J-CLASS OF FINITELY GENERATED ABELIAN SEMIGROUPS OF AFFINE MPAS ON $\mathbb{C}^n$ AND HYPERCYCLICITY

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ABSTRACT. We give a characterization of hypercyclic using (locally hypercyclic) of semigroup $G$ of affine maps of $\mathbb{C}^n$. We prove the existence of a $G$-invariant open subset of $\mathbb{C}^n$ in which any locally hypercyclic orbit is dense in $\mathbb{C}^n$.

1. Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices of order $n \geq 1$ with entries in $\mathbb{C}$ and $GL(n, \mathbb{C})$ be the group of all invertible matrices of $M_n(\mathbb{C})$. A map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called an affine map if there exist $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C}^n$ such that $f(x) = Ax + a$, $x \in \mathbb{C}^n$. We denote $f = (A, a)$, we call $A$ the linear part of $f$. The map $f$ is invertible if $A \in GL(n, \mathbb{C})$. Denote by $MA(n, \mathbb{C})$ the vector space of all affine maps on $\mathbb{C}^n$ and $GA(n, \mathbb{C})$ the group of all invertible affine maps of $MA(n, \mathbb{C})$.

Let $G$ be an abelian affine sub-semigroup of $MA(n, \mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of $G$ through $v$: $G(v) = \{f(v) : f \in G\} \subset \mathbb{C}^n$. Denote by $\overline{E}$ the closure of a subset $E \subset \mathbb{C}^n$. The semigroup $G$ is called hypercyclic if there exists a vector $v \in \mathbb{C}^n$ such that $\overline{G(v)} = \mathbb{C}^n$. For an account of results and bibliography on hypercyclicity, we refer to the book [4] by Bayart and Matheron. We refer the reader to the recent book [4] for a thorough account on hypercyclicity. Costakis and Manoussos in [5] “localize” the concept of hypercyclicity using J-sets. By analogy, we generalize this notion to affine case as follow: Suppose that $G$ is generated by $p$ affines maps $f_1, \ldots, f_p$ ($p \geq 1$) then for $x \in \mathbb{C}^n$, we define the extended limit set $J_G(x)$ to be the set of $y \in \mathbb{C}^n$ for which there exists a sequence of vectors $(x_m)_m$ with $x_m \to x$ and sequences of non-negative integers $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \ldots, p$ with

$$
(1.1) \quad k_m^{(1)} + k_m^{(2)} + \cdots + k_m^{(p)} \to +\infty
$$

such that

$$
f_1^{k_m^{(1)}} f_2^{k_m^{(2)}} \cdots f_p^{k_m^{(p)}} x_m \to y.
$$

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Note that condition (1.1) is equivalent to having at least one of the sequences \( \{k_m^{(j)} : m \in \mathbb{N}\} \) for \( j = 1, 2, \ldots, p \) containing a strictly increasing subsequence tending to \( +\infty \). We say that \( \mathcal{G} \) is \textit{locally hypercyclic} if there exists a vector \( v \in \mathbb{C}^n \setminus \{0\} \) such that \( J_\mathcal{G}(v) = \mathbb{C}^n \). So, the question to investigate is the following: When an abelian sub-semigroup of \( MA(n, \mathbb{C}) \) can be hypercyclic?

The main purpose of this paper is twofold: firstly, we give a general characterization of the above question for any abelian \textit{sub-semigroup} of \( MA(n, \mathbb{C}) \) using J-sets. Secondly, we generalize the results proved in [3], by A.Ayadi and H.Marzougi, for linear semigroups, which answers negatively, the question raised in the paper of Costakis and Manoussos [5]: is it true that a locally hypercyclic abelian semigroup \( H \) generated by matrices \( A_1, \ldots, A_n \) is hypercyclic whenever \( J_H(x) = \mathbb{C}^n \) for a finite set of \( x \in \mathbb{C}^n \) whose vector space is equal \( \mathbb{C}^n \)? Similarly for \( \mathbb{R}^n \).

Denote by \( \mathcal{B}_0 = (e_1, \ldots, e_n) \) the canonical basis of \( \mathbb{C}^n \). Let \( P \in \text{GL}(n, \mathbb{C}) \).

Let introduce the following notations and definitions. Denote by:

- \( \mathbb{C}^* = \mathbb{C}\setminus \{0\} \), \( \mathbb{R}^* = \mathbb{R}\setminus \{0\} \) and \( \mathbb{N}_0 = \mathbb{N}\setminus \{0\} \).
- \( \mathcal{B}_0 = (e_1, \ldots, e_{n+1}) \) the canonical basis of \( \mathbb{C}^{n+1} \) and \( I_{n+1} \) the identity matrix of \( GL(n+1, \mathbb{C}) \).

For each \( m = 1, 2, \ldots, n+1 \), denote by:

- \( T_m(\mathbb{C}) \) the set of matrices over \( \mathbb{C} \) of the form

\[
\begin{bmatrix}
\mu \\
 a_{2,1} & \mu \\
 \vdots & \ddots & \ddots \\
 a_{m,1} & \cdots & a_{m,m-1} & \mu
\end{bmatrix}
\]

- \( T_m^*(\mathbb{C}) \) the group of matrices of the form \( (1.1) \) with \( \mu \neq 0 \).

Let \( r \in \mathbb{N} \) and \( \eta = (n_1, \ldots, n_r) \in \mathbb{N}_0^r \) such that \( n_1 + \cdots + n_r = n + 1 \). In particular, \( r \leq n + 1 \).

- \( K_{\eta,r}(\mathbb{C}) := T_{n_1}(\mathbb{C}) \oplus \cdots \oplus T_{n_r}(\mathbb{C}) \). In particular if \( r = 1 \), then \( K_{\eta,1}(\mathbb{C}) = T_{n+1}(\mathbb{C}) \) and \( \eta = (n + 1) \).
- \( K_{\eta,r}^*(\mathbb{C}) := K_{\eta,r}(\mathbb{C}) \cap \text{GL}(n+1, \mathbb{C}) \).
- \( u_0 = (e_{1,1}, \ldots, e_{r,1}) \in \mathbb{C}^{n+1} \) where \( e_{k,1} = (1,0,\ldots,0) \in \mathbb{C}^{n_k} \), for \( k = 1, \ldots, r \). So \( u_0 \in \{1\} \times \mathbb{C}^n \).
- \( p_2 : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n \) the second projection defined by \( p_2(x_1, \ldots, x_{n+1}) = (x_2, \ldots, x_{n+1}) \).
- Define the map \( \Phi : GA(n, \mathbb{C}) \rightarrow \text{GL}(n+1, \mathbb{C}) \)

\[
f = (A,a) \mapsto \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix}
\]
We have the following composition formula
\[
\begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Ab + a & AB \end{bmatrix}.
\]

Then \( \Phi \) is an injective homomorphism of groups. Write
- \( G = \Phi(\mathcal{G}) \), it is an abelian subgroup of \( GL(n + 1, \mathbb{C}) \).

Let consider the normal form of \( G \): By Proposition 2.1, there exists a \( P \in \Phi(GA(n, \mathbb{C})) \) and a partition \( \eta \) of \((n + 1)\) such that \( G' = P^{-1}GP \subset K^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C})) \). For such a choice of matrix \( P \), we let
- \( v_0 = Pu_0 \). So \( v_0 \in \{1\} \times \mathbb{C}^n \), since \( P \in \Phi(GA(n, \mathbb{C})) \).
- \( w_0 = p_2(v_0) \in \mathbb{C}^n \). We have \( v_0 = (1, w_0) \).
- \( \varphi = \Phi^{-1}(P) \in GA(n, \mathbb{C}) \).

Denote by \( V' := \prod_{k=1}^{r} \mathbb{C}^* \times \mathbb{C}^{n_k-1}, V = P(V') \) and \( U' = p_2(V') \), then
\[
U' = \begin{cases} \mathbb{C}^{n_1-1} \times \prod_{k=2}^{r} \mathbb{C}^* \times \mathbb{C}^{n_k-1} & \text{if } r \geq 2 \\ \mathbb{C}^{n_1-1} & \text{if } r = 1. \end{cases}
\]

For such choice of the matrix \( P \in \Phi(GA(n + 1, \mathbb{C})) \), we can write
\[
P = \begin{bmatrix} 1 & 0 \\ d & Q \end{bmatrix}, \quad \text{with } Q \in GL(n, \mathbb{C}).
\]

We have \( \varphi = (Q, d) \in GA(n, \mathbb{C}), U = \varphi(U') \) and \( G^* = G \cap GL(n + 1, \mathbb{C}) \). We have \( G^* \) is an abelian semigroup of \( GL(n + 1, \mathbb{C}) \).

Our principal results are the following:

**Theorem 1.1.** Let \( \mathcal{G} \) be a finitely generated abelian semigroup of affine maps on \( \mathbb{C}^n \). If \( J_{\mathcal{G}}(v) = \mathbb{C}^n \) for some \( v \in U \) then \( \overline{\mathcal{G}(v)} = \mathbb{C}^n \).

**Corollary 1.2.** Under the hypothesis of Theorem 1.1 the following are equivalent:
- (i) \( \mathcal{G} \) is hypercyclic.
- (ii) \( J_{\mathcal{G}}(w_0) = \mathbb{C}^n \).
- (iii) \( \overline{\mathcal{G}(w_0)} = \mathbb{C}^n \).

**Corollary 1.3.** Under the hypothesis of Theorem 1.1 set \( A = \{x \in \mathbb{C}^n : J_{\mathcal{G}}(x) = \mathbb{C}^n \} \). If \( \mathcal{G} \) is not hypercyclic then \( A \subset \bigcup_{k=1}^{r} H_k \), \((r \leq n)\) where \( H_k \) are \( \mathcal{G} \)-invariant affine subspaces of \( \mathbb{C}^n \) with dimension \( n - 1 \).
2. Preliminaries and basic notions

**Proposition 2.1.** Let $\mathcal{G}$ be an abelian sub-semigroup of $MA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a sub-semigroup of $K_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n + 1$ and $\eta = (n_1, \ldots, n_r) \in \mathbb{N}_0^r$.

To prove Proposition 2.1, we need the following proposition.

**Proposition 2.2.** (2, Proposition 2.1) Let $\mathcal{G}$ be an abelian subgroup of $GA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a subgroup of $K^*_r(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n + 1$ and $\eta = (n_1, \ldots, n_r) \in \mathbb{N}_0^r$.

**Proof of Proposition 2.1.** Suppose first, $G \subset GL(n + 1, \mathbb{C})$. Let $\hat{G}$ be the group generated by $G$. Then $\hat{G}$ is abelian and by Proposition 2.2, there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}\hat{G}P$ is an abelian subgroup of $K^*_r(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n + 1$ and $\eta = (n_1, \ldots, n_r) \in \mathbb{N}_0^r$. In particular, $P^{-1}\hat{G}P \subset K^*_r(\mathbb{C})$.

Suppose now, $G \subset M_{n+1}(\mathbb{C})$. For every $A \in G$, there exists $\lambda_A \in \mathbb{C}$ such that $(A - \lambda_A I_{n+1}) \in GL(n + 1, \mathbb{C})$ (one can take $\lambda_A$ non eigenvalue of $A$). Write $\hat{L}$ be the group generated by $L := \{A - \lambda_A I_{n+1} : A \in G\}$. Then $\hat{L}$ is an abelian subsemigroup of $GL(n + 1, \mathbb{C})$. Hence by above, there exists a $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}\hat{L}P \subset K^*_r(\mathbb{C})$, for some $\eta \in (\mathbb{N}_0)^r$. As

$$P^{-1}LP = \{P^{-1}AP - \lambda_A I_{n+1} : A \in G\}$$

then $P^{-1}GP \subset K_{\eta,r}(\mathbb{C})$. This proves the proposition. □

Let $\mathcal{G}$ be the semigroup generated by $G$ and $\mathbb{C}I_{n+1} = \{\lambda I_{n+1} : \lambda \in \mathbb{C}\}$. Then $\mathcal{G}$ is an abelian sub-semigroup of $M(n + 1, \mathbb{C})$. By Proposition 2.1, there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a sub-semigroup of $K_{\eta,r}(\mathbb{C})$ for some $r \leq n + 1$ and $\eta = (n_1, \ldots, n_r) \in \mathbb{N}_0^r$ and this also implies that $P^{-1}\hat{G}P$ is a sub-semigroup of $K_{\eta,r}(\mathbb{C})$.

**Lemma 2.3.** Let $x \in \mathbb{C}^n$ and $G = \Phi(\mathcal{G})$. The following are equivalent:

(i) $\mathcal{G}(x) = \mathbb{C}^n$.
(ii) $G(1, x) = \{1\} \times \mathbb{C}^n$.
(iii) $G(1, x) = \mathbb{C}^{n+1}$. 

\textbf{Proof.} (i) \iff (ii) : is obvious since \(\{1\} \times \mathcal{G}(x) = \mathcal{G}(1, x)\) by construction.

(iii) \implies (ii) : Let \(y \in \mathbb{C}^n\) and \((B_m)_m\) be a sequence in \(\mathcal{G}\) such that
\[
\lim_{m \to +\infty} B_m(1, x) = (1, y).
\]
One can write \(B_m = \lambda_m \Phi(f_m)\), with \(f_m \in \mathcal{G}\) and \(\lambda_m \in \mathbb{C}^*\), thus \(B_m(1, x) = (\lambda_m, \lambda_m f_m(x))\), so \(\lim_{m \to +\infty} \lambda_m = 1\). Therefore,
\[
\lim_{m \to +\infty} \Phi(f_m)(1, x) = \lim_{m \to +\infty} \frac{1}{\lambda_m} B_m(1, x) = (1, y).
\]
Hence, \((1, y) \in \mathcal{G}(1, x)\).

(ii) \implies (iii) : Since \(\mathbb{C}^{n+1} \setminus \{(0) \times \mathbb{C}^n\} = \bigcup_{\lambda \in \mathbb{C}^*} (\{1\} \times \mathbb{C}^n)\) and for every \(\lambda \in \mathbb{C}^*, \lambda \mathcal{G}(1, x) \subset \tilde{\mathcal{G}}(1, x)\), we get
\[
\mathbb{C}^{n+1} = \overline{\mathbb{C}^{n+1} \setminus \{(0) \times \mathbb{C}^n\} = \bigcup_{\lambda \in \mathbb{C}^*} \lambda \mathcal{G}(1, x) \subset \tilde{\mathcal{G}}(1, x)}
\]
Hence \(\mathbb{C}^{n+1} = \overline{\mathcal{G}(1, x)}\). \hfill \Box

We will use the following theorem, to prove Lemma 2.5.

\textbf{Theorem 2.4.} (\cite{1}, Theorem 1.1) Let \(\tilde{\mathcal{G}}\) be a finitely generated abelian semigroup of matrices of \(M_{n+1}(\mathbb{C})\). If \(J_{\tilde{\mathcal{G}}}(v) = \mathbb{C}^{n+1}\) for some \(v \in V\) then \(\overline{\mathcal{G}(v)} = \mathbb{C}^{n+1}\).

\textbf{Lemma 2.5.} Let \(\mathcal{G}\) be an abelian semigroup of affine maps generated by \(f_1, \ldots, f_p\) and \(x \in \mathbb{C}^n\). Then the following assertion are equivalent:

(i) \(y \in J_{\tilde{\mathcal{G}}}(x)\).

(ii) \((1, y) \in J_{\tilde{\mathcal{G}}}(1, x)\).

\textbf{Proof.} (i) \implies (ii) : Since \(y \in J_{\tilde{\mathcal{G}}}(x)\), then there exists a sequence of vectors \((x_m)_m\) with \(x_m \to x\) and sequences of non-negative integers \(\{k_m^{(j)} : m \in \mathbb{N}\}\) for \(j = 1, 2, \ldots, p\) with
\[
(1.1) \quad k_m^{(1)} + k_m^{(2)} + \cdots + k_m^{(p)} \to +\infty
\]
such that
\[
f_1^{k_m^{(1)}} f_2^{k_m^{(2)}} \cdots f_p^{k_m^{(p)}} x_m \to y.
\]
Therefore, \((1, x_m) \to (1, x)\) and sequences of non-negative integers \(\{k_m^{(j)} : m \in \mathbb{N}\}\) for \(j = 1, 2, \ldots, p\) such that
\[
\Phi(f_1)^{k_m^{(1)}} \Phi(f_2)^{k_m^{(2)}} \cdots \Phi(f_p)^{k_m^{(p)}} (1, x_m) \to (1, y).
\]
Hence \((1, y) \in J_{\tilde{\mathcal{G}}}(1, x)\).

(ii) \implies (i) : Since \((1, y) \in J_{\tilde{\mathcal{G}}}(1, x)\), then there exists a sequence of vectors
(λ_m, x_m) ∈ ℂ × ℂ^n with (λ_m, x_m) → (1, x) and sequences of non-negative integers \( \{k_m^{(j)} : m \in \mathbb{N}\} \) for \( j = 1, 2, \ldots, p \) with

\[
(1.2) \quad k_m^{(1)} + k_m^{(2)} + \cdots + k_m^{(p)} \rightarrow +\infty
\]

such that

\[
\alpha k_m^{(1)} \cdots \alpha k_m^{(p)} \Phi(f_1)^{k_m^{(1)}} \Phi(f_2)^{k_m^{(2)}} \cdots \Phi(f_p)^{k_m^{(p)}} (λ_m, x_m) \rightarrow (1, y).
\]

Denote by \( c_m = \alpha k_m^{(1)} \cdots \alpha k_m^{(p)} \) and \( g_m = \alpha k_m^{(1)} \cdots \alpha k_m^{(p)} \Phi(f_1)^{k_m^{(1)}} \Phi(f_2)^{k_m^{(2)}} \cdots \Phi(f_p)^{k_m^{(p)}} \).

Then \( \lim_{m \to +\infty} c_m λ_m = 1 \). As \( \lim_{m \to +\infty} λ_m = 1 \), so \( \lim_{m \to +\infty} c_m = 1 \). It follows that \( \lim_{m \to +\infty} \frac{1}{c_m} g_m(1, x_m) = (1, y) \). Hence \( \lim_{m \to +\infty} f_1^{k_m^{(1)}} f_2^{k_m^{(2)}} \cdots f_p^{k_m^{(p)}} (x_m) = y \).

Proof of Theorem 1.1. Suppose that \( J_G(v) = ℂ^n \) with \( v \in U \). By Proposition 2.1, we can assume that \( G ⊂ K_η,ρ(ℂ) \) and \( P = I_{n+1} \). By Lemma 2.5, we have \( \tilde{J}_G(1, v) = ℂ^n+1 \). Then by Theorem 2.4, \( G(1, v) = ℂ^n+1 \). It follows by Lemma 2.3, that \( G(v) = ℂ^n \).

Now to prove Corollary 1.2 we need to use the following proposition:

**Proposition 2.6.** \[5\] Let \( G \) be an abelian sub-semigroup of \( M_{n+1}(ℂ) \) generated by \( A_1, \ldots, A_p, p ≥ 1 \). Then \( G \) is hypercyclic if and only if \( J_G(x) = ℂ^{n+1} \) for every \( x \in ℂ^{n+1} \).

Proof of Corollary 1.2. \( (i) \implies (ii) \) follows from Proposition 2.6 and \( (ii) \implies (iii) \) results from Theorem 1.1. \( (iii) \implies (i) \) is trivial.

Proof of Corollary 1.3. If \( G \) is not hypercyclic then by Theorem 1.1, \( J_G(v) ≠ ℂ^n \) for any \( v \in U' \), thus \( U \cap A = \emptyset \) and therefore \( A ⊂ ℂ^n \setminus U \). On the other hand, \( U = \varphi(U') \), so

\[
ℂ^n \setminus U = ℂ^n \setminus \varphi(U') = \varphi(ℂ^n \setminus U') = \varphi \left( \bigcup_{k=1}^{r} L_k \right) \text{ (by (1))}
\]

\[
= \bigcup_{k=1}^{r} \varphi(L_k)
\]
with $L_k = \{ x = [x_1, \ldots, x_r]^T, x_k \in \{ 0 \} \times \mathbb{C}^{n_k-1}, x_i \in \mathbb{C}^{n_i}, \text{ if } i \neq k \}$. It follows that $A \subset \bigcup_{k=1}^{r} H_k$ with $H_k = \varphi(L_k)$ is an affine space with dimension $n - 1$. □