Sound in relativistic superfluid with vorticity

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Abstract

We develop a theory of sound in a relativistic superfluid with quantum vortices. The vortices are presented by vortex fluid. For a particular separable model we find new modes of which a non-relativistic superfluid is deprived.

47.75.+f, 3.40.Kf, 67.40.Vs, 97.60.Jd

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I. INTRODUCTION

1. The basis of relativistic superfluid mechanics was emphasized by Israel \cite{1} and Dixon \cite{2} and their achievements were developed by several authors \cite{3,4,5,6,7}. Since the relativistic superfluid matter of neutron stars, being the main object of applications of this theory, contains quantum vortices, an ample discussion of a relativistic superfluid vortex prompted in diverse forms \cite{9,10,11,12,13,14} and taken into account by Carter and Langlois \cite{14} in a new method designed for general hydrodynamic description of a relativistic superfluid with quantum vortices \cite{13,15,16,17}. For the previous discussion did not explicitly include vorticity whose presence changes the equations of motion \cite{14}. The latter model shall be applied in the present study to investigation of the sound propagation through a superfluid containing quantum vortices. It should be noted that the sound \cite{19,20,7,13} and shock wave propagation \cite{20} in a two-constituent relativistic superfluid have been already discussed (as well as the recent analysis of the Cauchy problem \cite{?}); the sound in a ”vorticized” relativistic superfluid, however close it seems to the usual first-second sound, bears principal difference from the former, because vorticity $\omega_{\nu\rho}$ is interpreted as a tensor dynamical variable on which the Lagrangian function depends \cite{15}. So, in the model which we shall exploit below \cite{15,18} the role of the second constituent is played by the vorticity considered as a vortex ”liquid” rather than a vortex lattice where the so-called Tkachenko-Dyson waves \cite{21} can be observed (they are, evidently, inconceivable in the present model). Thus, we consider the waves in a relativistic superfluid in the ”continuous” or hydrodynamic limit.

We use a natural system of units $\hbar = c = 1$ and a metric $g_{\nu\rho} = \text{diag}(-1, 1, 1, 1)$.

II. RELATIVISTIC SUPERFLUID DYNAMICS WITH QUANTUM VORTICES

The equations of a relativistic superfluid with quantum vortices \cite{18} include the equation of motion

$$j^{\nu} w_{\nu\sigma} = 0$$

(1)
and the conservation law

$$\nabla_\nu n^\nu = 0$$  \hspace{1cm} (2)

of the particle number current \(n^\nu\). The total current

$$j^\nu = n^\nu + \nabla_\sigma \lambda^{\nu\sigma}$$  \hspace{1cm} (3)

is also conserved:

$$\nabla_\nu j^\nu = 0$$

The vorticity 2-form

$$w_{\nu\rho} = 2 \nabla[\nu, \mu\rho]$$  \hspace{1cm} (4)

satisfies the closure condition (square brackets imply antisymmetrization of indices)

$$\nabla_\alpha w_{\nu\rho} = 0$$  \hspace{1cm} (5)

but it is not trivially zero, for the vortices are presented.

Since the current \(n^\nu\) and the vorticity \(w_{\nu\rho}\) are considered as new dynamic variables, a variation of the Lagrangian \(L\) is given by formula [18]

$$dL = \mu_\nu dn^\nu - \frac{1}{2} \lambda^{\nu\rho} dw_{\nu\rho}$$  \hspace{1cm} (6)

Instead of the proper Lagrangian \(L\) we can use the pressure function

$$\Psi = L - \mu_\nu n^\nu$$  \hspace{1cm} (7)

whose variation

$$d\Psi = -n^\nu d\mu_\nu - \frac{1}{2} \lambda^{\nu\rho} dw_{\nu\rho}$$  \hspace{1cm} (8)

is expressed in terms of new primary variables \(\mu_\nu\) and \(w_{\nu\rho}\). In general the pressure depends on three invariants
\[ \mu^2 = -\mu^\nu \mu_\nu \quad W^2 = \frac{1}{2} w^{\nu \rho} w_{\nu \rho} \quad h^2 = h^\nu h_\nu \quad h^\nu = \frac{1}{2} \varepsilon^{\nu \alpha \beta \gamma} \mu_\alpha w_{\beta \gamma} \]  

(9)

with helicity vector \( h^\nu \), whose conservation

\[ \nabla_\nu h^\nu = 0 \]  

(10)

is a consequence of Eq. (4) and (5). The secondary variables \( n^\nu \) and \( \lambda^{\nu \rho} \) are expressed through primary variables by relation

\[
\begin{pmatrix}
  n^\nu \\
  \lambda^{\nu \rho}
\end{pmatrix} =
\begin{pmatrix}
  \bar{F}^{\nu \alpha} & \bar{Q}^{\nu \beta \gamma} \\
  \bar{R}^{\alpha \nu \rho} & \bar{G}^{\nu \beta \rho \gamma}
\end{pmatrix}
\begin{pmatrix}
  \mu_\alpha \\
  w_{\beta \gamma}
\end{pmatrix}
\]  

(11)

with

\[
\bar{F}^{\nu \alpha} = 2 \frac{\partial \Psi}{\partial \mu^2} g^{\nu \alpha} \quad \bar{Q}^{\nu \beta \gamma} = -2 \frac{\partial \Psi}{\partial h^2} h_\kappa \varepsilon^{\kappa \nu \beta \gamma}
\]  

(12)

\[
\bar{R}^{\alpha \nu \rho} = -2 \frac{\partial \Psi}{\partial h^2} h_\kappa \varepsilon^{\kappa \alpha \nu \rho} \quad \bar{G}^{\nu \beta \rho \gamma} = -2 \frac{\partial \Psi}{\partial W^2} g^{\nu \beta} g^{\rho \gamma}
\]  

(13)

It should be emphasized that the vorticity 2-form \( w_{\nu \rho} \) does not reflect the fine structure of the vortex cell, since the system of vortices is considered as a vortex "liquid". All variables are averaged over the vortex cell and vary over the macroscopic range. In the non-relativistic limit only space components of the vorticity form \( w_{\nu \rho} \) survive, while \( w_{\nu 0} \equiv 0 \).

### III. INFINITESIMAL DISCONTINUITIES

Let \( \Gamma^\nu \) be a space-like normal to the front of infinitesimal discontinuity. We shall apply the so-called Hadamard technique \[22\] for investigation propagation of small-amplitude perturbations. Recently Carter and Langlois \[7\] succeeded in calculating the first and second sound speed in a relativistic two-constituent superfluid. We use the same method to find the sound speed in a superfluid containing quantum vortices. The infinitesimal deviation \( \hat{A} \) of an arbitrary quantity \( A \) is proportional to the deviation of its gradient

\[ [\nabla_\nu A] = \hat{A} \Gamma_\nu \]  

(14)
Applying formula (14) to the conservation laws (10) and (2) we get, respectively

\[ \Gamma_\nu \hat{h}^\nu = 0 \]  

(15)

\[ \Gamma_\nu \hat{\nu} = 0 \]  

(16)

The closure condition (5) gives equation

\[ \Gamma_\alpha \hat{\nu}_{\nu \rho} = 0 \]  

(17)

while the equation of motion (1) and (4) in the linear approximation yield

\[ n^\nu \Gamma_\nu \hat{\nu}_{\nu \rho} + \Gamma_\sigma \hat{\nu}^\nu w_{\nu \rho} = 0 \]  

(18)

A relationship between the deviations of the primary and secondary variables is established by the formula

\[
\begin{pmatrix}
\hat{w}^\nu \\
\hat{\lambda}^\nu_{\nu \rho}
\end{pmatrix}
= \begin{pmatrix}
F^\nu_{\sigma} & Q^\nu_{\eta \rho} \\
R^\nu_{\rho \sigma} & G^\nu_{\rho \eta \sigma}
\end{pmatrix}
\begin{pmatrix}
\hat{\mu}_\sigma \\
\hat{w}_{\eta \rho}
\end{pmatrix}
\]

(19)

obtained by differentiation of equations (11)-(13) where

\[ F^\nu_{\sigma} = -2 \frac{\partial \Psi}{\partial \mu^2} g^\nu_{\sigma} - 4 \frac{\partial^2 \Psi}{\partial (\mu^2)\partial \mu^2} h^\nu_{\sigma} + 4 \frac{\partial^2 \Psi}{\partial \mu^2 \partial h^2} h_{\nu \sigma} w_{\beta \gamma} \left( \varepsilon^{\nu \rho \beta \gamma} \mu^\rho + \varepsilon^{\rho \sigma \nu \beta} \mu^\nu \right) \\
- 4 \frac{\partial^2 \Psi}{\partial (h^2)^2} h_{\nu \sigma} \varepsilon^{\nu \rho \lambda \xi} w_{\lambda \xi} h_{\alpha} \varepsilon^{\alpha \nu \beta \gamma} w_{\beta \gamma} - 2 \frac{\partial \Psi}{\partial h} \varepsilon^{\nu \beta \gamma} \varepsilon^{\sigma \nu \beta \gamma} w_{\beta \gamma} w_{\lambda \xi} \]  

(20)

\[ Q^\nu_{\eta \rho} = 4 \frac{\partial^2 \Psi}{\partial \mu^2 \partial W^2} h^\nu_{\rho} w_{\eta \sigma} + 4 \frac{\partial^2 \Psi}{\partial h^2 \partial W^2} h_{\nu \sigma} \varepsilon^{\nu \rho \gamma} \mu^\nu - 4 \frac{\partial^2 \Psi}{\partial h^2 \partial W^2} h_{\nu \sigma} \varepsilon^{\nu \rho \beta \gamma} w_{\alpha \beta} w_{\eta \sigma} \\
- \varepsilon^{\nu \rho \gamma} \mu_{\nu \beta} w_{\beta \gamma} \left( 4 \frac{\partial^2 \Psi}{\partial (h^2)^2} \varepsilon^{\nu \rho \beta \gamma} h_{\nu \lambda} + 2 \frac{\partial \Psi}{\partial h^2} \varepsilon^{\nu \beta \gamma} \right) - 2 \frac{\partial \Psi}{\partial h^2} h_{\nu \sigma} \varepsilon^{\nu \rho \eta \sigma} \]  

(21)

\[ R^\nu_{\rho \sigma} = Q^\nu_{\rho \sigma} \]  

(22)

\[ G^\nu_{\rho \eta \sigma} = -2 \frac{\partial \Psi}{\partial W^2} g^\eta g^\rho - 4 \frac{\partial^2 \Psi}{\partial (W^2)^2} w_{\rho \sigma} w_{\eta \tau} - 4 \frac{\partial^2 \Psi}{\partial W^2 \partial h^2} h_{\nu \sigma} \varepsilon^{\nu \rho \eta \sigma} \mu^\nu + \varepsilon^{\nu \rho \eta \sigma} w_{\nu \rho} + \varepsilon^{\nu \rho \eta \sigma} \mu^\nu \]  

\[ - \varepsilon^{\nu \rho \eta \sigma} \mu_{\alpha} \mu_{\beta} \left( 4 \frac{\partial^2 \Psi}{\partial (h^2)^2} h_{\nu \gamma} h_{\lambda} \varepsilon^{\nu \rho \gamma \eta \sigma} + 2 \frac{\partial \Psi}{\partial h^2} \varepsilon^{\beta \eta \sigma} \right) \]  

(23)
Substituting (19)-(23) in (15)-(18) we write ten equations

$$\varepsilon^{\nu\alpha\beta\gamma} \Gamma_\nu (\hat{\mu_\alpha w_{\beta\gamma}} + \mu_\alpha \hat{w}_{\beta\gamma}) = 0 \quad (24)$$

$$\Gamma_{[\alpha \hat{w}_{\nu\rho}]} = 0 \quad (25)$$

$$\Gamma_\nu \left( F^\nu_\sigma \hat{\mu}_\sigma + Q^\nu_\eta\vartheta \hat{w}_{\eta\vartheta} \right) = 0 \quad (26)$$

$$2n_\nu \Gamma_{[\nu \hat{\mu}_\sigma]} + \Gamma_\rho \left( R^\rho_\mu\nu \hat{\mu}_\mu + G^\rho_\eta\vartheta \hat{w}_{\eta\vartheta} \right) w_{\nu\sigma} = 0 \quad (27)$$

with ten independent unknowns, namely \(\hat{\mu}_0, \hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{w}_{01}, \hat{w}_{02}, \hat{w}_{03}, \hat{w}_{12}, \hat{w}_{13}, \hat{w}_{23}\). A linear system (24)-(27) is consistent if and only if its determinant vanishes, that yielding the characteristic equation for speed \(u\) of the wave.

IV. DILATONIC MODEL

In order to obtain an explicit analytic solution we shall confine ourselves with a particular dilatonic model [18] whose Lagrangian \(L\) and, hence, the pressure function \(\Psi\) do not depend on the helicity \(h\). Although this model is devoid of complete strictness, for weak vorticity is provided \((GW^2 \ll \mu n)\), it corresponds to the conditions inside the neutron stars and, hence, describes satisfactory the real superfluid matter. Thus, formulae (20)-(23) reduces to

$$F^\nu_\sigma = -2 \frac{\partial \Psi}{\partial \mu^2} g^{\nu\sigma} - 4 \frac{\partial^2 \Psi}{\partial (\mu^2)^2} \mu^\nu \mu^\sigma \quad (28)$$

$$Q^\nu_\eta\vartheta = 4 \frac{\partial^2 \Psi}{\partial \mu^2 \partial W^2} \mu^\nu \omega^{\eta\vartheta} \quad (29)$$

$$G^\nu_\rho\eta\vartheta = -2 \frac{\partial \Psi}{\partial W^2} g^{\nu\eta} g^{\rho\vartheta} - 4 \frac{\partial^2 \Psi}{\partial (W^2)^2} \omega^{\nu\rho} \omega^{\eta\vartheta} \quad (30)$$

while (22) remains the same.

For a dilatonic model it is convenient to practice with a dilaton \(\Phi^2\) instead of the set (9).

In the weak vorticity limit [18] we have
\[ \Psi [\Phi (\mu, W)] = \Phi^2 \frac{dV[\Phi]}{d\Phi^2} - V[\Phi] \]  

(31)

Since

\[ n^2 = 2\Phi^4 \frac{dV}{d\Phi^2} + 2KW\Phi^4 \quad \Phi^{-2} = \frac{\mu}{n} \]  

(32)

thereby,

\[ \mu^2 = 2\frac{dV}{d\Phi^2} + 2KW \]  

(33)

and

\[ d\Phi^2 = \left[ d\mu^2 - \frac{K}{W} dW^2 \right]/2V'' \]  

(34)

where

\[ V' = \frac{dV}{d\Phi^2} \]  

(35)

and quantity \( K \) (whose logarithmic dependance on vorticity is neglected), being proportional to the circulation quantum round an individual vortex, satisfies the inequalities

\[ KW \ll V' \quad \Phi^2 KW \ll V'' \]  

(36)

while the sound speed is given by the formula

\[ c_s^2 = \frac{n d\mu}{\mu d\Phi^2} = \frac{\Phi^2 V''}{\Phi^2 V'' + \mu^2} \]  

(37)

In the light of (31)-(35) we obtain the derivatives

\[ \frac{\partial \Psi}{\partial \mu^2} = \frac{d\Psi}{d\Phi^2} \left( \frac{\partial \mu^2}{\partial \Phi^2} \right) = \frac{1}{2} \Phi^2 [\mu, W] \]  

(38)

\[ \frac{\partial \Psi}{\partial W^2} = \frac{d\Psi}{d\Phi^2} \left( \frac{\partial \Phi^2}{\partial W^2} \right) = -\frac{1}{2} \Phi^2 [\mu, W] \frac{K}{W} \]  

(39)

and

\[ \frac{\partial \Psi}{\partial (\mu^2)^2} = \frac{1}{2} \frac{\partial \Phi^2}{\partial \mu^2} = \frac{1}{4V''} \]  

(40)

\[ \frac{\partial \Psi}{\partial \mu^2 \partial W^2} = -\frac{K}{4W V''} \]  

(41)

\[ \frac{\partial \Psi}{\partial (W^2)^2} = \frac{1}{4W^3} \left( \Phi^2 + \frac{K}{W V''} \right) \]  

(42)
V. EXPLICIT SOLUTION

Let the vortices be aligned along the axis $z$ and there is no dependence on $z$. The only non-zero components will be $w_{01}$, $w_{02}$, $w_{12}$. Also only $\mu_0$ and $n^0$ are not equal to zero in the local reference frame commoving the fluid (or one may merely consider the fluid at rest). We can choose the normal vector $\Gamma^\nu$ as

$$\Gamma^\nu = (u, \cos \chi, 0, \sin \chi) = (u, \Sigma, 0, \Omega)$$

where $\chi$ is the angle between the axis $z$ and the direction of the wave propagation. The coefficients (28)-(30) and (22), then, take the form

$$F^{\nu\sigma} = -\Phi^2 g^{\nu\sigma} - \frac{\mu^2 g^{\nu0} g^{\sigma0}}{V''}$$

$$Q^{\nu\eta\vartheta} = \Xi g^{\nu0} w^{\eta\vartheta}$$

$$G^{\nu\rho\eta\vartheta} \simeq G \left( g^{\nu\eta} g^{\rho\vartheta} - \frac{w^{\nu\rho} w^{\eta\vartheta}}{W^2} \right)$$

where $\Xi = \mu K/(W V'')$ and $G = \Phi^2 K/W$. Note that in Eq. (43) we do not add the negligible term $K W/V''$ incorporating Eq. (42). Substituting formulae (43)-(46) in equations (24)-(27) and taking into account (9), namely

$$W^2 = w^{10} w_{10} + w^{20} w_{20} + w^{12} w_{12}$$

we get

$$-\hat{\mu}_0 \Omega w_{12} - \hat{\mu}_1 \Omega w_{02} + \hat{\mu}_2 \Omega w_{01} - \hat{\mu}_3 U + \hat{\omega}_{23} \Sigma \mu_0 + \hat{\omega}_{12} \Omega \mu_0 = 0$$

$$- u \hat{\omega}_{12} + \Sigma \hat{\omega}_{20} = 0$$

$$- u \hat{\omega}_{13} + \Sigma \hat{\omega}_{30} + \Omega \hat{\omega}_{01} = 0$$

$$- u \hat{\omega}_{23} + \Omega \hat{\omega}_{02} = 0$$
\[ \Sigma \dot{w}_{23} + \Omega \dot{w}_{12} = 0 \quad (52) \]

\[ - u \dot{\mu}_0 F^{00} + \Sigma \dot{\mu}_1 F^{11} + \Omega \dot{\mu}_3 F^{33} - u \left( Q^{001} \dot{w}_{01} + Q^{002} \dot{w}_{02} + Q^{012} \dot{w}_{12} \right) = 0 \quad (53) \]

\[ \dot{\mu}_0 R_0 + \dot{w}_{\eta \vartheta} \alpha_0^{\eta \vartheta} = 0 \quad (54) \]

\[ \dot{\mu}_0 (-2n^0 \Sigma + R_1) - \dot{\mu}_1 2n^0 u + \dot{w}_{\eta \vartheta} \alpha_1^{\eta \vartheta} = 0 \quad (55) \]

\[ \dot{\mu}_0 R_2 - 2un^0 \dot{\mu}_2 + \dot{w}_{\eta \vartheta} \alpha_2^{\eta \vartheta} = 0 \quad (56) \]

\[ \dot{\mu}_0 \Omega + \dot{\mu}_3 u = 0 \quad (57) \]

where

\[ \alpha_j^{\eta \vartheta} = G a_j^{\eta \vartheta} \quad (58) \]

\[ R_0 = \Sigma w_{02} R_{120} - uw_{0k} R^{0k0} \quad R_2 = w_{02} \Sigma R^{010} + w_{12} u R^{010} \]

\[ R_1 = \Sigma w_{01} R^{010} + w_{12} (\Sigma R^{120} - u R^{020}) \quad (59) \]

\[ a_0^{01} = -B_0 U \quad a_0^{02} = -B_0 U \quad a_0^{03} = 0 \]

\[ a_0^{12} = (1 - B_0) U \quad a_0^{13} = -w_{01} \Omega \quad a_0^{23} = -w_{02} \Omega \quad (60) \]

\[ a_1^{01} = B_1 U \quad a_1^{02} = (1 + B_1) U \quad a_1^{03} = \Omega w_{01} \]

\[ a_1^{12} = B_1 U \quad a_1^{13} = 0 \quad a_1^{23} = -\Omega w_{12} \quad (61) \]

\[ a_2^{01} = (1 - B_2) U \quad a_2^{02} = -B_2 U \quad a_2^{03} = \Omega w_{02} \]

\[ a_2^{12} = -B_2 U \quad a_2^{13} = \Omega w_{12} \quad a_2^{23} = 0 \quad (62) \]

while

\[ B_0 = \frac{w^{01} w^{12}}{W^2} \quad B_1 = \frac{w^{01} w^{02}}{W^2} \quad B_2 = \frac{w^{01} w^{01}}{W^2} \quad (63) \]

\[ U = uw_{12} + \Sigma w_{02} \quad (64) \]

It should be noted that \( R_1 \) in (53), (54) and \( Q^{012} \) in (53) survive in the non-relativistic limit.
VI. WAVES

A. Horizontal waves

We call the wave propagating along the axis $x$ (i.e. at the right angle to the axis of vortices $z$) "horizontal" since for this case $\Omega = 0$, $\Sigma = 1$. The matrix of the linear system (65), then, has the form

$$\begin{pmatrix}
0 & 0 & 0 & -w_{02} - uw_{12} & 0 & 0 & 0 & 0 & 0 & \mu_0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & u & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & u & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-uF^{00} & F^{11} & 0 & 0 & Q^{01} & Q^{02} & 0 & Q^{12} & 0 & 0 \\
\bar{R}_0 & 0 & 0 & 0 & \alpha_0^{01} & \alpha_0^{02} & 0 & \alpha_0^{12} & 0 & 0 \\
-1 + \bar{R}_1 & -u & 0 & 0 & \tilde{\alpha}_1^{01} & \tilde{\alpha}_1^{02} & 0 & \tilde{\alpha}_1^{12} & 0 & 0 \\
\bar{R}_2 & 0 & -u & 0 & \tilde{\alpha}_2^{01} & \tilde{\alpha}_2^{02} & 0 & \tilde{\alpha}_2^{12} & 0 & 0 \\
0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

where for every variable $O$ we introduced the notation $\tilde{O} = O/(2n)$ and $Q^{\nu\rho} = -u Q^{0\nu\rho}$. The solution follows from a condition of the determinant (65) vanish that implies the possibility of longitudinal waves ($\hat{\mu}_1 \neq 0$ while $\hat{\mu}_2 = \hat{\mu}_3 = 0$, also $\hat{w}_{03} = \hat{w}_{13} = \hat{w}_{23} = 0$) with the relevant speed found from the characteristic equation

$$u^2 F^{00} - F^{11} U^2 \det \begin{pmatrix}
-B_0 & 1 - B_0 (1 - u) \\
1 - B_2 & B_2 (1 - u)
\end{pmatrix} = o(K)$$

which, according to Eq. (44), gives the usual sound speed (37), for

$$F^{00} = -\Phi^2 - \frac{\mu^2}{V_u} \quad F^{11} = -\Phi^2$$

We neglected in (66) the insufficient term $\bar{R}_1$ which reflects the negligible space anisotropy of the first sound; in the relevant equation for the first sound in vertical direction (see next section) this term is absent at all. The second solution of (66) is
In the non-relativistic limit \( w_{01} = w_{02} = 0 \) and the only non-zero elements of (60–62) will be \( a_1^{02} \) and \( a_1^{12} \). The matrix (65), then, determines the only ”first sound” solution, while (67) does not occur, for it pertains to a relativistic superfluid.

**B. Vertical waves**

Let the wave propagating along the axis \( z \) of the vortex cell, when \( \Sigma = 0 \) and \( \Omega = 1 \), be called ”vertical”. Eq. (62), then, gives \( \hat{w}_{12} = 0 \), while the variables \( \hat{w}_{01} \) and \( \hat{w}_{02} \) can be excluded by means of Eqs. (50) and (51). Then, we have to calculate the determinant of matrix

\[
\begin{pmatrix}
\hat{\mu}_0 & \hat{\mu}_1 & \hat{\mu}_2 & \hat{\mu}_3 & \hat{w}_{03} & \hat{w}_{13} & \hat{w}_{23} \\
-w_{12} & -w_{02} & w_{01} & -uw_{12} & 0 & 0 & 0 \\
-uF^{00} & 0 & 0 & F^{33} & 0 & -uQ^{01} & -uQ^{02} \\
-1 & 0 & 0 & -u & 0 & 0 & 0 \\
R_0 & 0 & 0 & 0 & 0 & \alpha_0^{13} + u\alpha_0^{01} & \alpha_0^{23} + u\alpha_0^{02} \\
\tilde{R}_1 & -u & 0 & 0 & \hat{G}w_{01} & u\tilde{\alpha}_1^{01} & \tilde{\alpha}_1^{23} + u\tilde{\alpha}_1^{02} \\
\tilde{R}_2 & 0 & -u & 0 & \hat{G}w_{02} & \tilde{\alpha}_2^{13} + u\tilde{\alpha}_2^{01} & u\tilde{\alpha}_2^{02}
\end{pmatrix}
\]

where the first row includes the independent variables. The speed of the longitudinal wave, characterized by \( \hat{\mu}_1 = \hat{\mu}_2 = 0 \), follows from the equation

\[
\left( u^2 F^{00} - F^{33} \right) \det \begin{pmatrix} 0 & b_0^1 & b_0^2 \\ w_{01} & b_1^1 & b_1^2 \\ w_{02} & b_2^1 & b_2^2 \end{pmatrix} = o(K^4) \]

\[ (69) \]

\[
b_0^1 = -w_{01} - uB_0U \quad b_0^2 = -w_{02} - uB_0U \]

\[ (70) \]

\[
b_1^1 = uB_1U \quad b_1^2 = -w_{12} + u(1 + B_1)U \]

\[ (71) \]
\[ b_2^1 = w_{12} + u (1 - B_2) U \quad b_2^2 = -u B_2 U \]  

(72)

and coincides approximately with the usual sound speed (37). For the determinant in (79), as one can check by means of (83) and (71)-(72), does not vanish.

As for the transversal waves ($\hat{\mu}_3 = 0$) they are determined by equation

\[
\det \begin{pmatrix}
-w_{02} & w_{01} & 0 & 0 & 0 \\
0 & 0 & 0 & -uQ^{01} & -uQ^{02} \\
0 & 0 & 0 & b_0^1 & b_0^2 \\
-u & 0 & w_{01} & \tilde{b}_1^1 & \tilde{b}_1^2 \\
0 & -u & w_{02} & \tilde{b}_2^1 & \tilde{b}_2^2 \\
\end{pmatrix} = 0
\]  

(73)

which trivially equals zero (due to the matrix degeneracy in (73), implying impossibility of transversal waves.

\section*{VII. CONCLUSION}

In order to investigate wave propagation in a relativistic superfluid with quantum vortices we have derived the linear system (24)-(27) by means of the Hadamard method [22] and have solved the appropriate characteristic equation. For a dilatonic model in the weak vorticity limit we found two types of waves propagating at the right angle to the axis \( z \) of vortices. Besides the usual sound (37), an additional horizontal (in direction orthogonal to \( z \) ) "second-sound" branch (57) exists, while only the usual sound may propagate in the direction \( z \). It should be noted that no wave propagation with the speed close to \( KW/V' \) or \( \Phi^2KW/V'' \) were found (the latter term determines the anisotropy of the first sound in the horizontal and vertical direction; we do not include this term in Eq. (66) because it is small and does change the result on qualitative level); for the second constituent was treated as a vortex "liquid", no transversal waves occur. On the other hand, due to orientation in \( z \) direction the condition of wave propagation along \( z \) and \( x \) (or \( y \)) is different. However, only the usual sound may propagate through a non-relativistic superfluid, and that follows from
Eqs. (48-57) taken in the non-relativistic limit \((w_0 = 0)\). Because the matrix of the relevant non-relativistic system

\[
\begin{pmatrix}
\hat{\mu}_0 & \hat{\mu}_1 & \hat{\mu}_2 & \hat{\mu}_3 & \hat{w}_{01} & \hat{w}_{02} & \hat{w}_{03} & \hat{w}_{12} & \hat{w}_{13} & \hat{w}_{23} \\
-\Omega w_{12} & 0 & 0 & -u w_{12} & 0 & 0 & 0 & \Omega \mu_0 & 0 & \Sigma \mu_0 \\
0 & 0 & 0 & 0 & 0 & -\Sigma & 0 & -u & 0 & 0 \\
0 & 0 & 0 & 0 & \Omega & 0 & -\Sigma & 0 & -u & 0 \\
0 & 0 & 0 & 0 & 0 & \Omega & 0 & 0 & 0 & -u \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega & 0 & \Sigma \\
-u F^{00} & \Sigma F^{11} & 0 & \Omega F^{33} & 0 & 0 & 0 & -u Q^{012} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u w_{12} & 0 & 0 \\
\Sigma \left( -n^0 + \frac{1}{2} w_{12} R^{120} \right) & -u & 0 & 0 & 0 & u w_{12} & 0 & 0 & 0 & -\Omega w_{12} \\
0 & 0 & -u n^0 & 0 & -u w_{12} & 0 & 0 & 0 & \Omega w_{12} & 0 \\
\Omega & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(74)

yields for the horizontal waves \((\Omega = 0, \Sigma = 1)\) the characteristic equation

\[
u^2 F^{00} + F^{11} \left( 1 - w_{12} \frac{R^{120}}{2n^0} \right) = 0
\]

(75)

where

\[
R^{120} = \frac{\mu K}{W V''} w_{12} \\
w_{12} \frac{R^{120}}{2n^0} = \frac{K W}{2\Phi^2 V''} \ll 1
\]

(76)

in accordance with (59), (13), (23), and merely

\[
u^2 F^{00} + F^{33} = 0 
\]

(77)

for the vertical waves \((\Omega = 1, \Sigma = 0)\). Indeed, Eq. (73), which we write here in the exact form with the negligible term (76), follows from (68), while Eq. (77) follows from (69) and they both determine the usual sound speed (37).

Nevertheless, formula (67) requires more discussion. What if (67) will be of order of the usual sound (37)? Then we have obviously missed the some information by vanishing \(w_{02}\).
We do not know the explicit form of $w_{02}$ and $w_{12}$; although $w_{02}$ vanishes in the non-relativistic limit, the ratio (37). The analysis of the exterior region of a vortex line [14] is not sufficient to determine the components of the vorticity 2-form. These components are generated by the vortex core and they can be derived from its internal structure; because outside the vortex core, i.e. outside the support of the vorticity 2-form $w_{\nu\rho}$, the irrotationality condition $w_{\nu\rho} = 0$ takes place. So, the complete answer requires explicit derivation of $w_{02}$ and $w_{12}$.

Of course, it seems reasonable to discuss in future the shock waves. However, equations (24)-(27) have already convey [or: reveal] the absence of any other modes except (37) and (37). For instance, the waves through the pure vortex constituent ($\hat{\mu}_\nu \equiv 0$) or those similar to the fourth sound [20] in superfluid ($\hat{w}_{\nu\rho} \equiv 0$) are impossible. This may serve a hint to search possible shock wave solutions. Moreover, the analysis of a more complicated model with sufficient vorticity $W$ and the pressure function $\Psi$ depending on the cross term $h$ may also be performed by means of Eqs. (24)-(27).
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