Compton Scattering off Relativistic Bound States: Two–Photon Vertices, Ward–Takahashi Identities, Gauge Invariance and Low–Energy Limit

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In a general framework that has been labeled the “gauging of equations method”, we study the diagrams that contribute to Compton scattering off a relativistic composite system. These contributions can be derived for $N$–particle bound states described by the covariant Bethe–Salpeter equation with a method equivalent to minimal substitution in the one–particle case and yield the correct contributions (including subtraction terms) in the order $\mathcal{O}(e^2)$. We give the Ward–Takahashi identities for the general two–photon vertex as well as the corresponding constraints for the two–photon irreducible interaction kernel and the Bethe–Salpeter amplitude describing the bound state. From this we can show that gauge invariance holds for the full two–photon vertex. We furthermore study in detail the low–energy limit of the Compton scattering tensor in this approach (including a discussion of the pole terms) and can prove that the full amplitude yields the correct Born–Thomson limit as we shall explicitly show for the spin–0 case.

The calculations are completed by the investigation of certain approximations that can be formulated for arbitrary $N$–particle bound states. We neglect for instance contributions from $n$–photon irreducible interaction kernels and show that in this case gauge invariance is only realized if either the interaction kernel in the Bethe–Salpeter equation is independent of the total momentum and additionally is of local type, or if the photon energies vanish; furthermore, we find the correct low–energy limit in this approximation. To clarify our approach, we also give the results in the order $\mathcal{O}(e)$; as examples, we will quote some resulting lowest order expressions for a $q\bar{q}$ system explicitly.

I. INTRODUCTION

The first order result for a Green’s function in an external field obtained by a minimal substitution prescription and the analogous first order result for Bethe–Salpeter amplitudes have been shown in refs. [1–3] by Kvinikhidze and Blankleider; the authors refer to this procedure as “gauging a hadronic system”. However, up to now nothing is known about the important question how this general scheme can be applied to Compton scattering off a hadronic system which is a second order process. In fact, this is a difficult task since electromagnetic gauge invariance is intimately related to the dynamics of the strong interaction inside extended composite hadrons (see [6] and references therein). This important question has been addressed by Ito and Gross in ref. [11] where the authors have derived various diagrams that have to be included to guarantee a gauge invariant description of the Compton scattering process. These considerations — based on a diagram analysis in a general four–dimensional Bethe–Salpeter framework — were explicitly applied only to the deuteron system (i.e. with only one charged constituent) as an example that keeps the equations as simple as possible. In contrast to this approach, the procedure of gauging the relevant bound state equations as applied here has the advantage that it allows for a complete overview of the possible contributions in a given order $\mathcal{O}(e^n)$ for any $N$–particle bound state; it thus provides a more systematic view on the numerous processes contributing to Compton scattering off a bound state. In this way, it is therefore possible to study certain approximations (like e.g. neglecting $n$–photon irreducible interaction kernels) on the basis of the knowledge of the full expressions; this seems to be a more firm way of introducing and handling these approximations than in the case where one does not know what terms are precisely suppressed under certain assumptions.

In this article, we aim — on a general level — at a complete classification of the diagrams that contribute to the Compton scattering process $A\gamma \rightarrow A\gamma$ for bound states $A$ irrespective of their composition. Although our approach

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can be considered as a quite technical study of this process, it seems worthwhile to present it in this publication since it provides a more transparent way of discussing this issue compared to the technically involved explicit calculations in ref. [3]; furthermore, it allows for a simple check of the correct low–energy limits. We thus demonstrate that the method of Blankleider and Kvinikhidze (see [1–3]) turns out to be a powerful tool to derive all contributions not only in first order of the electromagnetic interaction but also in the order $O(e^2)$. Herein, the relevant degrees of freedom are the constituent particles of the composite system; our results for the full two–photon vertices turn out to be independent of the interaction kernel used in the underlying Bethe–Salpeter equation describing the bound state. In order to make clear our notation and the scheme that we apply to the second order amplitudes, we will always give the full dependency of the internal relative momentum.

After some introductory remarks on fundamental equations, definitions and notations in section II, we will derive the second order expansions of Green’s functions, two–photon vertices and Bethe–Salpeter amplitudes in section III. Ward–Takahashi identities and differential expressions with respect to the bound state’s total momentum are given in section IV for vertices and $n$–photon irreducible interaction kernels. The issue of gauge invariance is addressed in section V before we study the low–energy limit of the Compton scattering tensor in section VI. In section VII, we summarize our results.

II. GREEN’S FUNCTIONS, T–MATRIX AND BETHE–SALPETER EQUATIONS

The fundamental Green’s function equation for a given number of $n$ fermions and $\tilde{n}$ anti–fermions — depicted in fig. 1 — is defined as follows:

$$G = G_0 - iG_0 KG$$

$$= G_0 - iGKG_0.$$  \hspace{1cm} (1)

\[\begin{array}{c}
\includegraphics[width=0.5\textwidth]{fig1.png}
\end{array}\]

FIG. 1. The equation for the Green’s function $G$ according to eq. (1). The vertical dots (here and in the following figures) denote the possible occurrence of additional fermion lines depending on the number of constituents.

Throughout this article, all $n + \tilde{n} - 1$ four–dimensional integrations over internal relative momenta are implicitly understood; furthermore, we omit all indices for internal quantum numbers such as flavour, colour and spin. To clarify this point, we write down the fundamental Green’s equation, i.e. the first line in eq. (1), for $G(P;\{k'_i\};\{k_j\}) = G(\{p'_i\};\{p_j\})$ with $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N - 1$ explicitly as

$$G(P;\{k'_i\};\{k_j\})_{\{\alpha_i\},\{\beta_j\}} = G_0(P;\{k'_i\};\{k_j\})_{\{\alpha_i\},\{\beta_j\}}$$

$$- i \int \frac{d^4 q'_i}{(2\pi)^4(N-1)} \int \frac{d^4 q_j}{(2\pi)^4(N-1)} [G_0(P;\{k'_i\};\{q'_j\})]_{\{\alpha_i\},\{k_j\}} [K(P;\{q'_j\};\{q_j\})]_{\{\kappa_j\},\{\alpha_i\}} [G(P;\{q_j\};\{k_j\})]_{\{\kappa_j\},\{\beta_j\}}$$  \hspace{1cm} (2)

where we have separated the total momentum $P = p_1 + \ldots + p_N = p'_1 + \ldots + p'_N$ ($N = n + \tilde{n}$) and we integrate over all relative variables $\{k_j\} = k_1, k_2, \ldots, k_{N-1}$; see also appendix A. Here, the index set $\{\alpha_i\} = \alpha_1, \ldots, \alpha_N$ represents the multi–indices for each constituent particle in flavour, colour and spin space. Since $G_0$ describes free propagation without any interaction between the constituents, it is defined as

$$G_0(P;\{k'_i\};\{k_j\}) := G_0(P;\{k_j\}) \cdot (2\pi)^4 \delta^4(k'_1 - k_1) \ldots (2\pi)^4 \delta^4(k'_{N-1} - k_{N-1})$$

in this explicit formula. Here and in what follows in this section, all quantities are functions of the total momentum $P$; furthermore, the dependence on the relative momenta $k_j$ of the constituents (aside from some examples) and all indices will be suppressed in this paper.

The kernel $K$ contains all irreducible interaction terms between fermions and anti–fermions, see fig. 2. In this publication, we will not make any assumption about this kernel; however, we will show in section IV.B that it has to obey certain — general — Ward–Takahashi identities depending on the approximations applied in order to preserve gauge invariance for electromagnetic processes.
The iteration of an infinite sum of irreducible diagrams in eq. (1) can also be written as follows:

\[ G = G_0 K G_0 - iG_0 K T = G_0 K G_0 - iT K G_0. \]

Let us stress that this kernel \( K \) includes not only \( N \)-particle irreducible diagrams but also diagrams in which some of the constituents are unaffected by the interactions between the other constituents; an example is given in the second diagram in fig. (2). In this sense, the kernel \( K \) in the Green’s function equation is the sum of all possible \( i \)-particle irreducible kernels \( K^{(i)} \) (see e.g. \([7]\) for a precise formulation of \( K = \sum_i K^{(i)} \) in the case of \( qqq \) baryons). The iteration of an infinite sum of irreducible diagrams in eq. (1) can also be written as \( G = G_0 - iT \) by introducing the \( T \) matrix as \( T := G_0 K G \) which then satisfies the following equations:

\[ T = G_0 K G_0 - iG_0 K T = G_0 K G_0 - iT K G_0. \]  

From the equation for the Green’s function, we can deduce the Bethe–Salpeter equation for the Bethe–Salpeter amplitude \( \chi \) describing on–shell bound states and its adjoint \( \bar{\chi} \) (see \([12,13]\)):

\[ \chi = -iG_0 K \chi \quad \text{and} \quad \bar{\chi} = -i\bar{\chi} K G_0. \]  

Here, \( \chi \) is a short–hand notation for the Bethe–Salpeter amplitude \( \chi_{\alpha_1,\ldots,\alpha_{n+\bar{n}}}(P;\{k_j\}) \) that in coordinate space is defined explicitly as

\[ \chi_{\alpha_1,\ldots,\alpha_{n+\bar{n}}}(x_1,\ldots,x_{n+\bar{n}}) := \langle 0 | \Gamma \psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_n}(x_n)\bar{\psi}_{\alpha_{n+1}}(x_{n+1})\ldots\bar{\psi}_{\beta_{n+\bar{n}}}(x_{n+\bar{n}}) | P \rangle \]

\[ = e^{-ipX} \int \frac{d^4 \{k_j\}}{(2\pi)^{4(N-1)}} e^{-ik_1 r_1} \ldots e^{-ik_{N-1} r_{N-1}} \chi_{\alpha_1,\ldots,\alpha_{n+\bar{n}}}(P;\{k_j\}) \]

for a bound state of \( n \) fermions and \( \bar{n} \) anti–fermions \((N = n + \bar{n})\). In this definition, \( \Gamma \) denotes the time–ordered product; see also appendix A for the definition of the coordinates \( X, r_1, \ldots, r_{N-1} \).

With the formal solution of eq. (1), i.e. \( G = (G_0^{-1} + iK)^{-1} \), the Bethe–Salpeter equations in eq. (4) can be formulated as follows:

\[ G^{-1} \chi = 0 \quad \text{and} \quad \bar{\chi} G^{-1} = 0. \]  

Note that these homogeneous Bethe–Salpeter equations are only valid (and thus the Bethe–Salpeter amplitudes are only defined) for on–shell bound states of momentum \( P \) and mass \( M \) with the constraint \( P^2 = M^2 \).

\[ \chi = -i K \chi. \]  

FIG. 2. Irreducible diagrams that contribute to the interaction kernel \( G_0 K G_0 \) with \( N = 4 \) constituents as an example.

FIG. 3. The equation for the \( T \) matrix in eq. (3).

FIG. 4. The Bethe–Salpeter equation for a bound states of \( N \) constituents in a graphical notation, see eq. (4).
III. SECOND ORDER EXPANSION IN THE ELECTROMAGNETIC FIELD

In the following we extend the fundamental Green’s equation and the Bethe–Salpeter equation by expanding operators 
\( O = G_0, G, K, \chi \) in the presence of an external electromagnetic field as

\[
O^A = O - i e O^\gamma - e^2 O^\gamma \gamma + \mathcal{O}(e^3) .
\]  

(7)

In this way, we find the expressions for the second order Green’s function \( G^\gamma \gamma \) and the second order Bethe–Salpeter amplitude \( \chi^\gamma \gamma \).

A. Green’s Functions

The Green’s function in eq. (1) can be transformed into a Green’s function in an external field via minimal substitution:

\[
G^A = G^A_0 - i G^A_0 K^A G^A
\]

(8)

where eq. (7) for the operators \( O = G_0, G, K \) is applied. In this way, each fermion line in the operators \( O \) will be attached by a number of photons in the order \( \mathcal{O}(e^n) \) similar to the well–known construction of the fermion propagator \( S^A(p, p') \) in an electromagnetic field, see fig. (5).

\[
\begin{align*}
\text{FIG. 5. The fermion propagator in an external field.}
\end{align*}
\]

1. First Order Expansion

With the expansion in eq. (8) and by using eq. (1), we can write down an equation for the Green’s function in first order of the electromagnetic coupling:

\[
G^\gamma = G_0^\gamma - i G_0^\gamma K G_0^\gamma K - i G_0^\gamma K G_0^\gamma K G_0^\gamma K
\]

\[
= \left(1 + i G_0 K\right)^{-1} \left(G_0^\gamma \left(1 - i K G_0^\gamma G_0\right)\right)
\]

\[
= G \left(G_0^{-1} G_0^\gamma G_0 - i K^\gamma G_0\right) G
\]

Defining the amputated full one–photon vertex \( \Gamma^\gamma := G^{-1} G^\gamma G^{-1} \) and the amputated free photon vertex \( \Gamma_0^\gamma := G_0^{-1} G_0^\gamma G_0^{-1} \), we can write

\[
G^\gamma = G \Gamma^\gamma G \quad \text{with} \quad \Gamma^\gamma = \Gamma_0^\gamma - i K^\gamma
\]

(9)

for the first order Green’s function in an external field; this equation is depicted diagrammatically in fig. (6).

\[
\text{FIG. 6. The one–photon vertex } G_0 \Gamma^\gamma G_0 \text{ according to eq. (9) in a graphical notation.}
\]
This full one–photon vertex can not solely be studied in models where the Bethe–Salpeter equation is used in its instantaneous approximation; it is also relevant in models including retardation effects (see [8]). The concept of an “interaction current” \( K^\gamma \) has been introduced in ref. [7]; its effects have been studied quantitatively in [9] where the electromagnetic pion form factor is calculated with the full vertex \( \Gamma^\gamma \) in a four–dimensional separable ansatz for \( K \) based on ‘t Hooft’s instanton–induced interaction (see also [6]).

To make clear the physical meaning of \( K^\gamma \), we give some diagrams contributing to this kernel explicitly in fig. [10] as an example. Note that — in analogy to the kernel without electromagnetic field in fig. [9] — all \( i \)–particle irreducible diagrams \((i = 2,\ldots,N)\) contribute to \( K^\gamma \), i.e. also processes where some of the constituents can be regarded as spectators.

\[
\begin{align*}
K^\gamma &= + + + + + \ldots
\end{align*}
\]

FIG. 7. The one–photon irreducible interaction kernel \( G_0K^\gamma G_0 \) with \( N = 4 \) charged constituents as an example. If the interaction kernel \( K \) includes charged exchange particles then additional diagrams contribute to \( K^\gamma \) in which the photon also couples to these charged exchange particles.

For bound states composed of more than two constituents (e.g. a nucleon described as a \( qq \) system), the method of gauging the Green’s function equation provides a solution to the notorious problem of overcounting diagrams in few–body systems. A detailed discussion of the role of subtraction terms emerging in this framework can be found in refs. [2,3] where the problem of ambiguous cuts in the related diagrams (as well as its cure) is intensively studied.

We remark moreover that our notion of a “lowest order vertex” with respect to \( \Gamma^\gamma \) (and as well to \( \Gamma_0^\gamma \) introduced in the next paragraph) is somewhat sloppy since — apart from coupling constants that might be absorbed in the interaction kernels \( K, K^\gamma \) and \( K^{\gamma\gamma} \) — we have no ordering or power counting scheme that is reflected in this term. In principal, all parts of \( \Gamma^\gamma \) (and \( \Gamma^{\gamma\gamma} \)) can contribute in the same magnitude to the full \( n \)–photon irreducible vertices. However, these lowest order vertices are minimal in the sense that they should be included in any framework that considers bound states like the pion or the nucleon as composite systems since they describe the simplest photon couplings to the constituents. Therefore, we will use this phrase now and then in the following discussion although it is not strictly exact.

2. Second Order Expansion

For the description of Compton scattering off a bound state described by the Bethe–Salpeter equation, we need the second order expansion of the Green’s function. Let us therefore again start with eq. [6] and evaluate it in the order \( O(e^2) \):

\[
G^{\gamma\gamma} = G_0^{\gamma\gamma} - iG_0^{\gamma\gamma} KG - iG_0 K^\gamma G - iG_0 K^{\gamma\gamma} G - iG_0^{\gamma\gamma} K^\gamma G - iG_0 K^{\gamma\gamma} G^\gamma
\]

\[
= \left( I + iG_0 K \right)^{-1} G_0^{\gamma\gamma} \left( I - iKG \right) - iG_0 K^{\gamma\gamma} G - iG_0^{\gamma\gamma} K^\gamma G - iG_0 K^{\gamma\gamma} G - iG_0 K^{\gamma\gamma} G^\gamma
\]

\[
= G \left( G_0^{-1} G_0^{\gamma\gamma} G_0^{-1} - iK^{\gamma\gamma} - iG_0^{-1} G_0^{\gamma\gamma} K^\gamma G - iG_0^{-1} G_0^{\gamma\gamma} K^\gamma G - iK^{\gamma\gamma} G^\gamma G^{-1} \right) G
\]

The last three terms can be simplified with the expression for the full one–photon vertex in eq. [8]:

\[
A^{\gamma\gamma} = -iG_0^{-1} G_0 \Gamma_0^\gamma G_0 K^\gamma - iG_0^{-1} G_0 \Gamma_0^\gamma G_0 K^\gamma G G^{-1}
\]

\[
= -i\Gamma_0^\gamma G_0 K^\gamma + \Gamma_0^\gamma GT^\gamma - \Gamma_0^\gamma G_0 \left( \Gamma_0^\gamma - iK^\gamma \right) - iK^{\gamma\gamma} G^\gamma G^{-1}
\]

\[
= \Gamma^\gamma \Gamma^\gamma - \Gamma_0^\gamma G_0 \Gamma_0^\gamma .
\]

We define — in analogy to the first order expansion — the amputated full two–photon vertex \( \Gamma^{\gamma\gamma} := G^{-1} G^{\gamma\gamma} G^{-1} \) and the amputated free two–photon vertex \( \Gamma_0^{\gamma\gamma} := G_0^{-1} G_0^{\gamma\gamma} G_0^{-1} \); the second order Green’s function then reads

5
\[ G^{\gamma\gamma} = G\Gamma^{\gamma\gamma}G \quad \text{with} \quad \Gamma^{\gamma\gamma} = \Gamma_0^{\gamma\gamma} - iK^{\gamma\gamma} + \Gamma^{\gamma}\Gamma^{\gamma} - \Gamma_0^{\gamma}\Gamma_0^{\gamma} \quad (10) \]

and is shown in terms of diagrams in fig. (8). This is the key result of this section since it provides — on the basis of rather simple considerations — the answer to the question which processes contribute to Compton scattering off a bound state. In this sense, eq. (10) can be regarded as a generalization of the results of ref. [11]: furthermore and in addition to it, it gives a more systematic overview and ordering of the various terms and by this means thus reveals the existence of a subtraction term besides the pole contributions in \( \Gamma^{\gamma}\Gamma^{\gamma} \). Note that this subtraction term \( \Gamma_0^{\gamma}\Gamma_0^{\gamma} \) naturally emerges in this framework; it prevents overcounting of the lowest-order parts in \( \Gamma^{\gamma}\Gamma^{\gamma} \) and turns out to be crucial for a proper description of Compton scattering off bound states. The two-photon irreducible interaction kernel \( K^{\gamma\gamma} \) in eq. (10) can be understood analogously to fig. (7).

As an approximation of this full two-photon vertex, we will introduce \( \hat{G}^{\gamma\gamma} := G\hat{\Gamma}^{\gamma\gamma}G \) where all \( n \)-photon irreducible interaction kernels, \( i.e. \) \( K^{\gamma} \) and \( K^{\gamma\gamma} \), are neglected. Then, by virtue of \( \hat{G}^{\gamma\gamma} = G_0 - iT \), the subtraction term can be evaluated explicitly — see fig. (8) — finally yielding

\[ \hat{G}^{\gamma\gamma} = G\hat{\Gamma}^{\gamma\gamma}G \quad \text{with} \quad \hat{\Gamma}^{\gamma\gamma} = \Gamma_0^{\gamma\gamma} - i\Gamma_0^{\gamma}T\Gamma_0^{\gamma} \quad . \quad (11) \]

FIG. 8. The two-photon vertex \( G_0\Gamma^{\gamma\gamma}G_0 \) according to eq. (10) in a graphical notation.

As an approximation of this full two-photon vertex, we will introduce \( \hat{G}^{\gamma\gamma} := G\hat{\Gamma}^{\gamma\gamma}G \) where all \( n \)-photon irreducible interaction kernels, \( i.e. \) \( K^{\gamma} \) and \( K^{\gamma\gamma} \), are neglected. Then, by virtue of \( \hat{G}^{\gamma\gamma} = G_0 - iT \), the subtraction term can be evaluated explicitly — see fig. (8) — finally yielding

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FIG. 9. The approximated two-photon vertex \( G_0\hat{\Gamma}^{\gamma\gamma}G_0 \) without inclusion of \( n \)-photon irreducible interaction kernels according to eq. (11) in a graphical notation.

\[ \hat{G}^{\gamma\gamma} = G\hat{\Gamma}^{\gamma\gamma}G \quad \text{with} \quad \hat{\Gamma}^{\gamma\gamma} = \Gamma_0^{\gamma\gamma} - i\Gamma_0^{\gamma}T\Gamma_0^{\gamma} \quad . \quad (11) \]

FIG. 9. The approximated two-photon vertex \( G_0\hat{\Gamma}^{\gamma\gamma}G_0 \) without inclusion of \( n \)-photon irreducible interaction kernels according to eq. (11) in a graphical notation.

3. Explicit Formulae for Compton Scattering

These results in first and second order of the coupling \( e \) were formulated for a hadron in an amorphous electromagnetic field without stating precisely the photon momenta and coupling indices in Dirac space. If we consider Compton scattering kinematics \( i.e. \) \( P + q_1 = P' + q_2 \), we now have to indicate not only the correct momentum transfers but also to specify coupling indices \( \mu, \nu, \ldots \) of the photon lines and make sure that in each order all possible diagrams including permutations are taken into account. For the second order fermion propagator, this yields two diagrams (their sum being explicitly crossing symmetric) as shown in fig. (10).

\[ \gamma_1 \quad P \quad \gamma_2 \quad = \quad \gamma_1 \quad P \quad \gamma_2 \quad + \quad \gamma_1 \quad P \quad \gamma_2 \]

FIG. 10. The fermion propagator in an external field in second order with Compton scattering kinematics \( P' = P + q_1 - q_2 \) for two photons \( \gamma_1 = \gamma_1(q_1, \mu) \) and \( \gamma_2 = \gamma_2(-q_2, \nu) \).
For the first order Green’s function in eq. (11), all terms can be identified uniquely and therefore no correction is
needed:

\[ G^{\mu}(P', P) = G(P') \Gamma^{\mu}(P', P) G(P) \]  

with \[ \Gamma^{\mu}(P', P) = \Gamma^{\mu}_0(P', P) - iK^{\mu}(P', P) \] .

Here we have introduced the momentum transfer \( P' = P + q \) for the photon \( \gamma = \gamma(q, \mu) \) involved in this process.

Considering the second order Green’s function in eq. (13), we find that additional terms appear due to crossing symmetry:

\[ G^{\mu\nu}(P', P) = G(P') \Gamma^{\mu\nu}(P', P) G(P) \]  

with \[ \Gamma^{\mu\nu}(P', P) = \Gamma^{\mu\nu}_\text{POLE}(P', P) + \Gamma^{\mu\nu}_\text{NoPole}(P', P) \]

\[ + \Gamma^{\mu}(P', P' - q_1) G(P - q_2) \Gamma^{\nu}(P - q_2, P) - \Gamma^{\mu}_0(P', P' - q_1) G_0(P - q_2, P) \]

\[ + \Gamma^{\nu}(P', P' + q_2) G(P' + q_2) \Gamma^{\mu}(P + q_1, P) - \Gamma^{\nu}_0(P', P' + q_2) G_0(P' + q_2) \Gamma^{\mu}_0(P + q_1, P) \] .

For Compton scattering with the incoming photon \( \gamma_1 = \gamma_1(q_1, \mu) \) and the outgoing photon \( \gamma_2 = \gamma_2(-q_2, \nu) \), energy–momentum conservation requires \( P + q_1 = P' + q_2 \) where \( P (P') \) denotes the incoming (outgoing) total four–momentum of the bound state. After multiplying out all contributions from \( \Gamma^\gamma G^\gamma \), all terms in eq. (13) can be identified with diagrams shown in [1] where Compton scattering off a deuteron system (i.e. with only one charged constituent) has been studied.

It will be helpful for some following remarks to split up the Compton vertex in eq. (13) into parts that include the pole terms induced by the full Green’s function \( G \) and the rest that is non–singular (see e.g. [22,24]):

\[ \Gamma^{\mu\nu}(P', P) = \Gamma^{\mu\nu}_\text{POLE}(P', P) + \Gamma^{\mu\nu}_\text{NoPole}(P', P) \]  

with \[ \Gamma^{\mu\nu}_\text{POLE}(P', P) = \Gamma^{\mu}(P', P' - q_1) G(P - q_2) \Gamma^{\nu}(P - q_2, P) \]

\[ + \Gamma^{\nu}(P', P' + q_2) G(P' + q_2) \Gamma^{\mu}(P + q_1, P) \] .

Let us note that from the equations above no constraint arises neither for the one–photon irreducible interaction kernel \( K^\mu(P', P) \) nor for the two–photon irreducible interaction kernel \( K^{\mu\nu}(P', P) \). However, they have to obey certain Ward–Takahashi identities in order to satisfy the constraints from gauge invariance as we will see in section [IV,B]. Concerning the interpretation of the \( n \)–photon irreducible interaction kernels, it is worthwhile to note that one can generally describe them in a different way than demonstrated above. Let us therefore first define

\[ G_\epsilon := G_0 - i\epsilon G_0 K G \]  

as a modified Green’s function that reduces to \( G_{(0)} = G_0 \) describing free propagation for \( \epsilon = 0 \). With this definition, we introduce

\[ \Gamma_\epsilon := G^{-1}_0 + i\epsilon K \]

\[ \Gamma_\epsilon^\mu := G^{-1}_0 - i\epsilon K^\mu \]

\[ \Gamma^{\mu\nu}_\epsilon := \Gamma^{\mu\nu}_0 - i\epsilon K^{\mu\nu} + \Gamma_\epsilon^\mu G_\epsilon(\epsilon) \Gamma_\epsilon^{\nu} + G_\epsilon^{\mu}(\epsilon) \Gamma_\epsilon^\nu \Gamma^{\mu\nu}_0 - \Gamma^{\mu}_0 G_0 \Gamma_\epsilon^\nu - \Gamma^{\nu}_0 G_0 \Gamma_\epsilon^\mu \]

and then finally

\[ G_\epsilon^{(1)} := \Gamma_\epsilon \]

\[ G_\epsilon^{(0)} := \Gamma_\epsilon^\mu \]

\[ G^{(\mu\nu)}_\epsilon := \Gamma^{(\mu\nu)}_\epsilon - \Gamma^{\mu}_0 G_\epsilon(\epsilon) \Gamma^{\nu}_0 + \Gamma^{\nu}_0 G_\epsilon(\epsilon) \Gamma^{\mu}_0 \]

which are quantities that are free from singularities. For \( \epsilon = 0 \), they obviously include no contributions from interaction kernels \( K, K^\mu \) and \( K^{\mu\nu} \) while for \( \epsilon = 1 \) we recover \( G_{(1)} = G^{-1} \), \( G^{(1)}_\mu = G^{\mu} \) and \( G^{(1)}_{\mu\nu} = \Gamma^{(\mu\nu)}_{\text{NoPole}} = \Gamma^{\mu\nu} - \Gamma^{\mu\nu}_{\text{POLE}} \). With these definitions, one can easily verify that the interaction kernels can simply be written as

\[ + iK = G_{(1)} - G_{(0)} \]

\[ - iK^\mu = G^{(1)}_\mu - G^{(0)}_\mu \]  

\[ - iK^{\mu\nu} = G^{(1)}_{\mu\nu} - G^{(0)}_{\mu\nu} \] .

We will come back to the definitions above in section [IV,A] since they obey Ward–Takahashi identities in a unified way due to their lack of poles; moreover, a derivation of Ward–Takahashi identities for the \( n \)–photon interaction kernels is straightforward with the help of eqs. (18)–(24).
With the systematic expansion in eq. (7), it is also possible to derive an “extended” Bethe–Salpeter equation
\[ \chi^A = -iG^A_0 K^A \chi^A \] (21)
for a bound state in an external field; here, we expand \( O = G_0, K, \chi \). For the first order result, one can easily show that
\[ \chi^\gamma = G \Gamma^\gamma \chi \] (22)
(see also \[1\]). With a little algebra, we find the second order result (see appendix B):
\[ \chi^{\gamma\gamma} = \tilde{\chi} \Gamma^{\gamma\gamma} \tilde{\chi} \] . (23)
Similar relations hold for the adjoint amplitudes if we start with the “extended” adjoint Bethe–Salpeter equation
\[ \tilde{\chi}^A = -i \tilde{\chi}^A K^A G^A_0 : \]
\[ \tilde{\chi}^\gamma = \tilde{\chi} \Gamma^\gamma G \] , (24)
\[ \tilde{\chi}^{\gamma\gamma} = \tilde{\chi} \Gamma^{\gamma\gamma} G \] . (25)
For the sake of completeness, we finally want to give these results for distinguishable photons:
\[ \chi^\mu(P', P) = G(P') \Gamma^\mu(P', P) \chi(P) \] , (26)
\[ \chi^{\mu\nu}(P', P) = G(P') \Gamma^{\mu\nu}(P', P) \chi(P) \] (27)
in first and second order, respectively, and analogously for the adjoint amplitudes \( \tilde{\chi}^\mu(P', P) \) and \( \tilde{\chi}^{\mu\nu}(P', P) \). In this sense, the expanded (adjoint) Bethe–Salpeter amplitudes \( \chi^\mu \) and \( \chi^{\mu\nu} \) (\( \tilde{\chi}^\mu \) and \( \tilde{\chi}^{\mu\nu} \)) are no “new” quantities obeying separately relations like \( n \)-th order (adjoint) Bethe–Salpeter equations but they can be formulated in terms of the full Green’s function, the full \( n \)-photon vertex and a “genuine” or “free” (adjoint) Bethe–Salpeter amplitude without any coupling to the electromagnetic field.

Let us stress that eqs. (26) and (27) represent well defined relations since the Green’s function \( G \) remains finite for \( q = P' - P \neq 0 \); we will come back to this point later.

IV. WARD–TAKAHASHI IDENTITIES

Ward–Takahashi identities (see \[14,15\]) relate amplitudes of electromagnetic processes of order \( O(e^{n+1}) \) with those of order \( O(e^n) \). In the following, we will derive constraints for the one–photon and two–photon irreducible interaction kernels from these relations as well as identities for the Bethe–Salpeter amplitudes in first and second order of the electromagnetic coupling.

Since we aim at a general description of bound states composed of an unspecified number of \( N \) constituents, we will again in general suppress all relative momenta and indicate only the dependency of the total four–momenta \( P \) and \( P' \); explicit examples of Ward–Takahashi identities for a \( q\bar{q} \) system are given in section IV C.

A. Identities for the Amputated Green’s Functions

In \[16\], a general form of the celebrated Ward–Takahashi identity is given (see also \[3\] and the seminal publications \[14,15\]):
\[ q_\mu M^\mu(P', P) = e \left( M(P' - q, P) - M(P', P + q) \right) \] . (28)
Here, \( M^\mu \) denotes the amplitude for the process with \( n + 1 \) photons and \( M \) is the amplitude for the process with \( n \) photons; the indices for these \( n \) other photons as well as the dependency of \( M^\mu \) and \( M \) of the internal relative momenta are suppressed in this notation. In fig. (11), this formula is depicted diagrammatically.
\[ q_\mu \cdot \begin{array}{c} \gamma \downarrow \\
 P \begin{array}{c} M^\mu \\ \end{array} \end{array} \begin{array}{c} \begin{array}{c} \cdots \\ \cdots \\ \end{array} \end{array} P' = e \left( \begin{array}{c} P \begin{array}{c} M \\ \end{array} \begin{array}{c} \cdots \\ \cdots \\ \end{array} \end{array} (P' - q) - \begin{array}{c} P + q \\ \begin{array}{c} M \\ \end{array} \begin{array}{c} \cdots \\ \cdots \\ \end{array} \end{array} \right) \]  

FIG. 11. The most general form of the Ward–Takahashi identity for a photon \( \gamma = \gamma(q, \mu) \) according to eq. (28); see also [16].

The amplitudes for processes with zero, one and two photons involved are given by

\[ M(P', P) := G(P') \Gamma(P', P) G(P) \]
\[ M^\mu(P', P) := (-ie) G(P') \Gamma^\mu(P', P) G(P) \]
\[ M^\mu\nu(P', P) := (-ie)^2 G(P') \Gamma^{\mu\nu}(P', P) G(P) \]

respectively. The vertices (or amputated Green’s functions) \( \Gamma^\mu \) and \( \Gamma^{\mu\nu} \) are given in eqs. (12) and (13). Furthermore, we have defined \( \Gamma(P', P) := G^{-1}(P) \cdot (2\pi)^4\delta^4(P' - P) \); for the charge factors in the equations above, see section VI A. For the electromagnetic current in the order \( \mathcal{O}(e) \) with \( P' = P + q \), the Ward–Takahashi identity reads explicitly

\[ q_\mu M^\mu(P + q, P) = e \left( M(P, P) - M(P + q, P + q) \right) \]
\[ \iff -iq_\mu \Gamma^\mu(P + q, P) = G^{-1}(P + q) - G^{-1}(P) \] (30)

In the case of Compton scattering, four–momentum conservation requires \( P + q_1 = P' + q_2 \); the Ward–Takahashi identity for this process in order \( \mathcal{O}(e^2) \) is therefore

\[ q_{1\mu} M^{\mu\nu}(P + q_1 - q_2, P) = e \left( M^{\nu}\mu(P - q_2, P) - M^{\nu\mu}(P + q_1 - q_2, P + q_1) \right) \]
\[ \iff -iq_{1\mu} \Gamma^{\mu\nu}(P + q_1 - q_2, P) = G^{-1}(P + q_1 - q_2)G(P - q_2)G^{\nu\mu}(P - q_2, P) - \Gamma^{\nu\mu}(P + q_1 - q_2, P + q_1)G(P + q_1)G^{-1}(P) \] (31)

It may seem surprising that not a simple difference of one–photon vertices occurs on the right–hand side of the last equation; however, we stress (as it will turn out in the next sections) that this combination of Green’s functions and their inverse being attached to \( \Gamma^{\nu\mu} \) is crucial for a correct description of the Compton scattering process. A similar Ward–Takahashi identity in the order \( \mathcal{O}(e^2) \) has also been used in ref. [22] in order to derive the lowest order contributions (i.e. at order \( \mathcal{O}(1) \) with respect to the photon momenta) for Compton scattering off spin–0 particles. The formulae in eqs. (30) and (31) provide general constraints for the explicit one–photon and two–photon vertices \( \Gamma^\nu \) and \( \Gamma^{\mu\nu} \) since a violation of these Ward–Takahashi identities leads to electromagnetic observables that are not gauge invariant; we will show this in section V.

Before we now insert the explicit expressions for the amputated Green’s functions given in eqs. (12) and (13), we briefly recall the well–known identities for the free one–fermion propagator

\[ -i(P' - P)_\mu \gamma^\mu = S^{-1}_F(P') - S^{-1}_F(P) \] (32)

and

\[ \gamma^\mu = i \frac{\partial}{\partial P^\mu} S^{-1}_F(P) \] (33)

that — because \( G_0 \) includes only free propagators — immediately lead to the following relations for the lowest order vertices \( \Gamma^{\mu}_0 \) and \( \Gamma^{\mu\nu}_0 \):

\[ -iq_\mu \Gamma^{\mu}_0(P + q, P) = G^{-1}_0(P + q) - G^{-1}_0(P) \]
\[ \text{and} \quad -iq_{1\mu} \Gamma^{\mu\nu}_0(P + q_1 - q_2, P) = G^{-1}_0(P + q_1 - q_2)G_0(P - q_2)\Gamma^{\nu\mu}_0(P - q_2, P) - \Gamma^{\nu\mu}_0(P + q_1 - q_2, P + q_1)G_0(P + q_1)G^{-1}_0(P) \] (34)
We stress that due to the definitions in eqs. (29), the Ward–Takahashi identities in this section do not include any charge factors in physical units. However, these charge factors $e$ and $e^2$, respectively, can be re–introduced by a simple re–definition of the $n$–photon irreducible vertices (see section IV C); see also the remarks on charge factors in section IV C.

Let us finally make some remarks on the apparently different form of the Ward–Takahashi identities of order $\mathcal{O}(e)$ and $\mathcal{O}(e^2)$, respectively. The difference between the one–photon and the two–photon irreducible vertex is that $\Gamma^{\mu\nu}$ in contrast to $\Gamma^{\mu}$ includes pole terms. To suppress the contributions from $\Gamma^{\mu}_{\text{POLE}}$, we have introduced $\Gamma^{\mu\nu}_{\text{NEWPOLE}} = \Gamma^{\mu\nu} - \Gamma^{\mu}_{\text{POLE}}$ in eq. (4). A generalization with and without inclusion of interaction kernels is given in eq. (17); it is straightforward to show that these quantities satisfy the following simple identities (see also eqs. (22)):

$$-i q_{\mu} G^{\mu}_{(e)}(P + q, P) = G^{(e)}(P + q) - G^{(e)}(P)$$  \hspace{1cm} (36)

and

$$-i q_{1\mu} G^{\mu\nu}_{(e)}(P + q_1 - q_2, P) = G^{\nu}_{(e)}(P - q_2, P) - G^{\nu}_{(e)}(P', P' + q_2)$$  \hspace{1cm} (37)

Both for $\epsilon = 1$ and $\epsilon = 0$, they can be derived analogously to the Ward–Takahashi identities given above. We should note in general that all identities in order $\mathcal{O}(e^2)$ can be re–formulated for the contraction with respect of the four–momentum $q_{2\nu}$ of the second (outgoing) photon by using crossing symmetry ($i.e.$ $q_1 \leftrightarrow -q_2$, $\mu \leftrightarrow \nu$).

B. Identities for the Interaction Kernels

As we will see in section IV C, we have to demand that the Ward–Takahashi identities in eqs. (24) and (25) hold in order to guarantee strict gauge invariance. From this principle, we can derive analogues identities for the irreducible interaction kernels. Let us start by inserting the explicit form of $\Gamma^{\mu}(P', P)$ into the left–hand side of the Ward–Takahashi identity of order $\mathcal{O}(e)$:

$$-i q_{\mu} \Gamma^{\mu}(P + q, P) = -i q_{\mu} (\Gamma^{\mu}_{0}(P + q, P) - i K^{\mu}(P + q, P))$$

$$= G^{0\dagger}_{(P + q)} - G^{0\dagger}_{(P)} - q_{\mu} K^{\mu}(P + q, P)$$  \hspace{1cm} (38)

We see from eq. (38) that the difference of two inverse full Green’s functions should appear; from this constraint and due to $G^{-1} = G^{0\dagger} + i K$, we can read off a Ward–Takahashi identity for the one–photon irreducible interaction kernel:

$$-i q_{\mu} K^{\mu}(P + q, P) = K(P) - K(P + q)$$  \hspace{1cm} (39)

This result is well–known for a long time and has been derived e.g. in refs. [14,17]. However, the same procedure can also be applied to the divergence of the two–photon vertex $\Gamma^{\mu\nu}(P', P)$ with $P' = P + q_1 - q_2$ and yields

$$-i q_{1\mu} \Gamma^{\mu\nu}(P + q_1 - q_2, P) = -i q_{1\mu} \Gamma^{\mu\nu}_{0}(P + q_1 - q_2, P) - i( -i q_{1\mu} K_{\nu}(P + q_1 - q_2, P))$$

$$+ \Gamma^{\nu}_{0}(P + q_1 - q_2, P + q_1) G^{0\dagger}(P + q_1) \Gamma^{\mu}_{0}(P + q_1) G^{0\dagger}(P + q_1)$$

$$- \Gamma^{\nu}_{0}(P + q_1 - q_2, P + q_1) G^{0\dagger}(P + q_1) \Gamma^{\mu}_{0}(P + q_1) G^{0\dagger}(P + q_1)$$

$$+ ( -i q_{1\mu} \Gamma^{\mu\nu}(P + q_1 - q_2, P - q_2)) G^{0\dagger}(P - q_2) \Gamma^{\nu}_{0}(P - q_2, P)$$

$$- ( -i q_{1\mu} \Gamma^{\mu\nu}(P + q_1 - q_2, P - q_2)) G^{0\dagger}(P - q_2) \Gamma^{\nu}_{0}(P - q_2, P)$$

$$= G^{0\dagger}(P + q_1 - q_2, P) G^{0\dagger}(P - q_2) \Gamma^{\nu}(P - q_2, P)$$

$$- G^{0\dagger}(P + q_1 - q_2, P) G^{0\dagger}(P - q_2) \Gamma^{\nu}(P - q_2, P)$$

$$- G^{0\dagger}(P + q_1 - q_2, P) G^{0\dagger}(P - q_2) \Gamma^{\nu}(P - q_2, P)$$

$$- q_{1\mu} K^{\mu\nu}(P + q_1 - q_2, P)$$

where we have used eqs. (34), (35) and (37). By comparison with eq. (34) we observe that the correct terms for the Ward–Takahashi identity in $\mathcal{O}(e^2)$ are already present and with $\Gamma^{0\nu} - \Gamma^{0\nu} = i K^{\nu}$ find the relation for the two–photon irreducible interaction kernel:

$$-i q_{1\mu} K^{\mu\nu}(P + q_1 - q_2, P) = K^{\nu}(P + q_1 - q_2, P) - K^{\nu}(P + q_1 - q_2, P + q_1)$$  \hspace{1cm} (39)

Note that we have assumed that $K^{\nu}$ obeys the relation in eq. (33) since we have used the full Ward–Takahashi identity in the order $\mathcal{O}(e)$. Let us remark that the identities in eqs. (38) and (39) can be derived in an even more transparent way by using eqs. (18)–(20) together with the Ward–Takahashi identities for the non–singular quantities $G^{\mu}_{(e)}$ and $G^{\mu\nu}_{(e)}$ in eqs. (26) and (27).
The two-body Ward–Takahashi identity in this case reads (see [4], [5] and references in [6])

\[-iq_{\mu}M^{\mu}(p_1', p_2'; p_1, p_2) = e_1\left(M(p_1' - q, p_2' - q_2; p_1, p_2) - M(p_1' - q_2; p_1, p_2)\right) - e_2\left(M(p_1' - q_2; p_1, p_2) - M(p_1' - q_2; p_1, p_2 - q_2)\right)\]  \hspace{1cm} (40)

note the details in the coupling of the photon to the anti-quark. For the sake of simplicity, we assume that the interaction will not be mediated by charged exchange particles (i.e. \(K, Q_f = 0\)). Furthermore, we define the total bound state momentum by \(p = q_1 p_1 - q_2 p_2\) (with \(q_1 + q_2 = 1\)), see also eq. (A0). With \(P' = P + q\), the two-body Ward–Takahashi identity in this new coordinates can be written as

\[-iq_{\mu}M^{\mu}(P', p'; P, p) = e_1\left(M(P' - q, P' - q_2; P, p) - M(P', p; P, p + q, p + q_2)\right) - e_2\left(M(P' - q, P' + q_1; P, p) - M(P', p; P, p + q_1 - q_2)\right)\]  \hspace{1cm} (41)

Now let us check the free Green’s function proportional to the free amplitude \(M_0^{\mu} = -ieG_0^{\mu} = -ieG_0\Gamma_0^{\mu}G_0\), explicitly written out in terms of one-particle propagators:

\[
iM_0^{\mu}(P', p'; P, p) = eG_0(P', p') \left[\Gamma_0^{\mu,1}(P', p'; P, p) + \Gamma_0^{\mu,2}(P', p'; P, p)\right] G_0(P, p)\]  \hspace{1cm} (42)

where \(\Gamma_0^{\mu}\) couples the photon to each fermion line of the free bound state propagator \(G_0\). Using the one-particle Ward–Takahashi identity in eq. (24), we thus find

\[
-iq_{\mu}M_0^{\mu}(P', p'; P, p) = -ie_1\left[S_F\left(\frac{P'}{2} + p\right) \otimes S_F\left(\frac{p'}{2} + p\right) - S_F\left(\frac{P'}{2} + p\right) \otimes S_F\left(\frac{p'}{2} + p\right)\right] - e_2\left[S_F\left(\frac{P'}{2} + p\right) \otimes S_F\left(\frac{p'}{2} + p\right) - S_F\left(\frac{P'}{2} + p\right) \otimes S_F\left(\frac{p'}{2} + p\right)\right] + e_1\left[S_F\left(\frac{P'}{2} + p\right) \otimes S_F\left(\frac{p'}{2} + p\right) - S_F\left(\frac{P'}{2} + p\right) \otimes S_F\left(\frac{p'}{2} + p\right)\right] \cdot (2\pi)^4\delta^4(p' - P - q)\delta^4(p' - p - q_2)\]  \hspace{1cm} (43)

We now define \(\tilde{e}_i = e_i / e\) and write down the related expressions for the amputated one-photon irreducible vertices \(e\Gamma^{\mu} = iG^{-1}\Gamma^{\mu}G^{-1}\) and \(e\Gamma_0^{\mu} = iG_0^{-1}\Gamma_0^{\mu}G_0^{-1}\):

\[
-iq_{\mu}\Gamma^{\mu}(P', p'; P, p) = \tilde{e}_1\left(G^{-1}(P', p'; P + q, p + q_2) - G^{-1}(P', p + q, p + q_2)\right) - \tilde{e}_2\left(G^{-1}(P', p'; P + q, p + q_2) - G^{-1}(P', p + q, p + q_2)\right)\]  \hspace{1cm} (44)

and

\[
-iq_{\mu}\Gamma_0^{\mu}(P', p'; P, p) = \tilde{e}_1\left(G_0^{-1}(P', p'; P + q, p + q_2) - G_0^{-1}(P', p + q, p + q_2)\right) - \tilde{e}_2\left(G_0^{-1}(P', p'; P + q, p + q_2) - G_0^{-1}(P', p + q, p + q_2)\right)\]  \hspace{1cm} (45)

here, the inverse free propagator \(\hat{G}_0\) (depending on primed and non-primed coordinates) is defined by
\[
\hat{G}_0^{-1}(P', p'; P, p) := \left[ S_{F}^{-1}\left(\frac{P}{2} + p\right) \otimes S_{F}^{-1}\left(-\frac{P}{2} + p\right) \right] \cdot (2\pi)^8 \delta^4(P' - P) \delta^4(p' - p) .
\]

(46)

If we recall \(G^{-1} = \hat{G}_0^{-1} + iK\) as well as the general relation \(\Gamma^\mu = \Gamma_0^\mu - iK^\mu\) for the one–photon irreducible vertex, then we easily find with \(P' = P + q\)

\[
- i\eta_\mu K^\mu(P', p'; P, p) = \hat{\epsilon}_1 \left( K(P' - q, p' - \eta_2q; P, p) - K(P', p'; P + q, p + \eta_2q) \right) - \hat{\epsilon}_2 \left( K(P' - q, p' + \eta_1q; P, p) - K(P', p'; P + q, p - \eta_1q) \right) .
\]

(47)

This is the full version of the constraint in eq. \(B\), formulated explicitly for a \(qq\) system without omitting the relative momenta. As we have pointed out in the last subsection, the divergence of \(K^\mu\) vanishes for \(q \to 0\). We now see in detail that this also holds if the interaction kernel \(K\) is independent of the total four–momentum \(P\) and is simultaneously of local type; then the terms on the right–hand side of eq. \(B\) mutually cancel if \(K \sim K(p' - p)\) is a convolution–like kernel.

In analogy to this derivation, we find the explicit formulation of the Ward–Takahashi identity for the two–photon irreducible interaction kernel of a \(qq\) system with \(P + q_1 = P' + q_2\):

\[
- i\eta_\mu K^{\mu\nu}(P', p'; P, p) = \hat{\epsilon}_1 \left( K^{\nu}(P' - q_1, p' - \eta_2q_1; P, p) - K^{\nu}(P', p'; P + q_1, p + \eta_2q_1) \right) - \hat{\epsilon}_2 \left( K^{\nu}(P' - q_1, p' + \eta_1q_1; P, p) - K^{\nu}(P', p'; P + q_1, p - \eta_1q_1) \right) .
\]

(48)

Note that for a deuteron described as a \(pn\) bound state \(i.e.\) with \(\hat{\epsilon}_1 = \hat{\epsilon}_p = 1\) and \(\hat{\epsilon}_2 = \hat{\epsilon}_n = 0\), the r.h.s. of this equation reduces to the difference in the first line. The resulting identity (up to a factor \(ie\) and in a slightly different notation) for the relative momentum parameter choice \(\eta_1 = \eta_2 = \frac{\pm 1}{2}\) is completely equivalent to eq. \(4.13\) in ref. \(\text{[1]}\); in this publication, the authors have found this constraint for the two–photon interaction current by demanding gauge invariance of the Compton scattering amplitude in a general Bethe–Salpeter framework. In fact, this represents a completely different approach to this issue compared to the one shown here. While we find the Ward–Takahashi identities for the \(n\)–photon irreducible interaction kernels in a rather general framework starting from the “gauging of equations method” applied to the Green’s function of a composite system, the authors of ref. \(\text{[1]}\) begin with the description of a special bound state \(i.e.\) the deuteron system; however, they keep a certain universality since they do not specify the interaction kernel in the Bethe–Salpeter equation. Then they consider Compton scattering off this system, include the lowest order contribution \(\sim \Gamma_0^{\mu\nu}\), re–scattering terms via \(T\) matrix diagrams and one–photon interaction kernels \(K^\mu\), and study the gauge invariance of the resulting tensor. They find that \(-i\eta_\mu K^{\mu\nu} = 0\) only holds if an additional two–photon irreducible term is included; this condition is then recasted in the condition quoted in eq. \(B\).

The direct derivation of the identity in eq. \(B\) is lengthy since for instance the (explicitly crossing–symmetric) subtraction terms in eq. \(\text{[3]}\) for a \(qq\) system read explicitly

\[
\left[ \Gamma_0^{\mu}(P', P' + q_2)G_0(P' + q_2)\Gamma_0^{\mu}(P + q_1, P) + \Gamma_0^{\mu}(P, P' - q_1)G_0(P' - q_1)\Gamma_0^{\mu}(P + q_2, P) \right]_{qq}
\]

\[\equiv \hat{\epsilon}_1^2 \Gamma_{0,1}^{\mu}(P', p' + \eta_2q; P' + q_2, p'' + \eta_2q_1)G_0(P' + q_2, p'' + \eta_2q_1; P + q_1, p''' + \eta_2q_1)\Gamma_0^{\mu}(P + q_1, p''' + \eta_2q_1; P, p) + \hat{\epsilon}_2^2 \Gamma_{0,2}^{\mu}(P', p' - \eta_1q; P' + q_2, p'' - \eta_1q_1)G_0(P' + q_2, p'' - \eta_1q_1; P + q_1, p''' - \eta_1q_1)\Gamma_0^{\mu}(P + q_1, p''' - \eta_1q_1; P, p) - \hat{\epsilon}_1\hat{\epsilon}_2 \Gamma_{0,1}^{\mu}(P', p' + Q; P' + q_2, p'' + \eta_2q_1)G_0(P' + q_2, p'' + \eta_2q_1; P + q_1, p''' + \eta_2q_1)\Gamma_0^{\mu}(P + q_1, p''' + \eta_2q_1; P, p) - \hat{\epsilon}_1\hat{\epsilon}_2 \Gamma_{0,1}^{\mu}(P', p' - Q; P' + q_2, p'' - \eta_1q_1)G_0(P' + q_2, p'' - \eta_1q_1; P + q_1, p''' - \eta_1q_1)\Gamma_0^{\mu}(P + q_1, p''' - \eta_1q_1; P, p) + \left\{ \begin{array}{c}
q_1 \leftrightarrow q_2 \\
\mu \leftrightarrow \nu
\end{array} \right\} ,
\]

where we have defined \(q := q_1 - q_2\) and \(Q := \eta_2q_1 + \eta_1q_2\); the definition of \(\Gamma_{0,i}^{\mu}\) can be read off eq. \(\text{[42]}\). Note that we still neither indicate the integrations \(\int d^4p''\) and \(\int d^4p'''\) nor the various indices; see eqs. \(\text{[1]}\) and \(\text{[2]}\) for comparison. Due to the complexity and the large number of terms occurring in the explicit \(qq\) formulation of the two–photon irreducible vertex \(\Gamma_{0,i}^{\mu}\), we feel that it might be more instructive to keep on omitting the relative momenta and the terms related to internal charges in the rest of this contribution since it will clarify the structure of the resulting expressions. We stress however that from contributions like those quoted above it is in fact straightforward to derive the explicit Ward–Takahashi identities for a \(qq\) system in the order \(O(e^2)\).
We have chosen the explicit example in eq. (49) not only to justify the suppression of relative momenta in this paper but also to illustrate the role of the subtraction terms \(-\Gamma_0^\mu G_0^\nu\Gamma_0^\mu - \Gamma_0^\mu G_0^\nu\). Note that if we write out the crossing symmetric terms in this example by exchanging \(q_1 \leftrightarrow -q_2\) and \(\mu \leftrightarrow \nu\) and by explicitly using the one-particle propagator formulation as in eq. (12), we find that the \(\tilde{e}_1 \tilde{e}_2\) parts are correctly produced but that a doubling of terms proportional to \(\tilde{e}_1 \tilde{e}_2\) occurs; these terms are essentially proportional to \(\gamma^\mu \otimes \gamma^\nu\) and \(\gamma^\nu \otimes \gamma^\mu\), respectively. This is precisely what happens in the lowest-order parts of \(+\Gamma^\mu G^\nu + \Gamma^\nu G^\mu\) — only with an opposite sign. By this mechanism, it is guaranteed that the only lowest-order terms surviving in the complete two-photon irreducible vertex \(\Gamma^\mu\) are collected in \(\Gamma_0^\mu\) without any double-counting.

**D. Identities for the Bethe–Salpeter Amplitudes**

Now that we have derived the Ward–Takahashi identities for the \(n\)-photon vertices and irreducible interaction kernels, we are able to give similar relations for the “extended” Bethe–Salpeter amplitudes of orders \(\mathcal{O}(e)\) and \(\mathcal{O}(e^2)\).

If we take the divergence of eq. (26), then we find immediately with the Ward–Takahashi identity for the one-photon vertex the relation

\[-iq_\mu \chi^\mu(P + q, P) = \chi(P)\]

(50)

here, we have used the Bethe–Salpeter equation \(G^{-1}(P)\chi(P) = 0\). Analogously, the relation for the second order Bethe–Salpeter amplitude reads

\[-iq_{1\mu} \chi^\mu\nu(P + q_1, P) = \chi^\nu(P - q_2, P)\]

(51)

As we have already stated in section III.B, the relations above are well defined for finite photon momenta.

**E. Differential Identities**

We will now consider differential forms of the relations in the preceding subsections. The limit \(q, q_1, q_2 \to 0\) yields \(P' \to P\); therefore we suppress all momentum dependencies in the following.

With \(\partial^\mu := \partial/\partial P_\mu\) or equivalently assuming infinitesimally small photon momenta, we find for the extended Green’s functions

\[\partial^\mu G = iG^\mu\]

(52)

and \[\partial^\nu G^\mu = iG^{\mu\nu}\]

(53)

From eqs. (28) and (31), we obtain the following relations for the one-photon and two-photon irreducible interaction kernels:

\[\partial^\mu K = iK^\mu\]

(54)

and \[\partial^\nu K^\mu = iK^{\mu\nu}\]

(55)

In order \(\mathcal{O}(e)\), the situation for the amputated Green’s function is simple. However, in the second order relation in eq. (31) no simple difference with shifted arguments occurs on the right-hand side. To derive a Ward identity for \(\Gamma^{\mu\nu}\), we therefore start with eq. (53) and recall that \(G^{\mu\nu} = GT^{\mu\nu}G\):

\[\partial^\nu G^\mu = \partial^\nu(GT^\mu G)\]

\[= (\partial^\nu G)T^\mu G + G(\partial^\nu T^\mu)G + GT^\mu (\partial^\nu G)\]

\[= (iG^{\mu\nu})T^\mu G + G(\partial^\nu T^\mu)G + GT^\mu (iG^{\mu\nu})\]

\[= G(i\Gamma^\mu T^\nu + i\Gamma^\nu T^\mu + \partial^\nu T^\mu)G\]

\[= iGT^{\mu\nu}G\]

For the one-photon and two-photon vertices, the corresponding relations with \(\Gamma := G^{-1}\) are therefore

\[\partial^\mu \Gamma = -i\Gamma^\mu\]

(56)

and \[\partial^\nu T^\mu = -i(\Gamma^\nu T^\mu + \Gamma^\mu T^\nu - \Gamma^{\mu\nu})\]

(57)
Again we find that the differential two-photon relation for the vertex is not a simple analogue to the one-photon case; however, both orders of the electromagnetic coupling have in common that exclusively non-singular terms appear since we can write \( \partial^\mu \Gamma^\nu = i \Gamma^\nu_{\text{NoPole}} \) for the last identity, see eq. (53). We will see in section VI that the non-singularity of the last equation will be crucial for the correct low-energy limit of the Compton scattering tensor; note that the sign in eq. (57) is essential for the related cancellation in the limit \( q_1 \to 0 \).

For the sake of completeness, we also quote the analogous results with \( \Gamma_0 := G_0^{-1} \) for the free vertices:

\[
\begin{align*}
\partial^\mu \Gamma_0 &= -i \Gamma_0^\mu \\
\partial^\nu \Gamma_0^\mu &= -i (\Gamma_0^\nu G_0 \Gamma_0^\mu + \Gamma_0^\mu G_0 \Gamma_0^\nu - \Gamma_0^\mu \Gamma_0^\nu) .
\end{align*}
\]

We will finish this section by studying differential identities for \( \chi \), \( \chi^\mu \) and \( \chi^{\mu\nu} \) comparable to those above. Let us anticipate that the following relations basing on eqs. (26) and (27) are derived in a quite formal way, i.e. regardless of the poles that will appear in the Green’s function \( G(P) \) for \( \tilde{P} \to P \) \( (P^2 = M^2) \) or equivalently for vanishing photon momenta.

To find the derivative of the Bethe–Salpeter amplitudes, we cannot start from the relation in eq. (50) since there no difference appears. However, we can exploit the Bethe–Salpeter equation and find

\[
\partial^\mu \chi = \partial^\mu (-iG_0 K \chi) = -i \left( (\partial^\mu G_0) K \chi + G_0 (\partial^\mu K) \chi + G_0 K (\partial^\mu \chi) \right)
\]

\[
= (\mathbf{1} + iG_0 K)^{-1} G_0 (\Gamma_0^\mu G_0 K \chi + K \chi)
\]

\[
= iG(\Gamma_0^\mu - iK^\mu \chi) = iG \Gamma^\mu \chi .
\]

The second derivative of \( \chi \) can be calculated analogously by using eq. (57) such that characteristic cancellations occur; we find with eqs. (26) and (27) the following relations:

\[
\partial^\mu \chi = i \chi^\mu = iG \Gamma^\mu \chi \\
\partial^\nu \chi^\mu = i \chi^{\mu\nu} = iG \Gamma^{\mu\nu} \chi .
\]

Similar equations hold for the adjoint amplitudes:

\[
\partial^\mu \bar{\chi} = i \bar{\chi}^\mu = i \bar{\chi} \Gamma^\mu G \\
\partial^\nu \bar{\chi}^\mu = i \bar{\chi}^{\mu\nu} = i \bar{\chi} \Gamma^{\mu\nu} G .
\]

Note that the bound state Bethe–Salpeter amplitudes are defined only for on-shell momenta \( P^2 = M^2 \) whereas the Green’s function \( G(P) \) generically includes off-shell contributions. Therefore the above-mentioned equations actually have to be considered with care: any derivative of \( \chi \) would infinitesimally shift the Bethe–Salpeter amplitude away from mass-shell in a region where \( \chi \) becomes ill-defined. However, we can formally introduce a quantity \( \chi(P) \) for off-shell momenta \( \tilde{P}^2 \neq M^2 \) analogous to the definition of \( \chi \) in eq. (5); then \( \chi(P) \) is not a solution of the Bethe–Salpeter equation but we can properly define the operation \( \partial^\mu \chi \). Additional terms due to \( G^{-1} \chi^{-1} \) will appear in eq. (51) that vanish in the limit \( \tilde{P}^2 \to M^2 \) in which we find the final relations in eqs. (52)–(54).

We shall remark here that the Green’s function \( G \) that enters in eqs. (51)–(54) exhibits a pole in the on-shell case and (for states with total momentum \( \tilde{P} \) and mass \( M \)) can be written (see e.g. (13)) as

\[
G(\tilde{P}) = -i \frac{\chi(P) \circ \bar{\chi}(P)}{P^2 - M^2} + \text{terms regular for } \tilde{P}^2 \to M^2 .
\]

Note that the Bethe–Salpeter amplitudes are on-shell quantities and therefore depend on \( P \) with \( P^2 = M^2 \). The physical implication of the dominant term in the Green’s function \( G(\tilde{P}) \) for the derivative of the Bethe–Salpeter amplitude and the related first order expanded amplitude \( \chi^{\mu} \) in the limit \( \tilde{P}^2 \to M^2 \) is depicted diagrammatically in fig. (12); an analogous diagram can be sketched for the second order Bethe–Salpeter amplitude \( \chi^{\mu\nu} \).

\[
\frac{\partial}{\partial P^\mu} \tilde{\chi} = i \tilde{\chi}^{\mu} = i \tilde{\chi} \tilde{\Gamma}^\mu G \sim \tilde{\chi} \tilde{\Gamma}^\mu \chi + \frac{1}{P^2 - M^2} \tilde{\chi} \tilde{\Gamma}^\mu \chi .
\]

FIG. 12. The Bethe–Salpeter amplitude expanded in first order of the electromagnetic field according to eq. (61).
We stress again that these differential relations for the Bethe–Salpeter amplitudes $\chi$ and $\bar{\chi}$ are derived with quite formal arguments and without discussing the apparent poles that enter the equations above. However, they allow for an alternative derivation of low–energy limit for the Compton scattering process as we shall show in section VTC.

V. GAUGE INVARIANCE

It is an important question whether the one–photon and two–photon vertices defined in the eqs. (12) and (13) satisfy gauge invariance if we evaluate them between Bethe–Salpeter amplitudes. To formulate this task more precisely, let us first define the electromagnetic current via

$$J^\mu := \langle P' | j^\mu(0) | P \rangle = -\bar{\chi}(P')\Gamma^\mu(P', P)\chi(P)$$

(66)

and the Compton scattering tensor via

$$T^{\mu\nu} := i \int d^4x \langle P' | Tj^\nu(x)j^\mu(0) | P \rangle e^{-iq_1x} = i\bar{\chi}(P')\Gamma^{\mu\nu}(P', P)\chi(P)$$

(67)

Here, $T$ denotes the time–ordered product. Gauge invariance implies that the divergences of $J^\mu$ and $T^{\mu\nu}$ vanish, i.e. that in momentum space

$$-iq_\mu J^\mu = 0 \quad \text{and} \quad -iq_\mu T^{\mu\nu} = -iT^{\mu\nu}q_{2\nu} = 0$$

holds. In the subsequent paragraphs, we will address this question for one–photon and two–photon amputated Green’s functions in various approximations. Let us furthermore stress that in the following two sections the momenta $P$ and $P'$ of the incoming and outgoing bound state are supposed to be on–shell (i.e. $P^2 = P'^2 = M^2$) since otherwise the Bethe–Salpeter equations could not be applied.

A. Electromagnetic Current

With the Ward–Takahashi identity for the one–photon vertex in eq. (30), it is easy to show that the divergence of the electromagnetic current vanishes (here, energy–momentum conservation requires $P' = P + q$):

$$-iq_\mu J^\mu = -\bar{\chi}(P')\left(-iq_\mu \Gamma^\mu(P', P)\right)\chi(P)$$

$$= -\bar{\chi}(P')(G^{-1}(P') - G^{-1}(P))\chi(P) = 0$$

(68)

Obviously it is trivial to show the gauge invariance of the electromagnetic current with the help of the Bethe–Salpeter equations $G^{-1}(P)\chi(P) = \bar{\chi}(P'G^{-1}(P')) = 0$. However, this requires that the one–photon vertex obeys the Ward–Takahashi identity in eq. (30) which is only true if the one–photon irreducible interaction kernel $K^\mu$ satisfies the relation in eq. (38). To see what happens if we neglect contributions of this one–photon kernel, i.e. if we only consider the lowest order vertex $\Gamma^\mu_0$, we again exploit the Bethe–Salpeter equations $G^{-1}_0(P)\chi(P) = -iK(P)\chi(P)$ and $\bar{\chi}(P')G^{-1}_0(P') = -i\bar{\chi}(P')K(P')$ and find

$$-iq_\mu J^\mu_0 = -\bar{\chi}(P')\left(-iq_\mu \Gamma^\mu_0(P', P)\right)\chi(P)$$

$$= -\bar{\chi}(P')(G^{-1}_0(P') - G^{-1}_0(P))\chi(P)$$

$$= i\bar{\chi}(P')(K(P') - K(P))\chi(P) = 0$$

This expression only vanishes if the interaction kernel does not depend on the bound states’ four–momenta. Furthermore it turns out that the kernel has to be of local type — i.e. $K(P; \{k_j\}, \{k'_j\}) = K(\{k_j - k'_j\})$ — if we explicitly write out the dependence on the relative momenta. This is true for kernels in ladder approximation but not necessarily for instantaneously approximated kernels (as e.g. used in [18][19]).

Note that, independent of the kernel type, we can state that $-iq_\mu J^\mu_0$ vanishes and therefore gauge invariance holds for $q = P' - P \rightarrow 0$. 

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B. Compton Scattering Tensor

As for the electromagnetic current, it is again easy to show that the divergence of the Compton scattering tensor vanishes if we take into account all contributions in \( \Gamma^{\mu\nu}(P', P) \). Then the Ward–Takahashi identity for the two–photon vertex in eq. (31) holds if the included \( n \)–photon irreducible interaction kernels obey eqs. (38) and (39). With the Bethe–Salpeter equations, we therefore find for \( P' = P + q_1 - q_2 \)

\[
-i q_{1\mu} T^{\mu\nu} = i \bar{\chi}(P') \left( -i q_{1\mu} \Gamma^{\mu\nu}(P', P) \right) \chi(P)
\]

\[
= i \bar{\chi}(P') (G^{-1}(P') G(P - q_2) \Gamma^{\nu}(P' - q_2, P) - \Gamma^{\nu}(P', P + q_1) G(P + q_1) G^{-1}(P)) \chi(P) = 0 .
\]

As an approximation, we now want to neglect the contributions from one–photon and two–photon irreducible interaction kernels. Then the subtraction term in eq. (43) cancels the lowest order part in \( \Gamma G \Gamma' \) — see eq. (11) — and we start with a modified two–photon vertex reading explicitly

\[
\hat{\Gamma}^{\mu\nu}(P', P) = \Gamma^{\mu\nu}_0(P', P) - i \Gamma^{\nu}(P', P' + q_2) T(P' + q_2) \Gamma^{\mu}_0(P + q_1, P) - i \tilde{\Gamma}^{\mu}_0(P', P' - q_1) T(P - q_2) \Gamma^{\nu}_0(P - q_2, P) ,
\]

where the \( T \) matrix satisfies eq. (11). The divergence of this vertex can easily be derived from the Ward–Takahashi identities for the free one–photon and two–photon vertices in eqs. (34) and (35):

\[
-i q_{1\mu} \hat{T}^{\mu\nu} = i \bar{\chi}(P') \left( -i q_{1\mu} \hat{\Gamma}^{\mu\nu}(P', P) \right) \chi(P)
\]

\[
= i \bar{\chi}(P') \left( -i K(P) G_0(P - q_2) - i(-i K(P')) T(P - q_2) + i G^{-1}_0(P - q_2, P) \chi(P) \right)
\]

\[
- i \bar{\chi}(P') \Gamma^{\nu}_0(P', P + q_1) \left( G_0(P + q_1) (-i K(P)) - i T(P + q_1) (-i K(P)) \right)
\]

\[
= \bar{\chi}(P') \left( K(P') G_0(P - q_2) - i T(P - q_2) G_0(P - q_2, P) \chi(P) \right)
\]

\[
- \bar{\chi}(P') \Gamma^{\nu}_0(P', P + q_1) \left( G_0(P + q_1) K(P) - i T(P + q_1) K(P) - G_0(P + q_1) K(P + q_1) \right)
\]

\[
+ i T(P + q_1) K(P + q_1) \chi(P)
\]

\[
= \bar{\chi}(P') \left( K(P') - K(P - q_2) \right) \left( G_0(P - q_2) - i T(P - q_2) \Gamma^{\nu}_0(P - q_2, P) \chi(P) \right)
\]

\[
+ \bar{\chi}(P') \Gamma^{\nu}_0(P', P + q_1) \left( G_0(P + q_1) - i T(P + q_1) \Gamma^{\nu}_0(P + q_1, P) \chi(P) \right) .
\]

With \( G = G_0 - i T \) and defining \( \bar{\chi}^{\mu} := G T^{\nu} \chi \) (where \( \tilde{\Gamma}^{\mu} = \Gamma^{\mu}_0 \)) in analogy to eqs. (22) and (23), we can therefore write the divergence of Compton scattering tensor without inclusion of \( n \)–photon irreducible interaction kernels as

\[
-i q_{1\mu} \tilde{T}^{\mu\nu} = \bar{\chi}(P') \left( K(P') - K(P - q_1) \right) \bar{\chi}^{\nu}(P - q_2, P)
\]

\[
+ \bar{\chi}^{\nu}(P', P' + q_2) \left( K(P + q_1) - K(P) \right) \chi(P) .
\]
This expression vanishes — as in the similar case of the lowest–order electromagnetic current, see eq. (68) — in the limit \( q_1, q_2 \to 0 \) (\( P' \to P \)). The gauge invariance condition \(-iq_1 \mu \hat{T}^{\mu\nu} = 0\) also holds if we use an interaction kernel \( K(P) \) that does not depend on the total four–momenta of the bound states. As it has been pointed out in the preceding subsection for the lowest order current \( J_0^\mu \), an inclusion of all relative momenta in the calculation shows that the kernel must also be local in order to satisfy gauge invariance in this case. This result has also been found in \([11]\) where an explicit calculation for deuteron Compton scattering has been presented. We agree completely with the authors who stress that this observation emphasizes the necessity of the inclusion of \( n\)–photon irreducible interaction kernels if the constituents of the composite system interact via non–local potentials.

The last point to study is the question of gauge invariance for the lowest order two–photon vertex \( \Gamma_0^{\mu\nu} \). If we recall the free Ward–Takahashi identity for this vertex in eq. (70), we find immediately

\[
- iq_1 \mu T_0^{\mu\nu} = i \bar{\chi}(P')( -iq_1 \mu \Gamma_0^{\mu\nu}(P', P)) \chi(P) \\
= i \bar{\chi}(P')(G_0^{-1}(P') G_0(P - q_2) \Gamma_0^\nu(P - q_2, P) - \Gamma_0^\nu(P', P' + q_1) G_0(P + q_1) G_0^{-1}(P)) \chi(P) \ .
\]

Without applying the Bethe–Salpeter equation, we see that this lowest order expression only vanishes for \( q_1, q_2 \to 0 \). It is clear that in this limit the re–scattering terms due to the \( T \) matrix will not contribute; therefore gauge invariance in the very–low–energy regime will be approximately restored even without explicit inclusion of intermediate states.

VI. LOW–ENERGY LIMITS

In this section, we will study the low–energy behaviour of the one–photon expression \( \bar{\chi} T^{\mu\nu} \chi \) and the two–photon expression \( \bar{\chi} \Gamma^{\mu\nu} \chi \). We will restrict our discussions to bound states with total angular momentum \( J = 0 \) so that we will not have to consider technical subtleties originating from non–zero spins; however, our statements remain valid also for spin–averaged amplitudes describing Compton scattering \( e.g. \) off a proton or a neutron with total spin \( \frac{1}{2} \).

In the first subsection, we will give a concise formalism how to include the correct charge factors. To clarify this point, we will pin down the lowest order results for the electromagnetic current and the Compton scattering tensor in explicit expressions for the bound state being a (pseudoscalar or scalar) \( q\bar{q} \) meson as an example.

A. Charge Factors

If we now study the low–energy limits of the electromagnetic current and the Compton scattering tensor, a caveat is in order at this point. Up to now, we have considered the photon coupling in Dirac space only, which was of the (correct) vector type \( \gamma^\mu \); we thus neglected the (in the general case iso–spin dependent) charge operator \( Q_f \) in flavour space that reads \( e.g. \) for hadronic bound states

\[
Q_f = \epsilon \begin{pmatrix}
\frac{2}{3} & 0 & 0 \\
0 & -\frac{1}{3} & 0 \\
0 & 0 & -\frac{1}{3}
\end{pmatrix}
\]  

(70)

for constituents being \( u, d \) and \( s \) quarks (see also \([2]\) for a different formulation of this issue). This observation does not spoil our results on Ward–Takahashi identities and gauge invariance but it turns out in this section that it will become important for the low–energy limits. Instead of the operator in eq. (70), we simply get \( Q_f = 1 \) if we introduce our vector coupling in Dirac space by a derivative of the one–particle propagators. However, these fermion propagators \( S_F(p_i) \) are functions of the momentum of the \( i \)–th constituent which depends in a simple way of the total momentum \( P \) (see appendix \([3]\) ). This dependence is given in eq. (73) and yields

\[
\frac{\partial}{\partial P_\mu} S_F(p_j) = -i \eta_j S_F(p_j) \gamma_\mu S_F(p_j) \ .
\]

As described in the appendix, the \( \eta_j \)'s are parameters that set a special choice of relative momenta; we can fix them by \( \eta_j = e_j/Q \) with \( Q = \sum_j e_j \) being the total charge and \( e_j \) being the charge of the \( j \)–th constituent.

We now have to specify what we mean in detail with the lowest order one–photon vertex \( \Gamma_0^\mu \) for a composite system of \( N \) constituents. Recalling \( G_0^{-1} =: \Gamma_0 \) where the total momentum is unchanged by \( G_0 \) (\( i.e. \) \( P = \sum_j p_j = \sum_j p'_j \)), we state with eq. (68) that
The coordinate choice $\eta$ with $i \in \mathbb{G}$ 

With the Ward–Takahashi in eq. (56), recalling terms of Bethe–Salpeter amplitudes and the one–photon vertex — see eq. (66) —, we easily find the correct form factor normalization for $q$-charged operator $Q$.

In the following subsections, we will give — together with the general results — also the explicit lowest order formulae holds. This procedure can also be applied to the full one–photon vertex $\Gamma$ to make clear that these re–definitions give the correct and complete matrix elements for electromagnetic interactions of $q\bar{q}$ mesons to make clear that these re–definitions give the correct and complete matrix elements for electromagnetic form factors and Compton scattering.

\[ i \frac{\partial}{\partial P_{\mu}} G_{0}^{-1}(P, \{k_{i}'\}, \{k_{j}\}) = i \frac{\partial}{\partial P_{\mu}} G_{0}^{-1}(\{p_{j}'\}, \{p_{j}\}) \]

\[ = \Gamma_{0}^{\mu}(\{p_{j}'\}, \{p_{j}\}) = \sum_{j} \Gamma_{0,j}^{\mu}(\{p_{j}'\}, \{p_{j}\}) \]

\[ = \eta_{1} \cdot \gamma^{\mu} \otimes S_{F}^{-1}(p_{2}) \otimes S_{F}^{-1}(p_{3}) \otimes \ldots \otimes S_{F}^{-1}(p_{N}) \cdot \delta_{1}(\{p_{j}' - p_{j}\}) + \eta_{2} \cdot S_{F}^{-1}(p_{1}) \otimes \gamma^{\mu} \otimes S_{F}^{-1}(p_{3}) \otimes \ldots \otimes S_{F}^{-1}(p_{N}) \cdot \delta_{2}(\{p_{j}' - p_{j}\}) + \eta_{3} \cdot S_{F}^{-1}(p_{1}) \otimes S_{F}^{-1}(p_{2}) \otimes \gamma^{\mu} \otimes \ldots \otimes S_{F}^{-1}(p_{N}) \cdot \delta_{3}(\{p_{j}' - p_{j}\}) + \ldots + \eta_{N} \cdot S_{F}^{-1}(p_{1}) \otimes \ldots \otimes \gamma^{\mu} \cdot \delta_{N}(\{p_{j}' - p_{j}\}) \]

with $i = 1, 2, \ldots, N - 1$ and $j, k = 1, 2, \ldots, N$ and the short–hand notation

\[ \delta_{k}(\{p_{j}' - p_{j}\}) := (2\pi)^{4(N-1)} \cdot \delta^{4}(p_{1}' - p_{1}) \delta^{4}(p_{2}' - p_{2}) \ldots \delta^{4}(p_{k-1}' - p_{k-1}) \delta^{4}(p_{k+1}' - p_{k+1}) \ldots \delta^{4}(p_{N}' - p_{N}) \]

The coordinate choice $\eta_{j} = e_{j}/Q$ induces a re–definition of the free vertex in eq. (71) by

\[ \Gamma_{0}^{\mu} = \sum_{i=1}^{N} \Gamma_{0,i}^{\mu} \quad \rightarrow \quad \frac{1}{Q} \Gamma_{0}^{\mu} = \frac{1}{Q} \sum_{i=1}^{N} \Gamma_{0,i}^{\mu} \]

such that now

\[ \Gamma_{0,i}^{\mu} = e_{j} \cdot S_{F}^{-1}(p_{1}) \otimes \ldots \otimes S_{F}^{-1}(p_{j-1}) \otimes \gamma^{\mu} \otimes S_{F}^{-1}(p_{j+1}) \otimes \ldots \otimes S_{F}^{-1}(p_{N}) \cdot \delta_{j}(\{p_{j}' - p_{j}\}) \]

holds. This procedure can also be applied to the full one–photon vertex $\Gamma^{\mu}$ where also contributions from the one–photon irreducible interaction kernel $K^{\mu}$ are included; in all terms that are considered the photon can be coupled to each fermion line in the corresponding diagram. Analogous re–definitions for the two–photon vertices, i.e.

\[ \Gamma^{\mu\nu} \rightarrow \frac{1}{Q} \Gamma^{\mu\nu} \]

\[ \text{B. Current Normalization} \]

From Lorentz invariance and charge conjugation symmetry, one can deduce the matrix element of the electromagnetic current to be of the on–shell form $J_{\mu} = Q \cdot (P + P')^{\mu} f(q^{2})$ where $f(q^{2})$ is the form factor with the normalization $f(0) = 1$ and $Q$ is the total charge of the bound state. In the limit of vanishing photon momentum $q = P' - P \rightarrow 0$, we therefore find

\[ \lim_{q \rightarrow 0} J_{\mu}^{\mu} = \lim_{q \rightarrow 0} (P' | j_{\mu}(0) | P) = Q \cdot 2P^{\mu} \]

Let us now consider the normalization condition of the Bethe–Salpeter amplitude describing a bound state of momentum $P$ and mass $M$ (see e.g. [12]):

\[ \bar{\chi}(P) \left( \frac{\partial}{\partial P_{\mu}} \left( G_{0}^{-1}(P) + iK(P) \right) \right) \big|_{P^{2} = M^{2}} = i 2P^{\mu} \]

With the Ward–Takahashi in eq. (56), recalling $G^{-1} =: \Gamma$ and by comparison with the definition of the current in terms of Bethe–Salpeter amplitudes and the one–photon vertex — see eq. (66) —, we easily find the correct form factor normalization for $q \rightarrow 0'$.
\[ - \bar{\chi}(P)\Gamma^\mu(P,P)\chi(P) = Q \cdot 2P^\mu. \]  

(73)

The total charge \( Q \) on the right–hand side comes from the re–definition of the vertex in eq. (73) acting such that a modified Ward–Takahashi identity \( \partial^\mu G^{-1} = -i \frac{1}{Q} \Gamma^\mu \) has to be applied. This low–energy limit is even correct if we neglect contributions from the one–photon irreducible interaction kernel \( K^\mu \). Due to the Ward identity \( \partial^\mu K = iK^\mu \), the interaction kernel \( K \) then must be independent of the total momentum \( P \) (as we have also found for the gauge invariance of the lowest order current \( J^\mu_0 \)). We thus can apply the Ward–Takahashi identity in eq. (58) for the free one–photon vertex:

\[ - \bar{\chi}(P)\Gamma^\mu_0(P,P)\chi(P) = Q \cdot 2P^\mu . \]  

(74)

As an example, we give the left–hand side of this last expression explicitly for a \( q\bar{q} \) bound state and a non–vanishing photon momentum \( q \neq 0 \) (note that here the anti–quark is defined with momentum \( -p_2 \) and charge \( -e_2 \)):

\[ J^\mu \vert_{q\bar{q}} = - e_1 \int \frac{d^4p}{(2\pi)^4} \bar{\chi}(P,p + \eta_2 q) \left( \gamma^\mu \otimes S_F^{-1}(-\eta_2 P + p) \right) \chi(P,p) \]

\[ + e_2 \int \frac{d^4p}{(2\pi)^4} \bar{\chi}(P,p - \eta_1 q) \left( S_F^{-1}(\eta_1 P + p) \otimes \gamma^\mu \right) \chi(P,p) . \]

The charge factors introduced in the preceding subsection are obviously correct; the indices of \( \bar{\chi}\Gamma^\mu_0\chi \) therefore must only be contracted in Dirac space (or, to be precise, the contraction in flavour space will yield a factor 1 due to the unaltered charge operator \( Q_f = 1 \)).

C. Born–Thomson Limit

For charged particles, the Compton scattering tensor includes not only structure dependent terms but also pole contributions in order \( O(\omega^0) \) with \( \omega = (q_1, q_2) \); then the Born part of \( T^{\mu\nu} \) in soft–photon approximation is known to be (see e.g. [22])

\[ T^{\mu\nu}_{\text{Born}} = Q^2 \cdot f(q_1^2) f(q_2^2) \left[ \frac{(2P + q_1)^\mu (2P' + q_2)^\nu}{s - M^2} + \frac{(2P' - q_1)^\mu (2P - q_2)^\nu}{u - M^2} \right] - 2g^{\mu\nu} \]  

(75)

with the Mandelstam variables \( s = (P + q_1)^2 = (P' + q_2)^2 \) and \( u = (P - q_2)^2 = (P' - q_1)^2 \) and the one–photon form factors \( f(q_i^2) \) defined as in the preceeding subsection. Note that \( T^{\mu\nu}_{\text{Born}} \) is explicitly gauge invariant due to the non–singular term proportional to \( g^{\mu\nu} \); the three terms in eq. (75) can be identified with the diagrams (a), (b) and (c) in fig. (13). The factor \( Q^2 \) forces the amplitude to vanish for neutral bound states in the low–energy limit.

![FIG. 13. Born terms for Compton scattering off a spin–0 particle.](image)

In the non–relativistic limit (i.e. for \( q_i \to 0 \)), only the Born terms in the full tensor \( T^{\mu\nu} \) survive. The amplitude \( T \) for Compton scattering off a bound state of mass \( M \) with spacelike photon polarization vectors \( \varepsilon_i = (0, \varepsilon_i) \) (and therefore \( q_i \cdot \varepsilon_i = 0 \) and \( P \cdot \varepsilon_i = P' \cdot \varepsilon_i = 0 \) for real photons) then yields the celebrated result of Low, Gell–Mann and Goldberger (see [21,23,24]):

\[ \lim_{q_i \to 0} T = - \frac{1}{2M} \lim_{q_i \to 0} \varepsilon_1^\mu T^{\mu\nu} \varepsilon_2^\nu = - \frac{Q^2}{M} (\varepsilon_1 \cdot \varepsilon_2) . \]  

(76)

Obviously, the pole terms in eq. (75) do not contribute to the Compton amplitude in the very–low energy limit; note furthermore that here and in the following we have used \( f(0) = 1 \) for real photons.

Now we want to show that the full two–photon vertex \( \Gamma^{\mu\nu} \) in fact yields this limit for \( q_1, q_2 \to 0 \) \( (P' \to P) \). Let us therefore consider eq. (73) and make some remarks on the right–hand side of this current normalization. Formally,
the factor $2P^\mu$ originates from a differentiation of the free boson propagator $\Delta_F(P) = (P^2 - M^2)^{-1}$ and a subsequent amputation of this vertex in the limit $q \to 0$ (see e.g. [26]):

$$
\dot{J}^\mu := \lim_{q \to 0} J^\mu = \Delta_F^{-1}(P) \left[ -Q \frac{\partial}{\partial P^\mu} \Delta_F(P) \right] \Delta_F^{-1}(P) = Q \cdot \frac{\partial}{\partial P^\mu} \Delta_F^{-1}(P) = Q \cdot 2P^\mu .
$$

(77)

The result is an amputated (point–form) boson–photon vertex; if we once more take the derivative $-Q \partial / \partial P_\nu$ then we end up with a two–photon vertex of non–singular form:

$$
\dot{T}_{\text{NoPole}}^{\mu\nu} := \lim_{q_i \to 0} T_{\text{NoPole}}^{\mu\nu} = -Q \cdot \frac{\partial}{\partial P_\nu} \dot{J}^\mu = -Q^2 \cdot \frac{\partial}{\partial P_\nu} \frac{\partial}{\partial P_\mu} \Delta_F^{-1}(P) = -Q^2 \cdot 2g^{\mu\nu} ;
$$

(78)

see diagram (e) in fig. (13). $\dot{T}_{\text{NoPole}}^{\mu\nu}$ is equivalent to the non–singular part of $T_{\text{Born}}^{\mu\nu}$ in the limit $q_i, q_i^2 \to 0$, see eq. (73). It can be recovered in the Bethe–Salpeter formulation by taking the derivative of the left–hand side of eq. (73) and applying $-Q \partial / \partial P_\nu$ only to the amputated (point–form) two–photon vertex:

$$
-\chi(P) \left( -Q \frac{\partial}{\partial P_\nu} \Gamma^\mu(P, P_\nu) \right) \chi(P)
$$

(79)

$$
= \lim_{q_i \to 0} i \chi(P) \left( \Gamma^{\mu\nu}(P', P) - \Gamma^{\nu\mu}(P', P' + q_2) G(P' + q_2) \Gamma^{\mu\nu}(P + q_1, P) \right. 
\left. - \Gamma^{\mu\nu}(P', P' - q_1) G(P - q_2) \Gamma^{\mu\nu}(P - q_2, P) \right) \chi(P)
$$

$$
= \lim_{q_i \to 0} i \chi(P) \Gamma_{\text{NoPole}}^{\mu\nu}(P', P) \chi(P)
$$

Here we have used that $\partial \Gamma^{\mu\nu} = \frac{\partial}{\partial P_\nu} \Gamma_{\text{NoPole}}^{\mu\nu}$; see eq. (57) and eq. (72) for the vertex’ re–definition. In analogy to eq. (4), we now split the Compton scattering tensor into two parts by defining $T^{\mu\nu} = T_{\text{Fole}}^{\mu\nu} + T_{\text{NoPole}}^{\mu\nu}$ where $T_{\text{Fole}}^{\mu\nu}$ includes all pole terms. Comparison with eq. (77) then yields

$$
\lim_{q_i \to 0} T_{\text{NoPole}}^{\mu\nu} = \lim_{q_i \to 0} i \chi(P) \Gamma_{\text{NoPole}}^{\mu\nu}(P', P) \chi(P) = -Q^2 \cdot 2g^{\mu\nu}
$$

(80)

which is the correct low–energy limit for the non–singular terms in the Compton scattering tensor. In the following, we will only work in the limit $q_i \to 0$ (i.e. $P' \to P$), therefore skip the dependence on the total four–momentum $P$ and use the abbreviation $\partial^\nu := \frac{\partial}{\partial P_\nu}$.

Let us now study the pole contributions in the low–energy limit of $T^{\mu\nu}$ which are necessary ingredients of the full tensor with respect to gauge invariance. First, we will investigate the second line of eq. (79) and find that the terms $\Gamma^{\mu\nu} G_{\mu\nu}$ accurately reveal the correct pole structure. This is due to the fact that the full Green’s function $G(\bar{P})$ has a pole for $\bar{P}^2 \to M^2$ (and therefore in the limit of vanishing photon momenta) with the residue $-i\chi \circ \bar{\chi}$, see eq. (33). Recasting the whole expression and using eq. (80) yields

$$
\lim_{q_i \to 0} \left( \check{\chi} \Gamma^{\mu\nu} \chi \cdot \frac{1}{s - M^2} \cdot \check{\chi} \Gamma^{\nu\mu} \chi \cdot \frac{1}{u - M^2} \cdot \check{\chi} \Gamma^{\mu\nu} + T_{\text{NoPole}}^{\mu\nu} \right) = \lim_{q_i \to 0} i \chi \Gamma^{\mu\nu} \chi .
$$

(81)

Since $J^\mu = -\check{\chi}(P') \Gamma^{\mu\nu}(P', P) \chi(P) = Q(P + P')^\mu f(q^2)$ with $q = P' - P$ holds for the electromagnetic current, we recover the full Born term $T_{\text{Born}}^{\mu\nu}$ of eq. (73) in the limit $q_i \to 0$ on the left–hand side of this equation; it includes all pole terms plus a non–singular contribution that restores gauge invariance. Only these Born terms contribute to the low–energy limit of Compton scattering because they are of zeroth order in the photon momenta; we therefore find

$$
\lim_{q_i \to 0} T_{\text{NoPole}}^{\mu\nu} = \lim_{q_i \to 0} \left( T_{\text{Fole}}^{\mu\nu} + T_{\text{NoPole}}^{\mu\nu} \right) = \lim_{q_i \to 0} i \chi \Gamma^{\mu\nu} \chi
$$

and thus end up with the correct Born–Thomson limit for the Compton scattering amplitude if we use the full two–photon irreducible vertex $\Gamma^{\mu\nu}$:

$$
\lim_{q_i \to 0} \mathcal{T}_{\mid_{\Gamma^{\mu\nu}}} = -\frac{1}{2M} \lim_{q_i \to 0} \varepsilon_1{\mu} T_{\mu\nu} \varepsilon_{2\nu} = -\frac{Q^2}{M} (\varepsilon_1 \cdot \varepsilon_2) .
$$

There is also an alternative way to demonstrate the correct low–energy behaviour of the full two–photon irreducible vertex $\Gamma^{\mu\nu}$. We recall that we have only used results from differentiations of amputated (point–form) operators such as $J^\mu$ and $\Gamma^\mu$. The pole terms can also be found if we define $\tilde{T}^{\mu\nu}$ in analogy to $\dot{J}^\mu$ as a twofold differentiation of the
free boson propagator $\Delta_P(P)$ with respect to the momentum $P$ followed by an amputation of the resulting vertex in the limit $q \to 0$ (see e.g. [20]):

$$\tilde{T}^{\mu \nu} := \lim_{q_i \to 0} T^{\mu \nu} = \Delta_P^{-1} \left[ (-Q \partial^\nu) (-Q \partial^\mu) \Delta_F \right] \Delta_F^{-1} = J^\mu \Delta_F J^\nu + J^\nu \Delta_F J^\mu - Q^2 \cdot 2 g^{\mu \nu}$$  \hspace{1cm} (83)

Inserting the explicit expressions for $J^\mu$ and $\Delta_F$, we find the low–energy limit of the Born term in eq. (74) including all pole contributions. This procedure is equivalent to a differentiation of the full expression $-\bar{\chi} \Gamma^\nu \chi$ in eq. (73); in this way, we do not only take into account the derivative of the amputated (point–form) vertex $\Gamma^\mu$ but also include terms proportional to $\partial^\nu \chi$ and $\partial^\nu \bar{\chi}$ which will then re–introduce the pole terms that are also present in $\tilde{T}^{\mu \nu}$. With the differential identities for $\partial^\mu \Gamma^\nu$ and for the Bethe–Salpeter amplitudes in eqs. (77), (81) and (83), and from the vertex’ re–definition in eq. (72), we recover

$$-(-Q \partial^\nu) \bar{\chi} \Gamma^\mu \chi = Q \left( (\partial^\nu \bar{\chi}) \Gamma^\mu + \bar{\chi} (\partial^\nu \Gamma^\mu) \chi + \bar{\chi} \Gamma^\mu (\partial^\nu \chi) \right)$$

$$= i \left( \bar{\chi} \Gamma^\mu G^\nu \chi + \bar{\chi} (\bar{\Gamma}^{\mu \nu} - \Gamma^\mu G^\nu \chi) + \bar{\chi} \Gamma^\mu G^\nu \chi \right)$$

$$= i \bar{\chi} \Gamma^\mu \chi .$$

For the relevance of the formal differentiation of the Bethe–Salpeter amplitudes $\chi$ and $\bar{\chi}$ which are properly defined only on mass shell, see the remarks in section IV E. It is crucial for obtaining this correct result that the Ward identity on mass shell, see the remarks in section IV E. It is crucial for obtaining this correct result that the Ward identity $\bar{\chi} \partial^\nu \chi$ in the static limit for interaction kernels $K^\mu$ that are independent of the total momentum, see also eq. (74), we find

$$\lim_{q_i \to 0} T^{\mu \nu} = \lim_{q_i \to 0} (T^{\mu \nu}_{\text{POLE}} + T^{\mu \nu}_{\text{NOPOLE}}) = \lim_{q_i \to 0} i \bar{\chi} \Gamma^\mu \chi .$$  \hspace{1cm} (84)

However, we shall note here that we prefer the first derivation which can be formulated without using terms like $\partial^\nu \chi$ and $\partial^\nu \bar{\chi}$ which are not strictly defined for on–shell quantities like Bethe–Salpeter amplitudes; note that the relations in eqs. (71) and (73) have been found on quite formal grounds.

The Born–Thomson limit can also be derived if we only include re–scattering terms due to $T$ matrix contributions but neglect one–photon or two–photon irreducible interaction kernels. This can be seen by applying the same procedure as before to the lowest order current matrix element in eq. (74):

$$\tilde{T}^{\mu \nu}_{\text{NOPOLE}} = -Q^2 \cdot 2 g^{\mu \nu} = -Q \partial^\nu \tilde{J}^\mu = -\bar{\chi} \left( -Q \partial^\nu \Gamma^\mu \right) \chi$$

$$= i \bar{\chi} (\Gamma^\mu \Gamma^\nu - \Gamma^\mu \Gamma^\nu - \Gamma^\mu G^\nu - \Gamma^\mu G^\nu - \Gamma^\mu G^\nu - \Gamma^\mu G^\nu) \chi .$$

here, we have only considered the differentiation of the amputated (point–form) operators and used that $G = G_0 - i T$. We thus find $\hat{\Gamma}^{\mu \nu} \to \hat{\Gamma}^{\mu \nu}$ as the modified tensor defined in eqs. (11), (84) and (82) as well as pole terms that can be re–arranged to the left–hand side. For $q_i \to 0$, the pole terms in the Green’s function $G$ dominate all other contributions; analogously to the derivation of the low–energy limit of the full two–photon vertex, we are led to

$$\lim_{q_i \to 0} \left( \bar{\chi} \Gamma^\mu_0 \chi \cdot \frac{1}{s - M^2} \cdot \bar{\chi} \Gamma^\mu_0 \chi + \bar{\chi} \Gamma^\mu_0 \chi \cdot \frac{1}{u - M^2} \cdot \bar{\chi} \Gamma^\mu_0 \chi + T^{\mu \nu}_{\text{NOPOLE}} \right) = \lim_{q_i \to 0} i \bar{\chi} \hat{\Gamma}^{\mu \nu} \chi .$$  \hspace{1cm} (86)

Since the one–photon irreducible interaction current $K^\mu$ (which is neglected in this approximation) does not contribute in the static limit for interaction kernels $K$ that are independent of the total momentum, see also eq. (74), we find the correct low–energy behaviour for the two–photon vertex $\hat{\Gamma}^{\mu \nu}$:

$$\lim_{q_i \to 0} T^{\mu \nu} = \lim_{q_i \to 0} (T^{\mu \nu}_{\text{POLE}} + T^{\mu \nu}_{\text{NOPOLE}}) = \lim_{q_i \to 0} i \bar{\chi} \hat{\Gamma}^{\mu \nu} \chi .$$  \hspace{1cm} (87)

This leads to the Born–Thomson limit for the Compton amplitude although we neglected contributions from $n$–photon irreducible interaction kernels in $\tilde{T}^{\mu \nu} := i \bar{\chi} \hat{\Gamma}^{\mu \nu} \chi$: 

$$\lim_{q_i \to 0} T |_{\Gamma^{\mu \nu}} = -\frac{1}{2 M} \lim_{q_i \to 0} \varepsilon_1 \varepsilon_2 \varepsilon_{\mu \nu} = -\frac{Q^2}{M} (\varepsilon_1 \cdot \varepsilon_2) .$$
Let us finally note that we cannot expect to recover the correct low–energy limit if we only consider the lowest order contribution to Compton scattering in $T_0^{\mu\nu} := i\bar{\chi}\Gamma_0^{\mu\nu}\chi$:

$$\lim_{q_i \to 0} T_{\gamma_i} = \frac{-1}{2M} \lim_{\xi_1 \to 0} \varepsilon_{1\mu} T_0^{\mu\nu} \varepsilon_{2\nu} \neq -\frac{Q^2}{M} (\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2).$$

This is obvious from the considerations given above: if we do not include pole contributions then we will not be able to produce the full gauge invariant Born term, see eq. (77). However, in contrast to the $-Q \cdot 2g^{\mu\nu}$ part, these pole terms do not contribute to the Compton amplitude $T$; but for the correct reproduction of the limit $-\frac{Q^2}{M} (\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)$ we must at least include the full non–singular tensor $T_{\mu\nu}^{\text{NoPole}}$ and not only its lowest–order part, see eq. (80).

As an example, we will finally write down explicitly the lowest–order Compton scattering tensor $T_0^{\mu\nu} = i\bar{\chi}\Gamma_0^{\mu\nu}\chi$ for a $q\bar{q}$ system with finite photon momenta $q_1, q_2 \neq 0$:

$$T_0^{\mu\nu} |_{q\bar{q}} = -ie_1 e_2 \int \frac{d^4p}{(2\pi)^4} \bar{\chi}(P, p + \eta_2 q_1 + \eta_1 q_2) (\gamma^\mu \otimes \gamma^\nu) \chi(P, p)$$

$$-ie_1 e_2 \int \frac{d^4p}{(2\pi)^4} \bar{\chi}(P, p - \eta_1 q_1 - \eta_2 q_2) (\gamma^\nu \otimes \gamma^\mu) \chi(P, p)$$

$$+ie_1^2 \int \frac{d^4p}{(2\pi)^4} \bar{\chi}(P, p + \eta_2 q_1 - \eta_2 q_2) (\gamma^\nu S_F(\eta_1 P + p + q_1) \gamma^\mu \otimes S_F^{-1}(-\eta_2 P + p)) \chi(P, p)$$

$$+ie_1^2 \int \frac{d^4p}{(2\pi)^4} \bar{\chi}(P, p + \eta_1 q_1 - \eta_2 q_2) (\gamma^\mu S_F(\eta_1 P + p - q_2) \gamma^\nu \otimes S_F^{-1}(-\eta_2 P + p)) \chi(P, p)$$

$$+ie_2^2 \int \frac{d^4p}{(2\pi)^4} \bar{\chi}(P, p - \eta_1 q_1 + \eta_1 q_2) (S_F^{-1}(\eta_1 P + p) \otimes \gamma^\nu S_F(-\eta_2 P + p - q_1) \gamma^\mu) \chi(P, p)$$

$$+ie_2^2 \int \frac{d^4p}{(2\pi)^4} \bar{\chi}(P, p - \eta_1 q_1 + \eta_2 q_2) (S_F^{-1}(\eta_1 P + p) \otimes \gamma^\mu S_F(-\eta_2 P + p + q_2) \gamma^\nu) \chi(P, p).$$

Again, the correct charge factors are found for the different terms contributing to $T_0^{\mu\nu}$; note that the anti–quark with momentum $-p_2 = -P/2 + p$ has the charge $-e_2$. Six diagrams emerge from this lowest order two–photon vertex; if $T$ matrix contributions are taken into account and interaction currents are still neglected, eight additional terms will have to be considered for a $q\bar{q}$ system.

**VII. SUMMARY AND CONCLUSIONS**

We have derived the second order Green’s function $G^{\gamma\gamma}$ for Compton scattering off a bound state in a scheme that the authors of ref. [1] (where it has been introduced for processes in first order of the electromagnetic coupling) have labelled “gauging a hadronic system”. The resulting full two–photon vertex $\Gamma^{\gamma\gamma} = G^{-1}G^{\gamma\gamma}G^{-1}$ includes the lowest order contribution $\Gamma_0^{\gamma\gamma}$ and an explicit two–photon irreducible interaction kernel $K^{\gamma\gamma}$ but also re–scattering terms via an intermediate full Green’s function $\Gamma^{\gamma\gamma}G^{\gamma\gamma}$ (with $G = G_0 - iT$) as well as subtraction terms $\Gamma_0^{\gamma\gamma}G_0\Gamma_0^{\gamma\gamma}$. Note that the singularities in $\Gamma^{\gamma\gamma}G^{\gamma\gamma}$ due to the poles in the Green’s function for vanishing photon momenta are responsible for the correct low–energy limit of the Compton scattering tensor including the pole contributions present in the Born terms.

By imposing that the amplitude of order $O(\epsilon^2)$ obeys the related Ward–Takahashi identity, we found a constraint for the two–photon irreducible interaction kernel similar to the one in first order (see also [1]). We also discussed the Ward–Takahashi identities for the “gauged” Bethe–Salpeter amplitudes $\chi^\mu$ and $\mu^{\mu\nu}$. Differential identities for Green’s functions, the vertices $\Gamma^\nu$ and $\Gamma^{\mu\nu}$ and the $n$–photon irreducible interaction kernels were derived. By using rather formal arguments, we found similar identities for the Bethe–Salpeter amplitudes; however, we stressed that these are strictly defined only on mass shell, i.e. for $P^2 = M^2$.

As we checked the gauge invariance condition for a matrix element proportional to $\bar{\chi}\Gamma^{\gamma\gamma}\chi$, we found that it is (as it should be) automatically satisfied if the two–photon vertex obeys the correct Ward–Takahashi identity. For an approximation in which we neglected all $n$–photon irreducible interaction kernels, we found that gauge invariance for the resulting two–photon vertex $\Gamma^{\gamma\gamma}$ is preserved only if the Bethe–Salpeter kernel $K$ does not depend on the bound state’s four–momentum $P$ and is of local type, i.e. $K(P; \{k_i\}, \{k_i'\}) = K((k_i - k_i'))$ where $\{k_i\}$ denote the relative momenta of the constituents. This is the same condition as for the related approximation in first order if we neglect the one–photon irreducible kernel, i.e. $\Gamma^{\gamma} = \tilde{\Gamma}^{\gamma} = \Gamma_0^{\gamma}$.  

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A similar result could be derived with regard to the low-energy limits: they are properly described by the full vertices $\Gamma^\gamma$ and $\Gamma^{\gamma\gamma}$. If we only neglect contributions from $K^\gamma$ and $K^{\gamma\gamma}$, then we also meet the correct limits for vanishing photon momenta. However, we do not find the correct Born–Thomson limit for the Compton scattering amplitude if we only include the lowest order irreducible vertex $\Gamma_0^{\gamma\gamma}$. Therefore we conclude that the corresponding vertex to the lowest order vertex $\hat{\Gamma}^\gamma = \Gamma_0^\gamma$ at $\mathcal{O}(e)$ is in second order of the electromagnetic coupling not $\Gamma_0^{\gamma\gamma}$ but merely $\Gamma^{\gamma\gamma} = \Gamma_0^{\gamma\gamma} - i\Gamma_0^\gamma T^{\gamma\gamma}_0$ as a vertex that also includes re-scattering terms due to $T$ matrix contributions.

Let us finally note that the scheme presented in this article can also be applied to processes like $\gamma\gamma \to A \bar{A}$, i.e. the production of bound state pairs (e.g. $\pi^+\pi^-$ or $\pi^0\pi^0$) in photon–photon collisions. The corresponding identities could as well be obtained by introducing new momentum variables $q_2 \to -q_2$ and $P \to -P$ that satisfy the four-momentum conservation $q_1 + q_2 = P + P'$. Furthermore, we want to stress that our results could also be of interest for Compton scattering off an electromagnetic bound state such as positronium since we didn’t specify the interaction kernel in the Bethe–Salpeter equation, and in this respect our considerations are quite general.

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APPENDIX A: COORDINATES FOR A COMPOSITE SYSTEM

Let us consider a system of \( n \) fermions and \( \bar{n} \) anti-fermions \( (N = n + \bar{n}) \) where the position of each constituent is described by the coordinate \( x_i \) \( (i = 1, 2, \ldots, N) \). Now we introduce new coordinates by

\[
X := \eta_1 x_1 + \eta_2 x_2 + \ldots + \eta_N x_N \\
r_1 := x_1 - x_2 \\
r_2 := \frac{1}{2}(x_1 + x_2) - x_3 \\
\vdots \\
r_{N-1} := \frac{1}{N-1}(x_1 + x_2 + \ldots + x_{N-1}) - x_N .
\]

The coefficients \( \eta_i \) have no direct geometrical meaning; the only constraint is that they have to sum up to unity:

\[
\eta_1 + \eta_2 + \ldots + \eta_N = 1 .
\]

Choosing them as \( \eta_i = m_i/M \) with \( M = \sum m_i \) gives \( X \) the meaning of a center–of–mass coordinate; however, a more common choice is simply \( \eta_i = 1/N \). Another possibility is to interpret \( X \) as a center–of–charge coordinate by adopting \( \eta_i = e_i/Q \) where \( e_i \) is the physical charge of the \( i \)-th constituent and \( Q = \sum e_i \) is the total charge; this choice will be applied in the last section of this article. Note that the fixing \( \eta_i = e_i/Q \) is formally also possible for neutral bound states since even for \( Q = 0 \) the relation \( \sum \eta_i = 1 \) is satisfied.

In a compact form, the transformation in eq. \((A1)\) can be written as a matrix equation like

\[
\alpha_x = A_x \beta_x \quad \text{with} \quad \alpha_x = \begin{pmatrix} X \\ r_1 \\ \vdots \\ r_{N-1} \end{pmatrix}, \quad \beta_x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}
\]

\[
\text{and} \quad A_x = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \ldots & \eta_N \\ 1 & -1 & 0 & \ldots & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N-1} & \frac{1}{N-1} & \frac{1}{N-1} & \ldots & -1 \end{pmatrix} .
\]

Now consider the canonical conjugated momenta \( p_1, p_2, \ldots, p_N \) and \( P, k_1, \ldots, k_{N-1} \), respectively, where \( P \) is the total momentum \( P = p_1 + p_2 + \ldots + p_N \). We can give a matrix equation for this transformation in momentum space as well:

\[
\alpha_p = A_p \beta_p \quad \text{with} \quad \alpha_p = \begin{pmatrix} P \\ k_1 \\ \vdots \\ k_{N-1} \end{pmatrix}, \quad \beta_p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} .
\]

\[
\text{Obviously, the } i\text{-th element in the first column of } A_p^{-1} \text{ gives the dependency of the momentum } p_i \text{ of the total momentum } P .
\]

The transformations given in eqs. \((A2)\) and \((A3)\) satisfy the following relations concerning their Jacobi determinant and their scalar product:

\[
\left| \frac{\partial (X, r_1, \ldots, r_{N-1})}{\partial (x_1, x_2, \ldots, x_N)} \right| = \left| \frac{\partial (P, k_1, \ldots, k_{N-1})}{\partial (p_1, p_2, \ldots, p_N)} \right| = 1
\]

and

\[
PX + r_1 k_1 + \ldots r_{N-1} k_{N-1} = p_1 x_1 + p_2 x_2 + \ldots + p_N x_N . \quad \text{(A4)}
\]

The last equation gives the transformation matrix \( A_p \) in momentum space in terms of the matrix \( A_x \) in position space.
\[\alpha_p^t \alpha_x = \beta_p^t \beta_x \quad \iff \quad \alpha_p^t A_x \beta_x = (A_p^{-1} \alpha_p)^t \beta_x = \alpha_p^t (A_p^{-1})^t \beta_x.\]

This yields \(A_p^{-1} = A_x^t\) so that we find the dependency of the momentum of the \(i\)-th constituent of the total momentum to be

\[p_i = \eta_i P + f(k_1, \ldots, k_{N-1})\]  \hspace{1cm} (A5)

where \(f(k_1, \ldots, k_{N-1})\) is a linear function that can be derived from \(A_p^{-1} = A_x^t\). The constituents of a \(q\bar{q}\) meson for instance have the momenta

\[p_1 = \eta_1 P + p \quad \text{and} \quad -p_2 = -\eta_2 P + p\]  \hspace{1cm} (A6)

with total momentum \(P = p_1 + p_2\) and relative momentum \(p := k_1 = \eta_2 p_1 - \eta_1 p_2\).

### APPENDIX B: BETHE–SALPETER AMPLITUDES IN SECOND ORDER

In this appendix, we want to derive the Bethe–Salpeter amplitude in second order of an external field. We start with eq. (21) and write down the expansion in eq. (7) for \(O = G_0, K, \chi\):

\[
\chi - ie\gamma - e^2\gamma \gamma + \ldots = -i(G_0 - ieG_0^t - e^2G_0^{\gamma t} + \ldots)(K - ieK^t - e^2K^{\gamma t} + \ldots)(\chi - ie\gamma - e^2\gamma \gamma + \ldots).
\]

In the order \(O(e^2)\) of the electromagnetic coupling, we find with the first order relation \(\chi^t = G\Gamma \chi\)

\[
\chi^{\gamma t} = -iG^t_0 \gamma K \chi - iG_0 \gamma K^t \chi - iG^t_0 \gamma K \chi - iG_0 \gamma K^t \chi
\]

\[= (1 + iG_0 K)^{-1} \left( -iG_0 \gamma K \chi - iG_0 \gamma K^t \chi - iG_0 \gamma G_0 \gamma K \chi - iG_0 \gamma G_0 \gamma K^t \chi - iG_0 \gamma G_0 \Gamma \gamma \chi - iG_0 \gamma G_0 \Gamma \gamma \chi \right)
\]

\[= iG_0 \gamma \chi - iG_0 \gamma G_0 (\gamma + iK) + (\Gamma_0^\gamma - iK^\gamma) G\Gamma \gamma \chi
\]

\[= G \left( \Gamma_0^\gamma - iK^\gamma + \Gamma^\gamma GT^\gamma - \Gamma_0^\gamma G_0 \Gamma_0^\gamma \right) \chi
\]

where we have used the fundamental equation for the Green’s function, the Bethe–Salpeter equation and some relations introduced in section III. With \(\Gamma^{\gamma t} := \Gamma_0^\gamma - iK^\gamma + \Gamma^\gamma GT^\gamma - \Gamma_0^\gamma G_0 \Gamma_0^\gamma\), we finally find the expression in eq. (23) for the second order Bethe–Salpeter amplitude \(\chi^{\gamma t} = G\Gamma^{\gamma t} \chi\).