Remarks on the solutions
of the
Maxwell- Chern-Simons theories

Z. NÉMETH (1)

Abstract.

The large distance behavior of the Maxwell- Chern-Simons (MCS) equations is analyzed, and it is found that the pure Chern-Simons limit, (when the Maxwell term is dropped from the equations), does not describe the large distance limit of the MCS model. This necessitates the solution of the original problem. The MCS gauge theory coupled to a nonrelativistic matter field, (governed by the gauged non-linear Schrödinger equation), is studied. It turns out, that there are no regular self-dual solutions as in the pure Chern-Simons case, but the model admits interesting, though singular self-dual solutions. The properties of these solutions, and their large distance limits are analyzed.

PACS numbers: 11.10.Lm, 11.10.Jj, 11.27.+d, 11.15.-q

(28/3/2022)

(1) Institute for Theoretical Physics, Eötvös University, H-1088 BUDAPEST, Puskin u. 5-7 (Hungary). e-mail: nemethz@ludens.elte.hu
It is a well known fact, that in the 2+1 dimensional electrodynamics, the Lagrange density of the gauge field can be modified by adding the Chern-Simons term \((\kappa/4)\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha\). This modification has many interesting consequences, like e.g. the existence of self-dual vortex solutions in the pure Chern-Simons limit, (when the gauge dynamics is governed by the CS term only)[1,2,3,4,5]. These self-dual solutions have many important applications in solid state physics, e.g. in the theory of fractional Quantum Hall Effect [6]. The Chern-Simons modified electrodynamics may describe anyons, and give a method for the perturbative description of the Aharonov-Bohm scattering [7]. Since in a real situation we must suppose the presence of the Maxwell term, it is an interesting task to find self-dual solutions of the Maxwell- Chern-Simons (MCS) system. The problem becomes more interesting, if we recognize, that the pure CS limit in general can not describe the large distance behavior of the MCS theory, as it was believed earlier.

In the first part of this paper the asymptotic properties of the MCS system are studied, and it is shown, that most of the known solutions of pure CS theory do not solve the large distance limit of the original field equations. Then I look for self-dual solutions in an example of the MCS theories, in which the gauge field is coupled with a nonrelativistic scalar field, described by the non-linear Schrödinger equation (NLSE). The same self-duality equations are used, which proved to be so useful in the examinations of the pure CS theories. It turns out, that only at very special values of the coupling constants, (when special relations exist between them), can one find self-dual solutions, but even these solutions are necessarily singular. Thus the model, governed by the NLSE and the MCS system, has no such non singular, finite energy self-dual solutions, as the solitons found by Jackiw and Pi in the pure CS case. Nevertheless there exist very interesting, though irregular solutions.

Self-duality in the nonrelativistic MCS theory has been already studied by G.V. Dunne and C.A. Trugenberger[8]. They introduce an extra neutral scalar field to find self-dual
solitons in the model. In the case of the applications in solid state physics I am interested in the existence of self-dual solutions without any extra fields.

One can say, that the pure CS limit is physically the large distance and low energy limit of the MCS model, where the lower-derivative CS term dominates the higher-derivative Maxwell term. It seems plausible, looking at the power like behavior of the vortex solutions for large values of the spatial coordinates, (this behavior is found in many models [3,4,5]) but actually it is misleading. The pure CS theory in general can’t be thought of as the large distance limit of the Maxwell-CS theory, e.g. for vortex solutions decreasing as \( r^{-n} \) at the spatial infinity, this truncation is invalidated at large distances. I show this for static system, because I will study static self-dual solutions in what follows, but the results are valid in general.

The field equations for the electromagnetic field, which contain both the Maxwell, and the Chern-Simons terms can be written as:

\[
\frac{\partial_i E^i}{\epsilon_{ij}} - \kappa B = e \rho, \\
\frac{\partial_0 E^j + \partial_j B + \kappa E^j}{\epsilon_{ij}} = -e \epsilon_{ij} J^j,
\]

(1)

where \( i, j = 1, 2 \), \( E^i = -\partial_0 A^i - \partial_i A^0 \) and \( B = \vec{\nabla} \times \vec{A} = \epsilon_{ij} \partial_i A^j \), and the indices are raised and lowered with the aid of the flat Minkowski metric with signature (+,-,-).

The static equations of the gauge field can be written as:

\[
\frac{\partial_i \partial^i A^0 - \kappa B}{\partial_i B + \kappa \partial_i A^0} = e \rho, \\
\partial_i B + \kappa \partial_i A^0 = -e \epsilon_{ij} J^j.
\]

(2)

If we want to neglect the first terms on the left hand sides in these equations, to get the pure CS limit, they have to be much smaller for large \( r \)-s, than the second terms. For the soliton solutions, when every quantity is decreasing as \( r^{-n} \) in the large \( r \) limit, these conditions can be written as:

\[
\frac{\partial_i \partial^i A^0}{\partial_i B} \propto \frac{r^{-2} A^0}{B} \propto r^{-k}, \\
\frac{\partial_i B}{\partial_i A^0} \propto \frac{B}{A^0/r} \propto r^{-l},
\]

(3)
with \( k \) and \( l \) positive integers. From these conditions it follows, that:

\[
1 = \frac{A^0}{B} \cdot \frac{B}{A^0} \propto r^{-(k+l-2)} \quad \Rightarrow \quad k = l = 1.
\]

Thus the first terms in (2) can be neglected only if \( B \propto A^0/r \) for large \( r \)-s. It is a very special condition, and there is no general reason why it should to be satisfied. (E.g. for the solutions in [4] and [5] this condition is not satisfied, and there is no reason to be satisfied for the non-topological solitons in [3].) In such situation only one of the two relations in (3) may apply but not both of them. Therefore, in general, we can not neglect the first terms in the two equations of (2) at the same time.

Now couple the gauge field, described by eq. (1), to a matter field, especially to a nonrelativistic matter field which is described by the nonlinear Schrödinger equation (NLSE):

\[
iD_0 \psi = -\frac{1}{2m} \vec{D}^2 \psi - g|\psi|^2 \psi,
\]

with

\[
D_\mu = \partial_\mu + ieA_\mu, \quad (\mu = 0, 1, 2),
\]

and identify the charge density \( \rho \) and current \( J^j \), which appear in the MCS equations, with the conserved “three current” \( j = (\rho, \vec{J}) \), where:

\[
\rho = |\psi|^2, \quad J^j = -\frac{i}{2m}(\psi^* D^j \psi - (D^j \psi)^* \psi).
\]

Using the identity:

\[
\vec{D}^2 \psi = (D_1 \pm iD_2)(D_1 \mp iD_2)\psi \pm eB\psi,
\]

the Schrödinger equation can be written in a new form as:

\[
i\partial_0 \psi = -\frac{1}{2m}(D_1 \pm iD_2)(D_1 \mp iD_2)\psi + \left( \mp \frac{e}{2m}B + eA_0 - g|\psi|^2 \right)\psi.
\]
Remarks on the solutions...

This equation admits static solutions, which solve the well known, and generally used self-duality equation:

\[(D_1 \mp iD_2)\psi = 0,\]  

(8)

if the algebraic equation:

\[\mp \frac{e}{2m} B - eA_0 - g|\psi|^2 = 0,\]  

(9)

is also satisfied. With \(\psi\) fields satisfying the ansatz, (8), the current density, (6), simplifies to:

\[J^j = \pm \frac{1}{2m} \varepsilon^{jk} \partial_k \rho.\]  

(10)

Thus, for static, self-dual solutions, the second eq. in (2) becomes:

\[\partial_i (A^0 - \frac{1}{\kappa} B \mp \frac{e}{2m\kappa} \rho) = 0.\]  

(11)

This equation can also be converted into an algebraic one:

\[A^0 - \frac{1}{\kappa} B \mp \frac{e}{2m\kappa} \rho = \text{const.} \]  

(12)

The well known solution of the self duality, (8), connects \(B\) with \(\rho\):

\[B = \pm \frac{1}{2} \nabla^2 \ln \rho.\]  

(13)

Once eq. (8) is imposed the three equations (9,11,12), together with the Chern-Simons modified Gauss law, (the first eq.in (2)), determine the behavior of the fields. Notice, that now we have four equations, of course, because the extra equation of self-duality is introduced, but we have only three unknown functions \((A^0, B \text{ and } \rho)\). Furthermore two of these four equations, namely (9) and (12), are algebraic, therefore, in the case of general coefficients in these equations, any pair of the three unknown functions can be expressed in terms of the third one, (e. g. \(A^0 \text{ and } B\) in terms of \(\rho\)). The remaining two equations, (13) and the first one in (2), constitute then two incompatible differential equations for this single
unknown. This outcome can be avoided only if the coefficients in (9) and (12) are such that the two equations become identical. (This requirement establishes a connection between the coupling constants (here: $e, g, \kappa$ and $m$), which is called the self-duality condition.)

To make eq (9) and (12) identical, we must choose the nonlinearity $g$ to be $g = \pm \frac{e^2}{2m\kappa}$, and tune the CS coupling constant $\kappa$ to the value $\mp 2m$. It is a very strong self-duality condition, which not only connects the coupling constants, but unusually fixes the value of $\kappa$. In the presence of the Maxwell terms we found a stronger self-duality condition, than for the pure CS theory, because in this case eq. (11) contains also the $B$ field in addition to $A^0$. With this condition, it can be seen, that $g = - (\frac{e}{2m})^2$ and the constant, which couples the magnetic field to the number density, is: $\pm \frac{\nu}{2m}$, the same as the magnetic moment of a spin one half electron. The negative sign of $g$ means, that the self-interaction of the matter field is repulsive. This sign has an important effect on the solutions, as we will see it.

Now the self-duality (13), the algebraic eq. (9) and the CS modified Gauss-law determine the system. Using the algebraic relation (9), the $\rho$ and $B$ fields may be eliminated from the CS modified Gauss-law, leaving a well known, solvable differential equation for the $A^0$ field:

$$\triangle A^0 = \kappa^2 A^0.$$  

(14)

Note, that this is consistent with the results found in topologically massive gauge theories [9], because this equation may describe the time component of a massive photon field. The general solution of eq (14) can be written as:

$$A^0 = \sum_{n=0}^{\infty} \exp(in\phi)(c_1^0 K_n(\kappa r) + c_2^0 I_n(\kappa r)),$$  

(15)

where $I_n$ and $K_n$ are the well known Bessel functions, having blowing up singularities at $r = 0$ (respectively $r = \infty$). Thus, for $c_1^0 \neq 0$, $c_2^0 \neq 0$ the solutions are necessarily singular either at the origin or at infinity.
Now $A^0$ is a known function and using eq. (9), the $B$ field may be eliminated from the CS modified Gauss law, to get the following differential equation for $\rho$:

$$\triangle \ln \rho = \frac{e^2}{m} \rho + 4mA^0. \tag{16}$$

It is an “inhomogeneous Liouville equation” with a known source $A^0$. If we are interested in axially symmetric, vortex like solutions, we must choose $c^0_{1,2}$ to be zero if $n > 0$, and $c^0_2 = 0$ guarantees the regularity of the solution at the spatial infinity. Now the $A^0$ field is just like a two dimensional Yukawa type potential, (the potential of a charged vortex interacting with a massive photon field), with singularity at the origin and decreasing as $\frac{1}{r^{1/2}} \exp(-\kappa r)$ for large $r$. With our choice of $A^0$, for spatial coordinates large enough, eq. (16) simplifies to the Liouville eq. which is known to be completely solvable. In the resulting Liouville equation, the relative sign of the right and left hand sides is positive, which means, that only singular solutions exist.

(There are some arguments from numerical calculations, which suggest, that the solutions of the complete inhomogenous eq. (16) are always singular too.)

In this paper it was shown, that the pure CS limit is not the large distance limit of the Maxwell-CS theories. Then the MCS-NLSE model was studied, using the self-duality condition, (8), which is generally used in CS theories. Then coupled non-linear system of differential equations reduced to one non-linear differential equation for the number density, namely to the inhomogeneous Liouville equation. It turns out, that the self-dual solutions of the MCS-NLSE model, (with self-duality (8)), must be singular. This is a surprising result, because the well known solutions of the pure CS theories are self-dual ones, but in that case there were regular solutions too: self-duality didn’t impose such a strong constraint on the couplings there, as in the MCS case. Here the self-duality fixes the value of the CS coupling $\kappa$. An interesting result of this fixing is, that the constant, which couples the magnetic field to the number density, is the same as the magnetic moment of a
spin one half electron. We have a lot of information about the solutions: the form of the $A^0$ field is exactly known, and, (in the axially symmetric case), it corresponds to the Yukawa type potential of a “point like” particle. The $r \to \infty$ asymptotic solution, resulting from it, is purely magnetic. The large distance behavior of the number density is determined by a Liouville equation, and so the solutions in this limit are completely known.

Finally, in the light of the findings of this paper, it would be interesting to inquire the existence of non self-dual nonsingular solutions in the MCS-NLSE model.

I would like to thank L. Palla for many helpful discussions.
Remarks on the solutions

References

[1] S. K. Paul, A. Khare, Phys. Lett. B 174, 420 (1986).

[2] I. Hong, Y. Kim, Y. Pac, Phys. Rew. Lett. 64 2230 (1990); R. Jackiw, E. Weinberg, Phys. Rew. Lett. 64 2234 (1990).

[3] R. Jackiw, K. Lee, E. Weinberg, Phys. Rew. D 42 3488 (1990).

[4] R. Jackiw and S.-Y. Pi, Phys. Rew. Lett. 64, 2969 (1990); Phys. Rew. D 42, 3500 (1990); see Prog. Theor. Phys. Suppl. 107, 1 (1992) for a review.

[5] C. Duval, P. A. Horváthy and L. Palla, Phys. Rew. D 52, 4700 (1995); Ann. of Phys. 249, 256 (1996).

[6] S. M. Girvin, The quantum Hall Effect, ed. R. E. Prange and S. M. Girvin. Chap. 10. (Springer Verlag, New York (1986); S. M. Girvin and A. H. MacDonald, Phys. Rew. Lett. 58, 1252 (1987); S. C. Zhang, T. H. Hansson and S. Kivelson, Phys. Rew. Lett. 62, 82 (1989).

[7] O. Bergman, G. Lozano, Ann. of Phys. 229, 416, (1994).

[8] G. V. Dunne, C. A. Trugenberger, Phys. Rew. D 43, 1323, (1991).

[9] S. Deser, R. Jackiw, S. Templeton, Ann. of Phys. 140, 372, (1982).