THE INDUCED CAPACITY AND CHOQUET INTEGRAL
MONOTONE CONVERGENCE

ROEE TEPER

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Abstract. Given a probability measure over a state space, a partial collection (sub-$\sigma$-algebra) of events whose probabilities are known, induces a capacity over the collection of all possible events. The induced capacity of an event $F$ is the probability of the maximal (with respect to inclusion) event contained in $F$ whose probability is known. The Choquet integral with respect to the induced capacity coincides with the integral with respect to a probability specified on a sub-algebra (Lehrer [7]). We study Choquet integral monotone convergence and apply the results to the integral with respect to the induced capacity. The paper characterizes the properties of sub-$\sigma$-algebras and of induced capacities which yield integral monotone convergence.

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School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel.
e-mail: teperroe@post.tau.ac.il.
1. Introduction

In many economic situations individuals face uncertainties regarding upcoming events. The probability of these events is often unknown, and decision making is left to subjective belief. The Ellsberg paradox [4] demonstrates a situation where (additive) expected utility theory (Savage [9] and Anscombe and Aumann [1]) is violated due to partial information obtained by the decision maker on the underlying probability.

Several proposed variations of the model relax the assumption of additivity of the subjective probability (e.g. Schmeidler [10], Gilboa and Schmeidler [2]). Here we adopt a recent model by Lehrer [7] which suggests a new approach to decision making under uncertainty. The model describes a decision maker who is partially informed about the underlying probability. The information consists of the probability of some (but maybe not all) events. According to Lehrer the decision maker then assess her alternatives with only the information obtained and completely ignores unavailable information.

What Lehrer actually suggests is a new integral for functions which are not measurable w.r.t. (with respect to) the available information given by a sub-$\sigma$-algebra. Given a probability measure space $(X, \mathcal{F}, P)$ and a sub-$\sigma$-algebra $\mathcal{A} \subseteq \mathcal{F}$, the integral of an $\mathcal{F}$-measurable function, say $f$, w.r.t. the probability $P$ restricted to $\mathcal{A}$, is the supremum over all integrals of $\mathcal{A}$-measurable functions that are smaller than or equal to $f$.

The integral induces a convex capacity over $\mathcal{F}$ as follows: the capacity of an $\mathcal{F}$-measurable set $F$ is the integral of the characteristic function of $F$, in other words, the probability of the maximal (w.r.t. inclusion) $\mathcal{A}$-measurable set contained in $F$. We call such a capacity the induced capacity by $\mathcal{A}$.

It turns out that the integral of an $\mathcal{F}$-measurable function w.r.t. the restriction of $P$ to $\mathcal{A}$ coincides with the Choquet integral (Choquet [3]) and the concave integral (Lehrer [6]) w.r.t. the induced capacity.

The theory of the Lebesgue integral of sequences of functions is well known. Choquet and concave integral convergence theorems were also proved in several versions (e.g. Li and Song [12], Murofushi and Sugeno [11], Lehrer and Teper [8]). In this analysis the precise definition of what is ‘almost everywhere’ w.r.t. a capacity is crucial. Several definitions have been suggested (e.g. Klir and Wang [5], Lehrer and Teper [8]), studied and applied to integral convergence theorems.

We focus on two definitions of ‘almost everywhere’ convergence w.r.t. a capacity which coincide with the usual definition in the additive case. Utilizing these definitions of
‘almost everywhere’ convergence we prove new Choquet and concave integral monotone convergence theorems.

Applying integral monotone convergence, we study the convergence of the integral w.r.t. the induced capacity. Sequences of functions converging in different ways require different properties of the induced capacity in order to obtain integral convergence. Since the induced capacity is determined by a sub-\(\sigma\)-algebra, different structures of the sub-\(\sigma\)-algebra would yield convergence theorems for different types of convergent sequences. We characterize these properties for several types of convergence.

The paper is organized as follows: Section 2 presents the notions of capacity, integration with respect to a capacity and ‘almost everywhere’ convergence. Then integral monotone convergence is studied. In Section 3 we give the definition of an induced capacity by a sub \(\sigma\)-algebra. The motivation behind the concept of the induced capacity is brought in Section 4. Sections 5 and 6 study the required properties of the sub-\(\sigma\)-algebra and of the induced capacity which yield integral monotone convergence for different types of converging sequences of functions. Finally, discussion and comments appear in section 7.

2. Preliminaries: Monotone Convergence of the Choquet Integral

2.1. The Choquet Integral for Capacities. Let \((X, \mathcal{F})\) be a measurable space. A finite set function \(v : \mathcal{F} \rightarrow [0, \infty)\) is a capacity if \(v(\emptyset) = 0\) and \(v(F) \leq v(E)\) whenever \(F \subseteq E\). A capacity \(v\) is convex (supermodular) if \(v(F) + v(E) \leq v(F \cup E) + v(F \cap E)\) for every \(F, E\).

Denote by \(\mathcal{M}\) the collection of all nonnegative \(\mathcal{F}\)-measurable functions. The Choquet integral (see Choquet [3]) of \(f \in \mathcal{M}\) w.r.t. a capacity \(v\) is defined by

\[\int_X f dv := \int_0^\infty v(\{x : f(x) \geq t\}) dt,\]

where the latter integral is the extended Riemann integral. By the definition of the Riemann integral,

\[\int_X f dv = \sup \left\{ \sum_{i=1}^N \lambda_i v(F_i) : \sum_{i=1}^N \lambda_i 1_{F_i} \leq f \text{ and } \{F_i\}_i \in \mathcal{F} \text{ is decreasing, } \lambda_i \geq 0, N \in \mathbb{N} \right\},\]

where for every \(F \in \mathcal{F}\), \(1_F\) is the characteristic function of \(F\), and by decreasing we mean that \(F_{i+1} \subseteq F_i\) for all \(i < N\).
2.2. Almost Everywhere Convergence. When discussing sequences of functions, then “almost everywhere” convergence arises naturally. We study two different definitions for almost everywhere convergence in the nonadditive case that coincide with the usual definition in the additive case.

When a capacity is a measure, a sequence converges almost everywhere if it converges over a set of full measure. Wang and Klir [5] proposed a definition for almost everywhere convergence when discussing a non-additive capacity \( v \). According to their definition, a sequence \( \{f_n\}_n \) converges to \( f \) \( v \)-a.e. iff

\[
v(\{x \in X : f_n(x) \not\to f(x)\}) = 0.
\]

Since we later define a stronger version of almost everywhere convergence, whenever \( v(\{x \in X : f_n(x) \not\to f(x)\}) = 0 \) we say that that \( \{f_n\}_n \) converges weakly \( v \)-a.e.

Lehrer and Teper [8] introduced a stronger definition to almost everywhere convergence. We say that \( \{f_n\}_n \) converges to \( f \) strongly \( v \)-a.e. iff

\[
v(\{x \in F : f_n(x) \to f(x)\}) = v(F)\text{ for all } F \in \mathcal{F}.
\]

It is also shown in [8] that if the capacity \( v \) is convex, then \( v(\{x \in X : f_n(x) \to f(x)\}) = v(X) \) implies that \( \{f_n\}_n \) converges to \( f \) strongly \( v \)-a.e.

**Definition 1.** A capacity \( v \) is null-additive if \( v(E \cup F) = v(F) \) for every \( E \) such that \( v(E) = 0 \) and every \( F \).

**Lemma 1.** Weak \( v \)-a.e. convergence coincides with strong \( v \)-a.e. convergence iff \( v \) is null-additive.

**Proof.** Clearly, strong almost everywhere convergence implies weak almost everywhere convergence.

Assume that \( v \) is null-additive, that \( \{f_n\}_n \) converges weakly \( v \)-a.e. to \( f \), and pick any \( F \in \mathcal{F} \). Since \( v \) is monotone, \( v(\{x \in F : f_n(x) \not\to f(x)\}) = 0 \).

Now, by null-additivity,

\[
v(\{x \in F : f_n(x) \to f(x)\}) = v(\{x \in F : f_n(x) \to f(x)\} \cup \{x \in F : f_n(x) \not\to f(x)\}) = v(F),
\]

that is \( \{f_n\}_n \) converges strongly \( v \)-a.e. to \( f \).

Conversely, assume that there exist \( F, E \in \mathcal{F} \) such that \( v(E) = 0 \) and \( v(F \cup E) > v(F) \). Let \( f = \mathbb{1}_{F \cup E} \) and \( f_n = \mathbb{1}_F \) for all \( n \). Now, \( v(\{x \in X : f_n(x) \neq f(x)\}) = v(E) = 0 \) whereas \( v(\{x \in F \cup E : f_n(x) \not\to f(x)\}) = v(F) < v(F \cup E) \). □

2.3. Monotone Convergence. Li and Song [12] characterized capacities which satisfy monotone Choquet integral convergence, when convergence of sequences of functions is considered to be weak almost everywhere convergence.
Definition 2. A capacity \( v \) is continuous from below (resp. from above) if \( \lim_n v(F_n) = v(\bigcup_n F_n) \) (resp. \( \lim_n v(F_n) = v(\bigcap_n F_n) \)) for every increasing (resp. decreasing) sequence \( F_1 \subseteq F_2 \subseteq \cdots \) (resp. \( F_1 \supseteq F_2 \supseteq \cdots \)).

Theorem 1 (Li and Song). Let \( v \) be a capacity. Then \( \lim_n \int_X f_n dv = \int_X f dv \) for any increasing sequence \( \{f_n\}_n \) converging weakly \( v \)-a.e. to \( f \) iff \( v \) is null-additive and continuous from below.

The following theorem is a variant of the previous one, considering strong almost everywhere convergence.

Theorem 2. Let \( v \) be a capacity. Then \( \lim_n \int_X f_n dv = \int_X f dv \) for any increasing sequence \( \{f_n\}_n \) converging strongly \( v \)-a.e. to \( f \) iff \( v \) is continuous from below.

The essence of the proof is in the next remark.

Remark 1. Assume that \( \{f_n\}_n \) is an increasing sequence converging strongly \( v \)-a.e. to a function \( f \). That is, \( v(\{x \in F : f_n(x) \to f(x)\}) = v(F) \), for every \( F \in \mathcal{F} \). If \( v \) is continuous from below, then for every \( F \in \mathcal{F} \), \( \varepsilon' > 0 \) and \( \delta > 0 \), there is \( N \in \mathbb{N} \) such that for every \( n > N \), \( v(\{x \in F : f(x) - f_n(x) < \delta\}) > v(F) - \varepsilon' \).

Proof of Theorem 2. If \( v \) is not continuous from below then there exist a sequence of increasing sets \( \{F_n\} \) such that \( \lim_n v(F_n) < v(\bigcup_n F_n) \). Since \( \int_X \mathbb{1}_F = v(F) \) for every \( F \in \mathcal{F} \), we have that \( \lim_n \int_X \mathbb{1}_{F_n} < \int_X \mathbb{1}_F \).

For the converse implication, assume first that \( \int_X \mathbb{1}_F dv < \infty \). Since \( f_n \leq f \), \( \lim_n \int_X f_n dv \leq \int_X f dv \). We will show that for every \( \varepsilon > 0 \), there exist \( M \) such that for every \( n \geq M \), \( \int_X f_n dv > \int_X f dv - \varepsilon \).

Fix \( \varepsilon > 0 \). There exist \( \sum_{k=1}^N \lambda_k \mathbb{1}_{F_k} \leq f \) such that \( \{F_k\}_k \) is decreasing and

\[
\int_X f dv - \sum_{k=1}^K \lambda_k v(F_k) < \varepsilon.
\]

Applying Remark 1 to \( F = F_k \), \( \varepsilon' = \frac{\varepsilon}{K \lambda_k} \) and \( \delta = \frac{\varepsilon}{v(X) K} \) \((k = 1, \ldots, K)\) one obtains an \( N_k \) and a set \( B_k = \{x \in F_k : f(x) - f_n(x) < \frac{\varepsilon}{v(X) K} \} \) that satisfy \( v(B_k) > v(F_k) - \frac{\varepsilon}{K \lambda_k} \) for every \( n \geq N_k \). Moreover, since \( \{F_k\}_k \) is decreasing and \( \delta \) is fixed, then \( \{B_k\}_k \) is decreasing as well. Set \( M := \max\{N_1, \ldots, N_K\} \). Now, for every \( n \geq M \) we get

\[
\int_X f_n dv > \sum_{k=1}^K \left( \lambda_k - \frac{\varepsilon}{v(X) K} \right) v(B_k) \geq \sum_{k=1}^K \lambda_k v(B_k) - \varepsilon >
\]
\[ \sum_{k=1}^{K} \lambda_k \left( v(F_k) - \frac{\varepsilon}{K \lambda_k} \right) - \varepsilon > \int_X^{\text{Cho}} f \, dv - 3\varepsilon. \]

Since \( \varepsilon \) is arbitrarily small, the result follows.

Now, if \( f \) is not integrable, that is \( \int_X^{\text{Cho}} f \, dv = \infty \), given a large \( L \), there exist \( \sum_{k=1}^{K} \lambda_k \mathbb{1}_{F_k} \leq f \) such that

\[ \sum_{k=1}^{K} \lambda_k v(F_k) > L. \]

The proof from this point is similar to the previous one. \( \square \)

The following is a monotone convergence theorem for sequences of functions that converge pointwise.

**Theorem 3.** Let \( v \) be a capacity. Then \( \lim_n \int_X^{\text{Cho}} f_n \, dv = \int_X^{\text{Cho}} f \, dv \) for any increasing sequence \( \{f_n\}_n \) converging pointwise to \( f \) iff \( v \) is continuous from below.

**Proof.** Continuity from below is necessary by Li and Song [12], and sufficient by Murofushi and Sugeno [11]. \( \square \)

### 2.4. The Concave Integral and Monotone Convergence

The concave integral (see Lehrer [6]) of \( f \in \mathcal{M} \) w.r.t. a capacity \( v \) is defined by

\[ \int_X^{\text{Cav}} f \, dv := \sup \left\{ \sum_{i=1}^{N} \lambda_i v(F_i) : \sum_{i=1}^{N} \lambda_i \mathbb{1}_{F_i} \leq f \text{ and } F_i \in \mathcal{F}, \lambda_i \geq 0, N \in \mathbb{N} \right\}. \]

Clearly, \( \int_X^{\text{Cav}} f \, dv \geq \int_X^{\text{Cho}} f \, dv \). It is shown in Lehrer and Teper [8] that the concave integral coincides with the Choquet integral iff the capacity \( v \) is convex.

The concave integral w.r.t. a capacity \( v \) induced a *totally balanced cover* \( \hat{v} \) over \( \mathcal{F} \), which is a capacity itself. The totally balanced cover is defined by

\[ \hat{v}(F) := \int_X^{\text{Cav}} \mathbb{1}_{F} \, dv, \quad \text{for every } F \in \mathcal{F}. \]

The following lemma states that in the view of the concave integral, all capacities are a totally balanced cover.

**Lemma 2** (Lehrer and Teper [8]). \( \int_X^{\text{Cav}} f \, dv = \int_X^{\text{Cav}} f \, d\hat{v} \) for every \( f \in \mathcal{M} \).

We now formulate monotone convergence theorems for the concave integral.

**Theorem 4** (Lehrer and Teper [8]). \( \lim_n \int_X^{\text{Cav}} f_n \, dv = \int_X^{\text{Cav}} f \, dv \) for any increasing sequence \( \{f_n\}_n \subset \mathcal{M} \) converging strongly \( v \)-a.e. to \( f \) iff \( \hat{v} \) is continuous from below.
Theorem 5. $\lim_{n} \int_{X} f_{n} dv = \int_{X} f dv$ for any increasing sequence $\{f_{n}\}_{n} \subset \mathcal{M}$ converging weakly $v$-a.e. to $f$ iff $\hat{v}$ is null-additive and continuous from below.

Proof. The ‘only if’ implication is clear. By Lemma 1 we have that $\{f_{n}\}_{n}$ converges strongly $\hat{v}$-a.e. to $f$, therefore by Theorem 4 we obtain the ‘if’ implication. □

Theorem 6. $\lim_{n} \int_{X} f_{n} dv = \int_{X} f dv$ for any increasing sequence $\{f_{n}\}_{n} \subset \mathcal{M}$ converging pointwise to $f$ iff $\hat{v}$ is and continuous from below.

Remark 2. The proof is similar to that of Theorem 2. Instead of applying Remark 1, the reader should apply the following argument: let $\{f_{n}\}_{n}$ be an increasing sequence converging pointwise to a function $f$. If $\hat{v}$ is continuous from below, then for every $F \in \mathcal{F}, \varepsilon' > 0$ and $\delta > 0$ there is $N \in \mathbb{N}$ such that for every $n > N$, $\hat{v}(\{x \in F : f(x) - f_{n}(x) < \delta\}) > \hat{v}(F) - \varepsilon'$.

3. Sub-σ-algebra and the induced capacity

Let $(X, \mathcal{F}, P)$ be a probability space. A sub-σ-algebra $\mathcal{A} \subseteq \mathcal{F}$ induces a convex capacity over $\mathcal{F}$ (see Lehrer [7]) by

$$v_{\mathcal{A}}(F) = \max\{P(A) : A \in \mathcal{A}, A \subseteq F\},$$

for every $F \in \mathcal{F}$. $\mathcal{A}$ is a σ-algebra therefore the maximum is attained and $v_{\mathcal{A}}$ is well defined. We denote by $A_{\mathcal{F}} = \arg\max\{P(A) : A \in \mathcal{A}, A \subseteq F\}$ the $\mathcal{A}$-measurable set (modulo a set of probability 0) at which the maximum is attained. We say that $v_{\mathcal{A}}$ is the induced capacity by $\mathcal{A}$.

Remark 3. Since the induced capacity is convex, the Choquet and concave integral w.r.t. it coincide. From this point forward, unless stated otherwise, the integral w.r.t. the induced capacity could be interpreted both as a Choquet and concave integral, and the “Cho” and “Cav” notation is therefore omitted.

We now present the main interest of this paper. The structure of a sub-σ-algebra could be varied to induce capacities with different properties. Now, assume that an increasing sequence of measurable nonnegative functions $\{f_{n}\}_{n} \subset \mathcal{M}$ converges in a certain way to a function $f$. We would like to address the following questions:

- Does $\lim_{n} \int_{X} f_{n} dv_{\mathcal{A}} = \int_{X} f dv_{\mathcal{A}}$?
- How to characterize the structure of a sub-σ-algebra $\mathcal{A}$ which would yield such convergent sequence of integrals?
• In what sense should sequences of functions converge to obtain convergence of the integrals?

Lemma 3. $v_A$ is continuous from above.

Proof. Assume that $\{F_n\}_n$ is decreasing to $F$. Obviously, $A_F \subseteq A_{F_n}$ for every $n$, therefore $A_F \subseteq \bigcap_n A_{F_n}$. Assume that $\emptyset \neq A' = \bigcap_n A_{F_n} \setminus A_F \in \mathcal{A}$. In particular, $A' \subseteq A_{F_n}$ for every $n$, therefore $A' \subseteq A_F$, a contradiction. \hfill \square

Continuity from below, which cannot be obtained for every induced capacity, plays a key property in integral convergence and will be discussed in detail in Section 5.

4. MOTIVATION: DECISION MAKING UNDER UNCERTAINTY

Expected utility is a customary theory to analyze the behavior of a decision maker (DM), where her preference order is described by the Lebesgue integral. The Ellsberg paradox [4] demonstrates a situation where expected utility theory is violated due to partial information that the DM obtains on the underlying probability. In a recent paper, Lehrer [7] suggests a new approach to this issue. According to Lehrer, the preference order is given by a new integral which utilizes only the information obtained by the DM and ignores completely unavailable information.

More formally, given a probability space $(X, \mathcal{F}, P)$ we describe the information obtained by the DM by a sub-$\sigma$-algebra $\mathcal{A} \subseteq \mathcal{F}$. The restriction of the probability $P$ to $\mathcal{A}$, denoted by $P_A$, is called a probability specified on a sub-algebra (PSA). The integral w.r.t. a PSA $P_A$ of an $\mathcal{F}$-measurable nonnegative function $f \in \mathcal{M}$ is defined by

$$\int_X f dP_A = \sup \left\{ \sum_{i=1}^N \lambda_i P(A_i) : \sum_{i=1}^N \lambda_i 1_{A_i} \leq f \text{ and } A_i \in \mathcal{A}, \lambda_i \geq 0, N \in \mathbb{N} \right\}.$$

The next Lemma relates the integral w.r.t. a PSA to the induced capacity by a sub-$\sigma$-algebra.

Lemma 4. 1. $v_A(F) = \int_X 1_F dP_A$ for all $F \in \mathcal{F}$.

2. $\int_X f dP_A = \int_X f dP_A$ for all $f \in \mathcal{M}$.

Proof. 1 is straight forward. As for 2, assume that $f$ is $v_A$-integrable and $P_A$-integrable. Fix $\varepsilon > 0$. There exists $\sum_{n=1}^N \lambda_n 1_{F_n} \leq f$ such that $\int_X f dP_A \leq \sum_{n=1}^N \lambda_n v_A(F_n) + \varepsilon$. Now,

$$\int_X f dP_A \leq \sum_{n=1}^N \lambda_n v_A(F_n) + \varepsilon = \sum_{n=1}^N \lambda_n P(A_{F_n}) + \varepsilon \leq \int_X f dP_A + \varepsilon.$$
In the same manner, there exist $\sum_{n=1}^{N} \lambda_n 1_{A_n} \leq f$ such that $\int_X f dP_A \leq \sum_{n=1}^{N} \lambda_n P(A_n) + \varepsilon$.

$$\int_X f dP_A \leq \sum_{n=1}^{N} \lambda_n P(A_n) + \varepsilon = \sum_{n=1}^{N} \lambda_n v_A(A_n) + \varepsilon \leq \int_X f d\nu_A + \varepsilon.$$  

Since $\varepsilon$ is arbitrarily small we obtain the expected result.

If, for example, $f$ is not $v_A$-integrable, then for every $L > 0$ there exist $\sum_{n=1}^{N} \lambda_n 1_{F_n} \leq f$ such that $\sum_{n=1}^{N} \lambda_n v_A(F_n) > L$. The proof that $f$ is not $P_A$-integrable is similar to the one above. $\square$

By Lemma 4 we can interpret the integral w.r.t. a PSA as the Choquet integral w.r.t. to the induced capacity.

5. Non-Atomic Probability Spaces

In this section we consider non-atomic probability spaces. Discrete probability spaces will be discussed later in section 6.

5.1. Weak Almost Everywhere Convergence. When considering weak almost everywhere convergence, then Theorem 1 (and Theorem 6) states that integral (Choquet and concave) monotone convergence is equivalent to null-additivity and continuity from below.

**Lemma 5.** If $v_A$ is null-additive then it is continuous from below.

**Proof.** Indeed, assume that $v_A$ is null-additive and let $\{F_n\}_n$ be a sequence of measurable sets increasing to $F$ such that $\lim_n v_A(F_n) < v_A(F)$. Set $C_n = (A_F \setminus \bigcup_n A_{F_n}) \cap F_n$. $\{C_n\}_n$ is increasing to $(A_F \setminus \bigcup_n A_{F_n})$ and $v_A(C_n) = 0$ for every $n$. Now, set $D_n = (A_F \setminus \bigcup_n A_{F_n}) \setminus C_n$. $\{D_n\}_n$ is decreasing to the emptyset, and by continuity from above $v_A(\bigcap_n D_n) = 0$. However, by null-additivity $v_A(D_n) = v_A(A_F \setminus \bigcup_n A_{F_n}) = P(A_F \setminus \bigcup_n A_{F_n})$ which is positive. $\square$

**Definition 3.** We say that a collection $\mathcal{C} \subseteq \mathcal{F}$ is dense in $\mathcal{F}$ iff for every $\varepsilon > 0$ and $F \in \mathcal{F}$ there exist $C \in \mathcal{C}$ such that $C \subseteq F$ and $P(F \setminus C) < \varepsilon$.

Since $\mathcal{A}$ is a $\sigma$-algebra, then being dense in $\mathcal{F}$ is equivalent to that, for every $F \in \mathcal{F}$, there exist $A \in \mathcal{A}$ included in $F$ such that $P(F \setminus A) = 0$.

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1. $F \in \mathcal{F}$ is an atom if $P(F) > 0$ and for every $G \in \mathcal{F}$ such that $G \subseteq F$, $P(G) = P(F)$ or $P(G) = 0$. A probability space is non-atomic if there are no atoms.
Proposition 1. The following are equivalent:
1. \( \mathcal{A} \) is dense in \( \mathcal{F} \);
2. \( \int_X f dv_A = \int_X f dP \) for every function \( f \in \mathcal{M} \);
3. \( \lim_n \int_X f_n dv_A = \int_X f dv_A \) for every increasing sequence of functions \( \{f_n\}_n \subset \mathcal{M} \) converging weakly \( v_A \)-a.e. to a function \( f \); and
4. \( v_A \) is null-additive.

Proof. (1) \( \Leftrightarrow \) (2). Pick \( F \in \mathcal{F} \). If \( \mathcal{A} \) is dense in \( \mathcal{F} \) then \( v_A(F) = \max\{P(A) : A \in \mathcal{A}, A \subseteq F\} = P(F) \), therefore the integrals coincide.

Now assume that \( \int_X f dv_A = \int_X f dP \) for every function \( f \in \mathcal{M} \). In particular, for all \( F \in \mathcal{F} \), \( \int_X \mathbb{1}_F dv_A = \int_X \mathbb{1}_F dP = P(F) \). But \( \int_X \mathbb{1}_F dv_A \) is equal to the probability \( P(A) \) of some \( A \in \mathcal{A} \) contained in \( F \). Thus \( \mathcal{A} \) is dense in \( \mathcal{F} \).

(1) \( \Rightarrow \) (3) is simply Levi’s monotone convergence theorem.

(3) \( \iff \) (4) by Theorem 1 (or Theorem 6) and Lemma 5.

(4) \( \Rightarrow \) (1). Assume that \( \mathcal{A} \) is not dense in \( \mathcal{F} \). Then there exist \( F \in \mathcal{F} \) of positive probability such that \( A_F = \arg \max \{P(A) : A \in \mathcal{A}, A \subseteq F\} \) satisfies \( P(A_F) < P(F) \). Denote \( B = A_F \cup F^c \). \( P(B^c) = P(F \setminus A) > 0 \) therefore \( P(B) < 1 \), thus \( v_A(B) < 1 \).

However, \( v_A(B^c) = v_A(F \setminus A) = 0 \), and by null additivity \( v_A(B) = v_A(B \cup B^c) = v_A(X) = 1 \). A contradiction.

\[ \square \]

5.2. \( P \)-Almost Everywhere Convergence.

Definition 4. A sub-\( \sigma \)-algebra \( \mathcal{A} \) satisfies property (A1) if for every \( F \in \mathcal{F} \) with \( P(A_F) > 0 \) there exist \( \delta > 0 \) such that for every \( G \in \mathcal{F} \) contained in \( F \) which \( P(F \setminus G) < \delta \) satisfies \( P(A_G) > 0 \).

Example 1. Consider the unit interval \([0, 1]\), \( \mathcal{B}^1 \) the Borel \( \sigma \)-algebra over \([0, 1]\) and \( m^1 \) the Lebesgue probability measure. Let \( C \subseteq [0, 1] \) be a set of probability 0 and of continuum cardinality. There exist a one-to-one correspondence \( f : C \to [0, 1] \setminus C \). Now, let \( \mathcal{A} = \{ A \subseteq [0, 1] : x \in A \iff f(x) \in A\} \). The induced capacity is continuous from below, however (A2) is not satisfied. Indeed, \( m^1_A([0, 1]) = 1 \) and \( m^1_A([0, 1] \setminus C) = 0 \) where \( m^1([0, 1] \setminus ([0, 1] \setminus C)) = 0 \). That is \( \mathcal{A} \) does not satisfy property (A1).

Example 2. Let \( \mathcal{A} \) be the sub-\( \sigma \)-algebra which contains all sets \( A \in \mathcal{B}^1 \) which satisfy \( A = A + \frac{1}{2} \pmod{1} \). Note that \( \mathcal{A} \) is not dense in \( \mathcal{B}^1 \) since, for example, \( m^1_A([0, 1/4]) = 0 \). For \( F \in \mathcal{B}^1 \), \( A_F = F \cap (F + \frac{1}{2} \pmod{1}) \). Now, assume that \( \{F_n\}_n \) is increasing to \( F \).
We will show that \( \{A_{F_n}\}_n \) is increasing to \( A_F \) which will prove that \( v_A \) is continuous from below. Indeed, if \( x \in A_F \) then \( x \in F \cap \{F + \frac{1}{2} \text{ (mod } 1)\} \). That is, there exist \( n \in \mathbb{N} \) such that \( x \in F_n \cap \{F_n + \frac{1}{2} \text{ (mod } 1)\} \), meaning that \( x \in A_{F_n} \), as desired. It follows by Proposition 1 that \( m^1_A \) is not null-additive. For example, \( m^1_A([0,1/2]) = 0 \) and \( m^1_A([1/2,1]) = 0 \), whereas \( m^1_A([0,1]) = 1 \). Furthermore, \( A \) satisfies (A2). For \( F \in \mathcal{F} \) with \( P(A_F) > 0 \), set \( \delta = \frac{P(A_F)}{4} \).

**Definition 5.** An induced capacity \( v_A \) is \( P \)-null-additive if \( v_A(G) = v_A(F) \) for every \( G, F \) such that \( G \subseteq F \) and \( P(F \setminus G) = 0 \).

**Proposition 2.** The following are equivalent:
1. \( A \) satisfies property (A1);
2. \( \lim_n \int_X f_n dv_A = \int_X f dv_A \) for every increasing sequence of functions \( \{f_n\}_n \subseteq \mathcal{M} \) converging \( P \)-a.e. to a function \( f \); and
3. \( v_A \) is continuous from below and \( P \)-null-additive.

**Proof.** (1) \( \iff \) (3). Assume that \( v_A \) is continuous from below and \( P \)-null-additive. Assume that there exist \( F \in \mathcal{F} \) with \( P(A_F) > 0 \) such that for every \( \delta > 0 \) there exist \( G \in \mathcal{F} \) contained in \( F \) such that \( P(F \setminus G) < \delta \) and \( P(A_G) = 0 \). Pick a sequence \( \{\delta_n\}_n \) such that \( \delta_n \to 0 \), then there is a sequence \( \{G_n\}_n \subseteq \mathcal{F} \) such that \( G_n \subseteq F \), \( P(F \setminus G_n) < \delta_n \) and \( P(A_{G_n}) = 0 \) for all \( n \). \( \mathbb{I}_{G_n} \) converges to \( \mathbb{I}_F \) in probability \( P \), therefore there exist a subsequence \( \mathbb{I}_{G_{n_m}} \) that converges to \( \mathbb{I}_F \) \( P \)-almost everywhere. The sequence \( \{H_m\}_m \) where \( H_m = \bigcap_{k \geq m} G_{n_k} \) is increasing and \( P(A_{H_m}) = 0 \) for every \( n \). Set \( \bar{F} = \bigcup_m H_m \). Showing that \( P(A_{\bar{F}}) > 0 \) will establish that \( v_A \) is not continuous from below. Since \( v_A \) is \( P \)-null-additive and \( P(F \setminus \bar{F}) = 0 \), we obtain that \( v_A(\bar{F}) = v_A(F) > 0 \), as desired.

Conversely, assume that there exist \( G \subseteq F \) such that \( P(F \setminus G) = 0 \) and \( v_A(F) > v_A(G) \).

Then \( A_F = A_G \cup E \cup H \) where \( E \subseteq F \setminus G, H \subseteq G \). Now, \( v_A(E \cup H) = P(H) > 0 \) where \( v_A(H) = 0 \) and \( P(E) = 0 \), therefore \( (A1) \) does not hold. Furthermore, if there is a sequence \( \{F_n\}_n \) increasing to \( F \) such that \( \lim_n v_A(F_n) < v_A(F) \). We obtain that \( \lim_n P(A_{F_n}) < P(A_F) \), in particular, \( \bigcup_n A_{F_n} \subseteq A_F \). Note that \( A' = A_F \setminus \bigcup_n A_{F_n} \in \mathcal{A}, v_A(F_n \cap A') = 0 \) for all \( n \) and \( v_A(F \cap A') > 0 \). \( P(A_{F'}) > 0 \) where \( F' = F \cap A' \). The sequence \( \{F'_n\}_n \) where \( F'_n = F_n \cap A' \) increases to \( F' \), thus since \( \mathcal{A} \) satisfies (A1) there is \( n \) large enough so that \( P(A_{F'_n}) > 0 \), a contradiction.

(2) \( \iff \) (3). Assume first that \( \lim_n \int_X f_n dv_A = \int_X f dv_A \) for every increasing sequence of functions \( \{f_n\}_n \subseteq \mathcal{M} \) converging \( P \)-a.e. to a function \( f \). The continuity from below
of \( v_A \) is obvious. If \( G \subseteq F \) such that \( P(F \setminus G) = 0 \), then \( P(\mathbb{1}_G = \mathbb{1}_F) = 1 \) therefore \( v_A(G) = v_A(F) \), and we obtain weak null-additivity as well.

Conversely, let \( \{f_n\}_n \) be an increasing sequence converging \( P \)-a.e. to a function \( f \). That is, \( P(\{x \in F : f_n(x) \to f(x)\}) = P(F) \), for every \( F \in \mathcal{F} \). If \( v_A \) assume that \( v_A \) is continuous from below and \( P \)-null-additive, then for every \( F \in \mathcal{F}, \varepsilon' > 0 \) and \( \delta > 0 \) there is \( N \in \mathbb{N} \) such that for every \( n > N, v(\{x \in F : f(x) - f_n(x) < \delta\}) = v(F) - \varepsilon' \). From this point the proof is similar to that of Theorem 2 (note that since the integral w.r.t. an induced capacity is the concave integral, using a collection of decreasing sets is not necessary).

\[ \square \]

5.3. Strong Almost Everywhere Convergence. Since an induced capacity \( v_A \) is convex, then \( \int_X \mathbb{1}_F dv_A = v_A(F) \) for all \( F \in \mathcal{F} \). Thus, Theorem 2 (and Theorem 4) can be applied to an induced capacity whenever strong almost everywhere convergence is in hand. The theorem states that monotone convergence holds for the integral w.r.t. an induced capacity iff it is continuous from below.

Example 3. Consider \( X = [0,1]^2, \mathcal{B}^2 \) the Borel \( \sigma \)-algebra over \([0,1]^2\) and \( m^2 \) the Lebesgue probability measure. Let \( \mathcal{A} = \{[0,1] \times B : B \in \mathcal{B}^1\} \). Consider any sequence of the form \( \{B_n \times [0,1]\}_n \) where \( \{B_n\}_n \) is increasing to \([0,1]\). \( m^2(\mathcal{A}(B_n \times [0,1])) = 0 \) for all \( n \) whereas \( m^2(\mathcal{A}([0,1]^2)) = 1 \). That is \( m^2_{\mathcal{A}} \) is not continuous from below.

Definition 6. A sub-\( \sigma \)-algebra \( \mathcal{A} \) satisfies property (A2) if for every \( F \in \mathcal{F} \) such that \( P(A_F) > 0 \) and every \( \{F_n\}_n \) increasing to \( F \) there is \( n \) such that \( P(A_{F_n}) > 0 \).

Proposition 3. The following are equivalent:
1. \( \mathcal{A} \) satisfies property (A2);
2. \( \lim_n \int_X f_n dv_A = \int_X f dv_A \) for every increasing sequence of functions \( \{f_n\}_n \subset \mathcal{M} \) converging strongly \( v_A \)-a.e. to a function \( f \);
3. \( \lim_n \int_X f_n dv_A = \int_X f dv_A \) for every increasing sequence of functions \( \{f_n\}_n \subset \mathcal{M} \) converging pointwise to a function \( f \); and
4. \( v_A \) is continuous from below.

Proof. (1) \( \iff \) (4). Clearly continuity from below implies (A2). As for the other implication, assume that \( \{F_n\}_n \) is increasing to \( F \) and that \( \lim_n v_A(F_n) < v_A(F) \). Setting \( A' = A_F \setminus \bigcup_n A_{F_n} \), then \( v_A(A') = P(A') > 0 \). Denote \( F'_n = F_n \cap A' \) for all \( n \), then \( \{F'_n\}_n \) is increasing to \( A' \) and \( v_A(F'_n) = 0 \) for all \( n \), that is \( v_A \) is not continuous from below.
(2) ⇔ (4) by Theorem 2 (and Theorem 4).
(3) ⇔ (4) by Theorem 3.

To conclude this section, which discusses monotone convergence of the integral w.r.t. the induced capacity in non-atomic probability spaces, we present the following diagram which summarizes the properties presented.

\[
\begin{array}{cccc}
\text{weak convergence} & \rightarrow & P - \text{a.e. convergence} & \rightarrow & \text{strong convergence} \\
\downarrow & & \downarrow & & \downarrow \\
\text{null-additivity} & \rightarrow & \text{cont.} + P - \text{null-additivity} & \rightarrow & \text{cont.} \\
\downarrow & & \downarrow & & \downarrow \\
\text{density} & \rightarrow & \text{property (A1)} & \rightarrow & \text{property (A2)} \\
\end{array}
\]

The top row shows to which type of converging sequences of functions monotone convergence holds. The middle row indicates the appropriate induced capacity which will suffice for the relevant type of convergence. The bottom row states the property of the sub-\(\sigma\)-algebra which would yield a corresponding property of the induced capacity. For instance, the arrow marked with (\(\ast\)) is simply the consequence of Proposition 1 that monotone convergence of the integral w.r.t. an induced capacity holds for every weak almost everywhere convergent sequence if and only if the induced capacity is null-additive.

6. Discrete Probability Spaces

Consider the case where \(X\) is a countable (possibly finite) space endowed with some probability \(P\). For the sake of convenience, if \(|X| = n\) then we will assume that \(X = \{1, \ldots, n\}\), otherwise \(X = \mathbb{N}\). Here \(\mathcal{A}\) is some \(\sigma\)-algebra generated by a partition of \(X\), \(\{A_i\}_{i \in \mathbb{N}}\). Namely, \(\{A_i\}_{i \in \mathbb{N}}\) are the atoms of \(\mathcal{A}\).

Note that in this case

\[
\int_X f \, dv_\mathcal{A} = \sum_i \left( \inf_{x \in A_i} f(x) \right) P(A_i)
\]

for all \(f \in \mathcal{M}\).

**Example 4.** Let \(P(k) \approx \frac{1}{k^2}\) for every \(k \in \mathbb{N}\), and \(\mathcal{A}\) be the \(\sigma\)-algebra generated by the partition \(\{\{2k, 2k-1\} : k \in \mathbb{N}\}\).
Let \( f = 1 \) and for every \( n \)

\[
f_n(k) = \begin{cases} 
1, & k \leq n, \\
0, & k > n.
\end{cases}
\]

By (1) \( \int f \, dv_{\mathcal{A}} = 1 \) and \( \int f_n \, dv_{\mathcal{A}} = \sum_{k \leq n} P(k) \) where the later converges to 1.

Denote by \( T = (\emptyset, \mathbb{N}) \) the trivial field. \( \int f_n \, dv_{T} = 1 \) but since \( \min f_n = 0 \) for all \( n \), \( \int f_n \, dv_{T} = 0 \) for all \( n \).

Example 4 might only reflect two particular structures of \( \mathcal{A} \). In the first example all atoms of \( \mathcal{A} \) are finite and we obtain a sequence of integrals which converge to the integral of the limit function. In the second example there is an infinite atom and we are unable to obtain integral convergence. The following proposition shows that in fact there are only two cases.

**Proposition 4.** The following are equivalent:

1. \( \mathcal{A} \) is generated by atoms consisting of finitely many elements of \( X \);
2. \( \lim_n \int_X f_n \, dv_{\mathcal{A}} = \int_X f \, dv_{\mathcal{A}} \) for every increasing sequence of functions \( \{ f_n \}_n \) converging pointwise to a function \( f \);
3. \( \lim_n \int_X f_n \, dv_{\mathcal{A}} = \int_X f \, dv_{\mathcal{A}} \) for every increasing sequence of functions \( \{ f_n \}_n \) converging strongly \( \nu_{\mathcal{A}} \)-a.e. to a function \( f \); and
4. \( \nu_{\mathcal{A}} \) is continuous from below.

**Proof.** (1) \( \Rightarrow \) (2). Assume at first that \( f \) is \( \nu_{\mathcal{A}} \)-integrable. Let \( \{ f_n \}_n \) be any increasing sequence converging to \( f \) and fix \( \varepsilon > 0 \). Let \( n_\ast \in \mathbb{N} \) be big enough so that

\[
\sum_{i > n_\ast} \left( \inf_{x \in A_i} f(x) \right) P(A_i) < \varepsilon.
\]

Define \( N_1 := \min \{ n \geq 1 : f(m) - f_n(m) < \frac{\varepsilon}{n_\ast P(A_1)}, m \in A_1 \} \). By induction, define \( N_i := \min \{ n \geq N_{i-1} : f(m) - f_n(m) < \frac{\varepsilon}{n_\ast \inf_{A_i} P(A_i)}, m \in A_i \} \) for all \( 2 \leq i \leq n_\ast \). Since every \( A_i \) is finite, it is guaranteed that \( N_i \) are finite for every \( i \leq n_\ast \). Now,

\[
\int_X f_n \, dv_{\mathcal{A}} \geq \sum_{i \leq n_\ast} \left( \inf_{x \in A_i} f(x) \right) P(A_i) - \varepsilon \geq \int_X f \, dv_{\mathcal{A}} - 2\varepsilon.
\]

Since \( \varepsilon \) is arbitrarily small, we have that \( \int f_n \, dv_{\mathcal{A}} \geq \int f \, dv_{\mathcal{A}} \). The inverse inequality is obvious.

If \( f \) is not \( \nu_{\mathcal{A}} \)-integrable, that is \( \int_X f \, dv_{\mathcal{A}} = \infty \), given a large \( L \), there exist \( n_L \)

\[
\sum_{i \leq n_L} \left( \inf_{x \in A_i} f(x) \right) P(A_i) > L.
\]

The proof from this point is similar to the one above.

(2) \( \Rightarrow \) (4) by the definition of continuity from below.
(4) ⇒ (1). Assume that there exist an atom $A = \{k_1, k_2, \ldots\}$ with infinite number of elements of $X$. Let $A_n = \{k_1, \ldots, k_n\}$ for every $n$. $v_A(A) = P(A)$ whereas $v_A(A_n) = 0$ for all $n$.

(3) ⇔ (4) by Theorem 4.

Remark 4. By Theorem 4, integral convergence where functions converge weakly almost everywhere is obtained iff the induced capacity is null-additive. To obtain null-additivity of the induced capacity it is easy to see that $A$ must be generated by the singletons of $X$, that is $A = 2^X$.

7. Discussion and Final Comments

7.1. Generalizing Induced Capacities. Lehrer [7] considers a second model of decision making with partially-specified probabilities (PSP). The PSP model illustrates the case where a DM obtains the information of the integrals of a sub collection $G \subseteq M$ of nonnegative measurable functions. The DM makers then approximates the integral of $f \in M$ by the supremum over all positive combinations of integrals of functions in $G$ that are smaller than or equal to $f$. Formally, the integral of $f$ w.r.t. $G$ is defined by

$$\int_X f dP_G = \sup \left\{ \sum_{i=1}^N \lambda_i \int_X g_i dP : \sum_{i=1}^N \lambda_i g_i \leq f \text{ and } g_i \in G, \lambda_i \geq 0, N \in \mathbb{N} \right\}.$$

The PSP model is indeed a generalization of the PSA model, since one could consider $G$ to be the collection of characteristic functions of some sub-$\sigma$-algebra $A$.

In the case of PSP, the analogous definition for the induced capacity is

$$v_G(F) := \int_X \mathbb{1}_F dP_G$$

for every $F \in \mathcal{F}$. Properties of the induced capacity in the PSA model do not hold for the induced capacity in the PSP model. For example $v_G$ is no longer convex, thus the Choquet and concave w.r.t. it do not coincide. It follows that Lemma 4 is no longer true. Integration type needs to be specified.

This discussion raises several questions: For which collections of functions would the induced capacity would be convex? For which collections of functions could Lemma 4 be formulated for the Choquet and concave integral? It would also be interesting to study the properties of such collections that yield integral convergence theorems.
7.2. Increasing Information. Assume that, at each period of time, a DM obtains more information regarding the underlying probability. That is \( \{A_n\}_n \) is an increasing sequence of sub-\( \sigma \)-algebras. We consider the case where the union of \( \{A_n\}_n \) generates a dense sub \( \sigma \)-algebra of \( \mathcal{F} \). For all \( n \) denote by \( v_n = v_{A_n} \) the induced capacity by \( A_n \). We would like to know whether \( \lim_n \int_X f d v_n = \int_X f d P \). \( \{v_n\}_n \) is an increasing sequence of capacities. We say that that it increases continuously to \( P \), if \( \lim v_n(F) = P(F) \) for all \( F \in \mathcal{F} \).

Lemma 6. \( \lim_n \int_X f d v_n = \int_X f d P \) for every \( f \in \mathcal{M} \) iff \( \{v_n\}_n \) increases continuously to \( P \).

Proof. The ‘only if’ direction is obvious. As for the ‘if’ direction, assume that \( \{v_n\}_n \) increases continuously to \( P \) and fix \( \varepsilon > 0 \). \( \int_X f d P \leq \sum_{i=1}^{N} \lambda_i P(F_i) + \varepsilon \) for some \( \sum_{i=1}^{N} \lambda_i 1_{F_i} \leq f \). For all \( i \leq N \) there exist \( N_i \) such that for every \( n \geq N_i \) \( v_n(F_i) \geq P(F_i) - \frac{\varepsilon}{N M_i} \). Denoting \( M = \max_i N_i \), we obtain that

\[
\int_X f d P \leq \sum_{i=1}^{N} \lambda_i P(F_i) + \varepsilon \leq \sum_{i=1}^{N} \lambda_i v_n(F_i) + 2 \varepsilon \leq \int_X f d v_n + 2 \varepsilon
\]

for all \( n \geq M \). Since \( \varepsilon \) is arbitrarily small we have that \( \int_X f d P \leq \lim_n \int_X f d v_n \). The converse inequality is trivial. \( \Box \)

For the next 2 examples consider the Borel \( \sigma \)-algebra over \([0,1]\) endowed with the Lebesgue measure.

Example 5. For every \( n \), \( A_n \) is the algebra generated by the diadic partition of length \( 2^{-n} \). The union of \( A_n \) generates the Borel \( \sigma \)-field. Now, let \( F \) be the set of all irrationals in \([0,1]\). \( P(F) = 1 \) where \( v_n(F) = 0 \) for every \( n \).

Example 6. Let \( A_n \) be the \( \sigma \)-algebra generated by all Borel measurable sets contained in \([0,a_n]\) and the set \([a_n,1]\), where \( \{a_n\}_n \) is increasing to 1. It is clear that \( \{v_n\}_n \) increases continuously to \( P \).

Remark 5. \( \{v_n\}_n \) increases continuously to \( P \) iff the union of \( \{A_n\}_n \) is dense in \( \mathcal{F} \).

In light of Lemma 4, a DM preference order obtaining partial information, as abundant as it might be, could be completely different from that of a DM obtaining the complete information. Consider Example 5. A fully informed DM would prefer \( 1_F \) to \( 1_{F^c} \), whereas a DM who is informed of \( A_n \) have no preference.
7.3. Families of Functionals. We have seen in Section 2 that for both Choquet and concave integrals, in order to obtain integral monotone convergence, the capacity needs to satisfy the same properties (considering of course a specific type of a converging sequence of functions).

Which properties of the Choquet and concave integral are essential for obtaining monotone convergence for the exact same capacities? In other words, assume that one defines a new functional w.r.t. $v$ over $\mathcal{M}$. Denote it by $I$. What characterizes $I$ so that monotone convergence would occur for the exact same properties of the capacities that yield monotone convergence for the Choquet and concave integrals?

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