We investigate the completeness of Gabor systems with respect to several classes of window functions on rational lattices. Our main results show that the time-frequency shifts of every finite linear combination of Hermite functions with respect to a rational lattice are complete in $L^2(\mathbb{R})$, thus generalizing a remark of von Neumann (and proved by Bargmann, Perelomov et al.). An analogous result is proven for functions that factor into certain rational functions and the Gaussian. The results are also interesting from a conceptual point of view since they show a vast difference between the completeness and the frame property of a Gabor system. In the terminology of physics we prove new results about the completeness of coherent state subsystems.

1. Introduction

We study the question when the set of time-frequency shifts

$$G(g, \alpha, \beta) = \{ e^{2\pi i \beta lx} g(x - \alpha k) : k, l \in \mathbb{Z} \}$$

is complete in $L^2(\mathbb{R})$, where $g \in L^2(\mathbb{R})$ and the lattice parameters $\alpha, \beta > 0$ are fixed. The completeness question arose first in J. von Neumann’s treatment of quantum mechanics [31] and remains relevant in physics and in applied mathematics. The motivation in signal analysis and time-frequency analysis comes from Gabor’s fundamental paper [14] on information theory. Gabor tried to expand a given function (signal) into a series of time-frequency shifts. Correspondingly, in mathematical terminology the set $G(g, \alpha, \beta)$ is called a Gabor system with window function $g$. In quantum mechanics the functions $e^{2\pi i \beta lx} g(x - \alpha k)$ are called phase-space shifts of a (generalized) coherent state, and $G(g, \alpha, \beta)$ can be interpreted as a discrete set of coherent states with respect to the Heisenberg group over a lattice $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ [3]. In fact, Perelomov’s book [33] on coherent states contains several sections devoted to the “completeness of coherent state subsystems.”

In the applied mathematics literature of the last 20 years the interest in Gabor systems has shifted to the frame property, mainly for numerical reasons [12] and because Gabor frames can be used to characterize function spaces [11] and to
describe pseudodifferential operators \[16\]. Here \(\mathcal{G}(g, \alpha, \beta)\) is a (Gabor) frame for \(L^2(\mathbb{R})\), if there exist constants \(A, B > 0\) such that
\[
(2) \quad A\|f\|_2^2 \leq \sum_{k,l \in \mathbb{Z}} |\langle f, e^{2\pi i \beta l} g(\cdot - \alpha k) \rangle|^2 \leq B\|f\|_2^2 \quad \forall f \in L^2(\mathbb{R}).
\]

Clearly (2) implies that \(\mathcal{G}(g, \alpha, \beta)\) is complete in \(L^2(\mathbb{R})\), but in general the frame property of \(\mathcal{G}(g, \alpha, \beta)\) is much stronger than its completeness. This difference is already present in von Neumann’s example \(\mathcal{G}(\phi, 1, 1)\) where \(\phi(x) = e^{-\pi x^2}\) is the Gaussian, i.e., the canonical coherent state in quantum mechanics, and \(\alpha = \beta = 1\). In this case it was proved 40 years after von Neumann that \(\mathcal{G}(\phi, 1, 1)\) is complete, but not a frame [5,32].

The main intuition for the results about the completeness and the frame property of \(\mathcal{G}(g, \alpha, \beta)\) is based on the uncertainty principle. According to the uncertainty principle every physical state \(g\), i.e., \(g \in L^2(\mathbb{R})\), \(\|g\|_2^2 = \int_{\mathbb{R}} |g(x)|^2 \, dx = 1\), occupies a cell in phase space (in the time-frequency plane) of minimal area one. The phase-space shift \(e^{2\pi i \beta l} g(x - \alpha k)\) is located roughly at position \(\alpha k\) and momentum \(\beta l\) in phase space \(\mathbb{R}^2\). Thus, in order to cover the entire phase space with a discrete set of coherent states \(\mathcal{G}(g, \alpha, \beta)\), we must have necessarily \(\alpha \beta \leq 1\), otherwise there would be gaps in phase space that cannot be reached by a phase-space shift of the form \(e^{2\pi i \beta l x} g(x - \alpha k)\). The physical intuition has been made mathematically rigorous in the form of numerous density theorems for Gabor systems: If \(\mathcal{G}(g, \alpha, \beta)\) is complete in \(L^2(\mathbb{R})\), then necessarily \(\alpha \beta \leq 1\) [9,23,34]. The converse holds only for special window functions. It was already noted in [5,32] that for the Gaussian \(\phi(x) = e^{-\pi x^2}\) the set \(\mathcal{G}(\phi, \alpha, \beta) = \{e^{2\pi i \beta l x} e^{-(x - k \alpha)^2} : k, l \in \mathbb{Z}\}\) is complete for \(\alpha \beta \leq 1\) and incomplete for \(\alpha \beta > 1\). In 1992 Lyubarskii [29] and Seip [36] strengthened this statement and showed that \(\mathcal{G}(\phi, \alpha, \beta)\) is frame, if and only if \(\alpha \beta < 1\). (In fact, they stated their results for arbitrary time-frequency shifts, not just lattice shifts.) The recent work [21] shows that an analogous result also holds for the class of so-called totally positive functions of finite type. Again, as in the case of the Gaussian, these functions possess good time-frequency localization, which in mathematical terms amounts to additional analyticity properties.

In this paper we return to the completeness problem for Gabor systems. In agreement with the physical description of quantum states, we will assume that the window functions possess strong localization properties in the time-frequency plane. Technically, we will assume that \(g\) and its Fourier transform have exponential decay.

Our main results provide a significant generalization of von Neumann observation on the completeness of coherent states (Gabor systems) and may be summarized as follows.

**Theorem 1.1.** Assume that \(g\) factors as \(g(x) = R(x)e^{-\gamma x^2}\), where \(\gamma > 0\), \(R\) is either a rational function with no real poles or a finite sum of complex exponentials \(R(x) = \sum_{j=1}^m c_j e^{\lambda_j x}\) with \(c_j, \lambda_j \in \mathbb{C}\). If \(\alpha \beta\) is rational and \(\alpha \beta \leq 1\), then the Gabor system \(\mathcal{G}(g, \alpha, \beta)\) is complete in \(L^2(\mathbb{R})\).
One may say that the physical intuition works far beyond the canonical coherent states. For rational lattices $\alpha \beta \in \mathbb{Q}$ and windows as in Thm. 1.1, the Gabor system (coherent state subsystem) $\mathcal{G}(g, \alpha, \beta)$ is complete, if and only if $\alpha \beta \leq 1$. Although it seems natural that Theorem 1.1 can be extended to arbitrary rectangular lattices $\alpha \mathbb{Z} \times \beta \mathbb{Z}$ with $\alpha \beta \leq 1$, we must leave this question open because there is no useful completeness characterization over irrational lattices.

To put this result in perspective, we note three special cases (Theorems 3.1, 3.4, and 3.6 and their corollaries in Section 3).

(a) Theorem 1.1 covers all Hermite functions $h_n = c_n e^{\pi x^2} \frac{d^n}{dx^n} (e^{-2\pi x^2})$ (with a suitable normalization constant $c_n$ and $n \in \mathbb{N}$). So far it was known that $\mathcal{G}(h_n, \alpha, \beta)$ is a frame, if $\alpha \beta < \frac{1}{n+1}$ [20]. However, for odd $n = 2m+1$ and $\alpha \beta = 1 - \frac{1}{N}$, $N = 2, 3, \ldots, \infty$, the Gabor system $\mathcal{G}(h_{2m+1}, \alpha, \beta)$ is not a frame [28]. The achievement of Theorem 1.1 is to assert the completeness of $\mathcal{G}(h_n, \alpha, \beta)$ for all Hermite functions and all rational lattices with $\alpha \beta \leq 1$.

The example of the Hermite functions also indicates the limits of the physical intuition which does not explain the difference between completeness (intuition is confirmed perfectly by Theorem 1.1) and the frame property (intuition is not correct). It would be interesting to understand from physical principles why the frame property can be destroyed by certain symmetries, as is the case for the odd Hermite functions.

Let us mention that, after a suitable transformation and choice of representation, the Gabor systems $\mathcal{G}(h_n, \alpha, \beta)$ can also be interpreted as a coherent state subsystem for the degenerate Landau levels of a particle in a constant magnetic field. Thus our results extend to higher Landau levels what was previously known only for the ground state. The standard complex analytic techniques that can be used to deal with the ground state do not work with more general Landau levels. See [2] for a detailed explanation of the connection between Gabor systems and the Landau levels.

(b) If $g$ is given by its Fourier transform $\hat{g}(\xi) = \prod_{j=1}^{m} (1 + i\delta_j \xi)^{-1} e^{-\gamma \xi^2}$ for $\delta_j \in \mathbb{R}$, then $g$ is a totally positive function by Schoenberg’s characterization [35]. It was conjectured in [18] that the Gabor system $\mathcal{G}(g, \alpha, \beta)$ for a totally positive function $g$ generates a frame, if and only if $\alpha \beta < 1$, but so far this statement is known to be true only for the class of totally positive functions of finite type [21]. Theorem 1.1 offers a similar result for the case of completeness for a complementary class of totally positive functions and supports the original conjecture.

(c) Finally, Theorem 1.1 covers the case when $g$ is a finite linear combination of time-frequency shifts of the Gaussian $\phi(x) = e^{-\pi x^2}$. Again, this is in line with the physical intuition.

The remainder of the paper will be organized as follows: In Section 2 we recall the abstract characterizations of complete Gabor systems over a rational lattice due to Zeevi and Zibulski [39] and derive a specialized criterium for functions in the Gelfand-Shilov class. We will juxtapose these characterizations with the corresponding characterizations of the frame property. The comparison of these
criteria reveals the fundamental difference between the completeness and the frame property and is particularly striking for window functions in the Gelfand-Shilov space. The completeness property hinges on the analyticity of \( g \) and \( \hat{g} \). It is easy to produce counter-examples to Theorem 1.1 belonging to the Schwartz class, where \( g \) and \( \hat{g} \) are \( C^\infty \).

In Section 3 we will prove several versions of Theorem 1.1. The main idea is to use our knowledge that the Gaussian window \( \phi(x) = e^{-\pi x^2} \) generates a complete system \( G(\phi, \alpha, \beta) \) for \( \alpha \beta \leq 1 \), and to subsequently show that the algebraic conditions that guarantee the completeness of \( G(g, \alpha, \beta) \) for a factorized function \( g(x) = R(x)e^{-\pi x^2} \) are satisfied. This is a new proof method in Gabor analysis that very likely can be sharpened and extended. Ultimately we hope that our results will also lead to a better understanding of several unsolved problems about Gabor frames that were formulated in [18].

Notation: We write \( f(x) \lesssim g(x) \) to say that there exists a constant independent of \( x \), such that \( f(x) \leq Cg(x) \) for all \( x \). Furthermore, \( f(x) \asymp g(x) \) means that \( f(x) \lesssim g(x) \) and \( g(x) \lesssim f(x) \).

2. Gabor systems

The translation and modulation operators act on a function \( f : \mathbb{R} \to \mathbb{C} \) by

\[
M_b f(x) := e^{2\pi ibx} f(x), \quad b \in \mathbb{R},
\]
\[
T_a f(x) := f(x-a), \quad a \in \mathbb{R}.
\]

The composition \( M_bT_a \) is called a time-frequency shift (or phase-space shift in the language of quantum mechanics). A Gabor system is a collection of time-frequency shifts of a given function (a window function in signal analysis, or a quantum mechanical state). Given \( g \in L^2(\mathbb{R}) \), \( \alpha, \beta > 0 \), we define then

\[
G(g, \alpha, \beta) = \{ M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z} \}.
\]

The frame operator associated with this family is given by

\[
Sf = \sum_{k, l \in \mathbb{Z}} \langle f, M_{\beta l} T_{\alpha k} g \rangle M_{\beta l} T_{\alpha k} g.
\]

Under mild assumptions on \( g \) the frame operator is bounded \([15,37]\). The standard assumption is that \( g \) belongs to the modulation space

\[
M^1(\mathbb{R}) := \{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} |\langle f, M_\xi T_x f \rangle| \, dx d\xi < +\infty \},
\]

also known as the Feichtinger algebra. If \( g \in M^1(\mathbb{R}) \), then \( S \) is bounded on \( L^2(\mathbb{R}) \). A Gabor system \( G(g, \alpha, \beta) \) is called a frame if the frame operator \( S \) is invertible on \( L^2(\mathbb{R}) \). This is equivalent to the frame inequalities \([2]\).

We say that \( G(g, \alpha, \beta) \) is complete (in \( L^2 \)) if the linear span of \( G(g, \alpha, \beta) \) is a dense subspace of \( L^2(\mathbb{R}) \). Equivalently, \( S \) is one-to-one on \( L^2(\mathbb{R}) \).
While the completeness of \( G(g, \alpha, \beta) \) means that any function \( f \in L^2(\mathbb{R}) \) can be approximated by linear combinations of elements of \( G(g, \alpha, \beta) \), the frame property implies the existence of a convergent expansion

\[
f = \sum_{k,l \in \mathbb{Z}} a_{k,l} M_{\beta l} T_{\alpha k} g,
\]

with \( \|a\|_2 \approx \|f\|_2 \). For the Gaussian \( \phi(t) := e^{-\pi t^2} \), \( G(\phi, \alpha, \beta) \) is complete if and only if \( \alpha \beta \leq 1 \) and \( G(\phi, \alpha, \beta) \) is a frame if and only if \( \alpha \beta < 1 \) \([29, 32, 36]\).

2.1. Completeness criteria for rational lattices. Our starting point is the characterization of the completeness of a Gabor system over a rational lattice by means of the Zak transform, due to Zeevi and Zibulski \([39]\).

We write \( \alpha \beta = p/q \) for relatively prime positive integers \( p \) and \( q \) and assume that \( p \leq q \). The Zak transform of \( f \in L^2(\mathbb{R}) \) with respect to a parameter \( \alpha \in \mathbb{R} \) is

\[
Z_\alpha f(x, \xi) = \sum_{k \in \mathbb{Z}} f(x - \alpha k) e^{2\pi i k \xi}.
\]

By quasi-periodicity this function is completely determined by its values on the rectangle \( I_\alpha = [0, \alpha] \times [0, 1/\alpha] \). Furthermore, the Zak transform is a unitary isomorphism from \( L^2(\mathbb{R}) \) to \( L^2(I_\alpha) \).

A note on terminology: In signal processing \( Z_\alpha \) is called the Zak transform in reference to one of its first applications by J. Zak \([38]\), in solid states physics \( Z_\alpha \) is usually called the Bloch-Floquet transform \([26]\), and in harmonic analysis it is often called the Weil-Brezin transform \([13]\).

We will also need the vector-valued version

\[
\overrightarrow{Z}_\alpha f(x, \xi) = (Z_\alpha f(x + \frac{a}{p} r, \xi))_{r=0}^{p-1}, \quad x, \xi \in [0, \alpha/p] \times [0, 1/\alpha),
\]

which is unitary from \( L^2(\mathbb{R}) \) onto the space \( L^2([0, \alpha/p] \times [0, 1/\alpha), \mathbb{C}^p) \) of vector-valued functions.

For rational lattices \( \alpha \mathbb{Z} \times \beta \mathbb{Z} \) with \( \alpha \beta \in \mathbb{Q} \), the characterizations of the frame property and of the completeness of \( G(g, \alpha, \beta) \) are in terms of the spectrum of a certain matrix containing Zak transforms of \( g \). Let \( Q_g(x, \xi) \) be the \( p \times q \)-matrix with entries

\[
Q_g(x, \xi)_{jk} = Z_\alpha g(x + \frac{\alpha j}{p}, \xi - \beta k) e^{2\pi i j k / q}, \quad j = 0, \ldots, p - 1, k = 0, \ldots, q - 1.
\]

and \( A_g(x, \xi) \) be the corresponding \( p \times p \) square matrix

\[
A_g(x, \xi) = Q_g(x, \xi) Q_g(x, \xi)^*.
\]

Then the frame operator has the representation

\[
\overrightarrow{Z}_\alpha S f(x, \xi) = \alpha^d A_g(x, \xi) \overrightarrow{Z}_\alpha f(x, \xi), \quad x, \xi \in [0, \alpha/p] \times [0, 1/\alpha),
\]

which is well defined for \( g \in M^1(\mathbb{R}) \). The following characterization of completeness is due to Zeevi and Zibulski \([39\) Theorem 2]. (We use a slightly different setup that follows \([15\) Chapter 8].)
Lemma 2.1. Let $g \in L^2(\mathbb{R})$ and $\alpha \beta = p/q \in \mathbb{Q}$. Then following are equivalent.

(i) The Gabor system $G(g, \alpha, \beta)$ is complete in $L^2(\mathbb{R})$.

(ii) $\det A_g(x, \xi) \neq 0$, for almost every $(x, \xi) \in \mathbb{R}^2$.

(iii) The matrix $Q_g(x, \xi)$ has full rank $p$ for almost every $(x, \xi) \in \mathbb{R}^2$.

It is instructive to compare this characterization of completeness with the corresponding characterization of the frame property in [39, Thm. 4]: $G(g, \alpha, \beta)$ is a frame, if and only if there exist $0 < \delta \leq \Delta$ such that $\delta \leq |\det A_g(x, \xi)| \leq \Delta$, for almost every $(x, \xi) \in \mathbb{R}^2$. When $g \in M^1(\mathbb{R})$, the continuity and quasi-periodicity of $Z_g$ yield the following simple characterization of the frame property.

Lemma 2.2. Let $g \in M^1(\mathbb{R})$ and $\alpha \beta = p/q \in \mathbb{Q}$. Then following are equivalent:

(i) The Gabor system $G(g, \alpha, \beta)$ is a frame in $L^2(\mathbb{R})$.

(ii) $\det A_g(x, \xi) \neq 0$, for all $(x, \xi) \in \mathbb{R}^2$.

(iii) The matrix $Q_g(x, \xi)$ has full rank for all $(x, \xi) \in \mathbb{R}^2$.

Note the subtle difference in conditions (ii)! A single zero of $\det A_g$ may destroy the frame property of $G(g, \alpha, \beta)$.

We mention that in both cases one may write a formal reconstruction of $f \in L^2(\mathbb{R})$ from the correlations $\langle f, M_{\beta l}T_{\alpha k}g \rangle$, $k, l \in \mathbb{Z}$ by inverting (3) as follows:

$$f = \alpha^{-d} \overline{Z}^{-1}_g A_g(x, \xi) \overline{Z}^{-1}_g Sf(x, \xi).$$

If $G(g, \alpha, \beta)$ is a frame, then this reconstruction is stable, whereas for a complete Gabor system this reconstruction may lead to instabilities on certain subspaces of $L^2(\mathbb{R})$, because $A_g$ is not invertible everywhere. For the classical coherent states $\phi(x) = e^{-\pi x^2}$ and $\alpha = \beta = 1$ explicit reconstruction formulas are known, see e.g. [30].

2.2. Windows in the Gelfand-Shilov class. For windows in the Gelfand-Shilov class $S^{1,1}$ the characterization of completeness can be reformulated in a useful way, which we now describe.

We say that a function $g : \mathbb{R} \to \mathbb{C}$ is in the Gelfand-Shilov class $S^{1,1}(\mathbb{R})$, if $g$ and its Fourier transform $\hat{g}$ have exponential decay, i.e.,

$$|g(x)| \lesssim e^{-a|x|} \quad \forall x \in \mathbb{R}$$

and

$$|\hat{g}(\xi)| \lesssim e^{-b|\xi|} \quad \forall \xi \in \mathbb{R}$$

for some decay constants $a, b > 0$. This is not the standard definition but a simpler equivalent condition due to [7]. In particular, every function of the form $g = R\phi$, where $R$ is a rational function with no poles on the real axis and $\phi(x) = e^{-\pi x^2}$, belongs to the Gelfand-Shilov class $S^{1,1}$. Likewise, $S^{1,1}$ includes functions of the form

$$g(x) = \sum_{k=1}^n c_k e^{2\pi ib_k x} \phi(x - a_k),$$
where \(a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}\).

The assumption \(g \in S^{1,1}\) implies that \(g\) is real analytic. More precisely, by the theorem of Paley-Wiener every \(g \in S^{1,1}\) extends to a function that is analytic on the strip \(S_b = \{z \in \mathbb{C} : |\Im(z)| < b\}\), where \(b\) is the decay constant in \([\mathbb{R}]\). Moreover, for every \(b' \in (0, b)\) there exists a constant \(C_{b'}\) such that
\[
|g(x + iy)| \leq C_{b'} e^{-a|x|} \quad x, y \in \mathbb{R} \text{ with } |y| \leq b'.
\]
See for example [8, Theorem 3.9]

We now observe that the Zak transform of \(g \in S^{1,1}\) can also be extended to an analytic function of two variables.

**Lemma 2.3.** If \(g \in S^{1,1}(\mathbb{R})\), then the Zak transform \(Z_\alpha g\) can be extended to an analytic function on \(S_b \times S_b \subseteq \mathbb{C}^2\) for some \(b > 0\).

**Proof.** Since \(g \in S^{1,1}\), there exists \(b > 0\) such that \(g\) extends analytically to \(S_b\) and satisfies the uniform decay estimate:
\[
|g(x + iy)| \lesssim e^{-2\pi a|x|} \quad x \in \mathbb{R}, |y| < b,
\]
for some \(a > 0\). Without loss of generality, we assume that \(b < a\). Let us show that \(Z_\alpha g\) extends analytically to \(S_b \times S_b\). It suffices to show that the series
\[
Z_\alpha g(z, w) = \sum_{k \in \mathbb{Z}} g(z - \alpha k) e^{2\pi i \omega k}
\]
converges uniformly and absolutely on compact subsets of \(S_b \times S_b\). Fix \(C > 0\) and let \((z, w) \in \mathbb{C}^2\) with \(|z|, |w| \leq C\) and \(|\Im(z)|, |\Im(w)| \leq b\). Then
\[
\sum_{k \in \mathbb{Z}} |g(z - \alpha k) e^{2\pi i \omega k}| \leq \sum_{k \in \mathbb{Z}} e^{-2\pi a|z - \alpha k|} e^{2\pi a|\Im(w)||k|}
\]
\[
\leq \sum_{k \in \mathbb{Z}} e^{-2\pi (a|k| - a|z - \alpha k|)} \leq e^{2\pi a C} \sum_{k \in \mathbb{Z}} e^{-2\pi a(b - |k|)} < +\infty,
\]
since \(a > b\). This completes the proof. \(\blacksquare\)

Using the analyticity of the Zak transform, we now obtain the following characterization for the completeness of a Gabor system with windows \(g \in S^{1,1}\).

**Proposition 2.4.** Let \(g \in S^{1,1}(\mathbb{R})\) and \(\alpha \beta = p/q \in \mathbb{Q}, \alpha \beta \leq 1\). Then the following are equivalent.

(i) \(\mathcal{G}(g, \alpha, \beta)\) fails to be complete.

(ii) \(\det A_g(x, \xi) = 0\) for all \(x, \xi \in \mathbb{R}\).

(iii) \(Q_g(x, \xi)\) has rank \(< p\) for all \(x, \xi \in \mathbb{R}\), in other words, all \(p \times p\) submatrices of \(Q_g(x, \xi)\) have determinant zero.

**Proof.** The equivalence between (ii) and (iii) is clear. In addition, by Lemma 2.1 (iii) implies (i). Let us show that (i) implies (iii). Assume that \(\mathcal{G}(g, \alpha, \beta)\) is...
We will show that \( E = \mathbb{R}^2 \). By Lemma 2.1, we know that \( E \) has positive Lebesgue measure, since, otherwise, \( Q_g(x, \xi) \) would have full rank almost everywhere. In addition, by Lemma 2.3, there exists \( b > 0 \) such that \( Z_\alpha \) extends analytically to \( S_b \times S_b \) and, therefore, so does \( \det Q_{g^0 \ldots g^{p-1}} \). Hence, \( E \) is the zero set of a real analytic function on \( \mathbb{R}^2 \), and, having positive measure, it must be equal to \( \mathbb{R}^2 \). We provide a short argument for this (known) fact.

By the continuity of \( \det Q_{g^0 \ldots g^{p-1}} \), we know that \( E \) is a closed set. Hence, in order to prove that \( E = \mathbb{R}^2 \), it suffices to show that \( |\mathbb{R}^2 \setminus E| = 0 \). Given \( x \in \mathbb{R} \), the section

\[
E_x := \{ \xi \in \mathbb{R} : (x, \xi) \in E \}
\]

is the zero set of the function \( \det Q_{g^0 \ldots g^{p-1}}(x, \cdot) : \mathbb{R} \to \mathbb{C} \), which admits an analytic extension to \( S_b \). Hence, \( E_x \) is either \( \mathbb{R} \) or has measure zero. Therefore, \( E = (X \times \mathbb{R}) \cup N_1 \), for some measurable sets \( X, N_1 \subseteq \mathbb{R} \) with \(|N_1| = 0 \) and \(|X| > 0 \). A similar argument, with the roles of \( x \) and \( \xi \) interchanged, shows that \( E = (\mathbb{R} \times Y) \cup N_2 \), with \(|Y| > 0 \) and \(|N_2| = 0 \). The conditions on \( E \) imply that \( 1_E(x, y) = 1_X(x) = 1_Y(y) \) for almost all \((x, y) \in \mathbb{R}^2 \). Since \(|E| > 0 \), this is only possible if \( 1_E \equiv 1 \) a.e. Therefore, \( |\mathbb{R}^2 \setminus E| = 0 \), as desired.

Motivated by Proposition 2.4, we compute explicitly the determinant of \( Q_{g^0 \ldots g^{p-1}}(x, \xi) \) for a given selection of columns \( L \equiv \{l_0, \ldots, l_{p-1}\} \subseteq \{0, \ldots, q-1\} \).

Let \( \text{Perm}(l_0, \ldots, l_{p-1}) \) denote the group of all permutations of the chosen columns. Then

\[
\det Q_{g^0 \ldots g^{p-1}}(x, \xi) = \sum_{\sigma \in \text{Perm}(l_0, \ldots, l_{p-1})} (-1)^{\text{sgn}(\sigma)} \prod_{j=0}^{p-1} Z_{\alpha j}(x + \alpha j/p, \xi - \beta \sigma(l_j)) e^{2\pi i \sigma(l_j)/q}
\]

\[
= \sum_{\sigma \in \text{Perm}(l_0, \ldots, l_{p-1})} (-1)^{\text{sgn}(\sigma)} \sum_{k_0, \ldots, k_{p-1} \in \mathbb{Z}} \prod_{j=0}^{p-1} g(x + \alpha j/p - \alpha k_j) \times e^{2\pi i \sum_{j=0}^{p-1} [(\xi - \beta \sigma(l_j))\alpha k_j + j \sigma(l_j)/q]} = 0.
\]

We sum over the permutations \( \sigma \) first and denote the resulting coefficients by

\[
c(k_0, \ldots, k_{p-1}) := \sum_{\sigma \in \text{Perm}(l_0, \ldots, l_{p-1})} (-1)^{\text{sgn}(\sigma)} e^{2\pi i \sum_{j=0}^{p-1} [-\rho \sigma(l_j)k_j/q + j \sigma(l_j)/q]}.
\]
Then
\[
\det Q_{l_0, \ldots, l_{p-1}}(x, \xi) = \sum_{k_0, \ldots, k_{p-1} \in \mathbb{Z}} \prod_{j=0}^{p-1} g(x + \frac{\alpha_j}{p} - \alpha k_j) c(k_0, \ldots, k_{p-1}) e^{2\pi i \alpha (k_0 + \cdots + k_{p-1}) \xi}.
\]

Thus for fixed \( x \in [0, \alpha] \) the determinant \( \det Q_{l_0, \ldots, l_{p-1}}(x, \xi) \) is a Fourier series with coefficients
\[
\Theta_g^L(x, N) := \sum_{k_0 + \cdots + k_{p-1} = N \in \mathbb{Z}} \prod_{j=0}^{p-1} g(x + \alpha_j/p - \alpha k_j) c(k_0, \ldots, k_{p-1}).
\]

Note that the selection of columns of \( Q_{l_0, \ldots, l_{p-1}}(x, \xi) \) enters only in the coefficients \( c(k_0, \ldots, k_{p-1}) \).

We now state one more reformulation of Lemma 2.1.

**Lemma 2.5.** Let \( g \in S^{1,1}(\mathbb{R}) \) and \( \alpha \beta = p/q \in \mathbb{Q} \), \( \alpha \beta \leq 1 \). Then the following are equivalent.

(i) The Gabor system \( G(g, \alpha, \beta) \) is incomplete in \( L^2(\mathbb{R}) \).

(ii) For all choices of subsets \( L \equiv \{l_0, \ldots, l_{p-1}\} \subset \{0, \ldots, q-1\} \), all \( x \in \mathbb{R} \) and all \( N \in \mathbb{Z} \), we have \( \Theta_g^L(x, N) = 0 \).

**2.3. Complete Gabor Systems versus Gabor Frames.** Although the formulations of Lemma 2.1 and 2.2 are deceptively similar, the difference between completeness and the frame property is dramatic. To highlight this difference, we show that the spanning properties of a Gabor family over a lattice are extremely stable and cannot be modified by naive changes. In order to state the result precisely (for arbitrary dimension), we recall that the lower Beurling density of a set \( N \subseteq \mathbb{R}^d \) is given by
\[
D^{-}(N) := \liminf_{R \to +\infty} \frac{\#(N \cap B_R(x))}{|B_R(x)|},
\]
where \( B_R(x) \) denotes the ball of radius \( R \) centered at \( x \in \mathbb{R}^d \). Hence \( D^{-}(N) = 0 \) if and only if \( N \) contains arbitrarily large holes.

**Proposition 2.6.** Assume that \( g \in M^1(\mathbb{R}^d) \), \( \Lambda \subseteq \mathbb{R}^{2d} \) is a lattice and \( G(g, \Lambda) \) is not a frame. If \( N \subseteq \mathbb{R}^{2d} \) is a set such that \( D^{-}(N) = 0 \), then \( G(g, \Lambda \cup N) \) fails to be a frame.

**Proof.** Assume on the contrary that \( G(g, \Lambda \cup N) \) is a frame. The proof is based on Theorem 5.1 from [19], which implies that for every set \( \Gamma \) that is a weak limit of translates of \( \Lambda \cup N \), \( G(g, \Gamma) \) is also a frame. (We refer the reader to [19] for the precise definition of weak convergence of sets.)
Since $D^-(N) = 0$, $N$ contains arbitrarily large holes centered at points of $\Lambda$. For every $n \in \mathbb{N}$ there exists $\lambda_n \in \Lambda$ such that $N \cap B_n(\lambda_n) = \emptyset$. This implies that the sequence of translates $\{(\Lambda \cup N) - \lambda_n : n \in \mathbb{N}\}$ converges weakly to $\Lambda$. Indeed

$$((\Lambda \cup N) - \lambda_n) \cap B_n(0) = (\Lambda \cup (N - \lambda_n)) \cap B_n(0) = \Lambda \cap B_n(0).$$

By [19, Theorem 5.1] $G(g, \Lambda)$ is a frame, contradicting our assumptions. ■

Proposition 2.6 means that in order to extend a complete Gabor family (that is not already a frame) into a (non-uniform) Gabor frame, we need to add a set of strictly positive density. This complements the results of Balan, Casazza, Heil and Landau that also stress the strong rigidity of the frame property: for a Gabor frame $G(g, \Lambda)$ with window $g \in M_1$, it is always possible to remove a certain infinite subset $\Lambda' \subseteq \Lambda$ while preserving the frame property. Moreover, it is possible to choose $\Lambda'$ so that the density of the remaining set $\Lambda \setminus \Lambda'$ is arbitrarily close to 1 [6].

3. Explicit Completeness Theorems

In this section we investigate the completeness of Gabor systems $G(g, \alpha, \beta)$ for several general, explicit classes of window functions in $S^{1,1}$ on a rectangular rational lattice. In all cases we will make use of the fact that the Gabor system $G(\phi, \alpha, \beta)$ for $\phi(x) = e^{-\pi x^2}$ is complete when $\alpha \beta \leq 1$ and then apply the algebraic characterization of Lemma 2.5.

We start with the class of windows that factor into a polynomial and the Gaussian. We first prove that, for an arbitrary polynomial $P$, the function $g(x) = P(x)e^{-\pi x^2}$ generates a complete system when $\alpha \beta$ is rational and $\alpha \beta \leq 1$.

**Theorem 3.1.** Let $g(x) = P(x)e^{-\pi x^2}$ with a non-zero polynomial $P$ and $\alpha \beta \in \mathbb{Q}$, $\alpha \beta \leq 1$. Then the Gabor system $G(g, \alpha, \beta)$ is complete in $L^2(\mathbb{R})$.

**Proof.** We set $d := \deg(P)$ and assume without loss of generality that the leading coefficient of $P$ is 1. As before, we write $\alpha \beta = p/q$ for relatively prime positive integers.

We recall the fundamental fact that the Gaussian $\phi(x) = e^{-\pi x^2}$ generates a complete Gabor system $G(\phi, \alpha, \beta)$, if and only if $\alpha \beta \leq 1$ [5,32]. We will show that this fact implies that $G(g, \alpha, \beta)$ with $g = P\phi$ is also complete. More precisely, we will verify the conditions of Proposition 2.4 and show that there exist column indices $L \equiv \{l_0, \ldots, l_{p-1}\}$ such that the determinant of $Q_{\phi}^{l_0, \ldots, l_{p-1}}$ does not vanish identically.

We argue by contradiction and assume that $G(g, \alpha, \beta)$ is not complete. By Lemma 2.5.

$$\Theta^L_g(x, N) = \sum_{k_0 + \ldots + k_{p-1} = N} \prod_{j=0}^{p-1} g(x + \frac{\alpha j}{p} - \alpha k_j) c(k_0, \ldots, k_{p-1}) = 0$$

for all $x \in \mathbb{R}$ and all $N \in \mathbb{Z}$ where the coefficients $c(k_0, \ldots, k_{p-1})$ are given by (6).
**Step 1.** We first compute the expression $\Theta^L_\phi(x, N)$ for the Gaussian $\phi(x) = e^{-\pi x^2}$ in place of $g$. The exponent of each product is

$$-\frac{1}{\pi} \log \prod_{j=0}^{p-1} \phi(x + \alpha_j/p - \alpha k_j)$$

$$= \sum_{j=0}^{p-2} (x + \frac{\alpha_j}{p} - \alpha k_j)^2 + (x + \frac{\alpha(p-1)}{p} - \alpha(N - k_0 - \cdots - k_{p-2}))^2$$

$$= px^2 + 2x\alpha \left[ \sum_{j=0}^{p-2} \left( \frac{j}{p} - k_j \right) + \frac{p-1}{p} - N + \sum_{j=0}^{p-2} k_j \right]$$

$$+ \alpha^2 \left[ \sum_{j=0}^{p-2} \left( \frac{j}{p} - k_j \right)^2 + \left( \frac{p-1}{p} - N + \sum_{j=0}^{p-2} k_j \right)^2 \right]$$

$$= px^2 + \frac{2x\alpha}{p} \left( \sum_{j=0}^{p-2} j \right) - 2x\alpha N + \alpha^2 \left[ \sum_{j=0}^{p-2} \left( \frac{j}{p} - k_j \right)^2 + \left( \frac{p-1}{p} - N + \sum_{j=0}^{p-2} k_j \right)^2 \right].$$

Since $g(\phi, \alpha, \beta)$ is complete for $\alpha \beta \leq 1$, there exists some $N \in \mathbb{Z}$, such that the quantity

$$e^{-\pi(px^2 + \alpha(p-1)x - 2\alpha x)} \sum_{k_0, \ldots, k_{p-2} \in \mathbb{Z}} \exp \left( -\pi \alpha^2 \left[ \sum_{j=0}^{p-2} \left( \frac{j}{p} - k_j \right)^2 + \left( \frac{p-1}{p} - N + \sum_{j=0}^{p-2} k_j \right)^2 \right] \right) \times$$

$$c(k_0, \ldots, k_{p-2}, N - k_0 - \cdots - k_{p-2}) \neq 0.$$  

(9)

Thus at least one of the coefficients

$$s_0(N) = \sum_{k_0, \ldots, k_{p-2} \in \mathbb{Z}} \exp \left( -\pi \alpha^2 \left[ \sum_{j=0}^{p-2} \left( \frac{j}{p} - k_j \right)^2 + \left( \frac{p-1}{p} - N + \sum_{j=0}^{p-2} k_j \right)^2 \right] \right) \times$$

$$c(k_0, \ldots, k_{p-2}, N - k_0 - \cdots - k_{p-2}) \neq 0$$  

(10)

must be non-zero.

**Step 2.** We next evaluate $\Theta^L_\phi(x, N)$ for $g(x) = P(x)e^{-\pi x^2}$. Since $P$ is a polynomial of degree $d$ with leading coefficient 1, the product $\prod_{j=0}^{p-1} P(x + \alpha j/p - \alpha k_j)$ is a polynomial of degree $dp$. The coefficient of $x^{dp}$ is one, whereas the coefficients of the lower order terms depend on $(k_0, \ldots, k_{p-1})$ in a complicated way. What matters is that the leading coefficient does not depend on $(k_0, \ldots, k_{p-1})$. After summing over $(k_0, \ldots, k_{p-1}) \in \mathbb{Z}^p$ with $k_0 + \cdots + k_{p-1} = N$, we obtain from (8) and the calculation in Step 1 that

$$e^{-\pi \left[ px^2 + x\alpha(p-1) - 2\alpha x \right]} \left[ x^{dp}s_0(N) + x^{dp-1}s_1(N) + \cdots + x s_{pd-1}(N) + s_{pd}(N) \right] = 0,$$

(11)
for some coefficients \( s_k(N) \in \mathbb{C} \), all \( N \in \mathbb{Z} \) and \( x \in \mathbb{R} \). The coefficient of the leading term is exactly \( s_0(N) \) from [10] obtained for the Gaussian. Since we have assumed that \( \mathcal{G}(g, \alpha, \beta) \) is incomplete, [11] vanishes identically for all \( N \in \mathbb{Z} \) by Lemma 2.5, and thus all coefficients \( s_m(N), m = 0, \ldots, pd \), must vanish. In particular, \( s_0(N) = 0 \) for all \( N \in \mathbb{Z} \), contradicting [10].

Altogether we have proved that \( \mathcal{G}(g, \alpha, \beta) \) is complete for rational \( \alpha \beta \leq 1 \).

We single out the special case of the Hermite functions \( h_n \) defined by \( h_n(x) = e^{\pi x^2} \frac{d^n}{dx^n}(e^{-2\pi x^2}) \).

**Corollary 3.2.** If \( \alpha \beta \in \mathbb{Q} \) and \( \alpha \beta \leq 1 \), then \( \mathcal{G}(h_n, \alpha, \beta) \) is complete in \( L^2(\mathbb{R}) \).

It is known that for \( \alpha \beta = 1 - 1/N \) for \( N = 2, 3, \ldots, \infty \) the Gabor system \( \mathcal{G}(h_{2m+1}, \alpha, \beta) \) cannot be a frame [28]. Furthermore, the frame property fails for \( \mathcal{G}(h_{4m+2}, \alpha, \beta) \) and \( \mathcal{G}(h_{4m+3}, \alpha, \beta) \) over certain rational lattices [27]. Corollary 3.2 shows that for all Hermite windows and the lattice parameters where the frame property has been shown to fail, the weaker property of completeness still holds.

The result concerning Hermite functions has also an interpretation in terms of complex analysis. We define the Bargmann-Fock spaces of polyanalytic functions as

\[
A^2_n := \{ f : \mathbb{C} \to \mathbb{C} : \int |f(z)|^2 e^{-\pi|z|^2} dA(z) < \infty, \tilde{\partial}^n f = 0 \},
\]

where \( dA(z) \) is the area measure on the complex plane and \( \tilde{\partial} = \frac{1}{2}(\partial_x + i\partial_y) \). The corresponding orthogonal difference spaces

\[
\delta A^2_n := A^2_n \ominus A^2_{n-1}
\]

are called true polyanalytic Bargmann-Fock spaces. These appear naturally in quantum mechanics [2] [22], where they are sometimes called the Landau levels. As in the case of the Gaussian \( n = 0 \), one verifies that \( \mathcal{G}(h_n, \alpha, \beta) \) is a complete system if and only if \( \alpha \mathbb{Z} \times \beta \mathbb{Z} \) is a uniqueness set for \( \delta A^2_n \) (see [1][2]).

According to the previous corollary, any rectangular lattice of rational density larger or equal to 1 is a uniqueness set for \( \delta A^2_q \). It would be interesting to obtain an alternative proof of this fact using complex analysis. The standard techniques for analytic functions do not seem to work in the polyanalytic setting.

Next we consider windows of the form \( g(x) = R(x) e^{-\pi x^2} \), where \( R = P/Q \) is a rational function with two polynomials \( P \) and \( Q \). We may assume without loss of generality that the leading coefficients of \( P \) and \( Q \) are 1. For well-posedness we must assume that \( R \) does not have any poles on the real axis. Then, in fact, the poles of \( g \) are outside a strip \( S_a = \{ z \in \mathbb{C} : |\Im z| < a \} \) for some \( a > 0 \). It is then easy to see that \( g \in S^{1,1} \).

Before stating a completeness theorem for windows of this type, we formulate a fact about rational functions required later.

**Lemma 3.3.** Let \( P, Q \in \mathbb{C}[X] \) be two monic polynomials of degree \( n \), and let \( M := \max \{|z| : z \in \mathbb{C}, P(z) = 0 \text{ or } Q(z) = 0\} \).
Then there exists a constant $C_n > 0$ depending only on the degree $n$, but not on $M$, such that

\[
\left| \frac{P(x)}{Q(x)} - 1 \right| \leq \frac{C_n M^n}{|x|}, \quad \text{for all } |x| \geq \max\{2M, 1\}.
\]

**Proof.** Let $|x| \geq 2M$ and let $Q(z) := \prod_{k=1}^{n} (z - z_k)$ with $z_k \in \mathbb{C}$. Then

\[
|x - z_k| \geq |x| - |z_k| \geq |x| - M \geq \frac{|x|}{2},
\]

and therefore $|Q(x)| \geq 2^{-n} |x|^n$.

Writing $P(x) = x^n + \sum_{k=0}^{n-1} a_k x^k$ and $Q(x) = x^n + \sum_{k=0}^{n-1} b_k x^k$, the coefficients obey the estimates $|a_k|, |b_k| \leq c_n M^n$. (This follows, for example, by expressing $a_k$ and $b_k$ as sums of products of the corresponding roots.)

For $|x| \geq \max\{2M, 1\}$, we now simply estimate

\[
\left| \frac{P(x)}{Q(x)} - 1 \right| = \left| \frac{\sum_{k=0}^{n-1} (a_k - b_k) x^k}{Q(x)} \right| \leq \frac{2^n \sum_{k=0}^{n-1} |a_k - b_k| |x|^k}{|x|^n} \leq \frac{2^{n+1} c_n M^n}{|x|},
\]

and we may take $C_n = 2^{n+1} c_n$. 

We then have the following completeness theorem. Note that it contains Theorem 3.1 as a special case, but the proof is more involved, so we presented the simpler case first.

**Theorem 3.4.** Let $g(x) = R(x) e^{-\pi x^2}$ with a non-zero rational function $R$ that does not have any poles on $\mathbb{R}$, and let $\alpha \beta = p/q \in \mathbb{Q}$, $\alpha \beta \leq 1$. Then the Gabor system $G(g, \alpha, \beta)$ is complete in $L^2(\mathbb{R})$.

**Proof.** The proof follows a similar outline as before for Theorem 3.1. In the proof of Theorem 3.1 we evaluated $\Theta^G_g(x, N)$ in (7) for both $g(x) = P(x) e^{-\pi x^2}$ and for the Gaussian $\phi(x) = e^{-\pi x^2}$ and then showed that the expression for the coefficient of the highest power in $x$ equaled, up to an exponential factor, precisely the expression $\Theta^G_{\phi}(x, N)$ for the Gaussian. Since $\Theta^G_{\phi}(x, N)$ cannot vanish identically, neither can $\Theta^G_g(x, N)$ for $g = P \phi$, whence the completeness of $G(g, \alpha, \beta)$.

In the case of a function $g(x) = R(x) e^{-\pi x^2}$ with rational $R$, we investigate the behavior of $\Theta^G_g(x, N)$ for $x \to \infty$. 
Assume that \( G(g, \alpha, \beta) \) is not complete. Then, by Lemma 2.5 and the calculation in Step 1 of the proof of Theorem 3.1, we conclude that, for all \( x \in \mathbb{R} \) and \( N \in \mathbb{Z} \),

\[
0 = \Theta^L_g(x, N)
\]

\[
= \sum_{k_0 + \cdots + k_{p-1} = N} \prod_{j=0}^{p-1} R(x + \frac{\alpha j}{p} - \alpha k_j) \prod_{j=0}^{p-1} \phi\left(x + \frac{\alpha j}{p} - \alpha k_j\right) c(k_0, \ldots, k_{p-1})
\]

\[
= e^{-\pi(px^2 + \alpha(p-1)x - 2xN)} \sum_{k_0 + \cdots + k_{p-1} = N} \prod_{j=0}^{p-1} R\left(x + \frac{\alpha j}{p} - \alpha k_j\right) \times \exp\left(-\pi \alpha^2 \sum_{j=0}^{p-2} \left(\frac{j}{p} - k_j\right)^2 + \left(\frac{p-1}{p} - N + \sum_{j=0}^{p-2} k_j\right)^2\right) c(k_0, \ldots, k_{p-1}).
\]

(12)

For \( (k_0, \ldots, k_{p-1}) \in \mathbb{Z}^p \), let

\[
d(k_0, \ldots, k_{p-1}) := \exp\left(-\pi \alpha^2 \sum_{j=0}^{p-1} \left(\frac{j}{p} - k_j\right)^2\right) c(k_0, \ldots, k_{p-1}),
\]

and drop one term in exponential in (12) to obtain

\[
\sum_{k_0 + \cdots + k_{p-1} = N} d(k_0, \ldots, k_{p-1}) \prod_{j=0}^{p-1} R\left(x + \frac{\alpha j}{p} - \alpha k_j\right) = 0, \quad x \in \mathbb{R}, \quad N \in \mathbb{Z}.
\]

(13) Now let \( R = P/Q \), with \( P, Q \in \mathbb{C}[X] \) monic, \( \Delta = \deg(P) - \deg(Q) \in \mathbb{Z} \) and note that

\[
\lim_{x \to \infty} \frac{P(x)}{x^{\Delta} Q(x)} = 1.
\]

(15)

After multiplying (14) by \( x^{-\Delta} \), we obtain

\[
0 = \sum_{k_0 + \cdots + k_{p-1} = N} x^{-\Delta} \prod_{j=0}^{p-1} R\left(x + \frac{\alpha j}{p} - \alpha k_j\right) d(k_0, \ldots, k_{p-1}).
\]

(16) Now we let \( x \) tend to infinity, formally interchange the sum and the limit - this is carefully justified below - and use (15) obtaining

\[
0 = \lim_{x \to \infty} \sum_{k_0 + \cdots + k_{p-1} = N} x^{-\Delta} \prod_{j=0}^{p-1} R\left(x + \frac{\alpha j}{p} - \alpha k_j\right) d(k_0, \ldots, k_{p-1})
\]

\[
= \sum_{k_0 + \cdots + k_{p-1} = N} \lim_{x \to \infty} x^{-\Delta} \prod_{j=0}^{p-1} R\left(x + \frac{\alpha j}{p} - \alpha k_j\right) d(k_0, \ldots, k_{p-1})
\]

\[
= \sum_{k_0 + \cdots + k_{p-1} = N} d(k_0, \ldots, k_{p-1}) = s_0(N),
\]
where the last expression is exactly $\Theta_\phi^T(x, N)$ for the Gaussian $\phi$ (apart for an exponential term). Since $s_0(N)$ must be non-zero for some $N$, we have arrived at a contradiction and conclude that $\mathcal{G}(g, \alpha, \beta)$ must be complete.

For a rigorous proof we need to justify the interchange of the limit and the sum. Let $M > 1$ and $x \in \mathbb{R}$ be arbitrary. Using (14), we write the partial sums of $s_0(N)$ as:

$$s_0^M(N) := \sum_{k_0 + \cdots + k_{p-1} = N \atop \| (k_0, \ldots, k_{p-1}) \|_2 \leq M} d(k_0, \ldots, k_{p-1})$$

$$= \sum_{k_0 + \cdots + k_{p-1} = N \atop \| (k_0, \ldots, k_{p-1}) \|_2 \leq M} d(k_0, \ldots, k_{p-1}) \left( 1 - x^{-p\Delta} \prod_{j=0}^{p-1} R(x + \frac{\alpha j}{p} - \alpha k_j) \right) +$$

$$\sum_{k_0 + \cdots + k_{p-1} = N \atop \| (k_0, \ldots, k_{p-1}) \|_2 \leq M} d(k_0, \ldots, k_{p-1}) x^{-p\Delta} \prod_{j=0}^{p-1} R(x + \frac{\alpha j}{p} - \alpha k_j)$$

$$- \sum_{k_0 + \cdots + k_{p-1} = N \atop \| (k_0, \ldots, k_{p-1}) \|_2 > M} d(k_0, \ldots, k_{p-1}) x^{-p\Delta} \prod_{j=0}^{p-1} R(x + \frac{\alpha j}{p} - \alpha k_j)$$

$$=: t_N^M(x) - u_N^M(x).$$

The numbers $t_N^M(x)$ and $u_N^M(x)$ depend on $x$, whereas the partial sum $s_0^M(N)$ is independent of $x$. To show that $\lim_{M \to \infty} s_0^M(N) = 0$, we will choose $x = x_M$ judiciously and obtain suitable bounds. First we note that the coefficients $d(k_0, \ldots, k_{p-1})$ satisfy the decay condition:

$$|d(k_0, \ldots, k_{p-1})| \lesssim e^{-\gamma \| (k_0, \ldots, k_{p-1}) \|_2^2}$$

for some $\gamma > 0$. Indeed, for each $j \in \{0, \ldots, p-1\}$ and $k_j \in \mathbb{Z}$, since $|j/p| \leq p^{-1} < 1$, it follows that $|k_j - j/p| \gtrsim |k_j|$.

We let $n := \max\{\deg(P), \deg(Q)\}$. For $M \gg 1$ and $\| (k_0, \ldots, k_{p-1}) \|_2 \leq M$, the function

$$x^{-p\Delta} \prod_{j=0}^{p-1} R(x + \frac{\alpha j}{p} - \alpha k_j)$$

is a quotient of two monic polynomials of degree $pn$ whose complex roots lie inside a ball of radius $\lesssim M$. Using Lemma 3.3, we now choose $x_M \in \mathbb{R}$ with $|x_M| \asymp M^{n+1}$.
such that
\[ \left| 1 - x_M^{-p\Delta} \prod_{j=0}^{p-1} R(x_M + \frac{\alpha j}{p} - \alpha k_j) \right| \lesssim \frac{1}{M}. \]
Combining this estimate with (17) we obtain
\[ (18) \quad |t_N^M(x_M)| \lesssim \frac{1}{M} \sum_{(k_0, \ldots, k_{p-1}) \neq 0} |d_{(k_0, \ldots, k_{p-1})}| \lesssim \frac{1}{M}. \]
Second, since \( R \) is a rational function without real poles, it satisfies the estimate
\[ |R(x)| \lesssim (1 + |x|)^l, \quad x \in \mathbb{R}, \]
for \( l := \max\{\Delta, 0\} \). This allows us to bound, for \( (k_0, \ldots, k_{p-1}) \neq 0 \),
\[ \left| x_M^{-p\Delta} \prod_{j=0}^{p-1} R(x_M + \frac{\alpha j}{p} - \alpha k_j) \right| \lesssim M^{(n+1)p\Delta} \prod_{j=0}^{p-1} (1 + |x_M + \frac{\alpha j}{p} - \alpha k_j|)^l \lesssim M^{(n+1)p\Delta} \left( M^{(n+1)p(p\Delta)} + |(k_0, \ldots, k_{p-1})|^p \right) \lesssim M^s |(k_0, \ldots, k_{p-1})|^2, \]
for some \( s > 0 \). Therefore
\[ |u_N^M(x_M)| \lesssim M^s \sum_{|(k_0, \ldots, k_{p-1})|^2 > M} |(k_0, \ldots, k_{p-1})|^2 e^{-\gamma |(k_0, \ldots, k_{p-1})|^2}. \]
The last bound shows that \( |u_N^M(x_M)| \to 0 \text{ as } M \to +\infty. \) Combining this with (18) we conclude that
\[ s_0(N) = \lim_{M \to +\infty} s_0^M(N) = 0, \]
and the proof is complete.

As a consequence of Theorem 3.4, we obtain the completeness of Gabor systems for a class of totally positive functions. Let \( \delta_j \in \mathbb{R}, \ j = 1, \ldots, M, \ \gamma > 0 \) and set
\[ (19) \quad \hat{g}(\xi) = e^{-\gamma \xi^2} \prod_{j=1}^{M} (1 + 2\pi i \delta_j \xi)^{-1}. \]
Schoenberg’s factorization theorem [35] asserts that \( g \) is a totally positive function. Since \( \hat{g} \) satisfies the assumptions of Theorem 3.4 and the completeness of a Gabor system is invariant under the Fourier transform, we obtain the following corollary.

**Corollary 3.5.** Let \( g \) be a totally positive function whose Fourier transform factors as in (19) and assume that \( \alpha \beta \leq 1 \) is rational. Then \( G(g, \alpha, \beta) \) is complete in \( L^2(\mathbb{R}) \).

To put the corollary into perspective, we note that it is conjectured that for an arbitrary totally positive function \( g \) the Gabor system \( G(g, \alpha, \beta) \) is a frame, if and only if \( \alpha \beta < 1 \). At this time, this conjecture is known to be true only for totally positive functions whose Fourier transform factors as \( \prod_{j=1}^{M} (1 + 2\pi i \delta_j \xi)^{-1} [21]. \)
Corollary 3.5 shows at least completeness for another class of totally positive functions.

As a final application of the method of proof used in Theorem 3.1 we treat windows of the type \( E(x)\phi(x) \), where \( E \) is an exponential polynomial
\begin{equation}
E(x) = \sum_{\lambda \in \Lambda} a_\lambda e^{\lambda x},
\end{equation}
for some finite set \( \Lambda \subset \mathbb{C} \) and non-zero coefficients \( a_\lambda \in \mathbb{C} \).

**Theorem 3.6.** Let \( g(x) = E(x)\phi(x) \) where \( E(x) \) is of the form (20), \( \alpha \beta \leq 1 \), and \( \alpha \beta \in \mathbb{Q} \). Then \( \mathcal{G}(g, \alpha, \beta) \) is complete in \( L^2(\mathbb{R}) \).

**Proof.** We proceed as in the proofs of Theorems 3.1 and 3.4. Writing \( \alpha \beta = p/q \) and with the notation from (13), the incompleteness condition of Lemma 2.5 becomes
\begin{equation}
0 = \sum_{\lambda_1, \ldots, \lambda_p \in \Lambda} a_{\lambda_1} \cdots a_{\lambda_p} e^{\sum_{j=0}^{p-1} \lambda_j x} \sum_{k_0 + \cdots + k_{p-1} = N} e^{-\alpha \sum_{j=0}^{p-1} \lambda_j k_j} d(k_0, \ldots, k_{p-1}),
\end{equation}
for all \( N \in \mathbb{Z} \). Next, let \( \lambda^* \in \Lambda \) be the minimal element of \( \Lambda \) in the lexicographic ordering of \( \mathbb{R}^2 \). In other words, let \( \hat{\Lambda} \) be the subset of \( \Lambda \) which contains the elements of \( \Lambda \) with minimal real part and let \( \lambda^* \) be the element of \( \hat{\Lambda} \) with minimal imaginary part. Then \( \sum_{j=0}^{p-1} \lambda_j = p\lambda^* \) if and only if \( \lambda_1 = \cdots = \lambda_p = \lambda^* \). Since the complex exponentials \( x \rightarrow e^{ax}, w \in \mathbb{C}, x \in \mathbb{R} \), are linearly independent, the coefficient of \( e^{p\lambda^* x} \) must vanish. Since \( a_\lambda \neq 0 \) for all \( \lambda \in \Lambda \), it follows from (21) that
\begin{equation}
0 = e^{-\alpha N p \lambda^*} \sum_{k_0 + \cdots + k_{p-1} = N} d(k_0, \ldots, k_{p-1}) = e^{-\alpha N p \lambda^*} s_0(N),
\end{equation}
for all \( N \in \mathbb{Z} \). As in the proofs of Theorems 3.1 and Theorem 3.4, this contradicts the completeness of \( \mathcal{G}(\phi, \alpha, \beta) \), with \( \phi \) the Gaussian. ■

**Corollary 3.7.** Let \( g = \sum_{j=1}^n d_j M_{b_j} T_{a_j} \phi \) be a non-zero finite linear combination of time-frequency shifts of the Gaussian \( \phi \), \( \alpha \beta \leq 1 \), and \( \alpha \beta \in \mathbb{Q} \). Then \( \mathcal{G}(g, \alpha, \beta) \) is complete in \( L^2(\mathbb{R}) \).

**Proof.** This is clear since sums of time-frequency shifts can be written in the form (20), as \( M_b T_a \phi(x) = e^{2\pi \lambda x} \phi(x) e^{-\pi b^2} \) for \( a, b \in \mathbb{R} \) and \( \lambda = a + ib \). ■

**Concluding remarks:** 1. In view of Theorem 1.1, it is tempting to conjecture that \( \mathcal{G}(g, \alpha, \beta) \) with \( \alpha \beta \leq 1 \) is complete for every function \( g \in S^{1,1} \). If true, the proof of this statement must heavily depend on the analyticity of \( g \) and \( \hat{g} \) and its Zak transform \( Z_a g \). The following simple counter-example shows what may happen without analyticity. Let \( g \) be a Schwartz function with support in \( \bigcup_{j \in \mathbb{Z}} [2j, 2j+1] \), then \( \mathcal{G}(g, \alpha, \beta) \) is incomplete whenever \( \alpha \in [0, 1) + 2\mathbb{Z} \) and \( \beta > 0 \), although \( g \) and \( \hat{g} \) are \( C^\infty \).

2. Furthermore we remark that the Gabor systems \( \mathcal{G}(g, \alpha, \beta) \) with \( \alpha \beta < 1 \) in Theorem 1.1, although being complete, cannot be Schauder bases. A Gabor
Schauder basis $G(g, \alpha, \beta)$ must satisfy $\alpha \beta = 1$ and the corresponding window $g$ is poorly localized either in the time or the frequency domain [10, 21].

3. The completeness results depend heavily on the lattice structure of the phase space shifts. Although the examples seem counter-intuitive from the point of view of quantum mechanics, one may construct complete Gabor systems without lattice structure that are of density zero [4].

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E-mail address: karlheinz.groechenig@univie.ac.at

E-mail address: antti.haimi@math.ntnu.no

E-mail address: jose.luis.romero@univie.ac.at

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria