HOMOTOPY DIMENSION OF PLANAR CONTINUA

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Abstract. We will show that the fundamental group determines the homotopy dimension for one-dimensional and planar Peano continua. This answers a question asked by Cannon and Conner and illustrates the rigidity of the fundamental group for locally planar continua.

1. Introduction

The homotopy dimension of a space $X$ is the smallest covering dimension of any space homotopy equivalent to $X$. Every contractible space has homotopy dimension zero and every planar continuum has homotopy dimension at most two. The space obtained by filling one removed square of the Sierpinski carpet gives an example of a planar continuum with homotopy dimension exactly two \[5\]. In fact there exist uncountably many non-homotopy equivalent planar continua with homotopy dimension exactly two \[12\].

Cannon and Conner have shown that the fundamental group of a planar Peano continuum always embeds into the fundamental group of a one-dimensional Peano continuum \[4\]. Thus there exists a planar continuum with homotopy dimension two whose fundamental group embeds into the fundamental group of a one-dimensional space. In contrast, we will show that the homotopy dimension of a planar Peano continuum is determined by its fundamental group.

**Theorem 1.1.** A planar Peano continuum $X$ has homotopy dimension at most one if and only if the fundamental group of $X$ is isomorphic to the fundamental group of a one-dimensional Peano continuum.

In general the fundamental group is too weak of an invariant to determine the homotopy dimension of a space. For example, the Warsaw circle is a simply connected continuum with homotopy dimension one, since it is not contractible. There are examples of simply connected planar continua with homotopy dimension two, as well as planar continua which are not locally homotopically one-dimensional at any point \[11\]. Thus in the absence of local connectivity the homotopy dimension is not determined by the fundamental group even in the case of planar sets.

Historically, homotopy dimension has been studied in the context of CW-complexes, see \[13, 14\]. However, when considering continua which are not locally simply connected the methods developed for CW-complexes are insufficient. We will use the following theorem which will allow us to translate topological properties of a space into algebraic properties of its fundamental group.

**Theorem 3.9.** Every homomorphism between the fundamental groups of one-dimensional or planar Peano continua is induced by a continuous map up to composition with a change of basepoint isomorphism.

The following is known for some cases but we will give a short proof here for all cases.

**Proposition 1.3.** If the homomorphism of Theorem 3.9 has uncountable image, then the path inducing the change of basepoint isomorphism is unique up to homotopy relative to its endpoints.

While the fundamental group determines the topology of the set of point at which a planar or one-dimensional Peano continuum is not locally simply connected \[7, 6\]; Theorem 3.11 shows
that there exist planar Peano continua with non-isomorphic fundamental groups and the sets at which they are not locally simply connected are homeomorphic.

2. Codiscrete subsets of \( S^2 \)

Cannon and Conner showed that every planar Peano continuum is homotopy equivalent to a nice subset of the 2-sphere which can be easier to manipulate than a general planar continuum.

A subset of the 2-sphere, \( S^2 \), is codiscrete if it is the complement of a discrete set.

**Theorem 2.1** ([4, Theorem 1.2]). Every Peano continuum \( M \) in the 2-sphere \( S^2 \) is homotopy equivalent to a codiscrete subset \( X \) of \( S^2 \). Conversely, every codiscrete subset \( X \) of \( S^2 \) is homotopy equivalent to a Peano continuum \( M \) in \( S^2 \).

Theorem 1.1 will be an immediate corollary of Theorem 2.1 and Theorem 3.16.

**Definition 2.2.** If \( X \) is a codiscrete subset of \( S^2 \), let \( D(X) \) be the discrete complement of \( X \) in \( S^2 \) and \( B(X) \) be the set of accumulation points of \( D(X) \). It is immediate that \( B(X) \) is the set of points at which \( X \) is not semilocally simply connected. For any space \( Y \), we will use \( B(Y) \) to denote the set of points at which \( B(Y) \) is not semilocally simply connected.

Cannon and Conner were able to prove the following topological characterization of codiscrete subsets of \( S^2 \) with homotopy dimension at most one [4].

**Theorem 2.3** (Cannon & Conner). Suppose that \( X \) is a codiscrete subset of the two-sphere \( S^2 \). Then \( X \) is homotopically at most one-dimensional if and only if the following two conditions are satisfied.

(i) Every component of \( S^2 \setminus B(X) \) contains a point of \( D(X) \).

(ii) If \( D \) is any closed disk in the two-sphere \( S^2 \), then the components of \( D \setminus B(X) \) that do not contain any point of \( D(X) \) form a null sequence.

The idea for the proof of Theorem 2.3 is to use the holes arising from \( D(X) \) to push the two-dimensional components of \( S^2 \setminus B(X) \) onto some nice one-dimensional core. The problem is that puncturing the Warsaw disc (the closure of the bounded component of the Warsaw circle in the plane) at any finite set of points is insufficient to be able to retract it onto any one-dimensional subset. Thus condition (ii) must be used to guarantee sufficient punctures to build nice retracts.

To prove Theorem 1.1, we will use Theorem 3.9 to find a continuous endomorphism of the corresponding the codiscrete subsets of the two-sphere which induces an isomorphism of the fundamental groups. The key step in the proof that is new here is the ability to recognize conditions (i) and (ii) from the fundamental group. Section 3 is dedicated to showing that if a continuous endomorphism of a codiscrete set induces an isomorphism on the fundamental group and factors through a map into a one-dimensional space then the conditions of Theorem 2.3 must be satisfied.

3. The following are standard definitions that we present here to fix notation.

**Definition 3.1.** Let \( B^X_r(x) = \{ y \in X \mid d(x,y) < r \} \) and \( S^X_r(x) = \{ y \in X \mid d(x,y) = r \} \). When no confusion will arise from suppressing the superscript in our notations for balls and spheres, we will do so. If \( X \) is a planar set then \( B^X_r(x) = B^{R^2}_r(x) \cap X \) and \( S^X_r(x) = S^{R^2}_r(x) \cap X \). If \( U \subset X \), we will use \( cl_X(U) \) to denote the topological closure of \( U \) in \( X \). When \( X \) is understood, the closure will be denoted simply by \( cl(U) \).
Lemma 3.4. It is locally simply connected.

A space is locally simply connected at \( x \) if every neighborhood of \( x \) contains a simply connected neighborhood of \( x \). A space \( X \) is semilocally simply connected at \( x \) if there exists a neighborhood \( U \) of \( x \) such that the inclusion induced homomorphism \( \iota_* : \pi_1(U, x) \to \pi_1(X, x) \) is trivial. A space is locally simply connected (or semilocally simply connected) if it is at each its points.

Lemma 3.2. Suppose that \( X \) is a topological space and \( x \in X \) has a planar or one-dimensional neighborhood. Then every neighborhood \( V \) of \( x \) contains a neighborhood \( U \) of \( x \) such that no essential loop in \( U \) can be freely homotoped out of \( V \).

Proof. Let \( V \) be a neighborhood of \( x \) in \( X \). Without loss of generality, we will assume that \( V \) is planar or one-dimensional. If \( V \) is planar, then \( x \) has a neighborhood \( U \) with closure contained in \( V \) that is either simply connected or \( U = B^2_r(x) \) and the components of \( S^2_\alpha(x) \) are intervals (possible degenerate). If \( V \) is one-dimensional, then \( x \) has a neighborhood \( U \) with closure contained in \( V \) that is either simply connected or \( U \) has 0-dimensional boundary. Thus in either case; if \( U \) is not simply connected, then the components of \( \partial U \) are intervals, possible degenerate.

If \( U \) is simply connected we are done. So we will assume that the components of \( \partial U \) are intervals. Suppose \( \alpha \) is a loop contained in \( U \) such that \( \alpha \) can be homotoped to a loop disjoint from \( V \).

Let \( A \) be an annulus in the plane with outer boundary component \( J_1 \) and inner boundary component \( J_2 \). Then there exists a map \( h : A \to V \) which takes \( J_1 \) to \( \alpha \) and \( J_2 \) to a loop in \( X \setminus V \). Then \( h^{-1}(\partial U) \) is a closed subset contained in the int(\( A \)) which separates \( J_1 \) from \( J_2 \) in \( A \). By the Phragmén-Brouwer properties (see [15, p. 47]), some component \( C \) of \( h^{-1}(\partial U) \) separates \( J_1 \) from \( J_2 \) in \( A \). Since the components of \( \partial U \) are intervals, \( h|_C \) maps into an arc in \( V \). By the Tietze Extension Theorem, we may alter \( h \) to a map of a disk into \( V \) with boundary mapping to \( \alpha \). Therefore \( \alpha \) is null-homotopic.

\[ \square \]

This also proves the following well known fact.

Corollary 3.3. A planar or one-dimensional set is semilocally simply connected if and only if it is locally simply connected.

Lemma 3.4. Let \( D \subset S^2 \). A simple closed curve in \( S^2 \setminus D \) is homotopically essential if and only if \( D \) intersects both components of the complement of the simple closed curve.

Lemma 3.5. Let \( A \) be a closed connected subset of \( S^2 \). Then each component of \( S^2 \setminus A \) is simply connected.

Sketch of proof. Let \( U \) be a component of \( S^2 \setminus A \) and suppose that \( \alpha : S^1 \to U \) is a loop in \( U \). Since \( A \) and \( \text{Im}(\alpha) \) are disjoint compact sets, \( \alpha \) is homotopic in \( U \) by a straight line homotopy to a polygonal path \( \beta \). Suppose that \( \beta' \) is a simply close subpath of \( \beta \). Since \( A \) is connected exactly one of the components of \( S^2 \setminus \text{Im}(\beta') \) can intersect \( A \) and the component not intersecting \( A \) must be contained in \( U \) which implies that \( \beta' \) is nullhomotopic in \( U \).

Since \( \beta \) is a polygonal path it can be reduced to the constant path by a finite process of replacing simply closed subpaths by constant paths. By the previous argument, this process preserves the homotopy class and thus \( \beta \) is nullhomotopic in \( U \).

\[ \square \]

When we say \( A \) separates \( B \) in \( C \), we mean that \( B \subset C \setminus A \) and \( B \) is not contained in a single connected component of \( C \setminus A \).
Lemma 3.6. Let $X$ be a codiscrete subset of $\mathbb{S}^2$ and $A \subset X \setminus B(X)$ such that $\text{cl}_{\mathbb{S}^2}(A) \cap D(X) = \emptyset$. Then $\text{cl}_{\mathbb{S}^2}(A) = \text{cl}_X(A)$ and if $A$ separates $B(X)$ in $X$ then $A$ separates $D(X)$ in $\mathbb{S}^2$.

Proof. Since $\text{cl}_{\mathbb{S}^2}(A) \setminus \text{cl}_X(A) \subset D(X)$ and $\text{cl}_{\mathbb{S}^2}(A) \cap D(X) = \emptyset$, we have $\text{cl}_{\mathbb{S}^2}(A) = \text{cl}_X(A)$.

Since $A$ separates $B(X)$ in $X$ there exists a continuous function $h : X \setminus A \to \{0, 1\}$ which is non-constant on $B(X)$. For each $d \in D(X)$ there exists $\epsilon_d > 0$ such that $\{B_{\epsilon_d}^2(d) \mid d \in D(X)\}$ is a cover of $D(X)$ by disjoint open balls each of which intersects $D(X)$ at a unique point and is disjoint from $A$. Thus $h$ is constant on $B_{\epsilon_d}^2(d) \setminus \{d\}$. Then we can define $\overline{h} : S^2 \setminus A \to \{0, 1\}$ by $\overline{h}(x) = h(x)$ on $X \setminus A$ and making $\overline{h}$ constant on each ball $B_{\epsilon_d}^2(d)$. It is immediate that $\overline{h}$ is continuous. Since $h$ was non-constant on $B(X)$ so is $\overline{h}$. Then $\overline{h}^{-1}(0)$ and $\overline{h}^{-1}(1)$ are disjoint open sets and both intersect $B(X)$ which implies they both intersect $D(X)$. Hence $\overline{h}$ is non-constant on $D(X)$.

Lemma 3.7. Let $X$ be a codiscrete subset of $\mathbb{S}^2$ and $U$ an open subset of $\mathbb{S}^2$ whose closure is a proper subset of $\mathbb{S}^2$. Fix $d \in U$ and $c \in S^2 \setminus \text{cl}_{\mathbb{S}^2}(U)$. For every $\epsilon > 0$ there exists a simply closed curve in $U \cap N_\epsilon(\partial U) \cap X$ which is essential in $S^2 \setminus \{c, d\}$.

Proof. Fix $d \in U$ and $c \in S^2 \setminus \text{cl}_{\mathbb{S}^2}(U)$ and $\epsilon > 0$. Let $A$ be a component of $\partial U$ which separates $c, d$. Let $V$ be the component of $S^2 \setminus A$ which contains $d$. After possible passing to a subset of $A$ which still separates $c$ and $d$, we may assume that $A = \partial V$. By the Riemann Mapping Theorem there exists a homeomorphism $f : V \to \mathbb{R}^2$. We may assume that $f(d) = (0, 0)$ and $2\epsilon < d(A, d)$.

Then $C = V \setminus N_\epsilon(A)$ is a compact set which contains $d$, hence $f(C)$ is also compact. We can find an $M > 1$ such that $f(C) \subset B_M((0, 0)) = B$.

Then $f^{-1}(\partial B)$ is a simple closed curve which separates $d, c$ and $f^{-1}(\partial B) \subset N_\epsilon(A) \subset N_\epsilon(\partial U)$. Notice that $f^{-1}(\partial B)$ might intersect $D(X)$. However after perturbing $f^{-1}(\partial B)$ near its intersections with $D(X)$, we may also assume that it is contained in both $X$ and $N_\epsilon(A)$. (However, $f^{-1}(\partial B)$ can not necessarily be homotoped off of $B(X)$.)

We will require the following technical lemma concerning separating components of point preimages.

Lemma 3.8. Suppose that $f : X \to Y$ is a continuous map where $X$ is a codiscrete subset of $\mathbb{S}^2$. Fix $y \in Y$ such that $f^{-1}(y)$ separates $D(X)$ in $\mathbb{S}^2$. Then either $\text{cl}_{\mathbb{S}^2}(A) \cap D(X) \neq \emptyset$ for every component $A$ of $f^{-1}(y)$ which separates $D(X)$ in $X$ and is disjoint from $B(X)$ or $f_*$ the induced map on fundamental groups in not injective.

Proof. Fix $y \in Y$ such that $f^{-1}(y)$ separates $D(X)$ in $\mathbb{S}^2$. Suppose that $A$ is a component of $f^{-1}(y)$ which separates $D(X)$ in $\mathbb{S}^2$ and is disjoint from $B(X)$ but $\text{cl}_{\mathbb{S}^2}(A) \cap D(X) = \emptyset$.

Notice that $\text{cl}_{\mathbb{S}^2}(A) \cap D(X) = \emptyset$ implies that $\text{cl}_{\mathbb{S}^2}(A) = \text{cl}_X(A)$. Since $A$ is a component of $f^{-1}(y)$, it is a maximal connected subset of the closed set $f^{-1}(y)$ which implies that $\text{cl}_X(A) \subset A$. Hence $A$ is a connected closed subset of $\mathbb{S}^2$.

Since $A \cap B(X) = A \cap D(X) = \emptyset$, there exists $\epsilon > 0$ such that $N_\epsilon(A) \cap D(X) = \emptyset$. Let $U$ be a component of $\mathbb{S}^2 \setminus A$ such that $U \cap D(X) \neq \emptyset$ and $(\mathbb{S}^2 \setminus \text{cl}_{\mathbb{S}^2}(U)) \cap D(X) \neq \emptyset$. Let $c \in U \cap D(X)$ and $d \in (\mathbb{S}^2 \setminus \text{cl}_{\mathbb{S}^2}(U)) \cap D(X)$. This is possible since $A$ separates $D(X)$ in $\mathbb{S}^2$.

By Lemma 3.7 there exists a simple closed curve $\alpha : S^1 \to N_{\epsilon/2}(A)$ which is homotopically essential in $\mathbb{S}^2 \setminus \{c, d\}$. Hence $\alpha$ is also homotopically essential in $X$.

Let $D$ be the disc in $\mathbb{S}^2$ with boundary parameterized by $\alpha$ and containing $A$. By construction, the components of $D \setminus A$ are either contained in $X$ or have boundary contained in $A$. Hence map $f \circ \alpha : S^1 \to Y$ extends to a map $h : D \to Y$ by sending each of the components bounded
by $A$ to $y$ and letting $h$ agree with $f$ on the components contained in $X$. Thus $f \circ \alpha$ extends to a map of the disc and $f_\alpha$ has a nontrivial kernel. 

We will use the following theorem that arbitrary homomorphisms of the fundamental groups of planar and one-dimensional Peano continua are induced by continuous maps.

**Theorem 3.9** ([8], [7], [12]). Let $X$ and $Y$ be one-dimensional or planar Peano continua and $\phi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ a homomorphism of their fundamental groups. Then there exists a continuous function $f : X \to Y$ and a path $T : (I, 0, 1) \to (Y, y_0, y)$ such that $f_\alpha = \tilde{T} \circ \phi$ where $\tilde{T}$ is the change of base-point isomorphism induced by $T$.

When $X$ and $Y$ are one-dimensional this was proved by Eda. When $X$ is one-dimensional and $Y$ is planar this was done by Conner and Kent. When $X$ is planar and $Y$ is one-dimensional or planar this was proved by Kent.

The following lemma is an easy exercise which will be left to the reader.

**Lemma 3.10.** Suppose that $\alpha : S^1 \to A \subset S^2$ is a continuous function and $f : A \to S^2 \setminus \{c, d\}$ is a continuous map such that $d(\alpha(t), f(\alpha(t))) < d(\alpha(t), \{c, d\})$ where the metric on the 2-sphere is the standard CAT(1) metric. If $f \circ \alpha$ is nullhomotopic, then $\alpha$ is nullhomotopic in $S^2 \setminus \{c, d\}$.

**Lemma 3.11.** Let $X$ be a one-dimensional continuum or a planar set. Suppose that $\alpha_n$ and $\beta_n$ are two null sequences of essential loops based at $x_0$ and $x_1$ respectively. If $\alpha_n$ is freely homotopic to $\beta_n$ for all $n$. Then $x_0 = x_1$ and if there exists a loop $\gamma$ such that $\gamma \ast \alpha_n \ast \overline{\gamma}$ is homotopic rel endpoints to $\beta_n$ for all $n$, then $\gamma$ is a nullhomotopic loop.

**Proof.** Let $\alpha_n, \beta_n$ be a null sequence of essential loops based at $x_0, x_1 \in X$. Suppose that there exists $n$ such that $\alpha_n$ is homotopic rel endpoints to $\beta_n$. Then $x_0 = x_1$, otherwise by Lemma 3.2. Suppose that there exists $n$ such that $\gamma \ast \alpha_n \ast \overline{\gamma}$ is homotopic rel endpoints to $\beta_n$ for all $n$.

By way of contradiction, suppose that $\gamma$ is an essential loop. If $X$ is a one-dimensional continuum, then we may assume that $X$ is embedded in $\mathbb{R}^3$. Let $X_\varepsilon = N_\varepsilon(X)$ be the $\varepsilon$-neighborhood of $X$ in the plane if $X$ is planar or in $\mathbb{R}^3$ if $X$ is one-dimensional.

Then there exists $\varepsilon > 0$ such that $i_\varepsilon([\gamma])$ is homotopically essential where $i : X \to X_\varepsilon$ is the inclusion map. (This is well known for one-dimensional spaces and a proof can be found in [9].) Let $K$ be the kernel of $i_\varepsilon : \pi_1(X, x_0) \to \pi_1(X_\varepsilon, x_0)$.

Then $K$ is an open normal subgroup of $\pi_1(X, x_0)$ endowed with standard quotient topology on the loop space endowed with the compact-open topology. Thus there exists a covering map $\rho : \tilde{X} \to X$ such that $\text{Im}(\rho_\bullet) = K$. (See [1] or [10].)

Since $\gamma \notin K$, $\gamma$ lifts to a path $\tilde{\gamma} : [0, 1] \to \tilde{X}$ between two distinct lifts of $x_0$. Let $\tilde{\alpha}_n$ be a lift of $\alpha_n$ based at $\tilde{\gamma}(0)$ and $\tilde{\beta}_n$ be a lift of $\beta_n$ based at $\tilde{\gamma}(1)$. For sufficiently large $n$, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ are loops. Since $\rho_\bullet$ is injective, $\tilde{\gamma} \ast \tilde{\alpha}_n \ast \overline{\tilde{\gamma}}$ is homotopic to $\tilde{\beta}_n$ for sufficiently large $n$. Thus $\tilde{\alpha}_n$ is freely homotopic to $\tilde{\beta}_n$ for sufficiently large $n$. Since $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$, Lemma 3.2 implies that $\tilde{\alpha}_n$ is nullhomotopic for sufficiently large $n$ which is a contradiction. Thus $\gamma$ must be a nullhomotopic loop.

**Proposition 3.12.** Let $X$ be a one-dimensional continuum or planar set and fix $x_0 \in B(X)$. If $\phi : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is a nontrivial inner automorphism, then $\phi$ is not induced by a continuous function.

**Proof.** Suppose that there exists a continuous function $f : X \to X$ such that $f_\alpha : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is a nontrivial inner automorphism. Then there exists a loop $\gamma$ at $x_0$ such that $f_\alpha([s]) = \tilde{T} \circ \phi([s]) = \tilde{T} \circ \phi([s])$ for all loops $s$ based at $x_0$. Since $x_0 \in B(X)$, there exists a null sequence of essential loops $\alpha_n$ at $x_0$. Then $\alpha_n$ and $f \circ \alpha_n$ are null sequences of loops which
are conjugate by \( \gamma \); hence Lemma 3.11 implies that \( \gamma \) is nullhomotopic. Thus \( f_* \) is the identity homomorphism.

The following is well known and there are several proofs in the literature, see for example [2, 4, 7].

**Lemma 3.13.** Suppose that \( X \) is a one-dimensional continuum or planar set and \( x_0 \in B(X) \). If \( f : X \to X \) is a continuous map fixing \( x_0 \) and inducing an inner automorphism of \( \pi_1(X, x_0) \) then \( f \) fixes \( B(X) \).

**Proof.** By Proposition 3.12 \( f_* \) is the identity homomorphism. Suppose that \( x \in B(X) \). Then there exists a null sequence of homotopically essential loops \( \alpha_n \) based at \( x \). Since \( f_* \) is the identity on \( \pi_1(X, x_0) \), \( \alpha_n \) is freely homotopic to \( f \circ \alpha_n \). Then Lemma 3.11 implies that \( x = 1 \in \pi_1(X, x_0) \).

**Proof of Main Theorem**

**Lemma 3.14.** If \( X,Y \) are planar or one-dimensional Peano continua with isomorphic fundamental groups then there exists continuous maps \( f : X \to Y \) and \( g : Y \to X \) such that \( g \circ f(x_0) = x_0 \) for some \( x_0 \in X \) and \( (g \circ f)_* = id_{\pi_1(X, x_0)} \).

**Proof.** It is a consequence of [4] Theorem 4.4 and Corollary 5.7, that \( B(X) = \emptyset \) if and only if \( \pi_1(X, x_0) \) is countable. Thus if \( B(X) = \emptyset \), then \( B(Y) = \emptyset \) and both \( X \) and \( Y \) are homotopy equivalent to a finite wedge of circles in which case the lemma is standard.

Thus we need only concern ourselves with the case that \( B(X) \neq \emptyset \). Fix \( x_0 \in B(X) \). By Theorem 3.9 there exists a continuous function \( f : (X, x_0) \to (Y, y_0) \) such that \( f_* \) is an isomorphism. We can apply Theorem 3.9 again to find continuous maps \( \alpha : (I, 0, 1) \to (X, x_0, x_1) \) and \( g : Y \to X \) such that \( f_*^{-1} = \alpha \circ g_* \). Then \( \alpha = (g \circ f)_*: \pi_1(X, x_0) \to \pi_1(X, x_1) \) is a change of basepoint isomorphism. Hence \( (g \circ f)_* \) defines an isomorphism from \( \pi_1(X, x) \) to \( \pi_1(X, g \circ f(x)) \) for all \( x \in X \).

Let \( s_n : I \to X \) be a null sequence of essential loops based at \( x_0 \). Notice that \( \alpha \circ s_n \circ \alpha \) is homotopic to \( g \circ f \circ s_n \) and freely homotopic to \( s_n \). Then applying Lemma 3.11 to \( g \circ f \circ s_n \) and \( s_n \) gives that \( x_1 = x_0 \). Thus \( \alpha \) is a loop based at \( x_0 \) and \( (g \circ f)_* \) is an inner automorphism. Lemma 3.12 implies that \( (g \circ f)_* \) is the identity on \( \pi_1(X, x_0) \) as desired.

The following is a technical lemma from Cannon and Conner’s proof of Theorem 2.3 which is a slight strengthening of condition 1.

**Lemma 3.15** (Annulus Lemma). Suppose that \( X \) is a codiscrete subset of the 2-sphere for which condition 1 of Theorem 2.3 fails. Then there exists an annulus \( A \) in \( S^2 \) and components \( U_1, U_2, \ldots \) of \( A \setminus B(X) \) such that \( U_i \) interests both boundary components of \( A \) and \( U_i \cap D(X) = \emptyset \) for all \( i \in \mathbb{N} \).

**Theorem 3.16.** Let \( X \) is a codiscrete subset of the 2-sphere, \( S^2 \). Then \( X \) is homotopy equivalent to a one-dimensional Peano continuum if and only if \( \pi_1(X, x_0) \) is isomorphic to the fundamental group of a one-dimensional Peano continuum.

**Proof of Theorem 3.16** For the remainder of this section \( X \) will be a codiscrete subset of \( S^2 \) such that \( \pi_1(X, x) \) is isomorphic to \( \pi_1(Y, y) \) where \( Y \) is a one-dimensional Peano continuum.

Note that if \( X \) is locally simply connected, then \( X \) is homotopy equivalent to the wedge of finitely many circles and the lemma is trivial. Thus we may assume that \( X \) is not locally simply connected. Fix \( x_0 \in B(X) \).
Proof. By Theorem 2.1, there exists a planar Peano continuum and homotopy equivalences $h : \bar{X} \to X$ and $k : X \to \bar{X}$. Since $h,k$ are homotopy equivalences $h \circ k(x_0) = x_0$.

By Lemma 3.14 there exists maps $f_1 : \bar{X} \to Y$ and $g_1 : Y \to \bar{X}$ such that $g_1 \circ f_1(k(x_0)) = k(x_0)$ and $(g_1 \circ f_1)_* = \text{id}_{\pi_1(X,x_0)}$.

Then $h \circ g_1 \circ f_1 \circ k(x_0) = x_0$ and $(h \circ g_1 \circ f_1 \circ k)_* = h_* \circ g_1_* \circ f_1_* \circ k_* = h_* \circ k_* = \text{id}_{\pi_1(X,x_0)}$. So $f_1 \circ k$ and $h \circ g_1$ are the desired maps.

We will now fix $f : X \to Y$ and $g : Y \to X$ such that $g \circ f(x_0) = x_0$ and $(g \circ f)_* = \text{id}_{\pi_1(X,x_0)}$.

**Lemma 3.18.** If $x \in B(X)$, then $g \circ f(x) = x$, any path $\alpha : (I,0,1) \to (X,x,0)$ is homotopic rel endpoints to $g \circ f \circ \alpha$, and $g \circ f$ induces the identity homomorphism on $\pi_1(X,x)$.

Proof. Fix $x \in B(X)$ and a path $\alpha : (I,0,1) \to (X,x,0)$. By Lemma 3.13 $g \circ f(x) = x$. Let $s : I \to X$ an essential loop based at $x$. Since $(g \circ f)_*$ is the identity on $\pi_1(X,x_0)$ we have $g \circ f \circ \alpha \ast g \circ f \circ s \ast g \circ f \circ \overline{\alpha}$ is homotopic rel endpoints to $\alpha \ast s \ast \overline{\alpha}$. Hence $g \circ f \circ s$ is homotopic rel endpoints to $g \circ f \circ \alpha \ast s \ast \overline{\alpha} \ast g \circ f \circ \alpha$. Thus $(g \circ f)_*$ defines an inner automorphism of $\pi_1(X,x)$. By Lemma 3.12 $g \circ f$ induces the identity homomorphism on $\pi_1(X,x)$. Hence $g \circ f \circ \alpha = 0$ is nullhomotopic and $g \circ f \circ \alpha$ is homotopic rel endpoints to $\alpha$.

**Lemma 3.19.** The codiscrete space $X$ satisfies condition (a) of Theorem 2.3.

Proof. Suppose there exists a component $U$ of $S^2 \setminus B(X)$ such that $U \cap D(X) = \emptyset$. Notice that this implies that $\text{cl}_X(U)$ is compact.

$U$ is not simply connected: Then $U$ has at least two distinct boundary components. Let $B(X) = D_1 \cup D_2$ where $D_1, D_2$ are disjoint nonempty closed sets and $\partial U$ intersects both sets.

Then $f(D_1), f(D_2)$ are disjoint closed subsets of a one-dimensional space. Hence there exists a 0-dimensional subspace $L$ of $Y$ which separates $f(D_1), f(D_2)$. Let $C_i$ be a boundary component of $U$ contained in $D_i$ for $i = 1,2$.

Since $\text{cl}(U)$ is connected any set which separates boundary components of $U$ must intersect $U$. Since $f^{-1}(L)$ is disjoint from $\partial U$ any component of $f^{-1}(L)$ which intersects $U$ must be contained in $U$. Therefore there exists a component $A$ of $f^{-1}(L)$ which separates $C_1,C_2$ and is contained in $U$.

By hypothesis $U \cap D(X) = \emptyset$ and $\partial U \subset B(X)$ which implies that $\text{cl}_Y(A) \cap D(X) = \emptyset$. Then Lemma 3.8 implies that $A$ separates $D(X)$ in $S^2$.

Since $L$ is 0-dimensional and $A$ is connected, $f$ is constant on $A$ and Lemma 3.8 implies that $f_*$ is not injective which is a contradiction.

$U$ is simply connected: Fix $x_1 \in U$ and $\epsilon = d(x_1, B(X)) > 0$. Since $U$ is simply connected, $d(x_1, D(X)) \geq d(x_1, B(X))$. Choose $0 < \delta < \epsilon/4$ such that $d(g \circ f(x), g \circ f(y)) < \epsilon/4$ for every $x,y \in \text{cl}_X(U)$ with $d(x,y) < \delta$.

By Lemma 3.7 there exists a simply closed curve $\alpha : S^1 \to N_{\delta}(\partial U) \cap U$ which is homotopically essential in $S^2 \setminus\{d, x_1\}$ for any fixed $d \in D(X)$.

For every $x \in \text{Im}(\alpha)$ there exists a $u \in B(X)$ such that $d(x,u) < \delta$ and we have

$$d(x,g \circ f(x)) \leq d(x,u) + d(u,g \circ f(u)) + d(g \circ f(u), g \circ f(x))$$

$$< \epsilon/4 + \epsilon/4 \leq \epsilon/2 < d(x,\{x_1,D(X)\}).$$

Thus the line segment from $x$ to $g \circ f(x)$ misses $x_0$ for every $x \in \text{Im}(\alpha)$. 
Since $\alpha$ bounds a disc in $X$, $f \circ \alpha$ must factor through a dendrite and hence so does $g \circ f \circ \alpha$. Thus $g \circ f \circ \alpha$ is nullhomotopic in $\Im(g \circ f \circ \alpha) \subset S^2 \setminus \{x_1, d\}$. However $\alpha$ is homotopically essential in $S^2 \setminus \{x_1, d\}$ which contradicts Corollary 3.10.

\[\square\]

Lemma 3.20. $X$ satisfies condition (ii) of Theorem 2.3.

Proof. Suppose that $X$ did not satisfy (ii) of Theorem 2.3.

By Lemma 3.15, there exists an annulus $A$ in $S^2$ and components $U_1, U_2, \ldots$ of $A \setminus B(X)$ such that $U_n$ interests both boundary components of $A$ and $U_n \cap D(X) = \emptyset$ for all $n \in \mathbb{N}$. Let $J_1$ and $J_2$ be the boundary components of $A$.

Suppose $U_n \cap D(X) = \emptyset$ and $U_n$ is a component of $A \setminus B(X)$, $\text{cl}_{\mathbb{K}}(U_n) \cap D(X) = \emptyset$. This implies that $\text{cl}_{\mathbb{K}}(U_n) = \text{cl}_X(U_n)$ and hence $\text{cl}_X(\bigcup U_n)$ is compact. (This follows since $\text{cl}_X(\bigcup U_n) \setminus (\bigcup U_n) \subset B(X)$.)

The boundary of $U_n$ has two types of points.

Type (1) Points that are contained in $B(X)$. When restricted to these points, the map $f$ is a homeomorphism and $g \circ f$ is the identity.

Type (2) Points which are on $J_1 \cup J_2 \setminus B(X)$. A priori we have not control over what $f$ or $g \circ f$ does on these types of points. So we have to insure that for sufficiently large $n$ the diameter of the set of points in $U_n$ of this type is sufficiently small.

Notice that the set of points of $U_n$ of Type (2) are contained in the unique boundary component of $U_n$, intersecting both $J_1$ and $J_2$.

Since each $U_n$ is open and has closure intersecting both $J_1$ and $J_2$, there exists an $\epsilon_1 > 0$ and a sequence of points $x_n \in U_n$ such that $d(x_n, J_1 \cup J_2) \geq 4\epsilon_1$ for some $\epsilon_1 > 0$. We may also fix $d \in D(X)$ such that $d(d, U_n) > 4\epsilon_1$ for all $n$ (by possible passing to a cofinal subsequence of $U_n$ and choosing a smaller $\epsilon_1$).

Fix $0 < \delta_1 < \epsilon_1/4$ such that $d(g \circ f(x), g \circ f(y)) < \epsilon_1$ for all $x, y \in \text{cl}_X(\bigcup U_n)$ with $d(x, y) < 4\delta_1$.

Fix $0 < \delta_2 < \delta_1$ such that $d(g \circ f(x), g \circ f(y)) < \delta_1$ for all $x, y \in \text{cl}_X(\bigcup U_n)$ with $d(x, y) < \delta_2$.

Since each $U_i$ is open and connected in $A$ and intersects both boundary components of $A$, $U_i \cap J_k$ is contained in a connected component of $J_k \setminus (\bigcup U_i)$ for $k = 1, 2$. Thus diam$(U_n \cap J_1)$ and diam$(U_n \cap J_2)$ both converge to $0$, i.e. the set of points of $U_n$ of Type (2) can be partitioned into two subsets with small diameter. Hence we can fix an $n_0$ such that $\max\{\text{diam}(U_{n_0} \cap J_1), \text{diam}(U_{n_0} \cap J_2)\} < \delta_2$. Let $W$ be the component of $U_{n_0} \setminus \text{cl}_X(\{N_{2\delta_1}(\partial A)\})$ containing $x_{n_0}$.

Suppose $W$ is not simply connected: Then $U_{n_0}$ has a boundary component $C_1$ such that $d(C_1, x) \geq 2\delta_1$ for all points $x$ of Type 2. Let $C_2$ be the component of $\partial U_{n_0}$ which intersects both $J_1$ and $J_2$. Since max$\{\text{diam}(U_{n_0} \cap J_1), \text{diam}(U_{n_0} \cap J_2)\} < \delta_2$ and $f$ restricts to a homeomorphism on $B(X)$, we see that $f(C_1)$ and $f(C_2)$ are disjoint closed sets.

Then $f(\partial U_{n_0})$ has at least two distinct components. Let $\partial U_{n_0} = D_1 \cup D_2$ where $D_1, D_2$ are disjoint nonempty closed sets with $C_i \subset D_i$ for $i = 1, 2$ and $f(D_1) \cap f(D_2) = \emptyset$.

Hence there exists a 0-dimensional subspace $L$ of $Y$ which separates $f(D_1), f(D_2)$. Then $f^{-1}(L)$ separates $C_1, C_2$ which implies that a single component $E$ of $f^{-1}(L)$ separates $C_1, C_2$. Since $f^{-1}(L)$ is disjoint from $\partial U_{n_0}$, $E$ is a subset $U_{n_0}$ and $E \cap B(X) = \emptyset$. By hypothesis $U_{n_0} \cap D(X) = \emptyset$ and $\partial U_{n_0} \subset X$ which implies that $\text{cl}_{\mathbb{K}}(E) \cap D(X) = \emptyset$.

By construction, $E$ separates $B(X)$ so Lemma 3.14 implies that $E$ separates $D(X)$ in $S^2$. Since $L$ is 0-dimensional and $E$ is connected, $f$ is constant on $E$ and Lemma 3.3 implies that $f_x$ is not injective which is a contradiction.

Suppose $W$ is simply connected.
Let \( 4\epsilon_2 = \min\{d(x_n_0, \partial U_n_0), 4\epsilon_2\} \). We can choose \( 0 < \delta_3 \leq \min\{\epsilon_2, \delta_2\} \) such that \( 4d(g \circ f(x), g \circ f(y)) < \epsilon_2 \) for all \( x, y \in cl_{S^2}(U_n_0) \) with \( d(x, y) < \delta_3 \).

By Lemma 3.7, there exists a simply closed curve \( \alpha : S^1 \to N_{\delta_3}(\partial W) \cap W \) which is homotopically essential in \( S^2 \setminus \{x_n_0\} \). Since \( \text{Im}(\alpha) \subset U_n_0 \) and \( d(U_n_0, d) > 4\epsilon_1 \) we have that \( d(\text{Im}(\alpha), d) \geq 4\epsilon_1 \). Points on \( \partial W \) are on \( \partial U_n_0 \) or at least \( 4\epsilon_1 - 2\delta_1 \) from \( x_n_0 \). We now need to show that \( d(x, g \circ f(x)) < d(x, \{x_n_0, d\}) \) for every \( x \in \text{Im}(\alpha) \). This breaks down into two cases corresponding to whether \( x \) is close to \( \partial U_n_0 \) or far from \( x_n_0 \) (at least \( 4\epsilon_1 - 2\delta_1 - \delta_3 \)).

**Case 1:** \( d(x, \partial U_n_0) \leq \delta_3 \). Then \( d(x, x_n_0) > 4\epsilon_2 - \delta_3 \) and \( u \in B(X) \) such that \( d(x, u) \leq \delta_3 \). Thus

\[
d(x, g \circ f(x)) \leq d(x, u) + d(u, g \circ f(u)) + d(g \circ f(u), g \circ f(x)) < \epsilon_2/2 + 0 + \epsilon_2/2 < d(x, x_n_0, d).
\]

**Case 2:** \( d(\alpha(t), \partial U_n_0) > \delta_3 \). Then there exists \( v \in \partial W \) such that \( d(v, x) \leq \delta_3 \) and \( d(v, \partial A \cap \partial U_n_0) \leq 2\delta_1 \). This implies that \( d(x, x_n_0) \geq 4\epsilon_1 - 2\delta_1 - \delta_3 > 2\epsilon_1 \). Since \( J_i \cap U_n_0 \) has diameter at most \( \delta_2 \) for \( i \in \{1, 2\} \), there exists \( u \in B(X) \cap \partial U_n_0 \cap \partial A \) such that \( d(x, u) \leq \delta_3 + \delta_2 + 2\delta_1 < 4\delta_1 \). Thus

\[
d(x, g \circ f(x)) \leq d(x, u) + d(u, g \circ f(u)) + d(g \circ f(u), g \circ f(x)) < \epsilon_1 + 0 + \epsilon_1 + \epsilon_1 \leq 2\epsilon_1 < d(x, x_n_0, d).
\]

Since \( \alpha \) bounds a disc in \( X \), \( f \circ \alpha \) must factor through a dendrite and hence so does \( g \circ f \circ \alpha \). However, \( \alpha \) is homotopically essential in \( S^2 \setminus \{x_n_0, d\} \) which contradicts Corollary 3.10.

**Proof of Theorem 1.1** Let \( X \) be a planar Peano continuum and \( Y \) a one-dimensional Peano continuum such that \( \pi_1(X, x_0) \) is isomorphic to \( \pi_1(Y, y_0) \). By Theorem 2.1, \( X \) is homotopy equivalent to a codiscrete subset of \( S^2 \) which by Theorem 3.16 is homotopy equivalent to \( Y \). Thus \( X \) is homotopy equivalent to \( Y \).

**4. Example**

In [5] Cannon, Conner, and Zastrow proved that the filled Sierpinski carpet is not homotopy equivalent to the standard Sierpinski carpet. Specifically they showed that the filled Sierpinski carpet has homotopy dimension two while the Sierpinski carpet has homotopy dimension one. Theorem 1.1 strengthens this result by showing that they cannot have isomorphic fundamental groups and gives the following corollary.

**Proposition 4.1.** The set of points at which a planar Peano continuum is not locally simply connected is not a perfect invariant of the homotopy type or of the fundamental group.

The fundamental group determines the set \( B(X) \) with its topology, as well as the homotopy dimension for a planar Peano continuum \( X \). Thus a natural question is what other topologically defined properties of a planar continuum are determined by the fundamental group.

The following is an example of two planar Peano continua with the same homotopy dimensions for which it is known if they have isomorphic fundamental groups or if they are homotopy equivalent.

**Example 4.2.** A Warsaw circle is a space homeomorphic to

\[
\{(x, \sin(\pi/x)) \mid 0 < x \leq 2\} \cup \{(x, y) \mid x \in \{0, 2\}, \ y \in [-2, 1]\} \cup \{(x, -2) \mid x \in [0, 2]\}.
\]
Notice that every Warsaw circle is tamely embedded into $S^2$, i.e. the complement has exactly two simply connected components. Let $W$ be a Warsaw circle in $S^2$ with complementary components $C_1, C_2$. Let $D(Y_i)$ be a null sequence of open discs in $C_i$ with limit set $W$. Let $X = S^2 \setminus D(Y_1)$ and $Y = S^2 \setminus D(Y_2)$.

Since there is no continuous map fixing $W$ which interchanges the complementary components of $W$ in $S^2$, it is not clear if $X$ and $Y$ should be homotopy equivalent or not.

**Question 1.** Are $X$ and $Y$ homotopy equivalent? Or possible weaker, do they have isomorphic fundamental groups?

**Question 2.** If $X$ and $Y$ are not homotopy equivalent, is there an invariant that can be used to distinguish between these two spaces?

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