Propagation inhibition and wave localization in a 2D random liquid medium

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Acoustic propagation and scattering in water containing many parallel air-filled cylinders is studied. Two situations are considered and compared: (1) wave propagating through the array of cylinders, imitating a traditional experimental setup, and (2) wave transmitted from a source located inside the ensemble. We show that waves can be blocked from propagation by disorders in the first scenario, but the inhibition does not necessarily imply wave localization. Furthermore, the results reveal the phenomenon of wave localization in a range of frequencies.

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When propagating through media containing many scatterers, waves will be repeatedly scattered by each scatterer, forming a multiple scattering process [1]. Multiple scattering of waves is responsible for many fascinating phenomena such as random laser [9], electronic transport in impure solids [3], and photonic or acoustic bandgaps [4,5]. Under proper conditions, multiple scattering leads to the unusual phenomenon of wave localization. That is, waves in a randomly scattering medium are trapped in space and will remain confined around the initial transmitting site until dissipated.

Over the past twenty years, tremendous efforts have been devoted to the investigation of the localization phenomenon of classical waves in random media (e.g. Ref. [6,7,11]). Observation of classical wave localization is a difficult task, partially because suitable systems are hard to find and partially because observation is often complicated by such effects as absorption and attenuation. In most previous experimental studies, the apparatus is set up in such a way that waves are transmitted at one end of a scattering ensemble, then the scattered waves are recorded either on the other end to measure the transmission or at the transmitting site to measure the reflection from the sample. The results are subsequently compared with the previous theory to infer possible localization effects. In this way, observations of wave localization effects have been reported, for example, for microwaves [6], acoustic waves [11], and arguably for light [12,16], showing propagation inhibition and an exponential decay in wave transmission along the propagation path, the indications for wave localization, and tending to support the prevailing view that waves are localized in two dimensional (2D) systems with any amount of disorders for all frequencies [12].

The purpose of this Letter is twofold. First, we would like to point out that traditional experimental methods have uncertainties in discerning localization effects, as the observation can be obscured by effects like reflection and deflection. These effects attenuate waves, resulting in a similar decay in transmission and thus making the data interpretation ambiguous. We show that while wave localization does lead to an inhibition in wave propagation, but the propagation inhibition does not necessarily imply wave localization. In other words, it is necessary to differentiate the situation that waves are blocked from transmission from the situation that waves can be actually localized in the medium. Second, we show that in the system we study, although waves are not localized for all frequencies, wave localization is evident in a range of frequencies and when there is a sufficient density of random scatterers.

The model in this Letter is acoustic propagation in water containing many parallel air-filled cylinders. Different from the common approach that derives approximately a diffusion equation for the ensemble-averaged energy, our method is to solve the wave propagation from the fundamental wave equation, without resort to approximations. The model has been studied previously for the coherent complete bandgaps [17].

Consider \( N \) straight cylinders located at \( \vec{r}_i \) with \( i = 1, 2, \ldots, N \) to form either a random or regular array. An acoustic line source transmitting monochromatic waves is placed at \( \vec{r}_s \). The scattered wave from each cylinder is a response to the total incident wave composed of the direct wave from the source and the multiply scattered waves from other cylinders. The final wave reaching a receiver located at \( \vec{r}_r \) is the sum of the direct wave from the source and the scattered waves from all the cylinders. Such a scattering problem can be solved exactly, following Twersky [18]. While the details are in [19], the essential procedures are summarized below.

The scattered wave from the \( j \)-th cylinder can be written as

\[
p_s(\vec{r}, \vec{r}_j) = \sum_{n=-\infty}^{\infty} i \pi A_n H^{(1)}_{n}(k|\vec{r} - \vec{r}_j|) e^{i \phi_{\vec{r} - \vec{r}_j} - r_j},
\]

where \( k \) is the wavenumber of the medium, \( H^{(1)}_{n} \) is the \( n \)-th order Hankel function of the first kind, and \( \phi_{\vec{r} - \vec{r}_j} \) is the azimuthal angle of the vector \( \vec{r} - \vec{r}_j \) relative to the positive \( x \)-axis. The total incident wave around the \( i \)-th cylinder \( (i = 1, 2, \ldots, N; i \neq j) \) is the summation of the
direct incident wave from the source and the scattered waves from all other scatterers, and can be expressed as

\[ p_i^n(\vec{r}) = p_0(\vec{r}) + \sum_{j \neq i}^N p_s(\vec{r}, \vec{r}_j) \]

\[ = \sum_{n=-\infty}^{\infty} B_n^i J_n(k|\vec{r} - \vec{r}_i|) e^{i\phi_{r-\vec{r}_i}}. \tag{2} \]

To solve for \( A_n^i \) and \( B_n^i \), we express the scattered wave \( p_s(\vec{r}, \vec{r}_j) \), for each \( j \neq i \), in terms of the modes with respect to the \( i \)-th scatterer by the addition theorem for Bessel functions \[20\]. The resulting formula for the scattered wave \( p_s(\vec{r}, \vec{r}_j) \) is

\[ p_s(\vec{r}, \vec{r}_j) = \sum_{n=-\infty}^{\infty} C_n^{j,i} J_n(k|\vec{r} - \vec{r}_j|) e^{i\phi_{\vec{r}-\vec{r}_i}}, \tag{3} \]

with

\[ C_n^{j,i} = \sum_{l=-\infty}^{\infty} i\pi A_l^j H_n^{(1)}(k|\vec{r}_l - \vec{r}_j|) e^{i(l-j+\phi_{\vec{r}_l - \vec{r}_j}). \tag{4} \]

The direct incident wave around the location of the \( i \)-th cylinder can be expressed in a Bessel function expansion with respect to coordinates centered at \( \vec{r}_i \)

\[ p_0(\vec{r}) = i\pi H_n^{(1)}(kr) = \sum_{l=-\infty}^{\infty} S_l^i J_l(k|\vec{r} - \vec{r}_i|) e^{i\phi_{\vec{r}-\vec{r}_i}}, \tag{5} \]

with the known coefficients \( S_l^i \) \[20\].

Using equations (2), (3) and (5), we have

\[ B_n^i = S_n^i + \sum_{j=1, j \neq i}^N C_n^{j,i}. \tag{6} \]

The boundary conditions state that the pressure and the normal velocity be continuous across the interface between a scatterer and the surrounding medium, leading to

\[ B_n^i = i\pi \Gamma_n^i A_n^i, \tag{7} \]

where \( \Gamma_n^i \) are the transfer matrices relating the acoustic properties of the scatterers and the surrounding medium, and have been given by Eq. (21) in Ref. \[20\].

The coefficients \( A_n^i \) and \( B_n^i \) can be inverted from Eqs. (4), (6), and (7). Once \( A_n^i \) are determined, the transmitted wave at any spatial point is given by

\[ p(\vec{r}) = p_0(\vec{r}) + \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} i\pi A_n^i H_n^{(1)}(k|\vec{r} - \vec{r}_i|) e^{i\phi_{\vec{r}-\vec{r}_i}}. \tag{8} \]

The acoustic intensity is represented by the squared module of the transmitted wave.

In the following computation, we assume \( N \) uniform air-cylinders of radius \( a \). The fraction of area occupied by the cylinders per unit area is \( \beta \). The average distance between nearest neighbors is therefore \( d = (\pi/\beta)^{1/2}a \), which is also the lattice constant for the corresponding square lattice array. Two situations are considered: (1) wave propagating through the array of cylinders, labelled hereafter as the ‘outside’ situation that imitates the traditional experimental setup, and (2) wave transmitted from a source located inside the ensemble, labelled hereafter as the ‘inside’ situation. Both cases are illustrated by Fig. 1. For the ‘outside’ case, all cylinders are randomly placed within a rectangular area with length \( L \) and width \( W \). The transmitter and receiver are located at some distance from the two opposite sides of the scattering area. For the ‘inside’ situation, all cylinders are placed within a circle of radius \( L \) with the transmitting source located at the center and the receiver located outside the scattering cloud. In the computation, the acoustic intensity is normalized in such a way that its value equals unity when there are no scatterers present; thus the uninteresting geometrical spreading effect is naturally eliminated. We scale all lengths by the parameter \( d \), and the frequency in terms of non-dimensional \( ka \); in this way, the computation becomes non-dimensional.

![FIG. 1. Conceptual layout: (a) acoustic propagation through a cloud of cylinders; (b) acoustic transmission from a line source located inside a cylinder cloud.](image)

A set of numerical computations has been performed for various area fractions \( \beta \), numbers \( N \), and dimensionless frequency \( ka \). The major controlling parameter is \( \beta \). The transmitter and receiver are placed at a distance of \( 2d \) from the sample; in fact, we found that as long as we keep the symmetry the results remain qualitatively un-
changed as the positions of transmitter and receiver vary. Though it is the line source that is used in computation, the features hold even when a beamed cylindrical plane wave is used.

![Graph](image)

**FIG. 2.** The middle and right panels, referring to the ‘Inside’ and ‘Outside’ cases respectively, show the normalized acoustic transmission versus frequency in terms of non-dimensional $ka$. Here the comparison is made between the results from the corresponding square arrays (solid lines) and from the complete random array of cylinders (dotted lines). Left panel: The band structures computed by the plane wave expansion method.

![Graph](image)

**FIG. 3.** Normalized acoustic transmission and its fluctuation versus the sample size for two frequencies: the left and right panels refer to the ‘Inside’ and ‘Outside’ cases respectively. The estimated slopes for the transmission are indicated in the figure. Here $T \equiv p/p_0$.

For $ka = 0.01$, the transmission decays exponentially with the sample size for both ‘Inside’ and ‘Outside’ situations. In the ‘Outside’ case, there are two decay slopes, i.e., -0.0427 and -0.0019. There is a transition regime separating the two slopes. The transmission starts to decay with slope $-0.0427$, followed by a milder decay of slope $-0.0019$. The slope of $-0.0427$ is nearly the same as the decay slope of $-0.0460$ in the ‘Inside’ case. Obviously, this is because at $ka = 0.01$, waves are localized. This exponential decay actually indicates that waves are trapped or localized near the transmitting source; this is clearly shown by the top portion of Fig. 3(b). The second slope in the ‘Outside’ case for $ka = 0.01$ is due to the finite width of the sample, as will be discussed later. Inside the localization regime, the transmission fluctuation is small, as expected from an earlier work [11]. Here we see that within the localization regime, wave localization can be observed in both ‘Outside’ and ‘Inside’ scenarios.

For $ka = 0.05$, the ‘Inside’ and ‘Outside’ scenarios differ significantly. While for the ‘Outside’ case the transmission decreases exponentially with a slope of 0.0027 along the path, the transmission in the ‘Inside’ situation does not decrease. For the ‘Outside’ case, the transmission fluctuation increases then drops along the path, emulating the localization effect. For the ‘Inside’ case, however, the transmission fluctuation representing the diffusive intensity $\ln<|T|^2>$ increases as more and more scattering occurs along the path, fully complying with the well-known non-localized Milne diffusion. The large fluctuation implies that the transmission is sensitive to the distribution of the cylinders, another indication of the non-localization property [13]. In fact, the apparent decay in the ‘Outside’ case is due to the scattering attenuation that the waves are reflected and scattered to the
sides. These results suggest that waves are actually not localized for $ka = 0.05$, and it would be a mistake to interpret the exponential decay shown in the ‘Outside’ situation as the indication of wave localization.

Now we consider the width effect in the ‘Outside’ situation. In Fig. 3, the ensemble averaged transmission is plotted as a function of the sample size ($L$) for three widths ($W$) at two frequencies.

For frequencies inside the localization regime, the second slop is due to the finite width, and can be interpreted as follows. At $ka = 0.01$, the localization effect is dominant, thus leading to a rapid decay in transmission along the path, giving rise to the first slop. As expected, this slop is almost independent of the width. As the sample size increases, the directly transmitted waves are almost completely blocked. But, due to the finite width, a small amount of the deflected waves around the sample sides can still reach the receiver. When increasing the sample size ($L$) for a fixed width ($W$), this effect gradually diminishes along the path, yielding the second slop. Increasing width ($W$) will reduce this finite-width effect, and thus the amount of deflected waves reaching the receiver will also decrease, as shown. The width effect disappears when the width approaches infinity. For $ka = 0.05$, outside the localization regime, the transmission decays nearly exponentially along the path. As the width increases, the slop will become smaller, and will be saturated to a value of -0.0008. These results indicate that in the ‘Outside scenario’, the path-dependent transmission behaves similarly for frequencies either inside or outside the localization regime as long as the width is sufficiently large; therefore the phenomenon of wave localization cannot be isolated in this scenario.

Finally, we stress that whether waves are localized or extended is an intrinsic property of the system that is supposed to be infinite. This property does not depend on the source, and should not depend on boundaries either. We believe that while the source is placed inside the medium with increasing sizes, the infinite system might be mimicked and the localization property could be probed without ambiguity.

FIG. 4. Normalized acoustic transmission across a scattering array as a function of the sample length for different sample widths. The solid line in (b) is the numerical extrapolation as the width approaches infinity.

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