INTEGRATION ON NON-COMPACT SUPERMANIFOLDS

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Abstract. We investigate the Berezin integral of non-compactly supported quantities. In the framework of supermanifolds with corners, we give a general, explicit and coordinate-free representation of the boundary terms introduced by an arbitrary change of variables. As a corollary, a general Stokes’s theorem is derived—here, the boundary integral contains transversal derivatives of arbitrarily high order.

Keywords: Berezin integral, Stokes’s theorem, change of variables, non-compact supermanifold, boundary term, manifold with corners.

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1. INTRODUCTION

Supermanifolds were introduced by Berezin, Leites and Konstant in the 1970s as a mathematical framework for the quantum theory of commuting and anticommuting fields. A remarkable contribution was Berezin’s definition of his integral, in Ref. [Ber66], predating the definition of supermanifolds by several years, and providing at the time sufficient indication that a reasonable supersymmetric analysis should exist.

Despite its utility, the integral suffers from a fundamental pathology: Only the integral of compactly supported quantities is well-defined in a coordinate independent form—changes of variables introduce, in general, so-called boundary terms. This can be seen as a major obstacle in the development of global superanalysis.

For example, although Stokes’s theorem

\[ \int_M d\omega = \int_{\partial M} \omega \]

has been extended to supermanifolds by Bernstein and Leites [BL77], this extension supposes that the supermanifold structure on the boundary \( \partial M \) enjoys a rather strong compatibility requirement. In fact, even for compactly supported integrands \( \omega \), the conclusion of the theorem fails in general, unless this assumption is made (cf. Example 3.9 below).

An invariant definition of the integral can however be made, on the basis of the following simple observation: For any supermanifold \( M \), there exist morphisms \( \gamma: M \to M_0 \)—which we call retractions—which are left inverse to the canonical embedding \( j_M: M_0 \to M \). Any retraction \( \gamma \) is a submersion whose fibres have compact base; thus, there is a well-defined fibre integral \( \gamma_! \) which takes Berezin

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forms on $M$ to volume forms on $M_0$, and one may define
\begin{equation}
\int_{(M,\gamma)} \omega = \int_{M_0} \gamma(\omega) .
\end{equation}

Taking pullback retractions, this definition is now trivially well-defined under coordinate changes. Furthermore, whereas retractions are non-unique in general, for certain classes of supermanifolds—e.g., Lie supergroups $G$, homogeneous $G$-supermanifolds, and superdomains—there exist canonical retractions.

This framework allows us to give an explicit description of the behaviour of the integral under coordinate changes. To state our main result (Theorem 5.15), let $N \subset M^{p|q}$ be an open subspace of a supermanifold whose underlying space $N_0 \subset M_0$ is a manifold with corners. That is, we have $N_0 = \{ \rho_i > 0 \mid i = 1, \ldots, n \}$ for some functions $\rho_i$ which define boundary manifolds $H_0 = \{ \rho_{i_1} = \cdots = \rho_{i_k} = 0, \rho_j > 0 \ (j \neq i_m) \}$. Let $\gamma, \gamma'$ be retractions on $N$. On each $H_0$, one considers the supermanifold structure $H$ induced by $\gamma^*(\rho_{i_m})$ and the retraction $\gamma_H$ induced by $\gamma$. Let $D_j$ be even vector fields such that $D_j(\gamma^*(\rho_j)) = \delta_{ij}$ on suitable neighbourhoods of $\{ \gamma^*(\rho_i) = \gamma^*(\rho_j) = 0 \}$.

Then, for any Berezin density $\omega$ such that the integrals exist,
\begin{equation}
\int_{(N,\gamma')} \omega = \int_{(N,\gamma)} \omega + \sum_{H \in B(\gamma^*(\rho))} \sum_{j \in J_H} \pm \int_{(H,\gamma_H)} (\omega_j, D_j) \big|_{H,\gamma^*(\rho)} .
\end{equation}

Here, we sum over all $H = \{ \gamma^*(\rho_{i_1}) = \cdots = \gamma^*(\rho_{i_k}) = 0 \}$ and all multi-indices $j \in J_H = \mathbb{N}^{1, \ldots, k}$; moreover, $\omega_j \coloneqq \frac{1}{j!} (\gamma^*(\rho) - \gamma^*(\rho))^j \omega$ and $j\downarrow$ denotes the multi-index $j$ with entries reduced by one. The differential operators on the right hand side are of degree up to $\frac{2}{2}$.

From this change of variables formula, we deduce a version of Stokes’s theorem which is valid for an arbitrary supermanifold structure on the boundary (Corollary 5.20). Compared to Equation (1.1), the right hand side depends not only on $\omega|_{\partial M}$, but on transversal derivatives up to order $\frac{2}{2}$.

The question of defining the integral of non-compactly supported Berezinians was first studied by Rothstein [Rot87] in his seminal paper. His fundamental insight was that the integral becomes well-defined if instead of the Berezinian sheaf, one considers the sheaf of super-differential operators with values in volume forms. This insight is vital—indeed, Rothstein’s techniques form the basis of our investigations, and one may view Equation (1.2) as an attempt to translate Rothstein’s definition of the Berezin integral via the ‘Fermi integral’ to the realm of ordinary Berezinians.

For applications to superanalysis, Rothstein’s sheaf is somewhat unwieldy, since it is an $\mathcal{O}_M$-module of infinite rank. For example, in the context of homogeneous supermanifolds, one frequently fixes integrands by invariance. Of course, this can only be done for $\mathcal{O}_M$-modules of rank one, which favours the Berezinian sheaf as a tool for superanalysis.

The applications we have in mind come from the spherical harmonic analysis on Riemannian symmetric supermanifolds, in particular, the study of orbital and Eisenstein integrals in the spirit of Harish-Chandra. Besides its relation to representation theory [All10], this subject is of high current interest in mathematical physics, in the study of $\sigma$-model approximations of invariant random matrix ensembles, as are applied to disordered metals and topological insulators [Zir91, HHZ05, LSZ08, DSZ10, GLMZ11].
Let us end with a brief synopsis of our paper. In Section 2, we recall some basic facts and define the integral of Berezin densities with respect to a retraction. In Section 3, we prove a version of Stokes’s theorem in this setting (Theorem 3.8). Here, the supermanifold structure on the boundary has to be chosen compatibly (see below). In Section 4, we prove a version of our change of variables formula in terms of coordinates (Theorem 4.4). Here, the ‘boundary’ nature of the ‘boundary terms’ is not yet evident. This is finally accomplished in Section 5, where the language and technique of supermanifolds with corners and boundary supermanifolds is introduced; here, the point of view of retractions proves particularly fruitful. By applying this machinery, we prove our main result (Theorem 5.15) and illustrate its use in some examples. Finally, we deduce a generalised Stokes’s theorem (Corollary 5.20) where the supermanifold structure on the boundary is arbitrary.

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2. The Berezin integral in the non-compact case

We use the standard definition of supermanifolds in terms of ringed spaces. For basic facts on these, we refer the reader to [Lei80, CdG94]. Let us fix our notation. Given an object in the graded category, we will denote the underlying ungraded object by a subscript 0. We denote supermanifolds as \( M = (M_0, O_M) \), \( N = (N_0, O_N) \), etc. Unless the contrary is stated explicitly, we will assume \( M, N \) to be of dimension \( (p, q) \). Manifolds will always be Hausdorff and second countable. By writing \( U \subseteq M \) we will mean that \( U \) is the ringed subspace \( M|_{U_0} := (U_0, O_M|_{U_0}) \) of \( M \) given by the open subset \( U_0 \subseteq M_0 \). Thus, unions and finite intersections of open subspaces are defined. Further, the set of superfunctions \( O_M(U_0) \) on \( U \) is abbreviated by \( O(U) \). Morphisms of supermanifolds \( M \to N \) are denoted \( \varphi = (\varphi_0, \varphi^*) \), with underlying smooth map \( \varphi_0 : M_0 \to N_0 \) and the sheaf morphism \( \varphi^* : O_N \to \varphi_0^* O_M \). For a given supermanifold \( M \) we denote the canonical embedding by \( j_M : M_0 \to M \). Given \( f \in O(U) \), we write \( f_0 \) for \( j_M^*(f) \).

Now we introduce a certain type of morphisms which will be central for the following developments.

**Definition 2.1.** A morphism \( \gamma : M \to M_0 \) is called a *retraction* if it is a right inverse of the canonical embedding \( j_M \), i.e.

\[
\gamma \circ j_M = \text{id}_{M_0}.
\]

**Remarks 2.2.** In the literature, the subalgebra \( \text{Im} \gamma^* \subseteq O(M) \) is called a *function factor*.

It is a known fact that retractions always exist on (real) supermanifolds [RS83, Lemma 3.2]. However, they are in general not unique. For superdomains there exists a canonical choice of retraction. Using exponential charts, one may also give canonical retractions in the case of Lie supergroups; this can also be extended to the case of homogeneous supermanifolds.

We will repeatedly use the following standard fact [Lei80, Theorem 2.1.7].
Proposition 2.3. Let $M, N$ be supermanifolds, $y = (v_1, \ldots, v_p, \eta_1, \ldots, \eta_q)$ a coordinate system on $N$, and $x = (u_1, \ldots, u_p, \xi_1, \ldots, \xi_q)$ a family of superfunctions on $M$ where the $u_i$ are even and the $\xi_j$ are odd. Then there exists a unique morphism $\varphi: M \to N$ such that $\varphi^*(y) = x$, if and only if the function $(u_{1,0}, \ldots, u_{p,0})$ takes its values in $v_0(N_0) = \{ (v_{1,0}(a), \ldots, v_{p,0}(a)) \mid a \in N_0 \}$.

Definition 2.4. If $\gamma$ is a retraction and $u_0 = (u_{1,0}, \ldots, u_{p,0})$ is a classical coordinate system, then $\gamma^*(u_0)$ is the even part of a coordinate system.

Conversely, if $s = (u, \xi)$ is a coordinate system, then there is a unique retraction $\gamma$ such that $\gamma^*(u_0) = u_0$ by the above proposition. We call this the retraction associated with $u$ (or $x$).

Let $x = (u, \xi)$ be a coordinate system and $\gamma$ be the retraction associated with $u$. Any superfunction $f$ possesses a unique decomposition

\begin{equation}
 f = \sum_{\nu \in \mathbb{Z}_2^p} \gamma^*(f_\nu) \xi^\nu, \quad f_\nu \in C^\infty,
\end{equation}

where $\xi^\nu = \xi_1^{\nu_1} \cdots \xi_q^{\nu_q}$. Observe that in the literature, one commonly writes this expansion in terms of functions $g_\nu(u)$, where $g_\nu$ are functions on the range of the chart associated with $u_0$. We note further that $j_M^*(f) = f_{(0,\ldots,0)}$, which explains the abbreviation $f_0$ for $j_M^*(f)$.

Using decomposition (2.1) we define derivations along the coordinates,

\begin{align*}
 \frac{\partial f}{\partial x_i} &:= \frac{\partial f}{\partial u_i} := \sum_\nu \gamma^*(\frac{\partial f_\nu}{\partial u_i}) \xi^\nu, & i = 1, \ldots, p, \\
 \frac{\partial f}{\partial x_{p+j}} &:= \frac{\partial f}{\partial \xi_j} := \sum_\nu \gamma^*(f_\nu) \xi^{\nu-j} (-1)^{\nu_1+\cdots+j-1}, & j = 1, \ldots, q.
\end{align*}

We abbreviate $\partial_{x_i} := \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, p + q$ and $\partial^i := \partial^i_{p+q} \circ \cdots \circ \partial^i_1$ for $i \in \mathbb{N}_0^p \times \mathbb{Z}_2^q$. Corresponding abbreviations for $u$ and $\xi$ are similarly defined.

Definition 2.5. Let $M$ be a supermanifold and $(U, x)$ a local coordinate system. A Berezin form $\omega$ on $U$ is an object of the form

$$
\omega = f \, Dx = (-1)^{|f|} f^{(b,p,q)} \, Dx \, f,
$$

where $f$ is a superfunction on $U$. We make here no choice for the parity $|Dx| := b(p,q)$; a common one is $b(p,q) = p + q$. If $y = (v, \eta)$ is another coordinate system on $U$, then one requires

$$
\omega = f \, Dx = f \frac{Dx}{Dy} \, Dy.
$$

Here, the Berezinian of the coordinate change is given by

$$
\frac{Dx}{Dy} := \text{Ber} \left( \frac{\partial x}{\partial y} \right) = \text{Ber} \left( \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial \eta} \right),
$$

where

$$
\text{Ber} \left( \begin{pmatrix} R & S \\ T & V \end{pmatrix} \right) := \det(R - SV^{-1}T) \det V^{-1}.
$$
The correspondence $U_0 \mapsto \text{Ber} U$ extends to an $\mathcal{O}_M$-module sheaf $\text{Ber}_M$ on $M_0$, as is well-known \cite{Man97, Che94, AH10}. The $\mathcal{O}(M)$-module of Berezin forms on $M$ is denoted by $\text{Ber}_M$.

One defines Berezin densities similarly, replacing the character Ber by $|\text{Ber}| = \text{sgn} f^*_{M} \cdot \text{Ber} \left( \begin{array}{cc} R & S \\ T & V \end{array} \right)$.

Thus, Berezin densities have the local form $f |Dx| = (-1)^{s(p,q)} f_{(1,\ldots,1)} |du_0|$, whenever the right hand side exists. Here, $|du_0|$ is the pullback of the standard Lebesgue density on $\mathbb{R}^p$ under $u_0$, and $f_{(1,\ldots,1)}$ is the top degree coefficient in Equation (2.1), where $\gamma$ is associated with $u$.

There is no uniform choice for the number $s(p,q) \in \mathbb{Z}_2$ in the literature. Customary are $s(p,q) = pq + \frac{q(q-1)}{2}$ or $s(p,q) = \frac{q(q-1)}{2}$. The definition of the integral of a Berezin form is similar.

We have the following classical theorem \cite[Theorem 2.4.5]{Lei80}.

**Theorem 2.7.** Let $U$ be a coordinate neighbourhood and $\omega$ be a Berezin density which is compactly supported on $U$. Then

$$\int_{(U,x)} \omega = \int_{(U,y)} \omega,$$

if $x = (u, \xi)$ and $y = (v, \eta)$ are coordinate systems on $U$. The same is true for Berezin forms if $u_0$ and $v_0$ are equally oriented.

As is well-known, the assumption of compact supports cannot be removed in the above theorem; the following classical counterexample is referred to as Rudakov’s example in the literature.
Example 2.8. Let $\Omega \subseteq \mathbb{R}^{1,2}$ be the superdomain with $\Omega_0 = [0,1]$. Let $x = (u, \xi_1, \xi_2)$ be a coordinate system on $\Omega$ with $u_0 = \text{id}_{\Omega_0}$. Let $y = (v, \eta_1, \eta_2)$ be the coordinate system given by $v = u + \xi_1 \xi_2$ and $\eta_i = \xi_i, \ i = 1, 2$. Set $\omega := vDy$.

We have

$$\frac{|Dy|}{|Dx|} = \text{Ber} \begin{pmatrix} \xi_2 & 0 & 0 \\ -\xi_1 & 0 & 1 \end{pmatrix} = 1,$$

hence $\omega = (u + \xi_1 \xi_2)Dx$. This leads to

$$\int_{(\Omega,x)} \omega = \pm 1 \neq 0 = \int_{(\Omega,y)} \omega.$$

However, Theorem 2.7 allows us to make the following observation.

Lemma 2.9. Let $\gamma$ be a retraction on $M$ and $\omega$ a Berezin density. Let $x = (u, \xi)$ and $y = (v, \eta)$ be coordinate systems on a coordinate neighbourhood $U$ with the same associated retraction $\gamma$. Let $\omega = f|Dx| = g|Dy|$ on $U$. Then

$$f_{(1,\ldots,1)}|du_0| = g_{(1,\ldots,1)}|dv_0|,$$

where $f_{(1,\ldots,1)}$ and $g_{(1,\ldots,1)}$ are the coefficients from Equation (2.1), applied to $f$ and $g$, respectively.

Proof. Choose a bump function $h \in C^\infty_c(U_0)$. Then by Theorem 2.7

$$\int_{U_0} hf_{(1,\ldots,1)}|du_0| = \pm \int_{(U,x)} \gamma^*(hf_{(1,\ldots,1)})\xi_1 \cdots \xi_q |Dx| = \pm \int_{(U,x)} \gamma^*(h)\omega$$

$$= \pm \int_{(U,y)} \gamma^*(h)\omega = \int_{U_0} hg_{(1,\ldots,1)}|dv_0|.$$

Since $h$ was arbitrary, this proves our claim.

Again, one can get the same result for Berezin forms. Thanks to this lemma, the following definition makes sense.

Definition 2.10. Let $\gamma$ be a retraction on the supermanifold $M$. We define the map $\gamma^! : |\text{Ber}|M \to |\Omega^p|M_0$ locally via

$$(\gamma^!(\omega))_U := (-1)^{s(p,q)} f_{(1,\ldots,1)}|du_0|,$$

where $\omega, U, x = (u, \xi)$ and $f_{(1,\ldots,1)}$ are as in Lemma 2.9.

Similar we define $\varphi^! : \text{Ber}M \to \Omega^pM_0$ via

$$\varphi^!(fDx) := (-1)^{s(p,q)} f_{(1,\ldots,1)}|du_0|.$$

In fact, $\varphi^!$ can be defined for any surjective submersion $\varphi$ [AH10]. Note that if one chooses $b(p,q) = p + q$ or $b(p,q) = q$ and fixes parity according to the sign rule, the morphism $\gamma^!$ becomes even.

One can easily check the following properties:

$$(2.3) \quad \gamma^!(\gamma^*(g)\omega) = g\gamma^!(\omega), \quad \text{supp} \gamma^!(\omega) \subseteq \text{supp} \omega$$

for any $g \in C^\infty(M_0)$ and $\omega \in |\text{Ber}|M$ (resp. $\omega \in \text{Ber}M$).
Definition 2.11. Let $\gamma$ be a retraction on $M$ and $\omega$ be a Berezin density on $M$. We call $\omega$ integrable with respect to $\gamma$ if $\gamma_!(\omega)$ is integrable on $M_0$ as density. In this case, we define
\[
\int_{(M, \gamma)} \omega := \int_{M_0} \gamma_! \omega.
\]
If $M_0$ is oriented, this definition can be extended to the case of Berezin forms.

On coordinate neighbourhoods $U$ this definition is compatible with the local definition, given in Definition 2.6:
\[
\int_{(U, x)} \omega = \int_{(U, \gamma)} \omega,
\]
where $\gamma$ is the retraction associated with $x$. In particular, the integral on the right hand side is the same for coordinate systems whose even parts induce the same retraction. Moreover, Theorem 2.7 generalises as follows.

Corollary 2.12. Let $\gamma, \gamma'$ be retractions on $M$ and $\omega$ be compactly supported on $M$. Then
\[
\int_{(M, \gamma)} \omega = \int_{(M, \gamma')} \omega.
\]

In this case, we will write $\int_M \omega$ for the integral.

Corollary 2.13. Let $\omega \in \{|\text{Ber}|M\}$ (resp. $\omega \in \text{Ber} M$) and $\gamma, \gamma'$ be retractions. The density (resp. volume form) $\gamma_!(\omega) - \gamma'_!(\omega)$ is exact.

Proof. If $\omega$ is compactly supported, then
\[
\int_{M_0} (\gamma_!(\omega) - \gamma'_!(\omega)) = \int_{(M, \gamma)} \omega - \int_{(M, \gamma')} \omega = 0,
\]
so $\gamma_!(\omega) - \gamma'_!(\omega)$ is exact.

In the general case, let $(\phi_\alpha) \subseteq \mathcal{O}(M)$ be a partition of unity with compact supports. By the above, $d\eta_\alpha = \gamma_!(\phi_\alpha \omega) - \gamma'_!(\phi_\alpha \omega)$ for some $\eta_\alpha$. One may assume the family of supports to be locally finite, so that $\eta = \sum_\alpha \eta_\alpha$ is well-defined, and one has $d\eta = \gamma_!(\omega) - \gamma'_!(\omega)$. \hfill \qed

Definition 2.14. Let $\varphi : M \to N$ be an isomorphism of supermanifolds.

(i) The pullback Berezin density $\varphi^\ast \omega$ of a Berezin density $\omega$ on $N$ is defined by writing $\omega|_U = f|Dx|$ on a coordinate neighbourhood $(U, x)$ on $N$ and setting
\[
(\varphi^\ast \omega)|_{\varphi^{-1}(U)} := \varphi^\ast(f)|D\varphi^\ast(x)|.
\]
Here, we observe that $\varphi^\ast(x) = (\varphi^\ast(x_1), \ldots, \varphi^\ast(x_{p+q}))$ is a coordinate system on $\varphi^{-1}(U) := M|_{\varphi^{-1}(U)}$.

This is well-defined, since
\[
\varphi^\ast \left( \frac{|Dx|}{|Dy|} \right) = \frac{|D\varphi^\ast(x)|}{|D\varphi^\ast(y)|}.
\]
The pullback of a Berezin form is defined analogously.
(ii) The pullback $\varphi^*\gamma$ of a retraction $\gamma$ on $N$ is defined by

$$\varphi^*\gamma := \varphi_0^{-1} \circ \gamma \circ \varphi : M_0 \to M.$$ 

**Corollary 2.15.** Let $\varphi : M \to N$ be an isomorphism of supermanifolds. Let $\gamma$ be a retraction on $N$ and let $\omega$ be a Berezin density or Berezin form on $N$ which is integrable with respect to $\gamma$. Then $\varphi^*\omega$ is integrable with respect to $\varphi^*\gamma$ and

$$\int_{(M,\varphi^*\gamma)} \varphi^*\omega = \int_{(N,\gamma)} \omega.$$ 

In the case of a Berezin form, $\varphi_0$ has in addition to be orientation preserving.

**Proof.** We only have to check that

$$(\varphi^*\gamma)_!(\varphi^*\omega) = \varphi_0^!(\gamma|\omega),$$

because then

$$\int_{(M,\varphi^*\gamma)} \varphi^*\omega = \int_{M_0} (\varphi^*\gamma)_!(\varphi^*\omega) = \int_{M_0} \varphi_0^!(\gamma|\omega) = \int_{N_0} \gamma|\omega = \int_{(N,\gamma)} \omega.$$ 

It suffices to check Equation (2.4) locally. So, we write $\omega = f|Dx$ and $f = \sum_{\nu} \gamma^*(f_{\nu})\xi^\nu$ for a coordinate system $x = (u, \xi)$ with which $\gamma$ is associated. Note that $\varphi^*\gamma$ is the retraction associated with $\varphi^*(x)$. We decompose $\varphi^*\omega$ with respect to this coordinate system:

$$\varphi^*\omega = \sum_{\nu} \varphi^*(\gamma^*(f_{\nu})\xi^\nu)|D\varphi^*(x)| = \sum_{\nu} (\varphi^*\gamma)^*(\varphi_0^!(f_{\nu}))\varphi^*(\xi)|D\varphi^*(x)|.$$ 

It follows that

$$(\varphi^*\gamma)_!(\varphi^*\omega) = (-1)^{(p,q)}\varphi_0^!(f_{(1,\ldots,1)})|d\varphi_0^*(u_0)| = \varphi_0^!(\gamma|\omega).$$

\[\square\]

### 3. Stokes's theorem

**Definition 3.1.** Recall [BL77, Man97] that the sheaf $\Sigma^k_M$ of integral forms of order $k \leq p$ is defined to be

$$\Sigma^k_M := \text{Ber}_M \otimes_{\mathcal{O}_M} S^{p-k}(\mathfrak{X}_M \Pi),$$

where $S^k(\mathfrak{X}_M \Pi)$ denotes the $k$-th supersymmetric power of the sheaf of parity changed super derivations. We will abbreviate $\Sigma^k_M(M_0)$ by $\Sigma^k M$. In the following we restrict to the case $k = p - 1$.

The *Cartan derivative* on $p - 1$ integral forms is given by

$$d : \Sigma^{p-1} M \to \Sigma^p M = \text{Ber}_M, \, \omega \otimes X \Pi \to (-1)^{|\omega||X\Pi|}\mathcal{L}_X \omega.$$ 

Here, $\mathcal{L}_X$ is the Lie derivative on Ber $M$, locally given by

$$\mathcal{L}_{\xi^i} (f \, dx) = (-1)^{|\xi^i||X\Pi|+|X|} \partial_{\xi^i} (gf) \, dx.$$ 

This does not depend on the chosen coordinate system [Lei80, Lemma 2.4.6].

For conceptual reasons we made here a choice for the sign which differs from Ref. [Man97]. (The sign there is given by $(-1)^{|\omega||X\Pi|+|X|}$.)
Remark 3.2. In the classical case \( M = M_0 \) integral forms and differential forms can be identified. For \( k = p - 1 \) this identification is given by

\[
\Psi : \Sigma^{p-1}M_0 \longrightarrow \Omega^{p-1}M_0, \quad \omega \otimes \mathcal{X} \longmapsto (-1)^{|\omega|} \iota_X \omega,
\]

where \( \iota_X \) is the contraction by \( X \). The definition of the Cartan derivative is compatible with this identification, as can be seen from

\[
(3.1) \quad d(\Psi(\omega \otimes \mathcal{X})) = (-1)^{|\omega|}d(\iota_X \omega) = (-1)^{|\omega|} \mathcal{L}_X \omega.
\]

Definition 3.3. Recall that a morphism \( \iota : N \to M \) is called an immersion in case the following is true: For each point \( o \in N_0 \) and some (any) coordinate system \( x = (x_1, \ldots, x_{p+q}) \) on a neighbourhood of \( \iota_0(o) \), there exists a coordinate neighbourhood \( U \) of \( o \), such that \((\iota^*(x_1), \ldots, \iota^*(x_i))\) is a coordinate system on \( U \) for certain \( i_1 < \cdots < i_k \).

Lemma 3.4. If \( \dim N = (p-k,q-l) \) and \( \dim M = (p,q) \), one can choose \( x = (u, \xi) \) such that \( \iota^*(u_i) = \iota^*(\xi_j) = 0 \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, l \).

For the remainder of this section we suppose \( N \) to be of dimension \( (p-1,q) \) and \( \iota : N \to M \) to be an immersion.

Definition 3.5. The pullback

\[
\iota^* : \Sigma^{p-1}M \longrightarrow \Sigma^{p-1}N = \text{Ber} \ N
\]

of integral forms of order \( p - 1 \) is defined as follows: For each point \( o \in M_0 \), choose a coordinate system \( x = (x_1, \dot{x}) \) at \( \iota_0(o) \) as in [Lemma 3.3] and set

\[
(3.2) \quad \iota^*(f \, Dx \otimes \partial_x, \Pi) := \begin{cases} (-1)^{|Dx|} \iota^*(f) D\iota^*(\dot{x}) & i = 1, \\ 0 & i \neq 1. \end{cases}
\]

Remark 3.6. [Definition 3.5] is compatible with the classical pullback via the identification \( \Psi \) from [Remark 3.2]. Let \( u_0 = (u_{1,0}, \ldots, u_{p,0}) \) be as in [Lemma 3.4] (i.e. \( \iota_0^*(u_{1,0}) = 0 \)). One computes

\[
\iota_0^*\left(\Psi(f \, du_0 \otimes \partial_{u_{i,0}}, \Pi)\right) = \iota_0^*\left((-1)^{p+i+1} f \, du_{1,0} \wedge \cdots \wedge \widehat{du_{i,0}} \wedge \cdots \wedge du_{p,0}\right) = \begin{cases} (-1)^p \iota_0^*(f) \, du_0^*(u_{2,0}) \wedge \cdots \wedge du_0^*(u_{p,0}) & i = 1, \\ 0 & i \neq 1. \end{cases}
\]

Proposition 3.7. The definition of the pullback at a certain point does not depend on the choice of the coordinate system and hence, the pullback of integral forms of order \( p - 1 \) is well-defined.

Proof. Let \( y = (y_1, \dot{y}) \) be another such coordinate system with \( \iota^*(y_1) = 0 \). We have to compute

\[
\iota^*(f \, Dy \otimes \partial_{y_i}, \Pi) = \iota^*\left(\sum_{j=1}^{p+q} (-1)^{|y_j|+|y_i|} \, D_{y_j} \frac{\partial}{\partial y_i} D_y \otimes \partial_{y_j}, \Pi\right)
\]

\[
= (-1)^{|y_i|+1} \iota^*\left(\frac{\partial}{\partial y_i} D_y \otimes \partial_{y_j}\right) D\iota^*(\dot{y}),
\]

\[
= (-1)^{|y_i|+1} \iota^*(f \, Dy \otimes \partial_{y_i}, \Pi).
\]
for $i = 1, \ldots, p + q$. For $i \geq 2$ we infer, using $\iota^*(y_1) = \iota^*(x_1) = 0$,
\[
0 = \frac{\partial \iota^*(x_1)}{\partial \iota^*(y_1)} = \sum_{j=1}^{p+q} \frac{\partial \iota^*(x_1)}{\partial \iota^*(y_j)} t^*(\frac{\partial x_1}{\partial y_j}) = \sum_{j=2}^{p+q} \delta_{ij} t^*(\frac{\partial x_1}{\partial y_j}) = t^*(\frac{\partial x_1}{\partial y_i}) .
\]
This implies $t^*(f D y \otimes \partial_{y_i} \Pi) = 0$ for $i \geq 2$. Now we examine
\[
t^*(\frac{Dx}{Dy}) = \text{Ber} \left( \begin{array}{ccc} t^*(\frac{\partial x_1}{\partial y_1}) & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ t^*(\frac{\partial x_1}{\partial y_{p+q}}) & \cdots & t^*(\frac{\partial x_1}{\partial y_{p+q}}) \end{array} \right) = \text{Ber} \left( \begin{array}{ccc} t^*(\frac{\partial x_1}{\partial y_1}) & \cdots & \cdots \\ 0 & t^*(\frac{\partial x_2}{\partial y_2}) & \cdots \\ \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots \end{array} \right). 
\]
Here, we have made use of
\[
(3.3) \quad \text{Ber} \left( \begin{array}{ccc} R_1 & * & * \\ 0 & R_2 & S \\ 0 & T & V \end{array} \right) = \det R_1 \text{Ber} \left( \begin{array}{ccc} R_2 & S \\ T & V \end{array} \right).
\]
We arrive at $\frac{D\iota^*(\tilde{y})}{D\iota^*(\tilde{x})} = t^*(\frac{\partial x_1}{\partial y_1}) t^*(\frac{Dx}{Dy})$ by inverting both sides of the above equation; hence
\[
t^*(f D y \otimes \partial_{y_i} \Pi) = (-1)^{|\bar{y}|} t^*(f) D \iota^*(\tilde{y}) . \quad \square
\]
For the formulation of Stokes’s theorem, we need to anticipate a later result [Proposition 5.9]. Let $U \subset M$ such that $U_0$ has smooth boundary $\partial U_0$ in $M_0$. Further, let $\gamma$ be a retraction on $M$.

Then there exists a unique supermanifold structure $\partial, U$ of dimension $(p - 1, q)$ on $\partial U_0$, together with an immersion $\iota: \partial, U \to M$ and a unique retraction $\partial \gamma$ on $\partial, U$ such that the following diagram commutes:
\[
\begin{array}{ccc}
\partial, U & \xrightarrow{\iota} & M \\
\partial \gamma \downarrow & & \downarrow \gamma \\
\partial U_0 & \xrightarrow{\iota_0} & M_0
\end{array}
\]

**Theorem 3.8** (Stokes’s theorem). Let $U \subset M$ such that $U_0$ is compact and has smooth boundary $\partial U_0$, and let $\gamma$ be a retraction on $M$. Let $M_0$ be oriented, and endow $\partial U_0$ with the usual boundary orientation. Then for $\bar{w} \in \Sigma^{p-1} M$ we have
\[
(3.5) \quad \int_{(U, \gamma)} d\bar{w} = (-1)^{s(p, q) + s(p-1, q) + q} \int_{\partial, U} t^*(\bar{w}),
\]
whenever the integral on the left hand side exists.

For the special choice $s(p, q) = pq + \frac{q(q-1)}{2}$, the sign in Stokes’s formula disappears. Therefore this choice might be reasonable in this context.

We make the following subtle point: The integral on the right hand side of Equation (3.5) does not depend on the boundary retraction $\partial \gamma$. However, one still has to take into account the boundary data $(\partial, U, i)$. In order to clarify this, we consider the following example.
Example 3.9. Let $M = \mathbb{R}^{1,4}$ and $U_0 = [0, \infty[$. Let $x = (u, \xi_1, \xi_2, \xi_3, \xi_4)$ be the standard coordinate system on $M$ ($u = u_0 = i\partial_{\lambda_0}$). We define another coordinate system $y = (v, \eta_1, \eta_2, \eta_3, \eta_4)$ by

$$v := u + \xi_1 \xi_2 + \xi_3 \xi_4, \quad \eta_j = \xi_j, \ j = 1, \ldots, 4.$$

Let $\gamma$ be the retraction associated with $y$, i.e. $\gamma^*(v_0) = v$. In this example, there is only one possible supermanifold structure of dimension $(0,4)$ on $\partial U_0 = 0$, namely $\partial U = \mathbb{R}^{0,4}$.

Now one might think that the immersion $\iota$ is just given by

$$\iota^*: C^\infty(M_0) \otimes \Lambda(\mathbb{R}^4)^* \to \Lambda(\mathbb{R}^4)^*, \sum f_\nu \xi^\nu \mapsto \sum f_\nu(0)\xi^\nu.$$

Let us examine where this leads to. Define $\varpi := \frac{1}{2}v^2 Dy \otimes \partial_v \Pi \in \Sigma^0 M$. We compute $d\varpi = \pm v Dy$, which implies that $\int_{(U, \gamma)} d\varpi = 0$.

Since $\iota^*(u) = 0$ we have to calculate $\varpi$ in the $x$-coordinates. We see $D_y u = 1$ and $\partial_v = \partial_u$, hence $\varpi = \left(\frac{1}{2}u^2 + u(\xi_1 \xi_2 + \xi_3 \xi_4) + \xi_1 \xi_2 \xi_3 \xi_4\right) \otimes \partial_u \Pi$. This means that $\iota^*(\varpi) = \pm \xi_1 \xi_2 \xi_3 \xi_4 D\xi$ and therefore

$$\int_{\partial U} \iota^*(\varpi) = \pm 1 \neq 0 = \pm \int_{(U, \gamma)} d\varpi.$$

The reason for this supposed contradiction is that with the chosen immersion $\iota$, Diagram (3.4) does not commute. The correct immersion is

$$\sum_\nu \gamma^*(f_\nu)(\xi^\nu) \mapsto \sum_\nu f_\nu(0)\xi^\nu.$$

Remark 3.10. Stokes’s theorem for supermanifolds was proved [BL77] for the case of domains with compact boundary. The domain of integration there is a closed superdomain which is characterised locally by an equation $u_1 \geq 0$. This corresponds to our choice of a retraction. The boundary of the closed superdomain is given locally by the equation $u_1 = 0$, similar to the unique structure on the boundary, which we get from diagram (3.4).

In [Man97], the theorem is stated as follows: One starts with a supermanifold structure on the boundary together with an immersion. It is remarked that the boundary is given locally by an equation $u_1 = 0$ (cf. Lemma 3.4), and $U$ by $u_1 > 0$. The conclusion as it is stated is correct only if the integral is evaluated by using a coordinate system which contains $u_1$. This means, that the integral of $U$ depends on the chosen immersion.

We feel that this formulation may easily be misunderstood, as in the above example, whereas the statement in terms of retractions might be more descriptive. As we shall see at the end of Section 5, Stokes’s theorem admits an extension to the case of an arbitrary immersion; however, in this case, additional terms will appear in the formula.

In the proof of Theorem 3.8 we need a generalisation of $\gamma_!$ to integral forms.

Definition 3.11. Define $\gamma_!: \Sigma^{p-1} M \to \Sigma^{p-1} M_0 = \Omega^{p-1} M_0$ locally via

$$\gamma_!(\omega \otimes \partial_x \Pi) := \begin{cases} \gamma_!(\omega) \otimes \partial_{a_i,0} \Pi & i \leq p, \\ 0 & i > p. \end{cases}$$
Here $x = (u, \xi)$ is a coordinate system with which $\gamma$ is associated. We check that the definition is independent of this choice. To that end, let $y = (v, \eta)$ be another coordinate system with $\gamma^*(\nu_0) = v$.

Then for $i = 1, \ldots, p$ we have $\partial v_i = \sum_{k=1}^p \gamma^*(\frac{\partial u_k,0}{\partial v_{i,0}}) \partial u_k$; hence

$$\gamma_i(\omega \otimes \partial v_i, \Pi) = \sum_k \gamma_i(\omega \gamma^*(\frac{\partial u_k,0}{\partial v_{i,0}}) \otimes \partial u_k \Pi)$$

$$= \sum_k \gamma_i(\omega) \frac{\partial u_k,0}{\partial v_{i,0}} \otimes \partial u_k,0 \Pi = \gamma_i(\omega) \otimes \partial v_{i,0} \Pi.$$ 

Similarly, we have $\gamma_j(\omega \otimes \partial \eta_j, \Pi) = 0$ for $j = 1, \ldots, q$.

**Proof of Theorem 3.8.** We only need to check the equations

(3.6) $$(\partial \gamma)_i(\iota^*(\varpi)) = (-1)^{b(p,q)+b(p,0)+s(p,q)+s(p-1,q)} \iota^*_0(\gamma_i(\varpi)),$$

(3.7) $\gamma_i(d\varpi) = (-1)^{b(p,q)+b(p,0)+q} d\gamma_i(\varpi).$

With these identities, we are able to apply the classical Stokes's theorem:

$$\int_{(\nu,\gamma)} d\varpi = \int_{U_0} \gamma_i(d\varpi) = \pm \int_{U_0} d\gamma_i(\varpi) = \pm \int_{\partial U_0} \iota^*_0(\gamma_i(\varpi))$$

$$= \pm \int_{\partial U_0} (\partial \gamma)_i(\iota^*(\varpi)) = (-1)^{s(p,q)+s(p-1,q)+q} \int_{(\partial, U, \partial \gamma)} \iota^*(\varpi).$$

The claim follows from Corollary 2.12.

Equations (3.6) and (3.7) can be checked locally. Let $u_0 = (u_{1,0}, \ldots, u_{p,0})$ be a coordinate system such that $\iota^*_0(u_{1,0}) = u_{1,0}|_{\partial U_0} = 0$ and set $u := \gamma^*(u_0)$. We supplement $u$ to a coordinate system $x = (u_1, \bar{x}) = (u, \xi)$.

Without loss of generality, we write $\varpi = f D x \otimes \partial x, \Pi$ and $f = \sum_\nu \gamma^*(f_\nu) \xi^\nu$. Noticing that $\partial \gamma$ is the retraction associated with $\iota^*(\bar{x})$, we get for $i = 1$:

$$(\partial \gamma)_i(\iota^*(\varpi)) = (-1)^{|Dx|}(\partial \gamma)_i(\sum_{\nu} \iota^*(\gamma^*(f_\nu)) \iota^*(\xi^\nu) D \iota^*(\bar{x}))$$

$$= (-1)^{|Dx|}(\partial \gamma)_i(\sum_{\nu} (\partial \gamma)^* (\iota^*_0(f_\nu)) \iota^*(\xi^\nu) D \iota^*(\bar{x}))$$

$$= (-1)^{|Dx|+s(p-1,q)} \iota^*_0(f_{(1,\ldots,1)}) d\iota^*_0(u_0)$$

$$= (-1)^{|Dx|+|du_0|+s(p-1,q)} \iota^*_0(f_{(1,\ldots,1)}) du_0 \otimes \partial u_{1,0}, \Pi$$

$$= (-1)^{b(p,q)+b(p,0)+s(p,q)+s(p-1,q)} \iota^*_0(\gamma_i(\varpi)).$$

In case $i > 1$, both sides of the equation vanish. As for the second equation, the case $i > p$ is easy, and we compute for $i \leq p$:

$$\gamma_i(d\varpi) = \gamma_i \left( (-1)^{|x,\Pi|} f_{Dx} \frac{\partial f}{\partial x_i} Dx \right)$$

$$= \gamma_i \left( \sum_{\nu} (-1)^{|x^\nu|} D x \gamma^*(\frac{\partial f_\nu}{\partial u_{i,0}}) \xi^\nu Dx \right)$$

$$= (-1)^{|Dx|+q+s(p,q)} \frac{\partial f_{(1,\ldots,1)}}{\partial u_{i,0}} du_0.$$
\[ (-1)^{|Dx|+|du_0|+q+s(p,q)} d \left(f_{(1,\ldots,1)} du_0 \otimes \partial_{u_{i_0}} \Pi \right) \]

= \[ (-1)^{b(p,q)+b(p,0)+q} d\gamma(\omega). \]

\[ \square \]

4. Boundary terms—the local picture

We will begin our examination of the behaviour of the Berezin integral under coordinate changes. In view of Corollary 2.15, what we need to understand is how the integrals for different retractions are related.

We start with the following observation on a coordinate neighbourhood \( U \). Let \( \gamma \) and \( \gamma' \) be retractions on \( U \). Choose a classical coordinate system \( u_0 \) on \( U_0 \) and define \( u := \gamma^*(u_0) \) and \( v := \gamma'^*(u_0) \). We complete these to coordinate systems \( x = (u, \xi) \) and \( y = (v, \eta) \) with \( \xi = \eta \).

Following Proposition 2.3, we know of the existence of a unique isomorphism \( \varphi : U \rightarrow U \) such that \( \varphi^*(x_i) = y_i \), \( i = 1, \ldots, p+q \). Of course, this implies \( \varphi_0 = \text{id}_{U_0} \) and \( \varphi^* \gamma = \gamma' \).

If \( \omega \) is a Berezin density on \( U \), Corollary 2.15 tells us that

\[ \int_{(U, \gamma')} \varphi^* \omega = \int_{(U, \gamma)} \omega, \]

whenever one of both integrals exists. One might interpret this as a first formula for coordinate changes. However, a more explicit expression is desirable. For this reason we take a closer look at \( \varphi \).

As one can conclude from the proof of Proposition 2.3, \( \varphi \) is given by

\[ \varphi^* = \sum_{j \in \mathbb{N}_0} \frac{1}{j!} (v - u)^i \partial^i_u, \]

with \( (v - u)^i := (v_1 - u_1)^{i_1} \cdots (v_p - u_p)^{i_p} \) and \( i! := i_1! \cdots i_p! \). Since \( u_0 = v_0 \), we have that \( v_s - u_s \) is nilpotent for each \( s \), so the sum is finite. Thus, \( \varphi^* \) is a differential operator of order at most \( \frac{p}{2} \).

There is a natural action of differential operators on Berezin densities; the following proposition can be found in Ref. [Che94].

Proposition 4.1. Let \( \text{Diff}(M) \) be the set of differential operators on \( M \). There is a unique \( \mathcal{O}(M) \)-right linear action of \( \text{Diff}(M) \) on \( |\text{Ber}||M| \) such that

\[ \omega.X = -(-1)^{|\omega|} L_X \omega \]

for all \( X \in \mathfrak{X}M \subseteq \text{Diff}(M) \) and \( \omega \in |\text{Ber}||M| \).

The corresponding statement for \( \text{Ber}M \) is also correct. Note that the additional minus sign in Equation \( 4.2 \) cannot be omitted.

The so defined action is compatible with restrictions and pullbacks, i.e.

\[ (\omega.A)|_U = \omega|_U.A|_U, \]

\[ \varphi^*(\omega.A) = \varphi^*(\omega).\varphi^*(A), \]

where \( \varphi^*(A) := \varphi^* \circ A \circ \varphi^{-1} \).

In the local picture this action has the form:

\[ \omega.A = |Dx| \sum_{j \in \mathbb{N}_0^p \times \mathbb{Z}_+^q} (-1)^{|j|+|f_{a_j}|} j_{j_{\text{odd}}} + \frac{|j_{\text{odd}}||j_{\text{odd}}|-1}{2} \partial^j(f_{a_j}). \]
Here, $\omega = |Dx|f$ and $A = \sum_j a_j \partial_j^i$; moreover, we set

$$j_{\text{odd}} := (j_{p+1}, \ldots, j_{p+q}) \quad \text{and} \quad |j| := j_1 + \cdots + j_{p+q}.$$  

With this definition, the pullback via a morphism and the action of differential operators are compatible.

**Proposition 4.2.** Let $\varphi : M \to M$ be a isomorphism such that $\varphi^*$ is a differential operator. Then we have for each Berezin density $\omega \in \text{Ber}_M$

$$\omega = \varphi^*(\omega^*).$$

The same is true for Berezin forms, if $\varphi_0$ is orientation preserving.

**Proof.** Let $h \in \mathcal{O}(M)$ be compactly supported. In our notation, integration by parts takes the form

$$\int_M (\omega.X)h = \int_M \omega X(h)$$

for any derivation $X \in \mathfrak{X}M$. To see this, one checks

$$\omega X(h) - (\omega.X)h = \omega X(h) + (-1)^{|\omega||X|}(\mathcal{L}_X\omega)h = (-1)^{|\omega||X|}\mathcal{L}_X(\omega h).$$

Since $\omega h$ is compactly supported, the integral of the right hand side vanishes \cite[Lemma 2.4.8]{Lei80}. Iteratively applying Equation (4.6), we get for any $A \in \text{Diff}(M)$

$$\int_M (\omega.A)h = \int_M \omega A(h).$$

Using Corollary 2.15, we conclude

$$\int_M \varphi^*(\omega,\varphi^*)h = \int_M (\omega,\varphi^*)\varphi^{*-1}(h) = \int_M \omega \varphi^*(\varphi^{*-1}(h)) = \int_M \omega h.$$  

Since $h$ was arbitrary, the assertion follows. \hfill $\square$

**Remark 4.3.** Proposition 4.2 is closely related to Ref. \cite[Theorem 3.2]{Rot87}. It is shown there that for $\varphi^* = \sum_j a_j \partial_j$, the inverse is

$$\varphi^{*-1} = \sum_j (-1)^{|j| + |j_{\text{odd}}(i_{j_{\text{odd}}(i+1)})/2} a_j D_y \frac{D}{Dx},$$

where $y = \varphi^*(x)$ for certain coordinate systems $x$ and $y$ (where $\varphi^*$ is denoted $e^Y$).

In fact, Equation (4.7) can be deduced from Equation (4.2) and Equation (4.5) by calculating for $f \in \mathcal{O}(U)$

$$Dx \varphi^{*-1}(f) = \varphi^{*-1}(Dy f) = \left(\frac{Dy}{Dx} D_x f\right) \varphi^*.$$  

As an aside, note that the coefficients are $a_j = \frac{1}{j!}(y_{p+q} - x_{p+q})^{j_{p+q}} \cdots (y_1 - x_1)^{i_1}$.

We use Proposition 4.2 to derive an explicit expression for the Berezin integral under the change of retractions.
Theorem 4.4. Let $U$ be a coordinate neighbourhood with two retractions $\gamma$ and $\gamma'$. Let $u_0$ be a coordinate system on $U_0$ and set $u = \gamma^*(u_0)$. Then $\omega \in |\text{Ber}|U$ (or $\omega \in \text{Ber}U$) is integrable with respect to $\gamma'$ and

$$\int_{(U,\gamma')} \omega = \int_{(U,\gamma)} \omega + \sum_{i \neq 0} \frac{1}{i!} \int_{(U,\gamma)} \omega \left(\gamma'^*(u_0) - \gamma^*(u_0)\right)^i \partial^i_u,$$

if the right hand side exists. Here, the sum is finite, and extends over $i \in \mathbb{N}_0$.

Proof. Let $x = (u, \xi)$, $y = (v, \eta)$ with $v = \gamma'^*(u_0)$ and $\xi = \eta$. With the morphism $\varphi^*$ from Equation (4.1) ($\varphi^*(x) = y$) we get

$$\int_{(U,\gamma')} \omega = \int_{(U,\gamma')} \varphi^*(\omega \varphi^*) = \int_{(U,\gamma)} \omega \varphi^* = \int_{(U,\gamma)} \omega + \sum_{i \neq 0} \frac{1}{i!} \int_{(U,\gamma)} \omega (v - u)^i \partial^i_u. \quad \square$$

We assemble our results in a general change of variables formula.

Corollary 4.5. Suppose given coordinate systems $x = (u, \xi)$ and $y = (v, \eta)$ on $U$ and $f \in \mathcal{O}(U)$. Let $\hat{u}_s = g_s(v)$ such that $u_s \equiv \hat{u}_s$ mod $\langle \eta \rangle$. Then

$$\int_{(U,y)} f|Dy| = \int_{(U,x)} f\left|\frac{Dy}{Dx}\right| |Dx| + \sum_{i \neq 0} \frac{1}{i!} \int_{(U,x)} \partial_u^i \left( f(u - \hat{u}) \left|\frac{Dy}{Dx}\right| \right) |Dx|,$$

if the integrals on the right hand side exist.

Note that we eliminated the sign $(-1)^{|i|}$ by replacing $(\hat{u} - u)$ with $(u - \hat{u})$. Further, $\hat{u}$ always exists: $\hat{u} = \gamma'^*(u_0)$, where $\gamma'$ is the retraction associated with $v$.

Remark 4.6. Observe that the above results bear a similarity to Ref. [Rot87, Theorem 3.2]. Compared to Rothstein’s result, the advantage of our theorem is that it is formulated in terms of Berezin densities, rather than of volume form valued differential operators. Whereas the former are a locally free $\mathcal{O}_M$-module of rank $(0, 1)^q$, the latter form one of infinite rank.

We apply these considerations in a few examples.

Examples 4.7.

(i) Recall the notation from Rudakov’s example [Example 2.8]. Here we have $u = v - \eta_1 \eta_2$, hence $\hat{u} = v$. For $f = \sum_{\nu} \gamma^*(f_{\nu}) \xi^\nu \in \mathcal{O}(\mathbb{R}^{1,2})$ and $\gamma^*(u_0) = u$ our recipe shows that

$$\int_{(\Omega,y)} f|Dy| = \int_{(\Omega,x)} f|Dx| + \int_{(\Omega,x)} \partial_u (f(u - v)) |Dx|$$

$$= \int_{(\Omega,x)} f|Dx| + \int_{(\Omega,x)} \partial_u (-f \xi_1 \xi_2) |Dx|$$

$$= \int_{(\Omega,x)} f|Dx| - \int_{(\Omega,x)} \gamma^* \left( \frac{\partial f_0}{\partial u_0} \right) \xi_1 \xi_2 |Dx|$$

$$= \int_{(\Omega,x)} f|Dx| - (-1)^{(1,2)} \int_{0}^{1} \frac{\partial f_0}{\partial u_0} |du_0|$$
\[ = \int_{(\Omega, y)} f|Dx| - (-1)^{s(1, 2)}(f_0(1) - f_0(0)). \]

Comparing to the computations in Example 2.8 \((f = v)\), this resolves the apparent contradictions.

(ii) Suppose \(\Omega \subset \mathbb{R}^{2, 2}\) with \(\Omega_0 = \{(o_1, o_2) \mid o_1^2 + o_2^2 < 1\}\). Let \(y = (v, \eta)\) be a coordinate system on \(\Omega\) with \(v_0 = \text{id}_{\Omega_0}\). We want to compute the \(y\)-related integral of a compactly supported \(f \in \mathcal{O}(\Omega)\), by using rotational symmetry. Thus, we consider on \(\Omega' \subset \Omega\) with \(\Omega'_0 = \Omega_0 \setminus (-\infty, 0) \times 0\) and a coordinate system \(x = (u, \xi)\) on \(\Omega'\), such that
\[
\begin{align*}
v_1 &= u_1 \cos(u_2)(1 - \xi^1 \xi^2), & \eta_1 &= u_1 \xi^1, \\
v_2 &= u_1 \sin(u_2)(1 - \xi^1 \xi^2), & \eta_2 &= u_1 \xi^2.
\end{align*}
\]

One computes \(v_1^2 + v_2^2 + 2\eta_1 \eta_2 = u_1^2\) and \(|\frac{Dy}{Dx}| = \frac{1}{u_1}\). It remains to find \(\hat{u}\).

We have
\[
\begin{align*}
u_1 &= \sqrt{v_1^2 + v_2^2} \eta_1 \eta_2 = \sqrt{v_1^2 + v_2^2} + \frac{\eta_1 \eta_2}{\sqrt{v_1^2 + v_2^2}},
\end{align*}
\]
hence \(\hat{u}_1 = \sqrt{v_1^2 + v_2^2}\) and
\[
\begin{align*}
u_1 - \hat{u}_1 &= \frac{\eta_1 \eta_2}{\sqrt{v_1^2 + v_2^2}} = \frac{u_1^2 \xi^1 \xi^2}{\sqrt{u_1^2(1 - 2\xi^1 \xi^2)}} = u_1 \xi^1 \xi^2.
\end{align*}
\]

Furthermore, we realise \(\frac{\eta_2}{\eta_1} = \tan(u_2)\), hence \(\hat{u}_2 = u_2\). This means that the second boundary term will vanish.

We write a compactly supported \(f \in \mathcal{O}(\Omega)\) as \(f = \sum_{\nu'} \gamma^*(f_\nu)\xi^\nu\) on \(\Omega'\), where \(\gamma\) is associated with \(u\). We obtain
\[
\begin{align*}
\int_{(\Omega, y)} f|Dy| &= \int_{(\Omega, x)} f \frac{1}{u_1} |Dx| + \int_{(\Omega, x)} \frac{\partial f_1}{\partial u_1} \left(f_1 u_1 \xi_2 \xi_2 \frac{1}{u_1}\right) |Dx| \\
&= \int_{(\Omega, x)} f \frac{1}{u_1} |Dx| + \int_{(\Omega, x)} \gamma^* \left(\frac{\partial f_0}{\partial u_1, 0}\right) \xi_1 \xi_2 |Dx| \\
&= \int_{(\Omega, x)} f \frac{1}{u_1} |Dx| + (-1)^{s(2, 2)} \int_0^\pi \int_0^1 \frac{\partial f_0}{\partial u_1, 0} du_1, 0 du_2, 0 \\
&= \int_{(\Omega, x)} f \frac{1}{u_1} |Dx| - (-1)^{s(2, 2)} 2\pi f_0(0).
\end{align*}
\]

A similar computation is contained in \cite{Zir91}, proof of Theorem 1.

(iii) We redo the calculation in the first example in a more general context. Suppose \(\Omega, \Omega' \subset \mathbb{R}^{2, 4}\), such that \(\Omega_0 = \Omega'_0 \cap \mathbb{R}_+^4\), where \(\mathbb{R}_+ = ]0, \infty[\). Let \(\gamma\) and \(\gamma'\) be retractions on \(\Omega'\) and let \(u_0 = \text{id}_{\Omega'_0}\) be the standard coordinate system on \(\Omega'_0\). Then Theorem 4.4 tells us for a compactly supported Berezin density \(\omega \in \mathcal{C}(\Omega)\)
\[
\int_{(\Omega, \gamma')} \omega = \int_{(\Omega, \gamma)} \omega + \sum_{i \neq 0} \int_{(\Omega, \gamma)} \omega_i \partial u_i,
\]
where \(u := \gamma^*(u_0)\) and
\[
\omega_i := \frac{1}{i!} \omega \left(\gamma^*(u_0) - \gamma^*(u_0)\right)^i = \sum_{\nu} \gamma^*(f_\nu) \xi^\nu |Dx|.
\]
Therefore,
\[
\int_{(\Omega,\gamma)} \omega, \partial_u u = (-1)^{s(p,q)+|\gamma|} \int_0^\infty \int_0^\infty \partial_u^\nu f_{(1,...,1)}(o) do_1 do_2.
\]

Applying the Fundamental Theorem of Calculus, we get
\[
\int_{(\Omega,\gamma')} \omega = \int_{(\Omega,\gamma)} \omega \pm \left( -\int_0^\infty \left( f_{(1,...,1)}^{(1,0)}(0,o_2) - \partial_2 f_{(1,...,1)}^{(2,0)}(0,o_2) \right) do_2 
- \int_0^\infty \left( f_{(1,...,1)}^{(0,1)}(o_1,0) - \partial_1 f_{(1,...,1)}^{(0,2)}(o_1,0) \right) do_1 
+ \partial_1,2 f_{(1,...,1)}^{(1,1)}(o)(0,0) \right).
\]

This shows explicitly that the ‘boundary terms’ indeed depend only on the values of \( \omega \), and its derivatives, on the boundary. We shall presently exploit this to derive a global expression for the boundary terms.

5. Boundary terms—the global picture

In this section, we will globalise the results of the previous section, using ideas from [Example 4.7](iii). A framework which is well suited to such a generalisation is that of supermanifolds built over manifolds with corners [Mel93, Mel96]. Locally, such spaces are modelled on \( \mathbb{R}^+ \times \mathbb{R}^{p-k} \).

To exclude strange example such as the drop (cf. Figure 1), we introduce the concept of boundary functions, cf. Ref. [Mel96, Chapter 2].

![Figure 1](image_url). The drop, which is not a manifold with corners (see below)

As before, \( M \) will denote a supermanifold of dimension \( (p, q) \) and \( M_0 \) will be the underlying manifold.

**Definition 5.1.** A family of smooth functions \( (\rho_1, \ldots, \rho_r) \) is called independent at \( o \in M_0 \), if the Jacobian \( J_{(\rho_1, \ldots, \rho_r)}(o) \) at \( o \) is of full rank.

A family \( (\rho_1, \ldots, \rho_n) \) is called a family of boundary functions, if the \( \rho_i \) are independent at each point at which they vanish. This means for each subfamily \( (\rho_{s1}, \ldots, \rho_{sk}) \) and every \( o \in M \):
\[
\rho_{si}(o) = 0 \text{ for } s = 1, \ldots, k \Rightarrow (\rho_{s1}, \ldots, \rho_{sk}) \text{ is independent at } o.
\]

This implies that at most \( p \) boundary functions can vanish simultaneously. Note that \( n \) does not have to be smaller than \( p \) (think of the case of an interval or a rectangle). Observe also that \( (\rho_{s1}, \ldots, \rho_{sk}) \) being independent at \( o \) implies that this family can be supplemented to a coordinate system \( (\rho_{s1}, \ldots, \rho_{sk}, f_{k+1}, \ldots, f_p) \) on sufficiently small neighbourhoods of \( o \).
**Definition 5.2.** A subset $N_0 \subset M_0$ is called a *manifold with corners*, if there exist boundary functions $\rho = (\rho_1, \ldots, \rho_n)$ such that

$$N_0 = \left\{ o \in M_0 \mid \rho_i(o) > 0, \; i = 1, \ldots, n \right\}.$$ 

For each subfamily $\rho' = (\rho_1, \ldots, \rho_k)$ we consider the set

$$H_0 := \left\{ o \in M_0 \mid \rho_i(o) = 0 \quad \text{for} \quad \rho_i \in \rho', \\
\rho_i(o) > 0 \quad \text{for} \quad \rho_i /\in \rho' \right\}.$$

Whenever $H_0$ is non-empty, it is called a *boundary manifold* of $N_0$ of codimension $k$. We set $\rho_{H_0} := \rho'$ and denote by $B_0(M_0, \rho) = B_0(\rho)$ the collection of all boundary manifolds. Each boundary manifold of $N_0$ is a submanifold of $M_0$. Furthermore, the disjoint union of all boundary manifolds coincides with the (topological) boundary of $N_0$ in $M_0$.

For later uses, we define for each boundary manifold $H_0 \in B_0(\rho)$ a set of multi-indices

$$J_{H_0} := J^\rho_{H_0} := \left\{ j \in \mathbb{N}_0^n \mid j_i = 0 \iff \rho_i \notin \rho_{H_0} \right\}.$$ 

Note that $\mathbb{N}_0^n \setminus \{0\}$ is the disjoint union of the $J_{H_0}$. Observe that for $j \in J_{H_0}$, the function $\rho^j = \rho_1^{j_1} \cdots \rho_n^{j_n}$ is a monomial in the boundary functions $\rho_{H_0}$.

**Examples 5.3.**

(i) The drop $N$ depicted in Figure 1 is *not* a manifold with corners. Indeed, suppose the contrary. Since there is only one codimension one boundary manifold, there can only be one boundary function. But then $N$ has only one boundary manifold, and is a manifold with boundary, contradiction.

(ii) Easy examples of manifolds with corners are displayed in Figure 2.

(iii) One can also consider $N_0 = \mathbb{R}_+^k \times \mathbb{R}^{p-k}$ (with $\mathbb{R}_+ = ]0, \infty[$) as a manifold with corners in $M_0 = \mathbb{R}^p$, with boundary functions $\rho = (\text{pr}_1, \ldots, \text{pr}_k)$. These are the model spaces for manifolds with corners.

We generalise this definition to the setting of supermanifolds.

**Definition 5.4.**

(i) A family of even superfunctions $\tau = (\tau_1, \ldots, \tau_n)$ on $M$ is called a family of *boundary superfunctions* if the family of underlying functions $\tau_0 = (\tau_{1,0}, \ldots, \tau_{n,0})$ is a family of boundary functions.

(ii) An open subspace $N \subset M$ is called a *supermanifold with corners* if there exist boundary superfunctions $\tau = (\tau_1, \ldots, \tau_n)$ such that $N_0 \subset M_0$ is a manifold with corners via the boundary functions $\tau_0$. 

![Figure 2. Manifolds with corners](image-url)
Remarks 5.5. Let \( o \in M_0 \) and \( \tau' \) be a subfamily of \( \tau \) such that \( \tau_{i,0}(o) = 0 \) for each \( \tau_i \in \tau' \). Similarly to the purely even case, one can augment the family \( \tau' \) to a coordinate system \( x = (u, \xi) = (\tau', \tilde{u}) = ((\tau', \tilde{u}), \xi) \) on a sufficiently small neighbourhood of \( o \).

If \( N \subset M \) is given such that \( N_0 \) is a manifold with corners with boundary functions \( \rho \), we are always able to turn \( N \) into a supermanifold with corners via the boundary superfunctions \( \tau = \gamma^*(\rho) \) for a retraction \( \gamma \) on \( M \).

For the remainder of this section, \( N \) will be a supermanifold with corners contained in the supermanifold \( M \); moreover, \( \tau = (\tau_1, \ldots, \tau_n) \) will be boundary superfunctions, and \( \rho = (\rho_1, \ldots, \rho_n) \) will be boundary functions.

Proposition 5.6. Let \( \tau \) be given and \( H_0 \in B_0(\tau_0) \). Then there exist

- a supermanifold \( H \) of dimension \((\dim H_0, q)\) with underlying space \( H_0 \),
- an immersion \( \iota_H : H \to M \) over the inclusion \( (\iota_H)_0 = \iota_{H_0} : H_0 \hookrightarrow M_0 \) such that \( \iota_H^*(\tau_i) = 0 \) whenever \( \tau_i \in \rho_{H_0} \).

The data \((H, \iota_H)\) are determined uniquely up to unique isomorphism by these conditions.

Proof. This follows from [Lei80, Propositions 3.2.6].

Concretely speaking, the condition \( \iota_H^*(\tau_i) = 0 \) means that \( H \) is the supermanifold obtained by setting the boundary coordinates to zero.

Definition 5.7. We call the supermanifolds \( H \) from Proposition 5.6 the boundary supermanifolds of \( N \) corresponding to \( \tau \). The set of all such \( H \) is denoted by \( B(M, \tau) = B(\tau) \). We define abbreviations \( \tau_H := (\tau_i)_{\tau_i \in \tau_{H_0}} \) and \( J_H^* := J_H^{\sigma} := J_H := J_{H_0} \).

We will need to integrate Berezin densities on \( M \) along the boundary supermanifolds \( H \). For that purpose, we define on \( H \) canonical retractions, as well as the restriction to \( H \) of Berezin densities on \( M \).

Lemma 5.8. Let \( \gamma \) be a retraction on \( M \) and \( H \in B(\gamma^*(\rho)) \). Then there exists a unique retraction \( \gamma_H \) on \( H \) such that the following diagram commutes:

\[
\begin{array}{ccc}
H & \xrightarrow{\iota_H} & M \\
\gamma_H \downarrow & & \gamma \\
H_0 & \xrightarrow{\iota_{H_0}} & M_0
\end{array}
\]  
(5.1)

Proof. The uniqueness of \( \gamma_H \) follows directly from \( \iota_{H_0} \circ \gamma_H = \gamma \circ \iota_H \), since \( \iota_{H_0} \) is a monomorphism.

For the existence, we choose \( o \in H_0 \) and complete the boundary functions \( \rho_{H_0} \) to a coordinate system \( u_0 = (\rho_{H_0}, \tilde{u}_0) \) on a sufficiently small neighbourhood \( U_0 \subset M_0 \) of \( o \). Since \( \iota_{H_0}^*(\rho_{H_0}) = \rho_{H_0}|_{H_0} = 0 \), it is clear that \( \iota_{H_0}^*(\tilde{u}_0) = \tilde{u}_0|_{U_0 \cap H_0} \) is a coordinate system on \( U_0 \cap H_0 \). We set

\[
\gamma_H^*(\tilde{u}_0|_{U_0 \cap H_0}) := \iota_H^*(\gamma^*(\tilde{u}_0)).
\]
Proposition 5.9. Let $H_0 \in B_0(\rho)$ be a boundary manifold and suppose that there is another family of boundary functions $\rho'$ which also determines $H_0$. Then the families of boundary superfunctions given by $\gamma^*(\rho)$ and $\gamma^*(\rho')$ determine the same supermanifold structure $H$ over $H_0$ with immersion $\iota_H$.

Proof. Let $H \in B(\gamma^*(\rho))$ be the supermanifold over $H_0$ associated with $\gamma^*(\rho)$. By the uniqueness in Proposition 5.6, it suffices to show $\iota_H^*(\gamma^*(\rho_{H_0})) = 0$. Since $\iota_H^*(\rho_{H_0}) = \rho_{H_0}|_{H_0} = 0$, Lemma 5.8 proves the claim. □

Proposition 5.9 shows that Diagram (5.1) uniquely determines the supermanifold $H$ for a given retraction $\gamma$ on $M$. We have already made use of this in the statement of Stokes’s theorem.

Definition 5.10. Let $H \in B(\tau)$ be a boundary supermanifold. We define a restriction map

$$|\operatorname{Ber}|M \longrightarrow |\operatorname{Ber}|H, \; \omega \longmapsto \omega|_{H,\tau}$$

as follows. For $o \in H_0$ supplement $\tau_H$ to a coordinate system $x = (\tau_H, \tilde{x})$ on a neighbourhood $U$ and write $\omega|_U = f | Dx |$. Since $\iota_H^*(\tau_H) = 0$, the family $\iota_H^*(\tilde{x})$ is a coordinate system on $U \cap H := H|_{U \cap H_0}$. Therefore, we may define

$$\omega|_{U \cap H,\tau} := \iota_H^*(f | D \iota_H^*(\tilde{x})|).$$

This definition depends on $\tau$, but it is independent of the choice of the $\tilde{x}$ (see below). Thus, this defines a morphism of sheaves $|\operatorname{Ber}|M \to |\iota_H| |\operatorname{Ber}|H$. By definition, $(f \omega)|_{H,\tau} = \iota_H^*(f \omega)|_{H,\tau}$, so it is in fact a morphism of $\mathcal{O}_M$-modules.

Lemma 5.11. The construction in Definition 5.10 does not depend on the choice of $x$.

Proof. This proof is similar to that of Proposition 3.7. Let $y = (\tau_H, \tilde{y})$ be another such coordinate system and let $k$ be the codimension of $H_0$. Then for $x_i \in \tilde{x}$ we obtain by the chain rule

$$\iota_H^* \left( \frac{\partial y_i}{\partial x_1} \right) = \frac{\partial \iota_H^*(y_i)}{\partial \iota_H^*(x_1)}.$$

Furthermore, $\frac{\partial y_0}{\partial x_1} = \delta_{il}$ for $y_i \in \tau_H$, hence

$$\iota_H^* \left( \frac{|Dy|}{Dx} \right) = \pm \operatorname{Ber} \left( \begin{array}{c} \iota_H^* \left( \frac{\partial y_1}{\partial x_1} \right) \\
\vdots \\
\iota_H^* \left( \frac{\partial y_q}{\partial x_1} \right) \\
\iota_H^* \left( \frac{\partial y_1}{\partial x_1} \right) \\
\vdots \\
\iota_H^* \left( \frac{\partial y_q}{\partial x_1} \right) \\
\iota_H^* \left( \frac{\partial y_1}{\partial x_1} \right) \\
\vdots \\
\iota_H^* \left( \frac{\partial y_q}{\partial x_1} \right) \\
\end{array} \right) = \pm \operatorname{Ber} \left( \begin{array}{c} 1_k \\
\vdots \\
1_k \\
\vdots \\
1_k \\
\end{array} \right) \begin{pmatrix} 0 \\
\delta_{il}^* \left( \frac{\partial y_0}{\partial x_1} \right) \\
\vdots \\
\delta_{il}^* \left( \frac{\partial y_q}{\partial x_1} \right) \\
\vdots \\
\delta_{il}^* \left( \frac{\partial y_0}{\partial x_1} \right) \\
\vdots \\
\delta_{il}^* \left( \frac{\partial y_q}{\partial x_1} \right) \\
\vdots \\
\delta_{il}^* \left( \frac{\partial y_0}{\partial x_1} \right) \\
\end{array} \right)$$

$$= \frac{|D \iota_H^*(y)|}{D \iota_H^*(x)}.$$
using Equation (3.3). Now suppose a Berezin density $f|Dy|$. We finish with
\[
(f|Dy|)|_{U\cap H,\tau} = \iota_H^* (f|Dy|) |\partial H \cap (\tilde{\mathcal{D}} H)^{\tau}| = \iota_H^* (f|\partial H \cap (\tilde{\mathcal{D}} H)^{\tau}|) = \iota_H^* (f|\partial H \cap (\tilde{\mathcal{D}} H)^{\tau}|).
\]

Remarks 5.12.

(i) The restriction can also be defined for Berezin forms; in this case, one has to fix an ordering on the family of boundary superfunctions $\tau$.

The restriction of Berezin forms and the pullback of integral forms are related as follows: Suppose $x = (u, \xi)$ is a coordinate system on a superdomain $\Omega$. Let $(H, \iota_H)$ be the boundary data given by the boundary function $u_1$. Then for any Berezin form $\omega \in \text{Ber} \Omega$ we have
\[
\omega|_{H,u_1} = \iota_H^* (-1)^{|Dx|} \omega \otimes \partial u_1).
\]

(ii) The restriction of Berezin densities is compatible with the notion of Riemannian measure; we elaborate this in the even case. Let $M_0$ carry a Riemannian density $\omega_g$ locally has the form
\[
\omega_g = \sqrt{|\det(g_{ij})(\tilde{\mathcal{D}} H)^{\tau}|} |\text{d}u_0|,
\]
for a coordinate system $u_0$, where $g_{ij}(\tilde{\mathcal{D}} H)^{\tau} = g_o(\partial u_{a,o}|_o, \partial u_{j,o}|_o)$ for $k, j = 1, \ldots, p$ and $o \in M_0$.

Consider $N_0 \subset M_0$ to be a manifold with corners and enumerate the boundary manifolds of codimension 1 by $H_1^1, \ldots, H_p^1$. The metric on $M_0$ determines on each $H_i^1$ uniquely the inner normal derivative $n_i$.

We assume $g$ to be such that the $H_i^1$ intersect each other orthogonally. This means that the corresponding $n_i$ are orthogonal to each other on the intersections of the $H_i^1$, which are just the boundary manifolds. Thus, we can choose boundary functions $\rho = (\rho_1, \ldots, \rho_n)$ which satisfy $n_i(p_j) = \delta_{ij}$ on $H_0^1$.

Let $H_0$ be any boundary manifold and $o \in H_0$. Choose a coordinate system $u_0 = (\rho_{H_0}, \tilde{u}_0)$ on a neighbourhood $U_0$ of $o$, which satisfies
\[
g_{ij}(\partial u_{a,o}|_o, \partial u_{j,o}|_o) = \delta_{kj} \quad \text{for all } o \in H_0 \cap U_0.
\]

Then $\tilde{g}_{ij}(\rho_{H_0}U_0) = 1$, and hence $\omega_g|_{H_0,\rho} = |\text{d}\tilde{u}_0|_{H_0 \cap U_0}|$.

On the other hand, $g$ induces a metric $g_{H_0}$ on $H_0$, which gives the canonical density $\omega_{g_{H_0}} = |\text{d}\tilde{u}_0|_{H_0 \cap U_0}|$ on $H_0 \cap U_0$. Thus,
\[
\omega_g|_{H_0,\rho} = \omega_{g_{H_0}}.
\]

Lemma 5.13. Let $\gamma$ be a retraction. The restriction to a boundary supermanifold is compatible with $\gamma_!$, in the following sense: for $H \in B(\gamma^!(\rho))$,
\[
\gamma_H! (\omega|_{H,\gamma^!(\rho)}) = (-1)^{s(dM) + s(dH)} (\gamma_H!(\omega))|_{H,\rho}
\]

Proof. We complete $\rho_{H_0}$ locally to a coordinate system $u_0 = (\rho_{H_0}, \tilde{u}_0)$ and choose $x = (u, \xi) = (\gamma^!(\rho_{H_0}), \tilde{x})$ to be a coordinate system with which $\gamma$ is associated. We write $\omega = f|Dx|$ with $f = \sum \gamma^!(\xi^\nu)$. Then
\[
\gamma_H! (\omega|_{H,\gamma^!(\rho)}) = \gamma_H! \left( \sum \iota_H^* \gamma^!(f^\nu) \iota_H^!(\xi^\nu)|\partial H \cap (\tilde{\mathcal{D}} H)^{\tau}| \right)
\]
Lemma 5.14. For the boundary superfunctions \(\tau\) there exists a family of derivations \(D = (D_1, \ldots, D_n)\) with the following properties:

- Each \(D_i\) is defined on a neighbourhood \(U_i^D\) of \(H_0\), where \(H_0\) is the boundary manifold of codimension 1, which is given by \(\tau_{i,0}\).
- \(D_i(\tau_j) = \delta_{ij}\) on \(U_i^D \cap U_j^D\).

We call a family \(D\) which satisfies these conditions a family of boundary superderivations for \(\tau\). Observe that \(D\) is not uniquely determined.

**Proof.** Let \(o \in M_0\) and choose a coordinate neighbourhood \(U\) with \(o \in U_0\). Let \(\tau_U\) be the subfamily of \(\tau\) of all boundary functions which vanish at any point of \(U_0\):

\[\tau_i \in \tau_U \iff \exists o' \in U_0 : \tau_{i,0}(o') = 0.\]

Possibly after shrinking \(U\), we may assume that \(\tau_U\) can be completed to a coordinate system on \(U\).

Since \(o\) was arbitrary, we can choose a locally finite covering \((U^\alpha)_{\alpha \in A}\) of \(M\) with such coordinate neighbourhoods. On each \(U^\alpha\) we supplement \(\tau_{U^\alpha}\) to a coordinate system \(x^\alpha = (\tau_{U^\alpha}, \tilde{x}^\alpha)\) and define for \(i = 1, \ldots, n\)

\[D_{i,\alpha} \coloneqq \begin{cases} \frac{\partial x^\alpha}{\partial x_i} & x_i = \tau_i \in \tau_{U^\alpha}, \\ 0 & \text{otherwise.} \end{cases}\]

This means \(D_{i,\alpha}(\tau_j) = \delta_{ij}\) for \(\tau_j \in \tau_{U^\alpha}\). Now we choose a partition of unity \((\phi_\alpha)_{\alpha \in A}\) subordinate to \((U^\alpha)_{\alpha \in A}\) and glue these local derivations to

\[D_i := \sum_{\alpha \in A} \phi_\alpha D_{i,\alpha}, \quad i = 1, \ldots, n.\]

It remains to define the neighbourhoods \(U_i^D\). We set for each \(i\)

\[B_i := \{\alpha \in A \mid 0 \in \tau_{i,0}(U_0^\alpha)\}, \quad C_i := \{\alpha \in A \mid 0 \notin \tau_{i,0}(U_0^\alpha)\}\]

to define

\[U_i^D := \bigcup_{\beta \in B_i} U_0^\beta \bigcap_{\alpha \in C_i} \supp \phi_\alpha = \bigcup_{\beta \in B_i} \bigcap_{\alpha \in C_i} U_0^\beta \bigcap_{\alpha \in C_i} \supp \phi_\alpha.\]

The so defined sets are open, since the covering was locally finite. By construction \(\supp \phi_\alpha \cap U_i^D \cap U_j^D = \emptyset\) for all \(\alpha \in C_i \cup C_j\), hence

\[1|_{U_i^D \cap U_j^D} = \sum_{\alpha \in A} \phi_\alpha|_{U_i^D \cap U_j^D} = \sum_{\alpha \in B_i \cup B_j} \phi_\alpha|_{U_i^D \cap U_j^D}.\]
Proof of Lemma 5.16. and \(\gamma\) are orthogonal to the boundary with respect to this metric (cf. Theorem 5.15). We will prove the formula in several steps.

**Proof.**

Note that \(\alpha \in B_i \cap B_j\) implies \(\tau_i, \tau_j \in \tau_{U^\alpha}\), hence \(D^\alpha(\tau_j) = \delta_{ij}\). With this observation we finish the proof by calculating on \(U^D_i \cap U^D_j\)

\[
D_i(\tau_j)|_{U^D_i \cap U^D_j} = \sum_{\alpha \in A} \phi_\alpha|_{U^D_i \cap U^D_j} D^\alpha(\tau_j) = \sum_{\alpha \in B_i \cap B_j} \phi_\alpha|_{U^D_i \cap U^D_j} D^\alpha(\tau_j) = \sum_{\alpha \in B_i \cap B_j} \phi_\alpha|_{U^D_i \cap U^D_j} \delta_{ij} = \delta_{ij}. \quad \square
\]

For \(j \in \mathbb{N}_0^n\) we define \(D^j := D^{i_1}_{1} \cdots D^{i_n}_{n}\) and the reduced multi-index \(j_\downarrow := (j_1, \ldots, j_n)\), where \(s_\downarrow := \max(s - 1, 0)\) for \(s \in \mathbb{N}_0\).

**Theorem 5.15** (Change of variables formula). Let \(\omega \in \text{Ber} M\) be a compactly supported, the same is true for the superderivations for \(\gamma^\ast\) as in Lemma 5.14 and

\[
\int_{(N,\gamma')} \omega = \int_{(N,\gamma)} \omega + \sum_{H \in B(\gamma^\ast(\rho))} \sum_{j \in J_H} (-1)^{s(\dim N) + s(\dim H)} \int_{(H,\gamma_H)} (\omega_j.D^{j_\downarrow})|_{H,\gamma^\ast(\rho)}\]

where \(s(\cdots)\) was defined in Definition 2.6. \(D = (D_1, \ldots, D_n)\) is a family of boundary superderivations for \(\gamma^\ast\) and

\[
\omega_j := \frac{1}{j!} (\gamma^\ast(\rho) - \gamma^\ast(\rho))^j \omega.
\]

Note \(\omega_j = 0\) if \(j > \frac{n}{2}\), so the sum over \(J_H\) is finite. Observe further that there are no summands for \(\text{codim} \, H_0 > \frac{n}{2}\), if \(q < 2p\). Moreover, the Berezin density \(\omega_j.D^{j_\downarrow}\) is defined on \(U^D_{i_1} \cap \cdots \cap U^D_{i_k}\), where \(\rho_{H_0} = (\rho_{i_1}, \ldots, \rho_{i_k})\). This set contains \(H_0\), so the restriction makes sense.

In general, there is no canonical choice for the boundary superderivations; given a super Riemannian metric on \(M\), one might take boundary superderivations which are orthogonal to the boundary with respect to this metric (cf. Remark 5.12(ii)).

**Proof.** We will prove the formula in several steps.

**Step 1.** We suppose \(M\) to be a superdomain, \(M \subset \mathbb{R}^{p,q}\), and \(N\) to satisfy \(N_0 = (\mathbb{R}_+^k \times \mathbb{R}^{p-k}) \cap M_0\). The boundary functions are chosen to be \(\rho = (\text{pr}_1, \ldots, \text{pr}_k)\). Furthermore, we consider a coordinate system \(x = (u, \xi)\) with \(u_0 = (\text{pr}_1, \ldots, \text{pr}_p)\) and \(\gamma^\ast(u_0) = u\).

**Lemma 5.16.**

(i) For \(s > k\) we have \(\int_{(N,\gamma)} \omega.\partial_x = 0\).

(ii) For \(H \in B(\gamma^\ast(\rho))\) and \(s > k\) we have \(\int_{(H,\gamma_H)} (\omega.\partial_x)|_{H,\gamma^\ast(\rho)} = 0\).

(iii) If \(i \in \mathbb{N}_0^n\) such that \(i_s \neq 0\) for some \(s > k\), then \(\int_{(N,\gamma)} \omega.\partial^{i_s}_u = 0\).

(iv) If \(H \in B(\gamma^\ast(\rho))\) and \(j \in J_H\) we get (with \((j,0) \in \mathbb{N}_0^n\))

\[
\int_{(N,\gamma)} \omega.\partial^{j_\downarrow}(0) = (-1)^{s(\dim N) + s(\dim H)} \int_{(H,\gamma_H)} (\omega.\partial^{j_\downarrow}_u)|_{H,\gamma^\ast(\rho)}.
\]

**Proof of Lemma 5.16.** Write \(\omega = f.Dx\) and \(f = \sum_\nu \gamma^\ast(f_\nu) \xi_\nu\). Since \(\omega\) is compactly supported, the same is true for the \(f_\nu\).
(i) If \( k < s \leq p \), we deduce
\[
\int_{(N,\gamma)} \omega \partial_u = \pm \int_0^\infty do_1 \cdots \int_0^\infty do_k \int_0^\infty do_{k+1} \cdots \int_0^\infty do_p \partial_s f_{(1,\ldots,1)}(\alpha).
\]

Thanks to the compact support, the right hand side vanishes. In the case of \( s > p \)
the claim is clear.

(ii) Similar to (i).

(iii) Follows directly from (i).

(iv) Write \( \tilde{u}_0 := (pr_{k+1}, \ldots, pr_p) \). As in Example 4.7 we apply the Fundamental
Theorem of Calculus in each direction in which a derivation occurs. So the
remaining integrals are the same as integrating along \( H_0 \).

In the following computation, we write \( \ell := \text{codim} H_0 \).

\[
\int_{(N,\gamma)} \omega \partial_u^i = (-1)^s(dim N) + |j| \int_0^\infty do_1 \cdots \int_0^\infty do_k \int_0^\infty do_{k+1} \cdots \int_0^\infty do_p \partial_{j} h(\alpha)
\]
\[
= (-1)^s(dim N) + \ell + |j| \int_{H_0} (\partial_{j} h \big|_{H_0}) d\tilde{u}_0 \big|_{H_0}
\]
\[
= (-1)^s(dim N) + \ell + |j| \int_{H_0} (\partial_{j} h \big|_{H_0} d\tilde{u}_0 \big|_{H_0})
\]
\[
= \int_{H_0} (\gamma(\omega, \partial_u^i)) \big|_{H_0, \rho} = (-1)^s(dim N) + s(dim H) \int_{H_0} \gamma_H \big( (\omega, \partial_u^i) \big|_{H, \gamma^s(\rho)} \big)
\]
\[
= (-1)^s(dim N) + s(dim H) \int_{(H, \gamma^s)} (\omega, \partial_u^i) \big|_{H, \gamma^s(\rho)}.
\]

In the second last equation, we applied Lemma 5.13. □

**Proof of Theorem 5.15 (continued).** Remembering \( h_0^{k} \{0\} = \bigcup_{H} J_H \), we see
that Lemma 5.16 applied to Theorem 4.4 proves Theorem 5.15 in the case of
\( D_s = \partial_u \), for \( s = 1, \ldots, k \).

**Step 2.** We stay in the setting of Step 1, but now we suppose general \( D = (D_1, \ldots, D_k) \) such that \( D_i(\gamma^s(\rho)) = \delta_{il} \) everywhere on \( M \) for \( i, l = 1, \ldots, k \). Then
we have for \( i = 1, \ldots, k \)
\[
D_i = \sum_{l=1}^{p+q} D_i(x_l) \partial_{x_l} = \partial_{u_i} + \sum_{l=k+1}^{p+q} D_i(x_l) \partial_{x_l}.
\]

Therefore we get for \( H \in B(\gamma^s(\rho)) \) and \( j \in J_H \)
\[
D^j = \partial_u^j + \sum_{l=k+1}^{p+q} A_l \partial_{x_l}
\]
for some (not further specified) differential operators \( A_k, \ldots, A_{p+1} \). Applying Lemma 5.16 we conclude
\[
\int_{(H, \gamma^s)} \omega \cdot (A_l \partial_{x_l}) = \int_{(H, \gamma^s)} (\omega, A_l) \partial_{x_l} = 0,
\]
hence
\[
\int_{(H, \gamma^s)} \omega \cdot D^j = \int_{(H, \gamma^s)} \omega \partial_u^j.
\]
This proves the claim for general $D = (D_1, \ldots, D_k)$.

Step 3. Now suppose $M$ to be a coordinate neighbourhood with boundary functions $\rho = (\rho_1, \ldots, \rho_k)$ such that $\gamma^*(\rho)$ can be completed to a coordinate system $x = (\gamma^*(\rho), \hat{x})$. Suppose further $D = (D_1, \ldots, D_k)$ to be such that $D_i(\gamma^*(\rho_i)) = \delta_{i1}$ everywhere on $M$. Let $\varphi : \Omega' \to M$ be the inverse of a chart such that $\rho_i \circ \varphi_0 = \text{pr}_i, \ i = 1, \ldots, k$. We set $\Omega := \Omega'|_{\varphi_0^{-1}(N_0)}$. The restriction $\varphi : \Omega \to N$ is also an isomorphism. Furthermore, $\Omega_0 = \Omega_0 \cap \mathbb{R}^k_+ \times \mathbb{R}^{p-k}$, meaning that $\Omega'$ and $\Omega$ are as in Steps 1 and 2. We recognise that the boundary manifolds are sent to each other by $\varphi_0$, more explicitly

$$B_0(\text{pr}_1, \ldots, \text{pr}_k) = \{\varphi_0^{-1}(H_0) \mid H_0 \in B_0(\rho)\}.$$  

Indeed, $\varphi_0$ restricts to diffeomorphisms of the boundary manifolds.

Similarly, $\varphi$ induces isomorphisms of the boundary supermanifolds. For each $H \in B(\gamma^*(\rho))$ we denote by $H_0 \in B(\varphi^*(\gamma^*(\rho)))$ the supermanifold over $\varphi_0^{-1}(H_0)$, which corresponds to the retraction $\varphi^*(\gamma)$.

Now fix $H \in B(\gamma^*(\rho))$. We recall that $\iota^*_H(\hat{x})$ is a coordinate system on $H$. Moreover, $\iota^*_{H_0}(\varphi^*(\hat{x}))$ is a coordinate system on $H_0$, since $\iota^*_{H_0}(\varphi^*(\gamma^*(\rho))) = 0$. Therefore

$$\varphi^*_H(\iota^*_H(\hat{x})) := \iota^*_{H_0}(\varphi^*(\hat{x}))$$

defines an isomorphism $\varphi_H : H_0 \to H$ making the following diagram commute:

$$\begin{array}{ccc}
H & \xrightarrow{\iota^*_H} & M \\
\varphi_H \downarrow & & \downarrow \varphi \\
H_0 & \xrightarrow{\iota^*_{H_0}} & \Omega'
\end{array}$$

The definition of $\varphi_H$ is also compatible with the pullback of retractions and with the restriction of Berezin densities. The former means just $\varphi^*_H(\gamma_H) = (\varphi^*(\gamma))_{H_0}$, which can be seen from the following diagram:

$$\begin{array}{ccc}
H & \xrightarrow{\iota^*_H} & M \\
\varphi_H \downarrow & & \downarrow \varphi \\
H_0 & \xrightarrow{\iota^*_{H_0}} & \Omega' \\
\varphi_{H_0}^{-1}(H_0) & \xrightarrow{\varphi_0} & \Omega_0
\end{array}$$

The left, right, upper, lower, and rear squares commute, and hence this has also to be true for the front side. The uniqueness condition in [Lemma 5.8] implies $\varphi^*_H(\gamma_H) = (\varphi^*(\gamma))_{H_0}$.

With $\omega = fDx$, the compatibility of $\varphi_H$ with restriction of Berezin densities is easily derived from

$$\varphi_H(\omega|_{H, \gamma^*(\rho)}) = \varphi^*_H(\iota^*_H(f)D\iota^*_H(\hat{x})) = \iota^*_{H_0}(\varphi^*(f))D\iota^*_{H_0}(\varphi^*(\hat{x}))$$

$$= (\varphi^*(\omega))_{H_0, \varphi^*(\gamma^*(\rho))}.$$
These properties lead to
\[ \int_{(H,\gamma)} (\omega|_{H,\gamma^*}(\rho)) = \int_{(H_0,(\varphi^*|_{\gamma^*}(\rho)))_{H_0}} (\varphi^*(\omega))|_{H_0,\varphi^*(\gamma^*(\rho))}. \]

Using the compatibility of pullbacks with the action of differential operators on Berezin densities (Equation (4.4)), we see for corresponding family of boundary derivations. Let \( \varphi \) be a family of boundary derivations for \( \varphi \) in a compactly supported in \( \omega \). Let \( (\varphi \cap H,\gamma) \) be arbitrary, we can take a covering \( (\varphi \cap H,\gamma) \) of \( \varphi \), where \( \varphi \cap H \) is compactly supported in \( \omega \). Possibly shrinking \( \omega \), we can choose a coordinate system on \( U \). Shrinking \( \omega \) further we can suppose \( U \), together with \( \varphi \cap H \) and \( D_U \), to be as in Step 3.

Possible shrinking \( U \) we may assume that \( \gamma^*(\varphi_U) \) can be completed to a coordinate system on \( U \). Shrinking \( U \) further we can suppose \( U \), together with \( \varphi \cap H \) and \( D_U \), to be as in Step 3.

Since \( \varphi \) was arbitrary, we can take a covering \( (U^\alpha)_\varphi \) of \( M \) with such coordinate neighbourhoods. Let \( (\varphi_{H^\alpha})_\varphi \) be a partition of unity subordinate to \( (U^\alpha)_\varphi \). Then \( \varphi_{H^\alpha} \) is compactly supported in \( U^\alpha \), hence by Step 3

\[ \int_{(N,\gamma)} \varphi_{H^\alpha} \varphi_{H^\alpha} = \sum_{\alpha} \int_{(V^\alpha,\gamma)} \varphi_{H^\alpha} \varphi_{H^\alpha} + \sum_{H \in J_H} \sum_{j \in J_H} \pm \int_{(H,\gamma^*)(\varphi_U)} ((\varphi_{H^\alpha})_\varphi \cap H,\gamma^*)|_{H,\gamma^*(\varphi_U)} \]

where \( V^\alpha := U^\alpha \cap N \) and \( H \) runs through \( B(U^\alpha,\gamma^*(\varphi_U)) \).

We recognise that
\[ B(U^\alpha,\gamma^*(\varphi_U)) = \left\{ U^\alpha \cap H \in B(M,\rho), U^\alpha \cap H = \emptyset \right\} \]
and similarly,
\[ B(U^\alpha,\gamma^*(\varphi_U)) = \left\{ H \cap U^\alpha \cap H = \emptyset \in B(M,\gamma^*(\rho)), U^\alpha \cap H = \emptyset \right\}. \]

Further, one trivially has for \( H_0 \in B(M,\rho) \) with \( U^\alpha_0 \cap H = \emptyset \) that
\[ J_{U^\alpha_0}^H = \left\{ (j_0, j_0)_{\rho,\varphi} \left| j \in J_{\gamma^*(\rho)}^H \right. \right\}. \]
Therefore, we get
\[ \int_{(N', \gamma')} \omega = \int_{(N, \gamma)} \omega + \sum_{\alpha} \sum_{H_\gamma \in J_H} \sum_{j \in J_H} \pm \int_{(H|_{U^\alpha \cap H_0, \gamma_H})} ((\phi_\alpha \omega_j).D^{j\gamma})|_{H, \gamma^*(\rho)}, \]
where \( H \) runs now through \( B(M, \gamma^*(\rho)) \). The supports of the integrands are contained in \( U_0^\alpha \cap H_0 \), which implies
\[ \int_{(N, \gamma')} \omega = \int_{(N, \gamma)} \omega + \sum_{\alpha} \sum_{H_\gamma \in J_H} \sum_{j \in J_H} \pm \int_{(H, \gamma_H)} ((\phi_\alpha \omega_j).D^{j\gamma})|_{H, \gamma^*(\rho)} \]
\[ = \int_{(N, \gamma)} \omega + \sum_{\alpha} \sum_{H_\gamma \in J_H} \sum_{j \in J_H} \int_{H_0} ((\gamma \omega_j).A^{j\gamma})|_{H_0, \rho^*}. \]
This completes the proof of Theorem 5.15 \( \Box \)

Sometimes, the following statement of the change of variables formula, which contains only ordinary boundary derivations, is more useful in applications.

**Corollary 5.17.** Let \( A = (A_1, \ldots, A_n) \) be a family of boundary derivations on \( M_0 \) corresponding to \( \rho \), i.e. \( A_i(\rho_l) = \delta_{i,l} \) on appropriate neighbourhoods. Then
\[ \int_{(N, \gamma')} \omega = \int_{(N, \gamma)} \omega + \sum_{\alpha} \sum_{H_\gamma \in J_H} \sum_{j \in J_H} \int_{H_0} ((\gamma \omega_j).A^{j\gamma})|_{H_0, \rho^*}, \]
where \( \omega_j \) is at in Theorem 5.15.

**Proof.** One only needs to check for \( H \in B(\gamma^*(\rho)) \) and \( j \in J_H \)
\[ ((\gamma \omega).A^j)|_{H_0, \rho} = (-1)^{s(\dim N) + s(\dim H)} \gamma_H^1((\omega.D^j)|_{H, \gamma^*(\rho)}), \]
where \( D = (D_1, \ldots, D_n) \) is a the family of boundary superderivations as in Theorem 5.15.

This can be done locally. Write \( \omega = fDx \) with \( x = (u, \xi) = (\gamma^*(\rho), \dot{x}) \). Then, similarly to Step 2 of the proof of Theorem 5.15 one sees
\[ D_s = \partial_{u_s} + \sum_{l=k+1}^{p+q} a_l \partial_{x_l}, \quad A_s = \partial_{u_s, o} + \sum_{l=k+1}^p b_l \partial_{u_l, o} \]
for some \( s \) and \( k = \text{codim} H_0 \), hence, using an analogous argument as in Step 3 of the proof of Theorem 5.15 one sees
\[ ((\gamma \omega).A^l)|_{H_0, \rho} = (\gamma \omega).\partial_{u_s, o}|_{H_0, \rho} = \gamma((\omega, \partial_{u_s})|_{H_0, \rho}) = \gamma((\omega.D^l)|_{H_0, \rho}). \]
It follows from Lemma 5.13 that
\[ ((\gamma \omega).A^j)|_{H_0, \rho} = \gamma((\omega.D^l)|_{H_0, \rho} = \pm \gamma H_1((\omega.D^l)|_{H, \gamma^*(\rho)}). \]

**Examples 5.18.**
(i) Let \( N \subset M = \mathbb{R}^{2,4} \) be the superdomain with \( N_0 = [0, 1]^2 \) and \( \gamma, \gamma' \) be retractions on \( M \). As we have seen earlier, boundary functions for \( N_0 \) are given by \( \rho = (\text{pr}_1, \text{pr}_2, 1 - \text{pr}_1, 1 - \text{pr}_2) \). Adequate boundary derivations might be \( D = (\partial_{u_1}, \partial_{u_2}, -\partial_{u_1}, -\partial_{u_2}) \), where we choose \( x = (u, \xi) = (\gamma^*(\text{pr}_1), \gamma^*(\text{pr}_2), \xi). \)
We count the number of summands which can be non-zero. Since \( q = 4 \) we obtain 1 summand for each dimension 0 boundary manifold, and 2 summands for each of dimension 1, resulting in 1 + 4 \( \cdot \) 2 + 4 \( \cdot \) 1 = 13 summands.

(ii) We reconsider Example 4.7 (ii) in light of the above theorem. Recall the simple calculation shows

\[
\begin{align*}
\gamma(r) &= \gamma^s(u_1,0) = u_1, \\
\gamma'(v_1,0) &= (1 + \frac{\eta_1 \eta_2}{v_1} + \frac{\eta_2}{v_2}), \\
\gamma'(v_2,0) &= (1 + \frac{\eta_1 \eta_2}{v_1} + \frac{\eta_2}{v_2}).
\end{align*}
\]

This shows that \( \gamma \) can be continued to a retraction on \( \Omega'' \), where \( \Omega'' = \Omega' \setminus \{0\} \). Unfortunately, \( \Omega'' \) cannot be considered as manifold with corners in the ambient space \( \Omega_0 \).

For this reason, let \( \Omega_\varepsilon \subset \Omega \) for \( 0 < \varepsilon < 1 \) be given by \( \Omega_\varepsilon = \{(v_1, v_2) \mid \varepsilon^2 < v_1^2 + v_2^2 < 1\} \). We turn \( \Omega_\varepsilon \) to a supermanifold with corners in \( \Omega \) via the boundary function \( \rho \) given by

\[
\rho = r - \varepsilon, \quad r = \sqrt{v_1^2 + v_2^2}.
\]

Of course, \( r = u_{1,0} \) on \( \Omega' \).

Since \( q = 2 \), we do not need to find any boundary derivation (although it is easy to see that the radial operator \( v_{1,0} \partial_{v_{1,0}} + v_{2,0} \partial_{v_{2,0}} \) is a boundary derivation). Let \( f \in \mathcal{O}(\Omega) \) be compactly supported. Using

\[
\gamma'(\rho) - \gamma^s(\rho) = \sqrt{v_1^2 + v_2^2} - \sqrt{v_1^2 + v_2^2}
\]

we get by Corollary 5.17

\[
\int_{(\Omega, y)} f|\mathcal{D}y| = \lim_{\varepsilon \to 0} \int_{(\Omega_\varepsilon, \gamma')} f|\mathcal{D}y| = \lim_{\varepsilon \to 0} \int_{(\Omega_\varepsilon, \gamma')} f|\mathcal{D}y| + \int_{\varepsilon S^1} \gamma_1 \left( -\frac{\eta_1 \eta_2}{\gamma^s(r)} f|\mathcal{D}y| \right) + \frac{\eta_2}{\gamma^s(r)} f_0|\mathcal{D}y|.
\]

For the application of \( \gamma_1 \) we need to use \( |\mathcal{D}y| \) with \( \hat{y} = (\gamma^s(v_0), \eta) \). A simple calculation shows \( \frac{|\mathcal{D}y|}{|\mathcal{D}y|} = 1 \). Furthermore, we recognise that \( |dv_0| \) is the standard density on \( \mathbb{R}^2 \). In Remark 5.12 (ii) we saw that this just means \( |dv_0|_{\varepsilon S^1} = dS \), which leads to

\[
\gamma_1 \left( -\frac{\eta_1 \eta_2}{\gamma^s(r)} f|\mathcal{D}y| \right) + \frac{\eta_2}{\gamma^s(r)} f_0|\mathcal{D}y| = -(-1)^{s(2,2)} \frac{f_0}{\varepsilon} dv_0|_{\varepsilon S^1, \rho} = -(-1)^{s(2,2)} \frac{f_0}{\varepsilon} dS.
\]

Since

\[
\lim_{\varepsilon \to 0} \int_{\varepsilon S^1} \frac{f_0}{\varepsilon} dS = 2\pi f_0(0),
\]

we have

\[
\int_{(\Omega, y)} f|\mathcal{D}y| = \lim_{\varepsilon \to 0} \int_{(\Omega_\varepsilon, \gamma')} f|\mathcal{D}y| = \lim_{\varepsilon \to 0} \int_{(\Omega_\varepsilon, \gamma')} f|\mathcal{D}y| + \int_{\varepsilon S^1} \gamma_1 \left( -\frac{\eta_1 \eta_2}{\gamma^s(r)} f|\mathcal{D}y| \right) + \frac{\eta_2}{\gamma^s(r)} f_0|\mathcal{D}y|.
\]
we see again
\[
\int_{(\Omega, y)} f|Dy| = \int_{(\Omega', \gamma)} f|Dy| - (-1)^{s(2,2)} 2\pi f_0(0).
\]

One may give a version of the change of variables formula for Berezin forms by considering induced orientations on the boundary manifolds. To do so, one has to fix an ordering of the boundary functions \(\rho\) and has to keep track of the boundary orientations; on the boundary manifolds of dimension 0, this leads to additional signs.

We do not state the resulting formula in full generality, since it is somewhat cumbersome. However, in the case of a supermanifold with boundary (i.e. for the case of only one boundary supermanifold), the theorem can be easily restated for Berezin forms, as follows.

**Corollary 5.19.** Let \(U \subset M\) with smooth boundary \(\partial U_0\) and let \(\gamma, \gamma'\) be retractions on \(M\). Then for compactly supported \(\omega \in \text{Ber} M\) we have
\[
\int_{(U, \gamma')} \omega = \int_{(U, \gamma)} \omega - \pm \sum_{j=1}^{\lfloor \frac{q}{2} \rfloor} \frac{1}{j!} \int_{(\partial_\gamma U, \partial_\gamma)} ((\gamma'^*(\rho) - \gamma^*(\rho))\omega, D^{j-1})|_{\partial_\gamma U, \gamma^*(\rho)}.
\]
Here, \(\rho\) is a boundary function for \(U_0\) and \(D\) is a boundary derivation corresponding to \(\gamma^*(\rho)\). The sign \(\pm\) is given by \((-1)^{s(p,q)+s(p-1,q)}\).

The additional minus sign occurring in the above formula comes from the fact that the boundary derivations define inner normals.

In [Example 3.9] we saw that it is important to choose the right immersion \(\iota: \partial U \to M\) to arrive at the usual formulation of Stokes’s theorem. In the case of an arbitrary boundary supermanifold \(\iota: U \to M\), one can apply the corollary to show the following generalisation of Stokes’s theorem (where, of course, additional boundary terms have to appear).

**Corollary 5.20.** Let \(U \subset M\) be a supermanifold with boundary such that \(U_0\) be compact, and let \(\omega = d\varpi\) be an exact Berezin form. Then
\[
\int_{(U, \gamma')} \omega = (-1)^{s(p,q)+s(p-1,q)}\left((-1)^q \int_{\partial U} \iota^*(\varpi) - \sum_{j=1}^{\lfloor \frac{q}{2} \rfloor} \int_{\partial U} (\omega_j, D^{j-1})|_{\partial U, \iota^*(\tau)}\right),
\]
where \(\tau\) is chosen such that \(\iota^*(\tau) = 0\) and \(\omega_j := \frac{1}{j!} (\gamma'^*(\tau_0) - \tau)^j \varpi\).

Note that in this formula, the retraction \(\gamma\) does not occur any longer.

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