Separable functions: symmetry, monotonicity, and applications

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Abstract

In this paper, we introduce concepts of separable functions in balls and in the whole space, and develop a new method to investigate the qualitative properties of separable functions. We first study the axial symmetry and monotonicity of separable functions in unit circles by geometry analysis, and we prove the uniqueness of the symmetry axis for nontrivial separable functions. Then by using reduction dimension and convex analysis, we get the axial symmetry and monotonicity of separable functions in high dimensional spheres. Based on the above results on unit circles and spheres, we deduce the axial symmetry and monotonicity of separable functions in balls and the radial symmetry and monotonicity of separable functions in the whole space. Conversely, the function with axial symmetry and monotonicity in the ball domain is separable function, and the function with radial symmetry and monotonicity in the whole space is also separable function. These enable us to provide easily some examples that separable functions in balls may be just axially symmetric not radially symmetric. Finally, as applications, we obtain the axial symmetry and monotonicity of all the positive ground states to the Choquard equation in a ball as well as the radial symmetry and monotonicity of all the positive ground states in the whole space.

Key words Separable function; Geometry analysis; Convex analysis, Monotonicity; Radial symmetry; Axial symmetry

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1 Introduction

As we know, symmetry and monotonicity are very important properties of solutions to elliptic partial differential equations, see [9, 11, 17]. They play an essential role in the uniqueness and dependence on parameters of solutions and hence have been extensively investigated, see [3, 6, 23, 30] and references therein. In these literatures, the maximum principle is vital to study various properties of solutions. Based on maximum principle, a solution \( u \) to elliptic equations is comparable with the mirror point about a hyperplane. To be precise, let \( H \subset \mathbb{R}^n \) be an open half-space. For \( x \in \mathbb{R}^n \), \( \sigma_H x \) denotes the symmetric point of \( x \) with respect to the hyperplane \( \partial H \). Then \( u \) keeps the larger value on \( H \) (or on \( \mathbb{R}^N \setminus H \)). It is natural to guess that sufficiently many such hyperplanes can lead to symmetry and monotonicity. This motivates us to introduce the following concept of separable functions.

**Definition 1.1.** Let \( \Omega \subset \mathbb{R}^N \). A function \( u : \Omega \rightarrow \mathbb{R} \) is said to be separable in \( \Omega \) if for any \( H \in \mathcal{H}_\Omega := \{ \text{open half space } H \subset \mathbb{R}^N | \sigma_H(H \cap \Omega) = \Omega \setminus \text{cl}(H) \} \),

\[
either u(x) \geq u(\sigma_H x) \text{ for all } x \in H \cap \Omega \text{ or } u(x) \leq u(\sigma_H x) \text{ for all } x \in H \cap \Omega. \tag{1.1}\]

In view of the above definition, we see that \( \Omega \) has better symmetry if the set \( \mathcal{H}_\Omega \) is larger, and then the symmetry and monotonicity of separable functions in \( \Omega \) can be simpler and richer. Since spheres, balls, and the whole space have rich symmetry, in the present paper, we mainly investigate symmetry and monotonicity of separable functions in these domains.

For motivations of this study, first we recall some tools used to study symmetry and monotonicity of solutions to elliptic equations, which include symmetric decreasing rearrangement (or Schwarz symmetrization), polarization method, the method of moving planes and its variants. The symmetric decreasing rearrangement mainly depends on rearrangement inequalities and minimizing method to obtain the existence and symmetry of the minimizer (see [20, 21]). For the polarization method, one can first establish polarization inequality and then show the relationship between the solution and its polarization, via which the symmetry can be proved, (see [1, 2, 29, 28] and references therein). As one of powerful tools in establishing symmetry and monotonicity of solutions to elliptic equations, the method of moving planes was proposed by the Soviet mathematician Alexanderoff in the early 1950s. Decades later, it was further developed by Gidas, Ni and Nirenberg [14], Chen and Li [3] and many others. Please see [4, 18, 26, 19, 8, 7, 16] and references therein. It is known that the three tools essentially rely on the elliptic equations and the specific solutions. Then a natural question is whether we can study the symmetry and monotonicity of the solutions by just using their separability instead of the elliptic equations and other properties of the solutions. In other words, whether we can study the symmetry and monotonicity only via separability. This paper will give an affirmative answer.
In this paper, we successively consider separable functions in circles, spheres, balls, and the whole space. We leave complicated domains for future study. Here we sketch the main ideas and approaches to study several separable functions in unit circle. Then by employing geometric analysis, we obtain that the set of global extremal points for a given positive and nonconstant separable function in unit circle is two arcs. One arc (max-arc, for short) is the set of maximum points and the other arc (min-arc, for short) is the set of minimum point of this function. This, combined with the separability of the function, implies that the centers of the max-arc and min-arc are the ends of the same diameter. By choosing suitable diameter and using the separability again, we prove the axial symmetry and monotonicity of separable functions in circles with the unique symmetry axis.

In what follows, in order to apply reduction dimension method, we give equivalent definitions of separable functions in high dimensional spheres. For a given positive and nonconstant separable function in a sphere, we have shown that the set of global extremal points of this function is two sphere caps by using reduction dimension method and convex analysis. One sphere cap (max-cap, for short) is the set of maximum points and the other sphere cap (min-cap, for short) is the set of minimum point of this function. Based on the separability of the function constrained in the unit circle through the centers of the max-cap and min-cap, we show that the centers of the max-cap and min-cap are in the same diameter. By constructing suitable circles and using the axial symmetry and monotonicity of this function in these circles, it is easy to check that this separable function in a given sphere is axially symmetric and monotone with the unique symmetry axis.

In the sequel, based on the fact that a ball is made up of homocentric spheres, by using the separability of functions in the ball, we point out that the centers of the max-caps and min-caps for all the homocentric spheres are in the same diameter. So we can deduce the axial symmetry and monotonicity of separable functions in balls by applying axial symmetry and monotonicity of separable functions in spheres. Conversely, the axially symmetric and monotone functions in balls are also separable. In other words, the separability is equivalent to the axial symmetry and monotonicity for a given function in balls. This observation enables us to give an example that separable functions in balls may be only axially symmetric but not radially symmetric.

Finally, we give the definitions of separable functions in the whole space. Note that a positive separable function in the whole space is separable in any ball. This fact, combined with the axial symmetry and monotonicity of the separable functions in balls, implies that the separable function in the whole space admits an unique symmetry axis passing through any given point, and all the symmetry axes are parallel to each other. Furthermore, suppose that the infimum of the separable function in the whole space is zero. Then we can deduce the radial symmetry and monotonicity of the separable
function in the whole space. Similarly, we also easily see that the separability is equivalent to the radial
symmetry and monotonicity for a given positive function with the infimum being zero.

As applications of symmetry and monotonicity of separable functions, we consider Choquard
equations in balls and in the whole space. Specifically, we obtain the axial symmetry and mono-
tonicity of all the positive ground states to Choquard equations in balls as well as the radial symmetry
and monotonicity of all the positive ground states to Choquard equations in the whole space.

To sum up, this paper provides a new perspective to study the symmetry and monotonicity of
solutions to elliptic equations. Roughly speaking, it involves two steps to obtain the symmetry and
monotonicity of solutions. In first step, we prove the symmetry and monotonicity of separable func-
tions. We emphasize that the proof of this step does not rely on the exact equations or properties of
specific solutions. In second step, we are concerned with the separability of a specific solution to a
concrete equation, and then deduce its symmetry and monotonicity.

Throughout this paper, we always assume that separable functions are only continuous.

For convenience, we introduce some notations as follows:

- \( \mathbb{N} \) is the set of all the positive integers.
- \( N \in \mathbb{N} \) and \( N \geq 2 \).
- \( 0_k := (0, \cdots , 0) \) for \( k \in \mathbb{N} \).
- Let \( \text{aff}(A) := \{ \sum_{i=1}^{m} \lambda_i x^i | m \in \mathbb{N}, x^i \in A, \lambda_i \in \mathbb{R} \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \} \)
  and \( \text{co}(A) := \{ \sum_{i=1}^{l} \lambda_i x^i | l \in \mathbb{N}, x^i \in A, \lambda_i \in [0, 1] \text{ and } \sum_{i=1}^{l} \lambda_i = 1 \} \).
- \( S^{N-1}(x) \) is the unit sphere centered at \( x \) and \( S^{N-1}_r(x) \) is the sphere centered at \( x \) with radius \( r > 0 \) in \( \mathbb{R}^N \). For simplicity of notations, we write \( S^{N-1}(0) \) and \( S^{N-1}_r(0) \) as \( S^{N-1} \) and \( S^{N-1}_r \), respectively.
- For \( r > 0 \), \( B_r(x) \) is the closed ball centered at \( x \in \mathbb{R}^N \) with radius \( r \). For simplicity of notations, we
  write \( B_r(0) \) as \( B_r \).
- \( O(N) \) represents the set of orthogonal transformations in \( \mathbb{R}^N \). For \( M \in O(N) \) and \( u \in C(\mathbb{R}^N, \mathbb{R}) \), we
define \( u_M(x) := u(M^{-1}x) \).
- Let \( H \subset \mathbb{R}^N \) be an open half-space. For any \( x \in \mathbb{R}^N \), \( \sigma_{\partial H}x \) is the symmetric point of \( x \) with respect to
\( \partial H \).

The remaining of this paper is organised as follows. In section 2, the axial symmetry and mono-
tonicity of separable functions in circles, spheres, and balls are established, by using reduction dimen-
sion method, geometric analysis, and convex analysis. Based on these results, section 3 is devoted to
the proof of radial symmetry and monotonicity of separable functions in the whole space. Finally, in
section 4, we apply our main theoretical results to Choquard type equations to obtain the axial symmetry and monotonicity of all the positive ground states in a ball as well as the radial symmetry and monotonicity of all the positive ground states in the whole space.

2 Separable functions in bounded domains

In this section, we investigate the symmetry and monotonicity of separable functions in bounded domains such as high dimensional balls $B_R \subset \mathbb{R}^N$ by using dimensionality reduction, geometry analysis, and convex analysis.

In the following, we first give some notations. Let $S_1$ be the unit circle in $\mathbb{R}^2$. Let $\overset{\frown}{xy}$ be an arc from $x$ to $y$ counterclockwise in $S_1$ and $s(\overset{\frown}{xy})$ be the arc length of $\overset{\frown}{xy}$. Let $S^1_\alpha = \{(\cos(\alpha + \theta), \sin(\alpha + \theta)) : \theta \in (0, \pi)\}$, $B^1_\alpha = \bigcup_{r \in [0,1]} rS^1_\alpha$, $l_\alpha = \partial B^1_\alpha \setminus S^1_\alpha$, where $\alpha \in \mathbb{R}$. For any $x \in S^1$, let $l_\alpha(x)$ be the axial symmetric point of $x$ with respect to $l_\alpha$.

For a given line $L \subseteq \mathbb{R}^N$, we say that $x, y \in \mathbb{R}^N$ are axially symmetric with respect to $L$ if there is $z^* \in L$ such that $||x - z^*|| = ||y - z^*||$ and $x - z^*, z^* - y \perp L$, respectively. Here $||x - z^*|| = \min\{||x - z|| : z \in L\}$, $||y - z^*|| = \min\{||y - z|| : z \in L\}$. A function $u \in C(\mathbb{R}^N, \mathbb{R})$ is said to be axially symmetric with respect to a line $L$ if $u(x) = u(y)$ for any $x, y \in \mathbb{R}^N$ that are axially symmetric with respect to $L$. Here $L$ is a symmetry axis of $u$.

Now we begin with the definition and properties of separable functions in $S^1$ in the following subsection.

2.1 Separable functions in unit circles

The following gives the definition of separable functions in unit circle $S^1$.

**Definition 2.1.** A function $v \in C(S^1, \mathbb{R})$ is said to be separable in $S^1$, if for any $\alpha \in [0, 2\pi)$, there holds

\begin{equation}
\text{either } v(l_\alpha(x)) \geq v(x) \text{ for all } x \in S^1_\alpha \text{ or } v(l_\alpha(x)) \leq v(x) \text{ for all } x \in S^1_\alpha.
\end{equation}

Now we show that the properties of separable functions in $S^1$, which plays a critical role in investigating the symmetry and monotonicity of separable functions in high dimensional spheres and balls.

**Lemma 2.1.** Let $v \in C(S^1, (0, \infty))$ be a separable function in $S^1$. Suppose that $\max_{S^1} v > \min_{S^1} v$. Then there exist $\alpha_0 \in [0, 2\pi)$ and $\theta_1, \theta_2 \in [0, \pi)$ such that
(i) \( \theta_1 + \theta_2 < \pi \).

(ii) \( v^{-1}(\max_{S^1} v) = \{(\cos(\alpha_0 + \theta), \sin(\alpha_0 + \theta)) : |\theta| \leq \theta_1\} \) and

\[
v^{-1}(\min_{S^1} v) = \{(\cos(\alpha_0 + \pi + \theta), \sin(\alpha_0 + \pi + \theta)) : |\theta| \leq \theta_2\}.
\]

(iii) \( v(x) = v(l_{x_0}(x)) \) for all \( x \in S^1_{\alpha_0} \).

(iv) \( v((\cos \alpha, \sin \alpha)) \) is not a constant function and is a nonincreasing function with respect to \( \alpha \in [\alpha_0, \alpha_0 + \pi] \).

Proof. Let \( A := v^{-1}(\max_{S^1} v) \) and \( B := v^{-1}(\min_{S^1} v) \). Then \( A \not= \emptyset \) and \( B \not= \emptyset \). We shall finish the proof by the following five steps.

Step 1. We claim that there exist \( x \in A \) and \( y \in B \) such that \( \|x - y\| = 2 \), that is, there exists \( \alpha_0 \in [0, 2\pi) \) such that \( x = (\cos \alpha_0, \sin \alpha_0) \in A \) and \( y = (\cos(\alpha_0 + \pi), \sin(\alpha_0 + \pi)) \in B \), where \( \| \cdot \| \) represents the Euclidean norm on \( \mathbb{R}^2 \).

Otherwise, according to the compactness of \( A \) and \( B \), there exist \( x \in A, y \in B \) such that \( \|x - y\| = \max d(A \times B) < 2 \) where \( d : A \times B \to \mathbb{R} \) by \( (x, y) \mapsto \|x - y\| \). Without loss of generality, we assume \( 0 < s(x, y) < \pi \). Clearly, there exists \( \alpha_0 \in [0, 2\pi) \) such that

\[
x = (\cos \alpha_0, \sin \alpha_0), \quad y = (\cos(\alpha_0 + s(x, y)), \sin(\alpha_0 + s(x, y))).
\]

By taking \( z = (\cos(\alpha_0 + 2s(x, y)), \sin(\alpha_0 + 2s(x, y))) \), we have \( s(x, y) = s(y, z) \in (0, \pi) \), and hence \( \hat{x} \cap A \subset (x, z] \). Let \( \alpha^* = \alpha_0 - \frac{2(\pi - s(x, y))}{3} \). Then \( 0 < \alpha_0 - \alpha^* < \pi - s(x, y) \), and thus \( l_{\alpha^*}(x) \in \hat{x}, s(x, y) \rightarrow s(x, y) = \frac{2\pi + s(x, y)}{3} \in (s(x, y), \pi) \), and \( x, y \in S^1_{\alpha^*} \). So, \( v(l_{\alpha^*}(x)) < \max_{S^1} v \) due to \( l_{\alpha^*}(x) \in \hat{x} \) and the choices of \( x, y \). It follows from (2.1) and \( v(l_{\alpha^*}(x)) < \max_{S^1} v = v(x) \) that

\[
v(\bar{x}) \geq v(l_{\alpha^*}(\bar{x})) \text{ for all } \bar{x} \in S^1_{\alpha^*}.
\]

In particular, \( v(y) \geq v(l_{\alpha^*}(y)) \) and hence \( l_{\alpha^*}(y) \in B \), a contradiction with \( s(\hat{x}, y) \in (s(x, y), \pi) \). The above arguments are illustrated in Figure 2.1

Therefore, we have finished the proof of Step 1.

Step 2. We shall prove \( A = \hat{x} \), where \( \bar{x} \in A \cap \hat{y}, \bar{x} \in A \cap \hat{y} \) with \( \|\bar{x} - y\| = \min_{z \in A \cap \hat{y}} \|z - y\| \) and

\[
\|\bar{x} - y\| = \min_{z \in A \cap \hat{y}} \|z - y\|.
\]

Clearly, \( A \subset \hat{x} \). We only need to prove \( \hat{x} \subset A \). It is clear that \( \hat{x} \subset A \) if \( s(\hat{x}) = 0 \). Now we suppose \( s(\hat{x}) > 0 \). Take \( \hat{x}, \hat{x}^* \in \hat{x} \) such that

\[
s(\hat{x}, \hat{x}) = \sup\{s(\hat{x}) : \hat{x} \subset A\}, s(\hat{x}, \hat{x}^*) = \sup\{s(\hat{x}) : \hat{x} \subset A\}.
\]
It suffices to prove $\bar{x}^* = \check{x}$. Otherwise, $\check{x}^* \neq \check{x}$. Without loss of generality, we may assume that $s(\check{x}^* \hat{x}) \geq s(\check{x} \hat{x})$. Take $\check{x}^0 \in \check{x} \hat{x}$ and $\alpha^* \in [0, 2\pi)$ such that $s(\check{x}^0 \hat{x}) = s(\check{x} \hat{x})$ and $\check{x} = l_{\alpha^*}(\check{x})$. Then min{$s(\check{x} \hat{x}), s(\check{x} \hat{x})$} > 0, $\check{x} \hat{x} \cup \check{x}^0 \hat{x} \subset A$, $\check{x}^0 \hat{x} \subset S^1$, and $\check{x} \hat{x}^* \subset S^1 \setminus S^1\!\alpha^*$. For any $\alpha \in [\alpha^*, \alpha^* + \min\{s(\check{x} \hat{x}), s(\check{x} \hat{x})\}$, we easily check that $\check{x}^0 \hat{x} \subset S^1\!\alpha$, $\check{x} \hat{x}^* \subset S^1 \setminus S^1\!\alpha$, $l_{\alpha}(\check{x}) \in \check{x} \hat{x}$, and thus $v(\check{x}) > v(\check{x}^0)$. It follows from (2.1) that $v(\check{x}^0) \geq v(\check{x}) = \max_{S^1} v$, (see Figure 2.2). As a result, $l_{\alpha}(\check{x}^0) \in A$ for all $\alpha \in (\alpha^*, \alpha^* + \min\{s(\check{x} \hat{x}), s(\check{x} \hat{x})\}$ and hence by the definition of $l_{\alpha}$ and the compactness of $A$, $\check{x} \hat{x}^* \subset \check{x} \hat{x} \subset S^1\!\alpha + \min\{s(\check{x} \hat{x}), s(\check{x} \hat{x})\} \subset A$, a contradiction with the choice of $\check{x}^*$. This proves $\check{x}^* = \check{x}$ and consequently $A = \check{x} \hat{x}$. 

**Step 3.** Show that there exist $\alpha_0 \in [0, 2\pi)$ and $\theta_1, \theta_2 \in [0, \pi)$ such that $A = \{(\cos(\alpha_0 + \theta), \sin(\alpha_0 + \theta)) : |\theta| \leq \theta_1\}$ and $B = \{(\cos(\alpha_0 + \pi + \theta), \sin(\alpha_0 + \pi + \theta)) : |\theta| \leq \theta_2\}$.

By **Step 2**, there exist $\alpha_0 \in [0, 2\pi)$ and $\theta_1 \in [0, \pi)$ such that $A = \{(\cos(\alpha_0 + \theta), \sin(\alpha_0 + \theta)) : |\theta| \leq \theta_1\}$. 

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**Figure 2.1:** schematic diagram for the proof of **Step 1**

**Figure 2.2:** schematic diagram for the partial proof of **Step 2**
By applying the claim in Step 2 again to \( \tilde{v}(x) := 1 + \max_{S^1} v - v(x) \), we have

\[ \tilde{v}^{-1}(\max_{S^1} \tilde{v}) = \{(\cos(\alpha_1 + \theta), \sin(\alpha_1 + \theta)) : |\theta| \leq \theta_2 \} \]

for some \((\alpha_1, \theta_2) \in [0, 2\pi) \times [0, \pi)\). In other words, \( B = \{(\cos(\alpha_1 + \theta), \sin(\alpha_1 + \theta)) : |\theta| \leq \theta_2 \} \). It suffices to prove that \((0, 0)\) belongs to the line segment \( \overline{x'y'} \), where \( x' := (\cos \alpha_0, \sin \alpha_0) \), \( y' := (\cos \alpha_1, \sin \alpha_1) \). Otherwise, there exists a diameter \( l \) such that \( x', y' \) are on the same side of \( l \) and \( l \cap \{x', y'\} = \emptyset \). By the choices of \( x' \) and \( y' \), there exist \( \tilde{x}', \tilde{y}' \in x'y' \) such that \( v(\tilde{x}') > v(l(\tilde{x}')) \) and \( v(\tilde{y}') < v(l(\tilde{y}')) \). This, combined with the separability of \( v \), implies a contradiction with the fact that \( v|_{S^1} \) is not constant.

**Step 4.** We show that \( v(l'(x)) = v(x) \) for any \( x \in \tilde{x}'y' \), where \( l' = x'y' \), and \( x', y' \) defined in Step 3 represent the centers of \( A \) and \( B \), respectively.

Otherwise, there exists \( \bar{x} \in \tilde{x}'y' \backslash (A \cup B) \) and \( v(l'(\bar{x})) \neq v(\bar{x}) \). Without loss of generality, we may assume that \( v(\bar{x}) > v(l'(\bar{x})) \) and \( l'(\bar{x}) \in y'x' \backslash (A \cup B) \). In view of the continuity of \( v \) and the compactness of \( A \), we know that there exists \( \bar{x} \in l'(\bar{x})x' \backslash A \) such that \( v(\bar{x}) > v(\bar{x}), \{x', x'', \bar{x}\} \) and \( \{y', \bar{x}\} \) locate on both sides of the line \( \bar{l} \), with \( x'' = (\cos(\alpha_0 - \theta_1), \sin(\alpha_0 - \theta_1)) \) and \( \bar{l} \) being the perpendicular bisector of \( \overline{x\bar{x}} \). It follows from \( \bar{l}(\bar{x}) = \tilde{x} \), \( v(\bar{x}) > v(\tilde{x}) \) and the separability of \( v \) that \( v(x'') \leq v(\bar{l}(x'')) \), where \( x'' = (\cos(\alpha_0 - \theta_1), \sin(\alpha_0 - \theta_1)) \), which yields a contradiction to \( \bar{l}(x'') \notin A \) (see Figure 2.3).

![Figure 2.3: schematic diagram for the proof of Step 4](image)

**Step 5.** We show that \( u : [0, \pi] \ni \theta \mapsto v(\cos(\alpha_0 + \theta), \sin(\alpha_0 + \theta)) \in (0, \infty) \) is decreasing at \( \theta \in [0, \pi] \).

Indeed, for any given \( \theta'_1, \theta'_2 \in [0, \pi] \) with \( \theta'_1 < \theta'_2 \), let

\[
\begin{align*}
\bar{x} &= (\cos(\alpha_0 + \theta'_1), \sin(\alpha_0 + \theta'_1)), \quad \bar{x} = (\cos(\alpha_0 + \theta'_2), \sin(\alpha_0 + \theta'_2))
\end{align*}
\]

and \( \bar{l} \) represent the perpendicular bisector of \( \overline{\bar{x}\bar{x}} \), that is, \( \bar{l}(\bar{x}) = \bar{x} \). Then \( \{x', \bar{x}\} \) and \( \{y', \bar{x}\} \) locate on both sides of the line \( \bar{l} \). It follows from (2.1) that

\[
v(\bar{x}) \geq v(\bar{l}(\bar{x})) = v(\bar{x}).
\]
In other words, \( u(\theta_1^*) \geq u(\theta_2^*) \). The arbitrariness of \( \theta_1^* \) and \( \theta_2^* \) implies that \( u \) is decreasing.

Therefore, **Step 3** gives (i) and (ii) while (iii) and (iv) follow from **Step 4** and **Step 5**, respectively. □

By Lemma 2.1, it is easily to check the following two corollaries, which are very useful in extending the conclusions in Lemma 2.1 to separable functions in high dimensional spheres and balls.

**Corollary 2.1.** Let \( v \) and \( \alpha_0 \) be choose in Lemma 2.1. For any \( \alpha \in \mathbb{R} \), if \( (\cos \alpha_0, \sin \alpha_0) \in S_{\alpha}^1 \) (or \( S_{\alpha}^1 \setminus S_{\alpha}^1 \)), then \( v(x) \geq v(l_\alpha(x)) \) (or \( v(x) \leq v(l_\alpha(x)) \)) for any \( x \in S_{\alpha}^1 \).

**Corollary 2.2.** Let \( v \in C(S^1, (0, \infty)) \). Suppose that \( v \) is separable and

\[
v((\cos \alpha, \sin \alpha)) = v((\cos (\alpha + \pi), \sin (\alpha + \pi))) \text{ for any } \alpha \in [0, 2\pi).
\]

Then \( v \) is a constant function on \( S^1 \).

**Proof.** By way of contradiction, we assume that \( v \) is not a constant function. In particular, \( \max_{S^1} v > \min_{S^1} v \). According to Lemma 2.1 there exists \( \alpha_0 \in [0, 2\pi) \) such that

\[
\begin{align*}
v((\cos \alpha_0, \sin \alpha_0)) &= \max_{S^1} v, \\
v((\cos (\alpha_0 + \pi), \sin (\alpha_0 + \pi))) &= \min_{S^1} v,
\end{align*}
\]

a contradiction with (2.2). This completes the proof. □

### 2.2 Separable functions in spheres

In this subsection, we study the axial symmetry and monotonicity of separable functions in high dimensional spheres.

First we list the following basic result, which indicates that every element in \( \text{aff}(A) \) and \( \text{co}(A) \) are a combination of at most \( N+1 \) points in \( A \) if \( A \subset \mathbb{R}^N \), which is standard and hence is omitted.

**Lemma 2.2.** Let \( A \subset \mathbb{R}^N \). Then we have the following results.

(i) \( \text{aff}(A) = \text{aff}_N(A) := \{ \sum_{i=1}^{N+1} \lambda_i x^i | x^i \in A, \lambda_i \in \mathbb{R} \text{ and } \sum_{i=1}^{N+1} \lambda_i = 1 \} \).

(ii) \( \text{co}(A) = \text{co}_N(A) := \{ \sum_{i=1}^{N+1} \lambda_i x^i | x^i \in A, \lambda_i \in [0, 1] \text{ and } \sum_{i=1}^{N+1} \lambda_i = 1 \} \).

Hence, \( \text{co}(A) \) is bounded and closed if \( A \) is bounded and closed.

Now we introduce the definition of separable functions in spheres \( S \subset \mathbb{R}^N \).
**Definition 2.2.** Assume $2 \leq k \leq N$ and $S \subset \mathbb{R}^N$ is a $k - 1$ dimensional sphere. We say $u \in C(S, \mathbb{R})$ is separable, if for any open half space $H \subset \mathbb{R}^N$ with $x^* \in \partial H$ and $\sigma_H x \in S$ for all $x \in S$, there holds

\[
either u(x) \geq u(\sigma_H x) \text{ for all } x \in H \cap S \text{ or } u(x) \leq u(\sigma_H x) \text{ for all } x \in H \cap S.
\]

Here $x^*$ is the center of the ball $\text{co}(S)$.

We can also define separable functions in spheres in another way.

**Definition 2.3.** Assume $2 \leq k \leq N$ and $S \subset \mathbb{R}^N$ is a $k - 1$ dimensional sphere. We say $u \in C(S, \mathbb{R})$ is separable, if there exist $r > 0$, $b \in \mathbb{R}^N$, and $M \in O(N)$ such that

\[
MS + b = rS^{k-1} \times \{0_{N-k}\} \subset \mathbb{R}^N, \quad MV + b = \mathbb{R}^k \times \{0_{N-k}\} \subset \mathbb{R}^N
\]

and $\bar{u}$ is separable in $S^{k-1} \subset \mathbb{R}^k$ in the sense of Definition 2.2. Here

\[
V := \text{aff}(S) = \left\{ \sum_{i=1}^{k+1} \lambda_i x_i \mid x_i \in S, \lambda_i \in \mathbb{R} \text{ and } \sum_{i=1}^{k+1} \lambda_i = 1 \right\},
\]

\[
\bar{u} : S^{k-1} \ni x \mapsto u_M((rx, 0_{N-k}) - b) = u(M^{-1}((rx, 0_{N-k}) - b)) \in (0, \infty).
\]

It is obvious that Definition 2.2 is equivalent to Definition 2.3. As a result, we say $u \in C(S^{N-1}, (0, \infty))$ be a separable function, however, we don’t have to emphasize the way we use the definition.

The next lemma is vital to investigate some basic properties of separable functions in spheres.

**Lemma 2.3.** Let $u \in C(S^{N-1}, (0, \infty))$ be a separable function in $S^{N-1}$ and $V$ be a $k \in [1, N - 1]$ dimensional hyperplane. If $(V \cap S^{N-1})^\# > 1$, then the following statements are true:

(i) $V \cap S^{N-1}$ is a $k - 1$ dimensional sphere;

(ii) $u|_{S^{N-1} \cap V}$ is separable in $S^{N-1} \cap V$.

Here $(V \cap S^{N-1})^\#$ represents the cardinality of elements contained in $V \cap S^{N-1}$.

**Proof.** (i) Take $b \in V$. Then $V - b$ is a $k$ dimensional linear subspace and thus there exists $M \in O(N)$ such that $M(V - b) = \mathbb{R}^k \times \{0_{N-k}\} \subset \mathbb{R}^N$, which implies that $M(V) = \mathbb{R}^k \times \{0_{N-k}\} + Mb$. Let $Mb := (a_1, a_2, \cdots, a_N)$. Then $MV = \mathbb{R}^k \times \{(a_{k+1}, \cdots, a_N)\}$. Note that $MS^{N-1} = S^{N-1}$ and

\[
V \cap S^{N-1} = M^{-1}(\mathbb{R}^k \times \{(a_{k+1}, \cdots, a_N)\}) \cap S^{N-1}.
\]

Hence, it suffices to prove that $(\mathbb{R}^k \times \{(a_{k+1}, \cdots, a_N)\}) \cap S^{N-1}$ is a $k - 1$ dimensional sphere. Indeed, we may conclude that $a_{k+1}^2 + \cdots + a_N^2 < 1$. Then

\[
(\mathbb{R}^k \times \{(a_{k+1}, \cdots, a_N)\}) \cap S^{N-1}
\]

\[
= \{(x_1, \cdots, x_N) \in \mathbb{R}^N \mid x_1^2 + \cdots + x_k^2 + a_{k+1}^2 + \cdots + a_N^2 = 1\}
\]

\[
= \{(x_1, \cdots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 = 1 - a_{k+1}^2 - \cdots - a_N^2\} \times \{(a_{k+1}, \cdots, a_N)\}
\]
is a \(k - 1\) dimensional sphere. So, the proof of (i) is complete.

(ii) By (i), we see \(V \cap S^{N-1}\) is a \(k - 1\) dimensional sphere whose center is denoted by \(x_0^*\). Then the vector \(\overrightarrow{Ox_0^*} \perp V\). Fix an open half space \(H \subset \mathbb{R}^N\) with \(x_0^* \in \partial H\) and \(\sigma_H(V \cap S^{N-1}) \subseteq V \cap S^{N-1}\). It follows that the vector \(\overrightarrow{Ox_0^*} \perp \partial H\) and \(\overrightarrow{Ox_0^*} \perp \partial H\) for all \(x \in V \cap S^{N-1}\). This, combined with \(\text{dim}(\partial H) = N - 1\), implies that \(\overrightarrow{Ox_0^*} \perp \partial H\). By \(x_0^* \in \partial H\), we deduce that \(O \in \partial H\). Applying the fact that \(u\) satisfies separability in \(S^{N-1}\), we have

\[
either u(x) \geq u(\sigma_H x) \text{ for all } H \cap S^{N-1} \text{ or } u(x) \leq u(\sigma_H x) \text{ for all } H \cap S^{N-1}.
\]

In particular, either \(u(x) \geq u(\sigma_H x)\) for all \(H \cap (S^{N-1} \cap V)\) or \(u(x) \leq u(\sigma_H x)\) for all \(H \cap (S^{N-1} \cap V)\), that is, the statement (ii) holds.

The following is devoted to the proof of symmetry and monotonicity of separable functions in \(S^{N-1}\).

**Lemma 2.4.** Let \(N \geq 2\) and \(u \in C(S^{N-1}, (0, \infty))\). Assume that \(u\) is nonconstant and separable in \(S^{N-1}\). Then \(u\) is axially symmetric and monotone in \(S^{N-1}\). To be precise, there exist \(M \in O(N)\) and \(h_1, h_2 \in [-1, 1]\) such that

(i) \(h_1 > h_2\);

(ii) \(u_M^{-1}(\max_{S^{N-1}} u_M) = \{x \in S^{N-1} | x_N \geq h_1\}\) and

\[
u_M^{-1}(\min_{S^{N-1}} u_M) = \{x \in S^{N-1} | x_N \leq h_2\}.
\]

(iii) For any fixed \(h \in [-1, 1]\), \(u_M|_{x \in S^{N-1} | x_N = h}\) is constant.

(iv) \(u_M(0_{N-2}, \cos \alpha, \sin \alpha)\) is decreasing with respect to \(\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\).

**Proof.** Since \(u\) is not constant, we have \(A = u^{-1}(\max_{S^{N-1}} u) \neq \emptyset\) and \(B = u^{-1}(\min_{S^{N-1}} u) \neq \emptyset\).

We shall finish the proof by the following two steps.

**Step 1.** We prove that \(A\) is a single set or an \(N - 1\) dimensional spherical cap as well as \(B\).

We shall argue it by inductive method.

It follows from Lemma 2.1 that the conclusion holds when \(N = 2\).

We assume that the conclusion holds for \(2 \leq N \leq k\).

Now we prove that the conclusion is also valid for \(N = k + 1\). Without loss of generality, we assume that \(A\) is not a single set. Note that \(A \neq S^{N-1}\). Then by Lemma 2.2, \(\text{co}(A) = \{\sum_{i=1}^{N+1} \lambda_i x^i | x^i \in A, \lambda_i \in [0, 1] \text{ and } \sum_{i=1}^{N+1} \lambda_i = 1\}\) and \(\text{co}(A)\) is a closed convex set. Clearly, \(\text{co}(A) \subsetneq B_1\) and \(A \subsetneq \partial(\text{co}(A))\).
where $B_1$ is the unit closed ball with the center at the origin. Take $x^* = (x_1^*, \ldots, x_N^*) \in \partial(\text{co}(A)) \setminus A$. By using the theorem of the separation of convex sets in [25, Chapter 3], we can find an $N$ dimensional open half space $H$ such that $x^* \in \partial H$, $\text{co}(A) \cap H = \emptyset$, and thus $A \cap H = \emptyset$. Then there exists $\hat{M} \in O(N)$ and $b \in \mathbb{R}^N$ such that $\hat{M} = \hat{M}H + b = \mathbb{R}^{N-1} \times (-\infty, 0)$. Let us define an affine transformation $T : \mathbb{R}^N \ni x \mapsto \hat{M}x + b \in \mathbb{R}^N$. So $\partial(T(H)) = \mathbb{R}^{N-1} \times \{0\}$ and $\text{co}(T(A)) \subset \mathbb{R}^{N-1} \times [0, \infty)$.

Next we show $A \cap \partial H \neq \emptyset$. Otherwise, $A \cap \partial H = \emptyset$ and hence by the convexity of $\mathbb{R}^N \setminus \text{cl}(H)$, we have $\text{cl}(H) \cap \text{co}(A) = \emptyset$, a contradiction with the fact that $x^* \in \text{co}(A) \cap \partial H$.

Let $y^* = (y_1^*, \ldots, y_N^*) \in A \cap \partial H$. In view of $x^* \in \partial H \cap (\partial(\text{co}(A)) \setminus A)$ and $y^* \in \partial H \cap A$, we see that $x^* \neq y^*$.

Now we claim that $A \cap \partial H \neq \{y^*\}$. Suppose on the contrary that $A \cap \partial H = \{y^*\}$. It follows that $(T(x^*))_N = (T(y^*))_N = 0$ and $x_N > 0$ for any $x = (x_1, \ldots, x_N) \in T(A) \setminus \{y^*\}$. By $T(x^*) \in T(\text{co}(A))$, there exists $x' \in T(A)$ and $\lambda_i \in [0, 1]$ with $\sum_{i=1}^{N+1} \lambda_i = 1$ such that $T(x^*) = \sum_{i=1}^{N+1} \lambda_i x_i'$ and so $0 = (T(x^*))_N = \sum_{i=1}^{N+1} \lambda_i x_i'_N \geq 0$. Hence $\lambda_i x_i'_N = 0$ for all $i = 1, \ldots, N + 1$. Clearly, $\lambda_i = 0$ or $x_i'_N = 0$ for all $i = 1, \ldots, N + 1$, which, together with $T(A) \cap (\mathbb{R}^{N-1} \times \{0\}) = T(A) \cap \partial(T(H)) = \{T(y^*)\}$, implies $T(x^*) = T(y^*)$ and thus $x^* = y^*$, a contradiction to $x^* \neq y^*$. Therefore, the claim holds and thus there exists $y^{**} \in (A \cap \partial H) \setminus \{y^*\}$.

Let $S = \partial H \cap S^{N-1}$. Clearly, $y^*, y^{**} \in A \cap S$. By Lemma 2.3, $S$ is an $N - 2$ dimensional sphere and $u$ is separable in $S$. Let $\tilde{x}, \tilde{r}$ be the center and radius of $S$, respectively, and let us define $\tilde{T} : S^{N-2} \ni z \mapsto T^{-1}(T(\tilde{x}) + \tilde{r}(z, 0)) \in S$ and $\bar{u} : S^{N-2} \ni z \mapsto u(\tilde{T}(z)) \in (0, \infty)$. Then we easily see that $\tilde{T}^{-1}(y^*), \tilde{T}^{-1}(y^{**}) \in S^{N-2} \cap \bar{u}^{-1}(\max \bar{u})$ and $\bar{u}$ is separable in $S^{N-2}$. By applying the inductive hypothesis to $\bar{u}^{-1}(\max \bar{u})$, we see that $\bar{u}^{-1}(\max \bar{u})$ is an $N - 2$ dimensional sphere cap and hence $u^{-1}(\max u)$ is an $N - 2$ dimensional sphere cap denoted by $S^*$. Without loss of generality, we can assume, in the remaining proof, that there exist $h \in (-1, 1)$ and $\delta \in [-\sqrt{1 - h^2}, \sqrt{1 - h^2})$ such that $H = \mathbb{R}^{N-1} \times (-\infty, h)$, $A \subset \mathbb{R}^{N-1} \times [h, \infty)$, $S = \{x \in S^{N-1} | x_N = h\}$ and $u^{-1}(\max u) = \{x \in S | x_N \geq \delta\}$.

Next we show $\delta = -\sqrt{1 - h^2}$, that is, $S^* = S$. Otherwise $|\delta| < \sqrt{1 - h^2}$. Let

$$z^*_+ = 0_{N-3} \times (-\sqrt{1 - h^2} - \delta, \delta, h), \quad z^*_- = 0_{N-3} \times (-\sqrt{1 - h^2} + \delta^2, \delta, h).$$

In addition, for $\epsilon \geq h$, let $z^*_+ = 0_{N-3} \times (-\sqrt{1 - \epsilon^2}, \epsilon)$ and $z^*_-$ be the point at which $S^{N-1}$ intersects the line containing $z^*_+$ and the point $0_{N-2} \times (\delta, h)$. Let us define

$$f_{\pm} : [h, 1] \ni \epsilon \mapsto u(z^*_\pm(\epsilon)) \in (0, \infty).$$

It is easy to check that $f_{\pm}(h) < \max u$, $(z^*_\pm)_N < \delta$ for all $\epsilon \in (h, 1)$, and $f_{\pm}$ is continuous and $f_{\pm}(h) < \max u$. So there exists $\epsilon^* > h$ such that

$$u(z^*_\pm(\epsilon^*)) = f_{\pm}(\epsilon^*) < \max u = u(z^*_\pm). \quad (2.5)$$

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Notice that the line segments \( \overline{z^*_+z^-} \) and \( \overline{z^*_+z^-} \) are coplanar (see Figure 2.4).

\[ V = \text{aff}(\{z^*_+, z^*_-\}) = \{\lambda_1 z^*_+ + \lambda_2 z^-_+ + \lambda_3 z^*_- + \lambda_4 z^-_+ | \lambda_i \in \mathbb{R}, \sum_{i=1}^{4} \lambda_i = 1\}. \]

Then \( \dim V = 2 \) and \( \max_{V \cap S^{N-1}} u = u(z^*_+ > u(z^*_-) \). By applying Lemma 2.3 and Lemma 2.1, we can obtain that \( V \cap S^{N-1} \) is a circle and \( u^{-1}(\max_{V \cap S^{N-1}} u) \) is an arc \( \Lambda \) containing \( z^*_\pm \). This implies that \( z^*_\pm \in \Lambda \) or \( z^*_\pm \in \Lambda \), a contradiction to (2.5). Hence \( S^* = S \), that is,

\[ \{x \in S^{N-1} | x_N = h\} \subset A \subset \{x \in S^{N-1} | x_N \geq h\} := A^*. \]

Now we shall prove \( A = A^* \). We argue it by contradiction as follows. Let \( x^* = (x^*_1, \cdots, x^*_N) \in A^* \setminus A \), \( w^*_h = (0_{N-1}, \pm \sqrt{1 - h^2}, h) \), and \( \tilde{V} = \text{aff}(\{w^*_+, w^*_-, x^*\}) \). Then \( w^*_+, w^*_- \in A \cap \partial H, x^* \notin A \setminus \text{cl}(H) \), \( \dim(\tilde{V}) = 2 \), and hence \( \tilde{V} \cap S^{N-1} \cap H \neq \emptyset \) due to \( \dim(\partial H) = N - 1 \) (see Figure 2.5). By applying Lemma 2.1, we may obtain that \( A \cap \tilde{V} \cap S^{N-1} \) is an arc \( \Gamma \) containing \( w^*_+, w^*_- \). It follows from \( \tilde{V} \cap S^{N-1} \cap H \neq \emptyset \) that
\(x^* \in \tilde{V} \cap S^{N-1} \setminus H \subseteq \Gamma \subset A\), a contradiction to \(x^* \not\in A\). As a result, we obtain that \(A = A^*\) is an \(N-1\) dimensional sphere cap.

By applying the above discussions to \(1 + \max u - u\), we obtain that \(B\) is a single point or an \(N-1\) dimensional spherical cap. This completes the proof of Step 1.

We denote the centers of two sphere caps \(A\) and \(B\) by \(a^*, b^*\), respectively. We next verify that \(a^*, b^*\), and the origin \(O\) are collinear. Otherwise, there exists an \(N\) dimensional open half space \(\tilde{H}\) such that \(a^*, b^* \in \tilde{H}\). Let \(V^* = \aff([a^*, b^*, O])\). Then \(V^*\) is a two dimensional plane and by Lemma \([3, 2]\), \(V^* \cap S^{N-1}\) is a circle, and \(u|_{V^* \cap S^{N-1}}\) is nonconstant and separable. Thus, it follows from the proof of Step 3 in Lemma \([2, 1]\) that \(a^*, b^*, O\) must be collinear, a contradiction. Since \(a^*, b^*\), and the origin \(O\) must be collinear, we know that there exists \(M \in O(N)\) such that \(M(a^*) = (0_{N-1}, 1), M(b^*) = (0_{N-1}, -1)\), and hence \(u_M\) satisfies (i) and (ii).

**Step 2.** In this step, we shall prove (iii) and (iv).

We shall finish the proof by distinguishing two cases.

**Case 1.** \(N = 2\).

In this case, (iii) and (iv) follow from Lemma \([2, 1]\).

**Case 2.** \(N \geq 3\).

(iii) Fix \(h \in (-1, 1)\). Then \(\{x \in S^{N-1}|x_N = h\}\) is an \(N-2\) dimensional sphere.

Letting \(\bar{x}, \bar{x} \in \{x \in S^{N-1}|x_N = h\}\) be any pair of symmetric points with respect to \((0_{N-1}, h)\), we easily see that \(W = \aff([\bar{x}, \bar{x}, (0_{N-1}, 1)])\) is a two dimensional plane and \(u_M|_{W \cap S^{N-1}}\) is nonconstant. Thus, by Lemma \([2, 3]\), \(W \cap S^{N-1}\) is a circle and \(u_M|_{W \cap S^{N-1}}\) is separable. Note that by (ii), \((0_{N-1}, 1)\) and \((0_{N-1}, -1)\) are centers of the arcs \(W \cap S^{N-1} \cup u_M^{-1}(\max u)\) and \(W \cap S^{N-1} \cup u_M^{-1}(\min u)\), respectively. By applying Lemma \([2, 1]\) to \(u_M|_{W \cap S^{N-1}}\) under some affine transformation, we know that \(u_M|_{W \cap S^{N-1}}\) is axial symmetric with respect to \(x_N\)-axis. In particular, we have \(u_M(\bar{x}) = u_M(\bar{x})\).

Let \(\bar{x}^*, \tilde{x}^* \in \{x \in S^{N-1}|x_N = h\}\). Then \(W^* = \aff([\bar{x}^*, \tilde{x}^*, (0_{N-1}, h)])\) is a two dimensional plane and hence by Lemma \([2, 3]\), \(W^* \cap S^{N-1}\) is a circle and \(u_M|_{W^* \cap S^{N-1}}\) is separable. These, together with Corollary \([2, 2]\) and the fact that \(u_M(x) = u_M(y)\) whence \(x, y \in W^* \cap S^{N-1}\) are given symmetric pairs with respect to \((0_{N-1}, h)\), implies that \(u_M|_{W^* \cap S^{N-1}}\) is constant. In particular, \(u_M(\bar{x}^*) = u_M(\tilde{x}^*)\). So by the arbitrariness of \(\bar{x}^*, \tilde{x}^* \in \{x \in S^{N-1}|x_N = h\}\), we get (iii).

(iv) By (ii), (iii), and by applying Lemma \([2, 1]\) (iv) to \(u_M(0_{N-2} \cdot)\) we easily see that \(u_M(0_{N-2}, \cos \alpha, \sin \alpha)\) is decreasing with respect to \(\alpha \in [\frac{\pi}{2}, \frac{3\pi}{2}]\).

The proof is completed. \(\square\)

**Corollary 2.3.** Let \(H\) be an open half space in \(\mathbb{R}^N\) with the origin \(O \in \partial H\). Under the assumptions of Lemma \([2, 4]\), we have the following statements:
(i) If \( M^{-1}(0, 0, \ldots, 0, 1) \in H \), then \( u(x) \geq u(\sigma H x) \) for any \( x \in H \cap S^{N-1} \);

(ii) If \( M^{-1}(0, 0, \ldots, 0, 1) \in \mathbb{R}^N \setminus \text{cl}(H) \), then \( u(x) \leq u(\sigma H x) \) for any \( x \in H \cap S^{N-1} \).

### 2.3 Separable functions in balls

In this subsection, we consider the axial symmetry and monotonicity of separable functions in high dimensional balls.

We first introduce the definition of separable functions in \( B_R \).

**Definition 2.4.** A function \( u : B_R \to \mathbb{R} \) is said to be separable if for any open half-space \( H \subset \mathbb{R}^N \) with \( O \in \partial H \),

\[
\text{either } u(x) \geq u(\sigma H x) \text{ for all } x \in H \cap B_R \text{ or } u(x) \leq u(\sigma H x) \text{ for all } x \in H \cap B_R. \tag{2.6}
\]

**Theorem 2.1.** Let \( u \in C(B_R, (0, \infty)) \) be a separable function. If \( u \) is not radially symmetric with respect to the origin \( O \), then there exists \( M \in O(N) \) such that the following statements are true:

(i) (Axial symmetry). For any \( \alpha \in (0, R) \) and \( h \in [-\alpha, \alpha] \), \( u_M|_{x=(x_1,x_2,\ldots,x_N) \in S^{N-1}_\alpha|_{x_N=h}} \) is constant, that is, \( u_M \) is axially symmetric with respect to \( x_N \)-axis;

(ii) (Monotonicity). For any given \( \alpha \in (0, R) \), \( u_M(0_{N-2}, \alpha \cos \theta, \alpha \sin \theta) \) is decreasing with respect to \( \theta \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \).

**Proof.** Fix \( \alpha \in (0, R) \). By applying Lemma 2.4 to \( u(\alpha \cdot)|_{S^{N-1}} \), we know that \( u^{-1}(\max u) \cap S^{N-1}_\alpha \) is a single set or a spherical cap of \( S^{N-1}_\alpha \). Let us write \( x_\alpha^\alpha \) for the centers of \( u^{-1}(\max u) \cap S^{N-1}_\alpha \).

We claim that there exists a radial with the peak at \( O \) passing through \( x_\alpha^\alpha \) and \( x_\beta^\beta \) for any \( \alpha, \beta \in (0, R) \). Indeed, if either \( u|_{S^{N-1}_\alpha} \) or \( u|_{S^{N-1}_\beta} \) is a constant function, then

\[
u^{-1}(\max u) \cap S^{N-1}_\alpha = S^{N-1}_\alpha \text{ or } u^{-1}(\max u) \cap S^{N-1}_\beta = S^{N-1}_\beta.
\]

So we can re-select \( x_\alpha^\alpha \) (or \( x_\beta^\beta \)) belonging to \( O x_\alpha^\alpha \cap S^{N-1}_\alpha \) (or \( O x_\beta^\beta \cap S^{N-1}_\beta \)).

If neither \( u|_{S^{N-1}_\alpha} \) nor \( u|_{S^{N-1}_\beta} \) is a constant function, then \( x_\alpha^\alpha, x_\beta^\beta \) are unique centers of \( u^{-1}(\max u) \cap S^{N-1}_\alpha \), \( u^{-1}(\max u) \cap S^{N-1}_\beta \), respectively. Suppose that \( x_\alpha^\alpha, x_\beta^\beta \) are not on the same radial with the peak at \( O \). Then there exists an open half space \( H \) in \( \mathbb{R}^N \) such that the origin \( O \in \partial H \), \( x_\alpha^\alpha \in H \), and \( x_\beta^\beta \not\in \text{cl}(H) \). It follows from Corollary 2.3 that \( u(x) \geq u(\sigma H x) \) for any \( x \in H \cap S^{N-1}_\alpha \) and \( u(x) \leq u(\sigma H x) \) for any \( x \in H \cap S^{N-1}_\beta \).

These, together with the separability of \( u \), implies either \( u(x) = u(\sigma H x) \) for any \( x \in H \cap S^{N-1}_\alpha \) or
It is easy to check that \( u \) satisfies the separable property. However, \( u \) is only axially symmetric with respect to the origin \( O \).

### Theorem 2.2

Let \( u \) are separable.

Observe that for any \( \alpha \in (0,R) \), let \( g(x_\alpha) \) be a nonconstant and nonincreasing function, and let \( h \in C([0,R],[0,\infty)) \) with \( h(R) = 0 \) and \( h([0,R)) \subset (0,\infty) \). Define \( u : B_R \to \mathbb{R} \) by

\[
    u(x_1, x_2, \cdots, x_N) = g(x_N) h(\sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}).
\]

It is easy to check that \( u \) satisfies the separable property. However, \( u \) is only axially symmetric with respect to the \( x_N \)-axis and is not radially symmetric with respect to the origin \( O \).
3 Separable functions in whole space

In this section, based on the results obtained in Section 2, we shall show that a separable function in $\mathbb{R}^N$ can imply its radial symmetry and monotonicity.

First we give the definition of separable functions in $\mathbb{R}^N$.

**Definition 3.1.** A function $u \in C(\mathbb{R}^N, \mathbb{R})$ is called separable if for any open half space $H \subset \mathbb{R}^N$, there holds

$$
either u(x) \geq u(\sigma_H x) \forall x \in H \text{ or } u(x) \leq u(\sigma_H x) \forall x \in H. \tag{3.1}$$

Let $u \in C(\mathbb{R}^N, \mathbb{R})$. A line $L \subset \mathbb{R}^N$ is a symmetry axis of $u$ if and only if for any given $\alpha > 0$, $z \in L \cap V$, and $N − 1$ dimensional hyperplane $V$ with $L \perp V$, $u|_{S^{N-1}_\alpha \cap V}$ is constant.

In the following lemma, we give some properties of separable functions in $\mathbb{R}^N$.

**Lemma 3.1.** Let $u \in C(\mathbb{R}^N, (0, \infty))$ be a separable function and let $\mathcal{L}$ be the set of all the symmetry axes of $u$. Assume that $u$ is not radially symmetric in $\mathbb{R}^N$. Then the following statements are true.

(i) For any $x \in \mathbb{R}^N$, there exists an unique $L_x := L(x) \in \mathcal{L}$ such that $x \in L_x$, and hence $L_x = L_y$ whence $y \in L_x$.

(ii) For any $x, y \in \mathbb{R}^N$, there holds either $L_x = L_y$ or $L_x \parallel L_y$ (that is, $L_x$ is parallel to $L_y$).

**Proof.** (i). Fix $x \in \mathbb{R}^2$. Since $u$ is not a radially symmetric function, there exists $\alpha_0 > 0$ such that $\max S^{N-1}_{\alpha_0}(x) > \min u$. By applying Lemma 2.4 to $u(x + \alpha_0 \cdot)_{|S^{N-1}}$, we know $u|_{S^{N-1}_{\alpha_0}(x)}$ is only an axially symmetric function, where $u(x + \alpha_0 \cdot)_{|S^{N-1}} : S^{N-1} \ni z \mapsto u(x + \alpha_0 z) \in (0, \infty)$. Let $L_x := L(x)$ be the line containing the symmetry axis of $u|_{S^{N-1}_{\alpha_0}(x)}$.

Now we prove $L_x \in \mathcal{L}$. Fix $\alpha > 0$, $z \in L_x \cap V$, and a $N − 1$ dimensional hyperplane $V$ with $L_x \perp V$. By applying Theorem 2.1 to $u|_{B_\alpha(x)}$, we obtain that $L_x$ is a unique symmetry axis of $u|_{B_\alpha(x)}$ and $u|_{S^{N-1}_{\alpha}(\cdot)}$ is constant, which implies $L_x \in \mathcal{L}$.

By the uniqueness of symmetry axis through one point, we easily see $L_x = L_y$ for any $y \in L_x$. This completes the proof of (i).

(ii) Fix $x, y \in \mathbb{R}^N$. By (i), we only consider the case of $y \notin L_x$. We prove it by contradiction. Suppose on the contrary that $L_x$ is not parallel to $L_y$.

We shall finish the proof by distinguishing two cases.

**Case 1.** $L_x \cap L_y \neq \emptyset$, that is, $L_x$ and $L_y$ are coplanar.

Take $x^* \in L_x \cap L_y$. Then $L_x, L_y$ are two different symmetry axes of $u$ through $x^*$, a contradiction with (i).

**Case 2.** $L_x \cap L_y = \emptyset$, that is $L_x$ and $L_y$ are not coplanar.
Since $u$ is not radially symmetric, it follows from Lemma 2.4 that there exist positive constants $R_1$ and $R_2$ such that both $u|_{S_{R_1}^{N-1}(x)}$ and $u|_{S_{R_2}^{N-1}(y)}$ are nonconstant and axially symmetric function with respect to the $L_x$ and $L_y$, respectively.

Take $x^* \in L_x \cap S_{R_1}^{N-1}(x), y^* \in L_y \cap S_{R_2}^{N-1}(y)$ with $u(x^*) = \max u$ and $u(y^*) = \max u$. Then $\dim(\text{aff}([x, y, \frac{x+y}{2}])) = 2$, $\dim(\text{aff}([x, y, x^*, y^*])) = 3$, and thus there exists a hyperplane $\hat{H} \subseteq \mathbb{R}^N$ such that $\text{aff}([x, y, x^*, y^*]) \times \hat{H} \subset \mathbb{R}^N$ and $\dim(\hat{H}) = N - 3$.

Let $H$ be open half space $H \subset \mathbb{R}^N$ with $\partial H = \text{aff}([x, y, \frac{x+y}{2}]) \times \hat{H}$. Then $x, y \in \partial H$, $x^*, y^* \notin \partial H$ and $\{x^*, y^*\} \setminus H \neq \emptyset$. Without loss of generality, we may assume that $x^* \in H$ and $y^* \notin \partial H$. By the choices of $x^*$ and $y^*$, there exist $\tilde{x}^* \in H \cap S_{R_1}^{N-1}(x)$ and $\tilde{y}^* \in S_{R_2}^{N-1}(y) \setminus \partial(H)$ such that $u(\tilde{x}^*) > u(\sigma_H \tilde{x}^*)$ and $u(\tilde{y}^*) > u(\sigma_H \tilde{y}^*)$. Note that $\tilde{x}^*, \tilde{y}^* \in B_{||x-y||+2(R_1+R_2)}(y)$ and $u|_{B_{||x-y||+2(R_1+R_2)}(y)}$ is nonconstant. Hence by the separability of $u$, we deduce a contradiction.

To sum up, the proof is completed. 

In what follows, we describe monotonicity of even separable functions in $\mathbb{R}$, which is important to obtain the monotonicity of radial separable functions in $\mathbb{R}^N$.

**Lemma 3.2.** Let $u \in C(\mathbb{R}, (0, \infty))$, $u(x) = u(-x)$ and $\lim \inf_{|x| \to \infty} u(x) = 0$. Suppose that for any $x \in \mathbb{R}$,

$$
either u(y) \geq u(2x - y) for all \; y \geq x or u(y) \leq u(2x - y) for all \; y \geq x. \quad (3.2)$$

Then $u$ is nonincreasing on $[0, \infty)$.

**Proof.** Let

$I = \{\alpha \geq 0\}$ there exists $x_\alpha > \alpha$ such that $u(x_\alpha) > u(2\alpha - x_\alpha)$,

$J = \{\alpha \geq 0\}$ there exists $x_\alpha > \alpha$ such that $u(x_\alpha) < u(2\alpha - x_\alpha)$,

$K = \{\alpha \geq 0|u(x) \equiv u(2x - x) for any \; x \in \mathbb{R}\}$.

Obviously, $0 \in K$ and $I \cup J \cup K = [0, \infty)$. By the continuity of $u$, for any $\alpha \in I$, there exists $\delta_\alpha \in (0, \alpha)$ such that $x_\beta > \beta$ and $u(x_\alpha) > u(2\beta - x_\alpha)$ for all $\beta \in (\alpha - \delta_\alpha, \alpha + \delta_\alpha)$, that is, $(\alpha - \delta_\alpha, \alpha + \delta_\alpha) \subset I$. Hence, $I$ is an open set. Similarly, $J$ is also an open set. It suffices to prove $J \cup K = [0, \infty)$, since $(3.2)$ and $J \cup K = [0, \infty)$ imply that $u$ is nonincreasing on $[0, \infty)$. If not, suppose $J \cup K \neq [0, \infty)$. Then $I \neq \emptyset$.

Note that $I \cup K \neq [0, \infty)$ since $I \cup K = [0, \infty)$ will yield a contradiction to $\lim \inf_{|x| \to \infty} u(x) = 0$. Then $I \neq \emptyset$ and $J \neq \emptyset$. Since $I \cap J = \emptyset$ and $I, J$ are open sets, we have $I \cup J \neq (0, \infty)$, and thus $K \setminus \{0\} \neq \emptyset$. To be precise, there exists $\alpha^* \in (0, \infty)$ such that $u(x) = u(2\alpha^* - x) = u(x - 2\alpha^*)$ for all $x \in \mathbb{R}$. Hence $u$ is a periodic function in $\mathbb{R}$, which contradicts with $\lim \inf_{|x| \to \infty} u(x) = 0$. This proves the claim and hence the proof is completed. 

Now we are ready to prove radial symmetry and monotonicity of separable functions in $\mathbb{R}^N$. 

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**Theorem 3.1.** Let \( u \in C(\mathbb{R}^N, (0, \infty)) \) be separable and \( \liminf_{|x| \to \infty} u(x) = 0 \). Then \( u \) is radially symmetric decreasing with respect to some point, that is, there exist \( x^* \in \mathbb{R}^N \) and a decreasing function \( v : [0, \infty) \to (0, \infty) \) such that

\[
u(x) = v(|x - x^*|)
\]

and \( \lim_{r \to \infty} v(r) = 0. \)

**Proof.** We shall argue it by contradiction. Suppose on the contrary that \( u \) is not radially symmetric. Then by Lemma 2.4 and Lemma 3.1-(i), there exists \( M \in O(N) \) such that \( u_M \) has a unique symmetry axis \( x_N \)-axis through \( O \) and \( (0, 0, \cdots, 0, s) \) is the center of the spherical cap consisting of the maximum points of \( u_M|_{S^{N-1}_s}. \)

Now we claim that for any \( (x_1, x_2, \cdots, x_{N-1}, x_N) \in \mathbb{R}^N \), we have

\[
u_M(x_1, x_2, \cdots, x_{N-1}, x_N) = u_M(0, 0, \cdots, 0, x_N).
\]

(3.3)

In fact, by taking \( y^* = (\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_{N-1}}{2}, 0) \), and by applying Lemma 3.1-(ii) to \( u_M \), we conclude that \( u_M \) is axially symmetric with respect to \( L_{y^*} \) and \( L_{y^*}//x_N \)-axis. Hence the claim follows from the symmetry pair \( (x_1, x_2, \cdots, x_{N-1}, x_N) \) and \( (0, 0, \cdots, 0, x_N) \) with respect to \( L_{y^*} \).

Now, fix \( s^* > 0 \). In view of \( u_M(0, 0, \cdots, 0, s^*) = \max u_M(S^{N-1}_s) \), we have

\[
u_M(0, 0, \cdots, 0, s^*) \geq \nu_M(\sqrt{s^2 - \tilde{s}^2}, 0, \cdots, 0, \tilde{s})
\]

for any \( |\tilde{s}| \leq s^* \). This, combined with (3.3), implies that

\[
u_M(0, 0, \cdots, 0, s^*) \geq \nu_M(0, 0, \cdots, 0, \bar{s}).
\]

Hence \( u_M(x_1, x_2, \cdots, x_{N-1}, x_N) \) is a nondecreasing function with respect to \( x_N \in (0, \infty) \), which contradicts with the assumption that \( \liminf_{|x| \to \infty} u_M(x) = 0 \). So \( u \) is radially symmetric with respect to some point in \( \mathbb{R}^N \).

Finally, by Lemma 3.2, \( u \) is a radially symmetric decreasing function. This completes the proof. \( \square \)

### 4 Applications

In this section, we illustrate our main results with the following nonlocal Choquard equation,

\[
-\Delta u + u = \left( \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x - y|^{N-\alpha}} dy \right) |u|^{p-2} u, \quad x \in \mathbb{R}^N
\]

(4.1)

with \( N \geq 3, \alpha \in (0, N), \frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2} \).
For the generalized Choquard equation (4.1), the existence and properties of solutions have been widely considered. See [20, 22, 27, 9, 30, 13, 12] and references therein. In particular, Moroz and Van Schaftingen [24] obtained the separability, radial symmetry and monotonicity of all the positive ground states of (4.1); Ma and Zhao [23] proved that positive solutions for (4.1) must be radially symmetric and monotonically decreasing about some point under appropriate assumptions on $N, \alpha, p$ by using the method of moving planes in integral form introduced by Chen et al. [6].

Let $H^1_0(B_R)$ be the usual Sobolev space with the standard norm $\|u\| := \left(\int_{B_R} (|\nabla u|^2 + |u|^2) dx\right)^{\frac{1}{2}}$. Let $\Omega \subset \mathbb{R}^N$. For any $1 \leq s < \infty$, the norm on $L^s(\Omega)$ is denoted by $|u|_{L^s(\Omega)} := \left(\int_{\Omega} |u|^s dx\right)^{\frac{1}{s}}$.

4.1 Choquard type equations in balls

It is well known that when $N \geq 3$ and $\alpha = 2$, by rescaling, (4.1) is equivalent to

$$
\begin{align*}
-\Delta u + u &= w|u|^{p-2}u \text{ in } \mathbb{R}^N, \\
-\Delta w &= |u|^p \text{ in } \mathbb{R}^N.
\end{align*}
$$

(4.2)

So the Dirichlet problem in a ball $B_R$ is

$$
\begin{align*}
-\Delta u + u &= w|u|^{p-2}u \text{ in } B_R, \\
-\Delta w &= |u|^p \text{ in } B_R, \\
w &= u = 0 \text{ in } \partial B_R.
\end{align*}
$$

(4.3)

It is clear that by using Green’s function (see [10]), (4.3) can be rewritten as

$$
- \Delta u + u = \left(\int_{B_R} G(x,y)|u|^p(y)dy\right)|u|^{p-2}u, \quad x \in B_R.
$$

(4.4)

Here

$$
G(x,y) = \frac{1}{|y-x|^{N-2}} - \frac{1}{(\frac{|x|}{R})^N |y-\tilde{x}|^{N-2}}, \quad (x,y \in B_R \text{ with } x \neq y),
$$

where $\tilde{x}$ is the dual point of $x$ with respect to $\partial B_R$ and can be defined by $\tilde{x} = \frac{R^2 x}{|x|^2}$.

The existence of positive ground states for (4.4) can be obtained by using variational methods. But the symmetry and monotonicity of the positive ground states for (4.4) are very difficult to deal with. Now the method of moving planes in integral form used in [23] is not applicable to (4.4) and the main obstacle is to establish the equivalence between the differential equation and the integral equation. Moreover, the arguments in [24, Proposition 5.2] is also not valid because the origin $O$ is required to belong to $\partial H$ in (4.4) for any half-space $H \subset \mathbb{R}^N$. But applying our main results can lead to axial symmetry and monotonicity of all the positive ground states to (4.4).
As usual, for $N \geq 3$ and $p \in \left(\frac{N+2}{N}, \frac{N+2}{N-2}\right)$, the corresponding energy functional $I : H^1_0(B_R) \to \mathbb{R}$ associated to (4.4) is

$$I(u) = \frac{1}{2} \int_{B_R} (|\nabla u|^2 + |u|^2) dx - \frac{1}{2p} \int_{B_R} \int_{B_R} G(x, y)|u(y)|^p|u(x)|^p dxdy,$$

(4.5)
due to the symmetry and positivity of $G(x, y)$ for $x, y \in B_R$ and $x \neq y$. By Hardy-Littlewood-Sobolev inequality and Sobolev inequality, we have

$$\int_{B_R} \int_{B_R} G(x, y)|u(y)|^p|u(x)|^p dxdy \leq \int_{B_R} \int_{B_R} \frac{|u(y)|^p|u(x)|^p}{|x-y|^{N+2}} dxdy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y)|^p|u(x)|^p}{|x-y|^{N+2}} dxdy \leq C|u|^{2p} \frac{2^p}{L^{2p}}(B_R),$$

where $\chi_{B_R}$ denotes the characteristic function on $\mathbb{R}^N$. It is easy to check that $I \in C^1(H^1_0(B_R), \mathbb{R})$ and its Gateaux derivative is given by

$$I'(u)v = \int_{B_R} (\nabla u \nabla v + uv) dx - \int_{B_R} \int_{B_R} G(x, y)|u(y)|^p|u(x)|^{p-2}u(x)v(x)dxdy$$

for any $v \in H^1_0(B_R)$. Recall that the critical points of $I$ are solutions of (4.4) in the weak sense. Let $c := \inf_{u \in \mathcal{N}} I(u)$, where $\mathcal{N} = \{u \in H^1_0(B_R) \setminus \{0\} : I'(u)u = 0\}$. For simplicity of notations, we denote

$$\mathcal{D}(u) = \int_{B_R} \int_{B_R} G(x, y)|u(y)|^p|u(x)|^p dxdy.$$

The proof of the following properties of the Nehari manifold $\mathcal{N}$ is standard and hence is omitted here.

**Lemma 4.1.** The following statements are true:

(i) $0 \notin \partial \mathcal{N}$ and $c > 0$;

(ii) For any $u \in H^1_0(B_R) \setminus \{0\}$, there exists a unique $t_u \in (0, \infty)$ such that $t_u u \in \mathcal{N}$ and $t_u = \left(\frac{\|u\|^2}{\mathcal{D}(u)}\right)^{\frac{1}{p-2}}$. Furthermore,

$$I(t_u u) = \sup_{t > 0} I(tu) = \left(\frac{1}{2} - \frac{1}{2p}\right) \left(\frac{\|u\|^2}{\mathcal{D}(u)}\right)^{\frac{2}{p-2}};$$

(4.6)

(iii) $c = \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in H^1_0(B_R) \setminus \{0\}} \sup_{t > 0} I(tu)$.

By using Nehari maifold methods, we can obtain the existence of ground states of (4.4) in $H^1_0(B_R)$. Recall that $u \in H^1_0(B_R)$ is said to be a ground state of (4.4), if $u$ solves (4.4) and minimizes the energy functional associated with (4.4) among all possible nontrivial solutions. Furthermore, by standard
elliptic regularity estimate and strong maximum principle, we conclude that any ground state of \((4.4)\) belongs to \(C^2(\tilde{B}_R)\), and \(u > 0\) or \(u < 0\) in \(B_R\). Since the nonlocal term of \((4.4)\) has some strong symmetrizing effect, by using the minimality property of the ground states, we shall deduce the separability property of the positive ground states. We start with the following lemma.

**Lemma 4.2.** Let \(H\) be an open half space in \(\mathbb{R}^N\) with \(0 \in \partial H\). Then the following statements are true:

(i) \(G(x,y) = G(\sigma_H x, \sigma_H y)\) for any \(x, y \in B_R\) and \(x \neq y\);

(ii) \(G(x, \sigma_H y) = G(\sigma_H x, y)\) for any \(x, y \in B_R\) and \(x \neq y\);

(iii) \(G(x, y) \geq G(\sigma_H x, y)\) for any \(x, y \in H \cap B_R\) and \(x \neq y\).

**Proof.** It is easy to check (i) and (ii). We shall prove (iii). Set

\[
a = |x - y|, \quad \bar{a} = |y - \sigma_H x|, \quad b = \frac{|x|}{R} |y - \bar{x}|, \quad \bar{b} = \frac{|\sigma_H x|}{R} |y - \sigma_H x|,
\]

where \(\bar{x}\) and \(\sigma_H x\) represent the dual points of \(x\) and \(\sigma_H x\) with respect to \(\partial B_R\). We will split the proof into two steps.

**Step 1.** We claim that \(\frac{\bar{a}}{a} \geq \frac{\bar{b}}{b} \geq 1\) for any \(x, y \in H \cap B_R\) and \(x \neq y\). Indeed, we only need to prove \(\bar{a}b \geq ab\). Note that for any \(x, y \in H \cap B_R\) and \(x \neq y\), we have \((y, \sigma_H x) < (y, x)\). Here \((\cdot, \cdot)\) is the inner product of \(\mathbb{R}^N\). Then,

\[
\bar{a}^2 b^2 - a^2 \bar{b}^2 = \frac{2|y|^2}{R} |y, \sigma_H x| + \frac{2|y|^2}{R} |y, x| + \frac{2|\sigma_H x|^2}{R} |\sigma_H x, y| + \frac{2|\sigma_H x|^2}{R} |\sigma_H x, \sigma_H x| + \frac{2|x|^2}{R} |x, y| + \frac{2|x|^2}{R} |x, x| + \frac{2|x|^2}{R} |\sigma_H x, \sigma_H x| + \frac{2|x|^2}{R} |x, \sigma_H x| + \frac{2|x|^2}{R} |x, x| - 1(R^2 - |y|^2)(y, x) - (y, \sigma_H x)\]
\[
\geq 0.
\]

**Step 2.** By **Step 1**, for any \(x, y \in H \cap B_R\) and \(x \neq y\), we have \(\frac{\bar{a}^{N-2} - \bar{a}^{N-2}}{\bar{b}^{N-2} - a^{N-2}} \leq \frac{\bar{a}^{N-2} - \bar{a}^{N-2}}{\bar{b}^{N-2} - a^{N-2}}\) and \(\bar{b}^{N-2} \geq b^{N-2}\). So

\[
G(x, y) - G(\sigma_H x, y) = (\frac{1}{\bar{a}^{N-2}} - \frac{1}{\bar{b}^{N-2}}) - (\frac{1}{a^{N-2}} - \frac{1}{\bar{b}^{N-2}})\]
\[
= (b^{N-2} - a^{N-2})(\frac{1}{(ab)^{N-2}} - \frac{1}{(ab)^{N-2}})\]
\[
\geq (b^{N-2} - a^{N-2})(\frac{1}{(ab)^{N-2}} - \frac{1}{(ab)^{N-2}})\]
\[
= \frac{1}{a^{N-2}} (b^{N-2} - a^{N-2})(\frac{1}{b^{N-2}} - \frac{1}{b^{N-2}})\]
\[
\geq 0.
\]

The proof is completed. \(\Box\)
Let $H$ be an open half-space in $\mathbb{R}^N$ with the origin $O \in \partial H$ and $u^H : B_R \rightarrow \mathbb{R}$ be the polarization of $u \in H_0^1(B_R)$ defined by

$$u^H(x) = \begin{cases} \max\{u(x), u(\sigma_H x)\}, & x \in H \cap B_R, \\ \min\{u(x), u(\sigma_H x)\}, & x \in B_R \setminus (H \cap B_R). \end{cases} \quad (4.7)$$

Let

$$A_u = \{x \in H \cap B_R : u(x) \geq u(\sigma_H x)\}, \quad B_u = \{x \in H \cap B_R : u(x) < u(\sigma_H x)\}.$$

Then we are ready to prove the separability property of positive ground states of \(\text{(4.4)}\).

**Proposition 4.1.** Suppose $u$ is a positive ground state of \(\text{(4.4)}\). Then

$$\mathcal{D}(u^H) \geq \mathcal{D}(u). \quad (4.8)$$

Moreover, $u$ is separable in $B_R$.

**Proof.** First for simplicity of notations, we write

$$a := |u(x)|^p, \quad b := |u(\sigma_H x)|^p, \quad c := |u(y)|^p, \quad d := |u(\sigma_H y)|^p.$$ 

It is easy to check

$$\mathcal{D}(u^H) - \mathcal{D}(u) := I_1 + I_2 + I_3 + I_4, \quad (4.9)$$

where

$$I_1 = \int_{A_u} \int_{A_u} G(x, y)(ac - ac) + G(\sigma_H x, y)(bc - bc) dxdy$$

$$+ \int_{A_u} \int_{A_u} G(x, \sigma_H y)(ad - ad) + G(\sigma_H x, \sigma_H y)(bd - bd) dxdy, \quad (4.10)$$

$$I_2 = \int_{A_u} \int_{B_u} G(x, y)(ad - ac) + G(\sigma_H x, y)(bd - bc) dxdy$$

$$+ \int_{A_u} \int_{B_u} G(x, \sigma_H y)(ac - ad) + G(\sigma_H x, \sigma_H y)(bc - bd) dxdy, \quad (4.11)$$

$$I_3 = \int_{B_u} \int_{A_u} G(x, y)(bc - ac) + G(\sigma_H x, y)(ac - bc) dxdy$$

$$+ \int_{B_u} \int_{A_u} G(x, \sigma_H y)(bd - ad) + G(\sigma_H x, \sigma_H y)(ad - bd) dxdy, \quad (4.12)$$

$$I_4 = \int_{B_u} \int_{B_u} G(x, y)(bd - ac) + G(\sigma_H x, y)(ad - bc) dxdy$$

$$+ \int_{B_u} \int_{B_u} G(x, \sigma_H y)(bc - ad) + G(\sigma_H x, \sigma_H y)(ac - bd) dxdy. \quad (4.13)$$

By Lemma \[4.2\] we have $I_1 = I_4 = 0$ and

$$I_2 = \int_{A_u} \int_{B_u} (G(x, y) - G(x, \sigma_H y))(a - b)(d - c) dxdy \geq 0 \quad (4.14)$$

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\[ I_3 = \int_{B_u} \int_{A_u} (G(x, y) - G(x, \sigma_H y))(b - a)(c - d) \, dx \, dy \geq 0. \quad \text{(4.15)} \]

Hence (4.8) holds.

In addition,
\[
\int_{B_R} |\nabla u^H(x)|^2 \, dx = \int_{x \in H \cap B_R} |\nabla u(x)|^2 \, dx + \int_{x \in H \cap B_R} |\nabla u(\sigma_H x)|^2 \, dx = \int_{A_u} |\nabla u(x)|^2 \, dx + \int_{A_u} |\nabla u(\sigma_H x)|^2 \, dx + \int_{B_u} |\nabla u(x)|^2 \, dx.
\]

Similarly,
\[
\int_{B_R} |u^H(x)|^2 \, dx = \int_{B_R} |u(x)|^2 \, dx.
\]

Since \( u \) is a ground state of (4.4), by Lemma 4.1, we deduce \( \mathcal{D}(u^H) \leq \mathcal{D}(u) \). This together with (4.8), implies that \( \mathcal{D}(u^H) = \mathcal{D}(u) \). So \( I_2 = I_3 = 0 \), that is, \( A_u = \emptyset \) or \( B_u = \emptyset \). Therefore, \( u^H = u \) or \( u^H = u \circ \sigma_H \), that is, (2.6) holds by the definition \( u^H \).

Now, Theorem 4.1 follows easily from Proposition 4.1 and Theorem 2.1.

**Theorem 4.1.** Let \( N \geq 3 \) and \( p \in (\frac{N+2}{N}, \frac{N+2}{N-2}) \). Assume that \( u \in H^1_0(B_R) \) is a positive ground state of (4.4). Then either \( u \) is radially symmetric with respect to the origin \( O \) or there exists \( M \in O(N) \) such that \( u_M \) is only axially symmetric with respect to \( x_N \)-axis and, for \( \alpha \in (0, R] \), \( u_M(\alpha \cos \theta, 0, \cdots, 0, \alpha \sin \theta) \) is decreasing with respect to \( \theta \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \).

### 4.2 Choquard type equations in whole space

In this subsection, we consider the Choquard equation (4.1) in \( \mathbb{R}^N \). The qualitative properties of ground states of (4.1) have been intensively studied in [24]. In particular, the separability of ground state is proved.

**Lemma 4.3.** Let \( N \geq 3, \alpha \in (0, N), p \in (\frac{N+2}{N}, \frac{N+2}{N-2}) \). Assume that \( u \in H^1(\mathbb{R}^N) \) is a positive ground state of (4.1). Then \( u \) is separable in \( \mathbb{R}^N \).

This lemma follows from the proof of Lemma 5.3 and Proposition 5.2 in [24]. Theorem 4.2 is a direct consequence of Theorem 3.1 and Lemma 4.3.
Theorem 4.2. Let $N \geq 3$, $\alpha \in (0,N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$. Assume that $u \in H^1(\mathbb{R}^N)$ is a positive ground state of (4.1). Then there exist $x^* \in \mathbb{R}^N$ and a nonnegative decreasing function $v : (0, \infty) \to \mathbb{R}$ such that $u(x) = v(|x - x^*|)$ and $\lim_{r \to \infty} v(r) = 0$.

This paper provides a new and different perspective to study the symmetry and monotonicity of solutions to elliptic equations. In the future work, on one hand, we want to investigate the symmetry and monotonicity of separable functions in other symmetric domains rather than $B_R$ and $\mathbb{R}^N$. On the other hand, we are interested in the radial symmetry and uniqueness of ground states of Choquard type equations in $B_R$. Furthermore, the relationship between the ground states of Choquard type equations in $B_R$ and that in $\mathbb{R}^N$ when $R \to \infty$ is also worth studying.

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