A note on the run length function for intermittency maps

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June 26, 2018

Abstract

We study the run length function for intermittency maps. In particular, we show that the
longest consecutive zero digits (resp. one digits) having a time window of polynomial (resp.
logarithmic) length. Our proof is relatively elementary in the sense that it only relies on the
classical Borel-Cantelli lemma and the polynomial decay of intermittency maps. Our results are
compensational to the Erdős-Rényi law obtained by Denker and Nicol in [7].

1 Introduction

Consider a piecewise monotone interval map $T : [0, 1) \to [0, 1)$ with a countable or finite partition
$\{I_j\}$, preserving a probability measure $\nu$. Given an $x \in [0, 1)$ and $k \in \mathbb{N}$, set $\varepsilon_k(x) = j$ if $T^{k-1}(x) \in I_j$ for some $j$. The run length function for this system is defined as the maximal length of consecutive
$j$ digits in the sequence $(\varepsilon_1(x), \ldots, \varepsilon_n(x))$. Namely,

$$r_n(x, j) = \max \{k \geq 0 : \varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+k}(x) = j \text{ for some } 0 \leq i \leq n - k\}.$$ 

In this note, we concern about the run length function for a class of intermittency maps, and show the existence of an appropriate scaling length $k(n)$, to quantify the asymptomatic behavior of $r_n(x, j)$ for $\nu$-typical $x$.

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The studies on the run length functions are motivated from both theoretical and practical aspects. First, it has been revealed that run length function of piecewise linear interval maps has several probabilistic interpretations, which have great applications in DNA string machine [1], reliability theory and non-parametric statistics (see e.g., [2, 3, 4, 14, 18, 21] and the references therein). For example, when specializing \( T(x) = 2x \mod 1 \) and \( \nu \) being the Lebesgue measure on \([0, 1)\), the unit interval endows with a finite partition \( \{I_0, I_1\} \), where \( I_0 = [0, 1/2) \) and \( I_1 = [1/2, 1) \).

The run length function for such a system corresponds to the longest length of consecutive terms of \( \lceil \cdot \rceil \) where \( \cdot \) denotes the smallest integer no less than some number. As an important by-product, the length of cylinder can be partially estimated in terms of run length function. This fact is critical in understanding many dynamical problems of \( \beta \)-transformations (see [10, 11]). Unfortunately, comparing to \( \beta \)-transformations, relatively limited work has been done in studying the run length function for interval maps with non-linear components. This forms the first goal of the note.

Besides, the run length function also associates with the so-called Erdős-Rényi law. Given an observation \( \varphi \) in \( L^1(\nu) \), denote by \( S_n(\varphi) := \sum_{i=1}^{n-1} \varphi \circ T^i \)

\[
\theta(n, K(n), \varphi) := \max_{0 \leq i \leq n - K(n)} \{ S_{i+K(n)}(\varphi) - S_i(\varphi) \} = \max \{ S_{K(n)}(\varphi) \circ T^i : 0 \leq i \leq n - K(n) \}, \quad \forall n \in \mathbb{N},
\]

i.e., the maximal Birkhoff sum gaining over a window of length \( K(n) \) up to time \( n \). The Erdős-Rényi law concerns about the existence of an appropriate scaling length \( K(n) \), such that the asymptotic behavior of the fraction \( \frac{\theta(n, K(n), \varphi)}{K(n)} \) has a non-degenerate limit for \( \nu \)-typical \( x \). Inspired by the work of Erdős and Rényi for i.i.d random valuables [9], it is believed that under certain hyperbolicity of \( T \) and regularity of \( \varphi \) hypothesis, the behavior of \( \theta(n, K(n), \varphi) \) asymptotically admit a dichotomy on exhibiting either ordinary law of large numbers or asymptotic law of long length function, subject to the scaling length of \( K(n) \).

Initialized by the work of Grigull [13] for hyperbolic rational maps, there are a number of results on the studies of Erdős-Rényi law (as well as answering the above dichotomy), particularly for the dynamical systems satisfies the large deviation principle. For examples, see the work of Chazottes and Collet [5] for uniformly expanding interval maps, and the work of Denker and Kabluchko [6] for Gibbs-Markov dynamics.

Later, Denker and Nicol [7] provide several extensions of Erdős-Rényi law to the non-uniformly expanding dynamical systems, including logistic-like maps and intermittency maps. Unfortunately, if the dynamical systems are absent of large derivation principle, only partial results can be obtained, and the regularity of the observation function \( \varphi \) has to be assumed Lipschitz/Hölder continuous (rather than integrable). Under this framework, the characteristic functions are inapplicable to Denker and Nicol’s result, and thus no results on the run length function have been known. Moreover, their proofs (e.g., [7, Theo 4.1] for the intermittency maps) heavily rely on the previous dynamical Borel-Cantelli lemma results obtained Gouezël [12, Theo 1.1], and large deviation results

\[ LXXIV \in \text{the year of 1738} \]
obtained by Pollicott and Sharp [20, Theo 4]. Our second goal is aiming to fulfil this gap of the run length function for the intermittency maps with a proof in a more elementary way (namely, without using the two technical machineries above). This will in our belief provide a more refined description in this research direction.

Let us now state our results more mathematically. Consider $0 < \alpha < 1$ and recall the intermittency maps $T_\alpha : [0, 1) \to [0, 1)$ is defined as

$$T_\alpha(x) = \begin{cases} x(1 + 2^n x^\alpha), & \text{if } 0 \leq x < 1/2; \\ 2x - 1, & \text{if } 1/2 \leq x < 1. \end{cases}$$

It is well known that $T := T_\alpha$ has a finite absolutely continuous invariant probability measure $\mu$.

Let $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1)$, and we are accordingly concerned with the length of the longest consecutive zero digits in $(\varepsilon_1(x), \varepsilon_2(x), \cdots, \varepsilon_n(x))$, and the length of the longest consecutive one digits for $\mu$-almost all $x \in [0, 1)$. Set

$$r_\alpha^n(x) = \max \left\{ k \geq 0 : \varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+k}(x) = 0 \text{ for some } 0 \leq i \leq n - k \right\},$$

and

$$R_\alpha^n(x) = \max \left\{ k \geq 0 : \varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+k}(x) = 1 \text{ for some } 0 \leq i \leq n - k \right\}.$$  

With this conventions, our main theorem is stated as follows.

**Theorem 1.** For $\mu$-almost all $x \in [0, 1)$,

$$\lim_{n \to \infty} \frac{\log r_\alpha^n(x)}{\log n^\alpha} = 1,$$

and

$$\lim_{n \to \infty} \frac{R_\alpha^n(x)}{\log_2 n} = 1.$$  

Recall that $S_n(\chi_{I_0}(x))$ is the number of positive numbers $0 \leq i \leq n - 1$ such that $T^i(x) \in I_0$, i.e.,

$$S_n(\chi_{I_0}(x)) = \sum_{i=0}^{n-1} \chi_{I_0} \circ T^i(x),$$

where $\chi_{I_0}$ is the indicator function of the interval $I_0$. One can easily obtain the following corollary, which says that for a particular observable the limit of the maximal average over a time window of the length $n^{\alpha_1}$ (with $\alpha_1 < \alpha$) is equal to $1$ $\mu$-almost surely.

**Corollary 2.** If $\alpha_1 < \alpha$, then for $\mu$-almost $x \in [0, 1)$,

$$\lim_{n \to \infty} \frac{\max \{ S_{n^{\alpha_1}}(\chi_{I_0} \circ T^i(x)) : 0 \leq i \leq n - n^{\alpha_1} \}}{n^{\alpha_1}} = 1.$$  

Similarly, we can obtain

**Corollary 3.** Let $S_n(\chi_{I_1}(x)) = \sum_{i=1}^{k(n)} \chi_{I_i} \circ T^i(x)$. Then for any integer sequence $k(n)$ that satifying $\limsup_{n \to \infty} \frac{k(n)}{\log_2 n} \leq 1$, and for $\mu$-almost $x \in [0, 1)$, we have

$$\lim_{n \to \infty} \frac{\max \{ S_{k(n)}(\chi_{I_i} \circ T^i(x)) : 0 \leq i \leq n - k(n) \}}{k(n)} = 1.$$
The theorem indicates that polynomial length consecutive digits is typical for the run length function of intermittency maps, and different order of the run length function can coexist (differing from the phenomenon in $\beta$-transformations). The result implies the [7, Theo 4.1(b)] for $\phi = \chi_{I_0}$ a indicator observable of $I_0$.

Our proof relies heavily on the classical Borel-Cantelli lemma and the polynomial decay of intermittency maps. We obtain the upper limit by using the equivalence between the absolutely continuous invariant measure $\mu$ and Lebesgue measure outside a small one-side neighborhood of 0. To get a result of other direction, we break the first $n$ iterations into disjoint blocks with a small scale, on which the digits are not all 0 or 1. Then using the correlations between different blocks, and combining the measure estimation of the consecutive 0 or 1 digits, we obtain the lower limit of the run length function.

2 Proof of theorem

Note that $T_\alpha$ has a finite absolutely continuous invariant probability measure $\mu$ with a density function $h(x) = d\mu/dx$ satisfying
\[ \lim_{x \to 0} x^\alpha h(x) = c \]
for some constant $c$ (see Theorem A of Hu [16]). As a consequence, we obtain there exists a constant $1 < C < \infty$ such that
\[ C^{-1} x^{1-\alpha} \leq \mu([0, x)) \leq C x^{1-\alpha} \]
for sufficiently small $x \in [0, 1)$. Moreover, $C \leq h(x) \leq C x^{1-\alpha}$ for $x \in [a, 1)$ with $a > 0$, then the measure $\mu$ and Lebesgue measure are uniform equivalent on every interval $[a, 1)$, see [17, 22]. It follows that there exists a constant $\tilde{C} < \infty$ such that for any interval $A \subset [0, 1)$, we have
\[ \frac{1}{C} |[1/2, 1) \cap A| \leq \mu([1/2, 1) \cap A) \leq \tilde{C} |A|. \]

We now turn to the proof of the theorem.

Proof. The proof is divided into three parts:
Part I. We will prove that for $\mu$-almost all $x \in [0, 1)$,
\[ \limsup_{n \to \infty} \frac{\log r_n^c(x)}{\log n} \leq \alpha. \]

Let $0 < \varepsilon < 1 - \alpha$. It suffices to show that $\mu \{ x \in [0, 1) : r_n^c(x) \geq n^{\alpha+\varepsilon} \ i.o. \} = 0$. In fact, if $r_n^c(x) \geq n^{\alpha+\varepsilon}$ holds for some $x \in [0, 1)$ and infinitely many $n \in \mathbb{N}$, then we obtain two cases that either $\varepsilon_i(x) = 0$ for all $i$, $1 \leq i \leq \lceil n^{\alpha+\varepsilon} \rceil$ and infinitely many $n$, or there exists $1 \leq i \leq n - \lceil n^{\alpha+\varepsilon} \rceil$ such that
\[ \varepsilon_i(x) = 1 \text{ and } \varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+\lceil n^{\alpha+\varepsilon} \rceil}(x) = 0 \text{ for infinitely many } n. \]

Put $a_0 = 1/2$ and $a_{n+1} = T^{-1}(a_n) \cap [0, 1/2)$. If $\varepsilon_i(x) = 0$ for all $1 \leq i \leq \lceil n^{\alpha+\varepsilon} \rceil$, then it follows from Lemma 3.2 of [17] that
\[ x \leq a_{\lceil n^{\alpha+\varepsilon} \rceil} \leq C \frac{1}{n^{1+\varepsilon}}. \]
Therefore, we obtain
\[ \mu \{ x : \varepsilon_i(x) = 0 \text{ for all } 0 \leq i \leq \lceil n^{\alpha + \varepsilon} \rceil \text{ i.o.} \} = 0. \]  
\[ (6) \]

For the second case, it means that there exist \( 1 \leq i \leq n - \lceil n^{\alpha + \varepsilon} \rceil \) and infinitely many \( n \) such that
\[ T^{i-1}(x) \in I_1 \cap T^{-1}[0, a^{\lceil i^{\alpha + \varepsilon} \rceil}] \subset I_1 \cap T^{-1}[0, a^{\lceil i^{\alpha + \varepsilon} \rceil}]. \]

Note that \( n^{\alpha + \varepsilon} \) is increasing to infinity as \( n \) goes to infinity. So, we obtain that either \( \varepsilon_i(x) = 1 \) and \( \varepsilon_j(x) = 0 \) for all \( j > i \), or \( T^{i-1}(x) \in I_1 \cap T^{-1}[0, a^{\lceil i^{\alpha + \varepsilon} \rceil}] \) for infinitely many \( i \). It is easy to show that
\[ \mu \{ x : \varepsilon_i(x) = 1 \text{ and } \varepsilon_j(x) = 0 \text{ for all } j \geq i \} = 0. \]
\[ (7) \]

Since
\[ \sum_{i=1}^{\infty} \mu \{ x : T^{i-1}(x) \in I_1 \cap T^{-1}[0, a^{\lceil i^{\alpha + \varepsilon} \rceil}] \} = \sum_{i=1}^{\infty} \mu \{ I_1 \cap T^{-1}[0, a^{\lceil i^{\alpha + \varepsilon} \rceil}] \} \leq \sum_{i=1}^{\infty} \frac{\tilde{C}}{2^{i+\alpha}} < \infty, \]
by Borel-Cantelli lemma, it follows that
\[ \mu \{ x : T^{i-1}(x) \in I_1 \cap T^{-1}[0, a^{\lceil i^{\alpha + \varepsilon} \rceil}] \text{ for infinitely many } i \} = 0. \]
\[ (8) \]

Combining (6, 7, 8), we obtain
\[ \mu \{ x \in [0, 1) : r_n^\alpha(x) \geq n^{\alpha + \varepsilon} \text{ i.o.} \} = 0. \]

Therefore, for \( \mu \)-almost all \( x \in [0, 1) \),
\[ \limsup \frac{\log r_n^\alpha(x)}{\log n} \leq \alpha + \varepsilon. \]

Letting \( \varepsilon \to 0 \), we complete the proof of Part I.

**Part II.** We will prove that for \( \mu \)-almost all \( x \in [0, 1) \),
\[ \liminf \frac{\log r_n^\alpha(x)}{\log n} \geq \alpha. \]

We shall use the following statement. Let \( \xi = \{ I_0, I_1 \} \) be a partition of [0, 1] and \( \xi_n = \xi \vee T^{-1} \xi \vee \cdots \vee T^{-n+1} \xi \), where \( \xi \vee \eta = \{ A \cap B : A \in \xi, B \in \eta \} \). In [16], Hu showed that there exists \( C_1 > 0 \) and \( l > 0 \) such that for any \( m \geq 0 \) and \( A \in \xi_m \) and for any measurable set \( B \subset [0, 1) \),
\[ |\mu(A \cap T^{-n-m}B) - \mu(A)\mu(B)| \leq \frac{C_1 m^{1/\alpha - 1}}{(n-l)^{1/\alpha - 1}} \mu(A)\mu(B), \text{ for } n > l. \]

Then by the invariance of \( \mu \), we obtain
\[ |\mu(A^c \cap T^{-n-m}B) - \mu(A^c)\mu(B)| \leq \frac{C_1 m^{1/\alpha - 1}}{(n-l)^{1/\alpha - 1}} \mu(A)\mu(B), \]
where $A^c := [0, 1) \setminus A$. It implies that

$$\mu(A^c \cap T^{-n-m}B) \leq \left( \frac{C_1 m^{1/\alpha - 1}}{(n-l)^{1/\alpha - 1}} \mu(A) + \mu(A^c) \right) \mu(B). \tag{9}$$

For any $x \in [0, 1)$ and $m, n \in \mathbb{N}$ with $m < n$, we define

$$r_{m,n}^\alpha(x) := \max\{k \geq 0 : \varepsilon_i(x) = \cdots = \varepsilon_{i+k}(x) = 0 \text{ for some } m - 1 \leq i \leq n - k\}.$$ 

Thus, $r_{1,n}^\alpha = r_n^\alpha$. For any $0 < \varepsilon < \alpha$, write $t_n := \lceil n^{\alpha - \varepsilon} \rceil$ and $\xi_n := \lceil \frac{n}{t_n^{1+\varepsilon}} \rceil$. Let

$$E = \{x \in [0,1) : r_{t_n}^\alpha(x) < t_n\}$$

and

$$F_j = \{x \in [0,1) : r_{j t_n^{1+\varepsilon} + 1, j t_n^{1+\varepsilon} + t_n}^\alpha(x) < t_n\}, \quad 1 \leq j \leq k_n - 1.$$ 

Then

$$\{x \in [0,1) : r_n^\alpha(x) < t_n\} \subseteq E \cap F_1 \cap \cdots \cap F_{k_n-1}$$

$$= E \cap T^{-t_n^{1+\varepsilon}}(E \cap F_1 \cap \cdots \cap F_{k_n-2}).$$

Let

$$I_k(0, \cdots, 0) := \{x \in [0,1) : \varepsilon_1(x) = \cdots = \varepsilon_k(x) = 0\} \text{ for some } k \geq 1.$$ 

Then $I_{t_n}(0, \cdots, 0) \subseteq \xi_{t_n}$ and $I_{t_n}(0, \cdots, 0) = E^c$. By (4), the similar arguments in Section 6.2 of [22] imply that

$$\tilde{C}^{-1} \frac{1}{t_n^{1/\alpha - 1}} \leq \mu(I_{t_n}(0, \cdots, 0)) \leq \tilde{C} \frac{1}{t_n^{1/\alpha - 1}},$$

where $\tilde{C} > 1$ is a constant. For $n$ big enough so that $t_n^{1+\varepsilon} - t_n > l$, combining this with (9), we deduce that

$$\mu\{x \in [0,1) : r_n^\alpha(x) < t_n\} \leq \mu(E) \cap T^{-t_n^{1+\varepsilon}}(E \cap F_1 \cap \cdots \cap F_{k_n-2})$$

$$\leq \left( \frac{C_1 t_n^{1/\alpha - 1}}{(t_n^{1+\varepsilon} - t_n - l)^{1/\alpha - 1}} \mu(I_{t_n}(0, \cdots, 0)) + \mu(E) \right)$$

$$\times \mu(E \cap F_1 \cap \cdots \cap F_{k_n-2})$$

$$\leq \cdots$$

$$\leq \left( \frac{C_1 \tilde{C}}{t_n^{1+\varepsilon}} + 1 - \frac{\tilde{C}^{-1}}{t_n^{1/\alpha - 1}} \right)^{k_n-1}. \tag{10}$$

Using the inequality: $e^{-x} \geq 1 - x$ for any $x \geq 0$, we have

$$\mu\{x \in [0,1) : r_n^\alpha(x) < t_n\} \leq \left( 1 - \frac{\tilde{C}^{-1}}{t_n^{1/\alpha - 1}} + o\left(\frac{1}{t_n^{1/\alpha - 1}}\right) \right)^{k_n-1}$$

$$\leq \exp\left(-\frac{\tilde{C}^{-1} n}{t_n^{1/\alpha - 1} t_n^{1+\varepsilon}}\right)$$

$$= \exp\left(-\tilde{C}^{-1} n^{-\frac{\varepsilon}{\alpha + \varepsilon}} \right).$$
Since $\frac{\varepsilon}{\alpha} + \varepsilon^2 > \alpha\varepsilon$, we eventually get that
\[
\sum_{n \geq 1} \mu\{x \in [0, 1) : r_n^\alpha(x) < t_n\} < \infty.
\]
By Borel-Cantelli lemma, we conclude that for $\mu$-almost all $x \in [0, 1)$,
\[
\liminf_{n \to \infty} \frac{\log r_n^\alpha(x)}{\log n} \geq \alpha - \varepsilon.
\]
Letting $\varepsilon \to 0^+$, the proof of Part II is completed.

**Part III.** We are concerned in this part with the run length function $R_n^\alpha(x)$. This part can be proved by the similar way as shown before. We first show that for $\mu$-almost all $x \in [0, 1)$,
\[
\limsup_{n \to \infty} \frac{R_n^\alpha(x)}{\log_2 n} \leq 1.
\]
In fact, let $\varepsilon > 0$, if $R_n^\alpha(x) \geq (1 + \varepsilon) \log_2 n$ holds for some $x \in [0, 1)$ and $n \in \mathbb{N}$, then there exists $0 \leq i \leq n - k$ (where $k = \lceil (1 + \varepsilon) \log_2 n \rceil$) such that
\[
\varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+k}(x) = 1.
\]
That is, $T^i(x) \in I_1$, $T^{i+1}(x) \in I_1$, \ldots, $T^{i+k-1}(x) \in I_1$. Hence
\[
1 - \frac{1}{(i + 1)^{1+\varepsilon}} \leq 1 - \frac{1}{2^k} \leq T^i(x) < 1.
\]
Since $(1 + \varepsilon) \log_2 n \to \infty$ as $n \to \infty$, $R_n^\alpha(x) \geq (1 + \varepsilon) \log_2 n$ for infinite many $n$ implies that either $T^i(x) \in [1 - \frac{1}{(i+1)^{1+\varepsilon}}, 1]$ for infinite many $i$ or there exists an $i \in \mathbb{N}$ such that $\varepsilon_j = 1$ for all $j \geq i$. Then it follows from Borel-Cantelli lemma that
\[
\mu\{x \in [0, 1) : R_n^\alpha(x) \geq (1 + \varepsilon) \log_2 n \ i.o. \} = 0.
\]
Therefore, for $\mu$-almost all $x \in [0, 1)$,
\[
\limsup_{n \to \infty} \frac{R_n^\alpha(x)}{\log_2 n} \leq 1 + \varepsilon.
\]
Taking $\varepsilon \to 0$, we obtain the proof of the first part.

Next we will show for $\mu$-almost all $x \in [0, 1)$,
\[
\liminf_{n \to \infty} \frac{R_n^\alpha(x)}{\log_2 n} \geq 1.
\]
We shall use the following facts. For any interval $A \subseteq (1/2, 1)$ and interval $B \subseteq (1/2, 1)$,
\[
|\mu(A \cap T^{-n}B) - c_n \mu(A) \mu(B)| \leq \frac{1}{n^{1/\alpha}} \mu(B), \quad \forall n \geq 1,
\]
where $c_n = 1 + \frac{s}{n^{1/\alpha-1}} + o\left(\frac{1}{n^{1/\alpha-1}}\right)$ for some non-zero constant $s$, see Eq. (1.3) of Gouëzel [12]. Then
\[
\mu(A^c \cap T^{-n}B) \leq \left(c_n \mu(A^c) + \frac{1}{n^{1/\alpha}}\right) \mu(B), \quad \forall n \geq 1. \tag{11}
\]
For any $x \in [0, 1]$ and $m, n \in \mathbb{N}$ with $m < n$, we define
\[
R^\alpha_{m,n}(x) := \max \{k \geq 0 : \varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+k}(x) = 1 \text{ for some } m - 1 \leq i \leq n - k \}.
\]
Thus, $R^\alpha_{1,n} = R^\alpha_n$. For $\varepsilon > 0$ small such that $\frac{\alpha}{1-\alpha} - \varepsilon - \varepsilon^2 \frac{1}{1-\alpha} > 0$, write $t_n := \lceil (1-\varepsilon) \log_2 n \rceil$, and
\[
l_n = \left\lceil \frac{\alpha}{1-\alpha} + \varepsilon \right\rceil .
\]
Set $\kappa_n := \lceil \frac{\alpha}{t_n} \rceil$. Let
\[
E = \{x \in [0, 1) : R^\alpha_{t_n}(x) < t_n \}
\]
and
\[
F_j = \{x \in [0, 1) : R^\alpha_{j_t_n + 1, j_t_n + t_n}(x) < t_n \}, \quad 1 \leq j \leq \kappa_n - 1.
\]
Then
\[
\{x \in [0, 1) : R^\alpha_{t_n}(x) < t_n \} \subseteq E \cap F_1 \cap \cdots \cap F_{\kappa_n - 1} = E \cap T^{-t_n}(E \cap F_1 \cap \cdots \cap F_{\kappa_n - 2}).
\]
Note that $I_{t_n}(1, \ldots, 1) \subseteq [1/2, 1)$ and $I_{t_n}(1, \ldots, 1) = E^c$, where
\[
I_{t_n}(1, \ldots, 1) := \{x \in [0, 1) : \varepsilon_1(x) = \cdots = \varepsilon_{t_n}(x) = 1 \}.
\]
By (5), it follows that
\[
\tilde{C}^{-1} \frac{1}{2^t_n} \leq \mu(I_{t_n}(1, \ldots, 1)) \leq \tilde{C} \frac{1}{2^t_n},
\]
where $\tilde{C} > 1$ is a constant. Combining this with (11), we deduce that
\[
\mu\{x \in [0, 1) : R^\alpha_{n}(x) < t_n \} \leq \mu(E \cap F_1 \cap \cdots \cap F_{\kappa_n - 1})
\]
\[
\leq \left( c_{t_n} \mu(E) + \frac{1}{t_n^{1/\alpha}} \right) \mu(E \cap F_1 \cap \cdots \cap F_{\kappa_n - 2})
\]
\[
\leq \cdots \cdots
\]
\[
\leq \left( c_{t_n} \mu(E) + \frac{1}{t_n^{1/\alpha}} \right)^{k_n - 1}
\]
\[
\leq \left( 1 + \frac{s}{2^t_n(1+\varepsilon)} + o\left( \frac{1}{2^t_n(1+\varepsilon)} \right) \right) \left( 1 - \tilde{C}^{-1} \frac{1}{2^t_n} \right) \left( 1 - \frac{1}{2^t_n} \right)^{k_n - 1}
\]
\[
\leq \left( 1 - \frac{\tilde{C}^{-1}}{2^t_n} + o\left( \frac{1}{2^t_n} \right) \right)^{k_n - 1}
\]
\[
\leq \exp \left\{ - \frac{\tilde{C}^{-1}}{2^t_n} \frac{n}{2^t_n(1+\varepsilon)} \right\}.
\]
Then the choosing of $\varepsilon$ implies that
\[
\sum_{n \geq 1} \mu\{x \in [0, 1) : R^\alpha_{n}(x) < t_n \} < \infty,
\]

by Borel-Cantelli lemma, we conclude that for \( \mu \)-almost all \( x \in [0,1) \),

\[
\liminf_{n \to \infty} \frac{R_n^\alpha(x)}{\log_2 n} \geq 1 - \varepsilon.
\]

Letting \( \varepsilon \to 0^+ \), we complete the proof.

\[\square\]

**Acknowledgments**

We would like to thank Prof. Manfred Denker and Prof. Huyi Hu for useful discussions, particularly for the proof of part I of our main theorem. This work is partially supported by grants from National Natural Science Foundation of China (Nos. 11701200, 11671395), Hubei Chenguang Talented Youth Development Foundation 2017 (Nos.0106011025), and Ky and Yu-Fen Fan traveling award, National Science Foundation 2018.

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