New Sufficient Conditions for Oscillation of Second-Order Neutral Delay Differential Equations

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Abstract: In this work, new sufficient conditions for the oscillation of all solutions of the second-order neutral delay differential equations with the non-canonical operator are established. Using a generalized Riccati substitution, we obtained criteria that complement and extend some previous results in the literature.

Keywords: delay differential equation; neutral; oscillation; noncanonical case

1. Introduction

Delay differential equation (DDE), as a branch of functional differential equations (FDEs), takes into account the system’s past, allowing for more accurate and efficient future prediction while also describing certain qualitative phenomena. Accordingly, this was a major incentive to study the qualitative properties of the solutions of these equations. In 1964, El’sgol’c laid out many of the foundations for the study of the qualitative methods of DEs in the book [1], and the substantially expanded edition of this book by El’sgol’c and Norkin [2] in 1964. On the other hand, the oscillatory theory of FDEs is a part of the qualitative theory of FDEs, which is concerned with the oscillatory and non-oscillatory properties of solutions. The interesting book by Györi and Ladas [3] summarizes some important work in this area, especially the relation between the distribution of the roots of characteristic equations and the oscillation of all solutions. Erbe et al. [4] contributed significantly to the development of the theory of oscillation and also dealt with some important topics such as estimates of the distance between zeros, and oscillation of equations with nonlinear neutral terms.

A neutral DDE is a DDE in which the highest order derivative of the solution appears both with and without delay. This type of equation appears in many electronic applications and physical problems, see [5].

This work is concerned with studying the oscillatory properties of the second-order neutral DDE

\[
\left( r(s)\left(z'(s)\right)^{\alpha}\right)'+q(s)x^{\alpha}\left(\theta(s)\right)=0, \tag{1}
\]

where \( s \geq s_0 \) and

\[
z(s) := x(s) + p_0 x(\theta(s)).
\]

Throughout this paper, we will assume the following:

(H1) \( \alpha \) is a quotient of odd positive integers;
(H2) \( r \in C([s_0, \infty), (0, \infty)) \), \( p_0 \geq 0 \), and
\[
\int_{s_0}^{\infty} r^{-1/\alpha}(h) \, dh < \infty;
\]
\[(2)\]

(H2) \( q \in C([s_0, \infty), [0, \infty)) \) and \( q(s) \) is not congruently zero for \( s \geq s_0 \geq s_0 \);

(H3) \( \theta \in C((s_0, \infty), \mathbb{R}) \), \( \theta(s) < s \), \( \theta(s) \) is non-decreasing and \( \lim_{s \to \infty} \theta(s) = \infty \);

(H4) \( \vartheta \in C((s_0, \infty), \mathbb{R}) \), \( \vartheta(s) \leq s \), and \( \lim_{s \to \infty} \vartheta(s) = \infty \).

For a proper solution of (1), we purpose a function \( x \in C([s_x, \infty), \mathbb{R}) \), \( s_x \geq s_0 \), which satisfies (1) on \([s_x, \infty)\), has the property \( z(s) \) and \( r(z')\) are continuously differentiable for \( s \in [s_x, \infty) \), and satisfies \( \sup\{|x(s)| : s_x \leq s\} > 0 \) for every \( s_x \geq s_x \). If \( x \) is neither eventually positive nor eventually negative, then \( x \) is called an oscillatory solution, otherwise it is called a non-oscillatory solution. Equation (1) is said to be oscillatory if all its solutions oscillate.

In 1985, Grammatikopoulos et al. [6] studied the oscillatory properties of the neutral DDE
\[
(x(s) + p_0(s)x(s - \theta_0))'' + q(s)x(s - \vartheta_0) = 0,
\]
where \( \theta_0, \vartheta_0 > 0 \), and proved that if \( p_0(s) \in [0, 1] \)
\[
\int_{s_0}^{\infty} q(v)(1 - p_0(v - \theta_0)) \, dv = \infty,
\]
then (3) is oscillatory. Grace and Lalli [7] obtained the oscillation condition for the neutral DDE
\[
(r(s)(x(s) + p_0(s)x(\theta(s)))')' + q(s)f(\theta(s)) = 0,
\]
with \( \theta(s) = s - \theta_0 \) and \( \vartheta(s) = s - \vartheta_0 \), and under the conditions
\[
\frac{f(x)}{x} \geq k > 0, \text{ and } \int_{s_0}^{\infty} r^{-1}(q) \, dq = \infty.
\]

Moreover, Han et al. [8] completed and generalized the results in [7], under the conditions
\[
0 \leq \theta(s) \leq \theta_0 < \infty \text{ and } \theta \circ \theta = \theta \circ \theta.
\]

By using the Riccati technique, Liu et al. [9] and Wu et al. [10] got the oscillation criteria for neutral DDE
\[
\left(r(s)|z'(s)|^{\alpha-1}z'(s)\right)' + \left(q(s)|x(\theta(s))|^{\beta-1}x(\theta(s))\right) = 0,
\]
when
\[
\int_{s_0}^{\infty} r^{-1/\alpha}(\epsilon) \, d\epsilon = \infty.
\]

Baculiková and Džurina [11] and Moaaz et al. [12] considered the neutral DDE
\[
\left(r(s)(z'(s))^{\alpha}\right)' + q(s)x^\beta(\theta(s)) = 0,
\]
where \( \beta \) is a quotient of odd positive integers, obtaining the criteria for oscillation under the conditions (5) and (6).

For the non-canonical case, which is (2) holds, Džurina and Jadlovská [13] established oscillation criteria for the DDE
\[
\left(r(s)(x'(s))^{\alpha}\right)' + q(s)x^\beta(\theta(s)) = 0.
\]

In [14,15], Chatzarakis et al. presented the oscillation results for the advanced case \( \theta(s) \geq s \). By many different techniques and approaches, Agarwal et al. [16], Bohner et al. [17], and Moaaz et al. [18,19] established criteria for oscillation of (7), or special cases
of it, in the non-canonical case. This development in the study of oscillation of second-order DDEs was followed by a great development in the study of even-order equations. The works [20–22] extended the results of second-order equations on the even-order, especially fourth-order.

In this paper, we study the oscillatory behavior of solutions to a class of neutral DDEs. By finding a new relationship between the solution $x$ and the corresponding function $z$, we obtain new oscillation criteria of an iterative nature.

To overcome the assumption $p_0 < 1$, we combine two forms of (1) and then use the inequalities in the following lemmas. The method used depends on the imposition of two Riccati substitutions, once in the traditional form and the other in the general form, and then obtaining from them the Riccati inequality. Thus, we find criteria that are applicable if $p_0 > 1$.

Below we present some lemmas that will be necessary to prove our main results.

**Lemma 1.** ([17], Lemma 2.6) Let $s(v) = A_1 v - A_2 (v - A_3)^{1+1/\alpha}$, where $A_i \in \mathbb{R}$ for $i = 1, 2, 3$, and $A_2 > 0$. Then, $s$ has a maximum value on $\mathbb{R}$ at $v^* = A_3 + ((A_1 A_3 / ((\alpha + 1) A_2))^\alpha$ and

$$s(v^*) = A_1 A_3 + \frac{A_3^{1+1/\alpha} A_1 \alpha}{(\alpha + 1)^{\alpha+1}}.$$

**Lemma 2.** ([23], Lemmas 1 and 2) Let $A, B \in [0, \infty)$. Then,

$$(A + B)^\alpha \leq \delta (A^\alpha + B^\alpha),$$

where

$$\delta := \begin{cases} 1 & \alpha \leq 1; \\ 2^{\alpha-1} & \alpha > 1. \end{cases}$$

2. Main Results

For simplicity, we will denote the set of all eventually positive solutions of (1) by $X^+$. Moreover, assuming $\theta_0 := s, \theta_k := \theta \circ \theta_{k-1}$, for $k = 1, 2, \ldots$ and

$$\eta(s_0) := \int_{s_0}^\infty r^{-1/\alpha}(h)dh.$$

The set of all solutions $x$ whose corresponding function $z$ satisfies $z(s)z'(s) < 0$, is denoted by $K$. In the following theorems, we obtain new criteria for the non-existence of solutions in the class $K$.

**Theorem 1.** If there exist an odd integer $n$ and a function $\rho \in C^1([s_0, \infty), (0, \infty))$ such that

$$p_+(s) := \sum_{k=0}^{(n-1)/2} p_0^{2k} \left( 1 - p_0^{\eta(\theta_{2k+1}(s))} / \eta(\theta_{2k}(s)) \right) > 0$$

and

$$\limsup_{s \to \infty} \frac{\eta^\alpha(s)}{\rho(s)} \int_{s_0}^s \left( p_0^\alpha(\theta(\mu))\rho(\mu)q(\mu) - \frac{r(\theta(\mu))(\rho(\mu))^{\alpha+1}}{(\alpha + 1)^{\alpha+1}(\rho(\mu))^{\alpha}(\theta(\mu))^{\alpha}} \right)d\mu > 1,$$

then $K = \emptyset$. 

Proof. Assume that (1) has a positive solution \( x \), and \( x \in K \). Then \( x(s), x(\vartheta(s)) \) and \( x(\vartheta(s)) \) are positive for \( s \geq s_1 \), for some \( s_1 \geq s_0 \). From the definition of \( z \), we have \( z(s) \geq x(s) > 0 \). From (1), we find that \((r(s)(z'(s))^a)' \leq 0 \). Thus,

\[
z(s) \geq - \int_s^\infty \frac{1}{\rho^{1/a}(\mu)} r^{1/a}(\mu)z'(\mu) d\mu \geq -r^{1/a}(s)z'(s)\eta(s),
\]

it follows that

\[
\left( \frac{z(s)}{\eta(s)} \right)' \geq 0,
\]

for \( s \geq s_1 \). Moreover, we have \( x(s) = z(s) - p_0z(\vartheta(s)) + \rho^2(\vartheta(s)) \). By repeating this step, we get

\[
x(s) = \sum_{k=0}^n (-1)^k p_0^k z(\theta_k(\vartheta(s))) + p_0^{n+1} x(\theta_{n+1}(\vartheta(s))) \geq \sum_{k=0}^{(n-1)/2} \left( p_0^{2k} z(\theta_{2k}(\vartheta(s))) - p_0^{2k+1} z(\theta_{2k+1}(\vartheta(s))) \right).
\]

From (12) and the fact that \( \theta_{2k+1}(\vartheta(s)) \leq \theta_{2k}(\vartheta(s)) \), we find

\[
z(\theta_{2k+1}(\vartheta(s))) \leq z(\theta_{2k}(\vartheta(s))) \left( \frac{(\theta_{2k+1}(\vartheta(s))}{\theta_{2k}(\vartheta(s))} \right),
\]

which with (13), gives

\[
x(s) \geq \sum_{k=0}^{(n-1)/2} p_0^{2k} \left( 1 - \frac{\eta(\theta_{2k+1}(\vartheta(s)))}{\eta(\theta_{2k}(\vartheta(s)))} \right) z(\theta_{2k}(\vartheta(s))) \geq z(s) \sum_{k=0}^{(n-1)/2} p_0^{2k} \left( 1 - \frac{\eta(\theta_{2k+1}(\vartheta(s)))}{\eta(\theta_{2k}(\vartheta(s)))} \right).
\]

Thus, from (1), we arrive at

\[
\left( r(s)(z'(s))^a \right)' = -q(s)x^a(\vartheta(s)) \leq -p^a(\vartheta(s))q(s)z^a(\vartheta(s)).
\]

Since \((r(s)(z'(s))^a)' < 0\), we get that

\[
r^{1/a}(s)z'(s) \leq r^{1/a}(\vartheta(s))z'(\vartheta(s)).
\]

Next, we define a generalized Riccati substitution as

\[
\omega(s) := \rho(s) \left( \frac{r(s)(z'(s))^a}{z^a(\vartheta(s))} + \frac{1}{\eta^a(s)} \right).
\]

Then, \( \omega(s) > 0 \). From (1), (14), and (15), we get

\[
\omega'(s) \leq \frac{\rho'(s)}{\rho(s)} \omega + \rho(s) \left( \frac{r(s)(z'(s))^a}{z^a(\vartheta(s))} \right)' - \frac{\alpha \theta'(s)}{(\rho(s)r(\vartheta(s)))^{1/a}} \left( \omega - \frac{\rho(s)}{\eta^a(s)} \right)^{1+1/a}
\]

\[
+ \frac{\alpha \rho(s)}{r^{1/a}(s)\eta^{a+1}(s)}
\]

\[
\leq -p^a(\vartheta(s))q(s) + \rho'(s)\omega(s) - \frac{\alpha \theta'(s)}{(\rho(s)r(\vartheta(s)))^{1/a}} \left( \omega - \frac{\rho(s)}{\eta^a(s)} \right)^{1+1/a}
\]

\[
+ \frac{\alpha \rho(s)}{r^{1/a}(s)\eta^{a+1}(s)}.
\]

Using Lemma 1 with

\[
A_1 = \frac{\rho'}{\rho}, \quad A_2 = \frac{\alpha \theta'}{(\rho r(\vartheta)))^{1/a}}, \quad A_3 = \frac{\rho}{\eta^a}, \text{ and } v = \omega,
\]
we obtain

\[
\omega'(s) \leq -p_{\alpha}(\vartheta(s))p(s)q(s) + \left(\frac{\rho(s)}{\eta(s)}\right)' + \frac{r(\vartheta(s))(\rho'(s))^a+1}{(\alpha+1)^{a+1}(\rho(s))^a(\vartheta'(s))^a}. \tag{17}
\]

Integrating (17) from \(s_2\) to \(s\) and using (16), we arrive at

\[
\int_{s_2}^{s} \left(p_{\alpha}(\vartheta(\mu))p(\mu)q(\mu) - \frac{r(\vartheta(\mu))(\rho'(\mu))^a+1}{(\alpha+1)^{a+1}(\rho(\mu))^a(\vartheta'(\mu))^a}\right) d\mu \leq \omega(s_2) - \omega(s) - \frac{\rho(s_2)}{\eta(s_2)} + \frac{\rho(s)}{\eta(s)} - \rho(s) s^a(s) \leq \frac{\rho(s_2) s'(s_2)^a}{\eta(s_2)^a} - \rho(s) s'(s)^a - \rho(s) s'(s)^a \leq \frac{\rho(s_2) s'(s_2)^a}{\eta(s_2)^a} - \rho(s) s'(s)^a.
\]

which, in view of (11), implies

\[
\frac{p_{\alpha}(s)}{\rho(s)} \int_{s_2}^{s} \left(p_{\alpha}(\vartheta(\mu))p(\mu)q(\mu) - \frac{r(\vartheta(\mu))(\rho'(\mu))^a+1}{(\alpha+1)^{a+1}(\rho(\mu))^a(\vartheta'(\mu))^a}\right) d\mu \leq 1.
\]

Taking the lim sup on both sides of this inequality, we arrive at a contradiction with (10). Hence, \(K = \emptyset\).

**Theorem 2.** Assume that

\[
\vartheta(s) \leq \vartheta(s), \, \vartheta \circ \vartheta = \vartheta \circ \vartheta \text{ and } \vartheta'(s) \geq \beta_0 > 0. \tag{18}
\]

If

\[
\limsup_{s \to \infty} \int_{s_0}^{s} \left(\frac{1}{\delta} \eta(s)Q(\mu) - \frac{\alpha + 1}{(\alpha+1)^{a+1}(\vartheta'(\mu))^a}\right) d\mu = \infty, \tag{19}
\]

then \(K = \emptyset\), where \(\delta\) is defined in Lemma 2, \(Q(s) = \min\{q(s), q(\vartheta(s))\}\) and

\[
\phi(s) := \frac{1}{r^{1+1/a}(s)\eta(s)} - \frac{p_{\alpha 0}}{\beta_0} \frac{(\vartheta'(s))^{a+1}}{\eta(\vartheta(s))r^{1+1/a}(\vartheta(s))}. \tag{19}
\]

**Proof.** Assume that (1) has a positive solution \(x\), and \(x \in K\). Then \(x(s), x(\vartheta(s))\) and \(x(\vartheta(s))\) are positive for \(s \geq s_1\), for some \(s_1 \geq s_0\). From the definition of \(z\), we have \(z(s) \geq x(s) > 0\). From (1), we find that \((r(s)z'(s)^a)' \leq 0, and\)

\[
\frac{1}{\vartheta'(s)} \left(r(\vartheta(s))(z'(\vartheta(s)))^a\right)' = -q(\vartheta(s))x^{a}(\vartheta(\vartheta(s)))
\]

and so

\[
\frac{1}{\beta_0} \left(r(\vartheta(s))(z'(\vartheta(s)))^a\right)' \leq -q(\vartheta(s))x^{a}(\vartheta(\vartheta(s))). \tag{20}
\]

Combining (1) and (20) and using Lemma 2, we get

\[
\left(r(s)(z'(s))^a\right)' + \frac{p_{\alpha}}{\beta_0} \left(r(\vartheta(s))(z'(\vartheta(s)))^a\right)' \leq -Q(s)[x^{a}(\vartheta(s)) + p_{\alpha}x^{a}(\vartheta(\vartheta(s)))] \leq -\frac{1}{\delta} Q(s)z^a(\vartheta(s)). \tag{21}
\]
Since \( \theta(s) < s \) and \((r(s)(z'(s))^a)' \leq 0\), we obtain
\[
z'(\theta(s)) \geq \left( \frac{r(s)}{r(\theta(s))} \right)^{1/a} z'(s). \tag{22}
\]

Next, we define
\[
w_1 := \frac{r(z')^a}{z^a(\theta)}. \tag{23}
\]

Thus, \( w_1(s) < 0 \). Using (22) and (23), we arrive at
\[
w_1'(s) \leq \frac{(r(s)(z'(s))^a)'}{z^a(\theta(s))} - \alpha \theta'(s) \frac{r^{1+1/a}(s)}{r^{1/a}(\theta(s))} \left( \frac{z'(s)}{z(\theta(s))} \right)^{a+1}
\leq \frac{(r(s)(z'(s))^a)'}{z^a(\theta(s))} - \alpha \theta'(s) \frac{r^{1+1/a}(s)}{r^{1/a}(\theta(s))} w_1^{1+1/a}. \tag{24}
\]

As in the proof of Theorem 1, we obtain (11) holds. Thus, from (23), we get
\[
w_1(s)\eta^a(s) \geq -1. \tag{25}
\]

Multiplying (24) by \( \eta^a(s) \) and integrating from \( s_1 \) to \( s \), we get
\[
1 + w_1(s_1)\eta^a(s_1) \geq - \int_{s_1}^{s} \left[ \frac{\alpha \eta^{a-1}(\mu)}{r^{1/a}(\mu)} w_1(\mu) - \frac{\alpha \theta'(\mu) \eta^a(\mu)}{r^{1/a}(\theta(\mu))} w_1^{1+1/a}(\mu) \right] d\mu
- \int_{s_1}^{s} \frac{(r(\mu)(z'(\mu))^a)'}{z^a(\theta(\mu))} \eta^a(\mu) d\mu.
\]

Using Lemma 1 with \( A_1 = \alpha \eta^{a-1}r^{-1/a} \), \( A_2 = \alpha \theta^{-1/a}(\theta)\eta^a \) and \( A_3 = 0 \), we obtain
\[
1 + w_1(s_1)\eta^a(s_1) \geq - \int_{s_1}^{s} \frac{\alpha^{a+1}r(\theta(\mu))(\theta'(\mu))^a}{(\alpha + 1)^{a+1}(\theta'(\mu))^a \eta(\mu)} d\mu
- \int_{s_1}^{s} \frac{(r(\mu)(z'(\mu))^a)'}{z^a(\theta(\mu))} \eta^a(\mu) d\mu. \tag{26}
\]

Now, we define another function
\[
w_2 := \frac{r(\theta)(z'(\theta))^a}{z^a(\theta)}. \tag{27}
\]

Proceeding exactly as in the previous part of the proof, we arrive at
\[
1 + w_2(s_1)\eta^a(\theta(s_1)) \geq - \int_{s_1}^{s} \frac{\alpha^{a+1}r(\theta(\mu))(\theta'(\mu))^a}{(\alpha + 1)^{a+1}(\theta'(\mu))^a \eta(\theta(\mu))r^{1+1/a}(\theta(\mu))} d\mu
- \int_{s_1}^{s} \frac{(r(\mu)(z'(\mu))^a)'}{z^a(\theta(\mu))} \eta^a(\mu) d\mu. \tag{27}
\]

Combining (26) and (27), we have
\[
1 + w_1(s_1)\eta^a(s_1) + \frac{\eta^a_0}{\eta_0} (1 + w_2(s_1)\eta^a(\theta(s_1))) \geq
\]
\[
- \int_{s_1}^{s} \frac{\alpha^{a+1}r(\theta(\mu))}{(\alpha + 1)^{a+1}(\theta'(\mu))^a} \left( \frac{1}{r^{1+1/a}(\theta(\mu))\eta(\theta(\mu))} - \frac{\eta^a_0}{\eta_0} \frac{(\theta'(\mu))^a}{r^{1+1/a}(\theta(\mu))} \right) d\mu
- \int_{s_1}^{s} \frac{\eta^a(\mu)}{z^a(\theta(\mu))} \left( (r(\mu)(z'(\mu))^a)' + \frac{\eta^a_0}{\eta_0} (r(\theta(\mu))(z'(\theta(\mu)))' \right) d\mu,
\]
which, with (21), gives
\[
\int_{s_1}^{s} \left( \frac{1}{\delta} \eta^a(\mu) Q(\mu) - \frac{\alpha^{a+1} r(\theta(\mu)) \phi(\mu)}{(\alpha + 1)^{a+1} (\theta'(\mu))^a} \right) d\mu \leq 1 + \frac{p_0^a}{\delta} (1 + w_2(s_1) \eta^a(\theta(s_1))).
\]

Taking the lim sup on the above inequality, we arrive at contradiction with (19). Hence, \(K = \emptyset\). □

**Theorem 3.** Assume that (18) holds. If there exists a function \(\rho \in C^1([s_0, \infty), (0, \infty))\) such that
\[
\limsup_{s \to \infty} \frac{\eta^a(s)}{\rho(s)} \int_{s_1}^{s} \left( \frac{1}{\delta} \rho(\mu) Q(\mu) - \frac{(\rho'(\mu))^{a+1} r(\theta(\mu))}{(\alpha + 1)^{a+1} (\rho(\mu))^{a} (\theta'(\mu))^a} \right) d\mu \geq \frac{p_0^a}{\delta} (1 + \frac{p_0^a}{\delta}),
\]
then \(K = \emptyset\), where \(\delta\) is defined in Lemma 2 and \(Q\) is defined as in Theorem 2.

**Proof.** Assume that (1) has a positive solution \(x\), and \(x \in K\). Then \(x(s), x(\theta(s))\) and \(x(\theta(s))\) are positive for \(s \geq s_1\), for some \(s_1 \geq s_0\). From the definition of \(z\), we have \(z(s) \geq x(s) > 0\). As in the proof of Theorem 2, we obtain (21) holds.

Now, we define the two generalized Riccati substitutions \(\omega\) as in (16) and
\[
W := \rho \left( \frac{r(\theta)(z'(\theta))^a}{z^a(\theta)} + \frac{1}{\eta^a(\theta)} \right).
\]

Proceeding exactly as in the proof of Theorem 1, we arrive, after differentiating and using Lemma 1, at
\[
\omega'(s) \leq \rho(s) \left( \frac{r(s)(z'(s))^a}{z^a(\theta(s))} \right)' + \left( \frac{\rho(s)}{\eta^a(s)} \right)' + \frac{(\rho'(s))^{a+1} r(\theta(s))}{(\alpha + 1)^{a+1} \rho^a(s)(\theta'(s))^a},
\]
and
\[
W'(s) \leq \rho(s) \left( \frac{r(\theta)(z'(\theta(s))^a)}{z^a(\theta(s))} \right)' + \left( \frac{\rho(s)}{\eta^a(\theta(s))} \right)' + \frac{(\rho'(s))^{a+1} r(\theta(s))}{(\alpha + 1)^{a+1} \rho^a(s)(\theta'(s))^a}.
\]

Combining (29) and (30) and using (21), we find
\[
\omega'(s) + \frac{p_0^a}{\rho(s)} W'(s) \leq \frac{-1}{\delta} \rho(s) Q(s) + \left( 1 + \frac{p_0^a}{\rho(s)} \right) \frac{(\rho'(s))^{a+1} r(\theta(s))}{(\alpha + 1)^{a+1} \rho^a(s)(\theta'(s))^a} + \left( \frac{\rho(s)}{\eta^a(s)} \right)' + \frac{p_0^a}{\rho(s)} \left( \frac{\rho(s)}{\eta^a(\theta(s))} \right)'.
\]

Integrating the above inequality from \(s_1\) to \(s\), we obtain
\[
\int_{s_1}^{s} \left( \frac{1}{\delta} \rho(\mu) Q(\mu) - \frac{(\rho'(\mu))^{a+1} r(\theta(\mu))}{(\alpha + 1)^{a+1} \rho^a(\mu)(\theta'(\mu))^a} \right) d\mu \leq -\omega(s) + \frac{\rho(s)}{\eta^a(s)} + \omega(s_1) - \frac{\rho(s_1)}{\eta^a(s_1)} - \frac{p_0^a}{\rho(s)} W(s) + \frac{\rho(s)}{\eta^a(\theta(s))} + \frac{p_0^a}{\rho(s)} W(s_1) - \frac{\rho(s_1)}{\eta^a(\theta(s_1))}.
\]
By completing the proof as in the proof of Theorem 1, we can verify that

\[
\eta^a(s) \frac{1}{p(s)} \int_{s_1}^{s} \left( \frac{d}{\delta(\mu)}Q(\mu) - \left( 1 + \frac{p_0}{\beta_0} \right) \frac{(p' (\mu))^a r(\theta(\mu))}{(\alpha + 1)\rho^a(\mu) (\theta'(\mu))^a} \right) d\mu \leq \left( 1 + \frac{p_0}{\beta_0} \right).
\]

Taking the lim sup on both sides of the above inequality, we arrive at a contradiction with (28). Hence, \( K = \emptyset \). \( \square \)

Next, by combining the results of previous theorem with existing ones in the literature, we set new oscillation criteria for the studied equations.

**Theorem 4.** Assume that there exists a function \( q \in C^1([s_0, \infty), (0, \infty)) \) such that

\[
\limsup_{s \to \infty} \int_{s_0}^{s} \left( 1 - p_0^a q(\mu) q(\mu) - \frac{(p'(\mu))^a r(\theta(\mu))}{(\alpha + 1)\rho(\mu) (\theta'(\mu))^a} \right) \leq \infty.
\]

Further, if one of the following sentences holds:

(a) There exist an odd integer \( n \) and a function \( \rho \in C^1([s_0, \infty), (0, \infty)) \) such that (9) and (10) hold;
(b) The conditions (18) and (19) hold;
(c) There exists a function \( \rho \in C^1([s_0, \infty), (0, \infty)) \) such that (18) and (28) hold,

then every solution of (1) is oscillatory.

**Proof.** Assume the contrary that \( x \in X^+ \). Then, \( x(s), x(\theta(s)) \) and \( x(\theta(s)) \) are positive for \( s \geq s_1 \), for some \( s_1 \geq s_0 \). From the definition of \( z \), we have \( z(s) \geq x(s) > 0 \). From (1), we find that \( (r(s)z'(s))^a \leq 0 \), and so \( z'(s) \) does not change sign eventually.

Let \( z'(s) > 0 \) for \( s \geq s_2 \geq s_1 \). The proof of this case is similar to that of theorem 2.1 in [24], and so we omit it.

Let \( z'(s) < 0 \) for \( s \geq s_2 \geq s_1 \). Using Theorems 1–3, we obtain a contradiction with (a)–(c), respectively. The proof is complete. \( \square \)

**Example 1.** Consider the second-order DDE

\[
\left( s^2 \left( x(s) + p_0 x \left( \frac{s}{2} \right) \right) \right)' + q_0 x(\lambda s) = 0,
\]

where \( s \geq 1, p_0 \in [0, 1/2], q_0 > 0 \) and \( \lambda \in (0, 1) \). By Theorem 1 and choosing \( \rho(s) = \eta^a(s) \), we obtain that (31) is oscillatory if

\[
q_0 > \frac{1}{4 \left( \sum_{k=0}^{-(n-1)/2} \frac{2k}{p_0^2 \rho(1 - 2p_0)} \right)}.
\]

Now, by using Theorem 3, we have (18) holds and the condition (28) reduced to

\[
q_0 > \frac{\delta \lambda}{4} \left( 1 + \frac{p_0}{\beta_0} \right).
\]

We note that the condition satisfies for all \( p_0 > 0 \).

**Remark 1.** Applying the results in [16], we get that (31) is oscillatory if

\[
q_0 > \frac{1}{4(1 - 2p_0)}.
\]

For a particular case, when \( p_0 = 0.4 \), the conditions (32) and (33) reduce to \( q_0 > 1.0501 \) and \( q_0 > 1.25 \), respectively.
3. Conclusions

Using a different approach, this paper deals with the problem of finding oscillation conditions for a class of neutral DDEs. We obtained criteria of an iterative nature that enable us to apply them to a wider area of equations. Then, using two Riccati substitutions, we get new criteria for oscillation of the studied equation, which can be used if condition (9) is not satisfied. It would be interesting to extend the obtained results to the more general superlinear equation

\[
\left( r(s) \left( (x(s) + p(s)x^\alpha (\theta (s)))^\beta \right) ' \right) ' + q(s)x^\gamma (\theta (s)) = 0,
\]

where \( \beta \) and \( \gamma \) are quotient of odd positive integers.

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