CONTROLLD FLOYD SEPARATION AND NON RELATIVELY HYPERBOLIC GROUPS

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Abstract. We introduce the notion of controlled Floyd separation between geodesic rays starting at the identity in a finitely generated group $G$. Two such geodesic rays are said to be Floyd separated with respect to quasigeodesics if the (Floyd) length of $c$--quasigeodesics (for fixed but arbitrary $c$) joining points on the geodesic rays is asymptotically bounded away from zero. The main purpose of this paper is to furnish an example of a finitely generated group $G$ such that
1) all finitely presented subgroups of $G$ are hyperbolic,
2) $G$ has an uncountable family of geodesic rays that are Floyd separated with respect to quasigeodesics,
3) $G$ is not hyperbolic relative to any collection of proper subgroups.
This provides a piece of negative evidence towards a question of Olshanskii, Osin and Sapir.

On the way towards constructing $G$, we construct a malnormal infinitely generated (and hence non-quasiconvex) subgroup of a free group, giving negative evidence towards a question of Swarup and Gitik.

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1. Introduction

In this paper, we introduce the notion of controlled Floyd separation between geodesic rays starting at the identity of a finitely generated group $G$. Here, the term ‘control’ refers in general to a choice of a $G$–invariant subcollection of paths in a Cayley graph $\Gamma$ of $G$ with respect to which the Floyd distance is computed (See Definition 2.15). We shall primarily be interested in subcollections given by $(\kappa, \kappa)$–quasigeodesics in $\Gamma$. The main purpose of this paper is to furnish an example of a finitely generated group $G$ which satisfies the following properties:

1. There are uncountably many geodesic rays starting at the identity in $G$, which are Floyd separated with respect to $(\kappa, \kappa)$–quasigeodesics for any $\kappa > 1$. See Theorem 7.1.
2. $G$ is not hyperbolic relative to any collection of proper subgroups. See Theorem 4.1.

This provides a piece of negative evidence towards a question of Olshanskii, Osin and Sapir [OOS09, Problem 7.11].

The group $G$ we construct has a number of other features:

1. $G$ is a graded small cancellation (and hence lacunary hyperbolic) group in the sense of Olshanskii, Osin and Sapir [OOS09]. See Theorem 5.7.
2. $G$ is a direct limit of cubulated hyperbolic groups $G_n$; each $G_n$ is hyperbolic and admits a geometric action on a non-positively curved square complex (in fact a $\mathcal{VH}$ complex in the sense of Wise [Wis12]). See Proposition 3.10.
3. Every finitely presented subgroup of $G$ is hyperbolic (Proposition 3.11). In particular, $G$ does not contain any Baumslag-Solitar group $BS(m,n)$.
4. $G$ has trivial Floyd boundary in the usual sense. See Theorem 8.1.

On the way towards constructing $G$, we construct (Proposition 3.2) a malnormal infinitely generated (and hence non-quasiconvex) subgroup of a free group, giving negative evidence towards a question of Swarup and Gitik [Bes04, Question 1.8].

A range of tools is used in proving the properties of $G$:

1. The theory of relative hyperbolicity and relative quasiconvexity [Far98, Bow12, Hru10].
2. Graphical small cancellation theory [Wis01, Wis02].
3. Graded small cancellation theory [OOS09, Ols91].
4. Tree graded spaces [DS05].
5. Asymptotic cones and lacunary hyperbolic groups [OOS09, Gro93].

**A Mnemonic:** In Theorem 7.1 we show that ideal points on the Gromov boundary of a qi-embedded free subgroup $K_2$ of $G$ are Floyd-separated with respect to quasigeodesics. To show this we pass to the asymptotic cone $\text{Con}^{\omega} G$ of $G$ which is
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a circle tree (Proposition 6.5). The asymptotic cone of $K_2$ is embedded as an $R$-tree $T$ in $\text{Con}^\omega G$. Theorem 6.7 shows that the intersection of any circle in $\text{Con}^\omega G$ has to be a single point or empty. In particular, it cannot be an arc. The reader may think of this as the statement ‘$D$’s are not allowed inside $\text{Con}^\omega G$’.

The proof of Theorem 8.1 shows that in fact it is possible for a circle in $\text{Con}^\omega G$ to touch $T$ at a point. Thus, ‘Circles tangential to lines is possible in $\text{Con}^\omega G$’.

1.1. History of the Problem. The notion of a Floyd function and Floyd boundary were introduced by Floyd in [Flo80] and generalized by Gromov in [Gro93] under the name of conformal boundary of a group. Gromov [Gro93, p. 264] and Karlsson [Kar03] showed that the action of a group on its Floyd boundary is a convergence action (in fact a geometric convergence action in the sense of Gerasimov) provided the boundary is non-trivial, i.e. if it contains more than two (and hence infinitely many) points. In [OOS09] Proposition 4.28, Olshanskii, Osin and Sapir showed that if $G$ is a finitely generated group whose Floyd boundary consists of at least 2 points, then all asymptotic cones of $G$ have cut points. These two facts may be regarded as evidence towards a positive answer to the following question:

**Question 1.1.** [OOS09, Problem 7.11] (see also [ACT14, p.18].) Suppose that a finitely generated group $G$ has a nontrivial Floyd boundary. Is $G$ hyperbolic relative to a collection of proper subgroups?

Theorem 7.1 shows that a weaker question has a negative answer. A very general and detailed treatment of relative hyperbolicity and its characterization in topological and dynamic terms has been carried out recently by Gerasimov and Potyagailo in a sequence of papers [Ger09, Ger12, GP11, GP13]. The notion of a controlled Floyd separation is related to, but different from the notion of a Karlsson function introduced in [GP11].

1.2. Notation.

1. $|h| = d(h, 1)$
2. $(h, H) = \max_{k \in H} (h, k)$
3. $\text{Rose}(F_2)$ will denote the rose on two petals labeled by the generators of $F_2$. $K_n$ will be a free subgroup of $F_2$ of rank $n$. $\text{Rose}(K_n)$ will denote the rose on $n$ petals labeled by the generators of $K_n$.
4. If $H$ is a subgroup of a (relatively) hyperbolic group $G$, the limit set of $H$ is denoted $\Lambda_H$.
5. $T(n)$ denotes the tower function of height $n$ with base 2, i.e. $T(n) = 2^{2^n}$, where the power is taken $n$ times.
6. $U(n)(a, b)$ is the word (element) in the free group $F_2$ on generators $a, b$ given by

$$U(n)(a, b) = [a^{T(n)+1}]b^{T(n)+1}\cdot\cdot\cdot[a^{T(n)+2}]b^{T(n)+2}\cdot\cdot\cdot[a^{T(n+1)}]b^{T(n+1)}].$$

We shall be using two equivalent definitions regarding quasigeodesics in this paper, which we elucidate here at the outset.
Definition 2.1. Let \((X,d)\) be a geodesic metric space.

1) A path \(\sigma : I \to X\) is a \((K,\varepsilon)\)-quasigeodesic with \(K \geq 1, \varepsilon \geq 0\) if for all \(t_1, t_2 \in I\),
\[
\frac{1}{K}|t_1 - t_2| - \varepsilon \leq d(\sigma(t_1), \sigma(t_2)) \leq K|t_1 - t_2| + \varepsilon.
\]

2) Alternately, \([\text{OOS09}]\) (Section 2.4), \(\sigma : I \to X\) is a \((\lambda, \varepsilon)\)-quasigeodesic with \(\lambda \in (0,1], \varepsilon \geq 0\) if for all \(t_1, t_2 \in I\),
\[
d(\sigma(t_1), \sigma(t_2)) \geq \lambda l(\sigma([t_1, t_2])) - \varepsilon,
\]
where \(l(\sigma([t_1, t_2]))\) denotes the length of the subpath from \(\sigma(t_1)\) to \(\sigma(t_2)\).

We use the first definition when we want \(K \geq 1\) and the second one when we want \(\lambda \in (0,1]\). The second one is used primarily in Section 3.

2.1. Relative Hyperbolicity. We quickly recall the notion of relative hyperbolicity introduced by Gromov \([\text{Gro85}]\). An equivalent notion was introduced by Farb \([\text{Far98}]\) and the equivalence of the two proven by Bowditch \([\text{Bow12}]\).

Let \((X,d)\) be a path metric space. A collection of closed subsets \(\mathcal{H} = \{H_\alpha\}\) of \(X\) will be said to be uniformly separated if there exists \(\varepsilon > 0\) such that \(d(H_1, H_2) \geq \varepsilon\) for all distinct \(H_1, H_2 \in \mathcal{H}\).

Definition 2.2. \([\text{Gro85}]\) For any geodesic metric space \((H,d)\), the hyperbolic cone (analog of a horoball) \(H^h\) is the metric space \(H \times [0,\infty) = H^h\) equipped with the path metric \(d_h\) obtained as follows:

1) \(d_h((x,t), (y,t)) = 2^{-t}d_H(x,y)\), where \(d_H\) is the induced path metric on \(H \times \{t\}\). Paths joining \((x,t), (y,t)\) and lying on \(H \times \{t\}\) are called horizontal paths.

2) \(d_h ((x,t), (x,s)) = |t - s|\) for all \(x \in H\) and for all \(t, s \in [0,\infty)\), and the corresponding paths are called vertical paths.

3) For all \(x, y \in H^h\), \(d_h(x, y)\) is the path metric induced by the collection of horizontal and vertical paths.

Definition 2.3. Let \(\Gamma\) be the Cayley graph of a group \(G\) and let \(\mathcal{H}\) be a collection of (left cosets of) finitely many subgroups. \(G\) is said to be hyperbolic relative to \(\mathcal{H}\) in the sense of Gromov, if \(G(\Gamma, \mathcal{H})\), obtained by attaching the hyperbolic cones \(aH^h\) to \(aH \in \mathcal{H}\) by identifying \((z, 0)\) with \(z\) for all \(aH \in \mathcal{H}\) and \(z \in aH\), is a complete hyperbolic metric space. The collection \(aH^h : H \in \mathcal{H}\) is denoted as \(\mathcal{H}^h\). The induced path metric is denoted as \(d_h\).

The boundary \(\partial G(\Gamma, \mathcal{H})\) is called the Bowditch boundary.

Here \(H \in \mathcal{H}\) are thought of as horosphere-like sets and \(H \times [0,\infty)\) as horoballs. The following Theorem of Bowditch \([\text{Bow12}]\) shall be useful (see also \([\text{Mj08}]\)).

Theorem 2.4. \([\text{Bow12}]\) Let \(G\) be a hyperbolic group without torsion. Then \(G\) is hyperbolic relative to \(\mathcal{P}\) if and only if

1. Each \(P_i\) is quasiconvex in \(G\).
2. For all \(P_i, P_j \in \mathcal{P}\) and \(g \in G\), \(gP_i g^{-1} \cap P_j = \{1\}\) unless \(i = j\) and \(g \in P_i\).

In particular if \(\mathcal{P}\) has exactly one subgroup \(P_i\), and \(G\) is hyperbolic relative to \(P_i\), the latter is malnormal quasiconvex in \(G\).
2.1.1. Relative Quasiconvexity. In [Hru10] Hruska gives a number of equivalent criteria for relative quasiconvexity for subgroups of relatively hyperbolic groups. We give one of these below to fix notions.

**Definition 2.5.** [Hru10] Let $G$ be hyperbolic relative to a collection $\mathcal{P}$ of parabolic subgroups. A subgroup $H \subset G$ is relatively quasiconvex if the following holds. Let $S$ be some (any) finite relative generating set for $(G, \mathcal{P})$, and let $E$ be the set containing all the elements of $P_i \in \mathcal{P}$ (for all $i$). Let $\Gamma$ be the Cayley graph of $G$ with respect to the generating set $S \cup E$ with all edges of length one. Let $d$ be some (any) proper, left invariant metric on $G$. Then there is a constant $D_0 = D_0(S, d)$ such that for each geodesic $\bar{c}$ in $\Gamma$ connecting two points of $H$, every vertex of $\bar{c}$ lies within a $d$–distance $D_0$ of $H$.

It is shown in [Hru10] that the above definition is independent of the choice of finite relative generating set $S$ and the choice of proper metric $d$.

**Theorem 2.6.** [Hru10] [Theorem 1.2] Let $G$ be a countable group that is relatively hyperbolic with respect to a finite family of subgroups $\mathcal{P} = \{P_1, \ldots, P_n\}$.

1. If $H \subset G$ is relatively quasiconvex, then $H$ is relatively hyperbolic with respect to a natural induced collection of subgroups.
2. If $H_1, H_2 \subset G$ are relatively quasiconvex, then $H_1 \cap H_2$ is also relatively quasiconvex.

**Theorem 2.7.** [Hru10] [Theorem 1.5] Let $G$ be a finitely generated relatively hyperbolic group and let $H$ be a finitely generated subgroup. If $H$ is undistorted in $G$, then $H$ is relatively quasiconvex.

Combining the equivalence of characterizations in [Hru10] with a Lemma of Yang we get

**Lemma 2.8.** [Yan12] [Lemma 2.6] Let $H$ be relatively quasiconvex in a relatively hyperbolic group $G$ such that $|\Lambda(H)| \geq 2$. Then for any subgroup $H \subset J \subset G$ satisfying $\Lambda(H) = \Lambda(J)$, we have that $H$ is of finite index in $J$. In particular, $J$ is relatively quasiconvex.

2.2. Floyd Boundary and Controlled Floyd Separation.

**Definition 2.9.** A function $f : \mathbb{N} \rightarrow \mathbb{R}$ is a Floyd scaling function if

1. $\sum_{n=0}^{\infty} f_n < \infty$,
2. There exists a positive $\lambda$ such that $1 \geq f_{n+1}/f_n \geq \lambda$ for all $n \in \mathbb{N}$.

In this paper we shall refer to a Floyd scaling function simply as a scaling function.

Let $G$ be a group with symmetric generating set $S$ and $\Gamma = \Gamma(G, S)$ denote the Cayley graph of $G$ with respect to $S$ and let $d$ denote the word-metric.

**Definition 2.10.** Let $f$ be a scaling function. Given an edge $[x, y]$ of $\Gamma$, define its Floyd length to be $f(d([x, y], 1))$. The resulting path metric on $\Gamma$ is called the Floyd metric with respect to the scaling function $f$ and is denoted as $d_f$.

The metric completion $(\overline{\Gamma, d_f})$ is called the Floyd completion and $\partial_f \Gamma = \overline{\Gamma, d_f} \setminus \Gamma$ is called the Floyd boundary. For $f(n) = \lambda^n$ with $\lambda \in (0, 1)$, $d_f$ will also be denoted as $d_{\lambda}$ and $\partial_f \Gamma$ will also be denoted as $\partial_{\lambda} \Gamma$.

We give a somewhat different but equivalent description of the Floyd boundary that will be better adapted to the applications we have in mind.
Lemma 2.11. Let $f$ be a (Floyd) scaling function for $\Gamma$. Let $r$ be a geodesic ray in $\Gamma$ starting at $1 \in \Gamma$. Then $r$ is a Floyd geodesic in $(\Gamma, d_f)$.

Proof. Let $p_m, p_n$ be points on $r$ such that $d(1, p_m) = m$ and $d(1, p_n) = n$. Then any path $\sigma$ from $p_m$ to $p_n$ must have, for $m \leq i < n$, at least one edge $E_i$ joining a point $q_i$ to a point $q_{i+1}$ where $d(q_i, 1) = i$ and $d(q_{i+1}, 1) = i + 1$ since any such path must travel between the $i-$ and $(i+1)-$spheres at least once. Hence $l_f(\sigma) \geq \sum_{m=1}^{n-1} l_f(E_i) = l_f(r)$. \hfill \Box

Definition 2.12. The set of geodesic rays in $\Gamma$ starting at the identity will be called the preboundary of $\Gamma$.

Since geodesic rays starting at the identity in $\Gamma$ are also Floyd geodesics, we shall denote the preboundary as $\text{preboundary}(\Gamma)$, emphasizing the scaling function $f$.

Remark 2.13. A sequence of points tending to infinity along a geodesic ray $r$ starting at the identity is necessarily Cauchy in the Floyd metric and hence $\text{preboundary}(\Gamma)$ can be identified with the ideal points of these rays. If $p$ denote such an ideal point, we shall also denote the ray from 1 corresponding to $p$ as $[1, p)$. It follows from the definition of Floyd boundary that the Floyd boundary of $\Gamma$ is a singleton set if and only if for any pair $p, q \in \text{preboundary}(\Gamma)$ and sequences $p_n \to p$, $q_n \to q$, there exist paths $\sigma_n$ in $\Gamma$ joining $p_n, q_n$ such that $l_f(\sigma_n) \to 0$ as $n \to \infty$.

The following Theorem will be used later.

Theorem 2.14. [Ger12, Proposition 3.4.6] Let $G$ be relatively hyperbolic with respect to a collection $\mathcal{P}$ of subgroups. Then there exists $\lambda \in (0, 1)$ such that the identity map $G \to G$ extends to a continuous equivariant map $F$ from the Floyd completion $(\overline{\Gamma, d_f})$ with respect to $f(n) = \lambda^n$ to the Bowditch completion of $G$ with respect to $\mathcal{P}$. In particular, the Floyd boundary of $G$ is non-trivial.

2.2.1. Controlled Floyd Separation. It is at this stage that we introduce an essentially new ingredient.

Definition 2.15. Let $f$ be a scaling function and let $l(\sigma)$ denote the Floyd length of a path $\sigma$ with respect to the scaling function $f$. Let $\Lambda$ denote a $G-$invariant collection of paths in $\Gamma$ such that for every $p, q \in \Gamma$, the geodesic joining $p, q$ belongs to $\Lambda$. Let $\Lambda(p, q)$ denote the subcollection of paths in $\Lambda$ joining $p, q$. We let $d_{f, \Lambda}(p, q) = \inf_{\sigma \in \Lambda(p, q)}(l(\sigma))$.

If $\Lambda$ consists of $(\kappa, \kappa)$-quasigeodesics then we also denote $d_{f, \Lambda}(p, q)$ by $d_{f, \kappa}(p, q)$.

For $p, q \in \text{preboundary}(\Gamma)$ we define $d_{f, \Lambda}(p, q)$ to be $\liminf_{p_n \to p, q_n \to q} d_{f, \Lambda}(p_n, q_n)$ where $p_n \in [1, p)$ and $q_n \in [1, q)$ are Cauchy sequences converging to $p, q$ respectively.

Note that for $p, q \in \text{preboundary}(\Gamma)$, $d_{f, \Lambda}(p, q)$ is independent of the Cauchy sequences $(p_n), (q_n)$. If $\Lambda$ consists of $(\kappa, \kappa)$-quasigeodesics and $p, q \in \text{preboundary}(\Gamma)$, $d_{f, \Lambda}(p, q)$ is also denoted as $d_{f, \kappa}(p, q)$.

Definition 2.16. If $d_{f, \Lambda}(p, q) > 0$ for $p, q \in \text{preboundary}(\Gamma)$, we shall say that $p, q$ are Floyd separated with respect to $\Lambda$. If $d_{f, \kappa}(p, q) > 0$ for $p, q \in \text{preboundary}(\Gamma)$ and for all $\kappa$, we shall say that $p, q$ are Floyd separated with respect to quasigeodesics.

Remark 2.17. (Gerasimov) Note that $d_{f, \kappa}(p, q)$ is not necessarily a metric as the triangle inequality need not be satisfied.

In future, if we want to emphasize that the Floyd length of a path $\sigma$ is being computed with respect to a Floyd function $f$ and $\Lambda$ consists of $(\kappa, \kappa)$-quasigeodesics, we shall use the suggestive notation $l^{f}_{f}(\sigma)$. 
2.3. Graphical Graded Small Cancellation. In this Subsection we give a discussion of graphical Graded Small Cancellation following Wise [Wis01, Wis02]. Our definitions are equivalent to those occurring in [Wis01, Wis02]. Let B be a metric graph with edges of length one. Let A be a (non-metric) topological graph (i.e. equipped with only a CW complex structure). A map \( \phi : A \to B \) is combinatorial if

1. \( \phi \) takes vertices of A to vertices of B.
2. \( \phi \) is an immersion restricted to the interior of every edge of A (where an immersion is a local injection).

A combinatorial map \( \phi \) induces a metric graph structure \( A_\phi \) on A after subdividing A (e.g. by pulling back the CW structure on B via \( \phi \)) making \( \phi \) a simplicial isometry restricted to the interior of every edge of \( A_\phi \).

A path \( \phi : P \to B \) is a combinatorial map where \( P \) is a real interval. \(|P|\) will denote the length of \( P \), i.e. the number of edges in the graph \( P_\phi \). When defined, \( PQ \to B \) will denote the concatenation of \( P \to B \) and \( Q \to B \). \( P^{-1} \) denotes the inverse of \( P \). In this paper, there will be a correspondence between combinatorial paths and words, and so the notation \(|W|\) for the length of a word, agrees with the notation \(|W|\) for the length of the corresponding path.

A (non-metric) graph A is graded if it is equipped with a function \( g : \text{Edges}(A) \to \mathbb{N} \), where \( g(J) \) is called the grade of \( J \). Let \( \phi : A \to B \) be a combinatorial map where \( A \) is a graded graph and \( B \) is a metric graph. Let \( A_\phi \) denote the induced metric graph structure on \( A \). Let \( J \) be a 1-cell of the (non-metric) graph \( A \) and \( J_\phi \), the induced metric graph structure on it. \( i_J \) denotes the inclusion of \( J \) in \( A \).

**Definition 2.18.** A combinatorial path \( \alpha : P \to J \) is a piece of \( J \) of grade \( n \) provided the following hold:

1. There exists a path \( \beta : P \to J' \) where \( J' \) is an edge of \( A \) with \( g(J') = n \) and \( n \) is the least such natural number.
2. \( \alpha : P \to J \) and \( \beta : P \to J' \) represent distinct paths in \( A \), i.e. \( i_J(\alpha(P)) \) and \( i_J(\beta(P)) \) are distinct edge-paths in \( A_\phi \).
3. \( \alpha : P \to J \) and \( \beta : P \to J' \) project to the same path in \( B \), i.e. \( \phi(i_J(\alpha(P))) \) and \( \phi(i_J(\beta(P))) \) give the same edge path in \( B \).
4. \( P \) is maximal with respect to the above conditions.

**Definition 2.19.** [Wis02][Definition 2.2] Let \( \phi : A \to B \) be a combinatorial map from a (non-metric) graded graph \( A \) to a metric graph \( B \). \( \phi \) satisfies the graded c\( (p) \) condition provided that for each edge \( J \) of the (non-metric) graded graph \( A \), and for any expression of \( J_\phi \) as the concatenation \( P_1, \ldots, P_r \) of pieces of \( J \) of grade \( \leq g(J) \), at least \( p \) of these pieces have grade equal to \( g(J) \).

**Theorem 2.20.** [Wis02][Theorem 1.7] Let \( \phi : A \to B \) be a combinatorial map between a graded (non-metric) graph \( A \) and a metric graph \( B \). If \( \phi \) satisfies the graded c\( (3) \) small-cancellation condition then \( \phi_1(\pi_1(A)) \subset \pi_1(B) \) is malnormal.

3. The Main Example

The main example is constructed in two steps:

1. We first construct a malnormal infinitely generated subgroup \( K = K_\infty \) of \( F_2 \) satisfying a number of properties.
2. Then we double \( F_2 \) along \( K \) to get \( G(\infty) = F_2 *_{K_\infty} F_2 \).
3.1. The Construction of $K$. We shall first construct a sequence of malnormal subgroups $\{K_n\} \subset F_2 = F_2(a, b)$ by choosing a sequence of elements $g_n$ satisfying a suitable small cancellation condition such that

1. No proper power of $a$ or $b$ is an element of $K_n$ and hence $K_n$ is a proper infinite index subgroup.
2. $K_{n+1} = \langle K_n, g_n \rangle$ where $g_{n+1} \in F_2 \setminus K_n$.
3. $(g_{n+1}, K_n) = \min_{h \in F_2 \setminus K_n} (h, K_n)$.
4. $g_{n+1}$ is cyclically reduced.

We shall define $K = K_\infty = \bigcup_n K_n$ and ensure that $K$ is of infinite index in $F_2$. The choice of $g_n$ satisfying a suitable small cancellation condition will ensure a few things:

1. $K = K_\infty$ is malnormal in $F_2$.
2. The limit set $\Lambda K$ of $K$ in $\partial F_2$ is all of $\partial F_2$.
3. $K$ is of infinite index in $F_2$.

Recall from the Introduction that

1. $T(n)$ denotes the tower function of height $n$ with base 2, i.e. $T(n) = 2^{2^2 \cdot n}$, where the power is taken $n$ times.
2. $U(n)(a, b)$ is the word (element) in the free group $F_2$ on generators $a, b$ given by
   $$U(n)(a, b) = [a^{T(n)+1} b^{T(n)+1}] [a^{(T(n)+2)} b^{(T(n)+2)}] \cdots [a^{T(n+1)} b^{T(n+1)}].$$

We choose $g_{n+1}$ inductively to be of the form $g_{n+1} = h_{n+1} W_{n+1}$, where $h_{n+1}$ satisfies the following:

1. $(h_{n+1}, K_n) = \min(r, K_n)$, where the minimum is taken over geodesic rays $r$ in $F_2$ starting at 1 and $h_{n+1}$ is an initial segment of $r$.
2. $|h_{n+1}| = (h_{n+1}, K_n) + 1$.

Here we have omitted, for convenience, the generators $a, b$ in the expression for the words $g, h$.

and $W_{n+1}$ is of the form $W_{n+1} = c_{n+1} U(N(n)) d_{n+1}$ where

A. $N(n)$ is at least twice the maximum value of $m$ for which $a^m$ or $b^m$ is a subword of some freely reduced word $w \in K_n$.
B. $N(n) > |h_{n+1}|$.
C. $c_{n+1}, d_{n+1}$ are chosen from $\{a, b, a^{-1}, a^{-1}\}$ to ensure that $g_{n+1}$ is cyclically reduced.

Define

$$K = K_\infty = \bigcup_n K_n.$$

By the choice of $g_n$, it follows that $K$ is of infinite index in $F_2$ since $K$ is infinitely generated free.

3.2. $K$ is malnormal.

Lemma 3.1. $K_n \subset F_2$ is malnormal.
Proposition 3.2. $K = K_{\infty} \subset F_2$ is malnormal.

Proof. First observe that a nested union of malnormal subgroups is malnormal. Suppose $H = \bigcup_i H_i$, where $H_i \subset H_{i+1}$ and each $H_i$ is malnormal in $G$. For some $h \in H, h \neq 1$ and $g \in G$ suppose that $ghg^{-1} = h_1 \in H$. Then there exists $n$ such that $h_i h_1 \in H_n$ forcing $g \in H_n$ since $H_n$ is malnormal in $G$. Hence $g \in H$ and $H$ is malnormal in $G$.

Since $K = K_{\infty} = \bigcup_n K_n$, and each $K_n$ is malnormal in $F_2$ by Lemma 3.1, it follows that $K$ is malnormal in $F_2$. 

Proposition 3.3. $\Lambda K = \partial F_2$.

Proof. This shall follow from the fact that in our construction, $(g_{n+1}, K_n)_1 = (h_{n+1}, K_n)_1 = \min \{h_0(F_2 \setminus K_0) | (h, K_0)_1 \}.$

It suffices to show that for every $g \in F_2$, there exists $n$ and $h \in K_n$ such that $g \in [1, h]$ i.e. $g$ lies on the geodesic from $1$ to $h$. Suppose not. Then $g \notin K_n$ for all $n$. Further, $(g, K_n)_1 \leq |g|$ for all $n$. Hence $(h_i, K_n)_1 \leq |g|$ for all $i \geq 2$. Hence $|h_i| \leq |g| + 1$ for all $i \geq 2$. This forces $h_i$ to range in a finite set – a contradiction.

3.4. Doubling. Define

1. $G(n) = F_2(a, b) *_{K_n(a,b)=K_n(x,y)} F_2(x, y)$.
2. $G = G(\infty) = F_2(a, b) *_{K_\infty(a,b)=K_\infty(x,y)} F_2(x, y)$.

We shall need the following fact from [BF92, Mit04]:

Theorem 3.4. Let $G_0$ be a hyperbolic group and $H$ a malnormal quasiconvex subgroup. Then $G = G_0 *_H G_0$ is hyperbolic and $H$ is quasiconvex in it.

The following is an immediate consequence:

Proposition 3.5. $G(n)$ is hyperbolic and $K_n$ is quasiconvex in it.

Proof. It suffices by Theorem 3.4 to show that $K_n$ is malnormal quasiconvex. Malnormality is the content of Lemma 3.1. Since $K_n$ is finitely generated in $F_2$, it is quasiconvex.

3.4.1. Geodesics in $G$. Fix a symmetric generating set $\{a, b, x, y, a^{-1}, b^{-1}, x^{-1}, y^{-1}\}$ of $G(n), G$. We first fix notation. Let $a, b$ denote the generators of the first copy of $F_2$ and let $x, y$ denote the generators of the second copy of $F_2$. Let $\Gamma$ denote the Cayley graph of $G$ with respect to (the symmetrization of) $a, b, x, y$. Let $\Gamma_1$ denote the (sub) Cayley graph of the subgroup $F_2(a, b) \subset G$. Similarly let $\Gamma_2$ denote the (sub) Cayley graph of the subgroup $F_2(x, y) \subset G$. Let $i_1, i_2$ denote the inclusion maps. There exists natural projection maps $\Pi_j : \Gamma \to \Gamma_j$ identifying the generators $a, b$ with $x, y$ respectively.

Lemma 3.6. $F_2(a, b)$ and $F_2(x, y)$ are isometrically embedded in $G(n)$ and $G$. 

Proof. It suffices by Theorem 2.20 to show that the map $\phi_n : \text{Rose}(K_n) \to \text{Rose}(F_2)$ between the roses corresponding to $K_n, F_2$ satisfies the graded $c(5)$ condition.

The $i$th petal (labeled by the generator $g_i$) of $\text{Rose}(K_n)$ is assigned grade $i$, $(i = 1 \cdots n)$. Then by Conditions (A) and (B) in the construction of $U(N(n))$ above, pieces of grade $n$ are of the form $a^m b^n$ or $b^n a^{m+1}$. Since the number of such pieces is at least 5, the map $\phi_n : \text{Rose}(K_n) \to \text{Rose}(F_2)$ satisfies the graded $c(5)$ condition. 

□
Proof. Clearly $d(\Pi_j(u), \Pi_i(v)) \leq d(u, v)$. Since $\Pi_j \circ i_j$ is the identity map (on the first or second $F_2$ respectively) it follows that $F_2(a, b)$, $F_2(x, y)$ are isometrically embedded in $G$. \hfill \Box

We shall need the following normal form Lemma (cf. [LS] p. 285-6) for free products with amalgamation. For the group $B \ast_A C$, let $i, j$ denote the inclusion maps of $A$ into $B, C$ respectively.

**Lemma 3.7.** Every element $g$ of $B \ast_A C$ which is not in the image of $C$ can be written in the normal form $v_1v_2 \cdots v_n$, where the terms $v_i$ lie in $B - i(A)$ or $C - j(A)$ and alternate between these two sets. The length $n$, called the normal length of the group element, is uniquely determined and two such expressions $v_1v_2 \cdots v_n$ and $w_1w_2 \cdots w_n$ give the same element of $B \ast_A C$ iff there are elements $a_1, \ldots, a_{n+1} \in A$ so that $w_k = a_{k-1}v_ka_k^{-1}$. (where $a_0 = a_{n+1} = 1$).

**Definition 3.8.** We shall say that a normal form word $g = v_1v_2 \cdots v_n$ is in reduced normal form if for each $i$, $v_iv_{i+1}$ is a geodesic word, i.e. amongst all representatives of the form $(v_0a_i)(a_i^{-1}v_{i+1})$ with each of $(v_0a_i)$ and $(a_i^{-1}v_{i+1})$ replaced by geodesic words, $v_iv_{i+1}$ is shortest.

**Proposition 3.9.** Given an element $g \in G(n)$ there exists a normal form geodesic word $v_1v_2 \cdots v_n$ representing $g$.

**Proof.** Let $T$ denote the Bass-Serre tree of the splitting. Observe first that a normal form word as defined above projects to a path in $T$ that does not return to a vertex after leaving it.

Now, let $u$ be any word that projects to a path in $T$ having a non-trivial subpath $\sigma$ that starts and ends at the same vertex of $T$. Let $u'$ be the subword of $u$ corresponding to $\sigma$. Then $u'$ corresponds to an element of either $F_2(a, b)$ or $F_2(x, y)$. Replace $u'$ by the geodesic word in $F_2(a, b)$ or $F_2(x, y)$ representing it. By Lemma 3.6, it follows that the resulting word is at most as long as $u$. Continuing this process we finally obtain a normal form word whose length is at most that of $u$. \hfill \Box

3.4.2. **Cyclic Geodesics in $G(n)$**. We shall now carefully choose a geodesic word $R_{n+1}$ in $G(n)$ denoting the new relator which when added to the relator set of $G(n)$ gives $G(n+1)$. $R_{n+1}$ will satisfy the condition that all its cyclic conjugates are geodesic words in $G(n)$.

We proceed inductively to define $R_n$ of the form $w^{-1}(x, y)w(a, b)$. Recall that $g_{n+1} = h_{n+1}g_{n+1}U(N(n))d_{n+1}$, where $h_{n+1}$ is a geodesic word in $F_2$. Let $R_n = R_{n-1}^{-1}(x, y)g_n(a, b)$. $R_n$ will be obtained from $R_n$ after cyclic reduction.

We shall define $w_{n+1}$ to be a reduced normal form geodesic representative of $g_{n+1}^{-1}(x, y)g_{n+1}(a, b)$ in $G(n)$ (of normal length 2 as per Lemma 3.7) obtained by replacing a maximal subword $w_0$ of the form $w^{-1}(x, y)w(a, b)$ by a subword $w'$ of the same form using the structure of $G(n)$ as a double along $K_n$. By Lemma 3.7, any normal form word representing $g_{n+1}^{-1}(x, y)g_{n+1}(a, b)$ is of the form $g_{n+1}^{-1}u^{-1}(x, y)g_{n+1}(a, b)$, where $u \in K_n$. In particular, the shortest normal form representative of $g_{n+1}^{-1}(x, y)g_{n+1}(a, b)$ is of the form $w^{-1}(x, y)w(a, b)$ where $w(a, b) \in (F_2(a, b) \setminus K_n(a, b))$ and $w(x, y) \in (F_2(x, y) \setminus K_n(x, y))$. Let $R_{n+1}$ denote such a representative.

Since Item(A) of the defining condition on $U(N(n))$ demands that $N(n)$ is at least twice the maximum value of $m$ for which $a^m$ or $b^m$ is a subword of some freely
reduced word in \( K_n \), it follows that after any reduction as above, \( w(a, b) \) contains a subword of the form \( V(n)(a, b) = [a^{(T(n)+1)} b^{(T(n)+2)}] \cdots [a^{T(n)+2}] b^{T(n)+1} \).

Next, \( g_{n+1}(a, b) \) is already a shortest normal form representative as it contains \( b^{T(n+1)} y^{−T(n+1)} \) as a subword which cannot be shortened further, again by Item(A) of the defining condition on \( U(n) \). It follows by the same reason that every cyclic conjugate of \( R_{n+1} \) is a geodesic in \( G(n+1) \).

Thus \( R_{n+1} \) satisfies the following properties:

Property 1: All cyclic conjugates of \( R_{n+1} \) are geodesics in \( G(n) \).

Property 2: \( R_{n+1} \) is of the form \( w_n(x, y)w_n(a, b) \) where \( w(a, b) \) contains a subword of the form \( V(n)(a, b) = [a^{(T(n)+1)} b^{(T(n)+2)}] \cdots [a^{T(n)+2}] b^{T(n)+1} \).

Property 3: \( |R_{n+1}| = O((V(n)(a, b))) = O(T(n + 1)^2) \)

Property 4: The largest piece of \( R_{n+1} \) is of the form \( a^{T(n+1)} \) which has length \( 2T(n + 1) − 1 \). (since \( N(n) > |h_{n+1} | \) by choice, initial segments cannot be maximal pieces).

The group \( G \) therefore admits a presentation \( G = \langle S | R \rangle \), where

1. \( S = \{a, b, x, y, a^−1, b^−1, x^−1, y^−1\} \).
2. \( R = \cup_i |R_i| \).

A more general statement is true. We learnt the proof of the following from Dani Wise. We refer the reader to [Wis12] for details on non-positively curved (npc) complexes.

**Proposition 3.10.** \( G = F *_A F \) is a double of a finitely generated free group \( F \) along a finitely generated subgroup \( A \), with generating set given by the union of the generating sets of the two copies of \( F \). Then \( G \) has cohomological dimension 2.

Then any reduced normal form word \( g \) of the form \( uwv \) (i.e. of normal length 3), where \( u, w \) are geodesic words in the first factor and \( v \) is a geodesic in the second factor are geodesics in \( G \).

More generally any reduced normal form word is a geodesic.

**Proof.** By Theorem 7.2 of [Wis12], \( G \) can be represented as the fundamental group of a non-positively curved (\( \mathcal{VH} \)) square complex formed by taking two graphs \( G \) (with \( F \) as fundamental group) as vertex spaces, and an edge space that is a \( G \times I \) that is attached by a local isometry on each side. It immediately follows that \( G \) has cohomological dimension 2.

Then \( g \) corresponds to a path \( \sigma \) labeled \( uvw \) consisting of three vertical paths labeled \( u, v, w \) joined by two horizontal edges, one from the terminal point of \( u \) to the initial point of \( v \) and one from the terminal point of \( v \) to the initial point of \( w \).

Consider a disk diagram \( D \) between \( \sigma \) and an arbitrary edge path \( \eta \) with the same end-points. The dual curves emanating from \( v \) cannot end on \( u \) or \( w \) because of the hypothesis that \( uv \) and \( vw \) are geodesics. The dual curves emanating from \( u \) cannot end on \( w \) since then the middle path \( v \) must have dual curves that start and end on itself, contradicting that it is itself reduced. Thus all dual curves emanating from \( \sigma \) must end on \( \eta \). Thus \( \eta \) has at least as many vertical edges as \( \sigma \), forcing \( \sigma \) to be a geodesic. This proves the first statement.

The proof of the last statement is now completed by a straightforward induction: We repeat the above argument after combining the first \( (n−1) \) words into a single geodesic. 

\( \square \)
3.5. Subgroups of $G$.

Proposition 3.11. All finitely presented subgroups $H$ of $G$ are hyperbolic.

Proof. Let $H = \langle S_1, R_1 \rangle$ be a finite presentation. Let $S_1 = \{s_1, \ldots, s_n\}$; $R_1 = \{r_1, \ldots, r_m\}$. Let $H_0$ be the subgroup of $G(0)$ generated by $S_1$. Then there exists $n$ such that each $r_i$ is trivial in $G(n)$ and hence in $G(m)$ for all $m \geq n$. Since $H$ is a subgroup of $G$, it follows that $H$ is a subgroup of $G(m)$ for all $m \geq n$. By the first part of Proposition 3.10, $G$ has cohomological dimension 2.

Hence $H$ is a finitely presented subgroup of a hyperbolic group $G(n)$ of cohomological dimension 2. It now follows from [Ger96] that $H$ is hyperbolic. $\square$

4. Non Relative Hyperbolicity

In this Section we show that:

Theorem 4.1. $G = F_2 *_K F_2$ is not hyperbolic relative to any proper collection of parabolic subgroups $\mathcal{P}$.

Proof. Suppose not, i.e. $G$ is hyperbolic relative to a proper collection of parabolic subgroups $\mathcal{P}$.

By Theorem 2.7, $F_2(a, b)$, $F_2(x, y)$ are relatively quasiconvex in $G$ and hence are relatively hyperbolic with respect to $\mathcal{P} \cap F_2(a, b)$, $\mathcal{P} \cap F_2(x, y)$ respectively. Since $K = F_2(a, b) \cap F_2(x, y) \subset G$, it follows that $K$ is also relatively quasiconvex in $G$ by Theorem 2.6. But $K \subset F_2(a, b) \subset G$. Hence $K$ is also relatively quasiconvex in $F_2(a, b)$, $F_2(x, y)$.

By Proposition 3.3, $K$ has full limit set in $F_2(a, b)$, $F_2(x, y)$, i.e. $\Lambda K = \partial F_2(a, b)$, $\Lambda K = \partial F_2(x, y)$. By Lemma 2.8, it follows that $K$ is of finite index in $F_2(a, b)$. This contradicts the construction of $K$ as an infinite index subgroup.

This final contradiction yields the Theorem. $\square$

Theorem 4 will also follow from Theorems 8.1 and 2.14 but the above proof is direct and simple.

5. Graded Small Cancellation Structure of $G$

In this section our primary goal is to show that the group $G = F_2(a, b) *_K F_2(x, y)$ constructed in Section 3 satisfies a certain (technical) graded small cancellation condition [OOS09, Ols91]. This will give (as usual with small cancellation theory) a number of geometric consequences.

5.1. Definitions and Preliminaries on Graded Small Cancellation.

Definition 5.1. A set $\mathcal{R}$ of words in an alphabet is symmetrized if for any $R \in \mathcal{R}$, all cyclic shifts of $R^{\pm 1}$ are contained in $\mathcal{R}$ and each $R$ is reduced. (Note that the feature that each $R$ is reduced is incorporated as part of the definition.)

Definition 5.2. [OOS09] Definition 4.2] Let $G$ be a group with a finite symmetric generating set $S$. Let $\mathcal{R}$ be a symmetrized set of words in $S$. Given $\epsilon > 0$, a subword $U$ of a word $R \in \mathcal{R}$ is called an $\epsilon$-piece if there exists $R' \in \mathcal{R}$ such that:

1. $R = UV$, $R' = U'V'$, for some words $V, U', V'$,
2. $U' = YUZ$ for words $Y, Z$ satisfying $\max\{|Y|, |Z|\} \leq \epsilon$.
3. $YRY^{-1} \neq R'$.
In the above definition ‘=’ is to be interpreted as equality as elements of G. In the following definition we rename condition SC of \[\text{[OOS09]}\] as the relative small cancellation condition.

**Definition 5.3.** [OOS09] Definition 4.3] Let \(\epsilon \geq 0, \mu \in (0,1), \text{and } \rho > 0\). A symmetrized set \(\mathcal{R}\) of words in a symmetrized generating set \(S\) for a group \(G\) satisfies the relative small cancellation condition \(C(\epsilon, \mu, \rho)\) if

1. The words in \(\mathcal{R}\) represent geodesics in \(G\).
2. \(|\mathcal{R}| \geq \rho\) for any \(\mathcal{R} \in \mathcal{R}\).
3. for any \(e\)-piece \(U\) and any \(R \in \mathcal{R}\), with \(U \subset R, |U| \leq \mu|\mathcal{R}|\).

**Definition 5.4.** [OOS09] Definitions 4.12, 4.14] Fix \(\alpha = 10^{-2}, K = 10^6\). The presentation

\[G = \langle S | \mathcal{R} \rangle = \left\langle S \bigg| \bigcup_{i=0}^{\infty} \mathcal{R}_i \right\rangle\]

is a graded small cancellation presentation if there exist sequences \(\epsilon = (\epsilon_n), \mu = (\mu_n), \text{and } \rho = (\rho_n)\) of positive real numbers \((n = 1, 2, \ldots)\) satisfying:

1. \((GSC_0)\) The group \(G_0 = \langle S | \mathcal{R}_0 \rangle\) is \(\delta_0\)-hyperbolic for some \(\delta_0\).
2. \((GSC_1)\) \(\epsilon_{n+1} > 8 \max |\mathcal{R}|, R \in \mathcal{R}_n = O(\rho_n)\).
3. \((GSC_2)\) \(\mu_n \to 0\) as \(n \to \infty\), \(\mu_n \leq \alpha\), and \(\mu_n \rho_n > K \epsilon_n\) for any \(n \geq 1\).
4. \((GSC_3)\) For all \(n \geq 1\), \(\mathcal{R}_n\) satisfies \(C(\epsilon_n, \mu_n, \rho_n)\) over \(G_{n-1} = \left\langle S \bigg| \bigcup_{i=0}^{n-1} \mathcal{R}_i \right\rangle\).

**Remark 5.5.** Though we have fixed \(\alpha = 10^{-2}, K = 10^6\) following \[\text{[OOS09]}\] above, any sufficiently large \(K\) and sufficiently small \(\alpha\) suffices.

We shall need the following:

**Lemma 5.6.** [OOS09] Lemmas 4.13, 4.18] Let

\[G = \langle S | \mathcal{R} \rangle = \langle S \bigg| \bigcup_{i=0}^{\infty} \mathcal{R}_i \rangle\]

be a graded small cancellation presentation. Then \(G_n\) is \(\delta_n\)-hyperbolic, where \(\delta_n \leq 4 \max_{\mathcal{R} \in \mathcal{R}_n} |\mathcal{R}|\).

Any subword of any word \(R_0 \in \mathcal{R}_0\) of length at most \(|\mathcal{R}_0|/2\) is a \((1-o(1), 0)\)-quasigeodesic in the Cayley graph of \(G\) with respect to the generating set \(S\).

### 5.2. Graded Small Cancellation for \(G\).

**Theorem 5.7.** The group \(G = F_2 \ast_K F_2\) equipped with the presentation \(G = \langle S | \mathcal{R} \rangle\), where \(S = \{a, b, x, y, a^{-1}, b^{-1}, x^{-1}, y^{-1}\}\) and \(\mathcal{R} = \bigcup_{i=0}^{\infty} \{R_i\}\) (cf. Section 3.4.2 and Proposition 3.10) is a graded small cancellation group and the presentation is a graded small cancellation presentation.

**Proof:** The proof will proceed by checking the properties in Definitions \[\text{5.4 and 5.3}\] of \(GSC_0, GSC_1, GSC_2, GSC_3\) for \(G\).
(1) \( \rho_n = |R_{n+k}| \).
(2) \( \epsilon_{n+1} = 10\rho_n \).

Condition GSC_1 immediately follows.

**Proof of Condition GSC_2:** It suffices to choose \( \mu_n > \frac{10^7 \rho_{n-1}}{\rho_n} \). Recall from Property 3 satisfied by \( R_{n+k} \) that \( \rho_n = O((T(n+k))^2) \). Since \( \frac{(T(n+k-1))^2}{(T(n+k))^2} \to 0 \) as \( n \to \infty \) very fast (more than exponentially), it will be easy to make a specific choice for \( \mu_n \) during the proof of GSC_3 below.

5.2.1. **Proof of GSC_3:** This is the crucial step where we shall prove that the set \( \mathcal{R}(n+k) \) of cyclic conjugates of \( R_{n+k} \) satisfies the **relative small cancellation condition**

\[ C(\epsilon_n, \mu_n, \rho_n) \text{ (Definition 5.3) over } G_{n-1} = \langle S|R = \bigcup_{i=0}^{n+k-1} \mathcal{R}(i) \rangle, \]

for suitable choices of \( \epsilon_n, \mu_n \) and \( \rho_n \) for all \( n \).

We observe that, by Proposition 3.5 all the groups \( G(i) \) are \( \delta_i \)-hyperbolic. We need to therefore choose parameters \( \epsilon_n, \mu_n \) and \( \rho_n \) for \( \mathcal{R}(n+k) \) satisfying \( C(\epsilon_n, \mu_n, \rho_n) \) over \( G_{n-1} = \langle S|R = \bigcup_{i=0}^{n+k-1} \mathcal{R}(i) \rangle \).

**Proof of RSC_1:** \( \mathcal{R}(n+k) \) consists of cyclic conjugates of exactly one relator \( R_{n+k} \) all of which are geodesics by their construction (Section 3.4.2 and Proposition 3.10).

**Proof of RSC_2:** Since all cyclic conjugates of \( R_{n+k} \) are geodesics (Section 3.4.2 and Proposition 3.10), all of them have the same length \( \rho_n \).

**Proof of RSC_3:** We will show that for the choice of \( \epsilon_n = 10\rho_{n-1} \) as in the Proof of Condition GSC_1, the length of an \( \epsilon_n \)-piece in \( \mathcal{R}(n+k) \) has length less than \( \mu_n|R_{n+k}| \) for suitably chosen \( \mu_n \).

Let \( U \) be any \( \epsilon_n \)-piece. By the form of the words \( R_{n+k} \), a subword \( W \) with length at least half that of \( U \) is a geodesic lying in one of the factors \( F(a, b) \) or \( F(x, y) \). Further, since \( U, U' \) are geodesics starting and ending at most \( \epsilon_n \)-apart, it follows that there exists a geodesic subword \( W' \) of \( U' \) such that the initial and final points of \( W, W' \) are at most \( (\epsilon_n + 4\delta_{n-1}) \) apart. Hence assume without loss of generality that \( U \) is an \( (\epsilon_n + 4\delta_{n-1}) \)-piece lying entirely in \( F_2(a, b) \). It follows from Definition 5.2 that

1. \( R = UV, R' = U'V' \), for some words \( V, U', V' \) and \( R, R' \in \mathcal{R}(n+k) \).
2. \( U' = YUZ \) for words \( Y, Z \) satisfying \( \max|Y|, |Z| \leq (\epsilon_n + 4\delta_{n-1}) \).
3. \( YRY^{-1} \neq R' \).

After reducing \( Y^{-1}U'Z^{-1} \) to reduced normal form, we obtain therefore a piece \( U_0 \) (in the usual small cancellation sense) of length at least \( |U| - 2(\epsilon_n + 4\delta_{n-1}) \). Recall that by Property 4 of \( R_{n+k} \), a maximal piece (in the usual small cancellation sense) has length at most \( 2T(n+k) \). Hence \( |U_0| < 2T(n+k) \) and so \( |U| < 2T(n+k) + 2(\epsilon_n + 4\delta_{n-1}) \).

**Choice of \( \mu_n \):** It therefore suffices to choose \( \mu_n \) such that \( 2T(n+k) + 2(\epsilon_n + 4\delta_{n-1}) \leq \mu_n \rho_n \). Define \( \mu_n = 2\max\{ \frac{2T(n+k) + 20\rho_{n-1}}{\rho_n}, \frac{10^7 \rho_{n-1}}{\rho_n} \} \). Since \( \rho_n = O((T(n+k))^2) \), and \( \delta_{n-1} = O(\rho_n) \) it follows that \( \mu_n \to 0 \) as \( n \to \infty \).

**Choice of \( k \):** It remains to choose \( k \). The choice of \( k \) is dictated by the requirement that \( \mu_n \leq a \) for all \( n \). Since \( \mu_n \to 0 \) as \( m \to \infty \) it follows that there exists \( k > 0 \) such that \( \mu_m \leq a \) for all \( m \geq k \). This decides the choice of \( k \) and completes the proof of Theorem 5.7. \( \square \)
6. Asymptotic Cone of $G = F_2 \ast K F_2$

We refer the reader to [DS05] for details on tree-graded spaces and [Gro93, OOS09] for details on asymptotic cones.

6.1. Tree-graded Spaces.

**Definition 6.1.** Let $X$ be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic subsets. Then $X$ is said to be tree-graded with respect to $\mathcal{P}$ if the following hold:

(1) Any two elements of $\mathcal{P}$ have at most one common point.
(2) Every simple geodesic triangle (i.e. a simple loop composed of three geodesics) in $X$ is contained in exactly one element of $\mathcal{P}$.

**Definition 6.2.** [OOS09] If $X$ is tree-graded with respect to $\mathcal{P}$ such that each element of $\mathcal{P}$ is a circle of radius uniformly bounded away from 0 and $\infty$, then $X$ is called a circle-tree.

6.2. Asymptotic Cones of Graded Small Cancellation Groups. We shall say that $a_i = O_\omega(b_i)$ if the $\omega$–limit of $\{a_i b_i\}$ is finite and non-zero. Similarly, $a_i = o_\omega(b_i)$ if the $\omega$–limit of $\{a_i b_i\}$ is zero.

**Definition 6.3.** [OOS09, Definitions 4.16,4.19] Given an ultrafilter $\omega$ and a scaling sequence $d = (d_n)$, a sequence of real numbers $f = (f_n)$ is said to be $(\omega, d)$–visible or simply asymptotically visible, if there exists a subsequence $(f_n)$ of $f$ such that $f_n = O_\omega(d_i)$.

For a group $G$ and its Cayley graph $\Gamma$, we shall say that a sequence of loops $\{\sigma_i\} \subset \Gamma$ is asymptotically visible if the sequences $\{d(1, \sigma_i)\}$ and $\{|\sigma_i|\}$ are asymptotically visible.

The following is one of the main Theorems of [OOS09].

**Theorem 6.4.** [OOS09] (Theorem 4.17) Let $G$ be a group having a graded small cancellation presentation with symmetrized relator set $\mathcal{R} = \bigcup_i [R_i]$, where $[R_i]$ is the grade $i$ symmetrized relator set. For any ultrafilter $\omega$, and any sequence of scaling constants $d = (d_n)$, the asymptotic cone $\text{Con}^\omega(G, d)$ is a circle-tree. $\text{Con}^\omega(G, d)$ is an $\mathcal{R}$–tree if and only if the sequence $(\rho_n)$ from Definition 5.7 is not $(\omega, d)$–visible.

Further any simple loop $C$ in $\text{Con}^\omega(G, d)$ can be realized as an $\omega$–limit of translates of $\{R_i\}$'s.

The last statement follows from Lemma 4.23 of [OOS09].

6.3. Properties of the Asymptotic Cone of $G$.

**Proposition 6.5.** $\text{Con}^\omega(G)$ is a circle-tree.

**Proof.** This follows from Theorems 5.7 and 6.4.

Let $\mathcal{G}$ denote $\text{Con}^\omega(G)$.

**Proposition 6.6.** $i : K_2 \to G$ induces a bi-Lipschitz embedding $\hat{i} : \text{Con}^\omega(K_2) \hookrightarrow \mathcal{G}$.

**Proof.** Since $K_2 \hookrightarrow F(a, b)$ is a quasi-isometric embedding and $F(a, b) \hookrightarrow G$ is an isometric embedding by Lemma 3.6, it follows that $i : K_2 \to G$ is a quasi-isometric embedding. The result follows.
exists a bigon.

Proof. Suppose not. Since \(6.3.1\), it follows that there exists a sequence \(\omega\) consisting of the union of two arcs:

1. a geodesic \(pq\) in \(T\).
2. an embedded path \(\gamma\) in \(G\) joining \(p, q\) with interior disjoint from \(pq\).

Since all elements of \(\mathcal{C}\) can be obtained as \(\omega\)-limits of translates of elements of \(\mathcal{R}\) by Theorem [6,4] we can assume that there exists a sequence of translates \(\{g_n, R_n\}\) whose \(\omega\)-limit is \(\sigma\). For the purposes of this proof, reindex by setting \(g_n R_n = R_n\). Geodesic segments in the Cayley graph of \(G\) will be denoted by \([E, F]\). Hence there exists a sequence \([R_n, A_n, B_n, C_n, D_n]\) such that (see Figure 1)

1. \(R_n \in \mathcal{R}\)
2. The scaling sequence \(d_n = O(d(n)) = O(\rho_n) = O(\delta_n)\), where \(\delta_n\) is the hyperbolicity constant of \(G(n)\) (The last equality follows from Lemma 5.6 and the asymptotic visibility of the sequence \(\{R_n\}\)).
3. \([A_n, B_n] \subset K_2\);  
4. \([D_n, C_n] \subset R_n\)
5. \([A_n, D_n], [B_n, C_n]\) are geodesics in normal form in \(G\).
6. \([|A_n, D_n|], [|B_n, C_n]|\) are \(\omega_n(d_n)\)
7. \([|A_n, B_n|], [|D_n, C_n]|\) are \(O(\omega_n(d_n))\)
8. \(\lim_{\omega_n}(A_n) = \lim_{\omega_n}(D_n) = p\)
9. \(\lim_{\omega_n}(B_n) = \lim_{\omega_n}(C_n) = q\)

By cutting off initial and final pieces from \([A_n, B_n], [D_n, C_n]\) of length \(o(d_n)\) if necessary and using thinness of quadrilaterals we can assume without loss of generality that \([|A_n, D_n|], [|B_n, C_n]|\) are at most \(O(d_{n-1}) = O(\delta_{n-1})\). More precisely, choose a subsegment \([A_n', B_n']\) such that \(d(A_n, A_n') = d(B_n, B_n') = 2max(|A_n, D_n|), [|B_n, C_n]|)\). Then there exist points \(D_n', C_n'\) on \([D_n, C_n]\) such that \(d(D_n', A_n') \leq 2\delta_{n-1}\) and \(d(C_n', B_n') \leq 2\delta_{n-1}\). Rename the four vertices \(A_n', B_n', C_n', D_n'\) as \(A_n, B_n, C_n, D_n\). Then as in the proof of Condition RSC3 in the proof of Theorem 5.7, we get a genuine (in the usual small cancellation sense) piece of length \(O(|A_n, B_n|)\) as a subword of \([A_n, B_n]\). Further, any maximal piece of \(R_n\) is \(o(\rho_n)\) and hence \(\frac{|A_n, B_n|}{\rho_n} \to 0\) as \(n \to \infty\). On the other hand, since \(\lim_{\omega_n}(A_n) = \lim_{\omega_n}(D_n) = p\) and \(\lim_{\omega_n}(B_n) = \lim_{\omega_n}(C_n) = q\), and since \(p \neq q\), it follows that \(\frac{|A_n, B_n|}{\rho_n}\) is bounded away from zero. This is a contradiction. \(\square\)
7. Controlled Floyd Separation

The main purpose of this section is to show:

**Theorem 7.1.** For any Floyd function \( f, c \geq 1, \) and \( p, q \in \partial K_2 \subset \text{Pre}(\partial)G, \) \( p, q \) are Floyd separated with respect to \( c \)-quasigeodesics.

**Proof.** Let \( p, q \in \partial K_2 \) be distinct points. Let \( \{p_n\}_{n=1}^\infty \) and \( \{q_n\}_{n=1}^\infty \) be Cauchy sequences in \( K_2, \) converging to \( p \) and \( q \) respectively. We shall show that \( d_{f_\infty}(p, q) > 0. \)

Suppose not. Then there exists a sequence of \( (c, c) \)-quasi geodesic paths \( \{\gamma_n\}, \) joining \( q_n \) to \( p_n \) in \( G, \) such that \( (\text{the } c\text{-controlled Floyd lengths}) f_\infty(\gamma_n) \to 0. \)

Let \( D_n \) be the Dehn-diagram obtained from the trivial word with label \( \{p_n, q_n, \gamma_n\} \) in \( G. \) We will construct a sequence \( \{D'_n\} \) of subdiagrams from \( \{D_n\} \) and a sequence of numbers \( h_n \) such that

1. \( h_n \) is the maximum possible distance of a point \( t_n \) on \( \gamma_n, \) from \( \{p_n, q_n\}. \)
2. The ultralimit \( \lim^\omega(\partial D'_n, [h_n]) \) contains a circle \( C \) with non-zero but finite length in the asymptotic cone \( \text{Con}^\omega(G, [h_n]). \)
3. The intersection \( C \cap T \) has nonzero length.

This will contradict Theorem 6.7.

Choose \( t_n \in \gamma_n \) such that \( d(t_n, [p_n, q_n]) = h_n \) is maximal. Next, choose \( p_n', q_n' \) to the left and right respectively of \( t_n \) such that the subarcs of \( \gamma_n \) joining \( p_n', t_n \) and \( t_n, q_n' \) each have length \( 5c h_n. \) It follows that the \( c\)-quasi geodesic arc \( (c > 1) \) joining \( p_n \) and \( q_n' \) has length \( 10c h_n \) and hence \( d(p_n', q_n') \geq 10c h_n - c. \)

Let \( p_n'' \) (resp. \( q_n'' \)) be points on \([p_n, q_n]\) closest to \( p_n' \) (resp. \( q_n' \)). By the choice of \( t_n, \)
\( d(p_n', p_n'') \leq h_n \) and \( d(q_n', q_n'') \leq h_n. \) Hence \( ||p_n'', q_n''|| \geq 8c h_n - c. \)

Let \( D''_n \) be the subdiagram of \( D_n \) with sides \([p_n', p_n''], [p_n', q_n''], [q_n'', q_n'']\) and the subarc of \( \gamma_n \) joining \( q_n' \) to \( p_n. \)

Let \( D' = \lim^\omega \partial D''_n \subset \text{Con}^\omega(G, [h_n]); p' = \lim^\omega p_n''; q' = \lim^\omega q_n''; p'' = \lim^\omega p_n''; q'' = \lim^\omega q_n''.\) Then \( ||p'', q''|| \geq 8, ||p', q'|| \leq 1, ||p', q''|| \leq 1. \) Also the top arc \( \eta \) of \( D', \) joining \( p', t, q' \) is an **embedded** \((c\text{-bi}\text{Lipschitz})\) arc of length \( 10c \) in \( \text{Con}^\omega(G, [h_n]). \) Hence \( D' \) contains a loop \( \theta \) such that \( 10c - 2 \leq ||\theta \cap \eta|| \leq 10c \)
and $6 \leq ||p'',q''||$. Since $\text{Cor}(G, \langle h_n \rangle)$ is a circle tree, and since $\eta, [p'',q'']$ are both embedded arcs, it follows that $\theta$ contains a simple loop $C$ (a circle) whose intersection with $[p'',q'']$ has non-zero length. This contradicts Theorem 6.7. □

8. Triviality of Floyd Boundary

For completeness, we show in this section that the usual Floyd boundary $\partial_f G$ consists of a single point and is hence trivial:

**Theorem 8.1.** $\partial_f G$ consists of a single point.

**Proof.** Let $\partial_f F_2(a,b) \subset \partial_f G$ denote the set of limit points of $F_2(a,b)$ in $\partial_f G$. Let $p,q \in \partial_f F_2(a,b)$ and let $(p,q)$ denote the label of the bi-infinite geodesic path between them. Assume without loss of generality that $1 \in (p,q)$. By the construction of the $h_n$'s in Section 3, there exist $p_n \rightarrow p, q_n \rightarrow q$ such that

1. $[p_n,q_n]$ is centered at 1
2. the geodesic path $[p_n,q_n]$ (i.e. the word $p_n^{-1}q_n$) in $a,b$ is a subpath of $[1,h_m]$ (cf. the construction in Section 3) for some $m = m(n)$, surviving in the construction of $R_m$.

We make condition (2) more precise. Choose $h_m$ and $c_n, d_n \in [1,h_m]$ such that

1. $c_n^{-1}d_n = p_n^{-1}q_n$
2. Any geodesic $[1,k]$ with $k \in K$, containing $[1,c_n]$ as a subsegment must have length greater than $2|c_n|$, i.e. $|k| > 2|c_n|$. Hence, the reduced normal form of $h_m^{-1}(x,y)h_m(a,b)$ contains $c_n^{-1}(a,b)d_n(a,b)$ (and hence also $c_n^{-1}(x,y)d_n(x,y)$) as a subword.

By ensuring the above, we are guaranteed the existence of a relator $R_m(= R_{m(n)}) \in \mathcal{R}$ such that $p_n^{-1}(a,b)q_n(a,b)$ (and hence also $p_n^{-1}(x,y)q_n(x,y)$) is a subword of $R_m$.

By Lemma 5.6 for $n$ sufficiently large, $R_{m(n)}$ can be described as the union of two (2,2) quasigeodesics $[1, t_n]$ and $[1, t_n]_2$ both starting at 1 and ending at an ‘antipodal’ point $t_n \in R_{m(n)}$. Further, $p_n \in [1, t_n]$ and $q_n \in [1, t_n]_2$ (See Figure 3).

Assume that the Floyd length of the geodesic rays $[1,p)$ (and $[1,q)$) are normalized to one. Hence for all $\epsilon > 0$, there exists $n$ sufficiently large such that the Floyd lengths of the subpaths $[p_n, t_n]_1$ and $[q_n, t_n]_2$ are each less than $\frac{\epsilon}{2}$. It follows that $d_f(p_n,q_n) < \epsilon$.

Since $\epsilon > 0$ is arbitrary, $p,q$ correspond to the same point on the Floyd boundary. Hence $\partial_f F_2(a,b)$ consists of a single point. Similarly, $\partial_f F_2(x,y)$ consists of a single point $z$.

Next, from the relators $R_n$ of the form $w_n(x,y)w_n^{-1}(a,b)$ centered at 1 one can extract subsequential limits $z_1 \in \partial_f F_2(x,y)$ and $z_2 \in \partial_f F_2(a,b)$ of $w_n(x,y)$ and $w_n(a,b)$ to conclude that $z_1 = z_2$ on $\partial_f G$. Hence $\partial_f F_2(x,y) = \partial_f F_2(x,y) = z$. Since the action of $G$ on $\partial_f G$ is minimal [Kar13 Theorem 2 and Section VI], and since $z$ is invariant under both $F_2(a,b)$ as well as $F_2(x,y)$ and hence under $G$, it follows that $\partial_f G = \langle z \rangle$. □

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Figure 3. $d_f(p_n, q_n)$ is small.

References

[ACT14] G. N. Arzhantseva, C. H. Cashen, and J. Tao. Growth tight actions. preprint, arXiv:1401.0499, 2014.

[Bes04] M. Bestvina. Geometric group theory problem list. M. Bestvina’s home page: http://math.utah.edu/bestvina, 2004.

[BF92] M. Bestvina and M. Feighn. A Combination theorem for Negatively Curved Groups. J. Diff. Geom., vol 35, pages 85–101, 1992.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. Internat. J. Algebra and Computation. 22, 1250016, 66pp, 2012.

[DS05] C. Drutu and M. Sapir. Tree-graded spaces and asymptotic cones of groups. Topology 44, pages 959–1058, 2005. With an appendix by Denis Osin and Sapir.

[Far98] B. Farb. Relatively hyperbolic groups. Geom. Funct. Anal. 8, pages 810–840, 1998.

[Flo80] W. J. Floyd. Group Completions and Limit Sets of Kleinian Groups. Invent. Math. vol.57, pages 205–218, 1980.

[Ger96] S. Gersten. Subgroups of hyperbolic groups in dimension 2. J. London Math. Soc., pages 261–283, 1996.

[Ger09] V. Gerasimov. Expansive convergence groups are relatively hyperbolic. Geom. Funct. Anal., 19:137–169, 2009.

[Ger12] V. Gerasimov. Floyd maps for relatively hyperbolic groups. Geom. Funct. Anal. 22 no. 5, pages 1361–1399, 2012.

[GP11] V. Gerasimov and L. Potyagailo. Quasiconvexity in the Relatively Hyperbolic Groups. preprint, arXiv:1103.1211, 2011.

[GP13] V. Gerasimov and L. Potyagailo. Quasi-isometric maps and Floyd boundaries of relatively hyperbolic groups. J. Eur. Math. Soc. 15, no. 6, pages 2115–2137, 2013.

[Gro85] M. Gromov. Hyperbolic Groups. in Essays in Group Theory, ed. Gersten, MSRI Publ., vol.8, Springer Verlag, pages 75–263, 1985.

[Gro93] M. Gromov. Asymptotic Invariants of Infinite Groups. in Geometric Group Theory, vol.2; Lond. Math. Soc. Lecture Notes 182, Cambridge University Press, 1993.

[Hru10] G. Christopher Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. Algebr. Geom. Topol. 10 no. 3, pages 1807–1856, 2010.
[Kar03] A. Karlsson. Free subgroups of groups with nontrivial Floyd boundary. *Communications in Algebra*, 31(11):5361–5376, 2003.

[LS] R. C. Lyndon and P. E. Schupp. *Combinatorial Group Theory*. Springer.

[Mit04] M. Mitra. Height in Splittings of Hyperbolic Groups. *Proc. Indian Acad. of Sciences, v. 114, no.1*, pages 39–54, Feb. 2004.

[Mj08] M. Mj. Relative Rigidity, Quasiconvexity and C-Complexes. *Algebraic and Geometric Topology* 8, pages 1691–1716, 2008.

[Ols91] A. Yu. Olshanskii. *Geometry of defining relations in groups*. Translated from the 1989 Russian original by Yu. A. Bakhturin. 70. Kluwer Academic Publishers Group, Dordrecht, 1991. Mathematics and its Applications (Soviet Series).

[OOS09] A. Yu. Olshanskii, D. V. Osin, and M. V. Sapir. Lacunary hyperbolic groups. *Geom. Topol.*, 13(4):2051–2140, 2009. With an appendix by Michael Kapovich and Bruce Kleiner.

[Wis01] D. T. Wise. The residual finiteness of positive one-relator groups. *Comment. Math. Helv.*, 76(2):314–338, 2001.

[Wis02] D. T. Wise. Residual finiteness of quasi-positive one-relator groups. *J. London Math. Soc. (2)* 66, pages 334–350, 2002.

[Wis12] D. T. Wise. From riches to raags: 3-manifolds, right-angled artin groups, and cubical geometry. *American Mathematical Soc.*, 2012.

[Yan12] W. Yang. Limit sets of relatively hyperbolic groups. *Geom. Dedicata 156*, pages 1–12, 2012.

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