Critical Edge States of Two-Dimensional Quantum Critical Magnets

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Based on large-scale quantum Monte Carlo simulations, we examine the correlations along the edges of two-dimensional semi-infinite quantum critical Heisenberg spin-1/2 systems. In particular, we consider coupled quantum spin-dimer systems at their bulk quantum critical points, including the columnar-dimer model and the plaquette-square lattice. The alignment of the edge spins strongly affects these correlations and the corresponding scaling exponents, with remarkably similar values obtained for various quantum spin-dimer systems. We furthermore observe subtle effects on the scaling behavior from perturbing the edge spins that exhibit the genuine quantum nature of these edge states. Our observations furthermore challenge recent attempts that relate the edge spin criticality to the presence of symmetry-protected topological phases in such quantum spin systems.

Quantum criticality in quantum many-body systems is a central aspect of current research in condensed matter physics [1]. In this respect, quantum spins systems in particular allow for a detailed comparison of experimental results to a quantitative computational modeling and analytical calculations of critical properties. Prominent examples are dimerized antiferromagnets, in which an explicit dimerization of the exchange couplings can be varied (e.g., by applying pressure [23]) in order to induce quantum phase transitions between quantum disordered phases and conventional antiferromagnetic order. In the absence of frustration, the quantum critical properties of such systems in d spatial dimensions are generally considered to be in accord with the universal-class class of the (d + 1)-dimensional classical Heisenberg model at its finite-temperature critical point, described by the Wilson-Fisher fixed point of the three-component $\phi^3$-theory [6]. This rationale is supported also by large-scale numerical studies of coupled spin dimer models on various two-dimensional (2D) lattices [7–12]. Within the non-linear $\sigma$-model description of quantum antiferromagnets [13–21], such an agreement with the critical $\phi^3$-theory suggests that for this purpose spin Berry-phase contributions [22] can be neglected in the effective action for 2D dimerized quantum antiferromagnets [21] (they may however give rise to additional scaling corrections from cubic terms in coupled dimer systems with reduced spatial symmetries [23]). As is well known, this is in stark contrast to the one-dimensional (1D) Heisenberg spin-1/2 chain, for which uncompensated spin Berry-phases lead to a non-vanishing topological $\theta$-term in the effective continuum action, associated with a gapless, quantum critical ground state [13–18]. Such a topological term can also emerge for a one-dimensional edge of 2D quantum spin systems: By appropriately cutting a 2D quantum antiferromagnet to a semi-infinite system, an effective 1D edge spin-1/2 system with similarly uncompensated Berry phases is generated. Such edge spins are furthermore susceptible to effective interactions induced by the coupling of the edge spins to the bulk. If the bulk system resides within the quantum disordered region, these effective interactions along the edge spins decay exponentially over a length scale set by the finite bulk correlation length. Due to the bipartite lattice structure, they respect a bipartite alignment of the edge spin chain, and thereby lead to long-distance ground state correlations as in a spin-1/2 Heisenberg chain [24]. The presence of such gapless edge states of dimerized bulk systems was furthermore found to be stable against various perturbations [25], whereas the scaling properties were found to strongly depend on the model parameters and the nature of the applied perturbation [25].

Here, we consider edge spin systems for which the bulk itself is tuned onto a quantum critical point: The long-ranged quantum critical bulk-fluctuations then dominate the effective interactions among the edge spins, which effects changes in the scaling properties of the correlations along the edge. As for bulk criticality, one may consider a comparison to surface critical phenomena in classical systems, for which several scenarios can be distinguished regarding the bulk vs. surface critical behavior [26–28]. In addition to the ordinary transition, at which the surface is critical due to the bulk transition, the surface may also order at a higher temperature scale than the bulk. Such a surface transition typically requires enhanced interactions at the surface with respect to those of the bulk, in order to compensate for the reduced coordination along the surface. At the bulk transition temperature, the ordered surface may in this case still exhibit additional singular behavior, known as the extraordinary transition. One may furthermore fine-tune the surface coupling to a multi-critical special transition, at which surface and bulk are critical simultaneously. Based on the quantum-to-classical correspondence, one would expect the edge spins of a semi-infinite quantum critical spin system to similarly exhibit genuine quantum criti-
FIG. 1. (Color online) (a) CD lattice with an edge of dangling spins shown on the bottom edge (CD-D) and non-dangling spins on the top edge (CD-N). (b) BS lattice with dangling (non-dangling) spins at the bottom square lattice. denoted BS-D and BS-N, respectively. (c) PS lattice with dangling (non-dangling) edge spins on the bottom (top) edge, for the DAF-D and DAF-N cases, respectively, and with dangling (non-dangling) edge spins on the top (bottom) edge, for the PAF-D and PAF-N cases, respectively. In all panels, the inter-dimer (intra-dimer) couplings $J$ (or $D$) are indicated by black (bold red) lines, the edge (bulk) spins by open (full) symbols, and periodic boundary conditions by open lines. (d) Phase diagram of the PS lattice with the antiferromagnetic region (AF), the dimer phase (D) for $J < J_{DAF} \approx 0.6 J_D$, and the plaquette phase (P) for $J > J_{PAF} \approx 1.1 J_D$.

We first consider the columnar-dimer (CD) lattice shown in Fig. 1(a). Its bulk quantum critical point has been located previously at $J = 0.52337(3)$ [8, 10]. Using periodic boundary conditions (PBC) along the lattice direction parallel to the dimers, we examine separately the two cases of cutting along the perpendicular direction, obtaining either an edge of dangling spins (with respect to the $J_D$ bonds), denoted CD-D in the following, or an edge of non-dangling spins, denoted CD-N, cf. Fig. 1(a). For both cases, we performed QMC simulations to measure the spin-spin correlations $\langle S_i^z S_j^z \rangle$, parallel, $C_{||}(r)$, and perpendicular, $C_{\perp}(r)$, to the row of edge spins as a function of distance $r$. In addition, we also measured the staggered susceptibility $\chi_s$ of the edge spin sub-system. The resulting data for $C_{||}(r)$ on a $L = 80$ system is shown (along with that for several other cases, discussed below) in Fig. 2. It shows $C_{||}(r)$ as a function of the conformal length (cord-distance) $\zeta(r) = \sin(\pi r/L)/L$, to account for the PBC along the edge. For both cases, we observe an algebraic decay, indicative of a quantum critical state of the edge spin system that can be quantified by the scaling $|C_{||}(r)| \propto r^{-2z-n}$, with an anomalous critical exponent $\eta_{||}$ and with $z = 1$, here and in the following. The drop of the correlation functions at large values of $\zeta(r)$, explicitly seen in the weaker-correlated non-dangling case, indicates residual finite-size effects [24].

To account for finite-size corrections, we thus estimate $\eta_{||}$ from the finite-size scaling of $C_{||}(L/2)$ vs. $L$ as $C_{||}(L/2) = (L/2)^{-\tilde{\omega}_0}(c_0 + c_1 L^{-\omega})$, including a sub-leading scaling correction (in practice, we fix $\omega = 1$ [24]), and $c_0$ and $c_1$ as non-universal fit-parameters [24]. We obtain this way the estimates $\tilde{\omega}_0 = -0.50(1)$ (CD-D) and $\tilde{\omega}_0 = 1.30(2)$ (CD-N), respectively, cf. also Tab. 1. We also observe scaling for $C_{\perp}(L/2)$, and from a corresponding fit to $C_{\perp}(L/2) = (L/2)^{-\tilde{\omega}_0 - \eta_{\perp}}(c_0 + c_1 L^{-\omega})$ obtain the estimates for $\eta_{\perp}$ provided in Tab. 1 [24]. Furthermore, from the finite-size scaling $\chi_s \propto L^{-(1+2z-2\eta_{\perp})}$, we estimate the scaling dimension $\eta_{\perp}$ of the (staggered) field along the
edge. With $z = 1$, this scaling corresponds to the standard form for classical surface critical behavior $\chi_s = c_{ns} + L^{-1+\varepsilon-2\eta_\perp}(c_0 + c_1 L^{-\omega})$ that includes an additive constant $c_{ns}$ to account for regular contributions in the non-dangling case $\chi_s$. The obtained estimates for $y_\perp$ are also listed in Tab. I.

For both edge spin configurations the critical exponents obey the above scaling relations to the precision of their estimated uncertainties. Moreover, we obtain similar values for these critical exponents for several other coupled spin-dimer systems. In particular, we examined the cases of dangling and non-dangling spins along the edge of a bilayer square (BS) lattice, cf. Fig. 1(b), which we denote as BS-D and BS-N, respectively. In the perpendicular direction, we again use PBC, such that in both cases the row of edge spins forms a periodic chain. The quantum critical point for the bilayer bulk system has previously been located at $J/J_D = 0.39651(2)$ [9]. The QMC data for the correlation function $C_{1\parallel}(r)$ for both cases are also shown in Fig. 2 and the extracted critical exponents from $C_{1\parallel}(L/2)$ as well as those from $C_{1\perp}(L/2)$, and $\chi_s$ are provided in Tab. I [24]. Within the estimated uncertainties, these values are in accord with those of the CD-D and CD-N case, respectively. We note that by reducing in an alternating fashion every second bond from the top and bottom square lattice layer, the BS model can be connected to the CD model [24].

Finally, we examined the plaquette-square (PS) lattice [7, 35], cf. Fig. 1(c). This model has been analyzed in the context of edge spin criticality in a recent publication [12], and we comment on the conclusions drawn by this work further below. Here, we consider PBC in the horizontal, and open boundary conditions in the vertical direction, cf. Fig. 1(c). As a function of the coupling ratio $J/J_D$, this system shows two quantum critical points, at $J = J_{DAF} = 0.603520(10)J_D$ and for $J = J_{PAF} = 1.064382(13)J_D$. They separate the antiferromagnetic phase obtained for $J \approx J_D$ from the quantum-disordered dimer-singlet (plaquette-singlet) dominated phase for $J < J_{DAF}$ ($J < J_{PAF}$), respectively (we consider $J, J_D > 0$). Noting the difference between the two quantum-disordered phases with respect to the pattern of the predominant singlet formation, we distinguish the following four different edge spin configurations: (i) For $J < J_{DAF}$, the system is quantum-disordered due to predominant singlet-formation along the $J_D$ dimer-bonds, and thus the systems exhibits dangling spins if we cut through a row of dimers, to obtain the bottom boundary in Fig. 1(c). At $J = J_{DAF}$, we hence denote this edge spin configuration as DAF-D. (ii) If for $J < J_{DAF}$ we instead consider the spins at the top edge in Fig. 1(c), we obtain non-dangling spins, and for $J = J_{DAF}$ we denote this edge spin configuration as DAF-N. (iii) For $J > J_{PAF}$, the system is instead quantum-disordered due to predominant four-site singlet-formation on the plaquettes formed by the $J$-bonds, and the system thus exhibits dangling spins at the top edge in Fig. 1(c). At $J = J_{PAF}$, we thus denote this edge spin configuration as PAF-D. (iv) If for $J > J_{PAF}$ we instead consider spins at the bottom edge in Fig. 1(c), we obtain non-dangling spins and for $J = J_{PAF}$ we denote this edge spin configuration as PAF-N. For the PS lattice, we can thus realize both the case of dangling and the non-dangling edge spins at either quantum critical point by considering appropriate edges. The QMC data for the correlation function $C_{1\parallel}(r)$ for both cases are also shown in Fig. 2. Performing again a finite-size scaling analysis of the correlation functions $C_{1\parallel}(L/2)$ and $C_{1\perp}(L/2)$ as well as the staggered susceptibility $\chi_s$ of the edge spins [24], we obtain scaling exponents that essentially correspond to those for the other considered cases, cf. Tab. I.

At the considered quantum critical points, which all belong to the 3D $O(3)$ universality class, the edge spins exhibit critical scaling exponents that apparently belong to two different classes, depending on whether the edge spins are dangling or not with respect to the predominant singlet formation in the neighboring quantum disordered phase. For the non-dangling case, the obtained critical exponent $y_\parallel$ is similar to the values $y_\parallel = 0.813(2)$ and $y_\parallel = 0.802(1)$ obtained from Monte Carlo [30] and conformal bootstrap [38] studies of the ordinary surface transition in the 3D $O(3)$ model, respectively. This is in accord with the expectation that the critical behavior for non-dangling edge spins is induced by the quantum critical fluctuations of the bulk system. The estimated exponents are comparable even to the values $y_\parallel = 1.307, y_\perp = 0.664$ and $y_\parallel = 0.846$, obtained for the ordinary surface transition from the second-order $\epsilon$-expansion of the $O(n)$-symmetric vector model ($\epsilon = 4 - d$) [28, 36], after a dégagé evaluation at $\epsilon = 1$ and $n = 3$ [37]. Regarding the dangling case, one observes a similar closeness of the critical exponents to the values $\eta_\parallel = -0.445, \eta_\perp = -0.212$.

### Table I. Critical exponents $\eta_\parallel, \eta_\perp$, and $y_\parallel$ for the edge spin configurations of 2D coupled spin-dimer systems in Fig. 1.

| Configuration | $\eta_\parallel$ | $\eta_\perp$ | $y_\parallel$ |
|---------------|-----------------|-------------|-------------|
| BS-N          | 1.32(8)         | 0.69(3)     | 0.87(2)     |
| CD-N          | 1.30(2)         | 0.69(4)     | 0.84(1)     |
| DAF-N         | 1.29(6)         | 0.65(3)     | 0.832(8)    |
| PAF-N         | 1.33(4)         | 0.65(2)     | 0.82(2)     |
| BS-D          | -0.49(2)        | -0.25(1)    | 1.733(3)    |
| CD-D          | -0.50(1)        | -0.27(1)    | 1.740(4)    |
| DAF-D         | -0.50(1)        | -0.228(5)   | 1.728(2)    |
| PAF-D         | -0.517(4)       | -0.252(5)   | 1.742(1)    |
and $\eta_1 = 1.723$ obtained from 2nd-order $\epsilon$-expansion for the special transition \cite{25,26}, evaluated at $\epsilon = 1$ and $n = 3$ \cite{37}; however, there is still some spread among the values in Tab. \ref{tab:1} and these estimates \cite{29}. Moreover, as mentioned above, the 3D O(3) model does not feature such a special transition, whereas the $\epsilon$-expansion is blind to this restriction \cite{28}. To assess if this apparent similarity of the critical exponents extends beyond a mere coincidence or whether further fine-tuning would be needed, requires, e.g., an $\epsilon$-expansion in the presence of a $\theta$-term from the dangling edge spins, to be compared to the $\epsilon$-expansion for the classical special transition, evaluated at $n = 3$. We are not aware of such an argument.

In Ref. \cite{12}, the observation that the scaling exponents for the DAF-D configuration differ from the ordinary transition is argued to be a consequence of symmetry-protected-topological (SPT) order \cite{40,41} in the ground state for $J < J_{DAF}$ in the form of an Affleck-Kennedy-Lieb-Tasaki state \cite{42} — in contrast to the trivial (non-SPT) nature of, e.g., the plaquette phase or the quantum-disordered phase of the CD model. However, we obtain such non-ordinary exponents also in the PAF-D configuration at $J = J_{PAF}$ as well as for the critical CD and the BS model with dangling spins. The non-ordinary edge criticality is thus not a characteristic feature of SPT phases but results from the dangling edge spin arrangement. Moreover, it is readily seen to be possible to adiabatically connect the quantum-disordered phase of the PS model for $J < J_{DAF}$ to the quantum-disordered regime of the BS lattice model without breaking any symmetries of the PS model \cite{24}.

In order to probe the stability of the scaling exponents with respect to variations of the edge spin couplings, we introduced modifications to the local environment of the dangling edge spins. We find that depending on the specific setting, different scenarios are realized. For example, enhancing the exchange couplings along the edge in the DAF-D configuration by a relative factor $\kappa$ does apparently not significantly alter the critical properties of the edge spins, cf. Fig. 3. Its main effect is a uniform overall reduction of the correlations, such as if the increased couplings quench the magnetic moments on the edge sites by forming effective spin-1/2 moments on the $\wedge$-shaped outer triangles. On the other hand, coupling each dangling spin in the DAF-D configuration to an additional spin, as shown in the inset of Fig. 4, strongly effects the scaling of the original edge spin correlations: while for small values of the additional coupling $\kappa J_D$, the edge spin correlations increase, they are eventually suppressed at large values of $\kappa$, as shown in Fig. 4. This non-monotonous behavior can be understood as follows: The additional couplings initially enhance antiferromagnetic tendencies, so that weak values of $\kappa$ lead to more extended correlations. On the other hand, a further increase of $\kappa$ leads to a predominant formation of local singlets on the new bonds, which eventually suppress the long-distance correlations. This perturbation, which allows us to tune from the DAF-D configuration ($\kappa = 0$) to the DAF-N configuration ($\kappa = 1$), thus exhibits explicitly the genuine quantum nature of these critical edge states.

It may also be intriguing to examine the possibility of a true phase transition within the edge states along this line. Based on the above findings, an analytical approach to the edge spin correlations in quantum critical bulk systems would be desirable, in particular in order to rationalize the apparent similarity of the exponents for the case of dangling spins with a naive extrapolation of the $\epsilon$-expansion for the special surface transition.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{(Color online) Lateral correlations $C_{ij}(r)$ as a function of the conformal distance $\zeta(r)$ for the DAF-D configuration for different values of the edge coupling enhancement $\kappa$, as shown in the inset.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{(Color online) Lateral correlations $C_{ij}(r)$ as a function of the conformal distance $\zeta(r)$ for the DAF-D configuration for different values of the coupling $\kappa J_D$ to additional spins, shown in the inset by semi-filled circles.}
\end{figure}
Furthermore, we thank the IT Center at RWTH Aachen University and the JSC Jülich for access to computing time through JARA-HPC. During the finalization of our investigation, we became aware of a related study [37], where consistent numerical findings for the CD model are reported.

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[24] See the supplemental material for (i) the edge spin correlations for quantum-disordered dimerized bulk systems, (ii) the examination of finite-size effects in $C_{\parallel}(r)$ as a function of $r$, (iii) the finite-size scaling plots of $C_{\parallel}(L/2)$, $C_{\perp}(L/2)$, and $\chi_s$, and details of the finite-size scaling analysis, (iv) the connection between the bilayer square lattice and columnar dimer models, (v) the connection between the dimerized phases of the bilayer square lattice and plaquette-square lattice models.
SUPPLEMENTAL MATERIAL

Edge correlations for a quantum-disordered bulk

As an example of the effective spin-1/2 Heisenberg chain-like correlations that emerge at the edge of a quantum disordered bulk system, we show in Fig. S1 the lateral correlations $C_\parallel(r)$ (circles) as a function of the distance $r$ between the edge spins for (i) the dangling edge spin configuration CD-D of the columnar dimer lattice (cf. the left inset), and (ii) the dangling edge spin configuration DAF-D of the plaquette-square lattice (cf. the right inset) for a coupling ratio of $J/J_D = 0.2$. For comparison, the corresponding correlation function of a spin-1/2 Heisenberg chain is also shown in this figure.

Figure S2 shows the lateral correlations $C_\parallel(r)$ as a function of the conformal distance $\zeta(r)$ along the edge spins of the CD-N configuration shown in Fig. 1 for different values of the linear system size $L$.

**FIG. S1.** Lateral correlations $C_\parallel(r)$ (circles) as a function of the cord distance $\zeta(r)$ between the edge spins for the two edge spin configurations shown in the insets: the dangling edge spin configuration CD-D of the columnar dimer lattice (left inset), and the configuration dangling edge spin DAF-D of the plaquette-square lattice (right inset), for a coupling ratio of $J/J_D = 0.2$, based on simulations with with $L = 80$, at a temperature of $T = 0.00125 J_D$, and $T = 0.0003125 J_D$, respectively. For comparison, the results of a spin-1/2 Heisenberg chain with 80 sites is also shown (squares).

**FIG. S2.** Lateral correlations $C_\parallel(r)$ as a function of the conformal distance $\zeta(r)$ along the edge spins of the CD-N configuration shown in Fig. 1 for different values of the linear system size $L$. The temperature is scaled as $T = J_D/(2L)$ in all cases.
**Finite-size scaling analysis and scaling plots**

In Fig. S3, we provide the QMC data for the lateral correlations $C_r(L/2)$, the transverse correlations $C_\perp(L/2)$, and the staggered susceptibility $\chi_s$, as functions of $L$ for the different edge spin configurations shown in Fig. 1, along with fits corresponding to the finite-size analysis, based on the finite-size scaling ansatz given in main text. Details concerning the range of system sizes from $L_{\text{min}}$ to $L_{\text{max}}$ that was accessible for the fitting procedure and whether inclusion of a scaling correction $c_1L^{-1}$ and a nonsingular contribution $c_{n_s}$ was required, are provided for each specific case in Tab. II. In particular, a nonsingular contribution $c_{n_s}$ is required for extracting $\eta_h$ from $\chi_s$ in the non-dangling cases, because the exponent of $\chi_s$ is negative and thus the background is the dominating term, in contrast to the dangling cases, where $\chi_s$ diverges and the background term would be a sub-leading correction, compared to the leading scaling correction $\propto L^{-1}$. The $L^{-1}$ correction term was included whenever a truncation of the interval from varying $L_{\text{min}}$ did not allow to compatibly fit the data to a simple power law. Also provided in Tab. II is the formula $N(L)$ for the number of lattice sites as a function of $L$ for the various lattices.

| Exponent | Config. | $L_{\text{min}}$ | $L_{\text{max}}$ | $N(L)$ | $c_1L^{-1}$ incl. | $c_{n_s}$ incl. |
|----------|---------|-----------------|-----------------|--------|-------------------|-----------------|
| $\eta_\parallel$ | BS-D | 36 | 140 | $L^2$ | Yes | No |
| CD-D | 30 | 140 | $L^2$ | Yes | No |
| DAF-D | 18 | 88 | $4L^2$ | Yes | No |
| PAF-D | 18 | 88 | $4L^2$ | Yes | No |
| BS-N | 36 | 140 | $L^2$ | No | No |
| CD-N | 30 | 140 | $L^2$ | No | No |
| DAF-N | 18 | 88 | $4L^2$ | No | No |
| PAF-N | 18 | 88 | $4L^2$ | No | No |
| $\eta_\perp$ | BS-D | 36 | 140 | $L^2$ | Yes | No |
| CD-D | 38 | 140 | $L^2$ | No | No |
| DAF-D | 18 | 88 | $4L^2$ | No | No |
| PAF-D | 18 | 80 | $4L^2$ | No | No |
| BS-N | 36 | 140 | $L^2$ | Yes | No |
| CD-N | 38 | 140 | $L^2$ | Yes | No |
| DAF-N | 18 | 80 | $4L^2$ | No | No |
| PAF-N | 18 | 88 | $4L^2$ | No | No |
| $\eta_h$ | BS-D | 24 | 140 | $L^2$ | Yes | No |
| CD-D | 22 | 140 | $L^2$ | Yes | No |
| DAF-D | 14 | 88 | $4L^2$ | Yes | No |
| PAF-D | 14 | 88 | $4L^2$ | Yes | No |
| BS-N | 24 | 140 | $L^2$ | No | Yes |
| CD-N | 22 | 140 | $L^2$ | No | Yes |
| DAF-N | 14 | 88 | $4L^2$ | No | Yes |
| PAF-N | 14 | 88 | $4L^2$ | No | Yes |

**TABLE II.** Details of the fitting range and fitting formula for the critical exponents $\eta_\parallel$, $\eta_\perp$, and $\eta_h$ for the different edge spin configurations of 2D coupled spin-dimer systems.
We furthermore monitored the dependence of the extracted exponents on the minimum lattice size \( L_{\text{min}} \) included in the fitting procedure. The dependence of the various critical exponents on \( L_{\text{min}} \) is shown in Fig. S4. We observe no significant \( L_{\text{min}} \)-dependence apart from strongly increasing uncertainties for the larger values of \( L_{\text{min}} \), reflecting the fact that data from fewer system sizes are then available for the fitting procedure.

Finally, we also considered a finite-size scaling ansatz, wherein the scaling correction \( \propto L^{-1} \) is replaced by a more general form \( \propto L^{-\omega} \), with a free exponent \( \omega \). For example, \( \omega \approx 0.8 \) corresponds to the correction-to-scaling exponent of the classical O(3) model at the 3D bulk phase transition [34]. As shown in Fig. S5, we observe only mild trends in the \( \omega \)-dependence for some of the exponents. Anticipating the statistical uncertainties on the accessible system sizes, no significant qualitative changes result for the estimated exponents. Based on the above considerations, we thus consider the exponents given in Tab. I to provide reliable estimates for the current purpose of distinguishing the two different cases of dangling vs. non-dangling edge spin configurations.
Connecting the bilayer and columnar dimer models

In Fig. S6 we demonstrate, how the columnar dimer lattice is obtained from the bilayer square lattice upon reducing intralayer couplings in an alternating way.

Connecting the plaquette-square and bilayer models

Figure S7 illustrates, how the bilayer lattice model is obtained from the plaquette-square lattice model upon increasing the additional exchange couplings $J'$ from 0 to the value of $J$. During this process neither the internal SU(2) symmetry nor the spatial symmetries of the original plaquette square lattice are broken. The dimer bonds $J_D$ thereby become the perpendicular inter-layer bonds. From explicit QMC calculations for different values of $J/J_D$ inside the dimerized phase, one indeed obtains no indication for a quantum phase transition during this increase of $J'$, neither from the ground state energy nor the fidelity susceptibility. To relate to the more conventional presentation of the bilayer model, one may consider shifting all the blue (green) bonds and plaquettes up (down) to form the upper (lower) square lattice.