Partial source separation from unknown correlation mixture for eliminating unknown periodic disturbances from random measured signals

Zu-Guang Ying and Yi-Qing Ni

1 Department of Mechanics, School of Aeronautics and Astronautics, Zhejiang University, Hangzhou 310027, People’s Republic of China
2 Hong Kong Branch of National Rail Transit Electrification and Automation Engineering Technology Research Centre; Department of Civil and Environmental Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

Abstract

Separating and eliminating periodic disturbances from measured signals are a key problem to obtain original responses used for further system identification and evaluation. Actual periodic disturbances are partial unknown sources in measured signals and have certain correlation with random noise sources in time domain. In this paper, a separation problem on partial unknown sources such as periodic sources correlated with random noises is introduced. A partial unknown source separation technique is proposed by combining signal eigenspace transformation, covariance joint diagonalization and decorrelation of correlation sources. The partial source separation procedure has two main stages: obtain uncorrelated sources by eigenspace transformation and joint diagonalization; and obtain partial periodic sources correlated with random noises from the uncorrelated sources by decorrelation. The proposed partial source separation technique is supported by several theorems. Under given assumptions, the separation technique will result in accurate partial sources. The separation technique has main features such as partial unknown sources separated from measured signals, separated periodic sources correlated with random noise sources, and being suitable for dominant random noises and non-dominant periodic disturbance sources in measured signals. Numerical results are presented to illustrate the effectiveness of the separation technique.

1. Introduction

Structural health monitoring or damage diagnosis is an important problem in engineering. Using structural dynamic responses for identifying modal feature variation is a common way to solve the problem [1–4]. A complex large-scale structure system in complicated dynamic surroundings has undeterminable model with excitation and only responses are usable. Disturbed or distorted response signals used will result in unreliable feature identification. For example, vibration response signals measured from a maglev train are inevitably disturbed by intensive electromagnetic fields which provide power and support. The electromagnetic interference is generally periodic with high frequency. Hence, separating and eliminating the periodic disturbances from the measured signals are a key problem to obtain original responses used for further identification. The periodic disturbances are partial unknown sources in the measured signals. The response signals commonly contain strong random noise sources which have certain correlation with deterministic sources in time domain. Therefore, that is a separation problem on partial unknown sources where there is correlation between noises and determinacy.

Conventional blind source separation problem has been studied and several separation techniques were presented as well as image recognition [5–13]. For example, the blind source separation technique using signal
second-order statistics can effectively extract simultaneous uncorrelated sources and mixture [7]. The separation procedure is composed of the singular value decomposition of signal covariance matrix and the joint diagonalization of transformed covariance matrices with various time lags. In this technique, it is assumed that all sources are fully uncorrelated and the intensity of white noises is relatively small for removal. The blind source separation technique has been applied to structural mode identification and anomaly detection based on extracted mixture by using response second-order statistics [14–20]. In those researches, all sources were separated except for small noises removed and particularly the sources were assumed as fully uncorrelated. However, actual structural vibration responses commonly contain strong random noises which characterize system dynamics. The random noises have certain correlation with deterministic sources such as periodic in time domain. These sources in measured response signals are dependent or correlative. Thus, the uncorrelated sources obtained by the blind source separation technique will be a combination of the correlated sources. That is each uncorrelated source may contain certain components of the correlated sources. A further decorrelation procedure is necessary, but all correlated sources are difficult to obtain without other conditions. Nevertheless, the separation of partial unknown correlated sources is possible, where deterministic sources such as periodic are uncorrelated and they have certain correlation with random noise sources. The periodic sources are expected to separate and eliminate from the measured signals.

In the present paper, a separation problem on partial unknown sources is introduced in which random noises have certain correlation with deterministic such as periodic sources. A new partial unknown source separation technique is proposed by combining signal eigenspace transformation, covariance joint diagonalization and decorrelation of correlation sources. The partial source separation procedure has two main stages: first, obtain uncorrelated sources by eigenspace transformation and joint diagonalization; and second, obtain partial periodic sources correlated with random noises from the uncorrelated sources by decorrelation. The proposed partial source separation technique is supported by several theorems. In section 2, the partial separation problem is described and basic assumptions are presented. In section 3, analytical expressions on signal eigenspace transformation and covariance joint diagonalization are presented. Several theorems are proved to support the expressions. A condition on the number of time lags in the joint diagonalization is explained. Section 4 proposes the decorrelation analysis including separating periodic disturbance sources correlated with random noises and extracting corresponding mixture matrix. The original response signals are recovered by eliminating the periodic disturbances from the measured signals based on relevant theorems. The effect of estimated correlation functions between periodic disturbances and random noises on the separation results is analyzed. The indices of relative differences and similarity for separation results are presented. Numerical results illustrate the effectiveness of the proposed separation technique presented in section 5.

2. Problem description

2.1. Partial source separation

Consider measured signal $x_i(t)$ ($i = 1, 2, \ldots, n$), which is a mixture of system vibration response source $e_{ij}(t)$ ($j = 1, 2, \ldots, N_s$) and periodic disturbance source $e_p(t)$ ($j = 1, 2, \ldots, N_p$) as follows:

$$X(t) = HE(t) = H_e(t) + H_p(t)$$

(1)

where $t$ is time and

$$X(t) = \begin{bmatrix} x_1(t), & x_2(t), & \ldots, & x_n(t) \end{bmatrix}^T, \quad E(t) = \begin{bmatrix} E_e^T(t), & E_p^T(t) \end{bmatrix}^T, \quad H = [H_e, \ H_p]$$

$$E_e(t) = \begin{bmatrix} e_{1}(t), & e_{2}(t), & \ldots, & e_{N_s}(t) \end{bmatrix}^T, \quad E_p(t) = \begin{bmatrix} e_{1p}(t), & e_{2p}(t), & \ldots, & e_{N_p}(t) \end{bmatrix}^T$$

$$H_e = \begin{bmatrix} h_{e,11} & h_{e,12} & \cdots & h_{e,1N_s} \\
     h_{e,21} & h_{e,22} & \cdots & h_{e,2N_s} \\
     \vdots & \vdots & \ddots & \vdots \\
     h_{e,N_s1} & h_{e,N_s2} & \cdots & h_{e,N_sN_s} \end{bmatrix}, \quad H_p = \begin{bmatrix} h_{p,11} & h_{p,12} & \cdots & h_{p,1N_p} \\
     h_{p,21} & h_{p,22} & \cdots & h_{p,2N_p} \\
     \vdots & \vdots & \ddots & \vdots \\
     h_{p,N_p1} & h_{p,N_p2} & \cdots & h_{p,N_pN_p} \end{bmatrix}$$

(2)

$E_e(t)$ and $E_p(t)$ are unknown response source vector and disturbance source vector, respectively. $H$ is an unknown mixture matrix representing source mixing weight. The partial unknown source separation problem is to use the measurement signal $X$ to extract the disturbance source $E_p$ and mixture $H_p$. The extracted $E_p$ and $H_p$ are eliminated from the signal $X$ to obtain original responses.

Signal $X$ contains generally periodic $w(t)$, oscillatory decayed $w(t)$ and random noise $w(t)$. Denote a correlation function in time domain by

$$R_{w_1w_2}(\tau) = \langle w_1(t), w_1^\tau(t + \tau) \rangle = \frac{1}{T_p} \int_0^{T_p} w_1(t)w_1^\tau(t + \tau)dt$$

(3)
where $\tau$ is a time lag and $T_R$ is time length. If a process has zero mean, its correlation function is equal to covariance. For an ergodic random process, its statistics in time domain will tend towards those over event space. The auto-correlation functions of the periodic, decayed and zero-mean random processes can be obtained respectively as

$$R_{w_iw_i}(\tau) = \frac{1}{2} a_i^2 \cos \omega_i \tau$$

(4)

$$R_{w_iw_j}(\tau) = 0$$

(5)

$$R_{w_iw_i}(\tau) = \sigma_i^2(\tau)$$

(6)

where $a_i$ and $\omega_i$ are amplitude and frequency, respectively, and $\sigma_i^2$ is covariance. For a pure random process, its correlation time is equal to zero and hence, $\sigma_i = 0$ for $\tau = 0$. If the periodic $w_i(t)$ and decayed $w_j(t)$ have different frequencies and the random $w(k)$ is uncorrelated with the others, the cross-correlation functions obtained are

$$R_{w_iw_j}(\tau) = R_{w_iw_j}(\tau) = R_{w_iw_j}(\tau) = 0$$

(7)

$$R_{w_iw_j}(\tau) = R_{w_iw_j}(\tau) = R_{w_iw_j}(\tau) = 0$$

(8)

As the system response can contain periodic and non-periodic sources, $N_p + N_d$ sources are divided into $N_p$ periodic and $N_d$ non-periodic uncorrelated sources ($N_p + N_d = N$). The number $n$ of signals is determined by number of sensors in measurement. The number $N_s$ of response sources is corresponding to number of excitation sources and $N_d$ is number of other disturbance sources such as measurement noises and electromagnetic interference. As the analysis with equations (3)–(8), two source processes are uncorrelated if their correlation function in time domain (3) is equal to zero. For example, periodic sources with different frequencies and infinite time lengths are uncorrelated. $N$ is corresponding to number of the uncorrelated sources. However, the sources and mixed signals used have actually finite time lengths with non-ideal factors and thus the sources have certain correlation. To separate sources from measured signals, many sensors are used and then the number $n$ of signals is usually not smaller than the number $N$ of sources.

2.2. Assumptions

For an actual random process, it has generally certain correlation with deterministic processes in time domain. Their cross-correlation functions are unequal to zero. However, a partial source separation procedure can be two stages: first, to obtain uncorrelated sources $E_i(t)$, and second, to obtain partial periodic sources $F_j(t)$ correlated with random noises. It is assumed for the first stage that: (I) all sources are uncorrelated with each other based on equations (7) and (8); (II) disturbances to system response are periodic sources; (III) number $n$ of measurement signals is not smaller than number $N$ of uncorrelated sources; (IV) mixture matrix in equation (1) has full column rank. For the second stage, the correlation between random and periodic sources will be considered to replace the first assumption. The assumptions are based on many practical problems including dynamic modelling under multiple periodic excitations with random noises and signal processing such as electromagnetic interference elimination of random vibration signals measured from maglev trains.

3. Signal eigenspace transformation and covariance joint diagonalization

The blind source separation technique using signal second-order statistics has been proposed. However, as a part of the partial source separation procedure, different expressions on signal eigenspace transformation and covariance joint diagonalization based on the assumptions (I)–(IV) are given in the following.

3.1. Signal eigenspace transformation

For time lag $\tau = 0$, the correlation function matrix of measured signals is expressed as

$$R_{XX}(0) = HR_{EE}(0)H^T = H\rho_1 R_{\rho_1}(0)H^T + H\rho_2 R_{\rho_2}(0)H^T$$

(9)

where subscripts $I$ and $J$ denote periodic and non-periodic sources, respectively, and

$$R_{EE}(0) = \text{diag}[\rho_1(0), \rho_2(0), ..., \rho_N(0)]$$

$$R_{\rho_1}(0) = \text{diag}[\rho_1(0), \rho_2(0), ..., \rho_{N_1}(0)]$$

$$R_{\rho_2}(0) = \text{diag}[\rho_1(0), \rho_2(0), ..., \rho_{N_1}(0)]$$

$$H = [H_1, H_2], E(t) = [E_1(t), E_2(t)]^T$$

(10)

in which $N$ is number of sources, $N_p$ and $N_d$ are numbers of periodic and non-periodic sources, respectively. By singular value decomposition, the correlation function matrix is expressed as
where \(U_o\) is unitary matrix and \(\Lambda_o\) is diagonal matrix with eigenvalues. The following proposition can be inferred.

**Proposition 1.** Based on the assumptions (I)–(IV), letting \(\Lambda_o\) be an eigenvalue matrix defined by singular value decomposition of the correlation function matrix \(R_{XX}(0)\) and \(R_{EE}(0)\) be the correlation function matrix of sources \(E(t)\), then the rank of \(\Lambda_o\) is equal to the rank of \(R_{EE}(0)\).

**Proof.** By combining equations (9) with (11), the correlation function of \(X\) is

\[
R_{XX}(0) = U_o \Lambda_o U_o^T = H R_{EE}(0) H^T
\]

\(U_o\) is a non-singular unitary matrix. Then the rank of \(R_{XX}(0)\) is equal to that of \(\Lambda_o\). The mixture matrix \(H\) has full column rank. Then the rank of \(R_{XX}(0)\) is equal to that of \(R_{EE}(0)\). Therefore, the rank of \(\Lambda_o\) is equal to the rank of \(R_{EE}(0)\).

From proposition 1, the number of non-zero elements or eigenvalues in \(\Lambda_o\) is equal to that in \(R_{EE}(0)\), which is \(N_I + N_J\).

Rearrange \(\Lambda_o\) and \(U_o\) as

\[
\Lambda_o = \text{diag} \{\Lambda_A, \Lambda_Z\}
\]

where subscripts \(Z\) and \(A\) denote zero and non-zero parts of eigenvalues in \(\Lambda_o\). The singular value decomposition (11) becomes

\[
U_A \Lambda_A U_A^T = U_A \Lambda_A U_A^T
\]

By normalizing sources \(E(t)\) with respect to correlation function matrix \(R_{EE}(0)\), equation (9) becomes

\[
H R_{EE}(0) H^T = H_n R_{EE}(0) H_n^T = H_n H_n^T
\]

where \(E_n(t)\) is normalized sources, \(H_n\) is corresponding mixture matrix and

\[
H_n = H R_{EE}^{1/2}(0)
\]

Introduce a transformation matrix

\[
T = U_A \Lambda_A^{1/2}
\]

Its generalized inverse is denoted by \(T^#\). There is the following proposition.

**Proposition 2.** Let \(\Lambda_A\) be non-zero eigenvalue diagonal matrix defined by singular value decomposition of the correlation function matrix \(R_{XX}(0)\), \(U_A\) be corresponding part of unitary matrix \(U_o\), \(R_{EE}(0)\) be diagonal correlation function matrix of normalized sources \(E_n\) and \(H_n\) be corresponding mixture matrix. The transformation matrix \(T\) is defined by equation (17). Then \(T^# H_n\) is a unitary matrix, i.e., \((I_N\) is identity matrix)

\[
T^# H_n (T^# H_n)^T = I_N
\]

**Proof.** The generalized inverse of \(T\) (17) is

\[
T^# = \Lambda_A^{-1/2} U_A^#
\]

where superscript \(^#\) denotes generalized inverse. Based on equations (12)–(17), \(T^# H_n\) is a square matrix. By using equations (16), (12) and (14),

\[
T^# H_n (T^# H_n)^T = T^# H_n H_n^T (T^#)^T = T^# H R_{EE}(0) H^T (T^#)^T
\]

\[
= T^# R_{XX}(0)(T^#)^T = T^# U_A \Lambda_A U_A^T (T^#)^T = T^# (T T^T)(T^#)^T
\]

\[
= I_N
\]

Thus, \(T^# H_n\) is a unitary matrix.

Matrix \(T\) can be used to transform signal \(X\) into its eigenspace, but sources are not separated yet. Mixture matrix \(H_n\) cannot be determined only by the unitary matrix \(T^# H_n\). It is necessary to use properties of the correlation function matrix \(R_{XX}(\tau)\) with time lag \(\tau\).

### 3.2. Covariance joint diagonalization

The measured signal \(X\) are firstly transformed into its eigenspace as

\[
Z(t) = T^# X(t) = T^# H E(t) = T^# H_n E_n(t)
\]

For a time lag \(\tau > 0\), the following proposition and theorems can be inferred.
Proposition 3. Based on the assumptions (I)–(IV), letting $R_{ZZ}(\tau)$ be the correlation function matrix of transformed signal $Z$, \((21)\), then $R_{ZZ}(\tau)$ has the singular value decomposition with unitary matrix $T^nH_n$ where $H_n$ is the mixture matrix \((16)\) corresponding to normalized sources $E_n(t)$.

Proof. Based on equation \((21)\), the correlation function of $Z$ is

$$R_{ZZ}(\tau) = T^nR_{XX}(\tau)(T^n)^\dagger = T^n\mu R_{EE}(\tau)(T^n)^\dagger$$

$$= T^nH_nR_{E,E}(\tau)(T^n)^\dagger$$ \((22)\)

As given by expression \((10)\), the correlation function of $E_n$ is

$$R_{E,E}(\tau) = diag[\rho_1(\tau)/\rho_1(0), \rho_2(\tau)/\rho_2(0), \ldots, \rho_N(\tau)/\rho_N(0)]$$ \((23)\)

From proposition 2, $T^nH_n$ is a unitary matrix. Thus, the right side of equation \((22)\) is the singular value decomposition of $R_{ZZ}(\tau)$ with unitary matrix $T^nH_n$.

The eigenvalue of the correlation function matrix $R_{ZZ}(\tau)$ generally varies with time lag $\tau$, and it is 1 for $\tau = 0$. Based on proposition 3, $T^nH_n$ can be determined by the condition of unitary matrix for all time lags. However, $T^nH_n$ is invariant to various time lags while $R_{ZZ}(\tau)$ varies with time lag. It may result in numerical indeterminacy that the unitary matrix is calculated directly by all singular value decomposition of $R_{ZZ}(\tau)$ with various time lags. The joint diagonalization of $R_{ZZ}(\tau)$ with various $\tau$ is an effective approach to determine $T^nH_n$.

The joint diagonalization of $R_{ZZ}(\tau)$ is to find a unitary matrix $V$ which makes $VR_{ZZ}(\tau)V^T$ diagonal or nondiagonal elements of $VR_{ZZ}(\tau)V^T$ close to zero. It can be implemented by minimizing the absolute nondiagonal elements of $VR_{ZZ}(\tau)V^T$, i.e.,

$$\sum_{1 \leq i < j \leq N} |[VR_{ZZ}(\tau)V^T]_{ij}|^2 \rightarrow \min$$ \((24)\)

The joint diagonalization with infinite various time lags is unpractical, and finite time lags $\tau = \tau_i, i = 1, 2, \ldots, N_d$ are considered. Under certain conditions, the joint diagonalization with adequate time lags is sufficient for determining the unitary matrix $V$. The diagonalization of $R_{ZZ}(\tau)$ is expressed as

$$VR_{ZZ}(\tau)V^T = \tilde{R}_{E,E}(\tau)$$ \((25)\)

Let

$$V^nH_n = [S_1, S_2, \ldots, S_N], S_j = [s_{ij}, s_{ij}, \ldots, s_{ij}]$$ \((26)\)

Then

$$VR_{ZZ}(\tau)V^T = V^nH_nR_{E,E}(\tau)(V^nH_n)^\dagger$$

$$= \sum_{j=1}^N \frac{\rho_j(\tau)}{\rho_j(0)} S_jS_j^\dagger$$ \((27)\)

For zero nondiagonal elements under $N_t$ time lags, equation \((27)\) has $N_d \times N \times (N - 1)$ algebraic equations of $s_{ij}(k = j)$. The condition of unitary matrix $V^nH_n$ without unit diagonal elements has $N \times (N - 1)$ algebraic equations of $s_{ij}$. These equations are solved to obtain variables from $N \times N \times (N - 1)$ products except for squares. $N$ sums of square variables are determined by $N$ equations from the condition of unit diagonal elements of the unitary matrix. The equations from joint diagonalization and unitary matrix may be dependent. Hence, the necessary condition for $N_d$ is derived as

$$N_d \geq N - 1$$ \((28)\)

Theorem 1. Based on the independence of equations from the joint diagonalization $VR_{ZZ}(\tau)V^T$ and unitary matrix $V^nH_n$ and condition \((28)\), let diagonal $\tilde{R}_{E,E}(\tau)$ be defined by the joint diagonalization \((25)\) and $R_{E,E}(\tau)$ be the diagonal correlation function matrix of normalized sources $E_n(t)$. The two diagonal matrices have corresponding diagonal elements otherwise only the row order of the unitary matrix is adjusted. Then the two diagonal matrices are identical, i.e.,

$$R_{E,E}(\tau) = \tilde{R}_{E,E}(\tau)$$ \((29)\)

Proof. According to the above joint diagonalization, there is a unitary matrix $V$ which satisfies equation \((25)\). Based on equations \((21)\) and \((22)\), \((25)\) becomes
VRZZ(τ)VT = VT†HnR_{E,E}(τ)(VT†Hn)† = \tilde{R}_{E,E}(τ)  \tag{30}

From proposition 2, T†Hn is a unitary matrix. Then VT†Hn is a unitary matrix.

The eigenvalues of the diagonal matrix \tilde{R}_{E,E}(τ) are equal to diagonal elements. Because the unitary matrix VT†Hn is nonsingular, the eigenvalues of VT†HnR_{E,E}(τ)(VT†Hn)† are equal to the eigenvalues of \tilde{R}_{E,E}(τ).

The eigenvalues of the diagonal matrix R_{E,E}(τ) are equal to diagonal elements. Therefore, the eigenvalues of VT†HnR_{E,E}(τ)(VT†Hn)† are equal to the diagonal elements of \tilde{R}_{E,E}(τ). Equation (30) leads to VT†HnR_{E,E}(τ)(VT†Hn)† or R_{E,E}(τ) and \tilde{R}_{E,E}(τ) have identical eigenvalues or diagonal elements. Thus, the two diagonal matrices are identical, that is equation (29).

**Theorem 2.** Based on the independence of equations from the joint diagonalization VRZZ(τ)VT and unitary matrix VT†Hn and condition (28), let diagonal \tilde{R}_{E,E}(τ) be defined by the joint diagonalization (25), where V is the unitary matrix, T† is the transformation matrix and Hn is the mixture matrix corresponding to normalized sources E(t). The diagonal matrices R_{E,E}(τ) and \tilde{R}_{E,E}(τ) have corresponding diagonal elements otherwise only the row order of the unitary matrix is adjusted. Then VT†Hn is an identity matrix with diagonal elements ±1, i.e.,

\[
VT†Hn = \pm I_N  \tag{31}
\]

**Proof.** From theorem 1, the joint diagonalization (30) leads to

\[
VRZZ(τ)VT = VT†HnR_{E,E}(τ)(VT†Hn)† = \tilde{R}_{E,E}(τ) = R_{E,E}(τ)  \tag{32}
\]

Equation (27) is rewritten as

\[
VRZZ(τ)VT = VT†HnR_{E,E}(τ)(VT†Hn)† = \sum_{j=1}^{N} \rho_j(τ)S_jS_j^T  \tag{33}
\]

with

\[
R_{E,E}(τ) = \text{diag}[\rho_1(τ)/\rho_1(0), \rho_2(τ)/\rho_2(0), ..., \rho_N(τ)/\rho_N(0)]
= \text{diag}[\mu_1(τ), \mu_2(τ), ..., \mu_N(τ)]  \tag{34}
\]

The combination of equations (33) with (32) yields

\[
\sum_{j=1}^{N} \mu_j(τ)S_jS_j^T = \text{diag}[\mu_1(τ), \mu_2(τ), ..., \mu_N(τ)]  \tag{35}
\]

or

\[
\mu_1(τ)s_1s_1^T + \mu_2(τ)s_2s_2^T + ... + \mu_N(τ)s_Ns_N^T = \mu_k(τ)\delta_{kl}
\]

\[
i = 0, 1, 2, ..., N, \quad \tau_0 = 0, \quad \mu_j(\tau_0) = 1  \tag{36}
\]

For k = l, equation (36) forms (N_d + 1) × N × (N – 1) linear algebraic equations for N × N × (N – 1) variables of s_{ij} × \delta_{ij}. Under the assumption of equation independence and condition (28), the equations have the unique solution

\[
s_{kj}\delta_{ij} = 0, \quad k, i = 1, 2, ..., N, \quad k = l  \tag{37}
\]

For k = l, equation (36) has (N_d + 1) × N linear algebraic equations for N × N variables of s_{ij} × \delta_{ij}. Under the same assumption as the above, the solution to the equations is

\[
s_{ij}^2 = \delta_{ij}, \quad k, j = 1, 2, ..., N  \tag{38}
\]

By combining equation (38) with (37), it is obtained that the unitary matrix VT†Hn with elements s_{ij} is an identity matrix with diagonal elements ±1, that is expression (31).

If the unitary matrix VT†Hn is restricted by such as positive definite, it has diagonal elements +1. Thus, VT†Hn = I_N. The mixture matrix is obtained as

\[
H_n = TVT = U_AA_d^{1/2}V^T  \tag{39}
\]

As the mixture matrix H_n has been extracted as equation (39) by the signal eigenspace transformation and covariance joint diagonalization, the corresponding normalized sources can be separated from the signal X using the generalized inverse of H_n, but the separated sources are fully uncorrelated as the assumption (I). However, actual random noises have certain correlation with deterministic or periodic sources and therefore, the expected original sources correlate are not the separated uncorrelated sources. The extracted mixture and separated sources are unexpected practical the original and further decorrelation analysis is necessary.
4. Decorrelation analysis of correlation sources and partial unknown source separation

In the present stage, the correlation between random and periodic sources is considered to replace the assumption (1). The mixture matrix $H_n$ (39) is just for the uncorrelated separated sources $E_n(t)$. However, the expected original sources denoted by $F(t)$ are correlative between random and deterministic sources. Then the uncorrelated sources $E_n(t)$ are a combination of the original sources $F(t)$, i.e., $E_n(t) = BF(t)$, where $B$ is a combination coefficient matrix. The $B$ will be estimated by the following decorrelation.

4.1. Decorrelation analysis

Based on equation (1) and disturbance $E_{p0}(t)$ chosen by practical problem, the uncorrelated separated sources are expressed as

$$E_n(t) = H_n^s X(t) = [E_n^T(t), E_{p0}^T(t)]^T$$

$$= BF(t) = B[F^T(t), F_p^T(t)]^T$$

$$= B_f(t) + B_p F_p(t)$$

where subscripts $s$ and $p$ denote response sources and periodic disturbance sources, respectively, and

$$H_n^s = VT^* = VA^{-1/2} U_n A^H, B = [B_s, B_p]$$

Each uncorrelated separated source in $E_n(t)$ has been normalized and then is suitable to further determine the original sources $F(t)$. To obtain periodic disturbance sources $F_p(t)$, the uncorrelation condition is used.

The periodic disturbance sources with different frequencies in $F_p(t)$ are uncorrelated with each other, but they have certain correlation with random noise in $F_s(t)$. Let the correlation function matrix be

$$R_{E,F_p}(0) = \langle B(E(t), F_p^T(t)) \rangle = B_s R_{E,F_s}(0)$$

Then the uncorrelation condition is expressed as

$$\langle E_n(t) - B_p F_p(t), F_p^T(t) \rangle = R_{E,F_p}(0) = 0$$

or

$$\langle E_n(t), F_p^T(t) \rangle = R_{E,F_p}(0) = \langle B_p F_p(t), F_p^T(t) \rangle = B_p A_p^2$$

where $A_p$ is a diagonal matrix representing periodic source amplitudes with

$$A_p^2 = \langle F_p(t), F_p^T(t) \rangle = R_{F,F_p}(0)$$

Introduce the following functions with respect to frequency $\omega$

$$\gamma_p(\omega) = \langle \gamma_p(t), \sin \omega t \rangle, \nu_p(\omega) = \langle \gamma_p(t), \cos \omega t \rangle$$

where $\gamma_p(t)$ is the $i$th element of $E_n(t)$. The amplitude and initial phase of the periodic sources $F_p(t)$ are determined by

$$a_j = B_p^{-1} \max_{\omega_j} \left\{ \sqrt{\gamma_p^2(\omega) + \nu_p^2(\omega)} \right\} = B_p^{-1} \sqrt{\gamma_n^2(\omega_j) + \nu_n^2(\omega_j)}$$

$$\theta_j = (B_{p,j} a_j)^{-1} \arccos \left\{ \gamma_n(\omega_j) \right\}$$

where $a_j$ is the $j$th diagonal element of $A_p$ and $B_{p,j}$ is the $(i \times j)$th element of $B_p$. By maximization of $a_j$ (47), the frequency $\omega_j$ of the periodic sources is obtained. The amplitude $a_j$ of the periodic sources is calculated by expression (45). The combination coefficient $B_{p,j}$ is calculated by expression (47) and amended by expression (44) with the correlation function (42). In general, the correlation function can be determined by prior statistics in the frequency band around $\omega_j$ or by reference signals without the periodic disturbance with frequency $\omega_j$.

The combination coefficient matrix $B_p$, with $H_n$, can be used for separating the periodic disturbance sources $F_p(t)$ from $E_n(t)$ or $X(t)$.

Theorem 3. Let the uncorrelated sources with signal $X(t)$, mixture matrix $H_n$, combination coefficient matrix $B$ and original sources $F(t)$ be defined by expressions (40) and the periodic sources $F_p(t)$ by expressions (45), (47) and (48). Then the recovered response signal $X_r(t) = X(t) - H_n B_p F_p(t)$ and sources $F_p(t)$ have the correlation function matrix $H_n R_{E,F_p}(0)$ or the uncorrelation condition is

$$\langle X_r(t) - H_n B_p F_p(t), F_p^T(t) \rangle = H_n R_{E,F_p}(0) = 0$$

(49)
4.2. Partial source separation procedure

Based on the above analysis and results on signal eigenspace transformation, covariance joint diagonalization and decorrelation of correlation sources, a partial unknown source separation procedure for eliminating periodic disturbances is proposed as follows: (I) calculate the correlation function matrix \( R_{XX}(0) \) of signal \( X(t) \) and the singular value decomposition of \( R_{XX}(0) \) to determine the transformation matrix \( T_I \); (II) transform signal \( X(t) \) into its eigenspace as \( Z(t) \), and implement the joint diagonalization of \( R_{ZZ}(\tau) \) with various time lags to determine the unitary matrix \( V \) or mixture matrix \( H_n = TV \) for the uncorrelated sources \( E_d(t) \); (III) transform signal \( X(t) \) into \( E_d(t) \), make its maximization analysis in frequency domain to determine the periodic disturbances \( F_p(t) \) and estimate the combination coefficient matrix \( B_p \) by decorrelation; (IV) eliminate the periodic disturbances from the measured signals to obtain the recovered response signals \( X_r(t) = X(t) - H_n B_p F_p(t) \) as correlated with the disturbances.

In the above procedure, the separation calculation is mainly singular value decomposition, joint diagonalization and decorrelation. For data statistics, only the correlation function matrices \( R_{XX} \) and \( R_{ZZ} \) in time domain are calculated and the other is basic matrix operation. The statistics and estimation based on probability densities are avoided. As the number of signals increases, the calculation time in the singular value decomposition will increase accordingly. However, the calculation time in the joint diagonalization and decorrelation will mainly depend on the number of sources. Then the calculation time increases nonlinearly with the size of separation problem and the number of signals used is determined mainly by practical problems and also constrained by the algorithm such as singular value decomposition.

The random noises may have a large intensity and certain correlation with the periodic disturbances. Under the given calculation accuracy, errors may be from the smaller signal time length, lower sampling frequency and inaccurate estimation of the correlation function between the periodic disturbances and random noises. Based on equation (40), the \( i \)th uncorrelated separated source is expressed as

\[
E_{\omega i}(t) = \sum_{j=1}^{N_x} H_{\omega i,j} E_{\omega j}(t) + \sum_{k=1}^{N_x} H_{\omega i,k} E_{\omega k}(t)
\]

(51)

where subscripts \( I \) and \( J \) denote periodic and non-periodic (or random noise) sources, respectively. \( E_{\omega j} \) and \( E_{\omega k} \) are uncorrelated sources, and then for the periodic source \( F_p \) correlated with non-periodic sources, the mixture \( H_{\omega i,j} \) of the corresponding periodic source \( E_{\omega j} \) contains a part \( \sum_k \alpha_{ijk} \) which belongs to the non-periodic sources. The equation for coefficient \( \alpha_{ijk} \) and the correlation function between the periodic \( (E_p) \) and non-periodic \( (E_j) \) sources can be derived from equation (51) as

\[
\alpha_{ijk} R_{E_{\omega i,j}}(0) = R_{E_{\omega i,k}E_{\omega j}}(0) \sqrt{H_{\omega i,k}^2 + \sum_j \alpha_{ijk}^2}
\]

(52)

The coefficient \( \alpha_{ijk} \) as a function of the correlation function \( R_{E_pE_d}(0) \) is obtained by solving equation (52), and thus the corresponding mixture \( H_{\omega i,j} = \sum_k \alpha_{ijk} \) can be calculated which is used to evaluate the correlation function effect.

To measure the accuracy of separated results, the indices of relative difference and similarity are used. The relative difference of the recovered response signal \( x_{\omega i}(t) \) and original response signal \( \tilde{x}_{\omega i}(t) \) is expressed as the normalized root-mean-square error

\[
\mathbf{e}_i = \frac{\|x_{\omega i}(t) - \tilde{x}_{\omega i}(t), x_{\omega i}(t) - \tilde{x}_{\omega i}(t)\|}{\|\tilde{x}_{\omega i}(t), \tilde{x}_{\omega i}(t)\|}
\]

(53)
The similarity is represented by the normalized correlation value

\[ \gamma_i = \frac{(x_{ni}(t), \tilde{x}_{ni}(t))^2}{(x_{ni}(t), x_{ni}(t)) (\tilde{x}_{ni}(t), \tilde{x}_{ni}(t))} \] (54)

When the recovered signals are equal to the original signals, the relative difference \( \varepsilon_i = 0 \) and similarity \( \gamma_i = 1 \). Then smaller \( \varepsilon_i \) and \( \gamma_i \) closer to 1 indicate that the recovered signals are more accurate.

5. Numerical example

To examine the proposed partial unknown source separation technique, consider five response signals \( (x_i, i = 1, 2, \ldots, 5) \) as shown in figures 1(a)–5(a) which are used for separating and eliminating periodic disturbances. The sampling frequency is 500 Hz and time length is 4s. The signals are the mixture of system responses and disturbances, where the responses include a random noise \((\varepsilon_1, \varepsilon_2)\) and an oscillatory decayed source \((\varepsilon_3)\) and the disturbances have two periodic sources \((\varepsilon_{p1}, \varepsilon_{p2})\) with different frequencies \((100 \text{ Hz}, 120 \text{ Hz})\) (number of periodic sources \(N_f = 2\) and number of non-periodic sources \(N_l = 2\)). The mixture matrix corresponding to normalized sources \((\varepsilon_{p1}, \varepsilon_{p2}, \varepsilon_2, \varepsilon_1)\) is

\[
H_o = \begin{bmatrix}
0.3000 & 0.3000 & 0.5924 & 3.2122 \\
0.3600 & 0.4350 & 0.7109 & 3.8546 \\
0.3600 & 0.3750 & 0.8294 & 3.8546 \\
0.3900 & 0.4125 & 0.7702 & 4.7112 \\
0.3300 & 0.2400 & 0.2962 & 4.0152
\end{bmatrix} \] (55)

It is seen from equation (55) that the random noise is relatively intensive in the signals. The mixture \(H_o\) has the full column rank of 4. The correlation function matrix of the sources \((\varepsilon_{p1}, \varepsilon_{p2}, \varepsilon_1, \varepsilon_2)\) (diagonal elements of auto-correlation, but the random noise has certain correlation with periodic disturbances.

\[
R_{XX}(0) = \begin{bmatrix}
0.5000 & 0.0000 & 0.0000 & 0.0149 \\
0.0000 & 0.5000 & 0.0000 & 0.0037 \\
0.0000 & 0.0000 & 0.0780 & -0.0085 \\
0.0149 & 0.0037 & -0.0085 & 1.1465
\end{bmatrix} \] (56)

It is seen from equation (56) that the non-diagonal elements of cross-correlation are small compared with the diagonal elements of auto-correlation, but the random noise has certain correlation with periodic disturbances. The random noise has the absolute correlation function with time lag smaller than 0.06 and then it has a little correlation time or wide-band spectrum.

By the singular value decomposition of the correlation function matrix \(R_{XX}(0)\) of the signals, the eigenspace transformation using \(T\) and the joint diagonalization of the correlation function matrix \(R_{ZZ}(\tau)\) of the transformed signals, the normalized uncorrelated sources \(E_i(t)\) and corresponding mixture matrix \(H_n\) are obtained. The extracted mixture matrix is

\[
H_n = \begin{bmatrix}
0.3645 & 0.3132 & 0.5102 & 3.2098 \\
0.4374 & 0.4508 & 0.6122 & 3.8518 \\
0.4374 & 0.3908 & 0.7307 & 3.8515 \\
0.4846 & 0.4318 & 0.6495 & 4.7080 \\
0.4107 & 0.2565 & 0.1933 & 4.0133
\end{bmatrix} \] (57)

The mixture coefficients in \(H_n\) have a good accuracy for the random noise \((\varepsilon_1, \varepsilon_2)\) and periodic disturbance \((\varepsilon_3)\) by comparing with equation (55), but certain errors for the decayed source \((\varepsilon_2)\) and periodic disturbance \((\varepsilon_3)\) which have relatively large correlation with the random noise. Based on the results, the two periodic disturbances are eliminated from the measured signals to obtain the response signals. The relative differences of each obtained response and original response (53) are

\[
[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_5] = [2.03\%, 2.03\%, 2.02\%, 2.04\%, 2.05\%] \] (58)

The total error \((\varepsilon_i)\) of the response signals by eliminating periodic disturbances is small and however, the response signals in time domain have certain difference from the original.

Further by the decorrelation with the maximization analysis of the normalized uncorrelated sources \(E_i(t)\) in frequency domain and the estimation of the combination coefficient matrix \(B_p\), the periodic disturbances \(F_p(t)\) correlated with the random noise and the corresponding mixture matrix \(H_nB_p\) are obtained. The mixture matrix is
The mixture matrix (59) has a better accuracy than that of equation (57) by comparing with the first two columns of equation (55) for the periodic disturbances correlated with the random noise. Based on the results, the two
periodic disturbances are eliminated from the measured signals to obtain the response signals \( x_{oi}, i = 1, 2, \ldots, 5 \) which are shown in figures 1(b)–5(b). The differences of each recovered and original responses \( \Delta x_{oi}, i = 1, 2, \ldots, 5 \) are shown in figures 1(c)–5(c) and the relative differences (53) are:

\[
[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_5] = [0.22\%, 0.28\%, 0.23\%, 0.06\%, 0.12\%]
\]
The total error ($\varepsilon_i$) of the recovered response signals is much smaller than that of equation (58). The similarity value between the recovered and original responses (54) is 1.0000 for all response signals. Therefore, the original response signals have been recovered well in time domain from the measured signals with periodic disturbances which are correlated with the random noise.

Figure 3. Measurement signal $x_3$, recovered response signal $x_{o3}$ and its error $\Delta x_{o3}$.
As inaccurate estimation of the correlation function between periodic disturbances and random noises will result in certain errors of the separation, for example, extracted mixture matrix, the effect of the estimation errors on the mixture matrix is discussed based on equations (51) and (52). Figure 6 shows the relative...
differences ($\varepsilon_{hep}$) of extracted and original mixture coefficients corresponding to the periodic source ($\varepsilon_p$) in signals ($x_i$) versus the relative difference ($\varepsilon_{Rpr}$) of estimated and original correlation functions between the periodic source and random noise. When the relative difference of the correlation function is smaller than 3%, it has very slight effect on the extracted mixture coefficients. When the relative difference of the correlation

Figure 5. Measurement signal $x_5$, recovered response signal $x_{o5}$ and its error $\Delta x_{o5}$. 

Phys. Scr. 97 (2022) 115204 Z.-G Ying and Y.-Q Ni
function is smaller than 20%, the relative differences of the extracted mixture coefficients are smaller than 4.5%. Thus the proposed separation technique is insensitive to the estimation errors of the correlation function between the periodic source and random noise.

6. Conclusion

Separating and eliminating periodic disturbances from measured signals are a key problem to obtain original responses used for further system identification and evaluation. Actual periodic disturbances are partial unknown sources in measured signals and have certain correlation with random noise sources. In this paper, a separation problem on partial unknown sources is introduced in which random noises have certain correlation with deterministic such as periodic sources. A new partial unknown source separation technique is proposed by combining signal eigenspace transformation, covariance joint diagonalization and decorrelation of correlation sources. The partial source separation procedure has two main stages: first, obtain uncorrelated sources by eigenspace transformation and joint diagonalization; and second, obtain partial periodic sources correlated with random noises from the uncorrelated sources by decorrelation. The proposed partial source separation technique is supported by several theorems. A condition on the number of time lags in the joint diagonalization is presented. Under given assumptions, the separation technique will result in accurate partial sources. Calculation errors may be from smaller signal time length, lower sampling frequency and inaccurate estimation of correlation functions between periodic disturbances and random noises. The effect of estimated correlation functions on the separation results is analyzed. The proposed separation technique has main features as follows: (I) partial unknown sources are separated from measured signals; (II) separated periodic sources are correlated with random noise sources; (III) random noises can be dominant in measured signals or have a large intensity; (IV) periodic disturbance sources can be non-dominant in measured signals; and (V) only correlation functions of measured signals with estimated correlation functions between periodic disturbances and random noises are used in calculation of statistics. Numerical results show that the original response signals can be recovered well (in terms of mixture coefficients, relative differences, similarity) in time domain from the measured signals in which periodic disturbances are correlated with random noises. The separation results are insensitive to the estimation errors of correlation functions between periodic disturbances and random noises. The proposed separation technique is applicable to, for example, signal processing such as electromagnetic interference elimination of random vibration signals measured from maglev trains and dynamic modelling under multiple periodic excitations with random noises.

Acknowledgments

This research was supported by the National Natural Science Foundation of China under Grant No. 12072312, the Research Grants Council of the Hong Kong Special Administrative Region under Grant No. R-5020-18, and the Innovation and Technology Commission of the Hong Kong Special Administrative Region to the Hong Kong Branch of the National Rail Transit Electrification and Automation Engineering Technology Research Centre under Grant No. K-BBY1.
Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

Zu-Guang Ying https://orcid.org/0000-0003-1795-6968
Yi-Qing Ni https://orcid.org/0000-0003-1527-7777

References

[1] Moughty J J and Casas J R 2017 A state of the art review of modal-based damage detection in bridges: development, challenges, and solutions Applied Sciences 7 310
[2] Seo J, Hu J W and Lee J 2016 Summary review of structural health monitoring applications for highway bridges J. Perform. Constr. Facil 30 014015072
[3] Yang Y B and Chen W F 2016 Extraction of bridge frequencies from a moving test vehicle by stochastic subspace identification J. Bridge Eng. 21 04015053
[4] Ying Z G, Ni Y Q and Kang L 2019 Mode localization characteristics of damaged quasiperiodically supported beam structures with local weak coupling Structural Control and Health Monitoring 26 e2351
[5] Tong L, Soon V C, Huang Y F and Liu R 1990 AMUSE: a new blind identification algorithm Proc. IEEE Int. Symp. on Circuits and Systems (New Orleans, LA, USA) 1784-1787
[6] Cardoso J F, Bose S and Friedlander B 1996 On optimal source separation based on second and fourth order cumulants Proc. 8th Workshop on Statistical Signal and Array Processing (Corfu, Greece) 198-201
[7] Belouchrani A, Abed-Meraim K, Cardoso J F and Moulines E 1997 A blind source separation technique using second-order statistics IEEE Trans. on Signal Processing 45 434-44
[8] Parra L and Sajda P 2003 Blind source separation via generalized eigenvalue decomposition Journal of Machine Learning Research 4 1261–9
[9] Gharieb R R and Cichocki A 2003 Second-order statistics based blind source separation using a bank of subband filters Digital Signal Process. 13 252–74
[10] Lukic T and Nagy B 2019 Regularized binary tomography on the hexagonal grid Phys. Scr. 94 025201
[11] Zhang Z and Liu H 2019 Nonlocal total variation based dynamic PET image reconstruction with low-rank constraints Phys. Scr. 94 065202
[12] Blanke S E, Hahn B N and Wald A 2020 Inverse problems with inexact forward operator: iterative regularization and application in dynamic imaging Inverse Prob. 36 124001
[13] Banggaard K O and Andersen M S 2021 A statistical reconstruction model for absorption CT with source uncertainty Inverse Prob. 37 085009
[14] Sadhu A, Narasimhan S and Antoni J 2017 A review of output-only structural mode identification literature employing blind source separation methods Mech. Syst. Sig. Proc. 94 415–31
[15] McNeill S J and Zimmerman D C 2008 A framework for blind modal identification using joint approximate diagonalization Mechanical Systems and Signals Processing 22 1526–48
[16] Antoni J and Chauhan S 2013 A study and extension of second-order blind source separation to operational modal analysis J. Sound Vib. 332 1079–106
[17] Sadhu A and Narasimhan S 2014 A decentralized blind source separation algorithm for ambient modal identification in the presence of narrowband disturbances Structural Control and Health Monitoring 21 262–302
[18] Brewick P T and Smyth A W 2017 Increasing the efficiency and efficacy of second-order blind identification (SOBI) method Structural Control and Health Monitoring 24 e1921
[19] Rainieri C, Magalhaes F, Gargaro D, Fabbrocino G and Cunha A 2019 Predicting the variability of natural frequencies and its causes by second-order blind identification Structural Health Monitoring 18 486–507
[20] Ying Z G, Wang Y W, Ni Y Q and Xu C 2021 Model-free identification of multiple periodic excitations and detection of structural anomaly using noisy response measurements Smart Structures and Systems 28 407–23