A SIMPLE TEST FOR CONSTANT CORRELATION MATRIX

Malay Bhattacharyya and Rajesh Kasa
Indian Institute of Management, Bangalore *

Abstract

We propose a simple procedure to test for changes in correlation matrix at an unknown point in time. This test requires constant expectations and variances, but only mild assumptions on the serial dependence structure. We test for a breakdown in correlation structure using eigenvalue decomposition. We derive the asymptotic distribution under the null hypothesis and apply the test to stock returns. We compute the power of our test and compare it with the power of other known tests.

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1 Introduction

An important problem in statistical modeling of financial time series is to analyze and detect structural changes in the relationship among stock returns. Pearson Correlation is one of the widely used metric in financial risk management to indicate the relationship among various returns. Long-term risk-averse investors tend to hold portfolios of assets whose returns are not positively correlated for diversification benefits. However, there is compelling empirical evidence that the correlation structure among returns of the assets cannot be assumed to be constant over time, see, e.g. [Forbes and Rigobon, 2002], [Krishnan et al., 2009], [Wied et al., 2012] and [Wied, 2017]. In particular, in periods of financial crisis, correlations among stock returns increase, a phenomenon which is sometimes referred to as diversification meltdown. In this article, we detect and test for these structural changes by considering the constancy of correlation matrix. [Wied et al., 2012] has shown that testing for changes of correlation can be more powerful than testing for changes in covariance, especially when there is more than one change point. However, one of the drawbacks of these existing tests ([Wied et al., 2012] and [Wied, 2017]) is that pairwise comparison of correlation matrix is not a scalable solution when there are multiple stocks

*Address correspondence to Malay Bhattacharyya, Professor, Decision Sciences and Information Systems, IIM Bangalore, India; e-mail:malayb@iimb.ac.in
involved in the portfolio. Instead of the vector of successively calculated pairwise correlation coefficients, we consider the largest eigenvalue of sample correlation matrix and derive its limiting distribution, based on assumptions from [Wied et al., 2012] and some proof ideas from [Anderson, 1963].

Outline  First, we start by showing how the assumptions of [Wied et al., 2012] lead to convergence (in distribution) of the time series to asymptotic normality. Next, we use this normality condition and results from [Anderson, 1963] to derive the asymptotic distribution of eigenvalues of the sample correlation matrix. In section 3 we test our hypothesis on stock returns on two-stock and three-stock portfolios. In section 4 we compare the power of our tests with those of [Wied et al., 2012]. Towards the end, in section 5 we discuss the merits and demerits of our approach.

2 Test statistic

First, we derive the asymptotic normality of the time series for a simple two-stock portfolio and a multi-stock portfolio, using results from [Wied et al., 2012] and [Wied, 2017] respectively. Then, we define the test statistic and derive its distribution using results from [Anderson, 1963]. Towards the end of this section, we give a simplified expression for the asymptotic distribution of test statistic for the cases of two-stock and three-stock portfolios.

2.1 Two-stock portfolio

Let \((X_t, Y_t), t = 0, \pm 1, \pm 2, \ldots\) be a sequence of bivariate random vectors with finite \((4 + \delta)\)th absolute moments for some \(\delta > 0\). We want to test whether the correlation between \(X_t\) and \(Y_t\) is constant over time in the observation period.

\[
H_0 : \rho_t = \rho_0 \quad \forall t \in 1, \ldots, T \quad \text{vs} \quad H_1 : \exists t \in \{1, \ldots, T - 1\} : \rho_t \neq \rho_{t+1}
\]

Where \(\rho_k = \frac{\sum_i^k (X_t - \bar{X}_k)(Y_t - \bar{Y}_k)}{\sqrt{\sum_i^k (X_t - \bar{X}_k)^2} \sqrt{\sum_i^k (Y_t - \bar{Y}_k)^2}}\)

and \(\bar{X}_k = \frac{\sum_i^k X_t}{k}, \bar{Y}_k = \frac{\sum_i^k Y_t}{k}\)

The following technical assumptions are required for the limiting null distribution. These assumptions are exactly the same ones used in [Wied et al., 2012].
(A1) For \( U_t := (X_t^2 - E(X_t^2), Y_t^2 - E(Y_t^2), X_t - E(X_t), Y_t - E(Y_t), X_t Y_t - E(X_t Y_t))' \) and \( S_j := \sum_{t=1}^j U_t \), we have
\[
\lim_{T \to \infty} E\left(\frac{S_T S_T'}{T}\right) =: D_1
\]
Essentially, \( D_1 = D_1' = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T \sum_{u=1}^T \)
\[
\begin{pmatrix}
\text{Cov}(X_t^2, X_u^2) & \text{Cov}(X_t^2, Y_u^2) & \text{Cov}(X_t^2, X_u) & \text{Cov}(X_t^2, Y_u) & \text{Cov}(X_t^2, X_u Y_u) \\
\text{Cov}(X_t^2, Y_u^2) & \text{Cov}(Y_t^2, X_u) & \text{Cov}(Y_t^2, Y_u) & \text{Cov}(Y_t^2, X_u Y_u) \\
\text{Cov}(X_t^2, X_u) & \text{Cov}(X_t^2, Y_u) & \text{Cov}(X_t^2, Y_u) & \text{Cov}(X_t^2, X_u Y_u) \\
\text{Cov}(X_t^2, Y_u) & \text{Cov}(Y_t^2, X_u Y_u) & \text{Cov}(Y_t^2, X_u Y_u) & \text{Cov}(Y_t^2, Y_u Y_u) \\
\text{Cov}(X_t^2, X_u Y_u) & \text{Cov}(X_t^2, Y_u Y_u) & \text{Cov}(X_t^2, Y_u Y_u) & \text{Cov}(X_t^2, Y_u Y_u)
\end{pmatrix}
\]

(A2) The \( r \)th absolute moments of the components of \( U_t \) are uniformly bounded for some \( r > 2 \).

(A3) The vector \( (X_t, Y_t) \) is \( L^2 \)-NED (near-epoch dependent) with size \(-r/(r-2)\), where \( r \) is from (A2), and constants \( (c_t) \), \( t \in \mathbb{Z} \), on sequence \( (V_t) \), \( t \in \mathbb{Z} \), which is \( \phi \)-mixing of size \( \phi^* := -r/(r - 2) \), i.e.,
\[
\left\| \left((X_t, Y_t) - E((X_t, Y_t)|\sigma(V_{t-m}, \ldots, V_{t+m})) \right) \right\| \leq c_t \nu_m
\]
with \( \nu_m \to 0 \), such that
\[
c_t \leq 2\|U_t\|
\]
Here, \( V_t = \frac{1}{\sqrt{T}} U_t^{***} \),
where \( U_t^{***} = \left(X_t^2 - \bar{X}_T^2, Y_t^2 - \bar{Y}_T^2, X_t - \bar{X}_T, Y_t - \bar{Y}_T, X_t Y_t - \bar{X}\bar{Y}_T\right)' \).

(A4) The moments \( E(X_t^2), E(Y_t^2), E(X_t), E(Y_t), E(X_t Y_t) \) are uniformly bounded and “almost” constant, in the sense that the deviations \( d_t \) from the respective constants satisfy
\[
\lim_{T \to \infty} \frac{\sum_{t=1}^T |d_t|}{T} = \lim_{T \to \infty} \frac{\sum_{t=1}^T d_t^2}{T} = 0
\]
The above condition allows for weak stationarity; i.e., \( d_t = 0 \) for all \( t \).

(A5) For a bounded function \( g \) that is not constant and that can be approximated by step functions such that the function \( \int_0^z g(u)du - z \int_0^1 g(u)du \) is different from 0 for at least \( z \in [0, 1] \)
\[
E(X_t^2) = a_2 + a_2 \frac{g(z)}{\sqrt{T}}
\]
\[
E(Y_t^2) = a_3 + a_3 \frac{g(z)}{\sqrt{T}}
\]
\[ E(X_t Y_t) = a_1 + a_1 \frac{g(\frac{t}{T})}{\sqrt{T}} \]

while \( E(X_t) \) and \( E(Y_t) \) remain constant.

Also, from Assumption (A5), it can be seen that

\[
\rho_k = \frac{\sum_1^k (X_t - \mu_x)(Y_t - \mu_y)}{\sqrt{(\sum_1^k (X_t - \mu_x)^2)}\sqrt{(\sum_1^k (Y_t - \mu_y)^2)}
\]

Here, \( \mu_x = E(X_t) \) and \( \mu_y = E(Y_t) \) as \( t \to \infty \)

Using the above assumptions (A1-A5), it is shown that (as lemma A.1. of [Wied et al., 2012])

on \( D([\epsilon, 1], \mathbb{R}^5) \), for arbitrary \( \epsilon > 0 \), we have

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{\tau(z)} (X_t - \mu_x) \xrightarrow{d} (D_{1/2}^{1/2})_3 W_5(.)
\]

Where, \( \tau(z) = [2 + z(T - 2)], z \in [0, 1] \), \( (D_{1/2}^{1/2})_3 \) is the third row of the \( D_{1/2}^{1/2} \) matrix, \( D_1 \) is a 5 x 5 matrix given in assumption (A1), and \( W_5(.) \) is 5-dimensional Brownian motion.

From the above lemma, say at sufficiently long time \( t_1 = \tau(z_1) \),

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{\tau(z_1)} (X_t - \mu_x) \xrightarrow{d} (D_{1/2}^{1/2})_3 W_5(z_1) \quad (a)
\]

Similarly, at time \( (t_1 + 1 = \tau(z_1 + \frac{1}{T-2})) \),

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{(t_1+1)} (X_t - \mu_x) \xrightarrow{d} (D_{1/2}^{1/2})_3 W_5(z_1 + \frac{1}{T-2}) \quad (b)
\]

From (a) and (b), we can see that \( \frac{X_{t_1+1-\mu_x}}{\sqrt{T}} \xrightarrow{d} (D_{1/2}^{1/2})_3 N_5(0, \tau_2^{-1}) \), where \( N_5(.) \) is a 5 dimensional normal vector. Essentially, for any large \( t \), \( X_t \) (similarly \( Y_t \)) is a linear combination of an iid Normal distributions, as we are taking the difference of a Brownian motion at two different times.

Because \( X_t \) and \( Y_t \) are linear combinations of independent normal distributions, the distribution of the vector \( (X_t, Y_t) \) converges to a bivariate normal distribution with a some correlation \( \Sigma_t \). We can standardize the vector \( (X_t, Y_t) \) by subtracting the mean vector \( \mu \) and dividing by the standard deviation. We assume \( E(X_T, Y_T) \) to a good estimate of \( \mu \); similarly, we assume that the long term covariance matrix \( \Sigma_T \) to be a good estimate of \( \Sigma \).
2.2 Multi-stock portfolio

For a multi-stock portfolio, with \( p \) stocks, say \((X_1, X_2, \ldots, X_p)\), our \( U_t \) will be a \((2p + \frac{p(p-1)}{2})\) vector, defined as follows,

\[
U_t = \begin{pmatrix}
X_{1,t} - E(X_{1,t}) \\
X_{2,t} - E(X_{2,t}) \\
\vdots \\
X_{p,t} - E(X_{p,t}) \\
X_{1,t}X_{2,t} - E(X_{1,t})E(X_{2,t}) \\
X_{1,t}X_{3,t} - E(X_{1,t})E(X_{3,t}) \\
\vdots \\
X_{p-1,t}X_{p,t} - E(X_{p-1,t})E(X_{p,t})
\end{pmatrix}
\]

Again, similar to the two-stock portfolio case, \( S_j := \sum_{t=1}^j U_t \), we have

\[
\lim_{T \to \infty} E\left( \frac{S_T S_T'}{T} \right) =: D_1
\]

Under assumptions A1-A6 of [Wied, 2017], it is shown in the appendix of [Wied, 2017] that at sufficiently long time \( t_1 = \tau(z_1) \),

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{\tau(z_1)} (X_{i,t} - \mu_{i,t}) \xrightarrow{d} (D_1^{1/2})_{p+i}W_{2p+\frac{p(p-1)}{2}}(z_1) \quad (c)
\]

where, \((D_1^{1/2})_{p+i}\) is the \((p+i)\)th row of the matrix \( D_1^{1/2} \), and, \( W_{2p+\frac{p(p-1)}{2}}(z_1) \) is a \((2p + \frac{p(p-1)}{2})\) dimensional Brownian motion. From (c), using a similar argument that was used in the two-stock portfolio case, it can be seen that \( X_{i,t} \) is a linear combination of normal distributions. Since the individual \( X_{i,t} \)'s are linear combinations of independent normal distributions, the vector \((X_1, X_2, \ldots, X_p)\) follows a \( p \)-variate normal distribution.

2.3 Asymptotic distribution of eigenvalues

Asymptotic distribution of the eigenvalues of sample covariance matrix is derived in equations (2.1 - 2.13) of [Anderson, 1963]. The theorem and results are as follows:

Say \( x_\alpha \) is a \( p \)-dimensional vector distributed according to \( N(0, \Sigma) \) and \( A = \sum_{\alpha} x_\alpha x_\alpha' \). From multivariate central limit theorem, \((1/n^{1/2})(A - n\Sigma)\) is asymptotically normally distributed. Let \((d_1, \ldots, d_p)\) be the eigenvalues of \( A \), where \( d_1 \geq d_2 \geq \ldots \geq d_p \), and \((\delta_1, \ldots, \delta_p)\) be the eigenvalues of \( \Sigma \), where \( \delta_1 \geq \delta_2 \ldots \geq \delta_p \).
First, we give the results for the simple case where all the characteristic roots of \( \Sigma \) are equal, that is, \( \delta_1 = \cdots = \delta_p = \lambda \), say. Define \( H := \sqrt{t}(D_t - \lambda I) \), where \( D_t \) is a \( p \times p \) diagonal matrix with \( (d_1, \ldots, d_p) \) as diagonal elements, and \( I \) is the \( p \times p \) identity matrix. Clearly, \( H \) is also a \( p \times p \) diagonal matrix with elements say, \( h_1, h_2, h_3, \ldots, h_p \). Asymptotic distribution of \( H \) is given by

\[
f(h_1, h_2, h_3, \ldots, h_p; \lambda, p) = \frac{K(p)}{\lambda^{p+1}} e^{-\frac{1}{4\lambda} \sum_{i<j} h_i h_j} \prod_{i=1}^{p} \Gamma\left[\frac{1}{2}(p + 1 - i)\right]
\]

where,

\[
\frac{1}{K(p)} = 2 \frac{p(p + 3)}{4} \prod_{i=1}^{p} \Gamma\left[\frac{1}{2}(p + 1 - i)\right]
\]

If the largest eigenvalue, \( \lambda \), has a multiplicity of \( q \) instead of \( p \), where \( q \leq p \), then test statistic is slightly modified as \( H = \sqrt{t}(D_t - \lambda I) \), where \( D_t \) is the \( q \times q \) sample eigenvalue diagonal matrix and \( I \) is the \( q \times q \) identity matrix. The distribution of \( H \) is same as the one given above in (1) with \( p \) replaced by \( q \). For example, when the maximum eigenmultiplicity is just 1 (the case when all the eigenvalues are unique), our equation (1) simplifies to

\[
f(h_i; \lambda, 1) = \frac{1}{2\sqrt{\pi} \lambda} e^{-\frac{h_i^2}{4\lambda}}
\]

### 2.3.1 Simplification for two-stock portfolio

We can simplify the above distribution (1) for the simple case of two-stock portfolio, i.e. \( p = 2 \). Let \( X_\alpha = (X_t, Y_t) \), where \( X_t, Y_t \) are standardized returns. Also, let \( A_t = \sum_1^t X_\alpha X_\alpha' \). The correlation matrix, \( A_T \), at time \( T \) can be represented as

\[
\begin{pmatrix}
1 & \rho_T \\
\rho_T & 1
\end{pmatrix}
\]

whose eigenvalues are \( 1 + \rho_T \) and \( 1 - \rho_T \) and the corresponding eigenvectors are \( \frac{1}{\sqrt{2}}(1, 1) \) and \( \frac{1}{\sqrt{2}}(1, -1) \). Let \( D_T = 1 + \rho_T \) be its largest eigenvalue.

The sample Correlation matrix, \( A_t \), at time \( t \) is

\[
\begin{pmatrix}
1 & \rho_t \\
\rho_t & 1
\end{pmatrix}
\]

Also, let \( D_t \) be the largest eigenvalue of \( A_t \)
The test statistic is

\[ h_t = \sqrt{t}(D_t - D_T) = \sqrt{t}((1 + \rho_t) - (1 + \rho_T)) \]

where, \( D_t, D_T \) are the maximum eigenvalues at times \( t \) and \( T \) respectively.

From equation (2), we can see that the asymptotic distribution of \( h_t \) is

\[ h_t = \sqrt{t}((1 + \rho_t) - (1 + \rho_T)) \xrightarrow{d} \frac{1}{2\sqrt{\pi}} \frac{1}{(1 + \rho_T)^{3/2}} e^{-\frac{s^2}{4(1+\rho_T)^2}} \]

### 2.3.2 Simplification for three-stock portfolio

In this subsection, we derive the expression for asymptotic distribution (1) for a three-stock portfolio, i.e. \( p = 3 \). Let \( X_\alpha = (X_t, Y_t, Z_t) \), where \( X_t, Y_t, Z_t \) are standardized returns. Also, let \( A_t = \sum_1^t X_\alpha X_\alpha' \). The correlation matrix, \( A_T \), at time \( T \) can be represented as

\[
\begin{pmatrix}
1 & \rho_{1,T} & \rho_{2,T} \\
\rho_{1,T} & 1 & \rho_{3,T} \\
\rho_{2,T} & \rho_{3,T} & 1
\end{pmatrix}
\]

The characteristic equation for the above matrix is

\[ (1 - \lambda)^3 - (1 - \lambda)(\rho_{1,T}^2 + \rho_{2,T}^2 + \rho_{3,T}^2) + 2\rho_{1,T}\rho_{2,T}\rho_{3,T} = 0 \quad (3) \]

Solving for \( \lambda \) in equation (3), gives the eigenvalues of the correlation matrix \( A_T \). Let \( \lambda^* = 1 - \lambda \), then equation (3) becomes

\[ (\lambda^*)^3 + p\lambda^* = q \quad (4) \]

where, \( p = -(\rho_{1,T}^2 + \rho_{2,T}^2 + \rho_{3,T}^2) \), \( q = -2\rho_{1,T}\rho_{2,T}\rho_{3,T} \)

Further, let \( Q = \frac{q}{3} \), \( R = \frac{p}{3} \) and \( D = Q^3 + R^2 \).

From AM-GM inequality, we can see that \( D \leq 0 \). If \( D \leq 0 \), [Weisstein, 2018]'s equations (57-73) give the real roots to equation (4) as

\[
\begin{align*}
D_{1,T} &= 2\sqrt{-Q} \cos\left(\frac{\theta}{3}\right) \\
D_{2,T} &= 2\sqrt{-Q} \cos\left(\frac{2\theta + 2\pi}{3}\right) \\
D_{3,T} &= 2\sqrt{-Q} \cos\left(\frac{4\theta + 2\pi}{3}\right)
\end{align*}
\]

where \( \theta = \cos^{-1}\left(\frac{R}{\sqrt{-Q^3}}\right) \)

Similarly, we can find the roots \( D_{1,t}, D_{2,t}, D_{3,t} \) corresponding to the correlation matrix, \( A_t \), at time \( t \). Let \( D_{1,T} \) and \( D_{1,t} \) be the smallest roots corresponding to \( A_T \) and \( A_t \) respectively. As this is the case of multiplicity being 1, from equation (2), we can see that the asymptotic distribution of \( h_t(= \sqrt{t}(D_{1,T} - D_{1,t})) \) is

\[ h_t \xrightarrow{d} \frac{1}{2\sqrt{\pi}} \frac{1}{2\sqrt{-Q} \cos\left(\frac{\theta}{3}\right)} e^{-\frac{s^2}{4(1-2\sqrt{-Q} \cos\left(\frac{\theta}{3}\right))^2}} \]
3 Testing on stock returns

First, we test our hypothesis on the same stocks considered in [Wied et al., 2012]; As seen from Figure 1, the highest value of the $|h_t|$ is 3.79, which coincides with the collapse of Lehman Brothers around 18 September, 2008, is slightly greater than the critical value for 90% confidence level.

Next, we test our hypothesis using a three-stock portfolio for two cases. In case (i), we have two American Indices (SNP 500 and DOWJones) and a German Index (DAX). In case (ii), we have indices from US, Germany and Japan (SNP, DAX and Nikkei respectively). As expected, it can be seen from Figure 2, the correlation structure has been disturbed more in case(ii), where all the indices are from different countries, as against case(i) where 2 indices are from the same country (US).
Figure 2: The figure (a), as evident from critical values, does not show considerable breakdown in correlation as against the case when all the stocks are from different countries in figure (b).

4 Local power

First, we derive the distribution of the test statistic $h_t = \sqrt{T}((1 + \rho_i) - (1 + \rho_T))$ for the alternative hypothesis under consideration for cases (1-4) mentioned below, where the correlation changes once at time $t_1$.

1. $\rho_i = 0.5$, $i \leq \frac{T}{2}$ and $\rho_i = 0.7$, $i > \frac{T}{2}$
2. $\rho_i = 0.5$, $i \leq \frac{T}{4}$ and $\rho_i = 0.7$, $i > \frac{T}{4}$
3. $\rho_i = 0.5$, $i \leq \frac{T}{2}$ and $\rho_i = -0.5$, $i > \frac{T}{2}$
4. $\rho_i = 0.5, i \leq \frac{T}{4}$ and $\rho_i = -0.5, i > \frac{T}{4}$

5. $\rho_i = 0.5, i \leq \frac{T}{4}$ and $\rho_i = 0.7, \frac{T}{4} < i < \frac{3T}{4}$ and $\rho_i = 0.5, i \geq \frac{3T}{4}$

Let $t_1$ be the time until which the correlation remains $\rho_1$ and from time $t_1$ to $t_1 + t_2$ the correlation remains $\rho_2$.

So, the correlation matrices for duration $t_1$ and $t_2$ can be written as

$$
\sum_{t=1}^{t_1} x^t x'_t \Bigg/ t_1 = E \begin{pmatrix} d_{1i_1} & 0 \\ 0 & d_{2i_1} \end{pmatrix} E'
$$

$$
\sum_{t=t_1}^{t_1+t_2} x^t x'_t \Bigg/ t_2 = E \begin{pmatrix} d_{1i_2} & 0 \\ 0 & d_{2i_2} \end{pmatrix} E'
$$

Therefore, the correlation matrix at time $t_1 + t_2$ can be written as

$$
\sum_{t=0}^{t_1+t_2} x^t x'_t \Bigg/ t_1 + t_2 = E \begin{pmatrix} (t_1(d_{1i_1} + t_2(d_{1i_2})) & 0 \\ 0 & t_1(d_{1i_1} + t_2(d_{1i_2})) \end{pmatrix} E'
$$

where $EE' = I$

From equation (1), we can get the distribution of $d_{1i_1}$, $d_{1i_2}$, and, hence, the distribution of $d_{1i_1} + d_{1i_2}$.

When the correlation changes only once at $t_1$, the form of our test statistic at $t_1$ is

$$
h_{t=t_1} = \sqrt{t_1} \left( d_{1i_1} - \frac{t_1(d_{1i_1}) + t_2(d_{1i_2})}{t_1 + t_2} \right) = \frac{\sqrt{t_1t_2}}{t_1 + t_2} (d_{1i_1} - d_{1i_2})
$$

Substituting the distributions of $d_{1i_1}$ and $d_{1i_2}$, we have,

$$
\frac{\sqrt{t_1t_2}}{t_1 + t_2} \left( (\rho_1 - \rho_2) + \sqrt{\left( \frac{(1 + \rho_1)^2}{0.5t_1} \right) + \left( \frac{(1 + \rho_2)^2}{0.5t_2} \right)} \right) N_s
$$

Where, $N_s$ is standard normal distribution

E.g. For case (1), we have $t_1, t_2 = 100$, $\rho_1 = 0.5$ and $\rho_2 = 0.7$

We can derive a similar expression for the case (v) where the correlation changes at two times, $t_1$ and $t_1 + t_2$, in the total duration $(t_1 + t_2 + t_3)$ under consideration.

The test statistic at $t_1$ is

$$
h_{t=t_1} = \sqrt{t_1} \left( d_{1i_1} - \frac{t_1(d_{1i_1}) + t_2(d_{1i_2}) + t_3(d_{1i_3})}{t_1 + t_2 + t_3} \right)
$$

Substituting the distributions of $d_{1i_1}$, $d_{1i_2}$, and, $d_{1i_3}$, we have the distribution of $h_{t=t_1}$ as
\[
\frac{\sqrt{t_1}}{t_1 + t_2 + t_3} \left( t_2 \left( \rho_1 - \rho_2 \right) + \left( \sqrt{\left( \frac{1 + \rho_1}{0.5t_1} \right)^2 + \left( \frac{1 + \rho_2}{0.5t_2} \right)^2} \right) N_s \right) \\
+ t_3 \left( \rho_1 - \rho_3 \right) + \left( \sqrt{\left( \frac{1 + \rho_1}{0.5t_1} \right)^2 + \left( \frac{1 + \rho_3}{0.5t_3} \right)^2} \right) N_s \right)
\]

We checked the power of our test for the conditions (i-v), in which variances constantly remain 1 and correlations change, and compared our results against that of [Wied et al., 2012] and [Aue et al., 2009].

| Alternative | T  | 1 | 2 | 3 | 4 | 5 |
|-------------|----|---|---|---|---|---|
| (a) Our test |    |   |   |   |   |   |
| 200         | 0.014 | 0.044 | 0.7 | 0.7 | 0.044 |   |
| 500         | 0.036 | 0.07 | 0.998 | 0.992 | 0.06 |   |
| 1000        | 0.085 | 0.14 | 1 | 1 | 0.08 |   |
| 2000        | 0.21  | 0.29 | 1 | 1 | 0.15 |   |
| (b) [Wied 2012]'s test |    |   |   |   |   |   |
| 200         | 0.309 | 0.255 | 0.953 | 0.88 | 0.101 |   |
| 500         | 0.255 | 0.488 | 0.996 | 0.989 | 0.207 |   |
| 1000        | 0.83  | 0.733 | 0.998 | 0.998 | 0.422 |   |
| 2000        | 0.967 | 0.928 | 1 | 0.999 | 0.75 |   |
| (c) [Aue 2009]'s test |    |   |   |   |   |   |
| 200         | 0.241 | 0.159 | 0.977 | 0.842 | 0.083 |   |
| 500         | 0.586 | 0.4 | 1 | 0.998 | 0.163 |   |
| 1000        | 0.858 | 0.69 | 1 | 1 | 0.284 |   |
| 2000        | 0.98  | 0.929 | 1 | 1 | 0.609 |   |

From Table 1, it is seen that, in general, the power of our test is lower compared to that of [Wied et al., 2012] and [Aue et al., 2009]. Also, in particular, for criteria 3 and 4, the power of our test is slightly higher - indicating that our test can detect large changes in the correlation structure more effectively. However, it should be noted that a correlation change of about (0.3 - 0.35) is common during the times of financial crisis as indicated in Figure 3. So, while the power of our test is comparatively lower, it still can be used to test the breakdown in correlation structure for practical purposes.
5 Conclusions

In this paper, we have proposed a new fluctuation test for constant correlation matrix under a multivariate setting in which the change points need not be specified apriori. Our approach is more simplified because it allows us to work with more standard operations like eigendecomposition and normal distributions against the pairwise comparison and Brownian bridges of [Wied et al., 2012] and [Wied, 2017].
Since we are dealing with only the largest eigenvalue, the power of our test is less compared to the pairwise comparison of the entire correlation matrix of [Wied et al., 2012] and [Wied, 2017]. Nevertheless, our test is simpler for practical application and is effectively able detect changes in correlation matrix, as indicated in Section 3. Moreover, our method can be generalized to detect any changes in covariance matrix structure, as a complementary technique to [Aue et al., 2009]. One drawback of our test, which is also shared by [Wied et al., 2012] and [Wied, 2017], is the assumption of finite fourth moments and constant expectations and variances. Another drawback, which is shared by most of correlation based tests, is the low power when there are multiple change points in the duration under consideration, as illustrated in [Cabrieto et al., 2018]. Hence, it may be worthwhile to consider techniques like prefiltering and/or other transformations to overcome these drawbacks.

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