Existence results for equilibrium problem

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January 3, 2017

Abstract

In this work, we introduce the notion of regularization of bifunctions in a similar way as the well-known convex, quasiconvex and lower semicontinuous regularizations due to Crouzeix. We show that the Equilibrium Problems associated to bifunctions and their regularizations are equivalent in the sense of having the same solution set. Also, we present new existence results of solutions for Equilibrium Problems.

Keywords: Equilibrium Problems, Convex Feasibility problems, Monotonicity generalized, Convexity generalized, Coercivity conditions, Upper sign property.

MSC (2000): 47J20, 49J35, 54C60, 90C37

1 Introduction

Given a real Banach space $X$, a nonempty subset $K$ of $X$ and a bifunction $f : K \times K \to \mathbb{R}$. The Equilibrium Problem, (EP) for short, is defined as follows:

Find $x \in K$ such that $f(x, y) \geq 0$ for all $y \in K$.

(EP)

Equilibrium Problems have been extensively studied in recent years (e.g., [4–6, 8, 9, 16, 17, 19–21]). Particularly, it is well known that many problems such as variational inequality problems, fixed-point problems, Nash equilibrium problems and optimization problems, among others, can be reformulated as equilibrium problems. (see for instance [6, 15, 21, 22]).

A recurrent subject in the analysis of this problem is the connection between the solution sets of (EP) and the solution set of the following problem:

Find $x \in K$ such that $f(y, x) \leq 0$ for all $y \in K$.

(CFP)

This can be seen as a dual formulation of (EP) and it corresponds to a particular case of the convex feasibility problem (cfr. [12,13]).

It was proved in [21] that if $f$ is upper semicontinuous in the first argument, convex and lower semicontinuous in the second one and it vanishes on the diagonal $K \times K$, then every solution of (CFP) is a solution of (EP), and moreover both solution sets trivially coincide under pseudomonotonicity of $f$.

In order to establish the nonemptiness of the solution set of (CFP) and the inclusion of this set in solution set of (EP) in [5], Bianchi and Pini introduced the concept of local convex feasibility problem and the upper sign continuity for bifunctions as an adaptation of the set-valued map introduced in [18], by Hadjisavvas. They adapted the existence result for variational inequalities developed by Aussel and Hadjisavvas in [2]. Basically, they proved that every solution of (CFP) is a local solution of (CFP) and all local solution of (CFP) is a solution of (EP). Following the same way, in [9], Castellani and Giuli introduced the concept of upper sign property for bifunction as a local property which is weaker than the upper sign continuity and they extend the result obtained by Bianchi and Pini.

Our aim in this paper is to provide sufficient conditions for the existence of solutions under weak assumptions on the bifunction and some coercivity conditions. We introduce, in Section 3, the regularization
of a bifunctor analogously of regularization of functions introduced in [11] by Crouzeix and we study the properties of such regularization. In section 4, we establish that the equilibrium problems associated to a bifunctor and its regularization are equivalent in the sense Castellani and Giuli. (cf. [8]). We provide, in Section 5, sufficient conditions for the existence of solutions for (EP).

2 Preliminary definitions and notations

Let $X$ be a real topological vector space, and let $A \subset X$. We denote by $\overline{A}$, $\text{co}(A)$ and $\text{epi}(A)$ the smallest closed set, convex set and closed convex set (in the sense of inclusion), respectively, which contains $A$. These sets are called the closure, convex hull and the closed convex hull, respectively. Given $h : X \to \mathbb{R}$, where $\mathbb{R} = [-\infty, +\infty]$, we consider the following sets:

- $\text{dom}(h) = \{ x \in X : h(x) < +\infty \}$;
- $\text{epi}(h) = \{(x, \lambda) \in X \times \mathbb{R} : h(x) \leq \lambda \}$;
- for each $\lambda \in \mathbb{R}$, $S_{\lambda}(h) = \{ x \in X : h(x) \leq \lambda \}$.

The sets $\text{dom}(h)$, $\text{epi}(h)$ and $S_{\lambda}(h)$ are called the domain, the epigraph and the lower level set of $h$ with respect to $\lambda$, respectively.

Considering the convention $+\infty - \infty = -\infty + \infty = +\infty$, recall that a function $h : X \to \mathbb{R}$ is said to be:

- convex if, for all $x, y \in X$ and all $t \in [0, 1]$, $h(tx) \leq th(x) + (1 - t)h(y)$,
- quasiconvex if, for all $x, y \in X$ and all $t \in [0, 1]$, $h(tx) \leq \max \{ h(x), h(y) \}$,
- semistrictly quasiconvex if $h$ is quasiconvex and for all $x, y \in X$
  \[ h(x) < h(y) \implies h(tx) < h(y), \forall t \in [0, 1], \]

where $x_t = tx + (1-t)y$. It is clear that a convex function is quasiconvex and that the domain of a quasiconvex function is convex.

We recall that $h$ is said to be lower semicontinuous (in short lsc) at $x_0 \in X$ if for all $\lambda < h(x_0)$, there exists a neighborhood $V$ of $x_0$ such that for all $x \in V$, it holds that $h(x) > \lambda$. Also, $h$ is said to be lower semicontinuous if it is lower semicontinuous at any $x_0 \in X$. A function $h$ is said to be upper semicontinuous if $-h$ is lower semicontinuous.

Crouzeix defined in [11] the regularizations of a function $h : X \to \mathbb{R}$ as:

- $h_+(x) = \inf \{ \lambda \in \mathbb{R} : (x, \lambda) \in \text{epi}(h) \}$,
- $h_-(x) = \inf \{ \lambda \in \mathbb{R} : (x, \lambda) \in \text{co}(\text{epi}(h)) \}$,
- $h_{\overline{\text{co}}}(x) = \inf \{ \lambda \in \mathbb{R} : (x, \lambda) \in \overline{\text{co}}(\text{epi}(h)) \}$,
- $h_{\text{q}}(x) = \inf \{ \lambda \in \mathbb{R} : x \in \text{co}(S_{\lambda}(h)) \}$ and
- $h_{\overline{\text{q}}}(x) = \inf \{ \lambda \in \mathbb{R} : x \in \overline{\text{co}}(S_{\lambda}(h)) \}$.

It results that $h_+, h_-, h_{\overline{\text{co}}}, h_{\text{q}}$ and $h_{\overline{\text{q}}}$ are the greatest lsc regularization (lsc regularization), the greatest convex function (convex regularization), the greatest lsc convex function (lsc convex regularization), the greatest quasiconvex function (quasiconvex regularization) and the greatest lsc quasiconvex function (lsc quasiconvex regularization) which are majorized by $h$, respectively. It is clear that $\text{epi}(h_+) = \text{epi}(h)$, $\text{epi}(h_{\overline{\text{co}}}) = \overline{\text{co}}(\text{epi}(f))$, $\text{epi}(h_{\overline{\text{q}}}) = \text{epi}(h_{\text{q}})$ and $h_{\overline{\text{q}}} \leq h_{\overline{\text{co}}} \leq h_{\text{q}} \leq h$.

\footnote{As is usual in convex analysis, we consider functions defined on the whole space; if it is not the case for some function $h$, we set $h(x) = +\infty$ for $x$ not in the domain of $h$.}
We say that a regularization \( h_i \) of \( h \) is well defined when \( h_i(x) \in \mathbb{R} \) for all \( x \in \text{dom}(h) \), where \( i \in \{c, \overline{c}, s, \overline{s}\} \).

We recall some different definitions of generalized monotonicity (the ones we will be use from now on) for some bifunction \( f : X \times X \rightarrow \mathbb{R} \):

- **Quasimonotone** if, for all \( x, y \in X \), \( f(x, y) > 0 \Rightarrow f(y, x) \leq 0 \).
- **Properly quasimonotone** if, for all \( x_1, x_2, \ldots, x_n \in X \), and all \( x \in \text{co}(\{x_1, x_2, \ldots, x_n\}) \), there exists \( i \in \{1, 2, \ldots, n\} \) such that \( f(x_i, x) \leq 0 \).
- **Pseudomonotone** if, for all \( x, y \in X \), \( f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \).
- **Monotone** if, for all \( x, y \in X \), \( f(x, y) + f(y, x) \leq 0 \).

Clearly, monotonicity implies pseudomonotonicity and this in turn implies quasimonotonicity. Nevertheless no relationship exists between quasimonotonicity and proper quasimonotonicity of bifunctions (e.g. [4]).

On the other hand, all the bifunctions \( f \) satisfying some property of generalized monotonicity mentioned above satisfy \( f(x, x) \leq 0 \) for all \( x \in X \).

Let \( K \) be a convex subset of \( X \). A bifunction \( f : K \times K \rightarrow \mathbb{R} \) is said to have the

- **local upper sign property** at \( x \in K \) if there exists \( r > 0 \) such that for every \( y \in K \cap B(x, r) \) the following implication holds:

\[
(f(x_t, x) \leq 0, \ \forall t \in [0, 1[) \Rightarrow f(x, y) \geq 0,
\]

where \( x_t = (1 - t)x + ty \).

For example, any positive bifunction has the upper sign property. Additionally, any bifunction such that \( f(x, x) \geq 0 \), \( f(\cdot, y) \) is upper semicontinuous and \( f(x, \cdot) \) is semistrictly quasiconvex, for all \( x, y \in K \), has the upper sign property. Clearly, every bifunction with the upper sign property has the local upper sign property. Moreover, in [1], Aussel et al. showed that these concepts are equivalent under the following condition:

\[
f(x, y) < 0 \text{ and } f(x, x) = 0 \Rightarrow f(x, x_t) < 0 \ \forall t \in [0, 1[,
\]

where \( x_t = tx + (1 - t)y \). In particular, this holds when \( f(x, \cdot) \) is a semistrictly quasiconvex bifunction.

### 3 Regularization of a bifunction

From now on, \( X \) stands for a real Banach space and \( f : K \times K \rightarrow \mathbb{R} \) for a bifunction defined on a nonempty and closed convex subset \( K \) of \( X \). For each \( x \in K \), we denote by \( f_s(x, \cdot) \), \( f_c(x, \cdot) \), \( f_l(x, \cdot) \), \( f_q(x, \cdot) \) and \( f_{\overline{q}}(x, \cdot) \) the lower semicontinuous, convex, convex and lower semicontinuous, quasiconvex and quasiconvex and lower semicontinuous regularizations of the function \( f(x, \cdot) \), respectively.

Clearly, for every \( x, y \in K \) holds that:

\[
f_c(x, y) \leq f_q(x, y) \leq f(x, y) \quad \text{and} \quad f_{\overline{q}}(x, y) \leq f_{\overline{s}}(x, y) \leq f_s(x, y) \leq f(x, y).
\]

In general, \( f_i(x, y) \) can be \(-\infty\), where \( i \in \{s, c, q, \overline{c}, \overline{q}\} \).

We define the following families of bifunctions depending on \( K \):

- **\( \mathcal{C}(K) = \{ f : K \times K \rightarrow \mathbb{R} : f_c(x, \cdot) \text{ is well defined for all } x \in K \} \)**
- **\( \mathcal{Q}(K) = \{ f : K \times K \rightarrow \mathbb{R} : f_q(x, \cdot) \text{ is well defined for all } x \in K \} \)**


• \( \mathcal{C}(K) = \{ f : K \times K \to \mathbb{R} : f(x, \cdot) \text{ is well defined for all } x \in K \} \).
• \( \mathcal{Q}(K) = \{ f : K \times K \to \mathbb{R} : f_{\mathcal{Q}}(x, \cdot) \text{ is well defined for all } x \in K \} \).
• \( \mathcal{S}(K) = \{ f : K \times K \to \mathbb{R} : f_s(x, \cdot) \text{ is well defined for all } x \in K \} \).

It is clear from (4) that:
\[
\mathcal{C}(K) \subset \mathcal{Q}(K) \quad \text{and} \quad \mathcal{C}(K) \subset \mathcal{S}(K).
\]

The following example shows that the previous inclusions are strict in general.

**Example 3.1.** Let \( K = \mathbb{R} \) and let \( f_1, f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) two bifunctions defined as \( f_1(x, y) = y^3 - x \) for all \((x, y) \in \mathbb{R}^2\) and
\[
f_2(x, y) = \begin{cases} -\ln(|y|), & y \neq 0, \\ 0, & y = 0. \end{cases}
\]

For each \( x \in \mathbb{R} \) we have the following graphs:

Clearly, \( f_1 \in \mathcal{Q}(K) \setminus \mathcal{C}(K) \) and \( f_2 \in \mathcal{S}(K) \setminus \mathcal{Q}(K) \).

The following result shows that under compactness of \( K \) the three families are the same.

**Proposition 3.1.** Let \( K \subset X \) be a nonempty and compact convex set. Then
\[
\overline{\mathcal{C}}(K) = \overline{\mathcal{Q}}(K) = \overline{\mathcal{S}}(K).
\]

**Proof.** In view of the inclusions (5), it is enough to show that \( \mathcal{S}(K) \subset \overline{\mathcal{C}}(K) \). Let \( f \in \mathcal{S}(K) \) and let \( x \in K \).

In view of the compactness of \( K \) and the lower semicontinuity of \( f_s(x, \cdot) \), there exists \( x_0 \in K \) such that
\[
\operatorname{epi}(f(x, \cdot)) \subset K \times [f_s(x, x_0), +\infty[.
\]

It follows that \( \operatorname{epi}(f_{\mathcal{Q}}(x, \cdot)) \subset K \times [f_s(x, x_0), +\infty[ \), concluding that \( f \in \overline{\mathcal{C}}(K) \).

Castellani et al. [8] considered the family of bifunctions \( f \) such that \( f(x, x) = 0 \) for all \( x \in K \) and satisfying the following condition:
\[
\forall x \in K, \exists x^* \in X^*, \exists a \in \mathbb{R} : \forall y \in K, \langle x^*, y \rangle + a \leq f(x, y),
\]
in a finite dimensional space. The following result shows that the family \( \overline{\mathcal{C}}(K) \) is also characterized by the condition (6) in an infinite dimensional space.

**Proposition 3.2.** The family \( \overline{\mathcal{C}}(K) \) is the set of bifunctions \( f \) satisfying the condition (6).
Proof. Let \( f \in \mathcal{C}(K) \) and let \( x \in K \). Without loss of generality we can assume that \( f(x, \cdot) \) is a convex and lower semicontinuous function, for all \( x \in K \). From [7, Theorem I.7] we have that for each \((x_0, \lambda) \in K \times \mathbb{R} \setminus \text{epi}(f(x, \cdot))\) there exists \((x_0^*, \lambda^*) \in X^* \times \mathbb{R}\) such that

\[
(x_0^*, x_0) + \lambda^* \lambda < \langle x_0^*, y \rangle + \lambda^* f(x, y) \quad \text{for all} \quad y \in K.
\]

By substituting \( y = x_0 \) into (7) we obtain \( \lambda^* > 0 \). Thus,

\[
f(x, y) \geq \left( \frac{x_0^*}{\lambda^*}, y \right) + \lambda + \left( \frac{x_0^*}{\lambda^*}, x_0 \right) \quad \text{for all} \quad y \in K.
\]

Therefore, the bifunction \( f \) satisfies the condition (6) with \( x^* = \frac{x_0^*}{\lambda^*} \) and \( a = \lambda + \left( \frac{x_0^*}{\lambda^*}, x_0 \right) \).

Conversely, let \( f : K \times K \to \mathbb{R} \) be a bifunction satisfying (6). In view of the convexity and lower semicontinuity of the function \( h_x : K \to \mathbb{R} \), defined as \( h_x(y) = \langle x^*, y \rangle + a \), the bifunction \( f \in \mathcal{C}(K) \).

It is natural to ask whether some kind of generalized monotonicity of a bifunction is shared with its regularizations. The following lemma is a key step towards this result.

**Lemma 3.1.** Let \( f, g : K \times K \to \mathbb{R} \) be two bifunctions such that

\[
g(x, y) \leq f(x, y) \quad \text{for all} \quad x, y \in K.
\]

If \( f \) is either monotone, pseudomonotone, quasimonotone or properly quasimonotone bifunction, then \( g \) is a bifunction of the same type of monotonicity.

**Proof.** In the case \( f \) is monotone [respectively pseudomonotone or quasimonotone], the inequalities

\[
g(x, y) \leq f(x, y) \quad \text{and} \quad g(y, x) \leq f(y, x) \quad \text{for all} \quad x, y \in K
\]

imply the monotonicity [respectively pseudomonotonicity or quasimonotonicity] of \( g \).

Now, assume that \( f \) is properly quasimonotone. Let \( x_1, x_2, \ldots, x_m \in K \) and let \( x \in \text{co}(x_1, x_2, \ldots, x_m) \). Then, there exists \( j_0 \in \{1, 2, \ldots, m\} \) such that

\[
f(x_{j_0}, x) = \min_{j \in \{1, 2, \ldots, m\}} f(x_j, x) \leq 0,
\]

and consequently by (8)

\[
\min_{j \in \{1, 2, \ldots, m\}} g(x_j, x) \leq g(x_{j_0}, x) \leq f(x_{j_0}, x) \leq 0,
\]

which shows that \( g \) is a properly quasimonotone bifunction. \( \square \)

Now, as a direct consequence of inequalities (4) and Proposition 3.1 we have the following corollary.

**Theorem 3.1.** If a bifunction is either monotone, pseudomonotone, quasimonotone or properly quasimonotone, then all its regularizations have the same type of generalized monotonicity.

**Remark 3.1.** The converse of the last result is not true in general. We consider for instance the bifunction \( f : K \times K \to \mathbb{R} \) defined as:

\[
f(x, y) = \begin{cases} 1, & (x, y) = (1, 0) \lor (x, y) = (0, 1) \\ 0, & (x, y) \neq (1, 0) \land (x, y) \neq (0, 1) \end{cases}
\]

where \( K = [0, 1] \). The following pictures represent the graphs of the functions \( f(x, \cdot) \):

[Graphs showing the behavior of the function]
The bifunction $f$ is not monotone, because $f(1, 0) + f(0, 1) = 2$. On the other hand, for all $x, y \in K$ and for all $i \in \{\pi, \bar{q}, s\}$ we have $f_i(x, y) = 0$. Therefore, $f_i$ is mononote for all $i \in \{\pi, \bar{q}, s\}$.

The following example shows that some bifunction can have the upper sign property without none of its regularizations having it.

**Example 3.2.** Let $K = [0, 1]$ and let $f : K \times K \to \mathbb{R}$ be a bifunction defined as

$$
\begin{align*}
f(x, y) = \begin{cases} 
0, & x, y \in K \cap \{x, y\} \subset \mathbb{Q} \lor \{x, y\} \cap \mathbb{Q} = \emptyset \\
1, & y \in K \setminus \mathbb{Q} \land \{x, y\} \cap \mathbb{Q} = \emptyset \\
-1, & x \in K \setminus \mathbb{Q} \land y \in \mathbb{Q} \cap K,
\end{cases}
\end{align*}
$$

where $\mathbb{Q}$ is the set of rational numbers. Let $x, y \in K$ such that

$$
f(x, x) \leq 0 \text{ for all } t \in [0, 1]. \quad (9)
$$

If $x \in \mathbb{Q}$, then $f(x, y) \geq 0$. However, if $x \notin \mathbb{Q}$, then

$$
f(x_t, x) = \begin{cases} 
0, & x_t \notin \mathbb{Q}, \\
1, & x_t \in \mathbb{Q}.
\end{cases}
$$

From (9) we have $x_t \notin \mathbb{Q}$, for all $t \in [0, 1]$. It follows that $x = y$ and consequently $f(x, y) = 0$. Therefore, the bifunction $f$ has the upper sign property.

On the other hand, for all $i \in \{\pi, \bar{q}, s\}$ it holds that

$$
f_i(x, y) = \begin{cases} 
0, & x \in \mathbb{Q} \cap K \\
-1, & x \in K \setminus \mathbb{Q}
\end{cases}
$$

and the regularization $f_i$ does not have the upper sign property on $K$, because taking $x = \sqrt{2}/2$ and $y = 0$, we have $f(x, y) = -1$ and $f(t\sqrt{2}/2, 0) \leq 0$, for all $t \in [0, 1]$.

In contrast to our result on generalized monotonicity, where from the inequality (8) the property is transmitted from the bifunction $f$ to $g$, the upper sign property is transmitted from $g$ to $f$.

**Lemma 3.2.** Let $f, g : K \times K \to \mathbb{R}$ be two bifunctions satisfying (8). If $g$ has the upper sign property on $K$, then $f$ also has this property.

**Proof.** Let $x, y \in K$ such that $f(x_t, x) \leq 0$ for all $t \in [0, 1]$. Since $g(x, y) \leq f(x, y)$ for all $x, y \in K$, then $g(x_t, x) \leq 0$ for all $t \in [0, 1]$. The upper sign property of $g$ implies that $g(x, y) \geq 0$, and therefore that $f(x, y) \geq 0$.

**Remark 3.2.** With the hypothesis of Proposition 3.2, if $g$ has local upper sign property, then $f$ also has the local upper sign property.

From Proposition 3.2 and the inequalities on (4) we have the following result.

**Theorem 3.2.** If some regularization of a bifunction has the (local) upper sign property, then the bifunction also has the (local) upper sign property.

The following result states that the quasiconvex regularization of a bifunction is upper semicontinuous on the second variable, provided that the bifunction also is.

**Proposition 3.3.** Let $f \in \mathcal{Q}(K)$. If $f(\cdot, y)$ is an upper semicontinuous function for every $y \in K$, then $f_q$ is also upper semicontinuous with respect to first argument.

**Proof.** For every $\varepsilon > 0$ and $\lambda \in \mathbb{R}$ such that $f_q(x, y) < \lambda < f_q(x, y) + \varepsilon$. Then $y \in \text{co}(S(\lambda(f(x, \cdot))))$ and this implies that there exists $y_1, \ldots, y_m \in S(\lambda(f(x, \cdot)))$ and $t_1, \ldots, t_m \in [0, 1]$ such that $\sum_{k=1}^m t_k = 1$ and $y = \sum_{k=1}^m t_k y_k$. Since $f(\cdot, y_k)$ is upper semicontinuous, there exists a neighborhood $V$ of $x$ such that for all $x' \in V$ and all $k \in \{1, \ldots, m\}$ hold $f(x', y_k) < f_q(x, y) + \varepsilon$. Therefore, $f_q(x', y_k) \leq f(x', y_k) < f_q(x, y) + \varepsilon$ for each $k \in \{1, \ldots, m\}$ and for all $x' \in V$. So, by the quasiconvexity of $f_q(x, \cdot)$, we have that $f_q(x', y) < f_q(x, y) + \varepsilon$ for all $x' \in V$.\qed
Remark 3.3. The previous result is also true for the convex regularizations.

We define the following subfamily of $Q(K)$:

$$SQ(K) = \{ f \in Q(K) : f_q(x, \cdot) \text{ is semistrictly quasiconvex for all } x \in K \}$$

Clearly, $C(K) \subset SQ(K) \subset Q(K)$.

As we have mentioned earlier, the local sign property and the sign property are equivalent under condition (3), a condition that holds for $f_q$ with $f \in SQ(K)$, but no for $f$ itself.

Example 3.3. Let $K = [0, 2]$ and let $f : K \times K \to \mathbb{R}$ be a bifunction defined by:

$$f(x, y) = \begin{cases} 
  y - 2, & y \neq 1 \\
  0, & y = 1
\end{cases}$$

It is not difficult to see that $f_q(x, y) = y - 2$ for all $x, y \in K$. Therefore, $f \in SQ(K)$. However $f$ does not satisfy the condition (3). Indeed, taking $y_1 = 0$ and $y_2 = 2$ we have $f(x, y_1) < 0$ and $f(x, y_2) = 0$, but $f(x, 1) = 0$.

Proposition 3.4. Let $f \in SQ(K)$ be a bifunction such that $f$ is upper semicontinuous with respect to first variable. Then $f_q$ has the upper sign property if, and only if, $f_q(x, x) \geq 0$, for all $x \in K$.

Proof. It is clear that if $f_q$ has the upper sign property then $f_q(x, x) \geq 0$, for all $x \in K$.

The converse, let $x, y \in K$ such that $f_q(x_t, x) \leq 0$, for every $x_t = tx + (1 - t)y$, $t \in [0, 1]$. If there exists $t \in [0, 1]$ such that $f_q(x_t, y) < 0$ then by semistrictly quasiconvexity of $f_q(x_t, \cdot)$ we have $f_q(x_t, x_t) < 0$, which is a contradiction. So, $f_q(x_t, y) \geq 0$ for all $t \in [0, 1]$. By Proposition 3.3 $f_q$ is upper semicontinuous with respect to first variable and this imply $f(x, y) \geq 0$.

4 Equilibrium Problems vs Convex Feasibility Problems

We denote by EP($f, K$) and CFP($f, K$) the solution sets of (EP) and (CFP), respectively.

Lemma 4.1. Let $f, g : K \times K \to \mathbb{R}$ be two bifunctions satisfying (3). Then EP($g, K$) $\subset$ EP($f, K$) and CFP($f, K$) $\subset$ CFP($g, K$).

Proof. It follows directly from definitions of (EP) and (CFP).

Clearly, (3) and Lemma 4.1 imply that if $f \in C(K)$ then

$$\begin{align*}
\text{EP}(f_{\sigma}, K) &\subset \text{EP}(f_{\tau}, K) \subset \text{EP}(f_{s}, K) \subset \text{EP}(f, K) \\
\text{CFP}(f_{s}, K) &\subset \text{CFP}(f_{s}, K) \subset \text{CFP}(f_{\eta}, K) \subset \text{CFP}(f, K).
\end{align*}$$

The following result says that if a bifunction admits convex and lower semicontinuous regularization, then the solution sets of equilibrium problem associated this bifunction and its regularizations are the same.

Proposition 4.1. If $f \in C(K)$ then $\text{EP}(f_{s}, K) = \text{EP}(f, K)$ for all $i \in \{ c, q, s, \tau, \eta \}$.

Proof. By (10) it is enough to show that EP($f, K$) $\subset$ EP($f_{\tau}, K$). For each $x \in \text{EP}(f, K)$ we have that epi($f(x, \cdot)$) $\subset K \times [0, +\infty]$ and consequently epi($f_{\tau}(x, \cdot)$) $\subset K \times [0, +\infty]$, i.e. $x \in \text{EP}(f_{\tau}, K)$.

Remark 4.1. In Proposition 4.1

- If $f \in C(K)$ then $\text{EP}(f_{q}, K) = \text{EP}(f, K)$, for all $i \in \{ q, s, \tau, \eta \}$.
- If $f \in S(K)$ then $\text{EP}(f_{s}, K) = \text{EP}(f, K)$.
- If $f \in C(K)$ then $\text{EP}(f_{s}, K) = \text{EP}(f, K)$, for all $i \in \{ c, q \}$.
- If $f \in Q(K)$ then $\text{EP}(f_{q}, K) = \text{EP}(f, K)$.
The inclusions in (11) and Proposition 4.1 motivate the following question: Do a bifunction and its regularizations have the same solution set for the convex feasibility problem? The following example gives a negative answer.

Example 4.1. Let \( K = [0, 1] \) and let \( f : K \times K \to \mathbb{R} \) defined as

\[
f(x, y) = \begin{cases} 
  y, & y \in [0, 1] \\
  0, & y = 1
\end{cases}
\]

For every \( x \in K \), the graph of \( f(x, \cdot) \) is

\[
\text{\[0, 1\]}
\]

It is not difficult to see that \( \text{CFP}(f, K) = \{0, 1\} \). On the other hand, \( f(x, y) = 0 \) for all \( x, y \in K \) and this implies \( \text{CFP}(f_s, K) = [0, 1] \).

Bianchi and Pini [5] considered a weaker concept of solution for \( \text{CFP} \), similar to the one proposed by Aussel and Hadjisavvas [2] in the setting of variational inequalities. They define the set of the local solutions

\[
\text{CFP}_{\text{local}}(f, K) = \{ x \in K : \exists r > 0 \text{ s.t. } f(y, x) \leq 0 \forall y \in K \cap B(x, r) \}.
\]

Clearly, \( \text{CFP}(f, K) \subset \text{CFP}_{\text{local}}(f, K) \).

In the following result, part (i) is from [9, Theorem 1], and part (ii) is an adaptation of [1, Proposition 3.1].

Proposition 4.2. Let \( f : K \times K \to \mathbb{R} \) be a bifunction.

(i) If \( f \) has the local upper sign property and satisfies (3) then \( \text{CFP}_{\text{local}}(f, K) \subset \text{EP}(f, K) \).

(ii) If \( f \) has the upper sign property then \( \text{CFP}(f, K) \subset \text{EP}(f, K) \).

The last result shows that, in order to obtain a solution of the equilibrium problem, it is enough to obtain a solution for the convex feasibility problem under the upper sign property, or under the local upper sign property and (3). As a consequence of Proposition 4.2, Proposition 3.1 and Remark 4.1 we have the following result.

Proposition 4.3. Let \( f \in \mathcal{S}(K) \).

(i) If \( f_s \) has the local upper sign property and satisfies (3) then \( \text{CFP}_{\text{local}}(f_s, K) \subset \text{EP}(f, K) \).

(ii) If \( f_s \) has the upper sign property then \( \text{CFP}(f_s, K) \subset \text{EP}(f, K) \).

(iii) If \( f \in \mathcal{SQ}(K) \) and \( f_q \) has the upper sign property then \( \text{CFP}_{\text{local}}(f_q, K) \subset \text{EP}(f, K) \).

The following examples show that the nonemptiness of the solution set of an equilibrium problem cannot be directly deduced from Proposition 4.2.

Example 4.2. Let \( K = \mathbb{R} \) and let \( f : K \times K \to \mathbb{R} \) be a bifunction defined by:

\[
f(x, y) = \begin{cases} 
  0, & x, y \in \mathbb{Q} \cap K \vee x = y \\
  1, & \text{otherwise}
\end{cases}
\]

The bifunction \( f \) has the upper sign property on \( K \), and it is not difficult to show that \( \text{CFP}_{\text{local}}(f, K) = 0 \).

Moreover, \( f_s(x, y) = 0 \) for all \( x, y \in K \), which implies that \( f_s \) is properly quasimonotone, it has the upper sign property on \( K \) and \( \text{CFP}_{\text{local}}(f_s, K) = K \). Therefore, by Proposition 4.2 (i) we have \( \text{EP}(f, K) \) is nonempty.
Example 4.3. Let \( K = [0, +\infty] \) and let \( f : K \times K \to \mathbb{R} \) defined by:

\[
f(x, y) = \begin{cases} 
0, & y = 0 \\
1/y, & y \neq 0.
\end{cases}
\]

It is not difficult to see that \( \text{CFP}_{\text{local}}(f, K) = \text{CFP}_{\text{local}}(f, K) = \emptyset \). On the other hand, \( f_q(x, y) = 0 \) for all \( x, y \in K \). Thus, \( f \in \mathcal{SQ}(K) \) and \( f_q \) has the upper sign property. Moreover, \( \text{CFP}_{\text{local}}(f_q, K) = K \) and by Proposition 4.3 (iii) we have that \( \text{EP}(f, K) \) is nonempty.

5 Existence results

In 1972, Ky Fan proved his famous minimax inequality (cf. [14, Theorem 1]).

Theorem 5.1 (Ky Fan, 1972). Let \( V \) be a real Hausdorff topological vector space and \( K \) a nonempty compact convex subset of \( V \). If \( f : K \times K \to \mathbb{R} \) satisfies:

(i) \( f(\cdot, y) : K \to \mathbb{R} \) is upper semicontinuous for each \( y \in K \),

(ii) \( f(x, \cdot) : K \to \mathbb{R} \) is quasiconvex for each \( x \in K \),

then there exists a point \( x^* \in K \) such that

\[
\inf_{y \in K} f(x^*, y) \geq \inf_{w \in K} f(w, w).
\]

The following result is a consequence of Theorem 5.1 and Proposition 4.1.

Corollary 5.1.1. Let \( K \) be a nonempty compact convex subset of \( X \) and \( f \in \mathcal{Q}(K) \). If the following assumptions hold:

(i) \( f_q(\cdot, y) \) is upper semicontinuous for all \( y \in K \),

(ii) \( f_q(x, x) = 0 \) for all \( x \in K \).

Then \( \text{EP}(f, K) \) is a nonempty set.

Remark 5.1. If \( f(\cdot, y) \) is upper semicontinuous for all \( y \in K \), by Proposition 3.3 the assumption (i) in Corollary 5.1.1 is satisfied.

Example 5.1. Let \( K = [0, 2] \) and let \( f : K \times K \to \mathbb{R} \) be a bifunction defined by

\[
f(x, y) = \begin{cases} 
0, & y \in [0, x] \\
-y + x, & y \in [x, 2] \land x \neq 1 \\
y - 1, & y \in [1, 2] \land x = 1 \\
1, & x = y = 1.
\end{cases}
\]

The following pictures represent the graphs of the functions \( f(x, \cdot) \):

Clearly, \( f(x, \cdot) \) is quasiconvex and continuous for all \( x \neq 1 \), but \( f(1, \cdot) \) is not quasiconvex. Nevertheless, \( f_q(1, \cdot) \) is quasiconvex and continuous. Therefore, \( f \neq f_q \) and moreover \( f_q(x, x) = 0 \) for all \( x \in K \). Applying Corollary 5.1.1 \( \text{EP}(f, K) \) is nonempty. Notice that the nonemptiness of \( \text{EP}(f, K) \) cannot be directly deduced from Theorem 5.1.
The following result is a consequence of [1 Proposition 2.1].

**Proposition 5.1.** Let $K$ be a weakly compact subset of $X$ and $f$ be a properly quasimonotone bifunction such that for every $x \in K$ the set $\{y \in K : f(x, y) \leq 0\}$ is weakly closed. Then $\text{CFP}(f, K)$ is nonempty.

Since every quasiconvex and lower semicontinuous function is lower semicontinuous in the weak topology, the application of Proposition 5.1 gives us the following result.

**Corollary 5.1.1.** Let $K$ be a weakly compact subset of $X$ and $f \in \mathcal{C}(K)$. If $f_\mathcal{C}$ is properly quasimonotone then $\text{EP}(f_\mathcal{C}, K)$ is nonempty. Moreover, if $f_\mathcal{C}$ has the upper sign property then $\text{EP}(f, K)$ is nonempty.

**Proof.** Clearly, by Proposition 5.1 $\text{CFP}(f, K)$ is a nonempty set. Since $f_\mathcal{C}$ has the upper sign property, Proposition 4.2(ii) implies that $\text{EP}(f_\mathcal{C}, K) \neq \emptyset$. The result follows from Remark 4.1.

Another consequence of Proposition 5.1 is the following one.

**Corollary 5.1.2.** Let $K$ be a nonempty compact convex subset of $X$ and $f \in \mathcal{S}(K)$ such that $f_\mathcal{S}$ is properly quasimonotone and it has the upper sign property. Then $\text{EP}(f, K)$ is nonempty.

**Proof.** By Proposition 5.1 we have that $\text{CFP}(f_\mathcal{S}, K)$ is nonempty. The result follows from Proposition 4.3(ii).

For each $n \in \mathbb{N}$, let $K_n = \{x \in K : \|x\| \leq n\}$ and $K_n^\circ = \{x \in K : \|x\| < n\}$.

**Proposition 5.2.** Suppose that for every $x, y_1, y_2 \in K$, the following implication holds:
\[
[f(x, y_1) \leq 0 \text{ and } f(x, y_2) < 0] \Rightarrow f(x, y_t) < 0, \forall t \in [0, 1],
\]
where $y_t = ty_1 + (1-t)y_2$. If for some $n \in \mathbb{N}$ and some $x \in \text{EP}(f, K_n)$ there exists $y \in K_n^\circ$ such that $f(x, y) \leq 0$, then $x \in \text{EP}(f, K)$.

**Proof.** Let $x \in \text{EP}(f, K_n)$ and $w \in K \setminus K_n$, if $f(x, w) < 0$ then by (12) $f(x, y_t) < 0$ for all $t \in [0, 1]$, where $y_t = ty_1 + (1-t)w$. On the other hand, since $y \in K_n^\circ$ there exists $t_0 \in [0, 1]$ such that $y_{t_0} \in K_n$, which is a contradiction.

**Remark 5.2.** Condition (12) is a technical assumption introduced by Farajzadeh and Zafarani in [19] in order to show the inclusion of $\text{CFP}_{\text{local}}(f, K)$ in $\text{EP}(f, K)$. Clearly, the semistrict quasiconvexity of $f(x, \cdot)$ guarantees the condition (12). So, in the Proposition 5.2 we can change condition (12) by $f \in \mathcal{SQ}(K)$ and use the Remark 4.1 to guarantee the nonemptiness of $\text{EP}(f, K)$.

As a direct consequence of previous result we have the following corollary.

**Corollary 5.2.1.** Suppose (12) holds and $f$ has the upper sign property (or $f \in \mathcal{SQ}(K)$ and $f_\mathcal{S}$ has the upper sign property). If for some $n \in \mathbb{N}$ and some $x \in \text{CFP}(f, K_n)$ there exists $y \in K_n^\circ$ such that $f(x, y) \leq 0$, then $x \in \text{EP}(f, K)$.

**Proof.** Is a direct consequence of Proposition 5.2 and Proposition 4.2(ii).

**Remark 5.3.** The Corollary 5.2.1 is an extension of [19 Lemma 4.1].

The following coercivity conditions were studied in [20,21] and [9]:

(C1) For every sequence $\{x_n\} \subset K \setminus \{0\}$ satisfying $\lim_{n \to +\infty} \|x_n\| = \infty$, there exists $u \in K$ and $n_0 \in \mathbb{N}$ such that $f(x_n, u) \leq 0$ for all $n \geq n_0$.

(C2) For every sequence $\{x_n\} \subset K \setminus \{0\}$ satisfying $\lim_{n \to +\infty} \|x_n\| = \infty$, there exists $n_0 \in \mathbb{N}$ and $u_{n_0} \in K$ such that $\|u_{n_0}\| < \|x_{n_0}\|$ and $f(x_{n_0}, u_{n_0}) \leq 0$.

(C3) For every sequence $\{x_n\} \subset K \setminus \{0\}$ such that $\lim_{n \to \infty} \|x_n\| = \infty$ and such that the sequence $\{|x_n|^{-1}x_n\}$ converges weakly to a point $x \in X$ such that $y+x \in K$ and $f(y, x+y) \leq 0$ for all $y \in K$, there exists another sequence $\{u_n\} \subset K$ such that, for $n$ large enough, $\|u_n\| < \|x_n\|$ and $f(x_n, u_n) \leq 0$. 

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It is not difficult to verify that (C1) implies (C2), which in turn implies (C3).

Clearly, if \( f \in \overline{C}(K) \) then (H) implies that \( f_i \) satisfies the coercivity conditions (C1) or (C2) for all \( i \in \{ s, c, \tau, q \} \) provided that \( f \) satisfies the same condition too.

We define the following subfamily of \( Q(K) \):

\[
\overline{SQ}(K) = \{ f \in \overline{C}(K) : f_\tau(x, \cdot) \text{ is semistrictly quasiconvex for all } x \in K \}
\]

Clearly, \( \overline{C}(K) \subset \overline{SQ}(K) \subset \overline{C}(K) \).

The following result extends the sufficient part of [20] Theorem 4.4 (i), and also [9] Theorem 5 with \( \mu = 0 \).

**Proposition 5.3.** Suppose \( X \) is a reflexive Banach space and \( K \) is closed convex. If \( f \in \overline{SQ}(K) \) is such that \( f_\tau \) is quasimonotone, it has the upper sign property on \( K \) and satisfies the coercivity condition (C3) then \( EP(f, K) \) is nonempty.

**Proof.** If \( f_\tau \) is not properly quasimonotone, then by [9] Theorem 3 and Corollary 1 \( EP(f_\tau, K) \) is nonempty and the result follows from Remark 5.1. Now, suppose that \( f_\tau \) is properly quasimonotone. Since \( K_n \) is a weakly compact set, Corollary 5.1 implies that \( CFP(f_\tau, K_n) \) is nonempty. If there exists \( x_n \in CFP(f_\tau, K_n) \) such that \( \|x_n\| < n \) then Corollary 5.2 implies that \( x_n \in EP(f, K) \). Thus, we may assume that \( \|x_n\| = n \) for all \( n \in \mathbb{N} \). Since the unit ball of \( X \) is weakly compact, without loss of generality we may assume that \( \{x_n/n\} \) converges weakly to some \( x \in X \). Fix \( y \in K \) and \( m > \|y\| \). For \( n \geq m, y \in K_n \). Since \( x_n \in CFP(f_\tau, K_n) \) we have that

\[
f_\tau(y, x_n) \leq 0.
\]

Let \( z_n = (1/n)x_n + (1 - 1/n)y \in K_n \). Then

\[
f_\tau(y, z_n) \leq 0
\]

Clearly, \( \{z_n\} \) converges weakly to \( x + y \in K \). Hence, the lower semicontinuity of \( f_\tau(y, \cdot) \) implies that

\[
f_\tau(y, x + y) \leq 0.
\]

Therefore, coercivity condition (C3) implies that there exists a sequence \( \{u_n\} \subset K \) such that \( \|u_n\| < \|x_n\| \) and \( f(x_n, u_n) \leq 0 \). From Corollary 5.2, we have that \( EP(f_\tau, K) \) is nonempty. The result follows from Remark 4.1.

**Proposition 5.4.** Suppose \( X \) is a finite dimensional space and \( K \) is closed convex subset of \( X \). If \( f \in \overline{SQ}(K) \) is such that \( f_q(\cdot, y) \) is upper semicontinuous for all \( y \in K, f_q(x, x) = 0 \) for all \( x \in K \), and \( f_q \) satisfies the coercivity condition (C2) then \( EP(f, K) \) is nonempty.

**Proof.** Since \( K_n \) is a compact set, then Corollary 5.1 implies that \( EP(f_q, K_n) \) is nonempty. If there exists \( n \in \mathbb{N} \) such that \( \|x_n\| < n \), then Proposition 5.2 with \( x = y = x_n \), implies that \( x_n \in EP(f_q, K) \) and the result follows from Remark 4.1. If \( \|x_n\| = n \) for all \( n \in \mathbb{N} \), condition (C2) implies that there exists \( n_0 \in \mathbb{N} \) and \( u \in K \) such that \( u \in K_{n_0}^c \) and \( f_q(x_n, u) \leq 0 \). Using Proposition 5.2 with \( x = x_n \) and \( y = u \), we have that \( x_n \in EP(f_q, K) \) and the result follows again from Remark 4.1.

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