A theta function for hyperbolic surfaces with cusps

Ulrich Bunke∗and Martin Olbrich†

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Abstract

For a Riemann surface with cusps we define a theta function using the eigenvalues of the Laplacian and the singularities of the scattering determinant. We provide its meromorphic continuation and discuss its singularities.

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1 Theta functions

Let $M$ be a complete Riemann surface of constant negative curvature $-1$ of finite volume. Consider the unique selfadjoint extension $\Delta_M$ of the Laplacian and form $A := \sqrt{\Delta_M - \frac{1}{4}}$, where we take the square root with positive imaginary part. Let $P_d$ be the projection onto the subspace of $L^2(M)$ spanned by the $L^2$-eigenfunctions of $\Delta_M$, i.e. onto the discrete subspace. The (eigenvalue) theta function of $M$ is defined by

Definition 1.1

$$\theta_d(t) := Tr P_d e^{itA} P_d, \quad Im(t) > 0.$$
It is known, that the number of eigenvalues of $\Delta_M$ (counted with multiplicity) being smaller than $\lambda$ is bounded by $C\lambda^2$ for some $C < \infty$. Hence, $\theta_d(t)$ is a well defined holomorphic function on the upper half plane.

If $M$ is compact, this $\theta$-function was introduced by Cartier/Voros [2]. They provide a meromorphic extension of $\theta_d(t)$ to the complex plane and give a complete description of its singularities. Their method starts with the Selberg Zeta-function of $M$. The theta-function is derived by a certain contour integral of the logarithmic derivative of the Selberg zeta-function.

In Bunke/Olbrich/Juhl [1] we developed another method to obtain the meromorphic extension starting with a distributional trace formula. In the present paper we use the trace formula in a similar way. If $M$ is non-compact, we are not able to provide an extension of $\theta_d(t)$ alone. The point is, that the eigenvalue theta-function enters into the trace formula in combination with another (scattering) theta-function obtained from the singularities of the scattering determinant.

Let $S(z)$ be the scattering determinant of $M$. $S(z)$ is a meromorphic function of finite order, satisfying

$$S(z) = S(-z)^{-1}, \quad S(\bar{z}) = S(z)^{-1}, \quad z \in \mathbb{C}.$$ 

Let $\Sigma$ be the set of singularities $\sigma$ of $S(z)$ with $\text{Re}(\sigma) \geq 0$, $\text{Im}(\sigma) > 0$. Let $m_\sigma (\overline{-m_\sigma}, 2m_\sigma, -2m_\sigma)$ be the order of the pole with $\text{Re}(\sigma) > 0$ (zero with $\text{Re}(\sigma) > 0$, pole with $\text{Re}(\sigma) = 0$, zero with $\text{Re}(\sigma) = 0$). It is known, that for some $C < \infty$ and all $\sigma \in \Sigma$ $\text{Im}(\sigma) < C$. Moreover, there are at most finitely many zeros on $\Sigma$ and the total multiplicity of all singularities with $\text{Re}(\sigma) < \lambda$ is bounded by $C\lambda^2$ for some $C < \infty$. Thus,

**Definition 1.2**

$$\theta_s(t) := \sum_{\sigma \in \Sigma} m_\sigma e^{it\sigma}, \quad \text{Im}(t) > 0$$

is a well defined holomorphic function on the upper half plane.

The main object of the present paper is the theta-function associated to $M$:

**Definition 1.3**

$$\theta(t) := \theta_d(t) + \theta_s(M),$$

which is initially defined on the upper half-plane.

In the second section we state the version of the Selberg trace formula suitable for our purpose. In the third section we give a meromorphic extension of $\theta(t)$ to the lower half-plane across $(0, \infty)$. Here, the trace formula can be applied immediately. In the forth section we discuss the necessary modifications needed to provide a meromorphic extension of the theta-function across the negative real axis. Finally, in the fifth section we collect all results together and state our main theorem about the meromorphic extension of the (modified) theta-function to the complex plane, and we give a complete description of its singularities.
2 The Selberg trace formula

Let \( \hat{\phi} \) be the Fourier transform
\[
\hat{\phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} \phi(t) dt
\]
of an even function \( \phi \) satisfying for some \( \delta > 0, \ C < \infty \)
\[
|\phi(t)| < Ce^{(-1/2+\delta)|t|}, \quad |\frac{d\hat{\phi}}{d\lambda}| < \frac{C}{(1 + |\lambda|)^{3+\delta}}.
\]

We introduce the hyperbolic contribution.

**Definition 2.1**
\[
c_h(\phi) = \sum_{c\text{-closed geodesic}} \frac{l_c}{2n_c sinh(l_c/2)} \phi(l_c),
\]
where \( l_c \) is the length of the geodesic \( c \) and \( n_c \) is its multiplicity.

The contribution of the identity is given by

**Definition 2.2**
\[
c_e(\phi) = -\frac{\text{vol}(M)}{4\pi} \int_{-\infty}^{\infty} \frac{d\phi(t)}{\sinh(t/2)} dt.
\]

Note that the integral is well defined at zero, since \( \phi \) is even. The Selberg trace formula, as it can be found e.g in Lax/Phillips [6], reads
\[
\sum_{\lambda \text{eigenvalue of } A} \hat{\phi}(\lambda) = c_e(\phi) + c_h(\phi)
- r \ln(2)\phi(0) + r/2\hat{\phi}(0)
- \frac{r}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{\phi}(\lambda)}{\Gamma(1 + i\lambda)} d\lambda
- \frac{r}{4\pi} \int_{-\infty}^{\infty} \frac{\hat{\phi}(\lambda)}{S(\lambda)} d\lambda.
\]

Set \( \phi(t) = \psi(t) + \psi(-t) \) for some \( \psi \in C^\infty_c(0, \infty) \). We can write all terms of the trace formula as applications of distributions to \( \psi \).

Let

**Definition 2.3**
\[
\tilde{\theta}_d(t) := Tr P_d e^{-itA} P_d, \quad Im(t) < 0.
\]
\( \tilde{\theta}_d(t) \) is a holomorphic function on the lower half-plane. The left-hand side of the trace formula can be written as

\[
< \theta_d(t + i0) + \tilde{\theta}_d(t - i0), \psi > .
\]

The hyperbolic contribution is

\[
< \sum_{c\text{-closed}\ geodesic} \frac{l_c}{2 n_c \sinh(l_c/2)} \delta(t - l_c), \psi >
\]

and the contribution of the identity is

\[
< -\frac{\text{vol}(M) \cosh(t/2)}{4\pi \sinh^2(t/2)}, \psi > .
\]

The last two terms of the trace formula are less trivial. We consider first

\[
-\frac{r}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(\lambda) \frac{\dot{\Gamma}(1 + i\lambda)}{\Gamma(1 + i\lambda)} d\lambda.
\]

Since \( \Gamma(1 + i\lambda) \) is of finite order and \( \hat{\psi} \) vanishes exponentially in the upper half-plane, we can apply the Cauchy integral formula. \( \Gamma(1 + i\lambda) \) has poles of first order in the points \( \lambda = ik, k = 1, 2, \ldots \). Thus

\[
-\frac{r}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(\lambda) \frac{\dot{\Gamma}(1 + i\lambda)}{\Gamma(1 + i\lambda)} d\lambda = \frac{r}{2\pi} \sum_{k=1}^{\infty} \hat{\psi}(ik) = < \frac{re^{-t}}{1 - e^{-t}}, \psi > .
\]

In an analogous manner

\[
-\frac{r}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}_-(\lambda) \frac{\dot{\Gamma}(1 + i\lambda)}{\Gamma(1 + i\lambda)} d\lambda = 0,
\]

where \( \psi_-(t) := \psi(-t) \). The term involving the scattering determinant is even more complicate. Fortunately, the following equation was shown by Müller [7]:

\[
\frac{i}{4\pi} \int_{-\infty}^{\infty} \hat{\phi}(\lambda) \frac{\dot{S}(\lambda)}{S(\lambda)} d\lambda = \sum_{\sigma \in \Sigma} m_\sigma (\hat{\psi}(\sigma) + \hat{\psi}(\bar{\sigma}))
\]

Let

**Definition 2.4**

\( \tilde{\theta}_s(t) := \tilde{\theta}_s(t), \quad \text{Im}(t) < 0. \)
\( \tilde{\theta}_s(t) \) is a holomorphic function on the lower half-plane. We can write

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} \phi(\lambda) \frac{\hat{S}(\lambda)}{S(\lambda)} d\lambda = \langle \theta_s (t + i0) + \tilde{\theta}_s(t - i0), \psi \rangle.
\]

Let \( \tilde{\theta} := \tilde{\theta}_d + \tilde{\theta}_s \). The following Lemma is now obvious.

**Lemma 2.5** The following equation of distributions holds on \((0, \infty)\)

\[
\theta(t + i0) + \tilde{\theta}(t - i0) = -\frac{\text{vol}(M) \cosh(t/2)}{4\pi} \frac{\cosh(t/2)}{\sinh^2(t/2)} + \sum_{c\text{-closed geodesic}} \frac{l_c}{2n_c \sinh(l_c/2)} \delta(t - l_c) + \frac{r}{1 - e^{-t}}.
\]

### 3 Analytic continuation across \((0, \infty)\)

**Theorem 3.1** \( \theta(t) \) has a meromorphic continuation across \((0, \infty)\) to the lower half-plane

\[
\theta(t) = -\tilde{\theta}(t) - \frac{\text{vol}(M) \cosh(t/2)}{4\pi} \frac{\cosh(t/2)}{\sinh^2(t/2)} + \frac{r}{1 - e^{-t}}, \quad \text{Im}(t) < 0.
\]

The singularities are

- first order poles at \( l_c, \) a closed geodesic, with residue \( \frac{l_c}{4\pi n_c \sinh(l_c/2)} \) and
- and second order poles at \(-2\pi i k, \) \( k = 1, 2, \ldots \) with the same singular part as

\[
-\frac{\text{vol}(M) \cosh(t/2)}{4\pi} \frac{\cosh(t/2)}{\sinh^2(t/2)} + \frac{r}{1 - e^{-t}}.
\]

**Proof:** We use (1) in order to define \( \theta(t) \) as a distribution on \( U := \mathbb{C} \setminus ((-\infty, 0] \cup -2\pi i \mathbb{N}) \). Then we compute \( \partial \tilde{\theta} \). The interesting region is near \((0, \infty)\). For \( \phi \in C_c^\infty(U) \) we define

\[
\langle \theta, \phi \rangle = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta(t + iu) \phi(t + iu) dt du
\]

\[
- \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta(t - iu) \phi(t - iu) dt du.
\]

In view of the polynomial bounds on the discrete spectrum and the singularities of the scattering determinant this defines a distribution. We compute, using Lemma 2.5,

\[
\langle \tilde{\theta}, \phi \rangle = \langle \theta, \tilde{\partial}^* \phi \rangle
\]

\[
= \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta(t + iu) \frac{1}{2} \left[ -\frac{\partial}{\partial(u)} - \frac{\partial}{\partial(t)} \right] \phi(t + iu) dt du
\]
4 \ θ NEAR \ ((-\infty, 0) \ni \theta \to 0)

\begin{align*}
&\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \theta(t - \epsilon \epsilon) \frac{1}{2} \left[ \frac{\partial}{\partial(\epsilon \epsilon)} - \frac{\partial}{\partial t} \right] \phi(t - \epsilon \epsilon) dt du \\
&\quad = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \theta(t + \epsilon) \frac{1}{2} \phi(t + \epsilon) dt \\
&\quad - \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \theta(t - \epsilon) \frac{1}{2} \phi(t - \epsilon) dt \\
&\quad = \frac{1}{2} < \theta(t + i0) + \tilde{\theta}(t - i0) + \frac{\text{vol}(M) \cosh(t/2)}{4\pi \sinh(t/2)} - \frac{r}{1 - e^{-t}}, \phi > \\
&\quad = \sum_{c \text{-closed geodesic}} \frac{l_c}{4n_c \sinh(l_c/2)} \delta(t - l_c), \phi > .
\end{align*}

Thus, \( \theta(t) \) is holomorphic on \( U \) except the points \( l_c, c \text{ closed geodesic} \). Since \( \tilde{\partial} \frac{1}{t} = \pi \delta(t) \), at these points are first order poles with the residues claimed in the Theorem. Since \( \tilde{\theta} \) is holomorphic for \( \text{Im}(t) < 0 \) the singularities in the lower half plane come from the additional terms

\[-\frac{\text{vol}(M) \cosh(t/2)}{4\pi \sinh(t/2)} + \frac{r}{1 - e^{-t}}.\]

\( \square \)

Unfortunately, this method cannot be applied in order to provide an extension of \( \theta \) across the negative real half-axis. We do not know the necessary trace formula as Lemma 2.5.

4 \ \theta \ near \ ((-\infty, 0) \ni \theta \to 0)

Note that the trace formula 2.5 involves the distributions

\[ \theta_s(t + i0) + \tilde{\theta}_s(t - i0), \ \theta_d(t + i0) + \tilde{\theta}_d(t - i0). \]

Obviously

\[ \theta_d(t + i0) + \tilde{\theta}_d(t - i0) = \theta_d(-t + i0) + \tilde{\theta}_d(-t - i0). \]

But

\[ \theta_s(t + i0) + \tilde{\theta}_s(t - i0) \neq \theta_s(-t + i0) + \tilde{\theta}_s(-t - i0). \]

We will apply the method of Cramer [3], Jorgenson/Lang [5] in order to provide a meromorphic extension of

\[ V(t) := \tilde{\theta}_s(-t) - \theta_s(t), \ \text{Im}(t) > 0. \]

Note that

\[ V(t) = \sum_{\sigma \in \Sigma} n_{\sigma}(e^{ilt\sigma} - e^{it\sigma}). \]

We write the scattering determinant as \( S(\lambda) = G(\lambda)L(\lambda) \), where

\[ G(\lambda) := e^{a + b\lambda} \left[ \sqrt{\pi} \frac{\Gamma(i\lambda)}{\Gamma(1/2 + i\lambda)} \right]. \]
(recall that \( r \) is the number of cusps). By Hejhal, \cite{4}, ch.8, 3.35, \( L(\lambda) \) has a representation as a certain Dirichlet series of the form

\[
L(\lambda) := 1 + \sum_{q \in Q} \frac{c_q}{\lambda^q}.
\]

Here, \( Q \) is a certain set of positive real numbers \( > 1 \) and \( c_q \) are real 'multiplicities'. In order to obtain the 1 in front of the sum one has to adjust \( a, b \in \mathbb{R} \) appropriately. The Dirichlet series converges for \( Im(\lambda) \) small enough. Let \( R := \{(n, q_1, \ldots, q_n) | n \in \mathbb{N}, q_i \in Q, i = 1, \ldots, n\} \). For \( p \in R \) define the multiplicity

\[
c(p) := \frac{(-1)^{n+1}}{n} \prod_{q \in p} c_q
\]

and

\[
|p| := \prod_{q \in p} q.
\]

Then \( \ln L(\lambda) \) also has a Dirichlet series representation

\[
\ln L(\lambda) := \sum_{p \in R} \frac{c(p)}{|p|^\lambda},
\]

which converges uniformly and absolutely in any half-plane \( Im(\lambda) \leq -A \) for some \( \infty > A > 0 \). Since \( S \) and \( G \) are meromorphic functions of finite order, \( L \) is also a meromorphic function of finite order. \( L \) satisfies the functional equation

\[
L(\lambda) = \frac{1}{G(-\lambda)G(\lambda)} \frac{1}{L(-\lambda)}.
\]

In Jorgenson/Lang it was shown that \( V(t) \) has a representation as a contour integral. Let \( \gamma \) be the piecewise straight path going from \( iA + \infty \) to \( iA \), then down to \( -iA \) and then back to \( -iA + \infty \). The down path avoids the singularities of \( S(z) \) on \( Re(\lambda) = 0 \) on positive real-part side. By Lang/Jorgenson \cite{5}, Thm. 2.3, we have

\[
2\pi i V(t) = \int_{\gamma} e^{it\lambda} \frac{\dot{S}(\lambda)}{S(\lambda)} + 2\pi i \sum_{\sigma \in \Sigma, Re(\sigma) = 0} m_\sigma (e^{-it\sigma} - e^{it\sigma}).
\]

We will study the integral further. Note that

\[
\frac{\dot{S}(\lambda)}{S(\lambda)} = \frac{\dot{G}(\lambda)}{G(\lambda)} + \frac{\dot{L}(\lambda)}{L(\lambda)}
\]

\( G \) has no singularities in the region encircled by \( \gamma \). Thus

\[
\int_{\gamma} e^{it\lambda} \frac{\dot{S}(\lambda)}{S(\lambda)} = \int_{\gamma} e^{it\lambda} \frac{\dot{L}(\lambda)}{L(\lambda)}.
\]
Let $h_1$ be the global holomorphic function

$$h_1(t) := -\int_{-A}^{A} e^{-t\lambda} \frac{\hat{L}(i\lambda)}{L(i\lambda)} d\lambda$$

(avoiding the singularities as above). Then

$$\int_\gamma e^{it\lambda} \frac{\hat{L}(\lambda)}{L(\lambda)} = h_1(t)$$

$$- e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{L}(iA + \lambda)}{L(iA + \lambda)} d\lambda = - e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{L}(-iA - \lambda)}{L(-iA - \lambda)} d\lambda$$

$$- e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{G}(-iA - \lambda)}{G(-iA - \lambda)} d\lambda$$

$$+ e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{G}(iA + \lambda)}{G(iA + \lambda)} d\lambda.$$ (3)

We employ now the functional equation (2) in order to rewrite (3).

$$- e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{L}(iA + \lambda)}{L(iA + \lambda)} d\lambda = - e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{L}(-iA - \lambda)}{L(-iA - \lambda)} d\lambda$$

$$- e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{G}(-iA - \lambda)}{G(-iA - \lambda)} d\lambda$$

$$+ e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{G}(iA + \lambda)}{G(iA + \lambda)} d\lambda.$$ (4)

In order to evaluate the integrals involving $L$ we use the representation of $\ln L$ as a Dirichlet series.

$$- e^{-At} \int_0^\infty e^{it\lambda} \frac{\hat{L}(-iA - \lambda)}{L(-iA - \lambda)} d\lambda$$

$$= - e^{-At} \ln L(-iA) - e^{-At} t \int_0^\infty e^{it\lambda} \ln L(-iA - \lambda) d\lambda$$

$$= - e^{-At} \ln L(-iA) - e^{-At} t \int_0^\infty e^{it\lambda} \sum_{p \in \mathbb{R}} \frac{c(p)}{|p|^{A+A}} d\lambda$$

$$= - e^{-At} \ln L(-iA) + e^{-At} t \sum_{p \in \mathbb{R}} \frac{c(p)}{|p|^{A}(t + \ln |p|)}.$$ (5)

Analogously

$$e^{At} \int_0^\infty e^{it\lambda} \frac{\hat{L}(-iA + \lambda)}{L(-iA + \lambda)} d\lambda$$

$$= - e^{At} \ln L(-iA) - e^{At} t \int_0^\infty e^{it\lambda} \ln L(-iA + \lambda) d\lambda$$
\[ -e^{At} \ln L(-iA) - e^{At} t \int_0^\infty e^{it\lambda} \sum_{p \in \mathbb{R}} \frac{c(p)}{|p|^{A+i\lambda}} d\lambda \]

\[ = -e^{At} \ln L(-iA) - e^{At} t \sum_{p \in \mathbb{R}} c(p) \int_0^\infty e^{it\lambda} e^{-(A-i\lambda)\ln |p|} d\lambda \]

\[ = -e^{At} \ln L(-iA) + e^{At} t \sum_{p \in \mathbb{R}} \frac{c(p)}{|p|^{A(t - \ln |p|)}}. \]

Define the global holomorphic function

\[ h_2(t) := -e^{-At} \ln L(-iA) - e^{At} \ln L(-iA) \]

and the global meromorphic function

\[ W(t) := e^{-At} t \sum_{p \in \mathbb{R}} \frac{c(p)}{|p|^{A(t + \ln |p|)}} + e^{At} t \sum_{p \in \mathbb{R}} \frac{c(p)}{|p|^{A(t - \ln |p|)}}. \quad (5) \]

Then

\[ -e^{-At} \int_0^\infty e^{it\lambda} \frac{\tilde{L}(iA + \lambda)}{L(iA + \lambda)} d\lambda + e^{At} \int_0^\infty e^{it\lambda} \frac{\tilde{L}(-iA + \lambda)}{L(-iA + \lambda)} d\lambda = h_2(t) + W(t). \]

Note that

\[ W(t) = W(-t), \quad h_2(t) = h_2(-t). \quad (6) \]

It remains to study the integrals involving the \( G \)-factors. We have

\[ \frac{\dot{G}(\lambda)}{G(\lambda)} = b + ir \left( \frac{\dot{\Gamma}(i\lambda)}{\Gamma(i\lambda)} - \frac{\dot{\Gamma}(1/2 + i\lambda)}{\Gamma(1/2 + i\lambda)} \right). \]

The term involving \( b \) will cancel out eventually (see (4)).

We consider

\[ -e^{-At} \int_0^\infty e^{it\lambda} \frac{\tilde{G}(-iA - \lambda)}{G(-iA - \lambda)} d\lambda. \]

We evaluate

\[ I_1(t) := -ie^{-At} \int_0^\infty e^{it\lambda} \frac{\dot{\Gamma}(A - i\lambda)}{\Gamma(A - i\lambda)} d\lambda. \]

Define the global holomorphic function

\[ h_3(t) := -e^{-At} \int_{-A}^0 e^{-t\lambda} \frac{\dot{\Gamma}(A + \lambda)}{\Gamma(A + \lambda)} d\lambda. \]

Since the singularities of \( \Gamma(A - i\lambda) \) are \(-A - ik, k = 0, 1, \ldots, \) we can change the path of integration obtaining

\[ I_1(t) = h_3(t) - ie^{-t} \int_0^\infty e^{it\lambda} \frac{\dot{\Gamma}(1 - i\lambda)}{\Gamma(1 - i\lambda)} d\lambda. \]
Assume $Re(t) > 0, Im(t) > 0$ for a moment. Define
\[ F(t) := \frac{1}{t} \int_0^\infty \frac{\lambda}{e^\lambda - 1} \frac{d\lambda}{\lambda + t} \]
and let $C$ be the Euler constant. It was shown by Cramér [3], eq. (12), that
\[ \int_0^\infty e^{it\lambda} \hat{\Gamma}(1 - i\lambda) \frac{d\lambda}{\Gamma(1 - i\lambda)} = \frac{C}{t} - F(t). \]
Moreover, there is a global meromorphic function
\[ M(t) := F(t) - \frac{\ln t}{e^t - 1}, \]
where the branch of the logarithm is chosen such that $\ln 1 = 0$ and the cut is at $(-\infty, 0]$. The poles of $M(t)$ are at $2\pi i \mathbb{Z}$ with residue $\text{sign}(k) \pi / 2$ at $t = k2\pi i$, $k \in \mathbb{Z}, k \neq 0$. Thus, we can write
\[ I_1(t) = h_3(t) - e^{-t}Ct + e^{-t}M(t) - \frac{\ln t}{e^t - 1}. \]
$I_1$ is a holomorphic function in $\mathbb{C} \setminus (-\infty, 0]$. The jump at the real negative axis is given by $\frac{2\pi i e^{-t}}{e^t - 1}$, if one crosses from above.

We evaluate
\[ I_2(t) := ie^{-At} \int_0^\infty e^{it\lambda} \hat{\Gamma}(1/2 + A - i\lambda) \frac{d\lambda}{\Gamma(1/2 + A - i\lambda)} d\lambda. \]
Define the global holomorphic function
\[ h_4(t) := e^{-At} \int_{1/2 - A}^{0} e^{-t\lambda} \hat{\Gamma}(A + 1/2 + \lambda) \frac{d\lambda}{\Gamma(A + 1/2 + \lambda)}. \]
Since the singularities of $\Gamma(A + 1/2 - i\lambda)$ are $-i(A + 1/2) - ik$, $k = 0, 1, \ldots$, we can change the path of integration obtaining
\[ I_2(t) = h_4(t) + ie^{-t/2} \int_0^\infty e^{it\lambda} \hat{\Gamma}(1 - i\lambda) \frac{d\lambda}{\Gamma(1 - i\lambda)} d\lambda. \]
Thus, we can write
\[ I_2(t) = h_4(t) + e^{-t/2} \frac{C}{t} - e^{-t/2}M(t) + \frac{e^{t/2} \ln t}{e^t - 1}. \]
$I_2$ is a holomorphic function in $\mathbb{C} \setminus (-\infty, 0]$. The jump at the real negative axis is given by $\frac{-2\pi i e^{t/2}}{e^t - 1}$, if one crosses from above.

Now we consider
\[ e^{-At} \int_0^\infty e^{it\lambda} \hat{G}(iA + \lambda) \frac{d\lambda}{G(iA + \lambda)}. \]
We evaluate
\[
I_3(t) := ie^{-At} \int_0^\infty e^{\mu\lambda} \frac{\hat{\Gamma}(-A + i\lambda)}{\Gamma(-A + i\lambda)} d\lambda.
\]
Assume, that \(A\) is not an integer or half-integer. Define the global holomorphic function
\[
h_5(t) := e^{-At} \int_0^{A+1} e^{\mu\lambda} \frac{\hat{\Gamma}(A + \lambda)}{\Gamma(A + \lambda)} d\lambda,
\]
where the path avoids singularities on the positive imaginary part side. The singularities of \(\Gamma(-A + i\lambda)\) are \(-iA + ik, \ k = 0, 1, \ldots\) We can change the path of integration obtaining
\[
I_3(t) = h_5(t) + ie^t \int_0^\infty e^{\mu\lambda} \frac{\hat{\Gamma}(1 + i\lambda)}{\Gamma(1 + i\lambda)} d\lambda.
\]
Assuming \(\Re(t) < 0\) for a moment we can write
\[
\begin{align*}
  ie^t \int_0^\infty e^{\mu\lambda} \frac{\hat{\Gamma}(1 + i\lambda)}{\Gamma(1 + i\lambda)} d\lambda &= e^t \int_0^\infty e^{\mu\lambda} \frac{\hat{\Gamma}(1 + \lambda)}{\Gamma(1 + \lambda)} d\lambda \\
  &= -te^t \int_0^\infty e^{\mu\lambda} \ln \Gamma(1 + \lambda) d\lambda \\
  &= te^t \int_0^\infty e^{-\mu\lambda} \ln \Gamma(1 - \lambda) d\lambda \\
  &= e^t \left( \frac{C}{t} + F(t) \right).
\end{align*}
\]
Now we can again apply a formula of Cramér (p. 114, top 3) and obtain
\[
I_3(t) = h_5(t) + e^t \frac{C}{t} + e^t M(-t) - \frac{\ln(-t)}{e^{-t} - 1}.
\]
\(I_3\) is a holomorphic function in \(\mathbb{C} \setminus [0, \infty)\). The jump at the real positive axis is given by \(\frac{2\pi i}{e^{\mu(1/2)} - 1}\), if one crosses from below.

We evaluate
\[
I_4(t) := -ie^{-At} \int_0^\infty e^{\mu\lambda} \frac{\hat{\Gamma}(-(A - 1/2) + i\lambda)}{\Gamma(-(A - 1/2) + i\lambda)} d\lambda.
\]
Define the global holomorphic function
\[
h_6(t) := -e^{-At} \int_0^{A+1/2} e^{\mu\lambda} \frac{\hat{\Gamma}(-(A - 1/2) + \lambda)}{\Gamma((A - 1/2) + \lambda)} d\lambda,
\]
where the path avoids the singularities on the positive imaginary part side. The singularities of \(\Gamma(-(A - 1/2) + i\lambda)\) are \(-i(A - 1/2) + ik, \ k = 0, 1, \ldots\) We can change the path of integration obtaining
\[
I_4(t) = h_6(t) - ie^{t/2} \int_0^\infty e^{\mu\lambda} \frac{\hat{\Gamma}(1 + i\lambda)}{\Gamma(1 + i\lambda)} d\lambda.
\]
Now we can again apply the formula of Cramér and obtain
\[ I_4(t) = h_6(t) - e^{t/2}C t - e^{t/2}M(-t) + \frac{e^{-t/2}ln(-t)}{e^{-t} - 1}. \]

\( I_4 \) is a holomorphic function in \( \mathbb{C} \setminus [0, \infty) \). The jump at the real positive axis is given by \( \frac{-2\pi i e^{-t/2}}{e^{-t} - 1} \), if one crosses from below.

Now we collect everything together. Define the global holomorphic function
\[
h(t) := h_1(t) + h_2(t) + rh_3(t) + rh_4(t) + rh_5(t) + rh_6(t) + 2\pi i \sum_{\sigma \in \Sigma, \Re(\sigma) = 0} m_\sigma(e^{-it\sigma} - e^{it\sigma}).
\]

**Theorem 4.1** If \( \Im(t) > 0 \) then
\[
2\pi i V(t) := 2\pi i(\tilde{\theta}_s(t) - \theta_s(t)) = h(t) + W(t) + r(-e^{-t/2}C t + e^{-t/2}C \frac{t}{\sqrt{t/2}} + e^{t/2}C t - e^{t/2}C \frac{t}{\sqrt{t/2}})
\]
\[
+ e^{-t}M(t) - e^{-t/2}M(t) + e^{t}M(-t) - e^{t/2}M(-t)
\]
\[
- \frac{ln t}{e^{t} - 1} + \frac{e^{t/2}ln t}{e^{t} - 1} - \frac{ln(-t)}{e^{-t} - 1} + \frac{e^{-t/2}ln(-t)}{e^{-t} - 1}.
\]

**Proposition 4.2**
\[
h(t) - h(-t) = 0
\]

**Proof** : We start with rewriting the integral defining \( h_1 \) using the functional equation for \( L \). We have
\[
h_1(t) = -2\int_{-A}^{A} e^{-t(\lambda - \alpha)} \frac{\dot{L}(\iota(\lambda + \iota 0))}{L(\iota(\lambda + \iota 0))} d\lambda
\]
\[
= -2\int_{0}^{A} e^{-t(\lambda - \alpha)} \frac{\dot{L}(\iota(\lambda + \iota 0))}{L(\iota(\lambda + \iota 0))} d\lambda
\]
\[
-2\int_{-A}^{0} e^{-t(\lambda - \alpha)} \frac{\dot{L}(\iota(\lambda + \iota 0))}{L(\iota(\lambda + \iota 0))} d\lambda,
\]
where we have indicated how the path avoids the singularities. We rewrite (7) as
\[
-2\int_{0}^{A} e^{t(\lambda + \iota 0)} \frac{\dot{L}(-\iota(\lambda + \iota 0))}{L(-\iota(\lambda + \iota 0))} d\lambda
\]
and apply the functional equation
\[
\frac{\dot{L}(-\iota(\lambda + \iota 0))}{L(-\iota(\lambda + \iota 0))} = \frac{\dot{L}(\iota(\lambda + \iota 0))}{L(\iota(\lambda + \iota 0))} + \frac{\dot{G}(\iota(\lambda + \iota 0))}{G(\iota(\lambda + \iota 0))} - \frac{\dot{G}(-\iota(\lambda + \iota 0))}{G(-\iota(\lambda + \iota 0))}
\]
in order to obtain \( h_1 = h_7 + r h_8 \) with
\[
\begin{align*}
    h_7(t) &= -i \int_0^A \left( e^{-t(\lambda - \sigma)} \frac{\hat{L}(\lambda - i0)}{L(\lambda - i0)} + e^{t(\lambda + i0)} \frac{\hat{L}(\lambda + i0)}{L(\lambda + i0)} \right) d\lambda \\
    r h_8(t) &= -i \int_0^A e^{t(\lambda + i0)} \left( \frac{\hat{G}(\lambda - 0)}{G(\lambda - 0)} + \frac{\hat{G}(-\lambda + 0)}{G(-\lambda + 0)} \right) d\lambda.
\end{align*}
\] (8)

The anti-symmetrization \( h_7(t) - h_7(-t) \) can be expressed in terms of the singularities of \( L \) on the interval \( (0, A) \). These singularities are contributed by the scattering matrix \( S \) as well as by the factor \( G \). We obtain
\[
    h_7(t) - h_7(-t) = -2\pi i \sum_{\sigma \in \Sigma, \text{Re}(\sigma) = 0} 2m_{\sigma} (e^{-t\sigma} - e^{t\sigma})
    -2\pi i r \sum_{0 < k < A} (e^{-tk} - e^{tk}) + 2\pi i r \sum_{0 < k < A - 1/2} (e^{-t(k+1/2)} - e^{t(k+1/2)}).
\]

Inserting the definition of \( G \) into (8) we obtain
\[
    h_8(t) = \int_0^A e^{t(\lambda + i0)} \left( \frac{\hat{G}(\lambda - 0)}{G(\lambda - 0)} - \frac{\hat{G}(1/2 - \lambda - i0)}{\Gamma(1/2 - \lambda - i0)} \right)
    - \frac{\hat{G}(\lambda + i0)}{\Gamma(\lambda + i0)} + \frac{\hat{G}(1/2 + \lambda + i0)}{\Gamma(1/2 + \lambda + i0)} d\lambda.
\]

We also have
\[
\begin{align*}
    h_3(t) &= -\int_1^A e^{-\lambda} \frac{\hat{G}(\lambda)}{\Gamma(\lambda)} d\lambda \\
    h_4(t) &= \int_{1/2}^A e^{-\lambda} \frac{\hat{G}(1/2 + \lambda)}{\Gamma(1/2 + \lambda)} d\lambda \\
    h_5(t) &= \int_{-1}^A e^{-\lambda} \frac{\hat{G}(\lambda + i0)}{\Gamma(\lambda + i0)} d\lambda \\
    h_6(t) &= -\int_{-1/2}^A e^{-\lambda} \frac{\hat{G}(1/2 - \lambda + i0)}{\Gamma(1/2 - \lambda + i0)} d\lambda.
\end{align*}
\]

Thus, defining
\[
    h_9(t) := h_8(t) - h_3(-t) - h_4(-t) - h_5(-t) - h_6(-t)
\]

and anti-symmetrizing we obtain
\[
    \begin{align*}
    h_9(t) - h_9(-t) &= 2\pi i \sum_{0 < k < A} (e^{-tk} - e^{tk}) - 2\pi i \sum_{0 < k < A - 1/2} (e^{-t(k+1/2)} - e^{t(k+1/2)}).
    \end{align*}
\]

Since \( h_2 \) is symmetric,
\[
    h(t) - h(-t) = h_7(t) - h_7(-t) + r(h_9(t) - h_9(-t)) + 2\pi i \sum_{\sigma \in \Sigma, \text{Re}(\sigma) = 0} 2m_{\sigma} (e^{-t\sigma} - e^{t\sigma}) = 0
\]
and the proposition follows. □

We can now write down a distributional trace formula on \((-\infty, 0)\) containing \(\theta(t + i0) + \tilde{\theta}(t - i0)\). In fact, for \(t > 0\)

\[
2\pi i(\theta_s(-t + i0) + \tilde{\theta}_s(-t - i0)) = 2\pi i(\theta_s(t + i0) + \tilde{\theta}_s(t - i0) + V(t + i0) - V(-t + i0))
\]

\[
= 2\pi i(\theta_s(t + i0) + \tilde{\theta}_s(t - i0)) + W(t + i0) - W(-t + i0)
\]

\[
+ 2\pi i r \left( \frac{1}{e^t - 1} - \frac{e^{-t/2}}{e^t - 1} \right).
\]

By (8) we have

\[
W(t + i0) - W(-t + i0) = -2\pi i \sum_{p \in \mathbb{R}} \ln |p| c(p) \delta(t - \ln |p|).
\]

We combine these two equations with the distributional trace formula and obtain

**Theorem 4.3** The following equation of distributions on \((-\infty, 0)\) holds:

\[
\theta(t + i0) + \tilde{\theta}(t - i0) = -r \frac{e^{t/2}}{e^t - 1} - \frac{\text{vol}(M) \cosh(t/2)}{4\pi \sinh^2(t/2)}
\]

\[
+ \sum_{c\text{-closed geodesic}} \frac{l_c}{2n_c \sinh(l_c/2)} \delta(t + l_c)
\]

\[
- \sum_{p \in \mathbb{R}} \ln |p| c(p) \delta(t - \ln |p|).
\]

We can now apply exactly the same technique as for Theorem 3.1 in order to prove

**Theorem 4.4** \(\theta(t)\) has a meromorphic continuation from the lower half-plane to another sheet of the upper half-plane across \((-\infty, 0)\). For \(\text{Re}(t) > 0\) this extension \(\theta_1\) is given by

\[
\theta_1(t) = \theta(t) + r \left( \frac{1}{1 - e^{-t}} - \frac{e^{t/2}}{e^t - 1} \right)
\]

\[
= \theta(t) + r \frac{e^{t/2} - 1}{2\sinh(t/2)}.
\]

The singularities on the negative real axis are first order poles in the points \(-l_c, c\text{ a closed geodesic, with residue } \frac{l_c}{4\pi n_c \sinh(l_c/2)}\) and in the points \(-\ln |p|, p \in \mathbb{R}, \text{ with residue } \frac{\ln |p| c(p)}{2\pi}\) (if two such points coincide, the residues add up). Moreover, it has first order poles in the points \((4k - 2)\pi i, k = 1, 2, \ldots\), with residue \(-2r\).
5 Conclusion

We have seen that $\theta(t)$ extends meromorphically to the Riemann surface of the logarithm. The difference between two sheets is

$$\theta_1(t) - \theta(t) = r \frac{e^{t/2} - 1}{2 \sinh(t/2)}.$$ 

Thus, the modified theta-function

$$\Theta(t) := \theta(t) + r \frac{\ln(t) e^{t/2} - 1}{2 \sinh(t/2)}$$

admits a meromorphic extension to the complex plane.

**Theorem 5.1** The meromorphic extension of the modified theta function has the following singularities:

- First order poles at $l_c$, $c$ a closed geodesic, with residue $\frac{l_c}{4\pi n_c \sinh(l_c/2)}$.
- First order poles at $l_c$, $c$ a closed geodesic, with residue $\frac{l_c}{4\pi n_c \sinh(l_c/2)}$ and at $-\ln|p|$, $p \in \mathbb{R}$, with residue $-\frac{\ln|p|e(p)}{2\pi}$ (if two such points coincide, the residues add up).
- First order poles at $(4k-2)\pi t$, $k = 1, 2, \ldots$, with residue $-r/2 - \ln(\pi(4k-2)) \frac{\pi}{\pi t}$.
- Second order poles at $(4k+2)\pi t$, $k = -1, -2, \ldots$, with residue $3r/2 - \ln(\pi(4k+2)) \frac{\pi}{\pi t}$ and second Laurent coefficient $-\frac{\text{vol}(M)}{\pi}$
- And second order poles at $4k\pi t$, $k = -1, -2, \ldots$, with residue $r$ and second Laurent coefficient $-\frac{\text{vol}(M)}{\pi}$.

Clearly, $\Theta$ has at most a pole of second order at $t = 0$.

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