The spherical $2+p$ spin glass model: an exactly solvable model for glass to spin-glass transition.

A. Crisanti$^\dagger$ and L. Leuzzi$^\ddagger$

Dipartimento di Fisica, Università di Roma “La Sapienza” and Istituto Nazionale Fisica della Materia, Unità di Roma, and SMC, P.le Aldo Moro 2, I-00185 Roma, Italy

We present the full phase diagram of the spherical $2+p$ spin glass model with $p \geq 4$. The main outcome is the presence of a new phase with both properties of Full Replica Symmetry Breaking (FRSB) phases of discrete models, e.g., the Sherrington-Kirkpatrick model, and those of One Replica Symmetry Breaking (1RSB). The phase, which separates a 1RSB phase from FRSB phase, is described by an order parameter function $q(x)$ with a continuous part (FRSB) for $x < m$ and a discontinuous jump (1RSB) at $x = m$. This phase has a finite complexity which leads to different dynamic and static properties.

PACS numbers: 75.10.Nr, 11.30.Pb, 05.50.+q

In the last years many efforts have been devoted to the understanding of complex systems such as spin glasses, structural glasses and others. Common denominator of all these systems is a large number of stable and metastable states whose complex structure determines their static or dynamic behaviors. In this framework mean-field models, and among them spherical models, represent a valuable tool of analytical and theoretical investigation since they can be largely solved. Up to now only spherical models with One Replica Symmetry Breaking (1RSB) phases were studied, mainly due to their relevance for the fragile glass transition [2, 3, 4].

To our knowledge the possibility infinite or Full Replica Symmetry Breaking (FRSB) phases in spherical models was first pointed out by Nieuwenhuizen [5] on the basis of the similarity between the replica free energy of some spherical models with multi-spin interactions and the relevant part of the free energy of the Sherrington-Kirkpatrick (SK) model [6, 7]. In this paper Nieuwenhuizen presented some results for the FRSB phase but a complete analysis was not provided. The problem was considered some years later [8] in connection with the following Ref. 3. The second approach starts from the microscopic dynamics and extend the results of Ref. 8 while the latter uses the Thouless-Anderson-Palmer approach [10]. In this Letter we shall mainly follow the replica approach, discussing differences with other approaches when necessary. A complete analysis of the properties of the model is beyond the scope of the Letter and will be presented elsewhere.

Applying the standard replica method the free energy per spin $f$ can be written as function of the symmetric $n \times n$ replica overlap matrix $Q_{\alpha\beta}$ as

$$-\beta f = -\beta f_0 + s(\infty) + \lim_{n \to 0} \frac{1}{n} \max_{Q} G[Q]$$

where $f_0$ is an irrelevant constant, $s(\infty) = (1 + \ln 2\pi)/2$ the entropy per spin at infinite temperature $T = 1/\beta$, and

$$G[Q] = \frac{1}{2} \sum_{\alpha\beta} g(Q_{\alpha\beta}) + \frac{1}{2} \ln \det Q$$

and

$$g(x) = \frac{\mu_q}{2} x^2 + \frac{\mu_p}{p} x^p.$$
The spherical constraint is ensured by the condition $Q_{\alpha \beta} = \gamma = 1$.

Following Parisi [11], the overlap matrix $Q_{\alpha \beta}$ for a number $R$ of steps in the replica symmetry breaking is divided into successive boxes of decreasing size $p_r$, with $p_0 = n$ and $p_{R+1} = 1$. The replica symmetric case and the FRSB case are obtained for $R = 0$ and $R \to \infty$, respectively. In the Parisi scheme the elements of $Q_{\alpha \beta}$ are then given by

$$Q_{\alpha \beta} = Q_{\alpha \cap \beta = r} = q_r, \quad r = 0, \cdots, R + 1$$

(6)

with $Q_{R+1} = \gamma$, where the notation $\alpha \cap \beta = r$ means that $\alpha$ and $\beta$ belong to the same box of size $p_r$ but to two distinct boxes of size $p_{r+1} < p_r$. The matrix obtained is conveniently expressed using the function

$$x(q) = p_0 + \sum_{r=0}^{R} (p_{r+1} - p_r) \theta(q - q_r)$$

(7)

which equals the fraction of pair of replicas with overlap less or equal to $q$. Inserting this structure into eqs. [9]-[10], neglecting terms of order $O(n^2)$, and replacing the sums by integrals, one gets after a little of algebra,

$$-2\beta f = 2s(\infty) - 2\beta f_0 + \int_0^1 dq x(q) \frac{d}{dq} g(q) + \ln(1 - q(1)) \int_0^1 dq \frac{d}{dq} x(q')$$

(8)

where $q(1) = q_R$ and $q(x)$ is the inverse of $x(q)$. Maximization of $f$ with respect to $q(x)$ leads to the self-consistent equation(s) for the order parameter function $q(x)$. Depending on the value of the coupling strengths $J^{(p)}$ and of the temperature $T$ the function $q(x)$ displays different forms which characterize the different phases of the model. Figure [11] shows the phase diagram in the space of the “natural” parameters $\mu_p - \mu_2$ for $p = 4$. In the following we shall limit ourself to the case $p = 4$, however the results are qualitatively valid for any $p \geq 4$. The analysis of the figure reveals four different phases, which will be discussed in the forthcoming part of this Letter.

The Paramagnetic phase (PM). This phase exists for not to large values of coupling parameters strengths and/or high temperature and is characterized by a null order parameter function: $q(x) = 0$ in the whole range $x \in [0,1]$. The phases becomes unstable above the line $\mu_2 = 1$ (DeAlmeida-Thouless line) where the “replicon” $\Lambda = 1 - \mu_2$ becomes negative. In this region for $p \geq 4$ and $\mu_p$ not too large a 1RSB solution is also unstable and a more complex phase (FRSB) appears. Below $\mu_2 = 1$ the PM phase remains stable for all values of $\mu_2$, similarly to what happens in the spherical $p$-spin model without a field [8], however as $\mu_p$ increases a more thermodynamically favorable 1RSB phase with a non vanishing order parameter appears.

The One Replica Symmetry Breaking phase (1RSB). This phase is characterized by a step-like order parameter function $q(x) = q_1 \theta(x - m)$ [12] and is stable as long as the replicon eigenvalue is positive:

$$\frac{1}{(1 - q_1 + m q_1)^2} - \frac{\partial^2}{\partial q^2} g(\bar{q}) \bigg|_{q=0} > 0$$

(9)

Maximization of $f$ with respect to $q_1$ and $m$ leads to the 1RSB equations whose solution can be conveniently expressed defining $q_1 = (1 - y)/(1 - y + my)$ in term of the function

$$z(y) = -2y \frac{1 - y + ln y}{1 - y}$$

(10)

introduced in Ref. [8] for the solution of the spherical $p$-spin glass model. For $p = 4$ the solution reads

$$\mu_4 = 2(1 - z(y)) (1 - y + my)^4 / m^2 y (1 - y)^2$$

(11)

$$\mu_2 = 2z(y) - (1 - y + my)^2 / m^2 y$$

By fixing the value of $m \in [0,1]$ and varying $y$ from $y_{\min}$: $z(y_{\min}) = 1/2 (\mu_2 = 0)$ to $y_{\max}$: $z(y_{\max}) = 1/2 (1 + y_{\max})$, where the replicon vanishes, one obtains the so called $m$-lines. The transition between the PM and the 1RSB phases corresponds to the $m = 1$-line. Along this line $q_1$ jumps discontinuously from zero (PM) to a finite value (1RSB) however, since $m = 1$, the thermodynamic quantities remain continuous. Inserting into [11] the value $y_{\max}$, for which the replicon vanishes, and varying $m$ from...
1 to 0 one obtains the critical line between the 1RSB and the 1-FRSB phase.

The static approach requires that \( f \) be maximal with respect to variations of \( m \). The dynamics, on the other hand, leads to the different conditions (marginal condition)

\[
\frac{1}{(1-q_1)^2} - \frac{d^2}{dq_1^2} \chi(q_1) = 0
\]

which can be stated by saying that the derivative of \( f \) with respect to \( m \) (the complexity) be maximal. As a consequence the transition lines for dynamics and statics do not coincide. Due to space limitations the equations for the dynamical transition lines will not be reported, but only drawn in Figure 1 for completeness.

The One-Full Replica Symmetry Breaking phase (1-FRSB). The analysis of the instability of the 1RSB solution reveals that in order to stabilize the phase above the line where the replica vanishes a non-zero \( q_0 \) would be needed. However in the absence of external fields the order parameter function must vanish as \( x \to 0 \), and hence a 1RSB solution is not possible. On the other hand the different location of the static and dynamic instability lines suggests that some sort of 1RSB-like form must survive in the solution. The way out is to look for a solution which below \( q_0 \) has a structure which vanishes as \( x \to 0 \). The most general form is an order parameter that has a discontinuity at \( x = m \), is continuous below it and vanish for \( x = 0 \):

\[
q(x) = \begin{cases} q_1 & \text{for } x > m \\ q_0(x) & \text{for } x < m \end{cases}
\]

with \( q(0) = 0 \) and \( \lim_{x \to m^-} q(x) = q_0 \neq \lim_{x \to m^+} q(x) = q(1) = q_1 \), see Figure 2. That this is the correct ansatz also follows from the numerical solution of the Parisi equations derived from stationarity of \( f \) with respect to \( q(x) \).

The 1-FRSB equations are obtained by inserting the form into the replica free energy and imposing stationarity with respect to \( q_0(x), q_1 \) and \( m \). The resulting equations can be solved in term of \( m \)-lines similarly to what done for the 1RSB case. For the \( p = 4 \) case the solution for the “discontinuous” part of \( q(x) \) reads

\[
\begin{align*}
\mu_4 &= \frac{[1 - y + my(1 - t)]^4}{m^2y(1 - y)^2(1 - t)^3(1 + 2t)} \\
\mu_2 &= \frac{[1 - y + my(1 - t)]^2}{m^2y(1 - t)^3(1 + 2t)}[y(1 + t + t^2) - 3t^2]
\end{align*}
\]

where

\[
t = \frac{q_0}{q_1} = \frac{1 + y - 2z(y)}{4z(y) - 3 - y}
\]

and \( q_1 = (1 - y)/(1 - y + my(1 - t)) \). The “continuous” part of \( q(x) \) is given by

\[
q = \int_0^q dq'[\mu_2 + 3\mu_4q'^2] \chi(q')^2, \quad 0 \leq q \leq q_0 = tq_1
\]

where

\[
\chi(q) = 1 - q_1 + m(q_1 - q_0) + \int_q^{q_0} dq' x(q')
\]

For any fixed value of \( m \in [0, 1] \) these equations can be solved varying \( y \) from \( y_{\text{min}} : x(y_{\text{min}}) = 0 \) (transition line to 1RSB phase) up to \( y = 1 \) \((t(y = 1) = 1)\) where the difference between \( q_1 \) and \( q_1 \) vanishes. These lines are the continuation into the 1-FRSB phase of the 1RSB \( m \)-lines. In particular the \( m = 1 \) line represents the transition between the 1-FRSB to the FRSB phase. For many aspects this transition is similar to the transition between the PM and the 1RSB phases, indeed \( q_1 \) jumps discontinuously from a null value (FRSB) to a finite value (1-FRSB) however the discontinuity appears at \( m = 1 \) so that the thermodynamic quantities are continuous across the transition. The critical \( m = 1 \)-line ends at the point where \( q_1 = q_0 \), which for \( p = 4 \) is

\[
q_1 = q_0 = \frac{1}{4}, \quad \mu_4 = \left(\frac{4}{3}\right)^4, \quad \mu_2 = \frac{32}{27}
\]

From this end-point on the transition between the FRSB and the 1-FRSB phases can only take place without a jump in in order parameter function. The value of \( t \), the ratio between \( q_1 \) and \( q_0 \), increases along the \( m \)-lines as one moves away from the transition line with the 1RSB phase, and the lines terminates when \( t = 1 \) \((q_0 = q_1)\). The set of all end-points for \( m \in [0, 1] \) defines the continuous critical line between the 1-FRSB and the FRSB phases, which for \( p = 4 \) reads

\[
\begin{align*}
\mu_4 &= \frac{1}{m^2} \left(\frac{1 + 3m}{3}\right)^4 \\
\mu_2 &= \frac{2}{3} \left(\frac{1 + 3m}{3m}\right)^2
\end{align*}
\]
On this line $q_1 = q_0 = 1/(1 + 3m)$ and $x_c \equiv x(q_0) = n$. It is worth to note that in going from the 1-FRSB to the FRSB phase the solution changes from stable to marginally stable, and remains marginally stable in the whole FRSB phase.

The presence of a discontinuity in the order parameter function leads to a finite complexity so that static and dynamics calculations leads to different solutions being the first associated with states of smallest (zero) complexity while the latter with states of largest complexity. As a consequence the $m$-lines in the two cases are different, as shown in Figure 1. We shall not report the expression for the dynamical $m$-line, this will be done elsewhere.

The inset of the figure shows the different transition lines between the FRSB and the 1-1RSB phases. The discontinuity, and hence the complexity, vanishes on the continuous transition line and two solutions coincide on this line in the whole FRSB phase.

The Full Replica Symmetry Breaking phase (FRSB). In this phase the order parameter function is always unstable [20], and has a finite complexity to the 1RSB, in particular it is stable, at difference with 1-FRSB. For many aspects the 1-FRSB phase is similar emphasizing its mixed nature this phase, which separates then both continuous jump at $m = 0$ can be computed [17, 19]. For the $2 + 4$ model it possesses a completely new phase with an order parameter function leads to a finite complexity so that static and dynamic approaches lead to the same results, in agreement with the conjecture made in Ref. [21] that the FRSB phase has vanishing complexity.

In conclusion we believe that this is a rather promising model since not only it can be fully solved, but it possess different phases which can be fully analyzed. Moreover it posses an interesting transition between two different glassy phases, similar to what found in some colloidal suspensions [22].

\begin{equation}
\begin{split}
q(x) = \frac{3/2}{3\mu_4} \frac{\mu_2}{6\mu_4} x + \frac{\mu_2}{6\mu_4} x^3 + \frac{13}{72\mu_4} x^7 + \cdots \quad (20)
\end{split}
\end{equation}

Using this expression one can show that as the PM-FRSB transition line is approached from above $\tau = 0^+ \mu_2 - 1 \rightarrow 0^+$ then both $q_0 = q(x_c)$ and $x_c$ vanishes linearly with $\tau$ as

\begin{equation}
q_0 = \frac{\tau}{2} + O(\tau^2), \quad x_c = \frac{3\mu_4}{2} \tau + O(\tau^2), \quad \tau \rightarrow 0^+. \quad (21)
\end{equation}

so that the transition between the PM and the FRSB phases takes place continuously without any jump in the order parameter function.

Conclusions. To summarize in this Letter we have provided the full phase diagram of the spherical $2 + p$ spin glass model with $p \geq 4$. Despite its simplicity the model has a rather rich phase. Indeed not only it presents a 1RSB phase similar to that of the spherical $p$-spin spin glass model and a FRSB phase similar to that of the SK model, but it also posses a completely new phase with an order parameter made of a continuous part for $x < m \leq 1$ much alike the FRSB order parameter and a discontinuous jump at $x = m$ resembling the 1RSB case. To emphasize its mixed nature this phase, which separates the FRSB phase and the 1RSB phase, has been called 1-FRSB. For may aspects the 1-FRSB phase is similar to the 1RSB, in particular it is stable, at difference with discrete models as the SK model where a solution of this form is always unstable [21], and has a finite complexity counting metastable states that are strict minima of the free energy landscape.

The transition between the FRSB and the 1-FRSB phase can be either continuous (for $\mu_2$ large enough) or discontinuous. In the first case the transition line is the continuation of the discontinuous transition between the PM and the 1RSB phases and as for the latter the discontinuity appears at $m = 1$. The presence of finite complexity in the 1-FRSB phase makes the static and dynamics transition different. The two lines join together at the end-point where the discontinuity at $m = 1$ in the order parameter function vanishes. From this point on the transition between the FRSB phase and the 1-FRSB can only take place continuously with the discontinuity of the 1-FRSB phase which vanishes at the transition. Along the the continuous transition line the complexity vanishes and the static and dynamic approaches lead to the same results, in agreement with the conjecture made in Ref. [21] that the FRSB phase has vanishing complexity.

\begin{thebibliography}{99}

[1] See for example Spin Glass Theory and Beyond, by M. Mézard, G. Parisi and M. Virasoro (World Scientific, Singapore 1987); Spin Glasses and Random Fields, edited by A. P. Young (World Scientific, Singapore 1998); Complex Behavior in Glassy Systems, edited by M. Rubí and C. Perez-Vicente (Springer-Verlag, Berlin 1996); C. A. Angell, Science 267, 1924 (1995).

[2] T.R. Kirkpatrick and D. Thirumalai, Phys. Rev. B, 36, 5388 (1987).

[3] A. Crisanti and H.-J. Sommers, Z. Phys. B 87, 341 (1992).

[4] For a review see: J. P. Bouchaud, L. F. Cugliandolo, J. Kurchan and M. Mezard in Spin glasses and random fields Ed. by A. P. Young (World Scientific, Singapore, 1997).

[5] Th. M. Nieuwenhuizen, Phys. Rev. Lett. 74, 4289 (1995).

[6] A.J. Bray and M.A. Moore, Phys. Rev. Lett. 41, 1086 (1978).

[7] E. Pytte and J. Rudnick, Phys. Rev. B 19, 3603 (1979).

[8] S. Ciuchi and A. Crisanti, Europhys. Lett. 49, 754 (2000).

[9] W. Götze and L. Sjörgen, J. Phys. Cond. Matt. 1, 4203 (1989).

[10] D. J. Thouless, P. W. Anderson and R. G. Palmer, Phil. Mag. 35, 593 (1977).

[11] G. Parisi, J. Phys. A 13, L115, 1101 and 1887 (1980).

[12] The most general form of the order parameter function in the 1RSB phase is $q(x) = q_0\theta(m - x) + q_1\theta(x - m)$. However in the absence of external fields $q_0 = 0$.

[13] A. Crisanti, H. Horner and H.-J. Sommers Z. Phys. B 92, 257 (1993).

[14] J. Kurchan, G. Parisi and M.A. Virasoro, J. Phys. I
(France) 3, 1819 (1993).
[15] A. Crisanti and H.-J. Sommers J. Phys. I (France) 5, 805 (1995).
[16] R. Monasson, Phys. Rev. Lett. 75, 2847 (1995).
[17] H.-J. Sommers and W. Dupont, J. Phys. C 17, 5785 (1984).
[18] A. Crisanti, T. Rizzo, Phys. Rev. E 65, 46137 (2002).

[19] H.-J. Sommers, J. Phys. (France) Lett. 46, L-779 (1985).
[20] A. Crisanti, L. Leuzzi, G. Parisi and T. Rizzo, cond-mat/0309256 Phys. Rev B (2004) (in press).
[21] A. Crisanti, L. Leuzzi, G. Parisi and T. Rizzo, Phys. Rev. Lett 92, 127203 (2004).
[22] K. Dawson et. al., Phys. Rev. E 63, 011401 (2001).