COMPLEXITY OF BEZOUT’S THEOREM VI: GEODESICS IN THE CONDITION (NUMBER) METRIC

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Abstract. We introduce a new complexity measure of a path of (problems, solutions) pairs in terms of the length of the path in the condition metric which we define in the article. The measure gives an upper bound for the number of Newton steps sufficient to approximate the path discretely starting from one end and thus produce an approximate zero for the endpoint. This motivates the study of short paths or geodesics in the condition metric.

1. Introduction

In a series of papers we have studied the complexity of solving systems of homogeneous polynomial equations by applying Newton’s method to a homotopy of (system, solution) pairs [6], [7], [8], [9], [10]. The latest word in this direction is [1], [2]. A key ingredient is an estimate of the number of Newton steps by the maximum condition number along the path multiplied by the length of the path of solutions. The main result of this paper is to show that the maximum condition number times the length may be replaced by the integral of the condition number times the length of the tangent vector to the path, Theorem 5. The result suggests using the condition number to define a Riemannian metric on the solution variety and to study the geodesics of this metric. Finding a geodesic is in itself not easy. So we do not immediately obtain a practical algorithm. Rather the study may help to understand in some systematic fashion the geometry of homotopy algorithms, especially those that attempt to avoid ill-conditioned problems. We note that recentering algorithms in linear programming theory may be seen as adaptively avoiding ill-conditioning. [5] compares the central path of linear programming theory to geodesics in an appropriate metric. In [3] we begin studying distances in the condition metric.

2. Definitions and Theorems

We begin by recalling the context. For every positive integer \( l \in \mathbb{N} \), let \( H_l \subseteq \mathbb{C}[X_0, \ldots, X_n] \) be the vector space of all homogeneous polynomials of degree \( l \). For \( d := (d_1, \ldots, d_n) \in \mathbb{N}^n \), let \( \mathcal{H}(d) := \prod_{i=1}^n H_{d_i} \) be the set of all systems \( f := (f_1, \ldots, f_n) \) of homogeneous polynomials of respective degrees \( \deg(f_i) = d_i, \ 1 \leq i \leq n \). So \( f : \mathbb{C}^{n+1} \to \mathbb{C}^n \). We denote by \( D := \max\{d_i : 1 \leq i \leq n\} \) the maximum of the degrees.

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The solution variety $\hat{V} \subset \mathcal{H}(d) \times (\mathbb{C}^{n+1} \setminus \{0\})$ is the set of points $\{(f, x)| f(x) = 0\}$. Since the equations are homogeneous, for all $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, $\lambda_1 f(\lambda_2 x) = 0$ if and only if $f(x) = 0$. So $\hat{V}$ defines a variety $V \subset \mathbb{P}(\mathcal{H}(d)) \times \mathbb{P}(\mathbb{C}^{n+1})$ where $\mathbb{P}(\mathcal{H}(d))$ and $\mathbb{P}(\mathbb{C}^{n+1})$ are the projective spaces corresponding to $\mathcal{H}(d)$ and $\mathbb{C}^{n+1}$ respectively. $\hat{V}$ and $V$ are smooth. We speak interchangeably of a path $(f_t, \zeta_t)$ in $\hat{V}$ and its projection $(f_t, \zeta_t)$ in $V$. Most quantities we define are defined on $\hat{V}$ but are constant on equivalence classes so are defined on $V$.

For $(g, x) \in \mathbb{P}(\mathcal{H}(d)) \times \mathbb{P}(\mathbb{C}^{n+1})$ let

$$
\mu_{\text{norm}}(g, x) = \frac{||g|| ||Dg(x)||^{-1}}{||N_x \Delta ||^{d-1}}
$$

where $||g||$ is the unitarily invariant norm defined by the unitarily invariant Hermitian structure on $\mathcal{H}(d)$ considered in [6] and sometimes called the Bombieri-Weyl or Kostlan Hermitian structure, $||x||$ is the standard norm in $\mathbb{C}^{n+1}$, $N_x$ is the Hermitian complement of $x$, and $\Delta(a_i)$ for $a_i \in \mathbb{C}, i = 1 \ldots n$ is the $n \times n$ diagonal matrix with $i$th diagonal entry $a_i$. If $Dg(x)||^{-1}$ does not exist we take $\mu_{\text{norm}} = \infty$. $\mu_{\text{norm}}$, also called $\mu_{\text{proj}}$ in some of our papers, is a normalized version of the condition number which we usually denote by $\mu$. We have also used various notions of distance in projective space. In this paper we use only the Riemannian distance inherited from the Hermitian structure on the vector space, i.e. the angle. In previous papers we have paid careful attention to the constants. In this paper we are more cavalier. We begin with an analysis of how the normalized condition number varies.

**Theorem 1.** Given $\epsilon > 0$ there is a constant $C > 0$ such that if

$$(g, x) \in \mathbb{P}(\mathcal{H}(d)) \times \mathbb{P}(\mathbb{C}^{n+1})$$

with

$$d(\zeta, \eta) < \frac{C}{D^{3/2} \mu_{\text{norm}}(g, \zeta)}$$

Then $\mu_{\text{norm}}(g, \eta) \leq (1 + \epsilon) \mu_{\text{norm}}(g, \zeta)$.

**Proof.** Use proposition 2.3 of [9].

Note that $u \leq D^{3/2} \mu_{\text{norm}}(g, \eta) \cdot d(\eta, \zeta) \leq C, r_0 \sim d(\eta, \zeta)$, the $\eta(f, x)$ in the definitions of $K$ is $\leq 1$ and $(\frac{d(\eta, \zeta)}{\eta(x)})^{D-1} \sim (1 + d(\zeta, \eta))^{D-1} < e^\epsilon$. □

**Theorem 2.** Given $\epsilon > 0$ there is a constant $C > 0$ such that if

$$f, g \in \mathbb{P}(\mathcal{H}(d)) \text{ and } \zeta \in \mathbb{P}(\mathbb{C}^{n+1})$$

with

$$d(f, g) < \frac{C}{D^{1/2} \mu_{\text{norm}}(f, \zeta)},$$

then

$$\mu_{\text{norm}}(g, \zeta) \leq (1 + \epsilon) \mu_{\text{norm}}(f, \zeta)$$
Proof. By Proposition 5b) of [6] SectionI-3.

\[
\mu_{\text{norm}}(g, \zeta) \leq \frac{\mu_{\text{norm}}(f, \zeta)(1 + d(f, g))}{1 - D^{1/2}d(f, g)\mu_{\text{norm}}(f, \zeta)} \leq \frac{\mu_{\text{norm}}(f, \zeta)(1 + \frac{C}{D^{1/2}\mu_{\text{norm}}(f, \zeta)})}{1 - C}
\]

Recall that \(\mu_{\text{norm}}(f, \zeta) \geq 1\). \(\square\)

**Theorem 3.** Given \(\epsilon > 0\) there is a constant \(C > 0\) such that if

\[ f, g \in P(\mathbb{H}(d)) \text{ and } \zeta, \eta \in P(\mathbb{C}^{n+1}) \]

and

\[ d(f, g) < \frac{C}{D^{1/2}\mu_{\text{norm}}(f, \zeta)} \]
\[ d(\zeta, \eta) < \frac{C}{D^{3/2}\mu_{\text{norm}}(f, \zeta)}, \]

then

\[
\frac{1}{1 + \epsilon}\mu_{\text{norm}}(g, \eta) \leq \mu_{\text{norm}}(f, \zeta) \leq (1 + \epsilon)\mu_{\text{norm}}(g, \eta).
\]

Proof. Apply Theorems 1 and 2 to prove the left hand inequality. Then given the left hand inequality apply Theorem 1 and 2 again to prove the right hand inequality applying the theorems with \(g, \eta\) in place of \(f, \zeta\). Adjust \(C\) and \(\epsilon\) as necessary. \(\square\)

The next proposition is useful for our Main Theorem [5]

**Proposition 1.** Given \(\epsilon > 0\) there is a \(C > 0\) with the following property:

Let \((f_t, \zeta_t)\) be a \(C^1\) path in \(V\) for \(t_0 \leq t \leq t_1\). Define \(S_0 = t_0\) and \(S_i\) to be the first value of \(t \leq t_1\) such that

\[
\int_{S_{i-1}}^{S_i} (\|f_t\| + \|\dot{\zeta}_t\|)dt = \frac{C}{D^{3/2}\mu_{\text{norm}}(f_{S_{i-1}}, \zeta_{S_{i-1}})} \text{ or } t_1.
\]

Then \(S_k = t_1\) for

\[
k \leq \max(1, \frac{(1 + \epsilon)}{C}D^{3/2}\int_{t_0}^{t_1} \mu_{\text{norm}}(f_t, \zeta_t)(\|f_t\| + \|\dot{\zeta}_t\|)dt)
\]

and \(\mu_{\text{norm}}(f_{t_1}, \zeta_{t_1}) \leq (1 + \epsilon)^k\mu_{\text{norm}}(f_{t_0}, \zeta_{t_0})\)
Proof.
\[
\int_{S_{i-1}}^{S_i} \mu_{\text{norm}}(f_t, \zeta_t)(\|\dot{f}_t\| + \|\dot{\zeta}_t\|)dt \geq \frac{1}{(1 + \epsilon)} \int_{S_{i-1}}^{S_i} \mu_{\text{norm}}(f_{S_i - 1}, \zeta_{S_i - 1})(\|\dot{f}_t\| + \|\dot{\zeta}_t\|)dt
\]
\[
\geq \frac{1}{1 + \epsilon} C \frac{D^{3/2}}{D^{3/2}} \text{ if } S_i < t_1.
\]
Consequently
\[
\int_{S_0}^{S_k} \mu_{\text{norm}}(f_t, \zeta_t)(\|\dot{f}_t\| + \|\dot{\zeta}_t\|)dt \geq \frac{(k - 1)C}{(1 + \epsilon)D^{3/2}} \text{ and}
\]
\[
\frac{(k - 1)}{(1 + \epsilon)D^{3/2}} \leq \int_{t_0}^{t_1} \mu_{\text{norm}}(f_t, \zeta_t)(\|\dot{f}_t\| + \|\dot{\zeta}_t\|)dt.
\]
so \( k - 1 \leq \frac{(1 + \epsilon)D^{3/2}}{C} \int_{t_0}^{t_1} \mu_{\text{norm}}(f_t, \zeta_t)(\|\dot{f}_t\| + \|\dot{\zeta}_t\|)dt. \)

\[\Box\]

Since we are working in the metric \( d \) we require an approximate zero theorem in this metric. First we prove a lemma. Recall the following quadratic polynomial \( \psi(u) = 1 - 4u + 2u^2 \) and the definition of the projective Newton iteration \( N_f(x) = x - (Df(x)|_{x^\perp})^{-1} f(x). \)

Here \( x^\perp \) is the Hermitian complement to \( x \). \( N_f : \mathbb{P}(\mathbb{C}^{n+1}) \to \mathbb{P}(\mathbb{C}^{n+1}) \) except that it fails to be defined where \((Df(x)|_{x^\perp})^{-1}\) does not exist. If \( f(\zeta) = 0 \) and \( d(N^k_f(x), \zeta) \leq \frac{1}{2^{k-1}} d(x, \zeta) \) for all positive integers \( k \), then \( x \) is called an \textit{approximate zero} of \( f \) with associated zero \( \zeta \).

**Lemma 1.** Let \( u < \frac{3 - \sqrt{7}}{4} \). Let \( f \in \mathcal{P}(\mathcal{H}(d)) \) and \( \zeta \in \mathbb{P}(\mathbb{C}^{n+1}) \) with \( f(\zeta) = 0 \). If \( d(x, \zeta) \leq \frac{4u}{D^{3/2} \mu_{\text{norm}}(f, \zeta) \psi(2u)} \), then
\[
d(N_f(x), \zeta) \leq \frac{4u}{\psi(2u)} d(x, \zeta).
\]

**Proof.** In the range of angles under consideration \( \tan d(x, \zeta) \leq 2d(x, \zeta) \). So apply Lemma 1 of P263 of [4] to conclude that
\[
d(N_f(x), \zeta) \leq \tan d(N_f(x), \zeta) \leq \frac{2u}{\psi(2u)} \tan d(x, \zeta)
\]
\[
\leq \frac{4u}{\psi(2u)} d(x, \zeta).
\]
Now let $u_0$ solve the equation

$$\frac{4u_0}{\psi(2u_0)} = \frac{1}{2} \quad \text{or} \quad u_0 = \frac{16 - \sqrt{232}}{16} \sim 0.048$$

**Theorem 4. (Approximate zero Theorem)**

Let $f \in \mathcal{P}(\mathcal{H}(d))$, $\zeta \in \mathbb{C}^{n+1}$ with $f(\zeta) = 0$. If $d(x, \zeta) < \frac{u_0}{D^{3/2} \mu_{\text{norm}}(f, \zeta)}$ Then $d(N^k_f(x), \zeta) \leq \frac{1}{2^k - 1} d(x, \zeta)$.

**Proof.** By induction

Suppose

$$d(N^k_f(x), \zeta) \leq \left(\frac{4u_0}{\psi(2u_0)}\right)^{2^k - 1} d(x, \zeta)$$

$$\leq \frac{4u_0}{\psi(2u_0)} \cdot \left(\frac{4u_0}{\psi(2u_0)}\right)^{2^k - 1} u_0$$

$$= \frac{4u_0}{\psi(2u_0)}^{2^k + 1 - 1} d(x, \zeta)$$

Let $u_1 = \left(\frac{4u_0}{\psi(2u_0)}\right)^{2^k - 1} u_0$. Note $u_1 \leq u_0$ and so $\psi(2u_1) \geq \psi(2u_0)$. By the lemma

$$d(N^{k+1}_f(x), \zeta) \leq \frac{4u_1}{\psi(2u_1)} d(N^k_f(x), \zeta)$$

$$\leq \frac{4u_1}{\psi(2u_0)} \cdot d(N^k_f(x), \zeta)$$

$$\leq 4 \left(\frac{4u_0}{\psi(2u_0)}\right)^{2^k - 1} \cdot d(x, \zeta)$$

$$= \left(\frac{4u_0}{\psi(2u_0)}\right)^{2^k + 1 - 1} d(x, \zeta)$$

\[\square\]

Let $(f_t, \zeta_t)$ be a (piecewise) $C^1$ path in $V$, $\frac{d}{dt}(f_t, \zeta_t)$ its tangent vector and $\|\frac{d}{dt}(f_t, \zeta_t)\|$ the length of its tangent vector.

**Theorem 5. (Main Theorem)** There is a constant $C_1 > 0$, such that: if $(f_t, \zeta_t)$ $t_0 \leq t \leq t_1$ is a $C^1$ path in $V$, then

$$C_1 D^{3/2} \int_{t_0}^{t_1} \mu_{\text{norm}}(f_t, \zeta_t) \|\frac{d}{dt}(f_t, \zeta_t)\| dt$$

steps of projective Newton method are sufficient to continue an approximate zero $x_0$ of $f_{t_0}$ with associated zero $\zeta_0$ to an approximate zero $x_1$ of $f_{t_1}$ with associated zero $\zeta_1$. 
Proof. Choose $C < \mu_0$ and $\epsilon$ small enough such that Theorem 3 and Theorem 4 apply, $u = 2C(1 + \epsilon) < \frac{3\sqrt{2}}{4}$ and $\frac{\log 4}{\psi(2n)} < \frac{1}{2(1 + \epsilon)}$. Hence, if $d(f, g) < \frac{C}{D^{3/2}\mu_{\text{norm}}(f, \zeta)}$

\[
d(\zeta, \eta) \leq \frac{C}{D^{3/2}\mu_{\text{norm}}(f, \zeta)}
\]

and $d(x, \zeta) \leq \frac{C}{D^{3/2}\mu_{\text{norm}}(f, \zeta)}$

then $d(N_g(x), \eta) \leq \frac{C}{D^{3/2}\mu_{\text{norm}}(g, \eta)}$ So $N_g(x)$ is an approximate zero of $g$ with associated zero $\eta$.

Now apply proposition 1 to produce $S_0, \ldots, S_k$ and $x_0$ such that

\[
d(x_0, \zeta_0) < \frac{C}{D^{3/2}\mu_{\text{norm}}(f_t, \zeta_t)}.
\]

Then $x_i = N_{f_{S_i}}(x_i - 1)$ is approximate zero of $f_{S_i}$ with associated zero $\zeta_{S_i}$ and

\[
d(x_i, \zeta_{S_i}) < \frac{C}{D^{3/2}\mu_{\text{norm}}(f_{S_i}, \zeta_{S_i})}.
\]

Corollary 1. There is a constant $C_2 > 0$, such that: if $(f_t, \zeta_t)$ $t_0 \leq t \leq t_1$ is a $C^1$ path in $V$, then

\[
C_2D^{3/2} \int_{t_0}^{t_1} \mu_{\text{norm}}^2(f_t, \zeta_t)\|\dot{f}_t\|dt
\]

steps of projective Newton method are sufficient to continue an approximate zero $x_0$ of $f_{t_0}$ with associated zero $\zeta_0$ to an approximate zero $x_1$ of $f_{t_1}$ with associated zero $\zeta_1$.

Proof. $\|\dot{f}_t\| \leq \mu_{\text{norm}}\|\dot{f}_t\|$\qed

Theorem 4 suggests that if we wish to continue a solution $\zeta \in P(\mathbb{C}^{n+1})$ of $f \in \mathcal{P}(\mathcal{H}(d))$ to a solution $\eta \in P(\mathbb{C}^{n+1})$ of $g \in \mathcal{P}(\mathcal{H}(d))$ an efficient way might be to follow a geodesic joining $(f, \zeta)$ to $(g, \eta)$ in the metric $\|\dot{f}, \dot{\zeta}\|_2^2 = \mu_{\text{norm}}(f, \zeta)^2(\|\dot{f}\|^2 + \|\dot{\zeta}\|^2)$. We call this Riemannian metric the condition (number)metric and quickly drop the "number" from the name.

Let $\sum' \subset V = \{f(\zeta) \in V \mid \mu_{\text{norm}}(f, \zeta) = \infty\}$ and $W = V - \sum'$. Note that $\mu_{\text{norm}}^2(\dot{f}, \dot{\zeta})$ is not differentiable everywhere on $W$.

Theorem 6. $W$ is complete in the metric $\|\cdot\|_k$.

Lemma 2. There is a constant $C > 0$ such that $(f_t, \zeta_t)$ in a $C^1$ path in $W$, $t_0 \leq t \leq t_1$ of length $L$ in the $\|\cdot\|_k$ metric, then $\mu_{\text{norm}}(f_{t_1}, \zeta_{t_1}) \leq C D^{3/2} L \mu_{\text{norm}}(f_{t_0}, \zeta_{t_0})$.

Proof of lemma
From proposition 1 it follows that

\[
\mu_{\text{norm}}(f_t, \zeta_t) \leq (1 + \epsilon)(\frac{L}{2n})D^{3/2} \sqrt{2} \mu_{\text{norm}}(f_{t_0}, \zeta_{t_0})
\]

For an appropriate $\epsilon, C$. Let $C = (1 + \epsilon)\sqrt{2} \frac{L}{2n}$

Proof of Theorem
Fix \((f_0, \zeta_0)\) for example \(f_0 = \frac{\mu^{1/2}}{n^{1/2}} X_i R_0^{d_i - 1} \) for \(i = 1, \ldots, n\) and \(\zeta_0 = (1, 0, \ldots, 0)\).

Then \(\mu_{\text{norm}}(f_0, \zeta_0) = n^{1/2}\).

Hence, \(\mu_{\text{norm}}(f, \zeta) \leq C D^{3/2} d((f, \zeta), (f_0, \zeta_0))^{1/2}\). So any Cauchy sequence in \(W\) stays a bounded distance away from \(\sum'\), hence in a compact region of \(W\) where it converges in the usual metric but also in the metric induced by \(\| \|_k\).

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