Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle

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Abstract

We consider a stabilization problem for a coupled wave equations on an exterior of bounded domain \( \Omega = \mathbb{R}^d \setminus \partial \Omega \) with interior stabilization. Under a geometrical control condition (BLR condition), for any initial data in the energy space, we show a result of exponential stability in odd dimensional case and polynomial stability in the case of even dimension.

Keywords: Defect measure; stabilization; Energy; Resolvent; Low and High frequencies

Introduction

In this paper we study the stabilization of a coupled wave equations. More precisely, we consider the following initial and boundary value problem :

\[
\begin{align*}
\partial_t^2 u_1 - \Delta u_1 + \alpha(x) \partial_t u_2 + \partial_t \mu_1 &= 0 \text{ in } \Omega \times (0, +\infty), \\
\partial_t^2 u_2 - \alpha \Delta u_2 - \partial_t u_1 &= 0 \text{ in } \Omega \times (0, +\infty), \\
u_1 &= 0 \text{ on } \partial \Omega \times (0, +\infty), \\
u_2 &= 0 \text{ on } \partial \Omega \times (0, +\infty), \\
u_1(x,0) &= u_1^0(x), \partial_t u_1(x,0) = u_2^1(x) \text{ in } \Omega, \\
u_2(x,0) &= u_2^0(x), \partial_t u_2(x,0) = u_1^1(x) \text{ in } \Omega,
\end{align*}
\]

where \( \Omega = \mathbb{R}^d \setminus \partial \Omega \) and \( \partial \Omega \) an open bounded set of \( \mathbb{R}^d \) with smooth boundary \( \partial \Omega \) and \( \theta \in C^2(\Omega) \) is a positive functions and \( \alpha \) is a positive constant.

The study of systems like (1)-(6) (and more generally coupled PDEs systems) is motivated by several physical considerations. In fact, there are many applied problems that can be modeled using coupled partial differential equations, for instance in heating processes, magnetohydrodynamics, quantum mechanics, optics, fluid dynamics,...

Among the nowadays many contributions, using different methods and techniques, are given, and relevant reference therein [1,2].

One of the earliest tools in the stabilization analysis of partial differential equation is the micro-local default action of Gérard [3], Tartar [4].

Such techniques have been used firstly to study and to explicit the value of the best decay rate of damped waves equation [5], reduce the boundary and the regularity of the initial data or to show that the geometric condition for control by the board is required [6,7].

Similar works, based on the use of microlocal defect measures in the spirit of the article, have been achieved [5]. In the large time behavior of solutions of the wave equation were studied. The microlocal defect measures have been used to provide estimates of energy was shown in particular how these demonstrate the results of exact controllability, observation and stabilization [8], without any assumption on the dynamics, the logarithmic decay of the local energy with respect to any Sobolev norm larger than the initial energy is proved [6]. In the two and three-dimensional system of linear thermoelasticity in a bounded smooth domain with Dirichlet boundary conditions were studied. In two space dimensions they proved a sufficient (and almost necessary) condition for the uniform decay under an assumption on the boundary of the domain and in three space dimensions sufficient conditions for the uniform decay are given [9].

Also, These techniques are used to study the stabilization of the wave equation in a domain with exterior Dirichlet condition [1], for the equation of damped waves equation in an outside field and under an "Exterior Geometric Control" condition inspired from the so-called microlocal condition of Bardos et al. [10] then for the stabilization of electromagnetic waves on an exterior bounded obstacle in 2D and 3D is treated, and under an exterior geometric control condition the behavior of the solution for large time is studied [11-13].

Later, in three dimension space and under a microlocal geometric condition, the rate of decay of the local energy for solutions of the Lamé system on exterior domain, with localized nonlinear damping was given in ref. [14].

Recently these techniques are also used to study the stabilization of different coupled equations and different results have been established in this domain, some results are given, by Duyckaerts [15] the exponential and the polynomial stabilization of a coupled hyperbolic-parabolic system of thermoelasticity are addressed with microlocal techniques, explained by Atallah-Baraket and Kammerer [16] the energy decay of thermoelasticity system with a degenerated second order operator in the Heat equation was studied, a stabilization problem for a coupled wave equations on a compact Riemannian manifold under a geometrical control condition was examined and a logarithmic decay result of the energy is given [13]. And finally in the exact controllability problem on a compact manifold for two coupled wave equations, with a control function acting on one of them only was treated [17].

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Our aim in this work is to establish the energy decay and to give the best rate of convergence of a coupled damped wave equation on an exterior bounded obstacle. We prove this result in a geometric hypothesis and by using the arguments of the analysis microlocal.

The organization of this paper is as follows. In section 2, we give the main result and recalled some preliminary results. In section 3, we study the poles of the resolvent, in the first, by means of conventional techniques is given a location on the low frequencies and by the defect measures theory we study the high frequencies. In section 4, the main results concerning the stability of systems are established. startsection

Let \( u=(u_1,u_2) \) then the system of equations (1)-(6) is equivalent to the following system

\[
\begin{align*}
\partial_t u - D_\mu K \partial_x u &= 0 \quad \text{in } \Omega \times (0, +\infty), \\
u &= 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
u(0,t) &= u^0, \quad \text{in } \Omega,
\end{align*}
\]

where

\[
D_\mu = \begin{pmatrix} \Delta & 0 \\
0 & \alpha \Delta \end{pmatrix}, \quad K_\mu = \begin{pmatrix} 2a(0) & 1 \\
-1 & 0 \end{pmatrix}.
\]

With respect to the norm

\[
\|u\|_n = \|\partial_x u\| + \|u\|_n + \|\Delta u\|_n + \|\Delta^2 u\|_n,
\]

where \( u^0 = (u_1^0, u_2^0) \) and \( u(t) = (u_1(t), u_2(t)) \).

Due to the nonlinear semi-group theory, it is well known that the problem (7) has an unique solution, obtained by using the Lumer-Philips theorem for an unbounded operator [18].

We consider the Hilbert space \( H = (H_0(\Omega))^2 \oplus (L^2(\Omega))^2 \), which is the closure of \((c_0(\Omega))' \times (c_0(\Omega))'\) with respect to the norm

\[
\int_\Omega \left| \nabla f_1 \cdot \alpha \nabla f_2 + \nabla g_1 \cdot \alpha \nabla g_2 \right|^2 dx.
\]

We define

\[
A_\mu = \begin{pmatrix} 0 & id \\
\mu_0 & -\mu_0 \end{pmatrix}
\]

and

\[
D(A_\mu) = \{(u_1,u_2) \in (u_1,u_2) \in H \text{ and } (u_1,D_\mu u_1 - K_\mu u_2) = (0,0)\}
\]

\[
= \{(u_1,u_2) \in (u_1,u_2) \in H \text{ and } (u_1,H_0(\Omega)) \cap H^2(\Omega) \|^2 \oplus (H_0(\Omega))^2 \|^2 \}.
\]

We can write the problem (7) as the following form

\[
\partial_t u(t) = A_\mu u(t), \quad u(0) = f,
\]

where \( u(t) = (u_1(t), u_2(t)) \). The problem (7) and (10) are equivalent if and only if that \( A_\mu \) has a domain \( D(A_\mu) \).

The problem (7) has an unique solution, obtained by using the Hille-Yosida theorem for an unbounded operator.

Let \( u(x,t) = (u_1(x),u_2(x),u_3(x)) \) solution of the problem (7) and set \( \nabla u = (\nabla u_1, \sqrt{\alpha} \nabla u_3) \). We define the energy functional at the time \( t \) by

\[
E(u,t) = \frac{1}{2} \int_{\Omega} \left( \partial_x u_1^2 + |\nabla u|^2 + \alpha |\nabla u_3|^2 \right) dx.
\]

that satisfy the following estimation

\[
E(u,0) - E(u,t) = \int_{\Omega} \partial_x u_1 u_2(\partial_x u_1 u_2 + \alpha |\nabla u_3|^2) dx.
\]

Let \( R > 0 \) such that \( \overline{\Omega} \subset B_R = \{ x \in R^3 ||x|| < R \} \), we set \( \Omega = \Omega \cap B_R \).

For \( u=(u_1,u_2) \) solution of (7), we denote \( E_s(u)(t) \) the local energy of at instant \( t > 0 \) define by

\[
E_s(u)(t) = \int_{\Omega} \left( \partial_x u_1 u_2(\partial_x u_1 u_2 + \alpha |\nabla u_3|^2) dx. \right)
\]

Now, according to the research of Moulahi [19], we recall that at the boundary point \((t,\eta) \in \overline{\Omega} \), let \( (t,\eta) \neq (0,0) \) be a tangential direction to \((t,\eta) \in \overline{\Omega} \); that is \( \eta \cdot v(x) = 0 \), \( v(x) \) being the exterior normal to \( \partial \Omega \) at \( x \) with the assumption \( \alpha > 1 \) we can consider \( (\tau,\eta) \) as an element of \( T_{(0,0)}(\overline{\Omega}) \) and, to look for its inverse image is the both characteristic sets means to look for \( \lambda \in R \) such that

\[
p_1(t,x,\tau,\eta) = 0, \quad p_2(t,x,\tau,\eta) = 0.
\]

That is

\[
p_1(t,x,\tau,\eta) = 0, \quad p_2(t,x,\tau,\eta) = 0.
\]

and we write

\[
\lambda = \pm \sqrt{r^2 - \eta^2}, \quad \text{ or } \lambda = \pm \sqrt{r^2 - \eta^2}.
\]

Hence, for the existence of such real \( \lambda \), one of the two relations

\[
\eta_1 = r^2 - \eta^2 \geq 0 \quad \text{or} \quad \eta_2 = r^2 - \eta^2 \geq 0
\]

must be fulfilled. From the geometrical point of view there are some possibilities for a tangential direction \( \zeta = (r,\eta) \neq (0,0) \) with different number of inverse image with respect to the projection \( \overline{T_{(0,0)}}(\overline{\Omega}) \rightarrow T(\overline{\Omega}) \).

We introduce the characteristic transversal manifold:

\[
\text{Char} T = \text{Char} T_{0,0} \cup \text{Char} T_{0,0},
\]

where

\[
\text{Char} T_{0,0} = \{(t,x,\tau,\xi) \in \overline{\Omega} | \xi = 0, t > 0 \}
\]

and the characteristic longitudinal manifold of the wave coupled system is

\[
\text{Char} L = \text{Char} T_{0,0} \cup \text{Char} T_{0,0},
\]

where

\[
\text{Char} L_{0,0} = \{(t,y,\tau,\eta) \in \overline{\Omega} | t > 0, \eta > 0 \}
\]

and the characteristic manifold of the system is

\[
\text{char} P_{\alpha} = \text{char} P_{\alpha} \cup \text{char} P_{\alpha},
\]

and the assumption on the coupled wave \((\alpha=1)\) one obtains

\[
\text{char} P_{\alpha} = \text{Char} T_{0,0} \cup \text{Char} L_{0,0},
\]

and

\[
\text{Char} P_{\alpha} = \text{Char} T_{0,0} \cup \text{Char} L_{0,0}, \quad \text{either}
\]

\[
\text{Char} P_{\alpha} = \text{Char} L_{0,0} \cup \text{Char} L_{0,0} \quad \text{if } \alpha > 1.
\]

According, we recall the following definition [12,14]
Definition 0.1
Let $\eta \in T^1 \partial \Omega$, we say that

1. $\eta$ is an elliptic (or $\eta \in E^1$) if and only if $\eta \not\in \text{Char} P^\alpha_{\omega^\alpha}$.
2. $\eta$ is a hyperbolic for the longitudinal wave (or $\eta \in H^\alpha$) if and only if $r^\alpha > 0$
3. $\eta$ is a glancing for the longitudinal wave (or $\eta \in G^\alpha$) if and only if $r^\alpha = 0$
4. $\eta$ is a hyperbolic for the transversal wave (or $\eta \in H^\alpha$) if and only if $r^\alpha > 0$
5. $\eta$ is a glancing for the transversal wave (or $\eta \in G^\alpha$) if and only if $r^\alpha = 0$

Now, we are going to make a description of a generalized bicharacteristic path and refer to the research of Lebeau G [5] for more details. The generalized bicharacteristic flow lives in $\text{Char} P^\alpha_{\omega^\alpha}$ and for $\rho \in \text{char} P^\alpha_{\omega^\alpha}$, we denote by $G(\rho, \rho)$ the generalized bicharacteristic path starting from $\rho$. Since $\text{char} P^\alpha_{\omega^\alpha}$ is the disjoint union of $\text{char} P^\alpha_{\omega^\alpha}$ and $G_\alpha$ if $\alpha > 1$ or $\text{Char} P^\alpha_{\omega^\alpha}$ and $G_\alpha$ if $\alpha < 1$. We shall consider separately the case where $\rho$ belongs to each one of these sets. Moreover all the description below holds for $|s|$ small, in the following we assume $\alpha > 1$.

Case 1. $\rho \in \text{char} P^\alpha_{\omega^\alpha}$
Here $\rho = (x, t, \xi, \nu)$ where $x \in \Omega$, $t \in (0, T)$, $\text{dept}_\alpha (x, t, \xi, \nu) = 0$. Then for $|s|$ small, we have

$$G(\rho, \rho) = \{(x(s), t(s), \tau, \xi) \mid t \in (0, T) \cap \{T \times (R \times \Omega)\}$$

Where $(x(s), \xi)$ is the characteristic starting from the point $(x, \xi)$ of

1. $p_1 = -r^\alpha + |\xi|^2$ if $\rho \in \text{Char} \Omega_\alpha$
2. $p_2 = -r^\alpha + a |\xi|^2$ if $\rho \in \text{Char} \Omega_\alpha$

Case 2. $\rho \in \text{char} P^\alpha_{\omega^\alpha}$ (i.e. $0 \leq p_1$)
Here $\rho = (x, t, \xi, \nu, \tau)$ where $x \in \partial \Omega$, $t \in (0, T)$ and the equation $p_{\rho}(x, t, \xi + \eta, \nu, \tau) = 0$ has roots $\zeta^\alpha = \Lambda \nu$ (described in ref. [12] and we have one of the two relation

$$r_1 = r^\alpha - \eta^2, \quad r_2 = r^\alpha - a \eta^2$$

For $s > 0$ (resp. $s < 0$), let $G^\alpha(\rho, \rho) = (x^\alpha(s), t(s), \xi^\alpha, \nu, \tau)$ (resp. $G^\alpha(\rho, \rho) = (x^\alpha(s), t(s), \xi^\alpha, \nu, \tau)$) be the outgoing (resp. incoming) bicharacteristic of $P^\alpha$. The generalized bicharacteristic path is such that $G(\rho, \rho) = \rho$ and

$$G(\rho, \rho) = \{(x(s), t(s), \nu, \tau) \mid t \in (0, T) \cap \{T \times (R \times \Omega)\}$$

For four possibilities may occur

1. $x^\alpha(s) = x + 2x \xi^\alpha s^2, \quad 0 < s < e,$  
   $x^\alpha(s) = x + 2x \xi^\alpha s^2, \quad e < s < 0$,
   where $\xi^\alpha = -\frac{\sqrt{\Lambda \nu}}{\sqrt{\alpha}} v(x)$ and $\tau = \frac{\sqrt{\Lambda \nu}}{\sqrt{\alpha}} v(x)$.

In particular, if $0 < r_1 < 1$ or one has $x(s) \in \Omega$ for small $|s|$.

2. If $0 \leq p_2$ (i.e. $\eta \in G_{\alpha} \cup H_{\alpha} \in H_{\alpha}$)
   $x^\alpha(s) = x + 2x \xi^\alpha s^2, \quad 0 < s < e,$  
   $x^\alpha(s) = x + 2x \xi^\alpha s^2, \quad e < s < 0$,
   where $\xi^\alpha = -\frac{\sqrt{\Lambda \nu}}{\sqrt{\alpha}} v(x)$ and $\tau = \frac{\sqrt{\Lambda \nu}}{\sqrt{\alpha}} v(x)$.

Theorem 0.4
Assume $\alpha \neq 1$ and under the hypothesis of (OGCC) above the $B_k$ for any $\delta < \rho = 2 \min(D(0), C(\infty))$, there exists $\varepsilon > 0$ such that for all $g \in H$ supported in $B_k$ we have the following estimate of the energy

$$E_{\rho}(u(t)) \leq e^{-\delta t} E(u(0)) \quad \forall t \geq 0 \text{ if } d \text{ odd}$$

or

$$E_{\rho}(u(t)) \leq \frac{1}{t^d} E(u(0)) \quad \forall t \geq 0 \text{ if } d \text{ even}$$

for all $t > 0$ and

$$t \in [0, \pi]$$

A generalized geodesic path is a projection of a generalized bicharacteristic path on $\Omega$. 

1
Location of the outgoing resolvent poles

We consider the operator \( R^\omega_\lambda (\lambda) \) define by the following expression

\[
R^\omega_\lambda (\lambda)f = \int_{\partial \Omega} e^{-i\lambda t} A^\omega_\lambda(t)dt \quad \text{for} \quad \text{Im}\lambda < 0
\]

\( A^\omega_\lambda \) is dissipative operator, by the Hille-Yosida theorem, generate a contraction semigroup \( \{e^{i\lambda t}\}_0^\infty \).

Then, it is clear that the relation (16) define a bounded family of operators from \((L^2(\Omega))^2\) onto \(H(\Omega)\) and it is holomorphic in \( \text{Im}\lambda<0 \).

Moreover, we have the following characterization of the resolvent \( R^\omega_\lambda \):

**Lemma 0.5** For all \( f \in (L^2(\Omega))^2 \) with support in \( B_\delta \) and for all \( \lambda \neq 0 \) and \( \text{Im}\lambda \leq 0 \) we have \( R^\omega_\lambda (\lambda)f \) is the unique solution satisfies the outgoing radiation condition (ORC) of the following problem:

\[
-D_\lambda \Psi - \lambda^2 \Psi + i\lambda K\Psi = f \quad \text{in} \quad \Omega,
\]

\[
\Psi = 0 \quad \text{on} \quad \partial \Omega.
\]

Firstly, we recall that \( w(\xi, \tau) \) satisfy the outgoing radiation condition if the identity follows

\[
\lim_{\delta \to 0} \int_{\partial \Omega} \left| \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) + \int_{\partial \Omega} \left| \lambda \partial_\tau \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) = 0.
\]

Now, let \( \psi \) the difference between two solutions of (17). Then, \( \psi \) satisfies the homogeneous problem with Dirichlet boundary. By integration on \( \Omega \), for \( R \) large enough, we have

\[
\int_{\partial \Omega} \partial_\tau \partial_\tau \psi \partial_\tau \partial_\tau d\sigma(\xi) + \int_{\partial \Omega} \partial_\tau \partial_\tau \partial_\tau \psi d\sigma(\xi) = 0.
\]

In particular, given that the real part of (19) is zero, gives

\[
\lim_{\delta \to 0} \int_{\partial \Omega} \left| \partial_\tau \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) + \int_{\partial \Omega} \left| \lambda \partial_\tau \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) = 0.
\]

Using the outgoing radiation condition, we get

\[
\lim_{\delta \to 0} \int_{\partial \Omega} \left| \partial_\tau \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) - \int_{\partial \Omega} \left| \lambda \partial_\tau \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) = 0.
\]

Therefore, if \( \text{Im}\lambda<0 \) then we have \( \int_{\partial \Omega} \left| \partial_\tau \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) = 0 \) that implies \( \Psi = 0 \) in \( \Omega \). Assuming that \( \text{Im}\lambda=0 \) and \( \lambda \neq 0 \) the equation (19) and

\[
\int_{\partial \Omega} \left| \partial_\tau \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) + \int_{\partial \Omega} \left| \lambda \partial_\tau \partial_\tau w(\xi, \tau) \right|^2 d\sigma(\xi) = 0.
\]

and combining the radiation condition, we conclude that \( \Psi(x, \tau) = 0 \).

**Lemma 0.6** Which proves the lemma.

In the following, we study the outgoing resolvent \( R^\omega_\lambda (\lambda) \) on the real axis. We show first that it has no real pole and secondly it is bounded in the neighborhood of 0 in any angular sector does not meet the imaginary axis \( iR \).

**Boundedness of the Resolvent Near Zero**

\[
(-D_\lambda - \lambda^2)w + i\lambda K w = f \quad \text{in} \quad \Omega,
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega.
\]

Before beginning the study of holomorphic of the resolvent \( R^\omega_\lambda (\lambda) \), Let us note that we can see (17) as a perturbation of the following problem in a free space

\[
(-D_\lambda - \lambda^2)w + i\lambda f = g \quad \text{in} \quad R^d, \quad g \in (L^2(\Omega))\}
\]

where \( f = \{0 \quad 1 \quad 0 \} \).

The solution of the eqn. (23) is given by \( w = R^\omega_\lambda (\lambda)g \) with \( R^\omega_\lambda (\lambda) \) is the free outgoing resolvent given by

\[
R^\omega_\lambda (\lambda)g = \int_{R^d} \Gamma^{\omega}(x-y, \lambda, t)g(x)dx
\]

and \( \gamma(x, \lambda) = \frac{1}{(2\pi)^{d/2}} H_{\omega}^{(2)l}(\lambda x) \) is the Hankel function and furthermore \( \gamma(x, \lambda) \approx e^{-i\frac{x}{\lambda}} \) for \( \lambda \) large [6].

Now, let

\[
u = w - \Theta v
\]

where

\[
D_\Theta v + \xi K v = 0 \quad \text{in} \quad \Omega,
\]

\[
u = w \quad \text{on} \quad \partial \Omega
\]

and \( \Theta \in C^\infty_0 \) equal to 1 on a neighborhood of \( \partial \Omega \) with support in \( B_\delta \). The parameter \( \xi \) being chosen and subsequently fixed the following discussion. And \( w \) is completely determined by \( g \) and \( v \) is completely determined by \( w \). The problem then is to determine the function \( g \) for which the function \( u \) verifies (22).

\[
f = (-D_\lambda - \lambda^2)w + i\lambda K w
\]

and \( \Theta \in C^\infty_0 \) equal to 1 on a neighborhood of \( \partial \Omega \) with support in \( B_\delta \). The parameter \( \xi \) being chosen and subsequently fixed the following discussion. And \( w \) is completely determined by \( g \) and \( v \) is completely determined by \( w \). The problem then is to determine the function \( g \) for which the function \( u \) verifies (22).

\[
f = (-D_\lambda - \lambda^2)w + i\lambda K w
\]

where

\[
T_\delta v = -\sigma(v)\Theta v - \nu(\xi - \xi)\sigma(\Theta v - i\theta K v)
\]

**Lemma 0.6**: We have

1. \( T_\delta \) is a bounded operator on \((L^2(\Omega))^2\) for any \( \lambda \neq 0 \).

2. \( T_\delta \) is a holomorphic function at \( \lambda \) in \( C \) on the Riemannian Logarithmic surface.

Proof. Let \((H^1(\Omega))^2\) the Sobolev space functions with the following norm
\[ \| g \|_{L^2} = \left\{ \sum_{E \in T} \left| \partial^2 g_E \right|^2 + \left| \partial g_E \right|^2 \right\}^{1/2}. \]

By (24) and the oscillatory integral theory we can see that
\[ \| \Theta w \| \leq C \| g \|_{L^2} \]
\[ \tag{25} \]

where \( C \) is bounded uniformly on any compact Riemannian Logarithmic surface \([20]\). Now we set \( \phi = \varphi - \Theta w \) satisfy the following problem
\[ D_\alpha \phi + \varepsilon^2 \phi - i \varepsilon K_\alpha \phi = -D_\alpha (\Theta w) - \varepsilon^2 \Theta w \text{ in } \Omega_\alpha, \]
\[ \phi = 0 \text{ on } \partial \Omega, \]
\[ \phi = 0 \text{ on } \{ x : |x| = R \}. \]
by the ellipticity argument we deduce that
\[ \| \phi \|_{L^2} \leq C \| \Theta w \| \]
we obtain by eqn. (25)
\[ \| \phi \|_{L^2} \leq C \| w \|_{L^2} \]
\[ \tag{27} \]
Moreover \( T_{\lambda} \) contains only derivations of order less than or equal to 1 of \( \varphi \),
\[ \| T_{\lambda} g \| \leq \| \phi \| \leq \| g \|_{L^2} \]
by the Rillich identity, we deduce that \( T_{\lambda} \) is compact operator on \((\mathcal{L}(\mathcal{H}))\) and this implies that \( R_{\lambda}(\lambda) \) is meromorphic on \( C \) ( resp. Riemannian logarithmic surface ) if \( d \) odd ( resp. \( d \) is even ).

### Low Frequencies

First we prove that the resolvent \( R_{\lambda}(\lambda) \) have not poles in the real axis and it is bounded in an angular sector contain the real axis at a neighborhood of zero. For this we begin by the following result

Now, we prove that in a neighborhood of the zero, the resolvent \( R_{\lambda}(\lambda) \) is bounded in an angular sector contain the real axis at a neighborhood of zero. For this we begin by the following result

**Proposition 0.7**: Let \( \gamma = e^{i \theta} \in [ -\pi/2, \pi/2 ] \) an \( d\Lambda \), the angular sector opening \( \pi/2 \)
\[ \Lambda_{\alpha} = \{ \lambda : \gamma_{\alpha} \not\in C^* \}; \]
\[ \text{Re}(\gamma_{\alpha}) \geq |\text{Im}(\gamma_{\alpha})|. \]
Then \( R_{\gamma}(\lambda) \) uniformly bounded in \( \Lambda_{\alpha} \).

**Proof**: Let \( f \in \mathcal{L}(\mathcal{H}) \) compact support in \( \mathcal{B}_\alpha \). By the previous lemma, the function \( \Phi = R_{\gamma}(\lambda) f \) is the unique solution satisfying the (OGRC)of the problem
\[ \left[ -D_\alpha - \lambda^2 + i \lambda K_\alpha \right] \Phi = f \text{ in } \Omega \]
\[ \Phi = 0 \text{ on } \partial \Omega \]
\[ \tag{30} \]
Let \( \lambda \in \Lambda_{\alpha} \) and \( u \) solution satisfy the outgoing radiation condition of the following system
\[ \left( D_\alpha + \lambda^2 - i \lambda K_\alpha \right) u = g \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial \Omega, \]
\[ \tag{31} \]
where \( \text{supp} u \subset \{ |x| < R \} \). We choose a function \( \varphi \in C^*(R) \) equal to 0 for \(|x|<R\) and to 1 for \(|x|>2R\). We follow the proof of ref. [6], we obtain for \( \lambda \in \Lambda_{\alpha} \)
\[ R_{\lambda}(\lambda) = \frac{i}{1 - \sqrt{d}} \left[ \frac{1}{\sqrt{d}} (R_{\lambda}(\lambda) - R_{\lambda}(0)) \right] \]
\[ \tag{32} \]
This implies that
\[ \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + \alpha | u_{\lambda t} |^2 \right] \leq C \lambda \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ + \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ + c | \lambda | \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \text{Then for } |\lambda| \leq 1, \]
\[ \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + \alpha | u_{\lambda t} |^2 \right] \leq C \lambda \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ + c | \lambda | \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \text{Since } \text{supp} G \subset \Omega \cap \{ |r < R \}, \lambda \in \Lambda_{\alpha} \text{ and } \lambda \text{ small enough} \]
\[ \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + \alpha | u_{\lambda t} |^2 \right] \leq c | \lambda | \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \leq c \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \times \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \leq c \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \times \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \text{Using the Hardy- Poincaré inequality for } d=2 \text{, we obtain} \]
\[ \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + \alpha | u_{\lambda t} |^2 \right] \leq c \int_{\Omega} e^{i \lambda t \theta \theta} \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \times \left[ | u_{\lambda t} |^2 + | u_{\lambda t} |^2 \right] \]
\[ \text{For } d=2 \text{ is used [6]. So in both cases we give a uniform bound of norm of the resolvent from } \mathcal{L}(\mathcal{H}) \text{ into } H^2_{\text{loc}} \text{ for } \lambda \text{ close to zero and in the } \Lambda_{\alpha}. \text{ By choosing a finite number of real } \gamma_{\alpha}, \text{ it covers a neighborhood of upper half-plane (which is excluded } 0\Lambda_{\alpha}, \text{ which leads to the conclusion that the resolvent is bounded near zero and we have the assumption (1.1) in ref. [6].} \]
\[ \text{which implies that one have to } \lambda \text{ goes to zero and } |\arg(\lambda) + \pi/2| \leq \pi, \text{ the following behavior:} \]

**Proposition 0.8**: \( R_{\lambda} \) does not allow the accumulation point, has no zero on the real axis and admits the following behavior
\[ R_{\lambda}(\lambda) = \frac{i}{1 - \sqrt{d}} \left[ \frac{1}{\sqrt{d}} (R_{\lambda}(\lambda) - R_{\lambda}(0)) \right] \]
\[ \text{if } d \text{ is odd} \]
and
\[ R_{\lambda}(\lambda) = \frac{i}{1 - \sqrt{d}} \left[ \frac{1}{\sqrt{d}} (R_{\lambda}(\lambda) - R_{\lambda}(0)) \right] \]
\[ \text{if } d \text{ is even} \]
where \( \text{rank} (M_{\lambda}) \leq 1 \) and \( F_{\lambda} \) is analytic at \( \lambda = 0. \)
We begin by the following lemma inspired from [2] which will be useful to the proof of our proposition.

which gives a good uniform bound on the norm of the resolvent from $(L^2(\Omega))^\perp$ onto $(H^1(\Omega))^\perp$ for a close to zero and in the sector.

**Proposition 0.9:** $R_{\alpha}^{\gamma}(\lambda)$ and $R_{\alpha}^{\gamma}\quad \text{have the same behavior near zero.}$

*Proof.* Let $R_{\alpha}^{\gamma}(\lambda)f\quad \text{is the unique solution of } (\Delta - \lambda^2 + iK_j)u = f \quad \text{in } \Omega, \ u_{\partial \Omega} = 0 \ \text{and } u \text{ satisfy (OGRC). Let } f \in (L^2(\Omega))^\perp \text{ supported in } B_{\alpha}$ and $u = R_{\alpha}^{\gamma}(\lambda)f$.

Then we obtain

$$-(\Delta - \lambda^2 + iK_j)u = f + i\lambda(K_j - J)u$$

$$u_{\partial \Omega} = 0$$

And $u$ satisfy the (OGRC). It follows that

$$u = R_{\alpha}^{\gamma}(\lambda)f \quad = R_{\alpha}^{\gamma}(\lambda) \cdot (f + i\lambda(K_j - J)u)$$

$$= R_{\alpha}^{\gamma}(\lambda) \cdot i\lambda(K_j - J)R_{\alpha}^{\gamma}(\lambda)f$$

so for any $f \in L^2(\Omega)$ supported in $B_{\alpha}$ we have

$$R_{\alpha}^{\gamma}(\lambda)f = R_{\alpha}^{\gamma}(\lambda) \cdot i\lambda(K_j - J)R_{\alpha}^{\gamma}(\lambda)f$$

where $R_{\alpha}^{\gamma} = \chi R_{\alpha}^{\gamma}\chi$ is the truncated free resolvent.

**Lemma 0.10** $\lambda R_{\alpha}^{\gamma}(\lambda)$ is analytic at $\lambda = 0$ and $i\lambda(K_j - J)R_{\alpha}^{\gamma}(\lambda) \to 0$ when $\lambda \to 0$.

Taking into account the Lemma 0.10 we deduce that $R_{\alpha}^{\gamma}(\lambda)$ and $R_{\alpha}^{\gamma}(\lambda)$ have the same behavior near zero.

**Studies of High Frequencies**

This section is devoted to the proof of Theorem 0.11.

**Theorem 0.11:** There exists $\delta_0 > 0$ and $\lambda_0 > 0$ such that the outgoing resolvent $R_{\alpha,\gamma}^{\gamma}$ extends to a holomorphic function in the region

$$G_{\lambda} = \{ \lambda \in C, \text{Im } \lambda \leq \delta_0 \text{ and } | \text{Re } \lambda | > \lambda_0 \}.$$  

Moreover, as there is a constant $c > 0$ such that for $f \in (L^2(\Omega))^\perp$, supp $f \subset B_{\alpha}$ and for all $\lambda \in G_{\lambda}$

$$\|\nabla R_{\alpha,\gamma}^{\gamma}(\lambda) f \|_{L^2(\Omega)} \leq c \| f \|_{L^2(\Omega)}$$

Firstly, we denote that the operator $R_{\alpha,\gamma}^{\gamma}(\lambda)$ defined by $(L^2(\Omega))^\perp$ in $H$ is meromorphic on $C$ (resp. the Riemann surface of the logarithm) if $R$ is even (resp. odd), holomorphic on $|\text{Im } \lambda| < \lambda_0$. Moreover, $c_0, \delta_0$ and $\lambda_0$ don't depend of and we can check that $R_{\alpha,\gamma}^{\gamma}(\lambda) = \bar{R}_{\alpha,\gamma}^{\gamma}(\bar{\lambda})$. This allows us to limit our study to $|\text{Re } \lambda| > 0$. The proof of (37) is based on a redaction ad absurdum argument. We assume that for any $\epsilon$ (in particular for $\epsilon = \min \{0, \text{Re } \lambda \}$, there exists $f \in (L^2(\Omega))^\perp$ and $supp f = (f_1, f_2) \subset B_{\alpha}$ such that $R_{\alpha,\gamma}^{\gamma}(\lambda) f = 0$ and $Re \lambda \geq \lambda_0$ (we assume for example $\text{Re } \lambda \geq \lambda_0$) such that

$$\|\nabla R_{\alpha,\gamma}^{\gamma}(\lambda) f \|_{L^2(\Omega)} \leq c \| f \|_{L^2(\Omega)}$$

We note that $u = (u_1, u_2) = R_{\alpha,\gamma}^{\gamma}(\lambda) f$ is normalized by

$$\|\nabla u_1 \|_{L^2(\Omega)} ^2 + \| \lambda u_2 \|_{L^2(\Omega)} ^2 \geq n \quad n \geq \| f \|_{L^2(\Omega)} ^2$$

We obtain

$$-\Delta u_1 - \lambda^2 u_1 + iK_j u_1 = f_1 \quad \text{in } \Omega$$

$$u_1 = 0 \quad \text{on } \partial \Omega$$

$$u_1 \text{ satisfies the outgoing radiation condition}$$

$$\|\nabla u_1 \|_{L^2(\Omega)} ^2 + \| \lambda u_2 \|_{L^2(\Omega)} ^2 = 1$$

$$\| f_1 \|_{L^2(\Omega)} ^2 = 0, \quad \| u_1 \|_{L^2(\Omega)} ^2 = 0 \quad \text{and } Im \lambda, \lambda_0 \to 0.$$
\[ f_s \to 0 \text{ in } L^2_0(\Omega) \text{ in fact we have:} \]
\[
\| f_s \|_{L^2_0(\Omega)}^2 \to 0
\]
\[
\| (\mathcal{I} + \mathcal{A}) u \|_{L^2_0(\Omega)}^2 \leq (\| \mathcal{I} + \mathcal{A} \|_2^2) \| u \|_{L^2_0(\Omega)}^2 \to 0
\]
\[
\| \mathcal{R}_b \mathcal{I} \|_2 \leq \| \mathcal{R}_b \mathcal{A} \|_2 \leq \frac{\| \mathcal{R}_b \|_2}{\| \mathcal{I} \|_2} \| \mathcal{I} \|_2 \| \mathcal{A} \|_2 \to 0
\]
\[
\| \mathcal{I} \|_2 \leq | \mathcal{I} | \| \mathcal{I} \|_2 \leq | \mathcal{I} \|_2 \| \mathcal{I} \|_2 \to 0
\]
\[
\| \mathcal{I} \|_2 \leq | \mathcal{I} | \| \mathcal{I} \|_2 \leq | \mathcal{I} \|_2 \| \mathcal{I} \|_2 \to 0
\]

We can associate a microlocal defect measure \( \mu \) in \( (H^0, \langle \cdot, \cdot \rangle) \), the support of which is a subset characteristic of the variety. On the other hand, \( \mu \) is even, and on the Riemann logarithmic surface if \( \mu \) is odd dimensional (resp. even dimensional). Theorem 15 and the bound of resolvent in a neighborhood of zero we deduce the decreasing exponential (resp. polynomial) of energy in odd dimensional (resp. even dimensional). Theorem 15 gives a stabilization result by the boundary for the local energy for a coupled wave equation, on the exterior domain \( \Omega = B^3 \setminus \Omega \). Some results of decreasing exponential has proved in ref. [1]. The proof is based on a method of the resolvent (Location of poles) in which we use a lemma recovery and a theorem of propagation for microlocal defect measures.

**Proof**

We will proceed in similar way to that one in ref. [21]. Let consider the function \( \phi \in C^0 \) such that:
\[
\phi(t) = \begin{cases} 0 & t \leq 1 \\ 1 & t \geq 2 \end{cases}
\]
and \( \psi(t) = e^{\phi(t)} \), where \( G^\omega_t = -iA^\omega_t \).

Note that by a simple calculation, one can find that for \( \Im \lambda < 0 \)
\[
(G^\omega_t - I)^{-1} = -i\lambda R^\omega_t (\lambda (K^\omega_t + i\lambda)) R^\omega_t (\lambda)
\]

Hence, \( (G^\omega_t - I)^{-1} : H_{loc}^\omega \to H_{loc}^\omega \) can be extended to an meromorphic operator on \( C \) if \( d \) is even, and on the Riemann logarithmic surface if \( d \) is even. Moreover, in view of Remark 3.2, \( (G^\omega_t - I)^{-1} \) is analytic at \( \lambda = 0 \) if \( n \) is odd and it has the following form, modulo an analytic function at \( \lambda = 0 \),
\[
(G^\omega_t - I)^{-1} = M^\omega_{2n+1} \ln(\lambda) + O(\lambda^{2n+1}) \ln(\lambda) \to 0, \quad \text{if } n \text{ is even.}
\]

Furthermore, it easy to see that under the assumptions of Theorem, \( (G^\omega_t - I)^{-1} \) can be extended by an analytical function on the set \( \Lambda = \{ \lambda \in C: 0 \leq \Im \lambda \leq C, \pm \Re \lambda > 0 \} \) and it satisfies the estimate\( \| (G^\omega_t - I)^{-1} f \|_{C^\omega} \leq C_{\| f \|} \| f \| \| \Im \lambda \|_{C^\omega} \| \Re \lambda \|_{C^\omega} \)

for every compactly supported \( f \in H(\Omega) \)

Now, the Fourier transform of the function \( V(t) \) is given by the integral:
\[
\hat{V}(\lambda) = \int_{-\infty}^{\infty} e^{ix} \hat{V}(t) \lambda(t)dt
\]

is well defined for \( \Im \lambda < 0 \) as a bounded operator on \( H. \) Furthermore, the inverse Fourier Transform of \( v \) is given by:
\[
V(t) = \frac{1}{2\pi} \int_{\Re \lambda = \varepsilon} e^{i\lambda} \hat{v}(\lambda) d\lambda, \quad \forall \varepsilon > 0
\]

and satisfy
\[
(\hat{v} - \hat{A}^\omega_t) \hat{V}(t) = \phi(t)e^{i\omega t}
\]

Then it follows that for \( \Im \lambda < 0 \):
\[
\hat{V}(\lambda) = i(G^\omega_t - \lambda)^{-1} \phi(t)U(\lambda)
\]

By the finite speed of the wave propagation, we have that for every compactly supported \( f \in H, \forall t \in \mathbb{R}, \phi(t)u(t) \) is supported in some
compact independent of $t$. Therefore, $\varphi(t)U(t): H_{u/w} \rightarrow H_{u/w}$ extends to an entire function on $C$.

$$V(\lambda) = \frac{1}{2\pi} \int_{|z|=\lambda} e^{\lambda z} \varphi(t)U(z+t) \varphi(t)U(z-t) \frac{dz}{z}$$

$$= \frac{1}{2\pi} e^{\lambda t} \int_{-\infty}^{\infty} e^{6\pi t} \varphi(t)U(z+t) \varphi(t)U(z-t) dz$$

$$+ \frac{1}{2\pi} \lim_{\lambda \to \infty} \int_{|z|=\lambda} e^{\lambda z} (A_\lambda - \lambda) \varphi(t)U(z+t) \varphi(t)U(z-t) \frac{dz}{z}$$

$$- \int_{|z|=\lambda} e^{\lambda z} \varphi(t)U(z+t) \varphi(t)U(z-t) \frac{dz}{z}$$

$$= \frac{1}{2\pi} e^{\lambda t} W_0(t) + \frac{1}{2\pi} W_0(t)$$

Clearly, $W_0(t) = 0$ if $n$ is odd, while for $n$ even, we have in view of (3.13), $W_0(t) = \delta y + O(t^{-n}) = O(t^{-n})$. In other words,

$$\|W_0(t)\| < C t^{-\alpha} \|f\|$$

for every compactly supported $f \in H$.

To estimate $\|W(t)\|$, we use Plancherel identity together with (3.12). We have

$$\int_{-\infty}^{\infty} \|W(t)\|^2 dt = \int_{-\infty}^{\infty} \|G(z)\|^2 \|\varphi(t)U(z+t)\|^2 dz$$

$$\leq C \int_{-\infty}^{\infty} \|\varphi(t)U(z+t)\|^2 dz$$

$$\leq \|\varphi(t)U(t)\|^2$$

Let $\chi \in C^\infty(\mathbb{R}^n), \chi = 1$ for $|x| \leq 1$. An easy computation gives

$$i \partial_\lambda \int_{-\infty}^{\infty} \varphi(t)U(z+t) \varphi(t)U(z-t) dz$$

and hence

$$\chi \varphi(t)U(t) = U(t) \chi \varphi(t)U(t) + \int_{0}^{t} U(t-s) \varphi(t)U(s) ds.$$

This implies

$$\|\varphi(t)U(t)\| \leq \|\chi \varphi(t)U(t)\| + \int_{0}^{t} \|U(t-s)\| \varphi(t)U(s) ds$$

$$\leq C \|\varphi(t)U(t)\| + (1-t)^{1/2} \left( \int_{0}^{t} \|U(t-s)\|^2 ds \right)^{1/2}.$$

It is easy to see that (3.18) holds with $\|W(t)\| \leq \|\varphi(t)U(t)\|$, replaced by $\|\varphi(t)U(t)\|$. For $t \geq 1$,

$$\|W(t)\| \leq \sqrt{2 \|\varphi(t)U(t)\|.$$}

Thus, (3.10) follows from (3.16), (3.17) and (3.19).

The Best Rate of Decay in Odd Dimension

Let $\beta$ be the best exponential decay rate defined by:

$$\beta = \sup \{\beta > 0; \exists c > 0, \forall \epsilon \in H, sup_{\epsilon \in B_{\epsilon}} E_{\epsilon}(u(t)) \leq C e^{-\beta \epsilon} E_{\epsilon}(u(0)), \forall \epsilon \geq 0\}$$

Then we have the following result

Theorem 0.14

Let $\beta > 2 \min(D(0), C(\infty))$.

If $\beta > 2 \min(D(0), C(\infty))$ then $\beta \geq 2 \min(D(0), C(\infty))$.

It remains to prove that $\beta \leq \alpha$.

Assume that $\beta > 2 \min(D(0), C(\infty))$, then there exists $\lambda$ such that $2 \min \lambda \leq \beta$. From $R(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda \epsilon} |f(\epsilon)| d\epsilon$ we obtain:

$$\|R(\lambda)\| \leq \int_{-\infty}^{\infty} e^{-\lambda \epsilon} |f(\epsilon)| d\epsilon$$

$$\leq \int_{-\infty}^{\infty} e^{-\lambda \epsilon} |f(\epsilon)| d\epsilon$$

$$\leq \int_{-\infty}^{\infty} e^{-\lambda \epsilon} e^{-\beta \epsilon} d\epsilon$$

$$= \frac{1}{\lambda + \beta} \int_{-\infty}^{\infty} e^{-\lambda \epsilon} e^{-\beta \epsilon} d\epsilon$$

$$= \frac{1}{\lambda + \beta} \int_{-\infty}^{\infty} e^{-\lambda \epsilon} e^{-\beta \epsilon} d\epsilon$$

which contradicts the fact that $\lambda$ is a pole of $R(\lambda)$. And therefore it follows that $\beta < 2 \min(D(0), C(\infty))$.

Indeed, let $\beta$ such that $\beta > 2 \min(D(0), C(\infty))$.

Then $\beta > 2 \min(D(0), C(\infty))$ and then

$$\rho(\beta, \eta) < e^{-\beta \epsilon} E(0), \quad \forall \epsilon \in H$$

and as $C(\eta) < \infty$, (note that if $C(\infty) < \infty$ the inequality is trivially satisfied, there exists such that:

$$C(\eta) \leq \frac{\beta - \eta}{2}$$

and

$$\rho(\beta, \eta) \leq e^{-\beta \epsilon} E(0)$$

which contradicts that $\rho(\beta, \eta)$ tends to zero in $L^1(0, \rho)$. This completes the proof of Theorem.

Conclusion

A stabilization problem for a coupled wave equations on an exterior bounded domain is derived through the research.

References

1. Aloui L, Khenissi M (2002) Stabilisation for the Wave Equation in an Exterior Domain. Rev Math Iberoamericana 18: 1-16.
2. Aloui L, Khenissi M (2002) Stabilisation for the Wave Equation in an Exterior Domain. Rev Math Iberoamericana 18: 1-16.
3. Aloui L, Khenissi M (2002) Stabilisation for the Wave Equation in an Exterior Domain. Rev Math Iberoamericana 18: 1-16.
4. Tartar L (1990) H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations. Proc Roy Soc Edinburgh A 115: 193-230.
5. Lebeau G (1996) Equation des ondes amorties. Algebraic and Geometric Methods in Mathematical Physics. Kluwer Academic Publishers 73-109.

6. Burq N (1998) Decay of the Local Energy of the Wave Equation for the External Problem and Absence of Resonance in the Vicinity of The Real. Acta Math 180: 1-29.

7. Burq N (1998) Control of the Equation of Waves in Open Spaces with Corners. Bull. Soc Math La France 126: 601-637.

8. Lebeau G (1996) Equation des ondes amorties. Algebraic and Geometric Methods in Mathematical Physics. Kluwer Academic Publishers 73-109.

9. Lebeau G and Zuazua E (1999) Decay rates for the three-dimensional linear system of thermoelasticity. Arch Rational Mech Anal 148:179-231.

10. Bardos C, Lebeau G, Rauch G (2006) Sharp Sufficient Conditions for the Observation, Control and Stabilization of Waves from the boundary. SIAM J Control Optim 30: 1024-1065.

11. Khenissi M (2003) Equation of Damped Waves in an External domain. Bull Soc Math France 131: 211-228

12. Moulahi A (2006) Stabilization of electromagnetic waves in a 2D external domain. Mathematical Rendering 342: 853-858.

13. Moulahi A (2013) Partial stabilization of a coupled wave equations. Palestine J Mathematics 2: 265-286.

14. Daoulatli M, Dehman B, Khenissi M (2010) Local Energy Decay for the Elastic System with Nonlinear Damping in an Exterior Domain. SIAM J Control Optim 48: 5254-5275.

15. Duyckaerts T (2004) Stabilization of the Linear System of Magnetoelasticity.

16. Atallah-Baraket A, Kammerer CF (2012) Analysis of the Energy Decay of a Degenerated Thermoelasticity System

17. Dehman BJ, Rousseau J, Leautaud M (2014) Controllability of Two Coupled Wave Equations on a Compact Manifold. Arch Rational Mech Anal 211: 113-187.

18. Pazy A (1983) Semigroups of linear Operators and applications to partial. Springer-Verlag, New York.

19. Moulahi A (2016) Boundary partial stabilization of a coupled wave equations on an exterior bounded obstacle. J Dynamical and Control Systems 22: 325-34.

20. Vainberg BR (1988) Asymptotique methods in equations of mathematical physics. Gordon and Breach, New York.

21. Morawetz CS (1975) Decay of solutions of the exterior problem for the wave equation. Comm Pure Appl Math 28: 229-264.