On the local vertex antimagic total coloring of some families tree

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Abstract. Let $G(V, E)$ be a graph of vertex set $V$ and edge set $E$. Local vertex antimagic total coloring developed from local edge and local vertex antimagic coloring of graph. Local vertex antimagic total coloring is defined $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ if for any two adjacent vertices $v_1$ and $v_2$, $w(v_1) \neq w(v_2)$, where for $v \in G$, $w(v) = \sum_{e \in E(v)} f(e) + f(v)$, where $E(v)$ and $V(v)$ are respectively the set of edges incident to $v$ and the set of vertices adjacent to $v$. Thus, any local vertex antimagic total coloring induces a proper vertex coloring of $G$ if each vertex $v$ is assigned the color $w(v)$. The chromatic number of local vertex antimagic total coloring denote $\chi_{lvat}(G)$ is the minimum number of colors taken over all colorings induced by local vertex antimagic total coloring of graph $G$. In this paper, we use some families tree graph. We also study the existence of local vertex antimagic total coloring chromatic number of some families tree namely star graph, double star graph, banana tree graph, centipede graph, and amalgamation of star graph.

Keywords: Local antimagic vertex total coloring, chromatic number, some families tree

1. Introduction

All graphs in this paper are finite, simple and undirected. A graph $G$ has vertex set $V$ and edge set $E$ is denoted by $G = (V, E)$. The vertex set and edge set of $G$ are often denoted by $V(G)$ and $E(G)$, respectively. The order of graph $G$ is notated $|V(G)|$ and the size of graph $G$ is notated $|E(G)|$. We refer the reader to \cite{4, 9, 11} and \cite{12} for all other terms and notation not provided in this paper. For graph theoretic terminology we refer to Chartrand and Lesniak \cite{3}.

Harsfield and Ringel \cite{10} introduced the concept of an antimagic labeling. By a
labeling of a graph, we mean any mapping that sends some set of graph elements to a set of positive integers. If the domain is a vertex set $V(G)$ or an edge set $E(G)$, the labelings are called respectively vertex labelings or edge labelings. If the domain is the set of all vertices and edges then the labeling is called a total labeling [5, 6, 7]. A general survey of graph labelings is found in [8].

Arumugam et al. [2] introduced the concept of local antimagic vertex coloring of graph $G$. Let $G = (V, E)$ be a graph of order $n$ and size $m$ having no isolated vertices. A bijection $f : E \rightarrow \{1, 2, ..., m\}$ is called a local antimagic labeling it for all $e \in E$ we have $w(u) \neq w(v)$ where $w(u) = \sum_{e \in E(u)} f(e)$. A graph $G$ is local antimagic if $G$ has a local antimagic labeling. The local antimagic chromatic number of graph $G$ denoted by $\chi_{la}(G)$. [2].

Local edge antimagic coloring developed from local antimagic vertex coloring of graph $G$. Agustin et al. [1] studied the existence of local edge antimagic coloring of some special graphs. By local edge antimagic coloring, we mean a bijection $f : V(G) \rightarrow \{1, 2, ..., |V(G)|\}$ is called a local edge antimagic coloring if for any two incident edges at the same vertices $e_1$ and $e_2$, $w(e_1) \neq w(e_2)$, where for $e = uv \in G$, $w(e) = f(u) + f(v)$. Thus, any local edge antimagic labeling induces a proper edge coloring of $G$ if each edge $e$ is assigned the color $w(e)$. The local edge antimagic chromatic number of graph $G$ denoted by $\gamma_{lea}(G)$. [1].

In this paper, we initiate to study a different type local antimagic coloring namely local vertex antimagic total coloring. A bijection of $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G)| + |E(G)|\}$ is called local vertex antimagic total coloring if for any two adjacent vertices $v_1$ and $v_2$, $w(v_1) \neq w(v_2)$, where for $v \in G$, $w(v) = \sum_{e \in E(v)} f(e) + f(v)$, where $E(v)$ and $V(v)$ are respectively the set of edges incident to $v$ and the set of vertices adjacent to $v$. Thus, any local vertex antimagic total coloring induces a proper vertex coloring of $G$ if each vertex $v$ is assigned the color $w(v)$. The chromatic number of local vertex antimagic total coloring denote $\chi_{levat}(G)$ is the minimum number of colors taken over all colorings induced by local vertex antimagic total coloring of graph $G$.

2. Main Results

In this paper, we have studied the existence of local vertex antimagic total coloring of some families tree and amalgamation of path and star. We have found the local vertex antimagic total chromatic number of star graph $S_n$, double star graph $S_{n,m}$, banana tree graph $B_{m,n}$, centipede graph $C_n$, and amalgamation of star graph $Amal(S_n,v,m)$.

**Observation 2.1.** If $\chi(G)$ is chromatic number vertex coloring then $\chi_{levat}(G) \geq \chi(G)$.

**Proof:** Based on the concept of local vertex antimagic total coloring that its coloring is proper coloring that is for every two vertex adjacent must have different colors or weights. Chromatic numbers are the number of different minimum color or weights variations. It depends not only on chromatic numbers. Thus the local vertex antimagic total chromatic number is obtained with more conditions than vertex chromatic number. Thus, $\chi_{levat}(G) \geq \chi(G)$. 

**Theorem 2.1.** For the graph star $S_n$ with $n \geq 2$, the local vertex antimagic total chromatic number of $S_n$ is $\chi_{levat}(S_n) = 2$. 


Example of local vertex antimagic total coloring for the double star graph $S_n$.

**Proof:** The graph $S_n$ is a connected graph with vertex set $V(S_n) = \{x\} \cup \{x_i, 1 \leq i \leq n\}$ and edge set $E(S_n) = \{xx_i, 1 \leq i \leq n\}$. Hence $|V(S_n)| = n + 1$ and $|E(S_n)| = n$. Define a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G)| + |E(G)|\}$ by the following:

$$f(v) = \begin{cases} 
2n + 1, & v = x \\
2n - i + 1, & v = x_i, \ 1 \leq i \leq n 
\end{cases}$$

$$f(e) = \begin{cases} 
i, & e = xx_i, \ 1 \leq i \leq n \\
j + 1, & e = yy_j, \ 1 \leq j \leq m 
\end{cases}$$

Clearly $f$ is a local vertex antimagic total labeling of $S_n$ and we formulate the vertex weights as follows:

$$w_i(v) = \begin{cases} 
2n + 1, & v = x_i, \ 1 \leq i \leq n \\
\frac{n}{2}(1 + n) + 2n + 1, & v = x 
\end{cases}$$

Hence, from the above vertex weight, it is easy to see that $f$ induces a proper local vertex antimagic total coloring of $S_n$ and it gives $\chi_{vat}(S_n) \leq 2$. Also it follow from Observation 2.1 that $\chi_{vat}(S_n) \geq 2$. Hence $\chi_{vat}(S_n) = 2$. \hfill $\Box$

**Figure 1.** Example of local vertex antimagic total coloring for $S_5$ and $S_8$

**Theorem 2.2.** For the double star graph $S_n,m$ with $n \geq 2$ and $m \geq 2$, the local vertex antimagic total chromatic number of $S_n,m$ is $\chi_{vat}(S_n,m) \leq 3$.

**Proof:** The graph $S_n$ is a connected graph with vertex set $V(S_n,m) = \{x\} \cup \{y\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq m\}$ and edge set $E(S_n,m) = \{xy\} \cup \{xx_i; 1 \leq i \leq n\} \cup \{yy_j; 1 \leq j \leq m\}$. Hence $|V(S_n,m)| = m + n + 2$ and $|E(S_n,m)| = m + n + 1$. Define a bijection $f : V(S_n,m) \cup E(S_n,m) \rightarrow \{1, 2, 3..., |V(S_n,m)| + |E(S_n,m)|\}$ by the following:

$$f(v) = \begin{cases} 
2m + 2n + 2, & v = x \\
2m + 2n + 3, & v = y \\
m + 2n - i + 2, & v = x_i, \ 1 \leq i \leq n \\
m + 2n - j + 2, & v = y_j, \ 1 \leq j \leq m 
\end{cases}$$

$$f(e) = \begin{cases} 
i, & e = xy \\
1 + m, & e = xx_i, \ 1 \leq i \leq n \\
1 + j, & e = yy_j, \ 1 \leq j \leq m 
\end{cases}$$
Clearly \( f \) is a local vertex antimagic total labeling of \( S_{n,m} \) and we formulate the vertex weights as follows:

\[
\begin{align*}
    w_t(v) &= \begin{cases} 
        2m + 2n + 3, & v = x_i = y_j, \ 1 \leq i \leq n, \ 1 \leq j \leq m \\
        2m + 2n + \frac{m}{2}(2m + n + 3) + 3, & v = x \\
        2m + 2n + \frac{m}{2}(m + 3) + 4, & v = y
    \end{cases}
\]

Hence, from the above the vertex weight, it is easy to see that \( f \) induces a proper local vertex antimagic total coloring of \( S_{n,m} \) and it gives \( \chi_{lat}(S_{n,m}) \leq 3 \).

\[\square\]

**Figure 2.** Example of local vertex antimagic total coloring for \( S_{2,7} \) and \( S_{7,7} \)

**Theorem 2.3.** For the banana tree graph \( B_{m,n} \) with \( n \geq 3 \) and \( m \geq 3 \), the local vertex antimagic total chromatic number of \( B_{m,n} \) is

\[
\chi_{lat}(B_{m,n}) \leq \begin{cases} 
    4 & \text{for, } n \text{ odd, } m \text{ odd} \\
    5 & \text{for, } n \text{ even, } m \text{ odd} \\
    6 & \text{for, } n \text{ even, } m \text{ even}
\end{cases}
\]

**Proof:** The graph \( B_{m,n} \) is a connected graph with vertex set \( V(B_{m,n}) = \{x\} \cup \{x_i; y_i, 1 \leq i \leq n\} \cup \{y_j, 1 \leq j \leq m - 1, 1 \leq i \leq n\} \) and edge set \( E(B_{m,n}) = \{xx_i, 1 \leq i \leq n\} \cup \{x_iy_i, 1 \leq i \leq n\} \cup \{iy_j, 1 \leq i \leq n, 1 \leq j \leq m - 1\} \). Hence \( |V(B_{m,n})| = mn + n + 1 \) and \( |E(B_{m,n})| = nm + n \).

**Case 1.** For \( n \) odd and \( m \) odd, define a bijection \( f : V(B_{m,n}) \cup E(B_{m,n}) \rightarrow \{1, 2, ..., |V(B_{m,n})| + |E(B_{m,n})|\} \) by the following:

\[
\begin{align*}
    f(v) &= \begin{cases} 
        2mn + 2m + 1, & v = x \\
        2mn + m + \frac{j-1}{2}, & v = x_j, \ 1 \leq j \leq m, \ j \text{ odd} \\
        2mn + m + \frac{m+j+1}{2}, & v = x_j, \ 2 \leq j \leq m - 1, \ j \text{ even} \\
        2mn + j, & v = y_j, \ 1 \leq j \leq m \\
        mn + (n-i)m + j, & v = y_i^j, \ 1 \leq j \leq m, \ 2 \leq i \leq n - 1, \ i \text{ even} \\
        mn + (n-i+1)m - j + 1, & v = y_i^j, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 2, \ i \text{ odd}
    \end{cases}
\]

for $i, j = 1, 2, ..., m, n$.

Clearly $f$ is a local vertex antimagic total labeling of $B_{m,n}$ and we formulate the vertex weights as follows:

$$w_t(x) = \frac{m}{2} (m + 1) + 2mn + 2m + 1$$

$$w_t(x_j) = \frac{m+1}{2} + 2mn + 3m + 1, \ 1 \leq j \leq m$$

$$w_t(y_j^i) = 2mn + 2m + 1, \ 1 \leq i \leq n - 1, \ 1 \leq j \leq m$$

$$w_t(y_j) = \frac{n-1}{2} (mn + m + 2) + \frac{n-1}{4} (mn + 5m) + 2mn + 2m + 1, \ 1 \leq j \leq m$$

Hence, from the above the vertex weight, it is easy to see that $f$ induces a proper local vertex antimagic total coloring of $B_{m,n}$ and it gives $\chi_{\text{leaf}}(B_{m,n}) \leq 4$. \hfill $\Box$

**Case 2.** For $n$ even and $m$ odd, define a bijection $f : V(B_{m,n}) \cup E(B_{m,n}) \rightarrow \{1, 2, ..., |V(B_{m,n})| + |E(B_{m,n})|\}$ by the following:

$$f(e) = \begin{cases} \frac{j}{2}, & e = xx_j, \ 2 \leq j \leq m - 1, \ j \text{ even} \\ \frac{m+j}{2}, & e = xx_j, \ 1 \leq j \leq m, \ j \text{ odd} \\ 2m - j + 1, & e = x_jy_j, \ 1 \leq j \leq m \\ 2m + \frac{j}{2}, & e = y_jy_j^i, \ i = 1, \ 2 \leq j \leq m - 1, \ j \text{ even} \\ 2m + \frac{m+j}{2}, & e = y_jy_j^i, \ i = 1, \ 1 \leq j \leq m, \ j \text{ odd} \\ (i+1)m + j, & e = y_jy_j^i, \ 1 \leq j \leq m, \ 2 \leq i \leq n - 2, \ i \text{ even} \\ (i+2)m - j + 1, & e = y_jy_j^i, \ 1 \leq j \leq m, \ 3 \leq i \leq n - 1, \ i \text{ odd} \end{cases}$$

$$f(v) = \begin{cases} 2mn + 2m + 1, & v = x \\ 2mn + m + \frac{j+1}{2}, & v = x_j, \ 1 \leq j \leq m, \ j \text{ odd} \\ 2mn + m + \frac{m+j+1}{2}, & v = x_j, \ 2 \leq j \leq m - 1, \ j \text{ even} \\ 2mn + \frac{j+1}{2}, & v = y_j, \ 1 \leq j \leq m, \ j \text{ odd} \\ 2mn + \frac{m+j+1}{2}, & v = y_j, \ 2 \leq j \leq m - 1, \ j \text{ even} \\ 2mn - m + \frac{j+2}{2}, & v = y_j^i, \ i = 1, \ 1 \leq j \leq m, \ j \text{ odd} \\ 2mn - m + \frac{2m-j+2}{2}, & v = y_j^i, \ i = 1, \ 2 \leq j \leq m - 1, \ j \text{ even} \\ mn + (n - i)m + j, & v = y_j^i, \ 1 \leq j \leq m, \ 3 \leq i \leq n - 1, \ i \text{ odd} \\ mn + (n - i + 1)m - j + 1, & v = y_j^i, \ 1 \leq j \leq m, \ 2 \leq i \leq n - 2, \ i \text{ even} \end{cases}$$

Clearly $f$ is a local vertex antimagic total labeling of $B_{m,n}$ and we formulate the vertex weights as follows:

$$w_t(x) = \frac{m}{2} (m + 1) + 2mn + 2m + 1$$

$$w_t(x_j) = \frac{m+1}{2} + 2mn + 3m + 1, \ 1 \leq j \leq m$$
vertex weights as follows:

\[ w_i(y_j) = 2mn + 2m + 1, \quad 1 \leq i \leq n - 1, \quad 1 \leq j \leq m \]

\[ w_i(y_j) = \frac{n-2}{4}(mn + 2m + 2) + \frac{n-2}{4}(mn + 6m) + \frac{m+1}{2} + 2mn + 4m + 1, \quad 1 \leq j \leq m \]

Hence, from the above the vertex weight, it is easy to see that \( f \) induces a proper local vertex antimagic total coloring of \( B_{m,n} \) and it gives \( \chi_{lvt}(B_{m,n}) \leq 4 \). \( \square \)

**Case 3.** For \( n \) odd and \( m \) even, define a bijection \( f : V(B_{m,n}) \cup E(B_{m,n}) \rightarrow \{1, 2, ..., |V(B_{m,n})| + |E(B_{m,n})|\} \) by the following:

\[
f(v) = \begin{cases} 
2mn + 2m + 1, & v = x \\
2mn + m + \frac{j+1}{2}, & v = x_j, \quad 1 \leq j \leq m - 1, \quad j \text{ odd} \\
2mn + m + \frac{m+j}{2}, & v = x_j, \quad 2 \leq j \leq m, \quad j \text{ even} \\
2mn + j, & v = y_j, \quad 1 \leq j \leq m \\
nm + (n - i)m + j, & v = y_i^j, \quad 1 \leq j \leq m, \quad 2 \leq i \leq n - 2, \quad i \text{ odd} \\
nm + (n - i + 1)m - j + 1, & v = y_i^j, \quad 1 \leq j \leq m, \quad 1 \leq i \leq n - 1, \quad i \text{ even}
\end{cases}
\]

\[
f(e) = \begin{cases} 
\frac{2}{m+j+1}, & e = xx_j, \quad 2 \leq j \leq m, \quad j \text{ odd} \\
2m - j + 1, & e = xx_j, \quad 1 \leq j \leq m - 1, \quad j \text{ even} \\
(i + 1)m + j, & e = y_jy_i^j, \quad 1 \leq j \leq m, \quad 1 \leq i \leq n - 1, \quad i \text{ odd} \\
(i + 2)m - j + 1, & e = y_jy_i^j, \quad 1 \leq j \leq m, \quad 2 \leq i \leq n - 2, \quad i \text{ even}
\end{cases}
\]

Clearly \( f \) is a local vertex antimagic total labeling of \( B_{m,n} \) and we formulate the vertex weights as follows:

\[ w_i(x) = \frac{m}{2} + 2mn + 2m + 1 \]

\[ w_i(x_j) = \frac{m+2}{2} + 2mn + 3m + 1, \quad 1 \leq j \leq m, \quad j \text{ odd} \]

\[ w_i(x_j) = \frac{m+2}{2} + 2mn + 3m, \quad 2 \leq j \leq m - 1, \quad j \text{ even} \]

\[ w_i(y_j) = 2mn + 2m + 1, \quad 1 \leq i \leq n - 1, \quad 1 \leq j \leq m \]

\[ w_i(y_j) = \frac{n-1}{4}(mn + m + 2) + \frac{n-1}{4}(mn + 5m) + 2mn + 2m + 1, \quad 1 \leq j \leq m \]

Hence, from the above the vertex weight, it is easy to see that \( f \) induces a proper local vertex antimagic total coloring of \( B_{m,n} \) and it gives \( \chi_{lvt}(B_{m,n}) \leq 5 \). \( \square \)

**Case 4.** For \( n \) even and \( m \) even, define a bijection \( f : V(B_{m,n}) \cup E(B_{m,n}) \rightarrow \{1, 2, ..., |V(B_{m,n})| + |E(B_{m,n})|\} \) by the following:

\[
f(v) = \begin{cases} 
2mn + 2m + 1, & v = x \\
2mn + m + \frac{j+1}{2}, & v = x_j, \quad 1 \leq j \leq m - 1, \quad j \text{ odd} \\
2mn + m + \frac{m+j}{2}, & v = x_j, \quad 2 \leq j \leq m, \quad j \text{ even} \\
2mn + \frac{j+1}{2}, & v = y_j, \quad 1 \leq j \leq m - 1, \quad j \text{ odd} \\
2mn + \frac{m+j}{2}, & v = y_j, \quad 2 \leq j \leq m, \quad j \text{ even} \\
2mn - m + \frac{m-j+1}{2}, & v = y_i^j, \quad i = 1, \quad 1 \leq j \leq m - 1, \quad j \text{ odd} \\
2mn - m + \frac{2m-j+2}{2}, & v = y_i^j, \quad i = 1, \quad 2 \leq j \leq m, \quad j \text{ even} \\
nm + (n - i)m + j, & v = y_i^j, \quad 1 \leq j \leq m, \quad 3 \leq i \leq n - 1, \quad i \text{ odd} \\
nm + (n - i + 1)m - j + 1, & v = y_i^j, \quad 1 \leq j \leq m, \quad 2 \leq i \leq n - 2, \quad i \text{ even}
\end{cases}
\]
For the centipede graph $C_n$ with $n \geq 4$, the local vertex antimagic total chromatic number of $C_n$ is $\chi_{lvat}(C_n) \leq 4$.

**Proof:** The graph $C_n$ is a connected graph with vertex set $V(C_n) = \{x_i; y_i, 1 \leq i \leq n\}$ and edge set $E(C_n) = \{x_ix_{i+1}, 1 \leq i \leq n-1\} \cup \{x_iy_i, 1 \leq i \leq n\}$. Hence $|V(C_n)| = 2n$ and $|E(C_n)| = 2n-1$.

**Case 1.** For $n$ even, define a bijection $f : V(C_n) \cup E(C_n) \rightarrow \{1, 2, 3, ..., |V(C_n)| + |E(C_n)|\}$ by the following:

$$f(v) = \begin{cases} \frac{3n-i}{2}, & v = x_i, \ 2 \leq i \leq n, \ i \ even \\ \frac{4n^2-i}{4}, & v = x_i, \ 1 \leq i \leq n-1, \ i \ odd \\ \frac{7n+1}{2}, & v = y_i, \ 1 \leq i \leq n-1, \ i \ odd \\ \frac{6n+1}{2}, & v = y_i, \ 2 \leq i \leq n, \ i \ even \end{cases}$$

$$f(e) = \begin{cases} \frac{i}{2}, & e = x_ix_{i+1}, \ 2 \leq i \leq n-2, \ i \ even \\ \frac{n-i-1}{2}, & e = x_ix_{i+1}, \ 1 \leq i \leq n-1, \ i \ odd \\ \frac{5n-i-1}{2}, & e = x_iy_i, \ 1 \leq i \leq n-1, \ i \ odd \\ \frac{6n-i}{2}, & e = x_iy_i, \ 2 \leq i \leq n, \ i \ even \end{cases}$$

Clearly $f$ is a local vertex antimagic total labeling of $B_{m,n}$ and we formulate the vertex weights as follows:

$$w_t(x) = \frac{mn}{2}(m+1) + 2mn + 2m + 1$$

$$w_t(x_j) = \frac{mn+2}{2} + 2mn + 3m + 1, \ 1 \leq j \leq m, \ j \ odd$$

$$w_t(x_j) = \frac{mn+2}{2} + 2mn + 3m, \ 2 \leq j \leq m-1, \ j \ even$$

$$w_t(y_j) = 2mn + 2m + 1, \ 1 \leq i \leq n-1, \ 1 \leq j \leq m$$

$$w_t(y_j) = \frac{n+2}{4}(mn+2m+2) + \frac{n-2}{4}(mn+6m) + \frac{mn+2}{2} + 2mn + 4m+1, \ 1 \leq j \leq m-1, \ j \ odd$$

$$w_t(y_j) = \frac{n+2}{4}(mn+2m+2) + \frac{n-2}{4}(mn+6m) + \frac{mn+2}{2} + 2mn + 4m, \ 2 \leq j \leq m, \ j \ even$$

Hence, from the above the vertex weight, it is easy to see that $f$ induces a proper local vertex antimagic total coloring of $B_{m,n}$ and it gives $\chi_{lvat}(B_{m,n}) \leq 6$. \hfill \Box
Clearly $f$ is a local vertex antimagic total labeling of $C_n$ and we formulate the vertex weights as follows:

$$w_t(v) = \begin{cases} \frac{9n - 2}{2}, & v = x_i, \ i = n \\ 5n - 1, & v = x_i, \ 2 \leq i \leq n - 2, \ i \text{ even} \\ 5n - 2, & v = x_i, \ 1 \leq i \leq n - 1, \ i \text{ odd} \\ \frac{12n - 2}{2}, & v = y_i, \ 1 \leq i \leq n \end{cases}$$

Hence, from the above the vertex weight, it is easy to see that $f$ induces a proper local vertex antimagic total coloring of $C_n$ and it gives $\chi_{lvat}(C_n) \leq 4$.  

**Case 2.** For $n$ odd, define a bijection $f : V(C_n) \cup E(C_n) \rightarrow \{1, 2, 3, \ldots, |V(C_n)| + |E(C_n)|\}$ by the following:

$$f(v) = \begin{cases} \frac{3n - i}{2}, & v = x_i, \ 1 \leq i \leq n, \ i \text{ odd} \\ \frac{4n - i}{2}, & v = x_i, \ 2 \leq i \leq n - 1, \ i \text{ even} \\ \frac{6n + i - 2}{2}, & v = x_i, \ 2 \leq i \leq n - 1, \ i \text{ even} \\ \frac{7n + i - 2}{2}, & v = y_i, \ 1 \leq i \leq n, \ i \text{ odd} \end{cases}$$

$$f(e) = \begin{cases} \frac{n + i}{2}, & e = x_ix_{i+1}, \ 1 \leq i \leq n - 2, \ i \text{ odd} \\ \frac{5n - i}{2}, & e = x_ix_{i+1}, \ 2 \leq i \leq n - 1, \ i \text{ even} \\ \frac{6n - i}{2}, & e = x_iy_i, \ 1 \leq i \leq n, \ i \text{ odd} \end{cases}$$

Clearly $f$ is a local vertex antimagic total labeling of $C_n$ and we formulate the vertex weights as follows:

$$w_t(v) = \begin{cases} \frac{7n - 1}{2}, & v = x_i, \ i = n \\ \frac{9n - 1}{2}, & v = x_i, \ 1 \leq i \leq n - 2, \ i \text{ odd} \\ \frac{11n - 1}{2}, & v = x_i, \ 2 \leq i \leq n - 1, \ i \text{ even} \\ \frac{12n - 2}{2}, & v = y_i, \ 1 \leq i \leq n \end{cases}$$

Hence, from the above the vertex weight, it is easy to see that $f$ induces a proper local vertex antimagic total coloring of $C_n$ for $n$ odd and it gives $\chi_{lvat}(C_n) \leq 4$.  

**Theorem 2.5.** For $n, m \geq 3$, the local vertex antimagic total chromatic number of graph $G = Amal(S_n, v, m)$ is:

$$\chi_{lvat}(G) \leq \begin{cases} 3, & \text{for } n \text{ odd or } n \text{ even } m \text{ odd} \\ 4, & \text{for } n \text{ even } m \text{ even} \end{cases}$$

**Proof.** The graph $G$ is a amalgamation of star graph with vertex set $V(G) = \{x\} \cup \{x_j, 1 \leq j \leq m\} \cup \{y_i, 1 \leq i \leq n - 1, \ 1 \leq j \leq m\}$ and edge set $E(G) = \{xx_j, 1 \leq j \leq m\} \cup \{x_jy_i, 1 \leq i \leq n - 1, \ 1 \leq j \leq m\}$. Hence $|V(G)| = mn + 1$ and $|E(G)| = mn$. 


**Case 1.** For \( n \) odd, define a bijection \( f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G)| + |E(G)|\} \) by the following:

\[
\begin{align*}
\text{Case 1:} & \quad f(v) = \begin{cases} 
1, & v = x \\
2m - j + 2, & v = x_j, \ 1 \leq j \leq m \\
(2n - i)m + j + 1, & v = y^j_i, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 1, \ i \text{ odd} \\
(2n + 1 - i)m - j + 2, & v = y^j_i, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 2, \ i \text{ even}
\end{cases} \\
& \quad f(e) = \begin{cases} 
i + 1, & e = xx_j, \ 1 \leq j \leq m, \ j \text{ even} \\
(n + 1)m - j + 2, & e = xjy^j_i, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 1, \ i \text{ odd} \\
(n + 1)m + j + 1, & e = xjy^j_i, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 2, \ i \text{ even}
\end{cases}
\end{align*}
\]

Clearly \( f \) is a local vertex antimagic total labeling of graph \( G \) and we formulate the vertex weights as follows:

\[
w_l(x) = \frac{(m-2)(m-3)}{2} + (m - 2)4 + 6
\]

\[
w_l(y^j_i) = 2mn + 2m + 3, \ 1 \leq i \leq n - 1, \ 1 \leq j \leq m
\]

\[
w_l(x_j) = \frac{n}{4}(mn + 4m + 2) + \frac{n^2}{4}(mn + 2m + 4) + \frac{3(m-1)}{2} + 5, \ 1 \leq j \leq m
\]

Hence, from the above the vertex weight, it is easy to see that \( f \) induces a proper local vertex antimagic total coloring of graph \( G = Amal(S_n, v, m) \) where \( n \) odd it gives \( \chi_{vat}(G) \leq 3 \)

**Case 2.** For \( n \) even and \( m \) odd, define a bijection \( f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G)| + |E(G)|\} \) by the following:

\[
\begin{align*}
\text{Case 2:} & \quad f(v) = \begin{cases} 
\frac{3m+j+3}{2}, & v = x, \ 1 \leq j \leq m, \ j \text{ even} \\
\frac{2m+j+3}{2}, & v = x_j, \ 1 \leq j \leq m, \ j \text{ odd} \\
(2n - i)m + j + 1, & v = y^j_i, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 1, \ i \text{ odd} \\
(2n + 1 - i)m - j + 2, & v = y^j_i, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 2, \ i \text{ even}
\end{cases} \\
& \quad f(e) = \begin{cases} 
\frac{j+2}{2}, & e = xx_j, \ 1 \leq j \leq m, \ j \text{ even} \\
\frac{m+j+2}{2}, & e = xx_j, \ 1 \leq j \leq m, \ j \text{ odd} \\
(i + 2)m - j + 2, & e = xjy^j_i, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 1, \ i \text{ odd} \\
(i + 1)m + j + 1, & e = xjy^j_i, \ 1 \leq j \leq m, \ 1 \leq i \leq n - 2, \ i \text{ even}
\end{cases}
\end{align*}
\]

Clearly \( f \) is a local vertex antimagic total labeling of graph \( G = Amal(S_n, v = x_1, m) \) and we formulate the vertex weights as follows:

\[
w_l(x) = \frac{(m-2)(m-3)}{2} + (m - 2)4 + 6
\]

\[
w_l(y^j_i) = 2mn + 2m + 3, \ 1 \leq i \leq n - 1, \ 1 \leq j \leq m
\]

\[
w_l(x_j) = \frac{n}{4}(mn + 4m + 2) + \frac{n^2}{4}(mn + 2m + 4) + \frac{3(m-1)}{2} + 5, \ 1 \leq j \leq m
\]
Hence, from the above the vertex weight, it is easy to see that \( f \) induces a proper local vertex antimagic total coloring of graph \( G = Amal(S_n, v, m) \) where \( n \) even and \( m \) odd and it gives \( \gamma_{lvt}(G) \leq 3 \).

**Case 3.** For \( n \) even and \( m \) even, define a bijection \( f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G)| + |E(G)|\} \) by the following:

\[
f(v) = \begin{cases} 
1, & v = x \\
\frac{3n+j+3}{2}, & v = x_j, \ 1 \leq j \leq m, \ \text{\( j \) odd} \\
\frac{2n+j+2}{2}, & v = x_j, \ 1 \leq j \leq m, \ \text{\( j \) even} \\
(2n-i)m+j+1, & v = y_j^i, \ 1 \leq j \leq m, \ 1 \leq i \leq n-1, \ \text{\( i \) odd} \\
(2n+1-i)m-j+2, & v = y_j^i, \ 1 \leq j \leq m, \ 1 \leq i \leq n-2, \ \text{\( i \) even}
\end{cases}
\]

\[
f(e) = \begin{cases} 
\frac{j+3}{2}, & e = xx_j, \ 1 \leq j \leq m \ j \ odd \\
\frac{m+j+2}{2}, & e = xx_j, \ 1 \leq j \leq m \ j \ even \\
(i+2)m-j+2, & e = x_jy_j^i, \ 1 \leq j \leq m, \ 1 \leq i \leq n-1, \ \text{\( i \) odd} \\
(i+1)m+j+1, & e = x_jy_j^i, \ 1 \leq j \leq m, \ 1 \leq i \leq n-2, \ \text{\( i \) even}
\end{cases}
\]

Clearly \( f \) is a local vertex antimagic total labeling of graph \( G = Amal(S_n, v, m) \) and we formulate the vertex weights as follows:

\[
w_t(x) = \frac{(m-2)(m-3)}{2} + (m-2)4 + 6
\]

\[
w_t(y_j^i) = 2mn + 2m + 3, \ 1 \leq i \leq n-1, \ 1 \leq j \leq m
\]

\[
w_t(x_j) = \frac{n}{4}(mn + 4m + 2) + \frac{n-2}{4}(mn + 2m + 4) + \frac{3(m-2)}{2} + 7, \ 1 \leq j \leq m, \ \text{\( j \) odd}
\]

\[
w_t(x_j) = \frac{n}{4}(mn + 4m + 2) + \frac{n-2}{4}(mn + 2m + 4) + \frac{3(m-2)}{2} + 6, \ 1 \leq j \leq m, \ \text{\( j \) even}
\]

Hence, from the above the vertex weight, it is easy to see that \( f \) induces a proper local vertex antimagic total coloring of graph \( G = Amal(S_n, v, m) \) where \( n \) even and \( m \) even and it gives \( \chi_{lvt}(G) \leq 4 \).

**3. Conclusion**

In this paper we have to study the local vertex antimagic total coloring os some families tree. We have found chromatic number of star graph \( S_n \), path graph \( P_n \), double star graph \( S_{n,m} \), banana tree graph \( B_{m,n} \), centipede graph \( C_n \), and amalgamation of star graph \( Amal(S_n, v, m) \).

**Open Problem 3.1.** Determine the lower bound of local vertex antimagic total coloring for any graph.

**Open Problem 3.2.** Determine the local vertex antimagic total chromatic number of other some families tree like caterpillar graph, firecracker graph and lobster graph.

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