Axionic cosmological constant

Katrin Hammer, a Pavel Jiroušek, b,c and Alexander Vikman b

aArnold Sommerfeld Center for Theoretical Physics, Ludwig Maximilian University Munich, Theresienstr. 37, D-80333, Munich, Germany

bCEICO-Central European Institute for Cosmology and Fundamental Physics, Institute of Physics of the Czech Academy of Sciences, Na Slovance 1999/2, 18221 Prague 8, Czech Republic

cInstitute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic

E-mail: vikman@fzu.cz, jirousek@fzu.cz, katrin.a.hammer@physik.uni-muenchen.de

Abstract. We propose a novel higher-derivative, Weyl-invariant and generally-covariant theory for the cosmological constant. This theory is a mimetic construction with gauge fields playing the role of dynamical variables. These fields compose Chern-Simons current instead of the vector field in the Henneaux and Teitelboim formulation of the unimodular gravity. The equations of motion exactly reproduce the traceless Einstein equations. We demonstrate that, reformulated in Weyl-invariant variables, this novel theory reduces to standard general relativity with the cosmological constant as a Lagrange multiplier. This Lagrange multiplier has an axion-like coupling.
1 Introduction and Discussion

More than a century ago Einstein proposed [1] the trace-free part of his now standard general relativity (GR) equations, as a foundation for the dynamics of spacetime. If one assumes energy-momentum conservation, this formulation is classically equivalent to GR with a cosmological constant that is now demoted from the status of a constant of nature to a mere constant of integration, for recent discussion see e.g. [2–4]. The value of this integration constant is fixed by initial conditions. Thus, the current cosmological constant may be a remnant from the early universe quantum gravity era, for discussion see e.g. [5, 6]. The main advantage of this formulation is that these traceless Einstein equations are invariant with respect to the vacuum shifts of the matter energy-momentum tensor

\[ T_{\mu\nu} \rightarrow T_{\mu\nu} + \Lambda g_{\mu\nu}. \]  

(1.1)

This rather desirable property [7] puts the cosmological constant problem in a rather different perspective.

The cosmological constant problem is one of the most important problems in modern physics, for reviews and recent pedagogical expositions see e.g. [2, 8–10]. The essence of the cosmological constant (CC) problem is a fine-tuning of the value of the observed acceleration of our expanding universe caused by this CC. However, any discussion of naturalness and anthropic reasoning implicitly assumes that the cosmological constant or vacuum energy can take different values for different solutions. Hence, theories where CC is not a constant of nature, but a dynamical variable are useful to apply anthropic reasoning.

As it was realized already by Einstein one can obtain such a formulation of gravitational dynamics form the Einstein-Hilbert action where one does not vary the determinant of the metric. Therefore very often such theories are referred to as “unimodular gravity”. There are
different formulations of the dynamics of the unimodular gravity, see e.g. [11, 12]. The most relevant for us is the generally-covariant construction by Henneaux and Teitelboim (HT) [11] with a vector field and our recently introduced mimetic reformulation [13] based on a vector field of conformal weight four.

In the current paper we upgraded our previous Weyl-invariant proposal of mimetic dark energy [13] with an unusual vector field of conformal weight four to a mimetic construction with common abelian and non-abelian /Yang-Mills gauge fields. In the current modification, the role of this vector field is played by a Chern-Simons current. Similarly to the original mimetic dark matter proposal [14], the theory has higher derivatives, but does not suffer from the Ostrogradsky ghosts because of the gauge degeneracy caused by the Weyl invariance. Our mimetic setup allows for a reformulation without higher derivatives. In this reformulation, the cosmological constant possesses an axion like coupling envisioned long time ago by Wilczek in [15]: “I would like to briefly mention one idea in this regard, that I am now exploring. It is to do something for the θ-parameter very similar to what the axion does for the A-parameter in QCD, another otherwise mysteriously tiny quantity. The basic idea is to promote these parameters to dynamical variables, and then see if perhaps small values will be chosen dynamically.” However, here we have not touched any dynamical mechanisms allowing to obtain value of CC corresponding to the current data.

The paper is organized in the following way. In section (2) we refresh the relevant elements from [13], introduce our mimetic construction for an abelian gauge field and connect this model to our previous proposal [13]. In section (3) we discuss identities which appear per construction in all mimetic theories and in particular in the theory proposed here. Then, in section (4), we discuss the correspondence between the U(1) gauge transformations in the current construction and gauge symmetry corresponding to the transverse shifts in the vector field in [13] and [11]. Then we derive the equations of motion and discuss their properties in section (5). In section (6) we find Weyl-invariant variables and reformulate the theory using Lagrange multiplier. Here we show that the field corresponding to the cosmological constant has an axion like coupling. After that in section (7) we follow the Faddeev-Jackiw procedure and analyze the dynamical degrees of freedom in the abelian case. Here we show that the cosmic time canonically conjugated to the cosmological constant is given by the Chern-Simons charge. It is easy to verify that this will also be the case for the further non-abelian extensions which we introduce in section (8). There we generalize this construction to arbitrary SU(N) gauge group and discuss a potential embedding of this theory as an IR-limit of a confined Yang-Mills theory (like QCD) with the axion. Finally, in section (9) we introduce other novel vector-tensor theories describing cosmological constant as an integration constant.

Clearly current luck of understanding of the tiny value of the cosmological constant encourages further investigation of these theoretic constructions.

2 Mimetic Construction with Chern-Simons Current

In the previous work [13] some of us proposed a new extension of the mimetic construction [14] incorporating a vector field $V^\alpha$. As usual in mimetic theories, we demoted the spacetime metric, $g_{\mu\nu}$, from its status of a dynamical variable and introduced the ansatz\footnote{After our previous work [13] was published and most of the current paper was completed, we became aware that this ansatz was introduced before in [16] from a completely different reasoning. We are thankful to Tony Padilla for pointing out this useful reference.} for $g_{\mu\nu}$:

\[ g_{\mu\nu} = \frac{1}{\sqrt{\text{det}(g)}} g^{\mu\nu}. \]
In this substitution, \( g_{\mu\nu} \) is the physical, free-fall, metric, while \( h_{\mu\nu} \) is an auxiliary metric and a new dynamical variable and the covariant derivative, \( \nabla^h_\alpha \), is the Levi-Civita connection compatible with this \[ \nabla^h_\alpha h_{\mu\nu} = 0 \] \[ (2.2) \]

The main idea behind the ansatz (2.1) was that any seed theory which originally has \( g_{\mu\nu} \) as a dynamical variable will be mapped into a new Weyl-invariant theory with dynamical variables \( h_{\mu\nu} \) and \( V^\mu \). The latter property can be realized only if the vector field \( V^\mu \) has conformal weight four under the Weyl transformations:

\[ V^\mu = \Omega^{-4}(x)V'^\mu, \quad \text{along with} \quad h_{\mu\nu} = \Omega^2(x)h'_{\mu\nu}. \] \[ (2.3) \]

On top of the Weyl symmetry above, the resulting theory will possess another gauge invariance

\[ V^\mu = V'^\mu + \xi^\mu, \quad \text{where} \quad \nabla^h_\mu \xi^\mu = 0, \] \[ (2.4) \]

as the latter preserves the ansatz (2.1).

As it was demonstrated in the previous work [13], if the seed theory is general relativity (GR) with the Einstein-Hilbert action, then the resulting theory, written in Weyl-invariant variables, reduces to the Henneaux and Teitelboim (HT) generally-covariant formulation [11] of the so-called “unimodular gravity”. Interestingly, this classical equivalence along with the Weyl-invariance of the resulting theory seem to be overlooked in [16].

In the resulting theory there is only one global additional dynamical degree of freedom (on top of the usual two graviton polarizations) which is given by the integral\(^2\)

\[ \mathcal{F}(t) = \int d^3x \sqrt{-h} V^t(t,x). \] \[ (2.5) \]

This dynamical degree of freedom - “cosmic time” - is canonically conjugated to the cosmological constant \( \Lambda \), and is shifted by a constant

\[ \mathcal{F}(t) = \mathcal{F}'(t) + c, \] \[ (2.6) \]

as a result of the gauge transformations (2.4). As usual in classical mechanics, shift-symmetry in a coordinate results in conservation of the canonical momentum: \( \Lambda = \text{const} \). Here it is crucial that similarly to current conservation, the divergence-free condition for the gauge transformations (2.4), \( \nabla^h_\mu \xi^\mu = 0 \), results in \( \int d^3x \sqrt{-\hat{h}} \xi^t(t,x) = \text{const} \) which we denoted as \( c \).

Contravariant vector fields of conformal weight four are rather unusual objects. A common conformal weight for a contravariant vector field is two. This suggests to look for more fundamental vector objects which can compose \( V^\mu \).

In this paper we propose a new vector mimetic construction where \( V^\mu \) is not a dynamical variable, but is composed out of usual gauge fields \( A_\mu \). This option was also briefly mentioned

\(^2\)We use: the standard notation \( \sqrt{-h} \equiv \sqrt{-\text{det}h_{\mu\nu}} \), the signature convention \((+,−,−,−)\), and the units \( c = \hbar = 1, M_{\text{Pl}} = (8\pi G_N)^{-1/2} = 1 \).
in [16], but again from a completely different perspective. Namely, we propose the ansatz
\( g_{\mu\nu} = h_{\mu\nu} \cdot \sqrt{F_{\alpha\beta} \tilde{F}^{\alpha\beta}} \),
(2.7)
for the spacetime metric in the Einstein-Hilbert action. In this ansatz
\( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \),
(2.8)
is the usual field strength tensor for a U(1) gauge potential \( A_\mu \), while \( \tilde{F}_{\alpha\beta} \) is the corresponding Hodge-dual tensor
\( \tilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \),
(2.9)
with the Levi-Civita symbol \( \epsilon^{0123} = +1 \).

Later in this paper we generalize this ansatz (2.7) to a non-abelian case. One can rewrite the physical metric as
\( g_{\mu\nu} = \frac{h_{\mu\nu} \cdot \sqrt{\mathcal{P}}}{(-h)^{1/4}} \),
(2.10)
where
\( \mathcal{P} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} = 2 \epsilon^{\alpha\beta\mu\nu} \partial_\alpha A_\beta \partial_\mu A_\nu \),
(2.11)
is the Chern-Pontryagin density, which is insensitive to the metric. Hence, in our ansatz, for the determinant of the physical metric we have
\( g = -\mathcal{P}^2 \).
(2.12)

The physical metric \( g_{\mu\nu} \) defined through (2.7) is invariant under the Weyl transformations of the auxiliary metric \( h_{\mu\nu} \). Indeed, as usual, the gauge field \( A_\mu \) is invariant under the Weyl transformations and so it is the case for the Chern-Pontryagin density, \( \mathcal{P} \). Hence, the invariance of \( g_{\mu\nu} \) directly follows from (2.10).

Now we can recall that
\( F_{\alpha\beta} \tilde{F}^{\alpha\beta} = F_{\alpha\beta} \left( \nabla^{(h)}_\alpha \right) A_\beta F_{\mu\nu} = \nabla^{(h)}_\alpha \left( F_{\alpha\beta} A_\beta F_{\mu\nu} \right) \),
(2.13)
as due to the Bianchi identity \( \nabla^{(h)}_\alpha \left( F_{\alpha\beta} F_{\mu\nu} \right) = 0 \). Hence, one can introduce the so-called Chern-Simons current, \( C^\alpha \), which is given by
\( C^\alpha = E^{\alpha\beta} A_\beta F_{\mu\nu} = 2 \tilde{F}^{\alpha\beta} A_\beta = 2 E^{\alpha\beta} A_\beta \nabla^{(h)}_\mu A_\nu = 2 E^{\alpha\beta} A_\beta \partial_\mu A_\nu \),
(2.14)
where we listed some of the useful identities. Under the Weyl transformations this pseudovector has conformal weight four as
\( C^\alpha = 2 \epsilon^{\alpha\beta\mu\nu} A_\beta \partial_\mu A_\nu = 2 \Omega^{-4} (x) \epsilon^{\alpha\beta\mu\nu} A_\beta \partial_\mu A_\nu = \Omega^{-4} (x) C^\alpha \).
(2.15)

We can now write our novel mimetic vector ansatz (2.7) as
\( g_{\mu\nu} = h_{\mu\nu} \cdot \sqrt{\nabla^{(h)}_\alpha C^\alpha} \),
(2.16)

\[ \text{It would be more proper to use the absolute value of the invariant } F_{\alpha\beta} \tilde{F}^{\alpha\beta} \text{ in this and consequent formulas, but for simplicity we omit the absolute value.} \]

\[ \text{We use the convention } \epsilon^{0123} = +1. \]
and identify the Chern-Simons current (ChS) with the vector field $V^\mu$ introduced in [13]. It is important to stress that the dynamical variables in the action are \( \{ A_\mu, h_{\alpha\beta} \} \). Contrary to ansatz (2.1) from our previous work [13], here, the physical metric $g_{\mu\nu}$ does not depend on the derivatives of the auxiliary metric $h_{\alpha\beta}$. The price for this simplification is that the field $V^\mu$ becomes the ChS current, which is a composite variable quadratic in the elementary dynamical variables $A_\mu$. Moreover, one should be rather vigilant in using this identification, as a straight field redefinition, $V^\mu \to A_\mu$, because (2.14) contains time derivatives $\partial_t A_\mu$. These time derivatives would usually imply that the equations of motion for \( \{ A_\mu, h_{\alpha\beta} \} \) may have more solutions than the equations of motion for \( \{ V^\mu, h_{\alpha\beta} \} \). Thus, the proposed construction should not necessarily result in a theory equivalent to the one introduced in [13] corresponding to the “unimodular gravity”. Nevertheless, we will show just a bit later that, indeed, both theories, with vector of conformal weight four (2.7) and with gauge field (2.1) have the cosmological constant as an integration constant and no other dynamical degrees of freedom that would be extra to those in standard GR. The key hint for this equivalence is that the true dynamical degree of freedom of the original theory (2.5), the cosmic time, only depends on $V^t$, which after identification becomes $C^t$ given by (2.14). However, the later does not contain dangerous time-derivatives of the gauge potential $A_\mu$.

Substitution of the ansatz (2.1) into any seed action functional $S[g, \Phi_m]$ (with some matter fields $\Phi_m$) induces a novel Weyl-invariant theory with the action functional

$$S[h, A, \Phi_m] = S[g(h, A), \Phi_m].$$

Now we can plug in the ansatz (2.1) into the Einstein-Hilbert action to obtain an action for a higher-derivative and U(1)-invariant vector-tensor theory

$$S_g[h, A] = -\frac{1}{2} \int d^4x \sqrt{-h} \left[ \left( F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right)^{1/2} R(h) + \frac{3}{8} \left( \nabla_\mu (F_{\alpha\beta} \tilde{F}^{\alpha\beta}) \right)^{2/3} \right].$$

From this action it is clear that the invariant $F_{\alpha\beta} \tilde{F}^{\alpha\beta}$ can never vanish on any physical solution. This is clearly a novel U(1)-invariant scalar-vector theory going beyond Horndeski and other more recent constructions. For details see [17, 18]. The total action is

$$S[h, A, \Phi_m] = S_g[h, A] + S_m[h, A, \Phi_m]$$

where the second term is the action for matter fields. It is convenient to introduce a scalar field $\varphi$

$$F_{\alpha\beta} \tilde{F}^{\alpha\beta} = \left( \frac{\varphi^2}{6} \right)^2,$$

so that all other matter fields are coupled to the physical metric

$$g_{\mu\nu} = \frac{\varphi^2}{6} \cdot h_{\mu\nu}.$$

An elegant representation of the ansatz (2.7) comes from the identity

$$\det F_{\mu\nu} = -\left( \frac{F_{\alpha\beta} \tilde{F}^{\alpha\beta}}{4} \right)^2 \cdot h,$$

implying that

$$g_{\mu\nu} = 2h_{\mu\nu} \cdot \left( \frac{\det F_{\mu\nu}}{h} \right)^{1/4}.$$
It is important to stress that the Weyl-invariance does not fix the form of the mimetic ansatz. Indeed, in [19, 20] it was proposed to use another invariant of the field strength tensor so that \( g_{\mu\nu} = h_{\mu\nu} \cdot \sqrt{F_{\alpha\beta} \tilde{F}^{\alpha\beta}} \) instead of (2.7). This ansatz also preserves the Weyl-invariance of the metric \( g_{\mu\nu} \), but instead of the cosmological constant the result mimics the spatial curvature, at least in cosmology.

Interestingly, similarly to these works [19, 20], one can show that among conformal substitutions \( g_{\mu\nu} = C \left( F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right) h_{\mu\nu} \) or \( g_{\mu\nu} = C \left( \nabla^h_{\alpha} V^\alpha \right) h_{\mu\nu} \), the only \( C \) that allows for a degeneracy and that induces a new theory and not just a mere field-redefinition\(^5\), is a square root. Thus the degeneracy and the appearance of new (global) degree of freedom is directly linked to the Weyl symmetry.

3 Built-in Constraints

Usually mimetic theories possess a built-in constraint involving the physical metric, \( g_{\mu\nu} \) and other Weyl-invariant quantities composed out of the dynamical variables. For instance, the original mimetic dark matter [14] has the built-in Hamilton-Jacobi equation

\[
g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1. \tag{3.1}
\]

While the mimetic dark energy from [13] has

\[
\nabla^g_\mu W^\mu = 1, \tag{3.2}
\]

where \( \nabla^g_\mu \) is the Levi-Civita connection compatible with the physical metric

\[
\nabla^g_\mu g_{\alpha\beta} = 0, \tag{3.3}
\]

and therefore Weyl-invariant with respect to \( h_{\mu\nu} = \Omega^2 (x) h'_{\mu\nu} \), while the Weyl-invariant vector field \( W^\mu \) was defined as

\[
W^\mu = \frac{V^\mu}{\sqrt{-g}} \nabla^h_{\alpha} V^\alpha, \tag{3.4}
\]

with \( h_{\mu\nu} \)-compatible Levi-Civita connection \( \nabla^h_{\alpha} \).

In the current reincarnation of mimetic dark energy it is useful to introduce \( F^{\star\alpha\beta} \) as the Hodge-dual tensor

\[
F^{\star\alpha\beta} = \frac{1}{2} \frac{\epsilon^{\alpha\beta\mu\nu}}{\sqrt{-g}} : F_{\mu\nu}, \tag{3.5}
\]

defined with respect to the physical metric, \( g_{\mu\nu} \). It is easy to check that, per construction, the analogue of (3.1) and (3.2) is

\[
F_{\alpha\beta} F^{\star\alpha\beta} = 1. \tag{3.6}
\]

All these constraints are just identities which also hold off-shell. Usually, a Weyl-invariant formulation of a mimetic theory boils down to just standard GR supplemented by these constraints that have to hold now only on-shell - on equations of motion. In section (6) we will see that this expectation is fulfilled also for the theory under consideration.

\(^5\)For the similar discussion in context of the scalar-field mimetic construction and more general disformal transformations see [21, 22], while for degeneracy in gauge field metric transformations see [23, 24].

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4 Matching Gauge Transformations?

It is useful to recall that the Chern-Simons current is not a gauge invariant object with respect to U(1) transformations. Indeed, under the usual U(1) gauge transformation

\[ A_\mu = A'_\mu + \partial_\mu \theta , \]  

the ChS current transforms inhomogeneously as

\[ C^\alpha = C'^\alpha + 2\tilde{F}^{\alpha\beta} \partial_\beta \theta . \]  

However, the good news are that

\[ \nabla^b \alpha \left( \tilde{F}^{\alpha\beta} \partial_\beta \theta \right) = 0 , \]

due to the Bianchi identity, commutativity of derivatives and antisymmetry of \( F_{\mu\nu} \). In this way, it seems that at least some of the gauge transformations (2.4) (and consequently the global shifts of the cosmic time (2.6)) can be generated by the usual U(1) gauge transformations (4.1) with a particular vector field

\[ \xi^\alpha_\theta = 2\tilde{F}^{\alpha\beta} \partial_\beta \theta . \]

However, not all of the gauge transformations (2.4) (global shifts of the cosmic time (2.6)) can be represented through the so-called small U(1) gauge transformations, with the gauge function vanishing at infinity. Indeed, a general divergence-free vector field \( \xi^\mu \) contains three independent functions, whereas in U(1) transformations there is only one free function \( \theta \).

Moreover, one can show that the global shifts in cosmic time (2.6) could be only generated by large gauge transformations that have \( \theta \) non-vanishing at the spatial boundary of spacetime, \( B \). Indeed,

\[ \int d^3 x \sqrt{-h} \xi^t_\theta = \int d^3 x \epsilon^{tikm} F_{km} \partial_i \theta = \int d^3 x \partial_i \left( \theta \epsilon^{tikm} F_{km} \right) = \oint_B ds_i \theta \epsilon^{tikm} F_{km} , \]

where we used (2.9), the Bianchi identity and the 3d Stokes theorem. Clearly, this integral can only be nonvanishing provided the gauge functions \( \theta \) and the component of magnetic field normal to the spatial boundary surface are both nonvanishing. Thus, it seems that off-shell properties of these two theories are not identical.

5 Equations of Motion

Let us derive the equations of motion for our novel vector-tensor theory (2.18). For the variation of the total action we get

\[ \delta S = \frac{1}{2} \int d^4 x \sqrt{-g} \left( T_{\mu\nu} - G_{\mu\nu} \right) \delta g^{\mu\nu} + \text{boundary terms} , \]

where \( G_{\mu\nu} \) is the Einstein tensor for the physical metric \( g_{\mu\nu} \) and the energy momentum tensor of matter is defined as usual with respect to the physical metric

\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} . \]
The variation of the contravariant physical metric gives
\[ \delta g^{\mu \nu} = \frac{6}{\varphi^2} \cdot \delta h^{\mu \nu} - 2g^{\mu \nu} \cdot \frac{\delta \varphi}{\varphi}, \] (5.3)
where the variation of the scalar can be expressed from (2.19) as
\[ 4F_{\alpha \beta} \tilde{F}^{\alpha \beta} \cdot \frac{\delta \varphi}{\varphi} = \delta \left( F_{\alpha \beta} \tilde{F}^{\alpha \beta} \right) = 4F_{\alpha \beta}^{\mu \nu} \nabla^h A_{\alpha} \nabla^h \delta A_{\beta} + \frac{1}{2} F_{\alpha \beta} \tilde{F}^{\alpha \beta} h_{\mu \nu} \delta h^{\mu \nu}. \] (5.4)

Then from (5.1) and (5.3), using the Bianchi identity, one obtains the equation of motion for the gauge potential
\[ \frac{1}{\sqrt{-h}} \delta S \delta A_{\nu} = \tilde{F}^{\mu \nu} \partial_{\mu} \left( T - G \right) = 0, \] (5.5)
where \( T = T_{\alpha \beta} g^{\alpha \beta} \) and \( G = G_{\alpha \beta} g^{\alpha \beta} \). The Hodge dual of the field tensor has an inverse provided \( F_{\alpha \beta} \tilde{F}^{\alpha \beta} \neq 0 \), for a recent discussion see [23]. Hence the equation of motion for the gauge field implies
\[ \partial_{\mu} \left( T - G \right) = 0. \] (5.6)

The equation of motion for the auxiliary metric reads
\[ \frac{1}{\sqrt{-g}} \delta S \delta h^{\alpha \beta} = 3 \varphi^2 \left[ T_{\alpha \beta} - G_{\alpha \beta} - \frac{1}{4} \left( T - G \right) g_{\alpha \beta} \right] = 0. \] (5.7)

Hence the equation of motion for the metric \( h_{\mu \nu} \) alone directly reproduces the trace-free part of the Einstein equations
\[ G_{\alpha \beta} - T_{\alpha \beta} - \frac{1}{4} g_{\alpha \beta} \left( G - T \right) = 0. \] (5.8)

We would like to stress again that these equations are clearly symmetric with respect to the vacuum shifts of the energy-momentum tensor for the matter fields
\[ T_{\mu \nu} \rightarrow T_{\mu \nu} + \Lambda g_{\mu \nu}, \] (5.9)
a property which was considered to be desirable already by Einstein [1] more than a century ago. For more recent discussions see e.g. [3, 4, 7].

Interestingly, the equation of motion for the gauge field \( A_{\mu} \) is a direct consequence of the equations of motion for the tensor \( h_{\mu \nu} \) and the equations of motion of the matter. This looks like a direct consequence of the second Noether theorem, see e.g. discussion in [25].

Crucially, both equations of motion (5.5) and (5.8) are second order PDE when written in terms of the manifestly Weyl-invariant composite metric \( g_{\mu \nu} \). However, considered as an equation of motion for original dynamical variables, (5.5) has fourth derivatives of \( A_{\mu} \) and third of \( h_{\mu \nu} \), while the trace-free part of the \( g \)–Einstein equations (5.8) has up to third derivatives of these original variables \( A_{\mu} \).

The resulting traceless Einstein equations correspond to those of the so-called \textit{unimodular} gravity. From the point of view of classical physics the only difference from standard GR is that the cosmological constant is an \textit{integration constant}. Indeed, integrating the equation of motion (5.5) for the gauge field one obtains
\[ G - T = 4\Lambda = \text{const}, \] (5.10)
which one can substitute into (5.8). However, from the point of view of quantum mechanics the cosmological constant is now promoted to an operator. Consequently, the observed value of \( \Lambda \) (in generic quantum state of the universe) will have quantum fluctuations, which are per definition impossible for a fixed constant of nature.
6 Gauge Invariant Variables and Scalar-Vector-Tensor Formulation

Now we can follow a similar procedure as in [26] and [13] and promote the scalar \( \varphi \) to an independent dynamical variable in order to eliminate the higher derivatives from the action. On this path we introduce a Lagrange multiplier, \( \lambda \), enforcing the definition (2.19) so that the action (2.18) transforms into

\[
S[h, \varphi, A, \lambda] = \int d^4x \sqrt{-h} \left[ -\frac{1}{2} (\partial \varphi)^2 - \frac{1}{12} \varphi^2 R(h) - \frac{\lambda}{72} \varphi^4 + \frac{\lambda}{2} \cdot F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right].
\]  

(6.1)

Hence, we have rewritten (2.18) as a scalar-vector-tensor theory in this way. This theory should be Weyl-invariant, as it was the case with the original action (2.18). This requirement forces the Lagrange multiplier \( \lambda \) to be invariant under the Weyl transformations. All other matter fields are coupled to the scalar \( \varphi \) through the physical metric

\[
g_{\mu\nu} = \frac{\varphi^2}{6} \cdot h_{\mu\nu}.
\]  

(6.2)

As we have already noticed in our previous work [13], the first three terms correspond to the Dirac theory of the Weyl-invariant gravity [27], see also [28]. These terms are also used in the so-called Conformal Inflation [29]. The auxiliary scalar field \( \varphi \) has a ghost-like kinetic term. The would be coupling constant \( \lambda \) is actually a Lagrange multiplier field which has a crucial axion-like coupling.

The form of the action is also closely related to those theories studied in [30] which are, however, only invariant with respect to transverse diffeomorphisms preserving the value of the metric determinant.

The form of the action suggests that the Weyl symmetry in this setup is in a sense empty (or as sometimes called fake or sham) and does not have any dynamical consequence, see e.g. [31–33].

The dynamical variables \( \{h_{\mu\nu}, A_\mu, \lambda, \varphi\} \) transform as

\[
\begin{align*}
  h_{\mu\nu} &= \Omega^2 (x) h'_{\mu\nu}, \\
  \varphi &= \Omega^{-1} (x) \varphi', \\
  A_\mu &= A'_\mu, \\
  \lambda &= \lambda'.
\end{align*}
\]  

(6.3)

Instead, of these dynamical variables one can introduce a new set of independent dynamical variables \( \{g_{\mu\nu}, A_\mu, \Lambda, \varphi\} \), where

\[
\begin{align*}
  g_{\mu\nu} &= \frac{\varphi^2}{6} \cdot h_{\mu\nu}, \\
  \Lambda &= \frac{\lambda}{2},
\end{align*}
\]  

(6.4)

are gauge invariant. These nonsingular field-redefinitions resemble the Weyl transformations with \( \Omega^2 = \varphi^2/6 \), except we do not reduce the dimensionality of the phase space, as \( \varphi \) is not affected by this field-redefinition.

Performing this field redefinition in (6.1) one rewrites the action as

\[
S[g, A, \Lambda, \Phi_m] = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R(g) + \Lambda \left( F_{\alpha\beta} \tilde{F}^{\alpha\beta} - 1 \right) \right] + S_m[g, \Phi_m],
\]  

(6.5)
where the matter is now minimally coupled to gravity and the Hodge dual $F^{*\alpha\beta}$ is now defined with respect to the physical metric $g_{\mu\nu}$, see (3.5). This action functional does not depend anymore on the conformal factor $\varphi$, but only on Weyl-invariant dynamical variables $\{g_{\mu\nu}, A_{\mu}, \Lambda\}$. Clearly the Lagrange multiplier $\Lambda$ has an axion-like coupling. However, contrary to a normal axion there is no kinetic term.

The corresponding equations of motion are: the constraint

$$F_{\alpha\beta} F^{*\alpha\beta} = 1,$$  \hspace{1cm} (6.6)

which for non-Weyl-invariant dynamical variables was just built-in off-shell, see (3.6) in section (3), while the variation with respect to the gauge field yields

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta A_\gamma} = -4\nabla_{\mu} \left( \Lambda E^{\alpha\beta\mu\gamma} \nabla_{\alpha} A_{\beta} \right) = 4 F^{*\gamma\mu} \partial_\mu \Lambda,$$  \hspace{1cm} (6.7)

and the Einstein equations are

$$\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}} = T_{\alpha\beta} + \Lambda g_{\alpha\beta} - G_{\alpha\beta} = 0.$$  \hspace{1cm} (6.8)

The constraint (6.6) enforces that $\tilde{F}^{\gamma\mu}$ is invertible so that $\Lambda = \text{const}$ which also follows from the Bianchi identity and the conservation of the total energy-momentum. Clearly these equations are equivalent to those obtained from the original action, (5.5) and (5.8).

One possible interpretation of the action (6.5), as well as the action of the Henneaux-Teitelboim generally-covariant formulation [11] of the so-called “unimodular gravity” is that they simultaneously describe all de Sitter and anti de Sitter universes. Interestingly, the dynamics of these systems is essentially mechanical and not field-theoretical, as the degrees of freedom are global and correspond to the cosmic time and the canonically conjugated cosmological constant.

7 Faddeev-Jackiw Procedure and Degrees of Freedom

A proper Hamiltonian analysis of the system would take too much space in this short paper, and will be reported elsewhere. Indeed, via the Dirac procedure this analysis can be quite involved and deserved a separate study in cases of the mimetic gravity e.g. [34] and of the unimodular gravity e.g. [35]. Therefore, here we will only provide a hint for the structure of the canonical degrees of freedom in this theory written in the Weyl-invariant variables (6.5). For this simplified analysis we will follow the Faddeev-Jackiw procedure [36, 37]. It is sufficient to consider the “constraint part of the action” (6.5) only. The corresponding Lagrangian density takes the form

$$\mathcal{L} = -\sqrt{-g} \Lambda + 2\Lambda \partial_0 \left( e^{ijklm} A_i \partial_k A_m \right) + 2 \partial_k \left( e^{k\beta\mu\nu} A_{\beta} \partial_\mu A_\nu \right),$$  \hspace{1cm} (7.1)

where the latin indices are purely spatial and run from 1 to 3. Furthermore, we can introduce a decomposition

$$A_i = \partial_i \chi + A_i^T,$$  \hspace{1cm} (7.2)

so that a U(1) gauge transformation leaves $A_i^T$ invariant and shifts $\chi$ as

$$\chi = \chi' + \theta,$$  \hspace{1cm} (7.3)
so that another invariant object is $A_0 - \chi$.

It is useful to introduce a "magnetic field"

$$B^i = \epsilon^{ikm} \partial_k A^T_m.$$  \hfill (7.4)

Substituting the decomposition into the Lagrangian, after some algebra, we obtain

$$\mathcal{L} = -\sqrt{-g} \Lambda + 2 \Lambda \left[ \partial_0 \left( B^i A^T_i \right) + 2 B^i \partial_i (\chi - A_0) + B^i A^T_i - A^T_i B^i \right].$$  \hfill (7.5)

In this Lagrangian $\chi$ and $A_0$ only enter as a gauge invariant combination $\chi - A_0$. Similarly to standard electrodynamics, $A_0$ enforces a constraint while a variation with respect to $\chi$ is superfluous, as it only enforces the time derivative of this constraint to vanish. Notably, the time derivatives enter this action only linearly, for this reason this action should be considered as a Hamiltonian action. Integrating the last two terms by parts and neglecting the boundary terms one obtains

$$\int d^3x \left( B^i A^T_i - A^T_i B^i \right) = -\int d^3x \epsilon^{ikm} A^T_i A^T_m \partial_k \Lambda.$$  \hfill (7.6)

Covariant equations of motion for the gauge potential (6.7) imply the constraint that the vacuum energy field is constant in space: $\partial_i \Lambda = 0$. Hence, if we follow the Faddeev-Jackiw procedure and plug in this constraint into the action we see that (7.6) is vanishing. Moreover, this implies that in the action one can move the global degree of freedom $\Lambda (t)$ out of the spatial integration

$$S = \int dt d^3x \mathcal{L} = \int dt 2 \Lambda \int d^3x \left[ \partial_0 \left( B^i A^T_i \right) + 2 B^i \partial_i (\chi - A_0) - \frac{1}{2} \sqrt{-g} \right].$$  \hfill (7.7)

Comparing this result with the usual Hamiltonian form of the action

$$S = \int dt \left( p \dot{q} - H (p, q) \right),$$  \hfill (7.8)

provides a hint (if not a strict proof) that $\Lambda$ is canonically conjugated to the Chern-Simons charge

$$2 \int d^3x B^i A^T_i = \int d^3x \sqrt{-g} C^i_T,$$  \hfill (7.9)

where $C^i_T$ is the U(1)-invariant time component of the of the Chern-Simons current, which is now constructed out of gauge-invariant $A^T_i$ instead of $A_i$. As we have already discussed in (4) small U(1) transformations do not change the value of the Chern-Simons charge, see (4.5). Thus the cosmic time is given by the Chern-Simons charge which is a quantity canonically conjugated to the cosmological constant. This resembles a construction from [38–40], where the Chern-Simons charge also appeared as time, but it was composed out of the imaginary part of the complex Ashtekar connection. Instead here we use a real connection $A_\mu$ which is a four covariant vector that we attribute to the matter sector.

**8 Non-abelian Generalization**

It is very easy to generalize this theory to any non-abelian gauge symmetry with an SU(N) gauge group. The motivation for this generalization is threefold. First, the very structure
of vacuum in the non-abelian theories can be highly involved in IR and due to confinement can possess nontrivial global degrees of freedom, see e.g. recent discussions in [41–43]. On the other hand, axionic couplings like the one in (6.5) are more typical for non-abelian field theories. And finally, contrary to the original system [13] with the unusual vector field $V^\mu$, here one can easily introduce standard couplings of the otherwise completely auxiliary gauge field $A^\mu$ to matter. In that case the structure of the theory will change, though in some regimes it can still approximate the cosmological constant. However, with the coupling to matter the gauge field configurations become relevant and have to respect the cosmological principle. Yet, it would not be possible to satisfy homogeneity and isotropy in case of just one abelian field, with nonvanishing magnetic and electric fields. The way out could be either to average over many fields like e.g. in vector inflation [44] or to introduce a triplet of mutually orthogonal vector fields like in [45, 46], or to invoke a non-abelian field theory, as in [47, 48]. The later was recently applied in a different mimetic construction in [49–51]. Especially interesting is the SU(2) case, as there three independent components of the gauge potential can play the role of the aforementioned triplet.

As it is common in the Yang-Mills theories, the non-abelian field can be expanded into generators $A^\mu = A^c_\mu T^c$, and the covariant derivative $D^\mu = \partial^\mu + ig A^\mu$ with the self-coupling constant $g$ yields a curvature

$$F^\mu_\nu = D^\mu A_\nu - D^\nu A_\mu = \partial^\mu A_\nu - \partial^\nu A_\mu + ig [A^\mu, A_\nu],$$

with the corresponding Hodge dual given by

$$\tilde{F}^{\mu\nu} = \frac{1}{2} E^{\mu\alpha\beta} F_{\alpha\beta}.$$

The ansatz (2.7) is then generalized to

$$g_{\mu\nu} = h_{\mu\nu} \cdot \sqrt{\text{Tr} F_{\alpha\beta} \tilde{F}^{\alpha\beta}}.$$

The corresponding Chern-Simons current is

$$C^\mu = \text{Tr} E^{\mu\alpha\beta\gamma} \left( F_{\alpha\beta} A_\gamma - \frac{2ig}{3} A_\alpha A_\beta A_\gamma \right),$$

so that (2.16) remains valid. The most important formulas (6.1) and (6.5) can be generalized by a simple substitution $F_{\alpha\beta} \tilde{F}^{\alpha\beta} \rightarrow \text{Tr} F_{\alpha\beta} F^{\alpha\beta}$ (or $\text{Tr} F_{\alpha\beta} F^{*\alpha\beta}$ when going to Weyl-invariant variables), in particular

$$S[g, A, \Lambda, \Phi_m] = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R(g) + \Lambda \left( \text{Tr} F_{\alpha\beta} F^{*\alpha\beta} - 1 \right) \right] + S_m [g, \Phi_m].$$

Clearly, the invariant $\sqrt{-g} \text{Tr} F_{\alpha\beta} F^{*\alpha\beta}$ is metric-independent so that the Einstein equations remain (6.8). As a trivial consequence of the Bianchi identity and the covariant conservation of the energy-momentum one obtains vacuum energy as an integration constant $\Lambda = \text{const}$.

This action literally realizes the axionic coupling of the cosmological constant envisioned long time ago by Wilczek in [15]: “I would like to briefly mention one idea in this regard, that I am now exploring. It is to do something for the $A$-parameter very similar to what the axion does for the $\theta$-parameter in QCD, another otherwise mysteriously tiny quantity. The basic idea is to promote these parameters to dynamical variables, and then see if perhaps
small values will be chosen dynamically.” However, here we do not discuss any dynamical selection mechanism.

There are two important differences of the non-abelian construction comparing to the abelian case described before. First of all, in the later, a nonvanishing $F_{\alpha\beta} \tilde{F}^{\alpha\beta}$ implies the Lorentz symmetry breaking, with both electric and magnetic field nonvanishing and not orthogonal to each other, whereas in the non-abelian case a nonvanishing $\text{Tr} F_{\alpha\beta} F^{\ast\alpha\beta}$ does not necessarily lead to any Lorentz violation.

One can speculate that the growing cosmic time can correspond to a growing winding number or growing complexity of the vacuum.

Another important distinction is that for the abelian field, there was no dimensional parameter entering the construction. However, here after a proper normalization, $F_{\mu\nu}$ is dimensionless so that $A_\mu$ has dimensions of length, and the coupling constant has dimensions $g = M^2$. Otherwise, one can canonically normalize the gauge fields and have the dimensionless coupling constant, however, in that case, one has to divide the expression under the square root in the ansatz (8.3) by the same $M^4$. Hence, an appearance of a mass scale $M$ is unavoidable.

Another intriguing conjecture is that action (8.5) can appear from a strongly coupled gauge field and a strongly coupled axion $\Lambda$ with a potential

$$ V(\Lambda) = M^4 \sin \left( \Lambda / M^4 \right) = \Lambda - \frac{1}{6} M^4 \left( \frac{\Lambda}{M^4} \right)^3 + \ldots, \quad (8.6) $$

expanded not around a minimum, but around zero under the assumption that $M \gg \Lambda$. In this case, the first term in the expansion generates the constraint (6.6). On the other hand the strong coupling does not allow $A_\mu$ and $\Lambda$ to have a wave-like propagation. This suggests that, in the strong-coupling regime, one can neglect both usual kinetic terms $\text{Tr} F_{\alpha\beta} F^{\ast\alpha\beta}$ and $(\partial \Lambda)^2$. The latter is the main condition for this approximation. This is somehow reminiscent to the dynamical regime which happens in completions of mimetic dark matter and k-essence /$P(X)$ by a globally-charged $U(1)$ scalar field, see e.g. [49–53]. There one can also neglect the kinetic term of the radial field. In the next section we pursue this analogy a bit deeper.

9 Nonlinear Extension

Of course, it is unusual to expand the potential (8.6) away from its minimum. Therefore, a natural question is whether one can neglect $\text{Tr} F_{\alpha\beta} F^{\ast\alpha\beta}$ and $(\partial \Lambda)^2$ on configurations around a minimum of the potential$^6$ $V(\Lambda)$ or maybe at some other point with nonvanishing $\Lambda$. Suppose this is the case. In that case the action will take the form

$$ S[g, A, \Lambda] = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R(g) + \Lambda \text{Tr} F_{\alpha\beta} F^{\ast\alpha\beta} - V(\Lambda) \right]. \quad (9.1) $$

In the action (9.1) the term $\sqrt{-g} \text{Tr} F_{\alpha\beta} F^{\ast\alpha\beta}$ is still metric-independent and does not contribute to the Einstein equations, while the last term adds to those only the CC contribution

$$ T_{\mu\nu} = V(\Lambda) g_{\mu\nu}, \quad (9.2) $$

which implies that $\partial_\mu \Lambda = 0$, because of

$^6$For this discussion, the form of the potential is not important and should not necessarily be (8.6). It may be even not periodic anymore.
For the theory (9.1), the axion field $\Lambda$ is not a Lagrange multiplier anymore, but rather an auxiliary field. This implies that one can integrate it out using the relation

$$\text{Tr} F_{\alpha\beta} F^{\alpha\beta} = V',$$  \hspace{0.5cm} (9.3)

provided $V'' \neq 0$, where prime denotes a derivative with respect to $\Lambda$. In that case we reduce the number of dynamical variables and obtain

$$S[g, A] = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R(g) + f \left( \text{Tr} F_{\alpha\beta} F^{\alpha\beta} \right) \right],$$  \hspace{0.5cm} (9.4)

where the function $f$ is given by a Legendre transformation

$$f \left( \text{Tr} F_{\alpha\beta} F^{\alpha\beta} \right) = \Lambda V' - V,$$  \hspace{0.5cm} (9.5)

where $\Lambda = \Lambda \left( \text{Tr} F_{\alpha\beta} F^{\alpha\beta} \right)$ is a solution of (9.3). On the other hand, one could take (9.4) with an arbitrary function $f$ as a starting point. In that case, the corresponding cosmological constant term would be still equal to (9.2), but written as

$$T_{\mu\nu} = \left( \text{Tr} F_{\alpha\beta} F^{\alpha\beta} f' - f \right) g_{\mu\nu}.$$  \hspace{0.5cm} (9.6)

Following our analogy between $W^\mu$ and the Chern-Simons current one can also write

$$S[g, W] = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R(g) + f \left( \nabla^\mu g^\mu \right) \right],$$  \hspace{0.5cm} (9.7)

instead of

$$S[g, W, \Lambda] = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R(g) + \Lambda \nabla^\mu g^\mu - V(\Lambda) \right],$$  \hspace{0.5cm} (9.8)

and check that (on equations of motion for $W^\mu$) both these actions also reproduce a cosmological constant term

$$T_{\mu\nu} = \left( \nabla^\mu g^\nu f' - f \right) g_{\mu\nu} = V(\Lambda) g_{\mu\nu}.$$  \hspace{0.5cm} (9.9)

Clearly, for $V(\Lambda) = \Lambda$, the action (9.8) corresponds to the original Henneaux and Teitelboim [11] construction for the “unimodular gravity”. The equations of motion both these systems imply $\nabla_\mu W^\mu = \text{const}$, which is up to a trivial rescaling reproduce the constraint equation in the original HT [11] action. Interestingly, a similar formulation of the “unimodular gravity” that was slightly more complicated than just (9.7) and (9.8), was given in [54].

Thus all these actions: the original Henneaux and Teitelboim [11] and (9.8), (9.7), (9.4), (9.1) and (8.5) along with our Weyl-invariant constructions (2.18) and [13] describe the cosmological constant as an integration constant which is also their global degree of freedom. A new feature that the formulations (9.8), (9.7), (9.4), (9.1) with free functions $f$ or $V$ can achieve is that these functions can be bounded in some range. This boundedness of $V$ or $f$ would imply that, for arbitrary values of $\Lambda$ (or $\nabla_\mu W^\mu$ or $\text{Tr} F_{\alpha\beta} F^{\alpha\beta}$), the resulting cosmological constant is limited to be not arbitrary, but to lie in some range. This feature is very interesting for theories where $\Lambda$ (or $\nabla_\mu W^\mu$ or $\text{Tr} F_{\alpha\beta} F^{\alpha\beta}$) are stochastic variables.

It is also worthwhile comparing the above transition from the Lagrange multiplier formulation to the auxiliary field with the similar transition in mimetic dark matter and k-essence, see e.g. [49–53]. There the theory with Lagrangian density

$$\mathcal{L} = \lambda \left( (\partial \phi)^2 - 1 \right),$$  \hspace{0.5cm} (9.10)
corresponds to a fluid-like dust with an identically vanishing sound speed, whereas
\[ \mathcal{L} = \lambda (\partial \phi)^2 - V(\lambda), \] (9.11)
represents k-essence with a nonvanishing speed of sound. Thus, there the promotion of the Lagrange multiplier to an auxiliary field substantially changes the theory. Notably, the fluid-like dust is just a singular case, which in the completion [49–53] is only realized under an extreme fine-tuning of identically vanishing self-interaction.

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References

[1] A. Einstein, “Spielen Gravitationsfelder im Aufbau der materiellen Elementarteilchen eine wesentliche Rolle?,” Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1919 (1919) 349–356. https://einsteinpapers.press.princeton.edu/vol7-doc/178.

[2] S. Weinberg, “The Cosmological Constant Problem,” Rev. Mod. Phys. 61 (1989) 1–23.

[3] G. F. R. Ellis, H. van Elst, J. Murugan, and J.-P. Uzan, “On the Trace-Free Einstein Equations as a Viable Alternative to General Relativity,” Class. Quant. Grav. 28 (2011) 225007, arXiv:1008.1196 [gr-qc].

[4] G. F. R. Ellis, “The Trace-Free Einstein Equations and inflation,” Gen. Rel. Grav. 46 (2014) 1619, arXiv:1306.3021 [gr-qc].

[5] A. D. Linde, “The Inflationary Universe,” Rept. Prog. Phys. 47 (1984) 925–986.

[6] A. Linde, “A brief history of the multiverse,” Rept. Prog. Phys. 80 no. 2, (2017) 022001, arXiv:1512.01203 [hep-th].

[7] T. Padmanabhan, “Gravity and Quantum Theory: Domains of Conflict and Contact,” arXiv:1909.02015 [gr-qc].

[8] J. Martin, “Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask),” Comptes Rendus Physique 13 (2012) 566–665, arXiv:1205.3365 [astro-ph.CO].
[9] C. P. Burgess, “The Cosmological Constant Problem: Why it’s hard to get Dark Energy from Micro-physics,” in Proceedings, 100th Les Houches Summer School: Post-Planck Cosmology: Les Houches, France, July 8 - August 2, 2013, pp. 149–197. 2015. arXiv:1309.4133 [hep-th].

[10] A. Padilla, “Lectures on the Cosmological Constant Problem,” arXiv:1502.05296 [hep-th].

[11] M. Henneaux and C. Teitelboim, “The Cosmological Constant and General Covariance,” Phys. Lett. B222 (1989) 195–199.

[12] W. Buchmuller and N. Dragon, “Einstein Gravity From Restricted Coordinate Invariance,” Phys. Lett. B207 (1988) 292–294.

[13] P. Jiroušek and A. Vikman, “New Weyl-invariant vector-tensor theory for the cosmological constant,” JCAP 1904 (2019) 004, arXiv:1811.09547 [gr-qc].

[14] A. H. Chamseddine and V. Mukhanov, “Mimetic Dark Matter,” JHEP 11 (2013) 135, arXiv:1308.5410 [astro-ph.CO].

[15] F. Wilczek, “Foundations and Working Pictures in Microphysical Cosmology,” Phys. Rept. 104 (1984) 143.

[16] I. Kimpton and A. Padilla, “Cleaning up the cosmological constant,” JHEP 12 (2012) 031, arXiv:1203.1040 [hep-th].

[17] L. Heisenberg, “A systematic approach to generalisations of General Relativity and their cosmological implications,” Phys. Rept. 796 (2019) 1–113, arXiv:1807.01725 [gr-qc].

[18] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, “Modified Gravity and Cosmology,” Phys. Rept. 513 (2012) 1–189, arXiv:1106.2476 [astro-ph.CO].

[19] M. A. Gorji, S. Mukohyama, H. Firouzjahi, and S. A. Hosseini Mansoori, “Gauge Field Mimetic Cosmology,” JCAP 1808 no. 08, (2018) 047, arXiv:1807.06335 [hep-th].

[20] M. A. Gorji, S. Mukohyama, and H. Firouzjahi, “Cosmology in Mimetic SU(2) Gauge Theory,” JCAP 1905 (2019) 019, arXiv:1903.04845 [gr-qc].

[21] M. Zumalacárregui and J. García-Bellido, “Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian,” Phys. Rev. D89 (2014) 064046, arXiv:1308.4685 [gr-qc].

[22] N. Deruelle and J. Rua, “Disformal Transformations, Veiled General Relativity and Mimetic Gravity,” JCAP 1409 (2014) 002, arXiv:1407.0825 [gr-qc].

[23] A. De Felice and A. Naruko, “On metric transformations with a $U(1)$ gauge field,” arXiv:1911.10960 [gr-qc].

[24] A. E. Gumrukcuoglu and R. Namba, “Role of matter in gravitation: going beyond the Einstein-Maxwell theory,” Phys. Rev. D100 no. 12, (2020) 124064, arXiv:1907.12292 [hep-th].

[25] K. Brading and H. R. Brown, “Noether’s theorems and gauge symmetries,” arXiv:hep-th/0009058 [hep-th].

[26] K. Hammer and A. Vikman, “Many Faces of Mimetic Gravity,” arXiv:1512.09118 [gr-qc].

[27] P. A. M. Dirac, “Long range forces and broken symmetries,” Proc. Roy. Soc. Lond. A333 (1973) 403–418.

[28] S. Deser, “Scale invariance and gravitational coupling,” Annals Phys. 59 (1970) 248–253.

[29] R. Kallosh and A. Linde, “Universality Class in Conformal Inflation,” JCAP 1307 (2013) 002, arXiv:1306.5220 [hep-th].

[30] E. Alvarez, D. Blas, J. Garriga, and E. Verdaguer, “Transverse Fierz-Pauli symmetry,” Nucl. Phys. B756 (2006) 148–170, arXiv:hep-th/0606019 [hep-th].
[31] N. C. Tsamis and R. P. Woodard, “No New Physics in Conformal Scalar - Metric Theory,” *Annals Phys.* **168** (1986) 457.

[32] R. Jackiw and S.-Y. Pi, “Fake Conformal Symmetry in Conformal Cosmological Models,” *Phys. Rev.* **D91** no. 6, (2015) 067501, arXiv:1407.8545 [gr-qc].

[33] I. Oda, “Fake Conformal Symmetry in Unimodular Gravity,” *Phys. Rev.* **D94** no. 4, (2016) 044032, arXiv:1606.01571 [gr-qc].

[34] A. Ganz, P. Karmakar, S. Matarrese, and D. Sorokin, “Hamiltonian analysis of mimetic scalar gravity revisited,” *Phys. Rev.* **D99** no. 6, (2019) 064009, arXiv:1812.02667 [gr-qc].

[35] A. Ganz, P. Karmakar, S. Matarrese, and D. Sorokin, “Hamiltonian analysis of mimetic scalar gravity revisited,” *Phys. Rev.* **D99** no. 6, (2019) 064009, arXiv:1812.02667 [gr-qc].

[36] L. D. Faddeev and R. Jackiw, “Hamiltonian Reduction of Unconstrained and Constrained Systems,” *Phys. Rev. Lett.* **60** (1988) 1692–1694.

[37] R. Jackiw, “(Constrained) quantization without tears,” in *Diverse topics in theoretical and mathematical physics*, pp. 163–175. 1993. arXiv:hep-th/9306075 [hep-th].

[38] L. Smolin and C. Soo, “The Chern-Simons invariant as the natural time variable for classical and quantum cosmology,” *Nucl. Phys.* **B449** (1995) 289–316, arXiv:gr-qc/9405015 [gr-qc].

[39] S. Alexander, J. Magueijo, and L. Smolin, “The Quantum Cosmological Constant,” *Symmetry* **11** no. 9, (2019) 1130, arXiv:1807.01381 [gr-qc].

[40] J. Magueijo and L. Smolin, “A Universe that does not know the time,” arXiv:1807.01520 [gr-qc].

[41] G. Dvali, “Three-form gauging of axion symmetries and gravity,” arXiv:hep-th/0507215 [hep-th].

[42] G. Dvali, R. Jackiw, and S.-Y. Pi, “Topological mass generation in four dimensions,” *Phys. Rev. Lett.* **96** (2006) 081602, arXiv:hep-th/0511175 [hep-th].

[43] G. Dvali, “A Vacuum accumulation solution to the strong CP problem,” *Phys. Rev.* **D74** (2006) 025019, arXiv:hep-th/0510053 [hep-th].

[44] A. Golovnev, V. Mukhanov, and V. Vanchurin, “Vector Inflation,” *JCAP* **0806** (2008) 009, arXiv:0802.2068 [astro-ph].

[45] C. Armendariz-Picon, “Could dark energy be vector-like?,” *JCAP* **0407** (2004) 007, arXiv:astro-ph/0405267 [astro-ph].

[46] M. C. Bento, O. Bertolami, P. V. Moniz, J. M. Mourao, and P. M. Sa, “On the cosmology of massive vector fields with SO(3) global symmetry,” *Class. Quant. Grav.* **10** (1993) 285–298, arXiv:gr-qc/9302034 [gr-qc].

[47] D. V. Gal'tsov and M. S. Volkov, “Yang-Mills cosmology: Cold matter for a hot universe,” *Phys. Lett.* **B256** (1991) 17–21.

[48] Y. Hosotani, “Exact Solution to the Einstein Yang-Mills Equation,” *Phys. Lett.* **147B** (1984) 44–46.

[49] E. Babichev and S. Ramazanov, “Caustic free completion of pressureless perfect fluid and k-essence,” *JHEP* **08** (2017) 040, arXiv:1704.03367 [hep-th].

[50] E. Babichev, S. Ramazanov, and A. Vikman, “Recovering P(X) from a canonical complex field,” arXiv:1807.10281 [gr-qc]. [JCAP1811,no.11,023(2018)].

[51] A. J. Tolley and M. Wyman, “The Gelaton Scenario: Equilateral non-Gaussianity from multi-field dynamics,” *Phys. Rev.* **D81** (2010) 043502, arXiv:0910.1853 [hep-th].

[52] D. T. Son, “Hydrodynamics of relativistic systems with broken continuous symmetries,” *Int. J. Mod. Phys.* **A16S1C** (2001) 1284–1286, arXiv:hep-ph/0011246 [hep-ph].
[53] D. T. Son, “Low-energy quantum effective action for relativistic superfluids,”
arXiv:hep-ph/0204199 [hep-ph].

[54] A. Padilla and I. D. Saltas, “A note on classical and quantum unimodular gravity,” Eur. Phys. J. C75 no. 11, (2015) 561, arXiv:1409.3573 [gr-qc].