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Topology of Foliations
given by the real part of holomorphic 1-forms

Abstract. Topology of Foliations of the Riemann Surfaces given by the real part of generic holomorphic 1-forms, is studied. Our approach is based on the notion of Transversal Canonical Basis of Cycles (TCB) instead of using just one closed transversal curve as in the classical approach of the ergodic theory. In some cases the TCB approach allows us to present a convenient combinatorial model of the whole topology of the flow, especially effective for $g=2$. A maximal abelian covering over the Riemann Surface provided by the Abel Map, plays a key role in this work. The behavior of our system in the Fundamental Domain of that covering can be easily described in the sphere with $g$ holes. It leads to the Plane Diagram of our system. The complete combinatorial model of the flow is constructed. It is based on the Plane Diagram and $g$ straight line flows in the planes corresponding to the $g$ canonically adjoint pairs of cycles in the Transversal Canonical Basis. These pairs do not cross each other. Making cuts along them, we come to the maximal abelian fundamental domain (associated with Abel Map and Theta-functions) instead of the standard $4g$-gon in the Hyperbolic Plane and its beautiful "flat" analogs which people used for the study of geodesics of the flat metrics with singularities.

Introduction. The family of parallel straight lines in the Euclidean Plane $\mathbb{R}^2$ gives us after factorization by the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ the standard straight line flow in the 2-torus $T^2$. It is a simplest ergodic system for the irrational direction. This system is Hamiltonian with multivalued Hamiltonian function $H$ and standard canonically adjoint euclidean coordinates $x, y$ (i.e. the 1-form $dH$ is closed but not exact, and $x_t = H_y, y_t = -H_x$ where $H_x, H_y$ are constant). Every smooth Hamiltonian system on the 2-torus without critical points with irrational "rotation number" is diffeomorphic to the straight line flow. Every $C^2$-smooth dynamical system on the 2-torus without critical points and with irrational rotation number is $C^0$-homeomorphic to the straight line flow according to the famous classical theorem (it is not

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true for $C^1$).

**Question:** What is a right analog of the straight line flow for the Riemann Surfaces of higher genus?

Certainly, it should be some special class of Hamiltonian Systems with multivalued Hamiltonian and trajectories given by foliation $dH = 0$. Which special subclass has the best properties? For the genus $g = 2$ the answer to this question will be given in the last section of this article.

In many cases we ignore time dependence of the trajectories and discuss only the properties of foliation $dH = 0$ given by the closed 1-form. The present author started to investigate such foliations in early 1980s as a part of the newborn Topology of The Closed 1-Forms. An important example was found in the Quantum Solid State Physics describing the motion of semiclassical electrons along the so-called Fermi Surface for the single crystal normal metals and low temperature in the strong magnetic field (see [1]). Good introduction to the corresponding Physics can be found in the textbook [2].

The Fundamental ”Geometric Strong Magnetic Field Principle” was formulated by the Kharkov school of I.Lifshitz many years ago (and fully accepted by physics community). It claims that following geometric picture gives a good description of electrical conductivity in the single crystal normal metals in the “reasonably strong” magnetic field (of the size $1t < B < 10^3t$ and temperature $T < 1K$ for the normal metal like gold): There is a Fermi Surface $M_F$ in the 3-torus of quantum “quasimomenta” $p \in T^3$ where $M_F \subset T^3$ is a nondegenerate level $\epsilon = \epsilon_F$ of the Morse function $\epsilon : T^3 \to R$ called the ”dispersion relation”. Every constant homogeneous magnetic field $B$ defines a 1-form $\sum_i B_idp_i$ whose restriction to the Fermi Surface is exactly our closed 1-form: $dH = \sum_i B_idp_i|_{M_F}$. The electron trajectories exactly coincide with connectivity components of the sections of Fermi Surface by the planes orthogonal to magnetic field. One might say that they are the levels of quasiperiodic function on the plane with 3 periods. An extensive study of that class was performed by the author’s Moscow Topological Seminar since early 1980s. These studies were continued in Maryland since the second half of 1990s. Among the author’s students who made important contribution here, let me mention A.Zorich, S.Tsarev, I.Dynnikov, A.Maltsev, R.Deleo (see the survey articles [3, 4, 5] describing topological, dynamical and physical results of our studies). This class of systems has remarkable ”topological complete integrability” and ”topological resonance” properties in the non-standard sense, for the set of directions of magnetic field of the full measure on the 2-sphere $S^2$. These properties play a key role in the physical applica-
tions. Numerical studies are described in [6]. According to the fundamental results of that theory, topological properties here generically can be reduced to the case of the genus 1. (Let us mention that the hamiltonian systems on 2-torus were studied in [23]; their ergodic properties were found finally in [24]). Our problem is more complicated. In particular, there exists a nonempty set of parameters with Hausdorff Dimension $d \leq 1$ (presumably, even strictly less than one) on the 2-sphere, where a complicated "chaotic" behavior was discovered.

Following class of systems was proposed by the various mathematicians (no applications outside of pure mathematics were found for them until now, unfortunately): Take any nonsingular compact Riemann Surface $V$, i.e complex algebraic curve, with genus equal to $g$. There exists a $g$-dimensional complex linear space $C^g$ of holomorphic differential forms with basis $\omega_1, \ldots, \omega_g \in C^g$. Every holomorphic 1-form $\omega \in C^g$ defines a Hamiltonian system (foliation) $F$ on the manifold $V$:  

$$F = \{ \omega^R = 0 \}$$

because $d(\omega^R) = 0$. Here $\omega = \omega^R + i\omega^I$. There is even more general class of "foliations $F$ with invariant transversal measure" on Riemann Surfaces: the existence of measure is required on the intervals transversal to the leaves (trajectories) invariant under the deformations such that every point is moving along trajectories. Take any holomorphic quadratic differential $\Omega$ and define foliation $F$ by the formula $(\sqrt{\Omega})^R = 0$. This is a locally hamiltonian foliation $\mathcal{F}$ (i.e. it admits a transversal measure) but may be non-orientable (it does not admit time direction globally), so we do not consider them.

The systems with transversal invariant measure were studied since early 1960s by the following method (see in the book [10]): Take any closed curve $\gamma$ transversal to our foliation $F$. Assume that almost every nonsingular trajectory is dense. For every point $Q \in \gamma$ except finite number there exists a first time $t_Q$ such that trajectory started in the point $Q$ returns to the curve $\gamma$ as a new point $P$ (in positive direction of time). We define a "Poincare map" $Q \rightarrow P$. This map by definition preserves a transversal measure which is a restriction of the form $dH$ on the curve $\gamma$. It is the Energy Conservation Law for Hamiltonian system. So our transversal closed curve $\gamma$ is divided into $k = k_\gamma$ intervals $\gamma = I_1 + I_2 + \ldots + I_k$. The Poincare map looks here as a simple permutation of these intervals on the circle; it is ill-defined in the finite number of points only. The time $t_Q$ varies continuously within each interval $Q \in I_j$. 

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No doubt, the use of closed transversal curves is extremely productive. At the same time, we are not satisfied by this approach; Following questions can be naturally asked:

1. This method essentially ignores time and length of trajectories starting and ending in $\gamma$. Where they are traveling and how long? We would like to see some sort of global topological description of the flow (or foliation $\mathcal{R}$) on the algebraic curve $V$ and its abelian coverings similar to the case of genus 1 as much as possible. The ergodic characteristics of foliation can be found here, but this model certainly in not enough for the description of topology.

2. There are many different closed transversal curves in the foliation $\mathcal{R}$. How this picture depends on the choice of transversal curve $\gamma$? Nobody classified them yet as far as I know. Indeed, in the theory of codimension 1 foliations developed by the present author in 1960s (see [7]) several algebraic structures were defined for the closed transversal curves:

   Fix any nonsingular point $Q \in V$ and consider all closed positively (negatively) oriented transversal curves starting and ending in $Q$. We can multiply them. Transversal homotopy classes of such curves generate a Transversal Semigroup $\pi_1^+(\mathcal{R}, Q)$ and its natural homomorphism into the fundamental group (even, into the fundamental group of the unit tangent $S^1$-bundle $L(V)$

   \[ \psi^\pm : \pi_1^\pm(\mathcal{R}, Q) \to \pi_1(L(V), Q) \to \pi_1(V, Q) \]

The set of all closed transversal curves naturally maps into the set of conjugacy classes in fundamental group. We denote it also by $\psi$. The Transversal Semigroups might depend of the leaf where the initial point $Q$ is chosen. For example, they are different for the separatrices entering critical points and for the generic nonsingular leaves.

**How to calculate these invariants for the Hamiltonian foliations described above on the algebraic curves?**

Let me point out the simplest fundamental properties of these foliations:

Property 1. They have only saddle type critical points. In the generic case such foliation has exactly $2g-2$ nondegenerate saddles.

Property 2. Every nonempty closed transversal curve $\gamma$ is non-homologous to zero. The period of the form $dH$ is positive $\int_\gamma dH > 0$ for every positively oriented transversal curve. So the composition

\[ \pi_1^+(\mathcal{R}, Q) \to \pi_1(V, Q) \to H_1(V, Z) \to \mathbb{R} \]
does not map any element into zero. Its image is strictly positive.

We present here a theory of the special classes $T$ and $T^k$ of these foliations. Some of them can be considered (in the generic case, at least) as a natural higher genus analog of the straight line flow, especially for $g = 2$. The definition of the classes is following:

**Definitions**

a. We say that foliation on the Riemann surface $V$ belongs to the **Class $T$** if there exist a **Transversal Canonical Basis** of curves

$$a_1, b_1, \ldots, a_g, b_g$$

such that

I. All these curves are nonselfintersecting and transversal to the foliation.

II. The curves $a_j$ and $b_j$ cross each other transversally for all $j = 1, \ldots, g$ exactly in one point. Other curves do not cross each other.

b. We say that foliation belongs to the **Class $T^0$** if there exists a canonical basis such that all a-cycles $a_1, \ldots, a_g$ are non-selfintersecting, do not cross each other and are transversal to the foliation.

c. We say that foliation belongs to the **Mixed Class** of the Type $T^k$ if there exists an incomplete canonical basis $a_j, b_q, j \leq g, q \leq k$, such that all these cycles are non-selfintersecting. The only pairs crossing each other are $a_j$ and $b_j$ for $j = 1, 2, \ldots, k$. The intersections of $a_j$ and $b_j$ are transversal and consist of one point each. All these curves $a_j, b_q, j \leq g, q \leq k$ are transversal to foliation. We have $T = T^g$

**Remark 1** A number of people including Katok, Hasselblatt, Hubbard, Mazur, Veech, Zorich, Konzevich, McMullen and others wrote a lot of works related to study the ergodic properties of foliations with "transversal measure" on the Riemann Surfaces, and their total moduli space (see [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]). People investigated recently closed geodesics of the flat Riemannian Metric $ds^2 = \omega \bar{\omega}$ singular in the critical points of the holomorphic 1-form $\omega$. These geodesics consists of all trajectories of the Hamiltonian systems $(\exp\{i\phi\} \omega)^R = 0$ on the algebraic curve $V$. Closed geodesics appear for the special number of angles $\phi_n$ only. Therefore we don't see them describing the generic Hamiltonian systems of that type. Beautiful analogs of the Poincare' 4-gons associated with these flat metrics with singularities were invented and used. Our intention is to describe completely topology of some specific good classes of the generic foliations given by
the equation $\omega^R = 0$ for the holomorphic form $\omega$. As A.Zorich pointed out to the author, considering very specific examples people certainly observed some features which may illustrate our ideas (see [10, 22]). It seems that our key idea of transversal canonical basis did not appeared before.

Section 1. Definitions. The Case of Hyperelliptic Curves.

First of all, our goal is to show the examples of concrete foliations in the classes $T, T^k$. Consider any real hyperelliptic curve of the form

$$w^2 = R_{2g+2}(z) = \prod_j (z - z_j), \ z_j \neq z_l$$

where all roots $z_j$ are real and ordered naturally $z_1 < z_2 < \ldots < z_{2g+2}$.

**Lemma 1** For every complex non-real and non-imaginary number $u + iv, u \neq 0, v \neq 0$, every polynomial $P_{g-1}(z)$ and number $\epsilon$ small enough, the Hamiltonian system defined by the closed harmonic 1-form below belongs to the class $T^0$ with possible choice of $a$-cycles as any subset of $g$ cycles out of $a_1, \ldots, a_g$ and $c_1, \ldots, c_{g+1}$ not crossing each other:

$$\omega = \frac{(u + iv + \epsilon P_{g-1}(z))dz}{\sqrt{R_{2g+2}(z)}}, \omega^R = 0$$

For $g = 1, 2$ this foliation belongs to the class $T$. For $g > 2$ it belongs to the class $T^2$.

Proof. The polynomial $R = R_{2g+2}(z)$ is real on the cycles $a_j = p^{-1}[z_{2j}z_{2j+1}], j = 1, \ldots, g$ and purely imaginary on the cycles $c_q$ located on the real line $x$ immediately before and after them $c_q = p^{-1}[z_{2q-1}z_{2q}], q = 1, 2, \ldots, g + 1$.

For $g = 1$ we take a canonical basis $a_1, b_1 = c_1$. For $g = 2$ we take a canonical basis $a_1, a_2, b_1 = c_1, b_2 = c_3$. We shall see that all these cycles are transversal to foliation.

For $\epsilon = 0$ we have

$$\omega^R = (udx - vdy)/R^{1/2} = 0, y = 0, x \in a_j$$

$$\omega^R = (vdx +udy)/iR^{1/2} = 0, y = 0, x \in c_j$$
In both cases we have for the second component \(dy \neq 0\) for the direction of Hamiltonian system. So the transversality holds for all points on the cycles except (maybe) of the branching points \(z_j\). The cycles \(c_l, a_k\) are orthogonal to each other in the crossing points (i.e. in the branching points). We need to check now that in all branching points \(z_j\) the angle between trajectory and both cycles \(a, c\) is never equal to \(\pm \pi/2\). because they meet each other with this angle exactly. After substitution \(w'^2 = z - z_j\) we have

\[
\omega = (u + iv)2w'dw'/w'F_j^{1/2} = 2(u + iv)dw'/F_j^{1/2}
\]

where \(F_j = \prod_{l \neq j} (z - z_l)\). We see that the real function \(F_j(x)\) for real \(x \in R\) near the point \(z_j \in R\), does not change sign passing through this point \(x = z_j\): \(F_j(z_j) \neq 0\). For \(w' = f + ig\) we have \(\omega^R = 2(udf - vdg)/F_j^{1/2}\) if \(F_j(z_j) > 0\) or \(\omega^R = 2i(udg + vdf)/F_j^{1/2}\) if \(F_j(z_j) < 0\). We have \(x = f^2 - g^2, y = 2fg\). The condition \(y = 0\) implies the equation \(fg = 0\), i.e. the union of the equations \(f = 0\) and \(g = 0\). It is exactly an orthogonal crossing of our cycles. In both cases our system \(\omega^R = 0\) implies transversality of trajectories to the both cycles \(df = 0\) and \(dg = 0\) locally.

We choose now the \(a\)-cycles as it was indicated above. All of them are transversal to foliation. Therefore we are coming to the class \(T^0\). Now we choose following two \(b\)-cycles: \(b_1 = c_1\) and \(b_g = c_{g+1}\). According to our arguments, this choice leads to the statement that our foliation belongs to the class \(T\) for \(g = 1, 2\) and to the class \(T^2\) for all \(g \geq 2\).

Lemma is proved now because small \(\epsilon\)-perturbation cannot destroy transversality along the finite family of compact cycles.

We choose now the generic perturbation such that all critical points became nondegenerate, and all saddle connections and periodic trajectories nonhomologous to zero, disappear.

Let now \(R = R_{2g+2} = \prod_{j=1}^{2g+2} (z - z_j)\) is a polynomial of even degree as above with real simple roots \(z_j \in R\), and \(\omega = P_{g-1}(z)dz/R^{1/2}\) is a generic holomorphic 1-form. Let \(P = u + iv\) where \(u, v\) are real polynomials in the variables \(x, y\). Only the zeroes \(v = 0\) in the segments \([z_{2q-1}z_{2q}]\), \(q = 1, ..., g+1\), and the zeroes \(u = 0\) in the segments \([z_{2j}z_{2j+1}]\), \(j = 1, ..., g\), are important now. We assume that \(u(z_k) \neq 0\) and \(v(z_k) \neq 0\) for all \(k = 1, ..., 2g + 2\).

**Lemma 2** Remove all open segments containing the important zeroes, i.e. the open segments \([z_{2q-1}z_{2q}]\) containing the zeroes \(v = 0\), and all open segments \([z_{2j}z_{2j+1}]\) containing the zeroes \(u = 0\). If remaining segments are
enough for the construction of the half-basis $a_1, \ldots, a_g$ (i.e. there exists at least $g$ disjoint closed segments between them), then our foliation belongs to the class $T^0$. In particular, it is always true for $g = 2$ where $u$ and $v$ are the linear functions (and have no more than one real zero each).

Let there are no "real" (i.e. located on the $x$-line) zeroes $v = 0$ in the segments $[z_1 z_2], [z_5 z_6]$, and no real zeroes $u = 0$ in the segments $[z_2 z_3], [z_4 z_5]$ for $g = 2$; Then this foliation belongs to the class $T$.

Let $g > 2$, all real zeroes of polynomial $v = 0$ belong to the open intervals

$$(-\infty, z_1), (z_3 z_4), (z_5 z_6), \ldots, (z_{2g-1} z_{2g}), (z_{2g+2}, +\infty)$$

and all real zeroes $u = 0$ are located on the $x$-line in the open segments

$$(-\infty, z_2), (z_3 z_4), \ldots, (z_{2g-1} z_{2g}), (z_{2g+2}, +\infty)$$

In this case foliation $\omega^R = 0$ belongs to the class $T^2$.

Proof is exactly the same as above.

For every foliation of the class $T^0$ we cut Riemann Surface $V$ along the transversal curves $a_j$. The remaining manifold $\tilde{V}$ has a boundary

$$\partial \tilde{V} = \bigcup_j a^\pm_j$$

where $a^\pm_j = S^1$ with foliation entering it from inside for $(a^-_j)$ and leaving it towards the inside of the surface $\tilde{V}$ for $(a^+_j)$. This system can be considered as a system on the plane in the domain $\tilde{V} = D^2_*$ where star means that $2g - 1$ holes removed from this disk inside: The external boundary is taken as $\partial D^2_* = a^+_1$. The boundaries of inner holes are

$$a^-_1, a^+_j, j > 1, \partial D^2 = a^+_1$$

All boundaries are transversal to our system (see Fig 1). They have numerical invariants

$$\oint_{a_j} \omega^R = |a_j| > 0, |a_1| \geq |a_2| \geq \ldots \geq |a_g| > 0$$

The system does not have critical points except $2g - 2$ nondegenerate saddles. Every trajectory starts at the in-boundary $\bigcup a^+_j$ and ends at the out-boundary $\bigcup a^-_j$ (see Fig 4).
Remark 2 Quite similar picture we obtain for the meromorphic 1-form $\kappa$ on the Riemann 2-sphere $S^2 = CP^1$ with $2g$ simple poles (one of them at infinity), and $2g - 2$ simple zeroes. Such systems $(\kappa)^R = 0$ probably produce all topological types of foliations in the domains like $\bar{V}$ obtained using holomorphic forms on the Riemann surfaces with genus $g$.

Consider a maximal abelian $Z^{2g}$-covering $V' \to V$ with basic shifts $a_j, b_j : V' \to V'$ for every class $T$ foliation. Its fundamental domain can be obtained cutting $V$ along the Transversal Canonical Basis. The connected pieces $A_j$ of the boundary exactly represent free abelian groups $Z_2^{2j}$ generated by the shifts $a_j, b_j$. At the covering space a boundary component $A_j$ near this place looks like standard square domain for 2-torus (see Fig 2). The foliation near the boundary $A_j$ looks like a standard straight line flow at the space $R^2_j$ with lattice $Z_2^{2j}$. Our fundamental domain $\bar{V} \subset V'$, $\partial \bar{V} = \bigcup A_j$, is restricted to the inner part of 2-parallelogram in every such plane $R^2_j$. Topologically this domain $\bar{V}$ is a 2-sphere with $g$ holes (squares), with boundaries $A_j$.

We can construct this covering analytically using the Abel Map $A = (A^1, \ldots, A^9) \in C^9$, with some initial point $P$:

$$A^j(Q) = \int_P^Q \omega_j, j = 1, \ldots, g, P, Q \in V$$

Here $\oint_{a_j} \omega_k = \delta_{jk}$ is a normalized basis of holomorphic forms:

$$\oint_{a_j} \omega_k = \delta_{jk}, \oint_{b_j} \omega_k = b_{jk} = b_{kj}$$

The form $\omega = \sum u_k \omega_k$ is generic here. It defines a one-valued function $F : V' \to C$:

$$F(Q') = A^\omega(Q') = \sum u_k A^k(Q')$$

where $Q' \to Q$ under the projection $V' \to V$. For every component of boundary of our fundamental domain we have for the basic shifts $a_j, b_j : V' \to V'$:

$$F(a_j(Q')) = F(Q') + u_j, F(b_j(Q')) = F(Q') + \sum_k u_k b_{kj}$$

The levels

$$F^R = const$$
are exactly the leaves of our foliation $\omega^R = 0$ on the covering $V'$. The map
$\pi_1(V) \to H_1(V, Z) = \mathbb{Z}^{2g} \to R$ is defined by the correspondence:

$$a_j \to u_k^R, b_j \to (\sum_k u_k b_{kj})^R$$

The map $\psi : \pi_1^+(\mathbb{R}) \to \pi_1(V) \to R^+$ of the positive transversal semigroup
(above) certainly belongs to the semigroup $\mathbb{Z}_+^{2g}$ where $(n, m) \in \mathbb{Z}_+^{2g}$ if

$$\omega^R(\sum_{m_k, n_l} m_k a_k + n_l b_l) > 0$$

There exists such choice of the phase vector $\eta_0$ that the of $V' = A(V) \subset C^g$ satisfies to the equation identically:

$$\Theta(A(Q) - \eta_0|B) = 0$$

for all points $Q$; the phase vector $\eta_0$ depends on the initial point $P$ only. Here

$$\Theta(\eta^1, \ldots, \eta^g|B) = \sum_{n \in \mathbb{Z}^g} \exp\{2\pi i \sum_{k,j} b_{jk} n_k n_j + \sum_j n_j \eta^j\}$$

For $g = 2$ it is a complete equation defining this submanifold as a $\Theta$-divisor
in the Jacobian variety.

Section 2. Some General Statements.

Consider now any compact nonsingular algebraic curve $V$ with holomor-
phic generic 1-form $\omega = \sum u_k \omega_k$ and foliation $\omega^R = 0$. Our foliation is defined
through the complex analytic function on the abelian covering $F : V' \to C$,
where this function is defined by the integral along the path joining the ini-
tial point $P$ with a variable-point $Q'$ which is a pair $(Q \in V, [\gamma])$ where $\gamma$ is
a homology class of paths joining $P, Q$:

$$F(Q') = \int_{\gamma} \omega, Q' \to Q$$

This is a restriction of the linear function $\sum u_k A_k$ in the space $C^g$ generated
by the normalized basis of holomorphic forms $\int_{a_q} \omega_j = \delta_{aq}$, to the complex
curve $V' = A(V) \subset C^g$. Here $A$ is a multivalued Abel Map. The levels
Definitions.

1. We call foliation **Generic** if it satisfies to the following requirements:
   - It has only nondegenerate critical points (i.e. saddles);
   - There exist no saddle connections (no separatrices joining two saddles);
   - Even more, no one line of our selected parallel family \( F = \text{const} \) in \( C \) crosses the critical value set \( F(S) \) twice: no one line of this family crosses twice also the "quasilattice" \( Z^{2g} \subset C \) generated by \( 2g \) complex numbers in \( C = R^2 \) where \( u_k, \sum u_kb_{kl}, k, l = 1, \ldots, g \); Every periodic trajectory is homologous to zero (i.e. it divides Riemann Surface into 2 pieces). Without any further quotations we are going to consider only **The Irreducible Generic Foliations** which do not have periodic orbits at all.

2. By the **Almost Transversal Curve** we call every parametrized piecewise smooth curve consisting of the two type smooth pieces:
   - First Type: Moving transversally to foliation in the same direction.
   - Second Type: Moving along the trajectories of foliation in any direction.

   A simple lemma known many years claims that every almost transversal curve can be approximated by the smooth transversal curve with the same endpoints (if there are any). In many cases below we construct closed almost transversal curves and say without further comments that we constructed smooth transversal curve.

3. We call by the **Plane Diagram** of foliation of the type \( T^k \) with transversal canonical basis a **Topological Type** of foliation on the Riemann Surface \( \bar{V} \) obtained from \( V \) by cuts along this basis (see Fig 10 for \( g = 2 \), and the descriptions below).

**Theorem 1** Every generic foliation given by the holomorphic 1-form \( \omega^R = 0 \) on the algebraic curve of genus 2 belongs to the class \( T \), i.e. admits a full Transversal Canonical Basis

Proof of this theorem follows from the following two lemmas:

**Lemma 3** Every generic foliation \( \omega^R = 0 \) belongs to the class \( T^0 \) for genus equal to 2

Proof. Take any trajectory such that its limiting set in both directions contains at least one nonsingular point. In fact, every nonseparatrix trajectory has this property: if its limiting set contains critical point, it also
contains a pair of separatrices entering and leaving it. Nearby of the limiting nonsingular point our trajectory appears infinite number of times. Take two such nearest returns and join them by the small transversal segment. Obviously, this closed curve consisting of the piece of trajectory and small transversal segment, is an almost transversal curve: it can be approximated by the closed non-selfintersecting smooth transversal curve. We take this curve as a cycle $a_1$. Now we cut $V$ along this curve and get the surface $\tilde{V}$ with 2 boundaries $\partial \tilde{V} = a_1^+ \cup a_1^-$. We take any trajectory started at the cycle $a_1^+$ and ended at $a_1^-$. Such trajectory certainly exists. Join the ends of this trajectory on the cycle $a_1^-$ by the positive transversal segment along the cycle $a_1^+$ in $V$. We get a transversal non-selfintersecting cycle $b_1$, crossing $a_1^+$ transversally in one point. Cut now $V$ along the pair $a_1^+, b_1$. We get a square $\partial D^2 = a_1^+ b_1^- a_1^- b_1^+ \subset \mathbb{R}^2$ with a 1-handle attached to the disk $D^2$ inside. We choose notations for cycles in such a way that our foliation enters this square along the piece $A_1^+ = a_1 b_1^- \subset \partial D^2$, and leaves it along the piece $A_1^- = a_1^+ b_1$ (see Fig 3). Nearby of the angles where these pieces $R^\pm$ are attached to each other, our trajectories spend small time inside of the square entering and leaving it (see Fig 3). So moving inside from the both ends of the segment $A_1^+$, we find 2 points $x_1^+, x_2^+ \in A_1^+$ where this picture ends (because the genus is more than 1): These points are the ends of the separatrices of the saddles. Very simple qualitative arguments show that $x_1^+ \neq x_2^+$. Take any point between $x_1^+$ and $x_2^+$ (nearby of $x_1^+$). The trajectory started at this point crosses $A_1^-$ somewhere (see Fig 3). Join the end-point $y'$ of this piece of trajectory by the transversal segment $\sigma = y'y$ along the curve $A_1^-$ in positive direction with the point equivalent to the initial one. This is a closed curve $c_1$ transversal to our foliation. If initial point is located on the cycle $a_1$, the curve $c_1$ does not cross this cycle $b_1$. We take cycles $b_1, c_1$ as a basis of the transversal $a$-cycles. If initial point is located on the cycle $b_1$, the curve $c_1$ does not cross $a_1$. In this case we take $a_1, c_1$ as a basis of the transversal $a$-cycles. It is easy to see that $c_1$ cannot be homologous to $b_1$ in the first case. Therefore it is a right basis of the $a$-cycles which is transversal. The second case is completely analogous. Our lemma is proved.

**Lemma 4** For every generic foliation $\omega^R = 0$ of the class $T^0$ on the algebraic curve of genus 2, the transversal basis of $a$-cycles can be extended to the full Transversal Canonical Basis $a, b$, so every $T^0$-class foliation belongs to the class $T$.

Proof. Cut the Riemann Surface $V$ along the transversal cycles $a_1, a_2$. 

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Assuming that \( f_{a_1}, \omega^R = |a_1| \geq |a_2| = f_{a_2}, \omega^R \), we realize this domain as a plane domain \( D^2 \) as above (see Fig 4 and the previous section). Here an external boundary \( \partial_{ext} D^2 = a_1^+ \) is taken as our maximal cycle. The elementary qualitative intuition shows that there are only two different topological types of the plane diagrams (see Fig 4, a and 4,b): The first case is characterized by the property that for each saddle all its separatrices end up in the four different components of boundary \( a_1^+, a_2^\pm \). We have following matrix of the trajectory connections of the type \((k, l) : a_k^+ \to a_l^-\) for the in- and out-cycles and their transversal measures:

\[
a_1^+ \to a_1^- \quad (\text{with measure } a), \quad a_k^+ \to a_l^-, \quad k \neq l \quad (\text{with measure } b \text{ for } (k, l) = (1, 2), (l, k) = (2, 1)), \quad a_2^+ \to a_2^- \quad \text{with measure } c. \quad \text{All measures here are positive. We have for the measures of cycles: } |a_1| = a + b, |a_2| = c + b. \text{ This topological type does not have any degeneracy for } a = c.
\]

The diagonal trajectory connections of the type \((l, l)\) generate the transversal \(b\)-cycles closing them by the transversal pieces along the end-cycles in the positive direction.

In the second case we have following matrix of trajectory connections:

\[
a_1^+ \to a_1^- \quad \text{with measure } a > 0, \quad a_1^+ \to a_2^+ \quad \text{with measure } b > 0, \quad a_2^+ \to a_1^- \quad \text{with the same measure } b > 0. \quad \text{We have } b = |a_2|, a + b = |a_1|. \quad \text{So we do have the trajectory-connection } a_1^+ \to a_1^-, \quad \text{but we do not have the second one, of the type } (2, 2). \quad \text{However, we may connect } a_2^+ \text{ with } a_2^- \text{ by the almost transversal curve as it is shown in the Fig 4,b), black line } \gamma. \quad \text{So we construct the cycles } b_1, b_2 \text{ as the transversal curves crossing the cycles } a_1, a_2 \text{ only.}
\]

Our lemma is proved.

Therefore the theorem is also proved.

The case \( g = 3 \). A lot of concrete foliations of the class \( T^2 \) were demonstrated above for the real nonsingular algebraic curves

\[
w^2 = \prod (z - z_1) \ldots (z - z_8) = R(z), \quad z_j \in \mathbb{R}
\]

, with the cycles

\[
a_1 = [z_2 z_3], \quad a_2 = [z_7 z_8], \quad a_3 = [z_4 z_5], \quad b_1 = [z_1 z_2], \quad b_2 = [z_6 z_7]
\]

and \( \omega = P_2(z)dz/R(z)^{1/2} \) The polynomial \( P_2(z) = u + iv \) should be chosen such that its real part does not have zeroes in the segments \( [z_2 z_3], [z_4 z_5], [z_6 z_7] \), and its imaginary part does not have zeroes in the segments \( [z_1 z_2], [z_7 z_8] \) (the proof is identical to one in the Section 1 for the constant \( u, v \)).
**Question:** Is it possible to extend this basis to the Transversal Canonical Basis? As we shall see below, the answer is negative in some cases: we need to reconstruct our incomplete basis in order to extend it to the full transversal canonical basis.

Let a nonsingular algebraic curve $V$ of the genus $g = 3$ is given with the generic foliation $\omega^R = 0$ belonging to the class $T^2$ with the incomplete basis $a_1, a_2, a_3, b_1, b_2$ transversal to the foliation. We construct its **Plane Diagram.** After cutting the Riemann Surface along these cycles we realize it as a plane domain with following components of the boundary:

The external boundary

$$A_1 = (a_1 b_1^{-1}) \cup (a_1^{-1} b_1) = A_1^+ \cup A_1^-$$

The internal boundary

$$A_2 = (a_2 b_2^{-1}) \cup (a_2^{-1} b_2) = A_2^+ \cup A_2^-$$

The interim boundary $a_3^+ \cup a_3^-$ inside.

Our notations are chosen in such a way that trajectories enter our domain through the piece with the sign $+$ and leave it through the pieces with the sign $–$. Our Hamiltonian provides a transversal measure. We make a numeration such that:

$$2|A_1^+| = A_1 > A_2 = 2|A_2^+|, |a_3| = |a_3^+| = a$$

Here $A_k$ means also the measure of this boundary component. Nearby of the ends of the segments $A_i^+$ the trajectories enter our domain and almost immediately leave it through the piece $A_i^-$. Therefore, there exist the first points in $A_i^\pm$ where this picture ends. These are the endpoints $x_{i,j}^\pm, x_{2,j}^\pm \in A_j^\pm$ of the pair of separatrices of saddles (see Fig 5). We don’t see here the other pair of separatrices for these saddles.

**Lemma 5** A complete list of topologically different types of the Plane Diagrams in the class $T^2$ for the genus $g = 3$ can be presented. It shows that there is only one type such that we cannot extend the incomplete transversal basis to the complete transversal basis (see Fig 6,a): no closed transversal curve exists in this case crossing the cycle $a_3$ and not crossing other curves $a_1, b_1, a_2, b_2$ (i.e. joining $a^+$ and $a^–$ on the plane diagram). In all other cases such transversal curve $b_3$ can be constructed.

Consider now a special case of the class $T^2$ where the transversal incomplete basis cannot be extended (Fig 6,a).
Lemma 6  There exists a reconstruction of this basis such that the new basis can be extended to the complete transversal canonical basis

For the proof of the second lemma, we construct a closed transversal curve $\gamma$ such that it crosses the cycle $a = a_3$ in one point and crosses the segment $A^+$ leaving our plane diagram (see Fig 7). It enters $A^-$ in the equivalent point, say, through the cycle $b_1$. We take a new incomplete transversal basis $a_2, b_2, a_3, \gamma, a_1$. If $\gamma$ crosses $A^+_1$ through the cycle $a_1$, we replace $a_1$ by the cycle $b_1$ as a last cycle in the new incomplete basis of the type $T^2$. The proof follows from the plane diagram of the new incomplete basis (we drop here these technical details, especially the list of the plane diagrams implying lemma 5).

Comparing these lemmas with the construction of special foliations in the previous section on the real Riemann Surface, we are coming to the following

Conclusion. For every real hyperelliptic Riemann Surface of the form $w^2 = \Pi (z - z_1) \ldots (z - z_8) = R(z), z_k \neq z_l \in R$, every generic form $\omega = P_2(z)dz/R(z)$ defines foliation $\omega^R = 0$ of the class $T$ if real and imaginary parts of the polynomial $P = u + iv$ do not have zeroes on the cycles indicated above in the lemma 2 for genus $g = 3$.

We prove in the Appendix that every generic foliation admit a Transversal Canonical Basis for $g = 3$. G. Levit communicated to the author another proof of this theorem for all $g \geq 2$ based on the construction of pant decomposition of Riemann surface with $3g - 3$ boundary curves transversal to dynamical system (his proof is also included in the Appendix).

Section 3. Topological Study of the Class $T$.

Let us describe here some simple general topological properties of the class $T$ foliations $\mathcal{R}$ and corresponding Hamiltonian Systems. After cutting the surface $V$ along the transversal canonical basis $a_j, b_j$, we are coming to the fundamental domain $\bar{V}$ of the group $\mathbb{Z}^{2g}$ acting on the maximal abelian covering $V' \subset C^g$ imbedded by the Abel Map. It leads to the Plane Diagram $D^2_*$ with $g$ boundary "squares" $\partial D^2_* = \bigcup_j A_j$ where $A_j = a_j b_j^{-1} a_j^{-1} b_j = A_j^+ \cup A_j^-$. Every piece $A_j^\pm$ consists of exactly two basic cycles $a_j, b_j$ attached to each other. These pieces are chosen such that trajectories enter $A_j^+$ from outside through $a_j b_j^{-1}$ and leave it into the fundamental domain $\bar{V}$ except the areas nearby of the ends. The trajectories enter $A_j^-$ from inside and leave fundamental domain. There is exactly $2g - 2$ saddle points inside of $\bar{V}$. They are not located on the selected transversal cycles $a_j, b_j$. Our foliation nearby
of each boundary square $A^\pm$ looks exactly as a straight line flow. It means in particular that there exist two pairs of points $x^\pm_{1,j}, x^\pm_{2,j} \subset A^\pm_j$ which are the endpoints of separatrices in $\bar{V}$, nearest to the ends at the each side $A^\pm_j$ (see Fig 3 and 5). We call them The Boundary Separatrices belonging to The Boundary Saddles $S_{j,1}, S_{j,2}$ for the cycle $A_j$. We see 2g of such saddles $S_{j,1}, S_{j,2}$ looking from the boundaries of $A_j$, but some of them are in fact the same. At least two of them should coincide leading to the saddles of the types $<jjkk>$ where all incoming and outcoming separatrices are of the boundary type (The "Double-boundary" saddles).

**Definitions**

1. We say that the saddle point $S \in \bar{V}$ has a type $<jklm>$ if it has two incoming separatrices starting in $A^+_j, A^+_l$ and two outcoming separatrices ending in $A^-k, A^-m$. The indices are written here in the cyclic order corresponding to the orientation of $\bar{V}$. Any cyclic permutation of indices defines an equivalent type. we normally write indices of the type starting from the incoming separatrix, as $<jklm>$ or $<lmjk>$.

2. We call foliation Minimal if all saddle points in $\bar{V}$ are contained in the set of boundary saddles. In particular, their types are $<jjkl>$. We call foliation Simple if there are exactly two saddle points of the double-boundary types $jjkk$ and $jjll$ correspondingly. The index $j$ we call Selected. We say that foliation has a rank equal to $r$, if there exists exactly $r$ saddles of the types $<jklm>$ where all four indices are nonboundary. In particular, we have $0 \leq r \leq g - 2$. There is exactly $t$ saddles of the double-boundary types like $<jjkk>$ where $t - r = 2$. There is also $2g - 2t$ other saddles of the types like $<jjkl>$ where only index $j$ corresponds to the boundary separatrices. The extreme cases are $r = 0, t = 2$ which we call special (above), and $r = g - 2, t = g$ which we call maximal. In the maximal case there exists a maximal number of saddles whose separatrices arrived from the nonboundary parts of $A^+_k$, and all boundary type saddles are organized in the pairs. One might say that for the maximal type every index is selected. **For maximal type the genus should be an even number** because the boundaries $A^\pm$ are organized in the cycles now, and every cycle should contain even number of them, by the elementary orientation argument. The relation $2g - t + r = 2g - 2$ for the total number of saddles gives $t - r = 2$. For the case $g = 2$ we obviously have $r = 0$. For the case $g = 3$ the only possible case is $t = 2, r = 0$ (the simple foliations); the case $t = 3, r = 1$ cannot be realized for $g = 3$ because it is maximal in this case: however, the maximal case corresponds to the even genus only. Therefore it is available only for the genus not less that 4.
How to build these systems topologically?

In order to answer this question, let us introduce following Building Data (see Fig 9):

I. The Plane Diagram consisting of the generic Hamiltonian System on the 2-sphere $S^2$ generated by the hamiltonian $H$ with nondegenerate critical points only (centers and saddles) sitting on different levels. Let one center is located in the point 0, and another one in $\infty$. It has $t$ centers and $r$ saddles. Let exactly $g$ transversal oriented segments are given $t_1, ..., t_g \subset S^2$ with transversal measures $m_1, m_2, ..., m_g$ provided by hamiltonian, such that:

a. They do not cross each other; the values of Hamiltonian in their ends, centers and saddles are distinct except that exactly two of them meet each other in every center; They do not touch any saddle point on the two-sphere.

b. Every cyclic and separatrix trajectory of the hamiltonian system on $S^2$ meets at least one of these segments.

We make cuts along these segments and define the sides $t^\pm_j$ where trajectories leave and enter it correspondingly.

II. The Torical Data consisting of the $g$ tori $T^2_j$, $j = 1, ..., g$ with distinct hamiltonian irrational straight line flows and selected oriented transversal segments $s_1, ..., s_g$ (one for each torus). Their transversal measures are equal to the same numbers $m_1, ..., m_g$. The Transversal Canonical Basis $a_k, b_k$ in every torus is selected where $a_k$ are positive, $b_k$ are negative, and $|a_k| + |b_k| > m_k$.

We make similar cuts along these segments in the tori, and define their sides $s^\pm$ in the same way.

Identifying the segments $s^+_j$ on the tori with $t^-_j$ on the plane and vice versa, we obtain a Riemann surface $M^2_g$ with foliation which has a transversal measure. The explanation should be given concerning the centers and the ends of the segments:

We construct our gluing in such a way that every end of the segment $t_j \subset S^2$ defines exactly one saddle of the boundary type $< j j k l >$. Here $t_k, t_l$ are the segments joined by the pieces of the same trajectory with the end of the segment $t_j$ on the 2-sphere $S^2$. These pieces of trajectory provide a pair of nonboundary separatrices for the saddle on the Riemann Surface. We assume that they meet these segments in the inner points because the foliation is generic.

By definition, every center generates a double-boundary saddle of the type $< j j k k >$. So we have $t - r = 2$.

In order to obtain this set of data from the generic foliation given by
the real part of holomorphic one-form with transversal canonical basis, we perform following operations:

1. Cut our surface along the TCB. The boundary of this domain $\bar{V}$ is equal to the union $\bigcup_j (A_j^+ \cup A_j^-) = \partial \bar{V}$. Every component is presented as lying in the Fundamental Parallelogram $P_j$ of the 2-torus $T_j^2$. Our flow covers the boundary of $P_j$ as a straight line flow: the trajectories enter through the path realized by the pair of cuts $a_jb_j^{-1}$ and leave through the path $a_j^{-1}b_j$ (see Fig 2).

2. Find for every 2-torus $T_j^2$ the pair of boundary saddles in $P_j$ and join them by the pair of transversal segments $s_j^\pm$. They should meet each other in the boundary saddles only (see Fig 5). We perform this operation in the fundamental parallelogram $P_j$ representing our torus. These segments should be chosen in such a way that outside of them in $P_j$ near the one-skeleton), we have a straight line flows.

3. Cut our surface $M_g^2$ along all these segments $s_j^+ \cup s_j^-$. It is divided now into the torical pieces $T_j^2$ and one plane piece $S^2$ whose boundary consists of the curves $\bigcup_j (t_j^+ \cup t_j^-)$ for $S^2$, and $s_j^+ \cup s_j^-$ for the tori $T_j^2$. After cutting the surface $M_g^2$ along the pieces $s_j^\pm$ we keep the notation $s_j^\pm$ for the curves in the tori, but for the plane part $S^2$ we change notations for these curves, and denote them by $t_j^\pm$.

4. Now we glue $t_j^+$ with $t_j^-$ for the sphere $S_j^2$, and $s_j^+$ with $s_j^-$ for the tori $T_j^2$, preserving the transversal measure. The system on the 2-sphere appears with $g$ selected transversal segments $t_j$. We have also $g$ 2-tori $T_j^2$ with the straight line flows and transversal pieces $s_j$ whose measures are equal $m_j$.

5. Near the double-boundary saddles we are coming to the picture topologically equivalent to the center, but this equivalence in non-smooth.

We can see that our construction allows to imitate all topology of foliation. It preserves also the measure-type invariants.

Therefore we are coming to the following

**Theorem 2** Every generic foliation given by the real part of holomorphic one-form, can be obtained by the measure-preserving gluing of the pieces $(S^2, H, t_1, \ldots t_g)$ and $(T_1^2, s_1), \ldots, (T_g^2, s_g)$ along the transversal segments $t_j$ and $s_j$, as it was described above. For the genus $g = 2$ we can remove sphere $S^2$ from the description: Every generic foliation given by the real part of holomorphic one-form, can be obtained from the pair of tori $(T_1^2, s_1)$ and $(T_2^2, s_2)$ with different irrational straight line flows and transversal segments $s_1, s_2$ with transversal measure $m_1 = m_2 = m$. 
Example. The Topological Types of the Minimal Foliations.

For the Minimal Foliations above we have \( t = 2, r = 0 \). The hamiltonian system on the 2-sphere is trivial (see Fig 9,a): It can be realized by the rotations around the point 0. There are no saddles on the sphere here, and the second center we take the point \( \infty \). For the simplest case \( g = 2 \) there are two segments \( t_1, t_2 \). Both of them join 0 and \( \infty \). So they form a cycle of the length 2. The difference between the values of Hamiltonian \( H(\infty) - H(0) \) is equal to \( m_1 = m_2 \). All possible pictures of the transversal segments can be easily classified here for every genus \( g \) (see Fig 9,a and b).

There exist following types of topological configurations only:

a. The Plane Diagram has exactly one ”cycle” \( t_1, t_2 \) of the length two (like for \( g = 2 \)) and \( g - 2 \) disjoint segments \( t_j, j \geq 3 \); This type is available for all \( g \geq 2 \).

b. The Plane Diagram has two pairs \( t_1, t_2 \) and \( t_3, t_4 \) where the members of each pair meet each other either in the center 0 or in \( \infty \), and \( g - 4 \) disjoint segments \( t_j, j \geq 5 \). In the second case we have \( g \geq 4 \).

c. The Plane Diagram has exactly one connected set consisting of 3 segments passing through both centers \( t_1t_2t_3 \) and \( g - 3 \) disjoint segment \( t_j, j \geq 4 \). For this type we have \( g \geq 3 \).

**Theorem 3** For \( g = 2, 3 \) every class \( T \) foliation is simple. A maximal type exists only for even genus \( g \geq 4 \).

Proof. For \( g = 2 \) this is obvious and was already established above: all generic (irreducible) hamiltonian foliations are simple. Both of indices \( k = 1, 2 \) are selected; at the same time they are maximal. Consider now the case \( g = 3 \). Our foliation can be either simple \( (t = 2, r = 0) \) or maximal \( (t = 3, r = 1) \) in this case. We have \( t = 3, 2g - 2t = 0 \) for the maximal case. If it is so, every boundary saddle should be paired with some other. So there is a cyclic sequence of boundary saddles containing all three boundary components. However, every cyclic sequence should contain even number of boundary components ( and the same number of boundary saddles), otherwise the orientation of foliation is destroyed. This is possible only for even number of indices which is equal to genus. Our conclusion is that \( g \geq 4 \). This theorem is proved.

The 2-sphere is covered by the nonextendable ”corridors” between two transversal segments \( t_j, t_k \subset S^2 \). They are the strips of nonseparatrix trajectories moving from the inner points of \( t_j \) to the inner points of \( t_k \) not
touching any points of \( t_i \) and the saddles. The right and left sides of these corridors are either separatrices of the saddles in \( S^2 \) or the trajectories passing on \( S^2 \) through the ends of some segments \( t_i \).

Classification of the generic Morse functions \( H \) on the sphere \( S^2 \) can be given easily (see Fig 9, c): Take any connected trivalent finite tree \( R \). It has vertices divided into the \( r \) Inner Vertices and \( t \) Ends. Assign to each vertex \( Q \in R \) a value \( H(Q) \); these values are not equal to each other \( H(Q) \neq H(P), P \neq Q \); therefore the edges become oriented, looking ”up”, to the direction of increasing of \( H \). Every inner vertex \( Q \) is a ”saddle”, i.e.

\[
\min_i H(Q_i) < H(Q) < \max_i H(Q_i), i = 1, 2, 3
\]

where \( Q_i \) are the neighbors. The function \( H \) on the graph should be such that for every edge \( [Q_1 Q_2] \subset R \) we have

\[
H(Q_1) \leq H(P) \leq H(Q_2)
\]

where \( H(P) \) is monotonic in this edge. We may take it linear.

For the description of the set of transversal segments \( t_j \subset S^2 \) we introduce a \textbf{locally constant set-valued function} \( \Psi(P), P \in R \) on the graph \( R \): the values \( \Psi(P) \) are the cyclically ordered nonempty finite sets

\[
\Psi(P) = \{t_1(P) < t_2(P) < ... < t_q(P) < t_1(P)\}
\]

such that: The number \( q = |\Psi(P)| \) of the points \( t_j(P) \) in this set, is equal to 2 nearby of the ends \( Q_j \) collapsing to one in the endpoint \( t_1(P) = t_2(P) \) if \( P = Q_j \). The number \( q \) may change \( q \to q \pm 1 \) in the isolated points \( P_i \in R \) such that \( H(P_i) \neq H(Q) \) for all inner vertices \( Q \). Passing ”up” through the inner vertex \( Q \), this set either splits into the pair of cyclically ordered sets \( \Psi(P) \to \Psi_1 \cup \Psi_2 \) inheriting orders where \( |\Psi_1| + |\Psi_2| = |\Psi(P)| \), or some pair of cyclically ordered sets is unified into the one ordered set choosing some initial points in each of them (the inverse process). Every continuous ”one-point branch” \( t_j(P) \) living in the vertical path between the points \( P_1, P_2 \in R \), defines the transversal segment \( t_j \). Its transversal measure is equal to \( m_j = H(P_2) - H(P_1) > 0 \)

One may imagine that the graph \( R \) is imbedded into the space \( \mathbb{R}^3 \). The sphere \( S^2 \) appears as a boundary of the small \( \epsilon \)-neighborhood of \( R \) in \( \mathbb{R}^3 \). The function \( H \) should be realized as a ”height function” \( S^2 \to R \). The points \( t_j(P) \in \Psi(P) \) are marked on the boundary of this small neighborhood. The
nearest component of the level $H = c$ is exactly a small circle near the noncritical point $P \in R$. Changing $c$, the points $t_j(P)$ are varying according to the rules above.

**Every trajectory** $\gamma \in V$ **defines a sequence of elements**

$$...W_{-M}W_{-M+1}...W_NW_{N+1}... = W(\gamma)$$

**where** $W_N \in H_1(V,S^2) = H_1(V)$ We compute these elements $W_N$ through the 3-street model and $m_j$-dependent new TCB in the next chapters.

**Section 4. The Three-Street Picture on the Torus**

**The Case** $g = 2$. **The Maximal Case for** $g = 4$

Let us describe the case $g = 2$ more carefully. We have here two selected levels $1, 2$. The reduced or working measure of the transition from $C_1$ to $C_2$ and back is equal to $m$. The transition map is an orientation preserving isometry of segments

$$\Phi : s = s_2 \rightarrow s_1 = s$$

We have two planes $C_1, C_2$ with two lattices $Z_1^2, Z_2^2$. A family of parallelograms starts in the selected points. The transversal segments $s = s_2, s = s_1$ are located in each of them. Everything is repeated periodically in each space $C_k$ with its own lattice. Our data include $|a_j|, |b_j|$ for $j = 1, 2$ and the transition measure $m$.

Consider the vertical flow in $C_k$.

**Question.** How long the trajectory can move in $C_k$ (i.e. in the torus $T_k^2$) until it hits some periodically repeated copy of the segment $s$?

It starts and ends in some segments of the selected periodic family generated by the segment $s$ in the corresponding parallelograms not crossing any segments in between. Such paths with fixed ends form the connected strip. We require that these strips cannot be extended to the left and right: every trajectory in the strip ends in the same segments. The extension of the strip to the right or to the left meets some saddles. They are presented by the ends of the segments of our family. Every such strip has a **Height** $h$ and **Width** $w$. The width is equal to the transversal measure. The height depends on the lattice periods. It has a meaning only as a topological quantity $h \in H_1(V,Z)$.

**Definition.** We call the unextendable strips by **The Streets** and denote them $p^r_k, k = 1, 2$. We denote a longest unextendable strip by $p^0_k$. It meets
the ends of some segments of our family strictly inside along the segment number zero (see Fig 11). The upper and lower segments of this strip should be located also strictly inside of the corresponding segments \(s, s''\). Their Heights and Widths we denote by \(h^\tau_k, |p^\tau_k|\) correspondingly. The street number 2 is located from the right side from the longest one, the street number 1–from the left side.

**Lemma 7** For every foliation with transversal canonical basis and for both planes \(k = 1, 2\) there exists exactly three streets \(p^\tau_k, \tau = 0, 1, 2\) such that \(\sum_\tau |p^\tau_k| = m\) and \(h^1_k + h^2_k = h^0_k\). Two smaller streets are attached to the longest one from the right and left sides. This picture is invariant under the involution changing time and orientation of the transversal segments. The union of the three streets started in the segment \(s\), is a fundamental domain of the group \(Z^2_k\) in the plane \(C_k\) (see Fig 11).

**Remark 3** Another fundamental domain associated with segment \(s\) can be constructed in the form of the "most thin parallelogram". It is generated by two vectors depending on the measure of this segment. The first vector corresponds to the shift \(s \rightarrow s'\) where \(s'\) is the second end of the street number 1 started in \(s\) (from the left sight of the longest street number 0–see Fig 11). The second vector corresponds to the shift \(s \rightarrow s''\) where \(s''\) is the second end of the street number 2 (from the right side of the street number 0–see Fig 11). We shall discuss this parallelogram later, in the last section. This \(m\)-dependent "thin" basis of the lattice \(Z^2\) can be canonically lifted to the free group with 2 generators \(a, b\). We denote them \(a^m(+) = a^*, b^m(+) = b^*\) such that the product path \(a^*b^*(a^*)^{-1}(b^*)^{-1}\) contains exactly one segment \(s\) inside. This new generators are also transversal to our foliation, so we can construct a new TCB out of them. We shall use them instead of the original TCB because they are adjusted to the idea of our description of foliations with TCB on the surfaces of higher genus \(g > 1\). Their transversal measures are following:

\[
|a^m_k| = |p^0_k| + |p^2_k|, |b^m_k| = |p^0_k| + |p^1_k|
\]

where \(\sum_\tau |p^\tau_k| = m, \tau = 0, 1, 2, k = 1, 2\)

Proof. Construct first the longest street \(p^0_k\). We start from any nonseparatrix trajectory ending in some segments inside both of them. Extending this strip in both (left and right) directions, we either meet the ends of upper or lower segments or meet some segment whose height is strictly between. In the
first case we see that after passing the most left end we can construct longer trajectories. Do it and start the same process with longer strip. Finally, we reach the locally maximal vertical length of the height. After that we extend it to the right and left. We necessarily meet from both sides some segments with heights in between, otherwise its height cannot be locally maximal. So the maximal provider is constructed. Consider the neighboring streets from the right and left sides. This is exactly two other streets. Their heights are smaller. Denote the left one by $p_k^1$ and the right one by $p_k^2$. What is important, is that the neighboring street can be extended till the left end of the upper segment of the locally longest one. This statement follows from the periodicity of the system of segments: we cannot meet any segment from the left until we reached the end of the upper one; the newly met segment should have the locally longest one from its right side. The same argument can be applied to the right extension and to the lower parts as well. We can see that our locally longest street is surrounded by the exactly four unextendable domains (two domains from each side). They are restricted from the right and left sides by the ends of the lower or upper segments (see Fig 11). The pairs of domains located across the diagonal of each other are equal. All our relations immediately follow from that. There are no other unextendable streets except these three.

Lemma is proved.

The Model of the Motion for $g = 2$ is following: The motion is vertical (up). Take two copies of the horizontal segment $s = s_k$ of the length $m$. Put each of them into the plane $C_k$ for $k = 1, 2$ inside of the parallelogram $P_k$. Construct three vertical streets $p_k^\alpha, k = 1, 2$ over $s = s_k$ from the upper side (the longest street lies in between). Assume that the segment $s = s_k$ belongs to the parallelogram $P_k$ with $\mathbb{Z}^2$-index $(0, 0)$ in both cases. Assign to the upper end of each street $p_k^\alpha$ an integer 2-vector $h_k^\alpha \in \mathbb{Z}^2$. It is exactly a $\mathbb{Z}^2$-index of the lattice parallelogram where it is located. We have $h_k^1 + h_k^2 = h_k^0$. The trajectory starts in $C_k$ at the segment $s_k$; it moves along the street vertically. After reaching the upper end, it jumps to $C_l, l \neq k$, exactly to the same point of the lower segment (modulo periods) as the endpoint in the street (but on the different plane). After that it moves along the corresponding street $p_l^\beta$ in the new plane $C_l$, and so on. Remember that the widths (i.e. the transversal measures) of the $l$-streets are different satisfying only to the conservation law that their sum is also equal to $m$. After each period of the straight-line motion along the street $p_k^\alpha$, the $\mathbb{Z}^2$-index of fundamental domain $\bar{V}$ changes: we add a vector $h_k^\alpha$ to the $\mathbb{Z}^{2n}$-number of our
domain $\bar{V}$ changing only the pair of components corresponding to the plane $C_k$. Other components remain unchanged. During the jump $\Phi$ from $C_1$ to $C_2$ and back all $Z^{2n}$-numbers of fundamental domain $\bar{V}$ remain unchanged.

This leads to the full topological description of foliation in the combinatorial form for $g = 2$ if we can calculate all incoming homotopy and homology classes of streets effectively.

**How to describe time dynamics? How much time is needed to pass every street?** For the precise definition of time intervals we mark a pair of segments $s_2, s_1$ transversal to foliation (except the ends) leading from one saddle to another in the fundamental domain $\bar{V}$. By definition, what is presented by the "streets" combinatorially in $C_k$, is presented in the actual dynamical system by the strips of trajectories which start in $s_k$ and end up in $s_l, l \neq k$. The boundary separatrices of shorter streets $p^\alpha_k, \alpha = 1, 2$, have exactly one saddle points at each of their boundaries in the lower or upper ends of the street. The longest streets $p^0_k$ have also one saddle at each of their boundaries located somewhere inside of the ends (see Fig 11).

We define **The Time Characteristic Functions** $t(x) > 0$ for $x \in (0, |p^k_0|)$ for passing every street where $t \to +\infty$ if $x \to 0$ or $x \to |p^k_0|$. In all cases the asymptotics is like $t(x) \sim -c \ln x$ or $t(p^\alpha_k - x) \sim -c \ln x$, but the constants are different. Every saddle provides two positive constants $c_m > 0, m = 1, 2$, where $m = 1$ corresponds to the left side, and $m = 2$ to the right side of the street. We have following asymptotics, $x \to +0$

1. For $x \in p^0_k$

   $$t(x) \sim (-c_2) \ln x; t(p^0_k - x) \sim (-c_1) \ln x$$

2. For $x \in p^1_k$

   $$t(x) \sim (-c_2/2) \ln x; t(p^1_k - x) \sim (-c_1/2) \ln x$$

3. For $x \in p^2_k$

   $$t(x) \sim (-c_1/2) \ln x; t(p^2_k - x) \sim (-c_2/2) \ln x$$

The exact value of the time characteristic functions $t(x)$ should be calculated numerically. Their singularities are important, for example, for the ergodic properties of hamiltonian systems. How much time the trajectory spends in this or that area? What size fluctuations might have? For $g = 1$ in the presence of saddles this problem was studied in several works [23, 24]. It

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was essentially solved in [21]. The ”mixing properties” were found for \( g = 1 \) as a consequence of time delays provided by saddles. Let us remind that the topology of typical open trajectory is the same here as in the straight line flow (after removal of domains influenced by the centers). Only saddle points deform the time functions for the essential part of the hamiltonian flow for \( g = 1 \). However, the role of these singularities is probably completely different for \( g = 2 \). In this case the topology of foliation was already mixing, so things like that should essentially remain unchanged.

Now let us consider another interesting example of **The Maximal Foliations** for \( g = 4 \).

For the maximal foliations of the class \( T \) all boundary saddles are paired with each other. It means that all of them organize a system of cycles where the next boundary saddle is paired with the previous one. The length of every cycle is equal to some even number \( 2l_q, q = 1, \ldots, f \), so \( 2l_1 + \ldots + 2l_f = g \). We say that the system has a cycle type \((l_1, \ldots, l_f)\). The maximal system contains total number of \( g \) saddles of the type \(< j j k k >\) and \( g - 2 \) saddles of the types \(< j k l m >\) where all 4 entries are distinct. For the case \( g = 4 \) we have two possibilities of the cycle types namely \((1,1)\) and \((2)\). The type \((2)\) is especially interesting. This cycle separates a 2-sphere on the South and North Hemisphere (see Fig 12) where \( A_1, A_2, A_3, A_4 \) are located along the equator. The additional two saddles are sitting in the poles exactly with separatix curves going to the each ”country” \( A_k, k = 1, 2, 3, 4 \) along the 4 selected meridians from the north and south poles.

**Lemma 8** The transitions \( A_k^+ \rightarrow A_l^- \) with measures \( m_{kl} \) visible from the poles (i.e. located in the corresponding hemisphere) are the following:

**From the North Pole** we can see the transitions

\[
A_k^+ \rightarrow A_l^-, k = 1, 3, l = 2, 4
\]

**From the South Pole** we can see the transitions

\[
A_k^+ \rightarrow A_l^-, k = 2, 4, l = 1, 3
\]

They satisfy to the Conservation Law \( \sum_k m_{kl} = |A_l|, \sum_l m_{kl} = |A_k| \), where \( k \) and \( l \) are neighbors in the cyclic order ...1234... . It implies in particular that \( A_1 - A_2 + A_3 - A_4 = 0 \) and ”the constant flux in one direction of the cycle” 1234 is defined provided by the asymmetry \( m \) of the trsition measures \( m_{kl} - m_{lk} \) which is constant:

\[
m_{12} - m_{21} = m_{32} - m_{23} = m_{34} - m_{43} = m_{41} - m_{14} = m
\]
Remark 4. We say that the system is rotating clockwise if \( m > 0 \). It is rotating counterclockwise if \( m < 0 \).

Section 4. The Homology and Homotopy Classes.
Trajectories and Transversal Curves: \( g = 2 \)

How to describe the image of transversal semigroups in fundamental group \( \pi^+_1(\mathcal{R}) \to \pi_1(V) \) and in homology group \( H_1(V, \mathbb{Z}) \)?

There are three types of the transversal curves generating all of them:
(I) The "Torical Type" transversal curves not touching the segments \( s \);
(II) The "Trajectory Type" transversal curves (or the Poincare Curves). They coincide with some trajectory of the Hamiltonian system started and ended in the transversal interval \( s^+ \) in the plane \( C_1 \). We make it closed joining the endpoints by the shortest transversal interval along \( s \);
(III) The general non-selfintersecting transversal closed curves.

Let a closed transversal curve \( \gamma \) is presented as a point moving in the combinatorial model. We realize its motion by the sequence of the almost transversal pieces \( \gamma_1 \gamma_2 \ldots \gamma_N \). The first piece \( \gamma_1 \subset C_1 \) starts in the point of the parallelogram \( P_{0,0}^1 \) just over the segment \( s^+ \). It travels in the plane \( C_1 \) and reaches first time one of the segments \( s^\pm \) located in the parallelogram \( P_{m_1,n_1}^1 \) of the same plane where \( (m_1, n_1) \neq (0, 0) \). The next path \( \gamma_2 \subset C_2 \) starts in the corresponding point of the same segment \( s \) but presented in another plane \( C_2 \). We start it also in the parallelogram \( P_{0,0}^2 \). It travels in the plane \( C_2 \) and ends up in the point of some another segment \( s^\pm \in P_{m_2,n_2}^2 \) and so on. Finally, the last path \( \gamma_N \) ends up in the same point where the first path started, after the last crossing of some segment \( s \subset P_{m_N,n_N}^2 \). We may think that our almost transversal path consists of the pieces of two kind: the trajectory pieces, passing streets \( P_k^0 \) from the initial segment to the end in positive or negative direction, and of the orthogonal positively oriented "jumps" along some segment \( s^\pm \). Only second type pieces are carrying the nonzero transversal measure. We can freely create such curves. Every such curve \( \gamma_q \) is almost transversal. However, these curves are not exactly closed: they start and end up in the equivalent parallelograms on the equivalent segments \( s \) but the ends are not coincide exactly (even after identification of the equivalent points). They approach the same segment \( s^\pm \) but (maybe) in the different points. **We can close the ends of the curve \( \gamma_q \) only if the interval from its initial point to the end one is negatively oriented.**
Let us call such almost transversal curves $\gamma_q$ semiclosed. Let us present every semiclosed factor $\gamma_q$ with fixed ends as a product $\gamma_q = \gamma_q^s \tilde{\gamma}_q$ where $\tilde{\gamma}_q$ is the maximal closed part, and $\gamma_q^s$ is the "shortest" part, i.e. it does not contain any closed piece inside. Therefore the curve $\gamma_q$ can be obtained as a product $\gamma_q \sim \gamma_q^s \tilde{\gamma}_q$ where $\tilde{\gamma}_q$ does not cross the segments $s^\pm$ at all (i.e. it is the transversal curve of the Plane Type (I)). Each path $\tilde{\gamma}_q$ is located completely within the plane $C_1$ or $C_2$.

Step 1: Classify all closed transversal paths $\tilde{\gamma} \in C_k, k = 1, 2$. We demonstrate below the description of their homology and homotopy types using the "3-street" combinatorics described above.

Let $|a_k|, |b_k|$ denote the measures of the basic transversal closed curves $a_k, b_k$. We made our pictures and notations above such that the cycles $a_k, b_k^{-1}$ are the positive transversal curves. According to our construction, the measure $m$ of the segments $s$ satisfies to the inequality

$$0 < m < |a_k| + |b_k|, k = 1, 2$$

Define the minimal nontrivial nonnegative pairs of integers $(u_k > 0, v_k \geq 0), (w_k \geq 0, y_k > 0)$ such that

$$m > u_k|a_k| - v_k|b_k| > 0$$

and

$$m > y_k|b_k| - w_k|a_k| > 0$$

(see Fig 13). It means exactly that these new lattice vectors represent the shifts $s', s''$ of the segment $s$ visible directly from $s$ along some shortest trajectories not crossing other shifted segments, looking to the positive time direction $(+)$. 

**Lemma 9** The new lattice vectors $h_k^1, h_k^2$ have transversal measures equal to $u_k|a_k| - v_k|b_k|$ and $y_k|b_k| - w_k|a_k|$ correspondingly. These measures are less than $m$ but their sum is greater than $m$. Their homology classes $h_k^1 = [a_k^m (+)] = u_k[a_k] + v_k[b_k]$ and $-h_k^2 = -[b_k^m (+)]$ for $[b_k^m (+)] = w_k[a_k] + y_k[b_k]$, generate homological image of the semigroup of positive closed transversal curves not crossing the segments $s$ (i.e. of the Torical Type (I)). These classes have canonical lifts $a^*_{k}, b^*_{k}$ to the free groups generated by $a_k, b_k$; The lifts $a^*_{k}, (b^*_{k})^{-1}$ of the homology classes $h_k^1, -h_k^2$ correspondingly generate semigroup of the homotopy classes of positive closed transversal curves not crossing the segments $s^k_\pm$, starting and ending in the street number 0. These semigroups are free with two generators depending on the measure $m$ only, whose

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transversal measures are smaller than \( m \). There exist also the similar classes \( a_k^m(-), b_k^m(-) \) constructed using the negative time direction: their homology classes are opposite to the positive ones \([a_k^m(-)] = -[a_k^m(+)], [b_k^m(-)] = -[b_k^m(+)]\). They represent the same streets going back. Their lifts \( a_k^m(-), b_k^m(-) \) to the fundamental group \( \pi_1(V, s_k^-) \) are defined as a mirror symmetry of the lifts \( a_k^m(+), b_k^m(-) \in \pi_1(V, s_k^+) \) where \( t \to -t, s_k^+ \to -s_k^- \).

The proof is given below after the reduction to the standard model.

Let us make following useful remark:

**Lemma 10** Every path starting and ending in the segment \( s_k^+, k = 1, 2 \) of the same plane \( C_k \), has well-defined homotopy class in the fundamental group \( \pi_1(V) \). Every curve with both ends in any segment of the type \( s_k^\pm \), has a well-defined homology class in the group \( H_1(V) \).

Proof. Every segment \( s_k^\pm \) is realized in the manifold \( V \) by the transversal segment joining two saddle points. After cutting along these segments, we split our surface into two disjoint pieces. Every piece has a boundary
\[
a_1^+ b_1^+(a_1^+)^{-1} (b_1^+)^{-1} \sim \kappa = s_k^+ \cup s_l^-, \ k \neq l
\]
. The transversal segments are identified with each other according to the rule
\[
s_1^+ = s_2^- \cup s_2^+, \ s_2^+ = s_1^-
\]
So the homology 1-classes modulo boundary \( H_1(V, \kappa) \) are the same as in \( H_1(V) \). We may assign homology class to every piece of trajectory passing any street from the beginning to the end.

Concerning homotopy classes, we assign the invariant \( \phi_{\alpha\beta} \in \pi_1(V, s_1^+) \) to every piece of trajectory passing two streets
\[
\gamma_q \subset p_2^\beta p_1^\alpha = \{ \alpha\beta \}, \phi : \gamma_q \to \pi_1(V, s_1^+)
\]
We use here the fact that the transversal segment \( s^\pm \) is contractible. Therefore the choice of initial point in it is unimportant. Every infinite trajectory \( \gamma \) is coded by the infinite sequence of pieces
\[
\gamma = \ldots \{ \alpha_q \beta_q \} \{ \alpha_{q-1} \beta_{q-1} \} \ldots
\]
So the can apply the homomorphism \( \phi \) into the fundamental group for all sequence. Every finite connected even piece \( \delta' \) of this sequence define the
element \( \phi(\delta') \in \pi_1(V, s_1^+) \). Its ends can be joined by the shortest transversal piece along the segment \( s \). Depending on orientation of this piece, either \( \phi(\delta') \) or \( \phi(\delta')^{-1} \in \pi_1(V, s_1^+) \) define a closed positive transversal curve.

**Reduction to the Standard Model:** The segment \( s \) of the length \( m \) is divided on 5 connected open pieces \( \tau_q \): There exist exactly 9 possible types of trajectory pieces \( \{\alpha\beta\} \), \( \alpha, \beta = 1, 0, 2 \) with measures \( p_{\alpha\beta} \) but 4 of them are in fact empty. They have measure equal to zero. In order to see this, we remind how these pieces were constructed. A segment \( s = s_1 \) of the total measure \( m \) is divided into 3 pieces by the points \( 0, 1, 2, 3 \) for \( k = 1 \) and by the points \( 0', 1', 2', 3' \) for \( k = 2 \), \( s = s_2' \): For the streets we have:

\[
\alpha = 1, 0, 2 = [01], [12], [23], \beta = 1', 0', 2' = [0'1'], [1'2'], [2'3']
\]

So, the index \( \alpha = 1, 0, 2 \) corresponds to the segments \( [01], [12], [23] \) with measures \( p_1^\alpha \), and the index \( \beta = 1', 0', 2' \) corresponds to the segments \( [0'1'], [1'2'], [2'3'] \) with measures \( p_2^\beta \). The positions of the points \( 0 = 0', 3 = 3' \) are fixed. Other points never coincide for the generic foliations.

Every jump from \( C_1 \) to \( C_2 \) is accompanied by the permutation of 3 segments: the left street number 1 in \( C_1 \) ends up in the extreme right part of \( s = s' \) before making jump to \( C_2 \). The right street number 2 in \( C_1 \) ends up in the extreme left part of \( s = s'' \) in \( C_2 \). So jumping from \( C_1 \) to \( C_2 \), we should permute the segments \( 1 = [01] = 1 \) and \( 2 = [23] = 2 \) preserving orientation:

\[
\eta_{12}: 2 \to 0 = 2^*, 3 \to 3^* \in s, 1 \to 3 = 1^*, 0 \to 0^* \in s
\]

Here \( |[23]| = |[2^*3^*]|, |[01]| = |[0^*1^*]| \). The segment \( s \) is divided by the points \( 0 = 2^*, 1', 2', 3^*, 0^*, 3 = 1^* \), so it is presented as a union of 5 sub-segments

\[
s = \tau_1 + ... + \tau_5
\]

In order to return back from the second torus (plane) \( C_2 \) back to \( C_1 \), we need to apply the similar map \( \eta_{21} \) based on the permutation of the streets \( 2' = [2'3'] \) and \( 1' = [0'1'] \). Our broken isometry \( i_\sigma \) based on the permutation \( \sigma \) of 5 pieces, is defined as a composition

\[
i_\sigma = \eta_{21} \eta_{12}
\]
Lemma 11 6 possibilities called the Topological Types of Foliations, for the nonzero set of 5 measures $p_{\alpha\beta}$ exist here (the measures $p_{\alpha\beta}$ of the sub-segments $\tau_q, q = 1, 2, 3, 4, 5$ are given in the natural order on the segment $s$):

(I) : $0 = 2^* < 1' < 2' < 3^* < 0^* < 1^* = 3; \sigma = (32541)$;

$$p_{12} + p_{02} + p_{21} + p_{22} = m$$

(II) : $0 = 2^* < 1' < 3^* < 2' < 0^* < 1^* = 3; \sigma = (24153)$;

$$p_{12} + p_{01} + p_{02} + p_{21} + p_{20} = m$$

(III) : $0 = 2^* < 1' < 3^* < 0^* < 2' < 1^* = 3; \sigma = (41523)$;

$$p_{10} + p_{11} + p_{00} + p_{21} + p_{20} = m$$

(IV) : $0 = 2^* < 3^* < 1' < 2' < 0^* < 1^* = 3; \sigma = (25314)$;

$$p_{12} + p_{01} + p_{00} + p_{20} + p_{21} = m$$

(V) : $0 = 2^* < 3^* < 1' < 0^* < 2' < 1^* = q3; \sigma = (31524)$;

$$p_{10} + p_{21} + p_{01} + p_{00} + p_{21} = m$$

(VI) : $0 = 2^* < 3^* < 0^* < 1' < 2' < 1^* = 3; \sigma = (52134)$;

$$p_{11} + p_{10} + p_{12} + p_{01} + p_{02} = m$$

All other measures $p_{\alpha\beta}$ are equal to zero. We have

$$\sum_{\alpha} p_{\alpha\beta} = p_{2}^{\beta}; \sum_{\beta} p_{\alpha\beta} = p_{1}^{\alpha}$$

As we can see, our 3-street model automatically creates the standard type broken isometry $\eta_{12} \eta_{12} = i_{\sigma} : s \rightarrow s$ generated by the permutation $\sigma$ of 5 pieces $\tau_q$ of the segment $s$. Such systems were studied by the ergodic people since 1970s. Let us point out that our combinatorial model easily provides full information about the topology lying behind this permutation. We know geometry of all pieces. Let us define also a Shift Function: The map $i_{\sigma} : \tau_q \rightarrow s$ for every sub-segment $q = 1, 2, 3, 4, 5$, is an orientation preserving isometry inside. So it is a shift $x \rightarrow x + r_q$ inside of these strips.

We call $r_j \in R$ a value of the Shift Function. Iterating our system, we have similar "Shift Function" for every piece of the trajectory $\gamma$ if it started and ended in the transversal segment $s^+$. We call $\gamma$ a Positive Piece if its Shift Function $r(\gamma)$ is positive. In the opposite case it is a Negative Piece.
For the irreducible generic systems there are no periodic solutions (i.e. no pieces with zero shift functions). Every piece defines a **Closed Transversal Curve of the Trajectory Type** closing it by the shortest path along $s$. Negative pieces define positive closed transversal curves and vice versa. Let us calculate the value of the shift function:

Let us define an algebraic object (no doubt, considered by the ergodic people many years ago): an **Associative Semigroup** $S_{\sigma,\tau}$ with “measure”. It is generated by the 5 generators $R_1, \ldots, R_5 \in S_{\sigma,\tau}$. There is also a zero element $0 \in S_{\sigma,\tau}$. We define multiplication in the semigroup $S_{\sigma,\tau}$ as in the free one but some “zero measure” words are equal to zero by definition. In order to define which words are equal to zero, we assign to every generator an interval $R_q \rightarrow \tau_q, q = 1, \ldots, 5$.

The word $R = R_p R_q$ is equal to zero in the semigroup $S_{\sigma,\tau}$ if and only if $\tau_p \cap i_{\sigma}(\tau_q) = \emptyset$. We assign to the word $R = R_p R_q$ the set $\tau_R$ such that

$$i_{\sigma}(\tau_{R_p R_q}) = i_{\sigma}(\tau_R) = \tau_p \cap i_{\sigma}(\tau_q)$$

if it is nonempty. By induction, we assign to the word $R_p R$ the set $\tau_{R_p R}$ where

$$\tau_{R_p R} = i_{\sigma}^{-N}(\tau_p) \cap \tau_R$$

if it is nonempty. We put $R_p R = 0$ otherwise. Here $N$ is the "length" of the word $R$. This semigroup is associative. For every word in the free semigroup $R = R_{q_1} \ldots R_{q_N}$ we assigned a set

$$i_{\sigma}^{-N}(-1) \cap i_{\sigma}(\tau_{q_1}) \cap \ldots \cap i_{\sigma}(\tau_{q_N}) = \tau_R$$

whose measure is well defined. We put $R$ equal to zero if the measure is equal to zero.

The ordered sequence $R_{\infty} = \prod_{p \in \mathbb{Z}} R_p$, infinite in both directions, defines trajectory of the flow if and only if every finite sub-word is nonzero. The measure is defined for the ”cylindrical” sets $U_R$ consisting of all ”trajectories” with the same sub-word $R$ sitting in the same place for all of them. It is equal to the ”measure” of the word $\tau_R$. The ”Shift Function” is also well-defined by the semigroup and every trajectory. It does not define a homomorphism because it can be nonzero for the word equivalent to zero. For the nonzero words its values are bounded $r(R) < m, 0 \neq R \in S_{\sigma,\tau}$.

We present below the calculation of representation of this semigroup in the fundamental group of the Riemann Surface generated by the real part of the holomorphic one-forms.
If we know the time characterization functions in each piece \( \{\alpha\beta\} \) (it is simply a sum of times in every street \( p_1^\alpha, p_2^\beta \)), we can also study the Hamiltonian system with corresponding natural time. All trajectories \( \gamma \in \{\alpha\beta\} \) have the same homotopy classes \( \phi_{\alpha\beta} \in \pi_1(V) \) known to us (see below). The global reduction of the flow is based on the non-closed transversal Poincaré section, i.e. on the transversal segment \( s \) leading from one saddle to another. This construction seems to be the best possible genus 2 analog of the reduction of straight line flow on the 2-torus to the rotation of circle. Many features of this construction certainly appeared in some very specific examples studied before.

We are going to classify now the homotopy and homology classes of the trajectories starting and ending in the transversal segments \( s \). Before doing that, we need to proof the lemma above in order to finish the first step.

Proof of the lemma. We denote the streets \( p_k^\alpha \) in the plane \( C_k \) simply by the symbols \( \alpha = 0, 1, 2 \) where the longest one is 0. It is located between two others. The right one in 2, and the left one is 1. For the sum of their homological ”lengths” in the simplified notations \( h_k^\alpha \to [\alpha] \) we have \([1] + [2] = [0]\). This relation will be treated also homotopically later. For the widths (i.e. transversal measures) we have \( \sum_{\alpha} |p_k^\alpha| = m \). Their bottom parts cover together the segment \( s \). This picture is invariant under the change of direction of time and simultaneous permutation of the lower and upper segments \( s \).

All transversal paths in the plane \( C_k \) can be written combinatorially in the form

\[
\ldots \to \alpha_q \to 0 \to \alpha_{q+1} \to 0 \to \alpha_{q+2} \to 0 \to \ldots
\]

where \( \alpha_m = 1, 2 \). We have a pair of positive ”basic cycles” \([0 \to 1 \to 0] = (b_k^m (+))^{-1} \) and \([0 \to 2 \to 0] = a_k^m (+), k = 1, 2 \). All other transversal cycles not touching the segment \( s \) and its shifts, starting and ending in the longest strip 0, have a form of the arbitrary word in the free semigroup generated by \( a_k^* = a_k^m (+), (b_k^*)^{-1} = (b_k^m (+))^{-1} \) in the plane \( C_k \). The measures of these new \( m \)-dependent basic cycles are

\[
|a_k^*| = |a_k^m| = |p_k^0| + |p_k^2|, |b_k^*| = |b_k^m| = |p_k^0| + |p_k^1|, k = 1, 2
\]

Topologically the cycles \( a_k^m (+), b_k^m (+) \) represent some canonical \( m \)-dependent basis of elementary shifts in the group \( Z_2^k \). We choose the simplest transversal paths joining the initial point with its image in order to lift this basis to the free group. These shifts map the segment \( s \subset P_{0,0}^k \) exactly into the the segments \( s', s'' \) attached to the middle part of the long street from the
left side (for \(a_k^m(+)\)) and from the right side (for \(b_k^m(+)\))—see Fig 13. The subgroup generated by \(a_k, b_k\) in fundamental group is free. Homologically the elements \([a_k^*] = h_k^1 = a_k^m(+)\), \([b_k^*] = h_k^2 = b_k^m(+)\) are calculated in the formulation of lemma using the transversal measures \(|a_k|, |b_k|\) of basic cycles and the measure \(m\). From geometric description of new cycles \(a_k^*, b_k^*\) as of the paths we can see that they satisfy to the same relation as the original \(a_k, b_k\): Their commutator path exactly surrounds one segment \(s\) on the plane. Lemma is proved. This is the end of the Step 1.

We always choose initial point on the segment \(s_1^+\) in the plane \(C_1\). It is the same as to choose initial point on the segment \(s_2^- = s_1^+\) in the plane \(C_2\). We are going to use a new basis of fundamental group \(\pi_1(V, s_1^+)\): The New \(m\)-Dependent Transversal Canonical Basis (see Fig 13) is

\[
a_1^* = a_1^m(+), b_1^* = b_1^m(+), a_2^* = a_2^m(-), b_2^* = b_2^m(-)
\]

We attach this basis to the segment \(s_1^+ = s_2^-\) in order to treat them as the elements of fundamental group \(\pi_1(V, s_1^+)\).

Lemma 12 The homology classes \(h_k^\alpha\) of the streets number \(\alpha\) in the plane \(C_k\), are following:

\[
h_k^1 = [a_k^m(+)], h_k^2 = [b_k^m(+)], h_k^0 = [a_k^m(+)] + [b_k^m(+)] \in H_1(V, Z) = H_1(V, s^+ \cup s^-)
\]

Here \(k = 1, 2\). We assume that the streets are oriented to the positive time direction. For the negative time we simply change sign \([a_k^\alpha(-)] = -h_k^1, [b_k^\alpha(-)] = -h_k^2\).

This lemma immediately follows from the description of the homology classes \([a_k^m(\pm)]\) and \([b_k^m(\pm)]\).

Let us describe the homotopy classes \(\phi_{\alpha\beta} \in \pi_1(V, s_1^+)\) of the two-street paths \(\alpha\beta\) starting and ending on the same open segment \(s = s_1^+ = s_2^- \in C_1 \cap C_2\). Here \(C_1 \cap C_2\) is a union of all shifts of the cycles \(s^+ \cup s^-\).

The streets in the plane \(C_k\) end up in the points of the segment \(s_k^-\). Consider first the plane \(C_1\). In order to make closed paths out of the streets \(p_k^\alpha\), we need to extend them: we go around the segment \(s\) from \(s_1^-\) to \(s_1^+\) from the right or from the left side of it (see Fig 14):

For the representation of street number 1 we use the path \(a_1^* = a_1^m(+)\) closed by passing \(s'\) from the right side along the path \(\kappa_1\), circling clockwise around \(s'\); For the street number 2 we use the path \(b_1^* = b_1^m(+)\)
closed by passing $s''$ from the left side along the path $\kappa_2$ circling clockwise around $s''$; So we have

$$p_1^1 \sim a_1^*\kappa_1^{-1}$$
$$p_1^2 \sim b_1^*\kappa_2^{-1}$$

Here and below the symbol \(\sim\) means "homotopic with fixed ends".

We define a closed path circling clockwise around the segment $s$ in $C_1$:

$$\kappa = \kappa_2\kappa_1^{-1} \sim (s^+ \bigcup s^-) \sim a_1^*b_1^*(a_1^*)^{-1}(b_1^*)^{-1}$$

For the street number 0 we assign the path

$$p_0^0 \sim a_1^*b_1^*\kappa_2^{-1}$$

The same description we have also for the streets $p_2^\alpha(-) = (p_2^\alpha)^{-1}$ in the plane $C_2$ going in the opposite direction, replacing (+) by (-) and the paths $\kappa, \kappa_1, \kappa_2$ by the similar paths $\delta, \delta_1, \delta_2$:

$$(p_2^1)^{-1} \sim a_2^\delta_1^{-1}, (p_2^2)^{-1}b_2^* \sim \delta_2^{-1}, (p_0^0)^{-1} \sim a_2^*b_2^\delta_2^{-1}$$

We have

$$\kappa_1 = \delta_2, \kappa_2 = \delta_1, \delta = \delta_1\delta_2^{-1} = \kappa^{-1}$$

We use the new basis $a_1^*, b_1^*, a_2^*, b_2^*$ defined above, dropping the measure $m$ and signs $\pm$, as it was indicated above. All Formulas are written in the new basis.

We are ready to present the two-street formulas in $\pi_1$, where at first we are passing the street $\alpha$ in $C_1$ and after that the street $\beta \in C_2$. Writing $\alpha\beta$ instead of $\phi_{\alpha\beta}^\ast$ and performing very simple multiplication of paths, we obtain following formulas in the group $\pi_1(V, s_1^\pm)$:

**Lemma 13** The homotopy types of all nonnegative almost transversal two-street passes in the positive time direction starting and ending in the segment $s_1^+=s_2^-$, including the trajectory passes, are equal to the following list of values of the elements $\phi_{\alpha\beta}^\ast \in \pi_1(V, s_1^\pm)$ where $\alpha = 1, 0, 2$ and $\beta = 1', 0', 2'$:

$$11' \sim a_1^*\kappa^{-1}(a_2^*)^{-1}; 10' \sim a_1^*b_1^*(a_2^*)^{-1}; 12' \sim b_1^*(a_2^*)^{-1}$$
$$01' \sim a_1^*(b_2^*)^{-1}(a_2^*)^{-1}; 00' \sim a_1^*b_1^*\kappa(b_2^*)^{-1}(a_2^*)^{-1}; 02' \sim b_1^*\kappa(b_2^*)^{-1}(a_2^*)^{-1}$$

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\[21' \sim a_1^*(b_2^*)^{-1}; 20' \sim a_1^*b_1^*\kappa(b_2^*)^{-1}; 22' \sim b_1^*\kappa(b_2^*)^{-1}\]

For the negative time direction the two-street passes have homotopy classes equal to \(\phi_{\alpha\beta}^{-1}\) in the same group \(\pi_1(V, s^-_2) = \pi_1(V, s^+_1)\)

For the Topological Types I–VI of Foliations following homotopy classes of almost transversal two-street passes have nonzero measure:

\[
\phi_{\alpha\beta} = p^\alpha_1x p^\beta_2, \phi_{\alpha\beta}^* = (p^\alpha_1)^{-1}x(p^\beta_2)^{-1}
\]

(\(x \geq 0\) means positive transversal shift along the segment \(s\)):

Type (I): All 9 classes \(\phi_{\alpha\beta}\), and all classes \(\phi_{\alpha\beta}^*\) except \(11', 10', 01', 00'\)

Type (II): All classes \(\phi_{\alpha\beta}\) except \(22'\), and all classes \(\phi_{\alpha\beta}^*\) except \(11', 10', 01'\)

Type (III): All classes \(\phi_{\alpha\beta}\) except \(22', 02'\), and all classes \(\phi_{\alpha\beta}^*\) except \(11', 01'\)

Type (IV): All classes \(\phi_{\alpha\beta}\) except \(20', 22'\), and all classes \(\phi_{\alpha\beta}^*\) except \(11', 10'\)

Type (V): All classes \(\phi_{\alpha\beta}\) except \(22', 20', 02'\), and all classes \(\phi_{\alpha\beta}^*\) except \(11'\)

Type (VI): All classes \(\phi_{\alpha\beta}\) except \(20', 22', 00', 02'\), and all 9 classes \(\phi_{\alpha\beta}^*\)

Next theorem follows the standard scheme of the ergodic theory, but the values of topological quantities are explicitly calculated within the combinatorial model of the flow on the Riemann Surface:

**Theorem 4** The combinatorial model of foliation defines an Ensemble consisting of the following ingredients:

1. The semigroup \(S_{\sigma, \tau}\) and its representation in the fundamental group are given. All positive finite words

\[R = R_{j_1}R_{j_2}\ldots R_{j_N} \in \pi_1(V)\]

are written in the new \(m\)-dependent Transversal Canonical Basis \(a_1^*, b_1^*, a_2^*, b_2^*\), of all lengths \(|R| = N \geq 1\), where every symbol \(R_j\) is equal to one of the elements \(\phi_{\alpha\beta} \in \pi_1(V), j = 1, 2, 3, 4, 5\).

2. The permutation \(\sigma\) and measures of the sub-segments are given. We have \(\tau_{q}, q = 1, 2, 3, 4, 5, \sum_{q} \tau_{q} = m\).

They represent the pairs \(\alpha\beta\) with nonzero measure, according to the types I, II, III, IV, V, VI above. The permutation \(\sigma\) defines a broken isometry \(i_\sigma: s \rightarrow s\) well-defined for the inner points of the sub-segments \(\tau_q\).

2. For every length \(N\), the set of words with nonzero measure is ordered in the following way: Every such word \(R \in \pi_1(V)\) is represented by a single
connected sub-segment $\tau_R \subset s$ with nonnegative length. Every word represented by the empty sub-segment $\tau_R = \emptyset$, is equivalent to zero. Nonzero segments do not intersect each other for $R \neq R'$. They are naturally ordered in the segment $s$ of the length $m$, and sum of their measures is equal to $m$ for every length $N$. In order to multiply any word $R$ from the left by the elementary word $R_q = \phi_{\alpha\beta}, q = 1, 2, 3, 4, 5$, we apply the map $i_{\sigma}^{-N+1}$ to the set $\tau_q$ segments of the nonzero measure and intersect it with segment $\tau_R$ representing $R_j$:

$$\tau_{R_q R} = i_{\sigma}^{-N+1}(\tau_q) \cap \tau_R$$

If intersection is empty, the product $R_q R$ is equivalent to zero by definition. The set of words $S$ factorized by the words equivalent to zero, forms an associative semigroup $S_{\sigma, \tau}$ with $0 \in S_{\sigma, \tau}$ defined by our foliation $dH = 0$. If intersection $i_{\sigma}^{-N+1}(\tau_q) \cap \tau_R$ is nonempty, we assign it to the corresponding product word $R_q R$ of the length $N + 1$. A width of this strip is treated as a nonzero measure of the new word. The Shift Function $r(R)$ for every nonzero word $R = R_{j_1} \ldots R_{j_N}$ is equal to the sum

$$r(R) = \sum_q r(R_{j_q})$$

It is bounded for all nonzero words $r(R) < m$.

3. Every finite word $R \in S_{\sigma, \tau}$ defines a connected strip of the nonseparatrix trajectories of the Hamiltonian System $dH = 0$ corresponding to our data, with the transversal measure $\tau_R$, starting and ending in the transversal segment $s^\dagger$. It defines a positive closed transversal curve $\gamma_R$ with transversal measure equal to $r(R)$ if $r(R) < 0$, and negative closed transversal curve if $r(R) > 0$. The transversal measure of these curves are equal to $-r(R)$. The individual infinite trajectories are presented by the infinite sequences of the symbols $R_j$ such that every finite piece of the sequence is nonzero as an element of the semigroup $S_{\sigma, \tau}$. The measure on the set of trajectories is defined by the transversal measure of the strips corresponding to the finite words in $S$. The basic measurable sets are "cylindrical" (i.e. they consists of all trajectories with the same finite word $R$, and measure is equal to $\tau_R$).

**Problem 1**: How to calculate effectively the words in the free group with 2 generators $a, b$ describing new $m$-dependent canonical transversal basis $a^*, b^*$ in the 2-torus with straight-line flow and obstacle $s \subset T^2$ of the transversal size $m$? We assume that $m \to 0$, and the initial transversal canonical basis $a, b$ remains fixed.
Therefore the Step 2 is realized. We described the homology and homotopy classes associated with trajectories starting and ending in the segments $s$. They correspond to the "Trajectory Type" closed transversal curves.

Now let us start to discuss the most complicated

**Problem 2:** How to describe all transversal curves?

We are going to study this problem extending the method of the previous section where the trajectory type curves were described.

It is easy to prove following statement:

**Lemma 14** Consider the transversal homotopy classes (with fixed ends $\gamma(0)$ and $\gamma(1)$) of positive almost transversal curves $\{\gamma\}$ starting and ending in the segment $s = s_1^+ = s_2^-$. Every such curve within its class can be presented by the almost transversal path $\gamma$ consisting of the following pieces: 1.Any full street $(p^\pm_k)\in C_k$, $k = 1,2$ passed in any direction; 2.Any jump $x \in (0,m)$ along the segments $s_{k}^\pm$, $k = 1,2$ in positive direction before passing the next street. Therefore every such curve can be presented by the word consisting of these symbols, starting and ending by the streets:

$$W' = (p^\pm_{k_1})_1^{\pm1} x_1(p^\pm_{k_2})_2^{\pm1} \ldots x_{N-1}(p^\pm_{k_N})_N^{\pm1}$$

The number $N$ is even. These pieces are satisfying to the following obvious relation: If the same street was passed in some direction and back immediately after that (maybe with some positive jump $x$ between them), it can be removed, and the jump between them can be divided into two parts $x = x' + x''$ added to the previous and to the next steps.

By the **Topological Type** $W$ of any Word $W'$ we call the same word where all transversal jumps $x_q$ are simply omitted. We say that $(x_1, \ldots, x_{N-1})$ belongs to the **Existence Domain of the given Topological Type** $\Delta_W$ if for all jumps belonging to this domain there exists an almost transversal curve of that type. We assign to every such Topological Type a positive measure equal to the measure of the domain $\Delta_W$ in the cube $[0m]^{N-1}$. This measure depends on the ends of the curve. We can close this curve and construct a closed transversal positive curve only if the last end $\gamma(1)$ is located from the left side of the initial point $\gamma(0)$. Taking $x_N = [\gamma(N), \gamma(0)]$. The transversal measure of this closed transversal curve is equal to $\oint \gamma dH = \sum_{q=1}^N x_q$.

We assign homology class to every transversal curve $\gamma$ with ends in the same segment $s_1^+ = s_2^-:

$$[\gamma] = \sum(\pm)[p^\alpha_k] \in H_1(V, Z)$$

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depending on the topological type only. The homology classes corresponding to the streets were calculated above.

In order to calculate the homotopy classes, we consider two-street passes as above. The list of homotopy classes of two-street passes in the group \( \pi_1(V, s^+_1) \) was done before, for the case where both of them are positively directed (or both are negatively directed like trajectories). For the trajectories, without transversal jumps, we had only 5 types of the trajectory type two-street passes \( \phi_{\alpha\beta} \) with nonzero measures. Here we have more as it was indicated in the lemma above. Besides that, we may have two-street transversal passes concentrated in one plane, of the types

\[
\psi'^0_1 = p^0_1 x (p^1_1)^{-1}, x \geq 0 \\
\psi'^1_2 = (p^2_2)^{-1} x (p^0_2), x \geq 0
\]

These transversal cycles in fact were described above: they form the new \( m \)-dependent Transversal Canonical Basis.

**Lemma 15** Following positive transversal two-street passes of the types \( \psi^* \alpha\alpha'_1 \), starting and ending in the segment \( s^+_1 = s^-_2 \), have nonzero measure only:

\[
\psi^{01}_1 = p^0_1 x (p^2_1)^{-1} \sim a^*_1; \psi^{10}_1 = p^1_1 x (p^0_1)^{-1} \sim (b^*_1)^{-1} \\
\psi^{12}_1 = p^1_1 x (p^2_1)^{-1} \sim (b^*_1)^{-1} a^*_1
\]

For the two-street passes of the type \( \psi^\beta\beta' \) we have

\[
\phi^{00'} = a^*_2, \phi^{22'} = (b^*_2)^{-1}; \phi^{21'} = (b^*_2)^{-1} a^*_2
\]

Here 1, 0, 2 are the numbers of streets for \( k = 1 \) and 1', 0', 2' are the numbers of streets for \( k = 2 \). Combining these formulas with homotopy classes of positive and negative almost transversal two-street passes \( \phi_{\alpha\beta} \) and \( \phi^*_{\alpha\beta} \) calculated above, we have a complete list of two-street transversal passes of all types jumping in positive transversal direction.

For every transversal curve \( \gamma \) we consider now the deformations within its topological type such that we allow to move the first end \( \gamma(0) \) to the left along the segment \( s \) (the last end \( \gamma(1) \) is staying). We allow to move left all streets passes except the last one changing the jumps \( x_q \). Finally we are coming to the following. We find an extremal (minimal) transversal distance between \( \gamma(0) \) and \( \gamma(1) \). If there is a position (within the given topological
type) such that the segment \( \tau \) starting in \( \gamma(1) \) and ending in \( \gamma(0) \), is positive, we say that this topological type represents a positive closed transversal curve \( \bar{\gamma} = \gamma \cup \tau \).

Every positive closed transversal curve \( \gamma \) starting and ending in the segment \( s_1^+ = s_2^- \), has a homotopy class which is a positive product of the following elements in the group \( \pi_1(V, s_1^+) \):

\[
\phi_{\alpha\beta}, \phi^*_{\alpha\beta}, \psi^{\alpha\alpha'}, \psi^{\beta\beta'}
\]

However, not all of them represent any closed transversal curve:

1. The measure of this type should be nonzero. 2. The initial point \( \gamma(0) \) for some representative should be located to the left from the endpoint \( \gamma(1) \).

The algorithm for finding such representative (or testing whether it exists or not) consists of the "left" deformations of street passes changing the jumps \( x_q \), starting from the initial point \( \gamma(0) \), and then repeating this process back starting from the motion of endpoint \( \gamma(1) \) to the right.

**How to classify all non-selfintersecting transversal closed curves?**

Let us describe first Solution of this Problem for the curves not touching the segment \( s \), i.e. belonging to one plane \( C_k \).

Consider first a 2-torus without one point \( T^* = T^2(\text{minus})P \) presented as a standard parallelogram \( P \subset C \) with one inner point missing, \( \partial P = aba^{-1}b^{-1} \).

Following facts are well-known:

1. Every non-selfintersecting closed curve \( \gamma \subset T^* \) representing homology class \( [\gamma] = ka + l(-b) \subset H_1(T^2, Z) = H_1(T^*, Z) \) where \( (k, l) = 1 \), can be deformed in \( T^* \) to the standard form such that it has \( k \) transversal intersection points with the cycle \( b \), and \( l \) transversal intersection points with \( a \).

Let us choose our notations for the basic cycles \( a, b \) and direction of curve \( \gamma \) such that \( k > |l| > 0 \). There are two cases here: (I) \( l > 0 \) and (II) \( l < 0 \). There are also trivial cases such that either \( k = 0 \), or \( l = 0 \), or \( |k| = |l| = 1 \).

2. Following algorithm allows to write effectively the presentation of a positive element \( k > 0, l > 0 \) as a positive word in the alphabet \( a = a, b = b^{-1} \), unique up to the cyclic permutation:

Let our curve \( \gamma \) starts at the vertex of the parallelogram \( P \). It is represented in \( P \) by the sequence of \( k + l \) segments \( t_j \subset P, j = 1, \ldots, k + l - 1 \), starting and ending in the crossing points of the curve \( \gamma \) with the boundary of parallelogram \( P \). Consider only the case (I). We choose notations for these points such that \( y_1, y_2, \ldots, y_l \subset a \) and \( y_{l+1}, \ldots, y_{k+l} \subset b \) and equivalent points.
\( y_j \) in the components \( a^{-1}, b^{-1} \). Here the points \( y_1 \) and \( y_{k+l} \) are the opposite vertices of \( P \) along the diagonal, all other points are distinct. All segments have positive length. We simply denote this crossing points by their numbers in the lemma below \( y_j \to j, y'_j \to j' \) (see Fig 15). The topological type of non-selfintersecting closed curve \( \gamma \) in \( T^* \) is completely determined by its homology class and by the number of the domain \( S_j \) between two neighboring segments containing the removed point \( P \in T^2 \). The list of segments beginning from the left upper vertex \( a \cap b \in P \), naturally ordered by the position in the Parallelogram \( P \), is following for the Case (I):

\[
t_1 = [l, l+1], t_2 = [l-1, l+2], ..., t_l = [1, 2l]
\]
\[
t_{l+1} = [(l+1)', 2l+1], ..., t_k = [k'(k+l)]
\]
\[
t_{k+1} = [(k+1)', l'], ..., t_{k+l-1} = [(k(l-1)', 2')]
\]

These segments divide the open parallelogram \( P \) into \( k+l \) domains

\[
P_{\text{minus}} (\bigcup_j t_j) = S_1 \cup S_2 ... \cup S_{k+l}
\]

where \( S_j \) is bounded by \( t_j, t_{j-1} \) and \( \partial P \).

Every segment \( t_j \) divides the parallelogram \( P \) into the upper and lower parts \( P_{\text{minus}}(t_j) = P^+_j \cup P^-_j \) where \( t_q \subset P^+_j \) for \( q < j \). Fixing the domain \( S_r \) containing the removed point, we see that all domains \( P^+_j \) for \( j \geq r \) also contain this point. We call them the Marked Domains. All domains \( P^+_j \) with \( j < r \) are not marked, i.e. they do not contain this point.

Consider now a naturally ordered sequence of the domains \( P^+_q, P^+_q, .., P^+_q \) along the closed non-selfintersecting curve \( \gamma \subset C \) not crossing the removed point and its shifts. The pieces \( t_j \subset \partial \tilde{P}^+_j \) exactly form a closed curve \( \gamma = \bigcup_j t_j \). We start from the point \( y_1 \), i.e. from the domain \( P^+_1 = P^+_q \), and end up with the domain \( P_k = P^+_q \) ending in the point \( y_{k+l} \).

Assign now to every domain \( P^+_j \) following element of the free group with two generators:

\[
\psi : P^+_j \to \kappa; \kappa = b^{-1}aba^{-1}
\]

if the boundary of its closure \( \partial \tilde{P}^+_j \) does not contain anyone of the full cycles \( a \) and \( b \)

\[
\psi : P^+_j \to a\kappa
\]

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if the boundary of its closure contains only a full cycle $a$, not $b$

$$\psi : P_j^+ \to \kappa^c b$$

if the boundary of its closure contains only a full cycle $b$ but not $a$

$$\psi : P_j^+ \to \alpha^c b$$

if the boundary of its closure contains fully both cycles $a, b$

Here $c = 0$ if our domain $P_j^+$ is non-marked, and $c = 1$ if it is marked.

The products of elements of free group with two generators

$$\psi(\gamma) = \psi(P_{q_1}^+) \cdots \psi(P_{q_{k+l-1}}^+)$$

is equal to the homotopy class of the closed curve $\gamma$ in the free group with two generators. We call this product \textbf{An Upper Triangle Decomposition}

of the class $[\gamma] \in \pi_1(T^*)$. This description gives a complete classification of the homotopy classes of non-selfintersecting closed curves in the punctured 2-torus $T^*$, representing the homology class

$$[\gamma] = k\alpha + l\beta \in H_1(T^*, Z) = H_1(T^2, Z), k > l > 0, (k, l) = 1$$

The homology class and the integer number $1 \leq r \leq k + l$ completely determine the homotopy class of the nonselfintersecting closed curve in $T^*$.

Therefore every indivisible positive homology class $k\alpha + l\beta \in H_1(T^*, Z), k > 0, l > 0$, can be represented by one positive word of the length $k + l$ in the free group with two positive generators $\alpha = a', b^{-1} = b'$ realized by the nonselfintersecting curve. This word is unique up to cyclic permutation (a source for the number $r$).

Let us assume now that we have a straight line flow on the 2-torus with irrational rotation number (i.e. it is generic) and fixed transversal canonical basis $a, b$ where $a, b^{-1}$ are positive as above. Every indivisible homology class in $H_1(T, Z)$ can be realized by the nonselfintersecting closed curve transversal to foliation. We can see that \textbf{semigroup of positive closed transversal curves has infinite number of generators containing the elements with arbitrary small transversal measure}. The same result remains true after removal of one point $T \to T^*$ but the semigroup became nonabelian. Remove now a transversal segment $s$ with positive measure $m > 0$ from the torus $T^* \to T^{(m)}$. As we shall see below, the situation changes drastically:
the semigroup became finitely generated (in fact, there are two generators in it).

Consider now the **Classification Problem** of the transversal closed curves for the hamiltonian foliations with Transversal Canonical Basis which do not touch the segment $s$. They are concentrated in the flat part $T^*_m = T^2 minus(s)$ in one plane $C_k$ only. Let $k = 1$ (we drop the number $k$ beginning from now). It is convenient to use a three-street model for this goal. As above, we denote the maximal street by the number 0, the left one by 1 and the right one by 2. Our Fundamental Domain consists of the union of these three streets. We use an extension of it adding 2 more streets— one more copy of the street 1 over 2 and one more copy of the street 2 over 1 (see Fig 16). Every positive non-selfintersecting transversal curve will be expressed in the $m$-dependent basis $a^*, b^* \in \pi_1(T^2_m)$ where $a^* = a_1^m(+)$, $b^* = b_1^m(+)$. Its homology class is $[\gamma] = ka^* + l(-b^*) \in H_1(T^*_m, Z)$, $k \neq 0, l \neq 0$. For our foliation the cycles $a = a^*, b^* = b^{-1}$ are the basic positive closed transversal curves.

Every non-selfintersecting positive closed transversal curve $\gamma \subset T^*_m = T^2 minus(s)$ crosses this fundamental domain several times (exactly $k + l$ times) from the right to the left. We denote these segments by $t_j, j = 1, ..., k + l$ counting them from the segment $s = s$ up, along the longest street 0. Its canonical representative can be chosen such that it crosses the domain 2 exactly $k$ times, entering the domain 0 in the points $y_1, ..., y_k$; they are naturally ordered by height. It crosses also the domain 1 exactly $l$ times, entering the domain 0 from 1 in the points $y_{k+1}, ..., y_{k+l}$. We denote the equivalent points from the left side of fundamental domain by the same figures with symbol '. The equivalent points on the left side of the street 0 are the points $y'_{k+1}, ..., y'_{k+l}$, on the segment separating street 0 from the street equivalent to 1.. The points $y'_1, y'_2, ..., y'_k$ are located on the common boundary segment separating the second (left) copy of the street 2 from the street 0. Therefore the sequence of segments ordered by height, is following:

The group (I):

$$t_1 = [1, (k + 1)', ..., t_l = [l, (k + l)']$$

The group (II):

$$t_{l+1} = [l + 1, 1'], ..., t_k = [k, (k - l)']$$

The group (III):

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We assign to every segment following homotopy class depending on the group I, II, III:

\[ \phi : t_j \rightarrow b', t_j \in (I) \]

\[ \phi : t_j \rightarrow a', t_j \in (II), (III) \]

We define

\[ \phi(\gamma) = \phi(t_{q_1})...\phi(t_{q_k}) \in \pi_1(T^*_m) \]

where \( \gamma \sim t_{q_1}...t_{q_k} \) in the natural order along the curve. As a result of the previous lemmas, we obtain following

**Theorem 5** The invariant \( \phi(\gamma) \) describes homotopy classes of the closed non-self-intersecting positive transversal curves in \( T^*_m \) as a positive words written in the free group with two \( m \)-dependent generators \( a' = a^*, b' = (b^*)^{-1} \) whose transversal measures are smaller than \( m \). In particular, every indivisible homology class \( k[a'] + l[b'], k > 0, l > 0 \), has unique \( m \)-dependent non-self-intersecting positive transversal representative. It defines a positive word unique up to the cyclic permutation (i.e. it defines exactly \( k + l \) positive words). This word is calculated above through the sequence of segments \( t_j \). Every such word can be taken as a part of new Transversal Canonical Basis in \( T^*_m \).

As a corollary, we can see that every integer-valued \( 2 \times 2 \)-matrix \( T, \det T = 1 \), with positive entries \( k, l, p, q \geq 0 \)

\[ a_1 = T(a') = ka' + lb', b_1 = T(b') = pa' + qb' \]

determines finite number of different transversal canonical bases \( a_1, b_1 \in \pi_1(T^*_m) \). They are represented by the curves \( a_1, b_1 \) crossing each other transversally in one point and representing the homology classes \( T(a'), T(b') \). We should locate the segments of both these curves in the domain number 0 as above. Let the curves \( a_1, b_1 \) are represented by the ordered sequences of pieces \( t_1, ...t_{k+l} \) and \( t'_1, ...t'_{p+q} \) correspondingly going from the right to the left side. We require existence of one intersection point for the selected pair
\( t_i \cap t_j' \neq \emptyset \). All other intersections should be empty. Every such configuration determines transversal canonical basis \( a_1, b_1 \). Its equivalence class is completely determined by the relative order of segments \( t, t' \) taking into account that \( t_i \) crosses \( t_j' \) only once, with intersection index equal to 1 from the right to the left side (i.e. \( t_i \) is "higher" than \( t_j' \) from the left side). Starting from the selected point \( t_i \cap t_j' \), we apply the procedure described above in the theorem. It gives us two positive words \( A, B \) in the free group \( F_2 \). By definition, the map \( \tilde{T} : a' \to A, b' \to B \) defines a lift from homology to fundamental group, for every pair of words \( A, B \) constructed in that way. Consider first a **Reducible** case where the left ends of the segments \( t_i \) and \( t_j' \) crossing each other, are located on the same part of boundary, where the domain 0 meets the same domain 1 or 2. If it is the domain 1, we can see that both words \( A \) and \( B \) start with the same letter \( b' \). We can deform our crossing point along the cycle \( b' \). After this step we are coming to the conjugated pair \( A', B' \) where \( b' \) is sent to the end:

\[
A = b' \tilde{A}, B = b' \tilde{B} \to A' = \tilde{A}b', B' = \tilde{B}b'
\]

The same argument we use if the pair \( t_i, t_j' \) ends in the left boundary of 0 with domain number 2 replacing \( b' \) by \( a' \). After the series of such steps, we are coming to the case where \( t_i \) ends in 1, \( t_j' \) ends in 2. This process cannot be infinite because the words \( A, B \) are not powers of the same word. So this process ends. We call **Irreducible** the case where the process ended up.

Example: The pair of words \( A = b'a'b'a'b', B = b'a' \) requires 5 steps to arrive to the irreducible case \( A'''' = A, B'''' = a'b' \).

In the final irreducible state the relation:

\[
ABA^{-1}B^{-1} = a'b'a'^{-1}b'^{-1}
\]

can be easily seen on the plane with periodic set of segments removed, looking on the 3-street decomposition of the plane.

The **Semigroup of unimodular 2 × 2-matrices with nonnegative integer-valued entries is free**, with two generators \( T_1, T_1 \) such that

\[
T_1(a') = a' + b', T_1(b') = b', T_2(a') = a', T_2(b') = a' + b'
\]

Their lifts to the automorphisms of the free group are following:

\[
\tilde{T}_1(a') = a'b', \tilde{T}(b') = b'; \tilde{T}_2(a') = a', \tilde{T}_2(b') = b'a'
\]
Obviously, they both preserve the word $a'b'a^{-1}b'^{-1}$, so all semigroup of non-negative unimodular matrices preserves this word. It is isomorphic to the semigroup of all positive automorphisms of free group $F_2$ preserving this word. (We clarified this question with I.Dynnikov. I don’t know where it is written).

Comparing this result with the previous arguments, we are coming to the following

**Conclusion:** The Semigroup $G$ of all positive automorphisms of the free group $F_2$ in the alphabet $a', b'$ preserving the conjugacy class of the word $\kappa = a'b'a^{-1}b'^{-1}$, contains following parts:

1. A free semigroup $G_\kappa$ consisting of transformations preserving this word exactly; It is isomorphic to the semigroup of matrices $T$ with $\det T = 1$ and nonnegative integer entries;

2. Every element $T \in G_\kappa$ defines finite number of positive transformations $T' \in G$ such that the corresponding pair of positive words $A', B'$ of $T' \in G$ are simultaneously conjugate to the words $A, B$ of $T \in G_\kappa$. All these conjugations should be performed by the simultaneous cyclic permutation of the words $A, B$ removing the same letter from the end and sending it to the beginning (until both words ends with the same letter). Total number of these elements (conjugations) is equal to the sum of matrix elements of $T$ minus 2. The natural projection of semigroup $G$ into $G_\kappa$ is such that the inverse image of each nonnegative unimodular matrix $T$ contains exactly $k + l + p + q - 2$ positive automorphisms. In the example above we have $k + l + p + q - 2 = 5$.

**Appendix. Existence of the Transversal Canonical Basis**

Consider any $C^1$-smooth vector field on the compact smooth Riemann Surface $M_g$ of the genus $g \geq 1$, with saddle (nondegenerate) critical points only. We introduce the **Class G of vector fields** requiring that there is no saddle connections and no periodic trajectories for the vector fields in this class. Let a finite family of smooth disjoint non-selfintersecting positively oriented segments $s_j \subset M_g$ is given, $j = 1, ..., N$, transversal to the vector field in every point, including the boundary points $P_{1,j} \cup P_{2,j} = \partial s_j$. We think that the points $P_{1,j}$ are left boundaries of the segments $s_j$.

**Lemma 16** There exists a non-selfintersecting smooth closed curve $\gamma$ transversal to our vector field, which does not cross any segment $s_j$.
Proof. Start the trajectory \( \kappa \) anywhere outside of the segments \( \bigcup s_j \). It either meets first time some segment \( s_j \) of that family at the moment \( t < \infty \), or never meets this family for \( t \to \infty \). In the second case we construct a closed transversal curve \( \gamma \) as usually, ignoring the segments \( s_j \). In the first case we make a turn contr-clockwise and move from the point \( \kappa(t - \epsilon) \) towards the end \( P_{1,j_1} \), parallel to the segment \( s_{j_1} \), constructing the almost transversal curve \( \kappa \). After passing a small transversal distance \( \delta \) beyond the point \( P_{1,j_1} \) to the left, we make a second turn and begin to move straight ahead extending the almost transversal curve \( \kappa \) along the trajectory of vector field. The small constants \( \epsilon, \delta \) are chosen a priori. They remain unchanged during this construction: they are the same for all steps. At some moment we either reach the segment \( s_{j_2} \) or never meet this family. The second case was already taken into account. In the first case we repeat our move with the same parameters \( \delta, \epsilon \): we make turn and move beyond the left end. After \( \delta \)-passing it, we make one more turn and extend our transversal curve along the trajectory, and so on. The infinite almost transversal curve \( \kappa \) constructed by this process, certainly has an \( \omega^+ \)-limiting point \( P \in \mathcal{M}_g \). It cannot be an inner point of any segment \( s_k \). There are 3 possibilities:

The case one: \( P \) is a left end of some segment \( s_k \).

The case two: \( P \) is a right end of some segment \( s_k \).

The case three: \( P \) is some point distant from \( \bigcup s_k \).

The first case is trivial: There is a sequence of points \( \kappa(t_q) \to P \) for \( t_q \to \infty \) which approach \( P \) from the left side of it. Therefore the small transversal intervals \( r_q = [\kappa(t_{q+1}), \kappa(t_q)] \) are positive. The curve \( \kappa(t), t_q \leq t \leq t_{q+1}, r_q \) is almost transversal, positive and closed.

In the second case our curve \( \kappa \) approaches \( P \) from the right side by sequence of points located inside of the pieces of trajectories of vector field. Assuming that these pieces are not infinite for \( t \to -\infty \), we conclude that they already met some segment of our family at negative moment of time. Therefore we see that all of them are coming after the \( \delta \)-passing some of the left ends of our segments. Therefore they start to cross each other already at some finite time \( t' < \infty \). The curve \( \kappa \) is selfintersecting. Taking this curve till the first self-crossing, we obtain a closed almost transversal curve \( \gamma \).

The third case is either trivial if trajectory passing the point \( P \) never meets our segments, or reduces to the cases one and two.

Lemma is proved.

Now we cut the surface \( \mathcal{M}_g \) along the transversal closed curve \( \gamma \to a^+ \bigcup a^- \). We obtain a surface \( \tilde{\mathcal{M}}_g \) such that \( \partial \tilde{\mathcal{M}}_g = a^+ \bigcup a^- \) as in the Fig
1. The trajectories start at the in-cycle $a^+$ and end up in the out-cycle $a^-$, but there is an obstacle $s$ inside.

**Lemma 17** There exists a closed curve $\gamma_1 \subset M_g$ transversal to foliation such that it crosses $\gamma$ transversally in one point only; it does not cross the segment $s$.

Proof. We are going to construct a trajectory which starts and ends up in the boundary $\partial \bar{M}_g = a^+ \cup a^-$ and does not touch the segment $s$. This construction immediately implies the existence of the transversal curve $\gamma_1$. Let every trajectory started in $a^+$, always meets $s$. Find the first point which trajectory meets and move it to the left along the transversal segment $s$ as far as possible. Either the trajectory joining us with $a^+$ meets the second end $P_2$, or we can continue moving left. Every time we jump (if needed) to the nearest point to the boundary $a^+$ if new point appears. Finally we are coming to conclusion that either the most left point $P_1 \in s$ can be reached directly from the boundary $a^+$, or the most right $P_2$. We denote this point $P_k$. Changing time direction, we prove that we can directly reach the second boundary $a^-$ from another boundary point $P_l \in \partial s, l \neq k$, not crossing $s$. So we construct a transversal connection from $a^+$ to $a^-$ moving from $a^+$ to $P_k$ along the trajectory found above, after that we move along the segment $s$ (not touching it), and finally we move from the point near $P_l$ to $a^-$ along the second trajectory found above. Here $k, l = 1, 2$. We make this curve closed in $M_{g-1}$ using additional transversal piece along the cycle $a^-$ of the desired orientation.

Lemma is proved.

Let us prove following

**Theorem 6** For every vector field of the class $G$ on the Riemann Surface $M_g$ with genus $g \geq 3$, there exists an incomplete Transversal Canonical Basis $a_1, b_1, a_2, b_2, a$ such that all basic cycles are non-selfintersecting and transversal to the vector field. All pairwise intersections are transversal and nonempty only for $a_k \cap b_k$ where they consist of one point, $k = 1, 2$.

Proof.

Step 1: Reduction of genus. We construct a first pair of the transversal basic cycles $a_1, b_1$. After that we cut $M_g$ along these cycles $a_1 \cup b_1$. This operation leads to the manifold $W$ with boundary $A_1 = A_1^+ \cup A_1^- = \partial W$.
consisting of parallelogram in the plane $C_1$ as before. We construct the transversal segments $s^\pm$ joining the pair of saddles as in the Fig 5. Now we remove all part of $W$ outside of the segments $s^\pm$. We are coming to the manifold

$$\bar{W}, \partial \bar{W} = s^+ \cup s^-$$

Finally, we identify the segments $s^+$ and $s^-$ and obtain from $\bar{W}$ a closed 2-manifold $M_{g-1}$ with vector field inherited from $M_g$ (which easily can be done $C^1$), and transversal segment $s^+ = s^- \subset M_{g-1}$

Step 2: Construction of the second transversal pair $a_2, b_2$. We use for that the lemmas above. They explain how to deal with the case of one transversal segment $s$. After that we reduce genus second time and come to the manifold $M_{g-2}$ with $C^1$-smooth vector field and two transversal segments $s_1, s_2 \subset M_{g-2}$

Step 3: Construction of the transversal cycle $a \subset M_{g-2}$ not crossing the previous segments. We use for that the lemma above.

Our Theorem follows from these steps.

Combining this theorem with lemmas 5 and 6 above (see Section 2), we obtain the following result for $g = 3$.

**Theorem 7** For every Hamiltonian System on the Riemann Surface given by the real part of the generic holomorphic one-form there exists a Transversal Canonical Basis.

As G. Levitt pointed out to the present author, his work [25] published in 1982, can be used to construct Transversal Canonical Bases on every Riemann Surfaces of any genus $g$ for the Dynamical Systems of the class $G$ (above). Let us present here this construction.

**Theorem 8** (G. Levitt, private communication) For every generic dynamical system of the class $G$ there exist a Transversal Canonical Basis

Proof. The result of [25] is following: For every dynamical system of this class there exist exactly $3g - 3$ non-selfintersecting and pairwise non-intersecting transversal cycles $A_i, B_i, C_i, i = 1, ..., g - 1$, where $C_{g-1} = C_1$. These cycle bound two sets of surfaces: $A_i \cup B_i \cup C_i = \partial P_i$ where $P_i$ is a genus zero surface ("pants"). The trajectories enter pants $P_i$ through the cycles $A_i, B_i$ and leave it through the cycle $C_i$. There is exactly one saddle inside of $P_i$ for each number $i$. The cycles $A_i, B_i, C_{i-1}$ bound also another
set of pants $Q_i$ where $\partial Q_i = A_i, B_i, C_{i-1}$. The trajectories enter $Q_i$ through $C_{i-1}$ and leave it through $A_i$ and $B_i$. There is also exactly one saddle inside $Q_i$.

The construction of Transversal Canonical Basis based on that result is following: Define $a_g$ as $a_g = C_1$. Choose a segment of trajectory $\gamma$ starting and ending in $C_1$ assuming that it passed all this "necklace" through $P_k, Q_k$. For each $i$ this segment meets either $A_i$ or $B_i$. Let it meets $B_i$ (it does not matter). We define $a_i$ as $a_i = A_i$, $i = 1, ..., g - 1$. Now we define $b_g$ as a piece of trajectory $\gamma$ properly closed around the cycle $C_1 = a_g$. This curve is almost transversal in our terminology (above), so its natural small approximation is transversal. We are going to construct the cycles $b_i$ for $i = 1, ..., g - 1$ in the union $P_i \cup Q_i$. Consider the saddle $q_i \in Q_i$. There are two separatrices leaving $q_i$ and coming to $A_i$ and $B_i$ correspondingly. Continue them until they reach $C_i$ through $P_i$. Denote these pieces of separatrices by $\gamma_{1,i}$, $\gamma_{2,i}$. Find the segment $S_i$ on the cycle $C_i$ not crossing the curve $\gamma$ chosen above for the construction of the cycle $b_g$. The curve $\gamma_{1,i}S_i\gamma_{2,i}^{-1}$ is closed and transversal everywhere except the saddle point $q_i$. We approximate now the separatrices $\gamma_{1,i}$ and $\gamma_{2,i}^{-1}$ by the two pieces of nonseparatrix trajectories $\gamma_i', \gamma_i''$ starting nearby of the saddle $q_i$ from one side of the curve $\gamma_{1,i}\gamma_{2,i}^{-1}$. Close this pair by the small transversal piece $s_i$. There are two possibilities here (two sides). We choose the side such that the orientation of the transversal piece $s_i$ near the saddle $q_i$ is agreed with the orientation of the segment $S_i$, so the whole curve $b_i = s_i\gamma_i'S_i\gamma_i''$ is closed and almost transversal. We define $b_i$ as a proper small transversal approximation of that curve. So our theorem is proved because every cycle $b_i$ crosses exactly one cycle $a_i$ for $i = 1, 2, ..., g$.

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Trajectory connections: $a_1^+ \rightarrow a_1^-, a_2^+ \rightarrow a_2^-$. 

Fig 4a.

Almost transversal curve $\gamma$ (boldface): $a_2^+ \rightarrow a_2^-$. 

Fig 4b.
Fig 5.

Non-extendable diagram of the type $T^2$.

Fig 6.

Reconstruction: new transversal curve $\gamma: a_1^+ \rightarrow a_1^-$ is passing through the boundary.

Fig 7.
Fig 8.

\[ g = 2 \]

(Spherical parts can be dropped)

Fig 9a.

\[ g = 3 \]

“Corridors” \( t_i^+ \to t_i^- \).

Fig 9b.
Graphs

Simple foliations  Trivalent Tree $R$

Fig 9c.

Fig 10.
The 3-street model.

Fig 11.

Fig 12.
Fig 13.

$$a'^m(+) : 0 \rightarrow 2 \rightarrow 0, \ b'^m(+) : 0 \rightarrow 1 \rightarrow 0.$$  

Fig 14.

$$\kappa = \kappa_1^{-1} \kappa_2$$
* = Removed point

$k = 4, \ l = 3.$

Fig 15.

\(\gamma\) crosses the street \([0]\)

$k = 3, \ l = 2.$

Fig 16.