Strategic Topology Switching for Security—Part I: Consensus & Switching Times

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Abstract—This two-part paper considers strategic topology switching for the second-order multi-agent system under attack. In Part I, we propose a strategy on switching times that enables the strategic topology-switching algorithm proposed in Part II to reach the second-order consensus in the absence of attacks. The control protocol introduced to the multi-agent system is governed only by the relative positions of agents. Based on the stability of switched linear systems, the strategy on the dwell time of topology-switching signal is derived. The primary advantages of the strategy in achieving the second-order consensus are: 1) the control protocol relies only on relative position measurements, no velocity measurements are needed; 2) the strategy has no constraint on the magnitude of coupling strength. Simulations are provided to verify the effectiveness of strategic topology switching in achieving the second-order consensus.

Index Terms—Multi-agent system, second-order consensus, strategic topology switching, dwell time.

I. INTRODUCTION

THE consensus of multi-agent systems with the first-order dynamics is a well-studied theoretical problem (see e.g., [1]–[4]) with many practical applications including decentralized computation [5], distributed optimization [6], [7], power sharing for droop-controlled inverters in islanded microgrids [8], clock synchronization for sensor network [9], and more. However, current and emerging systems, such as connected vehicles [1], [4], spacecraft [10], robot [11] and electrical power networks [12], rely on the second-order dynamics. This observation, coupled with the fact that the consensus algorithms designed for the first-order multi-agent systems cannot be directly applied to those with the second-order dynamics, is the main motivation of this work.

A second-order multi-agent system consists of a population of $n$ agents whose dynamics are governed by the following equations:

$$\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= u_i(t),
\end{align*}$$

where $x_i(t) \in \mathbb{R}$ is the position state, $v_i(t) \in \mathbb{R}$ is the velocity state, and $u_i(t) \in \mathbb{R}$ is the control protocol of the $i$th agent.

This first part of the two-part paper proposes a strategy on switching times that enables the strategic topology-switching algorithm (explained in detail in the second part of this series [13]) to reach the second-order consensus in the absence of attacks. In the following, we present the precise definition of the second-order consensus in this context.

Definition 1: [14] The second-order consensus in the multi-agent system (1) is achieved, if and only if the following holds

$$\begin{align*}
\lim_{t \to \infty} |x_i(t) - x_j(t)| &= 0, \\
\lim_{t \to \infty} |v_i(t) - v_j(t)| &= 0, \forall i, j = 1, \cdots, n.
\end{align*}$$

for any initial condition.

A. Related Work

Among the few studies [4], [14], [15] on this problem, two commonly studied control protocols are

$$\begin{align*}
u_i(t) &= \alpha \sum_{j=1}^{n} a_{ij} (x_j - x_i) - \beta v_i, \\
u_i(t) &= \alpha \sum_{j=1}^{n} a_{ji} (x_j - x_i) + \beta \sum_{j=1}^{n} a_{ji} (v_j - v_i),
\end{align*}$$

where $a_{ij}$ is the element of the coupling matrix describing the structure of the undirected/directed control network, and $\alpha$ and $\beta$ are the coupling strengths. In the past several years, based on the two consensus protocols (3) and (4), some exciting results are reported. For example, Mei et al. [16] proposed the adaptive control gain to relax the conditions on the coupling strengths and obtain a fully distributed consensus algorithm; to deal with the problem of limited agent interaction ranges, Song and You [17] proposed the range-based varying weighing; Qin et al. [18] studied the leaderless consensus and leader-following consensus and report some lower bounds for coupling strengths. The conditions of the well-studied second-order consensus are summarized in Table I which shows that the individual/relative velocity measurements are necessary for control protocols.

| Reference | Constraint on | Protocol |
|-----------|---------------|----------|
| [14], [15] | magnitude of coupling strength | Directed (4) |
| [16], [18] | magnitude of coupling strength | Directed weighted (3) |
| [16] | magnitude of coupling strength | Directed weighted (4) |
| [17] | magnitude of coupling strength, initial conditions | Directed weighted (3) |
| [4] | NONE | Undirected (3) |
B. Motivation of Part-I Paper

In practice, many network topologies are non-static, such as the communication networks of mobile agents [2] and brain networks [19]. Recent studies of dynamical networks have highlighted the important role played by the network topology [20]–[23]. For example, Menck et al. [20] find that in the numerical simulations of artificially generated power grids, tree-like connection schemes, so-called dead ends and dead trees, can strongly diminish the stability; Schultz et al. [23] show that how the addition of links can change the synchronization properties of the network. In power grids, a certain group of generators can be cut off or connected to prevent cascading instability [21]; and in the context of power systems, topology switching is equivalent to actively tripping or re-closing transmission lines, or adjusting active power output or transmission line reactance to improve the linear stability of the power system [24]. With the advance in the wireless communication networks, it is more feasible to set the topology of communication network as a control variable [25]. These inspire us to take the network topology as a control variable.

In past studies, unintentional topology changes in a networked control system are commonly handled as disturbances [2], [26]–[29] to the system. To the best of our knowledge, active/strategic topology switching for networked control systems has not been systemically studied, with exception being [30].

However, an intriguing question regarding the dynamic topology is whether the well-studied control protocols can be simplified significantly, and the advantage of control algorithm under fixed topology can still be maintained. This Part-I paper gives positive answers.

For multi-agent systems, especially the large scaled networked systems, due to the lack of centralized measurements, the systems are prone to attack [31]. The study of security in multi-agent systems is increasingly important. One of the fundamental problems of security is detection of stealthy attack. Recent experiment of stealthy false-data injection attacks on networked control system [32] showed the changes in the system dynamics could be used to reveal stealthy attack. To have changes in the system dynamics to reveal zero-dynamics attack, Teixeira et al. [33] considered the method of modifying input matrix or modifying output matrix. Obviously, topology switching works as another method such that the dynamics of multi-agent systems can have changes. Before using the topology switching to reveal zero-dynamics attack, the question that whether the changes on system dynamics can destroy system stability in the absence of attacks must be investigated. The strategy on switching times proposed in this Part-I paper can provide a positive answer that when the topology should strategically switch such that the agents in system can have the ability of achieving consensus in the absence of attacks.

Another interesting question pertains to the detectability of zero-dynamics attacks by strategic topology switching. This question is studied in detail in the second part of this two-part paper [13].

C. Contribution of Part-I Paper

The two-part paper comprises a study of a strategic topology-switching algorithm for the second-order multi-agent system under attack. Part I provides a basis for this strategic algorithm: when the topology should strategically switch such that the agents can have the ability of reaching consensus in the absence of attacks. The contribution of this paper is twofold, which can be summarized as follows.

- We propose a control protocol with only measurements of relative positions for the second-order multi-agent system. Based on the stability of switched linear systems, we obtain a strategy on dwell time of topology-switching signal that enables consensus in the absence of attacks.
- Based on the strategy on switching times, through employing a finite-time consensus network, we propose a decentralized topology-switching algorithm to achieve the second-order consensus without any constraint on the magnitude of coupling strength.

The remainder of this paper is organized as follows: Section II presents the preliminaries and problem formulation; the strategy on switching times is given in Section III; Section IV presents a topology-switching algorithm. Numerical examples are given in Section V; finally, Section VI concludes this Part-I paper.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Preliminaries

1) Notation: Let $i = \sqrt{-1}$ be the imaginary unit. $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the set of $n$-dimensional real vectors and the set of $m \times n$-dimensional real matrices, respectively. $\mathbb{N}$ represents the set of the natural numbers. Let I be identity matrix with compatible dimension. $1_n \in \mathbb{R}^n$ and $0_n \in \mathbb{R}^n$ denote the vector with all ones and the vector with all zeros, respectively. The superscript ‘$\top$’ stands for matrix transpose. $\mathcal{L}(\cdot)$ denotes the Laplace transform.

2) Graph Theory: The interaction among $n$ agents is modeled by an undirected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{\theta_1, \theta_2, \ldots, \theta_n\}$ is the set of vertices that represent $n$ agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges of the graph $G$. An undirected edge in $G$ is denoted by $a_{ij} = (\theta_i, \theta_j) \in \mathcal{E}$, where $a_{ij} = a_{ji} = 1$ if agents $i$ and $j$ interact with each other, and $a_{ij} = a_{ji} = 0$ otherwise. Assume that there are no self-loops, i.e., for any $\theta_i \in \mathcal{V}$, $a_{ii} \notin \mathcal{E}$. A path is a sequence of connected edges in a graph. A graph is a connected graph if there is a path between every pair of vertices.

Lemma 1: [34] If the undirected graph $G$ is connected, then its Laplacian $\mathcal{L} \in \mathbb{R}^{n \times n}$ has a simple zero eigenvalue (with eigenvector $1_n$) and all its other eigenvalues are positive and real.

Lemma 2: [35] The Laplacian of a path graph $P_n$ has the eigenvalues as

$$
\lambda_k = 2 - 2 \cos \left( \frac{(k - 1) \pi}{n} \right), \ k = 1, \ldots, n.
$$

(5)
Consider the following multi-agent system with control protocol under switching topology:

\begin{equation}
\dot{x}_i(t) = v_i(t), \quad i = 1, \ldots, n \tag{6a}
\end{equation}

\begin{equation}
\dot{v}_i(t) = u_i(t) = 2 \gamma \sum_{j=1}^{n} a_{ij}(t) (x_j(t) - x_i(t)), \tag{6b}
\end{equation}

where \( \gamma > 0 \) is the coupling strength, \( \sigma(t) : [0, \infty) \to \mathcal{S} \equiv \{1, 2, \ldots, s\} \), \( s \in \mathbb{N} \), is the switching signal of the interaction topology of communication network, i.e., \( \sigma(t) = p_k \in \mathcal{S} \) for \( t \in [t_k, t_{k+1}) \) means the \( p^k \) topology is activated over the time interval \([t_k, t_{k+1})\), and \( a_{ij}^{p_k} \) is the element of the coupling matrix that describes the activated \( p^k \) topology of undirected communication network.

For the undirected topology, considering the fact \( a_{ij} = a_{ji} \), from (6b) it is straightforward to verify that \( \sum_{i=1}^{n} \dot{v}_i(t) = 0 \), \( \forall t \geq 0 \), which implies that the average position

\begin{equation}
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i(0) + \frac{1}{n} \sum_{i=1}^{n} v_i(0) t, \tag{7}
\end{equation}

proceeds with the constant velocity

\begin{equation}
\bar{v} = \frac{1}{n} \sum_{i=1}^{n} v_i(t) = \frac{1}{n} \sum_{i=1}^{n} v_i(t_0). \tag{8}
\end{equation}

If the second-order consensus is achieved, the individual velocities will be equivalent to the average of initial of velocities, i.e., \( \lim_{t \to \infty} \dot{v}_i(t) - \bar{v} = 0, i = 1, \ldots, n \). Based on \( \bar{v} \), define fluctuations:

\begin{equation}
\tilde{x}_i(t) = x_i(t) - \frac{1}{n} \sum_{i=1}^{n} x_i(0) - \bar{v} t, \tag{9a}
\end{equation}

\begin{equation}
\tilde{v}_i(t) = v_i(t) - \bar{v}, \tag{9b}
\end{equation}

where \( \bar{v} \) is given by (8). Then the dynamics (6) can be transformed equivalently as

\begin{equation}
\dot{\tilde{x}}_i(t) = \tilde{v}_i(t), \quad i = 1, \ldots, n \tag{10a}
\end{equation}

\begin{equation}
\dot{\tilde{v}}_i(t) = u_i(t) = \alpha \sum_{j=1}^{n} a_{ij}^{p_k}(\tilde{x}_j(t) - \tilde{x}_i(t)). \tag{10b}
\end{equation}

It follows from (9) and (8) that

\begin{equation}
\mathbf{1}^T \bar{x}(t) = 0, \tag{11}
\end{equation}

\begin{equation}
\mathbf{1}^T \bar{v}(t) = 0. \tag{12}
\end{equation}

In practice, high switching frequency can result in increased switching costs hence, is undesirable. In the context of attack detection by strategic topology switching, low switching frequency is also undesirable, since it can result in attacks going undetected for a long period of time. Based on these considerations, we impose the minimum and maximum dwell times on the topology-switching signal \( \sigma(t) \).

**Definition 2:** For the second-order multi-agent system (10), the minimum dwell time \( \tau_{\min} \) and maximum dwell time \( \tau_{\max} \) of topology-switching signal satisfy

\begin{equation}
t_{k+1} - t_k \geq \tau_{\min}, \forall k \in \mathbb{N}_0, \tag{13}
\end{equation}

\begin{equation}
t_{k+1} - t_k \leq \tau_{\max}, \forall k \in \mathbb{N}_0. \tag{14}
\end{equation}

The following auxiliary lemmas will be used to prove the feasibility of the topology-switching algorithm studied in the following section.

**Lemma 3:** Consider the following system:

\begin{equation}
\dot{\tilde{x}}(t) = -\gamma L \int_0^t \tilde{x}(\tau) d\tau + \tilde{v}(t), \tag{15}
\end{equation}

where \( \gamma > 0 \), \( L \in \mathbb{R}^{n \times n} \) is the Laplacian matrix of a connected undirected graph; \( \tilde{x}(t) \in \mathbb{R}^n \) and \( \tilde{v}(t) \in \mathbb{R}^n \) satisfy (11) and (12), respectively. The individual position solutions \( \tilde{x}_i(t), i = 1, \ldots, n \), are

\begin{equation}
\tilde{x}_i(t) = \sum_{l=2}^{n} q_i q_l^T \left( \tilde{x}(0) \cos(\sqrt{\gamma \lambda_l} t) + \frac{\tilde{v}(0)}{\sqrt{\gamma \lambda_l}} \sin(\sqrt{\gamma \lambda_l} t) \right), t \geq 0 \tag{16}
\end{equation}

where \( \lambda_l (\lambda_1 = 0), l = 2, \ldots, n \), are the non-zero eigenvalues of \( L \), \( q_i = [q_{i1}, \ldots, q_{in}]^T \in \mathbb{R}^n \) is the eigenvector associated with the eigenvalue \( \lambda_i \) of \( L \).

**Proof:** See Appendix A.

**Lemma 4:** Consider the function

\begin{equation}
F(t) = \frac{\tilde{x}^T(t) \tilde{x}(t)}{2} + \frac{\tilde{v}^T(t) \tilde{v}(t)}{2}, \tag{17}
\end{equation}

with \( \tilde{v}(t) = \dot{\tilde{x}}(t) \in \mathbb{R}^n \). Along the dynamics (15), if the Laplacian matrix \( L \) has distinct eigenvalues and \( \varphi \) satisfies

\( 0 < \varphi \neq 0 \) \( \lambda_i (L) \), \( \forall i = 2, \ldots, n \),

then the following situation:

\begin{equation}
F(t) \equiv \varphi \neq 0, \quad \forall t \geq 0, \tag{19}
\end{equation}

would never happen.

**Proof:** See Appendix C.

**Remark 1 (Motivation of Lemma 4):** The solution in Lemma 3 implies that the multi-agent system (15), with \( \tilde{x}(t) \) and \( \tilde{v}(t) \) satisfying (11) and (12), can be viewed as one class of coupled oscillators. For this system, the inadmissible energy functions, i.e., nonzero constant positive functions exist, can easily exist. Take the positive function \( F(t) = \frac{\gamma \tilde{x}^T(t) L \tilde{x}(t) + \tilde{v}^T(t) \tilde{v}(t)}{2} \) as an example, its derivative along the system (15) is \( \dot{F}(t) = \gamma \tilde{v}^T(t) L \tilde{x}(t) + \tilde{v}^T(t) \tilde{v}(t) = \gamma \tilde{v}^T(t) (L \tilde{x}(t) - \gamma \tilde{v}(t) \tilde{x}(t)) \equiv 0 \), which means that with the nonzero initial conditions, the function is a nonzero constant over time, thus it is inadmissible. Lemma 4 provides a guide to construct an admissible energy function.

**Remark 2:** For the undirected communication network considered in this paper, there exists topologies with Laplacian matrices that have distinct eigenvalues. With the fact of \( 0 \leq \frac{(k-1)\pi}{n} < \pi \), \( \forall k = 1, \ldots, n \), Lemma 2 implies the Laplacian matrix of a path graph has distinct eigenvalues.

### III. STRATEGY ON SWITCHING TIMES

The multi-agent system under switching topology (10) can be modeled as a switched linear system:

\begin{equation}
\dot{z}(t) = A_{\sigma(t)} z(t) \tag{20}
\end{equation}

where \( z(t) \in \mathbb{R}^m \), \( A_{\sigma(t)} \in \mathbb{R}^{m \times m} \) and \( \sigma(t) \in \mathcal{S} \).
Lemma 3 shows that each subsystem of the switched system (10), i.e., the multi-agent system (10) under each fixed topology, is not stable. Hence, the problem of strategic topology switching studied in the following sections would be the stabilizing switching rule design for the switched systems (10) without stable subsystems. Before proceeding on, we present the following stability result which is useful in deriving the strategy on switching times that enables agents the ability of reaching the second-order consensus.

**Lemma 5**: [36] Given scalars \( \alpha \geq \alpha^* > 0, 0 < \beta < 1, 0 < \tau_{\text{min}} \leq \tau_{\text{max}} \). Consider switched linear system (20). If there exists a set of matrices \( P_{r,q} > 0, q = 0, 1, \cdots, L, r \in \mathfrak{S} \), such that \( \forall q = 0, 1, \cdots, L - 1, \forall r, s \in \mathfrak{S}, \forall q = 0, 1, \cdots, L - 1, \) we have

\[
\begin{align*}
A_r^T P_{r,q} + P_{r,q} A_r + \Psi_q - \alpha P_{r,q} &< 0, \\
A_r^T P_{r,q+1} + P_{r,q+1} A_r + \Psi_q - \alpha P_{r,q+1} &< 0, \\
A_r^T P_{r,L} + P_{r,L} A_r - \alpha P_{r,L} &< 0, \\
\ln(1 + \alpha) &< 0, \\
\end{align*}
\]

where \( \Psi_q = L(P_{q+1,q} - P_{q,q}) \). Then, the system (20) is globally uniformly asymptotically stable under any switching signal \( \sigma(t) \) (13) and (14).

Because the equilibrium of the multi-agent system (10) is \( (\bar{x}^*, \bar{v}^*) = (0_n, 0_n) \), we can use Lemma 5 to conclude stability of the second-order consensus. However, without considering additional system information, Lemma 5 is not feasible along the multi-agent system (10).

**Corollary 1**: Along the multi-agent system (10), Lemma 5 is infeasible.

**Proof**: See Appendix D

Let \( \sigma(t) = r \in \mathfrak{S} \) for \( t \in [t_k, t_{k+1}), k \in \mathbb{N} \), the dynamics in (10) can be rewritten as \( \ddot{\tilde{v}}(t) = -\gamma L_r \int_{t_k}^{t} \hat{v}(r) d\tau + \hat{v}(t), t \in [t_k, t_{k+1}). \) Therefore, Lemma 3 implies that the multi-agent agent system (10) under each fixed topology has a period \( \mathfrak{P} \) such that

\[
\begin{align*}
\tilde{v}_i(t) &= \tilde{v}_i(t + \mathfrak{P}), i = 1, \cdots, n \\
\bar{x}_i(t) &= \bar{x}_i(t + \mathfrak{P}), \forall i \in [t_k, t_{k+1}), k \in \mathbb{N}. 
\end{align*}
\]

**Remark 3**: The obtained solutions in Lemma 3 imply that the coupling strength \( \gamma > 0 \) can control the useful period \( \mathfrak{P} \).

The period \( \mathfrak{P} \) can be used to make Lemma 5 to be applicable to the multi-agent system (10) to derive the strategy on the switching times.

**Theorem 1**: Consider the second-order multi-agent system (10). For the given period \( \mathfrak{P} \) satisfying (26), scalars \( 0 < \beta < 0, 0 < \alpha < 1 \) and \( L \in \mathbb{N} \). If the dwell time \( \tau \) satisfies

\[
(\beta^{-\frac{1}{2}} - 1) \frac{L}{\alpha - \xi} < \tau \leq \tau_{\text{max}} = \tau_{\text{min}} + \frac{m \mathfrak{P}}{2}, m \in \mathbb{N},
\]

with

\[
\begin{align*}
\xi &< \alpha, \\
0 < \tau_{\text{max}} < & \frac{-\ln \beta}{\alpha}, \\
0 < \tau_{\text{max}} &< \frac{m \mathfrak{P}}{2} \left( \beta^{-\frac{1}{2}} - 1 \right) \frac{L}{\alpha - \xi}.
\end{align*}
\]

where \( \gamma \) is the coupling strength of the multi-agent system (10) and \( L_r \) is the \( i \)th eigenvalue of the Laplacian matrix \( L_r \). Then the second-order consensus can be achieved by Definition 1.

**Proof of Theorem 1**: Note the considered multi-agent system (10) can be described by switched system (20) where \( z(t) = [\bar{x}_1(t), \cdots, \bar{x}_n(t), \tilde{v}_1(t), \cdots, \tilde{v}_n(t)]^T \in \mathbb{R}^{2n} \) and

\[
A_{\sigma(t)}(t) = \begin{bmatrix} 0_{n \times n} & I \\ -\gamma L_{\sigma(t)} & 0_{n \times n} \end{bmatrix}.
\]

For each activated topology of the multi-agent system (10), consider the positive definite matrix

\[
P_{r,q} = \begin{bmatrix} P_{\bar{r},q} & 0_{n \times n} \\ 0_{n \times n} & P_{\dot{v},q} \end{bmatrix},
\]

where

\[
P_{\bar{r},q} = \beta^2 \tilde{h} I, q = 0, \cdots, L, \forall r \in \mathfrak{S}
\]

with \( h > 0 \). It follows from (34) that

\[
P_{\dot{v},0} = \beta \bar{P}_{\dot{v},q+1}, q = 0, \cdots, L - 1, \forall r \in \mathfrak{S},
\]

Substituting the matrices \( P_{r,q} \) (33) and \( A_{\sigma(t)}(t) \) (32) into the conditions (21), (22) and (23) yields, respectively,

\[
R_{r,q} \triangleq \begin{bmatrix} Q_{r,q} & (I - \gamma L_r \dot{P}_{\dot{v},q}) \\ (I - \gamma L_r) \dot{P}_{\bar{r},q} & Q_{r,q} \end{bmatrix} < 0, \quad (37)
\]

\[
\dot{R}_{r,q} \triangleq \begin{bmatrix} Q_{r,q} & (I - \gamma L_r) \dot{P}_{\dot{v},q+1} \\ (I - \gamma L_r) \dot{P}_{\bar{r},q+1} & Q_{r,q} \end{bmatrix} < 0, \quad (38)
\]

\[
S_{r,L} \triangleq \begin{bmatrix} -\alpha P_{\bar{r},L} & (I - \gamma L_r) \dot{P}_{\dot{v},L} \\ (I - \gamma L_r) \dot{P}_{\bar{r},L} & -\alpha P_{\bar{r},L} \end{bmatrix} < 0, \quad (39)
\]

where

\[
Q_{r,q} = \frac{L}{\tau_{\text{min}}} (\dot{P}_{r,q+1} - \dot{P}_{r,q}) - \alpha \dot{P}_{r,q},
\]

\[
\dot{Q}_{r,q} = \frac{L}{\tau_{\text{min}}} (\dot{P}_{r,q+1} - \dot{P}_{r,q}) - \alpha \dot{P}_{r,q+1}.
\]

Let \( W \) be the orthogonal matrix of the symmetric matrix \( L_r \), and denote \( W^T L_r W \triangleq \Lambda_r = \text{diag} \{ \lambda_1(L_r), \cdots, \lambda_n(L_r) \} \). Considering the matrices \( P_{r,q} \) (33) with \( P_{r,q} \) (34), the conditions (37), (38) and (39) can be equivalently expressed in the form of eigenvalue as

\[
\begin{align*}
\frac{L}{\tau_{\text{min}}} (\dot{P}_{r,q+1} - \dot{P}_{r,q}) - \alpha \dot{P}_{r,q} &\pm (1 - \gamma \Lambda_r) \dot{P}_{r,q} < 0, \\
\frac{L}{\tau_{\text{min}}} (\dot{P}_{r,q+1} - \dot{P}_{r,q}) - \alpha \dot{P}_{r,q+1} &\pm (1 - \gamma \Lambda_r) \dot{P}_{r,q+1} < 0,
\end{align*}
\]

\[
- \alpha \dot{P}_{r,L} \pm (1 - \gamma \Lambda_r) \dot{P}_{r,L} < 0,
\]

where
The rest proof is divided into four steps based on Lemma 5.

**Step One:** It follows from (33) and (36) that \( P_{s,0} = \beta P_{r,s}, r \neq s \in S \), thus the condition (24) in Lemma 5 holds.

**Step Two:** Consider the right-hand of (27) and a function \( \hat{V}(t) = \hat{x}^T(t) \hat{P} \hat{x}(t) + \hat{v}^T(t) \hat{Q} \hat{v}(t) \) with \( \hat{Q} > 0 \) and \( \hat{P} > 0 \). Lemma 3 shows that the multi-agent system (10) has period \( T \) satisfying (26), so \( \hat{V}(t + \tau_{max}) = \hat{V}(t + \tau_{max} + m T) \). Therefore, we can conclude that the right-hand of (27) maintains the original goal of the condition (25) in Lemma 5 by (29), which makes Lemma 5 applicable to the multi-agent system (10) by (30).

**Step Three:** Because of left-hand of (27), (28) and (31), \( 0 > -\alpha \hat{P}_{r,s} + \xi \hat{P}_{r,s} > -\alpha \hat{P}_{r,s} \pm (1 - \gamma \Lambda_r) \hat{P}_{r,s} \). From (44), the condition (23) in Lemma 5 is satisfied.

**Step Four:** It follows from (28) and the left-hand of (27) that
\[
\frac{(\alpha - \xi) \tau_{min}}{L} + 1 > \beta^{-\frac{1}{2}} = \beta^{(1-\xi)} \frac{1}{\beta}. 
\]
Considering \( h > 0 \), from (35) and (45) one has\[
\frac{(\alpha - \xi) \tau_{min}}{L} > \frac{P_{r+1,q} - P_{r,q}}{P_{r,q}} \]
which is equivalent to
\[
\frac{L}{\tau_{min}} \left( \frac{P_{r+1,q} - P_{r,q}}{P_{r,q}} \right) - (\alpha - \xi) P_{r,q} < 0, q = 0, \cdots, L - 1. 
\]
Because \( 1 > \beta > 0 \), result (35) implies \( \hat{P}_{r,q} < \hat{P}_{r+1,q} \).

Therefore, (46) implies
\[
\frac{L}{\tau_{min}} \left( \frac{P_{r+1,q} - P_{r,q}}{P_{r,q}} \right) + (\xi - \alpha) P_{r,q+1} < 0, q = 0, \cdots, L - 1. 
\]
Because of left-hand of (27) and (31), \( 0 > -\alpha \hat{P}_{r,q+1} + \xi \hat{P}_{r,q+1} > -\alpha \hat{P}_{r,q+1} \pm (1 - \gamma \Lambda_r) \hat{P}_{r,q+1} \) and \( 0 > -\alpha \hat{P}_{r,q} + \xi \hat{P}_{r,q} > -\alpha \hat{P}_{r,q} \pm (1 - \gamma \Lambda_r) \hat{P}_{r,q} \). So, (46) and (47) implies (42) and (43), respectively. Therefore, the conditions (21) and (22) in Lemma 5 hold.

Based on the above analysis, we can conclude the conditions conditions (21)–(25) in Lemma 5 always hold if the conditions in Theorem 1 are satisfied, thus the second-order consensus can be achieved, which completes the proof.

**Remark 4:** Theorem 1 shows that using the proposed control protocol (10b) without velocity measurements, for any coupling strength \( \gamma > 0 \), the second-order consensus can be achieved by strategy setting on the dwell time of switching topologies, which means the strategy has no constraint on the magnitude of coupling strength in achieving consensus. This maintains the advantage of the control protocol (4) studied in [4]. The condition (27) with (31) in Theorem 1 implies the coupling strength can affect the minimum dwell time and maximum dwell of topology switching signals, which further affects the convergence speed of consensus.

**Remark 5:** In the situation that the defender or the system operator has no knowledge of the attack-beginning time, Theorem 1 provides a guide when the system dynamics should have changes (caused by topology switching) to reveal zero-dynamics attack [32], such that the changes do not destroy the system stability in the absence of attacks.

IV. TOPOLOGY SWITCHING FOR CONSENSUS

A. Finite-Time Consensus Algorithm

The following finite-time consensus algorithm can be used to estimate the global coordinators precisely in finite time. Thus, it can be used to derive a decentralized topology-switching algorithm.

**Lemma 6:** [37] Consider the multi-agent system
\[
\dot{r}_i(t) = \alpha \sum_{j=1}^{n} b_{ij}(r_j(t) - r_i(t)) + \beta \sum_{j=1}^{n} b_{ij}(r_j(t) - r_i(t))^{\frac{\gamma}{2}}, i = 1, \cdots, n
\]
where \( b_{ij} \) is the element of the coupling matrix that describes topology of an undirected connected communication network and its corresponding Laplacian matrix is denoted as \( \mathcal{L}_A \), \( \alpha > 0 \), \( \beta > 0 \), the odd numbers \( m > 0 \), \( n > 0 \) \( p > 0 \) and \( \gamma > 0 \) that satisfy \( m \geq n \) and \( p < q \). Its global finite-time consensus can be achieved, i.e.,
\[
r_i(t) = \sum_{j=1}^{n} r_j(t) = 0, \forall t \geq T, i = 1, \cdots, n.
\]
Further, the setting time \( T \) is bounded by
\[
T < \frac{1}{\lambda_2(\mathcal{L}_A)} \left( \frac{n - \alpha}{\beta} \right) \left( \frac{n - m}{\alpha - m} + \frac{1}{\beta} \frac{q}{p} \right).
\]

**Remark 6:** Without considering the external disturbances, the considered finite-time consensus algorithm (48) is a simplified version studied in [37].

**Remark 7:** From (50) we can see that through adjusting the control gains \( \alpha \) and \( \beta \), we can obtain any desirable setting time \( \infty > T > 0 \).

**Remark 8:** Adjust parameters \( \alpha > 0 \) and \( \beta > 0 \) in the finite-time consensus network (48) such that
\[
\frac{1}{\lambda_2(\mathcal{L}_A)} \left( \frac{n - \alpha}{\beta} \right) \left( \frac{n - m}{\alpha - m} + \frac{1}{\beta} \frac{q}{p} \right) < \tau_{min},
\]
therefore, the setting time \( T \) in (50) satisfies \( T < \tau_{min} \). The setting (51) with (49) and (50) implies that if \( t_k \), the individual data
\[
\hat{F}_i(t_k) = \frac{\tau}{2} \hat{x}_i^T(t_k) + \frac{1}{2} \hat{v}_i^2(t_k).
\]

and \( \hat{F}_i(t_k) \) to the corresponding agent \( i \) in the finite-time consensus network (48), at time \( t_k + \tau_{min} \):
\[
\hat{F}_i(t_k + \tau_{min}) = \frac{1}{n} \sum_{i=1}^{n} \hat{F}_i(t_k) \triangleq \frac{1}{n} \hat{F}(t_k),
\]
\[
\hat{F}_i(t_k + \tau_{min}) = \frac{1}{n} \sum_{i=1}^{n} \hat{F}_i(t_k) \triangleq \frac{1}{n} \hat{F}(t_k).
\]

B. Decentralized Topology-Switching Algorithm

We make the following assumption on the topology set for topology-switching algorithm.

**Assumption 1:** The topology set \( S \) includes at least two topologies:
- at least one topology’s Laplacian matrix has distinct eigenvalues.
Based on Lemma 4 and Theorem 1, through employing the finite-time consensus network (48), the decentralized topology-switching algorithm—Algorithm 1—is proposed.

Algorithm 1: Decentralized Topology-Switching Algorithm

\begin{itemize}
  \item Input: Topology set $\mathcal{G}$ satisfying Assumption 1, individual functions $F_i(t_k)$ (52) with $\omega$ satisfying (18), initial time $t_k = 0$, initial topology $G_{\omega(t_k)}$, initial topology-switching time $t_{k+1} = t_k + \tau$ with $\tau$ generated by Theorem 1, loop-stopping criteria $\delta \geq 0$.
  \item while $F(t_{k-1}) > \delta$ do
    \item Input individuals $F_i(t_k)$ (52) and $\bar{F}_i(t_k)$ to agent $i$ in the finite-time consensus network (48) at time $t_k$;
    \item Output $F(t_k)$ (53) and $\bar{F}(t_k)$ (54) and from the finite-time consensus network (48) to the agents in (10) at time $t_k + \tau_{\text{min}}$;
    \item Run until the time $t_{k+1} = t_k + \tau$;
    \item if $\bar{F}(t_k) = 0$ then
      \item Switch the topology of network (10b) to $\sigma(t_{k+1})$ that satisfies:
        \begin{itemize}
          \item $\sigma(t_{k+1}) \neq \sigma(t_k)$,
          \item $L_{\omega(t_{k+1})}$ has distinct eigenvalues.
        \end{itemize}
    \item else
      \item Switch the topology of network (10b) to $\sigma(t_{k+1})$ that satisfies:
        \begin{itemize}
          \item $\sigma(t_{k+1}) \neq \sigma(t_k)$.
        \end{itemize}
  \item end
  \item Update the topology-switching time: $t_{k-1} \leftarrow t_k$;
  \item Update the topology-switching time: $t_k \leftarrow t_{k+1}$.
\end{itemize}

Theorem 2: Consider the multi-agent system (10). If the topology-switching signal is generated by Algorithm 1, then the following properties hold.

(i) If the loop-stopping criteria $\delta = 0$ (in Line 1 of Algorithm 1), the agents can achieve the second-order consensus by Definition 1.

(ii) If the loop-stopping criteria $\delta > 0$ (in Line 1 of Algorithm 1), by finitely topology switching the agents can achieve the second-order consensus under admissible consensus error $\delta$, i.e., $F(t_k) \leq \delta$ with $0 < k < \infty$ and $F(t)$ given by (17).

Proof of Theorem 2: (Proof of property (i)) The loop-stopping criteria $\delta = 0$ in Line 1 of Algorithm 1 means topology switching will stop when $F(t_k) = 0$. The definition of $F(t)$ in (17) implies that $\lim_{t \to \infty} F(t) = 0$ is equivalent to (2). This analysis means the topology switching will not stop until the second-order consensus by Definition 1 is achieved. Because the given $\tau$ in Input of Algorithm 1 is generated by (27) in Theorem 1. Hence, by Theorem 1 we can conclude the property (i) under Algorithm 1.

(Proof of property (ii)) If the function $F(t)$ is a non-zero constant over time, i.e., the situation (19) happens, and if $F(t_0) = \varphi > \epsilon$, from Line 1 of Algorithm 1 we know the topology switching will not stop even the consensus is achieved. The objective of Line 5 and Line 6 in Algorithm 1 is to switch to a topology that its Laplacian matrix has distinct eigenvalues when $F(t_k) = 0$; by Lemma 4, $F(t)$ cannot be constant over time if $F(t_k) = 0$ happens. Obviously, if $F(t_k) \neq 0$, $F(t)$ cannot be constant over time. The solution in Lemma 3 means the system states are bounded, which means once the consensus value is under an admissible consensus error of $\delta$, it will stay therein. Therefore, by Lemma 3 and Lemma 4 we can conclude the property (ii) under Algorithm 1.

Remark 9 (Motivation of Non-Zero Loop-Stopping $\delta$ (Property (ii) of Theorem 2)): The definition of the second-order consensus (2) has asymptotic behavior. In practical applications of consensus algorithm, such as decentralized computation and distributed optimization, the consensus algorithm would stop iteration/loop under admissible consensus error $\delta > 0$ rather that runs iteration/loop-switching infinitely over infinite time.

V. Simulation

The simulations on a second-order multi-agent system with $n = 4$ agents will be presented to demonstrate the effectiveness of the proposed topology switching algorithm. In the simulation setting, the initial position and velocity conditions are chosen as $x(0) = v(0) = [1, 2, 3, 4]^T$.

To convincingly illustrate the ability of topology-switching algorithm—Algorithm 1—in achieving the second-order consensus, the topology set $\mathcal{G} = \{1^*; 2^*\}$ where topologies $1^*$ and $2^*$ are described by Table II, provided to Algorithm 1 includes only two topologies: one path graph and one ring graph.

The eigenvalues of Laplacian matrices of the two topologies in Table II are solved as

\[
\begin{bmatrix}
\lambda_1 (L_1), \lambda_2 (L_1), \lambda_3 (L_1), \lambda_4 (L_1)
\end{bmatrix} = [0, 0.6, 2, 3.4],
\begin{bmatrix}
\lambda_1 (L_2), \lambda_2 (L_2), \lambda_3 (L_2), \lambda_4 (L_2)
\end{bmatrix} = [0, 2, 2, 4],
\]

Lemma 3 implies that the states of multi-agent system (10) under fixed topology are oscillating, which implies that even with very large coupling strength, the second-order cannot be achieved in the situation of fixed topology. This can be verified by the trajectories in Figure 1, where the coupling strength is set very large as $\gamma = 100$. Figure 1 also shows that the system states under each fixed topology have period.

A. The second-Order Consensus

Set the coupling strength as $\gamma = 2$. Using the solved eigenvalues in (55) and (56) and the state solutions (16), the period is calculated as $\text{period} \approx 6$. Using (55) and (56), we can also calculate $\xi$ (given by (31)) = 7. Following (28), set
\( \alpha = 7.5 \). Let \( \beta = 0.4 \) and \( L = 1 \). It follows from (29) and the left-hand of (27) that \( \tau_{\text{max}} < 0.1222 \) and \( \tau_{\text{min}} > 3 \). Let \( m = 1 \). Following (27) and (30), we can choose the dwell time \( \tau = \frac{3}{10} \), and \( \omega = 3.1 \). Then under Algorithm 1, the trajectories of position differences and velocity differences of multi-agent system (6) are shown in Figures 2 (a) and (b), respectively. Figure 2 shows that in the absence of attacks, Algorithm 1 succeeds in achieving the second-order consensus. Thus, the property (i) in Theorem 2 is verified.

B. Finitely Topology Switching

To satisfies the condition (18), we consider \( \gamma > \max_{i=2,\ldots,n;r=1,2} \{ \gamma \lambda_i (L_r) \} \). From the chosen \( \gamma = 2 \) and the calculated eigenvalues in (55) and (56), we can choose \( \gamma = 8.1 \). Let the loop-stopping criteria \( \delta = 2 \), then under Algorithm 1 the trajectory of \( F(t) \) and the topology-switching signal are shown in Figures 3 (a) and (b), respectively. Figures 3 well illustrates the property (ii) in Theorem 2.

VI. CONCLUSION

This Part-I paper explains how to take the network topology as a control variable for the second-order multi-agent system. The obtained results highlight the merits of topology switching in achieving the second-order consensus: (i) the control protocol does not need the velocity measurements, and (ii) the topology-switching algorithm has no constraint on the magnitude of coupling strength. The strategy on switching times provides a basis for the strategic topology-switching algorithm that is studied in Part-II paper [13]: when the topology of the multi-agent should switch to cause changes in the system dynamics to reveal zero-dynamics attacks, such that the changes do not destroy the agents’ ability of reaching consensus in the absence of attacks.

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Figure 3. Trajectory of $F(t)$ (17) and Topology-Switching Signal: finitely topology-switching in reaching the loop-stopping criteria $\delta = 2$, i.e., $F(t) < \delta = 2$.

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**APPENDIX A**

**PROOF OF LEMMA 3**

Let $Q = [q_1; \cdots ; q_n] \in \mathbb{R}^{n \times q}$ where $q_i, i = 1, \cdots , n$, is the eigenvector associated with the eigenvalue $\lambda_i$ of matrix $L$. By Lemma 1, arrange the eigenvalues of $L$ in the increasing order as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. Also note that the Laplacian
matrix $L$ is a symmetric real matrix, one has the following properties:

$$Q^T = Q^{-1},$$  \hspace{1cm} (57)

$$q_{11} = q_{12} = \ldots = q_{1n},$$  \hspace{1cm} (58)

$$Q^T LQ = \text{diag} \{ \lambda_2, \ldots, \lambda_n \} \geq \Lambda.$$  \hspace{1cm} (59)

Denote $X(s) = \mathcal{L} \{ \tilde{x}(t) \}$. The Laplace transform of the equation (15) can be obtained as

$$sX(s) - \tilde{x}(0) = -\frac{\gamma}{s}LX(s) + \frac{\bar{v}(0)}{s},$$  \hspace{1cm} (60)

which is equivalent to

$$X(s) = \frac{I_n}{s^2I_n + \gamma L} \left( \tilde{x}(0) + \frac{\bar{v}(0)}{s} \right).$$  \hspace{1cm} (61)

Let $\hat{\Lambda}(s) = \text{diag} \left\{ \frac{1}{\gamma s^2 + \alpha \lambda_1}, \ldots, \frac{1}{\gamma s^2 + \alpha \lambda_n} \right\}$ and $\hat{\Lambda}(s) = \text{diag} \left\{ \frac{1}{\gamma}, \frac{1}{\gamma s + \alpha \lambda_2}, \ldots, \frac{1}{\gamma s + \alpha \lambda_n} \right\}$. It follows from the properties (57) and (59) that

$$\frac{sI_n}{s^2I_n + \gamma L} = \frac{sI_n}{Q(s^2I_n + \gamma \Lambda)Q^T} = Q\hat{\Lambda}(s)Q^T,$$  \hspace{1cm} (62)

$$\frac{I_n}{s^2I_n + \gamma L} = \frac{I_n}{Q(s^2I_n + \gamma \Lambda)Q^T} = Q\hat{\Lambda}(s)Q^T.$$  \hspace{1cm} (63)

Let $X_i(s), i = 1, \ldots, n$, be the $i$th element of $X(s)$. From (61), (62) and (63),

$$X_i(s) = q_i q_i^T \left( \frac{1}{s} \tilde{x}(0) + \frac{1}{\gamma} \bar{v}(0) \right) + \sum_{l = 2}^n \frac{1}{s^2 + \alpha \lambda_l} q_i q_l^T \left( \tilde{x}(0) + \frac{1}{s} \bar{v}(0) \right).$$  \hspace{1cm} (64)

It follows from (11), (12) and (58) that

$$q_i^T \tilde{x}(0) = 0,$$  \hspace{1cm} (65)

$$q_i^T \bar{v}(0) = 0.$$  \hspace{1cm} (66)

Combining (64) with (65) and (66) yields

$$X_i(s) = \sum_{l = 2}^n \frac{1}{s^2 + \alpha \lambda_l} q_i q_l^T \tilde{x}(0) + \sum_{l = 2}^n \frac{1}{s^2 + \alpha \lambda_l} q_i q_l^T \bar{v}(0).$$  \hspace{1cm} (67)

Considering $\gamma > 0$ and $\lambda_i > 0, i = 2, \ldots, n$, the solution (16) can be obtained immediately from the inverse Laplace transform of (67).

**APPENDIX B**

**VANDERMONDE MATRIX**

In this section, we recall the determinant of Vandermonde matrix, which is used in the proof of Lemma 4.

**Lemma 7**: [38] The determinant of the Vandermonde matrix $H \in \mathbb{R}^{n \times n}$ is

$$\det (H) = \det \left( \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{array} \right) = (-1)^{n(n-1)/2} \prod_{i<j} (a_i - a_j).$$  \hspace{1cm} (68)

**APPENDIX C**

**PROOF OF LEMMA 4**

This proof can be finished by contradiction. Assume Equation (19) holds, from which one can obtain

$$\frac{\partial^4}{\partial t^4} F(t) \equiv 0, \forall t \in \mathbb{N}, \forall t \geq 0.$$  \hspace{1cm} (69)

It follows from the dynamics (15) that

$$\tilde{x}(t) = -\gamma L \tilde{x}(t), \forall t \geq 0,$$  \hspace{1cm} (70)

$$\bar{v}(t) = -\gamma L \bar{v}(t), \forall t \geq 0.$$  \hspace{1cm} (71)

Considering (69), the rest proof is divided into two steps. **Step One**: Equation (69) working together with (70) and (71) implies

$$\frac{\partial^4}{\partial t^4} F(t) = \frac{\partial^4}{\partial t^4} F(t) = 0.$$  \hspace{1cm} (72)

which implies

$$\tilde{x}(t) = 0,$$  \hspace{1cm} (73)

$$\bar{v}(t) = 0.$$  \hspace{1cm} (74)

It learns from (73), (57) and (59) that

$$\tilde{x}(t) = 0,$$  \hspace{1cm} (75)

$$\bar{v}(t) = 0.$$  \hspace{1cm} (76)

Consider (65) and (66). From (75) and (76), one has

$$\tilde{x}_1(t) \equiv 0 \text{ and } \bar{v}_1(t) \equiv 0,$$  \hspace{1cm} (77)

$$\tilde{x}_2(t) \equiv 0 \text{ and } \bar{v}_2(t) \equiv 0.$$  \hspace{1cm} (78)

Note matrix $\Lambda$ given in (59). Equation (74) is equivalent to

$$\sum_{i=2}^n \lambda_i \left( \bar{v} - \gamma \lambda_i \right) \bar{v}_i^2(t) \equiv \sum_{i=2}^n \lambda_i \bar{v}_i^2(t),$$  \hspace{1cm} (79)

$$\forall t \in \mathbb{N}, \forall t \geq 0,$$  \hspace{1cm} (79)
Denote 
\[ \hat{z}_i(t) = (\varpi - \gamma \lambda_i) (\gamma \lambda_i \hat{x}_i^2(t) - \hat{v}_i^2(t)), i = 2, \ldots, n, \]  
\[ H = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-2}^2 & \lambda_{n-2}^3 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \end{bmatrix}. \]  
(80)  
(81)

Considering (80) and (81), it can be obtained from (79) that
\[ H \hat{z}(t) = 0_{n-1}, \]  
(82)
where \( \hat{z}(t) = [\hat{z}_2(t), \ldots, \hat{z}_n(t)]^T \in \mathbb{R}^{n-1} \).

The condition that \( L \) has distinct eigenvalues means the elements \( \lambda_i, i = 2, \ldots, n \), in the Vandermonde matrix \( H \) are also distinct; then, by Lemma 7, \( \det(H) \neq 0 \). Therefore, one can conclude the solution of (82) is \( \hat{z}(t) = 0_{n-1} \).

Considering the condition (83), one can easily obtain from (80) that
\[ \gamma \lambda_i \hat{x}_i^2(t) = \hat{v}_i^2(t), \forall i = 2, \ldots, n, \forall t \geq 0, \]  
which is equivalent to
\[ \hat{v}(t) \equiv \Delta \hat{x}(t), \forall t \geq 0 \]  
(83)
where
\[ \Delta = \text{diag} \left\{ 0, \pm \sqrt{\gamma \lambda_2}, \ldots, \pm \sqrt{\gamma \lambda_n} \right\} \in \mathbb{R}^{n \times n}. \]  
(84)

**Step Two:** Along the dynamics (15), it follows from (17), (59), (69), (70), (75), (76) and (83) that
\[ \frac{\partial^m}{\partial t^m} F(t) = \frac{\partial^{m-1}}{\partial t^{m-1}} F(t) = \frac{\partial^{m-1}}{\partial t^{m-1}} (\hat{v}^T(t) \hat{z}(t) + \hat{z}^T(t) \hat{v}(t)) = \frac{\partial^{m-1}}{\partial t^{m-1}} (\hat{v}^T(t) (\varpi I - \gamma L) \hat{z}(t)) = \frac{\partial^{m-1}}{\partial t^{m-1}} (\hat{v}^T(t) Q (\varpi I - \gamma \Delta) Q^T \hat{z}(t)) = \frac{\partial^{m-1}}{\partial t^{m-1}} (\hat{v}^T(t) (\varpi I - \gamma \Delta) \hat{z}(t)) = \frac{\partial^{m-1}}{\partial t^{m-1}} (\hat{z}^T(t) \Delta (\varpi I - \gamma \Delta) \hat{z}(t)) = \frac{\partial^{m-1}}{\partial t^{m-1}} (2 \hat{v}^T(t) \Delta (\varpi I - \gamma \Delta) \hat{z}(t)) = 2^{m-2} \hat{v}^T(t) \Delta^m (\varpi I - \gamma \Delta) \hat{z}(t) = 0, \forall \varpi \in \mathbb{R}, \forall t \geq 0. \]  
(85)

Consider the matrices \( \Lambda \) and \( \Delta \) given in (59) and (84), respectively. Equation (85) is equivalent to
\[ \sum_{i=2}^{n} (\pm \sqrt{\gamma \lambda_i}) \hat{z}_i^2(t) = 0, \forall \varpi \in \mathbb{R}, \forall t \geq 0. \]  
(86)

Consider the Vandermonde matrix \( H \) given by (81). Let every \( m \) be an even number. Equation (86) implies
\[ \hat{A} \hat{z}(t) = 0_{n-1}, \forall t \geq 0. \]  
(87)
where \( \hat{A} = \text{diag} \left\{ \lambda_2, \ldots, \lambda_n \right\} \in \mathbb{R}^{(n-1) \times (n-1)} \) and \( \hat{z}(t) = [\hat{z}_2(t), \ldots, \hat{z}_n(t)]^T \in \mathbb{R}^{n-1} \) with
\[ \hat{z}_i(t) = (\varpi - \gamma \lambda_i) \hat{x}_i^2(t), i = 2, \ldots, n. \]  
(88)
As obtained in Step One, \( \det(H) \neq 0 \). Because \( \det(\hat{A}) \neq 0 \), one has \( \det(\hat{A} \hat{z}) = \det(H) \det(\hat{A}) \neq 0 \). Therefore, the solution of (87) is \( \hat{z}(t) \equiv 0_{n-1}, \forall t \geq 0 \). Condition (18) is equivalent to \( \varpi - \gamma \lambda_i \neq 0, \forall i = 2, \ldots, n, \) which working with (88) implies the obtained solution \( \hat{z}(t) \equiv 0_{n-1}, \forall t \geq 0, \) is equivalent to \( \hat{x}_i^2(t) \equiv 0, \forall t \geq 0, \forall i = 2, \ldots, n \); and considering (83), one has \( \hat{v}_i^2(t) \equiv 0, \forall t \geq 0, \forall i = 2, \ldots, n \). Finally, considering the matrix \( Q \) is full-rank, from (75), (76), (77) and (78) one can conclude that \( \hat{x}(t) \equiv 0_n \) and \( \hat{v}(t) \equiv 0_n, \forall t \geq 0, \) which contradicts with (19), thus the proof is finished.

**APPENDIX D**

**PROOF OF COROLLARY 1**

The proof is that can be finished by contradiction. Assume Lemma 5 is feasible. Write the multi-agent system (10) in the form of switched system (20) where \( \bar{A}(t) \) is given by (32).

Consider a positive define matrix \( P_{r,q} \in \mathbb{R}^{n \times n}, \) then we have
\[ \Gamma_{r,q} = \Delta^T P_{r,q} + P_{r,q} \Delta \]  
(89)
Let \( \Phi_{r,q} \) be the orthogonal matrix of \( \Gamma_{r,q} \). Since \( \Delta \), \( \forall r \in \mathcal{S} \), has eigenvalue zero, \( -2 \gamma \Delta \), \( \forall r \in \mathcal{S}, \forall q = 0, \ldots, L \), also has the eigenvalue zero. Then by denoting \( \Psi_{r,q}^g = \Phi_{r,q}^{-1} \Psi_{r,q} \Phi_{r,q}, \) \( \Psi_{r,q}^g \) is orthogonal, and \( \Psi_{r,q}^g = \text{diag} \{ 0, \lambda_1^g, \lambda_2^g, \ldots, \lambda_{n-1}^g \} \) where \( \lambda_i^g \) is the \( i \)th nonzero eigenvalue of matrix \( \Gamma_{r,q} \). (89) the conditions (21) and (24) can rewritten equivalently as
\[ X_q^g + \Psi_{r,q}^g - \alpha \hat{P}_{r,q} \leq 0, \forall r \in \mathcal{S}, \forall q = 0, 1, \ldots, L - 1 \]  
(90)
\[ \hat{P}_{r,0} - \alpha \hat{P}_{r,L} \leq 0, \forall s \neq r \in \mathcal{S}. \]  
(91)
Let \( \bar{P}_{r,q} \) be the element positioned at the first row and the first column of the matrix \( \hat{P}_{r,q} \). i.e., \( \hat{P}_{r,q} = \bar{P}_{r,q} \).

Considering \( \hat{P}_{r,q} = \hat{P}_{r,q} + \hat{P}_{r,q} \) from (90) and (91) we can conclude that if (21) and (24) hold, we have
\[ \frac{L}{\tau_{min}} \left( \bar{P}_{r,q+1} - \bar{P}_{r,q} \right) < \alpha \bar{P}_{r,q}, \forall r \in \mathcal{S}, \forall q = 0, 1, \ldots, L - 1 \]  
(92)
\[ \bar{P}_{r,0} \leq \beta \bar{P}_{r,L}, s \neq r, \forall s \neq r \in \mathcal{S}. \]  
(93)
Since $\hat{P}_{r,q}, \forall r \in \mathcal{S}, \forall q = 0, 1, \cdots, L,$ is positive definite, $\hat{p}_{r,q} > 0, \forall r \in \mathcal{S}, \forall q = 0, 1, \cdots, L.$ Thus, (92) is equivalent to
\[
\tau_{\min} > \frac{L}{\alpha} \left( \frac{\hat{p}_{r,q+1}}{\hat{P}_{r,q}} - 1 \right), \forall q = 0, \cdots, L - 1, \forall r \in \mathcal{S}. \quad (94)
\]

Condition (25) is equivalent to $\tau_{\max} < -\frac{\ln \beta}{\alpha},$ which combines with (94) and the fact of $\tau_{\max} \geq \tau_{\min}$ yields
\[
- \ln \beta > L \left( \frac{\hat{p}_{r,q+1}}{\hat{P}_{r,q}} - 1 \right), \forall q = 0, \cdots, L - 1, \forall r \in \mathcal{S}. \quad (95)
\]

Condition (93) implies $\frac{\hat{p}_{r,q+1}}{\hat{P}_{r,q}} \geq \frac{1}{\beta} > 1$ and $\frac{\hat{p}_{r,q}}{\hat{P}_{r,q}} \geq \frac{1}{\beta} > 1,$ which further implies
\[
\hat{p}_{s,0}^{-1} \hat{P}_{r,0} \hat{p}_{s,L} \hat{P}_{r,L}^{-1} = \prod_{q=0}^{L-1} \hat{p}_{s,q}^{-1} \hat{p}_{r,q} \hat{p}_{s,q+1} \hat{p}_{r,q+1} \geq \beta^{-2} > 1, \forall r \neq s \in \mathcal{S}. \quad (96)
\]

Considering (96), one can pick up one number $\hat{q} \in \{0, 1, \cdots, L - 1\}$ such that $\hat{p}_{s,\hat{q}}^{-1} \hat{P}_{r,\hat{q}} \hat{p}_{s,\hat{q}+1} \hat{p}_{r,\hat{q}+1} \geq \beta^{-\frac{1}{2}} > 1, \forall r \neq s \in \mathcal{S},$ which also implies that for one of the indices $s$ and $r,$ let us choose $r,$ that
\[
\hat{p}_{r,\hat{q}}^{-1} \hat{P}_{r,\hat{q}} \geq \beta^{-\frac{1}{2}} > 1, \forall \beta \in (0, 1). \quad (97)
\]

Combing (95) with (97) yields $- \ln \beta > L \left( \beta^{-\frac{1}{2}} - 1 \right)$ which is equivalent to
\[
\beta < e^{L \left( 1 - \beta^{-\frac{1}{2}} \right)}, \beta \in (0, 1). \quad (98)
\]

For (98), consider the function $g(\beta, L) = e^{L \left( 1 - \beta^{-\frac{1}{2}} \right)} - \beta$ with $g(0, L) = 0$ and $g(1, L) = 0.$ It is easy to verify that in term of $L \in \mathbb{N},$ $g(\beta, L)$ is strictly decreasing under fixed $0 < \beta < 1,$ so $g(\beta) = \max_{L \in \mathbb{N}} \{ g(\beta, L) \} = e^{(1 - \beta^{-1})} - \beta.$ It is also easy to verify that $e^{(1 - \beta^{-1})}$ is strictly increasing and $g(0) = 0$ and $g(1) = 0.$ So, for $\forall \beta \in (0, 1),$ $g(\beta) < 0,$ which is well illustrated by Figure 4. Hence, $g(\beta, L) = e^{L \left( 1 - \beta^{-\frac{1}{2}} \right)} - \beta < 0, \forall \beta \in (0, 1), L \in \mathbb{N}.$ Here, we can conclude that (98) never holds, thus a contradiction occurs which completes the proof.