Heavy hadrons on $N_f = 2$ and $2 + 1$ improved clover-Wilson lattices

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We present the masses of singly ($B$, $B_s$, $\Lambda_b$, $\Sigma_b$, etc.), doubly ($B_c$, $\eta_b$, $\Upsilon$, $\Xi_b$, $\Xi_b$, etc.), and triply ($\Omega_{bcc}$, $\Omega_{bdc}$, $\Omega_{bbc}$, etc.) heavy hadrons arising from (QCDSF-UKQCD) lattices with improved clover-Wilson light quarks. For the bottom quark, we use an $O(a, \alpha_s)$-improved version of lattice NRQCD. Part of the bottomonia spectrum is used to provide an alternative scale and to determine the physical quark mass and radiative corrections used in the heavy-quark action. Results for spin splittings, opposite parities, and, in some cases, excited states are presented. Higher lying states and baryons with two light quarks appear to be especially affected by the relatively small volumes of this (initially) initial study. This and other systematics are briefly discussed.

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I. INTRODUCTION

Ditto the above [1].

Here are some references concerning experimental and lattice-QCD results for heavy-hadron spectroscopy:

New bottomonia, including the ground-state $\eta_b$ [2–4] and the corresponding “radial” excitation $\eta_b(2S)$ [5, 6], the spin-singlet $P$-waves $h_b(1P)$ and $h_b(2P)$ [7], and a possible $\chi_b(3P)$ [8]:

- Orbitally excited $B$ and $B_s$ mesons [9, 12];
  - $\Sigma_b^{(*)}$ [13], $\Xi_b$ [14, 15], and $\Omega_b$ baryons [16, 18]. Excited $\Xi_b^{(*)}$ and $\Lambda_b$ baryons [20, 21].

- Lattice studies of bottomonia [22–24], including those with charm sea-quarks [25, 26], and predictions of $D$-wave [27] and higher states [28];
  - Lattice results for $B$ and $B_s$ mesons [29–36], including predictions for the $B_c$ system [37, 38];
  - Lattice results for $b$-baryons [31, 39, 40] and triply bottom baryons [41, 42].

Some more recent developments: Lattice studies of $bc$ baryons [43], bottomonia [44], and positive parity $B_s$ mesons [45]; and the experimentalists have been busy with the $\chi_b(3P)$ state and $\Xi_b$ baryons [46, 48].

For heavy-quark-model calculations, see, e.g., Ref. [49].

Once again [1].

II. LATTICE CALCULATION

In the present section we give the details of the simulations: the gauge configurations used, how the quark propagators are calculated, and how these are put together to form the correlators for the hadrons of interest.

A. Configurations

We use gauge configurations which include either $N_f = 2$ flavors of non-perturbatively improved clover-Wilson quarks [50] or $N_f = 2 + 1$ flavors of SLiNC quarks [51]. The relevant parameters of the ensembles used can be found in Table I. As can be seen in the last column, the spatial extent of the lattices is rather small. Whereas this may not strongly affect tightly bound, multiply heavy systems (e.g., $\Upsilon$, $B$, $B_c$, $\Omega_{bcc}$, etc.), higher excitations and hadrons with lighter valence quarks may “feel the pinch” [62] and this is therefore a source of systematic error which we must keep in mind.

B. Light and charm quark propagators

The propagators for the light ($u, d$), strange ($s$), and charm ($c$) quarks were produced using the Chroma software library [53]. The light-quark propagators on ensemble $a$ and the light- and strange-quark propagators on ensembles $b$ and $e$ were originally created for other projects (see [50] and [54], respectively) and we must work with the (rather severe) quark-source smearings chosen therein (see Table II of course, with great computational advantage of just having to read in the files). The light- and strange-quark propagators on ensemble $c$ and $d$ and the charm-quark propagators on ensembles $b$–$e$ were created with the aim of better resolving excited states (less smearing, see Table II). The charm-quark mass was

| $a$ | $b$ | $c$ | $d$ | $e$ |
|-----|-----|-----|-----|-----|
| $M_c$ | $10.49(17)$ | $10.55(17)$ | $10.49(17)$ | $10.55(17)$ |

TABLE I: Relevant lattice parameters [50, 51].
taken from a related study \cite{51,52} ($\kappa_c = 0.1109$). The lattice scale there, however, was set using a different observable (the flavor-singlet baryon-mass combination $X_N$ \cite{51}) than the one here ($M(1P) - M(1S)$ from $b\bar{b}$) and we therefore have a systematic shift in our $B_s$-system scale when using the bottomonia scale (see Sec. III B).

C. Non-relativistic quark propagators

For the bottom quark, we employ improved NRQCD \cite{59}, including terms up to $O(v^4)$, where $v$ is the heavy-quark velocity. We use the time-step symmetric form of the evolution equation:

$$
\phi(y, t + a) = \left(1 - \frac{a\delta H(t + a)}{2}\right) \left(1 - \frac{aH_0(t + a)}{2n}\right)^n \cdot U_4^+(t) \left(1 - \frac{aH_0(t)}{2n}\right)^n \left(1 - \frac{a\delta H(t)}{2}\right) \cdot \phi(x, t),
$$

(1)

where the binomial expression of the exponential of the lower-order (in $v$) terms is carried out to $n = 4$. $H_0$ handles the heavy-quark kinetic energy and the associated $O(a)$ time-step correction,

$$
H_0 = -\frac{\tilde{\Delta}}{2m_Q} - \frac{a}{4n} \frac{\tilde{\Delta}^2}{4m_Q^2},
$$

(2)

and $\delta H$ contains the $O(v^4)$ relativistic corrections,

$$
\delta H = -\frac{c_1}{8m_Q^2} \tilde{\Delta}^2 + \frac{igc_2}{8m_Q^2} (\nabla \cdot E - E \cdot \nabla) \tilde{\Delta} + \frac{gc_3}{8m_Q^2} \sigma \cdot (\nabla \times E - E \times \nabla) \tilde{\Delta} + \frac{gc_4}{2m_Q} \sigma \cdot B.
$$

(3)

D. Hadron correlators

As already mentioned above (Table II), we use a combination of source smearings for all propagators. For the heavy quark ($Q$), we use the smeared source, as well as another where a covariant Laplacian is also applied (giving a radial node; for $P$- and $D$-wave mesons, we use a local source as the second choice). For all quarks we use local sinks, while for the heavy quark we also consider the two smearings used at the source. This leads to a rather peculiar situation where the heavy-quarkonia and triply-heavy-baryon correlators form $2 \times 3$ matrices, with a $2 \times 2$ symmetric block, whereas all correlators involving light, strange, or charm quarks, together with heavy ones, form $2 \times 3$ off-diagonal blocks of a larger (mostly unknown) matrix. This is a not a major problem, however, as we can still fit such heavy-light correlators to the usual ansatz,

$$
C(t)_{ij} = \langle 0 | O_i(t) O_j^+(0) | 0 \rangle = \sum_{n=1}^{\infty} v_i^{(n)} v_j^{(n)*} e^{-t E(n)},
$$

(4)

except that we must fix one amplitude for each energy level considered (the fit then gives amplitude ratios, but the same energies). For some fits, we find it advantageous to consider a submatrix ($2 \times 2$ or $2 \times 1$) of the ones we have (this is likely due to limited statistics) and for most mass differences reported, we use appropriate (jackknifed) combinations of only the smeared-source, smeared-sink correlators (see below).

The interpolating operators that we use to combine the quarks together into the mesons of interest are shown in Tables III and IV. One needs to be careful when combining the $u, d, s$ quark propagators from Chroma with the nonrelativistic $Q$ propagators: a change in the spin-basis is needed \cite{64}.

The baryon operators can be found in Table V. In order to project out the desired spin and parity, the baryon

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**TABLE II: Quark source smearings.**

| lbl | quark | sm.type | params. |
|-----|-------|---------|---------|
| a   | $u, d$| Gauss 52 | $c_1 = 0.25, N = 400$; $f = 2.5, N = 25$ |
|     |       | APE 55  |         |
| b   | $u, d, s$ | Gauss 52 | $c_1 = 0.25, N = 150$; $f = 2, N = 20$ |
|     |       | APE 55  |         |
| c,d | $u, d, s$ | Gauss 52 | $c_1 = 0.25, N = 20$; $f = 2, N = 3$ |
|     |       | APE 55  |         |
| e   | $u, d, s$ | Gauss 52 | $c_1 = 0.3, N = 130$; $f = 2, N = 20$ |
|     |       | APE 55  |         |
| b-e | $c$    | Gauss 52 | $c_1 = 0.25, N = 12$; $f = 2, N = 3$ |
| a-e | $Q$   | Gauss 52 | $c_1 = 0.2, N = 16$ |
correlators should then be of the form
\[ B^{1/2\mp}_f (t) = \left\langle \frac{1}{2} (1 + \gamma_4) O \bar{O} \right\rangle \]  
(5)

or, for operators with an open Lorentz index,
\[ B^{1/2\mp}_{ij} (t) = \left\langle \frac{1}{2} (1 + \gamma_4) P^I_{ik} O_k \bar{O}_j \right\rangle , \]  
(6)

where the zero-momentum spin-projectors are \( P^{3/2}_{ik} = \delta_{ik} - \frac{1}{3} \gamma_i \gamma_k \) and \( P^{1/2}_{ik} = \frac{1}{3} \gamma_i \gamma_k \). In the end, we average over the nine remaining spatial indices \( (i, j) \). For the flavor projections, we follow lowest-order HQET and do not consider mixings between the different heavy-quark configurations (e.g., between \( \Lambda_Q \) and \( \Delta^3_Q \) in Table V we mostly consider the heavy-light diquark configurations in order to reach the negative-parity states). Although it is a poorer approximation, we do the same when considering different charm-quark configurations \( (q' = c) \). Depending on the desired state, however, it may be that one must take care with the appropriate flavor projections of the light quarks \( (q, q' = u, d, s) \) \[65\].

We also create non-zero-momentum correlators (smeared source / local sink only; \( \vec{p} = 2\pi \vec{n}/L \), where \( |\vec{n}| \leq 3 \)) for the \( \Upsilon, B^* \), and \( B^*_c \) in order to determine the kinetic masses of these mesons from their dispersion relations. This provides us with an absolute mass scale and a way to set (or interpolate to) the physical \( b \)-quark mass.

For many of the heavy baryons considered herein, we present an alternative analysis in which we try remove most of the remaining, leading uncertainties in the heavy-quark parameters. By considering appropriate combinations of (jackknifed) average correlators, we subtract \( E(\Upsilon)/2 \) for each \( b \) quark and \( E(B_c) - E(\Upsilon)/2 \) for each \( c \) quark. Inserting experimental values for \( M(\Upsilon) \) and \( M(B_c) \) into these mass differences, we arrive at more precise, absolute estimates of the heavy baryon masses, albeit via “less predictive” means.

### III. ANALYSIS

In the following subsections we present our analysis of the heavy hadron correlators, leading to our results for the associated masses \[66\].

#### A. Quarkonia

At the present stage in this project we have not yet achieved a high-precision analysis of the bottomonia system; more precise studies may be found in Refs. \[22\] - \[28\]. However, we need to start somewhere and, as such, we are interested in part of the \( b\bar{b} \) spectrum to set the lattice scale and the parameters used in the heavy-quark action \( (m_Q, c_i) \). We present a few other results (spin splittings and excitations) along the way. Further

![Plot](image-url)
quarkonia simulations (e.g., on higher-statistics, larger-volume $N_f = 2 + 1$ ensembles), hopefully leading to a more complete analysis, are currently underway\cite{11}.

For spin-averaged and spin-dependent splittings we use single-elimination jackknife to create appropriate combinations (e.g., ratios or ratios of products) of the smeared-source, smeared-sink correlators to extract the ground-state energy differences and to handle the associated error correlations. One such example is the spin-averaged $1P - 1S$ mass difference:

$$\Delta M_{PS} = \frac{5E(\chi_{b2}) + 3E(\chi_{b1}) + E(\chi_{b0})}{9} - \frac{3E(\bar{\chi}) + E(\eta_{b})}{4}. \quad (7)$$

It is this quantity which we use to set the scale for the lattices. Figure\cite{1} displays the results versus the pion mass. The leftmost point is the chirally extrapolated $24^3$, $N_f = 2 + 1$ difference (for $am_\gamma = 1.5 \approx am_b$, see below) and the closest black point is that for the $32^3$, $N_f = 2$ ensemble (a). Using the $\bar{b}b$ experimental value of 457 MeV\cite{67}, leads to $a^{-1} = 2726(206)$ MeV and 2837(55) MeV, respectively. Results from twice the heavy-quark mass ($am_\gamma = 3.0$) are also displayed, showing the relatively small dependence of this splitting on $m_Q$. Using ground-state energy levels from the finite-momentum $Q\bar{Q}$ vector correlators, we fit the dispersion relation to the following form:

$$E = E_0 + \frac{p^2}{2M_{kin}} - \frac{p^4}{8M_{kin}^3}. \quad (8)$$

Resulting values for $M_{kin}$ (in units of the $1P - 1S$ splitting, $\Delta M_{PS}$) as a function of the pion mass are presented in Fig.\cite{2}. The larger error bars on the chirally extrapolated $24^3$ result and on the $32^3$ result are those which also include the error in the lattice spacing determination (about 7.5% and 2%, respectively; the same applies to all following figures). The experimental value of $M_T/\Delta M_{PS}$ is also plotted and one can see agreement when $am_\gamma = 1.5$ on the $24^3$, $N_f = 2 + 1$ lattices. For the $32^3$, $N_f = 2$ lattice, a slightly lower value for the heavy-quark mass ($am_\gamma \approx 1.3$) may have been appropriate. With the lack of a chiral extrapolation and the quenching of the strange quark, however, it is difficult to determine which systematics would become absorbed into such an adjustment. We use $am_\gamma = 1.5$ as our “working value” of the physical bottom-quark mass on all ensembles.

In order to find the radiative corrections for the spin-dependent terms ($c_3$ and $c_4$) in the heavy-quark action, we look at the “spin-orbit” and “tensor” energies of the $1P \bar{b}b$ system:

$$E_{SO} = [-2E(\chi_{b0}) - 3E(\chi_{b1}) + 5E(\chi_{b2})] / 9, \quad (9)$$

$$E_T = [-2E(\chi_{b0}) + 3E(\chi_{b1}) - E(\chi_{b2})] / 9. \quad (10)$$

These are roughly proportional to $c_3$ and $c_4$, respectively (the corresponding experimental values are 18.20 MeV and 5.25 MeV). In Fig.\cite{3} we plot our values for these energies for the cases where $c_3 = c_4 = 1$ and $c_4 = 1, c_4 = 1.2$. Within the errors the tree-level $c_3 = 1$ appears to work fine for $E_{SO}$, whereas the correction $c_4 = 1.2$ leads to better agreement for $E_T$. For the $N_f = 2$ results, a slightly higher $c_4$ appears to be needed (at least without a chiral extrapolation); together with a lower value for heavy-quark mass ($am_\gamma = 1.3$ for better $M_{kin}$ and $E_{SO}$ as well), the value $c_4 = 1.23$ would bring the tensor energy in better agreement with experiment. Now that the lattice scale and parameters of the heavy-quark action have been handled, we can turn our attention to other results.

Another quantity which follows from a jackknife ratio analysis is the spin splitting in the ground-state $bb$ $S$-waves: $M(\bar{\chi}) - M(\eta_b)$. In Fig.\cite{4} one can see that the $24^3$, $N_f = 2 + 1$ results extrapolate to a value slightly below the experimental value. However, when we include the error in the lattice spacing, we see that this discrepancy is only a little more than 1σ. The $N_f = 2$ result is also low, even after naively extrapolating to $am_\gamma = 1.3, c_4 = 1.23$ (using the $am_\gamma = 3.0$ and $c_4 = 1$ results).

We also use jackknife ratios to look more closely at the $1P$ spin splittings, including the $h_b$, and the energy differences to (and among) the $1D$ states. These results for the $24^3$, $N_f = 2 + 1$ chiral extrapolation, the $N_f = 2 + 1$ ensemble closest to the chiral limit (b), and the $32^3$, $N_f = 2$ ensemble (a) can be found in Table\cite{11}. As eluded to earlier, we also fit the heavy-quarkonia correlator matrices to the form of Eq. (4). Depending upon the quantum numbers and time interval being considered, we fit anywhere from one to three energy levels.

\begin{center}
\includegraphics[width=\textwidth]{fig2.pdf}
\end{center}

**FIG. 2:** Ground-state vector $Q\bar{Q}$ kinetic mass versus pion mass.
possible (continuum-spin identification in Table III is the minimum be careful in interpreting the results of such states. The $N_f = 2$ operations in different lattice representations and look at the amplitudes as a way to confirm the associated spin

This provides us with estimates of the masses of first-excited states. The $N_f = 2$ and the chirally extrapolated $N_f = 2 + 1$ results are compiled in Table VI. One must be careful in interpreting the results of such states. The continuum-spin identification in Table III is the minimum possible ($J_{min}$) for the associated irreducible representation. Ideally, one should create the states of interest with operators in different lattice representations and look at the amplitudes as a way to confirm the associated spin (see, e.g., [28, 68]). For the $A_1$ operators, after $0$, the next lowest possible continuum spin is $4$. Such states should be much higher in mass and we believe that we can say with confidence that the first-excited states we see for such operators are still spin $0$. For the other representations, the separation in possible $J$ values is not so large: the next lowest values are $3$ for $T_1$ and $T_2$ and $4$ for $E$. Luckily, for the case of the $T_2$ operator we have the $E$ irreducible representation as well and can verify that the first excitation is consistent for both cases (in the end we average them). Otherwise, since the possible continuum-spin separation for the other operators is at least $2$ and we are only dealing with first-excited states, we assume that $J = J_{min}$ for these states as well.

B. $B_s, B_c$ mesons

Just as in the bottomonia case, we create jackknifed ratios of smeared-source, smeared-sink correlators to look at ground-state mass splittings of $B$ mesons. Table VII shows the results for the $N_f = 2 + 1$ chiral limit, the $N_f = 2 + 1$ ensemble closest to the chiral limit (b), and the $N_f = 2$ ensemble (a).

The spin splittings for the $S$- and $P$-wave $B$ mesons are shown in Figs. 5 and 6 as a function of the pion mass.

On all ensembles, the $B^+ - B$ difference agrees with experiment. Replacing the light quark with a strange or charm one leads to our results for $B_s$ and $B_c$ mesons. $N_f = 2 + 1$ ground-state splittings, as well as differences to some first-excited states are shown in Table VIII. One may note the fact that the $B^+ - B$ splitting appears to be slightly larger than that for $B^*_s - B_s$. This may be a sign of the small volumes’ effect on the light quarks.

| Splitting       | $N_f = 2 + 1$ (χ) | $N_f = 2 + 1$ (b) | $N_f = 2$ (a) |
|-----------------|-------------------|-------------------|--------------|
| $\Upsilon - \eta_b$ | 69.7(2.7)(4.5) 58.7(1.1)(4.5) | 50.76(90)(98) | |
| $1P - \chi_{00}$ | 46.8(7.5)(3.5) 43.9(3.1)(1.3) | 32.9(1.5)(0.6) | |
| $1P - \chi_{10}$ | 10.4(4.4)(0.8) 9.4(1.8)(0.3) | 9.04(86)(18) | |
| $\chi_{20} - 1P$ | 15.4(4.0)(1.2) 14.4(1.6)(0.4) | 12.01(78)(23) | |
| $1P - h_b$      | 1.3(1.8)(0.1)   2.13(64)(6) | 1.09(50)(2) | |
| $\eta_{b2} - 1S$ | 670(150)(50) 751(58)(23) | 788(35)(15) | |
| $T_{k2} - 1S$   | 720(190)(55) 718(77)(22) | 824(47)(16) | |
| $T_{k2} - \eta_{b2}$ | 130(95)(10) 60(40)(5) | 21(18)(1) | |
| $\Upsilon - \eta_b$ | 676(95)(51) 596(31)(18) | 485(36)(9) | |
| $T' - \Upsilon$ | 671(97)(51) 584(33)(18) | 474(38)(9) | |
| $\chi_{00} - \chi_{10}$ | 560(190)(40) 625(69)(19) | 668(93)(13) | |
| $\chi_{20} - \chi_{10}$ | 690(200)(50) 684(76)(21) | 729(97)(14) | |
| $\chi_{20} - \eta_{b2}$ | 510(220)(40) 583(86)(18) | 647(78)(13) | |
| $h_b - h_b$    | 620(200)(50) 630(79)(19) | 654(79)(13) | |
| $\eta_{b2} - \eta_{b2}$ | 590(350)(45) 810(170)(30) | 903(66)(18) | |
| $T'_{k2} - \eta_{b2}$ | 1550(400)(120) 1170(140)(40) | 900(190)(20) | |
Figure 5 displays the $B_s^0 - B$ mass difference as a function of pion mass. Indications are that the $B_s^0$ is below the $B^*K$ threshold. The same is true for the $B^*_s$ and the $B^{*+}K$ threshold (see Table VIII).

Due to the rather large amount of smearing used for the light- and strange-quark sources on ensembles $b$ and $e$ [54] (see also Table II), we are not able to resolve radially excited $B^{(*)}$ or $B^{(*)}_s$ mesons on these ensembles (and therefore, not in the chiral limit either). For the present study, light- and strange-quark sources with much less smearing were created on ensembles $c$ and $d$ to better
study such states. The results for the ensemble closer to the chiral limit (c) are shown in Table IX. The results for the corresponding pseudoscalar $B_x'$ splitting appear in Fig. 8, along with the $B_x'$ and $\eta_b'$ differences. Clearly, many more statistics are needed here, especially to reach a more reliable chiral limit.

C. Singly heavy baryons

The naming conventions for heavy baryons may not be strictly obeyed here, especially where negative-parity states are concerned. Therefore, for the purpose of notational clarity, we briefly point out the quantum numbers and the corresponding names of the baryon states discussed in this subsection:

$\cal P$-wave

$J^P(s_{dq}) = \frac{3}{2}^{-}(0) : \Lambda_b$, $\Xi_b$

$J^P(s_{dq}) = \frac{1}{2}^{-}(1) : \Sigma_b^*$, $\Xi_b^*$, $\Omega_b^*$

$J^P(s_{dq}) = \frac{3}{2}^{-}(1) : \Sigma_b^*$, $\Xi_b^*$, $\Omega_b^*$

where $s_{dq}$ is the spin of the diquark appearing in the associated baryon interpolator (not that of the state).

Table X lists the results for our mass splittings among the singly heavy baryons.

Figure 9 shows the $\Lambda_b - B$ mass difference as a function of $M_d^2$. The results on all lattices are high when compared to experiment, a sign that the small volumes (see Table I) may be drastically affecting our results for baryons with two light quarks. The $\Lambda_b - B$ splitting appears again in Fig. 10 along with the analogous $\Xi_b - B_s$ splitting, but here again our lattice results are too high, 536(78) MeV ($\chi$-extrap. $N_f = 2 + 1$), when compared to experiment (427 MeV). The $\Omega_b - B_s$ difference shows much better agreement.

Mass differences from alternative spin-flavor combinations appear in Fig. 11. Here, the (spin averaged) $\Sigma_b^{(s)} - \Lambda_b$ and $\Sigma_b^* - \Sigma_b$ splittings also show possible signs of finite-volume-induced enhancements (the dotted results are from chiral extrapolations without the heaviest pion mass, e). Other spin and multiplet splittings can be found in Table X (sometimes also with the same limited chiral extrapolation).

Figure 12 displays the mass differences between the

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**TABLE IX: Results for $B_{(s,c)}$-meson first-excited-ground state mass splittings (in MeV) using $am_d = 1.5$ and $c_s = 1.2$ on ensemble c.** The first error comes from the fit, the second from the scale setting ($\Delta M_{PS}$).

| splitting | $N_f = 2 + 1$ (c) |
|-----------|------------------|
| $B_s^* - B_s$ | 670(120)(30) |
| $B_s' - B_s^*$ | 640(90)(25) |
| $B_0^* - B_0^*$ | 790(160)(30) |
| $B_1^* - B_1^*$ | 760(110)(30) |
| $B_c^* - B_s$ | 697(100)(27) |
| $B_s' - B_s^*$ | 659(80)(25) |
| $B_{0'} - B_{0'}^*$ | 730(110)(30) |
| $B_{1'} - B_{1'}^*$ | 730(95)(30) |
| $B_c^* - B_c$ | 661(73)(25) |
| $B_s' - B_s^*$ | 662(73)(25) |
| $B_{0'} - B_{0'}^*$ | 630(110)(25) |
| $B_{1'} - B_{1'}^*$ | 675(95)(26) |
FIG. 10: Ξ bb − Υ, Ξ bc − B c, Ξ b − B s, and Λ b − B mass differences versus \( M_2^\pi \).

FIG. 11: Σ (∗) b − Λ b and Σ b − Σ b mass differences versus \( M_2^\pi \).

FIG. 12: Mass differences – involving the Λ b and Ξ bb negative-parity states – versus \( M_2^\pi \).

D. Doubly heavy baryons

As before, we point out the names of the states discussed in this subsection:

\[ J^P(s_dq) = \frac{1}{2}^+ (0) : \Xi_{bc}, \Omega_{bc} \]
\[ J^P(s_dq) = \frac{1}{2}^+ (1) : \Xi_{bc}', \Omega_{bc}', \Xi_{bb}, \Omega_{bb} \]

\begin{table}[h]
\begin{tabular}{|c|c|c|c|}
\hline
splitting & \( N_f = 2 + 1 \) (χ) & \( N_f = 2 + 1 \) (b) & \( N_f = 2 \) (a) \\
\hline
\( \Lambda_b - B \) & 450(99)(34) & 474(26)(14) & 444(12)(9) \\
\hline
\( \Xi_b - B_b \) & 536(78)(40) & 528(20)(16) & – \\
\hline
\( \Omega_b - B_b \) & 746(43)(56) & 695(16)(21) & – \\
\hline
\( \Xi_b - \Xi_b \) & 231(26)(17) & 87.2(8.2)(2.6) & – \\
\hline
\( \Xi_b^{(')} - \Lambda_b \) & 322(62)(24) & 274(19)(8) & 229(14)(4) \\
\hline
\( \Xi_b^{(')} - \Xi_b \) & 238(25)(18) & 190(10)(6) & – \\
\hline
\( \Omega_b^{(')} - \Xi_b \) & 154(40)(12) & 183(12)(6) & 274(15)(5) \\
\hline
\hline
\hline
\end{tabular}
\end{table}

\( \Lambda_b \) and its parity partner, \( \Lambda_b^{(')} \). Again, there is a clear overestimate here; the experimental value being around 300 MeV [20]. Mass differences involving this and other negative-parity states can be found in Table X [11].
TABLE XI: Results for \(bcq\) (where \(q = u, d, s\)) baryon mass splittings (in MeV) using \(amQ = 1.5\) and \(c4 = 1.2\) for the \(N_f = 2 + 1\) chiral limit (\(\chi\)) and the \(N_f = 2 + 1\) ensemble \(b\). The first error comes from the fit, the second from the scale setting (\(\Delta MPS\)).

| splitting | \(N_f = 2 + 1\) (\(\chi\)) | \(N_f = 2 + 1\) (\(b\)) | \(N_f = 2\) (\(a\)) |
|-----------|-----------------------------|-----------------------------|-----------------------------|
| \(\Xi_{bc}\) - \(B_c\) | 611(103)(46) | 681(41)(21) | |
| \(\Omega_{bc}\) - \(B_c\) | 152(34)(11) | 51(13)(2) | |
| \(\Xi'_{bc}\) - \(\Omega_{bc}\) | 50(19)(4) | 47.2(7.2)(1.4) | |
| \(\Omega'_{bc}\) - \(\Omega_{bc}\) | 38(15)(3) | 43.5(4.9)(1.3) | |
| \(\Xi''_{bc}\) - \(\Xi'_{bc}\) | 26(13)(2) | 26.7(4.6)(0.8) | |
| \(\Omega''_{bc}\) - \(\Omega'_{bc}\) | 21(11)(2) | 25.0(3.4)(0.8) | |
| \(\Xi''_{bc}\) - \(\Xi'_{bc}\) | 290(66)(22) | 314(21)(10) | |
| \(\Omega''_{bc}\) - \(\Omega'_{bc}\) | 342(53)(26) | 334(14)(10) | |
| \(\Xi''_{bc}\) - \(\Xi'_{bc}\) | 57(17)(4) | 48.1(5.4)(1.6) | |
| \(\Omega''_{bc}\) - \(\Omega'_{bc}\) | 52(14)(4) | 45.6(4.0)(1.4) | |

E. Triply heavy baryons

As before, we point out the names of the states discussed in this subsection:

\[ J^P(s_{dq}) = \frac{3}{2}^+ (1) : \Omega_{bc} , \Omega_{bb} \]

\[ J^P(s_{dq}) = \frac{3}{2}^- (0) : \Xi''_{bc} , \Omega''_{bc} \]

\[ J^P(s_{dq}) = \frac{1}{2}^- (1) : \Omega'_{bc} , \Omega'_{bb} \]

\[ J^P(s_{dq}) = \frac{1}{2}^+ (1) : \Xi'_{bc} , \Omega'_{bc} , \Omega_{bb} \]

and any “radial” excitations are denoted by a (2) after the name.

Mass differences involving the \(bc\)-baryons can be found in Table XII. Those for doubly bottom baryons appear in Table XIII.

Also present in the previously mentioned Fig. 10 are those for the (yet to be observed) \(\Xi_{bc}\) was found in Table XIII. Triply bottom results appear in Table XIV.

With the physical value for \(M(B_c)\) and \(M(\Upsilon)\) as input, we find:

\[ M(\Omega_{bc}) = 11182(27)(13) \text{ MeV (\(\chi\)-extrap. \(N_f = 2 + 1\)) ;} \]

\[ M(\Omega_{bb}) = 11182(27)(13) \text{ MeV (\(\chi\)-extrap. \(N_f = 2 + 1\)) ;} \]

\[ M(\Omega_{bcc}) = 11182(27)(13) \text{ MeV (\(\chi\)-extrap. \(N_f = 2 + 1\)) ;} \]

\[ M(\Omega_{bbb}) = 11182(27)(13) \text{ MeV (\(\chi\)-extrap. \(N_f = 2 + 1\)) ;} \]

FIG. 13: \(\Omega_{bc}, \Omega_{bb}, \) and \(\Omega_{bcc}\) masses versus \(M_{b}^{\star}\).
With the physical value for $M(\Upsilon)$ as input, we find:

\[
M(\Omega_{hbb}) = 14357.6(4.4)(3.3) \text{ MeV (} N_f = 2) ;
\]
\[
M(\Omega_{hbb}) = 14369(21)(14) \text{ MeV (} \chi\text{-extrap. } N_f = 2 + 1).\]

Figure 13 shows the $\Omega_{hbb}$ masses (referenced from an appropriate combination of $M(\Upsilon)$ and $M(B_s)$) versus the pion mass. Once the heavy-quark masses are “subtracted”, seemingly little difference remains between these triply heavy systems.

Figure 14 displays the $\Omega_{hbb}$ mass achieved via two schemes: from $E(\Omega_{hbb}) - 3E(\Upsilon)/2$ and $M(\Upsilon)$; or from $E(\Upsilon) + E(B^*) - E(\Omega_{hbb})$ and $M(\Upsilon) + M(B^*)$. The two estimates agree in the $N_f = 2 + 1$ chiral limit and on the $N_f = 2$ lattice.

For further results (e.g., spin and parity splittings) involving $bcc$ and $bbc$ baryons, see Table XIII. Estimates for the first-excited $\Omega_{hbb}$ appear in Table XIV.

Using (jackknifed) combinations of four correlators, we were able to compare the effect of trading one of the $b$ quarks for a lighter one in the mesons with the same effect in the baryons. These results appear in Table XV.

For the first meson difference, $\Upsilon - B_c$, the spin changes from 1 to 0, but these states are much better determined (experimentally) than the corresponding $\eta_c$ and $B^*_s$ (this extra spin jump adds about 50 MeV to the first four rows). Considering the errors, there appears to be little difference between the splittings involving $\Upsilon - B^*_s$ and $\Upsilon - B^*$, but exchanging the $b$ quark for a strange or light one appears to have a larger effect on the baryons than exchanging $b \rightarrow c$, especially when looking at baryons with fewer heavy quarks [1].

IV. DISCUSSION

We reserve this section for a brief discussion of possible sources of systematic errors affecting our results [1].

1) Finite-volume effects:

As can be seen in Table I all volumes are rather small when compared to the dynamical pion masses ($M_{fL} \lesssim 4$), with $L$ ranging from 1.7–1.8 fm for the $N_f = 2 + 1$ ensembles to 2.2 fm for the $N_f = 2$ one. Obvious energy enhancements can be seen in some hadrons containing

| splitting | $N_f = 2 + 1$ (c) | $N_f = 2 + 1$ (b) |
|-----------|------------------|------------------|
| $\Omega_{hbb} - B_c + \Upsilon/2$ | 164(27)(12) | 185(9)(6) |
| $\Omega_{hbc} - B_c - \Upsilon/2$ | 177(27)(13) | 179(9)(5) |
| $\Omega_{hbc} - \Omega_{hbb}$ | 28.5(5.6)(2.2) | 27.2(1.8)(0.8) |
| $\Omega_{hbc} - \Omega_{hbc}$ | 21.8(7.0)(1.7) | 26.5(2.2)(0.8) |
| $\Omega_{hbc} - \Omega_{hbb}$ | 449(41)(34) | 422(15)(13) |
| $\Omega_{hbc} - \Omega_{hbc}$ | 375(74)(28) | 382(26)(11) |
| $\Omega_{hbc} - \Omega_{hbc}$ | 30(12)(2) | 39.6(4.1)(1.2) |

TABLE XIV: Results for $bbb$ baryon mass splittings (in MeV) using $am_Q = 1.5$ and $c_4 = 1.2$ for the $N_f = 2 + 1$ chiral limit ($\chi$) and the $N_f = 2 + 1$ ensemble $b$. The first error comes from the fit, the second from the scale setting ($\Delta M_{PS}$). $^{(1)}$Constant fit, using only the three lightest $M_{PS}$ values.

| splitting | $N_f = 2 + 1$ (c) | $N_f = 2 + 1$ (b) |
|-----------|------------------|------------------|
| $\Omega_{hbb} - \Omega_{hbc}$ | 45(14)(3) | 42.3(3.5)(1.3) |
| $\Omega_{hbc} - \Omega_{hbc}$ | 16.7(9.8)(1.3) | 16.4(3.4)(0.5) |
| $\Omega_{hbc} - \Omega_{hbc}$ | 17(32)(1) | 18(12)(1) |
| $\Omega_{hbb} - \Xi_{bb}$ | 34(36)(3) | 31(23)(1) |
| $\Omega_{hbc} - \Xi_{bb}$ | -29(20)(2) | -29.3(8.4)(0.9) |
| $\Omega_{hbc} - \Xi_{bb}$ | -91(27)(7) | -84.8(8.8)(2.6) |
| $\Omega_{hbc} - \Xi_{bb}$ | -130(50)(10) | -137(17)(4) |
| $\Omega_{hbc} - \Xi_{bb}$ | -171(65)(13) | -154(27)(5) |
| $\Omega_{hbc} - \Xi_{bb}$ | -35(20)(3) | -29.7(9.6)(0.9) |
| $\Omega_{hbc} - \Xi_{bb}$ | -112(30)(8) | -96(10)(3) |
| $\Omega_{hbc} - \Xi_{bb}$ | -137(57)(10) | -139(21)(4) |
| $\Omega_{hbc} - \Xi_{bb}$ | -179(74)(13) | -153(33)(5) |
light quarks: e.g., $\Lambda_b^{(*)}, \Sigma_b^{(*)}, \Xi_b^{(*)}$, and possibly even $B^*-B$. We can only imagine how large such effects may be for higher excitations. However, there are no indications that hadrons containing strange quarks as their lightest constituents suffer from these small boxes.

(2) Different previous lattice scales; too heavy s and c quarks:

During the tuning and running stages of the present ensembles [50, 51], different scales were used than the one in the current study. The same was true during a tuning of the charm quark [53, 55]. The fact that we now use $\Delta M_{PS}$ from the bottomonia system to set the scale leads to a seemingly ~10% enhancement of the c and s (in the chiral limit) masses.

(3) For $N_f = 2$, too heavy b quark:

Looking at Fig. 2, one can see that on the $N_f = 2$ ensemble, $am_Q = 1.5$ is too large (by ~10%) for the bottom quark.

(4) Different smearings; different excited-state “contamination”:

On ensembles c and d, much less smearing was chosen for the light and strange quarks in an attempt to better identify excited states. However, this may lead to different levels of contamination from excited states when considering the ratios of smeared-smeared correlators on the different ensembles leading to the $N_f = 2 + 1$ chiral limit.

(5) Relatively heavy u, d quarks; long chiral extrapolations:

For many systems considered herein, this one is not so much a systematic error, outside of the fact that it systematically causes larger errors in the chiral limit. More trouble arises from this when the system considered should cross a threshold between ensemble b and the chiral limit.

V. CONCLUSIONS AND OUTLOOK

We have presented the heavy-hadron spectrum arising from NRQCD-approximated $b$ quarks and improved clover-Wilson $c, s, d, u$ quarks on $N_f = 2$ and $2 + 1$ lattices. Singly, doubly, and triply heavy $(b, c)$ systems were considered and results were found for spin splittings, alternate parities, and in some cases, “radial” excitations. Relative mass differences between mesons and baryons resulting from the exchange of one $b$ quark for a lighter one were presented as well. A number of systematics were pointed out (e.g., small volumes; see above), but these have not been fully quantified and the reader must use some caution with the results. Perhaps more trustworthy are the lower-lying spectra (no radial excitations) of hadrons containing only $b, c, or s$ valence quarks. A more careful analysis of all the data generated (e.g., with $am_Q = 3.0$ or the $40^3, N_f = 2$ correlators) and further runs on existing $32^3$, $N_f = 2 + 1$ ensembles could uncover more about the errors incurred, and an expansion of the code [61] could lead to a study of possible $bbqq$ states, but we must excuse ourselves from taking this project further as we have precious little free time and have found new ways in which to be wrong [60].

Acknowledgments

Light-quark propagators were generated with the use of the Chroma [53] software library. We thank our colleagues for saving the light- and strange-quark propagator files on ensembles a, b, and c [50, 54] and for making them available to the group at large. We would like to thank S. Gutzwiller and R. Schiel for help in checking that we were properly reading the light-quark propagator files and J. Najjar for providing initial plaquette results for our $u_0$ values. We would like to thank M. Göckeler for helpful suggestions on many topics and P. Pérez-Rubio for useful discussions involving heavy-hadron interpolators. Simulations were performed at the Uni-Regensburg Rechenzentrum and Institute for Theoretical Physics and we thank the Schäfer and Braun Chairs for continued access and the administrators for keeping the machines running smoothly. This work was supported in part by the DFG (SFB TR-55) and mostly by a patient family. Beyond the scope of this project, we would like to thank those who still deem it worthwhile to discuss physics with the unaffiliated. In the end, it must be admitted that the author has been in this place too long, rendering him a cynic [70]. Apologies for where it shows.

[1] Excuse the curt presentation in places, but this is as much exodus as it is catharsis and the later-written parts of the manuscript will show it. Initially intended as part of a larger study with larger lattices, these results have been sitting around partially analyzed for a while, but they may have retained some relevance. As it always should be, use or damn what you will.
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from an “alternative” chiral spin-basis to the NR one:
\[ M_{NR}^{-1} = U M^{-1} U^\dagger, \]
where
\[ U = \frac{1}{\sqrt{2}} (\sigma_y \sigma_y - \sigma_y \sigma_y). \]

[65] An error in the flavor projections in our code has left us without the possibility of exploring the \( J = \frac{1}{2} \) states arising from the \( \Sigma_{Q_0}^+ \) operator.

[66] For the sake of brevity, we leave out such things as effective-mass plots, fit choices, \( \chi^2 \) values, etc. The interested reader may peruse such details included in the tarball of the code [61].

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