NONCOMMUTATIVE CROSS-RATIO AND SCHWARZ DERIVATIVE

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Dedicated to Emma Previato.

Abstract. We present here a theory of noncommutative cross-ratio, Schwarz derivative and their connections and relations to the operator cross-ratio. We apply the theory to “noncommutative elementary geometry” and relate it to noncommutative integrable systems. We also provide a noncommutative version of the celebrated “pentagramma mirificum”.

1. Introduction

Cross-ratio and Schwarz derivative are one of the most famous invariants in mathematics (see [13], [16], [17]). Different versions of their noncommutative analogs and their various applications to integrable systems, control theory and other subjects were discussed in several publications including [4]. In this paper we recall some of these definitions, revisit the previous results and discuss their connections with each other and with noncommutative elementary geometry.

The paper is organized as follows. In Sections 1, 2 we recall a definition of noncommutative cross-ratios based on the theory of noncommutative quasi-Plücker invariants (see [7, 8]), in Section 3 we use the theory of quasideterminants (see [6]) to obtain noncommutative versions of Menelaus’s and Ceva’s theorems. In Section 5 we revisit an approach to noncommutative Schwarz derivative from [19]. In section 6 we compare our definition of cross-ratio with the operator version used in control theory [21] and show how Schwarz derivatives appear as the infinitesimal analogs of noncommutative cross-ratios appear.

It is our pleasure to dedicate this paper to Emma Previato, whose intelligence, erudition, interest to various domains of our science are spectacular and her friendship is constant and fidel. Her results ([4]) were one of important motives which inspired us to think once more about the role of non-commutative cross-ratio.

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2. Quasi-Plücker coordinates

We begin with a list of basic properties of noncommutative cross-ratios introduced in [18]. To this end we first recall the definition and properties of quasi-Plücker coordinates; observe that we shall only deal with the quasi-Plücker coordinates for $2 \times n$-matrices over a noncommutative division ring $\mathcal{R}$. The corresponding theory for general $k \times n$-matrices is presented in [7, 8].

Recall (see [5, 6] and subsequent papers) that for a matrix \( \begin{pmatrix} a_{ik} & a_{ij} \\ a_{2k} & a_{2j} \end{pmatrix} \) one can define four quasideterminants provided the corresponding elements are invertible:

\[
\begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} = a_{1k} - a_{1i}a_{2i}^{-1}a_{2k}, \quad \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix} = a_{1i} - a_{1k}a_{2i}^{-1}a_{2i}, \\
\begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix} = a_{2k} - a_{2i}a_{1i}^{-1}a_{1k}, \quad \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} = a_{2i} - a_{2k}a_{1i}^{-1}a_{1k}.
\]

Let $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \end{pmatrix}$ be a matrix over $\mathcal{R}$.

**Lemma 2.1.** Let $i \neq k$. Then

\[
\begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}^{-1} = \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} = \begin{vmatrix} a_{1k} & a_{1j} \\ a_{2k} & a_{2j} \end{vmatrix}^{-1}
\]

if the corresponding expressions are defined.

Note that in the formula the boxed elements on the left and on the right must be on the same level.

**Definition 2.2.** We call the expressions

\[
q_{ij}^k(A) = \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} \begin{vmatrix} a_{1k} & a_{1j} \\ a_{2k} & a_{2j} \end{vmatrix}^{-1} = \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} \begin{vmatrix} a_{1k} & a_{1j} \\ a_{2k} & a_{2j} \end{vmatrix}^{-1}
\]

the quasi-Plücker coordinates of matrix $A$.

Our terminology is justified by the following observation. Recall that in the commutative case the expressions

\[
p_{ik}(A) = \begin{vmatrix} a_{1i} & a_{1k} \\ a_{2i} & a_{2k} \end{vmatrix} = a_{1i}a_{2k} - a_{1k}a_{2i}
\]

are the Plücker coordinates of $A$. One can see that in the commutative case

\[
q_{ij}^k(A) = \frac{p_{jk}(A)}{p_{ik}(A)},
\]

i.e. quasi-Plücker coordinates are ratios of Plücker coordinates.

Let us list here the properties of quasi-Plücker coordinates over (noncommutative) division ring $\mathcal{R}$. We shall sometimes write $q_{ij}^k$ instead of $q_{ij}^k(A)$ where it cannot lead to a confusion.

1) Let $g$ be an invertible matrix over $\mathcal{R}$. Then

\[
q_{ij}^k(g \cdot A) = q_{ij}^k(A).
\]

2) Let $A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an invertible diagonal matrix over $\mathcal{R}$. Then

\[
q_{ij}^k(A \cdot \Lambda) = \lambda_i^{-1} \cdot q_{ij}^k(A) \cdot \lambda_j.
\]
3) If \( j = k \) then \( q_{ij}^k = 0 \); if \( j = i \) then \( q_{ij}^k = 1 \) (we always assume \( i \neq k \)).

4) \( q_{ij}^k \cdot q_{j\ell}^k = q_{i\ell}^k \). In particular, \( q_{ij}^i q_{ji}^i = 1 \).

5) "Noncommutative skew-symmetry": For distinct \( i, j, k \)
\[
q_{ij}^k \cdot q_{jk}^i \cdot q_{ki}^j = -1.
\]
One can also rewrite this formula as \( q_{ij}^k q_{ji}^i = -q_{ik}^j \).

6) "Noncommutative Plücker identity": For distinct \( i, j, k, \ell \)
\[
q_{ij}^k q_{\ell j}^i + q_{ik}^\ell q_{j\ell i}^k = 1.
\]

Remark 2.3. We presented here the theory of the left quasi-Plücker coordinates for 2 by \( n \) matrices where \( n > 2 \). The theory of the right quasi-Plücker coordinates for \( n \) by 2 or, more generally, for \( n \) by \( k \) matrices where \( n > k \) can be found in [7, 8].

3. Definition and basic properties of cross-ratios

3.1. Non-commutative cross-ratio: basic definition. We define cross-ratios over (non-commutative) division ring \( \mathcal{R} \) by imitating the definition of classical cross-ratios in homogeneous coordinates. Namely, if four points in (real or complex) projective plane can be represented in homogeneous coordinates by vectors \( a, b, c, d \) such that \( c = a + b \) and \( d = ka + b \), then their cross-ratio is \( k \).

So we let
\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}
\]
be four vectors in \( \mathcal{R}^2 \). We define their cross-ratio \( \kappa = \kappa(x, y, z, t) \) by equations
\[
\begin{cases}
t = x\alpha + y\beta \\ z = x\alpha\gamma + y\beta\gamma \cdot \kappa
\end{cases}
\]
where \( \alpha, \beta, \gamma, \kappa \in \mathcal{R} \).

In order to obtain explicit formulas, let us consider the matrix
\[
\begin{pmatrix}
x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2
\end{pmatrix}.
\]
We shall identify its columns with $x, y, z, t$. Then we have the following theorem (see [18])

**Theorem 3.1.**

\[ \kappa(x, y, z, t) = q_{zt}^y \cdot q_{tx}^x. \]

Note that in the generic case

\[
\kappa(x, y, z, t) = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & \end{vmatrix}^{-1} \begin{vmatrix} x_1 & t_1 \\ x_2 & t_2 \end{vmatrix},
\]

which shows that (3.1)

\[ \kappa \]

and also demonstrates the importance of conjugation in the noncommutative world.

**Corollary 3.2.** Let $x, y, z, t$ be vectors in $\mathcal{R}$, $g$ be a 2 by 2 matrix over $\mathcal{R}$ and $\lambda_i \in \mathcal{R}$, $i = 1, 2, 3, 4$. If the matrix $g$ and elements $\lambda_i$ are invertible then

\[
\kappa(gx\lambda_1, gy\lambda_2, gz\lambda_3, gt\lambda_4) = \lambda_3^{-1}\kappa(x, y, z, t)\lambda_3.
\]

Also, as expected, in the commutative case the right hand side of (3.1) equals $\kappa(x, y, z, t)$.

**Remark 3.3.** Note that the group $GL_2(\mathcal{R})$ acts on vectors in $\mathcal{R}^2$ by multiplication from the left: $(g, x) \mapsto gx$, and the group $\mathcal{R}^\times$ of invertible elements in $\mathcal{R}$ acts by multiplication from the right: $(\lambda, x) \mapsto x\lambda^{-1}$. These actions determine the action of $GL_2(\mathcal{R}) \times T_4(\mathcal{R})$ on $P_4 = \mathcal{R}^2 \times \mathcal{R}^2 \times \mathcal{R}^2 \times \mathcal{R}^2$ where $T_4(\mathcal{R}) = (\mathcal{R}^\times)^4$. The cross-ratios are relative invariants of the action.

The following theorem generalizes the main property of cross-ratios to the noncommutative case (see [18]).

**Theorem 3.4.** Let $\kappa(x, y, z, t)$ be defined and $\kappa(x, y, z, t) \neq 0, 1$. Then 4-tuples $(x, y, z, t)$ and $(x', y', z', t')$ from $P_4$ belong to the same orbit of $GL_2(\mathcal{R}) \times T_4(\mathcal{R})$ if and only if there exists $\mu \in \mathcal{R}^\times$ such that

\[
\kappa(x, y, z, t) = \mu \cdot \kappa(x', y', z', t') \cdot \mu^{-1}.
\]

The following corollary shows that the cross-ratios we defined satisfy cocycle conditions (see [13]).

**Corollary 3.5.** For all vectors $x, y, z, t, w$ the following equations hold

\[ \kappa(x, y, z, t) = \kappa(w, y, z, t)\kappa(x, w, z, t) \]

\[ \kappa(x, y, z, t) = 1 - \kappa(t, y, z, x), \]

if all the cross-ratios exist.

The last proposition can also be generalized as follows:

**Corollary 3.6.** For vectors $x, x_1, x_2, \ldots x_n, z, t \in \mathcal{R}^2$ one has

\[ \kappa(x, x, z, t) = 1 \]

and

\[ \kappa(x_{n-1}, x_n, z, t)\kappa(x_{n-2}, x_{n-1}, z, t) \ldots \kappa(x_1, x_2, z, t) = \kappa(x_1, x_n, z, t) \]

where we assume that all the cross-ratios exist.
3.2. Noncommutative cross-ratios and permutations. There are 24 cross-ratios defined for vectors \(x, y, z, t \in \mathcal{R}^2\), if we permute them. They are related by the following formulas:

**Proposition 3.7.** Let \(x, y, z, t \in \mathcal{R}\). Then
\[
\begin{align*}
q_{tx}^x \kappa(x, y, z, t)q_{yt}^x &= q_{tx}^y \kappa(x, y, z, t)q_{yt}^y = \kappa(y, x, z, t); \\
q_{xz}^y \kappa(x, y, z, t)q_{zy}^x &= q_{xz}^x \kappa(x, y, z, t)q_{zy}^y = \kappa(z, t, x, y); \\
q_{zy}^z \kappa(x, y, z, t)q_{yz}^x &= q_{zy}^x \kappa(x, y, z, t)q_{yz}^y = \kappa(t, z, x, y); \\
\kappa(x, y, z, t)^{-1} &= \kappa(y, x, z, t).
\end{align*}
\]

Note again the effect of conjugation in the noncommutative case since \(q_{ij}^k\) and \(q_{ji}^k\) are inverses to each other. Also observe that using Proposition 3.7 and the cocycle condition (corollary 3.5) one can get all 24 formulas for cross-ratios of \(x, y, z, t\) knowing just one of them.

3.3. Noncommutative triple ratio. Let \(\mathcal{R}\) be a division ring as above; we shall work with the plane \(\mathcal{R}^2\) with the right action of \(\mathcal{R}\). Consider the triangle with vertices \(O(0,0), X(x,0), Y(0,y)\). Let \(A(a_1,a_2)\) be a point on side \(XY\), \(B(b,0)\) be a point on side \(OX\) and \(C(0,c)\) be a point on side \(OY\). Recall that we have
\[x^{-1}a_1 + y^{-1}a_2 = 1.\]

Let \(P(p_1,p_2)\) be the point of intersection of \(XC\) and \(YB\). Then one has
\[p_1 = (y-c)(yb^{-1} - cx^{-1})^{-1}, \quad p_2 = (x-b)(by^{-1} - xc^{-1})^{-1}.\]

Let \(Q\) be the point of intersection of \(OP\) and \(XY\). The non-commutative cross ratio for \(Y, A, Q, X\) is equal to
\[x^{-1}(1 - p_1p_2^{-1}a_2a_1^{-1})^{-1}x\]
(compare it with the Ceva theorem in elementary geometry).

By changing the order of \(Y, A, Q, X\) we get up to a conjugation
\[p_1p_2^{-1}a_2a_1^{-1} = -(y-c)(yb^{-1} - cx^{-1})^{-1}(x-b)^{-1}xe^{-1}(yb^{-1} - cx^{-1})bx^{-1}(x-a_1)a_1^{-1}.\]

In the commutative case (up to a sign) we have
\[p_1p_2^{-1}a_2a_1^{-1} = (y-c)b(x-b)^{-1}(x-a_1)a_1^{-1}.\]

Note that \((x-a_1)^{-1}a_1^{-1} = YA/AX\) and (3.7) is a (non-commutative analogue of) triple cross-ratio (see section 6.5 in the book by Ovsienko and Tabachnikov [16]).

3.4. Noncommutative angles and cross-ratios. Let \(\mathcal{R}\) be a noncommutative division ring.

Recall that noncommutative angles (or noncommutative \(\lambda\)-lengths) \(T^i_j\) for vectors \(A_1, A_2, A_3, A_4 \in \mathcal{R}^2\), \(A_i = (a_{1i}, a_{2i})\) are defined by the formulas
\[T^i_j = x_{ji}^{-1}x_{jk}^{-1}x_{ik}^{-1}.
\]
Here \(x_{ij} = a_{1j} - a_{1i}a_{2i}^{-1}a_{2j}\), or \(x_{ij} = a_{2j} - a_{2i}a_{1i}^{-1}a_{1j}\) (see [3]). On the other hand the cross-ratio \(\kappa(A_1, A_2, A_3, A_4) = \kappa(1, 2, 3, 4)\) (see definition 2.2) is
\[\kappa(1, 2, 3, 4) = q_{54}^{21}q_{43}^{1}.\]

It implies that
\[\kappa(1, 2, 3, 4) = x_{43}^{-1}(T_{43}^{21})^{-1}T_{43}^{21}x_{43}.\]
In other words, cross-ratio is a ratio of two angles up to a conjugation.

Under transformation $x_{ij} \mapsto \lambda_i x_{ij}$ we have

$$T_i^{jk} \mapsto T_i^{jk} \cdot \lambda_i^{-1}.$$ 

Also note that

$$T_i^{jk}(T_i^{mk})^{-1} = q_j^i q_m^n$$

i.e. $T_i^{jk}(T_i^{mk})^{-1}$ is a cross-ratio. Further details on the properties of $T_i^{jk}$ can be found in [3].

4. Noncommutative Menelaus’ and Ceva’s theorems

4.1. Higher rank quasi-determinants: reminder. Let $A = (a_{ij})$, $i, j = 1, 2, \ldots, n$ be a matrix over a ring. Denote by $A^{pq}$ the submatrix of matrix $A$ obtained from $A$ by removing the $p$-th row and the $q$-th column. Let $r_p = (a_{p1}, a_{p2}, \ldots, a_{pq}, \ldots a_{pn})$ be the row submatrix and $c_q = (a_{1q}, a_{2q}, \ldots, a_{pq}, \ldots a_{nq})^T$ be the column submatrix of $A$. Following [5] we say that the quasideterminant $|A|_{pq}$ is defined if and only the submatrix $A^{pq}$ is invertible. In this case

$$|A|_{pq} = a_{pq} - r_p(A^{pq})^{-1}c_q .$$

In the commutative case $|A|_{pq} = (-1)^{p+q} \det A / \det A^{pq}$. It is sometimes convenient to use the notation

$$|A|_{pq} = \begin{vmatrix} \cdots & a_{1q} & \cdots \\ \cdots & \cdots & \cdots \\ a_{pq} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} .$$

4.2. Commutative Menelaus’ and Ceva’s theorems. We follow the affine geometry proof. Let the points $D, E, F$ lie on the straight lines $AB, BC$ and $AC$ respectively (see figure 1 (a)). Denote by $\lambda_D$ the coefficient for homothecy with center $D$ sending $B$ to $C$, by $\lambda_E$ the coefficient for homothecy with center $E$ sending $C$ to $A$, and by $\lambda_F$ the coefficient for homothecy with center $F$ sending $A$ to $B$. Note that

$$\lambda_D = (b_i - d_i)^{-1}(c_i - d_i), \ i = 1, 2,$$

$$\lambda_E = (c_i - e_i)^{-1}(a_i - e_i), \ i = 1, 2,$$

$$\lambda_F = (a_i - f_i)^{-1}(b_i - f_i), \ i = 1, 2.$$ 

Here $(a_1, a_2)$ are the coordinates of $A$ etc. We shall omit the indices and write $\lambda_D = (b - d)^{-1}(c - d)$, etc.

**Theorem 4.1.** Points $E, D, F$ belong to a straight line if and only if

$$(a - f)^{-1}(b - f) \cdot (c - e)^{-1}(a - e) \cdot (b - d)^{-1}(c - d) = 1 .$$

This is the Menelaus’ theorem in the commutative case.
**Figure 1.** The classical Menelaus (part (a)) and Ceva (part (b)) theorems.

**Proof.** The points belong to the same straight line iff the product of transformations $\lambda_D, \lambda_E, \lambda_F$ leave the point $B$ unchanged, so

$$\lambda_F \lambda_E \lambda_D = 1$$

i.e.

$$\left( a - f \right)^{-1} (b - f) \cdot \left( c - e \right)^{-1} (a - e) \cdot \left( b - d \right)^{-1} (c - d) = 1.$$  

\[ \blacksquare \]

Somewhat dually, one obtains Ceva theorem (see figure 1 (b)):

**Theorem 4.2.** Lines $AD, BE$ and $CF$ intersect each other in a point $O$ if and only if

$$\left( e - a \right)^{-1} (e - c) \cdot \left( f - b \right)^{-1} (f - a) \cdot \left( d - c \right)^{-1} (d - b) = -1.$$  

This is the Ceva’ theorem in the commutative case.

### 4.3. Non-commutative Menelaus’ and Ceva’s theorems.

Let $R$ be a noncommutative division ring. Consider $R^2$ as the right vector space over $R$. For a point $X \in R^2$ denote by $x_i$ its $i$-th coordinate, $i = 1, 2$. Here and below we shall use the properties of quasideterminants, see [5, 6]:

**Proposition 4.3.** Let points $X$ and $Y$ are in generic position, i.e. that matrix

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

is invertible. Then the points $X, Y, Z \in R^2$ belong to the same straight line (in the sense of linear algebra) in if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$  

**Proof.** From the general theory of quasideterminants it follows that that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} = 1 - \lambda - \mu$$
where \( \lambda, \mu \in R \) satisfy the equation \( X\lambda + Y\mu = Z \). Note that \( X, Y \) and \( Z \) belong to the same straight line if and only if there exists \( \lambda + \mu = 1 \).

\[ \text{Corollary 4.4.} \] Assume that \( x_i - y_i \in \mathcal{R}, \ i = 1,2 \) are invertible. Then \( X,Y,Z \) belong to one straight line if and only if

\[
(y_1 - x_1)^{-1}(z_1 - x_1) = (y_2 - x_2)^{-1}(z_2 - x_2).
\]

\[ \text{Proof.} \] Note that

\[
\begin{vmatrix}
 x_1 & y_1 & z_1 \\
 x_2 & y_2 & z_2 \\
 1 & 1 & 1 \\
\end{vmatrix}
= -\begin{vmatrix}
 x_1 & y_1 - x_1 \\
 x_2 & y_2 - x_2 \\
\end{vmatrix}
\begin{vmatrix}
 y_1 - x_1 & z_1 - x_1 \\
 y_2 - x_2 & z_2 - x_2 \\
\end{vmatrix}^{-1}
\]

and that \( (y_1 - x_1)^{-1}(z_1 - x_1) = (y_2 - x_2)^{-1}(z_2 - x_2) \) if and only if

\[
\begin{vmatrix}
 y_1 - x_1 & z_1 - x_1 \\
 y_2 - x_2 & z_2 - x_2 \\
\end{vmatrix} = 0.
\]

\[ \text{□} \]

4.4. NC analogy of Konopelchenko equations. Let again \( \mathcal{R} \) be a division ring. Consider \( \mathcal{R}^2 \) as the right module over \( \mathcal{R} \).

\[ \text{Proposition 4.5.} \] Let \( F_1 = (x_1,y_1), F_2 = (x_2,y_2) \) be two points in \( \mathcal{R}^2 \) in a generic position. Then the equation of the straight line \( L_{12} \) passing through \( F_1 \) and \( F_2 \) is

\[
(y_2 - y_1)^{-1}(y - y_1) = (x_2 - x_1)^{-1}(x - x_1).
\]

\[ \text{Corollary 4.6.} \] An equation of the line \( L'_{12} \) parallel to \( L \) and passing through \((0,0)\) is

\[
(y_2 - y_1)^{-1}y = (x_2 - x_1)^{-1}x,
\]

i.e. any point \( F_{12} \) on \( L'_{12} \) has coordinates \((x_2 - x_1)f_{12}, (y_2 - y_1)f_{12}\).

The proposition and the corollary are both straightforward consequences of the Proposition 4.3 and the Corollary 4.4.

Denote by \( L_{ij} \) the straight line passing through \( F_i = (x_i,y_i) \) and \( F_j = (x_j,y_j) \) and by \( L'_{ij} \) the parallel line though \((0,0)\). Consider now (additionally to the line \( L'_{12} \) and to a point \((x_2 - x_1)f_{12}, (y_2 - y_1)f_{12} \) on it) points \( F_{23} = ((x_3 - x_2)f_{23}, (y_3 - y_2)f_{23}) \) on line \( L'_{23} \) and \( F_{31} = (x_1 - x_3)f_{31}, (y_1 - y_3)f_{31} \) on line \( L'_{31} \).

\[ \text{Proposition 4.7.} \] For generic points \( F_1, F_2, F_3 \) the points \( F_{12}, F_{23}, F_{31} \) belong to a straight line iff

\[
f_{12}^{-1} + f_{23}^{-1} + f_{31}^{-1} = 0.
\]

\[ \text{Warning:} \] Note that before we considered points with coordinates \((x_1,x_2), (y_1,y_2), (z_1,z_2)\) and now with coordinates \((x_i,y_i)\).

\[ \text{Proof.} \] According to Proposition 4.3 in order to show that \( F_{12}, F_{23}, F_{31} \) lie on the same straight line it is necessary and sufficient to check that

\[
\theta := \begin{vmatrix}
(x_2 - x_1)f_{12} & (x_3 - x_2)f_{23} & (x_1 - x_3)f_{31} \\
(y_2 - y_1)f_{12} & (y_3 - y_2)f_{23} & (y_1 - x_3)f_{31} \\
1 & 1 & 1 \\
\end{vmatrix} = 0.
\]
According to the standard properties of quasideterminants
\[
\theta = \begin{vmatrix}
(x_2 - x_1) & (x_3 - x_2) & (x_1 - x_3) \\
(y_2 - y_1) & (y_3 - y_2) & (y_1 - y_3) \\
f_{12}^{-1} & f_{23}^{-1} & f_{31}^{-1}
\end{vmatrix} f_{31}.
\]

Adding first two columns to the third one does not change \(\theta\), so
\[
\theta = \begin{vmatrix}
(x_2 - x_1) & (x_3 - x_2) & 0 \\
(y_2 - y_1) & (y_3 - y_2) & 0 \\
f_{12}^{-1} & f_{23}^{-1} & f_{12}^{-1} + f_{21}^{-1} + f_{31}^{-1}
\end{vmatrix} f_{31} = (f_{12}^{-1} + f_{21}^{-1} + f_{31}^{-1}) f_{31}.
\]

This is a noncommutative generalization of formula (32) from Konopelchenko (12).

4.5. **Noncommutative Menelaus theorem and quasi-Plücker coordinates.** Let as above \(\mathcal{R}\) be a noncommutative division ring. Consider \(\mathcal{R}^2\) as the right vector space over \(\mathcal{R}\). For a point \(X \in \mathcal{R}^2\) denote by \(x_i\) its \(i\)-th coordinate, \(i = 1, 2\). Remind that points \(X, Y, Z\) are collinear if and only if
\[
\begin{vmatrix}
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
1 & 1 & 1
\end{vmatrix} = 0.
\]

**Corollary 4.8.** Points \(X, Y, Z\) are collinear if and only if
\[
(y_1 - x_1)^{-1}(z_1 - x_1) = (y_2 - x_2)^{-1}(z_2 - y_2)
\]
or, equivalently,
\[
(x_2 - y_2)^{-1}(z_2 - y_2) = q_{XZ}^Y.
\]

**Remark 4.9.** The second identity is equivalent to the equality
\[
\begin{vmatrix}
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
1 & 1 & 1
\end{vmatrix} = 0.
\]

**Proposition 4.10.** Let \(A, B, C\) be non-collinear points in \(\mathcal{R}^2\). Then any point \(P \in \mathcal{R}^2\) can be uniquely written as
\[
P = At + Bu + Cv, \quad t, u, v \in \mathcal{R}, \quad t + u + v = 1.
\]

We will write \(P = [t, u, v]\).

**Proposition 4.11.** Let \(P_i = [t_i, u_i, v_i], i = 1, 2, 3\). Then \(P_1, P_2, P_3\) are collinear if and only if
\[
\begin{vmatrix}
t_1 & t_2 & t_3 \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix} = 0.
\]

We follow now the book by Kaplansky, [11]. See pages 88-89. Consider a triangle \(ABC\) (vertices go anti-clock wise). Take point \(R\) at line \(AB\), point \(P\) at line \(BC\), and point \(Q\) at line \(AC\). Then
\[
P = B(1 - t) + Ct, \quad Q = C(1 - u) + Au, \quad R = A(1 - v) + Bv.
\]

Proposition 4.11 implies
Theorem 4.12. Points $A, B, C$ are collinear if and only if
\[ u(1 - u)^{-1} t(1 - t)^{-1} v(1 - v)^{-1} = -1 . \]

Note that $t(1 - t)^{-1} = (c_1 - p_1)^{-1} (p_1 - b_1)$. Corollary 18 implies that
\[ t(1 - t)^{-1} = -q^{P}_{CB} \]
where $q^{P}_{CB}$ is a quasi-Plücker coordinate. Similarly,
\[ u(1 - u)^{-1} = -q^{Q}_{AC}, \quad v(1 - v)^{-1} = -q^{R}_{BA} \]
and Theorem 4.12 implies
\[ q^{Q}_{AC} q^{P}_{CB} q^{R}_{BA} = 1 . \]

5. Non-commutative cross-ratio and Schwarzian

Consider the following “system of linear differential equations”:
\[
\begin{align*}
  f''_1 + a f'_1 + b f_1 &= 0 \\
  f''_2 + a f'_2 + b f_2 &= 0.
\end{align*}
\]
(5.1)

Here $a, b, f_1, f_2$ are elements of a division ring $\mathcal{R}$, and $'$ denotes a linear differentiation in this ring, i.e. a linear endomorphism of $\mathcal{R}$ verifying the Leibniz identity (a model example is the algebra of smooth operator-valued functions of one (real) variable).

Below we shall assume that all the elements we deal with are invertible if necessary. Using this assumption it is not difficult to solve the equations (5.1) as a linear system on $a$ and $b$: multiplying the equations by $f_1^{-1}$ and $f_2^{-1}$ respectively and subtracting the second one from the first one we obtain (see [3])

\[ a = -(f''_1 f_1^{-1} - f''_2 f_2^{-1})(f'_1 f_1^{-1} - f'_2 f_2^{-1})^{-1} \]

and similarly

\[ b = -(f''_1 (f'_1)^{-1} - f''_2 (f'_2)^{-1})(f_1 (f'_1)^{-1} - f_2 (f'_2)^{-1})^{-1} . \]

We can rewrite these formulas a little:

\[
\begin{align*}
  a &= -(f''_1 - f''_2 f_2^{-1} f_1)(f'_1 - f'_2 f_2^{-1} f_1)^{-1} \\
  b &= -(f''_1 - f''_2 (f'_2)^{-1} f'_1)(f_1 - f_2 (f'_2)^{-1} f'_1)^{-1}
\end{align*}
\]

so that now it is evident that $a$ and $b$ can be expressed as $a = -q_{32}^1$, $b = -q_{31}^2$, where $q_{jk}^i$ are right quasi-Plücker coordinates of the $3 \times 2$-matrix $\begin{pmatrix} f_1 & f'_1 & f''_1 \\ f_2 & f'_2 & f''_2 \end{pmatrix}^T$. See section 1 for details.

Observe that in the process of solving (5.1) we obtained the expression:
\[ -b = a f'_1 f_1^{-1} + f''_1 f_1^{-1} = a f'_2 f_2^{-1} + f''_2 f_2^{-1} . \]
(5.2)

(The expression is a special case for the formula from Proposition 4.8.1 from [2] rewritten for right quasi-Plücker coordinates. The proposition connects quasi-Plücker coordinates for matrices of different sizes.)

Thus
\[ a f'_1 f_1^{-1} f_2 + f''_1 f_1^{-1} f_2 = a f'_2 f_2^{-1} + f''_2 . \]
Hence
\[(5.3) \quad af_1(f_1^{-1}f_1'f_1^{-1}f_2 - f_1^{-1}f_2') = f_1(f_1^{-1}f'_2 - f_1^{-1}f'_1f_1^{-1}f_2).\]
Now one have the formulas:
\[
(f^{-1})'' = -(f^{-1}f'f^{-1})' = 2f^{-1}f'f^{-1}f' - f^{-1}f''f^{-1},
\]
and
\[
(fg)'' = f''g + 2f'g' + fg'',
\]
for all \(f, g \in A;\) so
\[
(f^{-1}g)'' = 2f^{-1}f'f^{-1}f'f^{-1}g - 2f^{-1}f'f^{-1}g' - f^{-1}f''f^{-1}g + f^{-1}g''.
\]
Thus on the right hand side of (5.3) we have
\[
f_1(f_1^{-1}f''_2 - f_1^{-1}f'_1f_1^{-1}f_2) = f_1(2f_1^{-1}f'_1f_1^{-1}f_2 - 2f_1^{-1}f'_1f_1^{-1}f_2 - f_1^{-1}f''_1f_1^{-1}f_2 + f_1^{-1}f''_2).
\]
On the other hand, on the left hand side of (5.3) we have \(-af_1(f_1^{-1}f_2)'\), so denoting \(\varphi = f_1^{-1}f_2\) we get:
\[
(5.4) \quad af_1 = -2f'_1 - f_1\varphi''(\varphi')^{-1},
\]
or equivalently
\[
(5.5) \quad a = -2f'_1f_1^{-1} - f_1\varphi''(\varphi')^{-1}f_1^{-1}.
\]
Here’s a simple corollary of the formula (5.5):

**Proposition 5.1.** When the elements \(f_i \in A\) are replaced by \(\tilde{f}_i = hf_i, i = 1, 2\) for some \(h \in \mathcal{R}\), then \(a\) in the system (5.1) should be replaced by \(\tilde{a} = -2h'h^{-1} + hah^{-1}\).

**Proof.** Observe that \(\varphi\) is not affected by the coordinate change \(f_i \leftrightarrow \tilde{f}_i, i = 1, 2\). Now direct calculation with formula (5.5) shows
\[
\tilde{a} = -2f'_1\tilde{f}_1^{-1} - f_1\varphi''(\varphi')^{-1}\tilde{f}_1^{-1}
\]
\[
= -2(h'f_1 + hf_1')\tilde{f}_1^{-1}h^{-1} - h(f_1\varphi''(\varphi')^{-1}f_1^{-1})h^{-1}
\]
\[
= -2h'h^{-1} + h(-2f'_1f_1^{-1} - f_1\varphi''(\varphi')^{-1}f_1^{-1})h^{-1}.
\]

\(\square\)

**Remark 5.2.** It is worth to observe a striking similarity of the expression in proposition 5.1 and the gauge transformation of a linear connection (the unnecessary \(2\) in front of \(h'h^{-1}\) can be eliminated by considering \(\alpha = \frac{1}{2}a\)).

It is now our purpose to find the way \(b\) changes, when \(f_1, f_2\) are multiplied by \(h\), at least under some additional assumptions on \(h\). We begin with the simple observation:

**Corollary 5.3.** If \(h\) verifies the “differential equation” \(h' = \frac{1}{2}ha\) then \(\tilde{a} = 0\).

Further, there’s another simple consequence of the formula (5.4):
Proposition 5.4. Assume that \( h \) verifies the equation \( h' = \frac{1}{2}ha \); denote \( \tilde{f}_1 = hf_1 \), \( \theta = \varphi''(\varphi')^{-1} \). Then

\[
\tilde{f}_1' = -\frac{\tilde{f}_1}{2}\theta.
\]

Proof.

\[
\tilde{f}_1' = (hf_1)' = \frac{1}{2}ha f_1 + hf_1' = \text{using equation (5.3)} = -hf_1' - \frac{1}{2}hf_1\theta + hf_1' = -\frac{\tilde{f}_1}{2}\theta.
\]

\[\square\]

Repeating the differentiation we see:

\[
(5.6) \quad \tilde{f}_1'' = \left(\frac{\tilde{f}_1}{2}\theta\right)' = \frac{\tilde{f}_1}{4}\theta - \frac{\tilde{f}_1}{2}\theta'.
\]

Finally, substituting these formulas in the first expression of (5.2) we obtain the following result:

Theorem 5.5. If \( h \) satisfy the equation \( h' = \frac{1}{2}ha \) then the coordinate change \( f_i \mapsto \tilde{f}_i = hf_i, \ i = 1, 2 \) transforms the system (5.1) in such a way that

\[
a \mapsto 0
\]

\[
b \mapsto \frac{1}{2}\tilde{f}_1 \left( \theta' - \frac{1}{2}\theta \right) \tilde{f}_1^{-1}
\]

where \( \theta = \varphi''(\varphi')^{-1}, \ \varphi = f_1^{-1}f_2 \) and the equation \( 2\tilde{f}_1' + \tilde{f}_1\theta = 0 \) holds.

Proof. Since \( a \mapsto 0 \), we obtain from (5.2):

\[
b = -\tilde{f}_1'' \tilde{f}_1^{-1} = \text{using (5.6)} = -\left( \frac{\tilde{f}_1}{4}\theta - \frac{\tilde{f}_1}{2}\theta' \right) \tilde{f}_1^{-1}
\]

\[\square\]

Remark 5.6. Observe that in the commutative case the expression \( \theta' - \frac{1}{2}\theta \) coincides with the classical Schwarz differential of \( \varphi \).

5.1. Generalized NC Schwarzian. Let \( f \) and \( g \) be two (invertible) elements of a division ring \( \mathcal{R} \), equipped with a derivation \( \cdot' \) (see previous section). We suppose that they satisfy so-called left coefficients equations \( f'' = F_1f, \ g'' = F_2g \) for some \( F_1, F_2 \in \mathcal{R} \). We set \( h := fg^{-1} \) and \( G := F_1h - hF_2 \).

Theorem 5.7. If \( G = 0 \) then we have the following relation:

\[
(5.7) \quad h''' = \frac{3}{2}h''(h')^{-1}h'' - 2h'F_2
\]

(a non-commutative analogue of the Schwarzian equation.)
Proof.

\[ h' = f'g^{-1} - hg'g^{-1}, \]
\[ h'' = G - 2h'g'g^{-1}, \]
\[ h''' = G' - 2h''g'g^{-1} - 2h'F_2 + 2h'(g'g^{-1})^2. \]

One can express \( g'g^{-1} = (1/2)(h')^{-1}(G - h'') \) and get

\[ h''' = (3/2)h''(h')^{-1}h'' - 2h'F_2 - (3/2)h''(h')^{-1}G + (1/2)G(h')^{-1}(G - h''). \]

Let \( f^{-1}f'' = g^{-1}g'' \), i.e. \( f, g \) are solutions of the same differential equation with right coefficients. Let \( g'' = Fg \), i.e. \( g \) is also a solution of a differential equation with a left coefficient. Let \( h = fg^{-1} \). Then

\[ h''' - (3/2)h''(h')^{-1}h'' = -2h'F. \]

Note that that the left-hand side is stable under Möbius transform

\[ h \rightarrow (ah + b)(ch + d)^{-1} \]

where \( a' = b' = c' = d' = 0. \]

Remark 5.8. Let us consider the commutative analogue of (5.7)

\[ (h')^{-1}h''' = (3/2)(h')^{-2}h'^2 - 2F_2. \]

In other words the Schwarzian \( \text{Sch}(h) \)

\[ \text{Sch}(h) := (h')^{-1}h''' - (3/2)(h')^{-2}h'^2 = -2F_2. \]

Hence, one can call a NC Schwarzian of \( h \) the following expression

\[ \text{NCSch}(h) := (h')^{-1}h''' - (3/2)(h')^{-1}h''(h')^{-1}h'' \]

Remark 5.9. In commutative case there exist the following famous version of KdV equation

\[ h_t = (h')\text{Sch}(h) \]

It is invariant under the projective action of \( SL_2 \) and, when written as an evolution on the invariant \( \text{Sch}(h) \) it becomes the "usual" KdV

\[ \text{Sch}(h)_t = \text{Sch}(h)''' + 3\text{Sch}(h)'\text{Sch}(h) \]

Introducing two commuting derivatives \( \partial_x = ' \) and \( \partial_t \) of our skew-field \( \mathcal{R} \) with respect to two distinguished elements \( x \) and \( t \) one can write the analogues of (5.10):

\[ h_t = (h')\text{NSch}(h) = h''' - (3/2)h''(h')^{-1}h'' \]

Remark 5.10. The equation (5.12) has an interesting geometric interpretation (specialisation) as the Spinor Schwarzian- KdV equation (see the equation (4.6) in ([2])).
6. Relation with matrix cross-ratio.

In this section we discuss the cross ratio in noncommutative algebras, introduced above in terms of \textit{quasideterminants} and its relation with the operator cross-ratio of Zelikin (see [21] and also Chapter 5 of [20]).

Recall that we defined the cross ratio of four elements \(a, b, c, d \in \mathcal{R}^{\oplus 2}\) by explicit formulas as follows:

\[
\kappa(a, b, c, d) = \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}^{-1} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}^{-1} \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix},
\]

under the assumption that all these expressions exist (in fact, except for the existence of the inverse elements of \(a_2\) and \(b_2\), it is enough to assume further that the matrices \(\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}\) and \(\begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix}\) are invertible).

The expression \(\kappa(a, b, c, d)\) has various algebraic properties (see sections 1-3). We are going now to compare it with the \textit{operator cross ratio} of Zelikin (see [21]). To this end we begin with the description of his construction.

Let \(\mathcal{H}\) be an even-dimensional (possibly infinite-dimensional) vector space; let us fix its polarization \(\mathcal{H} = V_0 \oplus V_1\), where the subspaces \(V_0, V_1\) have the same dimension (in infinite dimensional case one can assume that there is a fixed isomorphism \(\psi : V_0 \to V_1\) between them); let \((\mathcal{P}_1, \mathcal{P}_2)\) and \((\mathcal{Q}_1, \mathcal{Q}_2)\) be two other pairs of subspaces, polarizing \(\mathcal{H}\); i.e. \(\mathcal{P}_i, \mathcal{Q}_i\) are isomorphic to \(V_j\) and \(\mathcal{P}_1\) (resp. \(\mathcal{Q}_1\)) is transversal to \(\mathcal{P}_2\) (resp. to \(\mathcal{Q}_2\)). Then the cross ratio of these two pairs (or of the spaces \(\mathcal{P}_1\mathcal{P}_2, \mathcal{Q}_1\mathcal{Q}_2\)) is the operator

\[
\text{DV}(\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2) = (\mathcal{P}_1 \xrightarrow{\mathcal{P}_2} \mathcal{Q}_1 \xrightarrow{\mathcal{Q}_2} \mathcal{P}_1)
\]

Here we use the notation from [9], where \(\mathcal{P}_1 \xrightarrow{\mathcal{P}_2} \mathcal{Q}_1\) denotes the projection of \(\mathcal{P}_1\) to \(\mathcal{Q}_1\) along \(\mathcal{P}_2\) and similarly for the second arrow.

In the cited paper the following explicit formula for \(\text{DV}\) was proved: let \(\mathcal{P}_i\) be given by the graph of an operator \(P_i : V_0 \to V_1, i = 1, 2\) and similarly for \(\mathcal{Q}_j\), then the following formula holds:

\[
\text{DV}(\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2) = (P_1 - P_2)^{-1}(P_2 - Q_1)(Q_1 - Q_2)^{-1}(Q_2 - P_1) : V_0 \to V_0.
\]

The invertibility of the operators \(P_1 - P_2\) and \(Q_1 - Q_2\) is provided by the transversality of \(\mathcal{P}_1\) and \(\mathcal{P}_2\) (resp. \(\mathcal{Q}_1\) and \(\mathcal{Q}_2\)).

The first claim we are going to make is the following:

\begin{proposition}
The operator cross ratio \(\text{DV}(\mathcal{Q}_2, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{P}_1)\) (if it exists) is equal to \(\kappa(p_1, p_2, q_1, q_2)\), for \(p_1 = \begin{pmatrix} 1 \\ P_1' \end{pmatrix}\), \(p_2 = \begin{pmatrix} 1 \\ P_2' \end{pmatrix}\), \(q_1 = \begin{pmatrix} 1 \\ Q_1' \end{pmatrix}\), \(q_2 = \begin{pmatrix} 1 \\ Q_2' \end{pmatrix}\), where \(1\) is the identity operator on \(V_0\) and we identify \(V_0\) and \(V_1\) using the fixed map \(\psi\) so that \(P_i' = \psi^{-1} \circ P_i : V_0 \to V_0\).
\end{proposition}
Proof. This is a direct computation based on the explicit formula:

\[
\kappa(p_1, p_2, q_1, q_2) = \begin{vmatrix}
\frac{1}{P_1'} & \frac{1}{Q_1'} & \frac{1}{P_2'} & \frac{1}{Q_2'} \\
\frac{1}{P_1} & \frac{1}{Q_1} & \frac{1}{P_2} & \frac{1}{Q_2}
\end{vmatrix}
\]

\[
= (1 - (P_2'Q_2)^{-1}(1 - (P_2'Q_1)^{-1}(1 - (P_1'Q_1)^{-1}(1 - (P_1'Q_2)^{-1})
\]

\[
= (1 - P_2^{-1}Q_2)^{-1}(1 - P_1^{-1}Q_1)^{-1}(1 - P_1^{-1}Q_2)
\]

\[
= (P_2 - Q_2)^{-1}P_2P_2^{-1}(P_2 - Q_1)(P_1 - Q_1)^{-1}P_1^{-1}(P_1 - Q_2)
\]

\[
= (Q_2 - P_2)^{-1}(P_2 - Q_1)(Q_1 - P_1)^{-1}(P_1 - Q_2)
\]

\[
= \text{DV}(\mathcal{D}, \mathcal{P}, \mathcal{P}_1, \mathcal{P}_2).
\]

Observe, that the role of \(\psi\) is insignificant here: in effect, one can define the quasideterminants in the context of categories, i.e. for \(\mathcal{A}\) being a matrix of morphisms in certain category with its entries \(a_{ij}\) being maps from the \(i\)-th object to the \(j\)-th object (see \[\text{8}\]). This makes the use of \(\psi\) redundant.

One of the important properties of the operator cross ratio is that it represents the extension class of the tautological fibre bundle over the Grassmanian space of polarizations of \(\mathcal{H}\) (see \[\text{21}\]). In what follows we shall give this phenomenon a purely algebraic interpretation in terms of the descent data and a cocycle in a suitable cohomology theory.

More accurately, it is shown in \[\text{21}\] that the following equality holds for the \(\text{DV}\): let \((\mathcal{P}_1, \mathcal{P}_2)\) be a polarizing pair, and \(\mathcal{X}, \mathcal{Y}, \mathcal{Z}\) three hyperplanes, then

\[
\text{DV}(\mathcal{P}_1, \mathcal{X}, \mathcal{P}_2, \mathcal{Y}) \text{DV}(\mathcal{P}_1, \mathcal{Y}, \mathcal{P}_2, \mathcal{Z}) \text{DV}(\mathcal{P}_1, \mathcal{X}, \mathcal{P}_2, \mathcal{X}) = 1,
\]

or, using the algebraic properties of \(\text{DV}\),

\[
\text{DV}(\mathcal{P}_1, \mathcal{X}, \mathcal{P}_2, \mathcal{Y}) \text{DV}(\mathcal{P}_1, \mathcal{Y}, \mathcal{P}_2, \mathcal{Z}) = \text{DV}(\mathcal{P}_1, \mathcal{X}, \mathcal{P}_2, \mathcal{Z})
\]

if all three terms are well-defined. This corresponds to the following purely algebraic relation for the cross-ratio \(\kappa(a, b, c, d)\)

\[
\kappa(y, x, p_2, p_1)\kappa(z, y, p_2, p_1) = \kappa(z, x, p_2, p_1),
\]

see Section 2 above.

So, in order to give a homological interpretation of relations \((6.1)-(6.3)\), let us fix a vector \(\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathcal{R}^{\oplus 2}\); let \(\mathcal{R}_\omega^2\) denote the set of the elements \(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathcal{R}^{\oplus 2}\) such that the matrix

\[
\begin{pmatrix}
\omega_1 & a_1 \\
\omega_2 & a_2
\end{pmatrix}
\]

is invertible. Let \(a \in \mathcal{R}_\omega^2\) and let \(x \in \mathcal{R}_\omega^{\oplus 2}\) be such that both matrices

\[
\begin{pmatrix} a_1 & x_1 \\ a_2 & x_2 \end{pmatrix}\quad \text{and} \quad \begin{pmatrix} \omega_1 & x_1 \\ \omega_2 & x_2 \end{pmatrix}
\]

are invertible. It is clear that the set of such \(x\) is equal to the intersection \(\mathcal{R}_\omega^2 \cap \mathcal{R}_a^2\); we shall denote it by \(\mathcal{R}_{a,\omega}^2 = \mathcal{R}_a^2\) since \(\omega\) is fixed.
Consider now a Čech type simplicial complex $\tilde{C}(\mathcal{R};\omega)$: its set of $n$-simplices is spanned by the disjoint union of the intersections

$$\tilde{C}^n(\mathcal{R};\omega) = \coprod_{\alpha_0,\ldots,\alpha_n} \tilde{\mathcal{R}}^2_{\alpha_0} \cap \tilde{\mathcal{R}}^2_{\alpha_1} \cap \cdots \cap \tilde{\mathcal{R}}^2_{\alpha_n},$$

and the faces/degeneracies are given by the omitting/repeating the terms in the intersections respectively.

Then the formula

$$\phi = \{\phi_{\alpha_0,\alpha_1}\} : \tilde{C}_1(\mathcal{R};\omega) \to \mathcal{R}^*, \ \phi_{\alpha_0,\alpha_1}(x) = \kappa(a_1,a_0,x,\omega),$$

determines a map on the second term of this complex. Observe, that the cocycle condition now can be interpreted as the statement that $\phi$ can be extended to a simplicial map from $\tilde{C}(\mathcal{R};\omega)$ to the bar-resolution of the group $\mathcal{R}^*$ of invertible elements in $\mathcal{R}$. Namely: put

$$\phi_0 = 1 : \tilde{C}_0(\mathcal{R};\omega) \to [1] = B_0(\mathcal{R}^*);$$

$$\phi_1 = \phi : \tilde{C}_1(\mathcal{R};\omega) \to R^* = B_1(\mathcal{R}^*);$$

and for all other $n \geq 2$

$$\phi_n : \tilde{C}_n(\mathcal{R};\omega) \to (\mathcal{R}^*)^x = B_n(\mathcal{R}^*)$$

given by the formula

$$\phi_n(x) = [\phi_{\alpha_0,\alpha_1}(x)\phi_{\alpha_1,\alpha_2}(x)\cdots\phi_{\alpha_{n-1},\alpha_n}(x)],$$

for all $x \in \tilde{\mathcal{R}}^2_{\alpha_0,\ldots,\alpha_n}$. Then

**Proposition 6.2.** The collection of maps $\{\phi_n\}_{n \geq 0}$ determine a simplicial map from $\tilde{C}(\mathcal{R};\omega)$ to $B.(\mathcal{R}^*)$.

**Remark 6.3.** The construction we just described bears striking similarity with the well-known Goncharov’s complex (see [10]), so one can wonder if there are any relation with the actual Goncharov’s Grassmannian complex and higher cross ratios/polylogarithms in this case?

### 6.1. Schwarzian operator

Let us now describe the relation between the Schwarzian differential operator (see section 5) and the noncommutative cross ratios. In fact, we shall obtain this operator as an infinitesimal part of the deformation of the cross-ratio. It plays the role of the usual differentiation of vector-valued functions and is invariant with respect to the action of $GL_2(\mathcal{R})$ and the multiplication by invertible elements from $\mathcal{R}$.

Following the ideas in [21] we consider a smooth one-parameter family $Z(t) = \begin{pmatrix} Z(t)_1 \\ Z(t)_2 \end{pmatrix}$ of elements in $\mathcal{R}^{\oplus 2}$, such that for all different $t_1,t_2,t_3,t_4$ the cross ratio $\kappa(Z(t_1),Z(t_2),Z(t_3),Z(t_4))$ is well defined. Then, let us consider the function

$$f(t,t_1,t_2,t_3) = \kappa(Z(t_3),Z(t_1),Z(t),Z(t_2))$$

$$= (z(t_2) - z(t_1))^{-1}(z(t_1) - z(t))(z(t) - z(t_3))^{-1}(z(t_3) - z(t_2)),$$

where $z(t) = Z(t)^{-1}Z(t)_2$. Fix $t = 0$, and let $t_2 \to 0$. Then $f(0,t_1,t_2,t_3) \to 1$ and

$$\frac{\partial f}{\partial t_2}(0,t_1,0,t_3) = -(z(0) - z(t_1))^{-1}z'(0) + (z(0) - z(t_3))^{-1}z'(0).$$
Thus,

\[ f(t, t_1, t_2, t_3) = 1 - (t_2 - t) ((z(t) - z(t_1))^{-1}z'(t) - (z(t) - z(t_3))^{-1}z'(t)) + o(t_2 - t) \]

If \( t_1 = t_3 \), the derivative on the right vanishes; consider the second partial derivative:

\[ \frac{\partial^2 f}{\partial t_3 \partial t_2}(0, t_1, 0, t_1) = -(z(0) - z(t_1))^{-1}z'(t)(z(0) - z(t_1))^{-1}z'(0), \]

so that

\[ f(t, t_1, t_2, t_3) = 1 - (t_2 - t)(t_3 - t_1)(z(t) - z(t_1))^{-1}z'(t_1)(z(t) - z(t_1))^{-1}z'(t) + o((t_2 - t)(t_1 - t_3)). \]

This expression has a singularity at \( t_1 = 0 \). Now, using the Taylor series for \( z(t) \) we compute for \( t_1 \to 0 \):

\[ \frac{\partial^2 f}{\partial t_3 \partial t_2}(0, t_1, 0, t_1) = t_1^{-2} \left( 1 + \frac{t_1^2(z'(0))^{-1}z'''(0)}{6} - \frac{t_1^2((z'(0))^{-1}z''(0))^2}{4} + ... \right) \]

where \( ... \) denote the terms of degrees 3 and higher in \( t_1 \). So, we obtain

\[ (6.4) \]

\[ \frac{\partial^2 f}{\partial t_3 \partial t_2}(0, t_1, 0, t_1) = t_1^{-2}(1 + 6t_1^2 S(Z) + ...) + ... \]

where we put

\[ S(Z) = (z'(0))^{-1}z'''(0) - \frac{3}{2}((z'(0))^{-1}z''(0))^2. \]

Here \( Z \) and \( z \) are related as explained above. This differential operator is well-defined on functions with values in \( R^{\mathbb{C}^2} \) modulo the action of \( GL_2(R) \) and is conjugated by \( \lambda \in R^x \), when \( Z \) is multiplied by it on the right.

A more conceptual way to obtain the formula \((6.4)\) would be to consider the formal Taylor expansion of \( z(t_i) \) near \( t_i = 0 : z(t_i) = z(0) + z'(0)t_i + \frac{1}{2}z''(0)t_i^2 + \frac{1}{6}z'''(0)t_i^3 + ..., \; t_i = t, t_1, t_2, t_3. \)

Then (omitting the argument \( 0 \) from our notation)

\[ z(t_i) - z(t_j) = z'(t_i - t_j) + \frac{1}{2}z''(t_i^2 - t_j^2) + \frac{1}{6}z'''(t_i^3 - t_j^3) + ... \]

\[ = (t_i - t_j)(z' + \frac{1}{2}z''(t_i + t_j) + \frac{1}{6}z'''(t_i^2 + t_i t_j + t_j^2)) + ... \]

and similarly

\[ (z(t_i) - z(t_j))^{-1}(z(t_k) - z(t_l)) = \]

\[ \frac{t_k - t_l}{t_i - t_j} \left( 1 + \frac{1}{2}(z')^{-1}z''(t_i + t_j) + \frac{1}{6}(z')^{-1}z'''(t_i^2 + t_i t_j + t_j^2) \right)^{-1} \]

\[ \left( 1 + \frac{1}{2}(z')^{-1}z''(t_k + t_l) + \frac{1}{6}(z')^{-1}z'''(t_k^2 + t_k t_l + t_l^2) \right) + ... \]

\[ \frac{t_k - t_l}{t_i - t_j} \left( 1 - \frac{1}{2}(z')^{-1}z''(t_i + t_j) - \frac{1}{6}(z')^{-1}z'''(t_i^2 + t_i t_j + t_j^2) + \frac{1}{4}((z')^{-1}z'')^2(t_i + t_j)^2 \right) \]

\[ \left( 1 + \frac{1}{2}(z')^{-1}z''(t_k + t_l) + \frac{1}{6}(z')^{-1}z'''(t_k^2 + t_k t_l + t_l^2) \right) + ... \]

\[ \frac{t_k - t_l}{t_i - t_j} \left( 1 + \frac{1}{2}(z')^{-1}z''(t_k + t_l + t_i - t_j) + \frac{1}{6}(z')^{-1}z'''(t_k^2 + t_k t_l + t_l^2 - t_i t_j - t_j^2) \right. \]

\[ + \frac{1}{4}((z')^{-1}z'')^2((t_i + t_j)^2 - (t_k + t_l)(t_i + t_j)) \]
where we use ... to denote the elements of degree 3 and higher in $t_i$. In particular, taking $t_i = t_2$, $t_j = t_1$, $t_k = t_1$, $t_l = t$, we obtain
\[
(z(t_2) - z(t_1))^{-1} (z(t_1) - z(t)) = \frac{t_1 - t}{t_2 - t_1} \left( 1 + (t - t_2) \left( \frac{1}{2} (z')^{-1} z'' + \frac{1}{6} (z')^{-1} z''' (t + t_1 + t_2) - \frac{1}{4} ((z')^{-1} z'')^2 (t_2 + t_1) \right) \right) + ...
\]
Similarly, with $t_i = t$, $t_j = t_3$, $t_k = t_3$, $t_l = t_2$, we have:
\[
(z(t) - z(t_3))^{-1} (z(t_3) - z(t_2)) = \frac{t_3 - t_2}{t - t_3} \left( 1 + (t_2 - t) \left( \frac{1}{2} (z')^{-1} z'' + \frac{1}{6} (z')^{-1} z''' (t + t_2 + t_3) - \frac{1}{4} ((z')^{-1} z'')^2 (t + t_3) \right) \right) + ...
\]
Finally, taking the product of these two expressions we obtain
\[
f(t, t_1, t_2, t_3) = \frac{(t_1 - t)(t_3 - t_2)}{(t_2 - t_1)(t - t_3)} \left( 1 + (t_2 - t)(t_3 - t_1) \left( \frac{1}{6} (z'(0))^{-1} z'''(0) - \frac{1}{4} ((z(0))^{-1} z(0))'' \right) \right)
\]
Compare this formula with the formula (4.7) from the paper [1].

We call the expression $\text{Sch}(z) = (z')^{-1} z''' - \frac{3}{2} ((z')^{-1} z'')^2$ the noncommutative Schwarzian of $z(t)$. Just like the classical Schwarz derivative, this operator is invariant (up to conjugations) with respect to the Möbius transformations in $\mathcal{R}^2$; this is the direct consequence of the method we derived this formula from the (operator) cross-ratio.

6.2. **Infinitesimal Ceva ratio.** The following expression is intended as a 2-dimensional analog of the Schwarzian operator. More accurately, Schwarz derivative can be regarded as the infinitesimal transformation of the cross-ratio under a diffeomorphism of the projective line. It is natural to assume that the role of cross-ratio in projective plane should in some sense be played by the Ceva theorem (see figure 1, part (b)). Thus here we try to find the infinitesimal part of the transformation of the Ceva ratio under a diffeomorphism; in a general case this is quite a difficult question, so we do it under certain additional conditions.

Let $\xi$, $\eta$ be two commuting vector fields on a manifold $M$, and let $f : M \to M$ be a self-map of $M$ such that $df(\xi) = \kappa \cdot \xi$, $df(\eta) = \kappa \cdot \eta$ for some smooth function $\kappa \in C^\infty(M)$. It follows from this condition, that $f$ maps integral trajectories of both fields and of the fields, equal to their linear combinations with constant coefficients. One can imagine this map as a “change of coordinates along the 2-dimensional net”, or a generalized conformal map. However, we do not assume that these fields are linearly independent, they can even be proportional to each other.

Let us consider the following expression: take any point $x$; let $\phi(t)$ and $\psi(s)$ be the one-parameter diffeomorphism families, generated by $\xi$ and $\eta$ respectively. Since these fields commute, the composition $\phi(-r) \circ \psi(r) = \psi(r) \circ \phi(-r) =: \theta(r)$ is the one-parameter family, corresponding to their difference $\zeta = \eta - \xi$. Consider now the infinitesimal “triangle” at $x$: first we move from $x$ to $\phi(\epsilon)(x)$, then from this point to $\psi(2\epsilon)(x)$; then we apply to this point $\theta(\epsilon)$ and $\theta(2\epsilon)$; and finally we apply twice the diffeomorphism $\psi(-\epsilon)$. By definition, we come to the point $x$ again, having spun a “curvilinear triangle” $ABC$ ($A = x$, $B = \phi(2\epsilon)(x)$, $C = \psi(2\epsilon)(x)$) with points $K, L, M$ on its sides ($K = \phi(\epsilon)(x)$, $L = (\phi(\epsilon) \circ \psi(\epsilon))(x)$, $M = \psi(\epsilon)(x)$). If we use the inherent “time” along the trajectories of the vector fields to measure length along these trajectories, then the points $K, L$ and $M$ will be midpoints of the sides of $ABC$ and the standard
Ceva relation will be trivially 1:

\[ c(A, B, C; K, L, M) = \frac{AK}{KB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1. \]

Consider now the image of triangle \(ABC\) under \(f\): the points \(K, L, M\) will again fall on the “sides” of this image, however the lengths will be somehow distorted (in fact even the fields \(df(\xi) = \kappa \cdot \xi\) and \(df(\eta) = \kappa \cdot \eta\) need not be commuting). Let us now explore this “distortion” up to the degree 2 in \(\epsilon\):

**Proposition 6.4.** Up to degree 2 the difference between the distorted Ceva relation and 1 is trivial; we put

\[ c(f(A), f(B), f(C); f(K), f(L), f(M)) - 1 =: \epsilon^2 S_3(f, \xi, \eta; x) + o(\epsilon^2), \]

then

\[ S_3(f, \xi, \eta; x) = \frac{5 \kappa''_{\eta\eta}(x) - \kappa''_{\xi\xi}(x)}{6 \kappa(x)}, \]

where we use the standard notation \(\kappa'_{\xi} = \xi(\kappa)\), \(\kappa'_{\eta} = \eta(\kappa)\).

**Proof.** We compute:

\[
\begin{align*}
    f(A)f(K) &= \epsilon \kappa(x) + \frac{1}{2} \epsilon^2 \kappa'_{\xi}(x) + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x) + o(\epsilon^3), \\
    f(K)f(B) &= \epsilon \kappa(x + \epsilon \xi) + \frac{1}{2} \epsilon^2 \kappa'_{\xi}(x + \epsilon \xi) + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x + \epsilon \xi) + o(\epsilon^3) \\
        &= \epsilon \kappa(x) + \frac{3}{2} \epsilon^2 \kappa'_{\xi}(x) + \frac{5}{3} \epsilon^3 \kappa''_{\xi\xi}(x) + o(\epsilon^3), \\
    f(M)f(A) &= -\epsilon \kappa(x) - \frac{1}{2} \epsilon^2 \kappa'_{\eta}(x) - \frac{1}{6} \epsilon^3 \kappa''_{\eta\eta}(x) + o(\epsilon^3), \\
    f(C)f(M) &= -\epsilon \kappa(x) - \frac{3}{2} \epsilon^2 \kappa'_{\eta}(x) - \frac{5}{3} \epsilon^3 \kappa''_{\eta\eta}(x) + o(\epsilon^3),
\end{align*}
\]
\[ f(B)f(L) = \epsilon \kappa(x + 2\epsilon \xi) + \frac{1}{2} \epsilon^2 \kappa'_\xi(x + 2\epsilon \xi) + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x + 2\epsilon \xi) + o(\epsilon^3) \]

\[ = \epsilon \kappa(x) + 2\epsilon^2 \kappa'_\xi(x) + 2\epsilon^3 \kappa''_{\xi\xi}(x) \]

\[ + \frac{1}{2} \epsilon^2 \kappa'_\xi(x) + \epsilon^3 \kappa''_{\xi\xi}(x) + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x) + o(\epsilon^3) \]

\[ = \epsilon \kappa(x) + \frac{3}{2} \epsilon^2 \kappa'_\xi(x) + \frac{1}{2} \epsilon^2 \kappa'_\xi(x) \]

\[ + \frac{5}{6} \epsilon^3 \kappa''_{\xi\xi}(x) + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x) + \frac{2}{3} \epsilon^3 \kappa''_{\xi\xi}(x) + o(\epsilon^3) \]

\[ f(L)f(C) = \epsilon \kappa(x + \epsilon(\xi + \eta)) + \frac{1}{2} \epsilon^2 \kappa'_\xi(x + \epsilon(\xi + \eta)) \]

\[ + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x + \epsilon(\xi + \eta)) + o(\epsilon^3) \]

\[ = \epsilon \kappa(x) + \frac{3}{2} \epsilon^2 \kappa'_\xi(x) + \frac{1}{2} \epsilon^2 \kappa'_\xi(x) \]

\[ + \frac{1}{2} \epsilon^3 \kappa''_{\xi\xi}(x) + \frac{1}{2} \epsilon^3 \kappa''_{\xi\xi}(x) + \epsilon^3 \kappa''_{\xi\xi}(x) \]

\[ + \frac{1}{2} \epsilon^3 \kappa''_{\xi\xi}(x) - \frac{1}{2} \epsilon^3 \kappa''_{\xi\xi}(x) \]

\[ + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x) + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x) - \frac{1}{3} \epsilon^3 \kappa''_{\xi\xi}(x) + o(\epsilon^3) \]

\[ = \epsilon \kappa(x) + \frac{3}{2} \epsilon^2 \kappa'_\xi(x) + \frac{1}{2} \epsilon^2 \kappa'_\xi(x) \]

\[ + \frac{1}{6} \epsilon^3 \kappa''_{\xi\xi}(x) + \frac{5}{6} \epsilon^3 \kappa''_{\xi\xi}(x) + \frac{2}{3} \epsilon^3 \kappa''_{\xi\xi}(x) + o(\epsilon^3). \]

Plugging these expressions into the formula for \( c(A, B, C; K, L, M) \), we obtain the expression we need. \( \square \)

The analogy between this expression and the Schwarz derivative is quite evident. One can ask, if it is possible to extend it in any reasonable way to a more general situation when there are less restrictions on the diffeomorphism, and also if there exist a non-commutative version of this operator. We are going to address these questions in forthcoming papers.

7. Noncommutative Cross-ratios. Applications.

7.1. Noncommutative leapfrog map. Let \( \mathbb{P}^1 \) be the projective line over a noncommutative division ring \( \mathcal{R} \). Consider points five points \( S_{i-1}, S_i, S_{i+1}, S_i^- \) and \( S_i^+ \) on \( \mathbb{P}^1 \). The theory of noncommutative cross-ratios (see theorem 3.4) implies that there exists a projective transformation sending

\[ (S_{i-1}, S_i, S_{i+1}, S_i^-) \rightarrow (S_{i+1}, S_i, S_{i-1}, S_i^+) \]

(in this order!) if and only if the corresponding cross-ratios coincide:

\[ (S_{i+1} - S_i)^{-1} (S_i^- - S_i) (S_i^- - S_{i-1})^{-1} (S_{i+1} - S_{i-1}) = \]

\[ = \lambda^{-1} (S_{i-1} - S_i)^{-1} (S_i^+ - S_i) (S_i^+ - S_{i+1})^{-1} (S_{i-1} - S_{i+1}) \lambda \]

where \( \lambda \in R. \)
Note that the factor \((S_{i+1} - S_{i-1})\) appears in the both sides of the equation but with the different signs. It shows that in the commutative case one gets the identity (5.14) from [9].

7.2. Noncommutative cross-ratios and the pentagramma mirificum.

7.2.1. Classical 5-recurrence. There is a wonderful observation (known as the Gauss Pentagramma mirificum) that when a pentagramma is drawn on a unit sphere in \(\mathbb{R}^3\) with successively orthogonal great circles with the lengths of inner side arcs \(\alpha_i, \ i = 1, \ldots, 5\) and one takes \(y_i := \tan^2(\alpha_i)\), then the following recurrence relation satisfies:

\[
y_i y_{i+1} = 1 + y_{i+3}, \quad \text{mod} \ Z_5.
\]

Gauss has observed that the first three equations for \(i = 1, 2, 3\) in (7.1) completely define the last two equations for \(i = 4, 5\).

It was discussed in [15] (which is our main source of the classical data for the Gauss Pentagramma Mirificum) that the variables \(y_i\) can be expressed via the classical cross-ratios:

\[
y_i = \left[p_{i+1}, p_{i+2}, p_{i+3}, p_{i+4}\right] = \frac{(p_{i+4} - p_{i+1})(p_{i+3} - p_{i+2})}{(p_{i+4} - p_{i+3})(p_{i+2} - p_{i+1})},
\]

where \(p_i = p_{i+5}\) are five points on real or complex projective line.

**Proposition 7.1.** Suppose that two consecutive points \(y_i\) and \(y_{i+1}\) (cyclically) are differents. Then the five cross-ratios \(y_i\) satisfy the relation (7.1).

It was remarked in [15] that after renaming \(x_1 = y_1, x_2 = y_4, x_3 = y_2, x_4 = y_5, x_5 = y_3\), the variables \(x_i, \ i = 1, \ldots, 5\) satisfy the famous pentagon recurrence:

\[
x_{i-1} x_{i+1} = 1 + x_i.
\]

7.2.2. Non-commutative analogues. Let \(\mathcal{R}\) be an associative division ring. In [1] (see section 1) we defined the cross-ratio \(\kappa(i, j, k, l)\) four vectors \(i, j, k, l \in \mathcal{R}^2\). Recall that

\[
\kappa(i, j, k, l) = q_{kl}^i q_{ik}^l
\]

where \(q_{ij}^k\) is the corresponding quasi-Plücker coordinate. In particular, \(q_{ij}^k\) and \(q_{kj}^i\) are inverse to each other and

\[
\kappa(j, i, l, k) = q_{lk}^i \kappa(i, j, k, l) q_{ik}^l, \quad \kappa(k, l, i, j) = q_{ik}^j \kappa(i, j, k, l) q_{kl}^i.
\]

We set \(\kappa(i, j, k, l) = \kappa(j, i, k, l)\).

Let now \(i, j, k, l, m\) be five vectors in \(\mathcal{R}^2\). We start with multiplicative relations for their cross-ratios. All these relations are redundant in the commutative case.

\[
\kappa(i, j, k, l) q_{km}^i \kappa(i, k, m, l) q_{mk}^i = q_{kl}^i \kappa(i, k, m, l) \kappa(i, j, k, l) q_{ik}^l,
\]

\[
q_{lm}^i \kappa(i, j, k, l) q_{kl}^i \kappa(l, k, i, m) q_{im}^l = q_{il}^m \kappa(i, j, k, l) q_{im}^k.
\]

Noncommutative versions of the pentagramma mirificum relations can be written as follows:

\[
\kappa(i, j, k, l) q_{kj}^i \kappa(m, l, j, i) q_{jk}^i = 1 - \kappa(m, j, k, i),
\]

\[
\kappa(i, j, k, l) q_{kl}^i \kappa(l, k, i, m) q_{jk}^i = 1 - \kappa(l, j, k, m),
\]

\[
q_{jk}^i \kappa(i, j, k, l) q_{kj}^i \kappa(m, l, j, i) = 1 - \kappa(i, j, k, m).
\]

For five vectors \(1, 2, 3, 4, 5\) in \(\mathcal{R}^2\) we set

\[
x_1 = -\kappa(1, 2, 3, 4), \ x_2 = -\kappa(5, 2, 3, 1), \ x_3 = -\kappa(5, 4, 2, 1),
\]
\[ x_4 = -\kappa(3, 4, 2, 5), \quad x_5 = -\kappa(3, 1, 4, 5). \]

Then
\[ x_1 q_1^4 x_3 q_3^4 = 1 + x_2, \quad x_4 q_3^5 x_2 q_2^5 = 1 + x_3, \]
\[ x_3 q_2^5 x_5 q_4^5 = 1 + x_4, \quad x_6 q_4^5 x_4 q_2^5 = 1 + x_5, \]
\[ x_5 q_4^3 x_7 q_3^3 = 1 + x_6, \]
where \( x_6 := \bar{x}_1 \) and \( x_7 := \bar{x}_2 \). Note the different order for even and odd left hand sides. So, we have an \( 5 \)-antiperiodicity, i.e. the periodicity up to the anti-involution \( x_{k+5} = \bar{x}_k \)

Also, the relations with odd left hand parts imply the relations for even left hand parts as in the commutative case.

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