Variational Approximation of Factor Stochastic Volatility Models

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Abstract

Estimation and prediction in high dimensional multivariate factor stochastic volatility models is an important and active research area because they allow a parsimonious representation of multivariate stochastic volatility. Such factor models are usually estimated by Markov chain Monte Carlo or particle methods, which are usually slow for high dimensional or long time series because of the large number of parameters and latent states involved. Our article proposes fast batch and sequential variational methods to approximate the posterior distribution of the states and parameters in a factor stochastic volatility model. It also obtains one-step and multi-step ahead variational forecast distributions. The method is applied to simulated and real datasets and shown to produce good approximate inference and prediction compared to the latest particle Markov chain Monte Carlo approaches, but is much faster.
1 Introduction

Statistical inference and prediction in high dimensional time series models that incorporate stochastic volatility (SV) is an important and active research area. One of the main challenges of these models is that the number of parameters and latent variables involved often increases quadratically with dimension, while the length of each latent variable equal the number of observed time periods. These problems are somewhat lessened by using a factor stochastic volatility (FSV) model. See, e.g. Pitt and Shephard (1999) and Chib et al. (2006) who use independent and low dimensional latent factors to approximate a fully multivariate SV (MSV) model to achieve an attractive tradeoff between model complexity and model flexibility.

Even though FSV models are more tractable computationally than high dimensional multivariate SV (MSV) models, they are still challenging to estimate because they are relatively high dimensional in the number of parameters and latent factors; the likelihood of FSV models is also intractable as it is a high dimensional integral over the latent factors. Current state of the art approaches use Markov chain Monte Carlo (MCMC) such as Chib et al. (2006) and Kastner et al. (2017) were developed over a number of years and are very powerful. However, they are inexact as they use the approach proposed in Kim et al. (1998) (based on assuming normal errors) to approximate the distribution of the innovations in the log outcomes (usually the log of the squared returns) by a mixture of normals. These MCMC approaches are also less flexible than recent particle MCMC based approaches by Mendes et al. (2020), Li and Scharth (2020) and Gunawan et al. (2020), which are also exact in the sense of being simulation consistent. As well as estimating the FSV model, practitioners are often interested in updating estimates of the parameters and latent volatilities as new observations become available, as well as obtaining one or multiple step ahead forecasts of future volatilities and observations, as well as the corresponding prediction intervals. However, it is usually computationally very expensive to estimate a factor stochastic volatility by MCMC or particle MCMC. Sequential updating by MCMC or particle MCMC is even more expensive as a new MCMC simulation needs to be carried out as each new observation becomes available, although such updating is made simpler because good starting values for the MCMC are available from the previous MCMC run. Sequential Monte Carlo methods can of course also be used instead for estimation and sequential updating, but these can also be expensive.

Variational approximation (VA) (also known as Variational Bayes) has become in-
creasingly prominent as a method for conducting parameter inference in a wide range of challenging statistical models with a computationally difficult posterior \cite{Ormerod2010, Tan2018, Ong2018, Blei2017}. VA formulates the posterior estimation problem as an optimisation problem by approximating the target posterior density by a simpler distribution whose parameters are unknown, e.g., by a multivariate Gaussian with unknown mean and covariance matrix. VA methods usually produce posterior inference with much less computational cost compared to the exact methods such as MCMC or particle MCMC, although they are approximate.

Our article proposes fast batch and sequential variational approximations of the posterior distribution of the latent volatilities and parameters of the multivariate factor stochastic volatility model. The paper also studies the variational approach for generating probabilistic forecasts for high dimensional state space models. Sections 6.1.1 and 6.1.2 show that our variational approach produces forecasts that are very close to those obtained from the exact particle MCMC methods, even when the corresponding variational posterior variance is underestimated. A highlight of our work is that all variational predictions can be produced with much less computational cost compared to the time taken to produce the corresponding exact predictions using particle MCMC.

The current VA literature focuses on a Gaussian VA for approximating the posterior density of interest. The variational parameters to be optimised are the mean vector and the covariance matrix. It is challenging to perform Gaussian variational approximation with an unrestricted covariance matrix for high dimensional parameters because the number of elements in the covariance matrix of the variational density increases quadratically with the number of parameters in the model. It is then very important to find an effective and efficient parameterisation of the covariance structure of the variational density when the number of latent states and parameters are large as in the multivariate factor SV model.

Various suggestions in the literature exist for parsimoniously parameterising the covariance matrix in the Gaussian variational approximation (GVA). \cite{Titsias2014} consider a Cholesky factorisation of the full covariance matrix, but they do not consider sparsity in the parameterisation, except through diagonal approximations which lose any ability to capture posterior dependence in the variational approximation. \cite{Kucukelbir2017} also consider an unrestricted and diagonal covariance matrix and use similar gradient estimates to \cite{Titsias2014}, but use automatic differentiation. \cite{Tan2018} consider an approach which parameterises the precision matrix in terms of Cholesky factor and imposes a sparsity structure on it that reflects the conditional independence structure.
in the model. This approach is motivated because the presence of zeros in the pre-
cision matrix of a Gaussian distribution correspond to the conditional independence
between two variables. The computations can be done efficiently through fast sparse
matrix operations. Recently, Ong et al. (2018) considers a factor covariance structure
as a parsimonious representation of the covariance matrix in the Gaussian variational
approximation. They also use stochastic gradient ascent optimisation method with
the “reparameterisation trick”. Other parameterisations of the covariance matrix in
the Gaussian variational approximation are also considered by Opper and Archam-
beau (2009), Challis and Barber (2013), and Salimans and Knowles (2013). Quiroz
et al. (2018) combine the factor parameterisation and sparse precision Cholesky fac-
tors for capturing dynamic dependence structure in high-dimensional state space
model. They apply their method to approximate the posterior density of the mul-
tivariate stochastic volatility model via a Wishart process. The computational cost
of their variational approach increases with the length of the time series and their
variational approach is currently limited to short time series. Our proposed vari-
ational approach is scalable in terms of number of parameters, latent states, and
observations.

Tan et al. (2019) and Smith et al. (2019) propose more flexible variational ap-
proximations than GVA. Tan et al. (2019) extend the approach in Tan and Nott
(2018) by defining the posterior distribution as a product of two Gaussian densities.
The first is the posterior for the global parameters; the second is the posterior of
the local parameters, conditional on the global parameters. The method respects
a conditional independence structure similar to that of Tan and Nott (2018), but
allows the joint posterior to be non-Gaussian because the conditional mean of the
local parameters can be nonlinear in the global parameters. Similarly to Tan and
Nott (2018), they also exploit the conditional independence structure in the model
and improve the approximation by using the importance weights approach (Burda
et al., 2016). Smith et al. (2019) proposed an approximation that is based on im-
plcit copula models for original parameters with a Gaussian or skew-normal copula
function and flexible parametric marginals. They consider the Yeo-Johnson (Yeo and
Johnson, 2000) and G&H families (Tukey, 1977) of transformations and use sparse
factor structures proposed by Ong et al. (2018) as the covariance matrix of the Gaus-
sian or skew-normal densities. Our variational method for approximating posterior
distribution of the multivariate factor SV model makes use of both conditional inde-
pendence structures of the model and the factor covariance structure. In particular,
we make use of the non-Gaussian sparse Cholesky factor parameterisation proposed
by Tan et al. (2019) and implicit Gaussian copula with Yeo-Johnson transformation
and sparse factor covariance structure proposed by Smith et al. (2019) as our main
building blocks for our variational approximations. We use an efficient stochastic gradient ascent optimisation method using the so called “reparameterisation trick” for obtaining unbiased estimation of the gradients of the variational objective function (Paisley et al., 2012; Nott et al., 2012; Hoffman et al., 2013; Salimans and Knowles, 2013; Kingma and Welling, 2014; Rezende et al., 2014; Ranganath et al., 2014; Titsias and Lazaro-Gredilla, 2014). In practice, the reparameterisation trick greatly helps to reduce the variance of gradient estimates. Section 3 describes these methods further.

In related work, Tomasetti et al. (2019) propose sequential updating variational method, but they apply their method to a simple univariate autoregressive model with a tractable likelihood. Frazier et al. (2019) explore the use of Approximate Bayesian Computation (ABC) as a means of generating forecasts. ABC is also a method for doing approximate Bayesian inference, but it is very different to VA and is usually only effective in low dimensions. They prove that, under certain regularity conditions, approximate Bayesian forecasting produces forecasts that are asymptotically equivalent to those obtained from the exact Bayesian method with much less computational cost. However, they only apply their method to the univariate state space model.

The rest of the article is organised as follows. Section 2 introduces the multivariate factor stochastic volatility model. Section 3.1 gives a background on variational approximation. Sections 3.2 and 3.3 review the variational methods proposed by Tan et al. (2019) and Smith et al. (2019). Section 3.4 discusses our variational approximation for approximating the posterior of the multivariate factor SV model. Section 4 discusses the sequential variational algorithm for updating the posterior density. Section 5 discusses the variational forecasting method. Section 6 presents results from both simulated dataset and a US stock returns dataset. Section 7 concludes. The paper has an online supplement which contains some further empirical results.

## 2 The factor SV Model

Suppose that $P_t$ is a $S \times 1$ vector of daily stock prices and define $y_t := \log P_t - \log P_{t-1}$ as the vector of stock returns of the stocks. The factor model is

$$ y_t = \beta f_t + \epsilon_t, \ (t = 1, ..., T); \tag{1} $$

$f_t$ is a $K \times 1$ vector of latent factors (with $K \ll S$), $\beta$ is a $S \times K$ factor loading matrix of the unknown parameters. The latent factors $f_{k,t}, k = 1, ..., K$ are assumed independent with $f_t \sim N(0, D_t)$. The time-varying variance matrix $D_t$ is a diagonal
matrix with $k$th diagonal element $\exp(\tau_{f,k} h_{f,k,t})$. Each $h_{f,k,t}$ is assumed to follow an independent autoregressive process

$$
h_{f,k,1} \sim N\left(0, \frac{1}{1 - \phi_{f,k}^2}\right), \; h_{f,k,t} = \phi_{f,k} h_{f,k,t-1} + \eta_{f,k,t}\; \eta_{f,k,t} \sim N(0, 1), \; k = 1, \ldots, K.
$$

We model the idiosyncratic error as $\epsilon_t \sim N(0, V_t)$. The time-varying variance matrix $V_t$ is diagonal with $s$th diagonal elements $\exp(\tau_{\epsilon,s} h_{\epsilon,s,t} + \kappa_{\epsilon,s})$. Each $h_{\epsilon,s,t}$ is assumed to follow an independent autoregressive process

$$
h_{\epsilon,s,1} \sim N\left(0, \frac{1}{1 - \phi_{\epsilon,s}^2}\right), \; h_{\epsilon,s,t} = \phi_{\epsilon,s} h_{\epsilon,s,t-1} + \eta_{\epsilon,s,t}\; \eta_{\epsilon,s,t} \sim N(0, 1), \; s = 1, \ldots, S.
$$

Thus,

$$
y_t|\Sigma_t \sim N_S(0, \Sigma_t),
$$

with

$$
\Sigma_t = \beta D_t \beta^\top + V_t.
$$

The factor loading matrix $\beta$ is unidentified without further constraints. We assume that the factor loading matrix $\beta$ is lower triangular, i.e., $\beta_{s,k} = 0$ for $k > s$, similarly to Geweke and Zhou (1996). We also impose the restriction on the leading diagonal elements $\beta_{s,s} > 0$. The main drawback of the lower triangular assumption on the factor loading matrix $\beta$ is that the resulting inference can depend on the order in which the components of $y_t$ are chosen (Chan et al., 2017). We use the following transformations to map the constrained parameters to the real space $\mathcal{R}$ by letting $\alpha_{\epsilon,s} = \log(\exp(\tau_{\epsilon,s}) - 1)$, $\psi_{\epsilon,s} = \log(\phi_{\epsilon,s}/(1 - \phi_{\epsilon,s}))$ for $s = 1, \ldots, S$, $\alpha_{f,k} = \log(\exp(\tau_{f,k}) - 1)$, $\psi_{f,k} = \log(\phi_{f,k}/(1 - \phi_{f,k}))$ for $k = 1, \ldots, K$, and log-transform the diagonal elements of the factor loading matrix $\beta$, $\delta_k = \log(\beta_{k,k})$, for $k = 1, \ldots, K$. We follow Kim et al. (1998) and choose the prior for persistence parameter $\phi_{\epsilon,s}$ for $s = 1, \ldots, S$ and $\phi_{f,k}$ for $k = 1, \ldots, K$ as $(\phi + 1)/2 \sim Beta(a_0, b_0)$, with $a_0 = 20$ and $b_0 = 1.5$. The prior for each of $\tau_{\epsilon,s}$ and $\tau_{f,k}$ is half Cauchy, i.e. $p(\tau) \propto I(\tau > 0)/(1 + \tau^2)$ and the prior for $p(\kappa_{\epsilon,s}) \propto 1$. For every unrestricted element of the factor loadings matrix $\beta$, we choose independent Gaussian distribution $N(0, 1)$. The set of variables in the factor SV model is given by

$$
\tilde{\theta} = \left( \{h_{\epsilon,s,1:T}\}_{s=1}^S, \{h_{f,k,1:T}\}_{k=1}^K, \{f_{k,1:T}\}_{k=1}^K, \{\kappa_{\epsilon,s}, \alpha_{\epsilon,s}, \psi_{\epsilon,s}\}_{s=1}^S, \{\alpha_{f,k}, \psi_{f,k}\}_{k=1}^K, \beta \right).
$$
The posterior distribution of the multivariate factor SV model is given by

\[
p(\theta|y) = \left\{ \prod_{s=1}^{S} \prod_{t=1}^{T} p(y_{s,t}|\beta_s, f_t, h_{e,s,t}, \mu_{e,s}, \alpha_{e,s}, \psi_{e,s}) p\{h_{e,s,t-1}, \mu_{e,s}, \alpha_{e,s}, \psi_{e,s}\} \right\} \left\{ \prod_{k=1}^{K} \prod_{t=1}^{T} p(f_{k,t}|h_{f,k,t}, \alpha_{f,k}, \psi_{f,k}) p(h_{f,k,t-1}, \alpha_{f,k}, \psi_{f,k}) \right\} \left\{ \prod_{s=1}^{S} p(\mu_{e,s}) p(\phi_{e,s}) \frac{\partial \phi_{e,s}}{\partial \psi_{e,s}} | p(\tau_{e,s}) | \frac{\partial \tau_{e,s}}{\partial \alpha_{e,s}} \right\} \cdot \left\{ \prod_{k=1}^{K} p(\phi_{f,k}) \frac{\partial \phi_{f,k}}{\partial \psi_{f,k}} | p(\tau_{f,k}) | \frac{\partial \tau_{f,k}}{\partial \alpha_{f,k}} \right\} p(\beta) \prod_{k=1}^{K} \frac{\partial \beta_{k,k}}{\partial \delta_k}. \tag{4} \right.
\]

3 Variational Inference

3.1 Approximate Bayesian Inference

Let \( \theta \) and \( x_{1:T} \) be the vector of parameters and latent states in the model, respectively. Let \( y_{1:T} = (y_1, ..., y_T)^T \) be the observations and consider Bayesian inference for \( \theta \) and \( x_{1:T} \) with a prior density \( p(x_{1:T}|\theta) p(\theta) \). Denote the density of \( y_{1:T} \) conditional on \( \theta \) and \( x_{1:T} \) by \( p(y_{1:T}|\theta, x_{1:T}) \); the posterior density is \( p(\theta, x_{1:T}|y_{1:T}) \propto p(y_{1:T}|\theta, x_{1:T}) p(x_{1:T}|\theta) p(\theta) \). Denote \( h(\theta, x_{1:T}) := p(y_{1:T}|\theta, x_{1:T}) p(x_{1:T}|\theta) p(\theta) \). We consider the variational density \( q_\lambda(\theta, x_{1:T}) \), indexed by the variational parameter \( \lambda \), to approximate \( p(\theta, x_{1:T}|y_{1:T}) \). The variational Bayes (VB) approach approximates the posterior distribution of \( \theta \) and \( x_{1:T} \) by minimising over \( \lambda \) the Kullback-Leibler (KL) divergence between \( q_\lambda(\theta, x_{1:T}) \) and \( p(\theta, x_{1:T}|y_{1:T}) \), i.e.,

\[
\text{KL}(\lambda) = \text{KL}(q_\lambda(\theta, x_{1:T}) | p(\theta, x_{1:T}|y_{1:T})) = \int \log \frac{q_\lambda(\theta, x_{1:T})}{p(\theta, x_{1:T}|y_{1:T})} q_\lambda(\theta, x_{1:T}) d\theta dx_{1:T}.
\]

Minimizing the KL divergence between \( q_\lambda(\theta, x_{1:T}) \) and \( p(\theta, x_{1:T}|y_{1:T}) \) is equivalent to maximising a variational lower bound (ELBO) on the log marginal likelihood \( \log p(y_{1:T}) \) (where \( p(y_{1:T}) = \int p(y_{1:T}|\theta, x_{1:T}) p(x_{1:T}|\theta) p(\theta) d\theta dx_{1:T} \)) \cite{Blei2017}.

The ELBO is

\[
\mathcal{L}(\lambda) = \int \log \frac{h(\theta, x_{1:T})}{q_\lambda(\theta, x_{1:T})} q_\lambda(\theta, x_{1:T}) d\theta dx_{1:T}. \tag{5}
\]

In non-conjugate models, the variational lower bound \( \mathcal{L}(\lambda) \) may not have a closed form solution. When it cannot be evaluated in closed form, stochastic gradient methods are usually used \cite{Hoffman2013, Kingma2014, Nott2017}.  

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An initial value $\lambda^{(0)}$ is updated according to the iterative scheme
\[ \lambda^{(t+1)} = \lambda^{(t)} + a_t \circ \nabla_\lambda \mathcal{L} (\lambda^{(t)}), \tag{6} \]
where $\circ$ denotes the Hadamard (element by element) product of two random vectors. The update in Eq.(6) is continued until a stopping criterion is satisfied. The $a_t$, $t > 0$ is a sequence of vector valued learning rates, $\nabla_\lambda \mathcal{L} (\lambda)$ is the gradient vector of $\mathcal{L} (\lambda)$ with respect to $\lambda$, and $\nabla_\lambda \mathcal{L} (\lambda)$ denotes an unbiased estimate of $\nabla_\lambda \mathcal{L} (\lambda)$. The learning rate sequence is chosen to satisfy the Robbins-Monro conditions $\sum_t a_t = \infty$ and $\sum_t a_t^2 < \infty$ (Robbins and Monro, 1951), which ensures that the iterates $\lambda^{(t)}$ converge to a local optimum as $t \to \infty$ under suitable regularity conditions (Bottou, 2010). We consider adaptive learning rates based on the ADAM approach (Kingma and Ba, 2014) in our examples.

Reducing the variance of the gradient estimates $\nabla_\lambda \mathcal{L} (\lambda)$ is important to ensure stability and fast convergence of the algorithm. Our article uses gradient estimates based on the so-called reparameterisation trick (Kingma and Welling, 2014; Rezende et al., 2014), which is known to reduce the variance of the gradient estimates. To apply this approach, we represent samples from $q_\lambda (\theta, x_{1:T})$ as $(\theta, x_{1:T}) = u(\eta, \lambda)$, where $\eta$ is a random vector with a fixed density $f(\eta)$ that does not depend on the variational parameters. In the case of a Gaussian variational distribution parameterized in terms of a mean vector $\mu$ and the Cholesky factor $C$ of its covariance matrix, we can write $(\theta, x_{1:T}) = \mu + C\eta$, where $\eta \sim N (0, I)$.

Then,
\[
\mathcal{L} (\lambda) = E_q (\log h (\theta, x_{1:T}) - \log q_\lambda (\theta, x_{1:T}) ) \\
= E_f (\log h (u (\eta, \lambda)) - \log q_\lambda (u (\eta, \lambda))) .
\]

Differentiating under the integral sign
\[
\nabla_\lambda \mathcal{L} (\lambda) = E_f (\nabla_\lambda \log h (u (\eta, \lambda)) - \nabla_\lambda \log q_\lambda (u (\eta, \lambda))) \\
= E_f (\nabla_\lambda u (\eta, \lambda) \{ \nabla_{\theta,x_{1:T}} \log h (\theta, x_{1:T}) - \nabla_{\theta,x_{1:T}} \log q_\lambda (\theta, x_{1:T}) \} ) ;
\]
this is an expectation with respect to $f$ that can be estimated unbiasedly if it is possible to sample from $f$. Note that the gradient estimates obtained by the reparameterisation trick use gradient information from the model; it has been shown empirically that it greatly reduces the variance compared to alternative approaches. The rest
of this section is organised as follows. Section 3.2 discusses the non-Gaussian sparse Cholesky factor parameterisation of the precision matrix proposed by [Tan et al. (2019)]. Section 3.3 discusses the implicit Gaussian copula variational approximation with a factor structure for the covariance matrix proposed by [Smith et al. (2019)]. Section 3.4 discusses our proposed variational approximation for the multivariate factor stochastic volatility model.

3.2 Conditionally Structured variational Gaussian Approximation (CSGVA)

When the vector of parameters and latent variables is high dimensional, taking the variational covariance matrix $\tilde{\Sigma}$ in the Gaussian variational approximation to be dense is computationally expensive and impractical. An alternative is to assume that the variational covariance matrix $\tilde{\Sigma}$ is diagonal, but this loses any ability to model the dependence structure of the target posterior density. [Tan and Nott (2018)] consider an approach which parameterises the precision matrix $\Omega = \tilde{\Sigma}^{-1}$ in terms of its Cholesky factor, $\Omega = CC^T$, and impose on it a sparsity structure that reflects the conditional independence structure in the model. [Tan and Nott (2018)] note that sparsity is important for reducing the number of variational parameters that need to be optimised and it allows the Gaussian variational approximation to be extended to very high-dimensional settings. [Tan et al. (2019)] extend their previous approach to allow non-Gaussian variational approximation. For an $R \times R$ matrix $A$, let $\text{vec}(A)$ be the vector of length $R^2$ obtained by stacking the columns of $A$ under each other from left to right and $v(A)$ be the vector of length $R(R+1)/2$ obtained from $\text{vec}(A)$ by removing all the superdiagonal elements of the matrix $A$.

They consider the variational approximation $q_{\Lambda}^{TBN}(\theta, x_{1:T})$ of the posterior distribution $p(\theta, x_{1:T}|y_{1:T})$ of the form

$$q_{\Lambda}^{TBN}(\theta, x_{1:T}) = q_{\Lambda}^{TBN}(\theta)q_{\Lambda}^{TBN}(x_{1:T}|\theta),$$

(7)

where $q_{\Lambda}^{TBN}(\theta) = N(\mu_G, \Omega_G^{-1})$, $q_{\Lambda}^{TBN}(x_{1:T}|\theta) = N(\mu_L, \Omega_L^{-1})$, and $\Omega_G$ and $\Omega_L$ are the precision (inverse covariance) matrices of variational densities $q_{\Lambda}^{TBN}(\theta)$ and $q_{\Lambda}^{TBN}(x_{1:T}|\theta)$, respectively. Let $C_GC_G^T$ and $C_LC_L^T$ be the unique cholesky factorisations of $\Omega_G$ and $\Omega_L$, respectively, where $C_G$ and $C_L$ are lower triangular matrices with positive diagonal entries. We now explain the idea that imposing sparsity in $C_G$ and $C_L$ reflects the conditional independence relationship in the precision matrix $\Omega_G$ and $\Omega_L$, respectively. For example, for a Gaussian distribution, $\Omega_{L,i,j} = 0$, corresponds to variables $i$ and $j$ being conditionally independent given the rest. If $C_L$ is
a lower triangular matrix, Proposition 1 of Rothman et al. (2010) states that if $C_L$ is row banded then $\Omega_L$ has the same row-banded structure.

To allow for unconstrained optimisation of the variational parameters, they define $C^*_G$ and $C^*_L$ to also be lower triangular matrices such that $C^*_{G,ii} = \log(C_{G,ii})$ and $C^*_{G,ij} = C_{G,ij}$ if $i \neq j$. The $C^*_L$ can be defined similarly. In their parameterisation, the $\mu_L$ and $v(C^*_L)$ are linear functions of $\theta$:

$$\mu_L = d + C^*_L D (\mu_G - \theta), \quad v(C^*_L) = f + F\theta,$$

where, $d$ is a vector of length $T$, $D$ is a $T \times G$ matrix, $f$ is a vector of length $T (T + 1) / 2$, $F$ is a $T (T + 1) / 2 \times G$ matrix, and $G$ is the length of vector parameter $\theta$. The parametrisation of $q^T_{\lambda}(\theta, x_{1:T})$ in Eq. (7) is Gaussian if and only if $F = 0$ in Eq. (8). The set of variational parameters to be optimised is denoted as

$$\lambda = \left\{ \mu^T_G, v(C^*_G)^T, d^T, \text{vec}(D)^T, f^T, \text{vec}(F)^T \right\}^T. \quad (9)$$

The closed form reparameterisation gradients in Tan et al. (2019) can be used directly. Tan et al. (2019) discusses how the CSGVA approximation can be improved by maximising an importance weighted lower bound (IWLB) (Burda et al., 2016). They show that using IWLB leads to a tighter lower bound on the log of the marginal likelihood, and a variational approximation with larger posterior variance. The unbiased gradient estimate is given by

$$\nabla_\lambda L^I_B(\lambda) = \sum_{b=1}^B \tilde{w}_b \nabla_\lambda u(\epsilon_b, \lambda) \left\{ \nabla_{\theta_b, x_{b,1:T}} \log h(\theta_b, x_{b,1:T}) - \nabla_{\theta_b, x_{b,1:T}} \log q_\lambda(\theta_b, x_{b,1:T}) \right\}. \quad (10)$$

An unbiased estimate of $L^I_B(\lambda)$ is given by $\hat{L}^I_B(\lambda) = \log \frac{1}{B} \sum_{b=1}^B w_b$, where $w_b = w(\theta_b, x_{b,1:T}) = h(\theta_b, x_{b,1:T}) / q_\lambda(\theta_b, x_{b,1:T})$, for $b = 1, ..., B$, and $\tilde{w}_b = w_b / \sum_{b=1}^B w_b$.

### 3.3 Implicit Copula Variational Approximation through transformation

Another way to parameterise the dependence structure parsimoniously in a variational approximation is to use a sparse factor parameterisation for the variational covariance matrix $\tilde{\Sigma}$. This factor parameterisation is very useful when the prior and the density of $y_{1:T}$ conditional on $\theta$ and $x_{1:T}$ do not have a special structure. The variational covariance matrix is parameterised in terms of low-dimensional latent factor structure. The number of variational parameters to be optimised is reduced when the number of latent factors is much less than the number of parameters in...
the model. Ong et al. (2018) assume that the Gaussian variational approximation with sparse factor covariance structure \( q^{\text{ON}}_{\lambda} (\theta, x_{1:T}) = N (\mu, BB^T + D^2) \), where \( \mu \) is the mean vector, \( B \) is a \( R \times p \) full rank matrix \( p \ll R \) and \( D \) is a diagonal matrix with diagonal elements \( d = (d_1, ..., d_R) \), where \( R \) is the total number of parameters and latent states. Smith et al. (2019) extend their previous approach to go beyond Gaussian variational approximation using the implicit copula method, but still use the sparse factor parameterisation for the variational covariance matrix. They propose an implicit copula based on Gaussian and skew-Normal copulas with Yeo-Johnson (YJ) transformations for the marginals. Our article uses a Gaussian copula with a factor structure for the variational covariance matrix with Yeo-Johnson (Yeo and Johnson, 2000) and G&H families (Tukey, 1977) transformations for the marginals. Our article uses a Gaussian copula with a factor structure for the variational covariance matrix with Yeo-Johnson (YJ) transformations for the marginals.

Let \( t_\gamma \) be a Yeo-Johnson one to one transformation onto the real line with parameter vector \( \gamma \). Then, each parameter and latent state is transformed as \( \xi_\theta = t_\gamma (\theta) \) and \( \xi_{x_i} = t_\gamma (x_i) \) and we adopt a multivariate normal distribution function with mean \( \mu_\xi \) and covariance matrix \( \Sigma_\xi \), \( F (\xi; \mu_\xi, \Sigma_\xi) = \Phi_R (\xi; \mu_\xi, \Sigma_\xi) \), where \( \xi = (\xi_\theta, \xi_x) \).

We follow Ong et al. (2018) and adopt a factor structure for the covariance matrix \( \Sigma_\xi = B_\xi B_\xi^T + D_\xi^2 \), where \( B_\xi \) is an \( R \times p \) full rank matrix \( (p \ll R) \), with the upper triangle of \( B_\xi \) set to zero for identifiability; \( D_\xi \) is a diagonal matrix with diagonal elements \( d_\xi = (d_{\xi,1}, ..., d_{\xi,R}) \). If \( p (\xi; \mu_\xi, \Sigma_\xi) = (\partial^R/\partial \xi_1...\partial \xi_R) F (\xi; \mu_\xi, \Sigma_\xi) \) is the multivariate normal density with mean \( \mu_\xi \) and covariance matrix \( \Sigma_\xi \), then the density of the variational approximation is

\[
q^{\text{SRN}}_{\lambda} (\theta) = p (\xi; \mu_\xi, \Sigma_\xi) \prod_{i=1}^R t_\gamma^{-1} (\theta_i), \tag{11}
\]

where the variational parameters are \( \lambda = \left( \gamma_1^T, ..., \gamma_R^T, \mu_\xi^T, \text{vech} (B_\xi)^T, d_\xi^T \right) \). We can draw \( \xi \sim N (\mu_\xi, B_\xi B_\xi^T + D_\xi^2) \) by first drawing \( (z, \eta) \sim N (0, I) \) (where \( z \) is \( p \)-dimensional and \( \eta \) is \( R \)-dimensional) and calculating \( \xi = \mu_\xi + B_\xi z + d_\xi \circ \eta \), where \( \circ \) denotes the Hadamard (element by element) product of two random vectors. We then generate \( \theta_i = t_\gamma^{-1} (\xi_\theta) \) and \( x_i = t_\gamma^{-1} (\xi_{x_i}) \). If a parameter \( \theta_i \) is constrained, we first transform it to the real line. Smith et al. (2019) give details of the Yeo-Johnson transformation, its inverse, and derivatives with respect to the model parameters and the closed form reparameterisation gradients.

---

1The ‘vech’ operator is the half-vectorisation of a rectangular matrix, for example, for an \((r \times K)\) matrix \( A \) with \( r > K \) as \( \text{vech} (A) = (A^T_{1r,1}, ..., A^T_{K r,K})^T \) with \( A_{K \times r, K} = (A_{K,K}, ..., A_{r,K})^T \) for \( k = 1, ..., K \).
3.4 Variational Approximation for multivariate factor SV model

This section discusses our variational approach for approximating the posterior of the multivariate factor SV model described in Section 2. The main idea is to select variational approximations that balance the accuracy and the computational cost. We use the non-Gaussian sparse Cholesky factor parameterisation of the precision matrix defined in Section 3.2 and the Gaussian copula factor parameterisation of the variational covariance matrix defined in Section 3.3 as our main building blocks to approximate the posterior of the multivariate factor SV model.

The parameters and latent states of the idiosyncratic and factor log-volatilities are defined as \( \theta_{G,\epsilon,s} := (\kappa_{\epsilon,s}, \alpha_{\epsilon,s}, \psi_{\epsilon,s}) \); \( x_{\epsilon,s,1:T} := h_{\epsilon,s,1:T} \); \( \theta_{G,f,k} := (\alpha_{f,k}, \psi_{f,k}) \); and \( x_{f,k,1:T} := h_{f,k,1:T} \), for \( s = 1, ..., S \) and \( k = 1, ..., K \). From Eq. (4), \( h_{\epsilon,s,t} \) is conditionally independent of all the other states in the posterior distribution, given the parameters \( (\kappa_{\epsilon,s}, \alpha_{\epsilon,s}, \psi_{\epsilon,s}) \) and the neighbouring states \( h_{\epsilon,s,t-1} \) and \( h_{\epsilon,s,t+1} \) for \( s = 1, ..., S \); \( h_{f,k,t} \) is conditionally independent of all other states in the posterior distribution given the parameters \( (\alpha_{f,k}, \psi_{f,k}) \) and the neighbouring states \( h_{f,k,t-1} \) and \( h_{f,k,t+1} \) for \( k = 1, ..., K \).

It is therefore reasonable to take advantage of this conditional independence structure in the variational approximations, by letting \( q_{\lambda_{\epsilon,s}}^{\text{TBN}}(\theta_{\epsilon,s}, x_{\epsilon,s,1:T}) = q_{\lambda_{\epsilon,s}}^{\text{TBN}}(\theta_{G,\epsilon,s}) q_{\lambda_{\epsilon,s}}^{\text{TBN}}(x_{\epsilon,s,1:T}|\theta_{G,\epsilon,s}) \) for \( s = 1, ..., S \) and \( q_{\lambda_{f,k}}^{\text{TBN}}(\theta_{G,f,k}, x_{f,k,1:T}) = q_{\lambda_{f,k}}^{\text{TBN}}(\theta_{G,f,k}) q_{\lambda_{f,k}}^{\text{TBN}}(x_{f,k,1:T}|\theta_{G,f,k}) \) for \( k = 1, ..., K \). The sparsity structure imposed on \( \Omega_{L,\epsilon,s} \) and \( C_{L,\epsilon,s} \) for \( s = 1, ..., S \) is

\[
\Omega_{L,\epsilon,s} = \begin{bmatrix}
\Omega_{L,\epsilon,s,11} & \Omega_{L,\epsilon,s,12} & 0 & \cdots & 0 \\
\Omega_{L,\epsilon,s,21} & \Omega_{L,\epsilon,s,22} & \Omega_{L,\epsilon,s,23} & \cdots & 0 \\
0 & \Omega_{L,\epsilon,s,32} & \Omega_{L,\epsilon,s,33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \Omega_{L,\epsilon,s,TT}
\end{bmatrix},
\]

\[
C_{L,\epsilon,s} = \begin{bmatrix}
C_{L,\epsilon,s,11} & 0 & 0 & \cdots & 0 \\
C_{L,\epsilon,s,21} & C_{L,\epsilon,s,22} & 0 & \cdots & 0 \\
0 & C_{L,\epsilon,s,32} & C_{L,\epsilon,s,33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & C_{L,\epsilon,s,TT}
\end{bmatrix}.
\]

The number of non-zero elements in \( v(C_{L,\epsilon,s}^*) \) is \( 2T - 1 \). If we set \( f_{\epsilon,s,i} = 0 \) and
\( F_{e,s,ij} = 0 \) for all indices \( i \) in \( v(C^*_{l,e,s}) \) which are fixed at zero, then the number of variational parameters to be optimised is reduced from \( T(T + 1)/2 \) to \( 2T - 1 \) for \( f_{e,s,i} \), and from \( T(T + 1)G/2 \) to \( (2T - 1)G \) for \( F_{e,s,i} \). Similar sparsity structures are imposed on \( \Omega_{f,k} \) and \( C_{f,k} \) for \( k = 1, ..., K \). We can use the Gaussian copula with a factor structure for the factor loading matrix \( \beta \) and the latent factors \( (f_{1:T}) \) that do not have any special structures. We now propose two variational approximations for the posterior distribution of the multivariate factor SV model. The first variational approximation \( q^I_\lambda(\theta) \) is

\[
q^I_\lambda(\theta, x_{1:T}) = \prod_{s=1}^{S} q^{\text{TBN}}_{\lambda,s} (\theta_{e,s}, x_{e,s,1:T}) \prod_{k=1}^{K} q^{\text{TBN}}_{\lambda_{f,k}} (\theta_{G,f,k}, x_{f,k,1:T}) q^{\text{SRN}}_{\lambda_{\beta,f,k,1:T}} (f_{k,1:T}) q^{\text{SRN}}_\lambda (\beta).
\]

Our empirical work uses \( p = 0 \) factors for the variational density \( q^{\text{SRN}}_{\lambda_{f,k,1:T}} (f_{k,1:T}) \) and \( p = 4 \) factors for the variational density \( q^{\text{SRN}}_\lambda (\beta) \). The first VB approximation \( q^I_\lambda(\theta) \) ignores the posterior dependence between \( x_{e,s,1:T} \) and \( x_{e,j,1:T} \) for \( s \neq j \) and the posterior dependence of \( \theta_{G,e,s} \) and \( x_{e,s,1:T} \) with other parameters and latent states \( \left( \{\theta_{G,f,k}, x_{f,k,1:T}\}_{k=1}^{K}, \{f_{k,1:T}\}_{k=1}^{K} \right) \) for \( s = 1, ..., S; \) ignores the posterior dependence between \( x_{f,k,1:T} \) and \( x_{f,j,1:T} \) for \( k \neq j \) and the posterior dependence of \( \theta_{G,f,k} \) and \( x_{f,k,1:T} \) with other parameters \( \left( \{\theta_{G,e,s}, x_{e,s,1:T}\}_{s=1}^{S}, \{f_{k,1:T}\}_{k=1}^{K} \right) ; \) and (3) ignores the dependence between the latent factors \( f_{k,1:T} \) and \( f_{j,1:T} \) for \( k \neq j \) and the dependence between the latent factors \( f_{1:T} \) and \( \beta \).

The second variational approximation \( q^{II}_\lambda (\theta) \) is

\[
q^{II}_\lambda (\theta, x_{1:T}) = \prod_{s=1}^{S} q^{\text{TBN}}_{\lambda,s} (\theta_{e,s}, x_{e,s,1:T}) \prod_{k=1}^{K} q^{\text{TBN}}_{\lambda_{f,k}} (\theta_{G,f,k}, x_{f,k,1:T}) q^{\text{SRN}}_{\lambda_{\beta,f,k,1:T}} (f_{k,1:T}, \beta_{:,k}),
\]

where \( \beta_{:,k} = (\beta_{k,1}, ..., \beta_{S,k}) \). Our empirical work uses \( p = 4 \) for the variational density \( q^{\text{SRN}}_{\lambda_{\beta,f,k,1:T}} (f_{k,1:T}, \beta_{:,k}) \). The second variational approximation \( q^{II}_\lambda (\theta) \) takes into account the dependence between \( f_{k,1:T} \) and \( \beta_{:,k} \) but still ignores the dependence between \( (f_{k,1:T}, \beta_{k}) \) and \( (f_{j,1:T}, \beta_{:,j}) \) for \( k \neq j \). We also consider a version of \( q^I_\lambda (\theta) \) and \( q^{II}_\lambda (\theta) \) that maximises an importance weight lower bound (IWLB) \cite{Burda2016, Tan2019}.

Algorithm 1 below discusses the updates for the variational approximation.
It is straightforward to modify it to obtain the updates for the variational approximation $q_{\lambda}^T(\theta, x_{1:T})$.

Algorithm 1

Initialise all the variational parameters $\lambda$. At each iteration $t$,

For $i = 1 : S$, (Updates for the $q_{\lambda_{\epsilon,s}}^{TBN}(\theta_{\epsilon,s}, x_{\epsilon,s,1:T})$)

1. Generate $\eta_{\theta_{\epsilon,s}} \sim N(0, I_{G_\epsilon})$ and $\eta_{x_{1:T,\epsilon,s}} \sim N(0, I_T)$, where $G_\epsilon$ is the number of parameters of idiosyncratic log-volatilities

2. Generate $\theta_{\epsilon,s}^{(t)} = \mu_{\epsilon,s}^{(t)} + \left( C_{G_{\epsilon,s}}^{(t)} \right)^{-T} \eta_{\theta_{\epsilon,s}}$ and $x_{1:T,\epsilon,s}^{(t)} = \mu_{x_{\epsilon,s}}^{(t)} + \left( C_{L_{\epsilon,s}}^{(t)} \right)^{-T} \eta_{x_{1:T,\epsilon,s}}$.

3. Construct unbiased estimates of gradients $\nabla_{\lambda_{\epsilon,s}} L(\lambda_{\epsilon,s})$.

4. Set adaptive learning rates $a_{t,\epsilon,s}$ using ADAM method.

5. Set $\lambda_{\epsilon,s}^{(t+1)} = \lambda_{\epsilon,s}^{(t)} + a_{t,\epsilon,s} \nabla_{\lambda_{\epsilon,s}} L(\lambda_{\epsilon,s})^{(t)}$

Endfor

For $i = 1 : K$, (Updates for the $q_{\lambda_{f,k}}^{TBN}(\theta_{G_{f,k}}, x_{f,k,1:T})$)

1. Generate $\eta_{\theta_{G_{f,k}}} \sim N(0, I_{G_f})$ and $\eta_{x_{1:T,f,k}} \sim N(0, I_T)$, where $G_f$ is the number of parameters of factor log-volatilities

2. Generate $\theta_{G_{f,k}}^{(t)} = \mu_{G_{f,k}}^{(t)} + \left( C_{G_{f,k}}^{(t)} \right)^{-T} \eta_{\theta_{G_{f,k}}}$ and $x_{1:T,f,k}^{(t)} = \mu_{L_{f,k}}^{(t)} + \left( C_{L_{f,k}}^{(t)} \right)^{-T} \eta_{x_{1:T,f,k}}$.

3. Construct unbiased estimates of gradients $\nabla_{\lambda_{f,k}} L(\lambda_{f,k})$.

4. Set adaptive learning rates $a_{t,f,k}$ using ADAM method.

5. Set $\lambda_{f,k}^{(t+1)} = \lambda_{f,k}^{(t)} + a_{t,f,k} \nabla_{\lambda_{f,k}} L(\lambda_{f,k})^{(t)}$

Endfor

For $i = 1 : K$, (Updates for the $q_{\lambda_{f,k}}^{SRN}(f_{k,1:T})$)

1. Generate $\left( \eta_{f_k}^{(t)} \right) \sim N(0, I_T)$ and calculating $\xi_{f_k} = \mu_{\xi_{f_k}} + d_{\xi_{f_k}} \circ \eta_{f_k}$. 

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2. Generate $f_{k,i} = t_{\gamma_{f_{k,i}}}^{-1}(\xi_{f_{k,i}})$, for $i = 1, \ldots, T$.

3. Construct unbiased estimates of gradients $\nabla_{\lambda_{f_{k}}} \mathcal{L}(\lambda_{f_{k}})$.

4. Set adaptive learning rates $a_{t,f_{k}}$ using ADAM method.

5. Set $\lambda_{f_{k}}^{(t+1)} = \lambda_{f_{k}}^{(t)} + a_{t,f_{k}} \nabla_{\lambda_{f_{k}}} \mathcal{L}(\lambda_{f_{k}})^{(t)}$, where $\lambda_{f_{k}} = (\gamma_{f_{k,1}}, \ldots, \gamma_{f_{k,T}}, \mu_{\xi_{f_{k}}}, d_{\xi_{f_{k}}})$.

Endfor

(Updates for the $q_{\text{SRN}}^{(\beta)}$)

1. Generate $z_{\beta} \sim N(0, I_{p})$ and $\eta_{\beta} \sim N(0, I_{R_{\beta}})$; calculating $\xi_{\beta} = \mu_{\xi_{\beta}} + B_{\xi_{\beta}} z_{\beta} + d_{\xi_{\beta}} \circ \eta_{\beta}$; $R_{\beta}$ is the total number of parameters in the factor loading matrix $\beta$.

2. Generate $\beta_{i} = t_{\gamma_{\beta_{i}}}^{-1}(\xi_{\beta_{i}})$, for $i = 1, \ldots, R_{\beta}$.

3. Construct unbiased estimates of the gradients $\nabla_{\lambda_{\beta}} \mathcal{L}(\lambda_{\beta})$.

4. Set the adaptive learning rates $a_{t,\beta}$ using the ADAM method.

5. Set $\lambda_{\beta}^{(t+1)} = \lambda_{\beta}^{(t)} + a_{t,\beta} \nabla_{\lambda_{\beta}} \mathcal{L}(\lambda_{\beta})^{(t)}$, where $\lambda_{\beta} = (\gamma_{\beta_{1}}, \ldots, \gamma_{\beta_{R_{\beta}}}, \mu_{\xi_{\beta}}, B_{\xi_{\beta}}, d_{\xi_{\beta}})$.

The Learning Rate

Setting the learning rate in stochastic gradient algorithms is very challenging, especially when the vector of parameters is high dimensional. The choice of learning rate affects both the rate of convergence and the quality of the optimum attained. Learning rates that are too high can cause instability of the optimisation, while learning rates that are too low result in slow convergence and can lead to a situation where the parameters appear to converge, but have not. In all the examples, we set the learning rates adaptively using the ADAM method (Kingma and Ba, 2014) that gives different step sizes for each element of the variational parameter vector $\lambda$. At iteration $t + 1$, the variational parameter $\lambda$ is updated as

$$\lambda^{(t+1)} = \lambda^{(t)} + \Delta^{(t)}.$$ 

Let $g_{t}$ denote the stochastic gradient estimate at iteration $t$. ADAM computes (biased) first and second moment estimates of the gradients using exponential moving averages,

$$m_{t} = \tau_{1} m_{t-1} + (1 - \tau_{1}) g_{t}, \quad v_{t} = \tau_{2} v_{t-1} + (1 - \tau_{2}) g_{t}^{2};$$
$\tau_1, \tau_2 \in [0, 1)$ control the decay rates. The biased first and second moment estimates can be corrected by

$$\hat{m}_t = m_t \big/ \left(1 - \tau^*_1\right), \quad \hat{v}_t = v_t \big/ \left(1 - \tau^*_2\right).$$

The change $\Delta^{(t)}$ is computed as

$$\Delta^{(t)} = \frac{\alpha \hat{m}_t}{\sqrt{\hat{v}_t + \text{eps}}}.$$  \hspace{1cm} (12)

We set $\alpha = 0.001$, $\tau_1 = 0.9$, $\tau_2 = 0.99$, and $\text{eps} = 10^{-8}$ [Kingma and Ba, 2014].

4 Sequential Variational Approximation

This section discusses the extension of the variational approach in Section 3.4 to the situation where we are interested in obtaining sequential variational approximations of both the latent states and parameters of the multivariate factor SV model as new observations arrive. Here, using either MCMC or particle MCMC methods can become very expensive and time consuming as both methods need to repeatedly run the full MCMC as new observations arrive. In this setting, it is necessary to update the joint posterior of the latent states and parameters sequentially, to take into account the newly arrived data. In more detail, let 1, 2,...,$t$ be a sequence of time points, $x_{1:t}$ and $\theta$ be the latent states and parameters in the model, respectively. The goal is to estimate a sequence of posterior distributions of states and parameters, $p(\theta, x_{1:1}|y_{1:1})$, $p(\theta, x_{1:2}|y_{1:2})$, ..., $p(\theta, x_{1:T}|y_{1:T})$, where $T$ is the total length of the time series.

The usual application of Bayes rule at a time $t$ involves a likelihood computation from time 1 to $t$. However, with the availability of the posterior at time $t - 1$, given by its posterior density $p(\theta, x_{1:t-1}|y_{1:t-1})$, the updated posterior at time $t$ is given by

$$p(\theta, x_{1:t}|y_{1:t}) \propto p(y_t|\theta, x_t) p(x_t|x_{t-1}, \theta) p(\theta, x_{1:t-1}|y_{1:t-1}).$$  \hspace{1cm} (13)

The sequential algorithm, which we call, seq-VA, first estimates the variational approximation at time $t - 1$ as $q_{\lambda^*}(\theta, x_{1:t-1})$. Note that at this step of the algorithm, we simply use standard method given in Section 3.4 and approximate the posterior $p(\theta, x_{1:t-1}|y_{1:t-1})$ by $q_{\lambda^*}(\theta, x_{1:t-1})$. Then, after observing the additional data at time $t$, the seq-VA replaces the posterior construction defined in Eq. (13), with the available approximation $q_{\lambda^*}(\theta, x_{1:t-1})$,

$$\hat{p}(\theta, x_{1:t}|y_{1:t}) \propto p(y_t|\theta, x_t) p(x_t|x_{t-1}, \theta) q_{\lambda^*}(\theta, x_{1:t-1}).$$  \hspace{1cm} (14)
The term $\hat{p}(\theta, x_{1:t}|y_{1:t})$ is not the true posterior, and we refer to it as ‘approximate posterior’ at time $t$. The objective for each sequential update is to find the variational parameter $\lambda$ through the maximisation of the variational lower bound (ELBO) for each time period $t = 1, 2, ..., T$. The variational lower bound at time $t$ is

$$L(\lambda) = E_q \left( \log \hat{p}(\theta, x_{1:t}|y_{1:t}) - \log q_\lambda(\theta, x_{1:t}) \right).$$  \hfill (15)

Applying the reparameterisation trick as in Section 3.1 to Eq. (15) gives

$$L(\lambda) = E_f \left( \log \hat{p}(u(\eta, \lambda)) - \log q_\lambda(u(\eta, \lambda)) \right); \hfill (16)$$

we represent samples from $q_\lambda(\theta, x_{1:t})$ as $(\theta, x_{1:t}) = u(\eta, \lambda)$, where $\eta$ is a random vector with a fixed density $f(\eta)$ that does not depend on the variational parameters. Differentiating Eq. (16) under the integral sign gives

$$\nabla_\lambda L(\lambda) = E_f \left( \nabla_\lambda \log \hat{p}(u(\eta, \lambda)) - \nabla_\lambda \log q_\lambda(u(\eta, \lambda)) \right)$$

The following are three important features of this sequential variational approximation. At time $t$, the standard variational approximation in Section 3.4 targets the exact posterior $p(\theta, x_{1:t}|y_{1:t})$, whereas the seq-VA described in this section targets an approximate posterior distribution in Eq. (14). The seq-VA can start if only part of the data has been observed. The optimal value of the variational parameters $\lambda$ at time $t - 1$ can be used as the initial values of the variational parameters $\lambda$ at time $t$; Section 6.1.2 shows that this may substantially reduce the number of iterations required for the algorithm to converge.

## 5 Variational Forecasting

Let $Y_{T+1}$ be the unobserved value of the dependent variable at time $T+1$. Given the joint posterior distribution of all the parameters $(\theta)$ and latent state variables $(x_{1:T})$ of the multivariate factor SV model up to time $T$, the forecast density of $Y_{T+1} = y_{T+1}$ that accounts for the uncertainty about $\theta$ and $x_{1:T}$ is

$$p(y_{T+1}|y_{1:T}) = \int p(y_{T+1}|\theta, x_{T+1}) p(x_{T+1}|x_{T}, \theta) p(\theta, x_{1:T}|y_{1:T}) \, d\theta dx_{1:T}; \hfill (17)$$

$p(\theta, x_{1:T}|y_{1:T})$ is the exact posterior density that can be estimated from MCMC or particle MCMC. The draws from MCMC or particle MCMC can be used to produce
the simulation-consistent estimate of this predictive density

\[
p(y_{T+1}|y_{1:T}) = \frac{1}{M} \sum_{m=1}^{M} p(y_{T+1}|y_{1:T}, \theta^{(m)}, x^{(m)}_{T+1}); \quad (18)
\]

it is necessary to know the conditional density \( p(y_{T+1}|y_{1:T}, \theta^{(m)}, x^{(m)}_{T+1}) \) in closed form to obtain the estimates in Eq. (18). Alternatively, we can obtain \( M \) draws of \( y_{T+1} \) from \( p(y_{T+1}|y_{1:T}, \theta^{(m)}, x^{(m)}_{T+1}) \) that can be used to obtain the kernel density estimate \( p(y_{T+1}|y_{1:T}) \). Assuming that the MCMC has converged, these are two simulation-consistent estimates of the exact “Bayesian” predictive density.

The motivation for using the variational approximation in this setting is that obtaining the exact posterior density \( p(\theta, x_{1:T}|y_{1:T}) \) is expensive for high dimensional multivariate factor SV models. Variational approximation to obtain an approximation of the posterior density \( q_\lambda(\theta, x_{1:T}|y_{1:T}) \) is usually substantially faster than MCMC or particle MCMC. Given that we have the variational approximation of the joint posterior distribution of the parameters and latent variables, we can define

\[
g(y_{T+1}|y_{1:T}) = \int p(y_{T+1}|\theta, x_{T+1}) p(x_{T+1}|x_{T}, \theta) q_\lambda(\theta, x_{1:T}|y_{1:T}) d\theta dx_{1:T}, \quad (19)
\]

where \( q_\lambda(\theta, x_{1:T}|y_{1:T}) \) is the variational approximation to the joint posterior distribution of the parameters and latent variables of the multivariate factor SV model. As with most quantities in Bayesian analysis, computing the approximate predictive density in Eq. (19) can be challenging because it involves high dimensional integrals which cannot be solved analytically. However, it can be approximated through Monte Carlo integration,

\[
g(y_{T+1}|y_{1:T}) \approx \frac{1}{M} \sum_{m=1}^{M} p(y_{T+1}|y_{1:T}, \theta^{(m)}, x^{(m)}_{T+1}); \quad (20)
\]

where \( \theta^{(m)} \) and \( x_{1:T}^{(m)} \) denotes the \( m \)th draw from the variational approximation \( q_\lambda(\theta, x_{1:T}|y_{1:T}) \) up to time \( T \). For the factor SV model, we can obtain the approximate Bayesian predictive density \( g(y_{T+1}|y_{1:T}) \) by generating \( M \) draws of \( y_{T+1} \) from \( p(y_{T+1}|y_{1:T}, \theta^{(m)}, x^{(m)}_{T+1}) \) that can be used to obtain the kernel density estimate of \( g(y_{T+1}|y_{1:T}) \). Similarly, it is also straightforward to obtain multiple-step ahead predictive densities \( g(y_{T+h}|y_{1:T}) \), for \( h = 1, \ldots, H \).
6 Simulation Study and Real Data Application

6.1 Experimental Results

This section applies the variational approximation methods developed in Section 3 for the multivariate factor SV model to simulated and real datasets.

6.1.1 Simulation Study

We conducted a simulation study for the multivariate factor SV model discussed in Section 2 to compare our variational approximations $q_I(\theta, x_{1:T})$ and $q_{II}(\theta, x_{1:T})$ to the exact particle MCMC method of Gunawan et al. (2020) in terms of computation time and the accuracy of posterior and predictive densities approximation. We also investigate whether our variational approximations can be further improved using the importance weighting approach of Burda et al. (2016) in Section 3.2, considering $B \in \{28, 56, 140, 280\}$ samples to estimate the gradients of the importance weighted lower bound (IWLB) in Equation (10). Define $IW - q_I(\theta, x_{1:T})$ and $IW - q_{II}(\theta, x_{1:T})$ as the variational approximations $q_I(\theta, x_{1:T})$ and $q_{II}(\theta, x_{1:T})$ that maximise the IWLB, respectively. We initialised $IW - q_I(\theta, x_{1:T})$ and $IW - q_{II}(\theta, x_{1:T})$ using the fits from the variational approximations $q_I(\theta, x_{1:T})$ and $q_{II}(\theta, x_{1:T})$, respectively, and the algorithm is terminated after a short run of 5000 iterations as they are computationally intensive. The IWLB $L_{IW}^B(\lambda)$ reduces to $L(\lambda)$ with $B = 1$. Posterior distributions estimated using exact particle MCMC method of Gunawan et al. (2020) are treated as the ground truth for comparing the accuracy of the posterior and predictive density approximations. All experiments were done using Matlab on a single desktop computer with 6-CPU cores.

We simulated data with $T = 1000$, $S = 26$, and $K = 1$. We ran particle MCMC sampling scheme for 31000 iterations and discarded the initial 1000 iterates as warmup. We perform 50000 iterations of a stochastic gradient ascent optimisation algorithm with learning rates chosen adaptively according to ADAM approach discussed in Section 3.4. All the variational parameters are initialised randomly.

Figure 1 monitors the convergence of the variational approximations $q_I(\theta, x_{1:T})$ and $q_{II}(\theta, x_{1:T})$ via the estimated values of their lower bounds $\hat{L}_I(\lambda)$ and $\hat{L}_{II}(\lambda)$, using a single Monte Carlo sample. The figure shows that both variational approximations converge.
Figure 1: Plot of Lower Bound for variational approximations $q^I_{\lambda,x_{1:T}}(\theta)$ and $q^{II}_{\lambda,x_{1:T}}(\theta)$ for simulated dataset.

Table 1: Simulated data. Number of iterations $I$ (in thousands) and CPU times (in seconds) of the MCMC and variational methods. Average and standard deviation of the lower bound value over the last 1000 steps are also reported. PMCMC is particle MCMC. The $B$ is the number of samples to estimate the (importance weighted) lower bound.

| Methods          | $B$ | $I$ | Time per iteration | Total Time | $\mathcal{L}(\lambda)$          |
|------------------|-----|-----|--------------------|------------|----------------------------------|
| PMCMC            | 30  |     | 1.38               | 41400      | NA                               |
| $q^I_{\lambda}(\theta, x_{1:T})$ | 1   | 50  | 0.03               | 1500       | -33613.18 (17.60)                |
| $IW - q^I_{\lambda}(\theta, x_{1:T})$ | 28  | 5   | 0.62               | 3100       | -33577.91 (8.63)                 |
| $IW - q^I_{\lambda}(\theta, x_{1:T})$ | 56  | 5   | 1.18               | 5900       | -33574.37 (8.19)                 |
| $IW - q^I_{\lambda}(\theta, x_{1:T})$ | 140 | 5   | 2.60               | 13000      | -33569.30 (7.00)                 |
| $IW - q^I_{\lambda}(\theta, x_{1:T})$ | 280 | 5   | 5.84               | 27100      | -33561.51 (6.42)                 |
| $q^{II}_{\lambda}(\theta, x_{1:T})$ | 1   | 50  | 0.04               | 2000       | -33607.96 (17.87)                |
| $IW - q^{II}_{\lambda}(\theta, x_{1:T})$ | 28  | 5   | 0.96               | 4800       | -33575.34 (9.13)                 |
| $IW - q^{II}_{\lambda}(\theta, x_{1:T})$ | 56  | 5   | 1.80               | 9000       | -33571.29 (8.34)                 |
| $IW - q^{II}_{\lambda}(\theta, x_{1:T})$ | 140 | 5   | 4.52               | 22600      | -33566.91 (7.33)                 |
| $IW - q^{II}_{\lambda}(\theta, x_{1:T})$ | 280 | 5   | 8.50               | 42500      | -33563.36 (6.97)                 |

Table 1 shows the CPU times of the particle MCMC method and the variational approximations. It is clear that the CPU time of the more complex variational approximation $q^{II}_{\lambda}(\theta, x_{1:T})$ is higher than variational approximation $q^I_{\lambda}(\theta, x_{1:T})$, but with a higher lower bound. The variational approximations $q^I_{\lambda}(\theta, x_{1:T})$ and $q^{II}_{\lambda}(\theta, x_{1:T})$ are, respectively, 27.60 and 20.70 times faster than the exact particle MCMC method. The variational methods are potentially much faster than reported
as Figure 1 shows that the variational approximations $q^I_\lambda (\theta, x_{1:T})$ and $q^{II}_\lambda (\theta, x_{1:T})$ already converge after 20000 iterations. The table also shows variational approximations with importance weights are much more expensive but they have higher estimates and smaller standard deviations of the lower bounds.

Figure 2: Simulated dataset. Plots of the posterior mean estimates of the idiosyncratic log-volatilities $h_{\epsilon,4,1:T}$, $h_{\epsilon,8,1:T}$, $h_{\epsilon,20,1:T}$, and $h_{\epsilon,26,1:T}$ estimated using particle MCMC and the variational approximations $q^I_\lambda (\theta)$, and $q^{II}_\lambda (\theta)$, with different values of $B \in \{1, 28, 56, 140, 280\}$ samples to estimate the (importance weighted) lower bound.
Figure 3: Simulated dataset. Scatter plots of the posterior means of the idiosyncratic log-volatilities $h_{t,s,t}$, for $s = 1, \ldots, 25$ and $t = 10, 20, \ldots, 1000$ estimated using particle MCMC on the x-axis and the variational approximation $q_1^I(\theta)$ on the y-axis.

Figure 2 plots the estimated idiosyncratic log-volatilities $h_{t,s,1:T}$ for $s = 4, 8, 20$ and 26, estimated using the different variational approximation methods and the particle MCMC method. The figure shows that all variational approximations give posterior mean estimates that are very close to each other and close to the particle MCMC estimates. Figures 3 and S3 (in Appendix S1) compare the variational posterior means and standard deviations of $q_1^I(\theta, x_{1:T})$ with the posterior means and standard deviations computed using the particle MCMC for series 1 to 25. The figures show that the variational approximation $q_1^I(\theta, x_{1:T})$ estimates the posterior means accurately but slightly underestimates the posterior standard deviations when compared to the exact particle MCMC method.
Figure 4: Simulated dataset. The marginal posterior density plots of the parameters \( \{\alpha_{s,4}, \alpha_{s,8}, \alpha_{s,20}, \alpha_{s,26}\} \) estimated using particle MCMC and VB with the variational approximations \( q_1^I(\theta) \), and \( q_1^{II}(\theta) \), with different values of \( B \in \{1, 28, 56, 140, 280\} \) samples.

Figure 4 shows the marginal posterior densities of the parameters \( \{\alpha_{s,s}\} \) for \( s = 4, 8, 20, 26 \). All the variational approximations capture the posterior means quite well. There is a slight underestimation of the posterior variances of the \( \alpha \)'s. The variational approximations with the importance weights are quite similar to those without the importance weights, so there seems to be no gain in using importance weights lower bound. Similar conclusions can be made from Figures S1 and S2 in Appendix S1.

We now compare the predictive performance of the variational approximations with the (exact) particle MCMC method. Figure 5 shows the multiple-step ahead predictive densities \( \hat{p}(y_{T+h}|y_{1:T}) \) of an equally weighted combination of all series, for \( h = 1, \ldots, 10 \) obtained using variational and PMCMC methods. The figure shows that all the variational predictive densities are very close to the exact predictive densities obtained from the particle MCMC method. Using the multivariate factor SV model, we can also obtain the predictive densities of the time-varying correlation between any two series. Given the time-varying covariance matrix at time \( t \), the correlation matrix at time step \( t \) is

\[
\Gamma_t = \text{diag}(\Sigma_t)^{-\frac{1}{2}} \Sigma_t \text{diag}(\Sigma_t)^{-\frac{1}{2}}.
\]

Figure 6 shows the multiple-step ahead predictive densities of the time-varying correlation between series 23 and 26 \( \hat{p}(\Gamma(y_{23,T+h}, y_{26,T+h})|y_{1:T}) \) for \( h = 1, \ldots, 10 \). The
figure shows that all the variational predictive densities are very close to the exact predictive densities obtained from the PMCMC method. Both Figures 5 and 6 show that the variational predictive densities do not underestimate the predictive variances.

We conclude from the simulation study that the variational approximations are much faster than the exact MCMC approach; the variational approximations capture the posterior means of the parameters of the multivariate factor SV model quite accurately, but there is some slight underestimation in the posterior variance of some of the parameters; all variational approximations produce predictive densities for the returns and time-varying correlation between time series that are similar to the exact particle MCMC method; the variational approximation $q_\lambda^I(\theta, x_{1:T})$ is much faster than $q_\lambda^{II}(\theta, x_{1:T})$ and other variational approximations with importance weights and it gives very similar accuracy. Therefore, it is very important to balance the accuracy, CPU time, and also the complexity of the variational approximations when choosing a variational approximation method.

Figure 5: Simulated dataset. Plot of the predictive densities $\hat{p}(y_{T+h}|y_{1:T})$ of an equally weighted series, for $h = 1, \ldots, 10$, estimated using particle MCMC and the variational approximations $q_\lambda^I(\theta)$, and $q_\lambda^{II}(\theta)$, with different values of $B \in \{1, 28, 56, 140, 280\}$ samples.
6.1.2 Application to US Stock Returns Data

Our methods are now applied to a sample of weekly US industry stock returns data. The data, obtained from the website of Kenneth French, consists of weekly returns for \( S = 26 \) value weighted industry stock returns, which are listed in Table S1 in Appendix S2; the sample is from the 2nd of January 1990 to 21st of June 2019 (\( T = 1486 \) demeaned log returns in total). In this example, we estimate a two factor multivariate SV model. We note that the example merely illustrates our variational methods; they can easily handle more factors and most types of log-volatilities for both the factors and the idiosyncratic errors. We use similar setup and initial values as in the simulation study for both particle MCMC and the variational methods. This section considers the variational approximations without importance weights \( q^I_\lambda (\theta, x_{1:T}) \) and \( q^{II}_\lambda (\theta, x_{1:T}) \); the mean-field Gaussian variational approximation \( q^{MF}_\lambda (\theta, x_{1:T}) \) (Gaussian with diagonal covariance matrix; see Xu et al. (2018)) that ignores all posterior dependence structures in the model; and one example of the sequential variational approximation, \( \text{seq}-q^I_\lambda (\theta, x_{1:T}) \). By comparing our variational approximations with mean-field Gaussian variational approximation, we can investigate the importance of taking into account some of the posterior dependence structures in the model. For the sequential variational method, \( \text{seq}-q^I_\lambda (\theta, x_{1:T}) \), we use the standard variational method up to \( t = 1386 \) and use the sequential variational approach described in Section 4 from \( t = 1387 \) to \( 1486 \). We perform 5000 iterations of a stochastic gradient
ascent optimisation algorithm for each sequential update.

Figure 7 monitors the convergence of the three variational approximations $q_I^\lambda (\theta, x_{1:T})$, $q_{II}^\lambda (\theta, x_{1:T})$, and $q_{MF}^\lambda (\theta, x_{1:T})$ via the estimated values of their lower bounds, using a single Monte Carlo sample. The figure clearly shows that all variational approximations have converged. Table 2 reports the CPU times of the exact particle MCMC method and the variational approximations. The variational approximations $q_I^\lambda (\theta, x_{1:T})$, $q_{II}^\lambda (\theta, x_{1:T})$, and $q_{MF}^\lambda (\theta, x_{1:T})$ are 33.66, 11.40, and 86.25 times faster respectively than the exact particle MCMC method. The variational methods are potentially faster than reported in Table 2 as Figure 7 shows that all the variational approximations $q_I^\lambda (\theta, x_{1:T})$, $q_{II}^\lambda (\theta, x_{1:T})$, and $q_{MF}^\lambda (\theta, x_{1:T})$ converge around 20000 iterations. Surprisingly, the more complex variational approximation $q_{II}^\lambda (\theta, x_{1:T})$ has a smaller lower bound than the variational approximation $q_I^\lambda (\theta, x_{1:T})$. Figure 8 plots the lower bound for the sequential variational method seq-$q_{\lambda}^\lambda (\theta, x_{1:T})$ for the last 25 time periods. The figure shows that the sequential variational algorithm converges within 2500 iterations or less for all time periods. It is equivalent to 102.50 seconds for each time period update.

Figure 7: US stock return dataset. Plot of Lower Bound for variational approximations $q_I^\lambda (\theta, x_{1:T})$, $q_{II}^\lambda (\theta, x_{1:T})$, and $q_{MF}^\lambda (\theta, x_{1:T})$. 

![Figure 7: US stock return dataset. Plot of Lower Bound for variational approximations $q_I^\lambda (\theta, x_{1:T})$, $q_{II}^\lambda (\theta, x_{1:T})$, and $q_{MF}^\lambda (\theta, x_{1:T})$.](image)
Table 2: US stock return dataset. Number of iterations $I$ (in thousands) and CPU times (in seconds) of the exact PMCMC and variational methods. Average and standard deviation of the lower bound value over the last 1000 steps are also reported. PMCMC is particle MCMC.

| Methods          | $I$ | Time per iteration | Total Time | $\mathcal{L}(\tilde{\lambda})$ |
|------------------|-----|--------------------|------------|-------------------------------|
| PMCMC            | 30  | 2.300              | 69000      | NA                            |
| $q_I^L(\theta, x_{1:T})$ | 50  | 0.041              | 2050       | $-46983.71$ (28.87)           |
| $q_{II}^L(\theta, x_{1:T})$ | 50  | 0.121              | 6050       | $-46996.07$ (28.63)           |
| $q_{MF}^L(\theta, x_{1:T})$ | 50  | 0.016              | 800        | $-50123.99$ (49.69)           |

Figure 8: US stock return dataset. Plot of Lower Bound for variational approximations $seq-q^L(\theta, x_{1:T})$ for the last 25 time periods.

Figure S7 in Appendix S2 plots the estimated idiosyncratic log-volatilities $h_{\epsilon,s,1:T}$ for $s = 6, 8, 20$ and $26$, using different variational approximations and the exact particle MCMC method. Similarly to Section 6.1.1, the figure shows that all variational approximations give posterior mean estimates that are very close to each other and close to the particle MCMC estimates, except for the mean field variational approximation. This shows that the performances of the variational approximations $q^L(\theta, x_{1:T})$ and $q_{II}^L(\theta, x_{1:T})$ do not deteriorate when applied to real data. Figure 9 shows the estimated marginal posterior densities of the parameters $\{\alpha_{\epsilon,s}\}$ for $s = 6, 8, 20, 26$ using the different variational approximations and the exact particle MCMC method. The variational approximations $q^L(\theta, x_{1:T}), q_{II}^L(\theta, x_{1:T})$, and $seq-q^L(\theta, x_{1:T})$ are able to capture the posterior means quite well. There is some underestimation of the posterior variances of the $\alpha$’s in all the variational approximations. Note that the sequential variational approximation $seq-q^L(\theta, x_{1:T})$ has
smaller posterior variances compared to variational approximations \( q^I_\lambda (\theta, x_{1:T}) \) and \( q^{II}_\lambda (\theta, x_{1:T}) \). This occurs because the sequential variational approach targets the 'approximate posterior' and this leads to a loss of accuracy in the approximation. The mean field variational approximation gives very different estimates of the marginal posterior densities of the parameters \( \{\alpha_{\epsilon,s}\} \) for \( s = 6, 8, 20, 26 \) compared to the particle MCMC estimates and the other variational approximations. Similar conclusions can be made from Figures S4 and S5 in Appendix S2.

Figure 9: US stock return dataset. Plots of the estimated marginal posterior densities of the parameters \( \{\alpha_{\epsilon,s}\}, s = 6, 8, 20 \) and 26, for particle MCMC and the variational approximations \( q^I_\lambda (\theta, x_{1:T}), q^{II}_\lambda (\theta, x_{1:T}), q^{MF}_\lambda (\theta, x_{1:T}) \), and seq-\( q^I_\lambda (\theta, x_{1:T}) \).

The predictive performance of the variational approximations is now compared to that of the particle MCMC method. Figure S8 in Appendix S2 and figure 10 show the one-step ahead predictive densities \( \hat{p}(y_{T+1}|y_{1:T}) \) for different US stock returns and the multiple-step ahead predictive densities \( \hat{p}(y_{T+h}|y_{1:T}) \) of an equally weighted portfolio, for \( h = 1, \ldots, 10 \). Clearly, the predictive densities obtained from the variational methods are very close to the predictive densities obtained from the particle MCMC method, even for the mean field variational approximation. It is important to note that the variational predictive densities do not underestimate the posterior variance. Figure S6 in Appendix S2 and figure 11 show the multiple-step ahead predictive densities of the time-varying correlation between steel and food industries \( \hat{p}(\Gamma(y_{1:T+h},y_{2,T+h})|y_{1:T}) \) and automobiles and apparel industries \( \hat{p}(\Gamma(y_{15,T+h},y_{10,T+h})|y_{1:T}) \) for \( h = 1, \ldots, 10 \), respectively. The figures show that the variational predictive densities \( q^I_\lambda (\theta, x_{1:T}), q^{II}_\lambda (\theta, x_{1:T}), \) and seq-\( q^I_\lambda (\theta, x_{1:T}) \) are quite close to the exact predictive densities obtained from the particle MCMC method.
It is important to note that the mean-field variational predictive densities are very
different to the exact predictive densities obtained from the particle MCMC method.

Another important advantage of the sequential variational approximation seq-
$q_{\lambda}^I (\theta, x_{1:T})$ is that it can provide sequential one-step ahead predictive densities of
each stock return and any weighted portfolio as well as the time-varying correlation
between industry stock returns; these are difficult to obtain using MCMC or
particle MCMC approaches. Figure 12 plots the sequential one-step ahead predic-
tive densities of the equally weighted portfolio of US industry stock returns and
sequential one-step ahead predictive time-varying correlation estimates between au-
tomobiles and apparel industries and steel and food industries from 03/07/2017 to
28/06/2019. The figure shows that the steel industry is positively correlated with
the food industry, but the correlation decreases over time during that period. Similar
conclusions can be made for the relationship between the automobile and apparel
industries.

We can conclude this study that a) Variational approaches are much faster than
MCMC and particle MCMC approaches. b) The variational approximations capture
the posterior means of the parameters of the multivariate factor SV model quite
accurately, except for the mean-field variational approximation, but there is some
slight underestimation in the posterior variances for some of the parameters, espe-
cially with the estimates obtained from the sequential variational approximation. c)
The variational approximations $q_{\lambda}^I (\theta, x_{1:T})$, $q_{\lambda}^{II} (\theta, x_{1:T})$, and seq-$q_{\lambda}^I (\theta, x_{1:T})$ produce
similar one-step and multiple-step ahead predictive densities for the individual stock
returns, for an equally weighted portfolio, and for the time-varying correlation be-
tween any pair of stock returns as the exact particle MCMC method, without the
problem of underestimating the variance. d) Surprisingly, although the mean-field
variational approximation gives very different posterior mean estimates for the states
and parameters of the multivariate factor SV model, the mean field variational pre-
dictive densities for the individual stock returns and for an equally weighted portfolio
are not very different to the variational predictive densities obtained from the vari-
ational approximations $q_{\lambda}^I (\theta, x_{1:T})$, $q_{\lambda}^{II} (\theta, x_{1:T})$, and seq-$q_{\lambda}^I (\theta, x_{1:T})$ and the particle MCMC. e) However, the mean field variational approximation gives very different
estimates of the predictive densities of the time-varying correlation. f) The vari-
atational approximation $q_{\lambda}^I (\theta, x_{1:T})$ is much faster than the more complex variational
approximation $q_{\lambda}^{II} (\theta, x_{1:T})$. It is therefore very important to balance the accuracy,
CPU time, and also the complexity of the variational approximations when choosing
a variational approximation. g) The seq-VA can be applied even if only part of the
data is observed and the optimal value of the variational parameter $\lambda$ at the previous
time period $t - 1$ can be used as the initial value of the variational parameter $\lambda$ at
time \( t \); this is likely to reduce the number of iterations required for the algorithm to converge. 

h) The sequential variational approach can produce fast sequential one-step ahead predictive densities of the individual returns and portfolios of returns and the time-varying correlations between pairs of stock returns.

Figure 10: US stock return data. Plot of multiple-step ahead predictive densities \( \hat{p}(y_{T+h}|y_{1:T}) \) of equally weighted portfolios, for \( h = 1, \ldots, 10 \) estimated using particle MCMC and the variational approximations \( q_1^I(\theta, x_{1:T}) \), \( q_1^{II}(\theta, x_{1:T}) \), \( q_{MF}^{MF}(\theta, x_{1:T}) \), and seq-\( q_1^{I}(\theta, x_{1:T}) \).

Figure 11: US stock return data. Plot of the multiple-step ahead predictive densities of the time-varying correlation between Apparel and Automobiles and Trucks \( \hat{p}(\Gamma(y_{15,T+h}, y_{10,T+h})|y_{1:T}) \), for \( h = 1, \ldots, 10 \).
7 Conclusions

This paper proposes fast and efficient variational methods for estimating the posterior of the high dimensional factor SV model and obtains one-step and multi-step ahead variational forecast distributions. We show that the method works well for both simulated and real datasets and is much faster than exact MCMC approaches. Future work will consider developing variational approaches for other complex and high-dimensional time series models, such as the vector autoregressive model with stochastic volatility (VAR) of Chan et al. (2020) and multivariate financial time series model with recurrent neural network type architectures, e.g. the Long-Short term Memory model of Hochreiter and Schmidhuber (1997).

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S1 Additional Tables and Figures for the Simulation Study

Figure S1: Simulated data set. The plots of marginal posterior densities of the parameters $\{\kappa_{\epsilon,s}\}, s = 4, 8, 20$ and 26, estimated using particle MCMC and the variational approximations $q^I_\lambda(\theta, x_{1:T})$, $q^{II}_\lambda(\theta, x_{1:T})$, and $q^{MF}_\lambda(\theta, x_{1:T})$, with different values of $B \in \{1, 28, 56, 140, 280\}$.
Figure S2: Simulated data set. The plots of the marginal posterior densities of the parameters $\{\psi_{\epsilon,s}\}$, $s = 4, 8, 20$ and $26$, estimated using particle MCMC and the variational approximations $q^I_1(\theta, x_{1:T})$, $q^I_2(\theta, x_{1:T})$, and $q^MF_1(\theta, x_{1:T})$, with different values of $B \in \{1, 28, 56, 140, 280\}$.

Figure S3: Scatter plot of the posterior standard deviations of the idiosyncratic log-volatilities $h_{\epsilon,s,t}$, for $s = 1, \ldots, 25$ and $t = 10, 20, \ldots, 1000$ estimated using particle MCMC on the x-axis and variational approximation $q^I_1(\theta, x_{1:T})$ on the y-axis.
S2  Additional Tables and Figures for the US stock return dataset

| Stocks                      |
|-----------------------------|
| 1  Steel Works              |
| 2  Food Products            |
| 3  Recreation               |
| 4  Health Care and Equipment|
| 5  Coal                     |
| 6  Printing and Publishing  |
| 7  Retail                   |
| 8  Tobacco                  |
| 9  Consumer Goods           |
| 10 Apparel                  |
| 11 Chemicals                |
| 12 Textiles                 |
| 13 Fabricated Products      |
| 14 Electrical Equipment     |
| 15 Automobiles and Trucks   |
| 16 Aircraft, ships, and Railroad Equipment |
| 17 Industrial Mining        |
| 18 Petroleum and Natural Gas|
| 19 Utilities                |
| 20 Telecommunication        |
| 21 Personal and Business Services |
| 22 Business Equipment       |
| 23 Transportation           |
| 24 Wholesale                |
| 25 Restaurants, Hotels, and Motels |
| 26 Banking, Insurance, Real Estate |
Figure S4: US stock return data. Plots of the marginal posterior density estimates of the parameters $\{\kappa_{s,s}\}, s = 6, 8, 20$ and 26 using particle MCMC and the variational approximations $q^I_\lambda(\theta, x_{1:T}), q^{II}_\lambda(\theta, x_{1:T}), q^{MF}_\lambda(\theta, x_{1:T})$, and seq-$q^I_\lambda(\theta, x_{1:T})$.

Figure S5: US stock return data. Plots of marginal posterior density estimates of the parameters $\{\psi_{s,s}\}, s = 6, 8, 20$ and 26. using particle MCMC and the variational approximations $q^I_\lambda(\theta, x_{1:T}), q^{II}_\lambda(\theta, x_{1:T}), q^{MF}_\lambda(\theta, x_{1:T})$, and seq-$q^I_\lambda(\theta, x_{1:T})$. 
Figure S6: US stock return data. Plots of multiple-step ahead predictive densities of the time-varying correlations between the series Steel Works and Food Products $\hat{p}(\Gamma(y_{1,T+h}, y_{2,T+h} | y_{1:T}),$ for $h = 1, \ldots, 10$.

Figure S7: US stock return data. Plots of the posterior mean estimates of the idiosyncratic log-volatilities $h_{\epsilon,s,1:T}, s = 6, 8, 20, \text{and } 26$ using particle MCMC and the variational approximations $q^I_\lambda(\theta, x_{1:T}), q^{II}_\lambda(\theta, x_{1:T}), q^{MF}_\lambda(\theta, x_{1:T}), \text{and } \text{seq-q}^I_\lambda(\theta, x_{1:T})$. 
Figure S8: US stock return data. Plots of the one-step ahead predictive densities \( \hat{p}(y_{T+1}|y_{1:T}) \), estimated using particle MCMC and the variational approximations \( q^I_\lambda(\theta, x_{1:T}) \), \( q^{II}_\lambda(\theta, x_{1:T}) \), \( q^{M_F}_\lambda(\theta, x_{1:T}) \), and seq-\( q^I_\lambda(\theta, x_{1:T}) \)