1 Introduction

The incomplete gamma function and incomplete beta function are important special functions in statistics. In modern theory of special functions, we regard hypergeometric functions as pairings of twisted cycles and twisted cocycles [1, Chap 2]. However, domains of integrations of these incomplete functions are not cycles. Solutions of $\mathcal{A}$-hypergeometric systems ([3], [7, p.221]) have integral representations and domains of integrations are cycles. Some important integrals with parameters such as marginal likelihood functions in statistics, e.g., [4], look like $\mathcal{A}$-hypergeometric, but domains of integrations are not cycles. Being motivated with these functions, we want to generalize the theory of $\mathcal{A}$-hypergeometric systems to that for incomplete functions. We also hope that our theorems and formulas are useful for numerical and asymptotic evaluations of incomplete $\mathcal{A}$-hypergeometric functions.

This paper is a first step toward this direction. We make general discussions as well as detailed discussions on $\Delta_1 \times \Delta_1$-incomplete hypergeometric functions. A definition of incomplete generalized hypergeometric functions is given by Chardhry and Qadir [2]. A study of relations of these two definitions is a future problem.

2 General definition

Let $D$ be the Weyl algebra in $n$ variables. A multi-valued holomorphic function $f$ defined on a Zariski open set in $\mathbb{C}^n$ is called a holonomic function if there exists a left ideal $I$ of $D$ such that (1) $D/I$ is holonomic and (2) $I \cdot f = 0$.

We denote by $A = (a_{ij})$ a $d \times n$-matrix whose elements are integers. We suppose that the set of the column vectors of $A$ spans $\mathbb{Z}^d$. 

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Definition 1 We call the following system of differential equations $H_A(\beta, g)$ an incomplete $A$-hypergeometric system:

$$
\left( \sum_{j=1}^{n} a_{ij} x_j \partial_i - \beta_i \right) \cdot f = g_i, \quad (i = 1, \ldots, d)
$$

$$
\left( \prod_{i=1}^{n} \partial_i^{a_i} - \prod_{j=1}^{n} \partial_j^{b_j} \right) \cdot f = 0
$$

with $u, v \in \mathbb{N}_0^n$ running over all $u, v$ such that $Au = Av$.

Here, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{C}^d$ are parameters and $g = (g_1, \ldots, g_d)$ where $g_i$ are given holonomic functions which may depend on parameters $\beta$. We call solutions of the incomplete $A$-hypergeometric system incomplete $A$-hypergeometric functions.

Although we have introduced the incomplete $A$-hypergeometric system for arbitrary holonomic functions $g_i$, $g_i$ are often also solutions of smaller incomplete $A$-hypergeometric system or well-known special functions in interesting cases.

Example 1 The incomplete beta function is defined as

$$
B(\alpha, \beta; y) = \int_0^y s^{\alpha-1} (1 - s)^{\beta-1} ds
$$

Replace $s = yt$. Then, we have $B(\alpha, \beta; y) = y^\alpha \int_0^1 t^{\alpha-1}(1 - yt)^{\beta-1} dt$. Put $B(\alpha, \beta; y) = y^\alpha \tilde{B}(\beta-1, \alpha-1; 1, -y)$ where $B(\beta-1, \alpha-1; x_1, x_2) = \int_0^1 t^{\alpha-1}(x_1 + x_2 t)^{\beta-1} dt$. The function $\tilde{B}$ is a solution of an incomplete $A$-hypergeometric system $H_A(\beta, g)$ for $A = \left( \begin{array}{cc}
1 & 1 \\
0 & 0 \\
0 & 1
\end{array} \right)$, $\beta = (\beta-1, \alpha-1, 1)$, $g_1 = 0, g_2 = t^\alpha(x_1 + x_2 t)^{\beta-1}$.

We will revisit this example in Example 3.

Example 2 The incomplete elliptic integral of the first kind is defined as

$$
F(z; k) = \int_0^z \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.
$$

Replacing $x^2$ by $z^2 t$, we obtain

$$
F(z; k) = \frac{1}{2} z \int_0^1 t^{-\frac{1}{2}} (1 - z^2 t)^{-\frac{1}{2}} (1 - k^2 z^2 t)^{-\frac{1}{2}} dt.
$$

Hence, the incomplete elliptic integral of the first kind can be regarded as a solution restricted to a subvariety of incomplete $A$-hypergeometric system $H_A(\beta, g)$ for $A = \left( \begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0
\end{array} \right)$, $\beta = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. See the Section 4 on the incomplete $\Delta_1 \times \Delta_1$-hypergeometric system on $g_i$. We will revisit this example in Example 4. Add a picture of $\Delta_1 \times \Delta_1$ in $\mathbb{R}^3$. 

2
**Theorem 1** Solutions of the incomplete \( A \)-hypergeometric system are holonomic functions.

**Proof.** Since the \( A \)-hypergeometric left ideal \( H_A(\beta) \subset D \) is holonomic ideal, the function \( f \) satisfies the ordinary differential equation of the form

\[
\left( \sum_{k=0}^{r_i} a_k(\beta, x) \partial^k_i \right) \cdot f = \sum_{i=1}^{d} \ell_i \cdot g_i,
\]

where \( \ell_i \in D \). Since \( g := \sum_{i=1}^{d} \ell_i \cdot g_i \) is also a holonomic function, there exists an ordinary differential operator such that \( \left( \sum_{k=0}^{r_i} b_k(\beta, x) \partial^k_i \right) \cdot g = 0 \). Hence, \( f \) is annihilated by an ordinary differential operator with parameters of the order \( r_i r_i' \) with respect to \( x_i \) and then \( f \) is annihilated by a zero dimensional ideal \( J \) in \( C(x_1, \ldots, x_n)(\partial_1, \ldots, \partial_n) \) [7, p.33]. Since \( D/(J \cap D) \) is a left holonomic \( D \)-module (see, e.g., [9, Th 2.4], [7, p.34]), we have the conclusion. We note that this construction can be done algorithmically [5].

### 3 Algorithms deriving contiguity relations

In this section, we will give algorithms to obtain contiguity relations for incomplete \( A \)-hypergeometric functions under the condition that for all \( i \) \( (1 \leq i \leq d) \) there exists a constant \( c_k \in C \setminus \{0\} \) such that

\[
\partial_k \cdot g_i(\beta) = c_k^{-1} g_i(\beta - a_k).
\]

This condition holds in many interesting cases. We mean by a contiguity relation, a relation among two incomplete \( A \)-hypergeometric functions \( \Phi(\beta; x) \) and \( \Phi(\beta'; x) \) where \( \beta - \beta' \in \mathbb{Z}^d \) and \( \Phi(\beta; x) \) is a solution of \( H_A(\beta, g) \). Under the condition (1), we note that

\[
c_k^{-1} \partial_k \cdot \Phi(\beta; x) =: \Phi(\beta - a_k; x)
\]

is a solution of \( H_A(\beta - a_k, g) \). In other words, the operator \( \partial_k \) gives a contiguity relation for the parameter shift \( -a_k \). This contiguity relation follows from

\[
\partial_k \left( \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \right) \cdot f
\]

\[
= \left( \sum_{j=1}^{n} a_{ij} x_j \partial_j - (\beta_i + a_{ik}) \right) \partial_k \cdot f = \partial_k \cdot g_i(\beta)
\]

and the condition (1).

If we can find the inverse operator for \( \partial_k \), then it means that we find “generators” of the all contiguity relations because the matrix \( A \) is the full rank.
In [6] and [7], they give algorithms to find the inverse operator for $\partial_k$ in case of $A$-hypergeometric system. We utilize their algorithms to find the inverse operator. Let us recall Definition 1. Let $h_i (1 \leq i \leq d)$ be $\sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i$ respectively and $h_i (d+1 \leq i \leq s)$ be generators of the toric ideal $I_A$ of the form $\prod_{j=1}^n \partial_i^{\mu_j} - \prod_{j=1}^n \partial_j^{\nu_j}$. It follows from [6] and [7] that there exist $r, r_i \in D(\beta)$ such that
\[ r \partial_k + \sum_{i=1}^s r_i h_i = 1. \]
Applying the operator of the left hand side to $\Phi(\beta; x)$, we obtain the contiguity relation
\[ c_k r \cdot \Phi(\beta - a_k; x) + \sum_{i=1}^d r_i \cdot g_i = \Phi(\beta; x) \]
which contains the functions $g_i$.

Our second algorithm gives contiguity relations which do not contain the functions $g_i$. Suppose that we are given operators $\tilde{h}_i$ such that $\tilde{h}_i \cdot g_i = 0$ and operators $\partial_k, \tilde{h}_i h_i (1 \leq i \leq d)$, and $h_i (d+1 \leq i \leq s)$ generate a trivial ideal in $D(\beta)$. Then, we can construct operators $r, r_i \in D(\beta)$ such that
\[ r \partial_k + \sum_{i=1}^d r_i \tilde{h}_i h_i + \sum_{i=d+1}^s r_i h_i = 1 \]
by the Gröbner basis method. The operator $c_k r$ is the inverse of $\partial_k$.

We will apply these algorithms to obtain a complete list of contiguity relations of $\Delta_1 \times \Delta_1$-hypergeometric functions in Section 4.

4 Incomplete $\Delta_1 \times \Delta_1$-hypergeometric functions

We assume $0 < a < b$ for simplicity. We consider the integral
\[ \int_a^b t^\gamma (x_{11} + x_{21} t)^{\alpha_1} (x_{12} + x_{22} t)^{\alpha_2} dt \] (2)
for $x_{ij} > 0$ and $\Re \gamma, \Re \alpha_i > -1$. The integral and its analytic continuations satisfy the following incomplete $A$-hypergeometric system.
\[
\begin{align*}
\left( \theta_{11} \theta_{22} - \frac{x_{11} x_{22}}{x_{21} x_{12}} \theta_{21} \theta_{12} \right) \cdot f &= 0 \\
(\theta_{11} + \theta_{21} - \alpha_1) \cdot f &= 0 \\
(\theta_{12} + \theta_{22} - \alpha_2) \cdot f &= 0 \\
(\theta_{21} + \theta_{22} + \gamma + 1) \cdot f &= [g(t, x)]_{t=a}^{t=b} \quad (3)
\end{align*}
\]
Here, $g(t, x) = t^{\gamma+1} (x_{11} + x_{21} t)^{\alpha_1} (x_{12} + x_{22} t)^{\alpha_2}$. This fact can be shown by exchanging the integral and differentiations (see, e.g., [7, p.221]). When $[g(t, x)]_{t=a}^{t=b} = 0$, our system is essentially the Gauss hypergeometric equation (see, e.g., [7, Ch 1]).
Since the matrix $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ associated to the system can be regarded as $\Delta_1 \times \Delta_1$, we call it incomplete $\Delta_1 \times \Delta_1$-hypergeometric system and its solutions are incomplete $\Delta_1 \times \Delta_1$-hypergeometric functions.

In order to make a rigorous discussion, we need to specify branches of multivalued functions appearing in our discussion. In the sequel, $z^\alpha$ denotes the unique analytic continuation of the function $z^\alpha$ defined on $z > 0$ to the upper and the lower half plane as long as we make no annotation.

**Remark 1** Under this definition, we have $(zw)^\alpha = e^{2i\pi z^\alpha w^\alpha}$ for $\text{Im } z < 0, \text{Im } w < 0$ and $\text{Im } zw > 0$, $(zw)^\alpha = e^{-2i\pi z^\alpha w^\alpha}$ for $\text{Im } z > 0, \text{Im } w > 0$ and $\text{Im } zw < 0$, and $(zw)^\alpha = z^\alpha w^\alpha$ for other cases.

We take real numbers $x_{ij}^* > 0$ such that

$$0 < \frac{x_{21}^*}{x_{22}^*} < \frac{x_{11}^*}{x_{21}^*} < a < b < \frac{x_{21}^*}{x_{21}^*} < \frac{x_{22}^*}{x_{21}^*}$$

We consider the simply connected domain define by

$$(-1)^{d_1} \text{Im } x_{11} > 0, (-1)^{d_2} \text{Im } x_{21} > 0, (-1)^{d_3} \text{Im } x_{12} > 0, (-1)^{d_4} \text{Im } x_{22} > 0,$$

$$(-1)^{d_5} \text{Im } x_{21}/x_{11} > 0, (-1)^{d_6} \text{Im } x_{22}/x_{12} > 0, (-1)^{d_7} \text{Im } \frac{x_{21}x_{12}}{x_{11}x_{22}} > 0$$

Here, $d_i$ takes the values 0 or 1. Since we assume $a, b \in \mathbb{R}$ and the singular locus of the homogeneous $A$-hypergeometric system is $x_{11}x_{21}x_{12}x_{22}(x_{11}x_{22} - x_{21}x_{12}) = 0$, the solutions of our system are holomorphic on each of these domains. We denote by $D_d$ where $d = (d_1, \ldots, d_7) \in \{0, 1\}^7$ the simply connected domain standing for $d$.

We define the four domains as follows

$$D_{12}^{11} = \{x_{ij} \mid |x_{21}b/x_{11}| < 1, |x_{21}a/x_{11}| < 1, |x_{22}b/x_{12}| < 1, |x_{22}a/x_{12}| < 1\}$$

$$D_{12}^{12} = \{x_{ij} \mid |x_{21}b/x_{11}| < 1, |x_{21}a/x_{11}| < 1, |x_{12}/(x_{22}b)| < 1, |x_{12}/(x_{22}a)| < 1\}$$

$$D_{12}^{21} = \{x_{ij} \mid |x_{11}/(x_{21}b)| < 1, |x_{11}/(x_{21}a)| < 1, |x_{22}b/x_{12}| < 1, |x_{22}a/x_{12}| < 1\}$$

$$D_{12}^{22} = \{x_{ij} \mid |x_{11}/(x_{21}b)| < 1, |x_{11}/(x_{21}a)| < 1, |x_{12}/(x_{22}b)| < 1, |x_{12}/(x_{22}a)| < 1\}$$

The point $(x_{ij}^*)$ belongs to the last domain $D_{12}^{22}$. It is easy to see that each of these domains and $D_d$ has an open intersection since $a, b \in \mathbb{R}$.

**Remark 2** We note that

$$(x_{ii} + x_{2i}t)^{\alpha_i} = x_{ii}^{\alpha_i} \left(1 + \frac{x_{2i}t}{x_{ii}}\right)^{\alpha_i} = (x_{2i}t)^{\alpha_i} \left(\frac{x_{11}}{x_{2i}t} + 1\right)^{\alpha_i}$$

for $x_{ii} > 0$ and $t > 0$. This relation will be used to specify branches of $[g(t, x)]^{t=b}_{t=a}$. For example, when $(x_{ij}) \in D_{12}^{22}$, we regard $[g(t, x)]^{t=b}_{t=a}$ as

$$\left[t^{\gamma+1}(x_{21}t)^{\alpha_1} \left(\frac{x_{11}}{x_{21}t} + 1\right)^{\alpha_1} (x_{22}t)^{\alpha_2} \left(\frac{x_{12}}{x_{22}t} + 1\right)^{\alpha_2} \right]_{t=a}^{t=b}$$
of which series expansion converges on $D_{22}^{21}$. Since the domain $D_{22}^{21}$ has an open intersection with $D_d$, the function $[g(t, x)]_{t=a}^{t=b}$ has a unique analytic continuation to $D_d$.

4.1 Homogeneous system

As we have proved in Theorem 1, solutions of incomplete $\mathcal{A}$-hypergeometric systems are holonomic functions. The advantage of this point of view is that we can apply some algorithms for holonomic systems to study solutions of our system. For example, we can apply the algorithm given in the Chapter 2 of [7] to find candidates of series solutions. Holonomic systems which annihilate these functions can be obtained in an algorithmic way. However, outputs by the algorithm are sometimes tedious.

In the case of $\Delta_1 \times \Delta_1$-hypergeometric system, solutions satisfy the following relatively simple holonomic system.

$$\begin{cases}
(\theta_{11} \theta_{22} - \frac{x_{12} x_{22}}{x_{21} x_{12}} \theta_{21} \theta_{12}) \cdot f = 0 \\
(\theta_{11} + \theta_{21} - \alpha_1) \cdot f = 0 \\
(\theta_{12} + \theta_{22} - \alpha_2) \cdot f = 0 \\
(\partial_{22} - a \partial_{21})(\partial_{12} - b \partial_{11})(\theta_{21} + \theta_{22} + \gamma + 1) \cdot f = 0
\end{cases} \tag{4}$$

4.2 Contiguity relations

We will derive contiguity relations of incomplete $\Delta_1 \times \Delta_1$-hypergeometric functions by applying our two algorithms. In this section, we put $\beta = -\gamma - 1$ to make formulas of contiguity relations of the incomplete $\Delta_1 \times \Delta_1$-hypergeometric function simpler forms. We put

$$\Phi(\alpha_1, \alpha_2, \beta; x) = \int_a^b t^{-\beta - 1}(x_{11} + x_{21} t)^{\alpha_1} (x_{12} + x_{22} t)^{\alpha_2} dt.$$ 

Theorem 2 The incomplete $\Delta_1 \times \Delta_1$-hypergeometric function $\Phi(\alpha_1, \alpha_2, \beta; x)$ satisfies the following contiguity relations.

- Shifts with respect to $a_1 = (1, 0, 0)$

$$S(\alpha_1, \alpha_2, \beta; -a_1) \Phi(\alpha_1, \alpha_2, \beta) = \alpha_1 \Phi(\alpha_1 - 1, \alpha_2, \beta)$$

$$\hat{S}(\alpha_1 - 1, \alpha_2, \beta; +a_1) \Phi(\alpha_1 - 1, \alpha_2, \beta) = (\alpha_1 + \alpha_2 - \beta) \Phi(\alpha_1, \alpha_2, \beta) - [g(t, x)]_{t=a}^{t=b}$$

$$S(\alpha_1 - 1, \alpha_2, \beta; +a_1) \Phi(\alpha_1 - 1, \alpha_2, \beta) = \alpha_2(\alpha_1 + \alpha_2 - \beta) \Phi(\alpha_1, \alpha_2, \beta)$$

where

$$S(\alpha_1, \alpha_2, \beta; -a_1) = \partial_{11},$$

$$\hat{S}(\alpha_1 - 1, \alpha_2, \beta; +a_1) = (x_{21} x_{12} - x_{11} x_{22}) \partial_{22} + (\alpha_1 + \alpha_2) x_{11},$$

$$S(\alpha_1 - 1, \alpha_2, \beta; +a_1) = x_{12} x_{21} \{(a + b)x_{21} \partial_{21} \partial_{22} + x_{22} \partial_{22}^2 + (1 - \beta) \partial_{22} - ab(x_{21} \partial_{11} \partial_{22} + \partial_{12} \partial_{22} - \beta \partial_{12}) + x_{11} \partial_{21} \partial_{22} + x_{12} \partial_{22}^2 \} + (\alpha_1 + \alpha_2 - \beta)(\alpha_1 \alpha_2 x_{11} + x_{21} x_{12} \partial_{22})$$
• Shifts with respect to $a_2 = (1, 0, 1)$

$$S(a_1, a_2, \beta; -a_2) \Phi(a_1, a_2, \beta) = \alpha_1 \Phi(a_1 - 1, a_2, \beta - 1)$$

$$S(a_1 - 1, a_2, \beta - 1; +a_2) \Phi(a_1 - 1, a_2, \beta - 1) = \beta \Phi(a_1, a_2, \beta) + [g(t, x)]_{t=\alpha}$$

$$S(a_1 - 1, a_2, \beta - 1; +a_2) \Phi(a_1 - 1, a_2, \beta - 1) = ab \alpha_2 \beta \Phi(a_1, a_2, \beta)$$

where

$$S(a_1, a_2, \beta; -a_2) = \partial_{21},$$

$$S(a_1 - 1, a_2, \beta - 1; +a_2) = x_{11} x_{22} \partial_{12} + x_{21} x_{22} \partial_{22} + \alpha_1 x_{21},$$

$$S(a_1 - 1, a_2, \beta - 1; +a_2) = x_{11} x_{22} \partial_{12} + x_{21} x_{22} \partial_{22} + (a + b) x_{11} x_{22} (x_{21} \partial_{21} + x_{22} \partial_{22}) + ab x_{22} (x_{21} \partial_{21} - x_{11} \partial_{12}) + ab (a_2 + \beta - 1) x_{11} x_{22} \partial_{12} + (a_1 - \beta + 1) x_{11} x_{22} \partial_{22} - \alpha_2 (x_{11} + (a + b) x_{11} x_{21} + ab x_{21}) \partial_{21} + (a + b) \alpha_2 (\beta - 1) x_{11} + ab (a_1 + \beta + 1) x_{21}$$

• Contiguity relations with respect to $a_3 = (0, 1, 0)$ are obtained from those with respect to $a_1$ by the permutations $a_1 \leftrightarrow a_2, x_{11} \leftrightarrow x_{12}, \partial_{11} \leftrightarrow \partial_{12}$.

• Contiguity relations with respect to $a_4 = (0, 1, 1)$ are obtained from those with respect to $a_2$ by the same permutations as above.

Proof. Since the function $[g(t, x)]_{t=\alpha}^{t=b}$ satisfies the condition (1), we can apply the first algorithm for $A = (a_1, a_2, a_3, a_4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ to obtain the contiguity relations containing the function $[g(t, x)]_{t=\alpha}^{t=b}$. Contiguity relations which do not contain the function $[g(t, x)]_{t=\alpha}^{t=b}$ is obtained from (4). The generation condition of the trivial ideal generated by 1 is checked for (4) by a computer and then we can apply the second algorithm in section 3.

Theorem 2 gives contiguity relations for $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$, but it does not give these for $e_3 = (0, 0, 1)$. The set of vectors $\{e_1, e_2, e_3\}$ is the standard basis of $Z^3$. The contiguity relations for $e_3$ can be obtained from Theorem 2 as follows.

Corollary 1 The incomplete $\Delta_1 \times \Delta_1$-hypergeometric function $\Phi(a_1, a_2, \beta; x)$ satisfies the following contiguity relations.

• Shifts with respect to $e_3 = (0, 0, 1)$

$$S(a_1, a_2, \beta + 1; -e_3) \Phi(a_1, a_2, \beta + 1) = \alpha_1 \alpha_2 (a_2 + a_2 - \beta) \Phi(a_1, a_2, \beta)$$

$$S(a_1, a_2, \beta - 1; +e_3) \Phi(a_1, a_2, \beta - 1) = ab (a_2 + \beta) \Phi(a_1, a_2, \beta)$$

$$S(a_1, a_2, \beta + 1; -e_3) \Phi(a_1, a_2, \beta + 1) = \alpha_1 (a_1 + a_2 - \beta) \Phi(a_1, a_2, \beta) - \alpha_1 [g(t, x)]_{t=\alpha}^{t=b}$$

$$S(a_1, a_2, \beta - 1; +e_3) \Phi(a_1, a_2, \beta - 1) = \alpha_1 \beta \Phi(a_1, a_2, \beta) + \alpha_1 [g(t, x)]_{t=\alpha}^{t=b}$$

where
\[ S(\alpha_1, \alpha_2, \beta + 1; -e_3) = S(\alpha_1 - 1, \alpha_2, \beta; +a_1)S(\alpha_1, \alpha_2, \beta + 1; -a_2), \]
\[ S(\alpha_1, \alpha_2, \beta - 1; +e_3) = S(\alpha_1 - 1, \alpha_2, \beta - 1; +a_2)S(\alpha_1, \alpha_2, \beta - 1; -a_1), \]
\[ S(\alpha_1, \alpha_2, \beta + 1; -e_3) = S(\alpha_1 - 1, \alpha_2, \beta; +a_1)S(\alpha_1, \alpha_2, \beta + 1; -a_2), \]
\[ S(\alpha_1, \alpha_2, \beta - 1; +e_3) = S(\alpha_1 - 1, \alpha_2, \beta - 1; +a_2)S(\alpha_1, \alpha_2, \beta - 1; -a_1). \]

**Example 3** Let \( a = 0, b = 1 \). We consider the following degenerated incomplete \( \Delta_1 \times \Delta_1 \)-hypergeometric function.

\[ \Psi(\alpha_1, \beta; x) = \int_0^1 t^{-\beta-1}(x_{11} + x_{21} t)^{\alpha_1} dt \]

Then the last contiguity relation of Corollary 1 for this function is

\[ x_{21} \partial_{11} \Psi(\alpha_1, \beta - 1; x) \beta \Psi(\alpha_1, \beta; x) + (x_{11} + x_{21})^{\alpha_1}. \]

Multiplying both sides by \( x_{11} \) and by using the relation of the incomplete \( \mathcal{A} \)-hypergeometric system:

\[ x_{11} \partial_{11} \Psi(\alpha_1, \beta - 1; x) = (\alpha_1 - \beta + 1)\Psi(\alpha_1, \beta - 1; x) - (x_{11} + x_{21})^{\alpha_1}, \]

we have

\[ (\alpha_1 - \beta + 1)x_{21} \Psi(\alpha_1, \beta - 1; x) = \beta x_{11} \Psi(\alpha_1, \beta; x) + (x_{11} + x_{21})^{\alpha_1 + 1}. \]

Put \( x_{11} = 1, x_{21} = -y \) and replace \( \beta \) by \( -\alpha, \alpha_1 \) by \( \beta - 1, \) we have

\[ (\alpha + \beta)(-y)\Psi(\beta - 1, -\alpha - 1; y) = -\alpha \Psi(\beta - 1, -\alpha; y) + (1 - y)^{\beta}. \]

Multiplying both sides by \(-y^{\alpha}\), we obtain

\[ (\alpha + \beta)B(\alpha + 1, \beta; y) = \alpha B(\alpha, \beta; y) - y^{\alpha}(1 - y)^{\beta}. \]

This is a well-known relation of the incomplete beta function.
4.3 Series solutions

We define the following 4 series.

\[ f_{12}^{11} = x_{11}^{\alpha_1} x_{12}^{\alpha_2} \sum_{k,m \geq 0} \frac{(-1)^{k+m}}{\gamma + k + m + 1} \frac{(-\alpha_1)_k (-\alpha_2)_m}{(1)_k (1)_m} \]
\[ . \left( b^{\gamma+k+m+1} - a^{\gamma+k+m+1} \right) \left( \frac{x_{21}}{x_{11}} \right)^k \left( \frac{x_{22}}{x_{12}} \right)^m \]

\[ f_{22}^{11} = x_{11}^{\alpha_1} x_{22}^{\alpha_2} \sum_{k,m \geq 0} \frac{(-1)^{k+m}}{\gamma + \alpha_2 + k - m + 1} \frac{(-\alpha_1)_k (-\alpha_2)_m}{(1)_k (1)_m} \]
\[ . \left( b^{\gamma+\alpha_2+k-m+1} - a^{\gamma+\alpha_2+k-m+1} \right) \left( \frac{x_{21}}{x_{11}} \right)^k \left( \frac{x_{22}}{x_{22}} \right)^m \]

\[ f_{12}^{21} = x_{21}^{\alpha_1} x_{22}^{\alpha_2} \sum_{k,m \geq 0} \frac{(-1)^{k+m}}{\gamma + \alpha_1 - k + m + 1} \frac{(-\alpha_1)_k (-\alpha_2)_m}{(1)_k (1)_m} \]
\[ . \left( b^{\gamma+\alpha_1-k+m+1} - a^{\gamma+\alpha_1-k+m+1} \right) \left( \frac{x_{11}}{x_{21}} \right)^k \left( \frac{x_{12}}{x_{22}} \right)^m \]

\[ f_{22}^{21} = x_{21}^{\alpha_1} x_{22}^{\alpha_2} \sum_{k,m \geq 0} \frac{(-1)^{k+m}}{\gamma + \alpha_1 + \alpha_2 - k + m + 1} \frac{(-\alpha_1)_k (-\alpha_2)_m}{(1)_k (1)_m} \]
\[ . \left( b^{\gamma+\alpha_1+\alpha_2-k-m+1} - a^{\gamma+\alpha_1+\alpha_2-k-m+1} \right) \left( \frac{x_{11}}{x_{21}} \right)^k \left( \frac{x_{12}}{x_{22}} \right)^m \]

**Theorem 3** We assume that \( \gamma \notin \mathbb{Z} \).

1. The series \( f_{ij}^{kl} \) converges on the domain \( D_{ij}^{kl} \) and has a unique analytic continuation to \( D_d \). Here, \( D_{ij}^{kl} \) is defined in the beginning of this section.

2. The function \( f_{ij}^{kl} \) defined on \( D_d \) as above satisfies the incomplete \( \Delta_1 \times \Delta_1 \)-hypergeometric system for the branch of \( [g(t,x)]_{t=b}^{t=a} \) given in the Remark 2.

3. \( f_{12}^{11} \) can be expressed in terms of the Appell function \( F_1 \) as

\[ f_{12}^{11} = x_{11}^{\alpha_1} x_{12}^{\alpha_2} \left( \frac{b^{\gamma+1}}{\gamma + 1} \right) F_1 \left( \gamma + 1, -\alpha_1, -\alpha_2, \gamma + 2, \frac{-x_{21}}{x_{11}}, \frac{-x_{22}}{x_{12}} \right) \]
\[ \quad - a^{\gamma+1} \left( \frac{\gamma + 1}{\gamma + 1} \right) F_1 \left( \gamma + 1, -\alpha_1, -\alpha_2, \gamma + 2, \frac{-x_{21}}{x_{11}}, \frac{-x_{22}}{x_{12}} \right) \]

**Proof.** (Sketch) The item 1 is proved by utilizing majorant series. There exists a constant \( C \) such that

\[ C \left( \sum_{k=0}^{\infty} \frac{|(-\alpha_1)_k|}{k!} \frac{|b x_{21}|^k}{x_{11}^k} \right) \left( \sum_{m=0}^{\infty} \frac{|(-\alpha_2)_m|}{m!} \frac{|b x_{22}|^m}{x_{12}^m} \right) + C \left( \sum_{k=0}^{\infty} \frac{|(-\alpha_1)_k|}{k!} \frac{|a x_{21}|^k}{x_{11}^k} \right) \left( \sum_{m=0}^{\infty} \frac{|(-\alpha_2)_m|}{m!} \frac{|a x_{22}|^m}{x_{12}^m} \right) \]
is a majorant series of $f_{12}^{11}$. Other cases can be shown analogously.

The item 2 is proved by applying the algorithm to find series solutions for (4) given in the Chapter 2 of [7]. The Gröbner cone consists of 8 maximal dimensional cones. After constructing series solutions of the homogeneous system (4), we check if they satisfy the inhomogeneous system (3) and we find these four solutions.

The item 3 can be proved by utilizing the relation \[ \frac{1}{\gamma + k + m + 1} = \frac{1}{\gamma + 1} \frac{(\gamma + 1)k + m}{(\gamma + 2)k + m}. \]

**Example 4** As we have seen in Example 2, the incomplete elliptic integral of the first kind can be regarded as incomplete $\Delta_1 \times \Delta_1$-hypergeometric function. Let us apply Theorem 3 to obtain an expression in term of the Appell function $F_1$.

Put $x_{11} = 1, x_{21} = -z^2, x_{12} = 1, x_{22} = -k^2 z^2$ and $\alpha_1 = \alpha_2 = \gamma = -\frac{1}{2}$, $a = 0, b = 1$. Then we have

\[
F(z; k) = \frac{1}{2} z \cdot \frac{1}{-\frac{1}{2} + 1} F_1 \left( -\frac{1}{2} + 1, \frac{1}{2}, \frac{1}{2} - \frac{1}{2} + 2; \frac{z^2}{1}, \frac{k^2 z^2}{1} \right)
\]

\[= z F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{z^2}{1}, \frac{k^2 z^2}{1} \right). \]

This expression of the incomplete elliptic integral seems to be well-known [10].

Q. The common refinement of the Gröbner fan of $H_A(\beta)$ and that of Ann $[g(t, x)]_{t=0}^{t=b}$ is our fan with four cones? See Prog-ihg/gf1.sm1.

### 4.4 Connection formulas

Connection formulas for the Gauss hypergeometric functions are given on the upper half plane and on the lower half plane. We will give connection formulas of our series solutions in an analogous way.

The domain of convergence of our series solution $f_{ij}^{pq}$ and $D_d$ is non-empty and open set for any $d$, then there exists a unique analytic continuation of the series $f_{ij}^{pq}$ to the domain $D_d$. We will give connection formulas among our 4 series solutions on $D_d$.

**Theorem 4** We suppose $0 < a < b$ and exponents $\alpha_1, \alpha_2, \gamma$ are generic.

1. \begin{align*}
    f_{12}^{11} &= e^{2\pi i \alpha_1} f_{12}^{21} \quad \text{on } D_{(1,1,*,*,0,*,*)} \\
    f_{12}^{11} &= e^{-2\pi i \alpha_1} f_{12}^{21} \quad \text{on } D_{(0,0,*,*,1,*,*)} \\
    f_{12}^{11} &= f_{12}^{21} \quad \text{on other } D_d 's \\
\end{align*}

2. \begin{align*}
    f_{12}^{11} &= e^{2\pi i \alpha_2} f_{22}^{11} \quad \text{on } D_{(*,*,0,1,*,0,*)} \\
    f_{12}^{11} &= e^{-2\pi i \alpha_2} f_{22}^{11} \quad \text{on } D_{(*,*,1,0,*,1,*)} \\
    f_{12}^{11} &= f_{22}^{11} \quad \text{on other } D_d 's \\
\end{align*}
for the first hypergeometric function in (5), we utilize
placements by using the connection formula of the Gauss hypergeometric functions as follows.

I have not yet calculated the constants for other cases.

Intuitively speaking, the series \( f_{ij}^{kl} \) are different expansions of the same integral (2) in different domains and hence they will agree with some adjustments of constant factor as in the Theorem. Here, we will give a proof without using the integral representation. The advantage of this discussion is that we can avoid topological discussions about choices of branches of the integrant. Analogous discussion is used to study global behavior of solutions of the Euler-Darboux equation [8].

Proof. We note \( \frac{1}{\gamma + m + k + 1} = \frac{1}{\gamma + m + 1} \frac{(\gamma + m + 1)_{a}}{(\gamma + m + 2)_{a}} \). Then, the series \( f_{12}^{11} \) can be expressed as a superposition of contiguous family of Gauss hypergeometric functions as follows.

\[
x_{11}^{a_{1}} x_{12}^{a_{2}} \left( \sum_{m=0}^{\infty} \left( x_{22} \right)^{m} \frac{(m)_{a_{1}} (m)_{a_{2}}}{(\gamma + m + 1)m!} \right)_{F} \left( \begin{array}{c}
-a_{1}, \gamma + m + 1, -x_{21}b \\
\gamma + m + 2
\end{array} ; x_{11} \right)
- \sum_{m=0}^{\infty} \left( x_{22} \right)^{m} \frac{(m)_{a_{1}} (m)_{a_{2}}}{(\gamma + m + 1)m!} \right)_{F} \left( \begin{array}{c}
-a_{1}, \gamma + m + 1, -x_{21}a \\
\gamma + m + 2
\end{array} ; x_{11} \right)
\] (5)

The Gauss hypergeometric function has a unique analytic continuation to \( \text{Im} \ x_{21}/x_{11} > 0 \) and \( \text{Im} \ x_{21}/x_{11} < 0 \). We replace the Gauss hypergeometric functions in (5) with their series expansions around \( x_{21}/x_{11} = \infty \). In other words, we make replacements by using the connection formula of the Gauss hypergeometric function in (5). For the first hypergeometric function in (5), we utilize

\[
\begin{align*}
F \left( \begin{array}{c}
-a_{1}, \gamma + m + 1, -x_{21}b \\
\gamma + m + 2
\end{array} ; x_{11} \right)
&= \frac{\Gamma(\gamma + m + 2) \Gamma(\gamma + m + 1 + a_{1})}{\Gamma(\gamma + m + 1) \Gamma(\gamma + m + 2 + a_{1})} \left( \frac{x_{21}b}{x_{11}} \right)^{a_{1}} F \left( \begin{array}{c}
-a_{1}, 1 - a_{1} - \gamma + m - 2 \\
1 - a_{1} - \gamma - m - 1
\end{array} ; -x_{21}b \right) \\
&+ \frac{\Gamma(\gamma + m + 2) \Gamma(-a_{1} - \gamma - m - 1)}{\Gamma(-a_{1}) \Gamma(\gamma + m + 2 - \gamma - m - 1)} \left( \frac{x_{21}b}{x_{11}} \right)^{-a_{1}} F \left( \begin{array}{c}
\gamma + m + 1, 1 + \gamma + m + 1 - \gamma - m - 2, -x_{11} \\
1 + \gamma + m + 1 + a_{1}
\end{array} ; -x_{21}b \right)
\end{align*}
\]
and the analogous formula for the second Gauss hypergeometric function in (5). The terms obtained from the second terms of the connection formulas of the Gauss hypergeometric functions are canceled and we obtain

$$f_{i1}^{α_1, α_2} (x_{2i}^{α_1, α_2} \frac{F(x_{2i})}{x_{i1}}) = \sum_{m=0}^{∞} \left( \frac{-x_{22}^{α_1+γ+m+1}(-α_2)_m}{(γ + α_1 + m + 1)m!} \right) \left( \frac{-x_{21}^{1 - α_1 - γ - m - 2}}{-α_1 - γ - m} \right)^m\left( \frac{x_{21}^k}{x_{i1}} \right) \left( \frac{x_{22}^m}{x_{i2}} \right) ,$$

Expanding the Gauss hypergeometric functions, we see that the above sum equals to $x_{2i}^{α_1} f^{α_2} (x_{2i}/x_{i1}) α_{2i}^{-α_2} f^{α_2} (x_{2i}/x_{i2}).$ Applying the formulas in Remark 1, we obtain the first result 1. Other cases can be obtained analogously.

4.5 Monodromy formula

We study analytic continuation (monodromy) of the function $f_{ij}^{kl}$. We only give formula for $f_{12}^{kl}$. Formulas for other $f_{ij}^{kl}$ can be obtained analogously by symmetry. In order to give formulas, we define

$$f_{12}^{kl}(p, q; x) = x_{i1}^{α_1} x_{i2}^{α_2} \sum_{k,m≥0} \frac{(-1)^{k+m}}{γ + k + m + 1} \frac{(-α_1)_k(-α_2)_m}{(1)_{k}(1)_{m}} \cdot (pq)^{γ+k+m+1} \left( \frac{x_{21}^k}{x_{i1}} \right) \left( \frac{x_{22}^m}{x_{i2}} \right) ,$$

$$f(x) = x_{i1}^{α_1} x_{i2}^{α_2} \left( \frac{-x_{21}}{x_{2i}} \right)^{γ+1} F\left( \frac{-α_2}{γ + α_1 + 2}; \frac{x_{11}x_{22}}{x_{i2}x_{2i}} \right).$$

We note that $f(x)$ is a solution of the homogeneous system $H_A(β)$.

**Theorem 5** We fix $x_{i2}, x_{21}, x_{22}$ to real numbers and regarded the function as a function in one variable. Let $γ_α$ be a path which encircles the point $-ax_{21}$ in the positive direction and $γ_β$ be a path which encircles the point $-bx_{21}$ in the positive direction. We also suppose that exponents are generic. The analytic continuations of $f_{12}^{kl}$ along $γ_α$ and $γ_β$ are

$$f_{12}^{kl}(1, 1; x) \stackrel{γ_α}{\rightarrow} f_{12}^{kl}(1, e^{2πiα_1}; x) + \frac{-2πie^{πi(α_1+1)}}{Γ(-α_1)} f(x)$$

$$f_{12}^{kl}(1, 1; x) \stackrel{γ_β}{\rightarrow} f_{12}^{kl}(e^{2πiα_1}, 1; x) + \frac{-2πie^{πi(α_1+1)}}{Γ(-α_1)} f(x)$$

**Proof.** We replace the Gauss hypergeometric function in (5) with analytic continuations of them; we utilize the following formula of the analytic continuation of the Gauss hypergeometric function $F(a, b, c; x)$ along a path which encircles $x = 1$ positively.

$$F(a, b, c; x) \stackrel{γ}{\rightarrow} (1 - A)F(a, b, c; x) + Bx^{1-ε}F(a - c + 1, b - c + 1, 2 - c; x)$$
Here, we put

\[
A = \frac{(1 - e^{-2\pi i \alpha})(1 - e^{-2\pi i b})}{1 - e^{-2\pi i c}}, \quad B = \frac{2\pi i}{1 - c} \frac{\Gamma(c)^2}{\Gamma(a)\Gamma(b)\Gamma(c - a)\Gamma(c - b)}, e^{\pi i(c - a - b)}.
\]

We note that

\[
1 - A = 1 - \frac{(1 - e^{-2\pi i (-\alpha_1)})(1 - e^{-2\pi i (\gamma + m + 1)})}{1 - e^{-2\pi i (\gamma + m + 2)}} = 1 - (1 - e^{2\pi i \alpha_1}) = e^{2\pi i \alpha_1}.
\]

Then, we obtain the first term \( f_{12}^{11}(1, e^{2\pi i \alpha_1}; x) \) of the right hand side of the first formula by the replacement of the type \((1 - A)F(a, b, c; x)\). We make replacements of the type \( x^{-(\alpha_1 + \gamma - m - 1)} F(a - c + 1, b - c + 1, 2 - c; x) \). Then, we have

\[
F\left( -\alpha_1 - \gamma - m - 1, 0; -\frac{x_{11} b}{x_{11}} \right) = 1.
\]

Therefore, we have

\[
x_{11}^{\alpha_1} x_{12}^{\alpha_2} \sum_{m=0}^{\infty} \left( -\frac{x_{12}}{x_{22}} \right)^m \frac{b^{\gamma + m + 1}(-\alpha_2)_m}{(\gamma + m + 1)m!} = x_{11}^{\alpha_1} x_{12}^{\alpha_2} \sum_{m=0}^{\infty} \left( -\frac{x_{12}}{x_{22}} \right)^m \frac{b^{\gamma + m + 1}(-\alpha_2)_m}{(\gamma + m + 1)m!} \frac{\Gamma(\gamma + m + 2)}{\Gamma(-\alpha_1)\Gamma(\gamma + m + 1)\Gamma(\gamma + \alpha_1 + m + 2)\Gamma(1)} \cdot e^{\pi i (\alpha_1 + 1)} \left( -\frac{x_{11} b}{x_{11}} \right)^{-(\gamma + m + 1)}
\]

\[
= \frac{2\pi i}{-(\gamma + m + 1)} \frac{\Gamma(\gamma + m + 2)}{\Gamma(-\alpha_1)\Gamma(\gamma + m + 1)\Gamma(\gamma + \alpha_1 + m + 2)\Gamma(1)} \cdot e^{\pi i (\alpha_1 + 1)} \left( -\frac{x_{11} b}{x_{11}} \right)^{-(\gamma + m + 1)}
\]

\[
= \frac{2\pi i}{-(\gamma + m + 1)} \frac{\Gamma(\gamma + m + 2)}{\Gamma(-\alpha_1)\Gamma(\gamma + m + 1)\Gamma(\gamma + \alpha_1 + m + 2)\Gamma(1)} \cdot e^{\pi i (\alpha_1 + 1)} b^{-(\gamma + m + 1)} \left( -\frac{x_{11}}{x_{21}} \right)^{\gamma + m + 1}
\]

\[
= \frac{2\pi ie^{\pi i (\alpha_1 + 1)}}{\Gamma(-\alpha_1)} \left( -\frac{x_{11}}{x_{21}} \right)^{\gamma + m + 1} \sum_{m=0}^{\infty} \left( -\frac{x_{12}}{x_{22}} \right)^m \frac{(-\alpha_2)_m}{m!} \frac{\Gamma(\gamma + m + 1)}{\Gamma(\gamma + \alpha_1 + m + 2)} \left( -\frac{x_{11}}{x_{21}} \right)^m
\]

\[
= \frac{2\pi ie^{\pi i (\alpha_1 + 1)}}{\Gamma(-\alpha_1)} f(x)
\]

which is the second term of the right hand side of the first formula. The second formula in the Theorem is obtained analogously by exchanging the role of \( a \) and \( b \).

Remark 3 The function \([g(t, x)]_{t=a}^{t=b}\) is analytically continued as follows.

\[
[g(t, x)]_{t=a}^{t=b} \xrightarrow{q_1} e^{2\pi i \alpha_1} g(b, x) - g(a, x)
\]

\[
[g(t, x)]_{t=a}^{t=b} \xrightarrow{q_2} g(b, x) - e^{2\pi i \alpha_1} g(a, x)
\]
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