Research Article

Calculating Crossing Numbers of Graphs Using Combinatorial Principles

Zhenhua Su

School of Mathematics and Computing Sciences, Huaihua University, Huaihua 418000, China

Correspondence should be addressed to Zhenhua Su; szh820@163.com

Received 17 March 2022; Revised 24 April 2022; Accepted 10 May 2022; Published 7 June 2022

Academic Editor: Kamal Kumar

Copyright © 2022 Zhenhua Su. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The crossing numbers of graphs were started from Turán’s brick factory problem (TBFP). Because of its wide range of applications, it has been used in computer networks, electrical circuits, and biological engineering. Recently, many experts began to pay much attention to the crossing number of \( G \setminus e \), which obtained from graph \( G \) by deleting an edge \( e \). In this paper, by using some combinatorial skills, we determine the exact value of crossing numbers of \( K_{1,4,n} \setminus e \) and \( K_{2,3,n} \setminus e \). These results are an in-depth work of TBFP, which will be beneficial to the further study of crossing numbers and its applications.

1. Introduction

The concept of crossing number was introduced by the Hungarian mathematician Turán [1]. He encountered a practical problem in Budapest brick factory, which named “Turán’s brick factory problem” (TBFP). In fact, TBFP is to determine the minimal number of crossings among edges of the complete bipartite graph \( K_{m,n} \).

In the past five decades, it turned out that the crossing numbers have strong practical significance. And they can be widely used in various fields, such as the VLSI circuit layout [2], the identification and repaint of sketch [3], and automatic generation of ER diagram in software development [4] (see [5, 6]). One of important applications is to find the best location for a new electrical substation so that every two substations are directly connected and to do without overlapping power lines. Rach [7] has applied the crossing number to solve a problem of locating electrical substations in the city of Glencoe, MN.

With the depth of research, the crossing numbers of graphs have been investigated extensively in the mathematical, computer, and biological literature, often under different parameters, such as the parity [8], odd-crossing number [9], regular graphs [10], chromatic number [11], and genus [12]. For more results and its properties about crossing numbers, reader can refer to [13–16]. As for the complete bipartite graphs \( K_{m,n} \), Kleitman in [17] proved that

\[
\text{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor = Z(m,n),
\]

\[m \leq 6, m \leq n.\]  

Recently, the crossing numbers of complete multipartite graphs attracted much attention. In 1986, Asano [18] obtained the crossing numbers of graphs \( K_{1,4,n} \) and \( K_{2,3,n} \). In 2008, Huang and Zhao in paper [19] established that the crossing number of graph \( K_{1,4,n} \) is equal to \( n(n - 1) \). In 2020, the crossing numbers of graphs \( K_{1,1,4,n} \) as well as \( K_{1,1,4,T} \) have been proved in [20].

While studying the crossing number of the primal graph \( G \), many experts also began to follow with interest the crossing number of \( G \setminus e \), which is obtained by deleting an edge \( e \) from \( G \). This is an interesting problem worthy of consideration. For \( G \), it is a complete graph or complete bipartite graph. Ouyang in [21, 22] as well as Chia in [23], independently, established the precise values of crossing numbers of certain graphs \( G \setminus e \): (1) \( K_n \setminus e \) for \( n \leq 12 \), (2) \( K_{m,n} \setminus e \) for \( m = 3, 4 \) and \( n \geq 1 \).

Very recently, Huang and Wang in [24] by applying the method of edge-labeling, which is new and different from...
Proof. Let $D$ be an optimal drawing of $K_{n,n}$ having $Z(5,n)$ crossings due to Zarankiewicz [17]. In the drawing, we place the vertices of $K_{n,n}$ at coordinates $(0, i)$ and $(j, 0)$ where $-2 \leq i \leq 3$, $-[n/2] \leq j \leq [n/2]$, and $i, j \neq 0$. Then, we join $(0, i)$ to $(j, 0)$ with a straight line segment. Next, we join the edges of $K_{1,4}$ as shown in Figure 1, and an optimal drawing of the complete tripartite graph $K_{1,4,n}$ is obtained. Let us denote the drawing by $D'$. It is not difficult to see that $cr(K_{1,4,n}) = Z(5,n) + [n/2]$. In the following, we obtain the graph $K_{1,4,n}$ together with its drawing from $D'$.

(i) If $e \in XY$. By deleting the edge $e = xy_1$ from $D$, then a drawing of $K_{1,4,n}\setminus e_{xy}$ is obtained. We can easily check that there are $[n/2]$ crossings on the edge of $xy_1$. Therefore, we can verify that

$$cr(K_{1,4,n}\setminus e_{xy}) \leq Z(5,n) + 2\left(\frac{n}{2} - \frac{n}{2} + 1\right) = Z(5,n) + \left\lfloor \frac{n}{2}\right\rfloor.$$  

(ii) If $e \in XZ$. Then, by deleting the edge of $e = xz_{(n/2)}$, we can obtain a drawing of $K_{1,4,n}\setminus e_{xz}$. Likewise, the responsibility of the edge $xz_{(n/2)}$ is $2\left([n/2] - 1\right)$. So, we can obtain that

$$cr(K_{1,4,n}\setminus e_{xz}) \leq Z(5,n) + 2\left(\frac{n}{2} - 2\left(\frac{n}{2} - 1\right)\right) = Z(5,n) + 2\left(\frac{n}{2} - \left\lfloor \frac{n}{2}\right\rfloor + 1\right).$$  

(iii) If $e \in YZ$. In the optimal drawing of $K_{n,n}$, which have $Z(5,n)$ crossings given by Zarankiewicz, we reconnect the edges of $K_{1,4} = \{xy_i, 1 \leq i \leq 4\}$, as shown in Figure 2. And then, by deleting the edge of $e = y_1z_1$, a drawing of $K_{1,4,n}\setminus e_{y_1z_1}$ is obtained. Let us denote the drawing by $D$. Then, one can verify that the responsibility of the edges $K_{1,4}$ and $y_1z_1$ is $n$ and $2\left([n/2] - 1\right) + 1$, respectively. Therefore, we have

$$cr(K_{1,4,n}\setminus e_{y_1z_1}) \leq Z(5,n) + n - 2\left(\frac{n}{2} - 1\right) - 1 = Z(5,n) + \left\lfloor \frac{n}{2}\right\rfloor - \left\lfloor \frac{n}{2}\right\rfloor + 1.$$  

Combined with the above three cases, this completes the proof. □

Theorem 1. For any edge $e_{xy}$ in complete tripartite graph $K_{1,4,n}$, where $e_{xy} \in XY$, then

$$cr(K_{1,4,n}\setminus e_{xy}) = Z(5,n) + \left\lfloor \frac{n}{2}\right\rfloor.$$  

Proof. It is not difficult to know that $K_{1,4,n}\setminus e_{xy}$ contains a subgraph that is isomorphic to $(S_3 \cup K_1) + nK_4$. And it was shown from Lemma 3 that $cr((S_3 \cup K_1) + nK_4) = Z(5,n) + [n/2]$. Thus, $cr(K_{1,4,n}\setminus e_{xy}) \geq cr((S_3 \cup K_1) + nK_4) = Z(5,n) + [n/2]$. The reverse inequalities are confirmed by Lemma 4. This completes the proof. □

Theorem 2. For any edge $e_{yz}$ in complete tripartite graph $K_{1,4,n}$, where $e_{yz} \in YZ$, then

$$cr(K_{1,4,n}\setminus e_{yz}) = Z(5,n) + \left\lfloor \frac{n}{2}\right\rfloor.$$  

Proof. It is not difficult to know that $K_{1,4,n}\setminus e_{yz}$ contains a subgraph that is isomorphic to $(S_3 \cup K_1) + nK_4$. And it was shown from Lemma 3 that $cr((S_3 \cup K_1) + nK_4) = Z(5,n) + [n/2]$. Thus, $cr(K_{1,4,n}\setminus e_{yz}) \geq cr((S_3 \cup K_1) + nK_4) = Z(5,n) + [n/2]$. The reverse inequalities are confirmed by Lemma 4. This completes the proof. □
Proof. At first, according to Lemma 4, we have 
\[ \text{cr}(K_{1,4,n} \setminus e_{yz}) \leq Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1. \] (8)

Theorem 2 is true if the equality holds. For \( n = 1 \) since \( K_{1,4,1} \setminus e_{yz} \) is planar graph, therefore,\( \text{cr}(K_{1,4,1} \setminus e_{yz}) = 0 \geq Z(5,1) + \left\lfloor \frac{1}{2} \right\rfloor - \left\lceil \frac{1}{2} \right\rceil + 1 = 0 \) is true. Now, we suppose that 
\[ \text{cr}(K_{1,4,m} \setminus e_{yz}) \geq Z(5,m) + \left\lfloor \frac{m}{2} \right\rfloor - \left\lceil \frac{m}{2} \right\rceil + 1 \]
for any positive integer \( 2 \leq m \leq n - 1 \).

Let \( D \) be an optimal drawing of graph \( K_{1,4,n} \setminus e_{yz} \), which satisfies \( r_D(K_{1,4,n} \setminus e_{yz}) = \text{cr}(K_{1,4,n} \setminus e_{yz}) = c \). Without loss of generality, we say \( e_{yz} = y_1x_1 \). By deleting the edges of \( E_z \) from drawing \( D \), the graph \( K_{1,4,n-1} \) is obtained. Hence, we get that 
\[ r_D(z_i) \leq c - \text{cr}(K_{1,4,n-1}) \] (9)

Likewise, for any \( i = 2, 3, \ldots, n \), we have the following that 
\[ r_D(z_i) \leq c - \text{cr}(K_{1,4,n-1}). \] (10)

Otherwise, by deleting the edges of \( E_z \), for any \( i = 2, 3, \ldots, n \), we obtain the graph which is isomorphic to \( K_{1,4,n-1} \setminus e_{yz} \) and has less than \( \text{cr}(K_{1,4,n-1} \setminus e_{yz}) \) crossings. Therefore, from (9) and (10), summing up for \( 1 \leq i \leq n \), we can obtain that
\[ 2c = \sum_{i=1}^{n} r_D(z_i) \leq c - \text{cr}(K_{1,4,n-1}) + (n - 1)(c - \text{cr}(K_{1,4,n-1} \setminus e_{yz})). \] (11)

Thus, we simplify and conclude that
\[ \text{cr}(K_{1,4,n} \setminus e_{yz}) = c \geq \frac{(n-1)\text{cr}(K_{1,4,n-1} \setminus e_{yz}) + \text{cr}(K_{1,4,n-1})}{n-2} \] (12)
Finally, combining with inductive hypothesis and Lemma 1, we have that
\[ cr(K_{1,4n}) \geq Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1. \] (13)
Thus, the proof of Theorem 2 is finished. \(\square\)

**Theorem 3.** For any edge \(e_{xz}\) in complete tripartite graph \(K_{1,4n}\), where \(e_{xz} \in XZ\), then
\[ cr(K_{1,4n}) = Z(5,n) + 2\left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1\right). \] (14)

**Proof.** From Lemma 4, we can get that \(cr(K_{1,4n}) \leq Z(5,n) + 2([n/2] - [n/2] + 1)\). Thus, Theorem 3 is true; we need only to prove that \(cr(\phi(K_{1,4n})) \geq Z(5,n) + 2([n/2] - [n/2] + 1)\) for any drawing \(\phi\) of \(K_{1,4n}\).

Without losing generality, we assume that under the drawing \(\phi\) of \(K_{1,4n}\), the clockwise order of these four images \(\phi(x,y)\) around \(\phi(x)\) is \(\phi(x_{1}) \rightarrow \phi(x_{2}) \rightarrow \phi(x_{3}) \rightarrow \phi(x_{4})\) and we assume \(e_{xz} = xz_i\). Thus, the graph \(K_{1,4n}\) has an additional \(n - 1\) edges \(xz_i\) incident with \(x\) \((2 \leq i \leq n)\). Let \(A_1, A_2, A_3, A_4\) denote the sets of all those images \(xz_i\) each of which places in the angle \(\delta_i\) is formed between \(\phi(x_{1})\) and \(\phi(x_{i+1})\), where the indices are read modulo 4 (see Figure 3(a)). We note that \(|A_1| + |A_2| + |A_3| + |A_4| = n - 1\). Further, more, we see that in the plane \(R^2\), there exists a circular neighborhood around \(\phi(x)\) such that \(N(\phi(x), \epsilon) = \{s \in R^2 : ||s - \phi(x)|| \leq \epsilon\}\), where \(\epsilon\) is a positive number small enough such that for any other edge \(y_{i}z_{j} (i = 1, 2, 3, 4; j = 1, 2, \ldots, n)\) of \(K_{1,4n}\) not incident with \(x\), \(\phi(y_{i}z_{j}) \cap N(\phi(x), \epsilon) = \emptyset\). Next, we divide two cases to discuss. \(\square\)

**Case 1.** Assume \((n - 1)\) is odd. Thus, we have \(|A_1| \neq |A_3|\) or \(|A_2| \neq |A_4|\). Otherwise, \(n - 1 = \sum_{i=1}^{4} |A_i| = 2(|A_1| + |A_2|)\); this contradicts the fact that \((n - 1)\) is odd. Without loss of generality, we assume \(|A_1| \neq |A_3|\), and more precisely, let \(|A_1| \geq |A_3| + 1\). In the following, we produce the graph \(K_{5,n+1}\) together with its drawing \(\phi'\) by three steps.

**Step 1.** Add a new vertex \(z_{n+1}\) in some location of \(\phi(x_{1})\) in \(N(\phi(x), \epsilon)\).

**Step 2.** For all \(1 \leq i \leq 4\), delete the partition of \(\phi(x_{i})\) lying in \(N(\phi(x), \epsilon)\) (do not delete the vertex \(z_{n+1}\)).

**Step 3.** Connect \(z_{n+1}\) to each vertex in \(\{\phi(x), \phi(y_1), \ldots, \phi(y_4)\}\) in such a way as described in Figure 3(b).

Thus, we obtain a drawing of the graph \(K_{5,n+1}\) from the drawing \(\phi\) of \(K_{1,4n}\). It is easy to obtain that
\[ cr(\phi(K_{5,n+1})) = cr(\phi(K_{1,4n})) + 2|A_1| + |A_2| + |A_4|, \]
\[ \leq cr(\phi(K_{1,4n})) + n - 2. \] (15)
With the help of \(cr(K_{5,n+1}) = n(n - 1)\) by Lemma 2, we have that
\[ cr(\phi(K_{1,4n})) \geq cr(\phi(K_{5,n+1})) - n + 2 \]
\[ \geq n(n - 1) - n + 2 \]
\[ \geq Z(5,n) + 2\left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1\right). \] (16)

**Case 2.** Assume \((n - 1)\) is even. We consider arbitrarily a pair of numbers \(|A_1|\) and \(|A_3|\) or \(|A_2|\) and \(|A_4|\). Without losing generality, we say \(|A_1|\) and \(|A_3|\), again we say \(|A_1| \leq |A_3|\). Completely analogously to Case 1 above, we can get a drawing \(\phi'\) of graph \(K_{5,n+1}\) such that
\[ cr(\phi(K_{5,n+1})) = cr(\phi(K_{1,4n})) + 2|A_1| + |A_2| + |A_4| \]
\[ \leq cr(\phi(K_{1,4n})) + n - 1. \] (17)

Now, applying \(cr(K_{5,n+1}) = n(n - 1)\), we obtain the following that
\[ cr(\phi(K_{1,4n})) \geq cr(\phi(K_{5,n+1})) - n + 1 \]
\[ \geq n(n - 1) - n + 1 \]
\[ \geq Z(5,n) + 2\left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1\right). \] (18)

Thus, by the arguments derived in Cases 1 and 2, we have shown that \(cr(\phi(K_{1,4n})) \geq Z(5,n) + 2([n/2] - [n/2] + 1)\) for any drawing \(\phi\) of \(K_{1,4n}\). This completes the proof.
Therefore, from Theorems 1–3, we can get the following corollary immediately.

**Corollary 1.** For an edge \( e \) in complete tripartite graph \( K_{1,4,n} \), then

\[
\text{cr}(K_{1,4,n} \setminus e) = \begin{cases} 
Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor, & e \in XY, \\
Z(5,n) + 2 \left( \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil + 1 \right), & e \in XZ, \\
Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1, & e \in YZ.
\end{cases}
\]  

**3. Crossing Number of** \( K_{2,3,n} \) \( e \)

In graph \( K_{2,3,n} \), let \( \{X,Y,Z\} \) be the vertex partition, where \( X = \{x_1, x_2\}, Y = \{y_1, y_2, y_3\}, Z = \{z_1, z_2, \ldots, z_n\} \). We denote \( XY, XZ, YZ \) be the subgraphs of \( K_{2,3,n} \) induced by \( E_{XY}, E_{XZ}, E_{YZ} \), respectively. Clearly, \( XY \equiv K_3 \). For \( i = 1, 2, \ldots, n \), let \( E_x \) be the subgraph of \( K_{2,3,n} \) induced by five edges incident with \( z_i \). We can easily get that

\[
K_{2,3,n} = XY \cup XZ \cup YZ = XY \cup \left( \bigcup_{i=1}^{n} E_{z_i} \right).
\]  

**Lemma 5** (see [18]). \( \text{cr}(K_{2,3,n}) = Z(5,n) + n \) for any \( n \geq 1 \).

**Lemma 6** (see [26]). \( \text{cr}(K_{2,3} \setminus e + nK_1) = Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor \) for any \( n \geq 1 \).

**Lemma 7.** For an edge \( e \) in complete tripartite graph \( K_{2,3,n} \), then

\[
\text{cr}(K_{2,3,n} \setminus e) \leq \begin{cases} 
Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor, & e \in XY, \\
n^2 - 2n + 2, & e \in XZ, \\
n^2 - 2n + 1, & e \in YZ.
\end{cases}
\]

**Proof.** Let \( D \) be an optimal drawing of \( K_{2,3,n} \), having \( Z(5,n) \) crossings due to Zarankiewicz [17]. In the drawing, we place the vertices of \( K_{2,3,n} \) at coordinates \((0,i)\) and \((j,0)\) where \(-2 \leq i \leq 3, -1 \leq j \leq 1\) and \(i, j \neq 0\). Then, we join \((0,i)\) to \((j,0)\) with straight line segment.

Next, we join the edges of \( K_{2,3} = \{x_1y_1, 1 \leq i < j \leq 3\} \), as shown in Figure 4, and thus an optimal drawing of the complete tripartite graph \( K_{2,3,n} \) is obtained. Let us denote the drawing by \( D' \). We can easily check that \( \text{cr}(K_{2,3,n}) = \text{cr}(K_{2,3} \setminus e) = Z(5,n) + n \). In the following, we obtain the graph \( K_{2,3,n} \setminus e \) together with its drawing from \( D' \).

(i) If \( e \in XY \). By deleting the edge \( e = x_2y_2 \) from \( D' \), then a drawing of \( K_{2,3,n} \setminus e_{xy} \) is obtained. It is easily checked that there are \( \lfloor n/2 \rfloor \) crossings on the edge of \( x_2y_2 \). Therefore, we can verify that

\[
\text{cr}(K_{2,3,n} \setminus e_{xy}) \leq Z(5,n) + n - \left\lfloor \frac{n}{2} \right\rfloor = Z(5,n) + \left\lceil \frac{n}{2} \right\rceil.
\]  

(ii) If \( e \in XZ \). We can get a drawing of \( K_{2,3,n} \setminus e_{xz} \) by deleting the edge \( e = x_1z_{n/2} \) from \( D' \). Obviously, the responsibility of the edge of \( x_1z_{n/2} \) is \( 2(\lfloor n/2 \rfloor - 1) \). So, we have

\[
\text{cr}(K_{2,3,n} \setminus e_{xz}) \leq Z(5,n) + n - 2\left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) = n^2 - 2n + 2.
\]

(iii) If \( e \in YZ \). The drawing of \( K_{2,3,n} \setminus e_{yz} \) can be obtained by deleting the edge \( e = y_1z_{1} \) from \( D' \). We note that the edge \( y_1z_{1} \) crosses with the edges \( \langle YZ \cup x_2y_3 \rangle \) exactly \( 2(\lfloor n/2 \rfloor - 1) \) times. Thus, we obtain that

\[
\text{cr}(K_{2,3,n} \setminus e_{yz}) \leq Z(5,n) + n - 2\left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) - 1 = n^2 - 2n + 1.
\]

Therefore, according to the above analysis, we have completed the proof.

**Theorem 4.** For any edge \( e_{xy} \) in complete tripartite graph \( K_{2,3,n} \), where \( e_{xy} \in XY \), then

\[
\text{cr}(K_{2,3,n} \setminus e_{xy}) = Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor.
\]

**Proof.** It is not difficult to know that \( K_{2,3,n} \setminus e_{xy} \) is isomorphic to \( K_{2,3} \setminus e + nK_1 \). And it was shown by Lemma 6 that \( \text{cr}(K_{2,3} \setminus e + nK_1) = Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor \). Thus, \( \text{cr}(K_{2,3,n} \setminus e_{xy}) \geq \text{cr}(K_{2,3} \setminus e + nK_1) \geq Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor \). The reverse inequalities are confirmed by Lemma 7. Hence, the proof is done.

**Theorem 5.** For any edge \( e_{xz} \) in complete tripartite graph \( K_{2,3,n} \), where \( e_{xz} \in XZ \), then

\[
\text{cr}(K_{2,3,n} \setminus e_{xz}) = n^2 - 2n + 2.
\]

**Proof.** Without loss of generality, let \( e_{xz} = x_1z_n \). Therefore, according to (4), we note that

\[
K_{2,3,n} \setminus e_{xz} = XY \cup \left( \bigcup_{i=1}^{n-1} E_{x_i} \right) \cup E_{z_n} = K_{2,3} \cup \left( \bigcup_{i=1}^{n-1} E_{z_i} \right) \cup E_{z_n}.
\]

At first, we can obtain by Lemma 7 that \( \text{cr}(K_{2,3,n} \setminus e_{xz}) \leq n^2 - 2n + 2 \). It is not difficult to know that \( K_{2,3,1} \setminus e_{xz} \) contains a subgraph which is isomorphic to \( K_{3,3} \) and \( K_{2,3,2} \setminus e_{xz} \) contains a subgraph which is isomorphic to \( K_{3,4} \). Thus, we have \( \text{cr}(K_{2,3,1} \setminus e_{xz}) \geq 1 \) and \( \text{cr}(K_{2,3,2} \setminus e_{xz}) \geq 2 \), so the theorem is established for \( n = 1 \) and 2. Now, we suppose that \( n \geq 3 \) and that \( \text{cr}(K_{2,3,k} \setminus e_{xz}) \leq k^2 - 2k + 2 \) for any \( 3 \leq k \leq n - 1 \). We will derive contradiction to prove the
reverse inequality. We assume to contrary that $K_{2,3,n}/e_{xx}$ has a good drawing $D$ such that

$$cr_D(K_{2,3,n}/e_{xx}) \leq n^2 - 2n + 1. \quad (28)$$

In the subsequent proof process, we always deduce some contradictions to (\ast). We first have the following claims.

**Claim 1.** $cr_D(E_{zi}, E_{zj}) \geq 1$ for all $i, j = 1, 2, \ldots, n - 1$, and $i \neq j$.

**Proof.** Otherwise, without loss of generality, let $cr_D(E_{zi}, E_{zj}) = 0$. According to Lemma 5, $cr_D(K_{2,3,3}) = 2$ implies that $cr_D(K_{2,3}, E_{zi} \cup E_{zj}) \geq 2$. As $\langle E_{zi} \cup E_{zj} \rangle$ isomorphic to $K_{2,3}^\circ$, and it was shown by Lemma 2 that $cr(D^\circ) = 2$. Thus, $cr_D(E_{zi}, E_{zj}) \geq 2$. The known fact that $cr(K_{3,3}) = 4$ implies that $cr_D(E_{zi}, E_{zj} \cup E_{zj}) \geq 4$ for all $i = 3, 4, \ldots, n - 1$. Therefore, we have

$$cr_D(K_{2,3,n}/e_{xx}) = cr_D(K_{2,3} \cup \bigcup_{i=3}^{n-1} E_{zi} \cup E_{zj} \cup E_{zj})$$

$$\geq cr_D(K_{2,3,n-2}/e_{xx}) + cr_D(K_{2,3} \cup \bigcup_{i=3}^{n-1} E_{zi} \cup E_{zj} \cup E_{zj})$$

$$\geq Z(5, n - 2) + n - 2 - 2 \left\lfloor \frac{n - 2}{2} \right\rfloor + 2 + 4 + 4(n - 3)$$

$$= n^2 - 2n + 2. \quad (29)$$

Clearly, this contradicts to (\ast).

**Claim 2.** $cr_D(K_{2,3}) + cr_D(K_{2,3}, \cup_{i=3}^{n-1} E_{zi}) \leq n - 1$.

**Proof.** Otherwise, we have $cr_D(K_{2,3}) + cr_D(K_{2,3}, \cup_{i=3}^{n-1} E_{zi}) \geq n$. As $\cup_{i=3}^{n-1} E_{zi}$ is isomorphic to $K_{2,3}^\circ$, and it was proved by Lemma 2 that $cr(K_{2,3}^\circ) = (n - 1)(n - 2)$. Thus, we obtain

$$cr_D(K_{2,3,n}/e_{xx}) \geq cr_D(K_{2,3} + \bigcup_{i=3}^{n-1} E_{zi} + \bigcup_{i=3}^{n-1} E_{zi})$$

$$\geq cr_D(K_{2,3}) + cr_D(K_{2,3} \cup E_{zi} \cup E_{zj}) + cr_D(\bigcup_{i=3}^{n-1} E_{zi} \cup E_{zj})$$

$$\geq n + (n - 1)(n - 2)$$

$$= n^2 - 2n + 2. \quad (30)$$

This also contradicts to (\ast).

Furthermore, we have the following claim.

**Claim 3.** There exists a vertex $z_i (1 \leq i \leq n - 1)$ such that $cr_D(K_{2,3,d}) = 0$.

**Proof.** If $cr_D(K_{2,3}) = 0$, then the good drawing of $K_{2,3}$ induced by $D$ divides the plane into three quadrangular regions $\omega(y_1, y_2), \omega(y_2, y_3)$, and $\omega(y_3, y_1)$ depending on which two of the vertices $y_1, y_2, y_3$, and $y_4$ are placed on the corresponding boundary. Thus, under the drawing of $K_{2,3}$, we have $cr_D(K_{2,3}, E_{zi}) \geq 1$ and $cr_D(K_{2,3}, E_{zi}) \geq 1$ for all $i = 1, 2, \ldots, n - 1$. In other words, the edges of $K_{2,3}$ are crossed at least $n$ times by the subgraphs $\langle \cup_{i=1}^{n-1} E_{zi} \rangle$; this contradicts with Claim 2.
If $cr_D(K_{2,3}) \geq 1$, combining together with Claim 2, we can obtain that $cr_D(K_{2,3}) \cup \bigcup_{i=1}^{n-1} E_{z_i} \leq n - 2$. This forces that there exists a vertex $z_i(1 \leq i \leq n - 1)$ such that $cr_D(K_{2,3}, E_{z_i}) = 0$. Thus, Claim 3 is proved.

Now, we continue to prove the theorem. By Claim 3, without loss of generality, we assume $cr_D(K_{2,3}, E_{z_i}) = 0$. Thus, there is a disk such that the five vertices of $K_{2,3}$ are all placed on the boundary of disk. We assume the vertex $z_i$ placed in the external of the disk, and the edges of $K_{2,3}$ are all placed in the inner side of the disk. It is easy to obtain that two drawing of $\langle K_{2,3} \cup E_{z_i} \rangle$ as shown in Figure 5.

\[
\begin{align*}
    cr_D(K_{2,3,n}\backslash e_{xz}) &= cr_D(K_{2,3} \cup E_{z_i} \cup \bigcup_{i=2}^{n-1} E_{z_i} \cup E_{z_i}) \\
    &\geq cr_D(K_{2,3} \cup E_{z_i}) + cr_D(K_{2,3} \cup E_{z_i} \cup \bigcup_{i=2}^{n-1} E_{z_i} \cup E_{z_i}) + cr_D(\bigcup_{i=2}^{n-1} E_{z_i} \cup E_{z_i}) \\
    &\geq 1 + 3(n - 2) + 1 + (n - 2)(n - 3) \\
    &= n^2 - 2n + 2.
\end{align*}
\]

Clearly, this contradicts to (\#).

In summary, the hypothesis is not true, and the proof is done.

Using the method completely similar to Theorem 5, we can get the following Theorem 6. Thereby, the proof process of Theorem 6 is omitted here. \hfill \Box

**Theorem 6.** For any edge $e_{yz}$ in complete tripartite graph $K_{2,3,n}$ where $e_{yz} \in YZ$, then

\[
    cr(K_{2,3,n}\backslash e_{yz}) = n^2 - 2n + 1. \tag{32}
\]

Since $K_{2,3,n}\backslash e_{xz}$ is isomorphic to $K_{3,n}\backslash e + 2K_1$, thus we have the following corollary.

**Corollary 2.** $cr(K_{3,n}\backslash e + 2K_1) = n^2 - 2n + 1, n \geq 1$.

Together with Theorems 4–6, we can get the following corollary immediately.

**Corollary 3.** For an edge $e$ in complete tripartite graph $K_{2,3,n}$, then

\[
    cr(K_{2,3,n}\backslash e) = \begin{cases} 
    Z(5,n) + \frac{n}{2}, e \in XY, \\
    n^2 - 2n + 2, e \in XZ, \\
    n^2 - 2n + 1, e \in YZ.
\end{cases} \tag{33}
\]

In Figure 5, except for the region which marked with $a$, each of regions contains at most two vertices of $K_{2,3}$ in its boundary. Hence, we obtain that $cr_D(K_{2,3} \cup E_{z_i}, E_{z_i}) \geq 2$. When $z_i$ is placed in the region $a$, we have $cr_D(K_{2,3} \cup E_{z_i}, E_{z_i}) \geq 2$, if and only if $cr_D(K_{2,3}, E_{z_i}) = 2$ and the equality $cr_D(E_{z_i}, E_{z_i}) = 0$ holds. This together with Claim 1 implies that $cr_D(K_{2,3} \cup E_{z_i}, E_{z_i}) \geq 3$.

Moreover, each region contains at most three vertices of $K_{2,3}$ in its boundary in Figure 5. Thus, $cr_D(K_{2,3} \cup E_{z_i}, E_{z_i}) \geq 1$, $\langle \bigcup_{i=1}^{n-1} E_{z_i} \rangle \equiv K_{3,n}\backslash e$, and the assumption of the theorem that

\section{4. Conclusion}

The problem crossing numbers of graphs are originated in a practical application, whose theory has been widely applied in many fields. However, determining the crossing numbers of graphs are NP-complete. Because of its difficulty, the research progress is slow. In this paper, according to the structural characteristics of complete multipartite graph, using "drawing restriction method," "embedding method," and "point degree local modification method," we determine the exact value of crossing numbers of $K_{1,\alpha,n}$, $K_{2,\alpha,n}$, and $K_{3,\alpha,n}$. These results are an in-depth work of TFSP, which will be beneficial to the further study of crossing numbers and its applications.

Finally, we give some conjecture and open problems.

**Conjecture 1.** $cr(K_{6,n}\backslash e) = Z(6,n) - 2|n - 1/2|$ for $n \geq 1$.

**Problem 1.** For an edge $e$ in complete tripartite graph $K_{1,5,n}$, $K_{2,4,n}$, and $K_{3,3,n}$, then what are the precise values of crossing numbers of $K_{1,5,n}\backslash e$, $K_{2,4,n}\backslash e$, and $K_{3,3,n}\backslash e$?

**Data Availability**

No data were used to support the findings of the study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.
Acknowledgments

This work was supported by the scientific research project of Hunan Provincial Department of Education (2022).

References

[1] P. Turán, "A note of welcome," Journal of Graph Theory, vol. 1, no. 1, pp. 7–9, 1977.

[2] F. T. Leighton, “New lower bound techniques for VLSI,” Mathematical Systems Theory, vol. 17, no. 1, pp. 47–70, 1984.

[3] R. Cimikowski, “Crossing number bounds for the mesh of trees,” in Proceedings of the 29th. Southeastern Conference on Combinatorics, Graph Theory and Computing, pp. 9–13, Boca Raton, Florida, Oct.1998.

[4] J. Pach, F. Shahrokhi, and M. Szegedy, "Applications of the crossing number," Algorithmica, vol. 16, no. 1, pp. 111–117, 1996.

[5] D. Rach, “The Crossing Number of a Complete Graph and an Application,” http://facultypages.morris.umn.edu/math/Ma4901/F2011/Final/DannyRach-Final.pdf.

[6] Z. Su and Y. Huang, "The crossing numbers of $S_5 \cup K_1 + D_n$,” Acta Mathematicae Applicatae Sinica, vol. 44, no. 2, pp. 197–208, 2021.

[7] Z. Ouyang, "On the Crossing Number of a Graph," Phd Thesis, Hunan Normal University, Hunan, 2011.

[8] Z. Ding, "Rotation and crossing numbers for join products," Bulletin of the Malaysian Mathematical Sciences Society, vol. 43, no. 6, pp. 4183–4196, 2020.

[9] Z. Ding and Y. Huang, "A note on the crossing number of the cone of a graph," Graphs and Combinatorics, vol. 37, no. 6, pp. 2351–2363, 2021.