Tractable hierarchies of convex relaxations for polynomial optimization on the nonnegative orthant

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Abstract

We consider polynomial optimization problems (POP) on a semialgebraic set contained in the nonnegative orthant (every POP on a compact set can be put in this format by a simple translation of the origin). Such a POP can be converted to an equivalent POP by squaring each variable. Using even symmetry and the concept of factor width, we propose a hierarchy of semidefinite relaxations based on the extension of Pólya’s Positivstellensatz by Dickinson–Povh. As its distinguishing and crucial feature, the maximal matrix size of each resulting semidefinite relaxation can be chosen arbitrarily and in addition, we prove that the sequence of values returned by the new hierarchy converges to the optimal value of the original POP at the rate $O(\varepsilon^{-c})$ if the semialgebraic set has nonempty interior. When applied to (i) robustness certification of multi-layer neural networks and (ii) computation of positive maximal singular values, our method based on Pólya’s Positivstellensatz provides better bounds and runs several hundred times faster than the standard Moment-SOS hierarchy.

Keywords: Pólya’s Positivstellensatz; Handelman’s Positivstellensatz; basic semialgebraic set; sums of squares; polynomial optimization; moment-SOS hierarchy; factor width

1 Introduction

Polynomial optimization is concerned with computing the minimum value of a polynomial on a basic semialgebraic set. A well-known methodology is to apply positivity certificates (representations of polynomials positive on basic semialgebraic sets) to design a hierarchy of convex relaxations to solve polynomial optimization problems (POPs). Developed originally by Lasserre in [25], the hierarchy of semidefinite relaxations based on Putinar’s Positivstellensatz is called the Moment-SOS hierarchy. We utilize this approach in many applications arising from optimization, operations research, signal processing, computational geometry, probability, statistics, control, PDEs, quantum information, and computer vision. For more details, the interested reader is referred to, e.g., [56, 53, 60, 48, 47, 9, 7, 51, 36, 50] and references therein.

However, despite its theoretical efficiency (also observed in practice), the Moment-SOS hierarchy is facing a scalability issue mainly due to the increasing size of the resulting relaxations. Overcoming the scalability and efficiency issues has become a major scientific challenge in polynomial optimization. Many recent efforts in this direction are mainly developed around the following ideas:

1. **SDP-relaxations variants with small maximal matrix size solved efficiently by interior point methods.**
   - This includes correlative sparsity [21, 20], term sparsity [55, 57, 59], symmetry exploitation [17, 14], Jordan symmetry reduction [6], sublevel relaxations [5].
2. **Exploit low-rank structures of SDP-relaxations; see, e.g., [61, 62].**
3. **First-order methods to solve SDP-relaxations involving matrix variables of potentially large size with constant trace [33, 39].

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4. Develop convex relaxations that are based on alternatives to semidefinite cones. For example this includes linear programming (LP) \[27\] [1], second-order conic programming (SOCP) \[28\] [52] [1], copositive programming \[33\], non-symmetric conic programming \[34\], relative entropy programming \[11\] [35], geometric programming \[15\].

Let \( R[x] \) denote the set of real polynomials in vector of variables \( x = (x_1, \ldots, x_n) \). Given a real symmetric matrix \( M \), the notation \( M \succeq 0 \) denotes that \( M \) is positive semidefinite, i.e., all its eigenvalue are nonnegative. Given \( r \in \mathbb{N}_{>0}, \) denote \( [r] := \{1, \ldots, r\} \).

Sparsity exploitation is one of the notable methods to reduce the size of the Moment-SOS relaxations. For POPs in the form

\[
f^* := \min_{x \in \mathbb{R}^n} f(x),
\]

where \( f \) is a polynomial in \( R[x] \) and \( S(g) \) is the semialgebraic set associated with \( g = \{g_1, \ldots, g_m\} \subset R[x], \) i.e.,

\[
S(g) := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i \in [m]\},
\]

Waki et al. \[54\] (resp. Wang et al. \[55\]) have exploited correlative (resp. term) sparsity to define appropriate sparse-variants of the associated standard SOS-relaxations. Roughly speaking, in a given standard SOS-relaxation, they break each matrix variable into many blocks of smaller sizes and solve the new resulting SDP via an interior-point solver (e.g., \textit{Hosk} \[2\] or \textit{SDPT3} \[52\]). It is due to the fact that the most expensive part of interior-point methods for a standard SDP:

\[
\begin{align*}
\min_{\alpha \in \mathbb{R}^d} & \quad c^T z \\
\text{s.t.} & \quad z \in \mathbb{R}^n, \ A_j^{(i)} \in \mathbb{R}^{q \times q}, \\
& \quad A_j^{(i)} + \sum_{j=1}^d z_j A_j^{(i)} \succeq 0, \ t \in [u].
\end{align*}
\]

is solving a square linear system in every iteration. It has the complexity \( O(uq^2 + w^2q^2) + O(w^3) \), which mainly depends on the matrix size \( q \). Thus one can solve the above SDP efficiently by using interior-point methods if \( q, w \) are small, even when \( u \) is large. On one hand, correlative sparsity occurs for POP \((1.3)\) being such that the objective polynomial has a decomposition \( f = f_1 + \cdots + f_p \), where each polynomial \( f_i \) involves only a small subset of variables \( I_i \subset [n] \), and \( f_i \) together with the constraint polynomials \( g_{I_i}, i \in I_i \) (for some \( J_i \in I_i \)) share the same variables. On the other hand, term sparsity occurs for POP \((1)\) where \( f, g_1, \ldots, g_m \) have a few nonzero terms. To solve large-scale POPs, we simultaneously exploit correlative sparsity and term sparsity as in \[59\].

Denote by \( \|\cdot\|_2 \) the \( 2 \)-norm of a vector in \( \mathbb{R}^n \). A polynomial \( p \in \mathbb{R}[x] \) is written as \( p = \sum_{\alpha \in \mathbb{N}^n} \alpha p_\alpha x^\alpha \) with monomial \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) for some finite real sequence \( (p_\alpha)_{\alpha \in \mathbb{N}^n} \). Define \( \mathbb{N}^n_\alpha := \{\alpha \in \mathbb{N}^n : \alpha \leq t\} \) for each \( t \in \mathbb{N} \). Define \( \mathbb{R}^n_+ := [0, \infty) \). Let \( x_\alpha \) be the canonical basis of monomials for \( \mathbb{R}[x] \) (ordered according to the graded lexicographic order) and \( v_\alpha(x) \) be the vector of all monomials up to degree \( t \), with length \( b(n, t) := \binom{n+t}{n} \).

For each \( A \subset \mathbb{N}^n \), denote \( v_\alpha(x) = (x_\alpha)_{\alpha \in A} \). We say that a polynomial \( q \) is even in each variable if for every \( j \in [n] \), \( q(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = q(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \). A polynomial \( q \) is called a SOS of monomials if \( q = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha x^\alpha \) for some \( \lambda_\alpha \geq 0 \). Accordingly, if \( q \) is an SOS of monomials, then \( q = v_\alpha^T \text{diag}(u)v_\alpha \) for some \( d \in \mathbb{N} \) and \( u \in \mathbb{R}^{d(n-d)} \). For a given real-valued sequence \( y = (g_\alpha)_{\alpha \in \mathbb{N}^n} \), let us define the Riesz linear functional \( L_y : \mathbb{R}[x] \to \mathbb{R} \) by \( p \mapsto L_y(p) := \sum_\alpha p_\alpha y_\alpha \), for all \( p = \sum_\alpha p_\alpha x^\alpha \in \mathbb{R}[x] \).

**Factor width:** Originally defined in \[5\], the factor width of a real positive semidefinite matrix \( G \) is the smallest integer \( s \) for which there exists a real matrix \( P \) such that \( GG^T \) and each column of \( P \) contains at most \( s \) nonzeros. In this case, if \( u \) is a vector of several monomials in \( x \), the SOS polynomial \( u^T G u \) can be written as \( u(x)^T Gu(x) = \sum_i (q_i u(x))^2 \), where \( q_i \) is the \( i \)-th column of \( P \). It is not hard to prove that the Gram matrix of each square \( (q_i u(x))^2 \) has size at most \( s \) since \( q_i \) has at most \( s \) nonzeros. Thus, if an SOS polynomial has Gram matrix of factor width at most \( s \), it can be written as a sum of SOS polynomials with Gram matrix sizes at most \( s \). The converse also holds true thanks to eigen-decomposition. The applications of factor width for polynomial optimization can be found in, e.g., \[11\] [34].

**POP with nonnegative variables:** In the present paper, we focus on the following POP on the nonnegative orthant:

\[
f^* := \inf_{x \in S} f(x).
\]

2
where \( f \) is a polynomial and \( S \) is a semialgebraic set defined by

\[
S := \{ x \in \mathbb{R}^n : x_j \geq 0, \ j \in [n], \ g_i(x) \geq 0, \ i \in [m] \},
\]

for some \( g_i \in \mathbb{R}[x] \), \( i \in [m] \) with \( g_m := 1 \). Letting \( \tilde{q}(x) := q(x^2) \) (with \( x^2 := (x_1^2, \ldots, x_n^2) \)) whenever \( q \in \mathbb{R}[x] \), it follows immediately that problem (1.3) is equivalent to solving

\[
f^* = \inf_{x \in \tilde{S}} \tilde{f}(x),
\]

where \( \tilde{S} \) is a subset of \( \mathbb{R}^n \) defined by

\[
\tilde{S} := \{ x \in \mathbb{R}^n : \tilde{g}_i(x) \geq 0, \ i \in [m] \}.
\]

**Contribution.** Our contribution is twofold:

1. In our first contribution, we provide in Corollary 2 a degree bound for the extension of Pólya’s Positivstellensatz originally stated in [12]. Explicitly, if
   
   - \( \tilde{f}, \tilde{g}_1, \ldots, \tilde{g}_m \) are polynomials even in each variable,
   - \( \tilde{S} \) defined as in [16] has nonempty interior, \( \tilde{g}_i = R - \|x\|^2 \) for some \( R > 0 \),
   - \( \tilde{f} \) is of degree at most \( 2d_f \), each \( \tilde{g}_i \) is of degree at most \( 2d_{g_i} \), and \( \tilde{f} - f^* \) is nonnegative on \( \tilde{S} \),

   then there exist positive constants \( \bar{c}, \bar{c} \) and the cardinality of \( \mathcal{A} \) with the semidefinite relaxation (1.9). It is easy to see that the size of each Gram matrix

\[
(1 + \|x\|^2)^k (\tilde{f} - f^* + \varepsilon) = \sum_{i \in [m]} \bar{c}_i \tilde{g}_i,
\]

for some \( \sigma \), being SOS of monomials such that \( \deg(\sigma, \tilde{g}_i) \leq 2(k + d_f) \). (Here \( \tilde{g}_m := 1 \).

Consequently, the resulting LP-hierarchy of lower bounds (\( \rho_k^{\text{Pol}} \)) for POP (1.9),

\[
\rho_k^{\text{Pol}} := \sup_{\lambda, u_i} \lambda \quad \text{s.t.} \quad \lambda \in \mathbb{R}, \ u_i \in \mathbb{R}^{k(n,k_i)}, \ i \in [m], \quad \theta^k(\tilde{f} - \lambda) = \sum_{i \in [m]} \bar{c}_i \tilde{g}_i \frac{u_i^T}{u_i},
\]

where \( \theta := 1 + \|x\|^2 \) and \( k_i := k + d_f - d_{g_i} \) for \( i \in [m] \), converges to \( f^* \) with a rate at least \( O(\varepsilon^{-t}) \). This linear hierarchy was originally described in Dickinson and Povh [13] and the novelty w.r.t. [13] is that we now provide a convergence rate.

Unfortunately, for large relaxation order \( k \), this LP is potentially ill-conditioned (see for instance Example 2). In order to address this issue, we replace each diagonal Gram matrix \( \sum_{i \in [n]} \bar{c}_i \tilde{g}_i \) in POP (1.5) by a Gram matrix of factor width at most \( s \) to obtain a semidefinite relaxation, which is tighter than LP (1.8). Namely, consider the following SDP indexed by \( k \in \mathbb{N} \) and \( s \in \mathbb{N}_{>0} \):

\[
\rho_k^{\text{Pol}} := \sup_{\lambda, G_{ij}} \lambda \quad \text{s.t.} \quad \lambda \in \mathbb{R}, \ G_{ij} \succeq 0, \ j \in [b(n,k)], \ i \in [m], \quad \theta^k(\tilde{f} - \lambda) = \sum_{i \in [m]} \bar{c}_i \tilde{g}_i \frac{1}{2} \sum_{j \in [b(n,k)]} G_{ij} \frac{1}{2} \sum_{i \in [m]} \bar{c}_i \tilde{g}_i,
\]

where each \( \mathcal{A}_{s,d} \subseteq \mathbb{N}_d^2 \), chosen as in Section 3.2, is such that \( \mathcal{A}_{s,d} \) covers \( \mathbb{N}_d^2 \), i.e.,

\[
\cup_{r=1}^{|b(n,d)|} \mathcal{A}_{s,d} = \mathbb{N}_d^2,
\]

and the cardinality of \( \mathcal{A}_{s,d} \) at most \( s \). Here \( \tilde{g}_m := 1 \). We call \( s \) the factor width upper bound associated with the semidefinite relaxation (1.10). It is easy to see that the size of each Gram matrix \( G_{ij} \) in (1.9) is at most \( s \). In addition, due to (1.10), we obtain the following estimate for every \( s \in \mathbb{N}_d \):

\[
\rho_k^{\text{Pol}} = \rho_k^{\text{Pol}} \leq \rho_{k,s} \leq f^*,
\]

so that for every fixed \( s \in \mathbb{N}_d \), \( \rho_k^{\text{Pol}} \to f^* \) as \( k \) increases, with a rate at least \( O(\varepsilon^{-t}) \). Notice that when \( s = 2, (1.10) \) becomes an SOCP thanks to [23], Lemma 15.

We emphasize that in our semidefinite relaxation (1.10), for fixed \( k \) the size of Gram matrices \( G_{ij} \) can be bounded from above by any \( s \in \mathbb{N}_d \) while the maximal matrix size of the standard semidefinite relaxation for POP (1.3) is fixed for each relaxation order \( k \). Nevertheless, since we convert (1.3) to the form (1.5) (so as to use Corollary 2), the degrees of the new resulting objective and constraint polynomials are doubled, i.e., \( \deg(f) = 2 \deg(f) \) and \( \deg(\tilde{g}) = 2 \deg(g) \).

However, numerical experiments in Sections 4 and 6 strongly suggest that our method works significantly better than existing methods on examples of POPs with nonnegative variables. For instance, for 20-variable dense POPs on the nonnegative orthant, the standard SOS-relaxations based on Putinar’s Positivstellensatz
provide a lower bound for \( f^* \) in about 356 seconds while we can provide a better lower bound in about 5 seconds.

Next, in Sections 6.6.1 and 6.6.2 we provide two convergent hierarchies of linear and semidefinite relaxations for large scale POPs on the nonnegative orthant, that exploit \textit{correlative sparsity}, and with properties similar to those in [13] and [53]. Accordingly, for POPs on the nonnegative orthant with up to 1000 variables, we can provide lower bounds in no more than 19 seconds which are better than those obtained in about 56360 seconds with the sparsity-adapted version of the standard SOS-relaxations of Waki et al. [54].

II. In our second contribution, we provide a degree bound for a Handelman-type Positivstellensatz for arbitrary compact basic semialgebraic sets. More explicitly, Corollary 3 states the following result. If

- \( f, g_1, \ldots, g_m \) are polynomials even in each variable,
- \( S \) defined as in (1.6) has nonempty interior, \( g_i = R - \|x\|^2 \) for some \( R > 0 \),
- \( g_i \) is of degree at most \( 2d_{g_i} \) and \( f - f^* \) is nonnegative on \( S \),

then there exist positive constants \( \bar{c} \) and \( c \) depending on \( f, g_i \) such that for all \( \varepsilon > 0 \), for all \( k \geq \bar{c} \varepsilon^{-4} \),

\[
(f - f^*) + \varepsilon = \sum_{i \in [m]} \sum_{j=0}^{k-d_{g_i}} \sigma_{ij} \tilde{g}_i \tilde{g}_j^j
\]

for some \( \sigma_{ij} \) being SOS of monomials such that \( \deg(\sigma_{ij} \tilde{g}_i \tilde{g}_j^j) \leq 2k \). (Here \( \tilde{g}_m := 1 \).)

When compared with the extension of Pólya’s Positivstellensatz in (1.7), our Handelman-type Positivstellensatz (1.12) does not have the multiplier \( (1 + \|x\|^2)^k \) but its number of SOS of monomials is increased to \( \sum_{i=1}^{m} (k - d_{g_i} + 1) \), which becomes larger when \( k \) increases. In contrast, the extension of Pólya’s Positivstellensatz involves the same multiplier and its number of SOS of monomials is \( m + 1 \), which does not depend on \( k \).

As a consequence, in Section 6.6.2 we obtain a rate of convergence for the hierarchy of linear relaxations (1.13) based on (1.12). In addition, we also propose the new hierarchy (3.22) of semidefinite relaxations based on even symmetry and the concept of factor width similarly to the one relying on Pólya’s Positivstellensatz. A sparse version of this semidefinite hierarchy is also obtained in Section 6.6.2.

In Sections 3 and 6.7 we compare the numerical behavior of these new dense and sparse hierarchies of semidefinite relaxations with that of the ones based on the extension of Pólya’s Positivstellensatz. In almost all cases, the ones based on the Handelman-type Positivstellensatz are several times slower but provide slightly better bounds.

**Related works**

**Exploiting sparsity:** Structure exploitation in (1.9) is comparable to term sparsity and correlative sparsity but here we can deal with dense POPs of the form (1.3). Moreover, the maximal block sizes involved in the sparsity-exploiting SDP relaxations mainly depend on the POP itself as well as on the relaxation order. By comparison, the maximal block size of our SDP relaxations is controllable. Under mild conditions, the rate of convergence \( \rho_{k,n} \to f^* \) as \( k \) increases, is at least \( \mathcal{O}(\varepsilon^{-4}) \).

**Dickinson–Povh’s hierarchy of linear relaxations:** Dickinson and Povh state in [12] a specific constrained version of Pólya’s Positivstellensatz. Explicitly, if \( f, g_1, \ldots, g_m \) are homogeneous polynomials, \( S \) is defined as in (1.4), and \( f \) is positive on \( S \setminus \{0\} \), then

\[
(\sum_{i \in [n]} x_i^j)^k f = \sum_{i \in [m]} \sigma_i g_i
\]

for some homogeneous polynomials \( \sigma_i \) with positive coefficients. (Here \( g_m := 1 \).) They also construct a hierarchy of linear relaxations associated with (1.13).

The extension of Pólya’s Positivstellensatz restated in Corollary 2 is indeed analogous to (1.13). However, the approach is different and importantly, the result is more convenient as we provide \textit{degree bounds} for the SOS of monomials involved in the representation. Similarly, our corresponding linear relaxations (3.6) are the analogues to those of Dickinson and Povh [13]. As shown in Example 2 and other examples in Sections 4 and 6.7 this hierarchy of linear relaxations usually have a poor numerical behavior in practice when \( k \) is large. Our new hierarchy of semidefinite relaxations (3.19) is used to improve this issue.

**DSOS and SDSOS:** In their recent work [1], Ahmadi and Majumdar describe the two convex cones DSOS and SDSOS as an alternative to the SOS cone. As the factor width of DSOS and SDSOS is at most 2, they are more tractable than the SOS cone. In the unconstrained case of POP (1.5), our semidefinite hierarchy based on the extension of Pólya’s Positivstellensatz can be seen as a generalization of DSOS and SDSOS while using the notion of factor width, see Remark 18. In fact, to obtain our semidefinite relaxations for the constrained case (1.5), we replace each SOS of monomials involved in the certificate (1.7) by an SOS polynomial whose Gram matrix has factor width at most \( s \); see Remark 10.
2 Representation theorems

In this section, we derive representations of polynomials nonnegative on semialgebraic sets together with degree bounds.

2.1 Polynomials nonnegative on general semialgebraic sets

Extension of Pólya’s Positivstellensatz: We analyze the complexity of the extension of Pólya’s Positivstellensatz in the following theorem:

**Theorem 1.** (Homogenized representation) Let \(g_1, \ldots, g_m\) be homogeneous polynomials such that \(g_1, \ldots, g_m\) are even in each variable. Let \(S\) be the semialgebraic set defined by

\[
S := \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \}. 
\]  \hspace{1cm} (2.1)

Let \(f\) be a homogeneous polynomial of degree \(2d_f\) for some \(d_f \in \mathbb{N}\) such that \(f\) is even in each variable and nonnegative on \(S\). Then the following statements hold:

1. For all \(\varepsilon > 0\), there exists \(K_{\varepsilon} \in \mathbb{N}\) such that for all \(k \geq K_{\varepsilon}\), there exist homogeneous SOS of monomials \(\sigma_i\) satisfying

\[
\deg(\sigma_0) = \deg(\sigma_1g_1) = \cdots = \deg(\sigma_mg_m) = 2(k + d_f) \hspace{1cm} (2.2)
\]

and

\[
\|x\|^{2k}(f + \varepsilon\|x\|^{2d_f}) = \sigma_0 + \sigma_1g_1 + \cdots + \sigma_mg_m. \hspace{1cm} (2.3)
\]

2. If \(S\) has nonempty interior, then there exist positive constants \(\bar{c}\) and \(\varepsilon\) depending on \(f, g_i\) such that for all \(\varepsilon > 0\), one can take \(K_{\varepsilon} = \bar{c}\varepsilon^{-4}\).

The proof of Theorem 1 is postponed to Section 6.2.

Note that some other homogeneous representations for globally nonnegative polynomials even in each variable have been studied in [18, 21, 10].

**Remark 1.** The Gram matrix associated with each SOS of monomials is diagonal. In other word, it is a block-diagonal matrix with maximal block size one. It would be interesting to know for which types of input polynomials we could obtain other representations involving SOS with block-diagonal Gram matrices of very small maximal block size, similarly to Theorem 1. Some of them have been discussed in [18, 20] that includes SOS of binomials, trinomials, tetranomials and SOS of any \(s\)-nomials. We emphasize that such representations allow one to build up SDP relaxations of small maximal matrix size that can be solved efficiently by using interior-point methods as shown later in Section 4.

The following corollary is a direct consequence of Theorem 1.

**Corollary 1.** (Dehomogenized representation) Let \(g_1, \ldots, g_m\) be polynomials even in each variable. Let \(S\) be the semialgebraic set defined by (2.1). Let \(f\) be a polynomial even in each variable and nonnegative on \(S\). Denote \(d_f := \lceil \deg(f)/2 \rceil + 1\). Then the following statements hold:

1. For all \(\varepsilon > 0\), there exists \(K_{\varepsilon} \in \mathbb{N}\) such that for all \(k \geq K_{\varepsilon}\), there exist SOS of monomials \(\sigma_i\) satisfying

\[
\deg(\sigma_0) \leq 2(k + d_f) \hspace{1cm} \text{and} \hspace{1cm} \deg(\sigma_i g_i) \leq 2(k + d_f), \quad i \in [m], \hspace{1cm} (2.4)
\]

and

\[
\vartheta^k(f + \varepsilon\vartheta^{d_f}) = \sigma_0 + \sigma_1g_1 + \cdots + \sigma_mg_m, \hspace{1cm} (2.5)
\]

where \(\vartheta := 1 + \|x\|^2\).

2. If \(S\) has nonempty interior, there exist positive constants \(\bar{c}\) and \(\varepsilon\) depending on \(f, g_i\) such that for all \(\varepsilon > 0\), one can take \(K_{\varepsilon} = \bar{c}\varepsilon^{-4}\).

The proof of Corollary 1 is similar to the proof of [21 Corollary 1].

2.2 Polynomials nonnegative on compact semialgebraic sets

In this section, we provide a representation of polynomials nonnegative on semialgebraic sets when the input polynomials are even in each variable. We also derive in Section 6.5 some sparse representations when the input polynomials have correlative sparsity.
2.2.1 Extension of Pólya’s Positivstellensatz

The following corollary is deduced from Corollary [1]

Corollary 2. Let \( f, g, S, d_s \) be as in Corollary [7] such that \( g_i := R - \|x\|_2^2 \) for some \( R > 0 \). Then the following statements hold:

1. For all \( \varepsilon > 0 \), there exists \( K_\varepsilon \in \mathbb{N} \) such that for all \( k \geq K_\varepsilon \), there exist SOS of monomials \( \sigma_i \) satisfying (2.13) and

\[
(1 + \|x\|_2^2)^k (f + \varepsilon) = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_m g_m .
\]

(2.6)

2. If \( S \) has nonempty interior, there exist positive constants \( \tilde{c} \) and \( c \) depending on \( f, g_i \) such that for all \( \varepsilon > 0 \), one can take \( K_\varepsilon = \tilde{c} \varepsilon^{-a} \).

Corollary [2] can be proved in the same way as [31 Corollary 2].

Remark 2. If we remove the multiplier \((1 + \|x\|_2^2)^k\) in (2.6), Corollary [2] is no longer true. Indeed, let \( n = 1 \), \( f := (x^2 - \frac{1}{4})^2 \) and assume that \( f = \sigma_0 + \sigma_1 (1 - x^2) \) for some SOS of monomials \( \sigma_i \), \( i = 0, 1 \). Note that \( f \) is even and positive on \([-1, 1]\). We write \( \sigma_i := a_i + b_i x^2 + x^4 r_i(x) \) for some \( a_i, b_i \in \mathbb{R}_+ \) and \( r_i \in \mathbb{R}[x] \). It implies that

\[
x^4 - 3x^2 + \frac{9}{4} = (a_0 + b_0 x^2 + x^4 r_0(x)) + (a_1 + b_1 x^2 + x^4 r_1(x))(1 - x^2) .
\]

(2.7)

Then we obtain the system of linear equations: \( \frac{9}{4} = a_0 + a_1 \) and \(-3 = b_0 - a_1 + b_1 \). Summing gives \(-\frac{3}{2} = a_0 + b_0 + b_1 \). However, \( a_0 + b_0 + b_1 \geq 0 \) since \( a_i, b_i \in \mathbb{R}_+ \). This contradiction yields the conclusion. However, we are now able to exploit term sparsity/even symmetry for Putinar’s Positivstellensatz in this case as shown later in Proposition [10].

It is not hard to see that with the multiplier \((1 + x^2)^2\), we obtain the Pólya’s Positivstellensatz as follows:

\[
(1 + x^2)^2 f = \tilde{\sigma}_0 + \tilde{\sigma}_1 (1 - x^2) ,
\]

(2.8)

where \( \tilde{\sigma}_0 := x^8 \) and \( \tilde{\sigma}_1 := x^4 + \frac{15}{4} x^2 + \frac{9}{4} \) are SOS of monomials.

We prove in the following Proposition the existence of block-diagonal Gram matrices in Putinar’s Positivstellensatz when the input polynomials are even in each variable:

Proposition 1. Let \( f, g_1, \ldots, g_m \) be polynomials in \( \mathbb{R}[x] \) such that \( f, g_i \) are even in each variable. Assume that there exists a decomposition:

\[
f = \sum_{i=1}^m g_i v_i^\top G^{(i)} v_i ,
\]

(2.9)

for some \( d_i \in \mathbb{N} \) and real symmetric matrices \( G^{(i)} = (G^{(i)}_{\alpha, \beta})_{\alpha, \beta \in N_{d_i}} \). For every \( i \in [m] \), define \( \tilde{G}^{(i)} := (\tilde{G}^{(i)}_{\alpha, \beta})_{\alpha, \beta \in N_{d_i}} \), where:

\[
\tilde{G}^{(i)}_{\alpha, \beta} := \begin{cases} G^{(i)}_{\alpha, \beta} & \text{if } \alpha + \beta \in 2\mathbb{N}^n , \\
0 & \text{otherwise} .
\end{cases}
\]

(2.10)

Then \( \tilde{G}^{(i)} \) are block-diagonal up to permutation and

\[
f = \sum_{i=1}^m g_i v_i^\top \tilde{G}^{(i)} v_i .
\]

(2.11)

Moreover, if \( G^{(i)} \succeq 0 \), then \( \tilde{G}^{(i)} \succeq 0 \).

Proof. The proof is inspired by [17 Section 8.1]. Removing all terms in (2.9) except the terms of monomials \( x^{2\alpha}, \alpha \in \mathbb{N}^n \), we obtain (2.11). It is due to the fact that \( f, g_i \) only have terms of the form \( x^{2\alpha}, \alpha \in \mathbb{N}^n \) and

\[
\sum_{\alpha, \beta \in \mathbb{N}^n} G^{(i)}_{\alpha, \beta} x^{\alpha + \beta} .
\]

(2.12)

Next, we show the block-diagonal structure of \( \tilde{G}^{(i)} \). For every \( \gamma \in \{0, 1\}^n \), define

\[
\Lambda^{(i)}_{\gamma} := \{ \alpha \in \mathbb{N}^n : \alpha - \gamma \in 2\mathbb{N}^n \} .
\]

(2.13)

Then \( \Lambda^{(i)}_{\gamma} \cap \Lambda^{(i)}_{\eta} = \emptyset \) if \( \gamma \neq \eta \) and \( N_{d_i} := \bigcup_{\gamma \in \{0, 1\}^n} \Lambda^{(i)}_{\gamma} \). In addition, for all \( \alpha, \beta \in \Lambda^{(i)}_{\gamma} \), \( \alpha + \beta \in 2\mathbb{N}^n \). Moreover, if \( \alpha, \beta \in N_{d_i} \) and \( \alpha + \beta \in 2\mathbb{N}^n \), then there exists \( \gamma \in \{0, 1\}^n \) such that \( \alpha, \beta \in \Lambda^{(i)}_{\gamma} \). It implies that all blocks on the diagonal of \( \tilde{G}^{(i)} \) must be

\[
(\tilde{G}^{(i)}_{\alpha, \beta})_{\alpha, \beta \in \Lambda^{(i)}_{\gamma}} \in \{0, 1\}^n .
\]

(2.14)

This yields the desired results.

Remark 3. The block-diagonal structure in Proposition [1] can be obtained by using TSSOS [38]. For general input polynomials \( f, g_i \), we cannot ensure that the maximal block size in this form is upper bounded or possibly goes to infinity as each \( d_i \) increases. However, as shown in Remark [5], we cannot obtain blocks of size one for this form. In order to improve this, we provide another representation with diagonal Gram matrices in the next corollary.
2.2.2 A Handelman-type Positivstellensatz

The following corollary is a consequence of Theorem 1.

**Corollary 3.** (Dense representation without multiplier) Let \( f, g_i, S \) be as in Corollary 1 such that \( g_i := R - \|x\|_2^2 \) for some \( R > 0 \) and \( g_m := 1 \). Denote \( d_g := \lfloor \deg(g_i)/2 \rfloor \). Then the following statements hold:

1. For all \( \varepsilon > 0 \), there exists \( K_\varepsilon \in \mathbb{N} \) such that for all \( k \geq K_\varepsilon \), there exist SOS of monomials \( \sigma_{i,j} \) satisfying
   \[
   \deg(\sigma_{i,j}g_i) \leq 2k
   \]
   and
   \[
   f + \varepsilon = \sum_{i=1}^{m} \sum_{j=0}^{k-d_g} \sigma_{i,j}g_i^j
   \]

2. If \( S \) has nonempty interior, then there exist positive constants \( \bar{c} \) and \( c \) depending on \( f, g_i \) such that for all \( \varepsilon > 0 \), one can take \( K_\varepsilon = \bar{c} \varepsilon^{-1} \).

**Proof.** Denote \( d_f := \lfloor \deg(f)/2 \rfloor + 1 \). With an additional variable \( x_{n+1} \), we first define the following homogeneous polynomials:

\[
\bar{f} := \|(x, x_{n+1})\|_2^{2d_f} f\left(\frac{x}{\|x, x_{n+1}\|_2}\right) \text{ and } \bar{g}_i := \|(x, x_{n+1})\|_2^{2d_g} g_i\left(\frac{x}{\|x, x_{n+1}\|_2}\right).
\]

It is not hard to prove that \( \bar{f} \) is nonnegative on the semialgebraic set \( \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : \bar{g}_i(x, x_{n+1}) \geq 0, i \in [m]\} \), so that Theorem 1 yields the representation

\[
\|(x, x_{n+1})\|_2^{2k} (\bar{f} + \varepsilon \|(x, x_{n+1})\|_2^{2d_f}) = \sigma_1 \bar{g}_1 + \cdots + \sigma_m \bar{g}_m,
\]

for some SOS of monomials \( \sigma_i \). By replacing \( x_{n+1} \) by \( \sqrt{R - \|x\|_2^2} \), we obtain the results.

**Remark 4.** The number of SOS of monomials in the representation (2.19) is \( \sum_{i=0}^{m} (k - d_g + 1) \) which becomes larger when \( k \) increases, while the number of SOS of monomials in the representation (2.8) is \( m + 1 \), which does not depend on \( k \). However, a large number of Gram matrices is not a computational issue, since the complexity of interior-point methods mainly depend on the maximal block size of the Gram matrices and are still efficient even when their number is large.

**Remark 5.** With \( f \) being defined as in Remark 3, the following decomposition is an instance of the Handelman-type Positivstellensatz:

\[
f = \eta_0 + \eta_1(1 - x^2) + \eta_2(1 - x^2)^2,
\]

where \( \eta_0 = \frac{1}{4}, \eta_1 = \eta_2 = 1 \) are SOS of monomials. Note that the degrees of these SOS of monomials are zero while the degrees of the ones from (2.8) for the extension of Pólya’s Positivstellensatz are 8 and 4.

**Remark 6.** In Section 6.3, we provide some variations of Pólya’s and Handelman-type Positivstellensatz where the input polynomials are not required to be even in each variable. Moreover, the weighted SOS polynomials of these representations are still associated with Gram matrices of factor width one thanks to a change of monomial basis.

3 Polynomial optimization on the nonnegative orthant: Compact case

This section is concerned with some applications of (i) the extension of Pólya’s Positivstellensatz (2.9) and (ii) the Handelman-type Positivstellensatz (2.10), for polynomial optimization on compact semialgebraic subsets of the nonnegative orthant. The noncompact case is postponed to Section 6.4. Moreover, Section 6.5 is devoted to some applications of the sparse representation provided in Section 6.3 for polynomial optimization with correlative sparsity.

Consider the following POP:

\[
f^* := \inf_{x \in S} f(x),
\]

where \( f \in \mathbb{R}[x] \) and

\[
S = \{ x \in \mathbb{R}^n : x_j \geq 0, j \in [n], g_i(x) \geq 0, i \in [m] \},
\]

for some \( g_i \in \mathbb{R}[x], i \in [m] \), with \( g_m = 1 \). Throughout this section, we assume that \( f^* > -\infty \) and problem (3.1) has an optimal solution \( x^* \).
Remark 7. Every general POP in variable $x = (x_1, \ldots, x_n)$ can be converted to the form (3.1) with $S$ as in (2.2), by replacing each variable $x_i$ by the difference of two new nonnegative variables $x_i^+ - x_i^-$. If there are several constraints $x_i \geq a_i$, we can obtain an equivalent POP on the nonnegative orthant by defining new nonnegative variables $y_i := x_i - a_i$. In particular, we can easily convert a POP over a compact semialgebraic set to POP over the nonnegative orthant by changing the coordinate via an affine transformation. However, we restrict ourselves to POPs on the nonnegative orthant in this paper.

Recall that $\hat{q}(x) := q(x^2)$, for a given polynomial $q$. In this case, $\hat{q}$ is even in each variable. Then POP (3.1) is equivalent to

$$f^* := \inf_{x \in \hat{S}} \hat{f},$$

where

$$\hat{S} = \{ x \in \mathbb{R}^n : \hat{g}_i(x) \geq 0, i \in [m] \},$$

with $x^{2\theta}$ being an optimal solution.

Let $\theta := 1 + \|x\|_2^2$. Denote $d_f := \deg(f) + 1$, $d_{g_i} := \deg(g_i), i \in [m]$.

3.1 Linear relaxations

3.1.1 Based on the extension of Pólya's Positivstellensatz

Consider the hierarchy of linear programs indexed by $k \in \mathbb{N}$:

$$\tau_k^{Pol} := \inf_{y} L_y(\theta^k \hat{f})$$

s.t. $y = (y_{\alpha})_{\alpha \in \mathbb{N}^n_{d_f+k}} \subseteq \mathbb{R}^n, L_y(\theta^k) = 1,$

$$\text{diag}(M_{\lambda}(\hat{y}(y))) \in \mathbb{R}^{n(k+1)}_{+, i \in [m]},$$

where $k_i := k + d_f - d_{g_i}, i \in [m]$. Note that $\hat{g}_m = 1$.

Remark 8. The optimal value $\tau_k^{Pol}$ only depends on the subset of variables $\{y_{2\alpha} : \alpha \in \mathbb{N}^n_{d_f+k}\}$, i.e., the optimal value of LP (3.5) does not change when we assign each of the other variables with any real number. It is due to the fact that $\theta, \hat{f}, \hat{g}_i$ only have nonzero coefficients associated to the monomials $x^{2\alpha}$ for some $\alpha \in \mathbb{N}^n$.

Theorem 2. Let $f, g_i \in \mathbb{R}[x], i \in [m]$, with $g_m = 1$ and $g_1 := R - \sum_{j \in [n]} x_j$ for some $R > 0$. Consider POP (3.1) with $S$ being defined as in (3.2). For every $k \in \mathbb{N}$, the dual of (3.5) reads as:

$$\rho_k^{Pol} := \sup_{\lambda, u_i} \frac{\lambda}{\theta^k}$$

s.t. $\lambda \in \mathbb{R}, u_i \in \mathbb{R}^{n(k+1)}_{+, i \in [m]}$,

$$\theta^k(f - \lambda) = \sum_{i \in [m]} \hat{g}_i u_i \text{diag}(u_i) v_{k_i},$$

The following statements hold:

1. For all $k \in \mathbb{N}$,

$$\rho_k^{Pol} \leq \rho_{k+1}^{Pol} \leq f^*.$$  

2. The sequence $(\rho_k^{Pol})_{k \in \mathbb{N}}$ converges to $f^*$.

3. If $S$ has nonempty interior, there exist positive constants $\bar{c}$ and $c$ depending on $f, g_i$ such that $0 \leq f^* - \rho_k^{Pol} \leq (\bar{c}/c)^k$.

The proof of Theorem 2 relies on Corollary 2 and can be proved in almost the same way as the proof of [31] Theorem 4.1.

3.1.2 Based on the Handelman-type Positivstellensatz

Consider the hierarchy of linear programs indexed by $k \in \mathbb{N}$:

$$\tau_k^{Han} := \inf_{y} L_y(\hat{f})$$

s.t. $y = (y_{\alpha})_{\alpha \in \mathbb{N}^n_{2k}} \subseteq \mathbb{R}^n, y_0 = 1,$

$$\text{diag}(M_{\lambda}(\hat{g}_i, y(y))) \in \mathbb{R}^{n(k+1)}_{+, i \in [m], j \in \{0\} \cup [k - d_{g_i}]},$$

where $k_{ij} := k - d_{g_i} - j$, for $i \in [m], j \in \{0\} \cup [k - d_{g_i}]$. Note that $\hat{g}_m = 1$. 

8
**Theorem 3.** Let $f, g_i \in \mathbb{R}[x], \ i \in [m]$, with $g_m = 1$ and $g_1 := R - \sum_{i \in [m]} x_i$ for some $R > 0$. Consider POP \( [3.1] \) with $S$ being defined as in \([3.2]\). For every $k \in \mathbb{N}$, the dual of \([3.2]\) reads as:

$$
\rho_{k, n}^\text{Han} \coloneqq \sup_{\lambda, u_{ij}} \lambda \quad \text{s.t.} \quad \lambda \in \mathbb{R}, \ y_{ij} \in \mathbb{R}_{+}^{n_{x,j}(\alpha_k, \lambda_i)} \quad \text{for } i \in [m], \ j \in \{0\} \cup [k - d_{n_x}],
$$

$$
\frac{f - \lambda}{\sum_{i \in [m]} \sum_{j = 0}^{k - d_{n_x}} \frac{1}{2} g_{ij}^T \mathbf{v}_{ij}^T \text{diag}(u_{ij}) \mathbf{v}_{ij}.}
$$

The following statements hold:

1. For all $k \in \mathbb{N}$, $\rho_{k, n}^\text{Han} \leq \rho_{k+1, n}^\text{Han} \leq f^*$.

2. The sequence $(\rho_{k, n}^\text{Han})_{k \in \mathbb{N}}$ converges to $f^*$.

3. If $S$ has nonempty interior, there exist positive constants $\bar{c}$ and $c$ depending on $f, g_i$, such that $0 \leq f^* - \rho_{k, n}^\text{Han} \leq (\frac{1}{k})^{-\bar{c}}$.

The proof of Theorem 3 relies on Corollary 2 and can be proved in almost the same way as the proof of Theorem 1.

### 3.2 Semidefinite relaxations

In this subsection, we construct the sparsity pattern $A_1^{(s, d)} \subset \mathbb{N}^d_+$ inspired by even symmetry reduction in Proposition 1.

We write $\mathbb{N}^d = \{\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_{r+1}, \ldots\}$ such that

$$
\alpha_1 < \alpha_2 < \cdots \alpha_r < \alpha_{r+1} < \cdots
$$

Let

$$
W_j := \{i \in \mathbb{N} : i \geq j, \ \alpha_i + \alpha_j \in 2\mathbb{N}^d\}, \quad j \in \mathbb{N}_{>0}.
$$

Then for all $j \in \mathbb{N}_{>0}$, $W_j \neq \emptyset$ since $j \in W_j$. For every $j \in \mathbb{N}$, we write $W_j := \{i^{(j)}, i^{(j)}, \ldots\}$ such that $j = i^{(j)} < i^{(j)} < \ldots$. Let

$$
\mathcal{T}_j^{(s, d)} := \{\alpha_{i^{(j)}}, \ldots, \alpha_{i^{(j)}}\} \cap \mathbb{N}^d_+ \quad j \in \mathbb{N}_{>0}, d \in \mathbb{N}.
$$

For every $s \in \mathbb{N}_{>0}$ and $d \in \mathbb{N}$, define $A_s^{(s, d)} := \mathcal{T}_1^{(s, d)}$ and for $j = 2, \ldots, b(n, d)$, define

$$
A_s^{(s, d)} := \begin{cases}
\mathcal{T}_j^{(s, d)} & \text{if } \mathcal{T}_j^{(s, d)} \neq \emptyset, \forall \ell \in [j - 1], \\
\emptyset & \text{otherwise}.
\end{cases}
$$

Note that $\bigcup_{j=1}^{b(n, d)} A_j^{(s, d)} = \mathbb{N}^d_+$ and $|A_j^{(s, d)}| \leq s$. Here $| \cdot |$ stands for the cardinality of a set. Then the sequence

$$
(A + \beta)_{(A + \beta) \in [\alpha_{s+2}^{(s+2)}]} , j \in [b(n, d)]
$$

are overlapping blocks of size at most $s$ in $(\alpha + \beta)_{(\alpha + \beta) \in [\alpha_{s+2}^{(s+2)}]}$. Note that $\alpha + \beta \in \mathbb{N}^d$ for all $\alpha, \beta \in A_1^{(s, d)}$.

**Example 1.** Consider the case of $n = d = s = 2$. The matrix $(\alpha + \beta)_{(\alpha, \beta) \in A^{(s, d)}_1}$ can be written explicitly as

$$
\begin{bmatrix}
(0, 0) & (1, 0) & (0, 1) & (2, 0) & (1, 1) & (0, 2) \\
(1, 0) & (2, 0) & (1, 1) & (3, 0) & (2, 2) & (1, 2) \\
(0, 1) & (1, 1) & (2, 1) & (2, 2) & (1, 2) & (0, 3) \\
(2, 0) & (3, 0) & (2, 1) & (4, 0) & (3, 1) & (2, 2) \\
(1, 1) & (2, 1) & (2, 1) & (3, 1) & (2, 2) & (1, 3) \\
(0, 2) & (1, 2) & (0, 3) & (2, 2) & (1, 3) & (0, 4)
\end{bmatrix}.
$$

In this matrix, the entries in bold belong to $2\mathbb{N}^2$. Then $W_1 = \{1, 4, 6\}$. Since $s = 2$, we get $A_1^{(2, 2)} = \{(0, 0), (2, 0)\}$. Similarly, we can obtain $A_1^{(2, 2)} = \{(1, 0)\}$, $A_1^{(2, 2)} = \{(0, 1)\}$, $A_1^{(2, 2)} = \{(2, 0), (0, 2)\}$, $A_1^{(2, 2)} = \{(1, 1)\}$ and $A_1^{(2, 2)} = \emptyset$. The blocks $(\alpha + \beta)_{(\alpha + \beta) \in A_1^{(2, 2)}}$, $j \in [5]$, are as follows:

$$
\begin{bmatrix}
(0, 0) & (2, 0) \\
(2, 0) & (4, 0)
\end{bmatrix}, \quad
\begin{bmatrix}
(0, 2) \\
(2, 2)
\end{bmatrix}, \quad
\begin{bmatrix}
(2, 2) \\
(2, 2)
\end{bmatrix}, \quad
\begin{bmatrix}
(4, 0) & (2, 2) \\
(2, 2) & (0, 4)
\end{bmatrix}.
$$

For all $\mathcal{B} = \{\beta_1, \ldots, \beta_{r}\} \subset \mathbb{N}^d_+$ such that $\beta_1 < \cdots < \beta_{r}$, for every $h = \sum_{\gamma} h_{\gamma} x_{\gamma} \in \mathbb{R}[x]$ and for every $y = (y_{h})_{h \in \mathbb{N}^d} \subset \mathbb{R}$, let us define

$$
\mathbf{v}_{\mathcal{B}} := \begin{bmatrix} x_{\beta_1} \\ \vdots \\ x_{\beta_r} \end{bmatrix} \quad \text{and} \quad M_{\mathcal{B}}(hy) := (\sum_{\gamma} h_{\gamma} y_{\gamma + \beta_i + \beta_j})_{i, j \in [r]}.
$$
3.2.1 Based on the extension of Pólya’s Positivstellensatz

Consider the hierarchy of semidefinite programs indexed by \( s \in \mathbb{N}_{>0} \) and \( k \in \mathbb{N} \):

\[
\tau_{k,s}^{\text{Pol}} := \inf_{y} L_{y}(\theta^{k} \hat{f})
\]

\[\text{s.t. } y = (y_{\alpha})_{\alpha \in \mathbb{N}_{+}^{d(k)}} \subset \mathbb{R}, \quad L_{y}(\theta^{k}) = 1,
\]

\[
\mathbf{M}_{A_{\tau}^{(s,k)}}(\hat{g}_{i}y) \succeq 0, \quad i \in [b(n,k)], \quad i \in [m],
\]

where \( k_{i} := k + d_{f} - d_{y}, \quad i \in [m] \). Here \( \hat{g}_{m} = 1 \).

**Remark 9.** If we assume that \( \theta = 1 \), then (3.18) becomes a moment relaxation based on Putinar’s Positivstellensatz for POP \([55]\). Here each constraint \( \mathbf{M}_{A_{\tau}^{(s,k)}}(\hat{g}_{i}y) \succeq 0 \) is replaced by the constraint \( \mathbf{M}_{A_{\tau}^{(s,k)}}(\hat{g}_{i}y) \succeq 0 \). If \( s \) is large enough, (3.18) corresponds to an SDP relaxation obtained after exploiting term sparsity (see \([63]\)).

**Theorem 4.** Let \( f, g_{i} \in \mathbb{R}[x], \quad i \in [m] \), with \( g_{m} = 1 \) and \( g_{1} := R - \sum_{j \in [n]} x_{j} \) for some \( R > 0 \). Consider POP \([51]\) with \( S \) being defined as in \([52]\). For every \( s \in \mathbb{N}_{>0} \) and for every \( k \in \mathbb{N} \), the dual of (3.18) reads as:

\[
\rho_{k,s}^{\text{Pol}} := \sup_{\lambda} \mathbf{G}_{ij} \quad \text{s.t. } \lambda \in \mathbb{R}, \quad \mathbf{G}_{ij} \succeq 0, \quad j \in [b(n,k)], \quad i \in [m],
\]

\[
\theta^{k}(\lambda) = \sum_{i \in [m]} \hat{g}_{i}(\sum_{j \in [b(n,k)]} \mathbf{v}_{i}^{\top} A_{ij} v_{ij}^*(\lambda_{i} x_{ij})).
\]

The following statements hold:

1. For all \( k \in \mathbb{N} \) and for every \( s \in \mathbb{N}_{>0} \), \( \rho_{k,s}^{\text{Pol}} = \rho_{k,1}^{\text{Pol}} \leq \rho_{k,s}^{\text{Pol}} \leq f^{*} \).
2. For every \( s \in \mathbb{N}_{>0} \), the sequence \( (\rho_{k,s}^{\text{Pol}})_{k \in \mathbb{N}} \) converges to \( f^{*} \).
3. If \( S \) has nonempty interior, there exist positive constants \( \tilde{c} \) and \( \epsilon \) depending on \( f, g_{i} \) such that for every \( s \in \mathbb{N}_{>0} \) and for every \( k \in \mathbb{N} \), \( 0 \leq f^{*} - \rho_{k,s}^{\text{Pol}} \leq (\tilde{c})^{-\epsilon} \).
4. If \( S \) has nonempty interior, for every \( k \in \mathbb{N} \) and for every \( s \in \mathbb{N}_{>0} \), strong duality holds for the primal-dual problems (3.18)-(3.19).

**Proof.** It is not hard to prove the first statement. The second and third one are due to the first statement of Theorem 2. The final statement is proved similarly to the third statement of \([31]\) Theorem 4. \( \square \)

**Remark 10.** In order to construct the semidefinite relaxation (3.19), the SOS of monomials in the linear relaxation (3.6) are replaced by a sum of several SOS polynomials associated to Gram matrices of small sizes. This idea is inspired by \([71]\), where the authors replace the first nonnegative scalar by an SOS polynomial in the linear relaxation based on Krivine-Stengle’s Positivstellensatz.

**Remark 11.** At fixed \( s \in \mathbb{N}_{>0} \), the sequence \( (\rho_{k,s}^{\text{Pol}})_{k \in \mathbb{N}} \) may not be monotonic w.r.t. \( k \), and similarly at fixed \( k \in \mathbb{N} \).

**Example 2.** (AM-GM inequality) Consider the case where \( n = 3, \quad f = x_{1} + x_{2} + x_{3} \) and \( S = \{ x \in \mathbb{R}^{3} : x_{j} \geq 0, \quad j \in [3], \quad x_{1}x_{2}x_{3} - 1 \geq 0, \quad 3 - x_{1} - x_{2} - x_{3} \geq 0 \} \). Using AM-GM inequality, we have

\[
f(x) \geq 3(x_{1}x_{2}x_{3})^{1/3} \geq 3, \quad \forall x \in S,
\]

yielding \( f^{*} = 3 \). We solve SDP (3.18) with MOSEK and report the corresponding numerical results in Table 7. The table displays \( \tau_{3}^{\text{Pol}} = 2.9999 \) which is very close to \( f^{*} \). However, \( \tau_{1}^{\text{Pol}} = 1.5030 \) is smaller than \( \tau_{1}^{\text{Pol}} = 2.4000 \), which violates the theoretical inequality (3.7). The underlying reason is that the matrix \( A \) used to define the convex polytope \( P = \{ x \in \mathbb{R}^{n} : x \geq 0, \quad Ax \leq b \} \) in the equivalent form \( \min_{x \in P} c^{\top} x \) of LP (3.5) is ill-conditioned, and the solver is not able to accurately solve the LP corresponding to \( \tau_{1}^{\text{Pol}} \).

3.2.2 Based on the Handelman-type Positivstellensatz

Consider the hierarchy of semidefinite programs indexed by \( s \in \mathbb{N}_{>0} \) and \( k \in \mathbb{N} \):

\[
\tau_{k,s}^{\text{Han}} := \inf_{y} L_{y}(\hat{f})
\]

\[\text{s.t. } y = (y_{\alpha})_{\alpha \in \mathbb{N}_{+}^{d(k)}} \subset \mathbb{R}, \quad y_{0} = 1,
\]

\[
\mathbf{M}_{A_{\tau}^{(s,k)}}(\hat{g}_{i}g^{*}_{i}y) \succeq 0, \quad i \in [b(n,k)], \quad i \in [m], \quad j \in \{0\} \cup \{k - d_{y}\},
\]

where \( k_{i} := k - d_{y} - j, \quad i \in [m], \quad j \in \{0\} \cup \{k - d_{y}\} \). Note that \( \hat{g}_{m} = 1 \).
Table 1: Numerical values (in the first subtable) and computing time (in the second subtable) for $\tau_{k,s}^{Pol}$ in Example 2.

| $s$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|
| 0   | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1   | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2   | 0.0000 | 0.0000 | 0.4999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 3   | 1.0000 | 0.9999 | 0.9999 | 2.7454 | 2.8368 | 2.8383 | 2.9999 | 2.9999 |
| 4   | 1.4399 | 1.4999 | 1.4999 | 1.4999 | 1.4999 | 1.4999 | 2.9999 | 2.9999 |
| 5   | 1.8615 | 1.9961 | 1.9999 | 1.9999 | 1.9999 | 1.9999 | 2.9999 | 2.9999 |
| 6   | 2.1999 | 2.4526 | 2.4998 | 2.4999 | 2.4999 | 2.4999 | 2.4999 | 2.4999 |
| 7   | 2.3971 | 2.8663 | 2.9950 | 2.9996 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 8   | 2.4109 | 1.4999 | 1.4999 | 1.4999 | 1.4999 | 1.4999 | 2.9999 | 2.9999 |
| 9   | 2.5161 | 2.9372 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 10  | 2.5896 | 2.9520 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 11  | 2.6210 | 2.9607 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 12  | 2.6937 | 2.9615 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 13  | 2.7330 | 2.9662 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 14  | 2.7390 | 2.9687 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 15  | 2.7040 | 2.9607 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 16  | 2.4000 | 2.9710 | 2.9997 | 2.9997 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 17  | 1.5030 | 2.9723 | 2.9998 | 2.9998 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 18  | 0.5833 | 2.9732 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 19  | 0.8121 | 0.0000 | 0.0000 | 2.9995 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
| 20  | 0.7457 | 0.0000 | 0.0000 | 2.9994 | 2.9999 | 2.9999 | 2.9999 | 2.9999 |
Theorem 5. Let \( f, g_i \in \mathbb{R}[x], i \in [m] \), with \( g_m = 1 \) and \( g_1 := R - \sum_{j \in [n]} x_j \) for some \( R > 0 \). Consider POP (3.1) with \( S \) being defined as in (3.2). For every \( s \in \mathbb{N}_{>0} \) and for every \( k \in \mathbb{N} \), the dual of (3.2) reads as:

\[
\rho_{\text{Han}}^k := \sup_{\lambda, G_{ijr} \geq 0} \lambda
\begin{align*}
\text{s.t.} & \quad \lambda \in \mathbb{R}, G_{ijr} \geq 0, j \in [b(n,k)], i \in [m], j \in \{0\} \cup [k - d_i], \\
& \quad \hat{f} - \lambda = \sum_{i \in [m]} \sum_{j=0}^{k-d_i} g_j \hat{g}_j^i \left( \sum_{r \in [b(n,k)]} v_i^{(s,k)} G_{ijr} v_i^{(a)} \right).
\end{align*}
\]

(3.22)

The following statements hold:

1. For all \( k \in \mathbb{N} \) and for every \( s \in \mathbb{N}_{>0} \), \( \rho_{\text{Han}}^k = \rho_{\text{Han}}^{k+1} \leq \rho_{\text{Han}}^{k+2} \leq f^* \).

2. For every \( s \in \mathbb{N}_{>0} \), the sequence \( \rho_{\text{Han}}^k \) converges to \( f^* \).

3. If \( S \) has nonempty interior, there exist positive constants \( \bar{c} \) and \( \varepsilon \) depending on \( f, g_i \) such that for every \( s \in \mathbb{N}_{>0} \) and for every \( k \in \mathbb{N} \), \( 0 \leq f^* - \rho_{\text{Han}}^k \leq (\bar{c})^{-\varepsilon} \).

4. If \( S \) has nonempty interior, for every \( k \in \mathbb{N} \) and for every \( s \in \mathbb{N}_{>0} \), strong duality holds for the primal-dual problems (3.21)–(3.22).

The proof of Theorem 5 is based on Theorem 3 and similar to the proof of Theorem 4.

Remark 12. To make the use of the Handelman-type Positivstellensatz, we need at least one ball constraint \( g_1 := R - ||x||_2^2 \) for some \( R > 0 \). Thus, Theorem 5 is applicable only when the domain \( S \) of POP (3.1) is compact. To deal with the noncompact case, we might combine it with the so-called “big ball trick”.

3.3 Obtaining an optimal solution

A real sequence \( (y_\alpha)_{\alpha \in \mathbb{N}^n} \) has a representing measure if there exists a finite Borel measure \( \mu \) such that \( y_\alpha = \int_{\mathbb{R}^n} x^\alpha d\mu(x) \) is satisfied for every \( \alpha \in \mathbb{N}^n \).

Next, we discuss about the extraction of an optimal solution \( x^* \) of POP (3.1) from the optimal solution \( y = (y_\alpha)_{\alpha \in \mathbb{N}^n_{(d+k)}} \) of the semidefinite relaxations (3.18).

Remark 13. A naive idea is to define the new sequence of moments \( u = (u_\alpha)_{\alpha \in \mathbb{N}^n_{(d+k)}} \) given by \( u_\alpha := y_\alpha^2 \), for \( \alpha \in \mathbb{N}^n_{(d+k)} \). Obviously, if \( y \) has a representing Dirac measure \( \delta_{x^*} \), then \( u \) has a representing Dirac measure \( \delta_{x^*^2} \). In this case, we take \( x^* := x^2 \). However, there is no guarantee that \( u \) has a representing measure in general even if \( y \) has one.

Based on [33], Remark 3.4, we use the heuristic extraction algorithm presented in Algorithm 1.

4 Numerical experiments

In this section, we report results of numerical experiments obtained by solving the Moment-SOS relaxations of some random and nonrandom instances of POP (1.3). Other results for POP (1.3) with correlative sparsity can be found in Section 5.7. Notice that our relaxations from Section 3 are to deal with dense POPs while the ones from Section 4.6 are for POPs with correlative sparsity.

For numerical comparison purposes, recall the semidefinite relaxation based on Putinar’s Positivstellensatz for solving POP (1.3) indexed by \( k \in \mathbb{N} \):

\[
\tau_k^{\text{Put}} := \inf_{x} L_\lambda(f)
\begin{align*}
\text{s.t.} & \quad \mathcal{M}_{k-|g_i|}(g,y) \succeq 0, i \in [m].
\end{align*}
\]

(4.1)

Here \( m := m + n \), \( g_{m+k} := x_i \), \( j \in [n] \), and \( g_m := 1 \). As shown by Baldi and Mourrain [33], the sequence \( \tau_k^{\text{Put}} \) converges to \( f^* \) with the rate of at least \( O(e^{-\varepsilon}) \) when POP (1.3) has a ball constraint, e.g., \( g_1 := R - ||x||_2^2 \) for some \( R > 0 \). If \( g_1 := R - \sum_{j \in [n]} x_j \) for some \( R > 0 \), then \( \tau_k^{\text{Put}} \) still converges to \( f^* \) due to Jacobi-Prestel [23] Theorem 4.2) (see also [8] Theorem 1 (JP)).

Remark 14. If we assume that \( g_1 := R - \sum_{j \in [n]} x_j \) for some \( R > 0 \), SDP (4.1) may be unbounded when \( k \) is too small since its variable \( y \) is possibly unbounded. This issue occurs later on, see, e.g., Section 4.7. However, if we assume that \( g_1 := R - ||x||_2^2 \) for some \( R > 0 \), then SDP (4.1) is feasible for any order \( k \geq 1 \) (see Section 5.4).
Algorithm 1 Extraction algorithm for POPs on the nonnegative orthant

Input: precision parameter $\varepsilon > 0$ and an optimal solution $(\lambda, G_{ij})$ of SDP (3.19).
Output: an optimal solution $x^*$ of POP (3.1).

1. For $j \in [\lfloor n, km \rfloor]$, let $G_j = (w_{pq}^{(j)})_{p,q \in \mathbb{N}_m}$ such that $(w_{pq}^{(j)}) = G_j$ and $w_{pq} = 0$ if $(p,q) \notin (A^{(s,km)})^2$. Then $G_j \geq 0$ and

$$v_{N_{km}}^T G_j v_{N_{km}} = v_{A^{(s,km)}} v_{A^{(s,km)}};$$  

(3.23)

2. Let $G := \sum\{j \in [\lfloor n, km \rfloor] \mid G_j\}$. Then $G$ is the Gram matrix corresponding to $\sigma_m$ in the SOS decomposition

$$\theta^k(f - \lambda) = \sum_{i \in [m]} \bar{g}_i \sigma_i,$$

(3.24)

where $\sigma_i$ are SOS polynomials and $\bar{g}_m = 1$;

3. Obtain an atom $z^* \in \mathbb{R}^n$ by using the extraction algorithm of Henrion and Lasserre in [22], where the matrix $V$ in [22] (6) is taken such that the columns of $V$ form a basis of the null space $\{u \in \mathbb{R}^{\alpha_k} : Gu = 0\}$;

4. Verify that $z^*$ is an approximate optimal solution of POP (3.3) by checking the following inequalities:

$$|\bar{f}(z^*) - \lambda| \leq \varepsilon \|\bar{f}\|_{\text{max}}$$

and $\bar{g}_i(z^*) \geq -\varepsilon \|\bar{g}_i\|_{\text{max}}$, $i \in [m],$

(3.25)

where $\|q\|_{\text{max}} := \max_{q \in \mathbb{R}[x]}$ for any $q \in \mathbb{R}[x]$.

5. If the inequalities (3.25) hold, set $x^* := z^{*2}$.

The experiments are performed in Julia 1.3.1. We rely on TSSOS [53] to solve the Moment-SOS relaxations of sparse POPs.

The implementation of our method is available online via the link:

[https://github.com/maihoanganh/InterRelax](https://github.com/maihoanganh/InterRelax)

We use a desktop computer with an Intel(R) Core(TM) i7-8665U CPU @ 1.9GHz × 8 and 31.2 GB of RAM. The notation for the numerical results is given in Table 4.

### 4.1 Dense QCQPs

**Test problems:** We construct randomly generated dense quadratically constrained quadratic programs (QCQPs) in the form (1.3)-(1.4) as follows:

1. Take $a$ in the simplex

$$\Delta_n := \{x \in \mathbb{R}^n : x_j \geq 0, j \in [n], \sum_{j \in [n]} x_j \leq 1\}$$  

(4.2)

w.r.t. the uniform distribution.

2. Let $g_1 := 1 - \sum_{j \in [n]} x_j$ and $g_2 := 1$.

3. Take every coefficient of $f$ and $g_i$, $i = 2, \ldots, m$, in $(-1, 1)$ w.r.t. the uniform distribution.

4. Update $g_i(x) := g_i(x) - g_i(a) + 0.125$, for $i = 2, \ldots, m_{\text{ineq}}$.

5. Update $g_{i+m_{\text{ineq}}}(x) := g_{i+m_{\text{ineq}}}(x) - g_{i+m_{\text{ineq}}}(a)$ and set $g_{i+m_{\text{ineq}}+m_{\text{eq}}} = -g_{i+m_{\text{ineq}}}$, for $i \in [m_{\text{eq}}]$.

Here $m = m_{\text{eq}} + 2m_{\text{ineq}}$ with $m_{\text{ineq}}$ (resp. $m_{\text{eq}}$) being the number of inequality (resp. equality) constraints except the nonnegative constraints $x_j \geq 0$. If $m_{\text{eq}} = 2$ and $m_{\text{eq}} = 0$, we obtain the case of the minimization of a polynomial on the simplex $\Delta_n$. The point $a$ is a feasible solution of POP (1.3).

The numerical results are displayed in Table 3.

**Discussion:** Table 3 shows that Pól and Han are typically faster and more accurate than Put. For instance, when $n = 20$, $m_{\text{ineq}} = 5$ and $m_{\text{eq}} = 0$, Put takes 342 seconds to return the lower bound $-0.350601$ for $f^*$, while Pól only takes 9 seconds to return the better bound $-0.265883$ and an approximate optimal solution. It is due to the fact that Pól has 44 matrix variables with the maximal matrix size 20, while Put has 25 matrix variables with the maximal matrix size 231 in this case. In addition, Han provides slightly better bounds than Pól in Pb 3 and the same bounds with Pól in the others. Moreover, Pól runs about five times faster than Han in Pb 5 and 6.
Table 2: The notation

| Pb | the ordinal number of a POP instance |
| Id | the ordinal number of an SDP instance |
| n | the number of nonnegative variables in POP (1.3) |
| m_{ineq} | the number of inequality constraints of the form \( g_i \geq 0 \) in POP (1.3) |
| m_{eq} | the number of equality constraints of the form \( g_i = 0 \) in POP (1.3) |
| Put | the SDP relaxation based on Putinar’s Positivstellensatz (4.1) modeled by TSSOS and solved by Mosek 9.1 |
| Pól | the SDP relaxation based on the extension of Pólya’s Positivstellensatz (3.19) modeled by our software InterRelax and solved by Mosek 9.1 |
| Han | the SDP relaxation based on the Handelman-type Positivstellensatz (3.22) modeled by our software InterRelax and solved by Mosek 9.1 |
| k | the relaxation order |
| s | the factor width upper bound used in SDP (1.9) |
| nmat | the number of matrix variables of an SDP |
| msize | the largest size of matrix variables of an SDP |
| nscal | the number of scalar variables of an SDP |
| naff | the number of affine constraints of an SDP |
| val | the value returned by the SDP relaxation |
| * | there exists at least one optimal solution of the POP, which can be extracted by Algorithm 1 |
| − | the calculation runs out of space |
| ∞ | the SDP relaxation is unbounded or infeasible |

4.2 Stability number of a graph

In order to compute the stability number \( \alpha(G) \) of a given graph \( G \), we solve the following POP on the unit simplex:

\[
\alpha(G) = \min_{x \in \mathbb{R}^n_+} \{ x^T(A + I)x : \sum_{j=1}^n x_j = 1 \},
\]

where \( A \) is the adjacency matrix of \( G \) and \( I \) is the identity matrix.

Test problems: We take some adjacency matrices of known graphs from [45]. The numerical results are displayed in Tables 4 and 5. Note that in Table 5, we solve POP (4.3) with an additional unit ball constraint \( 1 - \|x\|^2 \geq 0 \). The columns under “val” show the approximations of \( \alpha(G) \).

Discussion: The graphs from Table 4 are relatively dense so that we cannot exploit term sparsity or correlative sparsity for POP (4.3) in these cases. For the graph GD02-a in Table 4, Pól and Han provide better bounds for \( \alpha(G) \) compared to the ones returned by the second order relaxations of Put. In Table 5, Put provides negative values for the first order relaxations. The additional unit ball constraint does not help to improve the bound for the second order relaxation for Id 2. Besides, Table 4 shows that Han provides slightly better bounds than Pól for johnson16-2-4, but its value is less accurate than the corresponding one from Table 4.

4.3 The MAXCUT problems

The MAXCUT problem is given by:

\[
\max_{x \in \{0,1\}^n} x^T W(e - x),
\]

where \( e = (1, \ldots, 1) \) and \( W \) is the matrix of edge weights associated with a graph (see [11, Theorem 1]).

Test problems: The data of graphs is taken from TSPLIB [42]. The numerical results are displayed in Table 6. Note that all instance of matrix \( W \) are dense.

Discussion: The behavior of our method is similar to that in Section 4.1.
Table 3: Numerical results for randomly generated dense QCQPs.

| Id | Pb | POP size | Put | Pöl | Han |
|----|----|----------|-----|-----|-----|
|    | n  | m_{ineq} | m_{eq} | k  | val | time | k  | s   | val | time |
| 1  | 1  | 20       | 2     | 0  | 1   | $\infty$ | 0  | 17  | -1.99792 | 1  | 2   | 5   | -1.99792 | 1  |
| 2  | 2  | 20       | 5     | 0  | 1   | $\infty$ | 0.03 | 342 | -0.265883 | 9  | 3   | 1   | -0.265883 | 1  |
| 3  | 3  | 20       | 4     | 0  | 1   | $\infty$ | 0.02 | 356 | -0.429442 | 5  | 3   | 7   | -0.429430 | 9  |
| 4  | 4  | 30       | 2     | 0  | 1   | $\infty$ | 0.0 | 3545 | -2.31695 | 2  | 2   | 10  | -2.31695 | 1  |
| 5  | 5  | 30       | 5     | 0  | 1   | $\infty$ | 0.2 | 15135 | -1.79295 | 45 | 3   | 20  | -1.79295 | 238 |
| 6  | 6  | 30       | 4     | 0  | 1   | $\infty$ | 0.1 | 12480 | -1.56374 | 54 | 3   | 15  | -1.56374 | 236 |

| Id | Put | Pöl | Han |
|----|-----|-----|-----|
|    | nmat | msize | nscal | naff | nmat | msize | nscal | naff | nmat | msize | nscal | naff |
| 1  | 1  | 22 | 21 | 231 | 22 | 1 | 231 | 10626 | 5 | 17 | 232 | 231 | 17 | 5 | 255 | 231 |
| 2  | 2  | 25 | 231 | 21 | 25 | 1 | 231 | 10626 | 44 | 20 | 1604 | 1771 | 0 | 1 | 2344 | 1771 |
| 3  | 3  | 25 | 231 | 29 | 231 | 1 | 231 | 10626 | 330 | 7 | 1688 | 1771 | 345 | 7 | 1945 | 1771 |
| 4  | 4  | 32 | 496 | 32 | 496 | 1 | 496 | 46376 | 11 | 21 | 497 | 496 | 22 | 10 | 530 | 496 |
| 5  | 5  | 37 | 496 | 37 | 496 | 1 | 496 | 46376 | 32 | 31 | 5116 | 5456 | 396 | 20 | 5650 | 5456 |
| 6  | 6  | 37 | 496 | 43 | 496 | 1 | 496 | 46376 | 32 | 31 | 5302 | 5456 | 561 | 15 | 5836 | 5456 |
Table 4: Numerical results for stability number of some known graphs in [45].

| Id | Pb         | POP size | Put | Pól  | Han |
|----|------------|----------|-----|------|-----|
|    |            |          | val | time |     |
| 1  | GD02_a     | 23       | 1   | 2    | 0.02 |
| 2  |            |          |     |      | 13.0110 | 394 |
| 3  | johnson8-2-4 | 28       | 1   | 2    | 0.03 |
| 4  |            |          |     |      | 7.00000 | 2098 |
| 5  | johnson8-4-4 | 70       | 1   | 2    | 1   |
| 6  |            |          |     |      | 13.00000 | 1  |
| 7  | hamming6-2 | 64       | 1   | 2    | 0.5  |
| 8  |            |          |     |      | 1.99999 | 3  |
| 9  | hamming6-4 | 64       | 1   | 2    | 0.6  |
| 10 |            |          |     |      | 12.00000 | 3  |
| 11 | johnson16-2-4 | 120      | 1   | 2    | 0.6  |
| 12 |            |          |     |      | 15.00001 | 5  |

| Id | 1     | 2     | nmat | msize | nscal | naff | nmat | msize | nscal | naff |
|----|-------|-------|------|-------|-------|------|------|-------|-------|------|
| 1  | 1     | 2     | 24   | 300   | 25    | 30   | 1    | 24    | 300   | 1    |
| 2  | 2     | 300   | 1    | 301   | 30    | 17550| 1    | 301   | 300   | 1    |
| 3  | 1     | 29    | 29   | 435   | 30    | 435  | 1    | 436   | 435   | 1    |
| 4  | 2     | 29    | 30   | 35960 | 36    | 35960| 1    | 436   | 435   | 1    |
| 5  | 1     | 71    | 72   | 2556  | 72    | 2556 | 1    | 2557  | 2556  | 1    |
| 6  | 2     | 71    | 72   | 1150626 | 72  | 1150626 | 1    | 2557  | 2556  | 1    |
| 7  | 1     | 65    | 66   | 2145  | 66    | 2145 | 1    | 2146  | 2145  | 1    |
| 8  | 2     | 65    | 66   | 2145  | 66    | 2145 | 1    | 2146  | 2145  | 1    |
| 9  | 1     | 65    | 66   | 2145  | 66    | 2145 | 1    | 2146  | 2145  | 1    |
| 10 | 2     | 65    | 66   | 2145  | 66    | 2145 | 1    | 2146  | 2145  | 1    |
| 11 | 1     | 121   | 122  | 7381  | 122   | 7381 | 1    | 7382  | 7381  | 1    |
| 12 | 2     | 121   | 122  | 7381  | 122   | 7381 | 1    | 7382  | 7381  | 1    |
Table 5: Numerical results for stability number of some known graphs in \([45]\) with an additional unit ball constraint.

| Id | Pb       | POP size | Put     | POP size | Pol     | Han     |
|----|----------|----------|---------|----------|---------|---------|
| 1  | GD02     | 23       | 1       | -0.62896 | 13.0170 | 0.02    | 442     | 0       | 13     | 13.0000 | 1       |
| 2  |          |          |         |          |         |         |         |         |        |         |         |
| 3  | johnson8-2-4 | 28       | 1       | -0.30434 | 7.00000 | 0.03    | 3010    | 0       | 23     | 7.00000 | 1       |
| 4  |          |          |         |          |         |         |         |         |        |         |         |
| 5  | johnson8-4-4 | 70       | 1       | -0.14056 | 1.00000 | 1       | 0       | 70      | 5.00000 | 10      | 2       | 70      | 5.00000 | 8 |
| 6  |          |          |         |          |         |         |         |         |        |         |         |
| 7  | hamming0-2 | 64       | 1       | -0.32989 | 1.00000 | 1       | 0       | 64      | 2.00000 | 7       | 2       | 64      | 2.00000 | 7 |
| 8  |          |          |         |          |         |         |         |         |        |         |         |
| 9  | hamming0-4 | 64       | 1       | -0.11764 | 1.00000 | 0.6     | 0       | 64      | 12.0000 | 6       | 2       | 64      | 12.0000 | 7 |
| 10 |          |          |         |          |         |         |         |         |        |         |         |
| 11 | johnson16-2-4 | 120      | 1       | -0.08982 | 2.00000 | 26      | 0       | 121     | 15.0000 | 75      | 2       | 121     | 15.0000 | 74 |
| 12 |          |          |         |          |         |         |         |         |        |         |         |

| Id | Put | Pol | Han |
|----|-----|-----|-----|
| 1  | nmat | msize | nscal | naff | nmat | msize | nscal | naff | nmat | msize | nscal | naff |
| 2  | 1    | 25   | 24    | 26   | 300  | 17550 | 12    | 13   | 302  | 300  | 12    | 13   | 327  | 300 |
| 3  | 1    | 30   | 29    | 31   | 435  | 35960 | 7     | 23   | 437  | 435  | 7     | 23   | 467  | 435 |
| 4  | 1    | 72   | 71    | 73   | 2556 | 1150626 | 2    | 70   | 2558 | 2556 | 2     | 70   | 2630 | 2556 |
| 5  | 1    | 66   | 65    | 67   | 2145 | 814385 | 2    | 64   | 2146 | 2145 | 2     | 64   | 2213 | 2145 |
| 6  | 1    | 66   | 65    | 67   | 2145 | 814385 | 2    | 64   | 2146 | 2145 | 2     | 64   | 2213 | 2145 |
| 7  | 1    | 122  | 121   | 123  | 7381 | 9381251 | 1    | 121  | 7383 | 7381 | 1     | 121  | 7505 | 7381 |
| 8  | 1    | 7381 | 7380  | 7381 | 9381251 | 1    | 121  | 7383 | 7381 | 1     | 121  | 7505 | 7381 |
Table 6: Numerical results for some instances of MAXCUT problems.

| Id | Pb  | POP size | Put val | time | Pol val | time | Han val | time |
|----|-----|----------|---------|------|---------|------|---------|------|
| 1  | burma14 | 14 | 1 30310.915 | 0.2 | 16 30302.000 | 1 3 16 30301.999 | 1 |
| 2  | gr17 | 17 | 1 25089.044 | 0.2 | 19 24986.001 | 1 3 19 24985.999 | 2 |
| 3  | fri26 | 26 | 1 22220.657 | 0.2 | 28 22218.001 | 12 3 28 22217.999 | 28 |
| 4  | att48 | 48 | 1 799281.420 | 1 | 1 798857.049 | 1129 3 50 798890.722 | 3600 |

| Id | Put nmat | msize | nscal | naff | Pol nmat | msize | nscal | naff | Han nmat | msize | nscal | naff |
|----|---------|-------|-------|------|---------|-------|-------|------|---------|-------|-------|------|
| 1  | 1 | 15 | 29 | 130 | 15 | 15 | 666 | 680 | 16 | 15 | 787 | 680 |
| 2  | 2 | 15 | 120 | 1681 | 3060 | 15 | 15 | 666 | 680 | 16 | 15 | 787 | 680 |
| 3  | 1 | 18 | 35 | 171 | 171 | 18 | 18 | 1123 | 1140 | 19 | 18 | 1295 | 1140 |
| 4  | 1 | 378 | 53 | 378 | 378 | 27 | 27 | 3628 | 3654 | 28 | 27 | 4007 | 3654 |
| 5  | 1 | 49 | 97 | 1225 | 1225 | 49 | 49 | 20777 | 20825 | 50 | 49 | 22003 | 20825 |

4.4 Positive maximal singular values

Test problems: We generate a matrix $M$ as in [16, (12)]. Explicitly,

$$M := \begin{bmatrix}
D & 0 & 0 & \ldots & 0 \\
CB & D & 0 & \ldots & 0 \\
CAB & CB & D & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
CA^{m-2}B & CA^{m-3}B & \ldots & CA^{-1}B & D
\end{bmatrix},$$

(4.5)

where $A, B, C, D$ are square matrices of size $r = m$. Every entry of $A, B, C, D$ is taken uniformly in $(-1, 1)$. In order to compute the positive maximal singular value $\sigma_+(M)$ of $M$, we solve the following POP on the nonnegative orthant:

$$\sigma_+(M)^2 = \max_{x \in \mathbb{R}^n_+} \{x^T(M^T M)x : \|x\|^2_2 = 1\}.$$  

(4.6)

Note that $n = m \times r = m^2$.

The numerical results are displayed in Table 7. The columns of val show the approximations of $\sigma_+(M)^2$.

Discussion: The behavior of our method is similar to that in Section 4.1.

4.5 Stability number of a graph

Let us consider POP [43] which returns the stability number of a graph $G$.

Test problems: We generate the adjacency matrix $A = (a_{ij})_{i,j \in [n]}$ of the graph $G$ by the following steps:

1. Set $a_{ii} = 0$, for $i \in [n]$.
2. For $i \in [n]$, for $j \in \{1, \ldots, i - 1\}$, let us select $a_{ij} = a_{ji}$ uniformly $\{0, 1\}$.

The numerical results are displayed in Table 8. Note that the columns of val show the approximations of $\alpha(G)$.
Table 7: Numerical results for positive maximal singular values.

| Id | Pb | POP size | Put | k | val | time | Pól | k | s | val | time |
|----|----|----------|-----|---|-----|------|-----|---|---|-----|------|
| 1  | 4  | 2 16     | 1   | 2 | 47.38110 | 0.02 | 16  | 30.18791 | 16  |
| 2  | 5  | 2 25     | 1   | 2 | 168.4450 | 0.04 | 877 | 91.28158 | 0.7 |
| 3  | 6  | 3 36     | 1   | 2 | 4759.12  | 0.2  | -   | -     | -   |
| 4  | 7  | 4 49     | 1   | 2 | 1777.53  | 0.5  | -   | -     | -   |
| 5  | 8  | 1 50     | 1   | 2 | 6.00000  | 1    | 1275| 6.00001 | 4   |

| Id | Put | Pól |
|----|-----|-----|
| 1  | 17  | 138 | 153 |
| 2  | 26  | 327 | 351 |
| 3  | 1   | 37  | 703 |
| 4  | 26  | 91390 | 703 |
| 5  | 1   | 50  | 1227| 1275|
| 6  | 50  | 292825 | 1275|

Table 8: Numerical results for stability number of randomly generated graphs.

| Id | Pb | POP size | Put | k | val | time | Pól | k | s | val | time |
|----|----|----------|-----|---|-----|------|-----|---|---|-----|------|
| 1  | 1  | 10       | 1   | 2 | ∞   | 0.01 | 0.6 | 11 | 3.00000 | 1    |
| 2  | 2  | 15       | 1   | 2 | ∞   | 0.01 | 10  | 16 | 5.00000 | 1    |
| 3  | 4  | 20       | 1   | 2 | ∞   | 0.02 | 119 | 21 | 5.00001 | 4    |
| 4  | 8  | 25       | 1   | 2 | ∞   | 0.04 | 1064| 26 | 6.00000 | 10   |

| Id | Put | Pól |
|----|-----|-----|
| 1  | 11  | 67  | 66  |
| 2  | 16  | 137 | 3876|
| 3  | 1   | 17  | 136 |
| 4  | 21  | 22  | 231 |
| 5  | 26  | 27  | 351 |
| 6  | 351 | 352 | 237 |

| Id | Put | Pól |
|----|-----|-----|
| 1  | 11  | 67  | 66  |
| 2  | 16  | 137 | 3876|
| 3  | 1   | 17  | 136 |
| 4  | 21  | 22  | 231 |
| 5  | 26  | 27  | 351 |
| 6  | 351 | 352 | 237 |

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Table 9: Numerical results for deciding the copositivity of a real symmetric matrix.

| Id | Pb | POP size | Put | Pól |
|----|----|----------|-----|-----|
|    |    | n        | k   | val | time | k   | s   | val | time |
| 1  | 2  | 10       | 2   | -1.45876 | 0.004 | 0   | 8  | -0.94862 | 1   |
| 2  | 2  | 15       | 2   | -1.14319 | -0.94862 | 0.007 | 10 | -0.94862 | 1   |
| 3  | 3  | 15       | 2   | -1.40431 | -0.65197 | 0.02 | 20 | -0.98026 | 1   |
| 4  | 2  | 20       | 2   | -1.34450 | -0.98026 | 0.03 | 19 | -0.97345 | 2   |

Discussion: The behavior of our method is similar to that in Section 4.1. Note that the graphs from Tables 8 are dense so that we cannot exploit term sparsity or correlative sparsity for POP (4.3) in these cases. Moreover, for all graphs in Table 8, spPól provides the better bounds for $\alpha(G)$ compared to the ones returned by the second order relaxations of Put.

Remark 15. In Pb 3, 4 of Table 8, Pól with $k = 1$ provides a better bound than Pól with $k = 0$. As shown in Remark 9, each SDP relaxation of Pól with $k = 0$ and sufficiently large $s$ corresponds to an SDP relaxation obtained after exploiting term sparsity.

4.6 Deciding the copositivity of a real symmetric matrix

Given a symmetric matrix $A \in \mathbb{R}^{n \times n}$, we say that $A$ is copositive if $u^T A u \geq 0$ for all $u \in \mathbb{R}^n_+$. Consider the following POP:

$$f^* := \min_{x \in \mathbb{R}_{+}^{n}} \{x^T A x : \sum_{j \in [n]} x_j = 1\}.$$  \hfill (4.7)

The matrix $A$ is copositive if $f^* \geq 0$.

Test problems: We construct several instances of the matrix $A$ as follows:

1. Take $B_{ij}$ randomly in $(-1,1)$ w.r.t. the uniform distribution, for all $i, j \in \{1, \ldots, n\}$.
2. Set $B := (B_{ij})_{1 \leq i, j \leq n}$ and $A := \frac{1}{2}(B + B^T)$.

The numerical results are displayed in Table 10.

Discussion: The behavior of our method is similar to that in Section 4.1. In all cases, we can extract the solutions of the resulting POP and certify that $A$ is not copositive since $f^*$ is negative.

4.7 Deciding the nonnegativity of an even degree form on the nonnegative orthant

Given a form $q \in \mathbb{R}[x]$, $q$ is nonnegative on $\mathbb{R}^n_+$ if $q$ is nonnegative on the unit simplex

$$\Delta := \{x \in \mathbb{R}_+^n : \sum_{j \in [n]} x_j = 1\}.$$  \hfill (4.8)
Table 10: Numerical results for deciding the nonnegativity of an even degree form on the nonegative orthant, with $d = 2$.

| Id | Pb | POP size | Put | Pól |
|----|----|----------|-----|-----|
|    | n  | k   | val | time | k   | s   | val | time |
| 1  | 1  | 5   | 2   | -1.87958 | 0.001 | 0   | 8   | -0.68020 | 1   |
| 2  | 2  | 10  | 2   | -1.87491 | 0.1   | 0   | 11  | -0.87524 | 5   |
| 3  | 2  | 15  | 2   | -2.01566 | 6     | 0   | 44  | -0.86938 | 79  |

| Id | Put | Pól |
|----|-----|-----|
|    | nmat | msize | nscal | naff | nmat | msize | nscal | naff |
| 1  | 6   | 21   | 22   | 126  | 31   | 6    | 72   | 126  |
| 2  | 6   | 56   | 232  | 462  | 67   | 8008 | 111  | 1001 |
| 3  | 11  | 66   | 2212 | 54264 | 3876 | 213  | 2637 | 3876 |

Given a form $f \in \mathbb{R}[x]$ of degree $2d$, we consider the following POP:

$$f^* := \min_{x \in \Delta} f(x).$$  (4.9)

Note that if $d = 1$, problem (4.9) boils down to deciding the copositivity of the Gram matrix associated to $f$. Thus, we consider the case where $d \geq 2$.

**Test problems:** We construct several instances of the form $f$ of degree $2d$ as follows:

1. Take $f_\alpha$ randomly in $(-1,1)$ w.r.t. the uniform distribution, for each $\alpha \in \mathbb{N}^n$ with $|\alpha| = 2d$.

2. Set $f := \sum_{|\alpha| = 2d} f_\alpha x_\alpha$.

The numerical results are displayed in Table 10.

**Discussion:** The behavior of our method is similar to that in Section 4.1. Note that PóI with order $k = 0$ provides worse bounds than Put with order $k = 2$. However, as shown in Table 11 PóI with order $k = 1$ provides the same bounds as Put with order $k = 2$.

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4.8 Minimizing a polynomial over the boolean hypercube

Consider the optimization problem:

$$\min_{x \in (0,1)^n} f(x),$$  (4.10)

where $f$ is a polynomial of degree at most $2d$. It is equivalent to the following POP on the nonnegative orthant:

$$\min_{x \in \mathbb{R}^n_+} \{ f(x) : x_j(1 - x_j) = 0, j \in [n] \},$$  (4.11)

**Test problems:** We construct several instances by taking the coefficients of $f$ randomly in $(-1,1)$ w.r.t. to the uniform distribution.

The numerical results are displayed in Table 11.

**Discussion:** The behavior of our method is similar to that in Section 4.1. Note that PóI with order $k = 0$ provides worse bounds than Put with order $k = 2$. However, as shown in Table 11 PóI with order $k = 1$ provides the same bounds as Put with order $k = 2$. 

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Table 11: Numerical results for minimizing polynomials over the boolean hypercube, with \(d = 1\).

| Id | Pb | POP size | Put | Polynomial | \(n\) | \(k\) | \(val\) | time | \(s\) | \(val\) | time |
|----|----|----------|-----|------------|-----|-----|-------|-----|-----|-------|-----|
| 1  | 1  | 10       | 1   | -4.61386   | 0.008| 1   | 11    | -4.34345 | 1   |
| 2  | 2  | 20       | 1   | -15.4584   | 0.02 | 1   | 21    | -14.9455 | 4   |
| 3  | 3  | 30       | 1   | -29.3433   | 0.1  | 1   | 31    | -27.6311 | 41  |

| Id | POP | nmat | nsize | nscal | naff | nmat | nsize | nscal | naff |
|----|-----|------|-------|-------|------|------|-------|-------|------|
| 1  | 11  | 6    | 56    | 21    | 66   | 11   | 276   | 286   |
| 2  | 21  | 21   | 231   | 462   | 231  | 21   | 1751  | 1771  |
| 3  | 31  | 31   | 496   | 14881 | 496  | 31   | 5426  | 5456  |

5 Conclusion

We have proposed in this paper semidefinite relaxations for solving dense POPs on the nonnegative orthant. The basic idea is to apply a positivity certificate involving SOS of monomials for a POP with input polynomials being even in each variable. It allows us to obtain a hierarchy of linear relaxations. Afterwards we replace each SOS of monomials by an SOS associated with a block-diagonal Gram matrix, where each block has a prescribed size. This ensures the efficiency of the corresponding hierarchy of SDP relaxations in practice. The convergence is still maintained, as it is based on the convergence guarantee of the hierarchy of linear relaxations. The resulting convergence rate of \(O(\varepsilon^{-c})\) is similar to the one of Baldi and Mourrain \([4]\).

As a topic of further applications, we would like to use our method for solving large-scale POPs for phase retrieval and feedforward neural networks.

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6 Appendix

6.1 Preliminary material

For each \(q = \sum_{\alpha} q_{\alpha} x^{\alpha} \in \mathbb{R}[x]\), we note \(|q| := \max_{\alpha} \frac{|q_{\alpha}|}{c_{\alpha}}\) with \(c_{\alpha} := \frac{|\alpha|!}{\alpha_{1}! \cdots \alpha_{n}!}\) for each \(\alpha \in \mathbb{N}^{n}\). This defines a norm on the real vector space \(\mathbb{R}[x]\). Moreover, for \(p_1, q_2 \in \mathbb{R}[x]\), we have
\[
\|p_1 q_2\| \leq \|p_1\| \|q_2\|,
\]
according to \([49]\) Lemma 8.

We recall the following bound for central binomial coefficient stated in \([24]\) page 590:

Lemma 1. For all \(t \in \mathbb{N}_{\geq 0}\), it holds that \(\left(\begin{array}{c}2t \\ t \end{array}\right) \frac{1}{2^{t}} \leq \frac{1}{\sqrt{\pi t}}\).

Define the simplex
\[
\Delta_{n} := \{x \in \mathbb{R}^{n} : x_{j} \geq 0, j \in [n], \sum_{j \in [n]} x_{j} = 1\}.
\]

We recall the degree bound for Pólya’s Positivstellensatz \([39]\):
Lemma 2. (Powers and Reznick [40]) If \( q \) is a homogeneous polynomial of degree \( d \) positive on \( \Delta_n \), then for all \( k \in \mathbb{N} \) satisfying
\[
| q(x_1, \ldots, x_n) | \leq | s(x_1, \ldots, x_n) | \leq \frac{d(d - 1)! | q |}{2 \min_{x \in \Delta_n} q(x)} - d,
\]
\((\sum_{j \in [n]} x_j)^k q \) has positive coefficients.

Let us recall the concept and the properties of polynomials even in each variable in [40] Definition 3.3. A polynomial \( q \) is even in each variable if for every \( j \in [n] \),
\[
q(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = q(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n).
\]

If \( q \) is even in each variable, then there exists a polynomial \( \tilde{q} \) such that \( q = \tilde{q}(x_1^2, \ldots, x_n^2) \). Indeed, let \( q = \sum_{\alpha \in \mathbb{N}^n} q_\alpha x^\alpha \) be a polynomial even in each variable. Let \( j \in [n] \) be fixed. Then \( q(x) = \frac{1}{q_j}(q(x) + q(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n)) \). It implies that \( q_\alpha = 0 \) if \( \alpha_j \) is odd. Thus, \( q = \sum_{\alpha \in \mathbb{N}^n} q_\alpha x^{2\alpha} \) since \( j \) is arbitrary in \( [n] \). This yields \( \tilde{q} = \sum_{\alpha \in \mathbb{N}^n} q_\alpha x^{2\alpha} \).

For convenience, we denote \( x^* := (x_1^2, \ldots, x_n^2) \). Moreover, if \( q \) is even in each variable and homogeneous of degree \( 2d_q \), then \( \tilde{q} \) is homogeneous of degree \( d_q \). Conversely, if \( q \) is a polynomial of degree at most \( 2d \) such that \( q \) is even in each variable, then the degree-\( 2d \) homogenization of \( q \) is even in each variable.

6.2 The proof of Theorem 1

Proof. Let \( \varepsilon > 0 \). By assumption, \( \deg(f) = 2d_j \), \( \deg(g_i) = 2d_{g_i} \) for some \( d_j, d_{g_i} \in \mathbb{N} \), for \( j \in [m] \).

Step 1: Converting to polynomials on the nonnegative orthant. We claim that \( \tilde{f} \) is non-negative on the semialgebraic set
\[
S := \{ x \in \mathbb{R}^n : x_j \geq 0, \ j \in [n], \ \tilde{g}_i(x) \geq 0, \ i \in [m] \}.
\]

Let \( y \in S \). Set \( z = (\sqrt{y_1}, \ldots, \sqrt{y_m}) \). Then \( g_i(z) = \tilde{g}_i(z^2) = \tilde{g}_i(y) \geq 0 \), for \( i \in [m] \). By assumption, \( f(y) = \tilde{f}(z^2) = \tilde{f}(z) \geq 0 \). It implies that \( \tilde{f} + \varepsilon (\sum_j x_j)^{d_j} \) is homogeneous and positive on \( S \setminus \{0\} \).

To prove the first statement, we proceed exactly as in the proof of [12] Theorem 2.4 for \( \tilde{f} + \varepsilon (\sum_j x_j)^{d_j} \) and derive the bound on the degree of polynomials having positive coefficients when applying Pólya’s Positivstellensatz. To obtain (2.2), we replace \( x \) by \( x^2 \) in the representation of \( \tilde{f} + \varepsilon (\sum_j x_j)^{d_j} \).

We shall prove the second statement. Assume that \( S \) has nonempty interior. Set \( \tilde{m} := m + n \) and \( g_{m+j} := x_j^2 \) with \( d_{m+j} := 1, \ j \in [n] \). Then \( \tilde{g}_{m+j} := x_j, \ j \in [n] \), and
\[
\tilde{S} := \{ x \in \mathbb{R}^n : \tilde{g}_i(x) \geq 0, \ i \in [\tilde{m}] \}.
\]

Note that \( \deg(\tilde{g}_i) = d_{g_i}, \ i \in [\tilde{m}] \). Since \( S \) has nonempty interior and \( \cup_{j=1}^n \{ x \in \mathbb{R}^n : x_j = 0 \} \) has zero Lebesgue measure in \( \mathbb{R}^n \), \( S \setminus (\cup_{j=1}^n \{ x \in \mathbb{R}^n : x_j = 0 \}) \) also has nonempty interior. Therefore \( a \in S \setminus (\cup_{j=1}^n \{ x \in \mathbb{R}^n : x_j = 0 \}) \) such that \( g_i(a) > 0, \ i \in [m] \). Let \( b = (\sqrt{|a_1|}, \ldots, \sqrt{|a_n|}) \). Then \( b \in (0, \infty)^n \) and \( b^2 = (|a_1|, \ldots, |a_n|) \). Since each \( g_i \) is even in each variable, \( \tilde{g}_i(b) = g_i(b^2) = g_i(a) > 0, \ i \in [m] \), yielding \( \tilde{S} \) has nonempty interior.

Step 2: Construction of the positive weight functions. We process similarly to the proof of [31] Theorem 1 (see [31] Appendix A.2.1)) to obtain functions \( \varphi_j : \mathbb{R}^n \to \mathbb{R}, \ j \in [\tilde{m}] \), such that,
1. \( \varphi_j \) is positive and bounded from above by \( C \varphi_j = r_j e^{-r_j} \) on \( B(0, \sqrt{n} + j) \) for some positive constants \( r_j \) and \( r_j \) independent of \( \varepsilon \).
2. \( \varphi_j \) is Lipschitz with Lipschitz constant \( L \varphi_j = \ell_j e^{-\ell_j} \) for some positive constants \( \ell_j \) and \( \ell_j \) independent of \( \varepsilon \).
3. The inequality
\[
\tilde{f} + \varepsilon - \sum_{i=1}^{\tilde{m}} \varphi_i^2 \tilde{g}_i \geq \frac{\varepsilon}{2n} \text{ on } [-1, 1]^n,
\]
holds.

Note that we do not need to prove the even property for each weight \( \varphi_i \) above.
Step 3: Approximating with Bernstein polynomials. For each \( i \in [\tilde{m}] \), we now approximate \( \tilde{\psi}_i \) on \([-1, 1]^n\) with the following Bernstein polynomials defined as in [31] Definition 1:

\[
B_i^{(l)}(x) = B_{y \rightarrow \tilde{\psi}_i(2y - e), d} \left( \frac{x + e}{2} \right), \quad d \in \mathbb{N},
\]

(6.8)

with \( e = (1, \ldots, 1) \in \mathbb{R}^n \). By using [31] Lemma 6], for all \( x \in [-1, 1]^n \), for \( i \in [\tilde{m}] \),

\[
|B_i^{(l)}(x) - \tilde{\psi}_i(x)| \leq L_{\tilde{\psi}_i} \left( \frac{n}{\bar{d}} \right), \quad d \in \mathbb{N},
\]

(6.9)

and for all \( x \in [-1, 1]^n \), for \( i \in [\tilde{m}] \):

\[
|B_i^{(l)}(x)| \leq \sup_{x \in [-1, 1]^n} |\tilde{\psi}_i(x)| \leq C_{\tilde{\psi}_i}.
\]

(6.10)

For \( i \in [\tilde{m}] \), let

\[
d_i := 2u_i \quad \text{with} \quad u_i = \frac{2C_0^2 C_2^n n L_{2, \tilde{\psi}_i} (\tilde{m} + 1)^2 2^{2m}}{\varepsilon^2},
\]

(6.11)

where \( C_{\tilde{\psi}_i} \) is an upper bound of \( |\tilde{\psi}_i| \) on \( B(0, \sqrt{\bar{\pi}} + \varepsilon) \). Set \( q_i := B_i^{(d_i)} \), \( i \in [\tilde{m}] \). Then for all \( x \in [-1, 1]^n \),

\[
|q_i(x) - \tilde{\psi}_i(x)| = |B_i^{(1)}(x) - \tilde{\psi}_i(x)|
\]

\[
\leq L_{\tilde{\psi}_i} \left( \frac{n}{\bar{d}_i} \right)^{\frac{1}{2}}
\]

\[
\leq L_{\tilde{\psi}_i} \left( \frac{2C_0^2 C_2^n n L_{2, \tilde{\psi}_i} (\tilde{m} + 1)^2 2^{2m}}{\varepsilon^2} \right)^{\frac{1}{2}}
\]

\[
= \frac{2C_0^2 C_2^n n L_{2, \tilde{\psi}_i} (\tilde{m} + 1) 2^{2m}}{\varepsilon^2}.
\]

(6.12)

Step 4: Estimating the lower and upper bounds of \( \tilde{f}(x) + \varepsilon - \sum_{i=1}^{\tilde{m}} q_i(x)^2 \tilde{g}_i(x) \) on \( \Delta_n \).

From these and (6.7), for all \( x \in \Delta_n \),

\[
\tilde{f}(x) + \varepsilon - \sum_{i=1}^{\tilde{m}} q_i(x)^2 \tilde{g}_i(x)
\]

\[
\geq \frac{\tilde{f}(x)}{\tilde{g}_i} - \sum_{i=1}^{\tilde{m}} \left| \frac{\tilde{g}_i(x)^2}{\tilde{\psi}_i(x)} \tilde{\psi}_i(x) + \sum_{i=1}^{\tilde{m}} \tilde{g}_i(x)|\tilde{\psi}_i(x)|^2 - q_i(x)|^2 \right|
\]

\[
\geq \frac{\tilde{f}(x)}{\tilde{g}_i} - \sum_{i=1}^{\tilde{m}} C_{\tilde{\psi}_i} (|\tilde{\psi}_i(x)| + |\tilde{\psi}(x)|) |\tilde{\psi}_i(x)| - q_i(x)|^2
\]

\[
\geq \frac{\tilde{f}(x)}{\tilde{g}_i} - \sum_{i=1}^{\tilde{m}} 2C_{\tilde{\psi}_i} C_{\tilde{\psi}_i} 2^{2m} \bar{g}_i (m + 1) 2^{2m}
\]

\[
= \frac{\tilde{f}(x)}{\tilde{g}_i} - \sum_{i=1}^{\tilde{m}} \frac{\tilde{g}_i (m + 1) 2^{2m}}{\tilde{g}_i (m + 1) 2^{2m}}.
\]

Thus,

\[
\tilde{f}(x) + \varepsilon - \sum_{i=1}^{\tilde{m}} q_i^2 \tilde{g}_i \geq \frac{\tilde{f}(x)}{\tilde{g}_i} - \sum_{i=1}^{\tilde{m}} \frac{\tilde{g}_i (m + 1) 2^{2m}}{\tilde{g}_i (m + 1) 2^{2m}} \quad \text{on} \quad \Delta_n.
\]

(6.14)

Step 5: Estimating the upper bound of \( |q_i| \). For \( i \in [\tilde{m}] \), we write

\[
q_i = B_i^{(2u_i)} = \sum_{k_1=0}^{2u_i} \cdots \sum_{k_n=0}^{2u_i} \tilde{\phi}_i \left( \frac{k_1-u_i}{u_i}, \ldots, \frac{k_n-u_i}{u_i} \right)
\]

\[
\times \prod_{j=1}^{m} \left[ \binom{2u_j}{k_j} \left( \frac{1-x_j}{2} \right)^{2u_j-k_j} \right].
\]

(6.15)

Then

\[
\deg(q_i) \leq 2nu_i,
\]

(6.16)

for \( i \in [\tilde{m}] \). From (6.1), we have

\[
|q_i| \leq \sum_{k_1=0}^{2u_i} \cdots \sum_{k_n=0}^{2u_i} \left| \tilde{\phi}_i \left( \frac{k_1-u_i}{u_i}, \ldots, \frac{k_n-u_i}{u_i} \right) \right|
\]

\[
\times \prod_{j=1}^{m} \left( \binom{2u_j}{k_j} \frac{1}{2^{2u_j}} \right) \|x_j + 1\|^2 \|1 - x_j\|^{2u_i-k_j}
\]

\[
\leq \sum_{k_1=0}^{2u_i} \cdots \sum_{k_n=0}^{2u_i} C_{\tilde{\phi}_i} \prod_{j=1}^{m} \left( \binom{2u_j}{k_j} \frac{1}{2^{2u_j}} \right)
\]

\[
= C_{\tilde{\phi}_i} \left( \binom{2u_i+1}{u_i} \right)^n.
\]

(6.17)

The second inequality is due to \( |x_j + 1| = |1 - x_j| = 1 \) and \( \binom{2u_j}{k_j} \geq \binom{2u_i}{k_j} \), for \( k_j = 0, \ldots, 2u_i \). The third inequality is implied from Lemma 1.
Step 6: Converting to homogeneous polynomials. Thanks to (6.14), we get
\[
\tilde{f} + 2\varepsilon - \sum_{i \in [m]} (\frac{\varepsilon}{mC_{\tilde{g}_i}}) \tilde{g}_i \geq \frac{\varepsilon}{(m+1)2^m} \quad \text{on } \Delta_n, 
\] (6.18)
since \(|\tilde{g}_i| \leq C_{\tilde{g}_i} \) on \( \Delta_n \). Note that \( \tilde{f}, \tilde{g}_i \) are homogeneous polynomials of degree \( d_f, d_{\tilde{g}_i} \), respectively.

For each \( q \in \mathbb{R}[x]_d \), \( q \) is a \( d \)-homogenization of \( q \) if
\[
\tilde{q} = \sum_{t=0}^{d} h(t) \left( \sum_{j \in [n]} x_j \right)^{d-t},
\] (6.19)
for some \( h(t) \) is the homogeneous polynomial of degree \( t \) satisfying \( q = \sum_{t=0}^{d} h(t) \). In this case, \( \tilde{q} = q \) on \( \Delta_n \).

Let \( p_i := \tilde{q}_i^2 + \frac{\varepsilon}{mC_{\tilde{g}_i}} \left( \sum_{j \in [n]} x_j \right)^{4nu_i} \), with \( \tilde{q}_i \) being a \( 2nu_i \)-homogenization of \( q_i \), for \( i \in [\tilde{m}] \). Then \( p_i \) is a homogeneous polynomial of degree \( 4nu_i \),
\[
p_i = \tilde{q}_i^2 + \frac{\varepsilon}{mC_{\tilde{g}_i}} \geq \frac{\varepsilon}{mC_{\tilde{g}_i}} \quad \text{on } \Delta_n,
\] (6.20)
and
\[
\|p_i\| \leq \|q_i\|^2 + \frac{\varepsilon}{mC_{\tilde{g}_i}} \leq T_{q_i}^2 + \frac{\varepsilon}{mC_{\tilde{g}_i}} =: T_{p_i}.
\] (6.21)
Set \( D := \max\{d_f, 4nu_i + d_{\tilde{g}_i} : i \in [\tilde{m}] \} \) and
\[
F := \left( \sum_{j \in [n]} x_j \right)^{D-d_f} \left( \tilde{f} + 2\varepsilon \left( \sum_{j \in [n]} x_j \right)^{d_f} \right) - \sum_{i \in [m]} \tilde{g}_i \left( \sum_{j \in [n]} x_j \right)^{D-4nu_i - d_{\tilde{g}_i}}.
\] (6.22)
Then \( F \) is a homogeneous polynomial of degree \( D \) and
\[
F = \tilde{f} + 2\varepsilon - \sum_{i \in [m]} \left( \frac{\varepsilon}{mC_{\tilde{g}_i}} \right) \tilde{g}_i \geq \frac{\varepsilon}{(m+1)2^m} \quad \text{on } \Delta_n,
\] (6.23)
Moreover,
\[
\|F\| \leq \left\| \sum_{j \in [n]} x_j \right\|^{D-d_f} \left( \|\tilde{f}\| + 2\varepsilon \left\| \sum_{j \in [n]} x_j \right\|^{d_f} \right) + \sum_{i \in [m]} \|\tilde{g}_i\| \left\| \sum_{j \in [n]} x_j \right\|^{D-4nu_i - d_{\tilde{g}_i}}
\leq \|\tilde{f}\| + 2\varepsilon + \sum_{i \in [m]} T_{p_i} \|\tilde{g}_i\| =: F_T,
\] (6.24)
since \( \|\sum_{j \in [n]} x_j\| = 1 \).

Step 7: Applying the degree bound of Pólya’s Positivstellensatz. Using Lemma 2 we obtain:

- For all \( k \in \mathbb{N} \) satisfying
  \[
  k \geq \frac{D(D-1)TF}{(m+1)2^m} =: K_0,
  \] (6.25)
  \( (\sum_{j \in [n]} x_j)^{k}F \) has positive coefficients.

- For each \( i \in [\tilde{m}] \) and for all \( k \in \mathbb{N} \) satisfying
  \[
  k \geq \frac{4nu_i(4nu_i-1)T_{p_i}}{mC_{\tilde{g}_i}} =: K_i,
  \] (6.26)
  \( (\sum_{j \in [n]} x_j)^{k}p_i \) has positive coefficients.

Notice that \( K_i, i = 0, \ldots, \tilde{m}, \) are obtained by composing finitely many times the following operators: “+”, “−”, “×”, “÷”, “◦”, “[−]”, “\( x, y \) → max\{x, y\}”, “\( x, y \) → min\{x, y\}”, “\( x, y \) → x \wedge y”, “\( x, y \) → x \vee y”, “\( x, y \) → x \uparrow y”, “\( x, y \) → x \downarrow y”, and “\( x, y \) → x \bullet y”, where all arguments possibly depend on \( \varepsilon \). Without loss of generality, let \( \varepsilon \) be positive constants independent of \( \varepsilon \) such that \( \varepsilon \leq 1 \).

Let \( k \geq \varepsilon^{-1} \) be fixed. Multiplying two sides of (6.22) with \( (\sum_{j \in [n]} x_j)^{k} \), we get
\[
s_0 = (\sum_{j \in [n]} x_j)^{D-d_f+k} \tilde{f} + 2\varepsilon (\sum_{j \in [n]} x_j)^{d_f} - \sum_{i \in [m]} \tilde{g}_i s_i (\sum_{j \in [n]} x_j)^{D-4nu_i-d_{\tilde{g}_i}},
\] (6.27)
where \( s_0 := (\sum_{j \in [n]} x_j)^{k}F \) and \( s_i := (\sum_{j \in [n]} x_j)^{k}p_i \) are homogeneous polynomials having nonnegative coefficients. Replacing \( x \) by \( x^2 \), we obtain:
\[
\|x\|^{2(D-d_f+k)}(f + 2\varepsilon \|x\|^{2d_f}) = s_0 + \sum_{i \in [m]} g_i s_i,
\] (6.28)
where
\[
\sigma_0 = s_0(x^2) + \sum_{j \in [n]} \hat{g}_{j+m}(x^2)^2 s_{j+m}(x^2) \|x\|_{2}^{2(D-4n_j + m - d_{j+m})}
\]
and
\[
\sigma_\iota = s_{\iota}(x^2) \|x\|_{2}^{2(D-4n_{\iota} - d_{\iota})}, \quad \iota \in [m],
\]
are SOS of monomials. Set \( K = D - d_\iota + K_1 \). Then \( \|x\|_2^2 (f + 2\varepsilon \|x\|_2^{2f}) = \sigma_0 + \sum_{\iota=1}^m g_\iota \sigma_\iota \) with \( \deg(\sigma_0) = \deg(g_\iota \sigma_\iota) = 2(K + d_\iota) \), for \( \iota \in [m] \). This completes the proof of Theorem \ref{theor:main} \( \square \)

6.3 Variations of Pólya’s and Handelman-type Positivstellensatz

For every \( \iota \in \mathbb{N} \), denote
\[
\hat{v}_\iota(x) := v_\iota(\underbrace{\frac{1}{2} \varepsilon \varepsilon(x - e)}_{\mathbb{R}^n}) = \left(\frac{1}{2\varepsilon\varepsilon}\right)^{\iota}(x - e)^{\mathbb{N}} \in \mathbb{N}^n,
\]
where \( e := (1, \ldots, 1) \in \mathbb{R}^n \).

As a consequence of Corollary \ref{corollary:main} the next proposition shows that the weighted SOS polynomials in Putinar–Vasilescu’s Positivstellensatz can be associated with diagonal Gram matrices via a change of monomial basis.

**Proposition 2.** (Putinar–Vasilescu’s Positivstellensatz with diagonal Gram matrices) Let \( g_1, \ldots, g_m \) be polynomials such that \( g_1 := R - \|x\|_2^{2f} \) for some \( R > 0 \) and \( g_m := 1 \). Let \( S \) be the semialgebraic set defined by
\[
S := \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \}.
\]
Let \( f \) be a polynomial of degree at most \( 2d_\iota \) such that \( f \) is nonnegative on \( S \). Denote \( d_{\iota} := |\deg(g_\iota)| / 2 \).

Then the following statements hold:

1. For all \( \varepsilon > 0 \), there exists \( K_\varepsilon \in \mathbb{N} \) such that for all \( k \geq K_\varepsilon \), there exist vectors \( \eta^{(k)} \in \mathbb{R}^{(2n+k+d_\iota-d_{\iota})} \)
satisfying
\[
\bigl(\|x\|_2 + n + 2\bigl)^{k}(f + \varepsilon) = \sum_{i=1}^m g_i \hat{v}_{k+d_\iota-d_{\iota}} \mathbb{R} \diag(\eta^{(k)} \hat{v}_{k+d_\iota-d_{\iota}}).
\]
2. If \( S \) has nonempty interior, then there exist positive constants \( \varepsilon_0 \) and \( \varepsilon_1 \) depending on \( f, g_i \) such that for all \( \varepsilon > 0 \), one can take \( K_\varepsilon = \varepsilon^{-1} \).

Proof. Take two vectors of \( n \) variables \( y = (y_1, \ldots, y_n) \) and \( z = (z_1, \ldots, z_n) \). Given \( q \in \mathbb{R}[x] \), denote the polynomial \( \hat{q}(y, z) = q(y^2 - z^2) \in \mathbb{R}[y, z] \). Let \( \hat{g}_{m+1} := \frac{1}{2}(L + n) - \|y, z\|^2_2 \) and \( d_{m+1} := 1 \). Define
\[
\hat{S} := \{ (y, z) \in \mathbb{R}^{2n} : \hat{g}_i(y, z) \geq 0, \ i \in [m+1] \}
\]
Note that \( \hat{g}_1 := R - \|y^2 - z^2\|^2_2 \) and \( \hat{g}_m := 1 \). Since \( f \geq 0 \) on \( S \), replacing \( x \) by \( y^2 - z^2 \) gives \( \hat{f} \geq 0 \) on \( \hat{S} \).

From this and Corollary \ref{corollary:main} there exist \( \eta^{(k)} \in \mathbb{R}^{(2n+k+d_\iota-d_{\iota})} \)
such that
\[
\bigl(\|y, z\|^2 + n + 2\bigl)^{k}(\hat{f} + \varepsilon) = \sum_{i=1}^{m+1} \hat{g}_i \hat{v}_{k+d_\iota-d_{\iota}} \mathbb{R} \diag(\eta^{(k)} \hat{v}_{k+d_\iota-d_{\iota}})^\top (y, z).
\]

With \( y = \frac{1}{2}(x + e) \) and \( z = \frac{1}{2}(x - e) \), it becomes
\[
\frac{1}{4}(\|x\|_2^2 + n + 2)^{k}(f + \varepsilon) = \sum_{i=1}^{m+1} g_i \hat{v}^\top_{k+d_\iota-d_{\iota}} \diag(\eta^{(k)} \hat{v}_{k+d_\iota-d_{\iota}}).
\]

Here \( g_{m+1} := \hat{g}_{m+1}(\frac{1}{2}(x + e), \frac{1}{2}(x - e)) = \frac{1}{2}g_1(x) \). Indeed, since \( y^2 - z^2 = x, \ \hat{f}(y, z) = f(x) \) and \( \hat{g}_i(y, z) = g_i(x) \), for \( i \in [m] \). Since \( y^2 + z^2 = \frac{1}{2}(x^2 + e) \), \( \|y\|^2 + \|z\|^2 = \frac{1}{2}(\|x\|^2 + n) \). This implies that
\[
\hat{g}_{m+1}(y, z) = \frac{1}{2}(L + n) - \|\hat{y}, \hat{z}\|^2_2 = \frac{1}{2}(R - \|x\|^2_2) = \frac{1}{2}g_1(x).
\]

Moreover, if \( a \) belongs to the interior of \( S \), then \( \frac{1}{2}(a + e), \frac{1}{2}(a - e) \) belongs to the interior of \( \hat{S} \). Thus, the desired result follows. \( \square \)

As a consequence of Corollary \ref{corollary:main} the next proposition states a new representation associated with diagonal Gram matrices for a polynomial positive on a compact semialgebraic set without assumption on even property.

**Proposition 3.** (Representation without even symmetry) Let \( f, g_i, S, d_{\iota} \) be as in Proposition \ref{prop:main} Then the following statements hold:
1. For all $\varepsilon > 0$, there exists $K_{\varepsilon} \in \mathbb{N}$ such that for all $k \geq K_{\varepsilon}$, there exist vectors $\eta^{(i,r)} \in \mathbb{R}_{+}^{(2n,k-d_{R},-r)}$ satisfying
\[
f + \varepsilon = \sum_{i=1}^{m} \sum_{r=0}^{k-d_{R}} g_{i} \hat{v}_{k-d_{R},-r} \text{diag}(\eta^{(i,r)}) \hat{v}_{k-d_{R},-r}.
\] (6.38)

2. If $S$ has nonempty interior, then there exist positive constants $\bar{c}$ and $\delta$ depending on $f, g_{i}$ such that for all $\varepsilon > 0$, one can take $K_{\varepsilon} = \bar{c} \varepsilon^{-\delta}$.

The proof of Proposition 6.4 relies on Corollary 4.4 and can be proved in almost the same way as the proof of Proposition 6.2.

Remark 16. Representation (6.38) in Proposition 6.4 is similar in spirit to the one of Roebers et al. in [44, (3), page 4]. The difference here is that the weight of each constrained polynomial $g_{i}$, $i \notin \{1, m\}$, in (6.38) is the polynomial
\[
\sum_{r=0}^{k-d_{R}} g_{i} \hat{v}_{k-d_{R},-r} \text{diag}(\eta^{(i,r)}) \hat{v}_{k-d_{R},-r},
\] (6.39)
which does not involve $g_{i}$. This is in contrast with the weight associated to each $g_{1}$ in [44 (3), page 4], which is of the form $\sigma_{i}(U_{i} - g_{i})$, where $U_{i}$ is an upper bound of $g_{i}$ on the ball $\{x \in \mathbb{R}^{n} : g_{i}(x) \geq 0\}$ and $\sigma_{i}$ is a univariate polynomial, e.g., $\sigma_{i}(t) = t^{2k}$ for some $\xi \in \mathbb{N}$.

Remark 17. In view of Propositions 6.4 and 6.3 replacing the standard monomial basis $v_{i}$ by the new basis $\hat{v}_{i}$ can provide a Positivstellensatz involving Gram matrix of factor width 1. Thus, ones can build up hierarchies of semidefinite relaxations with any maximal matrix size, based on both representations (6.38) and (6.39). However, expressing the entries of the basis $\hat{v}_{i}$ is a time-consuming task within the modeling process. A potential workaround is to impose (6.38) and (6.39) on a set of generic points similarly to [27 Section 2.3]. This needs further study.

6.4 Polynomial optimization on the nonnegative orthant: Noncompact case

6.4.1 Linear relaxations

Given $\varepsilon > 0$, consider the hierarchy of linear programs indexed by $k \in \mathbb{N}$:
\[
\tau_{k}^{\text{Pol}}(\varepsilon) := \inf_{y} L_{y}(\hat{f} + \varepsilon \hat{\theta}^{d_{f}}),
\] s.t. $y = (y_{\alpha})_{\alpha \in \mathbb{N}^{d_{f}}(\hat{y} \hat{g}_{i})} \subset \mathbb{R}_{+}, L_{y}(\hat{\theta}^{k}) = 1, \text{diag}(M_{k}(\hat{y}_{i}y)) \in \mathbb{R}_{+}^{d_{R},(n,k)}, i \in [m],
\] (6.40)
where $k_{i} := k + d_{f} - d_{R}, i \in [m]$. Here $\hat{g}_{m} = 1$. Note that
\[
\text{diag}(M_{k}(\hat{y}_{i}y)) = (\sum_{\gamma \in \mathbb{N}_{d_{R}}^{d_{R}}} y_{2\alpha+\gamma} \hat{g}_{i} \gamma)_{\alpha \in \mathbb{N}_{d_{R}}^{d_{R}}}.
\] (6.41)

Theorem 6. Let $f, g_{i} \in \mathbb{R}[x], i \in [m]$, with $g_{m} = 1$. Consider POP (6.4) with $S$ being defined as in (6.22). Let $\varepsilon > 0$ be fixed. For every $k \in \mathbb{N}$, the dual of (6.40) reads as:
\[
\rho_{k}^{\text{Pol}}(\varepsilon) := \sup_{\lambda, \hat{\lambda}_{m}} \lambda
\] s.t. $\lambda \in \mathbb{R}, \hat{\lambda}_{m} \in \mathbb{R}_{+}^{d_{R},(n,k)}, i \in [m], \theta^{k}(\hat{f} + \varepsilon \hat{\theta}^{d_{f}}) = \sum_{i=1}^{m} \hat{g}_{i} \hat{v}_{k_{i}} \text{diag}(u_{i}) \hat{v}_{k_{i}}.
\] (6.42)

Here $\hat{g}_{m} = 1$. The following statements hold:

1. For all $k \in \mathbb{N}$, $\rho_{k}^{\text{Pol}}(\varepsilon) \leq \rho_{k+1}^{\text{Pol}}(\varepsilon) \leq f^{*}$.

2. There exists $K \in \mathbb{N}$ such that for all $k \geq K$, $0 \leq f^{*} - \rho_{k}^{\text{Pol}}(\varepsilon) \leq \varepsilon(\hat{\theta}(x^{2}))^{d_{f}}$.

3. If $S$ has nonempty interior, there exist positive constants $\bar{c}$ and $\delta$ depending on $f, g_{i}$ such that for all $k \geq \bar{c} \varepsilon^{-\delta}$, $0 \leq f^{*} - \rho_{k}^{\text{Pol}}(\varepsilon) \leq \varepsilon(\hat{\theta}(x^{2}))^{d_{f}}$.

The proof of Theorem 6 relies on Corollary 1 and is exactly the same as the proof of [29 Theorem 7].
6.4.2 Semidefinite relaxations

Given $\varepsilon > 0$, consider the hierarchy of semidefinite programs indexed by $s \in \mathbb{N}_{>0}$ and $k \in \mathbb{N}$:

$$\tau_{k,s}^{\rho}(\varepsilon) := \inf_{\mathbf{y}} \frac{1}{\varepsilon} L_{\rho}(\theta^{k}(\tilde{f} + \varepsilon \theta^{d_f}))$$

s.t. \hspace{1cm} \mathbf{y} = (y_{a})_{a \in \mathbb{N}_{d_f+k}^{d_f+k}} \subseteq \mathbb{R}, \hspace{1cm} L_{\rho}(\theta^{k}) = 1, \hspace{1cm} M_{\mathcal{A}_{0}^{d_f+k}}(\mathbf{y}) \geq 0, \hspace{1cm} j \in [b(n,k)], \hspace{1cm} i \in [m], \hspace{1cm} (6.43)$$

where $k_i := k + d_f - d_{r_i}$, $i \in [m]$. Here $\hat{g}_m = 1$.

**Theorem 7.** Let $f, g \in \mathbb{R}[x], i \in [m]$, with $g_n = 1$. Consider POP (6.1) with $S$ being defined as in (6.2).

Let $\varepsilon > 0$ be fixed. For every $k \in \mathbb{N}$ and for every $s \in \mathbb{N}_{>0}$, the dual of (6.43) reads as:

$$\hat{\rho}_{k,s}^{\rho}(\varepsilon) := \sup_{\lambda, \mathbf{G}_{ij}} \lambda$$

s.t. \hspace{1cm} $\lambda \in \mathbb{R}, \hspace{1cm} \mathbf{G}_{ij} \succeq 0, \hspace{1cm} j \in [b(n,k)], \hspace{1cm} i \in [m], \hspace{1cm} (6.44)$$

$$\theta^{k}((\tilde{f} - \lambda + \varepsilon \theta^{d_f}) = \sum_{i \in [m]} \hat{g}_i \left( \sum_{j \in [b(n,k)]} \mathbf{v}_{\mathcal{A}_{0}^{d_f+k}}^{T} \mathbf{G}_{ij} \mathbf{v}_{\mathcal{A}_{0}^{d_f+k}} \right).$$

The following statements hold:

1. For all $k \in \mathbb{N}_{>0}$ and for every $s \in \mathbb{N}_{>0}$, $\hat{\rho}_{k,s}^{\rho}(\varepsilon) \leq \hat{\rho}_{k,s}^{\rho}(\varepsilon)$.
2. For every $s \in \mathbb{N}_{>0}$, there exists $K \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ satisfying $k \geq K$, $0 \leq f^{\ast} - \hat{\rho}_{k,s}^{\rho}(\varepsilon) \leq \varepsilon \theta(\mathbf{x}^{N})^{d_f}$.
3. If $S$ has nonempty interior, there exist positive constants $\bar{c}$ and $\bar{c}$ depending on $f, g$, such that for every $s \in \mathbb{N}_{>0}$ and for every $k < F_{n,m}^{\ast}$ satisfying $k \geq k$, $0 \leq f^{\ast} - \hat{\rho}_{k,s}^{\rho}(\varepsilon) \leq \varepsilon \theta(\mathbf{x}^{N})^{d_f}$.
4. If $S$ has nonempty interior, for every $s \in \mathbb{N}_{>0}$ and for every $k \in \mathbb{N}$ strong duality holds for the primal-dual problems (6.33–6.44).

The proof of Theorem 7 is based on Theorem 3 (31) and the inequalities $\hat{\rho}_{k,s}^{\rho}(\varepsilon) \leq \hat{\rho}^{\rho}(\varepsilon)$, where $\hat{\rho}^{\rho}(\varepsilon)$ is defined as in (31) (113). For each $q \in \mathbb{R}[x]_d$, denote the degree-$d$ homogenization of $q$ by $x_{d+1}^T q(x_{d+1}) \in \mathbb{R}[x, x_{d+1}]$.

**Remark 18.** Let $(\lambda, \mathbf{G}_{ij})$ be a feasible solution of (6.44) and consider the case of $m = 1$. Then the equality constraint of (6.44) becomes

$$\theta^{k}(\tilde{f} - \lambda + \varepsilon \theta^{d_f}) = \sum_{i \in [b(n,k)]} \mathbf{v}_{\mathcal{A}_{0}^{d_f+k}}^{T} \mathbf{G}_{ij} \mathbf{v}_{\mathcal{A}_{0}^{d_f+k}} \hspace{1cm} (6.45)$$

It implies that the degree-2$d_f$ homogenization of $\tilde{f} - \lambda + \varepsilon \theta^{d_f}$ belongs to the cone $k$-DSOS+$1, 2d_f$ (resp. $k$-SOS+$1, 2d_f$) when $s = 1$ (resp. $s = 2$) according to [1] Definition 3.10. More generally, the polynomial $\theta^{k}(\tilde{f} - \lambda + \varepsilon \theta^{d_f})$ belongs to the cone of SOS polynomials whose Gram matrix has factor width at most $s$ (see [1] Section 5.3).

6.5 Sparse representations

For every $I = \{i_1, \ldots, i_r\} \subseteq [n]$ with $i_1 < \cdots < i_r$, denote $x(I) = (x_{i_1}, \ldots, x_{i_r})$.

We will make the following assumptions:

**Assumption 1.** With $p \in \mathbb{N}_{>0}$, the following conditions hold:

1. There exists $(I_c)_{c \in [p]}$ being a sequence of subsets of $[n]$ such that $ \cup_{c \in [p]} I_c = [n] $ and $ \forall c \in \{2, \ldots, p\}, \exists c \in [c-1] : I_c \cap (\cup_{c-1}^{c-1} I_l) \subseteq I_{c-1}. \hspace{1cm} (6.46) $ Denote $n_c := |I_c|$, for $c \in [p]$.  
2. With $m \in \mathbb{N}_{>0}$ and $(g_c)_{c \in \mathbb{N}} \subseteq \mathbb{R}[x]$, there exists $(J_c)_{c \in [p]}$ being a finite sequence of subsets of $[m]$ such that $ \cup_{c \in [p]} J_c = [m] $ and $ \forall c \in [p], (g_c)_{c \in J_c} \subseteq \mathbb{R}[x(I_c)]. \hspace{1cm} (6.47) $  
3. For every $c \in [p]$, there exists $i_c \in J_c$ and $R_c > 0$ such that $ g_c := R_c - \|x(I_c)\|_2^2. \hspace{1cm} (6.48) $  

The condition (6.46) is called the running intersection property (RIP). Let $\theta_c := 1 + \|x(I_c)\|_2^2, c \in [p]$.  

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6.5.1 Extension of Pólya’s Positivstellensatz

We state the sparse representation in the following theorem:

**Theorem 8.** Let $g_1, \ldots, g_m$ be polynomials such that $g_1, \ldots, g_m$ are even in each variable and Assumption 4 holds. Let $S$ be the semialgebraic set defined by

$$S := \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \}. \quad (6.49)$$

Let $f = f_1 + \cdots + f_p$ be a polynomial such that $f$ is positive on $S$ and for every $c \in [p]$, $f_c \in \mathbb{R}[x(I_c)]$ is even in each variable. Then there exist $d, k \in \mathbb{N}$, $h_c \in \mathbb{R}[x(I_c)]$, $\sigma_0, \sigma_c, \sigma_c \in \mathbb{R}[x(I_c)]$ for $j \in J_c$ and $c \in [p]$, such that the following conditions hold:

1. The equality $f = h_1 + \cdots + h_p$ holds and $h_c$ is a polynomial of degree at most $2d$ which is even in each variable.
2. For all $i \in J_c$ and $c \in [p]$, $\sigma_0, \sigma_c, \sigma_c$ are SOS of monomials satisfying

$$\deg(\sigma_0, c) \leq 2(k + \theta_c) \quad \text{and} \quad \deg(\sigma_c, g_i) \leq 2(k + \theta_c) \quad (6.50)$$

and

$$\theta_c^k h_c = \sigma_0, c + \sum_{i \in J_c} \sigma_c, g_i. \quad (6.51)$$

**Proof.** Let $\varepsilon > 0$. Similarly as in Step 1 of the proof of Theorem 1, $\tilde{f} = \tilde{f}_1 + \cdots + \tilde{f}_m$ is positive on the semialgebraic set $S$ defined in (6.49). For every $c \in [p]$, let $J_c := J_c \cup (m + I_c)$. Recall that $\tilde{g}_{m+i} := x_j$, $j \in [n]$. By applying [20] Lemma 4, there exist polynomials $s_c, \tilde{g}_c, \tilde{g}_c \in \mathbb{R}[x(I_c)]$, for $j \in J_c$ and $c \in [p]$, such that

$$\tilde{f} = \sum_{c=0}^m (s_c + \sum_{i \in J_c} \tilde{g}_c \tilde{g}_i), \quad (6.52)$$

and for all $c \in [p]$, $s_c$ is positive on the set

$$\{ x(I_c) \in \mathbb{R}^{n_c} : x_j \geq 0, J \subseteq I_c, \tilde{g}_c, (x) = R_c - \sum_{j \in J_c} x_j \geq 0 \}. \quad (6.53)$$

Set $h_c := s_c(x^2) + \sum_{i \in J_c} \tilde{g}_c(x^2) \tilde{g}_i(x^2), c \in [p]$. Let $d \in \mathbb{N}$ such that $2d - 1 \geq \max \{ \deg(h_c) : c \in [p] \}$. Then $f = \sum_{c=0}^m h_c$ with $h_c \in \mathbb{R}[x(I_c)]_{2d}$ being even in each variable and positive on the semialgebraic set

$$S_c := \{ x(I_c) \in \mathbb{R}^{n_c} : g_c(x) \geq 0, i \in J_c \}. \quad (6.54)$$

Note that $g_c := R_c - \|x(I_c)\|_2^2$ with $i_c \in J_c$. By applying Corollary 2 there exists $k_c \in \mathbb{N}$ such that for all $K \geq k_c$, there exist $\sigma_0, \sigma_c, \sigma_c \in \mathbb{R}[x(I_c)]$, $i \in J_c$, such that $\sigma_0, \sigma_c, \sigma_c$ are SOS of monomials satisfying

$$\deg(\sigma_0, c) \leq 2(K + d) \quad \text{and} \quad \deg(\sigma_c, g_i) \leq 2(K + d) \quad (6.55)$$

for all $i \in J_c$, and

$$\theta_c^k h_c = \sigma_0, c + \sum_{i \in J_c} \sigma_c, g_i. \quad (6.56)$$

Set $k = \max\{ k(c) : c \in [p] \}$. Finally, we obtain the desired results.

**Remark 19.** In Theorem 8 it is not hard to see that $f$ has a rational SOS decomposition

$$f = \sum_{c=0}^p \frac{\sigma_0, c + \sum_{i \in J_c} \sigma_c, g_i}{\tilde{g}_c}. \quad (6.57)$$

This decomposition is simpler than the ones provided in [20] and thus is more applicable to polynomial optimization.

Another sparse representation without denominators can be found in the next theorem. However, the number of SOS of monomials is not fixed in this case.

6.5.2 Handelman-type Positivstellensatz

We process similarly to the proof of Theorem 8 and apply Corollary 5 to obtain the following theorem:

**Theorem 9.** (Sparse representation without multiplier) Let $f, g_1, \ldots, g_m$ be as in Theorem 8. Assume that $g_m := 1$ and $m \in J_c$, for all $c \in [p]$. Denote $d_{ni} := \lfloor \deg(g_i)/2 \rfloor$. Then there exist $k \in \mathbb{N}$, SOS of monomials $\sigma_{i, i}, \sigma_{i, c} \in \mathbb{R}[x(I_c)]$, for $j = 0, \ldots, k - d_{ni}$, $i \in J_c$ and $c \in [p]$, satisfying

$$\deg(\sigma_{i, i}, g_i) \leq 2k \quad (6.58)$$

and

$$f = \sum_{c=0}^p \sum_{i \in J_c} \sigma^{k-d_{ni}}_{i, i} g_i. \quad (6.59)$$
6.6 Sparse polynomial optimization on the nonnegative orthant

Consider the following POP:

\[ f^* := \inf_{x \in S} f(x), \tag{6.60} \]

where \( f \in \mathbb{R}[x] \) and

\[ S = \{ x \in \mathbb{R}^n : x_j \geq 0, \ j \in [n], g_i(x) \geq 0, \ i \in [m] \}, \tag{6.61} \]

for some \( g_i \in \mathbb{R}[x], \ i \in [m] \), with \( g_m = 1 \). Assume that \( f^* > -\infty \) and problem (6.60) has an optimal solution \( x^* \).

Then POP (6.60) is equivalent to

\[ f^* := \inf_{x \in \hat{S}} \hat{f}, \tag{6.62} \]

where

\[ \hat{S} = \{ x \in \mathbb{R}^n : \tilde{g}_i(x) \geq 0, \ i \in [m] \}, \tag{6.63} \]

with optimal solution \( x^{*\text{2}} \).

We will make the following assumptions:

**Assumption 2.** With \( p \in \mathbb{N}_{>0} \), the first two conditions of Assumption 2 and the following conditions hold:

1. For every \( c \in [p] \), there exist \( i_c \in J_c \) and \( R_c > 0 \) such that

\[ g_{i_c} = R_c - \sum_{j \in J_c} x_j. \tag{6.64} \]

2. There exist \( f_c \in \mathbb{R}[x(L_c)] \), for \( c \in [p] \), such that \( f = f_1 + \cdots + f_p \).

### 6.6.1 Linear relaxations

**Based on the extension of Pólya’s Positivstellensatz:** Consider the hierarchy of linear programs indexed by \( k, d \in \mathbb{N} \):

\[ T_{k,d}^{\text{poly}} := \inf_{y,y^{(c)}} L_y(\hat{f}) \]

s.t.

\[ \begin{align*}
    y = (y_0)_{\alpha \in \mathbb{N}^n_+} \subset \mathbb{R}, \ y^{(c)} = (y^{(c)}_\alpha)_{\alpha \in \mathbb{N}^{2(d+k)}_+} \subset \mathbb{R}, \ c \in [p], \ & \text{diag}(M_d(y, I_c)) = \text{diag}(M_d(\theta^{(c)}_k y^{(c)}, I_c)), \ c \in [p], \\
    \text{diag}(M_{k(d)}(\tilde{g}_i y^{(c)}, I_c)) \in \mathbb{R}_+^{l(n_c,k(d))}, \ i \in [m], \ c \in [p], \ y_0 = 1,
\end{align*} \tag{6.65} \]

where \( k^{(d)}_i := k + d - d_{y_i} \).

**Theorem 10.** Let \( f, g_i \in \mathbb{R}[x], \ i \in [m] \), with \( g_m = 1 \). Consider POP (6.60) with \( S \) being defined as in (6.61). Let Assumption 2 hold. The dual of SDP (6.65) reads as:

\[ \rho_{k,d}^{\text{poly}} := \sup_{\lambda, \mu_c, \nu_{c,d}^{(c)}} \lambda \]

s.t.

\[ \begin{align*}
    \lambda & \in \mathbb{R}, \ \mu_c \in \mathbb{R}^{l(n_c,d)}, \ \nu_{c,d}^{(c)} \in \mathbb{R}^{l(n_c,k(d))}, \ i \in [m], \ c \in [p], \\
    f - \lambda = \sum_{c \in [p]} h_c, \ h_c = v_{c,d}^{(c)} \text{diag}(u_c) v_{c,d}^{(c)}, \ c \in [p], \\
    \theta^{(d)}_k h_c = \sum_{i \in J_c} \tilde{g}_i v_{c,d}^{(c)} \text{diag}(w_{i,d}^{(c)}) v_{c,d}^{(c)}, \ c \in [p].
\end{align*} \tag{6.66} \]

The following statements hold:

1. For all \( k \in \mathbb{N} \) and for every \( s \in \mathbb{N}_{>0} \), \( \rho_{k,d}^{\text{poly}} \leq \rho_{k,d+1}^{\text{poly}} \leq \rho_{k,d+1}^{\text{poly}} \leq f^* \).

2. One has

\[ \sup(\rho_{k,d}^{\text{poly}} : (k, d) \in \mathbb{N}^2) = f^*. \tag{6.67} \]

**Proof.** It is fairly easy to see that the first statement holds. Let us prove the second one. Let \( \tilde{g}_{i_c} := R_c - \|x(I_c)\|_2^2 \) and \( \varepsilon > 0 \). Then \( \hat{f} - (f^* - \varepsilon) > 0 \) on \( S \). By applying Theorem 2 there exist \( d, k \in \mathbb{N}, \ h_c \in \mathbb{R}[x(I_c)], \alpha_{c,d}, \sigma_c \in \mathbb{R}[x(L_c)] \), for \( j \in J_c \) and \( c \in [p] \), such that the following conditions hold:

1. The equality \( \hat{f} - (f^* - \varepsilon) = h_1 + \cdots + h_p \) holds and \( h_c \) is a polynomial of degree at most \( 2d \) which is even in each variable.
2. For all $i \in J_c$ and $c \in [p]$, $\sigma_{0,e}, \sigma_{i,e}$ are SOS of monomials satisfying
\[
\deg(\sigma_{0,e}) \leq 2(k + d) \quad \text{and} \quad \deg(\sigma_{i,e}, \hat{h}_k) \leq 2(k + d)
\] (6.68)
and
\[
\hat{h}_k = \sigma_{0,e} + \sum_{i \in J_c} \sigma_{i,e}, \hat{h}_k.
\] (6.69)
It implies that there exists $u_c \in \mathbb{R}^{h(n_c,d)}$, $w^{(e)}_i \in \mathbb{R}^{h(n_c,k_d)}$ such that
\[
h_c = v^T_{n_c} \text{diag}(u_c)v_{n_c}^T \quad \text{and} \quad \sigma_{i,e} := v^T_{k_d} \text{diag}(w^{(e)}_i)v_{k_d}^T,
\] (6.70)
for $i \in J_c$ and $c \in [p]$. It implies that $(f^* - \varepsilon, u_c, w^{(e)}_i)$ is an optimal solution of LP (6.66). Thus $\rho_{k,d} \geq f^* - \varepsilon$, yielding (6.67).

Based on the Handelman-type Positivstellensatz: Consider the hierarchy of linear programs indexed by $k \in \mathbb{N}$:
\[
\tau_k^{\text{popHan}} := \inf_{y} L_y(f)
\]
s. t. $y = (y_0)_{a \in n_k} \subset \mathbb{R}$, $y_0 = 1$, $\text{diag}(M_{k}(y_{i,j}))_{i,j} \in \mathbb{R}^{h(n_c,k_{i,j})}$, $c \in [p]$, $i \in [m]$, $j \in [0] \cup [k - d_{y_i}]$, where $k_{ij} := k - d_{y_i} - j$, for $i \in [m]$, for $j \in [0] \cup [k - d_{y_i}]$.

Theorem 11. Let $f, g_i \in \mathbb{R}[x]$, $i \in [m]$, with $g_{m} = 1$. Consider POP (6.60) with $S$ being defined as in [6.61]. Let Assumption 3 hold. The dual of SDP (6.71) reads as:
\[
\bar{\rho}_k^{\text{popHan}} := \sup_{\lambda, w^{(e)}_i} \lambda
\]
s. t. $\lambda \in \mathbb{R}$, $w^{(e)}_i \in \mathbb{R}^{h(n_c,k_{i,j})}$, $c \in [p]$, $i \in J_c$, $j \in [0] \cup [k - d_{y_i}]$, $f - \lambda = \sum_{c \in [p]} \sum_{i \in J_c} \sum_{j=0}^{k-d_{y_i}} \hat{g}_i \hat{h}_k v^T_{k_d} \text{diag}(w^{(e)}_i)v_{k_d}^T$.

The following statements hold:
1. For all $k \in \mathbb{N}$, $\rho_k^{\text{popHan}} \leq \rho_{k+1}^{\text{popHan}} \leq f^*$.
2. The sequence $(\rho_k^{\text{popHan}})_{k \in \mathbb{N}}$ converges to $f^*$.

The proof of Theorem 11 relies on Theorem 3 and is similar to Theorem 5.

6.6.2 Semidefinite relaxations
For every $I \subset [n]$, we write $\mathbb{N}^I = \{\alpha_1^{(I)}, \alpha_2^{(I)}, \ldots, \alpha_i^{(I)}, \alpha_{i+1}^{(I)}, \ldots\}$ such that
\[
\alpha_1^{(I)} < \alpha_2^{(I)} < \cdots < \alpha_i^{(I)} < \alpha_{i+1}^{(I)} < \cdots.
\] (6.73)
Let
\[
W^{(I)}_j := \{i \in \mathbb{N} : i \geq j, \alpha_i^{(I)} + \alpha_j^{(I)} \in 2\mathbb{N}^I, \quad j \in N_{>0}, I \subset [n]\}.
\] (6.74)
Then for all $j \in N_{>0}$ and for all $I \subset [n]$, $W_j^{(I)} \neq \emptyset$ since $j \in W_j^{(I)}$. For every $j \in \mathbb{N}$ and for every $I \subset [n]$, we write $W_j^{(I)} := \{i^{(I)}_j, i^{(j)}_j, \ldots\}$ such that $i^{(I)}_j < i^{(j)}_j < \cdots$. Let
\[
T^{(s,d)}_{j,ij} = \{\alpha_i^{(I)}_{i^{(I)}_j}, \alpha_i^{(I)}_{i^{(j)}_j}, \ldots\} \cap N^I_d, \quad I \subset [n], \quad j, s \in N_{>0}, d \in \mathbb{N}.
\] (6.75)
For every $s \in N_{>0}$, for every $d \in \mathbb{N}$ and for every $I \subset [n]$, define $A_{j,ij}^{(s,d)} := T^{(s,d)}_{j,ij}$ and for $j = 2, \ldots, b(|I|, d)$, define
\[
A_{j,ij}^{(s,d)} := \begin{cases} T^{(s,d)}_{j,ij} & \text{if } T^{(s,d)}_{j,ij} \neq \emptyset, \forall l \in [j - 1], \\ \emptyset & \text{otherwise.} \end{cases}
\] (6.76)
Note that $A_{j,ij}^{(s,d)} = B_s$ and $|A_{j,ij}^{(s,d)}| \leq s$. Then the sequence
\[
(\alpha + \beta)_{\alpha, \beta \in A_{j,ij}^{(s,d)}}; \quad j \in b(|I|, d)
\] (6.77)
are overlapping blocks of size at most $s$ in $(\alpha + \beta)_{\alpha, \beta \in B_s}$. 

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Example 3. Consider the case of $n = d = s = 2$, $I_1 = \{1\}$ and $I_2 = \{2\}$. Matrix $(\alpha + \beta)_{(\alpha, \beta \in \mathbb{N}^2)}$ is written explicitly as in (3.15). We obtain two blocks:

$$(\alpha + \beta)_{(\alpha, \beta \in \mathbb{N}^2)} = \begin{bmatrix}
(0, 0) & (1, 0) & (2, 0) \\
(1, 0) & (2, 0) & (3, 0) \\
(2, 0) & (3, 0) & (4, 0)
\end{bmatrix}
$$

and

$$(\alpha + \beta)_{(\alpha, \beta \in \mathbb{N}^2)} = \begin{bmatrix}
(0, 0) & (0, 1) & (0, 2) \\
(0, 1) & (0, 2) & (0, 3) \\
(0, 2) & (0, 3) & (0, 4)
\end{bmatrix}
$$

Then $A^{(2,2)}_{i, t_1} = \{(0, 0), (2, 0)\}$, $A^{(2,2)}_{i, t_1} = \{(1, 0)\}$, $A^{(2,2)}_{t_2} = \emptyset$ and $A^{(2,2)}_{i, t_2} = \{(0, 0), (0, 2)\}$, $A^{(2,2)}_{t_2} = \{(0, 1)\}$.

For every $I \subset [n]$, with $\mathcal{B} = \{\beta_1, \ldots, \beta_s\} \subset \mathbb{N}^I$ such that $\beta_1 < \cdots < \beta_s$, for every $h = \sum_{\gamma \in \mathbb{N}^I} h_{\gamma} x^\gamma \in \mathbb{R}[x(I)]$ and $y = (y_\alpha)_{\alpha \in \mathbb{N}^2} \subset \mathbb{R}$, denote $M_\mathcal{B}(hy, I) := \left(\sum_{\gamma \in \mathbb{N}^I} h_{\gamma} y_{\gamma} + \alpha_i + \beta_s \right)_{i, j \in [r]}$.

Based on the extension of Pólya’s Positivstellensatz: Consider the hierarchy of linear programs indexed by $k, d \in \mathbb{N}$ and $s \in \mathbb{N}_{>0}$:

$$(\rho^{\text{PopPol}}_{k, d, s}) := \inf_{y \in \mathbb{R}^n} L_y(f)$$

s.t. $y = (y_\alpha)_{\alpha \in \mathbb{N}^2} \subset \mathbb{R}$, $y_0 = 1$, $\mathcal{M}_{A_{i, j}, t_2}((\tilde{g}, \tilde{v})_i, I_c) \geq 0$, $i \in J_c$, $c \in [p]$, $\tilde{f} - \lambda = \sum_{c \in [p]} h_c$, $h_c = v^T_{\alpha} \text{diag}(u_{\alpha}) v_{\alpha}$, $c \in [p]$, $A_{i, j, t_2} = 0$, $j \in [b(n_c, k_{ij})]$, $i \in J_c$, $c \in [p]$, $\vec{v}^T_{\alpha} v_{\alpha} = 1$

The following holds:

1. For all $k, d \in \mathbb{N}$ and for every $s \in \mathbb{N}_{>0}$, $\rho^{\text{PopPol}}_{k, d, s} = \rho^{\text{PopPol}}_{k, d, 1} \leq \rho^{\text{PopPol}}_{k, d, s} \leq f^*$.

2. For every $s \in \mathbb{N}_{>0}$, $\sup \rho^{\text{PopPol}}_{k, d, s} : (k, d) \in \mathbb{N}^2 = f^*$.

Proof. It is not hard to prove the first statement. The second one is due to the second statement of Theorem 11.

Based on the Handelman-type Positivstellensatz: Consider the hierarchy of linear programs indexed by $k \in \mathbb{N}$ and $s \in \mathbb{N}_{>0}$:

$$(\rho^{\text{HandHan}}_{k, s}) := \inf_{y} L_y(f)$$

s.t. $y = (y_\alpha)_{\alpha \in \mathbb{N}^2} \subset \mathbb{R}$, $y_0 = 1$, $\mathcal{M}_{A_{i, j}, t_2}((\tilde{g}, \tilde{v})_i, I_c) \geq 0$, $c \in [p]$, $i \in J_c$, $j \in \{0\} \cup \{k - d_{ij}\}$, $r \in [b(n_c, k_{ij})]$,

where $k_{ij} := k - d_{ij} - j$, for $i \in [m]$, for $j \in \{0\} \cup \{k - d_{ij}\}$.

Theorem 13. Let $f, g \in \mathbb{R}[x]$, $i \in [m]$, with $g_m = 1$. Consider POP (6.60) with $S$ being defined as in (6.61). Let Assumption 1 hold. The dual of SDP (6.62) reads as:

$$(\rho^{\text{HandHan}}_{k, s}) := \sup_{\lambda, \mathbf{G}^{(c)}_{ij}} \lambda$$

s.t. $\lambda \in \mathbb{R}$, $\mathbf{G}^{(c)}_{ij} \geq 0$, $c \in [p]$, $i \in J_c$, $j \in \{0\} \cup \{k - d_{ij}\}$, $r \in [b(n_c, k_{ij})]$, $\tilde{f} - \lambda = \sum_{e \in [p]} \sum_{i \in J_c} \sum_{j = 0}^{k - d_{ij}} \tilde{g} \tilde{v}^{(c)}_i \left(\sum_{r \in [b(n_c, k_{ij})]} v^T_{\alpha} \mathbf{G}^{(c)}_{ij} v_{\alpha} - \mathbf{A}^{(c, k_{ij})}_{i, j, t_2} \right)$.

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The following statements hold:
1. For all \( k \in \mathbb{N} \) and for every \( s \in \mathbb{N}_{>0} \),
   \[
   \rho_k^{\text{spHan}} = \rho_{k,1}^{\text{spHan}} \leq \rho_{k,s}^{\text{spHan}} \leq f^*. \tag{6.84}
   \]
2. For every \( s \in \mathbb{N}_{>0} \), the sequence \( (\rho_k^{\text{spHan}})_{k \in \mathbb{N}} \) converges to \( f^* \).

Proof. It is not hard to prove the first statement. The second one is due to the second statement of Theorem 11 and the inequalities \( \rho_k^{\text{spHan}} \leq \rho_{k,s}^{\text{spHan}} \leq f^* \).

\( \square \)

### 6.6.3 Obtaining an optimal solution

In other to extract an optimal solution of POP (6.60) with correlative sparsity, we first extract atoms on each clique similarly to Algorithm 1 and then connect them together to obtain atoms in \( \mathbb{R}^n \). Explicitly, we use the following heuristic extraction algorithm:

**Algorithm 2** Extraction algorithm for sparse POPs on the nonnegative orthant

**Input:** precision parameter \( \varepsilon > 0 \) and an optimal solution \((\lambda, u_c, G_{i,c})\) of SDP (6.81).

**Output:** an optimal solution \( x^* \) of POP (6.60).

1. For \( c \in [p] \), do:
   a. For \( j \in [b(n_c, k^{(d)})] \), let \( \tilde{G}_j^{(c)} = (u^{(c,j)}_{p,q})_{p,q \in N_c,} \) such that \( (w^{(c,j)}_{p,q})_{p,q \in \mathcal{A}_{i,c}^{(s,k^{(d))}}} = G_{m,j}^{(c)} \) and \( w^{(c,j)}_{p,q} = 0 \) if \( (p, q) \notin \mathcal{A}_{i,c}^{(s,k^{(d))}} \). Then \( \tilde{G}_j^{(c)} \geq 0 \) and \( \tilde{G}_j^{(c)} = \sum_{m,j} G_{m,j}^{(c)} v_{m,j} v_{m,j}^\top \).

2. Let \( \tilde{G}^{(c)} := \sum_{j \in [b(n_c, k^{(d))}]} \tilde{G}_j^{(c)} \). Then \( \tilde{G}^{(c)} \) is the Gram matrix corresponding to \( \sigma_m, c \) in the rational SOS decomposition
   \[
   \tilde{f} - \lambda = \sum_{c \in [p]} \sum_{i,c} \sigma_{i,c} \tilde{g}_i,
   \tag{6.86}
   \]
   where each \( \sigma_{i,c} \) is an SOS polynomial and \( \tilde{g}_m = 1 \);

3. Obtain an atom \( z^{*(c)} \in \mathbb{R}^{n_c} \) by using the extraction algorithm of Henrion and Lasserre in [22], where the matrix \( V \) in [22] (6) is taken such that the columns of \( V \) form a basis of the null space \( \{u \in \mathbb{R}^{n_c} : G^{(c)} u = 0\} \);

4. If \( z^* \) exists, verify that \( z^* \) is an approximate optimal solution of POP (6.62) by checking the following inequalities:
   \[
   |\tilde{f}(z^*) - \lambda| \leq \varepsilon ||f||_{\text{max}} \quad \text{and} \quad \tilde{g}_i(z^*) \geq -\varepsilon ||g_i||_{\text{max}}, \quad i \in [m],
   \tag{6.87}
   \]
   where \( ||q||_{\text{max}} := \max_{\alpha} |q_{\alpha}| \) for any \( q \in \mathbb{R}[x] \).

4. If the inequalities (6.87) hold, set \( x^* := z^*^2 \).

### 6.7 Numerical experiments

In this section we report results of numerical experiments for random instances with the same settings as in Section 4.

For numerical comparison purposes, recall the semidefinite relaxation based on the sparse version of Putinar’s Positivstellensatz for solving POP (133) (under Assumption 2) indexed by \( k \in \mathbb{N} \):

\[
\begin{align*}
\varepsilon_{k}^{\text{spPutar}} := \inf_{y} & \quad L_y(f) \\
\text{s.t.} & \quad y = (y_{\alpha})_{\alpha \in \mathbb{N}_k} \subset \mathbb{R}, y_0 = 1, \tag{6.88} \\
& \quad M_{i,c}^{(i-c)} (g_{i,y}) \geq 0, \quad i \in I_c, \ c \in [p].
\end{align*}
\]

Here \( I_c \subset [n], \ m := m + n, g_{m+j} := x_j, \ j \in [n], g_{m} := 1 \) and \( m \in J_c \subset [m], \) for \( c \in [p] \).

The notation for the numerical results is given in Table 12.
| Pb       | the ordinal number of a POP instance       |
|----------|------------------------------------------|
| Id       | the ordinal number of an SDP instance     |
| n        | the number of nonnegative variables in POP |
| m_{ineq} | the number of inequality constraints of the form $g_i \geq 0$ in POP |
| m_{eq}   | the number of equality constraints of the form $g_i = 0$ in POP |
| spPut    | the SDP relaxation for a sparse POP based on Putinar’s Positivstellensatz modeled by TSSOS and solved by Mosek 9.1 |
| spPół    | the SDP relaxation for a sparse POP based on the extension of Pólya’s Positivstellensatz modeled by our software InterRelax and solved by Mosek 9.1 |
| spHan    | the SDP relaxation for a sparse POP based on the Handelman-type Positivstellensatz modeled by our software InterRelax and solved by Mosek 9.1 |
| k        | the relaxation order                      |
| s        | the factor width upper bound used in SDP |
| d        | the sparsity order of the SDP relaxation  |
| nmat     | the number of matrix variables of an SDP  |
| msize    | the largest size of matrix variables of an SDP |
| nscal    | the number of scalar variables of an SDP  |
| naff     | the number of affine constraints of an SDP |
| val      | the value returned by the SDP relaxation  |
| *        | there exists at least one optimal solution of the POP, which can be extracted by Algorithm 1 or 6.6.3 |
| -        | the calculation runs out of space          |

### 6.7.1 Sparse QCQPs

**Test problems:** We construct randomly generated QCQPs in the form \([1.3]-[1.4]\) with correlative sparsity as follows:

1. Take a positive integer \(u, p := \lfloor n/u \rfloor + 1\) and let
   \[
   I_c = \begin{cases} \{u\}, & \text{if } c = 1, \\ \{u(c-1), \ldots, uc\}, & \text{if } c \in \{2, \ldots, p-1\}, \\ \{u(p-1), \ldots, n\}, & \text{if } c = p; \end{cases} \tag{6.89}
   \]

2. Generate a quadratic polynomial objective function \(f = \sum_{c \in [p]} f_c\) such that for each \(c \in [p]\), \(f_c \in \mathbb{R}[x(I_c)]_2\), and the coefficient \(f_{c, \alpha}, \alpha \in \mathbb{N}_2^n\) of \(f_c\) is randomly generated in \((-1, 1)\) w.r.t. the uniform distribution;

3. Take a random point \(a\) such that for every \(c \in [p]\), \(a(I_c)\) belongs to the simplex
   \[
   \Delta^{(c)} := \{x(I_c) \in \mathbb{R}^{n_c} : x_j \geq 0, j \in I_c, \sum_{j \in I_c} x_j \leq 1\} \tag{6.90}
   \]

4. Let \(q := \lfloor m_{ineq}/p \rfloor\) and
   \[
   J_c := \begin{cases} \{(c-1)q + 1, \ldots, cq\}, & \text{if } c \in [p-1], \\ \{(p-1)q + 1, \ldots, l\}, & \text{if } c = p. \end{cases} \tag{6.91}
   \]

For every \(c \in [p]\) and every \(i \in J_c\), generate a quadratic polynomial \(g_i \in \mathbb{R}[x(I_c)]_2\) by

(a) for each \(\alpha \in \mathbb{N}_2^n \setminus \{0\}\), taking a random coefficient \(G_{i, \alpha}\) of \(h_i\) in \((-1, 1)\) w.r.t. the uniform distribution;

(b) setting \(g_{i, 0} := 0.125 - \sum_{\alpha \in \mathbb{N}_2^n \setminus \{0\}} G_{i, \alpha} \alpha^{\alpha}.
\]

5. Take \(g_{i_c} := 1 - \sum_{c \in I_c} x_c\), for some \(i_c \in J_c\), for \(c \in [p]\);
Table 13: Numerical results for randomly generated QCQPs with correlative sparsity, \( n = 1000 \) and \( d = \deg(f) = 2 \).

| Id | Pb | POP size | spPut (\( u, m_{ineq}, m_{eq} \)) | spPol (\( k, val, time \)) | spHan (\( s, val, time \)) |
|----|----|----------|-------------------------------|--------------------------|-------------------------|
| 1  | 2  | 1 10 201 0 1 2                      | \( \infty \) 1.5 385 0 10 -12.906 15 | 2 7 -128.660 20         |
| 3  | 4  | 2 10 201 200 1 2                    | \( \infty \) 2.0 475 1 12 -65.319 51 | 3 10 -65.305 283        |
| 5  | 6  | 3 20 201 0 1 2                      | \( \infty \) 3.6 56360 0 15 -65.979 19 | 2 15 -65.864 24         |
| 7  | 8  | 4 20 201 200 1 2                    | \( \infty \) 9 1 22 -38.206 319 | 3 20 -38.2035 2146      |

| Id | nmat | msize | nscal | naff | spPut (\( nmat, msize, nscal, naff \)) | spPol (\( nmat, msize, nscal, naff \)) | spHan (\( nmat, msize, nscal, naff \)) |
|----|------|-------|-------|------|----------------------------------------|----------------------------------------|---------------------------------------|
| 1  | 100  | 12    | 1     | 7491 | 135641 299 10 7889 7491 599 7 9288 | 7491 |                        |
| 2  | 1300 | 78    | 1     | 1401 | 15577 1299 12 39920 43813 4184 10 41419 35926 |
| 3  | 100  | 12    | 1     | 1401 | 15577 1299 12 39920 43813 4184 10 41419 35926 |
| 4  | 1300 | 78    | 1     | 12481| 630231 399 15 25047 25109 399 15 13978 12481 |
| 5  | 50   | 22    | 1     | 12481| 630231 1149 22 108641 113428 3574 20 109990 100751 |
| 6  | 1250 | 253   | 1     | 12481| 630231 1149 22 108641 113428 3574 20 109990 100751 |

6. Let \( r := \lfloor m_{eq}/p \rfloor \) and

\[
W_e := \begin{cases} (c - 1)r + 1, \ldots , cr, & \text{if } c \in [p] \setminus \{p\}, \\ (p - 1)r + 1, \ldots , l, & \text{if } c = p. \end{cases}
\]  

For every \( c \in [p] \) and every \( i \in W_e \), generate a quadratic polynomial \( h_i, \alpha \) of \( h_i \) in \((-1,1)\) w.r.t. the uniform distribution;

(a) for each \( \alpha \in \mathbb{N}_2 \setminus \{0\} \), taking a random coefficient \( h_{i,\alpha} \) of \( h_i \) in \((-1,1)\) w.r.t. the uniform distribution;

(b) setting \( h_{i,0} := -\sum_{\alpha \in \mathbb{N}_2 \setminus \{0\}} h_{i,\alpha}a^\alpha \).

7. Take \( g_{i+m_{ineq}}(x) := h_i \) and set \( g_{i+m_{ineq}}(x) = -h_i \) for \( i \in [m_{eq}] \).

Here \( m = m_{ineq} + 2m_{eq} \) with \( m_{ineq} \) (resp. \( m_{eq} \)) being the number of inequality (resp. equality) constraints except the nonnegative constraints \( x_i \geq 0 \). The point \( a \) is a feasible solution of POP (1.3).

The numerical results are displayed in Table [13].

**Discussion:** Similarly to the previous discussion, spPol and spHan in Table [13] are also much faster and more accurate than spPut. For instance, when \( u = 20 \), \( m_{eq} = 201 \), and \( m_{eq} = 0 \), spPol takes 20 seconds to return the lower bound \(-65.9794\) for \( f^* \), while spPut takes 56360 seconds to return a worse bound of \(-66.1306\). In this case, spPol has 399 matrix variables with maximal matrix size 15, while spPut has 1250 matrix variables with maximal matrix size 253. In particular, spPol provides slightly better bounds than spP’ol for Pb 1, 2, 4 while it is seven (resp. five) times slower than spP’ol in Pb 4 (resp. Pb 2).

**6.7.2 Robustness certification of deep neural networks**

In [11], the robustness certification problem of a multi-layer neural network with ReLU activation function is formulated as the following QCQP for each \( y \):

\[
\begin{align*}
l^*_j(y, \hat{y}) := & \max_{x^L, \ldots, x^1} \left( c_y - c_{\hat{y}} \right)^\top x^L \\
& \text{s.t. } x^L (x^L - W_i^{-1}x^{i-1}) = 0, x^i \geq 0, x^i \geq W_i^{-1}x^{i-1}, \\
& \quad t \in [m], i \in [L], \quad -\varepsilon \leq x^i_t - \hat{x}^i_t \leq \varepsilon, t \in [m] \end{align*}
\]  

(6.93)
Table 14: Information for the training model (6.94).

| Dataset    | BHPD |
|------------|------|
| Number of hidden layers | $L = 2$ |
| Length of an input | 13 |
| Number of inputs | 506 |
| Test size | 20\% |
| Number of classes | $k = 3$ |
| Numbers of units in layers | $m = (13, 20, 10)$ |
| Number of weights | 490 |
| Optimization method | Adadelta algorithm |
| Accuracy | 70\% |
| Batch size | 128 |
| Epochs | 200 |

where we use the same notation as in [41, Section 2] and write $W_i^{-1} = \begin{bmatrix} W_{i-1}^1 \\ \vdots \\ W_{i-1}^m \end{bmatrix}$.

We say that the network is certifiably $\varepsilon$-robust on $(\bar{x}, \bar{y})$ if $l^*_y(\bar{x}, \bar{y}) < 0$ for all $y \neq \bar{y}$.

**Test problems:** To obtain an instance of weights $W^i$, we train a classification model by using Keras\footnote{https://keras.io/api/models/model_training_apis/}.

Explicitly, we minimize a loss function as follows:

$$\min_{W^0, \ldots, W^{L-1}} \frac{1}{2} \sum_{(x^0, y^0) \in D} \| f(x^0) - e_y^0 \|^2_2,$$

where the network $f$ is defined as in [41, Section 2] and $e_{y^0}$ has 1 at the $y^0$-th element and zeros at the others. Here the input set $D$ is a part of Boston House Price Dataset (BHPD). The class label $y^0$ is assigned to the input $x^0$. We classify the inputs from BHPD into 3 classes according to the MEDian Value of owner-occupied homes (MEDV) in $1000$ as follows:

$$y^0 = \begin{cases} 
1 & \text{if MEDV}<10, \\
2 & \text{if } 10 \leq \text{MEDV}<20, \\
3 & \text{otherwise}.
\end{cases}$$

We also take a clean input label pair $(\bar{x}, \bar{y}) \notin D$ with $\bar{y} = 3$ from BHPD.

As shown in [8, Section 4.2], POP (6.93) has correlative sparsity. To use our method, we convert (6.93) to a POP on the nonnegative orthant by defining new nonnegative variables $\tilde{z}_t := x_t^0 - \bar{x}_t + \varepsilon$. Doing so, the constraints $-\varepsilon \leq \bar{x}_t - x_t^0 \leq \varepsilon$ become $0 \leq \tilde{z}_t \leq 2\varepsilon$ in the new coordinate system. Here we choose $\varepsilon = 0.1$. More detailed information for our training model are available in Table 14.

The numerical results are displayed in Table 15.

**Discussion:** Compared to spPut, spPôl and spHan provide better upper bounds in less total time. Moreover, in Table 14 the values returned by spPut with $k = 1$ are positive and are much larger than the negative ones returned by spPut with $k = 2$. Since in Table 15, the upper bounds on $l_y^*(\bar{x}, \bar{y})$ are negative, for all $y \neq \bar{y}$, $l_y^*(\bar{x}, \bar{y})$ must be negative. Thus, we conclude that this network is certifiably $\varepsilon$-robust on $(\bar{x}, \bar{y})$.

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Table 15: Numerical results for robustness certification on BHPD, $n = 43$, $m_{ineq} = 43$, $m_{eq} = 30$ and $d = \deg(f) = 2$.

| Id | Pb | spPut | spPôl | spHan |
|----|----|-------|-------|-------|
|    | $y = 1$ | $y = 2$ |       |       |
| 1  | 1  | 1    | 1     |       |
| 2  | 2  | 0.4  | -     | 1364  |
| 3  | 3  | 0.4  | 1     | 1270  |

| Id | spPut | spPôl | spHan |
|----|-------|-------|-------|
| 1  | 23    | 35    | 46233 |
| 2  | 95    | 35    | 9670  |

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