Narrowing of the emission angle in high-intensity Compton scattering

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We consider the emission spectrum of high-energy electrons in an intense laser field. At high intensities ($a_0 \sim 200$) we find that the QED theory predicts a narrower angular spread of emissions than the classical theory. This is due to the classical theory overestimating the energy loss of the particles, resulting in them becoming more susceptible to reflection in the laser pulse.

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I. INTRODUCTION

Since the discovery of chirped pulse amplification [1] the powers and intensities of state of the art laser facilities have been exponentially increasing [2], the current record of $2 \times 10^{22}$ W cm$^{-2}$ having been set in 2008 [3]. With the advent of various new facilities over the next few years, such as the Vulcan 20 PW upgrade [4], the Extreme Light Infrastructure (ELI) Facility [5], and the XCELS project [6], this trend is expected to continue for the foreseeable future. The availability of such technology has driven a large field of research in the topic of nonlinear Thomson and Compton scattering, the understanding of which is important from a fundamental physics perspective [7,8]. Additionally, and at least as importantly, the process produces high-energy, tunable $\gamma$-ray beams, which are important for fundamental research [9], as well as more practical applications such as cancer radiotherapy [10] and the radiography of dense objects [11]. Recent experiments [12,13] have been pushing the limits of research in orientated crystals see, e.g., [17–19].

In this article we study nonlinear Thomson and Compton scattering at ultrahigh intensities, assessing the impact of classical radiation reaction and QED effects on the properties of the emitted photon spectra. We find that, because the classical theory overestimates the radiative energy loss, it predicts a broader angular spread of emissions than the QED theory.

II. THEORY

We consider the case of an electron in a head on collision with a laser pulse. We adopt natural units where $\hbar = c = 1$. To begin with we will take our laser to be a plane-wave field propagating in the $z$ direction described by the null wave vector $k_n = \omega_0(1,0,0,1)$, with central frequency $\omega_0$. We assume the field to be polarized in the perpendicular ($x$) direction and therefore introduce the polarization vector $\epsilon = (0,1,0,0)$. We define the dimensionless intensity in the usual manner, $a_0 = eE/\omega m$, where $E$ is the peak magnitude of the electrical-field strength. The electromagnetic field tensor of the wave is taken to depend arbitrarily on the phase $\phi \equiv k \cdot x = \omega_0(t-z)$; $F^{\mu\nu}(\phi) = a_0 f(\phi) f^{\mu\nu}$, where $f^{\mu\nu} = (k^\mu e^\nu - k^\nu e^\mu)/\omega_0$ and $f(\phi)$ is a function describing the pulse. Additionally, to aid future discussion, we also define the time-dependent intensity $a(\phi) = ef(\phi)E/\omega m$, which is equal to $a_0$ at the pulse peak.

The emission spectrum from a particle in the field can be decomposed into a sum of harmonics, corresponding to multiples of the laser frequency. In the quantum description these correspond to the number of laser photons involved in the scattering process [20]. During each scattering process, the electron will absorb an integer number $n$ of laser photons (each of momentum $k$) before emitting a photon of momentum $k'$. Conservation of momentum arguments it can be shown that the frequency of the scattered photon is given by [21]

$$\omega'_n = \frac{n\omega_0}{1 + j_n(1 - \cos \theta)},$$

where

$$j_n = \frac{n\omega_0/\gamma - \gamma \beta + a_0^2 (1 - \beta)/2}{\gamma (1 + \beta)}.$$  

It can be seen that when $j_n < 0$ the maximum emission frequency occurs when the photons are backscattered ($\theta = 180^\circ$). Conversely, when $j_n > 0$ the maximum frequency occurs for forward scattering ($\theta = 0$). Note that the support for a given harmonic depends on the harmonic number $n$ and that the amplitude of a given harmonic will be determined by the cross sections provided in the references. For high intensities the spectrum will be composed of a very large number of harmonics, with the spectrum decaying after the harmonic with number $n \sim 3a_0^2/2$ [22].

Now let us consider the angular directions of the emissions. From expressions (1) and (2) it can be seen that the angular range of each harmonic will depend on both $a_0$ and the $\gamma$ factor of the electron. It can be shown that the angle at which the emissions peak depends on the ratio of the sum of the laser photon momenta to the electron momenta (see [21] for further details). In the case where the electron momentum is greater than the sum of the laser photon momenta for all the photons involved in even the highest-order scattering processes, i.e.,
peak emissions will be backscattered relative to the laser axis. See a the laser axis. (b) a\00\`o \00\`o \00\`ø ≃ 2γ defines the “center-of-mass” frame for the collision where the peak emissions will be at 90°. (c) a\00\`o \00\`ø ≫ γ: the peak emissions will be backscattered relative to the laser axis. See Ref. [21] for further details.

When the background field is of high intensity the radiation emissions from the particle will be so strong that the resulting energy loss will begin to effect its motion [24]. This radiation reaction (RR) effect will cause the γ factor of the particle to decrease during the interaction with the laser field. This will mean that in order to estimate the peak emission angle we must consider the value of the γ factor when the particle is in the peak of the laser field (since the highest intensity region will produce the strongest emission signal strength and therefore dominate the total spectrum), rather than the initial value. This is illustrated in Fig. 2 where we show how the ratio of laser intensity to particle energy changes with time.

If we wish to consider quantum effects then it is instructive to introduce the dimensionless and invariant “quantum efficiency” parameter \( \chi_\text{e} \equiv m_\text{e}^2 F_\text{cr}^2 / m^2 \approx \gamma E / E_\text{cr} \), where \( E_\text{cr} = 1.3 \times 10^{16} \text{ V cm}^{-1} \) is the QED “critical” field (“Sauter-Schwinger” field) [25]. This can be interpreted as the work done on the electron by the laser field over the distance of a Compton wavelength. In the regime \( \chi_\text{e} \gtrsim 1 \) quantum effects, including pair production from the emitted photons, will dominate. For the purposes of this study we restrict ourselves to the regime where \( a_0, \gamma \gg 1 \), such that quantum effects play a role in the Compton spectra, but have \( \chi_\text{e} \lesssim 1 \), so that pair production can be neglected.

III. NUMERICAL TECHNIQUES

To generate the emission spectra we use the single particle code SIMLA [26], which runs in both classical and QED modes. In the classical case we propagate the particle through the fields using the Landau Lifshitz (LL) equation, which takes into account RR effects via the inclusion of some correctional terms to the Lorentz force equation [27]

\[
\dot{u}^\mu = F^{\mu\nu} u_\nu + r_0 \left( F^{\mu\nu} u_\nu + F^{\mu\sigma} F_\sigma^{\ \nu} u_\nu - u_\sigma F^{\sigma\nu} F_{\nu\mu} \right),
\]

where \( r_0 \equiv (2/3) e^2 / 4\pi m \) is the classical electron radius, and the overdot denotes differentiation with respect to proper time. We note that there are numerous alternative equations in the literature (for an overview, see Refs. [28,29] and for further discussions of the LL equation, see Refs. [30,31]). However, the LL equation has, along with some others, recently been shown to be consistent with QED to the order of the fine-structure constant \( \alpha \) [32].

The resulting classical emission spectra are calculated using a novel Monte Carlo method introduced in Ref. [33]. The method is simple and computationally efficient. As the particles in our simulation are ultrarelativistic the radiation due to transverse acceleration is dominant, since this is a factor \( \gamma^2 \) larger than that due to longitudinal acceleration [34]. Since the acceleration and velocity of the particle are perpendicular, we can approximate the radiation as synchrotron radiation. In our method we calculate the effective magnetic field, \( H_{\text{eff}} \), acting on the particle over each time step. This is the magnetic field which would cause the same acceleration as the electric and magnetic fields together. The typical frequency of synchrotron emission is then given by \( \omega_\text{e} = 3eH_{\text{eff}}^2 / 2m \). For an ultrarelativistic particle in an external, homogeneous magnetic field, the classical radiation cross section can be expressed in terms of the intensity given by [34]

\[
\frac{\partial \Gamma_\text{cl}}{\partial \omega'} = 1 \frac{\partial I}{\partial \omega'} = \frac{\sqrt{3} e^3 H_{\text{eff}}}{2\pi \omega' m} F_1(\omega'/\omega_\text{e}),
\]

where \( F_1(\xi) = \xi \int_{\xi}^\infty K_{5/3}(\xi')d\xi' \) is the first synchrotron function. [We note that (4) is integrable in the limit \( \omega' \rightarrow 0 \) and therefore the expression is well defined. For further details, see Ref. [35].] At each time step the quantity \( \omega_\text{e} \) is calculated and a Monte Carlo method is used to sample from the spectra. The direction of the emission is taken to be that of the particle velocity, a good approximation for the ultrarelativistic case [34].

In the QED case our code works as follows. We begin by introducing the (quantum) synchrotron parameter\(^1\) in terms of

\[\chi_\text{e} \equiv \gamma H_{\text{eff}} e/m^2, \text{ one can see that } \xi \approx \omega'/\omega_\text{e}, \text{ the argument of our classical expression (4).}\]

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\(^1\)Note that in the classical regime the energy of the emitted radiation is typically much lower than the electron energy \( m\gamma \) (since the radiation is emitted continuously rather than as discrete emissions) and so we can approximate \( \xi \approx (2/3) \chi_\text{e} \omega_\text{e} / \gamma \). Further approximating \( \chi_\text{e} \approx \gamma H_{\text{eff}} e/m^2 \), one can see that \( \xi \approx \omega'/\omega_\text{e} \), the argument of our classical expression (4).
the frequency of the emitted radiation \( \omega' \)

\[
\xi = \frac{2}{3\chi_e} \frac{\omega'}{\gamma m - \omega}.
\]  
(5)

In the case of high-energy emissions, \( \xi \gg 1 \), it can be shown that the photon coherence length goes like \([36]\]

\[
l_f \approx \sqrt{\frac{2}{3} \frac{\lambda}{\pi \omega_0} \sqrt{\frac{1}{\xi}}},
\]  
(6)

where \( \lambda = 2\pi/\omega_0 \) is the laser wavelength. Thus in the high-intensity limit \( \omega_0 \gg 1 \) the size of the radiation formation region is much smaller than the laser wavelength \([20]\) and so the laser background field can be approximated as locally constant and crossed \([37]\). We are then able to determine the probability of photon emission using the expression for the constant crossed field rate \( \Gamma_\xi \) per unit time,

\[
d\Gamma_\xi = \frac{a m}{\sqrt{3\pi} \gamma \chi_e} \left[ 2 \frac{x^2}{1 + x} \right] K_{2/3}(\tilde{x})
- \int_{\tilde{x}}^\infty dy K_{1/3}(y),
\]  
(7)

where \( K_\nu \) is the modified Bessel function and \( \chi_e = \sqrt{k_n F_n^2 k_n/m^2} \) for the emitted photon with momentum \( k_n \); note that \( x = \chi_e/(\chi_e - \chi_r) \), and \( \tilde{x} = 2x/(3\chi_e) \). Although \( d\Gamma_\xi/d\chi_r \) diverges at small \( \chi_r \), the total rate of photon emission \( \Gamma_\xi \), given by integrating (7) over all \( \chi_r \in [0, \chi_e] \), is finite. (This apparent softening of the usual infrared divergence in QED is explained in Ref. \([38]\).) A discussion of the relationship between quantum expression for the emission rate (7) and the classical expression (4) is given in the Appendix.

In the QED simulations the electron is propagated along a classical trajectory divided into discrete time steps. After each step \( \Delta t \) the code calls a statistical subroutine to calculate the probability of photon emission and to correct the electron’s momentum. The routine generates a random uniform number \( r \in [0,1] \) and, if the condition \( r \leq \Gamma_\xi \Delta t \) is satisfied, an emission is deemed to occur (under the requirement \( \Gamma_\xi \Delta t \ll 1 \)). Note that \( d\Gamma_\xi/d\chi_r \) (and \( \Gamma_\xi \)) are time-dependent quantities, due to the temporal variation of both the laser pulse and the electron motion. If an emission event occurs, a second uniform random number \( \xi \in [0,1] \) is generated and the photon’s \( \chi_r \) (and hence its frequency) is calculated as the root of the sampling equation

\[
\xi = \Gamma_\xi(t)^{-1} \int_0^{\chi_r} d\chi_r' \frac{d\Gamma_\xi(t)}{d\chi_r'}.\]

(8)

The photon momentum is then determined by \( \chi_r \) together with the assumption that at high \( \gamma \) the photon is emitted in the direction of the electron’s motion, just as in the classical method described above. Finally, the momentum of the photon is subtracted from the momentum of the electron, i.e., the electron is recoiled, imposing the conservation law \( \chi_e \to \chi_e - \chi_r \) \([20]\). The simulation then proceeds by propagating the electron (via the Lorentz equation) and the photon (on a linear trajectory) to the next time step. In this way, multiple emissions are described as sequential single photon emissions, as in Eq. (7), occurring at discrete time intervals (for further

![Figure 3](image3.png)

**FIG. 3.** Integrated frequency spectra (left panel) and angular spectra (right panel) for the case of an electron with initial \( \gamma_0 = 800 \) colliding with a laser of intensity \( \omega_0 = 200 \), \( \lambda = 0.8 \mu m \), and duration 30 fs. Blue line: classical. Red line: QED.

information on the multiphoton calculation, see Ref. \([39]\), and for discussions of double emissions, Refs. \([40,41]\)). This method has recently been tested against cases where we can calculate the Compton spectra analytically and found to perform extremely well \([42]\).

**IV. RESULTS**

**A. Plane wave model**

We begin by considering an electron with an initial \( \gamma_0 = 800 \) in a head on collision with a plane-wave laser of peak intensity \( \omega_0 = 200 \), \( \lambda = 0.8 \mu m \) and duration 30 fs FWHM. The emission spectra, calculated both classically and using the statistical QED routines, are plotted in Fig. 3. It can be seen that, although for these parameters there is little difference between the frequency spectra, the classical theory predicts a much stronger signal at small angles than the QED theory.

To understand the reason for this discrepancy we must consider how the electron energy changes with time. In Fig. 4 we plot the \( \gamma \) factor of the classical electron (white line). It can

![Figure 4](image4.png)

**FIG. 4.** Density plot showing how the electron \( \gamma \) factor changes with time (statistical distribution generated by recording the paths of 500 QED electrons all with the same initial condition \( \gamma_0 = 800 \)). Parameters are \( \omega_0 = 200 \), \( \lambda = 0.8 \mu m \), and duration 30 fs. The white line shows the \( \gamma \) factor for a classical electron.
be seen that the particle rapidly loses energy as it approaches the pulse focus. This means that the angular direction of the emitted radiation will be continuously changing as the ratio of the electron $\gamma$ factor to laser $\omega_0$ changes with time (see Fig. 1). This broadening of the angular range was first proposed in Ref. [43] as a signature of classical RR effects. (We also note that the rapid energy loss due to RR results in a natural resistance to the high-energy high-intensity regime, and is what prevents the use of lasers for distinguishing between RR models; see, e.g., Ref. [44].) Superimposed on the same plot we also show how the electron $\gamma$ factor changes in the QED case. To generate this statistical density we ran the code 500 times, using realistic, nonideal parameters, the difference in angular loss of the particle. The reason for this can be understood from the fact that the QED electron only emits at discrete times, whereas the classical electron is radiating and losing energy from an emission (an effect known as “straggling” [45]), whereas the QED theory predicts only minimal radiation at these angles. At the same time the integrated frequency spectra are still very similar for these values. Hence we find that these signatures are sufficiently robust to survive realistic beam focusing effects.

V. DISCUSSION

Our analysis shows a significant difference in the angular emission spectra predicted by the classical and QED theories. The intensities we have chosen are only slightly higher than the current state of the art, and well within the parameters expected for any given frequency, the classical radiation spectrum forms an upper bound to the (single photon) QED Compton spectrum.
at the new generation of facilities, such as ELI. For the regimes we have considered we find $\chi_e \sim 0.2-0.25$, indicating that this is an effect which becomes important before the onset of more explicit QED processes, such as pair production and runaway cascading. The difference in angular spectra won’t be visible for all sets of parameter values. One needs a large $a_0$ in order that RR effects are significant and, additionally, one needs for the electrons to become reflected roughly in the middle of the pulse. If one has $a_0 > a_0$, then the electrons will be reflected very early and thus spend a significant amount of time copropagating with the pulse—a regime in which the dynamics will be wholly classical. Also, if one has $\gamma > a_0$ the electrons will not be reflected and the angular broadening due to RR will be small. Nevertheless, there is a broad range of parameters where $a_0$ and $\gamma$ are of the same order of magnitude and the effect will be significant. (This parameter regime is found to be optimal for various studies of intense laser-particle interactions, including nonlinear Compton scattering [21], classical radiation reaction [43], and for the generation of attosecond gamma-rays [49].)

The angular narrowing we have predicted is important for two reasons. First, it provides a clear signal of strong-field RR effects, distinct from classical RR, at parameters that will soon be obtainable. Second, it is an effect which will have to be taken into consideration when planning applications of Compton scattering with the new generation of ultraintense lasers.

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**APPENDIX: DERIVING THE CLASSICAL RATE EXPRESSION FROM THAT USED IN THE QED CASE**

Here we demonstrate explicitly that the expression for photon emission in the QED form (7) coincides exactly with the one in the classical form (4) in the limit of low intensity and/or energy. Following the derivation of the classical synchrotron spectrum [27] we represent Eq. (7) via the first and the second synchrotron function [50]. First, we convert the rate of emission to the spectral intensity of emission, multiplying it by photon energy $\omega'$ (as previously, according to the choice of units $\hbar = c = 1$)

$$\frac{\partial I^q}{\partial \omega'} = \frac{\partial I^s}{\partial \omega} \frac{\omega'}{\partial \omega / \partial \gamma} = \frac{\omega' e^2}{\sqrt{3} \pi \gamma^2} \left[ 2 + \frac{x^2}{1 + x} \right] K_{2/3}(\tilde{x}) - \int_{\tilde{x}}^{\infty} dy K_{1/3}(y) .$$

(A1)

Next, we apply a recursive relation of the modified Bessel function

$$2 \frac{\partial}{\partial t} K_{\nu}(t) = -K_{\nu-1}(t) - K_{\nu+1}(t)$$

(A2)

for $\nu = 2/3$ to transform the integral in the right side [note, $K_{\nu}(t) = K_{-\nu}(t)$]

$$\frac{\partial I^q}{\partial \omega'} = \frac{\omega' e^2}{\sqrt{3} \pi \gamma^2} \left[ 2 + \frac{x^2}{1 + x} \right] K_{2/3}(\tilde{x}) + \int_{\tilde{x}}^{\infty} dy \left[ K_{5/3}(y) + 2 \frac{\partial}{\partial y} K_{2/3}(y) \right].$$

(A3)

Taking into account that $\lim_{\gamma \to \infty} K_{5/3}(y) = 0$ we obtain

$$\frac{\partial I^q}{\partial \omega'} = \frac{3 \omega' e^2}{2 \pi \gamma^2} \frac{\chi_e(\chi_e - \chi_y)}{\chi_y} \left[ \frac{x^2}{1 + x} F_2(\tilde{x}) + F_1(\tilde{x}) \right].$$

(A4)

where $F_{2}(\xi) = \xi K_{2/3}(\xi)$ is the second synchrotron function. In terms of the effective magnetic field $H_{\text{eff}}$ we can express

$$\chi_e = \gamma \frac{H_{\text{eff}}}{E_{\text{cr}}} , \quad \chi_y = \frac{\omega'}{m} \frac{H_{\text{eff}}}{E_{\text{cr}}} .$$

(A5)

Using the ratio of the photon energy to the initial electron energy $d = \omega'/(m\gamma') = \chi_y/\chi_e$ as a parameter, we obtain

$$\frac{\partial I^q}{\partial \omega'} = \frac{\sqrt{3} e^2 H_{\text{eff}}}{2 \pi m} \left[ \frac{d^2}{1 - d} F_2 \left( \frac{2}{3 \chi_e}, \frac{d}{1 - d} \right) \right] + F_1 \left( \frac{2}{3 \chi_e}, \frac{d}{1 - d} \right).$$

(A6)

This form of the expression makes it possible to see that in the case where the photon has much lower energy than the electron ($d \ll 1$) the expression tends exactly to the classical one (4).

[1] D. Strickland and G. Mourou, Opt. Commun. 56, 219 (1985).
[2] T. Tajima and G. Mourou, Phys. Rev. ST Accel. Beams 5, 031301 (2002).
[3] V. Yanovsky, V. Chvykov, G. Kalinchenko, P. Rousseau, T. Planchon, T. Matsuoka, A. Maksimchuk, J. Nees, G. Cheriaux, G. Mourou, and K. Krushelnick, Opt. Express 16, 2109 (2008).
[4] The Vulcan 10 Petawatt Project.
[5] The Extreme Light Infrastructure (ELI) Project.
[6] The Exawatt Center for Extreme Light Studies (XCELS).
[7] T. Heinzel, Int. J. Mod. Phys. A 27, 1260010 (2012).
[8] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, Rev. Mod. Phys. 84, 1177 (2012).
[9] Y. K. Wu, Overview of High Intensity Gamma-ray Source-Capabilities and Future Upgrades, 2013 International Workshop on Polarized Sources, Targets and Polarimetry, University of Virginia, 2013 (unpublished).
[10] T. S. Lawrence, R. K. Ten Haken, and A. Giaccia, Principles and Practice of Oncology, 8th ed. (Lippincott, Williams, and Wilkins, Philadelphia, 2008).
[11] Y. Glinec, J. Faure, L. Le Dain, S. Darbon, T. Hosokai, J. J. Santos, E. Lefebvre, J. P. Rousseau, F. Burgy, B. Mercier, and V. Malka, Phys. Rev. Lett. 94, 025003 (2005).
[12] S. Cipiccia, M. R. Islam, B. Ersfeld, R. P. Shanks, E. Brunetti, G. Vieux, X. Yang, R. C. Issac, S. M. Wiggins, G. H. Welsh.
M.-P. Anania, D. Maneuski, R. Montgomery, G. Smith, M. Hoek, D. J. Hamilton, N. R. C. Lemos, D. Symes, P. P. Rajeev, V. O. Shea, J. M. Dias, and D. Jaroszynski, Nat. Phys. 7, 867 (2011).

[13] G. Sarri, D. J. Corvan, W. Schumaker, J. M. Cole, A. Di Piazza, H. Ahmed, C. Harvey, C. H. Keitel, K. Krushelnick, S. P. D. Mangles, Z. Najmudin, D. Symes, A. G. R. Thomas, M. Yeung, Z. Zhao, and M. Zepf, Phys. Rev. Lett. 113, 224801 (2014).

[14] M. Vranic, J. L. Martins, J. Vieira, R. A. Fonseca, and L. O. Silva, Phys. Rev. Lett. 113, 134801 (2014).

[15] A. G. R. Thomas, C. P. Ridgers, S. S. Bulanov, B. J. Griffin, and S. P. D. Mangles, Phys. Rev. Lett. 113, 224801 (2014).

[16] D. G. Green and C. N. Harvey, Phys. Rev. Lett. 112, 164801 (2014).

[17] V. Baier, V. Katkov, and V. Strakhovenko, Electromagnetic Processes at High Energies in Oriented Single Crystals (World Scientific, Singapore, 1998).

[18] A. Baurichter, K. Kirsebom, Y. V. Kononets, R. Medenwaldt, U. Mikkelsen, E. Uggerhøj, K. Elsener, S. Ballestrero, P. Sona, S. H. Connell, J. P. F. Sellschop, and Z. Z. Vilakazi, Nucl. Instrum. Methods, Phys. Res., B 174, 274 (2001).

[19] V. Ritus, J. Sov. Laser Res. 6, 497 (1985).

[20] C. Harvey, T. Heinzl, and A. Ilderton, Phys. Rev. A 79, 063407 (2009).

[21] E. Esarey, S. K. Ride, and P. Sprangle, Phys. Rev. E 48, 3003 (1993).

[22] C. Harvey, T. Heinzl, A. Ilderton, and M. Marklund, Phys. Rev. Lett. 109, 100402 (2012).

[23] C. Harvey and M. Marklund, Phys. Rev. A 85, 013412 (2012).

[24] F. Sauter, Z. Phys. 69, 742 (1931); W. Heisenberg and H. Euler, ibid. 98, 714 (1936); J. Schwinger, Phys. Rev. 82, 664 (1951).

[25] D. Green and C. Harvey, Comput. Phys. Commun. 192, 313 (2015).

[26] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, Course of Theoretical Physics Vol. 2 (Butterworth-Heinemann, Oxford, 1987).

[27] D. A. Burton and A. Noble, Contemp. Phys. 55, 110 (2014).

[28] M. Vranic, J. L. Martins, R. A. Fonseca, and L. O. Silva, arXiv:1502.02432.

[29] C. Harvey, T. Heinzl, and M. Marklund, Phys. Rev. D 84, 116005 (2011).

[30] Y. Hadad, L. Labun, J. Rafelski, N. Elkina, C. Klier, and H. Ruhl, Phys. Rev. D 82, 096012 (2010).

[31] A. Ilderton and G. Torgrimsson, Phys. Lett. B 725, 481 (2013).

[32] E. Wallin, A. Gonoskov, and M. Marklund, Phys. Plasmas 22, 033117 (2015).

[33] J. D. Jackson, Classical Electrodynamics, 3rd ed. (Wiley, New York, 1999).

[34] M. K. Khokonov and H. Nitta, Phys. Rev. Lett. 89, 094801 (2002).

[35] M. K. Khokonov and I. Z. Bekulova, J. Tech. Phys. 55, 728 (2010).

[36] A. Nikishov and V. Ritus, Sov. Phys. JETP 19, 529 (1964).

[37] A. Ilderton and G. Torgrimsson, Phys. Rev. D 87, 085040 (2013).

[38] A. Di Piazza, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. Lett. 105, 220403 (2010).

[39] D. Seipt and B. Kampfer, Phys. Rev. D 85, 101701 (2012).

[40] F. Mackenroth and A. Di Piazza, Phys. Rev. Lett. 110, 070402 (2013).

[41] C. N. Harvey, A. Ilderton, and B. King, Phys. Rev. A 91, 013822 (2015).

[42] A. Di Piazza, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. Lett. 102, 254802 (2009).

[43] Y. Kravets, A. Noble, and D. Jaroszynski, Phys. Rev. E 88, 011201 (2013).

[44] C. S. Shen and D. White, Phys. Rev. Lett. 28, 455 (1972).

[45] I. Lerche and R. C. Tautz, J. Phys. A: Math. Theor. 43, 115207 (2010).

[46] L. W. Davis, Phys. Rev. A 19, 1177 (1979).

[47] J.-X. Li, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. Lett. 113, 044801 (2014).

[48] J.-X. Li, K. Z. Hatsagortsyan, B. J. Galow, and C. H. Keitel, Phys. Rev. Lett. 115, 204801 (2015).

[49] A. Gonoskov, S. Bastrakov, E. Efimenko, A. Ilderton, M. Marklund, I. Meyerov, A. Muraviev, A. Sergeev, I. Surmin, and E. Wallin, Phys. Rev. E 92, 023305 (2015).