Asymptotic approach for the nonlinear equatorial long wave interactions

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Abstract. In the present work we use an asymptotic approach to obtain the long wave equations. The shallow water equation is put as a function of an external parameter that is a measure of both the spatial scales anisotropy and the fast to slow time ratio. The values given to the external parameters are consistent with these computed using typical values of the perturbations in tropical dynamics. Asymptotically, the model converge toward the long wave approximation through intermediate realizable states. With this approach, the resonant nonlinear wave interactions are studied. To simplify, the reduced dynamics of a single resonant triad is used for some selected equatorial trios. It was verified by both theoretical and numerical results that the nonlinear energy exchange period increases smoothly as we move toward the long wave approach. The magnitude of the energy exchanges is also modified, but in this case depends on the particular triad used and also on the initial energy partition among the triad components. Some implications of the results for the tropical dynamics are discussed. In particular, we discuss the implications of the results for El Ni\~ no and the Madden-Julian in connection with other scales of time and spatial variability.

1. Introduction

It is common in theoretical works on large-scale tropical dynamics the use of the long wave approximation for studies related to the El Ni\~ no and/or the Madden-Julian oscillation ([1, 2]). This approximation filters out inertia-gravity waves of all wavelengths while being accurate for Kelvin and long Rossby waves. However, the long wave approximation is completely inaccurate for short Rossby waves ([10]). Furthermore, the completeness of the remaining waves has not been proven and consequently, the representation of any phenomenon on the basis of only these solutions is questionable. In the present work, an asymptotic method is used to obtain the long wave approximation as a limiting case of the shallow water equation. The difference with previous works is that the transformation of one regime to another is controlled by an external parameter ($\delta \in [0, 1]$) which is a measure of the anisotropy of the space and time scalings. The $\delta \to 1$ limit corresponds to the shallow water equation, whereas $\delta \to 0$ refers to the long wave approximation.
approximation. Any value in between corresponds to intermediate realizable states. The model equations developed here are valid for both limiting cases and also for the intermediate states. The advantage of this method is that it allows a continuous approach to the long wave equations while keeping the completeness of the solutions of the shallow water equations ([11]). With this approach we have studied the resonant nonlinear wave energy exchanges in the long wave like model. We have found an effective modification by the $\delta$ parameter of the conservation laws in both the physical and spectral space. The nonlinear coupling coefficients, which is valid for resonant and off-resonant interactions also depend on $\delta$. It was also verified by both theoretical and numerical results that the period of the nonlinear energy modulation increases as $\delta$ decreases. The magnitude of the energy exchange is also modified as a function of $\delta$.

In section 2 the basic model equations are presented together with the anisotropic scaling procedure. Section 3 deals with the conservation of energy and potential vorticity for the anisotropic model. In section 4 the spectral representation of the model equations using the eigensolutions of the linear problem as expansion basis is presented. Section 5 deals with the nonlinear evolution equation of the spectral coefficients and also with the modification of both the nonlinear coupling and nonlinear interaction coefficient as a function of the anisotropy parameter. In section 6 numerical experiments of the reduced dynamics of a single resonant triad are presented for some selected equatorial trios. In section 7 the nonlinear resonant three wave interactions are discussed and how these are modified by the anisotropy parameter. Finally, the main findings of the work are summarized in the Conclusions section.

2. Governing equations
We start with the unforced, inviscid and nonlinear shallow water equations in the equatorial $\beta$ plane in dimensional form (equation 1)

\[
\begin{align*}
\frac{\partial}{\partial x}u + v \cdot \nabla u - \beta y v + g \partial_x H &= 0 \quad (1a) \\
\frac{\partial}{\partial x}v + v \cdot \nabla v + \beta y u + g \partial_y H &= 0 \quad (1b) \\
\frac{\partial}{\partial x}H + v \cdot \nabla H + H \nabla \cdot v &= 0 \quad (1c)
\end{align*}
\]

In equation (1), $H = \bar{H} + \eta$, $\bar{H}$ is the mean thickness of the fluid layer, $\eta$ represents its perturbation and $\beta y$ is the equatorial Coriolis parameter. The equations can be nondimensionalized by taking units of length and time in the range presented in equation (2).

\[
\begin{align*}
L &= \left(\frac{C}{\beta}\right)^{1/2} \approx [500, 1500] \text{km} \quad (2a) \\
T &= \frac{1}{\left(\frac{C}{\beta}\right)^{1/2}} \approx [8, 18] \text{hrs} \quad (2b)
\end{align*}
\]

Where $C = \sqrt{g \bar{H}} \in [10, 50] \text{m/s}$ is the atmospheric baroclinic wave speed. The anisotropically scaled and nondimensionalized ($\delta$ dependant) variables are obtained as follows:

\[
\begin{align*}
x &= (L/\delta)x''; y = Ly''; t = (T/\delta)t'' \\
u &= Cu''; v = \delta C v''; H = (C^2/g)H''; \eta = \eta'' \quad (3a) \quad (3b)
\end{align*}
\]

We restrict the analysis to $0 < \delta < 1$. Therefore, $x''$ and $t''$ are large scale and slow time, respectively, whereas $v''$ is of small magnitude. With the substitution $(x,y,t,u,v,\eta)'' \rightarrow (x,y,t,u,v,\eta)$ equations (4) are obtained.

\[
\begin{align*}
\frac{\partial}{\partial x}u + u \partial_x u + v \partial_y u - y v + \partial_x \eta &= 0 \quad (4a) \\
\delta^2[\partial_x v + u \partial_x v + v \partial_y v] + y u + \partial_y \eta &= 0 \quad (4b) \\
\partial_t \eta + u \partial_x \eta + v \partial_y \eta + (1 + \eta)(\partial_x u + \partial_y v) &= 0 \quad (4c)
\end{align*}
\]
3. Conserved quantities

In what follows we will show that conserved quantities for equations (4) can also be expressed as a function of \( \delta \). The energy conservation is:

\[
\partial_t E + \text{div}(\eta + \frac{1}{2}\eta^2 + E)\vec{v} = 0
\]  

(5)

where

\[ E = E(u, v, \eta, \delta) = \frac{1}{2}[(1 + \eta)(u^2 + \delta^2v^2) + \eta^2] \]  

(6)

and the potential vorticity conservation law can also be obtained:

\[
\partial_t \left( \eta + \delta^2\partial_x v - \partial_y u \right) + \vec{v} \cdot \nabla \left( \frac{y + \delta^2\partial_x v - \partial_y u}{1 + \eta} \right) = 0
\]  

(7)

In the present study we will focus on the energy conservation as it is quadratic to the lowest order. Thus, its consequences for the nonlinear wave interactions are stronger than the imposed by the potential vorticity conservation law (see [13]).

4. Spectral representation

With \( \phi = (u, v, \eta)^T \) the modal representation of the dependent variables and the spectral coefficients are:

\[
\phi(x, y, t) = \sum_a \phi_a(x, y)Z_a(t);
\]  

(8a)

\[
Z_a(t) = \int \phi_a^+(x, y)\phi(x, y, t)dxdy
\]  

(8b)

where \( \phi_a(x, y) \) are the eigenfunctions of the linear operator \( \mathcal{L} \) defined below, and \( \phi_a^+ (x, y) \) is the transposed-conjugated of \( \phi_a \)

\[
(L - i\omega_a I)\phi_a(x, y) = 0
\]  

(9)

The asymptotic model (see equation (4)) can be written as:

\[
[\partial_t \mathcal{I}(\delta) + \mathcal{L}]\phi + \mathcal{I}(\delta)B(\phi, \phi) = 0
\]  

(10)

where

\[
\mathcal{I}(\delta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \delta^2 & 0 \\
0 & 0 & 1
\end{pmatrix}; \quad \mathcal{L}(\phi) = \begin{pmatrix}
-yv + \partial_x \eta \\
yu + \partial_y \eta \\
\partial_x u + \partial_y v
\end{pmatrix}; \quad B(\phi, \phi) = \begin{pmatrix}
\vec{v} \cdot \nabla u \\
\vec{v} \cdot \nabla v \\
\nabla (\eta \vec{v})
\end{pmatrix}.
\]  

(11)

5. Evolution equation

Projection of equation (10) onto an arbitrary eigenmode \( \phi_a \) results in:

\[
[d_t I + i\omega, I]Z_a(t) = 1/2 \sum_{bc} \delta_{abc}\sigma^a_b \mathcal{I}Z_b^*(t)Z_c^*(t).
\]  

(12)

In equation 12, \( \delta_{abc} \) indicates the interaction condition and establishes that in order for the interaction to take place, the modes must satisfy the relations \( k_a + k_b + k_c = 0 \) and \( n_a + n_b + n_c = \text{odd} \). Otherwise, the interaction is zero. The indentity matrix \( I(= \mathcal{I}(\delta=1)) \);
and \( \{ Z_a^*, Z_b^* \} \) are the complex conjugate of \( \{ Z_a, Z_b \} \), respectively. The coupling coefficient \( \sigma_{abc} \) can be put as a function of \( \delta \):

\[
\delta_{abc} \sigma_{abc} \mathcal{I}(\delta) = \int \phi_a^+(x,y) B(\phi_b(x,y), \phi_c^*(x,y)) \mathcal{I}(\delta) dxdy
\]  

(13)

Equation (12) is valid for both resonant and non-resonant nonlinear interactions. Using \( \xi = \partial x v_a - \partial y u_a - y \eta_a = -iv_a/\omega_a \) and \( T_{abc}^{bc} = u_a \xi_b \xi_c \) the coupling coefficient can be written as

\[
\delta_{abc} \sigma_{abc} \mathcal{I} = \omega_a \gamma_{abc} + (\omega_a + \omega_b + \omega_c) T_{abc}^{bc}
\]  

(14)

As can be seen in equation (15), the interaction coefficient \( \gamma_{abc} \) is clearly modified by \( \delta \)

\[
\gamma_{abc} = \delta \left[ \left( \frac{\eta_a \vec{v}_b \cdot \vec{v}_c + \eta_b \vec{v}_c \cdot \vec{v}_a + \eta_c \vec{v}_a \cdot \vec{v}_b}{\delta} - \frac{T_{abc}^{bc}}{T_a} \right) - \delta \{ T_{bc}^a + T_{bc}^a \} \right]
\]  

(15)

Taking into account the interaction \( (n_a + n_b + n_c = \text{odd}) \) and resonance \( (\omega_a + \omega_b + \omega_c = 0) \) conditions, it is possible to obtain the resonant coupling coefficient from equation (14). It can be readily seen that there is an effective reduction of the interaction frequency as \( \delta \) decreases, whereas the interaction strength is also modified, increasing in absolute values for \( (T_{bc}^{a} + T_{ab}^{c}) > 0 \) (see equations (16)).

\[
\omega_a > \delta \omega_a \quad \text{for} \quad 0 < \delta < 1
\]  

(16a)

\[
\lim_{\delta \to 0} |\gamma_{abc}| > \lim_{\delta \to 1} |\gamma_{abc}|
\]  

(16b)

6. Numerics

The numerical integration scheme of equation (12) for the resonant three-wave problem uses a semi-analytic method, which assumes that the non-linear terms are constant within a 2\( \Delta t \) time interval. Thus, if we know the values of the expansion coefficients \( Z_j, j = \{ a, b, c \} \) for \([ t - \Delta t, t] \), it is possible to obtain analytic solution at time \( t + \Delta t \). The nonlinear terms are defined at the central time \( t \). Zonal and meridional dependencies of the wave components, as well as their derivatives, are computed analytically.

The problem is initialized in such a way that the most energetically active component of a particular resonant triad (the mode with the highest absolute frequency) holds most of the initial energy, i.e., the initial amplitudes \( A_j(\omega_{\text{max}}) \gg A_i \) for defined in equation (17) below. According to [13] a more rigorous requirement to determine the most energetically active member can be obtained by using the mode with the intermediate slowness, which is the reciprocal of the phase speed. Initial phases are also consistent with the aim of maximizing energy exchange. Thus, it is required that \( \sum \lambda_j = \pi/2 \) (see [12,15,16]). The value of \( H (= 250m) \) used here is consistent with the first baroclinic mode of the dry atmospheric dynamics.

\[
Z_j(0) = A_j e^{i\lambda_j}; j = \{ a, b, c \} \quad \sum_j \lambda_j = \pi/2
\]  

(17)

7. Nonlinear equatorial long-wave like interactions

In this section the modified nonlinear dynamics of the anisotropically scaled shallow water model is studied. To simplify, the analysis is on the reduced dynamics of a single resonant triad. The nonlinear energy exchange is investigated for some of the resonant triads studied in [12], but in this case, the interaction coefficient (equation 15) is used for values of \( \delta = \{ 0.5, 1.0 \} \). The first value (0.5) is a conservative estimate of the anisotropy parameter; however, the long wave like
effect appears in equation (4) as the meridional momentum equation is multiplied by $\bar{\delta}^2 (\delta = 0.25)$. Meanwhile, the modification of the eigenfunctions is not such a dramatic, only of order of $\delta$, as can be seen in [11] (using the slowness space). A proof of the asymptotic completeness of the long wave solutions can also be found in [11]. Consequently, smaller values of $\delta$ could be used as well, but in such a case, the modifications of the eigensolutions must also be considered. Therefore, intermediate values of $\delta$ are used with the aim of not to filter any of the solutions, while keeping long wave like effects. Coincidentally, for the range of values of the baroclinic wave speed $10 - 50$ m/s, or the length scales $500 - 1500$ km, or time scales $8 - 18$ hrs; values of $\delta$ in the range of $0.3 - 0.5$ are reasonable.

Each resonant triad studied is labeled by three parameters $k, n, w$, where $k$ is the zonal wavenumber; $n$ refers to the meridional quantum number and $w$ label the wave type, say: R (Rossby); M (Mixed Rossby-Gravity); $G^+$ (Eastward Intertio-Gravity), $G^-$ (Westward Inertio-Gravity waves). In this way, two resonant triads are studied. The first one [[(5,1,R);(1,4,G $-$);(6,4,G $+$)] corresponds to an interaction involving on slow mode and two fast modes, whereas the second triad [(0,1,R);(5,2,R);(5,0,M)] is a representative example of slow modes interaction.

Fig.(1) displays the nonlinear energy modulations associated with the [(5,1,R);(1,4,G $-$);(6,4,G $+$)] triad. Most of the initial energy is projected onto the third component, which has the highest absolute frequency and therefore, it is the most energetically active member of the triad. More precisely, the initial energy distribution is given by $A_3 = 10.0, A_{1,2} = 1.0$. On the other hand, in Fig.(2) the [(0,1,R);(5,2,R);(5,0,M)] trio is displayed, for the same two values of $\delta = \{0.5, 1.0\}$. The first component of this trio is the zonally symmetric Rossby mode, which has zero frequency and therefore, null resonant interaction coefficient. This mode is necessary to catalyze the energy exchange between the other two components. The higher the energy given to this mode, the stronger the energy exchange between the other two modes (not shown). To maximize the energy exchange, most of the initial energy is projected onto the most energetically active mode (5,0,M), with the partition $A_3 = 10.0, A_1 = 2.5, A_2 = 1.0$. It must be noted that the frequency of the third mode is only marginally larger than the frequency of the second mode (5,2,R).

As can be noted in Fig.(1) there is a slow modulation of the energy, in a period longer than the period of the phase propagation of the involved components. This slow modulation is accentuated as $\delta$ decreases as was anticipated in equations (13-15). For instance, the maximum energy exchange occurs at periods of around 5.8; 16 and 25.4 for the shallow water and at around of 6.5; 17.1 and 28 for the long-wave like approach. Although the curves for $\delta = 0$ are not displayed, the slow energy modulation occurs at periods of almost twice as the differences between the curves for $\delta = 1$ and $\delta = 0.5$. The magnitude of the energy exchange among the components is also modified. More precisely, the amplitude related to the (5,1,R) wave is increased, whereas the amplitude of the (1,4,G $-$) is decreased. For this case, there is not an apparent modification of the energy associated with the most energetically active mode (6,4,G $+$).

For the other triad (see Fig.(2)), despite of involving only slow modes, it is also characterized by the presence of the zonally symmetric Rossby mode that does not exchange energy with the other modes but catalyzes the energy exchange between the other two components. For the initial condition considered here, this trio is energetically less active in the sense of energy exchange than the previous one. However, there is a modification of both the period and, although less evident, the strength of the energy modulation as a function of the anisotropy parameter $\delta$.

Thus, long wave like models tend to produce a longer nonlinear energy exchange when compared to the shallow water model. These results suggest some implications for the long term memory of the tropical region to disturbances. Physically, the long wave approach has shown to be useful to explain, in a simplified fashion, some key aspects of tropical phenomena such as El Niño and the MJO ([1, 5, 6, 2]). With the asymptotic approach used here it was possible
to see that there exist differences between the shallow water and the long wave approaches under the optics of the nonlinear wave interactions (resonant or not). The close link between long wave model and simplified models of large scale tropical variability such as the El Niño or the MJO suggests that there is a potential for applications of the results of the present study. The asymptotic completeness could be useful to study the connection of the above mentioned variability with small scale disturbances such as those associated with short Rossby waves.

8. Conclusions
The shallow water model is written in a parametric form as a function of an external parameter $\delta$. This parameter is a measure of both spatial and flow scales anisotropy and also measures the fast to slow timescales ratio. The model obtained is more general than the traditional long wave equations in the sense that it is valid for intermediate states between the shallow water and the long wave model. Using typical values of the perturbations as found in the tropical region it was possible to verify that the intermediate states are in fact realizable. For instance, values of $\delta$ between 0.3 - 0.5 are enough to produce a long wave like effect of the (order of $\delta^2$) in the meridional momentum equation (a near geostrophic balance), and at the same time only a modification of the order of $\delta$ in the eigensolutions (as discussed in [11] using the slowness space framework). A single proof on the completeness of the eigensolutions at the long wave like limit is found in [11]. Asymptotically the model and its solutions are also valid for the long wave approach. The value of the completeness was exploited to study the differences between the nonlinear wave interactions in the shallow water and the long wave like model approach. Model equations for the nonlinear interactions were developed. However, to simplify, the analysis is restricted to the reduced dynamics of single resonant triads. Two triads are studied: one of them formed by an slow mode and two fast modes, and the other formed by only slow modes. Both, the effective nonlinear coupling and interaction coefficient are computed for the general problem. It was verified that there is an effective reduction of the interaction frequency (independently of the triad studied) and consequently an increase of the energy exchange period of the triad components as $\delta$ decreases. This result has implications for the memory of the tropical region to wave disturbances. The larger the anisotropy (smaller $\delta$), the larger the nonlinear energy exchange period. However, an asymptotic upper limit is attained as $\delta$ tend to zero. The magnitude of the energy exchange is also modified but, in this case, depends on the specific resonant triad studied, as well as on the initial energy partition. Numerical experiments confirm the above results. Within this asymptotic framework there is a potential to better understand some tropical large scale phenomena such as the El Niño and the MJO in connection with smaller spatial scales of variability, as for example those related with small scales Rossby waves. The value of asymptotic methods was also demonstrated in other contexts. For instance, [14] used asymptotic methods to overcome the difficulty associated with the degeneracy of the eigenmodes at the zero frequency limit.

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Figure 1. Slow energy exchange for the \([(5,1,R);(1,4,G^-);(6,4,G^+)]\) trio. For each wave, two values of $\delta$ are plotted $\delta = 1.0$ and 0.5. These values correspond to the shallow water and to the long wave like approximation respectively. The initial energy partition is $[A_1^2 = 1.0; A_2^2 = 1.0; A_3^2 = 100.0]$. It is possible to note that nonlinear resonant triad interactions in the long wave like approach (thick lines) promotes an slower energy exchange than the same interactions in the shallow water approach (thin lines).
**Figure 2.** Same as Fig.(1) but for the [(0,1,R);(5,2,R);(5,0,M)] trio. The initial energy partition is also different \[ A_1^2 = 1.0; A_2^2 = 6.25; A_3^2 = 100.0 \]. Slower energy exchanges for the long wave like approach (thick lines) than for the shallow water (thin lines) are also noticeable.