SPECIAL VALUES OF DIRICHLET SERIES AND ZETA INTEGRALS

EDUARDO FRIEDMAN AND ALDO PEREIRA

Abstract. For $f$ and $g$ polynomials in $p$ variables, we relate the special value at a non-positive integer $s = -N$, obtained by analytic continuation of the Dirichlet series

$$\zeta(s; f, g) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_p=0}^{\infty} g(k_1, \ldots, k_p) f(k_1, \ldots, k_p)^{-s} \quad (\text{Re}(s) \gg 0),$$

to special values of zeta integrals

$$Z(s; f, g) = \int_{x \in [0, \infty)^p} g(x) f(x)^{-s} \, dx \quad (\text{Re}(s) \gg 0).$$

We prove a simple relation between $\zeta(-N; f, g)$ and $Z(-N; f_a, g_a)$, where for $a \in \mathbb{C}^p$, $f_a(x) = f(a + x)$. By direct calculation we prove the product rule for zeta integrals at $s = 0$, degree($fh$) $\cdot$ $Z(0; fh, g) = \text{degree}(f) \cdot Z(0; f, g) + \text{degree}(h) \cdot Z(0; h, g)$, and deduce the corresponding rule for Dirichlet series at $s = 0$, degree($fh$) $\cdot$ $\zeta(0; fh, g) = \text{degree}(f) \cdot \zeta(0; f, g) + \text{degree}(h) \cdot \zeta(0; h, g)$. This last formula generalizes work of Shintani and Chen-Eie.

1. Introduction

We shall deduce special values of Dirichlet series

$$\zeta(s; f, g) := \sum_{k_1, \ldots, k_p = 0}^{\infty} g(k_1, \ldots, k_p) f(k_1, \ldots, k_p)^{-s} \quad (\text{Re}(s) \gg 0) \quad (1)$$

from those of zeta integrals

$$Z(s; f, g) := \int_{x_1 = 0}^{\infty} \cdots \int_{x_p = 0}^{\infty} g(x_1, \ldots, x_p) f(x_1, \ldots, x_p)^{-s} \, dx_p \cdots dx_1. \quad (2)$$

Here $f$ and $g$ are polynomials in $p$ variables with complex coefficients, with some restrictions on $f$ to ensure the existence of an appropriate
branch of \( \log f \) and the convergence and analytic continuation of sums and integrals.

The use of integrals to express sums goes back to Euler’s invention of the Euler-MacLaurin formula to compute \( \zeta(2) = \sum_{n=1}^{\infty} n^{-2} \) numerically [16]. Later authors, such as Mellin [12], Mahler [11], Shintani [15], Cassou-Noguès [1], Sargos [14], Lichtin [10], Essouabri [7], Peter [13] and de Crisenoy [5], have used various integrals to ascertain the existence of a meromorphic continuation of \( \zeta(s; f, g) \) and to compute its residues and various expansions. As the Euler-MacLaurin formula already shows, at a general \( s \) the connection between Dirichlet series \( \zeta(s; f, g) \) and zeta integrals \( Z(s; f, g) \) is rather complicated. We will show, however, that at non-positive integers \( s = 0, -1, -2, \ldots \) the relationship becomes quite simple.

Consider, as a first easy case, the Riemann zeta function

\[
\zeta(s) = \zeta(s; x+1, 1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^s} \quad (\text{Re}(s) > 1)
\]

and its corresponding zeta integral (convergent for \( \text{Re}(s) > 1, \text{Re}(a) > -1 \))

\[
Z(s; x+1+a, 1) = \int_{0}^{\infty} \frac{dx}{(x+1+a)^s} = \frac{(1+a)^{1-s}}{s-1}.
\]

Here we have allowed ourselves to replace the polynomial \( f(x) = x + 1 \) by its shift \( f_a(x) = f(x+a) \). Using the above meromorphic continuation in \( s \) for \( Z(s; x+1+a, 1) \) we find

\[
Z(-N; x+1+a, 1) = \frac{(1+a)^{N+1}}{-N-1} = -\frac{1}{N+1} \sum_{\ell=0}^{N+1} \binom{N+1}{\ell} a^\ell,
\]

for \( N \geq 0 \) a non-negative integer. If we mindlessly replace every occurrence of \( a^\ell \) above by the Bernoulli number \( B_\ell \) we obtain

\[
\frac{-1}{N+1} \sum_{\ell=0}^{N+1} \binom{N+1}{\ell} B_\ell = -\frac{B_{N+1}(1)}{N+1} = (-1)^N \frac{B_{N+1}}{N+1} = \zeta(-N)
\]

[4, p. 76] [9, pp. 67–68], where the Bernoulli polynomials \( B_j(t) \) are defined by

\[
B_0(t) = 1, \quad \frac{dB_j}{dt} = jB_{j-1}(t), \quad \int_{0}^{1} B_j(t) \, dt = 0 \quad (j \geq 1), \quad (3)
\]

and the Bernoulli numbers as \( B_j = B_j(0) \). In short, to compute the value of the Dirichlet series \( \zeta(s) = \zeta(s; x+1, 1) \) at \( s = -N \), we simply take the polynomial (in \( a \)) giving the shifted zeta integral \( Z(-N; x+1+a, 1) \) and replace powers of \( a \) by Bernoulli numbers.
This simple relation between Dirichlet series and shifted zeta integrals holds quite generally, as we shall now describe. For \( h \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_p] \) and \( a = (a_1, \ldots, a_p) \in \mathbb{R}^p \), let \( h_a(x) := h(x + a) \). In §2 we prove (under some hypothesis on \( f \)) that the maps \( a \to Z(-N; f_a, g_a) \) and \( a \to \zeta(-N; f_a, g_a) \) are polynomials in \( a \). Here \( a \) ranges in a small enough ball in \( \mathbb{R}^p \) containing the origin, while \( N, f, \) and \( g \) are fixed. To express the relation between the two polynomials (in \( a \)) \( Z(-N; f_a, g_a) \) and \( \zeta(-N; f_a, g_a) \), write the former as a (finite!) sum of monomials

\[
Z(-N; f_a, g_a) = \sum_L c_L a^L \quad \left( a^L := \prod_{i=1}^p a_i^{L_i}, \ c_L \in \mathbb{C} \right).
\]

In §2, Proposition 4, we prove that

\[
\zeta(-N; f_a, g_a) = \sum_L c_L B_L(a) \quad \left( B_L(a) := \prod_{i=1}^p B_{L_i}(a_i) \right). \tag{4}
\]

Taking \( a = 0 \) we obtain the special value of the Dirichlet series

\[
\zeta(-N; f, g) = \sum_L c_L B_L \quad \left( B_L := \prod_{i=1}^p B_{L_i} \right) \tag{5}
\]

in terms of special values of zeta integrals and products of Bernoulli numbers. Equation (4) explains the profusion of Bernoulli polynomials in Shintani’s formulas [15].

Equation (4) follows rather formally from the “Raabe formula” (see Proposition 4)

\[
Z(s; f_a, g_a) = \int_{t \in [0,1]^p} \zeta(s; f_{a+t}, g_{a+t}) \, dt. \tag{6}
\]

This formula, though easily proved by an “unfolding” argument, provides a powerful link between zeta integrals and Dirichlet series. The Raabe formula (6) holds everywhere in \( s \) (save at the poles of \( Z(s; f, g) \)), but it can be inverted at special values \( s = -N \) to yield \( \zeta(-N; f_a, g_a) \) in terms of \( Z(-N; f_a, g_a) \). Quite generally (see Lemma 6), two polynomials \( Q(a) \) and \( P(a) = \sum_L d_L a^L \) in \( p \) variables are linked by a Raabe formula

\[
P(a) = \int_{t \in [0,1]^p} Q(a + t) \, dt
\]

1 We use this notation only when the subscript is the letter \( a \). For example, \( f_j \) in Theorem 1 below simply stands for one of \( n \) polynomials.

2 Raabe’s 1843 formula is \( \int_0^1 \log(\Gamma(x + t)/\sqrt{2\pi}) \, dt = x \log x \). See [8, p. 367] for the connection to (6). A \( p \)-adic version of Raabe’s formula was given in [9].
if and only if

\[ Q(a) = \sum_L d_L B_L(a). \]

Our main motivation for relating special values of zeta integrals and Dirichlet series is that integrals are usually easier to handle. In §3 we give (under some hypothesis on \( f \)) a slightly complicated formula for \( Z(s; f, g) \), for \( s \) a non-positive integer. For \( s = 0 \) we are able to simplify it enough to prove a formula for the special value \( Z(0; \prod_{j=1}^n f_j, g) \) in terms of the individual \( Z(0; f_j, g) \). In view of the relation between zeta integrals and Dirichlet series at special values, we deduce an analogous formula giving \( \zeta(0; \prod_{j=1}^n f_j, g) \) in terms of the individual \( \zeta(0; f_j, g) \).

A first case of this formula was proved by Shintani. Namely, if all the \( f_j(x) \) are polynomials of degree one, positive for all \( x \in [0, \infty) \), Shintani [15, p. 206] [8, p. 386] showed

\[ \zeta(0; \prod_{j=1}^n f_j, 1) = \frac{1}{n} \sum_{j=1}^n \zeta(0; f_j, 1). \]  

(7)

Shintani’s formula (7) cannot be expected to generalize literally to higher-degree polynomials.\(^3\) Besides correcting for the degree of the \( f_j \), we need some kind of irreducibility condition on the \( f_j \) not allowing them to factor into a product of polynomials in separate variables. Indeed, if we had

\[ f(x_1, \ldots, x_p) = h(x_1, \ldots, x_{\ell}) \cdot \tilde{h}(x_{\ell+1}, \ldots, x_p) \]  

(8)

for some \( 1 \leq \ell < p \), then from (11) we would find a corresponding factorization

\[ \zeta(s; f, 1) = \zeta(s; h, 1) \cdot \zeta(s; \tilde{h}, 1). \]

(9)

This kind of relation (when applied by analytic continuation at \( s = 0 \)) is inconsistent with simple generalizations of Shintani’s formula (7)\(^4\).

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\(^3\) To see this, assume (7) and consider two ways to associate \( \prod_{j=1}^3 f_j \),

\[ \zeta(0; f_1 f_2 f_3, g) = \frac{\zeta(0; f_1, g)}{2} + \frac{\zeta(0; f_2 f_3, g)}{2} = \frac{\zeta(0; f_1, g)}{2} + \frac{\zeta(0; f_2, g)}{4} + \frac{\zeta(0; f_3, g)}{4}, \]

\[ \zeta(0; (f_1 f_2) f_3, g) = \frac{\zeta(0; f_1 f_2, g)}{2} + \frac{\zeta(0; f_3, g)}{2} = \frac{\zeta(0; f_1, g)}{4} + \frac{\zeta(0; f_2, g)}{4} + \frac{\zeta(0; f_3, g)}{2}. \]

On subtracting, we find \( \zeta(0; f_1, g) = \zeta(0; f_3, g) \). This would imply that \( \zeta(0; f, g) \) does not depend on \( f \), contradicting a host of known facts, e.g. [15 Lemma 2].

\(^4\) More precisely, it is inconsistent with generalizations of the form

\[ \zeta(0; f_1 f_2 \cdots f_n, 1) = c_1 \zeta(0; f_1, 1) + c_2 \zeta(0; f_2, 1) + \cdots + c_n \zeta(0; f_n, 1), \]

where the \( c_\ell \) depend at most on the degrees of the \( f_j \)'s \( (1 \leq j \leq n) \).
Mahler [11] p. 385] gave a simple hypothesis on the polynomial \( f \) ensuring that it does not separate as in (8).

**Mahler’s Hypothesis.** The polynomial \( f(x) \in \mathbb{C}[x_1, \ldots, x_p] \) is non-constant and does not vanish anywhere in the closed “octant” \([0, \infty)^p\). Moreover, its top-degree homogeneous part \( f_{\text{top}}(x) \neq 0 \) for \( x \in [0, \infty)^p \), save at \( x = (0, 0, \ldots, 0) \).

Under his hypothesis Mahler [11] showed that \( Z(s; f, g) \) and \( \zeta(s; f, g) \) converge for \( \text{Re}(s) \gg 0 \), extend meromorphically in \( s \) to all of \( \mathbb{C} \) and are regular at the non-positive integers \( s = 0, -1, -2, \ldots \).

An advantage of Mahler’s Hypothesis for our purposes is that if \( f \) and \( h \) satisfy it, then so does \( fh \).

**Theorem 1.** Let \( g \) and \( f_j \) (\( 1 \leq j \leq n \)) be polynomials in \( p \) variables and assume that all the \( f_j \) verify Mahler’s Hypothesis above. Then the values at \( s = 0 \) obtained by analytic continuation of the Dirichlet series

\[
\zeta(s; f_j, g) := \sum_{k_1=0}^{\infty} \cdots \sum_{k_p=0}^{\infty} g(k_1, \ldots, k_p)f_j(k_1, \ldots, k_p)^{-s} \quad (\text{Re}(s) \gg 0)
\]

satisfy the product rule

\[
\text{degree}\left( \prod_{j=1}^{n} f_j \right) \cdot \zeta(0; \prod_{j=1}^{n} f_j, g) = \sum_{j=1}^{n} \text{degree } (f_j) \cdot \zeta(0; f_j, g). \quad (10)
\]

**Corollary 2.** (Chen-Eie) Assume furthermore that the polynomials \( f_j \) all have the same degree. Then

\[
\zeta(0; \prod_{j=1}^{n} f_j, g) = \frac{1}{n} \sum_{j=1}^{n} \zeta(0; f_j, g).
\]

Chen and Eie [2] p. 3219] do not explicitly mention Mahler’s Hypothesis, nor the condition of equal degree for the \( f_j \), but we have seen above that assumptions of this kind are unavoidable.\(^5\)

\(^5\) The branch of \( \log f \) used by Mahler in defining \( Z(s; f, g) \) and \( \zeta(s; f, g) \) is just any one that is continuous on \([0, \infty)^p\). It exists precisely because of the non-vanishing of \( f \) that he assumed. The value of \( Z(s; f, g) \) and \( \zeta(s; f, g) \) at integer values of \( s \) proves to be independent of the choice of (continuous) branch of \( \log f \).

\(^6\) Chen and Eie take their sums in (11) for \( k_i \geq 1 \) instead of \( k_i \geq 0 \), but this is just a matter of replacing \( f(x) = f(x_1, \ldots, x_p) \) by \( f(x_1 + 1, \ldots, x_p + 1) \), and similarly with \( g \).
2. Dirichlet series and zeta integrals at special values

Recall from §1 that a polynomial \( f \in \mathbb{C}[x] \) in \( p \) variables satisfies Mahler’s Hypothesis if \( m := \deg(f) > 0, \ f(x) \neq 0 \) for all \( x \in \mathbb{R}_{\geq 0}^p := [0, \infty)^p \), and if its top-degree homogeneous part \( f_{\text{top}} \) satisfies \( f_{\text{top}}(x) \neq 0 \) for all \( x \in \mathbb{R}_{\geq 0}^p - \{0\} \). We let \( \mathcal{M} = \mathcal{M}_{m,p} \) denote the set of all such \( f \). Certainly \( \mathcal{M} \) is non-empty, as it contains the polynomial \( x_1^m + x_2^m + \cdots + x_p^m + 1 \).

Together with \( f \in \mathcal{M} \), we will need to consider the shifted polynomial \( f_a \) defined as \( f_a(x) := f(x+a) \), where \( a \in \mathbb{C}^p \). Since \( (f_a)_{\text{top}} = f_{\text{top}} \), it is clear that for \( a \in \mathbb{R}_{\geq 0}^p \), \( f_a \in \mathcal{M} \) if \( f \in \mathcal{M} \). We will now show that \( f_a \in \mathcal{M} \) for all \( a \) in a small enough neighborhood of the origin in \( \mathbb{C}^p \). For this it suffices to show that \( \mathcal{M} \) is open in the finite-dimensional complex vector space of all polynomials of degree \( m \) in \( p \) variables (space of coefficients).

To show that \( \mathcal{M} \) is open we first estimate \( |f(x)| \) for \( x \in \mathbb{R}_{\geq 0}^p \). It proves convenient to switch away from cartesian coordinates \( x = (x_1, x_2, \ldots, x_p) \). Instead of the well-known spherical co-ordinates used by Mahler \[11\] for this purpose, we will use “cubical” co-ordinates \((\rho, \sigma)\),

\[
\rho = \rho(x) := \max(|x_1|, |x_2|, \ldots, |x_p|), \quad \sigma = \sigma(x) := \frac{x}{\rho(x)} \quad (x \neq 0).
\] (11)

We denote by \( \partial C_+^p \) the piece of boundary of the unit-hypercube \( C^p = [0, 1]^p \) where at least one co-ordinate is 1,

\[
\partial C_+^p := \{ x \in \mathbb{R}_{\geq 0}^p | \rho(x) = 1 \}. \quad (12)
\]

For \( f \in \mathcal{M} \) and \( x \in \mathbb{R}_{\geq 0}^p, \ x \neq 0 \), write

\[
\begin{align*}
  r(x) & = r_f(x) := \frac{f(x) - f_{\text{top}}(x)}{f_{\text{top}}(x)}, \\
  f(x) & = f_{\text{top}}(x)(1 + r(x)) = \rho^m f_{\text{top}}(\sigma)(1 + r(\rho \sigma)) \quad (x \neq 0). 
\end{align*}
\] (13)

Note that

\[
r(\rho \sigma) = r_f(\rho \sigma) = \frac{f_{m-1}(\sigma)}{\rho f_{\text{top}}(\sigma)} + \frac{f_{m-2}(\sigma)}{\rho^2 f_{\text{top}}(\sigma)} + \cdots + \frac{f_1(\sigma)}{\rho^{m-1} f_{\text{top}}(\sigma)} + \frac{f_0}{\rho^m f_{\text{top}}(\sigma)},
\] (14)

where \( f_j(\sigma) \) denotes the homogeneous part of \( f \) of degree \( j \). Hence for all \( \sigma \in \partial C_+^p \) and all \( \rho > \rho_f \) (for some large enough \( \rho_f \) we have \( |r_f(\rho \sigma)| < \frac{1}{3} \) (say). Similarly, for all polynomials \( \tilde{f} \) in a neighborhood of \( f \) we have \( 1 + r_f(\rho \sigma) \neq 0 \) for \( \rho > \rho_f \). Considering the factorization (13),
we see that non-vanishing conditions on compact sets insure Mahler’s Hypothesis for $\hat{f}$. Thus $\mathcal{M}$ is open.

Since $\mathbb{R}^p_{\geq 0}$ is simply connected and $f \in \mathcal{M}$ does not vanish there, we can choose a continuous branch $\log f : \mathbb{R}^p_{\geq 0} \to \mathbb{C}$. By the same token, locally around a given $f$ we can choose this branch so that it depends analytically on the coefficients of $f$. Any other continuous choice of $\log f$ will differ by $2\pi i \ell$ for some fixed integer $\ell$, introducing a factor of $e^{-2\pi i \ell s}$ in our zeta integrals (2) and series (1). Hence their values (or residues) at any integer $s$ are independent of the branch chosen.

Following Mahler [11] and considering (13), we choose our branch of $\log f$ so that for $x \neq 0$, $x \in \mathbb{R}^p_{\geq 0}$ and $f \in \mathcal{M}$,

$$\log f(x) = \log f(\rho \sigma) = m \log \rho + \log f_{\text{top}}(\sigma) + \log (1 + r(\rho \sigma)), \quad (15)$$

where $\log \rho$ is real-valued, $\log f_{\text{top}}(\sigma)$ is any continuous choice of $\log f_{\text{top}}$ on the (simply connected) hypersurface $\partial \mathbb{C}^p_+$, and $\log(1 + r)$ is the unique continuous branch which for large enough $\rho$ is given by the principal value

$$\log(1 + r) = - \sum_{\lambda = 1}^{\infty} \frac{(-r)^\lambda}{\lambda} \quad (r = r(\rho \sigma), \ \rho \gg 0).$$

In (15) we used Mahler’s Hypothesis to insure $1 + r \neq 0$ and $f_{\text{top}}(\sigma) \neq 0$.

We now state Mahler’s main result [11] concerning the meromorphic continuation of $\zeta(s; f, g)$ and of $Z(s; f, g)$.

**Theorem 3.** (Mahler) Suppose $f$ is a polynomial satisfying Mahler’s Hypothesis and let $g$ be any polynomial in the same number $p$ of variables. Then

$$Z(s; f, g) := \int_{\mathbb{R}^p_{\geq 0}} g(x) f(x)^{-s} \, dx \quad (16)$$

and

$$\zeta(s; f, g) := \sum_{k_1 = 0}^{\infty} \cdots \sum_{k_p = 0}^{\infty} g(k_1, \ldots, k_p) f(k_1, \ldots, k_p)^{-s} \quad (17)$$

both converge absolutely and uniformly on compact subsets of the right half-plane $\text{Re}(s) > (\deg(g) + p)/\deg(f)$, and extend to all of $\mathbb{C}$ as

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7 Here is a proof that the complement of such a set is closed. For $K \subset \mathbb{C}^p$ compact let

$$W = W_{K, m} := \left\{ f \in \mathbb{C}[x_1, \ldots, x_p] \mid \deg(f) \leq m, \ \exists k \in K, \ f(k) = 0 \right\}.$$ 

It suffices to show that if $\{f_n\} \subset W$ converges to $f$ (say, in the uniform norm), then $f \in W$. Let $f_n(k_n) = 0$, with $k_n \in K$. Since $K$ is compact, there exists a subsequence $k_{n_j}$ converging to $k \in K$. But then $0 = \lim_j f_{n_j}(k_{n_j}) = f(k)$. Thus $f \in W$. 

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meromorphic functions of $s$, regular at all non-positive integers $s = 0, -1, -2, \ldots$. Their poles are all simple and occur among the rational numbers of the form

$$s = \frac{\deg(g) + p - \ell}{\deg(f)},$$

with $\ell \geq 0$ an integer. Moreover, $Z(s; f, g)$ and $\zeta(s; f, g)$ are analytic in $s, f$ and $g$, as long as $s$ stays outside the above set of possible poles and $f$ stays in an open simply connected subset of $\mathcal{M}$.

Mahler does not explicitly address the analytic dependence on (the coefficients of) $f$ and $g$, but it is immediate from his proof. In §3 the reader will find a full proof of Mahler’s theorem for $Z(s; f, g)$. As Mahler showed, the analytic continuation for $\zeta(s; f, g)$ follows readily from that of $Z(s; f, g)$ and the Euler-MacLaurin formula.

We will need to go into Mahler’s proof to simplify his formulas for $Z(s; f, g)$ at special values. Mahler actually dealt with the slightly different sums

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_p=1}^{\infty} g(k_1, \ldots, k_p) f(k_1, \ldots, k_p)^{-s} \quad (\text{Re}(s) \gg 0),$$

i.e. Mahler summed over $k_i \geq 1$, whereas we use $k_i \geq 0$ in (17). Our series can be written as a finite sum of Mahler’s, and so his theorem gives the meromorphic continuation of (17). The only point worth noting here is that polynomials in fewer variables, obtained from $f$ (which we assume satisfies Mahler’s Hypothesis) by inserting 0 for some $x_i$’s again satisfy Mahler’s Hypothesis (in fewer variables) and have the same degree as $f$.

We can now prove

**Proposition 4.** Suppose $f$ and $g$ are polynomials in $p$ variables, and assume $f$ satisfies Mahler’s Hypothesis. Then,

1. (Raabe formula) For $s$ outside the possible pole set of $Z(s; f, g)$ given in Mahler’s Theorem above, we have

$$Z(s; f, g) = \int_{t \in [0,1]^p} \zeta(s; f_t, g_t) \ dt, \quad (18)$$

where $f_t(x) := f(t + x)$ and $dt$ is Lebesgue measure on $\mathbb{R}^p$.

2. For a fixed integer $N \geq 0$, the maps $a \rightarrow \zeta(-N; f_a, g_a)$ and $a \rightarrow Z(-N; f_a, g_a)$ are polynomials in $a = (a_1, \ldots, a_p) \in \mathbb{R}_{\geq 0}^p$ of degree at most $N \deg(f) + \deg(g) + p$. 
(3) If we write out the polynomial $Z(-N; f_a, g_a)$ as a sum of monomials,

$$Z(-N; f_a, g_a) = \sum_L c_L a^L \quad \left( a^L := \prod_{i=1}^{p} a_i^{l_i}, \ c_L = c_L(N; f, g) \in \mathbb{C} \right),$$

then

$$\zeta(-N; f, g) = \sum_L c_L B_L,$$

(19)

where $B_L := \prod_{i=1}^{p} B_{L_i}$ is a product of Bernoulli numbers. More generally, for $a \in \mathbb{R} \geq 0$ we have

$$\zeta(-N; f_a, g_a) = \sum_L c_L B_L(a),$$

where $B_L(a) = \prod_{i=1}^{p} B_{L_i}(a_i)$ is a product of Bernoulli polynomials.

We note that the Raabe formula (6) stated in §1 follows from (18) on replacing $f$ by $f_a$, noting that $(f_a)_t = f_{a+t}$.

Proof. Write $m := \deg(f)$, $q := \deg(g)$. For $\text{Re}(s) > \frac{p+q}{m}$, the integral and series defining $Z(s; f, g)$ and $\zeta(s; f, g)$ are absolutely convergent, as is clear from (14) and (13). Note also that if $t \in \mathbb{R}_{\geq 0}$, then $f_t$ satisfies Mahler’s Hypothesis and $f_t$ has the same degree as $f$. Let $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots \}$ and compute for $\text{Re}(s) > \frac{p+q}{m}$,

$$\int_{[0,1]^p} \zeta(s; f_t, g_t) \, dt = \int_{[0,1]^p} \sum_{k \in \mathbb{N}_0^p} g(k + t)f(k + t)^{-s} \, dt$$

$$= \sum_{k \in \mathbb{N}_0^p} \int_{k+[0,1]^p} g(t)f(t)^{-s} \, dt = \int_{\mathbb{R}_0^p} g(t)f(t)^{-s} \, dt = Z(s; f, g).$$

(20)

By analytic continuation, the Raabe formula

$$\int_{[0,1]^p} \zeta(s; f_t, g_t) \, dt = Z(s; f, g)$$

holds for all $s$ outside the possible pole set given in Mahler’s Theorem.

To prove the polynomial nature of $a \to \zeta(-N; f_a, g_a)$ we follow the proof sketched in [6]. Mahler’s Theorem implies that $\zeta(-N; f_a, g_a)$ is an analytic function of $a$ as $a$ ranges in some small open ball in $\mathbb{C}^p$ containing the origin. We shall show that all sufficiently high derivatives with respect to $a$ vanish.
Lemma 5. Let $\partial_L$ be the differential operator

$$\partial_L := \frac{\partial^{|L|}}{\partial x_1^{L_1} \cdots \partial x_p^{L_p}} \quad (L = (L_1, \ldots, L_p), \ |L| := \sum_{i=1}^p L_i),$$

and let $f$ and $g$ be polynomials in $p$ variables of degree $m$ and $q$, respectively. Then, for $x = (x_1, \ldots, x_p)$ in an open set in $\mathbb{C}^p$ where some branch of $\log f(x)$ is analytic, we have

$$\partial_L (g(x) f(x)^{-s}) = \sum_{\nu=0}^{|L|} \left( \prod_{j=0}^{\nu-1} (s + j) \right) P_{L,\nu}(x) f(x)^{-(s+\nu)}, \quad (21)$$

where $P_{L,\nu}(x)$ is a polynomial in $x$, independent of $s$, of degree at most $q + m\nu - |L|$, vanishing if $q + m\nu - |L| < 0$.

Proof. This is a straight-forward induction on $|L| := \sum_{i=1}^p L_i$. We omit the routine details. \qed

For $a \in \mathbb{R}^p_{\geq 0}$, Mahler’s theorem implies that

$$\zeta(s; f_a, (P_{L,\nu})_a) = \sum_{k \in \mathbb{N}_0^p} P_{L,\nu}(k + a) f(k + a)^{-(s+\nu)}$$

converges (absolutely and uniformly on compact subsets) in the right half-plane $\Re(s + \nu) > \frac{p + \deg(P_{L,\nu})}{\deg(f)}$. Taking $|L| \geq Nm + q + p + 1$ and using the bound on $\deg(P_{L,\nu})$ in the Lemma, this is the half-plane $\Re(s) > -N - \frac{1}{m}$. Hence, for such $L$ and $a$,

$$\frac{\partial^{|L|}}{\partial a_1^{L_1} \cdots \partial a_p^{L_p}} \left( \zeta(s; f_a, g_a) \right) = \sum_{\nu=0}^{|L|} \left( \prod_{j=0}^{\nu-1} (s + j) \right) \sum_{k \in \mathbb{N}_0^p} P_{L,\nu}(k + a) f(k + a)^{-(s+\nu)},$$

which is initially seen to be valid for $\Re(s) \gg 0$, actually gives absolutely convergent expressions for $\Re(s) > -N - \frac{1}{m}$. By analytic continuation, the above holds at $s = -N$. However, at $s = -N$ the right-hand side vanishes rather trivially. Indeed, for $\nu > N$, the product over $j$ vanishes at $s = -N$. For $0 \leq \nu \leq N$, the degree of $P_{L,\nu}$ is at most $q + m\nu - |L| \leq q + mN - (Nm + q + p + 1) < 0$, and so $P_{L,\nu}$ vanishes identically. We conclude that all $a$-derivatives of $\zeta(s; f_a, g_a)$ of order greater than $Nm + q + p$ vanish. Hence $\zeta(s; f_a, g_a)$ is a polynomial in $a$ of degree at most $Nm + q + p$.

Replacing $\sum_{k \in \mathbb{N}_0^p}$ by $\int_{x \in \mathbb{R}_{\geq 0}^p}$, the above proof shows that $Z(s; f_a, g_a)$ is also a polynomial in $a$ of degree at most $Nm + q + p$ (alternatively, this follows from Raabe’s formula $[6]$ and the polynomial nature of
$a \rightarrow \zeta(s; f_a, g_a)$. This completes the proof of the second claim in Proposition 4.

The third claim follows from the following lemma, with $Q(a) := \zeta(s; f_a, g_a)$ and $P(a) := \zeta(s; f_a, g_a)$.

**Lemma 6.** Let $P$ and $Q$ be two polynomials in $p$ variables linked by

$$P(a) = \int_{t \in [0,1]^p} Q(a + t) \, dt.$$ 

Write out

$$P(a_1, \ldots, a_p) = \sum_L d_L \prod_{i=1}^{p} a_i^{L_i},$$

where $d_L \in \mathbb{C}$ and $L = (L_1, \ldots, L_p) \in \mathbb{N}_0^p$ ranges over a finite set of multi-indices. Then

$$Q(a_1, \ldots, a_p) = \sum_L d_L \prod_{i=1}^{p} B_{L_i}(a_i),$$

where the $B_{L_i}(a_i)$ are Bernoulli polynomials, defined in [3]. Conversely, if $Q$ is given by (24), then (22) and (23) are equivalent formulas for $P$.

**Proof.** Let $V = V_{m,p}$ be the finite-dimensional complex vector space of polynomials in $p$ variables $a = (a_1, \ldots, a_p)$, with complex coefficients and having degree at most $m$. Note that both $\{a^L\}_L$ and $\{B_L(a)\}_L$ are $\mathbb{C}$-bases of $V$. Here $L = (L_1, \ldots, L_p)$ ranges over all multi-indices with $|L| := \sum_{i=1}^{p} L_i \leq m$ and $a^L := \prod_{i=1}^{p} a_i^{L_i}$. That $\{B_L(a)\}_L$ is a basis of $V = V_{m,p}$ can be proved by induction on $m$, since $a^L - B_L(a)$ has degree strictly less than $|L|$. Let $R : V \to V$ be the $\mathbb{C}$-linear map taking $Q = Q(a) \in V$ to

$$R(Q)(a) := \int_{t \in [0,1]^p} Q(a + t) \, dt.$$ 

The lemma can be restated as saying that the inverse map to $R$ exists and takes $a^L$ to $B_L(a)$. Hence, it will suffice to show that $R(B_L(a)) = a^L$, for then $R$ is an isomorphism (it takes one basis to another). Using [3, p. 4] [21, pp. 66–67]

$$\frac{d}{dx} B_{j+1}(x) = (j + 1)B_j(x) \quad \text{and} \quad B_j(x + 1) - B_j(x) = jx^{j-1},$$
we calculate

\[ R(B_L(a)) = \int_{t \in [0,1]} B_L(a + t) \, dt = \prod_{i=1}^{p} \int_{0}^{1} B_{L_i}(a_i + t_i) \, dt_i = \prod_{i=1}^{p} \frac{1}{L_i + 1} (B_{L_i+1}(a_i + 1) - B_{L_i+1}(a_i)) = \prod_{i=1}^{p} a_{i_i}^{L_i}. \]

This concludes the proof of the lemma and of Proposition \[4\] \hfill \Box

Using Proposition \[4\] we now show that Theorem \[1\] follows from the product formula for zeta integrals

\[ \deg(f) \cdot Z(0; f, g) = \sum_{j=1}^{n} \deg(f_j) \cdot Z(0; f_j, g), \quad (25) \]

which we prove in the next section. We always assume that all the \( f_j \) are polynomials in \( p \) variables and satisfy Mahler’s Hypothesis. Theorem \[1\] states that

\[ \deg(f) \cdot \zeta(0; f, g) = \sum_{j=1}^{n} \deg(f_j) \cdot \zeta(0; f_j, g) \quad (f := \prod_{j=1}^{n} f_j), \quad (26) \]

To prove this write

\[ Z(0; f_a, g_a) = \sum_{L} c_L(0; f, g)a^{L}, \quad Z(0; (f_j)_a, g_a) = \sum_{L} c_L(0; f_j, g)a^{L}. \quad (27) \]

Since \( \deg(f_a) = \deg(f) \), replacing \( f \) by \( f_a \), \( f_j \) by \( (f_j)_a \) and \( g \) by \( g_a \) in \( (25) \) gives

\[ \deg(f)c_L(0; f, g) = \sum_{j=1}^{n} \deg(f_j)c_L(0; f_j, g) \]

for all the coefficients \( c_L(0; f, g) \) and \( c_L(0; f_j, g) \) appearing in \( (27) \). Equation \( (19) \) now gives

\[ \deg(f)\zeta(0; f, g) = \sum_{L} B_L \deg(f)c_L(0; f, g) \]

\[ = \sum_{L} B_L \sum_{j=1}^{n} \deg(f_j)c_L(0; f_j, g) \]

\[ = \sum_{j=1}^{n} \deg(f_j) \sum_{L} B_L c_L(0; f_j, g) = \sum_{j=1}^{n} \deg(f_j)\zeta(0; f_j, g), \]

as claimed.
3. Special values of zeta integrals

In this section we first follow Mahler’s proof \[11\] of the meromorphic continuation of zeta integrals \(Z(s; f, g)\). We then show that Mahler’s formulas simplify when \(s\) is a non-positive integer. Finally, we consider \(s = 0\) and show the product formula \([25]\).

Let us prove the part of Mahler’s Theorem giving the meromorphic continuation of \(Z(s; f, g)\). Recall that in \([11]\) we introduced cubical coordinates \(x = \rho \sigma\) on \(\mathbb{R}^p_0\). Let \(d\sigma\) denote the natural \((p - 1)\)-dimensional volume element on \(\partial C^p_+\). A short Jacobian calculation shows that Lebesgue measure \(dx = \rho^{p-1} d\rho d\sigma\). We have also seen that a branch of \(\log f(x)\) can be chosen, continuous for \(x \in \mathbb{R}^p_0\) and analytic locally in \(f\). Note that the imaginary part \(\text{Im}(\log f(x))\) is uniformly bounded for \(x \in \mathbb{R}^p_0\).

For \(s\) in some compact set \(K \subset \mathbb{C}\) we have from \([15]\) the estimate

\[|g(x)f(x)^{-s}| \leq c_K \rho^{q-m\text{Re}(s)}\]

for some constant \(c_K\) independent of \(x\). Hence the zeta integral \([16]\) converges when \(q - m\text{Re}(s) < -p\), i.e. in the right half-plane \(\text{Re}(s) > (q + p)/m\). Moreover, for such \(s\) and polynomials \(h\) in a neighborhood of \(f\), the map \((s, h, g) \mapsto Z(s; h, g)\) is analytic in all three variables.

To prove Mahler’s theorem for \(Z(s; f, g)\) it suffices to show, for each integer \(N \geq 0\), that \(Z(s; f, g)\) extends meromorphically to the right half-plane \(\text{Re}(s) > -N - \frac{1}{m}\) is regular at \(s = -N\), and that all of its poles in the half-plane \(\text{Re}(s) > -N - \frac{1}{m}\) are at most simple and occur among \(s\) of the form \(s = (q + p - \ell)/m\), with \(\ell \geq 0\) an integer. First take \(\text{Re}(s) > (q + p)/m\), choose any \(w > 0\) (taken sufficiently large below) and write using cubical coordinates \([11]\),

\[
Z(s; f, g) = \int_{\rho=0}^\infty \int_{\sigma \in \partial C^p_+} \rho^{p-1} g(\rho \sigma)f(\rho \sigma)^{-s} d\sigma d\rho \\
= \int_{\rho=0}^w \int_{\sigma \in \partial C^p_+} \rho^{p-1} g(\rho \sigma)f(\rho \sigma)^{-s} d\sigma d\rho \\
+ \int_{\rho=w}^\infty \int_{\sigma \in \partial C^p_+} \rho^{p-1} g(\rho \sigma)f(\rho \sigma)^{-s} d\sigma d\rho =: Z_1(s, w) + Z_2(s, w). \tag{28}
\]

\[8\] This formula coincides formally with the Jacobian for spherical coordinates. This means that every formula below remains valid on replacing \(\partial C^p_+\) by the spherical piece

\[S^{p-1}_+ = \{(x_1, \ldots, x_p) \in \mathbb{R}^p_0 | x_1^2 + x_2^2 + \cdots + x_p^2 = 1\}\]

and letting \(d\sigma\) stand for the spherical \((p-1)\)-dimensional volume form. However, cubical coordinates will usually result in simpler integrals, since \(\partial C^p_+\) is flat.
The integral $Z_1(s, w)$ over the compact set $[0, w] \times \partial C^p_+$ gives an entire function of $s$, so we turn to the continuation of $Z_2(s, w)$. Using the factorization \eqref{13}, we find

$$Z_2(s, w) = \int_{\partial C^p_+} \int_w^\infty \rho^{p-1-m_s} f_{\text{top}}(\sigma)^{-s} g(\rho \sigma)(1 + r(\rho \sigma))^{-s} \, d\rho \, d\sigma.$$ 

Following Mahler we replace $(1 + r(\rho \sigma))^{-s}$ by its finite Taylor expansion. For $k$-times continuously differentiable $G : [0, 1] \to \mathbb{C}$ we have for $k \geq 1$ and $0 \leq y \leq 1$ \cite[§5-41]{17}

$$G(y) = \sum_{\lambda=0}^{k-1} \frac{G^{(\lambda)}(0)}{\lambda!} y^\lambda + \frac{1}{(k-1)!} \int_0^y G^{(k)}(t)(y-t)^{k-1} \, dt.$$ 

For $0 \leq t \leq 1$ and $r \in \mathbb{C}$ with $|r| < 1$, let

$$G(t) = G_r(t) := (1 + tr)^{-s},$$ 

where we use the principal branch. Then

$$\frac{G^{(\lambda)}(0)}{\lambda!} = r^\lambda \prod_{j=0}^{\lambda-1} (-s-j) = r^\lambda \binom{-s}{\lambda}.$$ 

The remainder term for $y = 1$ is

$$\frac{1}{(k-1)!} \int_0^1 G^{(k)}(t)(1-t)^{k-1} \, dt = k r^k \binom{-s}{k} \int_0^1 \frac{(1-t)^{k-1}}{(1+tr)^{s+k}} \, dt.$$ 

Hence, $G(1) = (1 + r)^{-s}$ is given by

$$(1 + r)^{-s} = \sum_{\lambda=0}^{k-1} \binom{-s}{\lambda} r^\lambda + k \binom{-s}{k} r^k \int_0^1 \frac{(1-t)^{k-1}}{(1+tr)^{s+k}} \, dt.$$ 

For $w \geq \rho_f$ (large enough that $|r(\rho \sigma)| \leq \frac{1}{2}$ for $\rho \geq w$), Re$(s) > (q + p)/m$ and $N \geq 0$ an integer, $Z_2(s, w)$ (see \eqref{23}) can now be written

$$Z_2(s, w) = k \binom{-s}{k} N_k(s, w) + \sum_{\lambda=0}^{Nm+q+p} \binom{-s}{\lambda} M_\lambda(s, w),$$  

where $k := Nm + q + p + 1$, and $N_k = N_k(s, w)$ and $M_\lambda$ are given by

$$N_k := \int_{\partial C^p_+} \int_w^\infty f_{\text{top}}(\sigma)^{-s} \rho^{p-1-m_s} g(\rho \sigma)r(\rho \sigma)^s \int_0^1 \frac{(1-t)^{k-1}}{(1+tr(\rho \sigma))^{s+k}} \, dt \, d\rho \, d\sigma,$$ 

$$M_\lambda(s, w) := \int_{\partial C^p_+} \int_w^\infty f_{\text{top}}(\sigma)^{-s} \rho^{p-1-m_s} g(\rho \sigma)r(\rho \sigma)^\lambda \, d\rho \, d\sigma.$$
We now extend $N_k$ analytically in $s$. Since $|r(\rho \sigma)|$ decreases at least like $\rho^{-1}$ as $\rho \to \infty$, the integrand in (30) decreases at least like
\[ \rho^{p+q-k-m \text{Re}(s)-1} = \rho^{-m(N+\frac{1}{m} \text{Re}(s))-1}. \]
Hence $N_k(s, w)$ extends to an analytic function in the right half-plane $\text{Re}(s) > -N - \frac{1}{m}$.

To get the meromorphic continuation of $M_\lambda$ expand
\[ g(\rho \sigma) r(\rho \sigma)^\lambda = \rho^{q-\lambda} \sum_{h=0}^{q+(m-1)\lambda} A_{\lambda,h}(\sigma) \rho^{-h} \ (q := \deg(g), \ m := \deg(f)), \tag{32} \]
where, in view of (14) and Mahler’s Hypothesis on $f$, the $A_{\lambda,h}$ are rational functions with no poles in a neighborhood of $\partial C^p_+$. From (31) and (32) we find, for $\text{Re}(s) > (q+p)/m$,
\[ M_\lambda(s, w) = \sum_{h=0}^{q+(m-1)\lambda} \int_{\partial C^p_+} \int_{\rho=w}^{\infty} \rho^{q+p-m \lambda - h-1} A_{\lambda,h}(\sigma) f_{\text{top}}(\sigma)^{-s} d\rho d\sigma \]
\[ = \sum_{h=0}^{q+(m-1)\lambda} \frac{w^{q+p-m \lambda - h}}{ms + \lambda + h - q - p} \int_{\partial C^p_+} A_{\lambda,h}(\sigma) f_{\text{top}}(\sigma)^{-s} d\sigma. \tag{33} \]
The above expression gives a meromorphic continuation of $M_\lambda(s, w)$ to all $s \in \mathbb{C}$, with at most simple poles at rational points of the form $s = \frac{q+p-\ell}{m}$, with $\ell = \lambda + h$ an integer in the range $\lambda \leq \ell \leq q + m \lambda$. As $\lambda \geq 0$, (29) shows that it only remains to prove that $Z_2(s, w)$ is regular at $s = -N$.

For $\lambda > N$ there is no pole of $Z_2(s, w)$ at $s = -N$ because of the factor $(-\lambda)^s$ multiplying $M_\lambda$ in (29). Finally, $M_\lambda$ has no pole at $s = -N$ for $0 \leq \lambda \leq N$ since its left-most pole occurs at $s = \frac{q+p-(q+m \lambda)}{m} > -\lambda \geq -N$.

The above proof (mainly equations (28) to (33)) shows that the analytic continuation obtained for $Z(s; h, g)$ depends analytically on the coefficients of the polynomials involved. This concludes the proof of Mahler’s Theorem for $Z(s; f, g)$. 
We now show that Mahler’s formulas above for $Z(s; f, g)$ simplify at non-positive integers $s = -N$.

**Theorem 7.** Let $f \in \mathbb{C}[x]$ be a polynomial in $p$ variables satisfying Mahler’s Hypothesis, let $g \in \mathbb{C}[x]$ be any polynomial in $p$ variables, and let $N \geq 0$ be a non-negative integer. Then the value of the analytic continuation of the zeta integral (16) at $s = -N$ is

$$Z(-N; f, g) = \frac{1}{m} \sum_{\lambda=N+\lceil p/m \rceil}^{q+p+Nm} \frac{(-1)^{\lambda-N}}{\lambda-N} \frac{1}{(\lambda N)} \int_{\sigma \in \partial C_p^+} C_{\lambda,N}(\sigma) f_{\text{top}}(\sigma)^N d\sigma,$$

where $m = \deg(f)$, $q = \deg(g)$, $\lceil p/m \rceil$ is the smallest integer $\geq p/m$, the integral is over $\partial C_p^+$ defined in (12), $C_{\lambda,N}(\sigma)$ is the coefficient of $\rho^{-p-mN}$ in the rational function $g(\rho \sigma) r_f(\rho \sigma)^\lambda$, with $r_f$ as in (14), and $f_{\text{top}}$ is the degree-$m$ part of $f$.

**Proof.** Combining (28) and (29) in the proof of Mahler’s theorem, we find for $\text{Re}(s) > -N - \frac{1}{m}$, $k := q + p + Nm + 1 > N$ and any large enough $w$,

$$Z(s; f, g) = Z_1(s, w) + k \binom{-s}{k} N_k(s, w) + \sum_{\lambda=0}^{Nm+q+p} \binom{-s}{\lambda} M_\lambda(s, w).$$

Since $N_k$ and $M_\lambda$ are analytic at $s = -N$ for $\lambda \leq N$, we find

$$Z(-N; f, g) = Z_1(-N, w) + \sum_{\lambda=0}^{N} \binom{N}{\lambda} M_\lambda(-N, w)$$

$$+ \lim_{s \to -N} \sum_{\lambda=N+1}^{Nm+q+p} \binom{-s}{\lambda} M_\lambda(s, w).$$
We shall now see that \( Z_1(-N, w) \) and the first sum above cancel. Using (13), the binomial expansion and (32) we have

\[
Z_1(-N, w) = \int_{\partial C_p^+} \int_0^w \rho^{p-1} g(\rho \sigma) f(\rho \sigma)^N d\rho d\sigma
\]

\[
= \int_{\partial C_p^+} f_{\text{top}}(\sigma)^N \int_0^w \rho^{mN+p-1} g(\rho \sigma) (1 + r(\rho \sigma))^N d\rho d\sigma
\]

\[
= \sum_{\lambda=0}^N \left( \frac{N}{\lambda} \right) \int_{\partial C_p^+} f_{\text{top}}(\sigma)^N \int_0^w \rho^{mN+p-1} g(\rho \sigma) (1 + r(\rho \sigma))^N d\rho d\sigma
\]

\[
= \sum_{\lambda=0}^N \left( \frac{N}{\lambda} \right) \sum_{h=0}^{\lambda(m-1)} \left( \frac{\lambda}{\lambda+q} \right) \int_{\partial C_p^+} f_{\text{top}}(\sigma)^N A_{\lambda,h}(\sigma) \int_0^w \rho^{mN+p-1+q-\lambda-h} d\rho d\sigma
\]

By (33), however, the above is just \(- \sum_{\lambda=0}^N \left( \frac{N}{\lambda} \right) M_\lambda(-N, w)\). Note that there is no singularity of the integrals above at \( \rho = 0 \) since

\[
mN+p-1+q-\lambda-h \geq mN+p-1+q-\lambda-(q+\lambda(m-1)) = m(N-\lambda)+p-1,
\]

which is clearly non-negative for \( \lambda \leq N \). Returning to (35) we now have

\[
Z(-N; f, g) = \sum_{\lambda=N+1}^{p+q+Nm} \lim_{s \to -N} \binom{-s}{\lambda} M_\lambda(s, w).
\] (36)

We have seen from (33) that \( M_\lambda \) may have simple poles at points \( s = \frac{q+\ell}{m} \) with \( \ell = \lambda + h \) in the range \( \lambda \leq \ell \leq q + m\lambda \). Setting \( s = -N \) we find \( \ell = q + p + Nm \), so \( h = q + p + Nm - \lambda \). As \( 0 \leq h \leq q + (m-1)\lambda \) in (33), \( M_\lambda \) can have a pole at \( s = -N \) only if

\[
h = q + p + Nm - \lambda \leq q + (m-1)\lambda, \quad \text{i.e.} \quad \lambda \geq N + \frac{p}{m}.
\]

Since \( \lambda \) is an integer, \( \lambda \geq N + \lfloor p/m \rfloor \), whence from (36) we have

\[
Z(-N; f, g) = \sum_{\lambda=N+\lfloor p/m \rfloor}^{p+q+Nm} \left( \frac{-s}{\lambda} \right) \lim_{s \to -N} \frac{\binom{-s}{\lambda}}{s+N} \cdot \text{Residue}_{s=-N}(M_\lambda(s, w)).
\]

Induction on \( \lambda \) (with \( \lambda \geq N + 1 \)) shows

\[
\lim_{s \to -N} \frac{\binom{-s}{\lambda}}{s+N} = \frac{(-1)^{\lambda-N}}{\lambda-N} \frac{1}{\binom{\lambda}{N}}.
\]
From (33), on the other hand, we find
\[
\text{Residue}_{s=-N} \left( M_\lambda(s, w) \right) = \frac{1}{m} \int_{\partial C_+^p} A_{\lambda q+p+Nm-\lambda}(\sigma) f_{\text{top}}(\sigma)^N d\sigma.
\]

Combining the last three equations we see that we have proved Theorem 7, but with \( C_{\lambda,N} \) replaced by \( A_{\lambda q+p+Nm-\lambda} \). Examining the definition of the \( A_{\lambda q} \) in (32), we see that \( A_{\lambda q} \) is the coefficient of \( \rho^{-\lambda-h} \) in \( g(\rho \sigma) r(\rho \sigma)^\lambda \). Thus \( A_{\lambda q+p+Nm-\lambda} \) is the coefficient of \( \rho^{-\lambda-(q+p+Nm-\lambda)} = \rho^{-p-Nm} \), i.e. \( C_{\lambda,N} \) in (34).

To apply Theorem 7 to \( N = 0 \), recall from (14) that for fixed \( \sigma \in \partial C_+^p \), \( f_{\text{top}}(\rho \sigma) = 1 + r(\rho \sigma) \) is an analytic function of \( 1/\rho \) (considering \( \rho \) now as complex with \( |\rho| \gg 0 \)) and that \( r(\rho \sigma) \to 0 \) as \( |\rho| \to \infty \). Thus the principal value \( \log \left( 1 + r(\rho \sigma) \right) \) is an analytic function of \( 1/\rho \) for \( 1/\rho \) in a disc near 0. In particular, the coefficient of any power of \( 1/\rho \) in the Laurent expansion of \( \log \left( 1 + r(\rho \sigma) \right) \) is well-defined.

**Corollary 8.** Let \( f \in \mathbb{C}[x] \) be a polynomial in \( p \) variables satisfying Mahler’s Hypothesis and let \( g \) be any polynomial in \( p \) variables. Then \( \deg(f) \cdot Z(0; f, g) \) is the coefficient of \( \rho^{-p} \) in the Laurent expansion of
\[
- \int_{\sigma \in \partial C_+^p} g(\rho \sigma) \log \left( \frac{f}{f_{\text{top}}}(\rho \sigma) \right) d\sigma.
\]

Before we prove Corollary 8, we use it to prove the product formula (25), i.e.
\[
\deg \left( \prod_{j=1}^n f_j \right) \cdot Z(0; \prod_{j=1}^n f_j, g) = \sum_{j=1}^n \deg(f_j) \cdot Z(0; f_j, g),
\]

where we assume that all the polynomials \( f_j \) satisfy Mahler’s Hypothesis (with the same \( p \)). Induction on \( n \) shows that it suffices to deal with the case \( n = 2 \). Suppose then that \( f_1 \) and \( f_2 \) satisfy Mahler’s Hypothesis. Since \( (f_1 \cdot f_2)_{\text{top}} = (f_1)_{\text{top}} \cdot (f_2)_{\text{top}} \), the product \( f_1 f_2 \) also satisfies Mahler’s Hypothesis. From
\[
\frac{f_1 f_2}{(f_1 f_2)_{\text{top}}} = \frac{f_1}{(f_1)_{\text{top}}} \cdot \frac{f_2}{(f_2)_{\text{top}}},
\]
and Corollary 8 we have (letting $\text{Coeff}_{\rho^{-p}}$ stand for the coefficient of $\rho^{-p}$)

$$\deg(f_1f_2)Z(0; f_1f_2, g) = -\text{Coeff}_{\rho^{-p}}\int_{\sigma \in \partial C^p_+} g(\rho \sigma) \log\left(\frac{f_1f_2}{(f_1f_2)_{\text{top}}} (\rho \sigma)\right) d\sigma$$

$$= -\text{Coeff}_{\rho^{-p}}\int_{\sigma \in \partial C^p_+} g(\rho \sigma) \left(\log\left(\frac{f_1}{(f_1)_{\text{top}}} (\rho \sigma)\right) + \log\left(\frac{f_2}{(f_2)_{\text{top}}} (\rho \sigma)\right)\right) d\sigma$$

$$= \deg(f_1)Z(0; f_1, g) + \deg(f_2)Z(0; f_2, g).$$

Note that it is legitimate to use

$$\log\left(\frac{f_1}{(f_1)_{\text{top}}} \cdot \frac{f_2}{(f_2)_{\text{top}}}\right) = \log\left(\frac{f_1}{(f_1)_{\text{top}}}\right) + \log\left(\frac{f_2}{(f_2)_{\text{top}}}\right)$$

in the second equation above since we only take (principal value) logarithms of complex numbers near 1 as $\rho \to \infty$.

**Proof (of Corollary 8).** Applying Theorem 7 at $s = 0$ we find

$$\deg(f)Z(0; f, g) = \int_{\sigma \in \partial C^p_+} \text{Coeff}_{\rho^{-p}} \left(g(\rho \sigma) \sum_{\lambda = [p/m]}^{q + p} \frac{(-1)^\lambda}{\lambda} r(\rho \sigma)^\lambda\right) d\sigma. \quad (37)$$

The above sum would be a logarithm if only we could extend the sum to all $\lambda \geq 1$. This is possible since

$$\text{Coeff}_{\rho^{-p}} \left(g(\rho \sigma)r(\rho \sigma)^\lambda\right) = 0 \quad (38)$$

for $1 \leq \lambda < p/m$ and also for $\lambda > q + p$. Indeed, the powers $\rho^t$ appearing in the $p$-expansion of $g(\rho \sigma)r(\rho \sigma)^\lambda$ in (32) are all in the range $-m\lambda \leq t \leq q - \lambda$. For $t = -p$, this amounts to $\frac{p}{m} \leq \lambda \leq p + q$. Thus, outside this range there is no coefficient of $\rho^{-p}$ in $g(\rho \sigma)r(\rho \sigma)^\lambda$.

Since $|r(\rho \sigma)| < \frac{1}{2}$ for large enough $\rho$, we obtain a convergent series on letting $\lambda$ in (37) range over all $\mathbb{N}$. From (37), (38) and (13) we get

$$\deg(f)Z(0; f, g) = \int_{\sigma \in \partial C^p_+} \text{Coeff}_{\rho^{-p}} \left(g(\rho \sigma) \sum_{\lambda = 1}^{q + p} \frac{(-1)^\lambda}{\lambda} r(\rho \sigma)^\lambda\right) d\sigma$$

$$= -\text{Coeff}_{\rho^{-p}} \left(\int_{\sigma \in \partial C^p_+} g(\rho \sigma) \log(1 + r(\rho \sigma)) d\sigma\right)$$

$$= -\text{Coeff}_{\rho^{-p}} \left(\int_{\sigma \in \partial C^p_+} g(\rho \sigma) \log\left(\frac{f}{f_{\text{top}}(\rho \sigma)}\right) d\sigma\right),$$

concluding the proof.
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