The Parabolic Mandelbrot Set

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Abstract

In this paper we prove that there is a natural dynamically defined homeomorphism between the parabolic Mandelbrot set $M_1$ and the Mandelbrot set $M$, as it was conjectured by J. Milnor in ‘93.

1 Introduction

A natural question in dynamical systems is whether one can read in the parameter space the dynamics of the underlying systems. More precisely the description that one wants to detect is the dichotomy between stability and bifurcations in the phase locus and then deduce a description of the dynamics of the corresponding map.

Pursuing this question in the generality of $M_2 := \text{Rat}_2/\text{PSL}(2, \mathbb{C})$ — the moduli space of rational maps of degree two acting on the Riemann sphere $\hat{\mathbb{C}}$ —, has been made possible thanks to the development of the analysis of J. Milnor in [Mi1]. It persistently allows for direct interaction between one parameter slices in $M_2 \simeq \mathbb{C}^2$ and the underlying dynamical system. He considers the particular affine lines

$$\text{Per}_1(\mu) = \{[f] \in M_2 \mid f \text{ has a fixed point with multiplier } \mu\}.$$ 

indeed a quadratic rational map has counting multiplicity three fixed points. By definition of $\text{Per}_1(\mu)$ one of the fixed point multipliers is $\mu$. The product of the two other fixed point multipliers define an isomorphism $\sigma_\mu$ between $\text{Per}_1(\mu)$ and $\mathbb{C}$. See [Mi1, Lemma 3.4] for details. To simplify notation we shall henceforth identify $\text{Per}_1(\mu)$ with $\mathbb{C}$ via this isomorphism. The most interesting dynamics of a rational map $f$ on the Riemann sphere $\hat{\mathbb{C}}$ takes place on its Julia set, $J(f)$, which is the minimal non trivial compact set, invariant by $f$ and $f^{-1}$. Moreover, the dynamics of a quadratic rational map $f$ is rich, when its Julia set is connected, since otherwise it is conjugated to the shift on a Cantor set.
Hence, in $Per_1(\mu)$, the interesting dynamical systems are located in the connectedness locus

$$M_\mu := \{[f] \in Per_1(\mu) \mid J(f) \text{ is connected}\}.$$

For $\mu = 0$, $M_0$ corresponds to the classical Mandelbrot set $M$. There is an extensive knowledge and literature about quadratic polynomials and the Mandelbrot set pioneered by Douady and Hubbard (see [DH1] and [DH2]). The family of quadratic polynomials $Q_c(z) = z^2 + c$, $c \in \mathbb{C}$ parametrizes $Per_1(0)$. The Julia set $J_c = J(Q_c)$ is the common boundary of the filled Julia set $K_c = K(Q_c) = \{z \in \mathbb{C} \mid Q^n_c(z) \text{ is bounded}\}$ and the basin of infinity $B_c(\infty) := \{z \in \mathbb{C} \mid Q^n_c(z) \to \infty\} = \mathbb{C} \setminus K_c$. The Mandelbrot set $M = \{c \in \mathbb{C} \mid J(Q_c) \text{ is connected}\}$ is also the set of parameters $c \in \mathbb{C}$ such that $c \in K_c$. The product $\sigma_0$ of the multipliers of the two finite fixed points of $Q_c$ equals $4c$, so that $M_0 = 4M$. The central hyperbolic component $Card$, i.e. the connected component of the interior of $M$ containing $0$ consists of parameters $c$ for which $Q_c$ has an attracting fixed point $\alpha_c \in \mathbb{C}$.

When $|\mu| < 1$, first Lyubich, Goldberg-Keen, then Uhre and, in a different spirit Bassanelli-Berteloot, proved that $M_\mu$ is a quasi-conformal deformation of $M_0$.

Milnor then conjectured (in [Mi1]) that for $\mu = 1$ there is still a homeomorphism:

**Conjecture** (Milnor ('93)). *The parabolic Mandelbrot set $M_1$ is homeomorphic to the Mandelbrot set $M$. Moreover the homeomorphism preserves the dynamics on the Julia sets.*

The aim of this paper is to prove this conjecture of Milnor, i.e. to construct such a homeomorphism. This conjecture is supported by computer pictures, see Fig.1 and Fig.2.

For $[g] \in Per_1(1)$, the filled Julia set $K(g)$ is the set of points whose orbit is not converging to the parabolic fixed point with multiplier 1.

**Main Theorem.** There exists a dynamically defined homeomorphism $\Phi_1 : M \to M_1$.

In this theorem, *dynamically defined* has the following sense. Let $c \in M$ and $[g] = \Phi_1(c)$.

- Either $c \in \partial Card$, then its dynamics is determined by the multiplier $\lambda \in S^1$ of its fixed point. The map $\Phi_1$ will be constructed such that the map $g$ has a fixed point with the same multiplier $\lambda$, when $\lambda \in \mathbb{D}$. Therefore $Q_c$ and $g$ have the same type of dynamics.
- In case $c \in M \setminus \partial Card$, there exists an homeomorphism

$$\rho_c : K(Q_c) \to K(g)$$
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conformal in the interior of $K(Q_c)$, conjugating $f$ and $g$ (see below Figure 2):

$$
\begin{array}{ccc}
K(Q_c) & \overset{f}{\rightarrow} & K(Q_c) \\
\rho_c & \downarrow & \rho_c \\
K(g) & \overset{g}{\rightarrow} & K(g)
\end{array}
$$

In fact, there are only some $c \in \partial \text{Card}$, for which it is not clear how to construct such a conjugacy on the Julia sets.

Figure 2: A parabolic Julia set (left) and the corresponding one in $M$ (right)

**Previous developpements**

Note that for $|\mu| < 1$, every rational map of $\text{Per}_1(\mu)$ has an attracting fixed point, say at infinity, so it is a quadratic-like map. This gives, using the work of Douady and Hubbard ([DH3]), a homeomorphism between $M$ and $M_\mu$. 

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conformal in the interior of $K(Q_c)$, conjugating $f$ and $g$ (see below Figure 1):

$$
\begin{array}{ccc}
\mathbb{C} & \overset{\mu}{\rightarrow} & \mathbb{C} \\
\mu & \overset{(\mu, z) = \mu(z)}{\rightarrow} & (\mu, z)
\end{array}
$$

• $\mu \rightarrow (\mu, z)$ is holomorphic on $D$ for any $z$,
• $z \rightarrow (\mu, z)$ is injective in $\mathbb{C}$ for any $\mu$,
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**Theorem** (Goldberg-Keen, Lyubich). For any $\mu \in D$, there exists a homeomorphism $\Phi_\mu : \mathbb{C} \to \mathbb{C}$ such that $\Phi_\mu(M_0) = M_\mu$. Moreover for $c \in M$, and $[g] = \Phi_\mu([Q_c])$ the maps $g$ and $Q_c$ are quasi-conformally conjugate on a neighborhood of their filled Julia set.

The maps $\Phi_\mu$ define a holomorphic motion:

**Theorem** (Lyubich, Uhre, Bassanelli-Berteloot). Those maps $\Phi_\mu$ form a holomorphic motion. $\Phi : D \times \mathbb{C} \to \mathbb{C}$ defined by $\Phi(\mu, z) = \Phi_\mu(z)$. It verifies

- $\mu \to \Phi(\mu, z)$ is holomorphic on $D$ for any $z$,
- $z \mapsto \Phi(\mu, z)$ is injective in $\mathbb{C}$ for any $\mu$,
- $\Phi(0, z) = z$ on $\mathbb{C}$.

**Challenges**

In the case at hand, where $\mu = 1$, the parabolic point creates completely different dynamics. The challenges to be overcome include:

1. There is no straightening result (like Douady-Hubbard straightening theorem) going from the parabolic world to the hyperbolic world (the disks defining the polynomial like map touch at their boundary). The work of Haissinsky (using Guy David’s theorem) applies to most but not all the family here. (there are restrictive conditions on the post-critical set), see [Ha1]. There is a notion of parabolic-like maps (see [Lo]), but it refers only to parabolic families;

2. There is no existing theory which allows us to extend the holomorphic motion $\Phi$ at the boundary point 1. There is definitely no extension of the holomorphic motion in neighbourhood of 1, because $M$ and $M_1$ are not quasi-conformally equivalent;

3. There is no complete description of the dynamics inside of the interior of $M$ that would allow us to compare the dynamics;

4. There could be queer components in $M_1$ which do not correspond to a queer component in $M$, i.e. components of the interior for which all periodic points are repelling for those maps. The above mentioned work of Haissinsky implies that every queer component in $M$ if any would correspond to a queer component in $M_1$ (Fatou’s conjecture says that there are none in $M$);

5. There is no complete description of the boundary $\partial M$ that describes the dynamics of the maps for instance by the exterior and that could have been transported. (It is conjectured that $M$ is locally connected).
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The strategy of the proof of the Main Theorem

Our strategy is to compare the representation of \( M \) given in \([PR1]\) in terms of combinatorial data together with analytic data with a similar representation of \( M_1 \). The representation of \( M \) was obtained with the use of Yoccoz’ theorem. More precisely, replacing every maximal, i.e. level one renormalization copy of \( M \) strictly inside \( M \) we obtain a tree. This tree is faithfully described by a space of stratified equivalence relations/laminations called towers. The union of the equivalence relations in a tower is an equivalence relation, which is forward invariant under \( Q_0 \) and which corresponds to the co-landing pattern of those rays which eventually land on the \( \alpha \)-fixed point.

In \([PR2]\) we provided a dynamically defined map \( \Psi^1 : M_1 \rightarrow M \). This was constructed by associating combinatorial and analytic data similar to those for quadratic polynomials to each element of \( M_1 \). The map \( \Psi^1 \) takes \( g \) in \( M_1 \) to \( c = \Psi^1(g) \) such that \( g \) and \( Q_c \) have the same combinatorial and analytic data.

In this paper we prove that \( \Psi^1 \) is a homeomorphism. We do this in two steps. First we prove an analogue of Yoccoz parameter theorem for \( M_1 \). From this theorem it easily follows that \( \Psi^1 \) is a bijection and is continuous, except possibly at the boundary of the level one renormalization copies of \( M \). The second and last step is to prove the continuity at those remaining parameters. The continuity of the inverse \( \Phi^1 : M \rightarrow M_1 \) then follows from abstract reasons.

In the course of the proof we prove the aforementioned parabolic Yoccoz parameter theorem, which falls in two parts shrinking of limbs along the disk and a parameter puzzle theorem, and we prove a shrinking carrots theorem for \( M_1 \).

2 Notation and description of the dynamics of Quadratic polynomials and maps in \( \text{Per}_{1}(1) \).

2.1 Basic notions for quadratic polynomials

An essential tool in the study of the dynamics of quadratic polynomials and the Mandelbrot set is the notion of external rays. For \( c \in \mathbb{C} \) we let \( \phi_c \) denote the Böttcher-coordinate conjugating \( Q_c \) to \( Q_0 \) in a neighbourhood of \( \infty \) normalized by being tangent to identity at \( \infty \). The Green’s function \( g_c \) for \( B_c(\infty) \) is the subharmonic function on \( \mathbb{C} \) defined by \( g_c(z) = \log|\phi_c(z)| \) on the domain of \( \phi_c \), the recursive relation \( g_c(Q_c(z)) = 2g_c(z) \) and \( g_c \equiv 0 \) on \( K_c \). The Böttcher-coordinate has a unique univalent extension \( \phi_c : \{z | g_c(z) > g_c(0)\} \rightarrow \hat{\mathbb{C}} \setminus \mathbb{D}(e^{g_c(0)}) \).

Denote by \( \psi_c \) the inverse of \( \phi_c \). The map \( \psi_c \) analytically extends along rays \( R_\theta \), where
Figure 3: In black, $M_1$, viewed in the $\sigma_1$ coordinate. Maps in $Per_1(1)$ have a degenerate fixed point, so that $\sigma_1([g])$ is also the multiplier of the unique third fixed point $\alpha_g$ of $g$. In particular the big black disk is the unit disk. It corresponds to $\alpha_g$ being attracting.

$R_\theta$ is the straight line of angle $\theta$, $\theta \in \mathbb{R}/\mathbb{Z} =: \mathbb{T}$. This extension stops when reaching $\mathbb{D}$ or a point whose image by $\psi_c$ is a pre-critical value of $Q_c$. In the rest of the paper we shall let $\psi_c$ mean the maximal radial extension. The external ray of angle $\theta$ is defined by $R_c = \psi_c(R_\theta)$. An external ray which stops at a pre-critical value is said to bump. The dynamics of $Q_c$ on rays is semi-conjugate to angle-doubling $m_2^{\theta} = 2\theta \mod 1$. Douady and Hubbard proved that a non-bumping (pre)-periodic ray lands at a (pre)-periodic repelling or parabolic orbit with the same pre-period and period dividing that of the ray (i.e. that of $\theta$). More precisely

**Theorem 2.1** (Douady-Hubbard). Let $c \in \mathbb{C}$. For any (pre-)periodic argument $\theta \in \mathbb{T}$, i.e. $m_2^k(m_2^l(\theta)) = m_2^l(\theta)$, if the external ray $R = R_{\theta}$ does not bump, then it converges to a $Q_c$ (pre-)periodic point $z \in J_c$ with $Q_c^k(Q_c^l(z)) = Q_c^l(z)$. If the argument is periodic (i.e. $l = 0$), let $k'$ denote the exact period of $z$ and let $q = k/k'$. Then the ray $R$ defines the combinatorial rotation number $p/q$, $(p, q) = 1$ for $z$. The periodic point $z$ is repelling or parabolic with multiplier $e^{2\pi p/q}$. Moreover any other external ray landing at $z$ is also $k$-periodic and defines the same rotation number.

**Theorem 2.2** (Douady). Let $c \in \mathbb{M}$ and suppose $z$ is a (pre)-periodic point, $Q_c^{k'}(Q_c^l(z)) = Q_c^l(z)$, $l \geq 0$ and $k \geq 1$. And suppose that $w = Q_c^l(z)$ is either repelling or parabolic. Then $w$ has a combinatorial rotation number in the sense above. That is $w$ is the landing point of at least one external ray and all rays landing at $w$ form a single cycle under $Q_c^{k'}$. 
Moreover $z$ is the landing point of the same number of rays, each one being a preimage of a ray landing at $w$.

For $c \in \mathbb{M}$, the map $\phi_c$ extends to an isomorphism between the basin of infinity and $\hat{\mathbb{C}} \setminus \mathbb{D}$, so that no ray bumps. The two fixed points of $Q_c(z)$ are labelled $\beta_c$ and $\alpha_c$, with the convention that $\beta_c$ is the landing point of the unique fixed ray, $R_0^c$. The other fixed point $\alpha_c$ can be attracting, neutral or repelling. It is non repelling precisely when $c \in \text{Card}$. Thus by Theorem 2.2, $\alpha_c$ is the landing point of $q > 1$ external rays that define a cycle of combinatorial rotation number $p/q$, and that thus assigns rotation number $p/q$ to $\alpha_c$. This leads to the following stratification of $\mathbb{M}$, (see [Mi4]).

**Theorem 2.3** (Douady-Hubbard).

\[ M = \overline{\text{Card}} \cup \bigcup_{\frac{p}{q} \neq 0} L^*_p, \]  

where the uprooted limb $L^*_p$ consists of those parameters $c \in \mathbb{M}$ for which the separating fixed point $\alpha_c$ is repelling and has combinatorial rotation number $p/q$.

### 2.2 Wakes

Since the Böttcher-coordinate $\phi_c$ extends univalently to the set of points of potential greater than $g_c(0)$ it follows that $\Phi_M : \mathbb{C} \setminus M \to \mathbb{C} \setminus \mathbb{D}$ given by $\Phi_M(c) := \phi_c(c)$ is well defined for $c \in \mathbb{C} \setminus M$. Douady and Hubbard proved that $\Phi_M$ is an isomorphism tangent to the identity at infinity. Let $\Psi_M$ denote its inverse. The parameter ray $R^M_{\theta}$ of argument $\theta \in \mathbb{T}$ is by definition $R^M_{\theta} := \Psi_M(R_0^\theta)$.

Rational parameter rays land at special parameters on the boundary of $\mathbb{M}$, see e.g. [DH1] and [DH2].

**Theorem 2.4** (Douady-Hubbard). Let $\theta \in \mathbb{T}$ be any (pre-)periodic argument under $m_2$, i.e. $m_2^k(m_2^l(\theta)) = m_2^l(\theta)$. Then $R^M_{\theta}$ lands a parameter $c \in \partial \mathbb{M}$.

If the argument is periodic (i.e. $l = 0$), then $Q_c$ admits a parabolic orbit of period $k'$ and multiplier $e^{2\pi i/p'}$ such that $k'q = k$. And the dynamical ray $R^M_{\theta}$ lands at a point in this parabolic orbit. Moreover if $k > 1$ then $c$ is also the landing point of precisely one more parameter ray $R^M_{\theta'}$. And $\theta$ and $\theta'$ belong to the same cycle if $q > 1$, the satelite case and to two different cycles if $q = 1$, the primitive case. In either case the critical value $c$ is contained in the wake $W_{\theta}(\theta, \theta')$, i.e. the domain bounded by the closure of the co-landing rays $R^M_{\theta}, R^M_{\theta'}$ and not containing 0.

If $l > 0$ then $R^M_{\theta}$ lands at a Misiurewicz parameter $c$, such that $Q^l_c(c)$ belongs to a repelling cycle of exact period $k$. Moreover for any $\theta' \in \mathbb{T}$ such that the dynamical ray $R^M_{\theta'}$ lands on $c$, the parameter ray $R^M_{\theta'}$ also lands on $c$. 
For $k > 1$ periodic arguments of parameter rays $\mathcal{R}^M_\alpha, \mathcal{R}^M_\beta$ co-landing on a parabolic parameter $c$ as in the theorem above, the parameter wake $\mathcal{W}^M(c, \alpha, \beta)$ is the domain bounded by the closure of these rays and not containing the origin. It is easy to see that for any $c' \in \mathcal{W}^M(c, \alpha, \beta)$ the dynamical rays $\mathcal{R}^c_\alpha, \mathcal{R}^c_\beta$ move holomorphically with $c'$ and co-land on a repelling periodic point $w_c$, which becomes the parabolic periodic point of $Q_c$ with the same rays landing, when $c'$ converges to the root $c$ of the wake. And that for $c' \notin \mathcal{W}^M(c, \alpha, \beta)$ the two rays $\mathcal{R}^c_\alpha, \mathcal{R}^c_\beta$ either land on different periodic orbits or at least one of them bump. For $c' \in \mathcal{W}^M(c, \alpha, \beta)$ we may thus also define the dynamical wake $\mathcal{W}_{c'}(c, \alpha, \beta)$ by the same description.

Without loss of generality we can assume $0 < \theta < \theta' < 1$. There exists $\theta, \theta'$, $0 < \theta < \theta' < \theta < \theta' < 1$ such that $m_2^k$ maps each of the intervals $[\theta, \theta']$ and $[\theta', \theta]$ diffeomorphically onto the full interval $[\theta, \theta']$. Let $C(\theta, \theta')$ denote the dyadic Cantor set consisting of those points which never escapes $[\theta, \theta'] \cup [\theta', \theta]$ under iteration by $m_2^k$. Then $C(\theta, \theta')$ naturally corresponds to the classical middle third Cantor set. As described by the Douady tuning algorithm the points of $C(\theta, \theta')$ are almost all of the arguments of external rays, which accumulates the filled-in Julia set $K_c$ with $w_c \in K_c \subset \mathcal{W}_{c'}(c, \alpha, \beta)$ of a polynomial-like restriction of degree 2 of $Q_k^c$ (see also Lemma 6.1). In the case of $q > 2$ each gap of the Cantor set contains $q - 2$ additional arguments of $K_c$. It is a deep theorem that the set of such parameters for which $K_c$ is connected, is a (derooted in the satellite case) copy $M^M_{\theta, \theta'}$ of the Mandelbrot set. And the set of arguments for parameter rays accumulating $M^M_{\theta, \theta'}$ contains $C(\theta, \theta')$.

The gaps of the Cantor set $C(\theta, \theta')$ naturally corresponds to the dyadic numbers $r/2^s$, $0 < r, s, r$ odd and $r < 2^s$ with the initial gap $[\theta', \theta']$ corresponding to $1/2$. The endpoints of the gap corresponding to $r/2^s$ map under $m_2^k$ to the $m_2^k$ fixed points $\theta, \theta'$ and so the corresponding parameter rays co-land on a Misiurewicz parameter $c'' = c(\theta, \theta', r, s)$ such that $Q_k^c(c'') = w_{c'}$. We denote by $\mathcal{W}^M(c, \alpha, \beta, r, s)$ the wake bounded by the closure of these rays. For $c' \in \mathcal{W}^M(c, \alpha, \beta, r, s)$ the corresponding dynamical rays co-land on a pre-image of $w_{c'}$ under $Q_k^c$ and so bound a corresponding dynamical wake $\mathcal{W}_{c'}(c, \alpha, \beta, r, s)$. We define the $r/2^s$-dyadic limb of $M^M_{\theta, \theta'}$ as the intersection

$$L^M(\theta, \theta', r, s) := \mathcal{W}_{c}(\theta, \theta', r, s) \cap M.$$ 

For a similar discussion see [PR3, Section 1].

In the special case where the $k$-periodic parameter rays $\mathcal{R}^M_\alpha, \mathcal{R}^M_\beta$ co-land on a parabolic parameter $c$ for which the parabolic periodic point of period $k' = 1$, i.e. equals $\alpha_c$, so that $\theta, \theta'$ belong to a $p/q$-cycle with $q = k$, we use the standard short hand $\mathcal{W}^M(p/q)$ for the parameter wake $\mathcal{W}^M(\theta, \alpha, \beta)$, $\mathcal{W}_{c}(p/q)$ for the dynamical wake $\mathcal{W}_{c'}(c, \alpha, \beta)$ when $c' \in \mathcal{W}^M(p/q)$, $M_{\theta, \theta'}$ for $M_{\theta, \theta'}$, $\mathcal{W}^M(p/q, r, s)$ for the dyadic parameter wake $\mathcal{W}^M(\theta, \alpha, \beta)$ and $\mathcal{W}_{c'}(p/q, r, s)$ for corresponding dyadic dynamical wakes $\mathcal{W}_{c'}(c, \alpha, \beta)$. Note that by definition the uprooted limb $L^*_{p/q} = M \cap \mathcal{W}^M(p/q)$. 

2.3 Basic notation for maps in \( \text{Per}_1(1) \).

The slice \( \text{Per}_1(1) \) does not admit a normal form which univalently parameterizes it. As a consequence there is not a universal choice of parametrization. We shall henceforth use several parametrizations interchangeably. First of all we shall write \([g]\) for the element in \( \text{Per}_1(1) \) represented by \( g \). Such a map \( g \) has a parabolic fixed point of multiplier 1 and one more fixed point of multiplier \( A \in \mathbb{C} \). This fixed point coincides with the two others precisely when \( A = 1 \). Thus \( \sigma_1([g]) = A \), because the parabolic fixed point of multiplier 1 fixed point is multiple. We invite the reader to think of \( g \) as taking the form \( g(z) = g_B(z) = z + B + 1/z \) for \( B \in \mathbb{C} \), in which case the fixed point of multiplier 1 is at infinity and the critical points are located at \( \pm 1 \) and the corresponding critical values are \( B \pm 2 \). For \( B = 0 \) the three fixed points coincide at \( \infty \) and otherwise \( g_B \) has a finite fixed point at \(-1/B\) with multiplier \( A = 1 - B^2 = \sigma_1([g_B]) \). As a consequence we have

**Remark 2.5.** The correspondence \( B \in \mathbb{C} \mapsto [g_B] \in \text{Per}_1(1) \) is a 2 to 1 branched covering. We shall use interchangeable the notations \( g = g_B, [g], B \) and \( A \). In particular we shall use \( g \in \mathcal{M}_1 \), \([g] \in \mathcal{M}_1 \), \( A \in \mathcal{M}_1 \) and \( B \in \mathcal{M}_1 \) in the obvious meaning.

However we shall mainly be interested in \( A \in \mathbb{C}\setminus[1, \infty[ \) which biholomorphically corresponds to \( B \in \mathbb{H}_+ := \{x + iy | x > 0\} \). For this reason our preferred representation of \( \text{Per}_1(1) \) shall be via the parameter \( B \in \mathbb{H}_+ \cup \{0\} \) or \( B \in \mathbb{H}_+ := \{B' | \Re(B') \geq 0\} \).

For \( B \in \mathbb{H}_+ \) the critical point 1 is the fastest escaping critical point. Indeed maps \( g_r \), where \( r \in \mathbb{R} \) commutes with reflection in the imaginary axis, so that the critical points \( \pm 1 \) escapes at equal rates for any \( r \) (for \( r = 0 \) the critical points however are in distinct components of the parabolic basin). Since the map \( B \mapsto 1 - B^2 \) is a two to one branch covering of \( \text{Per}_1(1) \), the parameters \( B = ir \) are the only maps for which the critical points escapes at equal rates. Thus in \( \mathbb{H}_+ \) either \( +1 \) is always the fastest escaping critical point or \(-1 \) is. For \( B = 1 \) we have \( g_1(-1) = -1 \) so that the critical point \(-1 \) is fixed and thus \(+1 \) is the only escaping and hence fastest escaping critical point.

For a map \( g = g_B \) the parabolic basin \( \Lambda_B \) is the maximal open subset of points converging to the parabolic fixed point \( \infty \) of multiplier 1. It is completely invariant : \( g^{-1}(\Lambda_g) = \Lambda_g \). The parabolic basin \( \Lambda_0 \) of \( g_0 \) has two connected components each one containing a critical point and value and the Julia set \( J(0) = i \mathbb{R} \cup \{\infty\} \). For \( B \neq 0 \) the parabolic basin \( \Lambda_B \) is connected and the filled Julia set is \( K(B) = \mathbb{C} \setminus \Lambda_B \). Moreover the Julia set is the common boundary \( J(B) = \partial K(B) = \partial \Lambda_B \). The Julia set is either connected and \( B \in \mathcal{M}_1 \) or it is a Cantor set (see [Mi1]). In the second case \( \Lambda_B \) is connected, infinitely connected, contains both critical points and the dynamics on the Julia set is conjugate to the one-side shift map on two symbols. For \( B \neq 0 \), \( g_B \) admits an attracting Fatou coordinate \( \phi_B \), which conjugates \( g_B \) to translation by 1 on some right half plane \( H \). The map \( \phi_B \) is unique up to post-composition by a translation. We denote also by \( \phi_B \) its analytic extension to all
of $\Lambda_B$. The extension is not injective and is thus a semi-conjugacy. For $B = 0$ there are two such coordinates one on each connected component of the basin $\Lambda_{g_0}$ of $\infty$. In the next section we shall describe another representative $Bl$ of $[g_0]$, which is more suitable for comparison with the polynomial $z^2$.

For $B \in \mathbb{H}_+$ the basin $\Lambda_B$ contains a unique maximal forward invariant sub-domain $\Omega^B$, which contains the critical value $B + 2$, which is mapped univalently onto some right half plane $H$ by a Fatou-coordinate and whose boundary contains the critical point 1. We denote by $v_B$ the other critical value $B - 2 = g_B(-1)$.

**Remark 2.6.** We normalise $\phi_B$ by $\phi_B(1) = 0$, so that the restriction

$$\phi_B : \Omega^B \longrightarrow \mathbb{H}_+$$

is a biholomorphic conjugacy.

For $B = 0$ there are two such coordinates one on each connected component of the basin $\Lambda_{g_0}$ of $\infty$. In the next section we shall describe another representative $Bl$ of $[g_0]$, which is more suitable for comparison with the polynomial $z^2$.

For $0 \neq B \in \mathbb{M}_1$, $\Lambda_B$ is isomorphic to $\mathbb{D}$ and contains only the critical point 1.

As remarked above, for $B \in i\mathbb{R}$ the map $g_B$ is conjugate to itself by reflection through the imaginary axis. Remark that this reflection exchanges the two critical points $\pm 1$. Thus for $B \in i\mathbb{R}\setminus\{0\}$ the holomorphic map $\overline{\phi_B(z)}$ is also a Fatou coordinate for $g_B$, which is univalent on $\Omega_{g_0}$ with co-domain $\mathbb{H}_+$, but sends 1 to 0 and 1 to $\overline{\phi_B(-1)}$. Thus in this case both critical points are on the boundary of $\Omega_B$ and we could have chosen to normalise $\phi_B$ using the critical point $-1$ in this case. The two choices for $\phi_B$ thus differ by a purely imaginary translation and the values of $\phi_B(c_1)$ for the two different choices are purely imaginary and complex conjugate. This fact is used by Shishikura, when constructing a natural isomorphism between $Per_1(1)\backslash \mathbb{M}_1$ and $\hat{\mathbb{C}}\backslash \mathbb{D}$, see e.g. [Mi2]. We shall in order to ease the notation not use this isomorphism here, but stop two steps before the end of the construction. This is the content of the following sub section.

### 2.4 Parametrization of $\mathbb{C}\setminus \mathbb{M}_1$

The idea of Shishikura’s proof is to parametrize $\mathbb{C}\setminus \mathbb{M}_1$ by the relative position of the critical values in suitable coordinates, namely in $\mathbb{C}\setminus \mathbb{D}$ viewed as the parabolic basin of the standard parabolic Blaschke product. For this purpose we introduce the parabolic Blaschke product $Bl \in [g_0]$. It acts as the external class of the maps $g_B$, similarly to $z^2$ for quadratic polynomials

$$Bl(z) = \frac{z^2 + 1/3}{1 + z^2/3}.$$
We have $Bl = \nu \circ g_0 \circ \nu^{-1}$, where $\nu$ denotes the Möbius transformation $\nu(z) = (z + 1)/(z - 1)$. It follows immediately from the above that the Julia set $J(Bl)$ is the unit circle, $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are the two components of the basin of the double parabolic fixed point 1. The critical points are 0 and $\infty$, with images $1/3$ and 3 respectively. The map $Bl$ admits $\tau(z) = 1/\pi$ as a symmetry interchanging the immediate basins. The arcs $[0, 1]$ and $[\infty, 1]$ form attracting axis for the attracting petals of $Bl$.

![Figure 4: The attracting lines of the Blaschke product $Bl$](image)

We denote by $\phi = \phi_0 \circ \nu^{-1} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \mathbb{C}$ the Fatou coordinate normalized as above, that is $\phi(\infty) = 0$ and $\phi$ is univalent on $\Omega = \phi^{-1}(\mathbb{H}^+)$, where $Bl(\Omega) \subset \Omega$. Let $D_0$ denote the connected component of $\phi^{-1}(\mathbb{C} \setminus -\infty, 0])$. Then $\phi^{-1}$ extends to a univalent map $\phi^{-1} : \mathbb{C} \setminus -\infty, 0] \to D_0 \supset \Omega$, because $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ contains only one critical point, $\infty$ for $Bl$.

![Figure 5: The tiling of the internal parabolic basin $Bl$ by the connected components of $\phi^{-1}(\mathbb{C} \setminus -\infty, 0])$.](image)
The Parabolic Mandelbrot Set

We write $D_{1/2}$ for the disk $-D_0$ and $D_1$ for the disk, which is the interior of the closure of $D_0 \cup D_{1/2}$. Then $D_1 = \mathcal{B}l^{-1}(D_0)$.

We define topological disks $\Omega_n = \mathcal{B}l^{-n}(\Omega)$ and $D_n = \mathcal{B}l^{-n}(D_0) \supset \Omega_n$ for each $n \geq 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{image6.png}
\caption{The image $\mathcal{B}l(\Omega)$ of the petal $\Omega$ in black.}
\end{figure}

The biholomorphic parametrization of $\mathbb{C} \setminus \mathbf{M}_1$ easily follows from the following construction. Let $B \in \mathbb{H}_+ \setminus \mathbf{M}_1$ so that both critical points $\pm 1$ of $g_B$ belong to $\Lambda_B$. For each $k \geq 0$ let $\Omega^B_k = g_B^{-k}(\Omega^B)$ and let $n \geq 0$ be minimal with $v_B = g_B(-1) \in \Omega^B_n$. Then each $\Omega^B_k$ with $k \leq n$ is simply connected. Define a biholomorphic conjugacy $h_B$ by

$$h_B : \Omega^B \rightarrow \Omega \quad z \mapsto \phi^{-1} \circ \phi_B(z).$$

Then since $h_B$ sends the critical value $g_B(1)$ to the critical value $3 = \mathcal{B}l(\infty)$, and the domains $\Omega^B_k$ are simply connected for $k \leq n$ the map $h_B$ can be univalently lifted iteratively to define a conjugacy between $g_B$ and $\mathcal{B}l$ on the domain $\Omega^B_n = g_B^{-n}(\Omega^B)$ containing the second critical value $v_B$, but not the second critical point $-1$. Then

\begin{align*}
\Omega^B_n & \longrightarrow g_B \longrightarrow \Omega^B_{n-1} \longrightarrow g_B \longrightarrow \Omega^B_{n-2} \longrightarrow g_B \longrightarrow \ldots \longrightarrow g_B \longrightarrow \Omega^B_1 \longrightarrow g_B \longrightarrow \Omega^B \\
\Omega_n & \longrightarrow \mathcal{B}l \longrightarrow \Omega_{n-1} \longrightarrow \mathcal{B}l \longrightarrow \Omega_{n-2} \longrightarrow \mathcal{B}l \longrightarrow \ldots \longrightarrow \mathcal{B}l \longrightarrow \Omega_1 \longrightarrow \mathcal{B}l \longrightarrow \Omega
\end{align*}

Hence we get the following Lemma.

**Lemma 2.7.** For every $B \in \mathbb{H}_+ \setminus \mathbf{M}_1$ there exist $n = n_B \in \mathbb{N}$ and $h_B : \Omega^B_n \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ a univalent conjugacy between $g_B$ and $\mathcal{B}l$ such that $h_B(B + 2) = 3$ and $v_B = B - 2 \in \Omega^B_n$.

**Remark 2.8.** Note that the map $(B, z) \mapsto h_B^{-1}(z)$ is complex analytic as a function of the pair of variables $(B, z)$. Because the Fatou-coordinates depend holomorphically on $B$ and the map $(B, z) \mapsto (B, h_B(z))$ is locally biholomorphic off the critical points of $h_B$. 
Definition 2.9. Let $\Omega = \mathcal{B}(\Omega)$ and define a holomorphic map

$$\Upsilon : \mathbb{H} \setminus M_1 \rightarrow \mathbb{C}(\mathbb{D} \cup \Omega) \quad \text{by} \quad \Upsilon(B) := h_B(v_B).$$

We shall see that this map is injective and analytically extends as homeomorphism

$$\Upsilon : \mathbb{H} \setminus M_1 \rightarrow \mathbb{C}(\mathbb{D} \cup \Omega).$$

Remark 2.10. Since $\sigma_1([g_B]) = 1 - B^2 = A$ the representation of the above map in the $A$-coordinate gives a holomorphic and in fact biholomorphic map (see also Remark 2.3)

$$\hat{\Upsilon} : \mathbb{C}(M_1 \cup [1, \infty[) \rightarrow \mathbb{C}(\mathbb{D} \cup \Omega').$$

Proposition 2.11. The map $\Upsilon$ is a proper holomorphic map of degree 1, hence an isomorphism.

Proof. We shall show that $B \rightarrow \partial(\mathbb{H} \setminus M_1)$ implies $\Upsilon(B) \rightarrow \partial(\mathbb{D} \cup \Omega')$. It follows that $\Upsilon$ is proper.

For $B \neq 0$ the linear map $z \mapsto z/B$ conjugates the map $g_B$ to $z \mapsto z + 1 + 1/(B^2z)$ with critical points at $\pm 1/B$ and corresponding critical values $1 \pm 2/B$. It follows that both critical values for $g_B$ belong to $\Omega_B$, when $|B|$ is sufficiently large and that $\Upsilon(B) = h_B(v_B)$ converge to $h_B(B+2) = 3$, when $|B| \rightarrow \infty$. If $\Re(B) \rightarrow 0$, then $\Upsilon(B) = h_B(v_B)$ converge to $\partial \Omega'$, since $B \mapsto h_B(v_B)$ is continuous and belongs to $\partial \Omega'$, when $\Re(B) = 0$. Finally suppose $\{B_k\}_k \in \mathbb{H} \setminus M_1$ is a sequence converging to $\partial M_1$, but $h_{B_k}(v_{B_k})$ does not converge to $\partial \mathbb{D}$. Then passing to a subsequence if necessary we can suppose that $B_k \mapsto B \in \partial M_1$ and $h_{B_k}(v_{B_k})$ converge to $w \in \hat{\mathbb{C}}(\mathbb{D} \cup \Omega')$. Choose $N$ such that $w \in \Omega_N$ and thus $\mathcal{B}^N(w) \in \Omega$. Then $g_{B_k}^N(v_{B_k}) \in \Omega_B^k$ for all $k$ large enough. But then also $g_{B_k}^N(v_B) \in \Omega_B$, contradicting that $B \in \partial M_1$.

Finally the degree is 1 because it extends continuously and injectively to $\partial \mathbb{H}$, which is mapped onto $\partial \Omega'$ and we showed above that if $B_n \mapsto \partial M_1$ then $\Upsilon(B_n)$ converge to $\partial \mathbb{D}$.

Lemma 2.12. If $\Upsilon(B) \notin D_0$ then $h_B^{-1}$ extends as a biholomorphic conjugacy

$$h_B^{-1} : D_1 \rightarrow h_B^{-1}(D_1).$$

Definition 2.13. In view of the Lemma above, when $\Upsilon(B) \notin D_0$ we define $D_1^B := h_B^{-1}(D_1)$, $D_{1/2}^B := h_B^{-1}(D_{1/2})$ and $D_0^B := h_B^{-1}(D_0)$.

Proof. Note that $\bigcup_n \Omega_n \cap D_0 = D_0$, $\bigcup_n \Omega_n \cap D_1 = D_1$ and that each set $\Omega_n \cap D_0$ is simply connected and does not contain $h_B(v_B)$. Hence we obtain an increasing sequence
of extensions of \( h_{B}^{-1} \) by iterated lifting:

\[
\begin{array}{cccccc}
\Omega_n \cap D_0 & \xrightarrow{Bl} & \Omega_{n-1} \cap D_0 & \xrightarrow{Bl} & \Omega_{n-2} \cap D_0 & \xrightarrow{Bl} & \cdots & \xrightarrow{Bl} & \Omega_1 \cap D_0 & \xrightarrow{Bl} & \Omega \\
\downarrow h_{B}^{-1} & & \downarrow h_{B}^{-1} & & \downarrow h_{B}^{-1} & & \cdots & & \downarrow h_{B}^{-1} & & \downarrow h_{B}^{-1} \\
\h_{B}^{-1}(\Omega_n^B \cap D_0^B) & \xrightarrow{g_B} & \h_{B}^{-1}(\Omega_{n-1}^B \cap D_0^B) & \xrightarrow{g_B} & \h_{B}^{-1}(\Omega_{n-2}^B \cap D_0^B) & \xrightarrow{g_B} & \cdots & \xrightarrow{g_B} & \h_{B}^{-1}(\Omega_1^B \cap D_0^B) & \xrightarrow{g_B} & \Omega^B \\
\end{array}
\]

\[\square\]

3 Parabolic Rays

3.1 Dynamical parabolic rays

We first define parabolic rays in \( \mathbb{D} \) and in \( \mathbb{C} \setminus \mathbb{D} \) for the model map \( Bl \). We then define parabolic rays in the basin of \( \infty \) for the maps \( g_B \). In the case where the Julia set is connected, the conjugacy \( h_B \) (between \( g_B \) and \( Bl \)) extends to the whole basin of infinity, so we just pull back the parabolic rays defined for the model map \( Bl \). In the non-connected case, we pull back when it is possible the beginning of the ray.

3.1.1 Parabolic ray for the Blaschke product

The notion of (external) rays is well defined for quadratic polynomials, since on their basin of \( \infty \) polynomials are conjugated (in the connected case) to \( z \mapsto z^2 \) on \( \mathbb{C} \setminus \mathbb{D} \). The (external) rays are the pull-back of straight lines in \( \mathbb{C} \setminus \mathbb{D} \).

The map \( Bl \) is a degree 2 map on \( \mathbb{D} \) and on \( \hat{\mathbb{C}} \setminus \mathbb{D} \), but it is not conjugate to \( z \mapsto z^2 \) on these domains. Nevertheless, \( Bl \) is conjugate to \( z^2 \) on \( S^1 \).

**Lemma 3.1.** There exists a unique homeomorphism \( h : S^1 \rightarrow S^1 \) fixing 1 and conjugating \( z \mapsto z^2 \) to \( Bl \), i.e. \( h(z^2) = Bl \circ h \). It commutes with \( z \mapsto \overline{z} \).

**Proof.** Indeed, the map \( Bl \) is weakly expanding on \( S^1 \), as \( |Bl'(z)| \geq 1 \) on \( S^1 \) with equality iff \( z^2 = 1 \). The rest of the proof is a classical theorem for strongly expanding maps, for which the proof passes over to the weakly expanding case with out any essential changes. (Define recursively \( h_n : S^1 \rightarrow S^1 \) by \( Bl \circ h_n = h_{n-1}(z^2) \) and \( h_n(1) = 1 \), with \( h_0 = id \). The maps \( h_n \) converge to an order preserving bijection between the two sets of iterated preimages of 1 and by the weakly expanding property both sets of iterated preimages are dense in \( S^1 \) so that the limit of the \( h_n \) exists on all of \( S^1 \) and is the required topological conjugacy.) \[\square\]
Similarly to the binary expansion of the angle, we will define rays for \( Bl \) using the itineraries.

Let \( \Sigma_2 := \{0, 1\}^\mathbb{N} \) denote the one-sided shift space on 2-symbols. The angle \( \theta \) is said to have binary expansion \( \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) \) if

\[
\theta = \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n}.
\]

Denote by \( \Pi_2 : \Sigma_2 \rightarrow S^1 \) the projection map: \( \Pi_2(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) = \exp(2\pi i \theta) \). Obviously \( \Pi_2 \) conjugates the shift \( \sigma_2 \) to \( z \mapsto z^2 \) on \( S^1 \) with \( \sigma_2 : \Sigma_2 \rightarrow \Sigma_2 \) the shift map:

\( \sigma_2(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) = (\epsilon_2, \epsilon_3, \ldots, \epsilon_{n-1}, \ldots) \). Moreover, we equip \( \Sigma_2 \) with the lexicographic ordering:

\( \epsilon_1 = (\epsilon_1^1, \epsilon_1^2, \ldots, \epsilon_1^n, \ldots) < (\epsilon_2^1, \epsilon_2^2, \ldots, \epsilon_2^n, \ldots) = \epsilon^2 \) if \( \epsilon^1_k = \epsilon^2_k \) for \( 1 \leq k < m \) and \( \epsilon^1_m < \epsilon^2_m \) for some \( m \in \mathbb{N} \).

Write the upper half-arc \( I_0 = [1, -1] \subset S^1 \) and lower \( I_1 = [-1, 1] \subset S^1 \). An itinerary of a point \( z \in S^1 \) under the map \( z \mapsto z^2 \) is a sequence \( \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) \) with the property that for all \( n \in \mathbb{N} \): \( z^{2^n} \in I_{\epsilon_{n+1}} \). The reader shall easily verify that for each \( \epsilon \in \Sigma_2 \) the point \( \Pi_2(\epsilon) \) is the unique point of itinerary \( \epsilon \) under \( z^2 \). Moreover two sequences \( \epsilon^1 < \epsilon^2 \) are the common itineraries of a point \( z \) if and only if \( z^{2^n} = 1 \) for some minimal \( n \geq 0 \) and equivalently for this \( n \) \( \epsilon^1_k = \epsilon^2_k \) for \( 1 \leq k < n \), \( 0 < \epsilon^1_k = \epsilon^2_k = 1 < 2 \) and \( \epsilon^1_k = 1, \epsilon^2_k = 0 \) for \( n < k \).

Defining itineraries for \( Bl \) by the same algorithm as for \( z^2 \) above, i.e. \( Bl^n(z) \in I_{\epsilon_{n+1}} \), we obtain exactly the same statements for \( Bl \). For example \( h \circ \Pi_2 \) conjugates the shift \( \sigma_2 \) to \( Bl \), any itinerary for \( Bl \) determines a unique point of \( S^1 \) and a point has two itineraries if and only if \( Bl^n(z) = 1 \) for some \( n \).

We shall now construct accesses to these points. Called parabolic rays, they sit in a tree. We explain the construction of this tree in \( \mathbb{D} \) instead of \( \mathbb{C} \setminus \mathbb{D} \) to be more visual.

For each \( j = 0, 1 \) the open sector \( S_j \) spanned by the arc \( I_j \), i.e. the interior of the convex hull of the union of \( I_j \) and 0, is mapped univalently onto \( \mathbb{D} \setminus [1/3, 1] \). The parts \([-1, 0] \subset \mathbb{R} \) and \([0, 1] \subset \mathbb{R} \) of the boundary are each mapped (homeomorphically) onto \([1/3, 1]\) which is forward invariant. Let \( z_0 = 0 \) and \( T_0 := Bl^{-1}([0, 1/3]) = [0, z_0] \cup [0, z_1] \), where \( z_0 = \frac{i}{\sqrt{3}} \) and \( z_1 = -z_0 \). Since \( T_0 \subset \mathbb{D} \setminus [1/3, 1] \), define \( T_j \) to be the connected component of \( Bl^{-1}(T_0) \) containing \( z_j \). Define recursively after \( n \in \mathbb{N} \) and for each \((\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \Sigma_2 \) the point \( z_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n} \) as the unique point of the preimage \( Bl^{-1}(z_{\epsilon_2, \ldots, \epsilon_n}) \) belonging to \( T_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n} \). Define then \( T_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n} \) to be the connected component of the preimage \( Bl^{-1}(T_{\epsilon_2, \ldots, \epsilon_n}) \) containing \( z_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n} \).

Define for each \( n \) the dyadic trees

\[
T_n := \bigcup_{k=0}^{n} Bl^{-k}(T_0) \quad \text{so that} \quad T_n = T_{n-1} \cup \bigcup_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{0, 1\}^n} T_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}.
\]
Then define
\[ T := \bigcup_{k=0}^{\infty} B\ell^{-k}(T_0) \]
with boundary (in) \( S^1 \).

**Definition 3.2.** For \( \xi \in \Sigma_2 \) a parabolic internal ray \( \hat{R}_\xi \) is the minimal connected subset of \( T \) containing the sequence of points \( z_{\xi_1,\xi_2,\ldots,\xi_n}, n \geq 0 \) (interpreting \( n = 0 \) as \( z_0 \)).

A parabolic external ray \( R_\xi \) is the image of \( \hat{R}_\xi \) by \( z \mapsto 1/z \).

In order to stay close to the notations for quadratic polynomials, it will be convenient to identify \( \mathbb{T} \) and the Julia set \( S^1 \) for \( \mathcal{B} \ell \). This motivates the following definition.

**Definition 3.3.** We shall say that \( \theta \in \mathbb{T} \) is the external angle of the point \( h(e^{i2\pi\theta}) \). And write \( R_\theta \) for the ray \( R_\xi \), where \( \xi \) is a binary expansion of \( \theta \mod 1 \). In the special case where \( \theta \) has two binary expansions \( \xi_1, \xi_2 \) we shall write \( R_\theta := R_{\xi_1} \cup R_{\xi_2} \).

For \( j \in \{0, 1\} \) the boundary \([-1, 1]\) of \( S_j \) in \( \mathbb{D} \) is forward invariant and \( z_j \in S_j \). It follows that the set \( S_j \cup \{0\} \) contains any of the rays \( R_{\xi} \) with \( \xi_1 = j \). Moreover as \( \Omega \) is also forward invariant and disjoint from \( T_0 \) we even have that \( S_j \cup \{0\} \setminus \Omega \) contains any of the rays \( R_\xi \) with \( \xi_1 = j \). It follows that we may define parabolic rays in parameter space by \( \mathcal{R}_{\xi}^{\mathcal{M}1} = \Upsilon^{-1}(R_\xi) \). See also Definition 3.5.

### 3.1.2 Parabolic rays for the rational map \( g_B \)

Parabolic rays for \( g_B \) are defined as pre-images of the external parabolic rays \( R_\xi \) and \( R_\theta \).
Figure 8: Construction of a parabolic ray for $Bl$ of rotation number $1/3$

**Definition 3.4.** Let $B \in M_1$ and $\xi$ be an itinerary. The parabolic dynamical ray for $g_B$ of itinerary $\xi$ is by definition $R^B_\xi = h^{-1}_B(R_\xi)$. And the parabolic dynamical ray for $g_B$ with angle $\theta$ is by definition $R^B_\theta = h^{-1}_B(R_\theta)$.

**Definition 3.5.** The parabolic parameter ray of itinerary $\xi$ is defined by $R^M_\xi = \Upsilon^{-1}(R_\xi)$. Similarly, the parabolic parameter ray of angle $\theta$ is defined by $R^M_\theta = \Upsilon^{-1}(R_\theta)$.

We say that a $q$ cycle of rays $R_0, \ldots, R_{q-1}$ for $g_B$ landing on a common $k$ periodic point $z$ and numbered in the counter clockwise order around $z$ defines the combinatorial rotation number $p/q$, $(p,q) = 1$ iff $g^*_B(R_{j}) = R_{(j+p) \mod q}$.

**Theorem 3.6.** Let $B \in M_1$. For any (pre-)periodic argument $\xi \in \Sigma_2$, i.e. $\sigma^k(\sigma^l(\xi)) = \sigma^l(\xi)$, the parabolic ray $\mathcal{R} = R^B_\xi$ converges to a $g_B$ (pre-)periodic point $z \in J(g_B)$ with $g^*_B(g_B^l(z)) = g_B^l(z)$. If the argument is periodic (i.e. $l = 0$), let $k'$ denote the exact period of $z$ and let $q = k/k'$. Then the ray $\mathcal{R}$ defines the combinatorial rotation number $p/q$, $(p,q) = 1$ for $z$. The periodic point $z$ is repelling or parabolic with multiplier $e^{2\pi i/p}$. Moreover any other external parabolic ray landing at $z$ is also $k$-periodic and defines the same rotation number.

This is a standard result which in its initial form is due to Sullivan, Douady and Hubbard, for the polynomials. See the proof in [2] Th. A and Prop. 2.1], it goes through for parabolic rays.

And conversely
3.1.2 Parabolic rays for the rational map $S$

Then define

For $\varepsilon \in \mathbb{C}$, $\varepsilon = h_1(B)(\varepsilon)$. The part $\delta = \delta_{\beta}^{-1}(R)$ containing

is any repelling or parabolic periodic point. Then there is a periodic parabolic ray landing at $z$. It defines for $z$ its combinatorial rotation number.

Proof. Since $g_B$ has degree 2 the parabolic basin for $\beta_B$ is completely invariant and thus the Theorem is a special case of [P, Th. B].

The non-connected case:

Assume now that $B \notin M_1$ so that the Julia set $J(g_B)$ is a Cantor set. The map $h_B$ (Lemma 2.7) is well defined on $\hat{\Omega}^B = \Omega^B$, so that $\delta_{\varepsilon} := h_B^{-1}(R_{\varepsilon})$ is well defined (it is the pull-back of the part in $\hat{\Omega} = h_B(\hat{\Omega}^B)$). The part $\delta_{\varepsilon}$ is the beginning of the dynamical ray (as before). Using the relation $g_B \circ h_B = h_B \circ B$, one can define the ray, until it bumps on an iterated pre-image of the second critical value $v_B$, as follows. Define recursively $\delta_{\varepsilon}^{n+1}$ as the connected component of $g_B^{-1}(\delta_{\varepsilon}^n)$ containing $\delta_{\varepsilon}^n$, with $\delta_{\varepsilon}^0 = \delta_{\varepsilon}$. For $n \geq 0$ define $\delta_{\varepsilon}^n$ as the connected component of $g_B^n(\delta_{\varepsilon}^{n-1})$ containing $\delta_{\varepsilon}$.

Lemma 3.8. Let $\xi \in \Sigma_2$. If the critical value $v_B$ does not belong to $\delta_{\varepsilon}^{n-1}$ for any $0 < j \leq n$, then the set $\delta_{\varepsilon}^n$ is a simple curve. Moreover $h_B$ has a univalent analytic extension to a
neighbourhood of $\delta^n$.

**Definition 3.9.** If the critical value $v_B$ does not belong to $\delta^\sigma(\xi)$ for any $n$, then we define the dynamical ray of itinerary $\xi$ by

$$R^B_\xi := \bigcup_n \delta^n_\xi.$$  

If the critical value $v_B$ belongs to $\delta^\sigma(\xi)$ for some $n$, then we say that the ray $R^B_\xi$ bumps on some (pre)-critical point in $g_B^{-n}(-1)$ and that the ray is defined until this (pre)-critical point by the same procedure.

**Remark 3.10.** As an immediate consequence of the definition of rays, the conjugacy $h_B$ has a unique analytic extension along rays. In particular we have

$$v_B \in R^B_\xi \iff B \in R^M_\xi.$$  

The landing property given by Theorem 3.6 in the connected case, translates in the non connected case as follows:

**Theorem 3.11.** Let $B \in \overline{H}_+$. For any (pre-)periodic argument $\xi \in \Sigma_2$, either the parabolic ray $R^B_\xi$ bumps on the critical point $-1$ or it converges to a $g_B$ (pre-) periodic point $z \in J(g_B)$ with $g_B^k(g_B^l(z)) = g_B^l(z)$. If periodic (i.e. $l = 0$), let $k'$ denote the exact period of $z$ and let $q = k/k'$. Then the ray $R^B_\xi$ defines a combinatorial rotation number $p/q$, $(p, q) = 1$ for $z$. The periodic point $z$ is repelling or parabolic with multiplier $e^{\frac{i2\pi p}{q}}$. Moreover any other external parabolic ray landing at $z$ is also $k$-periodic and defines the same rotation number.

The following stability statement will be crucial in the sequel:

**Lemma 3.12.** Let $B^* \in \overline{H}_+$ and assume that the critical point $-1$ is not on the forward orbit of $R^B_\xi$ and that $R^B_{\xi^*}$ is landing on either a pre-image of $\infty$ or on a point, which is pre-periodic to a repelling periodic point. Then, there exists a neighborhood $U$ of $B^*$ such that for any $B \in U$, the ray $R^B_\xi$ lands at a pre-periodic point and there exists a holomorphic motion $\psi : U \times R^B_{\xi^*} \to \mathbb{C}$ such that $\psi(B)(R^B_{\xi^*}) = R^B_\xi$.

**Proof.** In the case where the landing point is pre-repelling, the proof is similar to the one of Douady-Hubbard in the case of quadratic polynomials: it is based on the implicit function Theorem. Note that $R^B_\xi$ always lands at $\infty$ when it is defined. Hence, if $v_B$ is not on the closure of $R^B_\xi$, this ray varies holomorphically (in this family) and the pre-images $R^B_{\eta^*}, R^B_{\eta^0}$ cannot break on the critical point $-1$. Indeed, on any disk in $\mathbb{C} \setminus R^M_{\xi^0}$, we have a holomorphic motion of the arc $h_B^{-1}([0, 1])$ connecting the critical point 1 to its critical
value $g_B$, (1). Pullback by iteratively along $\mathcal{R}_B^1$ by the dynamics we never encounter the second critical value $v_B$ and so lifting the holomorphic motion gives a holomorphic motion of all the ray parameterized by this disk. By the $\lambda$-Lemma it extends to the closure of $\mathcal{R}_B^1$. Note that the only parameter $B \in M_1$ for which $v_B$ is on the closure of $\mathcal{R}_B^1$ is $B = 0$. The similar statement hold for $\mathcal{R}_B^{M_1}$.

![Figure 10: Parabolic chess board outside $M_1$, viewed in the $A$-parameter plane.](image)

**Corollary 3.13.** In any disk contained in the complement of $\hat{C} \setminus \bigcup_i \mathcal{R}_{2\theta}^{M_1}$, the ray $\mathcal{R}_\theta^B$ admits a holomorphic motion and so does its closure (by the $\lambda$-Lemma).

### 3.2 Limbs of $M_1$

Similarly to Douady and Hubbard description of $M$, the parabolic Mandelbrot set $M_1$ can be described in terms of limbs sprouting out of the central, period 1 (relative) hyperbolic component $H_0$. The aim of this section is to prove this result.

**Theorem 3.14.**

$$M_1 = \Pi_0 \cup \bigcup_{\substack{\frac{p}{q} \neq \frac{0}{1}}} \mathcal{L}_{p/q}^{M_1}$$
where \( \mathcal{L}_{p/q}^{M_1} \) are disjoint compact connected sets characterized by the fact that the fixed point in \( \mathbb{C} \) has rotation number \( p/q \).

Though this is similar to the Mandelbrot case we give in this section a complete proof following the approach by Milnor in [Mi4]. It is most conveniently discussed in the \( A \)-parametrization, where \( \mathbb{H}_0 = \mathbb{D} \). For \( A \in \mathbb{C} \setminus \{1\} \), the multiplier of the finite fixed point \( \alpha(A) \) is \( A \in \mathbb{C} \), so it is attracting when \( A \in \mathbb{D} = D(0,1) \), neutral when \( A \in \partial \mathbb{D} \) and repelling when \( A \in \mathbb{C} \setminus \overline{\mathbb{D}} \). For each \( p/q \neq 1 \), with \( (p, q) = 1 \), the parameter \( A_{p/q} = e^{i2\pi p/q} \in \partial \mathbb{D} \setminus \{1\} \) belongs to \( M_1 \), so that there is a parabolic external ray converging to \( \alpha(A) \) by Theorem 3.7 and Theorem 3.6. This ray has rotation number \( p/q \).

Let us denote by \( \mathcal{R}_{\theta,\{p/q\}}^{B} \) and \( \mathcal{R}_{\theta,\{p/q\}}^{D} \), recall \( A = 1 - B^2 \), the rays in the cycle that are adjacent to the critical value (i.e. to the Fatou component containing it). In Lemma 3.20 we prove that the corresponding parameter rays \( \mathcal{R}_{\theta,\{p/q\}}^{M_1} \) lands at \( A_{p/q} \) and that they cut off a wake \( \mathcal{W}^{M_1}(p/q) \). We call \( (p/q-) \) (derooted) limb of \( M_1 \), the set \( \mathcal{L}_{p/q}^* := M_1 \cap \mathcal{W}^{M_1}(p/q) \).

For \( \theta = p/2^l, l \geq 0 \) a dyadic angle we shall say that \( \mathcal{R}_{\theta}^{*} \in \{M_1, B\} \) lands if the two rays \( \mathcal{R}_{\theta}^{*} \) land on the same point, where \( \theta_0, \theta_1 \) are the two dyadic expansions of \( \theta \). We obtain quite precise properties of the landing in the parameter plane in the following:

**Theorem 3.15.** For every pre-periodic (i.e. rational) angle \( \theta \), the parameter ray \( \mathcal{R}_{\theta}^{M_1} \) lands. More precisely, suppose \( 2^{k+l} \theta \equiv 2^l \theta \mod 1 \) with (period) \( k > 0 \) and \( l \geq 0 \) minimal.

1. If \( \theta \) is periodic \( (l = 0) \) then \( \mathcal{R}_{\theta}^{M_1} \) lands on a parameter \( A = 1 - B^2 \) for which the corresponding dynamical ray \( \mathcal{R}_{\theta}^{B_\infty} \) lands at a parabolic periodic point \( z(B) \), with exact period \( k | k \) and with multiplier \( \lambda = (g_B^k)'(z(B)) \) a primitive \( k/k \)-th root of unity.

2. If \( l > 0 \) then \( \mathcal{R}_{\theta}^{M_1} \) lands at a parameter \( A \) for which the corresponding dynamical ray \( \mathcal{R}_{\theta}^{B_\infty} \) lands on \( v_B \) and for which \( g_B^{k+l}(v_B) = g_B^l(v_B) \) is a periodic point of exact period \( k \). This periodic point is repelling if \( k > 1 \) and is the parabolic fixed point \( \infty \) for \( k = 1 \). Moreover for any dynamical ray \( \mathcal{R}_{\theta}^{B_\infty} \) landing on \( v_B \), the corresponding parameter ray \( \mathcal{R}_{\theta}^{\rho} \) lands at \( A \in M_1 \).

**Remark 3.16.** In particular in point 1. \( \theta = 0 \) both rays \( \mathcal{R}_0^{B_\infty}, \mathcal{R}_1^{B_\infty} \) land at the parabolic fixed point \( \infty \) of multiplier 1, and there is no other ray landing at \( \infty \). In point 2., \( \theta = p/2^l \), \( l > 0 \) the landing point of \( \mathcal{R}_\theta^{M_1} \) is not on \( \partial \mathbb{H}_0 \) (but at the so called “\( \theta \)-dyadic tip” of \( M_1 \)), so that these two rays landing at the same point do not define a limb but bounds a disk.

### 3.2.1 Landing of parameter rays

**Lemma 3.17.** Let \( \theta \) be a \( k \)-periodic angle, then \( \mathcal{R}_{\theta}^{M_1} \) lands at a parameter \( A \in M_1 \). Moreover, the corresponding dynamical ray \( \mathcal{R}_{\theta}^{B_\infty} \) lands at a parabolic periodic point \( z(B) \).
Figure 11: The dyadic rays $R_{01}^{M_1}$, $R_{10}^{M_1}$ corresponding to the dyadic angle 1/2. They bound the disk $D_{1/2}^{M_1} := \Upsilon^{-1}(D_{1/2})$, viewed in the $A$-parameter plane.

where $A = 1 - B^2$. If $k = 1$, $z(B) = \infty$. If $k > 1$, $z(B)$ has exact period $k'|k$ and multiplier $\lambda = (g_B^{k'})'(z(B))$ a primitive $k/k'$-th root of unity.

Proof. The argument is classical. Let $A$ be any accumulation point of $R_{\theta}^{M_1}$. Since $A$ is in $M_1$ the Julia set is connected. For $B$ such that $A = 1 - B^2$, the ray $R = R_{\theta}^B$ lands at a $k'$-periodic point $z(B)$ of $J(g_B)$ with $k'|k$. It is either repelling or parabolic. If it is repelling or if it lands at the parabolic point $\infty$, Theorem 3.12 gives a holomorphic motion of $\mathcal{T}$ in a neighborhood of $A$ since $z(B)$ cannot be critical (it is periodic). But this contradicts the fact that if $A'$ is close to $A$ on $R_{\theta}^{M_1}$, the critical value is on the ray $R_{\theta}^{B'}$ (Lemma 3.10) so that the two preimages (one is in the cycle) bump on the critical point (since $\theta$ is periodic). Hence, $z(B)$ is a parabolic point of period $k'$ dividing $k \geq 1$, with multiplier $\lambda = (g_B^{k'})'(z(B))$ a primitive $k/k'$-th root of unity.
The Parabolic Mandelbrot Set

The set of parameters $B \in \mathbb{C}$ such that $g_B$ has a parabolic cycle of period $k' | k$ with $k > 1$ is included in $\{B \in \mathbb{C} | \exists z \in \mathbb{C}, g_B^k(z) = z, (g_B^k)'(z) = e^{2\pi i j k'/k}, j \in \{0, \ldots, k/k'\}\}$. This set is finite since it is defined by the equations in $(z, B)$ of two (relatively prime) polynomials.

Therefore, the accumulation set of $R^{M_1}_\theta$ is finite, so it reduces to one point. ☐

**Lemma 3.18.** Let $\theta$ be a strictly preperiodic angle: $2^{k+l} \theta \equiv 2^l \theta \mod 1$ with $k > 0$ and $l > 0$ minimal. The parameter ray $R^{M_1}_\theta$ lands. Moreover, if $k = 1$, the corresponding dynamical ray $R^B_\theta$ lands on the critical value $v_B$ and $g^l_B(v_B) = \infty$.

**Proof.** As before let $A$ be an accumulation point of the ray $R^{M_1}_\theta$. The dynamical ray $R^B_\theta$ lands at a strictly preperiodic point $z(B)$ of $J(g_B)$. The point $g^l_B(z(B))$ is periodic, either repelling or parabolic. Assume first that $k > 1$. As in previous lemma, the set of parameters such that $z(B)$ is parabolic is finite. Now if $z(B)$ is repelling, the critical point is in the orbit of $z(B)$, by the stability Lemma. This situation also corresponds to a finite number of $B \in \mathbb{C}$ since $B$ has to satisfy a polynomial equation (the critical point is pre-periodic). Since the accumulation set of $R^{M_1}_\theta$ is connected and finite, it reduces to one point. Therefore, the parameter ray lands.

We consider now the case $k = 1$. The angle is dyadic and $R^B_\theta$ lands at the critical value $v_B$, so $g^l_B(v_B) = \infty$. This equation also gives a finite number of parameters so that the parameter ray lands. ☐

To achieve the proof of Theorem 3.15 we need to define Wakes as in the Mandelbrot case.

### 3.2.2 Wakes

We consider now for $p/q \not\in \{0, 1\}$, the parameter $A_{p/q}$ and the $q$-cycle of external parabolic rays landing to the $\alpha$ fixed point, with angles $0 < \theta_0 < \theta_1 < \ldots < \theta_{q-1} < 1$ of combinatorial rotation number $p/q$ (i.e. $2\theta_i \equiv \theta_{(i+p) \mod q} \mod 1$) (defined at the beginning of the section). Denote by $\mathcal{I} = (\theta_-, \theta_+)$ the smallest interval in $S^1 \setminus \bigcup_{i \geq 0} \theta_i$.

**Lemma 3.19.** Let $A$ be a parameter outside of $M_1$ on some external ray of angle $t$. The dynamical rays $R^B_{\theta_0}, R^B_{\theta_1}, \ldots, R^B_{\theta_{q-1}}$ land at the repelling fixed point if and only if $t$ belongs to $\mathcal{I}$.

**Proof.** The proof is the same as Lemma 2.9 of [Mi4] (which deals with all kind of cycles of quadratic polynomials). We recall it briefly. Note that the two rays $R^B_{t/2} \cup R^B_{t/2+1/2}$ crash on the critical point and so partition the plane $\mathbb{C}$ in two sides. It follows that two
dynamical rays land at the same point if and only if they have the same itinerary with respect to this partition of \( \mathbb{C} \).

On the other hand, the arcs in the complement of the cycle \( \theta_0, \ldots, \theta_q \) are mapped injectively to another complementary arc, except for one complementary arc that double covers \( \mathcal{I} \). Therefore, if \( t \not\in \mathcal{I} \), its preimages \( t/2 \) and \( (t+1)/2 \) belong to different complementary arcs, or to the cycle, so that the rays of the cycle cannot land at the same point, or are even not defined until the end.

Now, for \( t \in \mathcal{I} \), \( t/2 \) and \( (t+1)/2 \) belong to a complementary interval of length greater than \( 1/2 \). So all the rays of the cycle are in the same component of the partition, they land at the same point.

\[ \square \]

**Figure 12: Wake 1/3**

**Lemma 3.20.** The parameter rays \( \mathcal{R}_{p/q}^{M_1}, \mathcal{R}_{q+}^{M_1} \) both land at \( A_{p/q} \). Moreover, the curve \( \mathcal{R}_{p/q}^{M_1} \cup \mathcal{R}_{q+}^{M_1} \cup A_{p/q} \) cuts the sphere into two connected components. Denote by \( \mathcal{W}^{M_1}(p/q) \) the one not containing \( \mathbb{D} \). The dynamical rays \( \mathcal{R}_{p/q}^j, 0 \leq i \leq q - 1 \), land at a common repelling fixed point if and only if \( A = 1 - B^2 \in \mathcal{W}^{M_1}(p/q) \).
Proof. The proof is similar to the one in Theorem 3.1 of [Mi4]. Let $W$ be the set of parameters in $\mathbb{C}$ such that the dynamical rays $R^B_{\theta_i}$ all land at the same point which is a repelling fixed point. Note that such rays cannot land at the fixed point $\infty$. By Lemma 3.19, $W$ is non empty; it is an open set by the stability property of such rays. From the charaterization given by Lemma 3.19, a parameter ray $R^M_\theta$ belongs to $W$ if and only if $t \in I$. The boundary of $W$ consists of parameters for which there is no stability of the dynamical rays. That is, either the critical point $c_1$ is on the cycle $R^B_{\theta_i}$ (it cannot be at the landing point which is periodic) or the landing point of the rays is parabolic (Lemma 3.12): $\partial W \subset (\bigcup_{0 \leq i \leq q-1} R^M_{\theta_i}) \cup \mathcal{F}$, where $\mathcal{F}$ corresponds to the finite set of parameters for which there is a parabolic point of period $q$. We deduce from this description of $\partial W$ and from Lemma 3.19 that $W$ is connected. Then the parameter rays $R^M_{\theta_-}$ and $R^M_{\theta_+}$ have to land at a common point $A$ of $\mathcal{F}$, since $W$ does not contain parameter rays of angle outside $I$. For this parameter $A$, the rays $R^B_{\theta_i}$ land at a parabolic $q'$-cycle different from $\infty$ by Lemma 3.17, with $q'$ dividing $q$. Assume that $A \neq A_{p/q}$, then the map $g_B$ has a repelling fixed point (since it can have only one parabolic cycle in $\mathbb{C}$). There is a cycle of external rays, different from $R^B_{\theta_i}$, landing at this fixed point. This cycle is stable (Lemma 3.12) in a small neighborhood since the critical point is in the basin of the parabolic cycle. This contradicts the fact that $A$ is in the boundary of $W$ where these rays do not land at the fixed point. Therefore $A = A_{p/q}$ and the statement follows. 

Definition 3.21. For $A = 1 - B^2 \in W^M_{p/q}$ we define the dynamical wake $W_B(p/q)$ as the connected component of $\mathbb{C} \setminus R^B_{\theta_-} \cup R^B_{\theta_+}$ containing the rays $R^B_t$ for $t \in (\theta_-, \theta_+)$. 

Note that when $A \notin W^M_{p/q}$, $p \neq 0$ then $R^B_{\theta_-} \cup R^B_{\theta_+}$ does separate $\mathbb{C}$ into two sub disks.

Corollary 3.22. The parameter $A$ in $W^M_{p/q}$, if and only if the second critical value $v_B$ is in the dynamical wake $W_B(p/q)$.

Proof. It follows from the construction. As explained in the proof of Lemma 3.19 the only non injective interval of angles double covers $I$. One deduces easily that the corresponding wake contains the critical value $v_B$. 

3.2.3 Proof of Theorem 3.14 and Theorem 3.15

Proof of Theorem 3.14. Let $A$ be any parameter in $M_1 \setminus \mathbb{D}$. The fixed point $\alpha \in \mathbb{C}$ is repelling and has rotation number $p/q$. It defines a cycle of external rays landing at it. By Lemma 3.20, this parameter belongs to $W^M_{p/q}$.

Corollary 3.23. The diameter of the limbs $L^M_{p/q}$ tends to zero, when $q$ tends to $\infty$. 
Proof. Assume to get a contradiction that there is a sequence of Limbs \( \mathcal{L}_{p_n/q_n}^{M_1} \) with \( q_n \to \infty \) whose diameter does not go to zero. Then, one can find points \( x_n, y_n \in \mathcal{L}_{p_n/q_n}^{M_1} \) converging to \( x \neq y \) respectively. By Theorem 3.14 these two points cannot both belong to \( \mathbb{D} \) (they would be separated by some wake). So one at least, say \( y \), correspond to a map that has a repelling fixed point and therefore is in some Limb \( \mathcal{L}(p/q) \). But this implies that the sequence \( y_n \) enters in \( \mathcal{W}_{M_1}^{M_1}(p/q) \) for \( n \) large. The contradiction comes from the fact that the wakes \( \mathcal{W}_{M_1}^{M_1}(p_n/q_n) \) and \( \mathcal{W}_{M_1}^{M_1}(p/q) \) are disjoint. Alternatively apply Yoccoz inequality in the form [P, Theorem C] to all \( g_B, B \in \mathcal{L}_{p/q}^{M_1} \). To obtain that log of the limb is contained in the closed Euclidean disk of radius \( r(q) = \log \frac{4}{q} \) Here the argument 4 comes from the inequality \( |B'(z)| \leq 4 \) on the unit circle. 

Proof of Theorem 3.15.

Note that this Corollary 3.22 together with Lemma 3.17 achieve the proof of part 1 of Theorem 3.15 in the case \( k = 1 \). Thus it suffices to consider the case \( k > 1 \).

Lemma 3.24. For a pre-periodic angle \( \theta \), i.e. \( 2^{k+l}\theta \equiv 2^l \theta \mod 1 \) with (period) \( k > 1 \) and \( l > 0 \) minimal, if \( A \) denotes the landing point of \( R_{M_1}^{M_1}(\theta) \), the corresponding dynamical ray \( R^{B}_{\theta} \) lands at \( v_B \) and \( g^{k+l}_{B}(v_B) = g^{l}_{B}(v_B) \) is a repelling periodic point of exact period \( k \).

Proof. Assume to get a contradiction that the external rays of the cycle of angles \( 2^{k+l}\theta \) land at a parabolic periodic point. The parameter \( A \) belongs to some wake \( \mathcal{W}_{\theta}^{M_1}(p/q) \), so that there is a cycle of external rays landing at the repelling fixed point. Let us denote by \( \Gamma \) the union of these external rays together with the fixed point. The iterated pre-images \( \Gamma_n = g_B^{-n}(\Gamma) \) give a partition of \( \mathbb{C} \) that separates for \( n \) large enough, the external rays of the cycle of angles \( 2^{i+l}\theta \) with \( i \geq 0 \) from the ray of angle \( \theta \). Therefore \( \Gamma_n \) separates the critical value \( v_B \) from the external ray of angle \( \theta \) (since it is in a Fatou component adjacent to one of the rays in the cycle \( 2^{i+l}\theta \)). Now the graphs \( \Gamma_n \) are stable, so for parameters \( A' \) in a neighborhood of \( A \), but on the ray \( R_{\theta}^{M_1} \), the graph still separates the critical value \( v_B' \) from the ray \( R_{\theta}^{B'} \) where \( A' = 1 - B'^2 \) (the elements stays in some different sectors). This contradicts the fact that \( v_B' \) has to be on \( R_{\theta}^{B'} \). Therefore \( R_{\theta}^{B} \) lands at \( v_B \). 

Recall from page 26 that the pair of periodic arguments \( \theta, \theta' \), \( 0 < \theta < \theta' < 1 \) of a pair of parameter rays \( R_{\theta}^{M} \) and \( R_{\theta}^{M} \) co-landing on a parabolic parameter defines a Cantor set \( C(\theta, \theta') \). And that this lead to the definition of dyadic wakes and limbs of the corresponding copy of the Mandelbrot set. In view of Lemma 3.17 andLemma 3.24 we generalize this to \( M_1 \) as follows. Let \( M_1(\theta, \theta') \) be the copy of \( \mathbb{M} \) in \( M_1 \) with root rays \( R_{\theta}^{M_1} \) and \( R_{\theta'}^{M_1} \).
Definition 3.25. Define the dyadic wake \( W_{M_1}(\theta, \theta', r, s) \) of \( M_{M_1}(\theta, \theta') \) as the set bounded by the pair of co-landing parameter rays \( R_{M_1}^{\theta_0}, R_{M_1}^{\theta_1} \), where \( \theta_0, \theta_1 \) are the the arguments bounding the \( r/2^s \) gap in the Cantor set \( C(\theta, \theta') \). And define the dyadic limb \( L_{M_1}(\theta, \theta', r, s) \) as the intersection\
\[ L_{M_1}(\theta, \theta', r, s) := W_{M_1}(\theta, \theta', r, s) \cap M_{M_1}. \]

Moreover as for polynomials in the special case where \( M_{M_1}(\theta, \theta') = M_{M_1}^{p/q} \) we shall write \( W_{M_1}^{p/q}(r, s) \) for \( W_{M_1}(\theta, \theta', r, s) \) and \( L_{M_1}^{p/q}(r, s) \) for \( L_{M_1}(\theta, \theta', r, s) \) dyadic wakes and limbs associated with \( M_{M_1}^{p/q} \).

The root \( B \) of the dyadic wake \( W_{M_1}(\theta, \theta', r, s) \) is the \( r/s \) dyadic tip of \( M_{M_1}(\theta, \theta') \), that is the parameter such that the \( q \)-renormalization of \( g_B \) is hybridly equivalent to the \( r/s \) tip, i.e. the landing point of the parameter ray \( R_{r/s} \) of the Mandelbrot set.

4 Parabolic Puzzles and Parabolic Para-puzzles

We shall state and prove in Section 8 a theorem for the parabolic Mandelbrot set \( M_1 \) analogous to the Yoccoz parameter puzzle theorem for the Mandelbrot set (see [R1]).

The idea underlying the proof is also in this case to transfer the result obtained in the dynamical plane to the parameter plane using the trick of Shishikura to control the dilatation of the holomorphic motion in puzzle pieces.

Yoccoz theorem for the parabolic map \( g_B \) was proved in [PR2]. We recall briefly the proof here since we need the detailed construction of the parabolic puzzle. Before let us recall the classical Yoccoz puzzle. Then, the construction of the parabolic puzzle will appear more natural even in the parameter plane.

4.1 Yoccoz puzzle for Quadratic polynomials

For \( c \in M \setminus \text{Card} \), \( c \) belongs to some derooted limb \( L_{p/q}^* \). For the rest of this section we fix the reduced rational \( p/q \), but we shall only occasionally make reference to \( p/q \). This motivates the following. Let \( 0 < \theta_0 < \theta_1 < \ldots \theta_{q-1} < 1 \) denote the arguments of the unique \( q \)-cycle of rotation number \( p/q \) for \( Q_0 \).

Recall that the wake parameter wake \( W_{M}(p/q) \) is the subset of parameter space \( \mathbb{C} \) bounded by the parameter rays of arguments \( \theta_{p-1}, \theta_p \). Recall further that \( c \in W_{M}(p/q) \) if and only if the cycle of dynamical rays \( R_{\theta_0}, \ldots, R_{\theta_{q-1}} \) co-land on \( \alpha_c \) and then also \( c \in W_{p/q}^c \), the dynamical wake bounded by the dynamical rays of arguments \( \theta_{p-1}, \theta_p \). Moreover the strictly pre-periodic pre-image rays of arguments \( \theta_0 + \frac{1}{2} < \ldots < \theta_{q-1} + \frac{1}{2} \subset]\theta_{q-1}, \theta_0[ \) co-lands
on \( \alpha'_c \). It follows immediately that all \( 2q \) rays together with their landing points \( \alpha_c, \alpha'_c \) move holomorphically with the parameter \( c \in \mathcal{W}_M(p/q) \).

We shall fix an arbitrary choice of potential \( l_0 = 1 \). For \( n \in \mathbb{N} \) we define the dynamical sets \( V^n_c := \{ z \in \mathbb{C} \mid g_c(z) < l_0/2^n \} \) bounded by the \( l_0/2^n \) level set \( \mathcal{E}^c_n = \{ z \in \mathbb{C} \mid g_c(z) = l_0/2^n \} \). And we define the restricted parameter wakes \( \mathcal{W}^M_n(p/q) := \{ c \in \mathcal{W}_M(p/q) \mid c \in V^n_c \} \).

For \( c \in \mathcal{W}_M^0(p/q) \) we define the Yoccoz puzzle as follows. Let \( \mathcal{G}_0^c \) denote graph

\[
\mathcal{G}_0^c = \mathcal{E}_0^c \cup \{ \alpha_c, \alpha'_c \} \cup \bigcup_{i=0}^{q-1} (R^c_{\theta_i} \cup R^c_{\theta_i+1/2} \cap V^c_0).
\]

That is the union of the equipotential \( \mathcal{E}_0^c \) together with \( \alpha_c, \alpha'_c \) and the segments, inside \( \mathcal{E}_0 \), of the external rays landing on these two points. (Note that the original construction involved only the cycle of rays landing on \( \alpha_c \) and not the preimages landing on \( \alpha'_c \), that we add here for convenience).

Following up on the remark above we note that the graph \( \mathcal{G}_0^c \) move holomorphically with \( c \in \mathcal{W}_M^0(p/q) \).

We define recursively the level \( n \)-Yoccoz graph \( \mathcal{G}_{n+1}^c := Q^{-1}_c(\mathcal{G}_n^c) \).

The level-0 puzzle pieces are the bounded connected components of \( \mathbb{C} \setminus \mathcal{G}^c_0 \). Denote by \( \mathcal{Y}_0^c \) the level-0 puzzle: the collection of these \( 2q - 1 \) puzzle pieces. Define the level-\( n \) puzzle \( \mathcal{Y}_n^c \) as the collection of connected components of \( Q^{-n}(Y) \), where \( Y \) ranges over all of the level-0 puzzle pieces or equivalently as the set of bounded connected components of \( \mathbb{C} \setminus \mathcal{G}_n^c \). The \( (p/q) \)-Yoccoz Puzzle for \( Q_c \) is the union \( \mathcal{Y}^c = \bigcup_{n \geq 0} \mathcal{Y}_n^c \) of the puzzles at all levels.

Denote by \( \mathcal{G}^c(n) \) the union \( \bigcup_{j=0}^n \mathcal{G}^c_j \) and let \( \mathcal{G}^c \) be the union of these graphs of all levels.

Any two puzzle pieces \( Y \in \mathcal{Y}_n^c \) and \( Y' \in \mathcal{Y}_m^c, m \leq n \) are either interiorly disjoint or nested with \( Y \subseteq Y' \) (because the potential is multiplied by two under the dynamics and the set of rays in the construction of \( \mathcal{Y}_0^c \) is forward invariant).

A nest, i.e. a sequence \( \mathcal{N} = \{ Y_n \}_n \), \( Y_n \in \mathcal{Y}_n^c \) with \( Y_{n+1} \subseteq Y_n \), is called convergent iff \( \text{End}(\mathcal{N}) := \bigcap_{n \in \mathbb{N}} \overline{Y}_n = \{ z \} \) is a singleton and is called divergent otherwise. When wanting to emphasise \( z \) we say the nest \( \mathcal{N} \) is convergent to \( z \). A nest \( \mathcal{N} \) is called critical iff \( 0 \in \text{End}(\mathcal{N}) \) and called a critical value nest iff \( c \in \text{End}(\mathcal{N}) \).

### Universal Yoccoz Puzzle

Associated to a rotation number \( p/q \), one defines the universal Yoccoz Puzzle on the complement of the disk. It is a model of all \( p/q \) Yoccoz Puzzle using the Böttcher
conjugacy. Let \( \mathcal{Z}_0 = \bigcup_{j=0}^{q-1} e^{2\pi j} \cup e^{2\pi(j+\frac{1}{2})} \) be the unique \( q \)-cycle for \( Q_0 \) of combinatorial rotation number \( p/q \) in \( S^1 \) and its preimage.

Let \( l_0 \) denote the choice of equipotential above and define \( E_0 = \{ z \mid |z| = e^{l_0} \} \). Let \( \mathcal{U}_0 \) denote the union of the equipotential \( E_0 \), the unit circle, together with the segments, of radial lines through the points of \( \mathcal{Z}_0 \) between \( E_0 \) and the unit circle.

The Universal Yoccoz graph is then
\[
\mathcal{U}_0 = E_0 \cup S^1 \cup \left( \bigcup_{i=0}^{q-1} R_{\theta_i} \cup \bigcup_{i=0}^{q-1} R_{\theta_i+1/2} \right) \cap \{ z \in \mathbb{C} \mid 1 < |z| < e^{l_0} \}
\]
and define the universal \((p/q\text{-Yoccoz})\) puzzle \( \mathcal{U}_0 \) as the set consisting of the \( 2q \) bounded connected components of the complement of \( \mathcal{U}_0 \) in \( \mathbb{C} \setminus \mathbb{D} \).

Define \( \mathcal{U}_n \) recursively as follows:
\[
\mathcal{U}_n = Q_0^{-1}(\mathcal{U}_{n-1})
\]
and the universal \((p/q\text{-Yoccoz})\) puzzle \( \mathcal{U}_n \) as the set consisting of the bounded connected components of the complement of \( \mathcal{U}_n \) in \( \mathbb{C} \setminus \mathbb{D} \).

Finally define \( \mathcal{U}_Y(n) = \bigcup_{j \leq n} \mathcal{U}_j \) and \( \mathcal{U}_Y = \bigcup_{n \in \mathbb{N}} \mathcal{U}_Y(n) \). We call \( \mathcal{U}_Y \) the universal \( p/q\text{-Yoccoz puzzle} \). Remark that if \( \phi_c(c) \notin \mathcal{U}_Y(n) \), in particular if \( c \in L^*_p/q \), then
\[
\mathcal{U}_Y^c(n) = \overline{\psi_c(\mathcal{U}_Y(n))}.
\]

### 4.2 Parabolic dynamical puzzle

Similarly to the polynomial case above, we have for each irreducible rational \( p/q \) defined in \([\mathbb{P}\mathbb{R}2]\) a universal parabolic \( p/q \)-puzzle using the parabolic rays described in section 3 above. We shall for completeness briefly review the construction here. The universal parabolic \( p/q \)-puzzle is the puzzle for the Blaschke product \( Bl \), which is the model map for the external class of the maps \( g \in Per_1(1) \). Recall that the notation \( R_\epsilon \) refers to the external parabolic ray of \( Bl \) with argument \( \epsilon \in \Sigma^2 \), as defined in section 3. Note that with the parabolics, the difficulty is that we do not have equipotentials. Therefore we will define the shortest path from one ray to the next (in the cycle) and use this path as an equipotential. In \([\mathbb{P}\mathbb{R}2]\) we compared the two universal puzzles and showed that there is a natural dynamics preserving bijection between the two puzzles. In fact there is a modified Universal Yoccoz puzzles inducing Yoccoz puzzles similarly as above, puzzles which yield the same puzzle results as standard Yoccoz puzzles associated with the Universal Yoccoz puzzle \( \mathcal{U}_Y \) and such that the modified Universal Yoccoz puzzle is homeomorphic to the Universal parabolic Yoccoz puzzle \( \mathcal{P} \) to be introduced below.
4.2.1 Shortcuts

Recall that the restriction $\phi^+ : D_0 \rightarrow \mathbb{C} \setminus [\infty, 0]$ of $\phi$ is a conformal isomorphism which extends continuously to the boundary. For $0 < n_0, n_1$ let $\tilde{\gamma}(n_0, n_1)$ be the arc which is mapped by $\phi^+$ to the Archimedean spiral/circle of center $0$ connecting $-n_0$ and $-n_1$ counter clockwise through $\mathbb{C} \setminus \mathbb{R}$. And let $\gamma(n_0, n_1) = -\tilde{\gamma}(n_0, n_1) \subset D_{1/2}$. Since any branch of $\mathcal{B}l^{-n}$ for any $n \geq 1$ is univalent on $D_{1/2}$, we may use such branches to define short-cuts in any of the pre-images $D_{r/2^n}$ of $D_{1/2}$ under $\mathcal{B}l^{n-1}$, where $n \geq 1$ and $r$ is the odd number such that $h(\exp(i2\pi r/2^n))$ belongs to the boundary of $D_{r/2^n}$. Short-cuts were introduced in [2R2] in order to produces parabolic Yoccoz graphs and puzzles, which are topologically similar to standard polynomial Yoccoz graphs and puzzles.

The basic observation is that if $\xi^0 \in \Sigma_2$ has $n_0 > 1$ leading $0$'s followed by a $1$ and $\xi^1 \in \Sigma_2$ has $n_1 > 1$ leading $1$'s followed by a $0$. Then the two rays $R_{\xi^0}$ and $R_{\xi^1}$ follow the boundary of the disk $D_0$ precisely down to times $-n_0$ and $-n_1$ respectively. When forming e.g. puzzles where the two rays $R_{\xi^0}, R_{\xi^1}$ are adjacent and so are destined to bound a puzzle piece we shall replace the subarc of $R_{\xi^0} \cup R_{\xi^1}$ between $R_{\xi^0}(-n_0)$ and $R_{\xi^1}(-n_1)$ with the short-cut $\tilde{\gamma}(n_0, n_1)$. Similarly if $\xi^0$ has a leading 1, followed by $n_0 - 1$ digits $0$ with $n_0 - 1 \geq 1$ and then a $1$ and $\xi^1$ has a leading 0 followed by $n_1 - 1 \geq 1$ leading $1$'s and then a $0$. Then the two rays $R_{\xi^0}$ and $R_{\xi^1}$ follow the boundary of the disk $D_{1/2}$ precisely down to times $-n_0$ and $-n_1$ respectively. In this case when the two rays $R_{\xi^0}, R_{\xi^1}$ are adjacent in a graph or puzzle we shall replace the subarc of $R_{\xi^0} \cup R_{\xi^1}$ between $R_{\xi^0}(-n_0)$ and $R_{\xi^1}(-n_1)$ with the short-cut $\gamma(n_0, n_1)$. And finally if $\xi^0, \xi^1$ coincide up to digit $n-1$, but differ on the $n$-th digit, say $\sigma_2(\xi^0)$ has a leading $1$, followed by $n_0 - 1 \geq 1$ $0$'s and then a $1$ and $\sigma_2(\xi^1)$ has a leading $0$ followed by $n_1 - 1 \geq 1$ leading $1$'s and then a $0$. Then the two rays $R_{\xi^0}$ and $R_{\xi^1}$ coincide down to time $n - 1$ and follow the boundary of the disk $D_{r/2^n}$ precisely down to times $-n_0 - n$ and $-n_1 - n$ respectively, where $r$ has the binary representation given by the first $n$ digits of $\xi^0$. Similarly to the above we can short-cut $R_{\xi^0} \cup R_{\xi^1}$ through $D_{r/2^n}$.

In any of the three cases we denote by $\tilde{\gamma}(\xi^0, \xi^1)$ the arc obtained from $R_{\xi^0} \cup R_{\xi^1}$ by short-cutting through the appropriate $D_{r/2^n}$.

4.2.2 The Universal Parabolic $p/q$ Yoccoz Puzzle.

As above let $Z_0 = \bigcup_{j=0}^{q-1} e^{2\pi \theta_j} \cup e^{2\pi (\theta_j + \frac{1}{2})}$ be the unique $q$-cycle for $Q_0$ of combinatorial rotation number $p/q$ in $S^1$ and its preimage. Then the set $h(Z_0)$ corresponds to the unique $p/q$ orbit of $\mathcal{B}l$ together with its preimage under $\mathcal{B}l$ (recall that $h$ is the conjugacy between $\mathcal{B}l$ and $z^2$ satisfying $h(z^2) = \mathcal{B}l \circ h$ defined in Lemma 3.1). Let $0 < \xi_0 < \xi_1 <
\[ \ldots < \epsilon_{2q-1} < 1 \] denote the unique itineraries of these points and let \( \mathcal{GP}_0 \) denote the graph
\[ \mathcal{GP}_0 = S^1 \cup \bigcup_{i=0}^{2q-1} \hat{\gamma}(\epsilon_i, \epsilon_{(i+1) \mod 2q}) \]
and define the universal parabolic \((p/q\text{-Yoccoz})\) puzzle \( P_0 \) as the set consisting of the \( 2q \) bounded connected components of the complement of \( \mathcal{GP}_0 \) in \( \mathbb{C} \setminus \overline{D} \). Denote by \( P_{1,0} \) and \( P_{-1,0} \) the puzzle pieces with 1 and \(-1\) respectively on the boundary. Then by construction all the level 0 puzzle pieces except \( P_{1,0} \) are pre-images of \( P_{-1,0} \) under some iterate of \( \mathcal{B}l^k \), \( 0 \leq k \leq q \). This is different from the level 0 universal Yoccoz-puzzle, where all puzzle pieces are bounded by the same equipotential.

Define \( \mathcal{P}_n \) recursively as follows:
\[ \mathcal{P}_n = \{ \mathcal{B}l^{-1}(P) \mid P \in \mathcal{P}_{n-1}, 1 \notin \partial P \} \cup \{ P_{1,n}, P_{-1,n} \}, \]
where \( P_{1,n} \), resp. \( P_{-1,n} \), is the component bounded by
\[ \hat{\gamma}((0,\ldots,0,\epsilon_0), (1,\ldots,1,\epsilon_{2q-1})) \] resp. by \( \hat{\gamma}((0,1,\ldots,1,\epsilon_{2q-1}), (1,0,\ldots,0,\epsilon_0)) \)
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together with the corresponding arc on the unit circle.

We shall write \( \tilde{\gamma}_n \) for the short-cut \( \partial P_{1,n} \cap D_0 \) and \( \gamma_n \) for the short-cut \( \partial P_{-1,n} \cap D_{1/2} \).

By construction the only non dynamical parts of the universal parabolic \( p/q \) graph and puzzle are the short-cuts \( \tilde{\gamma}_n \) and \( \gamma_n \), \( n \geq 0 \), i.e.

\[
B(\mathcal{GP}_{n+1}\setminus(\tilde{\gamma}_{n+1} \cup \gamma_{n+1})) = \mathcal{GP}_n \setminus \tilde{\gamma}_n.
\]

Finally define \( \mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n \). We call \( \mathcal{P} \) the (quadratic) universal parabolic \( p/q \)-Yoccoz puzzle. Denote by \( \mathcal{GP}_n = \bigcup_{P \in \mathcal{P}_n} \partial P \). Denote by \( \mathcal{GP}(n) \) the union \( \bigcup_{0 \leq k \leq n} \mathcal{GP}_k \); it coincides with the union of the boundaries of puzzle pieces of all levels up to and including \( n \). Let \( \mathcal{GP} \) be the union of these graphs of all levels.

Figure 14: In the model

For every \( p/q \), there is a correspondence between the Universal Yoccoz puzzle and Universal Parabolic puzzle. For any universal Yoccoz puzzle piece of depth \( n \) bounded by external rays of argument \( \{t_1, t_2\} \), the corresponding universal parabolic puzzle piece of depth \( n \) is bounded by the parabolic rays of argument \( \{h(t_1), h(t_2)\} \).

4.2.3 Parabolic \( p/q \)-Puzzle

Let \( p/q \) be an irreducible rational and let \( B \in \mathcal{W}_1^{M_1}(p/q) \) (for the definition of the \( p/q \) wake \( \mathcal{W}_1^{M_1}(p/q) \) see Lemma 3.20). The parabolic \( p/q \) puzzle for \( g_B \) is derived from the universal
parabolic $p/q$ puzzle, in a manner similar to how the Yoccoz-puzzle for $Q_c, c \in \mathcal{W}_1(p/q)$ is derived from the universal Yoccoz puzzle.

For each $n \geq 0$ let $V^P_n$ be the interior of the union of closures of level $n$ universal parabolic puzzle pieces. We define reduced wakes

$$W^M_n(p/q) := \mathcal{M}_n \cup \{B \in W^M_n(p/q) | h_B(v_B) \in V^P_n \},$$

which are similar to the reduced wakes $W^M_n(p/q)$ though the phrasing of the definition is different.

Recall from Corollary 3.22 that for $B \in \mathcal{W}_1^M(p/q)$, the unique $p/q$-cycle of parabolic rays for $g_B$ with rotation number $p/q$ co-lands on $\alpha_B$ the unique finite fixed point for $g_B$, which is repelling. And moreover the second critical value $v_B$ for $g_B$ belongs to the dynamical wake $\mathcal{W}_1^B(p/q)$. If $B \in \mathcal{W}_1^M(p/q) \setminus \mathcal{M}_1$ then we may extend $h_B^{-1}v_B$ analytically to a univalent map on a neighbourhood in $\hat{\mathbb{C}} \setminus B$ of $(\mathcal{G}^0_\mathcal{G}) \cup D_1 \cup U_{p/q}$, where $U_{p/q}$ is the unbounded connected component of $\hat{\mathbb{C}} \setminus B$. And if $B \in \mathcal{M}_1 \cap \mathcal{W}_1^M(p/q)$ then $h_B^{-1}$ even extends to a biholomorphic map $\hat{h}_B^{-1} : \mathbb{C} \setminus B \rightarrow \Lambda_B$. Thus for every $B \in \mathcal{W}_1^M(p/q)$ we may define short cuts $\tilde{\gamma}_n = h_B^{-1}(\gamma_n) \subset D_0^B$ and $\gamma_n = h_B^{-1}(\gamma_n) \subset D_{1/2}^B$. So we may define parabolic Yoccoz graphs:

**Definition 4.1.** For $B \in \mathcal{W}_1^M(p/q)$ we define the dynamical graph $\mathcal{G}^B_0$ as

$$\mathcal{G}^B_0 = h_B^{-1}(\mathcal{G}_0) \cup \{\alpha_B, \alpha'_B\}.$$ We define the parabolic $(p/q)$-Yoccoz puzzle $\mathcal{P}^B_0$ for $g_B$ as the set consisting of the $2q - 1$ connected components of the complement of $\mathcal{G}^B_0$ intersecting the Julia set of $g_B$.

Define $\mathcal{G}^B_n$ recursively by

$$\mathcal{G}^B_{n+1} := g^{-1}_B(\mathcal{G}^B_n \setminus \tilde{\gamma}_n) \cup \{\tilde{\gamma}_{n+1} \cup \gamma_{n+1}\}.$$ We denote by $\mathcal{G}^B(n)$ the union $\cup_{0 \leq k \leq n} \mathcal{G}^B_k$; and we write $\mathcal{G}^B$ for the union of these graphs of all levels.

And define the parabolic $(p/q)$-Yoccoz puzzle $\mathcal{P}^B_n$ for $g_B$ as the set of complementary components $P$ of $\mathcal{G}^B_n$ intersecting the Julia set for $g_B$. We take over the vocabulary from Yoccoz puzzles and write $\mathcal{P}^B_n(z)$ for the level $n$ puzzle piece containing $z$, if one such exists, $\mathcal{N} = \{P_n\}_n$ for a nest of puzzle pieces, $\text{End}(\mathcal{N}) = \cap_n \overline{P_n}$ for the end of $\mathcal{N}$. And moreover that $\mathcal{N}$ is convergent to $z$ iff $\text{End}(\mathcal{N}) = \{z\}$ and divergent if $\text{End}(\mathcal{N})$ is not a singleton.

**Remark 4.2.** Note that by construction the second critical point and value, $-1, v_B$ as well as $\beta_B$ are pairwise separated by $\mathcal{G}_0^B$ for every $B \in \mathcal{W}_1^M(p/q)$. Moreover the dynamical wake $W_B(p/q)$ is disjoint from $D_0^B$, for any $B \in \mathcal{W}_1^M(p/q)$ so that the second critical
point $-1$ does not belong to $D^B$ and hence for every $n$ the puzzle pieces $P_n(\beta_B)$ and $P_n(\beta'_B)$ are defined and the restrictions
\[
g_B : \partial P_{n+1}(\beta_B) \setminus \gamma_{n+1}^B \to \partial P_n(\beta_B) \setminus \tilde{\gamma}_n^B \quad \text{and} \quad g_B : \partial P_{n+1}(\beta'_B) \setminus \gamma_{n+1}^B \to \partial P_n(\beta_B) \setminus \tilde{\gamma}_n^B
\]
are diffeomorphisms.

Let $B_* \in W_1^{M_1}(p/q)$ and define the graph
\[
\Xi_{B_*} := \mathcal{G}P_0^B \cup \{\beta_B, \beta'_B\} \cup \bigcup_{n \geq 0} (\partial P_n(\beta_B) \cup \partial P_n(\beta'_B)).
\]

**Proposition 4.3.** Let $B_* \in W_1^{M_1}(p/q)$ be arbitrary. Then there is a holomorphic motion
\[
\psi^B_{\Xi_*} : W_1^{M_1}(p/q) \times \Xi_{B_*} \to \hat{\mathbb{C}}
\]
with base point $B_*$ such that $\psi^B_{\Xi_*}(B, \Xi_{B_*}) = \Xi_B$ and $g_B \circ \psi^B_{\Xi_*}(B, z) = \psi^B_{\Xi_*}(B, g_{B_*}(z))$ for every $B \in W_1^{M_1}(p/q)$ and $z \in \Xi_{B_*}$.

**Proof.** Since the family $g_B$, $B \neq 0$ has a persistent parabolic fixed point of fixed parabolic multiplicity, the normalized Fatou coordinates $\phi_B$ for $g_B$ depends holomorphically on both $B$ and $z$. Hence also the coordinates $h_B$ depends holomorphically on the two variables $(B, z)$. For the same price the short-cuts $\tilde{\gamma}_n^B, \gamma_n^B \subset D^B$ move holomorphically with $B$. Since $\mathcal{G}P_0^B$ is the closure of $h_B^{-1}(\mathcal{G}P_0 \setminus \overline{\mathbb{D}})$ the graph $\mathcal{G}P_0^B$ moves holomorphically with $B \in W_1^{M_1}(p/q)$. By construction $\beta_B \equiv \infty$ and $\beta'_B \equiv 0$ and so move holomorphically. Finally by Remark 4.2 above there is no critical point for $g_B$ on any of the boundary arcs $\partial P_{n+1}(\beta_B) \setminus \gamma_{n+1}^B, \partial P_{n+1}(\beta'_B) \setminus \tilde{\gamma}_{n+1}^B$ for any $n \geq 0$ and any $B \in W_1^{M_1}(p/q)$. Hence also these move holomorphically with $B$. From this the proof follows.

## 5 Tower of laminations, combinatorial invariants

### 5.1 Abstract Towers

The following presentation is an excerpt from [PRI]. For more details the reader is referred to this paper.

Let $E_0$ denote the unique $p/q$ cycle for $Q_0$, then denote by $Z_n = Q_0^{-(n+1)}(E_0)$ for $n \geq -1$ and $Z = \cup_{n \geq 0} Z_n$. Remark that $Z_0 = E_0 \cup (-E_0)$ and that $Z_n = Q_0^{-n}(Z_0)$ for $n \geq 0$.

For $E \subset S^1$ we let $H(E)$ denote $E$ union its hyperbolic convex hull in $\mathbb{D}$, $H(E)$ is therefore a closed set in $\overline{\mathbb{D}}$. 
Definition 5.1. A tuple of equivalence relations \((\sim_n)_{0 \leq n \leq N}\), with \(N \in \mathbb{N} \cup \{\infty\}\), is called a tower if it satisfies the following admissibility conditions (see also [K]):

i) For each \(n\): \(\sim_n\) is an equivalence relation on \(\mathbb{Z}_n\).

ii) \(\sim_0\) has the two classes \(E_0\) and \(-E_0\).

iii) For any class \(E\) of \(\sim_n\) with \(0 \leq n \leq N\) the set \(Q_0(E)\) is a class of \(\sim_{(n-1)}\);

iv) \(\sim_N = \bigcup_{n=0}^{N} \sim_n\) so that \(\sim_n |_{\mathbb{Z}_m} = \sim_m\) for any \(m, n\) with \(0 \leq m < n \leq N\);

v) For any two distinct classes \(E\) and \(E'\) of \(\sim_n\), with \(0 \leq n \leq N\), \(H(E) \cap H(E') = \emptyset\).

By property iv. \(\sim_N\) imposes \(\sim_n\) for \(n \leq N\). We shall thus abbreviate and write simply \(\sim_N\) for the tower \((\sim_n)_{0 \leq n \leq N}\).

The level of a class \(E\) is the minimal \(n \geq 0\) for which \(E \subset \mathbb{Z}_n\).

The finite towers \(\sim_n\) are the nodes of a tree with root \(\sim_0\) and with a branch connecting each child \(\sim_N\) back to its parent \(\sim_{N-1}\). We denote this tree by \(\mathcal{T}\). The infinite towers \(\sim_\infty\) on the other hand are the infinite branches of this tree starting at \(\sim_0\). We denote the set or space of all infinite branches \(\mathcal{T}_\infty\).

For a tower \(\sim_N\), we denote by the graph of \(\sim_N\) the set

\[
G_{\sim_N} = \bigcup_{E \text{ a class of } \sim_N} H(E) \subset \overline{D}
\]

A gap \(G\) of a finite tower \(\sim_n\) is any connected component of \(\overline{D} \setminus G_{\sim_n}\). We denote by essential boundary of a gap \(G\) the set \(\delta G = G \cap S^1\). The image of the gap \(G_n\) of \(\sim_n\) is defined as the gap \(G_{n-1}\) of \(\sim_{n-1}\) with \(\delta G_{n-1} = Q_0(\delta G_n)\).

A class \(E\) or a gap \(G\) is said to be critical iff \(0 \in H(E)\), resp. \(0 \in G\). Clearly any finite tower has either a (unique) critical class or gap. We shall denote the critical class/gap of \(\sim_n\) by \(E_n^*/G_n^*\) (or just \(E^*/G^*\)). The image of the critical class or gap will be called the critical value class or gap of \(\sim_n\) and denoted \(E_n^*/G_n^*\). Note that the critical value class or gap is a class or gap of \(\sim_{n-1}\) and (provided the level of the critical class is \(n\)) is a subset of the critical value gap of \(\sim_{n-1}\).

For a finite tower \(\sim_N\) with critical gap \(G_N^*\) define the critical period \(k \geq 1\) of \(\sim_N\) as the minimal \(k \geq 1\) for which \(Q_0(G_N^*)\) is again a critical gap (of \(\sim_{N-k}\)). Note that in fact \(k \geq q\) always. Also in order to ensure that a critical gap always has a critical period, we
may formally define \( Z_n = E_0 \) and \( \sim_n \) as the equivalence relation with only one class \( E_0 \) for any \( n \) with \(-q < n < 0\).

Let \( \sim_n \) be a finite tower. If \( \sim_n \) has a critical class \( E \) it has a unique child and we say that \( \sim_n \) is a terminal tower.

If \( \sim_n \) has a critical gap with critical value gap \( G_n' \) and if \( E \subset G_n' \) or \( G \subseteq G_n' \) is any class or gap of \( \sim_n \) within \( G_n' \). Then \( \sim_n \) has a unique extension \( \sim_{n+1} \) with critical value class \( E \) respective critical value gap \( G \). For this reason we say \( \sim_n \) is a fertile tower, when it has a critical gap.

An infinite tower \( \sim_\infty \) is said to be renormalizable with combinatorics \( \sim_N \) and renormalization period \( k \) if for every \( n \geq N \), \( \sim_n \) has critical period \( k \) and \( N \) is the minimal height with this period.

Suppose \( \sim_\infty^T \) is an infinite terminal tower with critical value class \( E_n' \), that is \( \sim_n^T = \sim_\infty^T |_{Z_n} \) has a critical class \( E'_n \) with image \( E'_n \) and \( \sim_n = \sim_\infty^T |_{Z_{n-1}} \) has a critical gap \( G_{n-1}' \) with image the critical value gap \( G_{n-1} \) containing \( H(E'_n) \). Then \( G_{n-1}' \) contains exactly \( q \) gaps \( G_{n-1}' \), \( G_n' \) of \( \sim_{n-1} \), which are adjacent to \( E_n' \), i.e. with \( H(E'_n) \cap \partial G_{n-1}' \neq \emptyset \), because \( H(E'_n) \) is a \( q \)-gon. In the light of the above discussion let \( \sim_n^j \) denote the unique extensions of \( \sim_{n-1} \) with critical value gaps \( G_n^j \) for \( j = 1, \ldots, q \). Define recursively for \( m > n \) unique extensions \( \sim_m^j \) of \( \sim_{m-1}^j \) with critical value gap \( G_m^j \subset G_{m-1}' \) adjacent to \( E_n' \). Finally denote by \( \sim_\infty^j = \bigcup_{m \geq n} \sim_m^j \) the corresponding infinite towers for \( j = 1, \ldots, q \).

We shall say that \( \sim_\infty^T \) is adjacent to any of the \( q \) towers \( \sim_\infty^1, \ldots, \sim_\infty^q \) and vice versa.

5.2 The natural tower puzzle relation

Recall that if \( c \in M \setminus \text{Card} \) then \( c \) belongs to a limb \( L_{p/q} \), so that the \( p/q \) cycle of rays co-land at the \( \alpha \) fixed point. Fix \( p/q \) with \((p, q) = 1\) and let \( Z_n, Z \) be given by \( p/q \) as in Subsection 5.1.

**Definition 5.2.** Let \( c \) belong to the limb \( L_{p/q} \). Define \( t, t' \in Z_n, n \in \mathbb{N} \cup \{\infty\} \) to be equivalent, \( t \sim_n^c t' \) if and only if the rays \( R^c_t \) and \( R^c_{t'} \) co-land. And define \( \sim^c \) as the corresponding tower of equivalence relations.

Note that the arguments \( t \in Z_n \) are precisely the arguments of external rays in the level \( n \) Yoccoz graph and puzzle. And moreover \( t \sim_n^c t' \) if and only if \( R^c_t \) and \( R^c_{t'} \) co-land on a point of \( Q^c_n(\alpha_c) \). We can thus view \( \sim_n^c \) as an abstract version of the Yoccoz graph/puzzle, where the gaps \( G \) corresponds to level \( n \) puzzle pieces and the hulls \( H(E) \) of classes \( E \) corresponds to the unions of segments of co-landing rays. In view of this for any class \( E \) of \( \sim^c \) say of level \( n \), we shall refer to the closure of the complete set of co-landing rays with arguments in \( E \) as \( R_E \). Then the Yoccoz graph \( \mathcal{GY}_n \) is also the first
graph containing the lower ends of the rays in $R_E$.

Similarly any $g \in M_1 \setminus \mathbb{D}$ belongs to a Limb $L_{p/q}^{M_1}$.

**Definition 5.3.** Let $g$ belong to the Limb $L_{p/q}^{M_1}$. Define $t, t' \in \mathbb{Z}_n$, $n \in \mathbb{N} \cup \{\infty\}$ to be equivalent, $t \sim_n^g t'$ if and only if the rays $R_{h(t)}^g$ and $R_{h(t')}^g$ co-land. And define $\sim$ as the corresponding tower of equivalence relations.

**Remark 5.4.** In both the polynomial and the parabolic case we shall extend the definition of $\sim$ to maps $Q_c$ with $c \in W_{n-1}^M(p/q)$ with $\not\in \mathcal{Y}_{n-1}^c$ respectively maps $g_B$ with $B \in W_{n-1}^M(p/q)$ with $v_B \notin \mathcal{G}_{n-1}^B$.

Recall that $h : \mathbb{S}^1 \to \mathbb{S}^1$ is the conjugacy between $Q_0$ and $B_l$ (see Lemma [3.1]).

In other words $t \sim_n^g t'$ are equivalent if and only if the parabolic rays of the level $n$ parabolic Yoccoz graph $\mathcal{G}_{n}^g$ corresponding to $h(u)$ and $h(v)$ co-land at the samepoint of $g^{-n}(\alpha(g))$. As in the polynomial case the equivalence relations $\sim_n^g$ can be viewed as abstract parabolic Yoccoz graphs/puzzles with the gaps $G$ corresponds to level $n$ puzzle pieces and the hulls $H(E)$ of classes $E$ corresponds to the unions of segments co-landing rays.

We proved in [PR2] Lemma 5.7 that among all $p/q$ towers adjacent to some terminal tower $\sim \infty$ only the renormalizeable tower $\sim \infty(p/q)$ with renormalisation period $q$ is realised as $\sim \infty$ for some $c \in M$ and similarly as $\sim \infty$ for some $g \in M_1$. Moreover we proved in [PR2] Theorem 5.6] that any other infinite tower is realized as $\sim \infty$ for some $c \in M$. We summarize this as the following Theorem

**Theorem 5.5.** Let $\sim \infty$ be any quadratic tower which is either equal to $\sim \infty(p/q)$ for some irreducible $p/q \neq 0/1$ or which is not adjacent to some terminal tower. Then there exists $c \in M$ such that $\sim \infty = \sim \infty^c$.

The following Proposition is a slightly more elegant formulation of [PR2] Lemma 5.1], whose proof can be easily adapted to prove the Proposition. It says that if $g \in M_1$ and $Q_c$ with $c \in M \setminus Card$ define the same infinite tower $\sim \infty$. Then their puzzles are similar:

**Definition 5.6.** Define an abstract map $\tilde{g}$ from $\mathcal{P}_B$ to itself given by $\tilde{g}(\beta') = \tilde{g}(P_n(\beta')) = P_n(\beta)$ for every $n \geq 1$ and $\tilde{g}(P_n) = g(P_n)$ for every other level $n \geq 1$ puzzle piece $P_n$.

**Proposition 5.7.** Let $p/q$ be an irreducible rational, let $B \in L_{p/q}^{M_1}$ and $c \in L_{p/q}$ be parameters such that $\sim \infty = \sim \infty^g$. Then there is a $1 : 1$ correspondence $\chi = \chi_g$ between the puzzle pieces of $\mathcal{Y}_c$ for $Q_c$ and $\mathcal{P}_B$ for $g_B$. Moreover

1. any annulus of the parabolic puzzle $\mathcal{P}_g$ is non degenerate if and only if the corresponding annulus in the Yoccoz puzzle $\mathcal{Y}_c$ is non degenerate,
2. For any puzzle piece $Y_n \in \mathcal{Y}_c$ of level $n \geq 1$ the puzzle piece $P_n = \chi(Y_n)$ also has level $n$ and $g \circ \chi(Y_n) = \chi \circ Q_c(Y_n)$.

3. in particular critical puzzle pieces correspond to critical puzzle pieces.

The proof is obtained through a natural correspondence $\chi_P$ between the corresponding universal puzzles.

Note that if $Y$ in the Proposition above contains either $\beta(c)$ or $\beta'(c) = -\beta(c)$, then $P = \chi(Y)$ contains the parabolic fixed point $\infty$ respectively its preimage 0.

6 Transfering Yoccoz results to maps in $\mathbb{M}_1$

We first recall the basic steps in the proof of Yoccoz theorem of local connectivity for quadratic polynomials. Then we show that a similar proof can be made for maps $g_B$ in $\mathbb{M}_1$. In the following chapters we use this setup to transfer results to the parameter spaces. Fix for the rest of this section an arbitrary irreducible rational $p/q$.

6.1 Basic Yoccoz puzzle theory and estimates

In this subsection we set-up the machinery for the proof of Yoccoz theorem on local connectivity of the Mandelbrot set at any non renormalizeable parameter in any de-rooted limb $L_{p/q}$ and for the same price local connectivity of the Julia set of such polynomials. (see for instance in [Mi3]). Recall that $l_0$ was the equipotential level in the definition of Yoccoz puzzles, that $V_n^c := \{ z \in \mathbb{C} \mid g_c(z) < l_0/2^n \}$ is the dynamical set bounded by the $l_0/2^n$ level set. And moreover $\mathcal{W}_n^M := \{ c \in \mathcal{W}_n^M(p/q) \mid c \in V_n^c \}, n \geq 0.$

Note that for $c \in \mathcal{W}_0^M$ the set $V_n^c$ is the interior of the union of closures of all level $n$ puzzle pieces.

Let $c \in \mathcal{W}_0^M(p/q)$. For $Y_n$ a puzzle piece of some level $n$ and $z \in Y_n$ we write $Y_n(z) := Y_n$. We shall furthermore use the abbreviations $Y^0_n := Y_n(0) \in \mathcal{Y}_n$ and $Y^c_n := Y_n(c) = Q_c(Y_{n+1}) \in \mathcal{Y}_n$ whenever there is such a puzzle piece, that is whenever $Q_c(Y_{n+1})$ belongs to a level 0 puzzle piece. If $K_c$ is connected, this only fails whenever $Q_c(Y_{n+1}) = \alpha_c$. Since the dynamics is quadratic any non-critical puzzle piece $Y$ has a unique dynamical twin $\tilde{Y}$ of the same level with $Q_c(\tilde{Y}) = Q_c(Y)$, in fact $\tilde{Y} = -Y$. And the critical puzzle pieces are siamese twins in the sense that $Y^0_n = \tilde{Y}^0_n$.

**Lemma 6.1.** Let $c \in \mathcal{W}_0^M(p/q)$. Then the map $Q_c^q$ has a quadratic-like restriction $f_c := Q_c^q : U \rightarrow U'$ with $V_q^0 \cap V_q^c \subset U \subset V_q^c$. 

Moreover the filled-in Julia set \( K'_c \) of \( f_c \) is contained in \( \{ \alpha, \alpha' \} \cup (Y_0^0 \cap V_q^-) \) and \( K'_c \) is connected if and only if \( f_n^c(0) = Q'^{aq}(0) \in \{ \alpha, \alpha' \} \cup (Y_0^0 \cap V_q^-) \) for all \( n \).

Proof. Apply a small thickening of \( \overline{Y_0^0} \cap V_q^- \subset U \) at the ends, see e.g. [Mi3, Corollary 1.7]. The proof given there in the case \( Q_{nq}^c(0) \in \overline{Y_0^0} \) for all \( n \) works for all \( c \in W_{q-1}(p/q) \).

In the following fix \( c \in L_{p/q} \). Then precisely one of the following two cases occur

D1. For all \( n \in \mathbb{N} : Q_{nq}^{aq}(0) \in \overline{Y_0^0} \).

D2. There exists \( m \geq 1 \) minimal such that \( Q_{mq}^{aq}(0) \notin \overline{Y_0^0} \).

In the first case D1 it follows from Lemma 6.1 above that \( Q_c \) is \( q \)-renormalizable. That is, there exists a quadratic like restriction \( Q_c^q : U \rightarrow U' \) with connected filled-in Julia set \( K'_c \subset Y_0^0 \cup \{ \alpha, \alpha' \} \). We shall henceforth focus on the second case D2.

In order to describe better the second case we set up some additional notation. For \( 0 < k < q \) let \( Y_k^c \) denote the level 0 puzzle piece contained in \( Q_k^c(Y_0^0) \). Then \( Q_k^c(0) \in Y_k^c \) for any \( c \in W_{q-2}(p/q) \). Each \( Y_k^c \) is adjacent to \( \alpha, Y_0^1 = Y_0^c \) and \( Y_0^q = Y_0(\beta') \). The corresponding twins \( \overline{Y_k^c} \) are adjacent to \( \alpha' \) and \( \overline{Y_0^q} = Y_0(\beta) \). Denote by \( X_0 \) the interior of \( \bigcup_{k=1}^{q-1} \overline{Y_k^c} \) and by \( \overline{X_0} \) its twin. Then the common univalent image \( Q_c(X_0) = Q_c(\overline{X_0}) \) covers all of the level 0 puzzle except \( \overline{Y_0} \).

Note that the condition \( Q_{mq}^{aq}(0) \notin \overline{Y_0^0} \) in D2 is equivalent to \( Q_{mq}^{aq}(0) \in \overline{X_0} \) and to \( Q_{mq}^{aq}(c) \notin \overline{Y_0^0} \).

When studying parameter space we shall also be interested in the following extension of condition D2 to all of \( W_0^M(p/q) \).

D2’. There exist \( m \geq 1 \) minimal with \( Q_{mq}^{aq}(0) \in \overline{X_0} \).

Parameters \( c \) satisfying D2’ belongs to a dyadic sub-wake of the satellite-copy \( M_{p/q} \):

Proposition 6.2. A parameter \( c \in W_0^M(p/q) \) satisfies D2’ if and only if

\[
c \in W_{mq-1}(p/q, r, m) := W_{mq-1}(p/q) \cap W_{mq}^M(p/q, r, m)
\]

where \( m \geq 1 \) is from D2’ and \( r \) is odd with \( 0 < r < 2^m \).
Proof. Let \( c \in \mathcal{W}_0^M(p/q) \) satisfy \( Q^m_c(0) = Q^{m-1}_c(c) \in \tilde{X}_0 \) for some minimal \( m \geq 1 \). Then \( c \in V^c_{m-1} \) and thus also \( c \in \mathcal{W}_{mq-1}^M \). Moreover \( Q^m_c(c) \notin \overline{V^c_0} \), hence \( Q^{(m-1)q}_c(c) \) belongs to the 1/2 dyadic wake \( \mathcal{W}_c(p/q, 1, 1) \) and thus \( c \in \mathcal{W}_c(p/q, r, m) \) for some odd \( r \) with \( 0 < r < 2^m \) by induction and minimality of \( m \). Hence also \( c \in \mathcal{W}_M(p/q, r, m) \). \( \square \)

Recall that for \( m \geq 1 \) and \( r \) odd \( 0 < r < 2^m \) the de-rooted dyadic decoration is given \( L^*(p/q, r, m) \) by

\[
L^*(p/q, r, m) = L^*_{p/q} \cap \mathcal{W}_M(p/q, r, m).
\]

This gives the following decomposition of the limb \( L_{p/q} \), first observed by Douady and Hubbard.

**Corollary 6.3.** The limb \( L_{p/q} \) has a natural stratification as

\[
L_{p/q} = M_{p/q} \cup \bigcup_{\tilde{r}} L^*(p/q, r, m).
\]

**Proof.** This follows immediately from the dichotomy, \( \square [1], \square [2] \) above. \( \square \)

For any \( c \in \mathcal{W}_0^M(p/q) \) the common image of puzzle pieces \( Q_c(Y_0(\beta')) = Q_c(Y_0(\beta)) \) univalently covers \( V_0 \setminus \overline{X}_0 \), and does not intersect the set \( X_0 \). It follows that the level 0 twin puzzle pieces \( Y_0(\beta') \) and \( Y_0(\beta) = \tilde{Y}_0(\beta') \) each contain \( q \) level 1 puzzle pieces, which are mapped homeomorphically onto \( Y_0^0 = \tilde{Y}_0^0 \) and \( \tilde{Y}_0^k \), \( 0 < k < q \). By similar reasoning every other level 0-puzzle piece contains a unique level 1 puzzle piece \( Y_1^k \subset Y_0^k \) respectively \( \tilde{Y}_1^k \subset \tilde{Y}_0^k \) which is mapped properly onto \( Y_0^{k+1} \).

**Proposition 6.4.** For any \( c \in \mathcal{W}_0^M(p/q) \) the boundaries of all puzzle pieces in the \( \beta \)-nest move holomorphically with \( c \in \mathcal{W}_0^M(p/q) \).

**Proof.** The level 0 Yoccoz graph \( Y_0 \subset \partial Y_0(\beta) \) moves holomorphically with \( c \) over \( \mathcal{W}_M(p/q) \), since the \( c \in \mathcal{W}_c(p/q) \cap V^c_0 \), see also Section \( \square [2.2] \). Moreover the restriction \( Q_c : Y_1(\beta) \rightarrow Y_0(\beta) \) is biholomorphic with a univalent extension to a neighborhood of \( Y_0(\beta) \). Hence by induction \( Y_{n+1}(\beta) = Q_c(Y_n(\beta)) \). From this the proposition follows by induction. \( \square \)

By construction there are \( q \) puzzle pieces of level \( n \) adjacent to \( \alpha_c \) for every \( n \). And thus \( q \) sequences of nested puzzle pieces \( \mathcal{N}^\alpha_{n,k} \) for every \( n \) and \( k \) and the degree is 1 unless \( k = 0 \) so that \( Y_{n+1}^0 = Y_n^0 \). It follows immediately that either all \( q \) nests are convergent to \( \alpha \) or none is convergent.
In order to create a fundamental system of nested neighbourhoods of \( \alpha_c \) we denote by \( Y_n^\alpha \) the interior of \( \cup_{k=0}^{n-1} Y_{n,k}^\alpha \), so that \( Y_n^\alpha \) is an open neighbourhood of \( \alpha_c \) for all \( n \). However no \( Y_n^\alpha \) is a puzzle piece. Let \( r/2^n \), \( r \) odd and \( 0 < r < 2^n \) be a dyadic rational and let \( c \in \mathcal{W}^M(p/q, r, m) \). Then \( Q_c \) maps \( Y_n^\alpha \) biholomorphically onto \( Y_{n-1}^\alpha \) and \( Y_n^\alpha \subset Y_{n+1}^\alpha \) for all \( n \geq (m+1)q \). Moreover the boundaries \( \partial Y_n^\alpha \) move holomorphically over \( \mathcal{W}^M(p/q, r, m) \) and continuously over the closure.

**Lemma 6.5.** Suppose \( c \in \mathcal{W}^M_0(p/q) \) satisfies \( \|Q_c\| < 2^q \), for some \( m \geq 1 \). Let \( f_c = Q_c^q : U \rightarrow U' \) be a quadratic like map as in Lemma 6.1 with \( Y_0^q \cap V_{mq} \subset U \subset V_{mq} \). Then

1. the filled Julia set \( K' \subset \overline{Y_{mq}^\alpha} \subset \overline{Y_{mq}^\alpha} \subset U \),
2. the restriction \( f_c : \overline{Y_{mq}^\alpha} \rightarrow \overline{Y_{m(q-1)}^\alpha} \) is a holomorphic diffeomorphism,
3. \( \text{diam}(\overline{Y}_{(n+m)q}) \rightarrow 0 \) as \( n \rightarrow \infty \) uniformly over all connected components \( Y_{(n+m)q} \) of \( f_c^{-n}(\overline{Y_{mq}^\alpha} \cup Y_{mq}^\alpha) \).
4. If \( Y_{(n+1+m)q} \subset Y_{(n+m)q} \) are nested puzzle pieces with \( f_c^n(Y_{(n+m)q}) \cup f_c^{n+1}(Y_{(n+1+m)q}) \in \{Y_{mq}^\alpha, Y_{mq}^-\} \), then \( \partial Y_{(n+1+m)q} \subset \partial Y_{(n+m)q} \subset f_c^{-(n+1)}(\alpha_c) \).
5. In particular for any \( z \in K' \) either \( z \) is not prefixed to \( \alpha_c \) under \( f \) and the nest \( \{Y_n(z)\} \) is convergent to \( z \) or \( Q_c^l(z) = \alpha_c \) for some minimal \( l \) and there are \( q \) nests \( \{Y_n^r(z), 0 \leq k < q \} \) convergent to \( z \), where \( Q_c^q(Y_n^r(z)) = Y_n^{\alpha,k} \) for each \( n \) and \( k \).

**Proof.** The set \( U \setminus (\overline{Y_{mq}^\alpha} \cup \overline{Y_{mq}^-}) \) consists of points \( z \) with \( f_c^n(z) \in X_0 \cup \tilde{X}_0 \) for some minimal \( n, 0 \leq n \leq m \). Thus all such points escapes and \( K' \subset (\overline{Y_{mq}^\alpha} \cup \overline{Y_{mq}^-}) \). The post-critical orbit \( \mathcal{O}_f \), the forward orbit of 0 under \( f_c \) is finite and disjoint from \( (\overline{Y_{mq}^\alpha} \cup \overline{Y_{mq}^-}) \). Hence the latter has finite hyperbolic diameter in \( U' \setminus \mathcal{O}_f \). Thus the hyperbolic diameter of \( \overline{Y_{(n+m)q}} \) for \( Y_{(n+m)q} \) any connected component of \( f_c^{-n}(\overline{Y_{mq}^\alpha} \cup \overline{Y_{mq}^-}) \) converges geometrically to 0, as \( n \rightarrow \infty \).

By construction \( \partial Y_{mq}^\alpha \cap \partial Y_{(m-1)q}^\alpha = \alpha_c \) and \( \partial \overline{Y_{mq}^-} \cap \partial Y_{(m-1)q}^\alpha = \alpha'_c \) so that \( \Box \) follows by induction.

Finally if \( z \in K' \), then \( z \in \text{End}(\mathcal{N}) \) for a unique nest \( \mathcal{N} = \{Y_{l}(z)\}_{l \geq 0} \) with \( f_c^{m-n}(Y_{mq}) \in \{Y_{mq}^\alpha, Y_{mq}^-\} \) for any \( n \geq m \). And by the above this nest is convergent to \( z \). Let \( \mathcal{N}^{\alpha,k} \) denote the \( q \) nests adjacent to \( \alpha \). Then the nest \( \mathcal{N}^{\alpha,0} \) is convergent to \( \alpha \) by the first part of the proof and hence all are. Thus if \( z \in K'_c \) is prefixed to \( \alpha \) by \( f_c \), then also all \( q \) nests adjacent to \( z \) are convergent. \( \Box \)
By Proposition 6.2, the hypothesis D2 of Lemma 6.5 is equivalent to \( c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \) for some odd \( r \) with \( 0 < r < 2^m \). Fix such \( r \) and define for \( c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \) the set

\[
\Gamma'_c := K'_c \cup \bigcup_{n \geq 0} f^{-n}_c (\partial Y^{a,0}_{mq} \cup \partial \tilde{Y}^{a,0}_{mq}) = \bigcup_{n \geq 0} f^{-n}_c (\partial Y^{a,0}_{mq} \cup \partial \tilde{Y}^{a,0}_{mq}).
\]

**Proposition 6.6.** Let \( m \geq 1 \), let \( r \) be odd with \( 0 < r < 2^m \) and fix \( c, \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \). Then there exists a holomorphic motion

\[
\psi_{r,m} : \mathcal{W}^{M}_{mq-1}(p/q, r, m) \times \Gamma'_c \rightarrow \mathbb{C}
\]

with base point \( c \) such that \( \psi_{r,m}^0(c, \Gamma'_c) = \Gamma'_c \) and \( f_c \circ \psi_{r,m}^0(c, z) = \psi_{r,m}^0(c, f_c(z)) \) for every \( c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \) and every \( z \in \Gamma'_c \).

**Proof.** For \( c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \) the twin boundaries \( \partial Y^{a,0}_{mq} \) and \( \partial \tilde{Y}^{a,0}_{mq} \) move holomorphically with \( c \), because \( f^{-m}_c \) maps each boundary onto the boundary of \( Y^0_0 \) by degree \( 2^{m-1} \) without passing the critical point, so that \( f^{-m}_c \) is a local diffeomorphism around each boundary point. Indeed, if the common forward orbit of the two boundaries were to pass the critical point 0, then the critical point would end up on the \( q \) periodic rays on the boundary of \( Y^0_0 \), so that \( Q^{mq}_c(0) \in \overline{Y}^0_0 \), which contradicts that \( c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \). The boundary of \( Y^0_0 \) moves holomorphically over the larger set \( \mathcal{W}^{M}(p/q) \), which compactly contains \( \mathcal{W}^{M}_{mq-1}(p/q, r, m) \). And \( f_c(z) \) is a holomorphic function of \( (c, z) \). Thus \( \partial Y^{a,0}_{mq} \) and \( \partial \tilde{Y}^{a,0}_{mq} \) move holomorphically with \( c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \). Secondly for \( c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \) the map \( f_c \) sends each puzzle piece \( Y^{a,0}_{mq} \) and \( \tilde{Y}^{a,0}_{mq} \) univalently onto the larger (i.e. containing) puzzle piece \( Y^0_{(m-1)q} \) and extends as a diffeomorphism of neighbourhoods of the closures.

Hence also all pre-images of \( \partial Y^{a,0}_{mq} \) and \( \partial \tilde{Y}^{a,0}_{mq} \) under iterates of \( f_c \) move holomorphically with \( c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m) \). Finally \( K'_c \) is contained in the closure of the union of all pre-image puzzle boundaries in \( \Gamma'_c \). Thus the proposition follows by the \( \lambda \)-lemma for holomorphic motions.

Note that \( z \in K_c \) with \( Q^{l}_c(z) = \alpha_c \) will be adjacent to \( 2q \) nests if \( 0 \) belongs to the orbit of \( z \).

For \( 0 < k < q \) let \( X_{1}^k \subset Y_0(\beta') \) denote the unique such level 1 puzzle piece with (univalent) image \( \overline{Y}_{0}^k \) and let \( X_1 \) denote the interior of \( \bigcup_k \overline{X}_1^k \), so that \( Q_c \) maps \( \overline{X}_1 \) diffeomorphically onto \( \overline{X}_0 \).

**Proposition 6.7.** Let \( c \in L_{p/q} \), then for any \( z \in K_c \) the orbit falls in precisely one of the following three categories:

i) There exists \( l \geq 0 \) such that \( Q^{l}_c(z) = \beta \).
\textit{ii}) There exists \( l \geq 0 \) such that \( Q_c^l(z) \in K_c' \).

\textit{iii}) There exists a strictly increasing sequence \( \{ l_n \} _{n \geq 0} \) with \( Q_c^{l_n}(z) \in X_1 \) for all \( n \).

\textit{Proof}. Notice at first that for any point \( z \in K_c \) which does not satisfy \( \textbullet \) there exists \( l \geq 0 \), such that \( Q_c^l(z) \in \overline{Y}_0(\beta') \). If \( Q_c^l(z) \not\in X_1 \), then \( Q_c^{l+1}(z) \in \overline{Y}_0(\beta') \), and thus \( Q_c^{l+nq}(z) \in \overline{Y}_0(\beta') \). Hence either \( Q_c^{l+1+mq}(z) \in \overline{Y}_0(\beta') \) for all \( n \geq 0 \) so that the orbit of \( z \) satisfies \( \textbullet \), by Lemma 6.1 or there exists some \( n \) such that \( Q_c^{l+nq}(z) \in X_1 \), set \( l_0 = l + nq \). Then apply the same argument recursively to first \( Q_c^{l_0} \), noting that \( Q_c(Q_c^{l_0}(z)) \not\in Y_0(\beta') \cup X_1 \), to obtain the desired strictly increasing sequence with \( Q_c^{l_n}(z) \in X_1 \).

**Proposition 6.8.** Let \( c \in L_{p/q} \) satisfy \( \blacksquare \). In the first two cases \( \blacksquare \) and \( \blacksquare \) of Proposition 6.7 any nest \( \{ Y_n \} _n \) such that \( z \in Y_n \) for every \( n \) is convergent to \( z \). Moreover if \( z = c \) then there exists \( N \geq 1 \) such that the restriction \( Q_c^1 : Y_N \rightarrow Q_c^1(Y_N) \) is univalent.

Recall that there is a unique \( Y_n = Y_n(z) \) with \( z \in \overline{Y_n} \) except if \( Q_c^k(z) = \alpha_c \) for some \( k \), in which case there are precisely \( q \) such nests if the orbit of \( z \) avoids the critical point 0 and 2q such nests if not.

**Proof.** If there exists \( l \geq 0 \) such that \( Q_c^l(z) = \beta \), take \( l \) minimal with this property. Then the restrictions \( Q_c^l : Y_{n+l} \rightarrow Y_n(\beta) \) are proper maps of non-increasing degrees \( d_n \) for every \( n \geq 0 \). Since the nest \( \{ Y_n(\beta) \} _n \) is convergent to \( \beta \) the nest \( \{ Y_n \} _n \) is convergent to \( z \). If \( 0 = Q_c^l(z) \) for some \( r \leq l \) then \( d_n = 2 \) for every \( n \geq 0 \). Otherwise, since the nest \( \{ Y_n \} _n \) is convergent to \( z \), there exists \( N \geq 1 \) so that \( 0 \not\in Q_c^k(Y_N) \) for any \( k < l \) and so the restriction \( f^l : Y_{N+l} \rightarrow Y_N(\beta) \) is univalent. In particular if \( z = c \), then this restriction is univalent.

For the remaining cases let \( m \geq 1 \) be given by \( c \) satisfying \( \blacksquare \).

If there exists \( l \geq 0 \) such that \( Q_c^l(z) := w \in K_c' \), with \( l \) minimal and \( z \) is not prefixed to \( \alpha \). Then the restrictions \( Q_c^l : Y_{n+l} \rightarrow Y_n(w) \) are proper maps of non-increasing degrees \( d_n \) for every \( n \geq 0 \). Since the nest \( \{ Y_n(w) \} _n \) is convergent to \( w \) by Lemma 6.5 the nest \( \{ Y_n \} _n \) is convergent to \( z \). If \( 0 = Q_c^r(z) \) for some \( r \leq l \) then \( d_n = 2 \) for every \( n \geq mq \). Otherwise, since the nest \( \{ Y_n \} _n \) is convergent to \( z \), there exists \( N \geq mq \) so that \( 0 \not\in Q_c^k(Y_N) \) for any \( r < l \) and thus \( d_n = 1 \) for \( n \geq N \). In particular if \( z = c \) then the restriction \( Q_c^l : Y_{N+l} \rightarrow Y_N(w) \) is univalent.

Finally if there exists \( l \geq 0 \) such that \( Q_c^l(z) = \alpha \in K_c' \), with \( l \) minimal. Then there exists \( k, 0 \leq k < q \) such that the restrictions \( Q_c^k : Y_{n+l} \rightarrow Y_{n+k} \) are proper maps of non-increasing degrees. Since the nest \( \{ Y_{n+k} \} _n \) is convergent to \( \alpha \) by Lemma 6.5 the nest \( \{ Y_n \} _n \) is convergent to \( z \). And hence also any other nest adjacent to \( z \) is convergent. \( \square \)

The above Proposition immediately gives the following Corollary for parameterspace.
Corollary 6.9. Let $c \in L_{p/q}$ and suppose there exists $m \geq 1$ minimal such that $Q_c^{m_0}(0) \notin \overline{Y}_0$. Then precisely one of the following three cases occur

i) There exists $l \geq mq$ such that $Q_c^l(0) = \beta$.

ii) There exists $l > mq$ such that $Q_c^l(0) \in K'$.

iii) There exists a strictly increasing sequence $\{l_n\}_{n \geq 0}$ with $l_0 = mq - 1$ with $Q_c^{l_n}(0) \in X_1$ for all $n$.

Moreover in both cases i) and ii) any nest $\{Y_n\}_n$ such that $c \in Y_n$ for every $n$ is convergent to $c$.

In the last statement there is a unique such $Y_n = Y_n(c)$ except if $Q_c^k(c) = \alpha c$ for some $k$, in which case there are precisely $q$ such nests, which all are convergent to $\alpha c$ by the discussion above.

Note that $Y_0^{q-1} \setminus X_1$ is a non degenerate annulus contained in any of the annuli $Y_0^{q-1} \setminus X_1^k$, $0 < k < q$.

Theorem 6.10. Let $c \in L_{p/q}$ satisfy the hypotheses of Corollary 6.9 and the property therein. Then there exists a non degenerate annulus $A_{n_0} = Y_{n_0} \setminus Y_{n_0+1}$ between nested puzzle pieces of the Yoccoz puzzle for $Q_c$ with $Q_c^{n_0}(Y_{n_0}) = Y_0^{q-1}$, $Q_c^{n_0}(Y_{n_0+1}) = X_1^k$ for some $0 < k < q$. And there exists a nested sequence of annuli $A_{n_i}^c = Y_{n_i}^c \setminus Y_{n_i+1}^c$, $i > 0$ with $n_0 < n_1 \nearrow \infty$ surrounding the critical value $c$ such that:

- the map $Q_c^{n_i-n_0} : A_{n_i}^c \to A_{n_0}^c$ is a covering map of degree $2^{d_i}$, $d_i \geq 0$ for $i \geq 1$;
- in particular also all the annuli $A_{n_i}^c$, $i > 0$ are non degenerate;
- either the sum $\sum_{i \geq 1} \mod(A_{n_i}^c) = \mod(A_{n_0}^c) \sum_{i \geq 1} \frac{1}{2^{d_i}}$ is infinite and
  the intersection $\bigcap_{n \geq 0} Y_n^c$ reduces to a point,
- or there exists $k > 0$ such that for all $n$ large enough the map $Q_c^k : Y_{n+k}^c \to Y_n^c$ is quadratic-like with connected filled-in Julia set.

Note that the annulus $A_{n_0}$ does not necessarily surround the critical value.

Theorem 6.10 follows from a classical tableaux argument, see e.g., [Mi3]. The degree $2^{d_0}$ of the restriction $Q_c^{n_0} : Y_{n_0} \to Y_0^{q-1}$ may be larger than the degree of the restriction $Q_c^{n_0} : Y_{n_0+1} \to X_1^k$. This happens precisely when $Y_0^{q-1} \setminus X_1^k$ contains one or more critical
values for the restriction of \( Q_c^n \) to \( A_{n_0} \). However as the long composition of \( Q_c \) with itself has degree either 1 or 2 in each step, it easily follows that

\[
\text{mod}(A_{n_0}) \geq \text{mod}(Y_0^{q-1} \setminus X_1^k)/2^{d_0} \geq \text{mod}(Y_0^{q-1} \setminus X_1)/2^{d_0}.
\]

Note that Theorem 6.10 can also be proved using the following results: [R2, Lemma 1.22] for the non-recurrent case and [R2, Lemma 1.25 case 1)] for the recurrent case.

### 6.2 Yoccoz-type estimates for the parabolic maps \( g_B \)

In this section we port the results above for the quadratic polynomials \( Q_c \) to the parabolic quadratic rational maps \( g = g_B = z + 1/z + B, \Re(B) > 0 \) with \( \beta_B = \infty \) a parabolic fixed point of multiplier 1 and a unique finite fixed point \( \alpha_B = -1/B \) of multiplier \( A(B) = 1 - B^2 \). We let \( \tau(z) = 1/z \) denote the covering involution for \( g \).

As for quadratic polynomials we denote by \( \beta' = \beta_B' = 0 \) the finite preimage of \( \beta \). Similarly we denote by \( \alpha' = \alpha_B' = -B \) the non fixed pre-image of \( \alpha_B \). The critical point 1 for \( g_B \) is first attracted in the sense that the extended attracting Fatou coordinate \( \phi_B \) for \( g_B \) maps 1 to 0 and the domain \( \Omega_B \) with 1 \( \in \partial \Omega_B \) univalently onto \( \mathbb{H}^+ \). The other or second critical point \( -1 \) for \( g_B \) and its critical value \( v_B = g_B(-1) = -2 + B \) play the same role for \( g_B \) as the critical point 0 and its critical value \( c \) plays for \( Q_c \). In particular the second critical point and value belong to the filled-in Julia set \( K_B \), if and only if \( K_B \) is connected and is otherwise in \( \Lambda_g \setminus g_B(\Omega_B) \).

In the rest of this subsection we shall fix an irreducible rational \( p/q \) and consider \( B \in \mathcal{W}_{M_1}(p/q) \). We setup notation for special puzzle pieces for \( g_B \) corresponding to the notation for special puzzle pieces for \( Q_c \).

Recall that \( V_n^P \) is the interior of the union of closures of level \( n \) universal parabolic puzzle pieces for \( n \geq 0 \). And that

\[
\mathcal{W}_n^{M_1}(p/q) := \mathcal{L}_{p/q}^{M_1} \cup \{ B \in \mathcal{W}_n^{M_1}(p/q) \mid h_B(v_B) \in V_n^P \}
\]

for \( n \geq 0 \).

Let \( B \in \mathcal{W}_0^{M_1}(p/q) \). We shall use the abbreviations \( P_n^0 := P_n(-1) \in \mathcal{P}_n \) for the critical puzzle piece of depth \( n \) and \( P_n^B := P_n(v_B) = g_B(P_n^0) \in \mathcal{P}_n \) for the critical value puzzle piece of depth \( n \), whenever there is such a puzzle piece. We shall use the symbol \( \bar{P} = \tau(P) \) for the dynamical twin of the puzzle piece \( P \), i.e. \( g_B(\bar{P}) = g_B(P) \).

**Lemma 6.11.** Let \( B \in \mathcal{W}_{q-1}^{M_1}(p/q) \). Then the map \( g_B^q \) has a quadratic-like restriction \( f = f_B = g_B^q : U \to U' \) with \( \bar{P}^0 \cap V_q^B \subset U \subset V_q^B \).
Moreover the filled-in Julia set $K'_B$ of $f$ is contained in $\{\alpha, \alpha'\} \cup (P^0_0 \cap V^B)$ and $K'_B$ is connected if and only if $f^n(-1) = g^m_B(-1) \in \{\alpha, \alpha'\} \cup (P^0_0 \cap V^B)$ for all $n$.

Proof. See the proof of the similar Lemma 6.1 above for the corresponding polynomials $Q_c$. $\square$

In the following fix $B \in \mathcal{L}^{M_1}_{p/q}$. Then just as for quadratic polynomials precisely one of the following two cases occur

DB1. For all $n \in \mathbb{N}: g^{mq}_B(-1) \in \overline{P^0_0}$.

DB2. There exists $m \geq 1$ minimal such that $g^{mq}_B(-1) \notin \overline{P^0_0}$.

In the first case DB1. it follows from Lemma 6.11 above that $g_B$ is $q$-renormalizable. That is, there exists a quadratic like restriction $f_B = g^q_B: U \rightarrow U'$ with connected filled-in Julia set $K'_B \subset P^0_0 \cup \{\alpha, \alpha'\}$.

As for polynomials we shall henceforth focus on the second case DB2.

We continue to set up notation analogous to the polynomial case. For $0 \leq k < q$ let $P^k_0$ denote the level 0 puzzle piece contained in $g^k_B(P^0_0)$. Then $g^k_B(-1) \in P^k_0$ for any $B \in \mathcal{W}^{M_1}_{q-1}(p/q)$. Each $P^k_0$ is adjacent to $\alpha$ and $P^{q-1}_0 = P_0(\beta')$. The corresponding twins $\tilde{P}^k_0$ are adjacent to $\alpha'$ and $\tilde{P}^{q-1}_0 = P_0(\beta)$. Denote by $S_0$ the interior of $\bigcup_{k=1}^{q-1} P^k_0$ and by $\tilde{S}_0$ its twin, that is $S_0$ plays the role of $X_0$. Then the common image $g_B(S_0) = g_B(\tilde{S}_0)$ covers the level 0 puzzle except for $\overline{P^B_0}$ and the subset $D^B_0$ between the shortcut $\gamma^B_0 = \partial P_0(\beta) \cap D^B_0$ and $g_B(\gamma^B_0) \subset D^B_0$.

Note that the condition $g^{mq}_B(-1) \notin \overline{P^0_0}$ in DB2 is equivalent to $g^{mq}_B(-1) \in \tilde{S}_0$.

When studying parameter space we shall as for polynomials also be interested in the following extension of condition DB2. on $B \in \mathcal{W}^{M_1}_{0}(p/q)$.

DB2'. There exist $m \geq 1$ minimal with $g^{mq}_B(-1) \in \tilde{S}_0$.

Parameters $B$ satisfying DB2. (see Definition 3.25 for the definition) belongs to a dyadic sub-wake of the satellite-copy $M^{M_1}_{p/q}$.

Proposition 6.12. A parameter $B \in \mathcal{W}^{M_1}_{0}(p/q)$ satisfies DB2. if and only if

$$B \in \mathcal{W}^{M_1}_{mq-1}(p/q, r, m) := \mathcal{W}^{M_1}_{mq-1}(p/q) \cap \mathcal{W}^{M_1}_{0}(p/q, r, m)$$

where $m \geq 1$ is from DB2. and $r$ is odd with $0 < r < 2^m$. 
Proof. Let \( B \in \mathcal{W}^{M_1}_0(p/q) \) satisfy \( g_B^{mq}(-1) = g_B^{mq-1}(v_B) \in S_0 \) for some minimal \( m \geq 1 \). Then \( v_B \in V^{B}_{mq-1} \) and thus also \( B \in \mathcal{W}^{M_1}_0 \). Moreover \( g_B^{m}(v_B) \notin \mathcal{P}_0^B \), hence \( g_B^{(m-1)q}(v_B) \) belongs to the 1/2 dyadic wake \( \mathcal{W}_B(p/q,1,1) \) and thus \( B \in \mathcal{W}_B(p/q,r,m) \) for some odd \( r \) with \( 0 < r < 2^m \) by induction and minimality of \( m \). Hence also \( B \in \mathcal{W}^{M_1}_0(p/q,r,m) \).

Recall that for \( m \geq 1 \) and \( r \) odd \( 0 < r < 2^m \) the derooted dyadic decoration \( \mathcal{L}^{M_1}_s(p/q,r,m) \) is the set of parameters

\[
\mathcal{L}^{M_1}_s(p/q,r,m) = \mathcal{L}^{M_1}_p(p/q,r,m) \cap \mathcal{W}^{M_1}_0(p/q,r,m).
\]

Let \( \mathcal{L}^{M_1}_p(p/q,r,m) \) denote the limb with root, i.e. \( \mathcal{L}^{M_1}_s(p/q,r,m) \) union the root point of \( \mathcal{W}^{M_1}_0(p/q,r,m) \) (see also Definition 3.25 and trailing comments). This gives the following decomposition of the limb \( \mathcal{L}^{M_1}_p \), corresponding to the decomposition of limbs of the Mandelbrot set.

**Corollary 6.13.** The limb \( \mathcal{L}^{M_1}_p \) has a natural stratification as

\[
\mathcal{L}^{M_1}_p = \mathcal{M}_{p/q} \cup \bigcup_{\beta} \mathcal{L}^{M_1}_s(p/q,r,m).
\]

**Proof.** This follows immediately from the dichotomy, DH1, DH2 above. 

As an immediate corollary of Proposition 4.3 we obtain

**Proposition 6.14.** The boundaries of the puzzle pieces in both the \( \beta_B \)-nest \( N(\beta_B) = \{P_n(\beta_B)\}_{n \geq 0} \) and the \( \beta'_B \)-nest \( N(\beta'_B) = \{P_n(\beta'_B)\}_{n \geq 0} \) move holomorphically with \( B \in \mathcal{W}^{M_1}_0(p/q) \).

It was proven in [PR2] that the \( \beta \) and \( \beta' \)-nest are convergent to \( \beta \) and \( \beta' \) respectively.

Similarly to the polynomial case, for any \( B \in \mathcal{W}^{M_1}_0(p/q) \) the level 0 twin puzzle pieces \( P_0(\beta) \) and \( P_0(\beta') \) each contain \( q \) level 1 puzzle pieces, which are mapped homeomorphically onto \( P^k_0 = \tilde{P}^k_0 \) and \( P_0^k \), \( 0 < k < q \) except for the slight variation that due to the short cuts \( \tilde{P}_1^{q-1} \subset P^{q-1}_0 \) and similarly for the twins, but \( g_B(P^{q-1}_1) = g_B(\tilde{P}_1^{q-1}) \cap D^B_0 \) differs in an inessential way from \( \tilde{P}_1^{q-1} \cap D^B_0 \). And every other level 0-puzzle piece contains a unique level 1 puzzle piece \( P^k_1 \subset P^k_0 \) respectively \( \tilde{P}_1^k \subset \tilde{P}_1^k \) which is mapped properly onto \( P^{k+1}_0 \).

By construction there are \( q \) puzzle pieces of level \( n \) adjacent to \( \alpha_B \) for every \( n \). And thus \( q \) sequences of nested puzzle pieces \( N^{\alpha,k} := \{P_n^{\alpha,k}\}_{n \geq 0}, P_0^{\alpha,k} = P_0^{k} \) adjacent to \( \alpha_B \). Moreover \( g_B \) maps \( P_{n+1}^{\alpha,k} \) properly onto \( P_n^{\alpha,(k+1)\mod q} \) for every \( n \) and \( k \) and the degree is 1 unless \( k = 0 \) so that \( P_n^{\alpha,k} = P_0^{n+1} \). It follows immediately that either all \( q \) nests are convergent to \( \alpha \) or none is convergent.
Lemma 6.15. Let $m \geq 1$ and suppose that $B \in \mathcal{W}_{mq-1}^{M_1}(p/q)$ satisfies $\text{DB}_2'$ with this $m$. Let $f : U \to U'$ be a quadratic like map as in Lemma 6.11 with $k = mq$, then

1. the filled Julia set $K'_B \subset \overline{P_{mq}^α} \cup \overline{P_{mq}^α} \subset U$,
2. the restriction $f_B : \mathcal{P}_{mq}^0 \to \mathcal{P}_{mq}^0$ is a holomorphic diffeomorphism,
3. $\text{diam}(\mathcal{P}_{(n+m)q}^0) \to 0$ as $n \to \infty$ uniformly over all connected components $P_{(n+m)q}$ of $f_B^{-n}(\overline{P_{mq}^α} \cup \overline{P_{mq}^α})$.
4. If $P_{(n+1+m)q} \subset P_{(n+m)q}$ are nested puzzle pieces with $f^n_B(P_{(n+m)q}), f^{n+1}_B(P_{(n+1+m)q}) \in \{\mathcal{P}_{mq}^0, \overline{\mathcal{P}_{mq}^α}\}$ then $\partial P_{(n+1+m)q} \cap \partial P_{(n+m)q} \subset f_B^{-1}(\partial B)$.
5. In particular for any $z \in K'_B$ either $z$ is not prefixed to $α_B$ under $g_B^l$ and the nest $\{P_n(z)\}_n$ is convergent to $z$ or $g_B^{lq}(z) = α_B$ for some minimal $l$ and there are $q$ nests $\{P_{n,k}^l\}_n, 0 \leq k < q$ convergent to $z$ and with $g_B^l(P_{n+1,k}^l) = P_{n,k}^l$ for each $n$ and $k$.

In order to create a fundamental system of nested neighbourhoods of $α_B$ we denote by $P_n^α$ the interior of $\bigcup_{k=0}^{q-1} \overline{P_n^k}$, so that $P_n^α$ is an open neighbourhood of $α_B$ for all $n$. However no $P_n^α$ is a puzzle piece. Let $r/2^m$, $r$ odd and $0 < r < 2^m$ be a dyadic rational and let $B \in \mathcal{W}_{mq-1}^{M_1}(p/q, r, m)$. Then for all $n \geq (m+1)q$ maps $P_n^α$ biholomorphically onto $P_{n-1}^α$ and $P_n^α \subset \mathcal{P}_{mq}^0$. Moreover the the union $\cup \partial P_n^α$ of boundaries of $\partial P_n^α$ move holomorphically over $\mathcal{W}_{mq-1}^{M_1}(p/q, r, m)$ and continuously over the closure.

Recall that $f$ similarly maps $\overline{P_{mq}^α}$ diffeomorphically onto $\overline{P_{mq}^0_{(q-1)}}$.

Proof. The proof is completely analogous to the proof of Lemma 6.5 above for polynomials and is left to the reader. \qed

By Proposition 6.12 the hypothesis $\text{DB}_2'$ of Lemma 6.15 is equivalent to $B \in \mathcal{W}_{mq-1}^{M_1}(p/q, r, m)$ for some odd $r$ with $0 < r < 2^m$. Fix such $r$ and define for $c \in \mathcal{W}_{mq-1}^{M_1}(p/q, r, m)$ the set

$$\Gamma'_B := K'_B \cup \bigcup_{n \geq 0} f_B^{-n}(\partial P_{mq}^0 \cup \partial \tilde{P}_{mq}^0) = \bigcup_{n \geq 0} f_B^{-n}(\partial P_{mq}^0 \cup \partial \tilde{P}_{mq}^0).$$

Proposition 6.16. Let $m \geq 1$, let $r$ be odd with $0 < r < 2^m$ and fix $B_0 \in \mathcal{W}_{mq-1}^{M_1}(p/q, r, m)$. Then there exists a holomorphic motion

$$\psi_{r,m}^{B_0} : \mathcal{W}_{mq-1}^{M_1}(p/q, r, m) \times \Gamma'_B \to \mathbb{C}$$

with base point $B_0$ such that $\psi_{r,m}^{B_0}(B, \Gamma'_B) = \Gamma'_B$ and $f_B \circ \psi_{r,m}^{B_0}(B, z) = \psi_{r,m}^{B_0}(B, f_B(z))$ for every $B \in \mathcal{W}_{mq-1}^{M_1}(p/q, r, m)$ and every $z \in \Gamma'_B$. 

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Proof. The proof is completely analogous to the proof of Proposition 6.6 above for polynomials and is left to the reader.

Note that $z \in K_B$ with $g^l_B(z) = \alpha_B$ will be adjacent to $2q$ nests if $-1$ belongs to the orbit of $z$.

For $0 < k < q$ let $S^k_1 \subset P_0(\beta')$ and $\tilde{S}^k_1 \subset \tilde{P}_0(\beta')$ corresponding to $X^k_1$ and $\tilde{X}^k_1$. As with $X_1$ in the polynomial case let $S^1_1$ denote the interior of $\bigcup_k S^k_1$.

Proposition 6.17. Let $B \in \mathcal{LM}_p/q$ and assume $g^{(m-1)q}(-1) \in \overline{P_0}$ and $g^mq(-1) \notin \overline{P_0}$ for some $m \geq 1$. Then for any $z \in \overline{K_B}$ the orbit falls in precisely one of the following three categories:

i) There exists $l \geq 0$ such that $g^l_B(z) = \beta$.

ii) There exists $l \geq 0$ such that $g^l_B(z) \in K'$.

iii) There exists a strictly increasing sequence $\{l_n\}_{n \geq 0}$ with $g^{l_n}_B(z) \in S^1_1$ for all $n$.

Moreover in both cases i) and ii) any nest $\{P_n\}_n$ such that $z \in \overline{P}_n$ for every $n$ is convergent to $z$.

In the last statement there is a unique such $P_n = P_n(z)$ except if $g^k_B(z) = \alpha_B$ for some $k$, in which case there are precisely $q$ such nests. By the above these $q$ nests are either all convergent or all divergent.

Proof. Again the proof is analogous to the proof of Proposition 6.7 and is left to the reader.

Proposition 6.18. Let $B \in \mathcal{LM}_p/q$ satisfy DE2. In the first two cases i) and ii) of Proposition 6.17 any nest $\{P_n\}_n$ such that $z \in \overline{P}_n$ for every $n$ is convergent to $z$. Moreover if $z = v_B$ then there exists $N \geq 1$ such that the restriction $g^{N-1}_B : P_N \to P_{N-1}(\beta')$ is univalent in i) and $g^l_B : P_N \to g^l_B(P_N)$ is univalent in ii).

Recall that there is a unique $P_n = P_n(z)$ with $z \in \overline{P}_n$ except if $g^k_B(z) = \alpha_B$ for some $k$, in which case there are precisely $q$ such nests if the orbit of $z$ avoids the critical point $-1$ and $2q$ such nests if not.
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Proof. The proof is mostly analogous to the proof of Proposition 6.8 we point out the difference and leave the rest to the reader. The difference is due to the fact that the relation between $P_{n+1}(\beta')$ and $P_n(\beta)$ is only partly dynamical because of the short-cuts on the boundary. However the only way to $\beta$ from $v_B$ is via $\beta'$ and all pre-images under iteration of any of the puzzle pieces $P_n(\beta')$ are dynamical, hence the $l-1$ in place of $l$ in the formula above in the case [3]. Other than this the proof is completely analogous to the proof of Proposition 6.8.

As in the polynomial case the above Proposition immediately gives the following Corollary for parameterspace.

Corollary 6.19. Let $B$ belong to a dyadic decoration $L_{M_1}(p/q, r, m)$ for some $r$ is odd and $0 < r < 2^m$. Then precisely one of the following three cases occur

i) There exists $l \geq mq$ such that $g^l_B(-1) = \beta_B$.

ii) There exists $l > mq$ such that $g^l_B(-1) \in K'_B$.

iii) There exists a strictly increasing sequence $\{l_n\}_{n \geq 0}$ with $l_0 = mq - 1$ with $g^{l_n}_B(-1) \in S_1$ for all $n$.

Moreover in both cases [i] and [ii] any nest $\{P_n\}$ such that the critical value $v_B \in \overline{P}_n$ for every $n$ is convergent to $v_B$.

In the last statement there is a unique such $P_n = P_n(v_B)$ except if $g^k_B(v_B) = \alpha_B$ for some $k$, in which case there are precisely $q$ such nests all of which are convergent by the discussion above.

Note that $P^{q-1}_0 \setminus S_1$ is a non degenerate annulus contained in any of the annuli $P^{q-1}_0 \setminus S^k_1$, $0 < k < q$.

For the rest of this paragraph we fix $B \in L_{M_1}^{p/q}$ and we let $c \in L_{p/q}$ be a parameter with $\infty = \infty$ as provided by Lemma 5.7.

Proposition 6.20. Let $B \in L_{M_1}^{p/q}$ and let $c \in L_{p/q}$ be a parameter with $\infty = \infty$. If $B \in M_{M_1}^{p/q}$ then $c \in M_{p/q}$. And if $B \in L_{M_1}(p/q, r, m)$ for some $r$ is odd and $0 < r < 2^m$, then $c \in L(p/q, r, m)$ and moreover

i) $g^l_B(-1) = \beta_B$ if and only if $Q^l_c(0) = \beta_c$.

ii) $g^l_B(-1) \in K'_B$ if and only if $Q^l_c(0) \in K'_c$.

iii) $g^l_B(-1) \in S_1$ if and only if $Q^l_c(0) \in X_1$. 

Proof. Let \( \hat{g}_B : \mathcal{P} \longrightarrow \mathcal{P} \) denote the map of puzzle pieces induced by \( g_B \) (defined in Definition 5.6). And let \( \chi : \mathcal{Y} \longrightarrow \mathcal{P} \) denote the dynamical correspondence between puzzles of Proposition 5.7. Then nests are mapped to nests and in particular the critical value nest \( \{ Y_{cn} \}_{n} \) is mapped to the critical value nest \( \{ P_{cn}^B \}_{n} \). From this it follows that \( c \in L(p/q, r, m) \) and \( iii) \) follows. Combining further with the descriptions of \( K'_c \) in Lemma 6.1 and \( K'_B \) in Lemma 6.11 yields \( ii) \). Finally the \( \beta \)-nest \( \{ Y_n(\beta_c) \}_{n} \) is easily seen to always be convergent. And the \( \beta \)-nest \( \{ P_n(\beta_B) \}_{n} \) was proven to always be convergent in [PR2, Prop 5.10]. So that also \( i) \) also follows from \( \chi \) conjugating puzzle dynamics.

We are now ready to state and prove a parabolic analog of Theorem 6.10:

\begin{theorem} \label{thm:parabolic_analog}
Let \( B \in \mathcal{L}^{M_i}_{p/q} \) satisfy the hypotheses of Corollary 6.19 and its property \( iii) \). Then there exists a non degenerate annulus \( A_{n_0}^B = P_{n_0} \setminus P_{n_0+1}^B \) between nested puzzle pieces of the parabolic Yoccoz puzzle for \( g_B \) with \( g_{n_0}^B(P_{n_0}) = P_0^{n_1} \), \( g_{n_0}^B(P_{n_0+1}) = S_1^k \) for some \( 0 < k < q \).

And there exists a nested sequence of annuli \( A_{n_i}^B = P_{n_i} \setminus P_{n_i+1}^B \), \( i > 0 \) with \( n_0 < n_1 \nearrow \infty \) surrounding the critical value \( v_B \) such that:

\begin{itemize}
  \item the map \( g_{n_i-n_0}^B : A_{n_i}^B \rightarrow A_{n_0}^B \) is a covering map of degree \( 2^{d_i} \), \( d_i \geq 0 \) for \( i \geq 1 \);
  \item in particular also all the annuli \( A_{n_i}^B \), \( i > 0 \) are non degenerate;
  \item either the sum \( \sum_{i \geq 1} \text{mod}(A_{n_i}^B) = \text{mod}(A_{n_0}^B) \sum_{i \geq 1} \frac{1}{2^{d_i}} \) is infinite, \( P_n^B \) is defined for all \( n \) and \( \text{End}(\{ P_n^B \}_{n}) = v_B \),
  \item or there exists \( k > 0 \) such that for all \( n \) large enough the map \( g_k^n : P_{n+k}^B \rightarrow P_n^B \) is quadratic-like with connected filled-in Julia set.
\end{itemize}

Note that the above sums are finite only when \( \sim \) is renormalizable of period \( k > q \). As with Theorem \ref{thm:main_theorem} the annulus \( A_{n_0}^g \) will in general not surround the critical value \( v_B \).

Proof. There are two immediate proof strategies. Either redo the usual puzzle argument or as we shall do here combine Proposition 5.7 with Theorem 6.10. And let \( c \in L(p/q) \) be a parameter with \( \sim^B = \sim^g \). Let \( \{ Y_{n_i} \} \) and \( \{ Y_{n_i+1} \} \) be the sequences of Yoccoz puzzle pieces given by Theorem \ref{thm:main_theorem}. And for each \( i \geq 0 \) let \( P_n = \chi(Y_n) \), \( P_{n+1} = \chi(Y_{n+1}) \).
Then by Proposition 5.7, the desired properties for the non-degenerate annuli $A^B_n$ follows from the similar properties of the annuli $A^c_n$ in Theorem 6.10. Moreover, the covering degree $d_i$ are the same so that
\[
\sum_{i \geq 1} \mod(A^B_n) = \mod(A^B_0) \sum_{i \geq 1} \frac{1}{2^{d_i}} = \frac{\mod(A^B_n)}{\mod(A^c_n)} \sum_{i \geq 1} \mod(A^c_n).
\]

Corollary 6.22. Let $B \in L_{p/q}^M$ satisfy the hypotheses of Corollary 6.19 and let $c \in L_{p/q}$ be a parameter with $\sim_{\infty} B = \sim_{\infty} c$. Then $c$ satisfies the hypotheses of Corollary 6.9 and for any nest $\{Y_n\}_n$ with $c \in Y_n$ for all $n$, $v_B \in P_n$ for all $n$ where $P_n = \chi(Y_n)$ and

- End($\{Y_n\}_n$) = \{c\} if and only if End($\{P_n\}_n$) = \{v_B\}
- End($\{Y_n\}_n$) is the filled Julia set of a quadratic-like restriction of $Q^c_k$

if and only if
- End($\{P_n\}_n$) is the filled Julia set of a quadratic like restriction of $g^B_k$.

7 Parabolic Parameter-Puzzles

7.1 Parabolic Parameter Puzzle

In the $p/q$-wake $W_{p/q}^M$, we define the parameter puzzle pieces using three different points of view. As a first definition, we take the universal parabolic $p/q$ graph $G_P^n$ and the parametrization $Y$ (see Definition 7.1) to define a parameter parabolic graph. This way, the complementary regions, the parameter puzzle pieces, of level $n$ parametrize a holomorphic motion of the level $n + 1$ dynamical graph $G_B^n$. In particular, this point of view allows to compare pieces and annuli in the dynamical plane and in the parameter plane. We characterize then the parameter puzzle pieces as the set of parameters such that the critical value stays in the holomorphic motion of the same puzzle piece. The third characterization is in terms of laminations. A parameter puzzle piece of level $n$ corresponds to the set of parameters sharing up to level $n + 1$ the same lamination associated to a center.

Definition 7.1. For $n \geq 0$, the parameter parabolic graph is defined by
\[
G_{PP}^n := W_{p/q}^M \cap Y^{-1}(G_P^n).
\]

Note that $Y^{-1}(G_P^n) \subset C \setminus M_1$. For this reason, we add the accumulation of $Y^{-1}(G_P^n)$ which consists of landing points of rays coming from the graph. Those rays have angles
which are pre-images of \( \theta \) and \( \theta' \), therefore they land at Misiurewicz parameters (see Lemma 3.24). Any \( B \in \mathcal{GPP}_n \cap L_{\frac{p}{q}}^M \) is a Misiurewicz parameter. It is common landing point of exactly \( q \) parabolic parameter rays in \( \mathcal{GPP}_n \) and the corresponding parabolic dynamical rays co-land at \( v_B \) (see Lemma 3.24). The graph \( \mathcal{GPP}_n \) consists in two parts: the sides which are parts of rays with landing points and the top which are short cuts.

**Definition 7.2.** Denote by \( \mathcal{PP}_n \) the set of parameter parabolic puzzle pieces of level \( n \), they are the connected components of \( \overline{W}_M(p/q) \setminus \mathcal{GPP}_n \) intersecting \( M_1 \). We write \( \mathcal{PP}_n(B) \) for the one containing the parameter \( B \).

Puzzle pieces are either disjoint or nested, in which case they have different levels.

**Remark 7.3.** There is a unique parameter puzzle piece of level \( 0 \) that we denote by \( \mathcal{PP}_0 \). It is the short-cutted version of the wake: \( W_{M_0}(p/q) \).

**Lemma 7.4.** Let \( B^* \in \mathcal{PP}_0 \). The graph \( \mathcal{GP}^{B^*}(1) \) admits a holomorphic motion

\[
\Psi_0 = \Psi_0^{B^*} : \mathcal{PP}_0(B^*) \times \mathcal{GP}^{B^*}(1) \to \mathcal{C} \quad \text{such that} \quad \Psi_0(B, \mathcal{GP}^{B^*}(1)) = \mathcal{GP}^{B}(1).
\]

**Proof.** From Proposition 4.3 the graph \( \mathcal{GP}_0^{B^*} \) admits a holomorphic motion parametrized by \( W_0^{M_1}(p/q) \).

The graph \( \mathcal{GP}^{B}(1) \) is defined by \( \mathcal{GP}^{B}(1) = \mathcal{GP}_0^{B} \cup \mathcal{GP}_1^{B} \) where

\[
\mathcal{GP}_1^{B} := g^{-1}_B(\mathcal{GP}_0^{B} \setminus z_0^B) \cup \{ z_1^B \cup \gamma_1^B \}.
\]

By Corollary 3.22 we have that \( B \in W_0^{M_1}(p/q) \) if and only if \( v_B \in W_B(p/q) \). Hence, \( v_B \notin \mathcal{GP}_0^{B} \) so that we can lift \( \Psi_0(B, \cdot) \) on \( \mathcal{GP}_0^{B} \setminus z_0^B \) to get a holomorphic motion of \( g^{-1}_B(\mathcal{GP}_0^{B^*} \setminus z_0^B) \) :

\[
\Psi_0(B, z) = g^{-1}_B(\Psi_0(B, g_B^*(z))).
\]

The holomorphic motion of the short cuts \( z_1^B \cup \gamma_1^B \), follows immediately from Proposition 4.3.

**Lemma 7.5.** Fix \( n \geq 0 \) and any \( B^* \) in a level \( n \) parameter puzzle piece \( \mathcal{PP}_n(B^*) \). There exists a holomorphic motion

\[
\Psi_n = \Psi_n^{B^*} : \mathcal{PP}_n(B^*) \times \mathcal{GP}_{n+1}^{B^*} \to \mathcal{C}
\]

such that for any \( B \in \mathcal{PP}_n(B^*) \), \( \Psi_n(B, \mathcal{GP}_{n+1}^{B^*}) = \mathcal{GP}_{n+1}^{B} \).
Moreover $\Psi_n$ extends to a holomorphic motion of the union $\mathcal{GP}_{B^*}(n+1)$ of all graphs up to and including $n+1$:

$$\tilde{\Psi}_n = \tilde{\Psi}^{B^*}_n : PP_n(B^*) \times \mathcal{GP}_{B^*}(n+1) \to \tilde{\mathbb{C}}$$

by setting $\tilde{\Psi}_n = \Psi_k$ on $PP_n(B^*) \times \mathcal{GP}^B_{k+1}$ for $-1 \leq k \leq n$.

Proof. The proof goes by induction, Lemma 7.4 provides a proof for $n = 0$. In the induction we prove that for $B \in PP_n(B^*)$ the graph is $\mathcal{GP}^B_{n+1} = h_B^{-1}(\mathcal{GP}_{n+1})$.

We define $\Psi_{n+1}$ by lifting of $\Psi_n$. Indeed, for $B \in PP_n(B^*)$, the critical value $v_B$ never crosses $\mathcal{GP}^B_{n+1}$. Otherwise, if $v_B \in \mathcal{GP}^B_{n+1}$, $h_B(v_B) \in \mathcal{GP}_{n+1}$ and $\Upsilon(B) = h_B(v_B)$ would be on $\mathcal{GP}_{n+1}$. Therefore we can define $g_B^{-1}(\mathcal{GP}^B_{n} \setminus \gamma_n^B)$ it coincides with $h_B^{-1}(\mathcal{GP}_{n+1} \setminus \gamma_n^B)$ by induction. Then

$$\mathcal{GP}^B_{n+1} = g_B^{-1}(\mathcal{GP}^B_{n} \setminus \gamma_n^B) \cup \{\hat{\gamma}^B_{n+1} \cup \gamma^B_{n+1}\}.$$ 

By hypothesis of induction, for $B \in PP_n(B^*)$ the graph $\mathcal{GP}^B_{n+1}$ equals $h_B^{-1}(\mathcal{GP}_{n+1})$ since $\hat{\gamma}^B_n = h_B^{-1}(\gamma_n)$ and $\gamma^B_n = h_B^{-1}(\gamma_n)$. The holomorphic motion follows from these considerations.

Let $PP_n$ be a parameter puzzle piece, $B^* \in PP_n$ and recall that $P_n^{B^*}$ denotes the puzzle piece of level $n$ containing the critical value. We want to compare the situation in the parameter plane around $B^*$ to the situation around the critical value $v_{B^*}$ in the dynamical plane of $g_{B^*}$: compare the puzzle pieces and the annuli. Following the graph through the holomorphic motion, we have seen that the puzzle pieces up to level $n$ are homeomorphic so the situation is stable. Nevertheless, the critical value might cross the graph. Therefore the notion of puzzle piece containing the critical value is not continuous. For this reason, we give the name $\hat{P}^B_n$ to the preferred puzzle piece, which is the holomorphic motion of this piece $P_n^{B^*}$. More precisely,

**Lemma 7.6.** For $i \leq n+1$ and $B \in PP_n(B^*)$, there is a unique parabolic puzzle piece $\hat{P}^B_i$ bounded by the holomorphic motion $\Psi_n(B, \partial P_i^{B^*})$ of the critical value puzzle piece $P_i^{B^*}$.

Proof. For $i \leq n+1$ let $C^{B^*}_i$ be the boundary of the puzzle piece $P^{B^*}_i$ containing the critical value for $g_{B^*}$. It is a Jordan curve separating the critical value from $\mathcal{GP}^{B^*}_i \setminus C^{B^*}_i$. Following $C^{B^*}_i$ in $PP_n(B^*)$ through the holomorphic motion of the graph $\mathcal{GP}^{B^*}_i$ defines a Jordan curve $C_i \subset \mathcal{GP}^B_i$ with $\mathcal{GP}^B_i \setminus C_i$ in a unique complementary component. Thus, we can define the connected component of the complement of $C_i$ which is disjoint from $\mathcal{GP}^B_i$, it is our preferred puzzle piece denoted by $P_i^B$.

**Lemma 7.7.** The puzzle piece $PP_{n+1}(B^*) \subset PP_n(B^*)$ is the set of parameters $B$ in $PP_n(B^*)$ such that the preferred puzzle piece $\hat{P}^B_{n+1} = P^B_{n+1}$. In particular $\hat{P}^B_i = P^B_i$ for all $0 \leq i \leq n$. 
Proof. For parameters \( B \in PP_{n+1}(B^*) \), the critical value is clearly in \( P_{n+1}^B \) since it never crosses the boundary of \( P_{n+1}^B \). Then, being on the boundary of \( PP_{n+1}(B^*) \) and using the coordinates in the Blaschke model, one see that locally, if the parameter crosses the boundary of \( PP_{n+1}(B^*) \) transversely, then the critical value follows the corresponding path in the dynamical plane. Hence it leaves the preferred puzzle piece \( \hat{P}_{n+1}^B \) and so has either to go into the puzzle piece adjacent to \( \hat{P}_{n+1}^B \) obtained by the holomorphic motion of the graph or to leave the level \( n+1 \) puzzle.

Corollary 7.8. If \( \widehat{P}_{n+1}^B \subset P_n^B \) then also \( \overline{PP}_{n+1}(B^*) \subset PP_n(B^*) \).

Proof. For a parameter \( B \) in \( PP_n(B^*) \), the critical value belongs to \( \hat{P}_{n+1}^B \subset \hat{P}_n^B \) and through the holomorphic motion we know that \( \hat{P}_{n+1}^B \subset \hat{P}_n^B \), so that if the critical value \( v_B \) belongs to the annulus \( P_n^B \setminus \hat{P}_{n+1}^B \), then parameter \( B \in PP_n B^* \setminus PP_{n+1}(B^*) \).

For \( B^* \in M_1 \), and \( p/q \) such that \( B^* \in \mathcal{W}^{M_1}(p/q) \), denote by \( \sim_{B^*}^n \) the lamination associated with the filled-in Julia set \( K_{B^*} \) (see Section 5).

Definition 7.9. Define \( PP_n(\sim_{\infty}^B) \) to be the set of parameters \( B \) in \( \mathcal{W}^{M_1}(p/q) \) such that \( \sim_{n+1}^B = (\sim_{\infty}^B)_{|n+1} \).

Recall from Section 4.2.3 that \( V_n^P \) is the interior of the union of closures of level \( n \) universal parabolic puzzle pieces and the reduced wakes \( \mathcal{W}_n^{M_1}(p/q) \) are

\[
\mathcal{W}_n^{M_1}(p/q) := \mathcal{L}_n^{M_1} \cup \{ B \in \mathcal{W}_n^{M_1}(p/q) | h_B(v_B) \in V_n^P \},
\]

so that

Lemma 7.10. \( PP_n(\sim_{\infty}^B) \cap \mathcal{W}_n^{M_1}(p/q) = PP_n(B^*) \).

Proof. By definition \( B^* \in PP_n(\sim_{\infty}) \). Now in \( PP_n(B^*) \) we have a holomorphic motion of the parabolic rays in the graph \( \mathcal{G}_{n+1}^B \) that gives the graph \( \mathcal{G}_{n+1}^B \). Therefore, we keep the landing relations for the parabolic rays in this graph in all the parameter puzzle piece. Hence, \( PP_n(B^*) \subset PP_n(\sim_{\infty}) \) and thus \( PP_n(\sim_{\infty}) \cap \mathcal{W}_n^{M_1}(p/q) \supset PP_n(B^*) \). For a parameter \( B \) on the boundary of \( PP_n(B^*) \) the critical value \( v_B \) is on the graph \( \mathcal{G}_n^B \), so either the second critical point \( -1 \) is on a pair of rays of the graph \( \mathcal{G}_{n+1}^B \) so that \( \sim_{n+1}^B \neq \sim_{n+1}^B \) or the critical value has escaped the level \( n+1 \) puzzle \( \mathcal{P}_n^B \).

Proposition 7.11. For every \( n \) there is

1. a 1 : 1 correspondence between the parameter puzzle pieces of level \( n \) and the set of distinct fertile towers \( \sim_{n+1} \) of level \( n+1 \).
2. a 1 : 1 correspondence between the set of level $n$ terminal towers and the set of points $\mathcal{L}_{p/q}^* \cap (\mathcal{GPP}_n \setminus \mathcal{GPP}_{n+1})$.

**Proof.** The proof is by induction. For level $n = 0$ there is only 1 tower of level $n+1 = 1$, it is fertile, the graph $\mathcal{GPP}^B(1)$ moves holomorphically over $\mathcal{W}_0^{M_1}(p/q)$ and induces the unique level 1 tower $\sim_1$ and finally $\mathcal{GPP}_0$ does not intersect $\mathcal{L}_{p/q}^{M_1}$. Suppose the statement holds for $n \geq 0$, let $\sim_{n+1}$ be any fertile tower, let $PP_n = PP(\sim_{n+1})$ be the corresponding level $n$ parameter puzzle piece and let $B^*$ be a parameter therein, i.e. $\sim_{n+1} \sim B^*$. By Lemma 7.5 the graph $\mathcal{GPP}^{B^*}(n+1)$ moves holomorphically over $PP_n$ and hence by definition of puzzle pieces $\Upsilon$ defines a homeomorphism between the parameter sub-graph $\mathcal{GPP}(n+1) \cap PP_n$ and the dynamical sub-graph $\mathcal{GPP}^{B^*}(n+1) \cap PP_n$ and hence induces a 1 : 1 correspondence between the points of $\mathcal{GPP}_{n+1} \cap PP_n \cap \mathcal{L}_{p/q}^*$ and the points of $\mathcal{GPP}^{B^*}_{n+1} \cap PP_n \cap K_{B^*}$. The (dynamical) puzzle pieces are in 1 : 1 correspondence with the gaps of $\sim_{n+1}$ contained in the critical value gap $G'_{n+1}$ of $\sim_{n+1}$, i.e. the gap of $\sim_n$, which is the image of the critical gap of $\sim_{n+1}$. And the graph points are in 1 : 1 correspondence with the set of level $n+1$ classes contained in $G'_{n+1}$. By definition of towers each gap $G' \subset G'_{n+1}$ of $\sim_{n+1}$ defines the unique level $n+2$ fertile tower extension $\sim_{n+2}$ of $\sim_{n+1}$ with critical value gap $G'$ and their totality enumerates all level $n+2$ fertile tower extensions of $\sim_{n+1}$. Similarly each class $K' \subset G'_{n+1}$ defines the unique terminal tower extension of $\sim_{n+1}$ with critical value class $K'$ and their totality enumerates all level $n+2$ terminal tower extensions of $\sim_{n+1}$.

**Definition 7.12.** In view of 2 of the above Proposition we shall abuse notation and for every terminal tower $\sim$ write $PP(\sim)$ for the singleton consisting of the unique parameter $B^*$ such that $\sim^B = \sim$. For this parameter $g_{B^*}^{n+1}(v_{B^*}) = \alpha_{B^*}$, where $n$ is the level of the critical value class for $\sim$.

8 Parabolic Parameter Yoccoz Theorem, transfer to the parameter space

This section is devoted to proving that $M_1$ is locally connected at any Yoccoz parameter, for a definition of such parameters see the item 3 below.

Following Yoccoz approach to local connectivity of the Mandelbrot set we distinguish 3 different types of parameters $B \in M_1$:

a. Parameters $B \in M_1$ such that the finite fixed point $\alpha_B$ is not repelling.
b. Parameters $B \in M_1$ such that some iterate $g_B^k$ is renormalizeable around the second critical point $-1$ or equivalently around the second critical value $v_B$

c. Parameters $B \in M_1$ which is not in any of the two previous categories, also called Yoccoz parameters.

Local connectivity of $M_1$ at a parameter $B$ is of type $A$ is most conveniently described in terms of the parameter $A = 1 - B^2 \in \mathbb{D}$. As a fundamental system of connected neighbourhoods of a parameter $|A| = 1$ is we may take a sequence of open intervals $J_n \subset S^1$ shring down to $A$ and with endpoints of irrational arguments, together with semi-disks in $\Delta_n \subset \mathbb{D}$, say bounded by the hyperbolic geodesic connecting the end-point of $J_n$ and together with the Limbs $\mathcal{L}_{p/q}^{M_1}$ with root in $J_n$. According to Corollary 3.23 such sets form a fundamental system of connected neighbourhoods of $A$.

We shall not here prove local connectivity of $M_1$ at renormalizeable parameters. It is not even known to be true in full generality for the corresponding parameters in $M$. In fact our proof that $M_1$ is homeomorphic to $M$ works because it essentially does not rely on properties of renormalization copies beyond the second renormalization level.

In order to handle Yoccoz parameters we look to Section 6. First we will consider one limb at a time, so we fix an irreducible rational $p/q$ and consider the limb $\mathcal{L}_{p/q}^{M_1}$. Secondly we have the basic Dichotomy for such parameters:

DB1. For all $n \in \mathbb{N} : g_B^{mq}(-1) \notin \overline{F}_0$.

DB2. There exists $m \geq 1$ minimal such that $g_B^{mq}(-1) \notin \overline{F}_0$.

Where the first is equivalent to $g_B$ is $q$-renormalizeable also denote immediate satelite type. And the second is equivalent to $B \in \mathcal{L}^{M_1}(p/q, r, m)$ for some odd $r$ with $0 < r < 2^m$ by Proposition 6.12.

Thirdly by Corollary 6.19 the second condition $B \in \mathcal{L}^{M_1}(p/q, r, m)$ for some odd $r$ with $0 < r < 2^m$ splits into three disjoint subsets or types of parameters:

i) There exists $l \geq mq$ such that $g_B^l(-1) = \beta_B$.

ii) There exists $l > mq$ such that $g_B^l(-1) \in K'_B$.

iii) There exists a strictly increasing sequence $\{l_n\}_{n \geq 0}$ with $l_0 = mq - 1$ with $g_B^{l_n}(-1) \in S_1$ for all $n$.

All three cases will be handled by using holomorphic motions to define a homeomorphism from the boundaries of puzzle pieces surrounding $v_B$ in dynamical space to the
boundaries of puzzle pieces surrounding \(B\) in parameter space, where \(B\) is any Yoccoz parameter in \(L^{M_{1}}(p/q, rm)\). In the two first types there are a sequence of boundaries puzzle pieces nesting down to \(v_{B}\) which move holomorphically over a fixed domain where as in the third case the domain of holomorphic motion of puzzle piece boundaries shrink, when the level increases. Moreover as it appears in Theorem 6.21 the third case splits-up further into two sub-cases renormalizable and not renormalizable.

We shall apply variations of the following Proposition from the book of applications of holomorphic motions.

For \(B \in \mathcal{W}_{0}^{M_{1}}(p/q)\) and \(G^{B} \subset \mathcal{G}P^{B}\) a sub-graph consisting of the boundary of one or more puzzle pieces, not necessarily of the same level, denote by \(X_{G^{B}}\) the connected component of \(\hat{\mathbb{C}} \setminus G^{B}\) containing the first critical point 1. So that for any puzzle piece \(P\) and \(G = \partial P\) we have \(X_{G^{B}} = \hat{\mathbb{C}} \setminus \mathcal{P}\).

**Proposition 8.1.** Let \(B^{*} \in \mathcal{W}_{0}^{M_{1}}(p/q)\) and let \(G^{B^{*}} \subset \mathcal{G}P^{B^{*}}\) be a sub-graph consisting of the boundary of one or more puzzle pieces \(P_{n}\) with \(v_{B^{*}} \in \mathcal{P}_{n}\). Suppose there exists a topological disk \(U \subset \mathcal{W}_{0}^{M_{1}}(p/q)\) with \(B^{*} \in U\) and a holomorphic motion \(H : U \times G^{B^{*}} \to \hat{\mathbb{C}}\) with base point \(B^{*}\) and with \(h_{B}(H(B, z)) = h_{B^{*}}(z)\) for every \((B, z) \in U \times (G^{B^{*}} \setminus J_{B^{*}})\). Let \(G^{B} := H(B, G^{B^{*}})\) and suppose that the second critical value \(v_{B} \in X_{G^{B}}\) on \(U \setminus M\) for some connected compact set \(M\). Then there exists a graph \(G \subset M\) consisting of boundaries of parameter puzzle pieces \(PP_{n}\) with \(B^{*} \in \mathcal{PP}_{n}\) such that \(v_{B} \in G^{B}\) for all \(B \in G\) and the map

\[B \mapsto \zeta(B) := H^{−1}_{B}(v_{B}) : G \to G^{B^{*}}, \text{ where } H^{−1}_{B}(H(B, z)) = z\]

is the restriction of a quasi-conformal homeomorphism \(\zeta\), which is asymptotically conformal at \(B^{*}\). Moreover for each \(P_{n}\) with \(v_{B^{*}} \in \mathcal{P}_{n}\) and \(\partial P_{n} \subset G^{B^{*}}\) the pre-image \(\zeta^{−1}(\partial P_{n}) \subset G\) is the boundary of a level \(n\) parameter puzzle piece \(PP_{n}\) with \(B^{*} \in \mathcal{P}_{n}\).

Note that the condition \(h_{B}(H(B, z)) = h_{B^{*}}(z)\) for every \((B, z) \in U \times (G^{B^{*}} \setminus J_{B^{*}})\) means that \(H\) is a holomorphic motion of puzzle piece boundaries, so that \(G^{B} \subset \mathcal{G}P^{B}\) for every \(B\). Thus if \(v_{B^{*}} \in P_{n}\) for some level \(n\) and \(\partial P_{n} \subset G^{B^{*}}\), then the holomorphic motion \(H\) coincides with the holomorphic motion \(\Psi^{B^{*}}_{n−1}\) of Lemma 7.5 where ever both are defined and \(H(B, \partial P_{n})\) is the boundary of the preferred puzzle piece in the sense of Lemma 7.7. And \(B \in PP_{n}(B^{*})\) precisely when the preferred puzzle piece equals the critical value puzzle piece \(\hat{P}_{n}^{B}\).

**Proof.** By Slodkowskis Theorem there exists a holomorphic motion extension \(\hat{H} : U \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}\) of \(H\). Define a quasi-regular map \(\zeta : U \to \hat{\mathbb{C}}\), which is asymptotically conformal at \(B^{*}\) by \(\zeta(B) := \hat{H}_{B}^{−1}(v_{B})\) and let \(G = \zeta^{−1}(G^{B^{*}})\). Then by construction \(G \subset M\) and \(G\) consists of boundaries of parameter puzzle pieces and \(\zeta\) has a non-zero degree over \(G^{B^{*}}\). Since \(\Upsilon\) is univalent, the degree is 1 so that the restriction \(\zeta : G \to G^{B^{*}}\) is a homeomorphism.
Finally for each \( P_n \) with \( v_{B^*} \in \overline{P}_n \) and \( \partial P_n \subset G^{B^*} \) the pre-image \( \zeta^{-1}(\partial P_n) \subset G \) is the boundary of a level \( n \) parameter puzzle piece \( PP_n \) with \( B^* \in \overline{PP}_n \).

**Corollary 8.2.** Suppose for some parameter \( B^* \in \mathcal{L}_{p/q}^{M_1} \) and some nest \( N = \{P_n\}_{n \geq 0} \) that \( \{v_{B^*}\} = \text{End}(N) \) and that for some increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) the graph \( G^{B^*} := \cup_k \partial P_{n_k} \) satisfies the hypotheses of Proposition 8.1 then the corresponding parameter nest \( \{PP_n\}_n \) with \( \partial PP_n := \zeta(\partial P_n) \) is convergent with

\[
\text{End}(\{PP_n\}_n) = \{B^*\}
\]

**Corollary 8.3.** Let \( B^* \in \mathcal{L}_{p/q}^{M_1} \) be a parameter satisfying \( \text{DF2} \) of type \( \text{i} \) or \( \text{ii} \) then the set \( M_1 \) is locally connected at \( B^* \). If \( v_{B^*} \) is not prefixed to \( \alpha_{B^*} \) then intersection \( \cap PP_n(\sim B^*) \) reduces to one point and thus \( \{M_1 \cap PP_n(\sim B^*)\}_{n \geq 0} \) is a fundamental system of connected neighbourhoods of \( B^* \) in \( M_1 \). And if \( v_{B^*} \) is prefixed to \( \alpha_{B^*} \) then \( B^* \) has a fundamental system of connected neighbourhoods consisting for each \( n \) of the interior of the union of closures of the \( q \) level \( n \) parameter puzzle pieces with \( B^* \) on the boundary.

For \( \text{DF2} \) type \( \text{iii} \) we need a refinement of Proposition 8.1 above due to Shishikura.

Fix any \( B^* \in M_1 \) of type \( \text{iii} \) for the rest of the section. Recall that Theorem 6.21 provides a non degenerate annulus \( A^{B^*}_{n_0} = P_{n_0} \setminus \overline{P}_{n_0+1} \) between nested puzzle pieces of the parabolic Yoccoz puzzle for \( g_{B^*} \) with \( g_{B^*}^{-1}(P_{n_0}) = P_{n_0-1}^{B^*} \), \( g_{B^*}^{n_0}(P_{n_0+1}) = S_k^i \) for some \( 0 < k < q \) and a nested sequence of annuli \( A^{B^*}_{n_i} = P_{n_i}^{B^*} \setminus \overline{P}_{n_i+1}^{B^*} \), \( i > 0 \) with \( n_0 < n_1 \searrow \infty \) surrounding the critical value \( v_{B^*} \) such that:

- the map \( g_{B^*}^{n_i-n_0} : A^{B^*}_{n_i} \to A^{B^*}_{n_0} \) is a covering map of degree \( 2^{d_i} \), \( d_i \geq 0 \) for \( i \geq 1 \);
- in particular also all the annuli \( A^{B^*}_{n_i}, i > 0 \) are non degenerate;
- either the sum \( \sum_{i \geq 1} \text{mod}(A^{B^*}_{n_i}) = \text{mod}(A^{B^*}_{n_0}) \sum_{i \geq 1} \frac{1}{2^{d_i}} \) is infinite,
  - or there exists \( k > 0 \) such that for all \( n \) large enough the map
    \[
    g_{B^*}^k : P_{n+k}^{B^*} \to P_n^{B^*}
    \]
    is quadratic-like with connected filled-in Julia set.

By corollary 7.8 the parameter \( B^* \) belongs to a complete sequence of puzzle pieces \( PP_n(\sim) \) defining non degenerate annuli for the subsequence \( A_{n_i}(\sim) \) where \( A_n(\sim) \) denotes the annulus \( PP_n(\sim) \setminus \overline{PP_{n+1}(\sim)} \).
From Lemma 7.5, for \( n = n_0 \), the parameter puzzle piece \( PP_{n_0}(B^*) \subset \mathcal{W}_0^{M_1}(p/q) \), parametrizes a holomorphic motion of the graph

\[
\mathcal{G}P^B_{n_0} \cup (\mathcal{G}P^B_{n_0+1} \cap P^B_{n_0}) \subset \mathcal{G} \mathcal{P}^B (n_0 + 1)
\]
as a restriction of

\[
\tilde{\Psi}_{n_0} = \tilde{\Psi}^B_{n_0} : PP_{n_0}(B^*) \times \mathcal{G}P^B_{n_0+1} \rightarrow \hat{\mathbb{C}}
\]

Then, applying Slodkowsky’s extension, we obtain a global holomorphic motion over \( PP_n(B^*) \subset \mathcal{W}_0^{M_1}(p/q) \) of \( \hat{\mathbb{C}} \) (we are however only interested in the part inside \( \mathcal{P}^B_{n_0} \)).

Therefore by restriction, we get a holomorphic motion of the annulus \( A^B_{n_0} \) which gives an annulus that coincides with the annulus \( A^B_{n_0} = P^B_{n_0} \setminus \mathcal{P}^B_{n_0+1} \).

Shishikura’s trick consists in lifting the holomorphic motion of the annulus to get a holomorphic motion of the annulus \( A^B_{n_0} \) defined in \( \mathcal{A}_n(\sim) \) with the same dilation. The lifting is possible since the map \( g^B_{n_0} : A^B_{n_0} \rightarrow A^B_{n_0} \) is a covering map of degree \( 2^{d_i} \), \( d_i \geq 0 \) for \( i \geq 1 \). Moreover, by Lemma 7.7, for parameters \( B \) in \( PP_n(B^*) \), the maps \( g^B_{n_0} : A^B_{n_0} \rightarrow A^B_{n_0} \) are all of the same type (covering map of degree \( 2^{d_i} \)) since the critical value \( v_B \) never passes though the boundary of \( P^B_{n_0} \).

**Lemma 8.4.** There exists a constant \( K \) such that for any integer \( n \in \{n_i \mid i \geq 0\} \)

\[
\frac{\text{mod} \left( A^B_{n_i} \right)}{K} \leq \text{mod} \left( \mathcal{A}_n(\sim) \right) \leq K \text{ mod } \left( A^B_{n_i} \right).
\]

**Proof.** The holomorphic motion of the boundary of \( A^B_{n_i} \) is defined in the whole parameter puzzle piece \( PP_{n_0}(\sim) \). By Slodkovski Theorem we can extend it to a holomorphic motion of \( \hat{\mathbb{C}} \) still parametrized by \( PP_{n_0}(\sim) \). Denote it by \( H^B(B, z) \). Now, one can lift \( H^B(B, z) \) to a holomorphic motion \( H^B(B, z) \) of \( \hat{A}^B_{n_i} \) using the unramified covering \( g^B_{n_i} \). This holomorphic motion defines a quasi-conformal homeomorphism \( H^B_{n_i}(z) := H^B(B, z) \), it has the same bound \( K \) on the dilatation. Now, it follows from DH3 Lemma IV.3, that the map \( \zeta(B) = (H^B_{n_i})^{-1}(v_B) \) is a quasi-conformal homeomorphism, it maps \( \mathcal{A}_n(\sim) \) to \( A^B_{n_i} \). The proof is exactly the same as in [RI].

**Corollary 8.5.** Let \( B^* \in \mathcal{L}^{M_1}_{p/q} \) be a parameter satisfying DI 2 and of type iiii. If \( g^B_{B^*} \) is not renormalizable or equivalently \( \sim^{B^*} \) is not renormalizable, then the intersection \( \cap PP_n(\sim^{B^*}) \) reduces to one point and thus \( \{M_1 \cap PP_n(\sim^{B^*})\}_{n \geq 0} \) is a fundamental system of connected neighbourhoods of \( B^* \) in \( M_1 \).

**Proof.** If \( \sim^{B^*} \) is not renormalizable, then the sum \( \sum_{i \geq 1} \text{mod} \left( A^B_{n_i} \right) = \text{mod} \left( A^B_{n_0} \right) \sum_{i \geq 1} \frac{1}{2^{d_i}} \)
is infinite, we deduce from previous Lemma that the sum $\sum_{i \geq 1} \mod(A_n(\sim B^i))$ is infinite. Then the result follows from Grötzsch inequality see [A].

\[ \sum_{i \geq 1} \mod(A_n(\sim B^i)) \]

8.1 The renormalizable case

We consider now a parameter $B^* \in \mathcal{L}^{M_1}_{p/q}$ satisfying DB2, of type iii, such that $g_{B^*}$ is renormalizable (equivalently $\sim B^*$ is renormalizable). We use the Douady-Hubbard theory of polynomial like mapping to get that the intersection $\cap PP_n(\sim B^*)$ is a copy of the Mandelbrot set and we obtain in this way a straightening map that will serve to construct the bijection $\Psi^1 : M_1 \rightarrow M$.

Definition 8.6. A subset $M_0$ of $M_1$ is a copy of $M$ if there exists a homeomorphism $\chi$ and an integer $k > 1$ (the period) such that

1. $M_0 = \chi^{-1}(M)$;
2. $\chi^{-1}(\partial M) \subset \partial M_1$; and
3. every $B \in M_0$ corresponds to a renormalizable map with $g^k$ topologically conjugated to $z^2 + \chi(B)$ on neighbourhoods of the filled Julia sets.

Proposition 8.7. Suppose $\sim$ is a renormalizable tower. Then the intersection

$$M_{\sim} = \bigcap_{n \geq 0} PP_n(\sim B^*)$$

is a copy of $M$.

Proof. For simplicity we write $\sim$ for $\sim B^*$. Let $k \geq q$ denote the period of the lamination $\sim$ (since it is renormalizable). We develop here the case where the period is $k \neq q$. If the period is $k = q$, it corresponds to the satellite renormalizable case, the proof is similar except that one should consider enlarged puzzle pieces at the $\alpha$ fixed point for $P^B_n$.

The proof for $k > q$ is as follows. The map $g_{B^*}$ satisfies the second case of the alternative of Proposition 6.21 and this is the case of all the maps $g_B$ for $B \in M_{\sim}$ since they have the same lamination: there exists $n_0$ such that for all $n \geq n_0$ the map

$$g^k_B : P^B_{n+k} \rightarrow P^B_n$$

is quadratic-like with connected filled-in Julia set.

For $n \geq n_0$ and $PP_n = PP_n(\sim)$, we consider the mapping $g : W' \rightarrow W$ defined by $W = \{(B, z) \mid B \in PP_n, z \in P^B_{n-k}\}$, $W' = \{(B, z) \mid B \in PP_n, z \in P^B_n\}$ and $g(B, z) = z^2 + \chi(B)$. The map $g$ extends to $PP_n$ and $PP_n(\sim)$ and the invariant curves $\cap PP_n(\sim)$ are also invariant under $g$ and $g^k$. Moreover, the filled-in Julia sets $P^B_n$ and $P^B_{n+k}$ are connected and have the same boundary. Therefore, $g^k(B, z) = z^2 + \chi(B)$ is topologically conjugate to $z^2 + \chi(B)$ on neighbourhoods of the filled Julia sets.
(\(B, g^k(z)\)). They form an analytic family of quadratic-like maps in the sense of Douady and Hubbard [DH3, p.304] since they satisfy the following three properties:

- the map \(g : W \to W\) is holomorphic and proper;
- the holomorphic motion of the disk \(P^B_n\), resp. \(P^B_{n-k}\), is a homeomorphism between \(W\), resp. \(W\), and \(PP_n \times \mathbb{D}\) which is fibered over \(PP_n\) (since \(B \in PP_n\));
- the projection \(\overline{W} \cap W \to PP_n\) (i.e. the first coordinate) is proper, since \(\overline{W} \cap W = \{(B, z) \mid B \in PP_n, z \in \overline{P^B_n}\}\).

Let \(M_g = \{B \in PP_n \mid K(g^k_B)\) is connected\} denote the connectedness locus of \(g_B\), where \(K(g^k_B) = \bigcap_{i \geq 0} (g^k_B)^{-i}(P^B_n)\) denote its filled Julia set. Then \(M_g\) coincides with \(M_\infty\). Indeed, for \(B \in M_\infty\), the critical point and its orbit under \(g_B\), do never cross the graphs. Therefore the critical point of \(g^k_B|_{P^B_n}\) never escapes the piece \(P^B_n\) (by iteration by \(g^n\)). Hence \(K(g^k_B)\) is connected and \(B \in M_g\). Conversely, for \(B \in PP_n \setminus PP_{n+1}\), the critical value belongs to \(A^B_n\). Thus \(g^k_B(v_B)\) is not in \(P^B_{n-k}\) and therefore the critical point of \(g^k_B\) escapes the domain; then the filled Julia set is not connected anymore so that \(B \notin M_g\).

Moreover, by Corollary 7.8 and Proposition 6.21 there exists a sequence \(n_i\) such that \(PP_{n_i+1} \subset PP_{n_i}\). Then \(M_\infty\) is also the intersection of the closed pieces: \(M_\infty = \bigcap_{n \geq 0} PP_n\) and therefore is compact.

Now, the theory of Mandelbrot-like families of Douady and Hubbard (see [DH3, Theorem II.2, Propositions II.14 and IV.21]) gives a continuous map \(\chi : PP_n \to \mathbb{C}\) such that the maps \(g^k_B\) and \(z^2 + \chi(B)\) are quasi-conformally conjugated on a neighbourhood of the filled Julia sets, for every \(B \in PP_n\).

Moreover, since \(M_\infty = M_g\) is compact, the map \(\chi\) induces a homeomorphism between \(M_g\) and the Mandelbrot set \(M\) if we are in the following situation (see [DH3]): for a closed disk \(\Delta \subset PP_n\) containing \(M_g\) in its interior, the quantity \(g^k_B(x_B) - x_B\), (where \(x_B\) denotes the critical point of \(g^k_B|_{P^B_n}\) should turn exactly once around 0 when \(B\) describes \(\partial \Delta\). We verify this property now.

Take some piece \(PP_{p}(B^*) = \Delta\), compactly contained \(PP_n(B^*)\) (see Corollary 7.8). It is a topological disk containing \(M_\infty\) in its interior. To compute the degree on \(\partial \Delta\) of \(\gamma(B) = g^k_B(x_B) - x_B\) we make a homotopy of this curve \(\gamma\) to the curve \(g^{k-1}_B(\zeta(B)) - x_B\) (see the proof of Lemma 8.4) as follows. Let \(h(B, z) = H^{p-1}(B, g^{k-1}_B(z)) - x_B\), where \(H^{p-1}(B, z)\) is the holomorphic motion of \(A^B_{m-1}\) then \(\gamma(B) = h(B, \zeta(B))\). Assume that \(PP_{p-1}\) is a round disk (if not use a conformal representation); then the homotopy is
simply \( G(t, B) = h(B^* + t(B - B^*), \zeta(B)) \) joining \( G(0, B) = \zeta(B) - x_{B^*} \) and \( G(1, B) = g(\varphi^{k-1}(x_B)) - x_B \).

Since \( H \) is a homeomorphism from \( \partial PP_p \) to \( \partial P^*_p \) (piece that surrounds \( x_{B^*} \)), the degree of \( H(B) - x_{B^*} \) around 0 is exactly 1, when \( B \) describes \( \partial PP_p \).

**Corollary 8.8.** For every \( c \in L^*_{p/q} \) there exists a \( B \in L^*_{p/q} \) with \( \sim c = \sim B \)

**Proof.** Let \( c \in L^*_{p/q} \) and let \( \sim = \sim c \). If \( \sim \) is a terminal tower then there exists a unique \( B \in \mathcal{L}_{p/q} \) with \( \sim B = \sim c \) by 2. of Proposition 7.11. If \( c \in \mathcal{M}^{M}_{p/q} \), i.e. \( Q_c \) is \( q \) renormalizable take any \( B \in \mathcal{M}^{M}_{p/q} \) i.e. the unique \( B \) such that \( f_c \) and \( f_B \) are hybrid equivalent.

Finally if \( c \) is any other parameter then \( Y^n_c \) is defined for all \( n \) and the parameter nest \( \{Y Y_n(c)\}_n \) is a system of nested neighbourhoods of \( c \). And for every \( n \) there is \( n' > n \) such that \( Y^n_{n'} \subset Y^n_c \) so that also \( \overline{Y^n_{n'}}(c) \subset Y Y_n(c) \). By the theorems of this section the similar statement \( \overline{PP}_{n'}(\sim) \subset PP_n(\sim) \) also holds so that for \( B \in \overline{PP}_n(\sim) \) we have \( \sim B = \sim = \sim c \).

9 **Proof of the Main Theorem**

In the paper [PR2, Subsection 5.4, the proof of the main theorem] we have constructed a projection \( \Psi^1 : \mathcal{M}_1 \to \mathcal{M} \) with the following properties, recall that \( A = 1 - B^2 \):

1. For \( B \) with \( A \in \overline{B} \) the define \( \Psi^1(B) := c \) where \( c \) is the unique parameter such that the fixed point \( \alpha_c \) for \( Q_c \) has multiplier \( A \).

2. For \( B \) with \( A \notin \overline{B} \) let \( p/q \) be the irreducible rational such that \( B \in \mathcal{L}^{\mathcal{M}M}_{p/q} \).

   a. If \( \sim B \) is renormalizable of period \( k \), then \( g_B \) is \( k \) renormalizable and \( B \in \mathcal{M}^{\mathcal{M}_1} = \mathcal{M}^{\mathcal{M}_1}(\sim B) \) a copy of \( \mathcal{M} \) in \( \mathcal{M}_1 \). Let \( \mathcal{M}^\mathcal{M} = \mathcal{M}^\mathcal{M}(\sim B) \) be the copy of \( \mathcal{M} \) in \( \mathcal{M} \) such that \( c \in \mathcal{M}^\mathcal{M} \) implies \( \sim c = \sim B \). Let \( \chi : \mathcal{M}^{\mathcal{M}_1} \to \mathcal{M}^\mathcal{M} \) be the homeomorphism induced by straightening. Define \( \Psi^1(B) = \chi(B) \).

   b. If \( \sim B \) is not renormalizable, i.e. a Yoccoz parameter let \( c \in \mathcal{M} \) be the unique parameter such that \( \sim B = \sim c \) and define \( \Psi^1(B) = c \).

In particular for any finite tower \( \sim_n \)

\[
\Psi^1(PP(\sim_n) \cap \mathcal{M}_1) = \mathcal{Y}(\sim_n) \cap \mathcal{M}
\]

where the parameter puzzles \( \mathcal{Y}(\sim_n) \) for \( \mathcal{M} \) are defined similarly to those for \( \mathcal{M}_1 \).
We shall show that this map is a homeomorphism. Note that by construction it is dynamic.

Injectivity of $\Psi^1$ is an immediate consequence of Corollary 8.3 and Corollary 8.5. For the surjectivity we need only consider the case $c \in L_{p/q}^*$ for some irreducible rational $p/q$. Let $\sim = \sim^c$.

If $\sim$ is renormalizable let $M_{M_1} = \cap Y_{\mu}(\sim)$, $M_{M} = \cap \mathcal{Y}_n(\sim)$ and $\chi : M_{M_1} \longrightarrow M_{M}$ be as in 2a., then $\Psi^c(\chi^{-1}(c)) = c$ and we are done.

If $\sim$ is not renormalizable then by Corollary 8.8 there exists $B^\star \in L_{p/q}^*$ with $\sim B^\star = \sim^c$ and $\Psi^1(B^\star) = c$ by construction of $\Psi^1$. Thus $\Psi^1$ is a bijection.

The continuity of $\Phi^1 = (\Psi^1)^{-1}$ follows from the first three properties by a standard result in topology.

For the continuity of $\Psi^1$ let us start by noting that $\Psi^1$ is continuous at any Yoccoz parameter, by construction and by local connectivity of both $M_1$ and $M$ at all Yoccoz parameters.

Thus we need only prove that $\Psi^1$ is continuous at the boundary of any top-level renormalization copy $M_{M_1}(\sim^B^\star)$ in $M_1$. So let $B^\star \in M_{M_1} \subset L_{p/q}^*$ be a boundary point, $M_{M} = M_{M_1}(\sim^B^\star)$ and $\chi : M_{M_1} \longrightarrow M_{M}$ be as in 2a.

We must show that $\Psi^1$ is continuous at $B^\star$. By construction $\Psi^1$ is continuous on $M_{M_1}$. Hence we only need to show that if $\{B_N\}_n \subset L_{p/q}^* \setminus M_{M_1}$ is a sequence converging to $B^\star$ then the sequence $c_n := \Psi^1(B_n)$ converges to $c^\star := \Psi^1(B^\star) = \chi(B^\star)$.

To this end we invoke the shrinking of dyadic decorations theorem. Recall that any renormalization copy comes equipped with dyadic limbs, which are the extremities of $M$ or $M_1$ beyound a renormalization copy (see e.g. Definition 3.25).

**Theorem 9.1 ([PR3, Theorem 4]).** For any copy $M$ of $M$ in $M$ or in $M_1$

$$\lim_{s \to \infty} \text{diam}(L_M(r, s)) = 0$$

where diam(·) denotes Euclidean diameter and $L_M(r, s)$ denotes the $r/2^s$ dyadic limb of $M$, i.e. if $M = M_{M}(\theta, \theta')$ then $L_M(r, s) = L_{M}(\theta, \theta', r, s)$ and if $M = M_{M_1}(\epsilon, \epsilon')$ then $L_M(r, s) = L_{M_1}(\epsilon, \epsilon', r, s)$.

For $\{B_n\}_n$ a sequence converging to $B^\star$ as above let $L_{M_1}(r_n, s_n)$ denote the dyadic limb of $M_{M_1}$ containing $B_n$ and let $B'_n \in M_{M_1}$ denote the root of that limb. Then by construction $c_n = \Psi^1(B_n) \in L_M(r_n, s_n)$ and $c'_n = \Psi^1(B'_n)$ is the root of that limb.

Now either $B_n$ and $(r_n, s_n)$ are eventually fixed or $s_n$ diverges to infinity, since any two distinct dyadic limbs of $M_{M_1}$ are strongly separated.
If the sequence $s_n$ diverges to infty, then both $|B_n - B_n'| \to 0$ and $|c_n - c_n'| \to 0$ as $n \to \infty$ by Theorem 9.1. Thus $B_n' \to B^*$ since $B_n \to B^*$ and hence $c_n' \to c^*$ as $n \to \infty$ since $\chi$ is continuous. And combining with $|c_n - c_n'| \to 0$ yields the desired $\Psi^1(B_n) = c_n' \to c^* = \Psi^1(B^*)$ as $n \to \infty$.

Next suppose $B_n' = B'$ and so $(r_n, s_n) = (r', s')$ for $n \geq N_0$. Moreover $c_n' = c' = \Psi^1(B')$ for $n \geq N_0$. Then $B^* = B'$ since $B^*, B' \in M^{M_1}$ and $B'$ is the only intersection of $M^{M_1}$ and $\mathcal{I}^{M_1}(r', s')$. And similarly $c' = c^*$.

If the renormalization period $k > q$ then $M^{M_1} = \cap P\kappa_n(\sim)$ and $M^M = \cap Y_n(\sim)$, where $\sim \equiv B^q \sim \sim c^*$. Thus for every $N_1$ there exists $N_2$ such that $B_n \in P\kappa_{N_1}(\sim)$ and hence $c_n \in Y_{N_1}(\sim)$ for every $n \geq N_2$. Hence $c_n \to c' = c^*$ as $n \to \infty$.

Finally if the renormalization period is $q$ so that $M^{M_1} = M^{p/q}$ and $M^M = M^{p/q}$. Then $g_{B^q}(v_{B^q}) = \alpha$, and similarly $Q_\alpha^q(c^*) = \alpha c^*$. Instead of developing augmented puzzles using more rays in the base puzzle, see e.g. [PR3] and its illustrations we shall give here an add hoc argument.

Recall that for $B \in \mathcal{W}^{M_1}_{mq-1}(p/q, r, m)$ we have defined a fundamental system of neighbourhoods $\{P_n^{\alpha}\}_n$ of $\alpha$, such that $P_n^{\alpha} \cap K_\alpha$ is connected, where $P_n^{\alpha}$ is the interior of $\cup_{k=0}^{n-1} P_m^{\alpha, k}$. Moreover the union of boundaries $\cup_{n} P_{n}^{\alpha, k}$ move continuously with $B$ in the closure of $\mathcal{W}^{M_1}_{mq-1}(p/q, r, m)$. Similarly for the quadratic polynomials $Q_c$, the union of boundaries $\cup_{n} Y_n^{\alpha}$ move continuously with $c \in \mathcal{W}^{M}_{mq-1}(p/q, r, m)$. By hypothesis $g_{B_n}^{mq}(v_{B_n}) \to g_{B_n}^{mq}(v_{B^*}) = \alpha$ as $n \to \infty$. Thus given $N_1$ there exists $N_2$ such that $g_{B_n}^{mq}(v_{B_n}) \in P_0^{n, \alpha, k} \cap P_0$ for every $n \geq N_2$. By construction $g_{B_n}^{mq}(v_{B_n}) \in P_0^{n, \alpha, k} \cap P_0$ if and only if $Q^{mq}_{c_n}(c_n) \in Y_0^{n, \alpha, k} \cap Y_0^{\alpha, k}$ for every $n$ and $N_1$. Thus $Q^{mq}_{c_n}(c_n) \in Y_0^{n, \alpha, k} \cap Y_0^{\alpha, k}$ for every $n \geq N_2$, i.e. also $Q^{mq}_{c_n}(c_n) \to Q^{mq}_{c_n}(c^*) = \alpha c_*$. Finally $Q^{mq}_{c_n}(c_n) \to Q^{mq}_{c_n}(c^*) = \alpha c_*$ implies $c_n \to c^*$ as $n \to \infty$. This completes the proof.

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