Lit-only sigma-game on nondegenerate graphs

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Abstract

A configuration of the lit-only \(\sigma\)-game on a graph \(\Gamma\) is an assignment of one of two states, on or off, to each vertex of \(\Gamma\). Given a configuration, a move of the lit-only \(\sigma\)-game on \(\Gamma\) allows the player to choose an on vertex \(s\) of \(\Gamma\) and change the states of all neighbors of \(s\). Given an integer \(k\), the underlying graph \(\Gamma\) is said to be \(k\)-lit if for any configuration, the number of on vertices can be reduced to at most \(k\) by a finite sequence of moves. We give a description of the orbits of the lit-only \(\sigma\)-game on nondegenerate graphs \(\Gamma\) which are not line graphs. We show that these graphs \(\Gamma\) are 2-lit and provide a linear algebraic criterion for \(\Gamma\) to be 1-lit.

Keywords: group action, lit-only \(\sigma\)-game, nondegenerate graph

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1 Introduction

The notion of the \(\sigma\)-game on finite directed graphs \(\Gamma\) without multiple edges was first introduced by Sutner [16] in 1989. A configuration of the \(\sigma\)-game on \(\Gamma\) is an assignment of one of two states, on or off, to each vertex of \(\Gamma\). Given a configuration, a move consists of choosing a vertex of \(\Gamma\), followed by changing the states of all of its neighbors. If only on vertex is allowed to choose in each move, we come to the variation: lit-only \(\sigma\)-game. Starting from an initial configuration, the goal of the lit-only \(\sigma\)-game on \(\Gamma\) is to minimize the number of on vertices of \(\Gamma\), or to reach an assigned configuration by a finite sequence of moves.

Given an integer \(k\), the underlying graph \(\Gamma\) is said to be \(k\)-lit if for any configuration, the number of on vertices can be reduced to at most \(k\) by a finite sequence of the moves. More precisely, we are interested in the orbits of the lit-only \(\sigma\)-game on \(\Gamma\) and the smallest integer \(k\), the minimum light number of \(\Gamma\) [18], for which \(\Gamma\) is \(k\)-lit. The notion of lit-only \(\sigma\)-games occurred implicitly in the study of equivalence classes of Vogan diagrams. The Borel-de Siebenthal theorem [2] showed that every Vogan diagram is equivalent to one with a single painted vertex, which implies that each simply-laced Dynkin diagram is 1-lit. The equivalence classes of Vogan diagrams were described by Chuah and Hu [7]. A conjecture made by Chang [5, 6] that any tree with \(k\) leaves is \([k/2]\)-lit was confirmed by Wang and Wu [18], where the name “lit-only \(\sigma\)-game” was coined.

The lit-only \(\sigma\)-game on a simple graph \(\Gamma\) is simply the natural action of a certain subgroup \(H_\Gamma\) of the general linear group over \(\mathbb{F}_2\) [18]. Under the assumption that \(\Gamma\) is the line graph of a simple graph \(G\), Wu [19] described the orbits of the lit-only \(\sigma\)-game on \(\Gamma\) and gave a characterization for the minimum light number of \(\Gamma\). Moreover, if \(G\) is a tree of order
\[ n \geq 3, \] Wu showed that \( H_G \) is isomorphic to the symmetric group on \( n \) letters. Weng and the author [12] determined the structure of \( H_G \) without any assumption on \( G \). The lit-only \( \sigma \)-game on a simple graph \( \Gamma \) can also be considered as a representation \( \kappa_\Gamma \) of the simply-laced Coxeter group \( W_\Gamma \) over \( \mathbb{F}_2 \) [11]. The dual representation of \( \kappa_\Gamma \) preserves a certain symplectic form \( B_\Gamma \). The two representations are equivalent whenever the form \( B_\Gamma \) is nondegenerate. From this viewpoint it is natural to partition simple connected graphs into two classes according as \( B_\Gamma \) is degenerate or nondegenerate.

In this paper we treat nondegenerate graphs \( \Gamma \) which are not line graphs. We show that \( H_\Gamma \) is isomorphic to an orthogonal group, followed by a description of the orbits of lit-only \( \sigma \)-game on \( \Gamma \) (Theorem 3.1). Moreover we show that these graphs \( \Gamma \) are 2-lit and provide a linear algebraic criterion for \( \Gamma \) to be 1-lit (Theorem 3.2). Combining Theorem 3.1, Theorem 3.2 and those in [12] and [19], the study of the lit-only \( \sigma \)-game on nondegenerate graphs is quite completed and the focus for further research is on degenerate graphs.

2 Preliminaries

From now on, let \( \Gamma = (S, R) \) denote a finite simple connected graph with vertex set \( S \) and edge set \( R \). Let \( \mathbb{F}_2 \) denote the two-element field \( \{0, 1\} \). Let \( V \) denote a \( \mathbb{F}_2 \)-vector space that has a basis \( \{\alpha_s \mid s \in S\} \) in one-to-one correspondence with \( S \). Let \( V^* \) denote the dual space of \( V \). For each \( s \in S \) we define \( f_s \in V^* \) by

\[
 f_s(\alpha_t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{else} \end{cases} \quad (1)
\]

for all \( t \in S \). The set \( \{f_s \mid s \in S\} \) forms a basis of \( V^* \) and is called the basis of \( V^* \) dual to \( \{\alpha_s \mid s \in S\} \). Each configuration \( f \) of the lit-only \( \sigma \)-game on \( \Gamma \) is interpreted as the vector

\[
 \sum_{\text{on vertices } s} f_s \in V^*, \quad (2)
\]

if all vertices of \( \Gamma \) are assigned the off state by \( f \), we interpret (2) as the zero vector of \( V^* \). Given \( s \in S \) and \( f \in V^* \) observe that \( f(\alpha_s) = 1 \) (resp. 0) if and only if the vertex \( s \) is assigned the on (resp. off) state by \( f \). For each \( s \in S \) define a linear transformation \( \kappa_s : V^* \to V^* \) by

\[
 \kappa_s f = f + f(\alpha_s) \sum_{st \in R} f_t \quad \text{for all } f \in V^*. \quad (3)
\]

Fix a vertex \( s \) of \( \Gamma \). Given any \( f \in V^* \), if the state of \( s \) is on then \( \kappa_s f \) is obtained from \( f \) by changing the states of all neighbors of \( s \), and \( \kappa_s f = f \) otherwise. Therefore we may view \( \kappa_s \) as the move of the lit-only \( \sigma \)-game on \( \Gamma \) for which we choose the vertex \( s \) and change the states of all neighbors of \( s \) if the state of \( s \) is on. In particular \( \kappa_s^2 = 1 \), the identity map on \( V^* \), and so \( \kappa_s \in \text{GL}(V^*) \), the general linear group of \( V^* \). The subgroup \( H = H_\Gamma \) of \( \text{GL}(V^*) \) generated by \( \kappa_s \) for all \( s \in S \) was first mentioned by Wu [18], which is called the flipping group of \( \Gamma \) in [11] and the lit-only group of \( \Gamma \) in [19].
The simply-laced Coxeter group $W = W_\Gamma$ associated with $\Gamma = (S, R)$ is defined by generators and relations. The generators are the elements of $S$ and the relations are

\[
\begin{align*}
\ s^2 &= 1, \\
(st)^2 &= 1 \quad \text{if } st \notin R, \\
(st)^3 &= 1 \quad \text{if } st \in R.
\end{align*}
\]

By [11, Theorem 3.2], there exists a unique representation $\kappa = \kappa_\Gamma : W \to \text{GL}(V^*)$ such that $\kappa(s) = \kappa_s$ for all $s \in S$. Clearly $\kappa(W) = H$. Given any $f, g \in V^*$ observe that $g$ can be obtained from $f$ by a finite sequence of the moves of the lit-only $\sigma$-game on $\Gamma$ if and only if there exists $w \in W$ such that $g = \kappa(w)f$. Given an integer $k$, the underlying graph $\Gamma$ is $k$-lit if and only if for each $\kappa(W)$-orbit $O$ on $V^*$, there exists a subset $K$ of $S$ with size at most $k$ such that $\sum_{s \in K} f_s \in O$.

A symplectic form $B = B_\Gamma$ on $V$ is defined by

\[
B(\alpha_s, \alpha_t) = \begin{cases} 
1 & \text{if } st \in R, \\
0 & \text{else}
\end{cases}
\]

for all $s, t \in S$ [15]. The radical of $V$ (relative to $B$) is the subspace of $V$ consisting of the vectors $\alpha$ that satisfy $B(\alpha, \beta) = 0$ for all $\beta \in V$. The form $B$ is said to be degenerate whenever the radical of $V$ is nonzero and nondegenerate otherwise. The graph $\Gamma$ is said to be degenerate whenever the form $B$ is degenerate, and nondegenerate otherwise. The form $B$ induces a linear map $\theta : V \to V^*$ given by

\[
\theta(\alpha)\beta = B(\alpha, \beta) \quad \text{for all } \alpha, \beta \in V.
\]

Since the kernel of $\theta$ is the radical of $V$ and the matrix representing $B$ with respect to the basis $\{\alpha_s \mid s \in S\}$ is the adjacency matrix of $\Gamma$ over $\mathbb{F}_2$, the following lemma is straightforward.

Lemma 2.1. Let $A$ denote the adjacency matrix of $\Gamma$ over $\mathbb{F}_2$. Then the following are equivalent:

(i) $\Gamma$ is a nondegenerate graph.

(ii) $\theta$ is an isomorphism of vector spaces.

(iii) $A$ is invertible.

The purpose of this paper is to investigate the lit-only $\sigma$-game on nondegenerate graphs which are not line graphs. It is natural to ask how to determine if a nondegenerate graph is a line graph. Here we give two characterizations of nondegenerate line graphs.

Lemma 2.2. Assume that $\Gamma$ is the line graph of a simple connected graph $G$ of order $n$. Then $\theta(V)$ has dimension $n - 1$ if $n$ is odd and has dimension $n - 2$ if $n$ is even.

Proof. Let $U$ denote the vertex space of $G$ over $\mathbb{F}_2$. Define a linear map $\mu : V \to U$ by

\[
\mu(\alpha_s) = u + v \quad \text{for all } s \in S,
\]
where \( u \) and \( v \) are the two endpoints of \( s \) in \( G \). Since \( G \) is connected, the image of \( \mu \) is the subspace of \( U \) consisting of these vectors each of which equals the sum of an even number of vertices of \( U \). Define a linear map \( \lambda : U \to V^* \) by

\[
\lambda(u)\alpha_s = \begin{cases} 
1 & \text{if } u \text{ is incident to } s \text{ in } G, \\
0 & \text{else}
\end{cases}
\]

for all \( u \in U \) and for all \( s \in S \). There is only one nonzero vector, the sum of all vertices of \( G \), in the kernel of \( \lambda \). Since \( \theta = \lambda \circ \mu \) and by the above comments, the result follows.

A claw is a tree with one internal vertex and three leaves. A simple graph is said to be claw-free if it does not contain a claw as an induced subgraph. A cut-vertex of \( \Gamma \) is a vertex of \( \Gamma \) whose deletion increase the number of components. A block of \( \Gamma \) is a maximal connected subgraph of \( \Gamma \) without cut-vertices. A block graph is a simple connected graph in which every block is a complete graph.

**Lemma 2.3.** [9, Theorem 8.5]. Let \( \Gamma \) denote a simple connected graph. Then \( \Gamma \) is the line graph of a tree if and only if \( \Gamma \) is a claw-free block graph.

The following proposition follows by combining Lemmas 2.1–2.3.

**Proposition 2.4.** Let \( \Gamma \) denote a simple connected graph. Then the following are equivalent:

(i) \( \Gamma \) is a nondegenerate line graph.

(ii) \( \Gamma \) is the line graph of an odd-order tree.

(iii) \( \Gamma \) is a claw-free block graph of even order.

### 3 Main results

A quadratic form \( Q \) on \( V \) (associated with \( B \)) is a function \( Q : V \to \mathbb{F}_2 \) satisfying

\[
Q(\alpha + \beta) = Q(\alpha) + Q(\beta) + B(\alpha, \beta)
\]

for all \( \alpha, \beta \in V \).

Let \( GL(V) \) denote the general linear group of \( V \). Given a quadratic form \( Q \) on \( V \), the orthogonal group with respect to \( Q \) is the subgroup of \( GL(V) \) consisting of all \( \sigma \in GL(V) \) such that \( Q(\sigma \alpha) = Q(\alpha) \) for all \( \alpha \in V \). Given a basis \( P \) of \( V \) we define \( Q_P \) to be the quadratic form on \( V \) satisfying \( Q_P(\alpha) = 1 \) for all \( \alpha \in P \).

For the rest of this paper, the form \( B \) is assumed to be nondegenerate. Moreover let \( Q = Q_P \) where \( P = \{ \alpha_s \mid s \in S \} \) and let \( O(V) \) denote the orthogonal group with respect to \( Q \). We now state the main results of this paper, which are Theorem 3.1, Theorem 3.2, and Corollary 3.3.

**Theorem 3.1.** Assume that \( \Gamma \) is a nondegenerate graph but not a line graph. Then \( \kappa(W) \) is isomorphic to \( O(V) \). Moreover the \( \kappa(W) \)-orbits on \( V^* \) are

\[
\{0\}, \quad \theta(Q^{-1}(0) \setminus \{0\}), \quad \theta(Q^{-1}(1)).
\]
Under the assumption that $B$ is nondegenerate, the number $|S| = 2m$ is even and there exists a basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of $V$ such that $B(\beta_i, \beta_j) = 0$, $B(\gamma_i, \gamma_j) = 0$ and

$$B(\beta_i, \gamma_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else} \end{cases}$$

for all $1 \leq i, j \leq m$. Such a basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of $V$ is called a *symplectic basis* of $V$.

The *Arf invariant* of $Q$ is defined to be

$$\text{Arf}(Q) = \sum_{i=1}^{m} Q(\beta_i)Q(\gamma_i),$$

which is independent on the choice of the symplectic basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of $V$ (for example see [1] or [8, Theorem 13.13]). Any two quadratic forms over $\mathbb{F}_2$ are equivalent if and only if they have the same Arf invariant and the underlying spaces have the same dimension (for example see [1] or [8, Proposition 13.14]). The order of $O(V)$ and the sizes of nontrivial $O(V)$-orbits on $V$ are given as follows (cf. [8, Chapter 14]). If $\text{Arf}(Q) = 0$ then

$$|O(V)| = 2^{m^2-m+1}(2^m-1)(2^2-1)(2^4-1)\ldots(2^{2m-2}-1),$$

$$|Q^{-1}(1)| = 2^{m-1}-2^{m-1},$$

$$|Q^{-1}(0) \setminus \{0\}| = 2^{m-1}+2^{m-1}-1.$$

If $\text{Arf}(Q) = 1$ then

$$|O(V)| = 2^{m^2-m+1}(2^m+1)(2^2-1)(2^4-1)\ldots(2^{2m-2}-1),$$

$$|Q^{-1}(1)| = 2^{m-1}+2^{m-1},$$

$$|Q^{-1}(0) \setminus \{0\}| = 2^{m-1}-2^{m-1}-1.$$

For each $s \in S$ there exists $\alpha_s^\vee \in V$ such that

$$B(\alpha_s^\vee, \alpha_t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{else} \end{cases} \quad (7)$$

for all $t \in S$. The set $\{\alpha_s^\vee \mid s \in S\}$ forms a basis of $V$ and is called the *basis of $V$ dual to* $\{\alpha_s \mid s \in S\}$ (with respect to $B$).

**Theorem 3.2.** Assume that $\Gamma = (S, R)$ is a nondegenerate graph but not a line graph. Then $\Gamma$ is 2-lit. Moreover the following are equivalent:

(i) $\Gamma$ is 1-lit.

(ii) The restriction of $Q$ to $\{\alpha_s^\vee \mid s \in S\}$ is surjective.

When the nondegenerate graph $\Gamma$ is bipartite, Theorem 3.2 can be reduced as follows.
Corollary 3.3. Assume that $\Gamma$ is a nondegenerate bipartite graph. Then $\Gamma$ is 2-lit. Moreover the following are equivalent:

(i) $\Gamma$ is 1-lit

(ii) $\Gamma$ contains a vertex with even degree or $\Gamma$ is a single edge.

As consequences of Corollary 3.3, we obtain two families of 1-lit graphs as follows.

- A tree is nondegenerate if and only if it has a perfect matching. By [10, Lemma 2.4], a tree with a perfect matching satisfies Corollary 3.3(ii) and is therefore 1-lit (cf. [13, Theorem 1.1]).

- For any two positive integers $m$ and $n$, the $m \times n$ grid is nondegenerate if and only if $m + 1$ and $n + 1$ are coprime [17]. By Corollary 3.3 any such $m \times n$ grid is 1-lit.

The following example shows that Corollary 3.3 is no longer true if the assumption of $\Gamma$ is the same as that of Theorem 3.2. Consider the graph $\Gamma = (S, R)$ as below.

The graph $\Gamma = (S, R)$ is nondegenerate and not a block graph. Therefore $\Gamma$ is not a line graph by Proposition 2.4. The basis $\{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_6^\vee\}$ of $V$ dual to $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ can be expressed as follows.

\[
\begin{align*}
\alpha_1^\vee &= \alpha_2 + \alpha_6, \\
\alpha_2^\vee &= \alpha_1 + \alpha_3 + \alpha_5 + \alpha_6, \\
\alpha_3^\vee &= \alpha_2 + \alpha_4 + \alpha_5, \\
\alpha_4^\vee &= \alpha_3 + \alpha_5, \\
\alpha_5^\vee &= \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6, \\
\alpha_6^\vee &= \alpha_1 + \alpha_2 + \alpha_3.
\end{align*}
\]

Using (6) and $Q(\alpha_s) = 1$ for all $s \in S$, we deduce that $Q(\alpha_s^\vee) = 0$ for all $s \in S$. Therefore $\Gamma$ is not 1-lit by Theorem 3.2, but the vertices 2, 5 have even degree in $\Gamma$.

4 Proof of Theorem 3.1

For $\alpha \in V$ the transvection on $V$ with direction $\alpha$ is a linear transformation $\tau_\alpha : V \to V$ defined by

\[ \tau_\alpha \beta = \beta + B(\beta, \alpha)\alpha \quad \text{for all } \beta \in V. \]

Observe that $\tau_\alpha$ preserves the form $B$ and that $\tau_\alpha \in \text{GL}(V)$ since $\tau_\alpha^2 = 1$. Here 1 denotes the identity map on $V$.  

6
For a subset $P$ of $V$ define $Tv(P)$ to be the subgroup of $\text{GL}(V)$ generated by $\tau_\alpha$ for $\alpha \in P$, and define $G(P)$ to be the simple graph whose vertex set is $P$ and where $\alpha, \beta$ in $P$ form an edge if and only if $B(\alpha, \beta) = 1$. For any two linearly independent sets $P$ and $P'$ of $V$, we say that $P'$ is elementary $t$-equivalent to $P$ whenever there exist $\alpha, \beta \in P$ such that $P'$ is obtained from $P$ by changing $\beta$ to $\tau_\alpha \beta$. The equivalence relation generated by the elementary $t$-equivalence relation is called the $t$-equivalence relation $[3]$.

**Lemma 4.1.** [3, Theorem 3.3]. Let $P$ denote a linearly independent set of $V$. Assume that $G(P)$ is a connected graph. Then there exists $P'$ in $t$-equivalence class of $P$ for which $G(P')$ is a tree.

**Lemma 4.2.** [14, Lemma 3.7]. Let $P$ denote a linearly independent set of $V$. Assume that $G(P)$ is the line graph of a tree. Then, for each $P'$ in the $t$-equivalence class of $P$, the graph $G(P')$ is the line graph of a tree.

A basis $P$ of $V$ is said to have orthogonal type $[4]$ if $P$ is $t$-equivalent to some $P'$ for which $G(P')$ is a tree containing the graph

\[
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}
\]

as a subgraph.

**Lemma 4.3.** Assume that $P$ is a basis of $V$ for which $G(P)$ is a tree but not a path. Then $P$ is of orthogonal type.

**Proof.** Since $G(P)$ is not a path it contains a vertex $\alpha$ with degree at least three. If any two neighbors of $\alpha$, say $\beta$ and $\gamma$, are leaves of $G(P)$, then $\beta + \gamma$ lies in the radical of $V$, which contradicts that $B$ is nondegenerate. Therefore at most one neighbor of $\alpha$ is a leaf in $G(P)$ and so $P$ is of orthogonal type. $\square$

**Lemma 4.4.** [4, Section 10]. Let $P$ denote a basis of $V$ which is of orthogonal type. Then $Tv(P)$ is the orthogonal group with respect to $Q_P$. Moreover the $Tv(P)$-orbits on $V$ are

\[
\{0\}, \quad Q_P^{-1}(0) \setminus \{0\}, \quad Q_P^{-1}(1).
\]

**Proof of Theorem 3.1.** For each $s \in S$ let $\tau_s$ denote the transvection on $V$ with direction $\alpha_s$. By [15, Section 5], there exists a unique representation $\tau = \tau_W : W \to \text{GL}(V)$ such that $\tau(s) = \tau_s$ for all $s \in S$. For each $w \in W$ the transpose of $\tau(w^{-1})$ is equal to $\kappa(w)$. Therefore $\kappa$ is the dual representation of $\tau$. Since $\tau$ preserves the form $B$ we have

\[
\theta \circ \tau(w) = \kappa(w) \circ \theta \quad \text{for all } w \in W. \tag{8}
\]

Let $P = \{\alpha_s \mid s \in S\}$. Clearly $Tv(P) = \tau(W)$ and $G(P)$ is (isomorphic to) $\Gamma$. By Lemma 4.1 there exists $P'$ in $t$-equivalence class of $P$ for which $G(P')$ is a tree. Since $G(P)$ is not a line graph, the tree $G(P')$ is not a path by Lemma 4.2. By Lemma 4.3 the basis $P'$ of $V$, as well as $P$, is of orthogonal type. By Lemma 4.4, the group $\tau(W) = O(V)$ and the $\tau(W)$-orbits on $V$ are $\{0\}$, $Q^{-1}(0) \setminus \{0\}$, and $Q^{-1}(1)$. Applying (8) and since $\theta$ is an isomorphism by Lemma 2.1, the result follows. $\square$
5 Proofs of Theorem 3.2 and its corollary

To prove Theorem 3.2 and Corollary 3.3, we introduce a simple graph which includes the information of the values $B(\alpha_s^\vee, \alpha_t^\vee)$ for all $s, t \in S$.

**Definition 5.1.** We define $R^\vee$ as the set consisting of all two-element subsets $\{s, t\}$ of $S$ with $B(\alpha_s^\vee, \alpha_t^\vee) = 1$. Define $\Gamma^\vee$ as the simple graph with vertex set $S$ and edge set $R^\vee$. We will refer to $\Gamma^\vee$ as the dual graph of $\Gamma$.

Note that the notion of dual graphs defined above is different from the usual ones in graph theory. The following lemma suggests why the graph $\Gamma^\vee$ is of interest.

**Lemma 5.2.** For each $s \in S$ we have $\theta(\alpha_s^\vee) = f_s^\vee$.

**Proof.** Let $s, t \in S$ be given. Using (5) and (7) we have $\theta(\alpha_s^\vee)\alpha_t = 1$ whenever $s = t$ and otherwise $\theta(\alpha_s^\vee)\alpha_t = 0$. Comparing this with (1) the result follows.

Recall from Section 2 that the symplectic form $B$ is defined on the basis $\{\alpha_s \mid s \in S\}$ of $V$. If the symplectic form associated with $\Gamma^\vee$ is defined on the basis $\{\alpha_s^\vee \mid s \in S\}$ of $V$, then the resulting form is $B$. Therefore $\Gamma^\vee$ is a nondegenerate graph. The dual graph of $\Gamma^\vee$ is $\Gamma$ since $\{\alpha_s \mid s \in S\}$ is the basis of $V$ dual to $\{\alpha_s^\vee \mid s \in S\}$.

**Lemma 5.3.** For each $s \in S$ we have

$$\alpha_s = \sum_{st \in R} \alpha_t^\vee. \tag{9}$$

**Proof.** Fix $s \in S$. By (1), (4) and (5), we find that the vector $\theta(\alpha_s)$ equals

$$\sum_{st \in R} f_t. \tag{10}$$

By Lemma 5.2, we find that (10) equals

$$\theta\left(\sum_{st \in R} \alpha_t^\vee\right).$$

Therefore (9) holds by Lemma 2.1(ii).

By duality and Lemma 5.3 the following lemma is straightforward.

**Lemma 5.4.** For each $s \in S$ we have

$$\alpha_s^\vee = \sum_{st \in R^\vee} \alpha_t. \tag{11}$$

**Lemma 5.5.** Let $A$ and $A^\vee$ denote the adjacency matrices of $\Gamma$ and $\Gamma^\vee$ over $\mathbb{F}_2$, respectively. Then $A$ and $A^\vee$ are inverses of each other.
Proof. We show that \( A^\vee A \) is equal to the identity matrix. Let \( s, t \in S \) be given. By the comment below Lemma 5.2 the \((s, t)\)-entry of \( A \) (resp. \( A^\vee \)) is equal to \( B(\alpha_s, \alpha_t) \) (resp. \( B(\alpha^\vee_s, \alpha^\vee_t) \)). By Definition 5.1 we find that the \((s, t)\)-entry of \( A^\vee A \) equals

\[
B\left( \sum_{su \in R^\vee} \alpha_u, \alpha_t \right). \tag{12}
\]

By (11) the vector in the first coordinate of (12) equals \( \alpha^\vee_s \). Therefore (12) equals 1 if and only if \( s = t \) by (7). The result follows.

We are now ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** In Lemma 5.2 we saw that \( \theta(\alpha^\vee_s) = \theta_s \) for all \( s \in S \). Therefore (i), (ii) are equivalent by Theorem 3.1. To show that \( \Gamma \) is 2-lit, it is now enough to consider the two cases: (a) \( Q(\alpha^\vee_s) = 0 \) for all \( s \in S \); (b) \( Q(\alpha^\vee_s) = 1 \) for all \( s \in S \).

(a) It suffices to show that there exist \( s, t \in S \) such that \( Q(\alpha^\vee_s + \alpha^\vee_t) = 1 \). Since the form \( B \) is nontrivial there exist \( s, t \in S \) such that \( B(\alpha^\vee_s, \alpha^\vee_t) = 1 \). Then the \( s \) and \( t \) are the desired elements in \( S \).

(b) It suffices to show that there exist two distinct \( s, t \in S \) such that \( Q(\alpha^\vee_s + \alpha^\vee_t) = 0 \). By our assumption, the graph \( \Gamma \) is not a complete graph. Using Lemma 5.5, we deduce that \( \Gamma^\vee \) is not a complete graph. Therefore there exist two distinct \( s, t \in S \) such that \( B(\alpha^\vee_s, \alpha^\vee_t) = 0 \). Such \( s \) and \( t \) are desired elements in \( S \).

To prove Corollary 3.3, we give a sufficient condition for Theorem 3.2(ii).

**Lemma 5.6.** Let \( \Gamma = (S, R) \) denote a nondegenerate graph. Assume that there exists \( s \in S \) with even degree in \( \Gamma \) such that

\[
\sum_{\{u, v\} \subseteq S, su, sv \in R} B(\alpha^\vee_u, \alpha^\vee_v) = 0, \tag{13}
\]

where the sum is over all two-element subsets \( \{u, v\} \) of \( S \) with \( su, sv \in R \). Then the restriction of \( Q \) to \( \{\alpha^\vee_t \mid st \in R\} \) is surjective.

**Proof.** Apply \( Q \) to either side of (9). Using (6), (13) and \( Q(\alpha_s) = 1 \) to evaluate the resulting equation, we obtain that

\[
\sum_{st \in R} Q(\alpha^\vee_t) = 1. \tag{14}
\]

By (14) there exists a neighbor \( u \) of \( s \) for which \( Q(\alpha^\vee_u) = 1 \). Since \( s \) has even degree in \( \Gamma \) there exists a neighbor \( v \) of \( s \) for which \( Q(\alpha^\vee_v) = 0 \). The result follows.

**Proof of Corollary 3.3.** By Proposition 2.4 a nondegenerate bipartite graph \( \Gamma \) is a line graph if and only if \( \Gamma \) is a path of even order. Since every path is 1-lit, this corollary holds for \( \Gamma \) as a line graph. We thus assume that \( \Gamma \) is not a line graph. By Theorem 3.2 the graph \( \Gamma \) is 2-lit. By Lemma 5.5 we deduce that the graph \( \Gamma^\vee \) is bipartite with bipartition as same as that of \( \Gamma \). We use this to show that (i), (ii) are equivalent.
(ii) $\Rightarrow$ (i): Let $s$ denote a vertex of $\Gamma$ with even degree. Since $\Gamma$ and $\Gamma^\vee$ are bipartite graphs with same bipartition, we deduce that $B(\alpha_u^\vee, \alpha_v^\vee) = 0$ for any neighbors $u, v$ of $s$ in $\Gamma$. Therefore (13) holds. By Lemma 5.6 the restriction of $Q$ on $\{\alpha_t^\vee \mid st \in R\}$ is onto. Therefore $\Gamma$ is 1-lit by Theorem 3.2.

(i) $\Rightarrow$ (ii): Suppose on the contrary that each vertex of $\Gamma$ has odd degree. Using Lemma 5.5 we deduce that each vertex of $\Gamma^\vee$ has odd degree. Let $s$ denote any element of $S$. By (11) the value $Q(\alpha_s^\vee)$ equals

$$Q\left(\sum_{st \in R^\vee} \alpha_t\right).$$

Since the bipartite graphs $\Gamma$ and $\Gamma^\vee$ have the same bipartition, we deduce that $B(\alpha_u, \alpha_v) = 0$ for any neighbors $u, v$ of $s$ in $\Gamma^\vee$. By (6) the summation in (15) can be moved out front. Since $Q(\alpha_s) = 1$ for all $s \in S$, it follows that (15) equals 1, contradicting to Theorem 3.2(ii). □

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