Multiparticle amplitudes at one-loop: an algebraic/numeric approach

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We discuss algebraic/numeric methods to compute one-loop corrections for multiparticle/jet production cross sections. By using efficient reduction algorithms a compact expression for the $gg\gamma\gamma \rightarrow 0$ amplitude is obtained. Further a numerical approach for 6-point 1-loop diagrams is presented.

1. INTRODUCTION

The theoretical description of multi-particle production at the one-loop level is a very challenging task, as the complexity of the Feynman diagrammatic approach grows exponentially with the number of external partons. No Standard Model process which has generic $2 \rightarrow 4$ kinematics is computed at the one-loop level although this is highly relevant for many Higgs boson search channels at the LHC, like gluon fusion and weak boson fusion, where additional jets have to be tagged to improve the signal to background ratio.

For signal reactions like $PP \rightarrow H + 0, 1, 2$ jets, with $H \rightarrow \gamma\gamma, WW^*, \tau^+\tau^-$ which are available at one-loop level, many backgrounds remain to be calculated. As an example for needed calculations consider $PP \rightarrow b\bar{b}b + X$, $PP \rightarrow \gamma\gamma + 2$ jets $+ X$ or $PP \rightarrow ZZ + \gamma\gamma + X$, which require the evaluation of hexagon graphs like the ones given in Fig. 1.

The computation of the related amplitudes relies on efficient methods for the evaluation of the corresponding Feynman graphs. In the next section we shortly review our reduction formalism. As an example for the efficiency of our methods we discuss the 5-point 1-loop amplitude $gg \rightarrow \gamma\gamma g$ in Section 3. It seems to be feasible to apply the presented techniques also for 6-point processes, as long as the internal masses of the problem can be neglected. In [1,2] we have shown that in the massless Yukawa model our formalism leads to compact expressions. Going to the massive case leads generally to much more involved expressions and in that case numerical methods seem to be preferable. An approach for the numerical evaluation of 6-point Feynman diagrams is outlined in Section 4.

2. REDUCTION FORMALISM

In the Feynman diagrammatic approach any one-loop amplitude can be represented as a linear combination of factors which contain the group theoretical information and tensor one-loop integrals:

$$\Gamma^{\{c\},\{\lambda\}}(p_j, m_j) = \sum_{\{c_i\}} f^{\{c_i\}} G^{\{\lambda\}}$$

where

$$G^{\{\lambda\}} = \int \frac{d^n k}{i \pi^{n/2}} \left( \frac{N^{\{\lambda\}}}{q_1^2 - m_1^2} \ldots (q_N^2 - m_N^2) \right)$$

Figure 1. Typical Hexagon graph for the 1-loop amplitudes $gg \rightarrow b\bar{b}b$ and $gg \rightarrow ZZ\gamma\gamma$. 

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In \[3\] we have derived a reduction formula for such maps rank \(N\) parts in parameter space. The derived formula can be expressed in terms of \(n = 4 - 2\epsilon\) dimensional bubble and triangle functions and \(n + 2\) dimensional boxes. More details on reduction formalisms can be found in \[31\].

### 3. THE LOOP AMPLITUDE \(gg \rightarrow \gamma\gamma g\)

To give an example for our algebraic approach we have considered the 5-point 1-loop amplitude \(gg \rightarrow \gamma\gamma g\). This amplitude is indirectly known from the 1-loop 5-gluon amplitude \[3\] by turning gluons into photons.

We define all particles as incoming.

\[
\gamma(p_1, \lambda_1) + \gamma(p_2, \lambda_2) + g(p_3, \lambda_3, c_3) + g(p_4, \lambda_4, c_4) + g(p_5, \lambda_5, c_5) \rightarrow 0
\]

(2)

(2)

In hadronic collisions this amplitude is relevant for the production of photon pairs in association with a jet and as such a contribution of the background to the Higgs boson search channel \(H \rightarrow \gamma + \text{jet}\). For a phenomenological analysis see \[78\]. The colour structure of this amplitude can be written as

\[
\Gamma^{(\lambda_1, c_1)}[\gamma gggg] \rightarrow 0 \quad \frac{Q^2_g g^3}{i\pi^2} \int_{\Delta c_1 c_5} A^{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5}
\]

\(A^{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5}\) are helicity dependent linear combinations of scalar integrals and a constant term which is a remnant of two-point functions with coefficients of order \((D - 4)\). Six independent helicity components exist: \(+++++, +++++-\), \(-++++-, +++++-\), \(-++++-\), \(-++++-\), \(-++++-\), \(-++++-\). As the amplitude is finite one expects that all 3-point functions which carry spurious infrared poles cancel. The function basis of the problem is thus reduced to 2-point functions

\[
I_2^{\Delta}(s_{ij}) = \frac{\Gamma(1 + \epsilon) \Gamma(1 - \epsilon)^2 (-s_{ij})^{-\epsilon}}{\Gamma(2 - 2\epsilon) \epsilon}
\]

4-point functions in 6 dimensions written as \[2\]

\[
F_1(s_{i_1 j_1}, s_{i_2 j_2}, s_{j_1 j_2}, s_{i_3 j_3}) = \frac{I_4^g(p_{j_1}, p_{j_2}, p_{j_3}, p_{j_4} + p_{j_5})}{s_{j_1 j_2}}
\]

and constant terms. From unitarity one expects that the \(+++++, +++++-\), \(-++++-\) amplitudes should be polynomial. The other helicity amplitudes will also contain non-polynomial functions.
like logarithms and dilogarithms contained in \( I_2^D \) and \( F_1 \).

To give an example for a compact helicity amplitude we show here the result for \( \mathcal{A}^{+-+-} \) only. The remaining ones which have also compact representations can be found in [5]. The result is expressed in terms of field strength tensors \( F_{\mu \nu}^a = p_0^a \epsilon_\mu^a - p_1^a \epsilon_\nu^a \) where \( \epsilon_\pm^a \) are the polarization vectors of the gluons and photons.

We split the result of \( \mathcal{A}^{+-+-} \) into three pieces with indices \( F, B, i \), which belong to the part proportional to 6-dimensional boxes \( F_1 \), a part containing bubble graphs \( I_2^D \), and a constant term, respectively.

\[
\mathcal{A}^{+-+-} = \mathcal{A}_F^{+-+-} + \mathcal{A}_B^{+-+-} + \mathcal{A}_i^{+-+-}
\]

We find

\[
\mathcal{A}_F^{+-+-} = \frac{\text{Tr}(F_1 F_2^+) \text{Tr}(F_3^+ F_4^+)}{s_{12}^2 s_{34}^2} \left[ C_{F}^{+-+-} p_1 \cdot F_5^+ \cdot p_3 - (3 \leftrightarrow 4) \right] F_1(s_{13}, s_{14}, s_{25})
\]

\[
- (4 \leftrightarrow 5) - (5 \leftrightarrow 3) + (1 \leftrightarrow 2)
\]

\[
- (1 \leftrightarrow 2, 4 \leftrightarrow 5) - (1 \leftrightarrow 2, 5 \leftrightarrow 3)
\]

\[
\mathcal{A}_B^{+-+-} = \frac{\text{Tr}(F_1^+ F_2^+) \text{Tr}(F_3 F_4^+)}{s_{12}^2 s_{34}^2} \left[ C_{B}^{+-+-} p_1 \cdot F_5^+ \cdot p_3 - (3 \leftrightarrow 4) \right] I_2^D(s_{15})
\]

\[
- (4 \leftrightarrow 5) - (5 \leftrightarrow 3) + (1 \leftrightarrow 2)
\]

\[
- (1 \leftrightarrow 2, 4 \leftrightarrow 5) - (1 \leftrightarrow 2, 5 \leftrightarrow 3)
\]

\[
\mathcal{A}_i^{+-+-} = \frac{\text{Tr}(F_1^+ F_2^+) \text{Tr}(F_3^+ F_4^+)}{s_{34} s_{45} s_{35}}
\]

The indicated permutations have to be applied to the indices of the field strength tensors, momenta and Mandelstam variables. The coefficients are

\[
C_{F}^{+-+-} = \frac{1}{2} \left[ s_{12} - 2 s_{13} s_{14} \right] - \frac{s_{14}}{s_{34}} - \frac{s_{14}}{s_{35}}
\]

\[
C_{B}^{+-+-} = \frac{s_{45}}{s_{15}} \left[ \frac{s_{13} + s_{35}}{s_{14} + s_{45}} + \frac{s_{14} + s_{45}}{s_{13} + s_{35}} \right] + \frac{s_{14}}{s_{15} s_{35}} + \frac{s_{14} s_{35}}{s_{15} s_{45}} + \frac{2 (s_{15} + s_{45})^2}{s_{35}^2}
\]

\[
- \frac{s_{14} s_{45}}{s_{15} s_{35}} + \frac{s_{14} + s_{45}}{s_{45}} + \frac{2 s_{14} (s_{15} + s_{45})}{s_{35}^2} + \frac{s_{12} - s_{14} - s_{35}}{s_{14} + s_{45}} + \frac{s_{25} s_{15}}{s_{35} (s_{13} + s_{35})} + \frac{2 s_{45} + s_{15}}{s_{13} + s_{35}}
\]

In the given expressions the \( S_2 \odot S_3 \) symmetry under exchange of the two photons and the three gluons is manifest after taking into account the omitted colour factor.

The result indicates that with our approach indeed a compact representation of complicated loop amplitudes can be obtained. The application of our approach to relevant 6-point amplitudes is presently under study.

4. NUMERICAL APPROACH

Due to the complexity of the analytic approach if massive particles are present, a numerical approach seems to be more appropriate to tackle different types of one-loop amplitudes in a unified and efficient way.

Recently a great activity in that direction with many new ideas can be observed [9,10,11,12].

4.1. Reduction to basic building blocks

As basic building blocks for an amplitude in our numeric approach, we choose scalar 2-point functions \( I_2^a \) and 3-point functions \( I_3^a \) and \( n + 2 \) dimensional box functions \( I_4^{n+2} \) with nontrivial numerators. The latter are infrared finite. Possible UV singularities are only contained in the 2-point functions and their subtraction is straightforward. The (soft and collinear) IR singularities are, as a result of the reduction, only contained in 2-point functions and 3-point functions with one or two light-like legs. In this form, they are easy to isolate and to subtract from the amplitude.

After reduction and separation of the divergent parts, we are left with finite integrals \( I_2^a(j_1, j_2, j_3) \) and \( I_4^{n+2}(j_1, j_2, j_3, j_4) \), with nontrivial numerators. As numerical stability problems are entirely from the denominators we discuss only the case of scalar integrals with trivial numerators here. Systematic methods for the combination of the IR divergences from the virtual corrections with their counterparts from the real emission contribution already exist ([13] and references therein).

In this section we focus on the evaluation of a
finite 6 point scalar integral. As a first step we reduce the hexagon integral to box and triangle functions which are the basic building blocks of the reduction.

4.2. Parameter representation of basic building blocks

To evaluate the box and triangle functions numerically, we first perform a sector decomposition.

\[ 1 = \Theta(x_1 > x_2, \ldots, x_N) + \Theta(x_2 > x_1, \ldots, x_N) + \ldots + \Theta(x_N > x_1, \ldots, x_{N-1}) \tag{3} \]

for the integration over \( N \) parameters (\( N = 3 \) for the triangle, \( N = 4 \) for the box). The step function \( \Theta \) is defined as 1 if the inequality of its argument is fulfilled, and 0 else. Now, we carry out one parameter integration explicitly. We show the explicit expressions only for the triangle integral, the ones for the box are analogous and can be found in [13]. We obtain

\[ I^b_3(s_1, s_2, s_3, m_1^2, m_2^2, m_3^2) = \]

\[ \left[ S^D_{Tri}(s_2, s_3, s_1, m_2^2, m_3^2, m_1^2) + S^D_{Tri}(s_3, s_1, s_2, m_3^2, m_1^2, m_2^2) + S^D_{Tri}(s_1, s_2, s_3, m_1^2, m_2^2, m_3^2) \right] \]

with

\[ S^D_{Tri}(s_1, s_2, s_3, m_1^2, m_2^2, m_3^2) = \int_0^1 dt_1 dt_2 \frac{1}{(1 + t_1 + t_2)} \frac{1}{At_2^2 + Bt_2 + C - i\delta} \tag{4} \]

\[ A = m_2^2 \]
\[ B = (m_1^2 + m_2^2 - s_2)t_1 + m_2^2 + m_3^2 - s_3 \]
\[ C = m_1^2t_1^2 + (m_1^2 + m_3^2 - s_1)t_1 + m_3^2 \]
\[ R = B^2 - 4AC + i\delta \]
\[ T = 2A(1 + t_1) - B \]

The discussion is also valid in the case of vanishing masses or invariants, as long as the functions remain IR finite. Note that if infrared divergences are present the triangle integrals can typically be treated analytically. The \((n+2)\)-dimensional box function are infrared finite for any physically relevant kinematics.

4.3. Singularity structure

Starting from [13] one integration is performed explicitly. In order to analyse the singularity structure of the integrands, we then separate imaginary and real part. One obtains

\[ S^{D=4}_{Tri}(s_1, s_2, s_3, m_1^2, m_2^2, m_3^2) \]

\[ = \int_0^1 dt \frac{4A}{T_2^2 - R} \left\{ \log(2A + B + T) - \log(B + T) \right\} \]
\[ + \Theta(R < 0) \left\{ \log(C) - \log(A + B + C) \right\} \]
\[ + \frac{T}{\sqrt{-R}} \left[ \arctan \left( \frac{\sqrt{-R}}{B} \right) - \arctan \left( \frac{\sqrt{-R}}{2A + B} \right) \right] \]
\[ + \Theta(R > 0) \left\{ \frac{T}{\sqrt{-R}} \left[ \log \left| 2A - B \right| - \log \left| B - \sqrt{R} \right| \right] \right\} \]
\[ - \pi \Theta(B < -\sqrt{R} < 2A + B) \]
\[ - \log \left| B - \sqrt{R} \right| \]
\[ - \log \left( \frac{2 + 4AC + i\delta}{B - 2A} \right) \]

Three regions which lead to an imaginary part can be distinguished:

**Region I:** \( A + B + C > 0, -2A < B < 0, C > 0 \leftrightarrow (B < \sqrt{R} < 2A + B) \).

**Region II:** \( A + B + C > 0, C < 0 \leftrightarrow (B < \sqrt{R} < 2A + B) \) and not \( (B < -\sqrt{R} < 2A + B) \).

**Region III:** \( A + B + C < 0, C > 0 \leftrightarrow (B < -\sqrt{R} < 2A + B) \) and not \( (B < -\sqrt{R} < 2A + B) \).

Region I is an overlap region where the imaginary part has two contributions. In regions II and III only one of the \( \Theta \)-functions contributes. Note that the box function \( I^{D=6}_{Tri} \) has the same singularity structure [13]. As \( I^D_{Tri} \) and \( I^{D=6}_{Tri} \) are the basic building blocks, this analysis of the singularity structure is done once and for all. Knowing the critical region of integration it is possible to map out the singularities by adequate parameter transformations.
4.4. Numerical integration

To demonstrate the practicality of our method to evaluate multi-leg integrals, we show in Fig. 2 a scan of the $2m_t = 350$ GeV threshold of the 4-dimensional scalar hexagon function for a realistic kinematical configuration. For details of the integration methods see [14,15].

5. CONCLUSION

To make reliable phenomenological studies for collider experiments operating at the TeV scale 1-loop calculations with many external particles are mandatory. In this talk I have outlined recent developments concerning the analytic and numeric evaluation of 1-loop Feynman diagrams. Using reduction methods a compact result for the 3-gluon 2-photon amplitude was presented. Concerning numerical methods we have developed an approach to successfully integrate hexagon functions numerically. Merging and applying these techniques to more challenging situations is presently under study.

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