from

Lemma 4.6. Assume that the action of $K$ on $V$ in the semidirect product $G = K \rtimes V$ is non-trivial. Then $G$ cannot be immersed in a compact Lie group.

Proof. Suppose there is a compact Lie group $C$ that contains $G$ as a subgroup. As the action of $K$ on $V$ is non-trivial, there exists an element $v \in V \subset C$ such that $\text{Ad}_{C}(v)$ has non-trivial unipotent Jordan part. But by compactness of $C$, every $\text{Ad}_{C}(g), g \in C$, is semisimple, a contradiction. □
Example 3.7 to a left-invariant tensor on $G_1$, and thus obtain a $G_1$-invariant pseudo-Riemannian metric of signature $(3, 3)$ on the quotient $M_1 = G_1/H = \tilde{SO}_3 \times T^3$. Here, $M_1$ is a non-simply connected manifold with a non-compact connected stabilizer.

In order to obtain a simply connected space, embed $T^3$ in a simply connected compact semisimple group $K_0$, for example $K_0 = \tilde{SO}_6$, so that $G_1$ is embedded in $G = (\tilde{SO}_3 \times \mathbb{R}^3) \times K_0$. As in Example 4.8, we can extend $\langle \cdot, \cdot \rangle$ from $g_1$ to $g$, and thus obtain a compact simply connected pseudo-Riemannian homogeneous manifold $M = G/H = \tilde{SO}_3 \times K_0$.

**Example 4.8.** Example 4.7 can be generalized by replacing $\tilde{SO}_3$ by any simply connected compact semisimple group $K$, acting by the coadjoint representation on $\mathfrak{d}$, where $d = \dim K$, and trivially on $T^d$. Define $H$ similarly as a graph in $\mathbb{R}^d \times T^d$, and embed $T^d$ in a simply connected compact semisimple Lie group $K_0$.

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