HEAT FLOW METHOD TO LICHNEROWICZ TYPE EQUATION ON CLOSED MANIFOLDS

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Abstract. In this paper, we establish existence results for positive solutions to the Lichnerowicz equation of the following type in closed manifolds

\[-\Delta u = A(x)u^{-p} - B(x)u^q, \quad \text{in} \ M,\]

where \( p > 1, q > 0, \) and \( A(x) > 0, B(x) \geq 0 \) are given smooth functions. Our analysis is based on the global existence of positive solutions to the following heat equation

\[
\begin{cases}
    u_t - \Delta u = A(x)u^{-p} - B(x)u^q, & \text{in} \ M \times \mathbb{R}^+, \\
    u(x,0) = u_0, & \text{in} \ M
\end{cases}
\]

with the positive smooth initial data \( u_0. \)

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1. Introduction

The aim of this paper is to give some existence results for positive solutions to the Lichnerowicz type equations in a compact Riemannian manifold \((M, g).\) In the interesting paper \([5]\), the authors have proved an existence result for Lichnerowicz equation on closed manifolds by the mini-max method. In \([9]\), Li Ma and Xingwang Xu have investigated the negative-positive mixed index case in elliptic equations, as \( \Delta u + f(u) = 0, \) where \( f(u) = u^{p-1} - u^{p-1}, \) they gained a uniform bound and a non-existence result for positive solutions to the Lichnerowicz equation in \( \mathbb{R}^n. \) Some further interesting results on the Lichnerowicz equation on a Riemannian manifold \((M, g)\) of dimension \( n \geq 3\) have been obtained in \([2, 3,\) and \([5].\) Motivated by these papers, we mainly consider heat flow methods for the Lichnerowicz type equations in the negative-positive mixed index cases.

Here we recall the Lichnerowicz equation on manifolds. Given a smooth symmetric 2-tensor \( \sigma, \) a smooth vector field \( W, \) and a triple data \((\pi, \tau, \varphi)\) of smooth functions on \( M.\) Set

\[
c_n = \frac{n - 2}{4(n - 1)}, \quad p = \frac{2n}{n - 2},
\]

and let

\[
R_{\gamma, \varphi} = c_n(R(\gamma) - |\nabla \varphi|_\gamma^2), \quad A_{\gamma, W, \pi} = c_n(|\sigma + DW|_\gamma^2 + \pi^2)
\]

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and
\[B_{\tau,\varphi} = c_n \left( \frac{n-1}{n} \tau^2 - V(\varphi) \right)\]

where \(V : \mathbb{R} \to \mathbb{R}\) is a given smooth function and \(R(\gamma)\) is the scalar curvature function of \(\gamma\). Then the Lichnerowicz equation for the Einstein-scalar conformal data \((\gamma, \sigma, \pi, \tau, \varphi)\) with the given vector field \(W\) is

\[
\Delta_{\gamma} u - R_{\gamma,\varphi} u + A_{\gamma, W, \pi} u^{p-1} - B_{\tau,\varphi} u^{p-1} = 0, \quad u > 0,
\]

where \(\Delta_{\gamma}\) is the Laplacian operator of \(\gamma\). As in [9], we use the convention that \(\Delta_{\gamma} u = u''\) on the real line \(\mathbb{R}\). Note that \(A_{\gamma, W, \pi} \geq 0\).

In this paper, we mainly deal with the heat equation of the following type

\[
\left\{
\begin{array}{l}
u_t - \Delta u = A(x)u^{-p} - B(x)u^q, \quad \text{in} \quad M \times \mathbb{R}^+,
\vspace{1mm}
u(x,0) = u_0, \quad \text{in} \quad M,
\end{array}
\right.
\]

where \(p > 1, q > 0; A(x) > 0\) and \(B(x)\) are both smooth functions in \(M\).

Applying sub-sup solution method on manifolds, we can prove

\textbf{Theorem 1.} If \(B(x) > 0\), and the initial data \(u_0\) is a smooth function with \(u_0 > 0\), then there exists a unique smooth solution to the problem (1.2). As \(t \to \infty\), we can find a subsequence \((t_j)\) such that

\[u(t_j) \to u_\infty, \quad \text{in} \quad H^1(M)\]

where \(u_\infty\) is the positive solution to

\[-\Delta u = A(x)u^{-p} - B(x)u^q, \quad \text{in} \quad M\]

If \(B(x) \geq 0\), and \(B(x)\) does have zero points in \(M\), then we have the following result

\textbf{Theorem 2.} Assume \(p > 1, 1 < q \leq \frac{n+2}{n-2}\). If \(B(x) \geq 0\), \(\int_M B(x) \frac{1}{q} dx < +\infty\), and the initial data \(u_0\) is a positive smooth function in \(M\), then there exists a unique smooth solution \(u(x,t)\) to problem (1.2). In addition, one can take a sequence \(t_i \to \infty\) such that \(u(x,t_i) \to u_\infty\) in \(H^1(M)\), and \(u_\infty \in C^2\) solves the Lichnerowicz type equation

\[
-\Delta u_\infty = A(x)u_\infty^{-p} - B(x)u_\infty^q, \quad \text{in} \quad M.
\]

We gain the existence results by the heat flow method. We point out that in E.Hebey’s recent work (see Appendix), he has explored the Lichnerowicz equation using sub-sup solution method, but this method can’t go through in Theorem 2. For some special cases we can give more detailed analysis of the heat equations.

This paper is organized as follows. In section 2 we prove Theorems 1 and 2. In section 3 some precise results are presented in special cases.
2. Heat flows methods to the Lichnerowicz type equation and proofs

This section is mainly devoted to the proofs of Theorems 1 and 2. To complete the proofs, we should make some preparations by establishing sub-sup solution method on manifolds. We take the same argument as in D.H. Sattinger’s paper [6], but the difference is that D.H. Sattinger has dealt with sub-sup solutions only in Euclidean spaces, and for convenience of readers, we present his some results on manifolds. We point out that the techniques in the proofs are almost the same as in D.H. Sattinger’s.

Consider the initial problem to the following heat equation in closed manifold

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= f(x,u), \quad \text{in } M \times \mathbb{R}^+, \\
u(x,0) &= u_0, \quad \text{in } M.
\end{align*}
\]

where \( f(x, u) = A(x)u^{-p} - B(x)u^q \), and \( A(x), B(x), u_0 > 0 \) are smooth functions.

We call the function \( \varphi_0(x,t) > 0 \) a sup-solution to (2.1) if

\[
\begin{align*}
\frac{\partial \varphi_0}{\partial t} - \Delta \varphi_0 - f(x, \varphi_0) &\geq 0, \quad \text{in } M \times \mathbb{R}^+, \\
\varphi_0(x,0) &\geq u_0, \quad \text{in } M.
\end{align*}
\]

Similarly the sub-solution \( \psi_0(x,t) > 0 \) is defined by if

\[
\begin{align*}
\frac{\partial \psi_0}{\partial t} - \Delta \psi_0 - f(x, \psi_0) &\leq 0, \quad \text{in } M \times \mathbb{R}^+, \\
\psi_0(x,0) &\leq u_0, \quad \text{in } M.
\end{align*}
\]

Next, we define the mapping \( \varphi_1 = \Lambda \varphi_0 \) by

\[
\begin{align*}
\frac{\partial \varphi_1}{\partial t} - \Delta \varphi_1 + \Omega \varphi_1 &= f(x, \varphi_0) + \Omega \varphi_0, \quad \text{in } M \times \mathbb{R}^+, \\
\varphi_1(x,0) &= u_0, \quad \text{in } M.
\end{align*}
\]

where \( \Omega \) is constant chosen large enough.

**Lemma 1.** Given a positive sup-solution \( \varphi_0(x,t) > 0 \) and a positive sub-solution \( \psi_0(x,t) > 0 \) to problem (2.1) in closed manifold M. Define sequences \( \{\varphi_n\} \) and \( \{\psi_n\} \) inductively by \( \varphi_{n+1} = \Lambda \varphi_n, \psi_{n+1} = \Lambda \psi_n \). If \( \Omega \) is large enough so that

\[
\frac{\partial f}{\partial u}(x,u) + \Omega > 0 \quad \text{on } \min_{M \times (0,T)} \psi_0 < u < \max_{M \times (0,T)} \varphi_0,
\]

then the sequences \( \{\varphi_n\} \) and \( \{\psi_n\} \) are monotone increasing and decreasing respectively. As \( n \to \infty \), they both tend to a unique fixed point \( u = \Lambda u \), which is a smooth solution of

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= f(x,u), \quad \text{in } M \times (0,T) \\
u(x,0) &= u_0, \quad \text{in } M.
\end{align*}
\]

**Proof.** We can take \( \Omega > 0 \) large enough, so that

\[
\frac{\partial f}{\partial u}(x,u) + \Omega > 0
\]

for all \( x \in M \), and \( \min \psi_0 \leq u \leq \max \varphi_0 \).
Recall that the sequence \( \{ \psi_k \} \) is defined as follows

\[
\begin{align*}
\partial_t \psi_{k+1} - \Delta \psi_{k+1} + \Omega \psi_{k+1} &= f(x, \psi_k) + \Omega \psi_k, \quad \text{in} \quad M \times \mathbb{R}^+, \\
\psi_{k+1}(x, 0) &= u_0, \quad \text{in} \quad M.
\end{align*}
\]  

(2.4)

and the sequence \( \{ \varphi_k \} \) is similarly defined.

We claim

\[
\psi_0 \leq \psi_1 \leq \cdots \leq \psi_k \leq \cdots \leq \varphi_k \leq \cdots \leq \varphi_1 \leq \varphi_0.
\]  

(2.5)

To confirm this, first note for \( k = 0 \) that

\[
\begin{align*}
\partial_t \psi_1 - \Delta \psi_1 + \Omega \psi_1 &= f(x, \psi_0) + \Omega \psi_0, \quad \text{in} \quad M \times \mathbb{R}^+, \\
\psi_1(x, 0) &= u_0, \quad \text{in} \quad M.
\end{align*}
\]  

(2.6)

Subtracting (2.6) from (2.2), we find

\[
\begin{align*}
\partial_t (\psi_1 - \psi_0) - \Delta (\psi_1 - \psi_0) + \Omega (\psi_1 - \psi_0) &\geq 0, \quad \text{in} \quad M \times \mathbb{R}^+, \\
(\psi_1 - \psi_0)(x, 0) &\geq 0, \quad \text{in} \quad M.
\end{align*}
\]  

(2.7)

Applying the maximum principle, we know

\[
\psi_0 \leq \psi_1, \quad \text{in} \quad M \times \mathbb{R}^+.
\]

(2.8)

We now assume inductively

\[
\psi_{k-1} \leq \psi_k, \quad \text{in} \quad M \times \mathbb{R}^+.
\]  

(2.9)

By (2.4) we have

\[
\begin{align*}
\partial_t (\psi_{k+1} - \psi_k) - \Delta (\psi_{k+1} - \psi_k) + \Omega (\psi_{k+1} - \psi_k) &= f(x, \psi_k) - f(x, \psi_{k-1}) + \Omega (\psi_k - \psi_{k-1}), \quad \text{in} \quad M \times \mathbb{R}^+, \\
(\psi_{k+1} - \psi_k)(x, 0) &= 0, \quad \text{in} \quad M.
\end{align*}
\]  

(2.10)

From (2.3), (2.7), using the maximum principle, we have

\[
\psi_k \leq \psi_{k+1}, \quad \text{in} \quad M \times \mathbb{R}^+.
\]

Hence similar argument can be applied to \( \{ \varphi_k \} \), we can deduce \( \varphi_{k+1} \leq \varphi_k \).

In light of this, we claim if \( \psi_k \leq \varphi_k \), then \( \psi_{k+1} \leq \varphi_{k+1} \). From (2.4), we have

\[
\begin{align*}
\partial_t (\varphi_{k+1} - \varphi_k) - \Delta (\varphi_{k+1} - \varphi_k) + \Omega (\varphi_{k+1} - \varphi_k) &= f(x, \varphi_k) - f(x, \psi_k) + \Omega (\varphi_k - \psi_k), \quad \text{in} \quad M \times \mathbb{R}^+, \\
(\varphi_{k+1} - \varphi_k)(x, 0) &= 0, \quad \text{in} \quad M.
\end{align*}
\]  

(2.11)

Then (2.5) is proved.
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Since \(\psi_0(x,t), \varphi_0\) are globally bounded, we define
\[
    u(x,t) := \lim_{k \to \infty} \psi_k(x,t)
\]
\[
    v(x,t) := \lim_{k \to \infty} \varphi_k(x,t).
\]

By (2.11), obviously we have
\[
    u(x,t) \leq v(x,t), \quad \text{in } M \times \mathbb{R}^+.
\]

Applying the monotone convergence theorem, we have
\[
    \psi_k(x,t) \to u(x,t), \quad \text{in } L^2(M),
\]
\[
    \varphi_k(x,t) \to v(x,t), \quad \text{in } L^2(M).
\]

By (2.3), we can deduce
\[
    \|f(x,\psi_k)\|_{L^2(M)} \leq C(1 + \|\psi_k\|_{L^2(M)}),
\]
where \(C\) depends only on \(A(x), B(x),\) and \(M\).

Furthermore letting \(k \to \infty\), we have
\[
    \begin{cases}
        u_t - \Delta u = f(x,u), & \text{in } M \times \mathbb{R}^+,
        \\
        u(x,0) = u_0, & \text{in } M,
    \end{cases}
\]

and
\[
    \begin{cases}
        v_t - \Delta v = f(x,v), & \text{in } M \times \mathbb{R}^+,
        \\
        v(x,0) = v_0, & \text{in } M.
    \end{cases}
\]

Subtract (2.15) from (2.16), we have
\[
    \begin{cases}
        (u - v)_t - \Delta (u - v) + \Omega(u - v) = f(x,u) - f(x,v) + \Omega(u - v), & \text{in } M \times \mathbb{R}^+,
        \\
        (u - v)(x,0) = 0, & \text{in } M.
    \end{cases}
\]

Observe that
\[
    f(x,u) - f(x,v) + \Omega(u - v) = (f_u(x,\tilde{\xi}) + \Omega)(u - v),
\]
where \(\tilde{\xi}\) is between \(u\) and \(v\).

From (2.3), (2.17), (2.12) and using the maximum principle, we deduce
\[
    u = v, \quad \text{in } M \times \mathbb{R}^+.
\]

By classical parabolic theory, we know that \(u\) is a positive smooth solution. \(\Box\)

**Proof of Theorem** Let \(\varphi_0 = S(\max \frac{A(x)}{B(x)})^{\frac{1}{p+q}}, \psi_0 = s(\min \frac{A(x)}{B(x)})^{\frac{1}{p+q}},\) where \(S \geq 1, 0 \leq s \leq 1\) is determined by the following
\[
    S(\max \frac{A(x)}{B(x)})^{\frac{1}{p+q}} \geq \max u_0,
\]
\[
    s(\min \frac{A(x)}{B(x)})^{\frac{1}{p+q}} \leq \min u_0.
\]
We know that \( \psi_0 \) and \( \varphi_0 \) constructed above are the sub-solution and sup-solution of the problem (1.2). By Lemma 1 we know that problem (1.2) has a positive smooth solution.

The uniqueness can be gained as below. If \( u, v \) are both smooth solutions to the problem (1.2). Then it follows that
\[
\begin{cases}
(u - v)_t - \Delta (u - v) = A(x)u^{-p} - B(x)u^q - A(x)v^{-p} + B(x)v^q, & \text{in } M \times \mathbb{R}^+ \\
(u - v)(x, 0) = 0, & \text{in } M.
\end{cases}
\]
We can just rewrite it as
\[
(2.18) \quad \begin{cases}
(u - v)_t - \Delta (u - v) + (pA(x)\xi^{-p-1} + qB(x)\eta^{q-1})(u - v) = 0, & \text{in } M \times \mathbb{R}^+ \\
(u - v)(x, 0) = 0, & \text{in } M,
\end{cases}
\]
where \( \xi, \eta \) are between \( u \) and \( v \).

Since \( u, v \) are positive smooth solutions, using the maximum principle, we can deduce
\[
u = v, \quad \text{in } M.
\]
Because of the uniform bound-ness of \( u(x, t) \), the convergence at \( t = \infty \) of \( u(x, t) \) is by now standard (see also the proof of Theorem 2 below), so we omit the detail. This completes the proof of Theorem 1.

**Proof of Theorem 2.** If the term \( B(x) \) in equation (1.2) has zero points, letting \( Z = \{ x \in M; B(x) = 0 \} \), the sub-sup solution method above can not be used. Then we add an positive epsilon to \( B(x) \) such that we can use the method to get an approximate solutions (1.2). Let \( B_1 = \sup_M B(x) > 0 \) and \( B_2 = \inf_M A(x) > 0 \).

First, for \( \epsilon > 0 \), we consider the following equation
\[
(2.19) \quad \begin{cases}
\partial_t u_\epsilon - \Delta u_\epsilon = A(x)u_\epsilon^{-p} - (B(x) + \epsilon)u_\epsilon^q, & \text{in } M \times \mathbb{R}^+ \\
u_\epsilon(x, 0) = u_0, & \text{in } M.
\end{cases}
\]
By the sub-super solution method we know that there is a positive smooth solution, which is denoted by \( u_\epsilon \), to (2.19). Sometimes we may write \( u = u_\epsilon \).

We claim that there is a uniform constant \( C := C(u_0) > 0 \) such that
\[
u_\epsilon(x, t) \geq C, \quad \text{in } M \times (0, \infty).
\]
In fact, for any fixed \( T > 0 \), let
\[
u(x_0, t_0) := u_\epsilon(x_0, t_0) = \inf_{M \times [0, T]} u(x, t).
\]
Then we have
\[
0 \geq (\partial_t - \Delta) u_\epsilon(x_0, t_0) = A(x_0)u_\epsilon(x_0, t_0)^{-p} - (B(x_0) + \epsilon)u_\epsilon(x_0, t_0)^q.
\]
Then we have
\[
A(x_0)u_\epsilon(x_0, t_0)^{-p} \leq (B(x_0) + \epsilon)u_\epsilon(x_0, t_0)^q.
\]
Then
\[
0 < B_2 \leq A(x_0) \leq (B(x_0) + \epsilon)u_\epsilon(x_0, t_0)^{q+p}
\]
\[
\leq (B_1 + 1)u_\epsilon(x_0, t_0)^{q+p},
\]
which implies that
\[
u_\epsilon(x_0, t_0) \geq C > 0
\]
for some uniform constant \( C = C(u_0) > 0 \). Then the claim is true and this fact will be used implicitly.

Multiplying (2.19) with \( \partial_t u_\epsilon \) and integrating in \( M \times (0, t) \) we have
\[
\int_M^{t} \int_M |\partial_t u_\epsilon|^2dxdt - \int_M^{t} \int_M \Delta u_\epsilon \partial_t u_\epsilon dxdt = \int_M^{t} \int_M A(x) u_\epsilon^{-p} \partial_t u_\epsilon dxdt
\]
\[
- \int_M^{t} \int_M (B(x) + \epsilon) u_\epsilon^q \partial_t u_\epsilon dxdt.
\]

Rearranging, we deduce
\[
\int_M^{t} \int_M |\partial_t u_\epsilon|^2dxdt + \frac{1}{2} \int_M |\nabla u_\epsilon|^2dx + \frac{1}{p-1} \int_M A(x) u_\epsilon^{p+1}dx
\]
\[
+ \frac{1}{q+1} \int_M (B(x) + \epsilon) u_\epsilon^{q+1}dx = \frac{1}{2} \int_M |\nabla u_\epsilon|^2dx + \frac{1}{p-1} \int_M A(x) u_\epsilon^{p+1}dx
\]
\[
+ \frac{1}{q+1} \int_M (B(x) + \epsilon) u_\epsilon^{q+1}dx
\]

(2.20)

By Poincare’s inequality, we obtain
\[
\int_M |u - \bar{u}|^2dx \leq C(M) \int_M |\nabla u|^2dx.
\]
Then it follows that
\[
\int_M u^2dx \leq |M| \bar{u}^2 + C(M) \int_M |\nabla u|^2dx,
\]
where \( |M| \) is the volume of manifold \( M \).

Note
\[
\bar{u} = \frac{1}{|M|} \int_M u dx = \frac{1}{|M|} \int_M B(x)^{-\frac{1}{q+1}} B(x)^{\frac{1}{q+1}} u dx
\]
\[
\leq \frac{1}{|M|} \left( \int_M B(x)^{-\frac{1}{q}} dx \right)^{\frac{q}{q+1}} \left( \int_M B(x) u^{q+1} dx \right)^{\frac{1}{q+1}}.
\]
(2.22)

Thus, from (2.20), (2.21), (2.22) and the assumption \( \int_M B(x)^{-\frac{1}{q}} dx \leq C(M) \),
we deduce for every \( \epsilon \),
\[
\begin{align*}
\int_M u_\epsilon^2dx &\leq C, \\
\int_M |\nabla u_\epsilon|^2dx &\leq C, \\
\int_t^0 \int_M |\partial_t u_\epsilon|^2dxdt &\leq C,
\end{align*}
\]
(2.23)
where \( C \) doesn’t depend on \( t, \epsilon \). It depends only on \( A(x), B(x), u_0, \) and \( M \).
Then we find a sequence $\epsilon_j$ with $\epsilon_j \to 0$, and the positive function $u \in H^1(M)$, such that

\begin{equation}
\begin{cases}
\epsilon_j \to u, \quad \text{in} \quad H^1(M), \\
\epsilon_j \to u, \quad \text{in} \quad L^2(M), \\
\partial_t \epsilon_j \to \partial_t u, \quad \text{in} \quad L^2(M \times \mathbb{R}^+).
\end{cases}
\end{equation}

(2.24)

Clearly $u$ satisfies

$$
\partial_t u - \Delta u = A(x)u^{-p} - B(x)u^q, \quad \text{in} \quad M \times \mathbb{R}^+
$$

with the initial data $u_0$ and the estimates (2.23).

Then we may take a subsequence $t_k \to \infty$ such that

$$
\partial_t u(x, t_k) \to 0, \quad \text{in} \quad L^2(M)
$$

(2.25)

and $u(x, t_k)$ converges weakly to $\tilde{u}$ in $H^1$, $\tilde{u} \in H^1(M)$ is a positive smooth solution to

$$
-\Delta u = A(x)u^{-p} - B(x)u^q, \quad \text{in} \quad M \times \mathbb{R}^+.
$$

(2.26)

Using the standard regularity theory we know that $\tilde{u}$ is the positive smooth solution to (2.26).

This completes the proof of Theorem 2.

3. Simple cases: global existence and asymptotic behavior

In this section we shall give more precise information in some simple cases. Firstly consider the following problem

\begin{equation}
\begin{cases}
\epsilon \to u, \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \\
u(x, 0) = u_0, \quad \text{in} \quad M,
\end{cases}
\end{equation}

(3.1)

where $p, q > 1$ and $u_0 > 0$ is a given smooth function in $M$.

When the initial data is small, we have

**Proposition 1.** Let $u(x, t)$ be the solution to the problem (3.1). If the initial data $u_0$ is such that $0 < u_0 \leq 1$, then $u(x, t) \to 1$ uniformly as $t \to \infty$.

For convenience we introduce some notations

$$
u_{\min} = \min_M u(x, t), \quad \nu_{\max} = \max_M u(x, t),
$$

if $A(x), B(x)$, and $u_0$ are smooth, by the classical parabolic theory we know the min, max functions are well defined.

**Proof.** Consider the evolitional trend of the function $u$ and we naturally study the $\min_M u, \max_M u$. Recall $u_0$ is a smooth function on $M$, by the classical parabolic theory, we know that $u$ is also smooth and the generalized derivatives of $\min_M u, \max_M u$ with respect to $t$ is well-defined.

Since $0 < u_0 \leq 1$, supposing $u$ first reaches $u(x, t) = 1$ at $(x_0, t_0)$, then from equation (3.1), we have

$$
u_{\max}(x_0, t_0) = \Delta \nu_{\max}(x_0, t_0) \leq 0,$$
then we deduce

\[ u \leq u_{\text{max}} \leq 1, \quad t \in \mathbb{R}^+, \]

(3.2)

Investigating

\[ u_{\text{min}} = \Delta u_{\text{min}} + u^{-p} - u^q \geq 0, \]

there exists a constant \( c_0 \) such that

(3.3)

\[ u \geq u_{\text{min}} \geq c_0 > 0, \quad t \in \mathbb{R}^+. \]

where \( c_0 \) can equal to \( \min_M u_0 \).

From (3.2) and (3.3), we know that the solution \( u \) is uniformly bounded, and the global existence result is obtained.

We claim that \( u(x,t) \to 1 \) uniformly as \( t \to \infty \). If \( u_{\text{min}} \leq 1 - \delta \) for some \( \delta > 0 \), we actually have

\[ u_{\text{min}} = C(\delta)u_{\text{min}} > 0, \]

which implies that

\[ u_{\text{min}}(t) \geq \exp(C(\delta)t)u_{\text{min}}(0) \to \infty \]

as \( t \to \infty \). Hence we have \( u \geq 1 - \delta \) for \( t \geq T \) for some large \( T > 0 \). Therefore we have uniform convergence property that \( u(x,t) \to 1 \) as \( t \to \infty \) uniformly. \( \square \)

When the initial data \( u_0 \) is large, we have

**Proposition 2.** Let \( u(x,t) \) be the solution to the problem (3.1). If the initial data \( u_0 \) is such that \( 0 < u_0 \leq L \), where \( L > 1 \), then we have that \( u(x,t) \to 1 \) uniformly in \( M \) as \( t \to \infty \).

**Proof.** Firstly consider the functions \( \min_M u, \max_M u \). If \( u_{\text{min}} \leq 1 - \delta \) for some \( \delta > 0 \), we then have

\[ u_{\text{min}} = C(\delta)u_{\text{min}} > 0, \]

which implies that

\[ u_{\text{min}}(t) \geq \exp(C(\delta)t)u_{\text{min}}(0) \to \infty \]

as \( t \to \infty \). Hence, we have \( u \geq 1 - \delta \) for \( t \geq T \) for some large \( T > 0 \).

Consider the property of \( u_{\text{max}} > 1 \) with

\[ u_{\text{max}} \leq u_{\text{max}}^{-p} - u_{\text{max}}^q \leq 0. \]

Then

\[ u_{\text{max}} \leq L. \]

By the similar argument as before, we have \( u \leq 1 + \delta \) for \( t \geq T_1 \) for some large \( T_1 > 0 \). Hence we have \( u(x,t) \to 1 \) uniformly as \( t \to \infty \). \( \square \)

We may also investigate the case where \( A(x) = C_0B(x) > 0 \). We have a similar result to Proposition \[1 \] [2]
Corollary 1. Suppose $A(x)/B(x) \equiv C_0$, and the initial data $u_0$ is a smooth function with $u_0 > 0$, then the problem (1.2) has a unique smooth solution $u(x,t)$. Furthermore, $u(x,t) \to C_0^{(p+q)}$ uniformly as $t \to \infty$.

Proof of Corollary 1 This proof is similar to the proofs of Propositions 1 and 2, so we may omit it.

4. Appendix

Consider the following PDE on a compact Riemannian manifold $(M,g)$

$$-\Delta_g u + h(x)u = A(x)u^{-p} + B(x)u^q.$$ 

Here $h(x), A(x),$ and $B(x)$ are smooth functions on $M$. Assume that $A(x) > 0$ and $B(x) < 0$.

Hebey’s result is below: one can get such a solution in a few lines with the sub and super solution method (at least if one assumes that $-\Delta_g + h$ is a positive operator). Assuming $-\Delta_g + h$ is a positive operator (e.g. $h > 0$) one lets $v > 0$ be arbitrary and $u > 0$ be the solution of $-\Delta_g u + hu = v$. Then (1) $u_0 = \epsilon u$, $0 < \epsilon \ll 1$ is a subsolution for the equation, and

(2) $u_1 = tu$, $t \gg 1$, is a supersolution for the equation. Noting that $u_0 < u_1$ one can apply the sub and super solution method and one gets a solution to the equation.

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