Scalar mesons in the chiral theory with quark degrees of freedom

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Abstract

The Chiral Confining Lagrangian, based on the chiral theory with quark degrees of freedom, is used to study the spectroscopy of scalar mesons. The theory does not contain arbitrary fitting parameters and takes into account infinite number of transitions from meson-meson to quark-antiquark states. For chiral mesons the transition coefficients are known and the resulting $\pi\pi$ and $K\bar{K}$ amplitudes $f_{\pi\pi}$ and $f_{K\bar{K}}$ are calculated in terms of the $q\bar{q}$ and the free meson Green’s functions. As a result the known $q\bar{q}$ pole at 1.05 GeV produces two resonances: a wide resonance $E_1$ in the range 500-700 MeV and narrow $E_2$ near 1 GeV, which can be associated with $f_{0}(500)$ and $f_{0}(980)$. A similar analysis, applied to the $I = 1$ channel, shows that in this case two very close poles in different sheets appear near $E = 980$ MeV, which can be associated with the $a_0(980)$ resonance. The resulting $\pi\pi$ interaction amplitudes, $\text{Re} f_{\pi\pi}(E)$ and $\text{Im} f_{\pi\pi}(E)$ are compared with the known data.

1 Introduction

Scalar mesons are in the center of experimental and theoretical interests for a long time (see summary of experimental data in Ref. [1] and a large amount of information about the scalars in the reviews [2, 3, 4, 5, 6, 7].
and recent comprehensive analysis in [8, 9]). The theoretical explanation of the scalar spectrum has faced difficulties and required the development of different approaches, like the tetraquark model [10], the chiral approach [11], the molecular approach [12], and the QCD sum rules [13], as well as lattice calculations [14] (see recent study in [15, 16]). One of realistic methods to treat the problem of the scalar mesons is the dispersive and analytic methods, which allow to calculate the pole parameters [11], [17], [18]. Effects of meson loops on the $q\bar{q}$ states were studied in [19, 20, 21, 22, 23], where also a convenient form of the $q\bar{q}$ propagators, available for the single $q\bar{q}$ channel, was used.

As it is, the situation with the scalar mesons, and first of all, with lowest scalar mesons, is still unclear and calls for new ideas. As one can see in [1], Table 2, the conventional opinion considers the resonances $a_0(1450)$ and $f_0(1370)$ as the lowest $3^P_0$ states for $I = 1, 0$ respectively. On the other hand, numerous exact calculations of the lowest $3^P_0$ $q\bar{q}$ states with realistic $q\bar{q}$ interaction, including spin-dependent forces refer to $a_0(980)(f_0(980))$ as the lowest $3^P_0$ states, see e.g. [24], while $a_0(1450)$ might be only connected to the first excited state.

There is no consensus on the lowest states ($f_0(500), f_0(980), a_0(980)$) in the modern approaches, including the attempts to derive these states in the molecular or tetraquark approaches. Unfortunately also in this latter approach a recent lattice calculation [25] of the $a_0(980)$ state with account of tetraquark ($q^2\bar{q}^2$) contribution does not show any explicit influence of the latter on the lowest states, thus calling for a new dynamics as a possible source of $f_0(500), f_0(980), a_0(980)$.

It is the purpose of the present paper to suggest a new approach to the solution of this problem and to demonstrate a new quark-chiral dynamics, which might explain the origin of the lowest scalar states. The essence of the method is as follows.

The full analysis of the scalars requires the multichannel approach to the problem, where several quark-antiquark ($q\bar{q}$) channels are present together with two or more Nambu-Goldstone boson channels ($\varphi\varphi$ channels). Therefore complete formulation requires the knowledge of 1) the Green’s functions both in $q\bar{q}$ and $\varphi\varphi$ channels; 2) the transition matrix elements between the channels. Without explicit knowledge of these entries one faces the multi-parameter and multi-channel situation with hardly possible informative output.

The treatment of the first point – the spectral representation of the $q\bar{q}$
Green’s function with accurate calculation of one-channel \(q\bar{q}\) poles and couplings, can be done in the framework of the Field Correlator Method (FCM) (see [26, 27] for reviews and [28] for recent calculations in different channels). The \(\varphi\varphi\) Green’s function in the initial one-channel set-up will be studied here, assuming that it can be replaced by the free two-body propagators and possible resonances exist only due to channel coupling, in particular, with the \(q\bar{q}\) channels, i.e. the problem 2) requires a new approach.

During the last 15 years one of the authors has succeeded to derive the Chiral Confining Lagrangian (CCL) - the powerful tool for the study of chiral effects in connection with quark d.o.f. [29, 30]. The latter contains both the quark and chiral d.o.f. and being nonlocal, tends in the local limit, when all NG momenta are small, to the standard Chiral Lagrangian [31]; all coefficients of CCL are easily calculated, as it was done in [30] in the order of \(p^4\). Moreover, the basic factors, like \(f_\pi, f_K\), are calculated within this method [32]. The only basic parameter, \(M(\lambda)\), which appears due to confinement, is a fixed quantity. In this way the CCL method allows to find analytically all entries 1) and 2), while the scalar decay constants \(f_s\) are calculated in the same way as \(f_\pi, f_K\) within the FCM, using the spectral representation of the Green’s function.

In principle, our method gives a possibility of treating any process with multiple \(q\bar{q}\) and any number of \(\varphi\varphi\) channels; the advantage of using the CCL is that for scalar mesons all transition coefficients are known. In the case of a single \(\varphi\varphi\) and a single \(q\bar{q}\) channel our results can be written in the form, comprising the Breit-Wigner resonance, similarly to results in Refs. [19, 20, 21, 22]. However, in the case of multiple \(\varphi\varphi\) channels more complicated expressions are obtained, using the \(K\)-matrix approach.

As will be seen, the essence of our approach is the summation of the infinite re-scattering series with multiple transitions between \(\varphi\varphi\) and \(q\bar{q}\) states, which yields several poles. In this way we obtain two poles exactly in the regions of \(f_0 (500)\) and \(f_0 (980)\).

The paper is organized as follows. In the next section the general structure of the coupled-channel Green’s function for a scalar meson is derived from CCL and we define basic quantities 1), 2) in terms of known standard coefficients. In section 3 we discuss the \(q\bar{q}\) Green’s functions in the spectral form and define the decay constants and the pole masses from the known confining, gluon exchange and spin-dependent interaction. Note, that this calculation does not use any parameters, beyond the string tension, the current quark masses, \(\Lambda_{QCD}\), and \(M(\lambda)\). In section 4 we discuss the \(\varphi\varphi\) Green’s
function and find the physical \( \varphi \varphi \) amplitudes (\( \pi \pi \) and \( K\bar{K} \)), containing two resonances, which can be associated with \( f_0(500) \) and \( f_0(980) \). In section 5 these results are augmented by the calculation of the real and imaginary parts of the \( \pi \pi \) amplitude, which are qualitatively similar to the results, known from theory and experiment [11, 12]. We also show that in the case of the isospin \( I = 1 \) our method gives a different picture of two nearby poles within 50 MeV in different sheets for \( a_0(980) \). Section 6 contains discussion and an outlook.

2 Coupled channel equations for the scalars from the Chiral Confining Lagrangian

In what follows we are using the Chiral Confining Lagrangian (CCL) [29, 30] with the scalar external currents \( s_0(x) \) and \( s_a(x)\hat{\lambda} \equiv \hat{s} \) for isospin \( I = 0 \) and \( I = 1 \), respectively.

\[
L_{ECCL} = -N_c tr \log(\hat{\partial} + \hat{\bar{m}} + s_0 + \hat{s} + M \hat{U}),
\]

In (1) one has the standard chiral operator,

\[
\hat{U} = \exp(i\gamma_5 \hat{\varphi}), \quad \hat{\varphi} = \frac{\varphi_a \lambda_a}{f_a},
\]

\[
\hat{\varphi} = \sqrt{2} \begin{pmatrix}
\frac{1}{f_\pi} \left( \frac{n}{\sqrt{6}} + \frac{n^0}{\sqrt{2}} \right), \\
\frac{n}{f_\pi}, \\
\frac{K^0}{f_{K^0}}, \\
\end{pmatrix}
\]

In (2) \( \lambda_a \) are the Gell-Mann matrices, \( tr \lambda_a \lambda_b = 2\delta_{ab} \). Using the scalar currents \( s_0, \hat{s} \), one can generate the scalar Green’s functions \( G^{s}_{q\bar{q}}, G_{\varphi \varphi}^{s} \).

\[
L_{CCL} = -N_c tr \log(\Lambda^{-1} + s_0 + \hat{s} + M(\hat{U} - 1)) = -N_c tr \log \Lambda^{-1}(1 + \Lambda(s_0 + \hat{s} + m(\hat{U} - 1))) = \]

\[
= \frac{N_c}{2} tr \left\{ (\Lambda(s_0 + \hat{s})\Lambda(s_0 + \hat{s})) + \ldots \right\} = \frac{N_c}{2} (G_{q\bar{q}}^{s} + G_{\varphi \varphi}^{s}) + \ldots
\]

Here \( \Lambda = \frac{1}{\hat{\partial} + \hat{\bar{m}} + M} \). The corresponding diagram is shown in Fig.1. One can write \( G^{s}_{q}(x, y) = tr \left( \hat{s}(x)g_{q\bar{q}}(x, y)s(y) \right) \), where \( g_{q\bar{q}} \) will be used later.

In (1) the confining kernel \( M(r) \) enters either inside the propagating \( q\bar{q} \) system, in which case it is equal to the confining potential, \( M(r) = \sigma r \), or
else it appears at the vertex of the $q\bar{q}$ Green’s function, connecting it to the $\varphi\varphi$ Green’s function. In this case the vertex $M(r)$ is taken at the effective distance $\lambda$, $M = M(\lambda) = \sigma\lambda$. One can consider this distance $\lambda$ as the spatial width of the transition vertex, connecting $\varphi\varphi$ and $q\bar{q}$ channels, and we take it approximately equal to the correlation length in the confining vacuum, $\lambda \approx 0.2$ fm. As a check of this approximation, this value was used to calculate $f_\pi$ and $f_K$ in good agreement with experimental and lattice data and therefore we fix $M(\lambda) = 0.15$ GeV in what follows. This factor $M(\lambda)$ appears to be the only parameter of our quark-chiral approach in (1) in addition to the quark masses $m_q$ and string tension $\sigma$.

![Figure 1: The scalar $q\bar{q}$ Green’s function $G_{q\bar{q}}$](image)

On the other hand, expanding the CCL (11) in powers of $s_0 + \hat{s} + M(\hat{U} - 1) \equiv \xi$, one obtains another term in the second order in $\xi$,

$$\Delta L = - N_c tr \Lambda s \Lambda M^2 \frac{\hat{s}^2}{2}, \quad s = s_0 + \hat{s}; \quad (5)$$

which corresponds to the diagram of Fig. 2

From (5) one can find the basic quantity, which will be used below, – the transition element $V_{q\bar{q}\varphi\varphi}$ which joins the $q\bar{q}$ Green’s function $g_{q\bar{q}}$ and the $\varphi\varphi$ Green’s function $g_{\varphi\varphi}$, see Fig.3 and its definition below. At this point it is important to understand which kind of the $q\bar{q}$ Green’s function is needed to join it with the $g_{\varphi\varphi}$, i.e. to annihilate at one vertex $q\bar{q}$ and create at this vertex two mesons $\varphi\varphi$. One clearly needs $g_{q\bar{q}}(x, y) \sim (S_q(x, y)S_{\bar{q}}(x, y)$, where $S_q(x, y)$ is the quark Green’s function, but with the definite total momentum, i.e. $g_{q\bar{q}}(P) = \int d^4(x - y)e^{iP(x-y)}tr(S_q(x, y)S_{\bar{q}}(x, y))$; originally $g_{q\bar{q}}(x, y)$ should be connected with $g_{\varphi\varphi}$ at the same point $x$ or $y$ and finally with $g_{\varphi\varphi}(P)$. However $g_{\varphi\varphi}(P)$ is divergent in its real part, which implies that
the transition from $q\bar{q}$ to $\varphi \varphi$ occurs not in one point, but at some distance between $q$ and $\bar{q}$, namely, at the same distance between $\varphi$ and $\varphi$ which we call $r_0 \sim \lambda \sim 0.2$ fm – the transition radius.

It is important that at this moment the $M(r)$ becomes $M(\lambda) = \sigma \lambda \approx 0.15$ GeV, and $\text{Re} \ g_{\varphi \varphi}$ should have a cut-off $N \sim \frac{1}{\lambda} \sim 1$ GeV. As will be shown below, this transition radius does not change much the $g_{\varphi \varphi}(r_0)$, which is anyhow convergent at $r_0 = 0$, but the variation of $\text{Re} \ g_{\varphi \varphi}(r_0)$ might be significant. In this approximation the total scalar Green’s function can be written as

$$G^s = G^s_{\varphi \varphi} + G^s_{q\bar{q}}V G^s_{\varphi \varphi} V G^s_{q\bar{q}} + \ldots = G^s_{q\bar{q}} \frac{1}{1 - VG^s_{\varphi \varphi} V G^s_{q\bar{q}}} \quad (6)$$

Here $V \equiv V_{q\bar{q}/\varphi \varphi}$ can be found from (5).

As it is seen from Fig.3 and (5), the transition coefficient $V$ is proportional to $\frac{M(\lambda)}{f^s_{\varphi}(n)}$, $\varphi = \pi, K$, and also to the quark decay constant of the scalar meson $f^s_{\varphi}(n)$, $(n = 1, 2, \ldots)$ to be found below.

Finally, to define how $V$ depends on isotopic indices, one can according to (5), project $\hat{\varphi}^2$ on a given isotopic state with $I = 0$ or 1.

$$\text{tr} \left( s_0 \frac{\hat{\varphi}^2}{2} \right) = s_0 (a_{11} + a_{22} + a_{33}); \quad (7)$$

$$\text{tr} \left( s_i \frac{\hat{\varphi}^2}{2} \right) = a_{11} \left( s_3 + \frac{1}{\sqrt{3}} s_8 \right) + a_{22} \left( -s_3 + \frac{1}{\sqrt{3}} s_8 \right) + a_{12} (s_1 + is_2) +$$

$$+ a_{21} (s_1 - is_2) - a_{33} \cdot \frac{2}{\sqrt{3}} s_8 \quad (8)$$
where $a_{ik}$ are

$$
a_{11} = \frac{1}{f_\pi^2} \left[ \left( \frac{\eta}{\sqrt{6}} + \frac{\pi^0}{\sqrt{2}} \right)^2 + \pi^+ \pi^- \right] + \frac{K^+ K^-}{f_K^2}, \quad (9)
$$

$$
a_{12} = \frac{2\eta \pi^+}{f_\pi^2 \sqrt{6}} + \frac{K^+ \bar{K}^0}{f_K^2}, \quad a_{21} = \frac{2\eta \pi^-}{f_\pi^2 \sqrt{6}} + \frac{K^0 K^-}{f_K^2}, \quad (10)
$$

$$
a_{22} = \frac{1}{f_\pi^2} \left[ \left( \frac{\eta}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} \right)^2 + \pi^+ \pi^- \right] + \frac{K^0 \bar{K}^0}{f_K^2}, \quad (11)
$$

$$
a_{33} = \frac{K^+ K^- + K^0 \bar{K}^0}{f_K^2} + \frac{2 \eta^2}{3 f_\pi^2}. \quad (12)
$$

Later we shall neglect the isotopic (SU(3)) dependence of the propagators $\Lambda$, apparent in the mass matrices $\hat{m}$, and take it into account at the end, since one can write $g_{q\bar{q}} \equiv g_1 = \begin{pmatrix} g_1(n\bar{n}) & 0 \\ 0 & g_1(s\bar{s}) \end{pmatrix}$.

### 3 The $q\bar{q}$ Green’s functions and the eigenvalues

To calculate the $q\bar{q}$ Green’s functions we shall use the exact relativistic formalism, based on the FCM [26] and essentially exploiting relativistic path integral methods [33, 34, 35, 24]; at the end we shall compare our results with those obtained in other methods.

The $q\bar{q}$ Green’s function $g_{q\bar{q}}^\Gamma(x, y) \equiv g_1(x, y)$ with the vertex $\Gamma$, defining the spin-parity, can be written as

$$
g_1(x, y) = tr \left( \frac{4Y}{(m_1^2 - \hat{D}_1^2)(m_2^2 - \hat{D}_2^2)} \right) \quad (13)
$$

where

$$
4Y = tr[\Gamma(m_1 - \hat{D}_1)\Gamma(m_2 - \hat{D}_2)]. \quad (14)
$$

Then using the relativistic path integral formalism (see [34] for a review) it can be written in the c.m. system and in the Euclidean time $T$

$$
\int d^3(x - y) g_1(x, y) = \frac{T}{2\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} (Y \langle 0 | e^{-H(\omega_1, \omega_2, P)T} | 0 \rangle). \quad (15)
$$
Here the c.m. Hamiltonian $H(\omega_1, \omega_2, p)$ depends on the virtual energies $\omega_1, \omega_2$ and includes all instantaneous interactions, including spin and angular momentum dependent,

$$H(\omega_1, \omega_2, p) = \sum_{i=1,2} \frac{p_i^2 + \omega_i^2 + m_i^2}{2\omega_i} + V_0(r) + V_{so} + V_T. \quad (16)$$

Here $V_0(r) = \sigma r - \frac{4}{3} \frac{\alpha V(r)}{r}$, $V_{so}$ is the spin-orbit interaction and $V_T$ is the tensor interaction, both in the relativistic form. Neglecting spin terms, one can rewrite the last term in (15) as

$$\langle 0|e^{H(\omega_1, \omega_2, p)T}|0 \rangle = \sum_{n=0} \varphi_n^2(0)e^{-M_n(\omega_1, \omega_2)T}, \quad (17)$$

where $\varphi_n(r)$ is the wave function. On the other hand one has a general relation

$$\int g_1(x, y)d^3(x - y) = \sum_n \int d^3(x - y) \langle 0|j_\Gamma|n \rangle \langle n|j_\Gamma|0 \rangle \times e^{i\mathbf{P}(x - y)-M_nT} \frac{d^3\mathbf{P}}{2M_n(2\pi)^3} = \sum_n \varepsilon_\Gamma \otimes \varepsilon_\Gamma \frac{(M_n f_\Gamma^{(n)})^2}{2M_n} e^{-M_nT}. \quad (18)$$

This relation allows to calculate the scalar decay constant $f_s^{(n)}$, which is done in Appendix 1.

Note, that using CCL, Eq. (1), one would have in (14) $m_i + M(\lambda)$ instead of $m_i$, which allows to obtain in the PS case ($\Gamma = \gamma_5$) the correct decay constants $f_\pi, f_K$, which otherwise would be zero in the zero quark mass limit.

In (15) it is convenient to integrate over $d\omega_1, d\omega_2$, using the stationary point method, and for vanishing quark masses $m_i = 0$ one obtains the so-called spinless Salpeter equation; if spin-dependent interactions are neglected. In the first approximation one has

$$\sqrt{p^2 + V_0(r)}\varphi_{nl}(r) = M_{cog}(nl)\varphi_{nl}(r), \quad (19)$$

where $M_{cog}$ means the center-of-gravity mass. Later we use only the fundamental parameters: $\sigma = 0.182(2)$ GeV$^2$ and $\Lambda_V(n_f = 3) = 0.465(15)$ GeV.
Figure 3: The $\pi\pi$ interaction amplitude in terms of the $q\bar{q}$ (solid lines) and $\pi\pi$ Green’s functions (broken lines). The filled and empty circles denote the transition matrix elements $V_{\pi 1} = V_{1\pi}$.

which are well established (see [36] for the definition of $\Lambda_V$ and an accurate perturbative treatment of scalar mesons), and obtain

$$M_{cog}(1P) = 1259(10) \text{ MeV}, \quad \omega_0(1P) = 499 \text{ MeV}.$$  \hfill (20)

Doing calculations in the same way as in [28, 34, 35], we give here the resulting mass of the $1^3P_0$ state with account of the tensor and spin-orbit forces

$$M(1^3P_0) = (1259(10) - 214) \text{ MeV} = 1045(10) \text{ MeV},$$  \hfill (21)

which defines the $q\bar{q}$ initial mass of $f_0$ and $a_0$, taken below, as $M_1 = 1 \text{ GeV}$. This mass can be compared with that obtained by other groups, where in [37] $M(0^{++}) = 1090 \text{ MeV}$, while in [38] $M(0^{++}) = 1176 \text{ MeV}$, and in [39], $M(0^{++}) = 970 \text{ MeV}$.

Note, that the first excited state in the $0^{++}, I = 0$ channel is obtained to be $M_2 = 1474 \text{ MeV}$ [35] and this state can be associated with the $a_0(1450)$.

Finally we can use (18) to calculate the full Fourier transform of $g_1(x, y)$ in the Minkovskian time, which yields

$$\tilde{g}_1(P) = \tilde{g}_1(E, P = 0) = \sum_n \frac{(f_0^{(n)})^2 M_n^2}{M_n^2 - E^2}. \quad (22)$$

This form with the lowest $n = 1$ will be used below to analyse the scalars $f_0$; it will be shown that the level $M_1 = 1045 \text{ MeV}$ generates both $f_0(500)$ and $f_0(980)$ resonances, connected respectively with the $\pi\pi$ and $K\bar{K}$ Green’s functions.
4 Analytic structure of physical amplitudes

We start with the structure of the meson-meson Green’s function, which at first we take as free two body relativistic Green’s function of two scalar particles with the total momentum \( P = 0 \) and the total c.m. energy \( E \). Then in the \( \varphi\varphi \) channel the free Green’s function is

\[
g_2(E) = \int \frac{d^4p}{(2\pi)^4(p^2 - m_1^2)((P - p)^2 - m_2^2)},
\]

with \( P = 0, P_0 = E \). Its imaginary part is

\[
\text{Im} g_2(E) = \frac{\sqrt{E^2 - (m_1 + m_2)^2)(E^2 - (m_1 - m_2)^2)}}{16\pi E^2},
\]

and for equal masses \( m_1 = m_2 = m \) one has for the real part

\[
\text{Re} g_2(E = 2m) = \frac{1}{8\pi^2} \ln \left( \frac{N + \sqrt{N^2 + m^2}}{m} \right),
\]

where a cut-off \( N \) was introduced in the integral \( d^3p \).

The term in (25) for the \( K\bar{K} \) Green’s function yields 0.0183 for \( N = 1 \) GeV and 0.027 for \( N = 2 \) GeV; the case with unequal masses \( m_1, m_2 \) and \( E < 2m \) is presented in Appendix 3.

Now we turn to the definition of the transition coefficients \( V_{\bar{q}q|\varphi\varphi} \), appearing in (6). Using the definition of \( \tilde{g}_1(P) \) (22) and leaving for \( \tilde{g}_1 \) only the combination \( \frac{M_2^2}{M_2^2 - E^2} \), one can associate the transition coefficient with the value

\[
V_{\bar{q}q|\varphi\varphi} V_{\varphi\varphi|\bar{q}q} = (V_{\bar{q}q|\varphi\varphi})^2 = \frac{C_i^2 m^2(\lambda)(f_{\varphi}^{(n)})^2}{f_{\varphi}^4}, \quad f_{\varphi} = f_\pi, f_K.
\]

Here the coefficient \( C_i \) can be found from (7-12). Note, however, that \( SU(3) \) symmetry is violated in the \( \bar{q}q \) sector, since the scalar meson masses in \( (n\bar{n}) = (u\bar{u}, d\bar{d}) \) channels are \( \sim 500 \) MeV lower than in the \( (s\bar{s}) \) channel. Introducing notation \( C_i = C_i^{\text{meson,meson}} \) one obtains from (5) and (7-12),

\[
(C_{\pi\pi}^{(0)})^2 = 3; \quad (C_{K\bar{K}}^{(0)})^2 = 2; \quad (C_{K\bar{K}}^{(1)})^2 = 2; \quad (C_{\pi\eta}^{(0)})^2 = \frac{2}{\sqrt{3}} |\sin \theta_P|.
\]

We have taken into account in (27) that the \( \eta \) can be decomposed as \( \phi_\eta = \psi_8 \cos \theta_P - \psi_1 \sin \theta_P, \quad \sin \theta_P \cong -0.26 \).
We start with the one-threshold situation and choose the channel $\pi\pi$, neglecting its connection to $K\bar{K}$. In this case one has the following basic elements, with notation $g^2(\pi\pi, E) \equiv g_{\pi}$, $\tilde{g}_1(E, \mathbf{P} = 0) = g_1$, where we keep the lowest pole $M_1$,

$$V_{\pi1} = V_\pi = \frac{C_{\pi\pi}(0) M(\lambda) f^{(1)}_\pi}{f^2_\pi}, \quad g_1 = \frac{M^2_1}{M^2_1 - E^2},$$

(28)

and the infinite series for the total $\pi\pi$ Green’s function reads, see Fig.3

$$G_{\pi\pi} = g_\pi + g_\pi V_{\pi1} g_1 V_\pi g_\pi + g_\pi V_{\pi1} g_1 V_\pi g_\pi V_{\pi1} g_1 V_\pi g_\pi + ..., $$

(29)

which can be summed up in the form

$$G_{\pi\pi} = g_\pi + g_\pi V_{\pi1} \frac{1}{1 - g_1 V_\pi g_\pi V_{\pi1}} g_1 V_\pi g_\pi.$$  

(30)

For the $\pi\pi$ scattering amplitude $A_\pi(e)$ one has

$$A_\pi(E) = V_{\pi1} \frac{1}{1 - \Box_\pi} g_1 V_\pi,$$  

(31)

where we have defined the 4-term code $\Box_\pi \equiv g_1 V_\pi g_\pi V_{\pi1}$.

In an analogous way one can define the one-channel $K\bar{K}$ Green’s function and amplitude

$$A_K(E) = V_{K1} \frac{1}{1 - \Box_K} g_1 V_K,$$  

(32)

where

$$\Box_K = g_1 V_{1K} g_K V_{K1}, \quad V_{K1} = V_{K1} = \frac{C_{K\bar{K}}(0) M(\lambda) f^{(1)}_K}{f_K}.$$  

(33)

Note, that the $q\bar{q}$ pole at $E^2 = M^2_1$ is cancelled in (31), (32); the only visible singularity is the unitary cut in $g_\pi$ and $g_K$ respectively.

One can check the unitarity of both amplitudes $A_\pi$ and $A_K$,

$$\text{Im} A_\pi(E) = \frac{k_\pi}{8\pi E} |A_\pi(E)|^2$$  

(34)

and the similar form for $A_K$ is valid with replacement $\pi \to K$. 

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At this point it is useful to introduce the dimensionless transition coefficient \( k(I) (q\bar{q} \text{ (meson, meson)}) \), which actually defines the coupling of \( q\bar{q} \) and meson-meson channels, e.g.,

\[
k(I) (q\bar{q} | \pi\pi) = \frac{(C^{I}_{\pi\pi})^2 M^2(\lambda)(f_{s}^{(n)})^2}{f_{\pi}^4}, \quad k(I) (q\bar{q} | K\bar{K}) = \frac{(C^{I}_{K\bar{K}})^2 M^2(\lambda)(f_{s}^{(n)})^2}{f_{K}^4}.
\]

(35)

One can also find the position of the pole in the amplitude \( A_{\pi}(E) \) from the denominator in (31), \( \Box_{\pi}(E) = 1 \). One has

\[
g_{\pi}k^{(0)}(n\bar{n}|\pi\pi) \frac{M_{1}^2}{M_{1}^2 - E^2} = 1, \quad g_{\pi}(E) = \text{Re} g_{\pi} + i \text{Im} g_{\pi}.
\]

(36)

We take here \( M_{1} = 1.05 \text{ GeV} \) as follows from (21).

In the real part of \( g_{\pi}(E) \) the cut-off \( N \) is taken at large momenta in (23), equal to the minimal length \( \lambda, N = 1/\lambda \), which yields \( (N = 1 \text{ GeV}) \).

\[
g_{\pi}(E) \approx 0.033 + i0.02 \sqrt{1 - \frac{0.078}{E^2}}.
\]

(37)

Inserting in (35) for \( I = 0 \) \( f_{\pi} = 93 \text{ MeV}, f_{s}^{(1)} = 125 \text{ MeV} \) (see Appendix 1), and \( M(\lambda) = 150 \text{ MeV}, (C_{\pi\pi}^{(0)})^2 = 3 \), one obtains the equation

\[
E^2 = M_{1}^2 (1 - 14.1 g_{\pi}(E))
\]

(38)

or using (37), one obtains the cut-off dependent resonance position \( E_{\pi} = (0.77 - i0.18) \text{ GeV (N = 1 GeV)} \), and \( E_{\pi} = (0.68 - i0.21) \text{ GeV (N = 2 GeV)} \). As a result one obtains the the resonance parameters in the range

\[
E_{\pi} = (0.6 \div 0.8) \text{ GeV} - i(0.18 \div 0.21) \text{GeV}.
\]

(39)

This can be favorably compared with the experimental values \( f_{0}(500), E = (400 - 550) \text{ MeV}, \Gamma = 400 \div 700 \text{ MeV} \) [1]. Note that we have obtained these values, however, with several simplifying approximations, including the neglect of higher levels in \( g_{1} \), possible coupling with the \( K\bar{K} \) channel and notably neglecting the \( 4\pi, 6\pi, \ldots \) vertices of the chiral theory, which imply the \( \pi\pi \) interaction in \( g_{\pi}(E) \). Therefore the resonance position and the width are subject to essential changes.

We now turn to the \( K\bar{K} \) channel, again neglecting connection to the \( \pi\pi \) channel and keeping only the lowest mass eigenvalue \( M_{1} = 1.05 \text{ GeV} \) in
\[ g_1(E). \] Inserting in (35) \((C_{KK}^{(0)})^2 = 2, f_K = 115 \text{ MeV}, M(\lambda) = 150 \text{ MeV}, \text{ and } f_s = 125 \text{ MeV}, \] one obtains \(k^{(0)}(n\bar{n} K \bar{K}) = 4.02\). From (32) one finds the equation for the pole position, \(\Box_{KK} = 1\), or

\[ E^2 = M_1^2(1 - 4.02 g_K(E)), \] (40)

where \(g_K(E)\) with the upper limit \(N = 1 \text{ GeV}\) in (25)

\[ g_K(E) = (0.018) + i0.02 \sqrt{\frac{E^2 - 4m_K^2}{E^2}}, \] (41)

which yields an approximate position of the pole

\[ E_K = (1.02-i0.011) \text{ GeV}(N = 0.5 \text{ GeV}); \ E_K(1.01-i0.009) \text{ GeV}(N = 1 \text{ GeV}). \] (42)

One can see that the pole \(E_K\) can be associated with the standard \(f_0(980)\) \[\square\]

\[ M(f_0(980) = (990 \pm 20) \text{ MeV}, \ \Gamma = (10 \div 100) \text{ MeV}, \] (43)

while the obtained width is inside the allowed region. It is interesting that in this case the cut-off \(N\) in the range \((0.5 \div 2) \text{ GeV}\) brings about only few percent change in the resulting resonance parameters. Taking into account the approximations made above, this agreement can be considered as reasonable, however, one should take into account, that both channels \(\pi \pi\) and \(K \bar{K}\) should be connected, as it is seen in the experimental measurements of the ratio for \(f_0(980), \frac{\Gamma(KK)}{\Gamma(\pi\pi)} = 0.69 \pm 0.32 \);\[\square\]

The standard way to include the \(\varphi\varphi\) channel coupling is to write for the amplitudes \(\hat{A}_{\alpha\beta} = \begin{pmatrix} A_{\pi\pi} & A_{\pi K} \\ A_{K\pi} & A_{KK} \end{pmatrix}\) the \(K\) matrix form,

\[ \hat{A}^{-1} = \begin{pmatrix} \frac{1-\Box_{\pi\pi}}{w_\pi} & a \\ b & \frac{1-\Box_{KK}}{w_K} \end{pmatrix}, \quad w_\pi = V_{\pi 1}g_1V_{1\pi}. \] (44)

As a result one obtains

\[ \hat{A} = \begin{pmatrix} \frac{1-\Box_{KK}}{w_K} & -b \\ -a & \frac{1-\Box_{\pi\pi}}{w_\pi} \end{pmatrix}, \] (45)

and in the limit \(ab = 0\) one returns to the two independent channels.
One can check that the amplitudes $A_{\alpha\beta}, \alpha, \beta = \pi, \pi, K\bar{K}$ satisfy the unitarity relations with the normalization factor $\text{Im} \ g_{\pi,K} = \frac{k_{\pi,K}(E)}{8\pi E}$. In particular for the $A_{K\bar{K}}$ one has in this channel coupling (CC) form

$$A_{K\bar{K}} = \frac{(1 - \Box_{\pi})w_{K}}{(1 - \Box_{\pi})(1 - \Box_{K}) - abw_{K}w_{\pi}}$$

Estimating the $w_{K}, w_{\pi}$ one finds that the CC can affect the positions and the widths of the uncoupled resonances (40), (42) and therefore this point should be studied in more detail.

We shall start with the $A_{\pi\pi}$ amplitude, which can be written as follows

$$A_{\pi\pi} = \frac{1}{w_{K} - g_{K}} \frac{(1 - \Box_{\pi})w_{K}}{(1 - \Box_{\pi})(1 - \Box_{K}) - abw_{K}w_{\pi}}$$

(47)

Numerically the coefficients in (47) are

$$w_{\pi} = V_{\pi}g_{1}V_{1\pi} = \frac{4M^{2}(\lambda)(f_{s}^{(1)})^{2}M_{1}^{2}}{f_{\pi}^{4}(M_{1}^{2} - E^{2})} = \frac{14.1M_{1}^{2}}{M_{1}^{2} - E^{2}};$$

(48)

$$w_{K} = V_{K}g_{1}V_{1K} = \frac{4M^{2}(\lambda)(f_{s}^{(1)})^{2}M_{1}^{2}}{f_{K}^{4}(M_{1}^{2} - E^{2})} = \frac{4.02M_{1}^{2}}{M_{1}^{2} - E^{2}};$$

(49)

$$g_{\pi}(E) = \text{Re} \ g_{\pi} + i \text{Im} \ g_{\pi} = 0.033 + i0.02\sqrt{\frac{E^{2} - 4m_{\pi}^{2}}{E^{2}}}$$

(50)

$$g_{K}(E) = \text{Re} \ g_{K} + i \text{Im} \ g_{K} = 0.011 + i0.02\sqrt{\frac{E^{2} - 4m_{K}^{2}}{E^{2}}}$$

(51)

and we can rewrite (47) as follows using $\gamma = 40.4 \ ab$ and the properly normalized amplitudes $\text{Im} \ f_{\pi}^{(0)} = \frac{2k}{E}|f_{\pi}^{(0)}|^2$

$$f_{\pi}^{(0)} = \frac{1}{16\pi}A_{\pi\pi}, \quad f_{K}^{(0)} = \frac{1}{16\pi}A_{K\bar{K}}$$

(52)

$$f_{\pi}^{(0)} = \frac{1}{3.56\frac{(E^{2} - E_{\pi}^{2})}{M_{1}^{2}} - \frac{\gamma M_{1}^{2}}{E_{K} - E_{\pi}}}$$

(53)
with
\[ E_\pi^2 = M_1^2 \left( 0.38 - i0.38 \sqrt{\frac{E^2 - 4m^2}{E^2}} \right), \quad E_K^2 = M_1^2 \left( 0.91 - i0.16 \sqrt{\frac{E^2 - 4m^2}{E^2}} \right). \]

(54)

Analogously for \( f_K^{(0)} \) one has
\[ f_K^{(0)} = \frac{1}{\frac{12.5}{M_1^2}(E_K^2 - E^2) - \frac{2.34\gamma M_1^2}{E^2 - E^2}}. \]

(55)

One can estimate the ratio of imaginary parts of the first and the second term in the denominator of (55), which yields the order of magnitude of the ratio of \( \Gamma_{KK}(f(980)) \) and \( \Gamma_{\pi\pi}(f(980)) \) at \( E = 1.00 \text{ GeV} \),

\[ \frac{\Gamma_{\pi\pi}(f(980))}{\Gamma_{KK}(f(980))} \approx \frac{2.4\gamma \sqrt{E^2 - 4m^2}}{\sqrt{E^2 - 4m^2_K}} \approx 17\gamma \]

and one can see that this ratio is around 1 for \( \gamma = 0.05 \), found in the next section by comparison with data. The resulting pole is near the \( KK \) threshold and satisfies the criteria of the \( f_K^{(0)}(980) \) resonance.

5 Results and discussion

From (53) one can see that the amplitude \( f_\pi^{(0)} \) can be expressed via the Green's functions \( g_\pi \) and \( g_K \) with the only parameter \( \lambda \), responsible for the coupling of channels \( \pi\pi \) and \( KK \). However, the real parts of both Green's functions contain the cut-off parameter \( N \approx 1/\lambda \approx O(1 \text{ GeV}) \), which makes these expressions approximate.

The main result of our approach, based on the CCL (1), is that the \( g\bar{q} \) pole at 1 GeV produces two quark-chiral resonances: \( f_0(500) \) due to coupling \( n\bar{n} - \pi\pi \), and \( f_0(980) \) due to coupling \( n\bar{n} - K\bar{K} \) \((n = u, d)\). This is a new feature, which is specific for our quark-chiral Lagrangian, and is obtained from the infinite sums of products of \( \Box_\pi \) and \( \Box_K \). It is interesting that these features are not very sensitive to the \( \pi\pi \) interaction at low energies, which is governed by the chiral Lagrangian. Indeed we have not taken it into account and as a result the \( \pi\pi \) scattering length is much higher than the experimental value: 0.36/m_\pi instead of the experimental number 0.23 m_\pi^{-1} [40]. The possible reason is that the low energy physics is only mildly

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connected to the $f^{(0)}$ resonance physics, and we have neglected the low energy \(\pi\pi\) interaction given by the chiral Lagrangian.

To make this point more quantitative, let us introduce the purely chiral \(\pi\pi\) amplitude via 4\(\pi\) vertex \(\alpha_{\pi}\), so that our equation for the \(\pi\pi\) amplitude in (31) will be changed by the addition of \(\alpha_{\pi}\) to the vertex \(V_{\pi 1} g_{1} V_{1\pi}\):

\[
V_{\pi 1} g_{1} V_{1\pi} \rightarrow V_{\pi 1} g_{1} V_{1\pi} + \alpha_{\pi}
\]

and Eq. (31) acquires the form (in the one-channel case)

\[
A_{\pi}(E) = \frac{(w_{\pi} + \alpha_{\pi})}{1 - g_{\pi}(w_{\pi} + \alpha_{\pi})}, \quad w_{\pi} = \frac{k^{(0)}(n\bar{n}|\pi\pi)}{1 - E^{2}}
\]

and the equation for the pole is

\[
E^{2} = 1 - \frac{14.1 \left( 0.033 + i0.02 \sqrt{\frac{E^{2} - 4m_{\pi}^{2}}{E^{4}}} \right)}{1 - \alpha_{\pi} \left( 0.033 + i0.02 \sqrt{\frac{E^{2} - 4m_{\pi}^{2}}{E^{4}}} \right)}. \quad (59)
\]

One can see in (59) that the main effect of \(\alpha_{\pi}\) is the change of \(g_{\pi}\), and first of all of its real part at low \(E\).

Therefore in our general two-channel form we include a possible modification of real part of \(g_{\pi}\), which is contained in the term \(E^{2}_{\pi}\) in (54), which leads to the following two-channel form, generalizing (53), with slightly shifted positions of resonances

\[
f_{\pi}(E) = \frac{1}{F_{1}(E) - \frac{\gamma}{F_{2}(E)}}, \quad F_{1}(E) = 2.67(1 - E^{2}) - x(E) - i\sqrt{1 - \frac{0.0784}{E^{2}}}
\]

\[
F_{2}(E) = 0.96 - 0.043 i\sqrt{1 - \frac{0.975}{E^{2}}} \theta(E^{2} - 0.975) - E^{2}
\]

Here \(x(E)\) is connected with Re \(g_{\pi}(E) = x(E)/50.2\) and we choose two possible cases:

1. \(x(E)\) is constant and corresponds to Re \(g_{\pi}\) with \(N = 1 \text{ GeV}\), Re \(g_{\pi} = 0.033\).

2. \(x(E)\) is chosen as a fitting parameter at small \(E < 0.6 \text{ GeV}\), ensuring the correct scattering amplitude at \(E = 2m_{\pi}\).
Figure 4: Re $f_\pi^{(0)}(E)$ as a function of $E$ in GeV from Eq. (53) (grey bands) in comparison with the resulting curves from the Pelaez et al. [41, 42] (broken lines) comprising the $\pi\pi$ data.

The two curves $f_\pi(E)$, corresponding to the conditions 1) and 2), are shown in Figs. 4 and 5 together with the curves from the paper of Pelaez et al. [41], obtained in the course of the analysis in [42]. As one can see, our Re $f_\pi(E)$ and Im $f_\pi(E)$ are in a qualitative agreement with the results of [41], with an exclusion of the region of relatively small energies, $E < 0.5$ GeV, where our lowest curve of Re $f_\pi(E)$ is made closer to the corresponding curve of [41], by changing Re $g_\pi(E)$, which actually is a fitting procedure.

We have checked that variation of $\alpha_\pi$ as in (59) leads to the same lower curves in Figs.4,5, implying that this is equivalent to the variation of Re $g_\pi$.

This means that the 4$\pi$ vertices are important in this region and the approximation of the free $\pi\pi$ Green’s function should be modified by inclusion of the purely chiral interactions.

However it was not the purpose of our study to reproduce exactly the $\pi\pi$ interaction amplitude in the whole region (280, 1000) MeV, but rather to discover the dynamical mechanism, producing the lowest scalar-isoscalar mesons $f_0(500)$ and $f_0(980)$. As shown, this mechanism can be reduced in its basic part to the interaction of the $q\bar{q}$ and free meson-meson channels, given by our quark-chiral interaction in the CCL, Eq. (1). Indeed, this interaction
provides the reasonable coupling $V_{q\bar{q}|\pi\pi}$ and $V_{q\bar{q}|K\bar{K}}$, in addition to the values of the $q\bar{q}$ Green’s functions and the corresponding poles $M_n(q\bar{q})$. In our case the lowest pole $M_1(q\bar{q})$ at 1 GeV produces a wide resonance $f_0(500)$ in “collaboration” with the $\pi\pi$ Green’s function and the $\pi\pi$ threshold, and a more narrow resonance $f_0(980)$ in “collaboration” with the $K\bar{K}$ Green’s function and the threshold. The interaction of these two channels, strongly shifted in energy from each other, only slightly modifies their individual properties, as can be seen comparing one-channel and coupled-channel characteristics.

This is the main concrete result of this paper, however, the general mechanism, described above, leads to many further possible discoveries.

At this point one can immediately ask: if the same $q\bar{q}$ level can create many resonances, roughly one for any $\varphi\varphi$ channel, what happens with $a_0(980)$ resonance, which can decay both to $\pi\eta$ and $K\bar{K}$, but in experiment one can see only one broad resonance near the $K\bar{K}$ threshold. Now we apply our technique to this case to understand the difference between the situation with $a_0(980)$ on one hand and $f_0(500), f_0(980)$ on another.

To this end we shall try to find separate resonances in the $\pi\eta$ and $K\bar{K}$ channels and write, as in (38), the resulting equation for the position of the assumed resonances $E^{(1)}(\pi\eta)$ and $E^{(1)}(K\bar{K})$, where the upper index refers to the isospin $I = 1$.

$$ (E^{(1)}(\nu))^2 = M_1^2 \left( 1 - (V_{\nu\nu} V_{\nu\nu})^2 g_\nu(E) \right); \; \nu = \pi\eta, K\bar{K}. \quad (62) $$
Now using (28) and (8)-(12) one can write:

\[ k^{(1)}(n\bar{n}|\pi\eta) \approx V_{\pi\eta,1}V_{1,\pi\eta} = (C_{\pi\eta}^{(1)})^2 \frac{M^2(\lambda)(f_\pi^{(1)})^2}{f_\pi^4} = \frac{1}{10}(C_{\pi\pi}^{(0)})^2 = 0.141 \]  

(63)

\[ k^{(1)}(n\bar{n}|K\bar{K}) = \frac{(C_{K\bar{K}}^{(1)})^2 M^2(\lambda)(f_\pi^{(1)})^2}{f_K^4} = 2.01 \]  

(64)

(We have neglected the difference between \( f_\pi \) and \( f_\eta \) for a rough estimate, in the real case the coefficient in (63) is smaller).

Now \( g_{\pi\eta}(E) \) has a smaller real and imaginary parts, (see Appendix 2 for details) as compared with \( g_{\pi\pi}(E) \) Eq. (37), while \( g_{K\bar{K}}(E) \) is the same as was used before, see Eq.(41). As a result, the solution of Eq. (62) gives two resonances

\[ E^{(1)}(\nu) = M_1(1 - \bar{a}_\nu - i\bar{b}_\nu), \]  

(65)

where a rough estimate yields

\[ \bar{a}_{\pi\eta} \approx 0.002, \quad \bar{b}_{\pi\eta} \approx 0.001 \]  

(66)

\[ \bar{a}_{K\bar{K}} = 0.022, \quad \bar{b}_{K\bar{K}} = 0.04\sqrt{\frac{E^2 - 4m_K^2}{E^2}}. \]  

(67)

One should take into account that \( M_1(I = 1) \approx M_1(I = 0) = 1.05 \) GeV and obtains \( E^{(1)}(\pi\eta) \approx (1.05) \) GeV, while \( E^{(1)}(K\bar{K}) \approx \left(1.04 - i0.02\sqrt{\frac{E^2 - 4m_K^2}{E^2}}\right) \) GeV, and \( E^{(1)}(\pi\eta) \) is on the second sheet with the \( \pi\eta \) threshold, while \( E^{(1)}(K\bar{K}) \) on the second sheet with the \( K\bar{K} \) threshold.

Thus one can see that the displacements of both resonances are small, being of the order of the width of resonances. This might be the reason why in experiment one actually observes one resonance \( a_0(980) \) near 1 GeV with two decay modes, while in the \( I = 0 \) channel with larger couplings \( k^{(0)}(n\bar{n}|\pi\pi) \) and more distant \( \pi\pi \) and \( K\bar{K} \) thresholds one observes two distinct resonances, and this example gives an additional support for our theory.

6 Conclusions and an outlook

Summarizing, the method suggested above has a general character and can be applied to any systems, consisting of several components, which can transform one into another. The only information, needed to describe the properties of such mixed systems, is the spectral properties of each component.
and transition coefficients. In the case of the quark-chiral system, $q\bar{q} - \varphi\varphi$, this information is given by the FCM approach plus quark-chiral CCL Lagrangian \[1\]. As it is, our method allows to resolve the old-standing problem of $f_0(500)$, $f_0(980)$ and $a_0(980)$ and to prove that these resonances belong to the lowest ($n = 1$) $q\bar{q} \, 3P_0$ states.

As applied to the lowest scalar resonances, we have shown that the resonances $f_0(500)$ and $f_0(980)$, as well as the $a_0(980)$ resonance, can be derived with the use of CCL, if one considers $n = 1, M \approx 1$ GeV $q\bar{q}$ resonance and takes into account its multiple transitions into chiral meson states and back. Then several questions arise:

1. Proving that $f_0(500), f_0(980)$ are produced by one $q\bar{q}$ state – the $3P_0$ ground state $n\bar{n}$ with mass around 1 GeV, one should consider the next $q\bar{q}$ state, $M_2(1474)$ as an excited $q\bar{q}$ state with $n = 2$, in contrast to an accepted view that this latter is a ground state. It is interesting to study consequences of this assignment.

2. What will be the result for excited $q\bar{q} - \varphi\varphi$ states, e.g. with $M_2 = 1474$ MeV, in connection with the same $\varphi\varphi$ thresholds and can one expect more additional resonances below $M_2$?

3. It is clear that taking into account the full sum $\sum_n \frac{M^2_n}{E^2_n}$ one meets with divergences and with the necessity of renormalization. This probably can be treated in the spirit of the formalism, developed in the method of Matrix Product States (MPS), see [43] for reviews.

4. We have considered above only one $q\bar{q}$ channel. However, for the $K\bar{K}$ system the $s\bar{s}$ channel provides bound states starting with $M_1 \approx 1400$ MeV, just near the first excited $n\bar{n}$ state. Therefore for the $K\bar{K}$ system one should take into account both $n\bar{n}$ and $s\bar{s}$ states, which requires an extension of our method with inclusion of several $q\bar{q}$ and one or more $\varphi\varphi$ channels to explain several extra resonances in the region 1300-1700 MeV, observed in experiment \[1\].

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Appendix A1. Decay constants of scalar mesons

In the framework of the path-integral formalism the decay constants of the $q\bar{q}$ meson states can be defined as in \cite{28,35}

\[
(f^{(n)}_\Gamma)^2 = \frac{2N_c\langle Y_\Gamma\rangle|\varphi_n(0)|^2}{\omega_1\omega_2 M_n}, \tag{A1.1}
\]

where $\omega_1, \omega_2$ are average energies of quarks with masses $m_1$ and $m_2$, $M_n$ is the mass of the meson, $\varphi_n(r)$ is the (relativistic) meson wave function of the relative distance $r$, while $\langle Y_\Gamma\rangle$ is

\[
\langle Y_\Gamma\rangle = \text{tr}(\left(m_1 - \hat{D}_1\right)\Gamma(m_2 - \hat{D}_2)\Gamma) = \text{tr}(\left(m_1 - i\hat{p}_1\right)\Gamma(m_2 + i\hat{p}_2)\Gamma). \tag{A1.2}
\]

Here $\Gamma$ is the vertex operator, for the scalar particle $\Gamma_s = 1$, but the momentum operators $\hat{p}_i$ are acting on the wave function $\varphi_n(r)$, namely

\[
i\hat{p}_i\varphi_n(r) = \partial_i\varphi_n(r).
\]

In our case

\[
\langle Y_s\rangle|\varphi_n(0)|^2 = (m_1m_2 - \omega_1\omega_2 - \hat{p}\hat{p}')|\Psi_S(0)|^2 \to (\partial_i\Psi_S(r)\partial'_i\Psi_S^*(r'))_{r\to0,r'\to0}.
\]

(A1.3)

Since $\Psi_S(r)$ is

\[
\Psi_S(r) = \sum \chi_{1m_1}\tilde{Y}_{1m_2}\frac{\varphi(r)}{r}C_{1m_1,1m_2}^{00}, \tag{A1.4}
\]

and $\tilde{Y}_{1m} \equiv rY_{1m}$, after summation over spin projections one finds

\[
\partial_i\Psi_S(r)\partial'_i\Psi_S^*(r') = \partial_i\partial'_i\frac{1}{4\pi}(xx' + yy' + zz') = \frac{1}{4\pi} \tag{A1.5}
\]

where we have taken into account, that the subscript $i$ refers to a fixed momentum direction. As a result one obtains

\[
(f^{(n)}_S)^2 = \frac{2N_c(R'_{nP}(0))^2}{4\pi\omega_1\omega_2 M_n}, \tag{A1.6}
\]

where $R'_{nP}(0) = \left(\frac{\varphi_n(r)}{r}\right)_{r\to0}$. Estimated in the same way as in \cite{28,35} for the $1P$ scalar state one has $R'_{1P}(0) = 0.086$ GeV$^{5/2}$, $\omega_1 = \omega_2 = 0.448$ GeV \cite{44} and according to (A1.6) one obtains

\[
(f^{(1)}_S)^2 = 0.01568 \text{ GeV}^2, \quad f^{(1)}_S = 0.125 \text{ GeV}. \tag{A1.7}
\]

For the first excited state, $2P$, one has for the scalar state $\omega(2P) \cong 0.5$ GeV, $R'_{2P}(0) = 0.0817$ GeV$^{5/2}$, $M(2P) = 1.474$ GeV \cite{44}. 21
As a result one obtains from (A1.6)

\[(f_s^{(2)})^2 = 0.00865 \text{ GeV}^2, \quad f_s^{(2)} = 0.093 \text{ GeV} .\]  

(A1.8)

Appendix A2. Meson-meson Green’s functions

The relativistic Green’s function of two scalar mesons with the total momentum $P$ can be written in the Euclidean space-time as

\[g(P) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{[(P-p)^2 + m_1^2][(p^2 + m_2^2)]}. \]  

(A2.1)

Integrating over $dp_4$ in the c.m. frame, $P = 0$, one obtains, with $P^4 = iE$, and $m_1 > m_2$,

\[\text{Re} \, g_{12}(E) = \int_0^N \frac{p^2 dp}{4\pi^2} \times \frac{E(\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}) + m_1^2 - m_2^2}{\sqrt{p^2 + m_1^2}} \frac{\sqrt{p^2 + m_2^2}[(\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2})^2 - E^2]}{E + \sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2}}.\]  

(A2.2)

Here we have introduced the cut-off $N$ in momentum $p$.

\[\text{Im} \, g(E) = \frac{1}{16\pi} \sqrt{(E^2 - (m_1 + m_2)^2)(E^2 - (m_1 - m_2)^2)} \frac{E}{E}.\]  

(A2.3)

In the equal mass limit one obtains

\[\text{Re} \, g(E) = \int_0^N \frac{p^2 dp}{8\pi^2 \sqrt{p^2 + m^2} (p^2 + m^2 - E^2/4)},\]  

(A2.4)

which for $E^2 = 4m^2$ reduces to a simple answer

\[\text{Re} \, g(2m) = \frac{1}{8\pi^2} \int_0^N \frac{dp}{\sqrt{p^2 + m^2}} = \frac{1}{8\pi^2} \ln \frac{N + \sqrt{N^2 + m^2}}{m}.\]  

(A2.5)

For $E^2 = 4m^2 - 4\Delta$, $\Delta > 0$ one has instead of (A2.5)

\[\text{Re} \, g(E) = \frac{1}{8\pi^2} \int_0^N \frac{dp}{\sqrt{p^2 + m^2}} - \frac{\Delta}{8\pi^2} \int_0^N \frac{dp}{\sqrt{p^2 + m^2} (p^2 + \Delta)}.\]  

(A2.6)
and for \( m^2 \gg \Delta \) the last integral in (A2.6) can be written as

\[
\Delta \text{Re} g(E) \cong -\frac{\sqrt{\Delta}}{16\pi m} \theta \left( m^2 - \frac{E^2}{4} \right) .
\]  

(A2.7)

Note, that \( \Delta \text{Re} g(E) \) is much smaller than \( \text{Re} g(2m) \), Eq. (A2.5) and can be neglected in the first approximation.

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