INTERPOLATION OF OPERATORS WITH TRACE INEQUALITIES RELATED TO THE POSITIVE WEIGHTED GEOMETRIC MEAN

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ABSTRACT

There are various generalizations of the geometric mean \((a, b) \mapsto a^{1/2}b^{1/2}\) for \(a, b \in \mathbb{R}^+_0\) to positive matrices, and we consider the standard positive geometric mean \((X, Y) \mapsto X^{1/2}(X^{-1/2}YX^{-1/2})^{1/2}X^{1/2}\). Much research in recent years has been devoted to relating the weighted version of this mean \(X\#Y := X^{1/2}(X^{-1/2}YX^{-1/2})^{1/2}X^{1/2}\) for \(t \in [0, 1]\) with operators \(e^{(1-t)X+ty}\) and \(e^{(1-t)X/2}e^{ty}e^{(1-t)X/2}\) in Golden-Thompson-like inequalities. These inequalities are of interest to mathematical physicists for their relationship to quantum entropy, relative quantum entropy, and Rényi divergences. However, the weighted mean is well-defined for the full range of \(t \in \mathbb{R}\). In this paper we examine the value of \(|||e^H \# e^K|||\) and variations thereof in comparison to |||\(e^{(1-t)H+tk}\)||| and |||\(e^{(1-t)H}e^{tK}\)||| for any unitarily invariant norm |||·||| and in particular the trace norm, creating for the first time the full picture of interpolation of the weighted geometric mean with the Golden-Thompson Inequality. We expand inequalities known for \(|||(e^H \# e^K)^{1/r}|||\) with \(r > 0\), \(t \in [0, 1]\) to the entire real line, and comment on how the exterior inequalities can be used to provide elegant proofs of the known inequalities for \(t \in [0, 1]\). We also characterize the equality cases for strictly increasing unitarily invariant norms.

Keywords operator interpolation · trace inequalities · geometric matrix mean · log majorization · quantum entropy

1 Introduction

Consider the following functions on positive definite complex matrices:

\[(X, Y) \mapsto e^{\log(X)+\log(Y)}, \quad (1.1)\]
\[(X, Y) \mapsto e^{\log(X)/2}e^{\log(Y)}e^{\log(X)/2} = X^{1/2}YX^{1/2}, \quad (1.2)\]
\[(X, Y) \mapsto X^{1/2}(X^{-1/2}YX^{-1/2})^{1/2}X^{1/2}. \quad (1.3)\]

Each can be seen as an operator analogue of the geometric mean function \((a, b) \mapsto a^{1/2}b^{1/2}\) defined on \([0, \infty) \times [0, \infty)\). In fact, the third is known as the geometric mean of positive matrices, first introduced by Pusz and Woronowicz [14] in 1975 as a way of generalizing \(\sqrt{xy}\) to sesquilinear forms.

Mathematicians and mathematical physicists will recognize the first two operator functions from their placement in the Golden-Thompson Inequality, proven by Golden for non-negative definite matrices in 1965 [7] and independently the same year by Thompson [16] for all Hermitian matrices \(H\) and \(K\),

\[\text{Tr}[e^{H+K}] \leq \text{Tr}[e^He^K]. \quad (1.4)\]

Characterizing the general relationship between these operator functions in a similar manner will be the goal of this paper.
Theorem 1.2. For Hermitian matrices $H$ and $K$ in $\mathbb{R}$, then
\[
\|e^{(1-t)H+tK}\| \leq \|e^{(1-t)H}e^{tK}\| \quad 0 \leq t \leq 1.
\]
(1.11)
\[
\|e^{(1-t)H+tK}\| \leq \|e^{(1-t)H}e^{tK}\| \quad 1 \leq t \leq 2.
\]
(1.12)
\[
\|e^{(1-t)H+tK}\| \leq \|e^{H}\#e^{K}\| \quad 2 \leq t.
\]
(1.13)

Theorem 1.1 is proven in Section 4. It should be noted that Theorem 1.1 can be deduced in the $t \in [-1, 0]$ in [12]; here we provide proof for the entire range together.

This allows us for the first time to characterize the relationship between the three operator functions for all $t \in \mathbb{R}$:

Theorem 1.2. For Hermitian matrices $H$ and $K$, and $t \in \mathbb{R}$ and any unitarily invariant norm $\|\cdot\|$, then
\[
\|e^{H}\#e^{K}\| \leq \|e^{(1-t)H+tK}\| \leq \|e^{(1-t)H}e^{tK}\| \quad 0 \leq t \leq 1.
\]
(1.11)
\[
\|e^{(1-t)H+tK}\| \leq \|e^{(1-t)H}e^{tK}\| \quad 1 \leq t \leq 2.
\]
(1.12)
\[
\|e^{(1-t)H+tK}\| \leq \|e^{H}\#e^{K}\| \quad 2 \leq t.
\]
(1.13)

Note that as the trace is a unitarily invariant norm for positive matrices, that the above inequalities all hold for the traces of the matrices. Complimentary negative inequalities can be found taking $X\#_{1-t} = Y\#_{1-t}X$.

Proof. Equation 1.11 comes from taking the $r = 1$ case of Equation 1.9 with the Golden-Thompson Inequality. Equation 1.12 comes from taking the $r = 1$ case in Equation 1.10 and Equation 4.12 (proven in [9] and discussed in Section 4) and the Golden-Thompson Inequality for the $t$ positive case, and using Relationship 2.6 to extend to the $t$ negative case. Equation 1.13 comes from taking the $r = 1$ case in Equations 1.10 and Equation 4.13 (also proven in [9] and discussed in Section 4) and the Golden-Thompson Inequality for the $t$ positive case, and once more using Relationship 2.6 to extend to the $t$ negative case.

Theorem 1 in the case of the trace norm is illustrated in Figure 1.
Figure 1: Comparison of $\text{Tr}[e^{H \#_t e^{K}}]$ in red, $\text{Tr}[e^{(1-t)H + tK}]$ in green, and $\text{Tr}[e^{(1-t)H \#_t e^{K}}]$ in blue calculated numerically for three sets of randomly generated positive matrices $H$ and $K$.

2 The Weighted Geometric Mean As Geodesics

Let $P_n \subseteq M_n$ denote the space of $n \times n$ positive definite complex matrices. We consider the Riemannian metric with arc length of the smooth path $\gamma : [a, b] \rightarrow P_n$ defined by

$$L(\gamma) := \int_a^b ||\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}||_2 dt,$$

(2.1)

where $|| \cdot ||_2$ denotes the Hilbert-Schmidt norm. Then the corresponding distance is

$$\delta(X, Y) = \inf \left\{ \int_0^1 ||\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}||_2 dt, \quad \gamma(0) = X, \quad \gamma(1) = Y \right\}.$$

(2.2)

This metric, first introduced by Skovgaard [15] for its applications in statistics, has the particular nice property that it is invariant under conjugation:

$$\delta(A^* X A, A^* Y A) = \delta(X, Y)$$

(2.3)

for all invertible matrices $A \in M_n$. It is shown in [4] [3] that there is a unique constant speed geodesic for any $X, Y \in P_n$ running between $X$ and $Y$ in unit time, namely

$$\gamma(t) = X \#_t Y := X^{1/2}(X^{-1/2}YX^{-1/2})^{t}X^{1/2}.$$

(2.4)

These geodesics satisfy [4] for all $t, t_0, t_1 \in \mathbb{R}$ and $X, Y \in P_n$

$$X \#_{(1-t)t_0 + t_1} Y = (X \#_{t_0} Y) \#_{t}(X \#_{t_1} Y).$$

(2.5)

Of particular interest to us will be the $t_0 = 1, t_1 = 0$, which gives the identity

$$X \#_{1-t} Y = Y \#_t X.$$

(2.6)

This will allow us to extend all inequalities proven for $t \in \mathbb{R}^+$ to $(1-t) \in \mathbb{R}^-$, and in particular to connect the positive and negative cases of Theorem 1.2.

Ando and Kubo prove in [13] that the map

$$(X, Y) \mapsto X \#_t Y$$

(2.7)

is jointly concave and monotone increasing in $X$ and $Y$ for $t \in [0, 1]$. Carlen and Lieb prove [4] that it is jointly convex for $t \in [-1, 0] \cup [1, 2]$. However, it is challenging to expand these results as $f(x) = x^t$ is not operator monotone for $t > 1$ and neither concave nor convex for $t \notin [-1, 2]$. Therefore, proofs of trace inequalities cannot rely solely on operator monotonicity or convexity, so we turn to our main tool: log majorization.
3 Log Majorization

Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \), with \( a_1 \geq \cdots \geq a_n \) and \( b_1 \geq \cdots \geq b_n \). Then \( b \) weakly majorizes \( a \), written \( a \prec_w b \), when

\[
\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \quad 1 \leq k \leq n
\]

(3.1)

and is majorized \( a \prec b \) when the final inequality is an equality. Weak log majorization \( a \prec_{w(\log)} b \) is similarly defined for non-negative vectors as

\[
\prod_{i=1}^{k} a_i \leq \prod_{i=1}^{k} b_i, \quad 1 \leq k \leq n
\]

(3.2)

with log majorization \( a \prec_{(\log)} b \) when the final inequality is an equality.

We define all of the above majorization for matrices, ie \( A \prec B \) and all variations, when the singular values in descending order considered as a vector \( (s_1(A), \ldots, s_n(A)) \prec (s_1(B), \ldots, s_n(B)) \).

We list the following results for convenience:

Lemma 3.1. Let \( A, B \in M^+_n \). Then the following conditions are equivalent: \( A \prec_{w(\log)} B, |||A||| \leq |||B||| \) for every unitarily invariant norm \( ||| \cdot ||| \) and \( |||f(A)||| \leq |||f(B)||| \) for every unitarily invariant norm \( ||| \cdot ||| \) and continuous increasing function \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(0) \geq 0 \) and \( f(e^t) \) is convex.

A full proof can be found in [11].

Corollary 3.1.1. When \( A \prec_{(\log)} B \), and for a function \( f \) that satisfies the hypotheses of Lemma 3.1 such that \( f(x_1 x_2) = f(x_1) f(x_2) \), then \( f(A) \prec_{(\log)} f(B) \).

Proof. Lemma 1 implies \( f(A) \prec_{w(\log)} f(B) \), then \( \prod_{i=1}^{n} \lambda(A) = \prod_{i=1}^{n} \lambda(B) \) gives \( \prod_{i=1}^{n} f(\lambda(A)) = f(\prod_{i=1}^{n} \lambda(A)) = f(\prod_{i=1}^{n} \lambda(B)) \).

\[ \square \]

Theorem 3.2. (Araki [2], Theorem 1) For all \( A, B \in P_n \),

\[
(A^{1/2} B A^{1/2})^r \prec_{(\log)} A^{r/2} B^r A^{r/2}, \quad (r \geq 1)
\]

(3.3)

or equivalently

\[
(A^{p/2} B^p A^{p/2})^{1/p} \prec_{(\log)} (A^{n/2} B^n A^{n/2})^{1/q}, \quad (0 < p \leq q).
\]

(3.4)

Corollary 3.2.1. For all self-adjoint \( H, K \in M_n \) and unitarily invariant norm \( ||| \cdot ||| \),

\[
|||e^{H+K}||| \leq |||(e^{H/2} e^{K} e^{H/2})^{1/q}|||,
\]

(3.5)

or equivalently

\[
e^{H+K} \prec_{(\log)} (e^{H/2} e^{K} e^{H/2})^{1/q}, \quad q > 0.
\]

(3.6)

Proof. Take \( p \rightarrow 0 \) in Theorem 3.2 and apply the Lie-Trotter product formula.

\[ \square \]

4 Trace Inequalities

Proof of Theorem 3.2. Let \( t \geq 1 \), and \( r > 0 \). For \( f(x) = x^{1/r} \), then \( f(0) = 0 \), \( f(e^t) \) is convex and increasing, and \( f(x_1 x_2) = f(x_1) f(x_2) \). Then applying Corollary 3.1.1 and Theorem 3.2

\[
(e^{rH} # e^{rK})^{1/r} = f(e^{rH/2} e^{-rH/2} e^{rH/2} e^{-rH/2})^{1/r}
\]

(4.1)

\[ \succ_{(\log)} \ f \left( (e^{rH/2} e^{-rH/2} e^{rH/2} e^{-rH/2})^{1/2} \right) \]

(4.2)

\[
= (e^{r(1-t)H/2t} e^{rK} e^{r(1-t)H/2t})^{1/r}.
\]

(4.3)

We explicitly write

\[
e^{r(1-t)H/2t} e^{rK} e^{r(1-t)H/2t} = e^{r(1-t)H/2t} e^{rK} e^{r(1-t)H/2t}
\]

(4.4)

As \( t/r > 0 \), we apply Corollary 3.2.1

\[
(e^{r(1-t)H/2t} e^{rK} e^{r(1-t)H/2t})^{1/r} \succ_{(\log)} e^{r(1-t)H+tK}.
\]

(4.5)
Then by Lemma 1,
\[ |||e^{rH}#t e^{rK}\rangle\rangle r \geq |||e^{(1-t)H+tK}||| \]  (4.6)
for any unitarily invariant norm. Relationship 2.6 extends this to \( t \leq 0 \).

**Proof of Theorem 4.2.** When \( H \) and \( K \) commute, all of the expressions being considered are equal, and hence their norms will be equal. Therefore, it remains to consider the implications of equality for strictly increasing unitarily invariant norms.

It is known from Hiai [8] (Theorem 3.1) that there is equality in the inequalities in Equation 4.11 for a strictly increasing unitarily invariant norm if and only if \( H \) and \( K \) commute, and in [9] for all the relationships between \( |||e^{rH}#t e^{rK}\rangle\rangle r \) and \( |||e^{(1-t)H+tK}||| \). Therefore, it remains to prove that there is equality in Equation 1.10 (re-stated above in Equation 4.6) if and only if \( H \) and \( K \) commute; then Relationship 2.6 extends the characterization to all \( t \), taking \( r = 1 \).

We use the following Lemma from Hiai [8]:

**Lemma 4.1.** (Hiai 1994) Let \( A, B \in M_n^+ \). Then \( ||(A^{p/2} B^{p/2})^{1/p}|| \) is not strictly increasing in \( p > 0 \) if and only if \( A \) and \( B \) commute.

From the proof of Theorem 1.1 we have
\[ (e^{rH}#t e^{rK})^{1/r} \gg_{\log} (e^{(1-t)H/2}e^{t rK/1} e^{(1-t)H/2})^{1/r} \gg_{\log} e^{(1-t)H+tK}. \]  (4.7)

Then for a strictly increasing unitarily invariant norm, it follows that we have
\[ |||e^{rH}#t e^{rK}\rangle\rangle r \geq |||e^{(1-t)H/2}e^{t rK/1} e^{(1-t)H/2})^{1/r}\rangle\rangle r \geq |||e^{(1-t)H+tK}||| \].  (4.8)

We then apply Lemma 4.1 to the second equality considering the Lie-Trotter product formula.

It is interesting to note that Equation 1.10 the \( t \in (-\infty, 0] \cup [1, \infty) \) case actually implies Equation 1.9 the \( t \in [0, 1] \) case. In [10], Hiai and Petz show that Equation 1.9 is equivalent to
\[ \frac{1}{r} \text{Tr} \left[ e^H \log(e^{rH/2} e^{-rK} e^{rH/2}) \right] \geq \text{Tr} \left[ e^H (H - K) \right]. \]  (4.9)

for all \( r > 0 \), \( t \in [0, 1] \). The forward implication comes from noting the inequality trivially becomes equality \( \text{Tr}[e^H] = \text{Tr}[e^H] \) at \( t = 0 \), so taking the limit from above one can relate the derivatives of each side. The backwards inequality is far more involved. Hiai and Petz prove Equation 1.9 by proving Equation 4.9 then making use of their equivalence.

The exact same reasoning behind the forward implication applies substituting in knowledge of the trace inequality for \( t \leq 0 \). Therefore, it is an immediate consequence of [10] that Equation 1.10 implies Equation 1.9 taking the derivative of Equation 1.10 at \( t = 0 \) with some changes of variable produces Equation 4.9 directly from Equation 1.10.

As mentioned in Section 1, Hiai [9] recently published work regarding log majorization and Rényi divergences, including that for \( t \geq 1 \),
\[ e^{(1-t)H/2} e^{t K} e^{(1-t)H/2} \gg_{\log} (e^{rH}#t e^{rK})^{1/r} \quad r \geq \max\{t/2, t-1\} \]  (4.10)
\[ (e^{rH}#t e^{rK})^{1/r} \gg_{\log} e^{(1-t)H/2} e^{t K} e^{(1-t)H/2} \quad r \leq \min\{t/2, t-1\}. \]  (4.11)

It then follows from Lemma 3.1 that for \( t \geq 1 \),
\[ \|e^{(1-t)H/2} e^{t K} e^{(1-t)H/2}\| \leq \|[e^{rH}#t e^{rK}]^{1/r}\| \quad r \geq \max\{t/2, t-1\} \]  (4.12)
\[ \|[e^{rH}#t e^{rK}]^{1/r}\| \leq \|e^{(1-t)H/2} e^{t K} e^{(1-t)H/2}\| \quad r \leq \min\{t/2, t-1\}. \]  (4.13)
for any unitarily invariant norm \( \|\cdot\| \).

It is possible to prove a simplified version of Equation 4.10 (the special case of \( r=1 \)) using similar methods as in [9], with the Furuta inequality:

**Theorem 4.2.** Let \( H, K \in M_n \) be Hermitian, and \( 1 \leq t \leq 2 \). Then
\[ e^{(1-t)H/2} e^{t K} e^{(1-t)H/2} \gg_{\log} e^H #t e^K. \]  (14.4)
Proof. We use the standard antisymmetric tensor power technique to prove log majorization, noting that for $A \in M_n$ and $k = 1, \ldots, n$, then
\[ \prod_{i=1}^{k} s_i(A) = s_1(A^{\wedge k}) = ||A^{\wedge k}||, \] (4.15)
and that the antisymmetric tensor product has the properties $(A^{\wedge k})^* = (A^*)^{\wedge k}$, $(AB)^{\wedge k} = A^{\wedge k}B^{\wedge k}$, and for all $A \geq 0$ and $r > 0$, $(A^*)^{\wedge k} = (A^{\wedge k})^r$ [2]. Then as
\[ \det(e^{(1-t)H/2} e^{tK} e^{(1-t)H/2}) = \det(e^{H})^{1-t} \det(e^{K})^{t} = \det(e^{H \#_te^K}), \] (4.16)
it suffices to show that
\[ ||e^{(1-t)H/2} e^{tK} e^{(1-t)H/2}|| \leq ||e^{H \#_te^K}||. \] (4.17)

Suppose $e^{H \#_te^K} \leq I$. We use the following relation [3] for all $A \in P_n$, $B \in M_n$ invertible, and real number $r$:
\[ (BAB^*)^r = BA^{1/2}(A^{1/2}B^r A^{1/2})^{r-1} A^{1/2} B^r. \] (4.18)
Then
\[ e^{H/2}(e^{-H/2} e^{K/2} e^{-H/2}) e^{H/2} \leq I \] (4.19)
\[ \Rightarrow (e^{-H/2} e^{K/2} e^{-H/2}) \leq e^{-H} \] (4.20)
\[ \Rightarrow e^{-H/2} e^{K/2} (e^{K/2} e^{-H} e^{K/2})^{t-1} e^{K/2} e^{-H} \leq e^{-H} \] (4.21)
\[ \Rightarrow (e^{K/2} e^{-H} e^{K/2})^{t-1} \leq e^{-K}. \] (4.22)

We now apply the Furuta Inequality [5] for $A \geq B \geq 0$, $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p + 2r$:
\[ (A^p B^p A^p)^{1/q} \leq A^{(p+2r)/q}. \] (4.23)

We choose $A = e^{-K}$, $B = (e^{K/2} e^{-H} e^{K/2})^{t-1}$, $r = \frac{1}{2}$, $p = (t-1)^{-1}$, $q = (t-1)^{-1}$. Note that as $t \in [1, 2]$ then $t-1 \in [0, 1]$, and $q \geq 1$. Furthermore,
\[ (1+2r)q - (p + 2r) = \frac{2}{t-1} - \frac{1}{t-1} - 1 = \frac{1}{t-1} - 1 \geq 0, \] (4.24)
so all the hypotheses of the inequality are satisfied. Then
\[ A^{(p+2r)/q} = e^{-\frac{1}{t+1}(t-1)K} = e^{-tK}, \] (4.25)
and
\[ (A^p B^p A^p)^{1/q} = (e^{-K/2}((e^{K/2} e^{-H} e^{K/2})^{t-1})^{1/(t-1)} e^{-K/2})^{t-1} = e^{-(t-1)H}. \] (4.26)
Therefore, we conclude
\[ e^{-(t-1)H} \leq e^{-tK}, \] (4.27)
which implies that
\[ e^{tK} \leq e^{(t-1)H}. \] (4.28)
and so
\[ e^{(1-t)H/2} e^{tK} e^{(1-t)H/2} \leq I. \] (4.29)

Hiai provides a full analysis on the connection to Rényi divergences by noting that in the $t = 2$ case, $e^{H \#_te^K} = e^{(1-t)H} e^{tK}$, and one can use the same technique taking the derivative:
\[ \frac{d}{dt} \text{Tr} \left[ e^{H \#_te^K} \right]_{t=2} = \text{Tr} \left[ e^{H/2} \log(e^{-H/2} e^{K/2} e^{-H/2}) (e^{-H/2} e^{K/2} e^{-H/2})^2 e^{H/2} \right] \] (4.30)
\[ = \text{Tr} \left[ e^{H/2} e^{-H/2} e^{K/2} e^{-H/2} \log(e^{-H/2} e^{K/2} e^{-H/2}) e^{-H/2} e^{K/2} e^{-H/2} e^{H/2} \right] \] (4.31)
\[ = \text{Tr} \left[ e^{K/2} e^{-H/2} \log(e^{-H/2} e^{K/2} e^{-H/2}) e^{-H/2} e^{K/2} e^{-H/2} \right] \] (4.32)
\[ = \text{Tr} \left[ e^{-H/2} e^{2K} e^{-H/2} \log(e^{-H/2} e^{K/2} e^{-H/2}) \right] \] (4.33)
which using
\[
\frac{d}{dt} \text{Tr} \left[ e^{(1-t)H} e^{tK} \right] \bigg|_{t=2} = \text{Tr} \left[ (K - H)e^{-H} e^{2K} \right]
\]
gives
\[
\text{Tr} \left[ e^{-H/2} e^{2K} e^{-H/2} \log(e^{-H/2} e^{-H/2}) \right] \leq \text{Tr} \left[ e^{-H} e^{2K} (K - H) \right]
\] (4.35)

Theorem 4.2 gives the same derivative identity at \( t = 2 \). We thus conjecture that similar methods to [10] it may be possible to show that
\[
\text{Tr} \left[ e^{(1-t)H} e^{tK} e^{(1-t)H/2} \right] \leq \text{Tr} \left[ e^{H} e^{tK} e^{H/2} \right]
\] (4.36)
also implies
\[
\text{Tr} \left[ e^{H} e^{tK} \right] \leq \text{Tr} \left[ e^{(1-t)H} e^{tK} e^{(1-t)H/2} \right]
\] (4.37)

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