Novel cloning machine with supplementary information

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Abstract

Probabilistic cloning was first proposed by Duan and Guo. Then Pati established a novel cloning machine (NCM) for copying superposition of multiple clones simultaneously. In this paper, we deal with the novel cloning machine with supplementary information (NCMSI). For the case of cloning two states, we demonstrate that the optimal efficiency of the NCMSI in which the original party and the supplementary party can perform quantum communication equals that achieved by a two-step cloning protocol wherein classical communication is only allowed between the original and the supplementary parties. From this equivalence it follows that NCMSI may increase the success probabilities for copying. Also, an upper bound on the unambiguous discrimination of two nonorthogonal pure product states is derived. Our investigation generalizes and completes the results in the literature.

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1. Introduction

Over the past decade, quantum computation and quantum information has been given extensively attention due to the more power in essence than classical computation [1]. While the characteristics of quantum principles such as quantum superposition and entanglement essentially enhance the power of quantum information processing, the unitarity and linearity of quantum physics also lead to some impossibilities—the no-cloning theorem [2,3,4] and the no-deleting principle [5]. The linearity of quantum theory makes an unknown quantum state unable to be perfectly copied [2,3] and deleted [5], and two nonorthogonal states are not allowed to be precisely cloned and deleted as a result of the unitarity [4,6,7], that is, for nonorthogonal pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, no physical operation in quantum mechanics can exactly achieve the transformation $|\psi_i\rangle \rightarrow |\psi_i\rangle|\psi_i\rangle$ ($i = 1, 2$). This has been generalized to mixed states and entangled states [8,9]. Remarkably, these restrictions provide a valuable resource in quantum cryptography [10], because they forbid an eavesdropper to gain information on the distributed secret key without producing errors.

Recently Jozsa [11] and Horodecki et al. [12] further clarified the no-cloning theorem and the no-deleting principle from the viewpoint of conservation of quantum information, and in light of this point of view two copies of any quantum state contain more information than one copy; in contrast, two classical states have only the same information as any one of the two states. Specifically, Jozsa [11] verified that if supplementary information, say a mixed state $\rho_i$ is supplemented, then there is a physical operation

$$|\psi_i\rangle \otimes \rho_i \rightarrow |\psi_i\rangle|\psi_i\rangle$$

if and only if there exists physical operation

$$\rho_i \rightarrow |\psi_i\rangle,$$

where by physical operation we mean a completely positive trace-preserving map, and $\{|\psi_i\rangle\}$ is any given finite set of pure states containing no orthogonal pairs of states. This result implies that the supplementary information must be provided as the copy $|\psi_i\rangle$ itself, since the second copy can always be generated from the supplementary information, independently
of the original copy. Therefore, this result may show the “permanence” of quantum information; that is, to get a copy of quantum state, the state must already exist somewhere. Notwithstanding, cloning quantum states with a limited degree of success has been proved always possibly. A natural issue is that if the supplementary information is added in a novel cloning machine (NCM) by Pati [13], then whether the optimal efficiency of the machine may be increased. This problem will be positively addressed in this paper.

Let us briefly recall the pioneers’ works regarding quantum cloning, and the more detailed references may be referred to Fiurášek [14] therein. In general, there are two kinds of cloners. One is the universal quantum-copying machine (UQCM) firstly introduced by Bužek and Hillery [15], and this kind of machines is deterministic and does not need any information about the states to be cloned, so it is state-independent. To be more precise, the UQCM obtained by Bužek and Hillery [15] is described by the following unitary transformation $U$:

$$|0\rangle_a |Q\rangle_x \rightarrow \frac{\sqrt{2}}{3} |00\rangle_{ab} |\uparrow\rangle + \frac{1}{3} |+\rangle_{ab} |\downarrow\rangle,$$

$$|1\rangle_a |Q\rangle_x \rightarrow \frac{\sqrt{2}}{3} |11\rangle_{ab} |\downarrow\rangle + \frac{1}{3} |+\rangle_{ab} |\uparrow\rangle,$$

(3)

where $|Q\rangle_x$ is the state of the copying device (auxiliary state), $|\uparrow\rangle$ and $|\downarrow\rangle$ are an orthonormal basis states, and $|+\rangle_{ab} = \frac{1}{\sqrt{2}} (|10\rangle_{ab} + |01\rangle_{ab})$. The “universal” means that for any pure state $|s\rangle_a = \alpha |0\rangle_a + \beta |1\rangle_a$ to be cloned, the distances $D_a = Tr[\rho^{(out)}_a - \rho^{(id)}_a]^2$, and, $D_{ab} = Tr[\rho^{(out)}_{ab} - \rho^{(id)}_{ab}]^2$ are independent of $\alpha$, that is to say, the efficiency of cloning under these measures does not rely on the original state $|s\rangle_a$, where by denoting $|\Psi^{(out)}_{abx}\rangle = U(|s\rangle_a |Q\rangle_x)$, then density operator $\rho^{(out)}_{abx} = |\Psi^{(out)}_{abx}\rangle \langle \Psi^{(out)}_{abx}|$, the real output in the system $ab$ is $\rho^{(out)}_{ab} = Tr_x[\rho^{(out)}_{abx}]$, the real output in system $a$ is $\rho^{(out)}_{a} = Tr_b[\rho^{(out)}_{abx}]$; by contrast, the ideal output in the system $ab$ is $\rho^{(id)}_{ab} = \rho^{(id)}_a \otimes \rho^{(id)}_b$, where $\rho^{(id)}_a = |s\rangle_a \langle s|$, $\rho^{(id)}_b = |s\rangle_b \langle s|$, in which $|s\rangle_b = \alpha |0\rangle + \beta |1\rangle$. (A direct calculation shows that $D_a = \frac{1}{18}$ for the above UQCM.) To date many authors have deeply dealt with this kind of cloning devices (for example, [16-26]). By the way, recently the universal quantum deleting machines have also been considered [27,28].

The other kind of cloners is state-dependent, since it needs some information from the states to be cloned. Furthermore, this kind of cloning machines may be divided into three fashions of cloning: First is probabilistic cloning machines proposed firstly by Duan and Guo.
[29,30], and then by Chefles and Barnett [31] and Pati [13], and Han et al. [32], that can
clone linearly independent states with nonzero probabilities. Duan and Guo’s machine can
be stated as follows: For states secretly chosen from the set \(S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\}\), there
is unitary operator \(U\) such that

\[
U(|\psi_i\rangle|\Sigma|P_0\rangle) = \sqrt{r_i}|\psi_i\rangle|\psi_i\rangle|P_0\rangle + \sum_{j=1}^{n} c_{ij}|\Phi_{AB}^{(j)}\rangle|P_j\rangle, \quad (i = 1, 2, \ldots, n),
\]

if and only if states \(|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\) are linearly independent, where \(r_i\) is the probability
of success for copying \(|\psi_i\rangle\), \(|\Sigma\rangle\) is a blank state, \(|P_0\rangle, |P_1\rangle, \ldots, |P_n\rangle\) are probe states and
orthonormal, and \(|\Phi_{AB}^{(j)}\rangle\) are \(n\) normalized states of the composite system \(ABP\). Therefore,
a general unitary evolution together with a post-selection by measurement results, yields
faithful copies of the input states with certain probabilities. Indeed, a more general unitary
evolution of the system \(ABP\) can be decomposed as the form:

\[
U(|\psi_i\rangle|\Sigma|P_0\rangle) = \sqrt{r_i}|\psi_i\rangle|\psi_i\rangle|P^{(i)}\rangle + \sqrt{1-r_i}|\Phi_{ABP}^{(i)}\rangle, \quad (i = 1, 2, \ldots, n),
\]

that can be stated as: The states \(|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\) can be probabilistically cloned with
efficiencies \(r_i\) if and only if the matrix \(X^{(1)} - \sqrt{T}X^{(2)}_P \sqrt{T}^+\) is positive semidefinite, where
matrices \(X^{(1)} = [|\psi_i\rangle\langle\psi_j|]\), \(\sqrt{T} = \text{diag}(r_1, r_2, \ldots, r_n)\), \(X^{(2)}_P = [|\psi_i\rangle\langle\psi_j|^2|P^{(i)}\rangle\langle P^{(j)}|]\); \(|P_0\rangle, |P^{(i)}\rangle\)
are normalized states of the probe \(P\) (not generally orthogonal) and \(|\Phi_{ABP}^{(i)}\rangle\) are \(n\) normalized
states of the composite system \(ABP\) (not generally orthogonal, but it is required that
\(\langle P^{(i)}|\Phi_{ABP}^{(j)}\rangle = 0\) for any \(i, j = 1, 2, \ldots, n\). The success probabilities \(r_i\) and \(r_j\) satisfy that

\[
\frac{r_i + r_j}{2} \leq \frac{1}{1 + |\langle \psi_i | \psi_j \rangle|},
\]

where \(|\langle \psi_i | \psi_j \rangle| \neq 1\) is assumed.

Second is deterministic cloners first investigated by Bruß et al. [33] and then by Chefles
and Barnett [34]. Such a deterministic cloning machine is described by the unitary operator
\(U\):  

\[
U(|\psi_i\rangle^\otimes M|\Sigma\rangle^\otimes (N-M)) = |\alpha_i\rangle, \quad (i = 1, 2, \ldots, n),
\]

where \(|\Sigma\rangle\) is a blank state and \(|\alpha_i\rangle\) are the output states cloned. According to [33] the global
fidelity \(F\) of this cloning device can be expressed as:

\[
F = \sum_{i=1}^{n} p_i |\langle \alpha_i | \psi_i \rangle^\otimes N |^2,
\]
where $p_i$ is the priori probability of the state $|\psi_i\rangle^\otimes M$ chosen. From [33,34] it follows that the optimal output state $|\alpha_i\rangle$ must lie in the subspace spanned by the exact clones $|\psi_1\rangle^\otimes N$, $|\psi_2\rangle^\otimes N$, $\ldots$, $|\psi_n\rangle^\otimes N$.

Third is hybrid cloner studied by Chefles and Barnett [32], that combines deterministic cloner with probabilistic one. The basic process of cloning is that firstly the initial states, say $|\psi_1\rangle$ and $|\psi_2\rangle$, are separated with certain probability $P_S$, i.e., a non-unitary transformation makes with certain probability $P_S$ the states $|\psi_1\rangle$ and $|\psi_2\rangle$ become states $|\phi_1\rangle$ and $|\phi_2\rangle$ [31], such that

$$|\langle \phi_1 | \phi_2 \rangle| \leq |\langle \psi_1 | \psi_2 \rangle|.$$  \hspace{1cm} (10)

Such a transformation is implemented by some linear operators $A_{Sk}$ and $A_{Fk}$ satisfying

$$\sum_k (A_{Sk}^\dagger A_{Sk} + A_{Fk}^\dagger A_{Fk}) = \hat{1},$$  \hspace{1cm} (11)

where $\hat{1}$ is identity operator, and

$$A_{Sk}|\psi_i^1\rangle = s_{ki}|\phi_i\rangle,$$

$$A_{Fk}|\psi_i^1\rangle = f_{ki}|\phi_i\rangle,$$

for $i = 1, 2$, where

$$P_S = \sum_{i=1}^{2} \frac{1}{2} \sum_k |s_{ki}|^2 \leq \frac{1 - |\langle \psi_1^1 | \psi_2^1 \rangle|}{1 - |\langle \phi_1 | \phi_2 \rangle|}.$$  \hspace{1cm} (12)

Whereafter, by utilizing deterministic cloner for copying the states $|\phi_1\rangle$ and $|\phi_2\rangle$, the states $|\psi_1^2\rangle$ and $|\psi_2^2\rangle$ are determinately obtained. Therefore, such a cloning scheme obtain the appropriate states $|\psi_i^{m+1}\rangle$ for copying $|\psi_i^1\rangle$ ($i = 1, 2$). (Notably, these quantum cloning machines stated above have been applied to many quantum cryptographic protocols [35-37].)

The probabilistic machine by Duan and Guo [29,30] can be thought of as $|\psi\rangle \rightarrow |\psi\rangle^\otimes 2$ cloning. A question addressed by many authors is that given a quantum state, whether it is possible for a device to produce $|\psi\rangle \rightarrow |\psi\rangle^\otimes 2$, $|\psi\rangle \rightarrow |\psi\rangle^\otimes 3$, $\ldots$, $|\psi\rangle \rightarrow |\psi\rangle^\otimes (m+1)$, in a deterministic or probabilistic way. Motivated by this proposal and the idea of probabilistic cloning, Pati [13] established a NCM that could produce $|\psi\rangle \rightarrow |\psi\rangle^\otimes (m+1)$ ($m = 1, 2, \ldots, k$) clones simultaneously, which appear in a linear superposition of all possible multiple copies.
with respective probabilities. Therefore, Pati’s NCM [13] generalizes Duan and Guo’s cloning machine [29,30]. For avoiding repetition, we will describe the NCM in Sections 2 and 3 in detail, and differentiate between our results and the previous those related. In this paper, we deal with the *NCM with supplementary information* (NCMSI), and present an equivalent characterization of such a quantum cloning device in terms of a two-step cloning protocol in which the original and the supplementary parties are only allowed to communicate with classical channel.

The remainder of the paper is organized as follows. In Section 2, we first introduce the existing results regarding probabilistic cloning with supplementary information, and then present our main contributions concerning NCMSI. Section 3 is the detailed demonstration of our major outcomes. In this section, we first provide a number of related unitary transformations describing cloning machines, and the corresponding inequalities characterizing the existence of these unitary transformations are then given; afterwards, we prove the main results expressed by Theorem 1 and Theorem 2. Also we derive an upper bound for unambiguous discrimination of the set \( \{|\psi_1\rangle|\phi_1\rangle, |\psi_2\rangle|\phi_2\rangle\} \) (Remark 1). Finally, in Section 4 we summarize our results obtained, mention some potential of applications, and address a number related issues for further consideration.

In addition, though some transformations describing cloning machines have been introduced in Section 1, in the interest of readability, we would like to present partially them again with somewhat different forms in Sections 2 and 3 to lead to our results.

2. Preliminaries and main results

In this section, we first give the existing results by Azuma et al. [38], and then present our main results.

As pointed out above, Jozsa [11] and Horodecki et al. [12] verified the no-cloning theorem and the no-deleting principle by utilizing supplementary information and conversation of quantum information, respectively. Then we may naturally address that if supplementary
information is added in the NCM, then whether the success probability for copying will be increased. Recently, Azuma et al. [38] suggested probabilistic cloning with supplementary information by combining probabilistic cloning and supplementary information. Specifically, for any two non-orthogonal states \( |\psi_1\rangle \) and \( |\psi_2\rangle \), and supplementary states \( |\phi_1\rangle \) and \( |\phi_2\rangle \), Azuma et al. [38] showed the following implication: If there exists unitary operator \( U \):

\[
U(|\psi_i\rangle|\phi_i\rangle|P_0\rangle) = \sqrt{r_i}|\psi_i\rangle \otimes (m+1)|P^{(i)}\rangle + \sqrt{1-r_i}|\Phi_{abp}^{(i)}\rangle, \quad (i = 1, 2),
\]

then there are corresponding unitary operators \( U_B \) and \( U_A \):

\[
U_B(|\phi_i\rangle|\Sigma|P_0\rangle) = \sqrt{r_i^B}|\psi_i\rangle \otimes m|P_B^{(i)}\rangle + \sqrt{1-r_i^B}|\Phi_{abp_B}^{(i)}\rangle, \quad (i = 1, 2),
\]

\[
U_A(|\psi_i\rangle|\Sigma|P_0\rangle) = \sqrt{r_i^A}|\psi_i\rangle \otimes (m+1)|P_A^{(i)}\rangle + \sqrt{1-r_i^A}|\Phi_{abp_A}^{(i)}\rangle, \quad (i = 1, 2),
\]

such that \( r_i^B + (1 - r_i^B)r_i^A \geq r_i \) (\( i = 1, 2 \)), where \( r_i^A \), \( r_i^B \), and \( r_i^A \) denote the success probabilities in the three machines, respectively, and \( \langle P^{(i)}|\Phi_{abp}^{(j)}\rangle = \langle P_B^{(i)}|\Phi_{abp_B}^{(j)}\rangle = \langle P_A^{(i)}|\Phi_{abp_A}^{(j)}\rangle = 0 \) for any \( i, j \in \{1, 2\} \). The above implication means that when the state chosen from two nonorthogonal states, the best efficiency of producing \( m + 1 \) copies is always achieved by a two-step cloning protocol in which the auxiliary party first tries to produce \( m \) copies from the supplementary state, and if it fails, then the original state is used to produce \( m + 1 \) copies by means of the probabilistic cloning device proposed by Duan and Guo [29,30]. For the sake of simplicity, we may represent the cloning devices described by Eqs. (13,14,15) as:

\[
|\psi_i\rangle|\phi_i\rangle \xrightarrow{r_i} |\psi_i\rangle^{m+1}, \quad (i = 1, 2),
\]

\[
\Rightarrow |\phi_i\rangle \xrightarrow{r_i^B} |\psi_i\rangle^m \text{ and } |\psi_i\rangle \xrightarrow{r_i^A} |\psi_i\rangle^{m+1}, \quad (i = 1, 2).
\]

However, when the state chosen from \( n \) states, with \( n > 2 \) and without orthogonal pairs of states, the above implication described by Eqs. (16,17) may not hold again, i.e., the best efficiency is not always reached by such a two-step cloning protocol [38].

In this paper, we will show the following equivalent relation: For any two non-orthogonal states \( |\psi_1\rangle \) and \( |\psi_2\rangle \), and supplementary states \( |\phi_1\rangle \) and \( |\phi_2\rangle \), there exists unitary operator \( U \):

\[
U(|\psi_i\rangle|\phi_i\rangle|P_0\rangle) = \sum_{k=1}^{m} \sqrt{r_k^{(i)}}|\psi_i\rangle \otimes (k+1)|0\rangle \otimes (m-k)|P_k^{(i)}\rangle + \sum_{l=m+1}^{N} \sqrt{f_l^{(i)}}|\Psi_l\rangle_{AB}|P_l\rangle, \quad (i = 1, 2),
\]

(18)
where $|P_1^{(i)}\rangle$, $|P_2^{(i)}\rangle$, ..., $|P_m^{(i)}\rangle$, $|P_{m+1}\rangle$, $|P_{m+2}\rangle$, ..., $|P_N\rangle$ are orthonormal for any $i \in \{1, 2\}$, if and only if there are unitary operators $U_B$ and $U_A$:

$$U_B(|\phi_i\rangle|\Sigma\rangle|P_0\rangle) = \sum_{k=1}^{m} \sqrt{r_{k,B}^{(i)}} |\psi_i^{(k)}\rangle \otimes |0\rangle^{\otimes(m-k+1)} |P_k^{(i)}\rangle + \sum_{l=m+1}^{N} \sqrt{f_{l,B}^{(i)}} |\Phi_{l,B}^{(i)}\rangle_{AB} |P_l,B\rangle, \quad (i = 1, 2),$$

(19)

$$U_A(|\psi_i\rangle|\Sigma\rangle|P_0\rangle) = \sum_{k=1}^{m} \sqrt{r_{k,A}^{(i)}} |\psi_i^{(k+1)}\rangle \otimes |0\rangle^{\otimes(m-k)} |P_k^{(i)}\rangle + \sum_{l=m+1}^{N} \sqrt{f_{l,A}^{(i)}} |\Phi_{l,A}^{(i)}\rangle_{AB} |P_l,A\rangle, \quad (i = 1, 2),$$

(20)

where $|P_{1,B}^{(i)}\rangle$, $|P_{2,B}^{(i)}\rangle$, ..., $|P_{m,B}^{(i)}\rangle$, $|P_{m+1,B}\rangle$, $|P_{m+2,B}\rangle$, ..., $|P_{N,B}\rangle$ are orthonormal, and, also, $|P_{1,A}^{(i)}\rangle$, $|P_{2,A}^{(i)}\rangle$, ..., $|P_{m,A}^{(i)}\rangle$, $|P_{m+1,A}\rangle$, $|P_{m+2,A}\rangle$, ..., $|P_{N,A}\rangle$ are orthonormal for any $i \in \{1, 2\}$; $r_{k,B}^{(i)}$, $r_{k,B}^{(i)}$, and $r_{k,A}^{(i)}$ represent the success probabilities for producing $|\psi_i^{(k+1)}\rangle$, $|\psi_i^{(k)}\rangle$, and $|\psi_i^{(k)}\rangle$, respectively, in three cloning devices.

Furthermore, it is satisfied that if the unitary transformation described by Eq. (18) holds, then there exist unitary transformations described by Eqs. (19,20) such that

$$\sum_{k=1}^{m} r_{k,B}^{(i)} + \left(1 - \sum_{k=1}^{m} r_{k,B}^{(i)}\right) \sum_{k=1}^{m} r_{k,A}^{(i)} \geq \sum_{k=1}^{m} r_{k}^{(k)}, \quad (i = 1, 2);$$

(21)

conversely, if Eqs. (19,20) hold, then there is unitary transformation by Eq. (18) satisfying

$$\sum_{k=1}^{m} r_{k,B}^{(i)} + \left(1 - \sum_{k=1}^{m} r_{k,B}^{(i)}\right) \sum_{k=1}^{m} r_{k,A}^{(i)} \leq \sum_{k=1}^{m} r_{k}^{(k)}, \quad (i = 1, 2).$$

(22)

In the interest of simplicity, we may represent the above Eqs. (18,19,20) as:

$$|\psi_i\rangle \sum_{k=1}^{m} r_{k,B}^{(i)} \sum_{k=1}^{m} |\psi_i^{(k)}\rangle^{\otimes(k+1)}, \quad (i = 1, 2),$$

(23)

$$\iff$$

$$|\phi_i\rangle \sum_{k=1}^{m} r_{k,A}^{(i)} \sum_{k=1}^{m} |\psi_i^{(k)}\rangle^{\otimes(k)}$$

(24)

and

$$|\psi_i\rangle \sum_{k=1}^{m} r_{k,A}^{(i)} \sum_{k=1}^{m} |\psi_i^{(k)}\rangle^{\otimes(k+1)}, \quad (i = 1, 2).$$

(25)

Note that transformation (20) is exactly the NCM studied by Pati [13] and stated above. The above equivalence shows that the optimal efficiency of the NCMSI in which the original party and the supplementary party can perform quantum communication equals the optimal efficiency achieved by the two-step cloning protocol wherein classical communication is only
allowed between the original and the supplementary parties. Therefore, in regard to the optimal success probabilities, if \( \sum_{k=1}^{m} r_{k,B}^{(i)} > 0 \), then \( \sum_{k=1}^{m} r_{k,A}^{(i)} > \sum_{k=1}^{m} r_{k,A}^{(i)} \), \( i = 1, 2 \), which implies that the NCMSI may increase the success probability. As well, if we take only one \( r_{k,B}^{(i)} \) and one \( r_{k,A}^{(i)} \) nonzero for some \( k \), then our right-implication reduces to the implication described by transformations (16,17). Therefore, our result generalizes and completes the result proved by Azuma et al. [38].

3. Proofs of main results

Firstly, for the sake of readability, we still quickly review the results by Azuma et al. [38], and present some transformations, some of which were indeed described before.

Probabilistic cloning machine firstly posed by Duan and Guo [29,30] describes that for any state set \{\( |ψ_1⟩ \), \( |ψ_2⟩ \), \ldots, \( |ψ_k⟩ \)\}, there exists unitary operator \( U \) such that

\[
U(|ψ_i⟩|Σ⟩|P_0⟩) = \sqrt{r_i} |ψ_i⟩|ψ_i⟩|^\otimes(m+1) + \sqrt{1 - r_i} |Φ_{ABP}⟩^i, \quad (i = 1, 2, \ldots, k),
\]

if and only if matrix \( X^{(1)} - \sqrt{Γ} X^{(2)} \sqrt{Γ}^\dagger \) is positive semidefinite, where \( X^{(1)} = [⟨ψ_i|ψ_j⟩], \)

\( X^{(2)} = [⟨ψ_i|ψ_j⟩^2|P(i)|P(j)⟩], \sqrt{Γ} = \sqrt{Γ}^\dagger = \text{diag}(\sqrt{r_1}, \sqrt{r_2}, \ldots, \sqrt{r_k}) \). The efficiency of cloning is as \( \sum_{i=1}^{k} p_i r_i \) if \( p_i \) are the probabilities for choosing states \( |ψ_i⟩ \) \( (i = 1, 2, \ldots, k) \).

Azuma et al. [38] showed that for two nonorthogonal states, \( |ψ_i⟩ \) \( (i = 1, 2) \), if there exists unitary operator \( U : |ψ_i⟩|φ_i⟩ → \sqrt{r_i} |ψ_i⟩|^\otimes(m+1) \) (for simplicity, they left out the failure item and the states of the probe device), then there also exist unitary operator \( U_A : |ψ_i⟩ → \sqrt{r_i} |ψ_i⟩^A|^\otimes(m+1) \) and unitary operator \( U_B : |φ_i⟩ → \sqrt{r_i} |ψ_i⟩^B|^\otimes(m) \) satisfying \( r_i^B + (1 - r_i^B) r_i^A \geq r_i \)

\( (i = 1, 2) \). For \( k \) states with \( k \geq 3 \), they verified that there exist state sets \{\( |ψ_i⟩ \)\} and \{\( |φ_i⟩ \)\}, as well as unitary operator \( U \) above, such that for any unitary operators \( U_A \) and \( U_B \) above, it holds that \( r_i^A = 0 \) \( (i = 1, 2, \ldots, n) \), and \( \sum_{i=1}^{k} \frac{1}{n} r_i > \sum_{i=1}^{k} \frac{1}{n} r_i^B \).

We enter on our discussion. Suppose Alice holds the original copy \( |ψ_i⟩ \) and Bob possesses the supplementary information \( |φ_i⟩ \) \( (i = 1, 2) \). If Alice and Bob are allowed to communicate with one-way quantum channel from Bob to Alice, then a single party holding both the original and the supplementary information \( |ψ_i⟩|φ_i⟩ \) performs the following cloning process.
described by a unitary operator $U$:

$$U(|\psi_i\rangle|\phi_i\rangle|P_0\rangle) = \sum_{k=1}^{m} \sqrt{r_k^{(i)}} |\psi_i\rangle \otimes |0\rangle \otimes (m-k)|P_k^{(i)}\rangle + \sum_{l=m+1}^{N} \sqrt{f_l^{(i)}} |\Psi_l\rangle_{AB}|P_l\rangle, \quad (i = 1, 2),$$

(27)

where $0 \leq r_k^{(i)} \leq 1$ for $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} r_k^{(i)} < 1$ (in terms of [13], $\sum_{k=1}^{m} r_k^{(i)} = 1$ is impossible), $|P_0\rangle$, $|P_k^{(i)}\rangle$, and $|P_l\rangle$ are the states of the probing device, satisfying that $|P_1^{(i)}\rangle$, $|P_2^{(i)}\rangle$, $\ldots$, $|P_m^{(i)}\rangle$, $|P_{m+1}\rangle$, $|P_{m+2}\rangle$, $\ldots$, $|P_N\rangle$ are orthonormal for $i = 1, 2$. Moreover, $N > m$, $|0\rangle$ is the state of the ancillary system $B$, $r_k^{(i)}$ and $f_l^{(i)}$ are the success and the failure probabilities, respectively. If $p_i$ are a priori probabilities for choosing $|\psi_i\rangle|\phi_i\rangle$ ($i = 1, 2$), then the global success probability $P_s$ for copying is

$$P_s = \sum_{i=1}^{2} p_i \sum_{k=1}^{m} r_k^{(i)}. \quad (28)$$

If Alice and Bob only can use classical channel for communication, they may respectively run the following machines described by unitary operators $U_A$ and $U_B$, where $U_A$ is exactly Pati’s NCM [13]:

$$U_A(|\psi_i\rangle|\Sigma\rangle|P_0\rangle) = \sum_{k=1}^{m} \sqrt{r_k^{(i)}} |\psi_i\rangle \otimes |0\rangle \otimes (m-k)|P_k^{(i)}\rangle + \sum_{l=m+1}^{N} \sqrt{f_l^{(i)}} |\Phi_l^{(A)}\rangle_{AB}|P_l\rangle, \quad (i = 1, 2),$$

(29)

such that $0 \leq r_k^{(i)} \leq 1$ for $k = 1, 2, \ldots, m$, where $|P_0\rangle$, $|P_k^{(i)}\rangle$, and $|P_l\rangle$ are the states of the probe device, satisfying that $|P_{1,A}^{(i)}\rangle$, $|P_{2,A}^{(i)}\rangle$, $\ldots$, $|P_{m,A}^{(i)}\rangle$, $|P_{m+1,A}^{(i)}\rangle$, $|P_{m+2,A}^{(i)}\rangle$, $\ldots$, $|P_{N,A}\rangle$ are orthonormal for $i = 1, 2$. If $p_i^{(A)}$ are a priori probabilities for choosing $|\psi_i\rangle$ ($i = 1, 2$), then the global success probability $P_s^{(A)}$ for copying is

$$P_s^{(A)} = \sum_{i=1}^{2} p_i^{(A)} \sum_{k=1}^{m} r_k^{(i)}. \quad (30)$$

$U_B$ is as follows:

$$U_B(|\phi_i\rangle|\Sigma\rangle|P_0\rangle) = \sum_{k=1}^{m} \sqrt{r_k^{(i)}} |\phi_i\rangle \otimes |0\rangle \otimes (m-k+1)|P_k^{(i)}\rangle + \sum_{l=m+1}^{N} \sqrt{f_l^{(i)}} |\Phi_l^{(B)}\rangle_{AB}|P_l\rangle, \quad (i = 1, 2),$$

(31)

such that $0 \leq r_k^{(i)} \leq 1$ for $k = 1, 2, \ldots, m$, where $|P_0\rangle$, $|P_k^{(i)}\rangle$, and $|P_l\rangle$ are the states of the probe device, satisfying that $|P_{1,B}^{(i)}\rangle$, $|P_{2,B}^{(i)}\rangle$, $\ldots$, $|P_{m,B}^{(i)}\rangle$, $|P_{m+1,B}^{(i)}\rangle$, $|P_{m+2,B}^{(i)}\rangle$, $\ldots$, $|P_{N,B}\rangle$
are orthonormal for \( i = 1, 2 \). If \( p_i^{(B)} \) are a priori probabilities for choosing \(|\psi_i\rangle\) \((i = 1, 2)\), then the global success probability \( P_s^{(B)} \) for copying is

\[
P_s^{(B)} = \frac{2}{m} \sum_{i=1}^{m} p_i^{(B)} \sum_{k=1}^{m} \bar{r}_{k,B}^{(i)}.
\]

(32)

If Alice and Bob only can use one-way classical channel for communication from Bob to Alice, then Bob first performs machine described by Eq. (31), and tells Alice the result of success or failure. If Bob succeeds, Alice only preserves her copy as is; otherwise, Alice runs the machine described by Eq. (29). Therefore, in this case, the success probability for producing quantum superposition of multiple clones \( \sum_{k=1}^{m} |\psi_i\rangle^{\otimes(k+1)} \) when inputting \(|\psi_i\rangle|\phi_i\rangle\), is

\[
\sum_{k=1}^{m} \bar{r}_{k,B}^{(i)} + \left( 1 - \sum_{l=1}^{m} \bar{r}_{l,B}^{(i)} \right) \sum_{l=1}^{m} \bar{r}_{l,A}^{(i)}.
\]

(33)

Similarly, if Alice and Bob can use only one-way classical channel for communication from Alice to Bob, then Alice first performs Pati’s machine described by Eq. (29), and then tells Bob the result of success or failure. If Alice succeeds, Bob does nothing; otherwise, Bob runs machine by Eq. (31). Thus, it is seen that the success probability for producing quantum superposition of multiple clones \( \sum_{k=1}^{m} |\psi_i\rangle^{\otimes(k+1)} \) with input \(|\psi_i\rangle|\phi_i\rangle\) is

\[
\sum_{k=1}^{m} \bar{r}_{k,A}^{(i)} + \left( 1 - \sum_{l=1}^{m} \bar{r}_{l,A}^{(i)} \right) \bar{r}_{k,B}^{(i)}.
\]

(34)

If Alice and Bob can use two-way classical channel for communication, i.e., they can communicate each other, then they first independently carry out machines described by Eqs. (29,31), and, afterwards, inform the other of the outcome produced. Therefore, the success probability for producing quantum superposition of multiple clones \( \sum_{k=1}^{m} |\psi_i\rangle^{\otimes(k+1)} \) with input \(|\psi_i\rangle|\phi_i\rangle\) will be

\[
1 - \left( 1 - \sum_{k=1}^{m} \bar{r}_{k,A}^{(i)} \right) \left( 1 - \sum_{k=1}^{m} \bar{r}_{k,B}^{(i)} \right) = \sum_{k=1}^{m} \bar{r}_{k,A}^{(i)} + \sum_{k=1}^{m} \bar{r}_{k,B}^{(i)} - \sum_{k=1}^{m} \bar{r}_{k,A}^{(i)} \sum_{k=1}^{m} \bar{r}_{k,B}^{(i)}.
\]

(35)

Notably, whichever classical communication we choose, it is clearly seen that with input \(|\psi_i\rangle|\phi_i\rangle\), the success probabilities for producing quantum superposition of multiple clones \( \sum_{k=1}^{m+1} |\psi_i\rangle^{\otimes(k+1)} \) are equal.

In what follows, we denote \( \alpha = \langle \psi_1 | \psi_2 \rangle \), \( \beta = \langle \phi_1 | \phi_2 \rangle \), \( p_k = \langle P_k^{(1)} | P_k^{(2)} \rangle \), \( p_{k,A} = \langle P_k^{(1)} | P_{k,A}^{(2)} \rangle \), \( p_{k,B} = \langle P_k^{(1)} | P_{k,B}^{(2)} \rangle \). Now we notice that Eqs. (27,29,31), hold if and only if the matrices
are positive semidefinite, respectively, where $Z^{(1)} = \langle \psi_i |\psi_j \rangle \langle \phi_i |\phi_j \rangle$, $X^{(1)} = \langle \psi_i |\psi_j \rangle$, and $Y^{(1)} = \langle \phi_i |\phi_j \rangle$; $G^{(m+1)} = \langle \psi_i |\psi_j \rangle^{m+1} \langle P^{(i)}_k |P^{(j)}_k \rangle$, $G^{(m+1)}_A = \langle \psi_i |\psi_j \rangle^{m+1} \langle P^{(i)}_{k,A} |P^{(j)}_{k,A} \rangle$, and $G^{(m)}_B = \langle \psi_i |\psi_j \rangle^m \langle P^{(i)}_{k,B} |P^{(j)}_{k,B} \rangle$; $\sqrt{\Gamma}_k = \text{diag}(r^{(1)}_k, r^{(2)}_k)$, $\sqrt{\Gamma}_{k,A} = \text{diag}(r^{(1)}_{k,A}, r^{(2)}_{k,A})$, and $\sqrt{\Gamma}_{k,B} = \text{diag}(r^{(1)}_{k,B}, r^{(2)}_{k,B})$. Furthermore, we note that the three matrices above are positive semidefinite if and only if their determinants are nonnegative, respectively, that is,

\[
\sqrt{\sum_{k=1}^m r^{(1)}_k (1 - \sum_{k=1}^m r^{(2)}_k)} - |\alpha| \beta - \sum_{k=1}^m \sqrt{r^{(1)}_k r^{(2)}_k} |\alpha|^{k+1} p_k | \geq 0, \tag{36}
\]

\[
\sqrt{\sum_{k=1}^m r^{(1)}_{k,A} (1 - \sum_{k=1}^m r^{(2)}_k)} - |\alpha| - \sum_{k=1}^m \sqrt{r^{(1)}_{k,A} r^{(2)}_k} |\alpha|^{k+1} p_{k,A} | \geq 0, \tag{37}
\]

\[
\sqrt{\sum_{k=1}^m r^{(1)}_{k,B} (1 - \sum_{k=1}^m r^{(2)}_k)} - |\beta| - \sum_{k=1}^m \sqrt{r^{(1)}_{k,B} r^{(2)}_k} |\beta|^{k+1} p_{k,B} | \geq 0. \tag{38}
\]

If $|\beta| > \sum_{k=1}^m \sqrt{r^{(1)}_k r^{(2)}_k} |\alpha|^{k}$, then, by taking appropriate amplitudes of $p_k$, Ineq. (36) is equivalent to

\[
\sqrt{\sum_{k=1}^m r^{(1)}_k (1 - \sum_{k=1}^m r^{(2)}_k)} - |\alpha| \beta + \sum_{k=1}^m \sqrt{r^{(1)}_k r^{(2)}_k} |\alpha|^{k+1} \geq 0; \tag{39}
\]

analogously, if $1 > \sum_{k=1}^m \sqrt{r^{(1)}_{k,A} r^{(2)}_{k,A}} |\alpha|^{k}$ and $|\beta| > \sum_{k=1}^m \sqrt{r^{(1)}_{k,B} r^{(2)}_{k,B}} |\alpha|^{k}$ hold, respectively, then correspondingly, Ineqs. (38,39) are respectively equivalent to

\[
\sqrt{\sum_{k=1}^m r^{(1)}_{k,A} (1 - \sum_{k=1}^m r^{(2)}_k)} - |\alpha| + \sum_{k=1}^m \sqrt{r^{(1)}_{k,A} r^{(2)}_k} |\alpha|^{k+1} \geq 0, \tag{40}
\]

\[
\sqrt{\sum_{k=1}^m r^{(1)}_{k,B} (1 - \sum_{k=1}^m r^{(2)}_k)} - |\beta| + \sum_{k=1}^m \sqrt{r^{(1)}_{k,B} r^{(2)}_k} |\beta|^{k} \geq 0. \tag{41}
\]

With input $|\psi_i \rangle \otimes |\phi_i \rangle$, the efficiency of producing quantum superposition of multiple clones $\sum_{k=1}^m |\psi_i \rangle \otimes (k+1)$ which Alice and Bob achieve via quantum channel can always be achieved by a two-step cloning protocol in which Alice and Bob are only allowed to execute one-way or two-way classical communication. This is described by the following Theorem 1.
Theorem 1. – If there exists unitary operator $U$ such that Eq. (27) holds, then there are unitary operators $U_A$ and $U_B$ satisfying Eqs. (29,31), respectively, such that

$$
\sum_{k=1}^{m} r_k^{(i)} \leq \sum_{k=1}^{m} r_{k,B}^{(i)} + (1 - \sum_{k=1}^{m} r_{k,B}^{(i)}) \sum_{k=1}^{m} r_{k,A}^{(i)},
$$

(42)

for $i = 1, 2$.

Proof: As above, denote $\alpha = \langle \psi_1 | \psi_2 \rangle$, $\beta = \langle \phi_1 | \phi_2 \rangle$.

Case 1. $|\beta| \leq \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)}} |\alpha|^k$. In this case, we only take any $r_{k,B}^{(i)}$ satisfying $r_{k,B}^{(i)} \geq r_k^{(i)}$ for $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} r_{k,B}^{(i)} = 1$ ($i = 1, 2$). Clearly, $|\beta| \leq \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)}} |\alpha|^k$ also holds. Then it suffices to take appropriate $p_{k,B}$ such that $\beta - \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)}} |\alpha|^k p_{k,B} = 0$. Thus, Ineq. (38) holds. By taking $r_{k,A}^{(i)} = 0$ ($1 \leq k \leq m, 1 \leq i \leq 2$), then Ineq. (37) holds. So, the theorem is proved in this situation.

Case 2. $|\beta| > \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)}} |\alpha|^k$. We set a function $F$ from $[0, 1]^m \times [0, 1]^m$ to $[0, +\infty)$ as:

$$
F(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_m) = \sqrt{\frac{(1 - \sum_{k=1}^{m} x_k)(1 - \sum_{k=1}^{m} y_k)}{|\beta| - \sum_{k=1}^{m} \sqrt{x_k y_k} |\alpha|^k}}.
$$

(43)

Clearly function $F$ is continuous on $[0, 1]^m \times [0, 1]^m$, and

$$
F(0, 0, \ldots, 0; 0, 0, \ldots, 0) = \frac{1}{|\beta|} \geq 1,
$$

(44)
as well as, by Ineq. (39),

$$
F(r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)}, r_1^{(2)}, r_2^{(2)}, \ldots, r_m^{(2)}) \geq |\alpha|.
$$

(45)

To prove the theorem, we somewhat change function $F$ to set up a new function $H$ that only has $m$ variables at most. The main idea to establish $H$ is to reduce the number $2m$ of the variables in $F$ to not more than $m$, and we present the way of constructing function $H$ from function $F$ in detail:

(i) For $1 \leq k \leq m$, if $0 \neq r_k^{(1)} \geq r_k^{(2)}$, then the pair of variables $(x_k, y_k)$ in $F$ will be replaced by $(x_k, c_k x_k)$, where $\frac{r_k^{(2)}}{r_k^{(1)}} = c_k \leq 1$; if $0 = r_k^{(1)} \geq r_k^{(2)}$, then the pair of variables $(x_k, y_k)$ in $F$ will be replaced by the pair $(0, 0)$ of constants.

(ii) For $1 \leq k \leq m$, if $r_k^{(1)} < r_k^{(2)}$, we replace the pair of variables $(x_k, y_k)$ in $F$ by $(c_k' y_k, y_k)$, where $c_k' = \frac{r_k^{(1)}}{r_k^{(2)}} \leq 1$. 

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By means of the above way to adjust and decrease those variables in function $F$, we obtain a new function $H$ whose number of variables is at most $m$, instead of $2m$, that is the form: For $z_k \in \{x_k, y_k\}, 1 \leq k \leq m$,

$$H(z_1, z_2, \ldots, z_m) = F(u_1, u_2, \ldots, u_m; v_1, v_2, \ldots, v_m), \quad (46)$$

where:

(i) If $0 \neq r_k^{(1)} \geq r_k^{(2)}$, then $z_k = x_k$, and, $u_k = x_k$, $v_k = c_kx_k$, where $c_k = \frac{r_k^{(2)}}{r_k^{(1)}} \leq 1$.

(ii) If $0 = r_k^{(1)} \geq r_k^{(2)}$, then $z_k = u_k = v_k = 0$.

(iii) If $\frac{r_k^{(1)}}{r_k^{(2)}} < 1$, then $z_k = y_k$, and, $u_k = c'_k y_k$, $v_k = y_k$, where $c'_k = \frac{r_k^{(1)}}{r_k^{(2)}}$.

Without loss of generality, we suppose that always $r_k^{(1)} \geq r_k^{(2)}$, $k = 1, 2, \ldots, m$. Then we have

$$H(x_1, x_2, \ldots, x_m) = F(x_1, x_2, \ldots, x_m; c_1x_1, c_2x_2, \ldots, c_mx_m), \quad (47)$$

where when $r_k^{(1)} = 0$, $x_k \equiv 0$, $(k = 1, 2, \ldots, m)$.

By Ineqs. (44,45),

$$H(0, 0, \ldots, 0) = F(0, 0, \ldots, 0; 0, 0, \ldots, 0) = \frac{1}{|\beta|} \geq 1, \quad (48)$$

and

$$H(r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)}) = F(r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)}; r_1^{(2)}, r_2^{(2)}, \ldots, r_m^{(2)}) \geq |\alpha|. \quad (50)$$

Next we consider two scenarios to complete the proof:

(I) If $H(r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)}) \geq 1$, then

$$F(r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)}; r_1^{(2)}, r_2^{(2)}, \ldots, r_m^{(2)}) = H(r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)}) \geq 1, \quad (52)$$
and, therefore, by Eq. (43) we have

\[
F(r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)}; r_1^{(2)}, r_2^{(2)}, \ldots, r_m^{(2)}) = \sqrt{(1 - \sum_{k=1}^{m} r_k^{(1)})(1 - \sum_{k=1}^{m} r_k^{(2)})} \\
\geq |\beta| - \sum_{k=1}^{m} \sqrt{r_k^{(1)} r_k^{(2)}} |\alpha|^k,
\]

(54)

Therefore, by taking \( r_{k,B}^{(i)} = r_k^{(i)} \), \((k = 1, 2, \ldots, m; i = 1, 2)\), Ineq. (41) holds. As a result, there exist unitary operators \( U_A \) and \( U_B \) such that Eqs. (29,31) hold, in which we can choose \( r_{k,A}^{(i)} = 0 \) and \( r_{k,B}^{(i)} = r_k^{(i)} \), \((k = 1, 2, \ldots, m; i = 1, 2)\). In this case, the theorem is proved.

(II) If \( H(r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)}) < 1 \), then, together with \( H(0, 0, \ldots, 0) \geq 1 \) (i.e., Eq. (49)), by intermediate value theorem of continuous functions, there exist \( r_{k,B}^{(i)} \) such that

\[
0 \leq r_{k,B}^{(i)} \leq r_k^{(i)}, \quad (k = 1, 2, \ldots, m),
\]

(55)

and

\[
H(r_{1,B}^{(1)}, r_{2,B}^{(1)}, \ldots, r_{m,B}^{(1)}) = 1.
\]

(56)

Now, for \( k = 1, 2, \ldots, m \), we take

\[
r_{k,B}^{(2)} = \begin{cases} 
0, & \text{if } r_k^{(1)} = 0, \\
\frac{r_k^{(2)}}{r_k^{(1)}} r_{k,B}^{(1)}, & \text{otherwise}.
\end{cases}
\]

(57)

Denoting \( c_k = \begin{cases} 
0, & \text{if } r_k^{(1)} = 0, \\
\frac{r_k^{(2)}}{r_k^{(1)}}, & \text{otherwise},
\end{cases} \) then clearly we have

\[
r_{k,B}^{(2)} = c_k r_{k,B}^{(1)}, \quad r_k^{(2)} = c_k r_k^{(1)},
\]

(58)

for \( k = 1, 2, \ldots, m \) and \( i = 1, 2 \); as well, by Ineqs. (55,58), \( \sum_{k=1}^{m} r_{k,B}^{(i)} \leq \sum_{k=1}^{m} r_k^{(i)} \) holds for \( i = 1, 2 \). Now we take

\[
r_{k,A}^{(i)} = \frac{r_k^{(i)} - r_{k,B}^{(i)}}{1 - \sum_{k=1}^{m} r_{k,B}^{(i)}}, \quad (i = 1, 2),
\]

(59)

then

\[
\sqrt{(1 - r_{k,A}^{(1)})(1 - r_{k,A}^{(2)})} = \frac{(1 - \sum_{k=1}^{m} r_k^{(1)})(1 - \sum_{k=1}^{m} r_k^{(2)})}{(1 - \sum_{k=1}^{m} r_{k,B}^{(1)})(1 - \sum_{k=1}^{m} r_{k,B}^{(2)})} \geq \frac{|\beta| - \sum_{k=1}^{m} \sqrt{r_k^{(1)} r_k^{(2)}} |\alpha|^k}{|\beta| - \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)}} |\alpha|^k} |\alpha|^k,
\]

(60)
and

\[ \sqrt{r_{k,A}^r r_{k,A}^r} = \frac{(r_k^{(1)} - r_{k,B}^{(1)})(r_k^{(2)} - r_{k,B}^{(2)})}{|\beta| - \sum_{k=1}^{m} r_{k,B}^{(1)} r_{k,B}^{(1)} |\alpha|^k}. \]  \hspace{1cm} (61)

By Ineq. (60) and Eq. (61) we have

\[ \sqrt{(1 - \sum_{k=1}^{m} r_{k,A}^{(1)})(1 - \sum_{k=1}^{m} r_{k,A}^{(2)}) - |\alpha| + \sum_{k=1}^{m} r_{k,A}^{(1)} r_{k,A}^{(2)} |\alpha|^k}. \]  \hspace{1cm} (62)

Due to Eq. (58), i.e., \( r_{k,B}^{(2)} = c_k r_{k,B}^{(1)} \), \( r_{k,B}^{(2)} = c_k r_{k}^{(1)} \), we have

\[ \sqrt{(r_k^{(1)} - r_{k,B}^{(1)})(r_k^{(2)} - r_{k,B}^{(2)})} = \sqrt{r_k^{(1)} r_k^{(2)}} - \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)}}. \]  \hspace{1cm} (63)

By combining Eq. (63) and Ineq. (62) above, we conclude that

\[ \sqrt{(1 - \sum_{k=1}^{m} r_{k,A}^{(1)})(1 - \sum_{k=1}^{m} r_{k,A}^{(2)}) - |\alpha| + \sum_{k=1}^{m} r_{k,A}^{(1)} r_{k,A}^{(2)} |\alpha|^k} \geq 0. \]  \hspace{1cm} (64)

Due to the above conditions, Ineq. (64) and Ineq. (37) are equivalent, and, therefore, the proof has been completed. \( \square \)

**Remark 1.** Theorem 1 shows that the two-step cloning protocol in terms of classical one-way or two-way communication can achieve the optimal efficiency by the NCMSI. This theorem generalizes Theorem 2 of [38]. Indeed, for \( i = 1, 2 \), given integer \( m > 0 \), if we take \( r_{k,B}^{(i)} = 0 \) for any \( k \neq m \), then from the above proof we can also take \( r_{k,B}^{(i)} = 0 \) and \( r_{k,A}^{(i)} = 0 \) for any \( k \neq m \). In this case, Theorem 1 reduces to Theorem 2 of [38] as stated in the beginning of this section. As well, due to \( \lim_{m \to \infty} \langle \psi_i |\psi_j \rangle^m = 0 \) for any \( i \neq j \), when \( m \to \infty \) the unitary transformation

\[ U(|\psi_i \rangle \langle \phi_i | P_0}) = \sqrt{r_{m}^{(i)} |\psi_i \rangle \langle 0 | P_0} + \sum_{l=m+1}^{N} \sqrt{f_{l}^{(i)} |\Psi_l \rangle AB} P_1} \]  \hspace{1cm} (65)

carries out the unambiguous discrimination of the set \( \{|\psi_1 \rangle |\phi_1 \rangle, |\psi_2 \rangle |\phi_2 \rangle \} \). Indeed, firstly, if \( |\phi_1 \rangle \) and \( |\phi_2 \rangle \) are orthogonal, then in Ineq. (36) we take \( r_{m}^{(i)} = r_{m}^{(2)} = 1 \) and \( p_m = 0 \), which is in accord with the result that \( \{|\psi_1 \rangle |\phi_1 \rangle, |\psi_2 \rangle |\phi_2 \rangle \} \) can be exactly discriminated thanks to
the orthogonality. If $|\phi_1\rangle$ and $|\phi_2\rangle$ are nonorthogonal, then $|\beta| > 0$, and we can take $m$ big enough such that $|\beta| > |\alpha|^m$. Therefore, by using Ineq. (36) we have that

$$\frac{r_m^{(1)} + r_m^{(2)}}{2} \leq \frac{1 - |\alpha\beta|}{1 - |\alpha|^m|p_m|}. \quad (66)$$

By taking $p_m = 0$ we obtain that

$$\frac{r_m^{(1)} + r_m^{(2)}}{2} \leq 1 - |\alpha\beta|. \quad (67)$$

This has been dealt with by Chen and Yang [39] for achieving the optimal unambiguous discrimination of any two nonorthogonal pure product multipartite states with any a priori probabilities via local operation and classical communication.

Next we may ask whether or not the two-step protocol is strictly stronger than the NCMSI. By the following Theorem 2 we show that the optimal efficiency obtained by the above two-step cloning protocol can also be achieved by some NCMSI. Therefore, they indeed have the same optimal efficiency.

**Theorem 2.** For any unitary operators $U_A$ and $U_B$ satisfying Eqs. (29,31), there is a unitary operator $U$ satisfying Eq. (27), such that

$$r_k^{(i)} = r_{k,B}^{(i)} + \left(1 - \sum_{l=1}^m r_{l,B}^{(i)}\right) r_{k,B}^{(i)}, \quad (68)$$

for $k = 1, 2, \ldots, m$ and $i = 1, 2$.

**Proof:** Leave $\alpha$ and $\beta$ as they are. If $|\beta| \leq \sum_{k=1}^m \sqrt{r_k^{(1)} r_k^{(2)}} |\alpha|^k$, where $r_k^{(i)} = r_{k,B}^{(i)} + (1 - \sum_{l=1}^m r_{l,B}^{(i)}) r_{k,B}^{(i)}$, then Ineq. (36) is always satisfied by taking appropriate $p_k$, i.e., the states $|P_k^{(i)}\rangle$ of the probe device for $k = 1, 2, \ldots, m$ and $i = 1, 2$. Hence, we assume that $|\beta| > \sum_{k=1}^m \sqrt{r_k^{(1)} r_k^{(2)}} |\alpha|^k$, in the following. First we note that

$$\sqrt{(1 - \sum_{k=1}^m r_{k,A}^{(1)})(1 - \sum_{k=1}^m r_{k,A}^{(2)})} \sqrt{(1 - \sum_{k=1}^m r_{k,B}^{(1)})(1 - \sum_{k=1}^m r_{k,B}^{(2)})} = \sqrt{(1 - \sum_{k=1}^m r_k^{(1)})(1 - \sum_{k=1}^m r_k^{(2)})}. \quad (69)$$

Since $|\beta| > \sum_{k=1}^m \sqrt{r_k^{(1)} r_k^{(2)}} |\alpha|^k$, Ineqs. (40,41) hold, and by these two inequalities, we have

$$\sqrt{(1 - \sum_{k=1}^m r_{k,A}^{(1)})(1 - \sum_{k=1}^m r_{k,A}^{(2)})} \sqrt{(1 - \sum_{k=1}^m r_{k,B}^{(1)})(1 - \sum_{k=1}^m r_{k,B}^{(2)})} \leq 1 - |\alpha\beta|. \quad (67)$$
\[ \alpha \leq \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k+1}} (|\beta| - \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k}}) \]

\[ = |\alpha \beta| - \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k+1}} - |\beta| \sum_{k=1}^{m} \sqrt{r_{k,A}^{(1)} r_{k,A}^{(2)} |\alpha|^{k}} + \left( \sum_{k=1}^{m} \sqrt{r_{k,A}^{(1)} r_{k,A}^{(2)} |\alpha|^{k+1}} \right) \left( \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k}} \right). \quad (70) \]

Therefore, to show Ineq. (39), it suffices to verify that

\[ (|\alpha| - \sum_{k=1}^{m} \sqrt{r_{k,A}^{(1)} r_{k,A}^{(2)} |\alpha|^{k+1}} (|\beta| - \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k}})) \geq |\alpha \beta| - \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k+1}}. \quad (71) \]

In terms of Eq. (70), Ineq. (71) is equivalent to

\[ \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k}} \geq \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k}} + |\beta| \sum_{k=1}^{m} \sqrt{r_{k,A}^{(1)} r_{k,A}^{(2)} |\alpha|^{k}} - \sum_{k=1}^{m} \sqrt{r_{k,A}^{(1)} r_{k,A}^{(2)} |\alpha|^{k}} \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k}}. \quad (72) \]

By using Ineq. (41), it is enough to show that

\[ \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k}} \geq \sum_{k=1}^{m} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} |\alpha|^{k}} + \sqrt{\left( 1 - \sum_{k=1}^{m} r_{k,B}^{(1)} \right) \left( 1 - \sum_{k=1}^{m} r_{k,B}^{(2)} \right)} \sum_{k=1}^{m} \sqrt{r_{k,A}^{(1)} r_{k,A}^{(2)} |\alpha|^{k}}. \quad (73) \]

We can easily check that for any \( k = 1, 2, \ldots, m, \)

\[ \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)}} \geq \sqrt{\left( 1 - \sum_{l=1}^{m} r_{l,B}^{(1)} \right) \left( 1 - \sum_{l=1}^{m} r_{l,B}^{(2)} \right)} \sqrt{r_{k,A}^{(1)} r_{k,A}^{(2)}} + \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)}}, \quad (74) \]

which follows from the inequality

\[ r_{k,B}^{(1)} (1 - \sum_{k=1}^{m} r_{k,B}^{(2)}) + r_{k,B}^{(2)} (1 - \sum_{k=1}^{m} r_{k,B}^{(1)}) r_{k,A}^{(1)} \]

\[ \geq 2 \sqrt{\left( 1 - \sum_{k=1}^{m} r_{k,B}^{(1)} \right) \left( 1 - \sum_{k=1}^{m} r_{k,B}^{(2)} \right)} \sqrt{r_{k,B}^{(1)} r_{k,B}^{(2)} r_{k,B}^{(1)} r_{k,B}^{(2)}}. \quad (75) \]

Therefore, we complete the proof. \( \square \)

**Remark 2.** Since cloning only one multiple copies is a special case of cloning superposition of multiple clones, Theorem 2 above shows that in Theorem 2 of [38], probabilistic cloning
with supplementary information and the two-step cloning protocol is equivalent. Therefore this completes Theorem 2 of [38].

Remark 3. If $|\psi_1\rangle, |\psi_2\rangle$ are linearly independent, and $|\phi_1\rangle, |\phi_2\rangle$ are linearly dependent, then by virtue of Lemma 1 in [38], the success probability of Bob running the cloning device described by unitary operator $U_B$ is zero. Therefore, in this case, the NCMSI has the same cloning efficiency as the NCM. However, if the supplementary information $|\phi_1\rangle, |\phi_2\rangle$ are linearly independent, then the success probabilities in the cloning machine described by $U_B$ are likely bigger than zero, and, thus, from Theorem 2 it follows that the success probability of the NCMSI for cloning is bigger than the NCM [13].

4. Concluding remarks

We have dealt with the novel cloning machine with the help of supplementary information (NCMSI) for producing quantum superposition of multiple copies. When two holders, say Alice and Bob, possess respectively the original and the supplementary information, we have derived that the optimal efficiencies of cloning achieved via quantum communication and via classical one-way or two-way communication between the two parties in these devices are indeed equivalent. Therefore, the NCMSI for producing quantum superposition of multiple copies may have bigger success probability than the NCM [13]. However, by classical communication we do not know how to obtain the all copies together in a quantum computer, so, in practice we may use the scenario of quantum communication, i.e., the NCMSI.

As stated in Section 1, probabilistic cloning may get precise copies with certain probability, so, improving the success ratio is of importance. We hope that our results would provide some useful ideas in preserving important quantum information, parallel storage of quantum information in a quantum computer, and quantum cryptography.

When cloning $n$ states with $n \geq 3$, Azuma et al. [38] demonstrated that the optimal efficiency of copying achieved via quantum communication between the original and the supplementary parties sometimes cannot be accomplished by using only classical channel.
Then an interesting problem is what is the sufficient and necessary condition for retaining the equivalence as we proved in this paper. A possible method is to combine matrix theory [40] and the present paper. Moreover, if the supplementary information is given as a mixed state or we have multiple supplementary information, then the probabilistic or novel cloning devices are still worth considering. We would like to explore these questions in future.

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