ONE REMARK ON BARELY $\dot{H}^{s_p}$ SUPERCRITICAL WAVE EQUATIONS

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Abstract. We prove that a good $\dot{H}^{s_p}$ critical theory for the 3D wave equation $\partial_{tt} u - \Delta u = -|u|^{p-1} u$ can be extended to prove global well-posedness of smooth solutions of at least one 3D barely $\dot{H}^{s_p}$ supercritical wave equation $\partial_{tt} u - \Delta u = -|u|^{p-1} u g(|u|)$, with $g$ growing slowly to infinity, provided that a Kenig-Merle type condition is satisfied. This result extends those \[26 \ 18\] obtained for the particular case $s_p = 1$.

1. Introduction

We shall consider the following wave equation

\[
\begin{cases}
\partial_{tt} u - \Delta u = -|u|^{p-1} u g(|u|) \\
u(0) := u_0 \in \dot{H}^2 \\
\partial_t u(0) := u_1 \in \dot{H}^1
\end{cases}
\]

(1.1)

where $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is a complex-valued scalar field, $p > 3$, $\dot{H}^2 := \dot{H}^2(\mathbb{R}^3) \cap H^{s_p}(\mathbb{R}^3)$, $\dot{H}^1 := \dot{H}^1(\mathbb{R}^3) \cap H^{s_p-1}(\mathbb{R}^3)$ and $g$ is a smooth, real-valued positive function defined on the set of nonnegative numbers and satisfying

\[
0 \leq g'(x) \lesssim x^{\frac{1}{2}} \tag{1.2}
\]

Condition (1.2) basically says that $g$ grows more slowly than any positive power of $u$.

We shall see that this equation (1.1) has many connections with the defocusing power-type defocusing wave equation

\[
\begin{cases}
\partial_{tt} u - \Delta u = -|u|^{p-1} u \\
u(0) := u_0 \in \dot{H}^{s_p}(\mathbb{R}^3) \\
\partial_t u(0) := u_1 \in \dot{H}^{s_p-1}(\mathbb{R}^3)
\end{cases}
\]

(1.3)

Here $s_p := \frac{3}{2} - \frac{2}{p-1}$. It is known that if $u$ satisfies (1.3) then $u_\lambda$ defined by

\[
u_\lambda(t, x) := \frac{1}{\lambda^{p-1}} u \left( \frac{t}{\lambda}, \frac{x}{\lambda} \right) \tag{1.4}
\]
satisfies the same equation, but with data \( u_\lambda(0, x) = \frac{1}{\lambda^{p-1}} u_0 \left( \frac{x}{\lambda} \right) \) and \( \partial_t u_\lambda(0, x) = \frac{1}{\lambda^{p-1}} u_1 \left( \frac{x}{\lambda} \right) \). Notice that (1.3) is \( \dot{H}^s \times \dot{H}^s \) critical, which means that the \( \dot{H}^s \times \dot{H}^s \) norm of \( (u(0), \partial_t u(0)) \) is invariant under the scaling defined above.

We recall the local existence theory: it is known (see for example [11, 22]) that there exists a positive constant \( \delta := \delta \left( \|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^s} \right) \) such that \( \|\cdot\|_{L^p_t L^q_x(\mathbb{R}^3)} \) is a solution of the integral equation, i.e.

\[
\int_0^t \sin \left( \frac{\pi}{2} \beta (t - \tau) \right) \frac{dt}{\pi} \int_0^\pi \sin \left( \frac{\pi}{2} \beta (\tau - \tau') \right) \left( |u|^{p-1} u \right) (t') \, dt' \leq \delta
\]

then there exists a unique solution \( \left( u, \partial_t u \right) \in C \left( [0, T], \dot{H}^s \times \dot{H}^s \right) \) with data \( \left( u_0, u_1 \right) \).

Moreover \( \|u\|_{L^p_t L^q_x(\mathbb{R}^3)} \leq C \|u_0\|_{\dot{H}^s} \), hence it is enough to find a finite upper bound of \( \|u\|_{L^p_t L^q_x(\mathbb{R}^3)} \) for all subinterval \( J \subset I_{max} \).

Now we turn to the global well-posedness theory of (1.3). In view of the local well-posedness theory, one can prove (see [11] and references), after some effort, that it is enough to find a finite upper bound of \( \|u\|_{L^p_t L^q_x(\mathbb{R}^3)} \) on arbitrary long time intervals \( I \), and, if this is the case, then the solution scatters to a solution of the linear wave equation. No blow-up has been observed for (1.3). Therefore it is believed that the following scattering conjecture is true.

**Conjecture 1.** "Scattering Conjecture" Assume that \( u \) is the solution of (1.3) with data \( (u_0, u_1) \in \dot{H}^s \times \dot{H}^s \). Then \( u \) exists for all time \( t \) and there exists \( C_1 := C_1 \left( \|(u_0, u_1)\|_{\dot{H}^s} \right) \) such that

\[
\|u\|_{L^p_t L^q_x(\mathbb{R}^3)} \leq C_1
\]

The case \( s_p = 1 \) (or, equivalently, \( p = 5 \)) is particular. Indeed the solution \( (u, \partial_t u) \in C \left( [0, T], \dot{H}^1(\mathbb{R}^3) \times \dot{H}^1(\mathbb{R}^3) \right) \) satisfies the conservation of the energy \( E(t) \) defined by

\[
E(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t u(t, x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 \, dx + \frac{1}{6} \int_{\mathbb{R}^3} |u|^6(t, x) \, dx
\]

In other words, \( E(t) = E(0) \). This is why this equation is often called energy-critical: the exponent \( s_p = 1 \) precisely corresponds to the minimal regularity required for (1.8) to be satisfied. The global well-posedness of (1.8) in the energy

\footnote{Notice that the \( L^2_t L^{2\tilde{p}}(\mathbb{R} \times \mathbb{R}^3) \) norm of \( u \) is invariant under the scaling (1.3). The choice of the space \( L^2_t L^{2\tilde{p}}(\mathbb{R} \times \mathbb{R}^3) \) in which we place the solution \( u \) is not unique. There exists an infinite number of spaces of the form \( L^p_t L^r \) scale invariant in which we can establish a local well-posedness theory.}
class and in higher regularity spaces is now understood. Rauch [17] proved the
global existence of smooth solutions of this equation with small data. Struwe [23]
showed that the result still holds for large data but with the additional assumption
of spherical symmetry of the data. The general case (large data, no symmetry
assumption) was finally settled by Grillakis [7,8]. Later Shatah and Struwe [20]
proved global existence of solutions in the energy class. Bahour and Gerard [1] re-
proved this result. Kapitanski [9] and, independently, Shatah and Struwe [21],
proved global existence of solutions of the energy-critical
quantity . Lately Tao [24] found an exponential tower type bound of this norm.

Notice that, in all these proofs of global existence of solutions of the energy-critical
wave equation, the conservation of energy, which leads, in particular, to the control
of the \( s \)-norm of the initial data. Unfortunately, the control of this norm is

indeed, Condition (1.2) basically says that for every \( \epsilon > 0 \), there exist two constants
\( c_1 := c_1(p) \) and \( c_2 := c_2(p,\epsilon) \) such that for \( |u| \) large

\[
(1.9) \quad c_1(p) \leq g(|u|) \leq c_2(p,\epsilon)|u|^\epsilon
\]
Since the critical exponent of the equation \( \partial_t u - \triangle u = -|u|^{p-1+\varepsilon} u \) is \( s_{p+\varepsilon} = s_p + O(\varepsilon) \), the nonlinearity of (1.1) is barely \( \dot{H}^s_p \) supercritical.

The goal of this paper is to check that this phenomenon, observed for \( s_p = 1 \), still holds for other values of \( s_p \). The standard local well-posedness theory shows us that it is enough to control the pointwise-in-time \( \dot{H}^2 \times \dot{H}^1 \) norm of the solution. In this paper, we will use an alternative local well-posedness theory. We shall prove the following proposition

**Proposition 1. "Local Existence for barely \( \dot{H}^s_p \) supercritical wave equation"** Assume that \( g \) satisfies (1.2) and

\[
\tag{1.10}
g''(x) = O \left( \frac{1}{x^2} \right)
\]

Let \( M \) be such that \( \|(u_0, u_1)\|_{\dot{H}^2 \times \dot{H}^1} \leq M \). Then there exists \( \delta := \delta(M) > 0 \) small such that if \( T_1 \) satisfies

\[
\tag{1.11}
\| \cos(tD)u_0 + \frac{\sin tD}{t}u_1 \|_{L_t^2(x_{p-1})L_x^2([0,T_1]) \times \mathbb{R}^3} \leq \delta
\]

then there exists a unique \((u, \partial_t u) \in C \left([0,T_1], \dot{H}^2 \right) \cap L_t^2(x_{p-1})L_x^{2(p-1)} \left([0,T_1]\right) \cap D^{1-s_p}L_t^4L_x^4 \left([0,T_1]\right) \cap D^{1-2s_p}L_t^2L_x^4 \left([0,T_1]\right) \times C \left([0,T_1], \dot{H}^1 \right) \) of (1.1) in the sense of the integral equation, i.e \( u \) satisfies the following Duhamel formula:

\[
\tag{1.12}
u(t) := \cos(tD)u_0 + \frac{\sin tD}{t}u_1 - \int_0^t \frac{\sin(t-s)D}{D} \left( |u(t')|^{p-1}u(t')g(|u(t')|) \right) \, dt'
\]

Notice that there are many similarities between Proposition 1 and the local well-posedness theory for (1.3).

This allows to define a maximum time interval of existence \( I_{\max,g} = [-T_{-g}, T_{+g}] \) such that for all \( J \subset I_{\max,g} \), we have \( \|u\|_{L_t^2(x_{p-1})L_x^{2(p-1)}(J)} < \infty \), \( \|D^{s_p-\frac{1}{2}}u\|_{W(J)} < \infty \), \( \|D^{2-\frac{1}{2}}u\|_{W(J)} < \infty \) and \( \|(u, \partial_t u)\|_{L_t^\infty \dot{H}^2(J) \times L_t^\infty \dot{H}^1(J)} < \infty \). Again, see [12] or [25] for more explanations.

Now we set up the problem. In view of the comments above for \( s_p = 1 \), we need to make two assumptions. First we will work with a "good" \( \dot{H}^s_p(\mathbb{R}^3) \) theory; therefore we will assume that Conjecture 1 is true. Then, we also would like to work with \( H^s_p(\mathbb{R}^3) \times H^{s_p-1}(\mathbb{R}^3) \) bounded solutions \((u(t), \partial_t u(t))\); more precisely we will assume that the following Kenig-Merle type condition holds

**Condition 1.1. "Kenig-Merle type condition"** Let \( g \) be a function that satisfies (1.2) and that is constant for \( x \) large. Then there exists \( C_2 := C_2 \left( \|(u_0, u_1)\|_{\dot{H}^2 \times \dot{H}^1}, g \right) \) such that

\[
\tag{1.13}
\sup_{t \in I_{\max,g}} \| (u(t), \partial_t u(t)) \|_{\dot{H}^s_p(\mathbb{R}^3) \times H^{s_p-1}(\mathbb{R}^3)} \leq C_2
\]

**Remark 1.2.** In the particular case \( s_p = 1 \), it is not difficult to see that Condition 1.1 is satisfied. Indeed \( u \) satisfies the energy conservation law.
\begin{equation}
E_0(t) := \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t u(t, x))^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 \, dx + \int_{\mathbb{R}^3} F(u(t, x), \bar{u}(t, x)) \, dx
\end{equation}
with
\begin{equation}
F(z, \bar{z}) = \left| z \right|^{5+1} \int_0^1 t^5 \Re \left( g(t|z|) \right) \, dt
\end{equation}
Since \( g \) is bounded then \( |F(z, \bar{z})| \lesssim |z|^6 \). By using the Sobolev embeddings \( \|u_0\|_{L^6} \lesssim \|u_0\|_{\dot{H}^2} \) and \( \|u(t)\|_{L^6} \lesssim \|u(t)\|_{\dot{H}^2} \), we easily conclude that Condition 1.1 holds. The energy conservation law was constantly used in [20, 13].

The main result of this paper is

**Theorem 1.3.** There exists a function \( \tilde{g} \) satisfying (1.2) and such that
\begin{equation}
\lim_{x \to -\infty} \tilde{g}(x) = \infty
\end{equation}
such that the solution of (1.1) (with \( g := \tilde{g} \)) exists for all time, provided that the scattering conjecture and the Kenig-Merle type condition are satisfied. Moreover there exists a function \( f \) depending on \( T \) and \( \|(u_0, u_1)\|_{\dot{H}^2 \times \dot{H}^1} \) such that
\begin{equation}
\|u\|_{L^\infty_t \dot{H}^2([-T, T])} + \|\partial_t u\|_{L^\infty_t \dot{H}^1([-T, T])} \leq f(T, \|(u_0, u_1)\|_{\dot{H}^2 \times \dot{H}^1})
\end{equation}

**Remark 1.4.**
- \( \tilde{g} \) is “universal”: it does not depend on an upper bound of the initial data
- \( \tilde{g} \) is unbounded: it goes to infinity as \( x \) goes to infinity

**Remark 1.5.** In fact, Theorem 1.3 also holds for a weaker version of Condition 1.1: there exists a function \( C_2 \) such that for all subinterval \( I \subset I_{\max, g} \)
\begin{equation}
\sup_{t \in I} \|((u(t), \partial_t u(t))\|_{\dot{H}^s_{\dot{H}^s-1}(\mathbb{R}^3)} \leq C_2
\end{equation}
with \( C_2 := C_2 \left( \|u_0, u_1\|_{\dot{H}^2 \times \dot{H}^1}, g, |I| \right) \). See the proof of Theorem 1.3 and, in particular, [21, 22] and [5, 4, 8].

We recall some basic properties and estimates. If \( t_0 \in [t_1, t_2] \), if \( F \in L^q_{t} L^r_x ([t_1, t_2]) \) and if \( (u, \partial_t u) \in C \left( [t_1, t_2], \dot{H}^{m}(\mathbb{R}^3) \right) \times C \left( [t_1, t_2], \dot{H}^{m-1}(\mathbb{R}^3) \right) \) satisfy
\begin{equation}
u(t) := \cos \left( t D \right) u_0 + \frac{\sin \left( t D \right)}{t} u_1 - \int_{t_0}^{t} \frac{\sin \left( t - t' \right) D}{t} F(t') \, dt'
\end{equation}
with data \( (u(t_0), \partial_t u(t_0)) \in \dot{H}^m(\mathbb{R}^3) \times \dot{H}^{m-1}(\mathbb{R}^3) \) then we have the Strichartz estimates [5, 10]
\begin{equation}
\|u\|_{L^q_t L^r_x ([t_1, t_2])} + \|u\|_{L^\infty_t \dot{H}^m(\mathbb{R}^3)([t_1, t_2])} + \|\partial_t u\|_{L^\infty_t \dot{H}^{m-1}(\mathbb{R}^3)([t_1, t_2])} \lesssim \|u(t_0), \partial_t u(t_0)\|_{\dot{H}^m(\mathbb{R}^3) \times \dot{H}^{m-1}(\mathbb{R}^3)}
\end{equation}
Here

- \((q, r)\) is m-wave admissible, i.e.

\[
\begin{aligned}
(q, r) &\in (2, \infty) \times [2, \infty] \\
\frac{1}{q} + \frac{2}{r} & = \frac{1}{2} - m
\end{aligned}
\]  

(1.21)

- \(\frac{1}{q} + \frac{2}{r} = \frac{1}{2} + \frac{1}{m} - 2\)

(1.22)

We set some notation that appear throughout the paper.

We write \(A \lesssim B\) if there exists a nonnegative constant \(C > 0\) such that \(A \leq C B\). \(A = O(B)\) means \(A \lesssim B\). More generally we write \(A \lesssim_{a_1, \ldots, a_n} B\) if there exists a nonnegative constant \(C = C(a_1, \ldots, a_n)\) such that \(A \leq C B\). We say that \(C\) is the constant determined by \(\lesssim\) in \(A \lesssim_{a_1, \ldots, a_n} B\) if \(C\) is the smallest constant among the \(C\)'s such that \(A \leq C B\). We write \(A \ll_{a_1, \ldots, a_n} B\) if there exists a universal nonnegative small constant \(c = c(a_1, \ldots, a_n)\) such that \(A \leq c B\).

Following [11], we define, on an interval \(I\),

\[
\|u\|_{S(I)} := \|u\|_{L^2_t L^{2(p-1)}_x(I)}
\]

\[
\|u\|_{W(I)} := \|u\|_{L^2_t L^4_x(I)}
\]

\[
\|u\|_{\tilde{W}(I)} := \|u\|_{L^2_t L^4_x(I)}
\]

(1.23)

We also define the following quantity

\[
Q(I, u) := \|D^{\alpha - \frac{1}{2}} u\|_{W(I)} + \|D^{2 - \frac{1}{2}} u\|_{W(I)} + \|\partial_t u\|_{L^\infty_t \dot{H}^2(I)} + \|\partial_t u\|_{L^\infty_t \dot{H}^1(I)}
\]

(1.24)

Let \(X\) be a Banach space and \(r \geq 0\). Then

\[
B(X, r) := \{ f \in X, \|f\|_X \leq r \}
\]

(1.25)

We recall also the well-known Sobolev embeddings. We have

\[
\|h\|_{L^\infty(\mathbb{R}^2)} \lesssim \|h\|_{\dot{H}^2}
\]

(1.26)

and

\[
\|h\|_{S(I)} \lesssim \|D^{\alpha - \frac{1}{2}} h\|_{L^2_t L^{2(p-1)}_x(I)}
\]

(1.27)

We shall combine (1.27) with the Strichartz estimates, since \(2(p - 1), \frac{6(p-1)}{2p-q}\) is \(\frac{1}{2}\)-wave admissible.

We also recall some Leibnitz rules [4, 13]. We have

\[
\|D^\alpha F(u)\|_{L^4_t L^4_x(I)} \lesssim \|F'(u)\|_{L^4_t L^4_x(I)} \|D^\alpha u\|_{L^4_t L^4_x(I)}
\]

(1.28)

with \(\alpha > 0\), \(r_1, r_2\) lying in \([1, \infty]\), \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\) and \(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}\).

The Leibnitz rule for products is
(1.29) \[ \|D^\alpha (uv)\|_{L_t^q L_x^r (I)} \lesssim \|D^\alpha u\|_{L_t^p L_x^s (I)} \|v\|_{L_t^q L_x^r (I)} + \|D^\alpha u\|_{L_t^q L_x^s (I)} \|v\|_{L_t^q L_x^r (I)} \]

with \( \alpha > 0, r, r_1, r_2 \) lying in \([1, \infty)\), \( \frac{1}{q} = \frac{1}{qr} + \frac{1}{qr_1} \), \( \frac{1}{p} = \frac{1}{qr} + \frac{1}{qr_1} + \frac{1}{r_2} \) and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \).

By using (1.28) and (1.29) the Leibniz rule for differences can be formulated as follows:

(1.30) \[ F(x) - F(y) = \int_0^1 F'(tx + (1-t)y)(x-y) \, dt \]

By using (1.28) and (1.29) the Leibniz rule for differences can be formulated as follows:

(1.31) \[ \|D^\alpha (F(u) - F(v))\|_{L_t^q L_x^r (I)} \lesssim \sup_{t \in [0,1]} \|F'(tu + (1-t)v)\|_{L_t^p L_x^s (I)} \|D^\alpha (u-v)\|_{L_t^q L_x^r (I)} \]

\[ + \sup_{t \in [0,1]} \|F''(tu + (1-t)v)\|_{L_t^p L_x^s (I)} \left( \|D^\alpha u\|_{L_t^q L_x^r (I)} + \|D^\alpha v\|_{L_t^q L_x^r (I)} \right) \]

with \( \alpha > 0, r, r_1, r_2, r_3 \) lying in \([1, \infty)\), \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \), \( \frac{1}{p} = \frac{1}{r_1} + \frac{1}{q_2} + \frac{1}{q_3} \), and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \).

We shall apply these formulas to several formulas of \( F(u) \), and, in particular, to \( F(u) := |u|^{p-1} u g(|u|) \). Notice that, by (1.22) and (1.10), we have \( F'(x) \sim |x|^{p-1} g(|x|) \) and \( F''(x) \sim |x|^{p-2} g(|x|) \). Notice also that, by (1.22) again, we have, for \( t \in [0,1] \)

\( g(|tx + (1-t)y|) \leq g(2 \max(|x|, |y|)) \)

(1.32) \[ \leq g(\max(|x|, |y|)) + 2g(|x|) + g(|y|) \]

This will allow us to estimate easily \( \sup_{t \in [0,1]} \|F'(tu + (1-t)v)\|_{L_t^q L_x^r (I)} \) and \( \sup_{t \in [0,1]} \|F''(tu + (1-t)v)\|_{L_t^q L_x^r (I)} \).

Now we explain the main ideas of this paper. We shall prove, in Section 3, that for a large number of \( g, s \), a special property for the solution of (1.1) holds:

**Proposition 2.** "control of \( S(I) \)-norm and control of norm of initial data imply control of \( L_{\tilde{t} \tilde{s}} \tilde{H}^2(I) \times L_{\tilde{t} \tilde{s}} \tilde{H}^1(I) \) norm" Let \( I \subset I_{max,g} \) and \( a \in I \). Assume that \( g \) satisfies (1.22), (1.10) and (1.32)

(1.33) \[ \int_1^\infty \frac{1}{9g^2(y)} \, dy = \infty \]

Let \( A \geq 0 \) such that \( \|(u_0, u_1)\|_{\tilde{H}^2 \times \tilde{H}^1} \leq A \). Let \( u \) be the solution of (1.1). There exists a constant \( C > 0 \) such that

(1.34) \[ \|(u, \partial_t u)\|_{L_{\tilde{t} \tilde{s}} \tilde{H}^2(I) \times L_{\tilde{t} \tilde{s}} \tilde{H}^1(I)} \leq (2C)^N A \]

\(^2\)Condition (1.33) basically says that \( g \) grows slowly on average.
with \( N := N(I) \) such that

\[
\int_{2CA}^{(2C)^N} \frac{1}{yg^2(y)} \, dy \geq \|v\|_{S(I)}^{2(p-1)}
\]

Moreover we shall give a criterion of global well-posedness (proved in Section 4)

**Proposition 3.** "criterion of global well-posedness" Assume that \(|I_{\max,g}| < \infty\). Assume that \(g\) satisfies (1.2), (1.10) and (1.33). Then

\[
\|u\|_{S(I_{\max,g})} = \infty
\]

The first step would be to prove global well-posedness of (1.1), with \(g_1\) a non decreasing function that is constant for \(x\) large (say \(x \geq C'_1\), with \(C'_1\) to be determined). By Proposition 3 it is enough to find an upper bound of the \(S([-T, T])\) norm of the solution \(u[i]\) for \(T\) arbitrary large. This is indeed possible, by proving that \(g_1\) can be considered as a subcritical perturbation of the nonlinearity. In other words, \(g_1(|u|)|u|^{p-1}u\) will play the same role as that of \(|u|^{p-1}u \left(1 - \frac{1}{|u|^\alpha}\right)\) for some \(\alpha > 0\). Once we have noticed that this comparison is possible, we shall estimate the relevant norms (and, in particular \(\|u[i]\|_{S([-T,T])}\)) by using perturbation theory, Conjecture 1 and Condition 1.1. By using Proposition 2 and (1.36), we can find a finite upper bound of \(|u|[i]\|_{L^\infty L^1([-T,T])}\). We assign the value that we have determined for \(C[i]\) to be determined

- it is an extension of \(g_{i-1}\) outside \([0, C[i-1]]\)
- it is increasing and constant (say equal to \(i+1\)) for \(x \geq C'_i\), with \(C'_i\) to be determined

Again, we shall prove that \(g_i\) may be regarded as a subcritical perturbation of the nonlinearity \((i+1)|u|^{p-1}u\). This allow us to control \(\|u[i]\|_{S([-i,i])}\), by using perturbation theory, Conjecture 1 and Condition 1.1. By using Proposition 2 and (1.26), we can find a finite upper bound of \(\|u[i]\|_{L^\infty L^1([-i,i])}\). We assign the value of this upper bound to \(C_i\). To conclude the argument we let \(\tilde{g} = \lim_{i \to -\infty} g_i\). Given
where $T > 0$, we can find a $j$ such that $[-T, T] \subset [-j, j]$ and $\| (u_0, u_1) \|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \leq j$. We prove that $u = u_{j|j}$ on $[-j, j]$ with $u$, solution of \((1.1)\) with $g := \tilde{g}$. Since we have a finite upper bound of $\| u_{j|j} \|_{H^j([-j, j])}$, we also control $\| u \|_{H^j([-T, T])}$ and $\| u \|_{H^j([-T, T])}$. Theorem \(1.3\) follows from Proposition \(3\).

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2. Proof of Proposition \(1\)

In this section we prove Proposition \(1\) for barely $\dot{H}^{s_p}(\mathbb{R}^3)$ supercritical wave equations \((1.1)\). The proof is based upon standard arguments. Here we have chosen to modify an argument in \(11\).

For $\delta, T_1, C, M$ to be chosen and such that \((1.1)\) holds we define

$$
B_1 := B \left( C([0, T_1], \dot{H}^2) \cap D^{\frac{s_p}{2}} W([0, T_1]) \cap D^{\frac{s_p}{2} - 2} W([0, T_1]), 2CM \right)
$$

(2.1)

$$
B_2 := B \left( S([0, T_1]), 2\delta \right)
$$

$$
B' := B \left( C([0, T_1], \dot{H}^1), 2CM \right)
$$

and

(2.2)

$$
X := \left\{ (u, \partial_t u) : u \in B_1 \cap B_2, \partial_t u \in B' \right\}
$$

Let

(2.3)

$$
\Psi(u, \partial_t u) := \left( \cos (tD)u_0 + \frac{\sin (tD)}{D} u_1 - \int_0^t \frac{\sin (t-t')D}{D} \left( |u(t')|^p - 1 u(t') g(|u(t')|) \right) dt' \right)
$$

$$
- D \sin (tD) u_0 + \cos (tD)u_1 - \int_0^t \cos (t-t')D \left( |u(t')|^p - 1 u(t') g(|u(t')|) \right) dt'
$$

Then

- $\Psi$ maps $X$ to $X$. Indeed, by \((1.11)\), \((1.20)\) and the fractional Leibnitz rule \((1.28)\) applied to $\alpha \in \left\{ s_p - \frac{3}{2}, 2 - \frac{3}{2} \right\}$ and $F(u) := |u|^{p-1} u g(|u|)$ and by applying the multipliers $D^{2-\frac{s_p}{4}}$ and $D^{s_p - \frac{s_p}{4}}$ to the Strichartz estimates with $m = \frac{s_p}{4}$ we have

(2.4)

$$
Q ([0, T_1]) \lesssim \| (u_0, u_1) \|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} + \| D^{s_p - \frac{s_p}{4}} (|u|^{p-1} u g(|u|)) \|_{\dot{W}([0, T_1])}
$$

$$
+ \| D^{2-\frac{s_p}{4}} (|u|^{p-1} u g(|u|)) \|_{\dot{W}([0, T_1])}
$$

$$
\leq CM + C \left( \| D^{s_p - \frac{s_p}{4}} u \|_{W([0, T_1])} + \| D^{2-\frac{s_p}{4}} u \|_{W([0, T_1])} \right) \| u \|_{S([0, T_1])} g(\| u \|_{L^{p} L^{\infty}(\mathbb{R}^3)}(0, T_1))
$$

$$
\leq CM + (2\delta)^{p-1} C(2CM) g(2CM)
$$

for some $C > 0$ and

(2.5)

$$
\| u \|_{S([0, T_1])} - \delta \lesssim \| D^{s_p - \frac{s_p}{4}} (|u|^{p-1} u g(|u|)) \|_{\dot{W}([0, T_1])}
$$

$$
\lesssim \| u \|_{S([0, T_1])} \| D^{s_p - \frac{s_p}{4}} u \|_{W([0, T_1])} g(\| u \|_{L^{p} L^{\infty}(\mathbb{R}^3)}(0, T_1))
$$

$$
\lesssim (2\delta)^{p-1} (2CM) g(2CM)
$$
Choosing $\delta = \delta(M) > 0$ small enough we see that $\Psi(X) \subset X$.

- $\Psi$ is a contraction. Indeed we get

\begin{equation}
(2.6) \quad \|\Psi(u) - \Psi(v)\|_X \lesssim \|D^{p-\frac{1}{2}}(|u|^{p-1}ug(|u|) - |v|^{p-1}vg(|v|))\|_{W([0,T])} + \|D^{2-\frac{1}{2}}(|u|^{p-1}ug(|u|) - |v|^{p-1}vg(|v|))\|_{\tilde{W}([0,T])}
\end{equation}

\begin{equation}
\lesssim g\left(\|u\|_{L^p L^\infty([0,T])}\times \left(\|u\|_{S([0,T])}^{p-1} + \|v\|_{S([0,T])}^{p-1}\right) \left(\|D^{p-\frac{1}{2}}(u-v)\|_{W([0,T])} + \|D^{2-\frac{1}{2}}(u-v)\|_{W([0,T])}\right)
+ \left(\|u\|_{S([0,T])}^{p-2} + \|v\|_{S([0,T])}^{p-2}\right) \left(\|D^{p-\frac{1}{2}}u\|_{W([0,T])} + \|D^{2-\frac{1}{2}}u\|_{W([0,T])} + \|D^{p-\frac{1}{2}}v\|_{W([0,T])}\right)\right)
\end{equation}

\begin{equation}
\lesssim (g(2CM)(2\delta)^{p-1} + (2\delta)^{p-2}(2CM)) \|u-v\|_X
\end{equation}

In the above computations, we applied the Leibnitz rule for differences to $\alpha \in \{s_p - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\}$ and $F(u) := |u|^{p-1}ug(|u|)$. Therefore, if $\delta = \delta(M) > 0$ is small enough, then $\Psi$ is a contraction.

3. Proof of Proposition 2

In this section we prove Proposition 2.

It is enough to prove that $Q(I) < \infty$. Without loss of generality we can assume that $A >> 1$. Then we divide $I$ into subintervals $(I_i)_{1 \leq i \leq N}$ such that

\begin{equation}
(3.1) \quad \|u\|_{S(I_i)} = \frac{\eta}{g^{-1/((2C)^N)A}}
\end{equation}

for some $C \geq 1$ and $\eta > 0$ constants to be chosen later, except maybe the last one. Notice that such a partition always exists since by (1.34) we get for $N := N(I)$ large enough

\begin{equation}
\sum_{i=1}^{N} \frac{1}{g^n((2C)^N)A} \geq \int_1^{N} \frac{g^{-1/((2C)^N)A}}{g^{-1/((2C)^N)A}} dx \\
\geq \int_{2CA \times 2^{x(1/2)^y}}^{1/2^x} \frac{1}{g^{-1/((2C)^N)A}} dy > \|u\|_{S(I_i)}
\end{equation}

We get, by a similar token used in Section 2

\begin{equation}
(3.3) \quad Q(I_1, u) \lesssim \|(u_0, u_1)\|_{H^2([0,T])} + \|D^{p-\frac{1}{2}}(|u|^{p-1}ug(|u|))\|_{W(I_1)} + \|D^{2-\frac{1}{2}}(|u|^{p-1}ug(|u|))\|_{W(I_1)} \lesssim A + \|D^{p-\frac{1}{2}}u\|_{W(I_1)} + \|D^{2-\frac{1}{2}}u\|_{W(I_1)} \|u\|_{S(I_1)}^{p-1}g(\|u\|_{L^p L^\infty(I_1)})
\end{equation}

We choose $C$ to be equal to the constant determined by $\lesssim$ in (3.3). Without loss of generality we can assume that $C > 1$. By a continuity argument, iteration on $i$, we get, for $\eta << 1$, (1.34).
4. Proof of Proposition 3

In this section we prove Proposition 3.

We argue as follows: by time reversal symmetry it is enough to prove that \( T_{+,g} < \infty \). If \( \|u\|_{S(I_{max,g})} < \infty \) then we have \( Q([0,T_{+,g}],u) < \infty \): this follows by slightly adapting the proof of Proposition 2. Consequently, by the dominated convergence theorem, there would exist a sequence \( t_n \to T_{+,g} \) such that \( \|u\|_{S([t_n,T_{+,g})]} < \delta \) and \( \|D^{p,-}\frac{1}{2}u\|_{W([t_n,T_{+,g})]} << \delta \) if \( n \) is large enough with \( \delta \) defined in Proposition 1. But, by (1.19) and (1.20)

\[
\| \cos((t-t_n)D)u(t_n) + \frac{\sin(t-t_n)D}{D}u_1 \|_{S([t_n,T_{+,g})]} \lesssim \|u\|_{S([t_n,T_{+,g})]} + \|u\|^p_{S([t_n,T_{+,g})]}\|D^{p,-}\frac{1}{2}u\|_{W([t_n,T_{+,g})]} g(Q([0,T_{+,g}],u)) \lesssim \|u\|_{S([t_n,T_{+,g})]} + \|u\|^p_{S([t_n,T_{+,g})]}\|D^{p,-}\frac{1}{2}u\|_{W([t_n,T_{+,g})]} \lesssim \|u\|_{S([t_n,T_{+,g})]} \lesssim \|u\|_{S([t_n,T_{+,g})]} < \delta
\]

and consequently, by continuity, there would exist \( \tilde{T} > T_{+,g} \) such that

\[
\| \cos((t-t_n)D)u(t_n) + \frac{\sin(t-t_n)D}{D}u_1 \|_{S([t_n,T_{+,g})]} \lesssim \|u\|_{S([t_n,T_{+,g})]} + \|u\|^p_{S([t_n,T_{+,g})]}\|D^{p,-}\frac{1}{2}u\|_{W([t_n,T_{+,g})]} \lesssim \|u\|_{S([t_n,T_{+,g})]} + \|u\|^p_{S([t_n,T_{+,g})]} < \delta
\]

which would contradict the definition of \( T_{+,g} \).

**Remark 4.1.** Notice that if we have the stronger bound \( \|u\|_{S(I_{max,g})} < C \) with

\[
C := C \left( \left\{ \|u_0,u_1\|_{\dot{H}^2 \times \dot{H}^1} \right\} \right) < \infty , \text{ then not only } I_{max,g} = (-\infty, +\infty) \text{ but also } u \text{ scatters as } t \to \pm \infty. \text{ Indeed by Proposition } 3, I_{max,g} = \mathbb{R}. \text{ Then by time reversal symmetry it is enough to assume that } t \to \infty. \text{ Let } v(t) := (u(t), \partial_t u(t)). \text{ We are looking for } v_{+,0} := (u_{+,0}, u_{+,1}) \text{ such that}

\[
\|v(t) - K(t)v_{+,0}\|_{\dot{H}^2 \times \dot{H}^1} \to 0 \quad \text{as } t \to \infty. \quad \text{Here}
\]

\[
K(t) := \begin{pmatrix}
\cos tD & \sin tD \\
-D\sin tD & \cos tD
\end{pmatrix}
\]

We have

\[
K^{-1}(t) = \begin{pmatrix}
\cos (tD) & -\frac{\sin (tD)}{D} \\
D\sin (tD) & \cos (tD)
\end{pmatrix}
\]

Notice that \( K^{-1}(t) \) and \( K(t) \) are bounded in \( \dot{H}^2 \times \dot{H}^1 \). Therefore it is enough to prove that \( K^{-1}(t)v(t) \) has a limit as \( t \to \infty. \) But since \( K^{-1}(t)v(t) = (u_0, u_1) - K^{-1}(t)(u_{nl}(t), \partial_u u_{nl}(t)) \) (with \( u_{nl} \) denoting the nonlinear part of the solution \( [1.12] \)), then it suffices to prove that \( K^{-1}(t)(u_{nl}(t), \partial_u u_{nl}(t)) \) has a limit. But

\[
u_{nl} := - \int_0^t \frac{\sin(t-t')D}{D} \left( |u(t')|^{p-1}u(t')g(|u(t')|) \right) dt'
\]

i.e.
\[ \| K^{-1}(t_1)u_{nl}(t_1) - K^{-1}(t_2)u_{nl}(t_2) \|_{\tilde{H}^2 \times \tilde{H}^1} \lesssim \| (u_{nl}, \partial_t u_{nl}) \|_{L_t^\infty \tilde{H}^2([t_1, t_2]) \times L_t^\infty \tilde{H}^1([t_1, t_2])} \]

\[
(4.6) \quad \lesssim \left( \| D^{\frac{p-1}{2}}|u|^{p-1}g(u)| \|_{W([t_1, t_2])} + \| D^{2-\frac{p}{2}}|u|^{p-1}g(u)| \|_{W([t_1, t_2])} \right) \lesssim \left( \| D^{\frac{p-1}{2}}u \|_{W([t_1, t_2])} + \| D^{2-\frac{p}{2}}u \|_{W([t_1, t_2])} \right) \| u \|_{S([t_1, t_2])}^p g(\| u \|_{L_t^\infty L_x^p(\mathbb{R})})
\]

It remains to prove that \( Q(\mathbb{R}) < \infty \) in order to conclude that the Cauchy criterion is satisfied, which would imply scattering. This follows from \( \| u \|_{S(\mathbb{R})} < \infty \) and a slight modification of the proof of Proposition 2.

5. Construction of the function \( \tilde{g} \)

In this section we prove Theorem 1.3. Let

\[ U_p(i) := \{(T, (u_0, u_1)) : 0 \leq T \leq i, \| (u_0, u_1) \|_{\tilde{H}^2 \times \tilde{H}^1} \leq i \}\]

As \( i \) ranges over \( \{1, 2, \ldots \} \) we construct, for each set \( U_p(i) \), a function \( g_i \) satisfying (1.2) and (1.10). Moreover it is constant for large values of \( |x| \). The function \( g_{i+1} \) depends on \( g_i \); the construction of \( g_i \) is made by induction on \( i \). More precisely we will prove that the following lemma

**Lemma 4.** Let \( A >> 1 \). Then there exist two sequences of numbers \( \{C_i\}_{i \geq 0}, \{C'_i\}_{i \geq 0} \) and a sequence of functions \( \{g_i\}_{i \geq 0} \) such that for all \( (T, (u_0, u_1)) \in U_p(i) \)

- \( g_0 := 1, C_0 := 0 \) and \( C'_0 = 0 \)
- \( \{C_i\}_{i \geq 0} \) and \( \{C'_i\}_{i \geq 0} \) are positive, non-decreasing. Moreover they satisfy

\[ AC_{i-1} < C'_i < AC_i \]

for \( i \geq 1 \) and

\[ C_i \geq i \]

- \( g_i \) is smooth, non-decreasing; it satisfies (1.2), (1.10),

\[ \int_1^{C'_i} \frac{1}{g_i(t)} \, dt \to i \to \infty \infty \]

and

\[ \begin{cases} g_i(|x|) = g_{i-1}(|x|), & |x| \leq AC_{i-1} \\ g_i(|x|) = i + 1, & |x| > C'_i \end{cases} \]

- the solution \( u_{[i]} \) of the following wave equation

\[ \begin{cases} \partial_t u_{[i]} - \triangle u_{[i]} = -|u_{[i]}|^{p-1}u_{[i]}g_i(|u_{[i]}|) \\ u_{[i]}(0) = u_0 \in \tilde{H}^2 \\ \partial_t u_{[i]}(0) = u_1 \in \tilde{H}^1 \end{cases} \]
satisfies

\[(5.7) \quad \max \left( \|u_{|j|}\|_{S([−t, t])}, \|u_{|j|}, \partial_t u_{|j|}\|_{L^\infty_t H^2([−T, T])} \times L^\infty_t \dot{H}^1([−T, T])} \right) \leq C_i \]

This lemma will be proved in the next subsection. Assuming that true let \( g = \lim_{i \to \infty} g_i \). Then clearly \( g \) is smooth; it satisfies \( (5.2) \) and \( (5.10) \). It also goes to infinity. Moreover let \( u \) be the solution of \( (1.1) \) with \( g := \tilde{g} \). We want to prove that the solution \( u \) exists for all time. Let \( T_0 \geq 0 \) be a fixed time. Let \( j := j(T_0, \|u_0\|_{H^2}, \|u_1\|_{H^1}) > 0 \) be the smallest positive integer such that \([−T_0, T_0]\) \( \cap [−j, j] \) and \( \|(u_0, u_1)\|_{H^2 \times H^1} \leq j \). We claim that \( \|(u, \partial_t u)\|_{L^\infty_t H^2([-T_0,T_0]) \times L^\infty_t \dot{H}^1([-T_0,T_0])} \leq C_j \) and \( \|u\|_{S([-T_0,T_0])} \leq C_j \). Indeed let

\[(5.8) \quad F_j := \{ t \in [0, j], \|(u, \partial_t u)\|_{L^\infty_t H^2([-t, t]) \times L^\infty_t \dot{H}^1([-t, t])} \leq C_j \text{ and } \|u\|_{S([-t, t])} \leq C_j \} \]

Then

- \( F_j \neq \emptyset: 0 \in F_j \), by \((5.3)\).
- \( F_j \) is closed. Indeed let \( \hat{t} \in \overline{F_j} \). Then there exist a sequence \((t_n)_{n \geq 1}\) such that \( t_n \to \hat{t} \), \( \|u\|_{S([-t_n, t_n])} \leq C_j \) and \( \|(u, \partial_t u)\|_{L^\infty_t H^2([-t_n, t_n]) \times L^\infty_t \dot{H}^1([-t_n, t_n])} \leq C_j \). It is enough to prove that \( \|u\|_{S([-\hat{t}, \hat{t}])} < \infty \) and then apply dominated convergence. There are two subcases:
  - \( \text{card}\{t_n, t_n \leq \hat{t}\} < \infty \): in this case, there exists \( n_0 \) large enough such that \( t_n \geq \hat{t} \) for \( n \geq n_0 \) and
    \[(5.9) \quad \|u\|_{S([-\hat{t}, \hat{t}])} \leq \|u\|_{S([-t_n, t_n])} < \infty \]
  - \( \text{card}\{t_n, t_n \leq \hat{t}\} = \infty \): even if it means passing to a subsequence, we can still assume that \( t_n \leq \hat{t} \). Let \( n_0 \geq 1 \) be fixed. Then, by the inequality
    \[(5.10) \quad \|\cos(t - t_{n_0})Du(t_{n_0}) + \frac{\sin(t - t_{n_0})D}{\partial t}\|_{S([t_{n_0}, \hat{t}])} \leq \|\cos(t_{n_0})Du(t_{n_0})\|_{\dot{H}^2} \leq C_j \]
    we conclude from the dominated convergence theorem that there exists \( n_1 := n_1(n_0) \) large enough such that
    \[(5.11) \quad \|\cos(t - t_{n_0})Du(t_{n_0}) + \frac{\sin(t - t_{n_0})D}{\partial t}\|_{S([t_{n_1}, \hat{t}])} \leq \delta \]
    with \( \delta := \delta(C_j) \) defined in Proposition \( \text{H} \). Therefore, by Proposition \( \text{H} \) we have \( \|u\|_{S([t_{n_1}, \hat{t}])} < \infty \). By a similar token, \( \|u\|_{S([-\hat{t}, t_{n_1}])} < \infty \). Combining these inequalities with \( \|u\|_{S([-t_{n_1}, t_{n_1}])} \leq C_j \), we eventually get \( \|u\|_{S([-\hat{t}, \hat{t}])} < \infty \).
- \( F_j \) is open. Indeed let \( \tilde{t} \in F_j \). By Proposition \( \text{H} \) there exists \( \alpha > 0 \) such that if \( t \in [\tilde{t} - \alpha, \tilde{t} + \alpha] \cap [0, j] \) then \([−t, t]\) \( \subset I_{\max, \delta} \) and \( \|u\|_{L^\infty_t H^2([-t, t])} \leq C_j \). Notice also that, by \((5.7)\), \([−t, t]\) \( \subset I_{\max, \delta} \). In view of these remarks, we conclude, after slightly adapting the proof of Proposition \( \text{2} \) that \( Q([-t, t], u) \lesssim_j 1 \) and \( Q([-t, t], u|_{j}) \lesssim_j 1 \). We divide
We consider \( w = u - u_j \). By applying the Leibnitz rules (1.28), (1.31) and (1.29), and by (5.12) we have

\[
(5.13)
\]

\[
Q(I_1, w) \lesssim \left( \sum_{i=0}^{j} \langle \| u \|_{L_t^p L_x^p(I_1)} \rangle \right) \left( \| D^{p-\frac{1}{2}} u \|_{W(I_1)} + \| D^{p-\frac{1}{2}} u \|_{W(I_1)} \right) \| u \|_{S(I_1)}^{p-1} + \| u \|_{S(I_1)}^{p-1} \left( \| D^{p-\frac{1}{2}} u \|_{W(I_1)} + \| D^{p-\frac{1}{2}} u \|_{W(I_1)} \right)
\]

\[
\lesssim (\hat{g} - g_j)(\| u \|_{L_t^p \tilde{H}^{2}(I_1)}) \left( \| D^{p-\frac{1}{2}} u \|_{W(I_1)} + \| D^{p-\frac{1}{2}} u \|_{W(I_1)} \right) \| u \|_{S(I_1)}^{p-1} + \| u \|_{S(I_1)}^{p-1} \left( \| D^{p-\frac{1}{2}} u \|_{W(I_1)} + \| D^{p-\frac{1}{2}} u \|_{W(I_1)} \right)
\]

since, by choosing \( A \) large enough and by construction of \( \hat{g} \) we have

\[
(5.14)
\]

\[
(\hat{g} - g_j)(\| u \|_{L_t^p \tilde{H}^{2}(I_1)}) = 0
\]

Therefore we conclude from a continuity argument that \( Q(I_1, w) = 0 \) and \( u = u_{[1]} \) on \( I_1 \). In particular \( u(b_1) = u_{[1]}(b_1) \). By iteration on \( i \), it is not difficult to see that \( u = u_{[1]} \) on \([-t, t]\). Therefore \( (\tilde{t} - \alpha, \tilde{t} + \alpha) \cap [0, j] \subset F_2 \), by (5.7).

Therefore, \( F_2 = [-j, j] \) and we have \( \| u \|_{S([-j, j])} \leq C_J \). This proves global well-posedness. Moreover, since \( j \) depends in \( T_0 \) and \( \| (u_0, u_1) \|_{\tilde{H}^{2} \times \tilde{H}^{1}} \), we get (1.17).
5.1. **Proof of Lemma 4** In this subsection we prove Lemma 4. We establish *a priori* bounds.

- **Step 1:** We construct $g_1$. $g_1$ is basically a nonnegative function that increases and is equal to two for $x$ large. Recall that $[-T, T] \subset [-1, 1]$ and $\|u_0, u_1\|_{H^2 \times H^1} \leq 1$. Let $I \subset [-T, T]$.

  Observe that the point $(\infty, 3+) := (3 + \epsilon, 3 + \epsilon)$ with $\epsilon << 1$ is $\frac{1}{2}$-wave admissible.

**Remark 5.1.** We would like to chop an interval $I$ such that $\|X_{L^\infty L^2(I)} < \infty$ into subintervals $I_j$ such that $\|X_{L^\infty L^2(I_j)}$ is as small as wanted. Unfortunately this is impossible because the $L^\infty_t$ norm is pathologic. Instead we will apply this process to $\|X_{L^\infty L^2(I)} - L^2\|$. This creates slight variations almost everywhere in the process of the construction of $g_1$. Details with respect to these slight perturbations have been omitted for the sake of readability: they are left to the reader, who should ignore the $+$ and $-$ signs at the first reading.

We define

$$X(I) := D^{\frac{1}{2}-sp} L^{\infty}_t L^2_x(I) \cap D^{\frac{1}{2}-sp} W(I) \cap S(I) \cap L^\infty_t \dot{H}^{sp}(I) \times L^\infty_t \dot{H}^{sp-1}(I)$$

Let $g_1$ be a smooth function, defined on the set of nonnegative real numbers nondecreasing and such that $h_1 := g_1 - 2$ satisfies the following properties: $h_1(0) = -1$, $h$ is non decreasing and $h_1(x) = 0$ if $|x| \geq 1$. It is not difficult to see that (1.2) and (1.10) are satisfied.

Observe that

$$|h_1(x)| \lesssim \frac{1}{|x|^{\frac{1}{2}}}$$

and

$$|h_1'(x)| \lesssim \frac{1}{|x|^{\frac{3}{2}}}$$

Let $u_{[1]}$ and $v_{[1]}$ be solutions of the equations

$$\begin{cases}
\partial_t u_{[1]} - \Delta u_{[1]} &= -|u_{[1]}|^{p-1} u_{[1]} g_1 \left(|u_{[1]}|\right) \\
\partial_t u_{[1]}(0) &= u_0 \in \dot{H}^2 \\
\partial_t u_{[1]}(0) &= u_1 \in H^1
\end{cases}$$

and

$$\begin{cases}
\partial_t v_{[1]} - \Delta v_{[1]} &= -2 |v_{[1]}|^{p-1} v_{[1]} \\
v_{[1]}(0) &= u_0 \\
\partial_t v_{[1]}(0) &= u_1
\end{cases}$$

There are several substeps

- **Substep 1:** We claim that $\|v_{[1]}\|_{X(\mathbb{R})} < \infty$. Indeed, since we assumed that Conjecture 11 is true, we first divide $\mathbb{R}$ into subintervals $(I_j = [t_j, t_{j+1}])_{1 \leq j \leq 1}$ such that $\|v_{[1]}\|_{S(I_j)} = \eta$ and $\|v_{[1]}\|_{S(I_j)} \leq \eta$ with $\eta << 1$. Then we have
Indeed, we will use Condition 1.1 in other parts of the argument: see (5.33).

\[ \sup_{t \in I_{max-g}} \| u(t) \|_{\dot{H}^s(R^3) \times \dot{H}^{s-1}(R^2)} \leq C_2 \left( \| (u_0, u_1) \|_{\dot{H}^2 \times \dot{H}^1} \right) \leq 1 \]

, following from Condition 1.1 and the assumption
\[ \| (u_0, u_1) \|_{\dot{H}^2 \times \dot{H}^1} \leq 1 \]

Therefore, by a standard continuity argument and iteration on \( j \) we have
\[ \| v[1] \|_{X(R)} \leq 1 \]

- Substep 2: we control \( \| w[1] - v[1] \|_{X([-\hat{t}, \hat{t}])} \) for \( \hat{t} < 1 \) to be chosen later.

By time reversal symmetry it is enough to control \( \| u[1] - v[1] \|_{X(u, \hat{a})} \).

To this end we consider \( w[1] := u[1] - v[1] \). We get
\[ \partial_t w[1] - \Delta w[1] = -|w[1] + v[1]|^{p-1}(v[1] + w[1]) \partial_t v[1] + 2|v[1]|^{p-1}v[1] \]

Let \( \eta' < 1 \). By (5.22), we can divide \([0, \hat{t}]\) into subintervals \((J_k)_{1 \leq k \leq m}\) that satisfy the following properties
\[ \begin{align*}
(1) & \quad \| D^{s} \frac{1}{2} w[1] \|_{L^\infty - L^{2+}(J_k)} = \eta' \quad \text{or} \quad \| D^{s} \frac{1}{2} w[1] \|_{W(J_k)} = \eta' \quad \text{for} \ 1 \leq k < m \\
(2) & \quad \| D^{s} \frac{1}{2} w[1] \|_{W(J_k)} \leq \eta' \quad \text{and} \quad \| D^{s} \frac{1}{2} w[1] \|_{L^\infty - L^{2+}(J_k)} \leq \eta' \quad \text{for} \ 1 \leq k \leq m
\end{align*} \]

We have
\[ \| w[1] \|_{X(J_{k+1})} \leq \| (w[1], \partial_t w[1]) \|_{\dot{H}^s(R^3) \times \dot{H}^{s-1}(R^2)} + \| D^{s} \frac{1}{2} (2|v[1]|^{p-1}v[1] - 2|v[1]|^{p-1}(v[1] + w[1])) \|_{W(J_{k+1})} + \| D^{s} \frac{1}{2} (h_1(|v[1]| + w[1])|v[1]| + w[1]|^{p-1}(v[1] + w[1])) \|_{L^1 L^2(J_{k+1})} \]

Let
\[ \begin{align*}
A_1 & := \| D^{s} \frac{1}{2} (2|v[1]|^{p-1}v[1] - 2|v[1]|^{p-1}(v[1] + w[1])) \|_{W(J_{k+1})} \\
A_2 & := \| D^{s} \frac{1}{2} (h_1(|v[1]| + w[1])|v[1]| + w[1]|^{p-1}(v[1] + w[1])) \|_{L^1 L^2(J_{k+1})}
\end{align*} \]

\[ \text{Notice that, at this stage, we only need to know that} \quad \| (u_0, u_1) \|_{\dot{H}^s(R^3) \times \dot{H}^{s-1}(R^2)} \leq \| (u_0, u_1) \|_{\dot{H}^2 \times \dot{H}^1} \leq 1 \quad \text{and apply Conjecture 1.1} \]

Therefore the introduction of \( \sup_{t \in I_{max-g}} \| (u(t), \partial_t u(t)) \|_{\dot{H}^s(R^3) \times \dot{H}^{s-1}(R^2)} \) in (5.24) is redundant. This is done on purpose.

Indeed, we will use Condition 1.1 in other parts of the argument: see (5.33).
By the fractional Leibnitz rule applied to \( q(x) := |x|^{p-1} x h(x) \), we have

\[
\tag{5.26}
A_2 \lesssim \| v_1 + w_1 \|^{\frac{1}{L^1_{L_2^p}(J_{k+1})}} \| D^{s_p-\frac{1}{2}} (v_1 + w_1) \|_{L^1_{L_2^p}(J_{k+1})}
\]

\[
\lesssim \| v_1 + w_1 \|^{\frac{1}{L^1_{L_2^p}(J_{k+1})}} \| D^{s_p-\frac{1}{2}} (v_1 + w_1) \|_{L^1_{L_2^p}(J_{k+1})}
\]

\[
\lesssim \xi \| D^{s_p-\frac{1}{2}} (v_1 + w_1) \|_{L^1_{L_2^p}(J_{k+1})}
\]

\[
\lesssim \xi \| D^{s_p-\frac{1}{2}} (v_1 + w_1) \|_{L^1_{L_2^p}(J_{k+1})}
\]

As for \( A_1 \) we follow [11], p 9

\[
\tag{5.27}
A_1 \lesssim \left( \| v_1 \|^{p-1}_{S(J_{k+1})} + \| w_1 \|^{p-1}_{S(J_{k+1})} \right) \| D^{s_p-\frac{1}{2}} w_1 \|_{W(J_{k+1})} + \left( \| v_1 \|^{p-2}_{S(J_{k+1})} + \| w_1 \|^{p-2}_{S(J_{k+1})} \right)
\]

\[
\left( \| D^{s_p-\frac{1}{2}} v_1 \|_{W(J_{k+1})} + \| D^{s_p-\frac{1}{2}} w_1 \|_{W(J_{k+1})} \right) \| w_1 \|_{S(J_{k+1})}
\]

\[
\lesssim \eta^{p-1} \| w_1 \|_{X(J_{k+1})} + \| w_1 \|_{X(J_{k+1})} + \eta^{p-2} \| w_1 \|_{X(J_{k+1})} + \eta \| w_1 \|_{X(J_{k+1})}
\]

This follows from \((1.31)\) and \((1.27)\). Therefore we have

\[
\tag{5.28}
\| w_1 \|_{X(J_{k+1})} \lesssim \| w_1 \|_{X(J_k)} + \eta^{p-1} \| w_1 \|^{p+1}_{X(J_{k+1})} + \eta^{p-2} \| w_1 \|^{p+1}_{Y(J_{k+1})} + \eta \| w_1 \|^{p+1}_{X(J_{k+1})}
\]

Let \( C \) be the constant determined by \((5.28)\). By induction, we have

\[
\| w_1 \|_{X(J_k)} \leq (2C)^k \eta^{\frac{p+1}{2}}
\]

provided that for \( 1 \leq k \leq m - 1 \) we have

\[
\tag{5.29}
C \eta^{\frac{p+1}{2}} \leq (2C)^k \eta^{\frac{p+1}{2}}
\]

\[
\leq (2C)^k \eta^{\frac{p+1}{2}}
\]

\[
\leq (2C)^k \eta^{\frac{p+1}{2}}
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\[
\leq (2C)^k \eta^{\frac{p+1}{2}}
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\[
\leq (2C)^k \eta^{\frac{p+1}{2}}
\]

\[
\leq (2C)^k \eta^{\frac{p+1}{2}}
\]

These inequalities are satisfied if \( \eta \ll 1 \)

\[
\tag{5.30}
\eta^{\frac{p+1}{2}} \ll 1
\]

since \( k \leq m - 1 \) and, by \((5.22)\), \( m \ll 1 \). We conclude that

\[
\| w_1 \|_{X([0,\bar{t}])} \lesssim 1
\]
Substep 3: we control \( \|u[1]\|_{X([-T,T])} \). By time reversal symmetry, it is enough to control \( \|u[1]\|_{X((0,T])} \). Recall that \( T \leq 1 \). We chop \( T \leq 1 \) into subintervals \( (J''_{k'} = [a_{k'}, b_{k'}])_{1 \leq k' \leq t} \) such that \( |J''_{k'}| = 1 \) for \( 1 \leq k' < t' \) and \( |J'_{t'}| \leq t' \). Notice that, by Condition 1.1, we have

\[
\|(u(a_{k'}), \partial_t u(a_{k'}))\|_{\dot{H}^{s_p}(\mathbb{R}^3) \times \dot{H}^{s_p-1}(\mathbb{R}^3)} \leq \sup_{t \in [1, t]} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p}(\mathbb{R}^3) \times \dot{H}^{s_p-1}(\mathbb{R}^3)} \leq C_2 \left( \|u[0, u_1]\|_{\dot{H}^{s_2} \times \dot{H}^{s_1}} \right) \lesssim 1
\]

\( k' \) we define \( v[1,k'] \) to be the solution of

\[
\begin{aligned}
&\partial_t v[1,k'] - \Delta v[1,k'] = -|v[1,k']|^{p-1} v[1,k'] \\
&v[1,k'](a_{k'}) = u[1](a_{k'}) \\
&\partial_t v[1,k'](a_{k'}) = \partial_t u[1](a_{k'})
\end{aligned}
\]

By slightly modifying the proof of the previous step and letting \( v[1,k'] \) play the role of \( v[1] \) (see previous step), this leads, by (5.33), to

\[
\|v[1,k']\|_{X(\mathbb{R})} \lesssim 1
\]

and

\[
\|v[1,k']\|_{X(J_{k'})} \lesssim 1
\]

with \( w[1,k'] := u[1] - v[1,k'] \). Therefore \( \|u[1]\|_{X(J_{k'})} \lesssim 1 \) and summing over the \( J_{k'} \)'s we have

\[
\|u[1]\|_{X((0,T])} \lesssim 1
\]

Substep 4: we control \( \|(u[1], \partial_t u[1])\|_{L^p_t \dot{H}^2([-1,1]) \times L^p_t \dot{H}^1([-1,1])} \) and \( \|u[1]\|_{S([-1,1])} \).

We get from (5.37)

\[
\|u[1]\|_{S([-1,1])} \lesssim 1
\]

By Proposition 2 and (5.38) we have

\[
\|(u[1], \partial_t u[1])\|_{L^p_t \dot{H}^2([-1,1]) \times L^p_t \dot{H}^1([-1,1])} \lesssim 1
\]

Therefore

\[
\max \left( \|u[1]\|_{S([-1,1])}, \|(u[1], \partial_t u[1])\|_{L^p_t \dot{H}^2([-1,1]) \times L^p_t \dot{H}^1([-1,1])} \right) \lesssim 1
\]

We let \( C'_1 \) (defined in the statement of Lemma 4) be equal to one. We can assume without the loss of generality that the constant determined by \( \lesssim \) in (5.40) is larger than 1. We let \( C'_1 \) (defined in the statement of Lemma 4) be equal to this constant. \( C'_1 \) and \( C_1 \) satisfy (5.2) and (5.3).  

\[5 \text{ in particular } v[1,k'] = v[1] \]
Step 2: construction of $g_i$ from $g_{i-1}$

Recall that $[-T, T] \subset [-i, i]$ and $\|(u_0, u_1)\|_{\tilde{H}^2 \times \tilde{H}^1} \leq i$. From (5.5) it is enough to construct $g_i$ for $|x| > AC_{i-1}$. It is clear that, by choosing $C'_i$ large enough, we can construct a function $\tilde{g}_i$ defined on $[AC_{i-1} - 1, C'_i]$ such that $g_i$, defined by

$$g_i(x) := \begin{cases} g_{i-1}(x), |x| \leq AC_{i-1} \\ \tilde{g}_i(x), |x| \geq AC_{i-1} \\ i + 1, |x| \geq C'_i \end{cases}$$

is smooth, slowly increasing; it satisfies (1.2), (1.10), and

$$\int_{AC_{i-1} - 1}^{C'_i} y g_i^2(y) \, dy \geq i$$

It remains to determine $C_i$ (defined in the statement of Lemma 4). To do that we slightly modify the step 1.

We sketch the argument. Let $h_i(x) := g_i(x) - (i + 1)$. Then $h_i(x) = 0$ if $|x| > C'_i$. It is not difficult to see that

$$|h_i(x)| \lesssim \frac{1}{|x|^{\frac{p-1}{2}}}$$

Let $u_{[i]}$ and $v_{[i]}$ be the solutions of the equations

$$\begin{cases} \partial_{tt} u_{[i]} - \Delta u_{[i]} = -|u_{[i]}|^{p-1} u_{[i]} g_i(|u_{[i]}|) \\ u_{[i]}(0) := u_0 \\ \partial_t u_{[i]}(0) := u_1 \end{cases}$$

and

$$\begin{cases} \partial_{tt} v_{[i]} - \Delta v_{[i]} = -(i + 1)|v_{[i]}|^{p-1} v_{[i]} \\ v_{[i]}(0) := u_0 \\ \partial_t v_{[i]}(0) := u_1 \end{cases}$$

We have

- **Substep 1**: we have

$$\|v_{[i]}\|_{X([0, \tilde{t}])} \lesssim 1$$

by adapting the proof of Substep 1, Step 2. Notice, in particular, that we can use Conjecture 1 and control $\|v_{[i]}\|_{S([0, \tilde{t}])}$ since $w_{[i]} := (i + 1)^\frac{1}{p+1} v_{[i]}$ satisfies $\partial_{tt} w_{[i]} - \Delta w_{[i]} = -|w_{[i]}|^{p-1} w_{[i]}$.

- **Substep 2**: we have $\|u_{[i]} - v_{[i]}\|_{X([0, \tilde{t})]} \lesssim 1$ for $\tilde{t} \ll 1$, by adapting the proof of Substep 2, Step 2. The dependance on $i$ basically comes from (5.43), (5.44) and (5.46).
\[\|(u_{[i]}(a_{k'}'), \partial_t u_{[i]}(a_{k'}'))\|_{\dot{H}^{r_2}(\mathbb{R}^2) \times \dot{H}^{r_1-1}(\mathbb{R}^2)} \leq \sup_{t \in I_{\max, g_1}} \|(u_{[i]}(t), \partial_t u_{[i]}(t))\|_{\dot{H}^{r_2}(\mathbb{R}^2) \times \dot{H}^{r_1-1}(\mathbb{R}^2)} \lesssim_1 1\]

We introduce

\[
\begin{align*}
\partial_t v_{[i,k']} - \Delta v_{[i,k']} &= -(i+1)|v_{[i,k']}|^{|p-1} v_{[i,k']} \\
v_{[i,k']}^{(a_{k'})} &= u_{[i]}(a_{k'}) \\
\partial_t v_{[i,k']}^{(a_{k'})} &= \partial_t u_{[i]}(a_{k'})
\end{align*}
\]

and, by using (5.48), we can prove

\[\|(u_{[i]}(a_{k'}), \partial_t u_{[i]}(a_{k'}))\|_{\dot{H}^{r_2}(\mathbb{R}^2) \times \dot{H}^{r_1-1}(\mathbb{R}^2)} \lesssim_1 1\]

Substep 4. By using Proposition 2 and (5.50) we get

\[\max\left(\|(u_{[i]}(a_{k'}'))\|_{S([-i,i])}, \|(u_{[i]}, \partial_t u_{[i]})\|_{L_T^\infty \dot{H}^2([-i,i]) \times L_T^\infty \dot{H}^2([-i,i])}\right) \lesssim_1 1\]

We can assume without loss of generality that the constant determined by \(\lesssim\) in (5.51) is larger than \(i\) and \(C_i\). We let \(C_i\) be equal to this constant. (5.2) and (5.3) are satisfied.

This ends the proof.

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