Quark Confinement and the Renormalization Group

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Abstract

Recent approaches to quark confinement are reviewed, with an emphasis on their connection to renormalization group methods. Basic concepts related to confinement are introduced: the string tension, Wilson loops and Polyakov lines, string breaking, string tension scaling laws, center symmetry breaking, and the deconfinement transition at non-zero temperature. Current topics discussed include confinement on $R^3 \times S^1$, the real-space renormalization group, the functional renormalization group, and the Schwinger-Dyson equation approach to confinement.

Keywords: renormalization group, qcd, quark confinement
I. INTRODUCTION

Quarks are the fermionic constituents of the hadrons, the strongly interacting particles such as the proton and the neutron. The problem of quark confinement, at its simplest, is to explain why quarks are not observed as physical states in high-energy scattering, but remain confined inside hadrons. The problem has been apparent for decades, and pre-dates quantum chromodynamics (QCD), the modern theory of the strong interactions \cite{1-3}. Lattice gauge theory \cite{4, 5} has given us a physical basis for quark confinement: the force between quarks is independent of distance at large separation, and an infinite amount of energy would be required to isolate a single quark. The force constant at infinite separation is the string tension $\sigma$. While lattice simulations have determined the relation of $\sigma$ to other observables of QCD with high precision, we are still far from a satisfactory understanding of quark confinement. We have neither a firm grasp of the mechanisms of a quark confinement nor the ability to calculate analytically even simple aspects of the physics of confinement.

Recent work in the study of confinement suggests that progress is now being made, and there are several complementary approaches under development. All of these approaches have common features: they use the renormalization group in an essential way, and are strongly conditioned by lattice gauge theory results. This review is based on talks given at the INT Workshop ”New applications of the renormalization group in nuclear, particle, and condensed matter physics”, held February 22-26 2010, and the emphasis is on current research rather than a comprehensive treatment of each approach. The review is organized as follows: Section II introduces the basic concepts involved in the study of quark confinement: the string tension, its measurement, string breaking, the finite-temperature deconfinement transition, center symmetry, and string tension scaling laws. Section III discusses recent progress in understanding confinement on $R^3 \times S^1$. Section IV describes recent work on a rigorous proof of confinement using real-space renormalization group methods. Section V discusses two related approaches to confinement based on the infrared behavior of quark and gluon propagators, the first using the functional renormalization group and the second using the Schwinger-Dyson equations. A final section attempts to summarize our current understanding and prospects for future progress.
II. FUNDAMENTALS OF CONFINEMENT

The fundamental theory of the strong interactions is quantum chromodynamics (QCD), a quantum field theory describing the interactions of quarks and gluons. QCD is a gauge theory: Gluons are massless spin-1 bosons whose interactions are tightly constrained by the requirement of local gauge invariance in a manner similar to the case of QED (quantum electrodynamics) and photons. In QED, the symmetry group is $U(1)$, the group of local phase transformations on charged fields: $\phi(x) \rightarrow e^{i\alpha(x)}\phi(x)$. In QCD, the corresponding symmetry group is $SU(3)$, and the symmetry acts on the three color charges carried by quarks, usually labeled as red, blue and green. There is a profound difference between QED and QCD: the gauge group $U(1)$ of QED is Abelian, but the gauge group $SU(3)$ of QCD is non-Abelian. Physically, this implies that the photon has no electric charge but gluons do carry color charge. The four-vector field of the photon is written as $A_\mu(x)$ where $\mu = 1, 2, 3, 4$, but an $SU(N)$ gauge field is written as $A_\mu^a(x)$, where the index $a$ runs over the $N^2 - 1$ one charges of the non-Abelian gauge field. While the photon is a free field with a trivial renormalization group beta function, $SU(3)$ gauge theory, even without quarks, is a theory with highly non-trivial interactions. The beta function of an $SU(N)$ gauge theory without quarks is

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \frac{11N}{3}$$

(1)

to lowest order in perturbation theory. Here $g$ is the strong-interaction coupling constant. It is actually a running coupling constant $g(\mu)$, depending on the momentum-space renormalization scale $\mu$, evolving according to

$$\mu \frac{dg}{d\mu} = \beta(g(\mu)).$$

(2)

The negative sign of $\beta(g)$ indicates that $SU(N)$ gauge theories are asymptotically free: There is an ultraviolet fixed point at $g = 0$, and the interactions become weak at high energies. There is a corresponding infrared fixed point at $g = \infty$, indicating that the running coupling $g(\mu)$ becomes large at low energies. This implies that the low-energy properties of QCD, unlike QED, cannot be analyzed via perturbation theory. The most effective tool we have for exploring the non-perturbative aspects of QCD is lattice gauge theory, in which continuous space-time is replaced by discrete lattice, with a fundamental cut-off scale set by the lattice spacing $a$. The renormalization group plays a crucial role in the taking of
the continuum limit, where $a \to 0$. In gauge theories, the gauge field has four-components but two physical polarization states. Two of the four components of the field correspond to redundant degrees of freedom, and must be eliminated by a choice of gauge, such as Coulomb gauge or Landau gauge, for continuum field theory methods to be applied. Lattice gauge theory, on the other hand, maintains exact gauge invariance: no gauge fixing is required, although it may be applied to lattice gauge theories. Physical predictions of theory, such as hadron masses, must of course be independent of the choice of gauge. A great deal of the technical problems in analytic treatments of non-perturbative aspects of gauge theories are ultimately associated with achieving gauge invariance and correct renormalization group scaling behavior simultaneously.

The static potential between two heavy quarks can be determined in a gauge-invariant way from the expectation value of the Wilson loop. This is a non-local operator associated with a closed curve $C$ in space-time parametrized as $x_\mu(\tau)$ where $\tau$ can be taken to be the unit interval and $x_\mu(1) = x_\mu(0)$. The Wilson loop is defined as

$$W[C] = \mathcal{P} \exp \left[ i \oint_C dx_\mu A_\mu(x) \right]$$

(3)

where $\mathcal{P}$ indicates path-ordering of the gauge fields $A_\mu(x)$ along the path $C$. Taking $W[C]$ to be an element of the abstract gauge group $G$, the basic observables are $\text{Tr}_R W[C]$, where the trace is taken over an irreducible representation $R$ of $G$. $\text{Tr}_R W[C]$ has a physical interpretation of the non-Abelian phase factor associated with a heavy particle in the representation $R$ moving adiabatically around the closed loop $C$. Typically, we are interested in rectangular Wilson loops of with sides $L$ and $T$. The loop can be associated with a process in which a particle-antiparticle pair are created at one time, move to a separation $L$, propagate forward in time for an interval $T$, and then annihilate. For $T \gg L$, we have

$$\langle \text{Tr}_R W[C] \rangle \simeq \exp \left[ -V_R(L) T \right]$$

(4)

where $V_R(L)$ is the heavy quark-antiquark potential for particles in the representation $R$. For large distances, the potential can grow no faster than linearly with $L$, and that linear growth defines the string tension $\sigma_R$ for the representation $R$:

$$V(L) \to \sigma_R L + \mathcal{O}(1)$$

(5)

as $L \to \infty$. The $\mathcal{O}(1)$ correction term represents a so-called perimeter law contribution, which is not physical; there are also $1/L$ corrections which are of physical interest. The
statement that quarks are confined is the statement that the string tension \( \sigma_F \) in the fundamental representation of \( SU(3) \) gauge theory is non-zero. A characteristic feature of pure gauge theories in four dimensions is dimensional transmutation: the classical action is scale-invariant, but the introduction of a mass scale \( \mu \) is necessary in order to define the running coupling constant \( g(\mu) \). It follows directly from the renormalization group that any physical mass in four-dimensional pure gauge theories must be proportional to a renormalization group invariant mass, usually written simply as \( \Lambda \). This implies that \( \sigma_R = c_R \Lambda^2 \), where \( c_R \) is a pure number.

Paradoxically, string tensions are generally determined in so-called pure gauge theories, where only the gauge fields are dynamical; quarks and particles exist only as static classical charges. This is done because of string breaking, also known as charge screening. Consider a theory with dynamical particles of mass \( m \) in the fundamental representation \( F \) of the gauge group. Then energy obtained by separating a pair of static sources in the fundamental representation is \( \sigma_F L \); when that energy becomes on the order of \( 2m \), it will become energetically favorable to produce a pair of dynamical particles at an energy cost \( 2m \), and no string tension will be seen for \( L \gtrsim \frac{2m}{\sigma_F} \).

If one or more directions in space-time are compact, the string tension may also be determined using the Polyakov loop \( P \), also known as the Wilson line. The Polyakov loop is essentially a Wilson loop that uses a compact direction in space-time to close the curve using a topologically non-trivial path in space time. The typical use for the Polyakov loop is for gauge theories at finite temperature, where space-time is \( R^3 \times S^1 \). The partition function being given by \( Z = Tr \left[ e^{-\beta H} \right] \), the circumference of \( S^1 \) is given by the inverse temperature \( \beta = 1/T \). In this case, we write

\[
P(\vec{x}) = \mathcal{P} \exp \left[ i \int_0^\beta dx_4 A_4(x) \right]
\]

and string tensions may be determined from a two-point function

\[
\langle Tr_R P(\vec{x}) Tr_R P^\dagger(\vec{y}) \rangle \sim e^{-\sigma_R |\vec{x} - \vec{y}|}
\]

a behavior that assumes that the one-point function \( \langle Tr_R P(\vec{x}) \rangle = 0 \). The Polyakov loop one-point function \( \langle Tr_R P(\vec{x}) \rangle \) can be interpreted as the Boltzmann factor \( e^{-\beta F_R} \), where \( F_R \) is the free energy required to add a static particle in the representation \( R \) to the system. Of course, \( \langle Tr_R P(\vec{x}) \rangle = 0 \) implies that \( F_R = \infty \), which is thus a fundamental criterion determining whether particles in the representation \( R \) are confined.
One of the most important concepts in our understanding of confinement is the role of center symmetry. The center of a Lie group is the set of all elements that commute with every other element. For $SU(N)$, this is $Z(N)$. Although the $Z(N)$ symmetry of $SU(N)$ gauge theories can be understood from the continuum theory, it is easier to understand from a lattice point of view. A lattice gauge theory associates link variable $U_\mu(x)$ with each lattice site $x$ and direction $\mu$. The link variable is considered to be the path-ordered exponential of the gauge field from $x$ to $x + \hat{\mu}$: $U_\mu(x) = \exp[iA_\mu(x)]$. Consider a center symmetry transformation on all the links in a given direction on a fixed hyperplane perpendicular to the direction. The standard example from $SU(N)$ gauge theories at finite temperature is $U_4(\vec{x},t) \rightarrow zU_4(\vec{x},t)$ for all $\vec{x}$ and fixed $t$, with $z \in Z(N)$. Because lattice actions such as the Wilson action consist of sums of small Wilson loops, they are invariant under this global symmetry. However, the Polyakov loop transforms as $P(\vec{x}) \rightarrow zP(\vec{x})$, and more generally

$$Tr_RP(\vec{x}) \rightarrow z^{k_R}Tr_RP(\vec{x})$$

where $k_R$ is an integer in the set $\{0, 1, ..., N - 1\}$ and is known as the $N$-ality of the representation $R$. If $k_R \neq 0$, then unbroken global $Z(N)$ symmetry implies $\langle Tr_RP(\vec{x}) \rangle = 0$. Thus global $Z(N)$ symmetry defines the confining phase of a gauge theory. For pure gauge theories at non-zero temperature, the deconfinement phase transition is associated with the loss of $Z(N)$ symmetry at the critical point $T_d$. Below that point $\langle Tr_FP(\vec{x}) \rangle = 0$ but above $T_d$, $\langle Tr_RP(\vec{x}) \rangle \neq 0$.

Notice that the case of zero $N$-ality representations is special within this framework: there is no requirement from $Z(N)$ symmetry that these representations are confined. This includes the adjoint representation, the representation of the gauge particles. However, lattice simulation indicate that $\langle Tr_RP(\vec{x}) \rangle$ is very small for these representations in the confined phase. Although screening by gauge particles must dominate at large distances, these zero $N$-ality representations have well-defined string tensions at intermediate distances scales, e.g., on the order of a few fermi for $SU(3)$, behaving in a manner very similar to representations with non-zero $N$-ality [6, 7].

It is known from lattice simulations that each representation apparently has its own distinct string tension at intermediate distance scales. To a good approximation, the scaling behavior observed in lattice simulations is consistent with Casimir scaling

$$\frac{\sigma_R}{\sigma_F} = \frac{C_R}{C_F}$$

(9)
where $C_R$ is the quadratic Casimir operator for the representation $R$. This behavior is
exact in two-dimensional pure gauge theories. It is generally believed that at sufficiently
large distance scales, screening by gauge particles will lead to a single string tension for all
representations within a given $N$-ality class, i.e., with the same value of $k$. Labeling the
lightest value of $\sigma_R$ within an $N$-ality class by $\sigma_k$, Casimir scaling predicts

$$\frac{\sigma_k}{\sigma_1} = \frac{k(N-k)}{N-1}$$  \hspace{1cm} (10)

However, the observed behavior is also consistent with so-called sine-Law scaling

$$\frac{\sigma_k}{\sigma_1} = \frac{\sin \left( \frac{\pi k}{N} \right)}{\sin \left( \frac{\pi}{N} \right)}$$  \hspace{1cm} (11)

and lattice results have not yet been definitive \[8\].

**III. CONFINEMENT ON $R^3 \times S^1$**

In the last few years, it has proven possible to construct four-dimensional gauge the-
ories for which confinement may be reliably demonstrated using semiclassical methods \[9, 10\]. These models combine $Z(N)$ symmetry, the effective potential for $P$, instantons,
and monopoles into a satisfying picture of confinement for a special class of models. All
of the models in this class have one or more small compact directions. Models with an
$R^3 \times S^1$ topology have been most investigated, and discussion here will be restricted to this
class. For recent work in other compactifications, see \[11\]. The use of one or more compact
directions will cause the running coupling constant of an asymptotically free gauge theory
to be small, so that semiclassical methods are reliable. For example, if the circumference $L$
of $S^1$ is small, i.e., $L \ll \Lambda^{-1}$, then $g(L) \ll 1$. However, this leads to an immediate problem:
generally speaking, one or more small compact directions lead to breaking of $Z(N)$ symme-
try in those directions. This has been clear for many years for the case of finite temperature
gauge theories, where $L = \beta = 1/T$. The effective potential for the Polyakov loop is easily
calculated to lowest order in perturbation theory; for a pure gauge theory it is given by

$$V_{\text{gauge}}(P, \beta) = \frac{-2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{T_{RA} P^n}{n^4}$$  \hspace{1cm} (12)

where the trace of $P$ in the adjoint representation is given by $T_{RA}P = |T_{RF}P|^2 - 1$. This
effective potential is minimized when $P = zI$ where $z \in Z(N)$, indicating that $Z(N)$
symmetry is spontaneously broken at high temperatures where the one-loop expression is valid. Note that when \( P = zI \), the effective potential is just the free energy density of a blackbody with \( 2 (N^2 - 1) \) degrees of freedom. It can easily be shown that any additional fields also act to break \( Z(N) \) symmetry at high temperatures, corresponding to small \( L = \beta \).

There are two broad approaches to maintaining \( Z(N) \) symmetry for small \( L \). The first approach deforms the pure gauge theory by adding additional terms to the gauge action \([9, 12, 13]\). The general form for such a deformation is

\[
S \rightarrow S + \beta \int d^3x \sum_{k=1}^{\infty} a_k \text{Tr} A^k P(\vec{x})^k.
\]

Such terms are often referred to as double-trace deformations. If the coefficients \( a_k \) are sufficiently large, they will counteract the effects of the one-loop effective potential, and \( Z(N) \) symmetry will hold for small \( L \). Strictly speaking, only the first \([N/2]\) terms are necessary to ensure confinement. It is easy to prove that for a classical Polyakov loop \( P \), the conditions \( Tr_F P^k = 0 \) with \( 1 \leq k \leq [N/2] \) determine the unique set of Polyakov loop eigenvalues that constitute a confining solution, i.e., one for which \( Tr_R P = 0 \) for all representations with \( k_R \neq 0 \) \([14]\). The explicit solution is interesting: up to a factor necessary to ensure \( \det P = 1 \), the eigenvalues of \( P \) are given by the set of \( N' \)th roots of unity, which are permuted by a global \( Z(N) \) symmetry transformation. The effective potential associated with \( S \) is given approximately by

\[
V_{1-\text{loop}} (P, \beta) = -\frac{2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{Tr_A P^n}{n^4} + \sum_{k=1}^{[N/2]} a_k Tr_A P^k.
\]

For sufficiently large and positive values of the \( a_k \)'s, the confined phase yields the lowest value of \( V_{1-\text{loop}} \). However, a rich phase structure emerges from the minimization of \( V_{1-\text{loop}} \). For \( N \geq 3 \), the effective potential predicts that one or more phases separate the deconfined phase from the confined phase. In the case of \( SU(3) \), a single new phase is predicted, and has been observed in lattice simulations \([9]\). For larger values of \( N \), there is a rich set of possible phases, including some where \( Z(N) \) breaks down to a proper subgroup \( Z(p) \). In such phases, particles in the fundamental representation are confined, but bound states of \( N/p \) such particles are not \([12]\).

Lattice simulations of \( SU(3) \) and \( SU(4) \) agree with the theoretical predictions based on effective potential arguments \([9]\). The phase diagram of \( SU(3) \) as a function of \( T = L^{-1} \) and \( a_1 \) has three phases: the confined phase, the deconfined phase, and another phase, the skewed
phase, where charge-conjugation symmetry is spontaneously broken. In the case of \( SU(4) \), a sufficiently large value of \( a_1 \) leads to a partially-confining phase where \( Z(4) \) is spontaneously broken to \( Z(2) \). Particles with \( k = 1 \) are confined in this phase, \( i.e., \langle Tr_F P (\vec x) \rangle = 0 \), but particles with \( k = 2 \) are not, as indicated by \( \langle Tr_F P^2 (\vec x) \rangle \neq 0 \). An important result obtained from the lattice simulation of \( SU(3) \) is that the small-\( L \) confining region, where semiclassical methods yield confinement, are smoothly connected to the conventional large-\( L \) confining region.

Another approach to preserving \( Z(N) \) symmetry for small \( L \) uses fermions in the adjoint representation with periodic boundary conditions in the compact direction [10]. In this case, it is somewhat misleading to use \( \beta \) as a synonym for \( L \), because the transfer matrix for evolution in the compact direction is not positive-definite. Periodic boundary conditions in the timelike direction imply that the generating function of the ensemble, \( i.e., \) the partition function, is given by

\[
Z = Tr \left[ (-1)^F e^{-\beta H} \right]
\]

where \( F \) is the fermion number. This graded ensemble, familiar from supersymmetry, can be obtained from an ensemble \( Tr \left[ \exp (\beta \mu F - \beta H) \right] \) with chemical potential \( \mu \) by the replacement \( \beta \mu \to i\pi \). This system can be viewed as a gauge theory with periodic boundary conditions in one compact spatial direction of length \( L = \beta \), and the transfer matrix in the time direction is positive-definite.

The use of periodic boundary conditions for the adjoint fermions dramatically changes their contribution to the Polyakov loop effective potential. In perturbation theory, the replacement \( \beta \mu \to i\pi \) shifts the Matsubara frequencies from \( \beta \omega_n = (2n + 1) \pi \) to \( \beta \omega_n = 2n\pi \). The one loop effective potential is now essentially that of a bosonic field, but with an overall negative sign due to fermi statistics [15]. The sum of the effective potential for the fermions plus that of the gauge bosons gives

\[
V_{1-looP} (P, \beta, m, N_f) = \frac{1}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{Tr_AP^n}{n^2} \left[ 2N_f \beta^2 m^2 K_2 (n\beta m) - \frac{2}{n^2} \right]
\]

where \( N_f \) is the number of adjoint Dirac fermions and \( m \) is their mass. Note that the first term in brackets, due to the fermions, is positive for every value of \( n \), while the second term, due to the gauge bosons, is negative.

The largest contribution to the effective potential at high temperatures is typically from
the $n = 1$ term, which can be written simply as
\[
\frac{1}{\pi^2 \beta^4} \left[ 2N_f \beta^2 m^2 K_2(\beta m) - 2 \right] \left[ |Tr_F P|^2 - 1 \right]
\]  
(17)
where the overall sign depends only on $N_f$ and $\beta m$. If $N_f \geq 1$ and $\beta m$ is sufficiently small, this term will favor $Tr_F P = 0$. On the other hand, if $\beta m$ is sufficiently large, a value of $P$ from the center, $Z(N)$, is preferred. Note that an $\mathcal{N} = 1$ super Yang-Mills theory would correspond to $N_f = 1/2$ and $m = 0$, giving a vanishing perturbative contribution for all $n$. In that case, non-perturbative effects lead to a confining effective potential for all values of $\beta$. In the case of $N_f \geq 1$, each term in the effective potential will change sign in succession as $m$ is lowered towards zero. For larger values of $N$, this leads to a cascade of phases separating the confined and deconfined phases. As $m$ increases, it becomes favorable that $Tr_F P^n \neq 0$ for successive values of $n$. If $N$ is even, the first phase after the confined phase will be a phase with $Z(N/2)$ symmetry. As $m$ increases, the last phase before reaching the deconfined phase will have $Z(2)$ symmetry, in which $k = 1$ states are confined, but all states with higher $k$ are not. Lattice simulations of $SU(3)$ with periodic adjoint fermions are completely consistent with this picture. For $N \geq 3$, there are generally phases intermediate between the confined and deconfined phases which are not of the partially-confined type. Careful numerical analysis appears to be necessary on a case-by-case basis to determine the phase structure for each value of $N$.

There are some interesting additional issues arising when periodic adjoint fermions are used to obtain $Z(N)$ symmetry for small $L$. There are strong indications from strong-coupling lattice calculations and the closely related Polyakov-Nambu-Jona Lasinio (PNJL) models that the mass $m$ that appears in the effective potential $V(P)$ should be regarded as a constituent mass that includes the substantial effects of chiral symmetry breaking. In strong-coupling lattice calculations and PNJL models, this effect is responsible for the coupling of $P$ and $\bar{\psi}\psi$ in simulations of QCD at finite temperature. Thus it is important that recent simulations of $SU(3)$ with $N_f = 2$ flavors of adjoint fermions show explicitly that the $Z(N)$-invariant confined phase is regained when the fermion mass is sufficiently small. In fact, the simulations show the behavior expected on the basis of effective potential arguments: a skewed phase separates the confined phase and deconfined phase. However, this raises another issue: For a simple double-trace deformation, the small-$L$ and large-$L$ regions are smoothly connected. Is this also the case if periodic adjoint fermions...
are used? A semi-phenomenological analysis based on a PNJL model \cite{23} suggests that the answer is yes. However, there may be problems in reaching sufficiently light fermion masses in lattice simulations to confirm this.

In addition to the overall phase structure, semiclassical methods may also be used to determine string tensions for small $L$; the same methods are used with double-trace deformations and with periodic adjoint fermions. There are two classes of string tensions, depending on whether the Polyakov loop or the Wilson loop is used. For simplicity, we will refer to the former case as electric and the latter as magnetic, borrowing the terminology from finite temperature gauge theory. Electric string tensions are calculable perturbatively in the high-temperature confining region from small fluctuations about the confining minimum of the effective potential \cite{16, 24}. The $m = 0$ limit has the simple form

$$
\left( \sigma_k^{(i)} L \right)^2 = \frac{\left( 2N_f - 1 \right) g^2 T^2}{3N} \left[ 3 \csc^2 \left( \frac{\pi k}{N} \right) - 1 \right]
$$

and is a good approximation for $\beta m \ll 1$. The confining minimum of the effective potential breaks $SU(N)$ to $U(1)^{N-1}$. This remaining unbroken Abelian gauge group naively seems to preclude spatial confinement, in the sense of area law behavior for spatial Wilson loops. However, as first discussed by Polyakov in the case of an $SU(2)$ Higgs model in $(2 + 1)$-dimensional gauge systems, instantons can lead to nonperturbative confinement \cite{25}. There are $N$ different fundamental monopoles in the gauge theory on $R^3 \times S^1$. This behavior was first seen in the construction of finite temperature instantons (calorons) with non-trivial Polyakov loop behavior at infinity \cite{26–30}. As in the case of Polyakov’s original model, the string tension is proportional to $\exp (-S_{\text{monopole}})$, where $S_{\text{monopole}}$ is the action of a single monopole. The other factors are essentially determined by the renormalization group up to an overall numerical factor \cite{13, 31}.

There remain, however, a number of interesting issues which require further analytic study, including string tension scaling laws for Wilson loops, the mechanism for chiral symmetry breaking, the approach to the large-$N$ limit of $SU(N)$, geometries with more than one compact direction \cite{11}, and gauge groups other than $SU(N)$. Lattice simulations have the potential to provide much useful information, for example on the connection between confinement at small $L$ and large $L$. Because of the small value of the running coupling constant at small $L$, gauge theories on $R^3 \times S^1$ are a natural area where lattice simulations and analytic calculation can work together effectively.
IV. CONFINEMENT VIA REAL-SPACE RENORMALIZATION GROUP

$SU(N)$ gauge theories in four dimensions have a single, Gaussian ultra-violet fixed point at $g^2 = 0$. This means that QCD with massless fermions has no free parameters once a renormalization-group invariant length scale $\Lambda$ is defined. However, non-perturbative features can so far only be determined in lattice simulations. The natural framework for dealing with such problems is a Wilsonian RG blocking procedure bridging the different scale regimes. A variety of lattice gauge theory approaches have been developed, including techniques for finding the action along the Wilsonian renormalization group trajectory (“perfect action”) and Monte Carlo renormalization group techniques. An alternative approach which has been extensively developed by Tomboulis is based on approximate real-space renormalization group transformations which satisfy rigorous bounds [32–35]. The general strategy is to employ approximate but easily explicitly computable renormalization group transformations that provide bounds on the ratios of partition functions, leading to results on free energy differences. Sufficiently strong bounds constrain will constrain the exact free energy differences.

The Migdal-Kadanoff approximation provides a starting point for rigorous analysis [36–39]. The Migdal-Kadanoff procedure uses a combination of decimation and bond-moving to construct an approximate renormalization group which gives a rigorous lower bound on free energies. Decimation refers to the procedure of integrating over some of the variable of a lattice system, e.g. a lattice spin system such as the Ising model or a lattice quantum field theory. Suppose the partition function for such a system is defined on lattice $\Lambda$ as $Z_\Lambda = Tr_\Lambda \exp(-S)$ where $S$ is the Euclidean action for a field theory or $\beta H$ for a classical system. The $Tr_\Lambda$ indicates that all variables on the lattice $\Lambda$ are to be integrated over. One way to construct a real-space renormalization group transformation is to integrate over some of the variables, leaving only the variables on a sub-lattice $\Lambda'$ and a new action $S'$ for those variables. This may be written as

$$\exp(-S') = Tr_{\Lambda'/\Lambda} \exp(-S)$$

where the trace is over variables on the complement $\Lambda'/\Lambda$ of $\Lambda$. A typical choice for $\Lambda'$ is to go from a lattice with lattice spacing $a$ to a sub-lattice of the same type with lattice spacing $ba$ with $b > 1$. However, this procedure can only be carried analytically for the simplest of models, notably $d = 1$ spin systems and $d = 2$ gauge theories with local interactions.
In higher dimensions, new, complicated quasi-local interactions are generated. The Migdal-Kadanoff procedure uses bond-moving to restrict the terms generated by decimation. Bond moving refers to modifying the lattice action from \( S \) to \( S + \Delta S \) in such a way that the average action is unchanged: \( \text{Tr}_\Lambda \left[ e^S \Delta S \right] = 0 \). In the simplest case, \( \Delta S \) decouples some of the variables on \( \Lambda/\Lambda' \) from those on \( \Lambda' \) to eliminate unwanted couplings. This procedure gives a bound \( Z'_{\Lambda'} \) on the original partition function

\[
Z'_{\Lambda'} = \text{Tr}_\Lambda \exp(-S - \Delta S) = Z_\Lambda \left( \frac{Z'_{\Lambda'}}{Z_\Lambda} \right) \geq Z_\Lambda
\]

where Jensen’s inequality has been used in the form \( \langle e^{-\Delta S} \rangle_\Lambda \geq e^{-\langle \Delta S \rangle_\Lambda} = 1 \). With a good choice for \( \Delta S \), this procedure leads to a relatively simple set of renormalization group equations for the coupling constants of the model. This procedure can be generalized in principle to include blocking of variables, in which a variable on \( \Lambda' \) represents a weighted average of variables on \( \Lambda \). The Migdal-Kadanoff approach has been usefully applied in lattice gauge theory to give an understanding of renormalization group flows \[40, 41\] and the deconfinement transition \[42, 43\] in lattice gauge theories.

Tomboulis has generalized the Migdal-Kadanoff procedure in such a way that not only is the partition function left invariant, but other quantities are left invariant as well. Suppose the goal is to calculate or estimate an expectation value

\[
\langle O \rangle = \frac{Z_\Lambda [O]}{Z_\Lambda}
\]

The goal is then to construct a renormalization group procedure consisting of a set of steps

\[
a \to ba \to b^2a \to ... \to b^n a \quad \Lambda \to \Lambda^{(1)} \to \Lambda^{(2)} \to ... \to \Lambda^{(n)}
\]

such that

\[
\langle O \rangle = \frac{Z_\Lambda [O]}{Z_\Lambda} = ... = \frac{Z_{\Lambda_m} [O]}{Z_{\Lambda_m}} = ... = \frac{Z_{\Lambda_n} [O]}{Z_{\Lambda_n}}
\]

After a sufficiently large number of renormalization group transformations, an asymptotically free theory will reach the strong-coupling region near the infra-red fixed point at \( g^2 = \infty \), where cluster expansion methods can be used reliably. Note that the construction is specific to a given operator \( O \): different operators will give rise to different renormalization group flows. In principle, the difference between the two flows would be irrelevant in the continuum limit.
The prototypical application of this approach is to the twisted partition function of an $SU(2)$ lattice gauge theory \[44\]. The twisted partition function $Z_{\Lambda}^{(-)}$ on an $L_1 \times L_2 \times L_3 \times L_4$ lattice $\Lambda$ is obtained from the usual partition function $Z_{\Lambda}$ by modifying the action such that plaquettes in the $x - y$ plane with fixed values $x_0$ and $y_0$ of $x$ and $y$ are multiplied in the action by $-1$ for all values of $z$ and $t$: $U_{xy}(x_0, y_0, z, t) \rightarrow -U_{xy}(x_0, y_0, z, t)$. The twisted partition function was originally defined in the continuum by 't Hooft and plays a central role in his analysis of the allowed phases of gauge theories \[45, 46\]. The vortex free energy $F_{\Lambda}$ is defined by the ratio of the two partition functions: $\exp \left[ -F_{\Lambda} \right] = Z_{\Lambda}^{(-)} / Z_{\Lambda}$; in this ratio the bulk free energy term proportional to space-time volume cancels. 't Hooft’s analysis shows that confinement holds if

$$F_{\Lambda} \sim cL_3L_4e^{-\rho L_1L_2}$$

with $\rho > 0$ as the infinite space-time volume limit is taken with $L_1L_2 \gg \log (L_3L_4)$. For lattice gauge theories, a theorem due to Tomboulis and Yaffe proves rigorously that the Wilson loop has area law behavior if $\rho$ is non-zero:

$$\langle W(C) \rangle \leq 2 \left[ \frac{1}{2} \left( 1 - e^{-F_{\Lambda}} \right) \right]^{A/L_1L_2}$$

where $A$ is the area associated with $C$ \[47\].

There remain some unresolved questions with this construction. There has been some criticism of both the implicit nature of the renormalization group construction and its possible inability to distinguish between asymptotically-free theories like $SU(2)$ and the non-asymptotically free $U(1)$ lattice gauge theory \[48, 50\]. It is true that the renormalization group flow is defined implicitly, and there is not yet a complete demonstration or working implementation of the specific renormalization group procedure proposed. However, this is related to the second, more physical criticism, which is based on the inability of Migdal-Kadanoff renormalization group equations to differentiate between non-Abelian and Abelian groups in the flow equations. This is in fact a very subtle issue. The four-dimensional $U(1)$ lattice gauge theory has two phases: a weak-coupling phase which is a lattice version of a free photon theory, and a strong-coupling phase where magnetic monopoles are responsible for confinement. After decades of effort, lattice simulations have not yet conclusively determined the order of the transition between these phases. It is believed that the weak-coupling phase consists of a line of renormalization group fixed points, similar to that found in the
\(d = 2\) XY model, but no reliable renormalization group calculation shows this. The Migdal-Kadanoff renormalization group shows a dramatic change in the renormalization group flow near the known critical point: the renormalization group beta function becomes very small, becoming proportional to \(\exp(-c/g^2)\). This behavior suggests that the renormalization group is trying to account for magnetic monopole effects. Exactly the same behavior is seen when the Migdal-Kadanoff scheme is applied to the \(d = 2\) XY model \[51\]. In this case, there is a reliable renormalization group calculation available, but it depends on the introduction of a vortex fugacity and a renormalization group flow with the coupling and the vortex fugacity as independent couplings. In the generalized Migdal-Kadanoff renormalization group developed by Tomboulis, the domain of decimation parameters is enlarged relative to the original Migdal-Kadanoff scheme. The issue of the existence of a fixed point versus a quasi-fixed point which fails by exponentially small corrections to be a true fixed point in the U(1) gauge theory thus requires renewed investigation in this more general context before it is settled. Thus the original debate points to the large, difficult issue of accounting for the infrared effects of topological excitations, particularly within a renormalization group framework.

Beyond issues concerning renormalization group flows in pure gauge theories, there are important issues arising from the inclusion of fermions within a real-space renormalization group framework. Fermions appear in functional integrals as Grassmann variables, i.e. anti-commuting \(c\)-numbers, and cannot be simulated or approximated in the same way as bosonic variables. Fermion fields that appear only quadratically in the action can be formally integrated over, yielding a functional determinant that acts as an additional weight in the bosonic functional integral. In lattice simulations, this functional determinant is included, essentially as an additional, non-local interaction that is expensive to simulate. In the context of real-space renormalization group calculations, any partial integration over fermions with light masses would introduce non-local interactions difficult to incorporate into a renormalization group calculation. Furthermore, the bounds on partition functions that hold for bosonic functional integrals no longer hold for fermionic degrees of freedom. Nevertheless, there has been some progress. Some analytical work has been carried out for free fermions \[52, 53\], and there has been more recent work in connection with the rooting problem of staggered lattice fermions \[54\]. In work reported at the workshop, Tomboulis discussed a blocking technique for fermions that is capable of producing a non-trivial fixed point in
SU(2) lattice gauge theory for a sufficiently large number of flavors $N_f$. The existence of such a fixed point is indicated by perturbation theory, but the fixed point is generally located outside the region of validity of perturbation theory. The presence of a non-trivial infrared fixed point indicates the existence of a “conformal window” which is of great interest for studies of particle physics beyond the standard model [55, 56].

V. CONFINEMENT VIA THE FUNCTIONAL RENORMALIZATION GROUP AND DYSON-SCHWINGER EQUATIONS

The closely related functional renormalization group and Dyson-Schwinger approaches to quark confinement have much in common with the research on confinement on $R^3 \times S^1$ described in section 3, including the construction of the Polyakov loop effective potential. However, these approaches focus on the infrared properties of conventional gauge and ghost propagators. In the functional renormalization group approach the renormalization group is used directly in attempting to determine the non-perturbative infrared properties of gauge theories from the renormalization group flow equation [57]. The Schwinger-Dyson approach [58, 59] aims at the calculation of the same quantities by a self-consistent solution of the Schwinger-Dyson equations, which are the equations of motion for correlation functions. Although beyond the scope of this review, both approaches can be used to study the interplay of confinement and chiral symmetry breaking [59, 60], with results similar to those obtained from more phenomenological approaches such as the PNJL model [22].

The functional renormalization group can be elegantly described for any field theory by the RG flow equation for the effective action $\Gamma[\phi]$ [61, 62]. The effective action $\Gamma[\phi]$ is the generator of one-particle irreducible (1PI) vertices, where $\phi$ is shorthand notation for one or more classical fields, one for each quantum field. The $n$-point 1PI vertices are obtained from $\Gamma[\phi]$ by functional differentiation:

$$\Gamma^{(n)}(x_1,...,x_n) = \frac{\delta^n \Gamma}{\delta \phi(x_1)...\delta \phi(x_n)}$$

(25)

The 2-point function $\Gamma^{(2)}(x_1,x_2)$ is the full propagator. The functional renormalization group generalizes $\Gamma[\phi]$ to $\Gamma_k[\phi]$, which is identical to $\Gamma[\phi]$, except that only momenta with $q^2 \gtrsim k^2$ are included in the $n$-point vertices. This exclusion of infrared modes is accomplished by inserting an infrared cutoff function $R_k$ into the functional integral that defines $\Gamma[\phi]$, so that
by definition $\Gamma_0 [\phi] = \Gamma [\phi]$. The RG equation for $\Gamma_k [\phi]$ is

$$\frac{\partial}{\partial k} \Gamma_k [\phi] = \frac{1}{2} Tr \left\{ \frac{1}{\Gamma_k^{(2)} [\phi]} + R_k \frac{\partial}{\partial k} R_k \right\} \tag{26}$$

where the trace involves an integration over momenta or coordinates as well as summation over any internal indices. For the study of spontaneous symmetry breaking, the key quantity is typically the effective potential $V (\phi)$, obtained by setting $\phi$ to a space-time independent value, and dividing $\Gamma [\phi]$ by the volume $\Omega$ of space-time: $V (\phi) = \Gamma [\phi] / \Omega$.

The functional renormalization group formalism can be applied to gauge theories with some modifications of the procedure for scalar fields: a gauge-fixing term must be added to the action and ghost fields must be introduced. In an asymptotically free theory, one can approximate $\Gamma_k$ by the classical action $S$ for large $k$, and evolve the functional RG equation down to $k = 0$. The effective potential for $A_4$ is given by $V (A_4) = \Gamma [A_4] / \Omega$, where $A_4$ is a constant, representing a constant Polyakov loop. This effective action can be evaluated using the background field method, and the effective potential is obtained within the functional renormalization group approach from

$$V_k (A_4) = \frac{\Gamma_k [A_4]}{\Omega} \tag{27}$$

in the limit $k \to 0$. In principal, the functional renormalization group is exact, but in practice, approximations are necessary. A scaling solutions is assumed in the infrared, in which the transverse gauge field propagator scales as $1 / (p^2)^{1+\kappa_A}$ and the ghost propagator scales as $1 / (p^2)^{1+\kappa_C}$. Then at sufficiently low temperature, the effective potential is given by

$$V (A_4) = \left\{ \frac{d-1}{2} (1 + \kappa_A) + \frac{1}{2} - (1 + \kappa_C) \right\} \cdot \frac{1}{\Omega} Tr \log \left[-D^2 [A_4] \right] \tag{28}$$

where $-D^2 [A_4]$ is the standard scalar propagator for a particle at finite temperature in a background field $A_4$. This should be compared with the standard high-temperature expression

$$V (A_4) = \left\{ \frac{d-1}{2} + \frac{1}{2} - 1 \right\} \cdot \frac{1}{\Omega} Tr \log \left[-D^2 [A_4] \right]. \tag{29}$$

If the solution of the functional renormalization group equations satisfies

$$\frac{d-1}{2} (1 + \kappa_A) + \frac{1}{2} - (1 + \kappa_C) < 0 \tag{30}$$

then the sign of the effective potential changes between high- and low-temperature, leading to the same confining minimum at low temperature found at high temperature in section 3.
This scenario was studied in Braun, Gies, and Pawlowski [57] using Landau-DeWitt gauge, and the values $\kappa_A = -1.19\ldots$ and $\kappa_C = 0.595\ldots$ were obtained, indicating confinement.

A closely related approach is based on the self-consistent solution of the Dyson-Schwinger equations in the infrared [58]. The Dyson-Schwinger equations are an infinite set of coupled equations relating $n$-point correlation functions to correlation functions of higher order. Again using a shorthand notation, the Dyson-Schwinger equations may be derived from functional integration by parts

$$\int [d\phi] \frac{\delta}{\delta\phi(y)} \left[\phi(x_1) \ldots \phi(x_n) e^{-S[\phi]}\right] = 0 \quad (31)$$

The simplest case of $n = 0$ reduces to $\langle \delta S/\delta\phi(y) \rangle = 0$, the statement that the classical equation of motion holds on the average. In both the functional renormalization group and Dyson-Schwinger approaches, an infinite tower of coupled functional equations for the correlation functions are solved, and both approaches include the same physics, albeit captured in different ways. At present, studies using the Schwinger-Dyson approach have restricted themselves mainly to the infrared regime relying on a general power counting analysis of the possible IR fixed points of the full system of equations of quenched QCD, i.e., without quarks, in the Landau gauge. Applying this approach to gauge theories requires extensive graphical analysis to identify the most important contributions to the Dyson-Schwinger equations in the infrared. Choice of gauge is important in simplifying the analysis. One finds both a decoupling solution that does not feature any infrared enhancement and a scaling solution where infrared divergences are induced in the ghost sector [58]. In the scaling solution, a long range interaction in gauge-dependent local correlation functions in the quark sector arises from a strong kinematic infrared divergence of the quark-gluon vertex. In this approach, the Wilson loop is then given by an infinite series of these local Green functions and the infrared divergence of the leading term, given by the quark-quark scattering kernel, yields in the quenched approximation area law behavior [59]. Recently work indicates that the corresponding long range interaction is a universal property of the gauge sector [63]. Although there are some significant differences at a technical level, the Dyson-Schwinger and functional renormalization group approaches are very similar, and in Landau gauge, the infrared behavior found for the ghost and gluon propagators is essentially the same [58, 64].
VI. CONCLUSIONS

The problem of quark confinement is an important and vital subject in theoretical physics, connected not just to hadronic physics, but to many areas of physics by concepts and techniques. It is the quintessential problem of dimensional transmutation, where renormalization introduces a length scale into a classically scale-invariant system. Fundamental questions of mechanism, i.e., the cause of confinement, have not yet been answered. In addition to questions of mechanism, there are also questions surrounding the implications of confinements, e.g., string tension scaling laws and other string properties.

Approaches have been developed in the last few years that are shedding new light on these problems. The discovery of a rich set of new phases in gauge theories on $R^3 \times S^1$ shows that there are new phenomena that remain to be uncovered. The topics discussed here, despite differences of approach and emphasis, have more in common with each other than the use of the renormalization group. For example, recent work using the functional renormalization group and the Dyson-Schwinger equation shares with work on $R^3 \times S^1$ an emphasis on the effective potential for the Polyakov model. As discussed in section 3, there are issues concerning magnetic monopoles raised by recent work using real-space renormalization group methods that are related to recent work on $R^3 \times S^1$. Beyond common technical issues, there are two basic themes all the approaches discussed here have in common: an incorporation of both the physics of continuum gauge theories indicated by the renormalization group and the increasingly precise results of lattice gauge theory. This marks a significant departure from previous approaches to confinement, which tended to neglect either lattice or continuum results, and should be welcomed as genuine progress towards greater understanding.

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[1] O. W. Greenberg, Phys. Rev. Lett. 13, 598 (1964).
[2] H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).
[3] D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973).
[4] K. G. Wilson, Phys. Rev. D 10, 2445 (1974).
[5] M. Creutz, Phys. Rev. D 21, 2308 (1980).
[6] S. Deldar, Phys. Rev. D 62, 034509 (2000) [arXiv:hep-lat/9911008].
[7] G. S. Bali, Phys. Rev. D 62, 114503 (2000) [arXiv:hep-lat/0006022].
[8] J. Greensite, Prog. Part. Nucl. Phys. 51, 1 (2003) [arXiv:hep-lat/0301023].
[9] J. C. Myers and M. C. Ogilvie, Phys. Rev. D 77, 125030 (2008) [arXiv:0710.0649 [hep-lat]].
[10] M. Unsal, Phys. Rev. Lett. 100, 032005 (2008) [arXiv:0708.1772 [hep-th]].
[11] T. Azeyanagi, M. Hanada, M. Unsal and R. Yacoby, [arXiv:1006.0717 [hep-th]].
[12] M. C. Ogilvie, P. N. Meisinger and J. C. Myers, PoS LAT2007, 213 (2007) [arXiv:0710.0649 [hep-lat]].
[13] M. Unsal and L. G. Yaffe, [arXiv:0803.0344 [hep-th]].
[14] P. N. Meisinger, T. R. Miller and M. C. Ogilvie, Phys. Rev. D 65, 034009 (2002) [arXiv:hep-ph/0108009].
[15] P. N. Meisinger and M. C. Ogilvie, Phys. Rev. D 65, 056013 (2002) [arXiv:hep-ph/0108026].
[16] P. N. Meisinger and M. C. Ogilvie, Phys. Rev. D 81, 025012 (2010) [arXiv:0905.3577 [hep-lat]].
[17] N. M. Davies, T. J. Hollowood, V. V. Khoze and M. P. Mattis, Nucl. Phys. B 559, 123 (1999) [arXiv:hep-th/9905015].
[18] N. M. Davies, T. J. Hollowood and V. V. Khoze, J. Math. Phys. 44, 3640 (2003) [arXiv:hep-th/0006011].
[19] G. Cossu and M. D’Elia, JHEP 0907, 048 (2009) [arXiv:0904.1353 [hep-lat]].
[20] J. C. Myers and M. C. Ogilvie, JHEP 0907, 095 (2009) [arXiv:0903.4638 [hep-th]].
[21] A. Gocksch and M. Ogilvie, Phys. Rev. D 31, 877 (1985).
[22] K. Fukushima, Phys. Lett. B 591, 277 (2004) [arXiv:hep-ph/0310121].
[23] H. Nishimura and M. C. Ogilvie, Phys. Rev. D 81, 014018 (2010) [arXiv:0911.2696 [hep-lat]].
[24] P. N. Meisinger and M. C. Ogilvie, Nucl. Phys. Proc. Suppl. 140, 650 (2005).
[25] A. M. Polyakov, Nucl. Phys. B 120, 429 (1977).
[26] K. M. Lee, Phys. Lett. B 426 (1998) 323 [arXiv:hep-th/9802012].
[27] T. C. Kraan and P. van Baal, Phys. Lett. B 428 (1998) 268 [arXiv:hep-th/9802049].
[28] K. M. Lee and C. H. Lu, Phys. Rev. D 58 (1998) 025011 [arXiv:hep-th/9802108].
[29] T. C. Kraan and P. van Baal, Nucl. Phys. B 533 (1998) 627 [arXiv:hep-th/9805168].
[30] T. C. Kraan and P. van Baal, Phys. Lett. B 435 (1998) 389 [arXiv:hep-th/9806034].
[31] K. Zarembo, Nucl. Phys. B 463 (1996) 73 [arXiv:hep-th/9510031].
[32] E. T. Tomboulis, [arXiv:0707.2179 [hep-th]].
[33] E. T. Tomboulis, PoS LAT 2007, 336 (2007) [arXiv:0710.1894 [hep-lat]].
[34] T. E. Tomboulis, PoS CONFINEMENT 8, 020 (2008).
[35] E. T. Tomboulis, Mod. Phys. Lett. A 24, 2717 (2009).
[36] A. A. Migdal, Sov. Phys. JETP 42, 743 (1975) [Zh. Eksp. Teor. Fiz. 69, 1457 (1975)].
[37] A. A. Migdal, Sov. Phys. JETP 42, 413 (1975) [Zh. Eksp. Teor. Fiz. 69, 810 (1975)].
[38] L. P. Kadanoff, Annals Phys. 100, 359 (1976).
[39] L. P. Kadanoff, Rev. Mod. Phys. 49, 267 (1977).
[40] K. M. Bitar, S. A. Gottlieb and C. K. Zachos, Phys. Rev. D 26, 2853 (1982).
[41] K. M. Bitar, D. W. Duke and M. Jadid, Phys. Rev. D 31, 1470 (1985).
[42] M. Ogilvie, Phys. Rev. Lett. 52, 1369 (1984).
[43] M. Imachi and H. Yoneyama, Prog. Theor. Phys. 78, 623 (1987).
[44] E. T. Tomboulis, Phys. Rev. Lett. 50, 885 (1983).
[45] G. 't Hooft, Nucl. Phys. B 138, 1 (1978).
[46] G. 't Hooft, Nucl. Phys. B 153, 141 (1979).
[47] E. T. Tomboulis and L. G. Yaffe, Commun. Math. Phys. 100, 313 (1985).
[48] K. R. Ito and E. Seiler, [arXiv:0711.4930 [hep-th]].
[49] E. T. Tomboulis, [arXiv:0712.2620 [hep-th]].
[50] K. R. Ito and E. Seiler, [arXiv:0803.3019 [hep-th]].
[51] J. V. Jose, L. P. Kadanoff, S. Kirkpatrick and D. R. Nelson, Phys. Rev. B 16, 1217 (1977).
[52] T. Balaban, M. O'Carroll and R. Schor, Commun. Math. Phys. 122, 233 (1989).
[53] U. J. Wiese, Phys. Lett. B 315, 417 (1993) [arXiv:hep-lat/9306003].
[54] Y. Shamir, Phys. Rev. D 75, 054503 (2007) [arXiv:hep-lat/0607007].
[55] T. Appelquist, J. Terning and L. C. R. Wijewardhana, Phys. Rev. Lett. 77, 1214 (1996) [arXiv:hep-ph/9602385].

[56] T. Appelquist, A. Ratnaweera, J. Terning and L. C. R. Wijewardhana, Phys. Rev. D 58, 105017 (1998) [arXiv:hep-ph/9806472].

[57] J. Braun, H. Gies and J. M. Pawlowski, Phys. Lett. B 684, 262 (2010) [arXiv:0708.2413 [hep-th]].

[58] R. Alkofer, M. Q. Huber and K. Schwenzer, Phys. Rev. D 81, 105010 (2010) [arXiv:0801.2762 [hep-th]].

[59] R. Alkofer, C. S. Fischer, F. J. Llanes-Estrada and K. Schwenzer, Annals Phys. 324, 106 (2009) [arXiv:0804.3042 [hep-ph]].

[60] J. Braun, L. M. Haas, F. Marhauser and J. M. Pawlowski, arXiv:0908.0008 [hep-ph].

[61] J. Berges, N. Tetradis and C. Wetterich, Phys. Rept. 363, 223 (2002) [arXiv:hep-ph/0005122].

[62] J. M. Pawlowski, Annals Phys. 322, 2831 (2007) [arXiv:hep-th/0512261].

[63] L. Fister, R. Alkofer and K. Schwenzer, Phys. Lett. B 688, 237 (2010) [arXiv:1003.1668 [hep-th]].

[64] C. S. Fischer, A. Maas and J. M. Pawlowski, Annals Phys. 324 (2009) 2408 [arXiv:0810.1987 [hep-ph]].