Residual calculation in He’s frequency–amplitude formulation

Yue Wu¹ and Yan-Ping Liu²

Abstract
He’s frequency–amplitude formulation and its modifications mainly depend upon the residual calculation. A suitable choice of a residual leads to an ideal result. This paper discusses some effective methods for the residual calculation, and a modification with a free parameter is suggested to effectively estimate the frequency of a nonlinear oscillator. Furthermore, an energy-based residual calculation is also suggested, which is deduced from a variational principle.

Keywords
He’s frequency–amplitude formulation, residual equation, Duffing oscillator, variational theory, semi-inverse method

Introduction
A nonlinear vibration problem arises everywhere from everyday life to an atom vibration. Recently, the nanoscale vibration for smart adhesion,¹,² nanoscale nonlinear transverse vibration in a nanofiber-reinforced concrete pillar,³ and the vibration of the micro-electromechanical systems⁴–⁷ have been caught much attention. The most important property of a nonlinear vibration system is the nonlinear relationship between the frequency and the amplitude. The self-cleaning of the Gecko’s feet and Lotus’ surface can be explained by the high frequency property of the water molecules on the surface.⁵ Gecko effect and Lotus effect can be explained by the geometric potential theory.⁸–¹⁴

There are many analytical methods to accurately determine the frequency–amplitude relationship of a nonlinear oscillator, for example, the variational iteration method (VIM),¹⁵–¹⁸ the homotopy perturbation method (HPM),¹⁹–²³ the Taylor series method,²⁴–²⁷ among which He’s frequency–amplitude formulation²⁸–³¹ has been proved to be the simplest and relatively effective, which was first proposed by a Chinese mathematician, Prof. Ji-Huan He, in his famous review article “Some asymptotic methods for strongly nonlinear equations” in 2006.²⁸ Though only few lines were given, the method caught an immediate attention due to its extreme simplicity and remarkable accuracy, and there was a very hot discussion in Computers & Mathematics with Applications (issue 8 of 2011) and Journal of Low Frequency Noise Vibration and Active Control (Issues 3–4 of 2019). There were many modifications available in literature, and this paper focuses on Ren–Hu’s modification.³²

Nonlinear oscillator and its variational formulation
We consider a nonlinear oscillator in the form

\[ u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0 \]  (1)

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where \( f \) is a continuous function satisfying \( f(-u) = -f(u) \) and \( A \) is the amplitude.

Its variational formulation can be established by the semi-inverse method, which is

\[
J(u) = \int_0^{T/4} \left\{ \frac{1}{2} u'^2 - F(u) \right\} dt
\]

(2)

where \( F \) is the potential energy defined as

\[
\frac{\partial}{\partial u} F(u) = f(u)
\]

(3)

In equation (2), \( \frac{1}{2} u'^2 \) is the kinetic energy, so the variational principle is the Hamilton principle, so we have the following Hamilton invariant.

\[
\frac{1}{2} u'^2 + F(u) = H
\]

(4)

where \( H \) is a Hamilton constant. Equation (4) is equivalent to equation (1). To prove this, we differentiate equation (4) with respect to time

\[
\frac{d}{dt} \left[ \frac{1}{2} u'^2 + F(u) \right] = \frac{d}{dt} (H)
\]

(5)

Considering \( H \) is a constant, we have

\[
u' u'' + \frac{\partial F(u)}{\partial u} u' = 0
\]

(6)

Equation (6) implies equation (1) or \( u' = 0 \), the latter has no physical meaning and can be ignored.

**He’s frequency–amplitude formulation and its modification**

He’s formulation is to choose two arbitrary frequencies \( \omega_1 \) and \( \omega_2 \) to produce two residual equations

\[
R_1(t) = f(A \cos \omega_1 t) - \omega_1^2 A \cos \omega_1 t
\]

(7)

and

\[
R_2(t) = f(A \cos \omega_2 t) - \omega_2^2 A \cos \omega_2 t
\]

(8)

The residuals in He’s frequency formulation are calculated as follows

\[
\bar{R}_1 = \frac{4}{T} \int_0^{T/4} R_1(t) \cos \omega_1 t dt
\]

(9)

and

\[
\bar{R}_2 = \frac{4}{T} \int_0^{T/4} R_2(t) \cos \omega_2 t dt
\]

(10)

The square of the frequency can be obtained as

\[
\omega_{He}^2 = \frac{\omega_2^2 \bar{R}_2 - \omega_1^2 \bar{R}_1}{\bar{R}_2 - \bar{R}_1}
\]

(11)
In Ren–Hu’s modification,\textsuperscript{32} the residuals are calculated as

\[
\bar{R}_1 = \frac{4}{T} \int_{0}^{T} R_1(t) \, dt \tag{12}
\]

and

\[
\bar{R}_2 = \frac{4}{T} \int_{0}^{T} R_2(t) \, dt \tag{13}
\]

Instead of equation (11), Ren and Hu suggested the following modification

\[
\omega_{\text{Ren–Hu}}^2 = \frac{\omega_1^2 \bar{R}_2 - \omega_2^2 \bar{R}_1}{R_2 - \bar{R}_1} \tag{14}
\]
The difference of He’s formulation and Ren–Hu’s modification lies on the calculation of the residuals. As illustrated in Figure 1, both methods can lead to a reasonable result.

**Calculation of the residuals**

Residual calculation plays an important role in He’s formulation, hereby gives an improvement

\[
R_1 = m\tilde{R}_1 + (1 - m)\tilde{R}_1 \quad (15)
\]

\[
R_2 = m\tilde{R}_2 + (1 - m)\tilde{R}_2 \quad (16)
\]

where \(m\) is a real number.

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**Figure 2.** Comparison of approximate solution \(u\) with the exact solution.

\[ A = 1 \text{ and } \alpha_0 = 1, \alpha_1 = 1, \alpha_2 = \alpha_3 = 0 \]
\[ A = 1 \text{ and } \alpha_0 = 1, \alpha_1 = 10, \alpha_2 = \alpha_3 = 0 \]
\[ A = 1 \text{ and } \alpha_0 = 1, \alpha_1 = 100, \alpha_2 = \alpha_3 = 0 \]
\[ A = 1 \text{ and } \alpha_0 = 1, \alpha_1 = 1000, \alpha_2 = \alpha_3 = 0 \]
\[ A = 1 \text{ and } \alpha_0 = 1, \alpha_2 = 10, \alpha_1 = \alpha_3 = 0 \]
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\[ A = 1 \text{ and } \alpha_0 = 1, \alpha_2 = 1000, \alpha_1 = \alpha_3 = 0 \]
As a result, He’s formulation can be modified as

$$
\omega^2 = \frac{\omega_2^2 R_2 - \omega_1^2 R_1}{R_2 - R_1} = \frac{\omega_2^2 (mR_2 + (1 - m)\tilde{R}_2) - \omega_1^2 (m\tilde{R}_1 + (1 - m)\tilde{R}_1)}{(mR_2 + (1 - m)R_2) - (mR_1 + (1 - m)\tilde{R}_1)}
$$  \hspace{1cm} (17)

By suitable choice of \( m \), a most accurate result can be obtained. For the simplest case, the frequency can be estimated by

$$
\omega = \frac{\omega_{He} + \omega_{Ren-Hu}}{2}
$$  \hspace{1cm} (18)

**An example**

Consider the following Duffing oscillator\textsuperscript{27,40–42}

$$
u'' + \alpha_0 \nu + \alpha_1 \nu^3 + \alpha_2 \nu^5 + \alpha_3 \nu^7 = 0, \quad u(0) = A, \quad u'(0) = 0
$$  \hspace{1cm} (19)

![Graphs showing the comparison between exact solutions and approximations for Duffing oscillator](image-url)
By a simple calculation, we have

\[ \omega_{He} = \sqrt{x_0 + \frac{3}{4} x_1 A^2 + \frac{5}{8} x_2 A^4 + \frac{35}{64} x_3 A^6} \]  
(20)

\[ \omega_{Ren-Hu} = \sqrt{x_0 + \frac{2}{3} A^2 x_1 + \frac{8}{15} A^4 x_2 + \frac{16}{35} A^6 x_3} \]  
(21)

\[ \omega = \sqrt{x_0 + \frac{1}{3} x_1 A^2 + \frac{5}{8} x_2 A^4 + \frac{35}{64} x_3 A^6 + \sqrt{x_0 + \frac{3}{4} A^2 x_1 + \frac{8}{13} A^4 x_2 + \frac{16}{35} A^6 x_3}} \]  
(22)

Table 1 shows the accuracy of each frequency estimation and good accuracy can be obtained; and equation (22) gives a much better result.

The approximate analytical solution and the exact numerical solution of equation (19) are shown in Figure 2. The error between the approximate analytical solution and the exact numerical solution of equation (19) are shown in Figure 3.

**Conclusion**

This paper discusses the effect of residual estimation on the accuracy of the frequency obtained by He’s frequency formulation and its modification. Considering the calculation process for residuals in He’s formulation given in equation (11) and Ren–Hu’s modification given in equation (14) are equally simple, a more general frequency formulation is suggested with a free parameter by taking into account both He’s residuals and Ren–Hu’s ones. By suitable choice of the free parameter, the accuracy can be remarkably improved. The free parameter given in equation (17) can be optimally identified by the either the HPM\textsuperscript{19} or the VIM\textsuperscript{15} by using equation (17) as an initial guess, after one iteration, the free parameter can be effectively identified by requirement of no secular terms in the solution.
The present modification of He’s frequency formulation with a free parameter provides an auxiliary tool to further improvement of the accuracy without losing its simplicity, the method can be easily extended to other nonlinear vibrations with fractal derivatives, and we will discuss the application in a forthcoming paper.

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