Homogenization of multiscale Maxwell wave equations

Van Tiep Chu and Viet Ha Hoang

Division of Mathematical Sciences,
School of Physical and Mathematical Sciences,
Nanyang Technological University, Singapore 637371

Abstract

We study homogenization of multiscale Maxwell wave equation that depends on \( n \) separable microscopic scales in a domain \( D \subset \mathbb{R}^d \) on a finite time interval \((0, T)\). Due to the non-compactness of the embedding of \( H_0(\text{curl}, D) \) in \( L^2(D)^d \), homogenization of Maxwell wave equation can be significantly more complicated than that of scalar wave equations in the \( H^1(D) \) setting, and requires analysis uniquely for Maxwell wave equations. We employ multiscale convergence. The homogenized Maxwell wave equation and the initial condition are deduced from the multiscale homogenized equation. When the coefficient of the second order time derivative in the multiscale equation depends on the microscopic scales, the derivation is significantly more complicated, comparing to scalar wave equations, due to the corrector terms for the solution \( u^\varepsilon \) of the multiscale equation in the \( L^2(D)^d \) norm, which do not appear in the \( H^1(D) \) setting. For two scale equations, we derive an explicit homogenization error estimate for the case where the solution \( u_0 \) of the homogenized equation belongs to \( L^\infty((0, T); H^1(\text{curl}, D)) \). When \( u_0 \) only belongs to a weaker regularity space \( L^s((0, T); H^r(\text{curl}, D)) \) for \( 0 < s < 1 \), we contribute an approach to deduce a new homogenization error in this case, which depends on \( s \). For general multiscale problems, a corrector is derived albeit without an explicit homogenization error estimate. These correctors and homogenization errors play an essential role in deriving numerical correctors for approximating the solutions to the multiscale problems numerically, as considered in our forthcoming publication.

1 Introduction

We consider multiscale Maxwell wave equation (2.3) in a domain \( D \subset \mathbb{R}^d \) which depends on \( n \) microscopic scales. Using multiscale convergence (11, 12), we deduce the multiscale homogenized equation, from which we deduce the homogenized equation. In comparison to multiscale wave equation in the \( H^1(D) \) setting, homogenization of a multiscale Maxwell wave equation is quite different due to the embedding of the space \( H_0(\text{curl}, D) \) in \( L^2(D)^d \) is not compact (see 3). It is far more complicated to derive the homogenized equation, and the initial condition for the solution of the homogenized equation than for a multiscale scalar wave equation when the coefficient \( b^\varepsilon \) of the second order time derivative in (2.3) depends also on the microscopic scales. This is due to the correctors \( u_i \) for \( u^\varepsilon \) in \( L^2(D)^d \) in (2.4), and the second order time derivative \( \frac{\partial^2}{\partial t^2} \nabla_{y_i} u_i \) in (2.4) which do not belong to \( L^2(D \times Y_1 \times \ldots \times Y_n) \), and can only be understood in the distribution sense. Derivation of the initial condition requires representations of the time derivative \( \frac{\partial}{\partial t} \nabla_{y_i} u_i \) in terms of the solutions of the cell problems, which are quite non-trivial. The main contributions of the paper are the new correctors and homogenization errors in two scale problems, especially when the solution to the homogenized equation possesses low regularity, and a general corrector for multiscale problems. Their derivation are quite non-trivial, comparing to elliptic and wave equations in the \( H^1(D) \) setting.

As is well known, for two scale wave equations, a corrector similar to that of two scale elliptic problems does not hold in general, as the energy of the multiscale problem does not converge to the energy of the homogenized problem (see 3). Therefore, to derive a corrector, we restrict our consideration to the case where the initial condition \( g_0 \) in (2.3) equals zero. For two scale elliptic problems in a domain \( D \subset \mathbb{R}^d \), it is well known that a \( O(\varepsilon^{1/2}) \) homogenization rate of convergence in the \( H^1(D) \) norm can be derived when the solution of the homogenized equation is smooth (see 3 and 10). In polygonal domains, which
are of interests in numerical discretization, this regularity may not hold. However, if the domain is convex, the solution to the homogenized equation belongs to \(H^2(D)\); the \(O(\varepsilon^{1/2})\) homogenization rate still holds with the same proof. For wave equation, the \(O(\varepsilon^{1/2})\) homogenization rate is deduced in \([15]\) for scalar wave equations and in \([16]\) for elastic wave equations. In this paper, for two scale Maxwell wave equations, we derive the \(O(\varepsilon^{1/2})\) homogenization rate when the solution \(u_0\) of the homogenized equation belongs to \(L^\infty(0, T; H^1(\text{curl}, D))\). However, in polygonal domains, this regularity condition normally does not hold. The solution \(u_0\) of the homogenized equation only belongs to a weaker regularity space \(L^\infty(0, T; H^s(\text{curl}, D))\) for \(0 < s < 1\). We develop an approach to deduce a new homogenization error for this case of low regularity, using the ideas of \([3]\). For multiscale problems, an explicit homogenization error is not available. However, we can deduce a general corrector which requires a procedure quite different from that for multiscale wave equation in \([15]\) due to the corrector terms \(u_i\) in \((2.7)\). The correctors and homogenization errors in this wave paper play a key role in establishing numerical correctors using finite element solutions for the multiscale problems as studied in Chu and Hoang \([6]\).

The paper is organized as follows. In the next section, we define the multiscale Maxwell wave equation, we review the concept of multiscale convergence, extended to functions that depend on the time variable, and use it to derive the multiscale Maxwell wave equation. We then derive the initial condition for the multiscale homogenized Maxwell wave equation, and show that this problem has a unique solution. In Section 3, we derive the homogenized equation from the multiscale homogenized equation, together with the initial condition. The derivation is quite non trivial in comparison to the multiscale wave equation in \([4]\) and \([15]\) due to the multiscale coefficient \(b^i\) in \((2.3)\) and the corrector terms \(u_i\) in \((2.7)\). In Section 4, we study the regularity of the solution \(u_0\) of the homogenized equation which is necessary for the derivation of the correctors and homogenization errors. We study correctors in Section 5. For two scale problems, when the solution \(u_0\) is in \(L^\infty(0, T; H^1(\text{curl}, D))\), we derive the \(O(\varepsilon^{1/2})\) homogenization error in the \(H(\text{curl}, D)\) norm. When the solution only belongs to a weaker regularity space \(L^\infty(0, T; H^s(\text{curl}, D))\) we deduce a weaker homogenization convergence rate. The proof requires substantial technical developments. For general multiscale problems, we derive in this section a general corrector, without an explicit convergence rate.

Throughout the paper, without indicating a variable, \(\text{curl}\) and \(\nabla\) denote the curl and gradient of a function of the variable \(x\), with respect to \(x\), and by \(\text{curl}_x\) and \(\nabla_x\), we denote partial curl and partial gradient with respect to \(x\), of a function depending on \(x\) and other variables. Repeated indices indicate summation.

## 2 Multiscale homogenization of multiscale Maxwell wave equation

### 2.1 Problem setting

Let \(D\) be a bounded domain in \(\mathbb{R}^d\) \((d = 2, 3)\). Let \(Y\) be the unit cube in \(\mathbb{R}^d\). By \(Y_1, \ldots, Y_n\) we denote \(n\) copies of \(Y\). We denote by \(Y\) the product set \(Y_1 \times Y_2 \times \ldots \times Y_n\) and by \(y = (y_1, \ldots, y_n)\). For \(i = 1, \ldots, n\), we denote by \(Y_i = Y_1 \times \ldots \times Y_{i-1} \times Y_{i+1} \times \ldots \times Y_n\). Let \(a\) and \(b\) be functions from \(D \times Y_1 \times \ldots \times Y_n\) to \(\mathbb{R}^{d \times d}_{\text{sym}}\). We assume that the symmetric matrix functions \(a\) and \(b\) satisfy the boundedness and coerciveness conditions: for all \(x \in D\) and \(y \in Y\), and all \(\xi, \zeta \in \mathbb{R}^d\),

\[
\begin{align*}
\alpha |\xi|^2 & \leq a_{ij}(x, y)\xi_i \xi_j, & a_{ij}\xi_i \zeta_j & \leq \beta |\xi||\zeta| \\
\alpha |\xi|^2 & \leq b_{ij}(x, y)\xi_i \xi_j, & b_{ij}\xi_i \zeta_j & \leq \beta |\xi||\zeta|
\end{align*}
\]

where \(\alpha\) and \(\beta\) are positive numbers. Let \(\varepsilon\) be a small positive value. Let \(\varepsilon_1, \ldots, \varepsilon_n\) be \(n\) functions of \(\varepsilon\) that denote the \(n\) microscopic scales that the problem depends on. We assume the following scale separation properties: for all \(i = 1, \ldots, n - 1\)

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon_{i+1}(\varepsilon)}{\varepsilon_i(\varepsilon)} = 0.
\]

Without loss of generality, we assume that \(\varepsilon_1(\varepsilon) = \varepsilon\). We define the multiscale coefficient of the Maxwell equation \(a^\varepsilon\) and \(b^\varepsilon\) which are functions from \(D\) to \(\mathbb{R}^{d \times d}_{\text{sym}}\) as

\[
a^\varepsilon(x) = a(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}), \quad b^\varepsilon(x) = b(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}).
\]
When \( d = 3 \) we define the spaces
\[
W = H_0(\text{curl}, D) = \{ u \in L^2(\Omega)^3, \, \text{curl} \, u \in L^2(\Omega)^3, \, u \times \nu = 0 \}, \quad H = L^2(D)^3
\]
and when \( d = 2 \)
\[
W = H_0(\text{curl}, D) = \{ u \in L^2(D)^2, \, \text{curl} \, u \in L^2(D), \, u \times \nu = 0 \}, \quad H = L^2(D)^2
\]
where \( \nu \) denotes the outward normal vector on the boundary \( \partial D \). We have the Gelfand triple \( W \subset H \subset W' \). We denote by \( \langle \cdot, \cdot \rangle \) the inner product in \( H \), extending to the duality pairing between \( W' \) and \( W \). We note that when \( d = 3 \), \( \text{curl} \, u^\varepsilon \) is a vector function in \( L^2(D)^3 \) and when \( d = 2 \), \( \text{curl} \, u^\varepsilon \) is a scalar function in \( L^2(D) \). Let \( f \in L^2(0, T; H) \), \( g^0 \in W \) and \( g^1 \in H \). We consider the problem: Find \( u^\varepsilon(t, x) \in L^2(0, T; W) \)
\[
\begin{cases}
\frac{\partial^2 u^\varepsilon}{\partial t^2}(t, x) + \text{curl}(\alpha^\varepsilon(x)\text{curl} u^\varepsilon(t, x)) = f(t, x), \quad (t, x) \in (0, T) \times D \\
u^\varepsilon(0, x) = g_0(x) \\
u^\varepsilon_t(0, x) = g_1(x)
\end{cases}
\tag{2.3}
\]
with the boundary condition \( u^\varepsilon \times \nu = 0 \) on \( \partial D \). We will mostly present the analysis for the case \( d = 3 \) and only discuss the case \( d = 2 \) when there is significant difference. For notational conciseness, we denote by
\[
H = L^2(D)^3, \quad H_i = L^2(D \times Y_i)^3, \quad i = 1, \ldots, n.
\tag{2.4}
\]
In variational form, this problem becomes: Find \( u^\varepsilon \in L^2(0, T; W) \cap H^1(0, T; H) \) so that
\[
\left\langle b^\varepsilon(x) \frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi(x) \right\rangle_{W', W} + \int_D a^\varepsilon(x)\text{curl} u^\varepsilon(t, x) : \text{curl} \phi(x)\,dx = \int_D f(t, x) : \phi(x)\,dx
\tag{2.5}
\]
for all \( \phi \in W \) when \( d = 3 \); and when \( d = 2 \) we need to replace the vector product for \( \text{curl} \) by the scalar multiplication. Problem \([2.3]\) has a unique solution \( u^\varepsilon \in L^2(0, T; W) \cap H^1(0, T; H) \cap H^2(0, T; W') \) that satisfies
\[
\|u^\varepsilon\|_{L^2(0, T; W)} + \|u^\varepsilon\|_{H^1(0, T; H)} + \|u^\varepsilon\|_{H^2(0, T; W')} \leq c(\|f\|_{L^2(0, T; H)} + \|g^0\|_W + \|g^1\|_H)
\tag{2.6}
\]
where the constant \( c \) only depends on the constants \( \alpha \) and \( \beta \) in \([2.1]\) and \( T \) (see Wloka [14]).

We will study this problem via multiscale convergence.

### 2.2 Multiscale convergence

We recall the definition of multiscale convergence (see Nguetseng [12], Allaire [1] and Allaire and Briane [2].

**Definition 2.1** A sequence of functions \( \{ w^\varepsilon \}_\varepsilon \subset L^2(0, T; H) \) \((n + 1)\)-scale converges to a function \( w_0 \in L^2((0, T); D \times Y) \) if for all smooth functions \( \phi(t, x, y) \) which are \( Y \) periodic w.r.t. \( y_i \) for all \( i = 1, \ldots, n \):
\[
\lim_{\varepsilon \to 0} \int_0^T \int_D w^\varepsilon(t, x) \phi(t, x, x_{\varepsilon_1}, \ldots, x_{\varepsilon_n})\,dx\,dt = \int_0^T \int_D \int_Y w_0(t, x, y) \phi(t, x, y)\,dy\,dx\,dt.
\]
We have the following result.

**Proposition 2.2** From a bounded sequence in \( L^2(0, T; H) \) we can extract an \((n + 1)\)-scale convergent subsequence.

We note that the definition above for functions which depend also on \( t \) is slightly different from that in [12] and [1] as we take also the integral with respect to \( t \). However, the proof of proposition 2.2 is similar.

For a bounded sequence in \( L^2(0, T; W) \), we have the following results which are very similar to those in [5] for functions which do not depend on \( t \). The proofs for these results follow the same lines of those in [8] so we do not present them here. As in [13] and [5], we denote by \( \hat{H}_b(\text{curl}, Y_i) \) the space of equivalent classes of functions in \( H_b(\text{curl}, Y_i) \) of equal curl.
Proposition 2.3 Let \( \{w^ε\}_ε \) be a bounded sequence in \( L^2(0,T;W) \). There is a subsequence (not renumbered), a function \( w_0 \in L^2(0,T;W) \), \( n \) functions \( w_i \in L^2((0,T) \times D \times Y_1 \times \ldots \times Y_{i-1}, H^1_b(Y_i)/\mathbb{R}) \) such that
\[
\lim_{ε \to 0} w^ε \overset{(n+1)-\text{scale}}{\to} w_0 + \sum_{i=1}^{n} \nabla_{y_i} w_i.
\]
Further, there are \( n \) functions \( u_i \in L^2((0,T) \times D \times Y_1 \times \ldots \times Y_{i-1}, H^1_b(Y_i)/\mathbb{R}) \) such that
\[
\lim_{ε \to 0} \nabla_y w^ε \overset{(n+1)-\text{scale}}{\to} \nabla_y w_0 + \sum_{i=1}^{n} \nabla_{y_i} u_i.
\]

2.3 Multiscale homogenized Maxwell wave problem

From (2.3) and Proposition (2.3), we can extract a subsequence (not renumbered), a function \( u_0 \in L^2(0,T;W) \), \( n \) functions \( u_i \in L^2(0,T;D \times Y_1 \times \ldots \times Y_{i-1}, H^1_b(Y_i)/\mathbb{R}) \) and \( n \) functions \( u_i \in L^2(0,T;D \times Y_1 \times \ldots \times Y_{i-1}, H^1_b(Y_i)/\mathbb{R}) \) such that
\[
\lim_{ε \to 0} u^ε \overset{(n+1)-\text{scale}}{\to} u_0 + \sum_{i=1}^{n} \nabla_{y_i} u_i,
\]
and
\[
\lim_{ε \to 0} \nabla_y u^ε \overset{(n+1)-\text{scale}}{\to} \nabla_y u_0 + \sum_{i=1}^{n} \nabla_{y_i} u_i.
\]
For \( i = 1, \ldots, n \), let \( W_i = L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^1_b(Y_i)/\mathbb{R}) \) and \( V_i = L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^1_b(Y_i)/\mathbb{R}) \). We define the space \( \mathbf{V} \)
\[
\mathbf{V} = W \times W_1 \times \ldots \times W_n \times V_1 \times \ldots \times V_n.
\]
For \( v = (v_0, \{v_i\}, \{u_i\}) \in \mathbf{V} \), we define the form
\[
|||v||| = ||v_0||_{H^1_b(Y,D)} + \sum_{i=1}^{n} ||v_i||_{L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^1_b(Y_i)/\mathbb{R})} + \sum_{i=1}^{n} ||u_i||_{L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^1_b(Y_i)/\mathbb{R})}.
\]
Let \( u = (u_0, \{u_i\}, \{u_i\}) \in \mathbf{V} \). We define the function
\[
\int_{\mathbf{V}_y} b(x,y) \left( u_0(t,x) + \sum_{i=1}^{n} \nabla_y u_i(t,x,y) \right) dy
\]
in \( \mathbf{W}' \) as
\[
\left\langle \int_{\mathbf{V}_y} b(x,y) \left( u_0(t,x) + \sum_{i=1}^{n} \nabla_y u_i(t,x,y) \right) dy, v_0 \right\rangle_{\mathbf{W}',\mathbf{W}} = \int_D \int_{\mathbf{V}_y} b(x,y) \left( u_0(t,x) + \sum_{i=1}^{n} \nabla_y u_i(t,x,y) \right) v_0 dy dx;
\]
and the function \( b(x,y)(u_0(t,x) + \sum_{i=1}^{n} \nabla_y u_i(t,x,y)) \) in \( \mathbf{V}'_j \) as
\[
\left\langle b(x,y) \left( u_0(t,x) + \sum_{i=1}^{n} \nabla_y u_i(t,x,y) \right), v_j \right\rangle_{\mathbf{V}'_j,\mathbf{V}_j} = \int_D \int_{\mathbf{V}_y} b(x,y)(u_0(t,x) + \sum_{i=1}^{n} \nabla_y u_i(t,x,y)) \nabla_y v_j(x,y) dy dx.
\]
We then have the following result.

Proposition 2.4 The function \( u = (u_0, \{u_i\}, \{u_i\}) \) satisfies
\[
\left\langle \frac{∂^2}{∂t^2} \int_{\mathbf{V}_y} b(x,y) \left( u_0(t,x) + \sum_{i=1}^{n} \nabla_y u_i(t,x,y) \right) dy, v_0 \right\rangle_{\mathbf{W}',\mathbf{W}} =
\int_D f(x) \cdot v_0(x) dx - \int_D \int_{\mathbf{V}_y} a(x,y) \left( \nabla_y u_0 + \sum_{i=1}^{n} \nabla_y u_i \right) \cdot \left( \nabla_y v_0 + \sum_{i=1}^{n} \nabla_y u_i \right) dy dx
\]
Passing to the two scale limit, using the scale separation (2.2), we have

\[ \langle \frac{\partial^2}{\partial t^2} \left( b(x, y) \left( u_0(t, x) + \sum_{i=1}^{n} \nabla_{y_i} u_i(t, x, y) \right) \right), v_j \rangle_{W', W} = 0, \quad j = 1, \ldots, n, \]

i.e. the function \( u \) satisfies the multiscale homogenized equation

\[
\left\langle \frac{\partial^2}{\partial t^2} \int_{\Omega} b(x, y) \left( u_0(t, x) + \sum_{i=1}^{n} \nabla_{y_i} u_i(t, x, y) \right) \, dy, v_0 \right\rangle_{W', W} + \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial t^2} b(x, y) \left( u_0(t, x) + \sum_{i=1}^{n} \nabla_{y_i} u_i(t, x, y) \right), v_j \right)_{V_j, V_j} + \int_{D} \int_{\Omega} a(x, y) \left( \text{curl } u_0 + \sum_{i=1}^{n} \text{curl}_{y_i} u_i \right) \cdot \left( \text{curl } v_0 + \sum_{i=1}^{n} \text{curl}_{y_i} v_i \right) \, dy \, dx = \int_{D} f(x) \cdot v_0(x) \, dx \tag{2.9}
\]

for all \( v = (v_0, \{v_i\}, \{v_i\}) \in V \).

**Proof** Let \( q \in \mathcal{D}(0, T) \). Let \( v_0 \in \mathcal{D}(D), v_i \in \mathcal{D}(D, C_\#^\infty(Y_1, \ldots, C_\#^\infty(Y_i) \ldots)) \) and \( n_i \in \mathcal{D}(D, C_\#^\infty(Y_1, \ldots, C_\#^\infty(Y_i) \ldots)) \) for \( i = 1, \ldots, n \). Choosing a test function of the form

\[ \phi(t, x) = \left( v_0(x) + \sum_{i=1}^{n} \varepsilon_i v_i(x, \frac{x}{\varepsilon_i}, \ldots, \frac{x}{\varepsilon_i}) + \sum_{i=1}^{n} \varepsilon_i \nabla_{y_i} v_i(x, \frac{x}{\varepsilon_i}, \ldots, \frac{x}{\varepsilon_i}) \right) q(t) \]

we obtain

\[
\int_{0}^{T} \int_{D} b(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}) u^\varepsilon(t, x) \cdot \left( v_0(x) + \sum_{i=1}^{n} \varepsilon_i v_i(x, \frac{x}{\varepsilon_i}, \ldots, \frac{x}{\varepsilon_i}) + \sum_{i=1}^{n} \varepsilon_i \nabla_{y_i} v_i(x, \frac{x}{\varepsilon_i}, \ldots, \frac{x}{\varepsilon_i}) \right) q''(t) \, dx \, dt \\
+ \int_{0}^{T} \int_{D} a(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}) \text{curl } u^\varepsilon(t, x) \cdot \left( \text{curl } v_0(x) + \sum_{i=1}^{n} \varepsilon_i \text{curl}_{y_i} v_i(x, \frac{x}{\varepsilon_i}, \ldots, \frac{x}{\varepsilon_i}) + \sum_{i=1}^{n} \varepsilon_i \nabla_{y_i} v_i(x, \frac{x}{\varepsilon_i}, \ldots, \frac{x}{\varepsilon_i}) \right) q(t) \, dx \, dt \\
= \int_{0}^{T} \int_{D} f(t, x) \cdot \left( v_0(x) + \sum_{i=1}^{n} \varepsilon_i v_i(x, \frac{x}{\varepsilon_i}, \ldots, \frac{x}{\varepsilon_i}) + \sum_{i=1}^{n} \varepsilon_i \nabla_{y_i} v_i(x, \frac{x}{\varepsilon_i}, \ldots, \frac{x}{\varepsilon_i}) \right) q(t) \, dx \, dt
\]

Passing to the two scale limit, using the scale separation (2.2), we have

\[
\int_{0}^{T} \int_{D} \int_{\Omega} b(x, y) \left( u_0(t, x) + \sum_{i=1}^{n} \nabla_{y_i} u_i(t, x, y) \right) \cdot \left( v_0(x) + \sum_{i=1}^{n} \nabla_{y_i} v_i(x, y) \right) q''(t) \, dy \, dx \, dt \\
+ \int_{0}^{T} \int_{D} \int_{\Omega} a(x, y) \left( \text{curl } u_0(t, x) + \sum_{i=1}^{n} \text{curl}_{y_i} u_i(t, x, y) \right) \cdot \left( \text{curl } v_0(x) + \sum_{i=1}^{n} \text{curl}_{y_i} v_i(x, y) \right) q(t) \, dy \, dx \, dt \\
= \int_{0}^{T} \int_{D} \int_{\Omega} f(t, x) \cdot \left( v_0(x) + \sum_{i=1}^{n} \nabla_{y_i} v_i(x, y) \right) q(t) \, dy \, dx \, dt \\
= \int_{0}^{T} \int_{D} f(t, x) \cdot v_0(x) q(t) \, dx \, dt.
\]
Proof As \( u^\varepsilon \) is bounded in \( H^1(0, T; H) \), \( u_0 \) belongs to \( H^1(0, T; H) \subset C([0, T]; H) \) and is the weak limit in this space of \( u^\varepsilon \). Let \( \phi \in C^\infty([0, T] \times \Omega) \) with \( \phi = 0 \) when \( t = T \). We have

\[
\int_0^T \int_\Omega \frac{\partial u^\varepsilon}{\partial t}(t, x) \phi(t, x) \, dx \, dt = \int_0^T \int_\Omega \left( \frac{\partial}{\partial t}(u^\varepsilon(t, x) \phi(t, x)) - u^\varepsilon(t, x) \frac{\partial \phi}{\partial t}(t, x) \right) \, dx \, dt
\]

\[
= - \int_0^T u^\varepsilon(0, x) \phi(0, x) - \int_0^T \int_\Omega u^\varepsilon(t, x) \frac{\partial \phi}{\partial t}(t, x) \, dx \, dt
\]

\[
= - \int_0^T g_0(x) \phi(0, x) \, dx - \int_0^T \int_\Omega u_0(t, x) \frac{\partial \phi}{\partial t}(t, x) \, dx \, dt.
\]

On the other hand

\[
\int_0^T \int_\Omega \frac{\partial u^\varepsilon}{\partial t}(t, x) \phi(t, x) \, dx \, dt \to \int_0^T \int_\Omega \frac{\partial u_0}{\partial t}(t, x) \phi(t, x) = - \int_0^T u_0(0, x) \phi(0, x) - \int_0^T \int_\Omega u_0(t, x) \frac{\partial \phi}{\partial t}(t, x) \, dx \, dt.
\]

Thus \( u_0(0, x) = g_0 \).

As \( \{\frac{\partial u^\varepsilon}{\partial t}\}_\varepsilon \) is bounded in \( L^2(0, T; H) \) so there is a subsequence that \((n + 1)\)-scale converges. Let \( \xi \in L^2((0, T); L^2(D \times Y)) \) be the \((n + 1)\)-scale limit. Let \( \phi(t, x, y) \in C_0^\infty((0, T); C_0^\infty(D, C_\#(Y))) \). We have that

\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \frac{\partial u^\varepsilon}{\partial t}(t, x, \frac{x}{\varepsilon}) \phi(t, x, \frac{x}{\varepsilon}) \, dx \, dt \to \int_0^T \int_\Omega \int_Y \xi(t, x, y) \phi(t, x, y) \, dy \, dx \, dt.
\]

On the other hand

\[
\int_0^T \int_\Omega \frac{\partial u^\varepsilon}{\partial t}(t, x, \frac{x}{\varepsilon}) \phi(t, x, \frac{x}{\varepsilon}) \, dx \, dt = - \int_0^T \int_\Omega u^\varepsilon(t, x) \frac{\partial \phi}{\partial t}(t, x, \frac{x}{\varepsilon}) \, dx \, dt
\]

which converges to

\[- \int_0^T \int_\Omega \left( u_0 + \sum_{i=1}^n \nabla_y u_i \right) \frac{\partial \phi}{\partial t}(t, x, y) \, dy \, dx dt = - \int_0^T \int_Y \left( u_0 + \nabla_y u_1 \right) \frac{\partial \phi}{\partial t}(t, x, y) \, dy \, dx dt.
\]

Thus

\[
\int_{Y_1} \ldots \int_{Y_n} \xi(t, x, y) \, dy_1 \ldots dy_n = \frac{\partial}{\partial t} (u_0 + \nabla_y u_1)
\]

so

\[
\frac{\partial}{\partial t} \nabla_y u_1 = \int_{Y_1} \ldots \int_{Y_n} \xi(t, x, y) \, dy_1 \ldots dy_n - \frac{\partial u_0}{\partial t} \in H_1
\]
As we refer to \(2.3\) for the definition of the spaces \(H_i\). Similarly, using a function \(\phi(t, x, y_1, y_2) \in C^\infty_0((0,T), C^\infty_0(D, C^\infty_0(Y_1, C^\infty_0(Y_2))))\), we have

\[
\frac{\partial}{\partial t} \nabla_y u_2 = \int_{Y_5} \ldots \int_{Y_n} \xi(t, x, y) dy_n \ldots dy_3 - \frac{\partial u_0}{\partial t} - \frac{\partial}{\partial t} \nabla_y u_1 \in H_2.
\]

Continuing this process, we have that for all \(i = 1, \ldots, n\),

\[
\frac{\partial}{\partial t} \nabla_y u_i \in H_i
\]

Finally, we have

\[
\xi(t, x, y) = \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y u_i.
\]

Let \(q \in C^\infty([0,T])\) with \(q(T) = 0\). Let \(\phi(x) = v_0(x) + \sum_{i=1}^n \varepsilon_i v_i(x, \frac{\varepsilon}{\varepsilon_1}, \ldots, \frac{\varepsilon}{\varepsilon_n}) + \sum_{i=1}^n \varepsilon_i \nabla v_i(x, \frac{\varepsilon}{\varepsilon_1}, \ldots, \frac{\varepsilon}{\varepsilon_n})\). We have

\[
\int_0^T \left( b'(x) \frac{\partial u^\varepsilon}{\partial t^2} \right)_{W^r, W} q(t) dt = \int_0^T \frac{\partial}{\partial t} \left( b' \frac{\partial u^\varepsilon}{\partial t} \phi \right)_{W^r, W} q(t) dt - \int_0^T \left( b' \frac{\partial u^\varepsilon}{\partial t} \phi \right)_{W^r, W} dq(t) dt
\]

\[
= - \left( b' \frac{\partial u^\varepsilon}{\partial t} \phi \right)_{W^r, W} q(0) - \int_0^T \left( b' \frac{\partial u^\varepsilon}{\partial t} \phi \right)_{W^r, W} dq(t) dt
\]

\[
= -\left( b' g^1, \phi \right) q(0) - \int_0^T \left( b' \frac{\partial u^\varepsilon}{\partial t} \phi \right)_{W^r, W} dq(t) dt
\]

\[
= -\int_0^T \int_D b(x, y) g_1(x)(v_0(x) + \sum_{i=1}^n \nabla_y v_i(x, y_i)) q(0) dy dx
\]

\[
\int_0^T \int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t}(x) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y u_i(t, x, y_i) \right) \cdot \left( v_0(x) + \sum_{i=1}^n \nabla_y v_i(x, y_i) \right) dq(t) dt dy dx dt \tag{2.13}
\]

when \(\varepsilon \to 0\). On the other hand, let \(q_n\) be a sequence in \(C^\infty_0(0,T)\) that converges to \(q(t)\) in \(L^2(0,T)\) when \(n \to \infty\). As \(b' \frac{\partial^2 u^\varepsilon}{\partial t^2} \) is bounded in \(L^2(0,T; W^r)\) so there is a constant \(c > 0\) such that

\[
\left| \int_0^T \left( b' \frac{\partial^2 u^\varepsilon}{\partial t^2} \phi \right)_{W^r, W} q_n(t) dt \right| \leq c \| q_n - q \|_{L^2(0,T)}.
\]

As \(q_n \in C^\infty_0(0,T), \) when \(\varepsilon \to 0,\)

\[
\lim_{\varepsilon \to 0} \int_0^T \left( b' \frac{\partial^2 u^\varepsilon}{\partial t^2} \phi \right) q_n(t) dt =
\]

\[
\int_0^T \int_D \int_Y b(x, y) (u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i)) \cdot (v_0(x) + \sum_{i=1}^n \nabla_y v_i(x, y_i)) q_n(t) dy dx dt
\]

\[
= \int_0^T \left( \frac{\partial^2}{\partial t^2} \int_Y b(x, y) (u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i)) dy y_{W, W} q_n(t) dt + \sum_{i=1}^n \int_0^T \left( \frac{\partial^2}{\partial t^2} b(x, y)(u_0(t, x) + \sum_{j=1}^n \nabla_y u_j(t, x, y_j), v_i) v_j W, W q_n(t) dt.
\]

Passing to the limit when \(n \to \infty,\) we have

\[
\lim_{\varepsilon \to 0} \int_0^T \left( b' \frac{\partial^2 u^\varepsilon}{\partial t^2} \phi \right) q(t) dt =
\]

\[
\int_0^T \left( \frac{\partial^2}{\partial t^2} \int_Y b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i)) dy y_{W, W} q(t) dt + \sum_{i=1}^n \int_0^T \left( \frac{\partial^2}{\partial t^2} b(x, y)(u_0(t, x) + \sum_{j=1}^n \nabla_y u_j(t, x, y_j), v_i) v_j W, W q(t) dt.
\]

\[\tag{2.14}\]

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From Proposition 2.4 as \( \frac{\partial}{\partial t} \int_V b(x, y)(u_0(t, x) + \sum_{i=1}^{n} \nabla_y u_i(t, x, y_i))dy \in L^2(0, T; W') \), \( \frac{\partial}{\partial t} \int_V b(x, y)(u_0(t, x) + \sum_{i=1}^{n} \nabla_y u_i(t, x, y_i))dy \in C([0, T]; W') \) so the initial condition \( \frac{\partial}{\partial t} \int_V b(x, y)(u_0(t, x) + \sum_{i=1}^{n} \nabla_y u_i(t, x, y_i))dy \) at \( t = 0 \) is well defined in \( W' \). Similarly, the initial condition \( \frac{\partial}{\partial t} \int_V b(x, y)(u_0(t, x) + \sum_{i=1}^{n} \nabla_y u_i(t, x, y_i))dy \) at \( t = 0 \) is well defined in \( V'_i \). The right hand side of (2.14) can be written as

\[
\int_0^T \frac{\partial}{\partial t} \left( \left( \frac{\partial}{\partial t} \int_V b(x, y)(u_0(t, x) + \sum_{j=1}^{n} \nabla_y u_j(t, x, y_j))dy, v_0 \right)_{W', W} dt - \int_0^T \left( \left( \frac{\partial}{\partial t} b(x, y)(u_0(t, x) + \sum_{j=1}^{n} \nabla_y u_j(t, x, y_j)), v_i \right)_{V'_j, V_j} q(t) dt - \right.
\]

\[
\sum_{i=1}^{n} \int_0^T \left( \left( \frac{\partial}{\partial t} b(x, y)(u_0(t, x) + \sum_{j=1}^{n} \nabla_y u_j(t, x, y_j)), v_i \right)_{V'_i, V_i} \frac{dq(t)}{dt} dt \right.
\]

\[
= - \left( \left( \frac{\partial}{\partial t} \int_V b(x, y)(u_0 + \sum_{j=1}^{n} \nabla_y u_j)dy \bigg|_{t=0}, v_0 \right)_{W', W} q(0) - \right.
\]

\[
\int_0^T \left( \left( \frac{\partial}{\partial t} b(x, y)(u_0(t, x) + \sum_{j=1}^{n} \nabla_y u_j(t, x, y_j))dy, v_0 \right)_{W', W} \frac{dq(t)}{dt} dt - \right.
\]

\[
\sum_{i=1}^{n} \int_0^T \left( \left( \frac{\partial}{\partial t} b(x, y)(u_0(t, x) + \sum_{j=1}^{n} \nabla_y u_j(t, x, y_j)), v_i \right)_{V'_i, V_i} \frac{dq(t)}{dt} dt \right.
\]

\[
- \sum_{i=1}^{n} \int_0^T \left( \left( \frac{\partial}{\partial t} (b(x, y)u_0(t, x) + \sum_{j=1}^{n} \nabla_y u_j(t, x, y_j)), v_i \right)_{V'_i, V_i} \frac{dq(t)}{dt} dt. \right.
\]

Comparing (2.13) and (2.15), we have

\[
\left( \frac{\partial}{\partial t} \int_V b(x, y)(u_0(t, x) + \sum_{i=1}^{n} \nabla_y u_i(t, x, y_i))dy \bigg|_{t=0}, v_0 \right)_{W', W} = \int_D \int_Y b(x, y)g_1(x) \cdot v_0(x)dydx,
\]

and

\[
\left( \frac{\partial}{\partial t} (b(x, y)u_0(t, x) + \sum_{j=1}^{n} \nabla_y u_j(t, x, y_j)) \bigg|_{t=0}, v_i \right)_{V'_i, V_i} = \int_D \int_Y b(x, y)g_1(x) \cdot \nabla_y u_i(x, y_i)dydx.
\]

for all \( i = 1, \ldots, n \). We get the desired result. \( \square \)

**Proposition 2.6** With the initial conditions (2.10), (2.11) and (2.12), problem (2.9) has a unique solution.

**Proof** We show that when \( f = 0, g_0 = 0 \) and \( g_1 = 0 \), the solution of (2.9) is \( u_0 = 0, u_i = 0 \) and \( u_i = 0 \) for all \( i = 1, \ldots, n \).

Following the procedure in [14] Theorem 19.1 for showing the uniqueness of a solution of a wave
equation, fixing $s \in (0, T)$, we define

$$w_0(t) = \begin{cases} \int_t^s u_0(\sigma) d\sigma, & t < s, \\ 0, & t \geq s; \end{cases}$$

$$w_1(t) = \begin{cases} \int_t^s u_1(\sigma) d\sigma, & t < s, \\ 0, & t \geq s; \end{cases}$$

$$w_1(t) = \begin{cases} \int_t^s u_1(\sigma) d\sigma, & t < s, \\ 0, & t \geq s. \end{cases}$$

We have

$$\frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \int_Y b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i) dy, w_0(t, \cdot)) \right)_{W', W}$$

$$+ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \int_Y b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i), w_1(t, \cdot)) \right)_{V', V}$$

$$= \left( \frac{\partial^2}{\partial t^2} \int_Y b(x, y)(u_0 + \sum_{i=1}^n \nabla_y u_i) dy, w_0 \right)_{W', W} + \left( \frac{\partial^2}{\partial t^2} b(x, y)(u_0 + \sum_{i=1}^n \nabla_y u_i, w) \right)_{V', V}$$

$$+ \int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y u_i \right) \cdot \left( \frac{\partial w_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y w_i \right) dy dx$$

$$= \int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y u_i \right) \cdot \left( \frac{\partial w_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y w_i \right) dy dx.$$

Integrating over $(0, s)$, we get

$$\left. \left( \frac{\partial}{\partial t} \int_Y b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i) dy, w_0(s, \cdot)) \right) \right|_{t=s} = u_0(s, \cdot)$$

$$+ \left. \left( \frac{\partial}{\partial t} b(x, y)(u_0 + \sum_{i=1}^n \nabla_y u_i(t, x, y_i), w_1(s, \cdot)) \right) \right|_{t=s} \right)_{V', V}$$

$$- \left. \left( \frac{\partial}{\partial t} \int_Y b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i)) dy \right) \right|_{t=0} = w_0(0, \cdot)$$

$$- \left. \left( \frac{\partial}{\partial t} b(x, y)(u_0 + \sum_{i=1}^n \nabla_y u_i(t, x, y_i)) \right) \right|_{t=0} \right)_{V', V}$$

$$= \int_0^s \int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y u_i \right) \cdot \left( \frac{\partial w_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y w_i \right) dy dx dt$$

$$- \int_0^s \int_D \int_Y a(x, y) \left( \text{curl } u_0 + \sum_{i=1}^n \text{curl}_y u_i \right) \cdot \left( \text{curl } w_0 + \sum_{i=1}^n \text{curl}_y w_i \right) dy dx dt$$

$$= \int_0^s \frac{1}{2} \int_D \int_Y b(x, y)(u_0 + \sum_{i=1}^n \nabla_y u_i) \cdot (u_0 + \sum_{i=1}^n \nabla_y u_i) dy dx dt$$

$$- \int_0^s \frac{1}{2} \int_D \int_Y a(x, y)(\text{curl } u_0 + \sum_{i=1}^n \text{curl}_y u_i) \cdot (\text{curl } w_0 + \sum_{i=1}^n \text{curl}_y w_i) dy dx dt$$
due to $u_0 = \frac{\partial w_0}{\partial t}$, $u_i = \frac{\partial w_i}{\partial t}$ and $u_t = \frac{\partial w_t}{\partial t}$ for $i = 1, \ldots, n$ from (2.10). Using (2.11) and (2.12), we have

$$0 = \frac{1}{2} \int_D \int_Y b(x, y)(u_0(s, x) + \sum_{i=1}^n \nabla_{y_i} w_i(s, x, y_i)) \cdot (u_0(s, x) + \sum_{i=1}^n \nabla_{y_i} w_i(s, x, y_i)) dy dx +$$

$$\frac{1}{2} \int_D \int_Y a(x, y) (\text{curl } w_0(0, x) + \sum_{i=1}^n \text{curl}_{y_i} w_i(0, x, y_i)) \cdot (\text{curl } w_0(0, x) + \sum_{i=1}^n \text{curl}_{y_i} w_i(0, x, y_i)) dy dx.$$

We thus deduce that $u_0(s) = 0$, $\nabla_{y_i} w_i(s) = 0 \forall s$, $\text{curl } w_0(0) = 0$ and $\text{curl}_{y_i} w_i(0) = 0$. This means that

$$\int_0^s \text{curl } u_0(\sigma) d\sigma = 0 \quad \text{and} \quad \int_0^s \text{curl}_{y_i} u_i(\sigma) d\sigma = 0$$

for all $s$. Thus for all $\sigma$, $\text{curl } u_0(\sigma) = 0$ and $\text{curl}_{y_i}(\sigma) = 0$. \hfill \Box

## 3 Homogenized equation

In this section, we use the multiscale homogenized problem (2.20) to deduce the homogenized equation. From (2.24), we have that for all $v_n \in V_n$ and all $q \in D(0, T)$

$$\int_D \int_Y b(x, y) \left( \int_0^T (u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} u_i) q''(t) dt + \nabla_{y_n} \int_0^T u_n q''(t) dt \right) \cdot \nabla_{y_n} v_n dy dx = 0.$$ 

Thus

$$\int_0^T u_n q''(t) dt = \left( \int_0^T (u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} u_i) q''(t) dt \right) w_n^k$$

where $w_n^k \in L^2(D \times Y_{n-1}, H^1_{\text{div}}(Y_n)/R)$ is the solution of the cell problem

$$\nabla_{y_n} \cdot (b(x, y)(e^k + \nabla_{y_n} w_n^k) = 0; \quad (3.1)$$

e
d is the unit vector in $\mathbb{R}^d$. Therefore

$$\int_0^T \left( \nabla_{y_n} u_n - (u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} u_i) k \nabla_{y_n} w_n^k \right) q''(t) dt = 0$$

so

$$\int_0^T \left( \frac{\partial}{\partial t} \nabla_{y_n} u_n - \frac{\partial}{\partial t} (u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} u_i) k \nabla_{y_n} w_n^k \right) q'(t) dt = 0.$$ 

From this we have

$$\frac{\partial}{\partial t} \nabla_{y_n} u_n = \frac{\partial}{\partial t} (u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} u_i) k \nabla_{y_n} w_n^k + G_n(x, y_n) \quad (3.2)$$

for a function $G_n(x, y_n)$ in $L^2(D \times Y_n)$. We then have $\forall v_{n-1} \in V_{n-1}$

$$\int_0^T \int_{Y_{n-1}} b^{n-1}(x, y_{n-1})(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} u_i) \cdot \nabla_{y_{n-1}} v_{n-1} q''(t) dy_{n-1} dx dt = 0,$$

where $b^{n-1}(x, y_{n-1})$ is the $(n-1)$th level homogenized coefficient which is defined by

$$b^{n-1}_{ij}(x, y_{n-1}) = \int_{Y_n} b_{kli}(x, y) \left( \delta_{jl} \frac{\partial w_{j}^n}{\partial y_{al}} + \frac{\partial w_{j}^n}{\partial y_{nk}} \delta_{ik} \right) dy_n. \quad (3.3)$$

Similarly we have

$$\frac{\partial}{\partial t} \nabla_{y_{n-1}} u_{n-1} = \frac{\partial}{\partial t} (u_0 + \sum_{i=1}^{n-2} \nabla_{y_i} u_i) k \nabla_{y_{n-1}} w_{n-1}^k + G_{n-1}(x, y_{n-1}),$$

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where \( G_{n-1}(x, y_{n-1}) \in L^2(D \times Y_{n-1}), \) and \( w_{n-1}^k \in L^2(D \times Y_{n-2}, H^1_{\#}(Y_{n-1})/\mathbb{R}) \) satisfies the cell problem
\[
\nabla y_{n-1} \cdot (b^{n-1}(\epsilon^k + \nabla y_{n-1} w_{n-1}^k)) = 0.
\]

Recursively, letting \( b^0(x, y_n) = b(x, y) \), we have for all \( i = 0, \ldots, n-1 \):
\[
\int_0^T u_i q''(t) dt = \left( \int_0^T (u_0 + \sum_{j=1}^{i-1} \nabla y, u_j) q''(t) dt \right) w_i^k,
\]

where \( w_i^k \in L^2(D \times Y_{i-1}, H^1_{\#}(Y_i)/\mathbb{R}) \) is the solution of the cell problem
\[
\nabla y_i \cdot (b^i(x, y_k)(\epsilon_i^k + \nabla y, w_i^k)) = 0.
\] (3.4)

From an argument as above, we have
\[
\frac{\partial}{\partial t} \nabla y_i u_i = \frac{\partial}{\partial t} (u_0 + \sum_{j=1}^{i-1} \nabla y, u_j) \nabla y_i w_i^k + G_i(x, y_i)
\]
for a function \( G_i(x, y_i) \in L^2(D \times Y_i) \). The positive definite matrix function \( b^i(x) \), which is defined as
\[
b^i_{pq}(x) = \int_Y b^i_{kl}(x, y_1) \left( \delta_{ql} + \frac{\partial u_q^i}{\partial y_1 l} \right) \left( \delta_{pk} + \frac{\partial u_p^i}{\partial y_1 k} \right) dy_1
\] (3.5)
is the homogenized coefficient. It satisfies
\[
\int_0^T \int_D \int_Y b(x, y) (u_0 + \sum_{i=1}^n \nabla y_i u_i) \cdot (v_0 + \sum_{i=1}^n \nabla y_i v_i) q''(t) d\mathbf{y} d\mathbf{x} dt = \int_0^T \int_D b^0(x) u_0 \cdot v_0 q''(t) d\mathbf{x} dt
\]
for all \( v_0 \in W \) and \( v_i \in V_i \).

From (2.9)
\[
\int_D \int_Y a(x, y) \left( \text{curl} u_0 + \sum_{i=1}^n \text{curl}_y u_i \right) \cdot \text{curl}_y v_n d\mathbf{y} d\mathbf{x} = 0
\]
for all \( v_n \in W_n \). For each \( l = 1, \ldots, d \), let \( N_l^i \in V_i \) be the solution of
\[
\text{curl}_y \cdot a(x, y)(\epsilon^i + \text{curl}_y N_l^i) = 0.
\]

We can write \( u_n \) as
\[
u_n = N_l^i \left( (\text{curl} u_0)_l + (\text{curl}_y u_1)_l + \ldots + (\text{curl}_y u_{n-1})_l \right).
\]

For all \( v_{n-1} \in W_{n-1} \),
\[
\int_D \int_Y a(x, y) \left( \text{curl} u_0 + \sum_{i=1}^{n-1} \text{curl}_y u_i + \text{curl}_y N_l^i \left( (\text{curl} u_0)_l + \sum_{i=1}^{n-1} (\text{curl}_y u_i)_l \right) \right) \cdot \text{curl}_y v_{n-1} d\mathbf{y} d\mathbf{x} = 0
\]
i.e.
\[
\int_D \int_Y a_{ij}(x, y) \left( \delta_{ij} + (\text{curl}_y N_l^i)_j \right) \left( (\text{curl} u_0)_l + \sum_{i=1}^{n-1} (\text{curl}_y u_i)_l \right) (\text{curl}_y v_{n-1})_j d\mathbf{y} d\mathbf{x} = 0
\]
for all \( v_{n-1} \in W_{n-1} \). Let
\[
a_{pq}^{n-1}(x, y_1, \ldots, y_{n-1}) = \int_{Y_n} a_{pq}(x, y) \left( \delta_{kq} + (\text{curl}_y N^q)_k \right) d\mathbf{y}.
\]
We have that
\[ u_{n-1} = N^i_{(n-1)}((\text{curl } u_0)i + (\text{curl}_u u_1)i + \ldots + (\text{curl}_{u_{n-2}} u_{n-2})i) \]
where \( N^i_{(n-1)} \) satisfies the cell problem
\[ \text{curl}_{y_{n-1}}(a^{n-1}(x, y_{n-1})(e^i + \text{curl}_y N^i_{n-1})) = 0. \]
Letting \( a^n = a \), we then have, recursively,
\[ u_i = N^i_{1}((\text{curl } u_0)i + (\text{curl}_u u_1)i + \ldots + (\text{curl}_{y_{i-1}} u_{i-1})i) \]
where \( N^i_1 \in W_i \) satisfies the cell problem
\[ \text{curl}_y (a^i(e^i + \text{curl}_y N^i_1)) = 0, \quad (3.6) \]
i.e.
\[ \int_D \int_Y a^i(e^i + \text{curl}_y N^i_1) \cdot \text{curl}_y v_i dy_i = 0 \]
for all \( v_i \in W_i \). For \( i = 1, \ldots, n-1 \), the \( i \)th level homogenized coefficient \( a^i \) is defined as
\[ a^i_{pq}(x, y_1, \ldots, y_i) = \int_{Y_{i+1}} a^{i+1}_{pk}(\delta_{kq} + (\text{curl}_{y_{i+1}} N^q_{i+1})) dy_{i+1}. \]
Continuing this process, we finally get the homogenized coefficient \( a^0(x) \) as
\[ a^0_{pq}(x) = \int_{Y_1} a^{i+1}_{pk}(\delta_{kq} + (\text{curl}_{y_i} N^q_i)) dy_i, \quad (3.7) \]
The homogenization equation is
\[ \int_0^T \int_D b^0(x_0(t), x) \cdot v_0(x_0(t), x) q(t) dx dt + \int_0^T \int_D a^i \text{curl } u_0 \cdot \text{curl } v_0 q(t) dx dt = \int_0^T \int_D f(t, x) \cdot v_0(x_0(t), x) q(t) dx dt \]
i.e.
\[ b^0(x) \frac{\partial^2 u_0}{\partial t^2} (t, x) + \text{curl } (a^0(x)\text{curl } u_0(x)) = f(t, x). \quad (3.8) \]
Now we derive the initial conditions. From (2.10), \( u_0(0) = y_0 \). As a distribution in \( V_n' \),
\[ \frac{\partial^2}{\partial t^2} b(x, y)(u_0 + \sum_{i=1}^n \nabla_y u_i) = 0, \]
so for all \( t \)
\[ \frac{\partial}{\partial t} b(x, y)(u_0 + \sum_{i=1}^n \nabla_y u_i) \bigg|_{t=0} = \frac{\partial}{\partial t} b(x, y)(u_0 + \sum_{i=1}^n \nabla_y u_i) \bigg|_{t=0}, \]
i.e.
\[ \int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y u_i \right) \cdot \nabla_y v_n dyc = \int_D \int_Y b(x, y) g_1 \cdot \nabla_y v_n dyc. \]
From (3.2) we have
\[ \int_D \int_Y b(x, y) g_1 \cdot \nabla_y v_n dyc \]
\[ = \int_D \int_Y b(x, y) \left( \frac{\partial}{\partial t} (u_0 + \sum_{i=1}^{n-1} \nabla_y u_i)(e^k + \nabla_y w^k_n) + G_n(x, y_n) \right) \cdot \nabla_y v_n(y_n) dy \]
\[ = \int_D \int_Y b(x, y) G_n(x, y_n) \cdot \nabla_y v_n(y_n) dy \]
due to (3.1). From (3.2) we have
\[
\text{curl}_{y_n} G_n(x, y_n) = 0, \quad \text{and} \quad \int_{Y_n} G_n(x, y_n) dy_n = 0.
\]
Thus there is a function \( \tilde{G}_n \in L^2(D \times Y_{n-1}, H^1_{y_n}(Y_n)/\mathbb{R}) \) such that \( G_n(x, y_n) = \nabla_{y_n} \tilde{G}_n(x, y_n) \). From
\[
\int_D \int_Y b(x, y)(-g_1 + \nabla_{y_n} \tilde{G}_n(x, y)) \cdot \nabla_{y_n} v(y_n) dydx = 0,
\]
we deduce that
\[
\tilde{G}_n(x, y) = -g_{1k} w_n^k
\]
where \( w_n^k \) is the solution of the cell problem (3.1). As a function in \( V'_{n-1} \),
\[
\frac{\partial}{\partial t} b(x, y)(u_0 + \sum_{i=1}^{n} \nabla_{y_i} u_i) = 0
\]
so
\[
\frac{\partial}{\partial t} b(x, y)(u_0 + \sum_{i=1}^{n} \nabla_{y_i} u_i) \bigg|_{t=0}.
\]
From (2.12), we have
\[
\int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial t} \nabla_{y_i} u_i \right) \cdot \nabla_{y_n-1} v_{n-1} dydx = \int_D \int_Y b(x, y) g_1 \cdot \nabla_{y_n-1} v_{n-1} dydx.
\]
From (3.2) and (3.9) we have
\[
\int_D \int_Y b(x, y)(e^k + \nabla_{y_n} w_n^k) \left( \frac{\partial u_0}{\partial t} + \sum_{j=1}^{n-1} \frac{\partial}{\partial t} \nabla_{y_j} u_j \right) \cdot \nabla_{y_n-1} v_{n-1} dydx
\]
\[
- \int_D \int_Y b(x, y) g_{1k} \nabla_{y_n} w_n^k \cdot \nabla_{y_n-1} v_{n-1} dydx = \int_D \int_Y b(x, y) g_1(x) \cdot \nabla_{y_n-1} v_{n-1} dydx.
\]
Thus
\[
\int_D \int_{Y_{n-1}} b^{n-1}(x, y_{n-1}) \left( \frac{\partial u_0}{\partial t} + \sum_{j=1}^{n-1} \frac{\partial}{\partial t} \nabla_{y_j} u_j \right) \cdot \nabla_{y_n-1} v_{n-1} dy_{n-1} dx
\]
\[
= \int_D \int_Y b(x, y)(e^k + \nabla_{y_n} w_n^k) g_{1k} \cdot \nabla_{y_n-1} v_{n-1} dydx
\]
\[
= \int_D \int_Y b^{n-1}(x, y_{n-1}) g_1(x) \cdot \nabla_{y_n-1} v_{n-1} dy_{n-1} dx.
\]
Continuing this, we find that
\[
G_t(x, y_t) = -g_{1k} \nabla_{y_i} w_i^k.
\]
Therefore, for all \( t \)
\[
\int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t}(t, x) + \sum_{i=1}^{n} \frac{\partial}{\partial t} \nabla_{y_i} u_i(t, x, y) \right) \cdot v_0 dydx
\]
\[
= \int_D b_0(x) \frac{\partial u_0}{\partial t}(t, x) \cdot v_0(x) dx - \sum_{i=1}^{n} \int_D \int_Y b_i(x, y_i) \nabla w_i^k g_{1k} \cdot v_0 dydx.
\]
Therefore for \( q \in C^\infty([0, T]) \) with \( q(T) = 0 \), using the fact that \( \frac{\partial}{\partial t} b(x, y)(u_0 + \sum_{i=1}^{n} \nabla_{y_i} u_i) \) is continuous.
as a map from \([0, T]\) to \(W'\), we have
\[
\int_0^T \left\langle \frac{\partial^2}{\partial t^2} \int_Y b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y))dy, v_0 \right\rangle \; q(t)dt
= \int_0^T \left\langle \int_Y b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y))dy, v_0 \right\rangle \; q(t)dt
- \int_0^T \left\langle \int_Y b(x, y)\left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y u_i\right), v_0 \right\rangle q'(t)dt
= - \left\langle b^0 \frac{\partial u_0}{\partial t} \bigg|_{t=0}, v_0 \right\rangle q(0) + \sum_{i=1}^n \int_D \int_Y b_i(x, y_i)\nabla w_i^k g_{1k} \cdot v_0 dy dx q(0)
- \int_0^T \left\langle \int_Y b(x, y)\left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y u_i\right), v_0 \right\rangle q'(t)dt.
\]

From (2.13) and (2.14), we deduce
\[
- \left\langle b^0 \frac{\partial u_0}{\partial t} \bigg|_{t=0}, v_0 \right\rangle q(0) + \sum_{i=1}^n \int_D \int_Y b_i(x, y_i)\nabla w_i^k g_{1k} \cdot v_0 dy dx q(0) = - \int_D \int_Y b(x, y)g_1(x) \cdot v_0(x)dy dx q(0).
\]

We then have
\[
\left\langle b^0 \frac{\partial u_0}{\partial t} \bigg|_{t=0}, v_0 \right\rangle = \int_D \int_Y b(x, y)g_1(x) \cdot v_0(x)dx + \sum_{i=1}^n \int_D \int_Y b_i(x, y_i)\nabla w_i^k g_{1k} \cdot v_0 dy dx.
\]

We note that
\[
\int_D \int_Y b(x, y)g_1(x) \cdot v_0(x)dx + \int_D \int_Y b(x, y)\nabla_y w_i^k g_{1k} \cdot v_0 dy dx = \int_D \int_Y b(x, y)\left(e^k + \nabla_y w_i^k\right) g_{1k} \cdot v_0 dy dx
= \int_D \int_{Y_{n-1}} b^{n-1}(x, y_{n-1}) g_1 \cdot v_0 dy_{n-1} dx.
\]

Continuing this we have
\[
\int_D \int_Y b(x, y)g_1(x) \cdot v_0(x)dx + \sum_{i=1}^n \int_D \int_Y b_i(x, y_i)\nabla w_i^k g_{1k} dy dx = \int_D b^0(x) g_1 \cdot v_0 dx.
\]

Thus as distribution in \(W'\)
\[
b^0 \frac{\partial u_0}{\partial t} \bigg|_{t=0} = b^0 g_1. \tag{3.10}
\]

Therefore, \(u^0\) is the solution of the problem (3.8) with the initial condition \(u^0(0) = g^0\) and (3.10) which has a unique solution.

The solution \(u\) is written in terms of \(u_0\) as
\[
u_i = N_i^{r_{i-1}}(\delta_{r_{i-1}r_{i-2}} + (\text{curl}_{y_{r_{i-1}}} N_i^{r_{i-1}})) \ldots (\text{curl}_{y_{r_0}}) \ldots (\text{curl} \; u_0), \tag{3.11}
\]
and
\[
\frac{\partial}{\partial t} \nabla_y u_i = \frac{\partial u_0}{\partial t}(\delta_{r_{i-1}} + \frac{\partial w_i^0}{\partial y_{r_{i+1}}})(\delta_{r_{i-2}} + \frac{\partial w_i^0}{\partial y_{r_{i+2}}}) \ldots \nabla_y u_i^{r_{i-1}} - g_{1r_{i}}(\delta_{r_{i-1}} + \frac{\partial w_i^0}{\partial y_{r_{i+1}}})(\delta_{r_{i-2}} + \frac{\partial w_i^0}{\partial y_{r_{i+2}}}) \ldots \nabla_y u_i^{r_{i-1}}. \tag{3.12}
\]

Given that at \(t = 0\), \(\nabla_y u_i = 0\), we then have
\[
\nabla_y u_i = u_0(\delta_{r_{i+1}} + \frac{\partial w_i^0}{\partial y_{r_{i+1}}})(\delta_{r_{i-2}} + \frac{\partial w_i^0}{\partial y_{r_{i+2}}}) \ldots \nabla_y u_i^{r_{i-1}} - g_{1r_{i}}(\delta_{r_{i+1}} + \frac{\partial w_i^0}{\partial y_{r_{i+1}}})(\delta_{r_{i-2}} + \frac{\partial w_i^0}{\partial y_{r_{i+2}}}) \ldots \nabla_y u_i^{r_{i-1}} t
- g_{0r_{i}}(\delta_{r_{i+1}} + \frac{\partial w_i^0}{\partial y_{r_{i+1}}})(\delta_{r_{i-2}} + \frac{\partial w_i^0}{\partial y_{r_{i+2}}}) \ldots \nabla_y u_i^{r_{i-1}} \tag{3.13}
\]
Without loss of generality, we let
\[ u_i = u_{0,i}(\delta_{r_0,i} + \frac{\partial u_{r_0,i}}{\partial y_{1,i}})(\delta_{r_1,i} + \frac{\partial u_{r_1,i}}{\partial y_{2,i}}) \ldots w_{r_{i-1},i}, \]
\[ - g_{0,i}(\delta_{r_0,i} + \frac{\partial u_{r_0,i}}{\partial y_{1,i}})(\delta_{r_1,i} + \frac{\partial u_{r_1,i}}{\partial y_{2,i}}) \ldots w_{r_{i-1},i}. \quad (3.14) \]

4 Regularity of the solution

To deduce the homogenization errors in the next section, we need regularity for the solution $u^0$ of the homogenized equation (3.8), and of the solutions of the cell problems (3.4) and (3.6). We assume:

**Assumption 4.1** The matrix functions $a$ and $b$ belong to $C^1(D, C^2(Y_1,\ldots,C^2(Y_n,\ldots))^{d \times d}$.

With this assumption, we have

**Proposition 4.2** Under Assumption 4.1, for all $i, r = 1,\ldots,d, \text{curl}_y N_i^r \in C^1(D, C^2(Y_1,\ldots,C^2(Y_{r-1}, H^2(Y_r)\ldots))$ and $w_i^r \in C^1(D, C^2(Y_1,\ldots,C^2(Y_{r-1}, H^2(Y_r)\ldots))$.

We refer to [3] for a proof of this proposition.

We have the following regularity results for the solution $u_0$ of the homogenized equation (3.8).

**Proposition 4.3** Under Assumption 4.1, assume

\[
\begin{align*}
&f \in H^2(0; T; H), \\
g_1 \in W, \\
&(b^0)^{-1}[f(0) - \text{curl}(a^0(x)\text{curl}g_0)] \in W, \\
&(b^0)^{-1}[\frac{\partial f}{\partial t}(0) - \text{curl}(a^0(x)\text{curl}g_1)] \in H, \\
\end{align*}
\]

\[ \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0; T; W), \quad \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0; T; H), \quad \text{and} \quad \frac{\partial^4}{\partial t^4} \nabla_y u_i \in L^\infty(0; T; L^2(D \times Y)). \quad (4.2) \]

Further, if

\[
\begin{align*}
&f \in H^3(0; T; H), \\
g_1 \in W, \\
&(b_0)^{-1}[f(0) - \text{curl}(a^0(x)\text{curl}g_0)] \in W, \\
&(b_0)^{-1}[\frac{\partial f}{\partial t}(0) - \text{curl}(a^0(x)\text{curl}g_1)] \in W, \\
&(b_0)^{-1}[\frac{\partial^2 f}{\partial t^2}(0) - \text{curl}(a^0(x)\text{curl}(b_0)^{-1}(f_0 - \text{curl}(a^0(x)\text{curl}g_0))))] \in H, \\
\end{align*}
\]

\[ \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0; T; V), \quad \frac{\partial^4 u_0}{\partial t^4} \in L^\infty(0; T; H), \quad \text{and} \quad \frac{\partial^4}{\partial t^4} \nabla_y u_i \in L^\infty(0; T; L^2(D \times Y)). \quad (4.4) \]

**Proof** We use the regularity theory of general hyperbolic equations (see, e.g., Wloka [14], Chapter 5).

From (4.1) we have that
\[ b^0 \frac{\partial^2}{\partial t^2} \left( \frac{\partial u_0}{\partial t} \right) + \text{curl} \left( a^0 \text{curl} \frac{\partial u_0}{\partial t} \right) = \frac{\partial f}{\partial t} \quad (4.5) \]
with compatibility initial conditions
\[ \frac{\partial u_0}{\partial t}(0) = g_1 \in W, \quad \frac{\partial^2 u_0}{\partial t^2}(0) = (b^0)^{-1}[f(0) - \text{curl} (a^0 \text{curl} g_0)] \in W \]
and
\[ b^0 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 u_0}{\partial t^2} \right) + \text{curl} \left( a^0 \text{curl} \frac{\partial^2 u_0}{\partial t^2} \right) = \frac{\partial^2 f}{\partial t^2}. \quad (4.6) \]
with compatibility initial conditions
\[ \frac{\partial^2 u_0}{\partial t^2}(0) = (\partial_0^{-1})[f(0) - \text{curl}(a^0 \text{curl} g_0)] \in W \quad \text{and} \quad \frac{\partial}{\partial t} \frac{\partial^2 u_0}{\partial t^2}(0) = (\partial_0^{-1})[\frac{\partial f}{\partial t}(0) - \text{curl}(a^0 \text{curl} g_1)] \in H. \]
We thus deduce that
\[ \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0,T; W) \quad \text{and} \quad \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0,T; H). \]
From (3.3) and Proposition 4.2, we deduce that equations (3.7) and (3.5), we have that
\[ \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0,T; L^2(D)). \]

Similarly, we deduce regularity (4.4) from (4.3).

For the regularity of \( u_0 \), we have the following result.

**Proposition 4.4** Under Assumption 4.1 if \( D \) is a Lipschitz polygonal domain, \( f \in H^1(0,T;H) \), \( g_0 \in H^1(\text{curl},D) \) and \( g_1 \in W \), \( \text{div} f \in L^\infty(0,T; L^2(D)) \), \( \text{div}(b^0 g_0) \in L^2(D) \) and \( \text{div}(b^0 g_1) \in L^2(D) \), there is a constant \( s \in (0,1] \) such that \( u_0 \in L^\infty(0,T; H^s(\text{curl},D)) \).

**Proof** Using Proposition 4.2 equations (3.7) and (3.5), we have that \( a^0, b^0 \in C^1(D)^{d \times d} \). As \( f \in H^1(0,T;H) \) and \( g_0 \in H^1(\text{curl},D) \), we have that \( (b^0)^{-1}[f - \text{curl}(a^0 \text{curl} g_0)] \in H \). The compatibility initial conditions hold so that
\[ \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0,T; H). \]

Let \( U(t) = a^0 \text{curl} u_0(t) \). Then
\[ \text{div}((a^0)^{-1}U(t)) = 0 \quad \text{and} \quad (a^0)^{-1}U(t) \cdot \nu = 0, \]
where there is a constant \( c \) and a constant \( s \in (0,1] \) which depend on \( a^0 \) and the domain \( D \) so that
\[ \|U(t)\|_{H^s(D)^3} \leq c(\|\text{curl} U(t)\|_{L^2(D)^3} + \|U(t)\|_{L^2(D)^3}) \]
so \( U \in L^\infty(0,T; H^s(D)^3) \). As \( \text{curl} u_0(t) = ((a^0)^{-1}U(t) \in C^1(D)^{d \times d} \), \( \text{curl} u_0 \in L^\infty(0,T; H^s(D)) \).

We note that
\[ \text{div}(b^0 \frac{\partial^2 u_0}{\partial t^2}) = \text{div}f, \]
so
\[ \text{div}(b^0 u_0(t)) = \int_0^t \int_0^s \text{div} f(s)ds + t \text{div}(b^0 g_1) + \text{div}(b^0 g_0) \in L^\infty(0,T; L^2(D)). \]
From Theorem 4.1 of Hiptmair [8], we deduce that there is a constant \( s \in (0,1] \) (we take it as the same constant as above), so that
\[ \|u_0(t)\|_{H^s(D)^3} \leq c(\|u(t)\|_{H^s(\text{curl},D)}) + \|\text{div}(b^0 u_0(t))\|_{L^2(D)^3}). \]
Thus \( u_0 \in L^\infty(0,T; H^s(\text{curl},D)) \).

Similarly, we can deduce the regularity for \( \frac{\partial^2 u_0}{\partial t^2} \).

**Proposition 4.5** Under Assumption 4.1 if \( D \) is a Lipschitz polygonal domain, if the compatibility conditions (4.3) hold, and if \( \text{div} f \in L^\infty(0,T; L^2(D)) \), then there is a constant \( s \in (0,1] \) such that \( \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0,T; H^s(\text{curl},D)) \).

**Proof** From equation (4.3), we have
\[ \text{curl} \left(a^0 \text{curl} \frac{\partial^2 u_0}{\partial t^2}\right) = \frac{\partial^2 f}{\partial t^2} - b^0 \frac{\partial^4 u_0}{\partial t^4} \in L^\infty(0,T; H) \]
as \( \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0,T; H) \) due to (4.3). Following a similar argument as in the proof of Proposition 4.4 we deduce that \( \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0,T; H^s(D)^3) \). We note that
\[ \text{div} b^0 \frac{\partial^2 u_0}{\partial t^2} = \text{div} f \in L^\infty(0,T; L^2(D)). \]
From Theorem 4.1 of [8], we deduce that \( \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0,T; H^s(D)^3) \).
5 Corrector for the homogenization problem

We derive homogenization correctors in the section. For two-scale problems, with sufficient regularity for the solution of the homogenized equation and the cell problems, we derive explicit homogenization errors in terms of the microscopic scales. We consider both the cases where \( u_0 \) and \( \frac{\partial u_0}{\partial x_0} \) belong to \( L^\infty((0, T); H^1(\text{curl}, D)) \) and where they only belong to a weaker regularity space \( L^\infty((0, T); H^s(\text{curl}, D)) \) for \( 0 < s < 1 \), which is normally the case in polygonal domains \( D \). For multiscale problems, we are not able to prove an explicit homogenization error but a general corrector is derived.

5.1 Corrector for two-scale problem

For conciseness, we denote the coefficients \( a(x, y) \) and \( b(x, y) \) as \( a(x, y) \) and \( b(x, y) \). The cell problems become

\[
\text{curl}_y a(x, y) (e^r + \text{curl}_y N^r(x, y)) = 0,
\]

and

\[
\nabla_y b(x, y) \cdot (e^r + \nabla_x \omega^r(x, y)) = 0.
\]

The homogenized coefficient is determined by

\[
ad^0_{pq}(x) = \int_Y a_{pq}(x, y) (\delta_{kl} + (\text{curl}_y N^q)_k) dy
\]

and

\[
b^0_{pq}(x) = \int_Y b_{pl}(x, y) \left( \delta_{qt} + \frac{\partial \omega^q}{\partial y_l} \right) dy.
\]

We then have the following

**Proposition 5.1** For two-scale problems, assume that \( g_0 = 0, g_1 \in H^1(D)^3 \bigcap W, f \in H^1(0, T; H) \), \( u_0 \in L^\infty(0, T; H^1(\text{curl}; D)), \frac{\partial u_0}{\partial x_0} \in L^\infty(0, T; H^1(\text{curl}, D)), \frac{\partial^2 u_0}{\partial x_0^2} \in L^\infty(0, T; H^1(D)^3), N^r \in C^1(\bar{D}, C(\bar{Y}))^3, \text{curl}_y N^r \in C^1(\bar{D}, C(\bar{Y}))^3 \) and \( \omega^r \in C^1(\bar{D}, C(\bar{Y})) \) for all \( r = 1, 2, 3 \). There exists a constant \( c \) that does not depend on \( \varepsilon \) such that

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} - \left[ \frac{\partial u_0}{\partial t} + \nabla_y \frac{\partial u_1}{\partial t} \left( \bullet, \cdot, \cdot \right) \right] \right\|_{L^\infty(0, T; H)} \leq c\varepsilon^{1/2}.
\]

**Proof** We note that \( \frac{\partial u^\varepsilon}{\partial t} \) satisfies

\[
\frac{\partial^2 u^\varepsilon}{\partial t^2} \left( \frac{\partial u^\varepsilon}{\partial t} + \text{curl} \left( a^\varepsilon \text{curl} \frac{\partial u^\varepsilon}{\partial t} \right) \right) = \frac{\partial f}{\partial t}
\]

with the initial condition \( \frac{\partial u^\varepsilon}{\partial t}(0) = g_1 \in W \) and \( \frac{\partial^2 u^\varepsilon}{\partial t^2}(0) = f(0) \in H \) (due to \( g_0 = 0 \)). We therefore deduce that

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^\infty(0, T; W)} \leq c \left( \left\| \frac{\partial f}{\partial t} \right\|_{L^\infty(0, T; H)} + \|g_1\|_V + \|f(0)\|_H \right)
\]

where \( c \) only depends on the constants \( \alpha \) and \( \beta \) in (2.1). Thus \( \frac{\partial u^\varepsilon}{\partial t} \) is uniformly bounded in \( L^\infty(0, T; W) \) for all \( \varepsilon \).

We consider the function

\[
u_1(t, x) = u_0(t, x) + \varepsilon N^r \left( x, \frac{x}{\varepsilon} \right) \left( \text{curl} u_0(t, x) \right)_r + \varepsilon \nabla u_1 \left( t, x, \frac{x}{\varepsilon} \right)\]

where we have from (3.14)

\[
u_1(t, x, \frac{x}{\varepsilon}) = w^r \left( x, \frac{x}{\varepsilon} \right) \left( u_{0r}(t, x) - g_{1r}(x) t - g_{0r}(x) \right).
\]

We first show that

\[
\left\| \text{curl} \left( a^\varepsilon \text{curl} \nu_1^\varepsilon \right) - \text{curl} \left( a^0(x) \text{curl} u_0 \right) \right\|_{L^\infty(0, T, W^*)} \leq c\varepsilon.
\]
We have
\[
\text{curl } u^\varepsilon_1 = \text{curl } u_0 + \varepsilon \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t,x)) + \varepsilon \nabla (\text{curl } u_0(t,x))_r \times N^r \left( x, \frac{x}{\varepsilon} \right)
\]
\[
+ \varepsilon \text{curl}_y N^r \left( x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t,x))_r
\]
(5.6)

Thus
\[
a^\varepsilon(x) \text{curl } u^\varepsilon_1(t,x) - a^0 \text{curl } u_0(t,x)
\]
\[
= a^\varepsilon \text{curl } u_0 + a^\varepsilon \text{curl}_y N^r \left( x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t,x))_r - a^0(x) \text{curl } u_0(t,x)
\]
\[
+ \varepsilon a^\varepsilon \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t,x)) + \varepsilon a^\varepsilon \nabla (\text{curl } u_0(t,x))_r \times N^r \left( x, \frac{x}{\varepsilon} \right)
\]
\[
= G_r(x, x, x, x)(\text{curl } u_0(x)) + \varepsilon \text{curl } I(t,x)
\]
where the vector functions $G_r(x,y)$ are defined by
\[
(G_r)_i(x,y) = a_{ir}(x,y) + a_{ij}(x,y) (\text{curl}_y N^r(x,y))_j - a^0_{ir}(x),
\]
(5.7)
and
\[
I(t,x) = a \left( x, \frac{x}{\varepsilon} \right) \left[ \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t,x))_r + \nabla (\text{curl } u_0(t,x))_r \times N^r \left( x, \frac{x}{\varepsilon} \right) \right]
\]
From (5.1), we have that $\text{curl}_y G_r(x,y) = 0$. Further from (5.3), $\int_{\Omega} G_r(x,y)dy = 0$. From these we deduce that there are functions $\tilde{G}_r(x,y)$ such that $G_r(x,y) = \nabla_y \tilde{G}_r(x,y)$. Since $\nabla_y \tilde{G}_r(x,\cdot) = G_r(x,\cdot) \in H^1(Y)^3$, we deduce that $\Delta_y \tilde{G}_r(x,\cdot) \in L^2(Y)$. From elliptic regularity, $\tilde{G}_r(x,\cdot) \in H^2(Y) \subset C(Y)$. As $G_r(x,\cdot) \in C^1(D, H^2(Y)^3)$, $\tilde{G}_r(x,y) \in C^1(D, H^2(Y)^3) \subset C^1(D, C(Y))$. We note that
\[
G_r(x, x, x, x) = \nabla_y \tilde{G}_r(x, x, x, x) = \varepsilon \nabla \tilde{G}_r(x, x, x, x) - \varepsilon \nabla_x \tilde{G}_r(x, x, x, x)
\]
Therefore, for all $\phi \in D(D)^d$ we have
\[
\langle \text{curl } (a^\varepsilon \text{curl } u^\varepsilon_1)(t) - \text{curl } (a^0 \text{curl } u_0)(t), \phi \rangle = \int_D G_r(x, x, x, x)(\text{curl } u_0(t,x))_r \text{curl } \phi(x)dx
\]
\[
+ \varepsilon \int_D I(t,x) \text{curl } \phi(x)dx = -\varepsilon \int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) \text{div}((\text{curl } u_0(t,x))_r) \text{curl } \phi(x)dx
\]
\[
- \varepsilon \int_D \nabla_x \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t,x))_r \text{curl } \phi(x)dx + \varepsilon \int_D I(t,x) \text{curl } \phi(x)dx.
\]
We note that
\[
\int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) \text{div}((\text{curl } u_0(t,x))_r) \text{curl } \phi(t,x)dx = \int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) \nabla (\text{curl } u_0(t,x))_r \cdot \text{curl } \phi(t,x).
\]
As $N^3 \in C^1(D, C(Y))$ and $\text{curl } u_0 \in L^\infty(0,T; H^1(D))$ we deduce that
\[
\left| \int_D I(t,x) \cdot \text{curl } \phi(x)dx \right| \leq c \|\text{curl } \phi\|_H
\]
where $c$ is independent of $t$. From these we conclude that
\[
|\langle \text{curl } (a^\varepsilon \text{curl } u^\varepsilon_1)(t) - \text{curl } (a^0(x) \text{curl } u_0)(t), \phi \rangle| \leq c \varepsilon \|\text{curl } \phi\|_{L^2(D)^d}.
\]
Using a density argument, we have that this holds for all $\phi \in W$. Thus
\[
\|\text{curl } (a^\varepsilon \text{curl } u^\varepsilon_1) - \text{curl } a^0 \text{curl } u_0\|_{L^\infty(0,T;W^*')} \leq c \varepsilon.
\]
(5.8)
Let $\tau^\varepsilon(x)$ be a function in $\mathcal{D}(D)$ such that $\tau^\varepsilon(x) = 1$ outside an $\varepsilon$ neighbourhood of $\partial D$ and $\sup_{x \in D} |\nabla \tau^\varepsilon(x)| < c$ where $c$ is independent of $\varepsilon$. Let
\[ w^\varepsilon_1(t, x) = u_0(t, x) + \varepsilon \tau^\varepsilon(x)(x, \frac{x}{\varepsilon}) (\text{curl } u_0(t, x)), + \varepsilon \nabla \left[ \tau^\varepsilon(x) u_1 \left( t, x, \frac{x}{\varepsilon} \right) \right]. \tag{5.9} \]

The function $w^\varepsilon_1(x)$ belongs to $L^2(0; T; H_0(\text{curl}, D))$. We note that
\[ u^\varepsilon_1 - w^\varepsilon_1 = \varepsilon (1 - \tau^\varepsilon(x)) N^\varepsilon \left( x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t, x)), + \varepsilon \nabla \left[ (1 - \tau^\varepsilon(x)) w^\varepsilon_r (x, \frac{x}{\varepsilon}) (u_{0r}(t, x) - g_{1r}(x)t - g_{0r}(x)) \right]. \tag{5.10} \]

From this,
\[ \text{curl } (u^\varepsilon_1 - w^\varepsilon_1) = \varepsilon \text{curl}_x N^\varepsilon \left( x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t, x)), (1 - \tau^\varepsilon(x)) + \varepsilon \nabla \tau^\varepsilon(x) \times N^\varepsilon \left( x, \frac{x}{\varepsilon} \right) + \varepsilon (1 - \tau^\varepsilon(x)) \nabla (\text{curl } u_0(t, x)), N^\varepsilon \left( x, \frac{x}{\varepsilon} \right). \tag{5.11} \]

Since $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0; T; H^1(D))$, we have
\[ \frac{\partial^2 u^\varepsilon_1}{\partial t^2} = \frac{\partial^2 u_0}{\partial t^2} + \varepsilon \tau^\varepsilon N^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial t^2} \text{curl } u_0(t, x), + \varepsilon \nabla \tau^\varepsilon w^r \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial t^2} (t, x)
+ \varepsilon \nabla \tau^\varepsilon w^r \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial t^2} (t, x) + \varepsilon \nabla \tau^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial t^2} (t, x).
\]

Therefore
\[ \frac{b^0}{\varepsilon \tau^\varepsilon} \frac{\partial^2 u_0}{\partial t^2} - b^r \frac{\partial^2 u^\varepsilon_1}{\partial t^2} = \frac{b^0}{\varepsilon \tau^\varepsilon} \frac{\partial^2 u_0}{\partial t^2} - b^r \frac{\partial^2 u_0}{\partial t^2} - \frac{b^r}{\varepsilon \tau^\varepsilon} \frac{\partial^2 u_0}{\partial t^2} (t, x)
+ b^r \varepsilon \nabla \tau^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial t^2} (t, x) + b^r \varepsilon \nabla \tau^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial t^2} (t, x) - b^r \varepsilon \nabla \tau^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial t^2} (t, x).
\tag{5.13} \]

For $\phi \in W$, we have
\[ \left\langle \frac{b^0}{\varepsilon \tau^\varepsilon} \frac{\partial^2 u_0}{\partial t^2} - b^r \frac{\partial^2 u_0}{\partial t^2}, \frac{\partial^2 u_0}{\partial t^2} (t, x) \phi \right\rangle_H
= \int_D b^0 \frac{\partial^2 u_0}{\partial t^2} - b^r \frac{\partial^2 u_0}{\partial t^2} - b^r \frac{\partial^2 u_0}{\partial t^2} (t, x) \phi \, dx
= \int_D \left( b^0 \frac{\partial^2 u_0}{\partial t^2} - b^r \frac{\partial^2 u_0}{\partial t^2}, \frac{\partial^2 u_0}{\partial t^2} (t, x) \phi \right) \, dx
= \int_D F_r (x, \frac{x}{\varepsilon}) \frac{\partial^2 u_0}{\partial t^2} (t, x) \cdot \phi (x) \, dx
\]

where the vector function $F_r$ is defined by
\[ (F_r)_i (x, y) = -b_{ir} (x, y) - b_{ij} (x, y) \frac{\partial w^r}{\partial y_j} (x, y) + b_{ir} (x). \tag{5.14} \]

Since $\text{div}_y F_r (x, y) = 0$, and $\int_y F_r (x, y) \, dy = 0$, there is a function $\tilde{F}_r (x, y)$ such that
\[ F_r (x, y) = \text{curl}_y \tilde{F}_r (x, y). \tag{5.15} \]
Detailed construction of the function $\tilde{F}_r$ is presented in [10] which implies that $F_r \in C^1(D, C(Y))^3$. We note that
\[
F_r(x, \frac{x}{\varepsilon}) = \text{curl}_y F_r(x, \frac{x}{\varepsilon}) = \varepsilon \text{curl} \tilde{F}_r(x, \frac{x}{\varepsilon}) - \varepsilon \text{curl}_y \tilde{F}_r(x, \frac{x}{\varepsilon}).
\]
Therefore
\[
\int_D F_r(x, \frac{x}{\varepsilon}) \frac{\partial^2 u_{0r}}{\partial t^2} \cdot \phi(x) dx = \varepsilon \int_D \tilde{F}_r(x, \frac{x}{\varepsilon}) \cdot \text{curl} (\frac{\partial^2 u_{0r}}{\partial t^2} \phi) dx - \varepsilon \int_D \text{curl}_y \tilde{F}_r(x, \frac{x}{\varepsilon}) \frac{\partial^2 u_{0r}}{\partial t^2} \cdot \phi dx \\
\leq \varepsilon \int_D \text{curl}_y \tilde{F}_r(x, \frac{x}{\varepsilon}) \cdot (\text{curl} \phi \frac{\partial^2 u_{0r}}{\partial t^2} + \phi \times \nabla \frac{\partial^2 u_{0r}}{\partial t^2}) dx \\
\leq c \varepsilon \|\text{curl} \phi\|_{(L^2(D))^3} \|\text{curl}_y \tilde{F}_r\|_{(L^2(D))^3} \]
due to the conditions $\frac{\partial^2 u_{0r}}{\partial t^2} \in L^\infty(0, T; H^1(D))^d$. Let $D^\varepsilon \subset D$ be the $\varepsilon$ neighbourhood of the boundary $\partial D$. We know that
\[
\|\phi\|_{L^2(D^\varepsilon)}^2 \leq c \varepsilon^2 \|\phi\|_{H^1(D)}^2 + c \varepsilon \|\phi\|_{L^2(\partial D)}^2 \\
\leq c \varepsilon \|\phi\|_{H^1(D)}^2
\]
(5.16) for $\phi \in H^1(D)$ (see Hoang and Schwab [11]). From the condition $\text{curl} u_0 \in L^\infty(0, T; H^1(D))^d$, $\frac{\partial u_{0r}}{\partial t} \in L^\infty(0, T; H^1(\text{curl}; D))$ and $\frac{\partial^2 u_{0r}}{\partial t^2} \in L^\infty(0, T; H^1(D)^3)$ we deduce that:
\[
\left\| \frac{\partial}{\partial t} \text{curl} u_0(t) \right\|_{L^2(D^\varepsilon)^3} \leq c \varepsilon^{1/2}, \quad \|\text{curl} u_0(x, t)\|_{(L^2(D^\varepsilon))^3} \leq c \varepsilon^{1/2} \quad \text{and} \quad \left\| \frac{\partial^2}{\partial t^2} u_{0r} \right\|_{(L^2(D^\varepsilon))^3} \leq c \varepsilon^{1/2}. \quad (5.17)
\]
From this and (5.13), we get
\[
\left\| b^r \frac{\partial^2 u_0}{\partial t^2} - b^r \frac{\partial^2 w_{0r}}{\partial t^2}, \phi \right\|_{H^1(D^\varepsilon)} \leq c \varepsilon \|\text{curl} \phi\|_{(L^2(D))^3} + c \varepsilon^{1/2} \|\phi\|_{(L^2(D))^3}. \quad (5.18)
\]
From
\[
b^r \frac{\partial^2 u_0}{\partial t^2} \text{curl} (a^r \text{curl} u_0) = b^r_0 \frac{\partial^2 u_0}{\partial t^2} + \text{curl} (a^0 \text{curl} u_0)
\]
we have
\[
b^r \frac{\partial (u_0^r - w_{0r})}{\partial t^2} + \text{curl} (a^r \text{curl} (u^r - w_{0r}))
\]
\[
= b^r_0 \frac{\partial u_0}{\partial t^2} - b^r \frac{\partial u_{0r}}{\partial t^2} + \text{curl} (a^r \text{curl} (u_0^r - w_{0r})) + \text{curl} (a^0 \text{curl} u_0) - \text{curl} (a^r \text{curl} u_0^r)
\]
We note that
\[
\frac{\partial u_{0r}}{\partial t} = \frac{\partial u_0}{\partial t} + \tau \text{curl} \mathcal{N}^r \left( x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial t} (\text{curl} u_0)_r + \left( \varepsilon \nabla \tau \left( x, \frac{x}{\varepsilon} \right) + \varepsilon \tau \left( x, \frac{x}{\varepsilon} \right) \nabla x \cdot \text{curl} \right) \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial}{\partial t} u_{0r}(t, x) - g_{1r}(x) \right)
\]
and
\[
\text{curl} \frac{\partial u_{0r}}{\partial t} = \text{curl} \frac{\partial u_0}{\partial t} + \tau (x) \left( \varepsilon \nabla \tau \left( x, \frac{x}{\varepsilon} \right) + \varepsilon \tau \left( x, \frac{x}{\varepsilon} \right) \nabla x \cdot \text{curl} \right) \left( x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial t} (\text{curl} u_0)_r + \left( \varepsilon \nabla \tau \left( x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial t} (\text{curl} u_0)_r + \varepsilon \tau \left( x, \frac{x}{\varepsilon} \right) \nabla \frac{\partial}{\partial t} (\text{curl} u_0)_r \right) \times \mathcal{N}^r \left( x, \frac{x}{\varepsilon} \right)
\]
As $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; W \cap H^1(\text{curl}, D))$, we deduce that $\frac{\partial u^I}{\partial t}$ is uniformly bounded in $L^\infty(0, T; W)$. Since $\frac{\partial u^I}{\partial t}(t)$ and $\frac{\partial w^I}{\partial t}(t)$ belong to $W$ and are uniformly bounded in the norm of $W$, from (5.15), we deduce that

\[
\int_0^t \left( b^0 \frac{\partial^2 u_0}{\partial t^2} - b^0 \frac{\partial^2 u^I}{\partial t^2} + \frac{\partial}{\partial s}(u^I(s) - w^I(s)) \right) \epsilon \leq \epsilon \left\| \frac{\partial}{\partial s}(u^I(s) - w^I(s)) \right\|_W + c \epsilon^{1/2} \max_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t}(u^I(t) - w^I(t)) \right\|_H
\]

From (5.8) we deduce

\[
\int_0^t \left\| (\text{curl}(a^0 \text{curl } u_0) - \text{curl}(a^I \text{curl } u^I), \frac{\partial}{\partial s}(u^I(s) - w^I(s))) \right\|_H \leq \epsilon \left\| \frac{\partial}{\partial s}(u^I - w^I) \right\|_{L^\infty((0,T), W)} \leq \epsilon
\]

where $c$ is independent of $t$. From (5.11) we get

\[
\int_0^t \left\| (\text{curl}(a^I \text{curl}(u^I - w^I)), \frac{\partial}{\partial s}(u^I(s) - w^I(s))) \right\|_H \leq \epsilon \frac{\partial}{\partial s}(u^I - w^I) \right\|_{L^\infty((0,T), W)} \leq \epsilon
\]

Using the initial condition $u_0(0) = w_0(0) = 0$, from (5.17), we deduce that

\[
\int_0^t \left\| (\text{curl}(a^I \text{curl}(u^I - w^I)), \frac{\partial}{\partial s}(u^I(s) - w^I(s))) \right\|_H \leq \epsilon \frac{\partial}{\partial s}(u^I - w^I) \right\|_{L^\infty((0,T), W)} \leq \epsilon
\]

Thus

\[
\int_0^t \left( b^0 \frac{\partial^2 (u^I - w^I)}{\partial s^2} + \text{curl}(a^I \text{curl}(u^I - w^I)), \frac{\partial}{\partial s}(u^I - w^I) \right) \epsilon \leq \epsilon \frac{\partial}{\partial s}(u^I - w^I) \right\|_{L^\infty((0,T), W)} \leq \epsilon
\]

The left hand side equals

\[
\frac{1}{2} \int_0^t \frac{d}{ds} \int_D b^0 \frac{\partial (u^I - w^I)}{\partial s} \cdot \frac{\partial (u^I - w^I)}{\partial s} \, dx + a^I \text{curl}(u^I(s) - w^I(s)) \cdot (u^I(s) - w^I(s)) \right\|_H \geq \alpha \left\| \frac{\partial (u^I - w^I)}{\partial s}(t) \right\|_H^2 - \beta \left\| \frac{\partial (u^I - w^I)}{\partial s}(0) \right\|_H^2 + \alpha \left\| \text{curl}(u^I(t) - w^I(t)) \right\|_H
\]

due to condition (2.1).
Similarly, from (5.11), we have

\[ \frac{\partial u^\varepsilon}{\partial t}(0) = \frac{\partial u_0}{\partial t}(0) - \frac{\partial w^\varepsilon}{\partial t}(0) \]

\[ = \| \frac{\partial u_0}{\partial t}(0) - \frac{\partial w^\varepsilon}{\partial t}(0) \|_H \]

\[ = \| \varepsilon \tau^r N^r(x, \frac{x}{\varepsilon}) - \varepsilon \nabla \left[ \tau^r w^r(x, \frac{x}{\varepsilon}) \left( \frac{\partial u_0}{\partial t} - g^r(x) \right) \right] \|_H \]

\[ = \varepsilon \tau^r N^r(x, \frac{x}{\varepsilon}) \frac{\partial u_0}{\partial t}(0, \cdot) \|_H \]

\[ \leq c \varepsilon \| \frac{\partial}{\partial t} \text{curl} u_0(0) \|_H \]

Thus, we deduce that for all \( t \in (0, T) \)

\[ \| \frac{\partial (u^\varepsilon - w^\varepsilon_1)}{\partial t}(0, \cdot) \|_H^2 + \| \text{curl} (u^\varepsilon - w^\varepsilon_1)(t) \|_H^2 \leq \]

\[ c \varepsilon + c \varepsilon^{1/2} \max_{0 \leq t \leq T} \| \frac{\partial (u^\varepsilon - w^\varepsilon_1)}{\partial t}(t) \|_H + c \varepsilon^{1/2} \max_{t \in [0, T]} \| \text{curl} (u^\varepsilon(t) - w^\varepsilon_1(t)) \|_H \]

which implies

\[ \| \frac{\partial (u^\varepsilon - w^\varepsilon_1)}{\partial t}(t) \|_H + \| \text{curl} (u^\varepsilon(t) - w^\varepsilon_1(t)) \|_H \leq c \varepsilon^{1/2}. \]

Due to \( \frac{\partial u_0}{\partial t} \in \mathcal{L}^\infty(0, T; H^1(D^3)) \) and \( g_1 \in H^1(D)^3 \),

\[ \| \frac{\partial u_0}{\partial t} \|_{\mathcal{L}^2(D^3)} \leq c \varepsilon^{1/2} \text{ and } \| g_1 \|_{\mathcal{L}^2(D^3)} \leq c \varepsilon^{1/2}. \]

From [5.17] and the facts that \( N^r \in C^1(\overline{D}, C(\overline{Y})) \), \( w^r \in C^1(\overline{D}, C(\overline{Y})) \) we have

\[ \| \frac{\partial (u^\varepsilon_1 - w^\varepsilon_1)}{\partial t} \|_{\mathcal{L}^\infty(0,T;H)} \]

\[ \leq \| \varepsilon (1 - \tau^r) N^r(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial t} \text{curl} u_0(t) \]

\[ + \varepsilon (1 - \tau^r) \nabla_x w^r(x, \frac{x}{\varepsilon}) + (1 - \tau^r) \nabla_y w^r(x, \frac{x}{\varepsilon}) \left( \frac{\partial u_0}{\partial t} - g_1^r \right) \]

\[ + \varepsilon (1 - \tau^r) w^r(x, \frac{x}{\varepsilon}) \left( \nabla \frac{\partial u_0}{\partial t} - \nabla g_1^r \right) \|_{\mathcal{L}^\infty(0,T;H)} \]

\[ \leq c \varepsilon^{1/2}. \]

Similarly, from [5.14], we have

\[ \| \text{curl} (u^\varepsilon_1 - w^\varepsilon_1) \|_{\mathcal{L}^\infty(0,T;H)} \leq c \varepsilon^{1/2}. \]

We therefore have

\[ \| \frac{\partial (u^\varepsilon - u^\varepsilon_1)}{\partial t} \|_{\mathcal{L}^\infty(0,T;H)} + \| \text{curl} (u^\varepsilon - u^\varepsilon_1) \|_{\mathcal{L}^\infty(0,T;H)} \leq c \varepsilon^{1/2}. \]

We note that

\[ \frac{\partial u^\varepsilon_1}{\partial t} = \frac{\partial u_0}{\partial t} + \varepsilon N^r(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial t} \text{curl} u_0(t, x) + \varepsilon \nabla_x w^r(x, \frac{x}{\varepsilon}) + \nabla_y w^r(x, \frac{x}{\varepsilon}) \left( \frac{\partial u_0}{\partial t}(t, x) - g_1^r(x) \right) \]

\[ + \varepsilon w^r(x, \frac{x}{\varepsilon}) \left( \nabla \frac{\partial u_0}{\partial t} - \nabla g_1^r \right). \]
Thus
\[
\| \frac{\partial u_1}{\partial t} - \frac{\partial u_0}{\partial t} - \nabla_y w^r (\cdot, \cdot) \left( \frac{\partial u_{0r}}{\partial t} (\cdot) - g_{1r} (\cdot) \right) \|_{L^\infty(0,T; H)} \leq c\varepsilon.
\]
Similarly, from (5.5), we have
\[
\| \text{curl } u_1 - \text{curl } u_0 - \text{curl}_y N^r (\cdot, \cdot) (\text{curl } u_{0r}) \|_{L^\infty(0,T; H)} \leq c\varepsilon.
\]
We then get the conclusion.

Next, we derive the homogenization error where \( u_0 \) only possesses the weaker regularity \( W^{2,\infty}(0, T; H^s(\text{curl } D)) \) for \( 0 < s < 1 \).

**Proposition 5.2** Assume that \( g_0 = 0, g_1 \in H^1(D) \cap W, f \in H^1(0,T; H), u_0 \) and \( \frac{\partial u_0}{\partial t} \) belong to \( L^\infty((0,T); H^s(\text{curl } D)) \), and \( \frac{\partial^2 u_0}{\partial t^2} \) belong to \( L^\infty(0,T; H^s(D)) \) for \( 0 \leq s < 1 \), \( N^r \in C^1(\overline{D}, C(\overline{Y})) \), \( \text{curl}_y N^r \in C^1(\overline{D}, C(\overline{Y})) \), \( r = 1, 2, 3 \). There exists a constant \( c \) that does not depend on \( \varepsilon \) such that
\[
\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} + \nabla_y \frac{\partial u_1}{\partial t} (\cdot, \cdot) \varepsilon \right\|_{L^\infty(0,T; H)} + \left\| \text{curl } u^\varepsilon - \text{curl } u_0 + \text{curl}_y u_1 (\cdot, \cdot) \varepsilon \right\|_{L^\infty(0,T; H)} \leq c\varepsilon^{-s}.
\]

**Proof** We consider a set of \( M \) open cubes \( Q_i \) (\( i = 1, \ldots, M \)) of size \( \varepsilon s_1, s_1 > 0 \) to be chosen later such that \( D \subset \bigcup_{i=1}^M Q_i \) and \( Q_i \cap D \neq \emptyset \). Each cube \( Q_i \) intersects with only a finite number, which does not depend on \( \varepsilon \), of other cubes. We consider a partition of unity that consists of \( \rho_i \) such that \( \rho_i \) has support in \( Q_i \), \( \sum_{i=1}^M \rho_i (x) = 1 \) for all \( x \in D \) and \( |\nabla \rho_i| \leq c\varepsilon^{-s_1} \). For \( r = 1, 2, 3 \) and \( i = 1, \ldots, M \), we denote by
\[
U_i^r (t) = \frac{1}{|Q_i|} \int_{Q_i} \text{curl } u_0 (t,x) \, dx
\]
and
\[
V_i^r (t) = \frac{1}{|Q_i|} \int_{Q_i} u_0 (t,x) \, dx
\]
(as \( u_0 \in H^s(D) \) and \( \text{curl } u_0 \in H^s(D) \), for the Lipschitz domain \( D \), we can extend each of them, separately, continuously outside \( D \) and understand \( u_0 \) and \( \text{curl } u_0 \) as these extensions (see Wloka [?]) Theorem 5.6)). Let \( U_i \) and \( V_i \) denote the vector \( (U_i^1, U_i^2, U_i^3) \) and \( (V_i^1, V_i^2, V_i^3) \) respectively. Let \( B \) be the unit cube in \( \mathbb{R}^3 \). From Poincare inequality, we have
\[
\int_B |\phi - \int_B \phi (x) \, dx|^2 \, dx \leq c \int_B |\nabla \phi (x)|^2 \, dx, \quad \forall \phi \in H^1(B).
\]
By translation and scaling, we deduce that
\[
\int_{Q_i} |\phi - \int_{Q_i} \phi (x) \, dx|^2 \, dx \leq c\varepsilon^{2s_1} \int_{Q_i} |\nabla \phi (x)|^2 \, dx, \quad \forall \phi \in H^1(Q_i)
\]
i.e.
\[
\| \phi - \int_{Q_i} \phi (x) \, dx \|_{L^2(Q_i)} \leq c\varepsilon^{s_1} \| \phi \|_{H^1(Q_i)}.
\]
Together with
\[
\| \phi - \int_{Q_i} \phi (x) \, dx \|_{L^2(Q_i)} \leq c \| \phi \|_{L^2(Q_i)}
\]
we deduce from interpolation that
\[
\| \phi - \int_{Q_i} \phi (x) \, dx \|_{L^2(Q_i)} \leq c\varepsilon^{s_1} \| \phi \|_{H^s(Q_i)}, \forall \phi \in H^s(Q_i).
\]
Thus for \( t \) fixed
\[
\int_{Q_i} |\text{curl } u_0 (t,x) - U_i^r (t)|^2 \, dx \leq c\varepsilon^{2s_1} ||(\text{curl } u_0)_r||_{H^s(Q_i)}^2. \quad (5.19)
\]
We consider the function
\[ u_1^\varepsilon(t,x) = u_0(t,x) + \varepsilon N^r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + \varepsilon \nabla \left[ \omega^r \left( x, \frac{x}{\varepsilon} \right) \left( V_j^r(t) \rho_j(x) - g_1^r(t) - g_0^r(x) \right) \right]. \]

We first show that
\[ \| \text{curl} \left( \alpha^\varepsilon \text{curl} u_1^\varepsilon \right) - \text{curl} \left( \alpha^0 \text{curl} u_0 \right) \|_{L^\infty(0,T,W^1)} \leq c(\varepsilon^{s_1} + \varepsilon^{s_2}). \]

We have
\[ \text{curl} u_1^\varepsilon = \text{curl} u_0 + \varepsilon \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + \varepsilon U_j^r(t) \nabla \rho_j(x) \times N^r \left( x, \frac{x}{\varepsilon} \right) + \varepsilon \text{curl}_y N^r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x). \]

Thus
\[ \alpha^\varepsilon \text{curl} u_1^\varepsilon(t,x) - \alpha^0 \text{curl} u_0(t,x) = \alpha^\varepsilon \text{curl} u_0 + \varepsilon \alpha^\varepsilon \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + \varepsilon U_j^r(t) \nabla \rho_j(x) \times N^r \left( x, \frac{x}{\varepsilon} \right) \]
\[ + \alpha^\varepsilon \text{curl}_y N^r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) - \alpha^0(\varepsilon) \text{curl} u_0(t,x) \]
\[ = \alpha^\varepsilon U_j^r(t) \rho_j(x) + \alpha^\varepsilon \text{curl}_y N^r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) - \alpha^0(\varepsilon) U_j^r(t) \rho_j(x) \]
\[ + \varepsilon \alpha^\varepsilon \left[ \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + U_j^r(t) \nabla \rho_j(x) \times N^r \left( x, \frac{x}{\varepsilon} \right) \right] \]
\[ + (\alpha^\varepsilon - \alpha^0) \left( \text{curl} u_0(t,x) - U_j(t) \rho_j(x) \right). \]

We have
\[ \text{curl} \left( \alpha^\varepsilon \text{curl} u_1^\varepsilon \right) - \text{curl} \left( \alpha^0 \text{curl} u_0 \right)(t,x) = \text{curl} \left[ G_r(x, \frac{x}{\varepsilon}) U_j^r(t) \rho_j(x) \right] + \varepsilon \text{curl} I(t,x) + \text{curl} \left[ \left[ \alpha^\varepsilon - a_0 \left( \text{curl} u_0(t,x) - U_j(t) \rho_j(x) \right) \right] \right] \]
where the vector functions \( G_r(x,y) \) are defined in (5.7) and
\[ I(t,x) = \alpha^\varepsilon \left( \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + U_j^r(t) \nabla \rho_j(x) \times N^r \left( x, \frac{x}{\varepsilon} \right) \right). \]

For all \( \phi \in \mathcal{D}(D)^d \) we have
\[ \langle \text{curl} \left( \alpha^\varepsilon \text{curl} u_1^\varepsilon \right)(t), \phi \rangle = \int_D G_r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \text{curl} \phi(x) \right) \text{dx} \]
\[ + \varepsilon \int_D I(t,x) \text{curl} \phi(x) \text{dx} + \int_D (\alpha^\varepsilon - a_0) \left( \text{curl} u_0(t,x) - U_j^r(t) \rho_j(x) \right) \cdot \text{curl} \phi(x) \text{dx} \]
\[ = -\varepsilon \int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) \text{div}(U_j^r(t) \rho_j(x) \text{curl} \phi(x)) \text{dx} + \varepsilon \int_D \nabla \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \text{curl} \phi(x) \text{dx} \]
\[ - \varepsilon \int_D I(t,x) \text{curl} \phi(x) \text{dx} + \int_D (\alpha^\varepsilon - a_0) \left( \text{curl} u_0(t,x) - U_j^r(t) \rho_j(x) \right) \cdot \text{curl} \phi(x) \text{dx}, \]
where \( G_r(x,y) = \nabla_y \tilde{G}_r(x,y) \) as shown above. Further,
\[ \int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) \text{div}(U_j^r(t) \rho_j(x) \text{curl} \phi(t,x)) \text{dx} = \int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) U_j^r(t) \text{curl} \phi(t,x) \cdot \text{curl} \phi(t,x). \]

We note that
\[ \left| \int_D \nabla \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) (U_j^r(t) \rho_j) \cdot \text{curl} \phi \text{dx} \right| \leq c\| (U_j^r(t) \rho_j) \|_{L^2(D)} \| \text{curl} \phi \|_{L^2(D)^3}. \]

From
\[ \| (U_j^r(t) \rho_j) \|_{L^2(D)}^2 = \int_D (U_j^r(t))^2 \rho_j(x)^2 \text{dx} + \sum_{i \neq j} \int_D U_i^r U_j^r \rho_i(x) \rho_j(x) \text{dx}, \]
and the fact that the support of each function $\rho_i$ intersects only with the support of a finite number 
(which does not depend on $\varepsilon$) of other functions $\rho_j$ in the partition of unity, we deduce
\[\|(U_j^r(t)\rho_j)\|_{L^2(D)} \leq c \sum_{j=1}^{M} (U_j^r(t))^2|Q_j| = c \sum_{j=1}^{M} \frac{1}{|Q_j|} \left( \int_{Q_j} \text{curl} u_0(t, x) \right)^2 \]
\[\leq c \sum_{j=1}^{M} \int_D \text{curl} u_0(t, x)^2 \, dx \leq c \int_D \text{curl} u_0(t, x)^2 \, dx.\]

Thus
\[\varepsilon \int_D \nabla G_r \left( x, \frac{x}{\varepsilon} \right) (U_j^r(t)\rho_j) \cdot \text{curl} \phi(t, x) \, dx \leq c \varepsilon \|\text{curl} \phi\|_{L^2(D)^3}.\]

We also have
\[\varepsilon \int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) \text{div} \left[ (U_j^r(t)\rho_j)\text{curl} \phi(t, x) \right] \, dx = \varepsilon \int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) \left[ U_j^r(t)\nabla \rho_j(x) \right] \cdot \text{curl} \phi \, dx \]
\[\leq c \varepsilon \|U_j^r(t)\nabla \rho_j\|_{L^2(D)^3} \|\text{curl} \phi\|_{L^2(D)^3}.\]

As the support of each function $\rho_i$ intersects with the support of a finite number of other functions $\rho_j$ 
and $\|\nabla \rho_j\|_{L^\infty(D)} \leq c \varepsilon^{-s_1}$, we have
\[\|U_j^r(t)\nabla \rho_j\|_{L^2(D)} \leq c \sum_{j=1}^{M} (U_j^r(t))^2|Q_j| \|\nabla \rho_j\|_{L^\infty(D)} \leq c \varepsilon^{-2s_1} \sum_{j=1}^{M} (U_j^r(t))^2|Q_j| \leq c \varepsilon^{-2s_1}\]

so
\[\varepsilon \int_D \tilde{G}_r \left( x, \frac{x}{\varepsilon} \right) \text{div} (U_j^r(t)\rho_j\text{curl} \phi(t, x)) \, dx \leq c \varepsilon \|U_j^r\nabla \rho_j\|_{L^2(D)^3} \|\text{curl} \phi\|_{L^2(D)^3} \leq c \varepsilon^{-s_1} \|\text{curl} \phi\|_{L^2(D)^3}.\]

We have further that
\[\langle \text{curl} \left( (a^r - a_0) (\text{curl} u_0(t, x) - U_j(t)\rho_j(x)) \right), \phi \rangle = \int_D (a^r - a_0) (\text{curl} u_0(t, x) - U_j(t)\rho_j(x)) \text{curl} \phi(t, x) \, dx \]
\[\leq c \|\text{curl} u_0(t, x) - U_j(t)\rho_j(x)\|_{L^2(D)^3} \|\text{curl} \phi\|_{L^2(D)^3}.\]

We note that, for $t$ fixed
\[\int_D \|\text{curl} u_0(t, x)\|_{L^2(D)}^2 - U_j^r(t)\rho_j(x) \|^2 \, dx = \int_D \sum_{j=1}^{M} \left( \langle \text{curl} u_0(t, x) \rangle_{r} - U_j^r \right) \rho_j \|^2 \, dx.\]

Using the support property of $\rho_j$, we have from (5.19)
\[\int_D \|\text{curl} u_0(t, x)\|_{L^2(D)}^2 - U_j^r(t)\rho_j(x) \|^2 \, dx \leq c \sum_{j=1}^{M} \int_{Q_j} \left( \langle \text{curl} u_0(t, x) \rangle_{r} - U_j^r \right) \rho_j \|^2 \, dx \]
\[= c \sum_{j=1}^{M} \left[ \int_{Q_j} \text{curl} u_0(t, x)^2 \, dx + \int_{Q_j \times Q_j} \frac{(\text{curl} u_0(t, x) - \text{curl} u_0(t, x'))^2}{|x - x'|^{2s}} \, dx \, dx' \right] \]
\[\leq c \sum_{j=1}^{M} \left[ \|\text{curl} u_0(t, x)\|_{L^2(D)}^2 + \int_{D\times D} \frac{(\text{curl} u_0(t, x) - \text{curl} u_0(t, x'))^2}{|x - x'|^{2s}} \, dx \, dx' \right] \]
\[= c \sum_{j=1}^{M} \|\text{curl} u_0(t, x)\|_{L^2(D)}^2.\]
Thus
\[
\langle \text{curl } [(a^2 - a_0) (\text{curl } u_0(t, x) - U_j(t,\rho_j(x))), \phi) \leq c\varepsilon^{-s_1} \|\text{curl } \phi\|_{L^2(D)^*}.
\]
From these we conclude that
\[
\|\text{curl } [(a^2 \text{curl } u_1^\varepsilon(t)) - (a^0 \text{curl } u_0)(t), \phi]\| \leq c(\varepsilon^{1-s_1} + \varepsilon^{s_s_1})\|\phi\|_{W}.
\]
Using a density argument, we have that this holds for all \(\phi \in W\). Thus
\[
\|\text{curl } [(a^2 \text{curl } u_1^\varepsilon(t)) - (a^0 \text{curl } u_0)]_{L^\infty(0,T;W^*)} \leq c(\varepsilon^{1-s_1} + \varepsilon^{s_s_1}).
\]
(5.23)
Since \(\frac{\partial u_0}{\partial t} \in L^\infty((0,T);H^s(D))\), by an identical argument, we deduce that
\[
\|\text{curl } [(a^2 \text{curl } u_1^\varepsilon(t)) - (a^0 \text{curl } u_0)]_{L^\infty(0,T;W^*)} \leq c(\varepsilon^{1-s_1} + \varepsilon^{s_s_1}).
\]
(5.24)
Choose \(s_1 = \frac{1}{s+1}\) we have
\[
\|\text{curl } [(a^2 \text{curl } u_1^\varepsilon(t)) - (a^0 \text{curl } u_0)]_{L^\infty(0,T;W^*)} \leq c\varepsilon^{-\frac{e}{e+t}}.
\]
and
\[
\|\text{curl } [(a^2 \text{curl } \frac{\partial}{\partial t} u_1^\varepsilon) - a^0(\text{curl } \frac{\partial}{\partial t} u_0)]_{L^\infty(0,T;W^*)} \leq c\varepsilon^{-\frac{e}{e+t}}.
\]
Let \(\tau^\varepsilon(x)\) be a function in \(D(D)\) such that \(\tau^\varepsilon(x) = 1\) outside an \(\varepsilon\) neighbourhood of \(\partial D\) and sup \(\varepsilon|\nabla \tau^\varepsilon(x)| < c\) where \(c\) is independent of \(\varepsilon\). Let
\[
u_1(t, x) = u_0(t, x) + \varepsilon\tau^\varepsilon(x)N^r \left( x, \frac{x}{\varepsilon} \right) U_j^\varepsilon(t)\rho_j(x) + \varepsilon \nabla \left[ \tau^\varepsilon(x) w^r \left( x, \frac{x}{\varepsilon} \right) \right] \left( V_j^\varepsilon(t)\rho_j(x) - g_1^\varepsilon(x)t - g_0^\varepsilon(x) \right).
\]
We then have
\[
u_1^\varepsilon - \nu_1 = \varepsilon(1 - \tau^\varepsilon(x))N^r \left( x, \frac{x}{\varepsilon} \right) U_j^\varepsilon(t)\rho_j(x) + \varepsilon \nabla \left[ (1 - \tau^\varepsilon(x)) w^r \left( x, \frac{x}{\varepsilon} \right) \right] \left( V_j^\varepsilon(t)\rho_j(x) - g_1^\varepsilon(x)t - g_0^\varepsilon(x) \right).
\]
From this,
\[
\text{curl } (\nu_1^\varepsilon - \nu_1) = \varepsilon\text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) U_j^\varepsilon(t)\rho_j(x)(1 - \tau^\varepsilon(x)) + \text{curl}_y N^r \left( x, \frac{x}{\varepsilon} \right) U_j^\varepsilon(t)\rho_j(x)(1 - \tau^\varepsilon(x))
\]
\[- \varepsilon U_j^\varepsilon(t)\rho_j(x) \nabla \tau^\varepsilon(x) \times N^r \left( x, \frac{x}{\varepsilon} \right) + \varepsilon (1 - \tau^\varepsilon(x)) U_j^\varepsilon(t)\nabla \rho_j(x) \times N^r \left( x, \frac{x}{\varepsilon} \right).\]
(5.27)
Let \(\tilde{D}^\varepsilon\) be the \(3\varepsilon^{s_1}\) neighbourhood of \(\partial D\). We note that \(\text{curl } u_0\) is extended continuously outside \(D\). As shown in Hoang and Schwab [9], for \(\phi \in H^1(D^\varepsilon)\),
\[
\|\phi\|_{L^2(\tilde{D}^\varepsilon)} \leq c\varepsilon^{-s_1/2} \|\phi\|_{H^1(D^\varepsilon)}.
\]
From this and
\[
\|\phi\|_{L^2(D^\varepsilon)} \leq M \|\phi\|_{L^2(\tilde{D}^\varepsilon)},
\]
using interpolation we get
\[
\|\phi\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{-s_1/2} \|\phi\|_{H^1(D^\varepsilon)} \leq c\varepsilon^{-s_1/2} \|\phi\|_{H^s(D)}.
\]
(5.28)
for all \(\phi \in H^s(D)\) extended continuously outside \(D\). We then have for \(t\) fixed,
\[
\|U_j^\varepsilon(t)\rho_j\|_{L^2(D^\varepsilon)}^2 \leq c \sum_{j=1}^{M} \int_{Q_j \cap D^\varepsilon} |U_j^\varepsilon(t)|^2 \rho_j^2 \,dx
\]
\[\leq c \sum_{j=1}^{M} |Q_j \cap D^\varepsilon| \frac{1}{|Q_j|^2} \left( \int_{Q_j} (\text{curl } u_0)^2 \,dx \right)^2
\]
\[\leq c \sum_{j=1}^{M} |Q_j \cap D^\varepsilon| \frac{1}{|Q_j|} \int_{Q_j} (\text{curl } u_0)^2 \,dx.
\]
As $\partial D$ is Lipschitz, $D^\varepsilon$ is the $\varepsilon$ neighbourhood of $\partial D$ and $Q_j$ has size $\varepsilon^{s_1}$, $|Q_j \cap D^\varepsilon| \leq \varepsilon^{1+(d-1)s_1}$ so 

$$|Q_j \cap D^\varepsilon| \leq \varepsilon^{1-s_1}. \text{ When } Q_j \cap D^\varepsilon \neq \emptyset, Q_j \subset D^\varepsilon. \text{ Thus}
$$

$$
\|U_j^\varepsilon(t)\|_{L^2(D^\varepsilon)} \leq \varepsilon^{1-s_1} \|\text{curl } u_0(t, x)\|_{L^2(D^\varepsilon)} \leq \varepsilon^{1-s_1+s_1} \|\text{curl } u_0(t, x)\|_{H^s(D)\varepsilon}^2
$$

Thus

$$
\|\text{curl } u_\varepsilon^\varepsilon(t) \cap D^\varepsilon \|_{L^2(D^\varepsilon)} \leq \varepsilon^{1-s_1} \|\text{curl } u_0\|_{H^s(D)\varepsilon}^2.
$$

Similarly, we have

$$
\|U_j^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2 \leq \varepsilon^{2s_1} \sum_{j=1}^M |Q_j \cap D^\varepsilon| \|U_j^\varepsilon(t)\|^2
$$

$$
\leq \varepsilon^{2s_1} \sum_{Q_j \cap D^\varepsilon \neq \emptyset} |Q_j| \int_{Q_j} \|\text{curl } u_0\|_{L^2(D^\varepsilon)}^2 dx
$$

$$
\leq \varepsilon^{2s_1+1} \|\text{curl } u_0\|_{L^2(D^\varepsilon)}^2
$$

Thus

$$
\|\varepsilon(1-\tau^\varepsilon(x))U_j^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2 \leq \varepsilon^{1-s_1+1} \|\text{curl } u_0\|_{L^2(D^\varepsilon)}^2.
$$

Therefore

$$
\|\text{curl } u_\varepsilon^\varepsilon(t) \cap D^\varepsilon \|_{L^2(D^\varepsilon)}^2 \leq \varepsilon(1-s_1+1)+\varepsilon^{1-s_1+1}/2.
$$

Arguing as above (with $t$ fixed) we deduce that

$$
\|V_j^\varepsilon(t)\|_{L^2(D^\varepsilon)} \leq \varepsilon^{(1-s_1+1)/2}, \quad \|V_j^\varepsilon(t)\|_{L^2(D^\varepsilon)} \leq \varepsilon^{(1-s_1+1)/2-s_1}.
$$

We have

$$
\frac{\partial^2 w_j^\varepsilon}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + \varepsilon \tau^\varepsilon N^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 U_j^\varepsilon}{\partial t^2}(t)\rho_j(x) + \varepsilon \nabla \tau^\varepsilon w^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^\varepsilon}{\partial t^2}(t)\rho_j(x)
$$

$$
+ \varepsilon \varepsilon \nabla \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^\varepsilon}{\partial t^2}(t)\rho_j(x) + \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 \varepsilon}{\partial t^2}(t)\rho_j(x).
$$

Thus

$$
\frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + \varepsilon \varepsilon \nabla \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 U_j^\varepsilon}{\partial t^2}(t)\rho_j(x) - \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^\varepsilon}{\partial t^2}(t)\rho_j(x)
$$

$$
+ \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^\varepsilon}{\partial t^2}(t)\rho_j(x) + \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 \varepsilon}{\partial t^2}(t)\rho_j(x).
$$

$$
\frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 U_j^\varepsilon}{\partial t^2}(t)\rho_j(x) - \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^\varepsilon}{\partial t^2}(t)\rho_j(x)
$$

$$
+ \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^\varepsilon}{\partial t^2}(t)\rho_j(x) + \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 \varepsilon}{\partial t^2}(t)\rho_j(x).
$$

$$
+ \varepsilon \varepsilon \nabla \rho_j^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 \varepsilon}{\partial t^2}(t)\rho_j(x).
$$

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For $\phi \in W$, we consider

$$J := \left( b^r \frac{\partial^2 V_j}{\partial t^2}(t) \rho_j(x) - b^c \frac{\partial^2 V_j}{\partial t^2}(t) \rho_j(x) - b^c(x) \nabla_y w^r \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x), \phi \right) \bigg|_{W'} W$$

$$= \int_D \left( b^r \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) - b^c \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \right) \phi_i \ dx$$

where the function $F_r(x,y)$ is defined in (5.14). Using the function $\tilde{F}_r$ defined in (5.15), we have

$$J = \int_D \left( \varepsilon \text{curl} \tilde{F}_r(x, \frac{x}{\varepsilon}) - \varepsilon \text{curl} x \tilde{F}_r(x, \frac{x}{\varepsilon}) \right) \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \cdot \phi \ dx.$$

As $\frac{\partial^2 w}{\partial t^2} \in L^\infty(0, T; H^s(D))$, there is a constant $c$ such that for all $t \in (0, T)$

$$\left| \int_D \left( b^c - b^0 \right) \left( \frac{\partial^2 u_j}{\partial t^2} - \frac{\partial^2 V_j}{\partial t^2} \right) \cdot \phi \ dx \right| \leq c \left\| \phi \right\|_H$$

Thus

$$\left\| \left( \frac{b^r \partial^2 u_j}{\partial t^2} - \frac{b^c \partial^2 w_i}{\partial t^2} \right), \phi \right\|_{W'} W$$

$$\leq \varepsilon \left\| \text{curl} \phi \right\|_{L^2(D)^s} + \varepsilon \left\| \text{curl} \phi \right\|_{L^2(D)^s}$$

when we choose $s_1 = 1/(1 + s)$. Using

$$b^c \frac{\partial u^c}{\partial t^2} + \text{curl} (a^c \text{curl} u^c) = b^r \frac{\partial^2 u_0}{\partial t^2} + \text{curl} (a^0 \text{curl} u_0)$$

we have

$$b^c \frac{\partial (u^c - w_i^r)}{\partial t^2} + \text{curl} (a^c \text{curl} (u^c - w_i^r))$$

$$= b^r \frac{\partial^2 u_0}{\partial t^2} - b^c \frac{\partial^2 w_i}{\partial t^2} + \text{curl} (a^c \text{curl} (u_i - w_i^r)) + \text{curl} (a^0 \text{curl} u_0) - \text{curl} (a^c \text{curl} u_i^r).$$

As shown in the proof of Proposition 5.1, $\frac{\partial^2 \rho_j}{\partial t^2}$ is uniformly bounded in $L^\infty(0, T; W)$ with respect to $\varepsilon$. For $w_i^r$, we have

$$\frac{\partial w_i^r}{\partial t} = \frac{\partial u_j}{\partial t}(t, x) + \varepsilon \tau^r(x) N^r \left( x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^r}{\partial t}(t) \rho_j(x)$$

$$+ \left( \varepsilon \nabla \tau^r(x) w^r \left( x, \frac{x}{\varepsilon} \right) + \varepsilon \tau^r(x) \nabla x w^r \left( x, \frac{x}{\varepsilon} \right) + \tau^r(x) \nabla y w^r \left( x, \frac{x}{\varepsilon} \right) \right) \left( \frac{\partial V_j^r}{\partial t}(t) \rho_j(x) - g_1 \right)$$

$$+ \varepsilon \tau^r(x) w^r \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial V_j^r}{\partial t}(t) \nabla \rho_j(x) - \nabla g_1 \right)$$

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We also have
\[
\nabla \partial_{tt} u = \nabla \frac{\partial u_0}{\partial t}(t, x) + \tau^* (x) \frac{\partial U^r_j (t)}{\partial t}(t) \rho_j(x) \left( \varepsilon \nabla_2 N^r \left( x, \frac{x}{\varepsilon} \right) + \nabla_1 N^r \left( x, \frac{x}{\varepsilon} \right) \right)
\]
\[
+ \frac{\partial U^r_j (t)}{\partial t} \left( \varepsilon \nabla_2 \tau^* (x) \rho_j(x) + \varepsilon \tau^* (x) \nabla \rho_j(x) \right) \times N^r \left( x, \frac{x}{\varepsilon} \right).
\]

As \( \frac{\partial u_0}{\partial t} \in L^\infty(0, T; H(\text{curl}, D)), \) \( \| \frac{\partial u_0}{\partial t} \|_{L^2(D)^3} \leq c, \) \( \| \frac{\partial U^r_j}{\partial t} \rho_j \|_{L^2(D)^3} \leq c, \) \( \| \frac{\partial U^r_j}{\partial t} \nabla \rho_j \|_{L^2(D)^3} \leq c \varepsilon^{-s_1} \) and \( \| \frac{\partial u_0}{\partial t} \|_{L^2(D)^3} \leq c \varepsilon^{-s_1}. \) Therefore \( \frac{\partial u_0}{\partial t} \) and \( \frac{\partial u_0}{\partial t} \) are uniformly bounded in \( H \) with respect to \( \varepsilon, \) i.e. \( \frac{\partial u_0}{\partial t} \) is uniformly bounded in \( L^\infty(0, T; H^1(\text{curl}, D)). \)

We have
\[
\int_0^t \left\langle \partial^2 u_0 - \beta \partial w^2(s) - \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_H dt 
\leq \int_0^t \left( c \varepsilon \| \nabla \frac{\partial(\tau^* - w^*)}{\partial s}(s) \|_H + c \varepsilon \sup_{0 \leq s \leq T} \| \frac{\partial(\tau^* - w^*)}{\partial s}(s) \|_H \right) dt 
\leq c \varepsilon + c \varepsilon \sup_{0 \leq s \leq T} \| \frac{\partial(\tau^* - w^*)}{\partial s}(s) \|_H.
\]

We also have
\[
\int_0^t \left\langle \nabla \partial_{tt} u_0(s) - \nabla \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{W', W'} ds = 
\int_0^t \partial \partial_{tt} \left\langle \nabla \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{W', W'} ds
\]
\[
\int_0^t \partial \partial_{tt} \left\langle \nabla \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{W', W'} ds
\]
\[
\int_0^t \left\langle \nabla \partial_{tt} u_0(s) - \nabla \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{W', W'} ds.
\]

Since \( u_0(0) = 0 \) and \( u_1(0) = 0, \) together with (5.24) we have that
\[
\left\| \int_0^t \left\langle \nabla \partial_{tt} u_0(s) - \nabla \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{H} ds \right\|
\leq \left\| \int_0^t \left\langle \nabla \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{H} ds \right\|

\leq c \varepsilon \sup_{0 \leq s \leq T} \| \frac{\partial(\tau^* - w^*)}{\partial s}(s) \|_{H}.
\]

Now we estimate
\[
\int_0^t \left\langle \nabla \partial_{tt} (u_1(s) - w^*_1), \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{H} ds
\]
using (5.24). We have that
\[
\left\| \int_0^t \left\langle \nabla \partial_{tt} (u_1(s) - w^*_1), \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{H} ds \right\|
\leq c \varepsilon \sup_{0 \leq s \leq T} \| \frac{\partial(\tau^* - w^*)}{\partial s}(s) \|_{H} \leq c \varepsilon;
\]
and
\[
\left\| \int_0^t \left\langle \nabla \partial_{tt} (u_1(s) - w^*_1), \partial \frac{\partial(\tau^* - w^*)}{\partial s}(s) \right\rangle_{W', W'} dx \right\|
\leq c \varepsilon \sup_{0 \leq s \leq T} \| \frac{\partial(\tau^* - w^*)}{\partial s}(s) \|_{H} \leq c \varepsilon ^{\frac{1}{3} + \frac{3}{2} - 1} = c \varepsilon ^{\frac{1}{3}}.
For the other two terms in (5.27), we have

\[
\int_0^t \left\langle \mathrm{curl} \left( a^\varepsilon \mathrm{curl}_y N^\tau \left( x, \frac{x}{\varepsilon} \right) U_j^\tau(s) \rho_j(x) (1 - \tau^\varepsilon(x)) \right), \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial s} \right\rangle \, ds
\]

\[
= \int_0^t \frac{\partial}{\partial s} \int_D a^\varepsilon(x) \mathrm{curl}_y N^\tau \left( x, \frac{x}{\varepsilon} \right) U_j^\tau(s) \rho_j(x) (1 - \tau^\varepsilon(x)) \cdot \mathrm{curl} (u^\varepsilon(s) - w_1^\varepsilon(s)) \, ds + \int_D a^\varepsilon(x) \mathrm{curl}_y N^\tau \left( x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^\tau}{\partial s}(s) \rho_j(x) (1 - \tau^\varepsilon(x)) \cdot \mathrm{curl} (u^\varepsilon(s) - w_1^\varepsilon(s)) \, ds.
\]

Thus

\[
\left| \int_0^t \left\langle \mathrm{curl} \left( a^\varepsilon \mathrm{curl}_y N^\tau \left( x, \frac{x}{\varepsilon} \right) U_j^\tau(s) \rho_j(x) (1 - \tau^\varepsilon(x)) \right), \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial s} \right\rangle \right| \leq \int_0^t \left\| \frac{\partial U_j^\tau}{\partial s}(s) \rho_j \right\|_{L^2(D')} \left\| \mathrm{curl} (u^\varepsilon(s) - w_1^\varepsilon(s)) \right\|_{L^2(D)} \, ds \leq c \varepsilon \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}} \sup_{0 \leq t \leq T} \left\| \mathrm{curl} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_{L^2(D)}.\]

as \( u_0 \) and \( \frac{\partial u_0}{\partial t} \) belong to \( L^\infty(0, T; H^1(D)) \). Similarly, we have

\[
\int_0^t \left\langle \varepsilon \mathrm{curl} \left( a^\varepsilon U_j^\tau(s) \rho_j(x) \nabla \tau^\varepsilon(x) \times N^\tau \left( x, \frac{x}{\varepsilon} \right) \right), \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial s} \right\rangle \, ds \leq c \varepsilon \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}} \sup_{0 \leq t \leq T} \left\| \mathrm{curl} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_{L^2(D)}.\]

Therefore,

\[
\left| \int_0^t \left\langle \mathrm{curl} (a^\varepsilon \mathrm{curl}_y (u_0^\varepsilon(s) - w_1^\varepsilon(s))), \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial s} \right\rangle \right| \leq c \varepsilon + c \varepsilon \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}} \sup_{0 \leq t \leq T} \left\| \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t} \right\|_{L^2(D)} + c \varepsilon \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}} \sup_{0 \leq t \leq T} \left\| \mathrm{curl} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_{L^2(D)}.
\]

We then deduce

\[
\left| \int_0^t \left\langle b^\varepsilon \frac{\partial^2 (u^\varepsilon - w_1^\varepsilon)}{\partial s^2} \right\rangle \left( \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial s} \right) \right| \leq c \varepsilon + c \varepsilon \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}} \sup_{0 \leq t \leq T} \left\| \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t} \right\|_{L^2(D)} + c \varepsilon \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}} \sup_{0 \leq t \leq T} \left\| \mathrm{curl} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_{L^2(D)}.
\]

Therefore

\[
\frac{1}{2} \int_D b^\varepsilon(x) \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \cdot \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \, dx + \frac{1}{2} \int_D a^\varepsilon(x) \mathrm{curl} (u^\varepsilon(t) - w_1^\varepsilon(t)) \cdot \mathrm{curl} (u^\varepsilon(t) - w_1^\varepsilon(t)) \, dx \leq c \varepsilon \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}} \sup_{0 \leq t \leq T} \left\| \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t} \right\|_{L^2(D)} + c \varepsilon \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}} \sup_{0 \leq t \leq T} \left\| \mathrm{curl} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_{L^2(D)}.
\]

Therefore

\[
1 \int_D b^\varepsilon(x) \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t}(0) \cdot \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t}(0) \, dx + 1 \int_D a^\varepsilon(x) \mathrm{curl} (u^\varepsilon(0) - w_1^\varepsilon(0)) \cdot \mathrm{curl} (u^\varepsilon(0) - w_1^\varepsilon(0)) \, dx.
\]
(note that \( u^\varepsilon(0) = w_1^\varepsilon(0) = 0 \)). We have
\[
\left\| \frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H
\leq \left\| \frac{\partial u_0}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H
\]
\[
+ \left\| \varepsilon \tau^r N^r \left( x, \frac{x}{\varepsilon} \right) \frac{\partial U^r}{\partial t}(0) \rho_j(x) + \varepsilon \nabla \left( \tau^r w^r \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial V^r}{\partial t}(0) \rho_j(x) - g_1^r(x) \right) \right) \right\|_H
\]
\[
+ \varepsilon \tau^r \frac{\partial V^r}{\partial t}(0) \rho_j(x)
+ \varepsilon \tau^r \frac{\partial V^r}{\partial t}(0) \nabla \rho_j(x) - \nabla g_1^r(x) \right\|_H.
\]
As \( g_1 \in H^1(D)^3 \), a similar argument as for showing \([5.22]\) for \( s = 1 \) shows that
\[
\left\| \frac{\partial V^r}{\partial t}(0) \rho_j(x) - g_1^r(x) \right\|_H \leq \varepsilon^{s_1} < \varepsilon \tau^{s_1}.
\]
Further,
\[
\left\| \frac{\partial V^r_j}{\partial t}(0) \nabla \rho_j(x) \right\|_{L^2(D^r)^3} \leq \varepsilon \tau^{-s_1}.
\]
Thus
\[
\left\| \frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H \leq \varepsilon \tau^{-s_1} + \varepsilon^{1-s_1} \leq \varepsilon \tau^{-s_1}.
\]
Using \([5.21]\), we get
\[
\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H^2 + \left\| \text{curl}(u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_H^2
\leq \varepsilon \tau^{-s_1} + \varepsilon \tau^{-s_1} \max_{0 \leq t \leq T} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H + \varepsilon \tau^{-s_1} \max_{0 \leq t \leq T} \left\| \text{curl}u^\varepsilon(t) - w_1^\varepsilon(t) \right\|_H.
\]
From this we deduce that for all \( t \in (0, T) \)
\[
\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H + \left\| \text{curl}(u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_H \leq \varepsilon \tau^{-s_1}.
\]  \hspace{1cm} (5.34)

From \([5.26]\) we have
\[
\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) = \varepsilon (1 - \tau^r(x)) N^r \left( x, \frac{x}{\varepsilon} \right) \frac{\partial U^r}{\partial t}(t) \rho_j(x)
\]
\[
+ \left[ -\varepsilon \nabla \tau^r(x) w^r \left( x, \frac{x}{\varepsilon} \right) + \varepsilon (1 - \tau^r(x)) \nabla_x w^r \left( x, \frac{x}{\varepsilon} \right) + (1 - \tau^r(x)) \nabla_y w^r \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial V^r}{\partial t}(t) \rho_j(x) - g_1^r \right) \right.
\]
\[
+ \varepsilon (1 - \tau^r(x)) w^r \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial V^r}{\partial t}(t) \nabla \rho_j(x) - \nabla g_1^r \right).
\]
Therefore, using \( g_1 \in H^1(D)^3 \) we get
\[
\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H \leq \varepsilon c \left\| \frac{\partial U^r}{\partial t}(t) \rho_j \right\|_{L^2(D^r)^3} + \left\| \frac{\partial V^r}{\partial t}(t) \rho_j \right\|_{L^2(D^r)^3} + \varepsilon ||g_1||_{L^2(D^r)^3} + \varepsilon \left\| \frac{\partial V^r}{\partial t}(t) \nabla \rho_j - \nabla g_1^r \right\|_{L^2(D^r)^3}.
\]
As \( \frac{\partial}{\partial t} \text{curl} u_0 \in L^\infty(0,T;H^s(D)^3) \), \( \frac{\partial}{\partial t} u_0 \in L^\infty(0,T;H^s(D)^3) \) and \( g_1 \in H^1(D)^3 \), we deduce that

\[
\begin{align*}
\left\| \frac{\partial U^r}{\partial t}(t) \rho_j \right\|_{L^2(D^3)^3} & \leq c, \\
\left\| \frac{\partial V^r}{\partial t}(t) \rho_j \right\|_{L^2(D^3)^3} & \leq c^\frac{1-s}{2}, \\
\left\| \frac{\partial V^r}{\partial t}(t) \nabla \rho_j \right\|_{L^2(D^3)^3} & \leq c^\frac{1-s}{2}, \\
\left\| g_1 \right\|_{L^2(D^3)^3} & \leq c^{1/2}.
\end{align*}
\]

Therefore

\[
\left\| \frac{\partial (u_1^\varepsilon - u_1^\varepsilon)}{\partial t}(t) \right\| \leq c^\frac{1-s}{2} + \varepsilon c^\frac{1-s}{2} + c^{1/2} \leq c^{1/\varepsilon}.
\tag{5.35}
\]

From (5.31), (5.34) and (5.35), we deduce that

\[
\left\| \frac{\partial (u_1^\varepsilon - u_1^\varepsilon)}{\partial t}(t) \right\|_H + \| \text{curl} (u_1^\varepsilon(t) - u_1^\varepsilon(t)) \|_H \leq c^{1/\varepsilon}.
\tag{5.36}
\]

We note that

\[
\left\| \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) U^r_j \rho_j \right\|_H \leq c\varepsilon, \quad \text{and} \quad \left\| \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) \times \left( U^r_j \nabla \rho_j \right) \right\|_H \leq c\varepsilon^{-s_1} = c^\frac{1}{1-s_1}
\]

so from (5.21)

\[
\| \text{curl} u_1^\varepsilon - [\text{curl} u_0 + \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) U^r_j \rho_j] \|_H \leq c^{1/\varepsilon}.
\]

Thus we deduce from (5.22) that

\[
\| \text{curl} u_1^\varepsilon - [\text{curl} u_0 + \text{curl}_x N^r \left( x, \frac{x}{\varepsilon} \right) u_0(x)] \|_H \leq c^\frac{1}{1-s_1}.
\tag{5.37}
\]

We further have

\[
\begin{align*}
\frac{\partial u_1^\varepsilon}{\partial t}(t) = \frac{\partial u_0}{\partial t}(t) + \varepsilon N^r \left( x, \frac{x}{\varepsilon} \right) \frac{\partial U^r}{\partial t}(t) \rho_j(x) + \left( \varepsilon \nabla_x w^r \left( x, \frac{x}{\varepsilon} \right) + \nabla_y w^r \left( x, \frac{x}{\varepsilon} \right) \right) \left( \frac{\partial V^r}{\partial t}(t) \rho_j - g_{1r} \right) + \\
\varepsilon w^r \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial V^r}{\partial t}(t) \nabla \rho_j(x) - \nabla g_{1r}(x) \right).
\end{align*}
\]

As

\[
\left\| N^r \left( \cdot, \frac{\cdot}{\varepsilon} \right) \frac{\partial U^r}{\partial t}(t) \rho_j \right\|_H \leq c, \quad \left\| \frac{\partial V^r}{\partial t} \rho_j \right\|_H \leq c, \quad \left\| \frac{\partial V^r}{\partial t} \nabla \rho_j \right\|_H \leq c^{-s_1}
\]

we have that

\[
\left\| \frac{\partial u_1^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y w^r \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial V^r}{\partial t}(t) \rho_j - g_{1r} \right) \right\|_H \leq c^{1/\varepsilon}.
\]

Using

\[
\left\| \frac{\partial V^r}{\partial t}(t) \rho_j(x) - \frac{\partial u_0}{\partial t}(t) \right\|_H \leq c^{s_1} = c^{1/s_1}
\]

we deduce that

\[
\left\| \frac{\partial u_1^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y w^r \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial u_0}{\partial t}(t) - g_{1r}(x) \right) \right\|_H \leq c^{1/\varepsilon}.
\tag{5.38}
\]

From (5.36), (5.37) and (5.38), we get

\[
\left\| \frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y w^r \left( \cdot, \frac{\cdot}{\varepsilon} \right) \left( \frac{\partial u_0}{\partial t}(t) - g_{1r} \right) \right\|_H + \\
\| \text{curl} u^\varepsilon - \text{curl} u_0 - \text{curl}_x N^r \left( \cdot, \frac{\cdot}{\varepsilon} \right) \text{curl} u_0(\cdot) \|_H \leq c^{1/\varepsilon}.
\]

\[\square\]
5.2 Corrector for multiscale problem

For the case of more than two scales, we cannot deduce an explicit homogenization error. However, we can deduce correctors for the case where $\varepsilon_{i-1}/\varepsilon_i$ is an integer for all $i = 2, \ldots, n$. We use the operator $T_n^\varepsilon$ and $U_n^\varepsilon$ defined as follows. We define the map

$$T_n^\varepsilon(\phi)(x, y) = \phi\left(\frac{x}{\varepsilon_1} + \varepsilon_2 \frac{y_1}{\varepsilon_1} + \ldots + \varepsilon_n \frac{y_{n-1}}{\varepsilon_{n-1}} + \varepsilon_n y_n\right)$$

for $\phi \in L^1(D)$ extended to 0 outside $D$. Letting $D^{\varepsilon_1}$ be the $2\varepsilon_1$ neighbourhood of $D$, we have

$$\int_D \phi dx = \int_{D^{\varepsilon_1}} \int_{Y_1} \cdots \int_{Y_n} T_n^\varepsilon(\phi) dy_n \cdots dy_1 dx$$

(5.39)

for all $\phi \in L^1(D)$. If a sequence $\{\phi^\varepsilon\}$ $(n+1)$-scale converges to $\phi(x, y_1, \ldots, y_n)$, then

$$T_n^\varepsilon(\phi) \rightarrow \phi(x, y_1, \ldots, y_n)$$

in $L^2(D \times Y_1 \times \ldots \times Y_n)$. We define the operator $U_n^\varepsilon$

$$U_n^\varepsilon(\Phi)(x) = \int_{Y_1} \cdots \int_{Y_n} \Phi\left(\frac{x}{\varepsilon_1} + \varepsilon_1 t_1, \frac{y_1}{\varepsilon_1} + \varepsilon_2 t_2, \ldots, \frac{x}{\varepsilon_n - 1} + \varepsilon_{n-1} t_{n-1}, \frac{x}{\varepsilon_n} + \varepsilon_n t_n\right) dt_n \cdots dt_1$$

for all functions $\Phi \in L^1(D \times Y)$. For each function $\Phi \in L^1(D \times Y)$ we have

$$\int_{D^{\varepsilon_1}} U_n^\varepsilon(\Phi) dx = \int_D \int_Y \Phi(x, y) dy dx.$$  

(5.40)

The proofs for (5.39) and (5.40) may be found in [7]. We then have:

$$T_n^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t}\right) \rightarrow \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla u_i,$$

(5.41)

and

$$T_n^\varepsilon(\text{curl } u^\varepsilon) \rightarrow \text{curl } u_0 + \sum_{i=1}^n \text{curl}_y u_i$$

(5.42)

in $L^2(D \times Y)$ when $\varepsilon \rightarrow 0$.

We have the following result.

**Theorem 5.3** Assume that $g_0 = 0$, $g_1 \in W$ and $f \in H^1(0, T; H)$. We have

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{\partial u^\varepsilon}{\partial t} - U_n^\varepsilon \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \nabla u_i \cdot \frac{\partial}{\partial t} u_i\right)\right\|_{L^\infty(0, T; H)} + \left\| \text{curl } u^\varepsilon - U_n^\varepsilon \left(\text{curl } u_0 + \sum_{i=1}^n \text{curl}_y u_i\right)\right\|_{L^\infty(0, T; H)} = 0.

**Proof** We consider

$$E^\varepsilon(t) = \int_D \int_Y T_n^\varepsilon(b^\varepsilon) \left(\frac{\partial u^\varepsilon}{\partial t}\right) (t) - \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla u_i\right) (t)$$

$$+ \int_D \int_Y T_n^\varepsilon(\text{curl } u^\varepsilon) \cdot \left(\text{curl } u^\varepsilon - \left(\text{curl } u_0 + \sum_{i=1}^n \text{curl}_y u_i\right)\right)$$

$$+ \int_D \int_Y T_n^\varepsilon(\text{curl } u^\varepsilon) - \left(\text{curl } u_0 + \sum_{i=1}^n \text{curl}_y u_i\right)$$

$$\cdot \left(\frac{\partial u^\varepsilon}{\partial t}\right) (t) - \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla u_i\right) (t) \, dy \, dx.$$

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We note that
\[
\lim_{\varepsilon \to 0} \int_D \int_Y \left[ \mathcal{T}_n(b^\varepsilon) \mathcal{T}_n^{\varepsilon} \left( \frac{\partial u^\varepsilon}{\partial t} \right) \cdot \mathcal{T}_n^{\varepsilon} \left( \frac{\partial u^\varepsilon}{\partial t} \right) + \mathcal{T}_n^{\varepsilon} (a^\varepsilon) \mathcal{T}_n^{\varepsilon} (\text{curl } u^\varepsilon) \cdot \mathcal{T}_n^{\varepsilon} (\text{curl } u^\varepsilon) \right] \, dy \, dx
\]
\[
= \lim_{\varepsilon \to 0} \int_D \left[ b^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial t} + a^\varepsilon(x) \text{curl } u^\varepsilon(t, x) \cdot \text{curl } u^\varepsilon(t, x) \right] \, dx \, dt
\]
\[
= \lim_{\varepsilon \to 0} \int_D b^\varepsilon(x) g_1(x) \cdot g_1(x) \, dx + 2 \int_0^t \int_D f(t, x) \frac{\partial u^\varepsilon}{\partial t} \, dx \, dt
\]
\[
= \int_D \left( \int_Y b(x, y) \, dy \right) g_1(x) \cdot g_1(x) \, dx + 2 \int_0^t \int_D f(t, x) \frac{\partial u^\varepsilon}{\partial t} \, dx \, dt
\]
where we have used the energy formula for wave equation (see Lions and Magenes [11]) and the initial condition \( g_0 = 0 \). Using (5.11) and (5.12), we have
\[
\lim_{\varepsilon \to 0} E^\varepsilon(t) = \int_D \left( \int_Y b(x, y) \, dy \right) g_1(x) \cdot g_1(x) \, dx + 2 \int_0^t \int_D f(t, x) \frac{\partial u_0}{\partial t} \, dx \, dt
\]
\[
- \int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y y_i u_i \right) \cdot \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y y_i u_i \right) \, dy \, dx
\]
\[
- \int_D \int_Y a(x, y) (\text{curl } u_0 + \sum_{i=1}^n \text{curl}_y y_i u_i) \cdot (\text{curl } u_0 + \sum_{i=1}^n \text{curl}_y y_i u_i).
\]
From (3.2) and (3.9), we have
\[
\int_D \int_Y b(x, y) \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y y_i u_i \right) \cdot \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_y y_i u_i \right) \, dy \, dx
\]
\[
= \int_D \int_Y b(x, y) \left[ \frac{\partial u_0}{\partial t} + \sum_{i=1}^{n-1} \nabla_y y_i u_i \right] + \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^{n-1} \nabla_y y_i u_i \right) \nabla_y w_n^k
\]
\[
\cdot \left[ \frac{\partial u_0}{\partial t} + \sum_{i=1}^{n-1} \nabla_y y_i u_i \right] + \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^{n-1} \nabla_y y_i u_i \right) \nabla_y w_n^k
\]
\[
= \int_D \int_Y b(x, y) \left[ \frac{\partial u_0}{\partial t} + \sum_{i=1}^{n-1} \nabla_y y_i u_i \right] \left( \frac{\partial}{\partial t} + \sum_{i=1}^{n-1} \nabla_y y_i u_i \right) \left( \frac{\partial}{\partial t} + \sum_{i=1}^{n-1} \nabla_y y_i u_i \right) \nabla_y w_n^k
\]
\[
- g_1 \nabla_y w_n^l \nabla_y w_n^l.
\]
From (3.1) and (3.3), this equals
\[
\int_D \int_Y b^{(n-1)} i \frac{\partial}{\partial t} \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^{n-1} \nabla_y y_i u_i \right) dx + \int_D \int_Y b_i(x, y_i) \nabla_y u_i^k \cdot \nabla_y w_i^j \nabla_y g_i \nabla_y g_i \, dy_i \, dx.
\]
Continuing this process, this expression equals
\[
\int_D b^n(x) \frac{\partial u_0}{\partial t} \frac{\partial u_0}{\partial t} dx + \sum_{i=1}^n \int_D \int_Y b_i(x, y_i) \nabla_y u_i^k \cdot \nabla_y w_i^j \nabla_y g_i \nabla_y g_i \, dy_i \, dx.
\]
On the other hand we have
\[
\int_D \int_Y a(x, y) (\text{curl } u_0 + \sum_{i=1}^n \text{curl}_y y_i u_i) \cdot (\text{curl } u_0 + \sum_{i=1}^n \text{curl}_y y_i u_i) \, dy \, dx = \int_D a^0(x) \text{curl } u_0 \cdot \text{curl } u_0 \, dx.
\]
Therefore
\[
\lim_{\varepsilon \to 0} E^\varepsilon(t) = \int_D \left( \int_Y b(x,y)dy \right) g_1 \cdot g_1 dx + 2 \int_0^t \int_D \frac{\partial u_0}{\partial t} dx dt - \int_D b_0(x) \frac{\partial u_0}{\partial t} dx
\]
\[= - \sum_{i=1}^n \int_D \int_Y b_i(x,y) \nabla_y w_i^k \cdot \nabla_y w_i^k g_{1k} g_{1l} dy_i dx - \int_D a^0(x) \text{curl } u_0 \cdot \text{curl } u_0 dx.
\]
From (3.3), we get
\[
\int_D b^0(x) \frac{\partial u_0}{\partial t} dx + \int_D a^0(x) \text{curl } u_0 \cdot \text{curl } u_0 dx = \int_D b^0(x) g_1 \cdot g_1 dx + 2 \int_0^t \int_D \frac{\partial u_0}{\partial t} dx dt.
\]
From (3.3) we have
\[
\int_{Y_i} b^i(x,y) \nabla_y w_i^k \cdot \nabla_y w_i^l dy_i = - \int_{Y_i} b^i(x,y) e^k \cdot \nabla_y w_i^l.
\]
Thus
\[
\int_D \left( \int_Y b(x,y)dy \right) g_1 \cdot g_1 dx = \sum_{i=1}^n \int_D \int_{Y_i} b^i(x,y) \nabla_y w_i^k \cdot \nabla_y w_i^l g_{1k} g_{1l} dy_i dx
\]
\[= \int_D \left( \int_Y b(x,y)dy \right) g_1 \cdot g_1 dx + \sum_{i=1}^n \int_D \int_{Y_i} b^i(x,y) e^k \cdot \nabla_y w_i^l g_{1k} g_{1l} dy_i dx.
\]
We consider
\[
\int_D \left( \int_Y b(x,y)dy \right) g_1 \cdot g_1 dx + \int_D \int_Y b(x,y) e^k \cdot \nabla_{y_n} w_{n}^I g_{1k} g_{1l} dy_i dx
\]
\[= \int_D \left( \int_Y b_{ij}(x,y)dy \right) g_{1j}(x) \cdot g_{1i}(x) dx + \int_D \int_Y b_{ij} \frac{\partial w_{n}^I}{\partial y_n} g_{1j}(x) g_{1i}(x) dx
\]
\[= \int_D \int_{Y_{n-1}} \left( \int_{Y_n} b_{ij}(\delta_{js} + \frac{\partial w_{n}^I}{\partial y_n}) dy_n \right) g_{1j}(x) g_{1i}(x) dy_{n-1} dx
\]
\[= \int_D \int_{Y_{n-1}} b_{ij}^{n-1}(x,y_{n-1}) g_{1j}(x) g_{1i}(x) dx dy_{n-1}.
\]
Continuing this, we get
\[
\int_D \left( \int_Y b(x,y)dy \right) g_1 \cdot g_1 dx + \sum_{i=1}^n \int_D \int_{Y_i} b^i(x,y_i) e^k \cdot \nabla_{y_i} w_i^I g_{1k} g_{1l} dy_i dx = \int_D b^0(x) g_1(x) \cdot g_1(x) dx.
\]
Thus
\[
\lim_{\varepsilon \to 0} E^\varepsilon(t) = 0.
\]
We show that the convergence is uniform. To make the notation concise, we denote by
\[
A(t) = T_n^\varepsilon \left( \frac{\partial u_0}{\partial t} \right)(t) - \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla u_i \right)(t).
\]
We consider
\[
\left| \int_D \int_Y [T_n^\varepsilon(b^I) A(t) \cdot A(t) - T_n^\varepsilon(b^I) A(s) \cdot A(s)] dy dx \right| = \int_D \int_Y T_n^\varepsilon(b^I) (A(t) - A(s)) \cdot (A(t) + A(s)) dy dx
\]
\[\leq c \|A(t) - A(s)\|_{L^2(D \times Y)^2} \|A(t) + A(s)\|_{L^2(D \times Y)^2}.
\]
We note that \( \|\partial u^\varepsilon/\partial t\|_{L^\infty(0,T;H)} \) is uniformly bounded for all \( \varepsilon, \partial u_0/\partial t \in L^\infty(0,T;H) \), and from (3.12), \( \partial/\partial t \nabla_{y,u_i} \in L^\infty(0,T;L^2(D \times Y, i)^3) \). Further

\[
A(t) - A(s) = \int_s^t \left[ T^\varepsilon_n \left( \frac{\partial^2 u^\varepsilon}{\partial t^2} \right)(\tau) - \left( \frac{\partial^2 u_0}{\partial t^2}(\tau) + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla_{y,u_i}(\tau) \right) \right] d\tau
\]

so

\[
\|A(t) - A(s)\|_{L^2(D \times Y)^3} \leq \int_s^t \left[ \left\| T^\varepsilon_n \left( \frac{\partial^2 u^\varepsilon}{\partial t^2} \right)(\tau) \right\|_{L^2(D \times Y)^3} + \left\| \frac{\partial^2 u_0}{\partial t^2}(\tau) \right\|_{L^2(D)^3} + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial t^2} \nabla_{y,u_i}(\tau) \right\|_{L^2(D \times Y)^3} \right] d\tau.
\]

From (5.3), we have that \( \partial^2 u^\varepsilon/\partial t^2 \) is uniformly bounded in \( L^2(0,T;H) \). By a similar argument using the compatible initial condition, we show that \( \partial^2 u_0/\partial t^2 \in L^2(0,T;H) \) which implies that \( \partial^2/\partial t^2 \nabla_{y,u_i} \in L^2(0,T;L^2(D \times Y, i)^3) \). We then have

\[
\int_s^t \left\| T^\varepsilon_n \left( \frac{\partial^2 u^\varepsilon}{\partial t^2} \right)(\tau) \right\|_{L^2(D \times Y)^3} d\tau \leq (t-s)^{1/2} \left( \int_0^t \left\| T^\varepsilon_n \left( \frac{\partial^2 u^\varepsilon}{\partial t^2} \right)(\tau) \right\|_{L^2(D \times Y)^3}^2 d\tau \right)^{1/2}
\]

\[
\leq c(t-s)^{1/2} \left( \int_0^t \left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau) \right\|_{L^2(D)^3}^2 d\tau \right)^{1/2} \leq c(t-s)^{1/2}.
\]

By the same argument, we have similar estimates for other terms in (5.13). We can perform similarly for the other terms in \( E^\varepsilon(t) \). From the Arzelà-Ascoli theorem, we deduce that \( E^\varepsilon(t) \) converges to 0 when \( \varepsilon \to 0 \) uniformly for all \( t \in [0,T] \). The conclusion of the proposition follows from the fact that

\[
\left\| \frac{\partial u^\varepsilon}{\partial t}(t) - U^\varepsilon_n \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y,u_i} \right)(t) \right\|_{L^2(D)^3} \leq \left\| T^\varepsilon_n \left( \frac{\partial u^\varepsilon}{\partial t} \right)(t) - \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y,u_i} \right)(t) \right\|_{L^2(D \times Y)^3},
\]

and

\[
\left\| \text{curl } u^\varepsilon(t) - U^\varepsilon_n \left( \text{curl } u_0 + \sum_{i=1}^n \text{curl}_{y,u_i} \right)(t) \right\|_{L^2(D)^3} \leq \left\| T^\varepsilon_n \left( \text{curl } u^\varepsilon \right)(t) - \left( \text{curl } u_0 + \sum_{i=1}^n \text{curl}_{y,u_i} \right)(t) \right\|_{L^2(D)^3}.
\]

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