THE RING OF MODULAR FORMS FOR THE EVEN UNIMODULAR LATTICE
OF SIGNATURE (2,10)

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ABSTRACT. We show that the ring of modular forms with characters for the even unimodular lattice of signature (2,10) is generated by forms of weights 4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42, and 252 with one relation of weight 504. The proof is based on the comparison of the orbifold quotient of the symmetric domain with the root stack of the coarse moduli space.

1. Introduction

Let $P$ be an even non-degenerate lattice of signature $(1,t)$ for some $0 \leq t \leq 19$. A $P$-polarized $K3$ surface is a pair $(Y,j)$ of a $K3$ surface $Y$ and a primitive lattice embedding $j: P \hookrightarrow \text{Pic} Y$. Lattice polarized $K3$ surfaces are introduced by Nikulin [Nik79a] and used by Dolgachev [Dol96] to study mirror symmetry for $K3$ surfaces. The mirror moduli space of $P$-polarized $K3$ surfaces is the moduli space of $\tilde{P}$-polarized $K3$ surfaces, where $\tilde{P} := (P \perp U)\perp$ is the orthogonal complement of the orthogonal sum of $P$ and the even unimodular hyperbolic lattice $U$ of rank 2 inside the $K3$ lattice $L := E_8 \perp E_8 \perp U \perp U \perp U$. Here we consider $P$ as a primitive sublattice of $L$ by the embedding $j$.

Let $T_{2,3,7}$ be the lattice determined by the Coxeter–Dynkin diagram shown in Figure 1.1. This lattice is isomorphic to $E_{8,\perp}U$ as an abstract lattice. This is the most symmetric lattice from the point of view of mirror symmetry, in the sense that the mirror dual lattice $\bar{T}_{2,3,7}$ is isomorphic to the original one; $L \cong T_{2,3,7} \perp T_{2,3,7} \perp U$. Let $M$ be the coarse moduli space of $T_{2,3,7}$-polarized $K3$ surfaces, which is isomorphic to the quotient $\mathcal{D}/\Gamma$ of a symmetric domain $\mathcal{D}$ of type IV by a discrete group $\Gamma$ (see Section 2 for the definition of $\mathcal{D}$ and $\Gamma$). Looijenga [Loo84] proved that the graded ring of modular forms (without characters) is a polynomial ring generated in degrees

$$w := (4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42),$$

(1.1)

and Brieskorn [Bri84] Theorem 5] proved that the period map induces an isomorphism from the weighted projective space $T^* := \mathbb{P}(w)$ of weight $w$ to the Satake–Baily–Borel compactification $M^*$ of $M$.

Let $\mathcal{M} := [\mathcal{D}/\Gamma]$ and $\mathbb{P}(w) := [(\mathbb{C}^{11} \setminus 0)/\mathbb{C}^\times]$ be the orbifold quotients, whose coarse moduli spaces are $M$ and $T^*$ respectively. For a pair of an orbifold and a divisor on it, one can perform the root construction [AGV08, Cad07] to introduce a generic stabilizer along the divisor. Let $T^*$ be the orbifold obtained from $\mathbb{P}(w)$ by the root construction of order two along a divisor $H_{\mathbb{P}}$ of degree 504.

The main result of this paper is the following:

Theorem 1.1. There is a bimeromorphic map $T^* \dashrightarrow \mathcal{M}$ of orbifolds, which is an isomorphism in codimension one.

The map in Theorem 1.1 is not an isomorphism, since the orbifold $T^*$ has singularities coming from the singularities of the divisor $H_{\mathbb{P}}$, whereas $\mathcal{M}$ is smooth (as an orbifold), and $T^*$ is proper, whereas $\mathcal{M}$ is not. Nevertheless, as a corollary to Theorem 1.1 one obtains the following structure theorem of the ring of modular forms:

Corollary 1.2. The graded ring of modular forms with characters for $\Gamma$ is generated by forms of weights 4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42, and 252 with one relation of weight 504.

The group of characters of $\Gamma$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the relation in Corollary 1.2 comes from the defining equation of the divisor $H_{\mathbb{P}} \subset \mathbb{P}(w)$. An explicit description of the generators in terms of Eisenstein series and a Borcherds product is given in [DKW19].

This paper is organized as follows: We collect basic definitions on moduli spaces of lattice polarized $K3$ surfaces and modular forms in Section 2. In Section 3, we recall the description of the coarse
moduli space of $T_{2,3,7}$-polarized K3 surfaces, which is a particular case of a much more general result of Looijenga \cite{Loo84}. The exposition in this section follows Shiga \cite{Shi} closely. In Section 4, we lift the period map $\Pi: T \to M$ between coarse moduli spaces to a meromorphic map of orbifolds and prove Theorem \[1.1\]. We prove Corollary \[1.2\] in Section 5 and give a description of the defining equation of the divisor $H_D$ in terms of a discriminant and a resultant in Section 6.

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2. Lattice polarized K3 surfaces

Let $L := E_8 \perp E_8 \perp U \perp U \perp U$ be the K3 lattice, and $P := T_{2,3,7}$ be the even unimodular lattice of signature $(1,9)$ appearing in Introduction. We can choose a subset $\Delta(P)^+ \subset \Delta(P)$ satisfying

(1) $\Delta(P) = \Delta(P)^+ \coprod (-\Delta(P)^+)$

and

(2) $\delta_1, \delta_2 \in \Delta(P)^+$ and $\delta_1 + \delta_2 \in \Delta(P)$, then $\delta_1 + \delta_2 \in \Delta(P)^+$.

There are ten indecomposable elements $\delta \in \Delta(P)^+$ (that is, $\delta$ cannot be written as $\delta_1 + \delta_2$ with $\delta_1, \delta_2 \in \Delta(P)^+$) corresponding to the ten vertices in Figure 1.1. The choice of $\Delta(P)^+$ is unique up to the action of the orthogonal group $O(P)$ of the lattice $L$, and $O(P)$ is generated by the reflections along $\delta \in \Delta(P)^+$. Define

\[(2.1) \quad C(P) := \{ h \in P \mid (h, \delta) \geq 0 \text{ for any } \delta \in \Delta(P)^+ \} ,\]

\[(2.2) \quad C(P)^o := \{ h \in P \mid (h, \delta) > 0 \text{ for any } \delta \in \Delta(P)^+ \} ,\]

and set

\[(2.3) \quad \text{Pic}(Y)^+ := C(Y) \cap H^2(Y; \mathbb{Z}) ,\]

\[(2.4) \quad \text{Pic}(Y)^{++} := C(Y)^o \cap H^2(Y; \mathbb{Z}) ,\]

where $C(Y)^o \subset H^{1,1}(Y) \cap H^2(Y; \mathbb{R})$ is the Kähler cone of $Y$ and $C(Y)$ is its closure.

\textbf{Definition 2.1} (Nikulin \cite{Nik79a}). A \textit{P-polarized K3 surface} is a pair $(Y, j)$ where $Y$ is a K3 surface and $j: P \hookrightarrow \text{Pic}(Y)$ is a primitive lattice embedding. An \textit{isomorphism} of $P$-polarized K3 surfaces $(Y, j)$ and $(Y', j')$ is an isomorphism $f: Y \to Y'$ of K3 surfaces such that $j = f^* \circ j'$. A $P$-polarized K3 surface is \textit{pseudo-ample} if $j(C(P)^o) \cap \text{Pic}(Y)^+ \neq \emptyset$, and \textit{ample} if $j(C(P)^o) \cap \text{Pic}(Y)^{++} \neq \emptyset$.

Fix a primitive lattice embedding $i_P: P \hookrightarrow L$ and let $Q := P^\perp$ be the orthogonal complement inside $L$, which is isomorphic to $P \perp U$ as an abstract lattice. The period domain $D$ is a connected component of

\[\{ [\Omega] \in \text{P}(Q \otimes \mathbb{C}) \mid (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0 \},\]

which is a bounded Hermitian domain of type IV. The global Torelli theorem \cite{PS71, BR75} and the surjectivity of the period map \cite{Tod80} show that the coarse moduli space of pseudo-ample $P$-polarized K3 surfaces is given by $M := D/\Gamma$, where $\Gamma := O(Q)^+$ is the index two subgroup of the orthogonal group of the lattice $Q$ preserving the connected component $D$, which acts naturally on $D$ through $\Gamma := PO(Q)^+ = \Gamma/\{ \pm \text{id} \}$. The coarse moduli space of ample $P$-polarized K3 surfaces is the subspace $(D \setminus H_D)/\Gamma$ of $M$, where $H_D := \bigcup_{\delta \in \Delta(Q)} \delta^\perp$ is the union of hyperplanes $\delta^\perp := \{ [\Omega] \in D \mid (\Omega, \delta) = 0 \}$.
Let $\overline{D}$ be the connected component of $\{\Omega \in Q \otimes \mathbb{C} \mid (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0\}$ projecting onto $D$. A modular form of weight $k \in \mathbb{Z}$ and character $\chi \in \text{Char}(\Gamma) := \text{Hom}(\Gamma, \mathbb{C}^*)$ is a holomorphic function $f: \overline{D} \to \mathbb{C}$ satisfying

(i) $f(\alpha z) = \alpha^{-k}f(z)$ for any $\alpha \in \mathbb{C}^*$, and
(ii) $f(\gamma z) = \chi(\gamma)f(z)$ for any $\gamma \in \Gamma$.

The vector spaces $A_k(\Gamma, \chi)$ of such modular forms constitute the ring

$$(2.5) \quad A(\Gamma) := \bigoplus_{k=0}^{\infty} \bigoplus_{\chi \in \text{Char}(\Gamma)} A_k(\Gamma, \chi)$$

of modular forms. We have $\text{Char}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ (see Lemmas 4.1 and 4.2).

3. Moduli space of $T_{2,3,7}$-polarized K3 surfaces

Let $P = P(6,14,21,1)$ be the weighted projective space of weight $(6,14,21,1)$ and consider the family

$$(3.1) \quad \varphi_T: \overline{\mathcal{Y}} = \big\{([x : y : z : w], t) \in P \times T^* \mid f(x, y, z, w; t) = 0\big\} \to T^* = A^{11} \setminus \emptyset$$

of hypersurfaces of $P$, where

$$(3.2) \quad f(x, y, z, w; t) = z^2 + y^3 + g_2(x, w; t)y + g_3(x, w; t),$$

$$g_2(x, w; t) = t_4x^4w^4 + t_{10}x^3w^{10} + t_{16}x^2w^{16} + t_{22}xw^{22} + t_{28}w^{28},$$

$$(3.3) \quad g_3(x, w; t) = x^7 + t_{12}x^5w^{12} + t_{18}x^4w^{18} + t_{24}x^3w^{24} + t_{30}x^2w^{30} + t_{36}xw^{36} + t_{42}w^{42},$$

and

$$(3.4) \quad t = (t_4, t_{10}, t_{16}, t_{18}, t_{22}, t_{24}, t_{28}, t_{30}, t_{36}, t_{42}) \in T^*.$$

The group $\mathbb{C}^*$ acts on $P$ and $\overline{T}^*$ in such a way that $\alpha \in \mathbb{C}^*$ sends $[x : y : z : w] \in P$ to $[x : y : z : \alpha^{-1}w]$ and $t = (t_i)_{i=4}^{12}$ to $\alpha \cdot t = (\alpha^i t_i)_{i=4}^{12}$. Since $f$ is invariant under the $\mathbb{C}^*$-action, the family $\varphi_T: \overline{\mathcal{Y}} \to T^*$ descends to a family $\varphi_T: \mathcal{Y} \to T^*$ over the weighted projective space $T^* = P(W)$ with weight

$$(3.5) \quad W = (4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42).$$

The fiber of $\varphi_T: \overline{\mathcal{Y}} \to T^*$ over $t \in T^*$ will be denoted by $\mathcal{Y}_t$. Let $T$ be the set of $t \in T^*$ such that $\mathcal{Y}_t$ has at worst rational double points, and $T := T/\mathbb{C}^*$ be the quotient variety.

The following fact is well-known:

**Proposition 3.1** (cf. **Mir89** Proposition III.3.2). An elliptic surface of the form

$$(3.6) \quad z^2 + y^3 + g_2(x)y + g_3(x) = 0$$

has a singularity worse than rational double points in the fiber over $x = a$ if and only if $\text{ord}_a(g_2) \geq 4$ and $\text{ord}_a(g_3) \geq 6$.

It follows that $\mathcal{Y}_t$ has a singularity worse than rational double points if and only if one can set

$$(3.7) \quad g_2(x, w) = ax^4w^4,$$

$$g_3(x, w) = x^7 + bx^6w^6$$

by a change of coordinates. By a simple change of coordinate from $(3.7)$ to $(3.3)$, one obtains the following:

**Corollary 3.2.** The complement $T^* \setminus T$ consists of points parametrized as

$$(3.8) \quad t = [a : -4ab : -21b^2 : 6ab^2 : 70b^3 : -4ab^3 : -105b^4 : ab^4 : 84b^5 : -35b^6 : 6b^7]$$

for $[a : b] \in P(4,6)$.

The adjunction formula shows that $\mathcal{Y}_t$ has the trivial canonical sheaf. Since the minimal resolution of a surface is crepant if and only if it has at worst rational double points, one obtains the following:

**Corollary 3.3.** The minimal model $Y_t$ of $\mathcal{Y}_t$ is a K3 surface if and only if $[t] \in T$. 

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Let $P = T_{2,3,7} = E_8 \perp U$ be the lattice appearing in Introduction.

**Proposition 3.4.** The minimal model $Y_t$ for any $[t] \in T$ has a natural structure of a pseudo-ample $P$-polarized K3 surface.

**Proof.** The divisor $\mathcal{Y}_t \cap \{w = 0\}$ of $\mathcal{Y}_t$ at infinity is given by
\begin{equation}
Y_\infty = \{ [x : y : z] \in \mathbb{P}(6, 14, 21) \mid x^7 + y^3 + z^2 = 0 \},
\end{equation}
which is a rational curve. The hypersurface $\mathcal{Y}_t$ has
- an $A_6$-singularity at $[0 : -1 : 1 : 0]$,
- an $A_2$-singularity at $[-1 : 0 : 1 : 0]$, and
- an $A_1$-singularity at $[1 : -1 : 0 : 0]$.

all coming from the singularity of the ambient space $P$. It follows that the minimal resolution $Y_t \to \mathcal{Y}_t$ has a configuration of $(-2)$-curves, whose dual intersection graph is given by $T_{2,3,7}$. □

We call the singularities appearing in the proof of Proposition 3.4 as generic singularities of the family.

**Proposition 3.5.** The $P$-polarized K3 surface $Y_t$ for $[t] \in T$ is ample if and only if $\mathcal{Y}_t$ has only generic singularities of the family.

**Proof.** A $P$-polarized K3 surface $Y_t$ is strictly pseudo-ample if and only if its period $\Omega = H^{2,0}(Y_t)$ is on the reflection hyperplane $H_8$ of a root $\delta \in \Delta(Q)$. This happens if and only if the element $\delta$ or $-\delta$ considered as a cohomology class by the embedding $Q \subset H^2(Y_t; \mathbb{Z})$ (which is induced by the $P$-polarization) is Poincaré dual to a $(-2)$-curve, since the orthogonal lattice of $\Omega$ inside $H^2(Y_t; \mathbb{Z})$ is the Néron-Severi lattice. This $(-2)$-curve is contracted in $\mathcal{Y}_t$ since $\delta$ is orthogonal to $P$, so that it appears as a singularity of $\mathcal{Y}_t$, which must be distinct from the generic singularities in the family since the class $\delta$ is not contained in $P$. □

**Proposition 3.6.** For any pseudo-ample $P$-polarized K3 surface $Y$, there exist $[t] \in T$ and an isomorphism $Y \cong Y_t$ of pseudo-ample $P$-polarized K3 surfaces.

**Proof.** We identify $P$ with its image by $j : P \to \text{Pic}(Y)$. Choose a basis $\{e, f\}$ of the orthogonal summand $U = U \perp E_8$ in such a way that $(e, e) = (f, f) = (e, f) - 1 = 0$ and $f \in C(P)$. The pseudo-amenplicity of $Y$ implies that $f$ is nef. Then one can show (cf. [PSS71 §3, Theorem 1]) that $Y$ admits a unique structure of an elliptic K3 surface with a section such that $f$ is the class of a fiber and $e - f$ is the class of a section.

An elliptic K3 surface with a section admits a Weierstrass model of the form
\begin{equation}
z^2 + y^3 + g_2(x, w)y + g_3(x, w) = 0
\end{equation}
in $\mathbb{P}(1, 4, 6, 1)$ (cf. e.g. [SS10 Section 4]). Since the sublattice $E_8 \subset P$ is orthogonal to $f \in U$, it is generated by irreducible components of a fiber of Kodaira type $\Pi^*$. One can choose a coordinate in such a way that this fiber lies over the point $x = \infty$ (or $w = 0$) in $\mathbb{P}^1$. In order for the elliptic surface (3.10) to have a singular fiber of type $\Pi^*$ at $\infty$, one needs
\begin{equation}
\text{ord}_\infty g_2(x, w) \geq 4, \quad \text{ord}_\infty g_3(x, w) = 5, \quad \text{ord}_\infty \Delta(x, w) = 10,
\end{equation}
where $\Delta = 4g_3^2 + 27g_3^2$ (cf. e.g. [Mir89 Table IV.3.1]). This requires
\begin{equation}
g_2(x, w) = \sum_{i=0}^{4} s_i x^i w^{8-i},
\end{equation}
\begin{equation}
g_3(x, w) = \sum_{i=0}^{7} s'_j x^j w^{12-j}.
\end{equation}
Since $\text{ord}_\infty g_3(x, w) = 5$, one can set $s'_7 = 1$ and $s'_6 = 0$ by a change of coordinates of $(x, w)$. The birational map
\begin{equation}
\left( \frac{x}{w^4}, \frac{y}{w^4}, \frac{z}{w^6} \right) \mapsto \left( \frac{x}{w^6}, \frac{y}{w^{14}}, \frac{z}{w^{21}} \right)
\end{equation}
of the ambient space from \( P(1, 4, 6, 1) \) to \( P(6, 14, 21, 1) \) sends \((3.10)\) to \((3.11)\). This map is compatible with \( P \)-polarizations because the map \( j: P \hookrightarrow \text{Pic}(Y) \) is uniquely determined by identifying the configuration of \((-2)\)-curves contained in the fiber over \( x = \infty \) with that in Figure 1.1. Hence Proposition \ref{prop:period} is proved.

The proof of Proposition \ref{prop:period} also shows the following:

**Proposition 3.7.** For an isomorphism \( \phi: Y_\ell \to Y_\ell' \) of pseudo-ample \( P \)-polarized K3 surfaces, there exists an element \( \alpha \in \mathbb{C}^* \) such that the following diagram commutes;

\[
\begin{array}{ccc}
Y_\ell & \xrightarrow{\varphi} & \overline{Y}_\ell \\
\downarrow & & \downarrow \\
Y_\ell' & \xrightarrow{\varphi'} & \overline{Y}_\ell'
\end{array}
\]

\[(3.15)\]

Here \( \varphi \) and \( \varphi' \) are minimal resolutions, \( \iota \) and \( \iota' \) are inclusions, and \( \phi_\alpha \) is the automorphism of \( P(6, 14, 21, 1) \) sending \([x : y : z : w] \) to \([x : y : z : \alpha w] \).

**Proof.** The proof of Proposition \ref{prop:period} shows that the embeddings \( \iota \) and \( \iota' \) given by the Weierstrass models \((3.1)\) are determined by the pseudo-ample \( P \)-polarization up to an automorphism of \( P(1, 6, 14, 21) \). The automorphism group of \( P(1, 6, 14, 21) \) consists of transformations of the form

\[
\begin{align*}
(3.16) & \quad x \mapsto \beta_1 x + \beta_2 w^6, \\
(3.17) & \quad y \mapsto \gamma_1 y + \gamma_2 xw^8 + \gamma_3 x^2 w^2, \\
(3.18) & \quad z \mapsto \delta_1 z + \delta_2 xyw + \delta_3 x^2 w^{15} + \delta_4 x^2 w^9 + \delta_5 x^3 w^3, \\
(3.19) & \quad w \mapsto \alpha w.
\end{align*}
\]

The only automorphism which preserves the Weierstrass model \((3.1)\) is \( w \mapsto \alpha w \).

By applying the global Torelli theorem \cite{PSS71, BR75} and the surjectivity of the period map \cite{Tod80}, one concludes that the period map \( \Pi_T: T \to M \), which is induced by \([t] \mapsto H^{2,0}(Y_t)\), is an isomorphism. We study the period map \( \Pi_T \) and its lifts in details in the next section.

4. **Orbifold structure in codimension one**

The natural projection \( \pi: \tilde{D} \to D \) is a principal \( \mathbb{C}^* \)-bundle, which is trivial since it admits a section

\[
\{ e - ((v, v)/2) f + v \in \tilde{D} \mid v = v_1 + \sqrt{-1} v_2 \in P \otimes \mathbb{C}, \ (v_2, v_2) > 0 \},
\]

It induces a principal \( \mathbb{C}^* \)-orbi-bundle

\[
(4.2) \quad [\pi]: \left[ \tilde{D}/\Gamma \right] \to \mathbb{M} := [D/\Gamma] \cong \left[ \tilde{D}/(\Gamma \times \mathbb{C}^*) \right]
\]

since \( \pi \) is equivariant with respect to the natural action of \( \Gamma \). The section \((4.1)\) of \( \pi \) is not a section of \([\pi]\) since it is not \( \Gamma \)-equivariant. The line bundle associated with the principal \( \mathbb{C}^* \)-orbi-bundle \((4.2)\) will be denoted by \( \mathcal{O}_{\mathbb{M}}(1) \).

The fixed locus \( D^g \) of an element \( g \in \Gamma \) is the intersection of a proper linear subspace and \( D \). The element \( g \) is said to be a reflection if \( D^g \) is the intersection of a hyperplane with \( D \).

**Lemma 4.1.** Any reflection in \( \Gamma \) is given by \( z \mapsto z + (z, \delta) \cdot \delta \) for some \( \delta \in \Delta(Q) \).

**Proof.** We may assume \( \text{rank}(Q^g) = 1 \) by taking \(-g\) instead of \( g \) if necessary. Let \( \delta \neq 0 \) be a primitive element in \((Q^g)^\perp\). Then we have \( g(\delta) = -\delta \) and \( g \) is given by \( z \mapsto z - (2(z, \delta)/(\delta, \delta)) \cdot \delta \).

Since \( Q \) is unimodular, there exists an element \( z \in Q \) such that \((z, \delta) = 1\). Since \( g(z) \in Q \), it follows that \( 2/(\delta, \delta) \in \mathbb{Z} \). Hence \((\delta, \delta) = \pm 2\). Since \( g \in O^+(Q) \), we have \((\delta, \delta) = -2\).

**Lemma 4.2.** The action of \( \Gamma \) on \( \Delta(Q) \) is transitive.
Proof. For any $\delta_1, \delta_2 \in \Delta(Q)$, it follows from Nikulin’s theory of discriminant forms of lattices [Nik79b] that $\delta_1^\perp \cong \langle 2 \rangle \perp U \perp E_8$ and that there exists an element $g \in O(Q)$ such that $g(\delta_1) = \delta_2$. We may assume $g \in O^+(Q)$ since $O^+(\delta_1^\perp) \subseteq O(\delta_1^\perp)$.

Now we construct the following diagram:

$$
\begin{array}{ccccccc}
\tilde{U} & \longrightarrow & \tilde{U} & \longrightarrow & U & \longrightarrow & V & \longrightarrow & T \backslash S_T \\
\downarrow \Pi_{\tilde{U}} & & \downarrow \Pi_U & & \downarrow \Pi_V & & \downarrow \Pi_T & \\
\tilde{D} \backslash S_{\tilde{D}} & \xrightarrow{\{\pm 1\}} & \tilde{D} \backslash S_{\tilde{D}} & \xrightarrow{C^* \times \{\pm 1\}} & D \backslash S_D & \xrightarrow{\gamma_2} & M_2 \backslash S_{M_2} & \xrightarrow{\mu} & M \backslash S_M \\
\end{array}
$$

where horizontal arrows are principal bundles and vertical arrows are isomorphisms.

Let $S_D \subset D$ be the locus where the stabilizer of the action of $\tilde{T}$ is non-trivial and does not coincide with a group of order two generated by a reflection. The locus $S_D$ contains not only intersections of more than two reflection hyperplanes, but also points where the corresponding lattice polarized K3 surface has an automorphism other than $[x : y : z : w] \mapsto [x : y : z : -w] = [x : y : -z : w]$. The action of $\tilde{T}$ on $D \setminus (H_D \cup S_D)$ is free, and the stabilizer of a point in $H_D \setminus (H_D \cap S_D)$ is a group of order two generated by a reflection.

Since the group $\tilde{T}$ is countable, the locus $H_T$ is a countable union of hyperplanes, and the locus $S_D$ is a countable union of linear subspaces of codimension greater than one. The images of $H_D$ and $S_D$ in $M$ will be denoted by $H_M$ and $S_M$, respectively.

Let $\tilde{T}_2$ and $\mu$ be the kernel and the image of the determinant map $\det : \tilde{T} \to \mathbb{C}^\times$. Set $M_2 := D/\tilde{T}_2$, $H_{M_2} := H_D/\tilde{T}_2$, and $S_{M_2} := S_D/\tilde{T}_2$. The map $\tilde{M}_2 \to M \setminus S_M$ is a double cover branched along $H_M \setminus S_M$, and the map $D \setminus S_D \to M_2 \setminus S_{M_2}$ is the universal cover since the action of $\tilde{T}_2$ on $D \setminus S_D$ is free.

Let $H_T$ and $S_T$ be the inverse images of $H_M$ and $S_M$ by the period map $\Pi : T \xrightarrow{\sim} M$. The pull-backs of $H_T$ and $S_T$ by the projection $\tilde{T} \to \tilde{M}$ will be denoted by $H_{\tilde{T}}$ and $S_{\tilde{T}}$. Let further $\Delta_{\tilde{T}} \in \mathbb{C}[t]$ be the defining equation of $H_{\tilde{T}}$, and

$$(4.4) \quad V = \tilde{V}/\mathbb{C}^\times, \quad \text{where } \tilde{V} = \left\{ (t, s) \in (\tilde{T} \setminus S_{\tilde{T}}) \times \mathbb{A}^1 \mid s^2 = \Delta_{\tilde{T}}(t) \right\},$$

be the double cover of $T \setminus S_T$ branched along $H_T \setminus S_T$. Here $\alpha \in \mathbb{C}^\times$ acts on $\mathbb{A}^1$ in $(4.4)$ by $s \mapsto \alpha^{\deg \Delta_{\tilde{T}}/2}s$. (We will show $\deg \Delta_{\tilde{T}} = 504$ in Proposition [5.2]). The restriction $\Pi|_{T \setminus S_T} : T \setminus S_T \xrightarrow{\sim} M \setminus S_M$ of the period map lifts to an isomorphism $\Pi_V : V \to M_2 \setminus S_{M_2}$, since both $V$ and $M_2 \setminus S_{M_2}$ are the double covers of isomorphic varieties branched along smooth divisors which are identified under the isomorphism.

The isomorphism $\Pi_V$ can further be lifted to an isomorphism $\Pi_U : U \to D \setminus S_D$, where $U$ is the universal cover of $V$. This isomorphism is equivariant under the action of the covering transformation group $\text{Gal}(\mathbb{D} \setminus S_D / (M \setminus S_M)) = \tilde{T}$.

Set $(x_1, x_2, x_3, x_4) = (x, y, z, w)$ and $(q_1, q_2, q_3, q_4) = (6, 14, 21, 1)$. The Griffiths–Dwork method provides the element

$$
\Omega = \text{Res} \sum_{i=1}^{4} (-1)^i q_i x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4
$$

of $H^0 \left( \Omega^2_{\tilde{T}/\tilde{T}} \right)$, which gives a 2-form $\Omega_t$ on $\tilde{T}_t$ for each $t \in \tilde{T}$, whose pull-back to $Y_t$ gives a holomorphic 2-form $\Omega_t$. Let $\mathcal{L}_{\tilde{T}} \to \tilde{T} \setminus (H_T \cup S_{\tilde{T}})$ be the local system whose fiber over $t \in \tilde{T} \setminus (H_T \cup S_{\tilde{T}})$ is the second homology group $H_2(Y_t; \mathbb{Z})$. For any $t \in \tilde{T} \setminus (H_T \cup S_{\tilde{T}})$, the fiber of the projection $\tilde{T} \to T$ above $[t] \in T$ can be identified with $\mathbb{C}^\times/\{\pm 1\}$, and the monodromy of $\mathcal{L}_{\tilde{T}}$ along the generator of the projection $\pi_1(\mathbb{C}^\times/\{\pm 1\}) \cong \mathbb{Z}$ is induced by the automorphism $[x : y : z : w] \mapsto [x : y : z : -w]$ of $\tilde{T}_t$, whose induced action on $Q$ is $-\text{id}_Q$.

The monodromy of $\mathcal{L}_{\tilde{T}}$ along $H_T$ is the Picard–Lefschetz transformation with respect to the vanishing cycle $C$, which is the reflection along the homology class $[C]$ of the vanishing cycle. Note that the class $[C]$ is equal to the class of the $(−2)$-curve which defines the reflection hyperplane. It follows
that the pull-back of $\Sigma_T$ to the double cover $\tilde{V}$ does not have a monodromy along the ramification divisor $H_T$, and hence extends to a local system $\Sigma_T$ on $\tilde{V}$.

Let $\Sigma_T$ be the pull-back of $\Sigma_T$ to $\tilde{U} := U \times_T \tilde{T}$. One has $\pi_1(\tilde{U}) \cong \pi_1(\mathbb{C}^\times / \{\pm 1\})$ since $U$ is the universal cover of $V$. The monodromy of $\Sigma_T$ along the generator of $\pi_1(\tilde{U}) \cong \pi_1(\mathbb{C}^\times / \{\pm 1\})$ is given by $-\text{id}_{\mathbb{H}_2(Y; \mathbb{Z})}$. It follows that the pull-back $\Sigma_{\tilde{U}}$ of $\Sigma_T$ to the non-trivial double cover $\tilde{U} \to \tilde{U}$ is trivial. The period map $\Pi_T : T \to M$ is defined in such a way that the lift $\Pi_{\tilde{T}} : \tilde{U} \to \tilde{D} \setminus S_{\tilde{D}}$ is given by integration of $\Omega$ along a basis of $\Sigma_{\tilde{U}}$ obtained by choosing a global trivialization of $\Sigma_{\tilde{U}}$. It follows from the construction that the map $\Pi_{\tilde{T}}$ is equivariant with respect to the natural action of the central extension $\Gamma$ of $\Gamma$ by $\{\pm 1\}$, where $\Gamma$ acts on the bases of the principal $\mathbb{C}^\times$-bundles $(\tilde{U} \to U) \cong (\tilde{D} \setminus S_{\tilde{D}} \to D \setminus D)$, and $\{\pm 1\}$ acts on the fibers. The map $\Pi_{\tilde{T}}$ induces an isomorphism $\Pi_{\tilde{T}} : \tilde{U} \to \tilde{D} \setminus S_{\tilde{D}}$, where $\tilde{D} := D / \{\pm 1\}$ and $S_{\tilde{D}} := S_D / \{\pm 1\}$.

The action of $\alpha \in \mathbb{C}^\times$ on $T$ sends a point $(t_i)$ to $(\alpha^i t_i)$. The change of $\Delta$ caused by this action can be absorbed by the coordinate change sending $w \to \alpha^{-1} w$ and keeping $x$, $y$, and $z$ fixed. This sends $\Omega$ to $\alpha^{-1} \Omega$, so that the period will be multiplied by $\alpha^{-1}$. This shows that $\Pi_{\tilde{T}}$ is $\mathbb{C}^\times$-equivariant.

Let $H_T$ be the hypersurface of $\mathbb{P}(w) := [\mathbb{T} / \mathbb{C}^\times]$ defined by $\Delta_T$. The orbifold $\mathbb{P}(w)$ has a generic stabilizer of order 2, and the natural morphism to the orbifold $\mathbb{P}(w)(2) := [\mathbb{T} / (\mathbb{C}^\times / \{\pm 1\})]$ without a generic stabilizer is a $B\{\pm 1\}$-bundle. Let $T^*$ be the stack obtained from $\mathbb{P}(w)$ by the root construction [AGV08 Cad07] of order 2 along the divisor $H_T$. The orbifold quotient

$$T := [\mathbb{V} / (\mu \times \mathbb{C}^\times)] \cong [\tilde{U} / (\Gamma \times \mathbb{C}^\times)]$$

is an open substack of $T^*$. Since $\Pi_{\tilde{T}}$ is $\Gamma \times \mathbb{C}^\times$-equivariant, one obtains an isomorphism

$$\Pi_T : T \cong M \setminus S_M$$

of orbifolds where $S_M := [S_{\tilde{D}} / (\Gamma \times \mathbb{C}^\times)]$. Since the codimension of $S_{\tilde{D}}$ in $\tilde{D}$ is greater than one, Theorem 1.1 is proved.

5. The canonical bundle and the total coordinate ring

A character of $\mathbb{C}^\times \times \Gamma$ gives a $\mathbb{C}^\times \times \Gamma$-equivariant structure on the trivial line bundle on $\tilde{D}$, which in turn gives a line bundle on the orbifold quotient $M \cong [\tilde{D} / (\mathbb{C}^\times \times \Gamma)]$. We write the line bundle on $M$ associated with the character $\mathbb{C}^\times \times \Gamma \ni (\alpha, g) \mapsto \alpha^{-k} \cdot (\det g)^l$ as $O_M(k) \otimes \det^l$.

**Proposition 5.1.** The canonical bundle on $M$ is given by $\omega_M \cong O_M(10) \otimes \det$.

**Proof.** We first consider the canonical bundle of $D \cong [\tilde{D} / \mathbb{C}^\times]$, which is an open subset defined by $(\Omega, \Omega) > 0$ of a quadratic hypersurface in $\mathbb{P}(Q \otimes \mathbb{C})$ defined by $(\Omega, \Omega) = 0$. In general, the canonical bundle of a degree $k$ hypersurface $X$ in $\mathbb{P}^n$ is given by $O_X(k - n - 1)$. We follow our convention for $O_M(k)$ and write the line bundle on $D \cong [\tilde{D} / \mathbb{C}^\times]$ associated with the character $\alpha \mapsto \alpha^{-k}$ of $\mathbb{C}^\times$ as $O_D(k)$. Since this is inverse to the usual convention, one has $\omega_D \cong O_D(10)$.

Now we take the action of $\Gamma$ into account and consider the canonical bundle of $M \cong [\tilde{D} / (\mathbb{C}^\times \times \Gamma)]$. Since a reflection changes the sign of a top differential form, one concludes that $\omega_M \cong O_M(10) \otimes \det$.

It follows from Theorem 1.1 that Pic $M$ is isomorphic to Pic $T^*$. Let $O_{T^*}(1) := p^* O_{\mathbb{P}(w)}(1)$ be the pull-back of the positive generator $O_{\mathbb{P}(w)}(1)$ of Pic $\mathbb{P}(w) \cong \mathbb{Z}$ by the structure morphism $p : T^* \to \mathbb{P}(w)$, and $O_{T^*}(H_{T^*})$ be the tautological bundle on the root stack satisfying

$$(5.1) \quad O_{T^*}(H_{T^*}) \otimes^2 \cong p^* O_{\mathbb{P}(w)}(H_T).$$

It follows from [Cad07] Section 3.1 that Pic $T^*$ is generated by $O_{T^*}(1)$ and $O_{T^*}(H_{T^*})$ with relation (5.1). This shows that Pic $T^*$, and hence Pic $M$, is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z}$. The free part is generated by $O_{T^*}(1)$, which is isomorphic to $\Pi^* O_M(1)$ since $\Pi_T$ comes from the $(\mathbb{C}^\times \times \Gamma)$-equivariant morphism
The identification of the torsion part is determined uniquely by the group structure of the Picard group as \( \Pi_t^* (\mathcal{O}_M \otimes \det) \cong \mathcal{O}_{T^*}(-\deg \Delta_T/2) \otimes \mathcal{O}_{T^*}(H_{T^*}) \).

**Proposition 5.2.** One has \( \deg \Delta_T = 504 \).

**Proof.** Since the root stack is the quotient of the branched double cover, the ramification formula for the canonical bundle gives

\[
\omega_{T^*} \cong p^* \omega_T \otimes \mathcal{O}_{T^*}(H_{T^*}) \\
\cong \mathcal{O}_{T^*}(-242) \otimes \mathcal{O}_{T^*}(H_{T^*}) \\
\cong \mathcal{O}_{T^*}(-242 + \deg \Delta_T/2) \otimes (\mathcal{O}_{T^*}(-\deg \Delta_T/2) \otimes \mathcal{O}_{T^*}(H_{T^*})).
\]

Since this is isomorphic to \( \Pi_t^* \omega_M \), one has \(-242 + \deg \Delta_T/2 = 10\) and hence \( \deg \Delta_T = 504 \). \( \square \)

One has

\[
(5.2) \quad A(\Gamma) := \bigoplus_{k=0}^{\infty} \bigoplus_{\chi \in \text{Char}(\Gamma)} A_k(\Gamma, \chi)
\]

\[
(5.3) \quad \cong \bigoplus_{\mathcal{L} \in \text{Pic} \mathcal{M}} H^0(\mathcal{L})
\]

\[
(5.4) \quad \cong \bigoplus_{\mathcal{L} \in \text{Pic} \mathcal{M}^*} H^0(\mathcal{L}),
\]

which is generated by the polynomial ring \( \bigoplus_{k=0}^{\infty} H^0(\mathcal{O}_{T^*}(k)) \cong \bigoplus_{k=0}^{\infty} H^0(\mathcal{O}_k(k)) \) and the canonical section \( s \in H^0(\mathcal{O}_{T^*}(H_{T^*})) \) satisfying \( s^2 = \Delta_T \). This concludes the proof of Corollary 1.2.

### 6. Discriminant and Resultant

The discriminant of \( y^3 + g_2(x, w; t)y + g_3(x, w; t) \) as a polynomial of \( y \) is given by \( 4g_2(x, w; t)^3 + 27g_3(x, w; t)^2 \), which defines a hypersurface of degree 84 in \( \mathbb{P}(6, 1) = \text{Proj} \mathbb{C}[x, w] \). In other words, \( [4g_2(x, w; t)^3 + 27g_3(x, w; t)^2]/w^{84} \) is a polynomial of degree 14 in \( x/w^6 \). Let \( k(t) \) be the discriminant of this polynomial in one variable, which is a homogeneous polynomial of degree 14 \( \cdot \) 13 \( \cdot \) 6 = 1092 in \( t \). A general point on the divisor \( D \subset \mathbb{P}(w) \) defined by \( k(t) \) corresponds to the locus where two fibers of Kodaira type I\(_1\) collapse into one fiber. The divisor \( D \) is a linear combination of two prime divisors \( D_1 \) and \( D_2 \). A general point on the component \( D_1 \) corresponds to the case when there exists a point \( p = [x : w] \) on \( \mathbb{P}(6, 1) \) such that neither \( g_2 \) nor \( g_3 \) vanishes at \( p \), and a general point on the other component \( D_2 \) corresponds to the case when both \( g_2 \) and \( g_3 \) vanish at \( p \). In the former case, the resulting singular fiber is of Kodaira type I\(_2\), and the surface \( \mathcal{Y}_t \) acquires an A\(_1\)-singularity. In the latter case, the resulting singular fiber is of Kodaira type II, and the surface \( \mathcal{Y}_t \) does not acquire any new singularity. The defining equation of \( D_1 \) is \( \Delta_T \). The defining equation of \( D_2 \) is the resultant of \( g_2 \) and \( g_3 \), which is given as the determinant

\[
(6.1) \quad f_2 = \begin{vmatrix}
  t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\
  t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\
  t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\
  t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\
  t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\
  t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\
  t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\
  t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\
  t_{42} & t_{36} & t_{30} & t_{24} & t_{18} & 0 & 1 \\
  t_{42} & t_{36} & t_{30} & t_{24} & t_{18} & 0 & 1 \\
  t_{42} & t_{36} & t_{30} & t_{24} & t_{18} & 0 & 1 \\
  t_{42} & t_{36} & t_{30} & t_{24} & t_{18} & 0 & 1 \\
  t_{42} & t_{36} & t_{30} & t_{24} & t_{18} & 0 & 1
\end{vmatrix}
\]

of the Sylvester matrix, which is homogeneous of degree \( d_2 = 196 \).

**Lemma 6.1.** For a polynomial \( f(x, y, t) = y^3 + g_2(x, t)y + g_3(x, t) \) in three variables, let \( h(x, t) = 4g_2(x, t)^3 + 27g_3(x, t)^2 \) be the discriminant of \( f(x, y, t) \) as a polynomial in \( y \), \( k(t) \) be the discriminant
of \( h(x, t) \) as a polynomial in \( x \), and \( r(t) \) be the resultant of the pair \((g_2(x, t), g_3(x, t))\) as polynomials of \( x \). Then one has \( k(t) = r(t)^3 \cdot \ell(t) \) for some polynomial \( \ell(t) \in \mathbb{C}[t] \).

**Proof.** We may assume that coefficients of \( g_2(x, t) \) and \( g_3(x, t) \) are generic. In general, if we set \( h(x, t) = g_2(x, t)^n - g_3(x, t)^m \) for \( n > m \geq 1 \), then the order of vanishing of the discriminant \( k(t) \) of \( h(x, t) \) along the resultant \( r(t) \) of \( g_2(x, t) \) and \( g_3(x, t) \) is given by \( n(m - 1) \). This follows from the fact that the set of solutions of

\[
(x - \alpha)^n - (x - \beta)^m = 0
\]

for \( n > m \) near \( \alpha = \beta \) consists of \( m \) solutions of the form

\[
a_i = \beta + c_i^n (\beta - \alpha)^n/m + o((\beta - \alpha)^n/m), \quad i = 0, \ldots, m - 1
\]

and \( n - m \) solutions of the form

\[
b_j = \alpha + c_j^{n-m} - \frac{m}{n - m}(\beta - \alpha) + o(\beta - \alpha), \quad j = 0, \ldots, n - m - 1,
\]

so that the leading term of the discriminant

\[
\prod_{i < j'} (a_i - a_{j'})^2 \cdot \prod_{j < j'} (b_j - b_{j'})^2 \cdot \prod_{i, j} (a_i - b_j)^2
\]

given by

\[
\prod_{i < j'} (a_i - a_{j'})^2 \sim ( (\beta - \alpha)^{n/m} )^{m-1} = (\beta - \alpha)^{n(m-1)}.
\]

Since \( \deg D - 3\deg D_2 = 1092 - 3 \cdot 196 = 504 = \deg D_1 \), one has \( D = D_1 + 3D_2 \) and

\[
\Delta_T(t) = \frac{k(t)}{r(t)^3}.
\]

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