CORRIGENDUM TO: “A RANK INEQUALITY FOR THE TATE CONJECTURE OVER GLOBAL FUNCTION FIELDS”

CHRISTOPHER LYONS

In [Lyo], we claimed the following:

**Theorem 2.1.** For a smooth, projective, geometrically connected variety $X$ over a global function field $k$, we have

$$r_{\ell,k}^{(m)} = r_{\text{an},k}^{(m)}$$

and thus

$$r_{\text{alg},k}^{(m)} \leq r_{\text{an},k}^{(m)}$$

for any $0 \leq m \leq \dim X$.

The equality $r_{\ell,k}^{(m)} = r_{\text{an},k}^{(m)}$ is presently unknown in general and should be corrected to read

$$r_{\ell,k}^{(m)} \leq r_{\text{an},k}^{(m)}.$$

In any case, the inequality

$$r_{\text{alg},k}^{(m)} \leq r_{\text{an},k}^{(m)},$$

which is the main focus of the paper, still holds since $r_{\text{alg},k}^{(m)} \leq r_{\ell,k}^{(m)}$.

To see that one has $r_{\ell,k}^{(m)} \leq r_{\text{an},k}^{(m)}$, one should correct the proof of Theorem 2.1 as follows. On p.104 of [Lyo], the line beginning “By (5b)...” is followed by a string of equalities. One should change the second equality to an inequality, so that string now reads:

$$-\text{ord}_{s=1} L^*(\rho_{\ell}(m), s) = -\sum_{i} \text{ord}_{s=1} L^*(\rho_{i}, s)$$

$$\geq \dim_{\overline{Q}_{\ell}} (V_{\ell}(m) \otimes \overline{Q}_{\ell})^{\Gamma_i}$$

$$= \dim_{\overline{Q}_{\ell}} V_{\ell}(m)^{\Gamma_i}$$

$$= r_{\ell,k}^{(m)}.$$

Date: March 2011.

We thank Uwe Jannsen and Dinakar Ramakrishnan for bringing the error in Theorem 2.1 to our attention.
The relevant point in introducing the inequality is that the order of the pole of $L^S(\rho_\ell(m), s)$ at $s = 1$ (which is $r^{(m)}_{\text{an}, k}$ by definition) is measuring the dimension of the $\Gamma_k$-invariants of the semisimplification, which in general is only an upper bound for the dimension of the $\Gamma_k$-invariants of the original representation.

**Remark 1.** If one knows that the Galois representation $\rho_\ell$ is semisimple (an assertion that is sometimes called the Serre–Grothendieck Conjecture), then one can deduce the stronger conclusion that $r^{(m)}_{\ell, k} = r^{(m)}_{\text{an}, k}$. In particular, this is known for products of curves and abelian varieties (by Zarhin [Zar] when $k$ is a global function field and by Faltings [Fal] when $k$ is a number field). Thus Proposition 6.1 is correct as stated, but one should note the use of semisimplicity of $\rho_\ell$ in its proof.

**Remark 2.** The corollary for the integers $r^{(m)}_{\text{an}, L}$ described in §2 does utilize the supposed equality $r^{(m)}_{\ell, k} = r^{(m)}_{\text{an}, k}$. Without this equality, one can only conclude that $r^{(m)}_{\text{an}, L} \geq r^{(m)}_{\text{alg}, L} \geq 1$ for all finite $L/k$, so the last three assertions in that section are still conjectural in general.

**References**

[Fal] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.* 73 (1983), 349–366.

[Lyo] C. Lyons. A rank inequality for the Tate conjecture over global function fields. *Expo. Math.* 27 (2009), 93–108.

[Zar] J. G. Zarhin. Endomorphisms of Abelian varieties over fields of finite characteristic. *Izv. Akad. Nauk SSSR Ser. Mat.* 39 (1975), 272–277, 471.