Lightcone Quantization of String Theory Duals of Free Field Theories

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We quantize in light cone gauge the bosonic sector of string theory on Anti-de Sitter space in the zero curvature radius limit. We find that the worldsheet falls apart into a theory of free partons and map the Hilbert space of the string theory to the Hilbert space of a free scalar in light-front description. We outline how the string worldsheet reproduces the field theory at weak coupling.
1 Introduction

Maldacena's duality \cite{1} was the first example of a string theory being exactly equivalent to a gauge theory. Both sides are governed by two parameters: one controlling the genus expansion ($1/N$ in the gauge theory and $g_s$ in the string theory) and one governing perturbation theory for the evaluation of the amplitude at a given genus ($\lambda = g_s^2/MN$, the 't Hooft coupling, and $R^4/l_s^4$, the curvature radius in string units respectively). While the genus expansion goes parallel on both sides, $g_s \sim 1/N$, in terms of the $\lambda$ expansion the gauge/gravity correspondence is a strong-weak duality: $\lambda \sim R^4/l_s^4$, for small $\lambda$ the gauge theory is perturbative, for large $\lambda$ string theory reduces to gravity. Latter fact suggests that the bulk string theory should simplify dramatically not just in the large $\lambda$ but also in the $\lambda \rightarrow 0$ limit. After all, it is equivalent to a free field theory.

In this note we give evidence that this is indeed the case. Going to light cone gauge, we quantize the bosonic string, or equivalently the bosonic part of the superstring, on AdS space in the limit where the curvature radius in string units vanishes. We obtain the exact spectrum of the theory. We show that the result agrees with what one obtains for a gauge theory in light-front quantization: the spectrum of the string reproduces the spectrum of free partons. The momentum in the extra dimension encodes the fraction of lightcone momentum carried by the individual partons.

To get a YM theory as the dual theory we have to include the fermions on the string side. The free, purely bosonic theory is claimed to be dual to a matrix valued free scalar field with $U(N)$ gauge invariance. It should also be possible to find an example where the bosonic string has a stable highly curved AdS background beyond zero coupling, which then should map to an interacting scalar field theory, e.g. in 3 dimensions. Such a string sized AdS seems to be a reasonable outcome of the closed string tachyon condensation, since it at least satisfies the naive intuition that the potential energy decreases along the condensation process. Since we are working in light cone gauge, the conformal anomalies are quite subtle to detect and we hope to be able to come back to this issue in the future. As far as the superstring goes, Tseytlin showed \cite{2} that for the full IIB string in $AdS_5 \times S^5$ the light cone worldsheet action in the $\lambda \rightarrow 0$ limit simplifies dramatically as well. However the analysis won't be quite as simple as the one we are going to present for the bosonic case.

Let us briefly comment on related work: Bardakci and Thorn have worked out a string representation for free scalars in \cite{3}. It is not clear yet how they work is related to our approach which is rooted in the AdS/CFT correspondence: their string does not seem to know about the extra dimension. It would be nice to make the connection more precise, since their approach was already extended to include fermions and perturbation theory by construction agrees with field theory \cite{4}. The idea that the tension of the string appears as a perturbation in strongly curved spacetimes already appeared in \cite{5}.

The paper is organized as follows: in the next section we will quantize the bosonic string (or the bosonic sector of the superstring) in light cone gauge on zero radius AdS and show how the theory can be treated perturbatively in a double expansion in $g_s$ and $\lambda$. In section 3 we give a very brief review of field theories in the light front and show that in the free limit, the field theory and the string theory give an identical spectrum. In section 4 we conclude.
2 The spectrum

2.1 Light Cone Gauge Hamiltonian

Consider the bosonic part of a string moving in a warped background geometry. The fermionic pieces of the superstring are not expected to alter the story significantly. Since we are working in light cone gauge, they don’t pose any conceptual problems: at least for type II strings we can treat them and their coupling to RR background fields in the GS formalism. Something similar should work for the non-critical type 0 examples as well. The $D+1$ dimensional background metric is taken to be of the general form

$$ds^2 = G_{\mu \nu} dZ^\mu dZ^\nu = e^{2A(Y)} (2dX^+dX^- + dX^i dX^j \delta_{ij} + dY^2)$$

where $\mu, \nu$ run from 0 to $D$ and $i, j$ run over the transverse $X$s, that is from 2 to $D-1$. We will also often use $M, N$ running from 2 to $D$ to label the transverse $X$s and $Y$. We will be mostly interested in AdS backgrounds with $e^{2A(Y)} = \frac{R_2^2}{Y^2}$, which describe conformal examples and should also be relevant for the asymptotic free regime of QCD. In the bosonic part of the string action

$$S = -\frac{1}{4 \pi \alpha'} \int_M d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a Z_\mu \partial_b Z_\nu G_{\mu \nu}$$

one can choose the light cone gauge \[6, 7\]

$$X^+ = \tau$$

$$\gamma_{01} = 0$$

(3)

(4)

to obtain a light cone Hamiltonian \[7\]

$$H = \frac{1}{2} \int_0^{2\pi P_-} d\sigma \left\{ P_X P_X + P_Y P_Y + \frac{e^{4A}}{\alpha'^2} (X'X' + Y'Y') \right\}.$$ (5)

We will mostly focus on the case of AdS where the Hamiltonian becomes

$$H = \frac{1}{2} \int_0^{2\pi P_-} d\sigma \left\{ P_X P_X + P_Y P_Y + \frac{\lambda}{Y^4} (X'X' + Y'Y') \right\}.$$ (6)

where $\lambda = R^4/l_s^4$ measures the curvature radius in string units and is what becomes the ’t Hooft coupling of the dual gauge theory living on the boundary of AdS as in the correspondence of \[1\].

2.2 The zero curvature radius limit

First let us discuss the $\lambda \to 0$ limit in which the string theory should reproduce free field theory in the light cone frame. The Hamiltonian simply becomes \[8\]

$$H = \frac{1}{2} \int_0^{2\pi P_-} d\sigma \left\{ P_X P_X + P_Y P_Y \right\}.$$ (7)

This is just the Hamiltonian of an infinite number of free particles. Since all $\sigma$ derivatives dropped out of the Hamiltonian, each spatial mode propagates on its own, independent of the others,
basically ripping apart the string. As we will soon see, this is the stringy manifestation of the partons in the gauge theory! Note that the same Hamiltonian would govern formally the $\alpha' \to \infty$ limit in flat space string theory. However, $\alpha'$ has dimension of length, so taking it to infinity just means we are zooming in on the high energy states of the theory. On the contrary, in AdS we have a dimensionless coupling constant $\lambda$. In the strict $\lambda \to 0$ the full string theory is captured by the free parton Hamiltonian. Duality to weakly coupled field theory ensures that this limit is well behaved.

There is one difference between (7) and $D-1$ towers of free particles: in the original metric the range of $Y$ was between $\infty$ and 0, where $Y = 0$ is the boundary of AdS. Even though the divergent $1/Y^4$ factor drops out in the $\lambda \to 0$ limit, we should still impose the finite range of $Y$. This is easiest done by imposing an orbifold projection on $Y$

$$Y \to -Y$$

which just identifies negative values of $Y$ to be positive values of $Y$. In the end we are only going to keep states that are invariant under the orbifold projection, that is the untwisted sector. We know that in general modular invariance forces us to introduce in addition a twisted sector living at the orbifold fixed plane, that is at $Y = 0$. So the twisted sector fields would look like singletons, living only at the boundary of AdS. In the usual AdS spirit one would suspect that they correspond to the “decoupled center of mass motion”, that is the dynamics of the $U(1)$ part of $U(N)$. However it is not clear that our limiting procedure is valid at $Y = 0$. We will leave a detailed analysis of the twisted sector for future work.

### 2.2.1 Open String:

For the open string a general solution to the equations of motion satisfying Neumann boundary conditions can be written as

$$Z^M(\sigma, \tau) = (z^M_0 + \frac{p^M_0}{2\pi P_{-}}) + \sum_{n>0}(2z^M_n + \frac{p^M_n}{\pi P_{-}})\cos(\frac{n\sigma}{2P_{-}}).$$

(9)

The canonical momentum is

$$P_M(\sigma, \tau) = \dot{Z}^M = \frac{p^M_0}{2\pi P_{-}} + \sum_{n>0}\frac{p^M_n}{\pi P_{-}}\cos(\frac{n\sigma}{2P_{-}}).$$

(10)

Imposing canonical commutation relations on $Z^M(\sigma, \tau)$ and $P_M(\sigma', \tau)$ shows that all the $z^M_n, p^M_n$ are canonical coordinates and momenta themselves

$$[z^M_n, p^N_m] = i\delta_{nm}\delta^{MN}.\quad (11)$$

Plugging back into the Hamiltonian (7), we indeed recover just a collection of free particles

$$2\pi H = \frac{1}{P_{-}}(p^M_0)^2 + \frac{1}{2P_{-}} \sum_{n>0,M} (p^M_n)^2.$$  

(12)

So the states are just given by a tensor product of free particle momentum eigenstates, that is they are of the form

$$\{ (p^M_0, p^M_{\perp}) ; \ldots ; (p^M_n, p^M_{\perp}) ; \ldots \}.$$  

(13)

3
where \((p^y)_n\) and \((p^\perp)_n\) are the \(y\) and transverse momentum of the particles respectively. Under the orbifold action all
\[(p^y)_n \rightarrow -(p^y)_n,\] (14)
so to impose the orbifold constraint we should only allow invariant states of the form
\[
\cdots; \{(p^y)_n, (p^\perp)_n\}; \cdots + \cdots; \{- (p^y)_n, (p^\perp)_n\}; \cdots
\] (15)
where we should now restrict to
\[p^y_n \geq 0\] (16)
in order to avoid redundancy.

### 2.2.2 Closed String:

For the closed string a general periodic solution to the equations of motion can be written as
\[
Z^M = \sum_{n=-\infty}^{\infty} \left[ (z^M_n + \frac{p^M_n}{2\pi P_-}) e^{\frac{in\sigma}{P_-}} \right] \] (17)
where reality demands \(p^\perp_n = p_n\) and again we obtain a collection of free particles
\[
2\pi H = \frac{1}{P_-} \sum_{n,M} p^M_n p^M_n = \frac{1}{P_-} (p^M_0)^2 + \frac{2}{P_-} \sum_{n>0,M} |p^M_n|^2. \] (18)
For the closed string we have to impose in addition the zero momentum constraint in order to insure gauge invariance under \(\sigma\) translations
\[
0 = P = \int_0^{2\pi P_-} d\sigma \sum_M P_M (Z^M)' = \sum_{n,M} \frac{in}{P_-} \left( x^M_n p^M_{-n} \right) \] (19)
The orbifold action, as in the open case, just restricts all the \(p_y\) to be non-negative.

### 2.3 Interactions

So far we have only quantized the theory in the \(g_s = 0, \lambda = 0\) limit. But it is obvious that perturbation theory in those two parameters is conceptually straightforward. The \(\lambda\) perturbation theory just amounts to good old fashioned quantum mechanical perturbation theory with the interaction Hamiltonian
\[
H_1 = \lambda \int_{0}^{2\pi P_-} d\sigma \frac{1}{4} \left( (X')^2 + (Y')^2 \right). \] (20)
In order to reproduce the \(g_s\) expansion we have to allow splitting and joining of strings. Given the simplicity of our Hamiltonian the corresponding string field theory vertex should be not too hard to construct. Note that the \(\lambda\) perturbation theory becomes singular whenever \(Y\) tends to zero. Some regularization procedure is necessary. One way would be to cut out regions from the worldsheet where \(Y < \epsilon\). Regularization is to be expected once we are describing the string theory dual to an interacting theory like YM: infinities arise and lead to dimensional transmutation of the coupling constant. In the case of \(\mathcal{N} = 4\) SYM all singularities in the \(\epsilon \to 0\) limit have to cancel between the bosons and fermions. We want to emphasize again that the dual field theory makes it clear, that the \(\lambda \to 0\) limit is smooth.
3 The dual field theory interpretation

3.1 Light Cone Quantization in Field Theory

The holographic dual is a free $D$-dimensional field theory of $N \times N$ matrices with $U(N)$ gauge invariance, that is we restrict physical observables to be $U(N)$ invariant. For the open string case there are in addition $M$ fundamental matter multiplets, where $M$ denotes the dimension of the Chan-Paton space (the number of spacetime filling D-branes). For the superstring case, this has to be augmented by free adjoint gauge fields, fermions and additional scalars, say to get $\mathcal{N} = 4$ SYM, but the bosonic sector we quantized above is already supposed to be dual to a the free scalar matrix valued field with gauge invariance.

Quantizing the string theory in the light-cone gauge forces us onto the light cone on the field theory side as well. So we briefly need to review the structure of YM theory in light cone quantization [8], see [9] for a recent review. As is known to most string theorists from the study of (M)atrix theory [10], light cone quantization leads to a non-relativistic structure with $P_-$ playing the role of mass in the Hamiltonian. In particular the vacuum is trivial. It’s whole complication gets hidden in the dynamics of the zero modes.

Let’s look at the quantization of a massless scalar field in the free limit. The light cone time translation generator becomes

$$P_+ = \frac{1}{2} \int dx_+ dx_\perp (\Phi_i (i\partial_\perp)^2 \Phi^i).$$

(21)

where $i$ is an adjoint $U(N)$ color index. One obtains a collection of free particles. One uses as a basis of states the free Fock space forming the eigenstates of the free Hamiltonian

$$|n : k_a^{1+}, k_a^{1-}\rangle$$

(22)

where $n$ denotes the number of particles in the Fock space and the $k_a$ are the momenta of the $a$-th particle. Usually one is only interested in gauge invariant states, but we kept the color label in order to distinguish partons with different contractions. At large $N$ the single trace states get singled out. It is a key property that all $k_a^+$ are positive

$$k_a^+ \geq 0$$

(23)

since this implies the triviality of the vacuum, modulo the complications with the zero mode. The total $P_+$ in a Fock space state is given by a sum over all the partons, each of which is on shell

$$P_+ = \sum_a \frac{(k_a^+)^2}{k_a^+}.$$  

(24)

We will treat the zero mode in a somewhat unusual fashion that resembles the treatment of soft quanta in standard perturbation theory. The standard light-front method would be to discretize the momenta and then to solve for the zero mode. In this case the ground state of the Fock space is the unique vacuum of the theory. The particles forming the Fock space are usually referred to as partons and it is precisely in this language that one can see the partonic behavior of QCD at
high energies: any hadron can be written as a wavefunction in the Fock-space basis, and at high energies the parton constituents become the scattering centers. The light cone Hamiltonian can be numerically diagonalized in the interacting theory by truncating the Fock space. The resulting spectrum gives the masses of the bound states. This program has been carried out with some success in QCD [9].

Instead, in order to incorporate all zero momentum quanta, we want to focus on the special subset of parton states of the form

\[ \text{tr} \left| N : (k_0^+, k_a^+) \right> \]  

The claim is that these span the complete physical meaningful Hilbert space of single trace states. If \( N - k \) of the momenta go to zero (the partons are on-shell, but since we are dealing with massless excitations the momenta can vanish), the state looks like a \( k \) parton state together with several “soft photons” (wee partons). Replacing the \( k \) parton state with the full \( N \) parton state is the analog of always calculating inclusive cross-section with an arbitrary number of soft photons in order to get well defined cross-sections. States with more than \( N \) partons can be rewritten as a multi-trace state due to \( U(N) \) identities.

### 3.2 Map between String Theory and Field Theory

It follows that the spectrum we found on the string theory side precisely reproduces the \( U(N) \) invariant single-trace spectrum of the free, massless scalar field theory, if we encode the momentum in the “5th dimension” (the \( Y \) direction) of the \( n \)-th particle in the string Fock space in the \( k^+ \) of the \( n \)-th particle in the light-cone Fock space, as we will do in detail below. The closed string maps to \( N \) gluon states,

\[ \text{tr} |ggggg \ldots ggggg> \]  

Including fundamental matter, color singlets involving two fundamentals and \( N \) adjoints,

\[ |gqggg \ldots gq> \],

map to the open string spectrum. Multitrace states are, as usual, interpreted as multi-string states. Note that this implies that in the string theory at finite coupling the number of independent free particles has to truncate at \( N = \frac{1}{g_s} \), that is it is a non-perturbative effect. This is the usual stringy exclusion principle [11].

Actually, at large \( N \) one would expect the partons to be ordered along \( \sigma \), whereas the free string modes we found are momentum modes in \( \sigma \). For the transverse momenta the right identification seems to be to identify the momenta on the string side as waves on the single trace \( N \) parton states, similar to the proposal of [12]. That is

\[ |0, 0, \ldots , 0, (p^+)_n, 0, \ldots , 0> = \frac{1}{\sqrt{N}} \sum_{a=1}^{N} e^{\frac{2\pi i a n}{N}} \text{tr} |0, 0, \ldots , 0, k_0^+, 0, \ldots , 0> . \]  

This actually vanishes by cyclicity of the trace, the string state does not satisfy the zero momentum constraint [19]. But in the same spirit we can identify

\[ |0, 0, \ldots , 0, p_n^+, 0, \ldots , 0, p_m^+, 0, \ldots , 0> = \frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} e^{\frac{2\pi i (an+bm)}{N}} \text{tr} |0, 0, \ldots , 0, k_0^+, 0, \ldots , 0, k_0^+, 0, \ldots , 0> . \]
Cyclicity of the trace means that the right hand side is invariant under translations in $\sigma = a/N$. This translation invariance was the origin of (19) and hence we precisely reproduce the closed string spectrum, as long as $p_y$ is given by such a discrete Fourier transform as well, as we will find it to be the case below. The partons of the field theory become string bits [13, 14] or a gluon chain [15]. The open spectrum works in an analogous fashion.

The map between the momentum in the 5th dimension and the $k^+$ carried by the individual partons is more subtle. Note that by group theory we only know that the total momenta $P_+, P_\perp$ and $P_-$ have to map into each other. The counting of parameters works nicely: string theory has for each mode $p^i_n$ and $p^y_n$, $P_-$ has no modes by gauge choice and the $p^-_n$ are solved for by the constraints. In field theory we have $k^\perp_a$ and $k^+_a$, while $k^-_a$ gets solved for as $(k^+_a)^2/k^+_a$. The total transverse momentum

$$P_\perp = p^0_\perp = \sum_a k^\perp_a$$

matches trivially. $P_-$ is just a parameter on the string side, it matches on the field theory side to

$$P^+ = \sum_a k^+_a.$$  

The map between $p^y_n$ and $k^+_a$ has to yield the matching of $P_\perp$. In detail, writing $X^- (\sigma, \tau)$ in the same form as the transverse coordinates, solving the constraints yields for the modes

$$(p^-)_n = \frac{1}{2\pi P^-} \sum_{m,M} p^M m p^M_{n-M}.$$  

These should be related to the $(k^-)_a = (k^+_a)^2/k^+_a$ by the Fourier transform

$$p^-_n = \frac{1}{\sqrt{N}} \sum_a e^{\frac{2\pi i a}{N}} k^-_a = \frac{1}{\sqrt{N}} \sum_a e^{\frac{2\pi i a}{N}} \frac{(k^+_a)^2}{k^+_a}.$$  

Requiring those two expressions for $p^-_n$ to agree yields the following map for $p^y_n$:

$$p^y_n = \frac{1}{\sqrt{N}} \sum_a e^{\frac{2\pi i a}{N}} \alpha_a$$

where

$$\alpha_a = \left| k^+_a \right| \left( \frac{1}{x_a} - 1 \right)^{\frac{1}{2}}$$

and $x_a$ denotes the $P^+$ momentum fraction carried by the $a$-th parton,

$$x_a = \frac{k^+_a}{P^+}.$$  

$x$ takes values between 0 and 1, so the $\alpha_a$ take values between $+\infty$ and 0 just as we expect from the dual to $p_y$. A consistent picture emerges.
4 Conclusions

We quantized the bosonic string on AdS backgrounds in light-cone gauge in the zero curvature radius limit and obtained the exact spectrum. The string falls apart into free partons, which map precisely to what one would expect from a free field theory with $U(N)$ gauge invariance in the light front formalism. In order to promote the field theory to an interacting theory, one possibility is to add in the fermions and study the full $\text{AdS}_5 \times S^5$ background of IIB around the zero radius limit in the GS formalism in the same spirit. The bosonic excitations will again show the partonic behavior of the dual theory. Another possibility that has to be explored is that maybe substringy AdS spaces are good backgrounds for the bosonic string in lower dimensions.

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