EQUIVARIANT PRINCIPAL BUNDLES AND LOGARITHMIC CONNECTIONS ON TORIC VARIETIES

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Let $M$ be a smooth complex projective toric variety equipped with an action of a torus $T$, such that the complement $D$ of the open $T$-orbit in $M$ is a simple normal crossing divisor. Let $G$ be a complex reductive affine algebraic group. We prove that an algebraic principal $G$-bundle $E_G \to M$ admits a $T$-equivariant structure if and only if $E_G$ admits a logarithmic connection singular over $D$. If $E_H \to M$ is a $T$-equivariant algebraic principal $H$-bundle, where $H$ is any complex affine algebraic group, then $E_H$ in fact has a canonical integrable logarithmic connection singular over $D$.

1. Introduction

Our aim is to give characterizations of the equivariant principal bundles on smooth complex projective toric varieties.

Let $M$ be a smooth complex projective toric variety equipped with an action

$$\rho : T \times M \to M$$

of a torus $T$. For any point $t \in T$, define the automorphism

$$\rho_t : M \to M, \quad x \mapsto \rho(t, x).$$

We assume that the complement $D$ of the open $T$-orbit in $M$ is a simple normal crossing divisor.

Let $G$ be a complex reductive affine algebraic group, and let $E_G$ be an algebraic principal $G$-bundle on $M$. In Proposition 4.1 we prove the following:

The principal $G$-bundle $E_G$ admits a $T$-equivariant structure if and only if the pulled-back principal $G$-bundle $\rho_t^*E_G$ is isomorphic to $E_G$ for every $t \in T$.

When $G = \text{GL}(n, \mathbb{C})$, this result was proved by Klyachko [1989, p. 342, Proposition 1.2.1].

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Using the above characterization of $T$-equivariant principal $G$-bundles on $M$, we prove the following (see Theorem 4.2):

*The principal $G$-bundle $E_G$ admits a logarithmic connection singular over $D$ if and only if $E_G$ admits a $T$-equivariant structure.*

The “if” part of Theorem 4.2 does not require $G$ to be reductive. More precisely, any $T$-equivariant principal $H$-bundle $E_H \to M$, where $H$ is any complex affine algebraic group, admits a canonical integrable logarithmic connection singular over $D$ (see Proposition 3.2).

2. Equivariant bundles

Let $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$ be the multiplicative group. Take a complex algebraic group $T$ which is isomorphic to a product of copies of $\mathbb{G}_m$. Let $M$ be a smooth irreducible complex projective variety equipped with an algebraic action of $T$

\[ \rho : T \times M \to M \]

such that

- there is a Zariski open dense subset $M^0 \subset M$ with $\rho(T, M^0) = M^0$,
- the action of $T$ on $M^0$ is free and transitive, and
- the complement $M \setminus M^0$ is a simple normal crossing divisor of $M$.

In particular, $M$ is a smooth projective toric variety. Note that $M^0$ is the unique $T$-orbit in $M$ with trivial isotropy.

Let $G$ be a connected complex affine algebraic group. A $T$-equivariant principal $G$-bundle on $M$ is a pair $(E_G, \tilde{\rho})$, where

\[ p : E_G \to M \]

is an algebraic principal $G$-bundle, and

\[ \tilde{\rho} : T \times E_G \to E_G \]

is an algebraic action of $T$ on the total space of $E_G$, such that

- $p \circ \tilde{\rho} = \rho \circ (\text{Id}_T \times p)$, where $\rho$ is the action in (2-1), and
- the actions of $T$ and $G$ on $E_G$ commute.

Fix a point $x_0 \in M^0 \subset M$. Let

\[ t : \rho(T, x_0) = M^0 \hookrightarrow M \]

be the inclusion map. Let $M^0 \times G$ be the trivial principal $G$-bundle on $M^0$. It has a tautological integrable algebraic connection given by its trivialization.
Let \((E_G, \tilde{\rho})\) be a \(T\)-equivariant principal \(G\)-bundle on \(M\). Fix a point \(z_0 \in (E_G)_{x_0}\). Using \(z_0\), the action \(\tilde{\rho}\) produces an isomorphism of principal \(G\)-bundles between \(M^0 \times G\) and the restriction \(E_G|_{M^0}\). This isomorphism of principal \(G\)-bundles is uniquely determined by the following two conditions:

- this isomorphism is \(T\)-equivariant (the action of \(T\) on \(M^0 \times G\) is given by the action of \(T\) on \(M^0\), and
- it takes the point \(z_0 \in E_G\) to \((x_0, e) \in M^0 \times G\).

Using this trivialization of \(E_G|_{M^0}\), the tautological integrable algebraic connection on \(M^0 \times G\) produces an integrable algebraic connection \(D^0\) on \(E_G|_{M^0}\). We note that the connection \(D^0\) is independent of the choice of the points \(x_0\) and \(z_0\). Indeed, the flat sections for \(D^0\) are precisely the orbits of \(T\) in \(E_G|_{M^0}\). Note that this description of \(D^0\) does not require choosing base points in \(M^0\) and \(E_G|_{M^0}\).

In Proposition 3.2 it will be shown that \(D^0\) extends to a logarithmic connection on \(E_G\) over \(M\) singular over the simple normal crossing divisor \(M \setminus M^0\).

3. Logarithmic connections

A canonical trivialization. The Lie algebra of \(T\) will be denoted by \(t\). Let

\[(3-1)\quad \mathcal{V} := M \times t \to M\]

be the trivial vector bundle with fiber \(t\). The holomorphic tangent bundle of \(M\) will be denoted by \(TM\). Consider the action of \(T\) on \(M\) in (2-1). It produces a homomorphism of \(\mathcal{O}_M\)-coherent sheaves

\[(3-2)\quad \beta : \mathcal{V} \to TM.\]

Let

\[(3-3)\quad D := M \setminus M^0\]

be the simple normal crossing divisor of \(M\). Let

\[(3-3)\quad TM(-\log D) \subset TM\]

be the corresponding logarithmic tangent bundle. Recall that \(TM(-\log D)\) is characterized as the maximal coherent subsheaf of \(TM\) that preserves \(\mathcal{O}_M(-D) \subset \mathcal{O}_M\) for the derivation action of \(TM\) on \(\mathcal{O}_M\).

**Lemma 3.1.**

1. The image of \(\beta\) in (3-2) is contained in the subsheaf \(TM(-\log D) \subset TM\).
2. The resulting homomorphism \(\beta : \mathcal{V} \to TM(-\log D)\) is an isomorphism.

*Proof.* The divisor \(D\) is preserved by the action of \(T\) on \(M\). Therefore, the action
of $T$ on $\mathcal{O}_M$, given by the action of $T$ on $M$, preserves the subsheaf $\mathcal{O}_M(-D)$. From this it follows immediately that the subsheaf $\mathcal{O}_M(-D) \subset \mathcal{O}_M$ is preserved by the derivation action of the subsheaf

$$\beta(\mathcal{V}) \subset TM.$$ 

Therefore, we conclude that $\beta(\mathcal{V}) \subset TM(-\log D)$.

It is known that the vector bundle $TM(-\log D)$ is holomorphically trivial. This follows from Proposition 2 in [Fulton 1993, p. 87], which says that $\Omega^1_M(\log D)$ is holomorphically trivial, together with the equality $\Omega^1_M(\log D)^* = TM(-\log D)$.

So, both $\mathcal{V}$ and $TM(-\log D)$ are trivial vector bundles, and $\beta$ is a homomorphism between them which is an isomorphism over the open subset $M^0$. From this it can be deduced that $\beta$ is an isomorphism over entire $M$. To see this, consider the homomorphism

$$\wedge^r \beta : \wedge^r \mathcal{V} \to \wedge^r TM(-\log D)$$

induced by $\beta$, where $r = \dim_{\mathbb{C}} T = \text{rank}(\mathcal{V})$. So $\wedge^r \beta$ is a holomorphic section of the line bundle $(\wedge^r TM(-\log D)) \otimes (\wedge^r \mathcal{V})^*$. This line bundle is holomorphically trivial because both $\mathcal{V}$ and $TM(-\log D)$ are holomorphically trivial. Fixing a trivialization of $(\wedge^r TM(-\log D)) \otimes (\wedge^r \mathcal{V})^*$, we consider $\wedge^r \beta$ as a holomorphic function on $M$. This function is nowhere vanishing because it does not vanish on $M^0$ and holomorphic functions on $M$ are constants. Since $\wedge^r \beta$ is nowhere vanishing, the homomorphism $\beta$ is an isomorphism. □

A canonical logarithmic connection on equivariant bundles. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$.

Let $p : E_G \to M$ be an algebraic principal $G$-bundle. Consider the differential

$$dp : T_{E_G} \to p^* TM,$$

where $T_{E_G}$ is the algebraic tangent bundle of $E_G$. The kernel of $dp$ will be denoted by $T_{E_G/M}$. Using the action of $G$ on $E_G$, the subbundle $T_{E_G/M} \subset T_{E_G}$ is identified with the trivial vector bundle over $E_G$ with fiber $\mathfrak{g}$.

The action of $G$ on $E_G$ produces an action of $G$ on $T_{E_G}$. So we get an action of $G$ on the quasicoherent sheaf $p_* T_{E_G}$ on $M$. The invariant part

$$\text{At}(E_G) := (p_* T_{E_G})^G \subset p_* T_{E_G}$$

is a locally free coherent sheaf; its coherence property follows from the fact that the action of $G$ on the fibers of $p$ is transitive, implying that a $G$-invariant section of $(T_{E_G})|_{p^{-1}(x)}$, $x \in M$, is uniquely determined by its evaluation at just one point of the fiber $p^{-1}(x)$. Also note that $\text{At}(E_G) = (T_{E_G})/G$. This $\text{At}(E_G)$ is known as the Atiyah bundle for $E_G$. Since $T_{E_G/M}$ is identified with $E_G \times \mathfrak{g}$, the invariant
direct image \((p_* T_{E_G/M})^G\) is identified with the adjoint vector bundle 

\[
\text{ad}(E_G) := E_G \times^G \mathfrak{g} \to M
\]

associated to \(E_G\) for the adjoint action of \(G\) on \(\mathfrak{g}\). We note that \(\text{ad}(E_G) = T_{E_G/M} / G\). Now the differential \(dp\) in (3-4) produces a short exact sequence of holomorphic vector bundles on \(M\)

\[
0 \to \text{ad}(E_G) \to \text{At}(E_G) \xrightarrow{\phi} TM \to 0,
\]

which is known as the Atiyah exact sequence. A holomorphic connection on \(E_G\) over \(M\) is a holomorphic splitting 

\[
TM \to \text{At}(E_G)
\]

of (3-5) [Atiyah 1957].

As before, setting \(D = M \setminus M^0\), define

\[
\text{At}(E_G)(-\log D) := \phi^{-1}(TM(-\log D)) \subset \text{At}(E_G),
\]

where \(\phi\) is the projection in (3-5) and \(TM(-\log D)\) is the subsheaf in (3-3). So (3-5) gives the following short exact sequence of holomorphic vector bundles on \(M\):

\[
0 \to \text{ad}(E_G) \to \text{At}(E_G)(-\log D) \xrightarrow{\phi} TM(-\log D) \to 0.
\]

A logarithmic connection on \(E_G\), with singular locus \(D\), is a holomorphic homomorphism 

\[
\delta : TM(-\log D) \to \text{At}(E_G)(-\log D)
\]

such that \(\phi \circ \delta\) is the identity automorphism of \(TM(-\log D)\), where \(\phi\) is the homomorphism in (3-6). Just like the curvature of a connection, the curvature of a logarithmic connection \(\delta\) on \(E_G\) is the obstruction for the homomorphism \(\delta\) to preserve the Lie algebra structure of the sheaf of sections of \(TM(-\log D)\) and \(\text{At}(E_G)(-\log D)\) given by the Lie bracket of vector fields. In particular, \(\delta\) is called integrable (or flat) if it preserves the Lie algebra structure of the sheaf of sections of \(TM(-\log D)\) and \(\text{At}(E_G)(-\log D)\) given by the Lie bracket of vector fields.

**Proposition 3.2.** Let \((E_G, \tilde{\rho})\) be a \(T\)-equivariant principal \(G\)-bundle on \(M\). Then \(E_G\) admits an integrable logarithmic connection that restricts to the connection \(D^0\) on \(M^0\) constructed in Section 2.

**Proof.** Let

\[
\tilde{\mathcal{V}} := E_G \times t \to E_G
\]

be the trivial vector bundle over \(E_G\) with fiber \(t\). Note that \(p^* \mathcal{V} = \tilde{\mathcal{V}}\), where \(\mathcal{V}\) is the vector bundle in (3-1), and \(p\), as before, is the projection of \(E_G\) to \(M\).
The action $\tilde{\rho}$ of $T$ on $E_G$ produces a homomorphism

\[(3-7) \quad \tilde{\beta} : \tilde{V} \to TE_G.\]

Since $p^{-1}(D)$ is preserved by the action of $T$ on $E_G$, the induced action of $T$ on $O_{E_G}$ preserves the subsheaf $O_{E_G}(-p^{-1}(D))$. Hence the image of $\tilde{\beta}$ lies inside the subsheaf

$$TE_G(-\log p^{-1}(D)) \subset TE_G.$$ 

Note that $p^{-1}(D)$ is a simple normal crossing divisor on $E_G$ because $D$ is a simple normal crossing divisor on $M$.

In Lemma 3.1(2) we saw that $\beta$ is an isomorphism. Consider

$$p^*\beta^{-1} : p^*(TM(-\log D)) \to p^*V = \tilde{V}.$$ 

Precomposing this with $\tilde{\beta}$ in (3-7), we have

$$\tilde{\beta} \circ (p^*\beta^{-1}) : p^*(TM(-\log D)) \to TE_G(-\log p^{-1}(D)).$$

We observe that the homomorphism $\tilde{\beta} \circ (p^*\beta^{-1})$ is $G$-equivariant for the trivial action of $G$ on $p^*(TM(-\log D))$ and the action of $G$ on $TE_G(-\log p^{-1}(D))$ induced by the action of $G$ on $E_G$. Therefore, taking the $G$-invariant parts of the direct images by $p$, the above homomorphism $\tilde{\beta} \circ (p^*\beta^{-1})$ produces a homomorphism

$$\beta' : TM(-\log D) = \left(p_*p^*(TM(-\log D))\right)^G \to \left(p_*TE_G(-\log p^{-1}(D))\right)^G = \text{At}(E_G)(-\log D).$$

It is now straightforward to check that the homomorphism $\beta'$ produces a holomorphic splitting of the exact sequence in (3-6). Therefore, $\beta'$ defines a logarithmic connection on $E_G$ singular on $D$. The restriction of this logarithmic connection to $M^0$ clearly coincides with the connection $\mathcal{D}^0$ constructed in Section 2. \qed

4. A criterion for equivariance

For each point $t \in T$, define the automorphism

$$\rho_t : M \to M, \quad x \mapsto \rho(t,x),$$

where $\rho$ is the action in (2-1). If $(E_G, \tilde{\rho})$ is a $T$-equivariant principal $G$-bundle on $M$, then clearly the map

$$E_G \to E_G, \quad z \mapsto \tilde{\rho}(t, z)$$

is an isomorphism of the principal $G$-bundle $\rho_t^*E_G$ with $E_G$. The aim in this section is to prove a converse of this statement.
Take an algebraic principal $G$-bundle
\[ p : E_G \to M. \]
Let $G$ be the set of all pairs of the form $(t, f)$, where $t \in T$ and where \[ f : E_G \to E_G \]
is an algebraic automorphism of the variety $E_G$ satisfying the following conditions:
\begin{itemize}
  \item $p \circ f = \rho_t \circ p$, and
  \item $f$ intertwines the action of $G$ on $E_G$.
\end{itemize}
Note that the above two conditions imply that $f$ is an algebraic isomorphism of the principal $G$-bundle $\rho_t^*E_G$ with $E_G$.

We have the following composition on the set $G$:
\[ (t_1, f_1) \cdot (t_2, f_2) := (t_1 \circ t_2, f_1 \circ f_2). \]
The inverse of $(t, f)$ is $(t^{-1}, f^{-1})$. These operations make $G$ a group. In fact, $G$ has the structure of an affine algebraic group defined over $\mathbb{C}$. Let $A$ denote the group of all algebraic automorphisms of the principal $G$-bundle $E_G$. So $A$ is a subgroup of $G$ with the inclusion map being $f \mapsto (e, f)$. We have a natural projection
\[ h : G \to T, \quad (t, f) \mapsto t \]
which fits in the following exact sequence of complex affine algebraic groups:
\[ 0 \to A \to G \xrightarrow{h} T. \]
We note that there is a tautological action of $G$ on $E_G$; the action of any $(t, f) \in G$ on $E_G$ is given by the map defined by $y \mapsto f(y)$.

Now assume that $E_G$ satisfies the condition that, for every $t \in T$, the pulled-back principal $G$-bundle $\rho_t^*E_G$ is isomorphic to $E_G$. This assumption is equivalent to the statement that the homomorphism $h$ in (4-1) is surjective.

In view of the above assumption, the sequence in (4-1) becomes the following short exact sequence of complex affine algebraic groups:
\[ 0 \to A \to G \xrightarrow{h} T \to 0. \]
Let $G^0 \subset G$ be the connected component containing the identity element. Since $T$ is connected and $h$ is surjective, the restriction of $h$ to $G^0$ is also surjective. Therefore, from (4-2) we have the short exact sequence of affine complex algebraic groups
\[ 0 \to A^0 \xrightarrow{i_A} G^0 \xrightarrow{h^0} T \to 0, \]
where $A^0 := A \cap G^0$ and $h^0 := h|_{G^0}$.
Take a maximal torus $T_G \subset G^0$. From (4-3) it follows that the restriction
\[ h' := h^0|_{T_G} : T_G \to T \]
is surjective. Define $T_A := \mathcal{A}^0 \cap T_G \subset T_G$ using the homomorphism $\iota_A$ in (4-3). Therefore, from (4-3) we have the short exact sequence of algebraic groups
\[ 0 \to T_A \xrightarrow{\iota_A} T_G \xrightarrow{h'} T \to \mathbb{0}. \] (4-4)
Recall that $G$ has a tautological action on $E_G$. Therefore, the subgroup $T_G$ has a tautological action on $E_G$ which is the restriction of the tautological action of $G$.

Now we assume that the group $G$ is reductive.

A parabolic subgroup of $G$ is a connected Zariski closed subgroup $P \subset G$ such that the variety $G/P$ is projective. For a parabolic subgroup $P$, its unipotent radical will be denoted by $R_u(P)$. A Levi subgroup of $P$ is a connected reductive subgroup $L(P) \subset P$ such that the composition $L(P) \hookrightarrow P \to P/R_u(P)$ is an isomorphism. Levi subgroups exist, and any two Levi subgroups of $P$ differ by conjugation by an element of $R_u(P)$ [Humphreys 1975, p. 184–185, §30.2; Borel 1991, p. 158, 11.22 and 11.23].

Let $\text{Ad}(E_G) := E_G \times^G G \to M$ be the adjoint bundle associated to $E_G$ for the adjoint action of $G$ on itself. The fibers of $\text{Ad}(E_G)$ are groups identified with $G$ up to an inner automorphism; the corresponding Lie algebra bundle is $\text{ad}(E_G)$. We note that $\mathcal{A}$ in (4-2) is the space of all algebraic sections of $\text{Ad}(E_G)$.

Using the action of $T_A$ on $E_G$, we have
\begin{itemize}
  \item a Levi subgroup $L(P)$ of a parabolic subgroup $P$ of $G$, and
  \item an algebraic reduction of structure group $E_{L(P)} \subset E_G$ of $E_G$ to $L(P)$ which is preserved by the tautological action of $T_G$ on $E_G$,
\end{itemize}
such that the image of $T_A$ in $\text{Ad}(E_G)$ (recall that the elements of $\mathcal{A}$ are sections of $\text{Ad}(E_G)$) lies in the connected component, containing the identity element, of the center of each fiber of $\text{Ad}(E_{L(P)}) \subset \text{Ad}(E_G)$ (see [Balaji et al. 2005; Biswas and Parameswaran 2006] for the construction of $E_{L(P)}$). The construction of $E_{L(P)}$ requires fixing a point $z_0$ of $E_G$, where $E_{L(P)}$ contains $z_0$. Using $z_0$, the fiber $(E_{L(P)})_{p(z_0)}$ is identified with $L(P)$. Moreover, the evaluation, at $p(z_0)$, of the sections of $\text{Ad}(E_G)$ corresponding to the elements of $T_A$ makes $T_A$ a subgroup of the connected component, containing the identity element, of the center of $E_{L(P)}$; in particular, this evaluation map on $T_A$ is injective (see the second paragraph in [Balaji et al. 2005, p. 230, Section 3]). We briefly recall (from [Balaji et al. 2005; Biswas and Parameswaran 2006]) the argument that the evaluation map on semisimple elements of $\mathcal{A}$ is injective. Let $\xi$ be a semisimple element of $\mathcal{A} = \Gamma(M, \text{Ad}(E_G))$. 


Since $\xi$ is semisimple, for each point $x \in M$, the evaluation $\xi(x)$ is a semisimple element of $\text{Ad}(E_G)_x$. The group $\text{Ad}(E_G)_x$ is identified with $G$ up to an inner automorphism of $G$. All conjugacy classes of a semisimple element of $G$ are parametrized by $T_G/W_{T_G}$, where $T_G$ is a maximal torus in $G$, and $W_{T_G} = N(T_G)/T_G$ is the Weyl group with $N(T_G)$ being the normalizer of $T_G$ in $G$. We note that $T_G/W_{T_G}$ is an affine variety. Therefore, we get a morphism $\xi': M \to T_G/W_{T_G}$ that sends any $x \in M$ to the conjugacy class of $\xi(x)$. Since $M$ is a projective variety and $T_G/W_{T_G}$ is an affine variety, we conclude that $\xi'$ is a constant map. So if $\xi(x) = e$ for some $x \in M$, then $\xi = e$ identically.

Let $Z^0_{L(P)} \subset L(P)$ be the connected component, containing the identity element, of the center. We note that $Z^0_{L(P)}$ is a product of copies of $\mathbb{G}_m$. Therefore, the above injective homomorphism $T_A \to Z^0_{L(P)}$ extends to a homomorphism

$$\eta : T_G \to Z^0_{L(P)}.$$ 

Define

(4-5) $$\eta' := \tau \circ \eta,$$

where $\tau$ is the inversion homomorphism of $Z^0_{L(P)}$ defined by $g \mapsto g^{-1}$.

Consider the action of $T_G$ on $E_{L(P)}$; recall that $E_{L(P)}$ is preserved by the tautological action of $T_G$ on $E_G$. We can twist this action on $E_{L(P)}$ by $\eta'$ in (4-5), because the actions of $Z^0_{L(P)}$ and $L(P)$ on $E_{L(P)}$ commute. For this new action, the group $T_A$ clearly acts trivially on $E_{L(P)}$.

Consider the above action of $T_G$ on $E_{L(P)}$ constructed using $\eta'$. Since $T_A$ acts trivially on $E_{L(P)}$, the action of $T_G$ on $E_{L(P)}$ descends to an action of $T$ on $E_{L(P)}$ (see (4-4)). The principal $G$-bundle $E_G$ is the extension of the structure group of $E_{L(P)}$ using the inclusion of $L(P)$ in $G$. Therefore, the above action of $T$ on $E_{L(P)}$ produces an action of $T$ on $E_G$. More precisely, the total space of $E_G$ is the quotient of $E_{L(P)} \times G$ where two elements $(z_1, g_1)$ and $(z_2, g_2)$ of $E_{L(P)} \times G$ are identified if there is an element $g \in L(P)$ such that $z_2 = z_1 g$ and $g_2 = g^{-1} g_1$. Now the action of $T$ on $E_{L(P)} \times G$, given by the above action of $T$ on $E_{L(P)}$ and the trivial action of $T$ on $G$, descends to an action of $T$ on the quotient space $E_G$. Consequently, $E_G$ admits a $T$-equivariant structure.

Therefore, we have proved the following:

**Proposition 4.1.** Let $G$ be reductive, and let $E_G \to M$ be a principal $G$-bundle such that, for every $t \in T$, the pulled-back principal $G$-bundle $\rho^*_t E_G$ is isomorphic to $E_G$. Then $E_G$ admits a $T$-equivariant structure.

For vector bundles on $M$, Proposition 4.1 was proved by Klyachko [1989, p. 342, Proposition 1.2.1].
Equivariance property from a logarithmic connection.

**Theorem 4.2.** Let $G$ be reductive, and let $p : E_G \to M$ be a principal $G$-bundle admitting a logarithmic connection whose singularity locus is contained in the divisor $D = M \setminus M^0$. Then $E_G$ admits a $T$-equivariant structure.

**Proof.** Since $E_G$ admits a logarithmic connection, by definition, there is a homomorphism of coherent sheaves

$$\delta : TM(-\log D) \to \operatorname{At}(E_G)(-\log D)$$

such that $\phi \circ \delta$ is the identity automorphism of $TM(-\log D)$, where $\phi$ is the homomorphism in (3-6). Let

$$\hat{\delta} : H^0(M, TM(-\log D)) \to H^0(M, \operatorname{At}(E_G)(-\log D))$$

be the homomorphism of global sections given by $\delta$. From Lemma 3.1(2) we know that $H^0(M, TM(-\log D))$ is the Lie algebra $\mathfrak{t}$ of $T$.

We will now show that there is a natural injective homomorphism

$$(4-6) \quad \theta : H^0(M, \operatorname{At}(E_G)(-\log D)) \to \operatorname{Lie}(G),$$

where $\operatorname{Lie}(G)$ is the Lie algebra of the group $G$ in (4-1).

The elements of $\operatorname{Lie}(G)$ are all holomorphic sections $s \in H^0(M, \operatorname{At}(E_G))$ such that the vector field $\phi(s)$, where $\phi$ is the projection in (3-5), is of the form $\beta(s')$, where $s' \in \mathfrak{t}$ and where $\beta$ is the homomorphism in (3-2). Now, if

$$s \in H^0(M, \operatorname{At}(E_G)(-\log D)) \subset H^0(M, \operatorname{At}(E_G)),$$

then $\phi(s)$ is a holomorphic section of $TM(-\log D)$ (see (3-6)). From Lemma 3.1(2) it now follows that $\phi(s)$ is of the form $\beta(s')$, where $s' \in \mathfrak{t}$. This gives us the injective homomorphism in (4-6).

Finally, consider the composition

$$\theta \circ \hat{\delta} : \mathfrak{t} = H^0(M, TM(-\log D)) \to \operatorname{Lie}(G).$$

From its construction it follows that

$$(dh) \circ \theta \circ \hat{\delta} = \operatorname{Id}_{\mathfrak{t}},$$

where $dh : \operatorname{Lie}(G) \to \mathfrak{t}$ is the homomorphism of Lie algebras given by $h$ in (4-1). In particular, $dh$ is surjective. Since $T$ is connected, this immediately implies that the homomorphism $h$ is surjective. Now from Proposition 4.1 it follows that $E_G$ admits a $T$-equivariant structure. □
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