Evolution of Higgs mode in a Fermion Superfluid with Tunable Interactions

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In this letter we present a coherent picture for the evolution of Higgs mode in both neutral and charged s-wave fermion superfluids, as the strength of attractive interaction between fermions increases from the BCS to the BEC regime. In the case of neutral fermionic superfluid, such as ultracold fermions, the Higgs mode is pushed to higher energy while at the same time, gradually loses its spectral weight as interaction strength increases toward the BEC regime, because the system is further tuned away from Lorentz invariance. On the other hand, when damping is taken into account, Higgs mode is significantly broadened due to coupling to phase mode in the whole BEC-BCS crossover. In the charged case of electron superconductor, the Anderson-Higgs mechanism gaps out the phase mode and suppresses the coupling between the Higgs and the phase modes, and consequently, stabilizes the Higgs mode.

The experimental search for Higgs boson in particle physics has made remarkable progresses 1,2. On the other hand, Higgs mode has also generated considerable interest in condensed matter and cold atom systems. Early in 1980s’, Raman scattering experiment has revealed an unexpected peak in a superconducting charge density wave compound NbSe2, 3, which was later attributed to the Higgs mode 4,5. Signal of Higgs mode has also been observed in antiferromagnet TiCuCl3 by the neutron scattering 6, and recently in superconducting NbN sample by terahertz pump probe spectroscopy in a nonadiabatic excitation regime 7,8. In cold atom system, Higgs mode has been observed near the superfluid to Mott insulator phase transition of bosonic atoms in optical lattices at integer filling 9,10.

Theoretically, the simplest field theory where Higgs mode emerges is a relativistic U(1) field theory with Lorentz invariance in the symmetry broken phase. This occurs, for example, in the weak coupling BCS superconductor 11,12 or in the Mott-superfluid transition of Bose-Hubbard model at integer filling 13,14. However, in most condensed matter systems, Lorentz invariance only emerges with fine tuning and the generic symmetry is usually Galilean 15. Thus, it is an interesting question to investigate how the Higgs mode evolves as the system is tuned away from the Lorentz invariance point. Moreover, in condensed matter systems, further complications often occur because the Higgs mode is usually coupled to other elementary excitations which leads to its damping 16-19. In this Letter, we investigate these issues in the context of the BEC-BCS crossover model. In the BCS limit, the system obeys approximate Lorentz symmetry due to particle-hole symmetry and is expected to host Higgs mode. In the BEC limit, it is a condensate of molecular bosons and obeys the Galilean invariance. It thus provides a unique system to describe the fate of Higgs mode as the system is tuned away from Lorentz invariant limit. In addition, due to tunable interactions

\[ S = \int dt d^3x \left[ -i \hbar \partial_t \phi + v \nabla \phi + \frac{m^*}{2} \phi - \frac{\nu^2}{2m^*} - r \right] \phi + \frac{b}{2} |\phi|^4, \]

where \( \phi \) is the Ginzburg-Landau order parameter. The various parameters \( u, v, r, b \) and \( m^* \) can be computed along BEC-BCS crossover in terms of the chemical potential \( \mu \), temperature \( T \) and \( \zeta = 1/(k_T a_S) \), where \( a_S \) is the s-wave scattering length. Within the Nozières-Schmitt-Rink 21 framework, this can be calculated as detailed in the supplementary material 22. The coefficients of the time derivative terms \( \sigma = u' + i u'' \) and \( v = v' + i v'' \) are complex in general. The real parts \( u' \) and \( v' \) describe

![FIG. 1: (Color online) \( v' \Delta_0 / u' \) and \( u'' / u' \) as functions of the scattering length \( \zeta = 1/k_F a_s \). In the inset we show \( v'' \Delta_0 / u'' \) as a function of \( \zeta \).](image)
the propagating behavior of the Cooper pair field, while the imaginary parts $u''$ and $v''$ describe its damping due to coupling to the fermionic quasiparticles. A plot of various parameters are given in Fig.\textbf{1}. We note the following features.

(i) Consider the real parts $u'$ and $v'$ in the BEC-BCS crossover. In the BCS limit, $u'/v'D_0 \to 0$ because of the approximate particle-hole symmetry in the weak-coupling BCS theory while $D_0 = \sqrt{\gamma/b}$ is the mean field value of order parameter. As a result, the system acquires an emergent Lorentz invariance, and one expects the emergence of Higgs mode, together with the standard Anderson-Bogoliubov mode for neutral fermion superfluid. In the BEC limit, however, $v'D_0/u' \sim D_0/|\mu| < 1$, and we can neglect the $v'$-term. This leads to a Galilean invariant neutral boson theory, for which only Bogoliubov mode exists.

(ii) The damping terms ($u''$) becomes important as one moves to the BCS side, because of the decreasing fermionic excitation gap and as a result, a stronger coupling of the pairing field to the quasi-particle excitations. This corresponds to finite lifetime of Cooper pairs at finite temperature. We will show that the damping $u''$-term itself will generate considerable effect for the appearance of the Higgs mode different from that in a pure Lorentz invariance theory. In the BEC limit, the imaginary parts vanishes within NSR. On the other hand, we find that whenever they are nonzero, $v''D_0/u'' < 1$ for the entire crossover regime and we shall thus neglect $v''$-term altogether in the following discussion.

Spectral Weight Transfer without Damping. To investigate the evolution of Higgs mode as the system is tuned gradually from its Lorentz-invariant BCS limit towards the Galilean invariant BEC limit, we shall first neglect the damping terms in Eq.\textbf{4} and study the transfer of spectral weight between the Higgs and Goldstone modes. In the symmetry broken state, we can write the order parameter $\phi = D_0 + \delta_\alpha + i\delta_p$, where $\delta_\alpha$ and $\delta_p$ describe amplitude and phase fluctuations, respectively. In terms of $\delta_\alpha$ and $\delta_p$ and with $u'' = v'' = 0$, we can write the action Eq.\textbf{4} in the Fourier space as

$$S = \int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3} \bar{\Phi}(\omega, -\mathbf{k})G^{-1}\Phi(\omega, \mathbf{k}),$$  

with $\bar{\Phi}(\omega, \mathbf{k}) = (\delta_\alpha(\omega, \mathbf{k}), \delta_p(\omega, \mathbf{k}))$ and the kernel $G$ is given by

$$G^{-1} = \begin{pmatrix} -v''\omega^2 + \xi_k & iu'\omega & iu''\omega \\ -iu'\omega & v''\omega^2 + \xi_k \\ -iu''\omega & v''\omega^2 + \xi_k \end{pmatrix},$$  

with $k = |\mathbf{k}|$ and $\xi_k = k^2/2m^*$. Two branches of spectrum can be identified, with mode frequencies given by

$$\omega^2 = \frac{v''}{u''} \frac{\xi_k + r}{\omega'} + \frac{u''^2}{2v''^2} \pm \sqrt{\frac{u''^2}{v''^2} + \frac{u'^2}{4v'^2} + \frac{u'^2}{v'^2}(\xi_k + r)}.$$

In the BCS limit, $\omega' D_0 \gg \omega'$ and solutions can be written as $\omega_-(k) = k/\sqrt{2m^*\omega'}$ and $\omega_+(k) = (\xi_k + 2r)/\sqrt{\omega'}$.

![FIG. 2: (Color online) Spectral function $A_{aa}(k, \omega)$ in the absence of damping term. $A_{aa}(k, \omega)$ as a function of $k$ (in unit of 1/$\xi$) and $\omega$ (in unit of $D_0$) for three different interaction strength $\zeta = -1/(k_F a)$, $\zeta = -7$ for (a), $\zeta = -3$ for (b) and $\zeta = -1$ for (c), corresponding to different gaps $D_0/\epsilon_F = 10^{-5}$, $D_0/\epsilon_F = 4 \times 10^{-3}$ and $D_0/\epsilon_F = 7 \times 10^{-3}$, respectively.](image-url)
sities associated with two modes \(\omega_-\) and \(\omega_+\). In Fig. 2 (a,b,c), we plot the spectral function \(A_{aa}(k, \omega)\) for three representative values of \(\zeta\) (corresponding to different \(\Delta_0/E_P\)). Two features can be noticed immediately. First, the Higgs gap increases beyond \(2\Delta_0\) of the BCS limit as interaction strength increases. Secondly, there is increasing spectral weight transfer from the gapped Higgs mode to the gapless mode. One can show explicitly that \(A_+/-A_- = 4v^r r^2 (u^d \sqrt{k^2/2m^* (k^2/2m^* + 2r)})^{-1}\), which indicates the gradual increasing of the mixing between phase and amplitude degrees of freedom.

**Including Damping Term.** Due to the presence of damping term, the time-dependent Ginzburg-Landau theory is not a pure Lorentz invariant \(U(1)\) theory. Thus, at any finite temperature, even in the BCS limit, the peak of Higgs excitation \(\omega_+(k)\) will not as sharp as discussed above. To calculate the equilibrium spectral weight in the presence of damping, we need to introduce the so-called Langevin force \(\eta(t, x)\), which satisfies the following conditions, \(\langle \eta(t', x') \eta(t, x) \rangle = \langle \eta^*(t', x') \eta(t, x) \rangle = 0\), and \(\langle \eta(t', x) \eta(t, x) \rangle = 2u'' k_B T \delta(t-t') \delta(x - x')\). Including the corresponding term in the action as \(S_L = \int dt dx(\phi^\dagger \eta + \phi \eta^*)\), we obtain the equations of motion for \(\delta_\alpha\) and \(\delta_p\), by setting \(\partial(S + S_L)/\partial \delta_\alpha = 0\) and \(\partial(S + S_L)/\partial \delta_p = 0\),

\[
(-v \omega^2 + \xi_k + 2r) \delta_\alpha - i u \omega \delta_p + \eta' = 0, \quad (6)
\]

\[
(-v \omega^2 + \xi_k) \delta_p + i u \omega \delta_\alpha + \eta'' = 0, \quad (7)
\]

where \(\eta'\) and \(\eta''\) are the real and imaginary parts of the Langevin force \(\eta\), respectively. The spectral functions for the amplitude fluctuation is given by, using fluctuation dissipation theorem,

\[
A_{aa} = \frac{u'' \omega}{2} \frac{|-v \omega^2 + \xi_k|^2 + |u \omega|^2}{(-v \omega^2 + \xi_k)(-v \omega^2 + \xi_k + 2r)^2}. \quad (8)
\]

By comparing Fig. 3 with Fig. 2 one can see three important features brought about by including the damping term. First, the spectral weight transfer is enhanced. For instance, for \(\zeta = -7\), there is almost no spectral weight transfer in the absence of damping (Fig. 2a) while in the presence of damping, for very small \(k < 1/\xi\), \(A_{aa}(k, \omega)\) exhibits a clear peak at the energy of Bogoliubov mode, with a weight proportional to \(u''\). Similar enhancement of spectral weight transfer can also be easily seen in Fig. 3(b) for \(\zeta = -3\). Secondly, also for \(k < 1/\xi\), in the BCS limit, the location of Higgs peak is substantially reduced from \(\sqrt{2r}/v'\) to \(\sqrt{2r}/v' - u''/v'^2\), as shown for \(\zeta = -7\) and \(-3\) in Fig. 3(a) and (b), respectively. Thirdly, as \(k\) starts to deviate from zero, the Higgs mode quickly loses its identity, due to strong hybridization with the Bogoliubov mode. For instance, even for \(k = 0.1/\xi\), as displayed by the purple dashed line in Fig. 3(a2-c2), no feature of sharp peak is observed in \(A_{aa}(k, \omega)\). And for \(\zeta = -1\), no sharp peak exists even for \(k = 0.01/\xi\).

**Effects of Coupling to External Gauge Fields.** Now we understand that, in the weakly interacting BCS side of a neutral superfluid, the appearance of Higgs mode suffers significant broadening due to finite \(u''\)-term at finite temperature, which couples the Higgs mode to the collective Bogoliubov excitations. Therefore, if we further consider the presence of coupling to external electromagnetic field for the case of charged fermions, the Bogoliubov mode is gapped out by the Anderson-Higgs mechanism. Thus, we expect that the Higgs mode is easier to observe in the charged case. To incorporate this effect of external electromagnetic field, we introduce the gauge potential \(\varphi(t, x)\) and extend the action as

\[
S_e = \int dt d^3x \left\{ \frac{\nabla^2 \varphi}{2m^*} - r |\varphi|^2 + \frac{b}{2} |\varphi|^4 - \frac{1}{8\pi} \varphi \nabla^2 \varphi \right\},
\]

where \(e\) is the charge of the electron. Following the same procedure as before, we find that the coupling between \(\delta_\alpha\) and \(\delta_p\) is modified and is now proportional to \(k^2\)

\[
\frac{2 i u \omega k^2}{k^2 + 32 \pi e^2 \Delta_0^2} \delta_\alpha(\omega, k) \delta_p(-\omega, -k).
\]

As a result, the original gapless phase mode is gapped to a finite frequency, \(\omega(k) = \sqrt{\xi_k/v' + 16 \pi e^2 \Delta_0^2/m^*}\), which is known as the Anderson-Higgs mechanism. Thus, the
large energy separation between this gapped phase mode and Higgs mode strongly suppresses their coupling. A further consequence of the modification is that at long wave length $k\to 0$, the coupling between phase and amplitude mode becomes small. The spectral function $A_{aa}(k,\omega)$ for charged case is plotted in Fig. 4. In sharp contrast to neutral case Fig. 3 the presence of damping term has almost no effect on Higgs mode, and there is always a peak located at $\omega = 2\Delta_0$. In this case, as attractive interaction increases and the system gradually loses its Lorentz invariance, the peak becomes more and more broad.

**Conclusion.** In summary, we have investigated the evolution of Higgs mode in the BEC-BCS crossover for both neutral and charged Fermi superfluid. Our main conclusions include: i) Towards the BEC side, as the system gradually loses the Lorentz invariance, the Higgs mode is pushed to very high energy and the spectral weight is transferred to Bogoliubov mode. ii) In the BCS side, damping terms arises in the Ginzburg-Landau theory, due to coupling between Cooper pair field and the fermionic quasi-particles, and strongly couples the Higgs mode to the gapless phase mode in the neutral superfluid, which enhances the spectral weight transfer and washes out features of Higgs mode at finite momentum. (iii) For the charged case, the phase mode is gapped out by coupling to external electromagnetic field, and the Higgs mode becomes much more stable.

Our results also deepen our understandings of Higgs mode in superconductor. The physical picture behind the observation of Higgs mode in a BCS superconductor is much more subtle and its observability is not merely guaranteed by Lorentz symmetry. While the damping terms broadens the Higgs peak, the Anderson-Higgs mechanism alleviate the coupling between Higgs and phase mode and as a result, Higgs mode remains at energy $2\Delta_0$. As for cold atom system, because of the cooling limit, so far we can not reach Fermi superfluid for $\zeta < -1$. However, our results show no Higgs feature in spectral function for $\zeta > -1$. On the other hand, with recent development of synthetic gauge field, there are many proposals to generate a synthetic dynamic gauge field in cold atom system. If such a dynamic gauge field can be experimentally realized and coupled to fermions, the Anderson-Higgs mechanism will be activated and a Higgs mode will be observed. This can be used as a way to test our theory.

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**FIG. 4:** (Color online) Spectral function $A_{aa}(k,\omega)$ for the charged case. $A_{aa}(k,\omega)$ as a function of $k$ (in unit of $1/\xi$) and $\omega$ (in unit of $\Delta_0$) for three different interaction strength $\zeta = -1/(k_F a_s)$, $\zeta = -7$ for (a), $\zeta = -3$ for (b) and $\zeta = -1$ for (c), corresponding to different $\Delta_0/E_F = 10^{-5}$, $\Delta_0/E_F = 4 \times 10^{-3}$ and $\Delta_0/E_F = 7 \times 10^{-2}$, respectively. (a2-c2): $A_{aa}(k,\omega)$ as a function of $\omega$ for $k = 0.1/\xi$ (purple dashed line) and $k = 0.01/\xi$ (blue solid line). $T/T_c = 0.9$ and $\delta$ is taken as $10^{-4}\Delta_0$. 

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A time-dependent Ginzburg-Landau theory can be constructed for the entire BEC-BCS crossover in the vicinity of $T_c$. The partition function takes the form $Z = \int D[\bar{\psi}_\sigma, \psi_\sigma] e^{-S[\bar{\psi}_\sigma, \psi_\sigma]}$, with

$$S[\bar{\psi}_\sigma, \psi_\sigma] = \int d\tau d^3x \left\{ \bar{\psi}_\sigma (\partial_\tau - \frac{\nabla^2}{2m} - \mu) \psi_\sigma - g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right\},$$

where $\psi_\sigma$ are Grassman fields and $g$ is the contact interaction between fermions of opposite spins. $\mu$ is the chemical potential which is determined by requiring the number density to be equal to $n$. To investigate the fluctuation effects in the Cooper channel, we use a Hubbard-Stratonovich transformation to decouple the interaction term in the Cooper channel and then integrating out the fermions. We obtain an effective theory for the bosonic field $\Delta(\tau, x)$, which represents the cooper pair field. Straightforward calculations yield the partition function in terms of field $\Delta$ as

$$Z = \int D(\bar{\Delta}, \Delta) \exp \left[ -\frac{1}{g} \int d\tau dx |\Delta|^2 + \ln \det \hat{G}^{-1} \right],$$

where

$$\hat{G}^{-1} = \left( \begin{array}{cc} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \frac{\Delta}{\Delta} \\ \frac{\Delta}{\Delta} & -\partial_\tau + \frac{\nabla^2}{2m} - \mu \end{array} \right)$$

is the Gor’kov Green function.

In the vicinity of the phase transition the gap parameter $\Delta$ is small and an expansion in terms of $\Delta$ becomes possible. Including both the spatial and time derivatives (after Wick rotation) and retaining the parameter $\Delta$ up to the forth order we obtain an effective action as

$$S[\Delta, \Delta] = \int dt d^3x \left\{ \Delta [-iu\partial_t + v\partial_t^2 - \frac{\nabla^2}{2m^*} - r] \Delta + \frac{b}{2} \Delta \Delta \Delta \Delta \right\},$$

where $u = u' + iu''$ and $v = v' + iv''$ are complex in general and all the parameters can be expressed in terms of
microscopic parameters as

\begin{align}
u' &= \frac{(2m)^{3/2}}{16\pi^2} \sum_{n=0}^{\infty} \frac{\sqrt{1 + (2n + 1)^2(\frac{\pi}{2m})^2 - \text{sgn}(\mu)}}{(2n + 1)^{3/2}} \frac{\pi \beta}{2 \sqrt{|\mu|} \theta(-\mu)}, \quad (5)
\end{align}

\begin{align}
u'' &= \frac{m^{3/2}}{8\sqrt{2\pi}} \beta \sqrt{|\mu|} \Theta(\mu), \quad (6)
\end{align}

\begin{align}
u' &= \frac{(2m)^{3/2}}{32\pi^2} \sum_{n=0}^{\infty} \frac{\sqrt{1 + (2n + 1)^2(\frac{\pi}{2m})^2 + \text{sgn}(\mu)}}{(2n + 1)^{3/2}} \frac{\pi \beta}{4 \sqrt{|\mu|} \theta(-\mu)}, \quad (7)
\end{align}

\begin{align}
u'' &= -\frac{m^{3/2}}{32\sqrt{2\pi}} \frac{\beta}{\sqrt{|\mu|}} \Omega(\mu), \quad (8)
\end{align}

\begin{align}
\frac{1}{2m^*} &= \frac{1}{2m} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1 - 2N(\xi_k)}{8\xi_k^2} + \frac{\partial N(\xi_k)}{\partial \xi_k} \frac{\partial N(\xi_k)}{\partial \xi_k} \frac{k^2}{2m} \right\}, \quad (9)
\end{align}

\begin{align}
r &= \frac{m}{4\pi a} + \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1 - 2N(\xi_k)}{2\xi_k} - \frac{1}{2\epsilon_k} \right\}, \quad (10)
\end{align}

\begin{align}
b &= \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1 - 2N(\xi_k)}{4\xi_k^2} + \frac{\beta N(\xi_k)N(\xi_k) - 1}{2\epsilon_k^2} \right\}. \quad (11)
\end{align}

In the above equations, \(N(\xi_k) = 1/\langle \exp(\beta \xi_k) + 1 \rangle\) is the Fermi distribution function and \(\xi_k = \epsilon_k - \mu\) with \(\epsilon_k = k^2/2m\). Function \(\Theta(2\mu)\) is the heaviside step function. Explicitly, the parameter \(b\) is the result of one-loop calculation with four fermion propagators

\begin{align}
b &= -\frac{1}{\beta^2} \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(-i\omega_n + k^2/2m - \mu)^2} \frac{1}{(i\omega_n + k^2/2m - \mu)^2}. \quad (12)
\end{align}

The other parameters \(u, v, \frac{1}{2m^*}\) and \(r\) are all derived from the inverse vertex function \(\Gamma^{-1}(\omega_n, k)\), which after the standard renormalization by replacing \(g\) with the two-body scattering length \(a_s\), is given by

\begin{align}
\Gamma^{-1}(\omega_n, k) = -\frac{m}{4\pi d} - \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{-i\omega_n + \epsilon_k + \epsilon_k - \frac{1}{2\epsilon_k}} \right\}. \quad (13)
\end{align}

To derive the time-dependent Ginzburg-Landau equation, we first analytically continue vertex function to real frequency \(i\omega_n \rightarrow \omega + i0^+\). This procedure generates a time-dependent term with parameter \(u\) and \(v\). The detailed derivation is as following.

The frequency dependent part of \(\Gamma^{-1}(\omega, k)\) is

\begin{align}
\Gamma^{-1}(\omega, 0) - \Gamma^{-1}(0, 0) &= -\frac{m}{4\pi a} - \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{-\omega - i\eta + 2\epsilon_k} \right\} \cdot (\omega) \cdot \left\{ \frac{1}{-\omega - i\eta + 2\epsilon_k - 2\mu} \right\} = \Gamma^{-1}(0, 0). \quad (14)
\end{align}

Then we expand it in series of small \(\omega\) as

\begin{align}
\Gamma^{-1}(\omega, 0) - \Gamma^{-1}(0, 0) &\simeq -\omega \cdot \int \frac{d^3k}{(2\pi)^3} \frac{1 - 2N(\epsilon_k - \mu)}{(2\epsilon_k - 2\mu - i\eta)^2} - \omega^2 \cdot \int \frac{d^3k}{(2\pi)^3} \frac{1 - 2N(\epsilon_k - \mu)}{(2\epsilon_k - 2\mu - i\eta)^3}. \quad (15)
\end{align}

We define the parameters as \(u \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1 - 2N(\epsilon_k - \mu)}{(2\epsilon_k - 2\mu - i\eta)^2}\) and \(v \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1 - 2N(\epsilon_k - \mu)}{(2\epsilon_k - 2\mu - i\eta)^3}\). They both can be calculated by contour integration.

\begin{align}
u \equiv \frac{\int d^3k}{(2\pi)^3} \frac{1 - 2N(\epsilon_k - \mu)}{\epsilon_k - 2\mu - i\eta^2} \left. \right|_{\epsilon_k - \mu} \frac{1}{2\epsilon_k - 2\mu - i\eta^2} \right. \quad (16)
\end{align}
\[ S = \int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3} \tilde{\Phi}(-\omega, -k) G^{-1} \Phi(\omega, k), \] (19)
At BCS limit we have $k = |\mathbf{k}|$ and $\xi_k = k^2/2m^*$. Then the amplitude-amplitude correlation function can be easily calculated as

$$G_{aa}(\omega, \mathbf{k}) = \frac{-v'\omega^2 + \xi_k}{-u^2\omega^2 + (-v'\omega^2 + \xi_k)(-v'\omega^2 + \xi + 2r)}.$$  

Straight forward calculation yields the spectral function as

$$A_{aa}(\omega, \mathbf{k}) = -\frac{1}{\pi} \text{Im} G_{aa}(\omega + i\delta, \mathbf{k}) = A_+(\mathbf{k})\delta(\omega - \omega_+(\mathbf{k})) + A_-(\mathbf{k})\delta(\omega - \omega_-(\mathbf{k})),$$

where the mode frequencies are given as

$$\omega_\pm^2 = \frac{\xi_k + r}{v'} + \frac{u'^2}{2v'^2} \pm \sqrt{\frac{1}{v'^2} + \frac{u'^4}{4v'^4} + \frac{u'^2}{4v'^2}(\xi_k + r)}$$

and the spectra weight density

$$A_+(\mathbf{k}) = \frac{v'\omega_+^2 - \xi_k}{2v'^2\omega_+(\omega_+^2 - \omega_+^2)},$$

$$A_-(\mathbf{k}) = \frac{-v'\omega_-^2 + \xi_k}{2v'^2\omega_-(\omega_+^2 - \omega_+^2)}.$$  

At BCS limit the ratio of the two spectral weight densities can be approximately calculated as

$$\frac{A_-}{A_+}(\mathbf{k}) = \frac{u'^2\sqrt{k^2/2m^*(k^2/2m^* + 2r)}}{4v'r'^2}.$$  

At BCS limit we have $u'/v' \to 0$, this ratio vanishes. This spectral weight transfer is shown in Fig. 2 in the main text.

### Spectral weight function in the case with damping term

The spectral weight function of the amplitude mode in the case with damping term $u''$ is

\[
A_{aa} = \frac{u''}{2} \cdot \frac{|v\omega^2 + \frac{k^2}{2m^*}|^2 + |u\omega|^2}{|-(u\omega)^2 + (v\omega^2 + \frac{k^2}{2m^*})(-v\omega^2 + \frac{k^2}{2m^*} + 2r)|^2} \\
= \frac{u''}{2} \cdot \frac{|v\omega^2 + \frac{k^2}{2m^*}|^2 + |u\omega|^2}{|v\omega^2 - \omega_+^2|(|\omega^2 - \omega_-^2| - 2iuv''\omega^2|^2)}.
\]
where the eigen mode frequencies are
\[ \omega_{\pm}^2 = \frac{\xi_k + r}{v'} + \frac{u'^2 - u''^2}{2v'^2} \pm \sqrt{\frac{v'^2}{v'^2} + \frac{(u'^2 - u''^2)^2}{4v'^4} + \frac{u'^2 - u''^2}{v'^3} (\xi_k + r)}. \] (27)

For small momentum they can be approximated as
\[ \omega_- = \sqrt{\frac{2r\xi_k}{2v'r + u'^2 - u''^2}}, \]
\[ \omega_+ = \sqrt{\frac{2v'r + u'^2 - u''^2}{v'^2} + \frac{2v'r + 2u'^2 - 2u''^2}{v'(2v'r + u'^2 - u''^2)} \xi_k}. \] (28)

Compared with the case without damping term we see that the gap of the Higgs mode is reduced from \( \sqrt{2r/v'} \) to \( \sqrt{2r/v' - u''/v'^2} \) at BCS limit. For small \( \xi_k \) the spectral weight on the Goldstone mode can be calculated as
\[ A_{aa}(\omega, \mathbf{k}) = \frac{u''}{8u'^2 \omega}. \] (29)

Different from the case without damping term, we see that in the case with damping term the spectral function has a weight proportional to \( u'' \) on the Goldstone mode.

The spectral weight function in the case with Coulomb interaction

A time-dependent Ginzburg-Landau theory with Coulomb interaction can be cast as [2]
\[ F = \int dt d^3x \left\{ -\frac{1}{8\pi} \phi \nabla^2 \phi + \Delta \left( -iu(\partial_t - 2e\phi) + v(\partial_t - 2e\phi)^2 - \frac{\Sigma^2}{2m^*} - r \right) \Delta + \frac{b}{2} \Delta \Delta \Delta \right\}, \] (30)

where \( e \) is the electric charge and \( \phi(t, \mathbf{x}) \) is the electric field. By taking a symmetry breaking \( \Delta \to \Delta_0 + \delta_a + i\delta_p \) we can have a free energy for the low energy excitations in the momentum space as
\[ F = \int \frac{d\omega}{2\pi} \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{ 2u\omega \delta_\omega(\omega, \mathbf{k}) \delta_p(-\omega, -\mathbf{k}) + \delta_a(-\omega, -\mathbf{k}) \right. \]
\[ \left. \quad (-\omega^2 + \frac{k^2}{2m^*}) \delta_\omega(\omega, \mathbf{k}) + 4ie\Delta_0 \delta_\omega(\omega, -\mathbf{k}) - 4ie\omega \Delta_0 \delta_p(-\omega, -\mathbf{k}) + 4ie^2 \Delta_0^2 \delta_p(-\omega, -\mathbf{k}) \phi(\omega, \mathbf{k}) \right\}. \] (31)

We integrate out the electric field \( \phi \) and obtain
\[ F = \int \frac{d\omega}{2\pi} \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{ 2u\omega \frac{k^2/8\pi}{k^2/8\pi + 4e^2\Delta_0^2} \delta_\omega(\omega, \mathbf{k}) \delta_p(-\omega, -\mathbf{k}) + \delta_a(-\omega, -\mathbf{k}) \right. \]
\[ \left. \quad (-\omega^2 + \frac{k^2}{2m^*} + 2r) \delta_\omega(\omega, \mathbf{k}) \right\}. \] (32)

Then the spectral functions can be calculated as
\[ \text{Im} \chi_{aa} = \frac{u''\omega}{2} \left( -\frac{\omega^2}{k^2/8\pi + 4e^2\Delta_0^2} + \frac{k^2}{2m^*} \right)^2 + \frac{u\omega}{k^2/8\pi + 4e^2\Delta_0^2} \left( -\omega^2 + \frac{k^2}{2m^*} + 2r \right)^2, \]
\[ \text{Im} \chi_{pp} = \frac{u''\omega}{2} \left( -\frac{\omega^2}{k^2/8\pi + 4e^2\Delta_0^2} + \frac{k^2}{2m^*} + 2r \right)^2 + \frac{u\omega}{k^2/8\pi + 4e^2\Delta_0^2} \left( -\omega^2 + \frac{k^2}{2m^*} + 2r \right)^2. \] (33)

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[2] Adriaan M. J. Schakel, Boulevard of Broken Symmetries: Effective Field Theories of Condensed Matter, World Scientific, Singapore, 2008.