LIFTING TO CLUSTER-TILTING OBJECTS IN HIGHER CLUSTER CATEGORIES

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Abstract. Let $d > 1$ be a positive integer. In this note, we consider the $d$-cluster-tilted algebras, the endomorphism algebras of $d$-cluster-tilting objects in $d$-cluster categories. We show that a tilting module over such an algebra lifts to a $d$-cluster-tilting object in this $d$-cluster category.

1. Introduction

Let $k$ be an algebraically closed field and $H$ be a finite-dimensional hereditary algebra. The associated cluster category $\mathcal{C}_H$ was introduced and studied in [3], and also in [6] for algebras $H$ of Dynkin type $A_n$. The cluster category $\mathcal{C}_H$ is the orbit category $D^b(H)/\tau^{-1}S$, where $S$ denotes the suspension functor and $\tau$ is the Auslander-Reiten translation in the bounded derived category $D^b(H)$. This is a certain 2-Calabi-Yau triangulated category which was invented in order to model some ingredients in the definition of cluster algebras introduced and studied by Fomin-Zelevinsky and Berenstein-Fomin-Zelevinsky in a series of articles [9, 10, 11]. For this purpose, a tilting theory was developed in the cluster category. This further led to the theory of cluster-tilted algebras initiated in [4].

For a positive integer $d > 1$, a certain $d + 1$-Calabi-Yau category, the $d$-cluster category $\mathcal{C}_d = D^b(H)/\tau^{-1}S^d$ was considered by Keller in [16]. This category is showed in [20] that it encodes the combinatorics of the $d$-clusters of Fomin and Reading [8] in a fashion similar to the way the cluster category encodes the combinatorics of the clusters of Fomin and Zelevinsky. For this reason, as a generalization of cluster categories, the $d$-cluster category and their (cluster-)tilting objects have been studied in [3, 15, 17, 20, 21, 22, 23] and so on.

It is an interesting problem to know the algebras derived equivalent to the cluster-tilted algebras. The study of their tilting modules is a step in this direction. In [19], a tilting module over a cluster-tilted algebra has been proved to lift to a cluster-tilting object in the cluster category. And in [7], the authors prove that this result holds generally in the 2-Calabi-Yau triangulated category, that is, a tilting module over the endomorphism algebra of a cluster-tilting object in a 2-Calabi-Yau triangulated category lifts to a cluster-tilting object in this 2-Calabi-Yau triangulated category. The aim of current note is to get similar result of identifying tilting modules over cluster-tilted algebras corresponding to the higher cluster category by using the $d + 1$-Calabi-Yau property. Namely, we prove the following.

Theorem. For a positive integer $d > 1$, let $\mathcal{C}_d$ be a $d$-cluster category and $T$ a $d$-cluster-tilting object, and let $\Gamma$ be the endomorphism algebra of $T$. Then a tilting $\Gamma$-module $L$ lifts to a $d$-cluster-tilting object in $\mathcal{C}_d$.

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We point out here that the methods used in this note are different from the ones used in the case of 2-Calabi-Yau category (where $d = 1$). Some descriptions on the relation between cluster and classical tilting are given in [12].

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2. Preliminaries

2.1. Tilting modules. Let $k$ be an algebraically closed field and $A$ be a finite-dimensional algebra. Let $\operatorname{mod}A$ be the category of finite-dimensional right $A$-modules. For an $A$-module $T$, let $\operatorname{add}T$ denote the full subcategory of $\operatorname{mod}A$ with objects all direct summands of direct sums of copies of $T$. Then $T$ is called a tilting module in $\operatorname{mod}A$ if
\begin{itemize}
  \item $\operatorname{pd}_A T \leq 1$,
  \item $\operatorname{Ext}^1_A(T, T) = 0$,
  \item there is an exact sequence $0 \to A \to T^0 \to T^1 \to 0$, with $T^0, T^1$ in $\operatorname{add}T$.
\end{itemize}
This is the original definition of tilting modules from [13], and it was proved in [2] that the third axiom can be replaced by the following:
\begin{itemize}
  \item the number of indecomposable direct summands of $T$ (up to isomorphism) is the same as the number of simple $A$-modules.
\end{itemize}

2.2. $d$-Calabi-Yau categories and higher cluster categories. Let $k$ be an algebraically closed field and $C$ be a Krull-Schmidt triangulated $k$-linear category with split idempotents and suspension functor $S$. We suppose that all Hom-spaces of $C$ are finite-dimensional and that $C$ admits a Serre functor $\Sigma$, cf. [18]. Let $i \geq 1$ be an integer. When we say that $C$ is Calabi-Yau of CY-dimension $i$ (or simply $i$-Calabi-Yau), we mean that there is an isomorphism of triangle functors
$$S^i \cong \Sigma.$$
For $X, Y \in C$ and $n \in \mathbb{Z}$, we put as usual
$$\operatorname{Ext}^n_C(X, Y) = \operatorname{Hom}_C(X, S^n Y).$$

Let $H$ be a hereditary algebra and the number of simple $H$-modules be $n$. Let $D = D^b(H)$ be the bounded derived category of $H$ with suspension functor $S$ and the Auslander-Reiten translate $\tau$. For a positive integer $d > 1$, the higher cluster category, $d$-cluster category is the orbit category $C_d = D/\tau^{-1}S^d$. It is shown in [16] that $C_d$ is a triangulated category and the canonical functor $D \to C$ is a triangle functor. We denote therefore by $S$ the suspension in $C_d$. The $d$-cluster category is also Krull-Schmidt and is Calabi-Yau of CY-dimension $d + 1$. That is, for any $X, Y$ in $C_d$,
$$\operatorname{Hom}(X, Y) \cong D \operatorname{Hom}(Y, S^{d+1} X),$$
or equivalently
$$\operatorname{Ext}^1(X, Y) \cong D \operatorname{Ext}^d(Y, X).$$

We recall the notation of $d$-cluster-tilting object from [17] [20] [22] [15]. This notation shares the same meaning as "maximal $d$-orthogonal subcategory" in the sense of Iyama [14]. Let $C$ be a $d + 1$-Calabi-Yau category. An object $X$ in $C$ is called rigid if
$$\operatorname{Ext}^i_C(X, X) = 0, \quad \text{for all } 1 \leq i \leq d.$$
A rigid object $T$ is called $d$-cluster-tilting if it satisfies the property: if $X \in C_d$ satisfies $\operatorname{Ext}^i_C(X, T) = 0$ for all $1 \leq i \leq d$, then $X \in T = \operatorname{add}T$. 

Let $\mathcal{C}$ be a $d+1$-Calabi-Yau category with a $d$-cluster-tilting object $T$. Let $\Gamma$ be the endomorphism algebra of $T$. For classes $\mathcal{U}, \mathcal{V}$ of objects, we denote by $\mathcal{U} \ast \mathcal{V}$ the full subcategory of all objects $X$ of $\mathcal{C}$ appearing in a triangle
\[ U \rightarrow X \rightarrow V \rightarrow SU. \]
Let $F : \mathcal{C} \rightarrow \text{mod} \Gamma$ be the functor which sends $X$ to $\text{Hom}_\mathcal{C}(T, X)$. There is an essential result in [17] as following.

**Theorem 2.1.** For each module $M \in \text{mod} \Gamma$, there is a triangle
\[ T_0 \rightarrow T_1 \rightarrow X \rightarrow ST_0 \]
such that $FX$ is isomorphic to $M$. The functor $F$ induces an equivalence
\[ T \ast ST/(ST) \sim \rightarrow \text{mod} \Gamma. \]

Thus when we say that a $\Gamma$-module $L$ lifts to the higher cluster category, we mean its preimage under the equivalence.

We need the following theorem which is shown in [22].

**Theorem 2.2.** Let $X$ be a rigid object in the $d$-cluster category $\mathcal{C}_d$, then $X$ is a $d$-cluster-tilting object if and only if $X$ has $n$ indecomposable summands, up to isomorphism.

## 3. Proof of the main result

First we prove the following crucial proposition.

**Proposition 3.1.** Let $\mathcal{C}$ be a $d+1$-Calabi-Yau category with a $d$-cluster-tilting object $T$ and let $\Gamma$ be the endomorphism algebra of $T$. Let $M, N$ be two objects in $T \ast ST$ and $FM, FN \in \text{mod} \Gamma$ be their images under the functor $F$. If $FM$ and $FN$ are of projective dimension at most one and satisfy $\text{Ext}^1_\Gamma(FM, FN) = 0$ and $\text{Ext}^1_\Gamma(FN, FM) = 0$, then $\text{Ext}^i_\mathcal{C}(M, N) = 0$ and $\text{Ext}^i_\mathcal{C}(N, M) = 0$ for all $0 < i < d + 1$.

**Proof.** We only need to show that the result holds for $M, N$ indecomposable. Since $FM$ is of projective dimension at most 1, we have the following exact sequence
\[ 0 \rightarrow P_1^M \rightarrow P_0^M \rightarrow FM \rightarrow 0. \]
As the assumption, $\text{Ext}^1_\Gamma(FM, FN) = 0$. By the definition of $\text{Ext}^1$, we have the following commutative diagram in $\text{mod} \Gamma$.

\[ \begin{array}{cccc}
P_1^M & \rightarrow & P_0^M & \rightarrow \quad FM & \rightarrow \quad 0. \\
\downarrow \exists f & & \downarrow \exists g & & \\
FN & \rightarrow & M & \rightarrow & ST_1^M.
\end{array} \]

Using the equivalence $T \ast ST/(ST) \sim \rightarrow \text{mod} \Gamma$ and because $\text{Hom}_\mathcal{C}(T, ST) = 0$, we have the following in $\mathcal{C}$.

\[ \begin{array}{cccc}
T_1^M & \rightarrow & T_0^M & \rightarrow \quad M & \rightarrow \quad ST_1^M. \\
\downarrow \exists j & & \downarrow \exists g & & \\
N & \rightarrow & T_1^M & \rightarrow & ST_1^M.
\end{array} \]

That is, for any $f : T_1^M \rightarrow N$, there exists $g : T_0^M \rightarrow N$ such that $f$ factors through $g$.

First, we claim that $\text{Hom}_\mathcal{C}(M, SN) = 0$. 
In fact, consider the following two triangles
\[ T_1^M \to T_0^M \xrightarrow{p_0^M} M \to ST_1^M \]
and
\[ ST_1^N \to ST_0^N \to SN \xrightarrow{\omega} S^2T_1^N. \]
Let \( \alpha \) be any morphism from \( M \) to \( SN \). Since \( T \) is a \( d \)-cluster-tilting object in \( C \), the composition
\[ \omega \cdot \alpha \cdot p_0^M \in \text{Hom}_C(T_0^M, S^2T_1^N) = 0. \]
Therefore there exists a morphism from \( T_0^M \) to \( ST_0^N \) which makes the following diagram of triangles commutative.

\[ \begin{CD}
T_1^M @>>> T_0^M @>>> M @>>> ST_1^M \\
@VVV @VV{p_0^M}V @V{\alpha}VV \\
ST_1^N @>>> ST_0^N @>>> SN @>>> S^2T_1^N
\end{CD} \]

Thus we get
\[ \alpha \cdot p_0^M = 0 \]
for the reason that \( \text{Hom}_C(T_0^M, ST_0^N) = 0 \). So there exist \( \beta : ST_1^M \to SN \) such that \( \alpha \) factors through \( \beta \). As described above, we get \( \gamma : ST_0^M \to SN \) such that
\[ \beta = \gamma \cdot h. \]
That is, we have the following commutative diagram

\[ \begin{CD}
T_1^M @>>> T_0^M @>>> M @>>> ST_1^M \\
@VVV @VV{p_0^M}V @V{h}VV @VV{\beta}V \\
ST_1^N @>>> ST_0^N @>>> SN @>>> S^2T_1^N
\end{CD} \]

where
\[ \alpha = \beta \cdot u = \gamma \cdot h \cdot u = 0. \]
This implies \( \text{Hom}_C(M, SN) = 0 \). Dually one can prove that \( \text{Hom}_C(N, SM) = 0 \).

Now let \( 1 < i < d \). As before, suppose that \( \alpha : M \to S^iN \). Here we consider the following two triangles
\[ T_1^M \to T_0^M \xrightarrow{p_0^M} M \to ST_1^M \]
and
\[ S^iT_1^N \to S^iT_0^N \to S^iN \xrightarrow{\omega} S^{i+1}T_1^N. \]
Again because \( T \) is a \( d \)-cluster-tilting object and \( i + 1 \leq d \), we have
\[ \omega \cdot \alpha \cdot p_0^M \in \text{Hom}_C(T_0^M, S^{i+1}T_1^N) = 0. \]
Thus there exists morphism from \( T_0^M \) to \( S^iT_1^N \), which is zero for the same reason that \( T \) is a \( d \)-cluster-tilting object, makes the diagram commutative.

\[ \begin{CD}
T_1^M @>>> T_0^M @>>> M @>>> ST_1^M \\
@VVV @VV{p_0^M}V @V{\omega}VV @V{\alpha}VV \\
S^iT_1^N @>>> S^iT_0^N @>>> S^iN @>>> S^{i+1}T_1^N
\end{CD} \]

That is \( \alpha \cdot p_0^M = 0 \), so there exists \( \beta : ST_1^M \to S^iN \) such that \( \alpha \) factors through \( \beta \). Note that
\[ \omega \cdot \beta \in \text{Hom}_C(ST_1^M, S^{i+1}T_1^N), \]
which is zero since $T$ is a $d$-cluster-tilting object and $i > 1$. So there exists $\gamma : ST_1^M \rightarrow S^iT_0^N$ such that $\beta$ factors through $\gamma$. But
\[
\text{Hom}_C(ST_1^M, S^iT_0^N) = 0
\]
for the reason that $T$ is a $d$-cluster tilting object and $i > 1$. This is to say that $\gamma$ is zero, further more $\beta$ is zero and so is $\alpha$.

Thus $\text{Hom}_C(M, S^iN) = 0$.

When $i = d$, thanks to the $d + 1$-CY property, we have
\[
\text{Hom}_C(M, S^dN) = D \text{Hom}_C(N, SM) = 0.
\]
Dually one can prove $\text{Ext}^i_C(N, M) = 0$ for all $0 < i < d + 1$.

Now our main result is an easy corollary.

**Theorem.** For a positive integer $d > 1$, let $C_d$ be a $d$-cluster category and $T$ a $d$-cluster-tilting object, and let $\Gamma$ be the endomorphism algebra of $T$. Then a tilting $\Gamma$-module $L$ lifts to a $d$-cluster-tilting object in $C_d$.

**Proof.** The tilting $\Gamma$-module $L$ lifts to a rigid object in $C_d$ by the proposition above. Note that the number of indecomposable direct summands of $L$ (up to isomorphism) is the same as the number of simple $\Gamma$-modules or equivalently the number of indecomposable summands of $T$, which is $n$ by Theorem 2.2. Thus $L$ lifts to a $d$-cluster-tilting object by Theorem 2.2 again.

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