On the General Solution to the Bratu and Generalized Bratu Equations

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Abstract: This work shows that the Bratu equation belongs to a general class of Liénard-type equations for which the general solution may be exactly and explicitly computed within the framework of the generalized Sundman transformation. In this perspective the exact solution of the Bratu nonlinear two-point boundary value problem as well as of some well-known Bratu-type problems have been determined.

Keywords: Bratu Equation, Boundary Value Problem, Initial Value Problem, General Solution, Generalized Sundman Transformation

Introduction

The mathematical modeling of many processes in physics as in other applied disciplines of science is often achieved in terms of differential equations supplemented by initial or boundary conditions. An initial problem requires the specification of the solution at a point, while several points of specification are needed for the solution to a boundary value problem. In this regard, solving explicitly and exactly a boundary value problem becomes more mathematically complicated than a problem with initial conditions. This complication is accentuated when there is a nonlinear boundary value problem since, up to now there is no explicit and exact general method that can account for the individual behavior of each nonlinear process. The Bratu nonlinear two-point boundary value problem is one of those nonlinear problems whose explicit and exact solution for a wide variety of initial and boundary conditions remains very difficult to formulate. The Bratu problem is also one of the most investigated boundary value problems in mathematics (Ascher et al., 1994; Wazwaz, 2005). This problem derives its importance first from the combustion theory where it has been used for several applications (Wazwaz, 2016; 2005) and secondly, from the fact that its exact solution is well known (Ascher et al., 1995; Wazwaz, 2005) so that it has been widely applied to test the accuracy and efficiency of many approximate methods of different complexity like the Adomian decomposition approach (Wazwaz, 2005), the Legendre wavelet method (Venkatesh et al., 2012), the perturbation technique (Aksoy and Pakdemirli, 2010) and the virial theorem (Amore and Fernández, 2009). This solution exhibits also a bifurcation pattern, which only characterizes nonlinear differential equations. The one-dimensional Bratu boundary value problem may be written (He, 2006; Wazwaz, 2005):

\[ u''(x) + \lambda e^{u(x)} = 0 \]  

where:

\[ u(0) = u(1) = 0, \]  

and \( \lambda \) is a constant. The Bratu type initial value problems have also been examined by a number of authors (Venkatesh et al., 2012; Wazwaz, 2005). Such an importance motivates the reason to investigate the explicit and exact general solution to the Bratu Equation.

The usual way to solve a boundary or initial value problem consists of computing first the general solution to the differential equation and secondly of finding the arbitrary parameters by applying the boundary or initial conditions (Bronson and Costa, 2010). So, several methods are developed in mathematics for finding explicit and exact solutions to nonlinear differential equations. In this way the variables change like the point transformation, the contact transformation and the generalized Sundman transformation may be mentioned.
The generalized Sundman linearization theory has been recently the object of many applications so that now first integrals (Partha et al., 2010) and general periodic solutions (Akande et al., 2017; Adjăi et al., 2017) for well-known nonlinear differential equations have been computed explicitly. As such the generalized Sundman transformation recently developed by the authors of this work (Akande et al., 2017; Adjăi et al., 2017) has successfully been applied to determine the explicit and exact general periodic solutions to various types of Liénard nonlinear differential equations. In this regard a general class of quadratic Liénard type equations whose exact general solutions are trigonometric functions has been for the first time highlighted by the application of the generalized Sundman transformation under consideration (Akande et al., 2017; Adjăi et al., 2017). It has been, particularly, possible to show in this context that the well-known Painlevé-Gambier XVIII equation and its inverted version admit, for the first time, a trigonometric function as explicit and exact general periodic solution but with amplitude dependent frequency (Akande et al., 2017). This is also shown for a reduced Painlevé-Gambier XII equation under an appropriate parametric choice but with a shift factor (Adjăi et al., 2017). The same generalized Sundman transformation has been used to compute successfully the explicit and exact general periodic solutions of the famous cubic Duffing equation in terms of Jacobian elliptic functions, as expected. In spite of this progress in explicit and exact methods for solving nonlinear differential equations, it seems that a century after, the general solution to the Bratu differential Equation 1 from which the exact solution to the Bratu boundary value problem may be determined? Such a general solution is of high interest from theoretical point of view since it may allow one not only, to better understand the analytical properties of the Bratu equation under various types of initial and boundary conditions but also to detect the connection between the Bratu equation and other differential equations. From a practical point of view, it may allow the use of the Bratu equation adequately and satisfactorily as a simulation model for a large variety of engineering applications under various types of initial and boundary conditions and may serve to better test the accuracy and effectiveness of various approximation theories. In this study, it is assumed that such a general solution may be computed explicitly and exactly by the application of the generalized Sundman transformation. To demonstrate, the generalized Sundman transformation theory needed is first reviewed (section 2) and secondly the generalized Bratu equation of interest (section 3) as well as its explicit and exact general solution are established (section 4) such that the well known exact solution to the one-dimensional Bratu boundary value problem may be deduced (section 5). Finally the explicit and exact solutions to some Bratu type initial and boundary value problems examined by Wazwaz (2005) using Adomian decomposition method and also by Boyd (2011) are easily computed (section 6) so that a discussion of results (section 7) and a conclusion may be addressed.

Review of the Generalized Sundman linearization theory

In this section the generalized Sundman linearization theory recently introduced by Akande et al. (2017) is considered. The application of this linearizing transformation requires to consider the general class of quadratic Liénard type nonlinear differential equations:

\[ u''(x) + \left( \frac{g'(u)}{g(u)} - \gamma \phi(u) \right) u^2(x) + \frac{a^2 \exp(2\gamma \phi(u))}{g(u)} \int g(u') \, du' = 0 \]  

(3)

which may be reduced under the conditions:

\[ y(\tau) = F(x, u), \quad d\tau = G(x, u) \, dx, \quad G(x, u) \frac{\partial F(x, u)}{\partial u} \neq 0 \]  

(4)

with:

\[ F(x, u) = \int g(u') \, du', \quad G(x, u) = \exp(y \phi(u)) \]
The Equation 5 admits the solution:

\[ y(\tau) = A_0 \sin(\alpha \tau + \alpha) \]  

(6)

where prime denote ordinary differentiation of the dependent variable with respect to the argument, \( A_0, \alpha, a, \ell \) and \( \gamma \) are arbitrary parameters. The functions \( \phi(u) \) and \( g(u) \neq 0 \), are arbitrary functions of \( u \). So with that the generalized Bratu equation of interest may be established.

**Generalized Bratu Equation**

This section is devoted to carry out the generalized Bratu equation under question. To that end it is required that:

\[ \ell \frac{g'(u)}{g(u)} - \gamma \phi'(u) = \ell \neq 0 \]  

(7)

that is:

\[ g(u) = e^{\frac{\ell}{\gamma} \phi(u)} \]  

(8)

such that (3) becomes:

\[ u^*(x) + a^2 e^{\gamma(x)} \int e^{\gamma(x)} du = 0 \]  

(9)

The application of \( \phi(u) = u \), to (9) yields as equation:

\[ u^*(x) + \frac{d^2}{\gamma} e^{2\gamma(x)} = 0 \]  

(10)

The above shows the following theorem.

**Theorem 1**

Let \( \phi(u) = u \) and \( g(u) = e^{\gamma u} \). Then Equation 3 reduces to (10). The Equation 10 is the desired generalized Bratu equation.

**Theorem 2**

Let \( \gamma = \frac{1}{2} \) and \( 2a^2 = \lambda \). Then (Equation 10) is reducible to (Equation 1).
\[ u(x) = \frac{1}{\gamma} \ln \left[ \gamma A_1 \sin \left( 2tg^{-1} \left( K e^{x A_0} \right) \right) \right] \]  

(16)

where, \( \gamma > 0 \). The Equation 16 is the desired explicit and exact general solution to the generalized Bratu Equation 10 for all \( x \in \mathbb{R} \). Therefore the following theorem results from the above.

**Theorem 3**

Equation 10 has the solution (Equation 16) if and only if (Equation 5) has the solution (Equation 6).

The parametric choice \( \gamma = \frac{1}{2} \), yields the general solution under question to the Bratu Equation 1, that is:

\[ u(x) = 2\gamma \ln \left[ \frac{A_1}{2} \sin \left( 2tg^{-1} \left( \frac{K e^{x A_0}}{K} \right) \right) \right] \]  

(17)

for all \( x \in \mathbb{R} \). As a consequence of the above, the following theorem may be formulated.

**Theorem 4**

Let \( \gamma = \frac{1}{2} \). Then Equation 16 becomes (Equation 17).

In this context the integration constants \( K \) and \( A_0 \) may be determined for various initial and boundary conditions. In other words, the behavior of \( u(x) \) depends on these conditions. The objective is now to show that the general solution (Equation 17) may yield the well-known exact solution to the one-dimensional Bratu boundary value problem under the conditions that \( u(0) = u(1) = 0 \).

**Exact Solution to the Bratu Boundary Value Problem**

This section is devoted to determine the exact solution of the one-dimensional Bratu boundary value problem, in other words to compute the two constants of integration \( A_0 \) and \( K \) under the conditions that \( u(0) = u(1) = 0 \). So the application of \( u(0) = 0 \) leads to:

\[ \ln \left[ A_1 \gamma \sin \left( 2tg^{-1} \left( K e^{x A_0} \right) \right) \right] = 0 \]

that is:

\[ \sin \left( 2tg^{-1} K \right) = \frac{1}{A_1 \gamma} \]  

(18)

Knowing that:

\[ \sin \left( 2tg^{-1} K \right) = \frac{2K}{1+K^2} \]  

(19)

the Equation 18 becomes:

\[ \frac{2K}{1+K^2} = \frac{1}{A_1 \gamma} \]  

(20)

On the other hand the application of the condition \( u(1) = 0 \), gives:

\[ \ln \left[ A_1 \gamma \sin \left( 2tg^{-1} \left( K e^{x A_0} \right) \right) \right] = 0 \]

that is:

\[ \sin \left[ 2tg^{-1} \left( K e^{x A_0} \right) \right] = \frac{1}{A_1 \gamma} \]  

(21)

which may be written in the form:

\[ \frac{2Ke^{x A_0}}{1+Ke^{2x A_0}} = \frac{1}{A_1 \gamma} \]  

(22)

Equating (Equation 20) and (Equation 22) yields:

\[ \frac{2Ke^{x A_0}}{1+Ke^{2x A_0}} = \frac{2K}{1+K^2} \]  

(23)

which leads after a little mathematical treatment to:

\[ K e^{x A_0} = \frac{1}{2} \]  

(24)

In this context the parameter \( A_0 \) may be computed, using (Equation 20) as:

\[ A_0 = \frac{1}{\gamma} \cosh \left( \frac{a_1 \gamma A_0}{2} \right) \]  

(25)

such that the general solution (Equation 16), that is, the exact solution to the generalized Bratu Equation 10 under the boundary conditions \( u(0) = u(1) = 0 \), may take the expression:

\[ u(x) = \frac{1}{\gamma} \ln \left[ 2tg^{-1} \left( 2Ke^{x A_0} \right) - \frac{1}{2} \right] \]  

(26)

or:

\[ u(x) = -\frac{1}{\gamma} \ln \left[ \frac{1 + e^{2x A_0} \left( \frac{1}{2} \right)}{2e^{x A_0} \left( \frac{1}{2} \right)} \right] \]  

(27)

Using the identity:
\[ \cosh q = \frac{e^q + e^{-q}}{2} \]  
(28)

the Equation 27 may be expressed as:

\[ u(x) = -\frac{1}{\gamma} \left\{ \frac{\cosh [\alpha \gamma A \left( x - \frac{1}{2} \right)]}{\cosh [\alpha \gamma A / 4]} \right\} \]  
(29)

The expression (Equation 29) is the desired explicit and exact solution to the generalized Bratu Equation 10 under the boundary conditions in consideration. In this situation the above results show the following theorem.

**Theorem 5**

Consider (Equation 2). Then (Equation 16) reduces to (Equation 29).

So the exact solution of the one-dimensional Bratu boundary value problem may, for the value \( \gamma = \frac{1}{2} \), take the expression:

\[ u(x) = -2\ell n \left\{ \frac{\cosh \left[ \frac{a A}{2} \left( x - \frac{1}{2} \right) \right]}{\cosh \left( \frac{a A}{4} \right)} \right\} \]  
(30)

so that for the parametric choice:

\[ \theta = a A \]

that is for the transcendental equation:

\[ \theta = 2 \alpha \cosh \left( \frac{\theta}{4} \right) \]  
(31)

the exact solution to the Bratu boundary value problem may be definitively written as:

\[ u(x) = -2\ell n \left\{ \frac{\cosh \left( \frac{\theta}{2} \left( x - \frac{1}{2} \right) \right)}{\cosh \left( \frac{\theta}{4} \right)} \right\} \]  
(32)

Therefore as a special case of the theorem 5, the following result may be expressed.

**Theorem 6**

Let \( \gamma = \frac{1}{2} \) and \( \theta = 2 \alpha \cosh \left( \frac{\theta}{4} \right) \). Then (Equation 29) is reducible to (Equation 32).

After showing that the exact solution of the one-dimensional Bratu boundary value problem may be calculated from the general solution to the Bratu differential equation, the purpose, now, is to show that the current general theory may also be used to compute the explicit and exact solutions to the initial and boundary value problems investigated by Wazwaz (2005) on the basis of Adomian decomposition method (Wazwaz, 2005) and Boyd (2011).

**Bratu Type Initial and Boundary Value Problems**

In the investigation of the Bratu boundary value problem by Adomian decomposition method, Wazwaz (2005) considered a number of Bratu type initial and boundary value problems. The results obtained by Wazwaz (2005) are later used by several authors (Venkatesh et al., 2012; Aksoy and Pakdemirli, 2010) to test the accuracy and efficiency of some approximate methods for solving differential equations. In this section, the explicit and exact solutions of the Bratu type boundary value problem investigated by Boyd (2011) and the Bratu type initial value problem considered by Wazwaz (2005) are determined using the general solution established in this study.

**Bratu-Type Problem 1**

The Bratu-type problem considered by Boyd (2011) may be written in the form:

\[ u''(x) + \lambda e^{u(x)} = 0, \quad -1 < x < 1 \]
\[ u(-1) = u(1) = 0 \]  
(33)

Although, here, the differential equation is that of Bratu, the boundary conditions are different from those usually used for the Bratu nonlinear two-point boundary value problem. In this regard under the condition \( u(-1) = 0 \), the general solution (Equation 16) yields:

\[ \ell n \left\{ A_{\gamma} \sin \left[ 2tg^{-1} \left( Ke^{-\gamma x} \right) \right] \right\} = 0 \]

that is:

\[ \sin \left[ 2tg^{-1} \left( Ke^{-\gamma x} \right) \right] = \frac{1}{A_{\gamma}} \]  
(34)

which may be reduced to:

\[ \frac{2Ke^{-\gamma x}}{1 + K e^{-2\gamma x}} = \frac{1}{A_{\gamma}} \]  
(35)

On the other hand, the application of \( u(1) = 0 \), turns (Equation 16) into:
\[ \ln \left( A_0 \gamma \sin \left( 2g^{-1} \left( Ke^{\gamma x} \right) \right) \right) = 0 \]

which leads to:

\[ \sin \left( 2g^{-1} \left( Ke^{\gamma x} \right) \right) = \frac{1}{A_0} \gamma \] (36)

such that:

\[ \frac{2Ke^{\gamma x}}{1 + Ke^{\gamma x}} = \frac{1}{A_0} \gamma \] (37)

The comparison of (Equation 35) with (Equation 37) allows, after a few mathematical treatments, to obtain:

\[ K = 1 \] (38)

so that the second integration constant \( A_0 \) may take the form:

\[ A_0 = \frac{1}{2} \left( e^{-\eta A_0} + e^{\eta A_0} \right) \]

which may also be written:

\[ A_0 = \frac{1}{2} \cosh \left( \eta A_0 \right) \] (39)

In this context the general solution (Equation 16) becomes:

\[ u(x) = \frac{1}{\gamma} \ln \left( \gamma A_0 \sin \left( 2g^{-1} \left( e^{\eta x} \right) \right) \right) \] (40)

where, \( A_0 \) is given by (Equation 39). Knowing that (Equation 1) is obtained for \( \gamma = \frac{1}{2} \), the exact solution to the boundary value problem (Equation 33) becomes:

\[ u(x) = 2 \ln \left( \frac{2A_0}{A_0} \cosh \left( \frac{aA_0}{2} x \right) \right) \] (41)

where:

\[ A_0 = 2 \cosh \left( \frac{aA_0}{2} \right) \] (42)

Using the identity:

\[ \sin \left( 2g^{-1} \left( \frac{aA_0}{2} x \right) \right) = \frac{1}{\cosh \left( \frac{aA_0}{2} x \right)} \] (43)

the solution (Equation 41) takes the form:

\[ u(x) = 2 \ln \left( \frac{2A_0}{A_0} \cosh \left( \frac{aA_0}{2} x \right) \right) \]

that is:

\[ u(x) = -2 \ln \left( \frac{\cosh \left( \frac{aA_0}{2} x \right)}{\cosh \left( \frac{aA_0}{2} \right)} \right) \] (44)

where, \( A_0 \) is given by (Equation 42). Now, a few algebraic manipulations is needed to compare the exact solution (Equation 44) to the problem (Equation 33) with the solution given by Boyd (2011). As \( 2a^2 = \lambda \), that is \( a = \pm \frac{\sqrt{\lambda}}{2} \), it suffices to set \( \frac{A_0}{2} = z \), which is equivalent to \( \cosh \left( \frac{A_0}{2} \sqrt{\lambda} \right) = z \), that is to say \( \cosh \left( \frac{\sqrt{\lambda}}{2} \right) = z \), to write (Equation 44) in the form:

\[ u(x) = -2 \ln \left( \frac{\cosh \left( \frac{z}{2} \sqrt{\lambda} \right)}{\cosh \left( \frac{\sqrt{\lambda}}{2} \right)} \right) \] (45)

or definitively under the expression:

\[ u(x) = -2 \ln \left( \frac{z^2 \text{ sech} \left( \frac{z}{2} \sqrt{\lambda} \right)}{\cosh \left( \frac{\sqrt{\lambda}}{2} \right)} \right) \] (46)

which is nothing but the form used by Boyd (2011) to express the solution to the boundary value problem (Equation 33).

**Bratu-Type Problem 2**

The third Bratu-type problem solved by Wazwaz (2005) is an initial value problem formulated as:

\[ u''(x) - 2e^{u(x)} = 0, \ 0 < x < 1 \]
\[ u(0) = u'(0) = 0 \] (47)

It is convenient before solving (Equation 47) to consider the general solution (Equation 16) under the general initial conditions \( u(0) = u_0 \) and \( u'(0) = v_0 \). In this perspective, the application of \( u(0) = u_0 \), gives:

\[ e^{u_0} = \gamma A_0 \sin \left( 2g^{-1}K \right) \] (48)
and the application of $u(0) = v_0$, leads to:

$$v_0 = \frac{2a_1 A K \cos(2g_1 K)}{1 + K^2} \sin(2g_1 K) \quad (49)$$

so that (49) may be rewritten as:

$$v_0 = \frac{2\gamma a_1 K \cos(2g_1 K)}{1 + K^2} e^{\gamma_0} \quad (50)$$

Therefore, for the initial conditions $u_0 = 0$ and $v_0 = 0$, one may find $K = 1$ and $A_0 = 2$, such that the general solution (Equation 16) under the above conditions reduces to:

$$u(x) = \frac{1}{\gamma} \ln\left[2\gamma \sin\left[2g_1 (e^{\gamma_0 x})\right]\right] \quad (51)$$

As the Bratu type initial value problem (Equation 47) is obtained from (10) for $\gamma = \frac{1}{2}$ and $a_0 = -1$, the exact solution to (Equation 47) may take the expression:

$$u(x) = 2i\ln\left[\sin\left[2g_1 (e^x)\right]\right] \quad (52)$$

where, $i$ is the purely imaginary number.

Using the identity:

$$\sin\left[2g_1 (e^x)\right] = \frac{1}{\cosh(x)} \quad (53)$$

the solution (52) may be written as:

$$u(x) = -2i\ln(\cosh(x)) \quad (54)$$

which takes the definitive expression:

$$u(x) = -2i\ln(\cos(x)) \quad (55)$$

This result (Equation 55) is identical to that obtained by Wazwaz using the Adomian decomposition method (Wazwaz, 2005).

**Conclusion**

While the exact solution of the one-dimensional Bratu boundary value problem is well known in the literature, the explicit and exact general solution of the Bratu nonlinear differential equation is unfortunately, after a century, an unresolved question. This constitutes a fundamental drawback in the understanding of analytical properties of Bratu equation. Fortunately this shortcoming has been, for the first time, overcome in this study, using the generalized Sundman transformation, which closely relates the Bratu differential equation to the linear harmonic oscillator differential equation. In so doing, it was possible to deduce from the computed general solution the well-known exact solution of the Bratu boundary value problem and to show that the Bratu equation is a special case of a more general equation. Therefore, one could compute the explicit and exact solutions of a large variety of Bratu type problems with a relative simplicity.

**Author’s Contributions**

All authors contributed to the development and formulation of this work. All authors read and approved the final manuscript for publication.
Ethics

The authors declare that there exists no competing interests.

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