The Hot Bang state of massless fermions

Benjamin Bahr
Institute for Theoretical Physics
University of Göttingen

March 28, 2022

Abstract

In 2002, a method has been proposed by Buchholz et al. in the context of Local Quantum Physics, to characterize states that are locally in thermodynamic equilibrium. It could be shown for the model of massless bosons that these states exhibit quite interesting properties. The mean phase-space density satisfies a transport equation, and many of these states break time reversal symmetry. Moreover, an explicit example of such a state, called the Hot Bang state, could be found, which models the future of a temperature singularity. However, although the general results carry over to the fermionic case easily, the proof of existence of an analogue of the Hot Bang state is not quite that straightforward. The proof will be given in this paper. Moreover, we will discuss some of the mathematical subtleties which arise in the fermionic case.

1 Introduction

In the framework of Local Quantum Physics, states that describe global thermodynamic equilibrium are well-known and are characterized by the KMS-condition [6]. For many models the KMS states are known and relatively simple to write down explicitly. On the other hand it is a nontrivial problem to obtain states that describe systems which are only locally in thermodynamic equilibrium, such as hydrodynamic flows and heat transfers. This is due to the fact that such a state should describe a system that thermalizes on a small, but not on a large scale, thus behaving significantly differently on these two scales.

In [2], Buchholz, Ojima and Roos proposed a method to make this idea mathematically precise. They tried to characterize states that describe situations of local thermodynamic equilibrium. This characterization uses the KMS states as a tool to locally compare a given state with global equilibrium states in order to assign
thermodynamic properties to that state. This comparison is point-dependent and thus delivers a way to describe notions of temperature or entropy that may vary from point to point.

This method has been applied to the model of massless, free bosons on $\mathbb{R}^4$, which has led to interesting results. Firstly, the microscopic dynamics induces a macroscopic transport equation for the phase-space density. Secondly, one finds that states that have a thermodynamic interpretation in a sufficiently large region break time-reversal symmetry, thus implementing a thermodynamic arrow of time [1, 2]. Thirdly, an example for a local equilibrium state is given by the so-called Hot-Bang state that describes the effects of a heat explosion at some point.

Most of the results stated above carry over to the fermionic case easily. They will only be mentioned in short in this paper. What poses a problem is to establish the existence of an analogy of the Hot-Bang state. The proof of the existence of such a state will cover the main part of this work.

## 2 Massless free fermions

The setting for the analysis will be the CAR-Algebra of massless free fermions. In the massless case, the Dirac equation decomposes into two independent equations, called Weyl equations. They describe the left-handed and the right-handed part of the Fermion separately. Thus, we will consider the smeared-out fields $\psi(f)$ and $\bar{\psi}(f)$, where $f$ is a smooth function with compact support and takes values in $\mathbb{C}^2$.

These $\psi(f)$ and $\bar{\psi}(f)$ create a $C^*$-algebra $\mathcal{F}$ subject to the relations:

\[
\{ \psi(f), \bar{\psi}(g) \} = 2\pi \int dp \delta(p^2) \epsilon(p_0) \bar{f}(p) P_\mathbb{M} \bar{g}(-p) \cdot 1 \tag{2.1}
\]

\[
\{ \psi(f), \psi(g) \} = \{ \bar{\psi}(f), \bar{\psi}(g) \} = 0, \tag{2.2}
\]

(where $\epsilon(x) = \theta(x) - \theta(-x)$ is the sign distribution) and

\[
\psi(-i(\partial_\mathbb{M})^T f) = \bar{\psi}(i\partial_\mathbb{M} f) = 0. \tag{2.3}
\]

Here, for a vector $a \in \mathbb{R}^4$ the $2 \times 2$-matrices $a_\mathbb{M}$ and $a^\mathbb{M}$ are defined by

\[
a_\mathbb{M} = \begin{pmatrix} a^0 + a^3 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 - a^3 \end{pmatrix} \quad \text{und} \quad a^\mathbb{M} = \begin{pmatrix} a^0 - a^3 & -a^1 + ia^2 \\ -a^1 - ia^2 & a^0 + a^3 \end{pmatrix}.
\]

Furthermore, the $*$-relation is given by

\[
\psi(f)^* = \bar{\psi}(\bar{f}) \quad \bar{\psi}(f)^* = \psi(\bar{f}), \tag{2.4}
\]

where $\bar{f}$ is the componentwise complex conjugate function of $f$. The double covering of the Poincaré group, the elements of which consist of pairs $(A, a)$ with $A \in SL(2, \mathbb{C})$, $a \in \mathbb{R}^4$, acts on $\mathcal{F}$ via

\[
\alpha_{(A,a)} \psi(f) = \psi((A^T)^{-1} f_{(A,a)}) \tag{2.5}
\]

\[
\alpha_{(A,a)} \bar{\psi}(f) = \bar{\psi}((A^\dagger)^{-1} f_{(A,a)}) \tag{2.6}
\]

2
with $f_{(Λ,a)}(x) = f(Λ^{-1}(x - a))$, the Lorentz transform $Λ$ being the corresponding one to $A ∈ SL(2, C)$. The global gauge group $U(1)$ acts on $F$ by

$$α_ϕψ(f) = e^{iϕψ(f)}$$

$$dϕ = ψ(f), \quad dϕ = e^{-iϕψ(f)}. \quad (2.7)$$

In the case of $F$, the KMS-states and their properties are known [4]. Since the theory is massless and free, the global equilibrium situations need to be labelled by inverse temperature $|β| > 0$, but not by chemical potential. Furthermore, every KMS state determines the rest system, with respect to which it is in equilibrium, due to the fact that Lorentz symmetry is spontaneously broken in KMS states. A rest system is uniquely defined by a future directed, timelike unit vector $e$. We combine these two parameters to a vector in the forward lightcone, which we denote by $β = |β|e$.

We will consider gauge-invariant KMS states only, and for each temperature-vector $β ∈ V^+$ there is a unique gauge-invariant KMS-state $ω_β [3]$. All these states are quasifree and thus completely determined by their two-point-function, which is given by

$$ω_β\left(\bar{ψ}(f)ψ(g)\right) = 2π \int_{ℝ^4} dp δ(p^2) ε(p_0) \frac{g^T(p)p_M f(-p)}{1 + e^{-β,p}} \quad (2.8)$$

$$ω_β\left(ψ(f)ψ(g)\right) = ω_β\left(\bar{ψ}(f)\bar{ψ}(g)\right) = 0. \quad (2.9)$$

A special case of this is the so-called vacuum state $ω_∞$. It is given by (2.8) and (2.9), where $β$ tends to timelike infinity, i.e. one has:

$$ω_∞\left(\bar{ψ}(f)ψ(g)\right) = 2π \int_{ℝ^4} dp δ(p^2) \theta(p_0) g^T(p)p_M f(-p). \quad (2.10)$$

### 3 Local equilibrium states

Let $B$ be a compact subset of $V^+$ and $dρ$ be a normalized measure on $B$. Due to (2.8) the function $β \mapsto ω_β(A)$ is continuous and thus one can form the statistical mixtures of KMS-states:

$$ω_B = \int_B dρ(β)ω_β. \quad (3.1)$$

The mixtures for all $B$ and all $dρ$ form the set $C$ of so-called reference states.

With the reference states at hand, we are able to compare a given state with them at a point in order to analyze the thermal properties of that state at that point. We do this by testing the states on a set of observables that correspond to measurements of thermal properties at a single point. It is obvious that the observables in $F$ cannot be used for this, since they consist of the field smeared over a finite region in spacetime. To proceed, we need to go over to idealized observables, that exist in the sense of forms.
Let $\mu = (\mu_1 \cdots \mu_m)$ be a multi-index. We define the following 'observables':

$$
\lambda^{\mu\nu}(x) \doteq \partial^\mu : \bar{\psi}_r(x)\sigma^{\nu rs}\psi_s(x) :
$$

$$
= \lim_{\zeta \to 0} \partial^\mu : \bar{\psi}_r(x + \zeta)\sigma^{\nu rs}\psi_s(x - \zeta) : \quad (3.2)
$$

where the normal ordering is performed with respect to the vacuum state $\omega_{\infty}$. The $\lambda^{\mu\nu}(x)$ are called thermal observables at $x \in \mathbb{R}^4$ and correspond to measurements at $x$. They are idealizations: One cannot expect the expression $\omega(\lambda^{\mu\nu}(x))$ to make sense for arbitrary states. This idealization is needed in order to distinguish thermal properties at different but arbitrary close points. We will only consider states in which the limit (3.2) exists.

For $x \in \mathbb{R}^4$, the linear span of the $\lambda^{\mu\nu}(x)$ (for all $\mu$) will be denoted by $S_x$. The elements in $S_x$ transform the way their tensor indices indicate:

$$
\alpha_{(S, a)}^{(S, a)} \lambda^{\mu\nu}(x) = \Lambda_{\mu_1}^{\mu_1} \cdots \Lambda_{\mu_m}^{\mu_m} \Lambda_{\nu}^{\nu} \lambda^{\mu\nu}(Ax + a).
$$

Thus, the spaces $S_x$ are transformed into each other by the action of the automorphisms via $\alpha_{y-x}S_x = S_y$. These thermal observables and the reference states are used to characterize local equilibrium states:

**Definition 3.1** Let $\omega$ be a state over $\mathcal{F}$ and $\mathcal{O} \subset \mathbb{R}^4$ be open. The state $\omega$ is called $S_\mathcal{O}$-thermal, if the following conditions hold:

(i) For every $x \in \mathcal{O}$ there is a reference state $\omega_{B_x} \in \mathcal{C}$ such that $\omega(\lambda^{\mu\nu}(x)) = \omega_{B_x}(\lambda^{\mu\nu}(x))$ for all $\lambda^{\mu\nu}(x) \in S_x$.

(ii) For every compact subset $U \subset \mathcal{O}$ there is a compact subset $B \subset V^+$ such that the regions $B_x$ for all $x \in U$ lie all in $B$.

One would think of an $S_\mathcal{O}$-thermal state as one being close to global equilibrium at every point $x \in \mathcal{O}$, because at this point it coincides with some global equilibrium state on the set of thermal observables.

For an element $\lambda^{\mu\nu}(x)$ the corresponding function

$$
V^+ \ni \beta \mapsto \omega_{\beta}(\lambda^{\mu\nu}(x)) = L^{\mu\nu}(\beta)
$$

is called thermal function. By straightforward calculation one shows that

$$
L^{\mu\nu}(\beta) = \omega_{\beta}(\lambda^{\mu\nu}(x)) = c_m \left( \partial^\nu_{\beta} \right) \left( \frac{1}{(\beta, \beta)} \right),
$$

with $m = \deg \mu$ and

$$
c_m = \begin{cases} 
  i \pi^{\frac{m+1}{2}} (2^{m+2} - 2^{m+1}) & \text{for odd } m \\
  0 & \text{for even } m
\end{cases}, \quad (3.5)
$$
where the $B_n$ are the Bernoulli numbers. Since the KMS-states $\omega_\beta$ are translation-invariant, the value of $L^{\mu\nu}(\beta)$ does not depend on $x$. It indicates what expectation value the thermal observables have in the global equilibrium states. It shows why the choice of (3.2) as thermal observables is sensible: By thermodynamic considerations [2, 8] one knows what value intensive thermal properties such as energy density, entropy current density and phase space density should have in the global equilibrium states. The thermal energy-momentum tensor in a system of massless, free fermions being in a state of constant temperature $\beta \in V^+$, for example, has the form:

$$E^{\mu\nu}(\beta) = \frac{\pi^2}{60} \left( \frac{4\beta^\mu \beta^\nu}{(\beta,\beta)^3} - \frac{\eta^{\mu\nu}}{(\beta,\beta)^2} \right)$$  \hspace{1cm} (3.6)

at every point $x \in \mathbb{R}^4$. In fact, the thermal observable

$$\theta^{\mu\nu}(x) := \frac{1}{2i}(\lambda^{\mu\nu}(x) + \lambda^{\nu\mu}(x))$$  \hspace{1cm} (3.7)

is not only the normal ordered, symmetrized energy-momentum tensor of the free, massless Dirac field, but we also have $\omega_\beta(\theta^{\mu\nu}(x)) = E^{\mu\nu}(\beta)$, as one can see by (3.4). So in the $S_x$ there is an observable for the thermal energy density at $x \in \mathbb{R}^4$. In fact, the $S_x$ contain enough elements to approximate all important thermal properties of a system, as will be shown in the following. This situation is similar to the bosonic case.

4 Admissible macroobservables and transport equations

Since $\beta \mapsto (\beta,\beta)^{-1}$ solves the wave equation on $V^+$, we see by (3.4) that all thermal functions do so too: $\Box_\beta L^{\mu\nu}(\beta) = 0$. In fact, if one introduces a family of seminorms on the space of continuous functions on $V^+$ via

$$||\Xi||_B \doteq \sup_{\beta \in B} |\Xi(\beta)|$$  \hspace{1cm} (4.1)

where $B \subset V^+$ is compact and indexes this family, then the set of smooth solutions of the wave equation on $V^+$ becomes a pre-Frechet space, call it $\mathcal{G}$. One can show [1, 3] that with respect to the seminorms (4.1) the space of all thermal functions $\beta \mapsto L^{\mu\nu}(\beta)$ is dense in $\mathcal{G}$. Thus in the spaces of thermal observables there are elements that approximate other idealized observables whose thermal functions are smooth solutions of the wave equation on $V^+$. From [8] and [2, 3] it is known that for massless, free fermions in an equilibrium state with inverse temperature $\beta \in V^+$ the entropy-current density $S^\mu$ is given by $S^\mu(\beta) = \frac{\pi^2}{60} \frac{\beta^\mu}{(\beta,\beta)}$. Furthermore, the phase-space density of such a system is $N_p(\beta) = (2\pi)^{-3}(1 + e^{(\beta,p)})^{-1}$ (i.e. both are constant in the space-time variable). Both are solutions of the wave-equation on $V^+ (in \beta)$. So, given any compact set on $B \subset V^+$, one can find elements in $S_x$ that approximate the observables entropy-current density and phase-space density on all $\omega_{B'}$ with $B' \subset B$ arbitrarily well. Condition (ii) in definition 3.1 guarantees that all $S_\Omega$-thermal states are continuous with respect to the seminorms (4.1), so
one can assign an expectation value of $N_p$ or $S^\mu$ to such a state $\omega$ at every point $x \in \mathcal{O}$ by the following rule: For $x \in \mathcal{O}$ let $\phi_n(x)$ be a sequence of elements in $S_x$ whose thermal functions $\Phi_n(\beta)$ tend to $N_p(\beta)$. Then define the phase-space density of the system in the state $\omega$ at $x \in \mathcal{O}$, $p \in \partial \mathcal{V}^+$ to be

$$\omega(N_p)(x) = \lim_{n \to \infty} \omega(\phi_n(x)). \quad (4.2)$$

By similar constructions, one can define the expectation values of $S^\mu$ or other desired properties, such as free energy or Gibbs-Potential, in $\omega$ at every $x \in \mathcal{O}$. One can show [1] that by this procedure every element $\Xi$ in $\mathcal{G}$ determines an observable (again called $\Xi$) commuting with all elements in $\mathcal{F}$. Let $\omega$ be an $\mathcal{S}_\mathcal{O}$-thermal state, then one can assign an expectation value of $\Xi$ in $\omega$ to every $x \in \mathcal{O}$ by

$$\omega(\Xi)(x) = \int_{B_x} d\rho_x(\beta) \Xi(\beta). \quad (4.3)$$

This generalizes (4.2). The $\Xi$ are called admissible macro-observables. So $S^\mu$, $N_p$ and $E^{\mu\nu}$ thus are such macroobservables interpreted as intensive thermal properties of the system, whose mean values in global equilibrium states are determined by their thermal function $\beta \mapsto \Xi(\beta)$. By (4.3) one can assign such a value to every point in $\mathcal{O}$ to a system being in an $\mathcal{S}_\mathcal{O}$-thermal state $\omega$.

Condition (ii) in definition 3.1 assures the thus constructed functions $x \mapsto \omega(\Xi)(x)$ to be differentiable in $x$ in the sense of distributions. Moreover, one would think of this function as the point-dependent mean value of the admissible macroobservable $\Xi$ in the $\mathcal{S}_\mathcal{O}$-thermal state $\omega$. So thermal properties such as energy density or entropy current density can vary from point to point. For instance, one would interpret the function

$$\mathcal{O} \times \partial \mathcal{V}^+ \ni (x,p) \mapsto \omega(N_p)(x) \quad (4.4)$$

to be the mean phase-space density of the system in the state $\omega$. As one can show [1, 3], the Weyl equations (2.3) determine an evolution equation for (4.4). Let $p \in \partial \mathcal{V}^+$ be a positive, lightlike vector, then one finds, for example, that

$$p_\mu \partial^\mu \omega(N_p)(x) = 0 \quad (4.5)$$

(where the derivative is to be taken with respect to $x$). This is the collisionless, free Boltzmann equation. So a transport equation for locally thermal states can be derived from first principles and does not need to be imposed on the system. Again, this feature quite mimics the situation in the case of massless bosons.

5 The Hot-Bang state

In the massless bosonic case, there is a special state $\omega_{Bos, hb}^{\mathcal{O}}$, called the Hot Bang state, whose features have been exhibited in [1]. It describes the effects of a heat explosion at the origin of Minkowski space, i.e. $\omega_{Bos, hb}^{\mathcal{O}}$ is an $\mathcal{S}_{\mathcal{V}^+}$-thermal state that describes a system with diverging temperature on the boundary of the forward lightcone. It is (up to reflections and translations) the only $\mathcal{S}_\mathcal{O}$-thermal state that has a KMS-state
as reference state at each point in \( O \). That is, the state describes a system with locally sharp temperature.

One would hope an analogous state \( \omega_{hb} \) to exist in the massless fermionic case, too. It can be shown [3] that the condition of local sharpness of \( \beta \) determines the two-point function of such a state to be

\[
\omega_{hb}(\bar{\psi}_r(x)\psi_s(y)) = (2\pi)^{-3} \int_{\mathbb{R}^4} dp \delta(p^2) \varepsilon(p_0) p_{s\bar{r}} \frac{e^{i(p,x-y)}}{1 + e^{\lambda(x+y,p)}} \quad (5.1)
\]

\[
\omega_{hb}(\psi_r(x)\bar{\psi}_s(y)) = \omega_{hb}(\bar{\psi}_r(x)\bar{\psi}_s(y)) = 0. \quad (5.2)
\]

It is straightforward to show that (5.1) and (5.2) define a linear, quasifree, gauge-invariant functional on a dense subset of the CAR-Algebra \( \mathcal{F} \). What is less clear is whether this functional is a state, that is if \( \omega_{hb}(A^*A) \geq 0 \) for all \( A \in \mathcal{F} \). So it is not clear, whether an analogue to the bosonic Hot-Bang-state exists. There, the proof of positivity for the corresponding functional is quite short and straightforward. The proof for the above functional to be positive, on the other hand, will cover the rest of this chapter.

What we will show in this chapter is that (5.1) is positive for \( x,y \in V^+ \), that is \( \omega_{hb}(A^*A) \geq 0 \) for all \( A \in \mathcal{F}(V^+) \). Here \( \mathcal{F}(V^+) \) denotes the sub-\( C^* \)-algebra of \( \mathcal{F} \) that is generated by all \( \psi(f) \) and \( \bar{\psi}(f) \) with \( \text{supp } f \subset V^+ \). This again is similar to the bosonic case, where the state \( \omega_{hb}^{\text{Bos}} \) exhibits thermal properties on \( V^+ \) only. In fact, neither in the bosonic nor in the fermionic case can this region be enlarged, by quite general arguments [1, 3].

By an argument in [4], one does not need to test this condition for all \( A \in \mathcal{F}(V^+) \), but only on \( \psi(f) \) and \( \bar{\psi}(f) \), \( f \in \mathcal{D}(V^+, \mathbb{C}^2) \), because the functional is quasifree. So we only need to show that

\[
\omega_{hb}(\bar{\psi}(f)\psi(f)) \geq 0
\]

\[
\omega_{hb}(\psi(f)\bar{\psi}(f)) \geq 0
\]

for all \( f \in \mathcal{D}(V^+, \mathbb{C}^2) \) to establish the result.

First of all, we consider some functional analytic arguments.

**Lemma 5.1** Let \( f \in \mathcal{D}(V^+, \mathbb{C}^2) \). Let \( \mathbb{C}_+ \doteq \{ z \in \mathbb{C} \mid \text{Im } z > 0 \} \), then

\[
F(z) = \int_{\mathbb{R}^3} \frac{d^3p}{2|p|} \bar{f}(zp') p'_M \bar{f}(z^{-1}p') \quad (5.3)
\]

exists for \( z \in \overline{\mathbb{C}_+} \setminus \{0\} \) and is continuous in \( z \). Furthermore, \( z \mapsto F(z) \) is holomorphic on \( \mathbb{C}_+ \).

**Proof:** First consider the complex Fourier transform of \( f \), with \( \zeta \in \mathbb{C}^4 \), which is an entire analytic function in \( \mathbb{C}^4 \):

\[
\hat{f}(\zeta) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} dx \, e^{i(\zeta,x)} f(x).
\]
Because of \( \text{supp} \ f \subset V^+ \), the theorem of Paley-Wiener can be written down like this:

\[
| \tilde{f}(zp') | \leq C_N \frac{e^{-\delta|p|} \text{Im} z}{(1 + |z| |p|)^N} \quad \text{for all } z \in \mathbb{C}_+.
\] (5.4)

For fixed \( p' = (|p|, \bar{p}) \in \partial V^+ \) the integrand

\[
z \mapsto \frac{1}{2|p|} \tilde{f}^T(zp') p'_M \overline{f(z^{-1}p')}
\]

\[
= \frac{1}{2|p|} \int_{\mathbb{R}^8} dx \, dy \, f^T(x) p'_M \overline{f(y)} e^{i(p' \cdot \bar{z} - z^{-1}y)}.
\]

is holomorphic on \( \mathbb{C} \setminus \{0\} \), since \( f \) has compact support. By the explicit form of \( p'_M \) one sees that every one of its components is bounded by \( 2|\bar{p}| \). Using this and the estimate (5.4), one sees that the integrand is dominated by

\[
\left| \frac{1}{2|p|} \tilde{f}^T(zp') p'_M \overline{f(z^{-1}p')} \right| \leq \frac{2C_N^2}{(1 + |p|^2)^N}
\] (5.5)

for \( z \in \mathbb{C}_+ \setminus \{0\} \). Thus, if \( z \) varies in some compact subset of \( \mathbb{C}_+ \), the integrand is uniformly bounded by an integrable function of \( \bar{p} \). Hence the integral exists and is holomorphic in \( \mathbb{C}_+ \). Furthermore, if \( \{z_n\}_{n \in \mathbb{N}} \) is a sequence in \( \mathbb{C}_+ \) converging to \( r \in \mathbb{R} \setminus \{0\} \), we may interchange integration and limit and get

\[
\lim_{z \to r} \int_{\mathbb{R}^3} \frac{d^3p}{2|p|} \tilde{f}^T(zp') p'_M \overline{f(z^{-1}p')} = \int_{\mathbb{R}^3} \frac{d^3p}{2|p|} \tilde{f}^T(rp') p'_M \overline{f(r^{-1}p')},
\] (5.6)

which was the actual claim.

**Theorem 5.1** Let \( f \in \mathcal{D}(V^+, \mathbb{C}^2) \). Then the function

\[
[0, \pi] \ni \phi \mapsto L(\phi) = \int_{\mathbb{R}^3} \frac{d^3p}{2|p|} \tilde{f}^T(e^{i\phi}p') p'_M \overline{f(e^{i\phi}p')} \in \mathbb{R}
\] (5.7)

is either identically zero or logarithmically convex and positive. Furthermore it is continuous on \([0, \pi]\) and smooth on \((0, \pi)\).

**Proof:** The claim about the continuity and smoothness is evident from Lemma (5.1) and \( L(\phi) = F(e^{i\phi}) \). By the Cauchy-Schwartz-inequality and scaling we get for \( z = re^{i\phi} \in \mathbb{C}_+ \setminus \{0\} \):

\[
|F(z)|^2 \leq \int_{\mathbb{R}^3} \frac{d^3p}{2|p|} \tilde{f}^T(re^{i\phi}p') p'_M \overline{f(re^{i\phi}p')} \int_{\mathbb{R}^3} \frac{d^3p}{2|p|} \tilde{f}^T(r^{-1}e^{i\phi}p') p'_M \overline{f(r^{-1}e^{i\phi}p')}
\]

\[
= \left( \int_{\mathbb{R}^3} \frac{d^3p}{2|p|} \tilde{f}^T(e^{i\phi}p') p'_M \overline{f(e^{i\phi}p')} \right)^2
\]

\[
= L(\phi)^2.
\]
Thus, if \( L \) is zero for some \( \phi \in [0, \pi] \), then \( F \) is zero on a ray emerging from the origin through \( e^{i\phi} \). If \( \phi \) is in \((0, \pi)\), then \( F \) is a holomorphic function that is zero on a set with accumulation points and hence must be zero entirely. If \( \phi = 0 \) or \( \phi = \pi \), then \( F \) has zero boundary values on a set that is open in the boundary and hence must be zero by the Schwartz reflection principle. So, since \( L \) is nonnegative by definition, either it is zero everywhere or nowhere.

It remains to show that in the latter case \( L \) is logarithmically convex. Let \( \alpha \in (0,1) \) and \( \mathbb{C}_{+,\alpha} \doteq \{ z \in \mathbb{C} \mid \text{arg} \, z < \frac{\pi}{1+\alpha} \} \). Consider the function

\[
\mathbb{C}_{+,\alpha} \ni z \mapsto F_\alpha(z) = \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|} \tilde{f}(z^{1+\alpha}p')p_M' \overline{f(z^{\alpha-1}p')} \in \mathbb{C}.
\]  

(5.8)

The integrand is holomorphic, as \( z \mapsto z^{1+\alpha} \) is on \( \mathbb{C}_{+,\alpha} \). Furthermore, if \( z \in \overline{\mathbb{C}_{+,\alpha}} \setminus \{0\} \), then \( z^{\alpha-1} \), \( z^{\alpha+1} \in \mathbb{C}_{+,\alpha} \setminus \{0\} \). So by (5.4) we have:

\[
\left| \frac{1}{2|\vec{p}|} \tilde{f}(z^{1+\alpha}p')p_M' \overline{f(z^{\alpha-1}p')} \right| \leq \frac{2C_3^2}{(1+|z^{\alpha}|^2)^N}
\]

for all \( z \in \overline{\mathbb{C}_{+,\alpha}} \setminus \{0\} \) and \( \vec{p} \in \mathbb{R}^3 \). Therefore the integral exists for all \( z \in \overline{\mathbb{C}_{+,\alpha}} \setminus \{0\} \).

The integrand is uniformly bounded by an integrable function if \( z \) varies in some compact subset of \( \mathbb{C}_{+,\alpha} \). So \( F_\alpha \) is holomorphic on \( \mathbb{C}_{+,\alpha} \) and has continuous boundary values for \( r \in \mathbb{R}^+ \) given by

\[
\lim_{z \to r} \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|} \tilde{f}(z^{1+\alpha}p')p_M' \overline{f(z^{\alpha-1}p')} = \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|} \tilde{f}(r^{1+\alpha}p')p_M' \overline{f(r^{\alpha-1}p')} = r^{-3\alpha} \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|} \tilde{f}(rp')p_M' \overline{f(r^{-1}p')} = r^{-3\alpha} F(r).
\]

So we see that the two functions \( z \mapsto F_\alpha(z) \) and \( z \mapsto z^{3\alpha} F(z) \) are both holomorphic on \( \mathbb{C}_{+,\alpha} \) and have the same continuous boundary values on \( \mathbb{R}^+ \). So, by an application of the Schwartz reflection principle, they have to be equal:

\[
F(z) = z^{-3\alpha} F_\alpha(z)
\]

(5.9)

on \( \overline{\mathbb{C}_{+,\alpha}} \setminus \{0\} \). So, for every \( 0 < \phi < \frac{\pi}{1+\alpha} \) we have

\[
L(\phi)^2 = |e^{-3\alpha} F_\alpha(e^{i\phi})|^2
\]

\[
= \left| \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|} \tilde{f}(e^{i(1+\alpha)\phi}p')p_M' \overline{f(e^{i(1-\alpha)\phi}p')} \right|^2
\]

\[
\leq \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|} \tilde{f}(e^{i(1+\alpha)\phi}p')p_M' \overline{f(e^{i(1+\alpha)\phi}p')} \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|} \tilde{f}(e^{i(1-\alpha)\phi}p')p_M' \overline{f(e^{i(1-\alpha)\phi}p')} = L(\phi(1+\alpha))L((1-\alpha)\phi).
\]

This means that for every \( \phi \in (0, \pi) \) there is a \( \delta > 0 \) such that

\[
L(\phi)^2 \leq L(\phi + \varepsilon)L(\phi - \varepsilon)
\]

9
for all \( \varepsilon < \delta \). Taking the logarithm on both sides, we get
\[
\frac{d^2}{d\phi^2} \ln L(\phi) = \lim_{\varepsilon \to 0} \frac{\ln L(\phi + \varepsilon) + \ln L(\phi - \varepsilon) - 2 \ln L(\phi)}{\varepsilon^2} \geq 0.
\]
So \( L \) is logarithmically convex, and thus the theorem is proven.

Now we relate \( L \) to the twopoint-function of \( \omega_{bb} \). Let \( z = re^{i\phi} \in \mathbb{C}_+ \setminus \{0\} \). Then by scaling we have
\[
\int_{\mathbb{R}^3} \frac{d^3 p}{2|p|} \frac{1}{r^3} \bar{f}(z p') p_M' f(z p') = \frac{1}{r^3} L(\phi).
\]

Now consider the sequence \( z_n = 1 + in\lambda \) for \( \lambda > 0 \). Then \( z_n = r_n e^{i\phi_n} \) with \( r_n = (\cos \phi_n)^{-1} \). With \( L \) as in (5.7) we see that the two series
\[
\sum_{n=0}^{\infty} (-1)^n \cos^3(\phi_n) L(\phi_n), \quad \sum_{n=1}^{\infty} (-1)^{n-1} |\cos^3(\pi - \phi_n)| L(\pi - \phi_n)
\]
are absolutely convergent. If we write \( L \) in its explicit integral form (5.7) and use (5.5), we may interchange integration and summation because of dominated convergence, and so we get:
\[
2\pi \sum_{n=0}^{\infty} (-1)^n \cos^3(\phi_n) L(\phi_n) + 2\pi \sum_{n=1}^{\infty} (-1)^{n-1} |\cos^3(\pi - \phi_n)| L(\pi - \phi_n)
\]
\[
= 2\pi \sum_{n=0}^{\infty} (-1)^n \int_{\mathbb{R}^3} \frac{d^3 p}{2|p|} \bar{f}(1 + i\lambda n) p_M' f(1 + i\lambda n)
\]
\[
+ 2\pi \sum_{n=1}^{\infty} (-1)^{n-1} \int_{\mathbb{R}^3} \frac{d^3 p}{2|p|} \bar{f}(-1 + i\lambda n) p_M' f(-1 + i\lambda n)
\]
\[
= (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{d^3 p}{2|p|} \int dx dy f^T(x)p_M f(y) e^{i(p',x-y)} \left( \sum_{n=0}^{\infty} (-e^{-\lambda(p',x+y)})^n \right)
\]
\[
+ (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{d^3 p}{2|p|} \int dx dy f^T(x)p_M f(y) e^{-i(p',x-y)} \left( e^{-\lambda(p',x+y)} \sum_{n=1}^{\infty} (-e^{-\lambda(p',x+y)})^{n-1} \right)
\]
\[
= (2\pi)^{-3} \int dp \delta(p^2) \epsilon(p_0) \int dx dy f^T(x)p_M f(y) \frac{e^{i(p,x-y)}}{1 + e^{\lambda(p,x+y)}}
\]
\[
= \omega_{bb} \langle \bar{\psi}(\bar{f}) \psi(f) \rangle.
\]
By making use of the anticommutation relations, one also gets
\[
\omega_{bb} \langle \psi(f) \bar{\psi}(\bar{f}) \rangle
\]
\[
= 2\pi \sum_{n=0}^{\infty} (-1)^n \cos^3(\phi_n) L(\phi_n) + 2\pi \sum_{n=1}^{\infty} (-1)^{n-1} |\cos^3(\pi - \phi_n)| L(\pi - \phi_n).
\]
\[
(5.11)
\]
So in order to check whether $\omega_{hh}$ is a state, we have to check whether the two series described above are nonnegative for every choice of $f \in \mathcal{D}(V^+, \mathbb{C}^2)$. This will be done in the following.

**Theorem 5.2** Let $L : [0, \pi] \to \mathbb{R}^+$ be a continuous, convex function that is smooth on $(0, \pi)$. Define $r : [0, \pi] \to \mathbb{R}$ by $r(\phi) = |\cos^3 \phi|$ and $g(\phi) = r(\phi)L(\phi)$. Let furthermore $\{\phi_n\}_{n \in \mathbb{N}}$ be a monotonically increasing sequence in $[0, \frac{\pi}{2})$ converging to $\frac{\pi}{2}$, such that

$$\sum_{n=0}^{\infty} g(\phi_n) < \infty.$$  \hfill (5.12)

Then the two series

$$A_L = \sum_{n=0}^{\infty} (-1)^n \left[ g(\phi_n) + g(\pi - \phi_{n+1}) \right]$$  \hfill (5.13)

$$B_L = \sum_{n=0}^{\infty} (-1)^n \left[ g(\pi - \phi_n) + g(\phi_{n+1}) \right]$$  \hfill (5.14)

converge absolutely and are both nonnegative.

**Proof:** Because $L$ is continuous at $\phi = \frac{\pi}{2}$, it follows from the convergence of (5.12) that $\sum_n g(\pi - \phi_n)$ converges, too. Since $L > 0$ we have $g \geq 0$, and therefore the two series (5.13) and (5.14) converge absolutely.

To establish positivity of the two series, we first show that the function $g$ is either monotonous on $[0, \frac{\pi}{2}]$ or on $[\frac{\pi}{2}, \pi]$: Assume $g$ not to be monotonous on $[0, \frac{\pi}{2}]$. Then there is a $\phi_N \in (0, \frac{\pi}{2})$ with $g'(\phi_N) = 0$. Since $L > 0$ and $r'(\phi_N) < 0$, we then have that

$$L'(\phi_N) = \frac{-r'(\phi_N)L(\phi_N)}{r(\phi_N)} > 0,$$

and thus, since $L$ is convex, $L' > 0$ on $[\frac{\pi}{2}, \pi)$. Therefore, for all $\phi \in [\frac{\pi}{2}, \pi)$, we have that

$$g'(\phi) = r(\phi)L'(\phi) + r'(\phi)L(\phi) > 0,$$

since $r$ and $r'$ are non-negative on $[\frac{\pi}{2}, \pi)$. So $g$ is monotonous on $[\frac{\pi}{2}, \pi]$.

Now assume $g$ to be not monotonous on $[\frac{\pi}{2}, \pi]$. Replace $L$ by $\overline{T}$ given by

$$\overline{T}(\phi) = L(\pi - \phi).$$  \hfill (5.15)

The function $\overline{T}$ is convex, too, and $\overline{g} = r \cdot \overline{T}$ is not monotonous on $[0, \frac{\pi}{2}]$. Thus, the above argument can be applied to $\overline{g}$ instead of $g$ and shows that $g$ is monotonous on $[0, \frac{\pi}{2}]$.

Since $g(0) > 0 < g(\pi)$ and $g(\frac{\pi}{2}) = 0$, we know that either $g$ is monotonically decreasing on $[0, \frac{\pi}{2}]$ or monotonically increasing on $[\frac{\pi}{2}, \pi]$ (or both). Without loss of
generality, we can assume the latter to be the case. Otherwise we could replace \( L \) by \( \mathcal{L} \) as in (5.15), since by (5.13) and (5.14) we see that \( A_L = B_L \) and \( B_L = A_L \). So by this replacement both series are just interchanged.

Thus, from now on, \( g \) will be monotonically increasing on \([\frac{\pi}{2}, \pi]\). There are two possibilities: \( L \) may or may not be monotonous on \([0, \frac{\pi}{2}]\).

- \( L \) is monotonous on \([0, \frac{\pi}{2}]\):

Let \( L \) be monotonically decreasing on \([0, \frac{\pi}{2}]\), then \( g \) is, too. This means that \( g \) is monotonous on \([0, \frac{\pi}{2}]\) and \([\frac{\pi}{2}, \pi]\). By reordering of (5.13) and (5.14), we get:

\[
A_L = \sum_{n=0}^{\infty} \left[g(\phi_{2n}) - g(\phi_{2n+1})\right] + \sum_{n=1}^{\infty} \left[g(\pi - \phi_{2n-1}) - g(\pi - \phi_{2n})\right] \quad (5.16)
\]

\[
B_L = \sum_{n=0}^{\infty} \left[g(\pi - \phi_{2n}) - g(\pi - \phi_{2n+1})\right] + \sum_{n=1}^{\infty} \left[g(\phi_{2n-1}) - g(\phi_{2n})\right]. \quad (5.17)
\]

Since \( \phi_m \leq \phi_{m+1} \) for all \( m \), we see that every expression in square brackets is non-negative, and so are \( A_L \) and \( B_L \).

Let \( L \) be monotonically increasing on \([0, \frac{\pi}{2}]\). Thus, since \( L'' > 0 \), we have, for all \( \phi \in (0, \frac{\pi}{2}) \) that \( 0 < L''(\phi) < L''(\pi - \phi) \) and \( 0 < L'(\phi) < L'(\pi - \phi) \). So, for such a \( \phi \) we have

\[
|g'(\phi)| \leq |r'(\phi)| \cdot |L'(\phi)|
\]

\[
\leq r'(\pi - \phi)L'(\pi - \phi) + r(\pi - \phi)L'(\pi - \phi)
\]

\[
= g'(\pi - \phi).
\]

Thus, for \( 0 \leq a \leq b \leq \frac{\pi}{2} \) we have

\[
|g(a) - g(b)| \leq \int_a^b |g'(\phi)|d\phi \leq \int_a^b g'(\pi - \phi)d\phi = g(\pi - a) - g(\pi - b).
\]

(5.18)

We rewrite (5.13) and (5.14) as follows:

\[
A_L = g(\phi_0) + \sum_{n=1}^{\infty} \left[g(\pi - \phi_{2n-1}) - g(\pi - \phi_{2n}) + g(\phi_{2n}) - g(\phi_{2n-1})\right]
\]

\[
B_L = g(\phi_0) + \sum_{n=0}^{\infty} \left[g(\pi - \phi_{2n}) - g(\pi - \phi_{2n+1}) + g(\phi_{2n+1}) - g(\phi_{2n})\right].
\]

By (5.18) and \( \phi_n \leq \phi_{n+1} \) for all \( n \in \mathbb{N} \), the expressions in square brackets are non-negative for all \( n \in \mathbb{N} \), and since \( g \) is positive, both series are positive as well.
• $L$ is not monotous on $[0, \frac{\pi}{2}]$:

Since $L$ is convex, there is a $\phi_{Null} \in (0, \frac{\pi}{2})$ such that $L$ is monotonously decreasing on $[0, \phi_{Null}]$ and monotonically increasing on $[\phi_{Null}, \frac{\pi}{2}]$. So $|g'(\phi)| \leq g'(\pi - \phi)$ for all $\phi \in [\phi_{Null}, \frac{\pi}{2}]$, by the same argument as above. Thus, relation (5.18) is valid for all $\phi_{Null} \leq a \leq b \leq \frac{\pi}{2}$. This means that for $\phi_0$ such that $\phi_{Null} \leq \phi_0$ we are done. If $\phi_0 < \phi_{Null}$, there is $p \in \mathbb{N}$ such that $\phi_p \leq \phi_{Null} \leq \phi_{p+1}$. Now consider the sequence $\{\phi\}_{n \in \mathbb{N}}$, which is given by

$$\tilde{\phi}_n = \phi_n \quad \text{for } n \leq p$$

$$\tilde{\phi}_{p+1} = \tilde{\phi}_{p+2} = \phi_{Null}$$

$$\tilde{\phi}_{n+3} = \phi_{n+1} \quad \text{for } n \geq p.$$

One easily sees by (5.13) and (5.14) that $A_L$ and $B_L$ evaluated with the sequence $\{\phi\}_n$ have the same values as evaluated with the sequence $\{\phi\}_n$. So, without loss of generality, we may assume $\phi_{Null}$ to be a member of $\{\phi_n\}_{n \in \mathbb{N}}$. By what we just said, we may also assume that $\phi_{Null} = \phi_{2m+1} = \phi_{2m}$ for some $m \in \mathbb{N}$. Again, we reorder the series (5.13) and (5.14) and get:

$$A_L = \sum_{n=0}^{m-1} \left[ g(\phi_{2n}) - g(\phi_{2n+1}) \right] + g(\phi_{2m}) + \sum_{n=m+1}^{\infty} \left[ g(\phi_{2n}) - g(\phi_{2n-1}) \right]$$

$$+ \sum_{n=1}^{m} \left[ g(\pi - \phi_{2n-1}) - g(\pi - \phi_{2n}) \right] + \sum_{n=m+1}^{\infty} \left[ g(\pi - \phi_{2n-1}) - g(\pi - \phi_{2n}) \right].$$

Since $L$ is monotonously decreasing on $[0, \phi_{2m}]$, so is $g$. Thus, the first sum is nonnegative. $L$ is increasing on $[\phi_{2m}, \frac{\pi}{2}]$ and therefore relation (5.18) is valid for all $\phi_{2m} \leq a \leq b \leq \frac{\pi}{2}$. So the second sum could be negative, but the fourth sum dominates it, so the sum of both is nonnegative. That the third sum is nonnegative is a consequence of the fact that $g$ is monotonically increasing on $[\pi - \phi_{2m}, \pi]$. So $A_L$ is nonnegative. Similarly we have:

$$B_L = \sum_{n=1}^{m} \left[ g(\phi_{2n-1}) - g(\phi_{2n}) \right] + g(\phi_{2m+1}) + \sum_{n=m+1}^{\infty} \left[ g(\phi_{2n+1}) - g(\phi_{2n}) \right]$$

$$+ \sum_{n=0}^{m} \left[ g(\pi - \phi_{2n}) - g(\pi - \phi_{2n+1}) \right] + \sum_{n=m+1}^{\infty} \left[ g(\pi - \phi_{2n}) - g(\pi - \phi_{2n+1}) \right].$$

Again, $L$ is monotonously decreasing on $[0, \phi_{2m+1}]$ and monotonically increasing on $[\phi_{2m+1}, \frac{\pi}{2}]$. So, by analogous arguments as above, $B_L$ is nonnegative, too.

This completes the proof of the theorem.

So, by theorems 5.1 and 5.2 we have shown that the expressions in (5.10) and (5.11) are nonnegative, and this means that (5.1) defines a gauge-invariant, quasifree state
on $\mathcal{F}(V^+)$. This is a closed sub-$C^*$-algebra of $\mathcal{F}$, and hence we can extend $\omega_{hb}$ to all of $\mathcal{F}$. Since due to the anticommutation relations positivity is equivalent to boundedness by one, the theorem of Hahn-Banach guarantees that such extension can be chosen to be positive and hence to be a state, too. The explicit form of such an extension will not be in our interest in the following.

6 The physical picture

A straightforward calculation shows that

$$\omega_{hb}(\lambda^{\mu\nu}(x)) = \omega_{2\lambda x}(\lambda^{\mu\nu}(x)), \quad (6.1)$$

where $\omega_{2\lambda x}$ is the KMS-state for $\beta = 2\lambda x$. Since the square of the temperature $\beta \mapsto (\beta, \beta)^{-1} = T^2$ solves the wave equation on $V^+$, $T^2$ is an admissible macroobservable that can be approximated by elements in $\mathcal{S}_x$ like in (4.2). Its expectation value in $\omega_{hb}$ is

$$\omega_{hb}(T^2)(x) = \frac{1}{4\lambda^2(x,x)}.$$

So the temperature diverges on the apex and the boundary of the forward lightcone $V^+$. The phase-space density in the state $\omega_{hb}$ is given by

$$\omega_{hb}(N_p)(x) = \frac{(2\pi)^{-3}}{1 + e^{2\lambda(x,p)}}.$$

This shows that at a point near $x$ the boundary of $V^+$, the dominant contribution of the particle density is made by the particles with momentum $p$ proportional to $x$. If $x$ approaches the boundary, the total number of particles increases and their speed tend to the speed of light. Thus, one would interpret this state as the result of a Hot Bang, after which a vast number of particles emanates into space with speed of light. Furthermore, for each observer 'inside the shockwave' the temperature of the system decreases with time as $T = (2\lambda t)^{-1}$. On can show that for each timelike vector $a \in V^+$ we have

$$\lim_{t \to \infty} \omega_{hb}(\alpha_{ta}(A)) = \omega_\infty(A),$$

so the state $\omega_{hb}$ approaches the vacuum in timelike infinity. Thus, the name Hot Bang-state is appropriate.

7 Less thermal observables

Let us now come to a mathematical subtlety of this approach of characterizing local equilibrium states. The choice of thermal observables (3.2) is not unique. In [2] a method is shown how to build a space of thermal observables $\mathcal{T}_x$ for generic models. In our case $\mathcal{S}_x$ is a proper subspace of $\mathcal{T}_x$. In the analysis, one could have chosen a larger or smaller subspace of $\mathcal{T}_x$ as space of thermal observables, thus admitting less or more states to be locally close to equilibrium in the sense of the definition
3.1. This is a potent method to impose a hierarchy onto states, by ordering them in terms of local closeness to reference states.

One can see from (3.2) and (3.4) that the thermal observables $\lambda^{\mu\nu}(x)$ are symmetric in the first $m = \deg \mu$ indices, but the last one has to be treated separately. The corresponding thermal functions $\beta \mapsto L^{\mu\nu}(\beta)$, though, are symmetric in all $m + 1$ indices. This indicates that there are thermal observables that are not zero by themselves but vanish in all reference states, for example $\lambda^{\mu\nu}(x) - \lambda^{\nu\mu}(x)$. Hence, in the sense of the approximation (4.2) the $\lambda^{\mu\nu}(x)$ contain redundancies. Therefore one could symmetrize the $\lambda^{\mu\nu}(x)$

$$\tilde{\lambda}^{\mu\nu}(x) \equiv \sum_{\pi \in P_{m+1}} \lambda^{\pi(\mu\nu)}(x).$$

(7.1)

in order to make the series used in (4.2) less ambiguous. For these new thermal observables one does not need to distinguish between different indices and is able to write $\tilde{\lambda}^{\mu}(x)$ instead. We denote the space of the $\tilde{\lambda}^{\mu}(x)$ by $S'_x$. So $S'_x$ is a proper subspace of $S_x$ for every $x$, thus there are potentially more $S'_\Omega$-thermal than $S_\Omega$-thermal states. The thermal functions induced by the $\tilde{\lambda}^{\mu}(x)$ are the same as the ones induced by the $\lambda^{\mu\nu}(x)$, of course, hence all the $S'_\Omega$-thermal states can be continued to the admissible macro-observables, too.

One can show [3] that the Weyl equations (2.3) induce differential equations on the $\lambda^{\mu\nu}(x)$, namely the following:

$$\partial_{\tau} \lambda^{r\tau\mu\nu}(x) = 0$$

(7.2)

$$\Box \lambda^{\mu\nu}(x) + \lambda_{\tau}^{\tau\mu\nu}(x) = 0$$

(7.3)

$$\sigma^{r,\tau, s} \sigma_{\tau, st} \lambda^{\mu\nu}(x) - \sigma^{r, \tau, s} \sigma_{\tau, st} \partial_{\nu} \lambda^{\mu\tau}(x) = 0$$

(7.4)

$$\sigma^{r, \tau, \tau} \sigma_{\tau, ts} \lambda^{\mu\tau}(x) + \sigma^{r, \tau, \tau} \sigma_{\tau, tr} \partial_{\nu} \lambda^{\mu\tau}(x) = 0.$$ 

(7.5)

These differential equations have implications for the space-time behavior of the expectation values of thermal macroobservables $\Xi$ in $S_\Omega$-thermal states $\omega$:

$$\Box \omega(\Xi)(x) = 0$$

(7.6)

$$\partial_{\mu} \omega(\partial_{\mu} \Xi)(x) = 0$$

(7.7)

$$\partial_{\mu} \omega(\partial_{\mu} \Xi)(x) - \partial_{\nu} \omega(\partial_{\nu} \Xi)(x) = 0.$$ 

(7.8)

Since the $\tilde{\lambda}^{\mu}(x)$ are less than the $\lambda^{\mu\nu}(x)$, not all of the equations (7.6) - (7.8) have to hold for $S'_\Omega$-thermal states. One can say something at least. If one takes the trace over the free spinor indices in (7.4) and (7.5), one obtains $\lambda_{\nu}^{\mu\nu}(x) = 0$ and $\partial_{\nu} \lambda^{\mu\nu}(x) = 0$. Let $\eta$ be the Minkowski metric, then we get
$$\eta_{\tau \rho} \sum_{\pi \in P_{m+3}} \chi^{\pi(\rho \tau \mu \nu)}(x) = \eta_{\tau \rho} \sum_{\pi \in P_{m+3}} \chi^{\pi(\rho \tau \mu \nu)}(x)$$

$$= \eta_{\tau \rho} (m + 2)(m + 1) \sum_{\pi \in P_{m+1}} \lambda^{\tau \rho \pi(\mu \nu)}(x),$$

since the $\lambda^{\mu \nu}(x)$ are symmetric in all but the last index. So we have because of (7.7):

$$0 = (m + 2)(m + 1) \sum_{\pi \in P_{m+1}} \Box \lambda^{\pi(\mu \nu)} + (m + 2)(m + 1) \eta_{\tau \rho} \sum_{\pi \in P_{m+1}} \lambda^{\tau \rho \pi(\mu \nu)}(x)$$

$$= (m + 2)(m + 1) \sum_{\pi \in P_{m+1}} \Box \lambda^{\pi(\mu \nu)} + \eta_{\tau \rho} \sum_{\pi \in P_{m+3}} \lambda^{\pi(\tau \rho \mu \nu)}(x),$$

and hence

$$\frac{1}{(m - 1)!} \Box \tilde{\lambda}^\mu(x) + \frac{1}{(m + 1)!} \tilde{\lambda}^{\tau \mu}(x) = 0. \quad (7.9)$$

With $\partial_\nu \lambda^{\mu \nu}(x) = 0$, on the other hand, it is easy to show that

$$\partial_\nu \tilde{\lambda}^{\nu \mu}(x) = 0. \quad (7.10)$$

Equations (7.9) and (7.10) suffice to show that the equations

$$\Box \omega(\Xi)(x) = 0, \quad \partial_\mu \omega(\partial^\mu \Xi)(x) = 0 \quad (7.11)$$

hold for $S'_0$-thermal states $\omega$, too. On the other hand, no equations of the form

$$A^{\nu, \dot{r}, s} \sigma_{\tau, st} \tilde{\lambda}_\nu^{\mu \tau}(x) - B^{\nu, \dot{r}, s} \sigma_{\tau, st} \partial_\nu \tilde{\lambda}^{\mu \tau}(x) = 0$$

$$A^{\nu, \dot{r}, s} \sigma_{\tau, t \dot{r}} \tilde{\lambda}_\nu^{\mu \tau}(x) + B^{\nu, \dot{r}, s} \sigma_{\tau, t \dot{r}} \partial_\nu \tilde{\lambda}^{\mu \tau}(x) = 0$$

for constants $A, B$ can hold. Otherwise, by taking the trace over the free spinor indices on both equations one would obtain $\tilde{\lambda}_\nu^{\nu \mu}(x) = 0$, violating equation (7.9). Nevertheless, the two equations (7.11) are enough to establish the main results of the bosonic case: The phase-space density $\omega(N_p)(x)$ of $S'_0$-thermal states satisfies the massless, free Boltzmann equation, and all statements about the breaking of time-reversal symmetry carry over to the fermionic case. So, in the analysis, one could look for $S'_0$-thermal states instead of $S_0$-thermal states. By restricting oneself to less thermal observables, one could find more states of interest that still exhibit interesting properties one would expect of local equilibrium states.

**Acknowledgements**

I would like to thank Professor Detlev Buchholz for fruitful discussions, his patience and time. Furthermore I am in debt to all members of the LQP-group in the Institute for Theoretical Physics in Göttingen for their time and support. Furthermore I would like to thank Thomas Thiemann for his help.
References

[1] Buchholz, D.: On Hot Bangs and the Arrow of Time in Relativistic Quantum Field Theory. Commun. Math. Phys. 237, 271-288 (2003)

[2] Buchholz, D., Ojima, I., Roos, H.: Thermodynamic Properties of Non-Equilibrium States in Quantum Field Theory. Annals Phys. 297, 219 (2002)

[3] Bahr, B.: Lokale Gleichgewichtszustände masseloser Fermionen. Diploma thesis http://www.theorie.physik.uni-goettingen.de/forschung/gft/theses/dipl/Bahr.pdf University of Göttingen 2004

[4] Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics. Vol 2: Equilibrium states. Berlin: Springer Verlag, 1996

[5] Ynduráin, F.J.: Relativistic Quantum Mechanics and Introduction to Field Theory. Berlin: Springer Verlag, 1996

[6] Haag, R.: Local Quantum Physics. Berlin: Springer Verlag, 1992

[7] Borchers, H.-J.: Translation Group and Particle Representations in Quantum Field Theory. Berlin: Springer Verlag, 1992

[8] Dixon, W.G.: Special Relativity. Cambridge University Press, 1979