Exact Solutions for Stationary and Unsteady Layered Convection of a Viscous Incompressible Fluid with the Specified Velocities at the Bottom

E Yu Prosviryakov$^{1,2,3}$ and L F Spevak$^1$

$^1$Institute of Engineering Science, Ural Branch of the Russian Academy of Sciences, 34 Komsomolskaya St., Ekaterinburg, 620049, Russia
$^2$Federal State Budgetary Educational Institution of Higher Education Kazan National Research Technical University named after A.N.Tupolev-KAI, 10, K.Marx Street, Kazan, Republic of Tatarstan 420111, Russia
E-mail: $^3$evgen_pros@mail.ru

Abstract. The layered convective flow of a viscous incompressible fluid is considered with the specified velocities at the bottom of an infinite layer. A new exact stationary and nonstationary solution of the Oberbeck-Boussinesq system is presented. The account of fluid velocity at the bottom is characterized by the presence of two stagnant points, this being indicative of the nonmonotonic kinetic energy profile with two local extrema.

1. Introduction

The layered convective flows of a viscous incompressible fluid in the standard nomenclature in the Cartesian coordinate system are described by the Oberbeck-Boussinesq equations [1–6]:

\[
\begin{align*}
\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} & = - \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right), \\
\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} & = - \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \right), \\
\frac{\partial P}{\partial z} & = \beta\frac{\partial T}{\partial z}, \\
\frac{\partial T}{\partial t} + V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} & = \chi \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right), \\
\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} & = 0.
\end{align*}
\]
The overdetermined system of hydrodynamics equations (1) is solvable with the use of the exact solution [1–4]

\[ V_x = U(z,t), \quad V_y = V(z,t), \]

\[ P = P_0(z,t) + xP_1(z,t) + yP_2(z,t), \quad T = T_0(z,t) + xT_1(z,t) + yT_2(z,t). \]  

The velocity field (2) identically satisfies the incompressibility equation (1). In this case, the unknown coefficients of the class of exact solutions (2) are calculated from the momentum conservation equation and the heat equation. The information on the use of exact solutions of form (2) can be found in [1–16].

Substituting the hydrodynamic fields (2), we obtain a loosely coupled system of equations. This system consists of heat-type evolution equations and stationary gradient relationships,

\[ \frac{\partial T_1}{\partial t} = \kappa \frac{\partial^2 T_1}{\partial z^2}, \quad \frac{\partial T_2}{\partial t} = \kappa \frac{\partial^2 T_2}{\partial z^2}, \]

\[ \frac{\partial P_1}{\partial z} = g\beta T_1, \quad \frac{\partial P_2}{\partial z} = g\beta T_2, \]

\[ \frac{\partial U}{\partial t} = -P_1 + \nu \frac{\partial^2 U}{\partial z^2}, \quad \frac{\partial V}{\partial t} = -P_2 + \nu \frac{\partial^2 V}{\partial z^2}, \]

\[ \frac{\partial T_1}{\partial t} + UT_1 + VT_2 = \kappa \frac{\partial^2 T_1}{\partial z^2}, \quad \frac{\partial P_1}{\partial z} = g\beta T_0. \]  

System (3) describes the nonlinear properties of the flow of a viscous incompressible fluid. The nonlinearity of system (3) is due to the conservation of the convective derivative in the caloric equation of system (1).

We formulate the initial-boundary value problem for finding a particular solution of system (3). Consider that, at the initial time \( t = 0 \), the velocities and the reference temperature are zero,

\[ U = V = 0, \quad T_0 = 0, \quad T_1 = 0, \quad T_2 = 0. \]  

At the lower fluid layer boundary described by the plane equation \( z = 0 \), the following boundary conditions are specified for \( t \geq 0 \):

\[ U = W \cos \alpha, \quad V = W \sin \alpha, \quad T_0 = 0, \quad T_1 = 0, \quad T_2 = B. \]  

The boundary conditions (5) determine the perturbation of the velocity field at the lower boundary. This condition can be interpreted as the non-satisfaction of the no-slip conditions, as well as the specification of the law of motion of the lower boundary in experimental hydromechanical studies.

On the free boundary (\( z = h \)), the boundary conditions of the form found in [8, 10] are valid,

\[ P_0 = 0, \quad P_1 = P_2 = 0, \quad T_0 = 0, \quad T_1 = A, \quad T_2 = 0, \quad \eta \frac{\partial U}{\partial z} = -\sigma T_1 = -\sigma A, \quad \eta \frac{\partial V}{\partial z} = -\sigma T_2 = 0. \]  

Here, \( \sigma \) is the temperature surface tension coefficient, \( \eta \) is the dynamic (shear) viscosity coefficient.

The curvature of the free boundary is negligible when large-scale processes are considered [8, 10]. Equations (1) describe large-scale flows through the use of a mathematical model in which the effect of vertical velocity is neglected for the motion of a viscous incompressible fluid.
2. Steady-state layered thermocapillary convection

To solve the initial-boundary value problems of layered thermocapillary convection, we reduce equation (3) and conditions (4)–(6) to the non-dimensional form. We select $h$, $l$, $\frac{h^2}{v}$, $\frac{gT_0h^2}{v}$, $T = Al$ and $g\beta T h$ as the measures of the lateral dimension, horizontal scale, time, velocity, temperature and reduced pressure, respectively. To make the analysis more convenient, we introduce the following non-dimensional complexes: $\Delta = \frac{B}{A}$, $\delta = \frac{h}{l}$ (a characteristic scale ratio), $Pr = \frac{v}{\nu}$, $We = \frac{\rho h^2 g\beta}{\sigma}$, $Gr = \frac{g\beta \Theta h^3}{v^2}$ and $Re = \frac{W l}{v}$ (the Prandtl, Weber, Grashof and Reynolds numbers) [8, 10]. Further dimensionless variables and dimensionless functions will be denoted by the same symbols as dimensional ones. The only exception is the dimensionless coordinate $Z = \frac{z}{h}$.

System (3) describing steady convective flows of a viscous incompressible fluid has the following nondimensional form:

$$
\begin{align*}
\frac{d^2 T_1}{dZ^2} &= 0, \quad \frac{d^2 T_2}{dZ^2} = 0, \\
\frac{dP_1}{dZ} &= T_1, \quad \frac{dP_2}{dZ} = T_2, \\
\frac{d^2 U}{dZ^2} &= \delta P_1, \quad \frac{d^2 V}{dZ^2} = \delta P_2, \\
\frac{d^2 T_0}{dZ^2} &= \frac{\delta Gr}{Pr} (UT_1 + VT_2), \quad \frac{dP_0}{dZ} = T_0.
\end{align*}
$$

(7)

The boundary conditions (5) and (6) in the dimensionless form are

$$
U = \frac{\delta Re}{Gr} \cos \alpha, \quad V = \frac{\delta Re}{Gr} \sin \alpha, \quad T_0 = 0, \quad T_1 = 0, \quad T_2 = \Delta.
$$

(8)

$$
P_0 = 0, \quad P_1 = P_2 = 0, \quad T_0 = 0, \quad T_1 = 1, \quad T_2 = 0, \quad \frac{\partial U}{\partial Z} = -\frac{1}{We}, \quad \frac{\partial V}{\partial Z} = 0.
$$

(9)

The boundary value problem (7)–(9) in the dimensionless form has the exact stationary solution

$$
T_1 = Z, \quad T_2 = \Delta(-Z + 1), \quad P_1 = \frac{1}{2}(Z^2 - 1), \quad P_2 = -\frac{\Delta}{2}(Z - 1)^2,
$$

$$
U = \delta \left( \frac{Z^4}{24} - \frac{Z^2}{4} + \frac{Z}{3} \right) - \frac{1}{We} Z + \frac{\delta Re}{Gr} \cos \alpha,
$$

$$
V = \Delta \delta \left( \frac{Z^4}{24} + \frac{Z^2}{4} - \frac{Z}{6} + \frac{Z}{2} \right) + \frac{\delta Re}{Gr} \sin \alpha,
$$

3
\[ T_0 = \text{GrPr} \left[ \frac{Z\delta}{12\text{We}} - \frac{Z^3\delta}{12\text{We}} - \frac{Z^4}{2520} - \frac{41Z\delta^2}{36} + \frac{Z^4\delta^2}{80} + \frac{Z^7\delta^2}{1008} - \frac{Z\delta^2}{126} + \frac{Z^7\delta^2}{36} - \frac{5Z^4\delta^2\Delta^2}{144} + \frac{Z^6\delta^2\Delta^2}{48} - \frac{Z^8\delta^2\Delta^2}{144} + \frac{Z^7\delta^2\Delta^2}{1008} + \frac{Z^3\text{Re}}{6\text{Gr} \cos \alpha} + \frac{Z^2\Delta\delta \text{Re}}{2\text{Gr} \sin \alpha} - \frac{Z^5\Delta\delta \text{Re}}{12\text{We} \sin \alpha} \right] \]

\[ P_0 = -\text{GrPr} \left[ -\frac{\delta}{40\text{We}} + \frac{Z^2\delta}{24\text{We}} - \frac{Z^4\delta}{60\text{We}} - \frac{Z^4\delta^2}{60\text{We}} + \frac{1}{13440} - \frac{61\delta^3}{5040} + \frac{41Z^2\delta^2}{180} - \frac{Z^6\delta^2}{480} \right. \]

\[ + \left. \frac{Z^8\delta^2}{8064} + \frac{1}{144} - \frac{Z^8\Delta^2}{252} + \frac{1}{144} \right] \frac{Z^4\delta^2\Delta^2}{1008} + \frac{Z^6\delta^2\Delta^2}{8064} + \frac{1}{12\text{Gr}} \frac{Z^8\delta^2\Delta^2}{1008} + \frac{1}{12\text{Gr}} \frac{Z^6\delta^2\Delta^2}{288} - \frac{Z^8\delta^2\Delta^2}{1008} \right] \frac{Z^4\Delta\delta \text{Re}}{6\text{Gr} \sin \alpha} + \frac{Z^2\Delta\delta \text{Re}}{24\text{Gr} \sin \alpha} + \left( \frac{1}{12\text{We}} - \frac{\delta \text{Re}}{6\text{Gr} \sin \alpha} \right) \frac{1}{15\text{We}} + \frac{\delta \text{Re}}{12\text{Gr} \sin \alpha} + \frac{5\Delta \delta \text{Re}}{24\text{Gr} \sin \alpha} \right]. \] (10)

The study of the temperature and pressure gradients is trivial; therefore, we will analyze the velocity field. The velocity \( V_x \) assumes one zero value (figure 1) when the inequality

\[ U(0)U(1) = \frac{\text{Re}}{\text{Gr} \sin \alpha} \left( \frac{1}{8} - \frac{1}{8\text{We}} + \frac{\text{Re}}{\text{Gr} \sin \alpha} \right) < 0 \] (11)

is satisfied.

For the velocity \( V_y \), the condition for the existence of a critical value has the form

\[ 0 < \frac{\delta \text{Re}}{\Delta \text{Gr}} < 1 \] (12)

Counterflows at \( \text{Re} = 0 \) (no slip at the bottom) were studied for thermal and concentration convection in [1, 2, 4, 16]. When the no-slip condition at the bottom is satisfied, the stagnant point is formed only with respect to the velocity \( V_y \). The velocity \( V_y \) is of a constant sign. If there is a slip on the bottom of the fluid layer, the velocity \( V_y \) can have one stagnant point or two stagnant points (see figures 1, 2).

3. Nonstationary layered thermocapillary convection

On a specified time interval, the parabolic equations of system (3) are sequentially solved in time steps with the corresponding boundary conditions (4) and (5) by the boundary element method, similarly to the way it was done in [14].

Figure 1 shows velocity graphs at different moments of time for the following parameter values: \( \text{Pr} = 6.7 \), \( \text{Gr} = 70.5 \), \( \text{Re} = 3.6 \), \( \delta = 0.1 \), \( \text{We} = 50 \), \( A = 1 \), \( \Delta = 0.5 \), \( \alpha = 1.3737 \). In this case, at each instant of time, one stagnant point is observed, similarly to stationary flow.
The velocity graphs are shown in figure 2 at different times for $\alpha = 1.7679$. The other parameters take their previous values. Here, two stagnant points are observed starting from a certain instant of time. The velocity profiles shown in figures 1, 2 testify to a considerable effect of dissipative mechanisms on the stress-strain state of a layered medium. The stratification of hydrodynamic fields results from the allowance made for diffusion effects in the motion equations and convective terms in the equation for heat conducting incompressible materials.

Figure 3 shows velocity hodographs for the solutions corresponding to figure 2.

**Figure 1.** Velocity components at different moments of time, one stagnant point: (a) $-V_{x}$, (b) $-V_{y}$

$Pr = 6.7$, $Gr = 70.5$, $Re = 3.6$, $\delta = 0.1$, $We = 50$, $\Lambda = 1$, $\Delta = 0.5$, $\alpha = 1.3737$.

**Figure 2.** Velocity components at different moments of time, two stagnant points: (a) $-V_{x}$, (b) $-V_{y}$

$Pr = 6.7$, $Gr = 70.5$, $Re = 3.6$, $\delta = 0.1$, $We = 50$, $\Lambda = 1$, $\Delta = 0.5$, $\alpha = 1.7679$. 

-1.2 -0.8 -0.4 0 0.4 0.8 1.2 100 * $-U$

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z

-1 -0.5 0 0.5 1 0 0.2 0.4 0.6 0.8 1 Z
4. Conclusion
The convective flow of a viscous incompressible fluid has been studied. The velocity is set at the lower boundary of the fluid layer. An exact solution of the overdetermined Oberbeck-Boussinesq equation system has been obtained, and this solution can be used with any dimensionless Reynolds and Grashof numbers. This exact solution describes incompressible fluid flows with a nonmonotonic kinetic energy profile. In this case, the velocity field can take a zero value at two points.

Acknowledgment
This work was supported by the grant “Numerical and physical modelling of aerodynamic and aeroacoustic characteristics of rotor systems of future concept aircraft” (No. 9.1577.2017/ P Ch) of the Ministry of Education and Science of the Russian Federation.

References
[1] Aristov S N and Prosviryakov E Yu 2013 Rus. J. Nonlin. Dyn. 9 651
[2] Aristov S N, Prosviryakov E Y and Spevak L F 2015 Computational Continuum Mechanics 8 445
[3] Aristov S N, Prosviryakov E Yu 2016 Theor. Found. Chem. Eng. 50 286
[4] Aristov S N, Prosviryakov E Y and Spevak L F 2016 Theor. Found. Chem. Eng. 50 132
[5] Sidorov A F 1989 J. Appl. Mech. Tech. Phy. 30 197
[6] Shvarz K G 2014 Fluid Dynamics 4 438
[7] Birich R.V. 1966 J. Appl. Mech. Tech. Phy. 3 69
[8] Gershuni G Z and Zhukhovitskii E M 1976 Convective stability of incompressible liquid (Jerusalem: Wiley)
[9] Napolitano L G 1980 Acta Astronaut. 7 461
[10] Landau L D and Lifshitz E M 1987 Fluid Mechanics (PergamonPress, Oxford)
[11] Goncharova O N and Kabov O A. 2009 Microgravity Sci. Tec 21 129
[12] Aristov S N, Knyazev D V and Polyannin A D 2009 Theor. Found. Chem. Eng. 43 642
[13] Pukhnachev V V 2011 The News of Altai State University 1-2 62
[14] Andreev V K and Bekezhanova V B 2013 J. Appl. Mech. Tech. Phy. 54 171
[15] Aristov S N and Knyazev D V. 2014 Fluid Dynamics 49 565
[16] Prosviryakov E Yu and Spevak L F AIP Conference Proceedings 1785 040048