Simplification of thermodynamic Bethe-ansatz equations

Minoru Takahashi
Institute for Solid State Physics, University of Tokyo,
Kashiwanoha 5-1-5, Kashiwa, Chiba, 277-8581 Japan

Abstract

Thermodynamic Bethe ansatz equations for XXZ model at \(|\Delta| \geq 1\) is simplified to an integral equation which has one unknown function. This equation is analytically continued to \(|\Delta| < 1\).

1. Introduction

Thermodynamic Bethe ansatz equations for exactly solvable one-dimensional systems have many unknown functions.\cite{1} About the XXZ model at \(|\Delta| \geq 1\), Gaudin-Takahashi equation contains infinite unknown functions.\cite{2,3} Here I can simplify this set of equations to an integral equation which contains only one unknown function. For Hamiltonian

\[
\mathcal{H}(J, \Delta, h) = -J \sum_{l=1}^{N} S_{l}^{x} S_{l+1}^{x} + S_{l}^{y} S_{l+1}^{y} + \Delta (S_{l}^{z} S_{l+1}^{z} - \frac{1}{4}) - 2h \sum_{l=1}^{N} S_{l}^{z}, \quad h \geq 0,
\]

thermodynamic Bethe ansatz equation at temperature \(T\) is

\[
\ln \eta_{1}(x) = \frac{2\pi J \sinh \phi}{T \phi} s(x) + s * \ln(1 + \eta_{2}(x)),
\]

\[
\ln \eta_{j}(x) = s * \ln(1 + \eta_{j-1}(x))(1 + \eta_{j+1}(x)), \quad j = 2, 3, \ldots,
\]

\[
\lim_{l \to \infty} \frac{\ln \eta_{l}}{l} = \frac{2h}{T}.
\]

Here we put

\[
\Delta = \cosh \phi, \quad Q \equiv \pi / \phi, \quad s(x) = \frac{1}{4} \sum_{n=-\infty}^{\infty} \text{sech}\left(\frac{\pi (x - 2nQ)}{2}\right), \quad s * f(x) \equiv \int_{-Q}^{Q} s(x-y)f(y)dy.
\]

The free energy per site is

\[
f = \frac{2\pi J \sinh \phi}{\phi} \int_{-Q}^{Q} a_{1}(x)s(x)dx - T \int_{-Q}^{Q} s(x) \ln(1 + \eta_{1}(x))dx, \quad a_{1}(x) \equiv \frac{\phi \sinh \phi / (2\pi)}{\cosh \phi - \cos(\phi x)}.
\]

On the contrary new equation is

\[
u(x) = 2 \cosh \left(\frac{h}{T}\right) + \int_{C} \frac{\phi}{2} \left(\cot \frac{\phi}{2} \cdot [x - y + 2i] \exp[-\frac{2\pi J \sinh \phi}{T \phi} a_{1}(y + i)]
\]

\[
+ \cot \frac{\phi}{2} \cdot [x - y + 2i] \exp[-\frac{2\pi J \sinh \phi}{T \phi} a_{1}(y - i)]\right) \frac{1}{u(y) 2\pi i},
\]

\[
(5)
\]
and free energy is given by
\[ f = -T \ln u(0). \]  
(6)

Contour \( C \) is an arbitrary closed loop counterclockwise around 0. \( 2nQ, n \neq 0 \) and \( \pm 2i + 2nQ \) should be outside of this loop. This loop should not contain zeros of \( u(y) \). It is expected that \( u(y) \) has not zero in region \( |\Re y| \leq 1 \). This equation can be calculated even at imaginary \( \phi \), namely, \( |\Delta| < 1 \) case. This equation converges numerically at least \( T/J > 0.07 \). The results coincide with those of older equations. Then this equation unifies Gaudin-Takahashi equation for \( \Delta \geq 1 \) and Takahashi-Suzuki equation for \( |\Delta| < 1 \).

2. Derivation

We should note that if \( g(x) = s \ast h(x) \),
\[ g(x + i) + g(x - i) = h(x). \]  
(8)

The Fourier transform of (7) is
\[ \tilde{g}(\omega) = \frac{1}{e^\omega + e^{-\omega}} \tilde{h}(\omega), \quad \omega = \frac{\pi}{Q} n. \]

Then we have
\[ (e^\omega + e^{-\omega}) \tilde{g}(\omega) = \tilde{h}(\omega), \]
and obtain (8).

Then set of equations (2) is rewritten as
\[ \eta_1(x + i) \eta_1(x - i) = \exp \left[ \frac{2\pi J \sinh \phi}{T \phi} \sum_n \delta(x - 2nQ) \right] (1 + \eta_2(x)), \]
\[ \eta_j(x + i) \eta_j(x - i) = (1 + \eta_{j-1}(x))(1 + \eta_{j+1}(x)), \quad j = 2, 3, \ldots, \]
\[ \lim_{l \to \infty} \frac{\ln \eta_l}{l} = \frac{2h}{T}. \]  
(9)

At \( J = 0 \) this set of equation becomes a difference equation and we have analytical solution,
\[ \eta_j = \left( \frac{\sinh(j + 1)h/T}{\sinh h/T} \right)^2 - 1. \]

We can expand perturbationally as power series of \( J/T \). Very surprisingly \( \eta_j(x) \) has singularity only at \( x = \pm ji, \pm (j + 2)i + 2nQ \). So we assume that \( \eta_j(x) \) is univalent on the complex plane of \( x \) and \( 1 + \eta_j(x) \) is factorized as follows:
\[ 1 + \eta_j(x) = A_j(x - ji) \overline{A}_j(x + ji) B_j(x - (j + 2)i) \overline{B}_j(x + (j + 2)i). \]  
(10)

Here functions \( A_j(x), B_j(x) \) are periodic with periodicity \( 2Q \) and have singularity only at \( x = 2nQ \) and
\[ \overline{A}_j(x) \equiv A_j(\overline{x}), \quad \overline{B}_j(x) \equiv B_j(\overline{x}). \]
One should note that $A_j(x)$ is not an analytic function of $x$ but $\overline{A_j(x)}$ is analytic. Then second equation of (9) becomes

\[
\eta_j(x + i)\eta_j(x - i) = \overline{A_j(x + (j - 1)i)A_j(x + (j - 1)i)B_j(x - (j + 1)i)B_j(x + (j + 1)i)}
\times \overline{A_j(x + (j + 1)i)B_j(x - (j + 3)i)B_j(x + (j + 3)i)}.
\] (11)

If we assume that $\eta_j(x)$ is factorized as

\[
\eta_j(x) = X(x - j'i)Y(x + j'i)Z(x - (j + 2)i)W(x + (j + 2)i),
\]

$\eta_j(x + i)\eta_j(x - i)$ is

\[
X(x - (j - 1)i)Y(x + (j - 1)i)X(x - (j + 1)i)Y(x + (j + 1)i)
\]

\[
Z(x - (j + 1)i)W(x + (j + 1)i)Z(x - (j + 3)i)W(x + (j + 3)i).
\]

Comparing the singularity at $x = \pm(j - 1)i$ and $x = \pm(j + 3)i$ we have $X \sim A_{j-1}$, $Y \sim A_{j-1}$, $Z \sim B_{j+1}$ and $W \sim \overline{B_{j+1}}$. Then $\eta_j(x)$ should be factorized as:

\[
\eta_j(x) = A_{j-1}(x - j'i)A_{j-1}(x + j'i)B_{j+1}(x - (j + 2)i)B_{j+1}(x + (j + 2)i).
\] (12)

Considering the singularity at $x = \pm(j + 1)i$ we have

\[
\frac{A_{j-1}(x)}{B_{j-1}(x)} = \frac{A_{j+1}(x)}{B_{j+1}(x)}.
\] (13)

Using the first equation of (9) and noting that

\[
\delta(x) = \frac{1}{2\pi} \left\{ \frac{-i}{x - i\epsilon} + \frac{i}{x + i\epsilon} \right\},
\]

we have

\[
A_0(x) = \exp\left(\frac{J\sinh \phi}{2T_i} \sum_n \frac{-i}{x - 2n\epsilon - i\epsilon} \right) = \exp\left(\frac{J\sinh \phi}{2T_i} \cot \frac{\phi}{2}(x - i\epsilon) \right),
\]

$B_0(x) = 1$. (14)

From (13) we have

\[
\frac{A_{2j}(x)}{B_{2j}(x)} = \frac{A_0(x)}{B_0(x)} = A_0(x),
\]

\[
\frac{A_{2j+1}(x)}{B_{2j+1}(x)} = \frac{A_1(x)}{B_1(x)} = A'_0(x).
\] (15)

One can show that $A_0(x) = A'_0(x)$. Consider $\eta_{2j+1}^{-1}(x + (2j + 1)i)$. This quantity approaches 0 in the limit of infinite $j$. Of course near the singularity at $2i, 0, -(4j + 2)i$ and $-(4j + 4)i$ the deviation from 0 becomes big. Nevertheless the region on the complex plane where $|\eta_{2j+1}^{-1}(x + (2j + 1)i)| > \epsilon$ is expected to be narrower as $j$ goes to infinity. So we have

\[
1 = \lim_{j \to \infty} \frac{1 + \eta_{2j+1}(x + (2j + 1)i)}{\eta_{2j+1}(x + (2j + 1)i)} = \lim_{j \to \infty} \frac{A_0(x)\overline{A_0(x)}}{A_0(x)\overline{A_0(x)}}
\]

\[
\times \left[ \frac{B_{2j+1}(x)\overline{B_{2j+1}(x + (4j + 2)i)}}{B_{2j}(x)\overline{B_{2j}(x + (4j + 2)i)}} \right].
\] (16)
The first fraction in the big bracket should go to $e^{h/T}$ and the second should go to $e^{-h/T}$. The bracket in (16) becomes 1 in the limit of infinite $j$. Then we have

$$\frac{A_0'(x)}{A_0(x)} = \frac{A_0(i\infty)}{A_0'(i\infty)} = \alpha.$$  

As $A_0(i\infty)/A_0'(i\infty) = A_0(-i\infty)/A_0'(-i\infty) = 1/\pi$, we have $|\alpha| = 1$. If we choose the phase factor $\alpha$ is 1, the ratio of $A_j(x)$ and $B_j(x)$ is always $A_0(x)$. From (10) and (12) we have

$$A_{j-1}(x - ji)A_{j-1}(x + ji)B_{j+1}(x - (j + 2)i)B_{j+1}(x + (j + 2)i) + 1
= A_j(x - ji)A_j(x + ji)B_j(x - (j + 2)i)B_j(x + (j + 2)i).$$  

At $j = 1$ we have

$$B_1(x - i)B_1(x + i)B_1(x - 3i)B_1(x + 3i) = \frac{1}{A_0(x - i)A_0(x + i)} + B_2(x - 3i)B_2(x + 3i).$$  

$B_1, B_2$ are unknown functions. But this equation and condition

$$\lim_{x \to i\infty} B_1(x - i)B_1(x + i)B_1(x - 3i)B_1(x + 3i) = (2 \cosh h/T)^2,$$  

are sufficient to determine these functions. Put

$$u(x) = B_1(x - 2i)B_1(x + 2i).$$  

Equation (18) is written as

$$u(x + i) = \frac{1}{A_0(x - i)A_0(x + i)u(x - i)} + \frac{B_2(x - 3i)B_2(x + 3i)}{u(x - i)}.$$  

The l.h.s. has singularity at $i, -3i$. The first term of r.h.s. has at $i, -i, 3i$. The second term of r.h.s has at $3i, -3i, -i$. Assume that $u(x)$ is expanded as follows

$$u(x) = 2 \cosh \left(\frac{h}{T}\right) + \sum_{j=1}^{\infty} \sum_{n} \frac{c_j}{(x - 2nQ - 2i)^j} + \sum_{j=1}^{\infty} \sum_{n} \frac{\bar{c}_j}{(x - 2nQ + 2i)^j}.$$  

Consider the contour integral around $x = i$. Coefficients $c_j$ is determined by

$$c_j = \oint A_0(x - i)A_0(x + i)u(x - i)^{j-1} \frac{dx}{2\pi i} = \oint A_0(y)A_0(y + 2i)u(y)^{j-1} \frac{dy}{2\pi i}.$$  

The first sum of r.h.s. of (22) is

$$\sum_{j=1}^{\infty} \sum_{n} \exp[-\frac{2\pi J \sinh \phi}{T}\frac{a_1(y + i)}{u(y)}]x^{j-1} \frac{dy}{2\pi i} = \sum_{n} \exp[-\frac{2\pi J \sinh \phi}{T}\frac{a_1(y + i)}{u(y)}] \frac{1}{u(y)} \frac{dy}{2\pi i}$$

$$= \oint \phi \cot \phi \left(\frac{a_1}{2}\frac{x}{x - y - 2i} \exp[-\frac{2\pi J \sinh \phi}{T}\frac{a_1(y + i)}{u(y)}] \frac{1}{u(y)} \frac{dy}{2\pi i}.$$  

The second sum is calculated in similar way. Thus we get (3). From equation (10) we have

$$1 + \eta_1(x) = \exp \left(\frac{2\pi J \sinh \phi}{T}\frac{a_1(x)}{u(x + i)}\right) u(x + i) u(x - i).$$  

(25)
Substituting this into eq. (4) we get eq. (6)

\[ f = -T \int_{-Q}^{Q} s(x) \ln u(x + i) + \ln u(x - i) \, dx = -T \ln u(0). \]  

(26)

3. Analytical solutions

**Ising limit**

In this limit we put \( \Delta \to \infty, J = J_z/\Delta \). As \( \phi \) goes to \( \infty \), \( 2Q \) becomes 0. Then function \( \frac{2\pi J_z \sinh \phi}{T} a_1(y) \) is 0 at \( |\Im y| > 1 \) and \( J_z/T \) at \( |\Im y| < 1 \). Function \( u(y) \) is \( 2 \cosh h/T \) at \( |\Im y| > 2 \) and also a constant \( u(0) \) at \( |\Im y| < 2 \). From eq. (26) we have

\[ u(0) = 2 \cosh(h/T) + \frac{1 - \exp(-J_z/T)}{u(0)}. \]  

(27)

So we have \( u(0) = \cosh(h/T) + \sqrt{\sinh^2(h/T) + \exp(-J_z/T)} \) and known free energy of the Ising model.

Kuniba, Sakai and Suzuki [5] showed that Takahashi-Suzuki equation for \( |\Delta| < 1 \), \( h = 0 \) can be derived from the quantum transfer matrix and its fusion hierarchy matrices. We are confident to derive eqs. (5, 6) from the quantum transfer matrix and its fusion hierarchies. Details will be published elsewhere.

This research was supported in part by Grants-in-Aid for the Scientific Research (B) No. 11440103 from the Ministry of Education, Science and Culture, Japan.

References

[1] Takahashi, M. (1999) Thermodynamics of One-Dimensional Solvable Models, Cambridge University Press.

[2] Takahashi, M. (1971) *Prog. Theor. Phys.* **46**, 401.

[3] Gaudin, M. (1971) *Phys. Rev. Lett.* **26**, 1301.

[4] Takahashi, M. and Suzuki, M. (1972) *Prog. Theor. Phys.* **46**, 2187.

[5] Kuniba, A., Sakai, K. and Suzuki, J. (1998) *Nucl. Phys.* B **525**, 597.