ON CENTRALITY OF $K_2$ FOR CHEVALLEY GROUPS OF TYPE $E_l$

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Abstract. For a root system $\Phi$ of type $E_l$ and arbitrary commutative ring $R$ we show that the group $K_2(\Phi, R)$ is contained in the center of the Steinberg group $St(\Phi, R)$. In course of the proof we also demonstrate an analogue of Quillen—Suslin local-global principle for $K_2(\Phi, R)$.

1. Introduction

Let $\Phi$ be a reduced irreducible root system and $R$ be a commutative ring. Denote by $G(\Phi, R)$ the simply connected Chevalley group of type $\Phi$ over $R$ and by $E(\Phi, R)$ its elementary subgroup. Taddei’s theorem asserts that $E(\Phi, R)$ is a normal subgroup of $G(\Phi, R)$ provided that $\text{rk}(\Phi) \geq 2$ (see [5]).

Let $St(\Phi, R)$ be the Steinberg group of type $\Phi$ over $R$. Denote by $\phi$ the canonical map $St(\Phi, R) \to G(\Phi, R)$ sending each formal generator $x_\alpha(\xi)$ to the elementary root unipotent $t_\alpha(\xi)$. The image of $\phi$ equals $E(\Phi, R)$. The unstable $K$-groups of M. Stein $K_1(\Phi, R)$ and $K_2(\Phi, R)$ are defined as the cokernel and the kernel of $\phi$ (see [13, § 1A]):

$$1 \longrightarrow K_2(\Phi, R) \longrightarrow St(\Phi, R) \xrightarrow{\phi} G(\Phi, R) \longrightarrow K_1(\Phi, R) \longrightarrow 1.$$ (1.1)

One of the standard conjectures in the theory of Chevalley groups over rings asserts that $St(\Phi, R)$ is a central extension of $E(\Phi, R)$ provided that the rank of $\Phi$ is sufficiently large. We refer to this conjecture as centrality of $K_2$. For groups of rank 2 centrality of $K_2$ fails (see [22, Theorem 1]).

W. van der Kallen in [3] and recently A. Lavrenov in [8] proved centrality of $K_2$ for $\Phi = A_l, C_l$, $l \geq 3$. The main technical ingredient in both proofs is so-called method of “another presentation” which consists in presenting $St(\Phi, R)$ as a group with generators modeling root unipotents of $E(\Phi, R)$. The key advantage of this method lies in the fact that it yields a well-defined action of $G(\Phi, R)$ on $St(\Phi, R)$ and shows that the above map $\phi$ defines the structure of a crossed module on the pair $(St(\Phi, R), G(\Phi, R))$. This simultaneously implies both centrality of $K_2$ and normality of the elementary subgroup.

However, the method of “another presentation” relies on rather bulky and technical reasoning and, moreover, requires separate analysis for each type of $\Phi$. This is caused by the fact that the description of root unipotents essentially depends on the “global geometry” of $G(\Phi, R)$ (e.g. the structure of the minimal representation of $G(\Phi, R)$). The description of root unipotents in the case $\Phi = E_l$ is particularly complicated and is only known for fields (see e.g. [11] and further references therein).

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The main goal of the present work is to obtain a short unified proof of centrality of $K_2$ for $\Phi = E_l$, $l = 6, 7, 8$ bypassing the description of root unipotents in these cases. More precisely, our main result is the following theorem.

**Theorem 1.** Let $\Phi = E_l$, $l = 6, 7, 8$. Then $K_2(\Phi, R)$ is contained in the center of the Steinberg group $\text{St}(\Phi, R)$.

The key ingredient in our proof of Theorem 1 is the following local-global principle.

**Theorem 2.** Let $\Phi = E_l$, $l = 6, 7, 8$. An element $\alpha \in \text{St}(\Phi, R[[t]], tR[[t]])$ is trivial if and only if its image in $\text{St}(\Phi, R_M[[t]])$ is trivial for every maximal ideal $M \trianglelefteq R$.

For $\Phi = A_l$, $l \geq 4$ similar result has been obtained by M. Tulenbaev (cf. [18, Theorem 2.1]). In turn, Tulenbaev’s theorem should be considered as a $K_2$-analogue of Quillen’s local-global principle. Its proof is heavily influenced by A. Suslin’s local-global principle for $K_1$ (see [15, Theorem 3.1]).

Centrality of $K_2$ has a number of important corollaries. Let us mention some of them pertaining to the theory of lower $K$-functors modeled on Chevalley groups (see [12], [13]).

**Corollary 3.** In the assumptions of Theorem 1 the following facts hold.

(a) The group $\text{St}(\Phi, R)$ is a universal central extension of $E(\Phi, R)$.

(b) The relative Steinberg group $\text{St}(\Phi, R, I)$ defined in section 3 is a universal relative central extension of the pair $(\text{St}(\Phi, R/I), \text{St}(\Phi, R))$ in the sense of [10, §3].

(c) If one defines $K_2(\Phi, R, I)$ as the kernel of $\phi: \text{St}(\Phi, R, I) \to E(\Phi, R, I)$ then there is an exact sequence

$$H_3(\text{St}(\Phi, R)) \to H_3(\text{St}(\Phi, R/I)) \to K_2(\Phi, R, I) \to K_2(\Phi, R) \to K_2(\Phi, R/I) \to \ldots$$

$$\ldots \to K_1(\Phi, R, I) \to K_1(\Phi, R) \to K_1(\Phi, R/I).$$

(d) Set $X(R) = BG(\Phi, R)[E(\Phi, R)\otimes R]$. The unstable groups $K_i(\Phi, R)$ of M. Stein coincide with $\pi_i(X(R))$, $i = 1, 2$ and the following unstable analogue of Gersten formula holds: $\pi_3(X(R)) \cong H_3(\text{St}(\Phi, R), \mathbb{Z})$.

Tulenbaev’s proof of [18, Theorem 2.1] crucially depends on a certain property of the linear Steinberg group which is in essence a particular case of excision property (see Remark 3.11). In turn, our proof of Theorem 2 goes as follows. First, we show that the relative Steinberg group of type $E_l$ can be presented as amalgamated product of several copies of Steinberg groups of type $A_4$. Next, we use this presentation to extend the specified property from pieces of type $A_4$ to the whole group $\text{St}(E_l, R, I)$.

The rest of the article is organized as follows. In 2 we introduce principal notation and prove certain facts about root systems. In 3 we define unstable relative Steinberg groups following the approach of F. Keune and J.-L. Loday. We also state some results comparing this definition with the one used by M. Tulenbaev in [18]. Finally, in 4 we prove our main results.

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2. Preliminaries

2.1. Root-theoretic lemmata. In what follows \( \Phi \) denotes a reduced irreducible root system and \( \Pi \subseteq \Phi \) denotes its basis (i.e. the set of simple roots). Denote by \( \tilde{\alpha}, \Phi^+ \) and \( \Phi^- \), respectively, the maximal root of \( \Phi \) and the subsets of positive and negative roots of \( \Phi \). The Dynkin diagram and the extended Dynkin diagram of \( \Phi \) corresponding to \( \Pi \) will be denoted by \( D(\Phi), \tilde{D}(\Phi) \), respectively.

A root subset \( S \subseteq \Phi \) is called parabolic (resp. reductive, resp. special) if \( \Phi = S \cup -S \) (resp. \( S = -S \), resp. \( S \cap -S = \emptyset \)). Any parabolic subset \( S \subseteq \Phi \) can be decomposed into the disjoint union of its reductive and special part, i.e. \( S = \Sigma_S \sqcup \Delta_S \), where \( \Sigma_S \cap (-\Sigma_S) = \emptyset \), \( \Delta_S = -\Delta_S \).

Denote by \( m_{\beta}(\alpha) \) the coefficient of \( \beta \) in the expansion of \( \alpha \) in \( \Pi \), i.e. \( \alpha = \sum_{\beta \in \Pi} m_{\beta}(\alpha)\beta \) for \( \beta \in \Pi \) denote by \( \Delta_\beta \) the subsystem of \( \Phi \) spanned by all simple roots except \( \beta \) and by \( \Sigma_\beta \) the set consisting of roots \( \alpha \in \Phi \) such that \( m_{\beta}(\alpha) > 0 \).

We denote by \( \langle \alpha, \beta \rangle \) the scalar product of roots and by \( \langle \alpha, \beta \rangle \) the integer \( 2(\alpha, \beta)/(\alpha, \alpha) \). The Weyl group \( W(\Phi) \) is a subgroup of isometries of \( \Phi \) generated by all reflections \( \sigma_\alpha \), where \( \sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \cdot \alpha \).

For a simply laced \( \Phi \) (i.e. \( \Phi \) such that \( \tilde{D}(\Phi) \) does not contain multiple bonds) denote by \( \Phi' \) the subsystem of \( \Phi \) consisting of all roots orthogonal to \( \tilde{\alpha} \). From a consideration of \( \tilde{D}(\Phi) \) it follows that \( \Phi' \) has the following type (depending on the type of \( \Phi \)).

| Type of \( \Phi \) | \( A_{l-2} \) | \( A_{l} + D_{l-2} \) | \( E_6 \) | \( E_7 \) | \( E_8 \) |
|-------------------|-------------|-----------------|------|------|------|
| Type of \( \Phi' \) | \( A_{l-2} \) | \( A_{l} + D_{l-2} \) | \( E_6 \) | \( E_7 \) | \( E_8 \) |

Lemma 1. Let \( \Phi \) be a simply laced irreducible root system of rank \( \geq 3 \). Every two roots \( \alpha, \beta \in \Phi \) are contained in some subsystem \( \Psi \subseteq \Phi \) of type \( A_3 \).

Proof. If \( \alpha, \beta \) are not orthogonal then they are contained in some subsystem of type \( A_2 \) and hence lie in some subsystem of type \( A_3 \).

Now assume that \( \alpha \perp \beta \). We can choose a basis of \( \Phi \) so that \( \alpha = -\tilde{\alpha} \). Denote by \( \gamma \) any of the simple roots adjacent to \( \alpha \) on \( \tilde{D}(\Phi) \).

If \( \Phi = D_l \), \( l \geq 4 \) and \( \beta \) is contained in a component of \( \Phi' \) of type \( A_1 \), then \( \Psi := \langle \alpha, \gamma, \beta \rangle \) satisfies the requirement of the lemma. If \( \text{rk} \Phi \geq 5 \) denote by \( \Phi'' \) either \( \Phi' \) itself (if \( \Phi \neq D_l \)) or the irreducible component \( D_{l-2} \) of \( \Phi' \) (if \( \Phi = D_l \)). In the remaining case we may assume that \( \beta \) lies in \( \Phi'' \). Denote by \( \delta \in \Phi'' \) the simple root adjacent to \( \gamma \). The Weyl group \( \tilde{W}(\Phi'') \) fixes \( \alpha \) and acts transitively on \( \Phi'' \). Therefore, there exists \( w \in \tilde{W}(\Phi'') \) such that \( w(\delta) = \beta \) and \( \Psi := w(\langle \alpha, \gamma, \delta \rangle) \) is the desired subsystem. \( \square \)

Denote by \( A_n(\Phi) \) the set of all subsystems \( \Psi \subseteq \Phi \) of type \( A_n \). The group \( \tilde{W}(\Phi) \) naturally acts on \( A_n(\Phi) \).

Lemma 2. Let \( \Phi \) be an irreducible simply laced root system of rank \( \geq 3 \). Let \( \Psi_0, \Psi_1 \subseteq \Phi \) be two subsystems of type \( A_3 \) containing a common root \( \alpha \). Then either \( \Psi_0 \cap \Psi_1 \) has type \( A_2 \), or there exists a subsystem \( \Psi \subseteq \Phi \) of type \( A_3 \) containing \( \alpha \) such that \( \Psi_0 \cap \Psi, \Psi_1 \cap \Psi \) both have type \( A_2 \).
Proof. As in the proof of the previous lemma we may assume that $\alpha = -\bar{\alpha}$. Denote by $\gamma$ any simple root adjacent to $\alpha$ on $\tilde{D}(\Phi)$ (it is unique if $\Phi \neq A_l$, otherwise there is only one other simple root, denote it by $\gamma'$).

For any $\beta_1, \beta_2 \in \Sigma_\gamma$ the subsystem $\Psi = \langle \alpha, \beta_1, \beta_2 \rangle$ has type either $A_2$ or $A_3$. Moreover, any $\Psi \in A_3(\Phi)$ containing $\alpha$ can be obtained in this fashion. Indeed, if $\Phi \neq A_l$ this follows from the fact that for any $\beta \in \Phi^+$ not orthogonal to $\alpha$ one has $m_\gamma(\beta) = \frac{(\beta, \alpha)}{(\gamma, \alpha)} = 1$.

In the case $\Phi = A_l$ it is possible that $m_\gamma(\beta) = 1$, $m_\gamma(\beta) = 0$. In this case one should additionally use the fact that $\beta' = \bar{\alpha} - \beta \in \Phi^+$ and $m_\gamma(\beta') = 1$.

Now if $\Psi_0 = \langle \alpha, \beta_1, \beta_2 \rangle$, $\Psi_1 = \langle \alpha, \beta_3, \beta_4 \rangle$ have type $A_3$ and $\Psi_0 \cap \Psi_1 = \langle \alpha \rangle$ then $\Psi = \langle \alpha, \beta_1, \beta_3 \rangle$ again has type $A_3$ while both intersections $\Psi \cap \Psi_0, \Psi \cap \Psi_1$ have type $A_2$.

\[\square\]

Lemma 3. For $\Phi = E_l$, $l = 6, 7, 8$ any subsystem $\Psi \subseteq \Phi$ of type $A_3$ is contained in some subsystem $\Psi' \subseteq \Phi$ of type $A_4$.

Proof. For $\Phi$ of type $E_l$ the action of $W(\Phi)$ on $A_3(\Phi)$ is transitive (see discussion after [3, Theorem 5.4]). It suffices to find on $\tilde{D}(\Phi)$ a subdiagram of type $A_3$ which is contained in a subdiagram of type $A_4$.

\[\square\]

Remark 2.1. Similar assertion for $\Phi = D_l$ is false even for $l \geq 5$. The reason for this lies in the fact there are two orbits of $A_3(D_l)$ under the action of $W(D_l)$, $l \geq 5$. In Tables 4–5 of [2] these orbits are labeled as $A_3$ and $D_3$. A subsystem contained in the orbit $A_3$ can be embedded into a subsystem of type $A_4$, while this is not true for a subsystem lying in the orbit $D_3$.

2.2. Steinberg groups.

Definition 2.2. If $\text{rk}(\Phi) \geq 2$ the Steinberg group $\text{St}(\Phi, R)$ can be defined as the group given by generators $x_\alpha(\xi), x \in R, \alpha \in \Phi$ and the following set of relations.

\[x_\alpha(s)x_\alpha(t) = x_\alpha(s + t),\]

\[[x_\alpha(s), x_\beta(t)] = \left(\prod x_{i\alpha + j\beta}(N_{\alpha, \beta, i, j}s^it^j)\right), \quad \alpha \neq -\beta, N_{\alpha, \beta, i, j} \in \mathbb{Z}.\]

The indices $i, j$ appearing in the right-hand side of the above formula range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The integers $N_{\alpha, \beta, i, j}$ are called structure constants of the Chevalley group $G(\Phi, R)$ and depend only on $\Phi$. More information on the structure constants and Chevalley groups can be found in [21, § 9].

Let $I \leq R$ be an additive subgroup and $\alpha \in \Phi$ be a root. Denote by $X_\alpha(I)$ the additive subgroup of $\text{St}(\Phi, R)$ generated by $x_\alpha(s), s \in I$. Such a subgroup will be called a root subgroup.

Generators $x_\alpha(\xi)$ should be thought of as formal symbols modeling elementary root unipotents $t_\alpha(\xi) \in E(\Phi, R) \leq G(\Phi, R)$, while relations (2.3)–(2.4) are elementary relations between $t_\alpha(\xi)$ which hold over arbitrary $R$.

Throughout this paper we will be mainly concerned with root systems whose Dynkin diagram is simply laced. In this case formula (2.4) further simplifies to

\[\text{[}x_\alpha(s), x_\beta(t)\text{]} = 1, \quad \alpha + \beta \notin \Phi;\]

\[\text{(2.5)}\]
(2.6) \[ [x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(N_{\alpha\beta}st), \quad \alpha + \beta \in \Phi, \text{ where } N_{\alpha\beta} = \pm 1. \]

The integers \( N_{\alpha\beta} \) mentioned above are precisely the structure constants of the simple complex Lie algebra \( L_c(\Phi) \) of type \( \Phi \) with respect to some positive Chevalley base \( \{e_\alpha\} \) (see \[21, \S 1\]), \[21, \S 2\]), i.e. \( [e_\alpha, e_\beta] = N_{\alpha\beta}e_{\alpha+\beta}, \alpha, \beta \in \Phi. \)

**Remark 2.7.** For \( \xi \in R^* \) set \( w_\alpha(\xi) := x_\alpha(\xi)x_{-\alpha}(-\xi^{-1})x_\alpha(\xi). \) If the rank of \( \Phi \) is at least 2 then the following identity is a consequence of 2.3–2.4:

\[
(2.8) \quad w_\alpha(\xi)x_\beta(s)w_\alpha(\xi)^{-1} = x_{\sigma_{\alpha}\beta}(\eta_{\alpha,\beta}\xi^{-(\beta,\alpha)}s), \quad \xi, s \in R.
\]

In the above formula \( \eta_{\alpha,\beta} \) are equal \( \pm 1 \) and can be deduced from the structure constants (see \[21, \S 13\]).

**Remark 2.9.** The definition 2.2 is not suitable in the case \( \Phi = A_1 \) since the set of relations of type 2.4 becomes empty. There exist different definitions of the Steinberg group of rank 1. For example, M. Stein defined \( \text{St}(A_1, R) \) as a group given by relations 2.3, 2.8 (cf. with \[12, \text{Definition 3.7}\]). Some authors further enforce all the standard relations for elements \( h_\alpha, \ w_\alpha \) (see e.g. relations 1–7 of \[1\]). The statements of our theorems below do not depend on the precise definition of \( \text{St}(A_1, R) \).

We now turn to functoriality properties of Steinberg groups. Clearly, there exists a well-defined Steinberg group functor \( \text{St}(\Phi, -): \text{Rings} \to \text{Grp} \). For a map of rings \( f: R \to S \) the induced map \( \text{St}(\Phi, f) \) sending each generator \( x_\alpha(\xi) \) to \( x_\alpha(f(\xi)) \) will be denoted either by the same letter \( f \) or by \( f_* \) (we choose more convenient notation depending on the context).

**Notation 2.10.** Let \( I \subseteq R \) be an ideal of \( R \). For \( s \in I, \xi \in R \) set \( Z_\alpha(s, \xi) := x_\alpha(s)x_{-\alpha}(\xi). \)

**Lemma 4.** Let \( \Phi \) be an irreducible root system of rank \( \geq 2 \). Let \( \pi: R \to R/I \) be the canonical projection. Set \( G := \text{Ker}(\pi_*: \text{St}(\Phi, R) \to \text{St}(\Phi, R/I)). \)

1. The group \( G \) is generated by elements \( Z_\alpha(s, \xi) \), where \( s \in I, \xi \in R, \alpha \in \Phi. \)
2. Let \( S \) be a parabolic set of roots of \( \Phi \) with special part \( \Sigma_S \). Then \( G \) is generated by the following two families of elements: \( x_\alpha(s) \), where \( s \in I, \alpha \in \Phi \) and \( Z_\alpha(s, \xi) \), where \( s \in I, \xi \in R, \alpha \in \Sigma_S. \)

**Proof.** The first statement of the lemma is well-known if one replaces \( \text{St}(\Phi, R) \) by \( E(\Phi, R) \) (see e.g. \[14, \text{Proposition 3.2}\], \[19, \text{Theorem 2}\]). Both these proofs rely solely on calculations with Chevalley commutator formula and hence can be reproduced verbatim for Steinberg groups. The stronger second statement is an analogue of \[14, \text{Theorem 3.4}\]. \( \square \)

Let \( \Phi \) be a simply laced root system. Denote by \( \text{Subsys}(\Phi) \) the category of root subsystems of \( \Phi \). Its objects are all root subsystems \( \Psi \subseteq \Phi \) and its morphisms are in one-to-one correspondence with inclusions of the form \( \Psi_1 \subseteq \Psi_2. \)

**Lemma 5.** The definition of the Steinberg group is functorial with respect to \( \Phi \) in the sense that there is a functor \( \text{St}(-, R): \text{Subsys}(\Phi) \to \text{Grp}. \)

**Proof.** Fix signs \( N_{\alpha, \beta}: \Phi^2 \to \{\pm 1\} \) of the structure constants for \( \Phi \). The definition of \( L_c(\Phi) \) is functorial with respect to \( \Phi \), therefore, one can obtain structure constants for any subsystem \( \Psi \subseteq \Phi \) by restricting \( N_{\alpha, \beta} \) on \( \Psi^2 \subseteq \Phi^2 \). \( \square \)
Definition 2.11. Denote by $G_{1,n}$ the full subcategory of $\text{Subsys}(\Phi)$ with $\text{Ob}(G_{1,n}) := A_1(\Phi) \sqcup A_n(\Phi)$. Clearly, $G_{1,n}$ is a directed bipartite graph whose only morphisms are inclusions of the form $i_{\gamma,\Psi} : \langle \gamma \rangle \hookrightarrow \Psi$, $\gamma \in \Psi$, $\Psi \in A_3(\Phi)$.

The next lemma will not be used directly in what follows. Nevertheless, it illustrates an important idea that Steinberg groups can be presented as amalgamated products of Steinberg groups of smaller rank.

Lemma 6. Let $\Phi$ be a simply laced irreducible root system of rank $\geq 3$. Then $\text{St}(\Phi, R)$ is isomorphic to $G := \varinjlim_{G_{1,3}} \text{St}(-, R)$.

Proof. Clearly, $G$ can be interpreted as the free product of $|A_3(\Phi)|$ copies of $\text{St}(A_3, R)$ modulo relations which identify the images of each generator $x_\alpha(\xi)$ with respect to $\text{St}(i_{\alpha,\Psi}, R)$. This shows that $G$ and $\text{St}(\Phi, R)$ have the same set of generators. All the defining relations of $G$ hold in $\text{St}(\Phi, R)$. It suffices to show the converse, namely that every defining relation of $\text{St}(\Phi, R)$ occurs in the set of defining relations of $\text{St}(\Psi, R)$ for some $\Psi \in A_3(\Phi)$. The latter statement is a consequence of the form of relations 2.5–2.6 and Lemma 1. \qed

Remark 2.12. A much stronger variant of Lemma 4 called Curtis–Tits presentation is known for Steinberg groups (see [1], Corollary 1.3]). This presentation is formulated in terms of subdiagrams of $D(\Phi)$ of ranks 1, 2 rather than subsystems of $\Phi$ of ranks 1, 3 and holds for arbitrary root systems (not necessarily simply laced). Moreover, this presentation remains valid in the context of Kac–Moody groups, i.e. it holds for Steinberg-like groups obtained from generalized Cartan matrices.

2.3. Van der Kallen’s “another presentation” of the linear Steinberg group.
Recall that a column $v = (v_1, \ldots, v_n)^t \in R^n$ is called unimodular if $Rv_1 + \ldots + Rv_n = R$. More generally, a matrix $v \in M_{n \times m}(R)$ is called unimodular if there exists $u \in M_{m \times n}(R)$ such that $uv$ is the identity matrix of size $m$. We denote the set of unimodular columns by $\text{Um}(n, R)$.

Let $v$ be a unimodular column of height $n$, and $u$ be a row of length $n$ such that $uv = 0$. Notice that under these assumptions matrix $e + vu$ is invertible and elementary (with inverse $e - vu$). Such matrices are called (linear) transvections.

Let $n \geq 4$. Consider the group $\text{St}(n, R)$ defined by generators $X_{v,u}$ and relations

\begin{equation} X_{v, u_1 + u_2} = X_{v, u_1} \cdot X_{v, u_2}; \end{equation}

\begin{equation} X_{v, u} X_{v', u'} = X_{(e - v'u')v, u(e + v'u')}. \end{equation}

The following relation follows from \ref{2.13}–\ref{2.14} (see discussion preceding Lemma 1.1 of [18]):

\begin{equation} X_{v_1 + bv_2, u} = X_{v_1, u} \cdot X_{v_2, bu}, \end{equation}

if $b \in R$, $(v_1, v_2) \in M_{n \times 2}(R)$ is a unimodular matrix and $w \in nR$ is a row such that $wv_1 = wv_2 = 0$.

The following theorem is the main result of [3].

Theorem 4. For $l \geq 3$ the map sending each generator $x_{ij}(\xi)$ to $X_{e_i, \xi e_j}$ defines an isomorphism of groups $\text{St}(A_l, R)$ and $\text{St}(l + 1, R)$. 

Remark 2.16. Below we continue to denote by St(n, −) linear Steinberg groups in “another presentation”. At the same time we retain notation St(A, −) for groups in the “usual presentation”.

3. Relative Steinberg groups

Recall that in the context of [12] an ordered pair (R, I) consisting of a ring R and an ideal I ⊆ R is called simply a pair. A morphism of pairs \( \varphi: (A, I) \to (A', I') \) is a ring morphism \( \varphi: A \to A' \) satisfying \( \varphi(I) \subseteq I' \). We denote the category of pairs by \( \text{Prs} \).

Let R be a ring and I ⊆ R be an ideal. Denote by \( D(R, I) \) the double of R along I, i.e. the pullback of two copies of R over \( R/I \).

\[
\begin{array}{c}
D(R, I) \xrightarrow{p_1} R \\
p_2 \downarrow \quad \pi \downarrow \\
R \xrightarrow{\pi} R/I.
\end{array}
\]

The elements of \( D(R, I) \) can be interpreted as ordered pairs \( a = (a_1, a_2) \in R^2 \) such that \( a_1 - a_2 \in I \). Maps \( p_i \) are defined by \( p_i(a) = a_i, i = 1, 2 \). Both \( p_1 \) and \( p_2 \) are split by the diagonal map \( \Delta: R \to D(R, I) \). Clearly, \( D \) is a functor from \( \text{Prs} \) to \( \text{Rings} \).

Definition 3.2. Let R be a ring and A be a (not necessarily unital) R-algebra. Consider the product \( R \times A \) as an abelian group with respect to componentwise addition. Give the ring structure on \( R \times A \) using the formula \( (a, b)(c, d) = (ac, ad + bc + bd) \). The resulting unital ring is denoted by \( R \ltimes A \). Clearly, \( 0 \times A \) is an ideal in \( R \ltimes A \).

It is easy to see that \( R \cong R/I \times I \) whenever quotient map \( R \to R/I \) admits section. Applying this consideration to the exact sequence \( 0 \times I \hookrightarrow D(R, I) \xrightarrow{p_1} R \) we get that \( D(R, I) \cong R \ltimes I \) for any \( R, I \).

3.1. The definition of the relative Steinberg group. In the current section we define relative Steinberg groups and study their basic properties. We follow the approach used by F. Keune and J.-L. Loday in the stable situation (cf. \[7\], \[10\]).

Set \( G_i := \text{Ker}(p_i) \). Denote by \( C \) the mixed commutator subgroup \( [G_1, G_2] \leq \text{St}(\Phi, D(R, I)) \). Clearly, \( C \) is a normal subgroup in \( G_1 \) contained in \( K_2(\Phi, D(R, I)) \).

Definition 3.3. Define the relative Steinberg group \( \text{St}(\Phi, R, I) \) as the quotient \( G_1/C \).

It is easy to see that the relative Steinberg group is a central extension of the normal subgroup of \( \text{St}(\Phi, R) \) spanned by elements \( x_\alpha(s) \) of level I (i.e. such that \( s \in I \)). More precisely, there is an exact sequence

\[
\begin{array}{c}
1 \longrightarrow (G_1 \cap G_2)/C \longrightarrow \text{St}(\Phi, R, I) \xrightarrow{p_2} \text{St}(\Phi, R) \xrightarrow{\pi} \text{St}(\Phi, R/I) \longrightarrow 1.
\end{array}
\]

Conjugation action of \( \text{St}(\Phi, R) \) on \( \text{St}(\Phi, R, I) \) induces an action on \( (G_1 \cap G_2)/C \). Notice that centrality of \( K_2(\Phi, R) \) would imply triviality of this action (cf. \[10\], Proposition 8]). The definition of \( \text{St}(\Phi, R, I) \) is functorial in both \( (R, I) \) and \( \Phi \), i.e. there is a functor \( \text{St: Subsys}(\Phi) \times \text{Prs} \to \text{Grp} \) (cf. Lemma [3]).

In one important case the relative Steinberg group will be a subgroup of the absolute group \( \text{St}(\Phi, R) \).
Lemma 7. Let $\Phi$ be a root system of rank $\geq 2$ and $I \leq R$ be an ideal such that the canonical map $R \rightarrow R/I$ splits. Then the map $\overline{\Phi}$ from sequence 3.4 is injective.

Proof. Clearly, one has $D(R, I) = R \times_{R/I} R \cong (R/I \times I) \times_{R/I} (R/I \times I) \cong R/I \times (I \times I)$. Consider two maps $\theta_1, \theta_2: R \rightarrow D(R, I)$ sending $(\xi, s)$ to $(\xi, (s, 0))$ and $(\xi, (0, s))$, respectively. Let $g$ be an element of $G_1 \cap G_2$. Clearly, $g$ lies in the kernel of $(\pi p_1)_* = (\pi p_2)_*$ and hence by Lemma 3 it can be presented as a finite product of generators of the form $Z_\alpha((0, (s_1, s_2)), \eta), s_1, s_2 \in I, \eta \in D(R, I)$. Without loss of generality, we may assume that $\eta = (\xi, (0, 0)) \in R/I$, indeed:

$$Z_\alpha((0, (s_1, s_2)), (\xi, (t_1, t_2))) = Z_\alpha((0, (s_1, s_2)), (\xi, (0, 0)))^{x-\alpha((0, (t_1, t_2)))} =$$

$$= Z_\alpha((0, (-t_1, -t_2)), 0) \cdot Z_\alpha((0, (s_1, s_2)), (\xi, (0, 0))) \cdot Z_\alpha((0, (t_1, t_2)), 0).$$

Each generator $Z_\alpha((0, (s_1, s_2)); (\xi, (0, 0)))$ can be factored into the product of elements $Z_\alpha((0, (s_1, 0)); (\xi, (0, 0)))$ and $Z_\alpha((0, (0, s_2)); (\xi, (0, 0)))$ lying in $G_2$ and $G_1$ respectively. These elements commute modulo relations in $C$, therefore we can rearrange factors in the decomposition of $g$ so that $gC = (\theta_1 p_1)_*(g)C \cdot (\theta_2 p_2)_*(g)C = C$, as required. 

From now on we will use simpler notation for certain elements of $St(\Phi, R, I)$. For $s \in I, \xi \in R$ we write $z_\alpha(s, \xi)$ instead of $Z_\alpha((0, s); \Delta(\xi))C = x_\alpha((0, s))^{x-\alpha(\Delta(\xi))}C$. We also set $y_\alpha(s) := z_\alpha(s, 0)$. From Lemma 4 it follows that elements $z_\alpha(s, \xi)$ generate $St(\Phi, R, I)$ as an abstract group if $\text{rk}(\Phi) \geq 2$.

Lemma 8. Let $\Phi$ be a simply laced root system and $\alpha, \beta \in \Phi$ be such that $\alpha + \beta \in \Phi$. Then elements $z_\alpha(s, \xi)$ satisfy the following relations:

1. $z_\alpha(a, \xi)^{x-\alpha(\eta)} = z_\alpha(a, \xi + \eta);
2. z_\alpha(a, \xi)^{x_\beta(\eta)} = y_\beta(-a\xi \eta) \cdot y_{\alpha + \beta}(N_{\alpha, \beta} \cdot a\eta) \cdot z_\alpha(a, \xi);
3. z_\alpha(a, \xi)^{x-\alpha(\eta)} = y_{-\beta}(a\xi \eta) \cdot y_{-\alpha - \beta}(N_{\alpha, \beta} \cdot a\xi^2 \eta) \cdot z_\alpha(a, \xi);
4. z_\alpha(a, \xi)^{x_\gamma(\eta)} = z_\alpha(a, \xi), \ \gamma \perp \alpha.$

Proof. These formulae can be checked directly using relations 2.3 2.4 and identities for structure constants (see [20, § 1] or [21, § 14]).

Let $X$ be a set and $G$ be a group. Denote by $F$ the group freely generated by $X \times G$. Let $R$ be a subset of $F$.

Definition 3.5. A group $H$ is said to be presented as a $G$-group by generators $X$ and relations $R$ if it is isomorphic to the quotient of $F$ modulo the smallest normal $G$-invariant subgroup containing $R$.

It is easy to see that $G$ naturally acts on $H$ on the right. Below we use more familiar notation $x^g$ for a generator $(x, g) \in X \times G$.

Now, we are ready to formulate an analogue of Swan’s presentation for relative Steinberg groups (cf. with [10, § 4], [7, Proposition 11]).

Proposition 5. Let $\Phi$ be a simply laced irreducible root system of rank $\geq 2$. Then $\text{St}(\Phi, R, I)$ can be presented as a $\text{St}(\Phi, R)$-group with generators $y_\alpha(s), s \in I, \alpha \in \Phi$ and relations:

1. $y_\alpha(s_1)y_\alpha(s_2) = y_\alpha(s_1 + s_2), \ s_1, s_2 \in I;$
(2) \([y_\alpha(s_1), y_\beta(s_2)] = 1, \alpha + \beta \not\in \Phi;\]
(3) \([y_\alpha(s_1), y_\beta(s_2)] = y_{\alpha+\beta}(N_{\alpha,\beta} \cdot s_1 s_2), \alpha + \beta \in \Phi;\]
(4) \(y_\alpha(s) x_{\alpha}(E) = y_\alpha(s), \alpha + \beta \not\in \Phi;\]
(5) \(y_\alpha(s) x_{\alpha}(E) = y_{\alpha+\beta}(N_{\alpha,\beta} \cdot \xi s), \alpha + \beta \in \Phi;\]
(6) \(y_\alpha(s_1) g x_{\alpha}(s_2) = y_{\beta}(-s_2) \cdot y_\alpha(s_1)^g \cdot y_\beta(s_2), s_1 \in I, g \in St(\Phi, R), \alpha, \beta \in \Phi.\]

Proof. Denote by \(H\) the \(St(\Phi, R)\)-group given by generators \(y_\alpha(s)\) and relations \([\Phi]\). One can repeat the argument of R. Swan (cf. [16, Lemma 7.8], [17, Lemma 8.4]) and show that \(G\) and \(H\) are isomorphic. One has to verify that the map \(\theta\) which sends \(x_\alpha((\xi_1, \xi_2)) \in St(\Phi, D(R))\) to \((y_\alpha(\xi_2-\xi_1), x_\alpha(\xi_1)) \in H \times St(\Phi, R)\) defines an isomorphism of these groups. By restricting \(\theta\) on \(G\) we obtain the needed isomorphism of \(G_1\) and \(H\).

Relation \([\Phi]\) corresponds under \(\theta\) to equation \([x_\alpha((0, s_1))^{\Delta_1(n)} x_\beta((s_2, 0))] = 1.\) It remains to notice that \(C = [G_1, G_2]\) is the normal closure of the subgroup spanned by all such commutators.

Remark 3.6. From Lemma \([\Phi]\) it follows that modulo relations \([\Phi]\) one can rewrite \(y_\alpha(s_1)^g\) as a product of elements of the form \(y_{\alpha^r}(s)^{x_{\alpha^r}(E)}\). Therefore, one can additionally assume in the statement of relation \([\Phi]\) that \(g = x_{-\alpha}(\xi), \xi \in R.\)

One can consider Proposition \([\Phi]\) as a universal property of relative Steinberg groups. Indeed, it allows one to construct a unique map \(St(\Phi, R, I) \rightarrow G\) whenever one chooses certain elements \(y_\alpha(s) \in G\) and there is an action of \(St(\Phi, R)\) on \(G\) which behaves on \(y_\alpha(s)\) according to relations \([\Phi].\]

3.2. Case \(\Phi = A_l, l \geq 3.\) The main goal of this subsection is to show that the abstract definition of the relative Steinberg group in the linear case agrees with Tulenbaev’s definition formulated in terms of another presentation (cf. [18, Definition 1.5]).

Definition 3.7. For \(n \geq 4\) the relative linear Steinberg group \(St(n, R, I)\) is the group defined by generators \(X_{v,w} (v \in \text{the orbit } E(n, R) \cdot e_1\) and \(w \in \text{a row such that } wv = 0)\) and relations \([\Phi,13]-[\Phi,15].\)

Proposition 6. Let \(I\) be an ideal such that \(R \rightarrow R/I\) splits, then \(St(n, R)\) is isomorphic to \(St(n) \ltimes St(n, R, I).\)

Proof. See [18, Proposition 1.6].

Throughout this subsection we identify vectors \(v \in D^n\) with pairs \((v, \tilde{v}) \in R^n \times R^n\) such that \(v - \tilde{v} \in I^n.\) If a column \((v, \tilde{v}) \in D^n\) is unimodular then both vectors \(v, \tilde{v} \in R^n\) are unimodular.

Consider the group \(St(A_{n-1}, D(R, I))\) in another presentation. Clearly, elements \(X_{(v, \tilde{v}), (0, \tilde{w})}\) and \(X_{(v, \tilde{v}), (w, 0)}\) belong to \(\text{Ker}(p_{1*})\) and \(\text{Ker}(p_{2*}),\) respectively. Moreover, the subgroup spanned by \(X_{(v, \bar{v}), (0, \bar{w})}\) for all \(\bar{w} \bar{v} = 0, v \in U(n, R)\) \(v-\bar{v} \in I^n\) is normal in \(St(A_{n-1}, D(R, I))\) and hence coincides with \(\text{Ker}(p_{1*}).\) Similar fact is true for \(\text{Ker}(p_{2*}).\)

It is easy to see that \(C \leq \text{Ker}(p_{1*})\) is generated by elements of the form \(X_{(v_1, \bar{v_1}), (0, \bar{w_1})}, X_{(v_2, \bar{v_2}), (w_2, 0)}\)

\(|X_{(v_1, \bar{v_1}), (0, \bar{w_1})}, X_{(v_2, \bar{v_2}), (w_2, 0)}| = X_{(v_1, \bar{v_1}), (0, \bar{w_1})} X_{(e+w_2 v_1, \bar{v_1})}, (0, -\bar{w_1})}.\)

The relative elementary subgroup \(E(n, R, I)\) is spanned by transvections of level \(I\) hence from the above equality it follows that

\[(3.8)\] 

\[X_{(v_1, \bar{v_1}), (0, \bar{w_1})} C = X_{(g v_1, \bar{v_1}), (0, \bar{w_1})} C \text{ for any } g \in E(n, R, I).\]
Proposition 7. Let $n \geq 4$, then $\text{St}(n, R, I)$ is isomorphic to $\text{St}(A_{n-1}, R)$.

Proof. Applying Proposition 3 to the map $p_1$ from diagram 3.1 we get that $\text{Ker}(p_1) \cong \text{St}(n, D(R, I), 0 \times I)$. In view of 3.8, the map $\psi: \text{St}(A_{n-1}, R) \to \text{St}(n, R, I)$ sending $X_{(v,v), (0,\omega)} C$ to $X_{\tilde{v},\tilde{w}} C$ is well-defined and preserves relations 2.13 and 2.13.

Now, let us construct a map inverse to $\psi$. Define $\varphi: \text{St}(n, R, I) \to \text{St}(A_{n-1}, R)$ using the formula $\varphi(X_{v,w}) = X_{(v,v), (0,\omega)}/C$. It is clear that $\varphi$ preserves relations 2.13 and 2.14. From identity 3.8 it follows that relation 2.14 is also preserved by $\varphi$:

$$\varphi(X_{v,w}) X_{v',w'} = X_{\Delta(v), (0,\omega)} X_{\Delta(v'), (0,\omega)} C = X_{(v,v), (0,\omega)}/C = \varphi(X_{(v,v'), (0,\omega)}/C).$$

Let us show that $\varphi$ and $\psi$ are inverse to each other. Only equality $\varphi \psi = 1$ is nontrivial. Let $(g, \tilde{g}) \in \text{E}(n, D(R, I))$ be such that $(g, \tilde{g})(v, \tilde{v}) = (e_1, e_1)$. Since $g^{-1} \cdot \tilde{g} \in \text{E}(n, R, I)$ by 3.8 we have that $X_{(v,\tilde{v}), (0,\omega)} C = X_{(g^{-1}, \tilde{g}), (0,\omega)} C = X_{(\tilde{v},\tilde{v}), (0,\omega)} C$, as required. \hfill $\square$

Remark 3.9. Recently, A. Lavrenov obtained a variant of another presentation for relative symplectic Steinberg groups (see [9]).

3.3. Tulenbaev’s map. Let $a \in R$ be a nonnilpotent element. Denote by $\lambda_a: R \to R_a$ the morphism of principal localization at $a$ (i.e. localization at $\{1, a, a^2, a^3, \ldots\}$). Similarly, if $\mathfrak{M} \leq R$ is a prime ideal we denote by $\lambda_{\mathfrak{M}}$ the morphism of localization at $R \setminus \mathfrak{M}$.

Let $R_a[t]$ be a polynomial algebra over $R_a$. Clearly, $t R_a[t]$ is an algebra over $R$. This allows us to form semidirect product $R \rtimes t R_a[t]$ (see Definition 3.2). Elements of this ring can be interpreted as polynomials whose free terms belong to $R$ and all other coefficients lie in $R_a$. There is a map $\theta: R[t] \to R \rtimes t R_a[t]$ which localizes all coefficients of terms of degree $\geq 1$ at $a$.

The key ingredient in Tulenbaev’s proof of the local-global principle for linear Steinberg groups is the following observation (cf. [15], Lemma 2.3).

Lemma 9. For $\Phi = A_{n-1}, n \geq 5$ the restriction of the map $\theta$ to the subgroup $\text{St}(\Phi, R[t], t R[t])$ factors through $\text{St}(\Phi, R_a[t], t R_a[t]), i.e. there exists a map $T \equiv T_\Phi$ such that the following diagram commutes:

$$\begin{array}{ccc}
\text{St}(\Phi, R[t], t R[t]) & \xrightarrow{\lambda_a} & \text{St}(\Phi, R_a[t], t R_a[t]) \\
\downarrow{\theta} & & \downarrow{T} \\
\text{St}(\Phi, R_a[t], t R_a[t]) & \xrightarrow{-} & \text{St}(\Phi, R \rtimes t R_a[t]).
\end{array}$$

In the remainder of this section we present some of the details pertaining to the construction of this map.

Let $A$ be an arbitrary commutative ring. Let $v \in A^n$ be a column. Denote by $O(v)$ the set of rows $w$ such that $wv = 0$. A row $w \in O(v)$ is called $v$-decomposable if it can be presented as a finite sum of rows $w^i \in O(v)$ each having at least two zero entries. Denote by $D(v)$ the set of all $v$-decomposable rows. For a column $v \in R^n$ denote by $I(v)$ the ideal of $A$ spanned by coordinates $v_1, \ldots, v_n$.

Let $u, w \in A^n$ be rows and $v \in A^n$ a column such that $wv = 0$. It is easy to check that $\langle uv \rangle \cdot w = \sum_{i<j} w_{ij}$, where $w_{ij} = (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \in A^n$ (cf. [9], Lemma 3.2)].
Such decomposition is called the *canonical* decomposition of \((uv) \cdot w\). In particular, this shows that row \(a \cdot w\) is always \(v\)-decomposable for \(a \in I(v)\), i.e. \(I(v) \cdot O(v) \subseteq D(v)\). It is also obvious that \(O(v) \subseteq O(bv)\), \(D(v) \subseteq D(bv)\) for \(b \in R\).

Let \(v\) be a (not necessarily unimodular) column of height \(n\), let \(w\) be a \(v\)-decomposable row and \(a\) be an element of \(I(v)\). In this context Tulenbaev defines elements \(X_{v,w}(a)\) of the Steinberg group \(St(n,A)\), \(n \geq 4\) (see [18, § 1]). The requirement \(a \in I(v)\) is needed in order to prove correctness of this definition.

Let us list certain useful properties of elements \(X_{v,w}(a)\).

**Lemma 10.** Let \(n \geq 4\). Elements \(X_{v,w}(a)\) satisfy the following relations:

1. \(X_{v,w}(ab) = X_{v,bw}(a)\) for \(w \in D(v)\), \(b \in R\), \(a \in I(v)\). If moreover \(a \in b \cdot I(v)\) then \(X_{v,w}(ab) = X_{bv,w}(a)\).
2. \(X_{v,w_1+w_2}(a) = X_{v,w_1}(a)X_{v,w_2}(a), w_1, w_2 \in D(v), a \in I(v)\).
3. For \(g \in St(n,A), w \in O(v), a,b \in I(v)\) one has \(g^{-1} \cdot X_{v,bw}(a) \cdot g = X_{g^{-1}v,bw}(a)\).
4. Let \(a\) be a non-nilpotent element of \(A\). For \(n \geq 5\) let \((v_1, v_2) \in M_{n \times 2}(A)\) be a matrix such that its image in \(M_{n \times 2}(A)\) is unimodular. Let \(w \in {}^nA\) be a row such that \(wv_1 = wv_2 = 0\). Then for sufficiently large natural numbers \(k, m\) one has

\[
X_{v_1+bv_2,a^k,w}(a^m) = X_{v_1,a^k,w}(a^m) \cdot X_{v_2,bv_2,w}(a^m), b \in A.
\]

**Proof.** The requirement \(b \in I(v)\) in the third statement of the lemma is necessary because we do not know whether \(wg\) is \(g^{-1}v\)-decomposable (even if we assume \(w \in D(v)\)).

The first statement is an immediate consequence of Tulenbaev’s definition of \(X_{v,w}(a)\) and [18, Lemma 1.1]. All other statements are contained in [18, Lemma 1.3].

Now, we briefly recall the details of Tulenbaev’s construction of \(T\).

**Proof of Lemma 10.** Let \(n \geq 5\), set \(A := R \rtimes tR_a[t], I := tA = tA_a = tR_a[t], \) clearly \(A_n = R_a[t]\). Let \(X_{v,w}\) be a generator of \(St(n, A, I)\) for \(v \in Um_n(A_a), w \in O(v) \cap {}^nI\). For sufficiently large \(m\) there exists \(\tilde{v} \in R^n\) such that \(\lambda_\tilde{v}(\tilde{v}) = a^m v\). Since \(v\) is unimodular, there exists \(k\) such that \(a^k \in I(\tilde{v})\). Define \(T\) using the formula \(T(X_{v,w}) = X_{\tilde{v},a^{-(n+k)}w}(a^k)\). In view of Lemma 11, \(T(X_{v,w})\) does not depend on the choice of \(k, m\) and the choice of lifting \(\tilde{v}\). From Lemma 10 it also follows that relations 2.13 and 2.14 are preserved by \(T\).

**Remark 3.11.** It is not hard to show that the image of \(T\) lies inside \(St(n, R \rtimes tR_a[t], tR_a[t])\) interpreted as a subgroup of \(St(n, R \rtimes tR_a[t])\) by virtue of Lemma 10. Moreover, it can be shown that \(T\) defines an isomorphism of \(St(n, R \rtimes tR_a[t], tR_a[t])\) and \(St(n, R \rtimes tR_a[t], tR_a[t])\) with the inverse map induced by \(\lambda_a : R \rtimes tR_a[t] \rightarrow R_a[t]\). This observation allows one to interpret Lemma 11 as a special case of excision property for Steinberg groups.

### 3.4. Relative Steinberg groups as amalgams.

Below we always assume that \(\Phi\) is an irreducible simply laced root system of rank \(\geq 3\).

The main purpose of the current section is to demonstrate the following relative analogue of Lemma 3.

**Theorem 8.** The group \(St(\Phi, R, I)\) is isomorphic to \(G := \varprojlim_{G_{1,3}} St(\cdot, R, I)\) as an abstract group.

The proof of this result is somewhat trickier than that of Lemma 3. The reason for this is that \(St(\Phi, R, I)\) comes with the natural action of \(St(\Phi, R)\) while a priori there is
no such action on $G$. Our immediate goal is to construct this action explicitly. The proof of Theorem 8 occupies the rest of the section.

**Remark 3.12.** Notice that from Lemma 2 and Lemma 3 it follows that $G$ can be interpreted as the free product of $|A_3(\Phi)|$ copies of $\text{St}(A_3, R, I)$ modulo relations which identify the images of elements $z_\alpha(s, \xi)$ with respect to $\text{St}(i_{\alpha, \psi}, R, I)$, $\alpha \in \Psi$, $\psi \in A_3(\Phi)$.

Denote by $j_\Psi$ the canonical map $\text{St}(\Psi, R, I) \to G$. There is a map $i : G \to \text{St}(\Phi, R, I)$ induced by $i_\psi : \text{St}(\Psi, R, I) \to \text{St}(\Phi, R, I)$. Clearly, $i_\psi = i \circ j_\psi$.

**Lemma 11.** Let $\Psi \subseteq \Phi$ be a subsystem of type $A_3$, let $\alpha \in \Phi$ be a root. For $\eta \in R$ there exists a map $c^\Psi_\alpha(\eta) : \text{St}(\Psi, R, I) \to G$ modeling conjugation with $x_\alpha(\eta)$ in $\text{St}(\Phi, R, I)$, i.e. such that the following diagram commutes:

$$
\begin{array}{ccc}
\text{St}(\Psi, R, I) & \xrightarrow{c^\Psi_\alpha(\eta)} & G \\
i_{\Psi} & \\ \xrightarrow{\varphi} \text{St}(\Phi, R, I)
\end{array}
$$

**Proof.** The cases when $\alpha \perp \Psi$ or $\alpha \in \Psi$ are obvious (set $c_{\alpha}(\eta) := j_{\Psi}$ or $c_{\alpha}(\eta) := j_{\Psi} \circ (-)x_{\alpha}(\eta)$, respectively).

Now consider the case $\alpha \not\perp \Psi, \alpha \not\in \Psi$. Denote by $S$ the closed subset of roots generated by $\alpha$ and $\Psi$. Let $\Psi'$ be the minimal root subsystem of $\Phi$ containing $S$. Clearly, $S$ is a parabolic subset of $\Psi'$. One can write $S = \Psi' \cup \Sigma$, where $\Sigma \ni \alpha$ is the special part of $S$. It is easy to see that $\Psi'$ has type $A_4$ or $D_4$ and that $\Sigma$ consists of 4 or 6 roots, respectively.

Denote by $\hat{U}(\Sigma, I)$ the subgroup of $G$ spanned by all $x_\gamma(s), \gamma \in \Sigma, s \in I$. It is not contained in a single factor of rank 3 however it is easy to show that $\hat{U}(\Sigma, I)$ is an abelian group isomorphic to $V_I := I^{[2]}$. Indeed, by Lemma 4 for every $\gamma_1, \gamma_2 \in \Sigma$ the root subgroups $X_{\gamma_1}(I), X_{\gamma_2}(I)$ are contained in $\text{St}(\Psi'', R, I)$ for some $\Psi'' \in A_3(\Phi)$ and hence commute. Denote by $\psi$ the isomorphism $V_I \cong \hat{U}(\Sigma, I)$.

Set $H := G(\Psi, R) \leq G(\Phi, R)$ and denote by $U$ the unipotent radical subgroup of $G(S, R)$, i.e. the subgroup spanned by $t_\alpha(\xi), \alpha \in \Sigma, \xi \in R$. Denote by $V$ the free $R$-module $R^{[2]}$ isomorphic to $U$ and by $e_\alpha$ the basis vector of $V$ corresponding to the root subgroup $X_\alpha(R) \leq U$ under this isomorphism. The conjugation action of $H$ on $U$ yields a representation $\rho : H \to \text{Aut}_R(V)$. By the definition of $\rho$ for $h \in H$ one has

$$h \cdot u = \rho(h)(u) \cdot h, \text{ for } u \in V_I \leq V.$$

The next step is to show that an analogue of the above relation hold in $G$. Set $\rho' := \rho \phi$, where $\phi$ denotes the projection $\text{St}(\Psi, R, I) \to G(\Psi, R, I) \leq H$. We argue that for $g \in \text{St}(\Psi, R, I)$ one has

$$j_\Psi(g) \cdot \psi(u) = \psi(\rho'(g)(u)) \cdot j_\Psi(g), \text{ for } u \in V_I \leq V. \quad (3.13)$$

Indeed, by Lemma 4 it suffices to check these relations for $g = z_\beta(a, \xi), u = s \cdot e_\gamma, s \in I, \gamma \in \Sigma$. Under these assumptions relation (3.13) has the same form as relations $[3, 13]$ of Lemma 3 and by Lemma 4 it can be found among the relations of some $\text{St}(\Psi'', R, I)$, $\Psi'' \in A_3(\Phi)$. Now we are ready to define the desired map $c^\Psi_\alpha(\eta)$. For $g \in \text{St}(\Psi, R, I)$ set

$$c^\Psi_\alpha(\eta)(g) := \psi(\rho'(g)(\eta e_\alpha) - \eta e_\alpha) \cdot j_\Psi(g) \in G.$$


This formula agrees with relations 2–3 from Lemma 8. Moreover, from 3.13 it follows that $c_\alpha^\psi(\eta)$ is a group homomorphism:

$$c_\alpha^\psi(\eta)(g_1) \cdot c_\alpha^\psi(\eta)(g_2) = \psi(\rho'(g_1)(\eta e_\alpha) - \eta e_\alpha) \cdot j_\psi(g_1) \cdot \psi(\rho'(g_2)(\eta e_\alpha) - \eta e_\alpha) \cdot j_\psi(g_2) =

\psi(\rho'(g_1)(\eta e_\alpha) - \eta e_\alpha) \cdot \psi(\rho'(g_2)(\eta e_\alpha) - \rho'(g_1)(\eta e_\alpha)) \cdot j_\psi(g_1) \cdot j_\psi(g_2) = c_\alpha^\psi(\eta)(g_1 \cdot g_2). \quad \Box$$

**Lemma 12.** Let $\alpha, \beta \in \Phi$ be roots and $\Psi_0, \Psi_1$ be subsystems of type $A_3$ containing $\beta$. Then the maps $c_0 := c_\alpha^{\Psi_0}(\eta)$ and $c_1 := c_\beta^{\Psi_1}(\eta)$ agree on $g = z_\beta(\alpha, \xi)$.

**Proof.** Since the action of both $c_0, c_1$ is consistent with identities of Lemma 3 and Proposition 5, the required statement follows from the construction of $c_0$ and $c_1$ if $\beta \neq \alpha$ or if $\xi = 0$.

Now consider the case when $\alpha = \beta$ and $\Lambda := \Psi_0 \cap \Psi_1$ has type $A_2$. By Lemma 3 we can choose a parabolic set of roots $S \subseteq \Lambda$ not containing $\beta$. Rewrite $g$ modulo relations of $\text{St}(\Psi_0 \cap \Psi_1, R, I)$ as a product of elements of the form $x_\gamma(s_0), \gamma \in \Lambda, z_\delta(s_1, \xi_1), \delta \in \Sigma_S$. In view of the first part of the argument, the values of $c_0$ and $c_1$ agree on each such factor, hence they agree on $g$.

In the last remaining case when $\Psi_0 \cap \Psi_1$ has type $A_1$ we can apply Lemma 2 and find a subsystem $\Psi$ containing $\beta$ such that $\Psi_0 \cap \Psi, \Psi_1 \cap \Psi$ have type $A_2$. By the first part of the argument we have $c_0(g) = c_\alpha^{\Psi_0}(\eta)(g) = c_1(g)$. \quad \Box

From the above lemma and the universal property of $G$ it follows that there exists a unique map $c_\alpha(\eta) : G \rightarrow G$ such that $c_\alpha(\eta) j_\psi = c_\alpha^{\psi}(\eta)$.

In order to define the action of $\text{St}(\Phi, R)$ on $G$ it remains to show that $c_\alpha(\eta)$ satisfy Steinberg relations 2.3, 2.3, 2.4.

**Lemma 13.** The maps $c_\alpha(s)$ satisfy the following relations:

1. $c_\alpha(s)c_\alpha(t)c_\alpha(-s-t) = \text{id}_G, s, t \in R$;
2. $c_\alpha(s)c_\beta(t)c_\alpha(-s)c_\beta(-t) = \text{id}_G, \alpha + \beta \notin \Phi$;
3. $c_\alpha(s)c_\beta(t)c_\alpha(s)c_\beta(-t)c_{\alpha+\beta}(-N_{\alpha\beta}st) = \text{id}_G, \alpha + \beta \in \Phi$.

**Proof.** Denote by $\theta = \prod c_\alpha(s_i)$ any of the maps in the left-hand side of the above relations. It suffices to check that $\theta$ fixes $g = z_\gamma(\alpha, \xi) \in G$ for all $\gamma \in \Phi$. Set $\Psi := \langle \alpha, \beta, \gamma \rangle$. Clearly, $\text{rk}(\Psi) \leq 3$ and we need to analyze several cases.

- Case $\Psi \subseteq \Psi' \subseteq A_3(\Phi)$. In this case the needed equality holds because it holds in $\text{St}(\Psi', R, I)$, indeed $\theta(g) = \theta(j_\psi(g)) = j_\psi(g^{\prod x_\alpha(s_i)}) = j_\psi(g) = 1$ as required.
- Case $\Psi = 3A_1$. In this case both $c_\alpha(\pm s)$ and $c_\beta(\pm t)$ fix $g$ by definition.
- Case $\Psi = A_2 + A_1$. If $\gamma \in \langle \alpha, \beta \rangle$ then, as before, $g$ is fixed by all factors of $\theta$. If $\alpha \perp \langle \beta, \gamma \rangle$ (only possible in the case of relation 2) then $c_\alpha(\pm s)$ act identically on $z_\delta(s, \xi), \delta \in \langle \beta, \gamma \rangle$, hence $\theta(g) = c_\beta(t)c_\beta(-t)(g) = g$. Case $\beta \perp \langle \alpha, \gamma \rangle$ is similar to the one just considered.

\[ \Box \]

**Proof of theorem.** From the above lemmata it follows that $G$ is a $\text{St}(\Phi, R)$-group and the map $i : G \rightarrow \text{St}(\Phi, R, I)$ is $\text{St}(\Phi, R)$-equivariant. In order to construct the inverse map we use the presentation of $\text{St}(\Phi, R, I)$ from the statement of Proposition 3. Indeed, every relation from Proposition 3 holds in $G$ since it occurs among the defining relations of some factor $\text{St}(\Psi, R, I)$ of $G, \Psi \in A_3(\Phi)$.

\[ \Box \]
4. Proof of main results

Throughout the current section we consider the relative Steinberg group \(\text{St}(\Phi, R[t], tR[t])\) as a subgroup of \(\text{St}(\Phi, R[t])\) thanks to Lemma 7.

4.1. Glueing Tulenbaev maps. The main goal of this subsection is to demonstrate an analogue of Lemma 9 for \(\Phi = E\) colimits. The idea of the proof is to construct \(T_\Phi\) by glueing maps \(T_\Psi\) for \(\Psi \in A_4(\Phi)\). First of all, we note that \(A_3\)'s in the statement of Theorem 8 can be replaced with \(A_4\)'s.

**Corollary 9.** For \(\Phi = E_l, l = 6, 7, 8\) there exists an isomorphism

\[
\text{St}(\Phi, R, I) \cong \lim_{\varphi_1, \varphi_2} \text{St}(-, R, I).
\]

**Proof.** This is a formal corollary of Theorem 8, Lemma 3 and the universal property of colimits. \(\square\)

Denote by \(i_\Psi\) the map \(\text{St}(\Psi, R \ltimes tR_0[t]) \to \text{St}(\Phi, R \ltimes tR_0[t])\) induced by embedding \(\Psi \hookrightarrow \Phi\) of root systems. The next step is to show that the maps \(i_\Psi \circ T_\Phi\) agree on \(z_\gamma(s, \xi)\).

**Lemma 14.** The value of \(i_\Psi \circ T_\Phi\) on \(z_\gamma(s, \xi), s \in tR_0[t], \xi \in R_0[t]\) does not depend on the choice of \(\Psi \in A_4(\Phi), \gamma \in \Psi\).

**Proof.** Let \(\xi'\) be an element of \(R[t]\). From the commutativity of diagram 3.10 it follows that \(T_\Psi(z_\gamma(s, \lambda_\alpha(\xi'))) = z_\gamma(s, \theta(\xi'))\). This shows that the statement of the lemma holds for \(\xi \in \text{Im}(\lambda_\alpha)\).

Now consider the case \(\xi \not\in \text{Im}(\lambda_\alpha)\). Let \(\Psi_0, \Psi_1 \in A_3(\Phi)\) be two subsystems containing \(\gamma\). Similarly to the proof of Lemma 12 we may restrict ourselves to the case when \(\Psi_0 \cap \Psi_1\) contains a subsystem of type \(A_2\). We can write \(\gamma = \alpha + \beta\) for \(\alpha, \beta \in \Psi_0 \cap \Psi_1\) such that \(N_{\alpha, \beta} = 1\). We can also choose \(n\) such that \(a^n \xi \in \text{Im}(\lambda_\alpha)\). Set \(b := a^n, u := a^{-n}s\). Direct computation with relations from Proposition 5 and Lemma 8 shows that

\[
z_\gamma(ub, \xi) = [y_{-\gamma}(u)^x,y_\alpha(ub^2)] = y_\alpha(-u\xi) \cdot y_{-\gamma}(u) \cdot y_{-\alpha}(ub^2) \cdot y_{\gamma}(ub) = z_{-\beta}(-u, -b\xi) \cdot z_{\beta}(-u, -b\xi^2) \cdot z_{\alpha}(u\xi, -b).
\]

Consequently, one can rewrite \(z_\gamma(s, \xi) = z_\gamma(ub, \xi)\) as a product of \(z_{\gamma'}(s', \lambda_\alpha(\xi'))\). By the first part of the argument, functions \(T_{\Psi_0}\) and \(T_{\Psi_1}\) coincide on each such factor, hence they are identical on \(z_\gamma(s, \xi)\). \(\square\)

Let \(A\) be a ring, \(B\) an \(A\) algebra and \(b \in B\). Denote by \(ev\left[\frac{A[t] \to B}{b}\right] : A[t] \to B\) the morphism of \(A\)-algebras evaluating each polynomial \(p(t) \in A[t]\) at \(b\), i.e. \(ev\left[\frac{A[t] \to B}{b}\right](p(t)) = p(b)\).

**Lemma 15.** Let \(h \in \text{St}(\Phi, R[t], tR[t])\) be such that \(\lambda_{a^n}(h) = 1\) in \(\text{St}(\Phi, R_0[t])\). Then for sufficiently large \(n\) one has \(ev\left[\frac{R[t] \to R[t]}{a^n \cdot t}\right](h) = 1\).

**Proof.** From the universal property of colimits and the previous lemma it follows that there exists a unique \(T_\Phi\) such that \(T_\Phi \cdot j_\Psi = i_\Psi \circ T_\Phi\) and \(\theta_0 = T_\Phi \cdot \lambda_{a^n}\). As before, \(\theta_0\)
denotes the map \( R[t] \to R \times tR_a[t] \) that localizes at \( a \) coefficients of terms of degree \( \geq 1 \) (cf. Lemma 15).

Let \( (A_i, f_{ij}) \) be the directed system of rings defined by

\[
A_i := R[t], \quad f_{ij} := ev \left[ \frac{R[t] \to R[t]}{t-a \cdot -t} \right], \quad i, j \in \mathbb{N}, \ i \leq j.
\]

It is easy to check that \( \varprojlim A_i \) coincides with \( R \times tR_a[t] \). Steinberg group functor commutes with colimits over directed systems (cf. [18, Lemma 2.2]), therefore

\[
St(\Phi, R \times tR_a[t]) = St(\Phi, \varprojlim A_i) \cong \varprojlim St(\Phi, R[t]).
\]

By hypothesis, \( \theta_s(h) = T_\Phi(1) = 1 \) and it remains to use the definition of colimit. \( \square \)

4.2. Quillen—Suslin local-global principle and centrality of \( K_2 \).

**Lemma 16.** Let \( a, b \) be elements of \( R \) which span \( R \) as an ideal and let \( \alpha \in St(\Phi, R[t], tR[t]) \) be such that \( \alpha_a = 1, \alpha_b = 1 \). Then \( \alpha = 1 \).

**Proof.** We reproduce the argument of [18, Lemma 2.3]. Set \( S := R[t, t_1] \). Consider the following element of \( St(\Phi, S[t_2]) \):

\[
\beta(t, t_1, t_2) := \alpha(t_1 t) \cdot \alpha^{-1}((t_1 + t_2)t) = ev \left[ \frac{R[t] \to S[t_2]}{t-a \cdot -t} \right] (\alpha) \cdot ev \left[ \frac{R[t] \to S[t_2]}{t-a \cdot -t} \right] (\alpha^{-1}).
\]

It is easy to see that \( \beta(t, t_1, t_2) \) can be rewritten as a product of conjugates of \( y_a(t \cdot f), \ f \in S[t_2] \) and hence lies in \( St(\Phi, S[t_2], t_2S[t_2]) \). On the other hand,

\[
\lambda_a(\beta(t, t_1, t_2)) = \left( \lambda_a \cdot ev \left[ \frac{R[t] \to S[t_2]}{t-a \cdot -t} \right] (\alpha) \cdot \left( \lambda_a \cdot ev \left[ \frac{R[t] \to S[t_2]}{t-a \cdot -t} \right] (\alpha^{-1}) \right) = ev \left[ \frac{R_a[t] \to S_a[t]}{t-a \cdot -t} \right] (\lambda_a(\alpha)) \cdot ev \left[ \frac{R_a[t] \to S_a[t]}{t-a \cdot -t} \right] (\lambda_a(\alpha^{-1})) = 1
\]

Similarly, \( \lambda_b(\beta(t, t_1, t_2)) = 1 \). In view of Lemma 15, there exists \( n \) such that \( \beta(t, t_1, a^n t_2) = 1, \beta(t, t_1, b^n t_2) = 1 \). By assumption, there exist \( r, s \in R \) such that \( ra^n + sb^n = 1 \). Finally,

\[
1 = \beta(t, 1, -sb^n)\beta(t, ra^n, -ra^n) = \alpha(t) \cdot \alpha^{-1}(ra^n \cdot t) \cdot \alpha(ra^n \cdot t) \cdot \alpha^{-1}(0) = \alpha(t). \quad \square
\]

**Proof of Theorem 1.** It suffices to show “if” part of the statement. Consider the set \( U \) of elements \( a \in R \) such that \( \alpha_a \) is trivial. Let us show that \( U \) is an ideal of \( R \). Indeed, let \( c \) be an element lying in the ideal \( \langle a, b \rangle \) spanned by \( a, b \in U \). Clearly, \( \lambda_a(\alpha_c) = \lambda_c(\alpha_a) = 1, \lambda_b(\alpha_c) = \lambda_c(\alpha_b) = 1 \). From the previous lemma it follows that \( \alpha_c = 1 \) and \( c \in U \).

If \( U \) is a proper ideal, then it is contained in some maximal ideal \( \mathfrak{M} \) and there exists \( s \in R \setminus \mathfrak{M} \) such that \( \lambda_s(\alpha) = 1 \) contrary to the definition of \( U \) and \( \mathfrak{M} \). This shows that \( U = R \) and \( \alpha = 1 \). \( \square \)

**Lemma 17.** Let \( \Phi \) be a simply laced root system of rank \( \geq 2 \). Then there exist a vertex \( \alpha_1 \) on \( D(\Phi) \) such that for every local ring \( R \) the map \( K_2(\langle \alpha_1 \rangle, R) \to K_2(\Phi, R) \) is surjective.

**Proof.** The result immediately follows from the surjective stability theorem for \( K_2 \) (see [13, Corollary 3.2], [13, Theorem 4.1]). \( \square \)
Proof of Theorem 1. Let $\alpha_1$ be the simple root from the statement of the previous lemma. Take any other simple root $\alpha \neq \alpha_1$. The subgroups $U_{\alpha}^\pm := \langle X_\beta(R), \beta \in \pm \Sigma_\alpha \rangle$ together generate $\text{St}(\Phi, R)$, therefore it suffices to show that $h := y_\beta(t^\beta) \cdot y_\beta(-t) \in \text{St}(\Phi, R[t], tR[t])$ is trivial for $\beta \in \pm \Sigma_\alpha$, $g \in K_2(\Phi, R)$ (we can specialize $t$ to any element of $R$).

By Quillen–Suslin local-global principle we are left to prove that $\lambda_M(h) = 1$ for every maximal ideal $M$ of $R$. But from the choice of $\alpha$ it follows that $\lambda_M(g)$ lies in the image of $K_2(\Delta_\alpha, R_M) \to K_2(\Phi, R_M)$ and therefore centralizes $U_{\alpha}^\pm$. \hfill \Box

Proof of Corollary 3. Statement \(a\) is a consequence of the fact that under the assumptions of Theorem 1 the group $\text{St}(\Phi, R)$ is superperfect, i.e. its first two homology groups are trivial (see \([12, \text{Theorem 5.3}]\)).

Statement \(b\) follows from \([10, \text{Proposition 6}]\) (compare with the proof of \([10, \text{Proposition 8}]\)). Statement \(c\) can be obtained by repeating the proof of \([10, \text{Theorem 4}]\). Finally, Gersten formula \(d\) can be obtained by repeating the proof of the main theorem of \([4]\) (see also discussion after \([8, \text{Corollary 2}]\)). \hfill \Box

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