RENORMALIZATION IN QUANTUM MECHANICS

K. S. Gupta and S. G. Rajeev.

Department of Physics,
University of Rochester,
Rochester, NY 14627, USA.

ABSTRACT

We implement the concept of Wilson renormalization in the context of simple quantum mechanical systems. The attractive inverse square potential leads to a $\beta$ function with a nontrivial ultraviolet stable fixed point and the Hulthen potential exhibits the crossover phenomenon. We also discuss the implementation of the Wilson scheme in the broader context of one dimensional potential problems. The possibility of an analogue of Zamolodchikov’s $C$ function in these systems is also discussed.
1. INTRODUCTION

Renormalization is a familiar concept in the context of field theory. Although the modern theory of renormalization as formulated by Wilson [1] is completely nonperturbative in nature, in many actual field theory calculation such a nonperturbative scheme is often impossible to implement in practice. It is therefore interesting to look for situations where the concept of renormalization can be actually implemented within a nonperturbative scheme. Quantum mechanics happens to provide us with such a situation.

The concept of renormalization is associated with a $\beta$ function whose zeroes correspond to the fixed points of the theory. However, in most field theories, the $\beta$ function can only be calculated within the perturbation theory. It would therefore be interesting to find examples where the $\beta$ function can be calculated nonperturbatively and its zeroes can be determined thereby. Quantum mechanics again promises to present such a framework.

The study of renormalization in quantum mechanics however is not entirely new. Such a concept has been used in the literature to treat singular power law type potentials [2, 3]. More recently Thorn [5] and Jackiw [6] considered the renormalization of a delta function potential in two and three dimensions. In two dimensions, this is a perfectly well posed eigenvalue problem, but the usual analysis does not yield a bound state. The concept of renormalization was introduced to produce a well defined bound state in that problem.

In this paper we carry out a similar analysis for the inverse square potential in one dimension and the Hulthen potential. The inverse square potential has been treated in the literature [3, 4] in the past. Such a potential is of interest in polymers [7]. However, as we show in Section 2, the usual analysis states produces a ground state with arbitrarily negative energy, thereby rendering the solution physically meaningless. Therefore, in this particular case, the problem as it stands is not well posed. This is an important conceptual difference between the inverse square potential problem and that treated by Thorn [5]. In
Section 3 we show how to remedy this using the concept of renormalization. Section 4 deals with the Hulthen potential. This approaches the Coulomb potential for short distances and therefore has no inherent instability. However there is an interesting renormalization of this problem which exhibits the crossover phenomenon. We would like to stress that renormalization is a general principle whose relevance is independent of the existence of any singularity in the problem. In Section 5 we sketch a method of implementing the full Wilson renormalization scheme in a more general context of one dimensional potential problems. Invariance of the full spectral data under scale transformation leads to a definite expression of the $\beta$ function. We indicate the method for obtaining the renormalized coupling constants using the inverse scattering method. We also speculate on the possibility of obtaining an analogue of the $C$ function of two dimensional conformal field theory in the context of quantum mechanics. Section 6 summarizes and concludes the paper.

2. THE INVERSE SQUARE POTENTIAL

The Hamiltonian for the one dimensional inverse square potential is given by

$$H = -\frac{d^2}{dx^2} - \frac{\alpha}{x^2},$$  

(2.1)

where a positive $\alpha$ correspond to an attractive potential. Eqn. 1 can be regarded as the radial equation for an s-wave state so that $x$ has a range $0 \leq x < \infty$. The above Hamiltonian can be decomposed as

$$H = \left( \frac{d}{dx} + \frac{\nu + 1/2}{x} \right) \left( -\frac{d}{dx} + \frac{\nu + 1/2}{x} \right),$$  

(2.2)

where $\alpha = \nu(1 - \nu)$. Therefore for $0 \leq \alpha \leq 1/4$, $H$ is a positive bilinear form on the space of smooth functions of compact support vanishing at the origin. We denote the latter by $C_0^\infty(R^+)$. 

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Next we note that the operator
\[ H_0 = -\frac{d^2}{dx^2} \]  
(2.3)
is self-adjoint in the domain \( D_0 = \{ \psi \in L^2(R^+) \mid \psi(0) = c \psi'(0), c \in R \} \). The operator \( H \) can be written as
\[ H = H_0 + H_1, \]  
(2.4)
where,
\[ H_1 = -\frac{\alpha}{x^2}. \]  
(2.5)
In general, \( H \) is not self-adjoint in the domain in the domain \( D_0 \). However, if \( H_1 \) is in some sense “small” compared to \( H_0 \), then \( H \) is actually self-adjoint in the domain \( D_0 \). [Technically \( H_1 \) is said to be relatively form-bounded with respect to \( H_0 \) \[8\].]

Let us next look at the case \( \alpha \geq 1/4 \). In this range \( H_1 \) is not form bounded by \( H_0 \). However, since \( H \) is real and symmetric in \( C_0^\infty(R^+) \), by von Neumann’s theorem \[8\] it would have self-adjoint extensions. Ordinarily, once an operator is self-adjoint it defines a well-posed eigenvalue problem. However in this case, even though the eigenvalue problem is well-posed mathematically and has been solved \[3\], the ground state has energy \(-\infty\) and hence is physically meaningless. There are normalizable wavefunctions belonging to \( C_0^\infty(R^+) \) of arbitrarily negative energy and no self-adjoint extension is going to remove this problem. To see this consider the expression
\[ \frac{(\psi, H\psi)}{(\psi, \psi)} = \frac{\int \mid \psi'(x) \mid^2 dx - \alpha \int \frac{\mid \psi(x) \mid^2 x^2}{x^2} dx}{\int \mid \psi(x) \mid^2 dx} \]  
(2.6)
where \( \psi \in C_0^\infty(R^+) \). Let \( \psi_\lambda(x) = \sqrt{\lambda} \psi(\lambda x) \). Then
\[ \frac{(\psi_\lambda, H\psi_\lambda)}{(\psi_\lambda, \psi_\lambda)} = \lambda^2 \frac{\int \mid \psi'(x) \mid^2 dx - \alpha \int \frac{\mid \psi(x) \mid^2 x^2}{x^2} dx}{\int \mid \psi(x) \mid^2 dx}. \]  
(2.7)
Thus if
\[ \alpha > \frac{\int \mid \psi'(x) \mid^2 dx}{\int \frac{\mid \psi(x) \mid^2 x^2}{x^2} dx}, \]  
(2.8)
then $H$ would be unbounded from below. So we can ask the question that what is the minimum value of $\alpha$ such that the above inequality holds. This is a problem of minimizing $\alpha$ over $\psi \in C_0^\infty(R^+)$. Let us choose as a trial wavefunction

$$\psi = \sqrt{x}h_1(x)h_2(x),$$

where $h_1(x)$ is 0 below $x = a/2$, rises from 0 to 1 smoothly between $a/2 \leq x \leq a$ and stays constant after that. Similarly $h_2(x)$ is equal to 1 below $x = b$, decreases smoothly from 1 to 0 between $b \leq x \leq 2b$ and stays constant after that. We also assume that $a < b$. Then it is easy to show that

$$\int |\psi'(x)|^2 \, dx \leq \frac{1}{4} \ln \frac{b}{a} + \frac{\ln \frac{b}{2}}{\ln \frac{a}{2}} + C$$

for any $\frac{b}{a}$ where $C$ is a constant independent of $a$ and $b$. If we now let $\frac{b}{a} \to \infty$, we get

$$\alpha_c \leq \frac{1}{4}$$

where $\alpha_c$ is the critical value of $\alpha$ for which the energy becomes negative. Eqn. (2.2) already shows that $\alpha_c > 1/4$. Hence we can conclude that $\alpha_c = 1/4$. Therefore we have shown that for $\alpha > 1/4$ the hamiltonian $H$ is unbounded from below on $C_0^\infty(R^+)$ as a quadratic form.

### 3. RENORMALIZATION

We have seen in the previous section that for $\alpha > 1/4$, $H$ is not bounded from below on $C_0^\infty(R^+)$. In this section we redefine the problem so that the ground state energy is finite. In what follows, $\alpha$ will be taken to be greater than 1/4 and we write $\alpha = \mu^2 + 1/4$. Next we introduce a cutoff $a$ and modify the Hamiltonian $H$ such that

$$H = -\frac{d^2}{dx^2} - \frac{\alpha(a)}{x^2}.$$
where $\alpha(a)$ is now considered to be a function of $a$ and $H$ is defined on the domain consisting of smooth functions in the range $[a, \infty]$ satisfying the condition

$$\psi(a) = 0.$$  \hfill (3.2)

$H$ is then self-adjoint and bounded below by $-\alpha(a)/a^2$. We will determine the dependence of $\alpha(a)$ on $a$ by requiring that the ground state energy be independent of $a$.

Let us now consider the eigenvalue problem of the Hamiltonian $H$ in the bound state sector of the theory. This is given by

$$H\psi = -\kappa^2 \psi, \quad \kappa \in \mathbb{R}. \hfill (3.3)$$

This can be written as

$$\frac{d^2 \psi}{dz^2} - \left[ \mu^2 + \frac{1}{4x^2} - 1 \right] \psi = 0, \quad z = \kappa a. \hfill (3.4)$$

The above equation is solved by

$$\psi(z) = \sqrt{z} K_{i\mu}(z), \hfill (3.5)$$

where $K_{i\mu}(z)$ is the modified Bessel function of the third kind (also called the Macdonald function) of order $i\mu$ which is purely imaginary. For small values of $z$ and $\mu$, the zeroes of this function, denoted by $z_n$ are given by

$$z_n = e^{\left(-\frac{n\pi}{\nu}\right)} \left(2e^{-\gamma}\right) [1 + O(\mu)], \hfill (3.6)$$

where $\gamma$ is the Euler’s constant and $n = 1, 2, \ldots, \infty$. Therefore the energy levels are given by

$$E_n = -e^{\left(-\frac{2n\pi}{\nu}\right)} \left[\frac{2}{ae^\gamma}\right]^2[1 + O(\mu)]. \hfill (3.7)$$

We can immediately see from the above equation that if the cutoff $a$ is taken to zero keeping $\mu$ fixed, the energy levels go to negative infinity. However we can now regard $\mu$ as a function of the cutoff $a$ and ask how $\mu(a)$ must depend on $a$ such that when the latter is
taken to zero, the ground state energy $E_1$ remains independent of $a$. This is going to fix the dependence of $\mu(a)$ on $a$ and hence give the $\beta$ function. Let us define the $\beta$ function as

$$\beta(\mu) = -a \frac{d\mu}{da}.$$  \hfill (3.8)

The condition that the ground state energy remains independent of the cutoff as the latter is taken to zero can be written as

$$a \frac{dE_1}{da} = 0 \quad (3.9)$$

This gives

$$\beta(\mu) = -\frac{\mu^2}{\pi} + .... \quad (3.10)$$

Thus we see that our system has an ultraviolet stable fixed point at $\mu = 0$ which is same as $\alpha = \frac{1}{4}$.

The renormalization carried out so far has used the bound state sector of the Hamiltonian. In the usual field theoretic treatments renormalization is performed on the scattering sector of the theory and the same analysis can be done here too. Below we show how to perform such an analysis.

In the scattering sector, the energy is positive and the Schrödinger’s equation looks like

$$H \psi = k^2 \psi, \quad k \in \mathbb{R}, \quad \psi(a) = 0.$$ \hfill (3.11)

This is the same as

$$\frac{d^2\psi}{dz^2} + \left[ \frac{\mu^2(a)}{z^2} + \frac{1}{z^2} + 1 \right] \psi = 0, \quad z = ka, \quad \psi(ka) = 0.$$ \hfill (3.12)

The solution of the above equation is given by

$$\psi(z) = \sqrt{z}(AJ_{i\mu}(z) + BJ_{-i\mu}(z)),$$ \hfill (3.13)

where $A$ and $B$ are constants. The boundary condition implies that

$$\frac{B}{A} = -\frac{J_{i\mu}(ka)}{J_{-i\mu}(ka)}.$$ \hfill (3.14)
Therefore up to an overall constant, the solution is given by

$$
\psi(z) = \sqrt{z}(J_{i\mu}(z)J_{-i\mu}(\kappa a) - J_{-i\mu}(z)J_{i\mu}(\kappa a)).
$$

(3.15)

Using the large $z$ asymptotic expression for $J_{\pm i\mu}(z)$ given by

$$
J_{\pm i\mu}(z) = \sqrt{\frac{2}{\pi z}} \cos\left( z \mp \frac{\pi}{2} i\mu - \frac{\pi}{4} \right) + ..., \quad (3.16)
$$

the solution can be written as

$$
\psi = \sqrt{\frac{1}{2\pi}} e^{iz} [e^{\pi(\frac{\mu}{2} - \frac{i}{4})} - e^{-\pi(\frac{\mu}{2} + \frac{i}{4})}] + e^{-iz} [e^{\pi(\frac{\mu}{2} + \frac{i}{4})} - e^{-\pi(\frac{\mu}{2} - \frac{i}{4})}] + ...
$$

(3.17)

Let us denote the phase shift by $\delta$. Using the fact that as $\mu \to 0$, $J_{i\mu}(\kappa a) \to (\kappa a)^{i\mu}$ we get

$$
e^{2i(\delta + \frac{\pi}{4})} = \frac{e^{\mu(i \log(\kappa a) - \frac{\pi}{2})} - e^{-\mu(-i \log(\kappa a) + \frac{\pi}{2})}}{e^{\mu(i \log(\kappa a) + \frac{\pi}{2})} - e^{-\mu(-i \log(\kappa a) - \frac{\pi}{2})}} + ...
$$

(3.18)

Let $\delta' = -\delta + \frac{\pi}{4}$. Then from the above expression we get

$$
\tan(\delta'(\kappa)) = \tan(\mu \log(\kappa a)) \coth(\mu \frac{\pi}{2}) + ...
$$

(3.19)

Eqn. (3.19) gives the phase shift at an arbitrary momentum $\kappa$. However the phase shift given by the above equation blows up as the cutoff is taken to zero. We can again regard $\mu$ as a function of $a$ and ask the question how $\mu$ must depend on $a$ such that the phase shift at any particular momentum $\kappa_0$ be independent of $a$ as the latter is taken to zero. This fixes the relation between $\mu(a)$ and $a$, which is now given by

$$
\mu(a) = -\frac{n\pi}{\log \kappa_0 a - \pi C},
$$

(3.20)

where $C$ is a constant given by $C = (\tan \delta') |_{\kappa=\kappa_0}$ and $n = 1, 2, 3,...$. From this we get the $\beta$ function as

$$
\beta = -\frac{\mu^2}{n\pi} + ...
$$

(3.21)
For \( n = 1 \), we recover the result given by eqn. (3.10).

We have performed the renormalization by keeping the phase shift at a particular value of \( \kappa = \kappa_1^2 \). Using eqn. (3.19), we can now compute the phase shift at any other arbitrary value of \( \kappa \) as the cutoff \( a \) is taken to zero. This is given by

\[
\tan(\delta'(\kappa)) = \tan(\delta'(\kappa_1)) + \frac{2}{\pi} \log \frac{\kappa}{\kappa_1}.
\]

(3.22)

The \( S \) matrix of the theory is given by \( S = e^{2i\delta} \). Using eqns. (3.18) and (3.19) this can be written as

\[
S = -e^{-\frac{i\pi}{2}} \frac{1 - i \tan(\delta'(\kappa))}{1 + i \tan(\delta'(\kappa))}.
\]

(3.23)

The bound states can be identified with the poles of the \( S \) matrix, which has a pole when \( \tan(\delta'(\kappa)) = i \). If the corresponding \( \kappa \) be denoted by \( \kappa_{\text{bound}} \) then the latter is given by

\[
\kappa_{\text{bound}} = \kappa_1 e^{-\frac{\pi}{2} \tan(\delta'(k_1))}.
\]

(3.24)

Therefore it seems that the \( S \) matrix has only one pole and correspondingly only one bound state. However, we also note that the \( S \) matrix has a branch point at \( \kappa = 0 \). Thus \( \kappa = 0 \) is an accumulation point of poles of the \( S \) matrix.

We note in the passing that for \( \alpha \leq 1/4 \) the scattering problem is perfectly well defined. If we write \( \alpha = 1/4 - \nu^2 \) then the phase shift in this case is given by \( \delta = -\frac{\pi}{2} (\nu + \frac{1}{2}) \), independent of the momentum. This is a consequence of scale invariance. Scale invariance is broken by renormalization for \( \alpha > \frac{1}{4} \). As \( x \to \infty \), \( \alpha \to \frac{1}{4} \) and \( \delta \to -\frac{\pi}{4} \) and the two results match.

4. THE HULTHEN POTENTIAL

The Hulthen potential is defined by

\[
V(r) = -\frac{\alpha}{a^2} \frac{e^{-\frac{r}{a}}}{1 - e^{-\frac{r}{a}}},
\]

(4.1)
where $\alpha$ and $a$ are constants. For $r$ small compared to $a$, the Hulthen potential approaches the Coulomb potential whereas for $r$ large compared to $a$ it approaches zero exponentially fast. Therefore $a$ can be thought of as an infrared regulator for the Coulomb problem. The bound states of this problem has energy levels given by

$$E_n = -\frac{1}{a^2} \frac{(\alpha - n^2)^2}{4n^2}. \tag{4.2}$$

Proceeding along the lines of the earlier section we now regard $\alpha$ as a function of $a$ and ask how $\alpha(a)$ must depend on $a$ so that the ground state energy is independent of $a$. This defines the $\beta$ function which is then given by

$$\beta(\alpha) = \alpha - 1. \tag{4.3}$$

Therefore the $\beta$ function has a zero at $\alpha = 1$ and that is an infrared stable fixed point.

In order to understand the crossover phenomenon, let us ask as to how the $\beta$ function behaves as $\alpha \to \infty$. Recall that the $\beta$ function is really a vector field, so that under a change of variables $\alpha \to w$, $\beta(\alpha) \frac{\partial}{\partial \alpha} = \tilde{\beta}(w) \frac{\partial}{\partial w}$ where $\tilde{\beta}(w)$ is the component of the $\beta$ vector field in the new variables. It is given by

$$\tilde{\beta}(w) = -w(1 - w). \tag{4.4}$$

As $\alpha \to \infty$, $w \to 0$ from above and therefore in that limit $1 - w$ is positive. Therefore $w = 0$ is an ultraviolet stable fixed point. In terms of the old coordinates this fixed point appears at an infinite value of $\alpha$. Thus we see that this system has two fixed points one of which is infrared stable and the other one is ultraviolet stable. Therefore, a system described by the Hulthen potential exhibits the crossover phenomenon.

5. GENERAL DISCUSSION OF THE $\beta$ FUNCTION

Until this point we have shown a couple of examples where the process of renormalization was performed in quantum mechanical systems by demanding the invariance of the
phase shift or the ground state energy under the renormalization group transformations. In this section we shall sketch a general method of implementing the renormalization group transformations in the space of one dimensional potential problems. The basic strategy for renormalization would be the same as displayed in the previous sections: given a potential we will first introduce a cutoff so that the solution to the Schrödinger’s equation depends on the cutoff. Here we consider the entire spectral data of the problem instead of just the phase shift or the ground state energy. The potential in general would be a function of many coupling constants and the corresponding spectral data would depend on the momentum, the cutoff and all the coupling constants. We can then ask as to how the potential or the coupling constants must depend on the cutoff so that the spectral data is independent of the cutoff as the latter is removed. This will fix the \( \beta \) function of the problem.

Let the system be described by the Schrödinger’s equation

\[
-\frac{d^2}{dr^2} \psi + V(r) \psi = k^2 \psi = \lambda \psi, \quad (5.1)
\]

where \( k^2 = \lambda \) is the energy. Let \( a \) denote the cutoff. Then the boundary conditions are given by \( \psi(k, a) = 0 \) and \( \psi'(k, a) = 1 \). We shall also assume (for technical reasons) that

\[
\int_a^\infty r^j |V(r)| \, dr < \infty, \quad j = 1, 2. \quad (5.2)
\]

A system described by (5.1) - (5.2) would in general have both bound states and scattering states. In the scattering sector \( k \) is real and \( > 0 \). Let \( f(k, r) \) be one of the two linearly independent solutions of (5.1) which in the limit of \( r \to \infty \) satisfies the condition \( e^{ikr} f(k, r) \to 1 \). Let \( f(k, a) \) be denoted by \( f(k) \). Then it can be shown that as \( r \to \infty \),

\[
\psi(k, r) \to \frac{|f(k)|}{k} \sin(k(r - a) + \eta(k)), \quad (5.3)
\]

where \( \eta(k) \) is the phase shift \([10]\). The bound state sector for the above problem will be described by a finite number (say \( n \)) \([10]\) of bound states with normalization factors
\(c_i, i = 1, \ldots, n\). With this information we can now define the spectral data. Consider a monotonically nondecreasing function \(\rho(\lambda)\), which for \(\lambda < 0\) is constant except for jumps at energies corresponding to those of the bound states, the magnitude of each jump being equal to the normalization constant for that bound state. For \(\lambda = 0\) let \(\rho(\lambda) = 0\) and for \(\lambda > 0\) let

\[
\rho(\lambda) = \frac{1}{\pi} \int_0^\lambda \frac{\sqrt{\lambda}}{|f(\sqrt{\lambda})|^2} d\lambda. \tag{5.4}
\]

This function \(\rho(\lambda)\) defines the spectral data \([10]\). The spectral data has a dimension of \(\lambda^{3/2}\) \([10]\) and we shall therefore divide it by \(\lambda^{3/2}\) and work with this dimensionless quantity denoted by \(\tilde{\rho}\). Let the cutoff be denoted by \(a\). Therefore \(\tilde{\rho}\) is a function of \(a, V(r)\) and \(\lambda\). Since this quantity is dimensionless, it is invariant under the simultaneous scaling of \(a, V(r)\) and \(\lambda\). This implies that

\[
-2\lambda \frac{\partial \tilde{\rho}}{\partial \lambda} + a \frac{\partial \tilde{\rho}}{\partial a} + \int \frac{\delta \tilde{\rho}}{\delta V(r)} \left(2V(r) + r \frac{\partial V(r)}{\partial r}\right) dr = 0. \tag{5.5}
\]

We must also impose the condition that the spectral data be invariant under the renormalization group transformations. This leads to the equation

\[
a \frac{\partial \tilde{\rho}}{\partial a} + \int \frac{\delta \tilde{\rho}}{\delta V(r)} a \frac{\partial V}{\partial a} dr = 0. \tag{5.6}
\]

Let us define the \(\beta\) function as

\[
\beta(r) = -a \frac{\partial V(r)}{\partial a} + (2V(r) + r \frac{\partial V(r)}{\partial r}) \tag{5.7}
\]

The above equations then imply that

\[
-2\lambda \frac{\partial \tilde{\rho}}{\partial \lambda} + \int \frac{\delta \tilde{\rho}}{\delta V(r)} \beta dr = 0. \tag{5.8}
\]

From this we get the \(\beta\) function as

\[
\beta(r) = \int 2\lambda \frac{\partial \tilde{\rho}}{\partial \lambda} \frac{\delta V(r)}{\delta \tilde{\rho}} d\lambda. \tag{5.9}
\]
Under an infinitesimal variation of the spectral data $\rho$, the change in the potential $V(r)$ is given by

$$\delta V(r) = -4 \int_{-\infty}^{\infty} \psi(\lambda, r) \psi'(\lambda, r) d(\delta \rho(\lambda)).$$

(5.10)

Integrating this by parts and dropping the boundary term (this is justified since at $\lambda = \pm \infty$, $\delta \rho(\lambda) = 0$) we get

$$\frac{\delta V(r)}{\delta \tilde{\rho}(\lambda)} = 4 \lambda^\frac{3}{2} \psi(\lambda, r) \psi'(\lambda, r).$$

(5.11)

Therefore the final form for the $\beta$ function is given by

$$\beta(r) = \int 8 \lambda^\frac{5}{2} \frac{\partial \tilde{\rho}}{\partial \lambda} \psi(\lambda, r) \psi'(\lambda, r) d\lambda.$$  

(5.12)

We end this section by making a few comments about the above set of formulae. The equation (5.7) can be written as

$$\beta(r) = -a \frac{dV(r, a)}{da} = -\frac{dV(r, t)}{dt}, \quad t = \log(a).$$

(5.13)

The left hand side of this equation is known from (5.12). However, integrating such an equation to obtain the renormalized potential is an enormously complicated task. We therefore approach this problem from a different point of view. Let us first note that what we are calling the $\beta$ function is really a vector field. The renormalization group transformations move along the integral curves of this vector field. Imagine now a “theory space” described in terms of all one dimensional potentials or correspondingly the collection of the spectral data (or $\tilde{\rho}$.) It is possible to express the components of the $\beta$ vector field in terms of either of these variables which can serve as coordinates for the “theory space”. Usually the renormalization group transformation in described in terms of the potentials or coupling constants. However, the components of the $\beta$ vector field in that coordinate system are very complicated objects. Therefore it is hard to integrate the $\beta$ vector field in this coordinate system to obtain the renormalized potential. However,
the renormalization group transformation in terms of the $\rho(\lambda)$ variable is very simple and is given by

$$\tilde{\rho}(\lambda) \rightarrow \tilde{\rho}(\lambda e^{-2t}).$$  \hfill (5.14)  

One can therefore start with a potential $V(r)$ and compute the spectral data $\rho(\lambda, a, V(r))$ and the corresponding $\tilde{\rho}(\lambda, a, V(r))$. The renormalization group scaling transformation is then applied to this $\tilde{\rho}$ to obtain a transformed $\tilde{\rho}$. The potential for this renormalized $\tilde{\rho}$ can then be computed using the inverse scattering method. This procedure would be equivalent to integrating the $\beta$ vector field in expressed in terms of the potentials and would therefore provide us with the renormalized potential. We would like to point out in the passing that the renormalization group transformation on $\tilde{\rho}(\lambda)$ is qualitatively different from the isospectral deformations used in the integrable models since the spectrum gets modified due to the renormalization group transformations.

In the examples discussed before, we have seen that the $\beta$ functions can vanish at certain values of the coupling constants thereby leading to interesting fixed points. The $\beta$ function given by (5.12) would in general have many different fixed points. It has been conjectured in the context of two dimensional conformal field theories that there exists a function, usually called the $C$ function, which is monotonic along the renormalization group flow and assumes specific values at the fixed points of the $\beta$ function $[11]$. The $\beta$ function that we compute is given in terms of the quantity $\tilde{\rho}$ which is the usual spectral data $\rho$ divided by $\lambda^{3/2}$. The usual spectral data is a monotonic nondecreasing function of $\lambda$ $[10]$. It is perhaps possible to construct in these systems an explicit expression of the $C$ function in terms of $\rho$ which would also exhibit monotonicity along the renormalization group flow. Such a function would be of great interest in the conceptual understanding of quantum mechanics and is currently under study.
6. CONCLUSION

We have studied the effect of renormalization on various quantum mechanical systems. It has been shown that renormalization can be used to make the bound state problem for the attractive inverse square problem physically well defined. The $\beta$ function in that cases has an ultraviolet stable fixed point. Next we studied the Hulthen potential which is an infrared regulated version of the Coulomb potential. Although there is no inherent instability in this problem, the process of renormalization in this case gives a $\beta$ function which has two fixed points, one being infrared stable and the other is ultraviolet stable. Thus we see an interesting example of the crossover phenomenon in quantum mechanics. This example also shows that renormalization is a completely general principle which is not necessarily related to the existence of instability or divergence in the problem.

We have also considered the general problem of renormalization in a broader context of one dimensional potential problems. A general formula for the $\beta$ function was derived which can be evaluated once the spectral data of the corresponding direct scattering problem is known. We also speculate on the possibility of constructing the analogue of the $C$ function of two dimensional conformal field theory. The construction of such a function which would have a monotonic gradient flow along the renormalization group transformations is currently under investigation.

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