AN EXTENSION OF BONNET-MYERS THEOREM

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Abstract. We give a large generalization of the extensions of Bonnet-Myers theorem obtained by Calabi and also Cheeger-Gromov-Taylor.

The Bonnet-Myers states that a complete Riemannian manifold with Ricci curvature $\text{Ric}_M \geq \delta > 0$ is compact. In [1] Calabi extended this by proving that, if for some point $p \in M$ every geodesic starting from $p$ has the property that
\[
\limsup_{a \to \infty} \int_0^a \text{Ric}_M(s) \frac{1}{2} ds - \frac{1}{2} \ln(a) = \infty,
\]
then $M$ is compact. In particular, this implies that (4) page 137) $M$ is compact provided
\[
(0.1) \quad \text{Ric}_M(x) \geq \frac{1}{(4 - \epsilon)(1 + r)^2}
\]
for $d(p, x) = r$ and all $r \geq 0$. Cheeger, Gromov and Taylor (c.f. [2] theorem 4.8) also proved a similar result in same spirit. The basic idea of their proof is to research carefully the index form (or the second variation). The condition on Ricci curvature insures that the index form is negative. Then $p$ must meet conjugate point along any geodesic and $M$ has to be compact.

In this short note we extend largely Calabi and Cheeger-Gromov-Taylor’s results. The main theorem is

Theorem 0.1. Let $M^n$ be a complete Riemannian manifold. If there exists $p \in M, k \geq 2$ and $r_0 > 0$ such that
\[
(0.2) \quad \text{Ric}_M(x) \geq C(n, k, r_0) \frac{(n - 1)(1 + \epsilon r_0)}{(k - 2)} \cdot r_0^{k-2}
\]
for all $r \geq 0$, where $d(p, x) = r$ and $C(n, k, r_0)$ is a constant depending on $n, k, r_0$, then $M^n$ is compact. In our situation, $C(n, k, r_0)$ can be chosen to equal to $(n - 1) \cdot \frac{(k - 1)^2}{(k - 2)^2} \cdot r_0^{k-2}$ for $k > 2$ and $(n - 1)(1 + \frac{\epsilon r_0}{2})$, $\epsilon > 0$ for $k = 2$.

Since the case $k < 2$ is covered by [0.1] we only consider $k \geq 2$. One is easy to see that $0.1$ or $0.2$ covers the classical Bonnet-Myers theorem. If $\text{Ric}_M \geq \delta > 0$, we can rescale the metric such that $\delta$ is bigger than the right hand of $0.1$ or $0.2$.

From the angle of index form or the second variation, the index 2 in $0.1$ is the best possible. To guarantee that $p$ meets conjugate points this is necessary. But to show that a complete Riemannian manifold is compact, “meeting conjugate points” is not need. Showing that the manifold has no ray is enough! This is our starting point. We would make use of $0.2$ to show that $M^n$ contains no ray.
1. A proof of the main theorem

Assume that $M^n$ is noncompact. Then for any $p \in M$ there is a ray $\sigma(t)$ issuing from $p$. Let $r(x) = d(p, x)$ be the distance function from $p$. We denote $A = \text{Hess}(r)$ outside the cut locus and write $A(t) = A(\sigma(t))$. The Riccati equation is given by

$$A' + A^2 + R = 0. \tag{1.1}$$

The $A(t)$ is smooth except at $t = 0$. Taking the trace we have

$$trA'(s) + \|A(s)\|^2 + Ric(T) = 0, \tag{1.2}$$

where $T = \sigma'(s)$.

Integrate\(1.2\) over the interval $[\epsilon, t]$,

$$\int_\epsilon^t \|A(s)\|^2 ds = trA(\epsilon) - trA(t) - \int_\epsilon^t Ric_M(T) ds$$

$$< \frac{n - 1}{\epsilon} - \int_\epsilon^t \frac{C}{(r_0 + s)^k} ds$$

$$= \frac{n - 1}{\epsilon} - \frac{C}{k - 1} \left[ \frac{1}{(r_0 + \epsilon)^{k-1}} - \frac{1}{(r_0 + t)^{k-1}} \right].$$

The “$<$” holds from $0 \leq trA(t) < \frac{n-1}{\epsilon}$ and condition\(1.2\). We claim that $0 \leq trA(t) < \frac{n-1}{\epsilon}$. Since $Ric_M > 0, trA(t) < \frac{n-1}{\epsilon}$ holds. To see $trA(t) \geq 0$, one can consider the excess function

$$e(x) = d(p, x) + d(\sigma(i), x) - i.$$ 

$e(x) \geq 0$ and $e(\sigma(t)) \equiv 0$ for $0 \leq t \leq i$. So

$$\Delta e(\sigma(t)) = \Delta(d(p, x) + d(\sigma(i), x))|_{\sigma(t)} \geq 0.$$ 

We have

$$trA(t) = \Delta d(\sigma(0), x)|_{\sigma(t)} \geq -\Delta d(\sigma(i), x)|_{\sigma(t)} \geq -\frac{n - 1}{t - i}.$$ 

Let $\epsilon \rightarrow +\infty, trA(t) \geq 0$.

Let $t \rightarrow \infty$. The above integral inequality becomes

$$0 \leq \int_\epsilon^\infty \|A(s)\|^2 ds < \frac{(n - 1)}{\epsilon} - \frac{C}{(k - 1)(r_0 + \epsilon)^{k-1}}.$$ 

We observe that when $C$ is very large, it is a contradiction. So $M$ must be compact. Now we work out the constant $C$ we need. Solving

$$\frac{(n - 1)}{\epsilon} - \frac{C}{(k - 1)(r_0 + \epsilon)^{k-1}} \leq 0,$$

we obtain

$$C \geq (n - 1)(k - 1)(r_0 + \epsilon)^{k-1}. \tag{1.3}$$

When $\epsilon = \frac{r_0}{k-2}, k > 2, \frac{(n+\epsilon)^{k-1}}{\epsilon}$ achieves minimal value. Substituting it into\(1.3\) we have

$$C \geq (n - 1) \cdot \frac{(k - 1)^k}{(k - 2)^{k-2}} \cdot r_0^{k-2}.$$ 

So we can choose $C(n, k, r_0) = (n - 1) \cdot \frac{(k - 1)^k}{(k - 2)^{k-2}} \cdot r_0^{k-2}$ for $k > 2$. When $k = 2, C(n, 2, r_0) = (n - 1)(1 + \frac{r_0}{2})$ for any $\epsilon > 0$. 

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Remark 1.1. The estimate on the integral of $\|A(s)\|^2$ is inspired by Dai and Wei’s paper [3]. In their work on Toponogov type comparison for Ricci curvature, the related estimate of Hessian plays an important role.

References
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