A Matrix-based Distance of Pythagorean Fuzzy Set and its Application in Medical Diagnosis

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Abstract

The pythagorean fuzzy set (PFS) developed based on intuitionistic fuzzy set, is more efficient in elaborating and processing uncertainties in indeterminate situations. How to measure the distance between two PFSs is still an open issue. Many kinds of methods have been proposed to address this problem. However, not all of existing methods can accurately display differences among PFSs and satisfy the property of similarity. Some methods neglect the relationship among three variables of PFS. To address the problems, a new method of distance measure is proposed which meets the requirements of axiom of distance measurement and is developed by generalizing the concept of score function in a matrix form to better measure parameters difference of PFS. In detail, our proposal takes into account the hesitance index and allocates its weight between membership and non-membership in a rational manner, enhancing the significance of both membership and non-membership in shaping the distances between PFSs. Some numerical examples are offered to show that the proposed method can avoid producing counter-intuitive results and is more reasonable than other previous methods. Besides, the proposed method is applied to real-world environments of application and compared with previous methods to demonstrate its superiority and efficiency.

Keywords: Pythagorean fuzzy set, Distance measure, Score function, Matrix

1. Introduction

There are lots of uncertainty existing in the real world, and how to measure the level of uncertainty has attracted incremental attention and interests from researches all around the world \cite{1,2,3}. Therefore, many meaningful and related theories have been proposed to be applied into practical applications, such as evidence theory \cite{4,5,6,7}, belief function \cite{8,9,10,11}, and structure \cite{12,13,14}, entropy theory \cite{15,16,17,18}, Z numbers \cite{19,20,21,22} and D numbers \cite{23,24,25,26}, which play important roles

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in all walks of life. Except for the theories mentioned above, fuzzy sets \cite{27, 28}, in seeking for useful information among uncertainties, becomes a key component of pattern recognition and decision making \cite{29, 30, 31}.

Obviously, when judging actual situations in real world, things are getting more complex, which indicates that a single value can not reflect the essence of certain objects. Therefore, a further concept based on the fuzzy set is proposed such as intuitionistic fuzzy set (IFS). It is developed by Atanassov \cite{32} and defined to contain 3 properties, namely membership, non-membership and hesitance, IFS is considered more efficient in handling ambiguity. It is further redesigned to satisfy different demands and be applicable in more complex situations, such as interval-valued intuitionistic fuzzy set \cite{33, 34} and quantum decision \cite{35, 36}. After that, a new extension is invented by Yager \cite{37} called pythagorean fuzzy set (PFS). It is a quadratic form of the original fuzzy set, which means that the new modality of fuzzy set has a larger range of the change of variables and therefore has more potential to express uncertainties. PFS is also extended into other forms, such as interval-valued pythagorean fuzzy set \cite{38} and applied in real-world problems \cite{39, 40, 41}.

For any kind of fuzzy sets, how to measure the differences among them is an unavoidable problem of practical significance. In order to express the distances properly, many methods of distance measure have been proposed and some of them have superb performance in classification problems. The most widely used method of measuring distances between IFSs are the Hamming distance \cite{42}, Euclidean distance \cite{42}, the Hausdorff metric \cite{43}, Own’s distance \cite{44}, De et al.’s distance \cite{45}, Szmidt et al.’s distance \cite{46}, Wei et al.’s distance \cite{47}, Mondal et al’s distance \cite{48} and others. However, not all of the methods can produce intuitive results under any circumstances and may be obviously conflicting with data given. Because PFS is a generalization of IFS, the method of measuring distance between IFSs can be rationally extended into the forms which can measure distance between PFSs. But similarly, the challenges that exist in measuring IFS are also present when measuring PFS.

Therefore, to address the potential counter-intuitive and irrational results that existing work may introduce into the measurement of PFS, a novel method of distance measure of PFSs is proposed in this paper. Different distances have different emphases, and there are varying advantages in measuring the degree of differences in PFSs. Due to the hesitation index being a feature of PFSs that describes the states of "membership or non-membership," unlike the first two parameters of PFSs, the information carried by the hesitation is uncertain. As a result, when comparing the differences between PFSs, the three parameters cannot be considered separately and assigned equal importance. In detail, the method transforms the elements contained in the PFSs into a form of vector to maximize the role every element plays in generating distances taking the concept of generalized score function. The newly defined matrix is utilized
to strengthen the membership and non-membership and the index of hesitance is also considered but its influence in distance measurement is more limited than the former two. Compared with the results produced by other methods, the ones from the proposed method is obviously more accurate and conforms to reality, which indicates it may be more efficient in practical applications.

The rest of the paper is organized as follows. In the section of preliminaries, some relative concepts are briefly introduced. Besides, details are clearly provided and some simple cases are utilized to introduce the proposed method more straightforward in the section of proposed method. For the next section, the correctness and validity of the proposed method in distance are verified. In the last, some details and advantages of the proposed method are concluded.

2. Preliminaries

In this section, some concepts related to Intuitionistic Fuzzy Set (IFS) and Pythagorean Fuzzy Sets (PFS) are recalled.

2.1. Intuitionistic Fuzzy Set

The IFS $A$ defined in space $X$ can be expressed as [32]:

$$A = \{(x, \mu(x), \nu(x))|x \in X\}$$ (1)

$\mu(x)$ and $\nu(x)$ satisfy:

$$\mu(x) : X \rightarrow [0,1]$$ (2)

$$\nu(x) : X \rightarrow [0,1]$$ (3)

$\mu(x)$ is the degree of membership of $x \in X$ and $\nu(x)$ is the degree of non-membership of $x \in X$. Both of them meet the condition that:

$$0 \leq \mu(x) + \nu(x) \leq 1$$ (4)

The hesitance function of an IFS $A$ in $X$ is defined as:

$$\pi(x) = 1 - \mu(x) - \nu(x)$$ (5)

The value of $\pi(x)$ can reflect the degree of hesitance of $x \in X$. 

3
2.2. *Pythagorean fuzzy sets*

The PFS $A$ defined in space $X$ can be expressed as [37]:

$$A = \{(x, A_Y(x), A_N(x))| x \in X\} \quad (6)$$

Besides, $A_Y(x)$ and $A_N(x)$ satisfy:

$$A_Y(x) : X \rightarrow [0, 1] \quad (7)$$

$$A_N(x) : X \rightarrow [0, 1] \quad (8)$$

$A_Y(x)$ is a representative of the degree of membership of $x \in X$ and $v(x)$ is a representative of the degree of non-membership of $x \in X$. Both of them meet the condition that:

$$0 \leq A_Y^2(x) + A_N^2(x) \leq 1 \quad (9)$$

The hesitance function of an PFS $A$ in $X$ is defined as:

$$A_H(x) = \sqrt{1 - A_Y^2(x) - A_N^2(x)} \quad (10)$$

The value of $A_H(x)$ can reflect the degree of hesitance of $x \in X$.

**Property:** Let $B$ and $C$ be two PFS, then both of them satisfy:

1. $B \subseteq C$ if $\forall x \in X B_Y(x) \leq C_Y(x)$ and $B_N(x) \geq C_N(x)$
2. $B = C$ if $\forall x \in X B_Y(x) = C_Y(x)$ and $B_N(x) = C_N(x)$
3. $B \cap C = \{(x, \min[B_Y(x), C_Y(x)], \max[B_N(x), C_N(x)])| x \in X\}$
4. $B \cup C = \{(x, \max[B_Y(x), C_Y(x)], \min[B_N(x), C_N(x)])| x \in X\}$
5. $C = \{(x, B_Y(x)C_Y(x), (B_Y^2(x) + C_Y^2(x) - B_Y^2(x)C_Y^2(x))^{1/2})| x \in X\}$
6. $B^n = \{(x, B_Y^n(x), (1 - (1 - B_N^2(x))^{1/2})| x \in X\}$

In the sequel, some distances between different IFSs and PFSs are briefly introduced. Related function is also mentioned, which is helpful in judging the validity of IFS and PFS in its extended form.

2.3. *Distances Related to Pythagorean Fuzzy Sets*

The Hamming distance which measures difference between IFSs $D$ and $E$ is [42]:

$$d_{H_m}(D, E) = \frac{1}{2} \cdot (|\mu_D(x) - \mu_E(x)| + |v_D(x) - v_E(x)| + |\pi_D(x) - \pi_E(x)|) \quad (11)$$
The normalized Hamming distance is expressed as [42]:

$$
\tilde{d}_{Hm}(D, E) = \frac{1}{2n} \cdot \sum_{i=1}^{n} (|\mu_D(x_i) - \mu_E(x_i)| + |\nu_D(x_i) - \nu_E(x_i)| + |\pi_D(x_i) - \pi_E(x_i)|)
$$

(12)

The Euclidean distance which measures difference between IFSs $D$ and $E$ is [42]:

$$
d_{Eu}(D, E) = \left( \frac{1}{2} \cdot ((\mu_D(x) - \mu_E(x))^2 + (\nu_D(x) - \nu_E(x))^2 + (\pi_D(x) - \pi_E(x))^2) \right)^{\frac{1}{2}}
$$

(13)

The normalized Euclidean distance is expressed as [42]:

$$
\tilde{d}_{Eu}(D, E) = \left( \frac{1}{2n} \cdot \sum_{i=1}^{n} ((\mu_D(x_i) - \mu_E(x_i))^2 + (\nu_D(x_i) - \nu_E(x_i))^2 + (\pi_D(x_i) - \pi_E(x_i))^2) \right)^{\frac{1}{2}}
$$

(14)

2.4. Extended Distance Measure of PFSs

The Hamming distance which measures difference between PFSs $F$ and $G$ is [49]:

$$
D_{Hm}(F, G) = \frac{1}{2} \cdot (|F_Y^2(x) - G_Y^2(x)| + |F_N^2(x) - G_N^2(x)| + |F_H^2(x) - G_H^2(x)|)
$$

(15)

The normalized Hamming distance is expressed as:

$$
\tilde{D}_{Hm}(F, G) = \frac{1}{2n} \cdot \sum_{i=1}^{n} (|F_Y^2(x) - G_Y^2(x)| + |F_N^2(x) - G_N^2(x)| + |F_H^2(x) - G_H^2(x)|)
$$

(16)

The Euclidean distance which measures difference between PFSs $F$ and $G$ is [49]:

$$
D_{Eu}(F, G) = \left( \frac{1}{2} \cdot ((F_Y^2(x) - G_Y^2(x))^2 + (F_N^2(x) - G_N^2(x))^2 + (F_H^2(x) - G_H^2(x))^2) \right)^{\frac{1}{2}}
$$

(17)

The normalized Euclidean distance is expressed as:

$$
\tilde{D}_{Eu}(F, G) = \left( \frac{1}{2n} \cdot \sum_{i=1}^{n} ((F_Y^2(x) - G_Y^2(x))^2 + (F_N^2(x) - G_N^2(x))^2 + (F_H^2(x) - G_H^2(x))^2) \right)^{\frac{1}{2}}
$$

(18)
The Chen’s distance which measures difference between PFSs $F$ and $G$ is \[ (19) \]

\[
D_C(F, G) = \left[ \frac{1}{2} \cdot (|F^2_Y(x) - G^2_Y(x)|^\beta + |F^2_H(x) - G^2_H(x)|^\beta + |F^2_N(x) - G^2_N(x)|^\beta) \right]^{\frac{1}{\beta}}
\]

The normalized Chen’s distance is expressed as:
\[
\tilde{D}_C(F, G) = \left[ \frac{1}{2n} \cdot \sum_{i=1}^{n} (|F^2_Y(x) - G^2_Y(x)|^\beta + |F^2_H(x) - G^2_H(x)|^\beta + |F^2_N(x) - G^2_N(x)|^\beta) \right]^{\frac{1}{\beta}}
\]

The parameter $\beta$ in Chen’s distance which satisfies the condition that $\beta \geq 1$. When $\beta = 1$, the method of Chen reduces to the method of Hamming distance. Besides, when $\beta = 2$, the method of Chen reduces to the method of Euclidean distance.

2.5. Score Function on IFS

A score function of IFS is initialized by Chen and Tan and it is utilized in the solution of problems which are produced in multi-attribute decision using intuitionistic sets. Let $A$ be an intuitionistic fuzzy set on the universe $X = \{x_1, x_2, ..., x_n\}$. IFS $A$ is given as $A = \{\langle x, \mu(x), \nu(x) \rangle\}$. The score function $S_A$ is defined as \[ (20) \]

\[
S_A(x_i) = \mu(x_i) - \nu(x_i)
\]

The value of the formula illustrates a degree that whether the intuitionistic fuzzy sets is comfortable enough for decision makers to have a clear and straightforward expectation to actual situations. Obviously, the value of $\mu(x)$ gets larger, the smaller the value of $\nu(x)$ is going to be and the probability of $x \in X$ gets greater, which means the value of $S_A(x_i)$ gets bigger. Therefore, the value of score function $S_A(x_i)$ can be regarded as a kind of level of support about the element $x \in X$. When $S_A(x_i) > 0$, it can be concluded that it is more believable to classify $x$ as a component of $X$. When $S_A(x_i) < 0$, it can be concluded that it is more believable to classify $x$ as an exception of $X$. More than that, the absolute value of the score function can manifest the situation of the degree of certainty of IFS. If the mass is getting bigger, then the IFS is more certain. If the mass is getting smaller, then the IFS is more uncertain. For example, let $A$ and $B$ be two PFSs and they are defined respectively as $A = \{\langle x, 0.35, 0.35 \rangle\}$ and $B = \{\langle x, 0.6, 0.1 \rangle\}$. It is worth noting that the degree of hesitance in these two IFSs is exactly identical. However, the IFS $B$ is more valuable and useful, because it offers more information and can clearly illustrate actual situations. Obviously, the values of $|S_A(x) = 0|$ and $|S_B(x) = 0.5|$ is exactly consistent with the conclusion mentioned above. As a consequence, the absolute value of $S_A(x)$ can be utilized as
an efficient method to measure the level of certainty.

3. Proposed method

How to measure the differences between PFSs is still an open issue. In this section, a proposed method of distance measure between PFSs. It aims to further improve the performance of the similarity matrix in managing the mass of membership, non-membership and the hesitance. Then, the properties of the proposed method are inferred and proven. In numerical examples, the proposed method designed for PFSs satisfies the distance measure axiom and is more efficient in producing intuitive and rational results.

3.1. Extended Score Function for PFS

On the base of the definition of score function which is developed on the concept of intuitionistic fuzzy set, a new score function $SP$ is on the notion of pythagorean fuzzy set which is defined as:

$$ SP_A(x_i) = A_Y^2(x_i) - A_N^2(x_i) $$

(22)

The extended PFS formula plays a role similar to that of a scoring function based on IFS when dealing with PFSs and it is utilized to serve as a component of the proposed distance measure.

3.2. The New Matrix-based Distance Measure

Let $A$ and $B$ be two PFSs in the finite universe of discourse $X$, the new distance measure is defined as follows:

$$ D_N(A, B) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \bar{m}_i M(\bar{m}_i M)^T} $$

(23)

$$ AB_Y = A_Y^4(x_i) + B_Y^4(x_i) $$

(24)

$$ AB_N = A_N^4(x_i) + B_N^4(x_i) $$

(25)

$$ AB_H = A_H^4(x_i) + B_H^4(x_i) $$

(26)

$$ \bar{m}_i = (A_Y^2(x_i) - B_Y^2(x_i), A_N^2(x_i) - B_N^2(x_i), A_H^2(x_i) - B_H^2(x_i)) $$

(27)

However, the definition of the matrix $M$ is still not clear. The most important efficacy of the matrix is to adjust and optimize the mass of membership, non-membership and the hesitance. Adopting the index of hesitance as a parameter to generating distance between different PFSs is not straightforward and concise, because the index of hesitance itself is a kind of uncertainty, which is very difficult to clarify the relationship between different hesitance. There is a kind of relationship between the hesitance and
membership and non-membership. For example, let $A$ and $B$ be two PFSs in the finite universe of discourse $X$, where $PFS_A = \{(0, 0, 1)\}$ and $PFS_B = \{(0.51, 0.49, 0.706)\}$. In $PFS_A$, the index of hesitance is equal to 1, which indicates that $PFS_A$ is totally uncertain. Besides, on the contrary, the index of hesitance of $PFS_B$ is 0, it means there is no uncertainty or vagueness commonly. However, the mass of membership and non-membership is so close that it is very difficult to make reasonable decisions.

It can be pointed out, the index of hesitance is not independent of the other two parameters, namely membership and non-membership, which means all of them have a similarity under certain relationship and is already sufficiently demonstrated in examples provided. Therefore, the definition of the new matrix $M$ is written as follows:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Y & N & H \end{pmatrix}$$

(28)

$$Y = \frac{(A_Y^2(x_i) + B_Y^2(x_i))}{(A_Y^2(x_i) + B_Y^2(x_i) + A_N^2(x_i) + B_N^2(x_i))}$$

(29)

$$N = \frac{(A_N^2(x_i) + B_N^2(x_i))}{(A_Y^2(x_i) + B_Y^2(x_i) + A_N^2(x_i) + B_N^2(x_i))}$$

(30)

$$H = \sqrt{1 - Y^2 - N^2}$$

(31)

The main idea of the definition is that because uncertainty cannot accurately show the real situation, then the mass of index is supposed to be distributed to membership and non-membership according to their mass in different PFSs to strengthen their ability of identification and not to change the original conditions, which enlarges the useful amount of information of pythagorean fuzzy sets and is helpful in target recognition. Due to the distribution to membership and non-membership, an adjustment of the index of hesitance should also be considered. Therefore, considering the mathematic form of PFS, it is proposed that $H = \sqrt{1 - Y^2 - N^2}$ is adopted as the remaining amount of information after the distribution to membership and non-membership. The operation of reduction in the index of hesitance improves the degree of identification, which is very helpful in measuring the distance of PFSs.

**Specific details about matrix:** The new distance proposed in this paper is based on the transformation of vectors from 3 parameters in PFSs. However, the role of the vector and matrix is not just showing the original mass of 3 parameters. What can be concluded is that membership and non-membership is independent of each other, so both of the parameters is considered as orthogonal. Though membership and non-membership are treated equally, the relationship between hesitance and the other 2 parameters is not just orthogonal. It presents a kind of proportion in membership and non-membership and also conforms to the operation of some methods which handles the uncertainties of multiple el-
elements propositions in evidence theory and is closely related to fuzzy sets. For example, two PFSs $A = \{0.6, 0.6, 0.529\}$ and PFS $B = \{0.3, 0.3, 0.905\}$ are given to explain the process of the proposed vector and the new matrix which adjust the distribution of 3 parameters. The process is presented as follows:

\[
\begin{align*}
\bar{m}_i &= (A^2_N(x_i) - B^2_N(x_i), A^2_H(x_i) - B^2_H(x_i)) = (0.27, 0.27, -0.54) \\
Y &= \frac{(A^2_Y(x_i) + B^2_Y(x_i))}{(A^2_Y(x_i) + B^2_Y(x_i) + A^2_N(x_i) + B^2_N(x_i))} = \frac{0.6^2 + 0.3^2}{2(0.6^2 + 0.3^2)} = 0.5 \\
N &= \frac{(A^2_Y(x_i) + B^2_Y(x_i))}{(A^2_Y(x_i) + B^2_Y(x_i) + A^2_N(x_i) + B^2_N(x_i))} = \frac{0.6^2 + 0.3^2}{2(0.6^2 + 0.3^2)} = 0.5 \\
H &= \sqrt{1-Y^2-N^2} = 0.707 \\
\bar{m}_i \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= (0.27, 0.27, -0.54) \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= (0, 0, 0.3818)
\end{align*}
\]

Then the construction of a crucial part of the numerator is completed.

### 3.3. Demonstration of Proposed Method Properties

In this part, many properties of the proposed method are verified, like symmetry and triangle inequality.

**Proof 1:** The commutative property is verified in this proof. Let $A$ and $B$ be two PFSs in the finite universe of discourse $X$. Considering $D_N(A, B)$ and $D_N(B, A)$, two equations can be obtained:

\[
\begin{align*}
D_N(A, B) &= \sqrt{\frac{1}{n} \sum_{i=1}^{n} \bar{m}_i M(\bar{m}_i M)^	op} \\
AB_Y &= A^4_Y(x_i) + B^4_Y(x_i) \\
AB_N &= A^4_N(x_i) + B^4_N(x_i) \\
AB_H &= A^4_H(x_i) + B^4_H(x_i) \\
\bar{m}_i &= (A^2_N(x_i) - B^2_N(x_i), A^2_H(x_i) - B^2_H(x_i)) \\
M &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
Y &= \frac{(A^2_Y(x_i) + B^2_Y(x_i))}{(A^2_Y(x_i) + B^2_Y(x_i) + A^2_N(x_i) + B^2_N(x_i))} \\
N &= \frac{(A^2_Y(x_i) + B^2_Y(x_i))}{(A^2_Y(x_i) + B^2_Y(x_i) + A^2_N(x_i) + B^2_N(x_i))} \\
D_N(B, A) &= \sqrt{\frac{1}{n} \sum_{i=1}^{n} \bar{m}_i M(\bar{m}_i M)^	op} \\
BA_Y &= B^4_Y(x_i) + A^4_Y(x_i) \\
BA_N &= B^4_N(x_i) + A^4_N(x_i) \\
BA_H &= B^4_H(x_i) + A^4_H(x_i) \\
\bar{m}_i &= (B^2_N(x_i) - A^2_N(x_i), B^2_H(x_i) - A^2_H(x_i))
\end{align*}
\]
\[
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
Y & N & H
\end{pmatrix}
\]

\[
Y = \frac{B^2_Y(x_i) + A^2_Y(x_i)}{(B^2_Y(x_i) + A^2_Y(x_i) + A^2_N(x_i) + A^2_N(x_i))}
\]

\[
N = \frac{B^2_N(x_i) + A^2_N(x_i)}{(B^2_N(x_i) + A^2_N(x_i) + B^2_H(x_i) + A^2_H(x_i))}
\]

It can be easily concluded that, when PFS \( A \) and \( B \) interchange their places, the attributes \( AB_Y, AB_N, \)
\( AB_H \), matrix \( M \), \( Y \) and \( N \) do not change their values. But how the numerator \( \bar{m}_i M(\bar{m}_i M)^T \) change is not very clear. Nevertheless, it can be also easily proved like this:

\[
\bar{m}_i M(\bar{m}_i M)^T
= (A^2_Y(x_i) - B^2_Y(x_i) + Y(A^2_H(x_i) - B^2_H(x_i)))^2
+ (A^2_N(x_i) - B^2_N(x_i) + N(A^2_H(x_i) - B^2_H(x_i)))^2
+ ((B^2_H(x_i) - A^2_H(x_i)) (√1 - Y^2 - N^2))^2
= (B^2_Y(x_i) - A^2_Y(x_i) + Y(B^2_H(x_i) - A^2_H(x_i)))^2
+ (B^2_N(x_i) - A^2_N(x_i) + N(B^2_H(x_i) - A^2_H(x_i)))^2
+ ((A^2_H(x_i) - B^2_H(x_i)) (√1 - Y^2 - N^2))^2
\]

Therefore, it is proved that the proposed method of distance measure between PFSs satisfy the property that \( D_N(A, B) = D_N(B, A) \).

**Proof 2:** A simple example is offered to verify a basic attribute that is when 2 PFSs are exactly the same, their distance is coincidently 0. Let \( A \) and \( B \) become two identical PFSs, namely \( \{0.5, 0.5, 0.707\} \), their distance is supposed to be 0 intuitively. According to the definition of the vector \( \bar{m}_i \), every component in the vector is equal to zero, which means the numerator \( \bar{m}_i M(\bar{m}_i M)^T \) of the method of distance measure is zero and the value of the whole formula is also zero. In a wider range, when two PFS are perfectly consistent, \( A^2_Y(x_i) = B^2_Y(x_i), A^2_N(x_i) = B^2_N(x_i), A^2_H(x_i) = B^2_H(x_i) \), it can be concluded that the value of \( \bar{m}_i \) should be exactly zero. Therefore, the distance of the proposed method satisfies the property that \( D_N(A, B) = 0 \) if and only if \( A = B \).

**Proof 3:** In this proof, the distance between 2 PFSs is demonstrated that \( 0 ≤ D_N(A, B) ≤ 1 \). Let \( A \) and \( B \) be two PFSs in the finite universe of discourse \( X \), then it can be obtained that:

\[
\bar{m}_i M
= (A^2_Y(x_i) - B^2_Y(x_i) + Y(A^2_H(x_i) - B^2_H(x_i))),
\]

\[
A^2_N(x_i) - B^2_N(x_i) + N(A^2_H(x_i) - B^2_H(x_i)),
\]

\[
(B^2_H(x_i) - A^2_H(x_i)) (√1 - Y^2 - N^2)
\]

\[
\bar{m}_i M(\bar{m}_i M)^T
= (A^2_Y(x_i) - B^2_Y(x_i) + Y(A^2_H(x_i) - B^2_H(x_i)))^2
\]
\[(A_N^2(x_i) - B_N^2(x_i) + N(A_H^2(x_i) - B_H^2(x_i)))^2\]
\[+ ((B_H^2(x_i) - A_H^2(x_i))(\sqrt{T} - Y^2 - N^2))^2\]
\[= (A_T^2(x_i) - B_T^2(x_i))^2 + (A_N^2(x_i) - B_N^2(x_i))^2\]
\[+ (A_H^2(x_i) - B_H^2(x_i))^2\]
\[+ 2Y(A_T^2(x_i) - B_T^2(x_i))(A_H^2(x_i) - B_H^2(x_i))\]
\[+ 2N(A_T^2(x_i) - B_T^2(x_i))(A_N^2(x_i) - B_N^2(x_i))\]
\[= A_T^2(x_i) + B_T^2(x_i) + A_N^2(x_i) + B_N^2(x_i) + A_H^2(x_i) + B_H^2(x_i)\]
\[- 2A_T^2(x_i)B_T^2(x_i) - 2A_N^2(x_i)B_N^2(x_i) - 2A_H^2(x_i)B_H^2(x_i)\]
\[+ (A_T^2(x_i) - B_T^2(x_i))\]
\[((2YA_T^2(x_i) + 2NA_N^2(x_i)) - (2YB_T^2(x_i) + 2NB_N^2(x_i))) \leq A_T^4(x_i) + B_T^4(x_i) + A_N^4(x_i) + B_N^4(x_i) + A_H^4(x_i) + B_H^4(x_i)\]

Obviously, when the entirety of \(2YA_T^2(x_i) + 2NA_N^2(x_i)\) is larger than that of \(2YB_T^2(x_i) + 2NB_N^2(x_i)\), \(A_T^2(x_i) - B_T^2(x_i)\) is definitely negative, according to the definition of pythagorean fuzzy set. Therefore, it can be inferred that

\[0 \leq \frac{\bar{m}_M(m_i,M)T}{A_T^4(x_i) + B_T^4(x_i) + A_N^4(x_i) + B_N^4(x_i) + A_H^4(x_i) + B_H^4(x_i)} \leq 1\]

Let \(T = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\bar{m}_M(m_i,M)T}{A_T^4(x_i) + B_T^4(x_i) + A_N^4(x_i) + B_N^4(x_i) + A_H^4(x_i) + B_H^4(x_i)}\)

Therefore, \(0 \leq \sqrt{T} \leq 1\)

Because \(\sqrt{T}\) has the same significance with \(D_n(A, B)\). Then, it can be concluded that \(0 \leq D_n(A, B) \leq 1\).

**Proof 4:** The triangle inequality is going to be proven in this part. Some assumptions are given as:

**Assumption 1:** \(A_T^2(x) \leq B_T^2(x) \leq C_T^2(x)\)

**Assumption 2:** \(C_T^2(x) \leq B_T^2(x) \leq A_T^2(x)\)

**Assumption 3:** \(B_T^2(x) \leq \min\{A_T^2(x), C_T^2(x)\}\)

**Assumption 4:** \(B_T^2(x) \geq \max\{A_T^2(x), C_T^2(x)\}\)

It can be easily demonstrated that the inequality:

\[|A_T^2(x) - C_T^2(x)| \leq |A_T^2(x) - B_T^2(x)| + |B_T^2(x) - C_T^2(x)|\]

which meets conditions of **Assumption 1** and **Assumption 2**. On the base of **Assumption 3**, it can be concluded that

\[A_T^2(x) \geq B_T^2(x), C_T^2(x) \geq B_T^2(x)\]

Therefore, it can be obtained that:

\[|A_T^2(x) - B_T^2(x)| + |B_T^2(x) - C_T^2(x)| - |A_T^2(x) - C_T^2(x)|\]
4.1. Examples and Discussions

Let $A$ and $B$ be two PFSs in the finite universe of discourse $X$, which are defined as $A = \{(x, A_Y(x), A_N(x))\}$ and $B = \{(x, B_Y(x), B_N(x))\}$. Besides, a further limitation is defined as:

- $A_Y^2(x) = \delta$, $A_N^2(x) = 1 - \delta$, $A_H^2(x) = \delta - 0.1$,
- $A_H^2(x) = 1.1 - \delta$, $A_H(x) = 0 = B_H^2(x)$

$$f(x) = \left\{ \begin{array}{ll}
A_Y^2(x) - B_Y^2(x) + C_Y^2(x) - B_N^2(x) - A_Y^2(x) \\
+C_Y^2(x) \text{ if } A_Y^2(x) \geq C_Y^2(x) \\
A_Y^2(x) - B_Y^2(x) + C_Y^2(x) - B_N^2(x) + A_Y^2(x) \\
-C_Y^2(x) \text{ if } A_Y^2(x) \leq C_Y^2(x) \\
\end{array} \right.$$

$$= 2 \cdot (\min \{A_Y^2(x), C_Y^2(x)\} - B_Y^2(x)) \geq 0$$

Similarly, on the base of Assumption 4, it can be summarized that:

$$|A_Y^2(x) - B_Y^2(x)| + |B_Y^2(x) - C_Y^2(x)| - |A_Y^2(x) - C_Y^2(x)|$$

$$f(x) = \left\{ \begin{array}{ll}
B_Y^2(x) - A_Y^2(x) + B_Y^2(x) - C_Y^2(x) - A_Y^2(x) \\
+C_Y^2(x) \text{ if } A_Y^2(x) \geq C_Y^2(x) \\
B_Y^2(x) - A_Y^2(x) + B_Y^2(x) - C_Y^2(x) + A_Y^2(x) \\
-C_Y^2(x) \text{ if } A_Y^2(x) \leq C_Y^2(x) \\
\end{array} \right.$$

$$= 2 \cdot (B_Y^2(x) - \max \{A_Y^2(x), C_Y^2(x)\}) \geq 0$$

Hence, the property of inequality is also satisfied under the condition provided by Assumption 3 and Assumption 4.

$$|A_Y^2(x) - C_Y^2(x)| \leq |A_Y^2(x) - B_Y^2(x)| + |B_Y^2(x) - C_Y^2(x)|$$

Because every component in the method of measuring distance is correspondingly proportionable to membership, non-membership and the index of hesitance. As a result, the triangle inequality has been proved. It can be concluded that:

$$D_N(A, B) + D_N(B, C) \geq D_N(A, C)$$

All of the required properties of the proposed which are expected to be satisfied have been demonstrated. In the next part, some examples are offered to further verify the validity of proposed method.

4. Numerical examples and applications

In this section, lots of numerical examples and actual applications are utilized to illustrate the effectiveness of the method proposed in this paper. And the accuracy and superiority of this method are also testified in this part.

4.1. Examples and Discussions

Example 1: Let $A$ and $B$ be two PFSs in the finite universe of discourse $X$, which are defined as $A = \{(x, A_Y(x), A_N(x))\}$ and $B = \{(x, B_Y(x), B_N(x))\}$. Besides, a further limitation is defined as:

- $A_Y^2(x) = \delta$, $A_N^2(x) = 1 - \delta$, $A_H^2(x) = \delta - 0.1$,
- $A_H^2(x) = 1.1 - \delta$, $A_H(x) = 0 = B_H^2(x)$

$A_Y^2(x) - B_Y^2(x) + C_Y^2(x) - B_N^2(x) - A_Y^2(x)
+C_Y^2(x) \text{ if } A_Y^2(x) \geq C_Y^2(x)
A_Y^2(x) - B_Y^2(x) + C_Y^2(x) - B_N^2(x) + A_Y^2(x)
-C_Y^2(x) \text{ if } A_Y^2(x) \leq C_Y^2(x)$
Table 1: Distances generated by Euclidean’s method and proposed method

| The change of \( \delta \) value | Distances     |
|----------------------------------|---------------|
|                                 | Euclidean     | Proposed method |
| 0.1                              | 0.1           | 0.1048          |
| 0.2                              | 0.1           | 0.1154          |
| 0.3                              | 0.1           | 0.1259          |
| 0.4                              | 0.1           | 0.1348          |
| 0.5                              | 0.1           | 0.1400          |
| 0.6                              | 0.1           | 0.1400          |
| 0.7                              | 0.1           | 0.1348          |
| 0.8                              | 0.1           | 0.1259          |
| 0.9                              | 0.1           | 0.1154          |
| 1.0                              | 0.1           | 0.1048          |

Table 2: The change of the value of score function

| \( \delta \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1  |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|----|
| \( S(A) \)  | 0.8 | -0.8| -0.6| -0.4| -0.2| 0   | 0.2 | 0.4 | 0.6 | 0.8|
| \( S(B) \)  | -1  | -0.8| -0.6| -0.4| -0.2| 0   | 0.2 | 0.4 | 0.6 | 0.8|
| \( |S(A)| \)  | 0.8 | 0.6 | 0.4 | 0.2 | 0   | 0.2 | 0.4 | 0.6 | 0.8| 1  |
| \( |S(B)| \)  | 1   | 0.8 | 0.6 | 0.4 | 0.2 | 0   | 0.2 | 0.4 | 0.6 | 0.8|

The results of distances with the change of \( \delta \) value and the change of the value of score function are provided in Fig 1 and 2. When setting the index of hesitance to 0, the difference between 2 PFSs is only about the values of membership and non-membership. The operation is carried to simplify the process of calculation. Of course, any parameter can be modified in this way, as long as the mass of each parameter satisfies the properties of pythagorean fuzzy set. The difference of classic Euclidean distance and the proposed is going to be discussed. All the results generated by the two methods are shown in Table 1. The values obtained by score function are also shown in Table 2 with the consistent increase of the value of parameter \( \delta \) which indicates a corresponding change in the distance of IFSs.

It can be easily noted that the distance generated by classic Euclidean’s method is a fixed value. However, this phenomenon is conflicting with the intuitive judgements. With the change of the \( \delta \), if the
results have not changed, then the results produced should be regarded irrational and counter-intuitive, which also indicates that the classic Euclidean distance is not sensitive to tiny variation among different PFSs. It may not accurately reveal potential changes in the relation of PFSs when they vary correspondingly but not symmetrically, which is crucial about whether a method of measuring distance can make reasonable judgements. Besides, when coming to analyze the variation of the variable $\delta$, the rationality and correctness of the proposed method is highlighted. With the increase of parameter $\delta$, when $\delta$ is in the range 0.1 to 0.5, the level of vagueness in these PFSs is getting higher, so the distance between two PFSs reaches its zenith. That is because the proposed method can better detect the changes of different PFSs and present the difference in the results produced, which improves the sensitivity of distance measure compared to classic Euclidean distance. The trend of the changes of PFSs can also be shown by the variation of the value of score function. While the parameter gradually reached the value of 0.5, both of the

| PFSs | Case1 | Case2 |
|------|-------|-------|
| $A_i$ | $\{ (x_1, 0.55, 0.45), (x_2, 0.63, 0.55) \}$ | $\{ (x_1, 0.55, 0.45), (x_2, 0.63, 0.55) \}$ |
| $B_i$ | $\{ (x_1, 0.39, 0.50), (x_2, 0.50, 0.59) \}$ | $\{ (x_1, 0.40, 0.51), (x_2, 0.51, 0.60) \}$ |

| PFSs | Case3 | Case4 |
|------|-------|-------|
| $A_i$ | $\{ (x_1, 0.71, 0.63), (x_2, 0.63, 0.55) \}$ | $\{ (x_1, 0.71, 0.63), (x_2, 0.63, 0.55) \}$ |
| $B_i$ | $\{ (x_1, 0.63, 0.63), (x_2, 0.71, 0.63) \}$ | $\{ (x_1, 0.77, 0.55), (x_2, 0.55, 0.45) \}$ |

| PFSs | Case5 | Case6 |
|------|-------|-------|
| $A_i$ | $\{ (x_1, 0.55, 0.45), (x_2, 0.63, 0.55) \}$ | $\{ (x_1, 0.30, 0.20), (x_2, 0.40, 0.30) \}$ |
| $B_i$ | $\{ (x_1, 0.67, 0.39), (x_2, 0.74, 0.50) \}$ | $\{ (x_1, 0.15, 0.25), (x_2, 0.25, 0.35) \}$ |

| PFSs | Case7 | Case8 |
|------|-------|-------|
| $A_i$ | $\{ (x_1, 0.30, 0.20), (x_2, 0.40, 0.30) \}$ | $\{ (x_1, 0.50, 0.40), (x_2, 0.40, 0.30) \}$ |
| $B_i$ | $\{ (x_1, 0.45, 0.15), (x_2, 0.55, 0.25) \}$ | $\{ (x_1, 0.40, 0.40), (x_2, 0.50, 0.40) \}$ |
Table 4: Distances generated by different methods

| Methods     | Case1 | Case2 | Case3 | Case4 | Case5 | Case6 | Case7 | Case8 |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(\tilde{d}_{Hm}\) | 0.1464 | 0.1351 | 0.1170 | 0.1350 | 0.1161 | 0.1500 | 0.1500 | 0.1500 |
| \(\tilde{d}_{Eu}\) | 0.1298 | 0.1180 | 0.1110 | 0.1261 | 0.1009 | 0.1323 | 0.1323 | 0.1414 |
| \(\tilde{D}_{Hm}\) | 0.1500 | 0.1400 | 0.1500 | 0.1500 | 0.1500 | 0.0825 | 0.1275 | 0.1250 |
| \(\tilde{D}_{Eu}\) | 0.1323 | 0.1217 | 0.1414 | 0.1414 | 0.1323 | 0.0740 | 0.1186 | 0.1170 |
| \(\tilde{D}_{C}(\beta = 1)\) | 0.1500 | 0.1400 | 0.1500 | 0.1500 | 0.1500 | 0.0825 | 0.1275 | 0.1250 |
| \(\tilde{D}_{C}(\beta = 2)\) | 0.1323 | 0.1217 | 0.1414 | 0.1414 | 0.1323 | 0.0740 | 0.1186 | 0.1170 |
| \(\tilde{D}_{N}\) | 0.1801 | 0.1756 | 0.1417 | 0.1517 | 0.1578 | 0.0746 | 0.0829 | 0.0982 |

absolute values of the score function of PFSs \(A\) and \(B\) are becoming smaller, which indicates the level of ambiguity of both PFSs is also getting higher. Because the PFSs are getting more uncertain, the distance between them is becoming bigger to indicate their distance can not be properly measured. Therefore, the proposed method reflects this factor in the results and can better embody underlying relationship between or among PFSs.

**Example 2:** In this part, some numerical examples are given to illustrate the superiority of proposed method compared to other previous methods. Let \(A_i\) and \(B_i\) be two PFSs in the finite universe of discourse \(X = \{x_1, x_2\}\) and all of the docimastic cases are presented in Table 3. And the distances generated by different methods are shown in Table 4. When coming to analyze the examples given in Table 3, it can be easily concluded that \(A_1 = A_2 = A_5\), but \(B_1 \neq B_2 \neq B_5\); \(A_3 = A_4\), but \(B_3 \neq B_4\); \(A_6 = A_7\), but \(B_6 \neq B_7\). However, the results produced by different methods show a completely different figure among provided methods of measuring distances. After inspecting all of results, it can be summarized that:

1. After comparing all of the method of measuring distances between different PFSs, every method can accurately detect the differences under the condition in Case1 and Case2.
2. However, the results produced by \(\tilde{D}_{Hm}, \tilde{D}_{Eu}, \tilde{D}_{C}(\beta = 1)\) and \(\tilde{D}_{C}(\beta = 2)\) seem not satisfying. They produce the exactly the same results in completely different cases, which is counter-intuitive and irrational.
3. Besides, when checking the results produced by \(\tilde{d}_{Hm}\) and \(\tilde{d}_{Eu}\) in Case6 and Case7, they are also unreasonable due to the same results in different cases.
4. More than what have been mentioned in point 3, \(\tilde{d}_{Hm}\) also dose not perform well in Case7 and Case8. Because the method produces another counter-intuitive results under different conditions.
5. All in all, when comparing all the cases mentioned above, it is very easy to distinguish that the distance of the proposed method is more sensitive to the changes in PFSs and significantly indicates the discrimination level of different PFSs, which is much more efficient than any other methods. It is more
6. A more important point should be noticed is that the proposed method performs well under any cases given in the table 3 in producing distances between PFSs, which other ones generate irrational and counter-intuitive results in the same cases.

Table 5: Properties satisfied by different methods

| Methods       | Non-degeneracy | Symmetry Triangle inequality | Boundness |
|---------------|----------------|-----------------------------|-----------|
| $d_{Hm}$      | ✓              | ✓                           | ✓         |
| $d_{Eu}$      | ✓              | ✓                           | ✓         |
| $D_{Hm}$      | ✓              | ✓                           | ✓         |
| $D_{Eu}$      | ✓              | ✓                           | ✓         |
| $D_C(\beta = 1)$ | ✓          | ✓                           | ✓         |
| $D_C(\beta = 2)$ | ✓          | ✓                           | ✓         |
| $D_N$         | ✓              | ✓                           | ✓         |

Notice: It can be inferred from Table 5 that except for the normalized Chen’s distance measure $D_C$, other methods of measuring distances like $d_{Hm}$, $d_{Eu}$, $D_{Hm}$, $D_{Eu}$, $D_N$ satisfy all the properties which the methods of distance measure are supposed to have. Moreover, the reason for the proposed method can generate more rational and intuitive results is that the proposed method considers discrepancies in PFSs and enlarges their influences in producing distances, which plays a significant role in manifesting differences. As a result, the proposed method is more acceptable and conforms to actual situations.

Example 3: Suppose there are three elements in the form of PFSs within the finite universe of discourse $A = \{a_1, a_2, a_3\}$. Then, we consider an application on the task of pattern recognition and the patterns $C = \{C_1, C_2, C_3, C_4\}$ expressed by PFSs are present in Table 6. The aim is to classify a unrecognized pattern which is given as:

$C_{un1} = \{\langle x_1, 0.45, 0.39\rangle, \langle x_2, 0.67, 0.22\rangle, \langle x_3, 0.71, 0.23\rangle\}$

$C_{un2} = \{\langle x_1, 0.56, 0.26\rangle, \langle x_2, 0.29, 0.22\rangle, \langle x_3, 0.87, 0.14\rangle\}$

Table 6: Symptoms expressed in the form of PFS of patients in example 3

| Patterns | Attribute1 | Attribute2 | Attribute3 |
|----------|------------|------------|------------|
| $C_1$    | $\langle x_1, 0.70, 0.20\rangle$ | $\langle x_2, 0.65, 0.12\rangle$ | $\langle x_3, 0.28, 0.60\rangle$ |
| $C_2$    | $\langle x_1, 0.05, 0.70\rangle$ | $\langle x_2, 0.55, 0.74\rangle$ | $\langle x_3, 0.68, 0.12\rangle$ |
| $C_3$    | $\langle x_1, 0.82, 0.19\rangle$ | $\langle x_2, 0.74, 0.19\rangle$ | $\langle x_3, 0.24, 0.73\rangle$ |
| $C_4$    | $\langle x_1, 0.58, 0.11\rangle$ | $\langle x_2, 0.67, 0.30\rangle$ | $\langle x_3, 0.38, 0.43\rangle$ |

After calculation, we can distinguish the differences between unrecognized patterns and reference patterns and the results are provided in Table 7 and 8. Through the analysis of tabular data, it is evident that the disparities between the first, second, and third patterns compared to the pattern to be identified...
Table 7: Distances generated by proposed method in example 3 for \( C_{un1} \)

| Pattern | \( x_1 \)  | \( x_2 \)  | \( x_3 \)  | Total   |
|---------|------------|------------|------------|---------|
| \( C_1 \) | 0.2049    | 0.0043    | 0.5322    | 0.4971  |
| \( C_2 \) | 0.3004    | 0.4499    | 0.0067    | 0.5023  |
| \( C_3 \) | 0.3813    | 0.0180    | 0.7042    | 0.5881  |
| \( C_4 \) | 0.0707    | 0.0040    | 0.3112    | 0.4343  |

Table 8: Distances generated by proposed method in example 3 for \( C_{un2} \)

| Pattern | \( x_1 \)  | \( x_2 \)  | \( x_3 \)  | Total   |
|---------|------------|------------|------------|---------|
| \( C_1 \) | 0.0691    | 0.1788    | 0.7478    | 0.5761  |
| \( C_2 \) | 0.5080    | 0.5697    | 0.1033    | 0.6274  |
| \( C_3 \) | 0.2041    | 0.3140    | 0.8518    | 0.6823  |
| \( C_4 \) | 0.0072    | 0.2771    | 0.5409    | 0.4977  |

are greater than those of the fourth pattern. In comparison to the reference pattern of the fourth type, the differences between the two types of unrecognized patterns are smaller. It can be concluded that the two patterns to be identified should belong to the reference fourth type.

4.2. Algorithm Designed for Decision-making Problems

**Problem statement:** Assume there are a finite universe of discourse \( X = \{x_1, x_2, x_3, ..., x_n\} \) and existing medical patterns \( P = \{P_1, P_2, P_3, ..., P_k\} \) consisting of \( n \) elements in the form of PFSs, expressed as \( P_j = \{(x_i, P_{ij}^Y(x_i), P_{ij}^N(x_i))\} \) \((1 \leq j \leq k)\) within the finite universe of discourse \( X \). Several examples \( E = \{E_1, E_2, E_3, ..., E_r\} \) which is composed of \( r \) samples is given to be recognized and testify the correctness of the new algorithm. All of the elements in example \( E \) is denoted as the form of PFS and the whole example is written as \( E_u = \{(x_i, E_{uY}(x_i), E_{uN}(x_i))\} \) \((1 \leq u \leq r)\). In sum, what are expected to be achieved is to decide or classify every element in example \( E_u \) whether belongs to the pattern \( P_j \). The algorithm is designed as:

**Step 1:** For every element in \( E_u \), the proposed method of measuring distances is utilized to produce the distance between \( P_j \) and \( E_u \).

\[
D_N(P_j, E_u) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( \tilde{\bar{m}}_i M(\tilde{\bar{m}}_i M)^T \right)}
\]

\[
P_j E_u Y = P_j^Y(x_i) + E_u^Y(x_i)
\]

\[
P_j E_u N = P_j^N(x_i) + E_u^N(x_i)
\]

\[
P_j E_u H = P_j^H(x_i) + E_u^H(x_i)
\]

\[
\tilde{\bar{m}}_i = (P_j^2 Y(x_i) - E_u^2 Y(x_i), P_j^2 N(x_i) - E_u^2 N(x_i), P_j^2 H(x_i) - E_u^2 H(x_i))
\]
\[
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
Y & N & H
\end{pmatrix}
\]

\[
Y = \frac{(P_j^2 Y(x)) + E_2^2 Y(x))}{(P_j^2 Y(x)) + E_2^2 Y(x) + P_j^2 N(x) + E_2^2 N(x)}
\]

\[
N = \frac{(P_j^2 N(x)) + E_2^2 N(x)}{(P_j^2 Y(x)) + E_2^2 Y(x) + P_j^2 N(x) + E_2^2 N(x)}
\]

\[
H = \sqrt{1 - Y^2 - N^2}
\]

**Step 2:** After calculating the distance of every pair of \(P_j\) and \(E_u\), the smallest value of the distances between two PFSs \(P_j\) and \(E_u\) is selected, which is written as:

\[
\bar{D}_N^{\text{chosen}} = \min_{1 \leq u \leq r} \bar{D}_N(P_j, E_u)
\]

**Step 3:** According to the results generated by **Step 2**, an element is classified into a pattern \(P_{\beta}\), which is written as

\[
\beta = \arg \min_{1 \leq u \leq r} \{\bar{D}_N(P_j, E_u)\}
\]

\(E_u \leftarrow P_{\beta}\)

In order to make the process of classifying more straightforward, a flow chart is offered as follows.

Except for that, the corresponding pseudocode is given in Algorithm 1.

**Algorithm 1** The details of the proposed algorithm

**Input:** The sets of every pattern \(P = \{P_1, P_2, P_3, ..., P_k\}\)
The sets of every sample \(E = \{E_1, E_2, E_3, ..., E_r\}\)

**Output:** The results of classification of samples \(E_u\)

**for** \(j = 1; j \leq k\) **do**

--- **for** \(u = 1; u \leq r\) **do**

----- Generate the distance \(\bar{D}_N(P_j, E_u)\) between different PFSs by using the proposed method

--- **end** /* Step 1 */

Choose the minimum value of \(\bar{D}_N(P_j, E_u)\) as the final distance /* Step 2 */

Classify the tested sample \(E_u\) into the corresponding pattern \(P_j\) /* Step 3 */

**end**

To clarify the specific process of the algorithm, an example is offered to illustrate details in handling data. Assume there are three medical patterns \(P_1, P_2\) and \(P_3\) which are expressed in the form of PFS in the finite universe of discourse \(X = \{x_1, x_2, x_3\}\) and the details of the three medical patterns are written as follows:

\(P_1 = \{\langle x_1, 0.4, 0.7 \rangle, \langle x_2, 0.5, 0.6 \rangle, \langle x_2, 0.7, 0.4 \rangle\}\)

\(P_2 = \{\langle x_1, 0.6, 0.7 \rangle, \langle x_2, 0.4, 0.5 \rangle, \langle x_2, 0.9, 0.1 \rangle\}\)

\(P_3 = \{\langle x_1, 0.3, 0.6 \rangle, \langle x_2, 0.7, 0.3 \rangle, \langle x_2, 0.2, 0.8 \rangle\}\)

Besides, two medical samples \(S_1\) and \(S_2\) are expressed in the form of PFS in the finite universe of discourse and defined as:

\(S_1 = \{\langle x_1, 0.6, 0.5 \rangle, \langle x_2, 0.3, 0.8 \rangle, \langle x_2, 0.2, 0.7 \rangle\}\)

\(S_2 = \{\langle x_1, 0.2, 0.6 \rangle, \langle x_2, 0.7, 0.6 \rangle, \langle x_2, 0.8, 0.3 \rangle\}\)
Table 9: Symptoms expressed in the form of PFS of patients in application on COVID-19 Recognition

| Patients | Symptom1     | Symptom2     | Symptom3     | Symptom4     | Symptom5     |
|----------|--------------|--------------|--------------|--------------|--------------|
| $V_1$    | $\langle a_1, 0.53, 0.70 \rangle$ | $\langle a_2, 0.63, 0.45 \rangle$ | $\langle a_3, 0.45, 0.56 \rangle$ | $\langle a_4, 0.42, 0.67 \rangle$ | $\langle a_5, 0.55, 0.61 \rangle$ |
| $V_2$    | $\langle a_1, 0.92, 0.15 \rangle$ | $\langle a_2, 0.86, 0.23 \rangle$ | $\langle a_3, 0.75, 0.26 \rangle$ | $\langle a_4, 0.77, 0.13 \rangle$ | $\langle a_5, 0.93, 0.20 \rangle$ |
| $V_3$    | $\langle a_1, 0.67, 0.47 \rangle$ | $\langle a_2, 0.89, 0.13 \rangle$ | $\langle a_3, 0.68, 0.30 \rangle$ | $\langle a_4, 0.81, 0.23 \rangle$ | $\langle a_5, 0.88, 0.29 \rangle$ |
| $V_4$    | $\langle a_1, 0.66, 0.45 \rangle$ | $\langle a_2, 0.75, 0.23 \rangle$ | $\langle a_3, 0.56, 0.87 \rangle$ | $\langle a_4, 0.57, 0.62 \rangle$ | $\langle a_5, 0.61, 0.43 \rangle$ |

What should be done is to categorize sample $S_1$ and $S_2$ into coincident classes. According to the procedure introduced above, the process of achieving the final outcomes is written as:

1. Generate the distance among $P_1$, $P_2$, $P_3$ and $S_1$, $S_2$ by using the proposed method respectively. The results are written as:
   
   $D_N(P_1, S_1) = 0.4470$, $D_N(P_2, S_1) = 0.5134$, $D_N(P_3, S_1) = 0.4511$

   $D_N(P_1, S_2) = 0.2076$, $D_N(P_2, S_2) = 0.3294$, $D_N(P_3, S_2) = 0.5127$

2. Choose the smallest value of the distances generated by the proposed method. And according to the regulation, the process is written as:
   
   $D_N^{\text{chosen}} = D_N(P_1, S_1) = 0.4470$

   $D_N^{\text{chosen}} = D_N(P_1, S_2) = 0.2076$

3. Classify the specific samples into corresponding patterns:

   $S_1 \leftarrow P_1$, $S_2 \leftarrow P_1$

This is the full process of the designed algorithm.

4.3. Application to COVID-19 Recognition

In early 2020, the world was engulfed by a vast pandemic triggered by a novel strain of Coronavirus Disease-2019 (COVID-19), leading to numerous fatalities due to respiratory failure and various complications. This virus, known for its heightened contagiousness, has a more severe impact on health compared to other viral infections. As a result, promptly detecting those infected and ensuring their isolation became crucial in managing and mitigating the spread of the outbreak. Suppose there are four patients, namely $V_1$, $V_2$, $V_3$ and $V_4$, who are denoted as $V = \{V_1, V_2, V_3, V_4\}$. Moreover, five attributes are symptoms given as $\text{fever}$, $\text{cough}$, $\text{fatigue}$, $\text{trouble breathing}$ and $\text{sore throat}$, which are denoted as $A = \{a_1, a_2, a_3, a_4, a_5\}$. The symptoms expressed in the form of PFS of patients in application on COVID-19 Recognition are provided in Table 9.

Suppose a scenario where data pertaining to an individual exhibiting symptoms closely associated with the condition is denoted by PFS $V_{ies}$ within A which is given as:

$V_{ies} = \{\langle x_1, 0.89, 0.18 \rangle, \langle x_2, 0.91, 0.17 \rangle, \langle x_3, 0.92, 0.10 \rangle, \langle x_4, 0.97, 0.05 \rangle, \langle x_5, 0.89, 0.19 \rangle\}$
The objective of this task is to rank the likelihood of illness for four patients, that is, if a patient’s condition is similar to the given examples of illness, then their probability of being ill is higher; if there is a significant difference from the examples of illness, then it is inclined to consider that the patient’s probability of being ill is lower. Specifically, judgements generated by different methods and distances generated by proposed method in application on COVID-19 recognition are provided in Table 10 and 11.

The method of comparison in the table involves scoring the probability of each patient being ill, with a higher score indicating a higher tendency towards illness. However, the method proposed in this article differs slightly; we measure the difference between the patient and the standard example. If the distance measurement obtained is small, it is considered that the patient has a higher probability of being ill; if the distance measurement is large, it is considered that the patient has a lower probability of being ill. Besides, the visualized results of distances generated by proposed method in application on COVID-19 Recognition is provided in Fig 3.

Overall, the solution proposed in this paper and the majority of the methods it was compared against achieved consistent results. Specifically, it was determined that Patient Two has the highest probability of being ill, followed by Patients Three, Four, and One, with a sequential decrease in the predicted probability of illness. This indicates the effectiveness of the distance measurement formula for Pythagorean
fuzzy sets introduced in this study, as well as the validity of the newly designed algorithm.

### 4.4. Applications to Symptom Diagnosis

In this section, the medical pattern recognition problems are provided to further verify the validity of the proposed distance and corresponding algorithm.

**Application 1** : Suppose there are four patients, namely Ragu, Mathi, Velu and Karthi, who are denoted as $P = \{P_1, P_2, P_3, P_4\}$. Besides, five attributes which are symptoms in fact are introduced as \textit{Headache}, \textit{Acidity}, \textit{Burning eyes}, \textit{Back pain} and \textit{Depression}, which are denoted as $A = \{a_1, a_2, a_3, a_4, a_5\}$. More than that, diagnostic results are divided into five categories, \textit{Stress}, \textit{Ulcer}, \textit{Vision problem}, \textit{Spinal problem} and \textit{Blood pressure}, which are also denoted as $D = \{D_1, D_2, D_3, D_4, D_5\}$. Specifically, symptoms expressed in the form of PFS of patient and diagnoses are provided in Table 12 and 13. With the help of the concept of PFS, all of the examples are presented in the form of PFS, which is effective in judging proper patterns. And every details of the patients are shown in Table 14. Additionally, the diagnoses in the form of PFS are also presented in Table 15.

By using the proposed method of measuring distance between PFSs, all the results generated by the proposed method are shown in table 16. According the standards raised in the algorithm and other methods, all of final judgements are presented in table 17. Besides, the visualized results of distances generated by proposed method in application 1 on Symptom Diagnosis is provided in Fig 4. After checking the results generated by the proposed methods, $P_1$ has the least value of $D_N = 0.1930; P_2$
Table 13: Symptoms expressed in the form of PFS of diagnoses in application 1

| Diagnoses | Symptom1          | Symptom2          | Symptom3          | Symptom4          | Symptom5          |
|-----------|-------------------|-------------------|-------------------|-------------------|-------------------|
| D_1       | ⟨a_1, 0.30, 0.00⟩ | ⟨a_2, 0.30, 0.50⟩ | ⟨a_3, 0.20, 0.80⟩ | ⟨a_4, 0.70, 0.30⟩ | ⟨a_5, 0.20, 0.60⟩ |
| D_2       | ⟨a_1, 0.00, 0.60⟩ | ⟨a_2, 0.20, 0.60⟩ | ⟨a_3, 0.00, 0.80⟩ | ⟨a_4, 0.50, 0.00⟩ | ⟨a_5, 0.10, 0.80⟩ |
| D_3       | ⟨a_1, 0.20, 0.20⟩ | ⟨a_2, 0.50, 0.20⟩ | ⟨a_3, 0.10, 0.70⟩ | ⟨a_4, 0.20, 0.60⟩ | ⟨a_5, 0.20, 0.80⟩ |
| D_4       | ⟨a_1, 0.20, 0.80⟩ | ⟨a_2, 0.10, 0.50⟩ | ⟨a_3, 0.70, 0.00⟩ | ⟨a_4, 0.10, 0.70⟩ | ⟨a_5, 0.20, 0.70⟩ |
| D_5       | ⟨a_1, 0.20, 0.80⟩ | ⟨a_2, 0.00, 0.70⟩ | ⟨a_3, 0.20, 0.80⟩ | ⟨a_4, 0.10, 0.80⟩ | ⟨a_5, 0.80, 0.10⟩ |

Table 14: Distances generated by proposed method in application 1

| Patients | Distances |
|----------|-----------|
|          | D_1  | D_2  | D_3  | D_4  | D_5  |
| P_1      | 0.1930| 0.3979| 0.2707| 0.6404| 0.6942|
| P_2      | 0.4230| 0.3648| 0.2882| 0.1181| 0.3191|
| P_3      | 0.3558| 0.4174| 0.0996| 0.4498| 0.5463|
| P_4      | 0.1658| 0.2727| 0.3270| 0.4664| 0.5125|

has the least value of $\tilde{D}_N = 0.1181$; $P_3$ has the least value of $\tilde{D}_N = 0.0996$; $P_4$ has the least value of $\tilde{D}_N = 0.1658$. All in all, the results produced by Sumuel and Rajakumar’s method and Xiao’s method conform to the ones produced by the proposed method, which proves that the accuracy and validity of the proposed method in real application and the feasibility of the proposed method in practical usage.

**Application 2**: Suppose there are four patients, namely Al, Bob, Joe and Ted, who are denoted as $P = \{P_1, P_2, P_3, P_4\}$. Besides, five attributes which are symptoms in fact are introduced as Temperature, Headache, Stomach pain, Cough and Chest pain, which are denoted as $A = \{a_1, a_2, a_3, a_4, a_5\}$. More than that, diagnostic results are divided into five categories, Viral fever, Malaria, Typhoid, Stomach problem and Chest, which are also denoted as $D = \{D_1, D_2, D_3, D_4, D_5\}$. Specifically, symptoms expressed in the form of PFS of patient and diagnoses are provided in Table 16 and 17. With the help of the concept of PFS, all of the examples are presented in the form of PFS, which is effective in judging proper patterns.

![Figure 4](image.png)
Table 15: Judgements generated by different methods in application 1

| Methods                      | Diagnoses          |
|------------------------------|--------------------|
| Smuel and Rajakumar [55]     | Stress, Spinal problem, Vision problem, Stress |
| Xiao [56]                    | Stress, Spinal problem, Vision problem, Stress |
| Proposed method              | Stress, Spinal problem, Vision problem, Stress |

Table 16: Symptoms expressed in the form of PFS of patients in application 2

| Patients | Symptom1     | Symptom2     | Symptom3     | Symptom4     | Symptom5     |
|----------|--------------|--------------|--------------|--------------|--------------|
| P1       | (a₁, 0.80, 0.10) | (a₂, 0.60, 0.10) | (a₃, 0.20, 0.80) | (a₄, 0.60, 0.10) | (a₅, 0.10, 0.60) |
| P2       | (a₁, 0.00, 0.80) | (a₂, 0.40, 0.40) | (a₃, 0.60, 0.10) | (a₄, 0.10, 0.70) | (a₅, 0.10, 0.80) |
| P3       | (a₁, 0.80, 0.10) | (a₂, 0.80, 0.10) | (a₃, 0.00, 0.60) | (a₄, 0.20, 0.70) | (a₅, 0.00, 0.50) |
| P4       | (a₁, 0.60, 0.10) | (a₂, 0.50, 0.40) | (a₃, 0.30, 0.40) | (a₄, 0.70, 0.20) | (a₅, 0.30, 0.40) |

And every details of the patients are shown in Table [18]. Additionally, the diagnoses in the form of PFS are also presented in Table [19] and the visualized results of distances generated by proposed method in application 2 on Symptom Diagnosis is provided in Fig [5].

By using the proposed method of measuring distance between PFSs, all the results generated by the proposed method are shown in table 8. According the standards raised in the algorithm, all of final judgements are presented in table 9. It can be concluded that Al (P₁) is diagnosed that he suffers from Malaria (D₂), Bob (P₂) is diagnosed that he suffers from Stomach problem (D₄), Joe (P₃) is diagnosed that she suffers from Typhoid (D₃) and Ted (P₄) is diagnosed that he suffers from Viral fever (D₁). And all the judgements produced by other methods and the results generated by the proposed method are placed together in table 9 to verify the correctness of the latter one. In the chart, what is the most obvious is that all of the methods have reach an agreement that Al (P₁) is diagnosed with Malaria (D₂) and Bob (P₂) is diagnosed with Stomach problem (D₄), which is satisfying and rational. However, when coming to judge the situation of patient Joe (P₃), the diagnosis vary among different methods. Five of them give judgements that Joe (P₃) is suffering from Typhoid (D₃) and only one of them considers that

![Figure 5: The visualized results of distances generated by proposed method in application 2 on Symptom Diagnosis](image)
Table 17: Symptoms expressed in the form of PFS of diagnoses in application 2

| Diagnoses | Symptom1   | Symptom2   | Symptom3   | Symptom4   | Symptom5   |
|-----------|------------|------------|------------|------------|------------|
| D1        | \langle a_1, 0.40, 0.00 \rangle | \langle a_2, 0.30, 0.50 \rangle | \langle a_3, 0.10, 0.70 \rangle | \langle a_4, 0.40, 0.30 \rangle | \langle a_5, 0.10, 0.70 \rangle |
| D2        | \langle a_1, 0.70, 0.00 \rangle | \langle a_2, 0.20, 0.60 \rangle | \langle a_3, 0.00, 0.90 \rangle | \langle a_4, 0.70, 0.00 \rangle | \langle a_5, 0.10, 0.80 \rangle |
| D3        | \langle a_1, 0.30, 0.30 \rangle | \langle a_2, 0.60, 0.10 \rangle | \langle a_3, 0.20, 0.70 \rangle | \langle a_4, 0.20, 0.60 \rangle | \langle a_5, 0.10, 0.90 \rangle |
| D4        | \langle a_1, 0.10, 0.70 \rangle | \langle a_2, 0.20, 0.40 \rangle | \langle a_3, 0.80, 0.00 \rangle | \langle a_4, 0.20, 0.70 \rangle | \langle a_5, 0.20, 0.70 \rangle |
| D5        | \langle a_1, 0.10, 0.80 \rangle | \langle a_2, 0.00, 0.80 \rangle | \langle a_3, 0.20, 0.80 \rangle | \langle a_4, 0.20, 0.80 \rangle | \langle a_5, 0.80, 0.10 \rangle |

Table 18: Distances generated by proposed method in application 2

| Patients | D1    | D2    | D3    | D4    | D5    |
|----------|-------|-------|-------|-------|-------|
| P1       | 0.2698| 0.2235| 0.2446| 0.5878| 0.6324|
| P2       | 0.2486| 0.2143| 0.2979| 0.0673| 0.5051|
| P3       | 0.2605| 0.4448| 0.1324| 0.4853| 0.5975|
| P4       | 0.1727| 0.2075| 0.3118| 0.5085| 0.5256|

this patient is suffering from Malaria (D2). Additionally, with respect to Ted (P4), four of the methods choose Viral fever (D1) as the diagnosis of Ted (P4) and the other two methods regard that Ted (P4) is suffering from Malaria (D2). All of the results demonstrate that it is very difficult to diagnose Ted (P4), because there may be a potential relationship between Viral fever (D1) and Malaria (D2) leading to a conflicting stage when comparing the results of different methods. Anyway, the results produced by Szmidt et al.’s method, Mondal et al.’s method and Wei et al.’s method conform to the ones produced by the proposed method, which proves that the accuracy and validity of the proposed method in real application and the feasibility of the proposed method in practical usage.

5. Conclusion

In the theory of pythagorean fuzzy sets, how to accurately and properly measure the distance between PFSs is still an open issue, which may lead to chaos in pattern recognition. To solve this problem,
in this paper, a completely new method of distance measure between different PFSs is proposed, which satisfy all of the properties required by the axiom of distance measurement. The main advantage of the proposed method is that it considers the index of hesitance and distribute its mass to membership and non-membership in a reasonable way to strengthen the role which membership and non-membership plays in generating distances between PFSs. Besides, the reduction in the index of hesitance is also very important in alleviating the vagueness in PFSs and helpful in recognizing corresponding targets or patterns. All in all, the proposed method produces much more rational results than previous methods, which is more closer to actual situations and conforms to intuitive judgements. Moreover, the algorithm which is developed on the basis of the proposed method offers a promising and reliable solution to address the recognition problems in pattern recognition applications. In future work, we will consider designing a PFS distance based on a learnable matrix, which can better utilize trainable data to set the values of matrix elements. This will enhance the distance measurement performance, allowing for improved accuracy in a broader range of applications.

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