Stability of KAM tori for nonlinear Schrödinger equation *

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Abstract: It is proved that the KAM tori (thus quasi-periodic solutions) are long time stable for nonlinear Schrödinger equation.

Key words: KAM tori, Normal form, Stability, \( p \)-tame property, KAM technique.

1 Introduction

Since Kuksin’s work \cite{20} in 1987, the infinite dimensional KAM theory has seen enormous progress with application to partial differential equations (PDEs). There are too many references to list all of them. Here we refer to two books \cite{21} and \cite{13} and two survey papers \cite{12} and \cite{22}.

As an object to which the infinite dimensional KAM theory applies, one of typical models is nonlinear Schrödinger equation (NLS)

\[
\sqrt{-1}u_t - \triangle u + V(x, \xi)u + |u|^2u + h.o.t. = 0, \tag{1.1}
\]

subject to Dirichlet boundary condition or periodic boundary condition \( x \in \mathbb{T}^d \), where the integer \( d \geq 1 \) is the spatial dimension of NLS.

Case 1. \( d = 1 \). In 1993, Kuksin \cite{21} proved that \eqref{1.1} possesses many invariant tori around the origin \( u = 0 \) (thus quasi-periodic solutions of small initial values) for “most” parameter vector \( \xi \) when the potential \( V \) depends on \( \xi \) in some non-degenerate way. This kind of invariant tori obtained by KAM theory are usually called KAM tori. In 1996, Kuksin-Pöschel \cite{23} further proved that the potential \( V = V(x, \xi) \) can be replaced by a fixed constant potential \( \equiv C \). All those results are obtained by KAM theory which involves the so-called second Melnikov conditions. By advantage of them the linearized equation along the KAM tori can be reduced to a linear equation of constant coefficient, thus the obtained KAM tori (thus quasi-periodic solutions) are linearly stable.

Case 2. \( d > 1 \). This case is significantly more complicated, since the second Melnikov conditions are violated seriously by multiplicity of the eigenvalues of \(-\triangle\). In his series of papers \cite{7}–\cite{13}, Bourgain developed a profound approach which was originally proposed by Craig-Wayne \cite{15}. It was proved by Bourgain \cite{10}(1998) and \cite{13}(2005) that \eqref{1.1} has many KAM tori for most \( \xi \) when \( V(x, \xi) \) is replaced by Fourier multiplier \( M_\xi \). This approach which is today called C-W-B method

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Theorem 1.1. Consider the nonlinear Schrödinger equation

\[ \sqrt{-1} u_t = u_{xx} - M_\xi u + \epsilon |u|^2 u, \]

subject to Dirichlet boundary conditions \( u(t, 0) = u(t, \pi) = 0 \), where \( M_\xi \) is a real Fourier multiplier, \( M_\xi \sin jx = \xi_j \sin jx, \quad \xi = (\xi_j)_{j \geq 1} \)

and

\[ \xi \in \Pi := \left\{ \bar{\xi} = (\xi_j)_{j \geq 1} \mid \xi_j \in [1, 2]/j, \ j \geq 1 \right\} \subset \mathbb{R}^N. \]

For a sufficiently small \( \epsilon > 0 \) depending on \( n \) and \( p \), there exists a subset \( \tilde{\Pi} \subset \Pi \), such that for each \( \xi \in \tilde{\Pi} \) equation (1.2) possesses an \( n \)-dimensional KAM torus \( \mathcal{T}_\xi \) in Sobolev space \( H^p_\Pi([0, \pi]) \). Moreover, the KAM torus \( \mathcal{T}_\xi \) of equation (1.2) is stable in long time, that is, for arbitrarily given \( \mathcal{M} \) with \( 0 \leq \mathcal{M} \leq C(\epsilon) \) (where \( C(\epsilon) \) is a constant depending on \( \epsilon \) and \( C(\epsilon) \to \infty \) as \( \epsilon \to 0 \)) and any \( p \geq 8(\mathcal{M} + 7)^d + 1 \), there exists a small positive \( \delta_0 \) depending on \( n \), \( p \) and \( \mathcal{M} \), such that for any \( 0 < \delta < \delta_0 \) and any solution \( u(t, x) \) of equation (1.2) with the initial datum satisfying

\[
\| u(0, x) \|_{H^p_\Pi([0, \pi])} \leq \delta, 
\]

then

\[
\| u(t, x) \|_{H^p_\Pi([0, \pi])} \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-\mathcal{M}}. 
\]
In §3, we will give some discussions and ideas of the proof of the abstract results.

In §4, we will discuss some properties of $p$-tame property.

In §5, we will discuss $p$-tame property of the solution of homological equation.

In §6, we will construct a partial normal form of order $M + 2$ in the $\delta$-neighborhood of the KAM tori and show that the KAM tori are stable in a long time (Theorem 2.10).

In §7, it is shown the existence and long time stability of KAM tori (i.e. quasi-periodic solutions) for the nonlinear Schrödinger equation (1.2) according to the above theorems and corollaries (Theorem 1.1).

In §8, it will be given some technical lemmas. In §9, the details of proof of Theorem 4.1 and Corollary 4.2 will be given, since it is a standard KAM proof, based on the results in Section 4 and Section 5.

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In the Autumn of 2007, Professor H. Eliasson gave a series of lectures on KAM theory for Hamiltonian PDEs in Fudan University. In those lectures, he proposed to study the normal form in the neighborhood of the invariant tori and the nonlinear stability of the invariant tori. The authors are heartily grateful to Professor Eliasson.

2 Some notations and the abstract results

2.1 Some notations

To finish the proof of Theorem 1.1 several abstract theorems will be given. To this end, we will introduce some notations firstly. Given positive integers $n$ and $p$, by $T^n = \mathbb{C}^n / 2\pi \mathbb{Z}^n$ denote the usual $n$-dimensional torus and let $\ell^2_p$ be the Hilbert space of all complex sequences $w = (w_1, w_2, \ldots)$ with

$$||w||^2_p = \sum_{j \geq 1} |w_j|^2 p_j \gamma < \infty.$$ 

Introduce an infinite dimensional symplectic phase space

$$(x, y, q, \bar{q}) \in \mathcal{P} := T^n \times \mathbb{C}^n \times \ell^2_p \times \ell^2_p$$

with the usual symplectic structure

$$dy \wedge dx + \sqrt{-1} dq \wedge \bar{q}.$$ 

Given a subset $\Pi \subset \mathbb{R}^\Pi$ with positive measure in some sense (for example, in the sense of Gauss or Kolmogorov), here $\Pi$ will be regarded a parameter set. Let $N(y, q, \bar{q}; \xi)$ be an integrable Hamiltonian which depends on parameter $\xi \in \Pi$ and is of the form

$$N(y, q, \bar{q}; \xi) = \sum_{j=1}^n \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) q_j \bar{q}_j,$$
where $\omega(\xi) = (\omega_1(\xi), \ldots, \omega_n(\xi))$ is called tangent frequency and $\Omega(\xi) = (\Omega_1(\xi), \Omega_2(\xi), \ldots)$ is called normal frequency. With the symplectic structure mentioned above, the motion equation of $N(y,q,\tilde{q};\xi)$ is

$$\dot{x} = \omega(\xi), \quad \dot{y} = 0, \quad \dot{q}_j = \sqrt{-1} \Omega_j(\xi) q_j, \quad \dot{\tilde{q}}_j = -\sqrt{-1} \Omega_j(\xi) \tilde{q}_j, \quad j \geq 1.$$  \hspace{1cm} (2.1)

It is clear that, for each $\xi \in \Pi$, $(x(t), y(t), q(t), \tilde{q}(t)) = (\omega(\xi) t, 0, 0, 0)$ is a quasi-periodic solution to equation (2.1) with rotational frequency $\omega(\xi)$. Moreover, let $\hat{T}^n = \mathbb{R}^n/2\pi \mathbb{Z}^n$, and

$$\mathcal{T}_0 = \hat{T}^n \times \{ y = 0 \} \times \{ q = 0 \} \times \{ \tilde{q} = 0 \}.$$  

Then $\mathcal{T}_0$ is an $n$-dimensional invariant torus with frequency $\omega(\xi)$ for equation (2.1).

Now consider a perturbation of the integrable Hamiltonian $N(y,q,\tilde{q};\xi)$:

$$H(x,y,q,\tilde{q};\xi) = N(y,q,\tilde{q};\xi) + R(x,y,q,\tilde{q};\xi),$$

where the perturbation $R(x,y,q,\tilde{q};\xi)$ depends on the parameter $\xi \in \Pi$ and the variable $(x,y,q,\tilde{q}) \in \mathcal{T}^n$, and $R(x,y,q,\tilde{q};\xi)$ is of small size in some sense ($p$-tame norm) which will be defined in the following steps.

**Definition 2.1.** Let $D(s) = \{ x \in \mathbb{T}^n \mid ||\text{Im} \; x|| \leq s \}$, where $|| \cdot ||$ denotes the sup-norm for complex vectors in $\mathbb{C}^n$ or $\mathbb{C}^\Pi$. Consider a function $W(x;\xi) : D(s) \times \Pi \to \mathbb{C}$ is analytic about the variable $x \in D(s)$ and $C^1$-smooth in the Whitney’s sense\(^1\) about the parameter $\xi \in \Pi$ with the Fourier series

$$W(x;\xi) = \sum_{k \in \mathbb{Z}^n} \hat{W}(k;\xi) e^{\sqrt{-1}T(k,x)},$$

where

$$\hat{W}(k;\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} W(x;\xi) e^{-\sqrt{-1}T(k,x)} dx$$

is the $k$-th Fourier coefficient of the function $W(x;\xi)$, and $(\cdot,\cdot)$ denotes the usual inner product, i.e.

$$\langle k, x \rangle = \sum_{j=1}^n k_j x_j.$$

Then define the norm

$$||W||_{D(s) \times \Pi} = \sup_{\xi \in \Pi, |j| \geq 1} \sum_{k \in \mathbb{Z}^n} \left( |\hat{W}(k;\xi)| + |\partial_{\xi_j} \hat{W}(k;\xi)| \right) e^{||kr||}.$$  \hspace{1cm} (2.2)

**Definition 2.2.** Let $D(s,r) = \{ (x,y) \in \mathbb{T}^n \times \mathbb{C}^n \mid ||\text{Im} \; x|| \leq s, \; ||y|| \leq r^2 \}$.

---

\(^1\)In the whole of this paper, the derivatives with respect to the parameter $\xi \in \Pi$ are understood in the sense of Whitney.
Consider a function $W(x, y; \xi) : D(s, r) \times \Pi \to \mathbb{C}$ is analytic about the variable $(x, y) \in D(s, r)$ and $C^{1}$-smooth about the parameter $\xi \in \Pi$ with the following form

$$W(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} W^{\alpha}(x; \xi)y^{\alpha}.$$ 

Define the norm

$$\|W\|_{D(s, r) \times \Pi} = \sum_{\alpha \in \mathbb{N}^n} ||W^{\alpha}(x; \xi)||_{D(s) \times \Pi} r^{2|\alpha|},$$

(2.3)

where $|\alpha| = \sum_{j=1}^{n} |\alpha_j|$. In this paper, always by $| \cdot |$ denotes 1-norm for complex vectors in $\mathbb{C}^n$ or $\mathbb{C}^N$.

**Definition 2.3.** Introduce the complex $\mathcal{T}_0$-neighborhoods

$$D(s, r_1, r_2) = \{(x, y, q, \bar{q}) \in \mathcal{D}^p \mid ||\text{Im } x|| \leq s, ||y|| \leq r_1^2, ||q||_p + ||\bar{q}||_p \leq r_2\}.$$ 

Let $r_1 = r_2 = r$. Consider a function $W(x, y, q, \bar{q}; \xi) : D(s, r) \times \Pi \to \mathbb{C}$ is analytic about the variable $(x, y, q, \bar{q}) \in D(s, r, r)$ and $C^{1}$-smooth about the parameter $\xi \in \Pi$ with the following form

$$W(x, y, q, \bar{q}; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n} W^{\alpha\beta\gamma}(x; \xi)y^{\alpha}q^{\beta}\bar{q}^{\gamma}.$$ 

Define the modulus $|W|_{D(s, r) \times \Pi}(q, \bar{q})$ of $W(x, y, q, \bar{q}; \xi)$ by

$$|W|_{D(s, r) \times \Pi}(q, \bar{q}) := \sum_{\beta, \gamma \in \mathbb{N}^n} ||W^{\beta\gamma}(x, y; \xi)||_{D(s, r) \times \Pi} q^{\beta}\bar{q}^{\gamma},$$

(2.4)

where

$$W^{\beta\gamma}(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} W^{\alpha\beta\gamma}(x; \xi)y^{\alpha}.$$ 

Denote $z = (q, \bar{q})$, i.e.

$$z = (z_j)_{j \in \mathbb{Z}} \in \ell^2_s \times \ell^2_s, \quad \tilde{\mathbb{Z}} = \mathbb{Z} \setminus \{0\},$$

where

$$z_{-j} = q_j, \quad z_j = \bar{q}_j, \quad j \geq 1.$$ 

[1] Here we consider $q$ and $\bar{q}$ as two independent variables and define $||z||_p = ||q||_p + ||\bar{q}||_p$. For a homogeneous polynomial $W(z)$ of degree $h > 0$, it is naturally associated with a symmetric $h$-linear form $\tilde{W}(z^{(1)}, \ldots, z^{(h)})$ such that $\tilde{W}(z, \ldots, z) = W(z)$. More precisely, given a monomial

$$W(z) = W^\beta z^\beta = W^{z_{j_1}} \cdots z_{j_h},$$

where $\beta = (\ldots, \beta_{-2}, \beta_{-1}, \beta_1, \beta_2, \ldots) \in \mathbb{N}^\infty$ and $|\beta| = \sum_{j \geq 1} \beta_j = h$, the symmetric $h$-linear form $W(z^{(1)}, \ldots, z^{(h)})$ is defined by

$$\tilde{W}(z^{(1)}, \ldots, z^{(h)}) = \tilde{W}^{\beta}z^{\beta} = \frac{1}{h!} \sum_{\gamma_1} W^{\beta}z_{j_1}^{\gamma_{(1)}} \cdots z_{j_h}^{\gamma_{(h)}},$$

(2.5)
Definition 2.6. Basing on the above notations, we will define \( p \)-tame norm of a Hamiltonian vector field. Firstly, consider a Hamiltonian

\[
W_h(x, y; z; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^3, |\beta| = h} W^\alpha_\beta(x; \xi)^\alpha z^\beta,
\]

where the modulus of \( W_h(x, y; z; \xi) \) is a homogeneous polynomial about \( z \) of degree \( h \) and \( W_h(x, y; z; \xi) \) itself is analytic about the variable \((x, y, z) \in D(x, r, r) \) and \( C^1 \)-smooth about the parameter \( \xi \in \Pi \).

To simplify the notation, we rewrite \( W_h(x, y; z; \xi) \) as \( W(x, y; z; \xi) \). By a little abuse of notation, let \( W = (\sqrt{-1} W_q, -\sqrt{-1} W_{\overline{q}}) \) here and later. Notice the Hamiltonian vector field \( X_W \) of \((x, y; z; \xi) \) is \((W_y, -W_x, W_z)\). Denote

\[
||| (z^h) |||_{p, 1} := \frac{1}{h} \sum_{j=1}^{h} ||| z^{(1)} |||_1 \cdots ||| z^{(j-1)} |||_1 ||| z^{(j)} |||_p ||| z^{(j+1)} |||_1 \cdots ||| z^{(h)} |||_1.
\]

### Definition 2.4.
In the normal direction of the Hamiltonian vector field \( X_W \), define the \( p \)-tame operator norm of \( W \) by,

\[
||| W \|||_{p, D(x, r) \times \Pi} := \sup_{0 \neq z \in \ell_2^4 \times \ell_2^3, 1 \leq j \leq h-1} \frac{||| (z^{(j-1)} \cdots z^{(h-1)}) |||_{p, 1}}{||| (z^h) |||_1},
\]

and define the \( p \)-tame norm of \( W \) by

\[
||| W \|||_{p, D(x, r) \times \Pi} = \max\left\{ ||| W \|||_{p, D(x, r) \times \Pi}, ||| W \|||_{1, D(x, r) \times \Pi}, 1 \right\}^{p^{-1}}.
\]

### Definition 2.5.
In the tangent direction of the Hamiltonian vector field \( X_W \), define the operator norm of \( W_u (u = x \text{ or } y) \) by,

\[
||| W_u \|||_{D(x, r) \times \Pi} := \sup_{0 \neq z \in \ell_2^4 \times \ell_2^3, 1 \leq j \leq h} \frac{||| (z^{(j-1)} \cdots z^{(h-1)}) |||_{1, 1}}{||| (z^h) |||_1},
\]

and define the norm of \( W_u (u = x \text{ or } y) \) by

\[
||| W_u \|||_{D(x, r) \times \Pi} := ||| W_u \|||_{D(x, r) \times \Pi}^{p^h}.
\]

### Remark 1.
Note that the dimension of the tangent direction is finite, so there is no so-called \( p \)-tame property. But \( ||| W_u \|||_{D(x, r) \times \Pi} \) is required as a bounded map from \( \ell_2^4 \times \ell_2^3 \) to \( \mathbb{C}^n \) to guarantee the persistence of \( p \)-tame property under Poisson bracket.

### Definition 2.6.
Define the \( p \)-tame norm of the Hamiltonian vector field \( X_W \) as follows,

\[
||| X_W \|||_{p, D(x, r) \times \Pi} = ||| W_x \|||_{D(x, r) \times \Pi} + \frac{1}{r^2} ||| W_x \|||_{D(x, r) \times \Pi} + \frac{1}{r} ||| W_z \|||_{p, D(x, r) \times \Pi}.
\]
**Definition 2.7.** Consider a Hamiltonian \( W(x,y,z;\xi) = \sum_{h \geq 0} W_h(x,y,z;\xi) \) is analytic about the variable \((x,y,z) \in D(s,r,\bar{r})\) and \(C^1\)-smooth about the parameter \(\xi \in \Pi\), where

\[
W_h(x,y,z;\xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^2, |eta| = h} W_h^{\alpha\beta}(x;\xi) y^\alpha z^\beta.
\]

Then define the \(p\)-tame norm of the Hamiltonian vector field \( X_W \) by

\[
|||X_W|||^T_{p,D(s,r,\bar{r}) \times \Pi} := \sum_{h \geq 0} |||X_{W_h}|||^T_{p,D(s,r,\bar{r}) \times \Pi}.
\] (2.13)

We say that a Hamiltonian vector field \( X_W \) (or a Hamiltonian \( W(x,y,z;\xi) \)) has \(p\)-tame property on the domain \( D(s,r,\bar{r}) \times \Pi \), if and only if \( |||X_W|||^T_{p,D(s,r,\bar{r}) \times \Pi} < \infty \).

**2.2 The abstract results**

Now we have the following theorems:

**Theorem 2.8.** (Normal form of order 2) Consider a perturbation of the integrable Hamiltonian

\[
H(x,y,q,\bar{q};\xi) = N(y,q,\bar{q};\xi) + R(x,y,q,\bar{q};\xi)
\] (2.14)

defined on the domain \( D(s_0,r_0,\bar{r}_0) \times \Pi \) with \( s_0, r_0 \in (0,1] \), where

\[
N(y,q,\bar{q};\xi) = \sum_{j=1}^n \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) q_j \bar{q}_j
\]

is a family of parameter dependent integrable Hamiltonian and

\[
R(x,y,q,\bar{q};\xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^2, \gamma \in \mathbb{N}} R^{\alpha\beta\gamma}(x;\xi) y^\alpha q^\beta \bar{q}^\gamma
\]

is the perturbation. Suppose the tangent frequency and normal frequency satisfy the following assumptions:

**Assumption A:** Frequency Asymptotics. There exists absolute constants \( c_1, c_2 > 0 \) such that

\[
|\Omega_j(\xi) - \Omega_j(\bar{\xi})| \geq c_1 |i - j|(i + j),
\] (2.15)

and

\[
|\Omega_j(\xi)| \leq c_2 j^2,
\] (2.16)

for all integers \( i, j \geq 1 \) uniformly on \( \xi \in \Pi \);

**Assumption B:** Twist conditions.

\[
\partial_{\xi_i} \omega_j(\xi) = \delta_{ij}, \quad \partial_{\xi_i} \Omega_j(\xi) = \delta_{j(i+n)}, \quad 1 \leq i \leq n, \quad j, j' \geq 1.
\] (2.17)

The perturbation \( R(x,y,q,\bar{q};\xi) \) has \( p\)-tame property on the domain \( D(s_0,r_0,\bar{r}_0) \times \Pi \) and satisfies the small assumption:

\[
\varepsilon := |||X_R|||^T_{p,D(s_0,r_0,\bar{r}_0) \times \Pi} \leq \eta^{12} \varepsilon, \quad \text{for some} \quad \eta \in (0,1),
\]
where \( \epsilon \) is a positive constant depending on \( s_0, r_0 \) and \( n \). Then there exists a subset \( \Pi_\eta \subset \Pi \) with the estimate

\[
\text{Meas} \, \Pi_\eta \geq (\text{Meas} \, \Pi) (1 - O(\eta)).
\]

For each \( \xi \in \Pi_\eta \), there is a symplectic map

\[
\Psi : D(s_0/2, r_0/2, r_0/2) \to D(s_0, r_0, r_0),
\]

such that

\[
\bar{H}(x, y, q, \bar{q}; \bar{\xi}) := H \circ \Psi = \bar{N}(y, q, \bar{q}; \bar{\xi}) + \bar{R}(x, y, q, \bar{q}; \bar{\xi}),
\]

where

\[
\bar{N}(y, q, \bar{q}; \bar{\xi}) = \sum_{j=1}^{n} \tilde{\alpha}_j(\bar{\xi})y_j + \sum_{j=1}^{\bar{q}} \tilde{\alpha}_j(\bar{\xi})q_j \bar{q}_j
\]

and

\[
\bar{R}(x, y, q, \bar{q}; \bar{\xi}) = \sum_{\alpha \in \mathbb{N}^n, \gamma \in \mathbb{N}^n} \tilde{R}^{\alpha \gamma}(x, \xi) y^\alpha q^\gamma.
\]

Moreover, the following estimates hold:

1. For each \( \xi \in \Pi_\eta \), the symplectic map \( \Psi : D(s_0/2, r_0/2, r_0/2) \to D(s_0, r_0, r_0) \) satisfies

\[
||\Psi - id||_{p, D(s_0/2, r_0/2, r_0/2)} \leq c \eta^6 \epsilon,
\]

where

\[
||\Psi - id||_{p, D(s_0/2, r_0/2, r_0/2)} = \sup_{w \in D(s_0/2, r_0/2, r_0/2)} ||(\Psi - id)w||_{\mathscr{P}^{p, D(s_0, r_0, r_0)}},
\]

and

\[
||w||_{\mathscr{P}^{p, D(s_0, r_0, r_0)}} = ||w_x|| + \frac{1}{r_0} ||w_y|| + \frac{1}{r_0} ||w_{\bar{q}}|| + \frac{1}{r_0} ||w_{\bar{q}}||_p
\]

for each \( w = (w_x, w_y, w_{\bar{q}}, w_{\bar{q}}) \in D(s_0, r_0, r_0) \); moreover,

\[
|||D\Psi - Id|||_{p, D(s_0/2, r_0/2, r_0/2)} \leq c \eta^6 \epsilon,
\]

where on the left-hand side hand we use the operator norm

\[
|||D\Psi - Id|||_{p, D(s_0/2, r_0/2, r_0/2)} = \sup_{0 \neq w \in D(s_0/2, r_0/2, r_0/2)} \frac{||(D\Psi - Id)w||_{\mathscr{P}^{p, D(s_0, r_0, r_0)}}}{||w||_{\mathscr{P}^{p, D(s_0/2, r_0/2, r_0/2)}}};
\]

2. The frequencies \( \bar{\omega}(\bar{\xi}) \) and \( \tilde{\Omega}(\bar{\xi}) \) satisfy

\[
||\bar{\omega}(\bar{\xi}) - \omega(\bar{\xi})|| + \sup_{j \geq 1} ||\partial_{\bar{q}_j} (\bar{\omega}(\bar{\xi}) - \omega(\bar{\xi}))|| \leq c \eta^8 \epsilon,
\]

and

\[
||\tilde{\Omega}(\bar{\xi}) - \Omega(\bar{\xi})|| + \sup_{j \geq 1} ||\partial_{\bar{q}_j} (\tilde{\Omega}(\bar{\xi}) - \Omega(\bar{\xi}))|| \leq c \eta^8 \epsilon;
\]

3. The Hamiltonian vector field \( X_\tilde{R} \) of the new perturbed Hamiltonian \( \bar{R}(x, y, q, \bar{q}; \bar{\xi}) \) satisfies

\[
|||X_\tilde{R}|||_{p, D(s_0/2, r_0/2, r_0/2) \times \Pi_\eta} \leq \epsilon (1 + c \eta^6 \epsilon),
\]

where \( c > 0 \) is a constant depending on \( s_0, r_0 \) and \( n \).

\[3\text{where } id \text{ denotes the identity map from } \mathscr{P}^{p} \to \mathscr{P}^{p} \text{ and } Id \text{ denotes its tangent map.} \]
Corollary 2.9. (The existence and the stability of KAM tori) Consider the Hamiltonian
\[ \tilde{H}(x,y,q,\tilde{q};\xi) = \tilde{N}(y,q,\tilde{q};\xi) + \tilde{R}(x,y,q,\tilde{q};\xi) \]
obtained in Theorem 2.8. For each \( \xi \in \Pi_\eta \), there is an analytic embedding invariant torus \( \mathcal{T}_0 = \hat{T}^n \times \{ y = 0 \} \times \{ q = 0 \} \times \{ \tilde{q} = 0 \} \) with frequency \( \delta_0(\xi) \) for the Hamiltonian \( \tilde{H}(x,y,q,\tilde{q};\xi) \), and \( \mathcal{T} := \Psi^{-1} \mathcal{T}_0 \) is an analytic embedding invariant torus (i.e. so-called KAM torus) for the original Hamiltonian \( H(x,y,q,\tilde{q};\xi) \).

Moreover, given any small positive \( \delta < r_0/10 \), if \( w(t) \) is a solution of Hamiltonian vector field \( X_H \) with the initial datum \( w(0) = (w_x(0),w_y(0),w_q(0),w_{\tilde{q}}(0)) \) satisfying
\[ d_p(w(0),\mathcal{T}) \leq \delta, \]
then
\[ d_p(w(t),\mathcal{T}) \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-\mathcal{M}}, \]
where the distance \( d_p(w,v) \) between any two points
\[ w = (w_x,w_y,w_q,w_{\tilde{q}}), v = (v_x,v_y,v_q,v_{\tilde{q}}) \in D(s_0/4,4\delta,4\delta) \]
is defined by
\[ d_p(w,v) = 4\delta||w - v||_{\mathcal{P},D(s_0/4,4\delta,4\delta)} \tag{2.28} \]
and
\[ d_p(w,\mathcal{T}) := \inf_{v \in \mathcal{T}} d_p(w,v). \tag{2.29} \]

Theorem 2.10. (The long time stability of KAM tori) Given any \( 0 \leq \mathcal{M} \leq (2c\eta^8\varepsilon)^{-1} \) (the same \( c, \eta \) and \( \varepsilon \) as stated in Theorem 2.8), there exist a small positive \( \delta_0 \) depending on \( s_0, r_0, n \) and \( \mathcal{M} \), and a subset \( \Pi_\eta \subseteq \Pi_\hat{\eta} \) satisfying
\[ \text{Meas } \Pi_\hat{\eta} \geq (\text{Meas } \Pi_\eta)(1 - O(\eta)), \tag{2.30} \]
where \( \hat{\eta} \) is some constant in \((0,1)\). For any \( p \geq 8(\mathcal{M} + 7)^4 + 1 \), \( 0 < \delta < \delta_0 \) and for each \( \xi \in \Pi_\hat{\eta} \), the KAM tori \( \mathcal{T} \) is stable in long time, i.e. if \( w(t) \) is a solution of Hamiltonian vector field \( X_H \) with the initial datum \( w(0) = (w_x(0),w_y(0),w_q(0),w_{\tilde{q}}(0)) \) satisfying
\[ d_p(w(0),\mathcal{T}) \leq \delta, \]
then
\[ d_p(w(t),\mathcal{T}) \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-\mathcal{M}}. \]

Remark 2. Theorem 2.8 is essentially due to Kuksin [20,21]. However, in [20,21], the symplectic map
\[ \Psi : D(s_0/2,0,0) \to D(s_0,r_0,r_0), \]
so the normal form \( H \circ \Psi \) is degenerate. One can extend the definition domain \( D(s_0/2,0,0) \) of \( \Psi \) to \( D(s_0/2,r_0,0) \) (even to the whole space) in view of a remark by Pöschel in [22,25] and an observation that \( \Psi \) is linear in \( y \) and quadratic in \( (q,\tilde{q}) \). In many recent KAM theorems by, for example, Eliasson-Kuksin [18], Grébert-Thomann [19] and Berti-Biasco [6], the extension is done in
this line. In particular, the detail is given out in [19]. Unfortunately, up to now we do not know how to fulfill the tame property of the perturbed vector field $X_\mathbb{R}$ in the extended domain in this line. The tame property of $X_\mathbb{R}$ is key ingredient in our present paper. On the other hand, in the earlier work by Wayne [26] there is another KAM iteration procedure which is a bit different from Kuksin's [20, 21]. In Wayne's procedure, the definition domain of $\Psi$ is just $D(s_0/2, r_0/2, r_0/2)$, not necessary to extend it to a larger domain. In the present paper, we adopt Wayne's iteration procedure directly so that the tame property of $X_\mathbb{R}$ can be verified explicitly. The proof of Theorem 2.8 is well known. The aim of providing the proof in §5 is to verify the tame property of $X_\mathbb{R}$. If the reader acknowledge the fact of the tame property of $X_\mathbb{R}$, the section §5 can be skipped.

Remark 3. Since the parameter set $\Pi \subset \mathbb{R}^N$ is of infinite dimension, the measure in the above theorems should be in the sense of Kolmogorov. Actually it is enough to assume the parameter set is of finite dimension. Write

$$\xi = (\xi^n, \xi^N, \tilde{\xi}^N) \in \mathbb{R}^n \times \mathbb{R}^N \times \ell^2$$

with a large $\mathcal{N}$ will be given in Theorem 2.10. In the proof of constructing normal form of order 2 in Theorem 2.8 it is enough to regard $\tilde{\xi}^n$ as parameters. In order to get partial normal form of order $\mathcal{N} + 2$ around the KAM torus, it suffices to take the $(\xi^n, \xi^N)$ as parameters in Theorems 2.10 and [17]. Therefore, the measure can be understood as Lebesgue measure in this paper.

Remark 4. Instead of equation (1.2), we also can prove the existence and long time stability of KAM tori for general nonlinear Schrödinger equations, such as

$$iu_t = u_{xx} - V(x)u - \frac{\partial g(x, u, \bar{u})}{\partial \bar{u}}, \quad x \in \mathbb{T}, \; t \in \mathbb{R},$$

where $V$ is a smooth and $2\pi$ periodic potential, and $g(x, u_1, u_2)$ is a smooth function on the domain $\mathbb{T} \times \mathcal{U}$, $\mathcal{U}$ being a neighborhood of the origin in $\mathbb{C} \times \mathbb{C}$, $g$ has a zero of order three at $(u_1, u_2) = (0, 0)$ and that $g(x, u, \bar{u}) \in \mathbb{R}$. Equations (2.31) were discussed in [4] and shown that the origin is stable in long time by the infinite dimensional Birkhoff normal form theorem.

3 Some discussions and ideas of the proof

We begin by discussing some basic observations in Bambusi-Grébert [4]. Consider an infinite dimensional Hamiltonian system

$$H(q, \bar{q}) = H_0(q, \bar{q}) + P(q, \bar{q}), \quad q, \bar{q} \in \ell^2_p,$$

with symplectic structure $\sqrt{-1} dq \wedge d\bar{q}$, where

$$H_0(q, \bar{q}) = \sum_{j \geq 1} \Omega_j q_j \bar{q}_j$$

is the quadratic part and

$$P(q, \bar{q}) = \sum_{\beta, \gamma \in \mathbb{N}^N \left| \beta \right| + \left| \gamma \right| \geq 3} p^\beta \gamma q^\beta \bar{q}^\gamma$$

is a smooth function having a zero of order at least three at the origin. Note that the Hamiltonian (3.1) is a normal form of order 2 around the origin. As a dynamical consequence, a solution starting
in the $\delta$-neighborhood of the origin stays in the $\delta$-neighborhood along the time $|t| \leq \delta^{-1}$. To show the origin is stable in a longer time such as $|t| \leq \delta^{-M}$ for any $M \geq 0$, a natural way is to construct a normal form of order $M + 1$ around the origin. To this end, split $P(q, \bar{q})$ into two parts, which is

$$P(q, \bar{q}) = P_1(q, \bar{q}) + P_2(q, \bar{q}),$$

where

$$P_1(q, \bar{q}) = \sum_{\beta, \gamma \in \mathbb{N}^n, |\beta| + |\gamma| \leq M + 1} p^{\beta} q^{\beta} \bar{q}^{\gamma}$$

is the part of low order, and

$$P_2(q, \bar{q}) = \sum_{\beta, \gamma \in \mathbb{N}^n, |\beta| + |\gamma| \geq M + 2} p^{\beta} q^{\beta} \bar{q}^{\gamma}$$

is the part of high order. In order to remove all non-normalized terms

$$\sum_{\beta, \gamma \in \mathbb{N}^n, |\beta| + |\gamma| \leq M + 1, |\beta - \gamma| \neq 0} p^{\beta} q^{\beta} \bar{q}^{\gamma}$$

in $P_1(q, \bar{q})$, the following non-resonant conditions:

$$|\langle \beta - \gamma, \Omega \rangle| \geq C(M) \quad (3.2)$$

are needed, where $\beta, \gamma \in \mathbb{N}^n$ satisfying

$$|\beta| + |\gamma| \leq M + 1, \quad |\beta - \gamma| \neq 0,$$

$\Omega = (\Omega_1, \Omega_2, \ldots)$ and $C(M) > 0$ is a constant depending on $M$. However, the conditions $(3.2)$ are hardly to hold for infinite dimensional Hamiltonian systems, since there are too many inequalities in $(3.2)$. A key idea in $(3.2)$ is that a large part of the nonlinearity is ‘not relevant’ according to tame property and all the remaining nonlinearity can be eliminated using a suitable non-resonant condition. More precisely, split the variable $q = (q_1, q_2, \ldots)$ into two parts with a given large $N$, i.e. let $q = (\bar{q}, \bar{q})$, where $\bar{q} = (q_1, \ldots, q_N)$ is called the low frequency variable and $\bar{q} = (q_{N+1}, q_{N+2}, \ldots)$ is called the high frequency variable. Rewrite $P_1(q, \bar{q})$ as

$$P_1(q, \bar{q}) = P_{11}(q, \bar{q}) + P_{12}(q, \bar{q}),$$

where

$$P_{11}(q, \bar{q}) = \sum_{\hat{\beta}, \hat{\gamma} \in \mathbb{N}^n} p^{\hat{\beta}} q^{\hat{\beta}} \bar{q}^{\hat{\gamma}} \bar{\bar{q}}^{\hat{\bar{\gamma}}}, \quad |\hat{\beta}| + |\hat{\gamma}| + |\hat{\bar{\gamma}}| \leq M + 1, \quad |\hat{\beta}| + |\hat{\bar{\gamma}}| \leq 2,$$

and

$$P_{12}(q, \bar{q}) = \sum_{\hat{\beta}, \hat{\gamma} \in \mathbb{N}^n} p^{\hat{\beta}} q^{\hat{\beta}} \bar{q}^{\hat{\gamma}} \bar{\bar{q}}^{\hat{\bar{\gamma}}}, \quad |\hat{\beta}| + |\hat{\bar{\gamma}}| + |\hat{\bar{\gamma}}| \leq M + 1, \quad |\hat{\beta}| + |\hat{\bar{\gamma}}| \geq 3,$$

with $\beta = (\hat{\beta}, \hat{\beta}), \gamma = (\hat{\gamma}, \hat{\gamma}), \hat{\beta} = (\beta_1, \ldots, \beta_N), \hat{\gamma} = (\gamma_1, \gamma_2, \ldots), \hat{\bar{\gamma}} = (\gamma_{N+1}, \gamma_{N+2}, \ldots)$ and $\hat{\bar{\gamma}} = (\gamma_{N+1}, \gamma_{N+2}, \ldots)$. As stated in $(3.2)$, $P_{12}(q, \bar{q})$ is the ‘non-relevant’ part. Moreover, the non-normalized terms in $P_{11}(q, \bar{q})$ can be removed by the non-resonant conditions

$$|\langle \hat{\beta} - \hat{\beta}, \Omega \rangle + \langle \hat{\gamma} - \hat{\bar{\gamma}}, \Omega \rangle| \geq C(M, N) \quad (3.3)$$
where

\[ |\tilde{\beta}| + |\tilde{\beta}| + |\tilde{\gamma}| + |\gamma| \leq \mathcal{M} + 1, |\tilde{\beta}| + |\gamma| \leq 2, |\tilde{\beta} - \gamma| + |\beta - \gamma| \neq 0, \]

where \( \tilde{\Omega} = (\Omega_1, \ldots, \Omega_N), \hat{\Omega} = (\Omega_{N+1}, \ldots) \) and \( C(\mathcal{M}, \mathcal{N}) \) is a positive constant depending on \( \mathcal{M} \) and \( \mathcal{N} \). Note that there are less inequalities in condition (3.3) than in conditions (3.2), because of \( |\tilde{\beta}| + |\gamma| \leq 2 \). More importantly, the non-resonant conditions (3.3) are satisfied for many infinite dimensional Hamiltonian systems. As a result in [4], it is shown that if the frequency \( \Omega = (\hat{\Omega}, \tilde{\Omega}) \) satisfies the non-resonant conditions (3.3), then there exists a symplectic transform \( \Phi \) such that

\[ H \circ \Phi = H_0 + Z + Q_1 + Q_2, \tag{3.4} \]

where \( Z \) depends on the actions \( I_1 = |q|^2 \), \( Q_1 = O(|q|^{\mathcal{M}+2}) \) and \( Q_2 = O(|\dot{q}|^3) \) for any \( \mathcal{M} \geq 0 \). Based on the partial normal form (3.4) and tame property, the following dynamical result in [4] is obtained: any solution with data in the \( \delta \)-neighborhood of origin still stays in the \( \delta \)-neighborhood of origin for time \( |t| \leq \delta^{-\mathcal{M}+1} \). Furthermore, the method in [4] can be used to construct almost-global existence solutions for many PDEs. For example, see Bambusi-Delort-Grébert-Szeftel [3].

In the present paper, we will prove long time stability of KAM tori for infinite dimensional Hamiltonian systems. Note that the standard KAM techniques constructs a non-degenerate normal form of order 2 (see Corollary 2.9). As a consequence, it is easy to show the existence, linear stability and time \( \delta^{-1} \) stability of KAM tori. In order to obtain a longer time stability such as \( \delta^{-\mathcal{M}} \) for any \( \mathcal{M} \geq 0 \), it is natural to construct a normal form of order \( \mathcal{M} + 1 \) around the KAM tori. To this end, we split \( \tilde{R}(x, y, q, \tilde{q}; \tilde{\xi}) \) (see (2.20)) into two parts, which is

\[ \tilde{R}(x, y, q, \tilde{q}; \tilde{\xi}) = \tilde{R}_1(x, y, q, \tilde{q}; \tilde{\xi}) + \tilde{R}_2(x, y, q, \tilde{q}; \tilde{\xi}), \]

where

\[ \tilde{R}_1(x, y, q, \tilde{q}; \tilde{\xi}) = \sum_{\alpha \in \mathbb{N}_0^\mathcal{M}, \beta \in \mathbb{N}_0^\mathcal{N}, 2|\alpha - |\beta + |\gamma| \leq \mathcal{M} + 1} \tilde{\alpha}^{\alpha/\gamma} q^\beta \tilde{q}^\gamma \]

is the part of low order, and

\[ \tilde{R}_2(x, y, q, \tilde{q}; \tilde{\xi}) = \sum_{\alpha \in \mathbb{N}_0^\mathcal{M}, \beta \in \mathbb{N}_0^\mathcal{N}, 2|\alpha - |\beta + |\gamma| \geq \mathcal{M} + 2} \tilde{\alpha}^{\alpha/\gamma} q^\beta \tilde{q}^\gamma \]

is the part of high order. Expand \( \tilde{R}^{\alpha/\gamma}(x, \tilde{\xi}) \) into Fourier series

\[ \tilde{R}^{\alpha/\gamma}(x, \tilde{\xi}) = \sum_{k \in \mathbb{Z}^n} \hat{\tilde{R}}^{\alpha/\gamma}(k; \tilde{\xi}) e^{\sqrt{-1}(k,x)}, \]

where \( \hat{\tilde{R}}^{\alpha/\gamma}(k; \tilde{\xi}) \) is the \( k \)-th Fourier coefficient of \( \tilde{R}^{\alpha/\gamma}(x, \tilde{\xi}) \). Then we must remove all non-normalized terms in \( \tilde{R}_1(x, y, q, \tilde{q}; \tilde{\xi}) \), which are

\[ \sum_{\alpha \in \mathbb{N}_0^\mathcal{M}, \beta \in \mathbb{N}_0^\mathcal{N}, 2|\alpha - |\beta + |\gamma| \leq \mathcal{M} + 1, |\alpha - |\beta + |\gamma| \neq 0} \sum_{k \in \mathbb{Z}^n} \hat{\tilde{R}}^{\alpha/\gamma}(k; \tilde{\xi}) e^{\sqrt{-1}(k,x)} y^\alpha q^\beta \tilde{q}^\gamma, \]

where the following non-resonant conditions are needed

\[ |\langle k, \omega \rangle + \langle \beta - \gamma, \Omega \rangle| \geq C(k, \mathcal{M}) \tag{3.5} \]
for any $k \in \mathbb{Z}^n$ and $\beta, \gamma \in \mathbb{N}^n$ satisfying

$$|\beta| + |\gamma| \leq \mathcal{M} + 1, \quad |k| + |\beta - \gamma| \neq 0,$$

where $C(k, \mathcal{M})$ is a positive constant depending on $k$ and $\mathcal{M}$. Note that there are more inequalities in conditions (3.5) than in conditions (3.2). Therefore, the non-resonant conditions (3.5) are more hardly to hold.

Following the idea in [4], we would like to construct a partial normal form around the KAM tori instead of a normal form. But we have to face the following problems: (1) are the 'weakened' non-resonant conditions satisfied when constructing a partial normal form of high order in the neighborhood of the KAM tori? (2) how should we define tame property in the case that the tangent direction exists ($n > 0$)? (3) does the tame property preserve under the KAM iteration (infinite number of symplectic transformations) and normal form iteration (finite number of symplectic transformations)? Since the number of the transformations is infinite and the transformations involve the action-angle variable in the KAM iteration, the problem (3) is very hard to solve.

We will solve the above problems as follows. Firstly, following [4] we split the normal variable $q$ into two parts with a given large $\mathcal{N}$, i.e. let $q = (\tilde{q}, \hat{q})$, where $\tilde{q} = (q_1, \ldots, q_{\mathcal{N}})$ is the low frequency normal variable and $\hat{q} = (q_{\mathcal{N} + 1}, q_{\mathcal{N} + 2}, \ldots)$ is the high frequency normal variable. Rewrite $\hat{R}_1(x, y, q, \bar{q}; \xi)$ as

$$\hat{R}_1(x, y, q, \bar{q}; \xi) = \hat{R}_{11}(x, y, q, \bar{q}; \xi) + \hat{R}_{12}(x, y, q, \bar{q}; \xi),$$

where

$$\hat{R}_{11}(x, y, q, \bar{q}; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n} \hat{R}^{\alpha \beta \gamma}(x, \xi) y^\alpha q^\beta \bar{q}^\gamma \hat{q}^{\gamma - \hat{\gamma}}$$

and

$$\hat{R}_{12}(x, y, q, \bar{q}; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n} \hat{R}^{\alpha \beta \gamma}(x, \xi) y^\alpha q^\beta \bar{q}^\gamma \hat{q}^{\gamma - \hat{\gamma}}.$$

Then we can obtain a partial normal form of order $\mathcal{M} + 1$ by removing the non-normalized terms in $\hat{R}_{11}(x, y, q, \bar{q}; \xi)$ which are

$$\sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n} \hat{R}^{\alpha \beta \gamma}(k; \xi) e^{\sqrt{-1} (k, x)} y^\alpha q^\beta \bar{q}^\gamma \hat{q}^{\gamma - \hat{\gamma}}$$

with

$$|k| + |\beta - \gamma| + |\hat{\beta} - \hat{\gamma}| \neq 0$$

under the following non-resonant conditions

$$|\langle k, \omega \rangle + (\beta - \gamma, \hat{\Omega}) + (\hat{\beta} - \hat{\gamma}, \hat{\Omega})| \geq \frac{\hat{\eta}}{C(\mathcal{M}, \mathcal{N}, |k| + 1)^2},$$

(3.6)

where $k \in \mathbb{Z}^n$ and

$$|\beta| + |\gamma| + |\beta| + |\gamma| \leq \mathcal{M} + 1, |\beta| + |\gamma| \leq 2, |k| + |\beta - \gamma| + |\hat{\beta} - \hat{\gamma}| \neq 0.$$
and $C(\mathcal{M}, \mathcal{N}) > 0$ is a constant depending on $\mathcal{M}$ and $\mathcal{N}$ will be given in (6.1). Note the conditions (3.6) are reduced to the non-resonant conditions (3.2) when $n = 0$, and they are similar to the standard non-resonant conditions

$$|\langle k, \omega \rangle + \langle l, \Omega \rangle| \geq \frac{\tilde{\eta}}{|k| + 1}, \quad k \in \mathbb{Z}^n, l \in \mathbb{Z}^n, |l| \leq 2, |k| + |l| \neq 0$$

in KAM technique while without splitting the normal variable $q$. In Section 6.2 we show the non-resonant conditions (3.6) are satisfied by removing the parameters of a small measure.

Secondly, note an important fact that when the dimensional of the tori is 0 ($n = 0$), the tori can be considered as a point, which is just the case discussed in [4]. We define $p$-tame norm ($p$-tame property) of Hamiltonian vector field where

$$\left| W(h, y, z; \xi) \right| = \sum_\beta |W^\beta (x, y; \xi)| \zeta^\beta$$

(See (2.4) in Definition 2.3 for the detail).

**Step 1.** Write $W(x, y, z; \xi) = \sum_\beta W^\beta (x, y; \xi) \zeta^\beta$ and take the norm $|| \cdot ||_{D(\sigma, r) \times \Pi}$ of $W^\beta (x, y; \xi)$:

$$|W|_{D(\sigma, r) \times \Pi} := \sum_\beta |W^\beta (x, y; \xi)||_{D(\sigma, r) \times \Pi} \zeta^\beta.$$

(See (2.4) in Definition 2.3 for the detail).

**Step 2.** Note that the modulus $|W|_{D(\sigma, r) \times \Pi}$ depends on normal variables $z = (q, \bar{q})$ only (independent of tangent variable $(x, y)$ and parameter $\xi$). Then we follow the method in [4] to define the $p$-tame norm of $W_c$ (the normal direction of Hamiltonian vector field $X_W$). More precisely, first consider a Hamiltonian $W(x, y, z; \xi)$, the modulus of which is a homogeneous polynomial about $z$ of degree $h$ (see (2.7)). Then define $p$-tame operator norm

$$||| \cdot |||_{p, D(\sigma, r) \times \Pi}$$

of $W_c$ (see (2.9) in Definition 2.4).

**Step 3.** For a general Hamiltonian

$$W(x, y, z; \xi) = \sum_{h \geq 0} W_h (x, y, z; x, l),$$

where $|W_h|_{D(\sigma, r) \times \Pi}$ is a homogeneous polynomial about $z$ of degree $h$. It is natural to define $p$-tame norm

$$|||W_c|||_{p, D(\sigma, r) \times \Pi} = \sum_{h \geq 1} |||W_h|||_{p, D(\sigma, r) \times \Pi} \ell_1^{h-1}.$$  

(3.7)

However, $p$-tame norm defined by (3.7) is not enough to show the persistence of $p$-tame norm under Poisson bracket. To this end, $|W_h|_{D(\sigma, r) \times \Pi}$ is required as a bounded map from $\ell_1^2 \times \ell_1^2$ into $\ell_1^2 \times \ell_1^2$, which is also used in [4]. Then we can define the $p$-tame norm of $(W_h)_c$ by

$$|||W_h|||_{p, D(\sigma, r) \times \Pi} = \max \left\{ |||W_h|||_{p, D(\sigma, r) \times \Pi}, |||W_h|||_{L^1, D(\sigma, r) \times \Pi} \right\} \ell_1^{h-1}$$

(see (2.10) in Definition 2.4).

**Step 4.** We deal with the tangent direction of Hamiltonian vector field $X_W$. Note that $W_x$ and $W_y$ are finite dimensional, so there is no so-called $p$-tame property. But to guarantee the persistence of $p$-tame property under Poisson bracket, define the operator norm

$$||| \cdot |||_{D(\sigma, r) \times \Pi}$$

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and the norm
\[ \| \cdot \|_{D(s,r,r) \times \Pi} \]
of \( W_x \) and \( W_y \) by (2.11) and (2.12) in Definition 2.5 respectively, where \( \| W_x \|_{D(s,r,r) \times \Pi} \) and \( \| W_y \|_{D(s,r,r) \times \Pi} \) are required as bounded maps from \( \mathbb{C}^2 \times \mathbb{C}^2 \) into \( \mathbb{C}^n \).

**Step 5.** give the definition of \( p \)-tame norm (\( p \)-tame property) of Hamiltonian vector field \( X_W \) for a general Hamiltonian \( W(x,y,z;\xi) \) in Definition 2.7.

Thirdly, we should prove \( p \)-tame property survives under KAM iteration and normal form iteration, which is the key part in this paper (see details in Section 4). The essential difference between this paper and [4] is that before constructing a partial normal form of \( \mathcal{M} + 1 \), we need to use infinite symplectic transformations (KAM iteration) to obtain a normal form of order 2 (see Theorem 2.8), while in [4] a normal form of order 2 is already there (see (3.1)). Thus, we should prove \( p \)-tame property is preserved under infinite symplectic transformations. To this end, we need estimate the \( p \)-tame norm of Poisson bracket of two Hamiltonian (Theorem 4.1), the composition of a Hamiltonian with a Hamiltonian flow (Theorem 4.2) and the solution of homological equation (Theorem 4.3).

Also, we have to face frequency shift in KAM iteration and more complicated small divisors than in [4] because of the existence of tangent direction of Hamiltonian vector field. Here, we point out that we can only obtain time \( |t| \leq \delta^{-\mathcal{M} + 1} \) stability of the KAM tori for \( \mathcal{M} \leq (2^c \eta^8 \varepsilon)^{-1} \), comparing to any \( \mathcal{M} \geq 0 \) in [4] because of the problem of the frequency shift.

Finally, basing on the partial normal form of high order and \( p \)-tame property, we get the dynamical consequence that solutions starting in the \( \delta \)-neighborhood of the KAM torus still remain in the \( \delta \)-neighborhood of the KAM torus for time \( |t| \leq \delta^{-\mathcal{M} + 1} \), i.e. most of KAM tori are long time stable.

At the end of this section, we will give some simple estimate in following remarks.

**Remark 5.** In view of (2.2) in Definition 2.1 it is easy to verify that
\[ \| \cdot \|_{D(s-\sigma,r) \times \Pi} \leq \| \cdot \|_{D(s,r) \times \Pi}, \]
for \( 0 < \sigma < s \). Moreover, in view of (2.3) in Definition 2.2 the following inequalities hold
\[ \| \cdot \|_{D(s-\sigma,r) \times \Pi} \leq \| \cdot \|_{D(s,r) \times \Pi}, \]
and
\[ \| \cdot \|_{D(s,r-\sigma') \times \Pi} \leq \| \cdot \|_{D(s,r) \times \Pi}, \]
where \( 0 < \sigma' < r \). Furthermore, (3.9) and (3.10) implies
\[ \| \cdot \|_{D(s-\sigma,r-\sigma') \times \Pi} \leq \| \cdot \|_{D(s,r) \times \Pi}. \]

**Remark 6.** Given a Hamiltonian \( W(x,y,z;\xi) \) has \( p \)-tame property on the domain \( D(s,r,r) \times \Pi \). It is easy to verify that
\[ \| X_W \|_{p,D(s-\sigma,r,r) \times \Pi} \leq \| X_W \|_{p,D(s,r,r) \times \Pi}, \]
where \( 0 < \sigma < r \). But the following inequality usually is false
\[ \| X_W \|_{p,D(s-\sigma',r-\sigma') \times \Pi} \leq \| X_W \|_{p,D(s,r,r) \times \Pi}. \]
where $0 < \sigma' < r$. However, if let $0 < \sigma' < r/2$, we have the following estimate
\[ |||X_W|||^T_{p,D(x,r)\times \Pi} \leq 4|||X_W|||^T_{p,D(x,r)\times \Pi}, \]  
(3.13)
since $0 < \sigma' < r/2$ implies $r/2 < r - \sigma' < r$.

Remark 7. Based on (2.9) in Definition 2.4 for each $(x, y, z) \in \mathcal{D}^p$ and $\xi \in \Pi$, the following estimate holds
\[ ||(W_h)_z(x, y, z; \xi)||_p \leq ||(W_h)_z||^T_{p,D(x,r)\times \Pi}||z||_p||z||_{\Pi}^{\max\{k-2,0\}}. \]  
(3.14)

Remark 8. Based on (2.11) in Definition 2.5 for each $(x, y, z) \in \mathcal{D}^p$ and $\xi \in \Pi$, the following estimates hold
\[ ||(W_h)_z(x, y, z; \xi)||_p \leq ||(W_h)_z||^T_{D(x,r)\times \Pi}||z||_1^h, \]  
(3.15)
and
\[ ||(W_h)_z(x, y, z; \xi)||_p \leq ||(W_h)_z||^T_{D(x,r)\times \Pi}||z||_1^h. \]  
(3.16)

Remark 9. Note that
\[ ||u||_{D(x,r)\times \Pi} \geq 0. \]
Then in view of Definition 2.4 and Definition 2.5 it is easy to verify that
\[ |||W_z|||^T_{p,D(x,r)\times \Pi} = \sup_{0 \neq z^{(j)} \in c^{(j)}_1, 1 \leq j \leq k, z^{(j)} \geq 0} \frac{|||W_z|||^T_{D(x,r)\times \Pi}(z^{(1)}, \ldots, z^{(h)})||_p}{||z^{(h)}||_1}, \]
and
\[ |||W_u|||^T_{D(x,r)\times \Pi} = \sup_{0 \neq z^{(j)} \in c^{(j)}_1, 1 \leq j \leq k, z^{(j)} \geq 0} \frac{||W_u||^T_{D(x,r)\times \Pi}(z^{(1)}, \ldots, z^{(h)})||_1}{||z^{(h)}||_1}, \]
where $u = x$ or $y$, $z^{(j)} \geq 0$ means $z^{(j)}_i \geq 0$ for $i \in \tilde{Z}$ and $\zeta^{(j)} = (\zeta^{(j)}_i)_{i \in \tilde{Z}}$. Without loss of generality, we always assume that each entry of $z^{(j)}$ is non-negative, when estimating $p$-tame norm of Hamiltonian vector field in the rest of this paper. Also see the same discussion in Remark 4.5 in [4].

4 Properties of the Hamiltonian with $p$-tame property

The following theorem will show that $p$-tame property persists under Poisson bracket.

Theorem 4.1. Suppose that both Hamiltonian functions
\[ U(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^d} U^\beta(x, y; \xi)z^\beta \]
and
\[ V(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^d} V^\beta(x, y; \xi)z^\beta, \]
satisfy $p$-tame property on the domain $D(s, r) \times \Pi$, where
\[ U^\beta(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^d} U^{\alpha\beta}(x, \xi)y^\alpha, \]
and
\[ V^\beta(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^d} V^{\alpha\beta}(x, \xi)y^\alpha, \]
and
\[ V^\beta(x,y;\xi) = \sum_{\alpha \in \mathbb{N}^n} V^{\alpha\beta}(x;\xi)y^\alpha. \]

Then the Poisson bracket \( \{U,V\}(x,y,z;\xi) \) of \( U(x,y,z;\xi) \) and \( V(x,y,z;\xi) \) with respect to the symplectic structure \( dy \wedge dx + \sqrt{-1} \sum_{j \geq 1} dz_j \wedge dz_j \) has \( p \)-tame property on the domain \( D(s - \sigma, r - \sigma', r - \sigma') \times \Pi \) for \( 0 < \sigma < s, 0 < \sigma' < r/2. \) Moreover, the following inequality holds
\[
|||X_{\{U,V\}}|||_{p,D(s-\sigma,r-\sigma',r-\sigma') \times \Pi} \\
\leq C \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} |||X_U|||_{p,D(s,r,r) \times \Pi} |||X_V|||_{p,D(s,r,r) \times \Pi}.
\]
where \( C > 0 \) is an absolute constant.

Proof. By a direct calculation,
\[
X_{\{U,V\}} = ([U,V]_y, -[U,V]_x, \sqrt{-1}[U,V]_q, -\sqrt{-1}[U,V]_q) = DX_U \cdot X_V - DX_V \cdot X_U,
\]
where
\[
DX_U \cdot X_V = \begin{pmatrix}
U_{xx} & U_{xy} & U_{xq} & U_{xq} \\
-U_{xx} & -U_{xy} & -U_{xq} & -U_{xq} \\
\sqrt{-1}U_{qy} & \sqrt{-1}U_{qy} & \sqrt{-1}U_{qy} & \sqrt{-1}U_{qy} \\
-\sqrt{-1}U_{qy} & -\sqrt{-1}U_{qy} & -\sqrt{-1}U_{qy} & -\sqrt{-1}U_{qy}
\end{pmatrix}
\begin{pmatrix}
V_y \\
-V_y \\
\sqrt{-1}V_q \\
-\sqrt{-1}V_q
\end{pmatrix}
\]
and
\[
DX_V \cdot X_U = \begin{pmatrix}
V_{xx} & V_{xy} & V_{xq} & V_{xq} \\
-V_{xx} & -V_{xy} & -V_{xq} & -V_{xq} \\
\sqrt{-1}V_{qy} & \sqrt{-1}V_{qy} & \sqrt{-1}V_{qy} & \sqrt{-1}V_{qy} \\
-\sqrt{-1}V_{qy} & -\sqrt{-1}V_{qy} & -\sqrt{-1}V_{qy} & -\sqrt{-1}V_{qy}
\end{pmatrix}
\begin{pmatrix}
U_y \\
-U_y \\
\sqrt{-1}U_q \\
-\sqrt{-1}U_q
\end{pmatrix}.
\]
Thus, there are 32 terms in \( X_{\{U,V\}} \), and we classify the 32 terms into 4 cases (for simplicity, we will omit the coefficients \( \pm \sqrt{-1} \) or \(-1 \) sometimes, which do not affect the estimate of \( p \)-tame norm below):

Case 1: finite – finite.
\[
\sum_{j=1}^{n} U_{xj} V_{yj}, \quad \sum_{j=1}^{n} U_{yj} V_{xj}, \quad \sum_{j=1}^{n} U_{xj} V_{yj}, \quad \sum_{j=1}^{n} U_{yy} V_{xj},
\]
and
\[
\sum_{j=1}^{n} V_{xx} U_{yj}, \quad \sum_{j=1}^{n} V_{xj} U_{yj}, \quad \sum_{j=1}^{n} V_{xj} U_{yj}, \quad \sum_{j=1}^{n} V_{yy} U_{xj}.
\]

Case 2: finite – infinite.
\[
\sum_{j \geq 1} U_{xj} V_{qj}, \quad \sum_{j \geq 1} U_{xj} V_{qj}, \quad \sum_{j \geq 1} U_{yj} V_{qj}, \quad \sum_{j \geq 1} U_{qj} V_{qj},
\]
and
\[
\sum_{j \geq 1} V_{xj} U_{qj}, \quad \sum_{j \geq 1} V_{xj} U_{qj}, \quad \sum_{j \geq 1} V_{qj} U_{qj}, \quad \sum_{j \geq 1} V_{qj} U_{qj}.
\]
Case 3: infinite – finite.

\[
\sum_{j=1}^{n} U_{qj}V_{sj}, \quad \sum_{j=1}^{n} U_{qsj}V_{sj}, \quad \sum_{j=1}^{n} U_{qsj}V_{sj}, \quad \sum_{j=1}^{n} U_{qsj}V_{sj},
\]

and

\[
\sum_{j=1}^{n} V_{qj}U_{sj}, \quad \sum_{j=1}^{n} V_{qsj}U_{sj}, \quad \sum_{j=1}^{n} V_{qsj}U_{sj}, \quad \sum_{j=1}^{n} V_{qsj}U_{sj};
\]

Case 4: infinite – infinite.

\[
\sum_{j=1}^{n} U_{qj}V_{qj}, \quad \sum_{j=1}^{n} U_{qsj}V_{qj}, \quad \sum_{j=1}^{n} U_{qsj}V_{qj}, \quad \sum_{j=1}^{n} U_{qsj}V_{qj},
\]

and

\[
\sum_{j=1}^{n} V_{qj}U_{qj}, \quad \sum_{j=1}^{n} V_{qsj}U_{qj}, \quad \sum_{j=1}^{n} V_{qsj}U_{qj}, \quad \sum_{j=1}^{n} V_{qsj}U_{qj};
\]

We will drop the index \( \Pi \) for shorten notations and regard \( x \) and \( y \) as scalar for simplicity, when estimating \( p \)-tame norm of Hamiltonian vector field \( X_{(U,V)} \) below.

Suppose \( U(x, y, z; \xi) \) and \( V(x, y, z; \xi) \) are homogeneous polynomials about \( z \), that is

\[
U(x, y, z; \xi) := U_h(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^2, |\beta| = h} U^\beta (x, y; \xi) z^\beta \quad (4.2)
\]

and

\[
V(x, y, z; \xi) := V_l(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^2, |\beta| = l} V^\beta (x, y; \xi) z^\beta, \quad (4.3)
\]

for some \( h, l \in \mathbb{N} \).

**Step 1. Estimate** \( ||\{U, V\}_x||_{D(s-\sigma, r-\sigma', r-\sigma')} \) and \( ||\{U, V\}_y||_{D(s-\sigma, r-\sigma', r-\sigma')} \).

In this step, we will give the following estimates

\[
\frac{1}{(r - \sigma')}^2 ||\{U, V\}_x||_{D(s-\sigma, r-\sigma', r-\sigma')} \leq \frac{32}{e^2} ||X_U||_{p, D(s, r, r)} ||X_V||_{p, D(s, r, r)}, \quad (4.4)
\]

and

\[
||\{U, V\}_y||_{D(s-\sigma, r-\sigma', r-\sigma')} \leq \frac{8r}{e} ||X_U||_{p, D(s, r, r)} ||X_V||_{p, D(s, r, r)}. \quad (4.5)
\]

Note that

\[
\{U, V\}_x = U_{sx}V_x - U_{xy}V_x - V_{sx}U_x + V_{sy}U_x \quad \text{(case 1)}
\]

\[
+ \sqrt{-1} \sum_{j=1}^{n} (U_{aq_j}V_{qj} - U_{aq_j}V_{qj} - V_{aq_j}U_{qj} + V_{aq_j}U_{qj}) \quad \text{(case 2)},
\]

and

\[
\{U, V\}_y = U_{sx}V_y - U_{sy}V_y - V_{sx}U_y + V_{sy}U_y \quad \text{(case 1)}
\]

\[
+ \sqrt{-1} \sum_{j=1}^{n} (U_{aq_j}V_{qj} - U_{aq_j}V_{qj} - V_{aq_j}U_{qj} + V_{aq_j}U_{qj}) \quad \text{(case 2)}.
\]
Without loss of generality, we just consider the term $U_{xy} V_s$, which is in case 1, and the term $\sum_{j \geq 1} (U_{yj} V_{qj} - U_{y\bar{j}} V_{qj})$, which is in case 2, and it is sufficient to give the following estimates

$$
\frac{1}{(r - \sigma)^2} \| U_{xy} V_s \|_{D(s, \sigma - r, r - \sigma')} \leq \frac{4}{e \sigma} \| X_U \| \| X_r \|_{p, D(s, r, r)},
$$

and

$$
\| \sum_{j \geq 1} (U_{yj} V_{qj} - U_{y\bar{j}} V_{qj}) \|_{D(s, \sigma - r, r - \sigma')} \leq \frac{r}{\sigma} \| X_U \| \| X_r \|_{p, D(s, r, r)}
$$

respectively.

Let $m = h + l$ and $\tau_m$ be an $m$-permutation. Since $U(x, y; z; \xi)$ and $V(x, y; z; \xi)$ have $p$-tame property on the domain $D(s, r, r) \times \Pi$, then by (2.11) in Definition 2.5 we get

$$
\| U_{xy} \|_{D(s, r, r)} \leq \frac{1}{e \sigma} \| U_x \|_{D(s, r)}.
$$

Hence, in view of (2.3) in Definition 3, the definition of symmetric linear form (see (2.6) and Remark 5), we have

$$
\| U_{xy} \|_{D(s, r, r)} \| U_x \|_{D(s, r)} \| U_y \|_{D(s, r)} \| V_s \|_{D(s, r)} \| V_r \|_{D(s, r)} \| \tau_m \|_{1, 1},
$$

Based on the inequalities (4.6), (4.7),

$$
\| U_{xy} \|_{D(s, r, r)} \| U_x \|_{D(s, r)} \| U_y \|_{D(s, r)} \| V_s \|_{D(s, r)} \| V_r \|_{D(s, r)} \| \tau_m \|_{1, 1}.
$$

Then we obtain

$$
\| U_{xy} V_s \|_{D(s, r, r, \sigma')} \| \tau_m \|_{1, 1} \leq \| U_{xy} V_s \|_{D(s, r, r, \sigma')} \| \tau_m \|_{1, 1} \leq \| U_{xy} V_s \|_{D(s, r, r, \sigma')} \| \tau_m \|_{1, 1} \leq \| U_{xy} V_s \|_{D(s, r, r, \sigma')} \| \tau_m \|_{1, 1}.
$$

where the last inequality is based on (4.9) and the fact that

$$
\| \tau_m \|_{1, 1} = \| \tau_1 \|_{1, 1} \cdots \| \tau_m \|_{1, 1} = \| \tau_m \|_{1, 1}.
$$
By (2.11) in Definition 2.5 and the inequality (4.11), it is easy to see that
\[
\| U_{x_}\|_{D(s-\sigma, r-\sigma') } \leq \frac{1}{e\sigma} \| U_{x_}\|_{D(s, r) } \| V_\|_{D(s, r) }.
\] (4.13)

Finally, we obtain
\[
\frac{1}{(r-\sigma')^2} \| U_{x_}\|_{D(s-\sigma, r-\sigma') } \leq \frac{1}{e\sigma} \frac{1}{(r-\sigma')^2} \| U_{x_}\|_{D(s, r) } \| V_\|_{D(s, r) } (r-\sigma')^{b+l}
\]
(by (2.12) in Definition 2.5)
\[
\leq \frac{1}{e\sigma} \| U_{x_}\|_{D(s, r) } \| V_\|_{D(s, r) } (r-\sigma')^{b+l}
\]
(based on the inequality (4.13))
\[
\leq \frac{1}{e\sigma} \frac{r^2}{(r-\sigma')^2} \| U_{x_}\|_{D(s, r) } \| V_\|_{D(s, r) } r^{l+h}
\]
\[
\leq \frac{4}{e\sigma} \left( \frac{1}{r} \| U_{x_}\|_{D(s,r) } \right) \| V_\|_{D(s,r) } r^{l+h}
\]
(based on $0 < \sigma' < r/2$ implying $r/2 < r - \sigma' < r$)
\[
\leq \frac{4}{e\sigma} \left( \frac{1}{r} \| U_{x_}\|_{D(s,r) } \right) \| V_\|_{D(s,r) } r^{l+h}
\]
(by (2.12) in Definition 2.5)
\[
\leq \frac{4}{e\sigma} \left( \| X_{\tau} \|_{T_{D(s, r)} } \right) \| X_{\tau} \|_{T_{D(s, r)} } r^{l+h}
\] (4.14)

Denote by
\[
U_{z_\tau} \cdot V_\| = \sqrt{-1} \sum_{j\geq 1} (U_{y_j} V_{q_j} - U_{y_j} V_{q_j}^r).
\]
Let $j = m - 2 = h + l - 2$ (here we assume $j \geq 0$, otherwise $U_{z_\tau} = 0$ or $V_\| = 0$). Then we obtain
\[
\| U_{z_\tau} \|_{D(s-\sigma, r-\sigma') } (z^{(1)} , \ldots, z^{(j)} )
\leq \frac{1}{j!} \sum_{\tau_j} \| U_{z_\tau} \|_{D(s-\sigma, r-\sigma') } (z^{(1)} , \ldots, z^{(h-1)} ) \cdot \| V_\|_{D(s-\sigma, r-\sigma') } (z^{(h)} , \ldots, z^{(j)} )
\]
(following the proof of the inequality (4.10) and $\tau_j$ is a $j$-permutation)
\[
\leq \frac{1}{j!} \sum_{\tau_j} \| U_{z_\tau} \|_{D(s-\sigma, r-\sigma') } (z^{(1)} , \ldots, z^{(h-1)} ) \| V_\|_{D(s-\sigma, r-\sigma') } (z^{(h)} , \ldots, z^{(j)} )
\]
(based on the inequality $|z \cdot \bar{z}| = |\sum_{j\in\mathbb{Z}} z_j \bar{z}_j| \leq \|z\|_0 \|\bar{z}\|_0 \leq \|z\|_1 \|\bar{z}\|_1$)
\[
\leq \frac{1}{j!} \sum_{\tau_j} \left( \| U_{z_\tau} \|_{D(s-\sigma, r-\sigma') } (z^{(1)} , \ldots, z^{(h-1)} ) \right) \| V_\|_{D(s-\sigma, r-\sigma') } (z^{(h)} , \ldots, z^{(j)} )
\]
(based on the generalized Cauchy estimate (8.2) in Lemma 8.4)
\[
\leq \frac{1}{r^\sigma} \| U_{z_\tau} \|_{D(s, r) } \| V_\|_{D(s, r) } (z^{(1)} ) \|_{1,1}
\] (4.15)
where the last inequality is based on the inequality (3.9) in Remark 5 the formula 4.1 for $m = j$ and the inequalities
\[
\| (U_{z_\tau} ) \|_{D(s, r) } (z^{(1)} ) \|_{1,1} \leq \left( \| U_{z_\tau} \|_{D(s, r) } \right) \| z^{(1)} ) \|_{1,1} \|.z^{(h-1)} \|_1,
\]
\[
\| V_\|_{D(s, r) } \| z^{(1)} ) \|_{1,1} \|.z^{(h-1)} \|_1,
\]
and
\[ \| [V_z]_{D(s,r)} (z^{(i)}, \ldots, z^{(j)}) \|_1 \leq \| [V_z]_{1,D(s,r)} \| \| z^{(i)} \|_1 \cdots \| z^{(j)} \|_1, \]

since \( U(x, y, z, \xi) \) and \( V(x, y, z, \xi) \) having \( p \)-tame property on the domain \( D(s, r) \times \Pi \) (see (2.9) in Definition 2.3 for \( p = 1 \)).

According to the estimate (4.15) and in view of (2.11) in Definition 2.5 we obtain
\[ \| [U]_{D(s, r)} \| \| z^{(i)} \|_1 \leq \frac{1}{r^{\sigma}} \| [U]_{1,D(s,r)} \| \| z^{(i)} \|_1, \] (4.16)

Furthermore,
\[
\| [U]_{D(s, r)} \| = \| [U]_{D(s, r)} \| (r - \sigma)^{h + l - 2} \quad \text{(by (2.12) in Definition 2.5)}
\leq \frac{1}{r^{\sigma}} \| [U]_{1,D(s,r)} \| \| z^{(i)} \|_1 \| [V_z]_{1,D(s,r)} \| r^{h + l - 2}
\quad \text{(based on the inequality (4.16), \( r - \sigma' < r \) and the assumption \( j = h + l - 2 \geq 0 \))}
\leq \frac{r}{\sigma} \left( \frac{1}{r} \| [U]_{1,D(s,r)} \| r^{h - 1} \right) \left( \frac{1}{r} \| [V_z]_{1,D(s,r)} \| r^{l - 1} \right)
\leq \frac{r}{\sigma} \| [X]_{D(s, r)} \| \| z^{(i)} \|_1 \| [V_z]_{1,D(s,r)} \| r^{l - 1} \quad \text{(by (2.10) in Definition 2.4)}
\leq \frac{r}{\sigma} \| [X]_{D(s, r)} \| \| z^{(i)} \|_1 \| [V_z]_{1,D(s,r)} \| r^{l - 1} \cdot
\]

Step 2. Estimate \( p \)-tame norm of the terms in case 3, which are
\[
U_{zV} = (U_{q1}V_y, U_{q2}V_y), \quad V_{zU} = (V_{q1}U_y, V_{q2}U_y)
\]
\[
U_{zz}V_y = (U_{q3}V_x, U_{q4}V_x), \quad V_{zz}U_x = (V_{q3}U_x, V_{q4}U_x).
\]

Firstly, we will estimate \( \| U_{zV} \|_{D(s, r)} \| z^{(i)} \|_1 \). Let \( \tilde{m} = h + l - 1 \). Following the proof of (4.11), we obtain
\[ \| U_{zV} \|_{D(s, r)} \| z^{(i)} \|_1 \leq \frac{1}{e^\sigma} \| U_{zV} \|_{D(s, r)} \| V_y \|_{D(s, r)} f(z), \] (4.17)

where
\[ f(z) = \frac{1}{\tilde{m}!} \sum_{j=1}^{h-1} \sum_{\tau_0} \| z_{\tau_0}^{(i)} \|_{1} \cdots \| z_{\tau_0}^{(j-1)} \|_{1} \| z_{\tau_0}^{(j)} \|_{p} |z_{\tau_0}^{(j+1)}|_{1} \cdots \| z_{\tau_0}^{(\tilde{m})} \|_{1}, \]

and using the inequalities
\[
\| U_{zV} \|_{D(s, r)} \| z^{(i)} \|_1 \leq \frac{1}{r^{\sigma}} \| U_{zV} \|_{1,D(s,r)} \| z^{(i)} \|_1 \| z^{(h+1)} \|_1 \]
and
\[
\| V_y \|_{D(s, r)} \| z^{(i)} \|_1 \leq \| V_y \|_{D(s, r)} \| z^{(h+1)} \|_1 \| z^{(\tilde{m})} \|_1, \]

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since $U(x,y,z; \xi)$ and $V(x,y,z; \xi)$ have $p$-tame property on the domain $D(s,r) \times \Pi$. If

$$f(z) = \|z^h\|_{p,1}$$

(4.18)

(which will be prove in Lemma 8.6), then in view of (2.9) in Definition 2.4 and the inequality (4.17), we obtain

$$\|U_{zx}V_y\|_{p,D(s-\sigma, r-\sigma')}^T \leq \frac{1}{e\sigma} \|U_{z}V_y\|_{p,D(s,r)}^T \|V_y\|_{D(s,r)},$$

(4.19)

In particular, when $p = 1$, (4.19) reads

$$\|U_{zx}V_y\|_{1,D(s-\sigma, r-\sigma')}^T \leq \frac{1}{e\sigma} \|U_{z}V_y\|_{1,D(s,r)}^T \|V_y\|_{D(s,r)}.$$

(4.20)

Hence,

$$\frac{1}{r-\sigma'} \|U_{zx}V_y\|_{p,D(s-\sigma, r-\sigma')}^T$$

$$= \frac{1}{r-\sigma'} \max \left\{ \|U_{zx}\|_{p,D(s, r-\sigma')}, \|U_{zx}V_y\|_{1,D(s,r)} \right\} (r-\sigma')^{b+l-1}$$

(by view of (2.10) in Definition 2.4)

$$\leq \frac{1}{e\sigma} \max \left\{ \|U_{zx}\|_{p,D(s,r)}, \|U_{zx}V_y\|_{1,D(s,r)} \right\} (r-\sigma')^{b+l-2}$$

(based on the inequalities (4.19) and (4.20))

$$\leq \frac{1}{e\sigma} \cdot \frac{r}{r-\sigma'} \cdot \frac{1}{r} \max \left\{ \|U_{zx}\|_{p,D(s,r)}, \|U_{zx}V_y\|_{1,D(s,r)} \right\} \|V_y\|_{D(s,r)} r^l$$

(based on $0 < \sigma' < r/2$ implying $r/2 < r - \sigma' < r$)

$$= \frac{2}{e\sigma} \left( \frac{1}{r} \|U_{zx}\|_{p,D(s,r)} \right) \|V_y\|_{D(s,r)} (r-\sigma')$$

(4.21)

Following the proof of the inequality (4.21), we obtain

$$\frac{1}{r-\sigma'} \|V_{xy}U_{z}V_y\|_{p,D(s-\sigma, r-\sigma')}^T \leq \frac{2}{e\sigma} \left( \frac{1}{r} \|V_{xy}\|_{p,D(s,r)} \right) \|U_{z}V_y\|_{D(s,r)},$$

(4.22)

$$\frac{1}{r-\sigma'} \|V_{xy}U_{z}V_y\|_{p,D(s-\sigma, r-\sigma')}^T \leq \frac{2}{\sigma} \|U_{z}V_y\|_{p,D(s-\sigma', r-\sigma')} \left( \frac{1}{r^2} \|V_y\|_{D(s,r)} \right),$$

(4.23)

$$\frac{1}{r-\sigma'} \|V_{xy}U_{z}V_y\|_{p,D(s-\sigma, r-\sigma')}^T \leq \frac{2}{r} \|U_{z}V_y\|_{p,D(s,r)} \left( \frac{1}{r^2} \|V_y\|_{D(s,r)} \right),$$

(4.24)

where, to prove the inequalities (4.23) and (4.24), we use the generalized Cauchy estimate (8.2) instead of (8.1) in Lemma 8.4.

Step 3. Estimate $p$-tame norm of the terms in case 4, which are

$$U_{zx} \cdot V_z = \sqrt{-1} \left( \sum_{j=1}^p (U_{\bar{q}j} V_{\bar{q}j} - U_{\bar{q}j} V_{\bar{q}j}) \right) \right) \sum_{j=1}^p (U_{\bar{q}j} V_{\bar{q}j} - U_{\bar{q}j} V_{\bar{q}j})$$

and

$$V_{zz} \cdot U_z = \sqrt{-1} \left( \sum_{j=1}^p (V_{\bar{q}j} U_{\bar{q}j} - V_{\bar{q}j} U_{\bar{q}j}) \right) \sum_{j=1}^p (V_{\bar{q}j} U_{\bar{q}j} - V_{\bar{q}j} U_{\bar{q}j}).$$
Firstly, note an important fact that \([U]_{D(x,r)}(z)\) and \([V]_{D(x,r)}(z)\) are two Hamiltonian depending only on the normal variable \(z\) (independent of the tangent variables \((x, y)\) and parameter \(\xi\)). Moreover,

\[
\{ [U]_{D(x,r)}(z), [V]_{D(x,r)}(z) \} = [U]_{D(x,r)} \cdot [V]_{D(x,r)} - [V]_{D(x,r)} \cdot [U]_{D(x,r)}.
\]

Following the proof of Lemma 4.12 in [4], we obtain

\[
\|\|\| U_{zz} \|D(x,r)\| [V]_{D(x,r)} \|T_pD(x,r)\| + \|\| V_{zz} \|D(x,r)\| [U]_{D(x,r)} \|T_pD(x,r)\| \\
\leq (h + l - 2) \|\| U_{z} \|D(x,r)\| \|V_{z} \|D(x,r)\| \|T_pD(x,r)\| \|T_pD(x,r)\|.
\]

In particular, when \(p = 1\), the inequality (4.25) reads

\[
\|\| U_{zz} \|D(x,r)\| [V]_{D(x,r)} \|T_1D(x,r)\| + \|\| V_{zz} \|D(x,r)\| [U]_{D(x,r)} \|T_1D(x,r)\| \|T_1D(x,r)\| \\
\leq (h + l - 2) \|\| U_{z} \|D(x,r)\| \|V_{z} \|D(x,r)\| \|T_1D(x,r)\| \|T_1D(x,r)\|.
\]

By Lemma 8.2 we have the following inequalities

\[
\|\| \tilde{U}_{z} \cdot \hat{V}_{z} \|D(x,r)\| (z^{(1)}, \ldots, z^{(h+l-3)}) \|p \leq \|\| U_{zz} \|D(x,r)\| [V]_{D(x,r)} \|T_pD(x,r)\| (z^{(1)}, \ldots, z^{(h+l-3)}) \|p,
\]

and

\[
\|\| \tilde{V}_{z} \cdot \hat{U}_{z} \|D(x,r)\| (z^{(1)}, \ldots, z^{(h+l-3)}) \|p \leq \|\| V_{zz} \|D(x,r)\| [U]_{D(x,r)} \|T_pD(x,r)\| (z^{(1)}, \ldots, z^{(h+l-3)}) \|p.
\]

Then in view of (2.9) in Definition 2.4 and the inequalities (4.25)–(4.28), we obtain

\[
\|\| U_{zz} \|D(x,r)\| + \|\| V_{zz} \|D(x,r)\| \|T_pD(x,r)\| \\
\leq (h + l - 2) \|\| U_{z} \|D(x,r)\| \|V_{z} \|D(x,r)\| \|T_pD(x,r)\| \|T_pD(x,r)\|.
\]

and

\[
\|\| U_{zz} \|D(x,r)\| + \|\| V_{zz} \|D(x,r)\| \|T_1D(x,r)\| \\
\leq (h + l - 2) \|\| U_{z} \|D(x,r)\| \|V_{z} \|D(x,r)\| \|T_1D(x,r)\| \|T_1D(x,r)\|.
\]
respectively. Hence, we have

\[ \frac{1}{r - \sigma'} \left( \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, D(s - \sigma, r - \sigma')}^T + \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, \tilde{D}(s - \sigma, r - \sigma')}^T \right) \]

\[ = \frac{1}{r - \sigma'} \max \left\{ \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, D(s - \sigma, r - \sigma')}^T, \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, \tilde{D}(s - \sigma, r - \sigma')}^T \right\} (r - \sigma')^{h+l-3} \]

\[ + \frac{1}{r - \sigma'} \max \left\{ \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, D(s - \sigma, r - \sigma')}^T, \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, \tilde{D}(s - \sigma, r - \sigma')}^T \right\} (r - \sigma')^{h+l-3} \]

(in view of (2.10) in Definition 2.4)

\[ \leq \max \left\{ \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, D(s, r)}^T, \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, \tilde{D}(s, r)}^T \right\} (r - \sigma')^{h+l-4} \]

\[ + \max \left\{ \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, D(s, r)}^T, \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, \tilde{D}(s, r)}^T \right\} (r - \sigma')^{h+l-4} \]

(in view of (3.11))

\[ \leq 2(h + l - 2) \max \left\{ \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, D(s, r)}^T, \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, \tilde{D}(s, r)}^T, \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, \tilde{D}(s, r)}^T, \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, D(s, r)}^T \right\} (r - \sigma')^{h+l-4} \]

(based on the inequalities (4.29) and (4.30))

\[ \leq \frac{2}{(r - \sigma')^3} \max \left\{ \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, D(s, r)}^T, \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, \tilde{D}(s, r)}^T, \| \mathcal{U}_{\ell}^T \mathcal{V}_{\ell} \|_{p, \tilde{D}(s, r)}^T, \| \mathcal{V}_{\ell}^T \mathcal{U}_{\ell} \|_{p, D(s, r)}^T \right\} r^{h+l-2} \]

(using the inequality \( k(r - \sigma')^{-k-1} \leq \frac{r^k}{\sigma'} \))

\[ \leq \frac{2}{(r - \sigma')^3} \| \mathcal{U}_{\ell} \|_{p, D(s, r)}^T \| \mathcal{V}_{\ell} \|_{p, \tilde{D}(s, r)}^T \]

(in view of (2.10) in Definition 2.4)

\[ \leq \frac{4r}{\sigma'} \left( \frac{1}{r} \| \mathcal{U}_{\ell} \|_{p, D(s, r)}^T \right) \left( \frac{1}{r} \| \mathcal{V}_{\ell} \|_{p, \tilde{D}(s, r)}^T \right), \]

(4.31)

where the last inequality is based on \( 0 < \sigma' < r/2 \) implies \( r/2 < r - \sigma' < r \).

**Step 4. Estimate**

\[ \| \{ \mathcal{U}, \mathcal{V} \} \|_{p, D(s - \sigma, r - \sigma')}^T \] \quad and \quad \| \mathcal{X} \|_{p, D(s - \sigma, r - \sigma')}^T \]

By the inequalities (4.21)-(4.24) and (4.31), we obtain

\[ \frac{1}{r - \sigma'} \| \{ \mathcal{U}, \mathcal{V} \} \|_{p, D(s - \sigma, r - \sigma')}^T \leq 12 \max \left\{ \frac{1}{\sigma}, \frac{1}{\sigma'} \right\} \| \mathcal{X} \|_{p, D(s, r)}^T \| \mathcal{X} \|_{p, \tilde{D}(s, r)}^T. \]

(4.32)

By Definition 2.6 and in view of the inequalities (4.4), (4.5) and (4.32), it is easy to see that

\[ \| \mathcal{X} \|_{p, D(s - \sigma, r - \sigma')}^T \leq C \max \left\{ \frac{1}{\sigma}, \frac{1}{\sigma'} \right\} \| \mathcal{X} \|_{p, D(s, r)}^T \| \mathcal{X} \|_{p, \tilde{D}(s, r)}^T, \]

(4.33)

where \( C > 0 \) is an absolute constant.

Finally, consider general Hamiltonian

\[ \mathcal{U}(x, y, z; \xi) = \sum_{k \geq 0} \mathcal{U}_k(x, y, z; \xi) \]

and

\[ \mathcal{V}(x, y, z; \xi) = \sum_{l \geq 0} \mathcal{V}_l(x, y, z; \xi), \]

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where

\[ U_h(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^2, |\beta| = h} U^\beta(x, y; \xi)z^\beta \]

and

\[ V_l(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^2, |\beta| = l} V^{a\beta}(x, y; \xi)z^\beta. \]

By a direct calculation, we have

\[
\begin{align*}
|||X_{U,V}|||^T_{p,D(s-\sigma,r-\sigma',r-\sigma') \times \Pi} &= |||X_{\sum_{h \geq 0} U_h \sum_{l \geq 0} V_l}|||^T_{p,D(s-\sigma,r-\sigma',r-\sigma') \times \Pi} \\
&= ||| \sum_{h,l \geq 0} X(U_h, V_l) |||^T_{p,D(s-\sigma,r-\sigma',r-\sigma') \times \Pi} \\
&\leq \sum_{h,l \geq 0} C \max \left\{ \frac{1}{\sigma + 1}, \frac{r}{\sigma'} \right\} \left( \sum_{h \geq 0} |||X_{U_h}|||^T_{p,D(s,r,r) \times \Pi} \right) \left( \sum_{l \geq 0} |||X_{V_l}|||^T_{p,D(s,r,r) \times \Pi} \right) \quad \text{(by (4.33))} \\
&= C \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} \left( \sum_{h \geq 0} \left( \sum_{l \geq 0} |||X_{U_h}|||^T_{p,D(s,r,r) \times \Pi} \right) \left( \sum_{l \geq 0} \left( \sum_{l \geq 0} |||X_{V_l}|||^T_{p,D(s,r,r) \times \Pi} \right) \right) \\
&= C \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} \left( \sum_{h \geq 0} \left( \sum_{l \geq 0} |||X_{U_h}|||^T_{p,D(s,r,r) \times \Pi} \right) \right)
\end{align*}
\]

where the last equality is based on Definition 2.6.

\[ \square \]

Remark 10. In view of the estimate (4.7), the coefficient

\[ C \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} \quad (4.34) \]

is necessary in normal form iteration (but not important in KAM iteration). The reason is that while constructing a partial normal form of higher order (see Section 6), \( \sigma \approx s, r \approx \rho \) and \( \sigma' \approx \rho' / M \), then (4.34) reads a constant independent of \( \rho \), but depending on \( n, s \) and \( M \), which make the first step of the normal form iteration work.

The theorem below show that \( p \)-tame property persists under Hamiltonian phase flow.

Theorem 4.2. Consider two Hamiltonian \( U(x, y, z; \xi) \) and \( V(x, y, z; \xi) \) satisfying \( p \)-tame property on the domain \( D(s,r) \times \Pi \) for some \( 0 < s, r \leq 1 \). Given \( 0 < \sigma < s, 0 < \sigma' < r/2 \), suppose

\[ |||X_U|||^T_{p,D(s,r,r) \times \Pi} \leq \frac{1}{2A}, \quad (4.35) \]

where

\[ A = 4Ce \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} \quad (4.36) \]

and \( C > 0 \) is the constant given in (4.7) in Theorem 4.1. Then for each \( |t| \leq 1 \), we have

\[ |||X_{V \circ \phi_t}|||^T_{p,D(s-\sigma,r-\sigma',r-\sigma') \times \Pi} \leq 2|||X_V|||^T_{p,D(s,r,r) \times \Pi}. \]
Proof. Let
\[ W^{(0)}(x,y,z;\xi) = V(x,y,z;\xi), \]
and
\[ W^{(j)}(x,y,z;\xi) = \{W^{(j-1)}, U\}(x,y,z;\xi), \quad j \geq 1, \]
Hence,
\[ W^{(1)}(x,y,z;\xi) = \{W^{(0)}, U\}(x,y,z;\xi) = \{V, U\}(x,y,z;\xi), \]
\[ W^{(2)}(x,y,z;\xi) = \{W^{(1)}, U\}(x,y,z;\xi) = \{\{V, U\}, U\}(x,y,z;\xi), \]
\[ \cdots \]
\[ W^{(j)}(x,y,z;\xi) = \{\ldots \{V, U\}, U\}, \ldots, U\}(x,y,z;\xi) \quad (\text{there are } j \text{ times } U). \]

For \( j \geq 1 \), let \( \sigma_j = \frac{\xi_j}{\sigma} \) and \( \sigma'_j = \frac{\xi'_j}{\sigma} \). Hence, we obtain
\[
\begin{align*}
|||X_W^{(0)}|||_{p,D(s-\sigma,r-\sigma,r-\sigma')}^T & = |||X_W^{(0)}|||_{p,D(s-\sigma,r-\sigma,r-\sigma')}^T \\
& \leq \left( 4C \max \left\{ \frac{1}{\sigma_j}, \frac{r}{\sigma_j'} \right\} \right)^j \left( |||X_V|||_{p,D(s,r,r)}^T \right)^j \left( |||X_U|||_{p,D(s,r,r)}^T \right)^j \\
& \quad \text{(based on Theorem 4.1 and (3.13))} \\
& = j^j \left( 4C \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} \right)^j \left( |||X_V|||_{p,D(s,r,r)}^T \right)^j \left( |||X_U|||_{p,D(s,r,r)}^T \right)^j. \quad (4.37)
\end{align*}
\]

Using the inequality
\[ j^j < j! e^j \]
and in view of the inequality (4.37), we have
\[
\begin{align*}
\frac{1}{j} |||X_W^{(0)}|||_{p,D(s-\sigma,r-\sigma,r-\sigma')}^T & \leq e^j \left( 4C \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} \right)^j \left( |||X_V|||_{p,D(s,r,r)}^T \right)^j \left( |||X_U|||_{p,D(s,r,r)}^T \right)^j \\
& = \left( |||X_V|||_{p,D(s,r,r)}^T \right)^j \left( |||X_U|||_{p,D(s,r,r)}^T \right)^j \left( A|||X_U|||_{p,D(s,r,r)}^T \right)^j \quad \text{(in view of (4.36))} \\
& = 2^{-j} |||X_V|||_{p,D(s,r,r)}^T \quad \text{(based on the inequality (4.35)).} \\
& \quad \text{(4.39)}
\end{align*}
\]

Expand the Hamiltonian \( V \circ X_U^{(j)}(x,y,z;\xi) \) into Taylor series about \( t \) at \( t = 0 \), and we have
\[
V \circ X_U^{(j)}(x,y,z;\xi) = \sum_{j \geq 0} \frac{t^j}{j!} W^{(j)}(x,y,z;\xi). \quad (4.40)
\]
Then
\[
|||X_{\alpha}|||_{p,D(s-\sigma,r)^{\times}\Pi}^{T} = \frac{1}{|||X_{\sum_{j=0}^{\infty} U_{j}}|||_{p,D(s-\sigma,r)^{\times}\Pi}^{T}} \quad \text{(in view of (4.40))}
\]
\[
\leq \sum_{j=0}^{\infty} \left|\frac{1}{j}|||X_{W_{j}}|||_{p,D(s-\sigma,r)^{\times}\Pi}^{T}\right| \quad \text{(in view of } |t| \leq 1\text{)}
\]
\[
\leq \sum_{j=0}^{\infty} 2^{-j}|||X_{r}|||_{p,D(s,r)^{\times}\Pi}^{T} \quad \text{(based on the inequality (4.39))}
\]
\[
= 2|||X_{r}|||_{p,D(s,r)^{\times}\Pi}^{T}.
\]

Based on the theorem below, we will estimate the \(p\)-tame norm of the solution of homological equation.

**Theorem 4.3.** Consider two Hamiltonian

\[
U(x,y,z;\xi) = \sum_{\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{2}} U_{\alpha \beta}(x;\xi) y^{\alpha} z^{\beta}
\]
and

\[
V(x,y,z;\xi) = \sum_{\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{2}} V_{\alpha \beta}(x;\xi) y^{\alpha} z^{\beta}.
\]

Suppose \(V(x,y,z;\xi)\) has \(p\)-tame property on the domain \(D(s,r) \times \Pi\), i.e

\[
|||X_{r}|||_{p,D(s,r)^{\times}\Pi}^{T} < \infty.
\]

For each \(\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{2}, k \in \mathbb{Z}^{n}, j \geq 1\) and some fixed constant \(\tau > 0\), assume the following inequality holds

\[
|\hat{U}_{\alpha \beta}(k;\xi)| + |\partial_{y_{j}} \hat{U}_{\alpha \beta}(k;\xi)| \leq (|k| + 1)^{\tau} (|\hat{V}_{\alpha \beta}(k;\xi)| + |\partial_{y_{j}} \hat{V}_{\alpha \beta}(k;\xi)|), \quad (4.41)
\]
where \(\hat{U}_{\alpha \beta}(k;\xi)\) and \(\hat{V}_{\alpha \beta}(k;\xi)\) are the \(k\)-th Fourier coefficients of \(U_{\alpha \beta}(x;\xi)\) and \(V_{\alpha \beta}(x;\xi)\), respectively. Then, \(U(x,y,z;\xi)\) has \(p\)-tame property on the domain \(D(s-\sigma,r) \times \Pi\) for \(0 < \sigma < s\). Moreover, we have

\[
|||X_{r}|||_{p,D(s-\sigma,r)^{\times}\Pi}^{T} \leq \frac{c}{\sigma^{\tau}} |||X_{r}|||_{p,D(s,r)^{\times}\Pi}^{T}, \quad (4.42)
\]
where \(c > 0\) is a constant depending on \(s\) and \(\tau\).

**Proof.** Without loss of generality, we suppose \(U(x,y,z;\xi) = U_{h}(x,y,z;\xi)\) and \(V(x,y,z;\xi) = V_{h}(x,y,z;\xi)\).

Firstly, we will estimate \(|||U_{h}|||_{D(s-\sigma,r)^{\times}\Pi}\). For \(1 \leq i \leq n\), note

\[
U_{i}(x,y,z;\xi) = \sum_{\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{2}, |\beta| = h} U_{\alpha \beta}(x;\xi) y^{\alpha} z^{\beta}
\]
and

\[
V_{i}(x,y,z;\xi) = \sum_{\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{2}, |\beta| = h} V_{\alpha \beta}(x;\xi) y^{\alpha} z^{\beta}.
\]

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Expand $U_{x_i}^{\alpha \beta}(x; \xi)$ and $V_{x_i}^{\alpha \beta}(x; \xi)$ into Fourier series, which are

$$U_{x_i}^{\alpha \beta}(x; \xi) = \sum_{k \in \mathbb{Z}^n} \widehat{U}_{x_i}^{\alpha \beta}(k, \xi)e^{i\mathbf{T}(k, x)}$$

and

$$V_{x_i}^{\alpha \beta}(x; \xi) = \sum_{k \in \mathbb{Z}^n} \widehat{V}_{x_i}^{\alpha \beta}(k, \xi)e^{i\mathbf{T}(k, x)}.$$

Note that

$$\widehat{U}_{x_i}^{\alpha \beta}(k, \xi) = k \widehat{U}^{\alpha \beta}(k, \xi),$$

and

$$\widehat{V}_{x_i}^{\alpha \beta}(k, \xi) = k \widehat{V}^{\alpha \beta}(k, \xi),$$

where $\widehat{U}^{\alpha \beta}(k, \xi)$ and $\widehat{V}^{\alpha \beta}(k, \xi)$ are $k$-th Fourier coefficients of $U^{\alpha \beta}(x; \xi)$ and $V^{\alpha \beta}(x; \xi)$ respectively. By the inequality (4.41), for $j \geq 1$, we have

$$|\widehat{U}_{x_i}^{\alpha \beta}(k, \xi)| + |\partial_j \widehat{U}_{x_i}^{\alpha \beta}(k, \xi)| \leq (|k| + 1)^\tau(|\widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| + |\partial_j \widehat{V}_{x_i}^{\alpha \beta}(k, \xi)|).$$

By a simple calculation, we obtain

$$\sum_{k \in \mathbb{Z}^n} \left( |\widehat{U}_{x_i}^{\alpha \beta}(k, \xi)| + |\partial_j \widehat{U}_{x_i}^{\alpha \beta}(k, \xi)| \right) e^{ik(s - \sigma)}$$

$$\leq \sum_{k \in \mathbb{Z}^n} (|k| + 1)^\tau \left( |\widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| + |\partial_j \widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| \right) e^{ik(s - \sigma)}$$

(according to the inequality (4.43))

$$= \sum_{k \in \mathbb{Z}^n} \left( |\widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| + |\partial_j \widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| \right) e^{ik(s - \sigma)}$$

$$\leq \left( \sum_{k \in \mathbb{Z}^n} \left( |\widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| + |\partial_j \widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| \right) \right) \left( \sup_{k \in \mathbb{Z}^n} (k| + 1)^\tau e^{-|k|\sigma} \right)$$

$$\leq \frac{c}{\sigma^\tau} \left( \sum_{k \in \mathbb{Z}^n} \left( |\widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| + |\partial_j \widehat{V}_{x_i}^{\alpha \beta}(k, \xi)| \right) \right),$$

where $c > 0$ is a constant depending on $s$ and $\tau$. By the inequality (4.43) and the definition of the norm $\| \cdot \|_{D(s) \times \Pi}$ (see (2.2) in Definition 2.1), we obtain

$$\|U_{x_i}^{\alpha \beta}\|_{D(s - \sigma) \times \Pi} \leq \frac{c}{\sigma^\tau} \|V_{x_i}^{\alpha \beta}\|_{D(s) \times \Pi}.$$ 

Hence, in view of (2.3) in Definition 2.2 we obtain

$$\|U_{x_i}^{\beta}\|_{D(s - \sigma, r) \times \Pi} \leq \frac{c}{\sigma^\tau} \|V_{x_i}^{\beta}\|_{D(s, r) \times \Pi},$$

where

$$U_{x_i}^{\beta}(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} U_{x_i}^{\alpha \beta}(x; \xi)y^\alpha$$

and

$$V_{x_i}^{\beta}(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} V_{x_i}^{\alpha \beta}(x; \xi)y^\alpha.$$
Then, following the proof of (4.14), it is easy to verify that
\[ ||| U_x |||_{D(s, r, r) \times \Pi} \leq \frac{c}{\sigma^r} ||| V_x |||_{D(s, r, r) \times \Pi}. \] (4.47)

Similar to the proof of (4.47), we get
\[ ||| U_y |||_{D(s, r, r) \times \Pi} \leq \frac{c}{\sigma^r} ||| V_y |||_{D(s, r, r) \times \Pi}, \] (4.48)
and
\[ ||| U_z |||_{D(s, r, r) \times \Pi} \leq \frac{c}{\sigma^r} ||| V_z |||_{D(s, r, r) \times \Pi}. \] (4.49)

Finally, using the inequalities (4.47)-(4.49), and in view of Definition 2.6, we obtain the estimate
\[ ||| X_U |||_{T_{p, D(s, r, r) \times \Pi}} \leq \frac{c}{\sigma^r} ||| X_V |||_{T_{p, D(s, r, r) \times \Pi}}. \] (4.51)

As in [25], define the weighted norm of Hamiltonian vector field \( X_U \) on the domain \( D(s, r, r) \times \Pi \) by
\[ ||| X_U |||_{\mathcal{P}, D(s, r, r) \times \Pi} = \sup_{(x, y; \xi) \in D(s, r, r) \times \Pi} ||| X_U |||_{\mathcal{P}, D(s, r, r) \times \Pi}. \] (4.50)

Then the following two theorems give the estimate of the norm of time-1 map.

**Theorem 4.4.** *Give a Hamiltonian*
\[ U(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{2}} U_{\beta}(x, y; \xi) \xi^{\beta} \]
satisfying p-tame property on the domain \( D(s, r, r) \times \Pi \) for some \( 0 < s, r \leq 1 \). Then we have
\[ ||| X_U |||_{\mathcal{P}, D(s, r, r) \times \Pi} \leq ||| X_U |||_{T_{p, D(s, r, r) \times \Pi}}. \] (4.51)

**Proof.** Without loss of generality, assume
\[ U(x, y, z; \xi) := U_{h}(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{2}, |\beta|=h} U_{\beta}(x, y; \xi) \xi^{\beta}. \]
Denote
\[ U_{j}^{\beta-1}(x, y; \xi) = \beta_{j} U_{h}(x, y; \xi), \]
where
\[ \beta - 1_{j} = (\ldots, \beta_{j-1}, \beta_{j} - 1, \beta_{j+1}, \ldots). \]
Based on the definition \( ||| \cdot |||_{D(s, r) \times \Pi} \) (see (2.3) in Definition 2.2), for each \((x, y; \xi) \in D(s, r) \times \Pi\), we have
\[ |U_{j}^{\beta-1}(x, y; \xi)| \leq ||| U_{j}^{\beta-1} |||_{D(s, r) \times \Pi}. \] (4.52)
Then

\[
|U_{\beta}(x, y; z; \xi)| = \left| \sum_{\beta \in \mathbb{N}^2, |\beta| = h} U_{\beta}(x, y; z; \xi) \right|
\]

\[
\leq \left| \sum_{\beta \in \mathbb{N}^2, |\beta| = h} |U_{\beta}(x, y; z; \xi)| \right|
\]

\[
\leq \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)|| \leq ||U_{\beta}(x, y; z; \xi)|| \leq ||U_{\beta}(x, y; z; \xi)||_{D(s, r) \times \Pi} \xi^{\beta - 1}
\]

(based on the inequality (4.52) and each entry of \( \xi^{(j)} \) is non-negative)

\[
= \left[ U_{\beta} \right]_{D(s, r) \times \Pi} (z^{(1)}, \ldots, z^{(h-1)}) |z^{(j)}| = 1 \leq j \leq h - 1.
\]

Moreover,

\[
||U_{\beta}(x, y; z; \xi)||_p \leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right) \leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right) \leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right)
\]

(based on the inequality (4.53))

\[
\leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right) \leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right)
\]

\[
\leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right) \leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right)
\]

(4.54)

For each \((x, y; z; \xi) \in D(s, r) \times \Pi\), we obtain

\[
||U_{\beta}(x, y; z; \xi)|| \leq ||U_{\beta}(x, y; z; \xi)||_{D(s, r) \times \Pi}.
\]

(4.55)

and

\[
||U_{\beta}(x, y; z; \xi)|| \leq ||U_{\beta}(x, y; z; \xi)||_{D(s, r) \times \Pi}.
\]

(4.56)

by following the proof of (4.54). Hence,

\[
||X_U||_{\mathbb{R}^p, D(s, r) \times \Pi} = \sup_{(x, y; z; \xi) \in D(s, r) \times \Pi} \left( ||U_{\beta}|| + \frac{1}{r} ||U_{\beta}|| + \frac{1}{r} ||U_{\beta}|| \right)
\]

(in view of the formula (4.50))

\[
\leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right) \leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right) \leq \left( \sum_{\beta \in \mathbb{N}^2, |\beta| = h} ||U_{\beta}(x, y; z; \xi)||_p \right)
\]

(4.54) - (4.56)

\[
= ||X_U||_{\mathbb{R}^p, D(s, r) \times \Pi}.
\]

Theorem 4.5. Suppose the Hamiltonian

\[
U(x, y; z; \xi) = \sum_{\beta \in \mathbb{N}^2} U^{\beta}(x, y; z; \xi) \xi^\beta
\]

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has \( p \)-tame property on the domain \( D(s,r) \times \Pi \) for some \( 0 < s,r \leq 1 \). Let \( X^I_U \) be the phase flow generalized by the Hamiltonian vector field \( X^I_U \). Given \( 0 < \sigma < s \) and \( 0 < \sigma' < r/2 \), assume

\[
|||X_U|||_{p,D(s,r,r)}^{T} < \min\{\sigma, \sigma'\}.
\]

Then, for each \( \xi \in \Pi \) and each \( |t| \leq 1 \), one has

\[
|||X^I_U - id|||_{p,D(s-\sigma, s-\sigma', r-r')} \leq |||X^I_U|||_{p,D(s,r,r)}^{T}.
\] (4.57)

**Proof.** The inequality (4.57) can be proven directly based on Theorem [24] and following the proof of Lemma A.4 in [25]. \( \Box \)

## 5 Proof of Theorem 2.8 and Corollary 2.9

### 5.1 The \( p \)-tame property of the solution of homological equation

#### 5.1.1 The derivation of homological equation

Recall the perturbation of the integrable Hamiltonian (see (2.14))

\[
H(x,y,q;\bar{q};\bar{\xi}) = N(y,q;\bar{q};\bar{\xi}) + R(x,y,q;\bar{q};\bar{\xi}).
\]

Denote \( R(x,y,q;\bar{q};\bar{\xi}) = R^{low}(x,y,q;\bar{q};\bar{\xi}) + R^{high}(x,y,q;\bar{q};\bar{\xi}) \), where

\[
R^{low}(x,y,q;\bar{q};\bar{\xi}) = \sum_{\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{n}, |\alpha| + |\beta| \leq 2} R^{\alpha \beta \gamma}(x;\bar{\xi}) y^{\alpha} q^{\beta} \bar{q}^{\gamma},
\]

and

\[
R^{high}(x,y,q;\bar{q};\bar{\xi}) = \sum_{\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{n}, |\alpha| + |\beta| \geq 3} R^{\alpha \beta \gamma}(x;\bar{\xi}) y^{\alpha} q^{\beta} \bar{q}^{\gamma}.
\]

The symplectic coordinate change will be produced by the time-1 map \( X^{F}_{s} |_{s=1} \) of the Hamiltonian vector field \( X_{F} \), where \( F(x,y,q;\bar{q};\bar{\xi}) \) is in the form

\[
F(x,y,q;\bar{q};\bar{\xi}) = F^{low}(x,y,q;\bar{q};\bar{\xi}) + \sum_{\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{n}, |\alpha| + |\beta| \leq 2} F^{\alpha \beta \gamma}(x;\bar{\xi}) y^{\alpha} q^{\beta} \bar{q}^{\gamma}.
\]

Using Taylor formula,

\[
H_{+} := H \circ X^{F}_{s} |_{s=1}
\]

\[
= H + \{H,F\} + \int_{0}^{1} (1-t) \{\{H,F\},F\} \circ X^{F}_{r} dt
\]

\[
= N + \{N,F\} + \int_{0}^{1} (1-t) \{\{N,F\},F\} \circ X^{F}_{r} dt
\]

\[
+ R^{low} + \int_{0}^{1} \{R^{low},F\} \circ X^{F}_{r} dt
\]

\[
+ R^{high} + \{R^{high},F\} + \int_{0}^{1} (1-t) \{R^{high},F\} \circ X^{F}_{r} dt.
\]

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Then we obtain the modified homological equation

\[ \{N,F\} + R^{\text{low}} + \{R^{\text{high}}, F\}^{\text{low}} = N_+ - N, \quad (5.1) \]

where \(N_+\) will be given in (5.16) below. If the homological equation (5.1) is solved, then the new perturbation term \(R_+\) can be written as

\[
R_+ = R^{\text{high}} + \{R^{\text{high}}, F\}^{\text{high}} \\
+ \int_0^1 (1-t) \{\{N + R^{\text{high}}, F\}, F\} \circ X_t \, dt \\
+ \int_0^1 \{R^{\text{low}}, F\} \circ X_t \, dt. \tag{5.2}
\]

Note that we do not need to eliminate the terms in (5.2) at next step of KAM procedure, so (5.2) is not necessary to be small. On the other hand, (5.3) is quadratic in \(F\) and (5.4) contains the terms \(R^{\text{low}}\) and \(F\), which guarantee \(R^{\text{low}}\) small enough. Therefore, the domain \(D(s, r, r)\) is not required to shrink too fast such that we can obtain a non-degenerate normal form of order 2 directly.

### 5.1.2 The solvability of homological equation (5.1)

To solve the homological equation (5.1), we should know what is the term \(\{R^{\text{high}}, F\}^{\text{low}}\) exactly. Following Kuksin’s notation in [21], write

\[
R^{\text{low}} = R^x + R^y + R^1 + R^2,
\]

where

\[
R^x = R^x(x; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n} R^{\alpha \beta \gamma}(x; \xi), \tag{5.5}
\]

\[
R^y = \sum_{j=1}^n R^{(i)}(x; \xi)y_j = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, |\alpha|+|\beta|+|\gamma|=0} R^{\alpha \beta \gamma}(x; \xi)y_\alpha, \tag{5.6}
\]

\[
R^1 = \sum_{j \geq 1} (R^{(2)}(x; \xi)q_j + R^{(3)}(x; \xi)q_j),
\]

\[
= \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, |\alpha|=0, |\beta|+|\gamma|=1} R^{\alpha \beta \gamma}(x; \xi)q_\beta \bar{q}^\gamma, \tag{5.7}
\]

\[
R^2 = \sum_{i,j \geq 1} (R^{(4)}(x; \xi)q_i q_j + R^{(5)}(x; \xi)q_i \bar{q}_j + R^{(6)}(x; \xi)q_i \bar{q}_j),
\]

\[
= \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, |\alpha|=0, |\beta|+|\gamma|=2} R^{\alpha \beta \gamma}(x; \xi)q_\beta \bar{q}^\gamma. \tag{5.8}
\]
Moreover, write $F = F_x + F_y + F_1 + F^2$ and $R^{high} = \sum_{j=0}^{4} R^{(j)}$, where

\[
R^{(0)} = \sum_{\alpha \in N^+, \beta, \gamma \in N^+, \alpha = 2, |\beta| + |\gamma| = 0} R^{\alpha \beta \gamma}(x; \xi) q^\alpha, \\
R^{(1)} = \sum_{\alpha \in N^+, \beta, \gamma \in N^+, \alpha = 1, |\beta| + |\gamma| = 1} R^{\alpha \beta \gamma}(x; \xi) q^\beta q^\gamma, \\
R^{(2)} = \sum_{\alpha \in N^+, \beta, \gamma \in N^+, \alpha = 1, |\beta| + |\gamma| = 2} R^{\alpha \beta \gamma}(x; \xi) q^\beta q^\gamma, \\
R^{(3)} = \sum_{\alpha \in N^+, \beta, \gamma \in N^+, \alpha = 0, |\beta| + |\gamma| = 3} R^{\alpha \beta \gamma}(x; \xi) q^\beta q^\gamma, \\
R^{(4)} = \sum_{\alpha \in N^+, \beta, \gamma \in N^+, \alpha = 2, |\beta| + |\gamma| \geq 5 \text{ or } |\beta| + |\gamma| \geq 4} R^{\alpha \beta \gamma}(x; \xi) q^\beta q^\gamma.
\]

By a direct calculation,

\[
\{R^{high}, F\}^{low} = \sum_{j=1}^{n} R^{(0)}_{x_j} F^{x}_{x_j} + \sum_{j=1}^{n} R^{(1)}_{x_j} F^{x}_{x_j} + \sum_{j=1}^{n} R^{(2)}_{x_j} F^{x}_{x_j} + \sum_{j=1}^{n} R^{(3)}_{x_j} F^{x}_{x_j} + \sqrt{-1} \sum_{j=1}^{n} \left( R^{(1)}_{\bar{q} j} F^{1}_{\bar{q} j} - R^{(1)}_{\bar{q} j} F^{1}_{\bar{q} j} + R^{(2)}_{\bar{q} j} F^{1}_{\bar{q} j} - R^{(2)}_{\bar{q} j} F^{1}_{\bar{q} j} \right).
\]

More precisely,

\[
\{R^{high}, F\}^{low} = \{R^{high}, F\}^{y} + \{R^{high}, F\}^{1} + \{R^{high}, F\}^{2},
\]

where

\[
\{R^{high}, F\}^{y} = \sum_{j=1}^{n} R^{(0)}_{x_j} F^{x}_{x_j} + \sqrt{-1} \sum_{j=1}^{n} \left( R^{(1)}_{\bar{q} j} F^{1}_{\bar{q} j} - R^{(1)}_{\bar{q} j} F^{1}_{\bar{q} j} \right), \\
\{R^{high}, F\}^{1} = \sum_{j=1}^{n} R^{(1)}_{x_j} F^{x}_{x_j}, \\
\{R^{high}, F\}^{2} = \sum_{j=1}^{n} R^{(2)}_{x_j} F^{x}_{x_j} + \sqrt{-1} \sum_{j=1}^{n} \left( R^{(3)}_{\bar{q} j} F^{1}_{\bar{q} j} - R^{(3)}_{\bar{q} j} F^{1}_{\bar{q} j} \right).
\]

Let $\partial_\alpha = \omega \cdot \partial_\alpha$ and $W = \{R^{high}, F^{x} + F^{1}\}$. Then the homological equation \([5.1]\) decomposes into

\[
\partial_\alpha F^x + \hat{N}^x = R^x, \\
(\partial_\alpha + \sqrt{-1}\Omega_j) F_q^x = R_q^x + \{R^{high}, F^{x}\}_q^x, \quad j \geq 1, \\
(\partial_\alpha - \sqrt{-1}\Omega_j) F_{\bar{q}}^x = R_{\bar{q}}^x + \{R^{high}, F^{x}\}_{\bar{q}}^x, \quad j \geq 1, \\
\partial_\alpha F^{y} + \hat{N}^y = R^y + W^{y}, \quad 1 \leq j \leq n, \\
(\partial_\alpha + \sqrt{-1}\Omega_i + \sqrt{-1}\Omega_j) F^{q_{i,j}} = R^{q_{i,j}} + W^{q_{i,j}}, \quad i, j \geq 1, \\
(\partial_\alpha + \sqrt{-1}\Omega_i - \sqrt{-1}\Omega_j) F^{q_{i,j}} = R^{q_{i,j}} + W^{q_{i,j}}, \quad i, j \geq 1, \\
(\partial_\alpha - \sqrt{-1}\Omega_i - \sqrt{-1}\Omega_j) F^{\bar{q}_{i,j}} = R^{\bar{q}_{i,j}} + W^{\bar{q}_{i,j}}, \quad i, j \geq 1,
\]

where

\[
\hat{N}^x(\xi) = \hat{R}^x(0; \xi), \quad \hat{N}^y(\xi) = \hat{R}^y(0; \xi) + W^y(0; \xi), \quad \hat{N}^{q_{i,j}}(\xi) = \hat{R}^{q_{i,j}}(0; \xi) + W^{q_{i,j}}(0; \xi).
\]
Hence, we can solve the homological equations (5.9)-(5.15) one by one, i.e. the homological equation (5.1) is solvable. Moreover, set
\[ N_+ = N + \tilde{N}(\xi) + \sum_{j=1}^n N_j(\xi) y_j + \sum_{j \geq 1} N_q(\xi) q_j \bar{q}_j, \]  
(5.16)
\[ \omega_+ = \omega_j + \tilde{N}_j(\xi), \quad 1 \leq j \leq n, \]  
(5.17)
and
\[ \Omega_+ = \Omega_j + \tilde{N}_q(\xi), \quad j \geq 1. \]  
(5.18)

5.1.3 The solution of homological equation (5.1)

Theorem 5.1. Consider a perturbation of the integrable Hamiltonian
\[ H(x, y, q, \bar{q}; \xi) = N(y, q, \bar{q}; \xi) + R(x, y, q, \bar{q}; \xi), \]
where
\[ N(y, q, \bar{q}; \xi) = \sum_{j=1}^n \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) q_j \bar{q}_j, \]
is a parameter dependent integrable Hamiltonian and
\[ R(x, y, q, \bar{q}; \xi) = R^{\text{low}}(x, y, q, \bar{q}; \xi) + R^{\text{high}}(x, y, q, \bar{q}; \xi). \]
Suppose assumptions A and B are fulfilled for \( \omega(\xi) \) and \( \Omega(\xi) \),
\[ |||X_{R^{\text{low}}}|||_{p, D(s, r, r) \times \Pi} \leq \varepsilon, \]  
(5.19)
and
\[ |||X_{R^{\text{high}}}|||_{p, D(s, r, r) \times \Pi} \leq 1, \]  
(5.20)
for some \( 0 < s, r \leq 1 \). For some fixed constant \( \tau > n+1 \), let
\[ \mathcal{R}_{kl} = \left\{ \xi \in \Pi : |\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle | \leq \frac{\eta}{(|k|+1)^{\tau}} \right\}, \quad k \in \mathbb{Z}^n, l \in \mathbb{Z}^n, \]  
(5.21)

and let
\[ \tilde{\Pi} = \Pi \setminus \bigcup_{k \in \mathbb{Z}^n, |l| \leq 2, |k| + |l| \neq 0} \mathcal{R}_{kl}. \]  
(5.22)
Then for each \( \xi \in \tilde{\Pi} \), the homological equation (5.1)
\[ \{N, F\} + R^{\text{low}} + \{R^{\text{high}}, F\}^{\text{low}} = N_+ - N \]
has a solution \( F(x, y, q, \bar{q}; \xi) \) with the estimates
\[ |||X_F|||_{p, D(s-\sigma/2, r-\sigma/2, r-\sigma/2) \times \tilde{\Pi}} \leq \frac{\varepsilon}{\eta^{s+\sigma+3}}, \]  
(5.23)
and
\[ |||X_{N_+ - N}|||_{p, D(s-\sigma/2, r-\sigma/2, r-\sigma/2) \times \tilde{\Pi}} \leq \frac{\varepsilon}{\eta^{s+\sigma+4}}, \]  
(5.24)

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where \( 0 < \sigma < \frac{1}{10} \min \{ s, r \} \) and \( a < b \) means there exists a constant \( c > 0 \) depending on \( n \) and \( \tau \) such that \( a \leq cb \). More precisely, we have

\[
\|X_{F^+}'\|^T_{p,D(s-\tilde{\sigma},r,r) \times \tilde{\Omega}} \lesssim \frac{\varepsilon}{\eta^3 \sigma^{2r+1}},
\]

(5.25) \[
\|X_{F^+}'\|^T_{p,D(s-3\sigma,r-3\sigma,r-3\sigma) \times \tilde{\Omega}} \lesssim \frac{\varepsilon}{\eta^3 \sigma^{4r+3}},
\]

(5.26) \[
\|X_{F^+}'\|^T_{p,D(s-5\sigma,r-5\sigma,r-5\sigma) \times \tilde{\Omega}} \lesssim \frac{\varepsilon}{\eta^6 \sigma^{6r+5}},
\]

(5.27) \[
\|X_{F^+}'\|^T_{p,D(s-\sigma,r-\sigma,r-\sigma) \times \tilde{\Omega}} \lesssim \frac{\varepsilon}{\eta^6 \sigma^{6r+5}},
\]

(5.28)

where \( \tilde{\sigma} = \sigma / 10 \).

Moreover, the new Hamiltonian \( H_+(x,y,q,\bar{q},\bar{\xi}) \) has the following form

\[
H_+(x,y,q,\bar{q},\bar{\xi}) := H \circ X'_F|_{t=1} = N_+(y,q,\bar{q},\bar{\xi}) + R_+(x,y,q,\bar{q},\bar{\xi}),
\]

where

\[
N_+(y,q,\bar{q},\bar{\xi}) = \sum_{j=1}^{n} \omega_+(\xi) y_j + \sum_{j \geq 1} \Omega_+(\xi) q_j \bar{q}_j
\]

and

\[
R_+(x,y,q,\bar{q},\bar{\xi}) = R^{\text{high}} + \{R^{\text{high}}, F\}^{\text{high}} + \int_0^1 (1-t) \{\{N,F\}, F\} \circ X'_F dt
\]

\[
+ \int_0^1 \{R^{\text{low}}, F\} \circ X'_F dt + \int_0^1 (1-t) \{\{R^{\text{high}}, F\}, F\} \circ X'_F dt,
\]

(5.29)

with the following estimates hold:

1. for each \( \xi \in \tilde{\Omega} \), the symplectic map \( \Phi = X'_F|_{t=1} \) satisfies

\[
\|\Phi - id\|_{p,D(s-\sigma/2,r-\sigma/2,r-\sigma/2)} \lesssim \frac{\varepsilon}{\eta^6 \sigma^{6r+6}},
\]

and

\[
\|D\Phi - Id\|_{p,D(s-\sigma,r-\sigma,r-\sigma)} \lesssim \frac{\varepsilon}{\eta^6 \sigma^{6r+6}};
\]

2. the frequencies \( \omega_+(\xi) \) and \( \Omega_+(\xi) \) satisfy

\[
\|\omega_+(\xi) - \omega(\xi)\| + \sup_{j \geq 1} \|\partial_{\xi_j}(\omega_+(\xi) - \omega(\xi))\| \lesssim \frac{\varepsilon}{\eta^4 \sigma^{4r+4}},
\]

and

\[
\|\Omega_+(\xi) - \Omega(\xi)\| + \sup_{j \geq 1} \|\partial_{\xi_j}(\Omega_+(\xi) - \Omega(\xi))\| \lesssim \frac{\varepsilon}{\eta^4 \sigma^{4r+4}};
\]

3. the perturbation \( R_+(x,y,q,\bar{q},\bar{\xi}) \) satisfies

\[
\|X_{R^+}^{\text{low}}\|^T_{p,D(s-\sigma,r-\sigma,r-\sigma) \times \tilde{\Omega}} \lesssim \left( \frac{\varepsilon}{\eta^6 \sigma^{6r+6}} \right)^2.
\]

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\[ |||X^\text{high}_{k+2}|||^T_{p,D(x,s,r,x,s,r,x,s,r)} \times \Omega \leq 1 + \frac{\varepsilon}{\eta^2 \sigma \sigma + 6} + \left( \frac{\varepsilon}{\eta^2 \sigma \sigma + 6} \right)^2; \]

(4) the measure of the subset \( \Pi \) of \( \Omega \) satisfies
\[ \text{Meas} \, \tilde{\Pi} \geq (\text{Meas} \, \Pi) (1 - O(\eta)). \] (5.30)

**Proof.** First of all, we will give two simple estimates.

**No. 1.** In view of (5.21) and (5.22), for each \( \xi \in \tilde{\Pi} \) and each \( k \in \mathbb{Z}^n, l \in \mathbb{Z}^n \) satisfying \( |l| \leq 2, |k| + |l| \neq 0 \), we have
\[ ||\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle || \geq \frac{\eta}{(|k| + 1)^2}. \]

Hence,
\[ ||\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle ||^{-1} \leq \frac{(|k| + 1)^2}{\eta}. \] (5.31)

Moreover, for each \( j \geq 1 \),
\[ \partial_{\xi_j} \left( \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \right)^{-1} = \frac{\langle k, \partial_{\xi_j} \omega(\xi) \rangle + \langle l, \partial_{\xi_j} \Omega(\xi) \rangle}{\left( \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \right)^2}. \]

Then in view of twist conditions (2.17), the inequality (5.31) and \( |l| \leq 2 \), we obtain
\[ ||\partial_{\xi_j} \left( \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \right)^{-1} || \leq \frac{2(|k| + 1)^2}{\eta^2}. \] (5.32)

**No. 2.** Recall the notations (5.5)–(5.8), and let \( U^0 = U^x + U^y \). Then in view of Definition 2.7, we have
\[ |||X_U|||^T_{p,D(x,s,r,x,s,r,x,s,r)} \times \Omega \leq |||X_U|||^T_{p,D(x,s,r,x,s,r)} \times \Omega \leq |||X_U|||^T_{p,D(x,s,r)} \times \Omega \quad |i| \leq 2, \] (5.33)

and
\[ |||X_U^\text{high}|||^T_{p,D(x,s,r,x,s,r,x,s,r)} \times \Omega \leq |||X_U|||^T_{p,D(x,s,r)} \times \Omega. \] (5.34)

Moreover,
\[ |||X_U^x|||^T_{p,D(x,s,r)} \times \Omega \leq |||X_U^0|||^T_{p,D(x,s,r)} \times \Omega \] (5.35)

and
\[ |||X_U^y|||^T_{p,D(x,s,r)} \times \Omega \leq |||X_U^0|||^T_{p,D(x,s,r)} \times \Omega. \] (5.36)

The inequalities (5.35) and (5.36) can be proven by following the proof of Theorem 4.3. Now we will prove the theorem by the following steps.
Step 1. Estimate $|||X_F|||_{p,D(s-\sigma/2,r-\sigma/2)}^T$ and $|||X_{N_c}|||_{p,D(s-\sigma/2,r-\sigma/2)}^T$ Based on (3.10) and in view of (5.25)-(5.28) hold. Since the proofs of (5.25)-(5.28) are similar, we only show (5.28) is right below and assume (5.25)-(5.27) is proven. Without loss of generality, consider the homological equations (5.14), which is

$$(\partial_\omega + \sqrt{-1}\Omega_i - \sqrt{-1}\Omega_j)F^{q\partial_j} + \delta_j N_{\overline{q}\partial_j} = R^{q\partial_j} + W^{q\partial_j}, \quad i, j \geq 1.$$ 

By passing to Fourier coefficients,

$$\hat{F}^{q\partial_j}(k, \xi) = \frac{\hat{R}^{q\partial_j}(k, \xi) + \hat{W}^{q\partial_j}(k, \xi)}{\sqrt{1(\xi, \omega(\xi) + \Omega_i(\xi) - \Omega_j(\xi))}}, \quad |k| + |i - j| \neq 0,$$

and

$$\hat{N}^{q\partial_j}(\xi) = \hat{R}^{q\partial_j}(0; \xi) + \hat{W}^{q\partial_j}(0; \xi). \quad (5.37)$$

Then in view of the inequalities (5.31) and (5.32), for each $\xi \in \Pi$ and $j' \geq 1$, we obtain

$$|F^{q\partial_j}(k; \xi)| + |\partial_{\xi_j} F^{q\partial_j}(k; \xi)|$$

$$\leq \frac{3(|k| + 1)^{2r+1}}{\eta^2} \left( |\hat{R}^{q\partial_j}(k; \xi) + \hat{W}^{q\partial_j}(k; \xi)| + |\partial_{\xi_j} \hat{R}^{q\partial_j}(k; \xi) + \partial_{\xi_j} \hat{W}^{q\partial_j}(k; \xi)| \right).$$

Similarly, we have

$$|F^{q\partial_j}(k; \xi)| + |\partial_{\xi_j} F^{q\partial_j}(k; \xi)|$$

$$\leq \frac{3(|k| + 1)^{2r+1}}{\eta^2} \left( |\hat{R}^{q\partial_j}(k; \xi) + \hat{W}^{q\partial_j}(k; \xi)| + |\partial_{\xi_j} \hat{R}^{q\partial_j}(k; \xi) + \partial_{\xi_j} \hat{W}^{q\partial_j}(k; \xi)| \right),$$

while considering homological equations (5.13) and (5.15). Hence, by the above three inequalities and Theorem 4.3, we get

$$|||X_{F^2}|||_p, D(s-5\delta, r-5\delta) \times \Omega$$

$$\leq \frac{1}{\eta^2 \sigma^2 r+1} \left( |||X_{F^2}|||_p, D(s-4\delta, r-5\delta) \times \Omega + |||X_{W^2}|||_p, D(s-4\delta, r-5\delta) \times \Omega \right)$$

$$\leq \frac{1}{\eta^2 \sigma^2 r+1} \left( |||X_{F^2}|||_p, D(s-\delta, r) \times \Omega + |||X_{W^2}|||_p, D(s-\delta, r-4\delta) \times \Omega \right)$$

(in view of the inequalities (3.13) and (5.35))

$$\leq \frac{\varepsilon}{\eta^6 \sigma^6 r+1}, \quad (5.38)$$

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where the last inequality is based on (5.19) and the following estimate

\[ |||X_{W}|||^T_{p,D(s-\sigma/2,r-\sigma/2,\epsilon)\times T} \leq \frac{\epsilon}{\eta^1 \sigma^{t+4}} \] (5.39)

which is based on \( W = \{ R_{high, F}, F^1 \} \), and in view of the inequalities (5.20), (5.23), (5.26) and Theorem 4.1.

The estimate of \( |||X_{N_{k}} - \nu|||^T_{p,D(s-\sigma/2,r-\sigma/2,\epsilon)\times T} \) is trivial, so we omit the details. **Step 2.**

**Estimate** \( |||\Phi - \text{id}|||_{p,D(s-\sigma/2,r-\sigma/2,\epsilon)\times T} \) and \( |||D\Phi - \text{id}|||_{p,D(s-\sigma,r-\sigma,\epsilon)\times T} \).

Note \( \Phi = X'_{\phi} \). Based on Theorem 4.5 and the inequality (5.23), we have

\[ |||\Phi - \text{id}|||_{p,D(s-\sigma/2,r-\sigma/2,\epsilon)\times T} \leq \frac{\epsilon}{\eta^1 \sigma^{t+5}}. \]

Moreover, by Cauchy inequality in Lemma 8.7, we obtain

\[ |||D\Phi - \text{id}|||_{p,D(s-\sigma,r-\sigma,\epsilon)\times T} \leq \frac{\epsilon}{\eta^0 \sigma^{t+6}}. \]

**Step 3.** Estimate the norm of the frequency shift

\[ |||\omega^*(\xi) - \omega(\xi)|| + \sup_{j \geq 1}|||\partial_{\xi_j}(\omega^*(\xi) - \omega(\xi))|||, \]

and

\[ |||\Omega^*(\xi) - \Omega(\xi)|| + \sup_{j \geq 1}|||\partial_{\xi_j}(\Omega^*(\xi) - \Omega(\xi))||| \]

Based on Theorem 4.4 and the estimate (5.24), it is easy to verify

\[ |||\omega^*(\xi) - \omega(\xi)|| + \sup_{j \geq 1}|||\partial_{\xi_j}(\omega^*(\xi) - \omega(\xi))||| \leq \frac{\epsilon}{\eta^1 \sigma^{t+4}}, \]

and

\[ |||\Omega^*(\xi) - \Omega(\xi)|| + \sup_{j \geq 1}|||\partial_{\xi_j}(\Omega^*(\xi) - \Omega(\xi))||| \leq \frac{\epsilon}{\eta^1 \sigma^{t+4}}. \]

**Step 4.** Estimate \( |||X_{N_{k, \text{low} , \text{high}}^*}|||_{p,D(s-\sigma, r-\sigma, \epsilon)\times T} \) and \( |||X_{N_{k, \text{high}}^*}|||_{p,D(s-\sigma, r-\sigma, \epsilon)\times T} \).

Firstly,

\[ \max \left\{ |||X_{N_{k, \text{low}}, \text{high}}^*|||^T_{p,D(s-\sigma, r-\sigma, \epsilon)\times T} \right\} \leq |||X_{N_{k, \text{high}}^*}|||^T_{p,D(s-\sigma/2, r-\sigma/2, \epsilon)\times T} \]

\[ \leq \frac{1}{\sigma} \min \{ s, r \} \]

\[ \leq \frac{1}{\sigma} |||X_{N_{k, \text{high}}}|||^T_{p,D(s-\sigma/2, r-\sigma/2, \epsilon)\times T} \]

\[ \leq \frac{\epsilon}{\eta^0 \sigma^{t+6}} \]

(in view of the inequalities (5.20) and (5.23)).

Secondly, estimate

\[ |||X_{N_{k, (1-t)}(\text{F}^1)}|||^T_{p,D(s-\sigma, r-\sigma, \epsilon)\times T} \]
Note
\[ \{N, F\} = N_+ - N - R^{\text{low}} + \{R^{\text{high}}, F\}^{\text{low}}. \]

Then in view of the inequalities (5.19), (5.24) and (5.40), we obtain
\[ \|X_{\{N,F\}}\|_p^{T} \leq \frac{\varepsilon}{\eta^6 \sigma^6 + 6}. \]

In view of the inequalities (5.23), (5.41) and Theorem 4.1, we have
\[ \|X_{\{N,F\}}\|_p^{T} \leq \frac{\varepsilon}{\eta^6 \sigma^6 + 6}. \]

By Theorem 4.2, for \(|t| \leq 1\), we get
\[ \|X_{\{N,F\}}\|_p^{T} \leq \frac{\varepsilon}{\eta^6 \sigma^6 + 6}. \]

Therefore, by (5.42) and (5.43),
\[ \|X_{\{N,F\}}\|_p^{T} \leq \frac{\varepsilon}{\eta^6 \sigma^6 + 6}. \]

In view of the inequalities (5.19), (5.20) and (5.23) and following the proof of the inequality (5.44), we obtain
\[ \|X_{\{N,F\}}\|_p^{T} \leq \frac{\varepsilon}{\eta^6 \sigma^6 + 6}. \]

In view of the formula (5.29), then
\[ R^{\text{low}}_+ = \left( \int_0^1 (1-t)\{N, F\} \circ X_t^i dt \right)^{\text{low}} + \left( \int_0^1 R^\text{low} \circ X_t^i dt \right)^{\text{low}} + \left( \int_0^1 (1-t)\{R^{\text{high}}, F\} \circ X_t^i dt \right)^{\text{low}}, \]
and
\[ R^{\text{high}}_+ = R^{\text{high}} + \{R^{\text{high}}, F\}^{\text{high}} + \left( \int_0^1 (1-t)\{N, F\} \circ X_t^i dt \right)^{\text{high}} + \left( \int_0^1 R^\text{low} \circ X_t^i dt \right)^{\text{high}} + \left( \int_0^1 (1-t)\{R^{\text{high}}, F\} \circ X_t^i dt \right)^{\text{high}}, \]

Hence, in view of the inequalities (5.20), (5.40), (5.44), (5.45) and (5.46), we obtain
\[ \|X_{R^{\text{low}}_+}\|_p^{T} \leq \frac{\varepsilon}{\eta^6 \sigma^6 + 6}. \]
Lemma 5.2. Suppose the assumptions A and B are fulfilled for the following iteration constants:

1. \( \eta_0 = \eta \) is given, \( \eta_m = \eta 2^{-m}, m = 1, 2, \ldots \);
2. \( \varepsilon_0 = \varepsilon = \eta^{12} \varepsilon, \varepsilon_m = \eta^{12} \varepsilon (4/3)^m, m = 1, 2, \ldots \);
3. \( \tau_0 = 0, \tau_m = (1^{-2} + \cdots + m^{-2})/2 \sum_{j=1}^{m} j^{-2}, m = 1, 2, \ldots \) (thus, \( \tau_m < 1/2 \));
4. Given \( 0 < s_0, r_0 \leq 1 \). Let \( 0 < \sigma = \min \{ s_0, r_0 \} \) and \( s_m = (1 - \tau_m) \sigma, r_m = (1 - \tau_m) \sigma, m = 1, 2, \ldots \) (thus, \( s_m > s_0/2, r_m > r_0/2 \)).

The proof of the inequality (5.30) (the estimate of the measure of \( \hat{\Pi} \)) is omitted, since the inequality (5.30) can be proven by following the proof of the measure of \( \Pi_{\hat{\eta}} \) in Section 6.2.

5.2 Iterative lemma

5.2.1 Iterative constants

As usual, the KAM theorem is proven by the Newton-type iteration procedure which involves an infinite sequence of coordinate changes. In order to make our iteration procedure run, we need the following iteration constants:

- 1. \( \eta_0 = \eta \) is given, \( \eta_m = \eta 2^{-m}, m = 1, 2, \ldots \);
- 2. \( \varepsilon_0 = \varepsilon = \eta^{12} \varepsilon, \varepsilon_m = \eta^{12} \varepsilon (4/3)^m, m = 1, 2, \ldots \);
- 3. \( \tau_0 = 0, \tau_m = (1^{-2} + \cdots + m^{-2})/2 \sum_{j=1}^{m} j^{-2}, m = 1, 2, \ldots \) (thus, \( \tau_m < 1/2 \));
- 4. Given \( 0 < s_0, r_0 \leq 1 \). Let \( 0 < \sigma = \min \{ s_0, r_0 \} \) and \( s_m = (1 - \tau_m) \sigma, r_m = (1 - \tau_m) \sigma, m = 1, 2, \ldots \) (thus, \( s_m > s_0/2, r_m > r_0/2 \)).

5.2.2 Iterative lemma

**Lemma 5.2.** Consider a perturbation of the integrable Hamiltonian

\[
H_m(x, y, q, \bar{q}; \xi) = N_m(y, q, \bar{q}; \xi) + R_m(x, y, q, \bar{q}; \xi),
\]

where

\[
N_m(y, q, \bar{q}; \xi) = \sum_{j=1}^{n} \omega m_j(\xi) y_j + \sum_{j \geq 1} \Omega m_j(\xi) q_j \bar{q}_j
\]

is a parameter dependent integrable Hamiltonian and

\[
R_m(x, y, q, \bar{q}; \xi) = R_m^{low}(x, y, q, \bar{q}; \xi) + R_m^{high}(x, y, q, \bar{q}; \xi)
\]

is the perturbation with the following form

\[
R_m^{low}(x, y, q, \bar{q}; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, 2|\alpha| + |\beta| + |\gamma| \leq 2} R_m^{\alpha\beta\gamma}(x, \xi) q^\alpha \bar{q}^\beta \gamma
\]

and

\[
R_m^{high}(x, y, q, \bar{q}; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, 2|\alpha| + |\beta| + |\gamma| \geq 3} R_m^{\alpha\beta\gamma}(x, \xi) q^\alpha \bar{q}^\beta \gamma.
\]

Suppose the assumptions A and B are fulfilled for \( \omega m(\xi) \) and \( \Omega m(\xi) \) with \( m = 0 \) and

\[
|\omega_m(\xi) - \omega_0(\xi)| + \sup_{j \geq 1} \left| \partial_{\xi_j} (\omega_m(\xi) - \omega_0(\xi)) \right| \leq \frac{\varepsilon_i^{2/3}}{\alpha_{i-1}},
\]

where
and
\[ ||\Omega_m(\xi) - \Omega_0(\xi)|| + \sup_{j \geq 1} ||\partial_{\xi_j}(\Omega_m(\xi) - \Omega_0(\xi))|| \leq \sum_{i=1}^{m} \varepsilon_i^{2/3}. \]

Suppose \( R_m^{\text{low}}(x, y, q, \bar{q}; \xi) \) satisfies the smallness assumption
\[ \|\| X_{R_m^{\text{low}}} \|\|_{p,D(x_m, r_m, r_m)} \times \Pi_m \leq \varepsilon_m, \]
and \( R_m^{\text{high}}(x, y, q, \bar{q}; \xi) \) satisfies
\[ \|\| X_{R_m^{\text{high}}} \|\|_{p,D(x_m, r_m, r_m)} \times \Pi_m \leq \varepsilon + \sum_{i=1}^{m} \varepsilon_i^{2/3}. \]

Let
\[ \mathcal{R}^m_{kl} = \left\{ \xi \in \Pi : |\langle k, \omega_m(\xi) \rangle + \langle l, \Omega_m(\xi) \rangle| \leq \frac{\eta_m}{(|k|+1)^{3/2}}, \quad k \in \mathbb{Z}^n, l \in \mathbb{Z}^n, \right\} \]
and let
\[ \Pi_{m+1} = \Pi_m \setminus \bigcup_{k \in \mathbb{Z}^n, |l| \leq 2, |k|+|l| \neq 0} \mathcal{R}^m_{kl}. \]

Then for each \( \xi \in \Pi_{m+1} \), the homological equation
\[ \{N_m, F_m\} + R_m^{\text{low}} + \{R_m^{\text{high}}, F_m\}^{\text{low}} = N_{m+1} - N_m \]
has a solution \( F_m(x, y, q, \bar{q}; \xi) \) with the estimates
\[ \|\| X_{F_m} \|\|_{p,D(x_m+1, r_m+1, r_m+1)} \times \Pi_{m+1} \leq \varepsilon_m^{2/3}, \]
and
\[ \|\| X_{N_{m+1} - N_m} \|\|_{p,D(x_m+1, r_m+1, r_m+1)} \times \Pi_{m+1} \leq \varepsilon_m^{2/3}. \]

Moreover,
\[ H_{m+1}(x, y, q, \bar{q}; \xi) := H \circ X_{F_m} |_{t=1} = N_{m+1}(y, q, \bar{q}; \xi) + R_{m+1}(x, y, q, \bar{q}; \xi), \]
where
\[ N_{m+1}(y, q, \bar{q}; \xi) = N_{m+1}(y, q, \bar{q}; \xi) = \sum_{j=1}^{n} \omega_{m+1}(j) \langle \xi \rangle y_j + \sum_{j \geq 1} \Omega_{m+1}(j) \langle \xi \rangle q_j \bar{q}_j, \]
and
\[ R_{m+1}(x, y, q, \bar{q}; \xi) = R_m^{\text{high}} + \{R_m^{\text{high}}, F_m\}^{\text{high}} + \int_0^1 (1-t) \{N_m, F_m\} \circ X_{F_m}^t dt \]
\[ + \int_0^1 \{R_m^{\text{low}}, F_m\} \circ X_{F_m}^t dt + \int_0^1 (1-t) \{R_m^{\text{high}}, F_m\} \circ X_{F_m}^t dt, \]
with the following estimates hold:
(1) for each \( \xi \in \Pi_{m+1} \), the symplectic map \( \Psi_m = X_{F_m}^t |_{t=1} \) satisfies
\[ \|\| \Psi_m - id \|\|_{p,D(x_{m+1}, r_{m+1}, r_{m+1})} \leq \varepsilon_m^{2/3}, \]

\[ \text{41} \]
and

\[ |||D \Psi_m - Id|||_{p,D(s_{m+1},r_{m+1},r_m)} \leq \varepsilon_m^{2/3}.\]

(2) the frequencies \( \omega_{m+1}(\xi) \) and \( \Omega_{m+1}(\xi) \) satisfy

\[ ||\omega_{m+1}(\xi) - \omega_0(\xi)|| + \sup_{j \geq 1} ||\partial_{\xi_j}(\omega_{m+1}(\xi) - \omega_0(\xi))|| \leq \sum_{j=1}^{m+1} \varepsilon_j^{2/3},\]

and

\[ ||\Omega_{m+1}(\xi) - \Omega_0(\xi)|| + \sup_{j \geq 1} ||\partial_{\xi_j}(\Omega_{m+1}(\xi) - \Omega_0(\xi))|| \leq \sum_{j=1}^{m+1} \varepsilon_j^{2/3}.\]

(3) the perturbation \( R_{m+1}(x,y,q,q;\xi) \) satisfies

\[ |||X^{R_{m+1}}\|\|_{p,D(s_{m+1},r_{m+1},r_m)} \leq \varepsilon_{m+1},\]

and

\[ |||X^{\Psi_{m+1}}\|\|_{p,D(s_{m+1},r_{m+1},r_m)} \leq \varepsilon + \sum_{j=1}^{m+1} \varepsilon_j^{2/3}.\]

(4) the measure of the subset \( \Pi_{m+1} \) of \( \Pi_m \) satisfies

\[ \text{Meas} \Pi_{m+1} \geq (\text{Meas} \Pi_m)(1 - O(\eta_m)). \] (5.47)

**Proof.** The proof is a standard KAM proof based on Theorem 4.1, Theorem 4.2, and Theorem 4.3. See details of KAM iteration in [25]. \( \square \)

### 5.3 Proof of Theorem 2.8

**Proof.** Let \( \Pi_\eta = \bigcap_{m=0}^\infty \Pi_m \cap D(s_0/2,r_0/2) \subset \bigcap_{m=0}^\infty D(s_m,r_m) \) and \( \Psi = \prod_{m=0}^\infty \Psi_m. \) By the standard argument, we conclude that \( \Psi, D\Psi, H_m, X_m \) converge uniformly on the domain \( D(s_0/2,r_0/2) \times \Pi_\eta. \) Let

\[ \hat{R}(x,y,q,q;\xi) := \lim_{m \to \infty} H_m = \hat{N}(y,q,q;\xi) + \hat{R}(x,y,q,q;\xi),\]

where

\[ \hat{N}(y,q,q;\xi) = \sum_{j=1}^n \hat{\omega}_j(\xi) y_j + \sum_{j \geq 1} \hat{\omega}_j(\xi) q_j q_j \]

and

\[ \hat{R}(x,y,q,q;\xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^4, |\alpha| + |\beta| + |\gamma| \geq 3} \hat{R}^{\alpha\beta\gamma}(x,\xi)y^\alpha q^\beta \bar{q}^\gamma.\]

Moreover, by the standard KAM proof, we obtain the following estimates:

(1) for each \( \xi \in \Pi_\eta, \) the symplectic map \( \Psi \) satisfies

\[ ||\Psi - Id||_{p,D(s_0/2,r_0/2)} \leq c\eta^6, \]

and

\[ |||D \Psi - Id|||_{p,D(s_0/2,r_0/2)} \leq c\eta^6. \]

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(2) the frequencies \( \hat{\omega}(\xi) \) and \( \hat{\Omega}(\xi) \) satisfy
\[
||\hat{\omega}(\xi) - \omega(\xi)|| + \sup_{j \geq 1}||\partial_{\xi_j}(\hat{\omega}(\xi) - \omega(\xi))|| \leq c\eta^8\varepsilon,
\]
and
\[
||\hat{\Omega}(\xi) - \Omega(\xi)|| + \sup_{j \geq 1}||\partial_{\xi_j}(\hat{\Omega}(\xi) - \Omega(\xi))|| \leq c\eta^8\varepsilon;
\]
(3) the Hamiltonian vector field \( X_{\hat{R}} \) satisfies
\[
|||X_{\hat{R}}|||_{T,s_0/2,r_0/2} \leq \varepsilon(1 + c\eta^6\varepsilon),
\]
where \( c > 0 \) is a constant depending on \( s_0, r_0, n \) and \( \tau \). Moreover, we fix \( \tau > n + 1 \) and then the constant \( c \) depends on \( s_0, r_0 \) and \( n \);
(4) the measure of \( \Pi_\eta \) satisfies
\[
\text{Meas} \Pi_\eta \geq (\text{Meas} \Pi)(1 - O(\eta)).
\]

5.4 Proof of Corollary 2.9

Proof. In view of (2.18),
\[
\hat{H}(x, y, q, \bar{q}; \xi) = \hat{R}(y, q, \bar{q}; \xi),
\]
where
\[
\hat{R}(y, q, \bar{q}; \xi) = \sum_{j=1}^n \hat{\omega}_j(\xi)y_j + \sum_{j \geq 1} \hat{\Omega}_j(\xi)q_j\bar{q}_j
\]
and
\[
\hat{R}(x, y, q, \bar{q}; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n, |\alpha|+|\beta|+|\gamma| \geq 3} \hat{R}^{\alpha\beta\gamma}(x, \xi)y^\alpha q^\beta \bar{q}^\gamma.
\]
It is easy to verify that
\[
\mathcal{T}_0 = \mathbb{T}^n \times \{y = 0\} \times \{q = 0\} \times \{\bar{q} = 0\}
\]
is an embedding torus with frequency \( \hat{\omega}(\xi) \in \hat{\omega}(\Pi_\eta) \) of the Hamiltonian \( \hat{H}(x, y, q, \bar{q}; \xi) \). Moreover, \( \Psi^{-1}\mathcal{T}_0 \) is an embedding torus of the original Hamiltonian \( H(x, y, q, \bar{q}; \xi) \). We finish the proof of the existence of KAM tori. The proof of time \( |t| \leq \delta^{-1} \) stability of KAM tori is omitted here, since it is similar to the proof of \( |t| \leq \delta^{-\infty} \) stability of KAM tori, which can be found in Section 6.3.

6 Proof of Theorem 2.10

6.1 Construct a partial normal form of order \( \mathcal{M} + 2 \)

Basing on the normal form of order 2 obtained in Theorem 2.8, we will construct a partial normal form of order \( \mathcal{M} + 2 \) in the neighborhood of KAM tori by \( \mathcal{M} \) times symplectic transformations.
Given a large $\mathcal{N} \in \mathbb{N}$, split the normal frequency $\hat{\Omega}(\xi)$ and normal variable $(q, \bar{q})$ into two parts respectively, i.e.

$$\hat{\Omega}(\xi) = (\hat{\Omega}(\xi), \hat{\Omega}(\xi)), \quad q = (\bar{q}, \bar{q}), \quad \bar{q} = (\bar{q}, \bar{q}),$$

where

$$\hat{\Omega}(\xi) = (\hat{\Omega}_1(\xi), \ldots, \hat{\Omega}_\mathcal{M}(\xi)), \quad \bar{q} = (q_1, \ldots, q_\mathcal{M}), \quad \bar{q} = (\bar{q}_1, \ldots, \bar{q}_\mathcal{M})$$

are the low frequencies and

$$\hat{\Omega}(\xi) = (\hat{\Omega}_{\mathcal{M}+1}(\xi), \hat{\Omega}_{\mathcal{M}+2}(\xi), \ldots), \quad \bar{q} = (q_{\mathcal{M}+1}, q_{\mathcal{M}+2}, \ldots), \quad \bar{q} = (\bar{q}_{\mathcal{M}+1}, \bar{q}_{\mathcal{M}+2}, \ldots)$$

are the high frequencies. Given $0 < \bar{\eta} < 1$, and $\tau > n + 3$, if the frequencies $\hat{\omega}(\xi)$ and $\hat{\Omega}(\xi)$ satisfy the following inequalities

$$\left| \langle k, \hat{\omega}(\xi) \rangle + \langle \hat{\Omega}_l(\xi), \hat{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle \right| \geq \frac{\bar{\eta}}{4\mathcal{M}(|k| + 1)^2(C(\mathcal{N}, \bar{\eta}))} \tag{6.1}$$

for any $k \in \mathbb{Z}^n, \bar{l} \in \mathbb{Z}^\mathcal{N}, \hat{l} \in \mathbb{Z}^\mathcal{N}$ with

$$|k| + |\bar{l}| + |\hat{l}| \neq 0, \quad |\bar{l}| + |\hat{l}| \leq \mathcal{M} + 2, \quad |\hat{l}| \leq 2,$$

where

$$C(\mathcal{N}, \bar{\eta}) = \mathcal{N}^{(|\bar{l}| + 4)^2}, \tag{6.2}$$

then we call that the frequencies $\hat{\omega}(\xi)$ and $\hat{\Omega}(\xi)$ are $(\bar{\eta}, \mathcal{N}, \mathcal{M})$-non-resonant. Define the resonant sets $\mathcal{R}_{k\bar{l}\hat{l}}$ by

$$\mathcal{R}_{k\bar{l}\hat{l}} = \left\{ \xi \in \Pi_{\bar{\eta}} : \left| \langle k, \hat{\omega}(\xi) \rangle + \langle \hat{\Omega}_l(\xi), \hat{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle \right| \leq \frac{\bar{\eta}}{4\mathcal{M}(|k| + 1)^2C(\mathcal{N}, \bar{\eta})} \right\}. \tag{6.3}$$

Let

$$\mathcal{R} = \bigcup_{|k| + |\bar{l}| + |\hat{l}| \neq 0, |\bar{l}| + |\hat{l}| \leq \mathcal{M} + 2, |\hat{l}| \leq 2} \mathcal{R}_{k\bar{l}\hat{l}}, \tag{6.4}$$

and

$$\Pi_{\bar{\eta}} = \Pi_\bar{\eta} \setminus \mathcal{R}, \tag{6.5}$$

where $\Pi_{\bar{\eta}}$ is defined in Theorem 2.8. Then it is easy to see that for each $\xi \in \Pi_{\bar{\eta}}$, the frequencies $\hat{\omega}(\xi)$ and $\hat{\Omega}(\xi)$ are $(\bar{\eta}, \mathcal{N}, \mathcal{M})$-non-resonant.

In this section, we always assume

$$\alpha \in \mathbb{N}, \quad \beta, \gamma \in \mathbb{N}^\mathcal{N}, \quad \mu, \nu \in \mathbb{N}^\mathcal{N}.$$

**Theorem 6.1.** (Partial normal form of order $\mathcal{M} + 2$) Consider the normal form of order 2

$$\hat{H}(x, y, q, \bar{q}; \xi) = \tilde{H}(y, q, \bar{q}; \xi) + \check{H}(x, y, q, \bar{q}; \xi)$$

obtained in Theorem 2.8. Given any positive integer $\mathcal{M}$ and $0 < \bar{\eta} < 1$, there exist a small $\rho_0 > 0$ and a large positive integer $\mathcal{N}_0$ depending on $s_0, r_0, n$ and $\mathcal{M}$. For each $0 < \rho < \rho_0$ and any integer $\mathcal{N}$ satisfying

$$\mathcal{N}_0 < \mathcal{N} < \left( \frac{\bar{\eta}^2}{2\rho} \right)^{\frac{1}{2(\mathcal{M} + 7)^2}}, \tag{6.6}$$

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and for each \( \xi \in \Pi_{\eta} \), then there is a symplectic map

\[
\Phi : D(s_0/4, 4p, 4p) \to D(s_0/2, 5p, 5p),
\]
such that

\[
\tilde{H}(x, y, q, \bar{q}; \xi) := H \circ \Phi = \tilde{\tilde{N}}(y, q, \bar{q}; \xi) + Z(y, q, \bar{q}; \xi) + P(x, y, q, \bar{q}; \xi) + Q(x, y, q, \bar{q}; \xi) \tag{6.7}
\]
is a partial normal form of order \( \mathcal{M} + 2 \), where

\[
Z(y, q, \bar{q}; \xi) = \sum_{4 \leq 2|\alpha|+2|\beta|+2|\mu| \leq \mathcal{M} + 2, |\mu| \leq 1} Z^{\alpha \beta \mu}(\xi) y^\alpha q^\beta \bar{q}^\mu \bar{\bar{q}}^\mu
\]
is the integrable term depending only on variables \( y \) and \( I_j = |q_j|^2, j \geq 1 \), and where

\[
P(x, y, q, \bar{q}; \xi) = \sum_{2|\alpha|+|\beta|+|\gamma|+|\nu| \geq \mathcal{M} + 3, |\mu| \leq 2} p^{\alpha \beta \gamma \nu}(x; \xi) y^\alpha q^\beta \bar{q}^\gamma \bar{\bar{q}}^\nu,
\]
and

\[
Q(x, y, q, \bar{q}; \xi) = \sum_{|\mu| \geq 3} Q^{\alpha \beta \gamma \nu}(x; \xi) y^\alpha q^\beta \bar{q}^\gamma \bar{\bar{q}}^\nu.
\]

Moreover, the following estimates hold:

1. the symplectic map \( \Phi \) satisfies

\[
\|\Phi - id\|_{p, D(s_0/4, 4p, 4p)} \leq \frac{c \mathcal{M}^{98} p}{\eta^2}, \tag{6.8}
\]

and

\[
\|D\Phi - Id\|_{p, D(s_0/4, 4p, 4p)} \leq \frac{c \mathcal{M}^{98} p}{\eta^2}; \tag{6.9}
\]

2. the Hamiltonian vector fields \( X_{\xi}, X_P \) and \( X_Q \) satisfy

\[
\|X_{\xi}\|_{p, D(s_0/4, 4p, 4p) \times \Pi_{\eta}} \leq c \rho \left( \frac{1}{\eta^2} \mathcal{M}^{2(\mathcal{M} + 6)^2} \right) \tag{6.10}
\]

\[
\|X_P\|_{p, D(s_0/4, 4p, 4p) \times \Pi_{\eta}} \leq c \rho \left( \frac{1}{\eta^2} \mathcal{M}^{2(\mathcal{M} + 7)^2} \right) \tag{6.11}
\]

and

\[
\|X_Q\|_{p, D(s_0/4, 4p, 4p) \times \Pi_{\eta}} \leq c \rho,
\]
where \( c > 0 \) is a constant depending on \( s_0, r_0, n \) and \( \mathcal{M} \).

To prove Theorem 6.1, we will give an iterative lemma first. Take

\[
s' = \frac{s_0}{12 \mathcal{M}} \quad \text{and} \quad \rho' = \frac{\rho}{2 \mathcal{M}}, \tag{6.10}
\]

Let \( 2 \leq j_0 \leq \mathcal{M} + 2 \) and denote

\[
\mathcal{D}_{j_0} = D(s_0/2 - 3(j_0 - 2)s', 5\rho - 2(j_0 - 2)\rho', 5\rho - 2(j_0 - 2)\rho'), \tag{6.11}
\]
$$\mathcal{D}_{j_0} = D(s_0/2 - (3(j_0 - 2) + 1)s', 5\rho - 2(j_0 - 2)\rho', 5\rho - 2(j_0 - 2)\rho')$$

and

$$\mathcal{D}''_{j_0} = D(s_0/2 - (3(j_0 - 2) + 2)s', 5\rho - (2(j_0 - 2) + 1)\rho', 5\rho - (2(j_0 - 2) + 1)\rho')$$

Then it is easy to see

$$\mathcal{D}_{j_1 + 1} \subset \mathcal{D}''_{j_0} \subset \mathcal{D}'_{j_0} \subset \mathcal{D}_{j_0},$$

and

$$\mathcal{D}_2 = D(s_0/2, 5\rho, 5\rho) \quad \text{and} \quad \mathcal{D}_{\mathcal{M} + 2} = D(s_0/4, 4\rho, 4\rho).$$

**Lemma 6.2.** Consider the partial normal form of order $j_0$ ($2 \leq j_0 \leq \mathcal{M} + 1$)

$$H_{j_0}(x, y, q, \bar{q}; \xi) = \tilde{N}(y, q, \bar{q}; \xi) + Z_{j_0}(y, \bar{q}; x; \xi) + P_{j_0}(x, y, q, \bar{q}; \xi) + Q_{j_0}(x, y, q, \bar{q}; \xi), \quad (6.12)$$

where

$$\tilde{N}(y, q, \bar{q}; \xi) = \sum_{j=1}^{n} \tilde{\varphi}_j(\xi) y_j + \sum_{j \geq 1} \tilde{\varphi}_j(\xi) q_j \bar{q}_j,$$

$$Z_{j_0}(y, q, \bar{q}; \xi) = \sum_{3 \leq j \leq j_0} Z_{j_0}(y, q, \bar{q}; \xi), \quad (6.13)$$

$$P_{j_0}(x, y, q, \bar{q}; \xi) = \sum_{j \geq j_0 + 1} P_{j_0}(x, y, q, \bar{q}; \xi), \quad (6.14)$$

$$Q_{j_0}(x, y, q, \bar{q}; \xi) = \sum_{j \geq 3} Q_{j_0}(x, y, q, \bar{q}; \xi), \quad (6.15)$$

with

$$Z_{j_0}(y, q, \bar{q}; \xi) = \sum_{2|\alpha| + 2|\beta| + 2|\mu| = |j|, |j_0| \leq 1}^{2|\alpha| + 2|\beta| + 2|\mu| = 1} Z_{j_0}^{\alpha \beta \mu}(y, q, \bar{q}; x; \xi) y^\alpha q^\beta \bar{q}^\mu,$$

$$P_{j_0}(x, y, q, \bar{q}; \xi) = \sum_{2|\alpha| + 2|\beta| + 2|\mu| = |j|, |\mu| + |\gamma| + 2 \leq 2} P_{j_0}^{\alpha \beta \gamma \nu}(x, \xi) y^\alpha q^\beta \bar{q}^\mu \bar{q}^\nu \bar{y}^\gamma,$$

$$Q_{j_0}(x, y, q, \bar{q}; \xi) = \sum_{2|\alpha| + 2|\beta| + 2|\mu| = |j|, |\mu| + |\gamma| \geq 3} Q_{j_0}^{\alpha \beta \gamma \nu}(x, \xi) y^\alpha q^\beta \bar{q}^\mu \bar{q}^\nu \bar{y}^\gamma. \quad (6.18)$$

Suppose $Z_{j_0}(y, q, \bar{q}; \xi), P_{j_0}(x, y, q, \bar{q}; \xi)$ and $Q_{j_0}(x, y, q, \bar{q}; \xi)$ satisfy the following estimates,

$$|||X_{Z_{j_0}}|||^T_{p, \mathcal{D}_{j_0} \times \Pi_\eta} \leq \rho \left( \frac{1}{\eta^2} \mathcal{N}^2(j_0 + 4)^2 \right)^{j-3}, \quad (6.19)$$

$$|||X_{P_{j_0}}|||^T_{p, \mathcal{D}_{j_0} \times \Pi_\eta} \leq \rho \left( \frac{1}{\eta^2} \mathcal{N}^2(j_0 + 5)^2 \right)^{j-3}, \quad (6.20)$$

$$|||X_{Q_{j_0}}|||^T_{p, \mathcal{D}_{j_0} \times \Pi_\eta} \leq \rho \left( \frac{1}{\eta^2} \mathcal{N}^2(j_0 + 2)^2 \right)^{j-3}, \quad (6.21)$$

where $a \leq b$ means there is a constant $c > 0$ depending on $s_0, n$ and $\mathcal{N}$ such that $a \leq cb$ (but independent of $\rho, \eta$ and $\mathcal{N}$). Then there exists a symplectic map $\Phi_{j_0}: \mathcal{D}_{j_0 + 1} \to \mathcal{D}_{j_0}$, such that

$$H_{j_0 + 1}(x, y, q, \bar{q}; \xi) = H_{j_0} \circ \Phi_{j_0}(x, y, q, \bar{q}; \xi) = \tilde{N}(y, q, \bar{q}; \xi) + Z_{j_0 + 1}(y, q, \bar{q}; \xi) + P_{j_0 + 1}(x, y, q, \bar{q}; \xi) + Q_{j_0 + 1}(x, y, q, \bar{q}; \xi).$$
where
\[
Z_{j_0+1}(y, q, \tilde{q}; \xi) = \sum_{3 \leq j \leq j_0+1} Z_{(j_0+1)j}(y, q, \tilde{q}; \xi),
\]
\[
P_{j_0+1}(x, y, q, \tilde{q}; \xi) = \sum_{j \geq j_0+2} P_{(j_0+1)j}(x, y, q, \tilde{q}; \xi),
\]
\[
Q_{j_0+1}(x, y, q, \tilde{q}; \xi) = \sum_{j \geq 3} Q_{(j_0+1)j}(x, y, q, \tilde{q}; \xi),
\]

with
\[
Z_{(j_0+1)j}(y, q, \tilde{q}; \xi) = \sum_{2|\alpha|+2|\beta|+2|\mu|=j, |\mu| \leq 1} Z^{\alpha\beta\mu\nu}_{j_0+1}(\xi)\alpha^\alpha \tilde{q}^\beta \tilde{q}^\mu \tilde{q}^\nu,
\]
\[
P_{(j_0+1)j}(x, y, q, \tilde{q}; \xi) = \sum_{2|\alpha|+|\beta|+|\mu|+|\nu|=j, |\mu|+|\nu| \leq 2} P^{\alpha\beta\mu\nu}_{j_0+1}(x; \xi)\alpha^\alpha \tilde{q}^\beta \tilde{q}^\mu \tilde{q}^\nu,
\]
\[
Q_{(j_0+1)j}(x, y, q, \tilde{q}; \xi) = \sum_{2|\alpha|+|\beta|+|\mu|+|\nu|=j, |\mu|+|\nu| \geq 3} Q^{\alpha\beta\mu\nu}_{j_0+1}(x; \xi)\alpha^\alpha \tilde{q}^\beta \tilde{q}^\mu \tilde{q}^\nu.
\]

Moreover, the following estimates hold:
(1) the symplectic map \( \Phi_{j_0} \) satisfies
\[
\|\Phi_{j_0} - id\|_{p, \mathcal{F}_{j_0}} \leq \left( \frac{1}{\eta^2} A^2(j_0+5)^2 \rho \right)^{j_0-1} \tag{6.22}
\]
and
\[
\|\mathcal{D}\Phi_{j_0} - Id\|_{p, \mathcal{F}_{j_0+1}} \leq \rho^{-1} \left( \frac{1}{\eta^2} A^2(j_0+5)^2 \rho \right)^{j_0-1} \tag{6.23}
\]
(2) the Hamiltonian vector fields \( X_{Z_{(j_0+1)j}}, X_{P_{(j_0+1)j}}, \) and \( X_{Q_{(j_0+1)j}} \) satisfy
\[
\|X_{Z_{(j_0+1)j}}\|_{p, \mathcal{F}_{j_0+1} \times \Pi_\eta} \leq \rho \left( \frac{1}{\eta^2} A^2(j_0+5)^2 \rho \right)^{j_0-1}, \tag{6.24}
\]
\[
\|X_{P_{(j_0+1)j}}\|_{p, \mathcal{F}_{j_0+1} \times \Pi_\eta} \leq \rho \left( \frac{1}{\eta^2} A^2(j_0+6)^2 \rho \right)^{j_0-1}, \tag{6.25}
\]
\[
\|X_{Q_{(j_0+1)j}}\|_{p, \mathcal{F}_{j_0+1} \times \Pi_\eta} \leq \rho \left( \frac{1}{\eta^2} A^2(j_0+6)^2 \rho \right)^{j_0-1}. \tag{6.26}
\]

Proof. Step 1. The derivative of homological equation.
Expand \( P_{(j_0+1)j}(x, y, q, \tilde{q}; \xi) \) into Fourier series
\[
P^{\alpha\beta\mu\nu}_{j_0}(x; \xi) = \sum_{k \in \mathbb{Z}^n} \overline{P^{\alpha\beta\mu\nu}_{j_0}(k; \xi)} e^{\sqrt{-1}k \cdot x}.
\]
To obtain the partial normal form of order \( j_0 + 1 \), we need to eliminate all non-integrable terms in \( P_{(j_0+1)(x, y, q, \tilde{q}; \xi)} \), which are
\[
\sum_{2|\alpha|+|\beta|+|\gamma|+|\mu|+|\nu|=j_0+1, |\mu|+|\nu| \leq 2} \sum_{k \in \mathbb{Z}^n} \overline{P^{\alpha\beta\mu\nu}_{j_0}(k; \xi)} e^{\sqrt{-1}k \cdot x} \alpha^\alpha \tilde{q}^\beta \tilde{q}^\mu \tilde{q}^\nu
\]
with \(|k| + |\beta - \gamma| + |\mu - \nu| \neq 0\). To this end, let

\[
F_{j_0}(x, y, q, \bar{q}; \xi) = \sum_{2|\alpha| + |\beta| + |\gamma| + |\mu| + |\nu| = j_0 + 1, |\mu| + |\nu| \leq 2} F_{j_0}^{\alpha\beta\gamma\mu\nu}(x, \xi)y^\alpha q^\beta \bar{q}^\gamma \bar{\mu}^\nu,
\]

(6.27)

and let \(\Phi_{j_0} = X_{F_{j_0}}^t|_{t=1}\) be the time-1 map of the Hamiltonian vector field \(X_{F_{j_0}}\).

Using Taylor formula,

\[
H_{j_0 + 1} := H_{j_0} \circ X_{F_{j_0}}^t|_{t=1} = \langle \bar{N} + Z_{j_0} + P_{j_0} + Q_{j_0} \rangle \circ X_{F_{j_0}}^t|_{t=1}.
\]

(6.28)

Then we obtain the homological equation

\[
\{\bar{N}, F_{j_0}\} + P_{j_0(j_0+1)} = \bar{Z}_{j_0},
\]

(6.31)

where

\[
\bar{Z}_{j_0}(y, q, \bar{q}; \xi) = \sum_{2|\alpha| + |\beta| + |\gamma| + |\mu| = j_0 + 1, |\mu| \leq 1} \bar{P}_{j_0}^{\alpha\beta\gamma\mu}(0; \xi)y^\alpha q^\beta \bar{q}^\gamma \bar{\mu}.
\]

(6.32)

If the homological equation (6.31) is solvable, then in view of (6.28)-(6.30) we can define

\[
Z_{j_0+1}(y, q, \bar{q}; \xi) = Z_{j_0}(y, q, \bar{q}; \xi) + \bar{Z}_{j_0}(y, q, \bar{q}; \xi),
\]

(6.33)

and

\[
P_{j_0 + 1} + Q_{j_0 + 1} = \int_0^1 (1 - t)\{\{\bar{N}, F_{j_0}\}, F_{j_0}\} \circ X_{F_{j_0}}^t dt
\]

(6.34)

\[
+ \int_0^1 \{P_{j_0(j_0+1)} + Z_{j_0}, F_{j_0}\} \circ X_{F_{j_0}}^t dt
\]

(6.35)

and

\[
H_{j_0 + 1}(x, y, q, \bar{q}; \xi) \text{ has the following form}
\]

\[
H_{j_0 + 1} = \bar{N} + Z_{j_0+1} + P_{j_0+1} + Q_{j_0+1}.
\]

(6.37)

**Step 2. The solution of homological equation (6.31).**

By passing to Fourier coefficients, (6.31) reads

\[
\hat{F}_{j_0}^{\alpha\beta\gamma\mu\nu}(k; \xi) = \frac{\hat{P}_{j_0}^{\alpha\beta\gamma\mu\nu}(k; \xi)}{\sqrt{1 - (\langle k, \bar{\omega}(\xi) \rangle + \langle \beta - \gamma, \bar{\Omega}(\xi) \rangle + \langle \mu - \nu, \bar{\Omega}(\xi) \rangle)}}
\]

(6.38)

for

\[
2|\alpha| + |\beta| + |\gamma| + |\mu| + |\nu| = j_0 + 1, |\mu| + |\nu| \leq 2, |k| + |\beta - \gamma| + |\mu - \nu| \neq 0.
\]
and otherwise
\[ F_{\alpha\beta\gamma\mu\nu}^{(k;\xi)} = 0. \]

Now we will estimate \[ \|X_{j_0}\|_{p,\partial_{j_0}^\gamma,P_{j_0}} \cdot P_{\tilde{n}}. \]

For each \( \xi \in \Pi_n \), the frequencies \( \hat{\omega}(\xi) \) and \( \hat{\Omega}(\xi) \) satisfy the \((\tilde{n},\mathcal{M},\mathcal{K})\)-non-resonant conditions \((6.1)\), i.e.
\[
|\langle k, \hat{\omega}(\xi) \rangle + \langle \beta - \gamma, \hat{\Omega}(\xi) \rangle + \langle \mu - \nu, \hat{\Omega}(\xi) \rangle| \geq \frac{\tilde{n}}{4^m(|k|+1)^2C(A,\beta - \gamma)}.
\]
Then
\[
\begin{align*}
\langle k, \hat{\omega}(\xi) \rangle + \langle \beta - \gamma, \hat{\Omega}(\xi) \rangle + \langle \mu - \nu, \hat{\Omega}(\xi) \rangle & = \frac{\tilde{n}}{4^m(|k|+1)^2C(A,\beta - \gamma)} - 1 \\
& \leq \frac{4^m}{\tilde{n}}(|k|+1)^2C(A,\beta - \gamma) \\
& \leq \frac{4^m}{\tilde{n}}(|k|+1)^2C(A^{(\beta - \gamma+4)}^2) \\
& \leq \frac{4^m}{\tilde{n}}(|k|+1)^2C(A^{(\beta + |\gamma|+4)}^2) \\
& \leq \frac{4^m}{\tilde{n}}(|k|+1)^2C(A^{(\beta + |\gamma|+4)}^2) \\
& \leq \frac{4^m}{\tilde{n}}(|k|+1)^2C(A^{(\beta + |\gamma|+4)}^2)
\end{align*}
\]

Moreover, in view of \((2.17)\) in Assumption B (twist conditions)
\[
\partial_{j_i} \hat{\omega}(\xi) = \delta_{j_i}, \quad \partial_{j_i} \hat{\Omega}(\xi) = \delta_{j_i(n+j')}, \quad 1 \leq i \leq n, \ j, j' \geq 1,
\]
and the estimates (see \((2.25)\) and \((2.26)\))
\[
\| \hat{\omega}(\xi) - \omega(\xi) \| + \sup_{j \geq 1} \| \partial_{j_i} (\hat{\omega}(\xi) - \omega(\xi)) \| \leq c\eta^8 \varepsilon,
\]
and
\[
\| \hat{\Omega}(\xi) - \Omega(\xi) \| + \sup_{j \geq 1} \| \partial_{j_i} (\hat{\Omega}(\xi) - \Omega(\xi)) \| \leq c\eta^8 \varepsilon,
\]
we have
\[
\begin{align*}
\partial_{\xi} \hat{\omega}(\xi) & \geq 1 - c\eta^8 \varepsilon, \quad \partial_{\xi} \hat{\Omega}(\xi) \geq 1 - c\eta^8 \varepsilon, \\
| \partial_{\xi} \hat{\omega}(\xi) | & \leq c\eta^8 \varepsilon, \quad j \neq i, \\
| \partial_{\xi} \hat{\Omega}(\xi) | & \leq c\eta^8 \varepsilon, \quad j \neq j'.
\end{align*}
\]
Then for each \( j \geq 1 \), we get
\[
\begin{align*}
\partial_{j_i} \left( \langle k, \hat{\omega}(\xi) \rangle + \langle \beta - \gamma, \hat{\Omega}(\xi) \rangle + \langle \mu - \nu, \hat{\Omega}(\xi) \rangle \right) & \leq \langle |k| + j_0 + 1 \rangle (c\eta^8 \varepsilon + 1) \quad \text{(in view of the estimates \((6.40)-(6.42)\))}
& \leq 2(|k| + j_0 + 1) \quad \text{(in view of \( 2 \leq j_0 \leq \mathcal{M} + 1 \))}
& \leq 2(|k| \mathcal{M} + 2) \quad \text{(in view of \( 2 \leq j_0 \leq \mathcal{M} + 1 \)).}
\end{align*}
\]

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Hence,

$$\begin{align*}
&\left| \partial_{\xi_j} \left( \langle k, \omega(\xi) \rangle + \langle \beta - \gamma, \hat{\Phi}(\xi) \rangle + \langle \mu - \nu, \hat{\Phi}(\xi) \rangle \right) \right|^{-1} \\
&\leq \frac{2 \cdot 4^{2M}}{\eta^2} (|k| + 1)2^{2\tau}M^{-2(j_0+5)^2}(|k|+M+2)
\end{align*}$$

(based on the inequalities (6.38) and (6.43))

$$\leq \frac{2 \cdot 4^{2M}}{\eta^2} (M+2) (|k| + 1)2^{2\tau+1}M^{-2(j_0+5)^2}, \quad (6.44)$$

where the last inequality is based on $|k|+M+2 \leq (|k|+1)(M+2)$. Then in view of the formula (6.38) and the inequalities (6.39) and (6.44), for each $j \geq 1$, we obtain

$$\left| \frac{\alpha_{\beta\gamma\mu\nu}}{\eta} p_{j_0}^\alpha p_{j_0}^\beta p_{j_0}^\gamma p_{j_0}^\mu \left( \langle \alpha\beta\gamma\mu\nu \rangle \right) \right| \leq \frac{(k+1)^{2\tau+1}M^{-2(j_0+5)^2}}{\eta^2} \left( \frac{1}{\eta^2} M^{-2(j_0+5)^2} \rho \right)^{j_0-1} \quad (6.45)$$

Noting $\tau > n+3$ will be fixed and recalling $s' = s_0/(12M)$ (see (6.10)), we obtain

$$\left\| X_{j_0} \right\|_{p, \mathscr{D}_{j_0}^s} \leq \frac{M^{-2(j_0+5)^2}}{\eta^2} \left\| X_{j_0(s_0+1)} \right\|_{p, \mathscr{D}_{j_0}^s} \quad (6.46)$$

Noting $\Phi_{j_0} = X_{j_0}^{l=1}$ and basing on the inequality (6.46) and Theorem 4.3, we obtain

$$\left\| \Phi_{j_0} - id \right\|_{p, \mathscr{D}_{j_0}^{s'}} \leq \frac{1}{\eta^2} M^{-2(j_0+5)^2} \rho \left( \frac{1}{\eta^2} M^{-2(j_0+5)^2} \rho \right)^{j_0-1} \quad (6.47)$$

Moreover,

$$\left\| D\Phi_{j_0} - Id \right\|_{p, \mathscr{D}_{j_0+1}^{s'}} \leq \rho^{-1} \left( \frac{1}{\eta^2} M^{-2(j_0+5)^2} \rho \right)^{j_0-1}$$

which follows the generalized Cauchy estimate in Lemma 8.7. We finish the proof of the inequalities (6.22) and (6.23).

Step 3. Estimate $\left\| X_{j_0+1} \right\|_{p, \mathscr{D}_{j_0+1}^s}$. rewrite $Z_{j_0+1}(y, \tilde{q}, \tilde{\xi})$ as

$$Z_{j_0+1}(y, \tilde{q}, \tilde{\xi}) = \sum_{3 \leq j \leq j_0+1} Z_{(j_0+1)}(y, \tilde{q}, \tilde{\xi})$$
Using Taylor formula again, we have
\[ Z_{(j_0+1)}(y,q,\bar{q};\xi) = \sum_{2|\alpha|+2|\beta|+|\mu|=j_0} Z_{j_0+1}^{\alpha\beta\mu}(\xi) y^\alpha \bar{q}^\beta q^\mu x^\nu. \]

In view of (6.13), (6.32) and (6.33), we have
\[ Z_{(j_0+1)}(y,q,\bar{q};\xi) = Z_{j_0}(y,q,\bar{q};\xi), \quad 3 \leq j \leq j_0 \] (6.47)
and
\[ Z_{(j_0+1)(j_0+1)}(y,q,\bar{q};\xi) = \tilde{Z}_{j_0}(y,q,\bar{q};\xi). \] (6.48)

For \( 3 \leq j \leq j_0 \),
\[ \begin{aligned}
|||X_{(j_0+1)}|||^T_{\mathcal{P};\mathcal{G}_{j_0+1} \times \Pi \tilde{q}} & \leq \rho \left( \frac{1}{\eta^2} \mathcal{N}^2(j_0+4)^2 \rho \right)^{j_3} \quad \text{(in view of the inequality (6.19))} \\
& \leq \rho \left( \frac{1}{\eta^2} \mathcal{N}^2(j_0+5)^2 \rho \right)^{j_3}. \quad \text{(6.49)}
\end{aligned} \]

When \( j = j_0 + 1 \), we have
\[ \begin{aligned}
|||X_{(j_0+1)(j_0+1)}|||^T_{\mathcal{P};\mathcal{G}_{j_0+1} \times \Pi \tilde{q}} & = |||X_{(j_0+1)}|||^T_{\mathcal{P};\mathcal{G}_{j_0+1} \times \Pi \tilde{q}} \quad \text{(in view of (6.48))} \\
& \leq |||X_{(j_0+2)}|||^T_{\mathcal{P};\mathcal{G}_{j_0+2} \times \Pi \tilde{q}} \quad \text{(in view of (6.32))} \\
& \leq \rho \left( \frac{1}{\eta^2} \mathcal{N}^2(j_0+5)^2 \rho \right)^{j_0-2} \quad \text{(in view of the inequality (6.20) for \( j = j_0 + 1 \)).} \quad \text{(6.50)}
\end{aligned} \]

Based on the inequalities (6.49) and (6.50), we finish the proof of the inequality (6.24).

To obtain the estimates (6.25) and (6.26), we will estimate the \( p \)-tame norm of the terms (6.34), (6.36) respectively.

Firstly we consider the term (6.34). Let
\[ \tilde{N}^{(0)} = \tilde{N} \circ X_{F_{j_0}}^L |_{t=0} = \tilde{N}, \] (6.51)
and
\[ \tilde{N}^{(j)} = \{ \tilde{N}^{(j-1)}, F_{j_0} \}, \quad j \geq 1. \] (6.52)

Using Taylor formula again, we have
\[ \int_0^1 (1-t)\{ \{ \tilde{N}, F_{j_0} \}, F_{j_0} \} \circ X_{F_{j_0}}^L dt = \sum_{j \geq 2} \frac{1}{j!} \tilde{N}^{(j)}. \] (6.53)

Moreover, note that \( \tilde{N}^{(j)}(x,y,q,\bar{q};\xi) \) has the following form
\[ \tilde{N}^{(j)}(x,y,q,\bar{q};\xi) = \sum_{2|\alpha|+|\beta|+|\mu|=j(j_0+1)+2-2j} \tilde{N}^{(j)}_{\alpha\beta\gamma\nu}(x,\xi) y^\alpha \bar{q}^\beta q^\mu \nu. \] (6.54)

In view of the homological equation (6.31)
\[ \{ \tilde{N}, F_{j_0} \} + P_{j_0(j_0+1)} = \tilde{Z}_{j_0}, \]

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and the formulas (6.51) and (6.52), we have
\[ \hat{N}^{(1)} = -P_{j_0(j_0+1)} + \hat{Z}_{j_0}. \]  

(6.55)

In view of the inequality (6.20) for \( j = j_0 + 1 \) and the formula (6.32),
\[ |||X_{\hat{G}(1)}|||_{P, \Sigma_{j_0} \times \Pi_{k}}^T \leq |||X_{P_{j_0(j_0+1)}}|||_{P, \Sigma_{j_0} \times \Pi_{k}}^T \leq \rho \left( \frac{1}{\eta^2} \mathcal{N}^{2(j_0+5)^2} \rho \right)^{j_0-2}. \]  

(6.56)

For \( j \geq 2 \), let
\[ s_j = \frac{s_0}{4j \mathcal{A}} \quad \text{and} \quad \rho_j = \frac{\rho}{2j \mathcal{A}}. \]

Then we have
\[ \frac{1}{j!} |||X_{\hat{G}(j)}|||_{P, \Sigma_{j_0+1} \times \Pi_{k}}^T \leq \frac{1}{j!} \left( 4C \max \left( \frac{4j \mathcal{A}}{s_0}, 10j \mathcal{A} \right) \right)^{j-1} \left( |||X_{\hat{G}(1)}|||_{P, \Sigma_{j_0} \times \Pi_{k}}^T \right)^{j-1}. \]

(based on Theorem (6.1))

\[ \leq \frac{j^{j-1}}{j!} \left( \rho \left( \frac{1}{\eta^2} \mathcal{N}^{2(j_0+5)^2} \rho \right)^{j_0-2} \right) \left( C_1 \left( \frac{1}{\eta^2} \mathcal{N}^{2(j_0+5)^2} \rho \right)^{j_0-1} \right)^{j-1} \]

(based on the inequalities (6.46) and (6.56),
and \( C_1 > 0 \) is a constant depending on \( s_0 \) and \( \mathcal{A} \))
\[ \leq \frac{e}{j} \left( \rho \left( \frac{1}{\eta^2} \mathcal{N}^{2(j_0+5)^2} \rho \right)^{j_0-2} \right) \left( e \left( \frac{1}{\eta^2} \mathcal{N}^{2(j_0+5)^2} \rho \right)^{j_0-1} \right)^{j-1} \]

(using the inequality \( j^j < j! e^j \))
\[ \leq \frac{e}{j} \left( \rho \left( \frac{1}{\eta^2} \mathcal{N}^{2(j_0+5)^2} \rho \right)^{j_0-2} \right) \left( \rho \left( \frac{1}{\eta^2} \mathcal{N}^{2(j_0+6)^2} \rho \right)^{j_0-1} \right)^{j-1} \]

(noting \( \mathcal{N} \) is large depending on \( s_0, r_0, n \) and \( \mathcal{A} \))
\[ \leq \rho \left( \frac{1}{\eta^2} \mathcal{N}^{2(j_0+6)^2} \rho \right)^{j_0(j_0+1)^{1-2j}}. \]  

(6.57)

In view of (6.54), note that the index \( \alpha, \beta, \gamma, \mu, \nu \) of \( \hat{G}^{(j)} \) satisfy
\[ 2|\alpha| + |\beta| + |\gamma| + |\mu| + |\nu| = j(j_0 + 1) + 2 - 2j \geq j_0 + 2 \]  (since \( j, j_0 \geq 2 \))
and
\[ j(j_0 + 1) - 1 - 2j = (j(j_0 + 1) + 2 - 2j) - 3. \]
Then in view of (6.53) and the inequality (6.57), we finish the estimate of \( p \)-tame norm of the term (6.34).

Next, we will give the estimate of \( p \)-tame norm of the term (6.35). Let
\[ W_i = Z_{j_0 i}, \quad \text{for} \ 3 \leq i \leq j_0, \]  

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and
\[ W_{j_0 + 1} = P_{j_0(j_0 + 1)} . \]

Then
\[ \sum_{3 \leq i \leq j_0 + 1} W_i = P_{j_0(j_0 + 1)} + Z_{j_0} . \tag{6.58} \]

Let
\[ W_i^{(0)} = W_i \circ X_{F_{j_0}}^t |_{t=0} = W_i , \]
and
\[ W_i^{(j)} = \{ W_i^{(j-1)} , F_{j_0} \} , \quad j \geq 1 . \]

In view of (6.58) and using Taylor formula, we have
\[ \int_0^1 \{ P_{j_0(j_0 + 1)} + Z_{j_0} , F_{j_0} \} \circ X_{F_{j_0}}^t dt = \sum_{3 \leq i \leq j_0 + 1} \left( \sum_{j \geq 1} \frac{1}{j!} W_i^{(j)} \right) . \tag{6.59} \]

Note that \( W_i^{(j)} (x, y, q, \tilde{q}; \xi) \) has the following form
\[ W_i^{(j)} (x, y, q, \tilde{q}; \xi) = \sum_{2|\alpha|+|\beta|+|\gamma|+|\mu|+|\nu|=j_0+1+i-2j} W_i^{(j)\alpha\beta\gamma\mu\nu} (x, \xi) y^\alpha \tilde{q}^\beta \tilde{q}^\gamma \tilde{q}^\mu \nu . \tag{6.60} \]

Using the proof of (6.57) and in view of (6.19) and (6.20), we have
\[ \frac{1}{j!} \frac{\| X_{W_i^{(j)}} \|^2 p_{j_0+1} , \tilde{q} \|}{\eta^2} \leq \rho \left( \frac{1}{\eta^2} \right)^{2(j_0+1)-i-2j-3} \tag{6.61} \]

In view of (6.60), note that the indices \( \alpha, \beta, \gamma, \mu, \nu \) of \( W_i^{(j)} \) satisfy
\[ 2|\alpha| + |\beta| + |\gamma| + |\mu| + |\nu| = j_0 + 1 + i - 2j \geq j_0 + 2 , \]

since \( j \geq 1, j_0 \geq 2, i \geq 3 \). Then in view of (6.59) and the inequality (6.61), we finish the estimate of \( p \)-tame norm of the term (6.35).

Finally, we will give the estimate of \( p \)-tame norm of the term (6.36). Let
\[ U_i = P_{j_0+i} , \quad \text{for } i \geq j_0 + 2 , \]
and
\[ V_i = Q_{j_0+i} , \quad \text{for } i \geq 3 . \]

Then
\[ \sum_{i \geq j_0 + 2} U_i + \sum_{i \geq 3} V_i = P_{j_0} - P_{j_0(j_0 + 1)} + Q_{j_0} . \tag{6.62} \]

For simplicity, denote \( T_i = U_i \) or \( V_i \). Let
\[ T_i^{(0)} = T_i \circ X_{F_{j_0}}^t |_{t=0} = T_i , \]
and
\[ T_i^{(j)} = \{ T_i^{(j-1)} , F_{j_0} \} , \quad j \geq 1 . \]
Using Taylor formula again, we have

\[
(P_{j_0} - P_{j_0(0, 0, 1)} + Q_{j_0}) \circ X_{P_{j_0}}^T|_{t = 1}
\]

\[
= P_{j_0} - P_{j_0(0, 0, 1)} + Q_{j_0} + \sum_{\ell \geq 2 j_0 + 2} \left( \sum_{j \geq 2 j_0 + 1} \frac{1}{j!} T^{(j)}(j) \right) + \sum_{\ell \geq 3} \left( \sum_{j \geq 2} \frac{1}{j!} V^{(j)}(j) \right).
\]  \hspace{1cm} (6.63)

For \( j \geq 1 \), note that \( T^{(j)}(x, y, q, \bar{q}, \bar{\xi}) \) has the following form

\[
T^{(j)}(x, y, q, \bar{q}, \bar{\xi}) = \sum_{2|\alpha| = |\beta| + |\gamma| + |\mu| + |\nu| = j(j_0 + 1) + i - 2} T^{(j)(\alpha\beta\gamma\mu\nu)}(x, \bar{\xi}) y^\alpha \bar{q}^\beta \bar{\xi}^\gamma \bar{q}^\mu \bar{\xi}^\nu. \]

Then basing on the inequalities (6.20) and (6.21), and following the proof of the inequalities (6.57) again, we have

\[
\frac{1}{j!} |||X^{(j)}|||_{p, \mathcal{D}_{j_0 + 1} \times \Pi_\eta} \leq \rho \left( \frac{\eta^2 N^2(j_0 + 6)^2 \rho}{j_0 + 1} \right)^{(j(j_0 + 1) - i - 2) - 3}. \]

In view of (6.64), note that the index \( \alpha, \beta, \gamma, \mu, \nu \) of \( T^{(j)} \) satisfy

\[
2|\alpha| + |\beta| + |\gamma| + |\mu| + |\nu| = j(j_0 + 1) + i - 2 \geq j_0 + 2,
\]

since \( j \geq 1, j_0 \geq 2, i \geq 3 \). Then in view of (6.63) and the inequality (6.65), we finish the estimate of \( p \)-tame norm of the term (6.36). Hence in view of the inequalities (6.57), (6.61) and (6.65), we obtain

\[
|||X^{(j)}|||_{p, \mathcal{D}_{j_0 + 1} \times \Pi_\eta} \leq \rho \left( \frac{\eta^2 N^2(j_0 + 6)^2 \rho}{j_0 + 1} \right)^{(j(j_0 + 1) - i - 2) - 3}.
\]

Then we finish the proof of Lemma 6.2.

\[ \square \]

**Proof of Theorem 6.1**

**Proof.** By Lemma 6.2 we will finish the proof of Theorem 6.1. By Theorem 2.8 we obtain a normal form of order 2 around KAM tori, which is

\[
\bar{H}(x, y, q, \bar{q}; \bar{\xi}) = \bar{N}(y, q, \bar{q}; \bar{\xi}) + \bar{R}(x, y, q, \bar{q}; \bar{\xi}),
\]

where

\[
\bar{N}(y, q, \bar{q}; \bar{\xi}) = \sum_{j = 1}^{\infty} J_j(\bar{\xi}) y_j + \sum_{j \geq 1} J_q(\bar{\xi}) y_j \bar{q}_j
\]

and

\[
\bar{R}(x, y, q, \bar{q}; \bar{\xi}) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, 2|\alpha| + |\beta| + |\gamma| \geq 3} \bar{R}_{\alpha\beta\gamma}(x, \bar{\xi}) y^\alpha q^\beta \bar{\xi}^\gamma.
\]

Then following the notations in Lemma 6.2 denote

\[
H^2(x, y, q, \bar{q}; \bar{\xi}) = \bar{H}(x, y, q, \bar{q}; \bar{\xi})
\]
and $R(x,y,q,q;\xi)$ can be rewritten as

$$R(x,y,q,q;\xi) = Z_2(y,q,q;\xi) + P_2(x,y,q,q;\xi) + Q_2(x,y,q,q;\xi), \quad (6.66)$$

where

$$Z_2(y,q,q;\xi) = \sum_{3 \leq j \leq 2} Z_{2j}(y,q,q;\xi) = 0, \quad (6.67)$$

$$P_2(x,y,q,q;\xi) = \sum_{j \geq 3} P_{2j}(x,y,q,q;\xi),$$

$$Q_2(x,y,q,q;\xi) = \sum_{j \geq 3} Q_{2j}(x,y,q,q;\xi),$$

and

$$Z_{2j}(y,q,q;\xi) = \sum_{2|\alpha|+2|\beta|+2|\mu|=j+\mu \leq 1} Z_2^{\alpha\beta\mu\nu}(\xi)^{\alpha\beta\mu\nu},$$

$$P_{2j}(x,y,q,q;\xi) = \sum_{2|\alpha|+|\beta|+|\gamma|+|\mu|=j+\mu \leq 2} P_2^{\alpha\beta\gamma\mu\nu}(x,\xi)^{\alpha\beta\gamma\mu\nu}, \quad (6.68)$$

$$Q_{2j}(x,y,q,q;\xi) = \sum_{2|\alpha|+|\beta|+|\gamma|+|\mu|=j+\mu \leq 3} Q_2^{\alpha\beta\gamma\mu\nu}(x,\xi)^{\alpha\beta\gamma\mu\nu}. \quad (6.69)$$

Denote

$$W_j(x,y,q,q;\xi) = P_{2j}(x,y,q,q;\xi) \text{ or } Q_{2j}(x,y,q,q;\xi), \quad j \geq 3.$$

Then

$$|||X_{W_j}|||^T \leq \epsilon(1 + c\eta^6\epsilon) \quad \text{(by (2.27))}. \quad (6.66)$$

Hence, by replacing $r_0/2$ with $5\rho$, we obtain

$$|||X_{W_j}|||^T \leq \epsilon(1 + c\eta^6\epsilon) \left(\frac{10\rho}{r_0}\right)^j. \quad (6.70)$$

Moreover, in view of the inequality (6.6) (i.e. noting $\mathcal{N}$ is large depending on $s_0, r_0, n$ and $\mathcal{M}$) and $0 < \eta_1 < 1$,

$$|||X_{W_j}|||^T \leq \rho \left(\frac{1}{\eta_1} \mathcal{N}\rho \right)^j. \quad (6.71)$$

In view of the formula (6.67) and the inequality (6.71), the assumptions (6.19)-(6.21) in Lemma (6.2) hold for $j_0 = 2$.

Let

$$\Phi = \Phi_2 \circ \Phi_3 \circ \cdots \circ \Phi_{\mathcal{M}+1}. \quad (6.72)$$

Then, based on Lemma (6.2), $\tilde{H} := \tilde{H} \circ \Phi$ has the form of

$$\tilde{H}(x,y,q,q;\xi) = \tilde{N}(y,q,q;\xi) + Z_{\mathcal{M}+2}(y,q,q;\xi) + P_{\mathcal{M}+2}(x,y,q,q;\xi) + Q_{\mathcal{M}+2}(x,y,q,q;\xi).$$
Therefore, recalling the definition of $\Phi$ (see (6.72)) and by the inequalities (6.22) for $j_0 \geq 3$, we have

$$|||\Phi - id|||_{p, D(x_0/4, 4p, 4p)} \leq \frac{\mathcal{N}^{98}}{\eta^2}.$$  (6.77)

Moreover, in view of the inequalities (6.23) for $j_0 \geq 3$, we obtain

$$|||D\Phi - Id|||_{p, D(x_0/4, 4p, 4p)} \leq \frac{\mathcal{N}^{98}}{\eta^2}.$$  (6.78)

Finally, let

$$Z = Z_{\mathcal{M}+2}, \quad P = P_{\mathcal{M}+2}, \quad Q = Q_{\mathcal{M}+2}.$$
Basing on the estimates (6.73)-(6.75), we have

\[
|||X|||_{p,D(s_0/4,4p,4p)}^{T} \leq \sum_{3 \leq j \leq \mathcal{M}+2} |||X_{(\mathcal{M}+2)j}|||_{p,\mathcal{D},\mathcal{M}+2}^{T} \times \tilde{\Pi} \eta \\
= \sum_{4 \leq j \leq \mathcal{M}+2} |||X_{(\mathcal{M}+2)j}|||_{p,\mathcal{D},\mathcal{M}+2}^{T} \times \tilde{\Pi} \eta \leq \sum_{j=4}^{\mathcal{M}+2} \rho \left( \frac{1}{\eta^2} N^{2(\mathcal{M}+6)^2} \rho \right)^{j-3} \\
\leq \rho \left( \frac{1}{\eta^2} N^{2(\mathcal{M}+6)^2} \rho \right)^{\mathcal{M}} \quad \text{(by (6.75))},
\]

\[
|||X_P|||_{p,D(s_0/4,4p,4p)}^{T} \leq \sum_{j \geq \mathcal{M}+3} |||X_P_{(\mathcal{M}+2)j}|||_{p,\mathcal{D},\mathcal{M}+2}^{T} \times \tilde{\Pi} \eta \\
\leq \sum_{j \geq \mathcal{M}+3} \rho \left( \frac{1}{\eta^2} N^{2(\mathcal{M}+7)^2} \rho \right)^{j-3} \leq \rho \left( \frac{1}{\eta^2} N^{2(\mathcal{M}+7)^2} \rho \right)^{\mathcal{M}} \quad \text{(by (6.75))},
\]

\[
|||X_Q|||_{p,D(s_0/4,4p,4p)}^{T} \leq \sum_{j \geq 3} |||X_Q_{(\mathcal{M}+2)j}|||_{p,\mathcal{D},\mathcal{M}+2}^{T} \times \tilde{\Pi} \eta \\
\leq \sum_{j \geq 3} \rho \left( \frac{1}{\eta^2} N^{2(\mathcal{M}+7)^2} \rho \right)^{j-3} \leq \rho \quad \text{(by (6.75))}.
\]

\[\square\]

6.2 The measure of the set \( \Pi_{\tilde{\eta}} \) satisfying \((\tilde{\eta}, \mathcal{N}, \mathcal{M})\)-non-resonant conditions

In this subsection, we will show

\[\text{Meas} \ \Pi_{\tilde{\eta}} \geq (\text{Meas} \ \Pi_{\eta})(1 - c\eta), \quad (6.79)\]

where \( c > 0 \) is a constant depending on \( n \).

Firstly, we will estimate the measure of the resonant sets \( R_{k\Pi} \).

Case 1.
For \( |k| \neq 0 \), without loss of generality, we assume

\[ |k_1| = \max_{1 \leq i \leq n} \{ |k_1|, \ldots, |k_n| \}. \quad (6.80)\]
Then
\[
|\partial_{\xi_1}(\langle \xi, \hat{\omega}(\xi) \rangle + \langle \tilde{\xi}, \hat{\Omega}(\xi) \rangle + \langle \tilde{\xi}, \hat{\Omega}(\xi) \rangle)|
\geq |k_1||\partial_{\xi_1}(\hat{\omega}_1(\xi))| - |\partial_{\xi_1}(\sum_{i=2}^{n} k_i \hat{\omega}_i(\xi) + \langle \tilde{\xi}, \hat{\Omega}(\xi) \rangle + \langle \tilde{\xi}, \hat{\Omega}(\xi) \rangle)|
\geq |k_1|(1 - c\eta^8e) - (\sum_{i=2}^{n} |k_i| + |\tilde{\xi}| + |\tilde{\xi}|)c\eta^8e \quad \text{(in view of the inequalities (6.40)-(6.42))}
\geq |k_1| - (|k| + \mathcal{M} + 2)c\eta^8e \quad \text{(in view of } |\tilde{\xi}| + |\tilde{\xi}| \leq \mathcal{M} + 2)\]
\geq \frac{1}{4}|k_1| \quad \text{(by (6.80) and } \mathcal{M} \leq (2c\eta^8e)^{-1})
\geq \frac{1}{4}.
\]
Hence,
\[
\text{Meas } \mathcal{R}_{k\tilde{l}} \leq \frac{4\tilde{\eta}}{4\mathcal{M}(|k| + 1)^2C(\mathcal{N}, \tilde{\xi})} \cdot \text{Meas } \Pi_{\eta}. \quad (6.81)
\]

**Case 2.**

If \(|k| = 0 \text{ and } |\tilde{\xi}| \neq 0\), without loss of generality, we assume
\[
|\tilde{\xi}| = \max_{1 \leq n \leq \mathcal{N}} \{|I_1|, \ldots, |I_{\mathcal{N}}|\}. \quad (6.82)
\]
Then
\[
|\partial_{\xi_1}(\langle \xi, \hat{\omega}(\xi) \rangle + \langle \tilde{\xi}, \hat{\Omega}(\xi) \rangle + \langle \tilde{\xi}, \hat{\Omega}(\xi) \rangle)|
\geq |\tilde{\xi}||\partial_{\xi_1}(\hat{\omega}_1(\xi))| - |\partial_{\xi_1}(\sum_{i=2}^{\mathcal{N}} |\tilde{\xi}| + |\tilde{\xi}|)c\eta^8e \quad \text{(in view of the inequalities (6.40)-(6.42))}
\geq |\tilde{\xi}| - (\mathcal{M} + 2)c\eta^8e \quad \text{(in view of } |\tilde{\xi}| + |\tilde{\xi}| \leq \mathcal{M} + 2)\]
\geq \frac{1}{4}|\tilde{\xi}|
\geq \frac{1}{4}.
\]
Hence,
\[
\text{Meas } \mathcal{R}_{\tilde{l}k} \leq \frac{4\tilde{\eta}}{4\mathcal{M}C(\mathcal{N}, \tilde{\xi})} \cdot \text{Meas } \Pi_{\eta}. \quad (6.83)
\]

**Case 3.**

If \(|k| = 0, |\tilde{\xi}| = 0\) and \(1 \leq |\tilde{\xi}| \leq 2\), then it is easy to see that \(|\langle \tilde{\xi}, \hat{\Omega}(\xi) \rangle|\) is not small, i.e.

the sets \(\mathcal{R}_{k\tilde{l}}\) are empty for \(|k| = 0, |\tilde{\xi}| = 0\) and \(1 \leq |\tilde{\xi}| \leq 2\). \quad (6.84)

Recall
\[
\mathcal{R} = \bigcup_{|k| + |\tilde{\xi}| = 0, |\tilde{\xi}| + |\tilde{\xi}| = \mathcal{M} + 2, |\tilde{\xi}| \leq 2} \mathcal{R}_{k\tilde{l}}. \quad \text{(see (6.4)).}
\]
In view of the estimates (6.81), (6.83) and (6.84), to estimate the measure of the resonant set \(\mathcal{R}\), we just need to count the number of the non-resonant sets \(\mathcal{R}_{k\tilde{l}}\). More precisely, for fixed \(k \in \mathbb{Z}\), we will show that the number of the sets \(\mathcal{R}_{k\tilde{l}}\) is finite.
In view of (2.15) in Assumption A
\[ |\Omega_i - \Omega_j| \geq c_1|i - j|(i + j), \]
and the estimates (6.40)-(6.42), then for \( i \neq j \), we have
\[ |\hat{\Omega}_i - \hat{\Omega}_j| \geq c_1|i - j|(i + j) - 2\eta^8 |l| \geq \frac{c_1}{2} |i - j|i + j|. \tag{6.85} \]
For \( |\hat{l}| = 1 \),
\[ |\langle \hat{l}, \hat{\Omega}(\xi) \rangle| = |\hat{\Omega}_i(\xi)|, \quad \text{for some } i \geq \mathcal{N} + 1. \]
Then in view of the inequality (6.85),
\[ |\langle \hat{l}, \hat{\Omega}(\xi) \rangle| \geq \frac{c_1}{2} |i| \geq \frac{c_1}{2} |i|. \tag{6.86} \]
For \( |\hat{l}| = 2 \),
\[ |\langle \hat{l}, \hat{\Omega}(\xi) \rangle| = |\hat{\Omega}_i(\xi) \pm \hat{\Omega}_j(\xi)|, \quad \text{for some } i, j \geq \mathcal{N} + 1, \]
then in view of the inequality (6.85) again,
\[ |\langle \hat{l}, \hat{\Omega}(\xi) \rangle| \geq \frac{c_1}{2} |i - j|(i + j) \geq \frac{c_1}{2} \max \{i, j\}. \tag{6.87} \]
If
\[ \max \{i, j\} \geq \frac{4}{c_1} ((|k| + 1)(||\omega(\xi)|| + 1) + c_2 j_0 \mathcal{N}^2 + 1), \]
where the constant \( c_2 \) is given in (2.16), then in view of the inequalities (6.86) and (6.87), we have
\[ |\langle \hat{l}, \hat{\Omega}(\xi) \rangle| \geq \frac{c_1}{2} \left( \frac{4}{c_1} ((|k| + 1)(||\omega(\xi)|| + 1) + c_2 j_0 \mathcal{N}^2 + 1) \right) \]
\[ = 2(|k| + 1)(||\omega(\xi)|| + 1) + 2c_2 j_0 \mathcal{N}^2 + 2. \tag{6.88} \]
Then for \( |\hat{l}| + |\hat{l}| = j_0 \) and \( 1 \leq |\hat{l}| \leq 2 \), we have
\[ |\langle \hat{k}, \omega(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle| \]
\[ \geq |\langle \hat{l}, \hat{\Omega}(\xi) \rangle| - |\langle \hat{k}, \omega(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle| \]
\[ \geq 2(|k| + 1)(||\omega(\xi)|| + 1) + 2c_2 j_0 \mathcal{N}^2 + 2 - (1 + c_1 \eta^8 \eta)(|k||\omega(\xi)|| + c_2 \hat{l}_i \mathcal{N}^2) \]
\[ \text{(in view of the estimates (6.88), (2.16), (2.25) and (2.26))} \]
\[ \geq 1, \]
which is not small.

Then the number of the non-empty non-resonant sets \( \mathcal{R}_{k\hat{l}} \) is less than
\[ A := (2.\mathcal{N} + 1)^{\hat{l}} \left( \frac{4}{c_1} ((|k| + 1)(||\omega(\xi)|| + 1) + c_2 j_0 \mathcal{N}^2 + 1) + 1 \right)^2. \tag{6.89} \]
Hence,
\[ \text{Meas } \mathcal{R} \leq \sum_{|k| + |\hat{l}| + |\hat{l}| \neq 0, |\hat{l}| + |\hat{l}| \leq \mathcal{N} + 2, |\hat{l}| \leq 2} \text{Meas } \mathcal{R}_{k\hat{l}} \]
\[ \leq \sum_{k \in \mathbb{Z}, |\hat{l}| \leq \mathcal{N} + 2} \frac{4\eta^A}{4^{\mathcal{N}}((|k| + 1)^{\mathcal{N} + 1})^C(\mathcal{N} + 1)} \cdot \text{Meas } \Pi_n \]
\[ \leq c \eta \cdot \text{Meas } \Pi_n, \]
where \( c > 0 \) is a constant depending on \( c_1, c_2 \) and \( n \). This finishes the proof.
6.3 Proof of Theorem 2.10

Proof. Based on Theorem 6.1 for given positive integer \(0 \leq \mathcal{M} \leq (2\eta^8 \varepsilon)^{-1}\) and \(0 < \tilde{\eta} < 1\), there exists a small \(\delta_0 > 0\) depending on \(s_0, r_0, \eta\) and \(\mathcal{M}\), such that, for each \(0 < \delta < \delta_0\), \(\xi \in \Pi_\eta\) and the positive integer \(\mathcal{N}\) satisfying
\[
\delta^{-\frac{\mathcal{M}+1}{\mathcal{M}}} \leq \mathcal{N} + 1 < \delta^{-\frac{\mathcal{M}+1}{\mathcal{M}}} + 1,
\]
there is a symplectic map
\[
\Phi : D(s_0/4, 4\delta, 4\delta) \to D(s_0/2, 5\delta, 5\delta),
\]
where
\[
\tilde{\mathcal{H}}(x, y, q, \tilde{q}; \xi) := \mathcal{H} \circ \Phi = \mathcal{N}(y, q, \tilde{q}; \xi) + Z(y, q, \tilde{q}; \xi) + P(x, y, q, \tilde{q}; \xi) + Q(x, y, q, \tilde{q}; \xi)
\]
is a partial normal form of order \(\mathcal{M} + 2\), where
\[
Z(y, q, \tilde{q}; \xi) = \sum_{4 \leq |\alpha| + |\beta| + 2|\mu| \leq \mathcal{M} + 2, |\mu| \leq 1} Z^{\alpha\beta\mu}(\xi) y^\alpha q^\beta \tilde{q}^\mu \xi^\alpha
\]
is the integrable term depending only on variables \(y\) and \(I_j = |q_j|^2, j \geq 1\),
\[
P(x, y, q, \tilde{q}; \xi) = \sum_{|\alpha| + |\beta| + |\gamma| + |\mu| + |\nu| \geq \mathcal{M} + 3, |\mu| + |\nu| \leq 2} P^{\alpha\beta\gamma\mu\nu}(x; \xi) y^\alpha q^\beta \tilde{q}^\gamma \xi^\mu \gamma^\nu,
\]
and
\[
Q(x, y, q, \tilde{q}; \xi) = \sum_{|\mu| + |\nu| \geq 3} Q^{\alpha\beta\gamma\mu\nu}(x; \xi) y^\alpha q^\beta \tilde{q}^\gamma \xi^\mu \gamma^\nu.
\]
Moreover, the following estimates hold:

1) the symplectic map \(\Phi\) satisfies
\[
\|\Phi - id\|_{p, D(s_0/4, 4\delta, 4\delta)} \leq \frac{cM^{-98}}{\tilde{\eta}^2}.
\]

2) the Hamiltonian vector fields \(X_Z, X_P\) and \(X_Q\) satisfy
\[
\|X_Z\|_{p, D(s_0/4, 4\delta, 4\delta) \times \Pi_\eta} \leq c\delta \left( \frac{1}{\tilde{\eta}^2} M^{2(\mathcal{M}+6)^2} \delta \right),
\]
\[
\|X_P\|_{p, D(s_0/4, 4\delta, 4\delta) \times \Pi_\eta} \leq c\delta \left( \frac{1}{\tilde{\eta}^2} M^{2(\mathcal{M}+7)^2} \delta \right)^{\mathcal{M}},
\]

and
\[
\|X_Q\|_{p, D(s_0/4, 4\delta, 4\delta) \times \Pi_\eta} \leq c\delta,
\]
where \(c > 0\) is a constant depending on \(s_0, r_0, n\) and \(\mathcal{M}\). Moreover, \(\tilde{\omega}(\xi)\) and \(\tilde{\Omega}(\xi)\) satisfy \((\tilde{\eta}, \mathcal{N}, \mathcal{M})\)-non-resonant conditions.
By a direct calculation,

$$c\delta \left( \frac{1}{\eta^2} \mathcal{M}^2(\mathcal{M} + 7)^2 \delta \right)^{\mathcal{M}}$$

$$= \delta^{\mathcal{M} + 1} \left( \frac{c \cdot \mathcal{M}^2(\mathcal{M} + 7)^2 \mathcal{M}}{\eta^{2\mathcal{M}}} \right)$$

$$\leq \delta^{\mathcal{M} + 1} \left( \frac{c \cdot \mathcal{M}^2(\mathcal{M} + 7)^2 \mathcal{M}}{\eta^{2\mathcal{M}}} \delta^{\frac{2(\mathcal{M} + 7)^2 \mathcal{M}(\mathcal{M} + 1)}{p-1}} \right)$$  \hspace{1cm} \text{(in view of the inequality (6.91))}

$$\leq \delta^{\mathcal{M} + \frac{\mathcal{M}}{p}}$$  \hspace{1cm} \text{(in view of } p \geq 8(\mathcal{M} + 7)^4 + 1)$$

$$\leq \delta^{\mathcal{M} + \frac{\mathcal{M}}{p}}$$  \hspace{1cm} \text{(by assuming } \delta \text{ is very small).} \hspace{1cm} (6.93)

In view of the inequalities (6.91) and (6.93), we have

$$|||X_P|||_{p, D(s_0/4, 4\delta, 4\delta) \times \Pi}\leq \delta^{\mathcal{M} + \frac{\mathcal{M}}{p}}. \hspace{1cm} (6.94)$$

Moreover, by the inequalities (4.51) and (6.94), we obtain

$$|||X_P|||_{\mathcal{P}, D(s_0/4, 4\delta, 4\delta) \times \Pi}\leq \delta^{\mathcal{M} + \frac{\mathcal{M}}{p}}. \hspace{1cm} (6.95)$$

On the other hand,

$$|||\tilde{z}|||_1 = \sqrt{\sum_{|j|\leq \mathcal{M} + 1} |z_j|^2 j^2}$$

$$= \sqrt{\sum_{|j|\leq \mathcal{M} + 1} |z_j|^2 j^2 p / j^{2(p-1)}}$$

$$\leq \frac{|||\tilde{z}|||_{p}}{(\mathcal{M} + 1)p^{-1}}$$

$$\leq \delta^{\mathcal{M} + 1} |||\tilde{z}|||_p$$  \hspace{1cm} \text{(in view of the inequality (6.90)).} \hspace{1cm} (6.96)

Then we have the following estimate

$$|||X_Q|||_{\mathcal{P}, D(s_0/4, 4\delta, 4\delta) \times \Pi}\leq \delta^{\mathcal{M} + \frac{\mathcal{M}}{p}}, \hspace{1cm} (6.97)$$

where we use that $Q(x, y, q, \bar{\bar{q}}; \bar{\bar{z}})$ has the form of

$$Q(x, y, q, \bar{\bar{q}}; \bar{\bar{z}}) = \sum_{|\mu| + |\nu| \geq 3} Q^{\alpha \beta \mu \nu} (x; \bar{\bar{z}}) y^\alpha \bar{\bar{q}}^\beta q^\mu \bar{\bar{z}}^\nu$$

and has $p$-tame property on the domain $D(s_0/4, 4\delta, 4\delta) \times \Pi$, and use the inequalities (6.92) and (6.96).

Assume $\tilde{w}(t)$ is a solution of Hamiltonian vector field $X_{\tilde{H}}$ with the initial datum

$$\tilde{w}(0) = (\tilde{w}_x(0), \tilde{w}_y(0), \tilde{w}_q(0), \tilde{w}_{\bar{\bar{q}}}(0)) \in D(s_0/8, 4\delta, 4\delta)$$

satisfying

$$d_p(\tilde{w}(0), \mathcal{P}_0) \leq \delta.$$
Noting \( \mathcal{R}_0 = \mathbb{T}^n \times \{ y = 0 \} \times \{ q = 0 \} \times \{ \tilde{q} = 0 \} \) and in view of (2.28) and (2.29), we have
\[
\frac{1}{4\delta} ||\tilde{w}_y(0)|| + ||\tilde{w}_q(0)|| + ||\tilde{w}_{\tilde{q}}(0)|| \leq \delta. \tag{6.98}
\]
Firstly, we will estimate \( ||\tilde{w}_q(t)|| \) and \( ||\tilde{w}_{\tilde{q}}(t)|| \). For each \((x, y, q, \tilde{q}) \in \mathscr{P}^p\), denote
\[
\bar{N}(x, y, q, \tilde{q}) := ||q||^2 = \sum_{j \geq 1} |q_j|^2 |j|^{2p}.
\]
In view of the inequality (6.98), we have \( \bar{N}(\tilde{w}(0)) \leq \delta^2 \). Define
\[
T := \inf \left\{ |t| : \bar{N}(\tilde{w}(t)) > 4\delta^2 \right\},
\]
and then \( \bar{N}(\tilde{w}(t)) \leq 4\delta^2 \) for all \(|t| \leq T\). Specially, we have
\[
\bar{N}(\tilde{w}(t)) = 4\delta^2, \quad \text{for } t = T \text{ or } t = -T. \tag{6.99}
\]
Without loss of generality, assume
\[
\bar{N}(\tilde{w}(T)) = 4\delta^2. \tag{6.100}
\]
Now we are position to show that \( T > \delta^{-\#} \). For each \((x, y, q, \tilde{q}) \in D(s_0/8, 4\delta, 4\delta)\),
\[
\left| \left\{ P, \bar{N} \right\} (x, y, q, \tilde{q}) \right| = \left| \sqrt{-1} \sum_{j \geq 1} (P_{q_j} \tilde{N}_{\tilde{q}_j} - P_{\tilde{q}_j} \tilde{N}_{q_j}) \right| \\
\leq \left( ||P_q||_p + ||P_{\tilde{q}}||_p \right) ||q||_p \\
\leq 4\delta ||X_P||_p ||q||_p \times P_{\mathbb{T}} ||q||_p \\
\leq 16\delta^{-\# + \frac{1}{2}} \quad \text{(in view of the inequality (6.98) and } ||q||_p \leq 4\delta), \tag{6.101}
\]
and
\[
\left| \left\{ Q, \bar{N} \right\} (x, y, q, \tilde{q}) \right| = \left| \sqrt{-1} \sum_{j \geq 1} (Q_{q_j} \tilde{N}_{\tilde{q}_j} - Q_{\tilde{q}_j} \tilde{N}_{q_j}) \right| \\
\leq \left( ||Q_q||_p + ||Q_{\tilde{q}}||_p \right) ||q||_p \\
\leq 4\delta ||X_Q||_p ||q||_p \times P_{\mathbb{T}} ||q||_p \\
\leq 16\delta^{-\# + \frac{1}{2}} \quad \text{(in view of the inequality (6.97) and } ||q||_p \leq 4\delta). \tag{6.102}
\]
Since $\tilde{N}(\tilde{v}(t)) \leq 4\delta^2$ for all $|t| \leq T$, then

$$
\begin{align*}
&\left| \tilde{N}(\tilde{v}(t)) - \tilde{N}(\tilde{v}(0)) \right| \\
&= \int_0^t \left\{ \tilde{H}, \tilde{N} \right\} (\tilde{v}(s)) ds \\
&= \int_0^t \left\{ P + Q, \tilde{N} \right\} (\tilde{v}(s)) ds \\
&\leq \int_0^t \left\{ P + Q, \tilde{N} \right\} (\tilde{v}(s)) ds \\
&\leq 32\delta^{2 - \frac{1}{2}}|t| \\
&\quad \text{(based on the inequalities \textcolor{red}{(6.101)} and \textcolor{red}{(6.102)})}
\end{align*}
$$

(6.103)

Assume by contradiction that $T \leq \delta^{-\frac{1}{2}}$, and then

$$
4\delta^2 = \tilde{N}(\tilde{v}(T)) \\
\leq \tilde{N}(\tilde{v}(0)) + |\tilde{N}(\tilde{v}(T)) - \tilde{N}(\tilde{v}(0))| \\
\leq \delta^2 + 32\delta^{2 - \frac{1}{2}} \delta^{-\frac{1}{2}} \\
< 2\delta^2,
$$

which is impossible.

Secondly, we will estimate $\|\tilde{Y}_j(t)\|$. For $1 \leq j \leq n$, let

$$
\tilde{Y}_j(x, y, q, \bar{q}) := y_j.
$$

Then in view of the inequality \textcolor{red}{(6.98)}, we have $|\tilde{Y}_j(\tilde{v}(0))| \leq 4\delta^2$. Define

$$
T_j := \inf \left\{ |t| : |\tilde{Y}_j(\tilde{v}(t))| > 8\delta^2 \right\},
$$

and then $|\tilde{Y}_j(\tilde{v}(t))| \leq 8\delta^2$ for all $|t| \leq T_j$. Specially, we have

$$
|\tilde{Y}_j(\tilde{v}(t))| = 8\delta^2, \quad \text{for } t = T_j \text{ or } t = -T_j.
$$

(6.104)

Without loss of generality, assume

$$
|\tilde{Y}_j(\tilde{v}(T_j))| = 8\delta^2.
$$

(6.105)

Now, we will show that $T_j > \delta^{-\frac{1}{2}}$. For each $(x, y, q, \bar{q}) \in D(s_0/8, 4\delta, 4\delta)$,

$$
\begin{align*}
\left| \left\{ P, \tilde{Y}_j \right\} (x, y, q, \bar{q}) \right| \\
&= \left| \sum_{1 \leq i \leq n} P_{qi} \tilde{Y}_{ji} \right| \\
&\leq ||P|| \\
&\leq 16\delta^2 ||X_P||_{\mathcal{D}(s_0/8, 4\delta, 4\delta) \times \Pi q} \\
&\leq 16\delta^{2 - \frac{1}{2}} (\text{in view of the inequality \textcolor{red}{(6.95)})},
\end{align*}
$$

(6.106)
\[ \left\{ Q, \tilde{Y}_j \right\} (x, y, q, \bar{q}) \]
\[ = \sum_{1 \leq i \leq n} Q_i \tilde{Y}_{j_i} \]
\[ \leq \| Q \| \]
\[ \leq 16 \delta^2 \| X_Q \| \rho^p D(s_0/8, 4\delta, 4\delta) \times \Pi_\eta \]
\[ \leq 16 \delta^{2 \#} + \frac{\delta}{2} \quad \text{(in view of the inequality (6.97))}. \quad (6.107) \]

In view of \( \tilde{Y}_j(\tilde{w}(t)) \leq 4 \delta^2 \) for all \( |t| \leq T_j \), then we have
\[ \left| \tilde{Y}_j(\tilde{w}(t)) - \tilde{Y}_j(\tilde{w}(0)) \right| \]
\[ = \int_0^t \left\{ \tilde{H}, \tilde{Y}_j \right\} (\tilde{w}(s)) ds \]
\[ = \int_0^t \left\{ P + Q, \tilde{Y}_j \right\} (\tilde{w}(s)) ds \]
\[ \leq \int_0^t \left\{ P + Q, \tilde{Y}_j \right\} (\tilde{w}(s)) ds \]
\[ \leq 32 \delta^{2 \#} + \frac{\delta}{2} |t| \quad \text{(by (6.106) and (6.107))} \]
\[ \leq 32 \delta^{2 \#} + \frac{\delta}{2} T. \quad (6.108) \]

Assume by contradiction that \( T_j \leq \delta^{-2 \#} \), and then
\[ 8 \delta^2 = \bar{Y}(\tilde{w}(T_j)) \]
\[ \leq \tilde{Y}_j(\tilde{w}(0)) + \left| \tilde{Y}_j(\tilde{w}(T_j)) - \tilde{Y}_j(\tilde{w}(0)) \right| \]
\[ \leq 4 \delta^2 + 32 \delta^{2 \#} + \frac{\delta}{2} T \]
\[ < 5 \delta^2, \]

which is impossible.

Hence, for all \( |t| \leq \delta^{-2 \#} \), we obtain
\[ \frac{1}{4\delta} \left( |\tilde{w}_y(t)| + |\tilde{w}_q(t)| \right)_p + |\tilde{w}_q(t)|_p \leq 2\delta, \]
i.e.
\[ d_p(\tilde{w}(t), \mathcal{S}_0) \leq 2\delta. \quad (6.109) \]

**Remark 11.** In view of (6.103) and (6.108), we can obtain a better estimates about the change of \( \tilde{N}(\tilde{w}(t)) \) and \( \tilde{Y}(\tilde{w}(t)) \), that is, for all \( |t| \leq \delta^{-2 \#} \),
\[ |\tilde{N}(\tilde{w}(t)) - \tilde{N}(\tilde{w}(0))|, |\tilde{Y}(\tilde{w}(t)) - \tilde{Y}(\tilde{w}(0))| \leq 32 \delta^{5/2}. \quad (6.110) \]

Based on the partial normal form (6.7) constructed in Theorem 2.10, for each \( \xi \in \Pi_\eta \) the KAM tori \( \mathcal{T} \) of the original Hamiltonian \( H(x, y, q; \bar{q}; \xi) \) can be defined by \( \mathcal{T} = (\Psi \circ \Phi)^{-1} \mathcal{S}_0. \)
Assume \(w(t)\) is a solution of original Hamiltonian vector field \(X_H\) with the initial datum \(w(0) = (w_x(0), w_y(0), w_q(0), w_{\bar{q}}(0))\) satisfying
\[
d_p(w(0), \mathcal{T}) \leq \delta.
\]

Then there exists \(w^* \in \mathcal{T}\) such that
\[
d_p(w(0), w^*) \leq \frac{9}{8} \delta. \quad (6.111)
\]

Hence,
\[
d_p(\Psi \circ \Phi \circ w(0), \mathcal{T}_0) \leq d_p(\Psi \circ \Phi \circ w(0), \Psi \circ \Phi \circ w^*) \quad \text{(in view of } \mathcal{T} = (\Psi \circ \Phi)^{-1} \mathcal{T}_0)\)
\leq d_p(\Psi \circ \Phi \circ w(0), w(0)) + d_p(w(0), w^*) + d_p(w^*, \Psi \circ \Phi \circ w^*)
\leq 4 \delta ||\Psi \circ \Phi \circ w(0) - w(0)||_{\mathcal{D}(\mathcal{T}_0/4,4\delta,4\delta)} + 9 \delta + 4 \delta ||\Psi \circ \Phi \circ w^* - w^*||_{\mathcal{D}(\mathcal{T}_0/4,4\delta,4\delta)}
\quad \text{(in view of } (2.28) \text{ and } (6.111))
\leq 4 \delta ||\Phi \circ \Psi - id||_{\mathcal{D}(\mathcal{T}_0/4,4\delta,4\delta)} (||w(0)||_{\mathcal{D}(\mathcal{T}_0/4,4\delta,4\delta)} + ||w^*||_{\mathcal{D}(\mathcal{T}_0/4,4\delta,4\delta)}) + \frac{9}{8} \delta
\leq \frac{4}{3} \delta, \quad (6.112)
\]
where the last inequality is based on the inequality (2.21) and (6.8) which imply \(\Phi \circ \Psi\) is close to an identity map. Hence, based on the estimate (6.109), Remark 11 and noting \(\tilde{w}(t) = \Psi \circ \Phi \circ w(t)\), we obtain
\[
d_p(\Psi \circ \Phi \circ w(t), \mathcal{T}_0) \leq \frac{5}{3} \delta, \quad \text{for all } |t| \leq \delta^{-\mu}. \quad (6.113)
\]

Moreover, we have
\[
d_p(w(t), (\Psi \circ \Phi)^{-1} \mathcal{T}_0) \leq 2 \delta, \quad \text{for all } |t| \leq \delta^{-\mu},
\]
which follows form the proof of (6.112).

7 Proof of Theorem 1.1

Proof. Consider the nonlinear Schrödinger equation (1.2)
\[
iu_t = u_{xx} - M_\xi u + \epsilon |u|^2 u
\]
subject to Dirichlet boundary conditions \(u(t, 0) = u(t, \pi) = 0\).

Step 1. Rewrite equation (1.2) as a Hamiltonian.

The eigenvalues and eigenfunctions of \(L_{M_\xi} = -\partial_{xx} + M_\xi\) with Dirichlet boundary conditions are \(\lambda_j = j^2 + \xi_j\) and \(\phi_j(x) = \sqrt{2/\pi} \sin jx\), respectively. Write
\[
u(t,x) = \sum_{j \geq 1} w_j(t) \phi_j(x),
\]
and then the Hamiltonian takes the form \(H(w, \tilde{w}) = H_0(w, \tilde{w}) + \epsilon P(w, \tilde{w})\), where
\[
H_0(w, \tilde{w}) = \sum_{j \geq 1} \lambda_j w_j \tilde{w}_j
\]
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Remark 12. It is easy to verify that $P_{ijkl} = 0$ unless $i \pm j \pm k \pm l = 0$, for some combination of plus and minus signs. Thus, only a codimension-one set of coefficients is actually different from zero, and the sum extends only over $i \pm j \pm k \pm l = 0$. In particular,

$$P_{ijij} = \frac{1}{2\pi} (2 + \delta_{ij}).$$

In view of example 3.2 in [4], it is proven that there exists a constant $c_p > 0$ such that

$$||X_p(z^{(1)}, z^{(2)}, z^{(3)})||_p \leq c_p ||(z^3)\||_{p,1}, \quad z = (w, \tilde{w}).$$

(7.1)

In particular, when $p = 1$, the inequality (7.1) reads

$$||X_p(z^{(1)}, z^{(2)}, z^{(3)})||_1 \leq c_1 ||(z^3)\||_{1,1}.$$

(7.2)

The inequalities (7.1) and (7.2) shows that the Hamiltonian vector filed $X_p(z)$ has $p$-tame property.

Step 2. Introduce the action-angle variables.

Now we choose any finite $n$ of modes $\phi_{j_1}, \phi_{j_2}, \ldots, \phi_{j_n}$ as tangent direction and the other as normal direction. Let

$$\tilde{w} = (w_{j_1}, \ldots, w_{j_n}) \quad \text{and} \quad q = (w_j)_{j \notin \{j_1, \ldots, j_n\}}$$

be the tangent variable and normal variable, respectively. Then rewrite $P(w, \tilde{w})$ in the multiple-index as

$$P(w, \tilde{w}) = \sum_{|\mu|+|\nu|+|\beta|=4} p^{\mu\nu\beta} \tilde{w}^\mu q^\nu z^\beta, \quad \mu, \nu \in \mathbb{N}^n, \beta \in \mathbb{N}^n,$$

(7.4)

where $z = (q, \bar{q})$ and $p^{\mu\nu\beta} = P_{ijkl}$ for some corresponding $i, j, k, l$. In the tangent direction, introduce the action-angle variables

$$w_{j_i} = \sqrt{2(\zeta_i + y_i)} e^{\sqrt{-1}T_{j_i}}, \quad \tilde{w}_{j_i} = \sqrt{2(\zeta_i + y_i)} e^{-\sqrt{-1}T_{j_i}}, \quad j_i \in \{j_1, \ldots, j_n\},$$

(7.5)

where $\zeta_i \in [1, 2]$ is the initial datum and will be considered as a constant. The symplectic structure is $dy \wedge dx + \sqrt{-1}dq \wedge d\bar{q}$, where $y = (y_1, \ldots, y_n)$ and $x = (x_1, \ldots, x_n)$. Hence, (7.4) is turned into

$$P(w, \tilde{w}) = P(x, y, z)$$

$$= \sum_{|\mu|+|\nu|+|\beta|=4} 2^{\frac{1}{2}(|\mu|+|\nu|)} p^{\mu\nu\beta} \sqrt{(\zeta + y)^\mu} \sqrt{(\zeta + y)^\nu} e^{\sqrt{-1}T(x)} z^\beta$$

$$= \sum_{|\mu|+|\nu|+|\beta|=4} 2^{\frac{1}{2}(|\mu|+|\nu|)} p^{\mu\nu\beta} e^{\sqrt{-1}T(\mu-v, x)} \sqrt{(\zeta + y)^\mu v^\nu z^\beta}$$

$$= \sum_{|\beta| \leq 4} p^{\beta} (x, y) z^\beta,$$

(7.6)
where
\[ \zeta = (\zeta_1, \ldots, \zeta_n) \] and 
\[ p^\beta(x, y) = \sum_{|\mu|+|\nu|=4-|\beta|} 2^{\frac{1}{2}(|\mu|+|\nu|)} p_{\mu \nu \beta} e^{\sqrt{-1}(\mu-y)} \sqrt{(\zeta+y)^{\mu-\nu}}. \]

**Step 3. Show that the Hamiltonian vector field \( X_P \) has \( p \)-tame property.**

In this step, we will show that the Hamiltonian vector field \( X_P(x, y, z) \) has \( p \)-tame property after introducing the action-angle variables. Assume \( P(x, y, z) = \sum_{|\beta|\leq 4} p^\beta(x, y)z^\beta \) is defined on the domain \((x, y, z) \in D(s_0, r_0, 0)\) for some \( 0 < s_0, r_0 \leq 1 \).

**Step 3.1. Estimate \( ||P||_{D(s_0, r_0, 0)} \times \mathbb{N} \).**

For \( 1 \leq i \leq n \),
\[ P_{w_i}(x, y, z) = P_{w_i}(w, \tilde{w}) = \sum_{|\mu|+|\nu|=4} \mu_i p_{\mu \nu \beta} \tilde{w}^\mu - 1_i \tilde{\nu}^\nu z^\beta, \] (7.7)

In view of (7.4) and (7.6), we have
\[ P_{w_i}(x, y, z) = P_{w_i}(w, \tilde{w}) = \sum_{|\mu|+|\nu|=4} \mu_i p_{\mu \nu \beta} \tilde{w}^{\mu - 1_i} \tilde{\nu}^\nu z^\beta, \] (7.8)

where
\[ \mu - 1_i = (\mu_1, \ldots, \mu_i - 1, \ldots, \mu_n). \]

Denote
\[ p^\beta_{w_i}(w, \tilde{w}) = \sum_{|\mu|+|\nu|=4-|\beta|} \mu_i p_{\mu \nu \beta} \tilde{w}^{\mu - 1_i} \tilde{\nu}^\nu. \]

Then in view of (7.8), we have
\[ P_{w_i}(x, y, z) = P_{w_i}(w, \tilde{w}) = p_{w_i}^\beta(\tilde{w}, \tilde{w})z^\beta. \] (7.9)

Furthermore, let
\[ p^\beta_{w_i}(w, \tilde{w}) = \sum_{|\mu|+|\nu|=4-|\beta|} \nu_i p_{\mu \nu \beta} \tilde{w}^{\mu - 1_i} \tilde{\nu}^\nu, \]

where \( \nu - 1_i = (\nu_1, \ldots, \nu_i - 1, \ldots, \nu_n) \), and then
\[ P_{w_i}(x, y, z) = P_{w_i}(w, \tilde{w}) = p_{w_i}^\beta(\tilde{w}, \tilde{w})z^\beta. \] (7.10)

Denote
\[ p^\beta_{w_i}(x, y) = p_{w_i}^\beta(\tilde{w}, \tilde{w})(-\sqrt{-1} \sqrt{\zeta_i + y_i e^{-\sqrt{-1}x_i}} + \sqrt{-1} \sqrt{\zeta_i + y_i e^{-\sqrt{-1}x_i}}). \] (7.11)

In view of the formulas (7.2) and (7.9)–(7.11), we have
\[ P_{w_i}(x, y, z) = \sum_{|\beta|\leq 4} p_{w_i}^\beta(x, y)z^\beta, \quad \beta \in \mathbb{N}^n. \] (7.12)

In view of the inequality (7.2) and for \( 1 \leq i \leq n \), we have
\[ \left| p_{w_i}^\beta(\tilde{z}^{(1)}, \tilde{z}^{(2)}, \tilde{z}^{(3)}) \right|, \left| \tilde{p}_{w_i}^\beta(\tilde{z}^{(1)}, \tilde{z}^{(2)}, \tilde{z}^{(3)}) \right| \leq c_1(\|z^{(1)}\|_1 \|z^{(2)}\|_1 \|z^{(3)}\|_1). \] (7.13)
In view of (7.12), based on the inequality (7.13) and the definition of $[\cdot]_{\mathcal{D}(s_0,r_0)\times\Pi}$ (see (2.3) for the details), we obtain
\[
[P_i]_{\mathcal{D}(s_0,r_0)\times\Pi}(z^{(1)},z^{(2)},z^{(3)}) \\
\leq c\sum_{1 \leq i \leq 3} ||z^{(i)}||_1 + \sum_{1 \leq i \neq j \leq 3} ||z^{(i)}||_1 ||z^{(j)}||_1 + ||z^{(1)}||_1 ||z^{(2)}||_1 ||z^{(3)}||_1,
\]
(7.14)
where $c > 0$ is a constant depending on $s_0, r_0$ and $n$. Based on the inequality (7.14) for $1 \leq i \leq n$ and Definition 2.5,
\[
|||P_i|||_{\mathcal{D}(s_0,r_0)\times\Pi} < \infty.
\]
(7.15)
Following the proof of the inequality (7.15), we obtain
\[
|||P_i|||_{\mathcal{D}(s_0,r_0)\times\Pi} < \infty.
\]
(7.16)
Now, we would like to show
\[
|||P_i|||_{P,D\mathcal{D}(s_0,r_0)\times\Pi} < \infty.
\]
In view of (7.4),
\[
P_x(x,y,z) = P_x(w,\tilde{w}) = \sum_{|\mu|+|\nu|+|\beta|=4} \beta \tilde{\nu}^{\mu} \tilde{w}^{\nu} z^{\beta-1}.
\]
(7.17)
Let
\[
P_x^{\beta-1}(x,y) = P_x^{\beta-1}(w,\tilde{w}) = \sum_{|\mu|+|\nu|=-|\beta|} \beta \tilde{\nu}^{\mu} \tilde{w}^{\nu},
\]
(7.18)
and then
\[
P_x(x,y,z) = \sum_{|\beta|\leq 4} P_x^{\beta-1}(x,y)z^{\beta-1}.
\]
(7.19)
Hence, we obtain
\[
|||\tilde{P}_x|||_{\mathcal{D}(s_0,r_0)\times\Pi}||p \leq c\sum_{1 \leq i \leq 3} ||z^{(i)}||_p + \sum_{1 \leq i \neq j \leq 3} ||z^{(i)}||_1 ||z^{(j)}||_p + ||z^{(3)}||_p1,
\]
(7.20)
where $c > 0$ is a constant depending on $s_0, r_0, n$ and $p$, and the above inequality is based on the inequality (7.1) and the definition of $[\cdot]_{\mathcal{D}(s_0,r_0)\times\Pi}$ (see (2.3) for the details), and noting that $\tilde{z} = (\tilde{w}, q, \tilde{w}, \tilde{q})$ and $z = (q, \tilde{q})$. In particular, when $p = 1$, the inequality (7.20) reads
\[
|||\tilde{P}_x|||_{\mathcal{D}(s_0,r_0)\times\Pi}||1 \leq \tilde{c}\sum_{1 \leq i \leq 3} ||z^{(i)}||_1 + \sum_{1 \leq i \neq j \leq 3} ||z^{(i)}||_1 ||z^{(j)}||_1 + ||z^{(3)}||_11,
\]
(7.21)
where $\tilde{c} > 0$ is a constant depending on $s_0, r_0$ and $n$. Based on the inequalities (7.20) and (7.21), we obtain
\[
|||P_i|||_{P,D\mathcal{D}(s_0,r_0)\times\Pi} < \infty.
\]
(7.22)
Finally, we obtain a Hamiltonian $H(x, y, q, \tilde{q}; \xi)$ having the following form

$$H(x, y, q, \tilde{q}; \xi) = N(x, y, q, \tilde{q}; \xi) + R(x, y, q, \tilde{q}; \xi),$$

(7.24)

where

$$N(x, y, q, \tilde{q}; \xi) = H_0(w, \tilde{w}) = \sum_{1 \leq j \leq n} \omega_j(\xi)y_j + \sum_{j \geq 1} \Omega_j(\xi)q_j \tilde{q}_j,$$

(7.25)

with the tangent frequency

$$\omega(\xi) = (\omega_1(\xi), \ldots, \omega_n(\xi)) = (f_1^2 + \xi_1, \ldots, f_n^2 + \xi_n),$$

(7.26)

and the normal frequency

$$\Omega(\xi) = (\Omega_1(\xi), \Omega_2(\xi), \ldots), \quad \Omega_j(\xi) = f_j^2 + \xi_j, \quad j \notin \{j_1, \ldots, j_n\},$$

(7.27)

and the perturbation

$$R(x, y, q, \tilde{q}; \xi) = \epsilon P(x, y, q, \tilde{q})$$

(7.28)

is independent of parameters $\xi$.

In view of the formulas (7.26) and (7.27), it is easy to show that Assumption A and Assumption B in Theorem 2.8 hold. Basing on the inequality (7.23) and noting $R(x, y, q, \tilde{q}; \xi) = \epsilon P(x, y, q, \tilde{q})$ (see (7.28)), we obtain $||X_{\epsilon}||_{L_p(D(s_0, r_0) \times \Pi)}$ satisfies the small assumption. Hence, all assumptions in Theorem 2.8 hold. According to Theorem 2.8 there exists a subset $\Pi_\eta \subset \Pi$ with the estimate

$$\text{Meas } \Pi_\eta \geq (\text{Meas } \Pi)(1 - O(\eta)).$$

For each $\xi \in \Pi_\eta$, there is a symplectic map

$$\Psi : D(s_0/2, r_0/2, r_0/2) \to D(s_0, r_0, r_0),$$

such that $H(x, y, q, \tilde{q}; \xi)$ can be transformed into a normal form of order 2 with the following form

$$H(x, y, q, \tilde{q}; \xi) = H \circ \Psi = \tilde{N}(y, q, \tilde{q}; \xi) + \tilde{R}(x, y, q, \tilde{q}; \xi),$$

where

$$\tilde{N}(y, q, \tilde{q}; \xi) = \sum_{j=1}^n \tilde{\omega}_j(\xi)y_j + \sum_{j \geq 1} \tilde{\Omega}_j(\xi)q_j \tilde{q}_j$$

and

$$\tilde{R}(x, y, q, \tilde{q}; \xi) = \sum_{\alpha, \beta, \gamma \in \mathbb{N}^3, |\alpha| + |\beta| + |\gamma| \geq 3} \tilde{R}^{\alpha, \beta, \gamma}(x, \xi)q^{\alpha} \tilde{q}^\beta.$$
where \( \phi_j(x) = \sqrt{2/\pi} \sin jx \) and
\[
\hat{v}(j) = \sqrt{\frac{2}{\pi}} \int_0^\pi v(x) \sin jx \, dx
\]
is the \( j \)-th Fourier coefficient of \( v(x) \).

Furthermore, for any solution \( u(t,x) \) of equation (1.2) with the initial datum satisfying
\[
d_{H_0^0([0,\pi])}(u(0,x), \mathcal{T}_\xi) \leq \delta,
\]
then
\[
d_{H_0^0([0,\pi])}(u(t,x), \mathcal{T}_\xi) \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-m}.
\]

8 Appendix: technical lemmas

Lemma 8.1. Consider two functions \( U(x; \xi) \) and \( V(x; \xi) \) defined on the domain \( D(s) \times \Pi \), which are analytic about the variable \( x \in D(s) \) and \( C^1 \)-smooth about the parameter \( \xi \in \Pi \), then the following inequality holds
\[
||UV||_{D(s) \times \Pi} \leq ||U||_{D(s) \times \Pi} ||V||_{D(s) \times \Pi}.
\]

Proof. Let \( W(x; \xi) = U(x; \xi)V(x; \xi) \) with its Fourier series \( W(x; \xi) = \sum_{k \in \mathbb{Z}^n} \hat{W}(k; \xi) e^{i\langle k, x \rangle} \). By a direct calculation,
\[
\hat{W}(k; \xi) = \sum_{k_1 + k_2 = k} \hat{U}(k_1; \xi) \hat{V}(k_2; \xi),
\]
and
\[
\partial_j \hat{W}(k; \xi) = \sum_{k_1 + k_2 = k} \left( \partial_j \hat{U}(k_1; \xi) \hat{V}(k_2; \xi) + \hat{U}(k_1; \xi) \partial_j \hat{V}(k_2; \xi) \right),
\]
where \( \hat{U}(k; \xi) \) and \( \hat{V}(k; \xi) \) are \( k \)-th Fourier coefficients of \( U(x; \xi) \) and \( V(x; \xi) \), respectively. Hence,
\[
\sum_{k \in \mathbb{Z}^n} |\hat{W}(k; \xi)| e^{i|k|s} = \sum_{k \in \mathbb{Z}^n} \left( \sum_{k_1 + k_2 = k} |\hat{U}(k_1; \xi) \hat{V}(k_2; \xi)| e^{i|k_1|s + |k_2|s} \right) \leq \sum_{k_1 \in \mathbb{Z}^n} \sum_{k_2 \in \mathbb{Z}^n} |\hat{U}(k_1; \xi)| |\hat{V}(k_2; \xi)| e^{i(|k_1| + |k_2|)s} = \left( \sum_{k_1 \in \mathbb{Z}^n} |\hat{U}(k_1; \xi)| e^{i|k_1|s} \right) \left( \sum_{k_2 \in \mathbb{Z}^n} |\hat{V}(k_2; \xi)| e^{i|k_2|s} \right).
\]
and

\[
\sum_{k \in \mathbb{Z}^n} |\partial_{\xi_j} \hat{W}(k; \xi)| e^{|k|s} = \sum_{k \in \mathbb{Z}^n} \left| \sum_{k_1 + k_2 = k} \partial_{\xi_j} \hat{U}(k_1; \xi) \hat{V}(k_2; \xi) + \hat{U}(k_1; \xi) \partial_{\xi_j} \hat{V}(k_2; \xi) \right| e^{|k_1 + k_2|s} 
\]

\[
\leq \sum_{k \in \mathbb{Z}^n} \sum_{k_1 + k_2 = k} \left( |\partial_{\xi_j} \hat{U}(k_1; \xi)||\hat{V}(k_2; \xi)| + |\hat{U}(k_1; \xi)||\partial_{\xi_j} \hat{V}(k_2; \xi)| \right) e^{(|k_1| + |k_2|)s} 
\]

\[
= \left( \sum_{k_1 \in \mathbb{Z}^n} |\partial_{\xi_j} \hat{U}(k_1; \xi)| e^{k_1s} \right) \left( \sum_{k_2 \in \mathbb{Z}^n} |\hat{V}(k_2; \xi)| e^{k_2s} \right) 
\]

\[
+ \left( \sum_{k_1 \in \mathbb{Z}^n} |\partial_{\xi_j} \hat{U}(k_1; \xi)| e^{k_1s} \right) \left( \sum_{k_2 \in \mathbb{Z}^n} |\hat{V}(k_2; \xi)| e^{k_2s} \right) 
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}^n} (|\hat{U}(k; \xi)| + |\partial_{\xi_j} \hat{U}(k; \xi)|) e^{k_s} \right) \left( \sum_{k \in \mathbb{Z}^n} (|\hat{V}(k; \xi)| + |\partial_{\xi_j} \hat{V}(k; \xi)|) e^{k|s|} \right) 
\]

Therefore,

\[
\sum_{k \in \mathbb{Z}^n} (|\hat{W}(k; \xi)| + |\partial_{\xi_j} \hat{W}(k; \xi)|) e^{k|s|} 
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}^n} (|\hat{U}(k; \xi)| + |\partial_{\xi_j} \hat{U}(k; \xi)|) e^{k|s|} \right) \left( \sum_{k \in \mathbb{Z}^n} (|\hat{V}(k; \xi)| + |\partial_{\xi_j} \hat{V}(k; \xi)|) e^{k|s|} \right) 
\]

Thus,

\[
||W||_{D(s) \times \Pi} = \sup_{\xi \in \Pi, j \geq 1} \sum_{k \in \mathbb{Z}^n} (|\hat{W}(k; \xi)| + |\partial_{\xi_j} \hat{W}(k; \xi)|) e^{k|s|} 
\]

\[
\leq \sup_{\xi \in \Pi, j \geq 1} \left( \sum_{k \in \mathbb{Z}^n} (|\hat{U}(k; \xi)| + |\partial_{\xi_j} \hat{U}(k; \xi)|) e^{k|s|} \right) \left( \sum_{k \in \mathbb{Z}^n} (|\hat{V}(k; \xi)| + |\partial_{\xi_j} \hat{V}(k; \xi)|) e^{k|s|} \right) 
\]

\[
\leq \sup_{\xi \in \Pi, j \geq 1} \left( \sum_{k \in \mathbb{Z}^n} (|\hat{U}(k; \xi)| + |\partial_{\xi_j} \hat{U}(k; \xi)|) e^{k|s|} \right) \sum_{k \in \mathbb{Z}^n} (|\hat{V}(k; \xi)| + |\partial_{\xi_j} \hat{V}(k; \xi)|) e^{k|s|} 
\]

\[
= ||U||_{D(s) \times \Pi} ||V||_{D(s) \times \Pi}. 
\]

\[\Box\]

**Lemma 8.2.** Consider two functions \( U(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} U^\alpha(x; \xi) y^\alpha \) and \( V(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} V^\alpha(x; \xi) y^\alpha \)

defined on the domain \( D(s, r) \times \Pi \), which are analytic about the variable \((x, y) \in D(s, r)\) and \(C^1\)-smooth about the parameter \( \xi \in \Pi \), then the following inequality holds

\[ ||UV||_{D(s, r) \times \Pi} \leq ||U||_{D(s, r) \times \Pi} ||V||_{D(s, r) \times \Pi}. \]
Proof. Let \( W(x, y; \xi) = U(x, y; \xi) V(x, y; \xi) \) with its Taylor series about the variable \( y \)

\[
W(x, y; \xi) = \sum_{\beta \in \mathbb{N}^n} W^\beta (x; \xi) y^\beta.
\]

By a direct calculation,

\[
W^\beta (x; \xi) = \sum_{\alpha + \alpha' = \beta} U^\alpha (x; \xi) V^{\alpha'} (x; \xi).
\]

Hence, by the definition of \( || \cdot ||_{D(s, r) \times \Pi} \),

\[
||W||_{D(s, r) \times \Pi} = \sum_{\beta \in \mathbb{N}^n} ||W^\beta||_{D(s) \times \Pi} 2^{\beta}
\]

\[
= \sum_{\beta \in \mathbb{N}^n} \left( \sum_{\alpha + \alpha' = \beta} ||U^\alpha V^{\alpha'}||_{D(s) \times \Pi} 2^{\alpha + \alpha'} \right)\]

\[
\leq \sum_{\beta \in \mathbb{N}^n} \left( \sum_{\alpha + \alpha' = \beta} ||U^\alpha V^{\alpha'}||_{D(s) \times \Pi} 2^{\alpha + \alpha'} \right)\]

\[
= \left( \sum_{\alpha \in \mathbb{N}^n} ||U^\alpha||_{D(s) \times \Pi} 2^{\alpha} \right) \left( \sum_{\alpha' \in \mathbb{N}^n} ||V^{\alpha'}||_{D(s) \times \Pi} 2^{\alpha'} \right)\]

\[
= ||U||_{D(s, r) \times \Pi} ||V||_{D(s, r) \times \Pi}.
\]

Now we will give two generalized Cauchy estimates:

**Lemma 8.3.** Consider a function \( W(x; \xi) = \sum_{k \in \mathbb{Z}^n} \hat{W}(k; \xi) e^{\sqrt{-1}k \cdot x} \) defined on the domain \( D(s) \times \Pi \), which is analytic about the variable \( x \in D(s) \) and \( C^1 \)-smooth about the parameter \( \xi \in \Pi \), then the following generalized Cauchy estimate holds

\[
||W||_{D(s, r) \times \Pi} \leq \frac{1}{e^\sigma ||W||_{D(s) \times \Pi}}.
\]

where \( 0 < \sigma < s \).

**Proof.** By a direct calculation,

\[
W_k (x; \xi) = \sum_{k \in \mathbb{Z}^n} \sqrt{-1}k_j \hat{W}(k; \xi) e^{\sqrt{-1}k \cdot x}.
\]

Hence,

\[
\sum_{k \in \mathbb{Z}^n} |k_j| (|\hat{W}(k; \xi)| + |\partial_{\xi_j} \hat{W}(k; \xi)|) e^{k_j (s-\sigma)}
\]

\[
\leq \sum_{k \in \mathbb{Z}^n} (|\hat{W}(k; \xi)| + |\partial_{\xi_j} \hat{W}(k; \xi)|) e^{k_j |x|} e^{-|k| |\sigma|}
\]

\[
\leq \left( \sup_{k \in \mathbb{Z}^n} k e^{-|k| \sigma} \right) \left( \sum_{k \in \mathbb{Z}^n} (|\hat{W}(k; \xi)| + |\partial_{\xi_j} \hat{W}(k; \xi)|) e^{k_j |x|} \right)
\]

\[
\leq \frac{1}{e^\sigma} \left( \sum_{k \in \mathbb{Z}^n} (|\hat{W}(k; \xi)| + |\partial_{\xi_j} \hat{W}(k; \xi)|) e^{k_j |x|} \right).
\]

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Hence,

\[ ||W_{x_j}||_{D(s-\sigma) \times \Pi} \]
\[ = \sup_{\xi \in \Pi, j \geq 1} \left( \sum_{k \in \mathbb{Z}^n} |k_j| (|\hat{W}(k; \xi)| + |\partial_{k_j} \hat{W}(k; \xi)|) e^{k |(s-\sigma)} \right) \]
\[ \leq \sup_{\xi \in \Pi, j \geq 1} \left( \frac{1}{e^{\sigma}} \left( \sum_{k \in \mathbb{Z}^n} (|\hat{W}(k; \xi)| + |\partial_{k_j} \hat{W}(k; \xi)|) e^{k |} \right) \right) \]
\[ = \frac{1}{e^{\sigma}} ||W||_{D(s) \times \Pi}. \]

Moreover,

\[ ||W_{y_i}||_{D(s-\sigma) \times \Pi} = \sup_{1 \leq j \leq n} ||W_{y_j}||_{D(s-\sigma) \times \Pi} \leq \frac{1}{e^{\sigma}} ||W||_{D(s) \times \Pi}. \]

\[ \square \]

**Lemma 8.4.** Consider a function \( W(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} W^\alpha(x; \xi) y^\alpha \) defined on the domain \( D(s, r) \times \Pi \), which is analytic about the variable \((x, y) \in D(s, r)\) and \( C^1 \)-smooth about the parameter \( \xi \in \Pi \), then the following generalized Cauchy estimate holds

\[ ||W_{x_j}||_{D(s-\sigma, r-\sigma') \times \Pi} \leq \frac{1}{r^{\sigma'}} ||W||_{D(s, r) \times \Pi}, \]

(8.1)

and

\[ ||W_{y_i}||_{D(s, r-\sigma) \times \Pi} \leq \frac{1}{r^{\sigma'}} ||W||_{D(s, r) \times \Pi}, \]

(8.2)

where \( 0 < \sigma < s \) and \( 0 < \sigma' < r/2 \).

**Proof.** The inequality (8.1) can be obtained directly by Lemma 8.3. Now, we will prove the inequality (8.2). By a direct calculation,

\[ W_{y_j}(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} \alpha_j W^\alpha(x; \xi) y^{\alpha-1}, \]

where

\[ \alpha - 1_j = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_n). \]

Hence,

\[ ||W_{y_j}||_{D(s, r-\sigma') \times \Pi} \]
\[ = \sum_{\alpha \in \mathbb{N}^n} ||\alpha_j W^\alpha||_{D(s, r-\sigma') \times \Pi} (r-\sigma')^{2(|\alpha|-1)} \]
\[ \leq \sum_{\alpha \in \mathbb{N}^n} ||\alpha|| ||W^\alpha||_{D(s, r-\sigma') \times \Pi} (r-\sigma')^{2(|\alpha|-1)} \]
\[ = \frac{1}{2(r-\sigma')} \sum_{\alpha \in \mathbb{N}^n} ||W^\alpha||_{D(s, r) \times \Pi} 2|\alpha|(r-\sigma')^{2|\alpha|-1} \]
\[ \leq \frac{1}{2(r-\sigma')} \sum_{\alpha \in \mathbb{N}^n} ||W^\alpha||_{D(s, r) \times \Pi} r^{2|\alpha|}/r^{\sigma'} \quad \text{(using the inequality } k(r-\sigma')^{k-1} \leq \frac{\lambda^k}{\sigma'}) \]
\[ \leq \frac{1}{r^{\sigma'}} \sum_{\alpha \in \mathbb{N}^n} ||W^\alpha||_{D(s, r) \times \Pi} 2|\alpha| \quad \text{(based on } 0 < \sigma' < r/2) \]
\[ = \frac{1}{r^{\sigma'}} ||W||_{D(s, r) \times \Pi}. \]
Therefore,
\[
\|W_j\|_{D(s,r-\sigma') \times \Pi} = \sup_{1 \leq j \leq n} \|W_j\|_{D(s,r-\sigma') \times \Pi} \leq \frac{1}{\|\sigma\|} \|W\|_{D(s,r) \times \Pi}.
\]

Assume \( F(z) \) is a homogeneous polynomial with the corresponding multi-linear form \( \widetilde{F}(z^{(1)}, \ldots, z^{(h)}) \).

Then the following lemma holds:

**Lemma 8.5.** Consider two homogeneous polynomials \( F(z) = \sum_{|\beta| = h_1} F^\beta z^\beta \) and \( G(z) = \sum_{|\beta'| = h_2} G^{\beta'} z^{\beta'} \).

Then
\[
\widetilde{F}G(z^{(1)}, \ldots, z^{(h)}) = \frac{1}{h!} \sum_{\tau_0} \widetilde{F}(z^{(\tau_0(1))}, \ldots, z^{(\tau_0(h_1))}) \widetilde{G}(z^{(\tau_0(h_1+1))}, \ldots, z^{(\tau_0(h))}),
\]
where \( h = h_1 + h_2 \) and \( \tau_0 \) is an \( h \)-permutation.

**Proof.** It suffices to consider \( F(z) \) and \( G(z) \) as two monomials with the following forms
\[
F(z) = F^\beta z^\beta = F^\beta z_{j_1} \cdots z_{j_{h_1}} \quad \text{and} \quad G(z) = G^{\beta'} z^{\beta'} = G^{\beta'} z_{i_1} \cdots z_{i_{h_2}}.
\]

Then
\[
W(z) = F(z)G(z) = F^\beta G^{\beta'} z_{j_1} \cdots z_{j_{h_1}} z_{i_1} \cdots z_{i_{h_2}} = F^\beta G^{\beta'} z_{k_1} \cdots z_{k_h},
\]
where \((k_1, \ldots, k_h) = (j_1, \ldots, j_{h_1}, i_1, \ldots, i_{h_2})\). Hence
\[
\frac{1}{h!} \sum_{\tau_0} \widetilde{F}(z^{(\tau_0(1))}, \ldots, z^{(\tau_0(h_1))}) \widetilde{G}(z^{(\tau_0(h_1+1))}, \ldots, z^{(\tau_0(h))})
= \frac{1}{h!} \sum_{\tau_0} \left( \sum_{j_1} F^\beta z^{(\tau_0(1))} \cdots z^{(\tau_0(h_1))} \right) \left( \sum_{i_1} G^{\beta'} z^{(\tau_0(h_1+1))} \cdots z^{(\tau_0(h))} \right)
= \frac{F^\beta G^{\beta'}}{h!h_1h_2!} \sum_{\tau_0} \sum_{j_1, k_1} \sum_{i_1, k_1} \sum_{j_2, k_2} \cdots \sum_{i_2, k_2} \quad \text{where} \quad \tau'_0 = (\tau_0(1), \ldots, \tau_0(h)) \circ \tau
= \frac{1}{h!h_1h_2!} \sum_{\tau_0, \tau_1, \tau_2} \frac{F^\beta G^{\beta'}}{h!} \sum_{j_1} \sum_{k_1} \sum_{j_2} \sum_{k_2} \cdots \sum_{i_2} \sum_{k_2} \quad \text{where} \quad \tau'_0 = (\tau_0(1), \ldots, \tau_0(h)) \circ \tau
= \frac{1}{h!h_1h_2!} \sum_{\tau_0, \tau_1, \tau_2} \widetilde{W}(z^{(1)}, \ldots, z^{(h)})
= \widetilde{W}(z^{(1)}, \ldots, z^{(h)}).
\]

\(\square\)

**Remark 13.** Usually, the permutation \( \tau_{h_1} \) is defined on the set \( \{1, \ldots, h_1\} \). However, \( \tau_{h_1}(1) \) may be larger than \( h_1 \), since \( \tau_{h_1} \) is an \( h = h_1 + h_2 \) permutation. In fact, we define \( \tau_{h_1} \) on the set \( \{\tau_0(1), \ldots, \tau_0(h_1)\} \) as follows. Assume \( \tau_0(l_1) < \cdots < \tau_0(l_i) < \tau_0(l_{i+1}) < \cdots < \tau_0(l_{h_1}) \) for \( \{l_1, \ldots, l_{h_1}\} = \{1, \ldots, h_1\} \).

Then define the map \( \wedge : \{\tau_0(1), \ldots, \tau_0(h_1)\} \rightarrow \{1, \ldots, h_1\} \) by \( \wedge(\tau_0(i)) = j \), if \( i = l_j \), and the inverse of \( \wedge \) by \( \wedge^{-1}(j) = \tau_0(i) \). Hence, we have \( \tau_{h_1} \circ \tau_0(i) := \wedge^{-1} \circ \tau_0 \circ \wedge(\tau_0(i)) \).
Lemma 8.6.

\[ f(z) := \frac{1}{h！} \frac{1}{h-1} \sum_{k=1}^{h-1} \sum_{j=1}^{h-1-1} \left| \sum_{k=1}^{h-1} \left| z_{\tau_{a_{1}}(1)} \right| \cdots \left| z_{\tau_{a_{j-1}}(1)} \right| \left| z_{\tau_{a_{j}}(1)} \right| \left| z_{\tau_{a_{j+1}}(1)} \right| \cdots \left| z_{\tau_{a_{(h-1)}}(1)} \right| \right|_{1} \]

\[ = \left| \left( z_{\tau_{a_{1}}} \right) \right|_{p,1}. \]

**Proof.** By a direct calculation,

\[ \tilde{m}! (h-1) f(z) \]

\[ = \sum_{j=1}^{h-1} \left( \sum_{\tau_{a_{j}}=j_{0}}^{h-1} \sum_{j=1}^{h-1-1} \left| z_{\tau_{a_{1}}(1)} \right| \cdots \left| z_{\tau_{a_{j-1}}(1)} \right| \left| z_{\tau_{a_{j}}(1)} \right| \left| z_{\tau_{a_{j+1}}(1)} \right| \cdots \left| z_{\tau_{a_{(h-1)}}(1)} \right| \right|_{1} \]

\[ = \sum_{j=1}^{h-1} \left( \sum_{\tau_{a_{j}}=j_{0}}^{h-1} \sum_{j=1}^{h-1-1} \left| z_{\tau_{a_{1}}(1)} \right| \cdots \left| z_{\tau_{a_{j-1}}(1)} \right| \left| z_{\tau_{a_{j}}(1)} \right| \left| z_{\tau_{a_{j+1}}(1)} \right| \cdots \left| z_{\tau_{a_{(h-1)}}(1)} \right| \right|_{1} \]

\[ = \sum_{j=1}^{h-1} \left( \tilde{m}! \sum_{j_{0}=1}^{h-1} \left| z_{\tau_{a_{j}}(1)} \right| \left| z_{\tau_{a_{1}}(1)} \right| \cdots \left| z_{\tau_{a_{j-1}}(1)} \right| \left| z_{\tau_{a_{j}}(1)} \right| \left| z_{\tau_{a_{j+1}}(1)} \right| \cdots \left| z_{\tau_{a_{(h-1)}}(1)} \right| \right) \]

\[ = \sum_{j=1}^{h-1} \left( \tilde{m}! \left| z_{\tau_{a_{j}}(1)} \right| \left| z_{\tau_{a_{1}}(1)} \right| \cdots \left| z_{\tau_{a_{j-1}}(1)} \right| \left| z_{\tau_{a_{j}}(1)} \right| \left| z_{\tau_{a_{j+1}}(1)} \right| \cdots \left| z_{\tau_{a_{(h-1)}}(1)} \right| \right) \]

\[ = \tilde{m}! (h-1) \left| \left( z_{\tau_{a_{1}}} \right) \right|_{p,1}. \]

Hence

\[ f(z) = \left| \left( z_{\tau_{a_{1}}} \right) \right|_{p,1}. \]

\[ \square \]

Let \( E \) and \( F \) be two complex Banach spaces with norm \( \| \cdot \|_E \) and \( \| \cdot \|_F \), and let \( G \) be an analytic map from an open subset of \( E \) into \( F \). The first derivative \( d_v G \) of \( G \) at \( v \) is a linear map from \( E \) into \( F \), whose induced operator norm is

\[ \|d_v G\|_{F,E} = \max_{u \neq 0} \frac{\|d_v G(u)\|_F}{\|u\|_E}. \]

The Cauchy inequality can be stated as follows.

**Lemma 8.7.** Let \( G \) be an analytic map from the open ball of radius \( r \) around \( v \) in \( E \) into \( F \) such that \( \|G\|_F \leq M \) on this ball. Then

\[ \|d_v G\|_{F,E} \leq \frac{M}{r}. \]  

(8.3)

**Proof.** See details in Lemma A.3 of [24]. \( \square \)
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