Hyperfunction Semigroups

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Abstract. We analyze Fourier hyperfunction and hyperfunction semigroups with non-densely defined generators and their connections with local convoluted $C$-semigroups. Structural theorems and spectral characterizations give necessary and sufficient conditions for the existence of such semigroups generated by a closed not necessarily densely defined operator $A$.

1. Introduction and preliminaries

The papers on ultradistribution semigroups, [32], [33] extend the classical theory of semigroups, (see [38], [6], [16], [22], [28] and [35]). S. Ōuchi [44] was the first who introduced the class of hyperfunction semigroups, more general than that of distribution and ultradistribution semigroups and in [45] he considered the abstract Cauchy problem in the space of hyperfunctions. Furthermore, generators of hyperfunction semigroups in the sense of [44] are not necessarily densely defined. A.N. Kochubei, [23] considered hyperfunction solutions on abstract differential equations of higher order. We analyze Fourier hyperfunction semigroups with non-densely defined generators continuing over the investigations of Roumieu type ultradistribution semigroups and constructed examples of tempered ultradistribution semigroups [32] as well as of Fourier hyperfunction semigroups with non-densely defined generators. An analysis of R. Beals [4, Theorem 2'] gives an example of a densely defined operator $A$ in the Hardy space $H^2(\mathbb{C}_+)$ which generates a hyperfunction semigroup of [44] but this operator is not a generator of any ultradistribution semigroup, and any (local) integrated $C$-semigroups, $C \in L(H^2(\mathbb{C}_+))$. Our main interest is the existence of fundamental solutions for the Cauchy problems with initial data being hyperfunctions.

In the definition of infinitesimal generators for distribution and ultradistribution semigroups in the non-quasi-analytic case, all authors use test functions supported by $[0, \infty)$. Such an approach cannot be used in the case of Fourier hyperfunction semigroups since in the quasi-analytic case only the zero function has this property. Because of that, we define such semigroups
on test spaces $P_*$ and $P_{*,a}$ ($a > 0$) but the axioms for such semigroups as well as the definition of infinitesimal generator are given on subspaces of quoted spaces consisting of functions $\phi$ with the property $\phi(0) = 0$ and $\phi'(0) = 0$. We note that the same can be done for the distribution and ultradistribution semigroups (we leave this for another paper).

Section 2 is devoted to Fourier hyperfunction semigroups. As we mentioned, the definition of such semigroups is intrinsically different than that of ultradistribution semigroups because test functions with the support bounded on the left cannot be used. Fourier hyperfunction semigroups with densely defined infinitesimal generators were introduced by Y. Ito [17] related to the corresponding Cauchy problem [16]. We give structural and spectral characterizations of Fourier- and exponentially bounded Fourier hyperfunction semigroups with non-dense infinitesimal generators, their relations with the convoluted semigroups and to the corresponding Cauchy problems. Spectral properties of hyperfunction semigroups give a new insight to S. Ōuchi’s results.

1.1. Hyperfunction and Fourier hyperfunction type spaces

The basic facts about hyperfunctions and Fourier hyperfunctions of M. Sato can be found on an elementary level in the monograph of A. Kaneko [18] (see also [41], [14], [19]-[20]). Let $E$ be a Banach space, $\Omega$ be an open set in $\mathbb{C}$ containing an open set $I \subset \mathbb{R}$ as a closed subset and let $\mathcal{O}(\Omega)$ be the space of $E$-valued holomorphic functions on $\Omega$ endowed with the topology of uniform convergence on compact sets of $\Omega$. The space of $E$-valued hyperfunctions on $I$ is defined as $\mathcal{B}(I, E) := \mathcal{O}(\Omega \setminus I, E)/\mathcal{O}(\Omega, E)$. A representative of $f = [f(z)] \in \mathcal{B}(I, E)$, $f \in \mathcal{O}(\Omega \setminus I, E)$ is called a defining function of $f$. The space of hyperfunctions supported by a compact set $K \subset I$ with values in $E$ is denoted by $\Gamma_K(I, \mathcal{B}(E)) = \mathcal{B}(K, E)$. It is the space of continuous linear mapping from $\mathcal{A}(K)$ into $E$, where $\mathcal{A}(K)$ is the inductive limit type space of analytic functions in neighborhoods of $K$ endowed with the appropriate topology [25]. Denote by $\mathcal{A}(\mathbb{R})$ the space of real analytic functions on $\mathbb{R}$: $\mathcal{A}(\mathbb{R}) = \text{proj lim}_{K \subset \subset \mathbb{R}} \mathcal{A}(K)$. The space of continuous linear mappings from $\mathcal{A}(\mathbb{R})$ into $E$, denoted by $\mathcal{B}_c(\mathbb{R}, E)$, is consisted of compactly supported elements of $\mathcal{B}(K, E)$, where $K$ varies through the family of all compact sets in $\mathbb{R}$. We denote by $\mathcal{B}_+(\mathbb{R}, E)$ the space of $E$-valued hyperfunctions whose supports are contained in $[0, \infty)$. As in the scalar case ($E = \mathbb{R}$) we have, if $f \in \mathcal{B}_c(\mathbb{R}, E)$ and $\text{supp}f \subset \{a\}$, then $f = \sum_{n=0}^{\infty} \delta^{(n)}(-a)x_n$, $x_n \in E$, where $\lim_{n \to \infty} (n!!|x_n|)^{1/n} = 0$. Let $\mathbb{D} = \{-\infty, +\infty\} \cup \mathbb{R}$ be the radial compactification of the space $\mathbb{R}$. Put $I_\nu = (-1/\nu, 1/\nu)$, $\nu > 0$. For $\delta > 0$, the space $\hat{\mathcal{O}}^{-\delta}(\mathbb{D} + iI_\nu)$ is defined as a subspace of $\mathcal{O}(\mathbb{R} + iI_\nu)$ with the property that for every $K \subset \subset I_\nu$ and $\varepsilon > 0$ there exists a suitable $C > 0$ such that $|F(z)| \leq Ce^{-(\delta - \varepsilon)|Rez|}$, $z \in \mathbb{R} + iK$. Then $P_*(\mathbb{D}) := \text{ind lim}_{n \to \infty} \hat{\mathcal{O}}^{-1/n}(\mathbb{D} + iI_n)$ is the space of all rapidly decreasing, real analytic functions (cf. [18] Definition 8.2.1) and the space of Fourier hyperfunctions $Q(\mathbb{D}, E)$ is the space of continuous linear mappings
from \( P_*(\mathbb{D}) \) into \( E \) endowed with the strong topology. We point out that Fourier hyperfunctions were firstly introduced by M. Sato in [16] who called them *slowly increasing hyperfunctions*. Let us note that the sub-index \(*\) in \( P_*(\mathbb{D}) \) does not have the meaning as in the case of ultradistributions. This is often used notation in the literature (cf. [18, Proposition 8.1.6, Lemma 8.1.7, Theorem 8.4.9]).

Suppose the structural theorem (cf. [18, Proposition 8.1.6, Lemma 8.1.7, Theorem 8.4.9]), \( L_h > 2 \) is of the form 

\[
|P| = ||P||_h > 0 \]

then we have the above representation with a corresponding local operator \( \text{Lemma 1.1} \). If \( \exists \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that \( |f(t)| \leq C_\varepsilon e^{\varepsilon|t|}, \ t \in \mathbb{R} \). More precisely, we have the following global structural theorem (cf. [18, Proposition 8.1.6, Lemma 8.1.7, Theorem 8.4.9]), reformulated here with a sequence \((L_p)_p\):

Let, formally,

\[
P_{L_p}(d/dt) = \prod_{p=1}^{\infty} (1 + \frac{L^2_p}{p^2} d^2/dt^2) = \sum_{p=0}^{\infty} a_p d^p/dt^p, \tag{1}
\]

where \((L_p)_p\) is a sequence decreasing to 0. This is a local operator and we call it hyperfunction operator. Then [18]:

Let \( T \in \mathcal{Q}(\mathbb{D},E) \). There is a local operator \( P_{L_p}(-\text{id}/dt) \) (with a corresponding sequence \((L_p)_p\)) and a continuous slowly increasing function \( f : \mathbb{R} \rightarrow E \), which means that, for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that \( ||f(x)|| \leq C_\varepsilon e^{\varepsilon|x|}, \ x \in \mathbb{R} \) and that \( T = P_{L_p}(-\text{id}/dt) f \).

If a hyperfunction is compactly supported, \( \text{supp} f \subset K, f \in \mathcal{B}(K,E) \), then we have the above representation with a corresponding local operator \( P_{L_p}(-\text{id}/dt) \) and a continuous \( E \)-valued function in a neighborhood of \( K \).

The spaces of Fourier hyperfunctions were also analyzed by J. Chung, S.-Y. Chung and D. Kim in [7–8]. Following this approach, we have that \( P_*(\mathbb{D}) \) (topologically) equal to the space of \( C^\infty \)-functions \( \phi \) defined on \( \mathbb{R} \) with the property: \( \exists h > 0 |||\phi|||_h < \infty \), where the norms \( ||| \cdot |||_h, h > 0 \), are defined by \( |||\phi|||_h := \sup\{|||\phi^{(n)}(x)||| e^{x/h} / (h^n n!) : n \in \mathbb{N}_0, x \in \mathbb{R}\} \), equipped with the corresponding inductive limit topology when \( h \rightarrow +\infty \). The next lemma can be proved by the standard arguments using the norms \( |||\phi|||_h \).

**Lemma 1.1.** If \( \phi, \psi \in \mathcal{P}_*(\mathbb{D}) \), then \( \phi \ast_0 \psi = \int_0^t \phi(t) \psi(t - \tau) d\tau, t > 0 \) is in \( \mathcal{P}_*(\mathbb{D}) \) and the mapping \( \ast_0 : \mathcal{P}_*(\mathbb{D}) \times \mathcal{P}_*(\mathbb{D}) \rightarrow \mathcal{P}_*(\mathbb{D}) \) is continuous.

**Proof.** Suppose \( x \in \mathbb{R}, n \in \mathbb{N} \) and \( h_1 > 0 \) fulfill \( |||\phi|||_{h_1} < \infty \). Suppose that \( h > 2h_1 \) satisfies \( |||\psi|||_{h_2} < \infty \) and put \( h_2 = \frac{\sqrt{h}h_1}{h-2h_1} \). We will use the next
inequality which holds for every \( t, \frac{|x|}{h} \leq \frac{|x-t|}{h} + \frac{|t|}{h} \leq \frac{|x-t|}{h} + \frac{|t|}{h_1} - \frac{|t|}{h_2}. \) We have

\[
\sup_{n \in \mathbb{N}_0, \ x \in \mathbb{R}} \frac{e^{|x|/h}}{h^n n!} \left| \left( \int_0^x \phi(t) \psi(x-t) \, dt \right)^{(n)} \right| \leq \sup_{n \in \mathbb{N}_0, \ x \in \mathbb{R}} \frac{e^{|x|/h} \int_0^x |\phi(t)\psi(x-t)| \, dt}{h^n n!} \left( \sum_{j=0}^{n-1} \sup_{n \in \mathbb{N}_0, \ x \in \mathbb{R}} \frac{e^{|x|/h} |\phi(j)(x)|}{h^n n!} |\psi(n-j)(0)| \right) = I + II.
\]

We will estimate separately \( I \) and \( II. \)

\[
I \leq \sup_{t \in \mathbb{R}} \left( |\phi(t)| e^{\frac{|t|}{h|t|}} \right) \left( \int_0^x e^{-\frac{|t|}{h|t|}} \, dt \right) \sup_{n \in \mathbb{N}_0, \ x, t \in \mathbb{R}} \frac{|\psi(n)(x-t)| e^{|x-t|/h}}{h^n n!} \sup_{n \in \mathbb{N}_0, \ x \in \mathbb{R}} \frac{|\psi(n-j)(0)|}{h^n n!} \left( \sum_{j=0}^{n-1} e^{\frac{|t|}{h|t|} |\phi(j)(x)|} \right),
\]

\[
II \leq \frac{1}{2^n} \sum_{j=0}^{n-1} \sup_{n \in \mathbb{N}_0, \ x \in \mathbb{R}} \frac{e^{|x|/h} |\phi(j)(x)|}{(h/2)^j j!} \sup_{n \in \mathbb{N}_0, \ x \in \mathbb{R}} \frac{|\psi(n-j)(0)|}{(h/2)^j n!} \left| |\phi(j)(x)| \right| \left( \sum_{j=0}^{n-1} e^{\frac{|t|}{h|t|} |\phi(j)(x)|} \right).
\]

This gives \( \phi * \psi \in \mathcal{P}_*(\mathbb{D}) \) while the continuity of the mapping \( \ast_0 : \mathcal{P}_*(\mathbb{D}) \times \mathcal{P}_*(\mathbb{D}) \to \mathcal{P}_*(\mathbb{D}) \) follows similarly. This completes the proof of the lemma. \( \square \)

Now we will transfer the definitions and assertions for Roumieu tempered ultradistributions to Fourier hyperfunctions.

**Definition 1.2.** Let \( a \geq 0. \) Then

\[ \mathcal{P}_{*,a}(\mathbb{D}) := \{ \phi \in C^{\infty}(\mathbb{R}) : e^{a \cdot \phi} \in \mathcal{P}_*(\mathbb{D}) \}. \]

Define the convergence in this space by

\[ \phi_n \to 0 \text{ in } \mathcal{P}_{*,a}(\mathbb{D}) \text{ iff } e^{a \cdot \phi_n} \to 0 \text{ in } \mathcal{P}_*(\mathbb{D}). \]

We denote by \( \mathcal{Q}_a(\mathbb{D}, E) \) the space of continuous linear mappings from \( \mathcal{P}_{*,a}(\mathbb{D}) \) into \( E \) endowed with the strong topology.

We have:

\[ F \in \mathcal{Q}_a(\mathbb{D}, E) \text{ iff } e^{-a} F \in \mathcal{Q}(\mathbb{D}, E). \quad (2) \]

**Proposition 1.3.** Let \( G \in \mathcal{Q}_a(\mathbb{D}, L(E)). \) Then there exists a local operator \( P \) and a function \( g \in C(\mathbb{R}, L(E)) \) with the property that for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that

\[ e^{-a x} |g(x)| \leq C_\varepsilon e^{\varepsilon |x|}, \ x \in \mathbb{R} \text{ and } G = P(d/dt)g. \]

**Proof.** From the structure theorem for the space \( \mathcal{Q}(\mathbb{D}, L(E)) \) and since \( e^{-a} G \in \mathcal{Q}(\mathbb{D}, L(E)) \), there exists a local operator \( P \) and a function \( g_1 \) with the property that for every \( \varepsilon > 0 \) there is corresponding \( C_\varepsilon > 0 \) such that

\[ |g_1(x)| \leq C_\varepsilon e^{\varepsilon |x|}, \ x \in \mathbb{R} \text{ and } G = e^{a x} P(d/dt)g_1. \]

We put \( g(x) = e^{a x} g_1(x), \ x \in \mathbb{R}. \) Using Leibnitz formula, we have

\[ e^{a x} P(d/dt)g_1(x) = \sum_{t=0}^{\infty} \left( \sum_{k=0}^{\infty} \binom{t+k}{k} (-1)^k a^k b_{k+t} (e^{a x} g_1(x))(t) \right). \]
The assertion will be proved if we show that \( \lim_{|t| \to \infty} (|c_t|t)^{\frac{1}{t}} = 0 \), where \( c_t = \sum_{k=0}^{\infty} \binom{k+t}{k} a^k b_{k+t} \). To prove this, we use
\[
\left( \frac{t+k}{k} \right) \leq (t+k)^k \leq 2^k k^k + 2^k t^k \leq 2^k (k^k + k^k e^t) = 2^k k^k (1 + e^t),
\]
where we used \( t^k \leq k^k e^t \). The last inequality is clear for \( k \geq t \). For \( k < t \), we put \( k = \nu t \). First let we note that \( \nu \ln \nu \in (-1,0) \). Then \( \nu t \ln t \leq \nu t \ln t + \nu t \ln \nu + t \). Hence \( t^k \leq k^k e^t \). Now,
\[
c_t = \sum_{k=0}^{\infty} 2^k k^k (1 + e^t) a^k b_{k+t} = \sum_{k=0}^{\infty} (2a)^k k^k (1 + e^t) b_{k+t}.
\]
The coefficients \( b_{k+t} \) are coefficients of a local operator, so for all \( \varepsilon > 0 \), exists \( M \in \mathbb{N} \) such that for all \( t + k > M \), \( |b_{k+t}|(t + k)! < \varepsilon^{t+k} \). With this we have
\[
|c_t| \leq (1 + e^t) \sum_{k=0}^{\infty} \frac{(2a)^k k^k (t + k)!|b_{k+t}|}{(t + k)!} \leq \sum_{k=0}^{\infty} \frac{(2a)^k (1 + e^t) e^k k! t!(t + k)!|b_{t+k}|}{(t + k)!} \leq \sum_{k=0}^{\infty} \frac{(2ae)^k (1 + e^t) \varepsilon^{t+k}}{t! k!} = (1 + e^t) \varepsilon^t \sum_{k=0}^{\infty} (2ae \varepsilon)^k
\]
and the assertion follows since we can choose \( \varepsilon \) arbitrary small. \( \square \)

**Remark 1.4.** By Lemma [1.7], one can easily prove that, if \( \phi, \psi \in \mathcal{P}_{*,a}(\mathbb{D}) \), then \( \phi \ast_0 \psi \in \mathcal{P}_{*,a}(\mathbb{D}) \) and the mapping \( \ast_0 : \mathcal{P}_{*,a}(\mathbb{D}) \times \mathcal{P}_{*,a}(\mathbb{D}) \to \mathcal{P}_{*,a}(\mathbb{D}) \) is continuous.

For the needs of the Laplace transform we define the space \( \mathcal{P}_*([-r, \infty]), r > 0 \). Note that \( [-r, \infty] \) is compact in \( \mathbb{D}. \)

\( \mathcal{P}_*([-r, \infty], h) \) is defined as the space of smooth functions \( \phi \) on \( (-r, \infty) \) with the property \( ||\phi||_{*, -r, h} < \infty \), where
\[
||\phi||_{*, -r, h} := \sup_{\alpha \in \mathbb{N}_0} \left\{ \frac{||\phi^{(\alpha)}(x)|| e^{|x|/h}}{h^{\alpha} \alpha!} : x \in (-r, \infty) \right\}.
\]
Then
\[
\mathcal{P}_*([-r, \infty]) := \lim_{h \to +\infty} \mathcal{P}_*([-r, \infty], h).
\]

**Lemma 1.5.** \( \mathcal{P}_*(\mathbb{D}) \) is dense in \( \mathcal{P}_*([-r, \infty]). \)

**Proof.** This is a consequence of Lemma 8.6.4 in [13]. \( \square \)

For \( a \geq 0 \), we define the space
\[
\mathcal{P}_{*,a}([-r, \infty]) := \{ \phi : e^{ax} \phi \in \mathcal{P}_*([-r, \infty]) \}.
\]
The topology of \( \mathcal{P}_{*,a}([-r, \infty]) \) is defined by:
\[
\lim_{n \to \infty} \phi_n = 0 \text{ in } \mathcal{P}_{*,a}([-r, \infty]) \text{ iff } \lim_{n \to \infty} e^{ax} \phi_n = 0 \text{ in } \mathcal{P}_*([-r, \infty]).
\]
If \( a \geq 0 \) and \( e^{-a} \mathbb{D} \in Q_{+}(\mathbb{D}, L(E)) \), then \( \mathbb{D} \) can be extended to an element of the space of continuous linear mappings from \( P_{*,0}([-r, \infty]) \) into \( L(E) \) equipped with the strong topology. This extension is unique because of Lemma [1.5] We will use this for the definition of the Laplace transform of \( \mathbb{D} \).

2. Fourier hyperfunction semigroups

The definition of (exponential) Fourier hyperfunction semigroup with densely defined infinitesimal generators of \( Y \). Ito (see [17, Definition 2.1]) is given on the basis of the space \( P_{0} \) whose structure is not clear to authors. Our definition is different and related to non-densely defined infinitesimal generators.

In the sequel, we use the notation \( Q_{+}(\mathbb{D}, L(E)) \) for the space of vector-valued Fourier hyperfunctions supported by \([0, \infty]\). More precisely, if \( \mathbb{D} \in Q_{+}(\mathbb{D}, L(E)) \) is represented by \( \mathbb{D}(t, \cdot) = \mathbb{D}_{+}(t + i0, \cdot) - \mathbb{D}_{-}(t - i0, \cdot) \), where \( \mathbb{D}_{+} \) and \( \mathbb{D}_{-} \) are defining functions for \( \mathbb{D} \) (see [18, Definition 1.3.6, Definition 8.3.1]) and \( \gamma_{+} \) and \( \gamma_{-} \) are piecewise smooth paths connecting points \(-a, (a > 0) \) and \( \infty \) such that \( \gamma_{+} \) and \( \gamma_{-} \) lie respectively in the upper and the lower half planes as well as in a strip around \( \mathbb{R} \) depending on \( \mathbb{D} \), then for any \( \psi \in P_{*}(\mathbb{D}) \),

\[
\int_{\mathbb{R}} f(t) \psi(t) \, dt = \int_{0}^{\infty} f(t) \psi(t) \, dt := \int_{\gamma_{+}} F_{+}(z) \psi(z) \, dz - \int_{\gamma_{-}} F_{-}(z) \psi(z) \, dz.
\]

Since we will use the duality approach of Chong and Kim, we will use notation \( \langle f, \psi \rangle \) for the above expression.

Let \( \varphi \in P_{*} \) and let \( \mathbb{D}(t, \cdot) = \mathbb{D}_{+}(t + i0, \cdot) - \mathbb{D}_{-}(t - i0, \cdot) \) be an element in \( Q_{+}(\mathbb{D}, L(E)) \). Then

\[
\varphi(t) \mathbb{D}(t, \cdot) := \varphi(t) \mathbb{D}_{+}(t + i0, \cdot) - \varphi(t) \mathbb{D}_{-}(t - i0, \cdot).
\]

We will denote by \( P_{*}^{0} \) a subspace of \( P_{*} \) consisting of functions \( \phi \) with the property \( \phi(0) = 0 \). Also, we will consider \( P_{*}^{00} \), a subspace of \( P_{*} \) consisting of functions \( \psi \) with the properties \( \psi(0) = 0 \) and \( \psi'(0) = 0 \). Note, any \( \psi \in P_{*} \) can be written in the form

\[
\psi(t) = \psi(0) \phi_{0}(t) + \theta(t), \quad t \in \mathbb{R}, \text{ respectively},
\]

\[
\psi(t) = \psi(0) \phi_{0}(t) + \psi'(0) \phi_{1}(t) + \bar{\theta}(t), \quad t \in \mathbb{R},
\]

where \( \phi_{0} \) and \( \phi_{1} \) are fixed elements of \( P_{*} \) with the properties \( \phi_{0}(0) = 1, \phi'_{0}(0) = 0, \phi_{1}(0) = 0, \phi'_{1}(0) = 1 \) and \( \theta \) varies over \( P_{*}^{0} \) respectively \( \bar{\theta} \) varies over \( P_{*}^{00} \). We define \( P_{*,a}^{0} \) as a space of functions \( \phi \in P_{*,a} \) with the property \( \phi(0) = 0 \) and \( P_{*,a}^{00} \), as a space of functions \( \phi \in P_{*,a} \) with the property \( \phi(0) = 0, \phi'(0) = 0 \) and note that the similar decompositions as [3] and [4] hold for elements of \( P_{*,a}^{0} \) and \( P_{*,a}^{00} \), respectively.

**Definition 2.1.** An element \( \mathbb{D} \in Q_{+}(\mathbb{D}, L(E)) \) is called a pre-Fourier hyperfunction semigroup, if the next condition is valid

\[
(H.1) \quad G(\phi \ast_{0} \psi) = G(\phi)G(\psi), \quad \phi, \psi \in P_{*}(\mathbb{D}).
\]
Further on, a pre-Fourier hyperfunction semigroup $G$ is called a Fourier hyperfunction semigroup, (FHSG) in short, if, in addition, the following holds
\((H.2)\) \(N(G) := \bigcap_{\phi \in \mathcal{P}^0_{+}\mathcal{D}} N(G(\phi)) = \{0\}.\)

If the next condition also holds:
\((H.3)\) \(\mathcal{R}(G) := \bigcup_{\phi \in \mathcal{P}^0_{+}\mathcal{D}} R(G(\phi))\) is dense in $E$, then $G$ is called a dense (FHSG).

If $e^{-\alpha}G \in \mathcal{Q}_{+}(\mathcal{D}, L(E))$, for some $a > 0$, and \((H.1)\) holds with $\phi, \psi \in \mathcal{P}_{+\alpha}(\mathcal{D})$ then $G$ is called exponentially bounded pre-Fourier hyperfunction semigroup. If \((H.2)\) and \((H.3)\) hold with $\phi \in \mathcal{P}^{00}_{+\alpha}(\mathcal{D})$, then $G$ is called a dense exponential Fourier hyperfunction semigroup, dense (EFHSG), in short.

Let $A$ be a closed operator. We denote by $[D(A)]$ the Banach space $D(A)$ endowed with the graph norm $\|a\|_{[D(A)]} = \|a\| + \|Ax\|$, $x \in D(A)$. Like in [16] Definition 2.1, Definition 3.1, we give the following definitions:

**Definition 2.2.** Let $A$ be a closed operator. Then $G \in \mathcal{Q}_{+}(\mathcal{D}, L(E, [D(A)]))$ is a Fourier hyperfunction solution for $A$ if $P \ast G = \delta \otimes I_{E}$ and $G \ast P = \delta \otimes I_{D(A)}$, where $P := \delta' \otimes I_{D(A)} - \delta \otimes A \in \mathcal{Q}_{+}(\mathcal{D}, L(D(A), E))$; $G$ is called an exponential Fourier hyperfunction solution for $A$ if, additionally,
\[e^{-\alpha}G \in \mathcal{Q}_{+}(\mathcal{D}, L(E, [D(A)])), \text{ for some } a > 0.\]

Similarly, if $G$ is an exponential Fourier hyperfunction solution for $A$ which fulfills \((H.3)\), then $G$ is called a dense, exponential Fourier hyperfunction solution for $A$.

Let $a \geq 0$ and $\alpha \in \mathcal{P}_{+\alpha}$, be an even function such that $\int a(t) dt = 1$. Let $\text{sgn} (x) := 1, x > 0$, $\text{sgn} (x) := -1, x < 0$ and $\text{sgn} (0) := 0$. A net of the form $\delta_{\varepsilon} = \alpha(\cdot / \varepsilon)/\varepsilon, \varepsilon \in (0,1)$, is called delta net in $\mathcal{P}_{+\alpha}$. Changing $\alpha$ with the above properties, one obtains a set of delta nets in $\mathcal{P}_{+\alpha}$. Clearly, every delta net converges to $\delta$ as $\varepsilon \to 0$ in $\mathcal{Q}(\mathcal{D})$. We define, for $x \in \mathbb{R}$,
\[
\delta \ast_0 \phi(x) := 2 \text{sgn} (x) \lim_{\varepsilon \to 0} \delta_{\varepsilon} \ast_0 \phi(x) = \phi(x), \; \phi \in \mathcal{P}^0_{+\alpha},
\]
\[
\delta' \ast_0 \phi(x) := 2 \text{sgn} (x) \lim_{\varepsilon \to 0} \delta'_{\varepsilon} \ast_0 \phi(x) = \phi'(x), \; \phi \in \mathcal{P}^{00}_{+\alpha}.
\]

**Definition 2.3.** Let $a \geq 0$ and $G$ be an (EFHSG). Then
1. $G(\delta)x := y$ iff $G(\delta \ast_0 \phi)x = G(\phi)y$ for every $\phi \in \mathcal{P}^0_{+\alpha}(\mathcal{D})$.
2. $G(-\delta')x := y$ iff $G(-\delta' \ast_0 \phi)x = G(\phi)y$ for every $\phi \in \mathcal{P}^{00}_{+\alpha}(\mathcal{D})$.

$A = G(-\delta')$ is called the infinitesimal generator of $G$.

Thus $G(\delta)$ is the identity operator. In order to prove that $G(-\delta')$ is a single-valued function, we have to prove that for every $x \in E$, $G(-\delta')x = y_1$ and $G(-\delta')x = y_2$ imply $y_1 = y_2$. This means that we have to prove that
\[G(\phi')x = G(\phi)y_1, \; G(\phi')x = G(\phi)y_2, \; \phi \in \mathcal{P}^{00}_{+\alpha} \implies y_1 = y_2.\]

**Proposition 2.4.** If $G(\phi')x = 0$ for every $\phi \in \mathcal{P}^{00}_{+\alpha}$, then $x = 0$. 

Proof. We shall prove that the assumption \( G(\phi)y = 0 \) for every \( \phi \in \mathcal{P}_{*,a} \) implies that \( y = 0 \). By (3), we have that for any \( \phi_0 \in \mathcal{P}_{*,a} \) such that \( \phi_0(0) = c \neq 0 \)

\[
G(\psi)y = \frac{\psi(0)}{c}G(\phi_0)y, \psi \in \mathcal{P}_{*,a}.
\]

Now let \( \phi, \psi \) be arbitrary elements of \( \mathcal{P}_{*,a} \). Since \( G(\phi \ast_0 \psi)y = G(\phi)G(\psi)y \) and \( \phi \ast_0 \psi(0) = 0 \), it follows, with \( z = G(\psi)y \),

\[
G(\phi \ast_0 \psi)y = G(\phi)z = 0, \quad \phi \in \mathcal{P}_{*,a} \implies z = 0.
\]

Thus, for any \( \psi \in \mathcal{P}_{*,a} \), we have \( G(\psi)y = 0 \) which finally implies \( y = 0 \).

Now, we will prove the assertion. By (4) we have that for every \( \psi \in \mathcal{P}_{*,a} \)

\[
G(\psi')x = \psi(0)G(\phi_0')x + \psi'(0)G(\phi_1')x = 0.
\]

Denote by \( P_{10} \) the set of all \( \phi_0 \in \mathcal{P}_* \) with the properties \( \phi_1(0) = c \neq 0, \phi_1'(0) = 0 \) and by \( P_{01} \) the set of all \( \phi_1 \in \mathcal{P}_* \) with the properties \( \phi_0(0) = 0, \phi_0'(0) = c \neq 0 \).

We have the following cases:

\[
(\forall \phi_0 \in P_{10})(\forall \phi_1 \in P_{01})(G(\phi_0)x = 0, G(\phi_1)x = 0); \quad (\forall \phi_0 \in P_{10})(\exists \phi_1 \in P_{01})(G(\phi_0)x = 0, G(\phi_1)x \neq 0); \quad (\exists \phi_0 \in P_{10})(\forall \phi_1 \in P_{01})(G(\phi_0)x \neq 0, G(\phi_1)x = 0); \quad (\exists \phi_0 \in P_{10})(\exists \phi_1 \in P_{01})(G(\phi_0)x \neq 0, G(\phi_1)x \neq 0).
\]

In the first case we have, by (4), \( G(-\psi')x = 0, \psi \in \mathcal{P}_{*,a} \). This implies, by the standard arguments, that \( G(\psi)x = C \int_{\mathbb{R}} \psi(t) dt x = 0, \psi \in \mathcal{P}_{*,a} \) and this holds for \( C = 0 \). Consider the fourth case. In this case we have that

\[
G(\psi')x = C_1(\delta, \psi)x + C_2(\delta', \psi)x
\]

and thus,

\[
G(\psi')x = C_1(\delta, \psi)x + C_2(\delta', \psi)x + C_3(1, \psi)x,
\]

where \( (1, \psi)x = \int_{\mathbb{R}} \psi(t) dt x \). Now, by the semigroup property it follows \( C_1 = C_2 = C_3 = 0 \) and with this we conclude as above that \( x = 0 \). We can handle out the second and the third case in a similar way. This completes the proof of the assertion.

\[ \square \]

2.1. Laplace transform and the characterizations of Fourier hyperfunction semigroups

The proofs of assertions of this section related to the Laplace transform are new but some of them are quite simple. They are based on the technics developed by Komatsu [24]-[27].

Note, for every \( r > 0, E_{\lambda} = e^{-\lambda r} \in \mathcal{P}_*([-r, \infty]) \), for every \( \lambda \in \mathbb{C} \) with \( \Re\lambda > 0 \). So, we can define the Laplace transform of \( G \in \mathcal{Q}_+(\mathbb{D}, L(E)) \) by

\[
\mathcal{L}G(\lambda) = \hat{G}(\lambda) := G(E_{\lambda}), \quad \Re\lambda > 0.
\]

**Proposition 2.5.** There exists a suitable local operator \( P \) such that

\[
|\hat{G}(\lambda)| \leq |P(\lambda)|, \quad \Re\lambda > 0.
\]
The proof of this assertion is even simpler than the proof of the corresponding assertion in the case of Roumieu ultradistributions.

If \( e^{-a'G} \in Q_+(D, L(E)) \), we define the Laplace transform of \( G \) by

\[
\mathcal{L}(G)(\lambda) = \hat{G}(\lambda) := G(E\lambda), \quad \text{Re}\lambda > a.
\]

It is an analytic function defined on \( \{ \lambda \in \mathbb{C} : \text{Re}\lambda > a \} \) and there exists a local operator \( P \) such that \( |\hat{G}(\lambda)| \leq |P(\lambda)|, \text{Re}\lambda > a \).

**Remark 2.6.** Similarly to the corresponding Roumieu case, one can prove the next statement:

If \( G \in Q_+(D, L(E, [D(A)]) \) is a Fourier hyperfunction solution for \( A \), then \( G \) is a pre-Fourier hyperfunction semigroup. It can be seen, as in the case of ultradistributions, that we do not have that \( G \) must be an \((FHSG)\).

Structural properties of the Fourier hyperfunction semigroups are similar to that of ultradistribution semigroups of Roumieu class. For the essentially different proofs of corresponding results we need the next lemma where we again use the Fourier transform instead of Laplace transform.

**Lemma 2.7.** Let \( P_{L_p} \) be of the form (1). The mapping

\[
P_{L_p}(\text{id}/dt) : P_*(D) \to P_*(D), \quad \phi \mapsto P_{L_p}(\text{id}/dt)\phi
\]

is a continuous linear bijection.

**Proof.** Due to [18, Proposition 8.2.2], \( \phi \in P_*(D) \) implies \( \mathcal{F}(\phi) \in P_*(D) \). Thus, for some \( n \in \mathbb{N} \), every \( \varepsilon > 0 \) and a corresponding \( C_\varepsilon > 0 \), \( |\mathcal{F}(\phi)(z)| \leq C_\varepsilon e^{-(1/n-\varepsilon)|\Re z|}, z \in \mathbb{R}+i\mathbb{N} \). By [18] Proposition 8.1.6, Lemma 8.1.7, Theorem 8.4.9, with some simple modifications, we have

\[
Ce^{-A(|\xi|+1)} \leq |P_{L_p}(\zeta)|, \quad |\eta| \leq \frac{|\xi|}{2} + \frac{1}{L_1}, \quad \zeta = \xi + i\eta,
\]

for some \( C, A > 0 \) and some monotone increasing function \( r \) with the properties \( r(0) = 1, \ r(\infty) = \infty \). This implies that there exists an integer \( n_0 \in \mathbb{N} \) such that

\[
\mathcal{F}(\phi)/P_{L_p} \in \tilde{O}^{-1/n_0}(\mathbb{R} + i\mathbb{N}_0).
\]

Thus, its inverse Fourier transform \( \mathcal{F}^{-1}(\mathcal{F}(\phi)/P_{L_p}) \) is an element of \( P_*(D) \).

Using the properties of local operators as well as norms \( || \cdot ||_{h,p,t} \), as in the case of Roumieu tempered ultradistributions, one obtains the following assertions.

**Theorem 2.8.** Suppose that \( f : \{ \lambda \in \mathbb{C} : \text{Re}\lambda > a \} \to E \) is an analytic function satisfying

\[
||f(\lambda)|| \leq C|P(\lambda)|, \quad \text{Re}\lambda > a,
\]

for some \( C > 0 \), some local operator \( P \) with the property \( |P(\lambda)| > 0, \text{Re}\lambda > a \). Suppose, further, that a local operator \( \tilde{P} \) satisfies (5). Then

\[
(\exists M > 0)(\exists h \in C^\infty([0, \infty); E))(\forall j \in \mathbb{N}_0)(h^{(j)}(0) = 0)
\]
such that $\|h(t)\| \leq Me^{\alpha t}$, $t \geq 0$, and

$$f(\lambda) = P(\lambda)\hat{P}(\lambda) \int_0^\infty e^{-\lambda t}h(t)\,dt, \text{ Re}\lambda > a.$$ 

**Theorem 2.9.** Let $A$ be closed and densely defined. Then $A$ generates a dense (EFHSG) iff the following conditions are true:

(i) $\{\lambda \in \mathbb{C} : \text{Re}\lambda > a\} \subset \rho(A)$,

(ii) There exist a local operator $P$ with the property $|P(\lambda)| > 0$, $\text{Re}\lambda > a$,

a local operator $\hat{P}$ with the properties as in the previous theorem and $C > 0$ such that

$$\|R(\lambda : A)\| \leq C|P(\lambda)\hat{P}(\lambda)|, \text{ Re}\lambda > a.$$ 

(iii) $R(\lambda : A)$ is the Laplace transform of some $G$ which satisfies (H.2).

**Proof.** We will prove the theorem for $a = 0$. ($\Leftarrow$): Theorem 2.8 implies that $R(\lambda : A)$ is of the form

$$R(\lambda : A) = P(\lambda)\hat{P}(\lambda) \int_0^\infty e^{-\lambda t}S(t)\,dt, \text{ Re}\lambda > 0,$$

where $S \in C^\infty([0, \infty))$, $S^{(j)}(0) = 0$, $j \in \mathbb{N}_0$ and for every $\varepsilon > 0$ there exists $M > 0$ such that $\|S(t)\| \leq M, \quad t \geq 0$ This implies $R(\lambda : A) = \mathcal{L}(G)(\lambda)$, $\text{Re}\lambda > 0$, where $G = P(-d/dt)\hat{P}(-d/dt)S$, and $G \in Q_+(\mathbb{R}, E)$. Since

$$(\delta' \otimes I_{D(A)} - \delta \otimes A) \ast G = \delta \otimes I_E,$$

$$G \ast (\delta' \otimes I_{D(A)} - \delta \otimes A) = \delta \otimes I_{D(A)} ,$$

and (iii) holds, we have that $G$ is a Fourier hyperfunction semigroup.

($\Rightarrow$): Put $E^+_\lambda = E_\lambda H, R^+_\lambda = R_\lambda H$, where $H$ is Heaviside’s function. Let $G \in Q_+(\mathbb{R}, \mathcal{L}(E, D(A)))$ and $\lambda \in \{z \in \mathbb{C} : \text{Re}\lambda > a\} \subset \rho(A)$ be fixed. Then

$$(\delta' + \lambda \delta) \ast E^+_\lambda = \delta.$$ 

Now let $\phi \in \mathcal{P}(\mathbb{R})$ and $x \in E$. Then

$$G((\delta' + \lambda \delta) \ast_0 E^+_\lambda \ast_0 \phi) = G(\phi)x,$$

and

$$G(\delta' \ast_0 R^+_\lambda \ast_0 \phi)x + \lambda G(\delta \ast_0 E^+_\lambda \ast_0 \phi)x = G(\delta')G(E^+_\lambda \ast_0 \phi)x + \lambda \hat{G}(\lambda)G(\phi)x.$$ 

Hence,

$$-A(\hat{G}(\lambda)G(\phi)x) + \lambda \hat{G}(\lambda)G(\phi)x = G(\phi)x.$$ 

Since (H.3) is assumed $(-A + \lambda)\hat{G}(\lambda) = I$, so $\|\hat{G}(\lambda)\| \leq C|P(\lambda)|$, $\text{Re}\lambda > a$, where $P$ is an appropriate local operator.

**Corollary 2.10.** Suppose $A$ is a closed linear operator. If $A$ generates an (EFHSG), (i), (ii) and (iii) of Theorem 2.9 hold.

If (ii) and (iii) of Theorem 2.9 hold, then $G$, defined in the same way as above, is a Fourier hyperfunction fundamental solution for $A$. If (iii) is satisfied, then $G$ is an (EFHSG) generated by $A$. 

We note that in Corollary 2.10 the operator $A$ is non-densely defined. Now we will prove a theorem related to Fourier hyperfunction semigroups. As in the case of ultradistributions, the theorem can be proved for (EFHSHG) but for the sake of simplicity, we will assume that $a = 0$.

We need one more theorem.

**Theorem 2.11.** Let $A$ be a closed operator in $E$. If $A$ generates a (FHSG) $G$, then $G$ is a Fourier hyperfunction fundamental solution for

$$P := \delta' \otimes I_{D(A)} - \delta \otimes A \in Q_+(\mathbb{D}, L([D(A)], E)).$$

In particular, if $T \in Q_+(\mathbb{D}, E)$, then $u = G * T$ is the unique solution of

$$-Au + \frac{\partial}{\partial t}u = T, \ u \in Q_+(\mathbb{D}, [D(A)]). \tag{6}$$

If $\text{supp} T \subset [\alpha, \infty)$, then $\text{supp} u \subset [\alpha, \infty)$.

Conversely, if $G \in Q_+(\mathbb{D}, L(E, [D(A)]))$ is a Fourier hyperfunction fundamental solution for $P$ and $N(G) = \{0\}$, then $G$ is an (FHSG) in $E$.

**Proof.** ($\Rightarrow$) One can simply check that $(G(\psi)x, G(-\psi')x - \psi(0)x) \in G(-\delta')$ and $G$ is a fundamental solution for $P$. The uniqueness of the solution $u = G * T$ of (6) is clear as well as the support property for the solution $u$ if $\text{supp} T \subset [\alpha, \infty)$.

The part ($\Leftarrow$) can be proved in the same way as in the [33, Theorem 3.3], part (d)$\Rightarrow$ (a). □

First, we list the statements:

1. $A$ generates an (FHSG) $G$.
2. $A$ generates an (FHSG) of the form $G = P_{L_p}(- \text{id}/d\text{t})S_{a,K}$, where $S_K : \mathbb{R} \rightarrow L(E)$ is exponentially slowly increasing continuous function and $S_K(t) = 0, \ t \leq 0$.
3. $A$ is the generator of a global $K$-convoluted semigroup $(S_K(t))_{t \geq 0}$, where $K = \mathcal{L}^{-1}\left(\frac{1}{P_{L_p}(-i\lambda)}\right)$.
4. The problem

$$(\delta \otimes (-A) + \delta' \otimes I_E) * G = \delta \otimes I_E, \ G * (\delta \otimes (-A) + \delta' \otimes I_{D(A)}) = \delta \otimes I_{D(A)}$$

has a unique solution $G \in Q_+(\mathbb{D}, L(E, [D(A)]))$ with $N(G) = \{0\}$.
5. For every $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

$$\rho(A) \supset \{\lambda \in \mathbb{C} : \text{Re}\lambda > 0\}$$

and

$$||R(\lambda : A)|| \leq K_\varepsilon e^{\varepsilon|\lambda|}, \ \text{Re}\lambda > 0.$$

**Theorem 2.12.** (1) $\iff$ (4); (1) $\Rightarrow$ (3); (3) $\Rightarrow$ (4); (4) $\Rightarrow$ (5);

**Proof.** The equivalence of (1) and (4) can be proved in the same way as in the case of ultradistribution semigroups, [33, Theorem 3.3].

One must use Lemma 2.7 in proving of (1) $\Rightarrow$ (3) (see [33, Theorem 3.3 ](a)$' \Rightarrow$ (c)$'$). The implication (4) $\Rightarrow$ (5) is a consequence of Theorem 2.9.

...
and Corollary 2.10. In the case when the infinitesimal generator is densely defined Y. Ito [16] proved the equivalence of a slightly different assertion (4), without the assumption $N(G) = \{0\}$, and (5). Our assertion is the stronger one since it is based on the strong structural result of Theorem 2.9.

Operators which satisfy (5) may be given using the analysis of P.C. Kunstmann [34, Example 1.6] with suitable chosen sequence $(M_p)_{p \in \mathbb{N}_0}$.

The definition of a hyperfunction fundamental solution $G$ for a closed linear operator $A$ can be found in the paper [44] of S. Ōuchi. For the sake of simplicity, we shall also say, in that case, that $A$ generates a hyperfunction semigroup $G$. The next assertion is proved in [44]:

A closed linear operator $A$ generates a hyperfunction semigroup if and only if for every $\varepsilon > 0$ there exist suitable $C_\varepsilon$, $K_\varepsilon > 0$ so that

$$\rho(A) \supset \Omega_\varepsilon := \{\lambda \in \mathbb{C} : \text{Re}\lambda \geq \varepsilon |\lambda| + C_\varepsilon\}$$

and

$$||R(\lambda : A)|| \leq K_\varepsilon e^{\varepsilon |\lambda|}, \lambda \in \Omega_\varepsilon.$$ We will give some results related to hyperfunction and convoluted semigroups in terms of spectral conditions and the asymptotic behavior of $\tilde{K}$. We refer to [2] for the similar results related to $n$-times integrated semigroups, $n \in \mathbb{N}_0$, to [15] for $\alpha$-times integrated semigroups, $\alpha > 0$ and to [40, Theorem 1.3.1] for convoluted semigroups. Since we focus our attention on connections of convoluted semigroups with hyperfunction semigroups, we use the next conditions for $K$:

(P1) $K$ is exponentially bounded, i.e., there exist $\beta \in \mathbb{R}$ and $M > 0$ so that

$$|K(t)| \leq Me^{\beta t}, \text{ for a.e. } t \geq 0.$$ (P2) $\tilde{K}(\lambda) \neq 0$, $\text{Re}\lambda > \beta$.

In general, the second condition does not hold for exponentially bounded functions, cf. [3] Theorem 1.11.1 and [31]. Following analysis in [10] and [29], Theorem 2.7.1, Theorem 2.7.2, in our context, we can give the following statements:

**Theorem 2.13.**

1. Let $K$ satisfy (P1) and (P2) and let $(S_K(t))_{t \in [0, \tau)}$, $0 < \tau \leq \infty$, be a $K$-convoluted semigroup generated by $A$. Suppose that for every $\varepsilon > 0$ there exist $\varepsilon_0 \in (0, \tau\varepsilon)$ and $T_\varepsilon > 0$ such that

$$\frac{1}{|\tilde{K}(\lambda)|} \leq T_\varepsilon e^{\varepsilon_0 |\lambda|}, \lambda \in \Omega_\varepsilon \cap \{\lambda \in \mathbb{C} : \text{Re}\lambda > \beta\}.$$ Then for every $\varepsilon > 0$ there exist $\overline{C}_\varepsilon > 0$ and $\overline{K}_\varepsilon > 0$ such that

$$\Omega^1_{\varepsilon} = \{\lambda \in \mathbb{C} : \text{Re}\lambda \geq \varepsilon |\lambda| + \overline{C}_\varepsilon\} \subset \rho(A) \text{ and } ||R(\lambda : A)|| \leq \overline{K}_\varepsilon e^{\varepsilon_0 |\lambda|}, \lambda \in \Omega^1_{\varepsilon}.$$ 2. Let $K \in L^1_{loc}([0, \tau))$ for some $0 < \tau \leq 1$ and let $A$ generate a $K$-convoluted semigroup $(S_K(t))_{t \in [0, \tau)}$. If $K$ can be extended to a function $K_1$ in $L^1_{loc}([0, \infty))$ which satisfies (P1) so that its Laplace transform has the same estimates as in Theorem 2.13 then $A$ generates $S$. Ōuchi’s hyperfunction semigroup.
3. Assume that for every $\varepsilon > 0$ there exist $C_\varepsilon > 0$ and $M_\varepsilon > 0$ so that $\Omega_\varepsilon \subset \rho(A)$ and that $||R(\lambda : A)|| \leq M_\varepsilon e^{C_\varepsilon |\lambda|}$, $\lambda \in \Omega_\varepsilon$.

(a) Assume that $K$ is an exponentially bounded function with the following property for its Laplace transform: There exists $\varepsilon_0 > 0$ such that for every $\varepsilon > 0$ exists $T_\varepsilon > 0$ with

$$|\tilde{K}(\lambda)| \leq T_\varepsilon e^{-\varepsilon_0 |\lambda|}, \quad \lambda \in \Omega_\varepsilon.$$  

If $\tau > 0$ and $K_{|[0,\tau)} \neq 0$ ($K_{|[0,\tau)}$ is the restriction of $K$ on $[0,\tau)$, then A generates a local $K$-semigroup on $[0,\tau)$.

(b) Assume that $K$ is an exponentially bounded function, $\tau > 0$ and $K_{|[0,\tau)} \neq 0$. Assume that for every $\varepsilon > 0$ there exist $T_\varepsilon > 0$ and $\varepsilon_0 \in (\varepsilon(1 + \tau), \infty)$ such that (7) holds. Then A generates a local $K$-semigroup on $[0,\tau)$.

Connections of hyperfunction and ultradistribution semigroups with (local integrated) regularized semigroups seems to be more complicated. In this context, there is a example (essentially due to R. Beals [4]) which shows that there exists a densely defined operator $A$ on the Hardy space $H^2(\mathbb{C}_+)$ which has the following properties:

1. $A$ is the generator of S. Ōuchi’s hyperfunction semigroup.
2. $A$ is not a subgenerator of a local $\alpha$-times integrated $C$-semigroup, for any injective $C \in L(H^2(\mathbb{C}_+))$ and $\alpha > 0$.

It is clear that there exists an operator $A$ which generates an entire $C$-regularized group but not a hyperfunction semigroup.

References

[1] W. Arendt, Vector-valued Laplace transforms and Cauchy problems. Israel J. Math. 59 (1987), 327–352.
[2] W. Arendt, O. El-Mennaoui and V. Keyantuo, Local integrated semigroups: evolution with jumps of regularity. J. Math. Anal. Appl. 186 (1994), 572–595.
[3] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, Vector-valued Laplace Transforms and Cauchy Problems. Birkhäuser Verlag, 2001.
[4] R. Beals, On the abstract Cauchy problem. J. Funct. Anal. 10 (1972), 281–299.
[5] R. Beals, Semigroups and abstract Gevrey spaces. J. Funct. Anal. 10 (1972), 300-308.
[6] J. Chazarain, Problèmes de Cauchy abstraites et applications à quelques problèmes mixtes. J. Funct. Anal. 7 (1971), 386–446.
[7] J. Chung, S.-Y. Chung and D. Kim, Characterization of the Gelfand-Shilov spaces via Fourier transforms. Proc. of AMS 124 (1996), 2101–2108.
[8] J. Chung, S.-Y. Chung and D. Kim, A characterization for Fourier hyperfunctions. Publ. Res. Inst. Math. Sci. 30 (1994), 203–208.
[9] I. Ciourănescu, Beurling spaces of class $(M_p)$ and ultradistribution semi-groups. Bull. Sci. Math. 102 (1978), 167–192.
[10] I. Ciorănescu, G. Lumer, *Problèmes d’évolution régularisés par un noyau général K(t).* Formule de Duhamel, prolongements, théorèmes de généralisation. C. R. Acad. Sci. Paris Sér. I Math. **319** (1995), 1273–1278.

[11] R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations.* Lect. Notes Math. **1570**, Springer 1994.

[12] H. A. Emami-Rad, *Les semi-groupes distributions de Beurling.* C. R. Acad. Sc. Sér. A **276** (1973), 117–119.

[13] H. O. Fattorini, *Structural theorems for vector valued ultradistributions.* J. Funct. Anal. **39** (1980), 381-407.

[14] L. Hörmander, *Between distributions and hyperfunctions.* Colloq. Honneur L. Schwartz, Ec. Polytech. 1983, Vol 1, Astérisque **131** (1985), 89-106.

[15] M. Hieber, *Integrated semigroups and differential operators on L^p spaces.* Math. Ann., **29** (1991), 1- 16.

[16] Y. Ito, *On the abstract Cauchy problems in the sense of Fourier hyperfunctions.* J. Math. Tokushima Univ. **16** (1982), 25-31.

[17] Y. Ito, *Fourier hyperfunction semigroups.* J. Math. Tokushima Univ. **16** (1982), 33-53.

[18] A. Kaneko, *Introduction to Hyperfunctions.* Kluwer, Dordercht, Boston, London, 1982.

[19] T. Kawai, *The theory of Fourier transformations in the theory of hyperfunctions and its applications.* Surikaiseki Kenkyusho, Kyoto Univ., **108** (1969), 84–288 (in Japanese).

[20] T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients.* J. Fac. Sci., Univ. Tokyo, Sec. IA. **17** (1970), 465–517.

[21] V. Keyantuo, *Integrated semigroups and related partial differential equations.* J. Math. Anal. Appl. **212** (1997), 135–153.

[22] J. Kisyński, *Distribution semigroups and one parameter semigroups.* Bull. Polish Acad. Sci. **50** (2002), 189–216.

[23] A.N. Kochubei, *Hyperfunction solutions of differential–operator equations.* Siberian Math. J., **20**, No. 4 (1979), 544-554.

[24] H. Komatsu, *Ultradistributions, I. Structure theorems and a characterization.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 25–105.

[25] H. Komatsu, *An introduction to the theory of generalized functions,* Iwanami Shoten, 1978, translated by L. S. Hahn in 1984., Department of Mathematics Sciences University of Tokyo.

[26] H. Komatsu, *Ultradistributions, III. Vector valued ultradistributions the theory of kernels.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), 653–718.

[27] H. Komatsu, *Operational calculus and semi-groups of operators.* Functional Analysis and Related topics (Kioto), Springer, Berlin, 213-234, 1991.

[28] M. Kostić, *C-Distribution semigroups.* Studia Math. **185** (2008), 201–217.

[29] M. Kostić, *Generalized semigroups and cosine functions.* Mathematical Institute, Belgrade, 2011.

[30] M. Kostić, *Convoluted C-cosine functions and convoluted C-semigroups.* Bull. Cl. Sci. Math. Nat. Sci. Math. **28** (2003), 75–92.
[31] M. Kostić and S. Pilipović, *Global convoluted semigroups*. Math. Nachr., **280**, No. 15 (2007), 1727–1743.

[32] M. Kostić and S. Pilipović, *Ultradistribution semigroups*. Siberian Math. J., **53**, No. 2 (2012), 232-242.

[33] M. Kostić, S. Pilipović and D. Velinov, *Structural theorems for ultradistribution semigroups* accepted in Siberian Math. J.

[34] P. C. Kunstmann, *Stationary dense operators and generation of non-dense distribution semigroups*. J. Operator Theory **37** (1997), 111–120.

[35] P. C. Kunstmann, *Distribution semigroups and abstract Cauchy problems*. Trans. Amer. Math. Soc. **351** (1999), 837–856.

[36] P. C. Kunstmann, *Banach space valued ultradistributions and applications to abstract Cauchy problems*, [http://www.math.kit.edu/iana1/~kunstmann/media/ultra-appl.pdf](http://www.math.kit.edu/iana1/~kunstmann/media/ultra-appl.pdf), preprint.

[37] M. Li, F. Huang and Q. Zheng, *Local integrated C-semigroups*. Studia Math. **145** (2001), 265–280.

[38] J. L. Lions, *Semi-groupes distributions*. Portugal. Math. **19** (1960), 141-164.

[39] G. Lumer and F. Neubrander, *The asymptotic Laplace transform: new results and relation to Komatsu’s Laplace transform of hyperfunctions*. Partial Differential Equations on Multistructures (Luminy, 1999), 147–162. Lect. Not. Pure Appl. Math., **219**, Dekker, New York, 2001.

[40] I. V. Melnikova and A. I. Filinkov, *Abstract Cauchy Problems: Three Approaches*. Chapman & Hall/CRC, 2001.

[41] M. Morimoto *An introduction to Sato’s hyperfunctions*. Translations of Mathematical Monographs, 129. American Mathematical Society, Providence, RI, 1993.

[42] F. Neubrander, *Integrated semigroups and their applications to the abstract Cauchy problem*. Pacific J. Math. **135** (1988), 111–155.

[43] Y. Ohya, *Le problème de Cauchy pour les équations hyperboliques à caractéristiques multiples*. J. Math. Soc. Japan **16** (1964), 268–286.

[44] S. Ōuchi, *Hyperfunction solutions of the abstract Cauchy problems*. Proc. Japan Acad. **47** (1971), 541–544.

[45] S. Ōuchi, *On abstract Cauchy problems in the sense of hyperfunctions in Hyperfunctions and Pseudo-Differential Equations*, Proc. Katata 1971, edited by H. Komatsu, Lect. Notes in Math., **287**, (1973), 135–152.

[46] M. Sato, *Theory of hyperfunctions*. Sûgaku **10** (1958), 1–27.

[47] S. Wang, *Quasi-distribution semigroups and integrated semigroups*. J. Funct. Anal. **146** (1997), 352–381.

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