Role of Externally Provided Randomness in Stochastic Teams and Zero-sum Team Games

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Abstract

Stochastic team decision problem is extensively studied in literature and the existence of optimal solution is obtained in recent literature. The value of information in statistical problem and decision theory is classical problem. Much of earlier does not qualitatively describe role of externally provided private and common randomness in stochastic team problem and team vs team zero sum game.

In this paper, we study the role of externally provided private or common randomness in stochastic team decision. We make observation that the randomness independent of environment does not benefit either team but randomness dependent on environment benefit teams and decreases the expected cost function. We also studied LQG team game with special information structure on private or common randomness. We extend these study to problem team vs team zero sum game. We show that if a game admits saddle point solution, then private or common randomness independent of environment does not benefit either team. We also analyze the scenario when a team with having more information than other team which is dependent on environment and game has saddle point solution, then team with more information benefits. This is also illustrated numerically for LQG team vs team zero sum game. Finally, we show for discrete team vs team zero sum game that private randomness independent of environment benefits team when there is no saddle point condition. Role of common randomness is discussed for discrete game.

I. INTRODUCTION

A team decision problem consists of two or more of decision makers (DMs) or players that make decisions in a random environment where the information of each DM is a (possibly partial) observation about the random environment. A DM takes an action as a function of the information; this function is referred to the as the decision rule. DMs choose the decision rule to jointly minimize an expected cost.

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If the decision makers had identical observations, then the multiple decision makers could be clumped together as a single decision maker and the problem reduces to that of a stochastic optimization or control problem. Of interest to us here is the case where information is asymmetric, whereby there is no obvious method of aggregating it. A team decision problem is in essence a decentralized stochastic optimal control problem. Problems with structure appear in a variety of settings for example in sensor networks. The decision makers could be sensors situated at different locations. These sensors observe the environment through different, possibly imperfect, channels and under this information structure, the sensors have to act collectively to minimize a certain cost function.

In this paper we consider the role of externally provided private and common randomness in stochastic teams and in stochastic team v/s team zero-sum games. In the setup described above, it is conceivable that an external source provides randomness to the players. This randomness may or may not be correlated with their observations, and it may or may not be correlated across players. This randomness increases the set of achievable joint distributions on the joint action space of the DMs. Our goal is to understand the role of such randomness in a team problem and a team v/s team game.

Qualitatively, there are three kinds of randomness that an external source may provide. First, the source of randomness could be a coordinator – namely an entity that mixes the actions of the DM by randomizing. Mathematically, this randomness is independent of the observations of the DMs, but it may be correlated across DMs. This correlation makes this randomness distinct from the usual notion of “randomized policies”, in which the randomization is independently performed by each DM. The second kind of randomness, may be imagined as a counsellor – this entity accesses the observations of each decision maker and provides a common message to all DMs. This kind of randomness is correlated with the environment. The sources of randomness mentioned above are relevant for team problems as well as team v/s team games. The third kind, which is relevant only in the team v/s team game, is that of a mole or a spy. This source of randomness provides information about observations of the opposite team.

Our interest is in qualitatively understanding the role of the kinds of randomness mentioned above and quantifying it. We make following contribution in this work.

1) A team decision problem:
   - We show that if a coordinator provides private or common randomness independent
of the environment, then it cannot improve the cost.

- We show that common randomness dependent on the environment can improve the team cost. For a certain class of LQG team problems, we show that if the information of each player is replaced by a convex combination of the information of all players, then the team improves its cost.

2) Team v/s team zero-sum game:

- We show randomness independent the an environment does not benefit teams if the zero-sum game admits saddle point solution.
- We prove that a team having more information than other team, benefit and decreases the cost function for minimizing team when randomness is dependent on environment.
- For LQG team zero-sum games we illustrate that common randomness dependent on the environment leads to an improvement in the optimal team cost.
- Finally, We give an example of a discrete team v/s team zero sum game, without a pure strategy saddle point, we also show that private and common randomness independent of an environment benefit teams. But it may not have Nash equilibrium.

A. Related Work

Early work on team decision problem in aspect of an organization theory studied in [1]; where author used the concepts from game theory and statistical decision theory. A general formulation of team decision problem are described in [2] and person by person optimality condition is established to solve the distributed team decision problem. Furthermore, the team decision problem extended to a LQG team problem in [3], [4]. They investigated static and dynamic LQG team decision problem and explored its connection with information theory and economics. In LQG team problem, there is a unique optimal solution, linear in information and it is obtained via solving person by person optimal condition. They also studied dynamic LQG team with partial-nested information structure. Moreover, the symmetric static team problem studied in [5] and have shown that the optimal strategy for a symmetric team problem not necessarily a pure strategy but it can have randomized strategy.

Two-team zero-sum game in LQG problem studied in [6], and they show that team having extra information not necessary ameliorate the expected loss. Apart from a team decision problem, the role of common randomness in multi-agent distributed control problem is analyzed in [7].
Our work is inspired from [3], [6], [7]. Role of common randomness is not quantified in [3], [6], whereas we discuss role of the private and common randomness in team decision problem.

The value of information for statistical problems is first introduced in [8], [9]. This is further extended to decision problems in [10], and author have shown that increasing information lead to increasing in utility. Early work on role of increasing information in two person game problem is presented in [11]. The surprising finding is presented in [11], where author finds that increasing informativeness leads to decreasing performance. The value of information available to players with two-person zero sum game is studied in [12]. As the additional information increased for a player, may lead to solution toward ideal optimality condition when there is a saddle point condition exists. This result further motivated study on value of information in team vs team zero sum game and similar result have shown for LQG team vs team zero sum game. In [13], the value of information for two players non zero sum game is developed, and they have show that in LQG model with better informed player, it decreases the average Nash cost for both players but in duopoly problem, the better informed player benefits only.

The great reference for stochastic team decision problem is [14]. In this book, authors discussed fundamental of team decision problem, sequential team decision problem, comparison of information structure, topological properties of information structure and its application to communication channel. It has motivated further research on team decision problem in recent time.

There are flurry of research activities on static team problems and their existence of solution. In [15], the class of sequential team problem is studied with a certain information structure and existence of optimal strategies are proved. Further, they have shown the existence of optimal solution for team problem under weaker assumptions, i.e., assumption on cost function to be bounded and continuous, action space of agent to be compact or not compact and observation satisfies technical condition. The ideas from weak convergence in probability theory is used to show convergence of measure of joint probability of actions. In [16], author extended study of [15], further weaken assumptions their. They have shown the existence of optimal strategies for static teams and topology on set of policies are introduced. In [17], authors studied convexity properties of strategy spaces and discussed redundancy of common or private information that is independent of randomness for static team. Though this result is similar to ours, their proof differs from our method. The role of common information in dynamic stochastic game is studied in [18],
where asymmetric of common information is considered among players. In [19], the existence of optimal solution to static team problem under private and common information structure is developed using topology of information and space of measures. Early ideas developed in [8]–[11], [13] on role of information are derived for zero sum game under slightly weaker assumptions in [20] and have shown existence of saddle point equilibrium.

But these paper do not provide qualitative comparison of role of externally provided private and common randomness in static team and team vs team zero sum game.

The rest of the paper is organized as follows. Private and common randomness in static team decision problem described in Section II. Role of private and common randomness in static team vs team zero-sum game developed in Section III. Finally, concluding remarks and future direction of research presented in Section IV.

II. PRIVATE AND COMMON RANDOMNESS IN STATIC TEAM PROBLEM

A. Team decision problem

Consider a team decision problem having $N$ decision makers $\text{DM}_1, \ldots, \text{DM}_N$ in a team and let $\mathcal{N} = \{1, \ldots, N\}$. Let $\xi$ be a random vector taking values in a space $\Xi$ denoting the state of nature or an environment; let its distribution be $P(\cdot)$. Define $y_i := \eta_i(\xi)$ for a measurable function $\eta_i$ to be the information observed by $\text{DM}_i$ and let $Y_i$ be the space of $y_i$. Let $U_i \subseteq \mathbb{R}^{m_i}$, $m_i \in \mathbb{N}$ denote the set of actions of $\text{DM}_i$. The strategy space of $\text{DM}_i$ is $\Gamma_i$, the space of measurable functions $\gamma_i$ mapping $Y_i$ to $U_i$ and an action $u_i$ is given by $u_i = \gamma_i(y_i)$. Without loss of generality we take $U_i \subseteq \mathbb{R}$ for all $i$, since a DM with a $\mathbb{R}^{m_i}$-valued strategy can be considered as $m_i$ separate DMs with $\mathbb{R}$-valued strategies; thus $m_i = 1$ for all $i \in \mathcal{N}$. Let

$$u := (u_1, \ldots, u_N), \quad \gamma := (\gamma_1, \ldots, \gamma_N),$$

$$u^{-i} := (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N), \quad \gamma^{-i} := (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_N)$$

The cost function is measurable function $\kappa : U \times \Xi \to \mathbb{R}$, where $U : \prod_{i \in \mathcal{N}} U_i$ and let

$$J(\gamma) = \mathbb{E}_\xi[\kappa(u_1 = \gamma_1(\eta_1(\xi)), \ldots, u_N = \gamma_N(\eta_N(\xi)), \xi)].$$

A team optimal solution of the above problem is defined as $\gamma^* \in \Gamma := \prod_{i \in \mathcal{N}} \Gamma_i$ such that

$$J_{\text{TO}} \triangleq J(\gamma^*) = \min_{\gamma \in \Gamma} J(\gamma) = \min_{\gamma \in \Gamma} \mathbb{E}_\xi[\kappa(u_1 = \gamma_1(\eta_1(\xi)), \ldots, u_N = \gamma_N(\eta_N(\xi)), \xi)]. \quad (1)$$
We assume throughout that a team optimal solution exists and use ‘min’ instead of ‘inf’. A related concept, called the person by person optimal solution is a \( \gamma \in \Gamma \) such that

\[
J_{PBP} = J(\gamma) = \min_{\gamma' \in \Gamma_i} J(\gamma_i', \gamma^{-i}) \quad \forall \, i \in \mathcal{N}.
\]

### B. Externally provided randomness

We now introduce externally provided randomness, beginning with private randomness. Suppose \( D_{Mi} \) chooses \( u_i \) randomly from \( U_i \) and let \( Q \) be the joint distribution of all variables involved, namely, \( \xi, y, u \). We say that the DMs have externally provided private randomness, if

\[
Q(u|y) = \prod_{i \in \mathcal{N}} Q(u_i|y_i).
\]  

(2)

This specification corresponds to the standard notion of randomized policies in stochastic control or behavioral strategies in stochastic games, wherein the action is chosen to be a random function of the information.

In general one has

\[
Q(\xi, y, u) = Q(u|\xi, y)Q(\xi, y),
\]

where \( Q(\xi, y, u) \) is the joint distribution of \( \xi, y, u \), \( Q(u|\xi, y) \) the conditional distribution of \( u \) given \( \xi, y \), and \( Q(\xi, y) \) is the marginal of \( \xi, y \) (evaluated at \( '\xi = \xi, y = y, u = u' \)). When the randomness provided to DMs is independent of \( \xi \), we have

\[
u|y \perp \xi,
\]

i.e, given \( y \) the choice of \( u \) is independent of \( \xi \). Furthermore, the joint distribution of \( (\xi, y) \) is known; denote this distribution by \( P(\xi, y) \). Consequently, any joint distribution of \( \xi, y, u \) is given by

\[
Q(\xi, y, u) = Q(u|y)P(\xi, y).
\]  

(3)

To describe externally provided common randomness, let \( w = (w_1, \ldots, w_N) \) be a random vector, \( w \perp \xi \), and assume that \( w_i \) is externally provided to \( D_{Mi} \) by a coordinator. With the additional information of \( w_i \), the strategies \( \gamma_i \) of \( D_{Mi} \) are deterministic \( y_i \times w_i \rightarrow u_i \) mappings and \( \Gamma_i \) is the space of such strategies. For a given random vector \( w \) with distribution \( \mathbb{P} \), the team optimal solution is defined analogously to (1), as follows:

\[
\min_{\gamma \in \Gamma} J(\gamma) = \min_{\gamma \in \Gamma} \mathbb{E}_{\xi, w}[\kappa(u_1 = \gamma_1(\eta_1(\xi), w_1), \ldots, u_N = \gamma_N(\eta_N(\xi), w_N), \xi)].
\]  

(4)
Since $\xi$ is independent of $w$, the expectation with respect to $(\xi, w)$ is well defined once the marginals of $\xi, w$ are defined.

C. Randomness independent of $\xi$

In this section, we study the case of externally provided randomness that is independent of the state of nature $\xi$. Our main result is that in a team problem, such randomness provides no benefit to the team. One may interpret this to mean that a team gains nothing by hiring a coordinator whose sole role is that of mixing the actions of the team members without the use of any knowledge of the underlying state of nature or of the observations made by team members.

Let $P(\cdots)$ be the set of joint distributions of on the space $\cdots$. Let $Q$ be the set of joint distributions of random variables $\xi, y, u$ that admit the decomposition above. i.e.,

$$Q = \{ Q \in P(\Xi \times Y \times U) \mid Q \text{ satisfies } (3), (2) \}.$$

Consider the following problem:

$$J^*_{\text{TO}} \triangleq \min_{Q \in Q} \mathbb{E}_Q[\kappa(u, \xi)].$$

(5)

From the decomposition of $Q$ provided by (3)-(2), it follows that (5) is a multilinear program with separable constraints. Classical results show that (5) admits a solution that is an extreme point, namely, one where $u_i$ is a deterministic function of $y_i$. Consequently, $J^*_{\text{TO}} = J^*_{\text{C}}$ and we have the following result.

Proposition 2.1: In a static stochastic team problem, externally provided private randomness that is independent of the state of nature cannot improve the team’s cost.

Proof is along the lines of proof of Proposition 2.2. We skip the proof details.

Consider the following cost:

$$J^*_{\text{TO}_c} = \min_{\gamma \in \Gamma, P} \mathbb{E}_{\xi,w}[\kappa(\gamma_1(\eta_1(\xi), w_1), \ldots, \gamma_N(\eta_N(\xi), w_N), \xi)].$$

This is the lowest cost that can be attained via common randomness. The common randomness is independent of environment $\xi$.

Proposition 2.2:

$$J^*_{\text{TO}} = J^*_{\text{TO}_p} = J^*_{\text{TO}_c}.$$
**Proof:** It is enough to show that $J^*_{TO} = J^*_{TO_C}$. Now consider

$$J^*_{TO_C} = \min_{\gamma \in \Gamma, P} \mathbb{E}_{\xi, w}[\kappa(\gamma_1(\eta_1(\xi), w_1), \ldots, \gamma_N(\eta_N(\xi), w_N), \xi)].$$

Assuming $\{y_1, \cdots, y_N\}$ are well defined. Rewriting above expression, we obtain

$$J^*_{TO_C} = \min_{\gamma \in \Gamma, P} \mathbb{E}_{w} \mathbb{E}_{\xi/w}[\kappa(\gamma_1(\eta_1(\xi), w_1), \ldots, \gamma_N(\eta_N(\xi), w_N), \xi)].$$

Since common randomness $w$ independent of $\xi$, we have

$$J^*_{TO_C} = \min_{\gamma \in \Gamma, P} \mathbb{E}_{w} \mathbb{E}_{\xi}[\kappa(\gamma_1(\eta_1(\xi), w_1), \ldots, \gamma_N(\eta_N(\xi), w_N), \xi)].$$

Now, we split the minimization $\min_{\gamma_1, \gamma_2 \in \Gamma, P(w)} = \min_{\gamma_1, \gamma_2 \in \Gamma, P(w)} \min_{\gamma_1, \gamma_2 \in \Gamma, P(w)}$, we can also interchange $\min_{\gamma_1, \gamma_2 \in \Gamma, P(w)}$ and expectation $\mathbb{E}_{w}$ since DMs can cooperate and communicate in team problem.

$$J^*_{TO_C} = \min_{\gamma \in \Gamma, P} \mathbb{E}_{w} \mathbb{E}_{\xi}[\kappa(\gamma_1(\eta_1(\xi), w_1), \ldots, \gamma_N(\eta_N(\xi), w_N), \xi)].$$

Next we have

$$J^*_{TO_C} = \min_{\gamma \in \Gamma, P} \mathbb{E}_{w}[J^*_TO(w)].$$

It is linear program, thus it has optimal at extreme points, that is, $w^* = \arg \min_w J^*_TO(w)$. Then

$$J^*_{TO_C} = J^*_TO(w^*).$$

Now consider that $J^*_TO_C$ is a convex function of decision rule $\gamma$. If the decision rule is linear in its information, that is, $\gamma_i(\eta_i(\xi), w_i) = \alpha_{i1}\eta_i(\xi) + \alpha_{i2}w_i$, then clearly cost function will convex in $\alpha_{i1}$ and $\alpha_{i2}$ for all $i = 1, \cdots, N$. Without loss of generality assume that $\mathbb{E}[w_i] = 0$ for all $i = 1, \cdots, N$. Since $w$ and $\xi$ are independent the cost function will be separable and minimization w.r.t. variable $\alpha_{i1}$ and $\alpha_{i2}$ for all $i = 1, \cdots, N$. It implies that cost will be minimum iff $\alpha_{i2} = 0$ for all $i = 1, \cdots, N$. Thus no weightage given to additional information under this decision rule.

For LQG team problem in Appendix A it is illustrated that if private and common randomness independent of the environment $\xi$, it does not improve the expected cost function.
D. Randomness dependent on \( \xi \)

Consider a scenario where consultant provides an extra randomness about an environment to decision makers. That means these extra randomness is correlated with an environment \( \xi \).

Let \( \omega = (\omega_1, \ldots, \omega_N) \) be a random vector represents an extra randomness provided to decision makers by consultant. Further assume that \( \omega \) is function of \( \xi \), i.e. \( \omega = f(\xi) = (f_1(\xi), \ldots, f_N(\xi)) \), where \( f, f_i \) be the measurable functions. The strategies of DM \( i \) are \( \gamma_i : y_i \times \omega_i \rightarrow u_i, \gamma_i \in \Gamma_i \) space of strategies and \( u_i \in U_i \) space of decision variables. The team optimal cost is defined as follows.

\[
\min_{\gamma \in \Gamma} J(\gamma) = \min_{\gamma \in \Gamma} \mathbb{E}_{\xi,\omega}[\kappa(\gamma_1(\eta_1(\xi), \omega_1), \ldots, \gamma_N(\eta_N(\xi), \omega_N))].
\] (7)

Note that \( \omega \) is function of \( \xi \). The optimal cost function is

\[
J^{*}_{TO_{ER}} = \min_{\gamma \in \Gamma} \mathbb{E}_{\xi}[\kappa(\gamma_1(\eta_1(\xi), f_1(\xi)), \ldots, \gamma_N(\eta_N(\xi), f_N(\xi)))].
\] (8)

In distributed team problem with no extra randomness, decision maker have only partial observation about \( \xi \). Thus an observations about \( \xi \) is distributed among decision makers and an optimal team cost \( J^{*}_{TO} \) found in section II-A. When a consultant provides an extra randomness about an environment \( \xi \) to the decision makers. Essentially, there is an increase in observation about \( \xi \) available at decision makers. Intuitively, we expect that optimal cost under extra randomness in distributed stochastic team will improve optimal cost functional. Thus we have following result.

**Proposition 2.3:** In distributed static stochastic team problem,

\[
J^{*}_{TO} \geq J^{*}_{TO_{ER}}.
\] (9)

**Proof:** We develop the proof using the ideas from [8]. Let \( B_1 = \{\eta_1(\xi), \eta_2(\xi), \ldots, \eta_N(\xi)\} \) be the information available at team and \( B_2 = \{(\eta_1(\xi), f_1(\xi)), (\eta_2(\xi), f_1(\xi)), \ldots, (\eta_N(\xi), f_N(\xi))\} \) be the another information available at team with extra common randomness. Thus \( B_1 \subset B_2 \), i.e., \( B_2 \) is more informative than \( B_1 \). Since DMs can cooperate and communicate in team problem.

The minimization problem \( \min_{\gamma \in \Gamma} \mathbb{E}_{\xi} [\kappa(u_1, u_2, \ldots, u_N) | B] \) As \( f_i \)s are measurable functions, \( \eta_i \)s are measurable functions, so \( \gamma_i \) are measurable and \( \Gamma \) is closed bounded convex set. The cost function is also convex and measurable, thus from [8 Theorem 2] we can have

\[
\min_{\gamma \in \Gamma} \mathbb{E}_{\xi} [\kappa(u_1, u_2, \ldots, u_N) | B_2] \leq \min_{\gamma \in \Gamma} \mathbb{E}_{\xi} [\kappa(u_1, u_2, \ldots, u_N) | B_1]
\]
This implies the desired result.

Consider LQG stochastic team problem which has decision maker DM$_1$ and DM$_2$ in a team, and we have following different variation of LQG team problem based on types of observation available at decision makers.

**Problem 1:** Let decision variable $u_1 = Ay$, where $A$ is diagonal matrix, $\text{diag}(A) = [\alpha_1, \ldots, \alpha_N]$, $y$ is observation available at decision makers, $y = [y_1, y_2]^T = [\mu_1, \mu_2]^T$, and $\Sigma$ is covariance matrix of random vector $y$. The expected team cost

$$J_{TOLQG,1}^* = \min_A \mathbb{E}[y^T BAy + 2y^T S \xi] = \min_A \text{Tr}[BA \Sigma + 2A^T S \Sigma].$$

Let $A^*$ be the matrix such that optimal cost function of team is

$$J_{TOLQG,1}^* = \text{Tr}[A^{*T} BA^* \Sigma + 2A^{*T} S \Sigma].$$

**Problem 2:** Let decision variable $u_2 = \tilde{A}\tilde{y}$, where $\tilde{A}$ is diagonal matrix, $\text{diag}(\tilde{A}) = [\alpha_1, \ldots, \alpha_N]$, $\tilde{y}$ is observation available at decision makers, $\tilde{y} = [y_2, y_1]^T = [\mu_2, \mu_1]^T$. Note that

$$\tilde{y} = \begin{bmatrix} y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$ 

Let $\tilde{I} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus $\tilde{y} = \tilde{I}y$ and $\xi = y = \tilde{I}\tilde{y}$.

The expected team cost is

$$J_{TOLQG,2}^* = \min_A \mathbb{E}[\tilde{y}^T \tilde{A}^T B \tilde{A}\tilde{y} + 2\tilde{y}^T \tilde{A}^T S \xi] = \min_A \text{Tr}[\tilde{A}^T B \tilde{A} \Sigma + 2\tilde{A}^T S \tilde{\Sigma}].$$

Here $\tilde{\Sigma} := S \tilde{I}$ and $\tilde{\Sigma}$ denote the covariance matrix of random vector $\tilde{y}$.

Let $A^{**}$ be the matrix such that

$$J_{TOLQG,2}^* = \text{Tr}[A^{**T} BA^{**} \tilde{\Sigma} + 2A^{**T} \tilde{S} \tilde{\Sigma}].$$

**Problem 3:** Let decision variable $u_3 = C\omega$, where $C$ is diagonal matrix, $\omega = [\omega_1, \omega_2]^T$, $\omega_1 = \beta y_1 + (1 - \beta)y_2$, $\omega_2 = (1 - \beta)y_1 + \beta y_2$, $\beta \in (0, 1)$. Hence $\omega = \beta y + (1 - \beta)\tilde{y}$. We assume that decision maker has available common randomness provided by a consultant. These common randomness is convex combination of observation available at decision maker that is $y_1$ and $y_2$. For example $\beta = \frac{1}{2}$, a consultant provides an average of observations. The optimal cost functional is
\[ J^*_{\text{TO-LQG},3} = \min_{u_3 \in U} \mathbb{E}_\xi[u_3^T B u_3 + 2u_3^T S \xi]. \]

**Proposition 2.4:**

1) \[ J^*_{\text{TO-LQG},3} \leq \beta J^*_{\text{TO-LQG},1} + (1 - \beta) J^*_{\text{TO-LQG},2}. \]

2) If \( \tilde{\Sigma} = \Sigma \) and \( \tilde{S} = S \), then

\[ J^*_{\text{TO-LQG},1} = J^*_{\text{TO-LQG},2}. \]

Furthermore, \( A^* = A^{**} \). Also,

\[ J^*_{\text{TO-LQG},3} \leq J^*_{\text{TO-LQG},1}. \]

**Proof:**

1) We have:

\[ J^*_{\text{TO-LQG},3} = \min_{u_3 \in U} \mathbb{E}_\xi[u_3^T B u_3 + 2u_3^T S \xi] \]

Now,

\[ u_3^T B u_3 = \omega^T C^T BC \omega = (\beta y + (1 - \beta) \tilde{y})^T C^T BC (\beta y + (1 - \beta) \tilde{y}) \leq \beta y^T C^T BC y + (1 - \beta) \tilde{y}^T C^T BC \tilde{y} \quad (10) \]

Since \( B \) is symmetric positive definite matrix, \((\beta y + (1 - \beta) \tilde{y})^T C^T BC (\beta y + (1 - \beta) \tilde{y})\) is quadratic convex function. Thus inequality in (10) follows from convexity property of function.

\[ J^*_{\text{TO-LQG},3} \leq \min_C \mathbb{E}_\xi[\beta y^T C^T BC y + (1 - \beta) \tilde{y}^T C^T BC \tilde{y} + 2\beta y^T C^T S \xi + 2(1 - \beta) \tilde{y}^T C^T S \xi] \]

\[ = \min_C \text{Tr}[\beta C^T BC \Sigma + 2\beta C^T S \Sigma + (1 - \beta) C^T BC \tilde{\Sigma} + 2(1 - \beta) C^T \tilde{S} \tilde{\Sigma}] \]

\[ = \beta \min_C \text{Tr}[C^T BC \Sigma + 2C^T S \Sigma] + (1 - \beta) \min_C \text{Tr}[C^T BC \tilde{\Sigma} + 2C^T \tilde{S} \tilde{\Sigma}] \]

\[ = \beta J^*_{\text{TO-LQG},1} + (1 - \beta) J^*_{\text{TO-LQG},2}. \]

2)

Let \( \tilde{\Sigma} = \Sigma \) and \( \tilde{S} = S \), we have:

\[ J^*_{\text{TO-LQG},1} = \min_A \text{Tr}[A^T BA \Sigma + 2A^T S \Sigma]. \]
J_{TOLQG,2}^* = \min \operatorname{Tr}[\bar{A}^T B \bar{A} \Sigma + 2 \bar{A}^T \bar{S} \Sigma].

Clearly, J_{TOLQG,1}^* = J_{TOLQG,2}^*. Consequently, A^* = A^{**}. Hence,

J_{TOLQG,3}^* \leq J_{TOLQG,1}^*.

So far, we studied role of common randomness (information) in a team problem. In next section, we describe the role of common randomness in two team zero-sum game.

III. PRIVATE AND COMMON RANDOMNESS IN STATIC TEAM VS TEAM ZERO-SUM GAME

We study role of private and common randomness in static two-team zero-sum game. We compare the static LQG team with zero-sum LQG team game under private and common randomness. Then We demonstrate the two team zero-sum discrete game.

Now consider the case where there are $N + M$ DMs. Let $\mathcal{M} = \{N + 1, \ldots, M\}$. DM$_i$, $i \in \mathcal{N}$ comprise of a single team, say Team 1, and DM$_j$, $j \in \mathcal{M}$ comprise of Team 2. Team 1 and Team 2 play a zero-sum game. Let $u = (u_1, \ldots, u_N)$, $\gamma = (\gamma_1, \ldots, \gamma_N)$ denote the actions of players of Team 1 and $v = (v_{N+1}, \ldots, v_M)$, $\delta = (\delta_{N+1}, \ldots, \delta_M)$ denote the actions of players in Team 2. Suppose the function the teams want to optimize is

$$\min_{u_i = \gamma_i(y_i), i \in \mathcal{N}} \max_{v_j = \delta_j(y_j), j \in \mathcal{M}} \mathbb{E}[\kappa(u, v, \xi)].$$

**Theorem 3.1:** If the zero-sum team game admits a saddle point, randomness independent of $\xi$ does not benefit either team.

**Proof:** We have:

$$\min_{u_i = \gamma_i(y_i), i \in \mathcal{N}} \max_{v_j = \delta_j(y_j), j \in \mathcal{M}} \mathbb{E}[\kappa(u, v, \xi)] = \max_{v_j = \delta_j(y_j), j \in \mathcal{M}} \min_{u_i = \gamma_i(y_i), i \in \mathcal{N}} \mathbb{E}[\kappa(u, v, \xi)].$$

$$\min_{u_i = \gamma_i(y_i), i \in \mathcal{N}} \max_{v_j = \delta_j(y_j), j \in \mathcal{M}} \mathbb{E}[\kappa(u, v, \xi)] \geq \min_{u_i = \gamma_i(y_i, w), i \in \mathcal{N}} \max_{v_j = \delta_j(y_j, z), j \in \mathcal{M}} \mathbb{E}[\kappa(u, v, \xi)] \geq \max_{v_j = \delta_j(y_j, z), j \in \mathcal{M}} \min_{u_i = \gamma_i(y_i, w), i \in \mathcal{N}} \mathbb{E}[\kappa(u, v, \xi)].$$

(11)

$$\geq \max_{v_j = \delta_j(y_j, z), j \in \mathcal{M}} \min_{u_i = \gamma_i(y_i), i \in \mathcal{N}} \mathbb{E}[\kappa(u, v, \xi)].$$
Eq \((\ref{11})\) follows from:

\[
\max_{v_j = \delta_j(y_j, z), j \in M} \mathbb{E}[\kappa(u, v, \xi)] \geq \min_{u_i = \gamma_i(y_i, w), i \in N} \mathbb{E}[\kappa(u, v, \xi)]
\]

. Consequently,

\[
\min_{u_i = \gamma_i(y_i, w), i \in N} \max_{v_j = \delta_j(y_j, z), j \in M} \mathbb{E}[\kappa(u, v, \xi)] \geq \max_{v_j = \delta_j(y_j, z), j \in M} \min_{u_i = \gamma_i(y_i, w), i \in N} \mathbb{E}[\kappa(u, v, \xi)]
\]

\. 

**Theorem 3.2:** If zero-sum game admits a saddle point, common randomness dependent of \(\xi\) is provided to one of team, then that team benefits. Suppose the consultant provides common randomness \(z\) which is dependent of \(\xi\) to decision makers of a team say, Team 2. Then we want to optimize

\[
J_{TOZS,CR} = \min_{u_i = \gamma_i(y_i), i \in N} \max_{v_j = \delta_j(y_j), j \in M} \mathbb{E}[\kappa(u, v, \xi)].
\]

Further, \(J_{TOZS,CR} = J_{TOZS}\), where

\[
J_{TOZS} = \min_{u_i = \gamma_i(y_i), i \in N} \max_{v_j = \delta_j(y_j), j \in M} \mathbb{E}[\kappa(u, v, \xi)].
\]

**Proof:**

We know from a team decision problem with common randomness dependent of \(\xi\), then

\[
\max_{v_j = \delta_j(y_j, z), j \in M} \mathbb{E}[\kappa(u, v, \xi)] \geq \max_{v_j = \delta_j(y_j), j \in M} \mathbb{E}[\kappa(u, v, \xi)]
\]

\[
\min_{u_i = \gamma_i(y_i), i \in N} \max_{v_j = \delta_j(y_j, z), j \in M} \mathbb{E}[\kappa(u, v, \xi)] \geq \min_{u_i = \gamma_i(y_i), i \in N} \max_{v_j = \delta_j(y_j), j \in M} \mathbb{E}[\kappa(u, v, \xi)]
\]

Since we assume saddle point solution of zero-sum game,

\[
\min_{u_i = \gamma_i(y_i), i \in N} \max_{v_j = \delta_j(y_j, z), j \in M} \mathbb{E}[\kappa(u, v, \xi)] = \max_{v_j = \delta_j(y_j, z), j \in M} \min_{u_i = \gamma_i(y_i), i \in N} \mathbb{E}[\kappa(u, v, \xi)]
\]

We also have

\[
\max_{v_j = \delta_j(y_j, z), j \in M} \min_{u_i = \gamma_i(y_i), i \in N} \mathbb{E}[\kappa(u, v, \xi)] \geq \max_{v_j = \delta_j(y_j), j \in M} \min_{u_i = \gamma_i(y_i), i \in N} \mathbb{E}[\kappa(u, v, \xi)]
\]

If two-team zero sum game without common randomness admits a saddle point, then

\[
\max_{v_j = \delta_j(y_j), j \in M} \min_{u_i = \gamma_i(y_i), i \in N} \mathbb{E}[\kappa(u, v, \xi)] = \min_{u_i = \gamma_i(y_i), i \in N} \max_{v_j = \delta_j(y_j), j \in M} \mathbb{E}[\kappa(u, v, \xi)].
\]

Hence result \(J_{TOZS,CR} = J_{TOZS}\) follows.  

**Remark:**
• If the common or private information is uncorrelated with an environment or uncertainty of world, no one can gain anything from this information in team vs team zero-sum game. This is also illustrated numerically for LQG zero sum team vs team game is illustrated in Appendix C2.
• In next subsection, we describe that a team having private information correlated with environment benefits. This implies that the team with more information manage to decrease the cost and even this is true in LQG teams decision problem. This is first observed by [12] and later this is extended to LQG teams problem in [6].
• We present results in our stochastic team vs team zero sum game. We illustrate role of common randomness in team vs team LQG zero sum game by numerical examples in Appendix C3.

A. Role of private randomness dependent on \( \xi \)

Let \( y_i = \eta_i(\xi) \) be the information available at player \( i \), and \( \tilde{y}^1 = (y_1, y_2, \cdots, y_N) \) be information available at team 1 and \( \tilde{y}^2 = (y_{N+1}, y_{N+2}, \cdots, y_{N+M}) \) be the information available at team 2. Note that a team 1 is minimizing using control \( u \) and team 2 is maximizing with control \( v \).

Define the cost function

\[
J(u, v) = \mathbb{E}[\kappa(u, v, \tilde{y}^1, \tilde{y}^2, \xi)]
\]

From saddle point condition at the information structure \( (\tilde{y}^1, \tilde{y}^2) \), we have

\[
J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*)
\]

The optimal decision pair is \( (u^*, v^*) \) at the information structure \( \tilde{y}^1 \), and \( \tilde{y}^2 \). Similarly, one can define saddle point condition for null information structure and has only prior knowledge about \( \xi \), information structure is \( (\tilde{y}^1, \tilde{y}^2) \) and optimal decision pair is \( (u^0, v^0) \).

The value of information for team 1 and team 2 is defined as follows.

\[
V_1(\tilde{y}^1, \tilde{y}^2) = J(u^*, v^*) - J(u^0, v^0)
\]
\[
V_2(\tilde{y}^1, \tilde{y}^2) = -V_1(\tilde{y}^1, \tilde{y}^2)
\]

Suppose the information at a team, say team 2 is fixed, i.e., \( \eta'_i(\xi) = \eta_i(\xi) \) for \( i = N+1, \cdots, N+M \). The opponent gets more information, say team 1, i.e., \( \eta'_i(\xi) \subseteq \eta_i(\xi) \) for \( i = 1, 2, \cdots, N \).
Thus the decision set for team 1 is $A_{\eta'} \subseteq A_{\eta}$ and that for team 2 is $C_{\eta'} = C_{\eta}$. We have the following result.

**Lemma 3.1:** If the information of team 1 is increasing, i.e., $\eta'_i(\xi) \subseteq \eta_i(\xi)$ for $i = 1, 2, \cdots, N$, and the information of team 2 is fixed, i.e., $\eta'_i(\xi) = \eta_i(\xi)$ for $i = N + 1, \cdots, N + M$, then the value of information satisfy the following inequality

$$V_1(\tilde{y}^1, \tilde{y}^2) \leq V_1(\hat{y}^1, \hat{y}^2).$$

Here $y_i = \eta_i(\xi)$, $\tilde{y}^1 = (y_1, \cdots, y_N)$, $\tilde{y}^2 = (y_{N+1}, \cdots, y_{N+M})$, and $y'_i = \eta'_i(\xi)$, $\tilde{y}^1 = (y'_1, \cdots, y'_N)$, $\tilde{y}^2 = (y'_{N+1}, \cdots, y'_{N+M})$.

The proof is analogous to [6, Lemma 3.3]. For clarity purpose we provide details is as follows.

The saddle point condition at information structure $\eta(\xi)$ implies that

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*) \quad (12)$$

for $u \in A_{\eta}$, $v \in C_{\eta}$. Another saddle point condition at information structure $\eta'(\xi)$ is

$$J(\hat{u}^*, \hat{v}) \leq J(\hat{u}^*, \hat{v}^*) \leq J(\hat{u}, \hat{v}^*) \quad (13)$$

for $\hat{u} \in A_{\eta'}$ and $\hat{v} \in C_{\eta'}$.

Since $C_{\eta} = C_{\eta'}$ we can have $\hat{v}^* \in C_{\eta}$ and then it implies that

$$J(u^*, v^*) = J(u^*, \hat{v}^*). \quad (14)$$

Because $A_{\eta'} \subseteq A_{\eta}$, and $\hat{u}^* \in A_{\eta'}$ implies $\hat{u}^* \in A_{\eta}$. Further,

$$J(\hat{u}, \hat{v}^*) \geq J(\hat{u}^*, \hat{v}^*) \geq J(u^*, \hat{v}^*) = J(u^*, v^*) \quad (15)$$

Thus we get $J(\hat{u}^*, \hat{v}^*) \geq J(u^*, v^*)$. As we note that $J(u^0, v^0)$ does not change. After substracting $J(u^0, v^0)$, we have desired inequality

$$V_1(\tilde{y}^1, \tilde{y}^2) \leq V_1(\hat{y}^1, \hat{y}^2).$$

**B. Discrete team vs team zero-sum game**

In this section, we investigate discrete team vs team zero-sum game and the role of extra randomness in the team and its decision makers.

**Claim 3.3:** In discrete team vs team zero-sum game,

1) it may not admit pure-strategy saddle point solution,
2) if a coordinator provides the private randomness independent of an environment to decision makers of team then it benefit both team and improves the team cost. But it may not achieve Nash equilibrium.

3) if a consultant provides the common randomness to decision makers of team, then it lead to improve in team cost. But it may not have Nash equilibrium.

Proofs of these are difficult to obtain but we provide examples in appendix B to support our claim.

IV. DISCUSSION AND CONCLUSIONS

The value of information is classic problem in decision theory. As information increases, we anticipated that the optimal cost decreases. This is first illustrated for statistical problems in [8]. In stochastic team problem and stochastic team vs team zero sum games, the value of private information to decision makers is not explicitly presented in earlier literature.

We analyzed a stochastic team decision problem when decision makers are provided with external private randomness which is correlated or independent of environment. The private randomness independent of environment does not decrease the cost function. But this randomness dependent on environment provided to DMs in a team decreases the cost function of team compare to no randomness. In stochastic LQG team decision problem under special information structure, we have shown that the correlated randomness decreases the cost function.

We next studied stochastic team vs team zero sum game, and showed that the randomness independent of environment does not benefit either time if a game admits a saddle point condition. In LQG team vs team zero sum game, we analyze the role of common randomness which is correlated with environment for one of team, then the optimal value function decreases with information. We further extended this finding to discrete team vs team zeros sum game when there is no saddle point condition and observed that common or private randomness independent of environment benefits both team. Even common randomness dependent on environment benefit a team and improves cost. This may not lead to saddle point condition.

It opens future research direction on problem of role of private or common randomness in stochastic teams with non zero sum games and sequential stochastic dynamic teams. Another research directions is on correlated equilibrium behaviors and common knowledge in sequential stochastic team vs team games.
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A. LQG Team Problem

Now we examine an example of a LQG team problem. Let an environment \( \xi := [\mu_1, \ldots, \mu_N]^T \) be random vector; it is Gaussian distributed zero mean and covariance \( \Sigma \). Let \( y_i = \eta_i(\xi) \) be the information observed by DM\(_i\), \( y = [y_1, \ldots, y_N]^T \) information vector observed by decision makers. In a static LQG team problem optimal action is linear in information observed by decision maker. Thus action of DM\(_i\) is \( u_i = \gamma_i(y_i) = \alpha_{i1}y_i \). Then

\[
\begin{align*}
    u &= (u_1, \ldots, u_N)^T = Ay,
\end{align*}
\]

where \( A \) is diagonal matrix of dimension \( N \times N \), \( \text{diag}(A) = [\alpha_{11}, \ldots, \alpha_{N1}] \). Standard LQG problem assumes cost function to be quadratic in nature. The cost function is \( \kappa(u, \xi) := u^T Bu + 2u^T S \xi \), here \( B \) is symmetric positive matrix.

The team optimal solution of LQG team problem is \( \gamma \in \Gamma \) such that

\[
\begin{align*}
    J_{\text{TO-LQG}}^* &= \min_{\gamma \in \Gamma} J(\gamma) = \min_{u \in U} \mathbb{E}_\xi[\kappa(u, \xi)] = \min_{u \in U} \mathbb{E}_\xi[u^T Bu + 2u^T S \xi].
\end{align*}
\]

Replacing \( u = Ay \), we obtain

\[
\begin{align*}
    J_{\text{TO-LQG}}^* &= \min_A \mathbb{E}_\xi[y^T A^T BAy + 2y^T A^T S \xi].
\end{align*}
\]

Further this can expressed as deterministic optimization problem as follows.

\[
\begin{align*}
    J_{\text{TO-LQG}}^* &= \min_A \text{Tr}[A^T BA \Sigma + 2A^T S \Sigma],
\end{align*}
\]

Note that Tr denote trace of matrix.

1) Private randomness independent of \( \xi \): We will show that in LQG team problem the private randomness provided by a coordinator do not benefit the team optimal cost functional.

Consider \( \omega = [\omega_1, \ldots, \omega_N]^T \) is private randomness available to decision makers, it is Gaussian distributed with zero mean and covariance matrix \( \Sigma_1 \) is diagonal; \( \omega_i \) is private randomness available at DM\(_i\). We suppose that \( \omega_i \) is independent of \( \omega_j \) for \( i \neq j \) and it is also independent of \( y \). (\( \mathbb{E}[\omega_i \omega_j] = 0 \) for \( i \neq j \) and \( \mathbb{E}[\omega_i y_k] = 0 \) for \( i \neq k, 1 \leq i, j, k \leq N \).)

The action \( u_i = \gamma_i(y_i, \omega_i) = \alpha_{i1}y_i + \alpha_{i2}\omega_i \). Let \( u = Ay + C\omega \), where \( A \) and \( C \) are diagonal matrix of dimension \( N \times N \), \( \text{diag}(A) = [\alpha_{11}, \ldots, \alpha_{N1}] \) and \( \text{diag}(C) = [\alpha_{12}, \ldots, \alpha_{N2}] \).
The optimal expected cost functional of LQG team problem with private randomness is
\[ J_{\text{TOP,LQG}}^* \triangleq \min_{Q \in Q} E_Q \left[ \kappa(u, \xi) \right] = \min_A \text{Tr}[A^TBA \Sigma + 2A^T S \Sigma] + \min_C \text{Tr}[C^T BC \Sigma_1]. \]  
(17)

From equation (17), \( \min_C \text{Tr}[C^T BC \Sigma_1] = 0 \) if and only if \( C \) is zero matrix. Hence \( J_{\text{TOP,LQG}}^* = J_{\text{TOP,LQG}}^* \).

2) Common randomness independent of \( \xi \): We study a LQG team problem with common randomness has structure similar to that of LQG team problem with private randomness. We demonstrate that common randomness provided to decision makers by the consultant is independent of \( \xi \), then it does not improve the expected cost functional.

Consider \( \omega = [\omega_1, \ldots, \omega_N]^T \) is common randomness available to decision makers, it is Gaussian distributed with zero mean and covariance matrix \( \Sigma_2 \); \( \omega_i \) is the common randomness at DM\( i \). We suppose that \( \omega_i \) is perfect correlation with \( \omega_j \) for \( i \neq j \) and it is also independent of \( y \). \( (E[\omega_i \omega_j] = 0 \) for \( i \neq j \) and \( E[\omega_i y_k] = 0 \) for \( i \neq k \), \( 1 \leq i, j, k \leq N \).) The action \( u_i = \gamma_i(y_i, \omega_i) = \alpha_{i1} y_i + \alpha_{i2} \omega_i \). Let \( u = Ay + C\omega \). The optimal expected cost function is
\[ J_{\text{TOC,LQG}}^* = \min_A \text{Tr}[A^TBA \Sigma + 2A^T S \Sigma] + \min_C \text{Tr}[C^T BC \Sigma_2]. \]  
(18)

Note that in LQG team problem, \( B \) is symmetric positive definite matrix. From (18), expression \( \min_C \text{Tr}[C^T BC \Sigma_2] \) attains minimum value = 0 if \( C \) is zero matrix. Thus we have following relation, \( J_{\text{TO,LQG}}^* = J_{\text{TOC,LQG}}^* \).

3) Common randomness dependent on \( \xi \): Next, we demonstrate the result in (9) via an example of LQG team problem. Further we show numerically for two decision maker LQG team problem that there is strict inequality between team optimal cost with and without extra randomness, that is \( J_{\text{TOP,LQG}}^* > J_{\text{TOER,LQG}}^* \).

Consider a LQG team problem consists of an environment \( \xi = [\mu_1, \ldots, \mu_N]^T \) as random vector with mean zero and covariance matrix \( \Sigma \). The information observed by DM\( i \) is \( y_i = \eta_i(\xi) = \mu_i \), \( y = [y_1, \ldots, y_N]^T \). Let \( \omega = [\omega_1, \ldots, \omega_N]^T \) be the extra randomness provided by a consultant to decision makers. Furthermore, assume that \( \omega = f(\xi) \) and \( f \) is linear function in \( \xi \). Thus \( \omega_i = \sum_j \phi_{ij} \mu_j \), \( \omega = \Phi \xi = \Phi y \), \( \Phi \) is matrix of dimension \( N \times N \), with entries in \( \phi_{ij} \geq 0 \) and \( \sum_j \phi_{ij} = 1 \). The cost function is \( \kappa(u, \xi) := u^T Bu + 2u^T S \xi \), the optimal expected cost under extra randomness is
\[ J_{\text{TOE,R}}^* = \min_{u \in U} E_{\xi}[u^T Bu + 2u^T S \xi]. \]
i, j \leq \omega \text{ consultant to decision makers. Consider E}\DM \text{observed at B}.

In this example, we suppose \xi \sim N(0, \Sigma), taking expectation and rewriting above expression, we obtain deterministic optimization problem as follows.

\mathcal{J}_{TOER,LQG}^* = \min_{A,C} \text{Tr}[A^TB^T\Sigma + 2A^T\Sigma + 2\Phi C^TBA + \Phi C^TBC\Phi \Sigma + 2\Phi C^T\Sigma]. \tag{19}

Intuitively, in LQG team problem with no extra randomness can described as \textit{incomplete information} static LQG team problem. Since extra randomness is linear function of an environment and under assumption of nonzero linear coefficient (\phi_{ij} \neq 0 \text{ for all } 1 \leq i, j \leq N), LQG team problem with extra randomness can be describe as \textit{complete information} static LQG team problem. Thus it is natural to expect that \mathcal{J}_{TOER}^* < \mathcal{J}_{TO}^*.

To support our claim of \mathcal{J}_{TOER}^* < \mathcal{J}_{TO}^*, we numerically evaluate the optimal cost functional with and without extra randomness which is dependent on \xi for LQG two team problem and show that our claim is indeed true. Further, we show impact of correlation coefficient \{\phi_{ij}, 1 \leq i, j \leq 2\} on optimal cost functional.

4) Numerical example–LQG team problem: Let \xi = [\mu_1, \mu_2]^T denote the state of nature or an environment having probability distribution \mathcal{N}(0, \Sigma). Let \eta_i(\xi) = \mu_i be the information observed at DM_i for 1 \leq i \leq 2. Let \omega = [\omega_1, \omega_2]^T be an extra randomness provided by a consultant to decision makers.

Thus we have \( A = \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{21} \end{bmatrix}, \quad C = \begin{bmatrix} \alpha_{12} & 0 \\ 0 & \alpha_{22} \end{bmatrix} \). Team optimal cost from (19) is

\mathcal{J}_{TOER}^* = \min_{A,C} \text{Tr}[A^TB^T\Sigma + 2A^T\Sigma + 2\Phi C^TBA + \Phi C^TBC\Phi \Sigma + 2\Phi C^T\Sigma].

In this example, we suppose B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_{\mu_1}^2 & \sigma_{\mu_1,\mu_2}^2 \\ \sigma_{\mu_1,\mu_2}^2 & \sigma_{\mu_2}^2 \end{bmatrix}.

We define \( \delta_1 = \mathbb{E}[y_1w_1] = \phi_{11}\sigma_{\mu_1}^2 + \phi_{12}\sigma_{\mu_1,\mu_2}^2, \quad \delta_2 = \mathbb{E}[y_1w_2] = \phi_{21}\sigma_{\mu_1}^2 + \phi_{22}\sigma_{\mu_1,\mu_2}^2, \quad \delta_3 = \mathbb{E}[y_2w_1] = \phi_{11}\sigma_{\mu_1,\mu_2}^2 + \phi_{12}\sigma_{\mu_1}^2, \quad \delta_4 = \mathbb{E}[y_2w_2] = \phi_{21}\sigma_{\mu_1,\mu_2}^2 + \phi_{22}\sigma_{\mu_1}^2, \quad \delta_5 = \mathbb{E}[u_1^2] = \phi_{11}^2\sigma_{\mu_1}^2 + \phi_{12}^2\sigma_{\mu_2}^2 + \phi_{11}\phi_{12}\sigma_{\mu_1,\mu_2}^2, \quad \delta_6 = \mathbb{E}[u_2^2] = \phi_{21}^2\sigma_{\mu_1,\mu_2}^2 + \phi_{22}^2\sigma_{\mu_1}^2 + \phi_{21}\phi_{22}\sigma_{\mu_1,\mu_2}^2]. \)
\[ \phi_{11}\phi_{12}\sigma_{\mu_1\mu_2}^2, \quad \delta_6 = \mathbb{E}[w_3^2] = \phi_{21}\sigma_{\mu_1}^2 + \phi_{22}\sigma_{\mu_2}^2 + \phi_{21}\phi_{22}\sigma_{\mu_1\mu_2}^2, \quad \delta_7 = \mathbb{E}[w_1w_2] = \phi_{11}\phi_{21}\sigma_{\mu_1}^2 + (\phi_{22}\phi_{11} + \phi_{12}\phi_{21})\sigma_{\mu_1\mu_2}^2 + \phi_{22}\phi_{12}\sigma_{\mu_2}^2, \quad \delta_8 = \mathbb{E}[w_1\xi_1] = \phi_{11}\sigma_{\mu_1}^2 + \phi_{12}\sigma_{\mu_2}^2, \quad \delta_9 = \mathbb{E}[w_2\xi_2] = \phi_{21}\sigma_{\mu_1\mu_2}^2 + \phi_{22}\sigma_{\mu_2}^2. \]

Now rewriting team optimal cost function we obtain,

\[ J_{TO\text{ER}}^* = \min_{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}} 2\alpha_{11}\sigma_{y_1}^2 - 2\alpha_{11}\alpha_{21}\sigma_{y_1y_2}^2 + \alpha_{21}\sigma_{y_2}^2 + 2\alpha_{11}\alpha_{12}\delta_1 - \alpha_{21}\alpha_{12}\delta_2 - \alpha_{11}\alpha_{22}\delta_3 + \alpha_{22}\alpha_{21}\delta_4 + 2\alpha_{12}\delta_5 - 2\alpha_{12}\alpha_{22}\delta_7 + \alpha_{22}\delta_6 + 2(\alpha_{11}\sigma_{y_1}^2 + \alpha_{21}\sigma_{y_2}^2) + 2(\alpha_{12}\delta_8 + \alpha_{22}\delta_9). \]

Differentiating above expression with respect to \( \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \) and equating to 0. We have

\[
\begin{bmatrix}
4\sigma_{y_1}^2 & -2\sigma_{y_1y_2}^2 & 2\delta_1 & -\delta_3 \\
-2\sigma_{y_1y_2}^2 & 2\sigma_{y_2}^2 & -\delta_2 & \delta_4 \\
2\delta_3 & -\delta_2 & 4\delta_5 & -2\delta_7 \\
-\delta_3 & \delta_4 & -2\delta_7 & 2\delta_6
\end{bmatrix}
\begin{bmatrix}
\alpha_{11} \\
\alpha_{21} \\
\alpha_{12} \\
\alpha_{22}
\end{bmatrix}
= 
\begin{bmatrix}
-2\sigma_{y_1}^2 \\
-2\sigma_{y_2}^2 \\
-2\delta_8 \\
-2\delta_9
\end{bmatrix}.
\]

Notice that computing optimal \( \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \) via solving linear systems of equations and finding optimal expected cost is computationally tedious. Without loss of generality, we suppose \( \sigma_{\mu_1}^2 = \sigma_{\mu_2}^2 = 1 \) and \( \sigma_{\mu_1\mu_2}^2 = \frac{1}{4} \). Furthermore, we fix \( \phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \) and evaluate the minimum team cost under optimal \( \alpha_{11}^*, \alpha_{12}^*, \alpha_{21}^*, \alpha_{22}^* \). Note that \( \phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \) determines the correlation of extra randomness with observations available at decision makers. From numerical computation in table (I), we make following concluding remarks.

1) In distributed static LQG team problem without extra randomness, the team optimal cost is highest.

2) In distributed static LQG team problem, only one decision maker having extra randomness which is correlated with \( \xi \) do not lead to improve in the team optimal cost. Instead it lead to increase in the team optimal cost.

3) In distributed static LQG team problem, all decision maker having extra randomness which is correlated with \( \xi \) lead to improvement in the team optimal cost. Thus we have strict inequality between \( J_{TO}^* \) and \( J_{TO\text{ER}}^* \), that means \( J_{TO\text{ER}}^* < J_{TO}^* \).

4) If an extra randomness provided by a consultant is an average of the observations \( \mu_1 \) and \( \mu_2 \), then team optimal cost is best than any other convex combination of the observations \( \mu_1 \) and \( \mu_2 \). Hence correlation coefficient \( \phi_{ij} \) for \( 1 \leq i, j \leq 2 \) plays significant role to attain minimal team optimal cost in distributed static LQG team problem with extra randomness dependent on \( \xi \).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
 & \((\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22})\) & \((\alpha_{11}^*, \alpha_{12}^*, \alpha_{21}^*, \alpha_{22}^*)\) & \min \mathbb{E}[\kappa(\alpha, \xi)] \\
\hline
No randomization & \((0, 0, 0, 0)\) & \((-0.6452, -1.1613, 0, 0)\) & \(-1.806\) \\
\hline
DM \(_1\) have randomness & \((\frac{1}{2}, \frac{1}{2}, 0, 0)\) & \((0, -1, -0.3024, 2.7513)\) & \(-0.477\) \\
\hline
Both DM have randomness & \((\frac{1}{3}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})\) & \((-0.3434, -0.7046, -2.7862, -4.0062)\) & \(-5.2974\) \\
\hline
Both DM have randomness & \((\frac{2}{3}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})\) & \((-0.5122, -1.4833, -2.6067, -3.2171)\) & \(-4.5211\) \\
\hline
Both DM have randomness & \((\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})\) & \((-0.7045, -0.7058, -0.6765, -1.522)\) & \(-3.6923\) \\
\hline
\end{tabular}
\caption{Comparison of expected cost with different randomization provided to DM}
\end{table}

**B. Proof of Lemma 3.3**

We prove our claim via illustrating an example of two-team discrete game.

Consider two-team label them as Team 1 and \(2\), Team 1 consists of a decision maker and Team 2 comprises two decision makers. Let \(\xi = [\mu_1, s_1, s_2]^T\) denote an environment or the state of nature; it is random vector with discrete distribution \(p(\xi)\). Each decision maker observes an environment partially since decision maker are situated distributed manner.

Let \(y_1 = \eta(\xi)\) denote an observation available at decision maker of Team 1; \(z_j = \zeta_j(\xi)\) represent an observation available at DM\(_j\) of Team 2. Decision rule at Team 1 and 2 is

\[ \gamma_1 : y_1 \to u_1 \]

and

\[ \delta_j : z_j \to v_j \]

\(j = 1, 2\).

Without loss of generality, we assume that \(\mu_1, s_j\) is binary random variable take values \(\{0, 1\}\); \(y_1 = \eta(\xi) = \mu_1, z_j = \zeta_j(\xi) = s_j\), for \(j = 1, 2\). Moreover, we consider \(u_1, v_j \in \{L, R\}, j = 1, 2\).

Binary random variable \(\mu_1, s_1\) and \(s_2\) defined as follows.

\[ \mu_1 = \begin{cases} 
1 \quad \text{with prob.} \quad p_1 \\
0 \quad \text{with prob.} \quad 1 - p_1.
\end{cases} \]

\[ s_1 = \begin{cases} 
\mu_1 \quad \text{with prob.} \quad p \\
0 \quad \text{with prob.} \quad 1 - p.
\end{cases} \]
\[ s_2 = \begin{cases} 1 - \mu_1 & \text{with prob. } q \\ s_1 & \text{with prob. } 1 - q. \end{cases} \]

The joint distribution of \((\mu_1, s_1, s_2)\) is \(P(\mu_1, s_1, s_2)\) and is written as

\[ P(\mu_1 = 0, s_1 = 0, s_2 = 0) = (1 - p_1)(1 - q) \]
\[ P(\mu_1 = 0, s_1 = 0, s_2 = 1) = (1 - p_1)q \]
\[ P(\mu_1 = 0, s_1 = 1, s_2 = 0) = 0 \]
\[ P(\mu_1 = 0, s_1 = 1, s_2 = 1) = 0 \]
\[ P(\mu_1 = 1, s_1 = 0, s_2 = 0) = p_1(1 - p) \]
\[ P(\mu_1 = 1, s_1 = 0, s_2 = 1) = 0 \]
\[ P(\mu_1 = 0, s_1 = 1, s_2 = 0) = p_1q \]
\[ P(\mu_1 = 1, s_1 = 1, s_2 = 1) = p_1p(1 - q) \]

There are four possible decision rule available at each decision maker. The decision rule of a decision maker is

\[ u_1^i = \gamma_1^i(y_1) = \gamma_1^i(\mu_1) = \begin{cases} L & \text{if } \mu_1 = 0 \\ R & \text{if } \mu_1 = 1 \end{cases} \]
\[ u_1^2 = \gamma_2^2(y_1) = \gamma_2^2(\mu_1) = \begin{cases} L & \text{if } \mu_1 = 1 \\ R & \text{if } \mu_1 = 0 \end{cases} \]
\[ u_1^3 = \gamma_3^3(y_1) = \gamma_3^3(\mu_1) = \begin{cases} L & \text{if } \mu_1 = 1 \text{ or } \mu_1 = 0 \\ R & \text{if } \mu_1 = 1 \text{ or } \mu_1 = 0 \end{cases} \]
\[ u_1^4 = \gamma_4^4(y_1) = \gamma_4^4(\mu_1) = \begin{cases} L & \text{if } \mu_1 = 1 \text{ or } \mu_1 = 0 \\ R & \text{if } \mu_1 = 1 \text{ or } \mu_1 = 0 \end{cases} \]

\[ v_1^1 = \delta_1^1(z_1) = \delta_1^1(s_1) = \begin{cases} L & \text{if } s_1 = 0 \\ R & \text{if } s_1 = 1 \end{cases} \]
\[ v_1^2 = \delta_1^2(z_1) = \delta_1^2(s_1) = \begin{cases} L & \text{if } s_1 = 1 \\ R & \text{if } s_1 = 0 \end{cases} \]
TABLE II
PAYOFF MATRIX: TEAM VS TEAM ZERO-SUM GAME

\[
\begin{array}{|c|c|c|c|}
\hline
& LL & LR & RL & RR \\
\hline
L & 20 & 0 & 1 & 30 \\
R & 20 & 1 & 0 & 30 \\
\hline
\end{array}
\]

We next formulate team vs team zero-sum game. Team 1 seeks to maximize the expected payoff whereas Team 2 seeks to minimize the expected payoff. We describe payoff matrix in table II. In table II row vector denotes actions of Team 1 and corresponding payoff; column vector denotes actions of Team 2 and corresponding payoff. Since observations available at each decision maker in team is function of state of nature \(\xi\) and \(\xi\) is random variable, we evaluate the expected payoff for different actions of decision makers and it is

\[
E \left[ \kappa \left( \gamma_1^l(\mu_1), \delta_1^m(s_1), \delta_2^n(s_2) \right) \right] = \sum_{\mu_1, s_1, s_2 \in \{0,1\}^3} \kappa \left( \gamma_1^l(\mu_1), \delta_1^m(s_1), \delta_2^n(s_2) \right) \mathbf{P}(\mu_1, s_1, s_2).
\]

where \(1 \leq l, m, n \leq 4\). Enumerating the expected payoff over all possible actions of decision makers, we obtain

\[
E \left[ \kappa \left( \gamma_1^1(\mu_1), \delta_1^1(s_1)\delta_2^1(s_2) \right) \right] = 20 - 20q + 20p_1q + 10p_1p - 30p_1pq
\]

\[
E \left[ \kappa \left( \gamma_1^2(\mu_1), \delta_1^1(s_1)\delta_2^2(s_2) \right) \right] = 40 - 40p_1 - 19q + 19p_1q - 29p_1pq + 30p_1p
\]
\[ \mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^1(s_1)\delta_2^1(s_2) \right) \right] = 20 - 20q + 30p_1p - 29p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^1(s_1)\delta_2^1(s_2) \right) \right] = 20 - 19q + 19p_1q + 10p_1p - 30p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^1(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = 1 - p_1 + 29q - 29p_1q + 19p_1pq + p_1p \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^2(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = 30 - 31p_1q + p_1 + 20p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = 1 + 29q - 30p_1q + 20p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = 30q - 30p_1q + 19p_1pq + p_1p \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^1(\mu_1), \delta_1^3(s_1)\delta_2^1(s_2) \right) \right] = 20 - 20q + 19p_1pq + p_1p \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^2(\mu_1), \delta_1^3(s_1)\delta_2^1(s_2) \right) \right] = 20 - 19q - p_1q + 20p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^3(s_1)\delta_2^1(s_2) \right) \right] = 20 - 20q + 20p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^3(s_1)\delta_2^1(s_2) \right) \right] = 20 - 19q - p_1q + 19p_1pq + p_1p \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^1(\mu_1), \delta_1^2(s_1)\delta_2^2(s_2) \right) \right] = 1 - p_1 + 29q - 29p_1q - 30p_1p - 30p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^2(\mu_1), \delta_1^2(s_1)\delta_2^2(s_2) \right) \right] = 30q - 31p_1q + p_1 - 29p_1q + 30p_1p \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^2(s_1)\delta_2^2(s_2) \right) \right] = 1 + 29q - 29p_1q - 30p_1p - 29p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^2(s_1)\delta_2^2(s_2) \right) \right] = 30q - 30p_1q - 30p_1p - 30p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^1(\mu_1), \delta_1^3(s_1)\delta_2^2(s_2) \right) \right] = 20q - 20p_1q + p_1 + 30p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^2(\mu_1), \delta_1^3(s_1)\delta_2^2(s_2) \right) \right] = 1 - p_1 + 19q - 19p_1q + 29p_1pq + p_1p \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^3(s_1)\delta_2^2(s_2) \right) \right] = 20q - 20p_1q + 29p_1pq + p_1p \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^3(s_1)\delta_2^2(s_2) \right) \right] = 1 + 19q - 19p_1q - p_1p + 30p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^1(\mu_1), \delta_1^2(s_1)\delta_2^2(s_2) \right) \right] = 30 - 29q + 29p_1q - 10p_1p - 19p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^2(\mu_1), \delta_1^2(s_1)\delta_2^2(s_2) \right) \right] = 30 - 10p_1 - 30q + 30p_1q - 20p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^2(s_1)\delta_2^2(s_2) \right) \right] = 30 - 29q + 29p_1q - 10p_1p - 20p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^2(s_1)\delta_2^2(s_2) \right) \right] = 30 - 30q + 30p_1q - 10p_1p - 19p_1pq \]

\[ \mathbb{E} \left[ \kappa \left( \gamma_1^1(\mu_1), \delta_1^3(s_1)\delta_2^2(s_2) \right) \right] = 20q - 20p_1q + p_1 + 19p_1p - 19p_1pq \]
$$\mathbb{E} \left[ \kappa \left( \gamma_1^2(\mu_1), \delta_1^3(s_1)\delta_2^3(s_2) \right) \right] = 1 - p_1 + 19q - 19p_1q + 20p_1p - 20p_1pq$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^4(s_1)\delta_2^2(s_2) \right) \right] = 20q - 20p_1q + 20p_1pq$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^4(s_1)\delta_2^2(s_2) \right) \right] = 1 + 19q - 19p_1q + 19p_1p - 19p_1pq$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^6(\mu_1), \delta_1^6(s_1)\delta_2^2(s_2) \right) \right] = 30 - 29q + 29p_1q - 29p_1p + 29p_1pq$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^8(\mu_1), \delta_1^8(s_1)\delta_2^2(s_2) \right) \right] = 30 - 30q + 30p_1q - 30p_1p + 30p_1pq$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^4(s_1)\delta_2^3(s_2) \right) \right] = 20 - 20p_1$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^6(\mu_1), \delta_1^6(s_1)\delta_2^3(s_2) \right) \right] = 20 + 19p_1p$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^8(\mu_1), \delta_1^8(s_1)\delta_2^3(s_2) \right) \right] = 20 + 19p_1p$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^4(s_1)\delta_2^3(s_2) \right) \right] = 20$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^4(s_1)\delta_2^3(s_2) \right) \right] = 20$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^4(s_1)\delta_2^3(s_2) \right) \right] = 20$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^6(\mu_1), \delta_1^6(s_1)\delta_2^3(s_2) \right) \right] = 20$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^4(s_1)\delta_2^3(s_2) \right) \right] = 1 - p_1$$
$$\mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^4(s_1)\delta_2^3(s_2) \right) \right] = 0$$
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^1(\mu_1), \delta_1^1(s_1)\delta_2^2(s_2) \right) \right] = p_1 + 29p_1p
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^2(\mu_1), \delta_1^1(s_1)\delta_2^2(s_2) \right) \right] = 1 - p_1 + 20p_1p
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^3(\mu_1), \delta_1^1(s_1)\delta_2^2(s_2) \right) \right] = 30p_1p
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^4(\mu_1), \delta_1^1(s_1)\delta_2^2(s_2) \right) \right] = 1 + 29p_1p
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^5(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = 30p_1 - 29p_1p
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^6(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = 30 - 30p_1p
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^7(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = 30 - 30p_1p
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^8(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = 30 - 29p_1p
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^9(\mu_1), \delta_1^2(s_1)\delta_2^1(s_2) \right) \right] = p_1
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^{10}(\mu_1), \delta_1^3(s_1)\delta_2^2(s_2) \right) \right] = 1 - p_1
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^{11}(\mu_1), \delta_1^3(s_1)\delta_2^2(s_2) \right) \right] = 0
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^{12}(\mu_1), \delta_1^3(s_1)\delta_2^2(s_2) \right) \right] = 1
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^{13}(\mu_1), \delta_1^4(s_1)\delta_2^2(s_2) \right) \right] = 30
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^{14}(\mu_1), \delta_1^4(s_1)\delta_2^2(s_2) \right) \right] = 30
\]
\[
\mathbb{E} \left[ \kappa \left( \gamma_1^{15}(\mu_1), \delta_1^4(s_1)\delta_2^2(s_2) \right) \right] = 30
\]

From above expression, it is difficult to make any comment on saddle point solution of zero-sum game. Thus we suppose \( p_1 = \frac{1}{4}, p = \frac{1}{3} \) and \( q = \frac{2}{3} \) but it is also possible that under different range of \( p_1, p, q \) our claim holds true. Rewriting expected payoff matrix for zero-sum game in table \( III \) in table \( III \) row vector denote strategies of a Team 2, column vector denote strategies of a Team 1 and corresponding expected payoff. Here, Team 2 wishes to minimize the expected payoff and Team 1 wishes to maximize the expected payoff. The security level of Team 1 is

\[
\overline{V}(A) = \max_i \min_j a_{ij} = 0.25
\]

Similarly, the security level of Team 2 is

\[
\underline{V}(A) = \min_i \max_j a_{ij} = 1.
\]

Notice that we have \( \overline{V}(A) > \underline{V}(A) \), it implies this game do not admit the pure strategy saddle point solution.
1) Role of the private randomness independent of \( \xi \): We are interested to understand the role of the private randomness in two-team zero-sum game. We assume a coordinator provides the private randomness to decision maker of a team, say Team 1 decision maker. Further we assume that these private randomization is independent of \( \xi \).

Consider Team 1 decision maker has private randomization over its strategies and plays strategy \( \gamma_1^i(\mu_1) \) with probability \( a_i \) for \( 1 \leq i \leq 4 \) and \( \sum_{i=1}^{4} a_i = 1 \). That is

\[
\gamma_1(\mu_1) = \begin{cases} 
\gamma_1^1(\mu_1) & \text{with prob. } a_1 \\
\gamma_1^2(\mu_1) & \text{with prob. } a_2 \\
\gamma_1^3(\mu_1) & \text{with prob. } a_3 \\
\gamma_1^4(\mu_1) & \text{with prob. } a_4.
\end{cases}
\]

Then the expected payoff is

\[
\mathbb{E} \left[ \kappa \left( \gamma_1(\mu_1)\delta^i_1(s_1)\delta^k_2(s_2) \right) \right] = \sum_{i=1}^{4} \mathbb{E} \left[ \kappa \left( (\gamma_1(\mu_1) = \gamma_1^i(\mu_1))\delta^i_1(s_1)\delta^k_2(s_2) \right) \right] a_i
\]
for \(1 \leq j, k \leq 4\). We have evaluated the expected payoff and given in table [IV]. From table [IV], notice that a Team 2 best response will be \((\delta^3_1(s_1)\delta^4_2(s_2))\) or \((\delta^4_1(s_1)\delta^3_2(s_2))\) depend on probability vector \(a = [a_1, a_2, a_3, a_4]\) at Team 1 (i.e. private randomization). Without loss of generality, we assume \(a_3 = a_4\), now observe that \(a_1\) and \(a_2\) determines the best response of Team 2. We demonstrate this as follows.

1) If \(a_1 < a_2\), the best response of team 2 will be \((\delta^3_1(s_1)\delta^4_2(s_2))\) and expected payoff will be \((0.75a_1 + 0.25a_2 + 1a_3)\). Further assume \(a_2 = 2a_1\), \(a_3 = a_4 = \frac{1}{12}\), then \(a_1 = \frac{5}{18}\) and expected payoff is 0.43.

2) If \(a_1 > a_2\), the best response of team 2 will be \((\delta^4_1(s_1)\delta^3_2(s_2))\) and expected payoff will be \((0.25a_1 + 0.75a_2 + 1a_4)\). Similarly, we assume \(a_1 = 2a_2\), \(a_3 = a_4 = \frac{1}{12}\), then \(a_2 = \frac{5}{18}\) and expected payoff is 0.43.

3) If \(a_1 = a_2\), the best response of team 2 will be \((\delta^3_1(s_1)\delta^4_2(s_2))\) or \((\delta^4_1(s_1)\delta^3_2(s_2))\) and expected payoff will be \(a_1 + a_3\). We assume \(a_3 = a_4 = \frac{1}{12}\), then \(a_2 = a_1 = \frac{5}{12}\) and expected

| \(\delta^1_1(s_1)\delta^2_1(s_2)\) | \(9.16a_1 + 22.3a_2 + 7.54a_3 + 9.66a_4\) |
| \(\delta^1_1(s_1)\delta^3_1(s_2)\) | \(16.39a_1 + 26.18a_2 + 16.45a_3 + 16.13a_4\) |
| \(\delta^1_2(s_1)\delta^3_1(s_2)\) | \(7.80a_1 + 8.27a_2 + 7.77a_3 + 8.30a_4\) |
| \(\delta^1_1(s_1)\delta^2_2(s_2)\) | \(16.08a_1 + 15.72a_2 + 16.30a_3 + 15.83a_4\) |
| \(\delta^1_2(s_1)\delta^3_2(s_2)\) | \(11.91a_1 + 11.94a_2 + 11.69a_3 + 12.08a_4\) |
| \(\delta^2_1(s_1)\delta^3_2(s_2)\) | \(13.61a_1 + 11.38a_2 + 13.55a_3 + 13.11a_4\) |
| \(\delta^2_1(s_1)\delta^3_2(s_2)\) | \(10.77a_1 + 10.80a_2 + 11.11a_3 + 11.02a_4\) |
| \(\delta^2_1(s_1)\delta^4_2(s_2)\) | \(14.69a_1 + 14.19a_2 + 14.69a_3 + 14.16a_4\) |
| \(\delta^2_1(s_1)\delta^4_2(s_2)\) | \(18.33a_1 + 18.41a_2 + 18.41a_3 + 18.33a_4\) |
| \(\delta^3_1(s_1)\delta^4_2(s_2)\) | \(2.41a_1 + 1.83a_2 + 1.83a_3 + 1.66a_4\) |
| \(\delta^2_1(s_1)\delta^3_2(s_2)\) | \(20\) |
| \(\delta^2_1(s_1)\delta^3_2(s_2)\) | \(0.75a_1 + 0.25a_2 + 1a_3\) |
| \(\delta^2_1(s_1)\delta^3_2(s_2)\) | \(2.66a_1 + 2.41a_2 + 2.5a_3 + 3.41a_4\) |
| \(\delta^3_1(s_1)\delta^3_2(s_2)\) | \(5.08a_1 + 27.5a_2 + 27.5a_3 + 27.5a_4\) |
| \(\delta^3_1(s_1)\delta^4_2(s_2)\) | \(0.25a_1 + 0.75a_2 + 1a_4\) |
| \(\delta^3_1(s_1)\delta^4_2(s_2)\) | \(30\) |

**TABLE IV**

Two-team zero-sum game expected payoff with team 1 has private randomization
TABLE V

| \(\delta_1(s_1)\delta_2(s_2)\) | payoff |
|-------------------------------|--------|
| \(\delta_1(s_1)\delta_1(s_2)\) | 16.33  |
| \(\delta_1(s_1)\delta_2(s_2)\) | 21.81  |
| \(\delta_2(s_1)\delta_2(s_2)\) | 8.10   |
| \(\delta_2(s_1)\delta_1(s_2)\) | 15.87  |
| \(\delta_3(s_1)\delta_2(s_2)\) | 11.92  |
| \(\delta_3(s_1)\delta_1(s_2)\) | 12.32  |
| \(\delta_2(s_1)\delta_2(s_2)\) | 10.83  |
| \(\delta_4(s_1)\delta_2(s_2)\) | 14.36  |
| \(\delta_1(s_1)\delta_2(s_2)\) | 18.38  |
| \(\delta_4(s_1)\delta_1(s_2)\) | 1.97   |
| \(\delta_1(s_1)\delta_2(s_2)\) | 20     |
| \(\delta_2(s_1)\delta_2(s_2)\) | 0.43   |
| \(\delta_1(s_1)\delta_2(s_2)\) | 2.57   |
| \(\delta_2(s_1)\delta_2(s_2)\) | 21.27  |
| \(\delta_1(s_1)\delta_2(s_2)\) | 0.57   |
| \(\delta_1(s_1)\delta_2(s_2)\) | 30     |

TWO-TEAM ZERO-SUM GAME WITH TEAM 1 PRIVATE RANDOMIZATION OVER ITS STRATEGIES \(a_3 = a_4 = \frac{1}{12}, a_1 = \frac{5}{12}, a_2 = \frac{10}{12}\).

This implies that under private randomization at one of team, it do not admit Nash equilibrium solution.

Observe that the expected payoff of Team 2 has improved from 1 to 0.43 if \(a_1 < a_2\) or \(a_1 > a_2\) and 0.5 if \(a_1 = a_2\) where Team 2 wishes to minimize the expected payoff.

Now from table [V], note that the best strategy of DM1 and DM2 in Team 2 would be to play pure strategy as \(\delta_1(s_1)\) and \(\delta_2(s_2)\) to minimize the expected payoff.

Furthermore, one of DM in Team 2 having private randomness may not lead to improve in the expected payoff. To demonstrate this, consider DM1 in Team 2 has private randomization
over his strategies.

\[
\delta_1(s_1) = \begin{cases} 
\delta^1_1(s_1) & \text{with prob. } b_1 \\
\delta^1_2(s_1) & \text{with prob. } b_2 \\
\delta^1_3(s_1) & \text{with prob. } b_3 \\
\delta^1_4(s_1) & \text{with prob. } b_4 
\end{cases}
\]

and \(0 \leq b_i \leq 1\) for \(1 \leq i \leq 4\), \(\sum_{i=1}^{4} b_i = 1\).

The expected payoff payoff is

\[
\mathbb{E} \left[ \kappa \left( \gamma_1(\mu_1) \delta_1(s_1) \delta^k_2(s_2) \right) \right] = \sum_{i,j} \mathbb{E} \left[ \kappa \left( \gamma_1(\mu_1) = \gamma^i_1(\mu_1) \right) \left( \delta_1(s_1) = \delta^j_1(s_1) \right) \delta^k_2(s_2) \right] a_i b_j,
\]

We illustrated the expected payoff matrix in table VI. If DM_1 in Team 2 do not play pure strategy, assume \(b_1 = b_3 = 0\), \(b_2 = \frac{1}{4}\), and \(b_4 = \frac{3}{4}\), then DM_2 of Team 2 will play strategy \(\delta^3_2(s_2)\) to minimize the expected payoff. Thus expected payoff 0.815. Note that 0.815 < \(\bar{V}_A\) but greater than pure strategy expected payoff (it is clear from table V) since expected payoff under pure strategy solution is 0.43. Here we assume if decision maker in Team 1 having private randomization with probability vector \(a_3 = a_4 = \frac{1}{12}\), \(a_1 = \frac{5}{18}\) \(a_2 = \frac{10}{18}\).

2) Role of common randomness independent of \(\xi\): Now consider common randomness independent of \(\xi\) is provided to DM_1 and DM_2 of team 2, i.e. Team 2 does joint randomization over its strategy then best for for team 2 to put positive mass on strategies \((\delta^1_1(s_1), \delta^3_2(s_2))\) or \((\delta^1_3(s_1), \delta^1_2(s_2))\). Otherwise its expected payoff more than pure strategy (it is clear from table III). In discrete team vs team zero-sum game with common randomness, do not admit Nash equilibrium solution. It also lead to improve in the expected payoff.
C. Example: LQG team vs team zero-sum game

Now, we illustrate an example of LQG zero-sum team vs team game and show that common randomness independent of environment $\xi$ does not benefit. We also demonstrate that common randomness dependent on $\xi$ benefit a team having extra randomness.

Consider two team LQG zero sum game, Team 1 and Team 2 consists of a decision maker and two decision makers, respectively. Let $\xi = [\mu_1, s_1, s_2]^T$ denote an environment or state of nature; it is random vector having probability distribution $N(0, \Sigma)$, $\Sigma$ is covariance matrix. Let $y_i = \eta_i(\xi)$ be the observations about $\xi$ available at decision maker $i$ of Team 1, for $i = 1$; $z_j = \zeta_j(\xi)$ represents the observations about $\xi$ available at decision maker $j$ of Team 2, for $j = 1, 2$. Mathematical simplicity, we assume $y_1 = \eta_1(\xi) = \mu_1$, $z_j = \zeta_j(\xi) = s_j$, $j = 1, 2$. In standard LQG two-team zero-sum game decision rule is defined as follows.

$$
\gamma_i : y_i \mapsto u_i, \\
\gamma_i \in \Gamma_i \text{ and } u_i \in U_i \text{ for } i = 1; \\
\delta_j : z_j \mapsto v_j, \\
\delta_j \in \Delta_j \text{ and } v_j \in V_j \text{ for } j = 1, 2.
$$

The optimal decision rule $(u_1^*, v_1^*, v_2^*)$ such that

$$
J_{ZS,LQG}(u_1, v_1^*, v_2^*) \leq J_{ZS,LQG}(u_1^*, v_1^*, v_2^*) \leq J_{ZS,LQG}(u_1^*, v_1^*, v_2^*),
$$

(20)

for all $u_1 \in U_1$, $v_1 \in V_1$ and $v_2 \in V_2$; $J_{ZS,LQG}(u_1, v_1, v_2) = E_\xi[\kappa(u_1, v_1, v_2, \xi)]$.

The cost function:

$$
\kappa(u_1, v_1, v_2, \xi) = \kappa(\theta, \xi),
$$

$$
= \theta^T B \theta + 2\theta^T S \xi,
$$

(21)

where $\theta = [u_1, v_1, v_2]^T$, $B = \begin{bmatrix} -1 & r_{11} & r_{12} \\ r_{11} & 1 & q_{12} \\ r_{12} & q_{12} & 1 \end{bmatrix}$, here $r_{11}$ and $r_{12}$ characterizes the coupling among teams, that is $r_{11}$ and $r_{12}$ is coupling of DM$_1$ of Team 1 with DM$_1$ and DM$_2$ of Team 2 respectively. And $q_{12}$ denotes coupling among DM$_1$ and DM$_2$ of Team 2. Moreover, we assume that Team 1 seeks to maximize the expected payoff and Team 2 seeks to minimize the expected
payoff. It is required that the cost function $\mathbb{E}_\xi[\kappa(u_1, v_1, v_2, \xi)]$ to be concave in $u_1$ and convex in $v_1$ and $v_2$. Hence, we assume $1 - q_{12}^2 > 0$ and $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Two-team LQG zero-sum game admits a saddle point solution (for which we refer the reader to [6] lemma 3.1, 3.2, theorem 3.1), i.e.

$$\max_{u_1 \in U_1} \min_{(v_1, v_2) \in V} \mathbb{E}_\xi[\kappa(u_1, v_1, v_2, \xi)] = \min_{(v_1, v_2) \in V} \max_{u_1 \in U_1} \mathbb{E}_\xi[\kappa(u_1, v_1, v_2, \xi)].$$

Equality in (23) follows from $\tilde{v} = V \cdot v$ in [6, lemma 3.1, 3.2, theorem 3.1].

Case I: Payoff. It is required that the cost function

$$\mathbb{E}_\xi[\kappa(u_1, v_1, v_2, \xi)]$$

will be satisfied for $\alpha$. This is not at all interesting. If $r_{11} = r_{12} = 0$, then there is no coupling among Team 1 and 2, as well as among decision makers of Team 2. This is not at all interesting. If $r_{11} = r_{12} = 0$, then there is no coupling among team 1 and 2. Problem becomes team decision problem. Hence we suppose $r_{11}, r_{12}, q_{12} \neq 0$.

Next, we analyze the role of common randomness in LQG two-team zero-sum game. We describe two cases as follows.

- **Case I:** Common randomness independent of $\xi$.
- **Case II:** Common randomness dependent on $\xi$. 
2) **Common randomness independent of** \( \xi \):

**Proposition A.1:** In LQG two-team zero-sum stochastic game, common randomness independent of \( \xi \) do not benefit the team.

**Proof:** Consider a coordinator provides common randomness which is independent of environment \( \xi \) to the decision makers of teams. For mathematical simplicity, we assume common randomness is available at one of team, say Team 2. The common randomness provided to decision maker DM\(_1\) and DM\(_2\) of team 2 is represented as \( \omega \), and also \( \omega \perp \xi \). The decision rule of a decision maker of Team 1 is

\[
\gamma_1 : y_1 \rightarrow u_1,
\]

and decision rule of Team 2 decision makers are

\[
\delta_j : z_j \times \omega \rightarrow v_j,
\]

\( j = 1, 2 \). Actions of decision makers are

\[
u_1 = \gamma_1(y_1) = \alpha_{11}y_1,
\]

\[
v_j = \delta_j(z_j, \omega) = \alpha_{2j}z_j + \beta_{2j}\omega,
\]

for \( j = 1, 2 \). Rewriting above expression, we obtain

\[
\theta = A\tilde{y} + \beta\omega,
\]

here, \( \theta = \begin{bmatrix} u_1 \\ v_1 \\ v_2 \end{bmatrix} \), \( A = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{21} & 0 \\ 0 & 0 & \alpha_{22} \end{bmatrix} \), \( \tilde{y} = \begin{bmatrix} y_1 \\ z_1 \\ z_2 \end{bmatrix} \), \( \beta = \begin{bmatrix} 0 \\ \beta_{21} \\ \beta_{22} \end{bmatrix} \).

The expected payoff of LQG two team zero-sum game with common randomness is

\[
J_{ZS,CR,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta) = \mathbb{E}_\xi[\tilde{y}^TBA\tilde{y} + 2\tilde{y}^TA^TS\xi + 2\tilde{y}^TA^TB\beta\omega + \omega^T\beta^T\beta\omega + 2\omega^T\beta^T\xi],
\]

\[
= \text{Tr}[A^TBA\Sigma + 2A^TS\Sigma + \beta^T\beta\Sigma_2].
\]

Equality in (26) because \( \omega \perp \xi \), \( \omega \sim N(0, \Sigma_2) \).

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Clearly, from above expression, minimization of $\text{Tr}[\beta^TB\beta\Sigma]$ attained at $\beta$ equals to zero, i.e. $\beta_1 = 0$, $\beta_{21} = 0$, $\beta_{22} = 0$ for given $B$ and $\Sigma_2 > 0$.

$$\max_{\alpha_{11}} \min_{\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}} J_{ZS,CR,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}) = \max_{\alpha_{11}} \min_{\alpha_{21}, \alpha_{22}} \text{Tr}[A^TBA\Sigma + 2A^T\Sigma S]$$

$$= \max_{\alpha_{11}} \min_{\alpha_{21}, \alpha_{22}} J_{ZS,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22})$$

$$= \min_{\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}} \max_{\alpha_{11}} J_{ZS,CR,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})$$

Hence we conclude that common randomness independent of $\xi$ do not benefit the team having common randomness.

3) **Common randomness dependent on $\xi$:** Suppose the common randomness available at decision makers of Team 2 of two-team LQG zero-sum game; it is denoted as $\omega$. The decision rule of a decision maker in Team 1 is

$$\gamma_1 : y_1 \rightarrow u_1,$$

and decision rule of Team 2 decision makers are

$$\delta_j : z_j \times \omega \rightarrow v_j,$$

$j = 1, 2$. Actions of decision makers are

$$u_1 = \gamma_1(y_1) = \alpha_{11}y_1,$$

$$v_j = \delta_j(z_j, \omega) = \alpha_{2j}z_j + \beta_{2j}\omega,$$

for $j = 1, 2$. We have

$$\theta = A\bar{y} + \beta\omega,$$

where $\theta = \begin{bmatrix} u_1 \\ v_1 \\ v_2 \end{bmatrix}$, $A = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{21} & 0 \\ 0 & 0 & \alpha_{22} \end{bmatrix}$, $\bar{y} = \begin{bmatrix} y_1 \\ z_1 \\ z_2 \end{bmatrix}$, $\beta = \begin{bmatrix} 0 \\ \beta_{21} \\ \beta_{22} \end{bmatrix}$.

Moreover it is assume that the common randomness is dependent on an environment $\xi$. Hence $\omega$ is function of $\xi$, that is $\omega = f(\xi)$; $f(\cdot)$ is measurable function. Let $f$ be the linear function, then

$$\omega = f(\xi) = \phi_{11}\mu_1 + \phi_{21}s_1 + \phi_{22}s_2$$

$$= \Phi^T\bar{y} = \Phi^T\xi.$$
Where $\Phi = [\phi_{11}, \phi_{21}, \phi_{22}]^T$, $\tilde{y} = \xi$ and $\xi \sim N(0, \Sigma)$. The expected cost functional is

$$J_{ZS,CR,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}) = \mathbb{E}_\xi[\tilde{y}^TBA\tilde{y} + 2\tilde{y}^TBS\xi + 2\tilde{y}^TBB\omega + \omega^T\beta^T\beta\omega + 2\omega^T\beta^T S\xi],$$

$$= \text{Tr}[BA\Sigma + 2AS\Sigma + 2AB\beta\Sigma + \beta^T\beta\Sigma + 2\beta^T S\Sigma].$$

(26)

In (26), $\tilde{\beta} = \beta \Phi^T$. Goal is to find $(\alpha^{*}_{11}, \alpha^{*}_{21}, \alpha^{*}_{22}, \beta^{*}_{21}, \beta^{*}_{22})$ such that

$$J_{ZS,CR,LQG}(\alpha^{*}_{11}, \alpha^{*}_{21}, \alpha^{*}_{22}, \beta^{*}_{21}, \beta^{*}_{22}) \leq J_{ZS,CR,LQG}(\alpha^{*}_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}) \leq J_{ZS,CR,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})$$

for $\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22} \in \mathbb{R}$.

Source of information (source of common randomness) can act as a mole or consultant depending on type of information it provides. If source of information is a mole then $\omega = \phi_{11}\mu_1$. It implies $\phi_{21} = 0$, and $\phi_{22} = 0$. If source of information is consultant, then $\omega = \phi_{21}s_1 + \phi_{22}s_2$. We will investigate two different cases based on source of information and types of information it provides.

a) Suppose the source of information is a mole or spy and it provide information (common randomness) $\omega = \phi_{11}\mu_1$. Let $J_{ZS,CR,LQG}^{a,*}$ denote the saddle point solution of LQG two-team zero-sum game with common randomness when source of common randomness to Team 2 decision makers is spy.

b) Let $J_{ZS,CR,LQG}^{b,*}$ represents the saddle point solution of LQG two-team zero-sum game with common randomness when source of common randomness to Team 2 decision makers is consultant and $\omega = \phi_{21}s_1 + \phi_{22}s_2$.

Intuitively, we expect to have following inequalities.

$$J_{ZS,CR,LQG}^{a,*} \leq J_{ZS,LQG}^{*}.$$  

(27)

$$J_{ZS,CR,LQG}^{b,*} \leq J_{ZS,LQG}^{*}.$$  

(28)

Note $J_{ZS,LQG}^{*}$ is saddle point solution of LQG two-team zero-sum game with no common randomness.

From (26), analytically, it is difficult to prove the inequalities in (27), (28). Hence we conjecture result in (27), (28). Now we present numerical results and show that above inequalities are true.
is
\[
J(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}) = -\alpha_{11}^2 \sigma_{\mu_1}^2 + \alpha_{21}^2 \sigma_s^2 + \alpha_{22}^2 \sigma_{s_2}^2 + 2r_{11} \alpha_{11} \alpha_{21} \sigma_{\mu_1,s_1}^2 + 2r_{12} \alpha_{11} \alpha_{22} \sigma_{\mu_1,s_2}^2 \\
+ 2q_{12} \alpha_{21} \alpha_{22} \sigma_{s_1,s_2}^2 + 2(\alpha_{21} \alpha_{22} \beta_{21} + r_{12} \alpha_{11} \beta_{22}) \sigma_{\mu_1,w}^2 + 2(\alpha_{21} \beta_{21} + q_{12} \alpha_{21} \beta_{22}) \sigma_{s_1,w}^2 \\
+ 2(q_{12} \alpha_{22} \beta_{21} + \alpha_{22} \beta_{22}) \sigma_{s_2,w}^2 + (\beta_{21}^2 + 2q_{12} \beta_{21} \beta_{22} + \beta_{22}^2) \sigma_{w}^2 + 2\alpha_{11} \sigma_{\mu_1}^2 \\
- 2\alpha_{21} \sigma_{s_1}^2 - 2\alpha_{22} \sigma_{s_2}^2 - 2\beta_{21} \sigma_{s_1,w}^2 - 2\beta_{22} \sigma_{s_2,w}^2. \tag{29}
\]

We know that LQG two-team zero-sum game has saddle point solution, that is
\[
\max_{\alpha_{11}} \min_{\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}} J_{ZS,CR,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}) = \min_{\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}} \max_{\alpha_{11}} J_{ZS,CR,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}). \tag{30}
\]

To evaluate \(\max_{\alpha_{11}} \min_{\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}} J_{ZS,CR,LQG}(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})\), we differentiate (29) with respect to \(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}\) and equate to 0. We obtain linear systems of equations as follows.

\[
\begin{bmatrix}
-\sigma_{\mu_1}^2 & r_{11} \sigma_{\mu_1,s_1}^2 & r_{12} \sigma_{\mu_1,s_2}^2 & r_{11} \sigma_{\mu_1,w}^2 & r_{12} \sigma_{\mu_1,w}^2 \\
 r_{11} \sigma_{\mu_1,s_1}^2 & \sigma_{s_1}^2 & q_{12} \sigma_{s_1,s_2}^2 & \sigma_{s_1,w}^2 & q_{12} \sigma_{s_1,w}^2 \\
r_{12} \sigma_{\mu_1,s_2}^2 & q_{12} \sigma_{s_1,s_2}^2 & \sigma_{s_2}^2 & q_{12} \sigma_{s_2,w}^2 & \sigma_{s_2,w}^2 \\
r_{11} \sigma_{\mu_1,w}^2 & \sigma_{s_1,w}^2 & q_{12} \sigma_{s_1,w}^2 & \sigma_{w}^2 & q_{12} \sigma_{w}^2 \\
r_{12} \sigma_{\mu_1,w}^2 & q_{12} \sigma_{s_1,w}^2 & \sigma_{s_2,w}^2 & q_{12} \sigma_{s_2,w}^2 & \sigma_{w}^2 \\
\end{bmatrix}
\begin{bmatrix}
\alpha_{11} \\
\alpha_{21} \\
\alpha_{22} \\
\beta_{21} \\
\beta_{22} \\
\end{bmatrix} = \begin{bmatrix}
-\sigma_{\mu_1}^2 \\
\sigma_{s_1}^2 \\
\sigma_{s_2}^2 \\
\sigma_{s_1,w}^2 \\
\sigma_{s_2,w}^2 \\
\end{bmatrix}.
\]

Numerically, we compare our result for different values matrix \(B\).

1) \(B = \begin{bmatrix}
-1 & 1/4 & 1/4 \\
1/4 & 1 & 1/2 \\
1/4 & 1/2 & 1
\end{bmatrix}\), 2) \(B = \begin{bmatrix}
-1 & 1/4 & 1/2 \\
1/4 & 1 & 1/2 \\
1/2 & 1/2 & 1
\end{bmatrix}\).

We assume \(\Sigma = \begin{bmatrix}
1/4 & 1/2 \\
1/2 & 1
\end{bmatrix}\) for all numerical results.

a) When source of information is a *mole* and \(\omega = \phi_{11} \mu_1\), we have \(\mathbb{E}[\omega] = 0\),

\[
\mathbb{E}[\omega^2] = \sigma_{\omega}^2 = \phi_{11} \sigma_{\mu_1}^2,
\]

\[
\sigma_{\mu_1,\omega}^2 = \phi_{11} \sigma_{\mu_1}^2,
\]

\[
\sigma_{s_1,\omega}^2 = \phi_{11} \mathbb{E}[\mu_1 s_1] = \phi_{11} \sigma_{\mu_1,s_1}^2,
\]

\[
\sigma_{s_2,\omega}^2 = \phi_{11} \mathbb{E}[\mu_1 s_2] = \phi_{11} \sigma_{\mu_1,s_2}^2.
\]
\[
\begin{array}{|c|c|c|}
\hline
(r_{11}, r_{12}, q_{12}) & (\phi_{11}, \phi_{21}, \phi_{22}) & J_{ZS,CR,LQG}^{a,*} \\
\hline
\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right) & (\frac{1}{2}, 0, 0) & 0.4012 \\
\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) & (\frac{1}{2}, 0, 0) & 0.2037 \\
\hline
\end{array}
\]

TABLE VII
WITH RANDOMIZATION: COMPARISON OF \( J_{ZS,CR,LQG}^{a,*} \) FOR DIFFERENT VALUES OF \( r_{11}, r_{12}, q_{12} \).

\[
\begin{array}{|c|c|c|}
\hline
(r_{11}, r_{12}, q_{12}) & (\phi_{11}, \phi_{21}, \phi_{22}) & J_{ZS,CR,LQG}^{b,*} \\
\hline
\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) & (0, \frac{1}{4}, \frac{1}{4}) & 0.1616 \\
\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right) & (0, \frac{1}{4}, \frac{1}{4}) & 0.2435 \\
\hline
\end{array}
\]

TABLE VIII
WITH RANDOMIZATION: COMPARISON OF \( J_{ZS,CR,LQG}^{b,*} \) FOR DIFFERENT VALUES OF \( r_{11}, r_{12}, q_{12} \).

Case 1) \( B = \begin{bmatrix} -1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \)

After solving linear systems of equation, we have
\[
\alpha_{11}^* = 0.9615, \quad \alpha_{21}^* = 0.8052, \quad \alpha_{22}^* = 0.8052, \quad \beta_{21}^* = -0.7103, \quad \beta_{22}^* = -0.7103.
\]

Team cost functional is
\[
J_{ZS,CR,LQG}^{a,*} = \max_{\alpha_{11}} \min_{\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}} J(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22}) = \max_{\alpha_{11}} \min_{\alpha_{21}, \alpha_{22}} J_{ZS,CR,LQG}^{a,*(\alpha_{11}, \alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})} = 0.4012.
\]

Case 2) \( B = \begin{bmatrix} -1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \)

Solving linear systems of equations we obtain \( \alpha_{11}^* = 0.8500, \quad \alpha_{21}^* = 0.8052, \quad \alpha_{22}^* = 0.8052, \quad \beta_{21}^* = -0.0693, \quad \beta_{22}^* = -1.7693. \)

Evaluating team cost functional
\[
J_{ZS,CR,LQG}^{a,*} = 0.2037.
\]

b) When a consultant provides an information, \( \omega = \phi_{21}s_1 + \phi_{22}s_2 \). Note that \( \mathbb{E}[w] = 0, \)
\[
\begin{array}{|c|c|c|}
\hline
(r_{11}, r_{12}, q_{12}) & (\phi_{11}, \phi_{21}, \phi_{22}) & J_{ZS,LQG}^* \\
\hline
(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) & (0, 0, 0) & 0.598 \\
(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) & (0, 0, 0) & 1.8991 \\
\hline
\end{array}
\]

**TABLE IX**

Without Randomization: Comparison of \(J_{ZS,LQG}^*\) for different values of \(r_{11}, r_{12}, q_{12}\).

\[
\sigma_w^2 = \mathbb{E}[w^2] = \phi_{21}^2 \sigma_{s_1}^2 + \phi_{22}^2 \sigma_{s_2}^2 + 2\phi_{21}\phi_{22} \sigma_{s_1,s_2}^2.
\]

\[
\sigma_{\mu_1,w}^2 = \mathbb{E}[\mu_1 w] = \phi_{21}^2 \sigma_{\mu_1,s_1}^2 + \phi_{22}^2 \sigma_{\mu_1,s_2}^2.
\]

\[
\sigma_{s_1,w}^2 = \mathbb{E}[s_1 w] = \phi_{21}^2 \sigma_{s_1}^2 + \phi_{22}^2 \sigma_{s_1,s_2}^2.
\]

\[
\sigma_{s_2,w}^2 = \mathbb{E}[s_2 w] = \phi_{21}^2 \sigma_{s_1,s_2}^2 + \phi_{22}^2 \sigma_{s_2}^2.
\]

We suppose \(\phi_{21} = \frac{1}{2}, \phi_{22} = \frac{1}{2}\).

Case 1) \(B = \begin{bmatrix}
-1 & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & 1 & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{2} & 1
\end{bmatrix}\)

Solving linear system of equation we have \(\alpha_{11}^* = 1.0381, \alpha_{21}^* = 2, \alpha_{22}^* = 2, \beta_{21}^* = -1.391, \beta_{22}^* = -1.391\) and team optimal cost \(J_{ZS,CR,LQG}^{b,*} = 0.1616\).

Case 2) \(B = \begin{bmatrix}
-1 & \frac{1}{4} & \frac{1}{2} \\
\frac{1}{4} & 1 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix}\).

Then \(\alpha_{11}^* = 1.0515, \alpha_{21}^* = 2, \alpha_{22}^* = 2, \beta_{21}^* = -1.3333, \beta_{22}^* = -1.5086\) and team optimal cost \(J_{ZS,CR,LQG}^{b,*} = 0.2435\).

From table VII, VIII, IX it clear that inequalities in (27), (28) satisfy numerically. Observe that common randomness dependent on \(\xi\) provided by either a mole or consultant benefits the team vs team zero-sum game.