Symmetry of massive Rarita-Schwinger fields

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Abstract

We derive the general lagrangian and propagator for a vector-spinor field in \( d \)-dimensions and show that the physical observables are invariant under the so-called point transformation symmetry. Until now the symmetry has not been exploited in any non-trivial way, presumably because it is not an invariance of the classical action nor is it a gauge symmetry. Nevertheless, we develop a technique for exploring the consequences of the symmetry leading to a conserved vector current and charge. The current and charge are identically zero in the free field case and only contribute in a background such as an electromagnetic or gravitational field. The current can couple spin-\( \frac{3}{2} \) fields to vector and scalar fields and may have important consequences in intermediate energy hadron physics as well as linearized supergravity. The consistency problem which plagues higher spin field theories is then discussed and some ideas regarding the possibility of solutions are presented.

1 Introduction

Theories of interacting high spin fields\(^1\) have been a subject of considerable interest for many years. This is partly due to the many particles with spin \( \geq \frac{3}{2} \) seen in accelerator laboratories and also because there is currently no general field theory description which is relativistic, interacting and also free of inconsistencies\(^2\). Over the years one interacting theory after another have been shown to be inconsistent leading many to suggest that all higher spin fields must be composite. On the other hand, higher spin elementary particles, such as the gravitino, play an important role in supersymmetry, which itself represents a fundamental building block of many modern unification schemes. Thus we would like to remain hopeful that a solution to the consistency problems can be found within point particle field theory. Perhaps our interpretation regarding the physical degrees of freedom is misguided or perhaps, as is our present concern, we have neglected symmetries or other

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\(^1\)By ‘high spin’ we mean particles of spin \( \geq \frac{3}{2} \).

\(^2\)See section 7 below.
aspects of nature that should be included. The hope being that if all of the symmetries are properly included, the result will be a consistent theory. That this hope is reasonable is exemplified by the fact that consistent solutions have already been found in restricted scenarios with curved backgrounds, cosmological constant tuning and Planck scale masses \[3, 4, 5, 6, 7, 8\].

The consistency problems seem to exist for most interacting higher spin field theories and are a main concern of many theorists working in the field and so we will devote the final section of this paper to a discussion of the problem and touch on some possible consequences of the symmetry. Perhaps the ideas that we present will inspire some new angles of attack on the problem.

The main goal of this paper is a review of the \(d\)-dimensional theory of interacting Rarita-Schwinger fields and an exploration of the symmetries. Our hope is that the general expressions and the new interactions that we present here will be of use in formulating effective theories of interacting hadrons as well as work involving the massive gravitinos of supergravity such as, for example, the AdS/CFT correspondence \[9, 10, 11, 12\]. Perhaps the AdS/CFT results can be extended to the case wherein the Rarita-Schwinger fields are not fixed at the start as non-interacting and onshell\(^3\).

We begin in sections 2 and 3 by (re-)deriving the most general lagrangian and propagator for a Rarita-Schwinger spin-\(\frac{3}{2}\) field\(^4\) using the method of Aurilia and Umezawa \[18\] extended to \(d\)-dimensions.

Since we are using the vector-spinor representation of spin-\(\frac{3}{2}\), we find the usual result that a lower spin content is retained in the field in order to maintain the desirable properties of the action such as hermiticity, linearity in derivatives and non-singular behavior. However, recently there have been other promising ideas where the lower spin content is given a physical interpretation \[16\] or where vector spinor description of spin-\(\frac{3}{2}\) is replaced by a pure spin-\(\frac{3}{2}\) field \[2, 13, 14, 15\].

Considering the lower spin components as unphysical, as we do here, leads to a non-unique action depending on an arbitrary complex parameter measuring the lower spin content of the theory. Various choices of the parameter are seen to reduce the general expression to the spin-\(\frac{3}{2}\) actions found in the literature. We formulate the equations in \(d\) spacetime dimensions in anticipation of diverse applications from effective theories of hadronic interactions involving the spin-\(\frac{3}{2}\) baryons to applications in arbitrary dimensional supergravity theories. For example, both the composite \(\Delta(1232)\) resonance found in low and intermediate energy nucleon scattering experiments and the gravitino of N-extended supersymmetry after spontaneous symmetry breaking are thought to be described by the massive, spin-\(\frac{3}{2}\), Rarita-Schwinger field that we study here.

In section 4 we examine the properties of the so-called ‘point’ or ‘contact’ transformations. These form a non-unitary group of transformations of the fields which shifts the parameter, amounting to a sort of rotation among the spin-\(\frac{3}{2}\) degrees of freedom. The path integral is seen to be invariant under point transformations which implies that physical correlation functions are invariant under a redefinition of the arbitrary parameter. That

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\(^3\)We say non-interacting since onshell fields are used and condition \(\Gamma \cdot \psi = 0\) is incompatible with the presence of additional non-gravitational interactions, for example a non-zero background electromagnetic field.

\(^4\)We follow the standard procedure of using 4-dimensional terminology for spin even when discussing fields in arbitrary dimensions.
the parameter is arbitrary is well known and this has caused many authors to simply fix it to a convenient value. Unfortunately, this has served to hide some of the freedom of the theory. We restore the explicit parameter dependence and, in sections 5 and 6, we derive and examine new conserved currents resulting from the the symmetry.

Finally, in section 7 we discuss the consistency problems and point out a few ideas of how the conserved charge found in section 6 might be useful in that context. The analysis we have used should also be generalizable to higher spins as well whenever the theory contains auxiliary fields of lower spin and has a similar symmetry group involving them.

2 Spin content of the Rarita-Schwinger field

In this section we give a decomposition of the Rarita-Schwinger field into separate spin blocks and derive some general formulas and identities that will be needed later. The result of this and the following section is the expression for the most general free lagrangian. The reader only interested in the result may want to turn immediately to equation (31) or (48) below.

A commonly used formulation of the spin-$\frac{3}{2}$ field is the vector-spinor representation given by Rarita and Schwinger in 1941 [17]. The vector-spinor transforms under the Lorentz group as

\[
(\frac{1}{2}, \frac{1}{2}) \otimes \left[ (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \right] = \left( \frac{1}{2}, \frac{1}{2} \right) \oplus \left( \frac{1}{2}, 1 \right) \oplus (0, \frac{1}{2}) \oplus \left( \frac{1}{2}, 0 \right)
\]  

whereas the spin decomposition of the field in the rest frame [15, 16] is

\[
\text{spin } \psi^\mu = (1 + 0) \otimes \frac{1}{2} = \frac{3}{2} + \frac{1}{2} + \frac{1}{2}.
\]  

The vector-spinor field thus contains two spin-$\frac{3}{2}$ components in addition to the physical spin-$\frac{3}{2}$ component. The decomposition of the spin-$\frac{3}{2}$ field that we will use is given by choosing \((0, \frac{1}{2}) = \psi^\mu, \text{where } p_\mu = i\partial_\mu\). The complimentary part is

\[
\left( \frac{1}{2}, \frac{1}{2} \right) = \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \psi^\nu,
\]

which can then be written in terms of spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ projectors as

\[
g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} = \left( P_{\frac{3}{2}}^{\frac{3}{2}} \right)_{\mu\nu} + \left( P_{\frac{1}{2}}^{\frac{1}{2}} \right)_{\mu\nu}.
\]

Defining \(\left( P_{\frac{3}{2}}^{\frac{3}{2}} \right)_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}\) we have an expansion of the identity

\[
g_{\mu\nu} = \left( P_{\frac{3}{2}}^{\frac{3}{2}} \right)_{\mu\nu} + \left( P_{\frac{1}{2}}^{\frac{1}{2}} \right)_{\mu\nu} + \left( P_{\frac{3}{2}}^{\frac{3}{2}} \right)_{\mu\nu}.
\]  

\(^5\)However there are other representations which describe spin-$\frac{3}{2}$ also, for example the 3 spinor representation [18] and also the \((\frac{3}{2}, 0) + (0, \frac{3}{2})\) representation of [13, 14].

\(^6\)See section 1.2.2 in van Nieuwenhuizen’s supergravity review [19].
Explicit expressions for the projectors can easily be found by contraction with $\gamma^\mu$ and $p^\mu$, but we will anticipate further applications\textsuperscript{7} and generalize to $d$-dimensions.

We define $d$-dimensional spin projection operators by requiring that they reduce properly to the usual 4 dimensional projection operators \textsuperscript{19,20,21} and yet remain projections in $d$-dimensions satisfying the following orthogonality relations,
\[
(P^I_{ij})_{\mu\nu} (P^J_{kl})^{\nu\rho} = \delta^{IJ} \delta_{jk} (P^I_{il})^{\rho}_{\mu}, \quad I,J \in \{1/2,3/2\}, \quad i,j,k,l \in \{1,2\}.
\] (6)
The result is
\[
(P^2)_{\mu\nu} = \frac{1}{p^2(d-1)} [(d-1)p^2g_{\mu\nu} - (d-2)p_\mu p_\nu - \not{p}(\gamma_\mu p_\nu - p_\mu \gamma_\nu) - \gamma_\mu \gamma_\nu p^2]
\]
\[
(P^2)_{\mu\nu} = \frac{1}{p^2(d-1)} [(d-2)p_\mu p_\nu + \not{p}(\gamma_\mu p_\nu - p_\mu \gamma_\nu) + \gamma_\mu \gamma_\nu p^2] = P^2_{11} + P^2_{22}
\]
\[
(P^2_{11})_{\mu\nu} = \frac{1}{p^2(d-1)} [-p_\mu p_\nu + \not{p}(\gamma_\mu p_\nu - p_\mu \gamma_\nu) + \gamma_\mu \gamma_\nu p^2]
\]
\[
(P^2_{22})_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}
\]
\[
(P^2_{12})_{\mu\nu} = \frac{1}{p^2\sqrt{d-1}} [p_\mu p_\nu - \not{p}\gamma_\mu p_\nu]
\]
\[
(P^2_{21})_{\mu\nu} = \frac{1}{p^2\sqrt{d-1}} [\not{p}p_\mu \gamma_\nu - p_\mu p_\nu].
\] (7)

From the definitions we see that the total spin-$\frac{1}{2}$ projection operator, $P^2$, reduces to the sum of the individual projection operators for the two different spin-$\frac{1}{2}$ components of the Rarita-Schwinger field. We also note the following convenient relations,
\[
[\not{p}, (P^2_{ij})_{\mu\nu}] = [\not{p}, (P^2)_{\mu\nu}] = \{\not{p}, (P^2_{12})_{\mu\nu}\} = \{\not{p}, (P^2_{21})_{\mu\nu}\} = 0,
\]
\[
\gamma^\mu (P^2)_{\mu\nu} = (P^2)_{\mu\nu} \gamma^\nu = p^\mu (P^2_{12})_{\mu\nu} = (P^2_{21})_{\mu\nu} p^\nu = 0,
\]
\[
\gamma^\mu (P^2_{12})_{\mu\nu} = (P^2_{12})_{\nu\mu} \gamma^\mu = \gamma_\nu,
\]
\[
\gamma^\mu (P^2_{21})_{\mu\nu} = (P^2_{21})_{\nu\mu} \gamma^\mu = \gamma_\nu - \frac{\gamma_\nu p_\mu}{p^2},
\]
\[
\gamma^\mu (P^2_{22})_{\mu\nu} = (P^2_{22})_{\nu\mu} \gamma^\mu = \frac{\gamma_\nu p_\mu}{p^2}.
\] (8)

These projection operators can now be used to derive the most general vector-spinor lagrangian and propagator in a flat\textsuperscript{8} background spacetime.

### 3 The free spin-$\frac{3}{2}$ lagrangian

The free lagrangian for the spin-$\frac{3}{2}$ field can be written as
\[
\mathcal{L} = \bar{\psi} \Lambda_{\alpha\beta} \psi^\beta,
\] (9)
\textsuperscript{7}For example one may want to set $d = 4 - \epsilon$ in dimensional regularization.
\textsuperscript{8}Curved space results can then be found by the usual technique of introducing vielbeins and a spin connection.
where $\Lambda_{\alpha\beta}$ is an operator and $\psi^\beta$ is a vector-spinor field with suppressed spin index\textsuperscript{9}. Using the projectors given in the previous section we will construct the most general operator $\Lambda_{\alpha\beta}$ subject to the following four conditions \textsuperscript{[17, 18, 22, 23]}:

1. The Euler-Lagrange equations derived from the free action should give the local Rarita-Schwinger equations for a spin-$\frac{3}{2}$ particle. These are a Dirac equation for each of the vector components as well as supplementary conditions to remove the lower spin degrees of freedom:

\begin{equation}
(i\partial - m)\psi^\mu = 0,
\gamma_\mu\psi^\mu = 0.
\end{equation}

(10)

2. The lagrangian should be non-singular in the limit $p \to 0$. In particular, we would like the pole of the propagator to occur at the mass of the particle.

3. The lagrangian should be linear in derivatives as it describes a fermionic field.

4. The operator $\gamma^0 \Lambda_{\alpha\beta}$ should be hermitian:

\begin{equation}
\gamma^0 (\Lambda_{\alpha\beta})^\dagger \gamma^0 = \Lambda_{\beta\alpha}.
\end{equation}

(11)

A consequence of equation (10) in condition 1 is the condition

\begin{equation}
\partial_\mu \psi^\mu = 0,
\end{equation}

(12)

as can be seen by multiplying the first equation in (10) on the left by $\gamma_\mu$ and using the second equation. The condition $\gamma_\mu^{AB}\psi^\mu_B = 0$ (where we now explicitly write the spinor indices $A, B$) represents a constraint equation for each value of the spin index $A$ whereas the condition $\partial_\mu \psi^\mu_B = 0$ is an equation of motion for the spinor components $\psi^\mu_B$. However, the Dirac equation (10) also gives an equation of motion for the same spinor components and when taken together, these result in another set of constraints. In four spacetime dimensions, these two sets of equations each constitute four constraints\textsuperscript{10} and serve to remove 8 components of the 16 component vector-spinor $\psi^\mu_A$, leaving $2(2s+1) = 8$ physical degrees of freedom as required for a massive spin $s = \frac{3}{2}$ particle\textsuperscript{11}.

The most general expression for the operator $\Lambda_{\alpha\beta}$ which obeys condition 1 is given by a combination of a spin-$\frac{3}{2}$ part plus an arbitrary amount of spin-$\frac{1}{2}$,

\begin{equation}
\Lambda_{\alpha\beta} = (\not{p} - m) P_{\alpha\beta}^{\frac{3}{2}} + \frac{a_1}{d}m \left( P_{11}^{\frac{1}{2}} \right)_{\alpha\beta} + \frac{a_2}{d}m \left( P_{22}^{\frac{1}{2}} \right)_{\alpha\beta},
\end{equation}

(13)

\textsuperscript{9}See \textsuperscript{24} and references therein for properties of spinors in $d$ dimensions, the $d$-dimensional Lorentz group and the group of $d$-dimensional general coordinate transformations.

\textsuperscript{10}This may not be true in the interacting theory where the interaction may cause a reduction in the number of constraint equations resulting in an increase in the number of degrees of freedom and leading to inconsistencies. This will be discussed further in section A.

\textsuperscript{11}The restriction of condition 2 is perhaps not necessary, both the photon and the gluon propagators also have extra singular terms before gauge fixing and indeed some of the popular spin-$3/2$ actions do not satisfy this condition. However, in our case we have a theory describing a massive field and so extra terms which are singular as $p \to 0$ give singularities which are not at the physical mass unlike the case of the photon or the gluon.
where we have included factors of \( m \) and \( 1/d \) in the two spin-\( \frac{3}{2} \) terms for later convenience.

The quantities \( P^3_{\alpha\beta} \), \( P^1_{11} \) and \( P^1_{22} \) are the projection operators defined previously which, respectively, project onto the spin-\( \frac{3}{2} \) part and the two spin-\( \frac{1}{2} \) parts of \( \psi^\beta \). The Euler-Lagrange equations \( \Lambda_{\alpha\beta} \psi^\beta = 0 \) separate because of these projection operators into the Rarita-Schwinger equation plus supplementary conditions as required by condition 1. The equation also contains two complex constants \( a_1 \) and \( a_2 \) which are arbitrary at the moment.

One can verify, using the projectors (7) that the operator (13) satisfies condition 1 for any choice of these constants and so we are free to fix them to whatever values are convenient. We will fix them by requiring the lagrangian to satisfy conditions 2 and 3.

Before proceeding with the other conditions let us take a moment to examine the general action given by (13). It contains possible gauge invariances which are already apparent. The field equations are seen to be

\[
(\not{p} - m) P^3_{\alpha\beta} \psi^\beta + \frac{a_1}{d} m \left( P^1_{11} \right)_{\alpha\beta} \psi^\beta + \frac{a_2}{d} m \left( P^1_{22} \right)_{\alpha\beta} \psi^\beta = 0
\]

and using (6) and (8) we notice that, if we could set \( a_1 = a_2 = 0 \), the equation would be invariant under the two variations \( \delta \psi^\beta = \left( P^1_{II} \right)_{\alpha\beta} \chi^\lambda \) for arbitrary spinor \( \chi^\lambda \). However, we will see that the remaining conditions require the parameters \( a_1 \) and \( a_2 \) to be related to each other in such a way that the vanishing of one implies the other is non-zero and so both parameters cannot be set to zero at the same time. If we set one of them to zero, i.e. \( a_I \to 0 \) for \( I \in 1, 2 \), the equations of motion become invariant under \( \delta \psi^\beta = \left( P^1_{II} \right)_{\beta\lambda} \chi^\lambda \).

For \( a_1 = a_2 \) this is the same as \( \delta \psi^\beta = \partial^\beta \epsilon \) as can be seen by inserting the explicit form of the projector from (7). We will see that our remaining conditions break this symmetry such that the gauge invariance is lost except in the massless limit (see equation 34). We will examine the lagrangian written in terms of projection operators again at the end of this section after we have imposed the remaining conditions 2 – 4.

Returning to our conditions, we see that the operator (13) as it is written, does not obey condition 2 since it is singular in the limit \( p \to 0 \). To remedy this, we use the method of Aurilia and Umezawa [18] and shift the spin-\( \frac{1}{2} \) components to form a new operator as

\[
\tilde{\Lambda}_{\alpha\beta} = (\eta_2 \eta_1)^{\lambda\mu} \Lambda_{\lambda\beta},
\]

with \( \eta_1 \) and \( \eta_2 \) given by

\[
\eta_1^{\mu\nu} = g^{\mu\nu} + \sqrt{d-1} \left[ \frac{g_1}{m} \not{p} + g_2 \right] \left( P^1_{12} \right)^{\mu\nu},
\]

\[
\eta_2^{\mu\nu} = g^{\mu\nu} + \sqrt{d-1} \left[ \frac{f_1}{m} \not{p} + f_2 \right] \left( P^1_{23} \right)^{\mu\nu}.
\]

The \( f_1, g_1, f_2, g_2 \) are new constants that we will fix by requiring the singular terms to vanish. The transformation (15) is allowed since we are only altering the coefficients of the separate spin-\( \frac{1}{2} \) projection operators in (13) and thus the equations of motion will still separate properly and condition 1 is still obeyed.
Substituting (16) into (15) and making use of the relations (8) we find that the singular terms will vanish if the constants $f_1, g_1, f_2, g_2, a_1$ and $a_2$ satisfy the following relations,

$$\frac{a_2}{d} = -\frac{d}{(d-1)} \left( 1 + \frac{1}{a_1} \right),$$

$$f_2 = \frac{a_1 + d}{(d-1)a_1} = -\left( 1 + \frac{a_2}{d} \right),$$

$$g_2 = \frac{a_1 + d}{(d-1)a_2} = -\frac{a_1}{d} \left( 1 + \frac{d}{a_2} \right),$$

$$a_1f_1 = a_2g_1 + \frac{(d-2)a_1}{(d-1)}.$$

(17)

The resulting non-singular operator is then

$$\tilde{\Lambda}^{\alpha\beta} = (\not{\!p} - m) g^{\alpha\beta} - \left[ \frac{a_2}{d} f_1 + \frac{1}{(d-1)} \right] \left( \gamma^\alpha p^\beta - p^\alpha \gamma^\beta \right)$$

$$+ \frac{(d-2)a_1}{(d-1)} \gamma^\alpha p^\beta - \frac{1}{(d-1)} \not{\!p} \gamma^\alpha \gamma^\beta - (d-1) \frac{a_2}{d} f_1 \frac{p^\alpha p^\beta}{m}.$$

(18)

This operator now satisfies conditions 1 and 2, and it is written in terms of two parameters, $f_1$ and $a_1$ (since the relations (17) can be used to write $a_2$ and $g_1$ in terms of $a_1$ and $f_1$).

We would like the field $\psi^\mu$ and its hermitian conjugate to appear symmetrically in the lagrangian, so that the variation of $\psi^\mu$ and that of $\bar{\psi}^\mu$ both give the field equations. To achieve this, we apply another shift to (18) via the transformation

$$\theta^{\mu\nu} = g^{\mu\nu} + \frac{k}{d} \gamma^\mu \gamma^\nu.$$

(19)

Again, we see that the equations of motion remain unaffected (since $\gamma^\nu \psi_\nu = 0$ onshell) and thus the resulting operator will still satisfy conditions 1 and 2. The purpose of this transformation is that now our operator contains a new constant, $k$, which we can use in satisfying the other conditions. Our general operator is then

$$\Lambda^{\alpha\beta} = \theta^{\alpha\lambda}(k) \tilde{\Lambda}^\beta_\lambda$$

$$= (\not{\!p} - m) g^{\alpha\beta} - \left[ \frac{a_2}{d} g_1 + \frac{1}{(d-1)} \right] \left( \gamma^\alpha p^\beta - p^\alpha \gamma^\beta \right) - \left[ \frac{ka_1 f_1}{d^2} + \frac{1}{(d-1)} \right] \not{\!p} \gamma^\alpha \gamma^\beta$$

$$+ \left[ 1 + \frac{a_1}{d} + k \left( \frac{1}{d} + \frac{a_1}{d} \right) \right] \frac{m \gamma^\alpha \gamma^\beta}{(d-1)} - \frac{(d-1)a_2 f_1 g_1}{md} \left[ \left( 1 + \frac{2k}{d} \right) p^\alpha p^\beta - \frac{k}{d} \not{\!p} \gamma^\alpha \gamma^\beta \right]$$

$$+ \frac{k}{d} \left[ \frac{(d-2)}{(d-1)} - a_2 g_1 \right] \gamma^\alpha p^\beta + \left[ \frac{2k}{d^2} a_1 f_1 + \frac{(d-2)}{d(d-1)} a_1 \right] p^\alpha \gamma^\beta.$$

(20)

Condition 3 requires the lagrangian to be linear in derivatives. Inspection of (20) reveals that the terms quadratic in $p^\alpha$ can be explicitly cancelled by setting $g_1 = 0$ or

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12See for example equation (4.2) in Johnson and Sudarshan [25] or equation (15) in van Nieuwenhuizen’s supergravity review [19].
\( f_1 = 0 \). This gives two paths to follow: setting \( g_1 = 0 \) gives

\[
\Lambda_1^{\alpha\beta} = (\not{\gamma} - m) \gamma^{\alpha\beta} - \frac{1}{(d-1)} (\gamma^{\alpha\beta} - p^{\alpha\gamma} p^{\gamma\beta}) - \frac{1}{(d-1)} \left[ k a_1 (d-2) \right] + 1 \not{\gamma}^{\alpha\gamma}.
\]

\[
+ \left[ 1 + \frac{a_1}{d} + \frac{k}{d} (1 + a_1) \right] d \gamma^{\alpha\beta} + \frac{k(d-2)}{d(d-1)} \gamma^{\alpha\beta} + \frac{a_1 (d-2)}{d(d-1)} \left[ \frac{2k}{d} + 1 \right] p^{\alpha\beta},
\]

whereas setting \( f_1 = 0 \) gives

\[
\Lambda_2^{\alpha\beta} = (\not{\gamma} - m) \gamma^{\alpha\beta} - \frac{1}{(d-1)} (\gamma^{\alpha\beta} - p^{\alpha\gamma} p^{\gamma\beta}) - \frac{1}{(d-1)} \not{\gamma}^{\alpha\gamma}.
\]

\[
+ \left[ \frac{\overline{a}_1}{d} + 1 \right] d \gamma^{\alpha\beta} + \frac{\overline{a}_1 (d-2)}{d(d-1)} \gamma^{\alpha\beta}.
\]

where we have defined \( \overline{a}_1 = a_1 + k (1 + a_1) \).

We now have two operators, the first (21) depending on two parameters, \( a_1 \) and \( k \), and the second (22) depending on only one, \( \overline{a}_1 \). Both operators satisfy conditions 1 through 3. The last condition to impose on our operators is condition 4, i.e. that the operators be hermitian. Imposing the hermitian requirement (11) on the first operator (21) fixes a relation between \( a_1 \) and \( k \):

\[
a_1^* = k
\]

thus reducing the number of parameters to one. The lagrangian (21) is then

\[
\Lambda_1^{\alpha\beta} = (\not{\gamma} - m) \gamma^{\alpha\beta} - \frac{1}{(d-1)} (\gamma^{\alpha\beta} - p^{\alpha\gamma} p^{\gamma\beta}) + \frac{(d-2)}{d^2 (d-1)} \left[ a_1^2 + \frac{d^2}{(d-2)} \right] \gamma^{\alpha\beta} \not{\gamma}^{\alpha\gamma}
\]

\[
+ \left[ d + a_1 + a_1^* + |a_1|^2 \right] \frac{d \gamma^{\alpha\beta}}{d(d-1)} + \frac{(d-2)}{d(d-1)} \left( a_1^* \gamma^{\alpha\beta} + a_1 p^{\alpha\gamma} \right).
\]

If we were to restrict \( a_1 \) to be real, and define the number \( A \) in terms of \( a_1 \) as

\[
a_1^* = a_1 = \frac{d(d-1)}{d-2} A + \frac{d}{d-2}
\]

then \( \Lambda_1^{\alpha\beta} \) would become

\[
\Lambda_1^{\alpha\beta} = (\not{\gamma} - m) \gamma^{\alpha\beta} + A (\gamma^{\alpha\beta} + p^{\alpha\gamma} p^{\gamma\beta}) + \frac{1}{(d-2)} \left[ (d-1) A^2 + 2A + 1 \right] \gamma^{\alpha\beta} \not{\gamma}^{\alpha\gamma}
\]

\[
+ \frac{m \gamma^{\alpha\gamma}}{(d-2)^2} \left[ d(d-1) A^2 + 4(d-1) A + d \right],
\]

which is the \( d \)-dimensional form of a common expression found in the literature [20, 22, 26, 27].

Imposing the hermitian requirement on our second operator (22), gives the condition

\[
\overline{a}_1 = 0
\]

\[
\Rightarrow a_1 = \frac{-k}{k+1}
\]
and the second lagrangian $\mathcal{L}_2$ becomes

$$\Lambda_2^{\alpha\beta} = (\not{p} - m) g^{\alpha\beta} - \frac{1}{(d-1)} \left( \gamma^\alpha p^\beta + p^\alpha \gamma^\beta \right) + \frac{1}{(d-1)} \gamma^\alpha (\not{p} + m) \gamma^\beta \equiv \Lambda_{RS}^{\alpha\beta} \quad (28)$$

which is the $d$-dimensional Rarita-Schwinger lagrangian [17] and has no arbitrary parameters.

We have thus found the most general set of $d$-dimensional lagrangians for a spin-$\frac{3}{2}$ field which satisfy our four conditions. These are given by (24) which depends on a complex parameter, and (28) which is the usual Rarita-Schwinger lagrangian and has no arbitrary parameters. In fact, the second lagrangian (28) corresponds to the case $a_1 \to 0$ of the first lagrangian (24). So the most general expression for the set of operators is given by the single expression (24) with the Rarita-Schwinger operator corresponding to the particular value $a_1 = 0$ of the arbitrary parameter. It will be shown in section 4 that this general operator can be written in a very simple manner as a transformation of the Rarita-Schwinger lagrangian with $a_1$ being the parameter of the transformation – a fact already noticed by Freedman and van Nieuwenhuizen [28] in 1976.

The propagator, $S_{\alpha\beta}$, for the spin-$\frac{3}{2}$ field is the inverse of the quadratic operator in the lagrangian

$$S_{\alpha\beta}\Lambda_{\beta\lambda} = \delta_{\lambda}^{\alpha}. \quad (29)$$

The definition of the projection operators (11) gives the equation

$$(p^2 - m^2) S_{\alpha\beta} = (\not{p} + m) P_{\alpha\beta} + \frac{(p^2 - m^2)}{m} \left[ \left( \frac{P_{11}^4}{a_1} \right)_{\alpha\beta} + \left( \frac{P_{22}^4}{a_2} \right)_{\alpha\beta} \right], \quad (30)$$

which can be solved in a similar way as we have done for the lagrangian, but we will skip this lengthy calculation and simply quote the result (32) below.

To summarize, the most general lagrangian for the spin-$\frac{3}{2}$ field consistent with our four conditions is

$$\mathcal{L} = \bar{\psi} \left\{ \left( \not{p} - m \right) g_{\alpha\beta} - \frac{1}{(d-1)} \left( \gamma_\alpha p_\beta + p_\alpha \gamma_\beta \right) + \frac{1}{(d-1)} \gamma_\alpha (\not{p} + m) \gamma_\beta 
+ \frac{(d-2)}{d^2(d-1)} |a|^2 \gamma_\alpha \not{p} \gamma_\beta + [a + a^* + |a|^2] \frac{m_\gamma_\alpha \gamma_\beta}{d(d-1)} + \frac{(d-2)}{d(d-1)} (a^* \gamma_\alpha p_\beta + a p_\alpha \gamma_\beta) \right\} \psi^\beta, \quad (31)$$

where $a$ is a complex parameter with the restriction $a \neq -1$. Notice that we have dropped the index on $a_1$ and will henceforth write it as simply $a$ since only this single parameter remains. The last line in (31) is a more compact way of writing the expression as will be

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13See also the equivalent, but much simpler, expression given by (48) and (51) in section 4 below.
shown in the next section. The propagator is

\[
S^{\alpha\beta} = \frac{1}{(p^2 - m^2)} \left[ g^{\alpha\beta} - \frac{1}{(d-1)} \gamma^\alpha \gamma^\beta - \frac{1}{(d-1)m} (\gamma^\alpha p^\beta - p^\alpha \gamma^\beta) - \frac{(d-2)p^\alpha p^\beta}{(d-1)m^2} \right] \\
+ \frac{(d-2)}{m^2(d-1)} (h^* \gamma^\alpha p^\beta + hp^\alpha \gamma^\beta - |h|^2 \gamma^\alpha \not{p} \gamma^\beta) + \frac{|h|^2d - h - h^* \gamma^\alpha \gamma^\beta}{(d-2)m} \\
= \theta_\alpha^\mu(\tilde{a}) S_{\mu\nu}^{RS} \theta^\nu_\beta(\tilde{a}^*),
\]

where \( h = \frac{d+a^*}{d(1+a^*)} \). Again, the last line in (32) is a more compact way of writing the expression as will be shown in the next section. The inverse parameter \( \tilde{a}^* \) is related to \( h \) in our expression as \( \tilde{a}^* = \frac{d(1-h)}{1-a} \). With real parameter \( A \) defined by (25) we get a propagator often used [20, 22, 23, 24, 29].

\[
S^{\alpha\beta} = \frac{(\not{p} + m)}{(p^2 - m^2)} \left[ g^{\alpha\beta} - \frac{1}{(d-1)} \gamma^\alpha \gamma^\beta - \frac{1}{(d-1)m} (\gamma^\alpha p^\beta - p^\alpha \gamma^\beta) - \frac{(d-2)p^\alpha p^\beta}{(d-1)m^2} \right] \\
+ \frac{A+1}{m^2(Ad+2)} \left\{ \frac{d - 4 - dA}{(d-2)(dA+2)} \right\} m\gamma^\alpha \gamma^\beta \\
+ \frac{(d-2)}{(d-1)} (\gamma^\alpha p^\beta + p^\alpha \gamma^\beta) - \frac{(d-2)(A+1)}{(d-1)(dA+2)} \gamma^\alpha \not{p} \gamma^\beta \\
= \theta_\alpha^\mu(\tilde{a}) S_{\mu\nu}^{RS} \theta^\nu_\beta(\tilde{a}^*),
\]

and setting \( A = -1 \) gives another common expression [21, 30, 31, 32, 33]. The Rarita-Schwinger lagrangian and propagator [17] correspond to the limits \( a \to 0 \) and \( h \to 1 \) whereas those of [19, 21, 30, 31, 33] correspond to \( a = -d \) (or \( h = 0 \)). The latter expression is often mistakenly cited as the original Rarita-Schwinger action in the supergravity literature.

Let us now take a moment to discuss the meaning of the arbitrary parameter \( a \). The vector-spinor field contains auxiliary lower spin components which are necessary in maintaining Lorentz covariance in our formalism. This is a generic feature of the treatment of higher spin fields in relativistic lagrangian field theory with constraints. In order to get a theory which satisfies our general conditions for the lagrangian, we introduced several arbitrary parameters. These parameters were not fully fixed by our conditions and instead we were left with one parameter, \( a \), remaining unfixed. We can see in (13) that \( a_1 \) and \( a_2 \) measure the relative strengths of the \( P_{11} \) and \( P_{22} \) parts of the field. Since \( a_1 \) and \( a_2 \) are inversely related by (17) increasing \( a \) corresponds to increasing the \( P_{11} \) part and decreasing the \( P_{22} \) part of the operator. Thus the parameter \( a \) is a measure of the proportion of the two different auxiliary spin-\( \frac{1}{2} \) components of the theory. The general conditions which we used in formulating the theory has resulted in this proportionality and transforming the parameter is equivalent to changing this proportionality. There is no choice of parameter which will eliminate both spin-\( \frac{1}{2} \) components simultaneously and hence our conditions have forced us to retain lower spin components in our theory, the ‘amount’ of each depending on the choice of parameter.

We can now return to the subject of gauge invariances that we touched upon previously and see how the invariances of equation (13) have been modified by our conditions. The
projection operators can be used to re-write the general lagrangian operator as
\[ \Lambda_{\alpha\beta} = (\not{\!m} - m) P^{2}_{\alpha\beta} + \left[ \frac{(d-2)}{(d-1)} |a|^2 (\not{\!m} - m) - m \left( a^* b + b^* a \right) \right] \frac{(d-1)}{d^2} (P^{*2}_{11})_{\alpha\beta} \\
+ \left[ (d-2) a^* b (\not{\!m} - m) - m \left( \frac{a^* (b + d)}{d - 1} - |b|^2 \right) \right] \frac{(d-1)}{d^2} (P^{*2}_{12})_{\alpha\beta} \\
+ \left[ (d-2) |b|^2 (\not{\!m} + m) + m \left( (b^* + d) b + b^* (b + d) \right) \right] \frac{(d-1)}{d^2} (P^{2}_{22})_{\alpha\beta} \\
- \left[ (d-2) a b^* (\not{\!m} + m) + m \left( \frac{(b^* + d) a}{d - 1} - |b|^2 \right) \right] \frac{(d-1)}{d^2} (P^{2}_{21})_{\alpha\beta}, \]

where we have defined \( b = \frac{a + d}{(1-d)} \). This is the analogous equation to (33) except that the action now satisfies our four conditions and so the symmetry of (33) has been reduced.

The free equations of motion \( \Lambda_{\alpha\beta} \psi^\beta = 0 \) are invariant under \( \psi^\beta \rightarrow \psi^\beta + \delta \psi^\beta \) whenever \( \delta \psi^\beta \) is annihilated by \( \Lambda_{\alpha\beta} \). Using (34) along with (3) we see that invariances occur only when the field is massless \( m = 0 \) and also either \( a = 0 \) or \( b = 0 \). The case \( a = 0 \) corresponds to the massless Rarita-Schwinger action and gives the invariance \( \delta \psi^\beta = (p^\beta - \gamma^\beta \not{\!m}) \epsilon \) for \( \epsilon \) an arbitrary spinor. This can be written as \( \delta \psi^\beta = \theta^{\beta\gamma}(-d) \partial_\gamma \epsilon \) using the results of the next section. The case \( b = 0 \) corresponds to \( a = -d \) and is the gravitino part of the linearized supergravity action. In this case, the projectors give the invariance \( \delta \psi^\beta = \partial^\beta \epsilon \) which is the linearized version of the gauge invariance in the gravitino action of supergravity\(^{14}\).

At first glance it may seem that the gauge invariance in the massless limit only occurs for two values of the parameter. In fact, these two invariances represent merely specific cases of the same general invariance\(^{15}\). We form the general gauge transformation by using the group properties of the point transformations which we discuss in the following sections \[31\], for example, \( \delta \psi^\alpha = \theta^{\alpha\beta} \psi^\beta \) is a gauge invariance for arbitrary parameter because the inverse transformation \( \theta^{\alpha\beta} \psi^\beta \) effectively sets \( b = 0 \), as will soon become clear.

### 4 The point transformation group

The transformation (19) that we have used in the previous section is sometimes referred to as a ‘point’ or ‘contact’ transformation in the literature. The Rarita-Schwinger equations (10) tell us that the onshell spin-\( \frac{3}{2} \) field \( \psi^\mu \) satisfies the following equation
\[ \theta^\alpha_\mu(a) \psi^\mu = \psi^\alpha \]

as can be seen by the form of the transformation (19): \( \theta^{\mu\nu}(a) = g^{\mu\nu} + \frac{2}{3} \gamma^\mu \gamma^\nu \), along with the onshell constraint \( \gamma \cdot \psi = 0 \). When \( a \neq -1 \) these transformations form a group with
\[ \theta^{\mu\nu}(a) \theta^\nu_\rho(b) = \theta^\rho_\nu(a + b + ab) \equiv \theta^\rho_\nu(a \circ b), \]
\[ (\theta^{\mu\nu})^{-1}(a) = \theta^{\mu\nu}(\frac{-a}{1+a}) \equiv \theta^{\mu\nu}(\bar{a}), \]

\(^{14}\)In curved space let \( \partial_\beta \rightarrow D_\beta = \partial_\beta + \frac{1}{2} \omega^a_\beta \gamma_a \) where \( a, b \) are flat indices, \( \omega^a_\beta \) is the spin connection and \( \gamma_a \equiv \frac{1}{2}[\gamma_a, \gamma_b]. \)

\(^{15}\)The author would like to thank V. Pascualtsa for pointing this out.
where we have defined the ‘circle’ operation \(a \circ b = a + b + ab\) and the inverse parameter \(\tilde{a} = \frac{-a}{1 + a}\). The lagrangian is tautologically invariant under the transformation \(\psi^\alpha \rightarrow \theta^\alpha_{\mu}(k)\psi^\mu\) if, in addition to transforming the fields, we also transform the parameter \(a\) as

\[
a' = \frac{a - k}{1 + k}.
\]

Our field transformation together with the transformation (37) of the parameter \(a\) are frequently called a point transformation.\(^{16}\)

It is worthwhile to notice that separate transformations from the left and from the right act separately on \(a\) and \(a^*\). If we recall how we formed our lagrangian by applying a transformation from the right and how the requirement that the operator be hermitian fixed \(k = a^*\), we can see that

\[
\theta^\alpha_{\mu}(k^*)\Lambda_{\alpha\beta}(a^*, a) = \theta^\alpha_{\mu}(k^*)\theta^\alpha_{\sigma}(a^*)\Lambda_{\alpha\beta}(a),
\]

\[
= \theta^\alpha_{\mu}(k^* \circ a^*)\Lambda_{\alpha\beta}(a)
\]

\[
= \Lambda_{\alpha\beta}(k^* \circ a^*, a).
\]

We find the left hand transformations by taking the hermitian conjugate,

\[
\Lambda_{\mu\nu}(a^*, a)\theta^\nu_{\beta}(k) = \Lambda_{\mu\beta}(a^*, a \circ k).
\]

Hence transforming \(\Lambda_{\mu\nu}(a^*, a)\) from the right affects only \(a\), whereas the transformation from the left affects only \(a^*\). We can use these left and right transformations, along with the definition of the Rarita-Schwinger lagrangian: \(\Lambda_{RS}^{\alpha\beta} = \Lambda_{RS}(0, 0)\), to write our most general lagrangian as

\[
\Lambda_{\alpha\beta}(a^*, a) = \Lambda_{\alpha\beta}(a^* \circ 0, 0 \circ a) = \theta^\alpha_{\mu}(a^*) \Lambda_{RS}^{\mu\nu} \theta^\beta_{\nu}(a).
\]

Substituting this expression into the relation (29) we have

\[
S_{\alpha\beta} \theta^\beta_{\mu}(a^*) \Lambda_{RS}^{\mu\nu} \theta^\nu_{\lambda}(a) = \delta^\lambda_{\alpha}
\]

\[
\Rightarrow \left[ \theta^{\sigma\alpha_{\mu}}(a) S_{\alpha\beta} \theta^{\beta\mu}(a^*) \right] \Lambda_{RS}^{\mu\nu} = \delta^\delta_{\nu}.
\]

and the expression in brackets in the last line must therefore equal the Rarita-Schwinger propagator. So the point transformation group properties have given us a convenient way of deriving the general propagator from the parameterless RS propagator which much simpler than the method used in section \(3\) namely:

\[
S_{\alpha\beta}^{RS} = \theta^\alpha_{\mu}(a) S_{\mu\nu} \theta^\nu_{\beta}(a^*)
\]

\[
\Rightarrow S_{\alpha\beta} = \theta^\alpha_{\mu}(\tilde{a}) S_{\mu\nu}^{RS} \theta^\nu_{\beta}(\tilde{a}^*).
\]

The point transformation becomes singular at the parameter value \(a = -1\) as can be seen by the fact that \(k \circ -1 = -1\) for any \(k\), so that

\[
\theta^{\mu\nu}(-1)\theta^\nu_{\lambda}(k) = \theta^\mu_{\lambda}(-1) \quad \forall k.
\]

\[\text{To compare with the notations of other authors, notice that in the case where } k \text{ and } a \text{ are real we can use } \Box \text{ to rewrite the transformation of the parameter } a \text{ in terms of } A \text{ as } A \rightarrow \frac{A - 2k/d}{1+k}. \text{ Also note our factor of } \frac{1}{d} \text{ in the group law.}\]
In the singular case $a = a^* = -1$ the transformations of the fields would no longer change the lower spin content of the operator and this choice may appear quite attractive. The problem with the choice $a = -1$ is that the operator in the lagrangian no longer has an inverse \[20\] as can be seen by the form of $h$ and $h^*$ in the propagator (32). Interestingly, the singular value, $a = -1$, generates the additive identity element (or ‘zero’) of a ring defined by the following addition rule

$$
\theta_{\mu\nu}(a) + \theta_{\mu\nu}(b) = \theta_{\mu\nu}(a + b + 1),
$$
$$
\theta_{\mu\nu}(a) - \theta_{\mu\nu}(b) = \theta_{\mu\nu}(a - b - 1),
$$

where the addition is defined \textit{modulo zero}, the two-sided ideal generated by $\theta_{\mu\nu}(-1)$. It is easily shown that the multiplication is distributive over addition.

One can redefine the parameter in various ways to make the group more convenient. For example, by shifting the singular point of the parameter space to $-\infty$ by letting $a \rightarrow e^{\alpha} - 1$ the group becomes \[31\]

$$
\theta_{\mu\nu}(\alpha) = g_{\mu\nu} + \frac{e^\alpha - 1}{d} \gamma_{\mu} \gamma^\nu = e^{\frac{\alpha}{d}} \gamma_{\mu} \gamma^\nu,
$$
$$
\theta_{\mu\nu}(\alpha) \theta_{\nu\lambda}(\beta) = \theta_{\lambda}(\alpha + \beta),
$$

$$
(\theta_{\mu\nu})^{-1}(\alpha) = \theta_{\mu\nu}(-\alpha).
$$

An even more convenient redefinition is so that the singular point is at 0. One has $a \rightarrow \alpha - 1$ and the ring is then defined by

$$
\theta_{\mu\nu}(\alpha) = g_{\mu\nu} + \alpha - 1 \gamma_{\mu} \gamma^\nu, \quad \text{(definition)}
$$
$$
\theta_{\mu\lambda}(\alpha) \theta_{\nu}^\nu(\beta) = \theta_{\lambda}(\alpha, \beta), \quad \text{(multiplication)}
$$
$$
\theta_{\mu\nu}(1) = g_{\mu\nu}, \quad \text{(multiplicative identity)}
$$
$$
(\theta_{\mu\nu})^{-1}(\alpha) = \theta_{\mu\nu}(\frac{1}{\alpha}), \quad \text{(multiplicative inverse)}
$$
$$
\theta_{\mu\nu}(\alpha) + \theta_{\mu\nu}(\beta) = \theta_{\mu\nu}(\alpha + \beta), \quad \text{(addition)}
$$
$$
\theta_{\mu\nu}(0) = g_{\mu\nu} - \frac{1}{d} \gamma_{\mu} \gamma^\nu, \quad \text{(additive identity)}
$$
$$
\theta_{\mu\nu}(\alpha) - \theta_{\mu\nu}(\beta) = \theta_{\mu\nu}(\alpha - \beta), \quad \text{(additive inverse)}
$$

where the addition is again defined modulo the additive identity.

5 Interactions

The path integral is invariant under a global point transformation of the fields since the functional determinant is trivial and factors out of the integral to be cancelled out of the generating functional by the identical factor in the denominator. Hence there are no path integral anomalies and all physical correlation functions are independent of the parameter $a$. This is also true in the interacting theory, as we will discuss shortly, and so all physical Green functions are independent of $a$ and it can be fixed to whatever value is convenient. As was pointed out by Nath, \textit{et al.} \[23\], the meaning of the invariance under point transformations is that the physical content of the theory does not depend on the parameter. However, the classical equations of motion in the presence of interaction are
not invariant under shifts of the parameter. This is due to the fact that the transformation is not unitary. This will be discussed further in the next section.

There has been some controversy about how to include consistent interactions involving spin-\(\frac{3}{2}\) fields \[20, 23, 31, 34\]. We find the most logical way is to require the interaction terms to transform the same way under point transformations as the free action as well as remaining consistent with the massless gauge invariance of the free action. In other words, we require that a shift of the parameter must leave the form of the entire action unchanged. This ensures that the path integral remains independent of the parameter and is the reason why many authors require the action to be invariant under a point transformation combined with a compensating parameter shift. This requirement means that we must have a factor of \(\theta_{\mu\nu}(a)\) for each \(\psi^\nu\) field and thus a factor of \(\theta_{\mu\nu}(a^*)\) in the hermitian conjugate so that a transformation \(\psi^\alpha \rightarrow \theta^\alpha_\beta(k)\psi^\beta\) will shift \(a \rightarrow a \circ k\) everywhere\(^{17}\).

Expression (40) shows that, for interactions which depend on the parameter in the same way as the lagrangian operator in (38), the inverse transformations attached to the RS propagator in (40) will cancel those of the interaction and will therefore lead to a theory whose Green functions are independent of the parameter. We can see schematically that the correlation functions are invariant under redefinitions of the parameter as follows. In a theory satisfying our requirements the propagator and vertex can be written, respectively, as (suppressing all indices for simplicity)

\[
S(a) = \theta^{-1} S \theta^{-1},
\]

\[
\Lambda(a) = \theta \Lambda \theta,
\]

where the \(a\) dependence on the right-hand-side is entirely contained in the \(\theta\) factors: \(S\) and \(\Lambda\) are independent of \(a\). This implies that products of these expressions found in Feynman diagrams will always reduce to the form

\[
G(a) \sim \Lambda(a) S(a) \Lambda(a) \cdots \Lambda(a) S(a) \Lambda(a),
\]

\[
= \theta G \theta.
\]

Similarly in the case of interactions with scalar fields, spinor fields, etc. For example, consider a diagram which might appear in resonant pion photoproduction at one-loop order shown in figure 1. The parameter dependence of the Green function is contained in the vertices and the spin-\(\frac{3}{2}\) resonance propagator. Our conditions require that the vertices and propagators can be written as

\[
\Lambda_{\pi N\Delta}(a) = \Lambda_{\pi N\Delta} \theta,
\]

\[
\Lambda_{\gamma N\Delta}(a) = \Lambda_{\gamma N\Delta} \theta,
\]

\[
\Lambda_{\pi\Delta\Delta}(a) = \theta \Lambda_{\pi\Delta\Delta} \theta,
\]

\[
S_\Delta(a) = \theta^{-1} S_\Delta \theta^{-1},
\]

\[
S_\pi(a) = S_\pi,
\]

\(^{17}\)Note that in order to be consistent with the massless gauge invariance of the theory, the interaction must depend only on \(d\psi\) rather than simply \(\psi\) and also there should be no ‘offshell parameter’ dependence. This is discussed by Pascalutsa and Timmermans in \[31\].
Figure 1: A loop diagram found in resonant photo-pion production exemplifying the cancellation of parameter dependence in Green functions. The thin lines are spin-$\frac{1}{2}$ (nucleons), the dashed lines are spin-0 (pions), the wavy line is spin-1 (photon) and the thick line is the Rarita-Schwinger, spin-$\frac{3}{2}$ (e.g. $\Delta(1232)$) field.

where again the $a$ dependence is entirely contained within the $\theta$ factors. The Green functions are found as in the following example

\[
G(a) \sim \Lambda_{\pi N\Delta}(a) S_\Delta(a) \Lambda_{\pi \Delta\Delta}(a) S_\Delta(a) S_\pi \Lambda_{\pi \Delta\Delta}(a) \Lambda_{\pi N\Delta}(a), \]

\[
= \Lambda_{\pi N\Delta} S_\Delta \Lambda_{\pi \Delta\Delta} S_\Delta S_\pi \Lambda_{\pi \Delta\Delta} S_\Delta \Lambda_{\pi N\Delta},
\]

(47)

This example shows the general pattern: all correlation functions for interacting theories are independent of the arbitrary parameter. This is true at every level of perturbation theory and at the non-perturbative level as well since the symmetry under $\theta$ is non-anomalous as we have discussed above.

### 6 Conserved vector current

We will now examine one of the ways in which the point transformation invariance of the correlation functions can be exploited. We will find that the analysis becomes simpler if we reparametrize our general action in terms of $\hat{b} = \frac{a + d}{1 - d}$ so that the general lagrangian

\[
\mathcal{L} = \bar{\psi}_\alpha \left( \Gamma^{\alpha\mu\beta} i \partial_\mu - m \Gamma^{\alpha\beta} \right) \psi_\beta,
\]

(48)

where

\[
\Gamma^{\alpha\mu\beta} = g^{\alpha\beta} \gamma^\mu - A_1 g^{\mu\beta} \gamma^\alpha - A_2 g^{\mu\alpha} \gamma^\beta + A_3 \gamma^{\alpha\beta} \gamma^\mu, \\
\Gamma^{\alpha\beta} = g^{\alpha\beta} - A_4 \gamma^{\alpha} \gamma^\beta.
\]

(49)

The coefficients are defined in terms of $b$ by

\[
A_1 = 1 + \frac{(d - 2)}{d} b^*, \quad A_2 = 1 + \frac{(d - 2)}{d} b \\
A_3 = 1 + \frac{(d - 2)}{d} \left[ \frac{(d - 1)}{d} |b|^2 + b^* + b \right] \\
A_4 = 1 + \frac{(d - 1)}{d} \left[ |b|^2 + b^* + b \right].
\]

(50)

The reason for this reparametrization is that we now have

\[
\mathcal{L} = \bar{\psi}^\alpha \theta_{\alpha\mu}(b^*) (\gamma^{\mu\nu} i \partial_\nu + m \gamma^{\mu\nu}) \theta_{\nu\beta}(b) \psi^\beta \equiv \bar{\psi}^\alpha \theta_{\alpha\mu}(b^*) \Lambda^{\mu\nu}_{SG} \theta_{\nu\beta}(b) \psi^\beta,
\]

(51)
where the totally antisymmetric combinations of gamma matrices can be written as 
\[ \gamma^{\mu\rho\nu} = \frac{1}{2} (\gamma^\mu \gamma^\rho \gamma^\nu - \gamma^\nu \gamma^\rho \gamma^\mu) \] and \( \gamma^{\mu\nu} = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \). The parameter choice \( b = 0 \) now corresponds to the expression, \( \bar{\psi}^\mu S_{\mu\nu}^{SG} \psi^\nu \), commonly found \[19, 31, 33\] as the massive gravitino action in linearized supergravity\(^{18}\). The propagator is then

\[ S^{\alpha\beta} = \theta^{\alpha\mu}(\tilde{b}) S_{\mu\nu}^{SG} \theta^{\nu\beta}(\tilde{b}^*) \],

(52)

where \( S_{\mu\nu}^{SG} \) is found from (32) by setting \( h = 0 \).

Notice that the choice of parameter giving the supergravity action is such that the dimension of spacetime, \( d \), does not explicitly appear. This, along with the antisymmetry of \( \gamma^{\mu\rho\nu} \) and \( \gamma^{\mu\nu} \), make many manipulations simpler and many results more transparent. Hence, we will use this expression for the general action from now on.

As we have seen, the observables of the quantum theory are independent of the parameter choice. We would like to explore the consequences of this invariance. However, the classical action is not invariant under the transformation since the symmetry transformation \( \theta(k) \) of the field is in general non-unitary, \( \theta(k^*) \neq \theta^{-1}(k) \). Under a global point transformation, \( \psi_\mu \to \theta_\mu(k) \psi_\nu \), the lagrangian \[18\] is not invariant, but transforms as \( \mathcal{L}(b) \to \mathcal{L}(b \circ k) \). In the absence of interactions the equations of motion are invariant since the free field equations of motion imply that \( \gamma \cdot \psi = 0 \) and this makes the point transformation \[19\] trivial. However, in the interacting theory this is no longer true. In order to explore the consequences of the symmetry we will therefore impose it on the classical action by using the following technique. We simply demand that:

- classical actions will be considered equivalent if they lead to the same physical observables.

Put in another way this says:

- classical actions will be considered equivalent if they are related by a circle-shift redefinition of the parameter.

This makes the point transformations a symmetry of equivalence classes of classical actions and we can examine the consequences.

Let us re-iterate what we mean by this equivalence to avoid any possible confusion. We have shown that \( \mathcal{L}(b) \) and \( \mathcal{L}(b \circ k) \) lead to exactly the same physical correlation functions. These lagrangians are not the same, nor are they related by any gauge symmetry -- point transformations are not gauge transformations. By considering these two different lagrangians as being equivalent, we are saying that any interaction whose only effect is to change \( \mathcal{L}(b) \) into \( \mathcal{L}(b \circ k) \), for some constant \( k \), will have no effect on observable physics. We now want to examine the consequences of this equivalence.

For simplicity, we will re-write our action so that it is symmetric in derivatives\(^{19}\) and we will restrict the parameter to be real. Thus

\[ \mathcal{L}(b) = \bar{\psi}_\alpha \left[ \frac{1}{2} \Gamma^{\alpha\beta}(b) i \overset{\leftrightarrow}{\partial}_\rho + m \Gamma^{\alpha\beta}(b) \right] \psi_\beta, \]

(53)

\(^{18}\)Many other expressions are found in the supergravity literature (for examples see Refs. \[3, 4, 35\]) but they are all found to be equivalent to ours by using some choice of parameter \( a \) and/or using the identity \( \gamma^{\mu\rho\alpha} = i \epsilon^{\mu\rho\alpha\beta\gamma} \gamma_\gamma \).

\(^{19}\)The reason is so that the analysis results in an hermitian current.
where we have written
\[
\Gamma^{\alpha\rho\beta} (b) = \theta^\alpha \mu (b) \gamma^{\mu\nu} \theta^\beta \nu (b), \\
\Gamma^{\alpha\beta} (b) = \theta^\alpha \nu (b) \gamma^{\mu\nu} \theta^\beta \nu (b), \\
\frac{\partial}{\partial \rho} = \frac{\partial}{\partial \rho} - \frac{\partial}{\partial \rho}. 
\] (54)

Under an infinitesimal local point transformation \( \theta (k(x)) \) the lagrangian varies as \( \mathcal{L}(b) \to \mathcal{L}(b \circ k) + \delta \mathcal{L} \) where \( \delta \mathcal{L} \) contains the derivative acting on the parameter and \( \mathcal{L}(b \circ k) \) is defined in exactly the same way as \( \mathcal{L}(b) \) in (53) with the derivatives acting only on the fields and not on the parameter. Explicit computation gives
\[
\delta \mathcal{L} = \frac{i}{2d} \psi_\alpha \left[ \Gamma^{\alpha\rho\beta} (b) \gamma_\rho \gamma_\nu - \gamma^{\alpha\gamma} \nu \gamma^{\beta\rho} \nu \right] \psi_\beta (\partial_\rho k). 
\] (55)

Integrating by parts and discarding the surface term we have
\[
\mathcal{L}(b) \to \mathcal{L}(b \circ k) - \frac{1}{2d} (\partial_\rho J^\rho) k(x), 
\] (56)

where \( J^\rho \) is given by
\[
J^\rho = i \psi_\alpha \left[ \Gamma^{\alpha\beta} \gamma_\rho \gamma_\nu - \gamma^{\alpha\gamma} \nu \gamma^{\beta\rho} \nu \right] \psi_\beta. 
\] (57)

Our symmetry says that \( \mathcal{L}(b) = \mathcal{L}(b \circ k) \) in the limit that \( k(x) \) becomes constant. This demands that \( \delta \mathcal{L} = 0 \) in the limit of constant \( k(x) \). Hence from (56) we find a conserved current \( J^\rho \) associated to the global symmetry: \( \partial_\rho J^\rho = 0 \). We see by (57) that the current changes under point transformations by a circle-shift of the parameter and is therefore invariant according to our symmetry. We can expand the \( \Gamma^{\alpha\rho\beta} \) to find a simpler expression of the current as follows
\[
J^\rho = i (1 + b) \psi_\alpha \left[ \gamma^\alpha g_\rho \gamma^\beta - g^{\alpha\gamma} \nu \gamma^{\beta\rho} \nu \right] \psi_\beta, \\
= i (1 + b) \left[ \bar{\psi} \cdot \gamma \psi^\rho - \bar{\psi} \gamma^\rho \cdot \psi \right]. 
\] (58)

Under a transformation \( \theta (k) \) the only change is the coefficient \( (1 + b) \to (1 + b \circ k) \). The conserved charge is given by
\[
Q = i (1 + b) \int d^{d-1} x \left[ \bar{\psi} \cdot \gamma \psi^0 - \bar{\psi}^0 \gamma \cdot \psi \right]. 
\] (59)

The charge can be put in a more suggestive form by defining \( \chi_1 = \gamma \cdot \psi \) and \( \chi_2 = \gamma^0 \psi^0 \) leaving
\[
Q = i (1 + b) \int d^{d-1} x \left[ \chi_1^\dagger \chi_2 - \chi_2^\dagger \chi_1 \right]. 
\] (60)

Since we have a conserved current, \( J^\rho \), we can couple a vector field such as the photon to it as follows
\[
\mathcal{L}_\gamma = g J_\mu A^\mu \\
= ig (1 + b) \left[ \bar{\psi} \cdot \gamma \gamma_\mu - \bar{\psi} \gamma_\mu \gamma \cdot \psi \right] A^\mu, 
\] (61)
where $g$ is a coupling constant. If this coupling is physically reasonable, then it should, among other things, have a measurable effect on the magnetic moment of the spin-$\frac{3}{2}$ particle. We can also form derivative interactions with scalar fields, such as the pion, as

$$\mathcal{L}_\pi = g_\pi J_\mu \partial_\mu \phi,$$

where $\phi$ is the scalar field. Furthermore, we also have the usual conserved vector current coming from electromagnetic gauge symmetry. This is given by

$$j^\mu = \overline{\psi}_\alpha \Gamma^{\alpha\beta} \psi_\beta = \overline{\psi} \theta_{\alpha\mu} (b^*) \gamma^{\mu\rho\sigma} \theta_{\nu\beta} (b) \psi^\beta. \quad (62)$$

We can use (48) to write this as

$$j^\mu = \overline{\psi}_\beta \gamma^\mu \psi^\beta - A_1 \left( \overline{\psi} \cdot \gamma \right) \psi^\mu$$

$$- A_1 \overline{\psi}^\mu \left( \gamma \cdot \psi \right) + A_3 \left( \overline{\psi} \cdot \gamma \right) \gamma^\mu \left( \gamma \cdot \psi \right), \quad (64)$$

where now $A_1 = A_2$ since the parameter is now real. The definitions of $\chi_1$ and $\chi_2$ allow us to write the charge as

$$Q_{\text{EM}} = \int d^{d-1}x \left[ \overline{\psi}^\dagger \psi^\beta - A_1 \left( \chi_1^\dagger \chi_2 + \chi_2^\dagger \chi_1 \right) + A_3 \chi_1^\dagger \chi_1 \right], \quad (65)$$

and we see that our new symmetry charge (60) involves only the cross terms contained in the usual electromagnetic charge (63).

The two currents (61) and (64) are separately conserved since they come from independent symmetries and hence we can form linear combinations of the two. Both currents couple to the lower spin components of the vector-spinor field and we adjust how much influence these lower spins have. It may be possible, with judicious choices of couplings, to eliminate the contribution of one or the other of the lower spins altogether by eliminating the cross term which contains both spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ components. Because of this new freedom, it seems that this new current will have some influence on the inconsistency problems that have been found in all interactions involving spin-$\frac{3}{2}$ fields [25, 36, 37]. Perhaps the inconsistencies can be made to cancel between the the different conserved currents so that the new symmetry can be used to find solutions to that long-standing problem. We will turn now to a bit of a review of the consistency problem, formulated in $d$-dimensions with arbitrary complex parameter as well as some ideas about the possibility of solutions (or lack thereof).

## 7 Consistency problems

As we have mentioned in the introductory section, the problem of finding consistent interactions for the spin-$\frac{3}{2}$ field is an old one. It was first pointed out in paper by Fierz and Pauli in 1939 [38] where the two-component spinor formalism was used to derive lagrangians for massive spin-$\frac{3}{2}$ and spin-2 with a minimally coupled electromagnetic field. They pointed out that subsidiary conditions are necessary to reduce the number of independent field components to the physical number. The subject of spin-$\frac{3}{2}$ was revisited by Rarita and Schwinger in 1941 [17] who developed the notation and action that are now
most often used. They also noticed that the theory was not unique and that a collection of actions all give equivalent theories. This is the same non-uniqueness that we have exploited in the present paper.

Consistency problems with the theory were further discussed in 1961 by Johnson and Sudarshan (JS) [25] who showed that in the presence of an electromagnetic field the field anti-commutator becomes indefinite, i.e. a Lorentz frame can always be found in which it is negative. A related discovery by Velo and Zwanziger (VZ) in 1969 [37] was that there are modes of the field which propagate faster than light. These problems derive from precisely the reasons given by Fierz and Pauli in 1939, namely that the lower spin particles should be removed by subsidiary conditions or they will give rise to negative energy states and indefinite charges\(^{20}\) and the number of subsidiary conditions must remain invariant to the presence of interaction.

In higher spin field theories involving auxiliary components and constraints, the time derivative operator of the lagrangian is a singular matrix. That this is necessary can be exemplified with our spin-3\(^{2}\) operator in (48). The dynamical term, which contains the time derivative operator in the equations of motion, is given by

\[ \Gamma^{a0\beta} i\partial_0 \psi_{\beta}, \tag{66} \]

where \(\Gamma^{a0\beta}\) has suppressed spin indices and can be viewed, in 4 dimensions, as a 16 \(\times\) 16 matrix. The equations of motion imply a set of constraints (equations among the field components which do not contain time derivatives) and so not all of the components of \(\psi_{\beta}\) are dynamical and these non-dynamical components will not appear in (66). Hence, there must exist vectors, corresponding to the non-dynamical components, which are annihilated by the above matrix. Therefore the matrix is necessarily singular and will have a determinant of zero.

In the case of spin-3\(^{2}\), the constraints implied by the equations of motion are: the primary constraint, found by applying \(\theta^a_{\alpha}(\tilde{b}^*)\) to the equations of motion (67), and the secondary constraint, found by taking the covariant derivative of the equations of motion. In four spacetime dimensions, as we have mentioned, these constitute a total of eight constraints needed to reduce the 16 component vector-spinor field to 2(2s + 1) = 8 on-shell degrees of freedom. In order to obtain a non-singular operator in (66) we must impose the constraint equations and eliminate the non-dynamical components in terms of dynamical ones, thus giving the on-shell equations of motions in terms of a new non-singular matrix.

In the free field case everything works perfectly and the non-dynamical components can be eliminated without trouble. However, a problem occurs with the constraints in the presence of interaction in that the secondary constraint becomes dependent on the external field. This fact can lead to inconsistencies as we will now show for the canonical example of electromagnetic minimal coupling. Our general action in the case of electromagnetic minimal coupling is found by inserting a covariant derivative into (51), giving the following equations of motion

\[ \theta^a_{\alpha}(b) (\gamma^{\mu\nu} iD_\rho + m\gamma^{\mu\nu}) \theta^b_{\beta}(b)\psi^\beta = 0, \tag{67} \]

where \(D_\rho = \partial_\rho + ieA_\rho\) is the covariant derivative. If we apply \(\theta^{a*}_{\alpha}\) to (67) we are left with an expression which contains no time derivatives (since \(\gamma^{\mu\nu}\) vanishes when two indices

\(^{20}\)However, see [2] for an interesting alternative to this.
are zero) and is the primary constraint. To derive the secondary constraint we apply $\gamma^\alpha$ to (67) to get the ‘useful relation’

$$iD \cdot \psi = \left( \frac{i(d-1)b+d}{d} p + \frac{(d-1)(1+b)}{d-2} m \right) \gamma \cdot \psi. \tag{68}$$

Now take the covariant derivative of (67), using (68) as well as the identities

$$\gamma_{\mu\nu} D^\mu D^\nu = \frac{ie}{2} \gamma_{\mu\nu} F_{\mu\nu},$$

$$\bar{p} p = \frac{ie}{2} \gamma_{\mu\nu} F_{\mu\nu} + \frac{1}{2} D^2,$$

to arrive at the secondary constraint

$$\left( \frac{ie}{2} \frac{(d-2)b+d}{d} \gamma_{\mu\nu} F_{\mu\nu} + \frac{(d-1)(1+b)}{d-2} m^2 \right) \gamma \cdot \psi = ie \gamma_{\mu} F^{\mu\nu} \psi_\nu. \tag{70}$$

In four spacetime dimensions we can use the relation$^{22}$

$$\frac{1}{2} \gamma_{\mu\nu} F^{\mu\nu} \gamma \cdot \psi = \gamma_{\mu} F^{\mu\nu} \psi_\nu - i \gamma^5 \gamma_{\mu} F^{\mu\nu} \psi_\nu,$$ \tag{71}

with the dual field strength $\star F^{\mu\nu} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}$ to write (70) in the simpler form

$$\gamma \cdot \psi = - \frac{ie}{3(1+b)m^2} \gamma_{\mu} F^{\mu\nu} \psi_\nu, \tag{72}$$

where

$$F^{\mu\nu} = b F^{\mu\nu} + i(b+2) \gamma^5 \star F^{\mu\nu}. \tag{73}$$

It should now be clear that the free field constraint $\gamma \cdot \psi = 0$ is no longer true unless the interaction vanishes.

Now that we have the constraints in the presence of electromagnetic minimal coupling we can describe the general case of the well known problem $^{36}$. The secondary constraint (72) can be written as

$$\left( \gamma^0 + \frac{ie}{3(1+b)m^2} \gamma_{\mu} F^{\mu0} \right) \psi^0 = \left( \gamma^k + \frac{ie}{3(1+b)m^2} \gamma_{\mu} F^{\mu k} \right) \psi^k, \tag{74}$$

and one can easily see from the equations of motion (67) that, in the case $b = 0$, the components $\psi^0$ are non-dynamical. In that case they must be fixed by the above condition so that they can be eliminated in terms of the dynamical field components. To do this, one must solve (74) for $\psi^0$. This requires that the matrix $\left( \gamma^0 + \frac{ie}{3(1+b)m^2} \gamma_{\mu} F^{\mu0} \right)$ be non-singular. This implies, since $b = 0$, that the matrix $\left( 1 - \frac{2e}{3m^2} \gamma^0 \gamma^5 \cdot \tilde{B} \right)$ is non-singular, which in turn requires that $\left( 1 + \frac{2e}{3m^2} \tilde{\sigma} \cdot \tilde{B} \right)$ is non-singular. Taking the determinant shows

$^{21}$A similar identity will hold in any even dimension where there exists a $\gamma^5$–like matrix $^{24}$.

$^{22}$We are using a mostly minus metric, diagonal $\gamma^0$, and $\gamma^5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3$. 

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that this latter matrix is singular when $|\vec{B}|^2 = \left(\frac{3m^3}{2e}\right)^2$. This is the famous result of [25] and [37] and means that if this equation is used to eliminate $\psi^0$ in the classical equations of motion one would find that, at this value of the external field, not all of the components of $\psi^0$ are determined and these undetermined components lead to space-like characteristic surfaces and the possibility of field modes propagating acausally. These non-physical characteristic surfaces indicate that the number of constraints has changed and that non-dynamical field components are still present in the field equations. For example, the operator coefficient to the time derivative in the equations of motion will still be a singular matrix at this external field value.

Note that we have conducted the analysis for $b = 0$, but we have lost no generality since the point transformation invariance can be used to transform this to the case of arbitrary $b$. To see this, let us briefly outline the method [37, 42] that is normally used to analyze the consistency of the classical field equations. The basic idea is to look at the characteristic surfaces given by the classical on-shell differential equation. To do this one plugs the constraints into the equations of motion, thus (hopefully) eliminating all of the non-dynamical degrees of freedom. Then the characteristics are found by replacing the derivative in the equation with a four-vector $n^\mu$. This vector will be a normal to a characteristic surface (a surface along which the maximum velocity solutions to the differential equation are restricted to propagate) if it is the solution to a certain equation (75). If we want only time-like solutions, so that causality is preserved, we would like the normals to the characteristic surfaces to remain always space-like.

The normals to the characteristic surfaces $n_\mu$ are given by the determinant

$$D(n) = \left| \hat{\Gamma}^{\alpha\mu\beta} n_\mu \right| = 0, \quad (75)$$

where we have replaced the derivatives in the field equations with $n_\mu$ and $\hat{\Gamma}^{\alpha\mu\beta}$ is the operator in the field equations after all of the constraints have been imposed. If there is a time-like normal we can use Lorentz invariance to write it as $n_\mu = (n, 0, 0, 0)$. Our characteristic equation is then

$$D(n) = n^{16} \left| \hat{\Gamma}^{\alpha0\beta} \right| = 0. \quad (76)$$

Notice that if this determinant vanishes for $b = 0$, then it also vanishes for arbitrary $b$ by the properties of the point transformations. The vanishing of the determinant means that there are characteristic surfaces which are space-like, indicating the possibility that field components propagate acausally. We say ‘indicating the possibility’ of acausal propagation because if the determinant was zero, so that there are space-like characteristics, one would still have to prove that there were actual physical field modes that propagate along these acausal characteristics. This is usually done with the method of shock discontinuities [42]. We will not discuss this method further since it has been established many times that there are field modes which propagate acausally in various cases. On the other hand, to prove that a theory is consistent one would only need to show that there are no space-like characteristic surfaces and it would follow that there is no acausal propagation, without need of the method of discontinuities.

Notice that if we didn’t impose constraints, the matrix in the determinant equation (76) would be the same one that we discussed earlier (66) which is necessarily singular.
Thus, if the constraints failed to eliminate a non-dynamical field component. This component would remain in the field equations but it would, by definition, not appear in the time derivative part since it is non-dynamical. Hence the matrix coefficient of the time derivative $\partial_0$ would annihilate the 16-component vector representing this component. The matrix then annihilates a non-zero vector and so it must have determinant equal to zero. Conversely, the physical components all have time derivatives which means the matrix coefficient of the time derivative does not annihilate them.

It is interesting that the inconsistency comes from the breakdown of constraints and is caused by non-dynamical, ‘unphysical’ components creeping back into the field equations at certain values of the external field and the presence of these components leads to physical information propagating acausally. Therefore, if we could find a covariant method of preventing these extra components from coming back the problems would be solved. Unfortunately, this seemingly simple task is a subtle one and has not been accomplished in general since the problem was discovered over 40 years ago although there have been many papers written on the subject\(^{23}\). We do not claim to have solved the problem here either, but we will present an idea that may be of some use in the search for a solution or a proof that one can’t exist.

We should mention that, in the context of supergravity, when the mass of the spin-$\frac{3}{2}$ field is tuned in certain ways to the background spacetime, there are theories which seem to be free of inconsistencies \(^7\). On the other hand, if we would like to use the theory to effectively model low mass spin-$\frac{3}{2}$ fields (such as the $\Delta(1232)$ nucleon resonance) at low energies one cannot expect the background spacetime or the existence of supersymmetry partners to be of much help.

We have exemplified the problem for the case $b = 0$ (i.e. the supergravity action) but, as we have said, the same is true for arbitrary $b$ since the lagrangian operators are related by a non-singular point transformation and so if the determinant (75) is zero for one value of the parameter, a point transformation will not alter this and it will be zero for any parameter value.

In any case, we can now ask: is it possible that the new symmetry current and charge can help us? The conserved charge arises from a symmetry of the path integral which is not shared by the usual classical theory, so if a solution to this problem could be found from this charge it would have the happy consequence that the inconsistency which is present in the classical theory would not affect the physical correlation functions. Conversely, it is also possible that the extra symmetry of the path integral may make things even worse.

Suppose we begin with no interaction and then slowly turn on an external field. In that case we can argue that the charge should remain zero since it is zero in the free case and is conserved. The charge \(^{59}\) is given in an electromagnetic background by

$$Q = i(1 + b) \int d^3 x \left[ \bar{\psi}_\nu (F^{\nu\mu})^* \gamma_\mu \psi^0 - \bar{\psi}^0 \gamma_\mu F^{\mu\nu} \psi_\nu \right].$$

The conservation of charge is related to the conservation of constraints since the spin-$\frac{1}{2}$ fields $\gamma \cdot \psi$ and $\psi^0$ in the charge are exactly the ones which are usually removed via the constraint structure of the free field Rarita-Schwinger equations. Hence it seems possible

\(^{23}\)We should mention the recent developments on a new type of spin 1/2 field with mass dimension one (called ‘Elko’) given in reference \(^{39}\). That work can be extended to spin 3/2 and may lead to new results for the consistency problem.
that the loss of constraints, which signals the onset of the inconsistency, will have a direct
effect on the charge. This gives us an idea for a possible new direction. When the external
field is such that there is a loss of constraints the charge should remain unaffected since
it is conserved. It is possible then, that in that case, the equation for the charge would
represent an additional constraint which could be used to eliminate the unphysical fields
which appear when the usual constraints breakdown. However, we do not want the extra
constraint in general, but only in the case when the usual constraints breakdown and then
only so as to exactly compensate for the breakdown. This seems like quite a demand. If
the conservation of charge implied additional contraints in general we would then have
too many constraints as soon as the background is turned on. So physically we want
the charge to be identically zero when the constraints are imposed. The only time the
situation changes is when the usual constraints break down. In that case we want the
charge to no longer vanish identically, but to itself become a constraint compensating for
the ones that were lost. The usual constraints are given by $\gamma \cdot \psi \sim \gamma_\mu F^{\mu\nu} \psi_\nu$ and this
combination then also appears in the expression for the charge. So perhaps it is possible
that a breakdown of this equation would have a compensating effect in the conserved
charge. In any case, we will leave this problem for now with the hope that these thoughts
may inspire new angles of attack so that either a new solution, or a new reason for the
lack of one, can be found.

8 Conclusion

We will now summarize the main features of our work. We have derived the general
lagrangian and propagator for the Rarita-Schwinger field in $d$-dimensions. These are given
by equations (48) and (32) respectively and should prove useful in calculating higher loop
effects in dimensional regularization as would occur in the effective resonance contribution
to the imaginary part of pion scattering amplitudes, anomalous magnetic moments, and
many other processes for which the $\Delta(1232)$ resonance or any other spin-$\frac{3}{2}$ particles play
a significant role.

We studied the point transformation algebra and explored the invariance properties
of the general action under rotations of the lower spin, off-shell fields. We found that this
invariance implies the existence of a conserved vector current and charge. The conserved
current leads to interactions involving spin-$\frac{3}{2}$ fields such as the electromagnetic couplings
that we have given in (61) above, possible couplings to other vector fields such as vector
mesons, derivative couplings to scalar fields such as the pion, etc. It is important to check
the predictions of these interactions.

Finally, we looked at the consistency problems and indicated two possible avenues
where progress might be made. The first is a possible cancellation of the problematic
terms by tuning interactions based on the conserved current. The second is by using
the conserved charge as an additional constraint to compensate for the loss of the usual
constraints at the ‘bad’ values of the external magnetic field.

It is important to attempt a generalization of these techniques in the theory of higher
spin fields in the same way as done for spin-$\frac{3}{2}$. It seems likely that similar (though more
complicated) groups will exist which rotate among the auxiliary lower spin components
in those cases as well, leading again to new conserved currents.
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