NONDEGENERACY OF THE POSITIVE SOLUTIONS FOR CRITICAL NONLINEAR HARTREE EQUATION IN $\mathbb{R}^6$

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ABSTRACT. We prove that any positive solution for the critical nonlinear Hartree equation
$$
- \Delta u(x) - \int_{\mathbb{R}^6} \frac{|u(y)|^2}{|x-y|^4} \, dy \, u(x) = 0, \quad x \in \mathbb{R}^6.
$$
is nondegenerate. Firstly, in terms of spherical harmonics, we show that the corresponding linear operator can be decomposed into a series of one dimensional linear operators. Secondly, by making use of the Perron-Frobenius property, we show that the kernel of each one dimensional linear operator is finite. Finally, we show that the kernel of the corresponding linear operator is the direct sum of the kernel of all one dimensional linear operators.

1. Introduction

The purpose of this paper is to derive the nondegeneracy property of the positive solutions to the following $H^1$-critical nonlinear Hartree (NLH) equation,
$$
\begin{align*}
- \Delta u(x) - \int_{\mathbb{R}^6} \frac{|u(y)|^2}{|x-y|^4} \, dy \, u(x) &= 0, \quad x \in \mathbb{R}^6, \\
\lim_{|x| \to +\infty} |u(x)| &= 0.
\end{align*}
$$ (1.1)

It is well-known that (see, for example [16]) any positive solution of the equation (1.1) belongs precisely to
$$
\{ \omega_{\lambda, z}(x) = \lambda^2 \omega(\lambda x + z) \mid \lambda > 0 \text{ and } z \in \mathbb{R}^6 \},
$$ (1.2)

where $\omega$ is the radial, positive ground state, and has the explicit form
$$
\omega(x) = \frac{12}{\pi^{7/2}} \frac{1}{(1 + |x|^2)^{2}}.
$$ (1.3)

On the one hand, since the function $\omega_{\lambda, z}(x)$, defined by (1.2), satisfies
$$
- \Delta \omega_{\lambda, z}(x) - \int_{\mathbb{R}^6} \frac{[\omega_{\lambda, z}(y)]^2}{|x-y|^4} \, dy \omega_{\lambda, z}(x) = 0, \quad x \in \mathbb{R}^6, \lambda > 0, \text{ and } z \in \mathbb{R}^6,
$$ (1.4)
by differentiating (1.4) with respect to the parameters $\lambda$ and $z$ at $\lambda = 1$ and $z = 0$ formally, we obtain that,
$$
L \omega(x) = 0, \quad \text{and} \quad L \frac{\partial \omega}{\partial x_j}(x) = 0, \quad 1 \leq j \leq 6,
$$

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where
\[
\Lambda \omega (x) := 2 \omega (x) + 2 x \cdot \nabla \omega (x),
\]
and the operator \( L \) is defined by
\[
L \varphi (x) := - \Delta \varphi (x) - \Phi (\omega^2) (x) \varphi (x) - 2 \Phi (\omega \varphi) (x) \omega (x),
\]
with
\[
\Phi f (x) := \int_{\mathbb{R}^6} \frac{f (y)}{|x - y|^4} d y.
\]
Moreover, by the linearity of the operator \( L \), if \( \varphi \) belongs to the set
\[
\mathcal{N} = \text{span} \left\{ \Lambda \omega, \frac{\partial \omega}{\partial x_1}, \frac{\partial \omega}{\partial x_2}, \frac{\partial \omega}{\partial x_3}, \frac{\partial \omega}{\partial x_4}, \frac{\partial \omega}{\partial x_5}, \frac{\partial \omega}{\partial x_6} \right\},
\]
we get, \( L \varphi (x) = 0 \).

On the other hand, an important question arising in the analysis of solutions for (1.1) is to study the kernel of the linearized operator \( L \) close to \( \omega \). More precisely, one can address the following question:

Is there any other function \( \varphi \) vanishing at infinity satisfies \( L \varphi (x) = 0 \), except that belongs to the set \( \mathcal{N} \) defined by (1.8)?

Our main result in this paper is devoted to a negative answer to the question above. More precisely, we will prove the following theorem which states that the solution \( \omega \) (see (1.3)) of (1.1) is nondegenerate.

**Theorem 1.1.** The solution \( \omega \), defined by (1.3), for the problem (1.1) is nondegenerate. More precisely, let \( L \) be defined by (1.6), if \( f \in L^2 (\mathbb{R}^6) \) satisfies \( Lf (x) = 0 \), then
\[
f \in \text{span} \left\{ \Lambda \omega, \frac{\partial \omega}{\partial x_1}, \frac{\partial \omega}{\partial x_2}, \frac{\partial \omega}{\partial x_3}, \frac{\partial \omega}{\partial x_4}, \frac{\partial \omega}{\partial x_5}, \frac{\partial \omega}{\partial x_6} \right\}.
\]

The nondegeneracy of the ground state for the nonlinear elliptic equations plays a key role in the analysis of long time dynamics of the solution to the corresponding evaluation equations. For example, in the context of \( H^1 \)-critical nonlinear Schr"{o}dinger (NLS) equation,
\[
i \frac{\partial u}{\partial t} (t, x) + \Delta u (t, x) + |u|^{4_N} u (t, x) = 0,
\]
and the \( H^1 \)-critical nonlinear wave (NLW) equation,
\[
\frac{\partial^2 u}{\partial t^2} (t, x) + \Delta u (t, x) + |u|^{4_N} u (t, x) = 0,
\]
the nondegeneracy of the ground state solution \( W \) to the corresponding elliptic equation
\[
- \Delta u (t, x) - |u|^{4_N} u (t, x) = 0,
\]
is crucial in the construction of blow-up solutions to the equations (1.9) and (1.10) (see, for instance, [6–10,13]). With the help of Theorem 1.1, we are able to construct blow-up solutions to the \( H^1 \)-critical nonlinear Schrödinger equation with Hartree terms,
\[
i \frac{\partial u}{\partial t} (t, x) + \Delta u (t, x) + \int_{\mathbb{R}^6} |u (y)|^2 d y u (x) = 0, \quad x \in \mathbb{R}^6,
\]
which will be considered in our subsequent work.
The nondegeneracy of the ground state solution to the semilinear elliptic equation (1.11) is also necessary in the construction of multi-bump solutions of the equation (1.11), see, for example [3–5,14,15]. We hope Theorem 1.1 can be used to construct multi-bump solutions to (1.1).

The paper is organized as follows. In Section 2, we give some notations and review several lemmas which will be frequently used in the remainder of the paper. In Section 3, we mainly prove Theorem 1.1.

2. Notation and useful lemmas

Notation and conventions. As usual, we use $S^5$ to denote 5-dimensional unit sphere in 6-dimensional Euclidean space $\mathbb{R}^6$,

$$S^5 = \{ x = (x_1, x_2, \cdots, x_6) \in \mathbb{R}^6 \mid |x|^2 = \sum_{j=1}^{6} x_j^2 = 1 \}.$$ 

For any $x, y \in \mathbb{R}^6$ with $|x| \neq |y|$, let us denote,

$$x \vee y := \begin{cases} x & \text{if } |x| > |y|, \\ y & \text{if } |x| < |y|, \end{cases} \quad \text{and} \quad x \wedge y := \begin{cases} y & \text{if } |x| > |y|, \\ x & \text{if } |x| < |y|. \end{cases} \quad (2.1)$$

An elementary calculation implies that,

$$|x \vee y| = \max \{|x|, |y|\}, \quad \text{and} \quad |x \wedge y| = \min \{|x|, |y|\}. \quad (2.2)$$

We use $L^2(\mathbb{R}^6)$ denote the real Hilbert space of measurable functions $f$ on $\mathbb{R}^6$ with the inner product,

$$(f, g) := \int_{\mathbb{R}^6} f(x) g(x) \, dx.$$ 

We shall also use $L^2(S^5)$ denote the space of measurable functions $f$ on $S^5$ for which

$$\int_{S^5} |f(x)|^2 \, d\sigma(x)$$

is finite, where $d\sigma$ is the surface area measure. With a slight abuse of notation, we write both $f(x)$ and $f(|x|)$ for radial functions $f$ on $\mathbb{R}^6$. Moreover, we will use $L^2((0, +\infty), r^5)$ denote the Hilbert space with measurable functions $f$ on $(0, +\infty)$ with the inner product,

$$\langle f, g \rangle := \int_0^{+\infty} f(r) g(r) \, r^5 \, dr.$$ 

Next, we recall some well-known results related to spherical harmonics. We use $H_k$ to denote the space of spherical harmonics of degree $k$ (i.e. the restrictions to $S^5$ of real, homogeneous harmonic polynomials of degree $k$). In fact, the dimension of $H_k$ is $[2,17]$ is

$$\dim H_k = \alpha_k := \begin{cases} 1, & \text{if } k = 0, \\ 6, & \text{if } k = 1, \\ \binom{k+5}{k} - \binom{k+3}{k-2}, & \text{if } k \geq 2. \end{cases}$$
We use $Y_{k,j}$ ($1 \leq j \leq \alpha_k$) to denote an orthogonal basis for $\mathcal{H}_k$, i.e.
\[
\int_{S^5} Y_{k,i}(\xi) Y_{k,j}(\xi) \, d\sigma(\xi) = \pi^3 \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
\]  
(2.3)

Specially, $\mathcal{H}_0 = \text{span} \{1\}$, and
\[
\mathcal{H}_1 = \text{span}\{ \frac{x_j}{|x|} \mid 1 \leq j \leq 6 \},
\]  
(2.4)

therefore, we may set $Y_{1,j}(\xi) = \sqrt{6} \xi_j$ with $\xi_j = \frac{x_j}{|x|}$. Moreover, the space $\mathcal{H}_k$

coincides with the eigenspace of the eigenvalue $-k(k+4)$ for the Laplace–Beltrami

operator $\triangle_{S^5}$ on $\mathbb{S}^5$, i.e.
\[
\triangle_{S^5} Y = -k(k+4) Y, \quad \text{for any } Y \in \mathcal{H}_k.
\]  
(2.5)

An elementary calculation implies that,
\[
\int_{S^5} Y_{k,i}(\xi) Y_{l,j}(\xi) \, d\sigma(\xi) = 0, \quad \text{for all } 1 \leq k \neq l < \infty, \ 1 \leq i \leq \alpha_k, \ 1 \leq j \leq \alpha_l.
\]  
(2.6)

Now, for any $f \in L^2(\mathbb{R}^6)$, let us denote
\[
f_{k,j}(r) := \frac{1}{\pi^3} \int_{S^5} f(r \xi) Y_{k,j}(\xi) \, d\sigma(\xi),
\]  
(2.7)

then we have the following direct sum decomposition (e.g. see [17]),
\[
f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} f_{k,j}(|x|) Y_{k,j}\left(\frac{x}{|x|}\right).
\]  
(2.8)

Moreover, the following identity holds,
\[
\int_{\mathbb{R}^6} |f(x)|^2 \, dx = \pi^3 \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} \int_{0}^{+\infty} |\varphi_{k,j}(r)|^2 r^5 \, dr.
\]  
(2.9)

In order to deal with the Newtonian potential appeared in (1.1), we need the following lemma, which related to Gegenbauer functions.

**Lemma 2.1** ([2, Lemma 1.2.3, Theorem 1.2.6]). For any $r \in (-1, 1)$, $t \in [-1, 1]$, we have,
\[
\frac{1}{(1 - 2rt + r^2)^2} = \sum_{k=0}^{\infty} C_k^{(2)}(t) r^k,
\]  
(2.10)

where $C_k^{(2)}(t)$ is the Gegenbauer polynomial of degree $k$ associated with 2 (see [2, p.418]). Moreover, the following addition formula holds,
\[
P_k^2(\xi \cdot \eta) = \frac{2}{k + 2} \sum_{j=1}^{\alpha_k} Y_{k,j}(\xi) Y_{k,j}(\eta), \quad \text{for any } \xi, \eta \in \mathbb{S}^5.
\]  
(2.11)

Now, we are able to give an explicit expression of the Newtonian potential in $\mathbb{R}^6$, by making use of spherical harmonics.

**Lemma 2.2.** For any $x, y \in \mathbb{R}^6 \setminus \{0\}$ with $|x| \neq |y|$, we have,
\[
\frac{1}{|x - y|^4} = \sum_{k=0}^{\infty} \frac{k}{k + 2} \frac{|x \wedge y|^k}{|x \vee y|^{k+4}} \sum_{m=1}^{\alpha_k} Y_{k,j}\left(\frac{x}{|x|}\right) Y_{k,j}\left(\frac{y}{|y|}\right),
\]  
(2.12)

$\pi^3$ is the volume of $\mathbb{S}^5$.  


Proof. First, for any \( x, y \in \mathbb{R}^6 \setminus \{0\} \) satisfying \( |x| \neq |y| \), we have,
\[
|x - y|^2 = |x|^2 - 2x \cdot y + |y|^2.
\] (2.13)

By making use of (2.1) and (2.2), we have, from (2.13),
\[
|x - y|^2 = |x \lor y|^2 - 2 |x \lor y| |x \land y| \frac{x}{|x|} \frac{y}{|y|} + |x \land y|^2.
\] (2.14)

Therefore, by (2.14), we obtain that,
\[
\frac{1}{|x - y|^2} = \frac{1}{|x \lor y|^2} \frac{1}{1 - 2 \frac{|x \land y|}{|x \lor y|} \frac{x}{|x|} \frac{y}{|y|} + \left(\frac{|x \land y|}{|x \lor y|}\right)^2}.
\] (2.15)

Now, by taking \( r = \frac{|x \land y|}{|x \lor y|} \) in (2.10), then taking \( \xi = \frac{x}{|x|} \) and \( \eta = \frac{y}{|y|} \) in (2.11), The formula (2.12) follows directly from (2.15). This ends the proof of Lemma 2.2. \( \square \)

**Remark 2.3.** The spherical harmonics expansion for the Newtonian potential in \( \mathbb{R}^n (3 \leq n \leq 4) \) and \( \mathbb{R}^4 \) is well-known, (see, for example, \([1, 11]\)).

The following proposition plays an important role in the nondegenerate analysis of positive solutions to (1.1).

**Proposition 2.4.** Let \( \Phi \) be defined by (1.7). For any \( f \in L^2 (\mathbb{R}^6) \), we have,
\[
\Phi f (x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} \int_0^{+\infty} \mathcal{K}_k (r, |x|) f_{k,j} (r) \, dr Y_{k,j} \left( \frac{x}{|x|} \right),
\] (2.16)

where \( f_{k,j} (r) \) is defined by (2.7), and
\[
\mathcal{K}_k (r, |x|) = \frac{2\pi^3}{k + 2} \left\{ \begin{array}{ll}
\frac{\alpha_k + 1}{|x|^{k+2}}, & \text{if } r < |x|, \\
\frac{\alpha_k}{|x|^{k+1}}, & \text{if } r > |x|.
\end{array} \right.
\] (2.17)

Proof. On the one hand, in view of Lemma 2.2,
\[
\frac{1}{|x - y|^2} = \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} \frac{k}{k + 2} |x \lor y|^{k+1} \sum_{m=1}^{\alpha_k} Y_{k,j} \left( \frac{x}{|x|} \right) Y_{k,j} \left( \frac{y}{|y|} \right),
\] (2.18)

where \( x, y \in \mathbb{R}^6 \setminus \{0\} \) satisfy \( |x| \neq |y| \).

On the other hand, by (2.8), we have
\[
f_{k,j} (r) = \frac{1}{\pi^3} \int_{\mathbb{S}^5} f (r \xi) Y_{k,j} (\xi) \, d\sigma (\xi),
\] (2.19)

where \( f_{k,j} (r) \) is defined by (2.7).

Combining with (2.18) and (2.19), we have,
\[
\begin{align*}
\int_{\mathbb{R}^6} f (y) |x - y|^2 \, dy &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=1}^{\alpha_k} \int_{\mathbb{R}^6} \frac{k}{k + 2} |x \lor y|^{k+1} Y_{k,j} \left( \frac{x}{|x|} \right) Y_{k,j} \left( \frac{y}{|y|} \right) f_{i,m} (|y|) Y_{i,m} \left( \frac{y}{|y|} \right) \, dy.
\end{align*}
\] (2.20)
By the definition of $x \land y$ and $x \lor y$ (see (2.1)), and (2.2), from (2.20), we find,

$$
\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^2} \, dy \\
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{\alpha_k} k \frac{k+2}{|y|^{k+4}} Y_{k,j} \left( \frac{x}{|y|} \right) f_{l,m} (|y|) Y_{l,m} \left( \frac{y}{|y|} \right) \, dy \\
+ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{\alpha_k} k \frac{k+2}{|y|^{k+4}} Y_{k,j} \left( \frac{x}{|y|} \right) f_{l,m} (|y|) Y_{l,m} \left( \frac{y}{|y|} \right) \, dy.
$$

(2.21)

With the change of variables $r = |x|$ and $\xi = \frac{x}{|x|}$ and using (2.3), (2.6), (2.21) becomes

$$
\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^2} \, dy \\
= \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} k \frac{k+2}{|x|^{k+4}} Y_{k,j} \left( \frac{x}{|x|} \right) \left( \int_{|x|}^{\infty} \pi^3 \frac{r^{k+5}}{|x|^{k+4}} f_{k,j} (r) \, dr \right) + \int_{|x|}^{+\infty} \pi^3 \frac{|x|^k}{r^{k+4}} f_{k,j} (r) \, dr,
$$

which means that (2.16) holds with $\mathcal{K}_k (r, |x|)$ defined by (2.17). □

As a consequence of Proposition 2.4, we obtain Newton’s theorem (see, for instance, [12, Theorem 9.7]).

**Corollary 2.5.** For any radial $f \in L^2 (\mathbb{R}^6)$, $\Phi f (x)$ is radial. Moreover, we have,

$$
\Phi f (x) = \int_{0}^{+\infty} \mathcal{K}_k (r, |x|) f (r) \, dr,
$$

(2.22)

where $\mathcal{K}_k (r, |x|)$ is defined by (2.17) with $k = 0$. Specially, the follows identity holds,

$$
\Phi \left( \omega^2 \right) (x) = 2 \pi^{3/2} \omega (x),
$$

(2.23)

$$
- \triangle \omega (x) - 2 \pi^{3/2} \omega (x) = 0.
$$

(2.24)

**Proof.** Since $f$ is radial, by (2.7), we have

$$
f_{0,1} (r) = f (r), \quad \text{and} \quad f_{k,j} (r) \equiv 0, \quad \text{for all} \quad k \geq 1 \quad \text{and} \quad 1 \leq j \leq \alpha_k.
$$

Therefore, (2.22) follows from (2.16). The identity (2.23) follows from elementary calculations and (2.22) by taking $f (r) = \omega^2 (r)$. Moreover, by inserting (2.23) into (1.4) with $\lambda = 1$ and $z = 0$, we obtain that (2.24) holds. □

The next theorem concerns the decomposition of the operator $L$ defined by (1.6) in terms of spherical harmonics.

**Theorem 2.6.** Let $L$ be defined by (1.6). For any $f \in L^2 (\mathbb{R}^6)$, we have,

$$
Lf (x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} \mathcal{L}_k f_{k,j} (|x|) Y_{k,j} \left( \frac{x}{|x|} \right),
$$

(2.25)
where \( f_{k,j} \) is defined by (2.7), and
\[
\mathcal{L}_k f (r) = - f''(r) - \frac{5}{r} f'(r) + \frac{k(k + 4)}{r^2} f(r) - 2\pi^{3/2} \omega(r) f(r) - 2\omega(r) \int_0^{+\infty} K_k(t,r) \omega(t) f(t) \, dt, \tag{2.26}
\]
with \( K_k \) defined by (2.17).

Proof. First, by (2.8) and (2.7), for any \( f \in L^2(\mathbb{R}^6) \), we have,
\[
f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} f_{k,j} (|x|) Y_{k,j} \left( \frac{x}{|x|} \right), \tag{2.27}
\]
where
\[
f_{k,j}(r) = \frac{1}{\pi} \int_{\mathbb{S}^5} f(r\xi) Y_{k,j}(\xi) \, d\sigma(\xi). \tag{2.28}
\]
Using the fact that,
\[
\Delta \left( f_{k,j} (|x|) Y_{k,j} \left( \frac{x}{|x|} \right) \right) = f_{k,j}'' (|x|) + \frac{5}{|x|} f_{k,j}' (|x|) + \frac{1}{|x|^2} \Delta_{\mathbb{S}^5} Y_{k,j} \left( \frac{x}{|x|} \right),
\]
and (2.5), we obtain that,
\[
- \Delta f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} \left( - f_{k,j}'' (|x|) - \frac{5}{|x|} f_{k,j}' (|x|) + \frac{k(k + 4)}{|x|^2} f_{k,j}(|x|) \right) Y_{k,j} \left( \frac{x}{|x|} \right). \tag{2.29}
\]
Next, since \( \omega \) is radial (see (1.3)), by Corollary 2.5, we have,
\[
- \Phi(\omega^2)(x)f(x) = - \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} 2\pi^{3/2} \omega(|x|) f_{k,j}(|x|) Y_{k,j} \left( \frac{x}{|x|} \right). \tag{2.30}
\]
Now, by (2.4), we have,
\[
-2\Phi(\omega f)(x) \omega(x) = -2 \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} \int_0^{+\infty} K_k(r,|x|) (\omega f)_{k,j}(r) \, dr Y_{k,j} \left( \frac{x}{|x|} \right) \omega(|x|). \tag{2.31}
\]
Since \( \omega \) is radial, it follows from (2.7) that,
\[(\omega f)_{k,j}(r) = \omega(r) f_{k,j}(r),\]
which, together with (2.31), implies that,
\[
-2\Phi(\omega f)(x) \omega(x) = -2 \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} \int_0^{+\infty} K_k(r,|x|) \omega(r) f_{k,j}(r) \, dr Y_{k,j} \left( \frac{x}{|x|} \right) \omega(|x|). \tag{2.32}
\]
Combining (2.29), (2.30) and (2.32), we obtain that (2.25) holds with \( \mathcal{L}_k \) defined by (2.26).

Lemma 2.7. Let \( \mathcal{L}_k \) be defined by (2.26). For any \( f \geq 0 \) with \( f \neq 0 \), we have,
\[
\langle \mathcal{L}_k f, f \rangle > \langle \mathcal{L}_{k-1} f, f \rangle. \tag{2.33}
\]
\textbf{Proposition 3.1.} Let $\mathcal{L}_0$ be defined by (2.26) with $k = 0$. If $\varphi \in L^2 \left( (0, +\infty), r^5 \right)$ with $\varphi \neq 0$ satisfies $\mathcal{L}_0 \varphi = 0$, then there exists $\eta \in \mathbb{R}$, such that

\begin{equation}
\varphi \left( r \right) = \eta \Lambda \omega \left( r \right), \tag{3.1}
\end{equation}

where $\Lambda \omega$ is defined by (1.5).

\textit{Proof.} First of all, for any $\varphi \in L^2 \left( (0, +\infty), r^5 \right)$ satisfies $\mathcal{L}_0 \varphi = 0$, a bootstrap argument implies that $\varphi$ is smooth and $\varphi' \left( 0 \right) = 0$.

Now, we claim that,

\begin{equation}
\mathcal{L}_0 \Lambda \omega \left( r \right) = 0. \tag{3.2}
\end{equation}
Indeed, for all $\lambda > 0$, $\omega_\lambda (x) := \lambda^2 \omega(\lambda x)$ satisfy (1.1), i.e.

$$- \Delta \omega_\lambda (x) - \int_{\mathbb{R}^6} \frac{|\omega_\lambda (y)|^2}{|x-y|^3} \, dy \omega_\lambda (x) = 0, \quad x \in \mathbb{R}^6,$$

by differentiation equation (3.3) with respect to $\lambda$, we obtain that,

$$L \omega_\lambda (x) = 0, \quad x \in \mathbb{R}^6,$$

where $L$ is defined by (1.6). Moreover, since $\Lambda \omega(x)$ is radial, it follows from (2.7) that,

$$(\Lambda \omega)_{0,1}(r) = \Lambda \omega (r), \quad \text{and} \quad (\Lambda \omega)_{k,j}(r) = 0, \quad \text{for all } k \geq 1 \text{ and } 1 \leq j \leq \alpha_k,$$

which, together with Theorem 2.6 implies that (3.2) holds.

Next, let us rewrite $\mathcal{L}_0 \varphi (r)$ as

$$\mathcal{L}_0 \varphi (r) = \mathcal{L}_0 \varphi (r) - 2\pi^3 \omega (r) \int_0^{+\infty} \varphi (t) \omega (t) \, dt,$$

where

$$\mathcal{L}_0 \varphi (r) = - \varphi'' (r) - \frac{5}{r} \varphi' (r) - 2\pi^3 \omega (r) \varphi (r) - 2\pi^3 \int_0^{r} \left( \frac{t^4}{r^4} - 1 \right) \varphi (t) \omega (t) \, dt.$$

Therefore, for any $\varphi \in L^2 ((0, +\infty), r^5)$ with $\varphi \neq 0$ satisfying $\mathcal{L}_0 \varphi = 0$, by (3.4) and (3.5), we have,

$$\mathcal{L}_0 \varphi (r) = 2\pi^3 \omega (r) \int_0^{+\infty} \varphi (t) \omega (t) \, dt,$$

and specially,

$$\mathcal{L}_0 \Lambda \omega (r) = 2\pi^3 \omega (r) \int_0^{+\infty} \Lambda \omega (t) \omega (t) \, dt.$$

By setting

$$\phi (r) = \varphi (r) - \frac{\int_0^{+\infty} \varphi (t) \omega (t) \, dt}{\int_0^{+\infty} \Lambda \omega (t) \omega (t) \, dt} \Lambda \omega (r),$$

we have $\phi \in L^2 ((0, +\infty), r^5)$ is smooth and $\phi'(0) = 0$, Moreover, an elementary calculation implies that,

$$\mathcal{L}_0 \phi (r) = 0.$$

Now, by Proposition A.1, we have

$$\phi (r) = 0 \quad \text{for all} \quad r \in [0, +\infty),$$

which implies that (3.1) holds with $\eta = \frac{\int_0^{+\infty} \varphi(t)\omega(t) \, dt}{\int_0^{+\infty} \Lambda \omega(t) \omega(t) \, dt}$. This ends the proof of Proposition 3.1.

\[\square\]

**Proposition 3.2.** Let $\mathcal{L}_k$ be defined by (2.26). For each $k \geq 1$, the operator $\mathcal{L}_k$ is bounded below and essentially self-adjoint on $C_0^\infty (0, +\infty) \subset L^2 ((0, +\infty), r^5 \, dr)$. Moreover, for each $k \geq 1$, the operator $\mathcal{L}_k$ enjoys the Perron-Frobenius property, i.e. if

$$\lambda_k^0 = \inf \{ \langle \mathcal{L}_k f, f \rangle \mid \int_0^{+\infty} |f (r)|^2 \, r^5 \, dr = 1 \}$$
is attained, then the lowest eigenvalue $\lambda^0_k$ of the operator $L_k$ is simple, and the corresponding eigenfunction $\chi^0_k(r)$ does not change sign on $(0, +\infty)$.

**Proof.** The proof of Proposition 3.2 is almost identical to that of [11, Lemma 7], therefore we omit the details. \(\square\)

**Proposition 3.3.** Let $L_1$ be defined by (2.26) with $k = 1$. If $\psi \in L^2 (\mathbb{R}^n, r^5)$ with $\psi \not\equiv 0$ satisfies $L_1 \psi = 0$, then there exists $\beta \in \mathbb{R}$, such that
\[
\psi(r) = \beta \omega'(r). 
\] (3.8)

Moreover,
\[
\lambda^0_1 = \inf \{ \langle L_1 f, f \rangle \mid \int_0^{+\infty} |f(r)|^2 r^5 dr = 1 \} = 0.
\] (3.9)

**Proof.** First, by differentiation (1.4) with respect to $z$, and taking $\lambda = 1$, $z = 0$, we obtain,
\[
L \frac{\partial \omega}{\partial x_j}(x) = 0, \quad 1 \leq j \leq 6.
\] (3.10)

Since $\omega$ is radial, by (2.4), we have,
\[
\frac{\partial \omega}{\partial x_j}(x) = 6 \omega'(|x|) Y_{1,j} \left( \frac{x_j}{|x|} \right), \quad 1 \leq j \leq 6,
\] (3.11)

which, together with (2.7) implies that,
\[
\omega'_{k,j}(r) = \begin{cases} 
0, & k = 0, j = 1, \\
\omega'(r), & k = 1, 1 \leq j \leq 6, \\
0, & k \geq 2, 1 \leq j \leq 6.
\end{cases}
\] (3.12)

By Theorem 2.6, using (3.10), (3.11) and (3.12), we get,
\[
L_1 \omega'(r) = 0,
\] which implies that, 0 is an eigenvalue of the operator $L_1$ with the eigenfunction $\omega'(r)$. Moreover, since $\omega'(r) < 0$ for all $r \in (0, +\infty)$, by Proposition 3.2,
\[
0 = \inf \{ \langle L_1 f, f \rangle \mid \int_0^{+\infty} |f(r)|^2 r^5 dr = 1 \}
\]
is the lowest eigenvalue of the operator $L_1$, and any function $\psi \in L^2 (\mathbb{R}^n, r^5)$ satisfying $L_1 \psi (r) = 0$ must belongs to the set $\{ \beta \omega' \mid \beta \in \mathbb{R} \}$. This ends the proof of Proposition 3.3. \(\square\)

**Proposition 3.4.** Let $k \geq 2$ and $L_k$ be defined by (2.26). For any $\varphi \in L^2 (\mathbb{R}^n, r^5)$ satisfying $L_1 \varphi = 0$, we have $\varphi \equiv 0$.

**Proof.** We argue by contradiction. For any $\varphi \in L^2 (\mathbb{R}^n, r^5)$, by letting
\[
f_{\rho,k}(x) = \varphi(|x|) Y_{k,1} \left( \frac{x}{|x|} \right),
\]
we have
\[
f_{\rho,k} \in L^2 (\mathbb{R}^n), \quad \text{and} \quad \langle f_{\rho,k}, \omega^2 \rangle = 0,
\]
which together with Lemma 2.8, implies that,
\[
\langle L_1 \rho, \rho \rangle = \langle L f_{\rho,k}, f_{\rho,k} \rangle \geq 0,
\]
therefore,

\[ \lambda^0 = \inf \{ (L_k \rho, \rho) : \int_0^{+\infty} |\rho(r)|^2 r^5 dr = 1 \} \geq 0. \]

If \( \lambda^0 = 0 \) is attained or \( \lambda^0 > 0 \), then for any \( \rho \in L^2((0, +\infty), r^5) \) with \( \rho \neq 0 \), we have \( (L_k \rho, \rho) > 0 \), which contradicts with \( L_k \rho = 0 \).

If \( \lambda^0 = 0 \) is attained, then by Proposition 3.2, the lowest eigenvalue 0 of the operator \( L_k \) is simple, and the corresponding eigenfunction \( \varphi \) with \( \int_0^{+\infty} |\rho(r)| r^5 dr = 1 \) does not change sign on \((0, +\infty)\). Without loss of generality, we may assume that \( \varphi(r) > 0 \) for all \( r \in (0, +\infty) \). By (2.7), we obtain that,

\[ (L_1 \varphi, \varphi) < (L_k \varphi, \varphi) = 0, \]

which contradicts with (3.9). This ends the proof of Proposition 3.4. \( \square \)

**Proof of Theorem 1.1.** First, by (2.7), (2.8), and Theorem 2.6, we have,

\[ f \in L^2(\mathbb{R}^6) \text{ satisfies } Lf = 0, \]

if and only if

\[ L_k f_{k,j}(r) = 0, \quad \text{for all } k \geq 0 \text{ and } 1 \leq j \leq \alpha_k, \]

where \( f_{k,j} \in L^2((0, +\infty), r^5) \) is defined by (2.7), and

\[ f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\alpha_k} f_{k,j}(|x|) Y_{k,j} \left( \frac{x}{|x|} \right). \]

Next, by Proposition 3.1, Proposition 3.3 and Proposition 3.4, there exist real numbers \( \eta \) and \( \beta_j, (1 \leq j \leq 6) \) such that,

\[ f(x) = \eta \Lambda \omega(x) + \sum_{j=1}^{6} \beta_j \omega'(|x|) \frac{x_j}{|x|}, \]

i.e.

\[ f \in \text{span} \left\{ \Lambda \omega, \frac{\partial \omega}{\partial x_1}, \frac{\partial \omega}{\partial x_2}, \frac{\partial \omega}{\partial x_3}, \frac{\partial \omega}{\partial x_4}, \frac{\partial \omega}{\partial x_5}, \frac{\partial \omega}{\partial x_6} \right\}. \]

This ends the proof of Theorem 1.1. \( \square \)

**Appendix A. Properties of the Linear Operator \( \mathfrak{L}_0 \)**

**Proposition A.1.** Let \( \mathfrak{L}_0 \) be define by (3.5). If \( \phi \in C^2[0, +\infty) \) with \( \phi(0) \neq 0 \) and \( \phi'(0) = 0 \) satisfies \( \mathfrak{L}_0 \phi(r) = 0 \), then \( \phi(r) \) does not change sign and

\[ |\phi(r)| > \frac{|\phi(0)|}{4}, \quad \text{for all } r \in [0, +\infty). \]  \( \text{(A.1)} \)

**Proof.** First, by setting

\[ \tilde{\varphi} = \frac{2\omega(0)}{\phi(0)} \varphi, \quad \text{and} \quad \lambda(r) = \frac{\tilde{\varphi}(r)}{\omega(r)}. \]  \( \text{(A.2)} \)

we obtain that \( \tilde{\varphi} \) satisfies the linear equation \( \mathfrak{L}_0 \tilde{\varphi}(r) = 0 \) as well. Moreover, since \( \phi \in C^2[0, +\infty) \) and \( \omega \in C^\infty[0, +\infty) \), by (A.2), we have \( \lambda \in C^2[0, +\infty) \) with \( \lambda(0) > 1 \). Let us define

\[ r^* = \sup \{ r > 0 : \lambda(t) > 1, \quad \text{for all } t \in [0, r] \}. \]
By the definition of $r^*$, it is obvious that,
\begin{equation}
\lambda' (r^*) \leq 0. \tag{A.3}
\end{equation}
We claim that
\begin{equation}
r^* = +\infty. \tag{A.4}
\end{equation}
For this, we argue by contradiction, assuming that $r^* < +\infty$ and obtaining a contradiction with the definition of $r^*$. By noting that $\omega$ satisfies
\begin{equation}
-\omega''(r) - \frac{5}{r} \omega'(r) - 2 \pi^2 \omega^2(r) = 0,
\end{equation}
we have,
\begin{equation}
\mathcal{L}_0 \omega(r) = -2 \pi^3 \omega(r) \int_0^r \left( \frac{t^4}{r^4} - 1 \right) \omega(t) \omega(t) \, dt. \tag{A.5}
\end{equation}
By combining $\mathcal{L}_0 \tilde{\phi} (r) = 0$ with (A.5), we obtain that,
\begin{align*}
2 \pi^3 \tilde{\phi} (r) & \omega (r) \int_0^r \left( \frac{t^4}{r^4} - 1 \right) \omega(t) \omega(t) \, dt \\
= & -\omega (r) \tilde{\phi}'' (r) - \frac{5}{r} \omega (r) \tilde{\phi}' (r) + \tilde{\phi} (r) \omega'' (r) + \frac{5}{r} \tilde{\phi} (r) \omega' (r) \\
& - 2 \pi^3 \omega^2 (r) \int_0^r \left( \frac{t^4}{r^4} - 1 \right) \tilde{\phi} (t) \omega(t) \, dt + 2 \pi^3 \tilde{\phi} (r) \omega (r) \int_0^r \left( \frac{t^4}{r^4} - 1 \right) \omega^2(t) \, dt.
\end{align*}
Moreover, it is elementary to check that,
\begin{equation}
[r^5 (\tilde{\phi}' (r) \omega (r) - \omega' (r) \tilde{\phi} (r))]' = -2 \pi^3 r^5 \omega^2 (r) \int_0^r \left( \frac{t^4}{r^4} - 1 \right) \tilde{\phi} (t) \omega(t) \, dt, \tag{A.6}
\end{equation}
which implies that,
\begin{equation}
[r^5 \omega^2 (r) \lambda' (r)]' = 2 \pi^3 r^5 \omega^2 (r) \int_0^r (r^4 - t^4) \tilde{\phi} (t) \omega(t) \, dt. \tag{A.7}
\end{equation}
By integrating (A.7) from 0 to $r^*$, we obtain that,
\begin{align*}
\lambda' (r^*) &= \frac{2 \pi^3}{r^* r^5 \omega^2 (r^*)} \int_0^{r^*} s \omega^2 (s) \int_0^s (s^4 - t^4) \tilde{\phi} (t) \omega(t) \, dt \, ds \\
&> \frac{2 \pi^3}{r^* r^5 \omega^2 (r^*)} \int_0^{r^*} s \omega^2 (s) \int_0^s (s^4 - t^4) \omega^2 (t) \, dt \, ds.
\end{align*}
Using the fact $\omega (r) > 0$, we have, $\lambda' (r^*) > 0$, which contradicts with (A.3). Therefore (A.4) holds.

Now, by integrating (A.7) from 0 to $r$ and an elementary calculation, we obtain that,
\begin{equation}
\lambda (r) - \lambda (0) > \frac{9}{5} r^4 + \frac{12}{5} r^2 - \frac{12}{5} \log (1 + r^2),
\end{equation}
which, together with $\lambda (0) > 1$, implies that, $\lambda (r) > r^4 + 1 \geq \frac{1}{2} (1 + r^2)^2$. By (1.3) and (A.2), we obtain that $\frac{\phi (0)}{\tilde{\phi} (0)} > \frac{1}{2}$. Therefore, (A.1) holds.

As a consequence of Proposition A.1, we have the following corollary.

**Corollary A.2.** Let $\mathcal{L}_0$ be define by (3.5). If $\phi \in L^2 ((0, +\infty), r^5) \cap C^2 (0, +\infty)$ satisfies $\mathcal{L}_0 \phi (r) = 0$ with $\phi' (0) = 0$, then $\phi (r) = 0$ for all $r \in [0, +\infty)$. 
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