The Mathematics of Text Structure

by Bob Coecke
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Abstract  In previous work we gave a mathematical foundation, referred to as DisCoCat, for how words interact in a sentence in order to produce the meaning of that sentence. To do so, we exploited the perfect structural match of grammar and categories of meaning spaces.

Here, we give a mathematical foundation, referred to as DisCoCirc, for how sentences interact in texts in order to produce the meaning of that text. First we revisit DisCoCat. While in DisCoCat all meanings are fixed as states (i.e. have no input), in DisCoCirc word meanings correspond to a type, or system, and the states of this system can evolve. Sentences are gates within a circuit which update the variable meanings of those words.

Like in DisCoCat, word meanings can live in a variety of spaces e.g. propositional, vectorial, or cognitive. The compositional structure are string diagrams representing information flows, and an entire text yields a single string diagram in which word meanings lift to the meaning of the entire text.

While the developments in this paper are independent of a physical embodiment (cf. classical vs. quantum computing), both the compositional formalism and suggested meaning model are highly quantum-inspired, and implementation on a quantum computer would come with a range of benefits.

We also praise Jim Lambek for his role in mathematical linguistics in general, and the development of the DisCo program more specifically.
1 Introduction

DisCoCat (cf. Categorical Compositional Distributional) [28] resulted from addressing the following question:

There are dictionaries for words.
Why aren’t there any dictionaries for sentences?

This question is both of academic interest as well as of practical interest. Firstly, it addresses how we can understand sentences that we never heard before, provided we understand the words. Secondly, it could enable machines to do so too—see e.g. [15] for a preliminary discussion.

Now, one obviously can extend the question to:

Why aren’t there any dictionaries for entire texts?

While there is no such thing like a grammar that very rigidly organises sentences in a text, there should be something else structuring sentences in a text, as, to put it naively, we can’t just swap sentences randomly around while retaining the meaning of the text. What exactly is the structure governing sentences? Also, how does it relate to the structure governing words in sentences? Overall, this paper has two purposes:

- To give a mathematical foundation for sentence meaning composition, and how sentence meaning interacts with word meaning composition, by modifying as well as elaborating upon our theory of grammar-driven word meaning composition known as DisCoCat.
- As this paper was commissioned for a volume dedicated to the great Jim Lambek, to praise Lambek for the development of mathematical linguistics in general, for his role in the development of the DisCoCat program more specifically, and for the role of physics in all of that [14].

Concerning the 1st goal, we believe that the results in this paper may constitute radical progress for the broad DisCoCat research program, as three important longstanding issues are addressed ‘in full generality’:

a. How sentence meanings compose in order to form the meaning of text.
b. How word meanings evolve in text, when learning new things.
c. What the type of the sentence meaning space is.

The key idea to achieve (a) is to not treat (all) word-meanings as static entities, like we did in DisCoCat, but as dynamic ones, so that they can evolve in the light of what is conveyed about that word within the text, i.e. (b). For example, using a movie analogy, we can learn more about the main actors of the text, and/or these actors themselves can learn about the rest of the world. Technically, rather than states, such actors will be represented by types, sentences then act on those types as I/O-processes, transforming some initial state of an actor into a resulting one, which yields (c). In other words, text made up of several sentences is organised as a circuit. Since:
1. Word-meanings: states $\mapsto$ types
2. Sentence-meanings: states $\mapsto$ I/O-processes
3. Text-meaning: $\emptyset$ $\mapsto$ circuit

are a radical departure from the state-focussed DisCoCat program, we name it DisCoCirc (cf. Circuit-shaped Compositional Distributional). Other than that, all of the attractive features of DisCoCat are all retained (see Sec. 4.4), such as sentences with different grammatical structure having the same type, model-flexibility, and the diagrammatic format (cf. Sec. 3). In fact, DisCoCat can be seen as an instance of DisCoCirc, in that in any DisCoCirc(uit) there will be multiple participating DisCoCat(s)—and the gloves are off too:

Jim Lambek’s legacy. We already started our praise for Lambek in the title of this paper, by tweaking the title of Lambek’s seminal paper “The Mathematics of Sentence Structure” [47]. Not much has changed in that Lambek’s story is pretty much still what mathematical linguists agree on today. Everyone? Well, Lambek himself certainly did not. Some 20 years ago Lambek decided to ‘do a von Neumann’,¹ and replaced Lambek grammar with pregroup grammar [52, 53]. Around 2004 I was giving a talk at the McGill category theory seminar about our then new diagrammatic description of quantum teleportation [1, 16]. Lambek immediately pointed out: “Those are pre-groups!” The compact closed category-theory underpinning of quantum theory, have indeed pregroups as the posetal instance. It was this connection, between grammar and teleportation diagrams, that inspired the DisCoCat model, making it look as if word-meaning being teleported around in sentences by means of the channels provided by the pregroup grammar [14]. Lambek himself explicitly stressed this connection between language and physics in a paper written in 2006 [54], which due to my all too slow editorial skills only appeared in 2011.

¹ von Neumann denounced his own quantum mechanical formalism merely three years after it was published as a book [81], and devoted a large portion of the remainder of his life to finding an alternative formalism—see [66] for a discussion.
Lambek made many more pioneering contributions, including significant contributions to linear logic, which we briefly get into in Sec. 5.2, and of course there also is the Curry-Howard-Lambek isomorphism \([48, 49, 50, 51]\). Just as in that context programming becomes an instance of category theory, something very similar is true in the case of DisCoCat for language.

**The need for structural understanding.** Being from a pure mathematician, Lambek’s work on the ‘real world’ was heavily structurally driven (and so is ours in this paper). These days prediction-driven work drawing from big data has clearly taken the forefront. Allow us to share some reflections on that, by looking at some key historical scientific developments.

In particular, undeniably, Natural Language Processing (NLP) has made great progress thanks to the great progress recently made in Machine Learning (ML). On the other hand it is fair to say that this hasn’t necessarily increased our structural understanding of language. It would be a major mistake to only follow the path where empirical success takes us, and ignore that which increases understanding, as we have learned the hard way from how (the most important ever) progress has been made in physics.

The discovery of the theory of Newtonian mechanics came from the study of movements of planets and stars. This study was data driven as these movements are vital for ships on sea to find their way home. The longest surviving model was Ptolemy’s epicycle model. While there were Copernicus, Kepler, Galilei and Newton, it took until Einstein for that line of research to match Ptolemy’s correct predictions, as the latter accounted for relativistic effects. The reason was simply that for any anomaly one could always add a sufficient number of epicycles. This shows that while structural insights (cf. Earth not in the middle of the universe), may take time to get the predictions right, but in the end even make the data driven models obsolete.\(^2\)

There is a deeper aspect to this story, namely: Why did we have to go to outer space to see mechanics in action in a manner that we could truly understand it? The simple reason is that the role of friction on earth is so imposing that one was rather led to take on the Aristotelian point of view that any movement requires force. Only in outer space we see frictionless movement. We think that the same is true for language: the manner it is used is full of deviations from the structural core. We believe that just like in the case of Newtonian mechanics, understanding that structural core will ultimately lead to much better predictive power. As an analogy, if ML would truly solve all of our problems, then one would expect it to be used to calculate trajectories of space ships and compute quantum spectra, learning from the available experimental data, but I personally don’t expect that we will dump non-Euclidean geometry and Hilbert space for these purposes, despite the

\(^2\) While neural networks have an ontological underpinning taking some inspiration from the human brain, the universal approximation theorem for neural networks \([30]\), which states that one can approximate any continuous function on compact subsets of \(\mathbb{R}^n\), seems to be somewhat on par with the unrestricted adding of epicycles.
fact that real physics calculations, for many reasons, all deviate from the ideal. That said, ML could probably have saved Copernicus a hell of a lot of time when analysing Brahe’s data.

**Why pregroups?** Pregroups are easily the simplest model of grammar, and have a very simple graphical presentation as wires-structures. We think that such a simplicity can be very helpful, just like how Copernicus’ simple circle model allowed Kepler and Newton to finally understand movement on Earth. More specifically, just like friction obstructed us from discovering the laws of mechanics here on Earth, more sophisticated features of language as well as all kinds of cultural aberrations may also obstruct us from seeing the foundational structures of meaning. There may of course also be more fundamental arguments for using pregroups, and Lambek took his conceptual motivation for pregroups from psychology. Some computational linguists have strong feelings about which grammatical algebra to be used, and many think that Lambek got it wrong. But even then, Copernicus was wrong too, since planets don’t move on circles around the sun, but without him we would not have been where we are now.

**Some related works.** The sentence type used in this paper was also used in the recent DisCoCat paper that introduces Cartesian verbs [24], and as discussed in [24] Sec. 2.3, precursors of this idea are in [35, 45, 44].

Also within the context of DisCoCat, the work by Toumi et al. [19, 77] involves multi-sentence interaction by relying on discourse representation structures [40], which comes at the cost of reducing meaning to a number. Textual context is also present in the DisCoCat-related papers [63, 86], although no sentence composition mechanism is proposed.

Within more traditional natural language semantics research, dynamic semantics [37, 79] models sentence meanings as I/O-transitions and text as compositions thereof. However, the approach is still rooted in predicate logic, just as Montague semantics is, hence not accounting for more general meaning spaces, and also doesn’t admit the explicit type structure of diagrams/monoidal categories. Dynamic semantics is a precursor of dynamic epistemic logic (DEL) [8, 7] which we briefly address in Sec. 5.1; we expect that DEL, and generalisations thereof, may in fact emerge from our model of language meaning by considering an epistemics-oriented subset of meanings. In [69], static and dynamic vector meanings are explicitly distinguished, taking inspiration for the latter from dynamic semantics. There are many other logic-oriented approaches to text e.g. [5], of text organisation e.g. [59], and of the use of categorical structure.³

³ Despite our somewhat provocative stance towards ML there is a clear scope for combining DisCoCirc with ML-methods. Some work in this direction, for the case of DisCoCat, is [57].
2 Background: DisCoCat

2.1 diagrams

Diagrams are made up of boxes:

```
|   |   |
|---|---|
| box |   |
|   |   |
```

each of which may have a number of inputs and outputs, and in this paper these boxes will typically be labeled by words or sentences. Boxes without either inputs or outputs may also occur, which we call states and effects respectively, and we depict them here as follows:

```
| state |   |
|       |   |
|   |   |
|   | effect |
|   |   |
```

The inputs and outputs of boxes can then be connected by wires yielding general diagrams e.g.:

```
| state | box 1 |
|       |       |
|       |   |
|       | box 2 |
```

What determines a diagram are (see also [23] Corollary 3.5 & Definition 3.8):

- the connectedness of the wire-structure, and
- the labels on wires and boxes.

Diagrams can either be read upward, like how we build structures from the ground up, or downward, like how gravity causes downfall. I prefer the constructive view in which diagrammatic structures are built rather than where they emerge by letting the forces have a go. Unfortunately, the elders of our portion of spoken language disagreed, and for that reason, in this paper, diagrams will be read from top to bottom.

In category-theoretic terms, diagrams live in monoidal categories where the wires correspond to objects and the boxes correspond to morphisms. Places where one can find easy-going introductions to the categories-diagrams connection include [26, 23]

There are two particular kinds of diagrams that will play a role in this paper, both discussed in great detail in [23].

**Circuits** are diagrams obtained by composing boxes in parallel and sequentially. In category-theoretic terms they live in a symmetric monoidal category. They admit a clear flow of time, from inputs to outputs, as a circuit carries a causal structure with boxes as nodes and wires as causal relationships (see [23] Theorem 3.22). This in particular implies that an output of a box will always be connected to an input of a box in its 'future'.

---

[23] Corollary 3.5 & Definition 3.8

[26] Theorem 3.22
String diagrams on the other hand allow for outputs to be connected to any input, and even to other outputs. Similarly, inputs can be connected to inputs. This craziness is enabled by the fact that string diagrams allow for cap- and cup-shaped wires:

![Cap and Cup Diagrams]

One can also think of these cap- and cup-shaped wires as boxes, respectively as a state and an effect, and then string diagrams can be given the shape of a circuit ([23] Theorem 4.19). Conveniently, every one-input-one-output-box can be transformed into a two-output-state, and vice versa:

\[
\begin{align*}
\text{box} & \quad \mapsto \quad \text{box} \\
\text{state} & \quad \mapsto \quad \text{state}
\end{align*}
\]

and this is a bijective correspondence called box-state duality (see [23] Sec. 4.1.2). More general uses of caps/cups for converting types are referred to as transposition. In category-theoretic terms string diagrams live in a compact closed category (also called autonomous category) [46, 74].

We also usually assume that string diagrams can be flipped vertically:

![Box Flip Diagram]

For example, if we flip a state, we get an effect, and vice versa, and if we flip a cap, we get a cup, and vice versa. Above we also flipped the text in the boxes, but this was just to make a point, and won’t do this anymore as this obstructs readability. In category-theoretic terms this flipping is called a dagger structure, or adjoints [2, 73] (see also [23] Sec. 4.3.1).

Another particular kind of box are spiders [22, 23]. In category-theoretic terms they correspond to so-called dagger special commutative Frobenius algebras. We represent them by a dot with some input and output wires:

![Spider Diagram]

The key property of spiders is that they fuse together:
An alternative way to think of these spiders is as multi-wires, which are generalised wires in that, rather than having two ends, can have multiple ends. What corresponds to fusion is that if two multi-wires are connected, then all the ends of one are also connected with all the ends of the other.

As we will justify in Sec 2.4, the spider with three legs should be thought of as the logical AND:

\[
\text{AND}
\]

The one with a single leg should be thought of as discarding.

2.2 From grammar to wirings

From the mathematics of sentence structure \([3, 10, 47]\) we know that the ‘fundamental particles’ making up phrases and sentences are not words, but some basic grammatical types. The noun-type \(n\), and the type of whole sentences \(s\) are examples of these basic types. On the other hand, the transitive-verb-type is not a basic type, but a composite one made up of two noun-types and one sentence-type. The precise manner in which these basic types interact depends on which categorial grammar one uses. We will adopt Lambek’s pregroups \([52, 53]\), since it can be formulated diagrammatically.

For each basic type \(x\) there are two corresponding ‘anti-types’, which we denote \(x^{-1}\) and \(-1x\). Think of these as a left and a right inverse. Then, in English, a transitive verb has type:

\[x^{-1} \cdot s \cdot n^{-1}\]

To understand this type, consider a transitive verb like hate. Simply saying hate doesn’t convey any useful information, until we also specify who hates who. That’s exactly the role of the anti-types: they specify that in order to form a meaningful sentence the transitive verb needs a noun on the left and a noun on the right, which then cancel out with the anti-types:

\[
\text{Alice hates Bob}
\]

So both \(n^{-1}n\) and \(n^{-1} \cdot n\) vanish, and what remains is \(s\), confirming that Alice hates Bob is a proper sentence (a.k.a. grammatically well-typed).

So where are the diagrams? They depict the cancelations:
For a more complex sentence (but with very similar meaning) like:

Alice does not love Bob

the wiring will be more complex [28]:

but the idea remains the same. In general, one can extract these wirings from the book [53], which assigns types to all grammatical roles.

The main idea of DisCoCat is to think of these wires not just as a representation of an algebraic computation, but as a representation of how the meanings of the words making up the sentence interact. Representing word-meanings as follows, also accounting for the types:

we can now apply the wiring to these as follows:

The wires now ‘feed’ the object and the subject into the verb in order to produce the meaning of the whole sentence.

One should contrast this compositional model for word and sentence meanings to the bag-of-word-model still employed in distributional NLP and information retrieval [38], where as the name says, words in a sentence are treated just like a structureless bag.

### 2.3 Internal wirings of meanings

Not only grammatical structure, but also certain meanings themselves can be represented using wiring. Examples include functional words, which play some kind of logical role and would carry no empirical data, and other words where a wiring structure provides a simplification. The internal structure of words can also help to better understand the overall meaning of a sentence.
2.3.1 Functional words

An example from [64, 28] is the sentence:

```
Alice does not love Bob
```

We see that both *does* and *not* have been represented using wires:

- **does** being entirely made up of wires,
- **not** involving a ¬-labeled box which represents negation of the meaning.

That these wirings are well-chosen becomes clear when we yank them:

```
Alice loves Bob
```

i.e. we get the negation of the meaning of *Alice loves Bob*.

An example from [67, 68] uses spiders for relative pronouns:

```
She who hates Bob
```

Simplification now yields:

```
[... ] hates Bob
```

i.e. the conjunction of *she* (i.e. being female) and the property *[... ] hates Bob*, which is indeed again the intended meaning of the sentence.

Building further on the idea that the merge spider represents AND, in [42] an account was given of coordination between identical syntactic types.

2.3.2 Adjectives and *to be*

*Intersective adjectives* [39] are adjectives which leave a noun unaltered except for specifying an additional property, e.g. *red car, hot chilies or sad Bob*, as opposed to *crashed car, rotten chilies or dead Bob*. While a general adjective has a composite type e.g.
the type of an intersective adjective can be reduced to a single wire [13]:

yielding a conjunction:

Closely related to adjective is the verb to be, since sad Bob and Bob is sad convey the same meaning. Of course, the overall type of these two statements is different, being a noun and a sentence respectively, but we will see later that this difference vanishes when we move from DisCoCat to DisCoCirc. Accepting Bob is sad to be a noun, the following internal wiring of the verb to be is induced by the one of intersective adjectives:

2.3.3 Compact verbs

Recently, in [24], an internal wiring for a special type of verbs was proposed. The main idea is that the verb’s only role is to impose an adjective \( \verb_s \) on the subject and an adjective \( \verb_o \) on the object. For example, paints will put a paintbrush in the subject’s hand, and makes the object change colour. For many verbs this description can be used as an adequate first approximation. So a transitive verb is semi-Cartesian if it has the following internal wiring:

\[
\begin{align*}
\verb_s & \circ \verb_o \\
\end{align*}
\]

As indicated in the picture, this wiring implies that its sentence type consists of two noun-types, which is a very natural choice to make.

However, a verb like being married to is clearly not of that form, as it expresses an entanglement of the specific subject and the specific object. The natural generalisation of the idea of semi-Cartesian verbs is then:
This representation was first used in [35], and also in [44], and we call it the compact representation of transitive verbs.

2.4 Models of meaning

Given that diagrams live in abstract (a.k.a. axiomatic) categories, they allow for a wide range of concrete models. It suffices to pick a concrete category that has cups and caps, and then wires become the objects (a.k.a. spaces) and boxes become the morphisms (a.k.a. maps between these spaces).

In NLP, the vector space model takes wires to be spaces of distributions and boxes to be linear maps. The distributions are empirically established, by means of counting co-occurrences with a selected set of basis words [71]. Adjusting this model to DisCoCat, the cups and caps are:

\[
\bigcap := \sum_i |i\rangle \quad \quad \bigcup := \sum_i \langle i|
\]

and spiders are in one-to-one correspondence with orthonormal bases [27]. Explicitly, given an orthonormal basis \{\ket{i}\}, the spiders arise as follows:

\[
\begin{array}{c}
| \quad | \\
\cdots \\
| \quad | \\
\end{array} := \sum_i \ket{i\ldots i} \bra{i\ldots i}
\]

Hence, caps and cups are instances of spiders, and so are copy and merge:

\[
\bigtriangleup := \sum_i |i\rangle \langle i| \quad \quad \bigtriangledown := \sum_i \ket{i} \bra{i}
\]

Another model employed in DisCoCat instead considers sets and relations [28]. By thinking of relations as Boolean-valued matrices this model is closely related to the previous one, and can be thought of as `possibilistic` distributions (contra `probabilistic`). This particular model also justifies to interpret merge as AND, as in the model states are subsets, and merge then corresponds to intersecting these states.

One can also take wires to be spaces of density matrices and boxes to be superoperators. This model was initially introduced to account for ambiguity of meaning [41, 61, 62], and was also used to capture lexical entailment [6, 9].
It will play a central role in this paper, although with a somewhat different interpretation. Again, density matrices can be established empirically [62].

Another recently developed model takes the convex subsets of certain convex spaces [13] to be the wires, following Gardenfors’ conceptual spaces program [32]. This model represents meanings in a manner that appeals directly to our senses. A plethora of generalisations thereof are in [20]. Again, empirical methods can be used to establish meanings.

2.5 Comparing meanings

Once we have computed the meanings of sentences in a concrete model, we can compare these meanings. Here are two examples of doing so:

**Similarity.** Establishing similarity is one of the standard tasks in NLP, and one does this in terms of a distance-measure: the less the distance, the more similar meanings are. One can use the inner-product (or some function thereof), which for sentences \( \sigma_1 \) or \( \sigma_2 \) is diagrammatically denoted as:

\[
\begin{array}{c}
\sigma_1 \\
\sigma_2 
\end{array}
\]

as one can indeed think of an inner-product as the composition of one state with the adjoint of another state (cf. Sec. 2.1). This manner of comparing meanings generalises to arbitrary dagger compact closed categories, such as the category of density matrices and completely positive maps. In the concrete representation of density matrices we obtain:

\[
\text{Tr} (\sigma_2 \sigma_1) = \begin{array}{c}
\sigma_1 \\
\sigma_2 
\end{array} = \begin{array}{c}
\sigma_1 \\
\sigma_2 
\end{array}
\]

where we used the fact that density matrices are self-adjoint, and that the transpose of the adjoint is the conjugate (which is indicated by the bar).

**Graded entailment.** One may want to know if one meaning entails another one. Given the noisiness of empirical data, a useful strict entailment relation might be hard to achieve. Instead, a graded entailment relation that tells us the degree to which one meaning entails another one is more useful. Strict entailment relations correspond to partial orderings, and a graded ones correspond to a labeled extension thereof. Still, many models of meaning in use, like the vector space model, don’t even admit a natural non-trivial graded entailment structure, and it is here that density matrices have a role to play. As shown in [9], for those such a structure does exist and is well-studied, namely the Löwner ordering for positive matrices [58]:
\( \sigma_1 \leq_k \sigma_2 \iff \sigma_2 - k \sigma_1 \) is positive

It is useful to play around a bit with the scaling of the density matrices. If one normalises density matrices by setting the trace to 1, then there are no strict comparisons. On the other hand, when one sets the largest eigenvalue to 1, then we get for the specific case of projectors (i.e. scaled density matrices with all non-zero eigenvalues the same):

\[ \sigma_1 \leq \sigma_2 \iff \sigma_1 \circ \sigma_2 = \sigma_1 \]

just like in Birkhoff-von Neumann quantum logic [12], which is then naturally interpreted as propositional inclusion. Some alternative scalings are in [78].

3 Features and flaws of DisCoCat

Here is a summary of the main features of DisCoCat:

**Feature 1.** The initially key identified feature of DisCoCat was that meanings of sentences with different grammatical structure still live in the same space, something that is crucial for comparing meanings (cf. Sec. 2.5). Earlier approaches that combined grammar and meaning, most notably, Smolensky’s connectionist cognitive architecture [75], did not have this feature. The quest for a model that does so was put forward in [15].

**Feature 2.** The DisCoCat algorithm that assigns meaning to sentences given the meanings of its words and its grammatical structure can be presented as an intuitive diagram that clearly shows how word-meanings interact to produce the meaning of the sentence.

**Feature 3.** Wire-structure can be used in DisCoCat to provide meanings of functional words as we did in Sec. 2.3 (while in standard NLP they are usually treated as noise), and to simplify the representation of words with composite types like adjectives and verbs as we did in Secs. 2.3.2 and 2.3.3.

**Feature 4.** While in this paper we used pregroups, as argued in [21, 34], DisCoCat also supports other categorial grammars such as standard Lambek calculus [47], Lambek-Grishin calculus [36] and CCG [76].

**Feature 5.** As discussed in Sec. 2.4, in DisCoCat word meanings can live in many kinds of spaces provided these organise themselves in a monoidal category that matches the structure of the grammar.

**Feature 6.** DisCoCat allows for integrating grammar and meaning in one whole. In the above we indeed had examples of compositional structure entering meaning-boxes, which then interact with the grammatical structure.\(^4\)

\(^4\) Admittedly, more work needs to be done for further exploiting this feature. Crucially, while initial formulations of DisCoCat either used a categorical product or a functor in order
Feature 7. Contra the bag-of-words model in NLP [38], the spaces for different grammatical types vary, which reflects the fact that their functionality within sentences is very different. In other words, meaning spaces are typed, with all the usual advantages. If words can play different grammatical roles, e.g. both as noun and adjective, then there also are canonical ways for inter-converting these, e.g. a noun becomes an adjective as follows:

\[
\text{red} \quad \rightarrow \quad \text{red}
\]

Feature 8. Proof-of-concept experiments showed that DisCoCat outperformed its competitors for certain academic benchmark tasks [35, 43].

As already indicated in the introduction, DisCoCat has some shortcomings:

Flaw 1. DisCoCat does not answer the question of how the meanings of sentences compose in order to provide the meaning of an entire text.

Flaw 2. DisCoCat assumes words to have a fixed meaning, while in text meanings will typically evolve.

Flaw 3. DisCoCat doesn’t determine the sentence type.

We will now resolve each of these flaws in one go!

4 Composing sentences: meet DisCoCirc

We will represent the \(|\sigma_i|\) words in a sentence \(\sigma_i\) as a horizontal string, and the \(|\tau|\) sentences in a text \(\tau\) as a vertical stack:

\[
\begin{align*}
\text{Word}_1 & \quad \ldots \quad \text{word}_{|\sigma_i|} \\
\vdots & \\
\vdots & \\
\text{Word}_1 & \quad \ldots \quad \text{word}_{|\tau|}
\end{align*}
\]

4.1 Naive composition of sentences for DisCoCat

In DisCoCat, each of the sentences (cf. those in Sec. 2) is a state, i.e. they have a single output of sentence-type, and no input. This substantially restricts the manner in which we can compose them. The structure available to us in to combine grammar and meaning [28, 64], the way forward is to assume a compositional structure encompassing both grammar and aspects of meaning.
DisCoCat are wires and spiders. The desirable thing to do is to also rely on this structure for composing sentences, so that word-meaning composition can interact with sentence-meaning composition. But then, pretty much the only thing one can do is to take the conjunction of all sentences:

\[
\sigma_1 \leftarrow \text{AND} \rightarrow \sigma_2 \leftarrow \text{AND} \rightarrow \ldots \leftarrow \text{AND} \rightarrow \sigma_n
\]  

We could call this the *bag-of-sentences model*, since without changing the diagram we can flip the vertical order of sentences, or not even give one:

\[
\sigma_n \leftarrow \ldots \leftarrow \sigma_2 \leftarrow \sigma_1 \leftarrow \ldots \leftarrow \sigma_n
\]

An example where this makes perfect sense is:

- It is cloudy.
- Liverpool has beaten Napoli.
- Brexit has become a total mess.

as these sentences clearly commute. A non-example is:

- Bob is born.
- Bob drinks beer.
- Bob dies.

Here one could still argue that the meaning of the sentences now dictates their ordering, so the latter could be extracted even if the sentences arrive in a bag, of course, at the cost of having to know the meanings of all words. The argument completely breaks down here:

- Add egg yolk and salt.
- Whisk mix for 20 seconds.
- Add mustard and acid.
- Whisk mix for 30 seconds.
- Slowly add oil while whisking.

as the order of adding ingredients is key to making good mayonnaise. Changing the order still would result in a meaningful recipe, but not mayonnaise.
More importantly, what is also clear from this example is that a lot more is going on besides the order of things: the ingredients and actions interact with each other similarly to how words in a sentence interact with each other as described in Sec. 2. Our bag-of-sentence-model doesn’t reflect any of that.

We will now make that interaction structure explicit, while still only making use of the structure available to us in DisCoCat. Of course, something will need to change, and rather than a conservative extension of the DisCoCat framework, we introduce a fundamentally modified framework, DisCoCirc, while retaining the features of the DisCoCat framework listed in Sec. 3.

### 4.2 Sentences as I/O-processes

Consider the following example:

Alice is a dog.
Bob is a person.
Alice bites Bob.

Clearly, the meaning of the third sentence crucially depends on what we learn about the meaning of the nouns Alice and Bob in the first two sentences, turning dog bites man into man bites dog if Bob were to be a dog and Alice were to be a person. Also, before the 1st sentence is stated, Alice is just a meaningless name, and the same goes for Bob until the 2nd sentence is stated. So the meaning of Alice and Bob evolves as the text progresses, and it is the sentences that update our knowledge about Alice and Bob.

What we propose is that the 3rd sentence, which would look like:

Alice is a dog.
Alice bites Bob.

in DisCoCat, would instead be drawn like this:

\[
\text{Alice} \rightarrow \text{bites} \rightarrow \text{Bob}
\]

So in particular, the nouns Alice and Bob are now not states but wires (a.k.a. types) and the sentence is an I/O-box:

Alice bites Bob

with the nouns Alice and Bob both as inputs and as outputs. In this way, the sentence can act on the nouns and update their meanings. Hence:
A sentence is not a state, but a process.

For the remainder of this paper we will restrict ourselves to updating nouns, but the same applies to other word-types. Using the wire-representation of the verb to be of Sec. 2.3.2, the wire-representation of Alice is a dog becomes:

\[
\begin{align*}
\text{Alice} & \quad \text{is} \quad \text{dog} \quad = \quad \text{Alice} \\
\end{align*}
\]

and similarly, that of Bob is a person becomes:

\[
\begin{align*}
\text{Bob} & \quad \text{is} \quad \text{person} \\
\end{align*}
\]

In Sec. 2.3 we mentioned that while our treatment of to be leads to a noun-type, this wouldn’t be a problem anymore in DisCoCirc. And indeed, within our new sentences-as-processes realm we obtain the same type as the sentence (4) simply by adjoining a ‘passive’ wire:

\[
\begin{align*}
\text{Alice} & \quad \text{is} \quad \text{dog} \\
\text{Bob} & \quad \text{is} \quad \text{person} \\
\end{align*}
\]

This passive wire stands for the fact that the ancillary noun is part of the text as a whole, but doesn’t figure in this particular sentence. By adding it, the I/O-types of all the sentences are the same, so they can be composed:

\[
\begin{align*}
\text{1st sentence} & \quad \text{2nd sentence} \quad \text{3rd sentence} \\
\end{align*}
\]

So in general, given a text, we end up with a wire diagram that looks like this:
where the sentences themselves also have a wire diagram. In particular, it’s a process, and this process alters our understanding of words in the text. This yields another slogan:

**Text is a process that alters meanings of words**

Using the form (2) for **biting** we obtain for *Alice bites Bob*:

![Diagram 7](image)

Diagram (5) now simplifies to:

![Diagram 8](image)

and is clearly distinct from the case of **man bites dog**, which would be:

![Diagram 9](image)

One thing we can also now do now is show that different texts can have the same meaning. For example, the single sentence:

*Alice who is a dog bites Bob who is a person.*

which has as its diagram:
can, using spider-fusion, be directly ‘morphed’ into the diagram:

One thing we may be aiming for is a network that shows how the different nouns are related, or maybe just whether they either are related, or not at all. How the network is connected will depend on, for example, the kinds of verbs that appear in the text, and which subject-object pairs they connect. A further simplification can be made if we only require a binary knowledge, e.g. *knows* vs. *doesn’t know*, which we can respectively represent as:

In the resulting network we then get clusters of connected nouns.

### 4.3 The use of states

In the main example of Sec. 4.2 we made use of states to represent *dog* and *person*. The reason we use states for them is that the text doesn’t help us understand these nouns. In contrast, the text is all about helping us understand *Alice* and *Bob* and their interactions. So we can clearly distinguish two roles for nouns:
• **Static nouns**: the text does not alter our understanding of them.
• **Dynamic nouns**: the text does alter our understanding of them.

This distinction between dynamic and static nouns may seem somewhat artificial, and indeed, it exists mainly for practical purposes. From a foundational perspective the natural default would be to let all nouns be dynamic, and not just nouns but all words, since also adjectives and verbs may be subject to change of meaning. However, taking only some nouns to be static is a very reasonable simplification given that in a typical text the meaning of many other words would not alter in any significant manner. Doing so significantly simplifies diagrams, and in particular their width. This does give rise to the practical question of how to decide on the ‘cut’ between the dynamic and static nouns. We briefly address this in Sec. 6.2.

Also in the main example of Sec. 4.2, we had no prior understanding of Alice nor Bob. In general we may already have some prior understanding about certain nouns. One way to specify this is by means of initial states, to which we then apply the circuit representing the text:

![Diagram](image)

where the initial states represented by plain dots stand for the case of no prior understanding (cf. Alice and Bob earlier). Without changing the circuit we can also put them where they enter the text:

![Diagram](image)

Of course, once we insert initial states, we cannot precompose with other text anymore. A straightforward way to avoid this problem is by instead using initial processes:

---

5 This is in particular the case given that the names Alice and Bob are gender neutral (cf. Alice Cooper and Bob in Blackadder II & IV).
4.4 DisCoCat from DisCoCirc

We now show that DisCoCat is an instance of DisCoCirc. Assuming that (1) text is restricted to a single sentence, and that (2) nouns are static, we exactly obtain DisCoCat sentences. Alternatively, assuming that (2') dynamic nouns have an initial state, we also obtain a DisCoCat sentence:

![Diagram showing DisCoCat from DisCoCirc]

4.5 Individual and subgroup meanings

In many cases one would just be interested in the meaning of a single dynamic noun, rather than the global meaning of a text. Or, maybe one is interested in the specific relation of two or more dynamic nouns. The way to achieve this is by discarding all others. For example, here we care about the meaning of the 2nd dynamic noun:

![Diagram showing individual meaning]

while here we care about the relationship between the 2nd and the 5th one:

![Diagram showing subgroup meaning]
The latter can for example teach us if two agents either agree or disagree, or, either cooperate or anti-cooperate. In the case of three or more agents it can tell us more refined forms of interaction, e.g. do they pairwise cooperate or globally (cf. the W-state vs. the GHZ-state in quantum theory [31]).

Subgroups may also arise naturally, when agents vanish from the story e.g. by being murdered. In that case discarding instead of being of an epistemic nature is actual ontic vanishing:

This vanishing can even be a part of the verb structure for those verbs that induce the vanishing of an object, for example:

Of course, if it remains of importance who the actual killer is, then we shouldn’t use the simple semi-Cartesian verb structure.

4.6 Example

Consider the following text.\(^6\)

Loosely adapted from “C’era una volta il West”, Sergio Leone, 1968.

---

\(^6\) Loosely adapted from “C’era una volta il West”, Sergio Leone, 1968.
Harmonica (is the brother of) Claudio.
Frank hangs Claudio.
Snaky (is in the gang of) Frank.
Harmonica shoots Snaky.
Harmonica shoots Frank.

As a diagram this becomes:

Using spider-fusion, transposition of states into effects, and identifying 2-legged spiders with either caps, cups or plain wires, simplifies this to:

Notice that one party induces the effect, while the other one is subjected to termination, matches the grammatical subject-object distinction. There
are indeed two dimensions to diagram (10), a static one, representing the connections between the dynamic nouns:

\[ \text{Harmonica} \rightarrow \text{Snaky} \]
\[ \text{brother} \rightarrow \text{Franks} \]
\[ \text{hangs} \rightarrow \text{shoots} \]

as well as a temporal-causal structure associated to these.

### 4.7 Other cognitive modes

The mathematical formalism presented here for text structure may be equally useful for modelling other cognitive modes, not just the linguistic one. One obvious example is the visual mode, which we can think of as movies. Here dynamic nouns correspond to the characters of the movie, and sentences to scenes. The grammatical structure then corresponds to interactions of characters. For example, the scene:

![Image of movie scene]

corresponds to the sentence:

**Harmonica shoots Frank.**

The subject corresponds to Harmonica (played by Bronson), the object to Frank (played by Fonda) and the verb is the shooting of Frank by Harmonica:
More broadly the verb corresponds to the interaction of the characters. Text corresponds to sequences of scenes, which ‘act’ on the characters that take part in it, hence forming a circuit.

Having a matching diagrammatic formalism for text and for movies allows one then to make translations between these, via the corresponding diagrams. For example, we can translate the example of Sec. 4.6 to a movie:

where the snapshots represent the entire scene they are part of.
5 Logic and language

Propositional logic emerged from language, translating words like and, or and not into logical connectives AND, OR and NOT. under the impetus of Aristotle and others. DisCoCirc re-enforces that link with several branches of modern non-classical logics. Here are two proof-of-concept examples.

5.1 Dynamic epistemic logic from language

Epistemic logic is concerned with how one represents knowledge in logical terms, and dynamic epistemic logic (DEL) [8, 7] how this logic gets updated when acquiring new knowledge, e.g. from communication, using language. Since in DisCoCirc we have a build-in update mechanism, one may expect that DEL-update could emerge from DisCoCirc-update, and hence DEL could directly emerge from language structure. This seems indeed to be the case.

In order to establish this, the types Alice and Bob will now represent the knowledge of those agents, rather than what we know about them. Sentences describing communication of knowledge typically involve a doubly transitive verb (i.e. one that both has a direct and an indirect object), or alternatively, a preposition like to. For example:

Alice tells Bob (a) secret.

Alice tells (a) secret to Bob.

As we haven’t proposed a wiring yet for a doubly transitive verb nor for to, we will do so now, and we will also give an internal wiring for tells specific to this epistemic context. Grammatical wirings are taken from [53]. Setting:

![Diagram](diagram.png)

indeed result in the same simplified diagram:
From this then follows an obvious wiring of \textit{knows}:

This is the same wiring as we had for \textit{is}, which makes sense, since \textit{being} in an ontic context translates to \textit{knowing} in an epistemic context.

We hope to further develop this link in a dedicated forthcoming paper, being guided by the conviction that a diagrammatic framework for DEL can be established that directly draws from spoken language, and that moreover allows us to accommodate a wide variety of models beyond the propositional and probabilistic ones.

5.2 Linear and non-linear and

Another feature of DisCoCirc is that it dictates different representations of \textit{and}, namely, when either conjoining properties that subjects possess, or, conjoining the subjects themselves. In linear logic (LL) lingo [33, 72], these respectively correspond to a \textit{linear conjunction} and a \textit{non-linear conjunction}. Hence, these different representations correspond to uses of \textit{and} with different meanings. Consider the sentence:

Alice wears (a) hat \textit{and} (a) scarf.

and the sentence:

Alice \textit{and} Bob wear (a) hat.

The fundamental difference between these sentences is that:

- Alice wearing both a hat and a scarf only requires one Alice, while,
- Alice and Bob both wearing a hat requires two hats.

In the case of the former we assign two properties to a single agent, namely \textit{wearing a hat} and \textit{wearing a scarf}, while in the case of the latter a single property, \textit{wearing a hat}, is attributed to two agents. This means that we require (a.k.a ‘consume’ in LL lingo) the property \textit{wearing a hat} twice, hence, it needs to be copied. Put differently, non-linear conjunction allows for copying, so it is the AND we have in classical propositional logic.

A physics analogy would be that \textit{and} in the 1st sentence refers to having two physical particles e.g. a proton \textit{and} electron, while \textit{and} in the 2nd sentence lists two properties of a single particle, e.g. position \textit{and} velocity.
The non-linear and is what we have been using until now all the time in this paper by means of spiders. So it is the linear and that needs an alternative treatment. We will do what is standard when representing two things in a string diagrams (see [23] Section 3.1.1), namely putting two wires side-by-side. So the different representations of AND look as follows:

Then we get for the 1st sentence:

and for the 2nd sentence we instead have:

where internally in wear some copying must happen:

The difference is also apparent in how each of the sentences can be decomposed in two sentences, which in the case of the 1st one yields a sequential composition, and in the case of the 2nd one a parallel composition:

6 Concrete models

We gave the beginnings of a compositional structure of word and sentence composition, with (multi-)wires and boxes as primitives, and illustrated how
one reasons with these in the absence of concrete models. We now provide some ideas for which kinds of models are particularly suitable for DisCoCirc.

### 6.1 Sketch of a concrete model

As we are dealing with updating and corresponding information gains, the vector space model of NLP (see Section 2.4) won’t do. Density matrices do have a clear notion of information gain, and for this reason form the basis of quantum information theory (see e.g. [11]). We now describe the ingredients of a DisCoCirc model based on density matrices. States, e.g. word meanings:

$$\text{state}$$

... are density matrices. An example of a state is the maximally mixed state of quantum theory, which has a density matrix corresponding to the (scaled) identity, and represents the state of no information whatsoever.

$$\text{\[1\]}$$

The states of perfect information correspond to the pure states of quantum theory, that is, matrices arising as doubled vectors $|\psi\rangle\langle\psi|$. The effect:

$$\text{\[2\]}$$

corresponds to the trace. General processes, e.g. sentence meanings:

$$\text{box}$$

... correspond to trace-preserving completely positive maps. Pure processes are those that send pure states to pure states, and these arise from linear maps as the Krauss forms $f^\dagger \circ \circ f$. In this manner, the spiders:

$$\text{\[3\]}$$

of Sec. 2.4 become part of this model too.

However, there are reasons to move away from these particular choices of spiders, and even part of the axioms for spiders. We give some suggestions here of some potential alternatives. The cups and caps of Sec. 2.4 remain perfectly ok, so they respectively become:

$$\text{\[4\]}$$

... := $\sum_{ij} |ii\rangle\langle jj|$ \hspace{1cm} \text{\[5\]}$$

... := $\sum_{ij} \langle ii| \circ \circ |jj\rangle$

The main role of merge is to assign properties. One could set:
where $P_x$ is the orthogonal projector on the subspace corresponding to property $x$. As suggested in [83, 82] it is indeed natural to think of subspaces as properties, just like in Birkhoff-von Neumann quantum logic [12] which we already mentioned earlier. This operation inherits associativity from diagram composition, which is an obvious minimal requirement. It is not commutative, but that also makes perfect sense when thinking of the changing colours of a chameleon, where post-composition should discard previous colours. After re-scaling projectors become a special case of density matrices, and using spectral decomposition $\rho_x = \sum_i p_i P_i$ one can associate properties to general density matrices, for example as follows:

$$x := \sum_i p_i P_i \circ \circ P_i$$

In the follow-up paper [25] we present a class of similar generalisations.

### 6.2 Computing text meaning

The following steps produce the data needed to derive text meaning:

1. Identify sentences using punctuation.
2. Establish grammatical types of all words using standard parsers.
3. Identify the dynamic nouns. As this concerns a new concept, this is also a new task and hence will need additional research. One could rely to some extent on grammar and multiplicity of occurrence throughout the text, but actual meaning will likely also play a role. For certain problems the dynamic nouns may be a given, namely those that are of particular interest as part of the statement of the problem, for example, when analysing the relationship of certain parties of particular interest.
4. Form a diagram (see Sec. 4.2):
   - The dynamic nouns are the systems of the circuit.
   - The sentences are the gates of the circuit.
   - The internal wiring of the gates is given by the grammar.
5. Establish meanings of states, which can be done using standard methods, or those previously developed for DisCoCat.

In order to obtain the actual meaning of the text:

6. Insert all meanings into the diagram.
7. For computing the resulting (possibly simplified) diagram, one way to do so is to decompose the diagram in tensor products and sequential compositions of boxes, caps/cups and dot operations. A more direct manner for computing diagrams is outlined in Theorem 5.61 of [23].

6.3 Comparing texts

To compare texts we can simply rely on what we did in DisCoCat (see Sec. 2.5), provided we use initial states (see Sec. 4.3) so that the meaning of the text as a whole becomes a state, or, compute similarity as follows:

\[
\tau_1 \tau_2
\]

using a generalisation of the Hilbert-Schmidt product, i.e. the inner-product applied to the states arising from box-state duality (see Sec. 2.1):

\[
\tau_i
\]

Also graded entailment is obtained as in DisCoCat (see Sec. 2.5) when using initial states, or, representation text as a state as in (13).

7 Physical embodiment

In the abstract we mentioned that while the developments in this paper are independent of a physical embodiment, most notably a classical vs. a quantum embodiment, both the compositional formalism and the suggested concrete model of meaning of Section 6.1 are highly quantum-inspired. The compositional structure is directly imported from quantum theory [14, 17, 18], the suggested concrete model of meaning employs the density matrices which von Neumann designed specifically for quantum theory [80], and also our suggested alternatives for spiders belong to an area of current activity in quantum foundations (see e.g. [55, 56, 29, 4, 70]), which aims for a quantum analog of Bayesian inference theory.

Therefore, it should come as no surprise that implementation of DisCoCirc on a quantum computer would come with a wide range of benefits. For example, as pointed out in [87], classically the required space resources grow exponentially in the number of dynamic nouns, and this exponential growth could vanish on a quantum computer. Similarly, density matrices substantially increase the space required to represent meanings, while for a quantum
computer they come for free. Regarding time resources, quantum computational speed-ups have already been identified for DisCoCat [87], by exploiting progress in quantum machine learning [85], and these straightforwardly carry over to DisCoCirc. Expect many dedicated publications on further advantages of implementing DisCoCirc on a quantum computer to be forthcoming, and in fact, quantum natural language processing (QNLP) may become one of the leading areas of the so-called NISQ era [65], given its tolerance for imperfection [87]. Currently, efforts are under way to implement the quantum algorithm of [87] on a simulator [60], and very recently, another paper appeared [84] that is entirely dedicated to QNLP.

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