How many matchings cover the nodes of a
graph?

Gauthier Stauffer

Abstract

Given an undirected graph, are there \( k \) matchings whose union
covers all of its nodes, that is, a matching-\( k \)-cover? A first, easy polyno-
modal solution from matroid union is possible, as already observed
by Wang, Song and Yuan (Mathematical Programming, 2014). How-
ever, it was not satisfactory neither from the algorithmic viewpoint nor
for proving graphic theorems, since the corresponding matroid ignores
the edges of the graph. We prove here, simply and algorithmically: all
nodes of a graph can be covered with \( k \geq 2 \) matchings if and only if for
every stable set \( S \) we have \( |S| \leq k|N(S)| \). When \( k = 1 \), an exception
occurs: this condition is not enough to guarantee the existence of a
matching-1-cover, that is, the existence of a perfect matching, in this
case Tutte’s famous matching theorem (J. London Math. Soc., 1947)
provides the right ‘good’ characterization. The condition above then
guarantees only that a perfect 2-matching exists, as known from an-
other theorem of Tutte (Proc. Amer. Math. Soc., 1953). Some results
are then deduced as consequences with surprisingly simple proofs, us-
ing only the level of difficulty of bipartite matchings. We give some
generalizations, as well as a solution for minimization if the edge-
weights are non-negative, while the edge-cardinality maximization of
matching-2-covers turns out to be already NP-hard. We have arrived
at this problem as the line graph special case of a model arising for
manufacturing integrated circuits with the technology called ‘Directed
Self Assembly’. We will show connections to results by Michel on the
fractional matching polytope and some implications.

Joint work with: Dehia Ait-Ferhat, Zoltan Kiraly and Andras Sebo