ONE IDEA AND TWO PROOFS OF THE KMT THEOREMS

MANJUNATH KRISHNAPUR

1. INTRODUCTION

1.1. The KMT theorems. Komlós, Major and Tusnády [8, 9] proved two “strong embedding” theorems: one for random walks and one for empirical processes.

- **KMT embedding for random walks (KMT-RW):** Let $X_i$ be i.i.d. random variables with zero mean, unit variance and finite moment generating function in a neighbourhood of zero. Then it is possible to couple the random walk $S_k = X_1 + \ldots + X_k$, with a standard Brownian motion $W$ in such a way that for some constant $C$ and any $x > 0$ and any $n \geq 1$,

$$\max_{0 \leq k \leq n} |S_k - W(k)| \leq C(\log n + x)$$

with probability at least $1 - e^{-x}$.

- **KMT embedding for empirical processes (KMT-EP):** Let $U_k$ be i.i.d. with uniform $[0, 1]$ distribution and let $F_n(t) = \frac{1}{n} \sum_{k=1}^{n} 1_{U_k \leq t}$. The uniform empirical process is the random function $G_n(t) = \sqrt{n}(F_n(t) - t)$ for $0 \leq t \leq 1$. It is possible to couple a standard Brownian bridge $W_0$ with $G_n$ so that for some constant $C$ and any $x > 0$ and any $n \geq 1$,

$$\sup_{0 \leq t \leq 1} \sqrt{n}|G_n(t) - W_0(t)| \leq C(\log n + x)$$

with probability at least $1 - e^{-x}$.

These theorems are counted among the most fundamental results of probability theory. There are many extensions to other situations (see the survey by Lifshits [10] or the ICM proceedings of Zaitsev [14] for some of these) but in this paper we just stick to the versions stated above. The original proofs being rather involved, there have been many efforts to simplify and streamline them. We are aware of two different kinds of proofs. In both, there are two big steps: (A) univariate coupling lemmas and (B) extension of the coupling to the level of paths. We describe the two approaches next. To see the results, one may jump directly to Sections 1.3 and 1.4.

The author is partially supported by UGC Centre for Advanced Study and the SERB-MATRICS grant MTR2017/000292.
1.2. The two approaches. The first one is the original proof of Komlós, Major and Tusnády with further developments due to Csörgő-Révész [6], Bretagnolle and Massart, Dudley, Massart (see [11] for a discussion of these papers), Carter-Pollard [3] and Pollard [12] (this is far from a complete list, see references in [12] and [3]). This works for both versions of KMT.

Step (A) in KMT-RW consists in proving a version of Tusnády’s lemma (Lemma 1), which gives a coupling of $S_n$ with a Gaussian. For KMT-EP, one needs this lemma only for Bernoulli steps (the connection is that $nF_n(t) \sim \text{Binomial}(n, t)$). This is achieved by a fine comparison of the tail probabilities of $S_n$ with that of a Gaussian, using Stirling’s approximation.

Step (B): For KMT-RW, the proof uses Step (A) to couple $S_n$ with $W(n)$, then $S_{n/2}$ with $W(n/2)$, then $S_{n/4}$ and $S_{3n/4}$, etc. For KMT-EP, the proof couples $G_n(t)$ and $W_0(t)$ for dyadic $t$. In either case, the dyadic procedure is carried up to a depth of about $\log n$ generations. An excellent exposition of the complete proof of the KMT theorem for empirical processes is the book by Pollard [12].

The second approach, due to Chatterjee [4], proves KMT-RW when the steps have symmetric Bernoulli distribution, by an approach that may be broadly described as Stein’s method. Bhattacharjee and Goldstein [2] extended this method of proof to a large class of step distributions.

Step (A) in this method consists in constructing couplings of Binomial and Hypergeometric distributions with Gaussian distributions. This is achieved by constructing the Stein coefficients for these distributions (Stein coefficient is a tool that measures how far a distribution or a random variable is from satisfying Stein’s equation for the Gaussian), and showing that they are close to constants. A key step is a general new result obtained by Chatterjee that deduces from this the existence of a good coupling. These couplings are weaker than Tusnády’s lemma, but suffice for the next step.

Step (B) comprises of a non-trivial induction on the number of steps of the random walk. To make the induction work, the hypothesis chosen is a careful statement about coupling the random walk bridge with a Brownian bridge having the same endpoint.

The goal of this paper is to give new and possibly simpler ways to carry out Step (A) in both methods, for the symmetric Bernoulli case. We have
little to add to Step (B), but for completeness, we sketch them in the two appendices. The appendices are not original and are taken from Chapter 10 of Pollard’s book [12] for KMT-EP and Sections 4,5 of Chatterjee’s paper [4] for the first KMT-RW (symmetric Bernoulli case only).

1.3. The univariate coupling lemmas. We now state the three Lemmas that we prove in this paper. These comprise Step (A) in the two proofs outlined above. Throughout the paper, unless indicated otherwise, Z will denote a standard Gaussian random variable and \( S_n = X_1 + \ldots + X_n \) where \( X_i \) are i.i.d. symmetric Bernoullis, i.e., \( X_i = \pm 1 \) with probability 1/2 each.

1.3.1. Step (A) in the proof of KMT-EP. This consists entirely in the following famous lemma of Tusnády (the usual reference is to [13] but what we understand comes from Pollard [12]).

**Lemma 1** (Tusnády-type lemma). For some \( n_0 \) and any \( n \geq n_0 \), there is a coupling of \( S_n \) with Z such that \( |S_n| \leq |Z| \sqrt{n} + 3 \) and \( |S_n - Z \sqrt{n}| \leq Z^2 + 11 \).

In the strongest known form of Tusnády’s lemma of this form, the result is valid for all \( n \geq 1 \) with \( |Z| \sqrt{n} + 2 \) and \( \frac{1}{n} Z^2 + 2 \) on the right sides of the two inequalities. Carter and Pollard [3] improved the main term on the right side of the second inequality to \( C_1 + C_2 Z^2 (1 + \frac{1}{n} |Z|) \), which is smaller for typical values of Z.

Our version has explicit and decent constants, and although suboptimal, they can be brought down further, see Section 6. We are not aware of any use of these constants though. In fact, in deriving the KMT for empirical processes, Pollard [12] (chapter 10 and Appendix D) uses a weaker lemma in which the right sides are \( C(1 + |Z| \sqrt{n}) \) and \( C(1 + Z^2) \) in the two inequalities, with unspecified constants. Of course, the statement of the Lemma [1] implies the same for all \( n \geq 1 \), albeit with larger constants.

1.3.2. Step (A) in the proof of KMT-RW for symmetric Bernoulli steps. Chatterjee proves the following lemmas in place of Tusnády’s and uses them to derive the KMT theorem for the Bernoulli random walk. These are Theorem 3.1 and Theorem 3.2 in [4].

The first lemma is about coupling a binomial distribution with a Gaussian distribution of the same mean and variance. Like in Tusnády’s lemma, the distance between the two does not grow with the variance of the variables.
Lemma 2 (Theorem 3.1 in [4]). For some $\theta_0 > 0$ and $\kappa_0 < \infty$, for every $n \geq 1$, there exists a coupling of $S_n$ with $Z$ such that $E[e^{\theta_0|S_n-Z\sqrt{n}|}] \leq \kappa_0$.

In fact, our proof allows any $\theta_0$ satisfying $8e^{2\theta_0}\theta_0^2 < 1$ and gives an explicit form for $\kappa_0$.

The second lemma that Chatterjee proves is a coupling of a hypergeometric distribution with a Gaussian distribution. The relevance of this is easy to see: In Step (B), when one proceeds by conditioning on position of the random walk at the end, the position of the walk at any intermediate time has a (shifted) hypergeometric distribution.

Some notation: For $0 \leq k \leq n$ and any probable value $s$ of $S_n$ (by that we mean $P\{S_n = s\} > 0$, or equivalently, that $|s| \leq n$ and $n-s$ is even), let $S_k[n,s]$ denote a random variable whose distribution is the conditional distribution of $S_k$ given $S_n = s$. An equivalent description is that $S_k[n,s]$ is the sum of the first $k$ coupons drawn without replacement, uniformly at random, from a box containing $n$ coupons of which $\frac{n+s}{2n}$ proportion are labeled $+1$ and the remaining $g = \frac{n-s}{2n}$ proportion are labeled $-1$. Thus $S_k[n,s]$ has (a the simple transformation of) hypergeometric distribution and has mean $k(p-q) = \frac{sk}{n}$ and variance $4pq\frac{k(n-k)}{n-1}$. Let $\sigma^2_{n,k} = \frac{1}{n}k(n-k)$.

Lemma 3. There exists a $\theta_1 > 0$ and $M_1 < \infty$ such that for any $n \geq 2$ and \( \frac{1}{3}n \leq k \leq \frac{2}{3}n \), and any probable value $s$ of $S_n$, there exists a coupling of $W := S_k[n,s] - \frac{sk}{n}$ with $Z \sim N(0,1)$ such that $E[e^{\theta|W-\sigma_{n,k}Z|}] \leq \exp\{1 + M_1\theta^2\frac{s^2}{n}\}$ for all $\theta \leq \theta_1$.

Note that $\sigma^2_{n,k}$ does not depend on $s$ and is comparable to the variance of $S_k[n,s]$ only when $s = O(\sqrt{n})$. For such $s$, the conclusion here is analogous to that of Lemma 2. For larger values of $s$, the coupling here is not between random variables of comparable variance and correspondingly, the right side could be large, but the point is that the controlling parameter is $s^2/n$.

1.4. Our approach. The single key idea of this paper is that Binomial distributions are similar objects (discrete, combinatorial), and comparing them with each other is easier than comparing one of them to the Gaussian distribution. This suggests that we approach the problem via the Cauchy criterion
and look for a coupling between $S_n/\sqrt{n}$ and $S_{4n}/\sqrt{4n}$. Then we can successively couple $S_n/\sqrt{n}$, $S_{4n}/\sqrt{4n}$, $S_{n^4}/\sqrt{n^4}$, ... If the coupling is sufficiently strong at each step, this sequence converges almost surely to a standard Gaussian variable $Z$ that is coupled well with $S_n/\sqrt{n}$.

This is clearly an approach that can be of more general use, for instance to get rates of convergence in limit theorems where often discrete combinatorial objects converge to a continuum object. In our context, this idea translates to proving the following theorems, which are natural analogues of Lemmas 1, 2, 3.

**Theorem 4.** For some $n_0$ and any even number $n \geq n_0$, there exists a coupling of $2S_n$ and $S_{4n}$ so that they have the same sign (meaning $S_nS_{4n} \geq 0$) and

$$|S_{4n}| - \frac{1}{8n}|S_{4n}|^2 - 9 \leq 2|S_n| \leq |S_{4n}| + 2.$$  

Equivalently, $2|S_n| \leq |S_{4n}| + 2$ and $|2S_n - S_{4n}| \leq \frac{1}{8n}|S_{4n}|^2 + 9$. This looks more similar to Lemma 1, with $\sqrt{4n}$ taking the place of $Z$.

**Theorem 5.** There exists $\theta_0 > 0$ and $\kappa_0 < \infty$ such that for any $n \geq 1$, there exists a coupling of $S_n$ with $S_{4n}$ such that $\mathbb{E}[e^{\theta_0|2S_n-S_{4n}|}] \leq \kappa_0$. In fact, any $\theta_0$ such that $8\theta_0^2e^{2\theta_0} < 1$ works.

The analogue of Lemma 3 is broken into two parts. In the unbiased case $s = 0$, we show that $S_k[n,0]$ can be coupled well with a Gaussian of the same variance. Then we show that $S_k[n,s]$ can be coupled with $S_k[n,0]$ so that the difference is controlled by $s^2/n$.

**Theorem 6.** There exists a $\Theta > 0$ and $M < \infty$ such that for any even number $n$ and $\frac{1}{3}n \leq k \leq \frac{2}{3}n$, and any probable value $s$ of $S_{nk}$, writing $W_1 := S_k[n,0]$, $W_2 := S_{4k}[4n,0]$ and $W := S_k[n,s] - \frac{s}{n}$, there are couplings.

---

1When we talk about a coupling of random variables $X, Y$ satisfying some statements, it means that there exists some probability space on which there are random variables having the same marginal distributions as $X$ and $Y$, and satisfying the mutual relationships in the statements. Given couplings $\theta_i$ of $X_i$ and $X_{i+1}$ for $i = 0, 1, \ldots$, one can get a sequence of random variables $(X'_i)_{i \geq 0}$ such that $(X'_i, X'_{i+1})$ has the joint distribution given by the coupling $\theta_i$ for each $i \geq 0$. This is done by running a Markov chain as follows: First sample $(X'_0, X'_1)$ from the distribution $\theta_0$, and then successively sample $X'_{i+1}$ from the conditional distribution of $X_{i+1}$ given $X_i$ in the distribution $\theta_i$. 

(1) of $W_1$ with $W_2$ such that $\mathbb{E}[e^{\theta|2W_1-W_2|}] \leq \frac{3}{\Theta}$ for all $\theta \leq \Theta$, and

(2) of $W_1$ with $W$ such that $\mathbb{E}[e^{\theta|W_1-W|}] \leq e^{1+M\theta^2+\frac{\pi}{2}}$ for all $\theta \leq \Theta$.

We assume $n$ to be even so that 0 is a probable value of $S_n$ and hence $S_k[n,0]$ makes sense. For odd $n$, the same holds if we replace 0 by 1 (or any fixed odd number). In another direction, we could have stated the theorem more generally for $\delta n \leq k \leq (1-\delta)n$ for any $\delta > 0$. Alternately from the point of view of its actual use in Step (B), we could have specialized to $k = \lfloor n/2 \rfloor$ and simplified the proofs a little. Since Chatterjee wrote it for $\frac{k}{n} \in \left[\frac{1}{3}, \frac{2}{3}\right]$, we do the same.

1.5. About the proofs. The potential to use combinatorial methods eliminates many technicalities to make the proofs of Theorems 4, 5, 6 simpler relative to Lemmas 1, 2, 3. And the deduction of those three lemmas from the corresponding theorems is also straightforward. While the simplicity may be in the eye of the beholder, given the fundamental nature of the KMT theorems, even improvement of exposition may be of some interest. Carter and Pollard [3] remark on the “...continuing perceived need for an accessible treatment of the coupling result that underlies the KMT construction”.

1.5.1. About the proof of Theorem 4. The main step are Lemmas 8 and 9, which give a comparison of Binomial coefficients of the form $\binom{n}{i}$ with $\binom{4n}{j}$. This comparison of the Binomial mass functions, together with some standard estimates on the tails of Binomial distribution allows us to prove the right kinds of inequalities between the tails of the two Binomial distributions. In contrast, earlier proofs (as in Pollard [12], Carter and Pollard [3], Massart [11], etc.) directly compare the Binomial tails to Gaussian tails. The estimates needed are more refined than the usual first-order Stirling’s formula used to prove the de Moivre-Laplace central limit theorem.

1.5.2. About the proofs of Lemma 2 and Lemma 3. First we outline Chatterjee’s approach - all references are to [4]. If there is a random variable $T$ (usually $T = T(W)$, although use of additional randomness is allowed) such that $\mathbb{E}[Wf(W)] = \mathbb{E}[Tf'(W)]$ for a large class of $f$, then Chatterjee calls $T$ a Stein coefficient for $W$. Stein’s famous characterization of the Gaussian
says that if $T = \sigma^2$ is a constant, then $W \sim N(0, \sigma^2)$. A key result in Chatterjee’s paper is that if $T$ is close to a constant in an appropriate sense, then $W$ is close to a Gaussian. Of course, this statement needs to be proved in a much finer form than usually needed to show central limit theorems by Stein’s method. After that, it is a matter of finding the Stein coefficients of the Binomial and Hypergeometric distributions (actually of some perturbations of those, since they are discrete and do not admit Stein coefficients) to prove Lemma 2 and Lemma 3.

In our proofs of Theorem 5 and Theorem 6 we work with finite state Markov chains and show that if two Markov chains on (segments of) integers have transitions $i \mapsto i \pm 1$ with rates $T(i) \mp i$ and $i \mapsto i \pm 1$ with rates $S(i) \mp i$, then if $T$ and $S$ are close, the stationary distributions of the two chains can be coupled well. This is achieved by constructing a joint Markov chain on $\mathbb{Z}^2$, whose stationary distribution gives a coupling of the stationary distributions of the given chains.

Although this may seem different from what was outlined as Chatterjee’s method, they are closely related. Stein’s equation is essentially a rephrasing of the forward equation for the Ornstein-Uhlenbeck process. And the proof of the main lemma of Chatterjee essentially boils down to showing the existence of an invariant measure for a certain generator of a joint Markov process. The technicalities are considerably reduced in our case as we are in the setting of finite state space Markov chains.

An alternate way to say the same thing is that we work with the Stein operator for the Binomial distributions. It is a fact that

$$E \left[ \left( \frac{n}{2} + X \right) (f(X - 1) - f(X)) \right] = E \left[ \left( \frac{n}{2} - X \right) (f(X + 1) - f(X)) \right]$$

for all functions $f$ if and only if $X$ has the same distribution as $S_n$. Thus, to prove Theorem 5, we show that $Y = 2(S_n + R)$ (where $R$ is independent of $S_n$ and takes values $-1, 0, 1$) satisfies

$$E[(T(Y) + Y)(f(Y - 1) - f(Y))] = E[(T(Y) - Y)(f(Y + 1) - f(Y))]$$

for all $f$, for a function $T$ that is close to the constant $2n$. This shows that $Y$ (and hence $2S_n$) can be coupled well with $S_{4n}$.

1.5.3. A minor point worth noting. In his famous paper where the well-known inequality comes, Hoeffding [7] also proved other less known but
remarkable results. One of them states that if $f : \mathbb{R} \to \mathbb{R}$ is convex, then $\mathbb{E}[f(\bar{X})] \leq \mathbb{E}[f(\bar{Y})]$ where $\bar{X}$ and $\bar{Y}$ are averages of samples drawn without and with replacement from a box of coupons. This allows one to get estimates on expectations of functions of a hypergeometric variable (see Lemma 21) without much effort. Not using this result of Hoeffding, direct and lengthier proofs are given in [4] (Lemmas 3.4 and 3.5 in [4] and Lemma 21 in this paper).

1.5.4. Some shortcomings. The first proof, although mainly combinatorial, could be made nicer if one could prove the main Lemmas 8 and 9 by bijective methods. We were unable to do that. Further, Lemma 9 strongly suggests that it should be possible to prove the improved version of Tusnády's lemma as found by Carter and Pollard. However, in trying to do that the remaining part of the proof got so bloated that we settled for the weaker form.

The second proof, by Markov chain coupling, appears to be amenable to proving the KMT theorem for random walks with more general step distributions. The reason is that the Stein coefficient has nice behaviour under convolutions, and it appears that Cramer's large deviation theorem should give a coupling of $S_n$ with $N(0, n)$ analogous to Theorem 5. Of course, this is only one of the key steps and there is more work needed. As of now, we do not have such a proof. As was mentioned earlier, Bhattacharjee and Goldstein [2] have already extended Chatterjee's method to more general random variables, although with some extra conditions.

1.6. Outline of the rest of the paper. In the next section, we deduce Lemmas 1, 2, 3 from Theorems 4, 5 and 6, respectively. Then the paper is split into two parts that can be read independently of each other. Part I proves Theorem 4 by a combinatorial method and Part II proves Theorems 5 and 6 by coupling Markov chains. The overall ideas of the proofs were outlined above. The appendices outline Step (B) in both proofs, i.e., the deduction of the KMT theorems from the coupling lemmas.

1.7. Acknowledgments. I first learned about the KMT theorem from Yuval Peres when (around 2007) he was asking for simpler proofs. That motivated me to think about the problem off and on. I would like to thank
Yogeshwaran D. for listening to my speculative ideas at various stages, for useful comments on the first draft, and for encouraging remarks at all times.

2. PROOFS OF LEMMAS 1, 2, 3 FROM THEOREMS 4, 5 AND 6

All three proofs are similar. For Lemmas 1, 2, we define $Z_j = S_{4jn} / \sqrt{4jn}$, for $j = 0, 1, 2 \ldots$. From the corresponding theorems, we can couple $Z_j$ with $Z_{j+1}$ for each $j$, and hence a couple all the $Z_j$s on a common probability space so that $Z_j$ and $Z_{j+1}$ are close for each $j$ (with high probability). Then we show that the sequence $Z_j$ converges almost surely to a standard Gaussian random variable $Z$, and that $Z_0$ and $Z$ are very close. That is, $S_n$ and $Z$ are coupled as required by the Lemmas. For Lemma 3, the idea is quite similar but the notation is different and will be introduced in the proof.

Proof of Lemma 1. Fix an even number $n \geq n_0$ as in Theorem 4, and couple all the $Z_k$s so that they all have the same sign and

$$|Z_k| \leq |Z_{k+1}| + \frac{1}{2k\sqrt{n}}, \quad |Z_k - Z_{k+1}| \leq \frac{1}{2k+2\sqrt{n}}|Z_{k+1}|^2 + \frac{9}{2k+1\sqrt{n}}.$$  (1)

By Bernstein’s inequality, $P\{|Z_{k+1}| \geq 2^{k/4}\}$ is summable, hence $|Z_{k+1}| \leq 2^{k/4}$ for all but finitely many $k$. Then the second inequality in (1) shows that $|Z_k - Z_{k+1}|$ is summable, and hence $(Z_k)_k$ is a Cauchy sequence, almost surely. The limiting variable, call it $Z$, must have $N(0, 1)$ distribution, by the central limit theorem. Evidently, $Z$ has the same sign as $Z_0$.

Summing the first inequality in (1) over $k \geq j$ gives $|Z_j| \leq |Z| + 2^{1-j}n^{-1/2}$. In particular $|Z_0| \leq |Z| + 2n^{-1/2}$ or equivalently $|S_n| \leq |Z|\sqrt{n} + 2$.

Use $|Z_{k+1}| \leq |Z| + 2^{-k-n^{-1/2}}$ in the second inequality in (1) and sum to get

$$|Z_0 - Z| \leq \sum_{k=0}^{\infty} \frac{1}{2^{k+2}\sqrt{n}} \left(|Z| + \frac{1}{2k\sqrt{n}}\right)^2 + \frac{9}{2k+1\sqrt{n}}$$

$$\leq \frac{1}{\sqrt{n}}Z^2 + \frac{4}{7n^{3/2}} + \frac{9}{\sqrt{n}}$$  (2)

where we used $(a + b)^2 \leq 2a^2 + 2b^2$ and summed the geometric series. Consequently, $|Z_0 - Z|\sqrt{n} \leq Z^2 + 10$ for large $n$. Equivalently, $|S_n - Z\sqrt{n}| \leq Z^2 + 10$.

This completes the proof of the Lemma for even $n$, with a saving of 1 on the right sides of both inequalities. If $n$ is odd, then $S_n$ can be coupled with
Proof of Lemma 2. Fix any \( Z \) using the coupling of \( \text{Recall that} \) for \( \theta \) for \( U \) hence \( |Z_0 - Z| \leq \sum_{k \geq 0} |Z_k - Z_{k+1}| \) for each \( k \geq 0 \).

By convexity of the exponential,

\[
\exp\{\theta_0 \sqrt{n} \sum_{k \geq 0} |Z_k - Z_{k+1}|\} \leq \sum_{k \geq 0} \frac{1}{2^{k+1}} \exp\{\theta_0 \sqrt{n} 2^{k+1} |Z_k - Z_{k+1}|\}.
\]

Therefore, the expectation of the left side quantity is bounded by \( \kappa_0 \). In particular, this shows that \( \kappa_0 \) implies that \( Z \) converges almost surely to some \( Z \). By the central limit theorem, \( Z \sim N(0, 1) \). As \( |Z_0 - Z| \leq \sum_{k \geq 0} |Z_k - Z_{k+1}| \) it follows that \( \mathbb{E}[e^{\theta_0 \sqrt{n}|Z_0 - Z|}] \leq \kappa_0 \). This is the same as \( \mathbb{E}[e^{\theta_0 |Z_n - Z\sqrt{n}|}] \leq \kappa_0 \).

Proof of Lemma 3. Recall that \( S_k[n, 0] \) has mean 0 and variance \( \frac{n}{n-1} \sigma_{n,k}^2 \). Set \( U_j = 2^{-j} S_{\lfloor k \rfloor}[4^j n, 0] \). Then \( U_j \) has zero mean and variance \((1 - \frac{1}{4^n})^{-1} \sigma_{n,k}^2 \). From the first part of Theorem 6, if \( n \) is an even number, we can construct random variables \( U_j \) on one probability space so that for any \( j \geq 0 \) and any \( \theta \leq \Theta \), we have \( \mathbb{E}[e^{\theta U_{j+1}^+} | U_j] \geq \frac{3}{2} \). Using convexity of the exponential,

\[
\mathbb{E}\left[e^{\theta \sum_{j \geq 0} |U_j - U_{j+1}|}\right] \leq \mathbb{E}\left[\sum_{j \geq 0} \frac{1}{2^{j+1}} e^{\theta 2 U_{j+1}^+} | U_j - U_{j+1}|\right] \leq \frac{3}{2}
\]

for \( \theta \leq \Theta \). This shows that \( \sum_j |U_j - U_{j+1}| \) converges almost surely and hence \( U_j \) converges almost surely to some random variable \( U \), as \( j \to \infty \). As \( |U_0 - U| \) is bounded by \( \sum_{j \geq 0} |U_j - U_{j+1}| \), we see that \( \mathbb{E}[e^{\theta |U_0 - U|}] \leq \frac{3}{2} \) for \( \theta \leq \Theta \). Further, this also shows that \( \{U_j\} \) is exponentially tight and hence its mean and variance converge to those of \( U \). Since \( U_j \) converges in distribution to \( N(0, \sigma_{n,k}^2) \) (this is an elementary fact, see Remark 7), it follows that \( U \sim N(0, \sigma_{n,k}^2) \). In other symbols, we have a coupling of \( W_1 = U_0 \) with \( Z := U/\sigma_{n,k} \) such that \( \mathbb{E}[e^{\theta |W_1 - \sigma_{n,k} Z|}] \leq \frac{3}{2} \) for all \( \theta \leq \Theta \).
Next, use the second part of Theorem 6 to construct \( W = S_k[n, s] - \frac{\sigma k}{n} \) coupled with \( W_1 \) in such a way that \( \mathbb{E}[e^{\theta|W-W_1|}] \leq \exp\{1 + M\theta^2 \frac{s^2}{n}\} \) for all \( \theta \leq \Theta \).

Now that we have \( W, W_1, Z \) on the same probability space, we just observe that for \( \theta \leq \frac{1}{2}\Theta \), by Cauchy-Schwarz,

\[
\mathbb{E}[e^{\theta|W-\sigma_n k Z|}] \leq \mathbb{E}[e^{2\theta|W_1-\sigma_n k Z|}]^{\frac{1}{2}} \mathbb{E}[e^{2\theta|W-W_1|}]^{\frac{1}{2}} \leq \sqrt{\frac{3}{2} e^{\frac{1}{2} + 2M\theta^2 \frac{s^2}{n}}}
\]

As this is less than \( e^{1 + 2M\theta^2 \frac{s^2}{n}} \), taking \( \theta_1 = \Theta/2 \) and \( M_1 = 2M \), this completes the proof for even \( n \).

If \( n \) is odd, we claim that \( S_k[n, s] \) can be coupled with \( S_{k+1}[n+1, s + \frac{1}{2}] \) so that the difference between the two is at most 2. To see this, consider the box of \((n + s)/2\) coupons labeled 1 and \((n - s)/2\) coupons labelled \(-1\) and a red coupon also labelled \(+1\). Drawing \( k \) coupons without replacement and adding them gives \( S_{k+1}[n, s] \). To get \( S_k[n, s] \), we do the same experiment, but if the red coupon comes up, discard it and draw a different one. It is clear that this can be done so that the difference between the two remains at most 2. Hence, using the result for even \( n \), we get a coupling such that

\[
\mathbb{E}[e^{\theta|W-\sigma_n k Z|}] \leq e^{2\theta} \sqrt{\frac{3}{2} e^{\frac{1}{2} + 2M\theta^2 \frac{(s+\frac{1}{2})^2}{n}}} \]

which is bounded by \( e^{1 + 4M\theta^2 \frac{s^2}{n}} \) if \( \theta \) is small enough. \( \blacksquare \)

Remark 7. We claimed that \( U_j \overset{d}{\rightarrow} N(0, \sigma_{n,k}^2) \) in the proof. This follows from the fact that if \( t \in (0, 1) \) is fixed, \( V_n := S_{\lfloor nt \rfloor}[n, 0]/\sqrt{n} \) converges in distribution to \( N(0, t(1-t)/4) \). A direct way to show this is to use Stirling’s approximation to write the probability that \( V_n = x \) as (here \( H(t) = -t \log t - (1-t) \log(1-t) \))

\[
\left(\frac{m + \frac{m}{2} \sqrt{2\pi}}{\left(\frac{2m}{2}\right)}\right)^m e^{-mH\left(t + \frac{\sqrt{2\pi}}{2m}\right) - H\left(t - \frac{\sqrt{2\pi}}{2m}\right) - 2H(t)} \sim \frac{1}{\sqrt{\pi m}} e^{m[H(t + \frac{\sqrt{2\pi}}{2m}) + H(t - \frac{\sqrt{2\pi}}{2m}) - 2H(t)]}.
\]

The exponent is \( -\frac{2x^2}{t(1-t)} + o(1) \), as the terms linear in \( x \) cancel. Thus, \( V_n \) converges in distribution \( N(0, \frac{1}{4}t(1-t)) \).
Part I. Coupling by comparison of binomial probabilities

3. The comparison between mass functions of $2S_{2m}$ and $S_{8m}$

In this section, assume that $n = 2m$ is even. Then $2S_{2m}$ is supported on multiples of 4 from $-2m$ to $2m$ and $S_{8m}$ is supported on even integers from $-8m$ to $8m$. Set $\alpha_m(k) = P\{2S_{2m} = 4k\}$ and $\beta_m(k) = P\{S_{8m} = 4k \text{ or } 4k-2\}$ for $k \geq 1$. Explicitly,

$$\alpha_m(k) = \left(\frac{2m}{m+k}\right) \frac{1}{2^{2m}}$$

$$\beta_m(k) = \left(\frac{8m}{4m+2k}\right) \frac{1}{2^{8m}} + \left(\frac{8m}{4m+2k-1}\right) \frac{1}{2^{8m}} = \left(\frac{8m+1}{4m+2k}\right) \frac{1}{2^{8m}}.$$

This are not mass functions, but if we double the values of $\alpha_m(k)$ and $\beta_m(k)$ for $k \geq 1$ and set $\alpha_m(0) = P\{S_{2m} = 0\}$ and $\beta_m(0) = P\{S_{8m} = 0\}$, we get the mass functions of $2\lfloor S_{2m}/4 \rfloor$ and $\lceil \lfloor S_{8m}/4 \rfloor \rceil$. The ultimate goal is to couple $2S_{2m}$ and $S_{8m}$ as closely as possible, which is possible if one gets a good comparison between the tails of the two distributions (see Lemma 13). As a first step towards this goal, in this section we get a comparison between the probability mass functions.

**Lemma 8.** For any $m \geq 1$, we have $\alpha_m(k) \leq \beta_m(k)$ for any $k \geq 1$.

**Lemma 9.** For any $m \geq 1$, we have $\alpha_m(k) \geq \beta_m(\ell)$ if $1 \leq k \leq \ell - \frac{1}{4} \left(1 + \frac{2}{m^2}\right)$.

In particular, $\alpha_m(\ell - 1) \geq \beta_m(\ell)$ for $\ell \leq (3m^2)^{1/3}$.

We now proceed to the proofs of the two lemmas, which are similar. For $h \geq 1$, define

$$f(m, k) := \frac{\beta_m(k)}{\alpha_m(k)} = \frac{\left(\frac{8m+1}{4m+2k}\right)}{2^{8m} \left(\frac{2m}{m+k}\right)},$$

$$g_h(m, k) := \frac{\beta_m(k)}{\alpha_m(k-h)} = \frac{\left(\frac{8m+1}{4m+2k}\right)}{2^{8m} \left(\frac{2m}{m+k-h}\right)}.$$

The assertion of the two lemmas is that $f(m, k) \geq 1$ for $1 \leq k \leq m$ and $g_h(m, k) \leq 1$ if $h+1 \leq k \leq m$ and $h \geq \frac{1}{4} \left(1 + \frac{k^3}{m^2}\right)$. It is tempting to try proving these by finding explicit injective maps between appropriate sets, since the numerators and denominators have obvious counting interpretations. We were unable to find such a proof and take a different route.
3.1. **Proof of Lemma** [8]. That $f(m, k) \geq 1$ for all $1 \leq k \leq m$, follows from three assertions:

1. $f(m, k + 1) \geq f(m, k)$ for any $1 \leq k \leq m - 1$ and any $m \geq 1$.
2. $f(m, 1) \geq f(m + 1, 1)$ for all $m \geq 1$.
3. $f(m, 1) \to 1$ as $m \to \infty$.

From the second and third assertions, it follows that $f(m, 1) \geq 1$ for all $m$ and then the first assertion gives $f(m, k) \geq f(m, 1)$, completing the proof.

Now we prove the three assertions.

**Step-1:** For $1 \leq k \leq m - 1$, canceling many factorials, $f(m, k + 1)/f(m, k)$ is seen to be equal to

\[
\frac{(m+k+1)(4m-2k)(4m-2k+1)}{(m-k)(4m+2k+1)(4m+2k+2)} = \frac{(4m+4k+4)(4m-2k)(4m-2k+1)}{(4m-4k)(4m+2k+1)(4m+2k+2)}.
\]

Write $x = 4m$ and $a = 2k$. If we subtract $(x-2a)(x+a+1)(x+a+2)$ (the denominator) from $(x+2a+4)(x-a)(x-a+1)$ (the numerator), we get $2x(x-2a)+2x+4a^2(a+2)$. As $x-2a = 4(m-k) > 0$, all three summands are positive if $1 \leq k \leq m-1$ and $m \geq 1$. Thus $f(m, k+1) \geq f(m, k)$.

**Step-2:** Observe that $f(m+1, 1)/f(m, 1)$ is equal to

\[
\frac{m(m+2)\times(8m+2)\ldots(8m+9)}{2^6 \times (2m+1)(2m+2) \times (4m+3)\ldots(4m+6) \times (4m+9)} = \frac{(m+2)(8m+3)(8m+5)(8m+7)(8m+9)}{2^4(2m+1)(2m+2)(4m+3)(4m+5)(4m+6)}
\]

by canceling five factors in the numerator and denominator. Multiplying by powers of 2 to write each terms as $8m + [\cdot]$, we see that

\[
\frac{f(m+1, 1)}{f(m, 1)} = \frac{(8m+16)(8m+3)(8m+5)(8m+7)(8m+9)}{(8m+4)(8m+8)(8m+6)(8m+10)(8m+12)} = \frac{(x+16)(x+3)(x+5)(x+7)(x+9)}{(x+4)(x+6)(x+8)(x+10)(x+12)}
\]

with $x = 8m$. Then the denominator minus the numerator is (we used Mathematica here) $15(528 + 257x + 40x^2 + 2x^3)$ which is positive for $x > 0$. Thus $f(m+1, 1) \leq f(m, 1)$ for all $m \geq 1$. 

Remark 10. Here is a less computational way to check the last point. For \( x > 0 \), the function \( \psi(a_1, \ldots, a_k) := (x + a_1) \cdots (x + a_k) \) is Schur-concave. This can be seen either from the well-known fact that elementary symmetric polynomials are Schur-concave (and \( \psi \) is a positive linear combination of those) or by directly checking the condition for Schur-concavity:

\[
(a_i - a_j)(\frac{\partial}{\partial a_i} - \frac{\partial}{\partial a_j}) \psi \leq 0.
\]

In our case, \((16, 9, 7, 5, 3)\) majorizes \((12, 10, 8, 6, 4)\), hence \( \psi(16, 9, 7, 5, 3) \leq \psi(12, 10, 8, 6, 4) \).

Step-3: That \( f(m, 1) \to 1 \) as \( m \to \infty \) is clear from the local central limit theorem, but can also argue directly from Stirlings’ approximation:

\[
f(m, 1) = \frac{(8m + 1)! \times (m + 1)! \times (m - 1)!}{2^{6m} \times (2m)! \times (4m + 2)! \times (4m - 1)!}
\]

\[
= \frac{(8m)! \times m! \times m!}{2^{6m} \times (2m)! \times (4m)! \times (4m)!} \times \frac{(8m + 1)(m + 1)(4m)}{m(4m + 1)(4m + 2)}
\]

\[
\sim \frac{(8m)^{8m + \frac{1}{2}} m^{2m + 1}}{2^{6m}(2m)^{2m + \frac{1}{2}}(4m)^{8m + 1}} \times 2
\]

\[
= 1.
\]

3.2. Proof of Lemma 9 The proof of the first statement in the Lemma will be achieved in three steps, similarly to the proof of Lemma 8. The second statement follows from the first by setting \( h = 1 \).

(1) \( g_h(m, k + 1) \leq g_h(m, k) \) if \( h + 1 \leq k \leq [(4h - 1)m^2]^\frac{1}{2} \).

(2) \( g_h(m + 1, h + 1) \geq g_h(m, h + 1) \) for \( m \geq h + 1 \).

(3) \( g_h(m, h + 1) \to 1 \) as \( m \to \infty \), for fixed \( h \).

The second and third assertions show that \( g_h(m, h + 1) \leq 1 \) for all \( m \geq h + 1 \) and the first assertion shows that \( g_h(m, k) \leq 1 \) if \( h + 1 \leq k \leq [(4h - 1)m^2]^\frac{1}{2} \).

The proofs of the assertions are similar to that of Lemma 8.

Step-1: Canceling many terms, \( g_h(m, k + 1)/g_h(m, k) \) is seen to be equal to

\[
\frac{(m + k + 1 - h)(4m - 2k)(4m - 2k + 1)}{(m - k + h)(4m + 2k + 1)(4m + 2k + 2)}
\]

\[
= \frac{(x + 2a - 4(h - 1))(x - a)(x - a + 1)}{(x - 2a + 4h)(x + a + 1)(x + a + 2)}
\]
with \( x = 4m \) and \( a = 2k \). Subtracting the numerator from the denominator gives
\[
2((4h - 1)x^2 - 2a^3) + 2x(8h + 2a - 1) + 8(a^2(h - 1) + ah + h)
\]
which is positive provided \( 2a^3 \leq (4h - 1)x^2 \). Thus, \( g_h(m, k + 1) \leq g_h(m, k) \) if \( k^3 \leq (4h - 1)m^2 \).

**Step-2:** Let \( (x)^{\uparrow}_m = x(x+1) \ldots (x+k-1) \) and \( (x)^{\uparrow}_m = x(x+2) \ldots (x+2k-2) \).
Consider
\[
g_h(m + 1, k) = \frac{(8m + 2)^{\uparrow}_8 \times (m + 1 + k - h) \times (m + 1 - k + h)}{2^6(2m + 1)(2m + 2) \times (4m + 2k + 1)^{\uparrow}_4 \times (4m - 2k + 2)^{\uparrow}_4}
\]
\[
= \frac{(8m + 2)^{\uparrow}_8 \times (8m + 8 + 8k - 8h) \times (8m + 8 - 8k + 8h)}{(8m + 4)(8m + 8) \times (8m + 4k + 2)^{\uparrow}_4 \times (8m - 4k + 4)^{\uparrow}_4}.
\]
Cancel \((8m+4)(8m+8)\) in the numerator and denominator, and set \( x = 8m \).
The above expression becomes \( \prod_{j=1}^8 (x + t_j) / \prod_{j=1}^8 (x + s_j) \) where
\[
t = (8k - 8h + 8, 9, 7, 6, 5, 3, 2, -8k + 8h + 8),
\]
\[
s = (4k + 8, 4k + 6, 4k + 4, 4k + 2, -4k + 10, -4k + 8, -4k + 6, -4k + 4).
\]
As \( k \geq h + 1 \geq 2 \), these vectors are written in decreasing order. The vector of partial sums of \( s - t \) is
\[
(8h - 4k, 8h - 3, 8h + 4k - 6, 8h + 8k - 10, 8h + 4k - 5, 8h, 8h - 4k + 4, 0)
\]
which is non-negative if and only if \( k \leq 2h \). Thus, for \( h + 1 \leq k \leq 2h \), we see that \( s \) majorizes \( t \), and (see Remark [10]) by Schur concavity \( \prod_j (x + t_j) \geq \prod_j (x + s_j) \). That is, \( g_h(m + 1, k) \geq g_h(m, k) \). In particular, this always holds for \( k = h + 1 \).

**Step-3:** \( g_h(m, k) \to 1 \) as \( m \to \infty \), for any fixed \( k, h \). This is immediate from Stirling’s formula or the local central limit theorem. Alternately one may observe that
\[
g_h(m, k) = f(m, k) \frac{(m - k + 1) \ldots (m - k + h)}{(m + k - h + 1) \ldots (m + k)}
\]
and use that \( f(m, k) \to 1 \) as \( m \to \infty \).
4. THE COMPARISON BETWEEN THE TAILS OF $2S_{2m}$ AND $S_{8m}$

We now prove the following crucial lemma comparing the tails of the two random variables.

**Lemma 11.** For some $m_0$ and any $m \geq m_0$, we have

1. $\alpha_m(k) \leq \beta_m(k)$ for all $k \geq 1$.
2. $\alpha_m(k) \geq \beta_m(\ell)$ if $1 \leq k \leq \ell - \frac{\ell^2}{4m} - 1$.

In Lemma 9, we had the relationship $k = \ell - C\ell^3m^2$. If the same relationship could be carried over to Lemma 11, we would have ended up with the Carter-Pollard [3] improvement of Tusnády's lemma. However, the proof (of the second part) got way longer, and hence we settled for the weaker form with $\ell^2/m$.

In addition to Lemma 9 and Lemma 8, we need the following basic estimates for binomial coefficients. Introduce the notation $D(p) := p\log(2p) + (1-p)\log(2-2p)$, usually written as $D(\text{Ber}(p)\|\text{Ber}(1/2))$, for the relative entropy of Bernoulli $(p)$ with respect to Bernoulli $(1/2)$.

1. Lemma 4.7.1 of Ash [1] states that with $A = 1/\sqrt{8}$ and $B = 1/\sqrt{2\pi}$, for any $1 \leq k \leq n - 1$,
   $$A \frac{\sqrt{n}}{\sqrt{k(n-k)}} e^{-nD(\frac{1}{2})} \leq \frac{1}{2^n} \binom{n}{k} \leq B \frac{\sqrt{n}}{\sqrt{k(n-k)}} e^{-nD(\frac{1}{2})}. \quad (3)$$

2. Lemma 4.7.2 of Ash [1] states that for $\frac{n}{2} < k < n$, with $A = 1/\sqrt{8}$,
   $$A \frac{\sqrt{n}}{\sqrt{k(n-k)}} e^{-nD(\frac{1}{2})} \leq \frac{1}{2^n} \sum_{j=k}^{n} \binom{n}{j} \leq e^{-nD(\frac{1}{2})}. \quad (4)$$

Both these estimates are proved using Stirling formula, but one needs more than the first term in the asymptotic expansion. For a somewhat simpler proof (that uses only the second term in Stirling’s formula) of (3) with the weaker constant $B = 1/\sqrt{\pi}$, see Lemma 17.5.1 of Cover and Thomas [5].

**The relative entropy function:** Let $s = 2t - t^2$, so that $t \mapsto s$ is an increasing bijection of $[0, 1]$ with itself. Define $Q(t) := 4D\left(\frac{1}{2} + \frac{s}{2}\right) - D\left(\frac{1}{2} + \frac{s}{2}\right)$, an object that will occur repeatedly in the proof. Note that $Q(t) = 2t^3 + O(t^4)$ as $t \to 0$ and that $Q(t)$ is strictly positive for all $t > 0$. Therefore, for any $\epsilon > 0$, we can write $Q(t) \geq (2 - \epsilon)t^3$ for $0 \leq t \leq \delta$ for some $\delta > 0$ and...
Q(t) ≥ c for δ ≤ t ≤ 1 for some c > 0. For the sake of simplifying the writing, we use the more convenient form

\[ Q(t) \geq \frac{3}{2}t^3 \text{ for } 0 \leq t \leq 1, \]  

which is easiest to check on a software, numerically or symbolically (in fact 3/2 can be replaced by 1.65 · · ·).

**Remark 12.** The precise constants do not matter in the big picture. If A were smaller or B were larger, or if the factor in front of \( m^{2/3} \) in Lemma 9 were less than \( 3^{1/3} \), the proof would still go through, but with a choice of \( s = t - \mu t^2 \) for a sufficiently large \( \mu \) and appropriate modifications that follow from it. This would only affect the constants in Lemma 1 and Theorem 4. The clean choice \( \mu = 1 \) happens to work, hence we fix it up front. See Section 6 for the best constant this proof can give.

With these preparations, we begin the proof of Lemma 11. Again we assume that \( n = 2m \) is even. For brevity of notation, write \( \alpha_k, \beta_k \) for \( \alpha_m(k), \beta_m(k) \) and define the tails, \( \overline{\alpha}_k = \sum_{j \geq k} \alpha_j \) and \( \overline{\beta}_k = \sum_{j \geq k} \beta_j \). Recall that \( \alpha_k = \mathbb{P}\{S_{2m} = 2k\} \) and \( \beta_k = \mathbb{P}\{S_{8m} = 4k - 2 \text{ or } 4k\} \) and hence

\[
\overline{\alpha}_k = \frac{1}{2^{2m}} \sum_{j \geq k} \binom{2m}{m + j} \quad \text{and} \quad \overline{\beta}_k = \frac{1}{2^{8m}} \sum_{j \geq 2k} \binom{8m}{4m + j}. \tag{6}
\]

**Proof of the first part of Lemma 11.** Lemma 8 immediately implies that \( \overline{\alpha}_k \leq \overline{\beta}_k \) if \( k \geq 1 \).\[\blacksquare\]

**Proof of the second part of Lemma 11.** We prove it in stages, starting with the larger values of \( \ell \) and proceeding to smaller values. We shall assume that \( m \) is sufficiently large (so that expressions like \( Cm^{2/3} \leq m \) hold) without further comment.

**Case** \( 2(m^2 \log m)^{1/3} \leq \ell < 2m \): Let \( t = \frac{\ell}{2m} \) and \( s = 2t - t^2 \). If \( k \leq \ell - \frac{\ell^2}{4m} \), then \( \frac{k}{m} \leq s \). Therefore, by (6) and (4)

\[
\frac{\overline{\alpha}_k}{\overline{\beta}_{\ell+1}} \geq \frac{A\sqrt{2m}}{\sqrt{(m+k)(m-k)}} e^{2m\left[4D\left(\frac{1}{4m}\right) - D\left(\frac{1}{4m}\right)\right]} e^{2mQ(t)} \geq \frac{1}{2\sqrt{m}} e^{2mQ(t)}
\]
as \( A = 1/\sqrt{8} \) and \((m - k)(m + k) \leq m^2\). By the bound (\ref{eq:Q}) \( Q(t) \geq \frac{3}{2}t^3 \), we see that \( 2mQ(t) \geq 2 \log m \) and hence the above expression is more than \( \frac{1}{2}m^{3/2} \). Thus \( \alpha_k \geq \beta_{\ell+1} \).

**Case** \((\frac{2}{3}m^2)^{1/3} \leq \ell \leq 2(m^2 \log m)^{1/3}\): Again let \( t = \frac{\ell}{2m} \) and \( s = 2t - t^2 \) and observe that the condition \( k \leq \ell - \frac{\ell^2}{4m} \) ensures that \( \frac{k}{m} \leq s \). Then by (\ref{eq:alpha}),

\[
\frac{\alpha_k}{\beta_{\ell+1}} \geq \frac{A}{B} \frac{4m - 2\ell + 1}{8m + 1} \sqrt{\frac{4(4m + 2\ell)(4m - 2\ell)}{2(4m + k)(m - k)}} e^{2m[4D(\frac{4}{m} + \frac{2}{m^2}) - D(\frac{4}{m} + \frac{2}{m^2})]} \geq \frac{\sqrt{3}A}{4B} e^{2mQ(t)}
\]

if \( m \) is large enough that \( \ell \leq m \), since in that case \((4m - 2\ell)(4m + 2\ell) \geq 12m^2 \) and \((m - k)(m + k) \leq m^2 \) and \( \frac{4m - 2\ell + 1}{8m + 1} \geq \frac{1}{4} \) when \( \ell \leq m \) (which holds for \( \ell \) in this range as we assume \( m \) is large enough). Plugging in the values of \( A, B \) and using the bound (\ref{eq:Q}), we see that for all \( \ell \) in this range

\[
\alpha_k \geq \frac{\sqrt{3\pi}}{8} e^{\frac{\ell^3}{4m^2}} \beta_{\ell+1} \text{ whenever } 1 \leq k \leq \ell - \frac{\ell^2}{4m}.
\]

Now suppose \( 1 \leq \ell \leq L := \lfloor 4(m^2(\log m))^{1/3} \rfloor \). Since we have shown that \( \alpha(L + 1) \geq \beta(L + 1) \) in the previous case, it suffices to prove that \( \sum_{j=k}^{L} \alpha_j \geq \sum_{j=\ell}^{L} \beta_j \) whenever \( k \leq \ell - \mu \frac{\ell^2}{m} - 1 \) and \( K = \lfloor L - \frac{\ell^2}{4m^2} \rfloor \).

Let \( \varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) be a continuous decreasing function such that \( \varphi(k) = \alpha_k \) for integer \( k \geq 0 \). Let \( \kappa(\ell) = \ell - \frac{\ell^2}{4m} \) (not necessarily an integer). From (\ref{eq:alpha}) we also have that

\[
\sum_{j=\ell}^{L} \varphi(\kappa(j)) \geq \frac{\sqrt{3\pi}}{8} e^{\frac{\ell^3}{4m^2}} \sum_{j=\ell}^{L} \beta_j.
\]

Although \( \kappa(j) \) ranges between \( k \) and \( K \), the sum on the left side cannot be simply bounded by \( \sum_{i=k}^{K} \alpha_i \), since a particular index \( i \) may occur as \( \lfloor \kappa(j) \rfloor \) for more than one \( j \) between \( \ell \) and \( L \). For \( j \) in this range, \( \kappa(j + 1) - \kappa(j) \geq
\[ \eta := 1 - \frac{2L+1}{m}. \] As \( \varphi \) is decreasing,
\[
\sum_{j=\ell+1}^{L} \varphi(\kappa(j)) \leq \frac{1}{\eta} \sum_{j=\ell+1}^{L} \varphi(\kappa(j)) \times (\kappa(j) - \kappa(j-1)) \leq \frac{1}{\eta} \int_{\kappa(\ell)}^{\kappa(L)} \varphi(x)dx \leq \frac{1}{\eta} \sum_{i=k}^{K} \varphi(i).
\]

In the last line we bounded the integral by the sum, valid since \( k \leq \kappa(\ell) \) and \( K \geq \kappa(L) \) are integers (and the spacings are of unit length). Plugging in the expression for \( \eta \) and using \( \varphi(i) = \alpha_i \),
\[
\sum_{i=k}^{K} \alpha_i \geq \frac{\sqrt{3\pi}}{8} \left( 1 - \frac{2L+1}{m} \right) e^{\frac{\alpha^3}{m^2}} \sum_{j=\ell}^{L} \beta_j.
\]

For \( m \) large, the product of the first two factors is more than \( 1/e \), hence it suffices to have \( \ell^3 \geq \frac{8}{3} m^2 \) to conclude that \( \overline{\alpha}_k \geq \overline{\beta}_\ell \).

**Case 1 \( \leq \ell \leq (\frac{8}{3} m^2)^{1/3} \):** Let \( L' := (\frac{8}{3} m^2)^{1/3} \) and \( L'' = (3m^2)^{1/3} \) and fix \( \ell \leq L' \). From the second statement in Lemma[9] we know that \( \beta_{\ell+2} + \beta_{\ell+2} + \ldots + \beta_{L''} \) can be bounded above by \( \alpha_{\ell+1} + \alpha_{\ell+2} + \ldots + \alpha_{L''-1} \). We claim that the entire tail \( \overline{\beta}_{L''+1} \) can be bounded by one term \( \alpha_{\ell} \). When added to the above inequality, this proves that \( \overline{\alpha}_k \geq \overline{\beta}_{\ell+2} \). This completes the proof, since \( k \leq \ell - 2 \).

To prove the claim, observe that by (4) and (3), and the bounds for \( D \),
\[
\overline{\beta}_{L''+1} \leq e^{-8mD(\frac{1}{2} + \frac{L''}{m})} \leq e^{-\frac{(L''/2)^3}{m}},
\]
\[
\alpha_{\ell} \geq \frac{A\sqrt{2m}}{\sqrt{(m-k)(m+k)}} e^{-2mD(\frac{1}{2} + \frac{L''}{m})} \geq \frac{1}{2\sqrt{m}} e^{-\frac{\overline{\alpha}^3}{m^2}} - 2\ell^3 e^{\frac{\alpha^3}{m^2}}.
\]
As \( \ell \leq L' \) and \( L'' \) is larger than \( L' \) by a factor more than 1, it is clear that \( \alpha_{\ell} \geq \overline{\beta}_{L''+1} \) for \( \ell \leq L' \).

5. PROOF OF THEOREM 4

Finally we deduce Theorem 4 from Lemma[11]. The basic idea is simple and stated as follows.
Lemma 13. Let \( \alpha = (\alpha_k)_{k \geq 0} \) and \( \beta = (\beta_k)_{k \geq 0} \) be probability mass functions. Let \( \overline{\alpha}(x) = \sum_{j \geq x} \alpha_j \) and the similarly defined \( \overline{\beta} \) denote their tails. Suppose that \( \overline{\alpha}(k - f(k)) \geq \overline{\beta}(k) \) and \( \overline{\beta}(k - g(k)) \geq \overline{\alpha}(k) \) for some \( f, g : \mathbb{N} \rightarrow \mathbb{N} \) and for all \( k \geq 1 \). Then, there is a coupling of \( X \sim \alpha \) and \( Y \sim \beta \) such that \( X \geq Y - f(Y) \) and \( Y \geq X - g(X) \).

Assuming this, let us deduce Theorem 4.

Proof of Theorem 4. Observe that what appear in Lemma 11 are (we drop the subscript \( m \) on \( \overline{\alpha}, \overline{\beta} \) to simplify notation) are \( \overline{\alpha}(k) = \mathbb{P}\{2S_{2m} \geq 4k\} \) and \( \overline{\beta}(k) = \mathbb{P}\{S_{8m} \geq 4k - 2\} \). Therefore, by the conclusion of that lemma, if \( m \) is large enough, we get

1. \( \mathbb{P}\{2S_{2m} \geq j\} \leq \mathbb{P}\{S_{2m} \geq j - 2\} \). To see this, observe that the left hand side equals \( \overline{\alpha}(k + 1) \) for \( 4k + 1 \leq j \leq 4k + 4 \), and by Lemma 11, it does not exceed \( \mathbb{P}\{S_{8m} \geq 4k + 2\} \) which is at most \( \mathbb{P}\{S_{8m} \geq j - 2\} \) as \( j - 2 \leq 4k + 2 \).

2. \( \mathbb{P}\{S_{8m} \geq j\} \leq \mathbb{P}\{2S_{2m} \geq j - 1 - 4\theta(j/4)\} \) where \( \theta(x) = \frac{x^2}{4m} + 2 \).

To see this, we use the bound \( \mathbb{P}\{S_{8m} \geq j\} \leq \overline{\beta}(k) \), for \( 4k - 2 \leq j \leq 4k + 1 \). By Lemma 11, this is at most \( \mathbb{P}\{2S_{2m} \geq 4(k - \theta(k))\} \). Then we observe that in each of the four cases, \( 4(k - \theta(k)) \) is at least as large as \( j - 1 - 4\theta(j/4) \).

By the symmetry of \( S_{2m} \) and \( S_{8m} \), the above probability comparisons hold for the absolute values. Lemma 13 assures that \( |S_{2m}| \) and \( |S_{8m}| \) can be coupled so that

\[
|S_{8m}| - 9 - \frac{|S_{8m}|^2}{16m} \leq 2|S_{2m}| \leq |S_{8m}| + 2.
\]

Conditional on the absolute values, choose the same sign for both to get a coupling of \( S_{2m} \) and \( S_{8m} \) as in the statement of the theorem.

Here is the obvious of the lemma on coupling.

Proof of Lemma 13. Let \( V \sim \text{uniform}[0, 1] \), find \( k, \ell \) such that \( \overline{\alpha}(k) \geq V > \overline{\alpha}(k + 1) \) and \( \overline{\beta}(\ell) \geq V > \overline{\beta}(\ell + 1) \) and set \( X = k \) and \( Y = \ell \). Then \( X \sim \alpha \) and \( Y \sim \beta \). Further, \( \overline{\alpha}(k) > \overline{\beta}(\ell + 1) \) and \( \overline{\beta}(k) > \overline{\alpha}(k + 1) \). But by the assumptions, \( \overline{\beta}(k - g(k)) \geq \overline{\alpha}(k) \) and \( \overline{\alpha}(\ell - f(\ell)) \geq \overline{\beta}(\ell) \). Therefore, we see that \( \overline{\beta}(k - g(k)) > \overline{\beta}(\ell + 1) \) and \( \overline{\alpha}(\ell - f(\ell)) > \overline{\alpha}(k + 1) \) which of course imply that \( k - g(k) \leq \ell \) and \( \ell - f(\ell) \leq k \).
6. A Remark on the Optimal Constant

What is the best constant that can be achieved by this proof? Fix \( \mu \leq 1 \) and make the choice \( s = (2t - \mu t^2) \land 1 \) (i.e., for \( t \geq (1 - \sqrt{1 - \mu})/\mu \) we take \( s = 1 \)), and use the expansion \( Q(t) = 2\mu t^3 - O(t^4) \). This is how various components of the proof change. Let \( a_m = a(m^2 \log m)^{1/3} \) and \( b_m = (bm^2)^{1/3} \) mark the boundaries between the three cases in the proof of Lemma 11. Below \( t = \ell/2m \). The first case offers no problems, we simply use \( Q(t) \geq \frac{1}{2}a_m^3 \) for all \( t \geq a_m \), and choose \( a \) large enough to ensure that \( 2mQ(a_m) \geq 2 \log m \). In the second case, we need \( \sqrt{8\pi} e^{2mQ(t)} \geq 1 \) for \( b_m \leq t \leq a_m \), which only needs to be checked at \( \ell = b_m \). As \( Q(t) \sim 2\mu t^3 \), this translates to \( b > \frac{2}{\mu} \log(8/\sqrt{3\pi}) \). For this to meet up with the third case and cover the entire range of \( \ell \), we require \( b < 3 \) (since the third case, that depends on Lemma 9, goes up to \( \ell \leq (3m^2)^{1/3} \)). This gives \( \mu > \frac{2}{3} \log(8/\sqrt{3\pi}) = 0.6385... \).

What does this mean for the constant in Lemma 1? With \( \mu = 1 \), we got \( 1 \times Z^2 \) in the statement and with \( \mu \), we could get \( \mu Z^2 \), following exactly the same proof. Actually we were wasteful in the proof of Lemma 1 when we bounded \( (Z + \sqrt{\frac{c}{n}})^2 \) by \( 2Z^2 + 2\frac{c^2}{n} \) in \( (2) \), thereby losing a factor of 2. We could instead use \( (a + b)^2 \leq a^2/p + b^2/q \) for any positive \( p, q \) that sum to 1, with \( a/p \) close to 1. This would improve the bound in Lemma 1 to \( cZ^2 + C \) for any \( c > 1/2 \) with the choice \( \mu = 1 \), and to 0.319... with the optimal choice of \( \mu \). This still falls short of Tusnady’s constant 1/4, not to mention our suboptimal additive constants.
Part II. Coupling via Markov chains or Stein coefficient

This part was inspired entirely by Chatterjee's proof \cite{4}, in an attempt to refashion it to compare Binomials among themselves than with the Gaussian. If a pair of probability distributions are stationary distributions of nearest neighbour Markov chains on integers, then we show how to construct a coupled Markov chain on $\mathbb{Z}^2$ whose stationary distribution provides a coupling of the two given distributions. We prove a general result that the coupling is good if the transition rates are close to each other (we write the result for the special class of Ehrenfest-like chains that are determined by one function on integers called a Stein coefficient). This is applied to suitable choices of Markov chains to prove Theorem 5 and Theorem 6.

7. Coupling two nearest-neighbour Markov chains on integers

Let $S = \{a, a + 1, \ldots, b\}$ and $T = \{c, c + 1, \ldots, d\}$ be shifted finite segments of integers ($b - a$ and $d - c$ must be integers, but $a, b, c, d$ need not be). Let $X$ be a continuous-time, nearest-neighbour Markov chain on $S$ with transitions $i \to i \pm 1$ with rates $\lambda_i^\pm$. Similarly, let $Y$ be a nearest-neighbour chain on $T$ with rates $\mu_j^\pm$. Naturally, $\lambda_b^+, \mu_d^+, \lambda_a^-, \mu_c^-$ are all zero. We assume that the chains are irreducible and denote the unique stationary probability distributions of $X$ and $Y$ by $\alpha$ and $\beta$ respectively.

7.1. A coupled pair of Markov chains. The goal is to construct a Markov chain on $U := S \times T$ such that the co-ordinates move like $X$ and $Y$, but stay close to each other. The most natural idea would be to make $X$ and $Y$ take a step to the right together or a step to the left together, to the extent possible, and avoid taking steps in opposing directions. This leads us to the Markov chain $Z$ on $U$ with rates

$$
\begin{align*}
\theta_{i,j}^{1,+} &= \lambda_i^+ \land \mu_j^+,
\theta_{i,j}^{1,-} &= (\lambda_i^- - \mu_j^-)_+,
\theta_{i,j}^{2,+} &= (\mu_j^+ - \lambda_i^+)_+,
\theta_{i,j}^{2,-} &= (\mu_j^- - \lambda_i^-)_+.
\end{align*}
$$

The notation is self-explanatory: For example, $\theta_{i,j}^{1,+}$ is the transition rate from $(i, j)$ to $(i + 1, j)$ while $\theta_{i,j}^{1,-}$ is the transition rate from $(i, j)$ to $(i - 1, j - 1)$. Observe that there are no transitions to $(i + 1, j - 1)$ or $(i - 1, j + 1)$. The
generator $L$ of $Z$ acts on $f : U \mapsto \mathbb{R}$ as

$$
Lf(i, j) = \theta_{i,j}^{+} [f(i + 1, j + 1) - f(i, j)] + \theta_{i,j}^{-} [f(i - 1, j - 1) - f(i, j)]
+ \theta_{i,j}^{+,o} [f(i + 1, j) - f(i, j)] + \theta_{i,j}^{-,o} [f(i - 1, j) - f(i, j)]
+ \theta_{i,j}^{o,+} [f(i, j + 1) - f(i, j)] + \theta_{i,j}^{o,-} [f(i, j - 1) - f(i, j)].
$$

7.2. **The stationary distribution of $Z$ is a coupling of $\alpha$ and $\beta$.** As $U$ is finite, $Z$ necessarily has a stationary distribution $\gamma$. Then $E_\gamma[Lf(Z)] = 0$ for all $f : U \mapsto \mathbb{R}$. In fact we shall show shortly that there is a unique stationary distribution, but the uniqueness will not play a role in the analysis.

Let $f(i, j) = \varphi(i)$ and $g(i, j) = \psi(j)$. From the formula for $L$, we see that $Lf(i, j) = L_1 \varphi(i)$ and $Lg(i, j) = L_2 \psi(j)$ where $L_1$ and $L_2$ are the generators of $X$ and $Y$, respectively. The equations $E_\gamma[Lf(Z)] = 0$ and $E_\gamma[Lg(Z)] = 0$ show that the co-ordinates of $Z$ move like $X$ and $Y$ and that the marginals of $\gamma$ are $\alpha$ and $\beta$. In other words, $\gamma$ is a coupling of $\alpha$ and $\beta$.

**Uniqueness of the stationary distribution:** In general, and even in examples of interest to us, $Z$ is not irreducible. But we claim that it has a unique recurrent class, and hence a unique stationary probability distribution. To prove this claim, choose $M'$ and $M''$ to be medians of $S$ and $T$ in such a way that the line $\ell^*$ in $\mathbb{R}^2$ having slope 1 and passing through $(M', M'')$ either intersects the top and bottom sides of $[a, b] \times [c, d]$ or the left and right sides of $[a, b] \times [c, d]$. This is trivial when the medians are unique, for then $(M', M'')$ is the center of the rectangle $[a, b] \times [c, d]$. A little case analysis shows that more generally, any choice of medians works, except when $S$ and $T$ have the same even number of elements, in which case two of the four choices (both being the smaller medians or both being the larger medians) work.

We claim that every state in $U$ leads to $(M', M'')$. Why so? Within each diagonal line $\ell_d := \{(i, j) \in S \times T : i - j = d\}$, every state leads to every other state. Further, on the top edge (respectively bottom, right, left) of $U$, there is a strictly positive rate to move to the right (respectively left, up, down). Therefore, if $(i, j)$ is to the “left of” $\ell_*$, then depending on whether $\ell_*$ intersects the bottom and top or left and right, move along the diagonal till you hit the top or left sides, and then move right or down to get to $\ell_*$,
and then move along ℓ_{\ast} to get to (M', M''). A similar argument works if (i, j) is to the right of the diagonal ℓ_{\ast}.

7.3. Tail bounds on the difference X − Y in the coupling \( \gamma \). Suppose \( f(i, j) = \varphi(i - j) \). Then writing \( k = i - j \), we get

\[
Lf(i, j) = (\theta_{i,j}^{+} + \theta_{i,j}^{-})[\varphi(k + 1) - \varphi(k)] + (\theta_{i,j}^{0,+} + \theta_{i,j}^{0,-})[\varphi(k - 1) - \varphi(k)]
\]

\[
= A(i, j)[\varphi(k + 1) - \varphi(k - 1)] + B(i, j)[\varphi(k + 1) - 2\varphi(k) + \varphi(k - 1)]
\]

where

\[
A(i, j) = \frac{\theta_{i,j}^{+} + \theta_{i,j}^{-} - \theta_{i,j}^{0,+} - \theta_{i,j}^{0,-}}{2} = \frac{1}{2} \left( \lambda_{i}^{+} + \mu_{j}^{-} - \mu_{j}^{+} - \lambda_{i}^{-} \right),
\]

\[
B(i, j) = \frac{\theta_{i,j}^{0,+} + \theta_{i,j}^{0,-} + \theta_{i,j}^{+} - \theta_{i,j}^{-}}{2} = \frac{1}{2} \left( |\lambda_{i}^{+} - \mu_{j}^{+}| + |\lambda_{i}^{-} - \mu_{j}^{-}| \right).
\]

Let \( Z = (X, Y) \) and \( H = X - Y \). The equation \( E_{\gamma}[Lf(Z)] = 0 \) becomes

\[
E_{\gamma}[A(Z)(\varphi(H - 1) - \varphi(H + 1))] = E_{\gamma}[B(Z)(\varphi(H + 1) - 2\varphi(H) + \varphi(H - 1))].
\]

Let \( \psi(x) = \varphi(x + 1) - \varphi(x) \) and rewrite this as

\[
E_{\gamma}[(B - |A|) \times (\psi(H) - \psi(H - 1))] = 2E_{\gamma}[A_{-}\psi(H - 1) - A_{+}\psi(H)]. \tag{8}
\]

Here and below, \( A_{+} = \max\{A, 0\} \) and \( A_{-} = (-A)_{+} \).

The basic idea now is to plug in various test functions to get bounds on the tail of the distribution of \( H \) under \( \gamma \). That is precisely what we want, a coupling of \( X \) and \( Y \) so that \( X - Y \) has light tails. Although it is possible to do this in general and write some bounds, we specialize to the class of Markov chains that we use in this paper.

7.4. Ehrenfest-like chains. Assume that \( X \) and \( Y \) have rates \( \lambda_{i}^{\pm} = S(i) \mp i \) and \( \mu_{j}^{\pm} = T(j) \mp j \) for some \( S: S \mapsto \mathbb{R}_{+} \) and \( T: T \mapsto \mathbb{R}_{+} \). When \( S = \{-\frac{1}{2}N, \ldots, \frac{1}{2}N \} \) and \( \lambda_{i}^{\pm} = \frac{1}{2}N \mp i \), this gives the usual Ehrenfest chain, which is why we refer to these more general chains as Ehrenfest-like. For Ehrenfest-like chains,

\[
A(i, j) = j - i
\]

\[
B(i, j) = \frac{1}{2}(|S(i) - T(j) + j - i| + |T(j) - S(i) + j - i|)
\]

\[
= |j - i| + (|T(j) - S(i)| - |j - i|)_{+}
\]
Consequently, (8) becomes
\[ 2 \mathbb{H} \text{ functions of } \psi. \]
By plugging in various test functions, one gets bounds on expectations of functions of \( H \). Similarly for tail bounds: we obtain a tail bound and a bound on expectations of functions in a straightforward way. But there is one place where it becomes necessary to have the tighter bound (13), hence we use that everywhere.

Tail bounds: Fix an integer \( a \geq 0 \) and let \( \psi(x) = 1_{x \geq a} \). Then (9) leads to
\[ 2 \mathbb{E}_\gamma[H_1 1_{H \geq a+1}] \leq \mathbb{E}_\gamma[(Q - a)_+]. \]
Similarly for \( H_- \). Add the two to get
\[ \mathbb{P}_\gamma[|H| \geq a + 1] \leq \frac{1}{a + 1} \mathbb{E}_\gamma[|H| 1_{|H| \geq a + 1}] \leq \frac{1}{a + 1} \mathbb{E}_\gamma[(Q - a)_+]. \]

Expectation bounds: Let \( g : \mathbb{Z} \to \mathbb{R}_+ \) be increasing with \( g(0) = 0 \). Let \( \psi(x) = \frac{1}{x + 1} g(x + 1) 1_{x \geq 0} \). Then \( \psi(x) - \psi(x - 1) \) is at most \( g(x + 1) - g(x) \) (which is zero for negative \( x \)). Therefore (9) implies that
\[ 2 \mathbb{E}_\gamma[g(H)] \leq \mathbb{E}_\gamma[(Q - |H|)_+ (g(H + 1) - g(H))] \leq \mathbb{E}_\gamma[Qg(Q)]. \]
Similarly for \( -H \). Adding the two gives \( \mathbb{E}_\gamma[g(|H|)] \leq \mathbb{E}_\gamma[Qg(Q)]. \)

In particular, for \( g(x) = (e^{\theta x} - 1) 1_{x \geq 0} \) this gives
\[ \mathbb{E}_\gamma[e^{\theta|H|}] \leq 1 + \mathbb{E}_\gamma[Q(e^{\theta Q} - 1)]. \]

Chatterjee’s method requires \( \theta^2 \) in the exponent on the right side, which motivates the more involved analysis that follows.

Exponential moment bounds: Let \( \psi(x) = e^{\theta(x+1)} 1_{x > 0} \) where \( 0 \leq \theta \leq 1 \). From (9),
\[ 2 \mathbb{E}_\gamma[H_1 e^{\theta H_+}] = (e^\theta - 1) \mathbb{E}_\gamma[(Q - |H|)_+ e^{\theta H} 1_{H > 0}]. \]
Adding it to the corresponding inequality holds for \( H_- \) gives
\[ 2 \mathbb{E}_\gamma[|H| e^{\theta|H|}] \leq (e^\theta - 1) \mathbb{E}_\gamma[Q e^{\theta|H|}]. \]
We bounded \( (Q - |H|)_+ \) by \( Q \).
We now use the fact that for any convex function $\varphi$ and its Legendre or convex dual $\varphi^*$ (both defined on subintervals of $\mathbb{R}$), $xy \leq \varphi(x) + \varphi^*(y)$, and hence $E[XY] \leq E[\varphi(X)] + E[\varphi^*(Y)]$ for any random variables $X, Y$ taking values in the domains of $\varphi$ and $\varphi^*$ respectively. One may recall that this is indeed the idea behind Hölder’s inequality, where one takes $\varphi(x) = x^p/p$ and $\varphi^*(x) = x^q/q$ for conjugate exponents $p, q$. But we apply it to a function $\varphi$ smaller than $x^p$ for all $p > 1$.

Let $\varphi(x) := \beta x \log x$ for $x > 0$, in which case $\varphi^*(y) := \beta e^{-1/y + 1}$ for $y \in \mathbb{R}$.

Applying the inequality to $X = e^{|H|}$ and $Y = Q$ on the right side of (12),

$$2E_\gamma \| H^{|H|} \| \leq \beta \theta(e^\theta - 1)E_\gamma \| H^{|H|} \| + \beta(e^\theta - 1)e^{-1}E_\gamma [e^{Q/\beta}]$$.

Rearranging, we get

$$E_\gamma [\| H^{|H|} \|] \leq \frac{\beta \theta(e^\theta - 1)}{e(2 - \beta \theta(e^\theta - 1))}E_\gamma [e^{Q/\beta}]$$.

as long as $\beta \theta(e^\theta - 1) < 2$. Writing $\beta \theta(e^\theta - 1) = 2(1 - \delta)$ with $0 < \delta < 1$, (as $\theta \mapsto \theta(e^\theta - 1)$ is a bijection from $\mathbb{R}_+$ to itself, for every $\theta > 0$ and $\delta \in (0, 1)$, there is a unique $\beta > 0$ such that $\beta \theta(e^\theta - 1) = 2(1 - \delta)$) we get

$$E_\gamma [\| H^{|H|} \|] \leq \frac{1 - \delta}{e\delta}E_\gamma [e^{\frac{1}{2(1 - \delta)}e^\theta Q}]$$

since $e^\theta - 1 = e^{\theta'}$ for some $\theta' \in (0, \theta)$. Now for any $\mu > 0$, we can write $e^x \leq \frac{1}{\mu}xe^x + e^\mu$ for $x \geq 0$ (the first term suffices for $x \geq \mu$ and the second for $x \leq \mu$). Therefore,

$$E_\gamma [e^{\theta|H|}] \leq e^\mu + \frac{1 - \delta}{\mu e}E_\gamma [e^{\frac{1}{2(1 - \delta)}e^{e\theta Q}}]$$

$$\leq \left( e^\mu + \frac{1 - \delta}{\mu e} \right)E_\gamma [e^{\frac{1}{2(1 - \delta)}e^{e\theta Q}}]$$ (13)

valid for any $0 < \delta < 1$ and $\mu > 0$ and $\theta > 0$. This is better than (11) when $\theta$ is small because of the power 2 on $\theta$ in the exponent. In examples, the right side is finite only for small $\theta$, hence (13) is actually the more useful one.

In applying (13), one can optimize over the right side, but it is perhaps more illuminating to observe the following.
(1) Given $\epsilon > 0$ and $\Theta < \infty$, choosing $\mu$ and $1 - \delta$ small, we see that there is some $M = M(\epsilon, \Theta)$ such that

$$E_\gamma[e^{\Theta |H|}] \leq (1 + \epsilon)E_\gamma[e^{M\theta^2Q}] \quad \text{for } \theta \leq \Theta.$$  

(14)

(2) Given $\epsilon > 0$, choosing $\delta$ small and $\mu$ arbitrarily, we see that there is some $\theta_\epsilon > 0$ and $M_\epsilon < \infty$ such that

$$E_\gamma[e^{\theta |H|}] \leq M_\epsilon E_\gamma[e^{(1+\epsilon)\theta^2Q}] \quad \text{for } \theta \leq \theta_\epsilon.$$  

(15)

**Remark 14.** Can one prove a Tusnády type lemma by this method? It is not possible in the naive sense, as the coupling $\gamma$ is supported on a union of diagonals of $\mathbb{U}$. A more plausible approach is to interpret the inequalities between expectations that we have obtained as a stochastic domination result such as $|H| \prec aQ + b$ (roughly speaking). That would imply a coupling of $|H|$ and $Q$ so that $|H| \leq aQ + b$. This can be done.

However, it is not clear to us that one can couple variables $X, Y$ having the marginal $\alpha, \beta$ so that $|X - Y| \leq |T(Y) - S(X)| + 1$. If this could be achieved in some way, that would be nice because, in the main example of coupling $2S_n$ and $S_{4n}$, the variable $Q = |T(Y) - S(X)|$ turns out to be exactly $O(1 + \frac{1}{n}S^2_{4n})$, as required by Theorem 4.

8. **Ehrenfest-like chains and the Stein coefficient**

To get couplings of given distributions on integers, we construct Ehrenfest-like Markov chains with these distributions and then use the general results of the previous section. The given distributions may need to be perturbed first to be able to do this. In this section we investigate Ehrenfest-like chains in general, although some of the discussions here are not needed in the proofs of Theorems 5 and 6.

8.1. **Ehrenfest-like chain with given stationary distribution.** Let $\alpha$ be a probability distribution whose support $\mathbb{S} = \{a, a + 1, \ldots, b\}$ is a shifted segment of integers ($a, b$ need not be integers). A nearest-neighbour Markov chain on $\mathbb{S}$ with transition rates $\lambda_i^\pm$ is reversible for $\alpha$ if and only if $\alpha(i)\lambda_i^+ = \alpha(i + 1)\lambda_{i+1}^-$ for all $i \in \mathbb{S}$ (with $\alpha(i)$ and $\lambda_i^\pm$ set to zero for $i \notin \mathbb{S}$). Since we assume that $\alpha(i) > 0$ for all $i \in \mathbb{S}$, it is always possible to find such rates, for example, set $\lambda_i^\pm = \alpha(i \pm 1)$. 
For the chain to be Ehrenfest-like, the condition is that $\lambda^\pm_i = T(i) \mp i$ for a function $T : \mathbb{S} \mapsto \mathbb{R}$. Does such a function exist? Unlike the general situation above where we had two parameters $\lambda^\pm_i$ per state to play with, now we have only one parameter $T(i)$. The equations for reversibility take the form

$$\alpha(i)(T(i) - i) = \alpha(i + 1)(T(i + 1) + (i + 1)). \tag{16}$$

This of course implies that $T(i) = |i|$ when $i \in \{a, b\}$, an end-point of $\mathbb{S}$. For $i \notin \mathbb{S}$, since $\alpha(i) = 0$ the above equations say nothing about $T(i)$, and it need not be defined. Our convention is to set $T(i) = |i|$ for $i \notin \mathbb{S}$.

Now assume that $\mathbb{S} = \{a, a + 1, \ldots, b\}$ is finite. Then we must set $T(b) = b$ and then successively solve for $T(b - 1), T(b - 2), \ldots, T(a)$. This is possible as $\alpha(i) > 0$ for all $i \in \mathbb{S}$. Inductively, the solution is seen to be

$$T(i) = i + \frac{2}{\alpha(i)} \sum_{j: j > i} \alpha(j)j \quad \text{for } i \in \mathbb{S}. \tag{17}$$

But there is one more equation, namely (16) for $i = a - 1$, that forces $T(a) = -a$. This is satisfied by the expression in (17) if and only if $\alpha$ has zero mean. In short, stationary distributions of Ehrenfest-like chains are precisely those that have zero mean and support equal to a shifted segment of integers. If $\alpha$ is such a distribution, then there is a unique Ehrenfest-like chain that keeps it stationary. The function $T$ will be called the Stein coefficient$^2$ of $\alpha$.

After presenting two examples that will be of use to us later, in the next two subsections we study the behaviour of Stein coefficient under convolution and scaling. The relevance is of course that we shall want to compare $S_{4n}$ with $2S_n$.

**Example 15.** Let $\alpha$ be the distribution of the centered Binomial$(n, p)$ distribution. The support is $\mathbb{S} = \{-np, -np + 1, \ldots, nq\}$ where $q = 1 - p$, and the

---

$^2$To reconcile with the language of Chatterjee’s paper [4], we could call it Stein coefficient with respect to the Binomial distribution, and what he uses as Stein coefficient with respect to the Gaussian. The connection to Stein's method is this: $\mathbb{E}[\sigma^2 f'(W)] = \mathbb{E}[W f(W)]$ for a rich class of functions $f$ if and only if $W \sim N(0, \sigma^2)$ and $\mathbb{E}[(n - X)f(X + 1) - Xf(X)] = 0$ for a rich class of functions if and only if $X \sim \text{Binomial}(n, \frac{1}{2})$. Chatterjee’s Stein coefficient for a random variable $W$ is a function $T$ that replaces $\sigma^2$ in the first equation and our Stein function is a function $T$ that replaces $n$ in the second. The essence of Stein’s method is that if $T$ is close to a constant, then the corresponding $W$ (respectively $X$) has a distribution close to Gaussian (respectively Binomial).
mass function is \( \alpha(x) = \binom{n}{j}p^j q^{n-j} \) where \( j = x + np \). The Stein coefficient of \( \alpha \) is \( T(x) = 2pqn + (q - p)x \). Indeed, the ratio of the right side to the left side of (16) is (again \( x = j - np \))
\[
\frac{2pqn + (q - p)(x + 1) + (x + 1)}{2pqn + (q - p)x - x} \times \frac{(n - j)p}{(j + 1)q} = 1.
\]
In particular, for the symmetric case \( p = \frac{1}{2} \) (which is the distribution of \( S_n/2 \)), we have \( T(x) = \frac{1}{2}n \) for \( x \in \{-\frac{1}{2}n, -\frac{1}{2}n + 1, \ldots, \frac{1}{2}n - 1, \frac{1}{2}n \} \). The corresponding chain is the usual Ehrenfest chain. Binomials are the only distributions for which the Stein coefficient is linear (this is the Stein characterizing equation for the Binomial).

**Example 16.** Let \( \tilde{S}_k[n, s] := \frac{1}{2}(S_k[n, s] + k) \) (see the text preceding Lemma 3 for the definition of \( S_k[n, s] \)). Equivalently, \( \tilde{S}_k \) is the sum of \( k \) coupons drawn without replacement from a box containing \( np = \frac{1}{2}(n + s) \) coupons labeled 1 and \( nq = \frac{1}{2}(n - s) \) coupons labeled 0. Let \( \alpha \) denote the distribution of the centered variable \( \tilde{W}_k[n, s] := \tilde{S}_k[n, s] - kp \). We refer to \( \alpha \) as the centered hypergeometric distribution.

Write \( s = n(q - p) \) with \( q = 1 - p \), and \( N = np \) and \( M = nq \). The support of \( \alpha \) is \( S := \{a, a + 1, \ldots, b \} \) where \( a = -[(kp) \land ((n - k)q)] \) and \( b = (kq) \land ((n - k)p) \). Therefore \( \alpha \) has a Stein coefficient. For \( x \in S \)
\[
\alpha(x) = \frac{\binom{N}{j} \binom{M}{k-j}}{\binom{n}{j}} \quad \text{where} \quad j = x + kp.
\]
Then,
\[
\frac{\alpha(x + 1)}{\alpha(x)} = \frac{(N - j)(k - j)}{(j + 1)(M - k + j + 1)}
\]
which shows that the Markov chain with \( \lambda^+_x = (N - j)(k - j) = ((n - k)p - x)(kq - x) \) and \( \lambda^-_x = j(M - k + j) = ((n - k)q + x)(kp + x) \), is reversible for \( \alpha \). Since \( \lambda^+_x - \lambda^-_x = nx \) by an easy calculation, we can write the rates as \( \lambda^+_x = T'(x) = \frac{k}{2}x \) where \( T'(x) = \frac{1}{2}(\lambda^+_x + \lambda^-_x) \). Scale down all the rates by \( n/2 \) to get an Ehrenfest-like chain with rates \( T(x) \equiv x \) where
\[
T(x) = \frac{\lambda^+_x + \lambda^-_x}{n} = 2pqk(n - k) - 2x^2 + (q - p)n - 2k - x.
\]
Observe that if \( k, p \) are fixed and \( n \to \infty \), this converges to the Stein coefficient of the centered Binomial\((k, p)\) distribution, as it should. When \( p = \frac{1}{2} \),
the Stein coefficient takes the simpler form

\[ T(x) = \frac{k(n - k)}{2n} + \frac{2x^2}{n}. \]

8.2. Stein coefficient under convolution. Let \( X \) and \( Y \) be independent random variables having Stein coefficients \( T_X \) and \( T_Y \) respectively and let \( Z = X + Y \). Let \( f, g, h \) denote the probability mass functions of \( X, Y \) and \( Z \) respectively. Then \( h(z) = \sum_x f(x)g(z - x) \) and by (17), the Stein coefficient of \( Z \) is

\[
T_Z(z) = z + \frac{2}{h(z)} \sum_{w > z} h(w)w = z + \frac{2}{h(z)} \sum_{x,y: x + y > z} (x + y)f(x)g(y)
\]

\[
= z + \frac{2}{h(z)} \left\{ \sum_x f(x) \sum_{y: y > z-x} yg(y) + \sum_y g(y) \sum_{x: x > z-y} xf(x) \right\}.
\]

Again using (17), the two inner sums are equal to \( \frac{1}{2}g(z - x)(T_Y(z - x) - (z - x)) \) and \( \frac{1}{2}f(z - y)(T_X(z - y) - (z - y)) \), respectively. Thus we get

\[
T_Z(z) = z + \frac{1}{h(z)} \sum_{(x,y): x + y = z} f(x)g(y)[T_X(x) - x + T_Y(y) - y]
\]

\[
= \frac{1}{h(z)} \sum_{(x,y): x + y = z} f(x)g(y)[T_X(x) + T_Y(y)]
\]

since \( x + y = z \) for all \( (x, y) \) in the sum and the sum of \( f(x)g(y) \) precisely gives \( h(z) \). We may write this as

\[
T_Z(z) = \mathbb{E} \left[ T_X(X) + T_Y(Y) \mid X + Y = z \right]. \tag{18}
\]

More generally, if \( X_i \) are independent and \( S_n = X_1 + \ldots + X_n \), then the Stein coefficient \( T_n \) of \( S_n \) is given by

\[
T_n(s) = \mathbb{E} \left[ \sum_{i=1}^n T_{X_i}(X_i) \mid S_n = s \right]. \tag{19}
\]

Example 17. If \( X_i \) are i.i.d. and take values \(-p, q\) with probabilities \( q, p \) respectively, then \( T_{X_i}(x) = |x| \) for \( x \notin \{-p, q\} \). From (19) the Stein coefficient of \( S_n \) is \( T_n(s) = \mathbb{E}[\sum_{i=1}^n |X_i| \mid S_n = s] \). But given \( S_n = s \), then exactly \( s + np \) of the \( X_i \)s are equal to \( q \) and the remaining \( nq - s \) are equal to \(-p\). Therefore, \( \sum_{i=1}^n |X_i| = 2pqn + (q - p)s \), reconfirming that \( T_n(s) \) is what was found by direct calculation in Example 15.
8.3. Stein coefficient under scaling (and an additive perturbation). Let $X$ be a zero mean random variable with support $S = \{a, a + 1, \ldots, b\}$, mass function $f$ and Stein coefficient $T_X$. We wish to consider $2X$, but its support has gaps of length 2. To get a random variable that has a Stein coefficient, we perturb it additively and consider $Y = 2X + R$ where $R$ is a non-degenerate random variable having a Stein coefficient. What is the Stein coefficient of $Y$? In Remark 20, we record the result for general $R$, but for our purposes it suffices to take the simple case of $R$ taking the values $-1, 0, 1$, with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively.

For this choice of $R$, the support of $Y$ is $T = \{2a - 1, 2a, \ldots, 2b, 2b + 1\}$ and mass function $g(2x) = \frac{1}{2}f(x)$ and $g(2x - 1) = \frac{1}{4}f(x) + \frac{1}{4}f(x - 1)$. Hence, if $y = 2x$, then by (17)

\[
T_Y(y) = y + \frac{2}{g(y)}\sum_{u>x}\left\{(2u - 1)\frac{f(u - 1) + f(u)}{4} + 2uf(u)\right\}
\]

\[
= y + \frac{4}{f(x)}\left\{\frac{1}{4}(2x + 1)f(x) + \sum_{u>x}f(u)\left[\frac{2u - 1}{4} + \frac{2u + 1}{4} + \frac{2u}{2}\right]\right\}
\]

\[
= y + (2x + 1) + \frac{8}{f(x)}\sum_{u>x}uf(u).
\]

The first two summands add up to $4x + 1$ and the third summand is $4(T_X(x) - x)$, by (17). Thus, $T_Y(y) = 4T_X(x) + 1$.

Next suppose $y = 2x - 1$. Apply the defining formula (16) for $T_Y$ with $i = y$ and $i = y - 1$ and average to get

\[
T_Y(y) = \frac{g(y + 1)}{2g(y)}(T_Y(y + 1) + y + 1) + \frac{g(y - 1)}{2g(y)}(T_Y(y - 1) - (y - 1))
\]

\[
= \frac{f(x)(4T_X(x) + y + 2) + f(x - 1)(4T_X(x - 1) - y + 2)}{f(x) + f(x - 1)}
\]

from the already worked out formulas for $T_Y$ at $y \pm 1$. In the above expression, we may replace $f(x)$ by $T_X(x - 1) - (x - 1)$ and $f(x - 1)$ by $T_X(x) + x$ (since the ratios are the same, by (16)), hence

\[
T_Y(y) = \frac{(T_X(x) - (x - 1))(4T_X(x) + y + 2) + (T_X(x) + x)(4T_X(x - 1) - y + 2)}{T_X(x) + T_X(x - 1) + 1}
\]

\[
= \frac{B(4A - y) + A(4B + y)}{A + B}
\]
where \( A = T_X(x) + x \) and \( B = T_X(x - 1) - (x - 1) \). The numerator is \( 8AB + y(A - B) \), which can be written as \( 2(A + B)^2 - 2(A - B)^2 + y(A - B) \). Therefore

\[
T_Y(y) = 2(A + B) - \frac{(A - B)(2(A - B) - y)}{A + B}
\]

\[= 2(T_X(x) + T_X(x - 1) + 1) - R(y)\]

where

\[
R(y) = \frac{(y + [T_X(x) - T_X(x - 1)])(y + 2[T_X(x) - T_X(x - 1)])}{T_X(x) + T_X(x - 1) + 1}.
\]

In conclusion

\[
T_Y(y) = \begin{cases} 
4T_X(x) + 1 & \text{if } x = \frac{1}{2}y, \in S, \\
2(T_X(x) + T_X(x - 1) + 1) - R(y) & \text{if } x = \frac{1}{2}(y + 1), \in S.
\end{cases}
\] (20)

Now we work out the two examples of centered binomial and hypergeometric distributions.

**Example 18.** If \( X \) is centered Binomial\((n, p)\), then we have seen in Example 15 that \( T_X(x) = 2pqn + (q - p)x \) for \( x \in S = \{-np, -np + 1, \ldots, nq\} \). Therefore, \( Y \) has Stein coefficient

\[
T_Y(y) = \begin{cases} 
8pqn + 2(q - p)y + 1 & \text{if } \frac{1}{2}y, \in S, \\
8pqn + 2(q - p)y + 2 - R_n(y) & \text{if } \frac{1}{2}(y + 1), \in S.
\end{cases}
\]

where \( R_n(y) = \frac{(y + [T_X(x) - T_X(x - 1)])(y + 2[T_X(x) - T_X(x - 1)])}{4pqn + (q - p)y + 1} \). For the balanced case \( p = \frac{1}{2} \),

\[
T_Y(y) = \begin{cases} 
2n + 1 & \text{if } y \in \{-n, -n + 2, \ldots, n - 2, n\}, \\
2n + 2 - \frac{y^2}{n+1} & \text{if } y \in \{-n - 1, -n + 1, \ldots, n - 1, n + 1\}.
\end{cases}
\]

**Example 19.** Let \( X = \hat{W}_k[n, s] \) be the centered hypergeometric variable of Example 16 and let \( Y = 2X + R \). We have seen the formula for the Stein coefficient of \( X \). It follows that if \( y = 2x - 1 \), then \( T_X(x - 1) - T_X(x) = \frac{-2y}{n} - (q - p)\frac{n-2k}{n} \) and hence

\[
R(y) = \frac{(y(1 + \frac{2}{n}) + (q - p)(1 - \frac{2k}{n}))(y(1 + \frac{2}{n}) + 2(q - p)(1 - \frac{2k}{n}))}{4pqk(n-k)^2 + \frac{y^2+1}{n} + (q - p)(1 - \frac{2k}{n})y + 1}.
\]
The expression looks complicated, but all that matters is that this is of smaller order than \( y^2/k \) (as long as \( k \) is away from 0 and \( n \)).

\[
T_Y(y) = \begin{cases} 
8pq \frac{k(n-k)}{n} + \frac{2y^2}{n} + 2(q-p)(1-\frac{2k}{n})y + 1, & \text{if } \frac{1}{2}y \in S, \\
8pq \frac{k(n-k)}{n} + \frac{2y^2+2}{n} + 2 + 2(q-p)(1-\frac{2k}{n})y - R(y), & \text{if } \frac{1}{2}(y+1) \in S.
\end{cases}
\]

For the balanced case \( p = \frac{1}{2} \), we see that \( R(y) = \frac{y^2(1+\frac{2}{n})(1+\frac{4}{n})}{k(n-k) + \frac{n^2+1}{n} + 1} \) and

\[
T_Y(y) = \begin{cases} 
\frac{2k(n-k)}{n} + \frac{2y^2}{n} + 1, & \text{if } y \in 2S, \\
\frac{2k(n-k)}{n} + \frac{2y^2+2}{n} + 2 - R(y), & \text{if } y \notin 2S.
\end{cases}
\]

**Remark 20.** For possible future use, we record the result for a general random variable \( R \) that has mass function \( r \) and Stein coefficient \( T_R \). Then \( Y = 2X + R \) has Stein coefficient given by

\[
T_Y(y) = \sum_{2u+t=y} f(u)r(t)(2T_X(u) + T_R(t)) + \sum_{2u+t = y \pm 1} f(u)r(t)(T_X(u) \pm u) \sum_{2u+t=y} f(u)r(t).
\]

### 9. Proof of Theorem 5

Recall that \( S_n \) is a sum of \( n \) i.i.d. symmetric Bernoulli random variables. Then \( \hat{S}_n := \frac{1}{2}S_n \) has the centered Bin \((n, \frac{1}{2})\) distribution. To couple \( S_n \) and \( S_{4n} \) we use the Markov chain coupling between the Ehrenfest-like chains associated to \( X = \hat{S}_{4n} \) and \( Y = 2\hat{S}_n + R = S_n + R \), where \( R \) is as before (and independent of \( S_n \)). If \( \theta > 0 \), then

\[
E[e^{\theta(2\hat{S}_n - S_{4n})}] \leq e^{2\theta} E[e^{2\theta(2\hat{S}_n - \hat{S}_{4n})}].
\]  

(21)

From the computation of Stein coefficients in Example 15 and Example 18 (this is the case \( p = \frac{1}{2} \)), we see that \( Q(x, y) = |T_Y(y) - T_X(x)| \) is equal to 1 if \( y/2 \) is in the support of \( \hat{S}_n \) and \( |2 - \frac{y^2}{n+1}| \) otherwise. Thus, \( Q \leq \frac{Y^2}{n+1} + 2 \) and as \( |Y| \leq |S_n| + 1 \), we can write \( Q \leq \frac{2S^2 + 2}{n+1} + 2 \leq \frac{2}{n} S_n^2 + 3 \).

The Bernstein/Hoeffding inequality says that \( \mathbb{P}\{S_n \geq t\} \leq e^{-t^2/2n} \), which can be interpreted as saying that \( (S_n)_+^2 \) and \( (S_n)_-^2 \) are stochastically dominated by \( 2n\xi \), where \( \xi \) is an exponential random variable with unit mean.
Therefore, $E[h(S_n^2)] \leq 2E[h(2n\xi)]$ for any increasing $h: \mathbb{R}_+ \to \mathbb{R}$. In particular, we get $E[h(Q)] \leq 2E[h(4\xi + 3)]$. In particular, for $\alpha < \frac{1}{4}$, we have

$$E[e^{\alpha Q}] \leq \frac{2e^{3\alpha}}{1 - 4\alpha} \leq \frac{5}{1 - 4\alpha}.$$ 

a uniform bound independent of $n$.

Now, in the bound (13) on $E[|e^{\theta |H|}|]$ in the Markov coupling, one can take any $\theta$ small enough that $\frac{1}{2(1-\delta)}e^{\theta^2} < \frac{1}{4}$. For this, if $2\theta^2 e^{\theta} < 1$, then we can always choose a $\delta > 0$ small enough. Then choose any $\mu$ to get a bound $\kappa$ which is independent of $n$. Thus, any $\theta_0$ with $2\theta_0^2 e^{\theta_0} < 1$ works. By (21), for the coupling of $S_n$ and $S_{4n}$, any $\theta_0$ with $8\theta_0^2 e^{2\theta_0} < 1$ suffices. ■

10. PROOF OF THE COUPLING THEOREM 6

As in the proof of Theorem 5, the variables $S_k[n, s]$ have spacing of 2 in their support, hence we consider the modified variable $\hat{S}_k[n, s] = (S_k[n, s] + k)/2$ and $\hat{W}_k[n, s] = \hat{S}_k[n, s] - kp$ that were defined in Example 16. Let $\hat{W}_1 = \hat{W}_1[n, 0]$ and $\hat{W}_2 = \hat{W}_{4k}[4n, 0]$ and $\hat{W} = \hat{W}_k[n, s]$. These are half of $W_1, W_2, W$ that occur in the statement of Theorem 6. Thus it suffices to prove the existence of a $\Theta > 0$ such that there exist couplings satisfying

$$E[|e^{\theta |2\hat{W}_1 - \hat{W}_2||}] \leq \frac{3}{2}$$ for all $\theta \leq \Theta$, \hspace{1cm} (22)

$$E[|e^{\theta |\hat{W}_1 - \hat{W}|}|] \leq e^{1+M\theta^2 \frac{\alpha}{\kappa}}$$ for all $\theta \leq \Theta$. \hspace{1cm} (23)

Proof of (22). Let $V = \hat{W}_1$ and $X = \hat{W}_2$ and $Y = 2V + R$ where $R$ takes values $-1, 0, 1$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. In Example 16 and Example 19 we have computed their Stein coefficients

$$T_X(x) = \frac{2k(n-k)}{n} + \frac{x^2}{2n},$$

$$T_Y(y) = \begin{cases} 
\frac{2k(n-k)}{n} + \frac{2y}{n} + \frac{1}{2} & \text{if } y \text{ is even,} \\
\frac{2k(n-k)}{n} + \frac{2y^2 + 2}{n} + 2 - \frac{y^2(1+\frac{3}{2})(1+\frac{1}{\kappa})}{2(k(n-k)) + 2y^2 + 1} & \text{if } y \text{ is odd.} 
\end{cases}$$

The complicated looking term in $T_Y(y)$ is bounded as follows ($n \geq 2$):

$$\frac{y^2(1+\frac{3}{2})(1+\frac{1}{\kappa})}{2(k(n-k)) + 2y^2 + 1} \leq \frac{27y^2}{n}$$
because \( k(n - k) \geq \frac{2}{3}n^2 \) (we just dropped the second and third terms in the denominator). Consequently, writing \( y = 2v + 1 \) (i.e., \( v \) is the value of \( V \)) and using \( y^2 \leq 8v^2 + 2 \) and \( k \leq n \) we get

\[
Q := |T_X(x) - T_Y(y)| \leq C \left( \frac{x^2}{4k} + \frac{v^2}{k} + \frac{1}{k} \right)
\]

for a constant \( C \). Consider the bound (13) for the Markov coupling of \( X \) and \( Y \), with \( H = |X - Y| \). Fix \( \epsilon > 0 \) and set \( \mu \) and \( \delta \) to be small positive numbers to get for some \( M > 0 \),

\[
\mathbb{E}_\gamma [e^{\theta H}] \leq (1 + \epsilon) \mathbb{E}_\gamma \left[ e^{CM\theta^2 e^\theta Q} \right] \\
\leq (1 + \epsilon) e^{\frac{1}{2}CM\theta^2 e^\theta} \mathbb{E} \left[ e^{2CM\theta^2 e^\theta \frac{y^2}{k}} \right] \mathbb{E} \left[ e^{2CM\theta^2 e^\theta \frac{x^2}{n}} \right].
\]

In the notation of Lemma 21, \( X = \frac{1}{2}W_{4k} \) and \( V = \frac{1}{2}W_k \). Hence if \( b = 2CMe^\theta \theta^2 < \frac{1}{2} \), part (1) of that Lemma shows that both expectations above are bounded by \( \frac{1}{\sqrt{1-4CM\theta^2 e^\theta}} \). Therefore,

\[
\mathbb{E}_\gamma [e^{\theta H}] \leq (1 + \epsilon) e^{\frac{1}{2}CM\theta^2 e^\theta} \frac{1}{\sqrt{1-4CM\theta^2 e^\theta}}.
\]

As \( |2\tilde{W}_1 - \tilde{W}_2| \leq |H| + 1 \), for \( \theta > 0 \),

\[
\mathbb{E}_\gamma [e^{\theta |2\tilde{W}_1 - \tilde{W}_2|}] \leq (1 + \epsilon) e^{\frac{1}{2}CM\theta^2 e^\theta} \frac{e^\theta}{\sqrt{1-4CM\theta^2 e^\theta}}.
\]

This can be made as close to 1 as desired by taking \( \theta \leq \Theta \) for a small enough \( \Theta > 0 \), and (22) follows. \( \blacksquare \)

**Proof of (23).** We want to find a good coupling between \( \tilde{W}_1 = S_k[n, 0] \) and \( \tilde{W} = S_k[n, s] - \frac{2k}{n} \). The Stein coefficients of these two variables were computed in Example 16 and Example 19 and are given by

\[
T_1(x) = \frac{1}{2} \frac{k(n - k)}{n} + \frac{2x^2}{n}, \\
T(y) = 2pq \frac{k(n - k)}{n} + \frac{2y^2}{n} + \frac{(q - p)n - 2k}{n} y.
\]
Therefore,

\[
|T_1(x) - T(y)| \leq \frac{1}{2} (1 - 4pq) \frac{k(n-k)}{n} + \frac{2x^2}{n} + \frac{2y^2}{n} + |q-p| |1 - \frac{2k}{n}| |y| \\
\leq \frac{1}{8} (p-q)^2 n + \frac{2x^2}{n} + \frac{2y^2}{n} + \frac{1}{3} |q-p| |y| \\
= \frac{s^2}{8n} + \frac{2x^2}{n} + \frac{2y^2}{n} + \frac{|s| |y|}{3\sqrt{n} \sqrt{n}}.
\]

Therefore, choosing \( \delta = 1 \) in (13) (it is here that we need this stronger form and (11) would not suffice), in the Markov coupling of \( \hat{W}_1 \) and \( \hat{W} \), with \( H = \hat{W}_1 - \hat{W} \) (and recall that \( \hat{W}_1 = \frac{1}{2} W_1 \) and \( \hat{W} = \frac{1}{2} W \)),

\[
E[e^{\theta H}] \leq 4E[e^{\theta^2 T_1(W_1) - T(\hat{W})}]
\]

\[
\leq 4e^{\theta^2 \frac{s^2}{8n}} E\left[e^{\theta^2 \hat{W}^2} \frac{2}{\sqrt{1 - 8\theta^2}} \frac{2}{\sqrt{1 - 8\theta^2}} e^{\frac{2\theta^4 y^2}{n}} \right]^{\frac{3}{2}}
\]

\[
= 4e^{\theta^2 \frac{s^2}{8n}} E\left[e^{\theta^2 \frac{s^2}{n}} \right] \frac{2}{\sqrt{1 - 8\theta^2}} \frac{2}{\sqrt{1 - 8\theta^2}} e^{\frac{2\theta^4 y^2}{n}} \]

\[
\leq 4e^{\theta^2 \frac{s^2}{8n}} \frac{1}{\sqrt{1 - 8\theta^2}} \frac{2}{\sqrt{1 - 8\theta^2}} e^{\frac{2\theta^4 y^2}{n}}.
\]

If \( \theta \leq \frac{1}{4} \), then \( 1 - 8\theta^2 \geq \frac{1}{2} \) and the whole thing is bounded by \( 12e^{\frac{4}{9} \frac{y^2}{n}} \).

Lastly \( |W_1 - W| = 2|H| \), hence (23) follows.

11. Some facts about the hypergeometric distribution

Consider a box that contains \( n \) coupons, of which \( np \) are labelled +1 and \( nq \) are labelled −1. The sum of all the labels is \( s := n(p - q) \). Let \( S'_k = X_1 + \ldots + X_k \) where \( X_i \) s are drawn without replacement and let \( S_k = Y_1 + \ldots + Y_k \) where \( Y_i \) s are drawn with replacement from the same box. We also write \( W'_k = S'_k - k(p - q) = \sum_{i=1}^{k} (X_i - (p - q)) \) and \( W_k = S_k - k(p - q) = \sum_{i=1}^{k} (Y_i - (p - q)) \) for the centered versions. Clearly, \( W'_k \) has a centered hypergeometric distribution with parameters \( (n, k, p) \) while \( W_k \) has a centered Binomial distribution with parameters \( (n, p) \).

We collect here certain results about these random variables, particularly bounds on expectations of certain functionals. While the Binomial is straightforward, the hypergeometric gets complicated if one tries a direct approach using the explicit mass function. Much difficulty can be avoided by using a wonderful result of Hoeffding [7] (this is the same paper where
the famous Hoeffding inequality is proved, but this result is in a somewhat less known Section 5 of the paper):

\[ \mathbb{E}[f(S'_k)] \leq \mathbb{E}[f(S_k)] \text{ for any convex function } f : \mathbb{R} \mapsto \mathbb{R}. \tag{24} \]

Now we collect the results we need.

**Lemma 21.**

1. For any real \(\lambda\) we have \(\mathbb{E}[e^{\lambda W'_k}] \leq \mathbb{E}[e^{\lambda W_k}] \leq e^{\frac{1}{2}\lambda^2 k} \).
2. For any real \(a\) and \(b < \frac{1}{2}\), we have
   \[ \mathbb{E}[e^{a \sqrt{k} W'_k + \frac{b}{k} W^2_k}] \leq \mathbb{E}[e^{a \sqrt{k} W_k + \frac{b}{k} W^2_k}] \leq \frac{1}{\sqrt{1 - 2b}} e^{\frac{1}{2}b k \frac{b}{n} \frac{2}{n}}. \]
3. For any real \(b < \frac{1}{2}\), we have
   \[ \mathbb{E}[e^{b S'_k}] \leq \mathbb{E}[e^{b S^2_k}] \leq \frac{1}{\sqrt{1 - 2b}} e^{\frac{b}{k} \frac{b}{n} \frac{2}{n}}. \]

In particular, given any \(\delta > 0\), there exists \(b_\delta > 0\) and \(c_\delta < 1\) such that for all \(k \leq (1 - \delta)n\) and any \(b < b_\delta\), these expectations are bounded by \(\exp\{1 + c_\delta \frac{2}{n}\} \).

**Proof.** The functions \(x \mapsto e^{\lambda(x-k(p-q))}\) and \(x \mapsto e^{a(x-k(p-q)) + b(x-k(p-q))^2}\) and \(x \mapsto bx^2/k\) are all convex. By (24) this implies the first inequalities in all three statements of the lemma. It only remains to prove the second inequalities.

1. This is really a part of the proof of case of the famous Hoeffding’s inequality, but let us quickly recap anyway. Since \(Y_i - (p-q)\) takes the values \(2q\) and \(-2p\) with probabilities \(p\) and \(q\), respectively, we see that \(\mathbb{E}[e^{\frac{1}{2}AW_k}] = (pe^{\lambda q} + qe^{-\lambda p})^k\). Let \(\varphi(\lambda) = \log(pe^{\lambda q} + qe^{-\lambda p})\) and observe that
   \[ \varphi'(\lambda) = \frac{pq(e^{\lambda q} - e^{-\lambda p})}{pe^{\lambda q} + qe^{-\lambda p}} \text{ vanishes at } \lambda = 0, \]
   \[ \varphi''(\lambda) = \frac{pe^{\lambda q} - qe^{-\lambda p}}{(pe^{\lambda q} + qe^{-\lambda p})^2} \leq \frac{1}{4} \text{ for any } \lambda, \]
   since \(4ab \leq (a + b)^2\) (here \(a = qe^{-\lambda p}\) and \(b = pe^{\lambda q}\)). Thus, \(\varphi(\lambda) \leq \frac{1}{8}\lambda^2\) for all \(\lambda \in \mathbb{R}\), by the second order Taylor expansion. Replacing \(\lambda/2\) by \(\lambda\), we have arrived at \(\mathbb{E}[e^{\frac{1}{2}AW_k}] \leq e^{\frac{1}{8}\lambda^2 k}\).

2. A useful and often used trick is to use the Parseval relation for characteristic functions to convert the quadratic in the exponent
into a linear term by introducing a new independent Gaussian variable \( Z \), independent of the other variables considered so far. As 
\[ E[e^{\lambda(Z+c)}] = e^{\lambda c + \frac{1}{2} \lambda^2}, \]
we can write 
\[
E[e^{aW_k + bW_k^2}] = E[e^{aW_k + \sqrt{2b}W_kZ}]
\leq E[e^{\frac{1}{2}k(a + \sqrt{2b}Z)^2}]
= e^{\frac{1}{2}ka^2} E[e^{ak\sqrt{2b}Z + bkZ^2}]
= \frac{1}{\sqrt{1 - 2kb}} e^{\frac{1}{2}ka^2 + \frac{b^2 a^2}{2(1 - 2kb)}}
\]
valid for any real \( a \) and \( b < \frac{1}{\sqrt{2}} \). Replace \( a \) and \( b \) by \( a/\sqrt{k} \) and \( b/k \) to get it in the form given in the statement of the Lemma.

(3) As \( S_k = W_k + k(p - q) \), we see that
\[
E[e^{bS_k^2}] = e^{bk^2(p-q)^2} E[e^{bkW_k^2 + 2bk(p-q)W_k}]
\leq \frac{1}{\sqrt{1 - 2kb}} \exp \left\{ \frac{bk^2(p-q)^2 + 4b^2 k^3(p-q)^2}{2(1 - 2kb)} \right\}
\leq \frac{1}{\sqrt{1 - 2kb}} \exp \left\{ \frac{bk^2(p-q)^2}{1 - 2kb} \right\}.
\]
Using \( s = n(p-q) \) and replacing \( b \) by \( b/k \) we get
\[
E[e^{bS_k^2/k}] = \frac{1}{\sqrt{1 - 2b}} \exp \left\{ \frac{b}{1 - 2b/n} k s^2 \right\}.
\]
This was the claimed inequality. Now it is clear that if \( k \leq n(1 - \delta) \), then choosing \( b_\delta < 2\delta \) ensures that the exponent is less than \( c_\delta s^2/n \) for \( c_\delta = (1 - \delta)/(1 - 2b_\delta) \). If \( b < \frac{1}{4} \), then \( 1/\sqrt{1 - 2b} \) is bounded by \( \sqrt{2} < e \), hence the whole of it is bounded by \( \exp\{1 + c_\delta s^2/n\} \).
Appendix 1: From Tusnady Type Lemma to KMT Theorem

This derivation of KMT-EP from Tusnady's lemma is, up to changes of notation, copied from Pollard [12] (chapter 10) and presented here for completeness.

**Dyadic intervals:** Let $D = \cup_{p \geq 0} D_p$, where $D_p$ denotes the set of dyadic intervals of generation $p$, of the form $I = [k2^{-p}, (k + 1)2^{-p}]$ with $p \geq 0$ and $0 \leq k \leq 2^p - 1$.

Dyadic intervals have a natural rooted binary tree structure, with $I_0 = [0, 1]$ as the root and $I = [k2^{-p}, (k + 1)2^{-p}]$ having two children, $I' = [(2k)2^{-p-1}, (2k+1)2^{-p-1}]$ and $I'' = [(2k+1)2^{-p-1}, (2k+2)2^{-p-1}]$. The ancestor of $I$ in the $q$th generation (for $0 \leq q \leq p$) is denote $I^q$. Hence $I^p = I$ and $I^0 = I_0$. The function $\psi_I : [0, 1] \to \mathbb{R}$ that is equal to $2^{p/2}$ on $I'$ and $-2^{p/2}$ on $I''$ and zero elsewhere, is called a Haar function.

Together with the constant function $1$, the Haar functions form an orthonormal basis of $L^2[0, 1]$. The function $\varphi_I(t) := \int_0^1 \psi_I(s)ds$, vanishes outside $I$, takes the value $2^{-\frac{1}{2}p-1}$ at the midpoint of $I$, and is linear on $I'$ and $I''$.

**Series expansion of Brownian bridge:** Let $Z(I)$ be i.i.d. standard Gaussians. Then, $W^{(m)}_0(t) := \sum_{p \geq 0} \sum_{I \in D_p} Z(I)\varphi_I(t)$ converges uniformly (as $m \to \infty$) over $t \in [0, 1]$ to the standard Brownian bridge $W_0$. If $t$ is a dyadic rational, then $W^{(m)}_0(t) = W_0(t)$ for large $m$.

**Constructing the uniform empirical process from $W_0$:** Fix $n$. Apply the coupling between Binomials and Gaussian assured by Lemma 1 and construct $\{N(I) : I \in D\}$ as follows.

Set $N(I_0) = n$. Inductively, suppose $I$ is an interval for which $N(I)$ has been defined but $N(I')$ and $N(I'')$ are not yet defined. If $N(I) = 0$, then set $N(I') = N(I'') = 0$. If $N(I) \geq 1$, then couple $Z(I)$ with $\hat{N}(I) \overset{d}{=} S_{N(I)}$ and set $N(I') = \frac{1}{2}(N(I) + \hat{N}(I))$ and $N(I'') = N(I) - N(I')$. Inductively, it is easy to see that $N(I) \sim \text{Bin}(n, |I|)$ where the length $|I| = 2^{-p}$ if $I \in D_p$.

From $N(I), I \in D$, we get $n$ i.i.d. uniform points in $[0, 1]$, and $N$ is just the counting measure of these points. Let $\nu = \sqrt{n}((\frac{1}{n}N - \lambda)$ (where $\lambda$ denotes Lebesgue measure on $[0, 1]$). The distribution function $G_n(t) := \nu[0, t]$ of the real measure $\nu$, is the uniform empirical process. Observe that $G'_n = \nu$, hence $\langle G'_n, \psi_I \rangle = \int_0^1 \psi_I(t)d\nu(t) = \frac{1}{\sqrt{n}}2^{p/2}\hat{N}(I)$. Thus $G_n(t) = \sum_{p \geq 0} \sum_{I \in D_p} \frac{2^{p/2}}{\sqrt{n}}\hat{N}(I)\varphi_I(t)$.

**Closeness of the coefficients of $G^{(m)}_n$ and of $W^{(m)}_0$:** The coupling in Lemma 1 ensures that $|\hat{N}(I) - Z(I)\sqrt{N(I)}| \leq A(N(I), Z(I))$ and $|\hat{N}(I)| \leq B(N(I), Z(I))$ where $A(n, t) = a(1 + t^2)$ and $B(n, t) = b(1 + |t|\sqrt{n})$ (in fact the statement of
Lemma [1] is stronger. For $I \in \mathcal{D}_p$,

$$
\frac{2^{p/2}}{\sqrt{n}} \hat{N}(I) - Z(I) \leq \frac{2^{p/2}}{\sqrt{n}} |\hat{N}(I) - Z(I) \sqrt{N(I)}| + |Z(I)| \times \sqrt{\frac{2^p}{n} N(I) - 1}
$$

$$
\leq \frac{2^{p/2}}{\sqrt{n}} A(N(I), Z(I)) + |Z(I)| \sum_{j=1}^{p} \left| \sqrt{\frac{2^j}{n} N(I)} - \sqrt{\frac{2^{j-1}}{n} N(I^{j-1})} \right|
$$

where in the second term we used the fact that $\frac{2^p}{n} N([0, 1]) = 1$ to write a telescoping series. For $a, b > 0$, we have $|\sqrt{a} - \sqrt{b}| \leq |a - b|/\sqrt{b}$, because of which the $j$th summand in the second term can be bounded by (assuming $N(I^{j-1}) > 0$, else the summand is zero)

$$
\frac{1}{\sqrt{2^{j-1} N(I^{j-1})}} \left| \frac{2^j}{n} N(I) - \frac{2^{j-1}}{n} N(I^{j-1}) \right| = \frac{2^{(j-1)/2}}{\sqrt{n}} \frac{|\hat{N}(I^{j-1})|}{\sqrt{N(I^{j-1})}}
$$

$$
\leq \frac{2^{(j-1)/2}}{\sqrt{n}} B(N(I^{j-1}), Z(I^{j-1}))
$$

Plugging in the expressions for $A(n, t)$ and $B(n, t)$ and changing $j - 1$ to $j$,

$$
\left| \frac{2^{p/2} \hat{N}(I)}{\sqrt{n}} - Z(I) \right|
$$

$$
\leq \frac{2^{p/2} a (1 + |Z(I)|^2)}{\sqrt{n}} + \frac{|Z(I)|}{\sqrt{n}} \sum_{j=0}^{p-1} 2^j \left( \frac{b}{\sqrt{n} N(I)} + |Z(I)| \right) 1_{N(I) > 0}
$$

$$
\leq a \frac{2^{p/2}}{\sqrt{n}} + \frac{2^{p/2}}{\sqrt{n}} (|Z(I)|^2 + \frac{b}{\sqrt{2} - 1} |Z(I)|) + \frac{1}{2 \sqrt{n}} \sum_{j=0}^{p-1} 2^j (|Z(I)|^2 + |Z(I^j)|^2)
$$

$$
\leq a' \frac{2^{p/2}}{\sqrt{n}} + \frac{b'}{\sqrt{n}} \sum_{j=0}^{p} 2^{j/2} |Z(I^j)|^2
$$

(25)

for some constants $a', b'$. In the second line we simply used $2xy \leq x^2 + y^2$ while to get to the last line, we first wrote $z^2 + \frac{6}{\sqrt{2 - 1}} |z| \leq 16(1 + z^2)$ and absorbed all the terms with $z^2$ into the last summand with $j = p$ (since the last term in the geometric series is of the same order as the sum).

Fix $m$ and use this in the series expansion for $t \in T(m)$, where $T(m)$ is the set of end-points of the intervals in $\mathcal{D}_m$. Then, the series for $G_n$ and $W_0$ run up to $p = m$,
and using (25) and the fact that $|\varphi_I(t)| \leq 2^{-|I|/2}$, we get

$$|G_n(t) - W_0(t)| \leq \frac{a'}{\sqrt{n}} m + \frac{b'}{\sqrt{n}} \sum_{j=0}^{m} \sum_{p=0}^{m} 2^{(j-p)/2} Z(I_{j,p})^2$$

$$\leq \frac{a'}{\sqrt{n}} m + \frac{4b'}{\sqrt{n}} \sum_{j=0}^{m} |Z(I_{j,t})|^2.$$ 

In the second line we used the fact that $I_{j,t} = I_{j,t}$ and interchanged the sums (the geometric series $\sum 2^{-i/2} \leq 4$). Hence, for $t \in T(m)$, we have $|G_n(t) - W_0(t)| = \frac{C}{\sqrt{n}} (m + S_m(t))$ where $S_m(t) = \sum_{p=0}^{m} |Z(I_{p,t})|^2$. Pollard (for all references to [12], see section 10.7 of that book) shows that with $T(m) = \{k2^{-m} : 0 \leq k \leq 2^{m}\}$,

$$P\{ \max_{t \in T(m)} S_m(t) \geq 10(m + x) \} \leq 2e^{-m-x}. \quad (26)$$

This is easy to see: For a fixed $t \in T(m)$, the variable $S_m(t)$ has $\chi_{m+1}^2$ distribution. Hence $E[e^{\lambda S_m(t)}] = (1 - 2\lambda)^{-m}$ for $\lambda < \frac{1}{2}$. Taking $\lambda = \frac{1}{2}$, we get $P\{S_m(t) \geq 8(m + x)\} \leq 2^{(m+1)-2(m+x)}$. The union bound gives something better than (26).

Now let $\Delta_m(f) = \max_{I \in D_m} \max_{t,s \in I} |f(t) - f(s)|$ denote the maximum oscillation of a function $f$ within any interval in $D_m$. Then (see p. 254 of [12]) using standard facts about Brownian bridge

$$P\{\Delta_m(W_0) \geq \frac{m}{\sqrt{n}} \} \leq 2^{m+1} e^{-\frac{2m^2}{2m}} \leq 2^{m+1} e^{-\frac{m^2}{2}}. \quad (27)$$

Next, since $\sqrt{n}(G_n(t) - G_n(s)) = (N_n(t) - N_n(s)) - n(t - s)$ for $s < t$,

$$P\{\Delta_m(G_n) \geq \frac{m + 2}{\sqrt{n}} \} \leq P\{\max_{t \in D_m} N_n(I) \geq m + 2 - \frac{n}{2m}\} \leq 2^m P\{\text{Bin}(n, 2^{-m}) \geq m\} \leq \frac{2^m}{m!}. \quad (28)$$

as $\text{Bin}(n, p)$ is stochastically dominated by $\text{Pois}(np)$. Now,

$$\max_{t \in [0,1]} |G_n(t) - W_0(t)| \leq \frac{C}{\sqrt{n}} \left\{ \max_{t \in T(m)} S_m(t) + \Delta_m(G_n) + \Delta_m(W_0) \right\}.$$

By (26), (27) and (28), one gets

$$P\left\{ \max_{t \in [0,1]} |G_n(t) - W_0(t)| \geq \frac{C(m + x)}{\sqrt{n}} \right\} \leq C'e^{-m-x}$$

which is the conclusion of KMT theorem, since $m = \log n + O(1)$. Usually it is written without the $m$ in the exponent, but as long as the constant $C$ is not specified in the left, it only makes sense to take $x \geq \log m$, which gives this term. It is neither stronger nor weaker to state it this way.
APPENDIX 2: CHATTERJEE’S PROOF OF KMT-RW BY INDUCTION

For any probable value $s$ of $S_n$, let $S[n, s] := (S_0[n, s], \ldots, S_n[n, s])$ (definition of $S_k[n, s]$ is in the paragraph preceding Lemma 3). Let $V[n] := (V_0[n], \ldots, V_n[n])$ be a centered Gaussian vector with covariances $E[V_i[n]V_j[n]] = \frac{i(n-j)}{n}$ for $0 \leq i \leq j \leq n$. When it is safe to do so without ambiguity, we shall drop the $n$ (and any probable value of $S$) in the notation.

Tools: The following facts will be used.

- By Lemma 3 for any $n \geq 2$ and any $k \in [n/3, 2n/3]$ and any $t$ in the support of $S_n$, there is a coupling such that for any $\theta \leq \theta_1$,
  \begin{equation}
  E \left[ e^{\theta |S_k[n,t]-\frac{k}{n}t-V_k[n]|} \right] \leq e^{1+M\theta^2\frac{s^2}{n}}. \tag{29}
  \end{equation}

- By part (3) of Lemma 21, there exists $\alpha_0 > 0$ and $\gamma < 1$ such that for any $n \geq 1$ and any $k \leq \frac{2}{3}n$, any probable value $t$ of $S_n$, and any $\alpha \leq \alpha_0$,
  \begin{equation}
  E \left[ e^{\alpha |S_k[n,t]|^2} \right] \leq e^{1+\gamma \alpha^2}. \tag{30}
  \end{equation}

Fix constants $A, B, \lambda_0$ satisfying: $A \geq \frac{1+\log 2}{\log \frac{n}{2}}$ and $B \geq \frac{2M}{1-\gamma}$ and $\lambda_0 \leq \frac{\theta}{2} \wedge \sqrt{\frac{\theta}{2M}}$ where $M, \gamma, \theta_1$ are those occurring in (29) and (30).

**Induction hypothesis:** For any $n \geq 1$ and any probable value $t$ of $S_n$, there is a coupling of $S[n, t]$ with $V[n]$ such that for any $\lambda \leq \lambda_0$,
  \begin{equation}
  E \left[ \exp \left( \lambda \max_{1 \leq i \leq n} |S_i[n, t] - \frac{it}{n} - V_i[n]| \right) \right] \leq e^{A \log n + B\lambda^2 \frac{s^2}{n}}. \tag{31}
  \end{equation}

**Base case:** If $A$ is large enough, then the statement is obvious for any fixed $n$. We just need to watch out that the induction step goes through without difficulty if $A$ is increased.

**Induction step:** Assume that the conclusion (31) holds for any $m < n$ in place of $n$ (and any probable value $t$ of $S_m$). Fix $n \geq 6$ and a probable value $t$ of $S_n$. Choose $k = \lfloor n/2 \rfloor$ so that $\frac{2}{3} \leq k \leq \frac{2n}{3}$ as $n \geq 6$.

1. Construct $(s, v)$ having the same marginals as $(S_k[n, t], V_k[n])$ and coupled so that (29) holds. That is, with $R = |s - \frac{k}{n}t - v|$, we have $E[\exp{\theta R}] \leq \exp{1 + M\theta^2\frac{s^2}{n}}$ for all $\theta \leq \theta_1$.
2. Conditional on $(s, v)$, construct independent pairs $(S', U')$ and $(S'', U'')$ such that $S' \overset{d}{=} S[k, s], U' \overset{d}{=} V[k], S'' \overset{d}{=} S[n-k, t-s]$ and $U'' \overset{d}{=} V[n-k]$ so that (31) holds for both pairs. That is, writing
   \begin{align*}
   T' &= \max_{1 \leq i \leq k} |S'_i - \frac{is}{k} - U'_i|, \quad \text{and} \quad T'' = \max_{1 \leq j \leq n-k} |S''_j - \frac{j(t-s)}{n-k} - U''_j|,
   \end{align*}
for all $\lambda \leq \lambda_0$ we have

$$E\left[e^{\lambda T} \mid s, v\right] \leq e^{\lambda \log k + B\lambda^2 \frac{2}{n}}$$

and

$$E\left[e^{\lambda T''} \mid s, v\right] \leq e^{A \log(n-k) + B\lambda^2 \frac{(n-k)^2}{n}}.$$

(3) Define $S = (S_0, \ldots, S_n)$ and $V = (V_0, \ldots, V_n)$ by setting

$$S_i = \begin{cases} S_i' & \text{if } i \leq k, \\ s + S''_{i-k} & \text{if } k \leq i \leq n, \end{cases} \quad \text{and} \quad V_i = \begin{cases} U_i' + \frac{1}{n}v & \text{if } i \leq k, \\ U''_{i-k} + \frac{n-i}{n}v & \text{if } k \leq i \leq n. \end{cases}$$

There is no ambiguity at $i = k$, as $S_k' = s$ and $S_k'' = U_k' = U_k'' = 0$. From the construction, it is clear that $S \overset{d}{=} S[n, t]$ and $V \overset{d}{=} V[n].$

(4) Now we observe that

$$S_i - \frac{it}{n} - V_i = \begin{cases} (S_i' - \frac{it}{n} - U_i') + \frac{i}{n}(s - \frac{kt}{n} - v) & \text{if } i \leq k, \\ (S''_{i-k} - \frac{(i-k)(t-a)}{n} - U''_{i-k}) + \frac{n-i}{n}(s - \frac{kt}{n} - v) & \text{if } k \leq i \leq n. \end{cases}$$

Consequently, if $T := \max_{i \leq n} |S(i) - \frac{1}{n}a - V(i)|$ then $T \leq (T' \lor T'') + R$ and hence $E[e^{\lambda T}] \leq E[e^{\lambda(T'+R)}] + E[e^{\lambda(T''+R)}].$ Now, for any $p \in (0, 1)$ (to be chosen depending on $\gamma$), we can write

$$E[e^{\lambda(T'+R)}] = E[e^{\lambda R} E[e^{\lambda T'} \mid s, v]]$$

$$\leq e^{A \log k} E\left[e^{\lambda R} e^{B\lambda^2 \frac{2}{n}}\right]$$

$$\leq e^{A \log k} E\left[e^{2\lambda R}\right]^\frac{1}{2} E\left[e^{B\lambda^2 \frac{2}{n}}\right]^\frac{1}{2}$$

by Hölder’s inequality. Since $2\lambda \leq \theta_1$ and $2B\lambda^2 \leq \alpha_0$,

$$E[e^{\lambda(T'+R)}] \leq e^{A \log k} e^{\frac{1}{2}(1 + 4M\lambda^2 \frac{2}{n})} e^{\frac{1}{2}(1 + 2B\lambda^2 \gamma^2 \frac{2}{n})}$$

$$= e^{A \log k} e^{1 + 2\lambda^2 \frac{2}{n}} (2M + \gamma B)$$

$$\leq e^{-A \log \frac{\theta_1}{2} + 1} e^{A \log n + 2B\lambda^2 \frac{2}{n}}$$

provided $2M \leq (1 - \gamma)B.$ By almost identical reasoning, we also get

$$E[e^{\lambda(T''+R)}] \leq e^{-A \log \frac{\theta_1}{2} + 1} e^{A \log n + 2B\lambda^2 \frac{2}{n}}.$$

Therefore,

$$E[e^{\lambda T}] \leq e^{A \log n + 2B\lambda^2 \frac{2}{n}} \left( e^{-A \log \frac{\theta_1}{2} + 1} + e^{-A \log \frac{n}{\theta_1} + 1} \right)$$

$$\leq e^{A \log n + 2B\lambda^2 \frac{2}{n}}$$

since the condition $k \leq \frac{\theta_1}{2}n$ ensures that second factor is bounded by $2e^{1 - A \log \frac{\theta_1}{2}} < 1.$
This completes the induction step.

**The KMT-RW theorem for symmetric Bernoulli steps** From (31), it is easy to deduce the KMT-RW theorem for symmetric Bernoullis (Chatterjee proves something stronger, keeping the same Brownian motion as $n$ varies, but we forego that strengthening now). For this, let $W$ be a standard Brownian motion, and recall that we may write $W(t) = W_0(t) + tZ$ where $Z$ is a standard Gaussian independent of the Brownian bridge $W_0$. Further, $V[n]$ has the same distribution as $W_0$ sampled at times $0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1$. Now the idea is clear.

1. Invoke Lemma 2 and couple $S_n$ and $Z$ so that $\mathbb{E}[e^{\theta|S_n - z\sqrt{n}|}] \leq \kappa_0$ for any $\theta \leq \theta_0$.

2. Conditional on $Z = z$, the distribution of $W$ is $W_0(t) + tz$ and conditional on $S_n = s$, the distribution of $(S_0, \ldots, S_n)$ is the same as $S[n, s]$. Now we can couple $S$ with the random vector $V = (W_0(0), W_0(1/n), \ldots, W_0(1))$ so that (31) holds.

3. We observe that

$$
\max_{0 \leq k \leq n} |S_k - W(k)| \leq \max_{0 \leq k \leq n} |S_k - \frac{k}{n}S_n - W_0(k/n)| + |S_n - W(1)|.
$$

Hence $\mathbb{E}[\exp\{\lambda \max_{0 \leq k \leq n} |S_k - W(k)|\}]$ is bounded by

$$
\mathbb{E}[\exp\{\lambda \max_{0 \leq k \leq n} |S_k - \frac{k}{n}S_n - W_0(k/n)|\}] + \mathbb{E}[\exp\{\lambda |S_n - W(1)|\}]
\leq e^{A \log n} \mathbb{E}[e^{B\lambda^2 S_n^2/n}] + \kappa_0
$$

if $\lambda \leq \lambda_0 \wedge \theta_0$. By Lemma 21 if $\lambda_0$ is sufficiently small, then this whole quantity is bounded by $\kappa e^{A \log n}$ for some constants $\kappa$ and $A$ that do not depend on $n$. By Markov’s inequality, we get KMT-RW.
References

[1] Ash, Robert, Information theory, *Dover publications* (1990).

[2] Bhattacharjee, C. and Goldstein, L., On strong embeddings by Stein’s method, *Electon. J. Probab.*, 21, paper 15, (2016).

[3] Carter, A. and Pollard, D. Tusnády’s inequality revisited, Technical report, Yale University (2000). [http://www.stat.yale.edu/~pollard](http://www.stat.yale.edu/~pollard).

[4] Chatterjee, Sourav, A new approach to strong embeddings, *Probab. Theory Relat. Fields*, 152, 231–264, (2012).

[5] Cover, T., and Thomas, J., Elements of information theory, second ed., *John Wiley and sons Inc.* (2006).

[6] Csörgő, M. and Révész, P., Strong Approximations in Probability and Statistics, Academic Press, New York (1981).

[7] Hoeffding, W., Probability inequalities for sums of bounded random variables, *Journal of the American Statistical Association*, 58 13–30, (1963).

[8] Komlós, J., Major, P., and Tusnády, G., An approximation of partial sums of independent rv-s, and the sample df. I, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 32, 111–131 (1975).

[9] Komlós, J., Major, P., and Tusnády, G., An approximation of partial sums of independent rv-s, and the sample df. II, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 34, 33–58 (1976).

[10] Lifshits, M.A., Lecture notes on strong approximation, *Publications IRMA Lille*, 53-XIII, 1–25, (2001). [https://sites.google.com/site/mlprobability/home10/ml09](https://sites.google.com/site/mlprobability/home10/ml09).

[11] Massart, P., Tusnády’s lemma, 24 years later, *Ann. Institut. Henri Poincaré (B) Probab. and Stat.*., 38, 991–1007, (2002).

[12] Pollard, David, A user’s guide to measure theoretic probability, *Cambridge university press* (2002).

[13] Tusnády, G., A study of Statistical Hypotheses. Ph. D. thesis, Hungarian Academy of Sciences, Budapest. In Hungarian. (1977).

[14] Zaïtsev, A.Yu., Estimates for the strong approximation in multidimensional central limit theorem. *In: Proceedings of the International Congress of Mathematicians*, vol. III, Higher Ed. Press, Beijing, 107–116 (2002).

Manjunath Krishnapur, Department of Mathematics, Indian Institute of Science, Bangalore, Karnataka, India.

E-mail address: manju@iisc.ac.in