The Relativistic Point Charge Revisited: Novel Features

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A fully relativistically covariant formulation of the classical Maxwell electrodynamics of an arbitrarily-moving point charge is presented, purely in terms of gauge invariant potentials without entailing any gauge fixing. A new, relativistically covariant energy-momentum tensor for the radiation fields is derived and yields results for the angular power distribution, in full agreement with results derived in a frame-dependent manner in standard texts of classical electrodynamics. This is then used to present a full derivation, not available in standard texts, of the energy-momentum of a relativistic point charge. Radiation backreaction is turned on and the system reanalyzed Lorentz-covariantly, including effects of mass renormalization. This leads us to reiterate earlier conclusions regarding the inherent inadequacy of classical Maxwell electrodynamics. The Abraham-Lorentz equation is derived en passant in appropriate limits without requiring any extraneous structural artifact, and the Landau-Lifschitz proposal for modification of the theory is also critically reviewed.

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I. INTRODUCTION

The standard textbook formulation of Maxwell electrodynamics, in vacua with sources, entails linear first order partial differential equations for electric and magnetic field strengths \( \vec{E} \) and \( \vec{B} \). Conventionally, the equations for these field strengths are first cast in terms of the scalar and vector potentials, \( \phi \) and \( \vec{A} \). The resulting second order equations for the potentials are found to be non-invertible because of the of gauge ambiguity of the potentials: gradients of arbitrary gauge functions generate gauge-equivalent classes of solutions for the same field strengths. This gauge ambiguity has often led to considering gauge potentials as unphysical, in comparison to the physical (gauge-invariant) field strengths. The standard procedure in getting to the solutions is to ‘gauge fix’ the potentials, i.e., impose subsidiary conditions on them so that the ambiguity may be resolved. There is a non-denumerably infinite set of such subsidiary ‘gauge conditions’, each one as ad hoc as the other, and none with any intrinsic physical relevance. This entire approach, tenuous as it is, avoids facing up to the central issue: why are the equations for the potentials non-invertible in the first place? Is Nature so unkind as to provide us with unique gauge-invariant equations for quantities which themselves are infinitely ambiguous?

As we shall show in section 3, this is indeed not the case. The reason that the equations for the potentials are non-invertible in the first place is because their intrinsic analytic structure involves a projection operator which has a non-trivial kernel of unphysical ‘pure gauge’ vector fields! This simple observation renders any ‘gauge fixing’ superfluous, since it is now obvious that the equations are to be interpreted in terms of projected physical, gauge invariant potentials not belonging to the kernel of the projection operator, hence obeying very simple wave equations which are immediately uniquely invertible without the need for any imposition of additional ‘gauge conditions’. We find surprising that this simple fact has not been clarified in any of the vast number of popular textbooks on classical electrodynamics.

Before taking up the gauge ambiguity issue, we discuss another lacuna in extant textbook treatments of Maxwell electrodynamics - the absence of a fully relativistically covariant formulation of the subject ab initio. Special relativity is intrinsically embedded in Maxwell electrodynamics with charge and current sources in empty space, as was discovered by Einstein in 1905. If, as per standard practice, the fundamental equations are written in terms of the electric and magnetic fields, the relativistic invariance of these equations is far from obvious. This emerges only after some effort is given to relate the electric and magnetic fields in different inertial reference frames connected by Lorentz boosts. In contrast, if the equations are cast in terms of the physical electromagnetic scalar and vector potentials introduced by Maxwell, then these potentials

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and the equations they obey, can be easily combined to yield a structure which is manifestly invariant under Lorentz boosts as well as spatial rotations, i.e., the full Lorentz transformations. Given that it is easier always to compute four, rather than six, field components for given source charge and current densities, it stands to reason to begin any formulation of electrodynamics from the (physical) potentials, rather than the field strengths.

Despite its antiquity, a formulation of classical electrodynamics that, from the outset, is fully relativistically covariant, is somehow not preferred in the very large number of excellent textbooks currently popular, with perhaps the sole exception of [2]. Even so, the issue of the gauge ambiguity and the full use of electromagnetic potentials rather than fields, has not been dealt with adequately, even in this classic textbook. Thus, while relativistically covariant Lienard-Wiechert potentials describing the solution of Maxwell’s equations due to an arbitrarily moving relativistic point charge have been obtained, the corresponding field strengths and the radiative energy-momentum tensor have not been given such a manifestly covariant treatment. The more widely used textbook [3] also fills this gap only in part. In this paper, we derive, manifestly covariantly, an energy-momentum tensor for the radiation due to an arbitrarily-moving relativistic point charge, expressed solely in terms of its 4-velocity and 4-acceleration, and its spacetime distance from the observer. The correctness of the energy-momentum tensor is ensured by its ability to reproduce the usual formulae for the angular distribution of the radiated power from the point charge given in the standard texts, after an appropriate Lorentz frame is chosen. Other essential properties like tracelessness, and vanishing when space is one dimensional, obtain trivially from our energy-momentum tensor, thus lending credence to our result.

While the energy and momentum of radiation due to a non-relativistic point charge are easily obtained from the full energy-momentum tensor for a relativistic charge in the appropriate approximation, with the angular integrations being reduced to quadratures, the same is not true for an ultra-relativistic point charge. In most of the afore-mentioned excellent textbooks, what is presented is a plausibility argument of a possible relativistic generalization from the non-relativistic formula for the energy and momentum of radiation from a point charge, giving a formula whose only significance is that it has the correct non-relativistic limit. One is left wondering that perhaps there are other relativistic formulas with an identical non-relativistic limit. We avoid such issues by presenting first in this paper a derivation of the radiant energy-momentum due to a relativistic accelerated charge, starting with the Lorentz-covariant radiant energy-momentum tensor derived earlier manifestly relativistically. A more complete, deductive derivation of this 4-momentum of radiation due to a relativistic, arbitrarily-moving point charge is also given, with inspiration from the formulation of S. Coleman [4] of a classical charge which employs Fourier-transformed distributions.

This beautiful formulation has the added advantage of the ability of inclusion of radiation backreaction, once again, manifestly Lorentz-covariantly. The relativistic charge equations of motion, with radiative backreaction included, immediately exhibit a pathology which forecast inherent problems of classical electrodynamics like acausality, from a completely relativistic standpoint. These issues are usually discussed in the literature in the non-relativistic limit [4]. Our fully relativistic formulation enables us to derive, in appropriate limits the well-known Abraham-Lorentz [5] and Landau-Lifschitz [2] equations, without the need to use any extraneous structural artifacts, like assuming the point charge to be a charged sphere of vanishingly small radius, as done in some contemporary papers (e.g., [2]).

The paper is organized as follows: in section 2, starting with the equations for the physical potentials we briefly discuss how Lorentz-invariance is manifest in these equations. To allay suspicions regarding the gauge ambiguity, we show how the equations for the potentials follow from the standard textbook equations written in terms of field strengths given in terms of these potentials. In section 3, we consider a single arbitrarily-moving point charge as the source, and obtain its potentials and field strengths thereof. Focusing on the radiative parts of the field strengths, we obtain the expression of our covariant energy-momentum tensor. In the next section, this energy-momentum tensor is used in its Fourier transformed version to derive the 4-momentum of a relativistic point charge. In section 5, we discuss the effect of including radiation reaction, and analyse the resulting equations leading to runaway solutions and/or preacceleration despite resorting to mass renormalization. This section follows the analysis of ref. [4] fairly closely. The inadequacy of Maxwell electrodynamics is thus reviewed in this section. In this section we also rederive the Abraham-Lorentz and Landau-Lifschitz equations for a point charge directly from our results, without having to make any ‘model’ of the point charge. We conclude briefly in section 6.
II. MAXWELL EQUATIONS REVISITED

A. Formulation in terms of physical potentials

The physical electromagnetic scalar potential \( \phi_P \) and the physical vector potential \( \vec{A}_P \) satisfy the Maxwell equations,

\[
\begin{align*}
\Box^2 \phi_P &= \rho \\
\Box^2 \vec{A}_P &= \vec{j} \\
\frac{\partial \phi_P}{\partial t} + \nabla \cdot \vec{A}_P &= 0,
\end{align*}
\]

where, \( \Box^2 \equiv (\partial^2/\partial t^2) - \nabla^2 \) is the d’Alembertian operator, and we are using units such that \( c = 1 \). All other constants are absorbed into redefinitions of \( \rho \) and \( \vec{j} \). Eqn. (1) can be motivated physically \textit{ab initio} from the most important characteristic of electrodynamics, namely that they must yield electromagnetic waves travelling through empty space. Indeed, the top two equations in (1) are nothing but \textit{inhomogeneous} wave equations, recalling that the d’Alembertian is the \textit{wave} operator. The last equation is a special characteristic property of the physical potentials \( \phi_P, \vec{A}_P \) which is relevant to ensure that the electromagnetic waves in empty space have transverse polarization. It is easy to see that the usual Maxwell equations for the electric and magnetic field strengths \( \vec{E} \) and \( \vec{B} \) result from (1) immediately, upon defining \( \vec{E} \equiv -\partial \vec{A}_P/\partial t - \nabla \phi_P, \vec{B} \equiv \nabla \times \vec{A}_P \).

Observe that these equations can be combined into the 4-vector potential \( \vec{A}_P \) with components \( A^i_P, i = 0, 1, 2, 3 \), with \( A^0_P = A_{P0} = \phi \), and \( A^\alpha_P = -A_{P\alpha}, \alpha = 1, 2, 3 \) where \( \vec{A}_P = \{ A^\alpha_P | \alpha = 1, 2, 3 \} \). Similarly, the charge and current densities can be combined into a current density 4-vector \( \vec{J} \) with components \( J^i (i = 0, 1, 2, 3) = (\{ \rho, J^\alpha \}, \alpha = 1, 2, 3) \).

Writing \( \dot{A} = \partial / \partial x^i \), eqn. (1) can now be summarised as

\[
\begin{align*}
\Box^2 A^i_P &= J^i \\
\partial_i A^i_P &= 0.
\end{align*}
\]

Raising and lowering of indices are effected by the Minkowski spacetime invariant metric tensor \( \eta_{ij} = \eta^{ij} = \text{diag}(1, -1, -1, -1) \).

It is to be noted that eqn.s (1, 2) are \textit{not} gauge-fixed versions of equations for potentials corresponding to standard Maxwell equations, even though they look enticingly similar. In other words, the second of the equations in (2) is \textit{not a gauge choice}, but a compulsion from Nature. We shall make this clear shortly. Thus, these equations are to be treated at the same physical footing as the standard Maxwell equations for the field strengths, containing the same physical information as the latter, without any ambiguity.

Observe now that the d’Alembertian operator \( \Box^2 \equiv \eta^{ij} \partial_i \partial_j \) is \textit{invariant} under the transformation \( \partial_i \rightarrow \partial'_i = \Lambda^i_\alpha \partial_j, \) provided the transformation matrix \( \Lambda \) satisfies

\[
\eta_{ij} \Lambda^i_\alpha \Lambda^j_\beta = \eta_{kl}.
\]

It follows that eqn.s (2) are invariant under these transformations, if \( A^i(x') = \Lambda^i_j A^j(x), J^i(x') = \Lambda^i_j J^j(x), \) with the transformation matrices \( \Lambda^i_j \) satisfying (3). As is expected, the coordinate transformations leave invariant the squared invariant interval in Minkowski spacetime \( ds^2 = \eta_{ij} dx^i dx^j \). Thus, the transformations that leave the equations of electrodynamics invariant are precisely the same transformations that constitute a \textit{symmetry} of Minkowski spacetime. It is obvious that these are the full \textit{Lorentz} transformations, including spatial rotations and Lorentz boosts. E.g., if \( \Lambda^0_0 = 1, \) \( \Lambda^\alpha_\alpha = 0 \), the remaining \( 3 \times 3 \) submatrix constitute the \textit{orthogonal} transformation matrix corresponding to rotations in 3-space. Likewise, if \( \Lambda^0_0 = \gamma = \Lambda^1_1; \) \( \Lambda^0_1 = -\beta \gamma = \Lambda^1_0 \) etc., that constitutes a Lorentz boost in the +x\textsuperscript{1} direction. The Lorentz factor \( \gamma = (1 - \beta^2)^{-1/2} \). Thus, \textit{all} Lorentz boosts and spatial rotations are just choices for the \( \Lambda \) matrix subject to the restriction (3).

The standard Lorentz-covariant equations of vacuum electrodynamics involving field strengths are easily obtained from eqn. (2) upon using the standard definition \( F_{ij} \equiv \partial_i A_{Pj} - \partial_j A_{Pi} \), leading immediately to the equations

\[
\begin{align*}
\partial_i F^{ij} &= J^j \\
\partial_i (\epsilon^{ijkl} F_{kl}) &= 0.
\end{align*}
\]
B. Derivation from the standard formulation

The purpose of this subsection is to link up the textbook Maxwell equations written in terms of \( F_{ij} \), with the contents of the foregoing subsection. We begin by defining the field strengths \( F_{ij} \) in terms of the standard gauge potential \( A_i \) (i.e., without the subscript \( \rho \)) : \( F_{ij} = \partial_i A_j - \partial_j A_i \). The second of the standard Maxwell equations \( [4] \) results immediately. The first of \( [4] \) is then either postulated on the basis of experiments, or derived from the Maxwell action \( [2] \). Be that as it may, one may substitute the above definition of the field strengths \( F_{ij} \) in terms of the gauge potentials, yielding

\[
\partial_i \partial^i A_j - \partial^j \partial_i A^i = J^j
\]

which, under Fourier transformation (in four dimensions) with Fourier variable \( k^i \), \( i = 0, 1, 2, 3 \), leads to the equation

\[
-k^2 P^j_i \tilde{A}(k)^i = \tilde{J}^j(k)
\]

\[
P^j_i = \delta^j_i - \frac{k^j k_i}{k^2}, \quad k^2 = k_i k^i \neq 0.
\]

We first confine to the inhomogeneous Maxwell equation, and take up the homogeneous case later. The projection operator \( P^j_i \) above possesses the standard properties

\[
P^j_i P^j_k = P^j_k
\]

\[
P^j_i k^j \tilde{f}(k) = 0 \forall \tilde{f}(k)
\]

where \( [9] \) characterizes the vectors in the kernel of the projection operator.

The fact that the vacuum Maxwell equation with sources is expressed uniquely and naturally in terms of a projection of the gauge potential, without having to make any choices, is of crucial importance, since the projection is clearly into the gauge-invariant physical subspace. This projected vector potential, defined as \( A_{\rho i} = P^j_i A_j \) has the following essential properties, which can be easily gleaned from Fourier space : (a) \( \partial_i A^j_{\rho i} = 0 \), i.e., it is spacetime transverse, (b) under gauge transformations \( A_i \rightarrow A^\omega_i = A_i + \partial_i \omega \), the projected (physical) vector potential \( A^\omega_{\rho i} = A_{\rho i} \), i.e., it is gauge-invariant and hence physical ! This implies that

\[
A_i = A_{\rho i} + \partial_i a
\]

so that the entire burden of gauge transformations of \( A_i \) is carried by \( a(x) : a(\omega) = a + \omega \), which underlines the complete unphysicality of the pure gauge part \( a \) of \( A \). It also follows trivially that \( F_{ij}(A) = F_{ij}(A_{\rho}) \), which means that invariance under gauge transformations does not represent a physical symmetry, but merely a redundancy in the gauge potential \( [8] \). One also sees that eqn. \( [2] \) results immediately from our consideration, so that we have come a full circle. In fact, in Fourier space, we have an explicit solution for the physical potential \( A_{\rho i} \) in terms of the sources :

\[
\tilde{A}_{\rho i}(k) = -\frac{\tilde{J}_i(k)}{k^2}
\]

so that, given the form of the 4-vector source, the physical potential and field strengths are determined in spacetime through appropriate inverse Fourier transforms.

Consider now the homogeneous or null case when \( J^i = 0 \) in eqn\( [5] \). It is obvious that in Fourier space, this corresponds to \( k^2 = 0 \), i.e., \( k \) is a null spacetime vector. In this case, it also follows from the Fourier transformed version of \( [6] \) that, for every nontrivial null vector \( k \), we must have

\[
k \cdot A = 0 \Rightarrow \partial_i A^i = 0.
\]

Observe that this is not a choice, but simply follows from the standard Maxwell equations written out in terms of the vector potential.

We know that 4 dimensional spacetime \( M(3,1) \) can be represented at every point as the Cartesian product \( M(1,1) \times \mathbb{R}^2 \), where the first factor is two dimensional Lorentzian spacetime, and the second is just the Euclidean plane. With this, we realize that there is another null vector \( n \), linearly independent of \( k \) which together with \( k \) span \( M(1,1) \). One can always choose \( n \) such that \( n \cdot k = 1 \) for our signature of the Lorentzian metric.
It follows that, for the unique projection operator \( P_{ab} \equiv \eta_{ab} - n_an_b - n_ank_b \), the projected vector potential \( \mathcal{A}_{ia} \equiv \mathcal{A}_a A^a_b \) has the properties that it is transverse to \( \mathcal{M}(1, 1) \), having two components both of which are tangential to the Euclidean plane, and also is gauge invariant. The latter property follows from the uniqueness of solutions of the two dimensional Laplace equation. To summarise therefore, the physical 4-vector potential still follows the same equations as (1), even when sources are absent.

III. RADIATION FROM A POINT CHARGE

A. Radiant energy-momentum tensor

The current density 4-vector can be easily written down for a relativistic point charge with the parametric equation of the trajectory \( \vec{x}^i(x) = e \int d\tau \ u^i(\tau) \delta^{(4)}(x - \vec{x}(\tau)) \).

where the 4-velocity of the charge \( u^i \equiv d\vec{x}/d\tau \). As is well-known [2], [3], substitution of this 4-current density into eqn. (2) leads to the Lienard-Wiechert 4-potentials (the subscript P has been dropped)

\[
A_i = \frac{e u_i(\tau_0)}{(\vec{R}(\tau_0) \cdot \vec{u}(\tau_0))}
\]

where, \( \tau = \tau_0 \) corresponds to the proper time at which the world line of the charge just enters the past light cone of the observer. Thus, the 4-vector \( \mathcal{R}(\tau) \equiv x - \vec{x}(\tau) \), giving the spacetime interval separating the charge from the observer, becomes null at \( \tau_0 : R^2(\tau_0) = 0 \). It is obvious that this defines the retarded time \( \tilde{t} = t - |\vec{r} - \vec{R}(\tau_0)| \).

The corresponding field strength tensor components can also be obtained [3]; it is easy to see that the field tensor can be decomposed into two parts, the ‘radiation’ and ‘Coulomb’ parts:

\[
F_{ij} = F_{ij}^{\text{rad}} + F_{ij}^{\text{Cou}}
\]

\[
F_{ij}^{\text{rad}} = e \left( \frac{\vec{R}_i(a_j - R_ja_i)(\vec{u} \cdot \vec{R}) + (u_iR_j - u_jR_i)(\vec{a} \cdot \vec{R})}{(\vec{u} \cdot \vec{R})^3} \right)
\]

\[
F_{ij}^{\text{Cou}} = e \frac{u^2(u_iR_j - u_jR_i)}{(\vec{u} \cdot \vec{R})^3}
\]

It is to be remembered that all quantities are evaluated at \( \tau = \tau_0 \) which corresponds to the retarded time. One obvious distinction between these two parts above is the dependence of the ‘radiation’ part on the 4-acceleration \( \vec{a} \) of the source charge of which the ‘Coulomb’ part is independent. Another distinction emerges if we choose a frame in which the null vector \( \vec{R} = |\vec{R}|(1, \vec{R}) \), where \( \vec{R} \equiv \vec{R}/|\vec{R}| \) : for large \( |\vec{R}| \), \( F_{ij}^{\text{Cou}} \) falls off as \( 1/|\vec{R}|^2 \), hence justifying the superscript ‘Coulomb’, while \( F_{ij}^{\text{rad}} \) falls off only as fast as \( 1/|\vec{R}| \), implying that the radiation field strength survives much farther away compared to the Coulomb field strength. From this point on, therefore, we shall ignore the Coulomb field strength.

The energy-momentum tensor of the electromagnetic field is given in terms of the field strength tensor by the well-known formula [2]

\[
T^{ij} = - \left[ F^{ik} F_{kj} - \frac{1}{4} \eta^{ij} F^{kl} F_{kl} \right].
\]

Substituting \( F_{ij}^{\text{rad}} \) from eqn (16), we obtain the fully Lorentz-covariant radiation energy-momentum tensor due to an arbitrarily-moving point charge

\[
T^{ij}_{\text{rad}} = e^2 \frac{\vec{R}_i \vec{R}_j}{(\vec{u} \cdot \vec{R})^6} \left[ a^2(\vec{u} \cdot \vec{R})^2 + u^2(\vec{a} \cdot \vec{R})^2 \right].
\]

We note that the expression (19) has not yet appeared in standard textbooks, to the best of our knowledge. It is also clear that in the frame where \( \vec{R} = |\vec{R}|(1, \vec{R}) \), the radiant energy-momentum tensor components fall off at large \( |\vec{R}| \) as \( 1/|\vec{R}|^2 \), so that the angular integral of \( |\vec{R}|^2 T^{ij} \) does not vanish on the surface of 2-sphere of very large radius, in this frame.
The angular distribution of the radiant power due to an accelerated, relativistic point charge can be computed from (19),
\[ \frac{dP}{d\Omega} = |\vec{R}|^2 \dot{R}_\alpha T^{\alpha \alpha}_{rad}. \] (20)

It is straightforward to check that this angular power distribution, expressed in terms of the 3-velocity \( \vec{\beta} \) and the 3-acceleration \( \vec{\alpha} \), where the ‘dot’ corresponds to the derivative with respect to the retarded coordinate time \( t \), coincides with the formula for angular power distribution due to a relativistic accelerated charge, in this frame, given in standard textbooks [2], [3].

In the non-relativistic limit \( \vec{\beta} \to 0 \), the angular integral can be performed, yielding the Larmor formula for the power radiated by a non-relativistic, accelerated, point charge
\[ P \simeq \frac{2}{3} e^2 |\vec{\beta}|^2. \] (21)

B. Radiant energy-momentum due to a relativistic point charge

It is clear that the energy-momentum of radiation due to a relativistic point charge is given by,
\[ P^i_{rad} = \int d^{3}s \, n_j \, T_{rad}^{ij} = \int d^3r \, T^{0i}_{rad} \] (22)
where, the first integral above is over a spacelike hypersurface \( S \) to which \( n \) is a timelike unit normal : \( n^2 = 1 \); here, \( T_{rad}^{ij} \) is given by eqn (19). The local conservation of the energy-momentum tensor guarantees that the energy-momentum components are independent of the choice of the spacelike hypersurface. The second equality results on choosing a frame in which \( n = (1, 0) \). However, unlike in the non-relativistic case, the angular integral in the relativistic case, in the chosen frame is difficult to perform, to give a closed form expression for the radiant energy-momentum. In standard textbooks [2] [3], the energy-momentum 4-vector of the radiation due to a relativistic charge is postulated by trying to ‘lift’ the Larmor formula (21) to spacetime, by expressing it in terms of 4-vectors \( a \) and \( u \), such that the Larmor formula ensues in the non-relativistic limit. This, clearly is not a satisfactory derivation, since there may be other Lorentz-covariant expressions with the same non-relativistic limit.

Using the first equality in (22), we write the radiant 4-momenta as
\[ P^i = e^2 \int_S d^3x \, R^i(n \cdot R) \left[ \frac{a^2(u \cdot R)^2 + u^2(a \cdot R)^2}{(u \cdot R)^6} \right]. \] (23)
Recall that the spacetime vector \( u \) is timelike with the squared norm \( u^2 = 1 \) for \( c = 1 \); if we set \( n = u \) without loss of generality, the above expression reduces to
\[ P^i = e^2 \int_S d^3x \, R^i \left[ \frac{a^2(u \cdot R)^2 + (a \cdot R)^2}{(u \cdot R)^5} \right] \] (24)
It follows that
\[ dP^i = e^2 \int_S d^3x \left[ -u^i \left\{ \frac{a^2}{(u \cdot R)^3} + \frac{(a \cdot R)^2}{(u \cdot R)^5} \right\} d\tau + R^i \frac{d}{d\tau} \left\{ \frac{a^2}{(u \cdot R)^3} + \frac{(a \cdot R)^2}{(u \cdot R)^5} \right\} \tau_0 d\tau \right]. \] (25)
We may, once again, without loss of generality, choose the spacelike hypersurface \( S \) to have a tangent vector given by \( a \) at the event where the normal to \( S \) is the 4-velocity \( u \). This point corresponds to the value \( \tau = \tau_0 \) which defines the retarded time. Observe, then, that the null 4-vector \( R \) has the resolution at this point
\[ R^i = (u \cdot R) u^i + \frac{(a \cdot R)}{a^2} a^i + R^i_{\perp} \] (26)
where, \( u \cdot R_{\perp} = 0 = a \cdot R_{\perp} \). It follows that the pullback to the hypersurface \( S \) can be written, choosing local Euclidean coordinates on \( S \), as,
\[ R^\alpha|_S = \left( \frac{a \cdot R}{a^2} \right) \frac{a^\alpha}{a^2}|_S + R^\alpha_{\perp}|_S \] (27)
Using these results, one can argue that the second integral in eqn (25) vanishes by symmetry, since apart from \( \mathbf{R} \), the rest of the integrand is rotationally invariant under space rotations in the hypersurface \( \mathcal{S} \). As for the first term, one can choose a frame in \( \mathcal{S} \) such that \( \mathbf{a} \cdot \mathbf{R} = -\hat{a} \cdot \hat{R} = -(-a^2)^{1/2}R_{\cos \theta} \), where \( \theta \) is the angle between the two 3-vectors. The angular integral then yields the result \( 4\pi/3 \); the radial integral can be absorbed into a mass renormalization (which we discuss in greater detail in the next section), yielding the relativistic formula \( 2 \), \( 3 \).

\[
dP^i = -\frac{2}{3} e^2 u^i a^2 d\tau = -\frac{2}{3} e^2 a^2 d\bar{x}^i
\]

While this passage from eqn (23) to (28) has a few caveats, we provide in the next section, a clearer derivation of the same formula, to justify that indeed it holds in general.

IV. ENERGY-MOMENTUM OF A RELATIVISTIC CHARGE REVISITED

A. In and Out fields

The four momentum of the radiation field can be written as

\[
P_i = P_i(A^{\text{out}}) - P_i(A^{\text{in}}) = P_i(A^{\text{out}}),
\]

since \( A_i^{\text{in}} = 0 \). Now, the radiation 4-potentials

\[
A_i = A_i^R = \int d^4x' D_R(x - x') J_i(x') = A_i^{\text{out}} + \int d^4x' D_A(x - x') J_i(x') \hat{A}_i^{\text{out}}
\]

Thus

\[
A_i^{\text{out}} = \int d^4x' J_i(x') [D_R(x - x') - D_A(x - x')] = \int D(x - x') J_i(x') d^4x'
\]

where \( D_{R(A)} \) is the retarded (resp advanced) Green’s function corresponding to the d’Alembertian operator, and \( D \) is one-half the difference between these Green’s functions. Upon Fourier transformation,

\[
\hat{A}_i^{\text{out}} = \hat{D}(k) \hat{J}_i(k) = 2\pi \epsilon(k_0) \delta(k^2) \hat{J}_i(k) \equiv \hat{a}_i(k) \delta(k^2).
\]

where \( \epsilon(k_0) \) is the sign function. To compute the radiant energy-momentum, we start with the integral

\[
t_{ij} = \int d^3x \, T_{ij}^{\text{rad}}(A^{\text{out}})
\]

It is obvious that the energy-momentum \( P_i \equiv t_{i0} \). Now, to compute \( t_{ij} \), it is convenient to evaluate first

\[
\int \partial_i A_i^{\text{out}} \partial_j A_j^{\text{out}} \, d^3x = \int \frac{d^4k d^4k'}{(2\pi)^8} k_i k'_j \hat{a}_i(k) \hat{a}_m(k') \delta(k^2) \delta(k'^2) e^{i\gamma(k_0 + k'_0)}
\]

The spatial integral is trivial, yielding

\[
\int \partial_i A_i^{\text{out}} \partial_j A_j^{\text{out}} \, d^3x = \int \frac{d^4k d^4k'}{(2\pi)^8} k_i k'_j \hat{a}_i(k) \hat{a}_m(-k') \delta((k')^2 - k_0^2) \delta((k)^2 - k_0^2) e^{i\gamma(k_0 + k'_0)}
\]

This immediately leads, upon using properties of Dirac delta functions, to perform the integral over \( k'^0 \), so that

\[
\int \partial_i A_i^{\text{out}} \partial_j A_j^{\text{out}} \, d^3x = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} |k_0| k_i k'_j \hat{a}_i(k) \hat{a}_m(-k) \delta(k^2)
\]

This implies that

\[
t_{ij} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} |k_0| k_i k'_j \hat{a}_i(k) \hat{a}_m(-k) \delta(k^2)
\]
B. Energy-momentum of relativistic charge

Going back to position space we get

\[ t_{ij} = \frac{1}{2} \int j_i(x) j_j(x') d^4x d^4x' \int \frac{k_i k_j}{|k_0|^2} e^{i k_0 (x_0 - x'_0)} \delta(k^2) \frac{d^4k}{(2\pi)^4}. \]  

(37)

This enables us to compute the four momentum of the radiation field

\[ P_i = t_{i0} = \frac{1}{2} \int j_i(x) j_j(x') d^4x d^4x' \int k_i e^{i k_0 (x_0 - x'_0)} \delta(k^2) \epsilon(k_0) \frac{d^4k}{(2\pi)^4}. \]  

(38)

Carrying out the integration over \( k \), we get

\[ P_i = \frac{1}{2} \int j_i(x) j_i(x') \partial_i D(x - x') d^4x d^4x', \]

where \( D(x - x') \) is the Green’s function:

\[ D(x - x') = \int \delta(k^2) e^{i k_0 (x_0 - x'_0)} \frac{d^4k}{(2\pi)^4}. \]

Equation 38 was stated by Coleman but the derivation was not given, which is original.

Substituting the expression [13] for the 4-current density for a point charge, we get

\[ P_i = \frac{e^2}{2} \int d\tau d\tau' u_i(\tau) u_i(\tau') \partial_i D(\bar{x}(\tau) - \bar{x}(\tau')) = \int d\tau u_i(\tau) \int u_i(\tau') \partial_i D(w) d\tau', \]

(40)

where \( \partial_i \equiv (\partial/\partial \bar{x}^i) \) and \( w \equiv ((\bar{x}(\tau) - \bar{x}(\tau'))^2)^{1/2} \). We evaluate the integral over \( \tau' \) first. Let

\[ I_{il} \equiv \int u_l(\tau') \partial_i D(w) d\tau', \]

(41)

Using

\[ \partial_i = \frac{\bar{x}_i}{w} \frac{d}{dw} \text{ and } u_l d\tau = \frac{d\bar{x}_l}{dw} dw \]

We get :

\[ I_{il} = \int_0^\infty \frac{d\bar{x}_l \bar{x}_i}{dw} D(w) dw. \]

Integrating by parts, we see that

\[ I_{il} = -\int_0^\infty dw D(w) \frac{d}{dw} \left( \frac{\bar{x}_i}{w} \frac{d\bar{x}_l}{dw} \right) \]

(42)

C. Regularization and Mass Renormalization [4]

Now, the Green’s function \( D(w) = \delta(w^2) \) and therefore, the maximal contribution to \( I_{il} \) must come from around \( w = 0 \), as it should, since the radiation emanates from a single relativistic accelerated charge. To enable a Taylor expansion around \( w = 0 \), we work with the regularized Green’s function \( D_\Lambda(w) \) defined by the requirement that \( \lim_{\Lambda \to \infty} D_\Lambda(w) = D(w) \). A particularly suitable regularized Green’s function is the Gaussian-regularized Green’s function

\[ D_\Lambda(w) = \frac{\Lambda}{\sqrt{2\pi}} \exp(-\Lambda^2 w^2). \]

(43)

Define \( x'_i = (d\bar{x}_i/dw)_{w=0} \), and likewise for the higher order derivatives, we obtain,

\[ \bar{x}_i \approx x'_i w + x''_i w^2 / 2 + x'''_i w^3 / 6 + \cdots \]

(44)
This implies that,

\[ I_\mu = \int_0^\Lambda dw \, D_\Lambda(w) \left[ x_i' x_i'' + \frac{1}{2} x_i'' x_i' + w \left( \frac{1}{3} x_i' x_i''' + x_i'' x_i'' + x_i''' x_i' \right) \right] + O(\Lambda^{-1}). \]  

(45)

Performing the integrals and evaluating the limits, we obtain,

\[ I_\mu = \left[ \Lambda (x_i'' x_i' + \frac{x_i''' x_i''}{2}) + \frac{1}{3} x_i' x_i''' + x_i'' x_i'' + x_i''' x_i' \right] + O(\frac{1}{\Lambda}). \]  

(46)

To reproduce the usual formula for the energy-momentum of a relativistic point charge, we need to convert the derivatives w.r.t. \( w \) to derivatives w.r.t. the affine parameter \( \tau \) evaluated at \( \tau = \tau_0 \). To this end, we use the relation

\[ \frac{d}{dr} \mid_{r=0} = \left( \frac{d\tau}{dr} \right) \frac{d}{d\tau} \mid_{r=0} \]  

(47)

to yield

\[ x_i' = \left( \frac{d\tau}{dr} \right) \mid_{r=0} \frac{dx_i}{d\tau} = \left( \frac{d\tau}{dr} \right) \mid_{r=0} u_i \]

\[ x_i'' = \left( \frac{d\tau}{dr} \right) \mid_{r=0}^2 a_i + \left( \frac{d^2\tau}{dr^2} \right) \mid_{r=0} u_i \]

\[ x_i''' = \left( \frac{d\tau}{dr} \right) \mid_{r=0}^3 \dot{a}_i + 3 \left( \frac{d^2\tau}{dr^2} \right) \mid_{r=0} \left( \frac{d\tau}{dr} \right) \mid_{r=0} a_i + \left( \frac{d^3\tau}{dr^3} \right) \mid_{r=0} u_i, \]

(48)

where the ‘dot’ signifies a \( \tau \) derivative. To evaluate the derivatives, we again expand \( r \) in a Taylor series in powers of \( \tau - \tau_0 \) about \( \tau \) assuming \( r=0 \) at \( \tau = \tau_0 \); to simplify the notation, we redefine \( \tau - \tau_0 \equiv \tau \), so that the expansion is about \( \tau = 0 \) in powers of \( \tau \). This leads to

\[ x_i \approx 0 + u_i(0) \tau + a_i(0) \frac{\tau^2}{2} + \dot{a}_i(0) \frac{\tau^3}{6} + \cdots \]  

(49)

Using this, we get

\[ r = (\bar{x}_c \bar{x}_c)^{\frac{1}{2}} \approx \sqrt{\tau^2 \left( 1 - \frac{\tau^2 a^2}{12} \right)} \approx \tau (1 - \frac{\tau^2 a^2}{24}) = \tau - \frac{\tau^3 a^2}{24} \]  

(50)

From here, it is straightforward to show that

\[ \frac{d\tau}{dr} \mid_{r=0} = 1; \quad \frac{d^2\tau}{dr^2} \mid_{r=0} = 0; \quad \frac{d^3\tau}{dr^3} \mid_{r=0} = -\frac{1}{4} (\bar{x}_c \bar{\bar{x}}_c) \]

Substituting in (49), one obtains,

\[ I_\mu = \left[ \Lambda (\bar{x}_c \bar{\dot{x}}_l + \frac{\bar{x}_c \ddot{x}_l}{2}) + \frac{1}{4\pi} (\bar{x}_c \dddot{x}_l + \frac{\bar{x}_c \ddot{x}_l}{3} + \bar{x}_c \dot{x}_l - \frac{\bar{x}_c \dot{x}_l}{3} \bar{x}_c \dddot{x}_c) \right] + O \left( \frac{1}{\Lambda} \right) \]  

(51)

Now, using \( a \cdot u = 0 \) and \( \tau \)-derivatives thereof

\[ P^i = \frac{e^2}{2} \int d\tau u_i I^\mu = \frac{e^2}{2} \int d\tau \left[ \Lambda a_i + \left( \frac{\dot{a}_i}{3} - \frac{2}{3} u^i a^2 \right) \right] + O \left( \frac{1}{\Lambda} \right) \]  

(52)

The first term in eqn (52) blows up in the physical limit \( \Lambda \to \infty \); this conundrum is taken care of by Mass Renormalization: one realizes that the measured mass \( m \) of the particle is not the same as the mass parameter \( m_0 \) inserted into the Lagrangian of the particle, because of the interaction of the particle with its own electromagnetic field. The physical mass differs from the ‘bare’ mass by the amount \( e^2 \Lambda / 2 \). Thus, the energy-momentum of the radiation field is equal in magnitude and opposite in sign to the energy-momentum lost by the particle. This implies that

\[ P^i = \int d\tau \, m \, a^i \]

\[ m = m_0 + \frac{e^2}{2} \Lambda \]  

(53)
It is clear that this procedure eliminates the $\Lambda$-dependent part of the radiant energy-momentum (52), so that, realizing that, of the remaining terms, the first term is a total divergence and therefore discarded, one is left with the expression,

$$dP^i = -\frac{2}{3} e^2 u^i a^2 d\tau = -\frac{2}{3} e^2 a^2 d\bar{x}^i$$ (54)

which is identical to the expression (28).

V. EQUATION OF MOTION OF CHARGE

A. Abraham-Lorentz Equation

Along its world line, the radiant charge loses energy, resulting in a deceleration, given by

$$\frac{dP_i}{d\tau} = ma_i = eF_{ij}u^j + e^2 \frac{2}{3} \left[ \dot{a}_i - 2u_ia^2 \right] ,$$ (55)

where, the first term on the rhs is the relativistic Lorentz force on the charge due to its own radiation field, the first term in the square-bracket is due to radiation backreaction, while the second is loss of energy due to radiation. Thus, the formalism in this section includes the effect of radiation backreaction on the source charge in the relativistic case, in contrast to the formulation in the previous section where this was ignored.

Recall that the charged particle follows a timelike trajectory with $u^2 = 1$ and $u \cdot a = 0$, signifying that the acceleration spacetime vector is spacelike. Notice however that there is a problem with these basic tenets, when we contract both sides of (55) with $u_i$, which immediately leads to $a^2 = 0$, upon using the relation $a^2 + u \cdot \dot{a} = 0$. Thus, eqn (55) is inconsistent with our earlier contention that the spacetime acceleration is a spacelike 4-vector, or alternatively, that the charge moves along a timelike trajectory. In other words, because of the effects of radiation on the source charge, including loss of energy and also radiation backreaction, the particle is driven to move towards superluminal propagation! This, if true, is a violation of causality, which might appear as a preacceleration, but is manifest relativistically here.

Indeed, the Abraham-Lorentz equation of motion \[5\] results immediately, if one takes the non-relativistic limit of eqn (55):

$$\frac{dP_i}{d\tau} = F_{rad} \approx \frac{2}{3} e^2 \dot{a}_i .$$ (56)

Notice that in arriving at eqn (56), no assumption regarding attributing any structure (like a sphere of vanishing radius) has been necessary, in contrast to certain recent assays \[7\]. The result is of course a non-relativistic approximation to the relativistic result (55), and inherits the consistency issues of that equation, in that it leads to a violation of causality in the form of a preacceleration \[4\].

B. Landau-Lifsitz Proposal

The key feature of the Landau-Lifsitz formulation \[2\] is the the modification of eqn (55) by extra terms so that it now reads

$$\frac{dP_i}{d\tau} = ma_i = eF_{ij}u^j + g_i ,$$ (57)

where, the radiation reaction 4-force $g_i$ is now constrained to obey the relation $g \cdot u = 0$. For this auxiliary terms have to be added to the second term of eqn (55);

$$g_i = e^2 \frac{2}{3} \dot{a}_i + \alpha u_i$$ (58)

the 4-scalar $\alpha$ is now determined by the requirement that the full $g$ must now obey the constraint $g \cdot u = 0$, yielding the final result \[2\]

$$\frac{dP_i}{d\tau} = ma_i = eF_{ij}u^j + \frac{2e^2}{3} \left[ \dot{a}_i + u_i a^2 \right] .$$ (59)
It is easy to verify that the pathology discussed in the last subsection, observed by first contracting (55) by the 4-velocity, disappears when the same procedure is repeated for eqn (59). Thus, at this relativistically covariant level, there is no prospect of any violation of causality. However, it is also clear that there is no possible manner in which the extra terms, by which equations (55) and (59) differ, can be generated by utilizing the ambiguities of mass renormalization, whereby the ‘counter-term’ added to the bare mass can be altered by terms that remain finite in the limit that the ultraviolet cutoff goes to infinity. What one needs are terms proportional to the four-velocity rather than the four acceleration, and these cannot be generated by tweaking the counterterm.

The issue of the Landau-Lifschitz proposal, thus, is one of interpretation. In a sense, the Landau-Lifschitz work evades any discussion of the physical origin of this extra term, even though it does identify this extra term as the one that is necessary to avoid the conundra mentioned in the last subsection. As discussed in [4], it is unlikely that such a term can be autonomously derived from the basic tenets of classical Maxwell electrodynamics, and only the inclusion of relativistic quantum field theoretic (quantum electrodynamics : QED) effects can truly provide a self-consistent formulation of electrodynamics of a point charge.

VI. SUMMARY AND DISCUSSION

The main purpose of this revisit of classical electrodynamics of an arbitrarily-moving point charge has been to exhibit its in-built special relativistic features, and to show how these lead to its essential results in a straightforward manner, including its inherent pathologies. Spacetime vector potentials have been accorded a central position in this formulation, and the often confusing issue of gauge invariance has been discussed within this relativistic framework, where a projection to the gauge-invariant subspace of vector potentials is shown to naturally arise from the basic equations without any need to ‘choose gauge’. A classical theory of gauge fields can thus be completely cast in terms of gauge-invariant projected potentials, without the burden of having to deal with unphysical degrees of freedom, at least in the Abelian case. Such a ‘gauge-free’ approach had already been discussed in [8], where possible non-Abelian generalizations have also been discussed.

Starting with the Lienard-Wiechert potentials for a relativistic, accelerated point charge, we have determined the corresponding field strengths and shown that they separate into a ‘radiative’ and a ‘Coulomb’ part. The radiative field strengths combine into a rather simple-looking radiant energy-momentum tensor expressed in terms of the 4-velocity and the 4-acceleration of the particle, and the spacetime distance of the charge from the observer. ‘Retarded time’ also emerges from this relativistic formulation, as it must. To the best of our knowledge, such a covariant energy-momentum tensor has not appeared in the literature. From this the radiant energy-momentum has also been derived directly, as also more rigorously from Fourier space analysis. The inclusion of radiation backreaction has been shown to lead to certain inherent lacunae of classical Maxwell electrodynamics. Even though it is clear what modifications to the theory must be made to eliminate these lacunae, what remains unknown is the absence of a clear mechanism as to how these terms may arise within a classical framework. This underlines the need to explore the QED framework in greater detail, to delineate precisely how the lacunae reappear in a semiclassical limit, and to ascertain what precisely the ameliorating features are, of the quantum field theory formulation. In other words, what one really needs to do is to recover the force equation using expectation values within the coherent states of quantum electrodynamics. What one knows with certainty is that, unlike the classical theory of a point electron, the QED treatment already exhibits improvement by showing that the counterterm to be chosen to renormalize the electron self-energy needs only to be logarithmic in the ultraviolet cutoff, rather than linear [9]. Whether this is just the ‘healing touch’ one needs, it is still not very clear. This task will be taken up in the near future.

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