Exact Partition Function and Boundary State of Critical Ising Model with Boundary Magnetic Field

R.Chatterjee †
Department of Physics and Astronomy
Rutgers University
P.O.Box 849, Piscataway, NJ 08855-0849

Abstract

We compute the exact partition function of the 2D Ising Model at critical temperature but with nonzero magnetic field at the boundary. The model describes a renormalization group flow between the free and fixed conformal boundary conditions in the space of boundary interactions. For this flow the universal ground state degeneracy $g$ and the full boundary state is computed exactly.

1. Introduction

A conformal field theory (CFT) [1] is understood to be the ultraviolet (UV) limit of a class of general quantum field theories (QFT). It is associated with fixed manifolds of the renormalization group (RG) flow in general QFT. And general QFT’s are understood as deformations of CFT’s and describing RG trajectories of their corresponding unstable manifolds. More precisely, they are interpreted as appropriate relevant perturbations of appropriate CFT’s [2]. For this interpretation to be viable, it should be possible, at least in principle, to compute infrared behavior of a QFT from its UV definition. In 2D, for certain “integrable perturbations” of CFT’s, this is made possible by the fact that the scattering (S) matrix is exactly calculable [2].

The S matrix is understood to encode all information about its underlying QFT. One partially successful way of reconstructing the QFT from the S matrix is the Thermodynamic Bethe Ansatz (TBA) approach [3-6]. In particular, the method yields a $c$ function, seemingly having all the characteristics of Zamolodchikov’s $c$ function [7].

All the above understanding has been and is still being extended to QFT with boundary. CFT on manifolds with boundary is studied in [8-10]. It is shown in [9] that, corresponding to each conformal family in the (bulk) CFT, there is a boundary condition which

† email: robin@physics.rutgers.edu
preserves conformal symmetry. Corresponding to each such conformal boundary condition, there is a boundary state containing all information about the boundary condition. This boundary state belongs to the Hilbert space of states in the Hamiltonian picture based on a spacetime coordinatization in which the boundary is a spacelike curve. In [9] it is shown how to classify and compute boundary states.

The understanding of CFT with boundary is extended to integrable field theory on manifolds with boundary in [11]. It is shown that, for a model which is integrable in the absence of boundary, when defined on manifolds with boundary, there are certain boundary conditions which preserve the integrability of the model. The notion of integrable boundary condition is made precise in [11] and a general theory to understand and compute exact boundary states and boundary scattering matrices for reflections off boundaries is formulated. Boundary scattering matrices for several models have since been computed [11-16].

Another aspect of boundary QFT and critical phenomena is the universal ground state degeneracy factor \( g \) [17]. In [9] it is shown how to calculate \( g \) for boundary CFT’s. In [17], \( g \) is conjectured to be monotonically decreasing from a less stable to a more stable fixed point along the RG flow of boundary interactions. However as of now, there exist no fundamental definition of the “\( g \) function” in terms of fundamental quantities of the boundary QFT like the correlation functions, and no proof for the “\( g \) theorem” conjecture.

In [19] and [20], the Thermodynamic Bethe Ansatz (TBA) method has been used to compute the boundary contribution to the free energy of boundary integrable models. However, because of certain difficulties discussed in these references, it has so far been possible to compute only ratios of the \( g \) function using the TBA approach.

One of the simplest conformally invariant boundary models is the Ising model with boundary, studied in [8-10]. In [9] the two essentially different conformally invariant boundary conditions are identified to be (1) free case: boundary spins are free; and (2) fixed case: all boundary spins are fixed to +1 or −1.

In this paper, we study more general boundary conditions breaking conformal invariance: the Ising spins interacting with an external magnetic field at the boundary. The model is solved on the lattice in [21,22] (see also [23]). In this paper, we demonstrate a short cut way to calculate the exact partition function. In addition, we compute the boundary state. We compute the partition function on a long cylinder of length \( L \) in one Hamiltonian representation and transform it to the cross channel representation. This enables us to compute the full boundary state exactly. In what follows, we describe the model, elaborate on this strategy and explicitly compute the partition function and boundary state.

2. Ising Model With Boundary Magnetic Field

We study the Ising model with its boundary spins interacting with a constant external magnetic field \( H_B \). This boundary condition interpolates between the free case when \( H_B = 0 \) and the fixed to \( \pm 1 \) case when \( H_B \to \pm \infty \). It thus drives the system from the free case in the UV down to the fixed case in the IR, which are the fixed points of this boundary RG flow [17]. The model is studied in [11,18]. At critical temperature, the continuum limit
of the model is described by the action:

\[
A_h = \frac{1}{2\pi} \int_D dx dy [\psi \partial_z \bar{\psi} + \bar{\psi} \partial_{\bar{z}} \psi] + \int_B dt \left[ -\frac{i}{4\pi} \bar{\psi} \psi + \frac{1}{2} a \dot{a} \right] + ih \int_B dt a(t) (e^{\frac{i}{2} \psi} + e^{\frac{i}{2} \bar{\psi}})(t) \tag{1}
\]

Here, \( z = x + iy, \bar{z} = x - iy \) are complex coordinates on our manifold \( D \); the boundary \( B \) is given by a parametric curve \( B: z = Z(t), \bar{z} = \bar{Z}(t), t \) being a real parameter; \( e(t) = \frac{d}{dt} Z(t) \), \( \bar{e}(t) = \frac{d}{dt} \bar{Z}(t) \) are components of the tangent vector \( (e, \bar{e}) \) to the boundary with \( e(t)\bar{e}(t) = 1 \); \( \psi, \bar{\psi} \) are free massless Majorana fermion fields; \( a(t) \) is auxiliary “boundary” fermionic degree of freedom (see [11,18]) with the two point function:

\[
\langle a(t) a(t') \rangle_{\text{free}} = \frac{1}{2} \text{sign}(t - t') \tag{3}
\]

and \( \dot{a} \equiv \frac{d}{dt} a \); \( h \) is the boundary coupling constant (appropriately rescaled external field \( H_B \)) with the dimension of \( [\text{Length}]^{-\frac{1}{2}} \). The first two terms in (1) describe the conformally invariant critical Ising model with free boundary conditions. The third boundary perturbation term is the boundary spin operator (see [11]):

\[
\sigma_B(t) = ia(t) [e^{\frac{i}{2} \psi} + e^{\frac{i}{2} \bar{\psi}}](t) \tag{2}
\]

The boundary spin operator was identified in [9,10]. Thus (1) can be interpreted as the free boundary Ising field theory perturbed by the boundary spin operator \( \sigma_B \).

It is easily seen from (1) that the fermion fields \( \psi, \bar{\psi} \) satisfy the following boundary condition at each boundary:

\[
\left( \frac{d}{dt} + i\lambda \right) \psi(t) = \left( \frac{d}{dt} - i\lambda \right) \bar{\psi}(t) \tag{4}
\]

where

\[
\lambda = 4\pi h^2 \tag{5}
\]

and

\[
\psi(t) = e^{\frac{i}{2} (t)} \psi(Z(t)), \quad \bar{\psi}(t) = e^{\frac{i}{2} (t)} \bar{\psi}(\bar{Z}(t)) \tag{6}
\]

In particular, from (6), we see that the free \((h = 0)\) and fixed \((h \to \pm\infty)\) boundary conditions correspond to:

\[
\psi(t) = \bar{\psi}(t), \quad \text{free case,}
\]

\[
\psi(t) = -\bar{\psi}(t), \quad \text{fixed case} \tag{7}
\]

At the left \((x = 0)\) and right \((x = L)\) boundaries of the cylinder (see Fig.1) the boundary condition (4) takes the forms:

\[
\left( \frac{d}{dy} + i\lambda \right) \omega \psi(-iy) = \left( \frac{d}{dy} - i\lambda \right) \omega \bar{\psi}(iy) \quad \text{at left boundary,} \tag{8a}
\]

\[
\left( \frac{d}{dy} + i\lambda \right) \omega \psi(iy) = \left( \frac{d}{dy} - i\lambda \right) \omega \bar{\psi}(-iy) \quad \text{at right boundary} \tag{8b}
\]
\[ \left( \frac{d}{dy} + i\lambda \right) \omega \psi(L + iy) = \left( \frac{d}{dy} - i\lambda \right) \bar{\omega} \bar{\psi}(L - iy) \] at right boundary,  

where \( \omega = e^{i\pi/4} \) and \( \bar{\omega} = e^{-i\pi/4} \).

3. Method Outline

Quantum Field Theory on the space-time of a cylinder of length \( L \) (later we will take \( L \) to be large) and circle length \( R \) (see Fig.1), coordinatized by \( x \) and \( y \) respectively, can be viewed in two alternate Hamiltonian pictures: In the first, one considers the (Euclidean) time coordinate to run along the length of the cylinder (i.e. \( x \)) and the space coordinate along the circle (i.e. \( y \)). A Hilbert space of states \( \mathcal{H}_R \) is associated with any time slice \( x = \text{const} \) across the cylinder with fields \( \psi, \bar{\psi} \) satisfying periodic (for Ramond sector) or antiperiodic (for Neveu-Schwarz sector) boundary condition along the circle. Naturally, \( \mathcal{H}_R \) is the same as in the bulk theory since in this picture there is no spatial boundary. The partition function is given by:

\[ Z = \langle B_{\text{right}} | e^{-L\mathcal{H}_R} | B_{\text{left}} \rangle \] (9)

where \( |B_{\text{left}}\rangle \) and \( |B_{\text{right}}\rangle \) are the boundary states corresponding to the boundary conditions at the \( x = 0 \) and \( x = L \) ends of the cylinder respectively, and \( \mathcal{H}_R \) is the Hamiltonian in this picture.

In the other picture, the space and time coordinates are interchanged (i.e. \( x \) becomes the space coordinate and \( y \) the time coordinate) and we now have a system of particles on a line of length \( L \) at temperature \( T = \frac{1}{R} \). The space of states \( \mathcal{H}_L (B_{\text{left}}, B_{\text{right}}) \) is associated with a \( y = \text{const} \) curve and these states must now satisfy boundary conditions \( B_{\text{left}}, B_{\text{right}} \) at \( x = 0 \) and \( x = L \) cylinder ends respectively. The partition function is given by

\[ Z_{\pm} = \text{tr} \left( (\pm 1)^F e^{-R\mathcal{H}_L} \right) \] (10)

where the + and − signs correspond to antiperiodic and periodic boundary conditions along \( y \), \( F \) is the fermion number operator and \( \mathcal{H}_L \) is the Hamiltonian in this picture. In this picture the free fields \( \psi, \bar{\psi} \) admit the following decomposition in terms of plane waves:

\[ \psi(z) = \sum_{k_l} b_{k_l} e^{ik_lz}, \quad \bar{\psi} = \sum_{k_l} \bar{b}_{k_l} e^{-ik_l\bar{z}} \] (11)

where the sum is over the free particle momenta \( k_l \) which are constrained to satisfy:

\[ 1 + X(k_l) = 0, \quad \text{where} \quad X(k) = e^{2ik_lL \left( \frac{k - i\lambda}{k + i\lambda} \right)^2} \] (12)

where (12) is obtained by imposing (8a) and (8b) on (11). In (11), \( b_{k_l}^\dagger = b_{-k_l} \) (notice that solutions of (12) occur in pairs \( \pm k_l \) and \( b_{k_l}, b_{k_l}^\dagger \) with \( k_l > 0 \) \( k_l=0 \) does not satisfy
(12)) are the usual free fermion creation and annihilation operators satisfying standard anticommutation relations \( \{ b_{k_n}, b_{k_m}^\dagger \} = \delta_{k_n,k_m} \) and \( \{ b_{k_n}, b_{k_m} \} = 0 = \{ b_{k_n}^\dagger, b_{k_m}^\dagger \} \). And, from (8a), \( \bar{b}_{k_l} = -i(\frac{k_l + \lambda}{k_l - \lambda})b_{k_l} \), implying that \( \bar{b}_{k_l} \) s do not represent independent degrees of freedom.

In the next sections we start with the partition function as sum over the states in (10) (described by modes in (11),(12)). We want to transform it so that it represents the partition function (9) in the cross channel picture. From this we can compute the boundary state.

4. Partition Function

The Hamiltonian of our model in the finite temperature field theory picture (\( x \) is space coordinate and \( y \) is time coordinate) is:

\[
H_L = E_L + \sum_{k_l > 0} k_l \ b_{k_l}^\dagger b_{k_l}
\]  

(13)

where \( k_l \) s are the allowed momenta satisfying (12), and \( E_L \) is the ground state energy. The partition function (9) is:

\[
Z_\pm = e^{-RE_L} \prod_{k_l > 0} (1 \pm e^{-Rk_l})
\]  

(14)

Here, as before, the \( \pm \) signs refer to antiperiodic and periodic boundary conditions along the cylinder circle satisfied by the fields \( \psi, \bar{\psi} \). At thermodynamic equilibrium the free energy \( F_\pm \) is given by:

\[
-RF_\pm = -RE_L + \sum_{k_l > 0} \log(1 \pm e^{-Rk_l})
\]  

(15)

This can be written as:

\[
-RF_\pm = -RE_L + \frac{1}{2\pi i} \int_C dk \frac{X'}{1 + X} \log(1 \pm e^{-Rk})
\]  

(16)

where the integration contour \( C = C_+ + C_- \) in the complex \( k \) plane and the positions of poles of the integrand are shown in Fig.2. In the above, \( X' = \frac{dX}{dk} \). We write the integral along contour \( C_- \) as

\[
\int_{C_-} dk \frac{X'}{1 + X} \log(1 \pm Y_R) = \int_{C_-} dk \frac{X'}{X} \log(1 \pm Y_R) - \int_{C_-} dk \frac{X'}{X^2(1 + \frac{1}{X})} \log(1 \pm Y_R)
\]  

(17)

where \( Y_R = e^{-kR} \). Now we change the integration variable \( k \to -k \) in the second integral on the RHS of (17), and noting that


\[ X(-k) = \frac{1}{X(k)} \, , \quad Y_R(-k) = \frac{1}{Y_R(k)} \, , \] (18)

we get:

\[
\frac{1}{2\pi i} \int_{C_-} dk \frac{X'}{1 + X} \log(1 \pm Y_R) = \frac{1}{2\pi i} \int_{C_-} dk \frac{X'}{X} \log(1 \pm Y_R) + \\
+ \frac{R}{2\pi i} \int_{C_-} dk \frac{kX'}{1 + X} \log(1 \pm Y_R) + \frac{\nu \pm 2}{2} \log 2
\] (19)

Here, the contour \(-C_-\) is shown in Fig.3; \(\nu_+ = 0\) and \(\nu_- = 1\). The last term in (19) arises as a result of writing \(\log(Y_R - 1)\) as analytic continuation of \(\log(1 - Y_R)\) and we have used

\[
\int_{-C_-} dk \frac{X'}{1 + X} = \log(1 + X) \bigg|_{i\infty}^0 = - \log 2
\] (20)

From (16), (19) we get:

\[
-RF_\pm = -RE_L + \frac{1}{2\pi i} \int_{C'} dk \frac{X'}{1 + X} \log(1 \pm Y_R) + \\
+ \frac{1}{\pi} \int_0^\infty dk (L + \frac{2\lambda}{k^2 + \lambda^2}) \log(1 \pm Y_R) + \frac{R}{2\pi i} \int_0^{i\infty} dk k \frac{X'}{1 + X} + \frac{\nu \pm 2}{2} \log 2
\] (21)

Here \(C' = C_+ + -C_-\) is shown in Fig.3. Now,

\[
H_L = \frac{1}{2} \sum_{k_l > 0} k_l (b_l^\dagger b_l - b_l b_l^\dagger) = \sum_{k_l > 0} (b_l^\dagger b_l - \frac{k_l}{2})
\] (22)

which when compared with (13) implies \(E_L = -\frac{1}{2} \sum_{k_l > 0} k_l\). This, as before, can be written as:

\[
E_L = -\frac{1}{4\pi i} \int_C dk k \frac{X'}{1 + X}
\] (23)

which yields the following:

\[
E_L = \frac{1}{2\pi i} \int_0^{i\infty} dk k \frac{X'}{1 + X} - \frac{1}{2\pi} \int_0^{i\infty} dk k (L + \frac{2\lambda}{k^2 + \lambda^2})
\] (24)

In (23), the integration contour is the same as in (16). In (24), the two terms in the integral in the RHS are the usual bulk and boundary intensive free energies respectively. We denote the boundary term as \(\varepsilon_\lambda\):
\[ 2\varepsilon_\lambda = -\frac{1}{\pi} \int_0^\infty dk \frac{2\lambda k}{k^2 + \lambda^2} \] 

(25)

\[ 2\varepsilon_\lambda \] goes as \( \lambda \log(\lambda^2) + \) nonuniversal terms which we set to 0. Likewise, we set the nonuniversal term in (24) \( \frac{1}{2\pi} \int_0^\infty dk k \) to zero. In (21), we evaluate the integral in the second term by parts and residues and get:

\[
\frac{1}{2\pi i} \int_{C'} dk \frac{X'}{1 + X} \log(1 \pm Y_R) = -\frac{1}{2\pi i} \int_{C'_W} dk \frac{\pm Y'_R}{1 \pm Y_R} \log(1 + X(k))
\]

(26)

where the Wick rotated contour \( C'_W \) is shown in Fig.4. Putting (21), (24), (26) together, we get:

\[
-RF_\pm = -2R\varepsilon_\lambda + \log \Sigma_\pm + \frac{L}{\pi} \int_0^\infty dk \log(1 \pm Y_R) +
\]

\[
\quad + \frac{1}{\pi} \int_0^\infty dk \frac{2\lambda}{k^2 + \lambda^2} \log(1 \pm Y_R) + \frac{\nu_\pm}{2} \log 2
\]

(27)

In (27), we recognize the third term to be the Casimir energy on the circle. The (+) sign corresponds to the Neveu-Schwarz sector and (-) sign to the Ramond sector. We denote these by \( E_R^{(\pm)} \):

\[
E_R^{(\pm)} = -\frac{1}{\pi} \int_0^\infty dk \log(1 \pm e^{-kR})
\]

(28)

The fourth and fifth terms in (27) are to be identified as related with the ground state degeneracy factor \( g(R) \).

\[
\log g_\pm(R) = \frac{\lambda}{\pi} \int_0^\infty dk \frac{1}{k^2 + \lambda^2} \log(1 \pm e^{-kR}) + \frac{\nu_\pm}{4} \log 2
\]

(29)

With this, we have:

\[
Z_\pm = e^{-LE_R^{(\pm)}} (g_\pm(R)e^{-R\varepsilon_\lambda})^2 \Sigma_\pm
\]

(30)
Notice that $\Sigma_\pm \to 1$ as $L \to \infty$. $g_\pm$ can easily be evaluated:

$$g_+(R) = \frac{\sqrt{2\pi}}{\Gamma(\alpha + \frac{1}{2})} \left( \frac{\alpha}{e} \right)^\alpha$$

$$g_-(R) = \frac{2^{\frac{1}{2}} \sqrt{\pi \alpha}}{\Gamma(\alpha + 1)} \left( \frac{\alpha}{e} \right)^\alpha$$

where

$$\alpha = 2h^2 R$$

4. Boundary States

Now, we expect that:

$$Z_\pm = Z_{hh} \pm Z_{h-h}$$

where

$$Z_{hh} = \langle B_h | e^{-LH_R} | B_h \rangle, \quad Z_{h-h} = \langle B_h | e^{-LH_R} | B_{-h} \rangle$$

Here $|B_h\rangle$ is the boundary state corresponding to boundary magnetic field $h$. In the large $L$ limit, the following expressions are valid:

$$Z_{hh} = e^{-LE_R^{(+)}(\langle 0 | B_h \rangle)^2} + e^{-LE_R^{(-)}(\langle \sigma | B_h \rangle)^2}$$

$$Z_{h-h} = e^{-LE_R^{(+)}(\langle 0 | B_h \rangle)^2} - e^{-LE_R^{(-)}(\langle \sigma | B_h \rangle)^2}$$

In (36) and (37), $|0\rangle$ is the ground state of energy $E_R^{(+)}$ of the Hilbert space $\mathcal{H}_R$ and it lies in the NS sector, while $|\sigma\rangle$ is the lowest energy state of energy $E_R^{(-)}$ in the R sector of $\mathcal{H}_R$: $|\sigma\rangle = \sigma |0\rangle$, $\sigma$ being the spin operator in the bulk theory. Also, in the above, we have used

$$\langle 0 | B_{-h} \rangle = \langle 0 | B_h \rangle \quad \text{and} \quad \langle \sigma | B_{-h} \rangle = -\langle \sigma | B_h \rangle$$

Moreover we have chosen the phase of $|0\rangle$ to be such that $\langle 0 | B_h \rangle$ and $\langle \sigma | B_h \rangle$ are real (see [17]). This can always be done. From (30), (34), (36), (37), it follows that:

$$\langle 0 | B_h \rangle = \frac{1}{\sqrt{2}} g_+(R)e^{-R\epsilon}$$

$$\langle \sigma | B_h \rangle = \frac{1}{\sqrt{2}} g_-(R)e^{-R\epsilon}$$

Now we construct the boundary states more explicitly. First we consider the picture in which $x = time$ and $y = space$ (it is in this picture that the boundary state $|B_h\rangle$ belongs
to the Hilbert space $\mathcal{H}_R$ and write down the mode expansions for the $\psi$ and $\bar{\psi}$ fermion fields. In the Neveu-Schwarz and Ramond sectors we have

**Neveu-Schwarz sector:**

\[
\psi(x, y) = \sum_{n=0}^{\infty} \left[ a_{n+\frac{1}{2}} e^{-k_{n+\frac{1}{2}}(x+iy)} + a_{n+\frac{1}{2}}^\dagger e^{k_{n+\frac{1}{2}}(x+iy)} \right]
\]

**(41)**

\[
\bar{\psi}(x, y) = \sum_{n=0}^{\infty} \left[ \bar{a}_{n+\frac{1}{2}} e^{-k_{n+\frac{1}{2}}(x-iy)} + \bar{a}_{n+\frac{1}{2}}^\dagger e^{k_{n+\frac{1}{2}}(x-iy)} \right]
\]

**(42)**

where

\[
k_{n+\frac{1}{2}} = \frac{2\pi(n + \frac{1}{2})}{R}, \quad n \in \mathbb{Z}
\]

**(43)**

Here $a_{n+\frac{1}{2}}^\dagger$ ($a_{n+\frac{1}{2}}$) and $\bar{a}_{n+\frac{1}{2}}^\dagger$ ($\bar{a}_{n+\frac{1}{2}}$) are the creation (annihilation) operators for right moving and left moving free Majorana fermions respectively. In the other sector we have:

**Ramond sector:**

\[
\psi(x, y) = \sum_{n=0}^{\infty} \left[ a_n e^{-k_n(x+iy)} + a_n^\dagger e^{k_n(x+iy)} \right]
\]

**(44)**

\[
\bar{\psi}(x, y) = \sum_{n=0}^{\infty} \left[ \bar{a}_n e^{-k_n(x-iy)} + \bar{a}_n^\dagger e^{k_n(x-iy)} \right]
\]

**(45)**

where

\[
k_n = \frac{2\pi n}{R}, \quad n \in \mathbb{Z}
\]

**(46)**

and with similar interpretation for the operators $a_n^\dagger$, $a_n$, $\bar{a}_n^\dagger$, $\bar{a}_n$ as in the NS case.

In terms of these mode creation and annihilation operators, the boundary condition \((4)\) can be expressed as the following set of constraints on any boundary state $|B_h\rangle$:

\[
a_l|B_h\rangle = \frac{i k_l - \lambda}{k_l + \lambda} \bar{a}_l^\dagger |B_h\rangle
\]

**(47)**

Here $l \in \mathbb{Z} + \frac{1}{2}$ if the $a_l$'s are in the NS sector and $l \in \mathbb{Z}$ if the $a_l$'s are in the R sector, and $k_l = \frac{2\pi l}{R}$. From equations \((39)\) and \((40)\) and mode expansions \((41)-(46)\), the boundary state takes the following form (following \([11]\)):
\[ |B_{\pm h}\rangle = \frac{1}{\sqrt{2}} g_{\pm h}(R) e^{-R\varepsilon_{\lambda}} \exp \left\{ \sum_{n=0}^{\infty} \mathcal{K}(k_{n+1/2}) \ a_{n+1/2}^\dagger \ a_{n+1/2}^\dagger \right\} |0\rangle \pm \]

\[ \pm \frac{1}{\sqrt{2}} g_{-h}(R) e^{-R\varepsilon_{\lambda}} \exp \left\{ \sum_{n=1}^{\infty} \mathcal{K}(k_n) \ a_n^\dagger \ a_n^\dagger \right\} |\sigma\rangle \]  

where

\[ \mathcal{K}(k_l) = \mathcal{R}(ik_l) = i \frac{k_l - \lambda}{k_l + \lambda} = i \frac{l - \alpha}{l + \alpha} , \quad \alpha = 2h^2 R \]  

where \( \mathcal{R}(k) = i \frac{k - i\lambda}{k + i\lambda} \) is the “massless boundary scattering matrix” for our model [11]; \( \mathcal{K} \) can also be obtained directly from (47). Putting (31) – (33), (48) and (49) together, we finally obtain the boundary state:

\[ |B_{\pm}\rangle = e^{-R\varepsilon_{\lambda}} \left( \frac{\alpha}{\epsilon} \right)^\alpha \sqrt{\pi} \left[ 1 \cdot \frac{1}{\Gamma(\alpha + \frac{1}{2})} \exp i \left\{ \sum_{n=0}^{\infty} \frac{n + \frac{1}{2} - \alpha}{n + \frac{1}{2} + \alpha} a_{n+1/2}^\dagger \ a_{n+1/2}^\dagger \right\} |0\rangle + \right. \]

\[ \left. \pm \frac{2^\frac{1}{2} \sqrt{\alpha}}{\Gamma(\alpha + 1)} \exp i \left\{ \sum_{n=1}^{\infty} \frac{n - \alpha}{n + \alpha} a_n^\dagger \ a_n^\dagger \right\} |\sigma\rangle \right] \]  

This expression (50) is the main result of this paper.

Now, we derive from (50) the boundary states corresponding to free and fixed boundary conditions. These correspond to the UV and IR limits of the boundary RG flow as discussed earlier.

**free case \( (h = 0) \):**

From (50), we readily obtain, putting \( \alpha = 0 \),

\[ |B_{free}\rangle = \exp i \left\{ \sum_{n=1}^{\infty} a_{n+\frac{1}{2}}^\dagger \ a_{n+\frac{1}{2}}^\dagger \right\} |0\rangle + i \exp i \left\{ \sum_{n=1}^{\infty} a_{n+\frac{1}{2}}^\dagger \ a_{n+\frac{1}{2}}^\dagger \right\} a_{\frac{1}{2}}^\dagger \ a_{\frac{1}{2}}^\dagger \ |0\rangle \]  

The above follows since the creation operators anticommute and since the second term in (50) is zero in this case. Hence,

\[ |B_{free}\rangle = \exp i \left\{ \sum_{n=1}^{\infty} a_{n+\frac{1}{2}}^\dagger \ a_{n+\frac{1}{2}}^\dagger \right\} (|0\rangle - |\varepsilon\rangle) \]  

since

\[ a_{\frac{1}{2}}^\dagger \ a_{\frac{1}{2}}^\dagger |0\rangle = i \ |\varepsilon\rangle \]  

\( \varepsilon \) being the energy density operator of the bulk critical Ising model.
fixed case \((h \to \pm \infty)\):

In this case, we have from (50):

\[
|B_{\text{fixed} \pm}\rangle = \frac{1}{\sqrt{2}} \exp \left\{ -i \sum_{n=1}^{\infty} a_{n+\frac{1}{2}}^{\dagger} \bar{a}_{n+\frac{1}{2}}^{\dagger} \right\} \left( |0\rangle + |\varepsilon\rangle \right) \pm \\
\pm \frac{1}{2^{\frac{3}{4}}} \exp \left\{ -i \sum_{n=1}^{\infty} a_{n}^{\dagger} \bar{a}_{n}^{\dagger} \right\} |\sigma\rangle
\]

Both (52) and (54) agree with well known results of Cardy [9].

Conclusion

The expression (50) for \(|B_{\pm h}\rangle\) is the main result of this paper. In the conformal limits, it matches with Cardy’s results [9]. Our method of computing the boundary state is the same in spirit as Cardy’s [9,10] in deriving boundary states in boundary CFT – namely, making use of a (modular) transformation to express the partition function in one channel in terms of states in the cross channel. An essentially similar calculation can be done for the off critical Ising model with boundary magnetic field; we will not deal with it here. It would be interesting to extend the method to general interacting boundary integrable field theories.

Acknowledgement

I am especially grateful to A.Zamolodchikov for his advice, inspiration and numerous illuminating discussions all along. I am also thankful to V.Brazhnikov for many helpful discussions.
References

1. “Conformal Invariance and Applications to Statistical Mechanics”, eds. C.Itzykson, H.Saleur and J.B.Zuber (World Scientific, 1988).
2. A.Zamolodchikov, Advanced Studies in Pure Mathematics, 19 (1989) 641
3. C.N.Yang and C.P.Yang, J. Math. Phys. 10 (1969) 1115
4. E.H.Lieb and W.Liniger, Phys. Rev. 130 (1963) 1605
5. A.Zamolodchikov, Nucl.Phys. B342 (1990) 695
6. T.Klassen and E.Melzer, Nucl. Phys. B350 (1991) 635
7. A.Zamolodchikov, Sov. J. Nucl. Phys. 46(6) (1987) 1090
8. J.Cardy, “Conformal Invariance and Surface Critical Behavior” in Ref.1.
9. J.Cardy, Nucl. Phys. B324 (1989) 581
10. J.Cardy and D.Lewellen, Phys. Lett. B259 (1991) 274
11. S.Ghoshal and A.Zamolodchikov, Int. J. Mod. Phys. A9, No.21 (1994) 3841
12. S.Ghoshal, Int. J. Mod. Phys. A9, No.27, (1994) 4801
13. S.Ghoshal, Phys. Lett. B334 (1994) 363
14. L.Chim, “Boundary S Matrix for Integrable q-Potts model”, RU-94-33, hep-th/9404118
15. A.Fring and R.Koberle, “Boundary States in Affine Toda Field Theory in the Presence of Reflecting Boundaries”, USP-IFQSC/TH/93-12, hep-th/9304144
16. A.Fring and R.Koberle, “Boundary Bound States in Affine Toda Field Theory”, USP-IFQSC/TH/94-03, hep-th/9404188
17. I.Affleck and A.Ludwig”, Phys. Rev. Lett. 67 (1991) 161
18. R.Chatterjee and A.Zamolodchikov, Mod. Phys. Lett. A9, No. 24 (1994) 2227
19. P.Fendley and H.Saleur, “Deriving Boundary S Matrices”, USC-94-001, hep-th/9402045
20. P.Fendley, H.Saleur and N.P.Warner, “Exact Solution of a Massless Scalar Field With a Relevant Boundary Interaction”, USC-94-10, hep-th/94-06125
21. B.M.McCoy and T.T.Wu, Phys. Rev. 162 (1967) 436
22. B.M.McCoy and T.T.Wu, Phys. Rev. 174 (1968) 546
23. M.M.McCoy and T.T.Wu, “The Two-Dimensional Ising Model”, Harvard University Press, (1973) Cambridge Mass.
