TRANSFINITE SEQUENCES OF CONTINUOUS AND BAIRE CLASS 1 FUNCTIONS

MÁRTON ELEKES AND KENNETH KUNEN

(Communicated by Alan Dow)

Abstract. The set of continuous or Baire class 1 functions defined on a metric space $X$ is endowed with the natural pointwise partial order. We investigate how the possible lengths of well-ordered monotone sequences (with respect to this order) depend on the space $X$.

Introduction

Any set $F$ of real valued functions defined on an arbitrary set $X$ is partially ordered by the pointwise order; that is, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. Then, $f < g$ iff $f \leq g$ and $g \not\leq f$; equivalently, $f(x) \leq g(x)$ for all $x \in X$ and $f(x) < g(x)$ for at least one $x \in X$. Our aim will be to investigate the possible lengths of the increasing or decreasing well-ordered sequences of functions in $F$ with respect to this order. A classical theorem (see Kuratowski [7], §24.III, Theorem 2') asserts that if $F$ is the set of Baire class 1 functions (that is, pointwise limits of continuous functions) defined on a Polish space $X$ (that is, a complete separable metric space), then there exists a monotone sequence of length $\xi$ in $F$ iff $\xi < \omega_1$. P. Komjáth [5] proved that the corresponding question concerning Baire class $\alpha$ functions for $2 \leq \alpha < \omega_1$ is independent of ZFC.

In the present paper we investigate what happens if we replace the Polish space $X$ by an arbitrary metric space.

Section 1 considers chains of continuous functions. We show that for any metric space $X$, there exists a chain in $C(X, \mathbb{R})$ of order type $\xi$ iff $|\xi| \leq d(X)$. Here, $|A|$ denotes the cardinality of the set $A$, while $d(X)$ denotes the density of the space $X$, that is,

$$d(X) = \max(\min\{|D| : D \subseteq X \& \overline{D} = X\}, \omega).$$

In particular, for separable $X$, every well-ordered chain has countable length, just as for Polish spaces.

Section 2 considers chains of Baire class 1 functions on separable metric spaces. Here, the situation is entirely different from the case of Polish spaces, since on some
separable metric spaces there are well-ordered chains of every order type less than \( \omega_2 \). Furthermore, the existence of chains of type \( \omega_2 \) and longer is independent of \( ZFC + \neg CH \). Under \( MA \), there are chains of all types less than \( c^+ \), whereas in the Cohen model, all chains have type less than \( \omega_2 \).

We note here that instead of examining well-ordered sequences, which is a classical problem, we could try to characterize all the possible order types of linearly ordered subsets of the partially ordered set \( \mathcal{F} \). This problem was posed by M. Laczkovich, and is considered in detail in [3].

1. SEQUENCES OF CONTINUOUS FUNCTIONS

**Lemma 1.1.** For any topological space \( X \): If there is a well-ordered sequence of length \( \xi \) in \( C(X, \mathbb{R}) \), then \( \xi < d(X)^+ \).

**Proof.** Let \( \{f_\alpha : \alpha < \xi \} \) be an increasing sequence in \( C(X, \mathbb{R}) \), and let \( D \subseteq X \) be a dense subset of \( X \) such that \( d(X) = \max(|D|, \omega) \). By continuity, the \( f_\alpha|D \) are all distinct; so, for each \( \alpha < \xi \), choose a \( d_\alpha \in D \) such that \( f_\alpha(d_\alpha) < f_\alpha+1(d_\alpha) \). For each \( d \in D \) the set \( E_d = \{ \alpha : d_\alpha = d \} \) is countable, because every well-ordered subset of \( \mathbb{R} \) is countable. Since \( \xi = \bigcup_{d \in D} E_d \), we have \( \xi \leq \max(|D|, \omega) = d(X) \). \( \square \)

The converse implication is not true in general. For example, if \( X \) has the countable chain condition (ccc), then every well-ordered chain in \( C(X, \mathbb{R}) \) is countable (because \( X \times \mathbb{R} \) is also ccc). However, the converse is true for metric spaces:

**Lemma 1.2.** If \( (X, \rho) \) is any non-empty metric space and \( \prec \) is any total order of the cardinal \( d(X) \), then there is a chain in \( C(X, \mathbb{R}) \) which is isomorphic to \( \prec \).

**Proof.** First, note that every countable total order is embeddable in \( \mathbb{R} \), so if \( d(X) = \omega \), then the result follows trivially using constant functions. In particular, we may assume that \( X \) is infinite, and then fix \( D \subseteq X \) which is dense and of size \( d(X) \). For each \( n \in \omega \), let \( D_n \) be a subset of \( D \) which is maximal with respect to the property \( \forall d, e \in D_n \ d \neq e \to \rho(d, e) \geq 2^{−n} \). Then \( \bigcup_n D_n \) is also dense, so we may assume that \( \bigcup_n D_n = D \). We may also assume that \( \prec \) is a total order of the set \( D \). Now, we shall produce \( f_d \in C(X, \mathbb{R}) \) for \( d \in D \) such that \( f_d < f_e \) whenever \( d \prec e \).

For each \( n \), if \( c \in D_n \), define \( \varphi^n_c(x) = \max(0, 2^{−n} − \rho(x, c)) \). For each \( d \in D \), let \( \psi^n_d = \sum \{\varphi^n_c : c \in D_n \cap c < d \} \). Since every \( x \in X \) has a neighborhood on which all but at most one of the \( \varphi^n_c \) vanish, we have \( \psi^n_d \in C(X, [0, 2^{−n}]) \), and \( \psi^n_d \leq \psi^n_e \) whenever \( d \prec e \). Thus, if we let \( f_d = \sum_{n<\omega} \psi^n_d \), we have \( f_d \in C(X, [0, 2]) \), and \( f_d \leq f_e \) whenever \( d \prec e \). But also, if \( d \in D_n \) and \( d \prec e \), then \( \psi^n_d(d) = 0 < 2^{−n} = \psi^n_e(d) \), so actually \( f_d < f_e \) whenever \( d \prec e \). \( \square \)

Putting these lemmas together, we have:

**Theorem 1.3.** Let \( (X, \rho) \) be a metric space. Then there exists a well-ordered sequence of length \( \xi \) in \( C(X, \mathbb{R}) \) iff \( \xi < d(X)^+ \).

**Corollary 1.4.** A metric space \( (X, \rho) \) is separable iff every well-ordered sequence in \( C(X, \mathbb{R}) \) is countable.

2. SEQUENCES OF Baire class 1 functions

If we replace continuous functions by Baire class 1 functions, then Corollary 1.4 becomes false, since on some separable metric spaces we can get well-ordered sequences of every type less than \( \omega_2 \). To prove this, we shall apply some basic facts
about \( C^+ \) on \( P(\omega) \). As usual, for \( x, y \subseteq \omega \), we say that \( x \subseteq^* y \) iff \( x \setminus y \) is finite. Then \( x \subseteq^* y \) iff \( x \setminus y \) is finite and \( y \setminus x \) is infinite. This \( \subseteq^* \) partially orders \( P(\omega) \).

**Lemma 2.1.** If \( X \subset P(\omega) \) is a chain in the order \( \subseteq^* \), then on \( X \) (viewed as a subset of the Cantor set \( 2^\omega \cong P(\omega) \)) there is a chain of Baire class 1 functions which is isomorphic to \( (X, \subseteq^*) \).

**Proof.** Note that for each \( x \in X \),

\[
\{ y \in X : y \subseteq^* x \} = \bigcup_{m \in \omega} \{ y \in X : \forall n \geq m \ [y(n) \leq x(n)] \},
\]

which is an \( F_\sigma \) set in \( X \). Likewise, the sets \( \{ y \in X : y \supseteq^* x \} \), \( \{ y \in X : y \subset^* x \} \), and \( \{ y \in X : y \supset^* x \} \), are all \( F_\sigma \) sets in \( X \), and hence also \( G_\delta \) sets. It follows that if \( f_x : X \to \{0, 1\} \) is the characteristic function of \( \{ y \in X : y \subset^* x \} \), then \( f_x : X \to \mathbb{R} \) is a Baire class 1 function. Then \( \{ f_x : x \in X \} \) is the required chain.

**Lemma 2.2.** For any infinite cardinal \( \kappa \), suppose that \( (P(\omega), \subseteq^*) \) contains a chain \( \{ x_\alpha : \alpha < \kappa \} \) (i.e., \( \alpha < \beta \to x_\alpha \subset^* x_\beta \}) \). Then \( (P(\omega), \subseteq^*) \) contains a chain \( X \) of size \( \kappa \) such that every ordinal \( \xi < \kappa^+ \) is embeddable into \( X \).

**Proof.** Let \( S = \bigcup_{1 \leq n < \omega} \kappa^n \). For \( s = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \in S \), let

\[
s^+ = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n + 1).
\]

Starting with the \( x_{(\alpha)} = x_\alpha \), choose \( x_s \in P(\omega) \) by induction on \( \text{length}(s) \) so that

\[
x_s = x_{s^0} \subset^* x_{s^0} \subset^* x_{s^0} \subset^* x_{s^*} \subset^* x_s^+
\]

whenever \( s \in S \) and \( 0 < \alpha < \beta < \kappa \). Let \( X = \{ x_s : s \in S \} \). Then, whenever \( x, y \in X \) with \( x \subset^* y \), the ordinal \( \kappa \) is embeddable in \( (x, y) = \{ z \in X : x \subset^* z \subset^* y \} \). From this, one easily proves by induction on \( \xi < \kappa^+ \) (using \( \text{cf}(\xi) \leq \kappa \)) that \( \xi \) is embeddable in each such interval \( (x, y) \).

Since \( P(\omega) \) certainly contains a chain of type \( \omega_1 \), these two lemmas yield:

**Theorem 2.3.** There is a separable metric space \( X \) on which, for every \( \xi < \omega_2 \), there is a well-ordered chain of length \( \xi \) of Baire class 1 functions.

Under \( CH \), this is best possible, since there will be only \( 2^\omega = \omega_1 \) Baire class 1 functions on a separable metric space, so there could not be a chain of length \( \omega_2 \). Under \( \neg CH \), the existence of longer chains of Baire class 1 functions depends on the model of set theory. It is consistent with \( \varepsilon = 2^\omega \) being arbitrarily large that there is a chain in \( (P(\omega), \subseteq^*) \) of type \( \varepsilon \); for example, this is true under \( MA \) (see [2]). In this case, there will be a separable \( X \) with well-ordered chains of all lengths less than \( \varepsilon^+ \). However, in the Cohen model, where \( \varepsilon \) can also be made arbitrarily large, we never get chains of type \( \omega_2 \). We shall prove this by using the following lemma, which relates it to the rectangle problem:

**Lemma 2.4.** Suppose that there is a separable metric space \( Y \) with an \( \omega_2 \)-chain of Borel subsets, \( \{ B_\alpha : \alpha < \omega_2 \} \) (so, \( \alpha < \beta \to B_\alpha \subsetneq B_\beta \)). Then in \( \omega_2 \times \omega_2 \), the well-order relation \( < \) is in the \( \sigma \)-algebra generated by the set of all rectangles, \( \{ S \times T : S, T \in P(\omega_2) \} \).
Proof. Each $B_\alpha$ has some countable Borel rank. Since there are only $\omega_1$ ranks, we may, by passing to a subsequence, assume that the ranks are bounded. Say each $B_\alpha$ is a $\Sigma^0_\mu$ set for some fixed $\mu < \omega_1$.

Let $J = \omega^\omega$, and let $A \subseteq Y \times J$ be a universal $\Sigma^0_\mu$ set; that is, $A$ is $\Sigma^0_\mu$ in $Y \times J$ and every $\Sigma^0_\mu$ subset of $Y \subseteq A$ in $Y \times J$ (see [7], §31). Now, for $\alpha, \beta < \omega_2$, fix $y_\alpha \in B_{\alpha+1}\setminus B_\alpha$, and fix $j_\beta \in J$ such that $A^{y_\alpha,j_\beta} = B_\beta$. Then $\alpha < \beta$ iff $(y_\alpha,j_\beta) \in A$. Thus, $\{(y_\alpha,j_\beta) : \alpha < \beta < \omega_2\}$ is a Borel subset of $\{(y_\alpha,j_\beta) : \alpha < \beta < \omega_2\}$, and is hence in the $\sigma$-algebra generated by open rectangles, so $\prec$, as a subset of $\omega_2 \times \omega_2$, is in the $\sigma$-algebra generated by rectangles. \hfill \Box

Theorem 2.5. Assume that the well-order relation $\prec$ on $\omega_2$ is not in the $\sigma$-algebra generated by the set of all rectangles. Then no separable metric space can have a chain of length $\omega_2$ of Baire class 1 functions.

Proof. Suppose that $\{f_\alpha : \alpha < \omega_2\}$ is a chain of Baire class one functions on the separable metric space $X$. Let $B_\alpha = \{(x,r) \in X \times \mathbb{R} : r \leq f_\alpha(x)\}$. Then the $B_\alpha$ form an $\omega_2$-chain of Borel subsets of the separable metric space $X \times \mathbb{R}$, so we have a contradiction by Lemma 2.4.

Finally, we point out that the hypothesis of this theorem is consistent, since it holds in the extension $V[G]$ formed by adding $\geq \omega_2$ Cohen reals to a ground model $V$ which satisfies $CH$. This fact was first proved in [3]. It also follows from the more general principle $HP_2(\omega_2)$ of Brendle, Fuchino, and Soukup [1]. They define this principle, prove that it holds in Cohen extensions (and in a number of other forcing extensions), and show the following:

Lemma 2.6. $HP_2(\kappa)$ implies that if $R$ is any relation on $\mathcal{P}(\omega)$ which is first-order definable over $H(\omega_1)$ from a fixed element of $H(\omega_1)$, then there is no $X \subseteq \mathcal{P}(\omega)$ such that $(X,R)$ is isomorphic to $(\kappa; <)$.

These matters are also discussed in [4], which indicates how such statements are verified in Cohen extensions. Here, $H(\omega_1)$ denotes the set of hereditarily countable sets.

Lemma 2.7. $HP_2(\omega_2)$ implies that in $\omega_2 \times \omega_2$, the well-order relation $\prec$ is not in the $\sigma$-algebra generated by the set of all rectangles, $\{S \times T : S,T \in \mathcal{P}(\omega_2)\}$.

Proof. Suppose that $\prec$ were in this $\sigma$-algebra. Then we would have fixed $K_\alpha \subseteq \omega_2$ for $\alpha < \omega$ such that $\prec$ is in the $\sigma$-algebra generated by all the $K_m \times K_n$. For each $\alpha$, let $u_\alpha = \{n \in \omega : \alpha \in K_n\}$. There is then a formula $\varphi(x,y,z)$ and a fixed $w \in H(\omega_1)$ such that for all $\alpha, \beta < \omega_2$, $\alpha < \beta$ iff $H(\omega_1) \models \varphi(u_\alpha, u_\beta, w)$; here, $w$ encodes the particular countable boolean combination used to get $\prec$ from the $K_\alpha$. Now, if $X = \{u_\alpha : \alpha < \omega_2\}$, then $\varphi$ defines a relation $R$ on $H(\omega_1)$ such that $(X,R)$ is isomorphic to $(\omega_2; <)$, contradicting Lemma 2.6. \hfill \Box

References

[1] J. Brendle, S. Fuchino, and L. Soukup, Coloring ordinals by reals, to appear.
[2] E. K. van Douwen, The integers and topology, in Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 111-167. MR [87f:54010X]
[3] M. Elekes, Linearly ordered families of Baire 1 functions, Real Analysis Exchange, 27 (2001/02), 49–63.
[4] I. Juhász and K. Kunen, The power set of \( \omega \), *Fundamenta Mathematicae*, Vol 170 (2001), 257-265.

[5] P. Komjáth, Ordered families of Baire-2-functions, *Real Analysis Exchange*, Vol 15 (1989-90), 442-444. MR [91d:26007]

[6] K. Kunen, *Inaccessibility Properties of Cardinals*, Doctoral Dissertation, Stanford, 1968.

[7] K. Kuratowski, *Topology*, Vol. 1, Academic Press, 1966. MR [36:840]

Department of Analysis, Eötvös Loránd University, Budapest, Pázmány Péter sétány 1/c, 1117, Hungary

E-mail address: emarci@cs.elte.hu

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

E-mail address: kunen@math.wisc.edu

URL: http://www.math.wisc.edu/~kunen