Deconfined quantum criticality and conformal phase transition in two-dimensional antiferromagnets

Flavio S. Nogueira and Asle Sudbø

Institut für Theoretische Physik III, Ruhr-Universität Bochum, Universitätsstraße 150, D-44801 Bochum, Germany
Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

PACS: 64.70.Tg – Quantum phase transitions
PACS: 11.10.Kk – Field theories in dimensions other than four
PACS: 75.10.Kt – Quantum spin liquids, valence bond phases and related phenomena

Abstract – Deconfined quantum criticality of two-dimensional SU(2) quantum antiferromagnets featuring a transition from an antiferromagnetically ordered ground state to a so-called valence-bond solid state, is governed by a non-compact CP\(^1\) model with a Maxwell term in 2 + 1 space-time dimensions. We introduce a new perspective on deconfined quantum criticality within a field-theoretic framework based on an expansion in powers of \(\epsilon = 4 - d\) for fixed number \(N\) of complex matter fields. We show that in the allegedly weak first-order transition regime, a so-called conformal phase transition leads to a genuine deconfined quantum critical point. In such a transition, the gap vanishes when the critical point is approached from above and diverges when it is approached from below. We also find that the spin stiffness has a universal jump at the critical point.

Many years have passed since a new paradigm for quantum phase transitions, the so-called deconfined quantum criticality (DQC) scenario, was introduced [1]. In this paradigm, the effective quantum field theory does not contain any elementary fields representing the order parameters associated with the underlying competing orders. It posits that in certain quantum phase transitions these order parameters are not elementary, but composed of more elementary fields in the same way that in elementary particle physics mesons are constituted by quarks. The precise context where this happens involves competing orders featuring broken internal and spacetime symmetries. This occurs, for example, in certain SU(2) quantum antiferromagnets (AF), where SU(2)-invariant interactions compete. A paradigmatic example is the so-called \(J - Q\) model [2],

\[
H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - Q \sum_{\langle i,j,k,l \rangle} \left( \mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} \right) \left( \mathbf{S}_k \cdot \mathbf{S}_l - \frac{1}{4} \right),
\]

where both \(J\) and \(Q\) are positive. Defining the dimensionless coupling \(g = Q/J\), we obtain the schematic phase diagram shown in fig. 1. For \(g \ll 1\) the first term in (1) dominates, favoring a Néel state. For \(g \gg 1\) the plaquette term in eq. (1) dominates, favoring a valence-bond solid (VBS) ordered state. The Néel state breaks an internal symmetry, namely \(SU(2)\). The VBS state preserves the \(SU(2)\) symmetry while breaking the symmetries of the square lattice. As one broken symmetry is internal (the \(SU(2)\) one) and the other one is spatial, quantum mechanics forbids their coexistence, since the VBS state is a long-range entangled state while the Néel state is long-range ordered.

In the DQC scenario the operators measuring both Néel and VBS order are comprised of more fundamental objects. These are the spinons, which are represented by an \(SU(2)\) doublet of complex fields \(\mathbf{z} = (z_1, z_2)\) satisfying the constraint \(|z_1|^2 + |z_2|^2 = 1\) at each lattice point. In terms of the spinon fields, the fields representing the Néel and VBS order parameters are \(U(1)\) gauge-invariant objects. The gauge field arising in such a theory is an emergent “photon” originally defined on the lattice, and hence it is necessarily compact. This leads to instanton excitations that gap the dual photon (defined as \(B_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda\)) in the phase where the expectation value of the Higgs field is zero, i.e., the paramagnetic phase. This gap also corresponds to the mass of the instantons [3]. Thus, the VBS phase is one where the spinons are confined (see the text in
spinons condense due to a spontaneous U(1) photon of the Higgs phase which is gapped. While in the confined phase it is the dual of the emergent excitations are gapped and the spinons are confined, leading to a Néel state. In the confinement phase, all excitations are gapped and the spinons are confined, leading to a VBS state. In the Néel phase the emergent photon is gapped, while in the confined phase it is the dual of the emergent photon of the Higgs phase which is gapped.

the caption of fig. 1). In the Néel phase, on the other hand, the photon is gapped due to the Higgs mechanism. One fundamental prediction of the DQC scenario is that the instanton-mass vanishes continuously for g approaching a quantum critical point \( g_c \) from above, thus suppressing them at the quantum critical point [1]. For a version of this theory with easy-plane anisotropy [4], the suppression of instantons has been confirmed by Monte Carlo (MC) simulations [5]. In the easy-plane case, the suppression occurs in a weak first-order phase transition, and no quantum criticality ensues [5,6].

For the SU(2) DQC model, early MC results indicated a weak first-order phase transition [7]. Simulations performed on the \( J-Q \) model have mostly yielded a second-order phase transition and signs of an emergent \( U(1) \) symmetry [2,8,9], although a weak first-order phase transition has also been reported [10]. Since the \( J-Q \) model is one of the emblematic lattice models for the DQC scenario, a recent MC study [11] made a comparative analysis of its phase diagram with the one obtained from the non-compact CP\(^1\) model. While both models agree over a substantial portion of the phase diagram for moderate system sizes, they behave differently at larger system sizes [11]. Furthermore, there are indications that none of the models become critical, which would corroborate a weak first-order phase transition scenario. Recent large scale simulations [12] on the non-compact Abelian Higgs model with CP\(^1\) constraint indicate that the existence of a tricritical point cannot be ruled out. It is also worth mentioning that a large \( N \)-like MC study of the \( J-Q \) and \( J_1-J_2 \) (\( J_1 \) nearest-neighbor and \( J_2 \) next-nearest-neighbor exchanges) models has been made recently [13], aiming to compare with large \( N \) limit of the CP\(^{N-1} \) model, where quantum criticality is known to occur. In this study strong evidence for DQC has been found for \( N > 4 \).

This brings us to the main topic of this paper, namely, a quantum field-theoretic analysis of the non-compact Abelian Higgs model with a global SU(\( N \)) symmetry. In the present context, there are two relevant versions of this theory, a non-linear and a linear one. The non-linear theory corresponds to a CP\(^{N-1} \) model with a Maxwell term [14],

\[
\mathcal{L}_{\text{CP}^{N-1}} = \frac{\Lambda^{d-2}}{g} \sum_{\alpha=1}^{N} |(\partial_\mu - i A_\mu) z_\alpha|^2 + \frac{1}{4e^2} F_{\mu\nu}^2, \tag{2}
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \sum_{\alpha=1}^{N} |z_\alpha|^2 = 1 \). The linear version softens this constraint and has the more standard Higgs model form [1]:

\[
\mathcal{L}_{\text{Higgs}} = \sum_{\alpha=1}^{N} \left[ |(\partial_\mu - i A_\mu) z_\alpha|^2 + r |z_\alpha|^2 \right] + \frac{u}{2} \left( \sum_{\alpha=1}^{N} |z_\alpha|^2 \right)^2 + \frac{1}{4e^2} F_{\mu\nu}^2. \tag{3}
\]

Both models have the same symmetries. In parameter regimes where a critical point exists, they should belong to the same universality class. In the limit \( e^2 \to \infty \), both \( \mathcal{L}_{\text{CP}^{N-1}} \) and \( \mathcal{L}_{\text{Higgs}} \) have exactly the same critical behavior for large \( N \) [15]. However, a recent calculation of the spin stiffness at large \( N \) and finite \( e^2 \) [16] showed that \( \rho_s \) exponentiates to a Josephson scaling form only when \( e^2 \to 0 \) or \( e^2 \to \infty \), corresponding to \( O(2N) \) or CP\(^{N-1} \) universality classes, respectively. For finite values of \( e^2 \), the spin stiffness exhibits logarithmic violations of scaling [16], which have been reported in recent MC simulations of the \( J-Q \) model [9,17].

Here, we address the actual nature of the phase transition in the gauge theory proposed to underpin deconfined quantum critical points. In gauge theories, Elitzur’s theorem [18] forbids the spontaneous breaking of a local gauge symmetry in any dimension. Therefore, there is no local order parameter available to distinguish phases. In MC simulations of the lattice version of model (2), one of the quantities studied is the spin stiffness [7], which provides a non-local order parameter. However, a jump in the spin stiffness does not necessarily imply a first-order phase transition in this case. One could have a situation where the gap vanishes continuously as the critical point is approached, while the spin stiffness vanishes discontinuously. Some theories behave precisely in this way, a prominent example being the Berezinskii-Kosterlitz-Thouless (BKT) transition [19], where the inverse correlation length features an essential singularity at the critical point and no local order parameter exists [20]. In the case of the BKT
transition, the superfluid stiffness has a universal jump at the critical point [21]. Theories with this type of behavior are said to undergo a conformal phase transition (CPT) [22]. Recent lattice simulations [23] show evidence of a CPT in SU(N) gauge theories in \( d = 3 + 1 \).

We provide arguments to support a CPT scenario in DQC gauge field theories. First, we show that the \( \epsilon \)-expansion for the model (3) contains a regime where the inverse correlation length has an essential singularity and show that the spin stiffness features a universal jump at the critical point. Then, we derive a similar behavior for the mass of instantons in the paramagnetic phase of the \( \text{CP}^{N-1} \) model (2).

The one-loop RG \( \beta \) functions are well known and were originally obtained using the Wilson RG [24]. In the classical RG-analysis of eq. (3) carried out in ref. [24], it was concluded that no stable infrared fixed point existed unless \( N \) exceeded some large value \( N_c \approx 185 \). For \( N < N_c \), runaway flows of the RG-equations were found, and this was originally interpreted as a signature of a first-order phase transition. A more modern interpretation of the same, is that it signals the existence of a strong-coupling fixed point, and it is this point of view we take. Contrary to the scope of ref. [24], in this paper we undertake a careful analysis of the precise character of this strong-coupling fixed point. This has, to our knowledge, not been carried out. Such an analysis is of paramount importance, given the proposed DQC-scenario.

To analyze eqs. (2) and (3), it is convenient to use the field theory RG with dimensional regularization in the minimal subtraction scheme [25], rather than the Wilson RG approach to the problem [24]. We introduce the renormalized dimensionless couplings \( f = N^{-\epsilon} \epsilon R^2/(8\pi^2) \) and \( g = m^{-\epsilon} u R/(8\pi^2) \), where \( \epsilon = 4 - d \) and \( m \) is the Higgs mass scale related to the inverse correlation length. Here, \( \epsilon R \) and \( u R \) are the renormalized counterparts of the bare couplings \( \epsilon^2 \) and \( u \). The asymptotic behavior of the renormalized gauge coupling will be crucial. In order to obtain it at one-loop order, we have to compute the vacuum polarization, \( \Pi_{\mu\nu}(p) \), which yields the lowest-order fluctuation correction to the Maxwell term in the action. In a \( d \)-dimensional spacetime, we have,

\[
S_{\text{Maxwell}} = \frac{1}{4\epsilon^2} \int d^d x F_\mu^2 + \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \Pi_{\mu\nu}(p) A_\mu(p) A_\nu(-p),
\]

where the vacuum polarization is obtained as

\[
\Pi_{\mu\nu}(p) = 2N \delta_{\mu\nu} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} - N \int \frac{d^d k}{(2\pi)^d} \frac{(2k-p)_\mu(2k-p)_\nu}{[(k-p)^2 + m^2][(k^2 + m^2)^2]}. \tag{5}
\]

If dimensional regularization is used, the gauge symmetry is preserved along with current conservation, and therefore \( \Pi_{\mu\nu}(p) \) is transverse. Thus, \( \Pi_{\mu\nu}(p) = \Pi(p)(p^2 \delta_{\mu\nu} - p_\mu p_\nu) \).

In the low-energy regime where \( |p| \ll m \), we can evaluate all integrals explicitly for arbitrary \( d \) to obtain \( \Pi(p) \approx N^d \epsilon^2 \epsilon R^2 / [(4\pi)^{d/2}(2 - d/2)] \). The effective Maxwell contribution to the Higgs Lagrangian then becomes,

\[
\mathcal{L}_{\text{Maxwell}} \approx \frac{1}{4\epsilon^2} \left[ 1 + \frac{N \epsilon^2}{3(4\pi)^{d/2}} \Gamma \left( 2 - \frac{d}{2} \right) \right] F_{\mu\nu}^2,
\]

\[
\equiv \frac{1}{4\epsilon^2} F_{\mu\nu}^2, \tag{6}
\]

which defines the renormalized gauge coupling \( \epsilon R \). Recall that for \( N = 2 \) and \( \epsilon^2 \to \infty \) the CP\(^{N-1} \) model (2) is equivalent to the O(3) non-linear \( \sigma \) model (see, for instance, sect. 7.9 in ref. [26]), which has a second-order phase transition. In fact, the CP\(^{N-1} \) model exhibits a second-order phase transition for any \( N \) if \( \epsilon^2 \to \infty \). The same is expected to be true for the abelian Higgs model (3). Thus, let us consider the limit \( \epsilon^2 \to \infty \) of eq. (6) and its approximate form for small \( \epsilon = 4 - d \),

\[
\mathcal{L}_{\text{Maxwell}} \approx \frac{N}{12(4\pi)^{d/2}} \epsilon R^2 \Gamma \left( 2 - \frac{d}{2} \right) m^{d-4} F_{\mu\nu}^2
\]

\[
\approx \frac{f_*}{4(8\pi^2 f_* m^2)} F_{\mu\nu}^2, \tag{7}
\]

where \( f_* = 3\epsilon/N \). After approximating eq. (6) for small \( \epsilon \), we obtain the one-loop \( \beta \) function for the dimensionless gauge coupling \( f \).

\[
\beta_f \equiv \frac{m \frac{df}{dm}}{f} = -\epsilon f + \frac{N}{3} f^2, \tag{8}
\]

and we identify \( f_* \) as the infrared stable fixed point. Hence, for \( 2 < d < 4 \) we have that \( f \) approaches \( f_* \) in two ways, namely, as \( \epsilon^2 \to \infty \) for fixed \( m \) or as \( m \to 0 \) for fixed \( \epsilon^2 \). Therefore, we can use \((\epsilon^2)^{1/(4-d)}\) as the ultraviolet cutoff \( \Lambda \).

Note that \( \beta_f \) is only a function of \( f \). A two-loop calculation does not change this [27], and \( \beta_f \) remains dependent only on \( f \). Within dimensional regularization in the minimal subtraction scheme one may show that this holds to all orders, since the only poles in \( \epsilon \) arise in diagrams containing uniquely powers of the gauge coupling. All other diagrams are finite for \( \epsilon \to 0 \). This follows from gauge invariance and Ward identities of the theory.

The \( \beta \) function for the coupling \( g \) is given at one-loop order by [15]

\[
\beta_g \equiv \frac{m \frac{dg}{dm}}{g} = -\epsilon g - 6 f g + (N + 4) g^2 + 6 f^2. \tag{9}
\]

There are two relevant regimes where critical points arise, depending on the value of the gauge coupling fixed point. For \( f = 0 \), we have a non-trivial fixed point \( g_* = \epsilon/(N + 4) \) governing the critical behavior corresponding to the O(2N) universality class, while the line \( f = f_* = 3\epsilon/N \) contains a critical \( (g_*) \) and a tricritical \( (g_-) \) fixed point for \( N > N_c = 6(15 + 4\sqrt{15}) \), given by \( g_\pm = \epsilon(18 + N \pm \sqrt{18}) \).
\[ \sqrt{\Delta}/[2N(N + 4)], \text{ where } \Delta = N^2 - 180N - 540. \] We are interested in analyzing the quantum critical behavior near the line \( f = f_\ast \). As we have seen, this corresponds to a regime of very strong bare gauge coupling. The behavior near the line \( f = f_\ast \) should correspond to a crossover to the critical behavior of the \( \mathrm{CP}^{N-1} \) model (2). In order to understand this quantum critical behavior, we recall that generally near a second-order phase transition \( \rho \approx (g - g_\ast)^{1/\omega} \), where \( g_\ast \) is the infrared stable fixed point and \( \omega \) is the exponent governing corrections to scaling [25]. In our case, \( g_\ast = g_{\ast c} \) for \( N > N_c \) and \( \omega = \partial \beta g(g_\ast, f_\ast)/\partial g = \epsilon \sqrt{\Delta}/N. \) Due to the presence of the tricritical point, we must have \( g \rightarrow g_\ast \) for \( m \ll \Lambda \) in addition to the usual behavior \( g \rightarrow g_\ast \) for \( m \ll \Lambda \). Thus, the solution of eq. (9) along the line \( f = f_\ast \) has the general form \( m/\Lambda = F(g)/F(g_\ast) \), where \( F(x) = [(g_\ast + x)/(x-g_\ast)]^{1/\omega} \) and \( g_\ast = g_{\ast c} \). For \( N < N_c \), we have that \( \beta g(g_\ast, f_\ast) \neq 0 \) for all \( g \in \mathbb{R} \), since the fixed points \( g_\ast \) and \( g_\ast \) both become complex. On the other hand, \( \partial \beta g_\ast(g_f, f_\ast)/\partial g \) vanishes for \( g = g_\ast = (g_\ast + g_\ast)/2 = \Re(g_\ast). \) Since for \( N < N_c \) \( g_\ast \) are complex conjugate to each other, \( m \) does not exhibit a power-law behavior any longer. Indeed, we obtain

\[
F(g) = \exp \left\{ -\frac{N}{\epsilon \sqrt{\Delta}} \arctan \left[ \frac{\sqrt{\Delta}e}{2N(N+4)(g-g_\ast)} \right] \right\}. \tag{10}
\]

The limit \( \epsilon \rightarrow 0 \) corresponds to a Gaussian fixed point. In the limit \( \epsilon \rightarrow 0 \) we have \( F(g) = \exp[-1/[(N+4)g]] \) for all \( N \). The \( \mathrm{CP}^{N-1} \) model has a similar behavior at its critical dimension, \( d = 1 + 1 \).

We note that \( m \) does not vanish at \( g = g_\ast \). As \( g \rightarrow g_\ast \), it approaches its minimum value, \( m_{\text{min}} \), and jumps abruptly to its maximum value, \( m_{\text{max}} \), which is attained as \( g \rightarrow g_{\ast c} \). The difference \( m_{\text{max}} - m_{\text{min}} \) is much larger than \( m_{\text{min}} \), showing that \( m \) almost vanishes as \( g \) is approached from above. Thus, adhering to the logic of the \( \epsilon \)-expansion, we can write approximately,

\[
F(g) \approx \exp \left\{ -\frac{1}{2(N+4)(g-g_\ast)} \right\}, \tag{11}
\]

which vanishes as \( g \rightarrow g_\ast \). On the other hand, approaching \( g_\ast \) from below causes \( m \) to grow to infinity. This is precisely the type of behavior arising in theories undergoing a CPT [22], associated with the breakdown of conformal symmetry. This aspect of gauge theories can be related to the so-called trace anomaly [28] of the stress tensor.

In order to find further signatures of a CPT, we search for universal behavior in physical quantities. The spin stiffness \( \rho_s \) is a crucial physical observable in DQC. In the case of a CPT, it must have a behavior similar to what is found in a BKT transition, where the superfluid stiffness exhibits a universal jump at the critical point [21].

To facilitate computing \( \rho_s \) within the present formalism, we observe that in the Higgs phase the renormalized photon mass is given by \( m_A^2 = 2\pi^2 g\rho_s \), and use the fact that \( m^2/m_A^2 = g/(2f) \) to derive an RG equation for \( \rho_s, m d\rho_s/dm = (2 - \epsilon - \beta g/g)\rho_s \), and solve it over the line \( f = f_\ast \). The solutions have the scaling form, \( \rho_s = m^2(\epsilon - \epsilon)/R(m/\Lambda) \). Consider first the case having \( N = N_c \), where a second-order phase transition takes place. We obtain the typical Josephson scaling, including corrections to scaling behavior

\[
\frac{\rho_s}{\Lambda^{2-\epsilon}} = \frac{[F(g)/F(g_\ast)]^{2-\epsilon}[1 + |F(g)|]}{g_\ast + \Lambda} \tag{12}
\]

When \( N < N_c \), on the other hand, we have

\[
\frac{\rho_s}{\Lambda^{2-\epsilon}} = \frac{2(N+4)[F(g)/F(g_\ast)]^{2-\epsilon} \cos \theta(g)}{(N+4)^2 g_\ast^2 + \omega^2 \cos \theta(g) + \theta_0^2}, \tag{13}
\]

where \( \theta(g) = (1/2) \arctan\{[\omega/[(N+4)(g-g_\ast)] \} \text{ and } \theta_0 = \arctan\{[\omega/[(N+4)g_\ast] \}. \) Since now \( m \) must approach \( g_\ast \) from below, it is not possible to use eq. (11) in eq. (13). As a consequence, \( g \rightarrow g_{\ast c} \) a universal jump arises, which is given by \( \rho_s/\Lambda^{2-\epsilon} = 2(N+4) |F(g_\ast)/(F(g_\ast))^{2-\epsilon}/[(N+4)g_\ast + \omega]| \). Thus, we have obtained another expected feature of a CPT reminiscent of the BKT behavior [21]. In fig. 2 we plot \( \rho_s \) for \( N = 2 \) and \( \epsilon = 1 \). When expressed in terms of \( m, \) eq. (13) includes logarithmic corrections to scaling, a behavior related to MC simulations of the \( J = Q \) model [9,17] and discussed recently in a large \( N \) context in ref. [16]. Indeed, recalling that \( m/\Lambda = F(g)/F(g_\ast) \), we can write \( \cos \theta(g) = \cos\{[\omega/2 \ln[F(g_\ast)m/\Lambda]\} \) and a similar expression for \( \cos\theta(g) + \theta_0 \). In order to see a possible connection with available numerical results, we may consider a finite-size scaling approach within the \( \epsilon \)-expansion framework [25]. Formally, an analysis would make use of an Abelian Higgs model in a periodic hypercube along with the results from the RG analysis in the continuum [25]. However, we can already predict the outcome of such a finite-size scaling analysis with the results obtained here. To this end, we consider a correlation length \( \xi = m_{\ast c}^{-1} = L \), where \( L \) represents the finite size of the system. For a finite size \( L \) and
to lowest order in $\epsilon$, we obtain,
\[
\rho_s \approx \frac{2(N + 4)L^{2-\epsilon}}{\cos \theta_0 \sqrt{(18 + N)^2 + |\Delta|^2/N^2}} \times \left\{ \frac{1}{\epsilon} + \frac{\sqrt{\Delta}}{2N} \tan \theta_0 \ln \left( \frac{\Delta L}{F'(g_\Lambda)} \right) \right\},
\]
which for $\Delta L \gg 1$ behaves like $\rho_s \sim L^{2-2\epsilon} \ln(L\Delta L)$, similarly to ref. [9] when $\epsilon = 1$, corresponding to $2 + 1$ dimensions. Thus, such an observed behavior in numerics may be a sign that for considerably larger system sizes a jump arises in the spin stiffness. It is worth emphasizing that such a logarithmic correction is tiny at large $N$. For $N \gg N_c$ we consider the explicit expression for the amplitude of $L^{2-\epsilon} \ln(L\Delta L)$ for large $L$,
\[
\rho_s \sim \frac{(N + 4)\sqrt{\Delta}}{8N^2(1 + 18 N^2)} L^{2-2\epsilon} \ln(L/L_0),
\]
where $L_0 = F(g_\Lambda)/\Lambda$. For $N \rightarrow \infty$, the coefficient in the expression above behaves like $\sim 1/N^2$, being therefore strongly suppressed. This is the reason why a large $N$ approach cannot easily predict a logarithmic correction in this case.

Another interesting quantity is the Néel magnetic susceptibility,
\[
\chi_N(x) = \langle \mathbf{n}(x) \cdot \mathbf{n}(0) \rangle,
\]
which is given by $\chi_N(x) = 2\langle \mathbf{z}^\dagger(x) \cdot \mathbf{z}(0) \mathbf{z}(x) \cdot \mathbf{z}(0) \rangle - N^{-1}\langle |\mathbf{z}(x)|^2 |\mathbf{z}(0)|^2 \rangle$. In order to calculate this quantity, we renormalize the action of composite operators to take into account, so that no renormalization constants are needed, namely, the wave function renormalization $Z_z$ for the spinon field and $Z_{\rho s}$ for the renormalization of the collective operator $\hat{\sigma}_s(x) = \hat{z}_s(x)$. Thus, at the critical point we have $\chi_0(x) = Z_z/Z_{\rho s}[\hat{z}_s(x);\hat{z}_s(0)]$ [9].

In this case it is useful to define the RG function $\gamma_s^2 = m\partial m/(Z_{\rho s}^2 Z_z)/\partial m$, which is given at one-loop order by $\gamma_s^2 = 3f - g$ [29]. The corresponding finite-size scaling susceptibility thus satisfies $L \partial \chi_N/L = 2(2 - \epsilon - \gamma_s^2)$. In general we then have $\chi_N = (L_0/L)^{2-2\epsilon + \eta_N} X(\ln(L/L_0))$, for $N > N_c$ a second-order phase transition takes place with $X(\ln(L/L_0)) \sim \text{const}$ and $\eta_N = 2 - \epsilon + 2g_\ast - 6f_\ast$. For $N < N_c$, on the other hand, we obtain
\[
\chi_N = \frac{\langle [1/F'(g_\Lambda)/\ln(L/L_0)] \rangle^{1/(N+4)}}{[F'(g_\Lambda)/\ln(L/L_0)]^{2-2\epsilon + \eta_N}},
\]
where $\eta_N = 2 - \epsilon + 2g_\ast - 6f_\ast$. Note, however, that in the latter expression positive values of $\eta_N$ arise only for $(13 + \sqrt{385})/2 < N < N_c$. Thus, a better approximation is necessary in order to access more physical values of $N$. Anyway, it is interesting to notice that a non-trivial logarithmic dependence arises in this case.

As a final calculation to support a CPT scenario in DQC, we consider the dynamics of instantons inside the VBS in the CP$^{N-1}$ model (2) at fixed dimensionality and large $N$. At leading order a standard calculation yields the mass gap, $M$, which due to the large $N$ limit exhibits, as expected, a conventional power-law behavior for $\hat{g} > \hat{g}_c$, i.e., $M/\Lambda = (2/\pi)(1 - \hat{g}_c/\hat{g})$, where $\hat{g}_c = 2\pi^2/N$. On the other hand, here the unconventional behavior arises in the gapped instanton excitations. To see this, we consider the explicit expression for the amplitude of $L^{2-2\epsilon} \ln(L\Delta L)$ for large $L$,
\[
\rho_s \sim \frac{(N + 4)\sqrt{\Delta}}{8N^2(1 + 18 N^2)} L^{2-2\epsilon} \ln(L/L_0),
\]
which for $\Delta L \gg 1$ behaves like $\rho_s \sim L^{2-2\epsilon} \ln(L\Delta L)$, similarly to ref. [9] when $\epsilon = 1$, corresponding to $2 + 1$ dimensions. Thus, such an observed behavior in numerics may be a sign that for considerably larger system sizes a jump arises in the spin stiffness. It is worth emphasizing that such a logarithmic correction is tiny at large $N$. For $N \gg N_c$ we consider the explicit expression for the amplitude of $L^{2-2\epsilon} \ln(L\Delta L)$ for large $L$,
\[
\rho_s \sim \frac{(N + 4)\sqrt{\Delta}}{8N^2(1 + 18 N^2)} L^{2-2\epsilon} \ln(L/L_0),
\]
where $L_0 = F(g_\Lambda)/\Lambda$. For $N \rightarrow \infty$, the coefficient in the expression above behaves like $\sim 1/N^2$, being therefore strongly suppressed. This is the reason why a large $N$ approach cannot easily predict a logarithmic correction in this case.

Another interesting quantity is the Néel magnetic susceptibility,
\[
\chi_N(x) = \langle \mathbf{n}(x) \cdot \mathbf{n}(0) \rangle,
\]
which in terms of spinon fields is given by $\chi_N(x) = 2\langle \mathbf{z}^\dagger(x) \cdot \mathbf{z}(0) \mathbf{z}(x) \cdot \mathbf{z}(0) \rangle - N^{-1}\langle |\mathbf{z}(x)|^2 |\mathbf{z}(0)|^2 \rangle$. In order to calculate this quantity, we renormalize the action of composite operators to take into account, so that no renormalization constants are needed, namely, the wave function renormalization $Z_z$ for the spinon field and $Z_{\rho s}$ for the renormalization of the collective operator $\hat{\sigma}_s(x) = \hat{z}_s(x)$. Thus, at the critical point we have $\chi_0(x) = Z_z/Z_{\rho s}[\hat{z}_s(x);\hat{z}_s(0)]$ [9].

In this case it is useful to define the RG function $\gamma_s^2 = m\partial m/(Z_{\rho s}^2 Z_z)/\partial m$, which is given at one-loop order by $\gamma_s^2 = 3f - g$ [29]. The corresponding finite-size scaling susceptibility thus satisfies $L \partial \chi_N/L = 2(2 - \epsilon - \gamma_s^2)$. In general we then have $\chi_N = (L_0/L)^{2-2\epsilon + \eta_N} X(\ln(L/L_0))$, for $N > N_c$ a second-order phase transition takes place with $X(\ln(L/L_0)) \sim \text{const}$ and $\eta_N = 2 - \epsilon + 2g_\ast - 6f_\ast$. For $N < N_c$, on the other hand, we obtain
\[
\chi_N = \frac{\langle [1/F'(g_\Lambda)/\ln(L/L_0)] \rangle^{1/(N+4)}}{[F'(g_\Lambda)/\ln(L/L_0)]^{2-2\epsilon + \eta_N}},
\]
where $\eta_N = 2 - \epsilon + 2g_\ast - 6f_\ast$. Note, however, that in the latter expression positive values of $\eta_N$ arise only for $(13 + \sqrt{385})/2 < N < N_c$. Thus, a better approximation is necessary in order to access more physical values of $N$. Anyway, it is interesting to notice that a non-trivial logarithmic dependence arises in this case.

As a final calculation to support a CPT scenario in DQC, we consider the dynamics of instantons inside the VBS in the CP$^{N-1}$ model (2) at fixed dimensionality and large $N$. At leading order a standard calculation yields the mass gap, $M$, which due to the large $N$ limit exhibits, as expected, a conventional power-law behavior for $\hat{g} > \hat{g}_c$, i.e., $M/\Lambda = (2/\pi)(1 - \hat{g}_c/\hat{g})$, where $\hat{g}_c = 2\pi^2/N$. However, here the unconventional behavior arises in the gapped instanton excitations. To see this, we consider the explicit expression for the amplitude of $L^{2-2\epsilon} \ln(L\Delta L)$ for large $L$,
$M_{\text{DH}}/(4\pi)$. Thus, we obtain,

$$\psi_{\text{VBS}} \approx \exp \left\{ -\frac{N\hat{g}}{48(\hat{g} - \hat{g}_c)} + \frac{\pi^2(N\hat{g})^{3/2}}{96\sqrt{3}(\hat{g} - \hat{g}_c)^{3/2}} \right\},$$

which vanishes continuously as $\hat{g} \to \hat{g}_c$ from above. In terms of the correlation lengths $\xi = M^{-1}$ and $\xi_{\text{mon}} = M_{\text{DH}}^{-1}$ for the instantons, we obtain in the present approximation and for $\hat{g}$ near $\hat{g}_c$, $\psi_{\text{VBS}} \sim \xi^{-\left(1+2N\rho_1\right)}\xi_{\text{mon}}^{-2}$. If $\xi_{\text{mon}}$ is ignored, it would appear that the VBS correlation length $\xi_{\text{VBS}}$ has a power-law behavior relative to $\xi$. However, due to $\xi_{\text{mon}}$, we find that the actual behavior near the critical point is highly peaked and vanishes quickly for $\hat{g}$ not much larger than $\hat{g}_c$. This result might be an artifact associated to the missing magnitude of $\psi_{\text{VBS}}$ in the above approximation.

To test numerically for a CPT, one needs to establish a universal jump in the stiffness at the transition. This is similar to the situation in the 2D XY model, which features a BKT transition, which is a CPT. This is done by considering higher-order response functions to phase-twists [31]. Similar techniques have been developed for searching for universal jumps in stiffnesses in 3D systems with proposed CPTs [32].

Summarizing, we have analyzed the $\epsilon$-expansion of the Abelian Higgs model in the allegedly first-order phase transition regime along a line in the RG flow diagram determined by the gauge coupling fixed points defining the strong-coupling regime. We have argued that within the accuracy of the $\epsilon$-expansion, a conformal phase transition associated with a deconfined quantum critical point occurs. We obtain a spinon mass gap featureing an essential singularity at the critical point. Similarly to the BKT transition in two dimensions, we find that the spin stiffness has a universal jump at the conformal phase transition critical point. We find further evidence for a conformal phase transition by analyzing the VBS phase at large $N$ in the presence of instantons, where the screening mass of the instantons also exhibits an essential singularity at the critical point.

***

FSN acknowledges the Deutsche Forschungsgemeinschaft (DFG) for the financial support via the collaborative research center SFB TR 12. AS acknowledges support from the Research Council of Norway, Grant Nos. 205591/V20 and 216700/F20.

REFERENCES

[1] Senthil T., Vishwanath A., Balents L., Sachdev S. and Fisher M. P. A., Science, 303 (2004) 1490; Senthil T., Balents L., Sachdev S., Vishwanath A. and Fisher M. P. A., Phys. Rev. B, 70 (2004) 144407.

[2] Sandvik A. W., Phys. Rev. Lett., 98 (2007) 227202.

[3] Polyaakov A. M., Nucl. Phys. B, 120 (1977) 429.

[4] Motrunich O. I. and Vishwanath A., Phys. Rev. B, 70 (2004) 075104.

[5] Kragset S., Smørbøv E., Hove J., Nogueira F. S. and Sudbø A., Phys. Rev. Lett., 97 (2006) 247201.

[6] Kuklov A. B., Prokof’ev N. V., Svistunov B. V. and Troyer M., Ann. Phys. (N.Y.), 321 (2006) 1602.

[7] Kuklov A. B., Prokof’ev N. V., Svistunov B. V. and Troyer M., Phys. Rev. Lett., 101 (2008) 050405.

[8] Melko R. G. and Saul R. K., Phys. Rev. Lett., 100 (2008) 017203.

[9] Sandvik A. W., Phys. Rev. Lett., 104 (2010) 177201.

[10] Jiang F.-J., Nyfeler M., Chandrasekharan S. and Wiese U.-J., J. Stat. Mech. (2008) P02009.

[11] Chen K., Huang Y., Deng Y., Kuklov A. B., Prokof’ev N. V. and Svistunov B. V., Phys. Rev. Lett., 110 (2013) 185701.

[12] Kaul R. K. and Sandvik A. W., Phys. Rev. B, 87 (2013) 134503.

[13] Sandvik A. W. and Kaul R. K., J. Phys. A, 45 (2012) 132701.

[14] Sawari I. D. and Athorne C., J. Phys. A: Math. Gen., 16 (1983) L587; 4428.

[15] Hirami S., Prog. Theor. Phys., 62 (1979) 226.

[16] Nogueira F. S. and Sudbø A., Phys. Rev. B, 86 (2012) 045121.

[17] Banerjee A., Damle K. and Alet F., Phys. Rev. B, 82 (2010) 155139.

[18] Elitzur S., Phys. Rev. D, 12 (1975) 3978.

[19] Berezinsky V. L., Sov. Phys. JETP, 32 (1971) 493 (Zh. Eksp. Teor. Fiz., 59 (1970) 907); Kosterlitz J. M. and Thouless D. J., J. Phys. C, 6 (1973) 1181.

[20] Mermin N. D. and Wagner H., Phys. Rev. Lett., 23 (1966) 1133.

[21] Nelson D. R. and Kosterlitz J. M., Phys. Rev. Lett., 39 (1977) 1201.

[22] Miransky V. A. and Yamawaki K., Phys. Rev. D, 55 (1997) 5051.

[23] Deuzeman A., Lombardo M. P. and Pallante E., Phys. Rev. D, 82 (2010) 074503.

[24] Halperin B. I., Lubensky T. C. and Ma S.-K., Phys. Rev. Lett., 32 (1974) 292.

[25] Zinn-Justin J., Quantum Field Theory and Critical Phenomena, 2nd edition (Oxford University Press, Oxford) 1993.

[26] Fradkin E., Field Theories of Condensed Matter Physics, 2nd edition (Cambridge University Press, Cambridge) 2013.

[27] Folk R. and Holovatch Y., J. Phys. A, 29 (1996) 3409.

[28] Collins J. C., Duncan A. and Joglekar S. D., Phys. Rev. D, 16 (1977) 438; Joglekar S. D. and Misra A., Phys. Rev. D, 38 (1988) 2546; Polchinski J., Nucl. Phys. B, 303 (1988) 226.

[29] Nogueira F. S., Phys. Rev. B, 77 (2008) 175101.

[30] Murthy G. and Sachdev S., Nucl. Phys. B, 344 (1990) 557.

[31] Minnhagen P. and Kim J. B., Phys. Rev. B, 67 (2003) 172509.

[32] Borkje K., Kragset S. and Sudbø A., Phys. Rev. B, 71 (2005) 085112.