The Application Law of Large Numbers That Predicts The Amount of Actual Loss in Insurance of Life

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Abstract. The law of large numbers is a statistical concept that calculates the average number of events or risks in a sample or population to predict something. The larger the population is calculated, the more accurate predictions. In the field of insurance, the Law of Large Numbers is used to predict the risk of loss or claims of some participants so that the premium can be calculated appropriately. For example there is an average that of every 100 insurance participants, there is one participant who filed an accident claim, then the premium of 100 participants should be able to provide Sum Assured to at least 1 accident claim. The larger the insurance participant is calculated, the more precise the prediction of the calendar and the calculation of the premium. Life insurance, as a tool for risk spread, can only work if a life insurance company is able to bear the same risk in large numbers. Here apply what is called the law of large number. The law of large numbers states that if the amount of exposure to losses increases, then the predicted loss will be closer to the actual loss. The use of the law of large numbers allows the number of losses to be predicted better.

1. Introduction
The law of large numbers describes the stability of large random variables. In the law of large numbers it is stated that if given a random sample of a distribution with its mean and its variance limited, then the average sample would be close to the population average [1]. The insurance business has a close relationship with statistical theory, especially the probability theory of managing the risks it assures [2].

The importance of mobile insurance to take over the risks that may occur due to destruction, theft or train accidents or when accidents cause direct damage to third parties. The principle brought forth from the theory of probability is named after a laws of large number [3]. The method undertaken by the insurance business in applicants of the laws of large numbers is increasing the number of risks themselves. If there are random variables so the random variable must be stable for a long time. That is, in a long time the value of observation will always be close to the value of expectation [4].

Insurance companies have principles in setting premiums. In calculating the possibility of certain disasters, such as accidents occurring with a home fire event, thereby increasing this value in return for being gained in such an event [5]. This amount is the amount that the company will give in paying the average for each person they cover. They then set interest rates at a rate that includes these "costs" in addition to their benefits. The policyholder acquires peace in mind because the insurance company can provide effective results in minimizing the risk of potential loss in a disaster [5].

Insurance companies will provide regular payments in exchange for massive benefits in the absence of substantial claims.
The Law of Large Numbers is a very powerful tool that is expected to say the definite things about the real-world results of unexpected unexpected events. This useful tool represents only one example of how the Great Numbers Law can be used to handle randomness. The Law of Large Numbers applies to specific results and their probabilities, but what about the overall range of possible outcomes and their associated possibilities? Just as the frequency of a particular event will tend toward its probability in the long run, a series of possible outcomes each tends to match its own probability. Studying the probability distribution of results and probabilities will give us more powerful tools to predict long-term average behavior [6].

2. Life insurance
Life insurance is a business. But it is only a business for those companies who are able to maintain their financial strength while paying out claims. While insurance helps you manage risk by protecting against things that would significantly impact your financial future if they occurred, the law of large numbers helps insurance companies by making predictable, with reasonable accuracy, the claims it will pay from year to year [7].

In the insurance business, the risks faced by each individual are transferred to the life insurance company, which agrees to indemnify the amount specified in the policy contract. To compensate for this loss, the insurer sets the premium to be paid by the insured individual. In setting a premium, there are a few things to note: possible losses, the value of any loss, administrative costs, required to run a business, such as collecting premiums from each member, measuring losses, paying claim, threshold errors that may arise when predicting losses, other factors such as financial, health, and social factors. Mistakes in measuring these factors can cause losses for life insurance companies, such as setting a smaller premium than they should [8]. Life insurance, as a tool to spread risk, can only work if a life insurance company is able to bear the same risk in large numbers. Here apply what is called the law of large numbers [9].

3. The Law of Large Numbers
The law of large numbers is the principle of statistics and probability theory which states that the more the number of samples used from an event, the monitoring results may be closer to the average population. Simply put in the world of insurance is: the more people who join the insurance, then the likelihood of losses will be close to the expected loss [4].

The law of large numbers if linked in insurance means that the insurer can determine the mortality rate (mortality rate) and morbidity level (the level of illness, injury and the occurrence of health failure) of the insured person. The mortality rate and morbidity are used in the calculation of the insurance company's premium rate to various types insurance products issued [10].

The purpose of this first requirement is for the insurance company to predict the potential for losses based on "the law of large number". Loss data can be complicated over time, and losses can be accurately predicted. So the cost of losses that can be distributed evenly distributed to all units of the insured in accordance with the class guaranteed by the insurance.

When given a random sample taken from an identical and free random variable. While the mean and its variance are, then the average of the sample will be close to the average of the population [11].

Let, \( Y_1, Y_2, \ldots, Y_n \) is a random variable array (for \( n \geq 1 \)) defined in the sample space \( \Omega \) the same one. Let \( Y \) Other random variables are defined in the same sample space. We will discuss what is meant by \( Y_n \) “toward (or convergent) to” \( Y \).

We say \( Y_n \) converges to \( Y \) with probability one if \( P(\lim_{n \to \infty} Y_n = Y) = 1 \). (this also called almost sure convergence, or almost everywhere, or strong convergence).

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Definition 2:
We say \( Y_n \) converges to \( Y \) in probability (\( Y_n \xrightarrow{p} Y \)) if for every \( \varepsilon > 0 \)
\[
\lim_{n \to \infty} P[|Y_n - Y| > \varepsilon] = 0
\]
(This is also called stochastic convergence, or convergence in measure, or weak convergence).

Definition 3:
We say \( Y_n \) converges to \( Y \) in mean square if
\[
\lim_{n \to \infty} E(|Y_n - Y|^2) = 0
\]
(we also say \( Y_n \to Y \) in \( L_2 \) or \( Y_n \xrightarrow{2} Y \))

Definition 4:
We say \( Y_n \) converges to \( Y \) in distribution (\( Y_n \xrightarrow{d} Z \)) if
\[
\lim_{n \to \infty} F_{Y_n}(t) = F_Y(t)
\]
at each point \( t \) where \( F_Y(.) \) is continuous. In such a case, we also say that \( F_n(t) = F_{Y_n}(t) \) converges weakly to \( F(t) = F_Y(t) \) and write \( F_n \to F \)

4. Proof of Weak Large Number Law with Chebyshev’s Inequality
The most important type of convergence in statistics is convergence in probability and distribution, these two concepts of convergence are called "weak" convergence types, and are generally easier to prove than "strong" convergence.

The weak large number law explain that mean sample converges in probability to expectation value \( \lim_{n \to \infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1 \)

It is shown to every \( \varepsilon > 0 \).
It is mentioned weak law because the converges in probability is indeed weak in convergence random variable.

The proof of weak large number law by using Chebyshev’s inequality

Known:
\[
\bar{X}_n = \frac{1}{n}(X_1 + X_2 + ... + X_n)
\]
Will be Proofed:
\[
\bar{X}_n \xrightarrow{p} \mu \quad \text{for} \quad n \to \infty
\]
\[
\text{var}(\bar{X}_n) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}
\]
\[
E(\bar{X}_n) = \mu
\]
\[
P(\left|\bar{X}_n - \mu\right| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}
\]
or:
\[
P(\left|\bar{X}_n - \mu\right| < \varepsilon) \geq 1 - \frac{\sigma^2}{n\varepsilon^2}
\]

As long as \( n \) to \( \infty \), according to definition the converges to probability can be:
\[
\bar{X}_n \xrightarrow{p} \mu \quad \text{for} \quad n \to \infty
\]
5. The Strong Law of Large Number

The strong law of large numbers explain that mean sample certainly will convergence to expectation value.

$$\bar{X}_n \xrightarrow{a.s.} \mu \ for \ n \to \infty$$

It is explained to every $\epsilon > 0$,

$$\lim_{n \to \infty} P(\bar{X}_n = \mu) = 1$$

The proof of this law is more complex than the weak law. This law makes true the interpretation that based on the intuition from expectation value of random variable as "Long-term average when sampling repeatedly". It is mentioned the Strong Law because the convergence almost sure is the strong convergence on random variable.

Definition 5:

The sequence random sample $Y_n$ is called converges in probability to $Y$, written $Y_n \xrightarrow{p} Y$ if:

(i) $\lim_{n \to \infty} P[|Y_n - Y| < \epsilon] = 1$

(ii) $\lim_{n \to \infty} P[|X_n - X| \geq \epsilon] = 0$

Theorem 1:

If $Y_n \xrightarrow{p} c$, then for any function $g(y)$ that continue on $c$,

$g(Y_n) \xrightarrow{p} g(c)$

Proof:

Because $g(y)$ is continue on $c$, then

$$\forall \epsilon > 0 \exists \delta > 0 \exists |y - c| < \delta \ where \ |g(y) - g(c)| < \epsilon$$

Because $|Y_n - c| < \delta$ $\subseteq |g(Y_n) - g(c)| < \epsilon$

Then $P(|g(Y_n) - g(c)| < \epsilon) \geq P(|Y_n - c| < \delta)$

Because $Y_n \xrightarrow{p} c$, this case means for $\delta > 0$

$P(|Y_n - c| < \delta) = 1$

$P(|g(Y_n) - g(c)| < \epsilon) \geq P(|Y_n - c| < \delta) = 1$

Because

$P(|g(Y_n) - g(c)| < \epsilon)$ impossible more than 1

Then $P(|g(Y_n) - g(c)| < \epsilon) = 1$

$\therefore \ g(Y_n) \xrightarrow{p} g(c)$

Theorem 2:

If $X_n$ and $Y_n$ is sequence of random variable where $X_n \xrightarrow{p} c$ and $Y_n \xrightarrow{p} d$, then:

1. $aX_n + bY_n \xrightarrow{p} ac + bd$
2. $X_nY_n \xrightarrow{p} cd$
3. $\frac{X_n}{c} \xrightarrow{p} 1$ $p(X_n \geq 0)$ $\forall n$, for $c \neq 0$
4. $\frac{1}{X_n} \xrightarrow{p},$ if $p(X_n \neq 0) = 1$, $\forall n$, $c \neq 0$
5. $\sqrt{X_n} \xrightarrow{p} \sqrt{c}$, if

Proof theorem 2.
3. If 
\( X_n \xrightarrow{p} c \neq 0 \) \( X_n \) is sequence of random variable

Conclusion \( \frac{X_n}{c} \xrightarrow{p} 1 \)

Proof :
Known : \( \lim_{n \to \infty} Pr(|X_n - c| \geq \varepsilon) = 0 \)
Will shown :
\( \lim_{n \to \infty} Pr\left(\left|\frac{X_n}{c} - 1\right| \geq \varepsilon\right) = 0 \)

Solution :
Take \( \varepsilon > 0 \) any
\( \lim_{n \to \infty} \left(\left|\frac{X_n}{c} - 1\right| \geq \varepsilon\right) = \lim_{n \to \infty} (|X_n - c| \geq \varepsilon|c|) \)
Example : \( \varepsilon = \varepsilon|c| \)
Applying \( \lim_{n \to \infty} Pr(|X_n - c| \geq \varepsilon') = 0 \)
\( \therefore \lim_{n \to \infty} Pr(|X_n - c| \geq \varepsilon) = \lim_{n \to \infty} Pr(|X_n - c| \geq \varepsilon'|c|) = 0 \)

4. Known : \( X_n \) is random variable
\( p(X_n \neq 0) = 1 \), \( \forall n \)

\( X_n \xrightarrow{p} c \) then \( \forall \varepsilon \geq 0 \ \exists \lim_{n \to \infty} Pr(|X_n - c| < \varepsilon) = 1 \)
Will proof :
\( - \frac{1}{X_n} \frac{1}{c}, c \neq 0 \)
\( - \lim_{n \to \infty} Pr(|X_n - c| < \varepsilon) = 1 \)

Take \( \varepsilon > 0 \) any
\( |X_n - c| < \varepsilon \)
\[ = \left(\frac{X_n - c}{X_n c}\right) \left|X_n c\right| < \varepsilon \]
\[ = \left|\frac{1}{c} - \frac{1}{X_n}\right| \left|X_n c\right| < \varepsilon \]
\[ Pr(|X_n - c| < \varepsilon) = Pr\left(\left|\frac{1}{c} - \frac{1}{X_n}\right| \left|X_n c\right| < \varepsilon\right) \]
\[ = Pr\left(\left|\frac{1}{X_n} - \frac{1}{c}\right| \left|X_n c\right| < \varepsilon\right) \]
\[ = Pr\left(\left|\frac{1}{X_n} - \frac{1}{c}\right| < \frac{\varepsilon}{|X_n c|}\right) \]

Because
\[ Pr\left(\left|\frac{1}{X_n} - \frac{1}{c}\right| < \frac{\varepsilon}{|X_n c|}\right) \leq Pr\left(\left|\frac{1}{X_n} - \frac{1}{c}\right| < \frac{\varepsilon}{|c|}\right) \leq 1 \]

Example
\( \varepsilon' = \frac{\varepsilon}{|c|} \), untuk \( c \neq 0 \)
\( \lim_{n \to \infty} Pr(|X_n - c| < \varepsilon) \leq \lim_{n \to \infty} Pr\left(\left|\frac{1}{X_n} - \frac{1}{c}\right| < \varepsilon'\right) = 1 \leq 1 \)

\[ 1 \leq \lim_{n \to \infty} Pr\left(\left|\frac{1}{X_n} - \frac{1}{c}\right| < \varepsilon'\right) \leq 1 \]
\[ \therefore \lim_{n \to \infty} Pr\left(\left|\frac{1}{X_n} - \frac{1}{c}\right| < \varepsilon'\right) = 1 \]
5. Known: $X_n$ is random variable

$$p[X_n \geq 0] = 1, \forall n$$

$X_n \xrightarrow{p} c$ means $\lim_{n \to \infty} Pr(|X_n - c| < \varepsilon) = 1$

Is asked $\sqrt{X_n} \xrightarrow{p} \sqrt{c} = \ldots ?$

The meaning will proof $\lim_{n \to \infty} Pr\left(\left|\sqrt{X_n} - \sqrt{c}\right| < \varepsilon\right) = 1$

Take $\varepsilon > 0$ any

$$Pr(|X_n - c| < \varepsilon) = Pr\left(\left|\sqrt{X_n} - \sqrt{c}\right| < \varepsilon\right)$$

$$\geq Pr\left(\left|\sqrt{X_n} - \sqrt{c}\right| < \frac{\varepsilon}{\sqrt{c}}\right)$$

For example $\varepsilon' = \frac{\varepsilon}{\sqrt{c}}$ for $c > 0$

Applying:

$$1 = \lim_{n \to \infty} Pr(|X_n - c| < \varepsilon) = \lim_{n \to \infty} Pr\left(\left|\sqrt{X_n} - \sqrt{c}\right| < \varepsilon'\right) \leq 1$$

Because $1 \leq \lim_{n \to \infty} Pr\left(\left|\sqrt{X_n} - \sqrt{c}\right| < \varepsilon\right) \leq 1$

So, $\lim_{n \to \infty} Pr\left(\left|\sqrt{X_n} - \sqrt{c}\right| < \varepsilon\right) = 1$

$\therefore \sqrt{X_n} \xrightarrow{p} \sqrt{c}$

6. Application of Law of Large Numbers in Life insurance

The gross premium collected by the insurance industry in 2014 reached the amount of Rp247.29 trillion, increased by 28.0% compared to the previous year figure of Rp193.06 trillion. Within the last five years, the average of gross premium annual growth was around 18.5%. If that gross premium is compared to Indonesian population in 2014, that is 252 million people, this will be resulted in insurance density of Rp981,297. It means, in average, each Indonesian person spent Rp981,297 for insurance premium.

In Life Insurance to know the amount of risk, widely used mathematical / statistical formulas, namely the theory called probability theory. The degree of risk depends on the size of the deviation between the expected and the actual event.

$$Degree \ of \ Risk = \frac{Area \ of \ Uncertainty}{Exposures} \times 100\%$$

The prescribed premium is based on the mortality table which is a table describing the number of possible deaths by age in which the table is based on a large group of people. In order for an insurance company to experience the exact mortality table, the submissions or risks received must be dispersed. This means that the risk should not be concentrated on a particular loss even more that is catastrophe (explosive risk). Meanwhile, the contribution of insurance industry to Gross Domestic Product (GDP), as measured by ratio of gross premium to GDP (penetration), increased by 0.22% from 2.13% in 2013 to 2.35% in 2014. Table 2 shows the ratio of gross premium relative to Indonesian GDP from 2010 to 2014 [12].
Based on Table 1 above shows that the gross premium collected by the insurance industry in 2014 reached the amount of Rp247.29 trillion, increased by 28.0% compared to the previous year figure of Rp193.06 trillion. Within the last five years, the average of gross premium annual growth was around 18.5%. If that gross premium is compared to Indonesian population in 2014, that is 252 million people, this will be resulted in insurance density of Rp981,297. It means, in average, each Indonesian person spent Rp 981,297 for insurance premium [12].

The highest increase of gross premium in 2014 was reached by agencies administering social insurance (69.4%), and followed by non life and reinsurance (27.8%), life insurance companies (-0.3%), companies administering mandatory Insurance (-36.9%). The biggest contribution to gross premium of insurance industry in 2014 was gross premium of life insurance companies (45.6%), followed by those of agencies administering social insurance (28.1%), gross premium of non life and reinsurance companies (22.1%); and premium collected [12].

7. The Application in insurance:
The probability of a 20-year-old and another 40 years old, both going to live another 20 years is 0.6. Of the 50,000 people who lived at the age of 20, 3000 of them died before the age of 25 years. So, the chances that a 25-year-old will now die before reaching the age of 60.

For example, lx = the exact number of people aged x

\[ l_{20} = 50,000 \text{ people} \]

\[ l_{25} = 50,000 - 3000 = 47,000 \text{ people} \]

\[ X_1 = \begin{cases} 
0.6, & \text{live another 20 years} \\
0, & \text{others} 
\end{cases} \]

To calculate the probability that a 25-year-old would die before the age of 60, it would take (1) 125 and (2) 125 - 160 (25 people die before the age of 60). Or the opportunity you want to look for is

\[ \frac{(l_{25} - l_{60})}{l_{25}} = \ldots? \]

Suppose too \[ nP_{x} = \text{probability of people aged } x \text{ years} \]

Is known, \[ 20P_{20,40} = 0.6, \text{ ie the probabilities of 20-year-olds will live 20 more years and 40-year-olds will live another 20 years, because both events are independent then multiplied. However, it can be said also: Opportunities of 20-year-olds will live 20 more years} \text{ to 40} \]
years) are \(P_{20}^{40}\) and the person will reach the age of 60 if he reaches the age of 40 years is \(P_{20}^{40}\).

Thus, the probability of a person aged 20 years reaching the age of 60 years is:

\[
P_{20}^{40} = P_{20}^{20} \cdot P_{40}^{20} = 0.6
\]

Obtained,

\[
l_{60} = l_{20} \cdot P_{20}^{40} = (50,000)(0.6) = 30,000 \text{ people}
\]

So the probability 25-year-old would die before reaching the age of 60 is \(\frac{47,000 - 30,000}{47,000} = \frac{17}{47}\).

Based on the above discussion, it is seen that to convergence while \(n \rightarrow \infty\) because even though the real \(n\) may be small or medium magnitude. However it is important to know whether the workings used are good enough for a very large \(n\) \((n \rightarrow \infty)\). Intuitively if a procedure is unlikely to be good although with so many observations it may also not be good with only a few observations.

8. Conclusion

Based on the above discussion can be concluded that:

(a) If random sampling of size \(n\) of an arbitrary population with a mean of \(\mu\) and finite variance, the greater the sample \(n\), the closer to the population, according to the law of large numbers definition, or in accordance with the equation:

\[
\overline{X}_n \xrightarrow{p} \mu \text{ for } n \rightarrow \infty
\]

(b) Law of large numbers is important because it "guarantees" a stable long-term outcome for the average of some random events. The greater the sample \(n\), the closer to the population, according to the law of large numbers definition, the more people who join the insurance, then the likelihood of losses will be close to the expected loss. Simply put in the world of insurance is: the more people who join the insurance, then the likelihood of losses will be close to the expected loss.

(c) To calculate the probability that a 25-year-old would die before the age of 60, it would take (1) \(l_{25}\) and (2) \(l_{25} - l_{60}\) (25 people die before the age of 60).

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