Method of the speed up calculations by the local Courant number for the filtration problem

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Abstract. We study the process of multiphase fluid-water flow in a heterogeneous formation in the presence of both producing and injection wells. The quasi-three-dimensional reservoir model is considered, comprising several seams, which characteristics are considered as constant thickness. Perforated wells are also can be presented. The main feature of the problem is that there are appear the filtration areas with high gradients in surroundings of wells. An approximate method is based on a preliminary finite-difference approximation of the problem. For its numerical realization there is proposed the algorithm, which allows to speed up the calculation time for the transfer equation using the explicit scheme, which is optimized taking into account the spatial inhomogeneity of the calculation time step.

1. Introduction
The filtration process is considered in the oil reservoir, which has a significant predominance of the size in the horizontal direction over the vertical. Along the strike the productive strata is limited by a cylindrical surface with a vertical generatrix. Along the vertical the reservoir is divided into the system of the interlayer, which has differ properties - not intersecting layers of variable thickness bordering on each other or separated by a cross connection with a low permeability. There are necessary to set certain transmission conditions or impermeability at these boundaries. The characteristics of the thickness are constant, so we can identify each interlayer with its middle surface. There are the system of the well, from a mathematical point of view, their presence is means the existence of boundaries relatively small size, which have the appropriate conditions. Wells are modeled as the sources/sinks. Note that the mathematical simulation of a multiphase filtration of high-viscous fluid in [1-10] is carried out.

2. Statement a problem
The problem of two-phase filtration is solved considering heterogeneity of associated oil and water according to field. This problem is described using the given equation to determine the reservoir pressure \( P(x,y,z,t) \):

\[
\text{div}(\sigma \text{grad}P) - \beta' \frac{\partial P}{\partial t} = 0,
\]

where \( \sigma(x,y,z,t) \) is the reservoir transmissibility, \( \beta' = m(s\beta_1 + (1-s)\beta_2) + \beta_e \) is the reservoir and liquids elastocapacity, \( m \) – porosity. The equation (1) will be named as the piezoconductivity equation.
The water saturation equation \( s = s_t(x, y, z, t) \) is given as the second equation of the system:

\[
\text{div} \left( f \sigma \text{grad} P \right) - \beta^t \frac{\partial P}{\partial t} - m \frac{\partial s}{\partial t} = 0
\]

(2)

where \( f(s) = k^t(s) / \left( \mu k^t(s) \right) \) is the J-function (Buckley–Leverett function), \( \beta^t = s(m \beta^t + \beta^t) \) is the reservoir and water elastocapacity.

The equations are valid in a multiply connected domain \( \Omega \) (in the reservoir) with the boundary \( \Gamma = \Gamma_h + \sum_{k=1}^N \Gamma_k \), \( k = 1...N \), where \( \Gamma \) is the external boundary (the cylindric surface) of the filtration area, \( \Gamma_h \) is the top and bottom of the reservoir, \( \Gamma_k \) is the boundary (the cylindric surface) of the well number \( k \), and \( N \) is the total amount of production and injection wells.

Initial and boundary conditions. Initially, the reservoir is in equilibrium. The fluid movement starts with the opening of its wells. At the initial time the distribution of pressure and saturation are known.

\[
P = \varrho(x, y, z), (x, y, z) \in \Omega
\]

(3)

\[
s = s_0(x, y, z), (x, y, z) \in \Omega
\]

(4)

Either the total flow rate is set to wells:

\[
\int_{\Gamma_k} \sigma \frac{\partial P}{\partial n} d\gamma dz = q_k, (k = 1,...,N_q)
\]

(5)

or bottomhole pressure on the perforated intervals:

\[
P = c_k(t), (k = N_q + 1,...,N)
\]

(6)

where \( N_q \) is the amount of the wells with the determined rate.

The pressure distribution is given on the external reservoir boundary:

\[
P = \varphi(x, y, z), (x, y, z) \in \Gamma
\]

(7)

or the condition of impermeability can be given:

\[
\frac{\partial P}{\partial n} = 0, (x, y, z) \in \Gamma
\]

(8)

And the top and bottom are impermeable:

\[
\frac{\partial P}{\partial n} = 0, (x, y, z) \in \Gamma
\]

(9)

Furthermore, the saturation is known in injection wells (on the sources):

\[
s = s^t, (x, y, z) \in \Gamma_k, (k = 1,...,N_i)
\]

(10)

where \( s^t \) is the cutoff water saturation, \( N_i \) is the amount injection wells.

The saturation is given on the part of the external contour \( \gamma(t) \):

\[
s = s_{ew} (x, y, z), (x, y, z) \in \gamma(t) \in \Gamma
\]

(11)

where \( \bar{n} \) is the external normal to the \( \Gamma \), \( s_{ew} \) is the water saturation distribution along the outer contour.

Thus, it is required to find the functions \( P(x, y, z, t) \) and \( s(x, y, z, t), (x, y, z) \in \Omega, t \in [0, T] \), that satisfy to the equations (1), (2) and additional conditions (3) – (11).

3. Approximation of the problem

The regular grid is constructed in the filtration area for the numerical solution. The layers of this grid situated on the middle of the reservoir surface.
where \( z_k \left( x_i, y_j \right) \) match with the vertical coordinates of the interlayer middle over the \( \left( x_i, y_j \right) \) point.

The difference schemes for the equations (1) and (2), integrated over the volume of the elementary grid cells are of the form:

\[
\beta_{i,j,k} V_{i,j,k} \frac{p_{i,j,k}^{N+1} - p_{i,j,k}^N}{\tau} - h(k\lambda H)_{i\pm1/2,j,k} \left( \frac{p_{i\pm1,j,k} - p_{i,j,k}}{h} \right) + h(k\lambda H)_{i\pm1/2,j,k} \left( \frac{p_{i,j,k} - p_{i-1,j,k}}{h} \right) -
\]

\[
-h^2(k\lambda)_{i,j,k+1/2} \left( \frac{p_{i,j,k+1} - p_{i,j,k}}{h} \right) + h^2(k\lambda)_{i,j,k+1/2} \left( \frac{p_{i,j,k} - p_{i,j,k-1}}{h} \right) -
\]

\[
-m_{i,j,k} V_{i,j,k} s_{i,j,k}^{n+1} - s_{i,j,k}^n = -q_{vi,j,k} - \tilde{\beta}_{i,j,k} V_{i,j,k} \frac{p_{i,j,k}^{N+1} - p_{i,j,k}^N}{\tau} -
\]

\[
-f_{i\pm1/2,j,k} u_{i\pm1/2,j,k} + f_{i-1/2,j,k} u_{i-1/2,j,k} -
\]

\[
-f_{i,j\pm1/2,k} u_{i,j\pm1/2,k} + f_{i,j-1/2,k} u_{i,j-1/2,k} -
\]

\[
-f_{i,j,k\pm1/2} u_{i,j,k\pm1/2} + f_{i,j,k-1/2} u_{i,j,k-1/2} -
\]

\[
 \text{where } \tau \text{ is the time step for the pressure calculation (the high index } N \text{), } \tau \text{ is the time step for saturation calculation – the part of the time step } t \text{ (the high index } n \text{), } \tilde{\beta} = m s(\beta_u + \beta_m) \text{ is the elasto-capacity, } H_{i,j,k} \text{ is the local thickness of the } k\text{-layer of the reservoir, } H_{i,j,k+1/2} = z_{k+1} \left( x_i, y_j \right) - z_k \left( x_i, y_j \right) \text{ is the distance between grid layers on vertical side, } V_{i,j,k} = h^2 H_{i,j,k} \text{ is the volume of the grid; } q_{vi,j,k} \text{ is the amount of liquid from the cell through the corresponding well part; } q_{vi,j,k} \text{ is amount of water from the cell. The amount filtration flows from are approximated in the following way:}
\]

\[
u_{i\pm1/2,j,k} = -h(k\lambda H)_{i\pm1/2,j,k} \left( \frac{p_{i\pm1,j,k} - p_{i,j,k}}{h} \right),
\]

\[
v_{i,j\pm1/2,k} = -h(k\lambda H)_{i,j\pm1/2,k} \left( \frac{p_{i,j\pm1,k} - p_{i,j,k}}{h} \right),
\]

\[
v_{i,j,k\pm1/2} = -h^2(k\lambda)_{i,j,k\pm1/2} \left( \frac{p_{i,j,k\pm1} - p_{i,j,k}}{H_{i,j,k+1/2}} \right).
\]
\[ f_{i+1/2,j,k} = \begin{cases} f_{i,j,k}, & u_{i+1/2,j,k} \geq 0, \\ f_{i+1,j,k}, & u_{i+1/2,j,k} < 0. \end{cases} \] (15)

For cells that contain wells difference approximation is different. You can use Peaceman approach [11]. In the right-hand sides of equations (12) and (13) are added, respectively, the terms:
- \( q_i l = l(i,j,k) \in L \) for perforated sections for a given production rate (for perforated sections assume \( q_i = 0 \), \( l = l(i,j,k) \notin L \) or \( P_i = P_{\text{well}}, l = l(i,j,k) \in L \) for non-perforated sections with a given bottom hole pressure;
- \( q_i f_i l = l(i,j,k) \in L \) for perforated sections for a given production rate (for non-perforated \( q_i = 0 \), \( l = l(i,j,k) \notin L \));

or more complicated variants of the approximation, taking into account the increasing number of cells (see., eg, [11]).

To take into account the features condition of formation in the near-well zone may use the concept of skin-factor, or a variant of effective corrections field reservoir parameters [11].

4. Algorithm of solving the problem

The IMPES method is using for solving the resulting system of difference equations, when the implicit scheme is using for pressure equation and the explicit scheme is using for saturation equation.

The reservoir pressure can be found from the (12) using a fully implicit scheme (\( P_{i,j,k} = P_{i,j,k}^{n+1} \)) with the use of the multigrid algorithms. As a result, in each pressure calculation step it is possible to determine approximate values velocities of flow through the respective faces (see (14)). Saturation will be determined by (13) and the explicit scheme with a known field of filtration flows \( u \). The Courant number for this equation is equal to:

\[ C = \tau \cdot F \cdot \max \left( \frac{u}{mV} \right) \] (16)

where the maximum should be taken as all relations of flows through the cell borders \( u \) to the pore volume \( mV \) and the value \( F \) is given by the formula:

\[ F = \max_s f'(s). \] (17)

The flows have the greatest impact on the value of time steps in small areas near the high performance wells. In most parts of the reservoir to satisfy the condition of stability is allowed to make calculations with a much larger time steps. The term of local Courant number is introduced:

\[ C_{i,j,k} = F_{i,j,k} \frac{u_{i,j,k}}{m_{i,j,k} V_{i,j,k}}, \] (18)

where \( u_{i,j,k} = \max\left( u_{i+1/2,j,k} \cdot u_{i,j+1/2,k} \cdot u_{i,j,k+1/2} \right) \).

The stability condition will be satisfied if the time step will be chosen so, that the local Courant number (18) was less than one for each cell:

\[ \tau_{i,j,k} < \frac{m_{i,j,k} V_{i,j,k}}{u_{i,j,k} F_{i,j,k}}. \]

However, the direct setting of time steps in each cell can not be realized, because it is necessary the agreed calculation of the flows component field to fulfill the conditions of conservativeness.

Here the algorithm of multiple matching the number of recalculations is proposed.
1. The time step is selected, equal to the minimum of steps for each cell from the condition (18)
2. The smallest time step among all of the cells is selected.
3. The coefficient of step increase is calculated.
4. The coefficients for the cells are rounded towards zero to the number of which constitutes a whole integer degree of the “d”. The number of small steps is determined, it will be used for recalculate the cell:

\[ K_{i,j,k} = \frac{\tau_{i,j,k}}{\tau_{\text{min}}}. \]

5. The time step is calculated for each cell:

\[ \tilde{\tau}_{i,j,k} = N_{i,j,k} \cdot \tau_{\text{min}} \quad (20) \]

6. The total saturation calculation is made with the number of steps, corresponding to the minimum step. But particular cell is "recalculated" for every step, and with its own "period" – one time for the \( N_{i,j,k} \) steps.

7. The period of recalculation of J-function on the each boundary of the cell is equal to the maximal value from periods of recalculation of the neighbor cells. It is obvious that in this case the balance of the transferred value in the cells is always complied, it means that the scheme is conservative.

5. Numerical experiment

The comparison of the results and calculation time using suggested algorithm was made in one-dimensional, two-dimensional and three-dimensional cases. During the test number “d” was equal two.

One dimensional case demonstrated the good agreement between the calculation results with the similar solution for the plane-parallel filtration. For the radially symmetric case it was compared the behavior of saturation front, which obtained using both algorithms (with acceleration and without).

6. Conclusion

It was found that using the suggested algorithm allows to reduce the calculation time of oil saturation field of about three times. Note that with increasing the size of the calculation grid the efficiency of the algorithm a bit improved. As a result the calculations by several million cells give a significant gain in real-time. The comparison with potentially possible "ideal" step allows us to hope that the improvement of the algorithm implementation will reduce the computation time is still about 3 times.

Acknowledgments

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