Optimal quantum violation of Clauser–Horne–Shimony–Holt like steering inequality

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Abstract
We study a recently proposed Einstein–Podolsky–Rosen steering inequality (Cavalcanti \textit{et al} 2015 \textit{J. Opt. Soc. Am. B} \textbf{32} A74–A81). Analogous to Clauser–Horne–Shimony–Holt (CHSH) inequality for Bell nonlocality, in the simplest scenario, i.e., two parties, two measurements per party and two outcomes per measurement, this newly proposed inequality has been proved to be necessary and sufficient for steering. In this article we find the optimal violation amount of this inequality in quantum theory. Interestingly, the optimal violation amount matches with optimal quantum violation of CHSH inequality, i.e., Cirel’son quantity. We further study the optimal violation of this inequality for different classes of 2-qubit quantum states.

Keywords: EPR steering, Bell nonlocality, CHSH inequality, Cirel’son bound

(Some figures may appear in colour only in the online journal)

1. Introduction

The phenomenal argument by Einstein, Podolsky and Rosen (EPR) in 1935 [1] to demonstrate the incompleteness of quantum mechanics, struck Schrödinger with the concept of ‘steering’ [2]. However, only recently, Wiseman \textit{et al} have formalized the concept of steering in the form of a task [3, 4]. The task of steering can be seen as one’s inability to construct a local hidden variable-local hidden state (LHV-LHS) model that reproduces a given bipartite correlation.

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The work of Wiseman et al has generated an immense interest in the study of this steering phenomenon [5–10]. On the other hand, the concept of steering has been extended for multipartite case and the idea of n-partite genuine multipartite steering has been explored [11]. Unlike the two well studied nonclassical correlations, namely nonlocality [12] and entanglement [13], there is an inherent asymmetry in the task of steering. This is because in case of steering, on one subsystem (which is being ‘steered’) the statistics must arise out of a valid measurement on a valid quantum state but no such constraint is required for the other subsystem. Here in addition to the simplicity of Bell’s assumptions of local causality one must also perform trusted measurements on one subsystem, whereas the other subsystem need not be trusted [14].

Apart from the foundational interest, the study of steering also finds applications in semi-device independent scenario where only one party has trust on his/her quantum device but the other party’s device is untrusted. As a concrete example it has been shown that steering allows for secure quantum key distribution when one of the parties’ device cannot be trusted [15]. One big advantage in this direction is that such scenarios are experimentally less demanding than fully device-independent protocols (where both of the parties distrust their devices) [15] and, at the same time, require less assumptions than standard quantum cryptographic scenarios [16].

In 1964, Bell sought a way to demonstrate that certain correlations appearing in quantum mechanics are incompatible with the notions of locality and reality aka local-realism, through an inequality involving measurement statistics [17]. A violation of such inequality implies the usefulness of correlations for EPR argument. In 1969, Clauser–Horne–Shimony–Holt (CHSH) proposed a set of simple Bell inequalities which are easy to realize experimentally [18]. In the same spirit of Bell’s inequality in nonlocality, several steering inequalities (SIs) have been proposed [19–23], so that a violation of any such SI can render a correlation to be steerable. But an unavoidable hindrance to formalize such SIs follows from the fact that steering scenario is device-independent only on one-side.

Recently Cavalcanti et al have proposed a CHSH-like inequality for quantum steering [24]. They have derived an EPR-SI that is necessary and sufficient for a set of correlations in the simplest two-party scenario involving two measurement settings per site and two outcomes per measurement, with mutually unbiased measurements by trusted party. In this article we have derived the tight optimal quantum violation of the EPR-SI proposed in [24]. We have also studied the violation amount of this inequality for different classes of 2-qubit states. We find that for different 2-qubit pure entangled states the optimal violation amount of this inequality differ. Note that, a known SI which exhibits different optimal violation for different pure entangled states is the one given in [23], whereas the inequality proposed in [25] does not show this feature.

The organization of this article goes as follows: we first briefly review few existing SIs along with the newly proposed CHSH like SI. Then we show that the optimal violation of this CHSH like SI in quantum theory is restricted to $2\sqrt{2}$. Then we study how the optimal violation amount of the CHSH like inequality varies for different entangled states.

2. Steering inequalities

To test EPR steering Reid first proposed a testable formulation for continuous-variable systems based on position-momentum uncertainty relation [19] which was experimentally tested by Ou et al [20]. Cavalcanti et al developed a general construction of experimental EPR-steering criteria based on the assumption of existence of LHS model [21]. Importantly this general construction is applicable to both the discrete as well as the continuous-variable observables and Reid’s criterion appears as a special case of this general formulation. On the
other hand, Walborn et al formulated a SI based on Bialynicki-Birula and Mycielski entropic position-momentum uncertainty relation [22, 23]. As the entropic uncertainty relation implies Heisenbergs uncertainty relation, hence the set of states violating Walborn et al’s SI contains all the states violating Reid’s inequality. Thus Walborn et al’s steering criterion is more powerful than Reid’s one. However this is true for continuous variable case only, not for the discrete case.

All of these SIs have been proved to be as sufficient conditions for witnessing steering in bipartite quantum systems. But none of these conditions are supposed to be necessary and sufficient conditions for steering. Search for such a necessary and sufficient condition has been culminated in a recent development by Cavalcanti et al [24]. They have proposed a CHSH like EPR-SI that is necessary and sufficient for the set of correlations in the simplest scenario involving two settings and two outcomes per setting, with mutually unbiased measurements at the trusted end. At this point it is interesting to ask the following questions:

(a) what is the optimal violation of this newly proposed CHSH like SI in quantum theory?
(b) how does the violation amount of the concerned inequality depend on the state?

We provide a definite answer for the first question and study the second one for some classes of states.

3. Maximum violation of CHSH-like SI

Let us first briefly review the steering scenario as introduced by Wiseman et al [3, 4]. Given a pair of systems at Alice and Bob, denote $\mathcal{D}_A$ and $\mathcal{D}_B$ the sets of observables in the Hilbert space of Alice’s and Bob’s systems, respectively. An element of $\mathcal{D}_A$ is denoted by $A$, with a set of outcomes labeled by $a \in \mathcal{L}(A)$, and similarly for Bob. The joint state $\rho_{AB}$ of the system is steerable by Alice iff it is not the case that for all $a \in \mathcal{L}(A), b \in \mathcal{L}(B), A \in \mathcal{D}_A, B \in \mathcal{D}_B$, the joint probability distributions can be written in the form

$$P(a, b|A, B; \rho_{AB}) = \sum_{\lambda} \varphi(\lambda) \varphi(a|A, \lambda) P(b|B; \rho_{\lambda}),$$

where $\varphi(a|A, \lambda)$ denotes an arbitrary probability distribution and $P(b|B; \rho_{\lambda})$ denotes the quantum probability of outcome $b$ given measurement $B$ on state $\rho_{\lambda}$. In other words the state $\rho_{AB}$ will be called steerable if it does not satisfy a LHV-LHS model. Note that, if for a given measurement strategy the correlation has a LHV-LHS model, this does not imply that the underlying state is not steerable, since there could be another strategy that does not. In the simplest scenario where Alice and Bob each has a choice between two dichotomic measurements to perform: $\{A_1, A_2\}, \{B_1, B_2\}$, and outcomes of $A$ are labeled $a \in \{-1, +1\}$ and similarly for the other measurements, the authors of [24] have derived a necessary and sufficient criterion for steering which reads as:

$$S = \sqrt{\left\langle (A_1 + A_2)B_1 \right\rangle^2 + \left\langle (A_1 + A_2)B_2 \right\rangle^2}$$
$$\quad + \sqrt{\left\langle (A_1 - A_2)B_1 \right\rangle^2 + \left\langle (A_1 - A_2)B_2 \right\rangle^2} \leq 2.$$  

We know that, in the simplest Bell scenario which involves two observers with 2 dichotomic measurements per site, the set of local correlations lie in a polytope (LHV polytope) with CHSH inequalities providing the nontrivial facets of the LHV polytope. In the steering scenario with similar settings the set of correlations having LHV-LHS description form a convex set.
One of the authors of this article, along with other collaborators, has shown that measurement incompatibility limits the Bell-CHSH inequality violation in quantum theory to Cirel’son bound \[26, 27\]. Adopting similar approach and using a recently established connection between measurement incompatibility and steering, we derive the optimal quantum violation of the SI (2). Before establishing this result we first briefly review the concept of measurement incompatibility and the concept of unsharp measurement in quantum theory.

**Measurement incompatibility**

In the case of projective measurements, compatibility is uniquely captured by the notion of commutativity. Noncommutative projective measurements in quantum mechanics do not admit unambiguous joint measurement \[28\]. With the introduction of the generalized measurements i.e. positive operator-valued measures (POVMs) \[29, 30\], it was shown that observables which do not admit perfect joint measurement, may allow joint measurement if the measurements are made sufficiently fuzzy \[31, 32\]. Therefore, for general measurements there is no unique notion of compatibility. In this article measurement incompatibility is captured by nonjoint measurability \[33\].

Mathematically, a POVM consists of a collection of operators \( \{ A_{\alpha} \} \), which are positive, \( A_{\alpha} \geq 0 \) \( \forall \alpha \), and sum up to the identity, \( \sum_{\alpha} A_{\alpha} = 1 \). Here \( \alpha \) denotes measurement outcome and \( x \) denotes measurement choice. Physically, any POVM can be realized by first letting the physical system interact with an auxiliary system and then measuring an ordinary observable on the joint system. A set of \( m \) POVMs \( \{ A_{\alpha} \} \) is called jointly measurable if there exists a measurement \( \{ A_{\alpha} \} \) with outcome \( \vec{a} = [a_{x=1}, a_{x=2}, ..., a_{x=m}] \) where \( a_{x} \in \{ 0, 1, ..., n \} \) gives the outcome of \( x \)th measurement, i.e.,

\[
A_{\vec{a}} \geq 0, \quad \sum_{\vec{a}} A_{\vec{a}} = 1, \quad \sum_{\vec{a} \setminus a_{x}} A_{\vec{a}} = A_{a_{x}} \quad \forall x, \tag{3}
\]

where \( \vec{a} \setminus a_{x} \) stands for the elements of \( \vec{a} \) except for \( a_{x} \). Hence, all POVM elements \( A_{a_{x}} \) are recovered as marginals of the mother observable \( A_{\alpha} \).

**Unsharp measurement**

Let us consider two dichotomic quantum measurements \( A_{1} \) and \( A_{2} \), which are not jointly measurable. Denoting eigenvalues of these operator as \( \pm 1 \), the expectation value over some state vector can be expressed as:

\[
\langle A_{k} \rangle_{\sigma} = p(1|A_{k}) - p(-1|A_{k}), \quad k \in \{ 1, 2 \},
\]

where \( p(\pm 1|A_{k}) = \text{Tr}(A_{k}^{\pm}|\sigma) \), with \( A_{k}^{\pm} \) being the POVM elements corresponding to \( \pm 1 \). The unsharp or fuzzy observable is given by \( A_{k}^{\pm(n)} = \{ A_{k}^{\pm(n)} | A_{k}^{\pm(n)} \geq 0 \text{ and } A_{k}^{+} + A_{k}^{-} = 1 \} \), with

\[
A_{k}^{\pm(n)} = \frac{1 \pm \eta}{2} A_{k}^{+} + \frac{1 \mp \eta}{2} A_{k}^{-}.
\]

Here \( \eta \in (0, 1] \) is known as ‘unsharpness parameter’ and the fraction \( \frac{1 \pm \eta}{2} \) is called ‘degree of reality’ \[32\]. It may happen that the observables \( A_{1} \) and \( A_{2} \) do not allow any joint measurement, but with introduction of sufficient amount of unsharpness, their unsharp versions \( A_{1}^{\pm(n)} \) and \( A_{2}^{\pm(n)} \) may allow joint measurement. In \[27\], the authors have proved that given any d-dimensional quantum system, joint measurement for unsharp versions of any two dichotomous observables \( A_{1} \) and \( A_{2} \) of the system is possible with the largest allowed value of the unsharpness parameter \( \eta_{opt} = \frac{1}{\sqrt{2}} \). Note that the expectation value of an unsharp
observable $A_k^{(0)}$ over some quantum state $\sigma$ is related to the expectation value of its sharp version in the following manner,

$$\langle A_k^{(0)} \rangle_\sigma = \eta \langle A_k \rangle_\sigma. \tag{4}$$

Similarly, if Alice performs unsharp measurement $A_k^{(0)}$ on her part and Bob performs sharp measurement $B_j$ on his part of a bipartite shared state $\rho_{AB}$ than we have,

$$\langle A_k^{(0)} B_j \rangle_{\rho_{AB}} = \eta \langle A_k B_j \rangle_{\rho_{AB}}. \tag{5}$$

Except from quantum entanglement, another necessary ingredient which is necessary for study of quantum nonlocality is the existence of incompatible set of measurements. In the simplest bipartite scenario Wolf et al have shown that any set of two incompatible POVMs with binary outcomes can always lead to violation of the CHSH-Bell inequality [34]. But, recently in [35, 36] the authors have proved that this result does not hold in the general scenario where numbers of POVMs and outcomes are arbitrary. However in this general settings the authors of [35, 36] have established a connection between measurement incompatibility and a weaker form of quantum nonlocality i.e., EPR–Schrödinger steering. They have shown that for any set of incompatible POVMs (i.e. not jointly measurable), one can find an entangled state, such that the resulting statistics violate a SI. Please note that, one of the authors of this article has recently proved that the connection between measurement incomparability and steering holds for a more general class of tensor product theories rather than just Hilbert space quantum theory [37].

Let Alice perform a measurement assemblage $\{A_{ax}\}$ on her part of a bipartite shared quantum state $\rho_{AB}$. Upon performing measurement $x$, and obtaining outcome $a$, the (un-normalized) state held by Bob is given by $\sigma_{a|x} = \text{Tr}(A_{ax} \otimes I_{\rho_{AB}})$. The normalized state on Bob’s side is given by $\sigma_{a|x}/\text{Tr}(\sigma_{a|x})$. Also we have $\sum_a \sigma_{a|x} = \sum_a \sigma_{a|x'}$ for $x \neq x'$, which actually ensure no signaling from Alice to Bob. The state assemblage $\{\sigma_{a|x}\}$ is un-steerable iff it admits a decomposition of the form

$$\sigma_{a|x} = \pi(\lambda)p(a|x, \lambda)\sigma_{a}, \quad \forall \, a, x, \tag{6}$$

where $\sum_a \pi(\lambda) = 1$. Existence of such decomposition for state assemblage on Bob’s side ensures that the statistics obtained from the state $\rho_{AB}$ admit a combined LHV-LHS model of the form of equation (1). The authors in [35, 36] have shown that the assemblage $\{\sigma_{a|x}\}$, with $\sigma_{a|x} = \text{Tr}(A_{ax} \otimes I_{\rho_{AB}})$, is un-steerable for any state $\rho_{AB}$ acting on $\mathbb{C}^d \otimes \mathbb{C}^d$ if and only if the set of POVMs $\{A_{ax}\}$ acting on $\mathbb{C}^d$ are jointly measurable. As a corollary of this result we can say that

**Corollary 1.** The assemblage $\{\sigma_{a|x}\}$, with $\sigma_{a|x} = \text{Tr}(A_{ax} \otimes I_{\rho_{AB}})$ and $x \in \{1, 2\}$, is un-steerable for any state $\rho_{AB}$ acting in $\mathbb{C}^d \otimes \mathbb{C}^d$ if and only if the set of POVMs $\{A_{ax}\}$ acting on $\mathbb{C}^d$ are jointly measurable.

At this stage, we are now in a position to prove our main result, which is described in the following theorem.

**Theorem 2.** Consider a composite quantum system composed of two subsystem with state spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. For any pair of dichotomic observables $A_1, A_2$ for the first system and the mutually unbiased dichotomic observables $B_1, B_2$ for the second system and the joint state $\rho_{AB}$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$, we have the following inequality:
where $\eta_{\text{opt}}$ is the optimal unsharpness parameter that allows joint measurement for any two dichotomic quantum observables.

**Proof.** Let us consider two arbitrary dichotomic observables $\{A_{x\pm1}\}$ on Alice’s side, $x \in \{1, 2\}$ and $a \in \{-1, +1\}$. These two observables in general may not allow joint measurement. However, introduction of unsharpness makes it possible to measure the unsharp versions of these two observables jointly. Let the optimal unsharpness be $\eta_{\text{opt}}$ which allows joint measurement for any two dichotomic observables.

Now according to corollary 1, as far as observables on Alice’s side are jointly measurable, they will not violate any SI and hence the SI $(2)$. Thus we have

$$\sqrt{\left(\left(A_1^{(\eta_{\text{opt}})} + A_2^{(\eta_{\text{opt}})}\right)B_1\right)^2 + \left(\left(A_1^{(\eta_{\text{opt}})} + A_2^{(\eta_{\text{opt}})}\right)B_2\right)^2} + \sqrt{\left(\left(A_1^{(\eta_{\text{opt}})} - A_2^{(\eta_{\text{opt}})}\right)B_1\right)^2 + \left(\left(A_1^{(\eta_{\text{opt}})} - A_2^{(\eta_{\text{opt}})}\right)B_2\right)^2} \leq 2.$$  

Now using the expressing from equation $(5)$ we get,

$$\sqrt{\left((A_1 + A_2)B_1\right)^2 + \left((A_1 + A_2)B_2\right)^2} + \sqrt{\left((A_1 - A_2)B_1\right)^2 + \left((A_1 - A_2)B_2\right)^2} \leq \frac{2}{\eta_{\text{opt}}}.$$ 

The value of $\eta_{\text{opt}}$ in quantum theory is proved to be $1/\sqrt{2}$ [27]. Therefore the upper bound of the SI $(2)$ in quantum theory is $2\sqrt{2}$, i.e., $S \leq 2\sqrt{2}$, which is numerically equal to the celebrated Cirel’son value [26]. Naturally the question arises whether this value is tight or not. It is indeed easy to show that this value can be achieved in quantum theory. Let Alice and Bob share a singlet state $|\psi^\text{−}\rangle = \frac{1}{\sqrt{2}}(|10\rangle_{AB} - |10\rangle_{AB})$. The joint expectation value of Alice’s spin measurement along direction $\hat{A}$ and Bob’s spin measurement along direction $\hat{B}$ on the singlet state reads $\langle \hat{A} \cdot \hat{B} \rangle_{\psi^\text{−}} = -1$. If Alice chooses her measurement directions $\hat{A}_1$ and $\hat{A}_2$ and Bob chooses his directions $\hat{B}_1$ and $\hat{B}_2$ from the same plane in such a way that $\hat{A}_1 \perp \hat{A}_2$ and $\hat{B}_1 \perp \hat{B}_2$ then the above inequality saturates. At this point it is important to note that not all choices of the measurement settings $\{\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2\}$ satisfying the said conditions provide the optimal Bell-CHSH violation and moreover in some cases they will not violate Bell-CHSH inequality at all. For optimal Bell-CHSH violation the additional condition required is that the angle between $\hat{A}_i$ and $\hat{B}_i$ is $\frac{\pi}{4}$, upto local relabeling of the measurement settings.

4. Violation of SI by different states

In this section we study the optimal violation of the SI $(2)$ for different given entangled states.

**Observation 3.** $S^{\text{opt}}$ is different for different 2-qubit pure entangled states.
Consider that an arbitrary 2-qubit pure entangled state $|\psi\rangle_{AB} = a|00\rangle_{AB} + b|11\rangle_{AB}$ ($|a|^2 + |b|^2 = 1$) is shared between Alice and Bob. Alice performs measurements $A_1 = \frac{1}{2}(I + \hat{m} \cdot \hat{\sigma})$ and $A_2 = \frac{1}{2}(I + \hat{n} \cdot \hat{\sigma})$ on her part of the entangled particle where $|\hat{m}|, |\hat{n}| \leq 1$. Similarly Bob performs measurements $B_1 = \frac{1}{2}(I + \hat{c} \cdot \hat{\sigma})$ and $B_2 = \frac{1}{2}(I + \hat{d} \cdot \hat{\sigma})$ on his part of the entangled particle where $|\hat{c}|, |\hat{d}| \leq 1$ and $\hat{c} \cdot \hat{d} = 0$, i.e. $B_1, B_2$ are mutually unbiased qubit measurements. Varying over the measurement directions of Alice and Bob we have numerically found the optimal violation amount of the SI (2) for a given 2-qubit pure entangled state and in figure 1 we have plotted it with respect to state parameter $a$. From figure 1 it is clear that all 2-qubit pure entangled states violate the SI (2). It is important to note that the optimal violation amount is different for different states.

**Observation 4.** $S_{\text{opt}}$ for 2-qubit Werner class of states.
Consider that Alice and Bob share the Werner state \( W_{AB} = w |\psi^\text{-}\rangle_{AB} \otimes \frac{1}{2} \) where \( |\psi^\text{-}\rangle_{AB} = \frac{1}{\sqrt{2}} (|10\rangle_{AB} - |10\rangle_{AB}) \) is the singlet state. Likewise in the previous case, varying over the measurement setup of Alice and Bob we find the optimal violation amount of the SI (2) for different Werner state and plot this values in figure 2. From figure 2 it is clear that Werner states violate the SI for \( w > \frac{1}{\sqrt{2}} \) and the violation amount increase with the parameter \( w \).

5. Conclusion

There have been several attempts to quantify quantum steering. Two most recent instances are ‘steerable weight’ \[38\] and ‘relative entropy of steering’ \[39\]. But not all of these proposed quantifiers assign different values to different pure entangled states \[38\]. At this point our observation becomes interesting. Since the amount of violation can also be shown to be nonincreasing under steering nonincreasing operations \[39\], one can take this amount of violation to be a valid quantifier of quantum steering. Our study stipulates further studies whether some semi device independent protocol(s) can be designed whose payoff scales with violation amount of the SI considered here. In such case different violation amount of the SI by different pure entangles states will have an operational explanation.

We also find the violation of the CHSH like SI by 2-qubit Werner states. The inequality is violated by Werner states if \( w > \frac{1}{\sqrt{2}} \) and beyond this value it follows a trait: the more entangled the state, more is the violation. We also find that the violation of this SI in QM is tightly upper bounded by \( 2\sqrt{2} \), the well known Cirel’son quantity.

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