Existence and Multiplicity results for the prescribed Webster Scalar Curvature Problem on three $CR$ manifolds

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Abstract. This paper is devoted to the existence of contact forms of prescribed Webster scalar curvature on a 3–dimensional CR compact manifold locally conformally CR equivalent to the unit sphere $S^3$ of $\mathbb{C}^2$. Due to Kazdan-Warner type obstructions, conditions on the function $H$ to be realized as a Webster scalar curvature have to be given. We prove new existence results based on a new type of Euler-Hopf type formula. Our argument gives an upper bound on the Morse index of the obtained solution. We also give a lower bound on the number of conformal contact forms having the same Webster scalar curvature. 

Mathematics Subject Classification (2000) : 53C15, 53C21, 35J65, 18G35.

Key words : Webster scalar curvature, Critical point at infinity, Gradient flow, Intersection number, Morse index, Topological methods

1 Introduction

Let $(M, \theta)$ be a strictly pseudoconvex CR compact manifold of dimension $2n + 1$ locally CR equivalent to the unit sphere $S^{2n+1}$ of $\mathbb{C}^{n+1}$ with a contact form $\theta$, and let $K : M \to \mathbb{R}$ be a $C^3$ positive function. The prescribed Webster scalar curvature on $M$ is to find suitable conditions on $K$ such that $K$ is the Webster scalar curvature for some contact form $\tilde{\theta}$ on $M$, CR equivalent to $\theta$. If we set $\tilde{\theta} = u^{\frac{2}{n}} \theta$, where $u$ is a smooth positive function on $M$, then the above problem is equivalent to solving the following equation

$$\begin{cases}
L_\theta u = \frac{n}{2(n+1)} K u^{1 + \frac{2}{n}} & \text{in } M \\
u > 0 & \text{in } M,
\end{cases}$$

$(P)$

where

$$L_\theta u = \Delta_\theta u + \frac{n}{2(n+1)} R_\theta u$$

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\( \Delta_\theta \) is the sublaplacian operator on \((M, \theta)\) and \( R_\theta \) is the Webster scalar curvature of \((M, \theta)\). Problem \((P)\) is the analogue of the prescribed scalar curvature problem on Riemannian manifolds. While the scalar curvature problem in the Riemannian framework was extensively studied (see for example the monograph [2] and the references therein), only few results were established for problem \((P)\) (see [13], [16], and [23]). On the contrary, the Yamabe problem on CR manifolds, that is when \( K \) is assumed to be constant, was widely studied by various authors (see [19], [20], [21], [15] and [17]).

The problem \((P)\) has a variational structure, however the associated Euler functional does not satisfy the Palais-Smale condition, that is, there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. Moreover, there are topological obstructions of Kazdan-Warner condition type to solve \((P)\), see [18]. Hence one does not expect to solve problem \((P)\) for all functions \( K \), and so it is natural to ask: under which conditions on \( K \) does \((P)\) have a solution?

In [23], Malchiodi and Uguzzoni considered the case where \( M = S^{2n+1} \) the unit sphere of \( C^{n+1} \) and gave a perturbative result for problem \((P)\), that is when \( K \) is assumed to be a small perturbation of a constant (see also [13]). Their approach uses a perturbation method due to Ambrosetti and Badiale [1]. In [16], N. Gamara noticed, in analogy with the 4-dimensional Riemannian case, that there is a balance phenomenon between the self interactions and the mutual interactions of the functions failing to satisfy Palais-Smale condition in the 3-dimensional CR case (see [9] and [11] for the Riemannian case). In [16] the case where \( M \) is locally conformally CR equivalent to the CR Sphere of \( C^2 \) was considered (thus when \( n = 1 \)), and a Euler-Hopf type criterion for \( K \) was provided to find solutions for \((P)\). The method used in [16] is due to Bahri and Coron [9]. It consists of studying the critical points at infinity of the associated variational problem, computing their total Morse index, and comparing this total index to the Euler-Poincaré characteristic of the space of variations.

In this paper we revisit the three dimensional case, namely the following equation

\[
(P_K) \quad \begin{cases}
L_\theta u = \frac{1}{4} Ku^3 & \text{in } M \\
u > 0 & \text{in } M.
\end{cases}
\]

Our goal here is to give new existence results which generalize the one obtained by N. Gamara [16] and also to give, in generic cases, a lower bound of the number of contact forms of prescribed Webster-Tanaka scalar curvature \( K \).

To state our results, we set the following notations. Let \( G(a, \cdot) \) be the Green’s function of \( L_\theta \) on \( M \) and \( A_y \) the value of the regular part of \( G \) at \( a \). Let \( \mathcal{K} \) the set of critical points of \( K \). We say that \( K \) satisfies the condition \((C_0)\) if it has only nondegenerate critical points such that

\[
\frac{-\Delta_\theta K(y)}{3K(y)} - 2A_y \neq 0 \quad \forall y \in \mathcal{K}
\]
Now, we introduce the following set
\[ \mathcal{K}_+ = \{ y \in \mathcal{K}; \frac{-\Delta_{\theta}K(y)}{3K(y)} - 2A_y > 0 \}. \tag{1.1} \]
For \( p \in \mathbb{N}^* \) and for any \( p \)-tuple \( \tau_p = (y_1, \ldots, y_p) \in (\mathcal{K}_+)^p \) such that \( y_i \neq y_j \) if \( i \neq j \), we define a matrix \( M(\tau_p) = (M_{ij})_{1 \leq i,j \leq p} \) by
\[
M_{ii} = \frac{-\Delta_{\theta}K(y_i)}{3K(y_i)^2} - \frac{2A_y}{K(y_i)}, \quad M_{ij} = -\frac{2G(y_i, y_j)}{(K(y_i)K(y_j))^{1/2}} \quad \text{for } i \neq j. \tag{1.2}
\]
We denote by \( \rho(\tau_p) \) the least eigenvalue of \( M(\tau_p) \) and we say that a function \( K \) satisfies the condition \( (C_1) \) if for every \( \tau_p \in (\mathcal{K}_+)^p \), we have that \( \rho(\tau_p) \neq 0 \).

We set
\[
\mathcal{F}_\infty := \{ \tau_p = (y_1, \ldots, y_p) \in (\mathcal{K}_+)^p; \rho(\tau_p) > 0 \}, \tag{1.3}
\]
and define an index \( \iota: \mathcal{F}_\infty \rightarrow \mathbb{Z} \) defined by
\[
\iota(\tau_p) := p - 1 + \sum_{i=1}^{p} (3 - m(K, y_i)),
\]
where \( m(K, y_i) \) denotes the Morse index of \( K \) at its critical point \( y_i \).

Now we state our main result.

**Theorem 1.1** Let \( 0 < K \in C^2(M) \) be a positive function satisfying the conditions \( (C_0) \) and \( (C_1) \).

If there exists \( k \in \mathbb{N} \) such that

1. \[
\sum_{\tau_p \in \mathcal{F}_\infty, \iota(\tau_p) \leq k-1} (-1)^{\iota(\tau_p)} \neq 1,
\]

2. \[
\forall \tau_p \in \mathcal{F}_\infty, \iota(\tau_p) \neq k.
\]

Then, there exists a solution \( w \) to the problem \( (P_K) \) such that:
\[
morse(w) \leq k,
\]
where \( morse(w) \) is the Morse index of \( w \), defined as the dimension of the space of negativity of the linearized operator:
\[
\mathcal{L}_w(\varphi) := L_\theta(\varphi) - 3w^2\varphi.
\]

Moreover, for generic \( K \) it holds
\[
\# \mathcal{N}_k \geq |1 - \sum_{\tau_p \in \mathcal{F}_\infty, \iota(\tau_p) \leq k-1} (-1)^{\iota(\tau_p)}|,
\]
where \( \mathcal{N}_k \) denotes the set of solutions of \( (P_K) \) having their Morse indices less than or equal to \( k \).
Please observe that, taking in the above \( k \) to be \( l_# + 1 \), where \( l_# \) is the maximal index over all elements of \( \mathcal{F}_\infty \), the second assumption is trivially satisfied. Therefore in this case, we have the following corollary, which recovers previous existence results for three dimensional CR manifolds locally CR equivalent to the sphere \( S^3 \) of \( \mathbb{C}^2 \) due to N. Gamara [16].

**Corollary 1.2** Let \( 0 < K \in C^2(M) \) be a positive function satisfying the conditions \((C_0)\) and \((C_1)\).

If

\[
\sum_{\tau_p \in \mathcal{F}_\infty} (-1)^{\ell(\tau_p)} \neq 1,
\]

Then the problem \((P_K)\) has at least one solution.

Moreover, for generic \( K \) it holds

\[
\# S \geq |1 - \sum_{\tau_p \in \mathcal{F}_\infty} (-1)^{\ell(\tau_p)}|,
\]

where \( S \) denotes the set of solutions of \((P_K)\).

We point out that the main new contribution of Theorem 1.1 is that we address here the case where the total sum in the above corollary equals 1, but a partial one is not equal 1. The main issue being the possibility to use such an information to prove existence of solution to the problem \((P_K)\). Please notice that an interpretation of the fact that the above sum is different from one, is that the topological contribution of the critical points at infinity to the topology of the level sets of the associated Euler-Lagrange functional is not trivial. In view of such an interpretation, we rise the following question: what happens if the total contribution is trivial, but some critical points at infinity induce a nontrivial difference of topology. Can we still use such a topological information to prove existence of solution?

With respect to the above question, theorem 1.1 gives a sufficient condition to be able to derive from such a local information, an existence as well as a multiplicity result together with information on the Morse index of the obtained solution. In Section 5, we give a more general condition. Since this condition involves the critical points at infinity of the variational problem, we have postponed its statement to the end of this paper, see please Theorem (5.1).

As pointed out above, our result does not only give existence results, but also, under generic conditions, gives a lower bound on the number of solutions of \((P_K)\). Such a result is reminiscent to the celebrated Morse Theorem, which states that, the number of critical points of a Morse function defined on a compact manifold, is lower bounded in terms of the topology of the underlying manifold. Our result can be seen as some sort of Morse Inequality at Infinity. Indeed it gives a lower bound on the number of metrics with
prescribed curvature in terms of the topology at infinity.
The remainder of this paper is organized as follows. In section 2 we set up the variational problem, its critical points at infinity are characterized in Section 3. Section 4 is devoted to the proof of the main result theorem 1.1 while we give in Section 5 a more general statement than theorem 1.1.

2 Variational setting and lack of compactness

In this section we recall the functional setting and the variational problem associated to $(P_K)$. Problem $(P_K)$ has a variational structure, the functional being

$$J(u) = \frac{\int_M Lu u \theta \wedge d\theta}{(\int_M K u^4 \theta \wedge d\theta)^{\frac{1}{2}}}$$

defined on the unit sphere of $S^2_1(M)$ equipped with the norm

$$||u||^2 = \int_M u Lu \theta \wedge d\theta,$$

where $S^2_1(M)$ is the Folland-Stein space (see [14] for definition).

Problem $(P_K)$ is equivalent to finding the critical points of $J$ subjected to the constraint $u \in \Sigma^+$, where

$$\Sigma^+ = \{u \in \Sigma / u \geq 0\}, \quad \Sigma = \{u \in S^2_1(M)/||u|| = 1\}$$

The Palais-Smale condition fails to be satisfied for $J$ on $\Sigma^+$. To characterize the sequences failing the Palais-Smale condition, we need to set some notations and constructions. Since $M$ is compact and locally CR equivalent to $S^3$, any point $a$ in $M$ has a neighborhood $U_a \supset B_r(a)$, $r$ is independent of $a$, where CR normal coordinates are defined, and such that the contact form of $M$ is conformal to the standard contact form $\theta_0$ of the Heisenberg group $\mathbb{H}^1$; that is there exists a positive function $\tilde{u}_a$ on $B_r(a)$ such that $\theta_0 = \tilde{u}_a^2 \theta$, ($\tilde{u}_a$ smoothly dependent on $a$). Let $u_a(x) = w_a(x)\tilde{u}_a(x)$, where $w_a(x) = \chi(|x|)$, $\chi$ is a cut-off function $\chi: \mathbb{R} \rightarrow [0,1]$ defined by

$$\chi(t) = 1 \quad \text{if} \quad 0 \leq t \leq r/2; \quad \chi(t) = 0 \quad \text{if} \quad t \geq r$$

and $|x| = |\exp^{-1}_a(x)|_{\mathbb{H}^1}$, where, letting $(z,t) = \exp^{-1}_a(x)$, $\exp_a$ being the parabolic exponential map based at $a$, then $|(z,t)|_{\mathbb{H}^1} = (|z|^4 + t^2)^{\frac{1}{2}}$ is the norm of the Heisenberg group $\mathbb{H}^1$ (see please [19], [20]).
Let $\lambda$ be a large positive parameter. We introduce on $B_r(a)$ the function
\[ \delta_{(a,\lambda)}(x) = c_1 \lambda |1 + \lambda^2 (|z|^2 - it)|^{-1}, \tag{2.3} \]
and the constant $c_1$ is chosen such that the following equation is satisfied
\[ L_{\theta_0} \delta_{(a,\lambda)} = \delta_{(a,\lambda)}^3 \quad \text{on} \quad B_r(a). \]
Let
\[ \hat{\delta}_{(a,\lambda)}(x) = \begin{cases} u_{a} \delta_{(a,\lambda)}(x) & \text{in} \quad B_r(a) \\ 0 & \text{in} \quad B_r(a)^c. \end{cases} \tag{2.4} \]
We define a family of "almost solutions" $\tilde{\delta}_{(a,\lambda)}$ to be the unique solution of
\[ L_\theta \tilde{\delta}_{(a,\lambda)}(x) = (\hat{\delta}_{(a,\lambda)}(x))^3 \quad \text{in} \quad M. \]
Setting $H_{a,\lambda} := \lambda(\tilde{\delta}_{a,\lambda} - \hat{\delta}_{a,\lambda})$,
we have that:

\begin{proposition} \cite{16} \end{proposition}
For $\lambda$ large, there exists a constant $C = C(\varrho)$ such that:
\[ |H_{a,\lambda}|_{L^\infty} \leq C; \quad \lambda |\partial H_{a,\lambda} / \partial \lambda|_{L^\infty} \leq C; \quad \lambda^{-1} |\partial H_{a,\lambda} / \partial a|_{L^\infty} \leq C. \]
Moreover for $\varrho$ small and $\lambda$ large there holds:
\[ H_{a,\lambda}(a) \to A_a \quad \text{as} \quad \lambda \to \infty \tag{2.5} \]
\[ H_{a,\lambda}(x) \to G(a, x) \quad \text{outside} \quad B_{2\varrho}(a) \quad \text{as} \quad \lambda \to \infty, \tag{2.6} \]
where $G(a, x)$ is the Green's function of the conformal sub-Laplacian $L_\theta$ and $A_a$ the value of its regular part evaluated at $a$.

We define now the set of potential critical points at infinity associated to the functional $J$.
For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define
\[
V(p, \varepsilon) = \left\{ u \in \Sigma / \exists a_1, \ldots, a_p \in M, \exists \lambda_1, \ldots, \lambda_p > 0, \exists \alpha_1, \ldots, \alpha_p > 0 \text{ s.t. } \| u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} \| \leq \varepsilon, \right. \|
\frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1 \| < \varepsilon, \quad \varepsilon_{ij} < \varepsilon \quad \lambda_i > \varepsilon^{-1} \right\},
\]
where \( \varepsilon_{ij}^{-1} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j d(a_i, a_j)^2 \right) \), and \( d(x, y) = |exp_x^{-1}(y)|_{\mathbb{H}} \) if \( x \) and \( y \) are in a small ball of \( M \) of radius \( r \), and \( d(x, y) \) is equal to \( \frac{r}{2} \) otherwise.

For \( w \) a solution of \((P_K)\) we also define \( V(p, \varepsilon, w) \) as

\[
\{ u \in \Sigma / \exists \alpha_0 > 0 \text{ s. t. } u - \alpha_0 w \in V(p, \varepsilon) \text{ and } |\alpha_0^2 J(u)^2 - 1| < \varepsilon \}. \tag{2.7}
\]

The failure of Palais-Smale condition can be described, following the ideas introduced in [12] [22] [25], as follows:

**Proposition 2.2** Let \((u_j) \in \Sigma^+ \) be a sequence such that \( J'(u_j) \) tends to zero and \( J(u_j) \) is bounded. Then, there exist an integer \( p \in \mathbb{N}^* \), a sequence \( \varepsilon_j > 0 \), \( \varepsilon_j \) tends to zero, and an extracted subsequence of \( u_j \)'s, again denoted \( u_j \), such that \( u_j \in V(p, \varepsilon_j, w) \) where \( w \) is zero or a solution of \((P_K)\).

If a function \( u \) belongs to \( V(p, \varepsilon) \), we consider the following minimization problem for \( u \in V(p, \varepsilon) \) with \( \varepsilon \) small

\[
\min \{ ||u - \sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i}||, \alpha_i > 0, \lambda_i > 0, a_i \in M \}. \tag{2.8}
\]

We then have the following proposition which defines a parameterization of the set \( V(p, \varepsilon) \). It follows from corresponding statements in [6], [8].

**Proposition 2.3** For any \( p \in \mathbb{N}^* \), there is \( \varepsilon_p > 0 \) such that if \( \varepsilon < \varepsilon_p \) and \( u \in V(p, \varepsilon) \), the minimization problem \((2.8)\) has a unique solution (up to permutation). In particular, we can write \( u \in V(p, \varepsilon) \) as follows

\[
u = \sum_{i=1}^{p} \alpha_i \delta_{\tilde{a}_i, \tilde{\lambda}_i} + v,
\]

where \((\tilde{a}_1, ..., \tilde{a}_p, \tilde{\lambda}_1, ..., \tilde{\lambda}_p)\) is the solution of \((2.8)\) and \( v \in \mathcal{S}^2_1(M) \) such that

\[
(V_0) \quad < v, \psi >_{\theta} = 0 \quad \text{for all } \psi \in \left\{ \delta_{\tilde{a}_i}, \frac{\partial \delta_{\tilde{a}_i}}{\partial \lambda_i}, \frac{\partial \delta_{\tilde{a}_i}}{\partial a_i}, \text{ for } i = 1, \ldots, p \right\}.
\]

Here, \(<,>_{\theta}\) denotes the \( L_{\theta}\)-scalar product defined on \( \mathcal{S}^2_1(M) \) by

\[
< u, v >_{\theta} = \int_M L_{\theta} u v \theta \wedge d\theta. \tag{2.9}
\]

Let \( \nabla_{\theta} \) be the CR gradient (or subelliptic gradient) which can be characterized by

\[
\int_M \nabla_{\theta} u \nabla_{\theta} v \theta \wedge d\theta = \int_M \Delta_{\theta} u v \theta \wedge d\theta. \tag{2.10}
\]

In the following we will say that \( v \in (V_0) \) if \( v \) satisfies \((V_0)\).
Proposition 2.4 [16] There exists a $C^1$ map which, to each $(\alpha_1, \ldots, \alpha_p, a_1, \ldots, a_p, \lambda_1, \ldots, \lambda_p)$ such that $\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$ with small $\varepsilon$, associates $\bar{\nu} = \bar{\nu}_{(\alpha_1, a_1, \lambda_1)}$ satisfying

$$J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \bar{\nu}\right) = \min_{v \in (V_0)} J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + v\right).$$

Moreover, there exists $c > 0$ such that the following holds

$$||\bar{\nu}|| \leq c \left(\sum_{i \leq p} \left|\frac{\nabla_\theta K(a_i)}{\lambda_i}\right| + \frac{1}{\lambda_i^2} + \sum_{k \neq r} \varepsilon_{kr} (\log(\varepsilon_{kr}^{-1}))^{1/2}\right).$$

Let $w$ be a solution of $(P_k)$. The following proposition defines a parameterization of the set $V(p, \varepsilon, w)$. Its proof follows from the same arguments used to prove similar statements in [6].

Proposition 2.5 There is $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $u \in V(p, \varepsilon, w)$, then the problem

$$\min_{\alpha_i > 0, \lambda_i > 0, a_i \in M, h \in T_w(W_u(w))} ||u - \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} - \alpha_0(w + h)||$$

has a unique solution $(\bar{\alpha}, \bar{\lambda}, \bar{\nu}, \bar{\bar{h}})$. Thus, we write $u$ as follows:

$$u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0(w + \bar{\bar{h}}) + v,$$

where $v$ belongs to $H^1(M) \cap T_w(W_u(w))$ and it satisfies $(V_0), T_w(W_u(w))$ and $T_w(W_u(w))$ are the tangent spaces at $w$ to the unstable and stable manifolds of $w$.

3 Critical points at infinity of the variational problem

In the sequel, $\partial J$ designates the gradient of $J$ with respect to the $L_\theta$-scalar product $< , >_\theta$, that is $\forall u, v \in S_1^2(M)$, we have $<\partial J(u), v >_\theta = J'(u) v$.

Following A. Bahri we set the following definitions and notations

Definition 3.1 A critical point at infinity of $J$ on $\Sigma^+$ is a limit of a flow line $u(s)$ of the equation:

$$\begin{cases}
\frac{\partial u}{\partial s} = -\partial J(u) \\
u(0) = u_0
\end{cases}$$
such that $u(s)$ remains in $V(p, \varepsilon(s), w)$ for $s \geq s_0$.

Here $w$ is either zero or a solution of $(P_K)$ and $\varepsilon(s)$ is some function tending to zero when $s \to \infty$. Using Proposition 2.5, $u(s)$ can be written as:

$$u(s) = \sum_{i=1}^{p} \alpha_i(s) \tilde{\delta}_{(a_i(s), \lambda_i(s))} + \alpha_0(s)(w + h(s)) + v(s).$$

Denoting $a_i := \lim_{s \to \infty} a_i(s)$ and $\alpha_i = \lim_{s \to \infty} \alpha_i(s)$, we denote by

$$(a_1, \cdots, a_p, w)_{\infty} \text{ or } \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \infty)} + \alpha_0 w$$

such a critical point at infinity. If $w \neq 0$ it is called of $w$-type.

### 3.1 Ruling out the existence of critical points at infinity in $V(p, \varepsilon, w)$ for $w \neq 0$

The aim of this subsection is to prove that, given a $C^2$ positive function $K$ satisfying the conditions of theorem 1.1 and a solution $w$ of $(P_K)$, then for each $p \in \mathbb{N}$, there are no critical point or critical point at infinity of $J$ in the set $V(p, \varepsilon, w)$. The reason is that there exists a pseudogradient of $J$ such that the Palais-Smale condition is satisfied along its decreasing flow lines.

In this section, for $u \in V(p, \varepsilon, w)$, using Proposition 2.5, we will write $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)} + \alpha_0(w + h) + v$.

**Proposition 3.2** For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)} + \alpha_0(w + h) + v \in V(p, \varepsilon, w)$, we have the following expansion

$$J(u) = \frac{S \sum_{i=1}^{p} \alpha_i^2 + \alpha_0^2 ||w||^2}{(S \sum_{i=1}^{p} \alpha_i^4 K(a_i) + \alpha_0^4 ||w||^2)^2} \left[ 1 - \frac{c_2 \alpha_0}{\gamma_1} \sum_{i=1}^{p} \frac{w(a_i)}{\lambda_i} \right]$$

$$- \frac{1}{\gamma_1} \sum_{i \neq j} \alpha_i \alpha_j c_{ij} \varepsilon_{ij} + f_1(v) + Q_1(v, v) + f_2(h) + \alpha_0^2 Q_2(h, h)$$

$$+ o \left( \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i} + ||v||^2 + ||h||^2 \right)$$
where
\[
Q_1(v, v) = \frac{1}{\gamma_1} ||v||^2 - \frac{3}{\beta_1} \int_M K \left( \sum_{i=1}^{p} (\alpha_i \tilde{\delta}_i)^2 + (\alpha_0 w)^2 \right) v^2,
\]
\[
Q_2(h, h) = \frac{1}{\gamma_1} ||h||^2 - \frac{3}{\beta_1} \int_M K(\alpha_0 w)^2 h^2,
\]
\[
f_1(v) = -\frac{1}{\beta_1} \int_M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^2 v,
\]
\[
f_2(h) = \frac{\alpha_0}{\gamma_1} \sum_{i} \alpha_i (\tilde{\delta}_i, h) - \frac{\alpha_0}{\beta_1} \int_M K \left( \sum_{i} \alpha_i \tilde{\delta}_i + \alpha_0 w \right)^2 h,
\]
\[
c_2 = c_1^2 \int_{\mathbb{H}^1} \frac{1}{|1 + |z|^2 - it|^3} \theta_0 \wedge d\theta_0,
\]
\[
\beta_1 = S + c_2 \frac{H_{a_i, \lambda_i}(a_i)}{\lambda_i^2} + o(1);
\]
\[
\gamma_1 = S + \sum_{i=1}^{p} \alpha_i^2 ||\tilde{\delta}_i||^2 + 2 \alpha_i \alpha_0 < \tilde{\delta}_i, w + h > + \alpha_0^2 \left( ||h||^2 + ||w||^2 \right) + ||v||^2 + \sum_{i \neq j} \alpha_i \alpha_j < \tilde{\delta}_i, \tilde{\delta}_j >.
\]

and where \(c_{ij} > 0\) bounded constants.

**Proof.** To prove the proposition, we need to estimate
\[
N(u) = ||u||^2 \quad \text{and} \quad D^2 = \int_M K(x) u^4 \theta \wedge d\theta,
\]
where
\[
||u|| := \int_M u L u \theta \wedge d\theta.
\]
Now expanding \(N(u)\), we get
\[
N(u) := \sum_{i=1}^{p} \alpha_i^2 ||\tilde{\delta}_i||^2 + 2 \alpha_i \alpha_0 < \tilde{\delta}_i, w + h > + \alpha_0^2 \left( ||h||^2 + ||w||^2 \right) + ||v||^2 + \sum_{i \neq j} \alpha_i \alpha_j < \tilde{\delta}_i, \tilde{\delta}_j >.
\]

Now it follows from [16] and elementary computations that
\[
||\tilde{\delta}_i||^2 = S + c_2 \frac{H_{a_i, \lambda_i}(a_i)}{\lambda_i^2} + o(1);
\]
\[
< \tilde{\delta}_i, \tilde{\delta}_j > = c_2 \frac{H_{a_i, \lambda_i}(a_i)}{\lambda_i \lambda_j} + c_i c_j (1 + o(1)), \quad \text{for } i \neq j,
\]
\[
< \tilde{\delta}_i, w > = \int_{B_{r}(a_i)} w \tilde{\delta}_i \theta \wedge d\theta = c_2 \frac{w(a_i)}{\lambda_i} + o(1).
\]
Therefore

\[ N = \gamma_1 + 2\alpha_0 \sum_{i=1}^{p} c_2\alpha_i \frac{w'(a_i)}{\lambda_i} + \alpha_i < \tilde{\delta}_i, h >_\theta + \sum_{i \neq j} \alpha_i \alpha_j c_{ij} \xi_{ij} \]  

(3.4)

\[ + \alpha_0^2 \|h\|^2 + \|v\|^2 + o\left(\sum_{i=1}^{p} \frac{1}{\lambda_i} + \sum_{i \neq j} \xi_{ij}\right). \]

Now concerning the denominator, we compute it as follows

\[ D^2 = \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^4 \theta \wedge d\theta + \int M K (\alpha_0 w)^4 \theta \wedge d\theta \]  

(3.5)

\[ + 4\alpha_0 \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^3 w \theta \wedge d\theta + 4\alpha_0^3 \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right) w^3 \theta \wedge d\theta \]

\[ + 4 \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + \alpha_0 w \right)^3 (\alpha_0 h + v) \theta \wedge d\theta \]

\[ + 12 \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + \alpha_0 w \right)^2 (\alpha_0^2 h^2 + v^2 + 2\alpha_0 hv) \theta \wedge d\theta \]

\[ + O\left( \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^3 w \theta \wedge d\theta \right) + O(||v||^2 + \|h\|^3). \]

Observe that

\[ \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^4 \theta \wedge d\theta = \sum_{i=1}^{p} \alpha_i^4 K(a_i) S \]

(3.6)

\[ + 4 \sum_{i \neq j} \alpha_i \alpha_j K(a_i) c_{ij} \xi_{ij} + O\left( \frac{1}{\lambda_i^2} \right) + o(\xi_{ij}), \]

\[ \int M K w^4 \theta \wedge d\theta = ||w||^2, \quad \int M K w^3 \tilde{\delta}_i \theta \wedge d\theta = c_2 \frac{w(a_i)}{\lambda_i} + o\left( \frac{1}{\lambda_i} \right), \]  

(3.7)

\[ \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^3 w \theta \wedge d\theta = c_2 \sum_{i=1}^{p} \alpha_i^3 K(a_i) \frac{w(a_i)}{\lambda_i} + o\left( \frac{1}{\lambda_i} \right), \]  

(3.8)

\[ \int M w^2 \alpha_i^2 \tilde{\delta}_i^2 + \alpha_0^2 w^2 \tilde{\delta}_i^2 \theta \wedge d\theta = o\left( \frac{1}{\lambda_i^2} \right), \]  

(3.9)

\[ \int M K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + \alpha_0 w \right)^2 v h \theta \wedge d\theta = O \left( \int M \left( \sum_{i=1}^{p} \tilde{\delta}_i^2 + w^{-1} \sum_{i=1}^{p} \tilde{\delta}_i \right) |v||h| \right) \]

\[ = O \left( ||v||^3 + \|h\|^3 + 1/\lambda_i^3 \right), \]  

(3.10)
where we have used that \( v \in T_w(W_s(w)) \) and \( h \) belongs to \( T_w(W_u(w)) \) which is a finite dimensional space. Hence it implies that \( \| h \|_\infty \leq c \| h \| \).

Concerning the linear form in \( v \), since \( v \in T_w(W_s(w)) \), it can be written as

\[
\int_M K(\sum_{i=1}^p \alpha_i \tilde{\delta}_i + \alpha_0 w)^3 v \theta \wedge d\theta = \int K(\sum_{i=1}^p \alpha_i \tilde{\delta}_i)^3 v \theta \wedge d\theta + O\left( \sum_{i=1}^p \int (\alpha_i^2 \alpha_0 \tilde{\delta}_i^2 w + \alpha_0^2 \alpha_i \tilde{\delta}_i w^2) |v| \right)
= f_1(v) + O\left( \frac{\| v \|}{\lambda_i} \right). \tag{3.11}
\]

Finally, we have

\[
\int K(\sum_{i=1}^p \alpha_i \tilde{\delta}_i + \alpha_0 w)^2 h^2 \theta \wedge d\theta = \alpha_0^2 \int K w^2 h^2 + o(\| h \|^2) \theta \wedge d\theta \tag{3.12}
\]

\[
\int K(\sum_{i=1}^p \alpha_i \tilde{\delta}_i + \alpha_0 w)^2 v^2 \theta \wedge d\theta = \sum_{i=1}^p \int K(\alpha_i \tilde{\delta}_i)^2 v^2 + \alpha_0^2 \sum_{i=1}^p \int K w^2 v^2
+ o(\| v \|^2). \tag{3.13}
\]

Combining (3.4) to (3.13), and the fact that \( \frac{\alpha_i^2 K_{(a_i)}}{\alpha_j^2 K_{(a_j)}} = 1 + o(1) \), the result follows.

Now, we state the following lemma whose proof follows the arguments used to prove similar statements in [5], see the Appendix of [16] were the necessary modifications are given.

**Lemma 3.3** [16] We have

(a) \( Q_1(v, v) \) is a quadratic form positive definite in \( E_v = \{ v \in S^2_1(M) / v \in T_w(W_s(w)) \text{ and } v \text{ satisfies } (V_0) \} \).

(b) \( Q_2(h, h) \) is a quadratic form negative definite in \( T_w(W_u(w)) \).

**Corollary 3.4** Let \( u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0 (w + h) + v \in V(p, \varepsilon, w) \). There is an optimal \((\overline{v}, \overline{h})\) and a change of variables \( v - \overline{v} \to V \) and \( h - \overline{h} \to H \) such that

\[
J(u) = J\left( \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0 w + \overline{h} + \overline{v} \right) + ||V||^2 - ||H||^2.
\]

Furthermore we have the following estimates

\[
||\overline{h}|| \leq \sum_i \frac{c}{\lambda_i} \quad \text{and} \quad ||\overline{v}|| \leq c \sum_i \frac{\left| \nabla \theta K_{(a_i)} \right|}{\lambda_i} + \frac{c}{\lambda_i^2} + c \sum_{k \neq r} \varepsilon_{kr} (\log \varepsilon^{-1})^{\frac{1}{2}},
\]
Proof. Now observe that the above expansion of $J$ with respect to $h$ (respectively to $v$) is almost equal to $Q_2(h,h) + f_2(h)$ (respectively $Q_1(v,v) + f_1(v)$). As $Q_2$ is negative definite (respectively $Q_1$ is positive definite), there is a unique maximum $\overline{h}$ in the space of $h$’s (respectively a unique minimum $\overline{v}$ in the space of $v$). Moreover, it is not difficult to prove that $||\overline{h}|| \leq c||f_2||$ and $||\overline{v}|| \leq c||f_1||$. Therefore the estimate of $\overline{\tau}$ follows from Proposition 2.4 while for the estimate of $\overline{h}$, we use the fact that for each $h \in T_w(W_u(w))$ which is a finite dimensional space, we have $||h||_\infty \leq c||h||$. Hence it follows that $||f_2|| = O(\sum \lambda_i^{-1})$ and our result follows.

Now we state the following corollary, which follows immediately from the above corollary and the fact that $w > 0$ in $M$.

Corollary 3.5 Let $K$ be a $C^2$ positive function and let $w$ be a nondegenerate critical point of $J$ in $\Sigma^+$. Then, for each $p \in \mathbb{N}^*$, there is no critical points or critical points at infinity in the set $V(p,\varepsilon,w)$, that means we can construct a pseudogradient of $J$ so that the Palais-Smale condition is satisfied along the decreasing flow lines.

Now, once the existence of mixed critical points at infinity is ruled out, it follows from [16], that the critical points at infinity are in one to one correspondence with the elements of the set $\mathcal{F}_\infty$ defined in (1.3). That is, a critical point at infinity corresponds to $\tau_p := (y_1, \ldots, y_p) \in (\mathcal{K}_+)^p$ such that the related Matrix $M(\tau_p)$ defined in (1.2) is positive definite. Such a critical point at infinity will be denoted by $\tau_p^\infty := (y_1, \ldots, y_p)^\infty$.

Like a usual critical point, it is associated to a critical point at infinity $x_\infty$ of the problem $(P_K)$, (which is a combination of classical critical points of $K$ with a $1$–dimensional asymptote), stable and unstable manifolds, $W^s_\infty(x_\infty)$ and $W^u_\infty(x_\infty)$. These manifolds can be easily described once a Morse type reduction is performed, see [16]. The stable manifold is, as usual, defined to be the set of points attracted by the asymptote, under the action of the flow (see below). The unstable one is a shadow object, which is the limit of $W_u(x_\lambda)$, $x_\lambda$ being the critical point of the reduced problem and $W_u(x_\lambda)$ its associated unstable manifolds. Indeed the flow in this case splits the variable $\lambda$ from the other variables near $x_\infty$. Notice that the flow of which it is question above is the flow of a pseudogradient at infinity of Morse-Smale type for $-J$. Such a pseudogradient at infinity, whose existence is ensured by the Proposition 4.3 in [16], is known to have a very nice behavior around the critical points at infinity.

We then may define the Morse index $\text{morse}(x_\infty)$ of the critical point at infinity $x_\infty$ to be equal to the dimension of $W^s_\infty(x_\infty)$. Observe that we have: $\text{morse}(\tau_p^\infty) = \ell(\tau_p)$.

In the following definition, we extend the notion of domination of critical points to critical points at infinity.
Definition 3.6 \( z_\infty \) is said to be dominated by another critical point at infinity \( z'_\infty \) if
\[
W_u(z'_\infty) \cap W_u(z_\infty) \neq \emptyset.
\]
We then write \( z_\infty < z'_\infty \). If we assume that the intersection is transverse, then we obtain
\[
morse(z'_\infty) \geq morse(z_\infty) + 1.
\]

4 Proof of the main result

This section is devoted to the proof of the main result of this paper, theorem 1.1.

Proof of Theorem 1.1

Setting
\[
l_\#: = \sup \{ \iota(\tau_p); \tau_p \in \mathcal{F}_\infty \}
\]
For \( l \in \{ 0, \cdots, l_\# \} \) we define the following sets:
\[
X_l^\infty := \bigcup_{\tau_p \in \mathcal{F}_\infty; \iota(\tau_p) \leq l} W_u^{\infty}(\tau_p),
\]
(4.1)
where \( W_u^{\infty}(\tau_p) \) is the unstable manifold associated to the critical point at infinity \( \tau_p^\infty \), and
\[
C(X_l^\infty) := \{ tu + (1 - t) (y_0)_\infty; t \in [0, 1], u \in X_l^\infty \},
\]
(4.2)
where \( y_0 \) is a global maximum of \( K \) on the manifold \( M \).

By a theorem of Bahri-Rabinowitz [10], it follows that:
\[
\overline{W_u^{\infty}(\tau_p)} = W_u^{\infty}(\tau_p) \cup \bigcup_{x_\infty < \tau_p^\infty} W_u^{\infty}(x_\infty) \cup \bigcup_{w < \tau_p^\infty} W_u(w),
\]
where \( x_\infty \) is a critical point at infinity dominated by \( \tau_p^\infty \) and \( w \) is a solution of \((P_K)\) dominated by \( \tau_p^\infty \). By transversality arguments we assume that the Morse index of \( x_\infty \) and the Morse index of \( w \) are not bigger than \( l \). Hence
\[
X_l^\infty = \bigcup_{\iota(\tau_p) \leq l} W_u^{\infty}(\tau_p) \cup \bigcup_{w < \tau_p^\infty} W_u(w).
\]
It follows that \( X_l^\infty \) is a stratified set of top dimension \( \leq l \). Without loss of generality, we may assume it equal to \( l \), therefore \( C(X_l^\infty) \) is also a stratified set of top dimension \( l + 1 \).

Now we use the gradient flow of \(-J\) to deform \( C(X_l^\infty) \). By transversality arguments we can assume that the deformation avoids all critical as well as critical points at infinity having their Morse indices greater than \( l + 2 \). It follows then, by a Theorem of Bahri and Rabinowitz [10], that \( C(X_l^\infty) \) retracts by deformation on the set
\[
U := X_l^\infty \cup \bigcup_{morse(x_\infty) = l+1} W_u^{\infty}(x_\infty) \cup \bigcup_{w < \tau_p^\infty} W_u(w).
\]
(4.3)
Now taking \( l = k - 1 \) and using the fact that, by assumption of theorem 1.1, there are no critical point at infinity with index \( k \), we derive that \( C(X_{k-1}^\infty) \) retracts by deformation onto
\[
Z_k^\infty := X_{k-1}^\infty \cup \bigcup_{w; J'(w)=0; \text{\( w \) dominated by \( C(X_{k-1}^\infty) \)}} W_u(w).
\]
(4.4)

Now, observe that it follows from the above deformation retract, that the problem \( (P_K) \) has necessary a solution \( w \) with \( \text{morse}(w) \leq k \). Otherwise it follows from (4.4) that
\[
1 = \chi(Z_k^\infty) = \sum_{\tau_p \in \mathcal{F}_\infty; \ell(\tau_p) \leq k-1} (-1)^{\ell(\tau_p)} + \sum_{w \in C(X_{k-1}^\infty); J'(w)=0} (-1)^{\text{morse}(w)}.
\]
where \( \chi \) denotes the Euler Characteristic. Such an equality contradicts the second assumption of the theorem.

Now for generic \( K \), it follows from the Sard-Smale Theorem that all solutions of \( (P_K) \) are nondegenerate solutions, in the sense that their associated linearized operator does not admit zero as an eigenvalue. See please [24] for a related discussion in the Riemannian setting.

We derive now from (4.4), taking the Euler Characteristic of both sides, that:
\[
1 = \chi(Z_k^\infty) = \sum_{\tau_p \in \mathcal{F}_\infty; \ell(\tau_p) \leq k-1} (-1)^{\ell(\tau_p)} + \sum_{w \in C(X_{k-1}^\infty); J'(w)=0} (-1)^{\text{morse}(w)}.
\]
It follows then that
\[
|1 - \sum_{\tau_p \in \mathcal{F}_\infty; \ell(\tau_p) \leq k-1} (-1)^{\ell(\tau_p)}| \leq \sum_{w; J'(w)=0, \text{morse}(w) \leq k} (-1)^{\text{morse}(w)} \leq \# \mathcal{N}_k,
\]
where \( \mathcal{N}_k \) denotes the set of solutions of \( (P_K) \) having their morse indices \( \leq k \). \hfill \Box

5 A general existence result

In this last section of this paper, we give a generalization of theorem 1.1. Namely instead of assuming that there are no critical points at infinity of index \( k \), we assume that the intersection number modulo 2, between the suspension of the complex at infinity of order \( k \), \( C(X_{k-1}^\infty) \) and the stable manifold of all critical points at infinity of index \( k \) is equal to zero. More precisely, for \( \tau_p \in \mathcal{F}_\infty \) such that \( \ell(\tau_p) = k \), we define the following intersection number:
\[
\mu_k(\tau_p) := C(X_{k-1}^\infty) \cdot W_s^\infty(\tau_p^\infty) \pmod{2}.
\]
Observe that this intersection number is well defined since we may assume by transversality that:
\[
\partial C(X_{k-1}^\infty) \cap W_s^\infty(\tau_p^\infty) = \emptyset.
\]
Indeed \( \dim(\partial C(X_{k-1}^\infty)) = k - 1 \), while \( \text{codim}(W_s^\infty(\tau_p^\infty)) = k \).

We are now ready to state the following existence result:
Theorem 5.1 Let $0 < K \in C^2(M)$ be a positive function satisfying the conditions $(C_0)$ and $(C_1)$. If there exists $k \in \mathbb{N}$ such that

1. \[ \sum_{\tau_p \in \mathcal{F}_\infty; \iota(\tau_p) \leq k-1} (-1)^{\iota(\tau_p)} \neq 1, \]

2. \[ \forall \tau_p \in \mathcal{F}_\infty, \text{ such that } \iota(\tau_p) = k, \text{ there holds } \mu_k(\tau_p) = 0. \]

Then, there exists a solution $w$ of the problem $(P_K)$ such that:

\[ \text{morse}(w) \leq k. \]

Moreover, for generic $K$ it holds

\[ \# \mathcal{N}_k \geq |1 - \sum_{\tau_p \in \mathcal{F}_\infty; \iota(\tau_p) \leq k-1} (-1)^{\iota(\tau_p)}|, \]

where $\mathcal{N}_k$ denotes the set of solutions of $(P_K)$ having their Morse indices less than or equal to $k$.

Proof. The proof goes along with the proof of theorem 1.1, therefore we will only sketch the differences. Keeping the notations of the proof of theorem 1.1, we observe that, since

\[ \forall \tau_p \in \mathcal{F}_\infty, \text{ such that } \iota(\tau_p) = k, \text{ there holds } \mu_k(\tau_p) = 0, \]

we may assume that the deformation of $C(X_{k-1}^\infty)$ along any pseudogradient flow of $-J$, avoids all critical points at infinity having their Morse indices equal to $k$. It follows then from (4.3) that $C(X_{k-1}^\infty)$ retracts by deformation onto

\[ Z_k^\infty := X_{k-1}^\infty \cup \bigcup_{w; J'(w)=0; w \text{ dominated by } C(X_{k-1}^\infty)} W_u(w). \] (5.1)

Now the remainder of the proof is identical to the proof of theorem 1.1. \hfill \square

References

[1] A. Ambrosetti and M. Badiale, *Homoclinics : Poincaré-Melnikov type results via a variational approach*, Ann. Inst. H. Poincaré Anal. Nonlinéaire 15 (1998), 233-252.

[2] T. Aubin, *Some nonlinear problem in differential geometry*, Springer-Verlag, New York 1997.
[3] T. Aubin and A. Bahri, *Méthode de topologie algébrique pour le problème de la courbure scalaire prescrite*, J. Math. Pures Appl., 76, 1997, 525–549.

[4] T. Aubin and A. Bahri, *Une hypothèse topologique pour le problème de la courbure scalaire prescrite*, J. Math. Pures Appl. **76** (1997), 843-850.

[5] A. Bahri, *Critical points at infinity in some variational problems*, Pitman Res. Notes Math, Ser **182**, Longman Sci. Tech. Harlow 1989.

[6] A. Bahri, *An invariant for Yamabe-type flows with applications to scalar-curvature problems in high dimension*, A celebration of John F. Nash, Jr. Duke Math. J. **81** (1996), 323-466.

[7] A. Bahri, *The scalar curvature problem on sphere of dimension n ≥ 7*, preprint 1996.

[8] A. Bahri and J. M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 255-294.

[9] A. Bahri and J. M. Coron, *The scalar curvature problem on the standard three dimensional spheres*, J. Funct. Anal. **95** (1991), 106-172.

[10] A. Bahri and P. Rabinowitz, *Periodic orbits of hamiltonian systems of three body type*, Ann. Inst. H. Poincaré Anal. Non linéaire **8** (1991), 561-649.

[11] M. Ben Ayed, Y. Chen, H. Chtioui and M. Hammami, *On the prescribed scalar curvature problem on 4-manifolds*, Duke Math. J. **84** (1996), 633-677.

[12] H. Brezis and J. M. Coron, *Convergence of solutions of H-systems or how to blow bubbles*, Arch. Rational Mech. Anal. **89** (1985), 21-56.

[13] V. Felli and F. Uguzzoni, *Some existence results for the Webster scalar curvature problem in presence of symmetry*, Ann. Mat. Pura Appl. **183** (2004), 469-493.

[14] G.B. Folland and E. Stein, *Estimates for $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429-522.

[15] N. Gamara, *The CR Yamabe conjecture, the case n = 1*, J. Eur. Math. Soc. **3** (2001), 105-137.

[16] N. Gamara, *The Prescribed scalar curvature on a 3-dimensional CR manifold*, Adv. Nonlinear Stud. **2** (2002), 193-235.

[17] N. Gamara and R. Yacoub, *CR Yamabe conjecture, the conformally flat case*, Pac. J. Math. **201** (2001), 121-175.
[18] N. Garofalo and E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. 41 (1992), 71-98.

[19] D. Jerison and J.M. Lee, *The Yamabe problem on CR manifolds*, J. Differential Geom. 25 (1987), 167-197.

[20] D. Jerison and J.M. Lee, *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Differential Geom. 29 (1989), 303-343.

[21] D. Jerison and J.M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. 1 (1988), 1-13.

[22] P. L. Lions, *The concentration compactness principle in the calculus of variations. The limit case*, Rev. Mat. Iberoamericana 1 (1985), I: 145-201; II: 45-121.

[23] A. Malchiodi and F. Uguzzoni, *A perturbation result for the Webster scalar curvature problem on the CR sphere*, J. Math. Pures Appl. 81 (2002), 983-997.

[24] R. Schoen and D. Zhang, *Prescribed scalar curvature on the n-sphere*, Calculus of Variations and Partial Differential Equations, 4 (1996), 1-25.

[25] M. Struwe, *A global compactness result for elliptic boundary value problems involving nonlinearities*, Math. Z. 187 (1984), 511-517.

[26] R. Yacoub, *The Webster scalar curvature problem on high dimensional CR manifolds*, to appear.