EVALUATING THE CRANE-YETTER INVARIANT

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I. INTRODUCTION

The purpose of this paper is to give an explicit formula for the invariant of 4-manifolds introduced by Crane and Yetter in [CR93]. For a closed 4-manifold $W$, this invariant will be denoted herein by $CY(W)$. Our main result is the following theorem, proved in section 4.

Theorem. Let $W$ be a closed 4-manifold. Let $\sigma(W)$ denote the signature of $W$, $\chi(W)$ denote the Euler characteristic of $W$ and $CY(W)$ denote the Crane-Yetter invariant of $W$. Let values $N$ and $\kappa$ be defined as in the beginning of section 4. Then $CY(W) = \kappa \sigma(W) N \chi(W)/2$.

This result is of general interest because it expresses the signature of a 4-manifold in terms of local combinatorial data (these data produce the state summation $CY(W)$ in terms of a triangulation of $W$). Our result should be compared with the work of Gelfand and Macpherson [GM92] where the Pontrjagin classes (and hence the signature) are produced by a combination of subtle combinatorics and geometric topology. Here we give a formula for the signature in terms of a topological quantum field theory that is based on $SU(2)_q$ and on $q$-deformed spin networks.

It should also be mentioned that the invariant $CY(W)$ is a rigorous version of ideas of Ooguri [OG92]. It is of interest to examine the implications of our result for the physics that is inherent in Ooguri’s work. In section 2 we recall the definition of the Crane-Yetter invariant. In section 3, $CY(W)$ is reformulated in terms of Temperley-Lieb recoupling theory. The Theorem is proved in section 4.

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II. A CONCISE DESCRIPTION OF THE INVARIANT $CY(W)$

Let $W = W^4$, a closed 4-manifold. Let $CY(W)$ denote the Crane-Yetter invariant of $W$, as defined in [CY93]. The formula for this invariant, in terms of a triangulation of $W$, is a state summation over colorings from the index set $\{0, 1, 2, \ldots, r-2\}$ of the two dimensional faces and three dimensional simplices of the triangulation of $W$. Here we use integer labellings for convenience. In [CY93] the labellings are by half integers, but the formulas are equivalent. Thus the invariant is a function of the integers $r = 3, 4, \ldots$.

To each colored face of the triangulation is assigned the quantum integer (quantum dimension)

$$\Delta(\text{face}) = \Delta(i) = (-1)^i[q^{i+1} - q^{i-1}]/[q - q^{-1}]$$

where $q = \exp(i\pi/r)$

where $i$ denotes the color assigned to that face.
To each colored tetrahedron is assigned $\Delta(\text{tet}) = \Delta(i)$, where $i$ is the color assigned to that tetrahedron.

To each 4-simplex is assigned the “15-j symbol” $\Phi(4\text{plex})$ that is associated to the coloration of its boundary. This 15-j symbol is an evaluation of a network associated with the boundary of the 4-simplex, obtained by having two interconnected 3-vertices in each tetrahedron, forming a network with four free ends corresponding to the boundary of the tetrahedron. These nets are then interconnected in the pattern of the joinings of the faces of the tetrahedra in the boundary of the 4-simplex. In [CY93] a specific convention for forming this net is given, and we refer to that paper for the details. The 3-vertices and network evaluations are done by Crane and Yetter in the Kirillov-Reshetikhin [KR88] diagrammatics for the recoupling theory of $SU(2)q$.

The formula for the invariant $CY(W)$ is then given as shown below.

$$CY(W) = N^{n_0-n_1} \sum \prod \Delta(\text{face}) \prod \Delta(\text{tet})^{-1} \prod \Phi(4\text{plex})$$

The summation is over all colorings of the faces and tetrahedra from the index set $\{0, 1, 2, \ldots, r-2\}$. The products are over all faces, tetrahedra and 4-simplices respectively. The values $n_0$ and $n_1$ are the number of 0-simplices and the number of 1-simplices in the triangulation. The value $N$ is equal to the sum of the squares of the quantum dimensions, and it has the specific value $-2r/(q-q^{-1})^2$:

$$N = \sum \Delta(i)^2 = -2r/(q-q^{-1})^2.$$

This completes our description of the invariant $CY(W)$.

### III. Translating $CY(W)$ into the Temperley-Lieb Format

In order to prove our result about the evaluation of $CY(W)$ it is useful to translate the state sum into the language of the Temperley-Lieb version of the recoupling theory. This recoupling theory is explained in detail in [KL93], and expositions of it are given in [K91] and [K92]. We shall refer to Temperley-Lieb recoupling as the TL theory, and to Kirillov-Reshetikhin recoupling as the KR theory.

The TL theory is rooted in combinatorics of link diagrams, and it is a direct generalization ($q$-deformation) of the Penrose spin network theory. Its advantage for us here is that there is no dependence in the diagrammatics of the TL theory on maxima and minima or on the orientation of the diagrams with respect to a direction in the plane. Thus TL networks can be freely embedded in handlebodies and 3-manifolds.

The basic information needed to transform KR nets into TL nets is the relationship of their 3-vertices. This is given by the formula below where the subscripts KR and TL discriminate the vertices in question.

$$[3 - \text{vertex}/a, b; c]_{KR} = \left(\frac{\sqrt{\Delta(c)}}{\sqrt{\theta(a, b, c)}}\right) [3 - \text{vertex}/a, b, c]_{TL}$$

Here $\theta(a, b, c)$ is the TL evaluation of a theta net with edges labelled $a$, $b$, and $c$ and $[3 - \text{vertex}/a, b; c]_{(-)}$ is the 3-vertex in the indicated recoupling theory with incoming edges labelled $a$ and $b$ and outgoing edge labelled $c$.

Note that the KR vertex is oriented with two legs up and one leg down. The TL vertex does not have a dependence on the leg placement. The value of a closed loop labelled $i = 0, 1, \ldots, r-2$ is $\Delta(i)$ in both theories. We omit further details of the relationship between TL and KR.

It is now a straightforward matter to translate the $CY(W)$ into the TL framework. The result is as shown below where $\phi$ denotes the TL $15-j$ symbol. The TL $15-j$ is described by exactly the same diagram as the KR $15-j$ symbol, but all its 3-vertices are of TL type.

$$CY(W) = N^{n_0-n_1} \sum \prod \Delta(\text{face}) \prod (\Delta(\text{tet})\theta_1(\text{tet})^{-1}\theta_2(\text{tet})^{-1}) \prod \phi(4\text{plex})$$
In this formula, \( n_d \) is the number of \( d \)-simplices in the triangulation of \( W \). The sum is over all colorings of the faces and tetrahedra of the triangulation.

\( \theta_1(\text{tet}), \theta_2(\text{tet}) \) are the two theta evaluations assigned to each tetrahedron. Each is of the form \( \theta(a, b, c) \) where \( a \), \( b \) and \( c \) are the colors assigned to one of the 3-vertices in the net associated with this tetrahedron. This means that we can take \( a \) and \( b \) to be the colors of two (paired) faces of the tetrahedron, and \( c \) to be the color associated with the tetrahedron itself.

\( \phi(\text{4plex}) \) is the TL evaluation of the \( 15 - j \) net associated with each tetrahedron.

This completes our translation of the invariant \( CY(W) \) into the Temperley-Lieb format.

IV. A Formula for \( CY(W) \)

In this section we use results of Justin Roberts [R93] to give an explicit formula for \( CY(W) \) in terms of the Euler characteristic and the signature of the closed 4-manifold \( W \).

(The reader should note that Roberts uses a notation \( CY(W) \), but that his \( CY(W) \) is not precisely the Crane-Yetter invariant. It differs from it by a factor involving the Euler characteristic of the 4-manifold.)

We need the following values

\[
\eta = (q - q^{-1})/i\sqrt{(2r)}
\]

\[
\kappa = \exp(i\pi(-3 - r^2)/2r) \exp(-i\pi/4).
\]

Note that \( N = \eta^{-2} \), where \( N \) is as defined in section 2.

Roberts [R93] proves that

\[
\kappa^{\sigma(W)} = \eta^{-n_0 + n_1 + n_2 - n_3 + n_4} S(W)
\]

where \( S(W) = \sum \prod \Delta(\text{face}) \prod \Delta(\text{tet}) \theta_1(\text{tet})^{-1} \theta_2(\text{tet})^{-1} \prod \phi(\text{4plex}) \) and \( \sigma(W) \) denotes the signature of the manifold \( W \).

Roberts’ state summation \( S(W) \) has exactly the same form, except for the power of \( N \), as our TL version of the Crane-Yetter invariant. The \( 15 - j \) evaluations of Roberts involve orientation conventions that are consistent with his use of TL networks in 3-dimensional handlebodies. This means that it follows from Robert’s work that different but consistent conventions for the \( 15 - j \) symbols will lead to the same results. The Crane-Yetter convention in the TL format is one such choice. Therefore \( CY(W) = N^{n_0 - n_1} S(W) \) by the results of section 3.

We can now prove the main theorem.

**Theorem.** Let \( W \) be a closed 4-manifold. Let \( \sigma(W) \) denote the signature of \( W \), \( \chi(W) \) denote the Euler characteristic of \( W \) and \( CY(W) \) denote the Crane-Yetter invariant of \( W \). Let values \( N \) and \( \kappa \) be defined as in the beginning of section 4. Then \( CY(W) = \kappa^{\sigma(W)} N^{\chi(W)/2} \).

**Proof.**

\[
\kappa^{\sigma(W)} = \eta^{-n_0 + n_1 + n_2 - n_3 + n_4} N^{n_1 - n_0} CY(W)
\]

\[
= \eta^{-n_0 + n_1 + n_2 - n_3 + n_4} q^{-2n_1 + 2n_0} CY(W)
\]

\[
= \eta^{n_0 - n_1 + n_2 - n_3 + n_4} CY(W)
\]

\[
= N^{-\chi(W)/2} CY(W). \quad \text{Q.E.D.}
\]

**Remark.** Note that if we choose \( r \) greater than the number of 2-simplices in \( W \), then \( \sigma(W) < r \) and is therefore determined by \( CY(W) \) and \( \chi(W) \) via the formula in the theorem. Thus it is quite correct to say that the Theorem produces a combinatorial formula for the signature of a compact 4-manifold in terms of local data from the triangulation.
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