SPACES ADMISSIBLE FOR THE STURM-LIOUVILLE EQUATION

N. A. CHERNYAVSKAYA
Department of Mathematics, Ben-Gurion University of the Negev
P.O.B. 653, Beer-Sheva, 84105, Israel

L. A. SHUSTER
Department of Mathematics, Bar-Ilan University
52900 Ramat Gan, Israel

(Communicated by Shoji Yotsutani)

Abstract. We consider the equation
\[-y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}\]
where \(f \in L^p_{\text{loc}}(\mathbb{R}), \ p \in [1, \infty)\) and \(0 \leq q \in L^1_{\text{loc}}(\mathbb{R})\). By a solution of (1) we mean any function \(y\), absolutely continuous together with its derivative and satisfying (1) almost everywhere in \(\mathbb{R}\). Let positive and continuous functions \(\mu(x)\) and \(\theta(x)\) for \(x \in \mathbb{R}\) be given. Let us introduce the spaces
\[L^p_{\mu}(\mathbb{R}) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}) : \|f\|_{L^p_{\mu}(\mathbb{R})} = \int_{-\infty}^{\infty} |\mu(x)f(x)|^p \, dx < \infty \right\},\]
\[L^p_{\theta}(\mathbb{R}) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}) : \|f\|_{L^p_{\theta}(\mathbb{R})} = \int_{-\infty}^{\infty} |\theta(x)f(x)|^p \, dx < \infty \right\}.

In the present paper, we obtain requirements to the functions \(\mu, \theta\) and \(q\) under which
1) for every function \(f \in L^p_{\mu}(\mathbb{R})\) there exists a unique solution (1) \(y \in L^p_{\mu}(\mathbb{R})\) of (1);
2) there is an absolute constant \(c(p) \in (0, \infty)\) such that regardless of the choice of a function \(f \in L^p_{\mu}(\mathbb{R})\) the solution of (1) satisfies the inequality
\[\|y\|_{L^p_{\mu}(\mathbb{R})} \leq c(p)\|f\|_{L^p_{\mu}(\mathbb{R})}.

1. Introduction. In the present paper, we consider the equation
\[-y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}\]
where \(f \in L^p_{\text{loc}}(\mathbb{R}), \ p \in [1, \infty)\) and
\[0 \leq q \in L^1_{\text{loc}}(\mathbb{R}).\]

Our general goal is to determine a space frame within which equation (1.1) always has a unique stable solution. To state the problem in a more precise way, let us fix two positive continuous functions \(\mu(x)\) and \(\theta(x)\), \(x \in \mathbb{R}\), a number \(p \in [1, \infty)\), and introduce the spaces \(L^p_{\mu}(\mathbb{R})\) and \(L^p_{\theta}(\mathbb{R})\):
\[L^p_{\mu}(\mathbb{R}) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}) : \|f\|_{L^p_{\mu}(\mathbb{R})} = \int_{-\infty}^{\infty} |\mu(x)f(x)|^p \, dx < \infty \right\}.

2000 Mathematics Subject Classification. 34B05, 34B24, 34K06.
Key words and phrases. Admissible spaces, Sturm-Liouville equation.
For brevity, below we write $L_{p,\mu}$ and $L_{p,\theta}$, $\|\cdot\|_{p,\mu}$ and $\|\cdot\|_{p,\theta}$, instead of $L_p(\mathbb{R}, \mu)$, $L_p(\mathbb{R}, \theta)$ and $\|\cdot\|_{L_p(\mathbb{R}, \mu)}$, $\|\cdot\|_{L_p(\mathbb{R}, \theta)}$, respectively (for $\mu = 1$ we use the standard notation $L_p (L_p := L_p(\mathbb{R}))$ and $\|\cdot\|_p (\|\cdot\|_p := \|\cdot\|_{L_p})$. In addition, below by a solution of (1.1) we understand any function $y$, absolutely continuous together with its derivative and satisfying equality (1.1) almost everywhere on $\mathbb{R}$.

Let us introduce the following main definition (see [12, Ch.5, §50-51]):

**Definition 1.1.** We say that the spaces $L_{p,\mu}$ and $L_{p,\theta}$ make a pair $\{L_{p,\mu}, L_{p,\theta}\}$ admissible for equation (1.1) if the following requirements hold:

I) for every function $f \in L_{p,\theta}$ there exists a unique solution $y \in L_{p,\mu}$ of (1.1);

II) there is a constant $c(p) \in (0, \infty)$ such that regardless of the choice of a function $f \in L_{p,\theta}$ the solution $y \in L_{p,\mu}$ of (1.1) satisfies the inequality

$$\|y\|_{p,\mu} \leq c(p)\|f\|_{p,\theta}. \quad (1.5)$$

Let us in addition we make the following conventions: For brevity we say “problem I–II)” or “question on I–II)” instead of “problem (or question) on conditions for the functions $\mu$ and $\theta$ under which requirements I–II) of Definition 1.1 hold.” We say “the pair $\{L_{p,\mu}; L_{p,\theta}\}$ admissible for (1.1)” instead of “the pair of spaces $\{L_{p,\mu}; L_{p,\theta}\}$ admissible for equation (1.1)”, and we often omit the word “equation” before (1.1). By $c, c(\cdot)$ we denote absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations. Our general requirement (1.2) is assumed to be satisfied throughout the paper, is not referred to, and does not appear in the statements.

Let us return to Definition 1.1. The question on the admissibility of the pair $\{L_{p}, L_p\}$ for (1.1) was studied in [3, 6] (in [3, 6] for $\mu \equiv \theta \equiv 1$ in the case where I–II) were valid, we said that equation (1.1) is correctly solvable in $L_p$. We maintain this terminology in the present paper.) Let us quote the main result of [3, 6] (in terms of Definition 1.1).

**Theorem 1.2 ([3]).** The pair $\{L_{p}, L_p\}$ is admissible for (1.1) if and only if there is $a \in (0, \infty)$ such that $q_0(a) > 0$. Here

$$q_0(a) = \inf_{x \in \mathbb{R}} \left[ \int_{x-a}^{x+a} q(t) dt \right]. \quad (1.6)$$

Below we continue the investigation started in [3, 6].

Our goal is as follows: given equation (1.1), to determine requirements to the weights $\mu$ and $\theta$ under which the pair $\{L_{p,\mu}; L_{p,\theta}\}$, $p \in [1, \infty)$, is admissible for (1.1). Such an approach to the inversion of (1.1) allows to study this equation also in the case where Theorem 1.2 is not applicable, for example, in the following three cases:

1) $q_0(a) > 0$ for some $a \in (0, \infty)$, $f \notin L_p$, $p \in [1, \infty)$;
2) $q_0(a) = 0$ for all $a \in (0, \infty)$, $f \in L_p$, $p \in [1, \infty)$;
3) $q_0(a) = 0$ for all $a \in (0, \infty)$, $f \notin L_p$, $p \in [1, \infty)$.

Our main result (see Theorem 2.4 in §2 below) reduces the stated problem to the question on the boundedness of a certain integral operator $S : L_p \rightarrow L_p$ (see (2.10) in §2). From this criterion, under additional requirements to the functions $\mu$, $\theta$ and $q$, one can deduce some concrete particular conditions which control the solution of our problem. See §2 for such restrictions.
The structure of the paper is as follows: In §2, we present our main results and relevant comments; all the proofs of these assertions are collected in §4; §3 contains various technical facts and their proofs; see §5 for an example or the main statements.

We thank the anonymous referee for his remarks that allowed us to substantially improve the exposition of the paper.

2. Main results. Throughout the sequel we assume that our standing requirements to the functions $q$ (see (1.2)), and $\mu$ and $\theta$ (see §1) are satisfied, and we do not mention them in the statements.

**Theorem 2.1.** Suppose that the function $q$ is nonnegative and continuous at every point of the real axis. Suppose that for a given $p \in [1, \infty)$ the following condition holds:

$$
\int_{-\infty}^{0} \mu(t)^p dt = \int_{0}^{\infty} \mu(t)^p dt = \infty.
$$

(2.1)

Then the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1) only if inequalities (2.2) hold:

$$
\int_{x}^{-\infty} q(t)dt > 0, \quad \int_{x}^{\infty} q(t)dt > 0, \quad \forall x \in \mathbb{R}.
$$

(2.2)

To make our a priori requirements independent of the parameter $p \in [1, \infty)$, throughout the sequel we assume that together with (1.2), condition (2.2) holds. Similar to (1.2), below this condition is not quoted and does not appear in the statements.

We will also need the following known facts.

**Theorem 2.2 ([2]).** Suppose that (2.2) holds. Then the equation

$$
z''(x) = q(x)z(x), \quad x \in \mathbb{R},
$$

(2.3)

has a fundamental system of solutions (FSS) $\{u(x), v(x)\}, \ x \in \mathbb{R}$, such that

$$
u(x) > 0, \quad v(x) > 0, \quad u'(x) < 0, \quad v'(x) > 0, \quad \forall x \in \mathbb{R},
$$

(2.4)

$$
v'(x)u(x) - u'(x)v(x) = 1, \quad \forall x \in \mathbb{R},
$$

(2.5)

$$
\lim_{x \to -\infty} \frac{v(x)}{u(x)} = \lim_{x \to \infty} \frac{u(x)}{v(x)} = 0,
$$

(2.6)

$$
|\rho'(x)| < 1, \quad \forall x \in \mathbb{R}, \quad \rho(x) \overset{\text{def}}{=} u(x)v(x).
$$

(2.7)

Let us introduce the Green function of equation (1.1):

$$
G(x,t) = \begin{cases}
u(x)v(t), & x \geq t \\
u(t)v(x), & x < t 
\end{cases}
$$

(2.8)

Our main results are the following lemma and theorem.

**Lemma 2.3.** Suppose that the following condition holds:

$$
\int_{-\infty}^{0} \mu(t)dt = \int_{0}^{\infty} \mu(t)dt = \infty.
$$

(2.9)

Then for every $p \in [1, \infty)$ equation (2.3) has no solutions $z \in L_{p,\mu}$ apart from $z \equiv 0$.

Note that for $\mu \equiv 1$ Lemma 2.3 was proved in [2].
Theorem 2.4. Suppose that condition (2.9) holds, \( p \in [1, \infty) \). Then the pair \( \{L_p, \mu; L_p, \theta\} \) is admissible for (1.1) if and only if the operator \( S : L_p \to L_p \) is bounded. Here

\[
(Sf)(x) = \mu(x) \int_{-\infty}^{\infty} \frac{G(x, t)}{\theta(t)} f(t) dt, \quad x \in \mathbb{R}, \quad f \in L_p. \tag{2.10}
\]

We also note that since the Green function \( G(x, t) \) (see (2.8)) is usually not known, Theorem 2.4 seems to be only a theoretical fact that is not applicable to a particular equation (1.1). Below we show by examples (see Theorem 2.10 and §5 below) that such an impression is deceptive. Specifically, in the same way that Cauchy’s criterion for the convergence of a number series yields various convenient and efficient “working” criteria, Theorem 2.4 allows one to obtain, in a similar way, particular conditions for the admissibility of a given pair for a given equation (1.1).

Before we state Theorem 2.10, we need new definitions, auxiliary assertions and comments.

Lemma 2.5 ([4]). For any given \( x \in \mathbb{R} \), the equation in \( d \geq 0 \)

\[
\int_0^{\sqrt{2d}} \int_{x-t}^{x+t} q(\xi) d\xi dt = 2 \tag{2.11}
\]

has a unique finite positive solution.

In the sequel, the solution of (2.11) will be denoted by \( d(x), x \in \mathbb{R} \). This function was introduced in [4] and is one of the versions of auxiliary functions which were first (and then repeatedly) used M.O. Otelbaev (see [13]). Although it is defined in a somewhat exotic way, it is useful to take into account that the function \( q^*(x) \equiv d^{-2}(x) \) \((d^{-2} := 1/d^2)\) can be interpreted as a composed (in the sense of function theory) average of the function \( q(\xi), \xi \in \mathbb{R} \), at the point \( \xi = x \) with step \( d(x) \). Indeed, denote

\[
S_\varepsilon(q)(t) = \frac{1}{2t} \int_{x-t}^{x+t} q(\xi) d\xi, \quad t > 0, \quad x \in \mathbb{R},
\]

\[
M(f)(\eta) = \frac{1}{\eta^2} \int_0^{\sqrt{2\eta}} t f(t) dt, \quad \eta > 0.
\]

Clearly, \( S_\varepsilon(q)(t) \) is the Steklov average with step \( t > 0 \) of the function \( q(\xi), \xi \in \mathbb{R} \), at the point \( \xi = x \), and \( M(f)(\eta) \) is the average of the function \( f(t), t > 0 \) with step \( \eta > 0 \) at the point \( t = 0 \). Now, using

\[
q^*(x) = \frac{1}{d^2(x)} = \frac{1}{2d^2(x)} \int_0^{\sqrt{2d(x)}} \int_{x-t}^{x+t} q(\xi) d\xi dt
\]

\[
= \frac{1}{d^2(x)} \int_0^{\sqrt{2d(x)}} t \left[ \frac{1}{2t} \int_{x-t}^{x+t} q(\xi) d\xi \right] dt = M(S_\varepsilon(q))(d(x)).
\]

Note two additional properties of \( d(x), x \in \mathbb{R} \). (The proofs of all assertions related to the properties of the function \( d \) are given in §3.)

Lemma 2.6. The function \( d(x) \) is continuously differentiable for all \( x \in \mathbb{R} \), and the following inequality holds:

\[
\sqrt{2}|d'(x)| \leq 1, \quad x \in \mathbb{R}. \tag{2.12}
\]

\[
2|d'(x)| \leq \nu(x), \quad x \in \mathbb{R}. \tag{2.13}
\]
Then the following assertions hold:

\[ \nu(x) = d(x) \int_0^{\sqrt{2}d(x)} (q(x + t) - q(x - t))dt, \quad x \in \mathbb{R}. \]  

(2.14)

Now we can give the following definition.

**Definition 2.7.** We say that the function \( q \) belongs to the class \( H \) (and write \( q \in H \)) if the following equality holds:

\[ \lim_{|x| \to \infty} \nu(x) = 0. \]  

(2.15)

In the next assertion, we state an important property of the functions \( q \in H \).

**Lemma 2.8.** Let \( q \in H \). Then for any \( \varepsilon > 0 \) there is a constant \( c(\varepsilon) \in [1, \infty) \) such that for all \( x, t \in \mathbb{R} \) the following inequalities hold:

\[ c(\varepsilon)^{-1} \exp \left( -\varepsilon \left| \int_x^t \frac{d\xi}{d(x)} \right| \right) \leq \frac{d(t)}{d(x)} \leq c(\varepsilon) \exp \left( \varepsilon \left| \int_x^t \frac{d\xi}{d(x)} \right| \right). \]  

(2.16)

In addition, for \( \varepsilon \geq 1/\sqrt{2} \) inequalities (2.16) hold regardless of condition (2.15).

The next definition is based on Lemma 2.8 and fixes the set of weight functions \( \mu \) and \( \theta \) which we study further (using Theorem 2.10 below) for pairs of functions \( \nu \) and \( \theta \) generating pairs \( \{L_{p,\mu}; L_{p,\theta}\} \) admissible for (1.1).

**Definition 2.9.** Let \( q \in H \). We say that a pair of weights (weight functions) \( \{\mu, \theta\} \) agrees with the function \( q \) if for any \( \varepsilon > 0 \) there is a constant \( c(\varepsilon) \in [1, \infty) \) such that for all \( t, x \in \mathbb{R} \) one has the inequalities

\[ c(\varepsilon)^{-1} \exp \left( -\varepsilon \left| \int_x^t \frac{d\xi}{d(x)} \right| \right) \leq \sqrt{\frac{d(t)}{d(x)} \mu(t)}; \quad \sqrt{\frac{d(t)}{d(x)} \theta(t)} \leq c(\varepsilon) \exp \left( \varepsilon \left| \int_x^t \frac{d\xi}{d(x)} \right| \right). \]  

(2.17)

In the latter case, we say that the pair \( \{L_{p,\mu}; L_{p,\theta}\}, p \in [1, \infty), \) agrees with equation (1.1).

The next theorem gives a solution of the main problem of the present paper for equation (1.1) with coefficient \( q \in H \).

**Theorem 2.10.** Suppose that conditions (2.9) hold. Let \( q \in H \). Suppose that the pair \( \{L_{p,\mu}; L_{p,\theta}\}, p \in [1, \infty) \) agrees with equation (1.1). Then this pair is admissible for (1.1) if and only if \( m(q, \mu, \theta) < \infty \). Here

\[ m(q, \mu, \theta) = \sup_{x \in \mathbb{R}} \left( \frac{\mu(x)}{\theta(x)} d^2(x) \right). \]  

(2.18)

The next assertions are convenient for the study of concrete equations. They are obvious and are given without proofs.

**Theorem 2.11.** Let \( q \in H \), and suppose that

\[ d_0 \overset{\text{def}}{=} \sup_{x \in \mathbb{R}} d(x) = \infty, \]  

(2.19)

\[ \int_{-\infty}^{0} q^*(x)dx = \int_{0}^{\infty} q^*(x)dx = \infty, \quad q^*(x) = \frac{1}{d^2(x)}, \quad x \in \mathbb{R}. \]  

(2.20)

Then the following assertions hold:

A) for \( p \in [1, \infty) \) the pair \( \{L_{p}; L_{p}\} \) is not admissible for (1.1);
B) for \( p \in [1, \infty) \) the pair \( \{L_{p,q}; L_p\} \) is admissible for (1.1).

**Theorem 2.12.** Let \( q \in H \), and suppose that the weight function \( \theta(x), x \in \mathbb{R} \), is such that \( m_0 > 0 \) where

\[
m_0 = \inf_{x \in \mathbb{R}} (q^*(x)\theta(x)), \quad q^*(x) = \frac{1}{d^2(x)}. \tag{2.21}
\]

Then for \( p \in [1, \infty) \) the pair \( \{L_{p,q}; L_{p,\theta}\} \) is admissible for (1.1).

Now we will apply the collection of definitions and assertions introduced above to particular equations (1.1) in order to present methods for checking relations (2.15), (2.17), (2.18), (2.20) and (2.21). All these relations include the implicit function \( d(x) \), \( x \in \mathbb{R} \) whose exact values can be found only special rare cases. However, it is easy to see that to check these relations, it is enough to have sharp by order two-sided estimates of the function \( d(x) \). Such estimates can easily be found using different methods for different requirements to the function \( q \). Below we present an assertion of such a type.

**Theorem 2.13.** Suppose that conditions (1.2) and (2.2) holds and the function \( q(x) \) can be written in the form

\[
q(x) = q_1(x) + q_2(x), \quad x \in \mathbb{R}, \tag{2.22}
\]

where \( q_1(x), x \in \mathbb{R} \), is positive and absolutely continuous together with its derivative, and \( q_2 \in L_{1 \text{loc}}^1(\mathbb{R}) \). Denote

\[
A(x) = \left[ 0, \frac{2}{\sqrt{q_1(x)}} \right], \quad x \in \mathbb{R}, \tag{2.23}
\]

\[
\kappa_1(x) = \frac{1}{q_1(x)^{3/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_1''(\xi)d\xi \right|, \quad x \in \mathbb{R}, \tag{2.24}
\]

\[
\kappa_2(x) = \frac{1}{\sqrt{q_1(x)}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi)d\xi \right|, \quad x \in \mathbb{R}. \tag{2.25}
\]

If we have the condition

\[
\kappa_1(x) \to 0, \quad \kappa_2(x) \to 0 \quad \text{as} \quad |x| \to \infty, \tag{2.26}
\]

then the following relations hold:

\[
d(x)\sqrt{q_1(x)} = 1 + \varepsilon(x), \quad |\varepsilon(x)| \leq 2(\kappa_1(x) + \kappa_2(x)), \quad |x| \gg 1, \tag{2.27}
\]

\[
c^{-1} \leq d(x)\sqrt{q_1(x)} \leq c \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.28}
\]

To prove inequalities (2.17), the following lemma can be useful.

**Lemma 2.14.** Suppose that a function \( \mu(x) \) is defined, positive and differentiable for all \( x \in \mathbb{R} \), let \( q \in H \), and let \( d(x), x \in \mathbb{R} \), denote the auxiliary function from Lemma 2.5. Then, if the equality

\[
\lim_{|x| \to \infty} \frac{\mu'(x)}{\mu(x)}d(x) = 0 \tag{2.29}
\]

holds, then for any given \( \varepsilon > 0 \) there is a constant \( c(\varepsilon) \in [1, \infty) \) such that for all \( t, x \in \mathbb{R} \) inequalities (2.17) hold.
3. Auxiliary assertions. In this section, we present the properties of the function \( d(x), \, x \in \mathbb{R} \) (see Lemma 2.5) and related properties of the FSS \( \{u, v\} \) of equation (2.3).

Recall that conditions (1.2) and (2.2) are assumed to hold, here and in the sequel, and are not mentioned in the statements.

Proof of Lemma 2.6. The existence of the derivative \( d'(x) \), \( x \in \mathbb{R} \) is a consequence of the theory of implicit functions [7, Ch.III, §1, no.3]. It is proven in the same way as in [5]. The following relations are deduced from (2.11):

\[
\int_0^{\sqrt{2}d(x)} \int_{x-t}^{x+t} q(\xi) d\xi dt = 2 \quad \Rightarrow 
\]

\[
0 = \sqrt{2}d'(x) \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi + \int_0^{\sqrt{2}d(x)} [q(x + t) - q(x - t)] dt 
\]

\[
= \sqrt{2}d'(x) \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi + \left[ \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi \right] 
\]

\[
\Rightarrow |d'(x)| = \frac{1}{\sqrt{2}} \left| \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi \right| \leq \frac{1}{\sqrt{2}} 
\]

\[
\Rightarrow (2.11) 
\]

Now, from (2.11) we obtain the inequality

\[
2 \leq \sqrt{2}d(x) \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi, \quad x \in \mathbb{R}. 
\]

Together with the formula for \( |d'(x)| \) and (2.14), this implies that

\[
|d'(x)| \leq \frac{d(x)}{\sqrt{2}} \left| \int_{x}^{x+\sqrt{2}d(x)} q(\xi) d\xi - \int_{x-\sqrt{2}d(x)}^{x} q(\xi) d\xi \right| \cdot \left( \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi \right)^{-1} 
\]

\[
\leq \frac{1}{2} d(x) \left| \int_{x}^{x+\sqrt{2}d(x)} q(\xi) d\xi - \int_{x-\sqrt{2}d(x)}^{x} q(\xi) d\xi \right| = \frac{\nu(x)}{2}, \quad x \in \mathbb{R} \Rightarrow (2.13) 
\]

Lemma 3.1. For \( x \in \mathbb{R} \), we have the inequalities

\[
4^{-1}d(x) \leq d(t) \leq 4d(x), \quad \text{if} \quad |t - x| \leq d(x). \quad (3.1) 
\]

Proof. Below we use Lagrange’s formula and (2.12):

\[
|d(t) - d(x)| = |d'(\theta)||t - x| \leq \frac{d(x)}{\sqrt{2}} \quad \Rightarrow 
\]

\[
d(t) \leq \left( 1 + \frac{1}{\sqrt{2}} \right) d(x) \leq 4d(x) \quad \text{for} \quad t \in [x - d(x), x + d(x)] 
\]

\[
d(t) \geq \left( 1 - \frac{1}{\sqrt{2}} \right) d(x) \geq \frac{d(x)}{4} \quad \text{for} \quad t \in [x - d(x), x + d(x)]. 
\]
Lemma 3.2. For \( x \in \mathbb{R} \), we have the inequalities (see Theorem 2.2):

\[
\frac{1}{e} \leq \frac{u(t)}{u(x)}; \quad \frac{v(t)}{v(x)} \leq c \quad \text{if} \quad |t - x| \leq d(x).
\] (3.2)

Proof. Below we use the following two assertions.

Theorem 3.3 ([8]). For \( x, t \in \mathbb{R} \), we have the Davies-Harrell representations for the solution \( \{u(x), v(x)\} \) and the Green function \( G(x, t) \) :

\[
u(x) = \sqrt{\rho(x)} \exp \left( -\frac{1}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right), \quad v(x) = \sqrt{\rho(x)} \exp \left( \frac{1}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right),
\] (3.3)

\[
G(x, t) = \sqrt{\rho(x)\rho(t)} \exp \left( -\frac{1}{2} \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right).
\] (3.4)

Here \( x_0 \) is a unique solution of the equation \( u(x) = v(x), x \in \mathbb{R} \) (see [2]), the function \( \rho \) is defined in (2.7).

Theorem 3.4 ([4]). Suppose that (1.2) and (2.2) hold. Then we have the Otelbaev inequalities:

\[
\frac{d(x)}{2\sqrt{2}} \leq \rho(x) \leq \sqrt{2d(x)}, \quad x \in \mathbb{R}.
\] (3.5)

Remark 3.5. Two-sided, sharp by order estimates of the function \( \rho \) were first obtained by M. Otelbaev (see [14]), and therefore all such inequalities are referred to by his name. Note that the inequalities given in [14] are expressed in terms of another auxiliary function, more complicated than \( d(x), x \in \mathbb{R} \), and are proven under auxiliary requirements to the function \( q \).

Let us now consider (3.2). Below we use (3.5) and (3.1):

\[
\int_{x-d(x)}^{x+d(x)} \frac{d\xi}{\rho(\xi)} = \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{\rho(\xi)} \cdot \frac{d(x)}{\rho(x)} \cdot \frac{d(\xi)}{d(x)} \leq 2\sqrt{2} \cdot 4 \cdot 2 = c < \infty,
\]

\[
\int_{x-d(x)}^{x+d(x)} \frac{d\xi}{\rho(\xi)} = \int_{x-d(x)}^{x+d(x)} \frac{d(\xi)}{\rho(\xi)} \cdot \frac{d(x)}{\rho(x)} \cdot \frac{d\xi}{d(x)} \geq \frac{1}{\sqrt{2}} \cdot \frac{1}{4} \cdot 2 \geq c^{-1} > 0.
\]

Now we use this together with (3.3) and obtain

\[
u(t) \leq \sqrt{\rho(t)} \exp \left( -\frac{1}{2} \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right),
\]

\[
\geq \sqrt{\frac{d(x)}{\rho(x)}} \cdot \frac{d(t)}{d(x)} \cdot \frac{\rho(t)}{d(t)} \exp \left( \frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{\rho(\xi)} \right),
\]

\[
\geq c^{-1} > 0;
\]

\[
u(t) \leq \sqrt{\rho(t)} \exp \left( \frac{1}{2} \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right) \leq \sqrt{\frac{d(x)}{\rho(x)}} \cdot \frac{d(t)}{d(x)} \cdot \frac{\rho(t)}{d(t)} \exp \left( \frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{\rho(\xi)} \right),
\]

\[
\leq c < \infty.
\]

Inequalities (3.2) for the solution \( v \) are checked similarly, and estimates (3.2) for \( \rho \) follow from the estimates of \( u \) and \( v \) and (2.7).
Lemma 3.6. For a given $x \in \mathbb{R}$, consider the function
\[ F(\eta) = \int_{0}^{\sqrt{2}q(\eta)} \int_{x-t}^{x+t} q(\xi)d\xi dt, \quad \eta \geq 0. \] (3.6)

The function $F(\eta)$ is differentiable and non-negative, together with its derivative, and
\[ F(0) = 0, \quad F(\infty) = \infty. \] (3.7)

In addition, the inequality $\eta \geq d(x)$ $(0 \leq \eta \leq d(x))$ holds if and only if $F(\eta) \geq 2$ ($F(\eta) \leq 2$).

Proof. To prove that the function $F(\eta)$ is differentiable and the functions $F(\eta)$ and $F'(\eta)$ are non-negative for $\eta \geq 0$, we use properties of integral. The last assertion of the lemma follows from Lagrange’s formula and the relations
\[ F(\eta) - 2 = F(\eta) - F(d(x)) = F'(\theta)(\eta - d(x)). \]

\[ \square \]

Lemma 3.7. Let a function $f$ be defined on $\mathbb{R}$ and absolutely continuous together with its derivative. Then for all $x \in \mathbb{R}$ and $t \geq 0$, we have the equality
\[ \int_{x-t}^{x+t} f(\xi)d\xi = 2f(x)t + \int_{0}^{t} \int_{t}^{t_2} f''(t_2)dt_3dt_2dt_1. \] (3.8)

Proof. To obtain (3.8), we use the following simple transformations
\[ \int_{x-t}^{x+t} f(\xi)d\xi = \int_{0}^{t} \left[ f(x + t_1) + f(x - t_1) \right] dt_1 \]
\[ = 2f(x)t + \int_{0}^{t} \left[ f(x + t_1) - f(x) \right] dt_1 - \int_{0}^{t} \left[ f(x) - f(x - t_1) \right] dt_1 \]
\[ = 2f(x)t + \int_{0}^{t} \left[ \int_{0}^{t_1} (f(x + t_2))'dt_2 \right] dt_1 - \int_{0}^{t} \left[ \int_{0}^{t_1} (f(x - t_2))'dt_2 \right] dt_1 \]
\[ = 2f(x)t + \int_{0}^{t} \int_{0}^{t_1} \left[ f(x + t_2)' - f(x - t_2)' \right] dt_2 dt_1 \]
\[ = 2f(x)t + \int_{0}^{t} \int_{t}^{t_2} f''(t_3)dt_3dt_2dt_1. \]

\[ \square \]

Proof of Theorem 2.13. Set
\[ \eta(x) = \frac{1 - \delta(x)}{\sqrt{q_1(x)}}, \quad \delta(x) = 2(\kappa_1(x) + \kappa_2(x)), \quad |x| \gg 1. \]

Then by (3.6), (3.8), (2.22), (2.23), (2.24), (2.25) and (2.26), we have
\[ F(\eta(x)) = \int_{0}^{\sqrt{2}\eta(x)} \int_{x-t}^{x+t} q_1(\xi)d\xi dt + \int_{0}^{\sqrt{2}\eta(x)} \int_{x-t}^{x+t} q_2(\xi)d\xi dt \]
\[ \leq \int_{0}^{\sqrt{2}\eta(x)} \left[ 2q_1(x)t + \int_{0}^{t} \int_{t}^{t_2} q''_2(t_3)dt_3dt_2dt_1 \right] dt \]
\[ + \sqrt{2}\eta(x) \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi)d\xi \right| \leq \left( \sqrt{2}\eta(x) \right)^2 q_1(x) \]
say, the second one, does not hold:

\[
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\]

Together with (2.27), this implies (2.28).

Proofs of the main results.

4. Without loss of generality, in what follows we assume \( x \geq 0 \). Let us introduce the functions \( \varphi \) and \( f_0 \).

1) \( \varphi \in C^\infty(\mathbb{R}) \), \( \text{supp} \varphi = [x_0, \infty) \), \( 0 \leq \varphi(x) \leq 1 \) for \( x \in \mathbb{R} \), \( \varphi(x) \equiv 1 \) for \( x \geq x_0 + 1 \) (4.3)
2) \( f_0(x) := -\varphi''(x) + q(x)\varphi(x), \ x \in \mathbb{R}. \) (4.4)

From 1)–2) we obtain the equality
\[
q(x)\varphi(x) \equiv 0, \quad x \in \mathbb{R} \quad \Rightarrow \quad f_0(x) = -\varphi''(x), \quad x \in \mathbb{R} \quad \Rightarrow \quad \text{supp} f_0 = [x_0, x_0 + 1]. \quad (4.5)
\]

According to (4.5), we conclude that \( f_0 \in L_{p,\theta} : \)
\[
\|f_0\|^p_{L_{p,\theta}} = \int_{-\infty}^{\infty} |\theta(x)f_0(x)|^p dx = \int_{x_0}^{x_0+1} |\theta(x)\varphi''(x)|^p dx = c(x_0) < \infty.
\]

Since the pair \( \{L_{p,\mu}; L_{p,\theta}\} \) is admissible for \( (1.1) \), we conclude that \( (1.1) \) for \( f = f_0 \) has a unique solution \( y_0 \in L_{p,\mu} \). Then (see (4.4) and (4.5))
\[
y_0(x) = \varphi(x) + z(x), \quad x \in \mathbb{R}, \quad (4.6)
\]
where \( z(x), x \in \mathbb{R}, \) is some solution of \( (2.3) \). From \( (2.3) \) and \( (4.1) \), we obtain the equality
\[
z''(x) = 0 \quad \text{for} \quad x \in [x_0, \infty) \quad \Rightarrow \quad z(x) = c_1 + c_2x \quad \text{for} \quad x \geq x_0. \quad (4.7)
\]

Let us show that \( c_2 = 0 \). Assume to the contrary that \( c_2 \neq 0 \). Choose \( x_1 \) so that to have the inequality
\[
\frac{1 + c_1}{c_2} \cdot \frac{1}{x} \leq \frac{1}{2} \quad \text{for} \quad x \geq x_1 \geq x_0 + 1. \quad (4.8)
\]

Then (see (4.7))
\[
\infty > \|y_0\|^p_{L_{p,\mu}} \geq \int_{x_1}^{\infty} \mu(x)^p|\varphi(x) + z(x)|^p dx = \int_{x_1}^{\infty} \mu(x)^p|1 + c_1 + c_2x|^p dx
\]
\[
\geq |c_2x_1|^p \int_{x_1}^{\infty} \mu(x)^p \left[ 1 - \left| \frac{1 + c_1}{c_2} \right| \frac{1}{x} \right]^p dx \geq \left| \frac{c_2x_1}{2} \right|^p \frac{\int_{x_1}^{\infty} \mu(x)^p dx}{\int_{x_1}^{\infty} \mu(x)^p dx} = \infty,
\]
and we get a contradiction. Hence \( c_2 = 0 \). Let us check that also \( c_1 = 0 \). Assume that \( c_1 \neq 0 \). Since \( \varphi \in C^\infty(\mathbb{R}) \), from (4.2) it follows that \( \varphi(x_0) = \varphi'(x_0) = 0 \) and therefore (see (4.7)):
\[
y(x_0) = \varphi(x_0) + z(x_0) = c_1,
\]
\[
y'(x_0) = \varphi'(x_0) + z'(x_0) = 0.
\]

In addition, \( \varphi(x) \equiv 0 \) for \( x \leq x_0 \), and therefore from (4.5) and (4.6) it follows that the function \( z \) is a solution of the Cauchy problem
\[
\begin{cases}
z''(x) = q(x)z(x), & x \leq x_0 \\
z(x_0) = c_1, & z'(x_0) = 0.
\end{cases} \quad (4.9, 4.10)
\]

Further, without loss of generality, we assume that \( c_1 = 1 \). Let us check that then we have the inequality
\[
z(x) \geq 1 \quad \text{for} \quad x \leq x_0. \quad (4.11)
\]

Towards this end, first note that since \( z(x_0) = 1 \), we have \( z(x) > 0 \) in some left half-neighborhood of the point \( x_0 \) (i.e., for \( x \in (x_0 - \varepsilon, x_0] \) for some \( \varepsilon > 0 \)). But then \( z(x) > 0 \) for all \( x < x_0 \). Indeed, if this is not the case, then \( z(x) \) has at least one zero on \((-\infty, x_0)\). Let \( \tilde{x} \) be the first zero of \( z(x) \) to the left from \( x_0 \). Then \( z' (\tilde{x}) \geq 0 \). Indeed, if \( z'(\tilde{x}) < 0 \) and \( z(\tilde{x}) = 0 \), then \( z(x) < 0 \) in some right half-neighborhood
of the $\dot{x}$. But $z(x_0) = 1$ and $\dot{x} < x_0$. Hence, the interval $(\dot{x}, x_0)$ contains a zero of $z(x)$, contrary to the definition of the point $\dot{x}$. Thus $z''(\dot{x}) \geq 0$. On the other hand,

$$z'(x_0) - z'(\dot{x}) = \int_{\dot{x}}^{x_0} q(\xi) z(\xi) d\xi \Rightarrow$$

$$z'(\dot{x}) = -\int_{x_0}^{\dot{x}} q(\xi) z(\xi) d\xi \leq 0.$$ 

Hence $z'(\dot{x}) = 0$. But then the function $z(x)$ is a solution of the Cauchy problem

$$z''(x) = q(x) z(x), \quad x \leq x_0 \quad \Rightarrow \quad z(x) \equiv 0, \quad x \leq x_0.$$ 

We get a contradiction because $z(x_0) = 1$. Thus $z(x) > 0$ for $x \leq x_0$. Then for $x \leq x_0$, we have

$$-z'(x) = z'(x_0) - z'(x) = \int_x^{x_0} q(\xi) z(\xi) d\xi \geq 0 \Rightarrow -z'(x) \leq 0, \quad x \leq x_0.$$ 

Hence $z(x) \geq z(x_0) = 1$ for $x \leq x_0$. This implies that

$$\infty > \|y_0\|_{p, \mu} = \int_{-\infty}^{\infty} |\mu(x)y_0(x)|^p dx \geq \int_{-\infty}^{x_0} |\mu(x)y_0(x)|^p dx = \int_{-\infty}^{x_0} |\mu(x)z(x)|^p dx \geq \int_{-\infty}^{x_0} |\mu(x)|^p dx = \infty.$$ 

We get a contradiction. Hence $c_1 = 0$, and we obtain the equality

$$y_0(x) = \varphi(x), \quad x \geq x_0 \quad \Rightarrow$$

$$\infty > \|y_0\|_{p, \mu}^p \geq \int_{x_0+1}^{\infty} |\mu(x)y_0(x)|^p dx = \int_{x_0+1}^{\infty} |\mu(x)\varphi(x)|^p dx = \int_{x_0+1}^{\infty} \mu(x)^p dx = \infty.$$ 

We get a contradiction. Hence (4.1) does not hold. \hfill \Box

**Proof of Lemma 2.3.** Let us show that in the case of (2.9) for all $p \in [1, \infty)$ we have the equalities

$$\int_{-\infty}^{\infty} (\mu(t)u(t))^p dt = \int_{-\infty}^{\infty} (\mu(t)v(t))^p dt = \infty. \quad (4.12)$$ 

We only consider the second equality because the first one can be proved in the same way. For $p = 1$ equality (4.12) follows from Theorem 2.2 and (2.9) is a straightforward manner. Let $p \in (1, \infty)$, $p' = p(p-1)^{-1}$. The following relations rely only on Theorem 2.2:

$$\int_0^\infty \frac{dt}{v(t)^{p'}} = \int_0^\infty \frac{v'(t)v(t)^{-p'}}{v(t)} dt \leq \frac{1}{v'(0)} \int_0^\infty v'(t)v(t)^{-p'} dt$$

$$= \frac{1}{p' - 1} \frac{1}{v'(0)} \left( \frac{1}{v(0)^{p'-1}} - \frac{1}{v(\infty)^{p'-1}} \right) \leq \frac{1}{p' - 1} \frac{1}{v'(0)v(0)^{p'-1}} = c(p) < \infty.$$ 

(4.13)

Let $A > 0$. Below we use Hölder’s inequality and (4.13):

$$\int_0^A \mu(t) dt \leq \left[ \int_0^A (\mu(t)v(t))^p dt \right]^{1/p} \cdot \left[ \int_0^A \frac{dt}{v(t)^{p'}} \right]^{1/p'} \leq c(p) \left[ \int_0^A (\mu(t)v(t))^p dt \right]^{1/p}.$$
Now, to obtain (4.12), in the last inequality we let \( A \) tend to infinity. Let us now go to the proof of the lemma. By Theorem 2.2, the general solution of (2.3) is of the form

\[ z(x) = c_1 u(x) + c_2 v(x), \quad x \in \mathbb{R}. \]

Let \( z \in L_{p,\mu} \). Then \( c_2 = 0 \). Indeed, if \( c_2 \neq 0 \), then denote \( x_1 \gg 1 \), a number such that for all \( x \geq x_1 \) we have the inequality (see (2.6)):

\[ \left| \frac{c_1}{c_2} \frac{u(x)}{v(x)} \right| \leq \frac{1}{2}, \quad x \geq x_1. \quad (4.14) \]

Now from (4.12), (4.14) and Theorem 2.2 it follows that

\[ \infty > \|z\|_{p,\mu}^p = \int_{-\infty}^{\infty} |\mu(x)(c_1 u(x) + c_2 v(x))|^p dx \]

\[ \geq |c_2|^p \int_{x_1}^{\infty} (\mu(x)v(x))^p \left| 1 - \left| \frac{c_1}{c_2} \frac{u(x)}{v(x)} \right| \right| dx \geq \frac{|c_2|^p}{2} \int_{x_1}^{\infty} (\mu(x)v(x))^p dx = \infty. \]

We get a contradiction. Hence \( c_2 = 0 \). The equality \( c_1 = 0 \) now follows from (2.4) and (4.12).

\[ \square \]

Remark 4.1. Below we use the following known theorems. It is convenient to formulate them separately from the main statements of the present paper.

**Theorem 4.2** ([11]). Let \( \mu \) and \( \theta \) be continuous positive functions in \( \mathbb{R} \), and let \( H \) be an integral operator

\[ (Hf)(t) = \mu(t) \int_{t}^{\infty} \theta(\xi)f(\xi) d\xi, \quad t \in \mathbb{R}. \quad (4.15) \]

For \( p \in (1, \infty) \), the operator \( H : L_p \rightarrow L_p \) is bounded if and only if \( H_p < \infty \). Here \( H_p = \sup_{x \in \mathbb{R}} H_p(x) \),

\[ H_p(x) = \left( \int_{-\infty}^{x} \mu(t)^p dt \right)^{1/p} \cdot \left( \int_{x}^{\infty} \theta(t)^{p'} dt \right)^{1/p'}, \quad p' = \frac{p}{p-1}. \quad (4.16) \]

In addition,

\[ H_p \leq \|H\|_{p \rightarrow p} \leq (p)^{1/p}(p')^{1/p'} H_p. \quad (4.17) \]

**Theorem 4.3** ([11]). Let \( \mu \) and \( \theta \) be continuous positive functions in \( \mathbb{R} \), and let \( \tilde{H} \) be an integral operator

\[ (\tilde{H}f)(t) = \mu(t) \int_{-\infty}^{t} \theta(\xi)f(\xi) d\xi, \quad t \in \mathbb{R}. \quad (4.18) \]

For \( p \in (1, \infty) \) the operator \( \tilde{H} : L_p \rightarrow L_p \) is bounded if and only if \( \tilde{H}_p < \infty \). Here \( \tilde{H}_p = \sup_{x \in \mathbb{R}} \tilde{H}_p(x) \)

\[ \tilde{H}_p(x) = \left[ \int_{-\infty}^{x} \theta(t)^{p'} dt \right]^{1/p'} \cdot \left[ \int_{x}^{\infty} \mu(t)^p dt \right]^{1/p}, \quad p' = \frac{p}{p-1}. \quad (4.19) \]

In addition,

\[ \tilde{H}_p \leq \|\tilde{H}\|_{p \rightarrow p} \leq (p)^{1/p}(p')^{1/p'} \tilde{H}_p. \quad (4.20) \]

Let us introduce the Green operator

\[ (Gf)(x) = \int_{-\infty}^{\infty} G(x,t)f(t) dt, \quad x \in \mathbb{R}. \quad (4.21) \]
Theorem 4.4 ([6]). Suppose that (2.2) holds, and let \( p \in [1, \infty) \). Then equation (1.1) is correctly solvable in \( L_p \) (or, in other words, the pair \( \{L_p, L_p\} \) is admissible for (1.1)) if and only if the operator \( G : L_p \to L_p \) is bounded. In the latter case, for \( f \in L_p \), the solution \( y \in L_p \) of (1.1) is of the form \( y = Gf \).

Theorem 4.5 ([3]). For \( p \in [1, \infty) \), equation (1.1) is correctly solvable in \( L_p \) (i.e., the pair \( \{L_p, L_p\} \) is admissible for (1.1)) if and only if inequalities (2.2) hold and \( \hat{d}_0 < \infty \). Here

\[
\hat{d}_0 = \sup_{x \in \mathbb{R}} \hat{d}(x) \tag{4.22}
\]

and \( \hat{d}(x) \) is the unique finite positive solution of the equation in \( d \geq 0 \):

\[
d \int_{x-d}^{x+d} q(\xi) d\xi = 2 \tag{4.23}
\]

Note that the function \( \hat{d}(x) \) was introduced by M. Otelbaev (see [13]). The following inequalities hold (see [4] and (2.11)):

\[
\frac{d(x)}{\sqrt{2}} \leq \hat{d}(x) \leq \sqrt{2d}(x), \quad x \in \mathbb{R}. \tag{4.24}
\]

Proof of Theorem 2.4 for \( p \in (1, \infty) \). Necessity.

We need the following lemma.

Lemma 4.6. Let \( p \in [1, \infty) \). Suppose that conditions (2.9) hold, and the pair \( \{L_{p,\mu}; L_{p,\theta}\} \) is admissible for (1.1). Then, if \( f \in L_p \) and \( \text{supp} \, f = [x_1, x_2] \), \( x_2 - x_1 < \infty \), then \( f \in L_{p,\theta} \) and the solution \( y \in L_{p,\mu} \) of (1.1) which corresponds to \( f \) that has the form \( y = Gf \) (see (2.8) and (4.21)).

Proof. Below we only consider the case \( p \in (1, \infty) \) (for \( p = 1 \) the arguments are similar). Let us continue the function \( f \) by zero beyond the segment \( [x_1, x_2] \) and maintain the original notation. From the obvious inequalities

\[
c^{-1} \leq \theta(x) \leq c, \quad x \in [x_1, x_2], \quad c = c(x_1, x_2), \tag{4.25}
\]

it follows that \( f \in L_{p,\theta} \). Set (see (2.8), (4.21))

\[
\tilde{y}(x) = \int_{-\infty}^{x} G(x, t)f(t)dt
\]

\[
= u(x) \int_{-\infty}^{x} v(t)f(t)dt + v(x) \int_{x}^{\infty} u(t)f(t)dt, \quad x \in \mathbb{R}. \tag{4.26}
\]

Let us estimate the integrals in (4.26):

\[
\int_{-\infty}^{x} v(t)|f(t)|dt \leq \left[ \int_{x_1}^{x_2} \left( \frac{v(t)}{\theta(t)} \right)^{p'} dt \right]^{1/p'} \cdot \left[ \int_{x_1}^{x_2} |\theta(t)f(t)|^p dt \right]^{1/p}
\]

\[
\leq c \left( \int_{x_1}^{x_2} v(t)^{p'} dt \right)^{1/p'} \cdot \|f\|_{p,\theta}, \quad x \in \mathbb{R}, \tag{4.27}
\]

\[
\int_{x}^{\infty} u(t)|f(t)|dt \leq \left[ \int_{x_1}^{x_2} \left( \frac{u(t)}{\theta(t)} \right)^{p'} dt \right]^{1/p'} \cdot \left[ \int_{x_1}^{x_2} |\theta(t)f(t)|^p dt \right]^{1/p}
\]

\[
\leq c \left( \int_{x_1}^{x_2} u(t)^{p'} dt \right)^{1/p'} \cdot \|f\|_{p,\theta}, \quad x \in \mathbb{R}. \tag{4.28}
\]
From (4.27) and (4.28) it follows that the function \( \hat{y}(x) \), \( s \in \mathbb{R} \), is well-defined. It is also easy to see that the function \( \hat{y}(x) \), \( x \in \mathbb{R} \) is a particular solution of (1.1). But, since \( f \in L_{p,\theta} \), (1.1) has a unique solution \( y \in L_{p,\theta} \). This means that we have the equality

\[
y(x) = \hat{y}(x) + c_1 u(x) + c_2 v(x), \quad x \in \mathbb{R}.
\]

Let us check that \( c_1 = c_2 = 0 \). Assume, say, that \( c_2 \neq 0 \). Then for \( x \geq x_2 \), we get

\[
|y(x)| \geq |c_2| v(x) - |c_1| u(x) - u(x) \int_{x_1}^{x_2} v(t) |f(t)| dt
\]

\[
= |c_2| v(x) \left[ 1 - \frac{c_1}{c_2} \frac{u(x)}{v(x)} - \frac{u(x)}{c_2 v(x)} \int_{x_1}^{x_2} v(t) |f(t)| dt \right].
\]

From (2.6) and (4.27) it follows that there exists \( x_3 \geq \max\{1, x_2\} \) such that

\[
|y(x)| \geq \frac{1}{2} |c_2| v(x) \quad \text{for} \quad x \geq x_3 \quad \Rightarrow \quad \text{(see (4.12))}:
\]

\[
\infty > \|y\|_{p,\mu}^p \geq \int_{x_3}^{\infty} |\mu(x) y(x)|^p dx \geq \frac{|c_2|^p}{2} \int_{x_3}^{\infty} |\mu(x) v(x)|^p dx = \infty.
\]

We get a contradiction. Hence \( c_2 = 0 \). Similarly, we prove that also \( c_1 = 0 \), and therefore \( y = \hat{y} \) (see (4.26)).

Let \([x_1, x_2]\) be any finite segment. Set

\[
f(t) = \begin{cases} \theta(t)^{-p'} \cdot u(t)^{p'-1}, & t \in [x_1, x_2] \\ 0, & t \notin [x_1, x_2] \end{cases}
\]

(4.29)

Then

\[
\|f\|_{L_{p,\theta}}^p = \int_{x_1}^{x_2} \theta(t)^p |f(t)|^p dt = \int_{x_1}^{x_2} \frac{\theta(t)^p u(t)^{p'-1}(t)}{\theta(t)^p} \frac{dt}{u(t)^p} = \int_{x_1}^{x_2} \left( \frac{u(t)}{\theta(t)} \right)^{p'} dt < \infty.
\]

(4.30)

Therefore, since the pair \( \{L_{p,\mu}; L_{p,\theta}\} \) is admissible for (1.1), in the case of (4.29) equation (1.1) has a solution \( y \in L_{p,\mu} \). This solution is of the form (4.21) (see Lemma 4.6). This implies that

\[
\infty > \|y\|_{p,\mu}^p = \int_{-\infty}^{\infty} |\mu(x) y(x)|^p dx
\]

\[
= \left\{ \int_{-\infty}^{\infty} \mu(x)^p \left[ u(x) \int_{-\infty}^{x} v(t) f(t) dt + v(x) \int_{x}^{\infty} u(t) f(t) dt \right]^p dx \right\}
\]

\[
\geq \int_{-\infty}^{\infty} (\mu(x) v(x))^p \left( \int_{x}^{\infty} u(t) f(t) dt \right)^p dx \geq \int_{x_1}^{x_1} (\mu(x) v(x))^p \left( \int_{x}^{\infty} u(t) f(t) dt \right)^p dx
\]

\[
\geq \int_{-\infty}^{x_1} (\mu(x) v(x))^p dx \left( \int_{x_1}^{x_2} u(t) f(t) dt \right)^p = \int_{-\infty}^{x_1} (\mu(x) v(x))^p dx \left( \int_{x_1}^{x_2} \left( \frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^p
\]

(4.31)

Now, using (4.31), (4.30) and (1.5), we obtain

\[
\left[ \int_{-\infty}^{x_1} (\mu(x) v(x))^p dx \right]^{1/p} \int_{x_1}^{x_2} \left( \frac{u(t)}{\theta(t)} \right)^{p'} dt
\]

\[
\leq \|y\|_{p,\mu} \leq c(p) \|f\|_{p,\theta} = c(p) \left[ \int_{x_1}^{x_2} \left( \frac{u(t)}{\theta(t)} \right)^{p'} dt \right]^{1/p} \Rightarrow
\]
our assertion now follows from the triangle inequality for norms.

Since in this inequality \( x_1 \) and \( x_2 \) \((x_1 \leq x_2)\) are arbitrary numbers, we conclude that

\[
M = \sup_{x \in \mathbb{R}} \left( \int_{-\infty}^{x} (\mu(t)v(t))^{p} dt \right)^{1/p} \left( \int_{x}^{\infty} \left( \frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \leq c(p) < \infty.
\]

This inequality means that the operator \( S_2 : L_p \rightarrow L_p \),

\[
(S_2 f)(x) = \mu(x)v(x) \int_{x}^{\infty} \frac{u(t)}{\theta(t)} f(t) dt, \quad x \in \mathbb{R}
\]

is bounded (see Theorem 4.2). Similarly, we use Theorem 4.3 to conclude that the operator \( S_1 : L_p \rightarrow L_p \),

\[
(S_1 f)(x) = \mu(x)u(x) \int_{-\infty}^{x} \frac{v(t)}{\theta(t)} f(t) dt, \quad x \in \mathbb{R}
\]

is bounded. Since we have the equality (see (2.8) and (2.10))

\[
S = S_1 + S_2
\]

our assertion now follows from the triangle inequality for norms.

\[\square\]

**Proof of Theorem 2.4. Sufficiency.**

**Lemma 4.7.** Let \( p \in [1, \infty) \), and let \( S, S_1, S_2 \) be operators (2.10), (4.33) and (4.32), respectively. Then we have the inequalities

\[
\frac{\|S_1\|_{p \rightarrow p} + \|S_2\|_{p \rightarrow p}}{2} \leq \|S\|_{p \rightarrow p} \leq \|S_1\|_{p \rightarrow p} + \|S_2\|_{p \rightarrow p}.
\]

**Proof.** The upper estimate in (4.35) follows from (4.34). To prove the lower estimate in (4.35), we use the following obvious relations:

\[
\|S_1(f)\|_p = \int_{-\infty}^{\infty} \mu(x)^p \left| u(x) \int_{-\infty}^{x} \frac{v(t)}{\theta(t)} f(t) dt \right|^p dx
\]

\[
\leq \int_{-\infty}^{\infty} \mu(x)^p \left( u(x) \int_{-\infty}^{x} \frac{v(t)}{\theta(t)} |f(t)| dt \right)^p dx
\]

\[
\leq \int_{-\infty}^{\infty} \mu(x)^p \left[ u(x) \int_{-\infty}^{x} \frac{v(t)}{\theta(t)} |f(t)| dt + v(x) \int_{x}^{\infty} \frac{u(t)}{\theta(t)} |f(t)| dt \right]^p dx
\]

\[
= \int_{-\infty}^{\infty} \mu(x) \int_{-\infty}^{\infty} \frac{G(x, t)}{\theta(t)} |f(t)| dt \right|^p dx = \|S(\|f\|)_p \|_p \leq \|S_1\|_{p \rightarrow p} \cdot \|f\|_p.
\]

This implies that \( \|S_1\|_{p \rightarrow p} \leq \|S\|_{p \rightarrow p} \). Similarly, we check that \( \|S_2\|_{p \rightarrow p} \leq \|S\|_{p \rightarrow p} \). These inequalities imply the lower estimate in (4.35).

\[\square\]

Let us now go to the proof of the theorem. Since (2.2) holds, equation (2.3) has a FSS \( \{u, v\} \) with the properties from Theorem 2.2. Since the operator \( S : L_p \rightarrow L_p \) is bounded, so are also the operators \( S_i : L_p \rightarrow L_p, i = 1, 2 \) (see (4.35)). Then, by Theorems 4.2 and 4.3, we obtain the inequalities

\[
\tilde{M}_p \overset{\text{def}}{=} \sup_{x \in \mathbb{R}} \left( \int_{-\infty}^{x} \frac{v(t)}{\theta(t)} \right)^{1/p'} \left( \int_{x}^{\infty} (\mu(t)u(t))^{p} dt \right)^{1/p} \leq \|S\|_{p \rightarrow p} < \infty.
\]

\[\text{(4.36)}\]
These inequalities imply that the function
\[ y(x) = (Gf)(x) = u(x) \int_{-\infty}^{x} v(t)f(t)dt + v(x) \int_{x}^{\infty} u(t)f(t)dt, \quad x \in \mathbb{R} \quad (4.38) \]
is well-defined because the integrals in (4.38) converge:
\[
\begin{align*}
\int_{-\infty}^{x} v(t)|f(t)|dt & \leq \left( \int_{-\infty}^{x} \left( \frac{v(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \cdot \|f\|_{p,\vartheta}, \quad x \in \mathbb{R}, \\
\int_{x}^{\infty} u(t)|f(t)|dt & \leq \left( \int_{-\infty}^{x} \left( \frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \cdot \|f\|_{p,\vartheta}, \quad x \in \mathbb{R}.
\end{align*}
\]

Further, one can check in a straightforward manner (see Theorem 2.2) that the function \( y(x), x \in \mathbb{R} \) is a solution of \((1.1)\). In addition,
\[
\|y\|_{p,\mu} = \left[ \int_{-\infty}^{\infty} \left( \mu(x) \left| \int_{-\infty}^{\infty} G(x,t)f(t)dt \right| \right)^{p} dx \right]^{1/p}
\]
\[
= \left[ \int_{-\infty}^{\infty} \left( \mu(x) \left| \int_{-\infty}^{\infty} \frac{G(x,t)}{\theta(t)} f(t)dt \right| \right)^{p} dx \right]^{1/p}
\]
\[
= \|S(\theta f)\|_{p} \leq \|S\|_{p \to p} \cdot \|\theta f\|_{p} = \|S\|_{p \to p} \cdot \|f\|_{p,\vartheta}.
\]
i.e., (1.5) holds. It only remains to refer to Lemma 2.3.

\( \square \)

Proof of Theorem 2.4 for \( p = 1 \). Necessity.
Let \([x_1, x_2]\) be an arbitrary finite segment, and let \( f \in L_1 \) be such that \( \text{supp} \ f = [x_1, x_2] \). Then (see (4.25)) \( f \in L_{1,\vartheta} \) and therefore equation \((1.1)\) with such a right-hand side has a unique solution \( y \in L_{1,\mu} \). By Lemma 4.6, this solution is given by formula \( y = Gf \) (see (4.21)) and satisfies (1.5). Let us introduce the operator \( \tilde{S} : \)
\[
(\tilde{S}g)(x) = \mu(x) \int_{x_1}^{x_2} \frac{G(x,t)}{\theta(t)} g(t)dt, \quad x \in [x_1, x_2], \quad g \in L_1(x_1, x_2)
\]
and the function \( g \) given on the segment \([x_1, x_2]\) by the formula
\[
g(x) = \theta(x)f(x), \quad x \in [x_1, x_2].
\]
Then we have
\[
\|\tilde{S}g\|_{L_1(x_1, x_2)} = \int_{x_1}^{x_2} \left| \mu(x) \int_{x_1}^{x_2} \frac{G(x,t)}{\theta(t)} g(t)dt \right| dx
\]
\[
= \int_{x_1}^{x_2} \mu(x) \left| \int_{x_1}^{x_2} G(x,t)f(t)dt \right| dx = \int_{x_1}^{x_2} \mu(x) \left| \int_{-\infty}^{\infty} G(x,t)f(t)dt \right| dx
\]
\[
\leq \int_{x_1}^{x_2} \mu(x) |g(x)|dx \leq \int_{-\infty}^{\infty} \mu(x) |g(x)|dx = \|y\|_{1,\mu} \leq c(1)\|f\|_{1,\vartheta}
\]
\[
= c(1) \int_{-\infty}^{\infty} \theta(t) |f(t)|dt = c(1) \int_{x_1}^{x_2} \theta(t) |f(t)|dt = c(1)\|g\|_{L_1(x_1, x_2)}.
\]
This implies that
\[
\|\tilde{S}\|_{L_1(x_1, x_2) \to L_1(x_1, x_2)} \leq c(1).
\quad (4.39)
\]
Theorem 4.8 ([10]). Let $-\infty < a < b \leq \infty$, let $K(x, t)$ be a continuous function for $s, t \in (a, b)$, and let $K$ be an integral operator
\[
(Kf)(t) = \int_a^b K(s, t)f(s)\,ds, \quad t \in (a, b). \tag{4.40}
\]
Then we have the inequality
\[
||K||_{L_1(a, b) \to L_1(a, b)} = \sup_{x \in (a, b)} \int_a^b |K(s, t)|\,dt. \tag{4.41}
\]
By Theorem 4.8, using (4.39), (4.40) and (4.41), we obtain
\[
\sup_{x \in [x_1, x_2]} \frac{1}{\theta(x)} \int_x^{x_2} \mu(t)G(x, t)dt = ||S||_{L_1(x_1, x_2) \to L_1(x_1, x_2)} \leq c(1).
\]
In the last inequality, $x_1$ and $x_2$ are arbitrary numbers. Hence
\[
\sup_{x \in \mathbb{R}} \frac{1}{\theta(x)} \int_{-\infty}^{\infty} \mu(t)G(x, t)dt \leq c(1) < \infty.
\]
But then by Theorem 4.8 we obtain that $||S||_{L_1 \to L_2} \leq c(1) < \infty$, as required.

Proof of Theorem 2.4 for $p = 1$. Sufficiency.

From (2.2) it follows that equation (2.3) has a FSS $\{u, v\}$ (see Theorem 2.2), the Green function and the operator $S$ are defined (see (2.8) and (2.10)). Further, the operators $S_i, i = 1, 2$ (see (4.32), (4.33)) are bounded because so is the operator $S : L_1 \to L_1$ (see Lemma 4.7). Let now $f \in L_{1, \vartheta}$ and $g = \vartheta \cdot |f|$. Then $0 \leq g \in L_1, S_i g \in L_1, i = 1, 2$, and one has the inequalities
\[
0 \leq (S_i g)(x) < \infty, \quad \forall x \in \mathbb{R}, \quad i = 1, 2. \tag{4.42}
\]
We will prove (4.42) for $i = 1$ (the case $i = 2$ is considered in a similar way). Assume to the contrary that there exists $x_1 \in \mathbb{R}$ such that $(S_1 g)(x_1) = \infty$. Let $x_2 > x_1$. Then, since the functions $\mu$ and $u$ are continuous, we have
\[
(S_1 g)(x_2) = \mu(x_2)u(x_2) \int_{x_2}^x \frac{v(t)}{\theta(t)} g(t)dt \\
\geq \frac{\mu(x_2)u(x_2)}{\mu(x_1)u(x_1)} \left(\mu(x_1)u(x_1) \int_{-\infty}^{x_1} \frac{v(t)}{\theta(t)} g(t)dt\right) = \frac{\mu(x_2)u(x_2)}{\mu(x_1)u(x_1)} (S_1 g)(x_1) = \infty
\]
\[
\Rightarrow \quad \infty > ||Sg||_1 = \int_{-\infty}^{\infty} \mu(x)u(x) \left|\int_{-\infty}^{x} \frac{v(t)}{\theta(t)} g(t)dt\right|\,dx \\
\geq \int_{x_1}^{\infty} \mu(x)u(x) \left(\int_{-\infty}^{x} \frac{v(t)}{\theta(t)} g(t)dt\right)\,dx = \int_{x_1}^{\infty} (S_1 g)(x)\,dx = \infty.
\]
We get a contradiction. Hence, inequalities (4.42) hold. From (4.42) and the definition of $g$ we obtain
\[
\int_{-\infty}^{x} v(t)|f(t)|dt < \infty, \quad \int_{-\infty}^{\infty} u(t)|f(t)|dt < \infty \quad \forall x \in \mathbb{R}. \tag{4.43}
\]
For instance,
\[
\int_{-\infty}^{x} v(t)|f(t)|dt = \frac{1}{\mu(x)u(x)} \left[\mu(x)u(x) \int_{\infty}^{x} \frac{v(t)}{\theta(t)} (\vartheta(t)|f(t)|)dt\right]
\]
Proof of Lemma 2.8. It is easy to see that inequalities (2.16) for differentiable, and satisfies the following relations (see Lemmas 2.5 and 2.6): regardless of condition (2.15).

It remains to note that by Lemma 2.3 this solution is unique in the class $L_{1,\mu}$. □

Proof of Lemma 2.8. It is easy to see that inequalities (2.16) for $\varepsilon \geq 1/\sqrt{2}$ hold regardless of condition (2.15).

Indeed, under conditions (1.2) and (2.2), the function $d(x)$, $x \in \mathbb{R}$ is well-defined, differentiable, and satisfies the following relations (see Lemmas 2.5 and 2.6):

$$
-\varepsilon \leq -\frac{1}{\sqrt{2}} \leq d'(\xi) \leq \frac{1}{\sqrt{2}} \leq \varepsilon, \; \xi \in \mathbb{R} \quad \Rightarrow
$$

$$
-\varepsilon \leq \frac{d'(\xi)}{d(\xi)} \leq \frac{\varepsilon}{d(\xi)}, \; \xi \in \mathbb{R} \quad \Rightarrow
$$

$$
\exp\left( -\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) \leq \frac{d(t)}{d(x)} \leq \exp\left( \varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right), \; x, t \in \mathbb{R} \quad \Rightarrow (2.16).
$$

Let now $\varepsilon \in (0, 1/\sqrt{2})$. Then there exists $x_0 = x_0(\varepsilon) \gg 1$ such that we have the inequality (see (2.13) and (2.15)):

$$
|d'(x)| \leq \varepsilon \quad \text{if} \quad |x| \geq x_0. \quad (4.44)
$$

It is easy to see that all possible cases of placing the numbers $t, x \in \mathbb{R}$ and the segments $(-\infty, -x_0]$, $[-x_0, x_0]$ and $[x_0, \infty]$ can be put in the following table:

| 1.1 | 1.2 | 1.3 |
|-----|-----|-----|
| $x \in (-\infty, -x_0)$ | $x \in (-\infty, -x_0)$ | $x \in (-\infty, -x_0)$ |
| $t \in (-\infty, -x_0)$ | $t \in [-x_0, x_0]$ | $t \in [x_0, \infty]$ |

| 2.1 | 2.2 | 2.3 |
|-----|-----|-----|
| $x \in [-x_0, x_0]$ | $x \in [-x_0, x_0]$ | $x \in [-x_0, x_0]$ |
| $t \in (-\infty, -x_0]$ | $t \in [-x_0, x_0]$ | $t \in [x_0, \infty)$ |

| 3.1 | 3.2 | 3.3 |
|-----|-----|-----|
| $x \in (x_0, \infty)$ | $x \in (x_0, \infty)$ | $x \in (x_0, \infty)$ |
| $t \in (-\infty, -x_0]$ | $t \in [-x_0, x_0]$ | $t \in [x_0, \infty)$ |

(4.45)

We check inequalities (2.16) separately in each case appearing in (4.45).
Cases 1.1 and 3.3. Both cases are treated in the same way. Let us introduce the standing notation for the whole proof:

\[ m(\varepsilon) = \min_{t \in [-x_0, x_0]} d(t), \quad M(\varepsilon) = \max_{t \in [-x_0, x_0]} d(t) \]

\[ c(\varepsilon) = \max \left\{ \frac{1}{m(\varepsilon)}, M(\varepsilon) \right\}, \quad a = \min \{ x, t \}, \quad b = \max \{ x, t \}. \]

Consider, say, Case 3.3. The following implications are obvious:

\[ -\varepsilon \leq d'(\xi) \leq \varepsilon \quad \text{for} \quad \xi \in [a, b] \Rightarrow -\varepsilon \int_a^b \frac{d\xi}{d(\xi)} \leq \ln \frac{d(b)}{d(a)} \leq \varepsilon \int_a^b \frac{d\xi}{d(\xi)} = \varepsilon \left| \int_a^b \frac{d\xi}{d(\xi)} \right| \Rightarrow \exp \left( -\varepsilon \left| \int_a^b \frac{d\xi}{d(\xi)} \right| \right) \leq \frac{d(b)}{d(a)}, \quad \frac{d(a)}{d(b)} \leq \exp \left( \varepsilon \left| \int_a^b \frac{d\xi}{d(\xi)} \right| \right) \Rightarrow (2.16). \]

Cases 1.2 and 2.1. Both cases are treated in the same way. For instance, in Case 1.2 we have

\[ \frac{d(t)}{d(x)} = \frac{d(t)}{d(-x_0)} \cdot \frac{d(-x_0)}{d(x)} \leq c(\varepsilon)^2 \exp \left( \varepsilon \left| \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} \right| \right) \]

\[ \leq c(\varepsilon)^2 \exp \left( \varepsilon \left| \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} + \int_{-x_0}^t \frac{d\xi}{d(\xi)} \right| \right) = c(\varepsilon)^2 \exp \left( \varepsilon \left| \int_{-x_0}^t \frac{d\xi}{d(\xi)} \right| \right); \]

\[ \frac{d(t)}{d(x)} = \frac{d(t)}{d(-x_0)} \cdot \frac{d(-x_0)}{d(x)} \geq c(\varepsilon)^{-2} \exp \left( -\varepsilon \left| \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} \right| \right) \]

\[ \geq c(\varepsilon)^{-2} \exp \left( -\varepsilon \left| \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} + \int_{-x_0}^t \frac{d\xi}{d(\xi)} \right| \right) = c(\varepsilon)^{-2} \exp \left( -\varepsilon \left| \int_{-x_0}^t \frac{d\xi}{d(\xi)} \right| \right) \]

\[ \Rightarrow (2.16). \]

Cases 1.3 and 3.1. Both cases are treated in the same way. For instance, in Case 1.3 we have

\[ \frac{d(t)}{d(x)} = \frac{d(-x_0)}{d(x)} \cdot \frac{d(x_0)}{d(-x_0)} \cdot \frac{d(t)}{d(x_0)} \leq \frac{M}{m} \exp \left( \varepsilon \left| \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} \right| + \varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right) \]

\[ \leq c(\varepsilon)^2 \exp \left[ \varepsilon \left( \int_{x_0}^{x_0} \frac{d\xi}{d(\xi)} + \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} + \int_{x_0}^t \frac{d\xi}{d(\xi)} \right) \right] \]

\[ = c(\varepsilon)^2 \exp \left( \varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right); \]

\[ \frac{d(t)}{d(x)} = \frac{d(-x_0)}{d(x)} \cdot \frac{d(x_0)}{d(-x_0)} \cdot \frac{d(t)}{d(x_0)} \geq \frac{m}{M} \exp \left( -\varepsilon \left| \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} \right| - \varepsilon \left| \int_{-x_0}^t \frac{d\xi}{d(\xi)} \right| \right) \]

\[ \geq c(\varepsilon)^{-2} \exp \left[ -\varepsilon \left( \int_{x_0}^{x_0} \frac{d\xi}{d(\xi)} + \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} + \int_{x_0}^t \frac{d\xi}{d(\xi)} \right) \right] \]

\[ \geq c(\varepsilon)^{-2} \exp \left( -\varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right). \]
Proof of Theorem 2.10 for Case 2.2.

We have
\[ \frac{d(t)}{d(x)} \leq \frac{M(\varepsilon)}{m(\varepsilon)} \leq c(\varepsilon)^2 \exp \left( \varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right); \]
\[ \frac{d(t)}{d(x)} \geq \frac{m(\varepsilon)}{M(\varepsilon)} \geq c(\varepsilon)^{-2} \exp \left( -\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right). \]

Cases 2.3 and 3.2.

Both cases are treated in the same way. For instance, in Case 2.3 we have
\[ \frac{d(t)}{d(x)} = \frac{d(x_0)}{d(x)} \cdot \frac{d(t)}{d(x_0)} \leq \frac{M(\varepsilon)}{m(\varepsilon)} \exp \left( \varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right) \leq c(\varepsilon)^2 \exp \left( \varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right); \]
\[ \frac{d(t)}{d(x)} = \frac{d(x_0)}{d(x)} \cdot \frac{d(t)}{d(x_0)} \geq \frac{m(\varepsilon)}{M(\varepsilon)} \exp \left( -\varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right) \geq c(\varepsilon)^{-2} \exp \left( -\varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right). \]

\[ \square \]

Proof of Theorem 2.10 for \( p \in (1, \infty) \). Necessity.

We need some auxiliary assertions.

Lemma 4.9. Let \( p \in [1, \infty) \), \( p' = p(p-1)^{-1} \). Denote
\[ M_p(x) = \left( \int_{-\infty}^x (\mu(t)v(t))^p dt \right)^{1/p} \cdot \left( \int_x^\infty \left( \frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'}, \quad x \in \mathbb{R}, \quad (4.46) \]
\[ \hat{M}_p(x) = \left( \int_{-\infty}^x \left( \frac{v(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \cdot \left( \int_x^\infty (\mu(t)u(t))^p dt \right)^{1/p}, \quad x \in \mathbb{R}. \quad (4.47) \]

Then we have the equalities (see (2.7)):
\[ M_p(x) = \left[ \int_{-\infty}^x (\mu(t)v(t))^p \exp \left( -\frac{p}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \]
\[ \cdot \left[ \int_x^\infty \left( \frac{\sqrt{\mu(t)}}{\theta(t)} \right)^{p'} \exp \left( -\frac{p'}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'}, \quad x \in \mathbb{R}, \quad (4.48) \]
\[ \hat{M}_p(x) = \left[ \int_{-\infty}^x \left( \frac{\sqrt{\mu(t)}}{\theta(t)} \right)^{p'} \exp \left( -\frac{p'}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \]
\[ \cdot \left[ \int_x^\infty (\mu(t)\sqrt{\mu(t)})^p \exp \left( -\frac{p}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p}, \quad x \in \mathbb{R}. \quad (4.49) \]
Proof. Equalities (4.48) and (4.49) are proved in the same way. Consider, say, (4.48). This equality can be obtained by substituting formulas (3.3) in (4.46):

\[ M_p(x) = \left[ \int_{-\infty}^{\infty} \left( \mu(t) \sqrt{\rho(t)} \right)^p \exp \left( \frac{p}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \]

\[ \cdot \left[ \int_{-\infty}^{\infty} \left( \frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left( -\frac{p'}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \]

\[ = \left[ \int_{-\infty}^{\infty} \left( \mu(t) \sqrt{\rho(t)} \right)^p \exp \left( -\frac{p}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) \cdot \exp \left( \frac{p}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \]

\[ \cdot \left[ \int_{-\infty}^{\infty} \left( \frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left( -\frac{p'}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) \cdot \exp \left( -\frac{p'}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \]

\[ = \left[ \int_{-\infty}^{\infty} \left( \mu(t) \sqrt{\rho(t)} \right)^p \exp \left( -\frac{p}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \]

\[ \cdot \left[ \int_{-\infty}^{\infty} \left( \frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left( -\frac{p'}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \]

Let us introduce some more notation:

\[ \varphi(x, t) = \begin{cases} \frac{\mu(x) v(x)}{\mu(t) v(t)}, & \text{if } x \leq t \\ \frac{\mu(t) v(t)}{\mu(x) v(x)}, & \text{if } x \geq t \end{cases}, \quad \psi(x, t) = \begin{cases} \frac{\theta(x) u(t)}{\theta(t) u(x)}, & \text{if } x \leq t \\ \frac{\theta(t) u(x)}{\theta(x) u(t)}, & \text{if } x \geq t \end{cases}. \]

\[ (4.50) \]

**Lemma 4.10.** Under the hypotheses of the theorem, for a given \( \varepsilon > 0 \) and for all \( t, x \in \mathbb{R} \), we have the inequality

\[ \max \{ \varphi(x, t); \psi(x, t) \} \leq c(\varepsilon) \exp \left( \left( \sqrt{2\varepsilon} - \frac{1}{2} \right) \left| \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right| \right). \]  

**Proof.** We will check inequality (4.51) for the function \( \varphi \) (for the function \( \psi \) the proof of (4.51) is similar). Below we use (3.3), (3.5) and (2.17). Let \( x \geq t \). Then

\[ \frac{\mu(t)}{\mu(x)} \cdot \frac{v(t)}{v(x)} = \frac{\mu(t)}{\mu(x)} \cdot \sqrt{\frac{\rho(t)}{\rho(x)}} \exp \left( -\frac{1}{2} \int_{t}^{x} \frac{d\xi}{\rho(\xi)} \right) \leq c \frac{\mu(t)}{\mu(x)} \cdot \sqrt{\frac{d(t)}{d(x)}} \exp \left( -\frac{1}{2} \int_{t}^{x} \frac{d\xi}{\rho(\xi)} \right) \]

\[ \leq c(\varepsilon) \exp \left( \varepsilon \int_{t}^{x} \frac{d\xi}{\rho(\xi)} - \frac{1}{2} \int_{t}^{x} \frac{d\xi}{\rho(\xi)} \right) \leq c(\varepsilon) \exp \left( \left( \sqrt{2\varepsilon} - \frac{1}{2} \right) \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right) \]

\[ = c(\varepsilon) \exp \left( \left( \sqrt{2\varepsilon} - \frac{1}{2} \right) \left| \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right| \right). \]

Similarly, for \( x \leq t \), we have:

\[ \frac{\mu(x)}{\mu(t)} \cdot \frac{v(x)}{v(t)} = \frac{\mu(x)}{\mu(t)} \cdot \sqrt{\frac{\rho(x)}{\rho(t)}} \exp \left( -\frac{1}{2} \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right) \leq c \frac{\mu(x)}{\mu(t)} \cdot \sqrt{\frac{d(x)}{d(t)}} \exp \left( -\frac{1}{2} \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right) \]
\[ \leq c(\varepsilon) \exp \left( \varepsilon \int_{t}^{\infty} \frac{d\xi}{d(\xi)} - \frac{1}{2} \int_{t}^{\infty} \frac{d\xi}{\theta(\xi)} \right) \leq c(\varepsilon) \exp \left( \left( \sqrt{2\varepsilon} - \frac{1}{2} \right) \int_{t}^{\infty} \frac{d\xi}{\rho(\xi)} \right) \]

\[ = c(\varepsilon) \exp \left( \left( \sqrt{2\varepsilon} - \frac{1}{2} \right) \int_{t}^{\infty} \frac{d\xi}{\rho(\xi)} \right) \].

\[ \text{Lemma 4.11.} \text{ Under conditions (1.1) and (2.2), we have the inequality} \]

\[ \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \leq 8 \quad \forall x \in \mathbb{R}. \quad (4.52) \]

\[ \text{Proof.} \text{ Estimate (4.52) follows from (3.1):} \]

\[ \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} = \int_{x-d(x)}^{x+d(x)} \frac{d(x)}{d(\xi)} \cdot \frac{d\xi}{d(x)} \leq \int_{x-d(x)}^{x+d(x)} 4 \frac{d\xi}{d(x)} = 8. \]

\[ \text{Lemma 4.12.} \text{ Under the hypotheses of the theorem, we have the inequalities} \]

\[ c^{-1} \leq \frac{\mu(t)}{\mu(x)}, \quad \frac{\theta(t)}{\theta(x) } \leq c \quad \text{if} \quad t \in [x - d(x), x + d(x)], \quad x \in \mathbb{R}. \quad (4.53) \]

\[ \text{Proof.} \text{ We will only check inequalities (4.53) for the function } \mu \text{ (the proof of (4.53)} \]

\[ \text{for the function } \theta \text{ is similar). In (2.17), set } \varepsilon = \frac{1}{2}. \text{ Now for } |t - x| \leq d(x), x \in \mathbb{R}, \text{ we use (3.1), (2.17) and (4.52):} \]

\[ \frac{\mu(t)}{\mu(x)} \leq c \sqrt{\frac{d(x)}{d(t)}} \exp \left( \frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) \leq c \exp \left( \frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) \leq c < \infty, \]

\[ \frac{\mu(t)}{\mu(x)} \geq c^{-1} \sqrt{\frac{d(x)}{d(t)}} \exp \left( -\frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) \geq c^{-1} \exp \left( -\frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) \geq c^{-1} > 0. \]

Let us now go to the theorem. Since condition (2.2) holds, by Theorem 2.2, a FSS \( \{u, v\} \) of equation (2.3) is defined, and thus the operator \( S \) (see (2.10)) is also defined. Since the pair \( \{L_{p, \mu}, L_{p, \theta}\} \) is admissible for (1.1), by Theorem 2.4 the operator \( S : L_{p} \to L_{p}, \quad p \in [1, \infty] \) is bounded. Then so are the operators \( S_{1} : L_{p} \to L_{p}, \quad i = 1, 2 \) (see (4.35)). Let \( p \in (1, \infty) \). Consider, say, the operator \( S_{2} : L_{p} \to L_{p} \). Since it is bounded, we have \( M_{p} < \infty \) by Theorem 4.2 (see (4.37) and (4.46). Below we use this fact together with Lemma 2.5, (3.5), (4.52) and (4.53):
we use Theorem 4.8, (4.32), Lemma 2.5, (3.3), (3.5), (4.52), (4.53) and (3.1):

\[ \geq \sup_{x \in \mathbb{R}} \left[ \int_{x-d(x)}^{x} \left( \sqrt{\rho(t)} \mu(t) \right)^p \exp \left( -\frac{p}{2} \int_{t}^{x} \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \]

\[ \cdot \left[ \int_{x}^{x+d(x)} \left( \frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left( -\frac{p'}{2} \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \]

\[ \geq c^{-1} \sup_{x \in \mathbb{R}} \left[ \int_{x-d(x)}^{x} \left( \sqrt{d(t)} \mu(t) \right)^p \exp \left( -\sqrt{2p} \int_{x-d(x)}^{x} \frac{d\xi}{d(\xi)} \right) dt \right]^{1/p} \]

\[ \cdot \left[ \int_{x}^{x+d(x)} \left( \frac{\sqrt{d(t)}}{\theta(t)} \right)^{p'} \exp \left( -\sqrt{2p'} \int_{x}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) dt \right]^{1/p'} \]

\[ \geq c^{-1} \sup_{x \in \mathbb{R}} \frac{\mu(x)}{\theta(x)} d^2(x) = c^{-1} m(q, \mu, \theta), \]

as required. Let \( p = 1 \). Since the operator \( S : L_1 \to L_1 \) is bounded (Theorem 2.4), so are the operators \( S_i : L_1 \to L_1, i = 1, 2 \) (see Lemma 4.7). Let, say, \( i = 2 \). Below we use Theorem 4.8, (4.32), Lemma 2.5, (3.3), (3.5), (4.52), (4.53) and (3.1):

\[ \infty > \|S_2\|_{1 \to 1} = \sup_{x \in \mathbb{R}} \frac{u(x)}{\theta(x)} \int_{-\infty}^{x} \mu(t)v(t)dt \geq \sup_{x \in \mathbb{R}} \frac{u(x)}{\theta(x)} \int_{x-d(x)}^{x} \mu(t)v(t)dt \]

\[ = \sup_{x \in \mathbb{R}} \frac{\sqrt{\rho(x)}}{\theta(x)} \int_{x-d(x)}^{x} \mu(t)\sqrt{\rho(t)} \exp \left( -\frac{1}{2} \int_{t}^{x} \frac{d\xi}{\rho(\xi)} \right) \int_{x-d(x)}^{x} \mu(t)\sqrt{d(t)} \exp \left( -\sqrt{2} \int_{x-d(x)}^{x} \frac{d\xi}{d(\xi)} \right) dt \]

\[ \geq c^{-1} \sup_{x \in \mathbb{R}} \frac{\sqrt{d(x)}}{\theta(x)} \int_{x-d(x)}^{x} \mu(t)\sqrt{d(t)} \exp \left( -\sqrt{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) dt \]

\[ \geq c^{-1} \sup_{x \in \mathbb{R}} \frac{\mu(x)}{\theta(x)} d^2(x) = c^{-1} m(q, \mu, \theta). \]

**Proof of Theorem 2.10. Sufficiency.**

It is easy to show that the operators \( S_i : L_p \to L_p, p \in [1, \infty), i = 1, 2 \), are bounded. Indeed, then so is the operator \( S : L_p \to L_p, p \in [1, \infty) \) (see (4.35)), and then by Theorem 2.4 the pair \( (L_p, \mu) \) is admissible for (1.1). Both operators \( S_i, i = 1, 2 \), are treated in the same way, and therefore below we only consider the operator \( S_2 \) (see (4.32)). Below, when estimating \( \|S_2\|_{p \to p}, p \in (1, \infty) \), we use Theorem 4.2, (4.32), (4.46), (4.50), (4.51), (2.17) for \( \varepsilon = 1/4\sqrt{2}, (3.5) \) and (2.18):
\[ \|S_2\|_{p \rightarrow p} \leq c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} (\mu(t)v(t))^{p} \, dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} \left( \frac{u(t)}{\theta(t)} \right)^{p'} \, dt \right]^{1/p'} \]

\[ = c(p) \sup_{x \in \mathbb{R}} (\mu(x)v(x))^{1/p'} \left[ \int_{-\infty}^{x} \left( \frac{\mu(t)v(t)}{\mu(x)v(x)} \right)^{p-1} (\mu(t)v(t)) \, dt \right]^{1/p} \]

\[ \cdot \left( \frac{u(x)}{\theta(x)} \right)^{1/p} \left[ \int_{x}^{\infty} \left( \frac{u(t)}{\theta(t)} \right)^{p'-1} \, dt \right]^{1/p'} \]

\[ = c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \phi(x,t)^{p-1} (\mu(t)v(t)) \, dt \right]^{1/p} \]

\[ \cdot \left[ \mu(x)v(x) \int_{x}^{\infty} \psi(x,t)^{p'-1} \left( \frac{u(t)}{\theta(t)} \right)^{1/p'} \, dt \right]^{1/p} \]

\[ = c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \psi(x,t)^{p'-1} \left( \frac{\mu(t)}{\theta(t)} \right)^{1/p} \, dt \right]^{1/p} \]

\[ \cdot \left[ \int_{x}^{\infty} \psi(x,t) \, dt \right]^{1/p'} \]

\[ \leq c(\varepsilon) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \left( \frac{\mu(t)}{\theta(t)} \right)^{2} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \right]^{1/p} \]

\[ \cdot \left[ \int_{x}^{\infty} \left( \frac{\mu(t)}{\theta(t)} \right)^{2} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \right]^{1/p'} \]

\[ \leq c(\varepsilon) m(q, \mu, \theta) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \right]^{1/p} \]

\[ \cdot \left[ \int_{x}^{\infty} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \right]^{1/p'} \]

\[ \leq cm(q, \mu, \theta) < \infty. \]

Consider the case \( p = 1 \). Below, when estimating \( \|S\|_{1 \rightarrow 1} \), we use (4.8), (2.10), (2.8) and (2.17) for \( \varepsilon = 1/4\sqrt{2} \), and (3.5):

\[ \|S\|_{1 \rightarrow 1} = \sup_{x \in \mathbb{R}} \frac{1}{\theta(x)} \int_{-\infty}^{\infty} \mu(t)G(x,t) \, dt \]

\[ = \sup_{x \in \mathbb{R}} \frac{\sqrt{\rho(x)}}{\theta(x)} \int_{-\infty}^{\infty} \mu(t) \sqrt{\rho(t)} \exp \left( -\frac{1}{2} \int_{x}^{t} \frac{d\xi}{\rho(\xi)} \right) \, dt \]

\[ = \sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} \left( \frac{\mu(t)}{\theta(t)} d^2(t) \right) \cdot \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \]

\[ \cdot \int_{x}^{\infty} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \]

\[ \cdot \int_{x}^{\infty} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \]

\[ \leq c(\varepsilon) m(q, \mu, \theta) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^{x} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \right]^{1/p} \]

\[ \cdot \left[ \int_{x}^{\infty} \left( \frac{\rho(t)}{\theta(t)} \right)^{2} \, dt \right]^{1/p'} \]

\[ \leq cm(q, \mu, \theta) < \infty. \]
From (4.54), one can easily deduce the estimates

Proof of Lemma 2.14. Fix $\varepsilon > 0$ and choose $x_0 = x_0(\varepsilon) >> 1$ in order to have the inequalities

$$- \frac{\varepsilon}{3} \leq \frac{\mu'(\xi)}{\mu(\xi)} d(\xi); \quad d'(\xi) \leq \frac{\varepsilon}{3} \quad \text{for all} \quad |\xi| \geq x_0. \quad (4.54)$$

From (4.54), one can easily deduce the estimates

$$- \frac{2\varepsilon}{3} \cdot \frac{1}{d(\xi)} \leq \frac{(\mu(\xi)d(\xi))'}{\mu(\xi)d(\xi)} \leq \frac{2\varepsilon}{3} \cdot \frac{1}{d(\xi)} \quad \text{for all} \quad |\xi| \geq x_0. \quad (4.55)$$

Let, say, $t \geq x \geq x_0$. Then from (4.55), we obtain

$$\exp\left(-\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)}\right) \leq \frac{\mu(t)d(t)}{\mu(x)d(x)} \leq \exp\left(\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)}\right), \quad t \geq x \geq x_0. \quad (4.56)$$

Let us write (4.56) in a different way:

$$\sqrt{\frac{d(x)}{d(t)}} \exp\left(-\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)}\right) \leq \frac{\mu(t)d(t)}{\mu(x)d(x)} \leq \sqrt{\frac{d(x)}{d(t)}} \exp\left(\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)}\right), \quad t \geq x \geq x_0.$$

We now combine the latter estimates with inequalities (2.16) written for $\frac{2\varepsilon}{3}$ instead of $\varepsilon$:

$$c\left(\frac{2\varepsilon}{3}\right)^{-1/2} \exp\left(-\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)}\right) \leq \sqrt{\frac{d(x)}{d(t)}} \leq c\left(\frac{2\varepsilon}{3}\right)^{1/2} \exp\left(\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)}\right).$$

We easily obtain that for $t \geq x \geq x_0$ we have the inequalities

$$c\left(\frac{2\varepsilon}{3}\right)^{-1/2} \exp\left(-\varepsilon \int_x^t \frac{d\xi}{d(\xi)}\right) \leq \frac{\mu(t)d(t)}{\mu(x)d(x)} \leq c\left(\frac{2\varepsilon}{3}\right)^{1/2} \exp\left(\varepsilon \int_x^t \frac{d\xi}{d(\xi)}\right),$$

as required. The cases $x \geq t \geq x_0$ and the cases $t \leq x \leq -x_0, x \leq t \leq -x_0$ are considered in a similar way. We then continue the proof as in Lemma 2.8, with obvious modifications, similar to those presented above.

\[ \square \]
5. Example. In this final section, we consider equation (1.1) with
\[ q(x) = \frac{1}{\sqrt{1 + x^2}} + \frac{\cos(e^{x|x|})}{\sqrt{1 + x^2}}, \quad x \in \mathbb{R}. \] (5.1)

Using the results obtained above, we show that the following assertions hold:

A) Equation (1.1) in the case of (5.1) is not correctly solvable in \( L_p \), for any \( p \in [1, \infty) \);

B) For equation (1.1) in the case of (5.1), for any \( p \in [1, \infty) \), the following pair of spaces \( \{L_{p,\mu}; L_{p,\theta}\} \) is admissible, where
\[ \mu(x) = \frac{1}{\sqrt{1 + x^2}\ln(2 + x^2)}, \quad \theta(x) = \frac{1}{\ln(2 + x^2)}, \quad x \in \mathbb{R}. \] (5.2)

Remark 5.1. Below we present an algorithm for the study of (1.1) for a given pair of spaces (cases (5.1) and \( \{L_p; L_p\} \) and \( \{L_{p,\mu}; L_{p,\theta}\} \) in the case of (5.2)). We do not consider the question of the description of all pairs of spaces admissible for (1.1) in the case of (5.1).

For the reader's convenience, we enumerate the main steps of the proof of assertions A) and B).

1) Checking condition (2.2).

Let us check that in the case of (5.1) condition (2.2) holds. Assume to the contrary that there is \( x_0 \in \mathbb{R} \) such that
\[ \int_{x_0}^{\infty} q(t)dt = 0. \] (5.3)

The function \( q \) in (5.1) is continuous and non-negative. Therefore, from (5.3) it follows that \( q(t) \equiv 0 \) for \( t \in [x_0, \infty) \) which is obviously false. This contradiction implies (2.2).

2) Existence of the function \( d(x) \), \( x \in \mathbb{R} \), and its estimates.

From 1) and Lemma 2.5, it follows that the function \( d(x) \) is defined for all \( x \in \mathbb{R} \). To obtain its estimates, we use Theorem 2.13. Denote (see (2.22) and (2.23))
\[ q_1(x) = \frac{1}{\sqrt{1 + x^2}}; \quad q_2(x) = \frac{\cos(e^{x|x|})}{\sqrt{1 + x^2}}, \quad x \in \mathbb{R}; \] (5.4)
\[ A(x) = [0, 2\sqrt{1 + x^2}]; \quad \omega(x) = \left[x - 2\sqrt{1 + x^2}, x + 2\sqrt{1 + x^2}\right], \quad x \in \mathbb{R}. \] (5.5)

Let us check (2.26) for the function \( \kappa_1 \) (see (2.24)),
\[ \kappa_1(x) = \frac{1}{q_1(x)^{3/2}} \sup_{t \in A(x)} \int_{x-t}^{x+t} q''_1(\xi)\omega_1(\xi)\,d\xi = (1 + x^2)^{3/4} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \left( \frac{1}{\sqrt{1 + \xi^2}} \right)'' \right| \omega_1(\xi)\,d\xi \]
\[ = (1 + x^2)^{3/4} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \frac{1 - 2\xi^2}{1 + \xi^2} \left( \frac{1}{1 + \xi^2}\right)^{3/2} \right| \omega_1(\xi)\,d\xi \] (5.6)

Note the obvious inequalities
\[ \left| \frac{1 - 2\xi^2}{1 + \xi^2} \right| \leq \frac{1 + 2\xi^2}{1 + \xi^2} \leq 2, \quad \xi \in \mathbb{R}. \] (5.7)
In addition, for $\xi \in A(x)$, $x \gg 1$, we have

\begin{align}
\frac{1 + \xi^2}{1 + x^2} \leq 1 + \frac{|\xi - x||\xi + x|}{1 + x^2} \leq 1 + \frac{c x^{3/2}}{1 + x^2} \leq 2; \\
\frac{1 + \xi^2}{1 + x^2} \geq 1 - \frac{|\xi - x||\xi + x|}{1 + x^2} \geq 1 - \frac{c x^{3/2}}{1 + x^2} \geq \frac{1}{2}.
\end{align}

From (5.6), (5.7), (5.8) and (5.9), it now follows that

\begin{align}
k_1(x) \leq (1 + x^2)^{3/4} \sup_{t \in A(x)} \left[ \int_{x-t}^{x+t} \frac{1 - 2\xi^2}{1 + \xi^2} \frac{1}{(1 + x^2)^{3/2}} \left( 1 + \frac{x^2}{1 + x^2} \right)^{3/2} d\xi \right] \\
\leq c (1 + x^2)^{3/4} \sup_{t \in A(x)} \left[ \int_{x-t}^{x+t} 1 d\xi \right] = \frac{c}{\sqrt{1 + x^2}} \to 0, \quad x \to \infty.
\end{align}

Let us now check (2.26) for $k_2(x)$, $x \gg 1$. First show that for $x \gg 1$ we have the inequality (see (5.5)):

\begin{align}
\sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_{\alpha}^{\beta} \frac{\cos e^t}{\sqrt{1 + t^2}} \right| \leq c \frac{e^{-x/2}}{\sqrt{1 + x^2}}, \quad x \gg 1.
\end{align}

We need the following simple assertions, given without proof:

a) $x - \sqrt{1 + x^2} \to \infty$ as $x \to \infty$;

b) the function $\varphi(\xi)$ where

\[ \varphi(\xi) = \frac{e^{-\xi}}{\sqrt{1 + \xi^2}}, \quad \xi \in \mathbb{R} \]

is monotone decreasing for all $\xi \in \mathbb{R}$.

Let $t$ be any point in the interval $(\alpha, \beta)$. Below we use assertions a), b) and the second mean theorem (see [15]):

\begin{align}
\sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_{\alpha}^{\beta} \frac{\cos e^t}{\sqrt{1 + \xi^2}} d\xi \right| &= \sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_{\alpha}^{\beta} \frac{e^{-\xi}}{\sqrt{1 + \xi^2}} (e^\xi \cos(e^\xi)) d\xi \right| \\
&\leq \sup_{[\alpha, \beta] \subseteq \omega(x)} \frac{e^{-\alpha}}{\sqrt{1 + \alpha^2}} \left| \int_{\alpha}^{\beta} e^\xi \cos(e^\xi) d\xi \right| \\
&\leq c \frac{e^{-\xi}}{\sqrt{1 + \xi^2}} \left| \int_{\xi = x-2}^{\xi = x} e^{\xi/2} d\xi \right| \leq c \frac{e^{-x/2}}{\sqrt{1 + x^2}}.
\end{align}

Now, from (5.11) for $x \gg 1$ we obtain

\begin{align}
k_2(x) = \frac{1}{\sqrt{q_1(x) \in A(x)}} \left| \int_{x-t}^{x+t} q_2(\xi) d\xi \right| = \sqrt{1 + x^2} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \frac{\cos e^t}{\sqrt{1 + \xi^2}} d\xi \right| \\
\leq \sqrt{1 + x^2} \sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_{\alpha}^{\beta} \frac{\cos e^t}{\sqrt{1 + \xi^2}} d\xi \right| \leq c \frac{e^{-x/2}}{\sqrt{1 + x^2}} \Rightarrow (2.26).
\end{align}

Since (2.26) is proven, by Theorem 2.13 we obtain

\begin{align}
d(x) = \sqrt{1 + x^2}(1 + \varepsilon(x)), \quad |\varepsilon(x)| \leq 2(k_1(x) + k_2(x)), \quad |x| \gg 1, \\
c^{-1} \sqrt{1 + x^2} \leq d(x) \leq c \sqrt{1 + x^2}, \quad x \in \mathbb{R}.
\end{align}
3) Proof of assertion A).

From (5.13), it follows that \( d_0 = \infty \) (see (4.24) and (4.22)). It remains to refer to Theorem 4.5. \( \square \)

Let us now go to assertion B).

4) Checking the inclusion \( q \in H \).

To prove (2.15), we need estimates of \( \tau_1(x) \) and \( \tau_2(x) \) for \( x > 1 \) where (see (5.1) and (5.4))

\[
\tau_1(x) = \left| \int_0^{\sqrt{2}d(x)} (q_1(x + t) - q_1(x - t))dt \right|; \tag{5.14}
\]

\[
\tau_2(x) = \left| \int_0^{\sqrt{2}d(x)} (q_2(x + t) - q_2(x - t))dt \right|. \tag{5.15}
\]

To estimate \( \tau_1(x) \), we use below (5.7), (5.8), (5.9) and (5.12):

\[
\tau_1(x) = \left| \int_0^{\sqrt{2}d(x)} \left( \int_{x-t}^{x+t} q_1'(\xi)d\xi \right)dt \right| \leq \sqrt{2}d(x) \sup_{|t| \leq \sqrt{2}d(x)} \left| \int_{x-\xi}^{x+\xi} q_1'(t)dt \right|
\]

\[
\leq c \frac{\sqrt{1 + x^2}}{1 + x^2} \sup_{|t| \leq \sqrt{2}d(x)} \left| \int_{x-\xi}^{x+\xi} t \frac{1 + x^2}{1 + x^2} dt \right|
\]

\[
\leq c \frac{\sqrt{1 + x^2}}{1 + x^2} \sup_{|t| \leq \sqrt{2}d(x)} |t| \leq \frac{c}{\sqrt{1 + x^2}}, \quad x \gg 1. \tag{5.16}
\]

The estimate for \( \tau_2(x) \), \( x \gg 1 \), follows from (5.10) and (5.1):

\[
|\tau_2(x)| \leq \left| \int_0^{\sqrt{2}d(x)} q_2(x + t)dt \right| + \left| \int_0^{\sqrt{2}d(x)} q_2(x - t)dt \right|
\]

\[
= \left| \int_{x-\sqrt{2}d(x)}^{x} q_2(\xi)d\xi \right| + \left| \int_{x}^{x+\sqrt{2}d(x)} q_2(\xi)d\xi \right|
\]

\[
\leq 2 \sup_{|\alpha, \beta| \leq x(\chi)} \left| \int_{\alpha}^{\beta} \frac{1 + \chi^2}{1 + \xi^2} d\xi \right| \leq c \frac{e^{-x}}{\sqrt{1 + x^2}}, \quad x \gg 1. \tag{5.17}
\]

From (5.16), (5.17) and (5.13), we obtain (2.15), and therefore \( q \in H \).

5) Checking that the weights \( \mu(x) \) and \( \theta(x) \) agree with the function \( q \).

Equalities (2.29) for the functions \( \mu(x) \) and \( 1/\theta(x) \) (see (5.2)) are easily proved with the help of estimates (5.13).

6) Proof of assertion B).

Below we use Theorem 2.10. Let us check that in case (5.2) requirements (2.9) are satisfied. Let \( x_0 \gg 1 \). Then

\[
\int_0^{\infty} \mu(t)dt = \int_0^{\infty} \frac{dt}{\sqrt{1 + t^2} \ln(2 + t^2)} \geq \frac{1}{\ln t} \int_0^{\infty} \frac{dt}{2 \ln t + \ln(1 + 2t^{-2})}
\]

\[
\geq c^{-1} \int_{x_0}^{\infty} \frac{dt}{t \ln t} = \infty \quad \Rightarrow (2.9).
\]

Since the weights \( \mu \) and \( \theta \) agree with the function \( q \), and one has the relations:

\[
m(q, \mu, \theta) = \sup_{x \in R} \left( \frac{\mu(x)}{\theta(x)} d^2(x) \right) \leq c \sup_{x \in R} \left( \frac{\mu(x)}{\theta(x)} \sqrt{1 + x^2} \right) = c < \infty,
\]

(see assertion (5.13)), assertion B) follows from Theorem 2.10. \( \square \)
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Received August 2016; revised November 2017.

E-mail address: nina@math.bgu.ac.il
E-mail address: miriam@macs.biu.ac.il