Estimating the Propagation of Several Cascading Outages with Multi-type Branching Processes

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Abstract—Branching processes can be applied to simulated cascading data to describe statistics of the cascades and quickly predict the distribution of blackout sizes. In this paper we apply the multi-type branching process to the line outage data and discretized load shed data. The branching process parameter such as the offspring mean matrix is estimated from simulated cascades by the Expectation Maximization (EM) algorithm and the multi-type branching process is then used to estimate the joint distribution of two blackout sizes based on the Lagrange-Good inversion. We test the estimated distributions with line outage and load shed data generated by the AC OPA cascading simulations on the IEEE 118-bus system.

Index Terms—Blackout, cascading failure, EM algorithm, joint distribution, Lagrange-Good inversion, multi-type branching process, power transmission reliability, simulation.

I. INTRODUCTION

Cascading blackouts are complicated sequences of dependent outages that lead to load shedding. They are rare and high impact events with substantial risk and pose great challenges in simulation, analysis, and mitigation [1]. Simulations of cascading outages from various cascading failure models, such as OPA model [2]–[5], AC OPA model [6], [7], OPA with slow process [8], can produce many samples of cascades. It is useful to be able to statistically describe these simulated cascades with high-level probabilistic models such as branching processes [9], [10], which have descriptive parameters that characterize the system resilience to cascading and can be used to efficiently predict the probability distribution of blackout size. First estimating the parameters of a branching process from a shorter simulation run and then predicting the distribution of blackout size using the branching process is much more time efficient than empirically estimating the distribution of blackout size which usually requires much longer simulation in order to accumulate enough cascades.

The branching processes have been validated by different simulations and real data. It is found that branching processes can match the distribution of number of line outages [11] and load shed [12] simulated by the OPA simulation on the IEEE 118-bus and 300-bus systems, the distribution of load shed for the TRELSS simulation on an industrial system of about 6250 buses [12], and the distribution of the number of cascading line outages in real data [13], [14].

Discrete blackout size, such as the number of line outages, can be directly analyzed by Galton-Watson branching process while continuously varying data such as load shed cannot. Analyzing the load shed data with the branching process is important because load shed is measure of blackout size that is of great significance to both utilities and society, whereas line outages are of direct interest only to utilities. In [12] the load shed data is discretized so that it becomes integer multiples of the chosen discretization unit and then the discretized load shed data is processed with a Galton-Watson branching process, which works with nonnegative integers. However in [12], the choice of the discretization unit is ad hoc and no systematic approach is given. To solve this problem, in [15] a systematic procedure is proposed to discretize continuously varying load shed data so that it can be analyzed as a Galton-Watson branching process.

Till now the branching process is only used to describe the blackout propagation in terms of one type of blackout size. But in real blackouts there are several different kinds of blackout size that can be used to track the cascading, such as the number of line outages, the number of tripped generators, the amount of load shed, and the number of customers affected. More importantly, these different kinds of blackout sizes are interdependent with each other and the propagation of cascading can be better understood when these different blackout sizes can be described at the same time, in which case their mutual influence can thus be analyzed.

In this paper, after discretizing the load shed data by using the method in [15] we apply Galton-Watson multi-type branching processes to analyze the overall statistics of the cascading of line outages and load shed simultaneously. The parameters of the multi-type branching process are estimated and used to quantify how line outages and load shed cause each other and the joint distributions of their total sizes are estimated by using multi-type branching processes.

The rest of this paper is organized as follows. Section II briefly introduces the multi-type Galton-Watson branching processes. Section III explains estimating branching process parameters. Section IV discusses the estimation of the joint distribution of the total size of various types with multi-type branching processes. Section V tests the proposed method with line outage data and load shed data generated by AC OPA simulation on IEEE 118-bus system. Finally the conclusion is drawn in Section VI.

II. MULTI-TYPE BRANCHING PROCESSES

An informal overview of branching processes, especially the multi-type branching processes, is given in this section...
and more details can be found in [11]–[15] and [9], [10].

As a high-level probabilistic model, the Galton-Watson branching process can describe how the number of outages in a blackout propagate. Its simplicity allows a high-level understanding of the cascading process without getting entangled in the various and complicated mechanisms of cascading, thus providing an alternative perspective that can be seen as complementary to detailed modeling of cascading outage mechanisms. When only one type of blackout size is considered, the propagation can be described in the following way. The initial outages propagate randomly to produce subsequent outages in generations and each outage in each generation (a “parent” outage) independently produces a random nonnegative integer number of outages (“children” outages) in the next generation. The distribution of the number of children from one parent is called the offspring distribution. The children outages then become parents to produce another generation until the number of outages in a generation becomes zero. The mean of the offspring distribution is the parameter $\lambda$, which is the average number of children outages for each parent outage and can quantify the tendency for the cascade to propagate in the way that larger $\lambda$ tends to cause the outages to grow faster. In the case of cascading failures in power systems $\lambda < 1$ and the outages will always eventually die out.

Galton-Watson multi-type branching process is a generalization of simple one-type Galton-Watson branching process that only considers one type of object by considering several types of objects. For cascading failures in power grid several types of objects can be different types of blackout size used to track the cascading. Different from the one-type branching process, for multi-type branching process, each outage of a type $i$ in one generation (a type $i$ “parent” outage) independently produces a random nonnegative integer number of outages of the same type (type $i$ “children” outages) and any other type (type $k$ “children” outages where $k \neq i$). All generated outages in different types comprise the next generation. The process ends if the numbers of generated outages in all types become zero. There are several offspring distributions, each of which is the distribution of the number of one type of children from one type of parent. Correspondingly, the parameters of multi-type branching processes become several offspring means and can be arranged into a matrix, which is named the offspring mean matrix $\Lambda$. Also the criticality of the multi-type branching process will not be directly determined by the offspring mean $\lambda$ in the one-type branching process but will be determined by the largest eigenvalue of $\Lambda$. The multi-type branching process will always eventually extinct if the largest eigenvalue of $\Lambda$ is less than or equal to 1.

The intent of the branching process modeling is not that each parent outage in some sense “causes” its children outages; the branching process simply produces random numbers of outages in each generation that can statistically match the outcome of the cascading [15]. For example, when used to track the number of line outages or the amount of load shed, the branching process does not specify which lines outage or which load is shed, or where the tripped lines or shed load are, or explain why the lines are tripped or the load is shed. This is different from OPA model and its variants [2]–[8], which retains information about the network topology, load flow, and the operator’s response, or the line interaction graph [10] or the interaction model [17], which aims at quantifying the interactions between components in the system. The branching process only describes the statistics of the number of outages in each generation and the statistics of the total number of outages. Since the underlying cascading processes that we are tracking can be complicated and varied (for example, there are situations in which load shed tends to inhibit the chance of further cascading and there are other situations in which load shed can tend to increase the chance of further cascading), it is not obvious that branching processes can summarize this complexity. In this paper, we show evidence that this can be done by simultaneously describing the statistics of line outages and load shed in each generation using multi-type branching process, summarizing the cascade propagation, and statistically estimating the joint distribution of the total line outages and total load shed.

Although the branching process model does not directly represent any of the physics or mechanisms of the outage propagation, after it is validated it can be used to predict the total number of outages. The parameters of a branching process model can be estimated from a much smaller data set, and then predictions of the total number of outages can be made based on the estimated parameters. The ability to do this via the branching process model with much less data is a significant advantage that enables practical applications.

### III. Estimating Multi-type Branching Process Parameters

In this section, we explain how the multi-type branching process parameters are estimated from the simulated data. The simulation of OPA and its variants can naturally produce line outages or load shed in generations; each iteration of the “main loop” of the simulation produces another generation. In each generation the number of line outages are counted and the continuously varying load shed amounts are processed and discretized as described in [15] to produce integer multiples of the chosen discretization unit.

A total of $M$ cascades are simulated to produce nonnegative integer data that can be arranged as

$$
\begin{align*}
\text{generation 0} & \quad \text{generation 1} & \quad \cdots \\
\text{cascade 1} & (Z_{0,1}^{1,1}, Z_{0,2}^{1,2}) & (Z_{1,1}^{1,1}, Z_{1,2}^{1,2}) & \cdots \\
\text{cascade 2} & (Z_{0,1}^{2,1}, Z_{0,2}^{2,2}) & (Z_{1,1}^{2,1}, Z_{1,2}^{2,2}) & \cdots \\
\vdots & \vdots & \vdots & \cdots \\
\text{cascade } M & (Z_{0,1}^{M,1}, Z_{0,2}^{M,2}) & (Z_{1,1}^{M,1}, Z_{1,2}^{M,2}) & \cdots
\end{align*}
$$

where $Z_{g}^{m,t}$ is the number of type $t$ outages produced by the simulation in generation $g$ of cascade number $m$. Each cascade has a nonzero number of outages in generation 0 for at least one type of outages. The shortest cascades stop in generation 1 by having no outages in generation 1 and
higher generations, but some of the cascades will continue for several or occasionally many generations before terminating. The assumption of a positive number of outages in generation 0 implies that all statistics assume a cascade starting. Note that in generation 0 the number of line outages is always nonzero but the discretized load shed can be zero; in the following generations both can be zero.

For \( n \)-type branching processes where \( n \geq 2 \) the average propagation or offspring mean \( \lambda \) will be generalized to the offspring mean matrix \( \Lambda \). As a special case the offspring mean matrix of the two-type branching process can be written as

\[
\Lambda = \begin{bmatrix}
\lambda_{11} & \lambda_{12} \\
\lambda_{21} & \lambda_{22}
\end{bmatrix}
\]

For branching processes with only one type the criticality is directly determined by the offspring mean \( \lambda \). By contrast, the criticality of multi-type branching processes is determined by the largest eigenvalue of \( \Lambda \), which is denoted by \( \rho \). If \( \rho \leq 1 \) the multi-type branching process will always eventually extinct. If \( \rho > 1 \) then the multi-type branching process will extinct with a probability \( 0 \leq q < 1 \) [10].

The largest eigenvalue \( \rho \) of the mean matrix can be estimated as the total number of all types of children divided by the total number of all types of parents by directly using the simulated cascades and ignoring the types [18]

\[
\hat{\rho} = \frac{M}{\sum_{m=1}^{\infty} \sum_{g=1}^{\infty} Z_{g}^{m,t}}.
\]

When the number of type \( j \) children to type \( i \) parents \( Y^{(i,j)} \) and the total number of type \( i \) parents \( Y^{(i)} \) are observed, \( \lambda_{ij} \) can be estimated by a maximum likelihood estimator that is the total number of type \( j \) children produced by type \( i \) parents divided by the total number of type \( i \) parents [19]

\[
\hat{\lambda}_{ij} = \frac{Y^{(i,j)}}{Y^{(i)}}
\]

where \( Y^{(i,j)} \) and \( Y^{(i)} \) can be described by using the simulated cascades as

\[
Y^{(i,j)} = \sum_{m=1}^{\infty} \sum_{g=1}^{\infty} Z_{g}^{m,i\rightarrow j}
\]

\[
Y^{(i)} = \sum_{m=1}^{\infty} \sum_{g=0}^{\infty} Z_{g}^{m,i}
\]

where \( Z_{g}^{m,i\rightarrow j} \) is the number of type \( j \) offspring generated by type \( i \) parents in generation \( g \) of cascade \( m \).

However, normally it is impossible to have so detailed information. In the case of cascading blackouts it is difficult to exactly determine the number of line outages or the amounts of load shed that are produced by line outages or load shed due to too many mechanisms for cascading outages. In other words, \( Z_{g}^{m,i\rightarrow j} \) in (3) cannot be determined and thus \( Y^{(i,j)} \) cannot be decided and the mean matrix cannot be estimated.

In order to solve this problem, we first assume the offspring distributions of branching processes are Poisson. There are general arguments suggesting that the choice of a Poisson offspring distribution is appropriate [12], based on the offspring outages being selected from a large number of possible outages that have small probability and are approximately independent. After the load shed data is discretized a Poisson offspring distribution is also used in [13] and the distribution of total size can be effectively predicted under this assumption.

The offspring mean matrix can be estimated by using the Expectation Maximization (EM) algorithm [20], which is a method for finding maximum likelihood estimates of parameters in statistical models, where the model depends on unobserved latent variables.

The EM algorithm mainly contains two steps, which are E-step and M-step. In our case of estimating the offspring mean matrix for two-type branching processes the EM algorithm can be formulated as follows.

1. **Initialization:** Set initial guess for the mean matrix as \( \hat{\Lambda}^{(0)} \). Since for cascading failures in power systems the outages will always eventually die out we have \( 0 \leq \lambda_{ij} \leq 1 \). Based on this all elements of the initial mean matrix are set to be the mid point of their possible ranges as

\[
\hat{\Lambda}^{(0)} = \begin{bmatrix}
0.50 & 0.50 \\
0.50 & 0.50
\end{bmatrix}
\]

2. **E-step:** Estimate \( Y^{(i,j)(k+1)} \) based on \( \hat{\Lambda}^{(k)} \).

Since the method for the estimation of \( Y^{(i,j)} \) for different \( i \) and \( j \) are the same we only take estimating \( Y^{(1,1)} \) as an example. Under the assumption that the offspring distributions are all Poisson, for generation \( g \geq 1 \) of cascade \( m \), the number of type 1 offspring produced by type 1 and type 2 parents separately follow Poisson distribution

\[
Z_{g}^{m,1\rightarrow 1} \sim \text{Pois}(Z_{g-1}^{m,1}\lambda_{11})
\]

\[
Z_{g}^{m,2\rightarrow 1} \sim \text{Pois}(Z_{g-1}^{m,2}\lambda_{21})
\]

Thus the probability that \( k \) of \( Z_{g}^{m,1\rightarrow 1} \) are produced by type 1 parents and the rest are produced by type 2 parents can be written as

\[
P(Z_{g}^{m,1\rightarrow 1} = k, Z_{g}^{m,2\rightarrow 1} = Z_{g}^{m,1} - k)
\]

\[
= (Z_{g-1}^{m,1}\lambda_{11})^{k} (Z_{g-1}^{m,2}\lambda_{21})^{Z_{g}^{m,1} - k} e^{-Z_{g}^{m,1}\lambda_{11} - Z_{g}^{m,2}\lambda_{21}}
\]

\[
\frac{k! (Z_{g}^{m,1} - k)!}{k! (Z_{g}^{m,1} - k)!}
\]

Then the expected number of type 1 offspring generated by type 1 parents for generation \( g \geq 1 \) of cascade \( m \) can be calculated as

\[
Z_{g}^{m,1\rightarrow 1} = \sum_{k=0}^{Z_{g}^{m,1}} P(Z_{g}^{m,1\rightarrow 1} = k, Z_{g}^{m,2\rightarrow 1} = Z_{g}^{m,1} - k) / P_{\text{total}},
\]

\[g = 1, 2, \ldots \]

where \( P_{\text{total}} \) is the total probability for all \( Z_{g}^{m,1\rightarrow 1} \) cases and can be written as

\[
P_{\text{total}} = \sum_{k=0}^{Z_{g}^{m,1}} P(Z_{g}^{m,1\rightarrow 1} = k, Z_{g}^{m,2\rightarrow 1} = Z_{g}^{m,1} - k).
\]
After obtaining $Z_g^{m_1 \cdots m_r}$ for all generations $g \geq 1$ of all cascades we are finally able to calculate $Y^{(1)}$ by using (7).

3) **M-step:** Estimate $\hat{\Lambda}^{(k+1)}$ based on $Y^{(i,j)}(k+1)$. After obtaining $Y^{(i,j)}(k+1)$ the updated mean matrix $\hat{\Lambda}^{(k+1)}$ can be estimated with the estimator given in (2).

4) **End:** Iterate the E-step and M-step until
\[
\max_{i,j \in [1,2]} |\hat{\lambda}_{ij}^{(k+1)} - \hat{\lambda}_{ij}^{(k)}| < \epsilon
\]
(5)
where $\epsilon$ is the tolerance that can be used to control the accuracy and $\hat{\lambda}_{ij}^{(k+1)}$ is the final estimate of $\lambda_{ij}$.

Note that it is desirable to have large initial outages and large number of cascades in order to get a reliable estimate. However, when the number of total initial outages is determined by the simulated cascades larger initial outages will surely lead to smaller number of cascades. Thus there is a tradeoff between them. Since the initial outages of the simulated cascades are not big enough we group the cascades to get cascades with larger initial outages. Similar to [15], every 20 cascades are grouped together to be one cascade. Then the EM algorithm is applied to the grouped cascades to obtain the offspring mean matrix.

**IV. ESTIMATING JOINT PROBABILITY DISTRIBUTION OF TOTAL BLACKOUT SIZES**

This section discusses how to estimate the joint distribution of two types of blackout size and we are most interested in statistics of the total number of outages produced by the cascades. The probability generating function for the type $i$ individual of a $n$-type branching process is
\[
f_i(s_1, \cdots, s_n) = \sum_{u_1, \cdots, u_n=0}^{\infty} p_i(u_1, \cdots, u_n)s_1^{u_1} \cdots s_n^{u_n}
\]
(6)
where $p_i(u_1, \cdots, u_n)$ is the probability that a type $i$ individual generates $u_j$ type 1, $\cdots$, $u_n$ type $n$ individuals.

According to (10) and (21), the probability generating function, $w_i(s_1, \cdots, s_n)$, of the total number of various types in all generations, starting with one individual of type $i$, is given by
\[
w_i = s_i f_i(w_1, \cdots, w_n), \quad i = 1, \cdots, n
\]
(7)

When the branching process starts with more than one type of individuals the total number of various types can be determined by using the Lagrange-Good inversion in [21], in which the following theorem is given.

**Theorem 1:** If the $n$-type random branching process starts off with $r_1$ individuals of type 1, $r_2$ of type 2, etc., then the probability that the whole process will contain precisely $m_1$ of type 1, $m_2$ of type 2, etc., is equal to the coefficient of $s_1^{m_1-1} \cdots s_n^{m_n-1}$ in
\[f_1^{m_1} \cdots f_n^{m_n} \left| \partial^\nu - s_\mu \frac{\partial f_\mu}{\partial s_\nu} \right|.
\]
(8)

In Theorem 1 $|a_{\mu\nu}|$ denotes the determinant of the $n \times n$ matrix whose entry is $a_{\mu\nu}(\mu, \nu = 1, \cdots, n)$ and $\delta_{\mu}^{\nu}$ is Kronecker’s delta ($= 1$ if $\mu = \nu$, otherwise $= 0$). We denote the coefficient of $s_1^{m_1-1} \cdots s_n^{m_n-1}$ as $c(r_1, \cdots, r_n, m_1, \cdots, m_n)$.

Given the joint probability distribution of initial sizes $P(Z_0^1, \cdots, Z_0^n)$ and the generating functions in (6), the formula for calculating the joint probability distribution of the total number of various types $(Y_1^n, \cdots, Y_n^n)$ can then be written as
\[
P[Y_1^n = y_1, \cdots, Y_n^n = y_n]
= \sum_{z_0^1=0}^{\infty} \cdots \sum_{z_0^n=0}^{\infty} \left[ P(Z_0^n = z_0^n, Z_0^1 = z_0^1) \cdot c(z_0^1, \cdots, z_0^n; y_1, \cdots, y_n) \right].
\]
(9)

When we consider line outages and load shed the empirical joint probability distribution of the number of initial outages $(Z_0^1, Z_0^2)$ can be obtained as
\[
P[Z_0^1 = z_0^1, Z_0^2 = z_0^2]
= \frac{1}{M} \sum_{m=1}^{M} I[Z_0^{m,1} = z_0^1, Z_0^{m,2} = z_0^2]
\]
(10)
where the notation $I[event]$ is the indicator function that evaluates to one when the event happens and evaluates to zero when the event does not happen.

As in section II we assume the offspring distribution for various types are all Poisson. Then the probability generating functions for a two-type branching process can be written as
\[
f_1(s_1, s_2) = \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \lambda_1^{u_1} \lambda_2^{u_2} e^{-\lambda_1 - \lambda_2} u_1! u_2! s_1^{u_1} s_2^{u_2}
\]
(11)
\[
f_2(s_1, s_2) = \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \lambda_2^{u_1} \lambda_2^{u_2} e^{-\lambda_1 - \lambda_2} u_1! u_2! s_1^{u_1} s_2^{u_2}
\]
(12)
where the parameters $\lambda_1$, $\lambda_2$, $\lambda_12$, and $\lambda_22$ can be estimated by using the method in section III.

In (8) the $2 \times 2$ matrix whose determinant needs to be evaluated is actually
\[
\begin{vmatrix}
1 - s_1 \frac{\partial f_1}{\partial s_1} - s_2 \frac{\partial f_1}{\partial s_2} & s_1 \frac{\partial f_1}{\partial s_2} \\
-s_2 \frac{\partial f_2}{\partial s_2} & 1 - s_2 \frac{\partial f_2}{\partial s_2}
\end{vmatrix}
\]

The joint probability distribution of the two-type branching process for line outages and load shed can be obtained by evaluating (9) with elementary algebra. Since the coefficients in (11) and (12) will decrease very fast with the increase of the order of $s_1$ and $s_2$, we can use a few terms to approximate the generating functions to reduce the calculation burden while guaranteeing accurate enough results. Furthermore, the probability obtained by (9) will also decrease with the increase of $y_1$ and $y_2$. We do not need to calculate the negligible probability for too large blackout size. Specifically, we can only calculate the joint probability for
\[
y_1 = z_0^1 + z_0^2 + 1, \cdots, z_0^1 + \tau_1
\]
(13)
TABLE I
ESTIMATED PARAMETERS BY USING 5000 CASCADES

| load level | $\lambda_{\text{line}}$ | $\lambda_{\text{load}}$ | $\rho$ | $\Lambda$ |
|------------|-----------------|-----------------|------|--------|
| 1.0        | 0.40            | 0.43            | 0.50 | 0.42   | 0.26 |
|            |                 |                 |      | 0.017  | 0.11 |
| 1.2        | 0.52            | 0.58            | 0.59 | 0.51   | 0.14 |
|            |                 |                 |      | 0.043  | 0.35 |

and

$$y_2 = z_0^2, z_0^2 + 1, \ldots, z_0^2 + \tau_2$$

(14)

where $\tau_1$ and $\tau_2$ are integers that are properly chosen. Too small $\tau_1$ or $\tau_2$ will make the probability of some blackout sizes that cannot be neglected not calculated. Too large $\tau_1$ or $\tau_2$ will lead to unnecessary calculation for negligible probability of some blackout sizes.

V. RESULTS

This section presents results of the branching process parameters computed from simulated cascades and and the joint distributions of outages predicted from these parameters. The cascading outage data is produced by the open-loop AC OPA simulation [6, 7] on the IEEE 118-bus test system, which is standard except that the line flow limits are determined with the same method in [15]. The probability for initial line outage is $p_0 = 0.0001$ and the load variability $\gamma = 1.67$, which are the same as [11], [15].

For testing the multi-type branching process model, the simulation is run so as to produce 5000 cascading outages with a nonzero number of line outages at two load levels, the base case and 1.2 times the base case.

The parameters for estimating parameters and joint distributions of multi-type branching processes are set as follows. The $\epsilon$ in (5) is chosen as 0.01. In (11) and (12) the highest orders for both $s_1$ and $s_2$ are chosen as 4. In (13) and (14) both $\tau_1$ and $\tau_2$ are set to be 15 for base case load level and 20 for 1.2 times base case load level. The parameters are chosen for a tradeoff between accuracy and calculation burden. For higher load levels the cascading tends to be able to propagate further and thus the selected $\tau_1$ and $\tau_2$ are larger to get acceptable accuracy.

A. Basic Results

The estimated branching process parameters for both load levels are shown in Table I, where $\lambda_{\text{line}}$ and $\lambda_{\text{load}}$ are separately the estimated offspring mean for only considering line outages and load shed. We can see that the estimated largest eigenvalue of the offspring mean matrix $\rho$ is greater than the estimated offspring means for only considering one type of cascading outage, indicating that the system is closer to criticality when we consider line outages and load shed simultaneously. This is because the line outages and load shed can mutually influence each other and thus aggregate the propagation of cascading.

From the offspring mean matrix $\Lambda$ we can see that line outages tend to have a greater influence on load shed but the influence of load shed on line outages is relatively weak. This is reasonable since in real cascading blackouts it is more possible for line tripping to cause load shed. Sometimes line outages directly cause load shed, for example, the simplest case occurs when a load is fed from a radial line.

The empirical and estimated joint distributions of line outages and load shed are shown in Figs. 1-2 in which the numbers beside the rows and columns are the number of line outages and the number of counts of the discretization unit of load shed. The estimated marginal distributions for line outages and load shed are shown in Figs. 3-4. The marginal distributions of total outages (dots) and initial outages (squares) are shown, as well as a solid line indicating the total outages predicted by the multi-type branching process. The branching process data is also discrete, but is shown as a line for ease of comparison. The branching process prediction matches the empirical distribution well.

B. Separating Data for Fitting and Validating

A reasonable objection to the above-mentioned comparison is that the same data is used both to estimate the distribution and to obtain the empirical distribution. To address this objection we divided the data into separate fitting and
validation sets. Specifically, we estimate the distribution from the odd numbered cascades and compare with the empirical
distribution for the even numbered cascades for the base case, and vice versa for the load level 1.2 times the base case. The estimated parameters are listed in Table II and are similar to those obtained by using 5000 cascades as listed in Table I. The comparisons between estimated and empirical marginal distributions for line outages and load shed are shown in Figs. 7–10 and the matchings are also satisfactory.

| load level | $\hat{\lambda}_{\text{line}}$ | $\hat{\lambda}_{\text{load}}$ | $\hat{\rho}$ | $\hat{\Lambda}$ |
|------------|--------|--------|--------|---------|
| 1.0        | 0.40   | 0.41   | 0.48   | 0.41 0.20 |
|            |        |        |        | 0.023 0.094 |
| 1.2        | 0.52   | 0.58   | 0.60   | 0.51 0.16 |
|            |        |        |        | 0.043 0.34 |

### C. Efficiency

It has been shown for one-type branching process that it is much more time efficient to estimate the parameters of

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Fig. 3. Estimated marginal probability distributions of number of line outages by AC OPA on IEEE 118-bus system at base case load level by using 5000 cascades. Dots indicate total outages and squares indicate initial outages; both distributions are empirically obtained from the simulated cascades. The solid line indicates the distribution of total outages predicted with the multi-type branching process.

Fig. 4. Estimated marginal probability distributions of load shed by AC OPA on IEEE 118-bus system at base case load level by using 5000 cascades.

Fig. 5. Estimated marginal probability distributions of number of line outages by AC OPA on IEEE 118-bus system at load level 1.2 times the base case by using 5000 cascades.

Fig. 6. Estimated marginal probability distributions of load shed by AC OPA on IEEE 118-bus system at load level 1.2 times the base case by using 5000 cascades.

Fig. 7. Estimated marginal probability distributions of number of line outages by AC OPA on IEEE 118-bus system at base case load level by using 2500 cascades.

Fig. 8. Estimated marginal probability distributions of load shed by AC OPA on IEEE 118-bus system at load level 1.2 times the base case by using 2500 cascades.

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TABLE II

| load level | $\hat{\lambda}_{\text{line}}$ | $\hat{\lambda}_{\text{load}}$ | $\hat{\rho}$ | $\hat{\Lambda}$ |
|------------|--------|--------|--------|--------|
| 1.0        | 0.40   | 0.41   | 0.48   | 0.41 0.20 |
|            |        |        |        | 0.023 0.094 |
| 1.2        | 0.52   | 0.58   | 0.60   | 0.51 0.16 |
|            |        |        |        | 0.043 0.34 |

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It has been shown for one-type branching process that it is much more time efficient to estimate the parameters of
we show the efficiency improvement for estimating the joint distribution of the number of line outages and the amount of load shed by using multi-type branching processes compared with empirically estimating the joint distribution. The estimated parameters for branching processes by only using 600 cascades are listed in Table III and they are quite close to those estimated by using all 5000 cascades. The estimated marginal distributions for line outages and load shed are shown in Figs. 11–14 and they match very well with the marginal distributions empirically obtained by using 5000 cascades.

| TABLE III | ESTIMATED PARAMETERS BY USING 600 CASCADES |
|-----------|-------------------------------------------|
| load level| $\hat{\lambda}_{\text{line}}$ | $\hat{\lambda}_{\text{load}}$ | $\hat{\rho}$ | $\hat{\Lambda}$ |
| 1.0       | 0.39          | 0.45          | 0.51 | 0.41 0.31 |
|           |               |               |     | 0.022 0.13 |
| 1.2       | 0.52          | 0.57          | 0.59 | 0.51 0.13  |
|           |               |               |     | 0.045 0.36 |

For the above 6 cases the EM algorithm that is used to estimate the offspring mean matrix of the multi-type branching
In this paper we apply multi-type branching processes to statistically describe the propagation of two cascading outages, which are the number of line outages and the amount of load shed. We used AC OPA simulation on the IEEE 118-bus system to produce cascading outage data. After discretizing the load shed data, we estimated the offspring mean matrix and the average size of the initial outage that are parameters of a multi-type branching process model. We then used the multi-type branching process model to estimate the joint distributions of lines outages and load shed. The estimated distributions are close to the empirical distributions, suggesting that the Galton-Watson multi-type branching process model can capture some overall statistical aspects of the cascading of line outages and load shed. In addition to providing a useful summary describing cascade statistics, the branching process enables the parameters and the distribution of blackout size to be estimated with much fewer simulated cascades, which is a significant advantage since simulation time is a limiting factor when studying cascading blackouts.

VI. CONCLUSION

The outcomes demonstrated in this study reinforce the critical analysis that cascading failures are a complex phenomenon that can significantly influence the stability and reliability of power systems. The use of multi-type branching processes provides an effective tool for estimating the marginal probability distributions of cascading events, which can be used for probabilistic risk assessment and decision-making in power system planning and operation.

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