On cosets weight distributions of the doubly-extended Reed-Solomon codes of codimension 4

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Abstract. We consider the \([q+1,q-3,5]_q 3\) generalized doubly-extended Reed-Solomon code of codimension 4 as the code associated with the twisted cubic in the projective space \(PG(3,q)\). Basing on the point-plane incidence matrix of \(PG(3,q)\), we obtain the number of weight 3 vectors in all the cosets of the considered code. This allows us to classify the cosets by their weight distributions and to obtain these distributions. For the cosets of equal weight having distinct weight distributions, we prove that the difference between the \(w\)-th components, \(3 < w \leq q + 1\), of the distributions is unambiguously determined by the difference between the 3-rd components. This implies an interesting (and in some sense unexpected) symmetry of the obtained distributions. To obtain the property of differences we prove a useful relation for the Krawtchouck polynomials. Also, we describe an alternative way of obtaining the cosets weight distributions on the base of the integral weight spectra over all the cosets of the fixed weight.

Keywords: Reed-Solomon codes, cosets weight distributions, MDS codes, twisted cubic, projective spaces.

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1 Introduction

Let $\mathbb{F}_q$ be the Galois field with $q$ elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $\mathbb{F}_q^+ = \mathbb{F}_q \cup \{\infty\}$. Let $\mathbb{F}_q^n$ be the space of $n$-dimensional vectors over $\mathbb{F}_q$. We denote by $[n, k, d]_qR$ an $\mathbb{F}_q$-linear code of length $n$, dimension $k$, minimum distance $d$, and covering radius $R$. If $d = n - k + 1$, it is a maximum distance separable (MDS) code. Let $\text{PG}(N, q)$ be the $N$-dimensional projective space over $\mathbb{F}_q$. For an introduction to coding theory see [1–3]. For an introduction to projective spaces over finite fields and connections between projective geometry and coding theory see [4–9].

In a parity check $(N + 1) \times (q + 1)$ matrix of a $[q + 1, q - N, N + 2]_q$ generalized doubly-extended Reed-Solomon (GDRS) code, the $j$-th column has the form $(v_j, v_j\alpha_j, v_j\alpha_j^2, \ldots, v_j\alpha_j^N)^{\text{tr}}$, where $j = 1, 2, \ldots, q$; $\alpha_1, \ldots, \alpha_q$ are distinct elements of $\mathbb{F}_q$; $v_1, \ldots, v_q$ are nonzero (not necessarily distinct) elements of $\mathbb{F}_q$. Also, this matrix contains one more column $(0, \ldots, 0, v)^{\text{tr}}$ with $v \neq 0$. The code, dual to a GDRS code, is a GDRS code too. Clearly, a GDRS code is MDS.

An $n$-arc in $\text{PG}(N, q)$, with $n \geq N + 1 \geq 3$, is a set of $n$ points such that no $N + 1$ points belong to the same hyperplane of $\text{PG}(N, q)$. An $n$-arc is complete if it is not contained in an $(n + 1)$-arc. Arcs and MDS codes are equivalent objects, see e.g. [2, 8, 9].

In $\text{PG}(N, q)$, $2 \leq N \leq q - 2$, a normal rational curve is any $(q + 1)$-arc projectively equivalent to the arc $\{(1, t^2, \ldots, t^{N-1}, t^N) : t \in \mathbb{F}_q\} \cup \{(0, \ldots, 0, 1)\}$. The points (in homogeneous coordinates) of a normal rational curve in $\text{PG}(N, q)$ treated as columns define a parity check matrix of a $[q + 1, q - N, N + 2]_q$ GDRS code [8, 9]. We say that this GDRS code is associated with the normal rational curve. In $\text{PG}(3, q)$, the normal rational curve is called a twisted cubic [7, 10]. We denote the cubic by $\mathcal{C}$, see Section 2.1 for detail.

Let $\mathcal{C}_\rho$ be the $[q + 1, q - 3, 5]_q3$ GDRS code associated with the cubic $\mathcal{C}$. Twisted cubics in $\text{PG}(3, q)$ have been widely studied; see e.g. [10, 13] and the references therein. In particular, in [13], the orbits of planes and points under the group $G_q$ of the projectivities fixing a cubic are considered. In [13], the structure of the point-plane incidence matrix in $\text{PG}(3, q)$ with respect to the orbits of points and planes under $G_q$ is described.

Also in [13], it is shown that twisted cubics can be treated as multiple $\rho$-saturating sets with $\rho = 2$ which, in turn, give rise to asymptotically optimal non-binary linear multiple covering $[q + 1, q - 3, 5]_q3$ codes of radius $R = 3$. This means that the code $\mathcal{C}_\rho$ can be viewed as an asymptotically optimal multiple covering. For an introduction to multiple covering codes and multiple saturating sets see [14–16].

A coset of a code is a translation of the code. A coset $\mathcal{V}$ of an $[n, k, d]_qR$ code $\mathcal{C}$ can be represented as

$$\mathcal{V} = \{x \in \mathbb{F}_q^n \mid x = c + v, c \in \mathcal{C}\} \subset \mathbb{F}_q^n$$

where $v \in \mathcal{V}$ is a vector fixed for the given representation; see [1–3, 17, 18] and the references therein.
The weight distribution of code cosets, including the number of the cosets with distinct distributions, is interesting itself; it is an important combinatorial property of a code. In particular, the distribution serves to estimate decoding results. There are many papers connected with distinct aspects of the weight distribution of cosets for codes over distinct fields and rings, see e.g. [1], [2, Sec. 5.5, 6.6, 6.9], [17, Sec. 7], [18, Sec. 10], [19–33], [34, Sec. 6.3] and the references therein. Nevertheless, as far as it is known to the authors, the weight distribution of GDRS code cosets is an open problem.

In this paper, we consider the weight distribution of the cosets of the $[q+1, q-3, 5]_q$ GDRS code of codimension 4. We consider it as the code $C$ associated with the twisted cubic. By a geometrical way, basing on the PG$(3, q)$ point-plane incidence matrix of [13], we obtain the number of weight 3 vectors in all the cosets of $C$. This allows us to use the approach of [17, Sec. 7], [18, Sec. 10]. As a result, we classify the cosets of $C$ by their weight distributions and obtain the relations that fully describe the needed weight distributions.

For the cosets of equal weight having distinct weight distributions, we prove the following property of differences: the difference between the $w$-th components, $3 < w \leq q+1$, of the distributions is unambiguously determined by the difference between the 3-rd components. This implies an interesting (and in some sense unexpected) symmetry of the weight distributions of the $[q+1, q-3, 5]_q$ code cosets. To obtain the property of differences we prove an useful relation for the Krawtchouck polynomials, see Lemma 20.

Also, we propose an alternative way of obtaining the cosets weight distributions on the base of the integral weight spectra of the cosets. (For a linear code of length $n$, we call integral weight spectrum the overall numbers of weight $w$ vectors, $0 \leq w \leq n$, in all the cosets of some fixed weight).

This paper is organized as follows. Section 2 contains preliminaries. In Section 3 we give the distribution of weight 3 vectors in the cosets of the code $C$ and classify the cosets. In Section 4 the relations that fully describe the weight distribution of the cosets of $C$ are obtained. In Section 5 we prove the property of differences and consider a symmetry in the weight distributions of the code $C$ cosets. An useful relation for the Krawtchouck polynomials is proven. Finally, in Section 6 we describe an alternative way of obtaining the cosets weight distributions on the base of the integral weight spectra.

Throughout the paper, we consider $q \geq 5$.

2 Preliminaries

2.1 Twisted cubic

In this subsection, we summarize some known results on twisted cubics from [10, Ch. 21].

Let $P(x_0, x_1, x_2, x_3)$ be a point of PG$(3, q)$ with the homogeneous coordinates $x_i \in \mathbb{F}_q$;
the leftmost nonzero coordinate is equal to 1. For \( t \in \mathbb{F}_q^+ \), let \( P(t) \) be a point such that
\[
P(t) = P(1, t, t^2, t^3) \quad \text{if} \quad t \in \mathbb{F}_q,
\]
\[
P(\infty) = P(0, 0, 0, 1).
\]
Let \( \mathcal{C} \subset \text{PG}(3, q) \) be the \textit{twisted cubic} consisting of \( q + 1 \) points \( P_1, \ldots, P_{q+1} \) no four of which are coplanar. We consider \( \mathcal{C} \) in the canonical form
\[
\mathcal{C} = \{ P_1, P_2, \ldots, P_{q+1} \} = \{ P(t)|t \in \mathbb{F}_q^+ \}. \tag{2}
\]
Let \( \pi(c_0, c_1, c_2, c_3) \subset \text{PG}(3, q) \) with \( c_i \in \mathbb{F}_q \) be the plane with equation \( c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 = 0 \). In every point \( P(t) \in \mathcal{C} \), there is an \textit{osculating plane} \( \pi_{\text{osc}}(t) \) such that \( \pi_{\text{osc}}(t) = \pi(−t^3, 3t^2, −3t, 1), P(t) \in \pi_{\text{osc}}(t) \).

A \textit{chord} of \( \mathcal{C} \) is a line through a pair of real points of \( \mathcal{C} \) or a pair of complex conjugate points. In the last case it is an \textit{imaginary chord}. If the real points are distinct, it is a \textit{real chord}. If the real points coincide with each other, it is a \textit{tangent}.

\textbf{Notation 1.} The following notation is used:

- \( G_q \): the group of projectivities in \( \text{PG}(3, q) \) fixing \( \mathcal{C} \);
- \( \Gamma \): the osculating developable to \( \mathcal{C} \);
- \( d_{\mathcal{C}} \)-plane: a plane containing \textit{exactly} \( d \) distinct points of \( \mathcal{C} \), \( d = 0, 1, 2, 3 \);
- \( 1_{\mathcal{C}} \setminus \Gamma \)-plane: a \( 1_{\mathcal{C}} \)-plane not in \( \Gamma \);
- \( \mathcal{C} \)-point: a point of \( \mathcal{C} \);
- \( \mu_\Gamma \)-point: a point off \( \mathcal{C} \) lying on \textit{exactly} \( \mu \) osculating planes, \( \mu_\Gamma = 0, 1, 3, q + 1 \);
- \( T \)-point: a point off \( \mathcal{C} \) on a tangent to \( \mathcal{C} \) for \( q \not\equiv 0 \pmod{3} \);
- \( TO \)-point: a point off \( \mathcal{C} \) on a tangent and one osculating plane for \( q \equiv 0 \pmod{3} \);
- \( RC \)-point: a point off \( \mathcal{C} \) on a real chord;
- \( IC \)-point: a point on an imaginary chord;
- \( \#S \): the cardinality of a set \( S \).

\textbf{Theorem 2.} [10, Ch. 21] \textit{Under} \( G_q \), \textit{there are five orbits} \( \mathcal{N}_i \) \textit{of planes and five orbits} \( \mathcal{M}_j \) \textit{of points}. \textit{These orbits have the following properties:}

\begin{enumerate}
  \item \textbf{(i)} For all \( q \), the orbits \( \mathcal{N}_i \) of planes are as follows: \( \mathcal{N}_1 = \{ \Gamma \text{-planes} \}, \mathcal{N}_2 = \{ 2_{\mathcal{C}} \text{-planes} \}, \mathcal{N}_3 = \{ 3_{\mathcal{C}} \text{-planes} \}, \mathcal{N}_4 = \{ 1_{\mathcal{C}} \setminus \Gamma \text{-planes} \}, \mathcal{N}_5 = \{ 0_{\mathcal{C}} \text{-planes} \} \).
  \item \textbf{(ii)} For \( q \not\equiv 0 \pmod{3} \), the orbits \( \mathcal{M}_j \) of points are as follows:
\end{enumerate}
\[
\mathcal{M}_1 = \mathcal{C}, \mathcal{M}_2 = \{ T \text{-points} \}, \mathcal{M}_3 = \{ 3_{\Gamma} \text{-points} \}, \mathcal{M}_4 = \{ 1_{\Gamma} \text{-points} \}, \mathcal{M}_5 = \{ 0_{\Gamma} \text{-points} \}; \tag{3}
\]
\[ M_1 = q + 1, \ M_2 = q(q + 1), \ M_3 = \frac{q^3 - q}{6}, \ M_4 = \frac{q^3 - q}{2}, \ M_5 = \frac{q^3 - q}{3}. \]

Also,

if \( q \equiv 1 \pmod{3} \) then \( M_3 \cup M_5 = \{ \text{RC-points} \}, \ M_4 = \{ \text{IC-points} \}; \)

if \( q \equiv -1 \pmod{3} \) then \( M_3 \cup M_5 = \{ \text{IC-points} \}, \ M_4 = \{ \text{RC-points} \}. \)

(iii) For \( q \equiv 0 \pmod{3} \), the orbits \( M_k \) of points are as follows:

\[ M_1 = \mathcal{C}, \ M_2 = \{ (q + 1)_{\Gamma} - \text{points} \}, \ M_3 = \{ \text{TO-points} \}, \ M_4 = \{ \text{RC-points} \}, \ M_5 = \{ \text{IC-points} \}; \]

\( \#M_1 = \#M_2 = q + 1; \ #M_3 = q^2 - 1; \ #M_4 = \#M_5 = \frac{q^3 - q}{2}. \)

(iv) No two chords of \( \mathcal{C} \) meet off \( \mathcal{C} \). Every point off \( \mathcal{C} \) lies on exactly one chord of \( \mathcal{C} \).

### 2.2 The number of \( 3\mathcal{K} \)-planes through points and lines of \( \text{PG}(3, q) \)

**Lemma 3.** [13] Lem. 4.3] The number of \( 3\mathcal{K} \)-planes and \( 2\mathcal{K} \)-planes through a real chord of \( \mathcal{C} \) is equal to \( q - 1 \) and \( 2 \), respectively.

**Lemma 4.** [13] Lem. 4.12] Through every point of the orbit \( \mathcal{M}_j \) we have the same number of planes from the orbit \( \mathcal{N}_j \).

Throughout the paper, for \( q \equiv \xi \pmod{3} \), let \( r_{3j}^{(\xi)} \) be the number of \( 3\mathcal{K} \)-planes from the orbit \( \mathcal{N}_3 \) through every point of the orbit \( \mathcal{M}_j \).

**Theorem 5.** [13] Th. 3.1, Tab. 1, 2] Let \( q \equiv \xi \pmod{3} \). Let \( r_{3j}^{(\xi)} \) be as above. The following holds:

(i) \( r_{31}^{(1)} = r_{31}^{(-1)} = \frac{1}{2}(q^2 - q), \ r_{32}^{(1)} = r_{32}^{(-1)} = \frac{1}{6}(q^2 - 3q + 2); \)

\( r_{33}^{(1)} = \frac{1}{6}(q^2 + q + 4), \ r_{34}^{(1)} = \frac{1}{6}(q^2 - q), \ r_{35}^{(1)} = \frac{1}{6}(q^2 - 4); \)

\( r_{33}^{(-1)} = \frac{1}{6}(q^2 - q + 4), \ r_{34}^{(-1)} = \frac{1}{6}(q^2 + q), \ r_{35}^{(-1)} = \frac{1}{6}(q^2 - q - 2). \)

(ii) \( r_{31}^{(0)} = \frac{1}{2}(q^2 - q), \ r_{32}^{(0)} = r_{35}^{(0)} = \frac{1}{6}(q^2 - q), \ r_{33}^{(0)} = \frac{1}{6}(q^2 - 3q), \ r_{34}^{(0)} = \frac{1}{6}(q^2 + q). \)

### 2.3 Cosets of a linear code

We give a few known definitions and properties connected with cosets of linear codes, see e.g. [13], [17], [18] and the references therein.

We consider a coset \( \mathcal{V} \) of an \([n, k, d]_q^R \) code \( \mathcal{C} \) in the form \( (\mathbb{F}_q^n) \). We have \#\( \mathcal{V} = \#\mathcal{C} = q^k \). One can take as \( v \) any vector of \( \mathcal{V} \). So, there are \#\( \mathcal{V} = q^k \) formally distinct representations of the form \( (\mathbb{F}_q^n) \); all they give the same coset \( \mathcal{V} \). If \( v \in \mathcal{C} \), we have \( \mathcal{V} = \mathcal{C} \). The distinct cosets of \( \mathcal{C} \) partition \( \mathbb{F}_q^n \) into \( q^{n-k} \) sets of size \( q^k \).
Notation 6. For an $[n, k, d]_q R$ code $C$ and its coset $V$ of the form (1), the following notation is used:

- $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ the number of correctable errors;
- $wt(x)$ the Hamming weight of a vector $x \in \mathbb{F}_q^n$;
- $A_w(C)$ the number of weight $w$ codewords of $C$;
- $S(C) = \#S(C)$ the number of nonzero weights in $C$; $S(C) = \{ w > 0 | A_w(C) \neq 0 \}$;
- $s(C) = \#S(C)$ the set of nonzero weights in $C$;
- $C^\perp$ the $[n, n-k, d^\perp]_q R^\perp$ code dual to $C$;
- $v + C$ the coset of $C$ of the form (1);
- $A_w(V)$ the number of weight $w$ vectors in $V$;
- $\mathcal{V}(W)$ a coset of weight $W$; if $w < W$ then $A_w(\mathcal{V}(W)) = 0$;
- a coset leader a vector in the coset of the smallest Hamming weight;
- $H(C)$ a parity check $(n-k) \times n$ matrix of $C$;
- $x^\text{tr}$ the transposed vector $x$;
- $H(C)x^\text{tr}$ the syndrome of a vector $x \in \mathbb{F}_q^n, H(C)x^\text{tr} \in \mathbb{F}_{q^{n-k}}$;
- a coset syndrome the syndrome of any vector of the coset.

In cosets of weight $> t$, a vector of the minimal weight is not necessarily unique. Cosets of weight $\leq t$ have a unique leader.

The code $C$ is the coset of weight zero. The leader of $C$ is the zero vector of $\mathbb{F}_q^n$.

All vectors in a code coset have the same syndrome; it is called the coset syndrome. Thus, there is a one-to-one correspondence between cosets and syndromes. The syndrome of $C$ is the zero vector of $\mathbb{F}_{q^{n-k}}$.

A linear $[n, k]_q$ code has covering radius $R$ if every column of $\mathbb{F}_{q^{n-k}}$ is equal to a linear combination of at most $R$ columns of a parity check matrix of the code, and $R$ is the smallest value with such property.

The covering radius $R$ of the code $C$ is equal to the maximum weight of a coset of $C$.

Theorem 7. (i) [17, Lem. 7.5.1] For a code $C$, the weight distributions of all cosets $\alpha v + C$, $\alpha \in \mathbb{F}_q^*$, are identical.

(ii) [2, Th. 6.20], [17, Th. 7.5.2], [18, Th. 10.10] For a code $C$, the weight distribution of any coset of weight $< s(C^\perp)$ is unambiguously determined if, in the coset, the numbers of vectors of weights $1, 2, \ldots, s(C^\perp) - 1$ are known.
As far as it is known to the authors, the weight distribution of cosets of GDRS codes, including the number of the cosets with distinct weight distributions, is an open combinatorial problem.

3 The distribution of weight 3 vectors in the cosets of the code $C_\xi$. Classification of the cosets

**Notation 8.** Let $C_\xi$ be the $(q+1, q-3, 5)_q3$ GDRS code associated with the twisted cubic $C$ of (2) so that columns of its parity check matrix are points of $C$ in homogeneous coordinates.

In $PG(3, q)$, every point off $C$ lies in a few $3_\xi$-planes, see Section 2.2; therefore, the covering radius of $C_\xi$ is equal to 3 and the cosets of $C_\xi$ have weight $\leq 3$.

**Remark 9.** Every point $P$ of $PG(3, q)$ gives rise to $q - 1$ nonzero syndromes of cosets of $C_\xi$; the syndromes can be generated by multiplying $P$ (in homogeneous coordinates) by elements of $F_q^\ast$. Points of each orbit $\mathcal{M}_j$ generate $\#\mathcal{M}_j \cdot (q - 1)$ syndromes corresponding to a set of cosets with the same weight distribution.

**Theorem 10.** Let $q \equiv \xi \pmod{3}$. The distribution of weight 3 vectors in all the cosets of the code $C_\xi$ is given by Table 1.

**Proof.** The columns of the parity check matrix $H(C_\xi)$ are points of $C$ in homogeneous coordinates. By Remark 9, the number of cosets in Table 1 is equal to 3 and the cosets of $C_\xi$ have weight $\leq 3$.

- Let a coset leader weight be 0. The coset is the code $C_\xi$ of minimum distance 5; it does not contain weight 3 words. We obtain row 1 of Table 1.
- Let a coset leader weight be 1. In this case, the syndrome of the coset can be represented as a linear combination of two columns of $H(C_\xi)$; it is a $C$-point. These cosets do not contain weight 3 vectors as minimum distance of $C_\xi$ is equal to 5. We have $\#\mathcal{M}_1 = q + 1$, see (3); therefore the number of cosets is $\#\mathcal{M}_1 \cdot (q - 1) = (q + 1)(q - 1)$. We obtain row 2 of Table 1.
- Let a coset leader weight be 2. In this case, the syndrome of the coset can be represented as a linear combination of three columns of $H(C_\xi)$; it is an intersection of $C$ of a real chord and a $3_\xi$-plane. To calculate the number $T$ of the intersections, we should take the total number of $3_\xi$-planes through an RC-point and subtract from it the number $q - 1$ of $3_\xi$-planes in which a real chord lies (see Lemma 3).
Table 1: The distribution of weight 3 words in the cosets of the $[q + 1, q - 3, 5]_q 3$ GDRS code $C_q$ associated with the twisted cubic $C$ of $[2]$, $q \equiv \xi \pmod{3}$. \{RC-p\} = \{RC-points\}, \{IC-p\} = \{IC-points\}, \{(q + 1)-p\} = \{(q + 1)-p\}

| no. | $\xi$ | Coset leader weight | Coset notation $A_3(V)$ of weight 3 vectors in a coset | The number of cosets of the given type | Orbits generating coset syndromes | $A_3(V)$ |
|-----|------|---------------------|---------------------------------|---------------------------------|---------------------------------|--------|
| 1   | any  | 0                   | $C_q$                           | 0                               | 1                               | $M_1 = \{C\text{-points}\}$ |
| 2   | any  | 1                   | $V^{(1)}$                       | 1                               | $M_1 = \{C\text{-points}\}$    | $r^{(1)}_{35} - (q - 1)$ |
| 3   | 1    | 2                   | $V^{(2)}_3$                     | 0                               | $M_5 = \{RC\text{-p}\}$ \setminus $M_3$ | $r^{(1)}_{35} - (q - 1)$ |
| 4   | 1    | 2                   | $V^{(2)}_6$                     | 1                               | $M_3 = \{RC\text{-p}\}$ \setminus $M_5$ | $r^{(1)}_{33} - (q - 1)$ |
| 5   | 1    | 3                   | $V^{(3)}_3$                     | 0                               | $M_2 = \{T\text{-points}\}$    | $r^{(1)}_{32} - (q - 1)$ |
| 6   | 1    | 3                   | $V^{(3)}_6$                     | 1                               | $M_4 = \{IC\text{-p}\}$ \setminus $M_3$ | $r^{(1)}_{34} - (q - 1)$ |
| 7   | -1   | 2                   | $V^{(2)}_3$                     | 0                               | $M_4 = \{RC\text{-p}\}$        | $r^{(1)}_{34} - (q - 1)$ |
| 8   | -1   | 3                   | $V^{(3)}_3$                     | 1                               | $M_2 = \{T\text{-points}\}$    | $r^{(1)}_{32} - (q - 1)$ |
| 9   | -1   | 3                   | $V^{(3)}_6$                     | 0                               | $M_5 = \{IC\text{-p}\}$ \setminus $M_3$ | $r^{(1)}_{33} - (q - 1)$ |
| 10  | -1   | 3                   | $V^{(3)}_6$                     | 1                               | $M_3 = \{IC\text{-p}\}$ \setminus $M_5$ | $r^{(1)}_{33} - (q - 1)$ |
| 11  | 0    | 2                   | $V^{(2)}_3$                     | 1                               | $M_4 = \{RC\text{-p}\}$        | $r^{(0)}_{34} - (q - 1)$ |
| 12  | 0    | 3                   | $V^{(3)}_6$                     | 0                               | $M_3 = \{TO\text{-points}\}$   | $r^{(0)}_{33}$ |
| 13  | 0    | 3                   | $V^{(3)}_6$                     | 1                               | $M_2 \cup M_5 = \{(q + 1)-p\} \cup \{IC\text{-p}\}$ | $r^{(0)}_{32} + r^{(0)}_{35}$ |

If $\xi = 1$, the syndromes are points of $M_5 \cup M_3 = \{RC\text{-points}\}$, see $[1]$. We take $r^{(1)}_{35} = \frac{1}{6}(q^2 + q - 2)$ and $r^{(1)}_{33} = \frac{1}{6}(q^2 + q + 4)$ from Theorem $[5]$(i), subtract $q - 1$, and obtain the values $T = \frac{1}{6}(q^2 - 5q + 4)$ and $\overline{T} = \frac{1}{6}(q^2 - 5q + 10)$ for rows 3 and 4 of Table $[1]$ respectively. The number of the corresponding cosets is $#M_5 \cdot (q - 1)$ and $#M_3 \cdot (q - 1)$, respectively. The values $#M_5$ and $#M_3$ can be taken from $[3]$. If $\xi \in \{-1, 0\}$, the syndromes are points of $M_4 = \{RC\text{-points}\}$, see $[1], [6]$. Therefore, the total number of $3\xi$-planes through an RC-point is the value $r^{(1)}_{34} = r^{(1)}_{33} = \frac{1}{6}(q^2 + q)$ of Theorem $[5]$ whence $\overline{T} = \frac{1}{6}(q^2 + q) - (q - 1) = \frac{1}{6}(q^2 - 5q + 6)$. The number of the cosets is $#M_4 \cdot (q - 1)$, see $[3]$. We obtain rows 7 and 11 of Table $[1]$. Let a coset leader weight be 3.

In this case, as every point off $C$ lies on exactly one chord of $C$ (see Theorem $[2]$(iv)), the syndrome of the coset belongs to $\text{PG}(3, q) \setminus \{\{C\text{-points}\} \cup \{RC\text{-points}\}\}$. In other words, the syndrome belongs to $\{T\text{-points}\} \cup \{IC\text{-points}\}$ for $\xi \neq 0$ (see Theorem $[2]$(ii)) or $\{TO\text{-points}\} \cup \{IC\text{-points}\}$.
\{(q + 1)r\text{-points}\} \cup \{IC\text{-points}\} for \(\xi = 0\) (see Theorem 2(iii)). Note that, for \(\xi = 0\), we have \(\{T\text{-points}\} = \{TO\text{-points}\} \cup \{(q + 1)r\text{-points}\}\). Thus, a weight 3 vector in the coset corresponds to an intersection of a tangent or an imaginary chord with a \(3_{e}\text{-plane}\). No tangent or imaginary chord lies in a \(3_{e}\text{-plane}\), otherwise we have an intersection of a tangent or an imaginary chord with a real chord lying in the \(3_{e}\text{-plane}\). Therefore, to calculate the number of the intersections, we directly use the needed values of \(r_{3j}^{(q)}\) from Theorem 5 without subtracting.

If \(\xi \in \{−1, 1\}\), we have \(M_{2} = \{T\text{-points}\}\), see (3). For rows 5 and 8 of Table 1 we take \(r_{32}^{(1)} = r_{32}^{(-1)}\) from Theorem 5 and use \(#M_{2} \cdot (q − 1)\).

If \(\xi = −1\), we have \(M_{3} \cup M_{5} = \{IC\text{-points}\}\), \(M_{3} = \{3r\text{-points}\}\), \(M_{5} = \{0r\text{-points}\}\), see (4), (3). For rows 9 and 10 of Table 1 we take, respectively, \(r_{35}^{(-1)} = \frac{1}{q}(q^2 − q − 2)\) and \(r_{33}^{(-1)} = \frac{1}{6}(q^2 − q + 4)\) from Theorem 5 and use \(#M_{3} \cdot (q − 1)\), \(#M_{5} \cdot (q − 1)\), see (3).

If \(\xi = 1\), we have \(M_{4} = \{IC\text{-points}\}\), see (4). For row 6 of Tab. 1 we take \(r_{34}^{(1)} = \frac{1}{6}(q^2 − q)\) from Theorem 5 and use \(#M_{4} \cdot (q − 1)\), see (3).

If \(\xi = 0\), we have \(M_{3} = \{TO\text{-points}\}\), \(M_{5} = \{(q + 1)r\text{-points}\}\), \(M_{5} = \{IC\text{-points}\}\), see (6). For rows 12 and 13 of Table 1 we take, respectively, \(r_{33}^{(0)}\) and \(r_{35}^{(0)}\) from Theorem 5(ii) and \(#M_{3} \cdot (q − 1)\), \(#M_{2} \cdot (q − 1)\), \(#M_{5} \cdot (q − 1)\), see (3).

\begin{theorem}
The weight distribution of any coset of \(C_{e}\) is unambiguously determined by Table 1.
\end{theorem}

\begin{proof}
The code \(C_{e}^\perp\) dual to \(C_{e}\) is an \([q + 1, 4, q − 2]_{q}R\) MDS code, \(R = q − 4\). By [35, Th. 10], \(C_{e}^\perp\) has 4 distinct nonzero weights, i.e. \(s(C_{e}^\perp) = 4\). By Theorem 10 we know the numbers of vectors of weights 1, 2, and 3 = \(s(C_{e}^\perp) − 1\) for all the cosets of \(C_{e}\), see Table 1 for weight 3 vectors. Recall also, that the cosets of weight \(\leq 2\) have a unique leader; the weight 1 cosets do not contain weight 2 vectors. Now the assertion follows from Theorem 7(ii).
\end{proof}

\begin{theorem}(classification of the cosets) Let \(q \equiv \xi \pmod{3}\). For the code \(C_{e}\), the following holds:
\end{theorem}

\begin{enumerate}[(i)]
\item All weight 1 cosets have the same weight distribution.
\item If \(\xi \neq 1\), all weight 2 cosets have the same weight distribution.
\hspace{1cm} For \(\xi = 1\), there are 2 distinct weight distributions of weight 2 cosets. The numbers of the corresponding cosets are \(\frac{1}{3}(q^3 − q)(q − 1)\) and \(\frac{1}{6}(q^3 − q)(q − 1)\).
\item For \(\xi = 1\), the are 2 distinct weight distributions of weight 3 cosets. The numbers of the corresponding cosets are \(q^2 + q)(q − 1)\) and \(\frac{1}{2}(q^3 − q)(q − 1)\).
\hspace{1cm} For \(\xi = −1\), there are 3 distinct weight distributions of weight 3 cosets. The numbers of the corresponding cosets are \((q^2 + q)(q − 1)\), \(\frac{1}{3}(q^3 − q)(q − 1)\), and \(\frac{1}{6}(q^3 − q)(q − 1)\).
\hspace{1cm} For \(\xi = 0\), there are 2 distinct weight distributions of weight 3 cosets. The numbers of the corresponding cosets are \((q^2 − 1)(q − 1)\) and \(\frac{1}{2}(q + 1)(q^2 − q + 2)(q − 1)\).
\end{enumerate}
4 On the weight distribution of the cosets of the code $C_\mathcal{C}$

We use the approach of [17, Th. 7.3.1, 7.5.2, Lem. 7.5.1], [18, Th. 10.7, 10.10].

For a code $C$ and its coset $V$, the approach includes the following stages:

- (A) Defining a special code $D$ on the base of $C$ and its cosets; searching of the needed properties of $D$.

- (B) Obtaining the weight distribution of the code $D^\perp$ dual to $D$ using the values of $A_i(V)$, $i = 1, 2, \ldots, s(C^\perp) - 1$.

- (C) Obtaining the weight distribution of the code $D$ using the Krawtchouck polynomials. Obtaining the weight distribution of the coset $V$ using the weight distributions of $D$ and $C$.

It is well known, see e.g. [2, Th. 11.3.6], [17, Th. 7.4.1], that for $w \geq d$, the weight distribution $A_w(C)$ of an $[n, k, d = n - k + 1]_q$ MDS code $C$ has the form

$$A_w(C) = \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w-d+1-j} - 1).$$  (7)

Remind that $C_\mathcal{C}$ is a $[q + 1, q - 3, 5]_q$ GDRS code. Its weight distribution $A_w(C_\mathcal{C})$ is given by (7) with $n = q + 1$, $d = 5$. The code $C_\mathcal{C}^\perp$ dual to $C_\mathcal{C}$ is a $[q + 1, 4, q - 2]_q R$ MDS code, $R = q - 4$. By [35, Th. 10], $C_\mathcal{C}^\perp$ has 4 distinct nonzero weights, i.e.

$$S(C_\mathcal{C}^\perp) = \{q - 2, q - 1, q, q + 1\}, \quad s(C_\mathcal{C}^\perp) = 4.$$

Remind also that for a coset $V(W)$ of $C_\mathcal{C}$, $V \neq C_\mathcal{C}$, the relation $1 \leq W \leq 3$ holds; so the coset can be written as $V(W) = v_W + C_\mathcal{C}$, where $v_W \in \mathbb{F}_{q+1} \setminus C_\mathcal{C}$ is a weight $W$ vector, $W \in \{1, 2, 3\}$. Therefore, in the rest of the paper, we assume $W \in \{1, 2, 3\}$.

For $\alpha \in \mathbb{F}_q^*$, the vector $\alpha v_W$ is also a weight $W$ vector. For $\alpha \in \mathbb{F}_q^*$, we denote by $V^{\alpha(W)}$ the weight $W$ coset of $C_\mathcal{C}$ such that $V^{\alpha(W)} = \alpha v_W + C_\mathcal{C}$. Thus, $\alpha v_W$ is a leader of $V^{\alpha(W)}$. By Theorem [7](ii), all the cosets $V^{\alpha(W)}$ have the same weight distribution.

We execute the stages (A) – (C).

- (A)

Let $D_W$ be the union of $C_\mathcal{C}$ and all the cosets $\alpha v_W + C_\mathcal{C}$ with $\alpha \in \mathbb{F}_q^*$, i.e.

$$D_w = \bigcup_{\alpha \in \mathbb{F}_q} (\alpha v_W + C_\mathcal{C}).$$  (9)

Lemma 13. For the code $D_W$ of (9) the following holds:

(i) $D_W$ is a linear $[q + 1, q - 2, W]_q$ code.
(ii) \( \mathcal{D}_W^⊥ \) is a \([q+1, 3, d_{\mathcal{D}_W^⊥}]_q \) code; \( \mathcal{D}_W^⊥ \subset \mathcal{C}_ϕ^⊥ \); \( d_{\mathcal{D}_W^⊥} \geq q - 2 \).

(iii) \( A_1(D_1) = q - 1 \), \( A_2(D_1) = A_3(D_1) = 0 \).

(iv) \( A_1(D_2) = 0 \), \( A_2(D_2) = q - 1 \), \( A_3(D_2) = (q - 1)A_3(\mathcal{V}_v^{(2)}) \).

(v) \( A_1(D_3) = A_2(D_3) = 0 \), \( A_3(D_3) = (q - 1)A_3(\mathcal{V}_v^{(3)}) \).

Proof. (i) By \([3]\), \( \mathcal{D}_W \) is a subspace of \( \mathbb{F}_q^{q+1} \) with size \#\( \mathcal{D}_W \) = \( q \cdot \#\mathcal{C}_ϕ = q \cdot q^{a-3} \). As the code distance of \( \mathcal{C}_ϕ \) is equal to 5, minimum weight codewords of \( \mathcal{D}_W \) are weight \( w \) leaders of the cosets \( \alpha \mathbf{v}_W + \mathcal{C}_ϕ \) with \( \alpha \neq 0 \).

(ii) The codewords of \( \mathcal{C}_ϕ^⊥ \) are orthogonal to \( \mathcal{C}_ϕ \) \([3]\) Sec. 2.3]. But some of these codewords can be non-orthogonal to the cosets \( \alpha \mathbf{v}_W + \mathcal{C}_ϕ \) with \( \alpha \neq 0 \). Therefore, \( \mathcal{D}_W^⊥ \subset \mathcal{C}_ϕ^⊥ \). Then we use \([3]\).

(iii) In \( \mathcal{D}_1 \), there are \( q - 1 \) weight 1 cosets of \( \mathcal{C}_ϕ \); it gives \( A_1(D_1) = q - 1 \). The equality \( A_2(D_1) = A_3(D_1) = 0 \) is obvious.

(iv) By part (i), the minimum distance of \( \mathcal{D}_2 \) is 2 whence \( A_1(D_2) = 0 \). The coset \( \mathcal{V}_v^{(2)} \) contains one weight 2 vector and \( A_3(\mathcal{V}_v^{(2)}) \) weight 3 vectors. By Theorem \([7]\)(i), all the cosets \( \mathcal{V}_v^{(2)} \), \( \alpha \in \mathbb{F}_q^* \), have the same weight distribution.

(v) By part (i), the minimum distance of \( \mathcal{D}_3 \) is 3 whence \( A_1(D_3) = A_2(D_3) = 0 \). The coset \( \mathcal{V}_v^{(3)} \) contains \( A_3(\mathcal{V}_v^{(3)}) \) weight 3 vectors. By Theorem \([7]\)(i), all the cosets \( \mathcal{V}_v^{(3)} \), \( \alpha \in \mathbb{F}_q^* \), have the same weight distribution.

• (B)

**Theorem 14.** The weight distribution \( A_j(\mathcal{D}_W^⊥) \) of the \([q+1, 3, d_{\mathcal{D}_W^⊥}]_q \) code \( \mathcal{D}_W^⊥ \) dual to \( \mathcal{D}_W \) is as follows:

(i) \( A_0(\mathcal{D}_W^⊥) = 1 \); \( A_i(\mathcal{D}_W^⊥) = 0 \), \( i = 1, 2, \ldots, q - 3 \).

(ii) The values of \( A_j(\mathcal{D}_W^⊥) \), \( j = q - 2, q - 1, q, q + 1 \), are given by the system of four linear equations

\[
\sum_{j=q-2}^{q+1} j^u A_j(\mathcal{D}_W^⊥) = G_{W,u}(q), \quad u = 0, 1, 2, 3, \tag{10}
\]

where

\[
G_{W,0}(q) = q^3 - 1, \tag{11}
\]

\[
G_{W,1}(q) = q^2(q^2 - 1 - A_1(\mathcal{D}_W)),
\]

\[
G_{W,2}(q) = q(q^2(q^2 - 1) - q(2q - 1)A_1(\mathcal{D}_W) + 2A_2(\mathcal{D}_W)),
\]

\[
G_{W,3}(q) = (q^2 - 1)q(q^3 + q - 1) - q(3q^3 - 3q^2 + 4q - 3)A_1(\mathcal{D}_W) + 6(q^2 - q + 1)A_2(\mathcal{D}_W) - 6A_3(\mathcal{D}_W).
\]

Proof. The part (i) follows from Lemma \([13]\) (ii). Also, the length of \( \mathcal{D}_W^⊥ \) is \( q + 1 \). Now we can obtain the system \([10]\), \([11]\) from \([17]\) Eq. (7.7), p. 259–260].

\[\square\]
Corollary 15. Let $q \equiv \xi \pmod{3}$. Let $G_{W,u}(q)$ be given by (11). The values of $G_{W,u}(q)$, $u = 1, 2, 3$, are as follows:

(i) $W = 1$. $G_{1,1}(q) = q^3(q - 1)$, 
$G_{1,2}(q) = q^2(q - 1)(q^2 - q + 1)$, 
$G_{1,3}(q) = q(q - 1)(q^4 - 2q^3 + 4q^2 - 4q + 2)$.

(ii) $W = 2$. $G_{2,1}(q) = q^2(q^2 - 1)$, 
$G_{2,2}(q) = q(q - 1)(q^3 + q^2 + 2)$, 
$G_{2,3}(q) = (q - 1) \left[ q^5 + q^4 + q^3 + 6q^2 - 7q + 6 - 6A_3(\mathcal{V}^{(2)}_{v_2}) \right]$ 
where $6A_3(\mathcal{V}^{(2)}_{v_2}) = \begin{cases} q^2 - 5q + 10 \text{ or } q^2 - 5q + 4 & \text{if } \xi = 1 \\ q^2 - 5q + 6 & \text{if } \xi = -1, 0 \end{cases}$.

(iii) $W = 3$. $G_{3,1}(q) = q^2(q^2 - 1)$, 
$G_{3,2}(q) = q^3(q^2 - 1)$, 
$G_{3,3}(q) = (q - 1) \left[ q^5 + q^4 + q^3 - q - 6A_3(\mathcal{V}^{(3)}_{v_3}) \right]$ 
where $6A_3(\mathcal{V}^{(3)}_{v_3}) = \begin{cases} q^2 - 3q + 2 \text{ or } q^2 - q & \text{if } \xi = 1 \\ q^2 - 3q + 2 \text{ or } q^2 - q + 4, \text{ or } q^2 - q - 2 & \text{if } \xi = -1 \end{cases}$.

Proof. We use (11), Lemma 13(iii)-(v), and Table 1. \square

Corollary 16. Let $A_3(\mathcal{V}^{(W)}_{w_3})$ be as in Table 1 and (13), (14); in particular, let $A_3(\mathcal{V}^{(1)}_{w_3}) = 0$. The weight distribution $A_j(D_{W}^\perp)$, of the $[q + 1, 3, d_{D_{W}^\perp}]_q$ code $D_{W}^\perp$ dual to $D_{W}$ is as follows:

$$A_j(D_{W}^\perp) = (q - 1) \left[ \Theta^{(W)}_j - (-1)^{q+1-j} \binom{3}{q+1-j} A_3(\mathcal{V}^{(W)}_{v_3}) \right]$$

where

$j = q - 2, q - 1, q, q + 1$,

$\Theta^{(1)}_{q-2} = \frac{1}{2}q(q - 1), \Theta^{(1)}_{q-1} = 2q, \Theta^{(1)}_q = \frac{1}{2}(q^2 - q + 2), \Theta^{(1)}_{q+1} = 0$;

$\Theta^{(2)}_{q-2} = q - 1, \Theta^{(2)}_{q-1} = \frac{1}{2}(q^2 - 3q + 6), \Theta^{(2)}_q = 2(q - 1), \Theta^{(2)}_{q+1} = \frac{1}{2}(q^2 - q + 2)$;

$\Theta^{(3)}_{q-2} = 0, \Theta^{(3)}_{q-1} = \frac{1}{2}q(q + 1), \Theta^{(3)}_q = q + 1, \Theta^{(3)}_{q+1} = \frac{1}{2}q(q - 1)$.

Proof. By the standard methods, we directly obtain the following solution of the linear system (10), (11):

$$A_{q-2}(D_{W}^\perp) = \frac{1}{6} \left[ q(q^2 - 1)G_{W,0}(q) - (3q^2 - 1)G_{W,1}(q) + 3qG_{W,2}(q) - G_{W,3}(q) \right],$$
\[ A_{q-1}(D_W^⊥) = \frac{1}{2} \left[ -q(q-2)(q+1)G_W,0(q) + (3q^2 - 2q - 2)G_W,1(q) - (3q - 1)G_W,2 + G_W,3(q) \right], \]
\[ A_q(D_W^⊥) = \frac{1}{2} \left[ (q^2 - 1)(q-2)G_W,0(q) - (3q^2 - 4q - 1)G_W,1(q) + (3q - 2)G_W,2(q) - G_W,3(q) \right], \]
\[ A_{q+1}(D_W^⊥) = \frac{1}{6} \left[ -q(q-1)(q-2)G_W,0(q) + (3q^2 - 6q + 2)G_W,1(q) - 3(q-1)G_W,2(q) + G_W,3(q) \right]. \]

Then we substitute the values \( G_{W,u}(q) \) from (11)–(14) and do simple transformations.

\[ \bullet \ (C) \]

Let
\[ K^{n,q}_w(x) = \sum_{i=0}^{w} (-1)^i(q-1)^{w-i} \binom{n-x}{i} \binom{w-i}{w-i} \] (16)

be the Krawtchouck polynomial [2,17].

**Theorem 17.** Let the weight distribution \( A_j(D_W^⊥) \) of the code dual to \( D_W \) be given by Corollary 16. The weight distribution \( A_w(D_W) \) of the \([q+1, q-2, W]_q \) code \( D_W \) is as follows:
\[ A_w(D_W) = \frac{1}{q^3} \left( K^{q+1,1}_w(0) + \sum_{j=q-2}^{q+1} A_j(D_W^⊥)K^{q+1,1}_w(j) \right), \quad w = W, W + 1, \ldots, q + 1. \] (17)

**Proof.** We use the well known relations to obtain the weight distribution of \( D_W \) from one of \( D_W^⊥ \), see e.g. [3, Sec. 4.4], [17, Eq. (K), p. 257].

**Theorem 18.** Let the weight distribution \( A_w(D_W) \) of the code \( D_W \) be given by (17). The weight distribution \( A_w(v_W + C_{\ell}) \) of the coset \( v_W + C_{\ell} \) is as follows:
\[ A_w(v_W + C_{\ell}) = \frac{A_w(D_W) - A_w(C_{\ell})}{q-1}, \quad w = W + 1, W + 2, \ldots, q + 1, \] (18)

where \( A_w(C_{\ell}) \) is given by [17] with \( n = q + 1, d = 5 \).

**Proof.** By [17, Lem. 7.5.1], we have \( A_w(v_W + C_{\ell}) = A_w(D_W \setminus C_{\ell})/(q-1) \). Also, by the construction [2], it is clear that \( A_w(D_W \setminus C_{\ell}) = A_w(D_W) - A_w(C_{\ell}) \).

**Example 19.** Examples of the weight distribution of the cosets of the code \( C_{\ell} \) are given in Table 19. Notation of the cosets are taken from Table 1. The distributions are obtained by the methods described above.
Table 2: Examples of the weight distribution of the cosets $\mathcal{V}(W)$ of the $[q + 1, q - 3, 5]_q$ GDRS code $\mathcal{C}_q$. $A_w = A_w(\mathcal{V}(W))$, $N$ is the number of the cosets of the given type.

| $q$ | coset | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ | $A_{11}$ | $A_{12}$ | $N$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----|
| 5   | $\mathcal{C}_q$ | 0 | 0 | 0 | 0 | 24 | 0 |   |   |   |   |   |   | 1 |
|     | $\mathcal{V}^{(1)}$ | 1 | 0 | 0 | 5 | 15 | 4 |   |   |   |   |   |   | 24 |
|     | $\mathcal{V}^{(2)}$ | 0 | 1 | 1 | 6 | 11 | 6 |   |   |   |   |   |   | 240 |
|     | $\mathcal{V}^{(3)}_a$ | 0 | 0 | 2 | 9 | 6 | 8 |   |   |   |   |   |   | 120 |
|     | $\mathcal{V}^{(3)}_b$ | 0 | 0 | 3 | 6 | 9 | 7 |   |   |   |   |   |   | 160 |
|     | $\mathcal{V}^{(3)}_c$ | 0 | 0 | 4 | 3 | 12 | 6 |   |   |   |   |   |   | 80 |
| 7   | $\mathcal{C}_q$ | 0 | 0 | 0 | 0 | 336 | 336 | 1056 | 672 |   |   |   |   |   | 1 |
|     | $\mathcal{V}^{(1)}$ | 1 | 0 | 0 | 35 | 217 | 490 | 966 | 692 |   |   |   |   |   | 48 |
|     | $\mathcal{V}^{(2)}_a$ | 0 | 1 | 3 | 40 | 182 | 541 | 935 | 699 |   |   |   |   |   | 672 |
|     | $\mathcal{V}^{(2)}_b$ | 0 | 1 | 4 | 35 | 192 | 531 | 940 | 698 |   |   |   |   |   | 336 |
|     | $\mathcal{V}^{(3)}_a$ | 0 | 0 | 5 | 45 | 162 | 566 | 921 | 702 |   |   |   |   |   | 336 |
|     | $\mathcal{V}^{(3)}_b$ | 0 | 0 | 7 | 35 | 182 | 546 | 931 | 700 |   |   |   |   |   | 1008 |
| 8   | $\mathcal{C}_q$ | 0 | 0 | 0 | 0 | 882 | 1764 | 7812 | 12411 | 9898 |   |   |   |   | 1 |
|     | $\mathcal{V}^{(1)}$ | 1 | 0 | 0 | 70 | 588 | 2268 | 7372 | 12606 | 9863 |   |   |   |   | 63 |
|     | $\mathcal{V}^{(2)}_a$ | 0 | 1 | 5 | 75 | 523 | 2399 | 7251 | 12661 | 9853 |   |   |   |   | 1764 |
|     | $\mathcal{V}^{(2)}_b$ | 0 | 0 | 7 | 84 | 483 | 2464 | 7197 | 12684 | 9849 |   |   |   |   | 504 |
|     | $\mathcal{V}^{(3)}_a$ | 0 | 0 | 9 | 72 | 513 | 2424 | 7227 | 12672 | 9851 |   |   |   |   | 1176 |
|     | $\mathcal{V}^{(3)}_b$ | 0 | 0 | 10 | 66 | 528 | 2404 | 7242 | 12666 | 9852 |   |   |   |   | 588 |
| 9   | $\mathcal{C}_q$ | 0 | 0 | 0 | 0 | 2016 | 6720 | 40320 | 113760 | 205040 | 163584 |   |   |   | 1 |
|     | $\mathcal{V}^{(1)}$ | 1 | 0 | 0 | 126 | 1386 | 8064 | 38760 | 114795 | 204669 | 163640 |   |   |   | 80 |
|     | $\mathcal{V}^{(2)}_a$ | 0 | 1 | 7 | 133 | 1267 | 8365 | 38389 | 115048 | 204577 | 163654 |   |   |   | 2880 |
|     | $\mathcal{V}^{(2)}_b$ | 0 | 0 | 9 | 147 | 1197 | 8505 | 38235 | 115146 | 204543 | 163659 |   |   |   | 640 |
|     | $\mathcal{V}^{(3)}_a$ | 0 | 0 | 12 | 126 | 1260 | 8400 | 38340 | 115083 | 204564 | 163656 |   |   |   | 2960 |
| 11  | $\mathcal{C}_q$ | 0 | 0 | 0 | 0 | 7920 | 55440 | 554400 | 3366000 | 15037000 | 45074040 | 81962880 | 68301200 | 1 |
|     | $\mathcal{V}^{(1)}$ | 1 | 0 | 0 | 330 | 5742 | 61908 | 543180 | 3378375 | 15028145 | 45078044 | 81961836 | 68301320 | 120 |
|     | $\mathcal{V}^{(2)}_a$ | 0 | 1 | 2 | 342 | 5424 | 63042 | 541080 | 3380763 | 15026408 | 45078837 | 81961628 | 68301344 | 6600 |
|     | $\mathcal{V}^{(2)}_b$ | 0 | 0 | 15 | 360 | 5292 | 63420 | 540450 | 3381435 | 15025940 | 45079044 | 81961575 | 68301350 | 1320 |
|     | $\mathcal{V}^{(3)}_a$ | 0 | 0 | 18 | 333 | 5400 | 63168 | 540828 | 3381057 | 15026192 | 45078936 | 81961602 | 68301347 | 4400 |
|     | $\mathcal{V}^{(3)}_b$ | 0 | 0 | 19 | 324 | 5436 | 63084 | 540954 | 3380931 | 15026276 | 45078900 | 81961611 | 68301346 | 2200 |
The property of differences and a symmetry in the weight distributions of the code $C\ell$ cosets

**Lemma 20.** Let $n \geq 0$. For the Krawtchouck polynomials $K_{w}^{n,q}(x)$ of (16) the following holds:

$$\sum_{x=n-m}^{n} (-1)^{n-x}\binom{m}{n-x}K_{w}^{n,q}(x) = (-1)^{w}q^{m}\binom{n-m}{w-m}, \quad 0 \leq m \leq w \leq n. \quad (19)$$

**Proof.** We write the left part of (19) directly by (10). In the product $(-1)^{n-x}\binom{m}{n-x}K_{w}^{n,q}(x)$ with $x = n - j - k$, the factor $(q - 1)^{j}$, $j = 0, 1, \ldots, m - k$, belongs to the term

$$(-1)^{j+k}\binom{m}{j+k}\left[(-1)^{w-j}(q - 1)^{j}\binom{n-j-k}{w-j}\binom{j+k}{j}\right], \quad k = 0, 1, \ldots, m - j.$$

By [36] Sec. 1.2, Eq. (IV), $(\binom{m}{j+k}) = (\binom{m}{j})$. Grouping together terms with the same factor $(q - 1)^{j}$, we transform the left part of (19) into the form

$$(-1)^{w}\sum_{j=0}^{m}(q - 1)^{j}\binom{m}{j}\sum_{k=0}^{m-j}(-1)^{k}\binom{n-j-k}{w-j}\binom{m-j}{k} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad **
Theorem 21. (property of differences) Let \( W = 2, 3 \). Let \( 3 \leq w \leq q + 1 \). Let \( \mathcal{V}_a^{(W)}, \mathcal{V}_b^{(W)} \) be two weight \( W \) cosets of \( \mathcal{C}_e \) with distinct weight distributions according to Table \( I \). We have

\[
A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)}) = -(1)^w \left[A_3(\mathcal{V}_a^{(W)}) - A_3(\mathcal{V}_b^{(W)})\right]\left(q - \frac{2}{w - 3}\right).
\]

(20)

Proof. Let \( \mathcal{D}_W^a \) (resp. \( \mathcal{D}_W^b \)) be the code \( \mathcal{D}_W \) of (9) such that the vector \( v_W \) is a leader of a coset \( \mathcal{V}_a^{(W)} \) (resp. \( \mathcal{V}_b^{(W)} \)) and let \( \mathcal{D}_W^a \) and \( \mathcal{D}_W^b \) be the corresponding dual codes. By Theorems \( I8 \) and \( I7 \) we have

\[
A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)}) = \frac{A_w(\mathcal{D}_W^a) - A_w(\mathcal{D}_W^b)}{q - 1}
\]

(21)

\[
= \frac{1}{q^3(q - 1)} \sum_{j=q-2}^{q+1} \left[A_j(\mathcal{D}_W^a) - A_j(\mathcal{D}_W^b)\right] K_{w}^{q+1,q}(j).
\]

In Corollary \( I6 \) the values of \( \Theta_{j}^{(W)} \) are the same for the codes \( \mathcal{D}_W^a \) and \( \mathcal{D}_W^b \). Therefore, from (21) and Corollary \( I6 \) with (15), it follows that

\[
A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)}) = \frac{(q - 1)\left[A_3(\mathcal{V}_a^{(W)}) - A_3(\mathcal{V}_b^{(W)})\right]}{q^3(q - 1)} \sum_{j=q-2}^{q+1} (-1)^{q+1-j} \left(\frac{3}{q + 1 - j}\right) K_{w}^{q+1,q}(j)
\]

whence we obtain the assertion using Lemma \( 20 \) with \( m = 3 \). \( \square \)

Theorem 22. (symmetry) Let \( w = 3, \ldots, \lfloor (q+3)/2 \rfloor \). Let \( q \equiv \xi \pmod{3} \). Let the notation of the cosets be as in Table \( I \). There is the following symmetry in the weight distributions of the code \( \mathcal{C}_e \).

(i) \( W = 2, \xi = 1; W = 3, \xi = 1; W = 3, \xi = 0 \) :

\[
A_{q+4-w}(\mathcal{V}_a^{(W)}) - (-1)^q A_w(\mathcal{V}_a^{(W)}) = A_{q+4-w}(\mathcal{V}_b^{(W)}) - (-1)^q A_w(\mathcal{V}_b^{(W)}).
\]

(ii) \( \xi = -1 \) :

\[
A_{q+4-w}(\mathcal{V}_a^{(3)}) - (-1)^q A_w(\mathcal{V}_a^{(3)}) = A_{q+4-w}(\mathcal{V}_b^{(3)}) - (-1)^q A_w(\mathcal{V}_b^{(3)})
\]

\[
= A_{q+4-w}(\mathcal{V}_c^{(3)}) - (-1)^q A_w(\mathcal{V}_c^{(3)}).
\]

Proof. It is obvious that \( \binom{q-2}{q+4-w-3} = \binom{q-2}{w-3} \) and

\[
(-1)^w = \begin{cases} 
(-1)^{q+4-w} & \text{if } q \text{ is even} \\
-(-1)^{q+4-w} & \text{if } q \text{ is odd}
\end{cases}
\]

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Therefore, by Theorem 21 and (20), we have
\[
A_{q+4-w}(V_a(W)) - A_{q+4-w}(V_b(W)) = \begin{cases} 
- \left[ A_w(V_a(W)) - A_w(V_b(W)) \right] & \text{if } q \text{ is even} \\
A_w(V_a(W)) - A_w(V_b(W)) & \text{if } q \text{ is odd}
\end{cases}
\]

We apply this relation in all cases of Table 1 (see also Theorem 12) when $C$ cosets of the same weight have distinct weight distributions. This implies the assertions.

6 The weight distributions of the $C$ code cosets on the base of the integral weight spectra

Let $A_w^\Sigma(V(W))$ (resp. $A_w^\Sigma(V^{\leq W})$) be the overall numbers of weight $w$ vectors in all cosets of weight $W$ (resp. weight $\leq W$).

**Definition 23.** For a fixed $W$, the set $\{A_w^\Sigma(V(W))|w = 0, 1, \ldots, n\}$ is an integral weight spectrum of the all code cosets of weight $W$.

Distinct representations of the integral weight spectra $A_w^\Sigma(V(W))$ and values of $A_w^\Sigma(V^{\leq W})$ are considered in the literature, see e.g. [1 Th. 14.2.2], [2 Th. 6.22], [22–24], [31, Lem. 2.14]. For instance, in [22 Eq. (11)–(13)], for an MDS code correcting $t$-fold errors, the value $D_u$ gives $A_w^\Sigma(V^{\leq t})$. In [24], the results of [22] are modified and Lemma 24 is obtained.

We denote
\[
\Phi_w^{(j)} = (-1)^{w-5} \left[ \begin{pmatrix} q+1 \\ w \end{pmatrix} \begin{pmatrix} w-1 \\ 3 \end{pmatrix} - \begin{pmatrix} q+1-j \\ w-j \end{pmatrix} \begin{pmatrix} w-1-j \\ 3-j \end{pmatrix} \right].
\]

**Lemma 24.** [24 Th. 3.4, 4.3, 5.1] (integral weight spectra) Let $C$ be an $[q+1, q-3, 5]_q3$ MDS code of minimum distance $d = 5$ and covering radius $R = 3$. For the integral weight spectra $A_w^\Sigma(V^{(w)})$ of the code $C$ cosets, the following holds:

\[
A_w^\Sigma(V^{(1)}) = (q^2 - 1) \left[ A_w(C) - \Phi_w^{(1)} \right], \quad 4 \leq w \leq q + 1.
\]
\[
A_w^\Sigma(V^{(2)}) = \left( q + 1 \atop 2 \right) (q-1)^2 \left[ A_w(C) - \Phi_w^{(2)} + (-1)^{w-5} \frac{1}{3} \left( \begin{pmatrix} q-2 \\ 2 \end{pmatrix} \begin{pmatrix} q-2 \\ w-3 \end{pmatrix} \right) \right], \quad 3 \leq w \leq q + 1.
\]
\[
A_w^\Sigma(V^{(3)}) = \left( q + 1 \atop w \right) (q-1)^w - \left[ A_w(C) + A_w^\Sigma(V^{(1)}) + A_w^\Sigma(V^{(2)}) \right], \quad 3 \leq w \leq q + 1.
\]

Thus, for the $[q+1, q-3, 5]_q3$ code $C$, the values of $A_w^\Sigma(V(W))$, $W = 1, 2, 3$, are known and easy to calculate.
Now we give direct formulas of the weight distributions of the cosets of \( \mathcal{C}_\xi \) on the base of the integral weight spectra \( \mathcal{A}_w^\Sigma(\mathcal{V}^{(W)}) \), \( W = 1, 2, 3 \).

Let \( \mathcal{V}^{(W)}_\Sigma(\mathcal{C}) \) be the total number of weight \( W \) cosets of a linear code \( \mathcal{C} \).

The following lemma is obvious.

**Lemma 25.** If all weight \( W \) cosets \( \mathcal{V}^{(W)} \) of a linear code \( \mathcal{C} \) have the same weight distribution, then the weight distribution \( \mathcal{A}_w(\mathcal{V}^{(W)}) \) of any coset \( \mathcal{V}^{(W)} \) has the form

\[
\mathcal{A}_w(\mathcal{V}^{(W)}) = \frac{\mathcal{A}_w^\Sigma(\mathcal{V}^{(W)})}{\mathcal{V}^{(W)}_\Sigma(\mathcal{C})}.
\]

**Theorem 26.** Let the weight distribution \( \mathcal{A}_w(\mathcal{C}_\xi) \) of the code \( \mathcal{C}_\xi \) be as in (7) with \( n = q + 1 \), \( d = 5 \). Let \( \Phi_w^{(j)} \) be as in (22). The following holds:

(i) **For all \( q \), the weight distribution \( \mathcal{A}_w(\mathcal{V}^{(1)}) \) of any weight 1 coset \( \mathcal{V}^{(1)} \) of the code \( \mathcal{C}_\xi \) is as follows**

\[
\mathcal{A}_w(\mathcal{V}^{(1)}) = \mathcal{A}_w(\mathcal{C}_\xi) - \Phi_w^{(1)}, \quad 4 \leq w \leq q + 1.
\]

(ii) **Let \( q \equiv 1 \pmod{3} \). The weight distribution \( \mathcal{A}_w(\mathcal{V}^{(2)}) \) of any weight 2 coset \( \mathcal{V}^{(2)} \) of the code \( \mathcal{C}_\xi \) is as follows**

\[
\mathcal{A}_w(\mathcal{V}^{(2)}) = \mathcal{A}_w(\mathcal{C}_\xi) - \Phi_w^{(2)} + (-1)^{w-5} \left( \frac{q - 2}{3} \right) \left( \frac{q - 2}{w - 3} \right), \quad 3 \leq w \leq q + 1.
\]

**Proof.** By Table 1 and Theorem 12 for all \( q \), all the code \( \mathcal{C}_\xi \) cosets of weight 1 have the same weight distributions. If \( q \equiv 1 \pmod{3} \), all the code \( \mathcal{C}_\xi \) cosets of weight 2 also have the same weight distributions. So, we may apply Lemma 25 where we put \( \mathcal{V}^{(1)}_\Sigma(\mathcal{C}_\xi) = (q + 1)(q - 1) \), \( \mathcal{V}^{(2)}_\Sigma(\mathcal{C}_\xi) = \left( \frac{q + 1}{2} \right)(q - 1)^2 \). Finally, we use Lemma 24. Note that for \( q \equiv 1 \pmod{3} \), the term \( \frac{1}{3}(q - 2) \) is an integer.

In Lemma 27 we denote the constant terms of equations by \( \Upsilon_j, \Lambda_j, \) and \( \Gamma_j \).

**Lemma 27.** Let \( q \equiv \xi \pmod{3} \). Let \( 4 \leq w \leq q + 1 \). Let \( W = 2, 3 \). Let the integral weight spectra \( \mathcal{A}_w^\Sigma(\mathcal{V}^{(W)}) \) be as in Lemma 24. Let \( \mathcal{V}_a^{(W)}, \mathcal{V}_b^{(W)}, \) and \( \mathcal{V}_c^{(W)} \) be the weight \( W \) cosets of \( \mathcal{C}_\xi \) with distinct weight distributions \( \mathcal{A}_w(\mathcal{V}_a^{(W)}), \mathcal{A}_w(\mathcal{V}_b^{(W)}), \) and \( \mathcal{A}_w(\mathcal{V}_c^{(W)}) \) according to Table 1. Then these weight distributions can be obtained from the following systems of linear equations.

(i) \( \xi = 1 \) : \[
\begin{align*}
2\mathcal{A}_w(\mathcal{V}_a^{(2)}) + \mathcal{A}_w(\mathcal{V}_b^{(2)}) &= 6\mathcal{A}_w^\Sigma(\mathcal{V}^{(2)})/[(q^3 - q)(q - 1)] := \Upsilon_1 \\
\mathcal{A}_w(\mathcal{V}_a^{(2)}) - \mathcal{A}_w(\mathcal{V}_b^{(2)}) &= (-1)^w \left( q^2 - 3 \right) := \Lambda_1
\end{align*}
\]
(ii) $\xi = 1$:
$$
\begin{align*}
2A_w(V^{(3)}_a) + (q-1)A_w(V^{(3)}_b) &= 2A^\Sigma_w(V^{(3)})/(q^3 - q) := \Upsilon_2 \\
A_w(V^{(3)}_a) - A_w(V^{(3)}_b) &= (-1)^w \frac{1}{3}(q - 1) \left( \frac{q-2}{w-3} \right) := \Lambda_2
\end{align*}
$$

(iii) $\xi = 0$:
$$
\begin{align*}
2(q-1)A_w(V^{(3)}_a) + (q^2 - q + 2)A_w(V^{(3)}_b) &= 2A^\Sigma_w(V^{(3)})/(q^2 - 1) := \Upsilon_3 \\
A_w(V^{(3)}_a) - A_w(V^{(3)}_b) &= (-1)^w \frac{1}{3} q \left( \frac{q-2}{w-3} \right) := \Lambda_3
\end{align*}
$$

(iv) $\xi = -1$:
$$
\begin{align*}
6A_w(V^{(3)}_a) + 2(q-1)A_w(V^{(3)}_b) + (q-1)A_w(V^{(3)}_c) &= 6A^\Sigma_w(V^{(3)})/(q^3 - q) := \Upsilon_4 \\
A_w(V^{(3)}_a) - A_w(V^{(3)}_b) &= (-1)^w \frac{1}{3} (q - 2) \left( \frac{q-2}{w-3} \right) := \Lambda_4 \\
A_w(V^{(3)}_b) - A_w(V^{(3)}_c) &= (-1)^w \left( \frac{q-2}{w-3} \right) := \Gamma_4
\end{align*}
$$

Proof. In each of the systems, the 1-st equation directly follows from Table 1 and the definition of $A^\Sigma_w(V^{(W)})$, see also Theorem 12. From Table 1 we take the number of the cosets of each type, e.g. $N_w^{(W)}(V^{(W)}_a)$ and $N_w^{(W)}(V^{(W)}_b)$, respectively, and obtain the equation
$$
N_w^{(W)}(V^{(W)}_a)A_w(V^{(W)}_a) + N_w^{(W)}(V^{(W)}_b)A_w(V^{(W)}_b) = A^\Sigma_w(V^{(W)}).
$$

For example, in the part (i), we have
$$
\frac{1}{3}(q^3 - q)(q-1)A_w(V^{(a)}_a) + \frac{1}{6}(q^3 - q)(q-1)A_w(V^{(W)}_b) = A^\Sigma_w(V^{(W)}).
$$

The 2-nd and 3-rd equations of the systems are based on Theorem 21 and relation (20) where the values $A_3(V^{(W)}_a)$, $A_3(V^{(W)}_b)$, and $A_3(V^{(W)}_c)$ are given in Table I. For example, in the part (i), we have
$$
A_w(V^{(2)}_a) - A_w(V^{(2)}_b) = -(-1)^w \left[ \frac{1}{6}(q^2 - 5q + 4) - \frac{1}{6}(q^2 - 5q + 10) \right] \left( \frac{q - 2}{w - 3} \right).
$$

Finally, in the equations, we do natural transformations. \qed

Theorem 28. Let $q \equiv \xi \pmod{3}$. Let $\Upsilon_j$, $\Lambda_j$, $\Gamma_j$, $j = 1, 2, 3, 4$, be the constant terms of the linear systems given in Lemma 27. Then the weight distributions $A_w(V^{(W)}_a)$, $A_w(V^{(W)}_b)$, $A_w(V^{(W)}_c)$ of the weight W cosets $V^{(W)}_a$, $V^{(W)}_b$, $V^{(W)}_c$ of $C^*_e$ are as follows:

(i) $\xi = 1$:
$$
A_w(V^{(2)}_a) = \frac{\Upsilon_1 + \Lambda_1}{3}, \quad A_w(V^{(2)}_b) = \frac{\Upsilon_1 - 2\Lambda_1}{3}.
$$

(ii) $\xi = 1$:
$$
A_w(V^{(3)}_a) = \frac{\Upsilon_2 + (q-1)\Lambda_2}{q+1}, \quad A_w(V^{(3)}_b) = \frac{\Upsilon_2 - 2\Lambda_2}{q+1}.
$$

(iii) $\xi = 0$:
$$
A_w(V^{(3)}_a) = \frac{\Upsilon_3 + (q^2 - q + 2)\Lambda_3}{q^2 + q}, \quad A_w(V^{(3)}_b) = \frac{\Upsilon_3 - 2(q-1)\Lambda_3}{q^2 + q}.
$$

(iv) $\xi = -1$:
$$
A_w(V^{(3)}_a) = \frac{\Upsilon_4 + 3(q-1)\Lambda_4 + (q-1)\Gamma_4}{3(q+1)}, \quad A_w(V^{(3)}_b) = \frac{\Upsilon_4 - 6\Lambda_4 + (q-1)\Gamma_4}{3(q+1)},
$$
$$
A_w(V^{(3)}_c) = \frac{\Upsilon_4 - 6\Lambda_4 - 2(q+2)\Gamma_4}{3(q+1)}.
$$
Proof. The assertions are the obvious solutions of the linear systems of Lemma 27.

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