A few properties of the ratio of Davenport-Heilbronn Functions

Tao Liu
Coherent Electron Quantum Optics Research Center, State Key Laboratory of Environment-friendly Energy Materials, Southwest University of Science and Technology, 59 Qinglong Road, Mianyang, Sichuan 621010, China
School of Science, Southwest University of Science and Technology, Mianyang, Sichuan 621010, China
Juhao Wu
Stanford University, Stanford, California 94309, USA

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Starting from the Davenport-Heilbronn function equation: \( f(s) = X(s) f(1-s) \), we discover the four properties of the meromorphic function \( X(s) \) defined as the ratio of the Davenport-Heilbronn functions: \( f(s)/f(1-s) \), and three corresponding lemmas. For the first time, we propose to study the distribution of the non-trivial zeros of the Davenport-Heilbronn function by exploring the monotonicity of the similarity ratio \( |f(s)/f(1-s)| \). We point out that for the \( f(s) \) which satisfies the Davenport-Heilbronn function equation, the existence of non-trivial zeros outside of the critical line \( \{ s_n | \sigma \neq 1/2 \} \) presents two puzzles: 1) \( f(s_n) \neq f(1-s_n) \); 2) the existence of non-trivial zeros \( \{ s_n | \sigma \neq 1/2 \} \) is in contradiction of the monotonicity of the similar ratio \( |f(s)/f(1-s)| \).

I. EQUATION FOR DA VENPORT-HEILBRONN FUNCTION

The Davenport-Heilbronn function satisfies the function equation:

\[
 f(s) = \left( \frac{s}{\pi} \right)^{\frac{1}{2} - s} \frac{\Gamma(1 - \frac{s}{2})}{\Gamma(\frac{1+s}{2})} f(1-s),
\]

where \( \Gamma(1-s) \) is the analytical continuation of the factorial and \( s = \sigma + it \), with \( \sigma \in \mathbb{R} \) and \( t \in \mathbb{R} \) both being real number. Notice that \( s = -2n \) \((n = 1, 2, 3, \ldots, \infty) \) are the trivial zeroes of \( f(s) \). Since \( f(s) \) is a Holomorphic function, \( f(s) \) is analytic everywhere on the complex \( s \)-plane.

II. THE MEROMORPHIC FUNCTION \( X(s) \)

A. Introduction of the meromorphic function \( X(s) \)

Based on Davenport-Heilbronn function equation (1), we can introduce a meromorphic function:

\[
 X(s) = \left( \frac{s}{\pi} \right)^{\frac{1}{2} - s} \frac{\Gamma(1 - \frac{s}{2})}{\Gamma(\frac{1+s}{2})} ,
\]

so that Eq. (1) is rewritten as:

\[
 f(s) = X(s) f(1-s).
\]

The distribution of the nontrivial zeros of the function \( f(s) \) satisfying Eq. (1) is closely related to the properties of the meromorphic function \( X(s) \). In the following, let us discuss the properties of \( X(s) \).

B. The properties of the meromorphic function \( X(s) \)

1. The reflection symmetry of \( X(s) \)

In the \( s \)-complex space, for arbitrary \( \varepsilon \in \mathbb{R} \), setting \( s_0 = 1/2 + it \), then the pair: \( s_{\pm} = s_0 \pm \varepsilon \) is a mirror symmetric pair with respect to \( s_0 \). The complex conjugates of \( s_+ \) and \( s_- \) are noted as \( s^+_\pm \) and \( s^-_\mp \). Under the \( X(s) \) map, \( s_{\pm} \) and their mirror reflected complex conjugate \( s^\pm_\pm \) are reciprocal pair.
\[ X(s_+)X(s_-) = 1. \]  \hspace{1cm} (4)

\textbf{Proof}

On the complex plane of \( s \), for arbitrary real number \( \epsilon \in R \), because:

\[ X(s_+) = \left( \frac{5}{\pi} \right)^{-\epsilon - it} \frac{\Gamma\left( \frac{3}{4} - \frac{\epsilon - \sigma}{2} \right)}{\Gamma\left( \frac{3}{4} + \frac{\sigma + it}{2} \right)} \]

and

\[ X(s_-) = \left( \frac{5}{\pi} \right)^{\epsilon + it} \frac{\Gamma\left( \frac{3}{4} + \frac{\sigma + it}{2} \right)}{\Gamma\left( \frac{3}{4} - \frac{\epsilon - \sigma}{2} \right)} = \frac{1}{X(s_+)} , \]

which is to say that in the \( X \)-space, \( X(s_-) \) must be equal to \( 1/X(s_+) \), which is the reciprocal of \( X(s_+) \); we have:

\[ X(s_+)X(s_-) = X(s_+) \frac{1}{X(s_-)} = 1. \]

Similarly, we can prove that in the \( X \)-space, \( X(s_+) \) must be equal to \( 1/X(s_-) \), which is the reciprocal of \( X(s_-) \), i.e., \( X(s_+)X(s_-) = 1 \). Therefore, property II B 1 is proven.

2. \( X(s) \) has trivial zeros at \( s = \{ -2n - 1 | n = 0, 1, 2, \ldots, \infty \} \) and pole \( s = \{ 2n + 2 | n = 0, 1, 2, \ldots, \infty \} \). \( X(s) \) at these trivial zeros and at these poles are reciprocal pairs: \( X(-2n-1)X(2+2n) = 1 \).

\textbf{Proof}

First of all, it is obvious that:

\[ X(-2n-1) = \left( \frac{5}{\pi} \right)^{\frac{3}{2}+2n} \frac{\Gamma\left( \frac{3}{4} + n \right)}{\Gamma(-n)} = 0 \]

for \( (n = 0, 1, 2, \cdots) \), and

\[ X(2n+2) = \left( \frac{5}{\pi} \right)^{-\frac{3}{2}-2n} \frac{\Gamma(-n)}{\Gamma\left( \frac{3}{4} + n \right)} = \infty \]

for \( (n = 0, 1, 2, \cdots) \).

Apparently, \( X(-2n-1) \) and \( X(2n+2) \) satisfy reflection symmetry relation:

\[ X(-2n-1)X(2n+2) = 1. \] \hspace{1cm} (5)

This then serves as the proof for property II B 2.

3. Monotonicity of the absolute value \( |X(s)| \) of \( X(s) \).

Excluding the zeros and the poles, the absolute value \( |X(s)| \) of the map \( X(s) \) is a monotonic function of \( t \) except when \( \sigma = 1/2 \):

1. in the range \( 0 < t < +\infty \) when \( \sigma > 1/2 \), \( |X(s)| \) monotonically decreases with the increase of \( t \); when \( \sigma < 1/2 \), \( |X(s)| \) monotonically increases with the increase of \( t \).

2. in the range \( -\infty < t < 0 \), when \( \sigma > 1/2 \), \( |X(s)| \) monotonically increases with the increase of \( t \); when \( \sigma < 1/2 \), \( |X(s)| \) monotonically decreases with the increase of \( t \).

\textbf{Proof} Based on the series representation of the Digamma Function \( \Psi(s) \), we have:

When \( t > 0 \),

\[ \frac{\partial}{\partial t} |f(1-s)| = \frac{\partial}{\partial t} |X(s)| = \sum_{n=1}^{\infty} \frac{8t (\frac{1}{2} - \sigma) (n - \frac{1}{2}) |X(s)|}{|it + 2n + \sigma - 1|^2 |it + 2n - \sigma|^2} \left\{ \begin{array}{ll} > 0, & (\sigma < \frac{1}{2}) \\ = 0, & (\sigma = \frac{1}{2}) \\ < 0, & (\sigma > \frac{1}{2}) \end{array} \right. \] \hspace{1cm} (6)
when \( t < 0, \)
\[
\frac{\partial}{\partial t} |f(s)| = \frac{\partial}{\partial t} |X(s)| = \sum_{n=1}^{\infty} \frac{8t (\frac{1}{2} - \sigma) (n - \frac{1}{2}) |X(s)|}{|it + 2n + \sigma - 1|^2 |it + 2n - \sigma|^2} \begin{cases} < 0, & (\sigma < \frac{1}{2}) \\ = 0, & (\sigma = \frac{1}{2}) \\ > 0, & (\sigma > \frac{1}{2}) \end{cases}. \tag{7}
\]

Therefore, in the range \( 0 < t < +\infty, \) when \( \sigma > 1/2, \) \(|X(s)|\) monotonically decreases with the increase of \( t; \) when \( \sigma < 1/2, \) \(|X(s)|\) monotonically increases with the increase of \( t. \)

in the range \( -\infty < t < 0, \) when \( \sigma > 1/2, \) \(|X(s)|\) monotonically increases with the increase of \( t; \) when \( \sigma < 1/2, \) \(|X(s)|\) monotonically decreases with the increase of \( t. \)

So, property [II B 3] is proven.

4. On the complex \( s-\)plane, in the range: \( 0 \leq \sigma \leq 1, \) but \( \sigma \neq 1/2, \) the \( t \) satisfying \(|X(s)| = 1 \) is bounded: \(|t| < \kappa.\)

Proof

1) Because of property [II B 3] \( i.e., \) the monotonicity of \(|X(s)|, \) in the range \( \sigma < 1/2, \) for arbitrary \( \sigma, \) \(|X(s)|\) always monotonically increases with the increase of \(|t|. \) Therefore, the \( t \) satisfying \(|X(s)| = 1 \) must be bounded.

2) According to property [II B 1] due to the reflection symmetry of \( X(s), \) it must be true that \(|X(s_+)||X(s_-)| = 1, \) \( i.e., \) when \( s_{\pm} = \frac{1}{2} \pm \varepsilon + it, \) \(|X(s_+)\) and \(|X(s_-)|\) have reflection symmetry. So, in the range of \( \sigma > 1/2, \) for an arbitrary \( \sigma, \) \(|X(s)|\) monotonically decreases when \(|t| \) increases. Therefore, the \( t \) satisfying \(|X(s)| = 1 \) must be bounded.

3) The implicit function curve on the \( s-\)plane for \(|X(s)| = 1 \) is shown as in Fig. 1.

FIG. 1. The implicit function curve \(|X(s)| = 1 \) on the \( s-\)plane for \(-6 \leq \sigma \leq 7; \) the yellow solid circle represents \(|X(\sigma)| = 1, \) the blue region represents \(|X(s)| > 1, \) and the red region represents \(|X(s)| < 1. \) The (b) and (c) are the zoom in details of (a) around the zero \((s = -5) \) and the pole \((s = 6) \) of \(|X(s)|. \)

In Fig. 1 we have a clear illustration of the distribution of \(|X(s)| = 1 \) on the \( s-\)plane: on both sides of \( \sigma = 1/2, \) it encloses all the zeros and poles of \( X(s), \) and they are symmetrically distributed. In the range \( 0 \leq \sigma \leq 1, \) the \( t \) satisfying \(|X(s)| = 1 \) is bounded \(|t| < \kappa = 1.21164. \)

So, property [II B 4] is proven.

III. THE DISTRIBUTION OF THE NONTRIVIAL ZEROS OF THE FUNCTION \( f(s) \)

Let us take the absolute value of the Davenport-Heilbronn function equation in (3) to have:
\[
|f(s)| = |X(s)||f(1-s)|. \tag{8}
\]

In the following, we will first prove a few lemmas of the function \( f(s) \) based on the properties of the meromorphic function \( X(s). \)
Lemma 1: functions $f(s)$ and $f(1-s)$ do not have nontrivial zeroes in the range: $0 < |X(s)| \neq 1$.

Proof

1) If $|X(s)| \neq 1$, then it must be true that $|f(s)| \neq |f(1-s)|$.

Assuming that $|f(s)| = |f(1-s)|$, based on the analytical properties of $f(s)$, it can be proven that $|X(s)| = 1$ (please refer to Appendix [A]). Then this is in conflict of $|X(s)| \neq 1$. Therefore, the assumption could not hold and it has to be true that $|f(s)| \neq |f(1-s)|$.

2) When $|X(s)| > 0$, for the function satisfying $|f(s)| \neq |f(1-s)|$, it is impossible to have $|f(s)| = 0$.

Assuming that $|f(s)| = 0$, i.e., $s$ is a nontrivial zero of $f(s)$, we would have $|X(s)| \neq 0$. Then due to $|f(s)| = |X(s)||f(1-s)|$, it must be true that $|f(1-s)| = 0$, so that $|f(s)| = |f(1-s)|$. Then this is in conflict with the statement that $|f(s)| \neq |f(1-s)|$. Therefore, the assumption $f(s) = 0$ can not hold.

3) When $|W(s)| > 0$, for the function satisfying $|f(s)| \neq |f(1-s)|$, $|f(1-s)| = 0$ can only be valid on the trivial zeroes.

Assuming that $|f(1-s)| = 0$, then due to $|f(s)| = |X(s)||f(1-s)|$, one would have $|f(s)| = 0$ with the only exception of $|X(s)| = \infty$, i.e., $s$ is a pole of $|X(s)|$. However, $|f(s)| = 0$ is in conflict with $|f(s)| \neq |f(1-s)|$. Now, the poles of $|X(s)|$ are $s = 2n + 2$, where $|f(1-s)| = |f(-2n-1)| = 0$, i.e., these are just the trivial zeroes.

Based on the above 1), 2), and 3), we know that the function does not have nontrivial zeroes for $0 < |X(s)| \neq 1$. So Lemma 1 is proven.

Lemma 2: $|X(s)| = 1$ is the necessary condition for the nontrivial zeroes of the function $f(s)$.

Proof

Based on Lemma 1: In the range: $0 < |X(s)| \neq 1$, there is no nontrivial zeroes of the function $f(s)$.

Therefore, the nontrivial zeroes of the function $f(s)$ can only be on $|X(s)| = 1$. Indeed, when $|X(s)| = 1$, we must have:

$$|f(s)| = |f(1-s)|. \quad (9)$$

If $|f(s)| = 0$, then $|f(1-s)| = 0$. However, $|f(s)| = |f(1-s)|$ does not guarantee $f(s)$ to be zero. Therefore, $|X(s)| = 1$ is only the necessary condition of the nontrivial zeroes of the function $f(s)$.

So Lemma 2 is proven.

Corollary 1: for $s = 1/2 + it$ being the nontrivial zeroes of $f(s)$, the necessary condition for $X(s)$ is that $|X(s)| = 1$.

Inserting $s = \frac{1}{2} + it$ and $s^* = \frac{1}{2} - it$ into Eq. (9), we have:

$$X(s)X(s) = \left(\frac{s}{\pi}\right)^{1-s-s^*} \Gamma\left(1 - \frac{s}{2}\right) \Gamma\left(1 - \frac{s^*}{2}\right) \left|\frac{\Gamma\left(1 - \frac{s}{2}\right)}{\Gamma\left(1 + \frac{s^*}{2}\right)}\right|_{s = \frac{1}{2} + it, s^* = \frac{1}{2} - it} = 1. \quad (10)$$

So we have: $|X(s)| = 1$. The geometric illustration is that $X(s)$ maps the straight line $s = 1/2 + it$ in $s$-space to the unit circle in the $X$-space.

Lemma 3: on the complex $s$-plane, except for $s = \frac{1}{2} + it$, the similarity ratio $\frac{|f(s)|}{|f(1-s)|}$ of the function $|f(s)|$ does not have the form of $\frac{0}{0}$ for finite $t$.

Proof

1) for $\sigma \neq \frac{1}{2}$, when and only when $t \to \pm \infty$, $\frac{f(s)}{|f(1-s)|}$ is in the form of $\frac{0}{0}$.

According to Eq. (8) and Eq. (9), we have:

$$|X(s)| = \frac{|f(s)|}{|f(1-s)|} = \left(\frac{s}{\pi}\right)^{1-s-s^*} \left|\frac{\Gamma\left(1 - \frac{s}{2}\right)}{\Gamma\left(1 + \frac{s^*}{2}\right)}\right|. \quad (11)$$

Therefore:

$$\frac{d}{dt} \left|\Gamma\left(1 - \frac{s}{2}\right)\right| = -t \sum_{n=1}^{\infty} \frac{|\Gamma\left(1 - \frac{s}{2}\right)|}{|\sigma + it + 2n|^2} \begin{cases} < 0 & (t > 0) \\ > 0 & (t < 0) \end{cases}, \quad (12)$$

$$\frac{d}{dt} \left|\Gamma\left(1 + \frac{s^*}{2}\right)\right| = -t \sum_{n=1}^{\infty} \frac{|\Gamma\left(1 + \frac{s^*}{2}\right)|}{|\sigma + it + 2n - 1|^2} \begin{cases} < 0 & (t > 0) \\ > 0 & (t < 0) \end{cases}. \quad (13)$$

For $t > 0$:

$|\Gamma\left(1 - \frac{s}{2}\right)|$ and $|\Gamma\left(1 + \frac{s^*}{2}\right)|$ are both continuous function monotonically decreasing with $t$ increasing. For an arbitrary $\sigma < \frac{1}{2}$, $(\frac{s}{\pi})^{1-s-s^*}$ is bounded and non-zero. When and only when $t \to \infty$, we have $|\Gamma\left(1 - \frac{s}{2}\right)| \to 0$, $|\Gamma\left(1 + \frac{s^*}{2}\right)| \to 0$. According to Eq. (11), we have:

$$\lim_{t \to \infty} \frac{|f(s)|}{|f(1-s)|} = \lim_{t \to \infty} \left(\frac{s}{\pi}\right)^{1-s-s^*} \left|\frac{\Gamma\left(1 - \frac{s}{2}\right)}{\Gamma\left(1 + \frac{s^*}{2}\right)}\right| \to 0.$$
Because both \( |\Gamma(1 - \frac{t}{2})| \) and \( |\Gamma(\frac{1-t}{2})| \) are continuous functions with none singular points, in the strict constraint of \( \sigma < \frac{1}{2} \), the similar ratio \( \frac{|f(s)|}{|f(1-s)|} \) continuously monotonically increases with \( t \) increasing. Therefore, when and only when \( t \to \infty \), it is in the form of \( 0^0 \).

Based on the same reasoning, in the strict constraint of \( \sigma > \frac{1}{2} \), the similar ratio \( \frac{|f(s)|}{|f(1-s)|} \) continuously monotonically decreases with \( t \) increasing. Therefore, when and only when \( t \to \infty \), \( \frac{|f(s)|}{|f(1-s)|} \) is in the form of \( 0^0 \).

For \( t < 0 \):
\[
|\Gamma(1 - \frac{t}{2})| \text{ and } |\Gamma(\frac{1-t}{2})| \text{ are both continuous function monotonically decreasing with } t \text{ decreasing. Based on the same reasoning, in the strict constraint of } \sigma \neq \frac{1}{2}, \text{ when and only when } t \to -\infty, \frac{|f(s)|}{|f(1-s)|} \text{ is in the form of } 0^0.
\]

2) when \( \sigma \neq \frac{1}{2} \), for an arbitrary finite \( t > 0 \), \( \frac{|f(s)|}{|f(1-s)|} \) can not be in the form of \( 0^0 \).

According to Eq. (10), when \( |X(s)| > 0 \) and \( \sigma < \frac{1}{2} \), we always have:
\[
\frac{d}{dt} \left( \frac{|f(s)|}{|f(1-s)|} \right) = \frac{|f(1-s)| \frac{df(s)}{dt} - |f(s)| \frac{df(1-s)}{dt}}{|f(1-s)|^2} = \frac{d}{dt} |X(s)| > 0.
\]
Therefore,
\[
|f(1-s)| \frac{df(s)}{dt} - |f(s)| \frac{df(1-s)}{dt} > 0.
\]

Using
\[
\frac{df(s)}{dt} = \frac{1}{2|f(s)|} \left( \frac{df(s)}{dt} f(s^*) + \frac{df(s^*)}{dt} f(s) \right).
\]
we have:
\[
\frac{d}{dt} \left( \frac{|f(s)|}{|f(1-s)|} \right) = \frac{1}{2|f(s)|} \left( \frac{df(s)}{dt} f(s^*) + \frac{df(s^*)}{dt} f(s) \right).
\]

By the same reasoning, we have:
\[
\frac{d}{dt} \left( \frac{|f(1-s)|}{|f(s)|} \right) = \frac{1}{2|f(s)|} \left( \frac{df(1-s)}{dt} f(s) + \frac{df(s)}{dt} f(s^*) \right).
\]

Therefore, Eq. (16) is finally written as:
\[
\frac{1}{2} \left\{ f(s^*) \frac{df(s)}{dt} + f(s) \frac{df(s^*)}{dt} - |X(s)| \left( f(1-s^*) \frac{df(1-s)}{dt} + f(1-s) \frac{df(1-s^*)}{dt} \right) \right\} > 0.
\]

For an arbitrary finite \( t > 0 \) and has none singular point; \( f(s), f(s^*), f(1-s), \text{ and } f(1-s^*) \) are all analytical functions. Furthermore, the derivatives: \( \frac{df(s)}{dt}, \frac{df(s^*)}{dt}, \frac{df(1-s)}{dt}, \frac{df(1-s^*)}{dt} \) do not have singular point, either.

If there exists a certain \( s_n \) satisfying \( f(s_n) = 0 \), then according to the function equation: \( f(s_n) = X(s_n) f(1-s_n) \), we must have: \( f(1-s_n) = 0 \). Therefore, we have \( f(s_n) = f(1-s_n) = 0 \). According to Eq. (17), we have:
\[
\frac{1}{2} \left\{ 0 \frac{df(s)}{dt} + 0 \frac{df(s^*)}{dt} - |X(s)| \left( 0 \frac{df(1-s)}{dt} + 0 \frac{df(1-s^*)}{dt} \right) \right\} = 0 \geq 0.
\]

That is to say, \( f(s_n) = 0 \), and \( f(1-s_n) = 0 \) must be in conflict with the inequality Eq. (17). Therefore, for \( \sigma < \frac{1}{2} \), and for a certain finite \( t > 0 \), the similar ratio \( \frac{|f(s)|}{|f(1-s)|} \) can not be in the form of \( 0^0 \). By the same reasoning, for \( \sigma > \frac{1}{2} \), and for a certain finite \( t > 0 \), the similar ratio \( \frac{|f(s)|}{|f(1-s)|} \) can not be in the form of \( 0^0 \), either.

3) when \( \sigma \neq \frac{1}{2} \), even if \( \frac{|f(s)|}{|f(1-s)|} = 1 \), it can not be in the form of \( 0^0 \). For the \( t \) satisfying \( \frac{|f(s)|}{|f(1-s)|} = 1 \), according to property [II B 4] we have \( |t| < \kappa = 1.21164 \), therefore:
\[
1 = |X(s)| = \frac{|f(s)|}{|f(1-s)|} = \frac{\left( \frac{5}{\pi} \right)^{1-\sigma} |\Gamma(1 - \frac{t}{2})|}{|\Gamma(\frac{1-t}{2})|} \to 0.
\]

So Lemma 3 is proven.
IV. SUMMARY

In this paper, starting from the Davenport-Heilbronn function equation, using the properties of the similar ratio $|X(s)|$ of function $f(s)$, we prove that the necessary condition for obtaining the nontrivial zeros of function $f(s)$ is $|X(s)| = 1$. Based on the monotonicity of $|X(s)|$ with respect to $t$, we rigorously prove that on the complex $s$-plane, expect $s = \frac{1}{2} + it$, the similar ratio of the Davenport-Heilbronn function $\frac{|f(s)|}{|f(1-s)|}$ can not be in the form of $\frac{2}{3}$ for an arbitrary finite $t$.

It needs to be pointed out in particular that: only the $s$ satisfying $f(s) = 0$ is the zero of $f(s)$. Those $s$ leading to $f(s) \to 0$ is not the zero of $f(s)$. In other words, the set of the nontrivial zeros of $f(s)$: $S_0 = \{s_n \mid f(s_n) = 0\}$ must be a subset of the set $S_1 = \{s \mid |f(s)| = |f(1-s)|\}$.

We notice that in 1994, R. Spira [4] found four nontrivial zeros outside of the limit line: $s = \frac{1}{2} + it$ of the Davenport-Heilbronn function:

$$s_1 = 0.808517 + 85.699348i,$$

$$s_2 = 0.650830 + 114.163343i,$$

$$s_3 = 0.574356 + 166.479306i,$$

$$s_4 = 0.724258 + 176.702461i.$$

From then on, there has been more work finding numerical results about the nontrivial zeros of the Davenport-Heilbronn function [5]. Almost all the so-called nontrivial zeros of the Davenport-Heilbronn function $f(s)$ share a common feature: their imaginary part $t$ is far larger than the up limit of $\kappa = 1.21164$ determined by the similar ratio $\frac{|f(s)|}{|f(1-s)|} = 1!$ Therefore, it does not satisfy the necessary condition of the nontrivial zero of $f(s)$. This leads to two puzzles:

**Puzzle 1:**
Theoretically, the nontrivial zeros of $f(s)$: $s_n(n = 1, 2, 3, 4)$, must satisfy $f(s_n) = f(1-s_n)$, but the numerical results reported in the literature give $f(s_n) \neq f(1-s_n)$.

**Puzzle 2:**
The nontrivial zeros of $f(s)$ outside of the limit line: $s_n(n = 1, 2, 3, 4)$ must be in conflict with the monotonicity of the similar ratio $\frac{|f(s)|}{|f(1-s)|}$ of the function $f(s)$.

Therefore, if we view the $s$ outside of the limit line and satisfying $f(s) \to 0$ as the nontrivial zeros, then the two puzzles mentioned above will not exist.

The ratio $\frac{|f(s)|}{|f(1-s)|}$ of the Davenport-Heilbronn function $f(s)$ has similar properties as the ratio $\frac{\zeta(s)}{\zeta(1-s)}$ of the Riemann function $\zeta(s)$. This shield light on understanding the nontrivial zeros of the Riemann function $\zeta(s)$.

**Appendix A:** When $|X(s)| > 0$, assuming $|f(s)| = |f(1-s)|$, it must be true that $|X(s)| = 1$

Proof

According to the function equation $f(s) = X(s)f(1-s)$, we have: $X(s) = \frac{f(s)}{f(1-s)}$. Because $X(s) = X(s^*)$, we have $\frac{\overline{f(s)}}{f(1-s)} = \frac{f(s^*)}{f(1-s^*)}$. Therefore:

$$|X(s)|^2 = X(s)\overline{X(s)} = \frac{f(s)\overline{f(s)}}{f(1-s)\overline{f(1-s)}} = \frac{f(s)f(s^*)}{f(1-s)f(1-s^*)}.$$ (A1)

Here, $s = \sigma + it$, and $s^* = \sigma - it$. For the convenience of discussion, let us denote:

$$P(\sigma, t) \equiv f(s)f(s^*),$$

and

$$Q(\sigma, t) \equiv f(1-s)f(1-s^*).$$

They are real functions of $\sigma$ and $t$. 
Because both \( f(s) \) and \( f(s^*) \) are analytical functions, furthermore, their partial derivatives with respect to \( \sigma \) and \( t \) exist to arbitrary orders; the partial derivatives of \( P(\sigma, t) \) with respect to \( \sigma \) and \( t \): \( \frac{\partial^m P(\sigma, t)}{\partial \sigma^m}, \frac{\partial^m P(\sigma, t)}{\partial t^m} \) also exist to arbitrary orders.

While both \( f(1 - s) \) and \( f(1 - s^*) \) are also analytical function, furthermore, their partial derivatives with respect to \( \sigma \) and \( t \) also exist to arbitrary orders; therefore the partial derivatives of \( Q(\sigma, t) \) with respect to \( \sigma \) and \( t \): \( \frac{\partial^m Q(\sigma, t)}{\partial \sigma^m}, \frac{\partial^m Q(\sigma, t)}{\partial t^m} \) also exist to arbitrary orders.

Assuming

\[
|f(s)| = |f(1 - s)|, \tag{A2}
\]

we have:

\[
P(\sigma, t) = Q(\sigma, t), \tag{A3}
\]

and

\[
\frac{\partial^m P(\sigma, t)}{\partial \sigma^m} = \frac{\partial^m Q(\sigma, t)}{\partial \sigma^m}; \tag{A4}
\]

\[
\frac{\partial^m P(\sigma, t)}{\partial t^m} = \frac{\partial^m Q(\sigma, t)}{\partial t^m}. \tag{A5}
\]

1) If \( f(s) \neq 0 \), because \( |X(s)| > 0 \), then it must be true that \( f(1 - s) \neq 0 \). According to Eq. (A1) and Eq. (A2), it must be true that:

\[
|X(s)| = \frac{|f(s)|}{|f(1 - s)|} = 1. \tag{A6}
\]

2) If \( f(s_n) = 0 \), then it must be true that \( f(1 - s_n) = 0 \). In this case, we have \( |X(s_n)| = \frac{|f(s_n)|}{|f(1 - s_n)|} = 0 \). Then it is easy to prove that:

\[
\lim_{t \to t_n} |X(\sigma_n + it)| = 1; \tag{A7}
\]

\[
\lim_{\sigma \to \sigma_n} |X(\sigma + it_n)| = 1. \tag{A8}
\]

Assuming that \( s_n = \sigma_n + it_n \) with \( (n = 1, 2, 3, \cdots) \) are nontrivial zeros of \( f(s) \) and \( f(1 - s) \). They must also be the zeros of the real functions \( P(\sigma, t) \) and \( Q(\sigma, t) \). Then there exists the Taylor expansion of \( P(\sigma, t) \) and \( Q(\sigma, t) \) around the zeros:

\[
P(\sigma_n, t) = \sum_{j=0}^{\infty} \frac{1}{j!} \left. \frac{d^j P(\sigma_n, t)}{dt^j} \right|_{t=t_n} (t - t_n)^j \tag{A9}
\]

\[
Q(\sigma_n, t) = \sum_{j=0}^{\infty} \frac{1}{j!} \left. \frac{d^j Q(\sigma_n, t)}{dt^j} \right|_{t=t_n} (t - t_n)^j. \tag{A10}
\]

The expansion coefficients of Eq. (A9) and Eq. (A10) with the constraint in Eq. (A5) satisfy:

\[
\left. \frac{d^j P(\sigma_n, t)}{dt^j} \right|_{t=t_n} = \left. \frac{d^j Q(\sigma_n, t)}{dt^j} \right|_{t=t_n} \quad (j = 0, 1, 2, \cdots, \infty) \tag{A11}
\]

and won’t be all equal to zero. Let us assume that the first nonzero coefficient comes as the \( k^{th} \) order, i.e.:

\[
\left. \frac{d^k P(\sigma_n, t)}{dt^k} \right|_{t=t_n} = \left. \frac{d^k Q(\sigma_n, t)}{dt^k} \right|_{t=t_n} \neq 0. \tag{A12}
\]

Then according to L’Hopital’s Rule, we have:

\[
\lim_{t \to t_n} |X(\sigma_n + it)|^2 = \lim_{t \to t_n} \frac{P(\sigma_n, t)}{Q(\sigma_n, t)} = \frac{\left. \frac{d^k P(\sigma_n, t)}{dt^k} \right|_{t=t_n}}{\left. \frac{d^k Q(\sigma_n, t)}{dt^k} \right|_{t=t_n}} = 1. \tag{A13}
\]
This leads to:

\[
\lim_{t \to t_n} |X(\sigma_n + it)| = 1.
\]

Similarly,

\[
\lim_{\sigma \to \sigma_n} |X(\sigma + it_n)|^2 = \lim_{\sigma \to \sigma_n} \frac{P(\sigma, t_n)}{Q(\sigma, t_n)} = \left. \frac{\partial^k P(\sigma, t_n)}{\partial \sigma^k} \right|_{\sigma = \sigma_n} = 1.
\]

Therefore, we have:

\[
\lim_{\sigma \to \sigma_n} |X(\sigma + it_n)| = 1.
\]

Following Eq. (A7) and Eq. (A8), if \( |f(s_n)| = |f(1 - s_n)| = 0 \), then \( |f(s)| \) and \( |f(1 - s)| \) approach the nontrivial zeros of the function \( f(s) \) with equivalent infinitesimals.

The above serves as a proof.

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