On a property of 2-dimensional integral Euclidean lattices

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Abstract

Let Λ be any integral lattice in the 2-dimensional Euclidean space. Generalizing the earlier works of Hiroshi Maehara and others, we prove that for every integer \( n > 0 \), there is a circle in the plane \( \mathbb{R}^2 \) that passes through exactly \( n \) points of \( \Lambda \).

Key Words and Phrases. quadratic fields, lattices.

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1 Introduction

We consider the following condition on 2-dimensional lattices \( \Lambda \subset \mathbb{R}^2 \).

Definition 1.1. If there is a circle in the plane \( \mathbb{R}^2 \) that passes through exactly \( n \) points of \( \Lambda \) for every integer \( n > 0 \), then \( \Lambda \) is called universally concyclic.

A lattice generated by \((a, b), (c, d)\) \( \in \mathbb{R}^2 \), \((ad - bc) \neq 0\) is denoted by \( \Lambda[(a, b), (c, d)] \). In [4], Maehara introduced the term “universally concyclic”. Then, he and others showed the following results. In [5] and [3], Schinzel, Maehara and Matsumoto proved that \( \mathbb{Z}^2 \), that is, \( \Lambda[(1, 0), (0, 1)] \) is universally concyclic. Moreover let \( a, b, c, d \in \mathbb{Z} \) be such that \( q := ad - bc \) is a prime and \( q \equiv 3 \pmod{4} \). Then \( \Lambda[(a, b), (c, d)] \) is universally concyclic.

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The equilateral triangular lattice $\Lambda[(1, 0), (-1/2, \sqrt{-3}/2)]$ and rectangular lattice $\Lambda[(1, 0), (0, \sqrt{-3})]$ are universally concyclic.

Let $\mathbb{Z}[x] := \{a + bx \mid a, b \in \mathbb{Z}\}$. We remark that for a positive integer $d$, a lattice $\Lambda[(1, 0), (a, b\sqrt{d})]$ is also given by $\mathbb{Z}[a + b\sqrt{-d}]$ in the complex plane.

We define the set $A(k)$ as follows:

$$A(k) := \{z \in \mathbb{Z}[\sqrt{-3}] \mid |z|^2 = 7^k\}.$$ 

In [4], Maehara proved the following result:

**Lemma 1.1** (cf. [4]). $\sharp A(k) = 2(k + 1)$.

Then, Maehara [4] proposed the following problems:

**Problem 1.1** (cf. [4]). For every square-free integer $d > 1$ and a prime $p$ such that $p = x^2 + y^2d$, we have $\sharp\{z \in \mathbb{Z}[\sqrt{-d}] \mid |z|^2 = p^k\} \geq 2(k + 1)$ for every $k$. Does equality always hold?

**Problem 1.2** (cf. [4]). Is $\Lambda[(a, b), (c, d)]$ universally concyclic if $a, b, c, d \in \mathbb{Z}$ and $ad - bc \neq 0$.

Here, we answer Problems 1.1 and 1.2 affirmatively. In fact, we prove a slightly stronger assertion in Theorems 1.1 and 1.2 below. Let $d$ be a square-free positive integer and $K$ be the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$. We define $\mathcal{O}_K$ as the integer ring of $K$. Let $\mathbb{Z} \cdot a + \mathbb{Z} \cdot b$ denote the linear combination of $a$ and $b$ with integer coefficients. Then $\mathcal{O}_K$ will be written as follows:

$$\mathcal{O}_K = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot w_K,$$

where

$$w_K = \begin{cases} \sqrt{-d} & \text{if } -d \equiv 2, 3 \pmod{4}, \\ -1 + \sqrt{-d} & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

We denote by $d_K$ the discriminant of $K$:

$$d_K = \begin{cases} -4d & \text{if } -d \equiv 2, 3 \pmod{4}, \\ -d & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

We review the concept of order in a quadratic field (for more details, see [2]). An order $\mathcal{O}$ in a quadratic field $K$ is a subset $\mathcal{O} \subset K$ such that

1. $\mathcal{O}$ is a subring of $K$ containing 1.
2. $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-module.
3. $\mathcal{O}$ contains a $\mathbb{Q}$-basis of $K$.

We can now describe all orders in a quadratic fields:

**Lemma 1.2** (cf. [2, page. 133]). Let $\mathcal{O}$ be an order in a quadratic field $K$ of discriminant $d_K$. Then $\mathcal{O}$ has a finite index in $\mathcal{O}_K$, and if we set $f = [\mathcal{O}_K : \mathcal{O}]$, then

$$\mathcal{O} = \mathbb{Z} + f \mathcal{O}_K = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot f w_K,$$

where $w_K$ is as in (3). Here $f$ is called a conductor of the order $\mathcal{O}$.

We denote $\mathcal{O}$ by $\mathcal{O}_f$ if $f = [\mathcal{O}_K : \mathcal{O}]$. Now, we introduce the concept of proper ideals of an order. For any ideal $a$ of $\mathcal{O}_f$, notice that

$$\mathcal{O}_f \subset \{ \beta \in K | \beta a \subset a \}$$

since $a$ is an ideal of $\mathcal{O}_f$. We say that an ideal $a$ of $\mathcal{O}_f$ is proper whenever equality holds, i.e., when

$$\mathcal{O}_f = \{ \beta \in K | \beta a \subset a \}.$$

A quadratic form $F$ is called integral if all the coefficients of $F$ are rational integers. A lattice $\Lambda$ is called integral if $(x, y) \in \mathbb{Z}$ for all $x, y \in \Lambda$, where $(x, y)$ is the standard inner product. Generally, it is well-known that there exists a one-to-one correspondence between the set of proper ideal classes of the order $\mathcal{O}_f$ and the equivalence class of primitive positive definite integral quadratic forms $F(x, y)$ with discriminant $f^2 d_K < 0$ (see Theorem 2.2 in Section 2, [1, Chapter 2, §7-6], [6, §11]). Hence, we consider the proper ideal classes of $\mathcal{O}_f$ to be the lattice in $\mathbb{R}^2$ corresponding to a quadratic forms $F(x, y)$. On the other hand, any 2-dimensional integral Euclidean lattice can be considered as some proper ideal class of $\mathcal{O}_f$. We define $\Lambda$ as the proper ideal classes of $\mathcal{O}_f$. Then, we prove the following theorems:

**Theorem 1.1.** Let $n \in \mathbb{N}$ and assume that $n \neq 1$. Let $p$ be a prime number such that there exists a $z \in \mathbb{Z}[\sqrt{-n}]$ with $|z|^2 = p$, $(\frac{d_K}{p}) = 1$ and $(p, f) = 1$, where $(\cdot)$ is the Legendre symbol. Then,

$$\sharp\{z \in \mathbb{Z}[\sqrt{-n}] | |z|^2 = p^k\} = 2(k + 1).$$

**Theorem 1.2.** All the 2-dimensional integral lattices in $\mathbb{R}^2$ are universally concyclic.
Remark 1.1. We remark that there exist some non-integral lattices which are not universally concyclic. Maehara also proved in [4] that if $\tau$ is a transcendental number, then $\Lambda[(1, \tau), (0, 1)]$ cannot contain four concyclic points, hence is not universally concyclic. The rectangular lattice $\Lambda[(\alpha, 0), (0, \beta)]$ does not contain five concyclic points if and only if $(\alpha/\beta)^2$ is an irrational number. Hence, some additional integrality conditions are necessary to ensure this property.

2 Preliminaries

In this paper, we consider the 2-dimensional integral Euclidean lattices. We shall always assume that $d$ denotes a positive square-free integer. Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and let $\mathcal{O}_K$ be its ring of algebraic integers defined by (1). As we mentioned in Section 1, there exists a one-to-one correspondence between the set of fractional ideal classes of the unique quadratic field $\mathbb{Q}(\sqrt{-d})$ and the equivalence class of primitive positive definite integral quadratic forms $F(x, y)$ with discriminant $d_K < 0$ [6, §10]. More generally, there exists a one-to-one correspondence between the set of fractional proper ideal classes of order $\mathcal{O}_f$ and the equivalence class of primitive positive definite integral quadratic forms $F(x, y)$ with discriminant $f^2d_K < 0$ [6, Chapter 2, §7-6], [1, Chapter 2, §7-6]. We remark that the value $f^2d_K$ is called the discriminant of the order $\mathcal{O}_f$. Finally, we give the well-known theorems needed later.

Theorem 2.1 (cf. [2, page 104]). We can classify prime ideals of a quadratic field as follows:

1. If $p$ is an odd prime and $\left(\frac{d_K}{p}\right) = 1$ (resp. $d_K \equiv 1 \pmod{8}$) then

   $$(p) = \mathfrak{p}\mathfrak{p}' \ (\text{resp.} \ (2) = \mathfrak{p}\mathfrak{p}') ,$$

   where $\mathfrak{p}$ and $\mathfrak{p}'$ are prime ideals with $\mathfrak{p} \neq \mathfrak{p}'$, $N(\mathfrak{p}) = N(\mathfrak{p}') = p$ (resp. $N(\mathfrak{p}) = 2$).

2. If $p$ is an odd prime and $\left(\frac{d_K}{p}\right) = -1$ (resp. $d_K \equiv 5 \pmod{8}$) then

   $$(p) = \mathfrak{p} \ (\text{resp.} \ (2) = p) ,$$

   where $\mathfrak{p}$ is a prime ideal with $N(\mathfrak{p}) = p^2$ (resp. $N(\mathfrak{p}) = 4$).

3. If $p \mid d_k$ then

   $$(p) = \mathfrak{p}^2 ,$$

   where $\mathfrak{p}$ is a prime ideal with $N(\mathfrak{p}) = p$. 


Theorem 2.2 (cf. [2, Theorem 7.7]). Let \( \mathcal{O} \) be an order of discriminant \( D \) in an imaginary quadratic field \( K \).

1. If \( F(x, y) = ax^2 + bxy + cy^2 \) is a primitive positive definite integral quadratic form of discriminant \( D \), then \( [a, (-b + \sqrt{D})/2] \) is a proper ideal of \( \mathcal{O} \).

2. The map sending \( F(x, y) \) to \( [a, (-b + \sqrt{D})/2] \) induces an isomorphism between the form class group and the ideal class group.

3. A positive integer \( m \) is represented by a form \( F(x, y) \) if and only if \( m \) is the norm \( N(a) \) of some ideal \( a \) in the corresponding ideal class mentioned in 2.

Lemma 2.1 (cf. [2, Lemma 7.18]). Let \( \mathcal{O}_f \) be an order of conductor \( f \). We say that a non-zero \( \mathcal{O}_f \)-ideal \( a \) is prime to \( f \) provided that \( a + f\mathcal{O}_f = \mathcal{O}_f \).

1. An \( \mathcal{O}_f \)-ideal \( a \) is prime to \( f \) if and only if its norm \( N(a) \) is relatively prime to \( f \).

2. Every \( \mathcal{O}_f \)-ideal prime to \( f \) is proper.

Proposition 2.1 (cf. [2, Proposition 7.20]). Let \( \mathcal{O}_f \) be an order of conductor \( f \) in an imaginary quadratic field \( K \). We say that a non-zero \( \mathcal{O}_K \)-ideal \( a \) is prime to \( f \) provided that \( a + f\mathcal{O}_K = \mathcal{O}_K \). If \( a \) is an \( \mathcal{O}_K \)-ideal prime to \( f \), then \( a \cap \mathcal{O}_f \) is an \( \mathcal{O}_f \)-ideal prime to \( f \) of the same norm.

Proposition 2.2 (cf. [2, Exercise 7.26]). Let \( \mathcal{O}_f \) be an order of conductor \( f \). Then \( \mathcal{O}_f \)-ideals prime to the conductor can be factored uniquely into prime \( \mathcal{O}_f \)-ideals (which are also prime to \( f \)).

Theorem 2.3 (cf. [2, Theorem 9.4]). Let \( n > 0 \) be an integer, and \( L \) be the ring class field of the order \( \mathbb{Z}[\sqrt{-n}] \) in the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \). If \( p \) is an odd prime not dividing \( n \), then

\[ p = x^2 + ny^2 \iff p \text{ splits completely in } L. \]

3 Proof of Theorem 1.1

Proof of Theorem 1.1. We remark that \( \mathbb{Z}[\sqrt{-n}] \) can be considered as the order \( \mathbb{Z}[\sqrt{-n}] = \mathcal{O}_f \subset K = \mathbb{Q}(\sqrt{-d}) \) for some \( f \) and \( d \) with the following condition \(-4n = f^2d_K\), namely,

\[
n = \begin{cases} 
  f^2d & \text{if } -d \equiv 2, 3 \pmod{4}, \\
  f^2d/4 & \text{if } -d \equiv 1 \pmod{4}.
\end{cases}
\]
Therefore, we remark that $\mathbb{Z}[^{\sqrt{-n}}] = \mathcal{O}_f$.

We fix a prime $p$ such that there exists a $z \in \mathbb{Z}[\sqrt{-n}]$ with $|z|^2 = p$.

Because of Theorem 2.1, $(p, f) = 1$. Moreover, the condition $z \in \mathbb{Z}[\sqrt{-n}]$ implies that the ideals $p$ and $p'$ are principal ideals. We set

$$q = p \cap \mathcal{O}_f,$$
$$q' = p' \cap \mathcal{O}_f.$$}

Then, by Proposition 2.1, the ideals $q$ and $q'$ are principal ideals of $\mathcal{O}_f$ prime to $f$. Because of Lemma 2.1, $\mathcal{O}_f$-ideal prime to $f$ is proper and using the unique factorization of proper ideals in Proposition 2.2, the ideals of norm $p^k$ are as follows:

$$q^k, q^{k-1}q', \ldots, q.$$  \hspace{1cm} (4)

Let $z_1$ be the element of $\mathbb{Z}[\sqrt{-n}]$ with norm $p^k$. Because of Lemma 2.1, $(z_1)$ is a proper $\mathcal{O}_f$-ideal. Moreover, for $-z_1 \in \mathbb{Z}[\sqrt{-n}]$, the ideals $(z_1)$ and $(-z_1)$ are same proper $\mathcal{O}_f$-ideals. Hence, there exists a one-to-one correspondence between the non-equivalent elements of $\mathbb{Z}[\sqrt{-n}]$ with norm $p^k$ under the action of $\{\pm 1\}$ and the set of proper $\mathcal{O}_f$-ideals of norm $p^k$ defined by (4). This completes the proof of Theorem 1.1. \hfill $\square$

4 Proof of Theorem 1.2

4.1 Setup

Proposition 4.1. For any positive integers $n$ and $a$, there exists a prime $p$ prime to $n$ such that

$$p = x^2 + ny^2$$

with $y \equiv 0 \pmod{4a}$.

Proof. We set $n' = 16a^2n$. Let $L$ be the ring class field of the order $\mathbb{Z}[\sqrt{-n}]$. (We refer to Cox [2] for the concept of ring class fields.) Because of Theorem 2.3, there exists a prime $p$ such that

$$p = x^2 + n'y^2 = x^2 + n(4ay)^2$$

if and only if $p$ splits completely in $L$. Then the primes that split completely in $L$ have density $1/[L : K]$, and in particular there are infinitely many of them (cf. [2, Corollary 5.21] and [2, Corollary 8.18]). Hence, there exists a prime $p$ prime to $n$. Therefore, we complete the proof of Proposition 1.1. \hfill $\square$
Because of Proposition 4.1, there exists prime \( p \) prime to \( n \) such that 
\[ p = x_1^2 + ny_1^2 \] with \( y_1 \equiv 0 \pmod{4a} \). We fix such a prime and denote it by \( p_{n,a} \). Then we define \( A_{n,a}(k) \) as follows:
\[
A_{n,a}(k) := \{ z \in \mathbb{Z}[\sqrt{-n}] \mid |z|^2 = p_{n,a}^k \}.
\]
By Proposition 4.1, if \( x + y\sqrt{-n} \in A_{n,a}(k) \) then \( y \equiv 0 \pmod{4a} \) and
\[
x + y \equiv \pm j \pmod{4a},
\] where \( j \equiv x_1^k \pmod{4a} \), \( 1 \leq j \leq 4a - 1 \). So, we define \( \tilde{A}_{n,a}(k) \) as follows:
\[
\tilde{A}_{n,a}(k) := \{ x + y\sqrt{-n} \in A_{n,a}(k) \mid x + y \equiv -j \pmod{4a} \}.
\]

**Lemma 4.1.** \( \#A_{n,a}(k) = 2(k + 1) \) and \( \#\tilde{A}_{n,a}(k) = k + 1 \).

**Proof.** Because of Proposition 4.1 \( (d_K/p_{n,a}) = 1 \) and \( (p_{n,a}, f) = 1 \). Hence, by Theorem 1.1 \( \#A_{n,a}(k) = 2(k + 1) \). If \( x + y\sqrt{-n} \in A_{n,a}(k) \), then \( x \neq 0 \), \( -x + y\sqrt{-n} \in A_{n,a}(k) \), and only one of them belongs to \( \tilde{A}_{n,a}(k) \). Therefore, \( \#\tilde{A}_{n,a}(k) = k + 1 \).

### 4.2 Proof of Theorem 1.2

Here, we start the proof of Theorem 1.2.

**Proof of Theorem 1.2** Let \( \Lambda \) be a 2-dimensional integral lattice and let the associated quadratic form be \( ax^2 + bxy + cy^2 \). Let \( \mathcal{O}_f \subset \mathbb{Q}\sqrt{-d} \) be the order corresponding to the lattice \( \Lambda \). We set \( n = -f^2d_K \) and \( \alpha := (-b + \sqrt{-n})/(2\sqrt{a}) \). It is enough to show that for each integer \( k > 0 \), there is a circle in the complex plane that passes through exactly \( k + 1 \) points of \( \Lambda \). For \( k > 0 \), define a circle \( \Gamma_k \) in the complex plane as follows:
\[
|4\sqrt{a}z - j|^2 = p_{n,a}^k,
\]
where \( j \) is defined by (5). Let \( C(k) \) be the subset of \( \Lambda \) lying on the circle \( \Gamma_k \). We show that \( \#C(k) = k + 1 \). If \( z = \sqrt{a}x + \alpha y \in C(k) \) then \( 4\sqrt{a}z - j = 4ax - 2by - j + 2y\sqrt{-n} \), so \( 4ax - 2by - j + 2y \equiv -j \pmod{4a} \). Therefore \( 4\sqrt{a}z - j \in A_{n,a}(k) \). Hence we can define the map \( \varphi : C(k) \to \tilde{A}_{n,a}(k) \) by:
\[
z \mapsto 4\sqrt{a}z - j.
\]
This map is a bijection. To see this, suppose \( x + y\sqrt{-n} \in \tilde{A}_{n,a}(k) \). Then \( x + y \equiv -j \pmod{4a} \), that is, \( x + by + j \equiv 0 \pmod{4a} \). Moreover, by
4 PROOF OF THEOREM ??

Proposition 4.1, \( y \equiv 0 \pmod{4a} \), and hence \( y \) is even. Therefore, we can define a map from \( \hat{A}_{n,a}(k) \) to \( C(k) \) as follows:

\[
x + y\sqrt{-n} \mapsto \frac{x + by + j}{4\sqrt{a}} + \frac{y}{2}\alpha.
\]

This gives the inverse of \( \varphi \). Therefore \( \varphi \) is surjective, that is, \( \sharp C(k) = \sharp \hat{A}_{n,a}(k) = k + 1 \).

Informing Hiroshi Maehara of Theorem 1.2, he proved the following fact:

**Corollary 4.1.** If \( (\alpha/\beta)^2 \in \mathbb{Q} \) then \( \Lambda[(\alpha, 0), (0, \beta)] \) is universally concyclic.

**Proof.** We assume that \( (\alpha/\beta)^2 = b/a \), where \( b/a \) is irreducible fraction. Then, the lattices \( \Lambda[(\alpha, 0), (0, \beta)] \) and \( \Lambda[(a, 0), (0, \sqrt{ab})] \) are similar under the similarity transformation \( \alpha/a \) and \( \Lambda[(a, 0), (0, \sqrt{ab})] \) is integral lattice. Because of Theorem 1.2, \( \Lambda[(a, 0), (0, \sqrt{ab})] \) is universally concyclic, so is \( \Lambda[(\alpha, 0), (0, \beta)] \). □

**Remark 4.1.** Finally, we generalize the definition of universally concyclic to higher dimensions.

**Definition 4.1.** Let \( \Lambda \subset \mathbb{R}^d \) be a \( d \)-dimensional lattice. If there is a spherical surface \( S^{d-1} \) in \( \mathbb{R}^d \) that passes through exactly \( n \) points of \( \Lambda \) for every integer \( n > 0 \), then \( \Lambda \) is called universally concyclic.

In [4], Maehara remark that \( \mathbb{Z}^3 \) is universally concyclic because the spherical surface \( (4x-1)^2 + (4y)^2 + (4z-\sqrt{2})^2 = 17k+2 \) passes through exactly \( k+1 \) points of \( \mathbb{Z}^3 \). We also remark that any integral lattices in higher dimension \( \mathbb{R}^d \) are universally concyclic:

**Corollary 4.2.** Any integral lattices in \( \mathbb{R}^d \) are universally concyclic.

**Proof.** Let \( \Lambda \) be an integral lattice in \( \mathbb{R}^d \). We define sublattices \( \{\Lambda^{(i)}\}_{i=2}^d \) such that

\[
\Lambda^{(2)} \subset \Lambda^{(3)} \subset \cdots \subset \Lambda^{(d)} = \Lambda
\]

and \( \Lambda^{(i)} \) spans \( \mathbb{R}^i \) which we denote by \( \mathbb{R}^{(i)} \) for all \( i \). Because of Theorem 1.2 for each \( k > 0 \), we can define the circle \( S^{(1)} \subset \mathbb{R}^{(2)} \) that passes through exactly \( k \) points of \( \Lambda^{(2)} \).

Let \( O^{(1)} \) be the center of \( S^{(1)} \) and let \( \ell \) be a half line in \( \mathbb{R}^{(3)} \) whose origin is \( O^{(1)} \), which is orthogonal to \( \mathbb{R}^{(2)} \). We define the sphere \( S^{(2)}(a) \), whose center \( O^{(2)}(a) \) lies on \( \ell \), the distance between \( O^{(1)} \) and \( O^{(2)}(a) \) is \( a \) and whose radius is \( \sqrt{a^2 + (\text{radius of } S^{(1)})^2} \). We assume that \( 0 \leq a \leq 1 \).

Since \( \Lambda \) is an integral lattice, the number of the points of \( \Lambda^{(3)} \) which intersect in \( S^{(1)}(a) \) is finite for any \( 0 \leq a \leq 1 \). Moreover, for \( a_1 \neq a_2 \),
the intersection of $S^{(1)}(a_1)$ and $S^{(1)}(a_2)$ is the points of $\Lambda^{(2)}$ in $\Lambda$, namely, the points of $S^{(1)}$. On the other hand, for $0 \leq a \leq 1$, the number of the spheres $S^{(2)}(a)$ is infinite. Therefore, there exists a number $a_0$ such that the intersection of $S^{(2)}(a_0)$ and $\Lambda$ is the points of $\Lambda^{(2)}$. We denote $S^{(2)}(a)$ by $S^{(2)}$ and $S^{(2)}$ passes through exactly $k$ points of $\Lambda^{(3)}$. We can define the spheres $S^{(3)}, \ldots, S^{(d-1)}$ recursively such that each of $\{S^{(i)}\}_{i=3}^{d-1}$ passes through exactly $k$ points of $\Lambda$, as we defined $S^{(2)}$ in $\mathbb{R}^{(3)}$.

So, we have shown that any integral lattices in $\mathbb{R}^d$ are universally concyclic. However, the points of lattice lying on the sphere constructed in the proof of Corollary 4.2 are on the plane $x_3 = \cdots = x_d = 0$. Hence, Maehara added some conditions to Definition 4.1 and showed the following theorem:

**Theorem 4.1** (cf. [4]). For $n > d \geq 2$, there is a sphere in $\mathbb{R}^d$ that passes through exactly $n$ lattice points on $\mathbb{Z}^d$, and moreover, the $n$ lattice points span a $d$-dimensional polytope.

Therefore, we can state the following problem:

**Problem 4.1.** Let $\Lambda$ be an integral lattice in $\mathbb{R}^d$. We assume $n > d \geq 2$. Is there a sphere in $\mathbb{R}^d$ that passes through exactly $n$ lattice points on $\Lambda$, which span a $d$-dimensional polytope?

A set of points in the $d$-dimensional Euclidean space is said to be in general position if no $d + 1$ of them lie in a $(d - 1)$-dimensional plane. Then, Maehara also proposed the following problem:

**Problem 4.2** (cf. [4]). Is there a sphere in $\mathbb{R}^3$ that passes through a given number of lattice points in general position on $\mathbb{Z}^3$?

It is also an interesting open problem to prove or disprove a similar conclusion as in Problem 4.1.2 for any integral lattices in higher dimension $\mathbb{R}^d$.

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