Research Article

The Hermite–Hadamard–Jensen–Mercer Type Inequalities for Riemann–Liouville Fractional Integral

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In this paper, we give Hermite–Hadamard type inequalities of the Jensen–Mercer type for Riemann–Liouville fractional integrals. We prove integral identities, and with the help of these identities and some other eminent inequalities, such as Jensen, Hölder, and power mean inequalities, we obtain bounds for the difference of the newly obtained inequalities.

1. Introduction

The concept of convex functions plays a vital role in both pure and applied mathematics. Convex functions also have many applications in other branches of science such as finance, economics, and engineering.

Definition 1 (see [1]). A function $\psi: [m, M] \rightarrow \mathbb{R}$ is convex if

$$
\psi(sx + (1 - s)y) \leq s\psi(x) + (1 - s)\psi(y),
$$

(1)

for all $x, y \in [m, M]$ and $s \in [0, 1]$.

If the inequality in (1) is strict for $x \neq y$, then $\psi$ is said to be a strictly convex function, and if $-\psi$ is convex, then $\psi$ is said to be a concave function [2, 3].

Many important inequalities such as Jensen, Jensen–Mercer, Hermite–Hadamard, and support line inequalities hold for convex functions. The classical Jensen’s inequality is among the most prominent inequalities stated as follows [4, 5].

If $\psi: [m, M] \rightarrow \mathbb{R}$ is convex, then

$$
\psi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \psi(x_i),
$$

(2)

for all $x_i \in [m, M]$ and $w_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} w_i = 1$.

In [6], Mercer presented a type of Jensen’s inequality called Jensen–Mercer inequality.

Theorem 1. If $\psi: [m, M] \rightarrow \mathbb{R}$ is convex, then

$$
\psi\left(m + M - \sum_{i=1}^{n} w_i x_i\right) \leq \psi(m) + \psi(M) - \sum_{i=1}^{n} w_i \psi(x_i),
$$

(3)

for each $x_i \in [m, M]$ and $w_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} w_i = 1$.

For a convex function, there exist at least one line lies on or below the graph of the function.

Definition 2 (see [7]). A function $\psi: I \rightarrow \mathbb{R}$ has a support at $x_0 \in I$ if

$$
\psi(x_0) + c(u - x_0) \leq \psi(u),
$$

(4)
for all \( x_0 \in I \) and for each \( u \in [m, M] \subset I \). Inequality (4) is
said to be the support line inequality.

The following theorem connects the support line inequality with convex functions.

**Theorem 2** (see [7]). A function \( \psi: [m, M] \longrightarrow \mathbb{R} \) is convex if and only if \( \psi \) has at least one line of support at each \( x_0 \in [m, M] \).

The Hermite–Hadamard inequality is one of the most investigated inequalities in the theory of convex functions due to its geometrical significance and applications. Because of the importance of the Hermite–Hadamard inequality, there is an ample amount of research work dedicated to the extensions, generalizations, refinements, and applications of the Hermite–Hadamard inequality. The Hermite–Hadamard inequality is given below [8].

**Definition 2** (see [26]). Let \( \psi: I \longrightarrow \mathbb{R} \) be a convex function, where \( I \) is an interval and \( m, M \in I \) such that \( m < M \). Then,

\[
\psi(\frac{m + M}{2}) \leq \frac{1}{M - m} \int_m^M \psi(x)dx \leq \frac{\psi(m) + \psi(M)}{2}.
\]

If \( \psi \) is concave, then (5) holds in the reversed direction. For more results associated with Hermite–Hadamard inequality, see [9–18].

The Hermite–Hadamard inequality has been extended by means of fractional integral operators. Most popular of them is the Riemann–Liouville fractional operator given in the following definition [19–22].

**Definition 3** (see [23, 24]). Let \( \psi \) be an integrable function defined on \([m, M]\). Then, the integrals \( f'_{m}^{\alpha} \psi(x) \) and \( f'_{M}^{\alpha} \psi(x) \) defined by

\[
f'_{m}^{\alpha} \psi(x) = \frac{1}{\Gamma(\alpha)} \int_{m}^{x} (x - s)^{\alpha - 1} \psi(s)ds, \quad x > m,
\]

\[
f'_{M}^{\alpha} \psi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{M} (s - x)^{\alpha - 1} \psi(s)ds, \quad x < M,
\]

are called the left and right Riemann–Liouville fractional integrals of order \( \alpha > 0 \) respectively. Here, \( \Gamma \) represents gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-s}s^{\alpha-1}ds \).

In [25, 26], authors used the following lemmas to obtain trapezoidal and midpoint type inequalities.

**Lemma 1** (see [25]). Let \( \psi: I^* \longrightarrow \mathbb{R} \) be a differentiable function and \( m, M \in I^* \) such that \( m < M \). If \( \psi' \in L[m, M] \), then

\[
\psi(m) + \psi(M) \quad \frac{1}{2} - \frac{1}{M - m} \int_m^M \psi(u)du = \frac{M - m}{2} - \int_0^1 (1 - 2s)\psi' \left( sm + (1 - s)M \right)ds.
\]

**Lemma 2** (see [26]). Let all the assumptions of Lemma 1 hold. Then,

\[
\frac{1}{M - m} \int_m^M \psi(u)du - \psi\left( \frac{m + M}{2} \right) = (M - m) \left( \int_0^{(1/2)} s\psi' \left( sm + (1 - s)M \right)ds + \int_0^{(1/2)} (s - 1)\psi' \left( sm + (1 - s)M \right)ds \right).
\]

In this article, we establish fractional Hermite–Hadamard–Jensen–Mercer type inequalities. We give identities involving fractional integrals, and from these identities, we derive trapezoidal and midpoint type inequalities.

Throughout this article, \( \alpha \) represents a positive real number.

**2. Main Results**

We begin this section with our first main result.

**Theorem 3.** Suppose \( \psi: [m, M] \longrightarrow \mathbb{R} \) is a convex function and \( x, y \in [m, M] \) such that \( x < y \). Then,
Since \( \psi \) is convex, it has support line at each point \( x_0 \in [m, M] \), that is,

\[
\psi(x_0) + c(u - x_0) \leq \psi(u),
\]

for each \( u \in [m, M] \). Substituting \( x_0 = m + M - \frac{(ax + y)}{(a + 1)} \) and \( u = m + M - sx - (1 - s)y \), where \( s \in [0, 1] \), in inequality (12), we obtain

\[
\psi\left(m + M - \frac{ax + y}{a + 1}\right) + c\left[\frac{ax + y}{a + 1} + \frac{ax + y}{a + 1}\right] \leq a \int_o^1 s^{a-1} \psi(m + M - sx - (1 - s)y)ds
\]

Multiplying (13) with \( as^{a-1} \) and integrate with respect to \( s \), we obtain

\[
\psi\left(m + M - \frac{ax + y}{a + 1}\right) \leq \psi(m + M - sx - (1 - s)y).
\]

Using Mercer’s inequality, we obtain

\[
\psi\left(m + M - \frac{ax + y}{a + 1}\right) \leq \alpha \int_o^1 s^{a-1} (\psi(m) + \psi(M) - (s\psi(x) + (1 - s)\psi(y)))ds
\]

Since \( \psi \) is convex, we have \( - (s\psi(x) + (1 - s)\psi(y)) \leq - \psi(sx + y(1 - s)) \) and (15) becomes

\[
\psi\left(m + M - \frac{ax + y}{a + 1}\right) \leq \psi(m) + \psi(M) - \alpha \int_o^1 s^{a-1} \psi(sx + (1 - s)y)ds.
\]
Substituting \( sx + (1-s)y = w \) in (16), we obtain

\[
\psi \left( m + M - \frac{ax + y}{a + 1} \right) \leq \psi(m) + \psi(M) - \frac{\alpha}{(y-x)^{a}} \int_{x}^{y} (y-w)^{a-1} \psi(w)dw
\]

\[
\Rightarrow \psi \left( m + M - \frac{ax + y}{a + 1} \right) \leq \psi(m) + \psi(M) - \frac{\Gamma(a+1)}{(y-x)^{a}f_{x}^{a}} \psi(y).
\]

Now, we prove the second inequality of (10). Put \( x_0 = (ax + y)/(a + 1) \) and \( u = sx + (1-s)y \) in (12), we obtain

\[
\psi \left( \frac{ax + y}{a + 1} \right) + c \left[ sx + (1-s)y - \frac{ax + y}{a + 1} \right] \leq \psi(sx + (1-s)y).
\]

Multiplying the above inequality with \( as^{a-1} \) and integrating and using \( \int_{0}^{s} s^{a-1}ds = 1/\alpha \) and \( \int_{0}^{1} s^{a}ds = 1/(a + 1) \), we obtain

\[
\psi \left( \frac{ax + y}{a + 1} \right) \leq \alpha \int_{0}^{1} s^{a-1} \psi(sx + (1-s)y)ds.
\]

Put \( sx + (1-s)y = w \), and we obtain

\[
\psi \left( \frac{ax + y}{a + 1} \right) \leq \frac{\alpha}{(y-x)^{a}} \int_{x}^{y} (y-w)^{a-1} \psi(w)dw
\]

\[
\Rightarrow \psi \left( \frac{ax + y}{a + 1} \right) \leq \frac{\Gamma(a+1)}{(y-x)^{a}f_{x}^{a}} \psi(y)
\]

\[
\Rightarrow - \frac{\Gamma(a+1)}{(y-x)^{a}f_{x}^{a}} \psi(y) \leq - \psi \left( \frac{ax + y}{a + 1} \right).
\]

Adding \( \psi(m) + \psi(M) \) on both sides of (20), we obtain

\[
\psi(m) + \psi(M) - \frac{\Gamma(a+1)}{(y-x)^{a}f_{x}^{a}} \psi(y) \leq \psi(m) + \psi(M) - \psi \left( \frac{ax + y}{a + 1} \right).
\]

and on combining (17) and (21), we obtain (10).

Now, we prove the inequalities in (11). Let \( u = m + M - sx - (1-s)y \Rightarrow s = (u - m - M + y)/(y - x) \) and (14) become

\[
\psi \left( m + M - \frac{ax + y}{a + 1} \right) \leq \alpha \int_{m+M-x}^{m+M-y} \left( \frac{u - m - M + y}{y - x} \right)^{a-1} \psi(u) \frac{du}{y-x}
\]

\[
= \frac{\alpha \Gamma(a)}{(y-x)^{a}f_{x}^{a}} \int_{m+M-y}^{m+M-x} (u-(m+M-y))^{a-1} \psi(u)du
\]

\[
\Rightarrow \psi \left( m + M - \frac{ax + y}{a + 1} \right) \leq \frac{\Gamma(a+1)}{(y-x)^{a}f_{(m+M-x)^{a}}^{a}} \psi(m + M - y).
\]

Now, we prove the other two inequalities of (11). As \( \psi \) is a convex function, we have

\[
\psi(m + M - sx - (1-s)y) = \psi(s(m + M - x) + (1-s)(m + M - y))
\]

\[
\Rightarrow \psi(m + M - sx - (1-s)y) \leq s \psi(m + M - x) + (1-s) \psi(m + M - y)
\]

\[
\leq \psi(m) + \psi(M) - s \psi(x) - (1-s) \psi(y).
\]
Multiplying with $\alpha s^{\alpha-1}$ and integrating, we obtain

\[
\alpha \int_0^1 s^{\alpha-1} \psi(m + M - sx - (1-s)y) ds \\
\leq \alpha \psi(m + M - x) \int_0^1 s^{\alpha} ds + \alpha \psi(m + M - y) \int_0^1 (s^{\alpha-1} - s^{\alpha}) ds \\
\leq \psi(m) + \psi(M) - \alpha \psi(x) \int_0^1 s^{\alpha} ds - \alpha \psi(y) \int_0^1 (s^{\alpha-1} - s^{\alpha}) ds \\
\Rightarrow \alpha \int_0^1 s^{\alpha-1} \psi(m + M - sx - (1-s)y) ds \\
\leq \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} \\
\leq \psi(m) + \psi(M) - \frac{\alpha \psi(x) + \psi(y)}{\alpha + 1}.
\]

(24)

By changing of variable, (24) becomes

\[
\alpha \int_{m+M-x}^{m+M-y} \left( \frac{u - m - M + y}{y - x} \right)^{\alpha-1} \psi(u) \frac{du}{y - x} \\
\leq \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} \\
\leq \frac{\alpha \psi(x) + \psi(y)}{\alpha + 1} \\
\Rightarrow \frac{\Gamma(\alpha)}{(y - x)^\alpha} \frac{\psi(m + M - y)}{\Gamma(\alpha)} \\
\leq \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} \\
\leq \psi(m) + \psi(M) - \frac{\alpha \psi(x) + \psi(y)}{\alpha + 1}.
\]

(25)

and on combining (22) and (25), we obtain (11). \qed
Remark 1. If we put $\alpha = 1$ in Theorem 3 and in the obtained expressions substitute $u = sx + (1-s)y$ and $u = m + M - v$, respectively, we obtain

$$
\psi\left(m + M - \frac{x + y}{2}\right) \leq \psi(m) + \psi(M) - \int_0^1 \psi(sx + (1-s)y)ds
$$

(26)

$$
\leq \psi(m) + \psi(M) - \psi\left(\frac{x + y}{2}\right).
$$

$$
\psi\left(m + M - \frac{x + y}{2}\right) \leq \frac{1}{y-x} \int_x^y \psi(m + M - v)dv
$$

(27)

$$
\leq \frac{\psi(m + M - x) + \psi(m + M - y)}{2}
\leq \psi(m) + \psi(M) - \frac{\psi(x) + \psi(y)}{2}.
$$

respectively. Inequalities (26) and (27) have been proved in [27].

Remark 2. Substituting $\alpha = 1$, $x = m$, and $y = M$ in (11), one can obtain Hermite–Hadamard inequality.

3. Bounds for the Difference of Hermite–Hadamard–Jensen–Mercer Type Inequalities

Throughout this section, we consider $\psi: [m, M] \rightarrow \mathbb{R}$ is a differentiable function. To give the bounds for the difference of Hermite–Hadamard–Jensen–Mercer type inequalities, first, we present the following lemmas.

Lemma 3. Let $x, y \in [m, M]$ such that $x < y$ and let $\psi' \in L[m, M]$. Then,

$$
\frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y-x)^\alpha} J_{(m+m-x)}^\alpha \psi(m + M - y)
$$

(28)

$$
= \frac{y-x}{\alpha + 1} \int_0^1 ((\alpha + 1)s^\alpha - 1)\psi'(m + M - sx - (1-s)y)ds.
$$
Proof. Using the techniques of integration, we have

\[
\frac{y - x}{\alpha + 1} \int_0^1 [(\alpha + 1)s^\alpha - 1] \psi'(m + M - sx - (1 - s)y) ds
\]

\[
= (y - x) \int_0^1 s^\alpha \psi'(m + M - sx - (1 - s)y) ds
\]

\[
- \frac{y - x}{\alpha + 1} \int_0^1 \psi'(m + M - sx - (1 - s)y) ds
\]

\[
= s^\alpha \psi(m + M - sx - (1 - s)y) \Big|_0^1 - \alpha \int_0^1 s^{\alpha - 1} \psi(m + M - sx - (1 - s)y) ds
\]

\[
- \frac{1}{\alpha + 1} \psi(m + M - sx - (1 - s)y) \Big|_0^1,
\]

\[
= \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \alpha \int_0^1 s^{\alpha - 1} \psi(m + M - sx - (1 - s)y) ds
\]

\[
= \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\alpha}{(y - x)^\alpha} \int_{m + M - x}^{m + M - y} (u - (m + M - y))^{\alpha - 1} \psi(u) du
\]

\[
= \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^\alpha} \int_{m + M - x}^{m + M - y} \psi(u) du.
\]

Remark 3. Substituting \( \alpha = 1, x = m, \) and \( y = M \) in (28), we obtain (8).

Lemma 4. Let all the assumptions of Lemma 3 hold. Then,

\[
\frac{\Gamma(\alpha + 1)}{(y - x)^\alpha} \psi(m + M - y) - \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right)
\]

\[
= (y - x) \left(\int_0^1 \frac{(1 - s^\alpha)}{\alpha^{(\alpha + 1)}} \psi'(m + M - sx - (1 - s)y) ds - \int_0^{\alpha/(\alpha + 1)} s^\alpha \psi'(m + M - sx - (1 - s)y) ds\right).
\]
Proof. Using techniques of integration, we have

\[
(y - x) \int_{a/(a+1)}^{1} (1 - s^a) \psi' (m + M - sx - (1-s)y) ds \\
- (y - x) \int_{0}^{a/(a+1)} s^a \psi' (m + M - sx - (1-s)y) ds \\
= (y - x) \left[ \int_{a/(a+1)}^{1} \psi' (m + M - sx - (1-s)y) ds \right] \\
- (y - x) \left[ \int_{0}^{1} s^a \psi' (m + M - sx - (1-s)y) ds \right] \\
= \psi (m + M - sx - (1-s)y) \bigg|_{a/(a+1)}^{1} - s^a \psi (m + M - sx - (1-s)y) \bigg|_{0}^{1} \\
+ a \int_{0}^{1} s^{a-1} \psi (m + M - sx - (1-s)y) ds \\
= -\psi \left( m + M - \frac{\alpha x + y}{\alpha + 1} \right) + a \int_{0}^{1} s^{a-1} \psi (m + M - sx - (1-s)y) ds \\
= \frac{\Gamma (\alpha + 1)}{(y - x)^{a}} \int_{(m+M-x)^{\alpha}}^{(m+M-y)^{\alpha}} \psi (m + M - y) - \psi \left( m + M - \frac{\alpha x + y}{\alpha + 1} \right). 
\]

Remark 4. If we put \( \alpha = 1, x = m, \) and \( y = M \) in (31), we obtain (9).

We use Lemmas 3 and 4 and obtain bounds for the difference of the inequalities in (11).

\[
\left| \frac{\alpha \psi (m + M - x) + \psi (m + M - y)}{\alpha + 1} - \frac{\Gamma (\alpha + 1)}{(y - x)^{a}} \int_{(m+M-x)^{\alpha}}^{(m+M-y)^{\alpha}} \psi (m + M - y) \right| \\
\leq \frac{y - x}{\alpha + 1} \left[ P_1 (\alpha) |\psi' (m)| + P_2 (\alpha) |\psi' (M)| - P_2 (\alpha) |\psi' (x)| - P_3 (\alpha) |\psi' (y)| \right]. 
\]

where

\[
P_1 (\alpha) = \frac{2\alpha}{(\alpha + 1)^{(\alpha+1)/\alpha}}, \\
P_2 (\alpha) = \frac{\alpha [2 + (\alpha + 1)^{2/\alpha}]}{2 (\alpha + 2) (\alpha + 1)^{2/\alpha}}, \\
P_3 (\alpha) = \frac{\alpha [4 (\alpha + 2) (\alpha + 1)^{(1/\alpha) - 1} - (\alpha + 1)^{2/\alpha} - 2]}{2 (\alpha + 2) (\alpha + 1)^{2/\alpha}}.
\]
Proof. From Lemma 3, we have

\[
\left| \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^\alpha} \int_{m+M-x}^{y} \psi(m + M - y) \right| \\
\leq \frac{y - x}{\alpha + 1} \int_{0}^{1} (\alpha + 1) s^\alpha - 1 ||\psi'(m + M - sx) - (1 - s)\psi(y)||ds.
\]

(35)

Since \(|\psi'|\) is convex, using Mercer’s inequality, we obtain

\[
\left| \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^\alpha} \int_{m+M-x}^{y} \psi(m + M - y) \right| \\
\leq \frac{y - x}{\alpha + 1} \int_{0}^{1} ((\alpha + 1)s^\alpha - 1)(||\psi'(m)|| + ||\psi'(M)|| - s||\psi'(x)|| - (1 - s)||\psi'(y)||)ds \\
= \frac{y - x}{\alpha + 1} \int_{0}^{1/(\alpha + 1)^{1/\alpha}} (1 - (\alpha + 1)s^\alpha)(||\psi'(m)|| + ||\psi'(M)|| - s||\psi'(x)|| - (1 - s)||\psi'(y)||)ds \\
\quad + \int_{1/(\alpha + 1)^{1/\alpha}}^{1} ((\alpha + 1)s^\alpha - 1)(||\psi'(m)|| + ||\psi'(M)|| - s||\psi'(x)|| - (1 - s)||\psi'(y)||)ds \\
\]

(36)

equivalent to

\[
\left| \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^\alpha} \int_{m+M-x}^{y} \psi(m + M - y) \right| \\
\leq \frac{y - x}{\alpha + 1} \left( [L_1(\alpha) + N_1(\alpha)]||\psi'(m)|| + [L_1(\alpha) + N_1(\alpha)]||\psi'(M)|| \\
- [L_2(\alpha) + N_2(\alpha)]||\psi'(x)|| - [L_3(\alpha) + N_3(\alpha)]||\psi'(y)|| \right),
\]

(37)

where
Lemma 4. \[ L_1(\alpha) = \int_0^{1/(\alpha+1)^\alpha} [1 - (\alpha + 1)s^\alpha] \, ds = \frac{\alpha}{(\alpha + 1)^{(\alpha+1)/\alpha}} \]

\[ L_2(\alpha) = \int_0^{1/(\alpha+1)^\alpha} [1 - (\alpha + 1)s^\alpha] s \, ds = \frac{\alpha}{2(\alpha + 2)(\alpha + 1)^{2/\alpha}} \]

\[ L_3(\alpha) = \int_0^{1/(\alpha+1)^\alpha} (1 - (\alpha + 1)s^\alpha)(1-s) \, ds = \frac{\alpha(2(\alpha + 2)(\alpha + 1)^{(1/\alpha)-1} - 1)}{2(\alpha + 2)(\alpha + 1)^{2/\alpha}} \]

\[ N_1(\alpha) = \int_0^1 [(\alpha+1)s^\alpha - 1] \, ds = \frac{\alpha}{(\alpha + 1)^{(\alpha+1)/\alpha}} \] \hspace{1cm} (38)

\[ N_2(\alpha) = \int_0^1 [(\alpha+1)s^\alpha - 1] s \, ds = \frac{\alpha((\alpha + 1)^{2/\alpha} + 1)}{2(\alpha + 2)(\alpha + 1)^{(2/\alpha)}} \]

\[ N_3(\alpha) = \int_0^1 [(\alpha+1)s^\alpha - 1] (1-s) \, ds \]

\[ = \frac{\alpha(2(\alpha + 2)(\alpha + 1)^{(1/\alpha)-1} - (\alpha + 1)^{2/\alpha} - 1)}{2(\alpha + 2)(\alpha + 1)^{2/\alpha}} \]

Substituting these values in (37), we get (33). \hfill \Box

Remark 5. If we put \( \alpha = 1, x = m, \) and \( y = M \) in (33), we get the inequality given in Theorem 2.2 of [25].

\[ \left| \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^q} \int_{m+M-x}^y \psi(m + M - y) \right| \]

\[ \leq \frac{y - x}{\alpha + 1} \left( P_1(\alpha) \right)^{1-1/q} \]

\[ \times \left( P_1(\alpha) \left| \psi'(m) \right|^q + P_1(\alpha) \left| \psi'(M) \right|^q - P_2(\alpha) \left| \psi'(x) \right|^q - P_3(\alpha) \left| \psi'(y) \right|^q \right)^{1/q}, \] \hspace{1cm} (39)

where \( P_1(\alpha), P_2(\alpha), \) and \( P_3(\alpha) \) are the same as defined in Theorem 4.

Proof. From Lemma 3, we have (35). Applying power mean inequality, we obtain

\[ \left| \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^q} \int_{m+M-x}^y \psi(m + M - y) \right| \]

\[ \leq \frac{y - x}{\alpha + 1} \left( \int_0^1 (\alpha+1)s^\alpha - 1 \right)^{1-1/q} \]

\[ \times \left( \left( \int_0^1 ((\alpha+1)s^\alpha - 1) \left| \psi'(m + M - s - y) \right| ds \right)^{1/q} \right). \] \hspace{1cm} (40)
Since $|\psi'|^q$ is convex, using Mercer’s inequality, we have

\[
\left| \frac{a\psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^{\frac{\alpha}{\alpha + 1}} f_{\alpha}^n(m + M - x) - \psi(m + M - y)} \right|
\]

\[
\leq \frac{y - x}{\alpha + 1} \left( \int_0^{1/(\alpha+1)^{\frac{1}{\alpha}}} \left[1 - (\alpha + 1)s^\alpha\right] ds + \int_1^{1/(\alpha+1)^{\frac{1}{\alpha}}} ((\alpha + 1)s^\alpha - 1) ds \right)^{1-1/q}
\]

\[
\times \left( \int_0^1 \left( (\alpha + 1)s^\alpha - 1 \right) \left( |\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q \right) ds \right)^{1/q}
\]

\[
\leq \frac{y - x}{\alpha + 1} \left( \frac{2\alpha}{(\alpha + 1)^{\frac{1}{\alpha + 1}/\alpha}} \left( \int_0^{1/(\alpha+1)^{\frac{1}{\alpha}}} \left(1 - (\alpha + 1)s^\alpha\right) \left( |\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q \right) ds \right) \right)^{1/q}
\]

\[
\times \left( \int_0^{1/(\alpha+1)^{\frac{1}{\alpha}}} \left(1 - (\alpha + 1)s^\alpha\right) \left( |\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q \right) ds \right)^{1/q}
\]

\[
+ \int_1^{1/(\alpha+1)^{\frac{1}{\alpha}}} ((\alpha + 1)s^\alpha - 1) \left( |\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q \right) ds \right)^{1/q}
\]

This implies that

\[
\left| \frac{a\psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^{\frac{\alpha}{\alpha + 1}} f_{\alpha}^n(m + M - x) - \psi(m + M - y)} \right|
\]

\[
\leq \frac{y - x}{\alpha + 1} \left( \frac{2\alpha}{(\alpha + 1)^{\frac{1}{\alpha + 1}/\alpha}} \left( \left( L_1(\alpha) + N_1(\alpha) \right) |\psi'(m)|^q + \left( L_1(\alpha) + N_1(\alpha) \right) |\psi'(M)|^q \right) \right)^{1-1/q}
\]

\[
- \left( L_2(\alpha) + N_2(\alpha) \right) |\psi'(x)|^q - \left( L_3(\alpha) + N_3(\alpha) \right) |\psi'(y)|^q \right)^{1/q}
\]

(41)

Substituting the values of $L_1$, $L_2$, $L_3$, $N_1$, $N_2$, and $N_3$ as given in the proof of Theorem 4 in (42), we get (39). \[ \square \]

**Remark 6.** If we put $\alpha = 1$, $x = m$, and $y = M$ in (39), we obtain the inequality proved in Theorem 1 of [28].

In the following theorem, we derive trapezoidal type inequality.

**Theorem 6.** Let $p, q > 1$ and $|\psi'|^q$ be a convex function, and let $x, y \in [m, M]$ such that $x < y$. Then,

\[
\left| \frac{a\psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^{\frac{\alpha}{\alpha + 1}} f_{\alpha}^n(m + M - x) - \psi(m + M - y)} \right|
\]

\[
\leq \frac{y - x}{\alpha + 1} \left( L_4(\alpha, p) \right)^{1/p} \left( |\psi'(m)|^q + |\psi'(M)|^q \right) - \frac{|\psi'(x)|^q + |\psi'(y)|^q}{2} \right)^{1/q},
\]

where $L_4(\alpha, p) = \int_0^{1/(\alpha+1)^{\frac{1}{\alpha}}} ((\alpha + 1)s^\alpha - 1)^p ds$ such that $(1/p) + (1/q) = 1$.

**Proof.** Using Lemma 3, we have (35). Applying Hölder’s inequality, we obtain
\[ \frac{\alpha \psi (m + M - x) + \psi (m + M - y)}{\alpha + 1} - \frac{\Gamma (\alpha + 1)}{(y - x)^\alpha} J^\alpha_{(m + M - x)} \psi (m + M - y) \]

\[ \leq \frac{y - x}{\alpha + 1} \left( \int_0^1 |(\alpha + 1)s^\alpha - 1|^p ds \right)^{1/p} \left( \int_0^1 |\psi' (m + M - sx - (1 - s)y)|^q ds \right)^{1/q} \]

\[ \leq \frac{y - x}{\alpha + 1} (L_4 (\alpha, p))^{1/p} \left( \int_0^1 (|\psi' (m)|^q + |\psi' (M)|^q - s|\psi' (x)|^q - (1 - s)|\psi' (y)|^q) ds \right)^{1/q} \]

\[ = \frac{y - x}{\alpha + 1} (L_4 (\alpha, p))^{1/p} \left( |\psi' (m)|^q + |\psi' (M)|^q - \frac{|\psi' (x)|^q + |\psi' (y)|^q}{2} \right)^{1/q}. \]

**Remark 7.** Substituting \( \alpha = 1, x = m, \) and \( y = M \) in Theorem 6, we obtain Theorem 2.3 of [25].

**Theorem 7.** Let \( |\psi' | \) be a convex function and let \( x, y \in [m, M] \) such that \( x < y \). Then,

\[ \left| \frac{\Gamma (\alpha + 1)}{(y - x)^\alpha} J^\alpha_{(m + M - x)} \psi (m + M - y) - \psi (m + M - \frac{\alpha x + y}{\alpha + 1}) \right| \]

\[ \leq (y - x) (P_5 (\alpha) |\psi' (x)| + P_6 (\alpha) |\psi' (y)|), \]

where

\[ P_5 (\alpha) = \frac{-\alpha}{2 (\alpha + 2) (\alpha + 1)^2}, \]

\[ P_6 (\alpha) = \frac{\alpha}{2 (\alpha + 2) (\alpha + 1)^2}. \]

**Proof.** Using Lemma 4, we have

\[ \frac{\Gamma (\alpha + 1)}{(y - x)^\alpha} J^\alpha_{(m + M - x)} \psi (m + M - y) - \psi \left( m + M - \frac{\alpha x + y}{\alpha + 1} \right) \]

\[ \leq (y - x) \int_{a (\alpha + 1)}^1 (1 - s^\alpha) \left( |\psi' (m)| + |\psi' (M)| - s|\psi' (x)| - (1 - s)|\psi' (y)| \right) ds \]

\[ - (y - x) \int_0^{a (\alpha + 1)} s^\alpha \left( |\psi' (m)| + |\psi' (M)| - s|\psi' (x)| - (1 - s)|\psi' (y)| \right) ds. \]

Since \( |\psi' | \) is convex, using Mercer’s inequality, we obtain

\[ \frac{\Gamma (\alpha + 1)}{(y - x)^\alpha} J^\alpha_{(m + M - x)} \psi (m + M - y) - \psi \left( m + M - \frac{\alpha x + y}{\alpha + 1} \right) \]

\[ \leq (y - x) \left( (N_5 (\alpha) - L_5 (\alpha)) |\psi' (x)| + (N_6 (\alpha) - L_6 (\alpha)) |\psi' (y)| \right). \]
where

\[
L_5(a) = \int_{a/(a+1)}^{1} (1 - t^a) dt = \frac{a(a + 1)^a + 2a^{a+2}}{2(a + 2)(a + 1)^{a+2}}.
\]

\[
L_6(a) = \int_{a/(a+1)}^{1} (1 - t^a) (1 - s) dt = \frac{4a^{a+1} - a(a + 1)^a}{2(a + 2)(a + 1)^{a+2}}.
\]

\[
N_5(a) = \int_0^{a/(a+1)} s^{a+1} ds = \frac{a}{(a + 1)^{a+2}}.
\]

\[
N_6(a) = \int_0^{a/(a+1)} s^a (1 - s) ds = \frac{2a^{a+1}}{(a + 2)(a + 1)^{a+2}}.
\]

Substituting these values in (49), we get (45).

In next theorem, we use power mean inequality and derive midpoint type inequality.

**Theorem 8.** Let \(|\psi'|^q\) be a convex function for \(q \geq 1\) and let \(x, y \in (m, M)\) such that \(x < y\). Then,

\[
\frac{\Gamma(a + 1)}{(y - x)^a} \int_{(m, M - x)} (\psi(m + M - y) - \psi(m + M - \frac{\alpha x + y}{\alpha + 1}))
\]

\[
\leq (y - x)(L_6(a))^{1/(1/q)}
\]

\[
\times \left( (L_6(a) \psi'(m))^{q} + L_6(a) \psi'(M)^q - L_5(a) \psi'(x)^q - L_6(a) \psi'(y)^q \right)^{1/q}
\]

\[
-(L_6(a) \psi'(m))^{q} + L_6(a) \psi'(M)^q - N_5(a) \psi'(x)^q - N_6(a) \psi'(y)^q \right)^{1/q},
\]

where

\[
L_6(a) = \int_{a/(a+1)}^{1} (1 - t^a) dt = \int_0^{a/(a+1)} s^{a+1} ds = \frac{a}{(a + 1)^{a+2}}.
\]

and \(L_6(a), L_5(a), N_5(a), \) and \(N_6(a)\) are given in the proof of Theorem 7.

**Proof.** Using power mean inequality in (47), we obtain

\[
\frac{\Gamma(a + 1)}{(y - x)^a} \int_{(m, M - x)} (\psi(m + M - y) - \psi(m + M - \frac{\alpha x + y}{\alpha + 1}))
\]

\[
\leq (y - x)\left( \int_{a/(a+1)}^{1} (1 - s^{a}) ds \right)^{1/(1/q)} \left( \int_{a/(a+1)}^{1} (1 - s^{a}) |\psi'(m + M - sx - (1 - s)y)|^{q} ds \right)^{1/q}
\]

\[
-(y - x)\left( \int_0^{a/(a+1)} s^{a} ds \right)^{1/(1/q)} \left( \int_0^{a/(a+1)} s^{a} |\psi'(m + M - sx - (1 - s)y)|^{q} ds \right)^{1/q}.
\]
Since \(|\psi'|^q\) is convex, using Mercer’s inequality, we obtain

\[
\frac{\Gamma(\alpha + 1)}{(y - x)^{\alpha}} \int_{(m,M-x)} \psi(m + M - y) - \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) \leq (y - x) \left( \frac{\alpha^{\alpha + 1}}{(\alpha + 1)^{\alpha + 2}} \right)^{1/(1/q)} \times \left( \int_0^1 \left( |\psi'(m)|^q + |\psi'(M)|^q - \frac{1}{\alpha} |\psi'(x)|^q - (1 - \frac{1}{\alpha}) |\psi'(y)|^q \right) ds \right)^{1/q}
\]

\[
= (y - x) (L_6(\alpha))^{1/(1/q)} \times \left( L_6(\alpha)|\psi'(m)|^q + L_6(\alpha)|\psi'(M)|^q - L_5(\alpha)|\psi'(x)|^q - L_6(\alpha)|\psi'(y)|^q \right)^{1/q}
\]

\[
- (L_6(\alpha)|\psi'(m)|^q + L_6(\alpha)|\psi'(M)|^q - N_5(\alpha)|\psi'(x)|^q - N_6(\alpha)|\psi'(y)|^q )^{1/q}.
\]

\[\square\]

**Remark 8.** Substituting \(\alpha = 1, x = m,\) and \(y = M\) in Theorem 8, we obtain the following midpoint type inequality:

\[
\left| \frac{1}{M - m} \int_m^M f(u) du - \psi\left(\frac{m + M}{2}\right) \right| \leq \frac{M - m}{8} \left( \left( \frac{|\psi'(m)|^q + 2|\psi'(M)|^q}{3} \right)^{1/q} - \left( \frac{2|\psi'(m)|^q + |\psi'(M)|^q}{3} \right)^{1/q} \right).
\]

Another midpoint type inequality is presented in the following theorem.

**Theorem 9.** Let \(p, q > 1\) and \(|\psi'|^q\) be convex, and let \(x, y \in [m, M]\) such that \(x < y\). Then,

\[
\frac{\Gamma(\alpha + 1)}{(y - x)^{\alpha}} \int_{(m,M-x)} \psi(m + M - y) - \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) \leq (y - x) (L_9(\alpha, p))^{1/p} \times \left( \frac{1}{\alpha + 1} |\psi'(m)|^q + \frac{1}{\alpha + 1} |\psi'(M)|^q - \frac{2\alpha + 1}{2(\alpha + 1)} |\psi'(x)|^q - \frac{1}{2(\alpha + 1)} |\psi'(y)|^q \right)^{1/q}
\]

\[
- (y - x) (L_{10}(\alpha, p))^{1/p} \times \left( \frac{\alpha}{\alpha + 1} |\psi'(m)|^q + \frac{\alpha}{\alpha + 1} |\psi'(M)|^q - \frac{\alpha^2}{2(\alpha + 1)} |\psi'(x)|^q - \frac{\alpha(\alpha + 2)}{2(\alpha + 1)^2} |\psi'(y)|^q \right)^{1/q}.
\]
where \( L_9(\alpha, p) = \int_{a/(\alpha+1)}^{1} (1 - s^\alpha) s^p \, ds \) and \( L_{10}(\alpha, p) = \int_{0}^{a/(\alpha+1)} s^p \, ds \) such that \((1/p) + (1/q) = 1.

Proof. Applying Hölder’s inequality in (47), we have

\[
\left| \frac{\Gamma(\alpha + 1)}{(y - x)^{\alpha}} f_{m+M-x}^\alpha \right| \leq (y - x) \left( \int_{a/(\alpha+1)}^{1} (1 - s^\alpha) s^p \, ds \right) \left( \int_{a/(\alpha+1)}^{1} \left| \psi' (m + M - sx - (1 - s)y) \right|^q \, ds \right)^{1/q}
- (y - x) \left( \int_{a/(\alpha+1)}^{1} s^p \, ds \right) \left( \int_{0}^{a/(\alpha+1)} \left| \psi' (m + M - sx - (1 - s)y) \right|^q \, ds \right)^{1/q}.
\]

As \(|\psi'|^q\) is convex, applying Mercer’s inequality, we obtain

\[
\left| \frac{\Gamma(\alpha + 1)}{(y - x)^{\alpha}} \psi (m + M - y) - \psi (m + M - \frac{ax + y}{\alpha + 1}) \right|
\leq (y - x) \left( \int_{a/(\alpha+1)}^{1} (1 - s^\alpha) s^p \, ds \right) \left( \int_{a/(\alpha+1)}^{1} \left| \psi' (m + M - sx - (1 - s)y) \right|^q \, ds \right)^{1/q}
- (y - x) \left( \int_{a/(\alpha+1)}^{1} s^p \, ds \right) \left( \int_{0}^{a/(\alpha+1)} \left| \psi' (m + M - sx - (1 - s)y) \right|^q \, ds \right)^{1/q}.
\]

Remark 9. Substituting \(\alpha = 1, x = m,\) and \(y = M\) in Theorem 9, we obtain

\[
\left| \frac{1}{M - m} \int_{m}^{M} f(u) \, du - \psi \left( \frac{m + M}{2} \right) \right|
\leq \frac{M - m}{4 (p + 1)^{1/p}} \left( \left( \frac{|\psi' (m)|^q + 3 |\psi' (M)|^q}{4} \right)^{1/q} - \left( \frac{3 |\psi' (m)|^q + |\psi' (M)|^q}{4} \right)^{1/q} \right).
\]
Theorem 10. Let $x, y \in [m, M]$ such that $x < y$. If $|\psi'|$ is concave on $[m, M]$, then

$$\frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1) f_{\alpha}^{\infty}(m + M - x) \psi(m + M - y)}{(y - x)^{\alpha}} \leq \frac{y - x}{\alpha + 1} \left\{ L_1(\alpha) |\psi'(m + M - L_{11}(\alpha)x + L_{12}(\alpha)y)| + N_1(\alpha) |\psi'(m + M - L_{13}(\alpha)x + L_{14}(\alpha)y)| \right\} \leq y - x \left( L_1(\alpha) \frac{(\alpha + 1)^{1-(1/\alpha)}}{2(\alpha + 2)} + N_1(\alpha) \frac{(\alpha + 1)^{1-(1/\alpha)}}{2(\alpha + 2)} \right) \leq y - x \frac{(\alpha + 1)^{1-(1/\alpha)}}{2(\alpha + 2)} \leq y - x \frac{(\alpha + 1)^{1-(1/\alpha)}}{2(\alpha + 2)}.$$ 

(60)

where $L_1$ and $N_1$ are given in the proof of Theorem 4 and

$$L_{11}(\alpha) = \frac{(\alpha + 1)^{1-(1/\alpha)}}{2(\alpha + 2)},$$

$$L_{12}(\alpha) = \frac{(\alpha + 1)^{1-(1/\alpha)}}{2(\alpha + 2)} \left\{ 2(\alpha + 2)(\alpha + 1)^{(1/\alpha) - 1} - 1 \right\},$$

$$L_{13}(\alpha) = \frac{(\alpha + 1)^{1-(1/\alpha)}}{2(\alpha + 2)} \left\{ (\alpha + 1)^{2/\alpha} + 1 \right\},$$

$$L_{14}(\alpha) = \frac{(\alpha + 1)^{1-(1/\alpha)}}{2(\alpha + 2)} \left\{ 2(\alpha + 2)(\alpha + 1)^{(1/\alpha) - 1} - (\alpha + 1)^{2/\alpha} - 1 \right\}.$$ 

(61)

Proof. Lemma 3 implies that

$$\frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1) f_{\alpha}^{\infty}(m + M - x) \psi(m + M - y)}{(y - x)^{\alpha}} \leq \frac{y - x}{\alpha + 1} \int_0^1 (\alpha + 1)^{1/\alpha} (1 - (\alpha + 1)s^\alpha) |\psi'(m + M - sx - (1 - s)y)| ds$$

$$= \frac{y - x}{\alpha + 1} \left( \int_0^{1/(\alpha + 1)^{1/\alpha}} (1 - (\alpha + 1)s^\alpha) |\psi'(m + M - sx - (1 - s)y)| ds + \int_{1/(\alpha + 1)^{1/\alpha}}^1 (\alpha + 1)^{1/\alpha} (1 - (\alpha + 1)s^\alpha) |\psi'(m + M - sx - (1 - s)y)| ds \right).$$ 

(62)
Since $|\psi'|$ is concave, using Jensen’s inequality, we obtain

$$\left| \frac{\alpha \psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^{\alpha}(m + M - x)^{\alpha}} \psi(m + M - y) \right|$$

$$\leq \frac{y - x}{\alpha + 1} \left( \int_0^{1/(\alpha + 1)^{1/\alpha}} (1 - (\alpha + 1)s^\alpha) ds \right) \left| \psi' \left( \frac{\int_0^{1/(\alpha + 1)^{1/\alpha}} (1 - (\alpha + 1)^{1/\alpha}) (m + M - sx - (1 - s)y) ds}{\int_0^{1/(\alpha + 1)^{1/\alpha}} (1 - (\alpha + 1)^{1/\alpha}) ds} \right) \right|$$

$$+ \int_0^1 (\alpha + 1)^{1/\alpha} s^\alpha ds \left| \psi' \left( \frac{\int_0^1 (\alpha + 1)^{1/\alpha} ((\alpha + 1)^{1/\alpha} - 1) (m + M - sx - (1 - s)y) ds}{\int_0^1 (\alpha + 1)^{1/\alpha} ((\alpha + 1)^{1/\alpha} - 1) ds} \right) \right| \right).$$

(63)

$$= \frac{y - x}{\alpha + 1} \left( L_1(\alpha) \right) \left| \psi' \left( m + M - L_{15}(\alpha)x - L_{16}(\alpha)y \right) \right|$$

$$+ N_1(\alpha) \left| \psi' \left( m + M - L_{17}(\alpha)x - L_{18}(\alpha)y \right) \right| \right).$$

Theorem 11. Let $|\psi'|$ be a concave function, and let $x, y \in [m, M]$ such that $x < y$. Then,

$$\left| \frac{\Gamma(\alpha + 1)}{(y - x)^{\alpha}(m + M - x)^{\alpha}} \psi(m + M - y) - \psi(m + M - \frac{ax + y}{\alpha + 1}) \right|$$

$$\leq (y - x) \left( \frac{\alpha^{\alpha + 1}}{(\alpha + 1)^{\alpha + 2}} \right) \left( |\psi'(m + M - L_{15}(\alpha)x - L_{16}(\alpha)y)| \right.$$  

$$- |\psi'(m + M - L_{17}(\alpha)x - L_{18}(\alpha)y)| \right),$$

(64)

where

$$\frac{\Gamma(\alpha + 1)}{(y - x)^{\alpha}(m + M - x)^{\alpha}} \psi(m + M - y) - \psi(m + M - \frac{ax + y}{\alpha + 1})$$

$$\leq (y - x) \left( \frac{\alpha^{\alpha + 1}}{(\alpha + 1)^{\alpha + 2}} \right) \left( \left| \psi' \left( \frac{\int_0^1 (1 - s^\alpha) (m + M - sx - (1 - s)y) ds}{\int_0^1 (1 - s^\alpha) ds} \right) \right|$$

$$- (y - x) \left( \frac{\alpha^{\alpha + 1}}{(\alpha + 1)^{\alpha + 2}} \right) \left| \psi' \left( \frac{\int_0^{\alpha/(\alpha + 1)} s^\alpha (m + M - sx - (1 - s)y) ds}{\int_0^{\alpha/(\alpha + 1)} s^\alpha ds} \right) \right| \right).$$

(66)

Proof. From Lemma 4, we have (47). As $|\psi'|$ is concave, using Jensen’s inequality, we obtain
4. Conclusion

In this paper, we establish the fractional Hermite–Hadamard type inequalities of Mercer type by using support line inequality. We expect that this work will lead to the new fractional integral studies for Hermite–Hadamard inequality. It is an open problem to prove inequalities (10) and (11) by any other method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to this paper. All authors read and approved the final manuscript.

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