On lower limits and equivalences for distribution tails of randomly stopped sums

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For a distribution $F^\tau$ of a random sum $S_\tau = \xi_1 + \cdots + \xi_\tau$ of i.i.d. random variables with a common distribution $F$ on the half-line $[0, \infty)$, we study the limits of the ratios of tails $F^\tau(x)/F(x)$ as $x \to \infty$, where $\tau$ is a counting random variable which does not depend on $\{\xi_n\}_{n \geq 1}$. We also consider applications of the results obtained to random walks, compound Poisson distributions, infinitely divisible laws, and subcritical branching processes.

Keywords: convolution tail; convolution equivalence; lower limit; randomly stopped sums; subexponential distribution

1. Introduction

Let $\xi, \xi_1, \xi_2, \ldots$, be independent identically distributed non-negative random variables. We assume that their common distribution $F$ on the half-line $[0, \infty)$ has an unbounded support, that is, $F(x) \equiv F(x, \infty) > 0$ for all $x$. Put $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n = 1, 2, \ldots$.

Let $\tau$ be a counting random variable which does not depend on $\{\xi_n\}_{n \geq 1}$ and which has finite mean. Denote by $F^\tau$ the distribution of a randomly stopped sum $S_\tau = \xi_1 + \cdots + \xi_\tau$.

In this paper, we discuss how the tail behavior of $F^\tau$ relates to that of $F$ and, in particular, under what conditions

$$
\liminf_{x \to \infty} \frac{F^\tau(x)}{F(x)} = E\tau. \tag{1}
$$

Relations on lower limits of ratios of tails were first discussed by Rudin [21]. Theorem 2* of that paper states (for an integer $p$) the following.

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Theorem 1. Suppose there exists a positive $p \in [1, \infty)$ such that $E \xi^p = \infty$, but $E \tau^p < \infty$. Then (1) holds.

Rudin’s studies were motivated by Chover, Ney and Wainger [7] who considered, in particular, the problem of existence of a limit for the ratio

$$\frac{F^{*\tau}(x)}{F(x)} \quad \text{as } x \to \infty. \quad (2)$$

From Theorem 1, it follows that if $F$ and $\tau$ satisfy its conditions and if a limit of (2) exists, then that limit must equal $E \tau$.

Rudin proved Theorem 1 via probability generating function techniques. Below, we give an alternative and more direct proof of Theorem 1 in the case of any positive $p$ (i.e., not necessarily integer). Our method is based on truncation arguments; in this way, we propose a general scheme (see Theorem 4 below) which may also be applied to distributions having all moments finite.

The condition $E \xi^p = \infty$ rules out many distributions of interest in, say, the theory of subexponential distributions. For example, log-normal and Weibull-type distributions have all moments finite. Our first result presents a natural condition on a stopping time $\tau$ guaranteeing relation (1) for the whole class of heavy-tailed distributions.

Recall that a random variable $\xi$ has a light-tailed distribution $F$ on $[0, \infty)$ if $E e^{\gamma \xi} < \infty$ with some $\gamma > 0$. Otherwise, $F$ is called a heavy-tailed distribution; this happens if and only if $E e^{\gamma \xi} = \infty$ for all $\gamma > 0$.

Theorem 2. Let $F$ be a heavy-tailed distribution and $\tau$ have a light-tailed distribution. Then (1) holds.

The proof of Theorem 2 is based on a new technical tool (see Lemma 2) and significantly differs from the proof of Theorem 1 in Foss and Korshunov [15], where the particular case $\tau = 2$ was considered. Theorem 2 is restricted to the case of light-tailed $\tau$, but here, we extend Rudin’s result to the class of all heavy-tailed distributions. The reasons for the restriction to $E e^{\gamma \tau} < \infty$ come from the proof of Theorem 2, but are, in fact, rather natural: the tail of $\tau$ should be lighter than the tail of any heavy-tailed distribution. Indeed, if $\xi_1 \geq 1$, then $F^{*\tau}(x) \geq P\{\tau > x\}$. This shows that the tail of $F^{*\tau}$ is at least as heavy as that of $\tau$. Note that in Theorem 1, in some sense, the tail of $F$ is heavier than the tail of $\tau$.

Theorem 2 may be applied in various areas where randomly stopped sums appear; see Sections 8–11 (random walks, compound Poisson distributions, infinitely divisible laws and branching processes) and, for instance, Kalashnikov [17] for further examples.

For any distribution on $[0, \infty)$, let

$$\varphi(\gamma) = \int_0^\infty e^{\gamma x} F(dx) \in (0, \infty], \quad \gamma \in \mathbb{R},$$
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and

\[ \hat{\gamma} = \sup\{\gamma : \varphi(\gamma) < \infty\} \in [0, \infty]. \]

Note that the moment generating function \( \varphi(\gamma) \) is increasing and continuous in the interval \((-\infty, \hat{\gamma})\) and that \( \varphi(\hat{\gamma}) = \lim_{\gamma \to \hat{\gamma}} \varphi(\gamma) \in [1, \infty] \). The following result was proven in Foss and Korshunov \cite{15}, Theorem 3. Let

\[ \frac{F*F(x)}{F(x)} \to c \quad \text{as } x \to \infty, \]

where \( c \in (0, \infty] \). Then, necessarily, \( c = 2\varphi(\hat{\gamma}) \). We state now a generalization to \( \tau \)-fold convolution.

**Theorem 3.** Let \( \varphi(\hat{\gamma}) < \infty \) and \( E(\varphi(\hat{\gamma}) + \varepsilon)^\tau < \infty \) for some \( \varepsilon > 0 \). Assume that

\[ \frac{F*\tau(x)}{F(x)} \to c \quad \text{as } x \to \infty, \]

where \( c \in (0, \infty] \). Then \( c = E(\tau \varphi^{\tau-1}(\hat{\gamma})) \).

For (comments on) earlier partial results in the case \( \tau = 2 \), see, for example, Chover, Ney and Wainger \cite{6, 7}, Cline \cite{8}, Embrechts and Goldie \cite{10}, Foss and Korshunov \cite{15}, Pakes \cite{19}, Rogozin \cite{20}, Teugels \cite{23} and further references therein. The proof of Theorem 3 follows from Lemmas 3 and 4 in Section 7.

**2. Preliminary result**

We start with the following result.

**Theorem 4.** Assume that there exists a non-decreasing concave function \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[ Ee^{h(\xi)} < \infty \quad \text{and} \quad E\xi e^{h(\xi)} = \infty. \quad (3) \]

For any \( n \geq 1 \), put \( A_n = E e^{h(\xi_1 + \cdots + \xi_n)} \). Assume that \( F \) is heavy-tailed and that

\[ E\tau A_{\tau-1} < \infty. \quad (4) \]

Then, for any light-tailed distribution \( G \) on \([0, \infty)\),

\[ \liminf_{x \to \infty} \frac{G*F*\tau(x)}{F(x)} = E\tau. \quad (5) \]

By considering \( G \) concentrated at 0, we get the following.
Corollary 1. In the conditions of Theorem 4, (1) holds.

In order to prove Theorem 4, first we restate Theorem 1* of Rudin [21] (in Lemma 1 below) in terms of probability distributions and stopping times.

Lemma 1. For any distribution $F$ on $[0, \infty)$ with unbounded support and any counting random variable $\tau$,
\[
\liminf_{x \to \infty} \frac{F^{*\tau}(x)}{F(x)} \geq E\tau.
\]

Proof. For any two distributions $F_1$ and $F_2$ on $[0, \infty)$ with unbounded supports,
\[
F_1 * F_2(x) \geq (F_1 \times F_2)((x, \infty) \times [0, x]) + (F_1 \times F_2)([0, x] \times (x, \infty)) \\
= F_1(x) + F_2(x) \quad \text{as } x \to \infty.
\]

By induction arguments, this implies that, for any $n \geq 1$,
\[
\liminf_{x \to \infty} \frac{F^{*n}(x)}{F(x)} \geq n.
\]

Applying Fatou’s lemma to the representation
\[
\frac{F^{*\tau}(x)}{F(x)} = \sum_{n=1}^{\infty} \mathbb{P}\{\tau = n\} \frac{F^{*n}(x)}{F(x)},
\]
completes the proof. \qed

Proof of Theorem 4. It follows from Lemma 1 that it is sufficient to prove the following inequality:
\[
\liminf_{x \to \infty} \frac{G * F^{*\tau}(x)}{F(x)} \leq E\tau.
\]

Assume the contrary, that is, that there exist $\delta > 0$ and $x_0$ such that
\[
G * F^{*\tau}(x) \geq (E\tau + \delta)F(x) \quad \text{for all } x > x_0. \quad (6)
\]

For any positive $b > 0$, consider a concave function
\[
h_b(x) \equiv \min\{h(x), bx\}, \quad (7)
\]
which is non-negative because $h \geq 0$. Since $F$ is heavy-tailed, $h(x) = o(x)$ as $x \to \infty$. Therefore, for any fixed $b$, there exists $x_1$ such that $h_b(x) = h(x)$ for all $x > x_1$. Hence, by condition (3),
\[
\mathbb{E}e^{h_b(\xi)} < \infty \quad \text{and} \quad \mathbb{E}\xi e^{h_b(\xi)} = \infty. \quad (8)
\]
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For any $x$, we have the convergence $h_b(x) \downarrow 0$ as $b \downarrow 0$. Then, for any fixed $n$,$$
A_{n,b} \equiv E e^{h_b(\xi_1+\cdots+\xi_n)} \downarrow 1 \quad \text{as } b \downarrow 0.
$$

This and condition (4) together imply that there exists $b$ such that
$$
E \tau A_{\tau-1,b} \leq E \tau + \delta/8. \tag{9}
$$

Let $\eta$ be a random variable with distribution $G$ which does not depend on $\{\xi_n\}_{n \geq 1}$ and $\tau$. Since $G$ is light-tailed,
$$
E \eta e^{h_b(\eta)} < \infty. \tag{10}
$$

In addition, we may choose $b > 0$ sufficiently small that
$$
E e^{h_b(\eta)} (E \tau + \delta/8) \leq E \tau + \delta/4. \tag{11}
$$

For any real $a$ and $t$, put $a^\lceil t \rceil = \min\{a,t\}$. Then
$$
\frac{E(\eta + \xi_{1}^\lceil t \rceil + \cdots + \xi_{n}^\lceil t \rceil)}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} e^{h_b(\eta + \xi_{1}^\lceil t \rceil + \cdots + \xi_{n}^\lceil t \rceil)} = \sum_{n=1}^{\infty} \frac{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\eta + \xi_{1} + \cdots + \xi_{n})}}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} \mathbb{P}\{\tau = n\} + \sum_{n=1}^{\infty} \frac{n \xi_1^\lceil t \rceil e^{h_b(\eta + \xi_{1} + \cdots + \xi_{n})}}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} \mathbb{P}\{\tau = n\}.
$$

By the concavity of the function $h_b$,$$
\sum_{n=1}^{\infty} \frac{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\eta + \xi_{1} + \cdots + \xi_{n})}}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} \mathbb{P}\{\tau = n\} \leq \sum_{n=1}^{\infty} \frac{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\eta + h_b(\xi_1) + \cdots + \xi_{n})}}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} \mathbb{P}\{\tau = n\} = \frac{E \eta e^{h_b(\eta)}}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} EA_{t,b} \to 0 \quad \text{as } t \to \infty,
$$
due to (10), (9) and (8). Again, by the concavity of the function $h_b$,$$
\sum_{n=1}^{\infty} \frac{n \xi_{1}^\lceil t \rceil e^{h_b(\eta + \xi_{1} + \cdots + \xi_{n})}}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} \mathbb{P}\{\tau = n\} \leq \sum_{n=1}^{\infty} \frac{n \xi_{1}^\lceil t \rceil e^{h_b(\eta) + h_b(\xi_1) + \cdots + \xi_{n})}}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} \mathbb{P}\{\tau = n\} = E e^{h_b(\eta)} \sum_{n=1}^{\infty} n A_{n-1,b} \mathbb{P}\{\tau = n\} \leq E \tau + \delta/4,
$$
by (9) and (11). Hence, for sufficiently large $t$,$$
\frac{E(\eta + \xi_{1}^\lceil t \rceil + \cdots + \xi_{n}^\lceil t \rceil) e^{h_b(\eta + \xi_{1} + \cdots + \xi_{n})}}{E_{\xi_{1}^\lceil t \rceil} e^{h_b(\xi_1)}} \leq E \tau + \delta/2. \tag{12}
$$
On the other hand, since \((\eta + \xi_1 + \cdots + \xi_r)^{[t]} \leq \eta + \xi_1^{[t]} + \cdots + \xi_r^{[t]}\),
\[
\frac{E(\eta + \xi_1^{[t]} + \cdots + \xi_r^{[t]})e^{h_0(\eta + \xi_1 + \cdots + \xi_r)}}{E\xi_1^{[t]} e^{h_0(\xi_1)}} \geq \frac{E(\eta + \xi_1 + \cdots + \xi_r)[e^{h_0(\eta + \xi_1 + \cdots + \xi_r)}]}{E\xi_1^{[t]} e^{h_0(\xi_1)}} = \frac{\int_0^\infty x^{[t]} e^{h_0(x)} (G * F^\tau)(dx)}{\int_0^\infty x^{[t]} e^{h_0(x)} F(dx)}.
\]
(13)

The right-hand side, after integration by parts, is equal to
\[
\frac{\int_0^\infty G * F^\tau(x) d(x^{[t]} e^{h_0(x)})}{\int_0^\infty F(x) d(x^{[t]} e^{h_0(x)})}.
\]
Since \(E\xi_1 e^{h_0(\xi_1)} = \infty\), both integrals in this fraction tend to infinity as \(t \to \infty\). For the non-decreasing function \(h_0(x)\), the latter fact and assumption (6) together imply that
\[
\liminf_{t \to \infty} \frac{\int_0^\infty G * F^\tau(x) d(x^{[t]} e^{h_0(x)})}{\int_0^\infty F(x) d(x^{[t]} e^{h_0(x)})} = \liminf_{t \to \infty} \frac{\int_0^\infty G * F^\tau(x) d(x^{[t]} e^{h_0(x)})}{\int_0^\infty F(x) d(x^{[t]} e^{h_0(x)})} \geq E \tau + \delta.
\]
Substituting this into (13), we get a contradiction of (12) for sufficiently large \(t\). The proof is thus complete. \(\square\)

3. Proof of Theorem 1

Take an integer \(k \geq 0\) such that \(p - 1 \leq k < p\). Without loss of generality, we may assume that \(E \xi^k < \infty\) (otherwise, we may consider a smaller \(p\)).

Consider a concave non-decreasing function \(h(x) = (p - 1) \ln x\). Then \(E e^{h(\xi_1)} < \infty\) and \(E \xi_1 e^{h(\xi_1)} = \infty\). Thus,
\[
A_n \equiv E e^{h(\xi_1 + \cdots + \xi_n)} = E(\xi_1 + \cdots + \xi_n)^{p-1} \leq (E(\xi_1 + \cdots + \xi_n)^k)^{(p-1)/k}
\]
since \((p-1)/k \leq 1\). Further,
\[
E(\xi_1 + \cdots + \xi_n)^k = \sum_{i_1, \ldots, i_k=1}^{n} E(\xi_{i_1} \cdots \xi_{i_k}) \leq c n^k,
\]
where
\[
c \equiv \sup_{1 \leq i_1, \ldots, i_k \leq n} E(\xi_{i_1} \cdots \xi_{i_k}) < \infty,
\]
due to the fact that \(E \xi^k < \infty\). Hence, \(A_n \leq c^{(p-1)/k} n^{p-1}\) for all \(n\). Therefore, we get \(E \tau A_{\tau^{-1}} \leq c^{(p-1)/k} E \tau^p < \infty\). All conditions of Theorem 4 are met and the proof is complete.
Proof. Without loss of generality, assume that ε the monotonicity of x ε choose ε h construct a piecewise linear function h \( \in R^+ \rightarrow R^+ \) such that \( Ee^{h(\xi)} \leq 1 + \delta \) and \( E\xi e^{h(\xi)} = \infty \).

In the sequel, we need the following existence result which strengthens a lemma in Rudin [21], page 989; and Lemma 1 in Foss and Korshunov [15]. Fix any \( \delta \in (0, 1] \).

**Lemma 2.** If a random variable \( \xi \geq 0 \) has a heavy-tailed distribution, then there exists a non-decreasing concave function \( h: R^+ \rightarrow R^+ \) such that \( Ee^{h(\xi)} \leq 1 + \delta \) and \( E\xi e^{h(\xi)} = \infty \).

**Proof.** Without loss of generality, assume that \( \xi > 0 \) a.s., that is, that \( F(0) = 1 \). We will construct a piecewise linear function \( h(x) \). For that, we introduce two positive sequences, \( x_n \uparrow \infty \) and \( \varepsilon_n \downarrow 0 \) as \( n \rightarrow \infty \), and let

\[
h(x) = h(x_{n-1}) + \varepsilon_n (x - x_{n-1}) \quad \text{if} \ x \in (x_{n-1}, x_n], \ n \geq 1.
\]

This function is non-decreasing since \( \varepsilon_n > 0 \). Moreover, this function is concave due to the monotonicity of \( \varepsilon_n \).

Put \( x_0 = 0 \) and \( h(0) = 0 \). Since \( \xi \) is heavy-tailed, we can choose \( x_1 \geq 2 \) so that

\[
E\{e^{\xi}; \xi \in (x_0, x_1]\} + e^{\xi_1} F(x_1) > 1 + \delta.
\]

Choose \( \varepsilon_1 > 0 \) so that

\[
E\{e^{\varepsilon_1 \xi}; \xi \in (x_0, x_1]\} + e^{\varepsilon_1 x_1} F(x_1) = e^{h(x_0)} F(0) + \delta/2 = 1 + \delta/2,
\]

which is equivalent to

\[
E\{e^{h(\xi)}; \xi \in (x_0, x_1]\} + e^{h(x_1)} F(x_1) = e^{h(x_0)} F(0) + \delta/2.
\]

By induction, we construct an increasing sequence \( x_n \) and a decreasing sequence \( \varepsilon_n > 0 \) such that \( x_n \geq 2^n \) and

\[
E\{e^{h(\xi)}; \xi \in (x_{n-1}, x_n]\} + e^{h(x_n)} F(x_n) = e^{h(x_{n-1})} F(x_{n-1}) + \delta/2^n
\]

for any \( n \geq 2 \). For \( n = 1 \), this is already done. Make the induction hypothesis for some \( n \geq 2 \). Due to heavy-tailedness, there exists \( x_{n+1} \geq 2^{n+1} \) sufficiently large that

\[
E\{e^{\varepsilon_n (\xi-x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_n (x_{n+1}-x_n)} F(x_{n+1}) > 1 + \delta.
\]

Note that

\[
E\{e^{\varepsilon_{n+1} (\xi-x_{n+1})}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1} (x_{n+1}-x_n)} F(x_{n+1})
\]

as a function of \( \varepsilon_{n+1} \) is continuously decreasing to \( F(x_n) \) as \( \varepsilon_{n+1} \downarrow 0 \). Therefore, we can choose \( \varepsilon_{n+1} \in (0, \varepsilon_n) \) so that

\[
E\{e^{\varepsilon_{n+1} (\xi-x_{n+1})}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1} (x_{n+1}-x_n)} F(x_{n+1}) = F(x_n) + \delta/(2^{n+1} e^{h(x_n)}).
\]
By definition of $h(x)$, this is equivalent to the following equality:

$$
E\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} + e^{h(x_{n+1})}F(x_{n+1}) = e^{h(x_n)}F(x_n) + \delta/2^{n+1}.
$$

Our induction hypothesis now holds with $n+1$ in place of $n$, as required.

Next,

$$
Ee^{h(\xi)} = \sum_{n=0}^{\infty} E\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\}
$$

$$
= \sum_{n=0}^{\infty} (e^{h(x_n)}F(x_n) - e^{h(x_{n+1})}F(x_{n+1}) + \delta/2^{n+1})
$$

$$
= e^{h(x_0)}F(x_0) + \delta/2 = 1 + \delta.
$$

On the other hand, since $x_k \geq 2^k$,

$$
E\{\xi e^{h(\xi)}; \xi > x_n\} = \sum_{k=n}^{\infty} E\{\xi e^{h(\xi)}; \xi \in (x_k, x_{k+1}]\}
$$

$$
\geq 2^n \sum_{k=n}^{\infty} E\{e^{h(\xi)}; \xi \in (x_k, x_{k+1}]\}
$$

$$
\geq 2^n \sum_{k=n}^{\infty} (e^{h(x_k)}F(x_k) - e^{h(x_{k+1})}F(x_{k+1}) + \delta/2^{k+1}).
$$

Then, for any $n$,

$$
E\{\xi e^{h(\xi)}; \xi > x_n\} \geq 2^n (e^{h(x_n)}F(x_n) + \delta/2^n) \geq \delta,
$$

which implies that $E\xi e^{h(\xi)} = \infty$. Also note that, necessarily, $\lim_{n \to \infty} \epsilon_n = 0$; otherwise, $\liminf_{x \to \infty} h(x)/x > 0$ and $\xi$ is light-tailed. The proof of the lemma is thus complete. □

5. Proof of Theorem 2

Since $\tau$ has a light-tailed distribution,

$$
E\tau(1 + \epsilon)^{\tau-1} < \infty
$$

for some sufficiently small $\epsilon > 0$. By Lemma 2, there exists a concave increasing function $h$, $h(0) = 0$, such that $Ee^{h(\xi)} \leq 1 + \epsilon$ and $E\xi e^{h(\xi)} = \infty$. Then, by concavity,

$$
A_n \equiv Ee^{h(\xi_1 + \cdots + \xi_n)} \leq Ee^{h(\xi_1 + \cdots + h(\xi_n))} \leq (1 + \epsilon)^n.
$$

Combining, we get $E\tau A_{\tau-1} < \infty$. All conditions of Theorem 4 are met and the proof is thus complete.
6. Fractional exponential moments

One can go further and obtain various results on lower limits and equivalences for heavy-tailed distributions $F$ which have all finite power moments (e.g., Weibull and log-normal distributions). For instance, we have the following result (see Denisov, Foss and Korshunov [9] for the proof).

**Suppose there exists** $\alpha$, $0 < \alpha < 1$, **such that** $E e^{c\xi^\alpha} = \infty$ **for all** $c > 0$. If $E e^{\delta\tau^\alpha} < \infty$ **for some** $\delta > 0$, **then** (1) holds.

7. Tail equivalence for randomly stopped sums

The following auxiliary lemma compares the tail behavior of the convolution tail and that of the exponentially transformed distribution.

**Lemma 3.** Let the distribution $F$ and the number $\gamma \geq 0$ be such that $\varphi(\gamma) < \infty$. Let the distribution $G$ be the result of the exponential change of measure with parameter $\gamma$, that is, $G(du) = e^{\gamma u}F(du)/\varphi(\gamma)$. Let $\tau$ be any counting random variable such that $E\varphi(\gamma) < \infty$ and let $\nu$ have the distribution $P\{\nu = k\} = \varphi^k(\gamma)P\{\tau = k\}/E\varphi(\gamma)$. Then

$$\liminf_{x \to \infty} \frac{G*\nu(x)}{G(x)} \geq \frac{1}{E\varphi^\gamma(\gamma)} \liminf_{x \to \infty} \frac{F*\tau(x)}{F(x)}$$

and

$$\limsup_{x \to \infty} \frac{G*\nu(x)}{G(x)} \leq \frac{1}{E\varphi^\gamma(\gamma)} \limsup_{x \to \infty} \frac{F*\tau(x)}{F(x)}.$$

**Proof.** Put

$$\hat{c} \equiv \liminf_{x \to \infty} \frac{F*\tau(x)}{F(x)}.$$

By Lemma 1, $\hat{c} \in [E\tau, \infty]$. For any fixed $c \in (0, \hat{c})$, there exists $x_0 > 0$ such that, for any $x > x_0$,

$$F*\tau(x) \geq cF(x). \quad (14)$$

By the total probability law,

$$G*\nu(x) = \sum_{k=1}^{\infty} P\{\nu = k\}G*^k(x)$$

$$= \sum_{k=1}^{\infty} \varphi^k(\gamma)P\{\tau = k\} \int_x^\infty e^{\gamma y}F*^k(dy) \varphi(\gamma)$$

$$= \frac{1}{E\varphi^\gamma(\gamma)} \sum_{k=1}^{\infty} P\{\tau = k\} \int_x^\infty e^{\gamma y}F*^k(dy).$$
Integrating by parts, we obtain
\[
\sum_{k=1}^{\infty} P\{ \tau = k \} \left[ e^{\gamma x} F^*k(x) + \int_x^{\infty} F^*k(y) \, dy \right] = e^{\gamma x} F^*(x) + \int_x^{\infty} F^*(y) \, dy.
\]
Also using (14) we get, for \( x > x_0 \),
\[
G^*(x) \geq c \frac{\varphi(\hat{\gamma})}{E \varphi(\hat{\gamma})} \int_x^{\infty} e^{\gamma y} F(dy) = c \frac{1}{E \varphi^{-1}(\hat{\gamma})} G(x).
\]
Letting \( c \uparrow \hat{c} \), we obtain the first conclusion of the lemma. The proof of the second conclusion follows similarly.

**Lemma 4.** If \( 0 < \hat{\gamma} < \infty \), \( \varphi(\hat{\gamma}) < \infty \) and \( E(\varphi(\hat{\gamma}) + \varepsilon)^\tau < \infty \) for some \( \varepsilon > 0 \), then
\[
\liminf_{x \to \infty} \frac{F^*(x)}{F(x)} \leq E \varphi^{-1}(\hat{\gamma})
\]
and
\[
\limsup_{x \to \infty} \frac{F^*(x)}{F(x)} \geq E \varphi^{-1}(\hat{\gamma}).
\]

**Proof.** We apply the exponential change of measure with parameter \( \hat{\gamma} \) and consider the distribution \( G(du) = e^{\gamma u} F(du)/\varphi(\hat{\gamma}) \) and the stopping time \( \nu \) with the distribution \( P\{ \nu = k \} = \varphi^k(\hat{\gamma})P\{ \tau = k \}/E \varphi^{-1}(\hat{\gamma}) \). From the definition of \( \hat{\gamma} \), the distribution \( G \) is heavy-tailed. The distribution of \( \nu \) is light-tailed because \( E e^{\kappa \nu} < \infty \) with \( \kappa = \ln(\varphi(\hat{\gamma}) + \varepsilon) - \ln \varphi(\hat{\gamma}) > 0 \). Hence,
\[
\limsup_{x \to \infty} \frac{G^*(x)}{G(x)} \geq \liminf_{x \to \infty} \frac{G^*(x)}{G(x)} = E \nu,
\]
by Theorem 2. The result now follows from Lemma 3 with \( \gamma = \hat{\gamma} \), since \( E \nu = E \tau \varphi^{-1}(\hat{\gamma})/E \varphi^{-1}(\hat{\gamma}) \).

**Proof of Theorem 3.** In the case where \( F \) is heavy-tailed, we have \( \hat{\gamma} = 0 \) and \( \varphi(\hat{\gamma}) = 1 \). By Theorem 2, \( c = E \tau \), as required.

In the case \( \hat{\gamma} \in (0, \infty) \) and \( \varphi(\hat{\gamma}) < \infty \), the desired conclusion follows from Lemma 4.

### 8. Supremum of a random walk

Hereafter, we need the notion of subexponential distributions. A distribution \( F \) on \( \mathbb{R}^+ \) is called subexponential if \( F^*F(x) \sim 2F(x) \) as \( x \to \infty \).
Let \( \{ \xi_n \} \) be a sequence of independent random variables with a common distribution \( F \) on \( \mathbb{R} \) and \( \mathbb{E} \xi_1 = -m < 0 \). Put \( S_0 = 0, S_n = \xi_1 + \cdots + \xi_n \). By the strong law of large numbers (SLLN), \( M = \sup_{n \geq 0} S_n \) is finite with probability 1.

Let \( F^I \) be the integrated tail distribution on \( \mathbb{R}^+ \), that is,

\[
F^I(x) \equiv \min\left(1, \int_x^\infty F(y) \, dy\right), \quad x > 0.
\]

It is well known (see, e.g., Asmussen [1], Embrechts, Klüppelberg and Mikosch [12], Embrechts and Veraverbeke [13] and references therein) that if \( F^I \) is subexponential, then

\[
P\{M > x\} \sim \frac{1}{m} F^I(x) \quad \text{as } x \to \infty.
\]

(15)

Korshunov [18] proved the converse: (15) implies subexponentiality of \( F^I \). We now supplement this assertion with the following result.

**Theorem 5.** Let \( F^I \) be long-tailed, that is, \( F^I(x + 1) \sim F^I(x) \) as \( x \to \infty \). If, for some \( c > 0 \),

\[
P\{M > x\} \sim c F^I(x) \quad \text{as } x \to \infty,
\]

then \( c = 1/m \) and \( F^I \) is subexponential.

**Proof.** Consider the defective stopping time

\[
\eta = \inf\{n \geq 1 : S_n > 0\} \leq \infty
\]

and let \( \{ \psi_n \} \) be i.i.d. random variables with common distribution function

\[
G(x) \equiv P\{\psi_n \leq x\} = P\{S_\eta \leq x \mid \eta < \infty\}.
\]

It is well known (see, e.g., Feller [14], Chapter XII) that the distribution of the maximum \( M \) coincides with the distribution of the randomly stopped sum \( \psi_1 + \cdots + \psi_\tau \), where the counting random variable \( \tau \) is independent of the sequence \( \{ \psi_n \} \) and is geometrically distributed with parameter \( p = P\{M > 0\} < 1 \), that is, \( P\{\tau = k\} = (1 - p)p^k \) for \( k = 0, 1, \ldots \). Equivalently,

\[
P\{M \in B\} = G^\tau(B).
\]

It follows from Borovkov [4], Chapter 4, Theorem 10, that if \( F^I \) is long-tailed, then

\[
G(x) \sim \frac{1 - p}{pm} F^I(x).
\]

(16)

The theorem hypothesis then implies that

\[
G^\tau(x) \sim \frac{cpm}{1 - p} G(x) \quad \text{as } x \to \infty.
\]
Therefore, by Theorem 3 with $\hat{\gamma} = 0$, $c = E\tau(1 - p)/pm = 1/m$. It then follows from Korshunov [18] that $F^I$ is subexponential. The proof is now complete. □

9. The compound Poisson distribution

Let $F$ be a distribution on $\mathbb{R}_+$ and $t$ a positive constant. Let $G$ be the compound Poisson distribution

$$G = e^{-t}\sum_{n \geq 0} \frac{t^n}{n!} F^*.$$ 

Considering $\tau$ in Theorem 3 with distribution $P\{\tau = n\} = t^n e^{-t}/n!$, we get the following result.

**Theorem 6.** Let $\varphi(\hat{\gamma}) < \infty$. If, for some $c > 0$, $G(x) \sim cF(x)$ as $x \to \infty$, then $c = te^{\varphi(\hat{\gamma}) - 1}$.

**Corollary 2.** The following statements are equivalent:

(i) $F$ is subexponential;
(ii) $G$ is subexponential;
(iii) $G(x) \sim tF(x)$ as $x \to \infty$;
(iv) $F$ is heavy-tailed and $G(x) \sim cF(x)$ as $x \to \infty$, for some $c > 0$.

**Proof.** Equivalence of (i), (ii) and (iii) was proven in Embrechts, Goldie and Veraverbeke [11], Theorem 3. The implication (iv) $\Rightarrow$ (iii) follows from Theorem 3 with $\hat{\gamma} = 0$. □

Some local aspects of this problem for heavy-tailed distributions were discussed in Asmussen, Foss and Korshunov [2], Theorem 6.

10. Infinitely divisible laws

Let $H$ be an infinitely divisible law on $[0, \infty)$. The Laplace transform of an infinitely divisible law $F$ can be expressed as

$$\int_0^\infty e^{-\lambda x} H(dx) = e^{-a\lambda} - \int_0^\infty (1 - e^{-\lambda x}) \nu(dx).$$

(see, e.g., Feller [14], Chapter XVII). Here, $a \geq 0$ is a constant and the Lévy measure $\nu$ is a Borel measure on $(0, \infty)$ with the properties $\mu = \nu(1, \infty) < \infty$ and $\int_0^1 x \nu(dx) < \infty$. Put $F(B) = \nu(B \cap (1, \infty))/\mu$.

Relations between the tail behavior of measure $H$ and of the corresponding Lévy measure $\nu$ were considered in Embrechts, Goldie and Veraverbeke [11], Pakes [19] and Shimura and Watanabe [22]. The local analog of that result was proven in Asmussen,
Foss and Korshunov [2]. We strengthen the corresponding result of Embrechts, Goldie and Veraverbeke [11] in the following way.

**Theorem 7.** The following assertions are equivalent:

(i) $H$ is subexponential;
(ii) $F$ is subexponential;
(iii) $\mathcal{N}(x) \sim H(x)$ as $x \to \infty$;
(iv) $H$ is heavy-tailed and $\mathcal{N}(x) \sim cH(x)$ as $x \to \infty$, for some $c > 0$.

**Proof.** Equivalence of (i), (ii) and (iii) was proven in Embrechts, Goldie and Veraverbeke [11], Theorem 1.

It remains to prove the implication (iv) $\Rightarrow$ (iii). It is pointed out in Embrechts, Goldie and Veraverbeke [11] that the distribution $H$ admits the representation $H = G * F * \tau$, where $G(x) = O(e^{-\varepsilon x})$ for some $\varepsilon > 0$ and $\tau$ has a Poisson distribution with parameter $\mu$. Since $H$ is heavy-tailed and $G$ is light-tailed, $F$ is necessarily heavy-tailed. Then, by Theorem 4, we get

$$
\liminf_{x \to \infty} \frac{H(x)}{F(x)} = \liminf_{x \to \infty} \frac{G * F * \tau(x)}{F(x)} = E\tau = \mu.
$$

On the other hand, for $x > 1$,

$$
\frac{\mathcal{N}(x)}{F(x)} = \mu \frac{\mathcal{N}(x)}{\mathcal{N}(x)} \to \mu c \quad \text{as } x \to \infty,
$$

by assumption (iv). Hence, $c = 1$. \qed

11. Branching processes

In this section, we consider the limit behavior of subcritical, age-dependent branching processes for which the Malthusian parameter does not exist.

Let $h(z)$ be the particle production generating function of an age-dependent branching process with particle lifetime distribution $F$ (see Athreya and Ney [3], Chapter IV, Harris [16], Chapter VI for background). We take the process to be subcritical, that is, $A = h'(1) < 1$. Let $Z(t)$ denote the number of particles at time $t$. It is known (see, e.g., Athreya and Ney [3], Chapter IV, Section 5, or Chistyakov [5]) that $E Z(t)$ admits the representation

$$
E Z(t) = (1 - A) \sum_{n=1}^{\infty} A^{n-1} F^{*n}(t).
$$

It was proven in Chistyakov [5] for sufficiently small values of $A$ and then in Chover, Ney and Wainger [6, 7] for any $A < 1$ that $E Z(t) \sim F(t)/(1 - A)$ as $t \to \infty$, provided $F$ is
subexponential. The local asymptotics were considered in Asmussen, Foss and Korshunov [2].

Applying Theorem 3 with \( \tau \) geometrically distributed and \( \hat{\gamma} = 0 \), we deduce the following.

**Theorem 8.** Let \( F \) be heavy-tailed, and, for some \( c > 0 \), \( E Z(t) \sim cF(t) \) as \( t \to \infty \). Then \( c = 1/(1 - A) \) and \( F \) is subexponential.

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