Intersection theoretic properties on the moduli space of genus 0 stable maps to a semipositive symplectic 4-manifold

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Abstract

We characterize transversality, non-transversality properties on the moduli space of genus 0 stable maps to a semipositive symplectic manifold of dimension 4, when GW([point], . . . , [point]) is enumerative. In particular, we show that the intersection theoretic property depends on the existence of a critical point on a stable map.

1 Introduction

The Gromov-Witten invariant is defined as an integration over a moduli space. Let $[a_i]$ be a cohomology class which is Poincarè dual to a point $a_i$, $i = 1, \ldots, c_1(f^*TX) - 1$, in a compact semipositive symplectic manifold $X$ of dimension 4. We call the Gromov-Witten invariant $GW([a_1], \ldots, [a_n])$ enumerative if it is a positive integer and counts the number of stable maps passing through $c_1(f^*TX) - 1$ points in general position. It implies the following:

- $ev_i^{-1}(a_i)$, $i = 1, \ldots, n := c_1(f^*TX) - 1$, meets transversally.
- The number of points in $\bigcap_{i=1}^n ev_i^{-1}(a_i)$ doesn’t vary depending on the general choice of configuration points $a_1, \ldots, a_n$.

In 2003, during his invitation, Gang Tian predicted the intersection theoretic properties on the moduli space of stable maps when the target space is $\mathbb{CP}^2$. His conjecture relates the non-transversality properties of the cycles $ev_i^{-1}(a_i)$, $i = 1, \ldots, n$, with the properties of the stable maps which represent...
the points in $\bigcap_{i=1}^{n} ev_i^{-1}(a_i)$. The main results in this paper prove his conjecture.

Usually, it is very hard to calculate the intersection multiplicity at an intersection point in $\bigcap_{i=1}^{n} ev_i^{-1}(s_i)$ straightforwardly. In this paper, we calculate the intersection multiplicities in a tricky way. Some technical motivations came from Tian’s suggestion. He related calculations to the deformation properties of stable maps. That is, an appearance of a skyscraper sheaf at a critical point can be used to show the intersection cycles’ non-transversality properties. Practically, studies on the local structure of a moduli space with a fixed generic almost complex structure $J$ and the singularity analysis of the product of the $i$-th evaluation maps enabled the author to prove his conjecture.

The main results of this paper are:

**Theorem 1.1** Let $n := c_1(f^*TX) - 1$. Let $f$ be in $\bigcap_{i=1}^{n} ev_i^{-1}(q_i)$, where $q_i$, $i = 1, \ldots, n$, are points in general position in the compact semipositive symplectic 4-manifold $X$. Then, the following holds.

(i) If $f$ is represented by a stable map which is an immersion and has an irreducible domain curve, then the intersection multiplicity at $f$ is one.

(ii) If $f$ is represented by a cuspidal stable map (Definition in 3.8) whose marked points are not critical points, then the intersection multiplicity at $f$ is two.

**Theorem 1.2** Let $n := c_1(f^*TX) - 1$. Let $X$ be a compact semipositive symplectic 4-manifold. The cuspidal stable maps locus is the unique equisingular locus in $M_n(X, \beta, J)$ of real codimension $\leq 2$ on which transversality uniformly fails.

This paper aims to exhibit the symplectic counterpart of the paper [Kwon2] in algebraic geometry category. For that purpose, we express the local structure of the moduli space algebraically. Preparations are done in sec 2 and sec 3.1. The author doesn’t claim any new results in sec 2 and sec 3.1. This paper’s main part starts from sec 3.2. In sec 3.2 we calculate the structure of the tangent space of $\overline{M}_n(X, \beta, J)$. In sec 4, we calculate the singularities of the $ev$ map and the index at the singularities. Some results in this section look similar to some results in [Shev]. However, the moduli space we consider in this paper is different. Shevchishin considered the total moduli
space and the singularity analysis were done on the total moduli space. If we fix the almost complex structure $J$ in the total moduli space in [Shev] as in this paper, then the moduli space Shevchishin worked become a set of discrete points where the singularity analysis is not possible. In Sec.5 we prove Tian’s conjecture on the transversality properties of the cycles $ev_i^{-1}(q_i)$, $i = 1, \ldots, c_1(f^*TX) - 1$.

2 Shevchishin’s D-cohomology group

Let $(X, \omega, J)$ be a compact semipositive symplectic $2n$-dimensional manifold with a $\omega$-compatible almost complex structure $J$ with $C^l$-smooth, $l \geq 2$. Assume $kp > 2$ and $k \leq l$. A parameter space of simple $J$-holomorphic maps, denoted by $P_{k,p}(X, \beta, J)$, is a space of continuous maps $f : \mathbb{CP}^1 \to X$ such that the $k$-th derivative of $f$ is of class $L^p$ and $f_*([\mathbb{CP}^1]) = \beta \in H_2(X; \mathbb{Z})$.

Let $L^p(\mathbb{CP}^1, \Lambda^{0,1}f^*TX)$ be a Banach space of $L^p$-integrable $f^*TX$-valued $(0,1)$-forms on $\mathbb{CP}^1$. Then, $\bigcup_f L^p(\mathbb{CP}^1, \Lambda^{0,1}f^*TX)$ is an infinite dimensional vector bundle over the space $P^{1,p}(X, \beta)$ of continuous maps $f : \mathbb{CP}^1 \to X$ of a class $L^{1,p}$ representing the homology class $\beta$. Let $\overline{\partial}_J$ be a complex anti-linear section on $P^{1,p}(X, \beta)$:

$$\overline{\partial}_J(f) := \frac{1}{2}(df + J \circ df \circ j) \in L^p(\mathbb{CP}^1, \Lambda^{0,1}f^*TX)$$

Then, the subspace $P^{1,p}(X, \beta, J)$ is an open subset of the intersection with the zero section $\overline{\partial}_J^{-1}(0)$.

The linearized operator $D_{f,J}$ of $\overline{\partial}_J$ is:

$$D_{f,J} : W^{1,p}(f^*TX) \to L^p(\mathbb{CP}^1, \Lambda^{0,1}f^*TX)$$

$$D_{f,J}(v) = \frac{1}{2}(\nabla v + J \circ \nabla v \circ j + \nabla_v J \circ df \circ j). \quad (2.1)$$

$D_{f,J}$ is an elliptic first order partial differential operator. The elliptic operator $D_{f,J}$ defines a two-step elliptic complex:

$$0 \to W^{1,p}(f^*TX) \xrightarrow{D_{f,J}} L^p(\Lambda^{0,1}f^*TX) \to 0. \quad (2.2)$$
The above two-step elliptic complex defines cohomology groups, which were named as \textit{D-cohomology groups} by Shevchishin.

\[ H^0_D(S, f^*TX) := \text{Ker}D_{f,J}, \quad H^1_D(S, f^*TX) := \text{Coker}D_{f,J}, \]

where \( S \) is a Riemann surface. \( H^0_D(S, f^*TX) \) and \( H^1_D(S, f^*TX) \) are finite dimensional vector spaces over \( \mathbb{R} \) because \( D_{f,J} \) is a Fredholm operator. We will call an element \( v \) in \( H^0_D(S, f^*TX) \) a pseudoholomorphic section.

\textbf{Remark 2.1} \( D \)-cohomology groups are defined with any \( k, p \) if \( kp > 2 \) and \( k \leq l \), where \( J \) is \( C^l \)-smooth. The elliptic regularity implies that \( H^0_D(\mathbb{CP}^1, f^*TX), H^1_D(\mathbb{CP}^1, f^*TX) \) don’t depend on the choice of the functional space. So, the resulting \( D \)-cohomology groups are independent of \( k, p \).

For generic \( J \), the linearized operator \( D_{f,J} \) at \( f \) is surjective if \( f \) is a simple map. It implies that the 1st \( D \)-cohomology group vanishes. The following Proposition is from the Implicit function theorem. See [M-S, Theorem A.3.3] or [Shev, Lemma 2.2.4].

\textbf{Proposition 2.2} \( P^{1,p}(X, \beta, J)^* \) is a smooth separable Banach manifold whose tangent space at \( f \) is:

\[ T_fP^{1,p}(X, \beta, J)^* = H^0_D(\mathbb{CP}^1, f^*TX) \]

\section{Local structures of the moduli space of genus 0 stable maps}

\subsection{D-cohomology Groups for Normal Sheaf}

\textbf{Lemma 3.1} Let \( X \) be a compact semipositive symplectic 4-manifold. Let \( f \) be a \( J \)-holomorphic map such that \( f_*([\mathbb{CP}^1]) \) is non-trivial. Let \( \mathcal{O}(V) \) be a sheaf of sections of a vector bundle \( V \). Then, the sequence of coherent sheaves

\[ 0 \to \mathcal{O}(T\mathbb{CP}^1) \stackrel{df}{\to} \mathcal{O}(f^*TX) \to \mathcal{O}(f^*TX)/df(\mathcal{O}(T\mathbb{CP}^1)) \to 0 \quad (3.1) \]

is exact.
Proof. It is obvious that the sequence is surjective to \( \mathcal{O}(f^*TX)/df(\mathcal{O}(T\mathbb{CP}^1)) \) and exact at \( \mathcal{O}(f^*TX) \). Thus, it is enough to show that \( df \) is injective. Since \( f \) is a \( J \)-holomorphic map and \( J \) is in the class \( C^l \), \( l \geq 2 \), the number of critical points of \( f \) is finite. See [M-W]. Thus, \( f \) is not locally constant by the unique continuation theorem. It implies that \( df(v) = 0 \) only if \( v \) is a trivial section. \( \square \)

Let \( c \in \mathbb{CP}^1 \) be a critical point of the \( J \)-holomorphic map \( f \). Let \( O(c) \) be an open neighborhood of \( c \) in \( \mathbb{CP}^1 \) and \( O(f(c)) \) be an open neighborhood of \( f(c) \). Then, by [M-S, Theorem E.1.1], there are \( C^2 \)-coordinate chart \( \phi : (O(c), c) \to (\mathbb{C}^1, 0) \) and a \( C^1 \)-coordinate chart \( \varphi : (O(f(c)), f(c)) \to (\mathbb{C}^2, 0) \) such that

- \( \varphi_*(J(f(c))) = J_0 \), where \( J_0 \) is a standard complex structure on \( \mathbb{C}^2 \).
- \( \phi(c) = 0 \) and \( \varphi(f(c)) = 0 \)
- \( f := \varphi \circ f \circ \phi^{-1} : \phi(O(c)) \to \mathbb{C}^2 \) is a polynomial in the variable \( z \).

The order at \( c \) is a unique integer \( k \) such that \( \varphi \circ f \circ \phi^{-1} \in O_k \setminus O_{k+1} \). Let \( o(c) \) denote [(the order at \( c \))-1].

Let \( H_c \) be a vector space defined as follows:

\[
H_c := \{ tv \mid t \in \mathbb{C}^1, \frac{f(z_i)}{|f(z_i)|} \to v \text{ as } z_i \to 0 \} \subset \mathbb{C}^2
\]

\( H_c \) is called a tangent cone at \( c \), which will be denoted by \( C_c \). By considering a normal coordinate chart, we can regard the tangent cone \( C_c \) at \( c \) as a subspace of \( T_{f(c)}X \). \( C_c \) is preserved by a \( J \)-action. See [M-S, p626]. The following Lemma is from [Shev, Sec.1.5].

**Lemma 3.2** Let \( c_i \) be a critical points of a \( J \)-holomorphic map \( f \). Then,

\[
\mathcal{O}(f^*TX)/df(\mathcal{O}(T\mathbb{CP}^1)) \cong N_f \oplus \bigoplus_i C_c^{o(c_i)}
\]

where \( N_f \) is a locally free sheaf.
$J$ structure on $f^*TX$ induces a $J$ structure on $N_f$ since $f$ is a $J$-holomorphic map. The elliptic operator $D_{f,J}$ induces the elliptic operator $D_{f,J}^N$ on $N_f$. Thus, we get a two-step elliptic complex:

$$0 \rightarrow W^{1,p}(N_f) \oplus \bigoplus_i C^0_{c_i(c_i)} \xrightarrow{D_{f,J}^N} L^p(\Lambda^{0,1}N_f) \oplus \bigoplus_i ((\Lambda^{0,1}c_i) \otimes C^0_{c_i(c_i)}) \rightarrow 0$$

(3.2)

$$(v, w) \mapsto (D_{f,J}(v), 0)$$

Note that $(\Lambda^{0,1}c_i) \otimes C^0_{c_i(c_i)}$ is trivial because $\Lambda^{0,1}c_i$ is trivial. The resulting non-trivial $D$-cohomology groups are the following:

$$H^0_D(CP^1, N_f \oplus \bigoplus_i C^0_{c_i(c_i)}) := \text{Ker} D_{f,J}^N \oplus \bigoplus_i C^0_{c_i(c_i)}$$

$$H^1_D(CP^1, N_f \oplus \bigoplus_i C^0_{c_i(c_i)}) := \text{Coker} D_{f,J}^N \cong H^1_D(CP^1, N_f)$$

There is a long exact sequence of $D$-cohomology group induced from the short exact sequence (3.1).

**Lemma 3.3** [Shev, Proposition 2.4.2] The following sequence is an exact sequence of $D$-cohomology groups:

$$0 \rightarrow H^0(CP^1, T_{CP^1}) \rightarrow H^0_D(CP^1, f^*TX) \rightarrow H^0_D(CP^1, N_f \oplus \bigoplus_i C^0_{c_i(c_i)}) \rightarrow$$

$$\rightarrow H^1(CP^1, T_{CP^1}) \rightarrow H^1_D(CP^1, f^*TX) \rightarrow H^1_D(CP^1, N_f) \rightarrow 0$$

Proof. The Lemma follows by applying the Snake Lemma and the definition of the $D$-cohomology groups to the following exact sequence of two-step elliptic complexes:

$$0 \rightarrow W^{1,p}(T_{CP^1}) \rightarrow W^{1,p}(f^*TX) \rightarrow W^{1,p}(N_f \oplus \bigoplus_i C^0_{c_i(c_i)}) \rightarrow 0$$

$$\bigoplus \downarrow \quad D_{f,J} \downarrow \quad D_{f,J}^N \downarrow$$

$$0 \rightarrow L^p(\Lambda^{0,1}T_{CP^1}) \rightarrow L^p(\Lambda^{0,1}f^*TX) \rightarrow L^p(\Lambda^{0,1}N_f) \rightarrow 0$$

$\blacksquare$
3.2 Tangent Space Splitting Theorem

Let \( P(X, \beta, J) \) be the parameter space of genus zero, smooth \((C^\infty)\), simple \(J\)-holomorphic maps, representing the homology class \( \beta \). Let \( \overline{M}_n(X, \beta, J) \) be the moduli space of genus zero, \( n \)-pointed, smooth \(J\)-holomorphic stable maps, representing the homology class \( \beta \). Let \( f: C \to X \) be a stable map from a reducible (arithmetic) genus 0 curve \( C \). Then, we call \( f \) a simple map if the map restricted to each irreducible component is simple and none of the irreducible components have the same image. We will denote the subset of irreducible, simple stable maps in \( M_n(X, \beta, J) \) by \( M_n(X, \beta, J) \), and the subset of any simple stable maps in \( M_n(X, \beta, J) \) by \( M_n(X, \beta, J) \). \( \overline{M}_n(X, \beta, J) \setminus M_n(X, \beta, J) \) consists of multiple cover stable maps and stable maps having reducible domain curves.

**Proposition 3.4** Let \( \text{Aut}(\mathbb{C}P^1) \) act on \( P(X, \beta, J) \) by \( f \mapsto f \circ \varphi^{-1} \), where \( \varphi \in \text{Aut}(\mathbb{C}P^1) \). Then, the following holds:

i. the quotient space \( P(X, \beta, J)/\text{Aut}(\mathbb{C}P^1) \) is diffeomorphic to \( M_0(X, \beta, J) \).

ii. The tangent space at \( [(f, \mathbb{C}P^1)] \in M_0(X, \beta, J) \) is \( H^0_D(\mathbb{C}P^1, N_f) \cong H^0_D(\mathbb{C}P^1, N_f) \oplus \bigoplus c_i \mathbb{C}o(c_i) \), where \( N_f, N_f, c_i \) are the normal sheaf, the normal bundle, the critical point of \( f \) respectively.

Proof. i. is obvious and well-known. See [M-S, Chap.6]. Let’s prove ii. Since \( P(X, \beta, J) \) is a smooth manifold, the action of \( \text{Aut}(\mathbb{C}P^1) \) is smooth, free and proper. Let \( (f, \mathbb{C}P^1) \) be an element in \( P(X, \beta, J) \) and \( O(f) \) be an orbit of \( (f, \mathbb{C}P^1) \). By [G-G-K, Lemma B.19], \( O(f) \) is a smooth, embedded submanifold. [G-G-K, Lemma B.20] implies that the tangent space of the orbit \( O(f) \) at \( (f, \mathbb{C}P^1) \) is the Lie algebra \( H^0(\mathbb{C}P^1, T\mathbb{C}P^1) \) of \( \text{Aut}(\mathbb{C}P^1) \). The result follows from the short exact sequence of \( D \)-cohomology groups (cf.Lemma 3.3):

\[
0 \to H^0(\mathbb{C}P^1, T\mathbb{C}P^1) \to H^0_D(\mathbb{C}P^1, f^*TX) \to H^0_D(\mathbb{C}P^1, N_f) \to 0
\]

and the diffeomorphism in i. \( \square \)

**Remark 3.5** The tangent space at \( [(f, \mathbb{C}P^1)] \) is \( H^0_D(\mathbb{C}P^1, N_f) \) if and only if \( f \) is an immersion.
Lemma 3.6 Let $F^n_i$ be an $i$-th forgetful map. The $i$-th forgetful map $F^n_i : M_n(X, \beta, J)^* \to M_{n-1}(X, \beta, J)^*$ is a submersion for any $n \geq 1$.

Proof. Let $[(f, \mathbb{CP}^1, a_1, \ldots, a_n)]$ be in $M_n(X, \beta, J)^*$. Let $\lambda : (-\epsilon, \epsilon) \to M_{n-1}(X, \beta, J)^*$ be a smooth path such that $\lambda(t) = [(f^t, \mathbb{CP}^1, a_1^t, \ldots, a_n^t)]$, $\lambda(0) = [(f, \mathbb{CP}^1, a_1, \ldots, a_n)]$ and $d\lambda(\frac{\partial}{\partial t} |_{t=0}) \neq 0$. There is a canonical lifting $\tilde{\lambda}$ of the path defined by $\tilde{\lambda}(t) = [(f^t, \mathbb{CP}^1, a_1^t, \ldots, a_n^t, a_n)]$. We have $\tilde{\lambda}(0) = [(f, \mathbb{CP}^1, a_1, \ldots, a_n)]$, $d\tilde{\lambda}(\frac{\partial}{\partial t} |_{t=0}) \neq 0$ and $d\lambda(\frac{\partial}{\partial t} |_{t=0}) = dF^n_i \circ d\tilde{\lambda}(\frac{\partial}{\partial t} |_{t=0})$. Other $i$-th forgetful maps can be proved in the same way. So, the Lemma follows. \hfill \blacksquare

Proposition 3.7 The tangent space at $[(f, \mathbb{CP}^1, a_1, \ldots, a_n)]$ in $M_n(X, \beta, J)^*$ is $H^*_D(\mathbb{CP}^1, N_f) \oplus \bigoplus_{c_i} \mathbb{C}c_i^{(c_i)} \oplus \bigoplus_{a_1} T_{a_1} \mathbb{CP}^1$, where $c_i$ is a critical point of $f$.

Proof. Assume the Proposition holds for $n-1 \geq 0$. Since $F^n_i$ is a submersion by Lemma 3.6, $(F^n_i)^{-1}([(f, \mathbb{CP}^1, a_1, \ldots, a_{n-1})])$ is smooth and diffeomorphic to $\mathbb{CP}^1 \setminus \{a_1, \ldots, a_{n-1}\}$ by $[(f, \mathbb{CP}^1, a_1, \ldots, a_{n-1})] \mapsto a_n$. Thus, the tangent space of the fibre $(F^n_i)^{-1}([(f, \mathbb{CP}^1, a_1, \ldots, a_{n-1})])$ at $[(f, \mathbb{CP}^1, a_1, \ldots, a_n)]$ is isomorphic to $T_{a_n} \mathbb{CP}^1$. The Proposition follows by the induction assumption and Proposition 3.4. \hfill \blacksquare

The type of stable maps in Definition 3.8 are based on the types of singularities on the image curve $f(\mathbb{CP}^1)$.

Definition 3.8 Let $f := (f, \mathbb{CP}^1, a_1, \ldots, a_n)$ be a simple pointed stable map. The node singularity of the stable map is the point $p \in f(\mathbb{CP}^1)$ such that $f^{-1}(p)$ consists of two points $p_1, p_2$ and $df(T_{p_1} \mathbb{CP}^1) \cap df(T_{p_2} \mathbb{CP}^1) = \{0\}$. We call $f$ a nodal stable map if the image curve $f(\mathbb{CP}^1)$ has only node singularities. The tacnode singularity of the stable map is the point $p \in f(\mathbb{CP}^1)$ such that $f^{-1}(p)$ consists of two points $p_1, p_2$ and $df(T_{p_1} \mathbb{CP}^1) = df(T_{p_2} \mathbb{CP}^1)$. We call $f$ a tacnode stable map if the singularities of the image curve $f(\mathbb{CP}^1)$ has a unique tacnode singularity and all other singularities in $f(\mathbb{CP}^1)$ are node singularities. The triple node singularity of the stable map is the point $p \in f(\mathbb{CP}^1)$ such that $f^{-1}(p)$ consists of three points $p_1, p_2, p_3$ and $df(T_{p_1} \mathbb{CP}^1) \cap df(T_{p_j} \mathbb{CP}^1) = \{0\}$ if $i \neq j$. We call $f$ a triple node stable map if $f$ has a unique triple node singularity and all other singularities are node singularities. The
A cuspidal singularity \( p \in f(\mathbb{CP}^1) \) is the image of an order 2 singularity of \( f \). We call the stable map \( f \) a cuspidal stable map if \( f(\mathbb{CP}^1) \) has a unique cuspidal singularity and all other singularities are node singularities. We will call the two stable \( J \)-holomorphic maps \( f, g \) are equi-singular if they have the same number of the irreducible components on the domain curve and they have the same type of singularities in their image curves.

Readers may refer to [Kwon2] for the local figures of singularities in \( f(\mathbb{CP}^1) \).

**Proposition 3.9** The nodal stable maps locus is open in \( \overline{M}_n(X, \beta, J) \).

Proof. The transversality is an open condition. Thus, a small perturbation of a nodal stable map doesn’t change the type of singularities in the image curve. A small perturbation doesn’t create other type of singularities because of the Symplectic adjunction formula and the type of singularities in the image curve of a nodal stable map. The result is obvious. \( \square \)

**Proposition 3.10** [Shen, Theorem 3.2.1]. The cuspidal stable map’s locus is a real codimension two locus in \( \overline{M}_n(X, \beta, J) \).

**Remark 3.11** (Deformation property near the degree 2 singularity) Let \( f \) be a cuspidal stable map with a critical point at \( c \). Then, the tangent space at \( f := [(f, \mathbb{CP}^1)] \) is \( H^0_D(\mathbb{CP}^1, N_f) \oplus \mathbb{C}_c \). After the coordinate changes on the neighborhood of \( c \) in the domain curve and the neighborhood of \( f(c) \) in the target space, \( f \) is represented by \( z \mapsto (z^2, z^3) \). Since \( H^0_D(\mathbb{CP}^1, N_f) \) parameterizes a first order deformation, any smooth path \( \xi : (-\epsilon, \epsilon) \to \overline{M}_n(X, \beta, J)^* \) tangential to a vector in \( H^0_D(\mathbb{CP}^1, N_f) \) and \( \xi(0) = f \) parameterizes a first order deformation \( f_\eta(z) = (z^2, z^3 + \eta z) \) for \( \eta \in (-\epsilon, \epsilon) \) after the coordinate changes. One can check that \( f_\eta \) has a node singularity if \( \eta \neq 0 \). Thus, Proposition 3.10 shows that \( \mathbb{C}_c \) is tangential to the cuspidal stable maps locus. By the definition of the cuspidal stable map, \( \mathbb{C}_c \) parameterizes a second order deformation near the singularity \( c \).

**Lemma 3.12** Let \( f := [(f, \mathbb{CP}^1, a_1, \ldots, a_n)] \) be in \( \overline{M}_n(X, \beta, J)^* \). Let \( ev_i \) denote the \( i \)-th evaluation map, defined by \( ev_i(f) = f(a_i) \). Then, the differential
of the \(i\)-th evaluation map at \(\mathbf{f}\) is given as follows:

\[
[H^0_\beta(\mathbb{CP}^1, N_f) \oplus \bigoplus_{c_j} C_{c_j}] \oplus \bigoplus_{a_j} T_{a_j} \mathbb{CP}^1 \to T_{f(a_i)} X
\]

\[(v_1 + v_2) + w \mapsto v_1(a_i) + df_{a_i}(\pi_i(w)) \tag{3.3}\]

where \(v_1, v_2, w\) belong to \(H^0_\beta(\mathbb{CP}^1, N_f)\), \(\bigoplus_{c_j} C_{c_j}\), \(\bigoplus_{a_j} T_{a_j} \mathbb{CP}^1\) respectively, and \(\pi_i : \bigoplus_{a_j} T_{a_j} \to T_{a_i} \mathbb{CP}^1\) is a natural projection map to the \(i\)-th component.

Proof. Since \(v_1, v_2, w\) are independent vectors, it is enough to check \(v_1 \mapsto v_1(a_i), v_2 \mapsto 0, w \mapsto df_{a_i}(w)\). Let \(\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}_n(X, \beta, J)^\ast\) be a smooth path tangential to \(v_1\) and \(\gamma(0) = \mathbf{f}\). \(d(ev_i \circ \gamma)(\frac{\partial}{\partial t} |_{t=0}) = \frac{d}{dt} \exp(t \cdot v_1(a_i)) = \pi_i(v_1)\). \(C_{c_j}\) parameterizes the 2nd order deformation. Thus, \(v_2 \mapsto 0\). It is obvious that if \(w \in T_{a_j} \mathbb{CP}^1, j \neq i\), then \(dev_i(w) = 0\) because \(T_{a_j} \mathbb{CP}^1\) parameterizes the deformation of the \(j\)-th marked point while we consider the \(i\)-th evaluation map. Let \(\lambda : (-\varepsilon, \varepsilon) \to \mathbb{CP}^1\) be a smooth path such that \(\lambda(0) = a_i, d\lambda(\frac{\partial}{\partial t} |_{t=0}) = w \in T_{a_i} \mathbb{CP}^1\). Then, the induced path \(ev_i \circ \lambda : (-\varepsilon, \varepsilon) \to X\) satisfies that \(ev_i \circ \lambda(0) = f(a_i)\) and \(d(ev_i \circ \lambda)(\frac{\partial}{\partial t} |_{t=0}) = df_{a_i}(\pi_i(w))\). \(\square\)

Let \(C\) be a pointed reducible curve which has two irreducible components \((C_1, a_1, \ldots, a_r), (C_2, b_1, \ldots, b_s)\), where \(r + s = n\). Let \(q_1 \in C_1, q_2 \in C_2\) be pregluing points. Then, a pointed stable map \((f, C, a_1, \ldots, a_r, b_1, \ldots, b_s)\) can be written as \(((f_1, (C_1, q_1), a_1, \ldots, a_r), (f_2, (C_2, q_2), b_1, \ldots, b_s))\).

Let \(\beta_1 \bigcup^r_s \beta_2\) denote the set of stable maps in \(\mathcal{M}_n(X, \beta, J)\) such that

- \([[(f_1, (C_1, q_1), a_1, \ldots, a_r), (f_2, (C_2, q_2), b_1, \ldots, b_s)]\), where \(C_i \cong \mathbb{CP}^1\)
- \(f_s([C_1]) = \beta_1, \quad f_s([C_2]) = \beta_2\)

\(\beta_1 \bigcup^r_s \beta_2\) is generically smooth and forms a (real) codimension two subspace in \(\mathcal{M}_n(X, \beta, J)\).

**Proposition 3.13** Let’s consider a stable map

\(f_1 \bigcup f_2 := [[[f_1, (C_1, q_1), a_1, \ldots, a_r), (f_2, (C_2, q_2), b_1, \ldots, b_s)]]) \in \beta_1 \bigcup^r_s \beta_2\).

(i) Let’s assume that \(f_i, i = 1, 2\), is an immersion. Then, the tangent space splitting at \(f_1 \bigcup f_2\) is:
\[ H^0_D(C_1, N_1) \oplus H^0_D(C_2, N_2) \oplus T_{a_1}C_1 \oplus \ldots \oplus T_{a_r}C_1 \oplus T_{b_1}C_2 \oplus \ldots \oplus T_{b_s}C_2 \oplus (T_{q_1}C_1 \otimes T_{q_2}C_2) \oplus T_{q_1}C_1 \oplus T_{q_2}C_2 \oplus T_{f(q)}X \]

where \( N_i \) is a normal bundle and \( q \) is a node in \( C \).

(ii) Let’s assume \( f_1([C_1]) \) is trivial and \( f_2 \) is an immersion. Then, the tangent space splitting at \( f_1 \vee f_2 \) is:

\[ H^1(C_1, TC_1(-q_1 - a_1 - \ldots - a_r)) \oplus H^0_D(C_2, N_2) \oplus T_{b_1}C_2 \oplus \ldots \oplus T_{b_s}C_2 \oplus (T_{q_1}C_1 \otimes T_{q_2}C_2) \oplus T_{q_2}C_2 \]

Proof. (i) Let \( M_{r+1}(X, \beta_1, J)^{**} \), \( M_{s+1}(X, \beta_2, J)^{**} \) be the open subset of \( M_{r+1}(X, \beta_1, J) \), \( M_{s+1}(X, \beta_2, J) \) consisting of stable maps without critical points. Let’s consider a smooth map \( ev_{\beta_1} \times ev_{\beta_2} : \)

\[ M_{r+1}(X, \beta_1, J)^{**} \times M_{s+1}(X, \beta_2, J)^{**} \to X \times X \]

\[ ([f_1, (C_1, q_1), a_1, \ldots, a_r], [(f_2, (C_2, q_2), b_1, \ldots, b_s)]) \mapsto (f_1(q_1), f_2(q_2)) \]

Let \( (\beta_1 \vee \beta_2)^{**} \) be the subset of \( \beta_1 \vee \beta_2 \) consisting of the stable maps whose restriction to each irreducible component is an immersion. Then, \( (\beta_1 \vee \beta_2)^{**} \) is diffeomorphic to \( ev_{\beta_1}^{-1} \times ev_{\beta_2}^{-1} \{ \{(q, q) \mid q \in X \} \} \). Let’s denote \( [(f_1, (C_1, q_1), a_1, \ldots, a_r)], [(f_2, (C_2, q_2), b_1, \ldots, b_s)] \) by \( f_1, f_2 \) respectively. By Proposition 3.7, we have:

\[ T_{f_1}M_{r+1}(X, \beta_1, J)^{**} \cong H^0_D(C_1, N_1) \oplus T_{q_1}C_1 \oplus \bigoplus_{a_i} T_{a_i}C_1 \quad (3.4) \]

\[ T_{f_2}M_{s+1}(X, \beta_2, J)^{**} \cong H^0_D(C_1, N_2) \oplus T_{q_2}C_2 \oplus \bigoplus_{b_i} T_{b_i}C_2 \quad (3.5) \]

The differential \( dev_{\beta_1} \times dev_{\beta_2} \) at \((f_1, f_2)\) is as follows:

\[ T_{f_1}M_{r+1}(X, \beta_1, J)^{**} \times T_{f_2}M_{s+1}(X, \beta_2, J)^{**} \to T_{f(q_1)}X \times T_{f(q_2)}X \]

\[ (\nu + \xi + \zeta, \nu' + \xi' + \zeta') \mapsto (\nu(q_1) + df_{p_1}(\xi), \nu'(q_2) + df_{p_2}(\xi')) \]

\[ 11 \]
where $v, v'$ are elements in $H^0_D(C_1, N_f_1)$, $H^0_D(C_2, N_f_2)$ respectively; $\xi, \xi'$ are elements in $T_q C_1, T_q C_2$ respectively; $\varsigma, \varsigma'$ are elements in $\bigoplus_a T_a C_1, \bigoplus_b T_b C_2$ respectively. By Lemma 3.12 $ev_{\beta_1}$ is a submersion. Thus, the differential $dev_{\beta_1}$ is surjective. Let $(f_1, f_2) = f_1 \vee f_2$, i.e., $f_1(q_1) = f_2(q_2)$. By combing with the natural surjective map $T_{f(q)}X \times T_{f(q)}X \rightarrow T_{f(q)}X$, $(w, w') \mapsto w - w'$, we get the following surjective map:

$$
T_{f_1^r} M_{r+1}(X, \beta_1, J)^{**} \times T_{f_2^s} M_{s+1}(X, \beta_2, J)^{**} \rightarrow T_{f(q)}X \quad (3.6)
$$

$$(v + \xi + \varsigma, v' + \xi' + \varsigma') \mapsto v(q_1) + df_{q_1}(\xi) - v'(q_2) - df_{q_2}(\xi')$$

By Implicit Function Theorem, $(\beta_1 \vee \beta_2)^{**}$ is smooth manifold of dimension $\dim M_{r+1}(X, \beta_1, J)^{**} + \dim M_{s+1}(X, \beta_2, J)^{**} - \dim X$ and the kernel of the map (3.6) is the tangent space of $(\beta_1 \vee \beta_2)^{**}$ at $(f_1 \vee f_2)^{**}$. The statement (i) is from a $K$-group expression of the orthogonal decomposition induced by the following short exact sequence (3.7), combining with (3.4), (3.5). The term $T_{q_1} C_1 \otimes T_{q_2} C_2$ parameterizes a smoothing node deformation.

$$
0 \rightarrow T_{f_1} \vee f_2 (\beta_1^r \vee \beta_2^s) \rightarrow 
T_{f_1^r} M_{r+1}(X, \beta_1, J)^{**} \times T_{f_2^s} M_{s+1}(X, \beta_2, J)^{**} \rightarrow T_{f(q)}X \rightarrow 0 \quad (3.7)
$$

(ii) Since $f_1^*([C_1])$ is trivial and a stable map, the irreducible component $C_1$ has at least 3 marked points. The map $M_{r+1}(X, 0, J) \rightarrow M_{r+1} \times X$, $f_1 \mapsto ([([C_1, q_1, a_1, \ldots, a_r]), f_1(q_1))]$ is a diffeomorphism, where $M_{r+1}$ is the Deligne-Mumford moduli space of $r + 1$-pointed smooth genus 0 curves. The tangent space at $([C_1, q_1, a_1, \ldots, a_r])$ in $M_{r+1}$ is $H^1(C_1, TC_1(-q_1 - a_1 - \ldots - a_r))$ which is the space of first order deformations of a pointed smooth Riemann surface. Thus, the tangent space at $f_1$ is:

$$
T_{f_1} M_{r+1}(X, 0, J) \cong H^1(C_1, TC_1(-q_1 - a_1 - \ldots - a_r)) \oplus T_{f(q)}X \quad (3.8)
$$

With the same notation we used in (i) for $v', \xi', \varsigma'$ and the similar argument, we get a submersion map:

$$
T_{f_1 M_{r+1}(X, \beta_1, J)} \times T_{f_2 M_{s+1}(X, \beta_2, J)^{**}} \rightarrow T_{f(q)}X \quad (3.9)
$$

$$(\varsigma + \xi, v' + \xi' + \varsigma') \mapsto \xi - v'(q_2) - df_{q_2}(\xi')$$
where \( \varsigma \in H^1(C_1, TC_1(-q_1 - a_1 - \ldots - a_r)), \xi \in T_{f(q)}X \). The rest of arguments are the same with (i).

Repeated similar calculations result in the following Theorem. See [Kwon 1] for the calculation in algebraic geometry.

**Theorem 3.14** (Tangent Space Splitting Theorem)

Let \([(f, C, a_1, \ldots, a_n)]\) be a point in \(\overline{M}_n(X, \beta, J)\) such that \(f\) is an immersion on each irreducible component. Let \(\tilde{p} : \tilde{C} := \bigsqcup_{i=1}^r C_i \to C\) be a normalization map of \(C\). Then, the tangent space splitting at \([(f, C, a_1, \ldots, a_n)]\) is:

\[
\bigoplus_{i=1}^r H^0_D(C_i, N_f) \oplus \bigoplus_{i=1}^n T_{a_i}C_i \oplus \left[ \bigoplus_{i=1}^\delta (T_{p_i}C_{\nu(p_i)} \otimes T_{p_i'}C_{\nu(p_i')}) \right] \oplus \bigoplus_{i=1}^\delta [T_{f(q)}X] \quad (3.10)
\]

where \( \delta \) is the number of gluing points, \( p_i \in C_{\nu(p_i)}, p_i' \in C_{\nu(p_i')}, i = 1, \ldots, \delta, \{\nu(p_i), \nu(p_i')\} \subset \{1, \ldots, r\} \) are pregluing points such that \(\tilde{p}(p_i) = \tilde{p}(p_i')\), and \(q_i, i = 1, \ldots, \delta\), is a gluing point.

### 4 Singularity Analysis and Calculations of Ramification Indices

**Lemma 4.1** Let \(f := [(f, \mathbb{CP}^1, a_1, \ldots, a_n)] \in \overline{M}_n(X, \beta, J)\) be a point represented by a stable map with \(l(\geq 0)\) singular points of order 2. The kernel of the differential \(\text{dev} := d(ev_1 \times \ldots \times ev_n)\) at \(f\) is isomorphic to \(H^0_D(\mathbb{CP}^1, N_f(-a_1 - \ldots - a_n)) \oplus \bigoplus_{j=1}^l \mathbb{C}c_j\), where \(a_i, i = 1, \ldots, n\), is a regular point of \(f\).

Proof. By Lemma [3.12] it is clear that \(\bigoplus_{j=1}^l \mathbb{C}c_j\) is in the kernel of \(\text{dev}\). \(v_1(a_i), df(\pi_i(w))\) in (3.3) are independent vectors. Thus, \(\text{dev}(v_1 + v_2 + w) = 0\) if and only if \(v_1(a_i) = 0\) and \(df(\pi_i(w)) = 0\) for all \(i = 1, \ldots, n\). If \(v_1(a_i) = 0\) for \(i = 1, \ldots, n\), then \(v_1 \in H^0_D(\mathbb{CP}^1, N_f(-a_1 - \ldots - a_n))\) and vice versa. Since \(a_i\) is a non-singular point of \(f, \pi_i(w)\) is a zero vector iff \(df(\pi_i(w))\) vanishes. Thus, the result follows. \(\square\)
Proposition 4.2 Let \( n := c_1(f^*TX) - 1 \). Let \( f := [(f, \mathbb{C}P^1, a_1, \ldots, a_n)] \) be a point in \( \overline{M}_n(X, \beta, J) \) such that \( f \) is an immersion. Then, the map \( \text{ev} := ev_1 \times \ldots \times ev_n \) is regular at \( f \).

Proof. Simple dimension count shows \( \dim H^0_D(\mathbb{C}P^1, N_f(-a_1 - \ldots - a_n)) = 0 \). Therefore, \( \text{dev} \) is one-to-one at \( f \). Since \( T_f \overline{M}_n(X, \beta, J) \) is a finite dimensional vector space, the result follows from Lemma 4.1. \( \square \)

Nodal stable map, triple node stable map, and tac node stable map are immersions. So, we get the following Corollary.

Corollary 4.3 Let \( n := c_1(f^*TX) - 1 \). Let \( f := [(f, \mathbb{C}P^1, a_1, \ldots, a_n)] \) be a point in \( \overline{M}_n(X, \beta, J) \). If \( f \) is a nodal stable map, triple node stable map, or tac node stable map, then the map \( \text{ev} := ev_1 \times \ldots \times ev_n \) is regular at \( f \).

Lemma 4.4 Let \( n := c_1(f^*TX) - 1 \). The cokernel of \( \text{dev} \) at \( f := [(f, \mathbb{C}P^1, a_1, \ldots, a_n)] \) \( \in \overline{M}_n(X, \beta, J) \) is isomorphic to \( H^1_D(\mathbb{C}P^1, N_f(-a_1 - \ldots - a_n)) \), where \( N_f \) is the normal bundle and \( a_i, i = 1, \ldots, n \), is a non-singular point of \( f \).

Proof. By Lemma 3.12 we have

\[
\text{coker} \text{dev} \cong \frac{\bigoplus_{i=1}^n T_{f(a_i)}X}{\text{dev}(H^0_D(\mathbb{C}P^1, N_f)) \oplus \text{dev}(\bigoplus_{i=1}^n T_{a_i} \mathbb{C}P^1)}
\]

Let \( B \) and \( C \) be subvector spaces of the vector space \( A \) such that \( B \cap C = \{0\} \). Then, one can easily check the elementary isomorphism \( \frac{A}{B \oplus C} \cong \frac{C^\perp \oplus C}{B \oplus C} \cong \frac{C^\perp}{B} \), where \( C^\perp \) is an orthogonal complement of \( C \) in \( A \).

Consider the short exact sequence of sheaves

\[
0 \to \mathcal{O}(N_f(-a_1 - \ldots - a_n)) \to \mathcal{O}(N_f) \to \bigoplus_{i=1}^n \nu_i \to 0, \quad (4.1)
\]

where \( \nu_i \) is a skyscraper sheaf supported by \( a_i, i = 1, \ldots, n \).

Since \( \text{dev}(H^0_D(\mathbb{C}P^1, N_f)) \cap \text{dev}(\bigoplus_{i=1}^n T_{a_i} \mathbb{C}P^1) \cong \{0\} \), we get
\[
coker dev \cong \left( \bigoplus_{i=1}^{n} N_{a_i} \right) / dev(H_D^0(\mathbb{CP}^1, N_f)) \quad (4.2)
\]
\[
\cong H_D^0(\mathbb{CP}^1, \bigoplus_{i=1}^{n} \nu_i) / dev(H_D^0(\mathbb{CP}^1, N_f)), \quad (4.3)
\]

By Lemma 3.3, (4.1) induces a long exact sequence of D-cohomology groups:

\[
0 \to H_D^0(\mathbb{CP}^1, N_f(-a_1 - \ldots - a_n)) \to H_D^0(\mathbb{CP}^1, N_f) \to
\]
\[
\to H_D^0(\mathbb{CP}^1, \bigoplus_{i=1}^{n} \nu_i) \to H_D^1(\mathbb{CP}^1, N_f(-a_1 - \ldots - a_n)) \to
\]
\[
\to H_D^1(\mathbb{CP}^1, N_f) \to \ldots .
\]

By Riemann-Roch’s Theorem, \(H_D^0(\mathbb{CP}^1, N_f(-a_1 - \ldots - a_n))\) and \(H_D^1(\mathbb{CP}^1, N_f)\) vanish. The Lemma follows. □

**Theorem 4.5** Let \(n := c_1(f^*TX) - 1\). Let \(f := [(f, \mathbb{CP}^1, a_1, \ldots, a_n)]\) be represented by a cuspidal stable map in \(\overline{M}_n(X, \beta, J)\), where \(a_i\) is a regular point. Then, \(f\) is a critical point of the \(ev\) map. The ramification index of the \(ev\) map along the cuspidal stable maps locus is 2.

Proof. Lemma 4.1 implies the kernel of \(dev_c := \tau\), where \(\tau\) is a skyscraper sheaf supported by the cuspidal singularity. The cokernel of \(dev\) is \(H_D^0(\mathbb{CP}^1, N_f(-a_1 - \ldots - a_n))\) by Lemma 4.4. The Kodaira-Serre duality in [Shev] Lemma 1.5.1 shows that \(H_D^1(\mathbb{CP}^1, N_f(-a_1 - \ldots - a_n))\) is isomorphic to \(H_D^0(\mathbb{CP}^1, \omega_{\mathbb{CP}^1} \otimes [N_f(-a_1 - \ldots - a_n)]^*)\), where \(\omega_{\mathbb{CP}^1}\) is a dualizing sheaf on \(\mathbb{CP}^1\) and \([N_f(-a_1 - \ldots - a_n)]^*\) is a dual vector bundle of \([N_f(-a_1 - \ldots - a_n)]\).

There is a canonical residue morphism of degree 2 from the local slice of the direction \(\tau\) to the direction \(H_D^1(\mathbb{CP}^1, N_f(-a_1 - \ldots - a_n))\) induced by the \(ev\) map. By Micallef-White’s Theorem, after the coordinate changes, the morphism can be written:

\[
\tau \simeq \tau^* \to \quad H_D^0(\mathbb{CP}^1, \omega_{\mathbb{CP}^1} \otimes [N_f(-a_1 - \ldots - a_n)]^*)^*
\]
\[
vdz \mapsto \frac{1}{2\pi i} \int_{\gamma} \frac{v^2}{z} dz
\]
because the local index of \( f \) is 2 and \( r \) parameterizes the 2nd order deformation. Thus, the Theorem follows. \( \square \)

**Proposition 4.6** Let \( n := c_1(f^*TX) − 1 \) and \( n = r + s \). Let the point
\[
f := [(f, C, z_1, \ldots, z_r, w_1, \ldots, w_s)] \\
:= [((f_1, (C, q_1), z_1, \ldots, z_r), (f_2, (C_2, q_2), w_1, \ldots, w_s))]
\]
be represented by a reducible stable map \( f \), where \( f_i \) is an immersion if \( f_i \) is not trivial. Then,
(i) If any of \( f_i \) is a degree 0 map, then the cokernel of dev at \( f \) has a (real) rank at least six.
(ii) If \( r \) or \( s \) is strictly bigger than \( c_1(f_1^*TX) − 2 \) or \( c_1(f_2^*TX) − 2 \) respectively, then the cokernel of dev at \( f \) has a (real) rank bigger than four.
(iii) If \( r \) or \( s \) is \( c_1(f_1^*TX) − 2 \) or \( c_1(f_2^*TX) − 2 \) respectively, then the cokernel of dev at \( f \) has a (real) rank two.
(iv) If \( r \) or \( s \) is \( c_1(f_1^*TX) − 1 \) or \( c_1(f_2^*TX) − 1 \) respectively, then the evaluation map \( ev \) at \( f \) is regular.

Sketch of the Proof. In Proposition [3.13] (i), the vector space \( T_{q_1}C_1 \oplus T_{q_2}C_2 \oplus T_{f(q)}X \) parameterizes the node deformation. So, it doesn’t contribute to the rank of the ev map. \( T_{q_1}C_1 \otimes T_{q_2}C_2 \) generates the first order deformation of smoothing node. During the smoothing node deformation, the stable map also changes because the stable maps are the same if they agree to infinite order at any point. Since the deformation space generating smoothing node deformation is smooth and parameterize the 1st order deformation, the image of any non-trivial vector in \( T_{q_1}C_1 \otimes T_{q_2}C_2 \) by dev is non-trivial. Therefore, the (real) rank dimension contribution from \( T_{q_1}C_1 \otimes T_{q_2}C_2 \) for the ev map is two. Let’s prove (i). Suppose that \( f_i \) is a trivial map. The stability condition implies that \( C_1 \) contains at least 2 marked points. The differential of the \( i \)-th evaluation map \( ev_i \) is zero on \( C_1 \). The maximum dimensional contribution to the rank of the ev map from \( C_2 \) is at most \( 2[(c_1(f^*TX) − 3)] \).

The first \( 2(c_1(f^*TX) − 3) \) is contributed by \( T_{w_1}C_2, \ldots, T_{w_s}C_2 \), where \( s = c_1(f^*TX) − 3 \). The second \( 2(c_1(f^*TX) − 3) \) is contributed by \( H^0_D(C_2, N_{f_2}) \).

The contribution toward the rank of the ev map from \( H^0_D(C_2, N_{f_2}) \) is limited by the number of marked points \( s \) in \( C_2 \). The comparison with the dimension \( 4(c_1(f^*TX) − 1) \) of the moduli space leads to the result (i). The proofs of (ii), (iii), (iv) are similar and straightforward. We will omit them. \( \square \)
5 Transversality Properties on $\overline{M}_n(X, \beta, J)$

**Lemma 5.1** Let $n := c_1(f^*TX) - 1$. Let $f := [(f, C, a_1, \ldots, a_n)]$ be in $\overline{M}_n(X, \beta, J)$. Assume that $f_j$ is regular at $a_i$ in the irreducible component $C_j$ in $C$. Let $c_k$, $k = 1, \ldots, m$, be critical points in $C_j$. If $\sum c_k (cf. def. of o(c_k) in sec. 3.1)$ is less than $c_1(f_j^*TX) - 1$, then the $i$-th evaluation map $ev_i$ is regular at $f$.

**Proof.** The condition in the sum of the order of the critical points and the dimension count by the Riemann-Roch’s Theorem with Proposition 3.7 implies $H^0_D(C_j, N_{f_j})$ has dimension at least two. Both $dev_i(Tf(a_i)X)$ and $dev_i(H^0_D(C_j, N_{f_j}))$ have dimension two because $f$ is regular at $a_i$. Moreover, $dev_i(Tf(a_i)X)$ and $dev_i(H^0_D(C_j, N_{f_j}))$ are independent vector spaces. Thus, $ev_i$ is regular at $f$. $\blacksquare$

Let $O(f)$ be a small open neighborhood of $f \in \bigcap_{i=1}^n ev_i^{-1}(q_i)$ such that $O(f) \cap (\bigcap_{i=1}^n ev_i^{-1}(q_i)) = \{f\}$. Let $O(q_i)$ be a small open neighborhood of $q_i \in X$. Then,

$$\text{Intersection multiplicity at } f := \# [O(f) \cap (\bigcap_{i=1}^n ev_i^{-1}(q_i))],$$

where $q_i'$ is a generic point in $O(q_i)$. Lemma 5.1 shows that the pull-back cycle $ev_i^{-1}(q_i)$ is smooth on general points of $ev_i^{-1}(q_i)$. Since the transversality property is an open condition, the intersection multiplicity is well-defined. We say the cycles $ev_1^{-1}(q_1), \ldots, ev_n^{-1}(q_n)$ meet transversally if the intersection multiplicity at any point in $\bigcap_{i=1}^n ev_i^{-1}(q_i)$ is one.

**Lemma 5.2** Let $n := c_1(f^*TX) - 1$. Let $f$ be in $\bigcap_{i=1}^n ev_i^{-1}(q_i)$, where $q_i$, $i = 1, \ldots, n$, are points in general position in the compact semipositive symplectic 4-manifold $X$. Then, the following holds.

(i) If $f$ is represented by a stable map which is an immersion and has an irreducible domain curve, then the intersection multiplicity at $f$ is one.

(ii) If $f$ is represented by a cuspidal stable map none of whose marked points is a critical point, then the intersection multiplicity at $f$ is two.

**Proof.** Note that $ev^{-1}(q_1, \ldots, q_n) = \bigcap_{i=1}^n ev_i^{-1}(q_i)$. (i) follows from Proposition 4.2 because $ev$ at $f$ is a local diffeomorphism. (ii) follows from Theorem 4.5 because the local ramification index is identical to the multiplicity. $\blacksquare$
All of the results so far sum up to the non-transversality property in Theorem 5.3 and prove the Tian’s deep conjecture.

**Theorem 5.3** Let \( n := c_1(f^*TX) - 1 \). The cuspidal stable maps locus is the unique equi-singular locus in \( M_n(X, \beta, J) \) of real codimension \( \leq 2 \) on which transversality uniformly fails.

Proof. The Theorem follows immediately from Lemma 5.2 \( \square \)

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