Groups with isomorphic fibered Burnside rings

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\textbf{ABSTRACT}

Let $G$ and $H$ be finite groups. We give a condition on $G$ and $H$ that implies that the $A$-fibered Burnside rings $B^A(G)$ and $B^A(H)$ are isomorphic. As a consequence, we show the existence of non-isomorphic groups $G$ and $H$ such that $B^0(G)$ and $B^0(H)$ are isomorphic rings. Here, the abelian fiber group $A$ can be chosen in a non-trivial way, that is, such that $B^0(G)$ and $B^0(H)$ are strictly bigger than the Burnside rings of $G$ and $H$, for which such counterexamples are already known.

\section{Introduction}

If $A$ is an abelian group, the $A$-fibered Burnside ring of a finite group $G$, denoted by $B^A(G)$, is the Grothendieck ring of the category of $A$-fibered $G$-sets. An $A$-fibered $G$-set is a $G \times A$-set which is free as $A$-set and has finitely many $A$-orbits. The category of $A$-fibered $G$-sets has coproducts and is symmetric monoidal with respect to a tensor product $\otimes_A$. These operations yield the ring structure on $B^A(G)$. The $A$-fibered Burnside ring was first introduced in greater generality by Dress in [5], where an action of the group $G$ on $A$ was allowed and the notation $\Omega_1(G,A)$ was used. We restrict ourselves to the case of the trivial action of $G$ on $A$ and use the notation $B^A(G)$.

The case $A = k^\times$ for a field $k$ is particularly interesting, since in this case the ring $B^{k^\times}(G)$ is also isomorphic to the Grothendieck ring $D^k(G)$ of the category of finite $G$-line bundles over $k$ (introduced in [3] and denoted by $R_{ab}^k(G)$), which is also sometimes called the category of monomial $k$-representations of $G$ (see [3, Section 5]). There is a direct connection to $kG$-modules: One has a natural ring homomorphism $D^k(G) \to R_k(G)$ into the Grothendieck ring of finitely generated $kG$-modules which is surjective if $k$ is algebraically closed. For $k = \mathbb{C}$, a natural section of $D^{\mathbb{C}}(G) \to R_{\mathbb{C}}(G)$, i.e., a canonical induction formula, was given in [2], and a categorification of the canonical induction formula (a monomial resolution in terms of $G$-line bundles over $\mathbb{C}$) was given in [3].

Recalling that $B^A(G) \cong B(G)$, the Burnside ring of $G$, if and only if $A$ has trivial $|G|$-torsion, we will say that two non-isomorphic finite groups $G$ and $H$ provide a non-trivial counterexample to the isomorphism problem of the $A$-fibered Burnside ring if $B^A(G)$ and $B^A(H)$ are isomorphic as rings and there is a non-trivial element $a$ in $A$ such that $a^{|G|} = 1$. Examples of non-isomorphic finite groups $G$ and $H$ with isomorphic Burnside rings have already been given for instance by Thévenaz in [7].

The notion of a species isomorphism for fibered Burnside rings was introduced by the second author in [6] for a ring isomorphism preserving the standard bases given by conjugacy classes of monomial pairs, which is analogous to an isomorphism of mark tables in the case of Burnside rings. In Section 2
of this note we present a sufficient condition on finite groups $G$ and $H$ for the existence of a species isomorphism between their fibered Burnside rings. In Section 3, we use this result to prove that Thévenaz' counterexamples to the isomorphism problem for Burnside rings (see [7]) of order $p^2q$ for primes $p$ and $q$ such that $q|(p-1)$, provide also non-trivial counterexamples for the $A$-fibered Burnside ring when the fiber $A$ has trivial $p$-torsion and elements of order $q$.

**Notation**

Throughout this note, the letters $G$ and $H$ stand for finite groups. We write $S_G$ for the set of subgroups of $G$, and $[S_G] \subseteq S_G$ for a set of representatives of the conjugacy classes. For an element $g \in G$, we denote the resulting conjugation map $c_g$ also by $g^{-1}: G \rightarrow G$, $x \mapsto gxg^{-1}$. For subgroups $K$ and $L$ of $G$ we write $K = G L$ and $K \leq G L$ if there exists $g \in G$ with $K = gL$ and $K \leq gL$, respectively. If $G$ acts on a set $X$, we write $G \setminus X$ for the set of its orbits and we write $x =_G y$ if two elements $x$ and $y$ of $X$ are in the same orbit.

**2. A criterion for species isomorphisms**

We first recall some basic definitions and results on fibered Burnside rings. We refer the reader to [4], [5] and [6] for further details.

Let $\mathcal{M}_G^A$ denote the set of all pairs $(K, \phi)$, also called subcharacters or monomial pairs, where $K \leq G$ and $\phi: K \rightarrow A$ is a group homomorphism. The group $G$ acts on $\mathcal{M}_G^A$ by $\delta(K, \phi) := (\delta K, \delta \phi)$, where $
abla \phi(x) := (g^{-1}xg)$ for $g \in G$, $(K, \phi) \in \mathcal{M}_G^A$, and $x \in \delta K$. By [5, Proposition 2.2], the transitive $A$-fibered $G$-sets are parametrized by the $G$-orbits of $\mathcal{M}_G^A$. More precisely, if $X$ is a transitive $A$-fibered $G$-set, one associates to it the pair $(K, \phi)$ consisting of the $G$-stabilizer $K$ of the $A$-orbit $Ax$ of a fixed element $x \in X$ and the homomorphism $\phi: K \rightarrow A$ defined as $gx = \phi(g)x$ for all $g \in K$. The orbit of $(K, \phi)$ does not depend on the choice of $x$ and is denoted by $[K, \phi]_G$. Thus, $B^A(G)$ can be regarded as the free abelian group with basis $G \setminus \mathcal{M}_G^A$. By [5, 2.3], the multiplication is given by $$[K, \phi]_G \cdot [L, \psi]_G = \sum_{KsL \in K \setminus G / L} [K \cap sL, \phi|_{K \cap sL}, \psi|_{K \cap sL}]_G,$$

with identity element $[G, 1]_G$. That this multiplication yields a commutative ring structure on $B^A(G)$ follows from the properties of the tensor product $\otimes_A$ of $A$-fibered $G$-sets, see [5, Section 1].

For $K \leq G$, $\mathbb{Z}\text{Hom}(K, A)$ is the group ring of $\text{Hom}(K, A)$ which is a group under point-wise multiplication. The group $G$ acts by conjugation on the resulting product ring $\prod_{K \leq G} \mathbb{Z}\text{Hom}(K, A)$. More precisely, for $g \in G$ and $(x_K)_{K \leq G} \in \prod_{K \leq G} \mathbb{Z}\text{Hom}(K, A)$, one has $$(\delta x)_K := (\delta x_K^{-1})_{K \leq G},$$

where, for $L \leq G$ and $y = \sum_{\phi} a_{\phi} \phi \in \mathbb{Z}\text{Hom}(L, A)$, we set $\delta y := \sum_{\phi} a_{\phi} \delta \phi \in \mathbb{Z}\text{Hom}(\delta L, A)$. Thus, if for $(K, \phi) \in \mathcal{M}_G^A$, the element $b_{(K, \phi)} \in \prod_{K \leq G} \mathbb{Z}\text{Hom}(K, A)$ is defined as $b_{(K, \phi)}(x_K)_{K \leq G}$ with $x_k = \phi$ if $L = K$ and $x_L = 0$ otherwise, then these elements form a $\mathbb{Z}$-basis of $\prod_{K \leq G} \mathbb{Z}\text{Hom}(K, A)$ and $\delta b_{(K, \phi)} = b_{(K, \phi)}$.

The ghost ring of $B^A(G)$ is defined as the subring $\widetilde{B}^A(G) := \left( \prod_{K \leq G} \mathbb{Z}\text{Hom}(K, A) \right)^G$ of $G$-fixed points under this action. Since the elements $b_{(K, \phi)}$, $(K, \phi) \in \mathcal{M}_G^A$, form a $\mathbb{Z}$-basis of $\prod_{K \leq G} \mathbb{Z}\text{Hom}(K, A)$ that is permuted by $G$, their orbit sums $\tilde{b}_{(K, \phi)} := \sum_{g \in [G/N_G(K, \phi)]} \delta b_{(K, \phi)}$, where $(K, \phi)$ runs through a set of representatives of the $G$-orbits of $\mathcal{M}_G^A$ and $N_G(K, \phi)$ denotes the $G$-stabilizer of $(K, \phi)$, form a $\mathbb{Z}$-basis of $\widetilde{B}^A(G)$.

The natural projection map $\pi|_{[S_G]}: \prod_{K \leq G} \mathbb{Z}\text{Hom}(K, A) \rightarrow \prod_{K \in [S_G]} \mathbb{Z}\text{Hom}(K, A)$ is a ring homomorphism, and it is injective when restricted to $\widetilde{B}^A(G)$, since the components indexed by elements outside $[S_G]$ are determined by the components indexed by elements in $[S_G]$. Moreover, one has
\[ B^A(G) \cong \pi_{[S_G]} \left( B^A(G) \right) = \prod_{K \in [S_G]} (\text{ZHom}(K, A))^N_{G(K)}. \]

In fact, each \( K \)-component of a \( G \)-fixed point of \( \prod_{K \leq G} \text{ZHom}(K, A) \) must be an \( N_{G(K)} \)-fixed point. For the converse, note that for \( K \in [S_G] \), the \( N_{G(K)} \)-orbit sums of the elements \( \phi \in \text{Hom}(K, A) \) form a \( \mathbb{Z} \)-basis of \( (\text{ZHom}(K, A))^N_{G(K)} \) and that the \( N_{G(K)} \)-orbit sum of \( \phi \) is equal to the \( K \)-component of \( \tilde{b}_{(K, \phi)} \). We set \( B^A(G) := \prod_{K \in [S_G]} (\text{ZHom}(K, A))^N_{G(K)} \) and, for \( K \in [S_G] \) and \((K, \phi) \in M^A_G \), we set \( \tilde{b}_{(K, \phi)} := \pi_{[S_G]}(\tilde{b}_{(K, \phi)}) \). Thus, the \( K \)-component of \( \tilde{b}_{(K, \phi)} \) is equal to the \( N_{G(K)} \)-orbit sum of \( \phi \) and its \( L \)-component is 0 for all \( L \in [S_G] \) with \( L \neq K \).

By [5, 2.5], the map
\[ \Phi^A_G : B^A(G) \to \widetilde{B}^A(G), \quad [L, \psi]_G \mapsto \left( \sum_{\phi \in \text{Hom}(K, A)} \gamma^G_{(K, \phi), (L, \psi)}(\phi) \right)_{K \leq G}, \]
is an injective ring homomorphism, also known as the mark morphism, where
\[ \gamma^G_{(K, \phi), (L, \psi)} = |\{sL \in G/L \mid (K, \phi) \leq_s (L, \psi)\}|, \]
for \((K, \phi), (L, \psi) \in M^A_G \). Several properties of these numbers are listed in [1, Section 1] and in [6, Lemma 2.2]. We set \( \widetilde{\Phi}^A_G := \pi_{[S_G]} \circ \Phi^A_G : B^A(G) \to \prod_{K \in [S_G]} (\text{ZHom}(K, A))^N_{G(K)} \).

If \( H \) is another finite group, a ring isomorphism \( \Theta : B^A(G) \to B^A(H) \) is called a species isomorphism if \( \Theta([K, \phi]_G) = [R, \rho]_H \in H \setminus M^A_H \) for every \([K, \phi]_G \in G \setminus M^A_G \), see [6, Def. 3.1]. We will make use of the following theorem which is part of the statement of Theorem 3.14 in [6].

**Theorem 2.1.** Let \( A \) be an abelian group and let \( G \) and \( H \) be finite groups. There exists a species isomorphism from \( B^A(G) \) to \( B^A(H) \) if and only if there exist bijections \( \Theta_S : S_G \to S_H \) and \( \Theta_K : \text{Hom}(K, A) \to \text{Hom}(\Theta_S(K), A) \), for \( K \leq G \), satisfying the following two conditions:

1. \( \gamma^H_{(\Theta_S(K), \Theta_K(\phi)), (\Theta_S(L), \Theta_L(\psi))} = \gamma^G_{(K, \phi), (L, \psi)} \) for all \((K, \phi), (L, \psi) \in M^A_G \).
2. The group homomorphism \( \widetilde{\Theta} : \widetilde{B}^A(G) \to \widetilde{B}^A(H) \) determined by mapping \( \tilde{b}_{(K, \phi)} \) to \( \tilde{b}_{(\Theta_S(K), \Theta_K(\phi))} \), for \((K, \phi) \in M^A_G \), is a ring isomorphism.

**Remark 2.2.** In order to clarify the statement in Theorem 2.1, we will show that the map \( \widetilde{\Theta} \) in (b) is well-defined, provided that the condition in (a) holds. That is, if \((K, \phi), (L, \psi) \in M^A_G \) are \( G \)-conjugate, then \( \tilde{b}_{(\Theta_S(K), \Theta_K(\phi))} = \tilde{b}_{(\Theta_S(L), \Theta_L(\psi))} \in B^A(H) \). For this it suffices to show that \((\Theta_S(K), \Theta_K(\phi)) \) and \((\Theta_S(L), \Theta_L(\psi)) \) are \( H \)-conjugate. By part 2 of [6, Lemma 2.2], \((K, \phi) =_G (L, \psi)\) if and only if \( \gamma^G_{(K, \phi), (L, \psi)} = 0 \). Thus, the condition in (a) implies that \((K, \phi) =_G (L, \psi)\) if and only if \((\Theta_S(K), \Theta_K(\phi)) =_H (\Theta_S(L), \Theta_L(\psi))\).

For later use in the proof of Theorem 2.3 we note that if \( K =_G L \) then \((K, 1) =_G (L, 1)\) and, by the above, \((\Theta_S(K), \Theta_K(1)) =_H (\Theta_S(L), \Theta_L(1))\), which implies that \( \Theta_S(K) =_H \Theta_S(L) \). Conversely, if \( \Theta_S(K) =_H \Theta_S(L) \), then \((K, \Theta_K^{-1}(1)) =_G (L, \Theta_L^{-1}(1))\), implying \( K =_G L \). Thus, the condition in (a) also implies that \( K =_G L \) if and only if \( \Theta_S(K) =_H \Theta_S(L) \).

Next we prove a slight modification of the above theorem.

**Theorem 2.3.** Let \( A \) be an abelian group and let \( G \) and \( H \) be finite groups. Then there exists a species isomorphism from \( B^A(G) \) to \( B^A(H) \) if and only if for any given \([S_G] \) and \([S_H]\), there are bijections \( \theta_{[S]} : [S_G] \to [S_H] \) and \( \theta_K : \text{Hom}(K, A) \to \text{Hom}(\theta_{[S]}(K), A) \) for \( K \in [S_G] \), satisfying the following two conditions:

1. \( \gamma^H_{\theta_{[S]}(K), \theta_K(\phi)), (\theta_{[S]}(L), \theta_L(\psi))} = \gamma^G_{(K, \phi), (L, \psi)} \) for all \( K, L \in [S_G], \phi \in \text{Hom}(K, A) \) and \( \psi \in \text{Hom}(L, A) \).
(b) The map $\overline{\theta}: B^k(G) \to B^k(H), \overline{b}(\theta, \phi) \mapsto \overline{b}(\theta S, K, \phi)$, for $K \in [S_G]$ and $\phi \in \text{Hom}(K, A)$, is a ring isomorphism.

**Remark 2.4.** With the same arguments as in Remark 2.2 one can show that in the situation of Theorem 2.3 the condition in (a) implies that the map $\overline{\theta}$ in (b) is well-defined. More precisely, (a) implies that for any $K \in [S_G]$, $\phi, \psi \in \text{Hom}(K, A)$, one has $(K, \phi) = G (K, \psi)$ if and only if $(\theta S, \theta K(\phi)) = H (\theta S, \theta K(\psi))$.

**Proof of Theorem 2.3.** First suppose that there exists a species isomorphism from $B^A(G)$ to $B^A(H)$. Then there exist bijections $\theta_S$ and $\theta_K$ for $K \leq G$ as in Theorem 2.1, satisfying the conditions (a) and (b) in Theorem 2.1. Let $[S_G]$ and $[S_H]$ be given. Then for every $K \in [S_G]$ there exists $h_K \in H$ such that $h_K \theta_S(K) \in [S_H]$. We define $\theta'_S: [S_G] \to [S_H]$ by $K \mapsto h_K \theta_S(K)$. The second part of Remark 2.2 implies that $\theta'_S$ is a bijection. Next, for $K \in [S_G]$, we define $\theta'_K: \text{Hom}(K, A) \to \text{Hom}([S_S](K), A)$ by $\phi \mapsto h_K \theta_K(\phi)$, noting that $\theta'_S(K) = h_K \theta_S(K)$. With $\theta_K$ being a bijection, also $\theta'_S$ is a bijection. By construction, we have $(\theta'_S(K), \theta'_K(\phi)) = \theta(S, \sigma)$ in $M^A_H \sigma$ for any $H$-conjugate pairs, and since the original maps $\theta_S$ and $\theta_K$, for $K \leq G$, satisfied condition (a) in Theorem 2.1, the bijections $\theta'_S$ and $\theta'_K$, for $K \in [S_G]$, now satisfy condition (a) in Theorem 2.3. Let $\overline{\theta}: B^A(G) \to B^A(H)$ denote the ring isomorphism associated to the bijections $\theta_S$ and $\theta_K$ for $K \leq G$ in Theorem 2.1(b) and let $\overline{\phi}: B^A(G) \to \overline{B}^A(H)$ denote the map associated to the bijections $\theta'_S$ and $\theta'_K$ for $K \in [S_G]$ in Theorem 2.3(b). Then, for all $K \in [S_G]$ and all $\phi \in \text{Hom}(K, A)$, we have

$$\overline{\phi} \circ \pi_S[G] = \pi_{[S_H]} \circ \overline{\theta}: B^A(G) \to \overline{B}^A(H).$$

Thus, $\overline{\phi} \circ \pi_S[G] = \pi_{[S_H]} \circ \overline{\theta}: B^A(G) \to \overline{B}^A(H)$. Since $\overline{\theta}, \pi_{[S_G]}: B^A(G) \to \overline{B}^A(G)$ and $\pi_{[S_H]}: \overline{B}^A(H) \to \overline{B}^A(H)$ are ring isomorphisms, also $\overline{\phi}$ is a ring isomorphism.

Conversely, for each $[K, \phi]_G \in G \setminus M^A_H$, we can assume $K \in [S_G]$, and mapping $[K, \phi]_G$ to $[\theta(S, \sigma)]_H$ gives a bijection from $G \setminus M^A_H$ onto $H \setminus M^A_H$. This bijection extends to an isomorphism of abelian groups $\Theta: B^A(G) \to B^A(H)$. Now the diagram

$$\begin{array}{ccc}
B^A(G) & \xrightarrow{\Theta} & B^A(H) \\
\overline{\phi} \downarrow & & \downarrow \overline{\pi}_H \\
\overline{B}^A(G) & \xrightarrow{\Theta} & \overline{B}^A(H)
\end{array}$$

commutes, and since the bottom and the vertical arrows are injective ring homomorphisms, also $\Theta$ is a ring isomorphism.

As remarked in [6, Cor. 3.12], some of the $\theta_K$ are necessarily group isomorphisms, but we ignore whether this has to be the case for all these maps. However, when the $\theta_K$ are isomorphisms, we can drop Condition (b) in Theorem 2.3.

**Proposition 2.5.** Let $A$ be an abelian group and let $G$ and $H$ be finite groups. Assume that there is a bijection $\theta_S: [S_G] \to [S_H]$ and, for each $K \in [S_G]$, a group isomorphism $\theta_K: \text{Hom}(K, A) \to \text{Hom}(\theta(S, \sigma), A)$ such that

$$\gamma_H^H(\theta(S, \sigma), \theta_K(\phi)) = \gamma_G^G(\theta(S, \sigma), \theta_K(\phi)).$$
for all $K, L \in [S_G], \phi \in \text{Hom}(K, A), \psi \in \text{Hom}(L, A)$. Then the assignment $[K, \phi]_G \mapsto [\theta_{[S]}(K), \theta_K(\phi)]_H$ for $K \in [S_G]$ and $\phi \in \text{Hom}(K, A)$ extends to a species isomorphism $\Theta : B^A(G) \to B^A(H)$.

**Proof.** For $K \in [S_G]$ and $\phi \in \text{Hom}(K, A)$, the assignment $\overline{b}_{(K, \phi)} \mapsto \overline{b}_{(\theta_{[S]}(K), \theta_K(\phi))}$ extends to an isomorphism $\overline{\Theta} : B^A(G) \to B^A(H)$ of abelian groups. Then the diagram

$$
\begin{array}{ccc}
B^A(G) & \xrightarrow{\overline{\Theta}} & B^A(H) \\
\downarrow & & \downarrow \\
\prod_{K \in [S_G]} \mathbb{Z}\text{Hom}(K, A) & \xrightarrow{(\theta_K)} & \prod_{L \in [S_H]} \mathbb{Z}\text{Hom}(L, A)
\end{array}
$$

where $\theta_K : \mathbb{Z}\text{Hom}(K, A) \to \mathbb{Z}\text{Hom}(\theta_{[S]}(K), A)$ is the $\mathbb{Z}$-linear extension of $\theta_K$ and the vertical arrows are the inclusions, commutes by Remark 2.4. Note that the bottom map is a ring isomorphism, since each $\theta_K$ was a group isomorphism. Therefore, as the bottom and the vertical arrows are injective, also $\overline{\Theta}$ is a ring isomorphism. By Theorem 2.3 and its proof, the maps $\theta_{[S]}$ and $\theta_K$ determine a species isomorphism. \hfill \square

### 3. Nontrivial counterexamples

We recall the construction of Thévenaz’ counterexamples in [7]. Let $p$ and $q \geq 3$ be prime numbers such that $q \mid (p-1)$, and take elements $a \neq b$ of order $q$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $P_a = \mathbb{Z}/p\mathbb{Z} = \langle x \rangle, P_b = \mathbb{Z}/p\mathbb{Z} = \langle y \rangle$ and $Q = C_q = \langle z \rangle$, then let $Q$ act on $P_a \oplus P_b$ by $x \mapsto ax$ and $y \mapsto by$, and consider the resulting semidirect product $G(a, b) = (P_a \oplus P_b) \rtimes Q$. A complete set of representatives of the conjugacy classes of subgroups of $G(a, b)$ is $\{1\}, P_a, P_b, P(j) = \langle x + jy \rangle$ for $j \in \{([\mathbb{Z}/p\mathbb{Z}]^\times / \langle a \rangle), P_a \oplus P_b, Q, P_a \rtimes Q, P_b \rtimes Q \}$ and $G(a, b)$. Taken in this order, the table of marks of $G(a, b)$ is independent of the choice of $(a, b)$. For fixed $p$ and $q$, there are precisely $\frac{q-1}{2}$ isomorphism classes of these groups; in fact, if also $c \neq d$ are elements of order $q$ in $(\mathbb{Z}/p\mathbb{Z})^\times$ then $G(a, b) \cong G(c, d)$ if and only if there exists $n \in \{1, \ldots, q-1\}$ such that $(c, d) = \{a^n, b^n\}$ (see [7]). Note that there are infinitely many choices for such $p$ and $q$: taking any prime $q \geq 3$, then by Dirichlet’s theorem there are infinitely many primes $p$ in the arithmetic progression $1 + q, 1 + 2q, 1 + 3q, \ldots$.

**Theorem 3.1.** Let $p$ and $q \geq 3$ be primes with $q \mid (p-1)$ and let $a \neq b$ and $c \neq d$ be elements of order $q$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. If $A$ has trivial $p$-torsion, then $B^A(G(a, b))$ and $B^A(G(c, d))$ are isomorphic rings.

**Proof.** Since we already know that these groups have isomorphic Burnside rings, we can assume that $A$ has elements of order $q$. For simplicity, set $G := G(a, b)$ and $H := G(c, d)$ and take $[S_G]$ and $[S_H]$ as in the first paragraph of this section. We let $\theta_{[S]} : [S_G] \to [S_H]$ be the obvious bijection inducing an isomorphism of the tables of marks and set $K' := \theta_{[S]}(K)$ for $K \in [S_G]$. Next we define group isomorphisms $\theta_K : \text{Hom}(K, A) \to \text{Hom}(K', A)$ for $K \in [S_G]$; if $K$ is a $p$-subgroup so is $K'$, and $\text{Hom}(K, A)$ and $\text{Hom}(K', A)$ are trivial, hence there is only one choice for $\theta_K$; if $K$ is not a $p$-subgroup, then a homomorphism $\phi : K \to A$ is determined by the value $\phi(z)$ which is either an element of order $q$ or 1, and we define $\phi' := \theta_K(\phi) : K' \to A$ by requiring $\phi'(z) = \phi(z)$.

We now compare $\gamma^G_{(K, \phi), (L, \psi)}$ and $\gamma^H_{(K', \phi'), (L, \psi')}$ for $K, L \in [S_G], \phi \in \text{Hom}(K, A)$ and $\psi \in \text{Hom}(L, A)$. First, since $\theta_{[S]}$ preserves the marks, we have

$$
\gamma^G_{(K, 1), (L, 1)} = |(G/L)^K| = |(H/L')^{K'}| = \gamma^H_{(K', 1), (L', 1)},
$$

for all $K$ and $L$ in $[S_G]$. Note that the subgroups $P_a, P_b$ and $P_a \oplus P_b$ are normal, while $N_G(P(j)) = P_a \oplus P_b$, and $P_a \rtimes Q, P_b \rtimes Q$ and $Q$ are self-normalizing. Moreover, it is straightforward to verify that, for $K$ and $L$ in $[S_G]$, if $K \not\subseteq L$ then $K \not\subseteq L$, and since $\theta_{[S]}$ preserves containments, then $\gamma^G_{(K, \phi), (L, \psi)} = \gamma^H_{(K', \phi'), (L, \psi')} = 0$ whenever $K \not\subseteq L$. 


Therefore, we are left with the case when $K \leq L$, $L$ is not a $p$-subgroup and $\psi \neq 1$, and we distinguish the following cases for $K$:

(i) If $K \in \{\{1\}, P_a, P_b, P_a \oplus P_b\}$ then $K \leq G$ and $\phi = 1$. Therefore $K \leq GL$ and $\delta \psi|_K = 1$ for all $g \in G$ and hence

$$\gamma^G_{(K,1),(L,\psi)} = [G : L] = [H : L'] = \gamma^H_{(K',1),(L',\psi')}.$$

(ii) If $K = P(j)$ we may assume that $L = G$. In this case, $\gamma^G_{(K,1),(G,\psi)} = 1 = \gamma^H_{(K',1),(H,\psi')}.$

(iii) If $K$ is not a $p$-subgroup, then $K, L \in \{Q, P_a \rtimes Q, P_b \rtimes Q, G\}$, and in particular $z \in K \leq L$. It is straightforward to verify that in all cases for $L$ the following holds: if $g \in G$ with $g \notin L$ then $z \notin GL$ and therefore $K \nleq GL$. We can conclude that

$$\gamma^G_{(K,\phi),(L,\psi)} = \begin{cases} 1 & \text{if } \phi(z) = \psi(z), \\ 0 & \text{otherwise}, \end{cases}$$

and by the way we have defined $\theta_K$, we have that $\gamma^H_{(K',\phi'),(L',\psi')} = \gamma^G_{(K,\phi),(L,\psi)}$.

We conclude that $\theta_{[S]}$ and the isomorphisms $\theta_K$ for $K \in [S_G]$ satisfy the condition of Proposition 2.5, hence they determine a species isomorphism.

\[ \square \]

**Remark 3.2.** The above theorem shows that Thévenaz’ counterexamples to the isomorphism problem for the Burnside ring are also non-trivial counterexamples for the $A$-fibered Burnside ring if $A$ is any abelian group with trivial $p$-torsion and having elements of order $q$. In particular, we have a negative answer to the isomorphism problem of the $C_q$-fibered Burnside ring for any prime $q \geq 5$. Moreover, for any field $k$ of characteristic $p$, we obtain $D^k(G(a,b)) \cong D^k(G(c,d))$, since in this case $D^k(G(a,b)) \cong B^C_k(G(a,b))$. However, the existence of non-isomorphic finite groups $G$ and $H$ with $D^C(G) \cong D^C(H)$ remains open.

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