On the conformal structure of the extremal Reissner–Nordström spacetime

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Abstract
We analyse various conformal properties of the extremal Reissner–Nordström spacetime. In particular, we obtain conformal representations of the neighbourhoods of spatial infinity, time-like infinity and the cylindrical end—the so-called cylinders at spatial infinity and at the horizon, respectively—which are regular with respect to the conformal Einstein field equations and their associated initial data sets. We discuss possible implications of these constructions for the propagation of test fields and non-linear perturbations of the gravitational field close to the horizon.

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1. Introduction
The analysis of the non-linear stability of stationary black holes is, no doubt, one of the key open problems in contemporary mathematical general relativity. Among stationary black hole spacetimes, extremal ones are of particular interest. The simplest example of an extremal black hole is given by the so-called extremal Reissner–Nordström—a static, spherically symmetric solution to the Einstein–Maxwell equations.

The mainstream approach to the analysis of the non-linear stability of black holes has consisted, in a first instance, of the study of the linear evolution equations (the wave equations or the Maxwell equations) on the black hole spacetime. This approach has also been pursued for the extremal Reissner–Nordström spacetime, and has provided valuable insights—see e.g. [2, 3, 9].

The conformal Einstein field equations and the conformal methods based upon them have been used with remarkable success to understand the existence and stability of asymptotically
simple spacetimes—see e.g. [16] for a review. In view of this success, it is natural to ask whether it is possible to adapt these ideas to analyse the stability of black hole spacetimes. One of the underlying strategies in this conformal approach is to obtain a detailed understanding of the geometric structure of the background solution under consideration in order to construct an evolution problem that is as simple as possible. In this respect, the conformal structure of the extremal Reissner–Nordström seems particularly amenable to a detailed analysis.

In [27] it has been shown that the domain of outer communication of the extremal Reissner–Nordström spacetime can be covered by a non-intersecting congruence of curves with special conformal properties (the so-called conformal curves). These curves are of special interest as they provide a simple expression for a conformal factor which, in turn, could be used to obtain a conformal representation of the spacetime. Moreover, these curves can be used as the cornerstone of a gauge for the conformal evolution equations. These ideas have been used in [34] to obtain global numerical evaluations of the extremal Reissner–Nordström spacetime.

One of the insights obtained from the analysis of [27] is the special role played in the conformal geometry by the points $i^\pm$ corresponding, respectively, to future and past time-like infinity. Recall that time-like geodesics that are not crossing the horizon start at $i^-$ and end at $i^+$. Although the analysis of the conformal curves does not allow us to conclude whether these points are regular points of the conformal geometry, it nevertheless shows that the extremal Reissner–Nordström is, in a particular sense, more regular at these points than, say, the Schwarzschild black hole or a non-extremal Reissner–Nordström solution. This statement is justified by the following observation: In the case of the extremal Reissner–Nordström solution, the conformal curves constructed in [27] that are passing through $i^\pm$ remain always time-like, while in the non-extremal case the curves become null at $i^\pm$. In view of the conformal properties of the conformal curves in the non-extremal case, this degeneracy in their causal character indicates a degeneracy of the conformal structure—see also the discussion in [17].

A property of the extremal Reissner–Nordström solution is that it possesses a conformal discrete isometry that, roughly speaking, maps the domain of outer communication into itself via an inversion of the radial coordinate and the horizon into null infinity—see [8]. Moreover, this conformal inversion maps the black hole region of the extremal Reissner–Nordström to the negative mass extremal Reissner–Nordström spacetime, and vice versa. A similar property is also present in the extremal Reissner–Nordström–de Sitter spacetime—see [5]. Initially this conformal isometry was viewed as a mere curiosity, but recently it has been used in [4] to put in correspondence a class of conserved quantities on the horizon observed in [2, 3] with the so-called Newman–Penrose constants defined at null infinity—see [11, 29, 30].

The purpose of this article is to exploit the discrete conformal isometry of the extremal Reissner–Nordström spacetime to obtain a representation of time-like infinity for which the conformal field equations and their associated initial data are regular. In [15] it has been shown that a conformal representation with these properties can be obtained for the spatial infinity of a vacuum spacetime with time-reflection symmetry—the so-called cylinder at spatial infinity $I$. In this article, it is shown that a similar construction can be implemented for the spatial infinity of the extremal Reissner–Nordström spacetime—the analysis being independent of the sign of the mass. By applying the conformal inversion to this construction, we obtain the cylinder at time-like infinity $\mathcal{H}_{c^+}$, a representation of the neighbourhood of the point $i^+$ for which the conformal field equations and their data are regular. Remarkably, this strategy can also be used to obtain a similar representation of a neighbourhood of the cylindrical end $c^0$ of the extremal Reissner–Nordström spacetime—the cylinder at the
singularity \( \mathcal{H}^0 \). We call these constructions collectively the cylinders at the horizon and denote them by \( \mathcal{H} \).

The cylinders at the horizon inherit, in a natural manner, many of the properties of cylinder at spatial infinity. Crucially, they are total characteristics of the evolution equations implied by the conformal field equations—that is, the evolution equations completely reduce to an interior system of transport equations. Consequently, no boundary conditions can be specified at the cylinders. The solution to the evolution equations is fully determined by its value at some section of the cylinder. A further analogy between the cylinder \( \mathcal{H} \) at the horizon and the cylinder \( \mathcal{I} \) at spatial infinity is given by the following: The cylinders terminate in sets with the topology of \( S^2 \) where one observes a degeneracy of the symmetric hyperbolic evolution equations deduced from spinorial field equations whose principal part is of the form

\[
\nabla^A \phi_{AB\cdots} - \phi_{AB\cdots} = \phi_{(AB\cdots)}
\]

In the case of the cylinder at spatial infinity, it has been possible to relate this degeneracy in the evolution equations with the appearance of potential obstructions to the smoothness of null infinity—see \([15, 35–38]\). It is conjectured that similar obstructions will arise from the analysis of transport equations at the cylinders at the horizon. This question and its potential implications for the analysis of non-linear perturbations of the extremal Reissner–Nordström spacetime will be analysed elsewhere.

There is, however, a crucial difference between the cylinder \( \mathcal{I} \) at spatial infinity and the cylinder \( \mathcal{H} \) at the horizon. While the conformal factor associated to the former representation vanishes at \( \mathcal{I} \), in the later representations the respective conformal factors do not vanish at \( \mathcal{H} \). Thus, strictly speaking, the cylinders at the horizon are not part of the conformal boundary.

A natural question to be asked at this point is how crucial is the conformal isometry of the extremal Reissner–Nordström in the construction presented in this article. While it is of great value in order to gain intuition about the underlying structures, we claim it is not essential. Once the key aspects of the construction have been identified, the results of this paper could have been obtained without using this isometry at the expense of lengthier arguments. This claim suggests the possibility of performing a similar analysis in other extremal black hole spacetimes, in particular the extremal Kerr solution.

**Outline of the article**

In section 2 we provide a summary of some basic facts concerning the extremal Reissner–Nordström spacetime (various types of coordinates, Penrose diagrams, properties of the conformal isometry) which will be used throughout this article. Section 3 discusses the basic properties of conformal geodesics in the electrovacuum spacetimes that will be used in our analysis. Section 4 provides an analysis of the conformal properties of time-symmetric hypersurfaces in the extremal Reissner–Nordström spacetime and the initial data for the conformal Einstein field equations. This analysis is key to identifying the singular behaviour of various conformal fields at time-like infinity and the cylindrical asymptotic end. Section 5 discusses the construction of the cylinder at spatial infinity for the extremal Reissner–Nordström spacetime. This construction is used, in turn, in section 6 to motivate and implement the representation of the cylinders at the horizon. Conclusions and possible implications of the present analysis to the propagation of fields close to time-like infinity and the cylindrical asymptotic end are discussed in section 7. Finally, an appendix provides some details about the transformation formulae relating objects in the cylinders at spatial infinity and at the horizon.
Notation

Our signature convention for spacetime (Lorentzian) metrics is \(+−−−\). In what follows \(a, b, c, \ldots\) denote spacetime tensorial indices while \(\sigma, \rho, \tau, \ldots\) correspond to spacetime frame indices taking the values \(0, 1, 2, 3\). Spatial tensorial indices will be denoted by \(i, j, k, \ldots\), while spatial frame indices will be denoted by \(i, j, k\). Part of the analysis will require the use of spinors. In this respect, we make use of the general conventions of Penrose & Rindler [31]. In particular, \(A, B, C, \ldots\) denote abstract spinorial indices, while \(\alpha, \beta, \gamma, \ldots\) indicate frame spinorial indices with respect to some specified spin dyad \(\{\delta_A\}\).

Index-free notation will also be used in many places. Given a 1-form \(\omega\), its pairing with a vector \(v\) will be denoted by \(\omega \langle v \rangle\). Given a metric \(g\), its contravariant counterpart will be denoted by \(\sharp g\). The operation of raising the index of the 1-form \(\omega\) will be denoted by \(\omega \equiv \sharp\sharp g(\cdot,\cdot)\). Similarly, the lowering of the index of the vector \(v\) will be denoted by \(\equiv \flat v g(\cdot,\cdot)\). Given a connection \(\nabla\), the covariant directional derivative along a curve with tangent \(v\) will be denoted by \(\nabla_v\).

Various connections will be used throughout. The connection \(\tilde{\nabla}\) will always denote the Levi–Civita connection of a Lorentzian metric \(g\) satisfying the Einstein–Maxwell field equations—hence, we call it the physical connection. Connections conformally related to \(\tilde{\nabla}\) will be denoted by \(\nabla\) and \(\tilde{\nabla}\) and will be called unphysical. Finally, \(\tilde{\nabla}\) will denote a Weyl connection in the conformal class of \(\tilde{\nabla}\).

2. Basic expressions

The extremal Reissner–Nordström spacetime is the solution to the Einstein–Maxwell field equations

\[ R_{\alpha \beta} - \frac{1}{2} R g_{\alpha \beta} = F_{\alpha \epsilon} F^\epsilon_{\beta} - \frac{1}{4} g_{\alpha \beta} F_{\epsilon \delta} F^{\epsilon \delta}, \]

\[ \tilde{\nabla}^b F_{ab} = 0, \]

\[ \tilde{\nabla}^a [F_{bc}] = 0, \]

given in standard spherical coordinates \((t, \tilde{r}, \theta, \phi)\) by

\[ g = \left(1 - \frac{m}{\tilde{r}}\right)^2 dt \otimes dt - \left(1 - \frac{m}{\tilde{r}}\right)^{-2} d\tilde{r} \otimes d\tilde{r} - r^2 \sigma \]

\[ F = \pm \frac{m}{2\tilde{r}^2} dt \wedge d\tilde{r}. \]

where

\[ \sigma \equiv (d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi) \]

is the standard metric of \(S^2\). The discussion in this article will be concerned with both \(m > 0\) and \(m < 0\) cases of the solution \((2a)–(2b)\). As it is well known, the case \(m > 0\) describes an extremal black hole spacetime (the extremal Reissner–Nordström spacetime proper), while the case \(m < 0\) corresponds to a spacetime containing a naked singularity. For ease of the presentation, in a slight abuse of nomenclature, we refer to the case \(m < 0\) as the negative mass extremal Reissner–Nordström spacetime despite the fact that it does not contain horizons. For the sake of generality in our analysis, we will use the general notation \(r, m, \tilde{g}, \) etc. so that both cases can be discussed simultaneously. However, where it is necessary to
avoid confusion or ambiguity, we distinguish the two mass cases by the corresponding subscript, i.e., on the positive mass Reissner–Nordström solution, we use $r_+, m_+, \tilde{g}_+$, etc., and on the negative mass spacetime, we use $r_-, m_-, \tilde{g}_-$, etc. In line with the above, we will denote the maximal analytic extension in the positive mass case ($m > 0$) by $(\tilde{\mathcal{M}}_+, \tilde{g}_+)$ (see e.g. Carter [6, 7]). Similarly, the maximal analytic extension in the negative mass case ($m < 0$) will be denoted by $(\tilde{\mathcal{M}}_-, \tilde{g}_-)$. 

2.1. Isotropic and null coordinates

In this article, it will be more convenient to make use of the isotropic coordinate $r$ defined by

$$r = \tilde{r} - m, \quad \tilde{r} = r + m.$$ 

In terms of this coordinate the metric of the extreme Reissner–Nordström spacetime is given by

$$\tilde{g} = \left( 1 + \frac{m}{r} \right)^2 dr \otimes dt - \left( 1 + \frac{m}{r} \right)^2 (dr \otimes dr + r^2 \sigma). \quad (3)$$

In the positive mass case, this metric is well defined for $r_+ \in (-m, 0) \cup (0, \infty)$: the region for which $r_+ > 0$ corresponds to the domain of outer communication of a static black hole spacetime while that for which $-m < r_- < 0$ corresponds to the black hole region. In the negative mass case there are no horizons, and $r = r_- \in [ml, \infty)$. The vector $\partial_t$ is clearly a Killing vector for the metric (3). Except at the horizon (in the positive mass case) it is always time-like—by contrast to the analogous vector field in the Schwarzschild spacetime or the non-extremal Reissner–Nordström spacetime.

Retarded and advanced null coordinates can be introduced via

$$u = t - \left( r - \frac{m^2}{r} + 2 m \ln |r| \right), \quad v = t + \left( r - \frac{m^2}{r} + 2 m \ln |r| \right),$$

to obtain the line elements

$$\tilde{g}(u, r) = \frac{r^2}{(r + m)^2} du \otimes du + (du \otimes dr + dr \otimes du) - (r + m)^2 \sigma, \quad (4a)$$

$$\tilde{g}(v, r) = \frac{r^2}{(r + m)^2} dv \otimes dv - (dv \otimes dr + dr \otimes dv) - (r + m)^2 \sigma, \quad (4b)$$

where in the previous expressions the notation $\tilde{g}(u, r)$ and $\tilde{g}(v, r)$ is intended to highlight the particular choice of coordinates being used to express the metric $\tilde{g}$. Similarly, in (2a) and (3) one could have written $\tilde{g}(t, \tilde{r})$ and $\tilde{g}(t, r)$, respectively.

2.2. Penrose diagrams

The Penrose diagrams of the extremal Reissner–Nordström solutions are well known. They were first discussed in [6]—see also [23]. The diagrams for both the positive and negative mass cases are given in figure 1 for reference.

For the purposes of the subsequent discussion, it is convenient to identify the following subsets of the positive mass extremal Reissner–Nordström spacetime:
$\mathcal{I} \equiv \{ p \in \mathcal{M}_+ \mid -\infty < v(p) < \infty, \ 0 < r(p) < \infty \}$,

$\mathcal{III} \equiv \{ p \in \mathcal{M}_+ \mid -\infty < v(p) < \infty, \ -m < r(p) < 0 \}$,

$\mathcal{H}^+ \equiv \{ p \in \mathcal{M}_+ \mid -\infty < v(p) < \infty, \ r(p) = 0 \}$,

$\mathcal{H}^- \equiv \{ p \in \mathcal{M}_+ \mid -\infty < u(p) < \infty, \ r(p) = 0 \}$,

Figure 1. Conformal diagram of the extremal Reissner–Nordström spacetime in the positive mass (left) and negative mass (right) cases. The definitions of the regions $\mathcal{I}$, $\mathcal{III}$ and $\mathcal{I}^*$ are given in the main text. The future and past null infinities of the positive mass case are denoted, respectively, by $\mathcal{I}^{\pm}$, the horizons by $\mathcal{H}^{\pm}$, the future and past time-like infinities by $\mathcal{I}^*$, the spatial infinities by $\mathcal{I}^*$ and the cylindrical end by $c^0$. A similar notation is used for the analogous sets in the negative mass case. The hypersurfaces $S$ and $R$ are time-symmetric slices in the positive mass case, while $S'$ is a time-symmetric slice in the negative mass case—see section 4.

Figure 2. Schematic two-dimensional diagram of the bundle manifold $\mathcal{M}_+$, giving rise to the cylinder at spatial infinity. The null hypersurfaces $\mathcal{I}^{\pm}$ correspond to the horizon, $\mathcal{I}$ is the cylinder at spatial infinity proper and $\mathcal{I}^*$ are the critical sets where the cylinder meets null infinity. The conformal geodesics on which this construction is based correspond to straight vertical lines as indicated. Notice that the diagram does not follow the conventions of Carter and Penrose—null hypersurfaces are not represented by straight lines with slope of $45^\circ$ or $135^\circ$. 
describing, respectively, the domain of outer communication, the black hole region, and the future and past horizons. In the negative mass case, there is no black hole region so that one finds only
\[ \bar{I}' \equiv \{ p \in \bar{\mathcal{M}}_\infty \mid -\infty < v(p) < \infty, \ |m| < r(p) < \infty \}. \]

**Remark.** The notation of \( \bar{I} \) and \( \bar{I}' \) was chosen to indicate their physical resemblance. Both regions describe the asymptotic region of the black hole; however, \( \bar{I} \) borders the horizon while \( \bar{I}' \) borders a naked singularity. In fact, as we will see below, in terms of its conformal geometry \( \bar{I}' \) is more alike that of \( \bar{I} \).

### 2.3. The discrete conformal isometry of the extremal Reissner–Nordström spacetime

In [8] it has been shown that the extremal Reissner–Nordström possesses a *conformal discrete isometry*—see also [4] for a more elaborated discussion. This isometry is best expressed in isotropic coordinates, and it is implemented by the inversion of the radial isotropic coordinate \( \iota \): \( r \rightarrow m^2/r \), so \( \iota \) is an involution. Let \( t_\iota \) denote the push-forward map implied by \( \iota \). As \( \iota \) is an involution for \( r \neq 0 \), its action is well defined on both covariant and contravariant tensors.

In order to discuss the effect of discrete conformal isometry it is convenient to introduce the *inversion radial coordinate* \( \varrho \equiv m^2/r \), so that \( \iota : r \rightarrow \varrho \) and \( \iota : \varrho \rightarrow r \). Moreover, it can be verified that the conformal inversion interchanges the null coordinates \( u \) and \( v \)—that is, one finds that
\[ t_\iota u \equiv u \ast \iota = v, \quad t_\iota v \equiv v \ast \iota = u. \]

A direct computation using the metric \( \tilde{g} \) in the form (3) shows that
\[ \tilde{g}(\iota, \varrho) = \left( \frac{m}{\varrho} \right)^2 \left( \frac{\varrho^2}{(\varrho + m)^2} \delta t \otimes dt - (\varrho + m)^2 \left( \delta \varrho \otimes \delta \varrho + \varrho^2 \sigma \right) \right). \]

Thus, the inversion \( r \mapsto m^2/r \) is implemented in the above metric by simply making the replacement \( \varrho \mapsto r \). Moreover,
\[ t_\iota \tilde{g} = \Omega^2 \tilde{g}, \quad \text{with} \quad \Omega \equiv \frac{m}{r} = \frac{\varrho}{m}. \quad (5) \]
A similar computation using the expressions of the extremal Reissner–Nordström metric in terms of the retarded and advanced coordinates \( u \) and \( v \), equations (4a)–(4b) shows that

\[
t_\Omega \tilde{g}(v, q) = \Omega^2 \tilde{g}(u, r), \quad t_\Omega \tilde{g}(u, q) = \Omega^2 \tilde{g}(v, r).
\]

For future use, it is convenient to define the \emph{unphysical} metric

\[
\tilde{g} \equiv \Omega^2 \tilde{g}, \quad \Omega \equiv \frac{m}{r},
\]

so that one can write

\[
t_\Omega \tilde{g} = \tilde{g} \quad \text{and} \quad t_\Omega \tilde{g} = \tilde{g}.
\]

\section*{Effect of the isometry on region \( \overline{I} \).}

From the discussion in the previous paragraphs, it follows that \((I, \tilde{g}_s(u, q))\) with

\[
I \equiv \left\{ p \in \mathbb{R} \times (0, \infty) \times S^2 \mid -\infty < u(p) < \infty, \ 0 \leq q(p) < \infty \right\},
\]

\[
\tilde{g}_s(u, q) = \frac{q^2}{(q + m)^2} du \otimes du - (du \otimes dq + dq \otimes du) - (q + m)^2 \sigma,
\]

is a conformal extension of \((\overline{I}, \tilde{g}_s(u, r))\) — in particular, \(\tilde{g}_s(u, q)\) is real analytic on \(I\). The null hypersurface

\[
\mathcal{I}^+ = \{ p \in I \mid q(p) = 0 \},
\]

on which \(\Omega = 0\), \(d\Omega \neq 0\) represents the future null infinity of the region \(\overline{I}\). From the discussion in the previous paragraphs it follows that \(\iota: \overline{I} \to \overline{I}\) can, in fact, be extended to a real analytic isometry

\[
\iota: I \to \overline{I} \cup \mathcal{H}^+.
\]

In particular, one finds that

\[
\iota(I^+) = \mathcal{H}^+.
\]

That is, the conformal isometry \(\iota\) sends future null infinity into the future horizon (and vice versa).

\section*{Effect of the isometry on \( \overline{M}_\pm \).}

Key for the purposes of this article is the effect of \(\iota\) on the negative mass extremal Reissner–Nordström spacetime. In analogy to the positive mass case, one can use the conformal metric (6) to construct a conformal extension of \((\overline{M}_-, \tilde{g}_\pm(u, r))\); namely, \((\overline{M}_-, \tilde{g}_\pm(u, q))\) with

\[
\tilde{g}_\pm(u, q) = \frac{q^2}{(q + m)^2} du \otimes du - (du \otimes dq + dq \otimes du) - (q + m)^2 \sigma.
\]

Due to the sign of the mass \(\Omega = m/r = -ml/r_+ < 0\). Observe that the same happens in \(\overline{II}\), since \(r_+ \in (-ml, 0)\). We note here that the negative sign of the conformal factor is of no
concern to us as this commonly happens when formulating the conformal Einstein field equations across null infinity

The null infinity of \( \mathcal{M}_- \) is given the set \( \mathcal{I}^+ \equiv \{ p \in \mathcal{M}_- \mid \Omega(p) = 0 \} \), on which \( \Omega = \rho/m = 0 \), \( d\Omega \neq 0 \) and which consists of two disjoint null hypersurfaces.

Applying the discrete conformal isometry to \( \tilde{g}_- \), one finds that

\[
i_\theta \tilde{g}_-(u, \rho) = \frac{r^2}{(r + m)^2} dv \otimes dv - (dv \otimes dr + dr \otimes dv) - (r + m)^2 \sigma.
\]

Hence, making the replacements \( \tilde{m} = -m \), \( \tilde{r} = -r \) and \( \tilde{v} = v \), one finds that \( i_\theta \tilde{g}_- \) can be identified with a positive mass extremal Reissner–Nordström metric in region \( \tilde{M} \). Consequently, in the negative mass case, the conformal inversion can be regarded as a real analytic conformal isometry

\[
i : \mathcal{M}_- \to \tilde{M} \cup \mathcal{H}^+ \quad \text{with} \quad \tilde{g}_- \to \tilde{g}_- = \tilde{g}_+ \]

such that

\[
i(\mathcal{I}^+) = \mathcal{H}^+.
\]

3. Conformal geodesics in electrovacuum spacetimes

In what follows, we will make use of conformal geodesics to probe the properties of the extremal Reissner–Nordström spacetime. In this section, we discuss some general properties of conformal geodesics in electrovacuum spacetimes. Unless explicitly stated, the properties discussed are not restricted to the extremal Reissner–Nordström spacetime and apply to general electrovacuum spacetimes with vanishing cosmological constant.

Following [14, 18, 21], a conformal geodesic is a pair \( (t_\tau, b_\tau) \) consisting of a curve \( x(\tau) \) in a spacetime \( (\mathcal{M}, \tilde{g}) \) with parameter \( \tau \) and a 1-form \( b_\tau \) along the curve—such that satisfies, in index-free notation, the equations

\[
\tilde{g}(\dot{x} \dot{x}) = -2 \langle b, \dot{x} \dot{x} \rangle + \tilde{g}(\dot{x} \dot{x} \tilde{b}).
\]

\[
\tilde{g}(\dot{b} \dot{b}) = \langle b, \dot{b} \rangle \tilde{b} - \frac{1}{2} \tilde{g}/(\tilde{b}, \tilde{b}) \dot{x} + \tilde{L}(\dot{x}, \dot{b}).
\]

Here \( \cdot \cdot \) denotes differentiation with respect to the parameter \( \tau \) and \( \tilde{L} \) denotes the Schouten tensor of the metric \( \tilde{g} \). In four dimensions, one sees that

\[
\tilde{L} \equiv \frac{1}{2} \text{Ric} \tilde{g} - \frac{1}{12} \tilde{R} \tilde{g},
\]

where \( \text{Ric} \tilde{g} \) and \( \tilde{R} \) denote, respectively, the Ricci tensor and Ricci scalar of the metric \( \tilde{g} \). The 1-form \( \tilde{b} \) can be thought of as an acceleration term that is driven by the Schouten tensor. This interpretation becomes more evident if one uses a parametrisation of the curve in terms of the \( \tilde{g} \)-proper time—see e.g. [18]. This point of view will not be required in the following.

In section 5.3, conformal geodesics will be used to construct a conformal Gaussian gauge system. This gauge system will be used, in turn, to obtain a symmetric hyperbolic reduction of

\footnote{The same happens when one conformally extends positive Reissner–Nordström across null infinity in terms of coordinates.}
the conformal field equations. Accordingly, in the remainder of this article we will only be interested in time-like conformal geodesics.

Given a (time-like) conformal geodesic \((x(\tau), \tilde{b}(\tau))\), it is convenient to consider a frame \(\{e_a\}\) propagated along curve according to the equation
\[
\tilde{V}_a e_a = -\langle \tilde{b}, e_a \rangle x - \langle \tilde{\rho}, x \rangle e_a + \tilde{g}(e_a, x) \tilde{b}^2.
\] (8)

Frames satisfying this equation are said to be Weyl propagated along the conformal geodesic. In the following we will only use frames for which \(e_0 = \tilde{x}\).

### 3.1. Canonical conformal factors

Time-like conformal geodesics allow one to single out a canonical representative of the conformal class \(\tilde{g}\). To see this, let
\[
g = \Theta^2 \tilde{g}
\]
and require that
\[
g(\tilde{x}, x) = \Theta^2 \tilde{g}(\tilde{x}, x) = 1.
\]

By repeatedly differentiating this last equation with respect to \(\tau\) and using the conformal geodesic equations (7a)–(7b), one finds that the following equations hold along a given conformal geodesic:
\[
\Theta = \langle \tilde{b}, \tilde{x} \rangle \Theta,
\] (9a)
\[
\dot{\Theta} = \frac{1}{2}\tilde{g}^2(\tilde{b}, \tilde{b})\dot{\Theta}^{-1} + \Theta \tilde{L}(\tilde{x}, \tilde{x}),
\] (9b)
\[
\ddot{\Theta} = \Theta \tilde{V}_x (\tilde{L}(\tilde{x}, \tilde{x})) + \Theta \tilde{L}(\tilde{x}, \tilde{x}) + \tilde{L}(\tilde{x}, \tilde{x})\Theta^{-1}.
\] (9c)

Moreover, one also sees that
\[
\tilde{V}_x \left( \Theta \langle \tilde{b}, e_a \rangle \right) = \Theta \tilde{L}(\tilde{x}, e_a) + \frac{1}{2} \Theta \tilde{g}^2(\tilde{b}, \tilde{b}) \tilde{g}(\tilde{x}, e_a),
\] (10a)
\[
\tilde{V}_x (g(e_a, e_b)) = 0.
\] (10b)

Hence, if the frame \(\{e_a\}\) is initially orthogonal, it remains orthogonal along the whole of the conformal geodesic. In what follows, it will be convenient to introduce the 1-form
\[
d \equiv \Theta \tilde{b}.
\]

Using equation (9a), it readily follows that
\[
d_0 = \dot{\Theta}.
\] (11)

That is, the time component of the 1-form \(d\) is known if the conformal factor \(\Theta\) is known. Moreover, from equation (10a) one concludes that
\[
\dot{d}_i = \Theta \tilde{L}_{0i},
\] (12)

where \(d_i \equiv (d, e_i)\) and \(\tilde{L}_{0i} \equiv \tilde{L}(\tilde{x}, e_i)\).

If the spacetime satisfies the vacuum field equation \(\tilde{R}ic[\tilde{g}] = \lambda \tilde{g}\), it follows readily from equation (9c) that \(\Theta = 0\). One can obtain an explicit expression for the conformal factor along the conformal geodesic, which is quadratic in \(\tau\)—see e.g. [14, 18, 21]. In the presence of matter, this simple expression for the conformal factor is no longer available, but one can still use equation (9c) to evolve \(\Theta\). For electrovacuum spacetimes with a vanishing
Cosmological constant, it follows readily from the Einstein field equations that \( \Lambda = \frac{1}{2} \tilde{T} \), where \( \tilde{T} \) is the physical energy-momentum tensor of the Maxwell field. In what follows, we will work mostly in the conformally extended spacetime \( (\mathcal{M}, \tilde{g}) \) and, accordingly, it is convenient to introduce the unphysical energy-momentum tensor \( T \equiv \Theta^{-2} \tilde{T} \). [13] It follows from the previous discussion that
\[
\Lambda = \frac{1}{2} \Theta^2 \tilde{T}.
\]

Defining the unphysical energy density by \( \mu \equiv T(\dot{x}, \dot{x}) \), one can recast equation (9c) in the more suggestive form
\[
\Theta = \frac{1}{2} \Theta^2 \mu + \frac{1}{2} \Theta^2 \dot{\Theta} \mu + \frac{1}{2} T \left( \dot{x}, \tilde{g}^2 (d, \cdot) \right).
\]

Thus, the evolution of the conformal factor is coupled to that of the matter content of the spacetime. Similarly, defining the components of unphysical electromagnetic flux vector by \( j_i \equiv T(x, e_i) \), one finds from (12) that
\[
\dot{d}_i = \frac{1}{2} \Theta_j j_i.
\]

Initial data for equation (13) is constrained by equations (9a)–(9b). One readily sees that
\[
\Theta_* = \langle d, \dot{x} \rangle_*,
\]
\[
\Theta_* \dot{\Theta}_* = \frac{1}{2} h^2 (d_*, d_*) + \Theta_*^2 \mu_*.
\]

where the subindex \( * \) indicates the value of the relevant quantities on a fiduciary hypersurface \( \tilde{S} \).

3.2. Behaviour under conformal isometry

In this section, we analyse the behaviour of the conformal geodesic equations and their solutions under conformal isometry (5). If \( (x(\tau), \dot{b}(\tau)) \) is a solution to equations (7a)–(7b), it follows readily from \( \Theta \dot{\Theta} \) \( \dot{g} = \tilde{g} \) and general properties of isometries that \( (t_o x(\tau), t_o b(\tau)) \) is a solution to
\[
\tilde{V}_{t_o x} t_o x = -2 \langle t_o \dot{b}, t_o \dot{x} \rangle t_o \dot{x} + \tilde{g}(t_o \dot{x}, t_o \dot{x}) t_o \dot{b}^2,
\]
\[
\tilde{V}_{t_o x} t_o b = \langle t_o \dot{b}, t_o \dot{x} \rangle t_o \dot{b} - \frac{1}{2} \tilde{g}^2 (t_o \dot{b}, t_o \dot{b}) t_o \dot{x}^2 + L(t_o \dot{x}, \cdot).
\]

—the conformal geodesic equations with respect to the connection \( \tilde{V} \). Finally, observing that \( \tilde{g} = \Omega^2 \tilde{g} \), and the properties of conformal geodesics under changes of connection—see [14, 18] for details—one concludes that \( (t_o x(\tau), b(\tau)) \) with
\[
b(\tau) \equiv t_o \dot{b} + \Omega^{-1} \dot{d} \Omega
\]
is a solution to the \( \tilde{V} \)-geodesic equations
\[
\tilde{V}_{t_o x} t_o x = -2 \langle b, t_o \dot{x} \rangle t_o \dot{x} + \tilde{g}(t_o \dot{x}, t_o \dot{x}) b^2,
\]
\[
\tilde{V}_{t_o x} b = \langle b, t_o \dot{x} \rangle b - \frac{1}{2} \tilde{g}^2 (b, b) t_o \dot{x}^2 + L(t_o \dot{x}, \cdot).
\]

Summarising, one sees the following:

**Lemma 1.** Conformal isometries map conformal geodesics into conformal geodesics.
3.3. Conformal geodesics and Weyl connections

Given a spacetime \((\mathcal{M}, \hat{g})\), a Weyl connection \(\hat{\nabla}\) is a torsion-free connection satisfying
\[
\hat{\nabla}_a \hat{\nabla}_b \hat{g} = -2 f_a \hat{g}_{bc}
\]
for some smooth 1-form \(f\). Conformal geodesics allow us to single out a canonical Weyl connection. More precisely, given a solution \(\tau_{\sigma b}(x)\) to the conformal geodesic equations (7a)–(7b) one can set \(f = \hat{b} - \theta^{\alpha \beta} \eta_{\alpha \beta}\) so that using the transformation rules of the conformal geodesic equations under changes of connection one concludes that
\[
\hat{\nabla}_a \hat{x} = 0, \quad \hat{L}(\hat{x}, \cdot) = 0, \quad \hat{\nabla}_a e_a = 0,
\]
where \(\hat{L}\) denotes the Schouten tensor of the Weyl connection \(\hat{\nabla}\). Hence, the curve \(x(\tau)\) is a (standard) geodesic with respect to \(\hat{\nabla}\) and the frame \(\{e_a\}\) is parallelly propagated with respect to \(\hat{\nabla}\) along the curve. For more details on Weyl connections and their relation to conformal geodesics, see [14, 18]. In section 5.3, the condition given in (17) will be used to construct a conformal Gaussian gauge system that, in turn, will give rise to a symmetric hyperbolic reduction of the conformal Einstein field equations.

4. Time-symmetric hypersurfaces of the extremal Reissner–Nordström spacetime

As a consequence of the time-like nature of the static Killing vector \(\partial_t\) everywhere (except at the horizon), it follows that the maximal analytical extension of the spacetime admits two types of time-symmetric hypersurfaces—that is, hypersurfaces with vanishing extrinsic curvature:

(a) The time-symmetric hypersurfaces \(\tilde{S}\) contained in the regions \(\tilde{I}\). These hypersurfaces are asymptotically Euclidean at \(i^0\) and have a cylindrical asymptotic end \(\partial\tilde{I}\). The cylindrical asymptotic end is shared by other black hole solutions, most notably the extremal Kerr solution—see e.g. [10] for further details.

(b) The time-symmetric hypersurfaces contained in the regions \(\tilde{I}^\pm\), to be denoted by \(\tilde{R}\), which start at \(i^+\) and end at the singularity. The conformal diagram of the extremal Reissner–Nordström spacetime suggests that these hypersurfaces are some sort of degenerate hyperboloid.

The location of these hypersurfaces is identified in figure 1, left.

By contrast, the negative mass extremal Reissner–Nordström contains a single asymptotically Euclidean time symmetric hypersurface \(\tilde{S}'\) which, in many senses, resembles the case (a) above—see also figure 1, right.

4.1. The conformal constraints

In what follows we analyse the properties of the hypersurfaces \(\tilde{S}\) and \(\tilde{R}\) in some detail from the perspective of the conformal Einstein field equations. The conformal Einstein–Maxwell field equations have been discussed in [13, 26, 32]. In particular, the constraint equations implied by the conformal Einstein–Maxwell equation in a spacelike hypersurface have been discussed in [32]. In view of the applications of this article, attention will be restricted to the time-symmetric conformal constraint equations.

In the rest of this section, let \(S\) denote an arbitrary hypersurface of an unphysical spacetime \((\mathcal{M}, g)\) and let \(\Xi\) denote the associated conformal factor—i.e., \(g = \Xi^2 \hat{g}\) where \(\hat{g}\)
denotes, as usual, the physical spacetime metric. Recall that Maxwell’s equations are conformally invariant; hence, the unphysical Faraday tensor $F_{ab}$ is given by

$$F_{ab} = \tilde{F}_{ab}.$$ 

The restriction of $\Xi$ to $S$ will be denoted by $\omega$. Let $n$ indicate the $g$-unit normal to $S$, and let $\{e_a\}$ denote a $g$-orthonormal frame such that $e_0 = n$. Consequently, $\{e_i\}$ constitutes a basis of orthonormal spatial vectors intrinsic to $S$. If $h$ denotes the intrinsic metric of $S$ implied by $g$, one sees that

$$h_{ij} \equiv h(e_i, e_j) = -\delta_{ij}.$$ 

In addition to $\omega$ as defined in the previous paragraph, the conformal constraint equations are expressed in terms of the fields

$$s \equiv \frac{1}{4} \nabla^a \nabla_a \Xi + \frac{1}{4} L \Xi,$$

$L_{ij}$ (the spatial part of the Schouten tensor of $g$),

$$d_{ij} = \Xi^{-1} C_{ij} \tilde{g}$$ (the electric part of the rescaled Weyl tensor),

$$l_{ij} = n_{ij} - \frac{1}{2} \tilde{h}_{ij}$$ (the Schouten tensor of $h$),

$$E_i = F_{i0}$$ (the electric part of the Faraday tensor),

$$\mu \equiv E_i E^i$$ (the unphysical energy density of the Faraday tensor),

where $n_{ij}$ denotes the components of the Ricci tensor of the 3-metric $h$, and $r$ is its Ricci scalar. Notice that there is no contribution to the initial data from the magnetic parts of the Faraday and Weyl tensors. This is a consequence of the time symmetry of the initial data sets being considered.

For hypersurfaces that are time symmetric and maximal in both the physical and unphysical spacetimes, the conformal Einstein constraint equations imply

$$D_i D_j \omega = -\omega L_{ij} + s h_{ij} + \omega^3 E_{(i} E_{j)}, \quad (18a)$$

$$6\omega s - 3D_k \omega D^k \omega = 0, \quad (18b)$$

$$l_{ij} = \omega d_{ij}, \quad (18c)$$

where $E_{(i} E_{j)}$ denotes the $h$-trace-free part of $E_i E_j$. The above equations imply, in particular, the time-symmetric Hamiltonian constraint

$$2\omega D_i D^i \omega - 3D_i \omega D^i \omega + \frac{1}{2} \omega^2 r = \omega^4 \mu.$$ 

The electric field $E_i$ satisfies the constraint

$$D^i E_i = 0, \quad (19)$$

where

$$E_i = \omega^{-1} \tilde{E}_i.$$ 

The constraint equations (18a)–(18c) can be solved to yield $d_{ij}$ and $L_{ij}$ in terms of $\omega$ and its derivatives, the intrinsic geometric fields and the electric field. One sees that:

$$d_{ij} = \frac{1}{\omega^2} D_i D_j \omega + \frac{1}{\omega} l_{ij} + \omega E_{(i} E_{j)}, \quad (20a)$$

The electric field $E_i$ satisfies the constraint

$$D^i E_i = 0, \quad (19)$$

where

$$E_i = \omega^{-1} \tilde{E}_i.$$ 

The constraint equations (18a)–(18c) can be solved to yield $d_{ij}$ and $L_{ij}$ in terms of $\omega$ and its derivatives, the intrinsic geometric fields and the electric field. One sees that:
These equations will be key for the remaining analysis in this section.

4.2. Basic conformal extensions of the time-symmetric initial hypersurfaces

A direct computation shows that the intrinsic metric $\tilde{h}$ implied by the extremal Reissner–Nordström metric (3) on any time-symmetric hypersurface of the spacetime is formally given by

$$\tilde{h} = \left(1 + \frac{m}{r}\right)^2 (dr \otimes dr + r^2 \sigma),$$

(21)

The key observation from expression (21) is that the 3-metrics of the time-symmetric hypersurfaces are conformally flat. This property simplifies many of the computations in the following. One also finds that the initial electric field is given by

$$\tilde{E} = \pm \frac{m}{r(r + m)} dr.$$

The precise properties of the initial data depend on the signs of $m$ and $r$.

4.2.1. The time-symmetric hypersurface of region $\bar{I}$. The intrinsic metric of the time-symmetric hypersurface $\bar{S}$ is given by the metric $\tilde{h}$ of (21) with the conditions $m > 0$ and $r > 0$. The asymptotically Euclidean end of $\bar{S}$ corresponds to the condition $r \rightarrow \infty$. Thus, it is natural to introduce the coordinate

$$\rho \equiv 1/r.$$

A computation shows that

$$h \equiv \omega^2 \tilde{h} = -\left(d\rho \otimes d\rho + \rho^2 \sigma\right) = -\delta,$$

with

$$\omega = \frac{\rho^2}{(1 + m\rho)}.$$

(23)

The rescaled (unphysical) electric field is given by

$$E = \omega^{-1} \tilde{E} = \mp \frac{m}{\rho^2} d\rho.$$

The metric $h = -\delta$ is clearly regular at $\rho = 0$. Thus, one obtains a conformal extension $(\bar{S}_\rho, -\delta)$ of the hypersurface $\bar{S}$ for which $\bar{S} \ni \{ p \in S_\rho \mid \rho = 0 \}$ corresponds to the point at infinity. One can readily verify that

$$\omega \big|_{\rho = 0} = 0, \quad d\omega \big|_{\rho = 0} = 0, \quad \text{Hess } \omega \big|_{\rho = 0} = 2\delta.$$

In order to analyse the behaviour of the fields $d\tilde{y}$ and $L_{ij}$, as given by equations (20a)–(20b), in a neighbourhood of $\bar{I}$ it is convenient to consider on $S_\rho$ a system of normal Cartesian coordinates $(x^i)$ centred at $\bar{I}$—that is, $x^i(\bar{I}) = 0$—with $\rho^2 = \delta_{ij} x^i x^j$. Moreover, on $S_\rho$ consider a frame $e_i$ such that $e_i = \partial/\partial x^i$. By construction, one readily sees that $h(e_i, e_j) = -\delta_{ij}$. Using equations (20a)–(20b) and keeping in mind that $h$ is flat so that $r_{ij} = r = 0$, it follows that
\[
d_{ij} = -\frac{3}{\rho^3(1 + mp)} x_{i} x_{j}, \quad (24a)
\]
\[
L_{ij} = \frac{3}{\rho^3(1 + mp)^2} x_{i} x_{j}, \quad (24b)
\]
\[
E_{i} = \pm \frac{m}{\rho^3} x_{i}, \quad (24c)
\]

with \(x_{i} \equiv -\delta_{ij} x^{j}\). Consequently, one sees that
\[
d_{ij} = O\left(\rho^{-3}\right), \quad L_{ij} = O\left(\rho^{-1}\right), \quad E_{i} = O\left(\rho^{-2}\right) \quad \text{on } S^{\infty}. \tag{25}
\]

One sees that the data is singular at \(i^{+}\). This singular behaviour is well known in the case of vacuum spacetimes—see e.g., [12, 15].

Now, the cylinder-like asymptotic end of the hypersurface \(\tilde{S}\) corresponds to the condition \(r \to 0\). A direct computation shows that the set of points for which \(r = 0\) lies at infinity with respect to the metric \(h\). Hence, in this case it is natural to use the coordinate \(r\) to construct a conformal extension of \(\tilde{S}\) near the cylinder end by letting

\[
h = \sigma^{2} \hat{h} = -\left(\rho r \otimes \rho r + r^{2}\sigma\right) = -\sigma,
\]

\[
\sigma = \frac{r}{r + m}.
\]

Clearly, \(h\) is smooth at \(r = 0\), so that one obtains a smooth conformal extension \((S^{\varphi}, -\sigma)\) of \((\tilde{S}, \hat{h})\). Letting \(c^{0} \equiv \{ p \in S_{\varphi} \mid r = 0\}\), one readily sees that

\[
\sigma^{0} \mid_{c^{0}} = 0, \quad d\sigma^{0} \mid_{c^{0}} \neq 0.
\]

Thus, the behaviour at the cylinder end resembles that of a hyperboloid. As in the analysis of \(i^{0}\), it is convenient to introduce a Cartesian system of normal coordinates \((x^{i})\) centred at \(c^{0}\) and an associated orthonormal frame \(\{e_{i}\}\) with \(e_{i} = \partial_{i}\). As a consequence of the flatness of the conformal metric, it is easy to compute the fields \(d_{ij}\) and \(L_{ij}\) on \(S^{\varphi}\). Using expressions \((20a)-(20b)\), one readily finds that

\[
d_{ij} = \frac{3}{r^{4}(r + m)} x_{i} x_{j}, \quad (20a)
\]
\[
L_{ij} = \frac{3}{r^{4}(r + m)^{2}} x_{i} x_{j}, \quad (20b)
\]
\[
E_{i} = \pm \frac{m}{r^{3}} x_{i}.
\]

The above data is singular at \(i^{+}\). More precisely, one sees that
\[
d_{ij} = O\left(\rho^{-2}\right), \quad L_{ij} = O\left(\rho^{-1}\right), \quad E_{i} = O\left(\rho^{-2}\right) \quad \text{on } S^{\infty}. \tag{26}
\]

4.2.2. The time-symmetric slice of region \(I^{+}\). The intrinsic metric of the time-symmetric hypersurface \(\tilde{S}^{*}\) is given by the metric \(\hat{h}\) of \((21)\) with the conditions \(m < 0\) and \(r > \text{ln} m\). This hypersurface has an asymptotically Euclidean end corresponding to the condition \(r \to \infty\). A conformal extension can be obtained in a similar way to what was done for the asymptotically Euclidean end of \(\tilde{S}\) by introducing the coordinate \(\rho = 1/r\). In particular, the conformal factor
\[ \omega = \frac{\rho^2}{1 + m\rho} \]

is formally identical to that of \( \omega \) as given in (23). The corresponding conformal extension, including \( \rho = 0 \), will be denoted by \((S^\rho, - \delta)\). It follows that one obtains the singular behaviour

\[ d_{ij} = O(\rho^{-3}), \quad L_{ij} = O(\rho^{-1}), \quad E_{ij} = O(\rho^{-2}) \quad \text{on} \quad S^\rho. \quad (27) \]

4.2.3. The time-symmetric slice of region \( \bar{III} \). Finally, we consider the case of the time-symmetric hypersurface \( \mathcal{R} \) in region \( \bar{III} \). In this case, the intrinsic metric is given by the expression for \( \mathbf{h} \) of equation (21) with the condition \( r < 0 \). This situation is completely analogous to the discussion of the cylinder-like end of region \( \bar{I} \). In particular, one can use the coordinate \( r \) to construct a conformal extension \((\mathcal{R}, - \delta)\) with a conformal factor

\[ \varpi = \frac{r}{r + m}, \]

from which one concludes the same singular behaviour as in (26) at \( i^+ = \{ \rho \in \mathcal{R} \mid r = 0 \} \).

4.3. The conformal isometry between the time-symmetric hypersurfaces

In order to gain further intuition into the behaviour of the extremal Reissner–Nordström spacetime around \( c^0 \) and \( i^* \), it is convenient to analyse the effects of the conformal isometry \( \iota \) on the various time-symmetric hypersurfaces.

4.3.1. The hypersurface \( \tilde{S} \). Taking into account that \( \rho \equiv 1/r \) and that \( \varrho \equiv m^2/r \), it follows readily that \( \varrho = m^2\rho \) and that

\[ m^4 \left( d\rho \otimes d\rho + \rho^2 \sigma \right) = d\varrho \otimes d\varrho + \varrho^2 \sigma. \]

Hence,

\[ \iota_{*} \mathbf{h} = \frac{1}{m^4} \left( d\rho \otimes d\rho + \rho^2 \sigma \right). \]

Thus, under the conformal isometry the limit \( \rho \to 0 \), respectively \( \varrho \to 0 \), correspond to \( r \to 0 \). Thus, one sees that

\[ \iota(S_{\rho}) = S_{\rho} \]

and, in particular

\[ \iota(c^0) = c^0. \]

4.3.2. The hypersurfaces \( \mathcal{R} \) and \( \bar{S} \). Using the expressions of the previous paragraph, and recalling that the conformal isometry \( \iota \) maps \( \bar{I} \) into \( \bar{III} \cup \mathcal{H}^* \), one concludes that

\[ \iota(S^\rho) = \mathcal{R}, \]

and, in particular

\[ \iota(i^0) = i^+. \]
4.4. Some remarks

The singular behaviour of the conformal fields $\delta g$ and $Lg$ at $i^0$ and $i^0$ given, respectively, by (25) and (27) is the main technical difficulty in the analysis of the conformal field equations in this region of spacetime—this is sometimes known as the problem of spatial infinity. For vacuum spacetimes it has been shown in [15] how one can introduce a conformal representation of this region of spacetime in which the equations and their data are regular—the so-called cylinder at spatial infinity (see the discussion in the introduction). We will show in section 6 that this construction can be extended, with minor modifications, to the electro-vacuum case.

The slightly milder singular behaviour at $c_0$ and $+i^0$ observed in (26) suggests that a similar regular conformal representation could be introduced for this part of the (positive mass) extremal Reissner–Nordström spacetime. Moreover, the correspondence between the asymptotically Euclidean ends and the singular hyperboloidal ends $c^0$ and $+i^0$ discussed in section 4.3 raises the following expectation: The regular representations can be simply mapped from the cylinders at infinity of $i^0$ and $i^0$ rather than built from scratch.

5. The cylinder at spatial infinity for the extremal Reissner–Nordström spacetime

The purpose of this section is to provide a discussion of the construction of the cylinder at infinity for the asymptotically Euclidean ends of the time-symmetric hypersurfaces $\tilde{S}$ and $S'$ of the extremal Reissner–Nordström spacetime. This construction follows closely the one for the Schwarzschild spacetime in [15]. The construction is local to a neighbourhood of spatial infinity, and thus, independent of the sign of the mass parameter. Accordingly, for ease of presentation, we consider the positive and negative mass cases simultaneously.

5.1. The bundle space $C_a$

The first step in the construction of the cylinder at infinity consists of the blowing up of spatial infinity $\tilde{i}^0$ to the 2-sphere $S^2$. Technically, the implementation of this idea requires the introduction of a bundle space $C_a$. The bundle space $C_a$ and its extensions play a central role in our subsequent analysis. Hence, we give a brief overview of the construction of $C_a$ and these extensions below—for details the reader is referred to [15].

Let $B_a(i^0) \subset S^2$ be the open ball of radius $a$ centred at $\tilde{i}^0$ for some sufficiently small $a > 0$. Further, let $SU(B_a(i^0))$ be the bundle of normalised spin frames over $B_a(i^0)$ with structure group $SU(2, \mathbb{C})$. To obtain the manifold $C_a$, one starts by choosing a fixed normalised spin dyad $\delta_A$ at $\tilde{i}^0$. Any other spin frame is of the form $\delta_A = \delta_A t^B A$ with $t = (t^B A) \in SU(2, \mathbb{C})$. For a given value of $t$, the spin frame $\delta_A(t)$ gives rise to an orthonormal frame

$$e_t(t) \equiv \sigma_i^{AB} \delta_A(t) \delta_B(t),$$

where $\sigma_i^{AB}$ indicates the spatial Infeld-van der Waerden symbols. For some values of $t$, the frame vector $e_t(t) = \sigma_i^{AB} \delta_A(t) \delta_B(t)$ corresponds to the radial vector at $\tilde{i}^0$. Keeping $t$ fixed, one then constructs on $B_a(i^0)$ the $h$-geodesic starting at $\tilde{i}^0$ with initial tangent vector $e_t(t)$. Because of the flatness of the conformal metric $h$, the coordinate $\rho$ is an affine parameter of this geodesic that vanishes at $\tilde{i}^0$. The spin dyad $\delta_A$ is then propagated along the geodesic. For a particular value of $\rho$, the spin dyad so constructed will be denoted by $\delta_A(\rho, t)$. One then sets...
The boundary of the bundle manifold $\mathcal{C}_a$ consists of the set
\[ I^0 \equiv \{ \delta_A(\rho, i) \in \mathcal{C}_a \mid \rho = 0 \}. \]

It can be verified that $I^0 \cong SU(2, \mathbb{C})$, so that the components of the boundary can be regarded as the blow up of the point at infinity $\bar{\rho}$.

For more details on the various aspects of this construction, the reader is referred to [15], section 3. An alternative, abridged discussion is given in [1].

5.1.1. Lifts to $\mathcal{C}_a$. Any smooth spinor field on $B_0(i^0) \subset S^\rho$ is represented on $\mathcal{C}_a$ by a spinor-valued function given at $\delta_A \in \mathcal{C}_a$ by the components of the spinor in the dyad defined by $\delta_A$. This procedure will be referred to as the lift of the spinor field. The lift to $\mathcal{C}_a$ of any symmetric valence 2 spinor on $\mathcal{A}_0$ can be spanned in terms of symmetric spinors $\chi_{AB}, \chi_{AB}, \chi_{AB}$ such that
\[ \epsilon = \frac{1}{\sqrt{2}} \epsilon_{AB} \]
and $y_A^Q y_B^Q = z_A^Q y_B^Q = 0$. Higher valence spinors can be spanned by suitable combinations of these spinors and the totally antisymmetric spinor $\epsilon_{AB}$—[15, 20].

5.1.2. Vector fields on $\mathcal{C}_a$. The manifold $\mathcal{C}_a$ has a dimension more than $S^\rho$. This extra dimension corresponds to the action of the subgroup $U(1)$ of $SU(2, \mathbb{C})$. In what follows, we will use $i_A^Q \in SU(2, \mathbb{C})$ and $\rho$ as coordinates on $\mathcal{C}_a$. In order to be able to compute derivatives on $\mathcal{C}_a$, one considers a basis $\{ X_+, X_-, X \}$ of the Lie algebra $su(2, \mathbb{C})$, such that $X$ is the generator of $U(1)$ and one finds the commutation relations
\[ [X, X+] = 2X_+, \quad [X, X-] = -2X_-, \quad [X_+, X_-] = -X, \]
with $X_+$ and $X_-$ complex conjugates of each other. These vector fields are extended to $\mathcal{C}_a$ by the requirements
\[ [\partial_\rho, X] = 0, \quad [\partial_\rho, X_+] = 0, \quad [\partial_\rho, X_-] = 0. \]
The vector fields $\{ \partial_\rho, X, X_+, X_- \}$ constitute a frame field on $\mathcal{C}_a$. A function $f$ is said to have spin weight $s$ if $X f = 2sf$, with $s$ an integer. Any spinor-valued function on $\mathcal{C}_a$ has a well-defined spin weight. To complete the discussion, one should consider forms $\alpha^+, \alpha^-$ and $\alpha$, which annihilate the vector field $\partial_\rho$ and, in addition, satisfy
\[ \alpha^+, X_+] = \alpha^-, X_- = \alpha, X = 1. \]
The normalisation conventions being used are such that $2(\alpha^+ \otimes \alpha^- + \alpha^- \otimes \alpha^+) \) pulls back to $\sigma$, the standard metric on $S^2$.

5.1.3. Frame fields, solder forms and connection forms. The vector fields and 1-forms on $\mathcal{C}_a$ introduced in the previous subsection will be used to span the following frame fields and corresponding solder forms:
\[ e_{AB} = x_{AB}\partial_{\rho} + \frac{1}{\rho}z_{AB}X_{+} + \frac{1}{\rho^2}y_{AB}X_{-}, \]
\[ \sigma^{AB} = -\chi^{AB}\partial_{\rho} - 2\rho y_{AB}\alpha^{+} - 2\rho z_{AB}\alpha^{-}. \]

The above fields have been chosen so that
\[ \sigma = \sigma^{AB}\sigma_{AB}, \quad \alpha = \alpha^{AB}\alpha_{AB}, \]
where in a slight abuse of notation \( h \) denotes the lift to \( C_0 \) of the conformal metric \( h = -\delta \), and \( h_{ABCD} \equiv -\epsilon_{(A}C_{CD)B} \) is the spinorial counterpart of \(-\delta_{ij}\). The associated spin connection coefficients \( \gamma_{ABCD} \) can be computed using the spinorial version of the Cartan structure equations. One sees that
\[ \gamma_{ABCD} = \frac{1}{2\rho}(\epsilon_{ACB}X_{D} + \epsilon_{DCA}X_{B}). \]

Covariant differentiation on \( C_0 \) is performed using the standard rules. Let \( F \) denote the lift to \( C_0 \) of a smooth function, \( f \) on \( \mathbb{R} \). The covariant derivative \( D_{AB}f \) is represented on \( C_0 \) by \( e_{AB}F \). In order to ease the notation, in what follows the same symbol will be used to denote a function on \( \mathbb{R} \) and its lift to \( C_0 \). Using this convention, let \( \mu_{AB} \) denote the lift to \( C_0 \) of the spinorial field \( \mu_{AB} \) on \( \mathbb{R} \). The lift of the covariant derivative \( D_{AB}C_{CD} \) is then given by
\[ D_{AB}\mu_{CD} = e_{AB}\mu_{CD} - \gamma_{ACB}^{CD} + \epsilon_{BDA}X_{C}. \]

Similar expressions hold for higher valence spinors.

5.2. The extended bundle space \( C_0 \)

The second step in the construction of the cylinder at spatial infinity consists of the introduction of an extended bundle space in which the spin dyads \( \delta \{ A \} \) are rescaled by a certain factor so that the components of singular fields at \( i^0 \) become regular. Given a non-negative smooth function \( \kappa \), one defines the extended bundle space
\[ C_{0,\kappa} \equiv \left\{ \kappa^{1/2}\delta_{A} \mid \delta_{A} \in C_0 \right\}. \]

In what follows, let \( \phi_{ABCD}, \Theta_{ABCD} \) and \( \varphi_{AB} \) denote the spinorial counterparts of the fields \( d_{ij}, \mathcal{L}_{ij} \) and \( E_{i} \) as discussed in section 4.1. Following the conventions of the previous sections, we denote their lifts to the bundle space \( C_0 \) by \( \Phi_{ABCD}, \Theta_{ABCD} \) and \( \varphi_{AB} \). Under the rescaling \( \delta_{A} \mapsto \kappa^{1/2}\delta_{A} \), the latter fields can be seen to transform as
\[ \phi_{ABCD} \mapsto \kappa^{3/2}\phi_{ABCD}, \quad \Theta_{ABCD} \mapsto \kappa^{3/2}\Theta_{ABCD}, \quad \varphi_{AB} \mapsto \kappa^{2}\varphi_{AB} \]
—see [15] for further details. In order to choose \( \kappa \) appropriately one needs to consider the lift of the above fields to \( C_0 \). A calculation using expressions (24a)-(24c) shows that
\[ \phi_{ABCD} = -\frac{6m}{\rho^3(1+m\rho)}e_{ABCD}^2, \quad \Theta_{ABCD} = \frac{6m}{\rho^2(1+m\rho)^2}e_{ABCD}, \quad \varphi_{AB} = \mp \frac{m}{\rho^2}x_{AB}. \]

where \( e_{ABCD}^2 \equiv \frac{1}{2}(\kappa_{(A}x_{BCD)}^2) \). Thus, in order to obtain rescaled fields which are finite at \( I^0 \), it follows that one should choose \( \kappa = O(\rho) \). For simplicity, we make the choice
\[ \kappa = \rho. \quad (29) \]

In what follows we will also require the transformation behaviour of the frame fields \( e_{AB} \), the soldering forms \( \sigma^{AB} \) and the spatial connection coefficients \( \gamma_{ABCD} \). These are given by
\[ e_{AB} \mapsto \kappa e_{AB}, \quad \sigma^{AB} \mapsto \kappa^{-1}\sigma^{AB}, \quad \gamma_{ABCD} \mapsto S_{ABCD} = \frac{1}{2} (\epsilon_{AC}\epsilon_{BD}\kappa + \epsilon_{BD}\epsilon_{AC}\kappa). \]

Thus, in particular, one sees that the lift of the 3-metric \( h = -\delta \) satisfies
\[ h \mapsto \frac{1}{\rho^2} d\rho \otimes d\rho + 2 (\alpha^+ \otimes \alpha^- + \alpha^- \otimes \alpha^+). \]

This is a feature not often appreciated of the construction of the cylinder at spatial infinity. Namely, that it renders a 3-metric that is singular at \( \rho = 0 \)—or in other words, the cylinder at spatial infinity is at an infinite distance as measured by the 3-metric \( \kappa^{-2}h \) so that the representation is not metrically compact. This does not cause problems with the conformal field equations as the metric (and the soldering forms) do not appear as unknowns in the equations—see the discussion in section 5.4.

### 5.3. Spacetime gauge considerations

The third step in the construction of the cylinder at spatial infinity consists of the implementation of a suitable spacetime gauge to analyse the evolution of initial data prescribed on \( C_{\rho} \). Again, following the ideas of [15], we rely on a gauge based on a congruence of conformal geodesics.

#### 5.3.1. Initial data for the congruence of conformal geodesics

Initial data for the congruence of conformal geodesics is prescribed for a given \( p \in \mathcal{B}_a(i^\pm) \setminus \{i^\pm\} \) so that
\[ x_* = x(p), \quad \dot{x}_* \perp \tilde{S}_{\dot{x}}, \quad \tilde{b}_* = \omega^{-1} d\omega, \quad \langle \tilde{b}, \dot{x} \rangle_* = 0. \tag{30} \]

The above initial data is supplemented by the following choice of data for the conformal factor—cf. equations (15a)–(15b):
\[ \Theta_* = \kappa^{-1}\omega, \quad \Theta_* = 0, \quad \Theta_* = -\frac{\kappa}{2}\delta(d\omega, d\omega) + \kappa^{-1}\omega \mu_* , \]
with \( \kappa \) given as in (29) and \( \mu_* \) given as the initial value of the energy density of the Maxwell field.

#### 5.3.2. Gauge conditions associated to the Weyl connection

As it will be seen in the next section, we will consider conformal field equations for the Einstein–Maxwell system that are expressed in terms of the Weyl connection \( \hat{\nabla} \) associated with the congruence of conformal geodesics with initial data given by (30). In addition to this connection, we also need to keep track of the Levi–Civita connections \( \tilde{\nabla} \) and \( \nabla \) of the metrics \( \tilde{g} \) and \( g = \Theta^2 \tilde{g} \), respectively. These three connections are related to each other via:
\[ \hat{\nabla} - \tilde{\nabla} = S(\tilde{b}), \]
\[ \nabla - \hat{\nabla} = S(\theta^{-1} d\theta), \]
\[ \tilde{\nabla} - \nabla = S(f), \]
where
\[ f \equiv \tilde{b} - \Theta^{-1} d\Theta, \]
and \( S \) is the connection transition tensor given in component notation by
\[ S_{ab}^{\cd} \equiv \delta_{a}^{\gamma} \delta_{b}^{\delta} + \delta_{a}^{\delta} \delta_{b}^{\gamma} - g_{ab} g^{\cd}. \tag{31} \]
In order to express tensorial objects, we use a frame \{e_a\} that is Weyl propagated along the congruence of conformal geodesics with initial data given by (30). This frame is adapted to the congruence by requiring \(e_0 = \dot{x}\). Moreover, it is also required that initially

\[ g_{ab}(e_a, e_b) = \Theta^2 \delta_{ab}(e_a, e_b) = \eta_{ab}. \]

In view of the Weyl propagation condition \( \dot{\nabla}_x \), it follows that the connection coefficients of the Weyl connection \( \tilde{\Gamma}_c^{ab} \equiv \langle a^b, \nabla_a e_c \rangle \) satisfy the gauge condition

\[ \tilde{\Gamma}_c^{00} = 0. \]

Using the transformation rule between the connections and the properties of the Levi–Civita connections, it can be readily verified that \( \tilde{\Gamma}_c = f \tilde{\Gamma}_c^0 \) from where it follows that

\[ f_0 = 0. \]

Finally, because of formulae (17), it follows that the components of the Schouten tensor of the Weyl connection \( \nabla_c \) with respect to the Weyl-propagated frame satisfy

\[ L_c^0 = 0. \]

5.3.3. Structure of the conformal boundary. As it will be seen in section 5.5, the radius \( a > 0 \) can be chosen small enough so that for each geodesic starting on \( p \in B_a(i^0) \) with initial data of the form given by (30), there exists \( \tau_f \geq 0 \) such that \( \Theta(\pm \tau_f(p)) = 0 \) and the conformal geodesic does not contain conjugate points in \([ - \tau_f(p), \tau_f(p) ]\). Moreover, it will be seen that if \( p = i^0 \), then \( \Theta = 0 \) for all \( \tau \). Accordingly, for \( p = i^0 \) we define \( \tau_f(i^0) \equiv \lim_{p- \to i^0} \tau_f(p) \). In an abuse of notation, in what follows, the lift of the point \( p \in B_a(i^0) \) to \( C_a \) will be denoted, again, by \( p \). Following these observations, the domain on which we will be looking for solutions to the conformal Einstein–Maxwell equations—see the next section—is of the form

\[ \mathcal{M}_{a,\kappa} \equiv \left\{ (\tau, p) \in \mathbb{R} \times C_{a,\kappa} \mid - \tau_f(p) \leq \tau \leq \tau_f(p) \right\}. \]

In a natural way, we define null infinity as \( \mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^- \) with

\[ \mathcal{I}^\pm \equiv \left\{ (\tau, p) \in \mathcal{M}_{a,\kappa} \mid \tau = \pm \tau_f, \ \rho(p) \neq 0 \right\}. \]

The cylinder at spatial infinity is given by

\[ \mathcal{J} \equiv \left\{ (\tau, p) \in \mathcal{M}_{a,\kappa} \mid \rho(p) = 0, \ - \tau_f(p) < \tau < \tau_f(p) \right\}. \]

Of interest are also the so-called critical sets

\[ \mathcal{I}^\pm \equiv \left\{ (\tau, p) \in \mathcal{M}_{a,\kappa} \mid \rho(p) = 0, \ \tau = \pm \tau_f(p) \right\}, \]

and

\[ \mathcal{I}^0 \equiv \left\{ (\tau, p) \in \mathcal{M}_{a,\kappa} \mid \rho(p) = 0, \ \tau = 0 \right\}. \]

the intersection of \( C_{a,\kappa} \) with \( \mathcal{I} \).

Coordinates on \( \mathcal{M}_{a,\kappa} \) are naturally dragged from \( C_{a,\kappa} \) along the conformal geodesics. Similarly, the vector fields \( \bar{\partial}_\rho, X_\tau \) extend in a unique way to vectors on \( \mathcal{M}_{a,\kappa} \) by requiring that they commute with \( \bar{\partial}_\tau \).
5.4. The extended conformal Einstein–Maxwell field equations

The conformal Einstein–Maxwell field equations have been first considered in [13] where they have been used to show the existence and stability of de Sitter-like spacetimes and the semiglobal existence and stability of asymptotically Minkowskian spacetimes. This formulation of the conformal field equations is formulated in terms of geometric quantities associated to the Levi–Civita connection of a conformally related (i.e., unphysical) metric $g$. In [14], the vacuum-conformal field equations have been rewritten in terms of an (in principle arbitrary) Weyl connection—we call these equations the extended conformal field equations. The extended conformal field equations with matter have been studied in [26], where applications to the global existence of various electrovacuum spacetimes have been considered.

5.4.1. The frame formulation of the equations. For completeness, we briefly review the general setting of the extended conformal field equations. In the remainder of this subsection, let $(\mathcal{M}, \tilde{g})$ denote a spacetime satisfying the Einstein–Maxwell field equations (1a)–(1c). Let $g$ denote the conformal metric defined by the relation $g = \Xi^2 \tilde{g}$ where $\Xi$ denotes a (yet undetermined) conformal factor and let $\nabla$ denote its Levi–Civita connection. Let $\{e_a\}$, $a = 0, \ldots, 3$ denote a frame field that is $g$-orthogonal so that $g(e_a, e_b) = \delta_{ab}$, and let $\{\omega^a\}$ denote its dual co-basis—i.e., $(\omega^a, e_a) = \delta^a_b$. As $\nabla$ is the Levi–Civita connection of $g$, one again understands that its connection coefficients, $\Gamma^a_{\alpha\beta} = \langle \alpha^a, \nabla_{\beta} e_{\alpha} \rangle$, satisfy the usual metric compatibility condition.

Let now $\hat{\nabla}$ denote a Weyl connection constructed from the Levi–Civita connection $\nabla$ and a 1-form $\hat{f}$ using formula $\hat{\nabla} = \nabla + \hat{f}$. If $\hat{\Gamma}^a_{\alpha\beta} = \langle \alpha^a, \hat{\nabla}_{\beta} e_{\alpha} \rangle$ denotes the connection coefficients of $\hat{\nabla}$ with respect to the frame $e_a$, one sees then that

$$f^a_{\alpha\beta} = \Gamma^a_{\alpha\beta} + S_{ab}^{\phantom{ab}cd} f^d_{\alpha\beta},$$

$$= \Gamma^a_{\alpha\beta} + \delta_a^{\alpha} \delta^b_{\beta} + \delta_b^{\alpha} \delta^a_{\beta} - \eta_{ab} \eta^{cd} f^d_{\alpha\beta}.$$  

In particular, one sees that $f_a = \frac{1}{2} \hat{f} a_{\beta} b$, as $\Gamma^a_{\alpha\beta} = 0$ in the case of a metric connection.

Let $\Sigma^a_{\alpha\beta}$ denote the torsion of the connection $\hat{\nabla}$. It is convenient to distinguish between the expression for the components of the Riemann tensor of the connection $\hat{\nabla}$ in terms of the connection coefficients $\hat{\Gamma}^a_{\alpha\beta}$ (the geometric curvature $\hat{\mathcal{P}}^a_{\alpha\beta}$) and the expression of the Riemann tensor in terms of the Schouten and Weyl tensors (the algebraic curvature $\rho^a_{\alpha\beta}$).

Explicitly, one sees that

$$\hat{\mathcal{P}}^a_{\alpha\beta} \equiv \mathcal{P}^a_{\alpha\beta} + 2 S_{d}^{\phantom{d}a \beta} \hat{f}^d_{\alpha} - 2 S_{d}^{\phantom{d}a \beta} \hat{f}^d_{\alpha},$$

$$\rho^a_{\alpha\beta} \equiv 2 \mathcal{S}_{d}^{\phantom{d}a \beta} \hat{L}^d_{\alpha},$$

Define the geometric zero quantities

$$\mathcal{S}_{a}^{\phantom{a}\alpha\beta} \equiv \langle \sigma^a, [e_a, e_b] \rangle - \langle \hat{f} a_{\beta} b, \hat{f}^d_{\alpha} \rangle,$$

$$\mathcal{S}_{\alpha\beta} \equiv \hat{\mathcal{P}}_{\alpha\beta} - \rho^a_{\alpha\beta},$$

$$\Delta_{\alpha\beta\gamma\delta} \equiv \hat{\nabla}_{\alpha} \hat{L}_{\beta\gamma\delta} - \hat{\nabla}_{\delta} \hat{L}_{\alpha\beta\gamma} - d a d^b_{\alpha\beta\gamma\delta} - \mathcal{S}_{\alpha\beta\gamma\delta},$$

$$\Lambda_{\alpha\beta\gamma} \equiv \hat{V}_{a}^{d} d^a_{\alpha\beta\gamma\delta} - \hat{T}_{\alpha\beta\gamma},$$
where $T_{cd} \equiv V_{[c} T_{d]}^{b}$ and the Maxwell zero quantities
\begin{align*}
M_a & \equiv V^b F_{ab}, \\
M_{abc} & \equiv V_{[a} F_{bc]}
\end{align*}
Then the extended conformal Einstein–Maxwell field equations are given by the conditions
\begin{align*}
\sum_e \varepsilon_{eb} = 0, \quad \hat{\Xi}_{dab} = 0, \quad \hat{\Delta}_{ab} = 0, \quad M_a = 0, \quad M_{abc} = 0. \quad (32)
\end{align*}

The fields $f_a$, $d_a$ and $\Xi$ are related to each other by the constraint
\begin{align*}
d_a = f_a + \nabla_a \Xi.
\end{align*}

The above conformal equations can be read as yielding differential conditions, respectively, for the frame components $e_a^b$, the spin coefficients $\Gamma^a_{\alpha\beta\gamma}$ (including the components $f_a$ of the 1-form $f$), the components of the Schouten tensor $\hat{L}_{ab}$, the components of the rescaled Weyl tensor $d^a_{bcd} \equiv \Xi^{-1} C^a_{bcd}$, and the components of the unphysical Faraday tensor $F_{ab}$.

Remark. Equations (32) have to be supplemented with gauge conditions or equations that determine the conformal factor $\Xi$ and the 1-form $d$. In the case of the particular applications to be considered in this article, these fields will be determined by means of a conformal Gaussian gauge fixed by the congruence of conformal geodesics discussed in section 5.3. Notice, however, that other choices are possible—e.g., a gauge based on the properties of the conformal curves discussed in [26].

5.4.2. Spinorial formulation of the equations. Hyperbolic reductions of the extended Einstein–Maxwell conformal field equations (32) and a subsequent reduction to spherical symmetry are best carried out using a space–spinor formalism based on a spinor $\tau^{AA}$ associated with a time-like vector such that $\tau_{AA} \cdot \tau^{BA} = e^{-2\alpha}$. The spinorial counterparts of the fields $e_{AB}$, $\Gamma_{ABCD}$, $\hat{L}_{ABCD}$, $F_{ab}$ are given, respectively, by the spinor fields
\begin{align*}
e_{AB}, \quad f_{a}^{b c}, \quad \hat{L}_{ab}, \quad d^a_{bcd}, \quad F_{ab}
\end{align*}
are given, respectively, by the spinor fields
\begin{align*}
e_{AB}, \quad \hat{f}_{ABCD}, \quad \hat{L}_{ABCD}, \quad \phi_{ABCD}, \quad \varphi_{AB}.
\end{align*}
Note that $\phi_{ABCD} = \phi_{(ABCD)}$, and $\varphi_{AB} = \varphi_{(AB)}$ so that they are already decomposed into irreducible terms. For the fields $e_{AB}$, $\hat{f}_{ABCD}$ and $\hat{L}_{ABCD}$ one sees the decompositions
\begin{align*}
e_{AB} & = \frac{1}{2} e_{AB} e^c_Q q^a_{(AB)}, \\
\hat{f}_{ABCD} & = -\frac{1}{2} \left( \varepsilon_{ABCD} - \chi_{ABCD} \right) + e_{AC} f_{DB}, \\
\hat{L}_{ABCD} & = \frac{1}{2} e_{AB} L_Q q_{CD} + \frac{1}{2} e_{CD} \hat{L}_{ABQ} q_{(AB)} + L_{(AB)} Q_{(CD)}
\end{align*}
where $f_{AB}$ is the space–spinor counterpart of the field $f_a$. It satisfies, in turn, the split
\begin{align*}
f_{AB} & = \frac{1}{2} e_{AB} f^c_Q q^a_{(AB)}.
\end{align*}
The spinor fields $\varepsilon_{ABCD}$ and $\chi_{ABCD}$ correspond, at least on an initial hypersurface, to the spinorial counterparts of the intrinsic connection to the hypersurface and the extrinsic curvature, respectively. It is recalled that in the space–spinor formalism, the trace part of spinorial fields corresponds to the time components of tensors—see e.g., [33].

Expressions of the spinorial version of the Einstein–Maxwell equations (32) in terms of the fields (33) can be given in terms of the following spinor zero-quantities:
\[ \dot{\Sigma}_{ABCD} = 0, \quad \dot{\Sigma}_{ABCDE} = 0, \quad \dot{\Delta}_{ABCD} = 0, \quad \dot{\Delta}_{ABCD} = 0, \quad \dot{M}_{AB} = 0. \]  

The explicit form of these zero-quantities will not be given here. The interested reader is referred to [26].

5.4.3. The hyperbolic reduction of the spinorial conformal Einstein–Maxwell field equations. In the remainder of this article we will consider evolution equations for the various spinorial conformal fields obtained through an hyperbolic reduction procedure based on the gauge properties of conformal geodesics discussed in section 5.3.2. In view of our particular application, the various spinor fields and their corresponding tensors will be lifted to the bundle space \( \mathcal{M}_{a,x} \). In a slight abuse of notation, we will denote the spinorial fields and their lifts to the bundle space with the same symbol. The precise nature of the object should be clear from the context.

The gauge conditions discussed in section 5.3.2, expressed in terms of spinorial objects, take the form

\[ \partial_i \Gamma = e_i^2 \tau, \quad f_q^0 = 0, \quad f_q^0 = 0, \quad \dot{L}_i^0 = 0, \]  

on \( \mathcal{M}_{a,x} \). Hence, in particular, one has that

\[ e_{AB} = \sqrt{2} e_{AB}^0 \partial_i + \left( e_{AB}^1 \partial_x + e_{AB}^2 x + e_{AB}^3 z \right), \]

where \( e_{AB}^\mu \) with \( \mu = 0, 1, +, - \).

Suitable evolution equations are obtained from the following components of (34)

\[ \dot{\Sigma}_{Q}^0 = 0, \quad \dot{\Sigma}_{Q}^0 = 0, \quad \dot{\Delta}_{Q}^0 = 0, \quad \dot{\Delta}_{Q}^0 = 0, \quad \dot{M}_{AB} = 0, \]  

(36)

together with the gauge conditions (35). The above evolution equations satisfy a suitable propagation of the constraints result. Namely, if equations (34) are satisfied initial on some hypersurface then they are also satisfied in the domain of dependence as long as the evolution equation (36) hold—see [26] for more details.

5.4.4. The evolution equations in spherical symmetry. A spherical symmetry reduction of the evolution equations (36) can be implemented by expressing the various quantities in terms of the spinors \( e_{AB}, x_{AB}, y_{AB} \) and \( z_{AB} \) introduced in section 5.1.1 and then making an Ansatz based on the spin-weight of the components. Save for the case of frame coefficients \( e_{AB}^\pm \), only components with spin-weight zero are considered. A computation shows that under these circumstances a suitable Ansatz for an Einstein–Maxwell field in the present formalism is given by

\[ e_{AB}^0 = e_{AB}^0 x_{AB}, \quad e_{AB}^1 = e_{AB}^1 x_{AB}, \quad e_{AB}^2 = e_{AB}^2 y_{AB}, \quad e_{AB}^3 = e_{AB}^3 y_{AB}, \]

\[ f_{AB} = f x_{AB}, \quad \xi_{ABCD} = \frac{1}{\sqrt{2}} \xi_{ABCD} (e_{AC} x_{BD} + e_{BD} x_{AC}), \]

\[ L_{ABCD} = \xi_{ABCD} \xi_{ABCD} + \frac{1}{\sqrt{2}} \xi_{ABCD} \xi_{ABCD}, \quad \phi_{ABCD} = \frac{1}{3} \theta_{h_{ABCD}} + \frac{1}{\sqrt{2}} \theta_{h_{ABCD}} \theta_{h_{ABCD}}, \]

where all the coefficients in the above Ansatz are real. For more details on the motivations behind the above spherical symmetric Ansatz, we refer the reader to [15, 34]. A lengthy computation using the suite \texttt{xAct} for tensorial and spinorial manipulations for \texttt{Mathematica}—see [22, 28]—yields the following \textit{spherically symmetric conformal}
evolution equations

\begin{align}
\dot{e}^0 &= \frac{1}{3}(\chi_2 - \chi_h)e^0 - f, \\
\dot{e}^1 &= \frac{1}{3}(\chi_2 - \chi_h)e^1, \\
\dot{e}^\pm &= -\frac{1}{6}(\chi_2 + 2\chi_h)e^\pm, \\
\dot{f} &= \frac{1}{3}(\chi_2 - \chi_h)f + \Theta h, \\
\dot{\xi}_s &= -\frac{1}{6}(\chi_2 + 2\chi_h)\xi_s - \frac{1}{2}\chi_2 f - \Theta h, \\
\dot{\chi}_s &= \frac{1}{6}(\chi_2 - 4\chi_h)\chi_h - \theta_2 + \Theta \phi, \\
\dot{\chi}_h &= -\frac{1}{6}\chi_2^2 - \frac{1}{3}\chi_2^2 - \Theta h, \\
\dot{\phi}_s &= \frac{1}{3}(\chi_2 - \chi_h)\phi_s - \frac{1}{2}\Theta^2d_s\phi - \frac{1}{2}\Theta^2\phi^2f, \\
\dot{\phi}_h &= -\frac{1}{3}\chi_2\phi_h - \frac{1}{2}\chi_2\Theta h - \Theta \phi^2\phi + \frac{1}{4}\Theta^2\phi^2\phi^2, \\
\dot{\phi} &= -\frac{1}{2}(\chi_2 + 2\chi_h)\phi + \frac{1}{2}(\chi_2 + 2\chi_h)\Theta \phi^2 - \Theta \phi^2, \\
\dot{\phi} &= -\frac{1}{2}(\chi_2 + 2\chi_h)\phi.
\end{align}

where, as before, \(\dot{\cdot}\) denotes differentiation with respect to the time coordinate \(\tau\). Notice that in the conformal Gaussian gauge used to write the above equations, the field unknowns associated to a spherically symmetric electrovacuum field are dynamic—that is, the evolution in this gauge does not follow the static Killing vector of the spacetime. Note, however, that the evolution equations are mere transport equations (i.e. ordinary differential equations) along the congruence of conformal geodesics—the evolution along a given conformal geodesic is decoupled from the evolution in nearby curves. This decoupling disappears when considering other (spherically symmetric) systems like the Einstein–Yang–Mills or the Einstein-conformally invariant wave equation as the solutions are not necessarily static—see e.g. [27].

The essential dynamics of the evolution equations (37a)–(37l) is steered by a subset thereof that decouples from the rest of the system—we call this subsystem the core system. Letting

\[ X \equiv \chi_h + \frac{1}{2}\chi_2, \quad L \equiv \theta_h + \frac{1}{2}\theta_2, \]

the core system is given by the equations

\begin{align}
\partial_s L &= -\frac{1}{3}XL - \frac{1}{2}\Theta \phi + \frac{3}{4}\Theta^2X\phi^2 - \frac{3}{2}\Theta \phi^2\phi, \\
\partial_s X &= -\frac{1}{3}X^2 - L + \frac{1}{2}\Theta \phi, \\
\partial_s \phi &= -X\phi + \Theta X\phi^2 - \Theta \phi\phi^2.
\end{align}
\[ \partial_{,\varphi} = -\frac{2}{3}X\varphi. \]  

(38d)

The evolution equations (37a)–(37l)—or (38a)–(38d)—are supplemented by the evolution equations for the fields \( \Theta, d, d_a \) provided by the conformal Gaussian gauge as discussed in section 3.1. Recalling that the spinorial counterpart of the unphysical energy–momentum tensor \( T \) is given by the spinor \( T_{\varphi\varphi} = \varphi^\dagger T \bar{\varphi} \), it follows that

\[ \varphi^\dagger T_{\varphi\varphi} \bar{\varphi} = -\frac{2}{3}X\varphi^2. \]

Note the overall factor in the expression arises from the normalisation \( \varphi^\dagger T_{\varphi\varphi} \bar{\varphi} = \frac{1}{2} \varphi^2 \). The spinorial counterpart of \( T_{0i} \) is given by

\[ \varphi^\dagger T_{x_0} \bar{\varphi} = -\frac{1}{2}X\varphi^2. \]

(39)

It then follows that the third order evolution equation for the canonical conformal factor \( \Theta \)—cf. (13)—can be written as

\[ \dot{\Theta} = \frac{1}{2} \Theta^2 \dot{\varphi} \varphi^2 - \frac{1}{3}X\Theta^3 \varphi^2. \]

(40)

If

\[ d_{AB} = \frac{1}{2}e_{AB} d + d_{(AB)} \]

denotes the space–spinor counterpart of the field \( d_a \), it follows from (11) that

\[ d = \dot{\Theta}, \]

(41)

while from (14) together with (39) one finds that \( d_{(AB)} = x_{AB} \) is constant along the conformal geodesics and can be specified directly from initial data—see equation (44b) in section 5.4.5 below.

**Remark 1.** Equations (37a)–(37l) together with (40) and (41) are the complete evolution system for spherically symmetric electrovacuum spacetimes. Strictly speaking, in order to obtain a system that is purely of the first order, one should introduce a further variable representing the second order derivative of the conformal factor. The hyperbolic reduction can be achieved without introducing the derivative of the Maxwell spinor \( \varphi_{AB} \) as a further unknown—see [13, 26]. This is a feature of the spherical symmetry where \( V_{AA} \varphi_{BC} \) has only two components—the time and the radial ones. The time derivative is readily available from the evolution equations, while the radial one can be obtained from the constraint implied by the Maxwell equations—see [34] for more details.

**Remark 2.** Given a solution to equations (37a)–(37l) together with (40) and (41), one can construct a metric tensor \( g \) on \( M_{a,\varphi} \) using the decomposition

\[ g = d\varphi \otimes d\varphi - \frac{e_0^0}{e_1^0} (d\varphi \otimes d\rho + d\rho \otimes d\varphi) - \left( \frac{1}{(e_1^0)^2} - \frac{e_0^0}{e_1^0} \right) d\rho \otimes d\rho - \frac{1}{e_0^0 e_1^-} \sigma. \]

(42)

as long as \( e_1^0, e_1^0 e_1^- \neq 0 \)—see [15, 34] for further details. This metric on a bundle space can be projected down to a (unphysical) spacetime metric that will be denoted, again, by \( g \).

**5.4.5. Initial data for the evolution equations.** Initial data for the evolution equations (37a)–(37l), (40) and (41) on the extended bundle space \( M_{a,\varphi} \) is obtained following the discussion of section 5.2 and observing the expressions in section 5.1.3. One finds that
The initial data for the conformal gauge unknowns is given by

\[ \Theta_\star = \frac{\rho}{1 + mp}, \quad \Theta_\star' = 0, \quad \Theta_\star'' = -\frac{\rho(2 + mp)^2}{2(1 + mp)^3}, \]

\[ d_{s\star} = \frac{2\rho + mp^2}{(1 + mp)^2}. \]

One observes that, in particular, \( \Theta_\star = \Theta_\star' = \Theta_\star'' = 0 \) at \( \rho = 0 \).

5.4.6. Conformal deviation equations. The gauge used in the construction of the cylinder at spatial infinity hinges on the properties of conformal geodesics. Hence, it is important to verify that the congruence of curves is free of conjugate points on \( \mathcal{M}_{a_x} \).

In what follows, let \( \eta_\star \) denote the conformal Jacobi field measuring the deviation of the curves in the congruence of conformal geodesics. Its spinorial counterpart has the space–spinor representation

\[ \eta_{AB} = \frac{1}{2} \eta^{CD} + \eta_{(AB)}, \]

where \( \eta \equiv \eta^0_\star \). In [25] it has been shown that \( \eta, \eta_{(AB)} \) satisfy the evolution equations

\[ \partial_\tau \eta = f_{(AB)}^{(CD)} \eta^{AB}, \]

\[ \partial_\tau \eta_{(AB)} = \chi_{(CD)(AB)} \eta^{(CD)}. \]

Conjugate points in the congruence arise if \( \eta_{(AB)} = 0 \). In view of the spherical symmetry of the setting, one sees that

\[ \eta_{(AB)} = \eta_\star x_{AB}. \]

Hence, the evolution equations for \( \eta \) and \( \eta_{(AB)} \) reduce to

\[ \partial_\tau \eta = -f_{AB}^{\star \star}, \]

\[ \partial_\tau \eta_{x} = \frac{1}{2} (\chi_h - \chi_2) \eta_{x}. \]

Without loss of generality, initial data for the above fields can be set to be

\[ \eta_\star = 0, \quad \eta_{x\star} = 1 \quad \text{on} \quad C_{a}. \]

In the next section, it will be shown that the solutions to equations (45a) and (45b) do not lead to critical points, at least for sufficiently small parameter \( a \).

5.5. Existence of solutions on \( \mathcal{M}_{a_x} \)

The evolution equations (37a)–(37l), (40) and (41) on the bundle manifold \( \mathcal{M}_{a_x} \) together with the initial data (43a)–(43c) and (44a)–(44b) on \( C_{a_x} \) are all regular for sufficiently small \( a > 0 \). Hence, they give rise to a regular finite initial value problem near spatial infinity. In view of the particular form of the equations, standard theory ensures the existence of a local
solution in a neighbourhood of $C_{a,k}$. Naturally, we are interested in extending the existence result to the whole of $M_{a,k}$. This requires a slightly more detailed analysis.

Letting

$$u \equiv \left( e^0, e^1, e^2, f, \xi_3, \xi_2, \chi_b, \theta_3, \theta_2, \phi, \varphi, \Theta, \Sigma, \Lambda \right)$$

with $\Sigma \equiv \Theta, \Lambda \equiv \Theta$, one can rewrite the initial value problem in the form

$$\partial \tau u = F(u, \tau, \rho; m), \quad u(0, \rho; m) = u_*(\rho; m),$$

where $F$ and $u_*$ are analytic functions of their arguments. The particular case $m = 0$ (i.e. the Minkowski spacetime) can be solved explicitly with the only non-vanishing geometric fields given by

$$e^0 = -\tau, \quad e^1 = \rho, \quad e^2 = 1, \quad f = 1,$$

while the fields associated to the conformal gauge are

$$\Theta = \rho \left(1 - \tau^2\right), \quad d_{AB} = 2\rho s_{AB}.$$

Consequently, this solution exists for all $\tau, \rho \in \mathbb{R}$. Moreover, from (45a)--(45b) it follows that

$$\eta = -\tau, \quad \eta_\rho = 1$$

so that no conjugate points arise in the congruence of conformal geodesics if $m = 0$.

Now, returning to the case $m \neq 0$, by Cauchy stability of ordinary differential equations —see e.g. [24]—given $\tau_0 > 1$ there exist $m_0 > 0, \rho_0 > 0$ such that the solution $u(\tau, \rho; m)$ is analytic in all variables and exists for

$$|\tau| \leq \tau_0, \quad |\rho| \leq \rho_0, \quad |m| \leq m_0.$$

By choosing $\tau_0$ sufficiently large and observing the properties of the reference $m = 0$ solution, one can ensure that for each conformal geodesic with $0 < \rho < \rho_0$ there exists a $\tau_\rho < \tau$ such that $\Theta_\rho \tau_\rho = 0, d\Theta_\rho \tau_\rho \neq 0$. In order to obtain a result that is valid for any value of $m$, it is noticed that equations (37a)--(37l), (40) and (41) together with the data (43a)--(43c), (44a)--(44b) are invariant under the rescaling

$$m \mapsto \frac{1}{\lambda} m, \quad \rho \mapsto \lambda \rho, \quad \phi \mapsto \frac{1}{\lambda} \phi, \quad \varphi \mapsto \frac{1}{\lambda} \varphi, \quad e^1 \mapsto \lambda e^1, \quad \Theta \mapsto \lambda \Theta,$$

for $\lambda > 0$. Consequently, for any arbitrary $m$ it is always possible to obtain a solution to the system (46) reaching null infinity if $a$, and hence $\rho$, is sufficiently small. By a similar argument based on Cauchy stability, one can conclude that if $\rho$ is sufficiently small, the congruence of conformal geodesics in $M_{a,k}$ is free of conjugate points. Once a solution to the conformal evolution equations has been obtained, the existence of a solution to the Einstein constraint equations follows from the analysis of the propagation of the constraints —see [14, 26]. In conjunction with the result on the propagation of the constraints given in [26] one obtains the following:

**Theorem 1.** Given $m > 0$, there exists $a > 0$ such that on $M_{a,k}$ there exists a unique smooth solution to the spherically symmetric initial value problem at spatial infinity for the extended conformal Einstein–Maxwell equations. For each $\rho \in [0, a]$ there exists $\tau_\rho(\rho) > 0$ such that $\Theta_\rho \tau_\rho(\rho) = 0$. For $\rho \neq 0$, the sets
correspond to the future and past null infinities of the metric $\Theta^{-2}g$ where the metric $g$ is given by expression (42).

Remark. It is important to emphasise, that the above result is independent of the sign of $m$.

5.5.1. The solution on the cylinder at spatial infinity. On the conformal geodesics with $\rho = 0$, the evolution equations (37a)–(37l), (40) and (41) simplify, and it is possible to obtain the solution in closed form. If $\rho = 0$, the data for the conformal factor satisfies

$$\Theta_* = \hat{\Theta}_* = \hat{\Theta}_* = 0.$$ 

It follows then from the evolution equation for $\Theta$ that

$$\Theta = \hat{\Theta} = 0$$

for all later times. Because of this observation, and taking into account the initial data for the various fields one finds, that the only non-vanishing unknowns are given by

$$e^0 = -\tau, \quad e^\pm = 1, \quad f = 1, \quad \phi = -6m, \quad \varphi = -m.$$  

(47)

6. The cylinders at the horizon

The purpose of this section is to show that it is possible to obtain regular representations of the points $i^+$ and $c_0$—see figures 2 and 3. This is achieved by combining the construction of the cylinder at spatial infinity discussed in the previous section with the conformal isometry of the extremal Reissner–Nordström spacetime. We call the resulting constructions the cylinders at the horizon. As it will be seen, the key features of the construction are independent of the sign of $m$. In order to distinguish quantities near the horizon from quantities near the conformal boundary, we will use an overbar for all variables related to the constructions of the cylinders at the horizon.

6.1. The bundle manifolds $\tilde{C}_a(i^*)$ and $\tilde{C}_a(e^0)$

In what follows, let $\tilde{r}$ indicate either $i^+$ or $e^0$ and let $(S_\delta, \delta)$ denote the conformal extensions of $(\tilde{R}, \tilde{h})$ or $(\tilde{S}, \tilde{h})$, as discussed in section 4.2 and

$$\delta = dr \otimes dr + r^2 \sigma.$$ 

By analogy to the construction of the cylinder at spatial infinity, we begin by considering the blow up of the point $\tilde{r}$. Given $a > 0$ and the open ball $B_a(\tilde{r}) \subset S_\delta$ one defines, in analogy to the definition of the bundle space $C_a$ given in equation (28), the bundle manifold

$$\tilde{C}_a \equiv \left\{ \delta^0(t, r) \in SU(B_a(\tilde{r})) \mid 0 \leq r < \tilde{a} \right\},$$

where $SU(B_a(\tilde{r}))$ is the bundle of normalised spin frames over $B_a(\tilde{r})$ with structure group $SU(2, C)$, and $\delta_a(t, r)$ corresponds to the frames such that the vector $e_3(t, r) = \sigma_3 A e_3(t, r)$ is tangent to the radial geodesics with the affine parameter $r$. Spinor fields on $B_a(\tilde{r})$ can be lifted to the bundle space $\tilde{C}_a$ in an analogous way to the lifts to $C_a$. Again, the resulting fields can be spanned in terms of the basic spinors $x_{AB}$, $y_{AB}$, $z_{AB}$ and $c_{AB}$. Moreover, the vector fields $\left\{ \partial, X, X_+, X_- \right\}$ with $X, X_+, X_-$ as the basis of the
Lie algebra \(\mathfrak{su}(2,\mathbb{C})\) discussed in section 5.1.2 constitute a set of frame fields on \(\hat{C}_\phi\). As in section 5.1.2, let \(\alpha\), \(\alpha^+\), \(\alpha^-\) denote the dual 1-forms to \(X\), \(X_+\), \(X_-\).

Frame fields and solder forms on \(\hat{C}_\phi\) are introduced in a natural manner via
\[
\mathbf{e}_{AB} = \hat{X}_{AB} \mathbf{e}_r + \frac{1}{r} \hat{Z}_{AB} \mathbf{e}_\theta + \frac{1}{r} \hat{Y}_{AB} \mathbf{e}_\phi,
\]
\[
\mathbf{\sigma}^{AB} = -\hat{X}^{AB} \mathbf{d}r - 2r\hat{Y}^{AB} \mathbf{d}\theta - 2r\hat{Z}^{AB} \mathbf{d}\phi,
\]
so that
\[
\mathbf{\check{h}} = h_{ABCD} \mathbf{\sigma}^{AB} \otimes \mathbf{\sigma}^{CD}, \quad \{\mathbf{\sigma}^{AB}, \mathbf{\check{e}}_{CD}\} = h_{AB}^{CD}.
\]

where \(\mathbf{\check{h}}\) denotes (again, in an abuse of notation) the lift to \(\hat{C}_\phi\) of the conformal metric \(\mathbf{h} = -\delta\).

The spin coefficients are given by
\[
\gamma_{\epsilon\epsilon}^{r\epsilon} \check{f}_{ABC}^{\epsilon\epsilon} = \frac{1}{r} \mathbf{e}_{AC} \mathbf{x}_{BD} + \mathbf{e}_{BD} \mathbf{x}_{AC}.
\]

In analogy to the discussion of section 5.2, given a non-negative smooth function \(\kappa\), we introduce the extended bundle space,
\[
\hat{C}_{\phi,\kappa} \equiv \left\{ \kappa^{1/2} \mathbf{d}^O \left\| \mathbf{d}^O \in \hat{C}_\phi \right\} \right\}.
\]

In what follows, \(\check{\phi}_{ABCD}\), \(\check{\Theta}_{ABCD}\) and \(\check{\varphi}_{AB}\) will now denote the spinorial counterparts of the fields \(d_{ij}\), \(\check{L}_i\) and \(E_i\) on \(S_t\) as discussed in section 4.1. Following our standard usage, we denote their lifts to the bundle space \(\hat{C}_\phi\) by \(\check{\phi}_{ABCD}\), \(\check{\Theta}_{ABCD}\) and \(\check{\varphi}_{AB}\). Under the rescaling \(\mathbf{d}^O \mapsto \kappa^{1/2} \mathbf{d}^O\), the latter fields can be seen to transform as
\[
\check{\phi}_{ABCD} \mapsto \kappa^{3/2} \check{\phi}_{ABCD}, \quad \check{\Theta}_{ABCD} \mapsto \kappa^2 \check{\Theta}_{ABCD}, \quad \check{\varphi}_{AB} \mapsto \kappa^{5/2} \check{\varphi}_{AB}.
\]

One also sees that
\[
\mathbf{e}_{AB} \mapsto \kappa^{1/2} \mathbf{e}_{AB}, \quad \mathbf{\sigma}^{AB} \mapsto \kappa^{-1} \mathbf{\sigma}^{AB}, \quad \mathbf{\check{e}}_{ABCD} \mapsto \kappa^{3} \mathbf{\check{e}}_{ABCD} - \frac{1}{2} (\epsilon_{AC} \mathbf{\check{e}}_{BD} \kappa + \epsilon_{BD} \mathbf{\check{e}}_{AC} \kappa).
\]

This is completely analogous to the rescaling behaviour for fields in \(\hat{C}_{\kappa,\phi}\) as discussed in section 5.2. A calculation readily shows that the lifts of the fields to \(\hat{C}_{\phi}\) is given by
\[
\check{\phi}_{ABCD} = \frac{6m}{r(r + m)} \mathbf{e}_{ABCD}, \quad \check{\Theta}_{ABCD} = \frac{6m}{r(r + m)^2} \mathbf{e}_{ABCD}^2, \quad \check{\varphi}_{AB} = \frac{m}{r^2} \mathbf{e}_{AB}.
\]

Thus, it follows that one should choose \(\kappa = O(r)\) in order to obtain rescaled fields that are finite at \(r=0\). In what follows, we make the simplest choice
\[
\kappa = r.
\]

As a result, one finds that the lift of the 3-metric \(\mathbf{h} = -\delta\) satisfies
\[
\mathbf{\hat{h}} \mapsto \frac{1}{r^2} \mathbf{d}r \otimes \mathbf{d}r + 2 \left( \mathbf{\alpha}^+ \otimes \mathbf{\alpha}^- + \mathbf{\alpha}^- \otimes \mathbf{\alpha}^+ \right).
\]

Hence, as in the case of the cylinder at spatial infinity, one obtains a 3-metric that is singular at \(r=0\). Again, this is not a problem for our general strategy of obtaining a regular representation of the spacetime region around \(i^+\) and \(\partial^0\) as the conformal field equations are expressed in terms of quantities that are chosen to be regular at \(r = 0\).
6.2. Spacetime gauge considerations

The implementation of a spacetime gauge on the domain of dependence of $\tilde{C}_{\alpha\kappa}$ is done in complete analogy to the construction of the cylinder at infinity. Recall that at the cylinder-like end of $\tilde{S}$ in region $I$ and $\tilde{R}$ in region $III$ the conformal factor was chosen to be $\sigma = r/(r + m)$. We prescribe the following initial data on $B_k(i) \setminus \{i\}$ for a congruence of conformal geodesics

$$\tilde{x}_* = \tilde{x}(p), \quad \tilde{x}_* \perp S_i, \quad \tilde{b}_* = \sigma^{-1}d\sigma, \quad \langle \tilde{b}, \tilde{x} \rangle_* = 0. \quad (49)$$

The above initial data is supplemented by the following choice of data for the conformal factor

$$\tilde{\Theta}_* = \sigma^{-1} \Theta, \quad \tilde{\Theta} = 0, \quad \tilde{\Theta}_* = -\frac{\kappa}{2m} \delta(d\sigma, d\sigma) + \sigma^{-3} \tilde{\mu}_*, \quad (48)$$

with $\sigma$ given as in (48). Hence, $\tilde{\Theta}_* = 1/(r + m)$. In addition, we consider a frame $\{\tilde{e}_a\}$ that is Weyl propagated along the congruence of conformal geodesics and set $\tilde{e}_0 = \tilde{x}$, as usual. The above initial data conditions ensure that

$$\tilde{g}(\tilde{x}, \tilde{x}) = \tilde{\Theta}^2 \tilde{g}(\tilde{x}, \tilde{x}) = 1$$

consistently with $\tilde{g} = \tilde{\Theta}^2 \tilde{g}$, where $\tilde{g}$ denotes the (physical) spacetime metric associated to the development of $S_i$. We also require that

$$(\tilde{g}(\tilde{e}_a, \tilde{e}_b))_* = \tilde{\Theta}^2 (\tilde{g}(\tilde{e}_a, \tilde{e}_b))_* = \eta_{ab}. \quad (49)$$

As in the case of the cylinder at spatial infinity, we consider 3 connections $\tilde{\nabla}$, $\tilde{\nabla}$ and $\hat{\nabla}$ —respectively, the physical Levi–Civita connection, the unphysical Levi–Civita connection and the canonical Weyl connection associated to the congruence of conformal geodesics. The relation between these various connections is given by expressions that are completely analogous to those described in section 5.3.2 for the cylinder at spatial infinity. In particular, one sees that

$$\hat{\nabla} - \hat{\nabla} = S(\tilde{f}),$$

where the 1-form $\tilde{f}$ satisfies

$$\tilde{f} = \tilde{b} - \Theta^{-1}d\Theta.$$

By analogy to the gauge construction in the cylinder at spatial infinity, one obtains, along the congruence of conformal geodesics, the gauge conditions

$$\tilde{L}_a = 0, \quad \tilde{f}_0 = 0, \quad \tilde{L}_{ab} = 0,$$

for the Weyl connection coefficients, the time component of $\tilde{f}$ and the components of the Schouten tensor of the Weyl connection $\hat{\nabla}$, respectively.

6.2.1. A representation of the horizon. In what follows it will be seen that using the conformal isometry of the Reissner–Nordström, the conformal geodesics in the construction of the cylinder at spatial infinity can be mapped to the conformal geodesics in a neighbourhood of $i$ arising from the data (49). As a consequence, given $q \in B_{a}(i)$, there exists $a > 0$ and $p \in B_{b}(i^0)$ such that $q = \iota(p)$. Recall that in section 5.3.3, $\tau_{\sigma}(p)$ was defined by the condition $\Theta(\pm \tau_{\sigma}(p)) = 0$, with $\Theta$ as the conformal factor associated with the congruence of conformal geodesics near the cylinder at spatial infinity. Let $\tau_{\sigma} \equiv \iota_{a}\iota_{\sigma}$. Because of the properties of the conformal isometry discussed in section 2.3, it follows that
Nevertheless, the function $\tau (q) = \tau * t$ will be useful in identifying the domain on which we will be looking for solutions to the conformal Einstein–Maxwell equations—namely:

$$\mathcal{M} = \{ (\tau, q) \in \mathbb{R} \times C | - \tau (q) \leq \tau \leq \tau (q) \}.$$

We also define the horizon $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ with

$$\mathcal{H}^\pm = \{ (\tau, q) \in \mathcal{M} | \tau = \pm \tau (q), \ r (q) \neq 0 \},$$

and the cylinder at the horizon as

$$\mathcal{H} = \{ (\tau, q) \in \mathcal{M} | r (q) = 0, - \tau (q) < \tau < \tau (q) \}.$$ 

Of interest are also the critical sets

$$\mathcal{H}^\pm = \{ (\tau, q) \in \mathcal{M} | r (q) = 0, \ \tau = \pm \tau (q) \},$$

and

$$\mathcal{H}^0 = \{ (\tau, q) \in \mathcal{M} | r (q) = 0, \ \tau = 0 \},$$

the intersection of $\mathcal{M}$ with $\mathcal{H}$.

Coordinates in $\mathcal{M}$ are naturally dragged from $\mathcal{C}$ along the conformal geodesics. We emphasise that as a result, the $r$ coordinate used below is constant along conformal geodesics and, hence, no longer represents the isotropic coordinate. Hence, statements involving $r = 0$ refer to the cylinder at the horizon $\mathcal{H}$, not the horizon $\mathcal{H}^\pm$. The former is ruled by the conformal geodesics starting at $r = 0$. Similarly, the vector fields $\partial$, $X^b$ extend in a unique way to vectors on $\mathcal{M}$ by requiring that they commute with $\partial_r$.

### 6.3. The regular initial value problems at $c^0$ and $i^*$

A regular initial value problem for the conformal field equations (34) with data on the bundle manifold $\mathcal{C}$ can be posed in analogy to that for the cylinder at spatial infinity discussed in sections 5.4.4 and 5.4.5. Analogous to section 5.4.4, one uses a spherically symmetric ansatz, and one obtains ordinary differential equations along the curves of the congruence of conformal geodesics for the fields $\xi^0, \xi^1, \xi^2, 2, 2, 2, 2$.

The regular data for these equations is given by

$$\bar{e}^0 = 0, \quad \bar{e}^1 = r, \quad \bar{e}^2 = 1,$$

$$\bar{f}^0 = 1, \quad \bar{f}^1 = 0, \quad \bar{f}^2 = 0, \quad \bar{f}_b = 0$$

$$\bar{\theta}_2 = \frac{6 m}{(r + m)^2}, \quad \bar{\theta}_2 = 0, \quad \bar{\theta}_b = 0, \quad \bar{\phi} = \frac{6 m}{r + m}, \quad \bar{\phi} = m.$$

The initial data for the conformal gauge unknowns is given by

$$\bar{\Theta_*} = \frac{1}{r + m}, \quad \bar{\dot{\Theta}_*} = 0, \quad \bar{\ddot{\Theta}_*} = \frac{(r + 2 m)^2}{(r + m)^2}.$$

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In particular, notice that \( \bar{\Theta}_*, \ \hat{\Theta}_* \neq 0 \) at \( r = 0 \).

The equations governing the evolution of the components of \( \bar{u} \) are formally identical to equations (37a)–(37l), (40) and (41). We write this system of equations and its initial data schematically as

\[
\partial_x \bar{u} = \bar{F}(\bar{u}, \tau, r; m), \quad \bar{u}(0, r; m) = \bar{u}_*(r; m)
\]

with \( \bar{F} \) and \( \bar{u}_*(r; m) \) analytic functions of its arguments. Also, observe that if \( m = 0 \), the initial data for the conformal factor is singular at \( r = 0 \). Thus, the behaviour of solutions to the system of transport equations (52) cannot be analysed directly by means of a perturbative argument as in the case of the cylinder at spatial infinity. Given a solution to system (52), one readily can construct an unphysical metric \( \bar{g} \) by analogy to (42) via

\[
\bar{g} = d\tau \otimes d\tau - \frac{e^0}{e^1_y} (d\tau \otimes d\rho + d\rho \otimes d\tau) - \left( \left( \frac{e^0}{e^1_y} \right)^2 - \left( \frac{\bar{e}^0}{\bar{e}^1_y} \right)^2 \right) d\rho \otimes d\rho = \frac{1}{\bar{e}^1_x e^1_y} \sigma. \tag{53}
\]

To obtain a suitable existence result that exhausts the whole of \( \bar{\mathcal{M}}_{a,\epsilon} \) for sufficiently small \( \bar{a} > 0 \), we exploit the conformal isometry \( \iota \) to map the solution \( u \) of system (46) on \( \mathcal{M}_{a,\epsilon} \) to the required solution \( \bar{u} \) of system (52). One obtains the following result:

**Theorem 2.** Given \( m > 0 \), there exists \( \bar{a} > 0 \) such that there exists a unique smooth solution to the spherically symmetric regular initial value problem at \( c^0 \) (or \( i^+ \)) for the extended conformal Einstein–Maxwell equations on the whole of \( \bar{\mathcal{M}}_{a,\epsilon} \). The surfaces \( \mathcal{H}^\pm \) corresponds to the horizons of the physical metric \( \bar{g} = \bar{\Theta}^{-2} \bar{g} \) with \( \bar{g} \) given by equation (53).

The proof of this, our main result, follows from the discussion of the following sections.

### 6.4. The action of \( \iota \) on \( \mathcal{M}_{a,\epsilon} \) and \( \bar{\mathcal{M}}_{a,\epsilon} \)

In view that the construction of the manifolds \( \mathcal{M}_{a,\epsilon} \) and \( \bar{\mathcal{M}}_{a,\epsilon} \) is based on the properties of conformal invariants, it is natural to expect a nice transformation between the objects on one manifold and the other under the action of the conformal isometry \( \iota \). In this section, we discuss these transformations. For conciseness, some of the details are put in an appendix at the end of the article. Recall that quantities near the horizon are distinguished from those near \( i^0 \) by an overbar, and physical quantities are denoted with a tilde over them. One can think of working with two identical manifolds, \( (\mathcal{M}, \bar{g}) \) and \( (\bar{\mathcal{M}}, \tilde{g}) \), where the points are interpreted differently. The horizon in one copy corresponds to the conformal boundary in the other, and so on. Under the conformal isometry \( \iota_* \bar{g} = \Omega^2 \tilde{g} \).

#### 6.4.1. The correspondence between the congruence of conformal geodesics

As already discussed in section 3.2, the conformal isometry maps conformal geodesics into conformal geodesics. More precisely, as a consequence of formula (16), conformal geodesics with initial data
are mapped into conformal geodesics satisfying
\[ q = i(p) \in \check{S}, \quad \check{x}_* = \check{x}(q), \quad \check{x}_* \perp \check{S}, \]
\[ \check{b}_* = i_\#(\omega^{-1}d\omega) + \Omega^{-1}d\mathcal{I} = \sigma^{-1}d\sigma. \]

This can be verified by an explicit computation noting that
\[ \omega^{-1}d\omega = \frac{2 + m\rho}{\rho(1 + m\rho)} d\rho = \frac{q + 2m}{q(Q + m)} dQ, \quad i_\#(\omega^{-1}d\omega) = \frac{r + 2m}{r(r + m)} dr, \quad \Omega^{-1}d\mathcal{I} = -\frac{1}{r} dr, \]
and that
\[ \sigma^{-1}d\sigma = \frac{m}{r(r + m)} dr. \]

As a consequence of the above discussion, the curves \( x(\tau) \) and \( \check{x}(\tau) \) with initial data as given above share the same affine parameter. Moreover, the radial coordinates \( \rho \) and \( r \) on the conformal extensions \( S^\rho \) and \( S^r \) are related to each other by
\[ \rho \mapsto r/m^2. \]

Thus, the conformal isometry \( i \) induces a correspondence between the conformal Gaussian coordinates of \( M_{a,\check{e}} \) and \( \check{M}_{a,\check{e}} \) which is given by
\[ (\tau, \rho, t^A_{\check{a}}) \mapsto (\tau, r/m^2, t^A_{\check{a}}), \quad t^A_{\check{a}} \in SU(2, \mathbb{C}). \]

Accordingly, one sees that
\[ i(F^\pm) = H^\pm, \quad i(I) = H, \quad i(I^\pm) = H^\pm, \]
and one has a natural bijection between \( M_{a,\check{e}} \) and \( \check{M}_{a,\check{e}} \) with \( \bar{a} = a/m^2 \). In what follows, it will be seen that this bijection is, in fact, an isometry.

6.4.2. The correspondence between the conformal factors. In order to relate the conformal factors \( \Theta \) and \( \check{\Theta} \) leading, respectively, to the representations of the cylinder at spatial infinity and the cylinders at the horizon, one starts by recalling that these are defined by the conditions
\[ \Theta^2\hat{g}(\hat{x}, \hat{x}) = 1, \quad \check{\Theta}^2\hat{\check{g}}(\check{x}, \check{x}) = 1. \tag{54} \]

Applying the inversion \( i \) to the first of these relations, and recalling that \( i_\#\hat{g} = i^2\hat{g} \), one sees that
\[ \left( \Theta^2\hat{g}(\hat{x}, \hat{x}) \right) \circ i = (\Theta \circ i)^2 i_\#\hat{g}(i_\#\hat{x}, i_\#\hat{x}) = (\Theta \circ i)^2 \Omega^2\hat{g}(\check{x}, \check{x}) = 1. \]

Comparing with the second relation in (54), one concludes that
\[ \check{\Theta} = (\Theta \circ i)\Omega. \tag{55} \]

In order to find a relation between the conformal metrics \( g \) and \( \hat{g} \), recall that
\[ g = \Theta^2\hat{g}, \quad \hat{g} = \check{\Theta}^2\hat{\check{g}}. \tag{56} \]
Writing the first of the relations in (56) in the form $\tilde{g} = \Theta^{-2} g$, one sees that
\[ t_0 \tilde{g} = t_0 \left( \Theta^{-2} g \right) = (\Theta \ast i)^{-2} t_0 g. \]

Recalling that $t_0 \tilde{g} = \Omega^2 \tilde{g}$, one concludes that
\[ t_0 g = (\Theta \ast i)^2 \Omega^2 \tilde{g}. \]

From the second relation in (56) in the form $\tilde{g} = \Theta^{-2} g$, one finds that
\[ t_0 g = (\Theta \ast i)^2 \Omega^2 \Theta^{-2} \tilde{g}. \]

Hence, using (55) one concludes that
\[ t_0 g = \tilde{g} \text{ and } t_0 \tilde{g} = g \] (57)

In other words, the conformal metrics $g$ and $\tilde{g}$ leading to the cylinder representations are related to each other by an isometry. In particular, if $\{ e_b \}$ denotes a $g$-orthogonal frame, then $\{ \tilde{e}_b \} \equiv \{ t_0 e_b \}$ is a $\tilde{g}$ orthogonal frame. If $\{ \omega_a \}$ is the associated co-frame of $\{ e_a \}$, then $\{ \omega^b \} \equiv \{ t_0 \omega^b \}$ is the associated co-frame of $\{ \tilde{e}_b \}$—i.e., $\langle \omega^b, \tilde{e}_b \rangle = \delta^b_a$.

6.4.3. Correspondence between the various conformal fields. In order to establish theorem 2, one needs to examine the explicit correspondence between the conformal fields appearing in the conformal field equations on $\mathcal{M}_{a,\kappa}$ and those appearing on the equations on $\mathcal{M}_{a,\bar{\kappa}}$.

Explicit computations show that
\[ f_a = t_0 f_a, \quad f^a_{\phantom{a}b} e = t_0 f^a_{\phantom{a}b} e, \]
\[ \tilde{a}^a_{\phantom{a}bcd} = \Omega^{-1} t_0 a^a_{\phantom{a}bcd}, \quad \tilde{L}_{ab} = t_0 L_{ab}, \quad \tilde{E}_a = t_0 E_a \]
where $t_0 f_a \equiv f_a \ast t$, etc. The details are given in the appendix. From these frame expressions, one readily obtains the transformation for the spinorial components. These are given by
\[ \tilde{f} = t_0 f, \quad \tilde{\xi}_1 = t_0 \xi_1, \quad \tilde{\xi}_2 = t_0 \xi_2, \quad \tilde{\xi}_h = t_0 \xi_h \]
\[ \tilde{\phi} = \Omega^{-1} t_0 \phi, \quad \tilde{\theta}_1 = t_0 \theta_1, \quad \tilde{\theta}_2 = t_0 \theta_2, \quad \tilde{\theta}_h = t_0 \theta_h, \quad \tilde{\varphi} = t_0 \varphi. \]

Remark. On the (unphysical) conformal fields, $t$ acts as an isometry, while for the physical variables $t$ acts as a conformal isometry.

As the fields
\[ f, \quad \xi_1, \quad \xi_2, \quad \xi_h, \quad \theta_1, \quad \theta_2, \quad \theta_h, \quad \varphi \]
are regular on $\mathcal{M}_{a,\kappa}$, it follows directly from the above transformation rules that
\[ \tilde{f}, \quad \tilde{\xi}_1, \quad \tilde{\xi}_2, \quad \tilde{\xi}_h, \quad \tilde{\theta}_1, \quad \tilde{\theta}_2, \quad \tilde{\theta}_h, \quad \tilde{\varphi} \]
are also regular on the whole of $\mathcal{M}_{a,\kappa}$. Moreover, it follows that $\eta_1 \neq 0$ on $\mathcal{M}_{a,\kappa}$ for suitably small $a > 0$, so no conjugate points arise in the congruence of conformal geodesics.

In order to conclude our analysis, all that is left to consider is the behaviour of $\tilde{\phi}$ and $\tilde{\Theta}$ at the horizon. This analysis requires information about the behaviour of the solutions at $H$.

6.5. The solution to the conformal field equations near the cylinders at the horizon

The behaviour of the solutions of the regular initial value problem formulated in section 6.3 on the cylinder $H$ is of particular interest.
Combining the expressions in (47) for the solutions of the regular initial value problem at spatial infinity with the transformation formulae discussed in the previous section, one readily finds that
\[ \xi = \xi_h = \chi_h = \partial_\xi = \partial_2 = \partial_h = 0 \quad \text{on } \mathcal{H}. \]

Under these conditions, the evolution equations for the conformal factor and the components of the Weyl and Maxwell spinors reduce to:
\begin{align*}
\dot{\phi} &= \dot{\Theta} \dot{\psi}^2, \quad (58a) \\
\dot{\phi} &= 0, \quad (58b) \\
\ddot{\Theta} &= 2\Theta^2 \ddot{\psi}^2. \quad (58c)
\end{align*}

Using the constraint equations for \( \Theta \) one finds that the initial data for the above equations is given by
\[ \bar{\phi}_s = -m, \quad \bar{\psi}_s = 0, \quad \bar{\Theta}_s = \frac{1}{m}, \quad \dot{\bar{\Theta}}_s = 0, \quad \ddot{\bar{\Theta}}_s = 0, \quad \text{on } \mathcal{H}^0. \]

It follows that the solution to the above equations is constant along \( \mathcal{H} \). Accordingly, one sees that
\[ \bar{\phi} = -m, \quad \bar{\psi} = 0, \quad \bar{\Theta} = \frac{1}{m}, \quad \text{on } \mathcal{H}. \]

Notice, in particular that the conformal factor \( \Theta \) does not vanish on the cylinder \( \mathcal{H} \). Thus, \( \mathcal{H} \) is not part of the conformal boundary. Moreover, by continuity, it follows that at least suitably close to \( \mathcal{H} \), \( \Theta \neq 0 \) on \( \mathcal{H}^+ \).

6.5.1. **Behaviour of \( \bar{\psi} \) at the horizon.** Finally, to conclude the analysis, we consider the behaviour of \( \bar{\psi} \) on \( \mathcal{H}^+ \). To this end, recall the transformation formula \( \bar{\psi} = \Omega^{-1} \psi_\ast \). To show that \( \bar{\psi} \) is regular on \( \mathcal{H}^+ \), consider the analogue of the core system equation (38b) on \( \mathcal{H}_\ast \):
\[ \partial_t \bar{X} = -\frac{1}{r} \bar{X}^2 - \bar{L} + \frac{1}{2} \Phi \bar{\psi}. \]

As \( \bar{X}, \bar{L} \) and \( \Phi \) are regular on \( \mathcal{H}^+ \), and, moreover, \( \Phi \neq 0 \) on \( \mathcal{H}^+ \) is at least sufficiently close to \( \mathcal{H} \), it follows that \( \bar{\psi} \) must be regular at \( \mathcal{H}^+ \), at least close enough to \( \mathcal{H} \). In fact, one can say a bit more by noticing that \( \Omega^{-1} = r/m \) so that \( \Omega^{-1} = 0 \) at the horizon. As \( \psi \) is regular on \( \mathcal{I}^+ \), it follows that \( \psi_\ast \phi \) is regular at \( \mathcal{H}^+ \) and \( \bar{\psi} \) vanishes at the horizon.

With this argument, we conclude the proof of theorem 2.

7. **Concluding remarks and perspectives**

The main conclusion of our analysis is that it is possible to obtain regular conformal representations of the extremal Reissner–Nordström spacetime in a neighbourhood of the points \( \iota^+ \) and \( \epsilon^0 \). These conformal representations have been obtained as the solution of the conformal field equations written in terms of a gauge based on conformal invariants of the spacetimes (conformal geodesics). It is important to point out that this representation is regular for the variables appearing in the conformal field equations, but not for the associated conformal (unphysical) metric. This peculiarity is not a major source of problems, as the metric is not one of the field unknowns for which one is solving.
Our construction of the cylinders at the horizon and the subsequent analysis have been eased by the conformal discrete isometry in the extremal Reissner–Nordström, inducing an isometry in the unphysical setting. However, we claim that the existence of this (conformal) isometry is not essential for the analysis. It is just a convenient property to shorten some of the arguments and to gain insight into the underlying structures. In particular, we claim it should be possible to obtain an analogous representation for the extremal Kerr spacetime.

In the remainder of this section, we discuss some possible implications of the present construction of the cylinders at the horizon.

Behaviour of test fields near the cylinders at the horizon

The key insight obtained from the construction of the cylinder at spatial infinity for vacuum spacetimes given in [15] is the existence of logarithmic singularities at the critical sets \( \pm \mathcal{I} \) for generic initial data. Given that null infinity is a characteristic of the field equations, it is expected that these singularities will spread along \( \mathcal{I} \), thus giving rise to a conformal boundary that is non-smooth. Although this picture seems quite plausible, there is no proof available for this conjecture. There is, however, some analysis with linear test fields that suggests how the full non-linear case could be controlled—see [19, 39].

Although an analysis such as the one described in the previous paragraph has not been carried out for electrovacuum spacetimes, it is reasonable to expect a similar behaviour at the critical sets of the cylinder at spatial infinity. More precisely, given the spin-1 zero-rest mass field equation

\[
\lambda^{\alpha} \lambda^{\beta} \lambda^{\gamma} = \partial_\alpha \lambda + A^{\alpha \beta} e^{\gamma}_{\alpha \beta} \partial_\beta \lambda = \mathbf{B}(I') \lambda.
\]

Here, \( \lambda \) is a column vector with complex valued components \( \lambda_0, \lambda_1, \lambda_2 \) (the independent components of \( \lambda_{AB} \)), and where \( A^{\alpha \beta} \) are constant matrices, \( \mathbf{B}(I') \) denotes a linear matrix value function of the connection coefficients, and \( E \) is the \( 3 \times 3 \) unit matrix. Using the information about the extremal Reissner–Nordström spacetime on the cylinder at spatial infinity, one readily finds that

\[
\left( \sqrt{2} E + A^{\alpha \beta} e^{\gamma}_{\alpha \beta} \right) \lambda^{\alpha} = \sqrt{2} \left( \begin{array}{ccc}
1 + \tau & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 - \tau
\end{array} \right) \quad \text{on} \quad H.
\]

Thus, the symmetric hyperbolic system (60) degenerates at the points for which \( \tau = \pm 1 \). It is likely that this degeneracy will give rise to logarithmic singularities in the components of \( \lambda \) as the solutions approach \( H^\pm \) along \( H \). This feature is expected to appear in both the cylinders at \( i^+ \) and \( e^0 \). The effects of this degeneracy in the evolution equations and the associated singular behaviour will be analysed in detail elsewhere.

If, as anticipated, the correspondence between the cylinders \( I \) at spatial infinity and the cylinders \( H \) at the horizon holds fully, it is to be expected that these singularities will give rise polyhomogeneous behaviour of the test fields at the horizon. This potential non-smoothness of generic test fields at the horizon may be related to the existence of conserved modes at the horizon in solutions of the wave equation on the extremal Reissner–Nordström spacetime [2, 3].

A more intriguing possibility is the idea of performing an analysis of the full conformal Einstein equations at the cylinders at the horizon in the manner of [15] for an initial data set.
that is a perturbation of data for the extremal Reissner–Nordström spacetime. Although purely asymptotic, this analysis would involve the full non-linearities of the Einstein field equations. Consequently, it should provide valuable insights and have implications for the question of the non-linear stability of the extremal Reissner–Nordström spacetime.

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Appendix A. Transformation formulae under the conformal inversion

The purpose of the present appendix is to provide some further details about the derivation of the transformation formulae of various conformal fields under the conformal isometry \( \iota \) given in section 6.4.3. The key observation in these computations is the isometry \( \iota = g \Sigma \bar{g}^* \).

Transformation of the Weyl tensor

Exploiting the conformal invariance of the Weyl tensor, one sees that

\[
C^a_{bcd}(\tilde{g}) = C^a_{bcd}[\tilde{\Theta}^2 \tilde{g}] = C^a_{bcd}[\iota \tilde{g}] = \iota_s C^a_{bcd}[\tilde{g}] = \iota_s C^a_{bcd}[g],
\]

where the last equality follows from the fact that \( \iota \) is an isometry. Of more interest for the conformal field equations is the rescaled Weyl tensor \( d^a_{bcd} \). Proceeding as in the case of the standard Weyl tensor, one sees that

\[
d^a_{bcd}(\tilde{g}) = d^a_{bcd}[\tilde{\Theta}^2 \tilde{g}] = \tilde{\Theta}^{-1} C^a_{bcd}[\tilde{\Theta}^2 \tilde{g}] = \tilde{\Theta}^{-1} \iota_s C^a_{bcd}[\tilde{g}]
\]

\[
= \Omega^{-1} (\Theta \ast \iota)^{-1} \iota_s C^a_{bcd}[g] = \Omega^{-1} \iota_s \left( \Theta^{-1} C^a_{bcd}[g] \right) = \Omega^{-1} \iota_s d^a_{bcd}[\tilde{\Theta}^2 \tilde{g}],
\]

so that one sees \( d^a_{bcd}(\tilde{g}) = \Omega^{-1} \iota_s d^a_{bcd}[g] \). From the above expressions, one can compute the transformation of \( d^a_{bcd} \) and the components of \( a^a_{bcd} \) with respect to the frame \( \{e_a\} \). One finds that

\[
\tilde{a}^a_{bcd} \equiv \tilde{a}^a_{bcd} \equiv \tilde{\omega}^a_{abcd} e^c \tilde{e}_d \tilde{d} \tilde{a}^a_{bcd} = \Omega^{-1} \iota_s \omega^a_{abcd} e^c \tilde{e}_d \tilde{d} a^a_{bcd}
\]

\[
= \Omega^{-1} (d^a_{bcd} \ast \iota) = \Omega^{-1} \iota_s d^a_{bcd}.
\]

The transformation rule for the spinorial components follows directly from these expressions.

Transformation law for the electric field

In order to analyse the transformation law of the electric field, it is recalled that the Faraday tensor is conformally invariant—that is, one sees that \( \tilde{F}_{ab} = F_{ab} \). Because of this invariance, one sees that.
\[ E_a \equiv E_a[\mathbf{g}] = F_{ab}[\mathbf{g}] x^b = F_{ab}[\Theta^2 \mathbf{g}] x^b = F_{ab}[t_a \mathbf{g}] x^b = t_a F_{ab}[\mathbf{g}] x^b = t_a E_a = t_a E_a. \]

A similar expression holds for the components with respect to \( \{ e_a \} \) and its spinorial counterpart:

\[ E_a = t_a E_a, \quad \phi_{AB} = t_a \phi_{AB}, \]

so that, in particular, one sees that \( \hat{\phi} = \phi \ast t = t_a \phi. \)

**Transformation law for the connection**

From the discussion in the main text, one sees that

\[ f = b - \Theta^{-1} d\Theta, \quad f = \hat{b} - \Theta^{-1} d\Theta. \]

Hence, it readily follows that

\[ \hat{f} = t_a \hat{b} - (\Theta \ast t)^{-1} d(\Theta \ast t), \]

so that

\[ \hat{f} = t_a f. \]

Hence, in particular, if \( \langle f, \hat{x} \rangle = 0 \), then \( \langle \hat{f}, \hat{x} \rangle = 0 \). To compute the transformation of the connection coefficients, recall that

\[ \Gamma^a_{\ b \ c} = \{ \omega_a, V_a e_c \}, \quad \hat{\Gamma}^a_{\ b \ c} = \{ \hat{\omega}_a, \hat{V}_a e_c \}. \]

Hence, using the results from the previous subsections, it readily follows that

\[ \hat{\Gamma}^a_{\ b \ c} = t_a \Gamma^a_{\ b \ c} = \Gamma^a_{\ b \ c} \ast t. \]

The spinorial components of \( \Gamma^a_{\ b \ c} \) transform accordingly.

**Transformation law for the Schouten tensor**

The transformation rule of the Schouten tensor follows a similar procedure as for the other geometric objects. One sees that

\[ L_{ab}[\mathbf{g}] = L_{ab}[t_a \mathbf{g}] = t_a L_{ab}[\mathbf{g}]. \]

We write the above more concisely as \( L_{ab} = t_a L_{ab} \). Moreover, for the Schouten tensors of the corresponding Weyl connections, one notices that

\[ \hat{L}_{ab} = \hat{L}_{ab} + \hat{V}_a f_b - \frac{1}{2} \hat{\mathbf{g}}_{ac} \hat{c}^d f_a f_d, \]

\[ = t_a L_{ab} + t_a \left( V_a f_b \right) - \frac{1}{2} t_a \mathbf{g}_{ab} \hat{c}^d t_a f_a t_a f_d = t_a \hat{L}_{ab}. \]

In particular for the components with respect to the Weyl propagated frame, one sees that \( \hat{L}_{ab} = t_a \hat{L}_{ab} = \hat{L}_{ab} \ast t. \)

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