ON THE CARTAN MODEL OF THE CANONICAL VECTOR
BUNDLES OVER GRASSMANNIANS

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Abstract. We give a representation of canonical vector bundles \( C_{n,p} \) over Grassmannian manifolds \( G(n,p) \) as non-compact affine symmetric spaces as well as their Cartan model in the group of the Euclidean motions \( SE(n) \).

MSC: 53C35, 53C30, Keywords: symmetric spaces, canonical vector bundles, Cartan model

1. Introduction

The Cartan model of Grassmannian manifolds \( G(n,p) \) in the special orthogonal group \( SO(n) \) is well known. Remarkably, we find that there is a representation of the canonical vector bundles \( C_{n,p} \) over \( G(n,p) \) as symmetric spaces, namely \( C_{n,p} = SE(n)/S(O(p) \times O(n-p)) \otimes \mathbb{R}^{n-p} \) and Cartan model realization in the group of Euclidean motions \( SE(n) \). To the author knowledge, this interesting fact is not observed yet (e.g., see [5, 4]).

The different homogeneous space representation of the canonical line bundles over projective spaces can be found in [3]. The Cartan-type model of the Möbius strip in \( SE(2) \) is recently obtained in [1]. Note that, due to [6], tangent bundles of Grassmannians have natural affine symmetric space structures.

2. Grassmannian Varieties

The points of the Grassmannian variety \( G(n,p) \) are by definition \( p \)-dimensional planes \( \pi \) passing through the origin of \( \mathbb{R}^n \). In particular, for \( p = 1 \), we have the projective space \( \mathbb{R}^{n-1} \), the set of lines through the origin in \( \mathbb{R}^n \).

Grassmannian manifolds are basic examples of compact symmetric spaces. The usual action of the group \( SO(n) \) on \( \mathbb{R}^n \) yields a transitive action on the set of all \( p \)-dimensional planes, i.e., on \( G(n,p) \). Let

\[
E_1 = (1,0,\ldots,0)^T, \quad E_n = (0,\ldots,0,1)^T.
\]

Take the plane \( \pi_0 = \text{span}\{E_1,\ldots,E_p\} \). Then the isotropy group of \( \pi_0 \) consists of matrices

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}, \quad A \in O(p), \quad B \in O(q), \quad \det A \cdot \det B = 1.
\]

It follows that \( G(n,p) \cong SO(n)/(S(O(p) \times O(q))) \). Further, let

\[
J_{p,q} = \begin{pmatrix}
-I_p & 0 \\
0 & I_q
\end{pmatrix},
\]

where \( I_k = \text{diag}(1,1,\ldots,1) \). Then \( \sigma_0 : SO(n) \to SO(n) \),

\[
\sigma_0(R) = J_{p,q}RJ_{p,q}^{-1}
\]
is an involutive automorphism with \( SO(n)^{\sigma_0} = SO(p) \times O(q) \) and the triple \((SO(n), SO(p) \times O(q)), \sigma_0\) is a symmetric space.

**Cartan model of Grassmannians.** Let

\[
\mathfrak{p}_p^0 = \text{span}\{E_i \wedge E_j \mid 1 \leq i \leq p < j \leq n = p + q\} \subset so(n).
\]

Then \( so(n) = so(p) + so(q) + \mathfrak{p}_p^0 \) is the symmetric pair decomposition of the Lie algebra \( so(n) \) on \((+1)\) and \((-1)\) eigenspaces of \( d\sigma_0 \) at the identity \( 1_n \).

Consider the \( \sigma_0 \)-twisted conjugation action \( A \mapsto R \sigma_0 (A)^{-1}, R, A \in SO(n) \).

Let

\[
Q_0 = \{ R \in SO(n) \mid \sigma_0(R) = R^{-1}\} = \{ R \in SO(n) \mid (R J_{p,q})^2 = 1_n\}.
\]

It can be easily verified that \( Q_0 \) is invariant under the \( \sigma_0 \)-twisted action.

The orbit through identity

\[
S_p^0 = SO(n) \bullet 1_n = \{ A \sigma_0 (A)^{-1} = AJ_{p,q}A^{-1} J_{p,q} \mid A \in SO(n)\},
\]

is isomorphic to \( G(n, p) \) as a \( SO(n) \)-space, relative to the \( \sigma_0 \)-twisted conjugation action and \( S_p^0 \) coincides with the identity connected component of \( Q_0 \) (the Cartan model of a symmetric space, e.g., see [2]). Furthermore, \( S_p^0 \) is equal to the image of \( \mathfrak{p}_p^0 \) under the exponential mapping.

Take the translation \( S_p^0 J_{p,q} = \{ AJ_{p,q}A^{-1} \mid A \in SO(n)\} \). The matrix \( AJ_{p,q}A^{-1} \) is symmetric and has \((-1)\) eigenvalue on the plane \( \pi = A \cdot \pi_0 = \text{span}\{A \cdot E_1, \ldots, A \cdot E_p\} \).

Thus, the diffeomorphism \( \rho_0 : S_p^0 \to G(n, p) \) can be seen as follows:

\[
\rho_0(R) = \pi,
\]

where \( \pi \) is the unique plane satisfying \( RJ_{p,q}(X) = -X, X \in \pi \).

**Projective Spaces.** For \( p = 1 \), \( J_{1,q} \) is the reflection \( S_1 \) with respect to the plane orthogonal to \( E_1 \). Further, the elements of \( \mathfrak{p}_1^0 \) can be taken to be of the form \(-\theta E_1 \wedge U, |U| = 1, U \perp E_1 \). Then \( R_{\theta,U} = \text{exp}(-\theta E_1 \wedge U) \) is the rotation in the plane spanned by \( E_1 \) and \( U \):

\[
R_{\theta,U}(E_1) = \cos \theta E_1 + \sin \theta U, \quad R_{\theta,U}(U) = -\sin \theta E_1 + \cos \theta U,
\]

which fix the orthogonal complement to \( \text{span}\{E_1, U\} \). The rotation can be represented as a composition: \( R_{\theta,U} = S_2 \circ S_1 \), where \( S_2 \) is the reflection with respect to the plane orthogonal to the vector \( V = \cos \frac{\theta}{2} E_1 + \sin \frac{\theta}{2} U \). Since \( R_{\theta,U} J_{1,q} = S_2 \circ S_1 \circ S_1 = S_2 \) and \( S_2(V) = -V \), we get

\[
\rho_0(R_{\theta,U}) = [V] = \left[ \begin{array}{c}
\cos \frac{\theta}{2} E_1 + \sin \frac{\theta}{2} U
\end{array} \right].
\]

Here \([V]\) denotes the line \( \{\mu V \mid \mu \in \mathbb{R}\}\).

3. **Cartan Model of the Canonical Vector Bundles**

Consider \( SE(n) \), the Lie group of the motions in the Euclidean space \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\). It is a semi-direct product of the special orthogonal group \( SO(n) \) (rotations) and the abelian group \( \mathbb{R}^n \) (translations) \( SE(n) = SO(n) \oplus \mathbb{R}^n \). We use the following usual matrix notation for the elements \( g \in SE(n) \):

\[
g = (R, X) = \begin{pmatrix}
R & X \\
0 & 1
\end{pmatrix}, \quad R \in SO(n), \quad X \in \mathbb{R}^n.
\]
The Lie algebra \( se(n) = so(n) \oplus \mathbb{R}^n \) consist of the \((n + 1) \times (n + 1)\) matrixes

\[
\xi = (\omega, v) = \left( \begin{array}{cc} \omega & v \\ 0 & 0 \end{array} \right), \quad \omega \in so(n), \quad v \in \mathbb{R}^n.
\]

The group multiplication and Lie bracket correspond to the usual multiplication and Lie bracket for the matrixes:

\[
(R_1, X_1) \cdot (R_2, X_2) = (R_1R_2, X_1 + R_1X_2),
\]

\[
[(\omega_1, v_1), (\omega_2, v_2)] = ([\omega_1, \omega_2], \omega_1v_2 - \omega_2v_1).
\]

**Lemma 1.** The mapping \( \sigma : SE(n) \rightarrow SE(n) \) given by

\[
\sigma((R, X)) = (\sigma_0(r), J_{p,q}X) = (J_{p,q} R J_{p,q}, J_{p,q}X).
\]

is an involutive automorphism and the set of fixed point consist of matrixes of the form

\[
\left( \begin{array}{ccc} A & 0 & 0 \\ 0 & B & X \\ 0 & 0 & 1 \end{array} \right), \quad A \in O(p), \quad B \in O(q), \quad \det A \cdot \det B = 1, \quad X \in \mathbb{R}^q,
\]

i.e., \( SE(n)^\sigma = S((O(p) \times O(q)) \oplus_s \mathbb{R}^q \)

Therefore, the triple \((SE(n), S((O(p) \times O(q)) \oplus_s \mathbb{R}^q, \sigma)\) is a non-compact affine symmetric space (we follow the notation of [5]). The differential \( d\sigma \) at the identity \((I_n, 0)\) is an involutive automorphism of the Lie algebra \( se(n) \). We have symmetric pair decomposition of \( se(n) \) on its \((+1)\) eigenspace (the Lie algebra of \( SE(n)^\sigma \)) and \((-1)\) eigenspace:

\[
\mathfrak{d}_p = \text{span}\{(E_i \wedge E_j, E_k) | 1 \leq i \leq p < j \leq n = p + q, 1 \leq k \leq p\} \cong \mathfrak{d}_p^0 \oplus \mathbb{R}^p.
\]

Let

\[
\mathcal{Q} = \{ g = (R, Y) \in SE(n) | \sigma(g) = g^{-1} \}.
\]

The set \( \mathcal{Q} \) is preserved under the \( \sigma \)-twisted conjugation action:

\[
(A, X) \bullet (R, Y) = (A, X) \cdot (R, Y) \cdot \sigma((A, X)^{-1})
\]

\[
= (AR J_{p,q} A^{-1} J_{p,q}, X + AY - AR J_{p,q} A^{-1} X).
\]

The Cartan model of symmetric spaces is usually given for reductive Lie groups. Similarly we have

**Theorem 1.** (The Cartan Model) The orbit through the identity

\[
\mathcal{S}_p = SE(n) \bullet (I_n, 0) = \{(AJ_{p,q} A^{-1} J_{p,q}, X - AJ_{p,q} A^{-1} X) | (A, X) \in SE(n)\}.
\]

is isomorphic to \( SE(n)/SE(n)^\sigma \) as a \( SE(n) \)-space, relative to the \( \sigma \)-twisted conjugation action. Furthermore, \( \mathcal{S}_p \) is equal to the identity component of \( \mathcal{Q} \) and it is equal to the image of \( \mathfrak{d}_p \) under the exponential mapping.

**Lemma 2.** The exponential mapping \( \exp : se(n) \rightarrow SE(n) \) is surjective.

**Proof.** A simple computation shows

\[
\xi^m = (\omega, v)^m = (\omega^m, \omega^{m-1}v), \quad m \in \mathbb{N}.
\]

Therefore

\[
\exp(\xi) = (\exp(\omega), Y),
\]
where the vector $Y = Y_\omega(v)$ is equal to

$$Y_\omega(v) = v + \frac{1}{2}\omega v + \frac{1}{3!}\omega^2 v + \cdots + \frac{1}{m!}\omega^{m-1} v + \ldots.$$  

Since $\exp: so(n) \to SO(n)$ is surjective, we only need to prove that the linear mapping $\hat{\sigma}$, for the fixed $R \in SO(n)$ and properly chosen $\omega$, $R = \exp(\omega)$, has the maximal rank.

Let $e_1, \ldots, e_n$ be the orthonormal base, in which the matrix $R$ has the canonical form

$$R = \text{diag}(R(\theta_1), R(\theta_2), \ldots, R(\theta_k), 1, 1, \ldots, 1),$$

where $R(\theta_i)$ are rotations in the planes $\text{span}\{e_{2i-1}, e_{2i}\}$:

$$R(\theta_i) = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix}, \quad |\theta_i| < 2\pi, \quad i = 1, \ldots, k.$$

Then we can take $\omega = \text{diag}(\Pi(\theta_1), \Pi(\theta_2), \ldots, \Pi(\theta_k), 0, 0, \ldots, 0)$, where

$$\Pi(\theta_i) = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}, \quad i = 1, \ldots, k.$$

Let $Y_i(v) = \langle Y_\omega(v), e_i \rangle$. For a given $v = v_1 e_1 + \ldots + v_n e_n$ we have

$$Y_i(v) = v_i, \quad i = 2k + 1, \ldots, n.$$

Further, from (4) we get that $Y_\omega(v)$ satisfies the relation

$$\omega Y_\omega(v) = (\exp(\omega) - I_n)v,$$

or, in coordinates:

$$-\theta_i Y_{2i}(v) = \cos \theta_i v_{2i-1} - \sin \theta_i v_{2i} - v_{2i-1}$$

$$\theta_i Y_{2i-1}(v) = \sin \theta_i v_{2i-1} + \cos \theta_i v_{2i} - v_{2i}$$

By using the trigonometric identities $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$, $1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$, we can write the components of the vector $Y$ in the compact form:

$$Y_{2i-1}(v) = 2 \sin \frac{\theta_i}{2} \left( \cos \frac{\theta_i}{2} v_{2i-1} - \sin \frac{\theta_i}{2} v_{2i} \right)$$

$$Y_{2i}(v) = 2 \sin \frac{\theta_i}{2} \left( \sin \frac{\theta_i}{2} v_{2i-1} + \cos \frac{\theta_i}{2} v_{2i} \right), \quad i = 1, \ldots, k.$$

From (5), (7), (8), it follows that $Y_\omega$ has no kernel. □

**Proof of the Theorem.** In proving the Theorem, we mainly follow standard arguments given for compact (or reductive) Lie groups (e.g., see [2]).

(i) Let $\tau : SE(n) \to S_p$ be the mapping defined by $\tau(g) = g\sigma(g^{-1})$. It is clear that $\tau$ is constant on left cosets modulo $SE(n)\sigma$ ($\tau(g_1) = \tau(g_2)$ if and only if $\sigma(g_1 g_2^{-1}) = g_1 g_2^{-1}$, i.e., $g_1 g_2^{-1} \in SE(n)\sigma$) and that the induced morphism $\hat{\tau} : SE(n)/SE(n)\sigma \to S_p$ is bijective and satisfies

$$\hat{\tau}(g_1 \cdot g_2 SE(n)\sigma) = g_1 \cdot \hat{\tau}(g_2 SE(n)\sigma).$$

Further, $\hat{\tau}$ is a diffeomorphism from the dimensional reasons. (It can be easily seen that the tangent space of $S_p$ at the identity of the group is $\mathfrak{p}$, so the differential $d\hat{\tau}|_{SE(n)\sigma}$ is surjective.)

(ii) Suppose that $(R, Y)$ belongs to the identity component of $Q$. Then

$$\sigma(R, Y) = (R^{-1}, -R^{-1}Y), \quad \text{i.e.,} \quad \sigma_0(R) = R^{-1} \quad \text{and} \quad J_{p,q}Y = -R^{-1}Y.$$
Remark 2 of RP to holonomic LL systems on SE bundles over projective spaces can be found in [3]. Namely, the SE space $\rho$ such that $SE(10)$ (10) $X - AJ_{p,q}A^{-1}X = 2\text{pr}_x X,$

where $\text{pr}_x$ denotes the orthogonal projection to $\pi$. Therefore, $(R,Y) \in S_p^1$. The another inclusion is trivial: $g = g^\prime\sigma(g^{-1})$ implies $\sigma(g) = \sigma(g^\prime)\sigma^{-1}(g) = g^{-1}$.

(iii) First, we shall prove the inclusion $\text{exp}(\mathfrak{d}_p) \subset S_p$. Let $g = \exp(\xi)$, $\xi \in \mathfrak{d}_p$. Consider the element $g^\prime = \exp(\xi/2)$. Then

$$\tau(g^\prime) = \exp(\xi/2)\sigma(\exp(-\xi/2)) = \exp(\xi/2)\exp(\xi/2) = (g^\prime)^2 = g,$$

that is $g \in S_p$. Here we used the identity $\sigma(\exp(\xi)) = \exp(d\sigma|_{(1,0)}\xi)$.

Now, let $R$ be an arbitrary element in $S_p^0$. From $S_p^0 = \exp(\mathfrak{d}_0^p)$ and Lemma 2, for a properly chosen $\omega \in \mathfrak{d}_0^p$, $R = \exp(\omega)$, we have that the linear mapping (11) define an isomorphism between $\text{span}\{E_1,\ldots,E_p\}$ and $\pi = \rho_0(R)$. Therefore $\exp : \mathfrak{d}_p \to S_p$ is a surjective map. □

Recall that the canonical vector bundle $C_{n,p}$ over $G(n,p)$ at the point $\pi \in G(n,p)$ has the fibre equal to $\pi$, now considered as a vector space:

$$C_{n,p} = \{(\pi, X) \in G(n,p) \times \mathbb{R}^n \mid X \in \pi\}.$$

**Lemma 3.** The variety $S_p$ is diffeomorphic to the canonical vector bundle $C_{n,p}$.

**Proof.** According to (11), the mapping $\rho : S_p \to C_{n,p}$, defined by

$$\rho(R, Y) = (\rho_0(R), Y)$$

establish the diffeomorphism between $S_p$ and $C_{n,p}$. □

From the above considerations, we see that the canonical vector bundles over Grassmannians in a canonical way can be considered as symmetric spaces.

**Theorem 2.**

$$(A, X) \ast (\pi, Y) = (A\pi, AY + 2\text{pr}_x X), \quad Y \in \pi \subset \mathbb{R}^n$$

defines a transitive $SE(n)$-action on the canonical vector bundle $C_{n,p}$ over $G(n,p)$ such that $\rho$ becomes a $SE(n)$-invariant diffeomorphism:

$$\rho((A, X) \ast (R, Y)) = (A, X) \ast (R, Y), \quad (A, X) \in SE(n), \ (R, Y) \in S_p.$$  

Therefore, the $SE(n)$-action (11) realizes $C_{n,p}$ as a non-compact affine symmetric space $(SE(n), S((O(p) \times O(q)) \otimes \mathbb{R}^q, \sigma)$.

**Remark 1.** The different homogeneous space representation of the canonical line bundles over projective spaces can be found in [3]. Namely, $\mathbb{R}P^n \setminus x_0$ is diffeomorphic to $C_{n-1,1}$, where $x_0 \in \mathbb{R}P^n$ is an arbitrary point. Then projective transformations of $\mathbb{R}P^n$ which leave $x_0$ invariant acts transitively on $\mathbb{R}P^n \setminus x_0 \approx C_{n-1,1}$.

**Remark 2.** The description of $\exp(\mathfrak{d}_p)$ is important in the study of discrete non-holonomic LL systems on $SE(n)$ (see [1]).

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1 The alternative proof of this statement is to show that the tangent space to a $\sigma$-twisted $SE(n)$-orbit coincides the tangent space to $Q$.  

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Canonical Line Bundles. For $p = 1$, there is a direct construction of a diffeomorphism between $\exp(\mathfrak{d}_1)$ and the canonical line bundle $C_{n,1}$. The elements of $\mathfrak{d}_1$ can be taken to be of the form $\xi = (-\theta E_1 \wedge U, \lambda E_1)$, $\theta, \lambda \in \mathbb{R}$, $|U| = 1$, $U \perp E_1$. Let $(R_{\theta, U}, Y) = \exp(-\theta E_1 \wedge U, \lambda E_1)$. Then the relation (6) reads
\begin{equation}
-\theta E_1 \wedge U(Y) = \lambda R_{\theta, U} E_1 - \lambda E_1.
\end{equation}
Therefore, taking into account (1) and (11) we obtain
\begin{equation}
-\theta \langle U, Y \rangle E_1 + \theta \langle E_1, Y \rangle U = \lambda (\cos \theta - 1) E_1 + \lambda \sin \theta U.
\end{equation}
From (4) we get that $Y$ belongs to $\text{span}\{E_1, U\}$ and (12) gives
\begin{equation}
Y = \lambda \frac{\sin \theta}{\theta} E_1 + \lambda \frac{1 - \cos \theta}{\theta} U = \lambda \frac{2 \sin \frac{\theta}{2}}{\theta} (\cos \frac{\theta}{2} E_1 + \sin \frac{\theta}{2} U).
\end{equation}
Finally, in the view of (2) and (13) we get $\exp(\mathfrak{d}_1) \approx C_{n,1}$.

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References

[1] Fedorov, Y. N. and Zenkov, D. V.: Discrete Nonholonomic LL Systems on Lie Groups, Nonlinearity, 18, no. 5, 2211–2241 (2005), arXiv: math.DS/0409415
[2] Fomenko, A. T.: Differential Geometry and Topology. Supplementary Chapters, Moscow University, Moscow, 1983, 217 p. (Russian)
[3] Gorbatsevich, V. V.: On three-dimensional homogeneous spaces, Sib. Mat. Zh. 18, no.2, 280-293 (1977) (Russian); English translation: Sib. Math. J. 18, 200-210 (1977).
[4] Helgason, S.: Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York 1978, 628 p.
[5] Kobayashi, S. and Nomizu, K.: Foundation of Differential Geometry, Volume II, John Willey & Sons, New York, 1969, 468 p.
[6] Yano, K. and Kobayashi, S.: Prolongation of tensor fields and connections to tangent bundles I, II, III, J. Math. Soc. Japan 18, no. 2, 194-210 (1966); 18, no. 3, 236-246 (1966); 19, no. 4, 486-488 (1967).

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2This example is motivated by [1] and was the staring point in writing this note.