Negative Imaginary Systems Theory in the Robust Control of Highly Resonant Flexible Structures

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Abstract—This paper covers recent developments in the theory of negative imaginary systems and their application to the control of highly resonant flexible structures. The theory of negative imaginary systems arose out of a desire to unify a number of classical methods for the control of lightly damped structures with collocated force actuators and position sensors including positive position feedback and integral force feedback. The key result is a stability result which shows why these methods are guaranteed to yield robust closed loop stability in the face of unmodelled spillover dynamics. Related results to be presented connect the theory of negative imaginary systems to positive real systems theory and a negative imaginary lemma has been established which is analogous to the positive real lemma. The paper also presents recent controller synthesis results based on the theory of negative imaginary systems.

I. INTRODUCTION

The problem of vibration control for flexible structures arises in a variety of applications such as large space structures, flexible dynamics in air vehicles, nano-positioning, the control of atomic force microscopes and the control of optical interfermetric sensing systems; e.g., see [1], [2]. A particular problem in vibration control for flexible structures is that unmodelled (spillover) dynamics can severely degrade the control system performance or lead to instability. Also, uncertainties in mode frequencies and damping can lead to similar problems of poor control system performance or instability; e.g., see [3], [4].

An approach to the control of flexible structures which addresses these robustness issues is positive-position feedback, which involves the use of collocated force actuators and position sensors [5], [6]. Roughly speaking a positive position feedback controller consists of a transfer function matrix which has the same form as a finite mode model of a flexible structure. This approach was originally proposed as an alternative to passivity based velocity feedback approaches in the control of large space structures. The original papers on positive position feedback included stability proofs based on Routh-Hurwitz and Nyquist techniques but the robustness properties were never fully quantified. The research presented in this paper aims to develop a general systems theory framework for positive position control and related control methods for flexible structures. An important special case of positive-position feedback involves the use of piezoelectric actuators and sensors. A positive-position feedback controller can be designed to increase the damping of the modes of a flexible structure. Furthermore, this controller is robust against uncertainty in the modal frequencies as well as unmodelled plant dynamics. We consider the theory of negative-imaginary systems to reveal the robustness properties of MIMO positive-position feedback controllers and related types of controllers for flexible structures; e.g., see [1], [7]–[15].

The negative-imaginary property of linear systems can be extended to nonlinear systems through the notion of counterclockwise input-output dynamics [16]–[18]. It is shown in [19] for the single-input, single-output (SISO) linear case that the results of [16], [17] guarantee the stability of a positive-position feedback control system in the presence of unmodelled dynamics and parameter uncertainties that maintain the negative-imaginary property of the plant.

A. Flexible Structure Modeling

In modeling an undamped flexible structure with collocated force actuators and position sensors, modal analysis can be applied to the relevant partial differential equation leading to the transfer function matrix

\[ P(s) = \sum_{i=1}^{\infty} \frac{1}{s^2 + \kappa_i s + \omega_i^2} \psi_i^T \psi_i, \]

where, for all \( i, \kappa_i > 0, \omega_i > 0 \), and \( \psi_i \) is an \( m \times 1 \) vector. Therefore, the Hermitian-imaginary part

\[ \Im[H(P(j\omega))] = -\frac{1}{2} \text{Tr}(P(j\omega) - P^*(j\omega)) \]

of the frequency response matrix \( P(j\omega) \) satisfies

\[ \Im[H(P(j\omega))] = -\omega \sum_{i=1}^{\infty} \frac{\kappa_i}{(\omega_i^2 - \omega^2)^2 + \omega^2 \kappa_i^2} \psi_i^T \psi_i \leq 0 \]

for all \( \omega \geq 0 \). That is, the frequency response matrix has a negative-semidefinite Hermitian-imaginary part for all \( \omega \geq 0 \). We thus refer to the transfer function matrix \( P(s) \) as negative imaginary. Any flexible structure with collocated force actuators and position sensors will have a negative imaginary transfer function matrix.

II. NEGATIVE IMAGINARY TRANSFER FUNCTION MATRICES

In this section, we present the formal definition of a negative imaginary transfer function matrix. We also present a number of extensions to this notion, which have appeared in the literature.
Definition 1 ([20]–[23]): A square real-rational proper transfer function matrix $R(s)$ is termed negative imaginary (NI) if

1) $R(s)$ has no poles at the origin and in $\mathbb{R}[s] > 0$;
2) $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of $\omega$ where $j\omega$ is a pole of $R(s)$;
3) If $j\omega_0, \omega_0 \in (0, \infty)$, is a pole of $R(s)$, it is at most a simple pole, and the residue matrix $R_{0} \triangleq \lim_{\omega \to j\omega_{0}} (s - j\omega_{0})R(s)$ is positive semidefinite Hermitian.

In the SISO case, a transfer function is negative imaginary if and only if all frequencies. Consequently, the Nyquist plot of a SISO negative-imaginary transfer function lies below the real axis; see Figure 1.

Remark 1: Note that this definition has been recently extended to allow for a simple pole at the origin; [24], [25]. However, in this case, this leads to a slightly different form of the stability condition. Also, in [26], a lossless version of this definition is given along with a corresponding negative imaginary lemma.

The following lemma from [21], [23] describes the relationship between negative imaginary transfer function matrices and positive real transfer function matrices.

Lemma 1 ([21], [23]): Given a square real-rational strictly proper transfer function matrix $R(s)$. Then $R(s)$ is negative imaginary if and only if

1) $R(s)$ has no poles at the origin;
2) $F(s) \triangleq sR(s)$ is positive real.

The following definition defines the strict version of the negative imaginary property.

Definition 2 (See [20], [21]): A square real-rational proper transfer function matrix $R(s)$ is termed strictly negative imaginary (SNI) if

1) $R(s)$ has no poles in $\mathbb{R}[s] \geq 0$;
2) $j[R(j\omega) - R^*(j\omega)] > 0$ for $\omega \in (0, \infty)$.

Remark 2: In the paper [27], the strict negative imaginary property is extended to a notion of strongly strict negative imaginary (SSNI) systems by imposing the additional conditions that

$$\lim_{\omega \to \infty} j\omega(R(j\omega) - R(j\omega)^*) > 0$$

and

$$\lim_{\omega \to 0} \frac{1}{j\omega}(R(j\omega) - R(j\omega)^*) > 0.$$ 

This extension is useful in characterizing the strict negative imaginary property using LMIs and in controller synthesis.

The following lemmas state some useful properties of negative imaginary transfer function matrices.

Lemma 2 ([21], [23]): A square real-rational proper transfer function matrix $R(s)$ is negative imaginary if and only if

$$R(\infty) = R^T(\infty)$$

and

$$R(s) \triangleq R(s) - R(\infty)$$

is negative imaginary.

Lemma 3 ([15]): If the $m \times m$ transfer function matrix $R(s)$ is NI and the $m \times m$ transfer function matrix $R_{s}(s)$ is NI, then the transfer function matrix

$$R_{T}(s) = R(s) + R_{s}(s)$$

is NI. Also, if in addition $R_{s}(s)$ is SNI, then $R_{T}(s)$ is SNI.

Lemma 4 ([15]): Consider the NI transfer function matrices $R(s)$ and $R_{s}(s)$, and suppose that the positive-feedback interconnection of this transfer functions is internally stable. Then the corresponding $2m \times 2m$ closed-loop transfer function matrix

$$T(s) = \begin{bmatrix}
R(s)(I - R_{s}(s)R(s))^{-1} & R_{s}(s)(I - R(s)R_{s}(s))^{-1} R(s)
\end{bmatrix}
\begin{bmatrix}
R(s)(I - R_{s}(s)R(s))^{-1} R_{s}(s)
\end{bmatrix}$$

is NI. Furthermore, if in addition, either $R(s)$ or $R_{s}(s)$ is SNI, then $T(s)$ is SNI.

We can also can also extend the above definitions of NI systems to the finite bandwidth case.

Definition 3 ([28]): A square real-rational proper transfer function matrix $R(s)$ is said to be finite frequency negative imaginary (FFNI) with bandwidth $\bar{\omega}$ if it satisfies the following conditions:

1) $R(s)$ has no poles at the origin and in the open right-half of the complex plane;
2) $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in \Omega$, where

$$\Omega = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}, j\omega \text{ is not a pole of } R(s)\};$$
3) Every pole of $R(s)$ on $j\Omega$, if any, is simple and the corresponding residue matrix of $jR(s)$ is positive semidefinite Hermitian, where $\Omega = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}\};$
4) $R(\infty) = R^T(\infty)$.

Based on the above definition, we have the following properties.

Lemma 5 ([28]): If $R(s)$ is an FFNI transfer function matrix with bandwidth $\bar{\omega}$, then the following properties hold:

1) $j[R(j\omega) - R^*(j\omega)] \leq 0$ for all $\omega \in \Omega_1$, where

$$\Omega_1 = \{\omega \in \mathbb{R} : -\bar{\omega} < \omega < 0, j\omega \text{ is not a pole of } R(s)\};$$
2) Every pole of $R(s)$ in $j\Omega_1$, if any, is simple and the corresponding residue matrix of $jR(s)$ is negative semidefinite Hermitian, where $\Omega_1 = \{\omega \in \mathbb{R} : -\bar{\omega} \leq \omega < 0\}$. 

![Nyquist plot of a negative-imaginary system.](image)
Note that the stability result given in [29], uses a notion similar to the FFNI property defined above along with a small gain property to establish robust stability.

### III. SOME NEGATIVE IMAGINARY LEMMAS

The following Theorem provides a state space condition for a system to be negative imaginary.

**Theorem 1 (Negative Imaginary Lemma, [20],[21],[22]):** Let $(A, B, C, D)$ be a minimal state-space realization of an $m \times m$ real-rational proper transfer function matrix $R(s)$, where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m}$. Then $R(s)$ is negative imaginary if and only if
1. $\det(A) \neq 0$, $D = D^T$, and
2. there exists a matrix $Y = Y^T > 0, Y \in \mathbb{R}^{n \times n}$, such that

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0.$$  

The following alternative form of the negative imaginary lemma also holds in the case that the system has a simple pole at the origin.

**Theorem 2 (Modified NI Lemma, [22],[23]):** Let $(A, B, C, D)$ be a minimal realization of $R(s) \in \mathbb{R}^{m \times m}$, where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m}$, $m \leq n$. Then $R(s)$ is negative imaginary if and only if $D = D^T$ and there exist matrices $P = P^T \geq 0, P \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{m \times m}$, and $L \in \mathbb{R}^{m \times n}$ such that the following LMI is satisfied:

$$
\begin{bmatrix}
PA + A^TP & PB - A^TC^T \\
BT - CA & -C(B + CT^D)
\end{bmatrix}
= \begin{bmatrix}
-LTL & -LTW \\
-WTL & -W^TW
\end{bmatrix} \leq 0.

**Remark 3:** Note that a version of this negative imaginary lemma for descriptor systems is presented in [30].

The following theorems provide corresponding conditions for strict negative imaginary properties.

**Theorem 3 (Weak SNI Lemma, [22],[23]):** Let $(A, B, C, D)$ be a minimal state-space realization of an $m \times m$ real-rational proper transfer function matrix $R(s)$, where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m}$. Then $R(s)$ is strictly negative imaginary if and only if
1. $A$ is Hurwitz, $D = D^T$, rank $(B) =$ rank $(C) = m$;
2. There exists a matrix $Y = Y^T > 0, Y \in \mathbb{R}^{n \times n}$, such that

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0;$$
3. The transfer function matrix $M(s) \sim \frac{A}{L^{-1}A^{-1}}B$ has full column rank at $s = j\omega$ for any $\omega \in (0, \infty)$. Here $L^{-1}L = -AY - YA^T$. That is, rank $(M(j\omega)) = m$ for any $\omega \in (0, \infty)$.

The following theorem provides a purely LMI approach to test the strong SNI property.

**Theorem 4:** [Strong SNI Lemma, [27],[31]] Given a square transfer function matrix $R(s) \in \mathbb{R}^{m \times m}$ with a state-space realization $(A, B, C, D)$, where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{m \times m}$. Suppose $R(s) + R(-s)^T$ has normal rank $m$ and $(C, A)$ is observable.

Then, $A$ is Hurwitz and $R(s)$ is Strongly Strict Negative Imaginary if and only if $D = D^T$ and there exists a matrix $Y = Y^T > 0$ such that

$$AY + YA^T < 0 \quad \text{and} \quad B = -AYC^T. \quad (4)$$

The following theorem from [28] gives a finite bandwidth version of the Negative Imaginary Lemma.

**Theorem 5 (FFNI Lemma [28]):**

Consider a real-rational proper transfer function matrix $R(s)$ with a minimal state-space realization $(A, B, C, D)$. Suppose all poles of $R(s)$ are in the closed left-half of the complex plane, and the poles on the imaginary axis, if any, are simple. Let a positive scalar $\tilde{\omega}$ be given. Also, suppose that if $A$ has eigenvalues $j\omega_i$ ($i \in \{1, \ldots, q\}$) such that $0 < \omega_i \leq \tilde{\omega}$, the residue of $(sI - A)^{-1}$ at $s = j\omega_i$ is given by $\Phi_i \triangleq \lim_{s \to j\omega_i} (s - j\omega_i)(sI - A)^{-1}$. Then the following statements are equivalent:

1. The transfer function matrix $R(s)$ is FFNI with bandwidth $\tilde{\omega}$.
2. $\det(A) \neq 0$, $D = D^T$, and the transfer function matrix $F(s)$ with a minimal state-space realization $(A, B, C, D, B)$ is FFPR with bandwidth $\tilde{\omega}$.
3. $\det(A) \neq 0$, $D = D^T$, and $CA \Phi_i B = (CA \Phi_i B)^* \geq 0$ for all $i \in \{1, \ldots, q\}$ if $A$ has any eigenvalues on $j\Omega$ where $\Omega = \{\omega \in \mathbb{R} : 0 < \omega \leq \tilde{\omega}\}$. Also, there exist real symmetric matrices $P = P^T \geq 0, Q = Q^T \geq 0$ such that

$$
PA + A^T P - A^T Q A + \omega^2 Q \leq 0, \quad (5)
$$

$$
C + B^T A^{-T} P \geq 0, \quad (6)
$$

$$
Q A^{-1} B \geq 0. \quad (7)
$$

4. $\det(A) \neq 0$, $D = D^T$, and $\Phi_i B = (CA \Phi_i B)^* \geq 0$ for all $i \in \{1, \ldots, q\}$ if $A$ has any eigenvalues on $j\Omega$ where $\Omega = \{\omega \in \mathbb{R} : 0 < \omega \leq \tilde{\omega}\}$. Also, there exist real symmetric matrices $Y = Y^T$, $X = X^T \geq 0$ such that

$$
AY + YA^T - AXA^T + \omega^2 X \leq 0, \quad (8)
$$

$$
B + AYC^T = 0, \quad (9)
$$

$$
CX = 0. \quad (10)
$$

In addition to the above negative imaginary lemmas, which involve LMI conditions to test if a given system is negative imaginary or strictly negative imaginary, the following result uses an eigenvalue test of a corresponding Hamiltonian matrix to test if a given system is NI.

**Theorem 6 ([32]):** Consider a transfer function matrix $R(s)$ with a minimal realization $(A, B, C, D)$. Also, suppose $A$ is a Hurwitz matrix, $Q = CB + BT^T C^T > 0$, and $D = D^T$. Then, $R(s)$ is NI if and only if the following conditions are satisfied:

1. The Hamiltonian matrix,

$$
N = \begin{bmatrix}
-A + BQ^{-1}CA & BQ^{-1}BT \\
-A^T C^T Q^{-1} CA & A^T - A^T C^T Q^{-1} B^T
\end{bmatrix}
$$

has no pure imaginary eigenvalues with odd multiplicity.
2. $j(\omega_1^2 + \omega_2^2 + \ldots + \omega_n^2) (R(j\omega_i) - R(j\omega_j))$ has only positive real eigenvalues for all $i \in \{1, k - 1\}$, where $\omega_i \in \Omega$, and $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ is the set of frequencies listed in strictly increasing order, such that $\det[F(j\omega_i) + F(j\omega_j)^*] = 0$. Here, $F(s) = sR(s)$ where $R(s) = R(s) - D$.

**Remark 4:** This approach has been extended in the paper [33] to consider the problem of perturbing a non-NI system model in order to obtain an NI system model.

Also, the paper [32] gives the following result for testing the NI property of SISO systems.

**Assumption 1:** Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}$, and suppose the transfer function $G(s) = C(sI - A)^{-1}B + D$ has all of its poles and zeros in the closed left half of the complex plane excluding poles at the origin. Also, any pole on the imaginary axis is assumed to be simple.

**Theorem 7 ([32]):** The SISO transfer function $G(s) = C(sI - A)^{-1}B + D$ with $CB > 0$, and satisfying Assumption 1 is NI if and only if the following conditions are satisfied:
1. The matrix $(I - \frac{1}{2}BC)$ has no eigenvalues of odd (algebraic) multiplicity on the open negative real axis.
2. All residues of $F(s) = s(G(s) - D)$ corresponding to poles on the imaginary axis are positive.

IV. **STABILITY OF INTERCONNECTIONS OF NEGATIVE IMAGINARY SYSTEMS**

The following theorem, which is the main stability result in the theory of negative imaginary systems, gives a necessary and sufficient condition for the stability of the positive feedback interconnection (see Figure 2) between an NI system and an SNI system.

**Theorem 8 ([20]–[23]):** Given a negative imaginary transfer function matrix $R(s)$ and a strictly negative imaginary transfer function matrix $R_s(s)$ that also satisfy $R(\infty)R_s(\infty) = 0$ and $R_s(\infty) \geq 0$. Then the positive feedback interconnection $[R(s), R_s(s)]$ is internally stable if and only if $\lambda_{\max}(\overline{R(0)}R_s(0)) < 1$.

**Remark 5:** Note that in the papers [34], [35], the robustness properties obtained by this theorem are compared with the robustness properties which are obtained from the $\mu$ analysis approach. Also, in the papers [36], [37], this stability robustness result is extended to allow for additional norm bounded uncertain parameters.

The above theorem can also be extended to give a sufficient condition for closed loop stability in the case in which the NI system has a simple pole at the origin.

**Theorem 9 ([24], [25]):** Consider a strictly proper negative imaginary transfer function matrix $R(s)$ with minimal state space realization $(A, B, C, 0)$, which may contain a simple pole at the origin. Also, consider a strictly negative imaginary transfer function matrix $R_s(s)$. Then the closed-loop positive feedback interconnection between $R(s)$ and $R_s(s)$ is internally stable if $R_s(s) < 0$ and the matrix $A + BR_s(0)C$ is not singular.

V. **NEGATIVE-IMAGINARY FEEDBACK CONTROLLERS**

In this section, we present a number of different classes of commonly used negative imaginary controllers; see also [15].

A. **Positive-Position Feedback Controllers**

In the SISO case, a positive-position feedback controller (e.g., see [5], [6], [12], [38]–[44]) is a controller of the form

$$C(s) = \sum_{i=1}^{M} \frac{k_i}{s^2 + 2\zeta_is + \omega_i^2},$$

where $\omega_i > 0$, $\zeta_i > 0$, and $k_i > 0$ for $i = 1, 2, \ldots, M$. Such controllers are always SNI.

B. **Resonant Controllers**

Resonant controllers are proper but not strictly proper SISO SNI controllers of the form

$$C(s) = \sum_{i=1}^{M} \frac{-k_is^2}{s^2 + 2\zeta_is + \omega_i^2},$$

where $\omega_i > 0$, $\zeta_i > 0$, and $k_i > 0$ for $i = 1, 2, \ldots, M$; e.g., see [8], [45]. The controller (12) can be implemented as the positive-position feedback controller (11) using an acceleration sensor rather than a position sensor. Alternatively, (12) can be implemented as the positive-real feedback controller

$$C(s) = \sum_{i=1}^{M} \frac{k_is}{s^2 + 2\zeta_is + \omega_i^2},$$

where $\omega_i > 0$, $\zeta_i > 0$, and $k_i > 0$ for $i = 1, 2, \ldots, M$, using a velocity sensor rather than a position sensor. Such controllers are always SNI.

C. **Integral Resonant Controllers**

MIMO transfer function matrices of the form

$$C(s) = [sI + \Gamma\Phi]^{-1}\Gamma$$

are SNI. Here, $\Gamma$ is a positive-definite matrix and $\Phi$ is a positive-definite matrix. The use of a controller of this form when applied to a flexible structure with force actuators and position sensors is referred to as integral resonant control, or integral force control [1], [9], [10], [46]–[51].
VI. STATE-FEEDBACK CONTROLLER SYNTHESIS

Consider the feedback control system in Figure 3 in the case that full state feedback is available. In this case, the negative imaginary lemma can be used to synthesize a state-feedback control law such that the resulting closed-loop system is NI. Indeed, suppose the uncertain system shown in Figure 3 is described by the state equations

\[ \dot{x} = Ax + B_1w + B_2u, \quad (13) \]
\[ z = C_1x, \quad (14) \]
\[ w = \Delta(s)z, \quad (15) \]

where the uncertainty transfer function matrix \( \Delta(s) \) is assumed to be SNI with \( |\max(\Delta(0))| \leq 1 \) and \( \Delta(\infty) \geq 0 \). Applying the state-feedback control law \( u = Kx \) yields the closed-loop uncertain system

\[ \dot{x} = (A + B_2K)x + B_1w, \quad (16) \]
\[ z = C_1x, \quad (17) \]
\[ w = \Delta(s)z. \quad (18) \]

The corresponding nominal closed-loop transfer function matrix is

\[ G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1. \quad (19) \]

Fig. 3. A feedback control system.

**Theorem 10 ([15],[52]):** Consider the uncertain system \([13],[14],[15]\) and suppose there exist matrices \( Y > 0, M \), and a scalar \( \varepsilon > 0 \) such that

\[
\begin{bmatrix}
AY + YAT + B_2M + MTB_2^T + \varepsilon I \\
B_1 + C_1YAT + C_1MTB_2^T \\
0
\end{bmatrix}
\leq 0,
\]

\[ C_1YCT - I < 0, \quad (20) \]

\[ Y > 0. \quad (21) \]

Here the parameter \( \varepsilon > 0 \) is chosen to be sufficiently small. Then the state-feedback control law \( u = MY^{-1}x \) is robustly stabilizing for the uncertain system \([13],[14],[15]\).

**Remark 6:** Note that in the papers \([53],[54]\), an alternative approach to controller synthesis is considered by connecting the negative imaginary synthesis problem to the \( H^\infty \) control problem.

VII. CONCLUSIONS

We have surveyed some recent results in the area of negative imaginary systems theory. This results characterize the class of negative imaginary systems and give methods for testing if a given system is negative imaginary. These results also show that the class of NI systems yields a robust stability analysis result, which broadly speaking can be captured by saying that if one system is negative imaginary and the other system is strictly negative imaginary, then a necessary and sufficient condition for internal stability of the positive-feedback interconnection of the two systems is that the dc loop gain is less than unity. This result provides a framework for the analysis of robust stability of lightly damped flexible structures with unmodelled dynamics.

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