Link Identifiability with Two Monitors: Proof of Selected Theorems

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I. INTRODUCTION

Selected lemmas and theorems in [1] are proved in detail in this report. We first list the theorems in Section II and then give the corresponding proofs in Section III. See the original paper [1] for terms and definitions. Table I summarizes all graph-theoretical notions used in this report (following the convention in [2]).

II. THEOREMS

Lemma II.1. Let $B$ be a biconnected component with monitoring agents $m_1'$ and $m_2'$. The set of identifiable links in $L(B)$ does not depend on whether $m_1'$ or $m_2'$ are monitors or not, except for link $m_1'm_2'$ (if it exists). Link $m_1'm_2'$ is identifiable if and only if $m_1'$ and $m_2'$ are both monitors.

Theorem II.2. Algorithm DIL-2M, Determining Identifiable Links under Two Monitors, can determine all identifiable links in a network with given 2-monitor placement.

III. PROOFS

A. Proof of Lemma II.1

1) Let $m_1'$ and $m_2'$ be the two monitoring agents of biconnected component $B$ in Fig. 1 and $m_1'(m_2')$ connects to the real monitor $m_1'(m_2')$ by path $P_1'(P_2')$, i.e., none of $m_1'$ and $m_2'$ are real monitors. In Fig. 1 it is impossible that $P_1'$ and $P_2'$ must have a common node; since otherwise $m_1'$ and $m_2'$ are not cut-vertices, contradicting the processing of localizing monitoring agents for a biconnected component (see DIL-2M). To identify link metrics in $B$, all measurement paths involving links in $B$ are of the following form

$$W_{P_1} + W_{m_1'a_i} + W_{P_{ij}} + W_{b_im_2'} + W_{P_2} = c_{ij}' ,$$

assuming $P_1' (P_2')$ is always selected to connect $m_1'$ and $m_1'$ $(m_2'$ and $m_2')$. We know that if $m_1'$ and $m_2'$ are real monitors, then each measurement (except direct link $m_1'm_2'$) path is of form

$$W_{m_1'a_i} + W_{P_{ij}} + W_{b_im_2'} = c_{ij}.$$  

Therefore, compared with (2), (1) is equivalent to abstracting each of $P_1', P_2'$, and $b_2$ as a single link. By Theorem III.1 [1], we know that none of the exterior links are identifiable. Thus, the link metrics of exterior links do not affect the identification of interior links. Therefore, $B$ can be visualized as a network with two monitors $m_1'$ and $m_2'$ but each exterior link in $\{m_1'a_i, m_2'b_j\}$ has an added weight from $W_{P_1}$ or $W_{P_2}$. The above argument also holds when $m_1'$ ($m_2'$) chooses another path, say $P_1'$ ($P_2'$), to connect to $m_1'$ ($m_2'$), then it simply implies that different exterior links in $\{m_1'a_i, m_2'b_j\}$ in $B$ may have different added path weights when regarding $m_1'$ and $m_2'$ as two monitors. Moreover, the above conclusion also applies to the case that one of $m_1'$ and $m_2'$ is a real monitor. Therefore, the identifiability of all links except for the direct link $l_d := m_1'm_2'$ (if any) remains the same regardless whether $m_1'$, $m_2'$ are monitors or not.

2) To identify direct link (if any) $l_d := m_1'm_2'$, all measurement paths traversing $l_d$ must utilize unidentifiable links incident to $m_1'$ or $m_2'$. To eliminate these unidentifiable links in linear equations, some other measurement paths in $B$ must be used; however, each measurement path in $B$ introduces two new uncomputable variables $W_{m_1'a}$ and $W_{b_im_2'}$, and thus each newly added path for identifying $l_d$ involves new unknown variables. Therefore, $l_d$ cannot be identified when one of $m_1'$ and $m_2'$ is not a real monitor, i.e., $m_1'$ and $m_2'$ must be both real monitors such that $l_d$ is identifiable.

Fig. 1. Monitoring agents $m_1'$ and $m_2'$ wrt biconnected component $B$. 

TABLE I. NOTATIONS IN GRAPH THEORY

| Symbol | Meaning |
|--------|---------|
| $V(\mathcal{G})$, $L(\mathcal{G})$ | set of nodes/links in graph $\mathcal{G}$ |
| $|\mathcal{G}|$ | degree of $\mathcal{G}$: $|\mathcal{G}| = |V(\mathcal{G})|$ (number of nodes) |
| $||\mathcal{G}||$ | order of $\mathcal{G}$: $||\mathcal{G}|| = |L(\mathcal{G})|$ (number of links) |
| $\mathcal{G} \cup \mathcal{G}'$ | union of graphs: $\mathcal{G} \cup \mathcal{G}' = (V \cup V', L \cup L')$ |
| $\mathcal{H}$ | interior graph |
| $P$ | simple path |
| $m_i$ | $m_i \in V(\mathcal{G})$ is the $i$-th ($i = \{1, 2\}$) monitor in $\mathcal{G}$ |
| $W_{l_i}$, $W_{l_j}$ | metric on link $l_i$ and sum metric on path $P$ |
| $m_1', m_2'$ | two monitoring agents in a biconnected component |
Fig. 2. Triconnected component $\mathcal{T}$ in biconnected component $\mathcal{B}$, where \{a,b\} is the 2-vertex-cut, $\mathcal{B}_T$ is the neighboring biconnected component connecting to $\mathcal{T}$ via \{a,b\} and $m'_1$ is a monitoring agent.

Fig. 3. Triconnected component containing no monitoring agents.

Fig. 4. Virtual link replacement.

B. Proof of Theorem II.2

1) completeness of four categories. DIL-2M only processes the biconnected components with 2 monitoring agents as none of the links in biconnected components with 1 or 0 monitoring agent are identifiable. Since only 2 monitors are used in $\mathcal{G}$, the number of monitoring agents for each biconnected component cannot be greater than 2; therefore, it is correct for DIL-2M to only process the biconnected components with 2 monitoring agents. If a triconnected component contains only a single link, then this triconnected component is also a biconnected component, whose identifiability is determined by line 2-4 in DIL-2M based on Lemma II.1. Therefore, the four identifiability categories do not consider the case of a triconnected component which is a single link. Now we discuss triconnected components (with at least 3 nodes) as follows.

(i) A triconnected component $\mathcal{T}$ containing only one monitoring agent. In this case, $\mathcal{T}$ must contain one 2-vertex-cut as $\mathcal{T}$ contains 2 monitoring agents otherwise. This case is illustrated in Fig. 2 where \{a,b\} is the 2-vertex-cut, $\mathcal{T}_N$ is the neighboring biconnected component connecting to $\mathcal{T}$ via \{a,b\} and $m'_1$ is a monitoring agent. Since the associated biconnected component contains two monitoring agents, the neighboring component $\mathcal{T}_N$ must contain one monitoring agent, which cannot be the same as $a$ or $b$ as $\mathcal{T}$ involves two monitoring agents otherwise. Thus, \{a,b\} is of Type-1-VC. If $m'_1 \not\in \{a,b\}$, then $\mathcal{T}$ belongs to Category 1. If $m'_1 = a$ or $m'_1 = b$, then $\mathcal{T}$ belongs to Category 2. If $m'_1 \in \{a,b\}$, the number of monitoring agents for each biconnected component $\mathcal{T}'$ within the same parent biconnected component, thus resulting $\mathcal{T}$ to be of Category 2. Fig. 3 b illustrates the case that each neighboring component ($\mathcal{B}_{T1}$ and $\mathcal{B}_{T2}$) contains one monitoring agent. According to the connectivity of $\mathcal{T}$ in Fig. 3 b, $\mathcal{T}$ belongs to either Category 3 or 4.

Therefore, excluding the triconnected component containing a single link, Category 1-4 are complete to cover all cases of triconnected component within biconnected components with 2 monitoring agents.

2) identification of each category. In Theorem III.2 [1], the prerequisite for network identifiability is that all involved links can be used for constructing measurement paths. In DIL-2M, we sequentially consider each triconnected component which possibly contains virtual links (see [1]). These virtual links, however, do not exist in real networks. To tackle with this issue, we have the following Claim.

Claim 1. A triconnected component $\mathcal{T}$ may contain multiple virtual links. For each involved virtual link whose end-points $\{v_1, v_2\}$ (the end-points of a virtual link must form a vertex cut) are neither Type-1-VC nor Type-2-VC (used to determine the category of $\mathcal{T}$) wrt $\mathcal{T}$, there exists a simple path $\mathcal{P}_r$ with the same end-points in a neighboring biconnected component which connects to $\mathcal{T}$ via $\{v_1, v_2\}$. $\mathcal{P}_r$ can be used to replace the associated virtual link in $\mathcal{T}$ if this virtual link is chosen to construct measurement paths for identifying real links in $\mathcal{T}$. This replacement operation does not affect all existing path construction policies or the identification properties of real links in $\mathcal{T}$.

Proof: Fig. 4 illustrates a triconnected component $\mathcal{T}$ with two Type-1-VCs $\{a,b\}$ and $\{c,d\}$. For vertex cut $\{v_1, v_2\}$ (which is neither $\{a,b\}$ nor $\{c,d\}$), there exists a simple path $\mathcal{P}_r$, connecting $v_1$ and $v_2$ in the neighboring biconnected component $\mathcal{B}_{\mathcal{T}}$ of $\mathcal{T}$ as $\mathcal{B}_{\mathcal{T}}$ contains at least 3 nodes. We know that $\mathcal{B}_{\mathcal{T}}$ connects to the $\mathcal{T}$-involved component by only $\{v_1, v_2\}$; therefore, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ do not have common nodes with $\mathcal{B}_{\mathcal{T}}$ except $a$ which is equal to $v_1$ or $v_2$. Hence, for virtual link $v_1v_2$, if it is used for identifying real links in $\mathcal{T}$ based on Theorem III.2 [1], it can be replaced by $\mathcal{P}_r$ which is a simple path and can be abstracted as a real link in $\mathcal{T}$.
Thus, it is equivalent to the case in Fig.1. We also have that $T$ is a triconnected component. Therefore, for $T$, all real links incident to $a$ and $b$ are unidentifiable and the rest links are identifiable. On top of the discussed scenario shown in Fig.6 now we consider the case that there exists real link $ab$ in $T$, i.e., $ab$ is a real link in Fig.6. Since $\{a,b\}$ is a Type-2-VC wrt $T$ and none of $a$ or $b$ are monitoring agents, $\{a,b\}$ is not a Type-1-VC or Type-2-VC or two monitoring agents wrt the neighboring components connecting to $T$ via $\{a,b\}$. Thus, in those neighboring components, $ab$ is only an ordinary link (not incident to the two vertices of a Type-1-VC or Type-2-VC), the identifiability of which can be determined in identifying the neighboring components. Therefore, we do not need to consider the identifiability of $ab$ in the current triconnected component $T$.

Suppose $T$ contains one Type-2-VC and one monitoring agent, i.e., $a = m'_1$ or $b = m'_2$. Then it is still equivalent to the case in Fig.11 for identifying links (except direct link $ab$) in $T$. Since one of $a$ or $b$ is a monitoring agent, the link $ab$ incident to this monitoring agent is unidentifiable according to Theorem III.1 [1] unless both end-points of $ab$ are monitoring agents (Lemma III.1), which contradicts the assumption that $T$ contains only one monitoring agent.

Suppose $T$ contains two monitoring agents. Then $T$ itself satisfies the interior graph identifiability conditions (Theorem III.2 [1]) and exterior link unidentifiability conditions (Theorem III.1 [1]). For the direct link connecting to these two monitoring agents, the identifiability is already determined by line 2-4 of DIL-2M.

In sum, for Category 2, the effective interior (exterior) links are identifiable (unidentifiable) and the effective direct link is determined in the identification of other triconnected components.

(iii) Category 3.

Category 3 is illustrated in Fig.4. Due to the 2-vertex-connectivity of the corresponding biconnected component, there exist pairwise internally vertex disjoint paths $P_1$, $P_2$, $P_3$ and $P_4$. Then similar arguments for discussing Category 1 can be applied. Therefore, $T$ is the effective interior graph of $T \cup P_1 \cup P_2 \cup P_3 \cup P_4$ and thus all involved links in $T$ are identifiable when $T$ is 3-vertex-connected.

(iv) Category 4 (processed by auxiliary algorithm - Algo-
Let the two Type-1-VCs be \( \{v_1, v_2\} \) and \( \{v_1, v_3\} \) and \( S_1 \) (\( S_2 \)) the set of immediately neighboring triconnected components connecting to \( T \) via \( \{v_1, v_2\} \) (\( \{v_1, v_3\} \)), as illustrated in Fig. 7. Based on the above discussions for Category 1-3, we know that there exist internally vertex disjoint paths \( P_1, P_2, P_3 \) and \( P_4 \). If \( v_1v_2 \) (or \( v_1v_3 \)) is a real link, then \( v_1v_2 \) (or \( v_1v_3 \)) is known as a Cross-link [3] since \( v_2v_3 \) can be replaced by a path in neighboring biconnected component if \( v_2v_3 \) is virtual. Therefore, \( v_1v_2 \) (or \( v_1v_3 \)) is identifiable. Now we focus on the identification of \( v_2v_3 \) in \( T \) (when \( v_2v_3 \) is a real link). The conditions to guarantee the identifiability of \( v_2v_3 \) is:

- (link \( v_1v_2 \) is real OR \( |S_1| \geq 2 \)) OR one component in \( S_1 \) is 3-vertex-connected AND (link \( v_1v_3 \) is real OR \( |S_2| \geq 2 \)) OR one component in \( S_2 \) is 3-vertex-connected.

Suppose the above condition is not satisfied. Then we can prove \( v_2v_3 \) is unidentifiable as follows.

We first consider the condition: link \( v_1v_3 \) is real OR \( |S_2| \geq 2 \) OR one component in \( S_2 \) is 3-vertex-connected. If not satisfied, then it means there is no real link \( v_1v_3 \) and the only immediately neighboring triconnected component is a triangle, i.e., \( v_1a-a-v_3 \) in Fig. 8 (\( v_1a \) and \( v_3a \) can be virtual links as well).

(iv-a) If the monitoring agent \( m^1_1 \) is in the location shown in Fig. 8a, then all paths from \( m^1_1 \) to \( m^2_3 \) traversing \( v_2v_3 \) must use one simple path in \( D_1 \). Therefore, the best case is that we can compute the sum metric of link \( v_2v_3 \) and another link which is incident to \( v_3 \) in \( D_1 \), but cannot compute them separately.

(iv-b) If the monitoring agent \( m^2_3 \) is in the location shown in Fig. 8b, then \( P_3 \) and \( v_2v_3 \) become a “double bridge” connecting \( D_2 \) and \( D_1 \). Abstracting \( P_3 \) as a single link, [3] proves that none of the links in a double bridge is identifiable when constraining the measurement paths to simple paths. If we choose other paths as \( P_3 \) in \( D_3 \), then the same argument applies. Therefore, based on (iv-a) and (iv-b), \( v_2v_3 \) is unidentifiable.

Analogously, we can prove that \( v_2v_3 \) is unidentifiable when condition (link \( v_1v_2 \) is real OR \( |S_1| \geq 2 \) OR one component in \( S_1 \) is 3-vertex-connected) is not satisfied.

When the required conditions are satisfied, we can prove that \( v_2v_3 \) is identifiable as follows:

If \( v_1v_2 \) (or \( v_1v_3 \)) is a virtual link, then it can be replaced by a path in a neighboring component. For instance, if \( |S_1| \geq 2 \), then one replacement path can be found in one component of \( S_1 \). If one component in \( S_1 \) is 3-vertex-connected, then there exist 2 internally vertex disjoint paths (each with the order greater than 1) connecting \( v_1 \) and \( v_3 \). Thus, we can choose one of them as a replacement path. Note that the virtual links possibly involved in the replacement paths can be further replaced by the paths in their neighboring components recursively. After these replacement operations, \( v_2v_3 \) in Fig. 8 is a Shortcut (defined in [3]), which is proved to be identifiable in [3].

Therefore, the auxiliary algorithm (Algorithm 3) of DIL-2M can determine all identifiable/unidentifiable links in a triangle triconnected component.

Consequently, with the complete coverage of four categories and the identification efficacy of each category, DIL-2M can determine all identifiable/unidentifiable links. ■

**REFERENCES**

[1] L. Ma, T. He, K. K. Leung, A. Swami, and D. Towsley, “Link identifiability in communication networks with two monitors,” in *IEEE Globecom*, 2013.

[2] R. Diestel, *Graph theory*. Springer-Verlag Heidelberg, New York, 2005.

[3] L. Ma, T. He, K. K. Leung, A. Swami, and D. Towsley, “Identifiability of link metrics based on end-to-end path measurements,” in *ACM IMC*, 2013.