A division theorem for nodal projective hypersurfaces

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Abstract

Let \( V_{n,d} \) be the variety of equations for hypersurfaces of degree \( d \) in \( \mathbb{P}^n(\mathbb{C}) \) with singularities not worse than simple nodes. We prove that the orbit map \( G' = SL_{n+1}(\mathbb{C}) \to V_{n,d}, \ g \mapsto g \cdot s_0, s_0 \in V_{n,d} \) is surjective on the rational cohomology if \( n > 1, d \geq 3, \text{ and } (n, d) \neq (2, 3) \).

As a result, the Leray–Serre spectral sequence of the map from \( V_{n,d} \) to the homotopy quotient \((V_{n,d})_hG'\) degenerates at \( E_2 \), and so does the Leray spectral sequence of the quotient map \( V_{n,d} \to V_{n,d}/G' \) provided the geometric quotient \( V_{n,d}/G' \) exists. We show that the latter is the case when \( d > n + 1 \).

We write \( \Pi_{n,d} \) for the space of homogeneous complex polynomials of degree \( d \) in \( n + 1 \) variables, \( U_{n,d} \subset \Pi_{n,d} \) for the affine subvariety of those homogeneous polynomials which give smooth hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \), and \( \Sigma_{n,d} = \Pi_{n,d} \setminus U_{n,d} \) for its complement. Define for \( l = 1, \ldots, n + 1 \)

\[
\Sigma_{n,d}^{(l)} = \{ f \in \Sigma_{n,d} \mid V(f)_{\text{sing}} \cap \Lambda_{n-l+1} \neq \emptyset \}.
\]

Here, \( V(f) \subset \mathbb{P}^n(\mathbb{C}) \) is the zero locus of \( f \), \( V(f)_{\text{sing}} \) is the singular locus of the hypersurface \( V(f) \), and \( \Lambda_{n-l+1} \) is a fixed linear subspace in \( \mathbb{P}^n(\mathbb{C}) \) of codimension \( l - 1 \). In other words, \( f \in \Sigma_{n,d}^{(l)} \) if and only if \( V(f) \) has at least one singular point in \( \Lambda_{n-l+1} \). Note that \( \Sigma_{n,d}^{(l)} \subset \Sigma_{n,d} \) is an irreducible subvariety of codimension \( l \) in \( \Pi_{n,d} \), so it defines a fundamental class in the singular Borel-Moore homology:

\[
[\Sigma_{n,d}^{(l)}] \in H_{BM}^{n-l+1}(\Sigma_{n,d}, \mathbb{Z}).
\]

We denote by \( \text{Lk}_l \in H^{2l-1}(U_{n,d}, \mathbb{Z}) \) the Alexander dual cohomology class of \( [\Sigma_{n,d}^{(l)}] \). We point out that \( \text{Lk}_l \) does not depend on the choice of \( \Lambda_{n-l+1} \subset \mathbb{P}^n(\mathbb{C}) \).

Note that the group \( G = GL_{n+1}(\mathbb{C}) \) acts on \( \Pi_{n,d} \) by change of variables and this action preserves \( U_{n,d} \). We will denote by \( U_{n,d}/G \) the geometric quotient of the affine variety \( U_{n,d} \) by the \( G \)-action, which exists if \( d \geq 3 \) by [11, Proposition 4.2] and is affine. (In this paper we use the definition of the geometric quotient given in [11].)

Fix an element \( s_0 \in U_{n,d} \) and consider the orbit map \( O : G \to U_{n,d}, g \mapsto g \cdot s_0 \). C. Peters and J. Steenbrink [12] proved the following theorem:

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Theorem 1 ([12]) The induced map \( O^*: H^*(U_{n,d}, \mathbb{Q}) \to H^*(G, \mathbb{Q}) \) does not depend on \( s_0 \), and the classes \( O^*(Lk_l) \in H^*(G, \mathbb{Q}), l = 1, \ldots, n + 1 \) are multiplicative generators if \( d \geq 3 \). In particular, there is an isomorphism of rings:

\[
H^*(U_{n,d}, \mathbb{Q}) \cong H^*(G, \mathbb{Q}) \otimes H^*(U_{n,d}/G, \mathbb{Q})
\]

compatible with mixed Hodge structures. \( \square \)

Later this theorem was generalized by A. Gorinov and the author in [5] to the case of a general reductive group action on the space of regular sections of an equivariant vector bundle over a smooth complex compact variety. In this paper, we study a generalization in a different direction. Namely, we consider a group action not only on the space of regular sections, but on a space of sections with singularities of a certain type.

Let \( V_{n,d} \) be the open subvariety of \( \Pi_{n,d} \) formed by all homogeneous polynomials \( f \) such that the kernel of the Hessian matrix of \( f \) at any non-zero singular point is one-dimensional (or, equivalently, the hypersurface \( V(f) \) has no singularities other than simple nodes.) The special linear group \( G' = SL_{n+1}(\mathbb{C}) \) acts on \( V_{n,d} \). Fix \( s_0 \in V_{n,d} \) and consider the orbit map \( O': G' \to V_{n,d}, g \mapsto g \cdot s_0 \). In Theorem 2, we will show that the induced map of the rational cohomology

\[
O'^*: H^*(V_{n,d}, \mathbb{Q}) \to H^*(G', \mathbb{Q})
\]

is surjective if \( n > 1, d \geq 3, \) and \( (n, d) \neq (2, 3) \). This implies that there is an isomorphism of rings

\[
H^*(V_{n,d}, \mathbb{Q}) \cong H^*(G', \mathbb{Q}) \otimes H^*_G(V_{n,d}, \mathbb{Q})
\]

compatible with mixed Hodge structures, where \( H^*_G(V_{n,d}, \mathbb{Q}) \) is the equivariant cohomology ring. Finally, we show in Proposition 3 that a geometric quotient \( V_{n,d}/G' \) exists if \( d > n + 1 \), and so in this case we have \( H^*_G(V_{n,d}, \mathbb{Q}) \cong H^*(V_{n,d}/G', \mathbb{Q}) \), see Corollary 3.

We also note that Theorem 2 was used in [3] to compute the cohomology ring \( H^*(V_{n,d}, \mathbb{Q}) \) if \( (n, d) = (2, 4) \).

We begin with generalities on spaces of regular sections and sections with nodal singularities. Let \( L \) be a line bundle over a smooth complex projective variety \( X \). Let us denote by \( J^rL \) the \( r \)-th jet bundle of \( L \), cf. [6, Chapter 16.7]. We recall that \( J^0L = L \) and there are maps of vector bundles over \( X \):

\[
j_r: X \times \Gamma(X, L) \to J^rL.
\]

Moreover, there are short exact sequences:

\[
0 \to \text{Sym}^r(\Omega_X^1) \otimes L \to J^rL \to J^{r-1}L \to 0,
\]

and the right map is compatible with (1). If \( L \) is very ample, then \( j_1 \) is surjective, cf. [7, Proposition II.7.3]; we denote its kernel by \( \Sigma(L) \subset X \times \Gamma(L) \). By (2), the map \( j_2 \) restricted to \( \Sigma(L) \) lifts to a map

\[
\tilde{j}_2: \Sigma(L) \to L \otimes \text{Sym}^2(\Omega_X^1) \hookrightarrow L \otimes \Omega_X^1 \otimes \Omega_X^1.
\]
and

\[ p_*(p^*(\Omega_X^1)) \cong \Omega_X^1. \]

These isomorphisms and the projection formula give a canonical isomorphism:

\[ L \otimes \Omega_X^1 \otimes \Omega_X^1 \cong p_*(p^*(L \otimes \Omega_X^1) \otimes O_{\mathbb{P}(T_X)}(1)). \]

We now apply the adjunction \( p^* \dashv p_* \) to (3) and obtain a map:

\[ h : p^*(\tilde{\Sigma}(L)) \rightarrow p^*(L \otimes \Omega_X^1) \otimes O_{\mathbb{P}(T_X)}(1). \] (4)

Informally, \( h \) works as follows. If \( x \in X \) is a point, then over \( x \) we have

\[ h_x : \mathbb{P}(T_{x,X}) \times \tilde{\Sigma}_X(L) \rightarrow \mathbb{P}(T_x, X) \times (L_x \otimes \Omega_{x,X}^1) \]

\[ h_x([l], s) = ([l], j_2(s)(x)(l)) \] (5) (6)

Here, \( l \in T_{x,X}, [l] \in \mathbb{P}(T_x, X) \) is its equivalence class, and we consider \( j_2(s)(x) \) as a map from \( T_{x,X} \) to \( L_x \otimes \Omega_{x,X}^1 \) using \( j_1(s)(x) = 0 \). We note that the twist by \( O_{\mathbb{P}(T_X)}(1) \) is necessary to make (6) independent of a choice of a representative \( l \) for \([l]\).

Suppose now that \( h \) is surjective and denote its kernel by \( \tilde{N}(L) \). Note that

\[ \varphi : \tilde{N}(L) \rightarrow \tilde{\Sigma}(L) \]

is a proper map between the total spaces of vector bundles. We wish to compute

\[ \varphi_* : H_{BM}^*(\tilde{N}(L), \mathbb{Z}) \rightarrow H_{BM}^*(\tilde{\Sigma}(L), \mathbb{Z}), \]

where \( H_{BM}^*(-, \mathbb{Z}) \) are the singular Borel-Moore homology groups with integer coefficients. Since \( \tilde{N}(L) \) and \( \tilde{\Sigma}(L) \) are oriented vector bundles over \( \mathbb{P}(T_X) \) and \( X \) respectively, we have the Thom isomorphisms

\[ H_*(X, \mathbb{Z}) \cong H_{*+2rk\Sigma(L)}(\tilde{\Sigma}(L), \mathbb{Z}), \] (7)

\[ H_*(\mathbb{P}(T_X), \mathbb{Z}) \cong H_{*+2rk\tilde{N}(L)}(\tilde{N}(L), \mathbb{Z}). \] (8)

Here, \( \text{rk} \) denotes the complex rank of a vector bundle. The next proposition is well known, cf. [10].

**Proposition 1** Under the Thom isomorphisms (7), (8), the map

\[ \varphi_* : H_{BM}^*(\tilde{N}(L), \mathbb{Z}) \rightarrow H_{BM}^*(\tilde{\Sigma}(L), \mathbb{Z}) \] (9)

identifies with

\[ H_*(\mathbb{P}(T_X), \mathbb{Z}) \rightarrow H_{*-2(r_1-r_2)}(X, \mathbb{Z}) \]

\[ y \mapsto p_*(y - e), \] (10) (11)

where \( r_1 = \text{rk} \tilde{\Sigma}(L), r_2 = \text{rk} \tilde{N}(L), \) and \( e \in H^{2(r_1-r_2)}(\mathbb{P}(T_X), \mathbb{Z}) \) is the Euler class of the quotient bundle

\[ p^*(\tilde{\Sigma}(L))/\tilde{N}(L) \cong p^*(L \otimes \Omega_X^1) \otimes O_{\mathbb{P}(T_X)}(1). \]

\[ \square \]
In the sequel, we want (11) to be surjective. However, it is more convenient to check that the dual map

$$y \mapsto p^*(y) \sim e, \quad y \in H^*(X, \mathbb{Z}). \quad (12)$$

is injective on (singular) cohomology. Recall that the projection $p: \mathbb{P}(TX) \to X$ is a map between compact oriented manifolds, so we have the pushforward map on cohomology

$$p_*: H^*(\mathbb{P}(TX), \mathbb{Z}) \to H^*\!-\!2n+2(X, \mathbb{Z})$$

defined via the Poincaré duality. Here $n = \dim X$ is the complex dimension of $X$.

**Proposition 2** $p_!(e) = nc_1(L) - 2c_1(TX)$.

**Proof.** Recall that

$$H^*(\mathbb{P}(TX)) = H^*(X)[c]/(c^n + c_1(TX)c^{n-1} + \ldots + c_n(TX)),$$

where $c = c_1(\mathcal{O}_{\mathbb{P}(TX)}(1))$ is the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}(TX)}(1)$. Note that $p_!(e^k) = 0$ if $k < n - 1$, $p_!(e^{n-1}) = 1$, and $p_!(e^n) = -c_1(TX)$. Set $E = L \otimes \Omega^1_X$, then by the splitting principle we have

$$e = c_n(p^*(E) \otimes \mathcal{O}_{\mathbb{P}(TX)}(1)) = \sum_{i=0}^n c^{n-i} p^*(c_i(E)).$$

Finally, by the projection formula:

$$p_!(e) = p_!(e^n + e^{n-1} p^* c_1(E)) = -c_1(TX) + c_1(\Omega^1_X \otimes L)$$
$$= -c_1(TX) + c_1(\Omega^1_X) + nc_1(L) = -2c_1(TX) + nc_1(L). \quad \square$$

**Corollary 1** Let $X = \mathbb{P}^n(\mathbb{C})$, $L = \mathcal{O}(d)$, $n > 1$, $d > 2$. Then the morphism $h$ (see (4)) is a surjective map of vector bundles over $\mathbb{P}(TX)$. Moreover, if $(n, d) \neq (2, 3)$ and $* < \dim(\Sigma(L))$, then the map

$$\varphi_*: H^*_{BM}(\widetilde{N}(L), \mathbb{Q}) \to H^*_{BM}(\widetilde{\Sigma}(L), \mathbb{Q})$$

is surjective on the rational Borel-Moore homology.

**Proof** The map $h$ is surjective by a straightforward computation. Indeed, $s \in \Gamma(\mathbb{P}^n(\mathbb{C}), \mathcal{O}(d))$ is a homogeneous polynomial of degree $d$ in $n + 1$ variables. The condition that $j_1(s)([x]) = 0$, $[x] \in \mathbb{P}^n(\mathbb{C})$ is equivalent to $x \neq 0$ being a critical point of $s$. Then $j_2(s)([x])$ is the matrix of the second derivatives of $s$ at $x$, i.e. the Hessian matrix of $s$ at $x$. Hence it suffices to show that for each $[x] \in \mathbb{P}^n(\mathbb{C})$ there exists a polynomial $s$ of degree $d$ such that $x$ is a critical point of $s$ and the Hessian matrix of $s$ at $x$ has kernel of dimension $\geq 2$. The latter is clear.

By Proposition 1 and the projection formula, it is enough to check that $p_!(e) \neq 0$ if $(n, d) \neq (2, 3)$. By Proposition 2, we have

$$p_!(e) = (nd - 2(n + 1))H \in H^2(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}),$$

where $H = c_1(\mathcal{O}(1))$ is a multiplicative generator of $H^*(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$. This expression is zero if and only if $(n, d) = (1, 4)$ or $(n, d) = (2, 3). \quad \square$

Recall that $\Pi_{n,d}$ denotes the space of homogeneous polynomials of degree $d$ in $n + 1$ variables $z_0, \ldots, z_n$, i.e. $\Pi_{n,d} = \Gamma(\mathbb{P}^n(\mathbb{C}), \mathcal{O}(d))$. We let

$$\Sigma_{n,d} = \{ f \in \Pi_{n,d} \mid f \text{ has a critical point outside 0} \},$$
and we set $N_{n,d}$ to be the subvariety of $\Sigma_{n,d}$ formed by all $f$ such that the kernel of the Hessian matrix of $f$ at some nonzero critical point $x$ contains a 2-plane $P \ni x$ (or, equivalently, the hypersurface $V(f)$ has a singularity other than a simple node).

**Corollary 2** Suppose that $n > 1$, $d > 2$, and $(n, d) \neq (2, 3)$. For $l > 1$ there exists a cohomology class $a_l \in H^*(\Pi_{n,d} \setminus N_{n,d}, \mathbb{Q})$ such that $a_l|_{\Pi_{n,d} \setminus \Sigma_{n,d}} = Lk_l$.

**Proof** Let $L = \mathcal{O}(d)$. We have that

$$\tilde{\Sigma}(L) = \{(p, f) \in \mathbb{P}^n(\mathbb{C}) \times \Pi_{n,d} \mid df|_p = 0\}.$$ 

We denote by $\tilde{\Sigma}^{(l)}(L)$ the subvariety of $\tilde{\Sigma}(L)$ given by

$$\tilde{\Sigma}^{(l)}(L) = \{(p, f) \in \Lambda^{n-l+1} \times \Pi_{n,d} \mid df|_p = 0\} \subset \tilde{\Sigma}(L),$$

where $\Lambda^{n-l+1}$ is the fixed linear subspace of $\mathbb{P}^n(\mathbb{C})$. The projection map

$$\pi : \tilde{\Sigma}(L) \to \Sigma_{n,d}$$

is proper and generically finite of degree 1. Moreover,

$$[\Sigma^{(l)}_{n,d}] = \pi_*(b_l),$$

where $b_l = [\tilde{\Sigma}^{(l)}(L)] \in H^*_{BM}(\tilde{\Sigma}(L))$. By Corollary 1, $b_l = \varphi_*(c_1)$, $c_1 \in H^*_{BM}(\tilde{N}(L))$ if $l \neq 1$. Finally, let $\pi_1 : \tilde{N}(L) \to N_{n,d}$ be the projection map and $\iota : N_{n,d} \hookrightarrow \Sigma_{n,d}$ be the embedding. Then $[\Sigma^{(l)}_{n,d}] = \iota_*\pi_1*(c_1)$ if $l > 1$ and $a_l$ is the Alexander dual of $\pi_{1*}(c_1)$. \hfill \Box

Let $G' = SL_{n+1}(\mathbb{C}) \subset G$ be the special linear group. Recall that the *universal $G'$-bundle* $EG'$ is a contractible space with a free continuous right $G'$-action such that $EG' \to EG'/G'$ is a locally trivial fiber bundle. We denote by $H^*_G(V_{n,d}, \mathbb{Q})$ the equivariant cohomology of $V_{n,d}$, that is the cohomology of the homotopy quotient

$$(V_{n,d})_{hG'} = EG' \times_{G'} V_{n,d}.$$ 

For a more detailed account of equivariant cohomology we refer the reader e.g. to Part I of [13]. Furthermore, since $G'$ is an algebraic group and $V_{n,d}$ is an algebraic $G'$-variety, we endow equivariant cohomology groups $H^*_G(V_{n,d}, \mathbb{Q})$ with a (functorial) mixed Hodge structure. The construction of this mixed Hodge structure seems to be well-known and it is implicitly contained in [4]; see e.g. [3, Proposition A.5] for more details.

There is a fiber sequence

$$G' \to EG' \times_{G'} V_{n,d} \to EG' \times_{G'} V_{n,d},$$

and the second component of the first map is $O' : G' \to V_{n,d}$ given by $g \mapsto g \cdot s_0$ for some $s_0 \in V_{n,d}$. Using Theorem 1 and Corollary 2, we immediately obtain:

**Theorem 2** Suppose $n > 1$, $d \geq 3$, and $(n, d) \neq (2, 3)$. Then the classes $O'^*(a_l) \in H^*(G', \mathbb{Q})$, $l = 2, \ldots, n+1$ are multiplicative generators. In particular, the Leray–Serre spectral sequence associated with (13) degenerates at $E_2$, i.e. there is an isomorphism of rings

$$H^*(V_{n,d}, \mathbb{Q}) \cong H^*_G(G', \mathbb{Q}) \otimes H^*_{G'}(V_{n,d}, \mathbb{Q})$$

compatible with mixed Hodge structures. \hfill \Box
Remark 1 The fact that \( O^{rs}(\text{Lk}_i) \in H^*(G', \mathbb{Q}) \), \( l = 2, \ldots, n + 1 \) are multiplicative generators implies that the connected component \((G'_s)^0\) of the stabilizer subgroup \( G'_s \subset G' \) is unipotent for all \( s \in V_{n,d} \). Using that one can see that Theorem 2 is false for \((n, d) = (2, 3)\). Indeed, the polynomial \( s = z_0z_1z_2 \) is in \( V_{2,3} \) and the stabilizer \( G'_s \) contains a torus.

Remark 2 If a geometric quotient \( V_{n,d}/G' \) exists and the quotient map \( q : V_{n,d} \to V_{n,d}/G' \) is affine, then, under the assumptions of Theorem 2, the stabilizer subgroups \( G'_s, s \in V_{n,d} \) are finite, see \([5, \text{Theorem 3.1.1 and Proposition 3.1.3}]\). Furthermore, by ibid., Section 4.2.1, the order \( |G'_s| \) divides in this case the following number
\[
\prod_{i=2}^{n+1} ((d - 1)^{n+1} + (-1)^{i+1}(d - 1)^{n+1-i}).
\]

Corollary 3 Suppose that \((n, d)\) are as in Theorem 2, there exists a geometric quotient \( V_{n,d}/G' \), and the stabilizer subgroups \( G'_s, s \in V_{n,d} \) are finite. Then there is an isomorphism:
\[
H^*_G(V_{n,d}, \mathbb{Q}) \cong H^*(V_{n,d}/G', \mathbb{Q}),
\]
and the Leray spectral sequence for \( q \) degenerates at \( E_2 \).

Proof The first statement seems to be folklore; for details see e.g. Proposition A.4 and the remark after Theorem A.3 in \([3]\). For the second part we again apply Theorem 1 and Corollary 2 to show that the orbit map \( O^{rs} \) is a surjective map of the rational cohomology. Thus the Leray spectral sequence for \( q \) degenerates at \( E_2 \) by \([12, \text{Theorem 2}]\) and the example after it, cf. \([3, \text{Theorem A.3}]\). \( \Box \)

Remark 3 In Corollary 3, the finiteness of the stabilizer subgroups is not a restrictive assumption. Indeed, by the Matsumura-Monsky theorem \([9, \text{Theorem 1}]\), the stabilizer subgroups \( G'_s \) are finite if \( s \in U_{n,d}, d \geq 3, \) and \( n \geq 3 \). Moreover, if a geometric quotient \( V_{n,d}/G' \) exists, then \( \dim(G'_s) \) is a constant function on \( V_{n,d} \) by the remark before Proposition 0.2 in \([11]\). Thus \( G'_s, s \in V_{n,d} \) are finite groups if \( d \geq 3, n \geq 3, \) and \( V_{n,d}/G' \) exists.

Finally, we show that the geometric quotient \( V_{n,d}/G' \) exists if \( d > n + 1 \).

Proposition 3 There exists a geometric quotient \( V_{n,d}/G' \) if \( d > n + 1 \). Moreover, \( q : V_{n,d} \to V_{n,d}/G' \) is affine.

Proof The proof is based on the geometric invariant theory and we will use the terminology and notation from \([11]\). Namely, we show that \( V_{n,d} \) is contained in the subset of \( G' \)-stable points \( \Pi^s_{n,d} \subset \Pi_{n,d} \). More precisely, \( \Pi^s_{n,d} \) is the preimage of the set of properly \( G' \)-stable points in the projective space \( \mathbb{P}(\Pi_{n,d}) \) with respect to the standard \( SL_{n+1}(\mathbb{C}) \)-linearization of \( O(1) \). Then the geometric quotient \( \Pi^s_{n,d}/G' \) exists (by Theorem 1.10 in ibid.), and so does \( V_{n,d}/G' \). Moreover, the morphism \( V_{n,d} \to V_{n,d}/G' \) is affine by ibid., Theorem 1.10(i).

We will show that if \( f \in \Pi_{n,d} \) is not properly stable, then the hypersurface \( V(f) \) has a singularity worse than a simple node. Suppose that
\[
f = \sum a_{i_0, \ldots, i_n} z_0^{i_0} \cdots z_n^{i_n} \in \Pi_{n,d}
\]
is not a properly stable point. By Theorem 2.1 ibid., there exists a (non-trivial) one-parametric subgroup \( \lambda : \mathbb{C}^* \to G' \) such that \( \mu(f, \lambda) \leq 0 \). Any one-parametric reductive subgroup of
$G'$ is conjugate to a one in the subgroup of diagonal matrices, so we can assume that
\[
\lambda(t) = \begin{pmatrix}
 t^{r_0} & 0 & \cdots & 0 \\
 0 & t^{r_1} & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & t^{r_n}
\end{pmatrix},
\]
where $r_0, \ldots, r_n \in \mathbb{Z}$ and $r_0 + \ldots + r_n = 0$. By permuting coordinates, we assume further that $r_0 \geq r_1 \geq \ldots \geq r_n$, cf. [11, Section 4.2, pp. 81-82]. In this case,
\[
\mu(f, \lambda) = \max\{i_0 r_0 + \ldots + i_n r_n \mid a_{i_0}, \ldots, a_{i_n} \neq 0\}.
\]

Since $d > n + 1$, we have following inequalities for any sequence $r_0 \geq \ldots \geq r_n$, $r_0 + \ldots + r_n = 0$:
\[
d r_0 > 0,
\]
\[
(d - 1)r_0 + r_i > nr_0 + r_i \geq r_0 + \ldots + r_n = 0, \text{ if } 0 < i \leq n;
\]
\[
(d - 2)r_0 + 2r_i > (n - 1)r_0 + r_i + r_{i+1} \geq r_0 + \ldots + r_n = 0, \text{ if } 0 < i < n;
\]
\[
(d - 2)r_0 + r_i + r_j > (n - 1)r_0 + r_i + r_j \geq r_0 + \ldots + r_n = 0, \text{ if } 0 < i, j \leq n, \text{ } i \neq j.
\]
Since $\mu(f, \lambda) \leq 0$, these inequalities imply that the coefficients of $f$ for $z_0^d, z_0^{d-1} z_i, z_0^{d-2} z_i^2$ ($i \neq n$, $z_0^{d-2} z_i z_j$ are zero. Therefore, the point $p = (1, 0, \ldots, 0)$ is a critical point for $f$ and $\text{dim ker Hess}_p(f) \geq 2$. \hfill $\Box$

**Remark 4** It seems likely that the geometric quotient $V_{n,d}/G'$ exists for $3 \leq d \leq n + 1$ as well; at least this was proven in several particular cases. For instance, $V_{n,d}/G'$ exists if $(n, d)$ is $(3, 3)$, $(4, 3)$, and $(5, 3)$ by [2, Proposition 6.5], [1], and [8] respectively.

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