Lifting graph $C^*$-algebra maps to Leavitt path algebra maps

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Abstract
Let $\xi : C^*(E) \rightarrow C^*(F)$ be a unital $*$-homomorphism between simple purely infinite Cuntz–Krieger algebras of finite graphs. We prove that there exists a unital $*$-homomorphism $\phi : L(E) \rightarrow L(F)$ between the corresponding Leavitt path algebras such that $\xi$ is homotopic to the map $\hat{\phi} : C^*(E) \rightarrow C^*(F)$ induced by completion. We show moreover that $\hat{\phi}$ is a homotopy equivalence in the $C^*$-algebraic sense if and only if $\phi$ is a homotopy equivalence in the algebraic, polynomial sense. We deduce, in particular, that any isomorphism between simple purely infinite Cuntz–Krieger algebras is homotopic to the completion of a unital algebraic homotopy equivalence.

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1 INTRODUCTION

A finite, regular, directed graph consists of finite sets $E^0$ and $E^1$ of vertices and edges and maps $r, s : E^1 \rightarrow E^0$ with $s$ surjective. We write $L(E)$ for the Leavitt path algebra [2] of $E$ over $C$, which is a $*$-algebra, and $C^*(E)$ for its $C^*$-completion. We say that $E$ is simple purely infinite, or spi, if it satisfies the hypothesis of the purely infinite simplicity theorem [2, Theorem 3.1.10], which by that theorem are equivalent to the purely infinite simplicity of $L(E)$ and also to that of $C^*(E)$ (see [2, Section 5.6]). A theorem of Cuntz and Rørdam [21, Theorem 6.5], which is now a particular case of the Kirchberg–Phillips theorem [19, Theorem 4.1.1 and Corollary 4.2.2], says that the group $K_0(C^*(E))$ scaled by the class $[C^*(E)]$ of its free module of rank one, is a complete invariant for the isomorphism class of the $C^*$-algebra of a finite spi graph $E$. The classification question for Leavitt path algebras [1], restricted to $C$-algebras, asks whether a similar result holds for the Leavitt path
algebra $L(E)$. Since the scaled groups $(K_0(C^*(E)), [C^*(E)])$ and $(K_0(L(E)), [L(E)])$ are isomorphic, the above question can be restated as follows:

**Question 1.** For finite spi graphs $E$ and $F$, does the existence of a $*$-isomorphism $C^*(E) \cong C^*(F)$ imply that of an algebra isomorphism $L(E) \cong L(F)$?

By [19, Theorem 4.1.1 and Corollary 4.2.2], one may equivalently substitute unital $C^*$-homotopy equivalence for $C^*$-algebra isomorphism in the question above. By [10, Theorem 6.1], $C^*(E) \cong C^*(F)$ implies that $L(E)$ and $L(F)$ are homotopy equivalent in the algebraic sense; by [8, Theorem 14.1] one can further conclude that an involution preserving homotopy equivalence exists upon hyperbolic matricial stabilization. A main result of the current paper is the following.

**Theorem 1.1.** Let $E$ and $F$ be finite spi graphs and let $\xi : C^*(E) \to C^*(F)$ be a unital $*$-homomorphism. Then there exists a unital $*$-homomorphism $\phi : L(E) \to L(F)$ whose completion $\hat{\phi} : C^*(E) \to C^*(F)$ is $C^*$-homotopic to $\xi$. Moreover if $\xi$ is a homotopy equivalence in the $C^*$-algebra sense, then $\phi$ is a homotopy equivalence in the algebraic, polynomial sense.

We point out that in the theorem above, algebraic homotopy equivalence is understood in the sense of not necessarily involution preserving algebra homomorphisms. Thus the $*$-homomorphism $\phi$ of the theorem is a homotopy equivalence if there exists an — not necessarily involution preserving — algebra homomorphism $\psi : L(F) \to L(E)$ such that the composites $\psi \circ \phi$ and $\phi \circ \psi$ are algebraically homotopic to the respective identity maps.

The main tools we use to prove the theorem above are Kasparov's bivariant $K$-theory of $C^*$-algebras and the algebraic bivariant $K$-theory introduced in [12]. The latter consists of a triangulated category $kk$ and a functor $j : \text{Alg}_C \to kk$ from the category of algebras, which is algebraically homotopy invariant, maps algebra extensions to distinguished triangles and is matricially stable, and is universal initial with these properties. Similarly, Kasparov's $K$-theory can also be described as consisting of triangulated category $KK$ and a functor $k : C^* - \text{Alg} \to KK$ from the category of separable $C^*$-algebras, which is homotopy invariant and stable in the $C^*$-algebra sense, and maps those extensions admitting completely positive splittings to distinguished triangles, and is universal initial with these properties [18]. Write $\text{Alg}^*_C$ for the category of $*$-algebras and $*$-homomorphisms and

$$\text{Leavitt}^* \subset \text{Alg}^*_C \text{ and } \langle \text{Leavitt} \rangle_{kk} \subset kk$$

for the full subcategories on the Leavitt path algebras of finite regular graphs.

Recall that a functor $F : C \to D$ is called conservative [20, p. 180] if for any morphism $\phi$ in $C$, $F(\phi)$ being an isomorphism implies that $\phi$ is an isomorphism. A key step in proving Theorem 1.1 above is the following, which is also a main result of the article.

**Theorem 1.2.** Let $\hat{\cdot} : \text{Leavitt}^* \to C^* - \text{Alg}$ be the completion functor. There is a $\mathbb{Z}$-linear, full and conservative functor $\text{comp} : \langle \text{Leavitt} \rangle_{kk} \to KK$ such that the following diagram commutes:

$$\begin{array}{ccc}
\text{Leavitt}^* & \xrightarrow{\hat{\cdot}} & C^* - \text{Alg} \\
j & & k \\
\langle \text{Leavitt} \rangle_{kk} & \xrightarrow{\text{comp}} & KK
\end{array}$$
The rest of this article is organized as follows. In Section 2 we consider, for a finite regular graph \( E \) and a \( C^* \)-algebra \( \mathcal{A} \), the canonical bijection \( \text{hom}_{\text{Alg}_{C^*}}(L(E), \mathcal{A}) \overset{\sim}{\rightarrow} \text{hom}_{\text{C}^*-\text{Alg}}(C^*(E), \mathcal{A}) \). Proposition 2.4 extends the latter to a surjective natural transformation of functors \( C^* - \text{Alg} \rightarrow \mathcal{A} \mathcal{B} \),

\[
\text{comp}_{\mathcal{A}} : kk(L(E), \mathcal{A}) \rightarrow KK(C^*(E), \mathcal{A}).
\]

which is an isomorphism in certain cases, for example, when \( \mathcal{A} \) is either stable or properly infinite (2.5). In Section 3 we prove Theorem 3.2, which contains Theorem 1.2. Because \( j \) and \( k \) are the identity on objects, the prescription that the diagram of the latter theorem be commutative dictates the definition of \( \text{comp} \) on objects as \( \text{comp}(L(E)) = C^*(E) \). To define \( \text{comp} \) on homomorphisms we compose the map \( kk(L(E), L(F)) \rightarrow kk(L(E), C^*(F)) \) with the natural transformation \( \text{comp}_{C^*(F)} \) of (1.3).

Two \( C^* \)-algebra homomorphisms \( \phi, \psi : \mathcal{A} \rightarrow \mathcal{B} \) are \( M_2 \)-homotopic if they become \( C^* \)-homotopic upon composing with the inclusion \( t_2 : \mathcal{B} \rightarrow M_2 \mathcal{B} \) into the upper left hand corner; algebraic \( M_2 \)-homotopy between maps in \( \text{Alg}_{C^*} \) is defined similarly. In Section 4 we prove Theorem 4.10, which contains Theorem 1.1, and in addition says that completion sends \(*\)-homomorphisms satisfying a strong fullness assumption onto \( M_2 \)-homotopy classes of nonzero \( C^* \)-algebra homomorphisms \( C^*(E) \rightarrow C^*(F) \), and that \( \hat{\phi} \) is an \( M_2 \)-homotopy equivalence in \( C^* - \text{Alg} \) if and only if \( \phi \) is one in \( \text{Alg}_{C^*} \).

In the Appendix we prove the technical Lemma A.5 which says that if \( \mathcal{A} \) and \( \mathcal{B} \) are \( C^* \)-algebras with \( \mathcal{B} \) properly infinite, then the monoid of homotopy classes of \(*\)-homomorphisms from \( \mathcal{A} \) to the stable \( C^* \)-algebra \( \mathcal{K} \sim \otimes \mathcal{B} \) is isomorphic to the monoid of \( M_2 \)-homotopy classes of \(*\)-homomorphisms \( \mathcal{A} \rightarrow \mathcal{B} \).

2 | EXTENDING COMPLETION TO A NATURAL TRANSFORMATION \( kk(L(E), \mathcal{A}) \rightarrow KK(C^*(E), \mathcal{A}) \)

All algebras considered in this paper are over \( \mathbb{C} \). If \( A \) is an algebra, we write \( K_*(A), KV_*(A) \) and \( KH_*(A) \) for its Quillen, Karoubi–Villamayor and Weibel’s homotopy algebraic \( K \)-theory ([17],[26],[7]). If \( \mathcal{A} \) is a \( C^* \)-algebra, we further consider its Bott-periodic topological \( K \)-theory \( K_{\text{top}}^*(\mathcal{A}) \).

Lemma 2.1. Let \( \mathcal{A} \) be a \( C^* \)-algebra. For \( n \in \mathbb{Z} \), let \( K_n(\mathcal{A}) \rightarrow K_n^{\text{top}}(\mathcal{A}) \) and \( K_n(\mathcal{A}) \rightarrow KH_n(\mathcal{A}) \) be the canonical comparison maps. Then there is a natural map \( KH_n(\mathcal{A}) \rightarrow K_{n+1}^{\text{top}}(\mathcal{A}) \) making the following diagram commute.

\[
\begin{array}{ccc}
K_n(\mathcal{A}) & \rightarrow & K_n^{\text{top}}(\mathcal{A}) \\
& & \\
& KH_n(\mathcal{A}) & \downarrow \\
\end{array}
\]

Proof. Here we use the notation for path and loop functors and the description of \( KH \) of [7, Sections 3–5]. Let \( r + n \geq 0 \). Restriction of polynomials to maps on the unit interval yields a map of
The boundary map \( K_{-r}^\text{top}(\mathcal{A}(0,1)^{r+n}) \to K_{-r-1}^\text{top}(\mathcal{A}(0,1)^{r+n+1}) \) is an isomorphism since \( \mathcal{A}(0,1)^{r+n+1} \) is contractible. Hence we have a natural map

\[
KH_n(\mathcal{A}) = \text{colim}_r K_{-r}(\Omega^{n+r} \mathcal{A}) \to \text{colim}_r K_{-r}^\text{top}(\mathcal{A}(0,1)^{n+r}) = K_n^\text{top}(\mathcal{A}).
\]

For \( n \leq 0 \), \( K_n(\mathcal{A}) \) is the first term of the inductive system in the colimit defining \( KH_n(\mathcal{A}) \) and the factorization of the lemma is clear. For \( n \geq 1 \) the map \( K_n(\mathcal{A}) \to KH_n(\mathcal{A}) \) factors through Karoubi–Villamayor \( K \)-theory \( KV_n(\mathcal{A}) \) and the comparison maps fit in a commutative diagram

\[
\begin{array}{ccc}
K_n(\mathcal{A}) & \to & KV_n(\mathcal{A}) = KV_1(\Omega^{n-1} \mathcal{A}) \to K_0(\Omega^n \mathcal{A}) \\
& \downarrow & \downarrow \\
K_n^\text{top}(\mathcal{A}) = K_1^\text{top}(\mathcal{A}(0,1)^{n-1}) & \longleftarrow & KH_n(\mathcal{A}).
\end{array}
\]

This finishes the proof. \( \square \)

**Remark 2.2.** The composite \( K_0(\mathcal{A}) \to KH_0(\mathcal{A}) \to K_0^\text{top}(\mathcal{A}) = K_0(\mathcal{A}) \) is the identity map; thus \( KH_0(\mathcal{A}) \to K_0(\mathcal{A}) \) is a split surjection.

**Lemma 2.3.** Let \( \mathcal{A} \) be a properly infinite \( C^* \)-algebra. Then \( \mathcal{A} \) is \( K \)-regular and the natural map \( KH_*(\mathcal{A}) \to K_*^\text{top}(\mathcal{A}) \) is an isomorphism.

**Proof.** We know from [11, Theorem 3.2] that \( K_*(\mathcal{A}) \to K_*^\text{top}(\mathcal{A}) \) is an isomorphism, so by Lemma 2.1 it suffices to show that \( \mathcal{A} \) is \( K \)-regular. This follows from [11, Corollary 2.3] and the argument of [24, Theorem 20]. \( \square \)

In what follows we write \( \mathcal{A} \otimes \mathcal{B} \) for the minimal tensor product of \( C^* \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \). Let \( kk \) and \( KK \) be the triangulated bivariant \( K \)-theory categories of \( C \)-algebras and of \( C^* \)-algebras, and let \( j : \text{Alg}_C \to kk \) [12] and \( k : C^* \to \text{Alg} \to KK \) [18] be the canonical functors.

**Proposition 2.4.** Let \( E \) be a regular finite graph. Then there is a surjective natural transformation of functors \( C^* \to \text{Alg} \to \mathcal{A} \mathcal{B} \),

\[
\text{comp}_{\mathcal{A}} : kk(L(E), \mathcal{A}) \to KK(C^*(E), \mathcal{A}),
\]
making the following diagram commute.

\[
\begin{array}{ccc}
\text{hom}_{\text{Alg}_C}(L(E), \mathcal{A}) & \xrightarrow{\sim} & \text{hom}_{C^*-\text{Alg}}(C^*(E), \mathcal{A}) \\
\downarrow && \downarrow \\
kk(L(E), \mathcal{A}) & \xrightarrow{\text{comp}_\mathcal{A}} & KK(C^*(E), \mathcal{A})
\end{array}
\]

If moreover $\mathcal{A}$ is $K_0$-regular and $KV_1(\mathcal{A}) \to K_1^{\text{top}}(\mathcal{A})$ is injective, then $\text{comp}_\mathcal{A}$ is an isomorphism.

\textbf{Proof.} Let $E'$ be the essential graph obtained from $E$ by source removal [2, Definition 6.3.26]; there is a full projection $p \in L(E)$ such that $L(E')$ and $C^*(E')$ are $\ast$-isomorphic to $pL(E)p$ and $pC^*(E)p$. Hence we may assume that $E = E'$, by Morita invariance. Let $E_t$ be the transpose graph. The map $\text{comp}_\mathcal{A}$ is defined as the following composite, where the individual maps are explained below.

\[
\text{kk}(L(E), \mathcal{A}) \cong KH_1(\mathcal{A} \otimes_C L(E_t)) \to KH_1(\mathcal{A} \tilde{\otimes} C^*(E_t)) \to K_1^{\text{top}}(\mathcal{A} \tilde{\otimes} C^*(E_t)) \cong KK(C^*(E), \mathcal{A}).
\]

The isomorphisms above come from Poincaré duality, proved for $KK$ and graph $C^*$-algebras in [15] and for $kk$ and Leavitt path algebras in [8]. The second map in the sequence is induced by the completion map $L(E_t) \to C^*(E_t)$ and the third is the comparison map from homotopy algebraic to topological $K$-theory, as in Lemma 2.1. Write $e_t \in E_t^1$ for the edge corresponding to an edge $e \in E^1$. By the proof of [8, Theorem 11.2], the algebraic Poincaré dual of a $\ast$-homomorphism $\phi : L(E) \to \mathcal{A}$ in $KH_1(L(E_t) \otimes \mathcal{A})$ is the class of the unitary

\[
1 \otimes 1 - \sum_{v \in E^0} \phi(v) \otimes v + \sum_{e \in E^1} \phi(e) \otimes e_t^*,
\]

which maps to the class of the same unitary in $K_1^{\text{top}}(\mathcal{A} \tilde{\otimes} C^*(E_t))$. This shows that the diagram of the proposition commutes, since the class just mentioned is also the image under $C^*$-Poincaré duality of the $\ast$-homomorphism $\hat{\phi} : C^*(E) \to \mathcal{A}$ of [15, Section 4]. Next observe that passage to completion induces a map of extensions from the presentation of $L(E)$ as a quotient of the Cohn algebra [2, Definition 1.5.1] and Proposition 1.5.5 to that of $C^*(E)$ as a quotient of the Toeplitz algebra; this map in turn induces a map of exact sequences of abelian groups

\[
0 \to \mathcal{B}(E)^\vee \otimes KH_1(\mathcal{A}) \longrightarrow \text{kk}(L(E), \mathcal{A}) \longrightarrow \text{hom}(\mathcal{B}(E), KH_0(\mathcal{A})) \rightarrow 0
\]

\text{comp}_\mathcal{A}

\[
0 \to \mathcal{B}(E)^\vee \otimes K_1^{\text{top}}(\mathcal{A}) \longrightarrow KK(C^*(E), \mathcal{A}) \longrightarrow \text{hom}(\mathcal{B}(E), K_0(\mathcal{A})) \rightarrow 0
\]

Here $\mathcal{B}(E)^\vee$ is the dual Bowen–Franks group of [8, Section 12], whose definition is recalled in (3.1) below. The vertical map on the right is surjective by Remark 2.2; that of the left is surjective because $K_1(\mathcal{A}) \to K_1^{\text{top}}(\mathcal{A})$ is. Diagram chasing shows that $\text{comp}_\mathcal{A}$ is surjective and that it is an isomorphism whenever the map $KH_i(\mathcal{A}) \to K_i^{\text{top}}(\mathcal{A})$ is an isomorphism for $i = 0, 1$. If $\mathcal{A}$ is $K_0$-regular then $KH_1(\mathcal{A}) = KV_1(\mathcal{A})$, and the last assertion of the proposition follows, since $K_0(\mathcal{A}) = K_0^{\text{top}}(\mathcal{A})$ and $KV_1(\mathcal{A}) \to K_1^{\text{top}}(\mathcal{A})$ is always surjective. \(\Box\)
Example 2.5. Jonathan Rosenberg has conjectured [23, Conjecture 2.1] that any C*-algebra is $K_0$-regular. It was shown in [13, Theorem 8.1] that commutative C*-algebras are K-regular. If $\mathfrak{A}$ is a stable C*-algebra, then $\mathfrak{A}$ is $K$-regular by [23, Theorem 1.5] and the comparison map $K_*(\mathfrak{A}) \to K_*^{\text{top}}(\mathfrak{A})$ is an isomorphism by [25, Theorem 10.9] and [16, Théorème 4.9]. By Lemma 2.3 the same is true if $\mathfrak{A}$ is properly infinite. Hence the map $\text{comp}_\mathfrak{A}$ of Proposition 2.4 is an isomorphism if $\mathfrak{A}$ is either stable or properly infinite.

3 A FUNCTOR FROM $\langle \text{Leavitt} \rangle_{kk}$ to KK

A *-algebra is a C-algebra $A$ equipped with a semilinear involution $*: A \to A^{\text{op}}$; a *-homomorphism of *-algebras is an involution preserving algebra homomorphism. Write $\text{Alg}_C^*$ for the category of *-algebras and *-homomorphisms and

$$\text{Leavitt}^* \subset \text{Alg}_C^*$$

for the full subcategories on the Leavitt path algebras of finite regular graphs. If $E$ is a finite regular graph, we write $A_E$ for its incidence matrix and

$$\mathfrak{B}(E) = \text{Coker}(I - A_E^t) \quad \text{and} \quad \mathfrak{B}(E) = \text{Coker}(I - A_E)$$

for its Bowen–Franks and dual Bowen–Franks groups.

Theorem 3.2. Let $\hat{}: \text{Leavitt}^* \to C^* - \text{Alg}$ be the completion functor. Then there is a $\mathbb{Z}$-linear functor $\hat{\text{comp}} : \langle \text{Leavitt} \rangle_{kk} \to KK$ with the following properties.

(i) The following diagram commutes

$$\begin{array}{ccc}
\text{Leavitt}^* & \xrightarrow{\hat{}} & C^* - \text{Alg} \\
\downarrow j & & \downarrow k \\
\langle \text{Leavitt} \rangle_{kk} & \xrightarrow{\text{comp}} & KK
\end{array}$$

(ii) We have an exact sequence

$$0 \to \mathfrak{B}(E) \otimes \mathfrak{B}(F) \otimes C^* \to kk(L(E), L(F)) \xrightarrow{\hat{\text{comp}}} KK(C^*(E), C^*(F)) \to 0.$$ 

In particular, $\hat{\text{comp}}$ is a full functor.

(iii) $\hat{\text{comp}}$ is a conservative functor.

Proof. In order that the diagram in (i) commutes, we must set $\hat{\text{comp}}(L(E)) = C^*(E)$; this defines $\hat{\text{comp}}$ on objects. To define it also on homomorphisms, let $i : kk(L(E), L(F)) \to kk(L(E), C^*(F))$ be the map induced by the inclusion, let $\text{comp}$ be as in Proposition 2.4 and set

$$\hat{\text{comp}} = \text{comp}_{C^*(F)} \circ i : kk(L(E), L(F)) \to KK(C^*(E), C^*(F)).$$
It follows from Proposition 2.4 that the diagram of part (i) commutes. In particular \( \tilde{\text{comp}} \) preserves identity maps. Next we have to check that if \( E, F \) and \( G \) are finite regular graphs, \( \xi \in kk(L(F), L(G)) \) and \( \eta \in kk(L(E), L(F)) \), then

\[
\tilde{\text{comp}}(\xi \circ \eta) = \text{comp}(\xi) \circ \text{comp}(\eta).
\] (3.3)

First consider the case when \( E \) and \( F \) are essential graphs. Recall from [5, Lemma 6.1], that Leavitt path algebras of finite graphs are regular supercoherent, so the tensor product of two such algebras is \( K \)-regular by [4, Theorems 7.6 and 8.6]. Let \( \omega(\eta) \in K_1(L(F) \otimes L(E)) = KH_1(L(F) \otimes L(E)) \) and \( \omega(\xi) \in K_1(L(G) \otimes L(F)) \) be the elements associated to \( \eta \) and \( \xi \) under the Poincaré duality isomorphism described in [8, Theorem 11.2]. Let \( u_E = 1 - \sum_{v \in E_0} v \otimes v + \sum_{e \in E_1} e \otimes e^* \in K_1(L(E) \otimes L(E)) \) and let \( \rho_E : L(E) \otimes L(E) \to \Sigma_X \) be as in the proof of [8, Theorem 11.2]. One checks, using the explicit formulae of [8, Theorem 11.2] for the Poincaré duality isomorphisms (which are in turn those of [15]) and the identification \( K_n(\Sigma X A) = K_{n-1}(A) \), that for the cup-product \( \star \) of [14, Lemmas 4.5 and 8.3], we have

\[
\omega(\xi \circ \eta) = (LG \otimes \rho_F \otimes L(F))((\omega(\xi) \star ((LF \otimes \rho_E \otimes L(E))(\omega(\eta) \star u_E))).
\]

Since passage to the completion followed by the comparison map \( K_* \to K_*^{\text{top}} \) preserves cup-products, and since the formulae for the Poincaré duality isomorphism in the \( C^* \)-algebra setting are the same as in the algebraic one, we get the identity (3.3), under our current assumption that \( E \) and \( F \) are essential. Next we consider the general case. Let \( E' \) be as in the proof of Proposition 2.4 and let \( \text{inc}_E : L(E') \to L(E) \) and \( \hat{\text{inc}}_E : C^*(E') \to C^*(E) \) be the full corner inclusions. It is straightforward from the definition of \( \tilde{\text{comp}} \) that

\[
\tilde{\text{comp}}(\eta) = \text{comp}(\eta \circ \text{inc}_E) \circ (\text{inc}_E)^{-1}.
\]

Hence using naturality and what we have just proved, we obtain

\[
\tilde{\text{comp}}(\xi) \circ \tilde{\text{comp}}(\eta) = \text{comp}(\xi \circ \text{inc}_F) \circ (\text{inc}_F)^{-1} \circ \text{comp}(\eta \circ \text{inc}_E) \circ (\text{inc}_E)^{-1} = \text{comp}(\xi \circ \eta \circ \text{inc}_E) \circ (\text{inc}_E)^{-1} = \text{comp}(\tilde{\text{comp}}(\xi \circ \eta)).
\]

Thus \( \tilde{\text{comp}} \) is a functor; this finishes the proof of part (i). To prove part (ii), observe that, by construction, we have a commutative diagram with exact rows

\[
0 \to \mathcal{B} \mathcal{F}(E)^{\vee} \otimes K_1(L(F)) \longrightarrow kk(L(E), L(F)) \longrightarrow \text{hom}(\mathcal{B} \mathcal{F}(E), \mathcal{B} \mathcal{F}(F)) \to 0
\]

\[
0 \to \mathcal{B} \mathcal{F}(E)^{\vee} \otimes \text{Ker}(I - A^*_F) \longrightarrow KK(C^*(E), C^*(F)) \longrightarrow \text{hom}(\mathcal{B} \mathcal{F}(E), \mathcal{B} \mathcal{F}(F)) \to 0.
\]

The vertical map on the left is induced by the surjection in the following exact sequence

\[
0 \to \mathcal{B} \mathcal{F}(F) \otimes C^* \to K_1(L(F)) \to \text{Ker}(I - A^*_F) \to 0.
\]
This sequence splits because $\text{Ker}(I - A'_E)$ is a free abelian group. Hence by the snake lemma, $\text{comp}$ is surjective and $\text{Ker}(\text{comp}) = \mathcal{B} \mathcal{F}^\vee(E) \otimes \mathcal{B} \mathcal{F}(F) \otimes \mathbb{C}^*$, finishing the proof of part (ii). It follows that for finite regular $E$, we have a surjective ring homomorphism

$$\text{comp} : kk(L(E), L(E)) \rightarrow KK(C^*(E), C^*(F)).$$

To prove that $\text{comp}$ is conservative it suffices to show that the kernel $\mathfrak{F}$ of (3.4) is a nilpotent ideal. We will in fact show that $\mathfrak{F}^2 = 0$. Let $C(E)$ be the Cohn algebra and $\pi : C(E) \rightarrow L(E)$ the projection. Then

$$\mathcal{B} \mathcal{F}(E) \otimes \mathbb{C}^* = \text{Ker}(K_1(L(E)) \rightarrow \text{Ker}(I - A'_E)) = \text{Im}(K_1(C(E)) \overset{\pi}{\rightarrow} K_1(L(E)))$$

$$\subset kk_1(C, L(E)).$$

Besides, by [9, Formula (6.4)], $\mathcal{B} \mathcal{F}^\vee(E) = kk_{-1}(L(E), \mathbb{C})$, and by [9, Lemma 7.21], $\mathfrak{F}$ is generated as an abelian group by the composites $\eta \circ \xi$ with $\xi \in kk_{-1}(L(E), \mathbb{C})$ and $\eta \in \text{Ker}(K_1(L(E)) \rightarrow \text{Ker}(I - A'_E))$. Hence it suffices to show that $\xi \circ \eta = 0$ for any such two elements. By (3.5) there is $\eta' \in kk_1(C, C(E))$ such that $\eta = \pi \circ \eta'$. But by [9, Theorem 4.2], $C(E)$ is $kk$-isomorphic to $\mathbb{C}(E^0)$, and so $\xi \circ \eta \in kk_{-1}(C(E), C) = K_{-1}(C)^{E^0} = 0$. Thus $\xi \circ \eta = 0$; this finishes the proof of the theorem. □

**Corollary 3.6.** If either $\mathcal{B} \mathcal{F}(E)$ or $\mathcal{B} \mathcal{F}(F)$ is finite, then $\text{comp} : kk(L(E), L(F)) \rightarrow KK(C^*(E), C^*(F))$ is an isomorphism.

**Proof.** Observe that $\mathcal{B} \mathcal{F}(E)$ and $\mathcal{B} \mathcal{F}(E)^\vee$ are finitely generated and have the same rank, so one is finite if and only if the other is. Now use part (ii) of Theorem 3.2 and the fact that $C^*$ is a divisible group. □

### 4 PROPERLY INFINITE AND SIMPLE PURELY INFINITE ALGEBRAS

In this section we consider algebraic (that is, polynomial) homotopy between algebra homomorphisms and continuous, involution preserving homotopy between $C^*$-algebra homomorphisms; we write $\sim$ for the former and $\approx$ for the latter. Let $t_2 : A \rightarrow M_2A$ be the upper left hand corner inclusion of an algebra into the $2 \times 2$ matrices with entries in $A$. We say that two algebra homomorphisms $f, g : A \rightarrow B$ are $M_2$-homotopic, and write $f \sim_{M_2} g$, if $t_2 \circ f \sim t_2 \circ g$. We put $[A, B] = \text{hom}_{\text{Alg}}(A, B) / \sim$ and $[A, B]_{M_2} = \text{hom}_{\text{Alg}}(A, B) / \sim_{M_2}$. The relation $\approx_{M_2}$ and the sets $[[\mathfrak{A}, \mathfrak{B}]]$ and $[[\mathfrak{A}, \mathfrak{B}]]_{M_2}$ of $(M_2)$- $C^*$-homotopy classes of $C^*$-algebra homomorphisms $\mathfrak{A} \rightarrow \mathfrak{B}$ are defined similarly.

Recall from [6, Section 6.11] that an idempotent $p$ in a unital algebra $A$ is *very full* if there are elements $x \in pA$ and $y \in Ap$ such that $yx = 1$. A homomorphism of unital algebras $\phi : A \rightarrow B$ is *very full* if $\phi(1)$ is a very full idempotent. We write $[L(E), A]_{M_2}^{vf}$ for the set of $M_2$-homotopy classes of very full homomorphisms $L(E) \rightarrow A$. 
Example 4.1. Any nonzero idempotent of an sp ring is very full, and any idempotent $M_2$-homotopic to zero is itself zero. Hence if $E$ is a finite graph and $A$ an sp algebra, we have

$$[L(E), A]^{vf}_{M_2} = [L(E), A]_{M_2} \setminus \{0\}.$$  \hfill (4.2)

If $E$ is an sp and $A$ is any unital algebra containing elements $x_1, x_2, y_1, y_2$ such that $y_i x_j = \delta_{i,j}$, then by [8, Example 13.19], the set $[L(E), A]^{vf}_{M_2}$ is naturally a group, and there is a group isomorphism

$$[L(E), A]^{vf}_{M_2} \cong kk(L(E), A).$$  \hfill (4.3)

Recall that a $*$-homomorphism of $C^*$-algebras $\phi : \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ unital is called full if $\phi(1)$ is a full projection in $\mathcal{B}$. We write $[[\mathcal{A}, \mathcal{B}]]' \subset [[\mathcal{A}, \mathcal{B}]]$ and $[[\mathcal{A}, \mathcal{B}]]'_{M_2} \subset [[\mathcal{A}, \mathcal{B}]]_{M_2}$ for the subsets of homotopy classes of full homomorphisms.

Example 4.4. If $\mathcal{A}$ and $\mathcal{B}$ are separable and unital, and $\mathcal{A}$ is also simple and nuclear, then by [19, Proposition 3.1.2 and Theorem 4.1.1], and Lemma A.5, $[[\mathcal{A}, \mathcal{B}]]'_{M_2}$ is a group under direct sum, and there is a group isomorphism

$$[[\mathcal{A}, \mathcal{B}]]'_{M_2} \cong KK(\mathcal{A}, \mathcal{B}).$$  \hfill (4.5)

If in addition $\mathcal{B}$ is simple, then the same is true of $\mathcal{B}$, and thus

$$[[\mathcal{A}, \mathcal{B}]]'_{M_2} = [[\mathcal{A}, \mathcal{B}]]_{M_2} \setminus \{0\}.$$  \hfill (4.6)

If furthermore $\mathcal{B}$ is purely infinite, then $\mathcal{B} \cong \mathcal{B}$ by Kirchberg’s theorem [19, Theorem 2.1.5] and any nonzero projection in $\mathcal{B}$ is very full.

Proposition 4.7. Let $E$ be an sp graph and let $\mathcal{A}$ be a unital separable $C^*$-algebra. Then there is a natural isomorphism

$$[L(E), \mathcal{O}_\infty \widetilde{\otimes} \mathcal{A}]^{vf}_{M_2} \cong [[C^*(E), \mathcal{O}_\infty \widetilde{\otimes} \mathcal{A}]]'_{M_2}.$$  \hfill (4.8)

Proof. Properly infinite $C^*$-algebras are $K$-regular by Lemma 2.3. In particular this applies to $\mathcal{O}_\infty \widetilde{\otimes} \mathcal{A}$, and thus (4.3) provides an isomorphism between the left-hand side of (4.8) and $kk(L(E), \mathcal{O}_\infty \widetilde{\otimes} \mathcal{A})$. The proof follows from this, together with Proposition 2.4, Lemma 2.3 and (4.5). \hfill \square

Corollary 4.9. Let $E$ and $F$ be finite sp graphs. Then there are

(i) an isomorphism of abelian groups

$$[L(E), C^*(F)]_{M_2} \setminus \{0\} \cong [[C^*(E), C^*(F)]]_{M_2} \setminus \{0\}$$

and
(ii) an exact sequence

\[ 0 \to \mathcal{B}(E)^\vee \otimes \mathcal{B}(F) \otimes C^* \to [L(E), L(F)]_{M_2} \setminus \{0\} \to [[C^*(E), C^*(F)]]_{M_2} \setminus \{0\} \to 0. \]

Proof. Graph \( C^* \)-algebras of finite graphs are unital, separable and nuclear; moreover, because \( F \) is spi, the same is true of \( C^*(F) \). Hence \([L(E), C^*(F)]_{M_2} \setminus \{0\} \cong [[C^*(E), C^*(F)]]_{M_2} \setminus \{0\}\) by Proposition 4.7 together with Examples 4.4, (4.2) and (4.6). The exact sequence of the corollary follows from that of Theorem 3.2 and the isomorphisms (4.3) and (4.5).

We say that a \(*\)-homomorphism \( \phi : A \to B \) of unital \(*\)-algebras has property (P) if \( \phi(1_B) \) contains an isometry. Put

\[ \text{hom}_{\text{Alg}^*}(A, B) \supset \text{hom}_{\text{Alg}^*}(A, B)^P = \{ \phi \text{ has property (P)} \}. \]

Theorem 4.10. Let \( E \) and \( F \) be finite, spi graphs. Then

(i) The map \( \text{hom}_{\text{Alg}^*}(L(E), L(F))^P \to [[C^*(E), C^*(F)]]_{M_2} \setminus \{0\}, \phi \mapsto [\hat{\phi}] \) is onto and maps the subset of unital \(*\)-homomorphisms onto the subset of homotopy classes of unital \( C^* \)-algebra homomorphisms.

(ii) Let \( \phi \in \text{hom}_{\text{Alg}^*}(L(E), L(F))^P \). Then \( \hat{\phi} : C^*(E) \to C^*(F) \) is an \( M_2 \)-continuous homotopy equivalence if and only if \( \phi \) is a polynomial \( M_2 \)-homotopy equivalence. If furthermore \( \phi \) is unital, then \( \hat{\phi} \) is a continuous homotopy equivalence if and only if \( \phi \) is a polynomial homotopy equivalence.

Proof. By Example 4.4, we may replace \([[[C^*(E), C^*(F)]]_{M_2} \setminus \{0\}\) by \( KK(C^*(E), C^*(F)) \) in the statement of the theorem. Let \( \xi \in KK(C^*(E), C^*(F)) \) and let \( \xi_0 = K_0(\xi) : \mathcal{B}(E) \to \mathcal{B}(F) \). Let \( K^*_0 \) be the Grothendieck group of projections as defined in [8, Section 7] and let \( \text{can}'_E : \mathcal{B}(E) \to K_0(L(E))^* \) be the split monomorphism of [8, Remark 7.5]. By [8, Theorem 9.4] applied to \( \text{can}'_E \circ \xi_0 \) there is a \(*\)-homomorphism \( \phi : L(E) \to L(F) \) with property (P) such that \( K_0(\phi)^* \circ \text{can}'_E = \text{can}'_F \circ \xi_0 \), which can be taken unital if \( \xi_0 \) is. Thus \( K_0(\phi) = \xi_0 \) by [8, Remark 7.5] and therefore

\[ \eta = \xi - k(\hat{\phi}) \in L = \text{Ker}(K_0 : KK(C^*(E), C^*(F)) \to \text{hom}(\mathcal{B}(E), \mathcal{B}(F))). \]

Let \( L(F)_{\phi} = \bigoplus_{e \in E^1} \phi(ee^*)L(F)\phi(ee^*) \) and let \( K_1^* \) be unitary \( K_1 \) as defined in [8, equation (10.1)]. Put

\[ K = \text{Ker}(K_0 : \text{kk}(L(E), L(F)) \to \text{hom}(\mathcal{B}(E), \mathcal{B}(F))). \]

We have a commutative diagram

\[
\begin{array}{ccc}
K_1(L(F)_{\phi})^* & \xrightarrow{1} & (K_1(L(F)))^*_{\text{Bi}} \\
\downarrow & & \downarrow 2 \\
K_1(L(F)_{\phi}) & \xrightarrow{3} & K_1(L(F))_{\text{Bi}} \xrightarrow{4} K \\
\text{Ker}(I - \Lambda_{\text{Bi}}) & \xrightarrow{4} & L
\end{array}
\]
The map labelled 1 and the parallel map below are isomorphisms by Morita invariance [8, Lemma 10.2]. By [10, Proposition 4.6], the composite of the maps labelled 2 and 3 is onto. Moreover, because $F$ is spi, then by [8, Lemma 8.7], $L(F)_{\phi}$ is strictly properly infinite in the sense of [8, Section 8] and thus, by [8, Proposition 10.4] $K_1(L(F)_{\phi})$ is a quotient of the unitary group of $L(F)_{\phi}$. Summing up, there is a unitary $u \in U(L(F)_{\phi})$ whose class in $K_1(L(F)_{\phi})$ maps to $\eta$ through the composite of the maps labelled 1 through 4. By [8, Lemma 13.7] applied to $R = L(F) \oplus L(F)$ equipped with the involution $(x, y)^* = (y^*, x^*)$, the sum of the image of $u$ in $\text{kk}(L(E), L(F))$ with the class of $\phi$ is the class of the $*$-homomorphism $\phi^n : L(E) \to L(F)$ that maps $e \mapsto u\phi(e)$ which again has property (P). By construction, $k(\hat{\phi}u) = \xi$. This proves part (i). Let $\phi : L(E) \to L(F)$ be a $*$-homomorphism with property (P) such that $k(\phi)$ is an isomorphism. Then $j(\phi)$ is an isomorphism by parts (i) and (iii) of Theorem 3.2 and therefore $\phi$ is an $M_2$-homotopy equivalence by [10, Theorem 5.8]. If $\phi$ is unital and a polynomial $M_2$-homotopy equivalence, then it is a homotopy equivalence, by the argument at the end of the proof of [10, Theorem 6.1]. This concludes the proof.

**Corollary 4.11.** If $E$ and $F$ are finite spi graphs and either of $\mathcal{B}(E), \mathcal{B}(F)$ is finite, then for every algebra homomorphism $\xi : L(E) \to L(F)$ there exists a $*$-homomorphism $\phi : L(E) \to L(F)$ such that $\phi \sim_{M_2} \xi$. If moreover $\xi$ is unital, then $\phi$ can be chosen unital and so that $\phi \sim \xi$.

**Proof.** Immediate from Corollary 3.6 and Theorem 4.10.

**Example 4.12.** Let $R_n$ be the graph consisting of 1 vertex and $n$ loops and $R_n^-$ its Cuntz’ splice [3, Definition 2.11]. Write $L_n = L(R_n), O_n = C^*(R_n), L_n^- = L(R_n^-)$ and $O_n^- = C^*(R_n^-)$. Then by Theorem 4.10 and Corollary 3.6, every $*$-isomorphism $O_n \sim O_n^-$ is homotopic to the completion of a unital $*$-homomorphism $\phi : L_n \to L_n^-$, any such $\phi$ is an algebraic homotopy equivalence, and its homotopy class depends only on the $C^*$-homotopy class of $\xi$.

**APPENDIX: THE $M_2$-HOMOTOPY RELATION**

**Lemma A.1.** Let $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ be $C^*$-algebras and let $\text{inc}_{\mathcal{A}}$ and $\text{inc}_{\mathcal{B}}$ be the inclusion maps. Let $x \in \mathcal{C}$ such that $x\mathcal{A}x^* \subset \mathcal{B}$ and $ax^*xa' = aa'$ for all $a, a' \in \mathcal{A}$. Then $\text{ad}(x) : \mathcal{A} \to \mathcal{B}, \text{ad}(x)(a) = xax^*$ is a $*$-homomorphism and $\text{inc}_{\mathcal{B}} \circ \text{ad}(x) \approx_{M_2} \text{inc}_{\mathcal{A}}$. If moreover $\mathcal{A} = \mathcal{B}$ and $x \subset \mathcal{A}$, then $\text{ad}(x) \approx_{M_2} \text{id}_{\mathcal{A}}$.

**Proof.** This is the $C^*$-algebra version of [8, Lemma 8.12]; it is proved in the same way, using the fact that the upper left and lower right corner inclusions of a $C^*$-algebra into the $2 \times 2$ matrices are $C^*$-homotopic.

Let $\mathcal{B}$ be a properly infinite $C^*$-algebra and $s = (s_1, s_2) \in \mathcal{B}^2$ a pair of orthogonal isometries. The map

$$\square : \mathcal{B} \oplus \mathcal{B} \to \mathcal{B}, (a, b) \mapsto s_1 as_1^* + s_2 bs_2^*$$

(A.2)

is a $C^*$-algebra homomorphism, and if $\mathcal{A} \subset \mathcal{B}$, it restricts to a homomorphism $\mathcal{A} \oplus \mathcal{A} \to \mathcal{A}$. 

□
Lemma A.3. Let \( \mathcal{A} \), \( \mathcal{B} \) and \( \mathcal{C} \) be \( C^* \)-algebras with \( \mathcal{C} \) properly infinite and \( \mathcal{B} \triangleleft \mathcal{C} \) an ideal. Let \( t_1, t_2 \in \mathcal{C} \) be orthogonal isometries. Then \( \boxplus \) makes \( [[\mathcal{A}, \mathcal{B}]]_{M_2} \) into an abelian monoid.

Proof. Straightforward from Lemma A.1.

Lemma A.4. Let \( \mathcal{B} \) be a properly infinite \( C^* \) algebra and let \( s = (s_1, s_2) \) and \( t = (t_1, t_2) \) such that \( s_i s_j = t_i^* t_j = 1 \). Let \( \mathcal{A} \triangleleft \mathcal{B} \) be a closed ideal. Then \( \boxplus, \boxplus \) : \( \mathcal{A} \oplus \mathcal{A} \to \mathcal{A} \) are \( M_2 \)-homotopic.

Proof. The element \( u = t_1 s_1^* + t_2 s_2^* \) is a partial isometry with domain projection \( p = \boxplus_s(1, 1) = \sum_{i=1}^{2} s_i^* s_i \). By Lemma A.1, \( \text{ad}(u) : \mathcal{A} p \to \mathcal{A} \) is \( M_2 \)-homotopic to the inclusion \( \mathcal{A} p \subset \mathcal{A} \). Hence \( \boxplus \approx_{M_2} \text{ad}(u) \circ \boxplus \). □

Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \)-algebras. By [22, Section 8.1 and formula (8.1.3)], the set

\[
[[\mathcal{A}, \mathcal{K} \sim \otimes \mathcal{B}]]
\]

has a natural monoid structure.

Lemma A.5. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \)-algebras, with \( \mathcal{B} \) properly infinite. Let \( p \in \mathcal{K} \) be a rank one projection; set \( \iota : \mathcal{B} \to \mathcal{K} \sim \otimes \mathcal{B} \), \( \iota(b) = p \sim \otimes b \). Then

(i) \( \iota \) induces an isomorphism of monoids

\[
\iota_* : [[\mathcal{A}, \mathcal{B}]]_{M_2} \to [[\mathcal{A}, \mathcal{K} \sim \otimes \mathcal{B}]]_{M_2}.
\]

(ii) The canonical surjection is a monoid isomorphism

\[
[[\mathcal{A}, \mathcal{K} \sim \otimes \mathcal{B}]] \to [[\mathcal{A}, \mathcal{K} \sim \otimes \mathcal{B}]]_{M_2}.
\]

Moreover both isomorphisms above restrict to bijections between the subsets of homotopy classes of full homomorphisms.

Proof. It is clear that \( \iota_* \) is a monoid homomorphism. Let \( s \in \mathcal{B}^{1 \times \infty} \) be a row matrix of orthogonal isometries. Then \( \text{ad}(s) : M_{\infty} \mathcal{B} \to \mathcal{B} \), \( \text{ad}(s)(x) = sxs^* \) is a \( * \)-homomorphism and thus induces a \( C^* \)-algebra homomorphism \( \mathcal{K} \sim \otimes \mathcal{B} \to \mathcal{B} \). Composition with \( \text{ad}(s) \) defines a map \( \text{ad}(s)_* : [[\mathcal{A}, \mathcal{K} \sim \otimes \mathcal{B}]]_{M_2} \to [[\mathcal{A}, \mathcal{B}]]_{M_2} \). It follows from Lemma A.1 that \( \text{ad}(s)_* \circ \iota \) and \( \iota \circ \text{ad}(s)_* \) are \( M_2 \)-homotopic to the identity maps. This proves (i). The surjection of part (ii) is a monoid homomorphism by Lemma A.4; to prove that it is injective, it suffices to show that the composite of the corner inclusion \( \mathcal{K} \to M_2 \mathcal{K} \) with an isomorphism \( M_2 \mathcal{K} \cong \mathcal{K} \) is homotopic to the identity. This follows from [22, Proposition 1.3.4(i) and the implications (8.1.2)]. □

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**REFERENCES**

1. G. Abrams, P. N. Ánh, A. Louly, and E. Pardo, *The classification question for Leavitt path algebras*, J. Algebra *320* (2008), no. 5, 1983–2026. [https://doi.org/10.1016/j.jalgebra.2008.05.020.MR2437640](https://doi.org/10.1016/j.jalgebra.2008.05.020.MR2437640)

2. G. Abrams, P. Ara and M. Siles Molina, *Leavitt path algebras*, Lecture Notes in Math. vol. 2008, Springer, London, 2017.

3. G. Abrams, A. Louly, E. Pardo, and C. Smith, *Flow invariants in the classification of Leavitt path algebras*, J. Algebra *333* (2011), 202–231.

4. P. Ara, M. Brustenga, and G. Cortiñas, *K-theory of Leavitt path algebras*, Münster J. Math. *2* (2009), 5–33. MR2545605.

5. P. Ara and G. Cortiñas, *Tensor products of Leavitt path algebras*, Proc. Amer. Math. Soc. *141* (2013), no. 8, 2629–2639. [https://doi.org/10.1090/S0002-9939-2013-11561-3](https://doi.org/10.1090/S0002-9939-2013-11561-3)

6. B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications, vol. 5, Springer, New York, 1986. MR859867.

7. G. Cortiñas, *Algebraic v. topological K-theory: a friendly match*, Topics in algebraic and topological K-theory, Lecture Notes in Math. vol. 2008, Springer, Berlin, 2011, 103–165. MR2762555.

8. G. Cortiñas, *Classifying Leavitt path algebras up to involution preserving homotopy*, arXiv:2101.05777.

9. G. Cortiñas and D. Montero, *Algebraic bivariant K-theory and Leavitt path algebras*, J. Noncommut. Geom. *15* (2021), 113–146. [https://doi.org/10.1142/S0219498823500846](https://doi.org/10.1142/S0219498823500846).

10. G. Cortiñas and D. Montero, *Homotopy classification of Leavitt path algebras*, Adv. Math. *362* (2020), 106961, 26 pp. [https://doi.org/10.1016/j.aim.2019.106961](https://doi.org/10.1016/j.aim.2019.106961)

11. G. Cortiñas and N. C. Phillips, *Algebraic K-theory and properly infinite C*-algebras*, arXiv:1402.3197.

12. G. Cortiñas and A. Thom, *Bivariant algebraic K-theory*, J. Reine Angew. Math. *610* (2007), 71–123. [https://doi.org/10.1515/CRELLE.2007.068](https://doi.org/10.1515/CRELLE.2007.068).

13. G. Cortiñas and A. Thom, *Algebraic geometry of topological spaces I*, Acta Math. *209* (2012), no. 1, 83–131. [https://doi.org/10.1007/s11511-012-0082-6](https://doi.org/10.1007/s11511-012-0082-6).

14. G. Cortiñas, S. Vega, *Hermitian bivariant K-theory and Karoubi’s fundamental theorem*, J. Pure Appl. Algebra *226* (2022), 107124. [https://doi.org/10.1016/j.jpaa.2022.107124](https://doi.org/10.1016/j.jpaa.2022.107124).

15. J. Kaminker and I. Putnam, *K-theoretic duality for shifts of finite type*, Commun. Math. Phys. *187* (1997), 505–541.

16. M. Karoubi, *Homologie de groupes discrets associés à des algèbres d’opérateurs*, J. Operator Theory *15* (1986), no. 1, 109–161. With an appendix in English by Wilberd van der Kallen.

17. M. Karoubi and O. Villamayor, *K-théorie algébrique et K-théorie topologique. I*, Math. Scand. *28* (1972), no. 1971, 265–307. [https://doi.org/10.7146/math.scand.a-11024](https://doi.org/10.7146/math.scand.a-11024).

18. R. Meyer, *Universal coefficient theorems and assembly maps in KK-theory*, Topics in algebraic and topological K-theory, Lecture Notes in Math. vol. 2008, Springer, Berlin, 2011, pp. 45–102.

19. N. C. Phillips, *A classification theorem for nuclear purely infinite simple C*-algebras*, Doc. Math. *5* (2000), 49–114.

20. E. Riehl, *Category theory in context*, Aurora: Modern Math Originals, Dover, 2016, 048680903X.

21. M. Redam, *Classification of Cuntz-Krieger algebras*, K-theory *9* (1995), no. 1, 31–58.
22. M. Rødam and E. Stømer, *Classification of nuclear C*-algebras. Entropy in operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 126, Springer, Berlin, 2002, Operator Algebras and Non-commutative Geometry, 7.

23. J. Rosenberg, *The algebraic K-theory of operator algebras*, K-Theory 12 (1997), no. 1, 75–99. [https://doi.org/10.1023/A:100736420938](https://doi.org/10.1023/A:100736420938).

24. J. Rosenberg, *Comparison between algebraic and topological K-theory for Banach algebras and C*-algebras*, Handbook of K-theory, vol. 1, 2, Springer, Berlin, 2005, pp. 843–874. [https://doi.org/10.1007/978-3-540-27855-9_16](https://doi.org/10.1007/978-3-540-27855-9_16).

25. A. A. Suslin and M. Wodzicki, *Excision in algebraic K-theory*, Ann. of Math. (2) 136 (1992), no. 1, 51–122. [https://doi.org/10.2307/2946546](https://doi.org/10.2307/2946546).

26. C. A. Weibel, *Homotopy algebraic K-theory*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), Contemp. Math. vol. 83, Amer. Math. Soc., Providence, RI, 1989, 461–488. MR991991 (90d:18006).