ON THE TOPOLOGICAL ESSENTIAL RANGE AND
REGULARITY OF COCYCLES OVER COMPACT AND
GENERIC SYSTEMS

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Abstract. We consider the notions of topological essential range and regularity for continuous cocycles over minimal \( \mathbb{Z} \)-systems introduced in [GH] and discuss relations with their generic counterparts. The alternative generic definitions can be given by using the notion of generic Mackey action associated with a cocycle. We further present a description of recurrent cocycles over minimal rotations with values in discrete groups and derive several consequences.

1. Introduction and preliminaries

The concept of topological essential range for continuous cocycles over compact minimal systems was introduced in [Al, LM] where the case of abelian groups of values has been under research. A generalization to a non-abelian case was made in [GH]. It was applied to obtain several results on regularity of topological cocycles which permit to describe the topological ergodic decomposition and structure of orbit closures in skew product actions.

Our aim is to describe the interplay between the above mentioned notions and their generic analogues. We consider Polish group valued cocycles over Polish minimal systems and suggest a parallel approach based on the notion of generic Mackey action associated with a cocycle ([GK]). It appears that several important properties of continuous cocycles related with essential ranges and regularity can be derived using such a generic approach but in more general situations.

We further proceed with the study of regularity problem for cocycles over a minimal rotation on a compact monothetic group. We completely describe the case when the group of values is discrete. It is applied for getting several reduction results for cocycles taking values in locally compact groups. Unlike the situation in [GH], where rotations on locally connected compact groups are considered, the most of our results are referred to a disconnected or arbitrary base space.

Let \( X \) be a perfect Polish space, \( T \) a homeomorphism of \( X \) and \( G \) a Polish group. A continuous map \( f : X \to G \) defines a \( \mathbb{Z} \)-cocycle by:

\[
f(n, x) = \begin{cases} 
  f(T^{n-1}x) \cdot \ldots \cdot f(Tx) \cdot f(x) & \text{if } n \geq 1, \\
  e & \text{if } n = 0, \\
  f(-n, T^n x)^{-1} & \text{if } n < 0,
\end{cases}
\]

A skew product action is a continuous \( \mathbb{Z} \)-action on \( X \times G \) defined by:

\[
T^n_f(x, g) = (T^n x, f(n, x) \cdot g)
\]

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A cocycle $f$ is said to be recurrent if the skew product $T_f$ is topologically conservative (i.e. for every open nonempty $O \subset X \times G$ there exists an integer $n \neq 0$ with $T^nO \cap O \neq \emptyset$). $f$ is called ergodic if the skew product $T_f$ is topologically ergodic.

We will denote by $\mathcal{R}_T$ the equivalence relation on $X$ generated by $T$: $\mathcal{R}_T = \{(T^n x, x) : x \in X, n \in \mathbb{Z}\}$ and by $\overline{\mathcal{R}}_T$ the equivalence relation (called the generic ergodic decomposition $(\mathbb{W}_G, \mathbb{K}e(0))$) on $X$ defined by: $(x, y) \in \mathcal{R}_T \iff T^2 x = T^2 y$.

Let us recall the definition of generic Mackey action associated with a cocycle (see [GK]). We formulate it here for our situation when the base transformation group is $\mathbb{Z}$ (acting by $T$) and the cocycle $f : X \to G$ is continuous. Let $\Omega = (X \times G)/\overline{\mathcal{R}}_T$, denote the factor-space with the factor-topology and $\phi : X \times G \to \Omega$ the factor-map. The following properties of $\Omega$ are described in [GK]: $\Omega$ is a Baire, second countable $T_0$-space (not necessarily Hausdorff). The map $\phi$ is open, for any meager $S \subset \Omega$, $\phi^{-1}(S)$ is meager too, and for any meager $\overline{\mathcal{R}}_T$-invariant $L \subset X \times G$, $\phi(L)$ is meager. The Borel structure on $\Omega$ generated by its topology is standard. Let $V(G)$ be the right translation action on $X \times G$: $V(g)(x, h) = (x, hg^{-1})$.

**Definition 1.1.** ([GK]) The action $W_f(G)$ of the group $G$ on the space $\Omega$ defined by

$$W_f(g)\omega = \phi(V(g)\omega),$$

where $y \in \phi^{-1}(\omega)$, $\omega \in \Omega, g \in G$, is called the generic Mackey action associated with the cocycle $f$.

It is shown in [GK] that $W_f(G)$ is a continuous action and it is minimal iff $T$ is. We will denote by $G_\omega$, where $\omega \in \Omega$, the stability group at the point $\omega$: $G_\omega = \{g \in G : W_f(g)\omega = \omega\}$. Regardless of the fact that $\Omega$ can be even not $T_1$-space every $G_\omega$ is closed (see [GK]).

From now we will suppose that $T$ is minimal. A cocycle $f$ is called generically regular if, modulo a meager subset of $X$, it is (Borel) cohomologous to a (generally Borel) ergodic cocycle taking values in a closed subgroup of $G$ ([GK]). This property has been investigated in [GK] and it was shown there that the generic regularity of a cocycle $f$ is equivalent to the essential transitivity of the generic Mackey action $W_f(G)$ (i.e., modulo a meager subset of $\Omega$, $W_f(G)$ is a transitive action).

2. Regularity and essential ranges: a generic approach

The following definition of the notion of topological essential range comes from [GH] but we give it here in a more general situation.

Let $G$ be a Polish group, $f : X \to G$ a continuous cocycle of a minimal Polish system $(X, T)$.

**Definition 2.1.** The local essential range $E_x(f)$ at the point $x \in X$ is defined by: $g \in E_x(f)$ if for any open neighborhood $U = U(g)$ and any open neighborhood $O = O(x)$ there exists $n \neq 0$ such that the set $O \cap T^{-n}O \cap \{x : f(n, x) \in U\}$ is nonempty.

As in [GH] we will denote by $P_x(f)$ the set $\{g \in G : (x, g) \in T^2_f(x, e)\}$. Then $P_x(f) \subset E_x(f)$ for every $x \in X$.

The unit $e$ belong to the essential range $E_x$ iff the cocycle $f$ is recurrent. We will always consider below recurrent cocycles.

Now let us consider the following family of closed subgroups of $G$:

$$\{G_{\phi(x, e)}\}_{x \in X}$$
which are stabilizers of the generic Mackey action associated to \( f \) at points of the form \( \phi(x,e) \). One easily sees that it satisfies the conjugacy equation:
\[
G_{\phi(T^n x,e)} = f(n,x) \cdot G_{\phi(x,e)} \cdot f(n,x)^{-1},
\]
for all \( x \in X \) and \( n \in \mathbb{Z} \).

The following proposition shows that on a dense \( G_0 \)-subset of \( X \) this family coincides with the family of essential ranges \( \{ E_x \}_{x \in X} \). Thus, we get an alternative generic definition of the notion of local essential range given in terms of an associated generic Mackey action. By the way, the proof presented below is a different proof of the fact that the topological essential ranges are groups on a comeager subset of \( X \) and it fits for a general Polish situation (cf. [GH, Proposition 1.1]), where it is assumed \( X \) is compact and \( G \) is locally compact).

**Proposition 2.2.** There exists an invariant dense \( G_0 \)-set \( X_0 \subset X \) such that for every \( x \in X_0 \) one has \( E_x = P_x = G_{\phi(x,e)} \).

**Proof.** Observe that \( G_{\phi(x,e)} \subset P_x \) for any \( x \in X \). Indeed, \( g^{-1} \in G_{\phi(x,e)} \) means that \( V(g^{-1})R_f(x,e) = R_f(x,e) \) so \( (x,g) \in T^n_f(x,e) \). Thus it suffices to prove that \( E_x \subset G_{\phi(x,e)} \) on some dense \( G_0 \)-subset. Let \( \Omega = (X \times G)/R_f \) be the topological factor-space. By [GK, Proposition 6] there exists a dense \( G_0 \)-subset \( \Omega_0 \subset \Omega \) such that \( \Omega_0 \) is a Polish space. Then \( Y_0 = \phi^{-1}(\Omega_0) \) is dense \( G_0 \) in \( X \times G \). By the topological Fubini theorem there exists a dense \( G_0 \)-set \( X_0 \subset X \) with the property \( \{ g \in G : (x,g) \in Y_0 \} \) is comeager in \( G \) for every \( x \in X_0 \). We claim that \( E_x \subset G_{\phi(x,e)} \) on \( X_0 \). Fix \( x_0 \in X_0 \) and suppose \( g \in E_{x_0} \). Let us denote \( A_x = \{ g \in G : (x,g) \in Y_0 \} \). It follows from the definition of the essential range that there exist sequences \( \{ y_k \}, \{ n_k \} \) with \( y_k \to x_0, T^{n_k} y_k \to x_0 \) and \( f(n_k, y_k) \to g \). Moreover, one can choose \( \{ y_k \} \subset X_0 \) as for open \( O \subset X \), \( U \subset G \) and \( n \in \mathbb{Z} \) the set \( O \cap T^{-n} O \cap \{ x \in X : f(n,x) \in U \} \) is open, so in case of it is nonempty it intersects \( X_0 \). Then \( A = \bigcap_A A_{y_k} \cap A_{x_0} \) is comeager in \( G \). By Pettis theorem (see [Ke93, 9.9]) \( A \cdot A^{-1} = G \), so \( g \) can be represented of the form \( g = g'' \cdot g'''^{-1} \) where \( g', g'' \in A \). Then all the points \( (y_k, g'), T^{n_k}_f(y_k, g'), (x_0, g'), (x_0, g'') \) belong to \( Y_0 \) so their corresponding images under \( \phi \) belong to \( \Omega_0 \). Since \( \phi(y_k, g') \to \phi(x_0, g') \), \( \phi(T^{n_k}_f(y_k, g')) \to \phi(x_0, g'') \) and \( \phi(y_k, g') = \phi(T^{n_k}_f(y_k, g')) \) we conclude that \( \phi(x_0, g') = \phi(x_0, g'') \) as \( \Omega_0 \) is already a Hausdorff space. This yields \( W(g) \phi(x,e) = \phi(x,e) \), i.e. \( g \in G_{\phi(x,e)} \). \( \square \)

Thus, the only difference between \( G_{\phi(x,e)} \) and \( E_x \) can occur on a meager subset of \( X \). But even for these exceptional points we have the inclusion \( G_{\phi(x,e)} \subseteq E_x \). Note also that the properties \( E_x = E_x^{-1} \), \( E_x \) is closed and \( E_x T^n = f(n,x) \cdot E_x \cdot f(n,x)^{-1} \) are holds on the whole space \( X \).

Assume now that \( G \) is locally compact. Then the mapping \( x \mapsto G_{\phi(x,e)} \) from \( X \) to the space \( (S, \mathcal{G}) \) of closed subgroups of \( G \) with the Fell topology is Borel and therefore it is continuous on a dense \( G_0 \)-subset of \( X \).

We recall the definition of regularity for cocycles introduced in [GH]: let \( (X,T) \) be a minimal compact system, then a continuous cocycle \( f : X \to G \) is called regular if the skew product \( T_f \) admits a surjective orbit closure closure, i.e. there exists a point \( (x_0, g_0) \in X \times G \) such that \( \pi_X(T_f(x_0, g_0)) = X \). Such a definition differs from that given in [LM], where a stronger condition must be fulfilled.
Proposition 2.3. Let $f$ be a continuous cocycle over a minimal compact system $(X,T)$ with values in a locally compact group $G$. Then $f$ is regular if and only if it is generically regular.

Proof. Let $\Omega = (X \times G)/{\hat{R}_f}$. Suppose $f$ is regular. Let $C$ be a surjective orbit closure so that $C = \overline{T^x_f(x,e)}$ for some $x \in X$. It follows $V(G)C = X \times G$ and it is routine to verify that then $V(G){\hat{R}_f}|[(x,e)]$ is comeager in $X \times G$. This implies that each essential range is an almost connected subgroup. Then, as the generic Mackey action $W_f(G)$ is essentially transitive and hence $f$ is generically regular (see [GK]).

Conversely, suppose $f$ is generically regular. Then, again by [GK] Prop. 17, the action $W_f(G)$ is essentially transitive so there exists $\omega_0 \in \Omega$ with $W_f(G)\omega_0$ comeager in $\Omega$. This yields $V(G){\hat{R}_f}|[(x_0,g_0)]$ is comeager in $X \times G$ for some $(x_0,g_0) \in \phi^{-1}((x,0))$. Hence $\pi({\hat{R}_f}|[(x_0,g_0)])$ is comeager in $X$, where $\pi$ denotes the projection $X \times G \longrightarrow X$. Let $C = \overline{T^x_f(x_0,g_0)}$. Then $\pi(C)$ is $T$-invariant and comeager subset of $X$. Let $\{K_n\}_{n \geq 1}$ be a countable family of compact subsets of $G$ with $G = \bigcup_n K_n$. Then each set $F_n = \pi((X \times K_n) \cap C)$ is compact and their union $\bigcup_{n \geq 1} F_n = \pi(C)$ is comeager. It follows from Baire’s category theorem that for some $m \in \mathbb{N}$ $F_m$ contains a non-empty open subset $O$ of $X$. Since $(X,T)$ is a minimal compact system we have $X = \bigcup_{i=1}^k T^i(O)$ for some $k \in \mathbb{N}$ and as $\pi(C)$ is $T$-invariant we conclude $\pi(C) = X$. The latter means that $f$ is regular. □

It easily follows from the above proposition and the generic definition of essential range that for a regular cocycle there exists a dense $G_δ$-subset of $X$ on which all the essential ranges are conjugate (cf. [GH] Theorem 2.2).

Remark 2.4. Note that the smoothness of a generic Mackey action would imply regularity of a cocycle in our situation. Furthermore, let us consider the case when $G$ is a connected Lie group. Suppose we know that, modulo a meager subset of $X$, each essential range is an almost connected subgroup. Then, as the generic Mackey action $W_f(G)$ is ergodic with respect to the $σ$-ideal of meager sets, we are in the assumptions of [Da] Corollary 4.3] which imply that on a comeager subset of $X$ all essential ranges are automorphic in $G$. If, additionally, one of the conditions (i), (ii), (iii), (iv) of [Da] Corollary 4.4] is satisfied (for instance, $G$ is almost algebraic or $E_x$ is compact for all $x$ from a comeager subset of $X$) we conclude that all essential ranges are conjugate on a comeager subset of $X$.

Corollary 2.5. Suppose $f$ is a regular continuous cocycle over a compact minimal system $(X,T)$ with values in a locally compact group $G$. Then the following is true:

1. $f$ is Borel cohomologous to a (Borel) cocycle $\tilde{f}$ which takes values in a closed subgroup $H$ of $G$ and is ergodic in $H$.

2. If $\tilde{f}$ is a continuous cocycle which is Borel cohomologous to $f$ then $\tilde{f}$ is regular.

Proof. The first part follows immediately from 2.3. The second part is a consequence of 2.3 and the fact that the generic Mackey action is invariant under the cohomology equivalence ([GK] Prop. 11)]. □

Remark 2.6. Since the generic Mackey action is an invariant of orbit equivalence for cocycles (see [GK]) we have that the assertion (2) of the corollary remains to be true even in case of $f$ is (continuously or generically) orbit equivalent to $\tilde{f}$.
The next technical result permits us to reduce the question of regularity to consideration of a factor cocycle in the special situation of factorization by a compact subgroup. For the proof of it we provide here the generic arguments which, actually, work also for the case of a general Polish group $G$.

**Proposition 2.7.** Let $G$ be a locally compact group, $f : X \to G$ a continuous cocycle over a minimal compact system $(X, T)$. Suppose $K$ is a compact normal subgroup of $G$ such that the factor cocycle $f_K = f/K$ is regular. Then $f$ is regular.

**Proof.** The cocycle $f_K$ is cohomologous to an ergodic cocycle $\varphi = b(Tx)f_K(x)b(x)^{-1}$ with values in a closed subgroup $H$ of $G/K$. Let $\mathcal{R}_T$ be the equivalence relation on $X$ generated by $T$. It follows from [GKS Th. 3.3, 4.3] that there exists a cocycle $\psi$ defined on $\mathcal{R}_T$ which is orbit equivalent to $\varphi$ with $\text{Ker}\psi$ being an ergodic subrelation of $\mathcal{R}_T$. Let $s : G/K \to G$ be a Borel section with $\pi_K \circ s = \text{id}$, where $\pi_K : G \to G/K$ is the projection. Then the values of the cocycle $\tilde{\psi} = (s \circ b)(Tx)\psi(x)(s \circ b)(x)^{-1}$ on the ergodic subrelation $\text{Ker}\psi$ belong to $K$. This yields $\tilde{\psi}|_{\text{Ker}\psi}$ is generically regular ([GK Prop. 20]) and hence $\tilde{\psi}$ is. Note that $f$ is generically orbit equivalent to $\tilde{\psi}$ to complete the proof. $\square$

### 3. Minimal Rotations

We turn here to the case when the base dynamical system is a minimal rotation on a compact monothetic group $X$, i.e. $Tx = ax$, where $\{a^n : n \in \mathbb{Z}\}$ is dense in $X$.

**Theorem 3.1.** Let $G$ be a discrete countable group. Suppose $f : X \to G$ is a continuous recurrent cocycle over a minimal compact group rotation $(X, T)$. Then $f$ is regular. Moreover, $f$ is (continuously) cohomologous to a cocycle $\hat{f}$ taking values in a finite subgroup $K \subset G$ and $\hat{f}$ is ergodic.

**Proof.** Let $\rho$ be the invariant metric on $X$ and $d_0$ a metric on $G$ defined by: $d_0(g_1, g_2) = 1$ if $g_1 \neq g_2$, and $d_0(g_1, g_2) = 0$ if $g_1 = g_2$. Let $d$ be a metric on $X \times G$ by:

$$d((x_1, g_1), (x_2, g_2)) = \rho(x_1, x_2) + d_0(g_1, g_2)$$

Note that since $f$ is uniformly continuous there exists $\delta > 0$ such that for all $x_1, x_2 \in X$ with $\rho(x_1, x_2) < \delta$ and $g_1, g_2 \in G$ one has: $d(T_f(x_1, g_1), T_f(x_2, g_2)) = d((x_1, g_1), (x_2, g_2))$.

Suppose now that $T_f^{n_k}(x, g) \to (y, h)$ when $n_k \to \infty$ for some $x, y \in X, g, h \in G$. Given any $0 < \varepsilon < \delta$ there exists $N > 1$ with $\rho(T^{n_k}x, y) < \varepsilon$ and $f(n_k, x)g = h$ for all $k > N$. Then, as $\rho(T^jx, T^{-n_k+j}y) = \rho(T^{n_k}x, y)$ for all $0 \leq j \leq n_k$, we have

$$d((x, g), T_f^{n_k}(y, h)) = d(T_f(x, g), T_f^{n_k+1}(y, h)) = \ldots = d(T_f^{n_k}(x, g), (y, h)) < \varepsilon$$

It follows $T_f^{n_k}(y, h) \to (x, g)$. The latter yields that every orbit closure $\overline{T_f(x, g)}$, $((x, g) \in X \times G)$ is minimal under $T_f$. Since $f$ is recurrent one may assume without loss of generality that there exists a positive sequence $\{m_k\}_{k>1} \subset \mathbb{Z}^+$ such that $T_f^{-m_k}(x, g) \to (x, g)$ for some $(x, g) \in X \times G$. By the same argument as above $T_f^{-m_k}(x, g) \to (x, g)$ so the point $(x, g)$ is recurrent in the terminology of [GoHe] (i.e. $(x, g)$ belongs to both positive and negative orbit closures of $(x, g)$). This together with minimality of $(\overline{T_f(x, g)}, T_f)$ implies $\overline{T_f(x, g)}$ is compact ([GoHe] Theorem
7.05), so \( f \) is regular \((GH)\). By virtue of \([LM\) Proposition 2.1\] there exists a compact subgroup \( K \) (which equals \( E_x \) in our case) of \( G \) and a continuous map \( \gamma : X \to G/K \) such that \( \gamma(Tx) = f(x)\gamma(x) \) for all \( x \in X \). Let \( s : G/K \to G \) be a section (so that \( \pi_K \circ s = id \), where \( \pi_K : G \to G/K \) is the projection). Put \( b = s \circ \gamma \) and \( \hat{f} = b(Tx)^{-1}f(x)b(x) \). Then one verifies that \( \hat{f} \) satisfies the conditions of the theorem. \( \square \)

**Corollary 3.2.** Let \( G \) be a discrete countable group without finite subgroups. Let \( f \) be any continuous cocycle over a minimal compact group rotation \((X,T)\) with values in \( G \). Then either \( f \) is a coboundary or the skew product action is an action with only discrete orbits.

**Corollary 3.3.** Let \((X,T)\) be a minimal compact group rotation. Suppose \( G \) is a locally compact s.c. group such that there exists a compact normal subgroup \( K \subset G \) with \( G/K \) discrete (in particular, when \( G \) is a Lie group with the compact identity component or a totally disconnected abelian group). Then every recurrent cocycle \( f : X \to G \) is regular and is (continuously) cohomologous to an ergodic cocycle taking values in a compact subgroup of \( G \).

**Proof.** The proof repeats the arguments of \(3.1\) \( \square \)

**Corollary 3.4.** Let \((X,T)\) be a minimal compact group rotation. Suppose \( G \) is a Lie group or a locally compact s.c. abelian group. Then every recurrent cocycle \( f : X \to G \) is (continuously) cohomologous to a cocycle taking values in an almost connected (closed) subgroup \( H \) of \( G \). In particular, there are no continuous ergodic cocycles over \((X,T)\) taking values in a Lie group with infinitely many connected components.

**Proof.** Let \( G^0 \) be the identity component of \( G \) and \( f_0 : X \to G/G^0 \) the factor cocycle. Suppose, for a start, \( G \) is a Lie group. By \(3.1\) we have \( f_0 = b(Tx)f_0b(x)^{-1} \), where \( f_0 \) is a cocycle taking values in a finite subgroup \( F \) of \( G/G^0 \) and \( b : X \to G/G^0 \) is continuous. Let \( s : G/G_0 \to G \) be the section. Let \( s \circ b \). Let    \( (s \circ b)(Tx)f(x)(s \circ b)(x)^{-1} \). Then evidently \( f \) satisfies the conditions of our assertion.

Now suppose \( G \) is a locally compact abelian group. Then \( G/G^0 \) is totally disconnected so there exists a compact subgroup \( K \subset G/G^0 \) with \((G/G^0)/K \) being discrete. By virtue of \(3.1\) one may assume that the cocycle \( f_K \) defined by \( f_K = f_0|K \) takes all its values in a finite subgroup of \((G/G^0)/K \). Thus \( f_0 \) takes values in a compact subgroup of \( G/G^0 \) and the similar argument as above implies our assertion. \( \square \)

We complete with the following result on regularity of cocycles over an arbitrary minimal rotation on a compact group with values in an arbitrary locally compact abelian group. In \([Me\) the similar result was proved for the case when \( G \) does not contain compact subgroups, however it was used there a little different notion of regularity (see \([Me\ 2.7\]). The case of locally connected compact group \( X \) is covered by \([GH\) Theorem 4.1].

**Theorem 3.5.** Let \((X,T)\) be a minimal compact group rotation, \( G \) a locally compact s.c. abelian group. Then every recurrent cocycle \( f : X \to G \) is regular.

**Proof.** There exists a compact subgroup \( K \subset G \) with \( G/K \) being a Lie group. So, in view of \(2.4\) it suffices to prove our assertion for the case \( G = \mathbb{R}^n \times \mathbb{T}^k \times D \), where
$D$ is discrete. The application of [3.4] allows us to suppose that $D$ is finite and, again, by [2.7] the situation is reduced to the case $G = \mathbb{R}^n$. The latter was shown in [Me] but we give here a slightly different argument to illustrate our approach. Recall that in the abelian case all the essential ranges are the same and equal to a closed subgroup $E(f)$ of $G$ ([LM, GH]). So $E(f) \cong \mathbb{R}^k \times \mathbb{Z}^m$. If $k \neq 0$ lets consider the factor cocycle $\tilde{f} : X \to G/E(f) \cong \mathbb{R}^{n-m-k} \times \mathbb{T}^m$. Evidently, the existence of a surjective orbit closure for $\tilde{f}$ would imply the regularity of $f$. So, arguing as before, one may assume without loss of generality that $\tilde{f}$ is a cocycle with values in $\mathbb{R}^{n-m-k}$. As we know that any recurrent $\mathbb{R}$-valued cocycle is regular ([LM, Theorem 1]) the application of a standard inductive argument completes the proof.

\[\square\]

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