A CARLEMAN TYPE ESTIMATE OF VARIABLE ORDER SPACE-FRACTIONAL DIFFUSION EQUATIONS AND APPLICATIONS TO SOME INVERSE PROBLEMS

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Abstract. Variable order space-fractional diffusion equation derived as an important model to describe anomalous diffusion phenomenon. In this article, well-posedness has been proved for equations with the “Dirichlet” or the “Neumann” type volume-constrained conditions by using the technique of the nonlocal vector calculus. Then some regularity properties have been obtained under the variable-order Sobolev space framework. By choosing a space-independent weight function and using the technique of the nonlocal vector calculus, a Carleman type estimate has been obtained. At last, the Carleman type estimate has been used both to a backward diffusion problem and an inverse source problem to obtain some uniqueness and stability results.

1. Introduction

Nonlocal diffusion equations have been wildly studied for its important applications in physics [17], image processing [3] and so on. From the mathematical point of view, there are also a lot of investigations [1, 15, 16, 18]. In this paper, we focus on a special type of nonlocal diffusion equations that is the variable order space-fractional diffusion equations. In a simple setting, the model we studied has the following form

$$\partial_t u(x, t) + (-\Delta)^{\beta(x)} u(x, t) = f(x, t) \quad \text{in } \Omega \times (0, T),$$

accompanied with appropriate initial and boundary conditions. Here $\beta(\cdot)$ is a continuous function between 0 and 1. Actually, we study a general model with anisotropic diffusion coefficient but, in this section, we only present this simple model for the general model could not be clearly shown in a few words.

When $\beta(\cdot)$ is a constant, a lot of results have been obtained recently. Du, Gunzburger, Lehouc and Zhou [4, 5] develop the technique of the nonlocal vector calculus and construct weak solutions for some general nonlocal equations. In 2013, a Harnack’s inequality has been obtained by Felsinger and Kassmann [8] for a general space nonlocal diffusion equation which include equation (1.1) with constant $\beta$. In 2015, Felsinger, Kassmann and Voigt [9] construct weak solutions for general nonlocal equations which include non-symmetric kernels. In 2016, Fernández-Real and Ros-Oton [10] prove some results about the boundary regularity, which have been generalized to a very general setting in a recent paper [11].

When $\beta(\cdot)$ is a function, to the best of our knowledge, the studies seem to be limited. When $\beta(\cdot)$ is Lipschitz continuous, Tsucchiya [24] study a stochastic
differential equation related to $(−∆)^{β}$. In a recent book \cite{2}, there are some studies from the operator semigroup perspective. In \cite{19}, Silvestre study the Hölder regularity of the elliptic equations with a variable order fractional Laplace operator.

Because the studies of diffusion equations with variable order space-fractional Laplace operator are little, there are also rare studies on the related inverse problems. However, for constant $β$ case, there are already some investigations. Uniqueness and some numerical results have been obtained in \cite{21,20,22,23} for equations with fractional Laplace operator. For the diffusion equation with the fractional Laplace operator on a periodic domain, backward diffusion problem has been studied in \cite{13} under the Bayesian statistical framework and the same backward diffusion problem has also been studied by using the variable total variation regularization method in \cite{14}.

In this paper, we firstly construct a weak solution, then an improved regularity result has been obtained. Using a space-independent two parameter weight function and the first Green’s identity for the general nonlocal operators, we prove a Carleman type estimate. Let us recall that the Carleman type estimates concerned with the integer-order parabolic equations are useful tools in studying inverse problems e.g. \cite{7,12,25,27}. However, for the fractional diffusion equations, there seems only one paper \cite{26} provide some results on one-dimensional time-fractional space-integral order diffusion equations. Hence, our result on the Carleman type estimate may be a useful tool in studying inverse problems related to the space-fractional diffusion equations. In the last part of this paper, we provide two applications of the Carleman type estimate. Specifically speaking, a stability result for backward diffusion problem has been proved and a uniqueness result for an inverse source problem has also been obtained.

The organization of this paper is as follows. In Section 2 some equivalence relation of function space with variable derivative have been proved. Then, by using Lax-Milgram theorem and Galerkin-type arguments, well-posedness results have been constructed for variable order space-fractional diffusion equation. In the last part of Section 2 some weak regularity properties have been studied for our purpose on the inverse problems. In Section 3 a Carleman type estimate has been constructed, which provide a powerful tool for studying inverse problems. In Section 4 the stability of a backward diffusion problem has been obtained by using the Carleman type estimate. As another application, an inverse source problem has also been studied.

\section{Nonlocal equations}

In this section, a short review of some important concepts in nonlocal vector calculus will be provided, then we propose our general variable order space-fractional diffusion equation with the “Dirichlet” and the “Neumann” type volume constrains. After that, well-posedness and regularity properties will be studied which provide a foundation for our investigations on some Carleman type estimates. Before going further, let us provide some notations used in all of this paper:

- $n$ denotes the space dimension; $Ω ⊂ \mathbb{R}^n$ is a bounded open set;
- $L^p(Ω)$ ($1 \leq p < ∞$) denotes the usual Lebesgue integrable function space; $W^{s,p}(Ω)$ denotes the usual Sobolev space for functions with $s$-order derivative belong to $L^p(Ω)$; If $p = 2$, we denote $H^s(Ω)$ as $W^{s,2}(Ω)$;
• For a function $u$ depend on the time variable $t$, $u'$ stands for the time derivative of $u$;
• Denote $A$ as a matrix, then $A^T$ stands for the transpose of the matrix $A$;

2.1. A brief review of the nonlocal vector calculus. In [4, 5], a nonlocal vector calculus is developed. Here, we briefly review the aspects of that calculus that are useful in what follows.

Given vector mapping $\nu(x, y), \alpha(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ with $\alpha$ antisymmetric, i.e., $\alpha(y, x) = -\alpha(x, y)$, the action of the nonlocal divergence operator $\mathcal{D}$ on $\nu$ is defined as

$$
\mathcal{D}(\nu)(x) := \int_{\mathbb{R}^n} (\nu(x, y) + \nu(y, x)) \cdot \alpha(x, y) dy \quad \text{for } x \in \mathbb{R}^n,
$$

where $\mathcal{D}(\nu) : \mathbb{R}^n \to \mathbb{R}$. Given the mapping $u(x) : \mathbb{R}^n \to \mathbb{R}$, the adjoint operator $\mathcal{D}^*$ corresponding to $\mathcal{D}$ is the operator whose action on $u$ is given by

$$
\mathcal{D}^*(u)(x, y) = -(u(y) - u(x)) \alpha(x, y) \quad \text{for } x, y \in \mathbb{R}^n,
$$

where $\mathcal{D}^*(u) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$. With $\mathcal{D}^*$ denoting the adjoint of the nonlocal divergence operator, we view $-\mathcal{D}^*$ as a nonlocal gradient.

From (2.1) and (2.2), one easily deduces that if $a(t, x, y) = a(t, y, x)$ denotes a second-order tensor satisfying $a = a^T$, then

$$
\mathcal{D}(a \cdot \mathcal{D}^* u)(x) = -2 \int_{\mathbb{R}^n} (u(y) - u(x)) \alpha(x, y) \cdot (a(t, x, y) \cdot \alpha(x, y)) dy \quad \text{for } x \in \mathbb{R}^n,
$$

where $\mathcal{D}(a \cdot \mathcal{D}^* u) : \mathbb{R}^n \to \mathbb{R}$. In the following, sometimes, we denote $\gamma(t, x, y) := \alpha(x, y) \cdot (a(t, x, y) \cdot \alpha(x, y))$. Because $\mathcal{D}$ and $\mathcal{D}^*$ are adjoint operators, if $a$ is also positive definite, the operator $\mathcal{D}(a \cdot \mathcal{D}^*)$ is nonnegative.

Given an open subset $\Omega \subset \mathbb{R}^n$, the corresponding interaction domain is defined as

$$
\Omega_I := \{ y \in \mathbb{R}^n \setminus \Omega \quad \text{such that } \alpha(x, y) \neq 0 \text{ for some } x \in \Omega \},
$$

so that $\Omega_I$ consists of those points outside of $\Omega$ that interact with points in $\Omega$. Note that the situation $\Omega_I = \mathbb{R}^n \setminus \Omega$ is allowable, as is $\Omega = \mathbb{R}^n$, in which case $\Omega_I = \emptyset$. Then, corresponding to the divergence operator $\mathcal{D}(\nu) : \mathbb{R}^n \to \mathbb{R}$ defined in (2.1), we define the action of the nonlocal interaction operator $\mathcal{N}(\nu) : \mathbb{R}^n \to \mathbb{R}$ on $\nu$ by

$$
\mathcal{N}(\nu)(x) := -\int_{\Omega \cup \Omega_I} (v(x, y) + \nu(y, x)) \cdot \alpha(x, y) dy \quad \text{for } x \in \Omega_I.
$$

It is shown in [4, 5] that $\int_{\Omega_I} \mathcal{N}(\nu) \, dx$ can be viewed as a nonlocal flux out of $\Omega$ into $\Omega_I$.

With $\mathcal{D}$ and $\mathcal{N}$ defined as in (2.1) and (2.4), respectively, we have the nonlocal Gauss theorem

$$
\int_{\Omega} \mathcal{D}(\nu) \, dx = \int_{\Omega_I} \mathcal{N}(\nu) \, dx.
$$

Next, let $u(x)$ and $v(x)$ denote scalar functions. Then it is a simple matter to show that the nonlocal divergence formula (2.5) implies the generalized nonlocal
Green’s first identity
\[
\int_{\Omega} v D(a \cdot D^* u) dx - \int_{\Omega \cap \partial I} \int_{\Omega \cap \partial I} D^* v \cdot (a \cdot D^* u) dy dx = \int_{\Omega} v N(a \cdot D^* u) dx.
\]

2.2. Nonlocal diffusion equations. In this subsection, we will describe our model precisely which include equation (1.1) as a special case. With the nonlocal vector calculus, space nonlocal diffusion equation with the “Dirichlet” volume-constrained condition can be written as
\[
\begin{aligned}
\frac{\partial u}{\partial t} + D(a \cdot D^* u) &= f(t, x) \quad \text{on } \Omega \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(0, x) &= u_0(x) \quad \text{on } \Omega \cup \Omega_\mathcal{I}.
\end{aligned}
\]

Space nonlocal diffusion equation with the “Neumann” volume-constrained condition could be written as
\[
\begin{aligned}
\frac{\partial u}{\partial t} + D(a \cdot D^* u) &= f(t, x) \quad \text{on } \Omega \times (0, T), \\
N(a \cdot D^* u) &= 0 \quad \text{on } \Omega_\mathcal{I} \times (0, T), \\
u(0, x) &= u_0(x) \quad \text{on } \Omega \cup \Omega_\mathcal{I}, \\
\int_{\Omega_\mathcal{I}} u dx &= 0.
\end{aligned}
\]

Let us specify the functions \(\alpha(\cdot, \cdot), a(\cdot, \cdot)\) in the definitions of \(D\) and \(D^*\). Assume \(\beta(x)\) to be a continuous function with upper bound \(\beta^* < 1\) and lower bound \(\beta_* > 0\). We choose
\[
\alpha(x, y) = \frac{y - x}{\|y - x\|^{\alpha/2 + \beta(x) + \frac{1}{2}}} \mathbf{1}_{B_\epsilon(x)}(y)
\]
and
\[
\alpha(t, x, y) = (a_{ij}(t, x, y))_{1 \leq i, j \leq n}
\]
with \(a_{ij}(t, x, y) \in C^1([0, T] \times \mathbb{R}^n \times \mathbb{R}^n)\), \(a_{ij} = a_{ji}\) and in addition, we assume
\[
0 < a_* |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(t, x, y)\xi_i\xi_j \leq a^* |\xi|^2,
\]
\[
\sum_{i,j=1}^n \partial_a a_{ij}(t, x, y)\xi_i\xi_j \leq a^* |\xi|^2
\]
for all \(\xi \in \mathbb{R}^n\) and \(y \in B_\epsilon(x)\), where \(B_\epsilon(x) := \{y \in \Omega \cup \Omega_\mathcal{I} : |y - x| \leq \epsilon\}\). Denote \(\gamma(t, x, y) := \alpha(x, y) \cdot a(t, x, y) \cdot \alpha(x, y)\), then we know that
\[
\gamma(t, x, y) = \frac{\gamma(y - x) \cdot a(t, x, y) \cdot \gamma(y - x)}{|y - x|^{n + 2\beta(x) + 2}} \mathbf{1}_{B_\epsilon(x)}(y).
\]

Remark 2.1. If we choose \(\epsilon = \infty\), then we easily know that \(\Omega_\mathcal{I} = \mathbb{R}^n \setminus \Omega\). Assume \(a(t, x, y)\) to be the identity matrix, then the operator \(D(a \cdot D^*)\) could be simplified as \((-\Delta)^{\beta(\cdot)}\) which is appeared in equation (1.1). Hence, equations (2.7) and (2.8) include equation (1.1) as a special case.
As in [4], we define the nonlocal energy norm, nonlocal energy space, and nonlocal volume-constrained energy space by

\begin{align}
\|u\| &:= \left(\frac{1}{2} \int_{\Omega_0 \cup \Omega_T} \int_{\Omega_0 \cup \Omega_T} \mathcal{D}^*(u)(x, y) \cdot (a(t, x, y) \cdot \mathcal{D}^*(u)(x, y)) dy dx\right)^{1/2}, \\
V(\Omega_0 \cup \Omega_T) &:= \{u \in L^2(\Omega_0 \cup \Omega_T) : \|u\| < \infty\}, \\
V_c(\Omega_0 \cup \Omega_T) &:= \{u \in V(\Omega_0 \cup \Omega_T) : E_c(u) = 0\},
\end{align}

respectively, where \(E_c(u)\) given by

\[E_c(u) := \int_{\Omega_T} u^2 dx \quad \text{if} \ \Omega_T \neq \emptyset\]

in the “Dirichlet” volume-constrained condition case and given by

\[E_c(u) := \left(\int_{\Omega_0 \cup \Omega_T} u dx\right)^2 \quad \text{if} \ \Omega_T \neq \emptyset\]

in the “Neumann” volume-constrained condition case. Define \(\|u\|_{V_c^*(\Omega_0 \cup \Omega_T)}\) to be the norm for the dual space \(V_c^*(\Omega_0 \cup \Omega_T)\) of \(V_c(\Omega_0 \cup \Omega_T)\) with respect to the standard \(L^2(\Omega_0 \cup \Omega_T)\) duality pairing.

For a Banach space \(X\), we define

\[W(0, T; X) := \{u \in L^2(0, T; X) : u' \text{ exists and } u' \in L^2(0, T; X^*)\}\]

where \(W(0, T; X)\) is a Banach space endowed with the norm

\[\|u\|_{W(0, T; X)}^2 := \int_0^T \|u(t)\|_X^2 \, dt + \int_0^T |||u'(t)|||_{X^*}^2 \, dt.\]

In addition, we denote

\[L_c^2(\Omega_0 \cup \Omega_T) := \{u \in L^2(\Omega_0 \cup \Omega_T) : E_c(u) = 0\}.\]

**Definition 2.2.** (Parabolic variational formulation) Let \(u_0 \in L_c^2(\Omega_0 \cup \Omega_T)\) and \(f \in L^2(0, T; V_c^*(\Omega))\). We say that \(u \in W(0, T; V_c(\Omega_0 \cup \Omega_T))\) is a solution of (2.7) or (2.8) if for all \(\phi \in V_c(\Omega_0 \cup \Omega_T)\)

\[\int_{\Omega} \partial_t u \phi dx + \int_{\Omega} \int_{\Omega} \mathcal{D}^*(u) \cdot (a(t, x, y) \cdot \mathcal{D}^* \phi) dy dx = \int_{\Omega} f \phi dx, \quad \text{for a.e. } t \in (0, T), u(0, x) = u_0(x).\]

For convenience, we always denote \(\tilde{\Omega} := \Omega \cup \Omega_T\), \(Q := \Omega \times (0, T)\) and \(\tilde{Q} := \tilde{\Omega} \times (0, T)\) in the following parts of this article.

### 2.3. Equivalence of spaces and well-posedness

For a continuous function \(\beta(\cdot)\) satisfying \(0 < \beta_s \leq \beta(x) \leq \beta^* < 1\), the variable-order Sobolev space is defined as

\[H^{\beta(\cdot)}(\Omega_0 \cup \Omega_T) := \{u \in L^2(\Omega_0 \cup \Omega_T) : \|u\|_L^2(\Omega_0 \cup \Omega_T) + |u|_{H^\beta(\cdot)(\Omega_0 \cup \Omega_T)} < \infty\},\]

where

\[|u|_{H^\beta(\cdot)(\Omega_0 \cup \Omega_T)}^2 := \int_{\Omega_0 \cup \Omega_T} \int_{\Omega_0 \cup \Omega_T} \frac{(u(y) - u(x))^2}{|y - x|^{n+2\beta(x)}} dy dx.\]

Moreover, we define the subspaces

\[H_{c}^{\beta(\cdot)}(\Omega_0 \cup \Omega_T) := \{u \in H^{\beta(\cdot)}(\Omega_0 \cup \Omega_T) : E_c(u) = 0\}.\]
Now, let us provide some useful properties about the variable-order Sobolev space defined in (2.18) and (2.19).

**Lemma 2.3.** Let $\beta(\cdot)$ to be a continuous function and there exist $\beta_*$ and $\beta^*$ such that $0 < \beta_* \leq \beta(x) \leq \beta^* < 1$. Let $\Omega$, $\Omega_T$ are open sets in $\mathbb{R}^n$, and $u : \Omega \cup \Omega_T \rightarrow \mathbb{R}$ be a measurable function. Then

$$C^{-1}\|u\|_{H^{\beta_*}(\Omega \cup \Omega_T)} \leq \|u\|_{H^{\beta^*}(\Omega \cup \Omega_T)} \leq C\|u\|_{H^{\beta^*}(\Omega \cup \Omega_T)}$$

for some suitable positive constant $C = C(n, \beta_*, \beta^*) \geq 1$. In particular,

$$H^{\beta_*(\Omega \cup \Omega_T)} \subset H^{\beta^*(\Omega \cup \Omega_T)} \subset H^{\beta^*}(\Omega \cup \Omega_T).$$

**Proof.** Denote $\tilde{\Omega} := \Omega \cup \Omega_T$. Firstly, we have

$$\int_{\tilde{\Omega}} \int_{\tilde{\Omega} \cap \{|y-x| \geq 1}\} \frac{|u(x)|^2}{|y-x|^{n+2\beta_*}} \, dx \, dy \leq C\|u\|_{L^2(\tilde{\Omega})}^2,$$

where the integrability of the kernel $1/|z|^{n+2\beta_*}$ has been used. If using $\beta^*$ instead of $\beta_*$, the above estimate also holds. Because

$$\int_{\tilde{\Omega}} \int_{\tilde{\Omega} \cap \{|y-x| \geq 1\}} \frac{|u(y) - u(x)|^2}{|y-x|^{n+2\beta_*}} \, dy \, dx = \int_{\tilde{\Omega}} \int_{\tilde{\Omega} \cap \{|y-x| \geq 1\}} \frac{|u(y) - u(x)|^2}{|y-x|^{n+2\beta_*}} \, dy \, dx + \int_{\tilde{\Omega}} \int_{\tilde{\Omega} \cap \{|y-x| < 1\}} \frac{|u(y) - u(x)|^2}{|y-x|^{n+2\beta_*}} \, dy \, dx = I_1 + I_2,$$

$$I_1 \leq 2 \int_{\tilde{\Omega}} \int_{\tilde{\Omega} \cap \{|y-x| \geq 1\}} \frac{|u(y)|^2 + |u(x)|^2}{|y-x|^{n+2\beta_*}} \, dy \, dx \leq C\|u\|_{L^2(\tilde{\Omega})}^2$$

and

$$I_2 \leq \int_{\tilde{\Omega}} \int_{\tilde{\Omega} \cap \{|y-x| < 1\}} \frac{|u(y) - u(x)|^2}{|y-x|^{n+2\beta(x)}} \, dy \, dx \leq \|u\|_{H^{\beta^*}(\tilde{\Omega})}^2,$$

we infer that

$$\|u\|_{H^{\beta_*}(\tilde{\Omega})} \leq C\|u\|_{H^{\beta^*}(\tilde{\Omega})},$$

where (2.20) has been used when we estimate $I_1$. Similarly, we could obtain

$$\|u\|_{H^{\beta^*}(\tilde{\Omega})} \leq C\|u\|_{H^{\beta^*}(\tilde{\Omega})}.$$

Hence, the proof is completed. \qed

**Lemma 2.4.** Let the function $\gamma$ defined as in (2.13). Then

$$|u|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 \leq a_*^{-1} \||u||^2 + C\epsilon^{-2\beta_*} \|u\|_{L^2(\tilde{\Omega})}^2,$$

and

$$\||u||^2 \leq a_* |u|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2.$$
Proof. We have
\[ |u|^2_{H^\beta(\Omega)} = \int_\Omega \int_\Omega \frac{|u(y) - u(x)|^2}{|y - x|^{n+2\beta}} \, dy \, dx + \int_\Omega \int_\Omega \frac{|u(y) - u(x)|^2}{|y - x|^{n+2\beta}} \, dy \, dx. \]
\[ \leq \int_\Omega \int_\Omega \frac{|u(y) - u(x)|^2}{|y - x|^{n+2\beta}} \, dy \, dx + \int_\Omega \int_\Omega \frac{|u(y)|^2 + |u(x)|^2}{|y - x|^{n+2\beta}} \, dy \, dx. \]

Noticing
\[ \int_\Omega \int_\Omega \frac{|u(x)|^2}{|y - x|^{n+2\beta}} \, dy \, dx \leq C \int_\Omega \int_\Omega \frac{1}{|z|^{n+2\beta}} \, dz \, |u(x)|^2 \, dx \]
\[ \leq C \epsilon^{-2\beta_n} \|u\|^2_{L^2(\Omega)}, \]
and
\[ \int_\Omega \int_\Omega \frac{|u(y)|^2}{|y - x|^{n+2\beta}} \, dy \, dx = \int_\Omega \int_\Omega \frac{1}{|y - x|^{n+2\beta}} \, dx \, |u(y)|^2 \, dy \]
\[ \leq C \epsilon^{-2\beta_n} \|u\|^2_{L^2(\Omega)}, \]
we conclude that
\[ |u|^2_{H^\beta(\Omega)} \leq a_0^{-1} \|u\|^2 + C \epsilon^{-2\beta_n} \|u\|^2_{L^2(\Omega)}. \]
The second inequality follows directly from
\[ \int_\Omega \int_\Omega \nabla u \cdot (a(t, x, y) \cdot \nabla u) \, dy \, dx \leq a^* \int_\Omega \int_\Omega \frac{|u(y) - u(x)|^2}{|y - x|^{n+2\beta}} \, dy \, dx. \]

\[ \square \]

Lemma 2.5. (Poincaré type inequality) Let the function \( \gamma \) defined as in (2.12). Then there exists a positive constant \( C \) such that
\[ \|u\|^2_{L^2(\Omega)} \leq C \|u\|^2 \quad \forall u \in V_c(\hat{\Omega}). \]

Proof. From Lemma 2.4, we know that \( H^\beta(\Omega) \subset H^\beta(\hat{\Omega}) \). Then, because \( H^\beta(\hat{\Omega}) \) is compactly embedded in \( L^2(\Omega) \), the embedding \( H^\beta(\hat{\Omega}) \hookrightarrow L^2(\hat{\Omega}) \) is compact. With this compact embedding result, this lemma could be proved by standard contradiction arguments. For the standard contradiction arguments, we refer to [1] [6] [7].

From Lemma 2.4 and Lemma 2.5, we obtain the following theorem.

Theorem 2.6. Let the function \( \gamma \) defined as in (2.12). Then there exists a constant \( C > 0 \) such that
\[ C^{-1} \|u\|^2_{H^\beta(\Omega \cup \Omega_2)} \leq \|u\|^2 \leq C \|u\|^2_{H^\beta(\Omega \cup \Omega_2)}, \quad \forall u \in V_c(\Omega \cup \Omega_2). \]

In addition, the constrained spaces \( H^\beta_c(\Omega \cup \Omega_2) \) and \( V_c(\Omega \cup \Omega_2) \) are equivalent.

With these preparations, and recalling the parabolic variational formulation given in Definition 2.2, we could obtain the following theorem by using the Lax-Milgram theorem and the Galerkin-type arguments. For the Galerkin-type arguments, we refer to [6] [9]. Since the arguments are standard, we omit the details here.
Theorem 2.7. Let $\beta(\cdot)$ to be a continuous function and there exist $\beta_*$ and $\beta^*$ such that $0 < \beta_* \leq \beta(x) \leq \beta^* < 1$. Assume that $f \in L^2(0,T; (H^2_c(\Omega \cup \Omega_T))^*)$ and $u_0 \in L^2_c(\Omega \cup \Omega_T)$, then the initial volume-constrained problem (2.7) or (2.12) with nonlocal operator defined by (2.22) and (2.12) has a unique solution $u \in C(0,T; H^2_c(\Omega \cup \Omega_T)) \cap H^1(0,T; (H^2_c(\Omega \cup \Omega_T))^*)$.

2.4. Regularity improvement. In this subsection, we provide more regularity properties of the solution constructed in Theorem 2.7. Define $L^2(0,T; \tilde{H}^{2\beta(\cdot)}(\tilde{\Omega}))$ as a space which includes functions in the following set

$$\left\{ u \in L^2_c(\tilde{\Omega}) : \int_0^T \int_{\tilde{\Omega}} \left| (u(y,t) - u(x,t))\gamma(t,x,y)\right|^2 \, dx \, dt < \infty \right\}$$

with

$$\|u\|^2_{L^2(0,T; \tilde{H}^{2\beta(\cdot)}(\tilde{\Omega}))} := \int_0^T \int_{\tilde{\Omega}} \left| (u(y,t) - u(x,t))\gamma(t,x,y)\right|^2 \, dx \, dt.$$

Then we show the main result in the following theorem.

Theorem 2.8. Assume

$$u_0 \in H^2_c(\tilde{\Omega}), \quad f \in L^2(0,T; L^2(\tilde{\Omega})).$$

Suppose $u$ is the weak solution stated in Theorem 2.7, then we have

$$u \in C(0,T; H^2_c(\tilde{\Omega})) \cap L^2(0,T; \tilde{H}^{2\beta(\cdot)}(\tilde{\Omega})), \quad \partial_t u \in L^2(0,T; L^2(\Omega)), \quad \partial_t \gamma \in L^2(0,T; L^2(\Omega)),$$

and in addition, we have the estimate

$$\sup_{0 \leq t \leq T} \|u(\cdot,t)\|_{H^\beta(\tilde{\Omega})} + \|\partial_t u\|_{L^2(0,T; L^2(\tilde{\Omega}))} + \|u\|_{L^2(0,T; \tilde{H}^{2\beta(\cdot)}(\tilde{\Omega}))} \leq C \left( \|f\|_{L^2(0,T; L^2(\tilde{\Omega}))} + \|u_0\|_{H^\beta(\tilde{\Omega})} \right),$$

where the constant $C$ depending on $T$, $\Omega$ and $a$.

By Galerkin approximations, we can prove this theorem rigorously. However, the proof seems to be standard, so we refer to the proof of Theorem 5 in Chapter 7 in [9] as a prototype and only provide a simple formal derivation for concise.

Proof. From (2.7) or (2.8), we could obtain

$$\int_{\Omega} f^2 \, dx = \int_{\Omega} |\partial_t u + D(a \cdot D^* u)|^2 \, dx \quad \Rightarrow \quad \int_{\Omega} |\partial_t u|^2 + |D(a \cdot D^* u)|^2 + 2D(a \cdot D^* u) \cdot \partial_t u \, dx.$$

Using the generalized nonlocal Green’s first identity (2.9), we have

$$\int_{\Omega} f^2 \, dx = \int_{\Omega} |\partial_t u|^2 + |D(a \cdot D^* u)|^2 + 2 \int_{\Omega} \int_{\Omega} D^* \partial_t u \cdot a(t, x, y) \cdot D^* u \, dy \, dx.$$
We have
\[
2 \int_{\Omega} \int_{\Omega} \partial_t (u(y, t) - u(x, t)) \gamma(t, x, y) (u(y, t) - u(x, t)) dy dx = \frac{d}{dt} \int_{\Omega} \int_{\Omega} (u(y, t) - u(x, t))^2 \gamma(t, x, y) dy dx.
\]
(2.24)

Integrating (2.23) from 0 to \(T\), and using (2.24), we could obtain
\[
\int_{0}^{T} \int_{\Omega} \left| \partial_t u \right|^2 + |D(a \cdot D^* u)|^2 dx dt + \int_{\Omega} \int_{\Omega} (u(y, t) - u(x, t))^2 \gamma(t, x, y) dy dx
\]
\[
\leq \int_{0}^{T} \int_{\Omega} f^2 dx dt + \int_{\Omega} \int_{\Omega} (u(y, 0) - u(x, 0))^2 \gamma(0, x, y) dy dx
\]
\[
+ \int_{0}^{T} \int_{\Omega} \int_{\Omega} (u(y, t) - u(x, t))^2 \partial_t \gamma(t, x, y) dy dx dt
\]

From the above estimate and remembering the estimate of weak solutions, we obtain the desired result.

3. Carleman type estimate

In this section, we focus on a Carleman type estimate, which is an important tool for researches about inverse problems. We denote
\[
L(u) = \partial_t u + D(a \cdot D^* u).
\]
(3.1)

In order to obtain a Carleman estimate, we adopt the following weight function
\[
\varphi(t) = e^{\lambda t}
\]
where \(\lambda > 0\) is fixed suitably. This weight function has been used in [27] to obtain a Carleman type estimate for some integer order diffusion equations. Since our spatial derivative is an anisotropic variable-order fractional Laplace operator, we use the nonlocal vector calculus compared with the integer order diffusion equation case.

**Theorem 3.1.** (Carleman estimate) We set \(\varphi(t) = e^{\lambda t}\). Then there exists \(\lambda_0 > 0\) such that for any arbitrary \(\lambda \geq \lambda_0\) we can choose a constant \(s_0(\lambda) > 0\) satisfying:
\[
\text{there exists a constant } C = C(s_0, \lambda_0) > 0 \text{ such that}
\]
\[
\int_{\Omega} \int_{0}^{T} \left\{ \frac{1}{8 \varphi} \left( |\partial_t u|^2 + |D(a \cdot D^* u)|^2 \right) + s \lambda^2 \varphi^2 u^2 \right\} e^{2s\varphi} dx dt
\]
\[
+ \lambda \int_{0}^{T} \int_{\Omega} \int_{\Omega} \frac{|v(y, t) - v(x, t)|^2}{|y - x|^{n+2\beta(x)}} e^{2s\varphi} dy dx dt
\]
\[
\leq C \int_{\Omega} |L(u)|^2 e^{2s\varphi} dx dt + C e^{C(\lambda)s} \left( \|u(\cdot, T)\|_{H^{\beta(\cdot)}(\Omega)}^2 + \|u(\cdot, 0)\|_{H^{\beta(\cdot)}(\Omega)}^2 \right)
\]
for all \(s > s_0\) and all \(u \in C(0, T; H^{\beta(\cdot)}(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^{2\beta(\cdot)}(\Omega))\)

(3.2)
\[ u = 0 \text{ in } \Omega \times (0, T) \]
(3.3) \[ N(a \cdot D^* u) = 0 \quad \text{in} \quad \Omega_T \times (0, T). \]

Proof. Set \[ v = e^{s\varphi} u, \quad P v = e^{s\varphi} L(e^{-s\varphi} v) = e^{s\varphi} f. \]

Assume that \( u|_{\Omega_T} = 0 \) or \( N(a \cdot D^* u)|_{\Omega_T} = 0 \). Obviously, we obtain

(3.4) \[ P v = \partial_t v - (s\lambda \varphi v - D(a \cdot D^* v)) = e^{s\varphi} f. \]

In addition, we have

\[
\|e^{s\varphi} f\|_{L^2(Q)}^2 = \int_Q |\partial_t v|^2 dxdt + 2 \int_Q \partial_t v (-s\lambda \varphi v + D(a \cdot D^* v)) dxdt \\
+ \int_Q |s\lambda \varphi v - D(a \cdot D^* v)|^2 dxdt \\
\geq \int_Q |\partial_t v|^2 dxdt + 2 \int_Q \partial_t v D(a \cdot D^* v) dxdt + 2 \int_Q \partial_t v (-s\lambda \varphi v) dxdt
\]

(3.5) \[ \equiv \int_Q |\partial_t v|^2 dxdt + I_1 + I_2. \]

Thus

(3.6) \[ \int_Q f^2 e^{2s\varphi} dxdt \geq I_1 + I_2 \]

and

(3.7) \[ \int_Q |\partial_t v|^2 dxdt \leq \int_Q f^2 e^{2s\varphi} dxdt + |I_1 + I_2|. \]

In the following, \( C_j > 0 \ (j \in \mathbb{N}) \) denote generic constants which are independent of \( s, \lambda \). Since \( s, \lambda \) are assumed to be a large enough constant, without loss of generality, we can assume \( s > 1 \) and \( \lambda > 1 \). For \( I_1 \), we have

\[
|I_1| \geq 2 \int_0^T \int_{\Omega_\tau} \partial_t v D(a \cdot D^* v) dxdt \\
= 2 \int_0^T \int_{\tilde{\Omega}} D^* \partial_t v \cdot a \cdot D^* v dydx dt + 2 \int_0^T \int_{\Omega_T} \partial_t v N(a \cdot D^* v) dydx dt \\
= 2 \int_0^T \int_{\tilde{\Omega}} (\partial_t v(y, t) - \partial_t v(x, t)) \gamma(t, x, y) (v(y, t) - v(x, t)) dydx dt \\
= \int_0^T \int_{\tilde{\Omega}} \partial_y \left[ [v(y, t) - v(x, t)]^2 \gamma(t, x, y) \right] dydx dt \\
\]

\[
= - \int_0^T \int_{\tilde{\Omega}} \partial_t \gamma(t, x, y) (v(y, t) - v(x, t))^2 dydx dt \\
\quad + \int_{\tilde{\Omega}} \int_{t=0}^{t=T} \gamma(t, x, y) (v(y, t) - v(x, t))^2 dydx \bigg|_{t=0}^{t=T} \\
\]

(3.8) \[ \leq C_4 \|v\|_{L^2(0, T; H^2(\tilde{\Omega}))}^2 + C_4 \|\gamma(\cdot, T)\|^2_{H^2(\tilde{\Omega})} + C_1 \|v(\cdot, 0)\|^2_{H^2(\tilde{\Omega})} + C_1 \|v(\cdot, 0)\|_{H^2(\tilde{\Omega})}. \]
where (2.11) has been used for the last inequality. For $I_2$, we have

$$I_2 = -s\lambda \int_{Q} 2\partial_t v v \varphi dxdt = s\lambda \int_{Q} v^2 \partial_t \varphi dxdt - s\lambda \left( \int_{\Omega} \varphi v^2 dx \right)_{t=0}^{t=T}$$  

(3.9)

$$\geq s\lambda^2 \int_{Q} \varphi v^2 dxdt - s\lambda \int_{\Omega} (e^{AT}|v(x,T)|^2 + |v(x,0)|^2) dx.$$  

From (3.6), (3.8) and (3.9), we obtain

$$\int_{Q} (Pv) v dxdt = \int_{Q} v \partial_t v dxdt - \int_{Q} s\lambda \varphi v^2 dxdt + \int_{Q} v D(a \cdot D^* v) dxdt$$  

(3.10)

$$\equiv J_1 + J_2 + J_3.$$  

For $J_1$, we obtain

$$|J_1| = \left| \int_{Q} v \partial_t v dxdt \right| = \frac{1}{2} \int_{Q} \partial_t (v^2) dxdt \leq \frac{1}{2} \int_{\Omega} (|v(x,T)|^2 + |v(x,0)|^2) dx.$$  

For $J_2$, we have

$$|J_2| = \left| - \int_{Q} s\lambda \varphi v^2 dxdt \right| \leq C_2 \int_{Q} s\lambda \varphi v^2 dxdt.$$  

At last, for $J_3$, we have

$$J_3 = \int_{0}^{T} \int_{Q} v \cdot D(a \cdot D^* v) dxdt = \int_{0}^{T} \int_{\Omega} \int_{\Omega} D^* v \cdot a(t, x, y) \cdot D^* v dy dxdt$$  

$$= \int_{0}^{T} \int_{\Omega} \int_{\Omega} (v(y, t) - v(x, t))^2 \gamma(t, x, y) dy dxdt$$  

$$\geq a_1 \int_{0}^{T} \int_{\Omega} \int_{\Omega} \frac{(v(y, t) - v(x, t))^2}{|y - x|^{n+2\beta(x)}} dy dxdt.$$  

From (3.11) and the above estimates on $J_1$, $J_2$ and $J_3$, we obtain

$$\int_{Q} (Pv) v dxdt \geq a_1 \int_{0}^{T} \int_{\Omega} \int_{\Omega} \frac{(v(y, t) - v(x, t))^2}{|y - x|^{n+2\beta(x)}} dy dxdt - C_2 \int_{Q} s\lambda^2 \varphi v^2 dxdt$$  

(3.12)

$$- \frac{1}{2} \lambda \int_{\Omega} (|v(x,T)|^2 + |v(x,0)|^2) dx.$$  

On the other hand,

$$\int_{Q} \chi(Pv) v dxdt \leq \|Pv\|_{L^2(Q)} (\chi \|v\|_{L^2(Q)}) \leq \frac{1}{2} \|Pv\|_{L^2(Q)}^2 + \frac{\lambda^2}{2} \|v\|_{L^2(Q)}^2$$  

(3.13)

$$\leq \frac{1}{2} \|f e^{\alpha^2} \|_{L^2(Q)}^2 + \frac{\lambda^2}{2} \|v\|_{L^2(Q)}^2.$$  

\[ \text{VARIABLE ORDER SPACE-FRACTIONAL DIFFUSION EQUATION 11} \]
Hence, \((3.12)\) and \((3.13)\) yield
\[
a_\ast \int_{0}^{T} \int_{\bar{\Omega}} \lambda \left( \frac{(v(y,t) - v(x,t))^2}{|y-x|^{n+2\beta(x)}} \right) dy dx dt \leq C_2 \int_{Q} s \lambda^2 \varphi v^2 dx dt + \frac{1}{2} \| f e^{s \varphi} \|_{L^2(\bar{\Omega})}^2
\]
\[
+ \frac{\lambda^2}{2} \| v \|_{L^2(\bar{\Omega})}^2 + \frac{\lambda}{2} \int_{\bar{\Omega}} \left( |v(x,T)|^2 + |v(x,0)|^2 \right) dx.
\]
Estimating the first term on the right-hand side by \((3.10)\), we obtain
\[
a_\ast \int_{0}^{T} \int_{\bar{\Omega}} \lambda \left( \frac{(v(y,t) - v(x,t))^2}{|y-x|^{n+2\beta(x)}} \right) dy dx dt \leq C_3 \| f e^{s \varphi} \|_{L^2(\bar{\Omega})}^2 + C_3 \| v \|_{L^2(0,T;H^{\beta(\cdot)}(\tilde{\Omega}))}^2
\]
\[
+ C_3 \lambda \left( \| v(\cdot, T) \|_{L^2(\bar{\Omega})}^2 + \| v(\cdot, 0) \|_{L^2(\bar{\Omega})}^2 \right)
\]
\[
+ C_3 \| v(\cdot, T) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 + \| v(\cdot, 0) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2
\]
\[
+ C_3 \lambda \left( e^{\lambda T} \| v(\cdot, T) \|_{L^2(\bar{\Omega})}^2 + \| v(\cdot, 0) \|_{L^2(\bar{\Omega})}^2 \right).
\]
Considering both \((3.10)\) and \((3.14)\), we obtain
\[
s \lambda^2 \int_{Q} \varphi v^2 dx dt + a_\ast \int_{0}^{T} \int_{\bar{\Omega}} \lambda \left( \frac{(v(y,t) - v(x,t))^2}{|y-x|^{n+2\beta(x)}} \right) dy dx dt
\]
\[
\leq C_4 \| f e^{s \varphi} \|_{L^2(\bar{\Omega})}^2 + C_4 \| v \|_{H^{\beta(\cdot)}(\bar{\Omega})}^2 + C_4 \lambda \left( \| v(\cdot, T) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 + \| v(\cdot, 0) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 \right)
\]
\[
+ C_4 \left( \| v(\cdot, T) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 + \| v(\cdot, 0) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 \right)
\]
\[
+ C_4 \lambda \left( e^{\lambda T} \| v(\cdot, T) \|_{L^2(\bar{\Omega})}^2 + \| v(\cdot, 0) \|_{L^2(\bar{\Omega})}^2 \right).
\]
Now, we take \(s > 0, \lambda > 0\) large to absorb the second and third terms on the right-hand side into the left-hand side, then we obtain
\[
\int_{Q} s \lambda^2 \varphi v^2 dx dt + \int_{0}^{T} \int_{\bar{\Omega}} \lambda \left( \frac{(v(y,t) - v(x,t))^2}{|y-x|^{n+2\beta(x)}} \right) dy dx dt
\]
\[
\leq C_5 \| f e^{s \varphi} \|_{L^2(\bar{\Omega})}^2 + C_5 e^{C(\lambda)s} \left( \| v(\cdot, T) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 + \| v(\cdot, 0) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 \right).
\]
Since \(v = e^{s \varphi} u\), in addition, we have
\[
\int_{Q} s \lambda^2 \varphi u^2 e^{2s\varphi} dx dt + \int_{0}^{T} \int_{\bar{\Omega}} \lambda \left( \frac{|u(y,t) - u(x,t)|^2}{|y-x|^{n+2\beta(x)}} \right) e^{2s\varphi} dy dx dt
\]
\[
\leq C_6 \| f e^{s \varphi} \|_{L^2(\bar{\Omega})}^2 + C_6 e^{C(\lambda)s} \left( \| u(\cdot, T) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 + \| u(\cdot, 0) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 \right).
\]
Since \(\partial_t u = -s \lambda \varphi e^{-s \varphi} v + e^{-s \varphi} \partial_t v\), we obtain
\[
\frac{1}{s \varphi} |\partial_t u|^2 e^{2s\varphi} \leq 2s \lambda^2 \varphi v^2 + \frac{2}{s \varphi} |\partial_t v|^2.
\]
By \((3.27)\), \((3.10)\) and estimates about \(I_1, I_2\), we find
\[
\int_{Q} \frac{1}{s \varphi} |\partial_t u|^2 e^{2s\varphi} dx dt \leq C \int_{Q} f^2 e^{2s\varphi} dx dt
\]
\[
+ C e^{C(\lambda)s} \left( \| u(\cdot, T) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 + \| u(\cdot, 0) \|_{H^{\beta(\cdot)}(\tilde{\Omega})}^2 \right).
\]
From \(D(a \cdot D^* u) = f - \partial_t u\), we could finish the proof by using \((3.16)\) and \((3.18)\). □
Remark 3.2. From Theorem 2.8, we know that the regularity condition required in Theorem 3.1 could be satisfied.

Remark 3.3. From Section 5.3 in [4], we know that a variable-order space fractional advection-diffusion equation could be formulated as follow

\begin{equation}
\partial_t u + D(a \cdot D^s u) + \frac{1}{2} D(\mu u) = f \quad \forall x \in \Omega, t > 0.
\end{equation}

And we have

\begin{equation}
D(\mu u) = uD(\mu) - \int_{R^n} \mu \cdot D^s(u) dy.
\end{equation}

Since

\begin{equation}
\int_\Omega \int_\Omega -\mu \frac{(u(y,t) - u(x,t))(y - x)}{|y - x|^{n+2\beta(x)+1}} dydx
\end{equation}

\begin{equation}
\leq \left( \int_\Omega \int_\Omega \mu^2 dydx \right)^{1/2} \left( \int_\Omega \int_\Omega \frac{|u(y,t) - u(x,t)|^2}{|y - x|^{n+2\beta(x)}} dydx \right)^{1/2},
\end{equation}

by some simple calculations, we can easily see that if we take $s, \lambda$ large enough, a Carleman estimate also holds for the variable-order space fractional advection-diffusion equation as illustrated for the integer-order diffusion equations in [27].

In the last part of this section, we show the following theorem which could be proved similar to Theorem 3.1.

**Theorem 3.4.** We set $\phi(t) = e^{-\lambda t}$. Then there exists $\lambda_0 > 0$ such that for any arbitrary $\lambda \geq \lambda_0$ we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that

\begin{equation}
\int_Q \left\{ \frac{1}{s\phi} \left( |\partial_t u|^2 + |D(a \cdot D^s u)|^2 \right) + s\lambda^2 \phi u^2 \right\} e^{2s\phi} dxdt
+ \lambda \int_0^T \int_\Omega \int_\Omega \frac{|v(y,t) - v(x,t)|^2}{|y - x|^{n+2\beta(x)}} e^{2s\phi} dydxdt
\leq C \int_Q |L(u)|^2 e^{2s\phi} dxdt + Ce^{C(s)}|u(\cdot,0)|_{H^{2\beta(\lambda)}(\tilde{\Omega})}^2
+ C \int_{\Omega^2 \times (0,T)} s\lambda(|u| + |\partial_t u|| \mathcal{N}(a \cdot D^s u)| e^{2s\phi} dxdt
\end{equation}

for all $s > s_0$ and all $u \in C(0,T; H^{2\beta(\lambda)}(\tilde{\Omega})) \cap H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^{2\beta(\lambda)}(\tilde{\Omega}))$

satisfying

\begin{equation}
\int_0^T \int_\Omega u(x,t)dxdt = 0 \quad \text{in} \ \Omega.
\end{equation}

4. APPLICATIONS TO TWO INVERSE PROBLEMS

In the first part this section, we apply Theorem 3.1 to a backward diffusion problem and establish a conditional stability estimate. Let us now specify the problem as follows.

**Backward in time problem:** Let $0 \leq t_0 < T$. For system (2.7) or (2.8), determine $u(x,t_0), x \in \Omega$ from $u(x,T), x \in \Omega \cup \Gamma_T$.

For this backward in time problem, we have the following theorem.
Theorem 4.1. Let \( u \) to be a solution of system (2.7) or (2.8) satisfying
\[ u \in C(0, T; \mathcal{H}^{1}(\Omega)) \cap L^{2}(0, T; \mathcal{H}^{2}(\Omega)), \quad \partial_{t}u \in L^{2}(0, T; L^{2}(\Omega)). \]
For any \( t_{0} \in (0, T) \), there exist constants \( \theta \in (0, 1) \) and \( C > 0 \) depending on \( t_{0} \), \( a_{*}, a^{*}, T, \Omega \) and \( \Omega_{2} \) such that
\[ \| u(\cdot, t_{0}) \|_{L^{2}(\Omega)} \leq C\| u \|^{\frac{1}{2} - \theta}_{L^{2}(\bar{\Omega})} \| u(\cdot, T) \|^{\theta}_{H^{\theta}(\Omega_{2}, \Omega_{2})}. \]
If in addition, \( \| u(\cdot, 0) \|_{L^{2}(\Omega_{2}, \Omega_{2})} < \infty \), then we have
\[ \| u \|_{L^{2}(\bar{Q})} = O \left( \log \frac{1}{\| u(\cdot, T) \|_{H^{\theta}(\Omega_{2}, \Omega_{2})}} \right)^{-1/2} \]
as \( \| u(\cdot, T) \|_{H^{\theta}(\Omega_{2}, \Omega_{2})} \to 0 \).

Proof. As in the proof of Theorem 9.2 in [27], we could choose same cut-off function and by using Theorem 3.1 to conclude that
\[ \| u(\cdot, t_{0}) \|_{L^{2}(\Omega)} \leq C\| u \|^{\frac{2 + \delta_{k} - \delta_{1}}{2 + \delta_{k} - \delta_{1}}}_{L^{2}(\bar{Q})} \| u(\cdot, T) \|^{2(\delta_{k} - \delta_{1})}_{H^{\theta}(\Omega_{2}, \Omega_{2})}, \]
where \( \delta_{k} = e^{\lambda t_{k}}, k = 0, 1, 2 \) and \( 0 < t_{2} < t_{1} < t_{0} \). Now, integrating the above inequality with respect to \( t_{0} \) from 0 to \( T \), we find that
\[ \| u \|_{L^{2}(\bar{Q})} \leq C(1 + \| u(\cdot, 0) \|_{L^{2}(\bar{Q})}) \left( \log \frac{1}{\| u(\cdot, T) \|_{H^{\theta}(\Omega_{2}, \Omega_{2})}} \right)^{-1/2}. \]
Thus the proof is completed. \( \square \)

In the second part of this section, let us focus on a special inverse source problem. Let \( x = (x_{1}, x') \in \mathbb{R}^{n} \) and \( x'(x_{2}, \cdots, x_{n}) \in \mathbb{R}^{n-1} \), \( \Omega = (0, t) \times D', D' \subset \mathbb{R}^{n-1} \) be a bounded domain with smooth boundary \( \partial D' \). We consider
\[ \begin{cases}
\partial_{t}u(x, t) + \mathcal{D}(a \cdot \mathcal{D}^{*}u)(x, t) &= f(x', t), \quad x \in \Omega, t > 0 \\
u(x, 0) &= 0, \quad x \in \Omega \cup \Omega_{t} \\
\mathcal{N}(a \cdot \mathcal{D}^{*}u)(x, t) &= 0, \quad x \in \Omega_{t}, t > 0.
\end{cases} \]
Here, we only consider a simple case that is \( \beta \) is a constant between 0 and 1, \( \epsilon \) appeared in the definition of \( \mathcal{D}(a \cdot \mathcal{D}^{*}) \) is equal to \( \infty \) and function \( a(t, x, y) \equiv 1. \)

Inverse heat source problem: Let \( t_{0} > 0 \). Determine \( f(x', t) \), \( x' \in D' \), \( 0 < t < t_{0} \) by \( u|_{\Omega \times (0, t_{0})} \).

For this problem, we have the following result.

Theorem 4.2. We assume that \( u, \partial_{t}u \in C(0, T; \mathcal{H}_{x}^{\beta}(\bar{\Omega})) \cap H^{1}(0, T; L^{2}(\Omega)), \) For \( t_{0} > 0 \), if \( u|_{\mathbb{R}^{n} \times (0, t_{0})} = 0 \), then \( f(x', t) = 0, x' \in D', 0 \leq t \leq t_{0} \).

Proof. For arbitrary small \( \epsilon > 0 \), we choose \( t_{1}, t_{2} \) such that \( 0 < t_{0} - \epsilon < t_{1} < t_{2} < t_{0} \). We set \( \delta_{k} = e^{-\lambda_{k}}, k = 0, 1, 2 \). Let \( \chi \in C^{\infty}(\mathbb{R}) \) be a cut-off function such that \( 0 \leq \chi \leq 1 \) and
\[ \chi(t) = \begin{cases} 
1, & t < t_{1}, \\
0, & t \geq t_{2}.
\end{cases} \]
For \( u \), we have
\[ \partial_{t}u + \mathcal{D}^{*}D' = f(x', t), \quad x \in \Omega, 0 < t < t_{2} \]
and

\[ u|_{(\mathbb{R}^n \setminus \Omega) \times (0,t_0)} = \mathcal{N}(D^*u)|_{(\mathbb{R}^n \setminus \Omega) \times (0,t_0)} = 0. \]

Differentiating both sides of (4.5) with respect to \( x_1 \) and setting \( v = \partial_{x_1} u \), we obtain

\[ \partial_t v + D D^* v = 0, \quad x \in \Omega, \ 0 < t < t_2. \]

(4.6)

In the above calculation, we used the property \( \partial_{x_1} DD^* = DD^* \partial_{x_1} \), which relies on our assumptions stated below (4.4). Setting \( w = \chi v \), we have

\[ \partial_t w + D D^* w = -\chi'(t)v, \quad x \in \Omega, \ 0 < t < t_2 \]

and

\[ w(x,t_0) = w(x,0) = 0, \quad w|_{(\mathbb{R}^n \setminus \Omega) \times (0,t_0)} = 0. \]

Using Theorem 3.4, we find that

\[ \int_0^{t_0} \int_\Omega |\chi' v|^2 e^{2s\varphi} dxdt + \int_0^{t_0} \int_\Omega \int_\Omega \frac{|w(y,t) - w(x,t)|^2}{|y-x|^{n+2\beta}} e^{2s\varphi} dydxdt \leq C \int_0^{t_0} \int_\Omega |\chi' v|^2 e^{2s\varphi} dxdt. \]

Denote \( M = \|\partial_{x_1} u\|_{L^2(Q)} \). Using the properties of \( \chi \), we obtain

\[ \int_0^{t_0} \int_\Omega |\chi' v|^2 e^{2s\varphi} dxdt \leq CM^2 e^{2s\delta_1}. \]

(4.8)

Setting \( \delta_\varepsilon = e^{-\lambda(t_0-\varepsilon)} \), using the properties of \( \chi \) and (4.8), there holds

\[ e^{2s\delta_\varepsilon} \left\{ \int_0^{t_0} \int_\Omega sw^2 dxdt + \int_0^{t_0} \int_\Omega \int_\Omega \frac{|w(y,t) - w(x,t)|^2}{|y-x|^{n+2\beta}} dydxdt \right\} \leq \int_0^{t_0} \int_\Omega sw^2 e^{2s\varphi} dxdt + \int_0^{t_0} \int_\Omega \int_\Omega \frac{|w(y,t) - w(x,t)|^2}{|y-x|^{n+2\beta}} e^{2s\varphi} dydxdt \leq CM^2 e^{2s\delta_1}. \]

(4.9)

Hence, we easily obtain

\[ \|w\|_{L^2(0,t_0-\varepsilon; H^{\beta}(\Omega))} \leq CM^2 e^{-2s(\delta_\varepsilon - \delta_1)} \]

for all large \( s > 0 \). Since \( \delta_\varepsilon - \delta_1 > 0 \), letting \( s \to \infty \), we obtain \( w = 0 \) in \( \Omega \times (0,t_0-\varepsilon) \). Obviously, we have \( v = 0 \) in \( \Omega \times (0,t_0-\varepsilon) \). Considering the assumptions, we have \( u = 0 \) in \( \Omega \times (0,t_0-\varepsilon) \), which implies \( f = 0 \) in \( D' \times (0,t_0-\varepsilon) \) by noticing (4.5). Because \( \varepsilon > 0 \) is arbitrary, the proof is completed.

\[ \square \]

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