Multiple coefficient identification in electrical impedance tomography with energy functional method

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Abstract In this paper we investigate the problem of simultaneously identifying the conductivity and the reaction in electrical impedance tomography with available measurement data on an accessible part of the boundary. We propose an energy functional method and the total variational regularization combining with the quadratic stabilizing term to tackle the identification problem. We show the stability of the proposed regularization method and the convergence of the finite element regularized solutions to the identification in the $L^s$-norm for all $s \in [0, \infty)$ and in the sense of the Bregman distance with respect to the total variation semi-norm. To illustrate the theoretical results, a numerical case study is presented which supports our analytical findings.

Key words and phrases Electrical impedance tomography, simultaneous identification, energy functional method, finite element method, Bregmann distance, conductivity coefficient, reaction coefficient.

AMS Subject Classifications 35R25; 47A52; 35R30; 65J20; 65J22.

1 Introduction

Electrical impedance tomography is a noninvasive type of electroencephalography and medical imaging, where the tomographic image of the electrical conductivity, permittivity, and impedance of a body part is desired to infer from surface electrode measurements. This problem attracted a great deal of attention from many applied scientists in the last decades. For surveys on the subject, we refer the reader to, e.g., [1, 9, 14, 15, 20] and the references given there.

Mathematically, assume that the electric potential or voltage $u$ in the body $\Omega$ is governed by the equation

$$\nabla \cdot (q \nabla u) = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \quad d \geq 2$$

with a free source. Here $q = q(x), \ x \in \Omega$ is the electrical conductivity which must be identified from some measurements of the state $u$ on the boundary $\partial \Omega$ of the body $\Omega$. In an ideal situation we know all outward pointing normal components of the current density $q \frac{\partial u}{\partial n} = j$ and the voltage $\Lambda_{\partial \Omega} j := u_{|\partial \Omega} := g$ as well, i.e. the knowledge of the Neumann-to-Dirichlet map

$$\Lambda_{\partial \Omega} : H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)$$

is described. This is the continuum model which is commonly used in mathematical researches on the question of the solution uniqueness

$$p, q \in L^\infty(\Omega) \quad \text{with} \quad \Lambda_{\partial \Omega}^q = \Lambda_{\partial \Omega}^p \quad \Rightarrow \quad p = q.$$

In dimensions three and higher the uniqueness result has been investigated by Sylvester and Uhlmann [24], Päivärinta et al. [21], and Brown and Torres [10]. Meanwhile for the two dimensional setting it can be found in Nachman [19], Brown and Uhlmann [11], and Astala and Päivärinta [3].

In practice we however do not know all current density and the voltage $(j, g)$, we measure them at some discrete electrodes on the relatively open subset $\Gamma$ of the boundary $\partial \Omega$ only. An interpolation process is then required to derive the measured current density and voltage $(j_\delta, g_\delta)$ on $\Gamma$, where $\delta > 0$ refers to the error level of the interpolation process and/or the measurements. The identification is now to reconstruct the electrical conductivity $q$ distributed inside the body $\Omega$ from boundary measurements of the current density and electric potential, i.e. from the pair $(j_\delta, g_\delta)$. This problem is known to be non-linear and severely ill-posed, due to the lack of data.

In the present paper we investigate the problem of simultaneously identifying the conductivity (or diffusion) and the reaction (or absorption), subjecting several sets of measurement data on an accessible part of the...
boundary are available. More precise, assume Ω is open, bounded and connected. We consider the elliptic system
\begin{align*}
-\nabla \cdot (q \nabla u) + au &= f \quad \text{in } \Omega, \\
q \nabla u \cdot \vec{n} + \sigma u &= j^\dagger \quad \text{on } \Gamma, \\
q \nabla u \cdot \vec{n} + \sigma u &= j_0 \quad \text{on } \partial \Omega \setminus \Gamma, \\
u &= g^\dagger \quad \text{on } \Gamma,
\end{align*}
where the source term \( f \in H^{-1}(\Omega) := H^1(\Omega)^* \), the Neumann boundary condition \( j_0 \in H^{-1/2}(\partial \Omega \setminus \Gamma) := (H^{1/2}(\partial \Omega \setminus \Gamma))^\dagger \), and the Robin coefficient \( \sigma \) are assumed to be known with \( \sigma \in L^\infty(\partial \Omega) \) and \( \sigma(x) \geq 0 \) a.e. on \( \partial \Omega \). The identification problem is to seek the pair \( (q, a) \) in the aforementioned system \((1.1) - (1.4)\) assuming the full knowledge of the Neumann-to-Dirichlet map \( \Lambda \) a.e. on \( \partial \Omega \) and \( (q,a) \) solvable in the class of piecewise constant functions.

Nevertheless, in [17] Harrach proved that the identification problem is uniquely solvable in the class of piecewise constant functions.

Our aim in this work is to reconstruct the pair \( (q, a) \in Q \times A \) from several sets of measurement data \((j^\delta, g^\delta)_{\delta=1,...,I} \subset H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)\) of the exact data \((j^1, g^1)\) satisfying the noise model
\begin{equation}
\frac{1}{I} \left( \|j^\delta - j^1\|_{H^{-1/2}(\Gamma)} + \|g^\delta - g^1\|_{H^{1/2}(\Gamma)} \right) \leq \delta
\end{equation}
with \( \delta > 0 \) standing for the error level of the observations. Here the admissible sets are assumed to be constrained of the general type
\begin{equation}
Q := \{ q \in L^\infty(\Omega) \mid q \leq q(x) \leq \overline{q} \text{ a.e. in } \Omega \}
\end{equation}
and
\begin{equation}
A := \{ a \in L^\infty(\Omega) \mid a \leq a(x) \leq \overline{a} \text{ a.e. in } \Omega \},
\end{equation}
where \( q, \overline{q}, \overline{a}, \overline{a} \) are given with \( 0 < q \leq \overline{q} \) and \( 0 < a \leq \overline{a} \). Furthermore, for simplicity of exposition, hereafter we assume that \( I = 1 \), i.e. only one Neumann-Dirichlet pair \((j^\delta, g^\delta)\) available. We also discuss the multiple measurement in Section 6.

With the pair \((j^\delta, g^\delta)\) at hand we examine the Neumann boundary value problem
\begin{equation}
-\nabla \cdot (q \nabla u) + au = f \quad \text{in } \Omega, \quad q \nabla u \cdot \vec{n} + \sigma u = j^\delta \quad \text{on } \Gamma, \quad q \nabla u \cdot \vec{n} + \sigma u = j_0 \quad \text{on } \partial \Omega \setminus \Gamma
\end{equation}
as well as the mixed boundary value problem
\begin{equation}
-\nabla \cdot (q \nabla v) + av = f \quad \text{in } \Omega, \quad v = g^\delta \quad \text{on } \Gamma, \quad q \nabla v \cdot \vec{n} + \sigma v = j_0 \quad \text{on } \partial \Omega \setminus \Gamma
\end{equation}
whose weak solutions are denoted by \( N_{j^\delta}(q, a) \) and \( M_{g^\delta}(q, a) \), respectively. We propose the non-negative misfit energy functional
\begin{equation}
J^\delta(q, a) := \int_\Omega q \left| \nabla (N_{j^\delta}(q, a) - M_{g^\delta}(q, a)) \right|^2 \, dx + \int_\Omega a (N_{j^\delta}(q, a) - M_{g^\delta}(q, a))^2 \, dx
\end{equation}
for the identification problem and consider its minimizers over \( Q \times A \) as reconstructions. However, since the identification problem is ill-posed, we make use a regularization method to seek stable solutions. Furthermore, for interests in estimating piecewise constant coefficients we therefore utilize total variation regularization combining with the quadratic stabilizing term, i.e. we consider the minimization problem
\begin{equation}
\min_{(q,a)\in\mathcal{Q}_{ad}\times\mathcal{A}_{ad}} \mathcal{Y}_{\delta,\rho}(q, a) := J^\delta(q, a) + \rho R(q, a), \quad (\mathcal{P}_{\delta,\rho})
\end{equation}
with

$$R(q, a) := \int_\Omega |\nabla q| + \int_\Omega |\nabla a| + \frac{1}{2} \|q\|^2_{L^2(\Omega)} + \frac{1}{2} \|a\|^2_{L^2(\Omega)}$$

and

$$Q_{ad} := Q \cap BV(\Omega) \quad \text{and} \quad A_{ad} := A \cap BV(\Omega),$$

where $BV(\Omega)$ is the space of all functions of bounded total variation with the semi-norm $\int_\Omega |\nabla(\cdot)|$ and the norm $\int_\Omega |\nabla(\cdot)| + \|\cdot\|_{L^1(\Omega)}$ (cf. [4]), and $\rho > 0$ is the regularization parameter. Total variation regularization originally introduced in image denoising [23]. Somewhat later, it has been used to treat several ill-posed and inverse problems over the last decades. This method is of particular interest for problems with possibility of discontinuity in the solution, see, e.g., [12, 13].

Let $V_1^h$ be the finite dimensional space of piecewise linear, continuous finite elements, and $N_{j_0}^h(q, a)$ and $M_{g_0}^h(q, a)$ be respectively the finite element approximations of $N_{j_0}(q, a)$ and $M_{g_0}(q, a)$ in $V_1^h$, where $h > 0$ is the mesh size of the triangulation. We then approximate the problem $(P_{\delta, \rho})$ by the discrete one

$$\min_{(q, a) \in Q_{ad}^h \times A_{ad}^h} \mathcal{Y}_{\delta, \rho}^h(q, a) := J_{h}^\delta(q, a) + \rho R(q, a),$$

where

$$J_{h}^\delta(q, a) := \int_\Omega q |\nabla (N_{j_0}^h(q, a) - M_{g_0}^h(q, a))|^2 \, dx + \int_\Omega a (N_{j_0}^h(q, a) - M_{g_0}^h(q, a))^2 \, dx$$

$$+ \int_{\partial \Omega} \sigma (N_{j_0}^h(q, a) - M_{g_0}^h(q, a))^2 \, ds$$

and

$$Q_{ad}^h := Q \cap V_1^h \subset Q \cap BV(\Omega) \quad \text{and} \quad A_{ad}^h := A \cap V_1^h \subset A \cap BV(\Omega).$$

As the identification problem is non-linear and severely ill-posed, the stable analysis and convergence result of finite dimensional regularized solutions to the identification are crucial. The contributions of the paper are as follows.

### 1.1 Stability

Let $\partial \left( \int_\Omega |\nabla(\cdot)|(q) \right)$ stand for the sub-differential of the semi-norm $\int_\Omega |\nabla(\cdot)|$ of the space $BV(\Omega)$ at $q \in BV(\Omega)$ defined by

$$\partial \left( \int_\Omega |\nabla(\cdot)| \right)(q) := \left\{ q^* \in BV(\Omega)^* \mid \int_\Omega |\nabla p| - \int_\Omega |\nabla q| - \langle q^*, p - q \rangle_{(BV(\Omega)^*, BV(\Omega))} \geq 0, \forall p \in BV(\Omega) \right\}.$$  

Since $q \mapsto \int_\Omega |\nabla q|$ is a continuous functional on the space $BV(\Omega)$, the set $\partial \left( \int_\Omega |\nabla(\cdot)| \right)(q) \neq \emptyset$, see, e.g., [16]. Then for a fixed element $q^* \in \partial \left( \int_\Omega |\nabla(\cdot)| \right)(q)$ the non-negative quality

$$D_{TV}^q(p, q) := \int_\Omega |\nabla p| - \int_\Omega |\nabla q| - \langle q^*, p - q \rangle_{(BV(\Omega)^*, BV(\Omega))}$$

is called the Bregman distance with respect to $\int_\Omega |\nabla(\cdot)|$ and $q^*$ of two elements $p, q$ (see [12]). The Bregman distance is not a metric on $BV(\Omega)$, since, e.g., $D_{TV}^q(p, q) \neq D_{TV}^q(p, q)$. However, $D_{TV}^q(p, q) \geq 0$ and $D_{TV}^q(q, q) = 0$.

Let the regularization parameter $\rho$ and the observation data $(j_0, g_0)$ be fixed and $(q_n, a_n) := (q_n^{\delta, \rho}, a_n^{\delta, \rho})$ denote an arbitrary minimizer of $(P_{\delta, \rho}^h)$ for each $n \in \mathbb{N}$. We then show that the sequence $(q_n, a_n)$ has a subsequence not relabeled converging to an element $(q_\delta, a_\delta) \in Q_{ad} \times A_{ad}$ in the $L^*(\Omega)$-norm for all $s \in [1, \infty)$. Furthermore,

$$\lim_{n \to \infty} \int_\Omega |\nabla q_n| = \int_\Omega |\nabla q_\delta| \quad \text{and} \quad \lim_{n \to \infty} \int_\Omega |\nabla a_n| = \int_\Omega |\nabla a_\delta|,$$

$$\lim_{n \to \infty} D_{TV}^q(q_n, q_\delta) = \lim_{n \to \infty} D_{TV}^q(a_n, a_\delta) = 0.$$
for all \((\ell, \kappa) \in \partial (f_0 \mid \nabla (\cdot)) \mid (q_{\delta, \rho}) \times \partial (f_0 \mid \nabla (\cdot)) \mid (a_{\delta, \rho})\), where \((q_{\delta, \rho}, a_{\delta, \rho})\) is a minimizer of \((P_{\delta, \rho})\). If the uniqueness for solutions of the problem \((P_{\delta, \rho})\) is satisfied, then the above convergences hold true for the whole sequence \((q_n, a_n)\).

1.2 Convergence

Let \((q^\dagger, a^\dagger)\) denote the unique \(TV - L^2\)-minimizing solution of the identification problem

\[
(q^\dagger, a^\dagger) = \arg \min_{(q, a) \in \{ (q, a) \in Q_{ad} \times A_{ad} \mid N_{\ell j}(q, a) = M_{\ell j}^\dagger(q, a) \}} R(q, a) \tag{IP}
\]

Assume that the weak solution \(u(q^\dagger, a^\dagger)\) of the system \((1.1)-(1.4)\) is in \(H^2(\Omega)\). Let \((h_n)\) and \((\delta_n)\) be any positive sequences converging to zero and the sequence of the regularization parameters \((\rho_n)\) is chosen such that \(\rho_n \to 0\), \(\frac{\delta_n}{\sqrt{\rho_n}} \to 0\) and

\[
\left( \frac{h_n \log h_n}{\sqrt{\rho_n}} \right)^r \to 0 \quad \text{with} \quad \begin{cases} r < 1/2 & \text{if} \quad d = 2 \quad \text{and} \\ r = 1/3 & \text{if} \quad d = 3 \end{cases}
\]

as \(n \to \infty\). Moreover, assume that \((j_{\delta_n}, g_{\delta_n})\) is a sequence satisfying

\[
\| j_{\delta_n} - j^\dagger \|_{H^{-1/2}(\Gamma)} + \| g_{\delta_n} - g^\dagger \|_{H^{1/2}(\Gamma)} \leq \delta_n
\]

and that \((q_n, a_n) := (q_{\rho_n, \delta_n, \rho, a_{\rho_n, \delta_n}})\) is an arbitrary minimizer of \((P_{\rho_n, \delta_n})\) for each \(n \in \mathbb{N}\). Then,

(i) The whole sequence \((q_n, a_n)\) converges to the identification \((q^\dagger, a^\dagger)\) in the \(L^s(\Omega)\)-norm for all \(s \in [1, \infty)\) and

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla q_n| = \int_{\Omega} |\nabla q^\dagger| \quad \text{and} \quad \lim_{n \to \infty} \int_{\Omega} |\nabla a_n| = \int_{\Omega} |\nabla a^\dagger|,
\]

\[
\lim_{n \to \infty} D^\ell_{TV}(q_n, q^\dagger) = \lim_{n \to \infty} D^\kappa_{TV}(a_n, a^\dagger) = 0
\]

for all \((\ell, \kappa) \in \partial (f_0 \mid \nabla (\cdot)) \mid (q^\dagger) \times \partial (f_0 \mid \nabla (\cdot)) \mid (a^\dagger)\).

(ii) The sequences \((N_{j_{\delta_n}}^h(q_n, a_n))\) and \((M_{g_{\delta_n}}^h(q_n, a_n))\) converge in the \(H^1(\Omega)\)-norm to the solution \(u(q^\dagger, a^\dagger)\).

Furthermore, we show that the misfit term \(J^\ell_{\delta_n}(q, a)\) is Fréchet differentiable and for each \((q, a) \in Q_{ad} \times A_{ad}^h\), the Fréchet differential in the direction \((q_n, a_n) \in V_{1}^h \times V_{1}^h\) given by

\[
J^\ell_{\delta_n}(q, a)(q_n, a_n) = \int_{\Omega} \eta_q \left( |\nabla M_{g_{\delta_n}}^h(q, a)|^2 - |\nabla N_{j_{\delta_n}}^h(q, a)|^2 \right) dx + \int_{\Omega} \eta_a \left( |M_{j_{\delta_n}}^h(q, a)|^2 - |N_{g_{\delta_n}}^h(q, a)|^2 \right) dx.
\]

Based on this fact, we perform some numerical results for the simultaneous coefficient identification problem, which illustrate the efficiency of the proposed variational method.

To complete this introduction, we wish to mention that the problem of identifying the sole coefficient has been investigated in our recent papers [18, 22] and many others in the literature. We have not yet found numerical analysis for the multiple coefficient identification problem so far. By using a non-standard version of the misfit term combining with an appropriate regularized technique we could here outline that two coefficients distributed inside the physical domain can be simultaneously reconstructed from a finite number of observations on a part of the boundary.

The paper is organized as follows. In Section 2 we introduce some useful notations and show the existence of a minimizer of the regularized minimization problem. Finite element method for the identification problem is presented in Section 3. Stability analysis of the proposed regularization approach and convergence of the finite dimensional approximations to the identification are enclosed in Section 4. We in Section 5 perform the differentials of the discrete coefficient-to-solution operators and of the associated cost functional. Finally, some numerical examples supporting our analytical findings are presented in Section 6.
2 Preliminaries

The expression
\[ \int_\Omega q \nabla u \cdot \nabla v dx + \int_\Omega au v dx + \int_{\partial \Omega} \sigma u v ds \]
generates an inner product on the space \( H^1(\Omega) \) which is equivalent to the usual one, i.e. there exist positive constants \( c_1, c_2 \) such that
\[
c_1 \|u\|_{H^1(\Omega)} \leq \int_\Omega q \nabla u \cdot \nabla u dx + \int_\Omega au^2 dx + \int_{\partial \Omega} \sigma u^2 ds \leq c_2 \|u\|_{H^1(\Omega)} \tag{2.1}\]
for all \( u \in H^1(\Omega) \). Therefore, for each \((q, a) \in Q \times A\) the Neumann boundary value problem \((1.8)\) defines a unique weak solution denoted by \( N_{js}(q, a) \) in the sense that \( N_{js}(q, a) \in H^1(\Omega) \) and the equation
\[
\int_\Omega q \nabla N_{js}(q, a) \cdot \nabla \phi dx + \int_\Omega a N_{js}(q, a) \phi dx + \int_{\partial \Omega} \sigma N_{js}(q, a) \phi ds \tag{2.2}
\]

Thus we define the non-linear coefficient-to-solution operators
\[
N_{js}, M_{gs} : Q \times A \subset L^\infty(\Omega) \times L^\infty(\Omega) \rightarrow H^1(\Omega)
\]
which are uniformly bounded, due to \((2.3)\) and \((2.5)\).

We here present some properties of the coefficient-to-solution operators.

Lemma 2.1. Assume that the sequence \((q_n, a_n) \subset Q \times A\) converges to \((q, a)\) almost everywhere in \(\Omega\). Then \((q, a) \in Q \times A\) and the sequence \((N_{js}(q_n, a_n), M_{gs}(q_n, a_n))\) converges to \((N_{js}(q, a), M_{gs}(q, a))\) in the \(H^1(\Omega) \times H^1(\Omega)\)-norm.

Proof. By the equation \((2.2)\), we for each \(n \in \mathbb{N}\) get that
\[
\int_\Omega q_n \nabla (N_{js}(q_n, a_n) - N_{js}(q, a)) \cdot \nabla \phi dx + \int_\Omega a_n (N_{js}(q_n, a_n) - N_{js}(q, a)) \phi dx \\
+ \int_{\partial \Omega} \sigma (N_{js}(q_n, a_n) - N_{js}(q, a)) \phi ds = \int_\Omega (q - q_n) \nabla N_{js}(q_n, a_n) \cdot \nabla \phi dx + \int_{\partial \Omega} (a - a_n) N_{js}(q_n, a_n) \phi dx.
\]
Taking $\phi = N_{js}(q_n, a_n) - N_{js}(q, a)$ and using the inequality \((2.1)\), we arrive at
\[
\|N_{js}(q_n, a_n) - N_{js}(q, a)\|_{H^1(\Omega)} \leq C \left( \int_\Omega |q - q_n|^2 |\nabla N_{js}(q, a)|^2 dx + \int_\Omega |a - a_n|^2 |N_{js}(q, a)|^2 dx \right)
\]
By the Lebesgue dominated convergence theorem, we conclude that $\lim_{n \to \infty} \|N_{js}(q_n, a_n) - N_{js}(q, a)\|_{H^1(\Omega)} = 0$. Similarly, we also obtain $\lim_{n \to \infty} \|M_{gs}(q_n, a_n) - M_{gs}(q, a)\|_{H^1(\Omega)} = 0$, which finishes the proof. \(\square\)

Next, let us quote the following useful results.

**Lemma 2.2** (\([1]\)). (i) Let $(w_n)$ be a bounded sequence in the $BV(\Omega)$-norm. Then a subsequence not relabeled and an element $w \in BV(\Omega)$ exist such that $(w_n)$ converges to $w$ in the $L^1(\Omega)$-norm.

(ii) Let $(w_n)$ be a sequence in $BV(\Omega)$ converging to $w$ in the $L^1(\Omega)$-norm. Then $w \in BV(\Omega)$ and
\[
\int_\Omega |\nabla w| \leq \liminf_{n \to \infty} \int_\Omega |\nabla w_n|.
\]

**Lemma 2.3** (\([5]\)). Assume that $w \in BV(\Omega)$. Then for all $\epsilon > 0$ an element $w^\epsilon \in C^\infty(\Omega)$ exists such that
\[
\int_\Omega |w - w^\epsilon| \leq \epsilon \int_\Omega |\nabla w|, \quad \int_\Omega |\nabla w^\epsilon| \leq (1 + C\epsilon) \int_\Omega |\nabla w| \quad \text{and} \quad \int_\Omega |D^2 w^\epsilon| \leq C\epsilon^{-1} \int_\Omega |\nabla w|,
\]
where the positive constant $C$ is independent of $\epsilon$.

We are now in the position to prove the main result of the section.

**Theorem 2.4.** The problem $(P_{\delta, \rho})$ attains a solution $(q_{\delta, \rho}, a_{\delta, \rho})$, which is called the regularized solution of the identification problem.

**Proof.** Let $(q_n, a_n) \subset Q_{ad} \times A_{ad}$ be a minimizing sequence of the problem $(P_{\delta, \rho})$, i.e.
\[
\lim_{n \to \infty} \Psi_{\delta, \rho}(q_n, a_n) = \inf_{(q, a) \in Q_{ad} \times A_{ad}} \Psi_{\delta, \rho}(q, a).
\]
Therefore, the sequence $(q_n, a_n)$ is bounded in the $BV(\Omega)$-norm. By Lemma 2.2, a subsequence which is not relabeled and an element $(q, a) \in Q_{ad} \times A_{ad}$ exist such that
\[
(q_n, a_n) \text{ converges to } (q, a) \text{ in the } L^1(\Omega) \times L^1(\Omega)-\text{norm},
\]
\[
(q_n, a_n) \text{ converges to } (q, a) \text{ almost everywhere in } \Omega,
\]
\[
\int_\Omega |\nabla q| \leq \liminf_{n \to \infty} \int_\Omega |\nabla q_n| \quad \text{and} \quad \int_\Omega |\nabla a| \leq \liminf_{n \to \infty} \int_\Omega |\nabla a_n|.
\]
Furthermore, by the inequality
\[
\|q_n - q\|_{L^2(\Omega)}^2 + \|a_n - a\|_{L^2(\Omega)}^2 \leq 2 \max(\bar{q}, \bar{a}) \left( \|q_n - q\|_{L^1(\Omega)} + \|a_n - a\|_{L^1(\Omega)} \right),
\]
the sequence $(q_n, a_n)$ also converges to $(q, a)$ in the $L^2(\Omega) \times L^2(\Omega)$-norm. We thus have
\[
R(q, a) \leq \liminf_{n \to \infty} R(q_n, a_n). \tag{2.6}
\]
Furthermore, an application of Lemma 2.1 deduces that the sequence $(N_{js}(q_n, a_n), M_{gs}(q_n, a_n))$ converges to $(N_{js}(q, a), M_{gs}(q, a))$ in the $H^1(\Omega) \times H^1(\Omega)$-norm and then
\[
J_\delta(q, a) = \lim_{n \to \infty} J_\delta(q_n, a_n). \tag{2.7}
\]
Therefore, we obtain from \((2.6)\)–\((2.7)\) that
\[
\Psi_{\delta, \rho}(q, a) \leq \lim_{n \to \infty} J_\delta(q_n, a_n) + \liminf_{n \to \infty} \rho R(q_n, a_n)
\]
\[
= \liminf_{n \to \infty} (J_\delta(q_n, a_n) + \rho R(q_n, a_n))
\]
\[
= \inf_{(q, a) \in Q_{ad} \times A_{ad}} \Psi_{\delta, \rho}(q, a)
\]
and $(q, a)$ is hence a solution of the problem $(P_{\delta, \rho})$, which finishes the proof. \(\square\)
3 Finite element discretization

Let \((T^h)_{0 < h < 1}\) be a quasi-uniform family of regular triangulations of the domain \(\Omega\) with the mesh size \(h\) and
\[
V_1^h := \{ v^h \in C(\Omega) \mid v^h \big|_T \in \mathcal{P}_1, \forall T \in T^h \}
\]
\[
V_{1,0}^h := V_1^h \cap H^1_0(\Omega),
\]
where \(\mathcal{P}_1\) consists of all polynomial functions of degree less than or equal to 1. For each \((q, a) \in Q \times A\) the variational equations
\[
\int_\Omega q \nabla u^h \cdot \nabla \phi^h \, dx + \int_\Omega a v^h \phi^h \, dx + \int_{\partial \Omega} \sigma u^h \phi^h \, ds = (f, \phi^h)_{H^{-1}(\Omega), H^1(\Omega)} + (j_0, \phi^h)_{(-1/2)(\partial \Omega \setminus \Gamma), H^{1/2}(\partial \Omega \setminus \Gamma)} + (j_s, \phi^h)_{(-1/2)(\partial \Omega \setminus \Gamma), H^{1/2}(\partial \Omega \setminus \Gamma)}
\]
for all \(\phi^h \in V_1^h\) and
\[
\int_\Omega q \nabla v^h \cdot \nabla \phi^h \, dx + \int_\Omega a v^h \phi^h \, dx + \int_{\partial \Omega} \sigma v^h \phi^h \, ds = (f, \phi^h)_{H^{-1}(\Omega), H^1(\Omega)} + (j_0, \phi^h)_{(-1/2)(\partial \Omega \setminus \Gamma), H^{1/2}(\partial \Omega \setminus \Gamma)} + (g_s, \phi^h)_{(-1/2)(\partial \Omega \setminus \Gamma), H^{1/2}(\partial \Omega \setminus \Gamma)}
\]
for all \(\phi^h \in V_{1,0}^h\) and \(v^h \big|_T = g_s\) admit unique solutions \(u^h := N_{j_s}^h(q, a) \in V_1^h\) and \(v^h := M_{g_s}^h(q, a) \in V_{1,0}^h\), respectively. Furthermore, the estimates
\[
\| N_{j_s}^h(q, a) \|_{H^1(\Omega)} \leq C( \| f \|_{H^{-1}(\Omega)} + \| j_s \|_{H^{-1/2}(\partial \Omega \setminus \Gamma)} + \| j_0 \|_{H^{1/2}(\partial \Omega \setminus \Gamma)} )
\]
\[
\| M_{g_s}^h(q, a) \|_{H^1(\Omega)} \leq C( \| f \|_{H^{-1}(\Omega)} + \| g_s \|_{H^{1/2}(\partial \Omega \setminus \Gamma)} + \| j_0 \|_{H^{1/2}(\partial \Omega \setminus \Gamma)} )
\]
hold true, where the positive constant \(C\) is independent of \(h\).

Remark 3.1. Due to the standard theory of the finite element method for elliptic problems (cf. [8]), we for any fixed \((q, a) \in Q \times A\) get the limits
\[
\lim_{h \to 0} \| N_{j_s}^h(q, a) - N_{j_s}(q, a) \|_{H^1(\Omega)} = \lim_{h \to 0} \| M_{g_s}^h(q, a) - M_{g_s}(q, a) \|_{H^1(\Omega)} = 0.
\]
Furthermore, under additional assumptions \(q \in W^{1,\infty}(\Omega), a \in L^\infty(\Omega), f \in L^2(\Omega), j_0 \in H^{1/2}(\partial \Omega \setminus \Gamma), j_s \in H^{1/2}(\Gamma), g_s \in H^{3/2}(\Gamma)\) and either \(\Omega\) is convex or of the class \(C^{0,1}\), the weak solutions \(N_{j_s}(q, a), M_{g_s}(q, a) \in H^2(\Omega)\) (see, e.g., [25]) satisfying
\[
\| N_{j_s}(q, a) \|_{H^2(\Omega)} \leq C( \| f \|_{L^2(\Omega)} + \| j_s \|_{H^{1/2}(\partial \Omega \setminus \Gamma)} + \| j_0 \|_{H^{1/2}(\partial \Omega \setminus \Gamma)} )
\]
\[
\| M_{g_s}(q, a) \|_{H^2(\Omega)} \leq C( \| f \|_{L^2(\Omega)} + \| g_s \|_{H^{3/2}(\Gamma)} + \| j_0 \|_{H^{1/2}(\partial \Omega \setminus \Gamma)} )
\]
which yield the error bounds
\[
\| N_{j_s}^h(q, a) - N_{j_s}(q, a) \|_{L^2(\Omega)} + h \| N_{j_s}^h(q, a) - N_{j_s}(q, a) \|_{H^1(\Omega)} \leq C h^2 \| N_{j_s}(q, a) \|_{H^2(\Omega)}
\]
\[
\| M_{g_s}^h(q, a) - M_{g_s}(q, a) \|_{L^2(\Omega)} + h \| M_{g_s}^h(q, a) - M_{g_s}(q, a) \|_{H^1(\Omega)} \leq C h^2 \| M_{g_s}(q, a) \|_{H^2(\Omega)}.
\]

We introduce the Lagrange nodal value interpolation operator
\[
I_1^h : C(\Omega) \to V_1^h.
\]
By the continuous embedding \(W^{1,p}(\Omega) \hookrightarrow C(\Omega)\) with \(p > d\), the operator \(I_1^h : W^{1,p}(\Omega) \to V_1^h\) is well defined. Furthermore, see, e.g., [8], it holds the limit
\[
\lim_{h \to 0} \| I_1^h \phi - \phi \|_{W^{1,p}(\Omega)} = 0
\]
and the estimate
\[
\| I_1^h \phi - \phi \|_{L^p(\Omega)} \leq C h \| \phi \|_{W^{1,p}(\Omega)}.
\]
We have the following existence result. Its proof exactly follows as in the continuous case, is therefore omitted here.

Theorem 3.2. The discrete regularized problem \((\mathcal{P}_{\delta,\rho}^h)\) attains a minimizer \((q_{\delta,\rho}^h, a_{\delta,\rho}^h)\), which is called the discrete regularized solution of the identification problem.
4 Convergence analysis

The aim of this section is to prove the stability of the proposed regularization approach and the convergence of finite element approximations to the identification.

**Theorem 4.1.** Assume that the regularization parameter $\rho$ and the observation data $(j_\delta, g_\delta)$ are fixed. For each $n \in \mathbb{N}$ let $(q_n, a_n) := (q^{h_n}_{\delta, \rho}, a^{h_n}_{\delta, \rho})$ denote an arbitrary minimizer of $(P_{\delta, \rho})$. Then the sequence $(q_n, a_n)$ has a subsequence not relabeled converging to an element $(q_{\delta, \rho}, a_{\delta, \rho}) \in Q_{ad} \times A_{ad}$ in the $L^p(\Omega)$-norm for all $s \in [1, \infty)$. Furthermore,

$$
\lim_{n \to \infty} \int_{\Omega} |\nabla q_n| = \int_{\Omega} |\nabla q_{\delta, \rho}| \quad \text{and} \quad \lim_{n \to \infty} \int_{\Omega} |\nabla a_n| = \int_{\Omega} |\nabla a_{\delta, \rho}|, \quad (4.1)
$$

$$
\lim_{n \to \infty} D_{TV}^q(q_n, q_{\delta, \rho}) = \lim_{n \to \infty} D_{TV}^\infty(a_n, a_{\delta, \rho}) = 0 \quad (4.2)
$$

for all $(\ell, \kappa) \in \partial (\int_{\Omega} |\nabla (\cdot)|, q_{\delta, \rho}) \times \partial (\int_{\Omega} |\nabla (\cdot)|, a_{\delta, \rho})$, where $(q_{\delta, \rho}, a_{\delta, \rho})$ is a minimizer of $(P_{\delta, \rho})$.

**Proof.** Let $(q, a) \in Q_{ad} \times A_{ad}$ be arbitrary but fixed. Due to Lemma 2.3 for any fixed $\epsilon \in (0, 1)$ an element $(q^\epsilon, a^\epsilon) \in C^\infty(\Omega) \times C^\infty(\Omega)$ exists such that

$$
\frac{1}{2} \|q - q^\epsilon\|_{L^2(\Omega)} + \frac{1}{2} \|a - a^\epsilon\|_{L^2(\Omega)}^2 \leq \epsilon \max(q, \overline{q}) \left( \int_{\Omega} |\nabla q| + \int_{\Omega} |\nabla a| \right) \quad (4.3)
$$

for some positive constant $C$ independent of $\epsilon$. We denote by

$$
r^\epsilon(q) := r^\epsilon := \max_q(q, \min(q^\epsilon, \overline{q})) \quad \text{and} \quad b^\epsilon(a) := b^\epsilon := \max_a(a, \min(q^\epsilon, \overline{q}))
$$

that satisfy that

$$(r^\epsilon, b^\epsilon) \in (Q \cap W^{1, \infty}(\Omega)) \times (A \cap W^{1, \infty}(\Omega))$$

and

$$(r^\epsilon_n, b^\epsilon_n) := (I^h_1 r^\epsilon, I^h_1 b^\epsilon) \in Q^h_{ad} \times A^h_{ad}.$$  

Let $p > d$ and $p^*$ be the adjoint number of $p$, i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$. We get

$$
|\Omega|^{-1/p^*} \int_{\Omega} |\nabla r^\epsilon_n| = |\Omega|^{-1/p^*} \int_{\Omega} |\nabla r^\epsilon_n(x)| dx \leq \left( \int_{\Omega} |\nabla r^\epsilon_n(x)|^p dx \right)^{1/p} = |r^\epsilon_n|_{W^{1, p}(\Omega)} \quad (4.4)
$$

for $n$ large enough, by the limit (3.8). Furthermore, using (4.3), we have the estimate

$$
|r^\epsilon|_{W^{1, p}(\Omega)} = \int_{\Omega} |\nabla r^\epsilon(x)| dx = \int_{\{x \in \Omega \mid r^\epsilon(x) = q^\epsilon(x)\}} |\nabla r^\epsilon(x)| dx \leq \int_{\Omega} |\nabla q^\epsilon(x)| dx \leq (1 + C\epsilon) \left( \int_{\Omega} |\nabla q| + \int_{\Omega} |\nabla a| \right), \quad (4.5)
$$

by the fact that $r^\epsilon$ is constant on $\{x \in \Omega \mid r^\epsilon(x) = q^\epsilon(x)\}$. Combining (4.4) and (4.5), we have the boundedness

$$
\int_{\Omega} |\nabla r^\epsilon_n| + \int_{\Omega} |\nabla b^\epsilon_n| \leq C \quad (4.6)
$$

for all $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$.

Now, by the definition of $(q_n, a_n)$, we for all $n \in \mathbb{N}$ get that

$$
J_{\delta}^{h_n}(q_n, a_n) + \rho R(q_n, a_n) \leq J_{\delta}^{h_n}(r^\epsilon_n, b^\epsilon_n) + \rho R(r^\epsilon_n, b^\epsilon_n). \quad (4.7)
$$
By \((3.3)\) and \((3.4)\), it holds \(J^h_n(r_n', b_n') \leq C\). We thus deduce from \((4.6)-(4.7)\) that
\[
R(q_n, a_n) \leq C
\]
for all \(n \in \mathbb{N}\). An application of Lemma \(2.2\) then follows that a subsequence of \((q_n, a_n)\) not relabeled and an element \((\tilde{q}, \tilde{a}) \in Q_{ad} \times A_{ad}\) exist such that \((q_n, a_n)\) converges to \((\tilde{q}, \tilde{a})\) in the \(L^s(\Omega)\)-norm for all \(s \in [1, \infty)\)
\[
R(\tilde{q}, \tilde{a}) \leq \liminf_{n \to \infty} R(q_n, a_n).
\] (4.8)

Using Lemma \(2.1\) and the identities \((3.5)\), we get that
\[
J_\delta(\tilde{q}, \tilde{a}) = \lim_{n \to \infty} J^h_n(q_n, a_n).
\] (4.9)

Furthermore, since
\[
\lim_{n \to \infty} \|r_n^r - r^r\|_{L^1(\Omega)} = \lim_{n \to \infty} \|b_n^r - b^r\|_{L^1(\Omega)} = 0,
\]
we also have
\[
J_\delta(r^r, b^r) = \lim_{n \to \infty} J^h_n(r_n^r, b_n^r).
\] (4.10)

On the other hand, by the definition of \((r^r, b^r)\), we get
\[
|r^r - q| \leq |q - q| \quad \text{and} \quad |b^r - a| \leq |a - a|
\]
a.e. in \(\Omega\). Integrating the above inequalities over the domain \(\Omega\), it gives
\[
\|r^r - q\|_{L^1(\Omega)} + \|b^r - a\|_{L^1(\Omega)} \leq \|q - q\|_{L^1(\Omega)} + \|a - a\|_{L^1(\Omega)} \leq C\epsilon
\]
together with the limit
\[
J_\delta(q, a) = \lim_{\epsilon \to 0} J_\delta(r^r, b^r).
\] (4.11)

We mention that
\[
\|r^r - q\|_{L^2(\Omega)} + \|b^r - a\|_{L^2(\Omega)} \leq 2 \max(q, a)^{1/2} \left(\|r^r - q\|_{L^1(\Omega)} + \|b^r - a\|_{L^1(\Omega)}\right)^{1/2}
\]
and then
\[
\|r^r\|_{L^2(\Omega)}^2 + \|b^r\|_{L^2(\Omega)}^2 \leq C\epsilon + \|q\|_{L^2(\Omega)}^2 + \|a\|_{L^2(\Omega)}^2.
\]
Combining this with \((4.5)\), it gives
\[
R(r^r, b^r) \leq C\epsilon + R(q, a).
\] (4.12)

Furthermore, with the aid of \((3.8)\), we get
\[
R(r^r, b^r) = \lim_{n \to \infty} R(r_n^r, b_n^r).
\] (4.13)

Therefore, we obtain from by \((4.8), (4.9), (4.7), (4.10), (4.13)\) and \((4.12)\) that
\[
\Psi_{\delta, \rho}(\tilde{q}, \tilde{a}) = J_\delta(\tilde{q}, \tilde{a}) + \rho R(\tilde{q}, \tilde{a})
\]
\[
\leq \lim_{n \to \infty} J^h_n(q_n, a_n) + \liminf_{n \to \infty} \rho R(q_n, a_n)
\]
\[
= \liminf_{n \to \infty} \left( J^h_n(q_n, a_n) + \rho R(q_n, a_n) \right)
\]
\[
\leq \liminf_{n \to \infty} \left( J^h_n(r_n^r, b_n^r) + \rho R(r_n^r, b_n^r) \right)
\]
\[
= J_\delta(r^r, b^r) + \rho R(r^r, b^r),
\]
\[
\leq J_\delta(r^r, b^r) + \rho R(q, a) + C\epsilon\rho.
\]
Sending $\epsilon \to 0$, by (4.11), we arrive at

$$\mathcal{Y}_{\delta, \rho}(\hat{q}, \hat{a}) \leq J_\delta(q, a) + \rho R(q, a).$$

Since $(q, a)$ is arbitrarily taken in the admissible set $Q_{ad} \times A_{ad}$, the last relation shows that $(\hat{q}, \hat{a})$ is a solution to $(\mathcal{P}_{\rho, \delta})$.

Now, denoting

$$(\hat{r}^*, \hat{b}^*) := (\hat{r}^*(\hat{q}), \hat{b}^*(\hat{a})) \quad \text{and} \quad (\hat{r}_n^*, \hat{b}_n^*) := (\hat{r}_n^*(\hat{q}), \hat{b}_n^*(\hat{a})),$$

we have

$$\limsup_{n \to \infty} \rho R(q_n, a_n) = \limsup_{n \to \infty} J_\delta(q_n, a_n) + \limsup_{n \to \infty} \rho R(q_n, a_n) - J_\delta(\hat{q}, \hat{a})$$

$$= \limsup_{n \to \infty} \left( J_\delta(q_n, a_n) + \rho R(q_n, a_n) \right) - J_\delta(\hat{q}, \hat{a})$$

$$\leq \limsup_{n \to \infty} \left( J_\delta(q_n, \hat{b}_n) + \rho R(q_n, \hat{b}_n) \right) - J_\delta(\hat{q}, \hat{a})$$

$$= \lim_{n \to \infty} J_\delta(q_n, \hat{b}_n) + \rho \lim_{n \to \infty} R(q_n, \hat{b}_n) - J_\delta(\hat{q}, \hat{a})$$

$$\leq J_\delta(\hat{r}^*, \hat{b}^*) + \rho R(\hat{q}, \hat{a}) + C\epsilon - J_\delta(\hat{q}, \hat{a}).$$

Sending $\epsilon \to 0$, we get

$$\limsup_{n \to \infty} \rho R(q_n, a_n) \leq J_\delta(\hat{q}, \hat{a}) + \rho R(\hat{q}, \hat{a}) - J_\delta(\hat{q}, \hat{a}) = \rho R(\hat{q}, \hat{a}).$$

This together with (4.8) infers

$$R(\hat{q}, \hat{a}) \leq \liminf_{n \to \infty} R(q_n, a_n) \leq \limsup_{n \to \infty} R(q_n, a_n) \leq R(\hat{q}, \hat{a})$$

and thus

$$\lim_{n \to \infty} \left( \int_{\Omega} |\nabla q_n| + \int_{\Omega} |\nabla a_n| \right) = \int_{\Omega} |\nabla \hat{q}| + \int_{\Omega} |\nabla \hat{a}|.$$

Utilizing Lemma 2.2 again, we have

$$\int_{\Omega} |\nabla \hat{q}| \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla q_n|$$

$$= \lim_{n \to \infty} \left( \int_{\Omega} |\nabla q_n| + \int_{\Omega} |\nabla a_n| \right) - \liminf_{n \to \infty} \int_{\Omega} |\nabla a_n|$$

$$= \int_{\Omega} |\nabla \hat{q}| + \int_{\Omega} |\nabla \hat{a}| - \liminf_{n \to \infty} \int_{\Omega} |\nabla a_n|$$

and arrive at

$$\liminf_{n \to \infty} \int_{\Omega} |\nabla a_n| \leq \int_{\Omega} |\nabla \hat{a}| \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla a_n|. $$

This leads to the identity (4.11). Finally, since $(q_n, a_n)$ converges to $(\hat{q}, \hat{a})$ in the $L^1(\Omega)$-norm and (4.11), we conclude that $(q_n, a_n)$ weakly converges to $(\hat{q}, \hat{a})$ in $BV(\Omega) \times BV(\Omega)$ (see [3], Proposition 10.1.2, p. 374). Therefore, (4.2) follows. The theorem is proved.

We now introduce the notion of the unique $TV - L^2$-minimizing solution of the identification problem.

**Lemma 4.2.** The problem

$$\min_{(q, a) \in \{(q, a) \in Q_{ad} \times A_{ad} \mid N_j(q, a) = M_j'(q, a)\}} R(q, a) \quad (\mathcal{I}P)$$

admits a unique solution, which is called the $TV - L^2$-minimizing solution of the identification problem.
Proof. It is straightforward to check that the problem (IP) has a solution. Furthermore, since the functional $R$ is strictly convex on the convex set under consideration, the uniqueness of the solution is then followed, which finishes the proof.

**Lemma 4.3.** For any fixed $(q, a) \in Q_{ad} \times A_{ad}$ an element $(\hat{q}^h, \hat{a}^h) \in Q_{ad}^h \times A_{ad}^h$ exists such that

$$
\|\hat{q}^h - q\|_{L^1(\Omega)} + \|\hat{a}^h - a\|_{L^1(\Omega)} \leq C h |\log h| \tag{4.14}
$$

and

$$
\lim_{h \to 0} R(\hat{q}^h, \hat{a}^h) = R(q, a). \tag{4.15}
$$

Proof. The existence of the pair $(\hat{q}^h, \hat{a}^h) \in Q_{ad}^h \times A_{ad}^h$ satisfying the inequality (4.14) follows from Lemma 4.6 of [18], where

$$
\lim_{h \to 0} \int_\Omega |\nabla \hat{q}^h| = \int_\Omega |\nabla q| \quad \text{and} \quad \lim_{h \to 0} \int_\Omega |\nabla \hat{a}^h| = \int_\Omega |\nabla a|.
$$

Since $\lim_{h \to 0} h |\log h| = 0$, the identity (4.15) now follows by (4.16). The proof completes.

For any $(q, a) \in Q_{ad} \times A_{ad}$ let $(\hat{q}^h, \hat{a}^h) \in Q_{ad}^h \times A_{ad}^h$ be arbitrarily generated from $(q, a)$. We have the limit

$$
\chi_{j_{3},g_{3}}^h(q, a) := \|N_{j_{3}}^h(\hat{q}^h, \hat{a}^h) - N_{j_{3}}(q, a)\|_{H^1(\Omega)} + \|M_{g_{3}}^h(\hat{q}^h, \hat{a}^h) - M_{g_{3}}(q, a)\|_{H^1(\Omega)} \to 0 \quad \text{as} \quad h \to 0
$$

and the estimate

$$
\chi_{j_{3},g_{3}}^h(q, a) \leq C_R (h |\log h|)^r \quad \text{with} \quad \begin{cases} r < 1/2 & \text{if} \quad d = 2 \\ r = 1/3 & \text{if} \quad d = 3 \end{cases}
$$

in case $N_{j_{3}}(q, a), M_{g_{3}}(q, a) \in H^2(\Omega)$ (see [18] Lemma 4.8]).

**Theorem 4.4.** Let $(h_n)$, $(\delta_n)$ and $(\rho_n)$ be any positive sequences such that

$$
\rho_n \to 0, \quad \delta_n \to 0, \quad \text{and} \quad \chi_{j_{3},g_{3}}^{h_n}(q^1, \rho_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{4.18}
$$

where $(q^1, \rho^1)$ is the unique $TV - L^2$-minimizing solution of the identification problem (IP). Moreover, assume that $(j_{\delta_n}, g_{\delta_n}) \subset H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ is a sequence satisfying

$$
\|j_{\delta_n} - j\|_{H^{-1/2}(\Gamma)} + \|g_{\delta_n} - g\|_{H^{1/2}(\Gamma)} \leq \delta_n
$$

and that $(q_n, a_n) := (q_{\rho_n, \delta_n}^h, a_{\rho_n, \delta_n}^h)$ is an arbitrary minimizer of $(\mathcal{P}_{\rho_n, \delta_n}^h)$ for each $n \in \mathbb{N}$. Then,

(i) The whole sequence $(q_n, a_n)$ converges to $(q^1, \rho^1)$ in the $L^s(\Omega)$-norm for all $s \in [1, \infty)$ and

$$
\lim_{n \to \infty} \int_\Omega |\nabla q_n| = \int_\Omega |\nabla q^1| \quad \text{and} \quad \lim_{n \to \infty} \int_\Omega |\nabla a_n| = \int_\Omega |\nabla a^1|, \tag{4.19}
$$

$$
\lim_{n \to \infty} D_{TV}^\ell (q_n, q^1) = \lim_{n \to \infty} D_{TV}^\kappa (a_n, a^1) = 0 \tag{4.20}
$$

for all $(\ell, \kappa) \in \partial (\int_\Omega |\nabla (\cdot)|) (q^1) \times \partial (\int_\Omega |\nabla (\cdot)|) (a^1)$.

(ii) The sequences $(N_{j_{\delta_n}}^h(q_n, a_n))$ and $(M_{g_{\delta_n}}^h(q_n, a_n))$ converge in the $H^1(\Omega)$-norm to the unique weak solution $u(q^1, \rho^1)$ of the boundary value problem (1.1)–(1.4).

Proof. We have from the equation $N_{j_{\delta_n}}^h(q_n, a_n) = M_{g_{\delta_n}}^h(q_n, a_n)$ and the optimality of $(q_n, a_n)$ that

$$
J_{\delta_n}^h(q_n, a_n) + \rho_n R(q_n, a_n) \leq J_{\delta_n}^h(\hat{q}_{\delta_n}^h, \hat{a}_{\delta_n}^h) + \rho_n R(\hat{q}_{\delta_n}^h, \hat{a}_{\delta_n}^h), \tag{4.21}
$$
where \((\hat{q}^{h_n}_i, \hat{a}^{h_n}_i)\) is generated from \((q^i, a^i)\) according to Lemma 4.3 and
\[
J_{b_n}^{h_n}(\hat{q}^{h_n}_i, \hat{a}^{h_n}_i) \leq C \left\| J_{j_n}^{h_n}(\hat{q}^{h_n}_i, \hat{a}^{h_n}_i) - M_{g_n}^{h_n}(\hat{q}^{h_n}_i, \hat{a}^{h_n}_i) \right\|_{H^1(\Omega)}^2
\]
\[
\leq C \left( \left\| N_{j_n}^{h_n}(\hat{q}^{h_n}_i, \hat{a}^{h_n}_i) - N_{j_i}^{h_n}(\hat{q}^{h_n}_i, \hat{a}^{h_n}_i) \right\|_{H^1(\Omega)}^2 + \left\| M_{g_i}^{h_n}(\hat{q}^{h_n}_i, \hat{a}^{h_n}_i) - M_{g_n}^{h_n}(\hat{q}^{h_n}_i, \hat{a}^{h_n}_i) \right\|_{H^1(\Omega)}^2 \right)
\]
\[
\leq C \left( \left\| j_n - j_i \right\|^2_{H^{-1/2}(\Gamma)} + \left\| g_n - g_i \right\|^2_{H^{1/2}(\Gamma)} \right) + C \chi_{j_i, g_i}^{h_n}(q^i, a^i)^2
\]
\[
\leq C \left( a^2 + \chi_{j_i, g_i}^{h_n}(q^i, a^i)^2 \right).
\]
Therefore it follows from (4.21), (4.18) and (4.15) that
\[
\lim_{n \to \infty} J_{b_n}^{h_n}(q_n, a_n) = 0
\]
(4.22)

and
\[
\limsup_{n \to \infty} R(q_n, a_n) \leq \limsup_{n \to \infty} R(\hat{q}^{h_n}_i, \hat{a}^{h_n}_i) = R(q^i, a^i).
\]
(4.23)

With the aid of Lemma 2.2 a subsequence of \((q_n, a_n)\) not relabeled and an element \((q^0, a^0) \in Q_{ad} \times A_{ad}\) exist such that \((q_n, a_n)\) converges to \((q^0, a^0)\) in the \(L^s(\Omega)\)-norm for all \(s \in [1, \infty)\) and
\[
R(q^0, a^0) \leq \liminf_{n \to \infty} R(q_n, a_n).
\]
(4.24)

Thus, due to Lemma 2.1 we obtain that \((N_{j_n}^{h_n}(q_n, a_n), M_{g_n}^{h_n}(q_n, a_n))\) converges to \((N_{j_i}^{h_n}(q^0, a^0), M_{g_i}^{h_n}(q^0, a^0))\) in the \(H^1(\Omega) \times H^1(\Omega)\)-norm. This yields the equation
\[
\left\| N_{j_i}^{h_n}(q^0, a^0) - M_{g_i}^{h_n}(q^0, a^0) \right\|_{H^1(\Omega)} = \lim_{n \to \infty} \left\| N_{j_n}^{h_n}(q_n, a_n) - M_{g_n}^{h_n}(q_n, a_n) \right\|_{H^1(\Omega)}
\]
\[
\leq C \lim_{n \to \infty} \sqrt{J_{b_n}^{h_n}(q_n, a_n)} = 0,
\]
by (4.22). Thus, \((q^0, a^0)\) belongs to the set \(\{(q, a) \in Q_{ad} \times A_{ad} \mid N_{j_i}^{h_n}(q, a) = M_{g_i}^{h_n}(q, a)\}\).

On the other hands, it follows from (4.23) – (4.24) that
\[
R(q^0, a^0) \leq \liminf_{n \to \infty} R(q_n, a_n) \leq \limsup_{n \to \infty} R(q_n, a_n) \leq R(q^i, a^i).
\]

By the uniqueness of the \(TV - L^2\)-minimizing solution \((q^i, a^i)\), we then have that \((q^0, a^0) = (q^i, a^i)\) and
\[
R(q^i, a^i) = \lim_{n \to \infty} R(q_n, a_n).
\]

Therefore, using the arguments included in the proof of Theorem 4.1 we arrive at (4.19) and (4.20). Since \((q^i, a^i)\) is uniquely defined, the above convergences are fulfilled for the whole sequence, which finishes the proof.

5) Differentials

In this section we present the differentials of the discrete coefficient-to-solution operators and of the associated cost functional.

Lemma 5.1. The discrete operators \(N_{j_i}^{h} \) and \(M_{g_i}^{h} \) are infinitely Fréchet differentiable. For \((q, a) \in Q \times A\) and \(((\eta^1_q, \eta^1_a), \ldots, (\eta^m_q, \eta^m_a)) \in L^\infty(\Omega)^{2m},\) the \(m\)-th order differentials
\[
D_{\eta}^{h}(m) := N_{j_i}^{h}(m)(q, a)((\eta^1_q, \eta^1_a), \ldots, (\eta^m_q, \eta^m_a)) \in V^h_1
\]
Proof. The proof is based on standard arguments, is therefore omitted here.

Below we present the gradient of the cost functional. For \((q, a) \in Q_{ad}^h \times A_{ad}^h\) and \((\eta_q, \eta_a) \in V_1^h \times V_1^h\) we get that

\[
J_\delta^h(q, a)(\eta_q, \eta_a) = \frac{\partial J_\delta^h(q, a)}{\partial q} \eta_q + \frac{\partial J_\delta^h(q, a)}{\partial a} \eta_a,
\]

where

\[
\frac{1}{2} \frac{\partial J_\delta^h(q, a)}{\partial q} \eta_q = \frac{1}{2} \int_\Omega \eta_q \left( \nabla (N_{^j_{js}}^h(q, a) - M_{^j_{gs}}^h(q, a)) \right)^2 \, dx
+ \int_\Omega q \nabla (N_{^j_{js}}^h(q, a) - M_{^j_{gs}}^h(q, a)) \cdot \nabla (N_{^j_{js}}^h(q, a)(\eta_q, 0) - M_{^j_{gs}}^h(q, a)(\eta_q, 0)) \, dx
+ \int_\Omega a (N_{^j_{js}}^h(q, a) - M_{^j_{gs}}^h(q, a)) (N_{^j_{js}}^h(q, a)(\eta_q, 0) - M_{^j_{gs}}^h(q, a)(\eta_q, 0)) \, dx
+ \int_{\partial \Omega} \sigma (N_{^j_{js}}^h(q, a) - M_{^j_{gs}}^h(q, a)) (N_{^j_{js}}^h(q, a)(\eta_q, 0) - M_{^j_{gs}}^h(q, a)(\eta_q, 0)) \, ds
\]

and

\[
\frac{1}{2} \frac{\partial J_\delta^h(q, a)}{\partial a} \eta_a = \int_\Omega q \nabla (N_{^j_{js}}^h(q, a) - M_{^j_{gs}}^h(q, a)) \cdot \nabla (N_{^j_{js}}^h(q, a)(0, \eta_a) - M_{^j_{gs}}^h(q, a)(0, \eta_a)) \, dx
+ \frac{1}{2} \int_\Omega \eta_a (N_{^j_{js}}^h(q, a) - M_{^j_{gs}}^h(q, a))^2 \, dx
+ \int_\Omega a (N_{^j_{js}}^h(q, a) - M_{^j_{gs}}^h(q, a)) (N_{^j_{js}}^h(q, a)(0, \eta_a) - M_{^j_{gs}}^h(q, a)(0, \eta_a)) \, dx
+ \int_{\partial \Omega} \sigma (N_{^j_{js}}^h(q, a) - M_{^j_{gs}}^h(q, a)) (N_{^j_{js}}^h(q, a)(0, \eta_a) - M_{^j_{gs}}^h(q, a)(0, \eta_a)) \, ds.
\]
Thus,

\[
\frac{1}{2} J^{h'}_g(q,a)(\eta_q, \eta_a) = \frac{1}{2} \int_{\Omega} \eta_q \left| \nabla \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \right|^2 \, dx + \frac{1}{2} \int_{\Omega} \eta_a \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right)^2 \, dx \\
+ \int_{\Omega} q \nabla \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \cdot \nabla \left( N^{h'}_{js}(q,a)(\eta_q, \eta_a) - M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \right) \, dx \\
+ \int_{\Omega} a \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \left( N^{h'}_{js}(q,a)(\eta_q, \eta_a) - M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \right) \, dx \\
+ \int_{\partial \Omega} \sigma \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \left( N^{h'}_{js}(q,a)(\eta_q, \eta_a) - M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \right) \, ds.
\]

Denoting by \( \sum \) the last three terms in the above sum, we have

\[
\sum := \int_{\Omega} q \nabla N^{h'}_{js}(q,a)(\eta_q, \eta_a) \cdot \nabla \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, dx \\
+ \int_{\Omega} a N^{h'}_{js}(q,a)(\eta_q, \eta_a) \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, dx \\
+ \int_{\partial \Omega} \sigma N^{h'}_{js}(q,a)(\eta_q, \eta_a) \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, ds \\
+ \int_{\Omega} q \nabla M^{h}_{gs}(q,a) \cdot \nabla M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \, dx + \int_{\Omega} a M^{h}_{gs}(q,a) M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \, dx \\
+ \int_{\partial \Omega} \sigma M^{h}_{gs}(q,a) M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \, ds \\
- \int_{\Omega} q \nabla N^{h}_{js}(q,a) \cdot \nabla M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \, dx - \int_{\Omega} a N^{h}_{js}(q,a) M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \, dx \\
- \int_{\partial \Omega} \sigma N^{h}_{js}(q,a) M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \, ds.
\]

With the aid of Lemma 5.1 together with (3.1) and (3.2) we get

\[
\sum = -\int_{\Omega} \eta_q \nabla N^{h}_{js}(q,a) \cdot \nabla \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, dx - \int_{\Omega} \eta_a N^{h}_{js}(q,a) \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, dx \\
+ \langle f, M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\Omega)} + \langle j_0, M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\Omega)} \\
- \langle f, M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\Omega)} - \langle j_0, M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\Omega)} \\
= -\int_{\Omega} \eta_q \nabla N^{h}_{js}(q,a) \cdot \nabla \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, dx - \int_{\Omega} \eta_a N^{h}_{js}(q,a) \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, dx,
\]

due to the fact \( M^{h'}_{gs}(q,a)(\eta_q, \eta_a) \in V^{h}_{1,1} \). Consequently, we obtain that

\[
J^{h'}_g(q,a)(\eta_q, \eta_a) = \int_{\Omega} \eta_q \left| \nabla \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \right|^2 \, dx + \int_{\Omega} \eta_a \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right)^2 \, dx \\
- 2 \int_{\Omega} \eta_q \nabla N^{h}_{js}(q,a) \cdot \nabla \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, dx \\
- 2 \int_{\Omega} \eta_a N^{h}_{js}(q,a) \left( N^{h}_{js}(q,a) - M^{h}_{gs}(q,a) \right) \, dx.
\]

Therefore, we arrive at the following result.

**Lemma 5.2.** The differential of the functional \( J^{h}_g \) at \( (q,a) \in Q^{h}_{ad} \times A^{h}_{ad} \) in the direction \( (\eta_q, \eta_a) \in V^{h}_{1} \times V^{h}_{1} \) given by

\[
J^{h'}_g(q,a)(\eta_q, \eta_a) = \int_{\Omega} \eta_q \left( \left| \nabla M^{h}_{gs}(q,a) \right|^2 - \left| \nabla N^{h}_{js}(q,a) \right|^2 \right) \, dx + \int_{\Omega} \eta_a \left( \left| M^{h}_{gs}(q,a) \right|^2 - \left| N^{h}_{js}(q,a) \right|^2 \right) \, dx. \quad (5.1)
\]
The main computational challenge of the total variation regularization method is non-differentiable of the $BV$-semi-norm. To overcome this difficulty, we replace the total variation by a differentiable approximation

$$
\int_\Omega |\nabla q| \approx \int_\Omega \sqrt{|\nabla q|^2 + \epsilon^h} dx \quad \text{and} \quad \int_\Omega |\nabla a| \approx \int_\Omega \sqrt{|\nabla a|^2 + \epsilon^h} dx,
$$

where $\epsilon^h$ is a positive function of the mesh size $h$ satisfying $\lim_{h \to 0} \epsilon^h = 0$. Thus, the regularization term is approximated by

$$
R(q, a) \approx R^\epsilon(q, a) := \int_\Omega \sqrt{|\nabla q|^2 + \epsilon^h} dx + \int_\Omega \sqrt{|\nabla a|^2 + \epsilon^h} dx + \frac{1}{2} \|q\|_{L^2(\Omega)}^2 + \frac{1}{2} \|a\|^2_{L^2(\Omega)}.
$$

For all $(\eta_q, \eta_a) \in V_1^h \times V_1^h$ we get that

$$
R^\epsilon(q, a)(\eta_q, \eta_a) = \int_\Omega \frac{\nabla \eta_q \cdot \nabla q}{\sqrt{|\nabla q|^2 + \epsilon^h}} dx + \int_\Omega \frac{\nabla \eta_a \cdot \nabla a}{\sqrt{|\nabla a|^2 + \epsilon^h}} dx + \int_\Omega \eta_q q dx + \int_\Omega \eta_a a dx. \tag{5.2}
$$

The discrete cost functional $\Upsilon_{\delta, \rho}^h(q, a)$ of the problem $(P_{\delta, \rho}^h)$ is then approximated by

$$
\Upsilon_{\delta, \rho}^{h, \epsilon}(q, a) := J_{\delta}^h(q, a) + \rho R^\epsilon(q, a).
$$

**Lemma 5.3.** The differential of the approximated cost functional $\Upsilon_{\delta, \rho}^{h, \epsilon}$ at $(q, a) \in Q_{\text{ad}}^h \times A_{\text{ad}}^h$ in the direction $(\eta_q, \eta_a) \in V_1^h \times V_1^h$ fulfilled the identity

$$
\Upsilon_{\delta, \rho}^{h, \epsilon}(q, a)(\eta_q, \eta_a) = \int_\Omega \eta_q \left( |\nabla M_{\delta}^h(q, a)|^2 - |\nabla N_{\delta}^h(q, a)|^2 \right) dx + \int_\Omega \eta_a \left( |M_{\delta}^h(q, a)|^2 - |N_{\delta}^h(q, a)|^2 \right) dx + \rho \left( \int_\Omega \frac{\nabla \eta_q \cdot \nabla q}{\sqrt{|\nabla q|^2 + \epsilon^h}} dx + \int_\Omega \frac{\nabla \eta_a \cdot \nabla a}{\sqrt{|\nabla a|^2 + \epsilon^h}} dx + \int_\Omega \eta_q q dx + \int_\Omega \eta_a a dx \right).
$$

**Proof.** The affirmation directly follows from the definition of the functional $\Upsilon_{\delta, \rho}^{h, \epsilon}$ and the identities [5.1]−[5.2].

### 6 Numerical examples

Our numerical case study is the system

$$
\begin{align*}
-\nabla \cdot (q^I \nabla u) + a^I u &= f & \text{in} & \Omega, \tag{6.1} \\
q^I \nabla u \cdot \vec{n} + \sigma u &= j^I & \text{on} & \Gamma, \tag{6.2} \\
q^I \nabla u \cdot \vec{n} + \sigma u &= j_0 & \text{on} & \partial \Omega \setminus \Gamma, \tag{6.3} \\
u &= g^I & \text{on} & \Gamma \tag{6.4}
\end{align*}
$$

with the domain $\Omega = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid -1 < x_1, x_2 < 1 \}$ and the observation boundary $\Gamma := (-1, 1) \times \{-1\} \cup \{-1\} \times [-1, 1]$ (the bottom edge and the left edge).

The known functions are given as: the Robin coefficient $\sigma = 1$ on $\partial \Omega$, the Neumann data on $\partial \Omega \setminus \Gamma$

$$
j_0 := 4\chi_{\{1\} \times [-1, 1]} - 3\chi_{[-1, 1] \times \{1\}},
$$

and the source term

$$
f := \chi_D - \chi_{\Omega \setminus D},
$$

where $\chi_D$ is the characteristic function of the Lebesgue measurable set

$$
D := \{(x_1, x_2) \in \Omega \mid |x_1| + |x_2| \leq 1/2 \}.
$$
The sought diffusion and reaction coefficients \( q^\dagger \) and \( a^\dagger \) are respectively assumed to be discontinuous and given by
\[
q^\dagger := 2\chi_{\Omega_1^a} + \chi_{\Omega_2^a}^2 + 3\chi_{\Omega_3^a}
\]

with
\[
\Omega_1^a := (-1, -1/2) \times (-1, 1), \quad \Omega_2^a := (-1/2, 1/2) \times (-1, 1), \quad \Omega_3^a := (1/2, 1) \times (-1, 1)
\]

and
\[
a^\dagger := 3\chi_{\Omega_1^a} + 5\chi_{\Omega_2^a},
\]

where
\[
\Omega_1^a := (-1, 1) \times (-1, 0), \quad \Omega_2^a := (-1, 1) \times (0, 1).
\]

The exact Neumann data on \( \Gamma \) given by
\[
j^\dagger := -\chi_{(0,1)\times(-1)} + \chi_{(-1,0)\times(-1)} - 2\chi_{(-1)\times(-1,0)} + 3\chi_{(-1)\times(0,1)}
\]

and the exact Dirichlet data \( g^\dagger := \gamma_\Gamma \gamma_j(q^\dagger, a^\dagger) \), where \( \gamma \) is the Dirichlet trace operator on \( \Gamma \).

The constants appearing in the sets \( Q \) and \( A \) are chosen as \( q := a := 0.1 \) and \( \overline{q} := \overline{a} := 8 \). The interval \((-1, 1)\) is divided into \( \tau \) equal segments, the computational process will be started with \( \tau = 4 \), and so on for \( \tau = 8, 16, 32, 64 \). The regularization parameter \( \rho = \rho_\tau := 10^{-3}\sqrt{h_\tau} \), where the mesh size \( h = h_\tau = \sqrt{8}/\tau \). We utilize the variable metric projection type method which is described in [8] for reaching the numerical solutions of the problem \( \mathcal{F}_{\delta, \rho}^\tau \), where the initial approximations are the constant functions defined by \( q_0 = 1.5 \) and \( a_0 = 4 \).

The noisy observation data is assumed to be available in the form
\[
(j_\delta, g_\delta) = (j^\dagger + r\theta, g^\dagger + r\theta), \tag{6.6}
\]

where \( r \) is randomly generated in \((-1, 1)\).

With respect to the level \( \tau \), the pair \((q^\tau, a^\tau)\) is denoted the obtained numerical solution and errors
\[
E_{q,a} = \|q^\tau - I^{h^\tau}q^\dagger\|_{L^2(\Omega)} + \|a^\tau - I^{h^\tau}a^\dagger\|_{L^2(\Omega)}, \quad E_N = \|N_{\delta^\tau}^{h^\tau}(q^\tau, a^\tau) - N_{\delta^\tau}^{h^\tau}(q^\dagger, a^\dagger)\|_{L^2(\Omega)}, \quad E_M = \|M_{\delta^\tau}^{h^\tau}(q^\tau, a^\tau) - M_{\delta^\tau}^{h^\tau}(q^\dagger, a^\dagger)\|_{L^2(\Omega)}, \quad E_D = \|D_{\delta^\tau}^{h^\tau}(q^\tau, a^\tau) - D_{\delta^\tau}^{h^\tau}(q^\dagger, a^\dagger)\|_{L^2(\Omega)},
\]

where
\[
\delta^\tau = \begin{cases} g_\delta & \text{on } \Gamma, \\ \gamma_{\partial\Omega\setminus\Gamma} \gamma_j^{h^\tau}(q^\dagger, a^\dagger) & \text{on } \partial\Omega \setminus \Gamma \end{cases}
\]

and \( D_{\delta^\tau}^{h^\tau}(q, a) \) denotes the numerical solution of the problem \(-\nabla \cdot (q \nabla u) + au = f \) in \( \Omega \), supplemented with the Dirichlet boundary condition \( u = g \) on the boundary \( \partial\Omega \).

The numerical result is summarized in Table 1 where the measurement noise level \( \delta_\tau := \|j_\delta - j^\dagger\|_{L^2(\Gamma)} + \|g_\delta - g^\dagger\|_{L^2(\Gamma)} \) and \( \theta = h_\tau \sqrt{10\rho_\tau} \).

| \( \tau \) | \( \delta_\tau \) | \( E_{q,a} \) | \( E_N \) | \( E_M \) | \( E_D \) |
|---|---|---|---|---|---|
| 4 | 0.1912 | 2.3025 | 0.5545 | 0.3908 | 0.1827 |
| 8 | 7.0192e-2 | 0.7771 | 0.1508 | 0.1238 | 6.9520e-2 |
| 16 | 2.6801e-2 | 0.2712 | 6.9911e-2 | 4.9141e-2 | 2.4563e-2 |
| 32 | 1.0663e-2 | 0.1377 | 3.5084e-2 | 2.7613e-2 | 1.6002e-2 |
| 64 | 4.3377e-3 | 5.9782e-2 | 1.6918e-2 | 1.3122e-2 | 8.0575e-3 |

Table 1: Refinement level \( \tau \) and errors \( E_{q,a}, E_N, E_M, E_D \) corresponding the noise level \( \delta_\tau \).
Figure 1: Computed diffusion $q_\tau$ and the difference $I^{h^*}q^\dagger - q_\tau$.

Figure 2: Computed reaction $a_\tau$ and the difference $I^{h^*}a^\dagger - a_\tau$.

Figure 3: Differences $N^{h^*}_{g^\dagger}(q^\dagger, a^\dagger) - N^{h^*}_{g}(q_\tau, a_\tau)$ and $D^{h^*}_{g^\dagger}(q^\dagger, a^\dagger) - D^{h^*}_{g}(q_\tau, a_\tau)$. 
Next, we assume that \( I \) multiple measurements \( (j_i^g, q_i^g)_{i=1, \ldots, I} \) on \( \Gamma \) are available. With these datum at hand, we examine the minimization problem

\[
\min_{(q,a)\in Q_{\text{ad}} \times A_{\text{ad}}} \tilde{\mathcal{Y}}_{\delta, \rho}(q, a),
\]

where

\[
\tilde{\mathcal{Y}}_{\delta, \rho}(q, a) := \tilde{J}_{\delta}(q, a) + \rho R(q, a)
\]
and

\[
\tilde{J}_{\delta}(q, a) := \frac{1}{I} \sum_{i=1}^{I} \left( \int_{\Omega} q \left| \nabla (N_{j_i^g}^h(q, a) - M_{g_i^g}^h(q, a)) \right|^2 \, dx + \int_{\Omega} a (N_{j_i^g}^h(q, a) - M_{g_i^g}^h(q, a))^2 \, dx + \int_{\partial\Omega} \sigma (N_{j_i^g}^h(q, a) - M_{g_i^g}^h(q, a))^2 \, ds \right),
\]

which admits a minimizer \( (\tilde{q}_{\delta, \rho}^h, \tilde{a}_{\delta, \rho}^h) \).

Table 2 presents the errors, where with respect to \( \theta = 0.1 \), i.e. \( \delta = 0.2317 \). We observe that the use of multiple measurements improves the obtained numerical solutions in case of the large noise level.

| Number of measurements \( I \) | \( E_{q,a} \) | \( E_N \) | \( E_M \) | \( E_D \) |
|-------------------------------|----------------|---------|---------|---------|
| 1                            | 0.3921         | 0.1485  | 0.1121  | 5.214e-2|
| 6                            | 0.2849         | 8.4842e-2 | 6.0352e-2 | 3.8542e-2 |
| 16                           | 0.1609         | 4.7708e-2 | 3.3418e-2 | 1.9260e-2 |

Table 2: Multiple measurements \( I = 1, 6, 16 \) and the measurement error level \( \delta = 0.2317 \).

Figure 4: \( I = 1 \): differences \( I^h q^t - q^t \) and \( I^h a^t - a^t \).
Figure 5: $I = 6$: differences $I^{h^r} q^\dagger - q_{\tau}$ and $I^{h^r} a^\dagger - a_{\tau}$.

Figure 6: $I = 16$: differences $I^{h^r} q^\dagger - q_{\tau}$ and $I^{h^r} a^\dagger - a_{\tau}$.

References

[1] B. J. Adesokan, B. Jensen, B. Jin and K. Knudsen, Acousto-electric tomography with total variation regularization, *Inverse Problem* 35(2019), 035008 pp. 25.

[2] S. R. Arridge and W. R. B. Lionheart, Nonuniqueness in diffusion-based optical tomography, *Opt. Lett.*, 23(1998), pp. 882–884.

[3] K. Astala and L. Päivärinta, Calderón’s inverse conductivity problem in the plane, *Ann. Math.* 163(2006), pp. 265–299.

[4] H. Attouch, G. Buttazzo and G. Michaille, *Variational Analysis in Sobolev and BV Space*, Philadelphia: SIAM, 2006.

[5] S. Bartels, R. H. Nochetto and A. J. Salgado, Discrete TV flows without regularization, *SIAM J. Numer. Anal.* 52(2014), pp. 363–385.

[6] L. Blank and C. Rupprecht, An extension of the projected gradient method to a Banach space setting with application in structural topology optimization, *SIAM J. Control Optim.* 55(2017), pp. 1481–1499.
[7] L. M. Bregman, The relaxation of finding the common points of convex sets and its application to the solution of problems in convex programming, USSR Comput. Math. Math. Phys. 7(1967), pp. 200–217.

[8] S. Brenner and R. Scott, The Mathematical Theory of Finite Element Methods, New York: Springer, 1994.

[9] B. H. Brown, Electrical impedance tomography (EIT): a review, Journal of Medical Engineering & Technology 27(2003), pp. 97–108.

[10] R. M. Brown and R. H. Torres, Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in $L^p$, $p > 2n$, J. Fourier Anal. Appl. 9(2003), pp. 563–574.

[11] R. M. Brown and G. A. Uhlmann, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, Comm. Partial Differential Equations 22(1997), pp. 1009–1027.

[12] M. Burger and S. Osher, A guide to the TV zoo, in: Level-Set and PDE-based Reconstruction Methods, M. Burger, S. Osher, eds.: Springer, 2013.

[13] G. Chavent and K. Kunisch, Regularization of linear least squares problems by total bounded variation, SIAM Contr. Optim. Calc. Var. 2(1997), pp. 359–376.

[14] M. Cheney, D. Isaacson and J. C. Newell, Electrical impedance tomography, SIAM Rev. 41(1999), pp. 85–101.

[15] D. C. Dobson, Recovery of blocky images in electrical impedance tomography, in Inverse Problems in Medical Imaging and Nondestructive Testing, H. W. Engl, A. K. Louis, and W. Rundell, eds.: Springer, 1997, pp. 43–64.

[16] I. Ekeland and R. Temam, Convex Analysis and Variational Problems, New York: American Elsevier Publishing Company Inc., 1976.

[17] B. Harrach, On uniqueness in diffuse optical tomography, Inverse Problems 25 (2009), 055010 pp. 14.

[18] M. Hinze, B. Kaltenbacher and T. N. T. Quyen, Identifying conductivity in electrical impedance tomography with total variation regularization, Numerische Mathematik 138(2018), pp. 723–765.

[19] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. Math. 143(1996), pp. 71–96.

[20] K. Niinimäki, M. Lassas, K. Hämäläinen, A. Kallonen, V. Kolehmainen, E. Niemi and S. Siltanen, Multi-resolution parameter choice method for total variation regularized tomography, SIAM J. Imag. Sci. 9(2016), pp. 938–974.

[21] L. Päivärinta, A. Panchenko, and G. Uhlmann, Complex geometric optics solutions for Lipschitz conductivities, Rev. Mat. Iberoamericana 19(2003), pp. 57–72.

[22] T. N. T. Quyen, Finite element analysis for identifying the reaction coefficient in PDE from boundary observations, Appl. Numer. Math. 145(2019), pp. 297–314.

[23] L. I. Rudin, S. J. Osher and E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D 60(1992), pp. 259–268.

[24] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. Math. 125(1987), pp. 153–169.

[25] G. M. Troianiello, Elliptic differential equations and obstacle problems, New York: Plenum, 1987.