Orientifolds of Matrix theory and Noncommutative Geometry

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Abstract

We study explicit solutions for orientifolds of Matrix theory compactified on noncommutative torus. As quotients of torus, cylinder, Klein bottle and Möbius strip are applicable as orientifolds. We calculate the solutions using Connes, Douglas and Schwarz's projective module solution, and investigate twisted gauge bundle on quotient spaces as well. They are Yang-Mills theory on noncommutative torus with proper boundary conditions which define the geometry of the dual space.

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1 Introduction

According to the Matrix theory conjecture [1][2], discrete lightcone quantization (DLCQ) of M-theory is described by the maximally supersymmetric gauge U(N) quantum mechanics, where N is the lightlike momentum, or the number of D-particles when interpreted as effective dynamics of D0-branes.

Toroidal compactification of M-theory using Matrix theory formulation can be performed by considering D0-brane dynamics on the covering space and imposing periodicity on the variables [3]. It is shown to lead to Yang-Mills gauge theory on dual torus [3], when we consider $T^d$ with $d < 4$. Additional moduli from winding mode of extended objects in M-theory should be taken into account when we consider compactification on higher dimensional tori [4]. Also the supersymmetric Yang-Mills (SYM) theory on $T^2$ is modified when the three-form potential of eleven dimensional supergravity is turned on along the lightlike direction. It is described by SYM theory on noncommutative torus [3, 4]. Noncommutative torus $T^2_\theta$ has additional SL(2,Z) symmetry, which corresponds to the T-duality in the DLCQ direction.

In general it is in mathematical language Morita equivalence of noncommutative tori [7] which governs the duality of Matrix theory compactifications with nonvanishing expectation value of NS-NS two form potential. When compactified on $T^d$ the complete Morita group is found to be SO(d,d|Z) [8]. The noncommutative $T^d_\theta$ is defined by $d \times d$ matrix $\Theta$, which transforms with fractional linear transformations with respect to SO(d,d|Z). The rank of the gauge group, magnetic flux numbers and instanton or other higher topological characters together comprise spinor representation, which means that under this duality SYM theories with different rank of the gauge group are related. Various related topics such as D-brane dynamics and noncommutative geometry, Morita equivalence of noncommutative tori and the duality symmetry of Matrix theory action and the BPS spectrum are studied further by various authors [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

In this paper we study orientifolds of Matrix theory compactification. As the simplest but still very illuminating examples, which are important on their own, we consider orientifolds of SYM on noncommutative $T^2_\theta$. Matrix theory compactification on non-orientable surfaces had been studied in [25, 26] before the noncommutative geometry nature was noted. After that it was noted that when we introduce the concept of noncommutativity, the dual space may not be determined uniquely [4]. When we mod out the torus to make cylinder, we can have the dual space either cylinder of Klein bottle. In the same way both cylinder and Klein bottle can be assigned as the dual space of Klein bottle. Dual space of Möbius strip is always again Möbius strip. This kind of ambiguity is generic with M-theory [27, 28] and Matrix theory [29, 30, 31]. We have to resort to physical arguments to decide which is the right
answer. Usually we have to introduce properly chosen twisted sector to cancel the anomaly. For example we know that Matrix theory on cylinder is SYM on cylinder with twisted sector fermions on the boundary. These correspond to the $E_8$ gauge field sector introduced on the end of the world when we consider M-theory compactified on $S^1/Z_2$. Similarly the dual of Klein bottle cannot be cylindrical, since it is generically anomalous without twisted sector fields which M-theory compactification lacks.

In this paper we aim to solve the orientifold compactification equations, first using the projective module solution presented in [5], and then we construct twisted gauge bundle on noncommutative torus and Klein bottle.

2 Compactification of Matrix Theory

The procedure of compactification in Matrix theory is straightforward. It is the philosophy of the Matrix theory, or D-brane dynamics, that the positions of the D-branes are encoded in the matrices as the eigenvalues. To realize periodicity we demand the following conditions. We consider compactification on two-torus in this paper.

\[ X_1 + 2\pi R_1 = U_1 X_1 U_1^{-1}, \]
\[ X_2 + 2\pi R_2 = U_2 X_2 U_2^{-1}. \] (1)

That is, we identify $X_i + 2\pi R_i$ with $X_i$, up to a certain similarity transformation. There are 8 other directions in Matrix theory but they are intact with above similarity transformations. So we have as well

\[ X_1 = U_2 X_1 U_2^{-1}, \]
\[ X_2 = U_1 X_2 U_1^{-1}, \]
\[ X_a = U_i X_a U_i^{-1}, \quad i = 1, 2, \quad a = 3, \ldots 10. \]

It is obvious that with matrices of finite size one cannot satisfy above conditions. The original solution [3] was based on the assumption that the translation operators commute

\[ [U_1, U_2] = 0. \]

The standard solution is taking $U_i$ as the generators of the algebra of the functions on dual torus $\tilde{T}^2$,

\[ U_i = e^{2\pi i R_i x_i}. \]

The particular solutions to Eq.(1) are partial derivatives, and general solutions should be covariant derivatives

\[ X_1 = i\partial_1 + A_1(U_1, U_2), \]
\[ X_2 = i\partial_2 + A_2(U_1, U_2). \] (2)

Looking at \( U_i \), we can see that dual space with size \( \tilde{R}_i = 1/R_i \) is created. While \( X_1, X_2 \) become the covariant derivatives on the dual torus, other components become scalar fields on the dual torus.

In general we can consider the case when the translation operators \( U_i \) do not commute each other, but satisfy

\[ U_1 U_2 = e^{2\pi i\theta} U_2 U_1. \] (3)

The physical meaning of the parameter \( \theta \) was first studied in [5]. It is the integral of the three-form potential of eleven dimensional supergravity on a three-cycle including the lightlike direction,

\[ \theta = R \int C_{12-} dx^1 dx^2 dx^{-}. \]

Interpreted as functions on torus again, i.e. \( U_i = e^{i\sigma_i} \), Eq.(3) defines a quantum plane algebra,

\[ [\sigma_1, \sigma_2] = -2\pi i \theta. \]

Then this gives SYM theory on noncommutative torus, where the multiplication is defined as

\[ fg \rightarrow \exp \left( \pi i \theta \epsilon^{ij} \frac{\partial}{\partial x'_i} \frac{\partial}{\partial x''_j} \right) f(x')g(x'') \bigg|_{x'=x''=x}. \] (4)

### 3 CDS’ projective module solution

We start by reviewing the compactification solution for \( e^{2\pi i\theta} \neq 1 \), presented by Connes, Douglas and Schwarz (CDS) in [3]. After we fix \( U_i \), the general solution has the form of \( X_i = \tilde{X}_i + A_i \), where \( \tilde{X}_i \) are particular solutions and \( A_i \) are operators commuting with \( U_i \).

We consider operators on the space of functions on \( \mathbb{R} \otimes \mathbb{Z}_q \), where \( \mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z} \). We define \( U_i \) as operators acting on function \( f(s, k) \) where \( s \in \mathbb{R}, k \in \mathbb{Z}_q \) and transforming them as

\[ U_1 f(s, k) = e^{2\pi i k/q} f(s, k - p), \]
\[ U_2 f(s, k) = e^{-2\pi i k/q} f(s + 1, k). \] (5)

They satisfy

\[ U_1 U_2 = e^{-2\pi i \gamma + 2\pi i p/q} U_2 U_1, \] (6)

and we set \( \theta = p/q - \gamma \). We define the operators \( \tilde{X}_i \) as follows

\[ \tilde{X}_1 f(s, k) = i\nu \frac{\partial f}{\partial s}, \]
\[ \tilde{X}_2 f(s, k) = \tau s f(s, k). \] (7)
and find they are in fact particular solution, with $R_1 = \nu \gamma$ and $R_2 = \frac{\tau}{2\pi}$. We note their commutator is

$$[\bar{X}_1, \bar{X}_2] = \frac{2\pi i}{\gamma} R_1 R_2. \quad (8)$$

Later we will interpret the solution as a gauge bundle on the dual space of compactification, and above particular solution is one with constant curvature.

Now we are looking for two independent operators which commute with both $U_1, U_2$, which will be generators for the gauge field.

$$Z_1 f(s, k) = e^{2\pi is/q} f(s, k - 1),$$
$$Z_2 f(s, k) = e^{i\nu k} f(s + \sigma, k), \quad (9)$$

with $\sigma = \frac{1}{\gamma q}, \nu = -\frac{2\pi a}{q}$ where $ap + bq = 1$ and $a, b \in \mathbb{Z}$. They satisfy

$$Z_1 Z_2 = e^{2\pi i \theta'} Z_2 Z_1, \quad (10)$$

with

$$\theta' = \frac{a\theta + b}{p - q\theta}. \quad (11)$$

Thus the homogeneous solution of compactified directions $A_i$, and $X_a$ for uncompactified directions are thought to be fields on the dual noncommutative torus with parameter $\theta'$. Above discussion constitutes rough sketch of Morita equivalence; U($q$) theory on $T_{-\theta}$ is identical to U(1) theory on $T_{\theta'}$, where $\theta$ and $\theta'$ are related by SL(2, $\mathbb{Z}$) transformation (11).

For U(1) theory the two generators $Z_1, Z_2$ will be interpreted as

$$Z_1^m Z_2^n \rightarrow e^{i(m\sigma_1 + n\sigma_2 + \pi \theta'mn)}, \quad (12)$$

where $\sigma_i$ are coordinates of the dual noncommutative torus, satisfying

$$[\sigma_1, \sigma_2] = -2\pi i \theta'. \quad (13)$$

And the general solution of the compactification condition is

$$X_1 = \bar{X}_1 + \sum c_{mn} Z_{mn},$$
$$X_2 = \bar{X}_2 + \sum d_{mn} Z_{mn}, \quad (14)$$

where

$$Z_{mn} = e^{-\pi i mn\theta'} Z_1^m Z_2^n. \quad (15)$$

Since

$$[\bar{X}_1, Z_1] = -\frac{2\pi \nu}{q} Z_1,$$
$$[\bar{X}_2, Z_2] = -\tau \sigma Z_2, \quad (16)$$
We can identify as
\[ X_i \rightarrow i \frac{2\pi R_i}{\gamma q} D_i, \]  
with constant curvature
\[ [D_1, D_2] = \frac{\gamma q^2}{2\pi i}. \]  
And the general solutions should be identified as
\[ X_1 = i \frac{2\pi R_1}{\gamma q} D_1 + A_1(\sigma_1, \sigma_2), \]
\[ X_2 = i \frac{2\pi R_2}{\gamma q} D_2 + A_2(\sigma_1, \sigma_2), \]
where \( A_i \) are gauge field defined on a noncommutative torus.

4 Orientifolds of CDS’ solution

4.1 Cylinder

In addition to the toroidal compactification condition (1), we impose one more condition to make it orientifold on cylinder,
\[ \mathcal{M} X_1 \mathcal{M}^{-1} = -X_1^T, \]
\[ \mathcal{M} X_2 \mathcal{M}^{-1} = +X_2^T. \]

This is Matrix theory realization of the involution giving rise to cylinder from torus,
\[ (\sigma_1, \sigma_2) \sim (-\sigma_1, \sigma_2). \]

Considering successive transformations we can find consistency condition, following [9].
\[ U_1 U_2 = e^{2\pi i \theta} U_2 U_1, \]
\[ U_1^* \mathcal{M} = \epsilon_1 \mathcal{M} U_1^{-1}, \]
\[ U_2^* \mathcal{M} = \epsilon_2 \mathcal{M} U_2, \]
\[ \mathcal{M} \mathcal{M}^* = \epsilon 1. \]

All new parameters introduced here are complex numbers with unit magnitude. It is obvious that \( \epsilon_2 \) can be scaled away by redefining \( U_2 \). It turns out that to satisfy the consistency conditions, we can choose only from \( \epsilon_1 = \pm 1, \epsilon = \pm 1 \). It was found that \( \epsilon_1 = 1 \) corresponds to the case that the dual space is cylinder, while when \( \epsilon = -1 \) we have Klein bottle instead [9]. \( \epsilon \) seems to select the gauge group on the boundary, when the dual space is cylinder. We will study this further in the following sections on quantum bundle.
Now that we have found the solutions for $U_1, U_2$ as operators on functions defined on $\mathbb{R} \otimes \mathbb{Z}_q$, our next task here is to find $\mathcal{M}$. Eq.(25) means that when $\epsilon = -1$, $\mathcal{M}$ is antisymmetric. We know that antisymmetric matrices of odd dimensionality cannot be unitary. And Eq.(23) amounts to finding unitary transformation between $U_1$ and $U_1^T$, when $\epsilon_1 = 1$. But it turns out that this cannot be done when $\mathcal{M}$ is antisymmetric. So we have solutions for only three cases.

Let’s begin with the case of $(\epsilon_1, \epsilon) = (1, 1)$. The solution is

$$\mathcal{M} f(s, k) = f(s, -k),$$

(26)

Under which

$$\mathcal{M} Z_1 \mathcal{M}^{-1} = Z_1^T,$$

$$\mathcal{M} Z_2 \mathcal{M}^{-1} = Z_2^*,$$

(27)

which means

$$\mathcal{M}(Z_{mn}) \mathcal{M}^{-1} = (Z_{m,-n})^T.$$  

(28)

So if we identify the operators $Z_1, Z_2$ as generators of $U(1)$ bundle on the noncommutative torus, functions which are invariant under the projection condition should satisfy

$$A_1(\sigma_1, \sigma_2) = -A_1(\sigma_1, -\sigma_2)$$

$$A_2(\sigma_1, \sigma_2) = +A_2(\sigma_1, -\sigma_2)$$  

(29)

So the dual space is cylindrical, as expected.

Now turn to the case of $(\epsilon_1, \epsilon) = (-1, 1)$. Looking at Eq.(23) and taking determinant, we can show that $q$ should be even. And since $p$ is prime to $q$, it is odd. The solution is

$$\mathcal{M} f(s, k) = (-1)^k f(s, -k).$$

(30)

And under that

$$\mathcal{M} Z_1 \mathcal{M}^{-1} = -Z_1^T,$$

$$\mathcal{M} Z_2 \mathcal{M}^{-1} = Z_2^*.$$  

(31)

So on the dual noncommutative two torus we have the following conditions,

$$A_1(\sigma_1, \sigma_2) = -A_1(\sigma_1 + \pi, -\sigma_2),$$

$$A_2(\sigma_1, \sigma_2) = +A_2(\sigma_1 + \pi, -\sigma_2).$$  

(32)

So we have Klein bottle.
Last solution for \((-1, -1)\). Again \(q\) is even, while \(p\) odd. The solution is

\[
\mathcal{M} f(s, k) = (-1)^k f(s, 1 - k),
\]

and

\[
\begin{align*}
\mathcal{M} Z_1 \mathcal{M}^{-1} &= - Z_1^T, \\
\mathcal{M} Z_2 \mathcal{M}^{-1} &= e^{2\pi i \frac{q}{a}} Z_2^*.
\end{align*}
\] (34)

So on the dual noncommutative two torus we have the following conditions,

\[
\begin{align*}
A_1(\sigma_1, \sigma_2) &= -A_1(\sigma_1 + \pi, 2\pi q/a - \sigma_2), \\
A_2(\sigma_1, \sigma_2) &= +A_2(\sigma_1 + \pi, 2\pi q/a - \sigma_2).
\end{align*}
\] (35)

Above boundary conditions give Klein bottle as well. Or the coefficient in front of \(Z_2^*\) here may be thought to be irrelevant, because we can scale it away redefining \(Z_2\). The convenient fundamental region of the half-torus could be different, but the topology is intact.

### 4.2 Klein Bottle

Now we consider orientifolding on a Klein bottle.

\[
\begin{align*}
\mathcal{M} X_1 \mathcal{M}^{-1} &= - X_1^T, \\
\mathcal{M} X_2 \mathcal{M}^{-1} &= + X_2^T + \pi R_2,
\end{align*}
\] (36)

which are Matrix theory realization of the following involution making Klein bottle from torus,

\[
(\sigma_1, \sigma_2) \sim (-\sigma_1, \pi + \sigma_2).
\] (37)

Now we follow the same procedure we used for the case of cylinder. The consistency consideration gives

\[
\begin{align*}
U_1 U_2 &= e^{2\pi i \theta} U_2 U_1, \\
U_1^* \mathcal{M} &= \epsilon_1 \mathcal{M} U_1^{-1}, \\
U_2^* \mathcal{M} &= \epsilon_2 \mathcal{M} U_2, \\
\mathcal{M}^* \mathcal{M} &= \epsilon U_2.
\end{align*}
\] (38)\(\quad\) (39)\(\quad\) (40)\(\quad\) (41)

This time both \(\epsilon_2, \epsilon\) can be absorbed into \(U_2\), so irrelevant. And consistency consideration gives us \(\epsilon_2 = e^{-2\pi i \theta}\). A solution can be found only if \(q\) is odd, with \(\epsilon_1 = (-1)^p e^{-\pi i \theta}, \epsilon_2 = \epsilon = 1, \epsilon_1 = e^{2\pi i \theta}\).

\[
\mathcal{M} f(s, k) = (-1)^k e^{\frac{\pi i}{a} k} f(s + 1/2, -k),
\] (42)
this transforms the basis

\[ \mathcal{M}Z_1\mathcal{M}^{-1} = -Z_1^T, \]
\[ \mathcal{M}Z_2\mathcal{M}^{-1} = Z_2^*. \]  

(43)

It is evident that we have Klein bottle for the dual space. This is a rather surprising result, since for U(1) theory on noncommutative torus, the orientifold projection apparently could give cylindrical or Klein bottle topology for the dual space [9]. But when we actually try to find the solution, we have only one case, which gives SYM theory on Klein bottle.

4.3 Möbius strip

The involution we have to realize in terms of matrices is

\[ (\sigma_1, \sigma_2) \sim (\sigma_2, \sigma_1). \]  

(44)

The orientifold condition is

\[ \mathcal{M}X_1\mathcal{M}^{-1} = \frac{R_1}{R_2}X_2^T, \]
\[ \mathcal{M}X_2\mathcal{M}^{-1} = \frac{R_2}{R_1}X_1^T. \]  

(45)

The consistency condition gives

\[ U_1U_2 = e^{2\pi i\theta}U_2U_1, \]  

(46)
\[ U_1^*\mathcal{M} = \epsilon_1\mathcal{M}U_2, \]  

(47)
\[ U_2^*\mathcal{M} = \epsilon_2\mathcal{M}U_2, \]  

(48)
\[ \mathcal{M}^*\mathcal{M} = \epsilon \mathbf{1}. \]  

(49)

We can find one solution which is Fourier transformation operator on all the variables,

\[ \mathcal{M}f(s, k) = \int dt \sum_{l=1}^{q} e^{2\pi i\gamma st - i\nu kl} f(t, l) \]  

(50)

with \( \epsilon_1 = \epsilon_2 = \epsilon = 1 \), which gives

\[ \mathcal{M}Z_1\mathcal{M}^{-1} = Z_2^*, \]
\[ \mathcal{M}Z_2\mathcal{M}^{-1} = Z_1^*. \]  

(51)

It is straightforward to check that the general should satisfy

\[ A_1(\sigma_1, \sigma_2) = A_2(-\sigma_2, -\sigma_1), \]
\[ A_2(\sigma_1, \sigma_2) = A_2(-\sigma_2, -\sigma_1), \]  

(52)

which defines dual Möbius strip through boundary condition.
5 Twisted Quantum Bundle on T^2

In this section we review the construction of twisted quantum U(q) bundle on noncommutative torus with constant abelian curvature. This is studied first in [14] and generalized later in [20]. Quantum torus T^2_θ is defined in terms of two noncommuting coordinates, i.e.

\[ [\sigma_1, \sigma_2] = 2\pi i \theta. \]

Using gauge invariance, any connection with constant curvature can be written as

\[ D_1 = \partial_1 + i F \sigma_2, \quad D_2 = \partial_2 - i F \sigma_1, \quad (53) \]

with field strength

\[ F \equiv i[D_1, D_2] = 2(F + \pi \theta F^2). \quad (54) \]

The connections satisfy the periodic boundary condition up to unitary transition functions \( \Omega_i \).

\[ D_i(\sigma_1 + 2\pi, \sigma_2) = \Omega_i(\sigma_2) D_i(\sigma_1, \sigma_2) \Omega_i^{-1}(\sigma_2), \quad \]
\[ D_i(\sigma_1, \sigma_2 + 2\pi) = \Omega_i(\sigma_1) D_i(\sigma_1, \sigma_2) \Omega_i^{-1}(\sigma_1). \quad (55) \]

The solutions for \( \Omega_i \) can be found easily,

\[ \Omega_1 = e^{i P \sigma_2} U, \]
\[ \Omega_2 = e^{-i P \sigma_1} V, \quad (56) \]

where

\[ P = \frac{2\pi F}{1 + 2\pi \theta F}, \quad (57) \]

and \( U, V \) are q-dimensional 't Hooft matrices satisfying

\[ UV = \omega VU, \]

with \( \omega = e^{-2\pi i p/q} \). In this paper we choose \( U_{ij} = \omega^i \delta_{ij} \) and \( V_{ij} = \delta_{i,j+1} \). Without loss of generality we assume \( q, p \) are relatively prime.

Due to the requirement of consistency the transition functions \( \Omega_i \) must satisfy the cocycle condition

\[ \Omega_1(\sigma_2 + 2\pi) \Omega_2(\sigma_1) = \Omega_2(\sigma_1 + 2\pi) \Omega_1(\sigma_2). \quad (58) \]

This imposes the following conditions

\[ \frac{p \theta}{q} = 1 - Q^{-2}, \]
\[ Q = 1 + 2\pi \theta F. \quad (59) \]
Now we can find the adjoint section of this quantum bundle, which satisfy (53):

\[ Z_1 = e^{iQ\sigma_1/qV^b}, \]
\[ Z_2 = e^{iQ\sigma_2/qU^{-b}}, \]

where \( a, b \) are integers and satisfy \( aq - bp = 1 \). \( Z_1, Z_2 \) generate the algebra of sections on the adjoint bundle. They satisfy

\[ Z_1Z_2 = e^{2\pi i\theta'}Z_2Z_1, \]

with \( \theta' = \frac{a(-\theta) + b}{p(-\theta) + q} \).

The general solution of U(q) quantum bundle on noncommutative torus \( T^2_{-\theta} \) can be written as

\[ A_i(\sigma_1, \sigma_2) = \sum_{i,j} c_{mn} J_{mn}(\sigma_1, \sigma_2), \]

where

\[ J_{mn}(\sigma_1, \sigma_2) = e^{-\pi imn\theta'}Z_1^m(\sigma_1)Z_2^n(\sigma_2), \]
\[ = J_{mn}e^{iQ(m\sigma_1+n\sigma_2)/q}, \]

with

\[ J_{mn} = e^{-\pi imnb/qV^bU^{-b}}. \]

The duality of SYM on noncommutative torus comes from the fact that \( J_{mn} \) can be treated as U(1) bundle on \( T^2_{\theta'} \) as well as U(q) bundle on \( T_{-\theta} \). Note that \( J_{mn} \) generate U(q). It is obvious that \( \theta \) and \( \theta' \) are related by SL(2,\( \mathbb{Z} \)). Actually it is proved [8] that in general the duality group is SO(\( d, d \mid \mathbb{Z} \)) on \( d \)-dimensional noncommutative torus. To be correct what was shown is that two noncommutative tori \( T_{\theta}^d \) and \( T_{\hat{\theta}}^d \) are Morita equivalent when \( \theta \) and \( \hat{\theta} \) belong to the same orbit of the group SO(\( d, d \mid \mathbb{Z} \)), the and equivalence of action functionals [9] and BPS spectrum of SYM theories on Morita equivalent tori are proved [10, 11, 12]. For two dimensional case SO(2, 2|\( \mathbb{Z} \)) = SL(2,\( \mathbb{Z} \) \( \times \) SL(2,\( \mathbb{Z} \)), where one SL(2,\( \mathbb{Z} \)) is the ordinary symmetry for two dimensional torus, and the other SL(2,\( \mathbb{Z} \)) is T-duality which involves the lightlike direction. In the following two sections we will study the orientifolding of this quantum twisted bundle on cylinder and Klein bottle respectively.

6 Twisted Quantum bundle on Cylinder

The twisted boundary condition should be

\[ D^T_1(\sigma_1, -\sigma_2) = -\mathcal{M}D_1(\sigma_1, \sigma_2)\mathcal{M}^{-1}, \]
\[ D^T_2(\sigma_1, -\sigma_2) = +\mathcal{M}D_2(\sigma_1, \sigma_2)\mathcal{M}^{-1}, \]
We can introduce coordinate dependence into $\mathcal{M}$, but it turns out it that does not give us any genuine physical difference. Now $\mathcal{M}$ is a unitary matrix and acts only on the gauge group part, and it is straightforward to check that the constant curvature connection (53) satisfies above conditions with chosen sign convention, which is consistent with our previous result (29).

Here we must consider additional consistency conditions which is similar to the cocycle condition. First we act the orientifold condition twice and get

$$\mathcal{M}^* \mathcal{M} = \pm 1,$$

which means $\mathcal{M}$ is either symmetric or antisymmetric.

Mingled with $\Omega_i$, we also get

$$\mathcal{M} \Omega_1(\sigma_2) \mathcal{M}^{-1} = e^{i\phi_1} \Omega_1^*(-\sigma_2),$$
$$\mathcal{M} \Omega_2^{-1}(\sigma_1) \mathcal{M}^{-1} = e^{i\phi_2} \Omega_2^*(\sigma_1),$$

where $\phi_i$ are arbitrary phases. Coordinate dependence is trivially satisfied, and the gauge part is essentially the same with the solutions we found before for orientifolding of CDS’ projective module solution.

As symmetric one, we have

$$\mathcal{M} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}. \quad (68)$$

Then the adjoint section we found before transforms as

$$\mathcal{M}Z_1(\sigma_1, \sigma_2) \mathcal{M}^{-1} = Z_1^T(\sigma_1, -\sigma_2),$$
$$\mathcal{M}Z_2(\sigma_1, \sigma_2) \mathcal{M}^{-1} = Z_2^{-1T}(\sigma_1, -\sigma_2).$$

(69)

Thus we find the solution after orientifolding as

$$A_i(\sigma_1, \sigma_2) = \sum_{m,n \in \mathbb{Z}} c_{m,n} \left( J_{m,n} + (-1)^i J_{m,-n} \right).$$

(70)

It is important to check what happens on the boundary of the cylinder, i.e. $\sigma_2 = 0$.

$$A_i(\sigma_1, 0) = \sum_{m,n \in \mathbb{Z}} c_{mn} \left( J_{m,n} + (-1)^i J_{m,-n} \right) e^{iQm\sigma_1/q}. \quad (71)$$

It is known that $J_{m,n} - J_{m,-n}$ generate $\text{SO}(q)$ [29]. $A_1$ is in adjoint and $A_2$ is in symmetric tensor representation.
Next two choices apply only when $q$ is even. First we have

$$
\mathcal{M} = \begin{pmatrix}
1 & -1 \\
& & \\
& & \\
& & \\
-1 & & & & 1
\end{pmatrix}.
$$

(72)

Then we have

$$
\mathcal{M} Z_1(\sigma_1, \sigma_2) \mathcal{M}^{-1} = (-1)^b Z_1^T(\sigma_1, -\sigma_2),
$$

$$
\mathcal{M} Z_2(\sigma_1, \sigma_2) \mathcal{M}^{-1} = Z_2^{-1T}(\sigma_1, -\sigma_2).
$$

(73)

Since $q$ is even, $b$ is always odd, which is obvious from $aq - bp = 1$. Thus we have the general solution as following.

$$
D_i(\sigma_1, \sigma_2) = \sum_{m,n \in \mathbb{Z}} c_{m,n} \left( J_{m,n} + (-1)^i(-1)^m J_{m,-n} \right).
$$

(74)

On the boundary

$$
D_i(\sigma_1, 0) = \sum_{m,n \in \mathbb{Z}} c_{m,n} \left( J_{m,n} + (-1)^i(-1)^m J_{m,-n} \right) e^{iQm\sigma_1/q}.
$$

(75)

Within our choice of $U, V$ this subset corresponds to SO$(q)$ [29].

Finally we have to try the case when $\mathcal{M}$ is antisymmetric,

$$
\mathcal{M} = \begin{pmatrix}
1 & & & \\
& -1 & & \\
& & & \\
& & & 1
\end{pmatrix}.
$$

(76)

Since we have

$$
\mathcal{M} U \mathcal{M}^{-1} = e^{2\pi i p/q} U^{-1},
$$

$$
\mathcal{M} V \mathcal{M}^{-1} = -V^{-1}.
$$

(77)

We get

$$
\mathcal{M} Z_1(\sigma_1, \sigma_2) \mathcal{M}^{-1} = (-1)^b Z_1^T(\sigma_1, -\sigma_2),
$$

$$
\mathcal{M} Z_2(\sigma_1, \sigma_2) \mathcal{M}^{-1} = e^{-2\pi i b p/q} Z_2^{-1T}(\sigma_1, -\sigma_2).
$$

(78)

And we again make use of the fact that $b$ is odd, and $bp = aq - 1$ to get the general solution

$$
A_i(\sigma_1, \sigma_2) = \sum_{m,n \in \mathbb{Z}} c_{m,n} \left( J_{mn} + (-1)^i(-1)^m e^{2\pi i n/q} J_{m,-n} \right).
$$

(79)
On the boundary

\[ A_i(\sigma_1, 0) = \sum_{m,n \in \mathbb{Z}} c_{mn} (J_{mn} + (-1)^i(-1)^m e^{2\pi i n/q} J_{m,-n}) e^{iQm\sigma_1/q}. \]  

(80)

And with our choice we have USp(q) on the boundary \cite{29}.

To summarize, twisted U(q) bundle on cylinder can have SO(q) or USp(q) on the boundary according to the solutions.

7 Twisted Quantum bundle on Klein Bottle

We consider

\[ D^T_1(\sigma_1 + \pi, -\sigma_2) = -\mathcal{M}(\sigma_1, \sigma_2) D_1(\sigma_1, \sigma_2) \mathcal{M}^{-1}(\sigma_1, \sigma_2), \]

\[ D^T_2(\sigma_1 + \pi, -\sigma_2) = +\mathcal{M}(\sigma_1, \sigma_2) D_2(\sigma_1, \sigma_2) \mathcal{M}^{-1}(\sigma_1, \sigma_2). \]  

(81)

We have to consider the consistency condition. As the first one we act \( \mathcal{M} \) twice on \( D_i \) and identify with \( \Omega_2 \). Then we have

\[ \mathcal{M}(\sigma_1, \sigma_2) = e^{i\sigma_2/2} \mathcal{M}_{KB}, \]  

(82)

with

\[ \mathcal{M}^*_{KB} \mathcal{M}_{KB} = U. \]  

(83)

where \( U \) is the gauge part of \( \Omega_2 \). Again if we consider successive transformations with \( \Omega_i \) and \( \mathcal{M} \), we get

\[ \mathcal{M} \Omega_1(\sigma_2) \mathcal{M}^{-1} = e^{i\phi_1} \Omega_1^*(\sigma_2), \]

\[ \mathcal{M} \Omega_2^{-1}(\sigma_1) \mathcal{M}^{-1} = e^{i\phi_2} \Omega_2^*(\sigma_1 + \pi), \]  

(84)

where \( \phi_i \) are arbitrary phases. When \( q \) is odd we can easily find the solution to (83). It is done as follows. \( U, V \) are related by unitary transformation, so assume \( KUK^{-1} = V \), then we can easily check \( \mathcal{M}_{KB} = K^T V^{q+1} K \) satisfies (83). It turns out that when \( q \) is even one cannot find any solution, which is consistent with the fact that with the conventional choice of \( U, V \) the determinant of \( U \) is \(-1\) when \( q \) is even, while from the left hand side it should be always positive. Our choice in this paper is \( U = U^T \) and \( V^T = V^{-1} \) so the solution for \( \mathcal{M} \) is found to be

\[ \mathcal{M}_{KB} = \begin{pmatrix} 1 & & & \\ & \cdots & \eta^{q-1} & \\ & \eta^2 & \cdots & \\ \eta & & & \end{pmatrix}, \]  

(85)
where $\eta = \omega^{2+1}$. Then we have

\[
\mathcal{M}_{KB} U \mathcal{M}_{KB}^{-1} = U^{-1},
\]
\[
\mathcal{M}_{KB} V \mathcal{M}_{KB}^{-1} = \eta V^{-1}.
\] (86)

And after some calculation the transformation for $Z_i$ are simplified as

\[
\mathcal{M}(\sigma_1, \sigma_2) Z_1(\sigma_1, \sigma_2) \mathcal{M}^{-1}(\sigma_1, \sigma_2) = -Z_1^T (\sigma_1 + \pi, -\sigma_2),
\]
\[
\mathcal{M}(\sigma_1, \sigma_2) Z_2(\sigma_1, \sigma_2) \mathcal{M}^{-1}(\sigma_1, \sigma_2) = Z_2^{-1T} (\sigma_1 + \pi, -\sigma_2).
\] (87)

Thus the general solution should be written as

\[
A_i(\sigma_1, \sigma_2) = \sum_{m,n \in \mathbb{Z}} c_{mn} \left( J_{mn} + (-1)^i(-1)^m J_{m,-n} \right).
\] (88)

8 Discussions

In this paper we studied aspects of Matrix theory orientifolds on noncommutative torus. It was important to note that Matrix compactifications can allow ambiguity in determining the dual space, but in this paper concrete solutions may not exist in some cases. For example when compactified on Klein bottle, U(1) theory on Klein bottle can be related to U($q$) with odd $q$ only. Here we used the simplest projective module solution presented by Connes, Douglas and Schwarz [5] to investigate the dual space of Matrix theory orientifold compactifications. Obviously we could extend to more general cases. For example if we consider functions $f(s_1, s_2, k)$ instead, we surely have noncommutative four-torus. And we can also introduce more than one gauge indices, e.g. if we consider $f(s, k_1, k_2)$ where $k_1 \in \mathbb{Z}_{q_1}, k_2 \in \mathbb{Z}_{q_2}$, it turns out we have Morita equivalence between U(1) and U($q$), where $q$ is the least common multiple of $q_1, q_2$. The orientifold operator $\mathcal{M}$ can act on either of the gauge indices, and there is a possibility to obtain Morita equivalent pairs which were excluded by our analysis in this paper. Or if our result persists even with more general projective module solutions there should be physical reason for that. This is an open problem at this stage, and we hope to report in due time.

Now that we have studied orientifold compactification on $\mathbb{T}^2$, it should be very interesting to study higher dimensional cases following the procedure presented here, for example ALE spaces $\mathbb{C}^2/\mathbb{Z}_n$ [32]. We expect to be able to interpolate the topology of dual space again, but when we actually find out the solution some might be excluded as was the case with lower dimensional examples studied here. We also suggest the study of heterotic Matrix string theory, which is Matrix theory compactified on cylinder $S^1 \times S^1/\mathbb{Z}_2$ [33, 34, 35], with noncommutativity in more detail. To cancel the gauge anomaly on the boundaries of cylinder
we have to introduce fermion fields which correspond to the D8-branes in type IIA string theory. It is known that when the D8-branes are displaced from the boundary we have to introduce Chern-Simons term to cancel the gauge anomaly \[34\] \[35\]. Of course the study of Chern-Simons term on noncommutative torus should be very interesting on its own. These issues are under investigation and we will report somewhere else \[36\].

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