Homoclinic Bifurcations for the Hénon Map

D. Sterling, H.R. Dullin & J.D. Meiss
Department of Applied Mathematics
University of Colorado
Boulder, CO 80309

Abstract

Chaotic dynamics can be effectively studied by continuation from an anti-integrable limit. We use this limit to assign global symbols to orbits and use continuation from the limit to study their bifurcations. We find a bound on the parameter range for which the Hénon map exhibits a complete binary horseshoe as well as a subshift of finite type. We classify homoclinic bifurcations, and study those for the area preserving case in detail. Simple forcing relations between homoclinic orbits are established. We show that a symmetry of the map gives rise to constraints on certain sequences of homoclinic bifurcations. Our numerical studies also identify the bifurcations that bound intervals on which the topological entropy is apparently constant.

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1 Introduction

The problem of determining the sequence of bifurcations that result in the creation of a Smale horseshoe is an interesting one [1, 2, 3]. In this paper we use a continuation technique based on an “anti-integrable” (AI) limit [4] to study some of these bifurcations from the opposite side, that is, as bifurcations that destroy the horseshoe.

As a simple example, we study the family of Hénon maps [5, 6]

\[
\begin{pmatrix}
{x'} \\
{y'}
\end{pmatrix} = \begin{pmatrix}
y - k + x^2 \\
- bx
\end{pmatrix}.
\]

Apart from the fact that the Hénon maps are the simplest, non-trivial maps of the plane, they are of more general interest as well, since vector fields

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in the neighborhood of certain codimension-two homoclinic bifurcations can be reduced to Hénon-like maps \[7, 8\].

As we recall in §2, the AI limit for this map is essentially \( k \to \infty \). In order to represent this with finite parameters, we need only rescale the map, letting

\[ z = \epsilon x, \text{ where } \epsilon = \frac{1}{\sqrt{k}}. \]

As was shown by Devaney and Nitecki \[9\], the Hénon map has a hyperbolic horseshoe when

\[ k > (1 + |b|)^2 \frac{5 + 2\sqrt{5}}{4}. \]  \hspace{1cm} (2)

The Hénon map has at most \( 2^n \) \[10\] periodic points of period \( n \), and when the map has a hyperbolic horseshoe, all these orbits exist and are easily identified by their symbolic labels.

We showed earlier \[11\] that a contraction mapping argument implies there is a one-to-one correspondence between orbits in the AI limit and bounded orbits of the Hénon map in precisely the range Eq. (2). In §5 we show that if we consider a particular subset of orbits, this bound can be increased. Moreover, in §6, we present results of numerical investigations for all \( b \) that give what we believe are optimal bounds.

In general, the existence of an anti-integrable limit leads to a natural symbolic characterization of orbits—for the Hénon map this is the same as the horseshoe coding. We use this coding, and as discussed in §3, a predictor-corrector continuation method \[12\], to give each orbit a global code. That is, we label an orbit with the AI code, and use this designation for the family of orbits until it collides with a family with a different code. For the Hénon map, this gives a map from the bounded orbits of the map to sequences of symbols \( s \in \mathbb{Z}^2 \) modulo cyclic permutations, providing only that every orbit can be smoothly connected to the AI limit. This is the working hypothesis for our numerical method, even though we know that it is probably not true in general. It is certainly valid when the hyperbolic horseshoe exists. We give an example of a periodic orbit not smoothly connected to the AI limit in the dissipative case. This illustrates a general anti-monotonicity result \[13\], stating that when \( b \neq -1, 0, 1 \) the map generically creates but also destroys orbits when \( k \) is increased, so that the topological entropy is not necessarily monotone. We extend this anti-monotonicity result to the area preserving case by a quite different argument concerning the vanishing of twist in the neighborhood of the period tripling bifurcations in a separate paper \[14\]. Even though the area preserving case exhibits anti-monotonicity, we still conjecture that there are no isolated bubbles in the bifurcation diagram, i.e. that every orbit is continuously connected to the AI limit.

Our global code contrasts with other methods for constructing symbolic dynamics for maps, which rely on some attempt to obtain a generating partition \[15, 2, 16, 17, 18\]. These methods rely on somewhat ad hoc techniques
for constructing the partition, especially when there exist elliptic orbits. Our method gives a natural partition that is smoothly connected to the horseshoe, though it does rely on our working hypothesis.

In our computations of the Hénon map, we observe that the horseshoe destroying bifurcation appears to be a homoclinic saddle-node bifurcation when the map is orientation preserving, and a heteroclinic saddle-node when it is not. In §7 we study in detail the homoclinic orbits of the area preserving Hénon map, and show how the AI code directly gives other properties of the orbits, such as their “type,” “transition time,” and “Poincaré signature.”

For an area preserving map, the destruction of a horseshoe by a homoclinic bifurcation gives rise generically to elliptic orbits. Specifically, if \( f \) is a \( C^1 \) area preserving diffeomorphism with a homoclinic tangency at \( x \) then for any neighborhood \( U \) of \( x \), there is an area preserving diffeomorphism \( C^1 \)-close to \( f \) that has an elliptic periodic point in \( U \).

Much more is known about the behavior of area-contracting maps near a homoclinic tangency. Gavrilov and Šilnikov proved that if a \( C^3 \) map has a quadratic homoclinic tangency at a parameter \( k^* \) then there exists a sequence of parameter values \( k_n \to k^* \) such that at \( k_n \) there is a saddle-node bifurcation creating orbits of period \( n \) [20, 21]; because one of the created orbits is a sink, this is called a cascade of sinks. Robinson extended these results to the real analytic case where the intersection is created degenerately [22]. In our computations we will find a similar cascade of saddle-node bifurcations for the area preserving Hénon map—this gives a sequence of elliptic orbits that limit on the homoclinic bifurcation. Thus the destruction of the horseshoe is associated with the creation of the first stable “island.”

The ordering on the invariant manifolds poses severe restrictions on the possible bifurcations. In §8 we use it to prove which homoclinic bifurcation of the hyperbolic fixed point is the first one. We observe that the forcing relations between homoclinic orbits up to type 6 is essentially like the unimodal ordering of one dimensional maps. Generically a homoclinic bifurcation corresponds to a quadratic tangency of the stable and unstable manifolds—a “homoclinic saddle-node bifurcation.” There are two more generic bifurcations in maps with a symmetry: a homoclinic pitchfork when the manifolds exhibit a cubic tangency, and a simultaneous pair of asymmetric saddle node bifurcations. In §9 we show that a symmetric homoclinic bifurcation forbids certain other bifurcations to occur after it, leading to a natural mechanism to create homoclinic pitchfork bifurcations or asymmetric saddle node pairs. We observe all three of these bifurcations for the area preserving Hénon map, which has a time-reversal symmetry.

Davis, MacKay, and Sannami [3] conjectured that there are intervals of \( k \) below the horseshoe for which the Hénon map is a hyperbolic Markov shift. They also identified the Markov partitions for these cases. Their conjecture was based on computing all the periodic orbits up to a period 20 using the technique of Biham and Wenzel [23, 24]. In §11 we confirm their computations with our continuation technique and extend them to period 24—an
order of magnitude more orbits. Moreover, we identify the bifurcations responsible for the creation and destruction of these apparently hyperbolic intervals; as befits with the theme of this paper, they are homoclinic bifurcations.

2 Anti-Integrable Limit

Dynamics in discrete time can be represented by a relation, \( F(x, x') = 0 \) where \( x \) and \( x' \) are points in some manifold. Normally, we can explicitly solve for \( x' = f(x) \), giving a map on the manifold, with orbits defined by sequences \( x_t = f(x_{t-1}) \). Suppose, however, that \( F \) depends upon a parameter \( \epsilon \), in such a way that this is not always possible; for example, \( F(x, x') = \epsilon G(x, x') + H(x) \). In this case the implicit equation \( F = 0 \) can no longer be solved for \( x' \) when \( \epsilon = 0 \); instead “orbits” correspond to arbitrary sequences of points, \( x_t \) that are zeros of \( H \)—the dynamics are not deterministic. In this case we say that \( \epsilon = 0 \) corresponds to an anti-integrable (AI) limit of the map \( f \). If the derivative of \( H \) is nonsingular, then a straightforward implicit function argument can be used to show that the AI orbits can be continued for \( \epsilon \neq 0 \) to orbits of the map \( f \). An AI limit with this property is called nondegenerate.

For example, consider the Hénon map Eq. (1). Denoting points on an orbit by a sequence \( x_t, t \in \mathbb{Z} \), we can rewrite Eq. (1) as a second order difference equation

\[
    x_{t+1} + bx_{t-1} + k - x_t^2 = 0.
\]

Introducing the scaled coordinate \( z = \epsilon x \) and choosing \( \epsilon = k^{-1/2} \) gives an implicit map in the variable \( z \) with parameter \( \epsilon \)

\[
    \epsilon(z_{t+1} + bz_{t-1}) + 1 - z_t^2 = 0. \tag{3}
\]

With this choice of \( \epsilon \), we can study only the range \( 0 < k < \infty \); however, one could redefine \( \epsilon \) to shift this range. A period \( n \) orbit of the Hénon map is given by a sequence \( z_0, z_1, \ldots, z_{n-1} \) that satisfies Eq. (3), together with the condition that \( z_{t+n} = z_t \). The corresponding family of periodic orbits is denoted by \( z(\epsilon) \).

At the AI limit, the map Eq. (3) reduces to

\[
    z_t^2 = 1.
\]

Thus “orbits” in this limit are arbitrary sequences of \( \pm 1 \), which we abbreviate with \( + \) and \( - \). Let \( \Sigma \) denote the space of such sequences

\[
    \Sigma \equiv \{-, +\}^\mathbb{Z} = \{s : s_t \in \{-, +\}, t \in \mathbb{Z}\}. \tag{4}
\]

\(^1\) For example, choosing \( \hat{\epsilon} = (k + \delta)^{-1/2} \), maps positive values of \( \hat{\epsilon} \) to the range \(-\delta < k < \infty \). Our numerical routines typically use \( \delta = 1 \) so that we can cover the entire parameter range where there are bounded orbits. In the text we always use \( \delta = 0 \).
For ease of notation we denote the corresponding sequence of \( \{+1, -1\} \subseteq \mathbb{R}^\mathbb{Z} \) by the same symbol \( s \). Hence every sequence \( s \in \Sigma \) is an orbit \( s \in \mathbb{R}^\mathbb{Z} \) at the anti-integrable limit, and each of these can be continued to an orbit of the Hénon map for small enough \( \epsilon \) [22, 23]. Previously we gave an explicit upper bound on \( \epsilon \) for the existence of orbits for every symbol sequence [11]:

**Theorem 1.** For every symbol sequence \( s \in \Sigma \) there exists a unique orbit \( z(\epsilon) \) of the Hénon map Eq. (3), such that \( z(0) = s \) providing

\[
|\epsilon|(1 + |b|) < 2\sqrt{1 - 2/\sqrt{5}} \approx 0.649839. \tag{5}
\]

The basic idea of the proof of this theorem is as follows [11]. Let \( B_M \) be the closed ball of radius \( M \) around the point \( s \in \Sigma \),

\[
B_M(s) = \{ z : ||z - s||_\infty \leq M \}. \tag{6}
\]

For each symbol sequence \( s \in \Sigma \) and small enough \( M \), define a map \( T : B_M \to B_M \) by

\[
T_i(z) = s_i \sqrt{1 + \epsilon(z_{i+1} + bz_{i-1})}, \tag{7}
\]

then the corresponding Hénon map orbit \( z(\epsilon) \) is a fixed point of \( T \). The conclusion of Theorem [3] follows from finding the maximum value of \( \epsilon \) for which there is an \( n \) such that \( T^n \) is a contraction mapping (i.e., \( T^n : B_M \to B_M \) and \( ||DT^n|| < 1 \)).

The fact that \( T \) is a contraction implies that there are no bifurcations in the range Eq. (5). This is the statement of

**Corollary 2.** There are no bifurcations in the Hénon map when \( \epsilon \) and \( b \) are in the range given in Theorem [3].

**Proof:** Denote the system of equations (3) by \( H(z, \epsilon) = 0 \). This infinite continuation problem has a unique solution \( z(\epsilon) \) if the inverse of \( D_z H \) is bounded. The \( t \)-th component of \( T \) is related to the \( t \)-th component of \( H \) by

\[
H_t = T_t^2 - z_t^2.
\]

Differentiating this at the fixed point that exists according to Theorem [3] gives

\[
D_z H = \text{diag}(2z_t)(DT - \text{id}).
\]

The operator \( D_z H \) has bounded inverse because \( z_t \) is bounded away from zero by Eq. (3) and the inverse of \( DT - \text{id} \) exists because \( ||DT|| < 1 \). \( \square \)

This result can be extended to imply that the invariant set is uniformly hyperbolic for the case \( b = 1 \) when the operator \( D_z H \) is symmetric [28].
In §6 we will use numerical continuation to estimate the parameters at which the first bifurcation occurs, giving an improvement in this bound, albeit a numerical one.

It is interesting that Devaney and Nitecki [9] obtained precisely the same bound, Eq. (5), for the parameter domain in which the non-wandering set of the Hénon map is a hyperbolic horseshoe. Nevertheless the AI continuation argument has two advantages over the geometrical arguments of Devaney and Nitecki. First, it easily generalizes to higher dimensions allowing one to compute parameter bounds for the existence of horseshoes in higher dimensional maps [29]. Second, it allows us to easily bound the parameter range for which certain subsets of orbits exist (i.e. the parameter range where the map is conjugate to a subshift of finite type). We present such a bound for a subshift of finite type in §3.

3 Numerical Method

In this section we formulate the problem of following Hénon map orbits away from the anti-integrable limit as a classical continuation problem [30].

A period $n$ orbit family of the Hénon map with coordinates given by $z(\epsilon)$ is a zero of the function $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ whose $t^{th}$ component is given by the left hand side of Eq. (3). The zeros of $G$ are generically smooth curves in $\mathbb{R}^n \times \mathbb{R}$ defined by the continuation problem

$$\begin{cases}
G(z, \epsilon) = 0 \\
z(0) = s
\end{cases}$$

Practically the continuation is always started at the AI limit. The curve might either extend to some maximal $\epsilon_{\text{max}}$ and return the the limit or just continue indefinitely. Since for the Hénon map there are no orbits for $k < -1$ we expect that all the curves do return to the AI limit.

This is a standard continuation problem, which we solve using a predictor-corrector method with a linear tangent predictor and a Newton’s method corrector. For numerical linear algebra we use the Meschach library [31]. The algorithm incorporates an adaptive step size control with bisection backtracking if the corrector fails to converge. The algorithm terminates when a user-specified value of $\epsilon$ is reached or, when the tangent direction is not uniquely defined. The process of continuing a sequence of orbits is trivially parallelizable since the operations performed on each orbit are completely independent of each other. We use a “divide and conquer” strategy to spread the total computational effort across several different machines running simultaneously. This is especially advantageous when the number of orbits continued reaches into the millions.

Continuation methods are based on the assumption that the orbits of interest are actually connected to the limit at which the continuation starts. Since the Hénon map does not have an integrable limit, the natural starting
point would be to continue all the orbits that bifurcate off the fixed points. But this would only yield a small fraction of all orbits: many of them are born in saddle-node bifurcations that are not connected to any other orbit. Therefore the AI limit is a much better limit from which to continue. However, the same general restriction applies, i.e. only orbits that are connected to the limit (or their parents, grandparents etc.) can be found. We formulate this central hypothesis as the “no-bubble-conjecture”:

**Conjecture 1 (No Bubbles).** For the area preserving Hénon map every orbit is (at least) continuously connected to the anti-integrable limit.

We could only hope for continuous connection in \( \mathbb{R}^n \times \mathbb{R} \) because the branches corresponding to the children in a bifurcation are in general not smoothly connected to the parent. Our numerics currently does not perform any branch-switching from parents to children. Therefore, in practice, we are actually using the working hypothesis: “Every orbit of the Hénon map can be smoothly connected to the anti-integrable limit.” Unfortunately our working hypothesis is not true in general. For \( b \neq -1, 0, 1 \) it has been shown \cite{13} that periodic orbits are both created and destroyed when the map parameter \( k \) is increased (the authors of \cite{13} call this “antimonotonicity”). Moreover, in \cite{14} we show that this conclusion holds for the cases \( b = \pm 1 \) as well. Consequently the topological entropy is not necessarily a monotone increasing function, as it is for the logistic map, \( b = 0 \). In the following we will elucidate the relation between antimonotonicity, our working hypothesis and the no-bubble conjecture.

Even though we have antimonotonicity whenever \( b \neq 0 \), this does not readily imply that our working hypothesis is false. In particular the smallest period orbit that is antimonotonic when \( b = 1 \) is of period 10, and it is still smoothly connected with the AI limit \cite{14}. However, orbits that bifurcate from this one may violate the working hypothesis.\footnote{This orbit is created in a 3/10 rotational bifurcation of the elliptic fixed point, and it initially moves towards smaller \( k \) values. While it is traveling in the “wrong” direction, the elliptic 3/10 orbit has a winding number that does not exceed \( 1/5000 \); therefore orbits that bifurcate from it are of period 50000 or higher. We suspect that these would be orbits that are not smoothly connected to the AI limit, and therefore violate our working hypothesis. Since they bifurcate off the 3/10 orbit which in turn is smoothly connected, they are, however, at least continuously connected to the AI limit, so that the no-bubble conjecture could still be true.}

The worst possible case from the point of view of continuation is an orbit that neither smoothly nor continuously connects to the limit, i.e. an “isolated bubble.” Note that in the area preserving case, one orbit of this type implies an infinite number of them because it must be born in a saddle node bifurcation and the stable orbit of the created pair generically passes through an infinite number of rational winding numbers. In order to constitute a violation of our conjecture, none of these orbits would be allowed to reach the AI limit; otherwise the original orbit would be continuously connected.
In the dissipative case, the lowest period example we were able to find of a periodic orbit that is not smoothly connected occurs for \( b = -0.46 \), where at \( k \approx 1.0346 \) the period 8 orbit with sequence \((-5 + -+)^\infty\) has a period doubling bifurcation that creates a period 16 orbit that is not smoothly connected to the AI limit. Since this is the start of a period doubling sequence resulting in a what appears to be a strange attractor at a slightly smaller value of \( k \), we expect there are many saddle-node bifurcations in this region that create orbits as \( k \) decreases that are (presumably) not connected to the AI limit. This is reason to believe that the “no bubbles” conjecture is false when \( |b| < 1 \).

In any case, continuation from the anti-integrable limit has the advantage that at the beginning point all orbits exist, and they all continue nondegenerately.

## 4 Symbolic Dynamics

In this section we introduce some notation for symbol sequences and bifurcations. For simplicity we concentrate mostly on the area preserving case, \( b = 1 \), though many results apply generally.

Orbits in the anti-integrable limit are bi-infinite sequences \( s \in \Sigma \). When it is needed, we will indicate the current time along an orbit using a “\(.\)”, so that \( s = \ldots s_{-2} s_{-1} s_0 s_1 s_2 \ldots \). The dynamics on \( s \in \Sigma \) are given by the shift map, \( \sigma : \Sigma \to \Sigma \) defined as

\[
\sigma(\ldots s_{-1} s_0 s_1 s_2 \ldots) = \ldots s_{-1} s_0 s_1 s_2 \ldots
\]

An orbit of the symbolic dynamics is periodic if the sequence \( s \) is periodic. We will denote an orbit of least period \( n \) by the string of \( n \) symbols and a superscript \( \infty \) to represent repetition:

\[
(s_0 s_1 \ldots s_{n-1})^\infty = \ldots s_{n-2} s_{n-1} s_0 s_1 \ldots s_{n-1} s_0 \ldots
\]

Of course any cyclic permutation of a periodic orbit gives another point on the same orbit.

Trivially, the map \( \sigma \) has two fixed points, \((+)^\infty\) and \((-)^\infty\), and these correspond to the two fixed points of the Hénon map. These are born in a saddle-node bifurcation at \( k = -(1 + b)^2 / 4 \), which we denote by

\[
\text{sn} \{(+)^\infty, (-)^\infty\}.
\]

We denote bifurcations with the general template

\[
\text{parent} \to \text{type} \{\text{children}\},
\]

where \text{parent} refers to the orbit that is undergoing the bifurcation, if any, and \text{type} is one of sn, pf, pd, or \( m/n \), corresponding to a saddle-node, pitchfork, period doubling, or rotational bifurcation, respectively. The set
of orbits created in the bifurcation is listed as the *children*. When the
stability of these differ, we adopt the convention that the unstable child is
listed first, and the stable one second.

When \( b = 1 \) the fixed points of the Hénon map are located at
\[
z_\pm = \pm \sqrt{1 + \epsilon^2} + \epsilon = \epsilon \left(1 \pm \sqrt{1 + k}\right).
\]
The stability of a period \( n \) orbit of an area preserving map \( f \) is conveniently
classified by the “residue” defined as
\[
R = \frac{1}{4} \left(2 - \text{Tr} (Df^n)\right),
\]
so that an orbit is hyperbolic if it has negative residue, elliptic when \( 0 < R < 1 \) and is reflection hyperbolic when \( R > 1 \) \[32\]. The residues of the
fixed points are
\[
R_\pm = \pm \frac{1}{2\epsilon} \sqrt{1 + \epsilon^2} = \pm \frac{1}{2} \sqrt{1 + k},
\] (8)
so the sign of the symbol is opposite to the sign of the residue of the fixed
point. Thus the orbit \((+)^\infty\) is always hyperbolic, while the orbit \((-)^\infty\) is
reflection hyperbolic for small \( \epsilon \), or large \( k \), but becomes elliptic at \( \epsilon = 1/\sqrt{3} \),
or \( k = 3 \).

For \( b = 1 \) the sequence \((+-)^\infty\) corresponds to the period two orbit
\[
(+-)^\infty : (z_0, z_1) = (\sqrt{1 - 3\epsilon^2} - \epsilon, -\sqrt{1 - 3\epsilon^2} - \epsilon).
\]
This orbit exists only for \( \epsilon < 1/\sqrt{3} \), and is created by a period doubling of
the elliptic fixed point (when \( R_- = 1 \)). We denote this bifurcation by
\[
(-)^\infty \rightarrow \text{pd} \{ (+)^\infty \}.
\]
Similarly there are two period three orbits,
\[
(-+)^\infty : (z_0, z_1, z_2) = (-\sqrt{1 - \epsilon^2}, -\sqrt{1 - \epsilon^2}, \sqrt{1 - \epsilon^2} - \epsilon)
\]
\[
(-+)^\infty : (z_0, z_1, z_2) = (-\sqrt{1 - \epsilon^2} - \epsilon, \sqrt{1 - \epsilon^2}, \sqrt{1 - \epsilon^2}).
\]
These are created in a saddle-node bifurcation at \( k = \epsilon = 1; \)
\[
\text{sn} \{ (--^\infty, (-+^\infty) \}.
\]
We list the low period orbits and their bifurcation values in Table \[4\].
Another class of bifurcations shown in the table are rotational bifurcations.
A rotational bifurcation occurs when the winding number of an elliptic orbit
becomes \( \omega = m/n \); we denote such bifurcations by the winding number of
the parent orbit. For example the birth of orbits with winding number 1/\( n \)
at the fixed point \((-)^\infty\) is denoted
\[
(-)^\infty \rightarrow 1/n \{ (-+)^{n-2}, (-+)^{n-2} \}.
\] (9)
This particular rotational bifurcation occurs when the multipliers of the fixed point are $e^{2\pi \omega}$ or using Eq. (8), when $k$ is given by
\begin{equation}
    k_\omega = \cos(2\pi \omega)(\cos(2\pi \omega) - 2) .
\end{equation}

We have empirically identified the symbol sequences for rotational bifurcations, and will present the general symbolic formula for these and for rotational “island around island” orbits in [29].

The residue of any periodic orbit of a Lagrangian system is easily computed from the matrix $M$ formed from the second variation of the action [33]. For a period $n$ orbit of the Hénon map this formula gives:
\begin{equation}
    R(z(\epsilon)) = -\frac{1}{4} \frac{\det(M)}{\epsilon^n} ,
\end{equation}

where $M$ is the periodic tridiagonal matrix with elements
\begin{equation}
    M_{t,t-1} = -b \epsilon , \quad M_{t,t} = 2z_t(\epsilon) , \quad M_{t,t+1} = -\epsilon .
\end{equation}
As we approach the anti-integrable limit, \( z(\epsilon) \to s \) as \( \epsilon \to 0 \) and \( M \) approaches the diagonal matrix \( \text{Diag}(2s_i) \). Thus we see that the residue becomes infinite at the anti-integrable limit and its sign is given by \(-\prod_{t=0}^{n-1} s_t\). Hence,

\[
\text{sign}(R(s)) = -(-1)^j,
\]

where \( j \) is the number of minus signs in the symbol sequence \( s \).

5 A Subshift of Finite Type

In this section, we extend Theorem 1 to the case of a subshift of finite type. In particular, the biggest restriction in the proof of the theorem arises from the fact that the lower bound on the operator \( T \) given in Eq. (7) is weakest when the signs \( s_{i+1} = s_{i-1} = -1 \) for positive \( b \). We can improve the bound by restricting the set of admissible symbol sequences to forbid this particular case. The shift map restricted to this subspace is a subshift of finite type with the forbidden set \( \mathcal{F} = \{--, --, --\} \); that is, we define the shift space

\[
\Sigma_{\mathcal{F}} = \Sigma \setminus \{ s : \exists t \in \mathbb{Z} \text{ such that } s_{t-1} = s_{t+1} = -1 \}.
\]

This subshift can be easily described as a subshift on 2-blocks represented by \( \{--, --, ++\}^\mathbb{Z} \). The subshift on the two-block space is represented by a vertex graph with the state transition matrix

\[
S = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix},
\]

which indicates by \( S_{ij} = 1 \) an allowed transitions from state \( j \) to state \( i \). The two zeros \( S_{11} \) and \( S_{32} \) come from the forbidden sequence in \( \Sigma_{\mathcal{F}} \), the remaining are obtained because successive two-blocks overlap in one symbol, i.e., \( (s_{t-1}s_t) \) has to be followed by \( (s_ts_{t+1}) \). The number of fixed points of period \( n \) for the subshift is given by

\[
\text{Tr}(S^n) = \gamma^n + (1 - \gamma)^n + 2(-1)^{n/2}(n - 1 \mod 2),
\]

where \( \gamma = (1 + \sqrt{5})/2 \) is the golden mean. Thus the topological entropy for \( \Sigma_{\mathcal{F}} \) is \( \ln \gamma \). The number of distinct periodic orbits can be obtained from the trace formula by subtracting the number of periodic orbits for all factors of \( n \) and then dividing by the number of cyclic permutations, \( n \). For comparison with the full shift and with the numerical results below, we give a list of these in Table 2. For example there are a total of 1,465,020 periodic points of the full shift with period \( n \leq 24 \), while there are only 12,216 in the subshift \( \Sigma_{\mathcal{F}} \).

When \( b \) is non-negative, orbits with symbol sequences in the subspace \( \Sigma_{\mathcal{F}} \) can be shown to persist longer than a general orbit:
Table 2: Number of orbits with minimal period $n$ of the 2-shift and the subshift $\Sigma_F$.

| Period | $\Sigma$ | $\Sigma_F$ | Period | $\Sigma$ | $\Sigma_F$ |
|--------|----------|------------|--------|----------|------------|
| 1      | 2        | 1          | 13     | 630      | 40         |
| 2      | 1        | 0          | 14     | 1161     | 58         |
| 3      | 2        | 1          | 15     | 2182     | 90         |
| 4      | 3        | 2          | 16     | 4080     | 135        |
| 5      | 6        | 2          | 17     | 7710     | 210        |
| 6      | 9        | 2          | 18     | 14532    | 316        |
| 7      | 18       | 4          | 19     | 27594    | 492        |
| 8      | 30       | 5          | 20     | 52377    | 750        |
| 9      | 56       | 8          | 21     | 99858    | 1164       |
| 10     | 99       | 11         | 22     | 190557   | 1791       |
| 11     | 186      | 18         | 23     | 364722   | 2786       |
| 12     | 335      | 25         | 24     | 698870   | 4305       |

Theorem 3 (Existence and Uniqueness of $\Sigma_F$ orbits). Suppose $0 \leq b \leq 1$. For every symbol sequence $s \in \Sigma_F$ there exists a unique orbit $z(\epsilon)$ of the Hénon map Eq. (4) such that $z(0) \equiv s$ providing $0 \leq \epsilon < \epsilon_{\text{max}}$, where

$$
\epsilon_{\text{max}} \equiv \frac{2}{1+b} \sqrt{-b^2 + 2b + 5 - 2\sqrt{5 + 4b}} \frac{1}{(1-b)(5-b)}. \tag{12}
$$

This theorem follows from the same argument that gave Theorem 1 with only minor modifications. We summarize the changes in the argument in the following discussion.

**Proof:** When $0 \leq b \leq 1$, $s \in \Sigma_F$, and $z \in B_M$, we can bound the norm of iterates of $T$ in Eq. (7) using the inequalities

$$
\alpha_k \leq ||T^k(z)||_{\infty} \leq \beta_k,
$$

where the coefficients $\alpha_k$ and $\beta_k$ are determined by the recursions

$$
\beta_k = \frac{1 + \epsilon(1+b)\beta_{k-1}}{1 + \epsilon(5 + 4b)} ,
$$

with $\beta_0 = 1 + M$ and $\alpha_0 = 1 - M$. The sequence $\beta_k$ is the same as that in (11); it has the unique attracting fixed point

$$
\beta_{\infty} = \frac{1}{2} \left\{ \epsilon(1+b) + \sqrt{\epsilon^2(1+b)^2 + 4} \right\}.
$$

Since the recursion for $\{\alpha_n\}$ depends on $\beta$, but not vice versa, the coupled system also has a unique attracting fixed point, which is given by $(\alpha_{\infty}, \beta_{\infty})$ with

$$
\alpha_{\infty} = \frac{1}{2} \left\{ \epsilon b + \sqrt{\epsilon^2 b^2 + 4(1 - \epsilon \beta_{\infty})} \right\}.
$$
6 HORSESHOE BOUNDARY

This implies that for large enough \( n \), \( T^n \) maps the ball \( B_{1−\alpha_\infty} \) into itself. The map \( T \) is a contraction map on this ball providing \( \|DT^k\|_\infty < 1 \). This leads to the same bound as that in [11], namely:

\[
\epsilon_{\text{max}}(1 + b) < 2\alpha_\infty.
\]

After some simplification, this inequality yields the formula for \( \epsilon_{\text{max}} \). The condition that the operator \( T \) be real, \( \epsilon < 1/(\beta_\infty - b\alpha_\infty) \), is satisfied whenever the map is a contraction. \( \square \)

Similar arguments for negative \( b \) lead to bounds for a subshift with the forbidden subsequence \(+\ast−\) (where \( \ast \) is any symbol). However, this subshift is not of much interest, since there are only three periodic orbits in it.

6 Horseshoe Boundary

Theorem [2] provides an analytical bound on the parameter range for which the Hénon map has a hyperbolic horseshoe. This bound corresponds to the dark grey region in Fig. 1. According to Theorem [3] the subshift \( \Sigma_F \) exists, in addition, in the lighter shaded area in the figure. This bound is valid only for \( b \geq 0 \), and meets the former at \( b = 0 \).

Here, we use our continuation method to estimate the boundary of existence of the horseshoe by following all orbits up to period 24 from the anti-integrable limit. In order for the numerical boundary to be valid we only need to assume that we can extrapolate from 24 to \( \infty \). Since there are at most \( 2^n \) periodic points of period \( n \) in the Hénon map [10], we know that we are not missing any orbits because we have them all at the AI limit. This is no longer true after the first bifurcation because orbits that have disappeared might reappear for smaller \( k \).

To construct a numerical approximation for the boundaries, we first generate all symbol sequences for orbits of periods up to 24. Then, for fixed \( b \), we numerically continue each orbit in \( \epsilon \) away from the anti-integrable limit and monitor its multipliers to detect bifurcations. For each \( b \) we record the smallest value of \( \epsilon \) at which a bifurcation occurs. The resulting numerical bounds in Fig. 1 are shown as solid curves.

The numerical bound for the full shift is similar in shape to the analytical one, but shifted to the right in \( \epsilon \). While the analytical bound is symmetric under \( b \to -b \), the numerical results are not. For example the first bifurcation at \( b = 1 \) occurs for \( \epsilon \approx 0.41888 \), while at \( b = -1 \) it occurs for \( \epsilon \approx 0.40167 \). In the logistic limit \( (b = 0) \), Eq. (1) reduces to the logistic map,

\[
z_{t+1} = \frac{1}{\epsilon}(z_t^2 - 1),
\]

for which the first bifurcation occurs at \( \epsilon = 1/\sqrt{2} \), where the orbit of the critical point becomes bounded.
When $b$ is positive, the symbol sequences for the first pair of orbits destroyed up to period 24 extrapolate to orbits that are homoclinic to the fixed point $(+)^\infty$; we conjecture that these are the first orbits destroyed as $\epsilon$ increases from 0:

**Conjecture 2.** For positive $b$, the first bifurcation as $\epsilon$ increases from 0 corresponds to the homoclinic saddle-node bifurcation

$$\text{sn } \left\{ +^\infty - (+) - +^\infty, +^\infty - (-) - +^\infty \right\}.$$  

(14)

The parenthesis in the middle enclose the “core” of the homoclinic orbit, see the next section. A theorem of Smillie [35] implies that the first bifurcation destroying the Hénon horseshoe must be a quadratic homoclinic tangency for some orbit. Our observations imply that it is a homoclinic bifurcation of $(+)^\infty$. When $b < 0$, however, the most natural description of the first bifurcation is as a heteroclinic tangency, which leads to

**Conjecture 3.** For negative $b$, the first bifurcation as $\epsilon$ increases from 0 corresponds to the heteroclinic saddle-node bifurcation

$$\text{sn } \left\{ -^\infty + (-) - +^\infty, -^\infty + (+) - +^\infty \right\}.$$ 

This does not contradict Smillie’s theorem, as there are many homoclinic bifurcations that accumulate on this heteroclinic bifurcation. For example, for each $m$ the orbits $(-m+++)^\infty - +(-m+++)^\infty$, are homoclinic

Figure 1: First bifurcations for the Hénon Map. The dark shaded region represents Theorem 1 and the lighter that of Theorem 2. The curves represent the numerical results for the first orbits destroyed up to period 24. Bounds for the subshift $\Sigma_F$ are indicated with a triangle symbol.
to the periodic orbit \((-m + + - +^m)\infty\), and the bifurcation points of these homoclinic orbits limit on that of the heteroclinic orbits as \(m \to \infty\).

Filtering the symbol sequences to choose only those in the subshift \(\Sigma_F\), we can use the same numerical data previously described to find the first bifurcation amongst the orbits in \(\Sigma_F\). This gives the solid curve marked with triangles in Fig. 1. This curve has a qualitatively different shape than the analytical bound.

For reference we indicate in Fig. 1 the point \(k = 1.4\), and \(b = -0.3\), corresponding to the much studied Hénon attractor. We also sketch the parameter range (\(b\) small enough, \(1 < k < 2\)) for which the theorem of Benedicks and Carleson [36] implies that the Hénon map has a transitive attractor with positive Lyapunov exponent.

Note that the numerical horseshoe boundary does not depend on the no bubble conjecture. This is so because it is known that there can be at most \(2^n\) periodic points of period \(n\) [10]. Since we follow all of them up to period 24 there can be no other orbits up to that period. In other words, orbits first have to be destroyed before they can be reborn.

7 Homoclinic Orbits

In this section we use the symbolic dynamics to classify orbits of the Hénon map that are homoclinic to the hyperbolic fixed point \(p = (+)^\infty\) and study their bifurcations. We begin with some general terminology, referring to the Hénon map as an example.

Let \(f\) be an orientation preserving map of the plane with hyperbolic fixed point \(p\). The stable and unstable manifolds of \(p\) are denoted by \(W^u\) and \(W^s\), and a closed segment of such a manifold between two points \(\alpha\) and \(\beta\) by \(W^u[\alpha, \beta]\). We use a parenthesis to denote an open endpoint of a segment. A segment that extends to the fixed point, e.g. \(W^u(p, \alpha]\), is called an initial segment of the manifold. The set of homoclinic orbits is the set of intersections \(W^s \cap W^u\). A point \(\alpha\) is on a primary (or principal) homoclinic orbit if the two initial segments to \(\alpha\) touch only at \(\alpha\), i.e.,

\[
W^u(p, \alpha] \cap W^s[p, \alpha] = \{\alpha\}.
\]

Thus the initial segments to a primary homoclinic orbit define a Jordan curve; we call the interior of this curve a resonance zone. More generally, a resonance zone is a region bounded by alternating initial segments of stable and unstable manifolds [37, 38].

For example, in Fig. 3 we sketch the left-going branches of the manifolds from \(p = (+)^\infty\) for the area preserving Hénon map. There are precisely two primary homoclinic orbits; in the figure, we label points on these orbits with \(\alpha\) and \(\zeta\). We choose to use \(\alpha\) to construct the resonance zone.

\footnote{The orientation reversing case could be included by considering \(f^2\), since its manifolds have the same geometry as those of \(f\).}
The stable manifold is divided into two invariant branches by the fixed point. An ordering is defined on each branch of $W^s$, so that $\beta <_s \gamma$ for two points on a branch of $W^s$ if $\beta$ is nearer to $p$ along $W^s$ than $\gamma$, i.e., $\beta \in W^s(p, \gamma)$. We similarly define an ordering $<_u$ on each branch of $W^u$.

A segment of a manifold from a point to its iterate, $W^s(\beta, f(\beta))$, is called a fundamental segment \[15\]. The union of the iterates of a fundamental segment is the entire branch of the manifold that contains $\beta$. Moreover, since the iterates are disjoint, every homoclinic orbit on this branch must have precisely one point on each fundamental segment.

For the Hénon map, we focus on the left-going branches of $W^s$ and $W^u$ and the fundamental segments between $f^{-1}(\zeta)$ and $\zeta$. These also form the boundaries of the incoming and exit sets for the resonance zone defined by $\alpha$. The exterior halves of these segments, $W^s(\alpha, \zeta)$ and $W^u(f^{-1}(\zeta), \alpha)$, contain no homoclinic points since orbits on these segments are unbounded, so it is sufficient to look for homoclinic points on the interior halves,

\[
U \equiv W^u[\alpha, \zeta], \\
S \equiv W^s[f^{-1}(\zeta), \alpha].
\]

Every homoclinic orbit must have exactly one point on both $S$ and $U$.

Homoclinic orbits can be classified in a number of ways. The type \[38\],

Figure 2: Stable and unstable manifolds for the Hénon map at $k = 5$ and $b = 1$ shown in $(z, z')$ coordinates.
of a homoclinic point $\beta$ is

$$\text{type}(\beta) = \sup\{j \geq 0 : W^s(p, f^j(\beta)) \cap W^u(p, \beta) \neq \emptyset\} ;$$

i.e., the number of iterates for which the stable initial segment to $f^j(\beta)$ intersects with the unstable initial segment to $\beta$. The type of a homoclinic point is invariant along its orbit. Primary homoclinic points have type 0.

Homoclinic orbits on particular branches of $W^s$ and $W^u$ can also be classified by their transition time. In general this is defined relative to a choice of a primary homoclinic point, $\zeta$ and the fundamental segments $W^u(f^{-1}(\zeta), \zeta]$ and $W^s(f^{-1}(\zeta), \zeta]$. Any homoclinic orbit on these branches has exactly one point, $\beta$, on the unstable segment. The transition time is the number of iterates required for $\beta \in W^u(f^{-1}(\zeta), \zeta]$ to reach the stable segment:

$$t_{\text{trans}}(\beta) = k \text{ if } f^k(\beta) \in W^s(f^{-1}(\zeta), \zeta]$$

The value of the transition time depends upon the choice of fundamental segments, so it is not as basic a property as the type.

In the simplest case, the transition time is easily related to the type of the orbit \footnote{Our definition of the type differs from Easton’s slightly, to comply with his definition that type 1 is equivalent to the horseshoe. Rom-Kedar \cite{40} uses the term Birkhoff signature instead of type.}:

**Lemma 4.** Assume there are exactly two primary homoclinic orbits, $\zeta$ and $\alpha$, and the segments $S$ and $U$ defined in Eq. (13) contain all of the homoclinic orbits. Then for each homoclinic point in $\beta \in U$, $t_{\text{trans}}(\beta) = \text{type}(\beta)$.

**Proof:** If $\beta \in U$ is of type $t$, then by definition $W^s(p, f^t(\beta)) \cap W^u(p, \beta) \neq \emptyset$. Now since $\alpha <_u \beta <_u \zeta$ and $W^s(p, \zeta) \cap W^u(p, \zeta) = \emptyset$, this implies that $\alpha <_s f^t(\beta)$. However, $W^s(p, f^{t+1}(\beta)) \cap W^u(p, \beta) = \emptyset$, which means that $f^{t+1}(\beta) <_s \alpha$, but there are no homoclinic points on $W^s(\zeta, \alpha)$, so actually $f^{t+1}(\beta) <_s \zeta$. Now $S$ contains every homoclinic point that reaches $W^s(p, \zeta)$ in one iteration, so $f^t(\beta) \in S$. \qed

Each homoclinic orbit has a Poincaré signature that determines the direction of crossing of $W^u$ and $W^s$ at points on the orbit. We define the signature to be +1 if, looking along the unstable manifold in the direction of motion, the stable manifold crosses the unstable from the left to the right side. Crossings in the opposite direction have signature −1. If the manifolds do not cross but only touch (a topologically even intersection), the signature is defined to be zero. Since the map is orientation preserving, the signature is invariant along an orbit. Thus in Fig. 2 $\alpha$ and $\zeta$ have signatures −1 and +1, respectively. The signature of a particular homoclinic orbit is typically not preserved in a bifurcation, but the total signature of the bifurcating orbits must be the same on each side of the bifurcation value.
a saddle-node bifurcation creates a zero signature orbit that splits into one positive and one negative signature orbit.

For the Hénon map, the AI symbol sequence can be used for the classification of homoclinic orbits. It is easy to construct homoclinic and heteroclinic orbits using the symbolic dynamics: an orbit heteroclinic from a periodic orbit \((s)^\infty\) to a periodic orbit \((s')^\infty\) has a symbol sequence that begins with a head sequence \((s)^\infty\) and ends with a tail sequence \((s')^\infty\) with some arbitrary, finite symbol sequence separating the head and tail. For example, the simplest orbits homoclinic to \(p = (+)^\infty\) are the primary homoclinic orbits:

\[
\begin{align*}
\zeta &= +^\infty - +^\infty, \\
\alpha &= +^\infty - - +^\infty,
\end{align*}
\]

(16) corresponding to those we labeled in Fig. 2. These symbol sequences arise because as \(\epsilon \to 0\) the point \(\alpha\) moves to the point \(- -\), while \(\zeta\) moves to \(- . +\) and \(f^{-1}(\zeta)\) to \(+ . -\).

All other orbits homoclinic to \(p\) can be written in the form \(+^\infty - (s) - +^\infty\), where \(s\), the core, is any finite sequence—thus there is a one-to-one correspondence between finite symbol sequences and potential homoclinic orbits (all of which exist in the AI limit). This implies, for example, that near the anti-integrable limit there are \(2^k\) homoclinic orbits with core length \(k\). We will often denote an orbit homoclinic to \(p\) simply by writing the core in parenthesis, \((s)\). Note that a given core \((s)\) is not equivalent to any core with the same \(s\) cyclically permuted.

The classification of homoclinic orbits by their symbol sequence can be used to compute other invariants. To determine the type of an orbit, we simply note that the AI symbols give exactly the same coding for an orbit as the standard symbolic coding for the horseshoe. This implies that the point \(+^\infty - (s) - +^\infty\) corresponds to a phase point on \(U\), and the point \(+^\infty - (s) - +^\infty\) is on \(S\), thus

**Lemma 5.** The transition time of the homoclinic orbit close to the AI limit is given by the length of the core sequence.

For example, the homoclinic orbit \(+^\infty - (s) - +^\infty\) has the core sequence \((- - +)\), and therefore has transition time 3.

Similarly the signature of a homoclinic orbit in the horseshoe is given by simply counting the number of \(-\) signs in the core sequence.

**Lemma 6.** The signature of a homoclinic orbit with core \((s)\) close to the AI limit is given by \(-(-1)^j\) where \(j\) is the number of \(-\) signs in \(s\).

Thus the orbit \((- - +)\) has signature \(-1\).\footnote{Eq. (11) implies that the signature is the same as the limiting sign of the residue of periodic orbits that approximate the homoclinic orbit.} We will see that, when \(b = 1\), some homoclinic orbits undergo pitchfork bifurcations, which change their signature, so this rule is not valid for all parameter values.
The positions of the homoclinic orbits on $U$ for orbits of type 1, 2 and 3, labeled by their core sequences, are shown in Fig. 3. The order of the homoclinic orbits on the segments $S$ and $U$ close to the AI limit must be the same as the corresponding ordering of homoclinic points in the complete horseshoe by continuity. This ordering is equivalent to that of the logistic map, Eq. (13), for the orbits forward asymptotic to the $+$ fixed point. This gives an easy way to compute the ordering, see Fig. 4. In the logistic limit, all of the sequences forward asymptotic to the fixed point are destroyed when the orbit of the critical point becomes bounded at $\epsilon = 1/\sqrt{2}$. The fixed point $\cdot + \infty$ has a single preimage, which is $\alpha = \cdot - + \infty$. Every other orbit that is forward asymptotic to $\cdot + \infty$ has the form $\cdot (s) - + \infty$.

In the area preserving case the ordering of the orbits along $S$ is equivalent to that on $U$ upon time reversal. Thus a type $t$ point $+ \infty - (s_1 s_2 \ldots s_t) \cdot - + \infty$ on $S$ is in the same relative position as the point $+ \infty - \cdot (s_t s_{t-1} \ldots s_1) - + \infty$ on $U$. Close to the AI limit we always have this ordering on the manifolds, which is just another way of saying that the map is conjugate to the horseshoe map.

So long as there are no homoclinic bifurcations, then the orderings $>_u$ and $>_s$ are just given by the usual unimodal ordering as stated in

**Lemma 7.** The ordering $>_u$ on $U$ and $>_s$ on $S$ close to the AI limit is
Figure 4: Ordering of pre-periodic points for the + fixed point of the logistic limit of the Hénon map. The symbols are determined by the itinerary of the orbit relative to the critical point at \( z = 0 \).

Given by

\[
\begin{align*}
+\infty - \ast e + \ldots & > u  \\
+\infty - \ast o + \ldots & < u  \\
\cdots + e & > \gamma  \\
\cdots + o & < \gamma
\end{align*}
\]

where \( e / o \) are finite sequences with an even / odd number of minus signs, respectively.

The ordering shown in Fig. 3 and Fig. 4 is exactly this one upon appending the “homoclinic tail” \(-+\infty\) to the cores. The maximal orbit on \( U \) is \( \zeta \), corresponding to the tail \(+\infty\), the minimal orbit is \( \alpha \), corresponding to \(-+\infty\).

8 Homoclinic Bifurcations

Homoclinic bifurcations are bifurcations between homoclinic orbits. Compared to ordinary bifurcations of periodic orbits they possess additional
structure because the invariant manifolds (with their ordering) must be involved in the bifurcation process. To make this explicit, we say that two homoclinic orbits $\beta$ and $\gamma$ are double neighbors if the segments $W^u[\beta, \gamma]$ and $W^s[\beta, \gamma]$ contain no other homoclinic orbits. Three ordered homoclinic points $\beta <_u \gamma <_u \delta$ are triple neighbors if both $\beta, \gamma$ and $\gamma, \delta$ are double neighbors. An obvious observation with nevertheless important consequences is the “double neighbor” lemma:

**Lemma 8.** Two homoclinic orbits $\beta$ and $\gamma$ cannot bifurcate unless they are double neighbors.

The converse gives a simple forcing relation: before $\beta$ and $\gamma$ can bifurcate any homoclinic orbit on either segment between them must have disappeared.

Another consequence is the transition time lemma:

**Lemma 9.** If two homoclinic orbits $\beta$ and $\gamma$ bifurcate then they must have the same transition time $t_{\text{trans}}$.

**Proof:** Let $\beta$ and $\gamma$ be neighbors on $U$. If their transition time is different then they are not neighbors on $S$, so they cannot bifurcate. □

This allows us to extend Lemma 5 away from the AI limit, so that one can take the transition time as an adequate replacement of the period:

**Corollary 10.** The transition time of a homoclinic orbit never changes.

**Proof:** Since the transition time is an integer it cannot change under smooth deformations. It could only change at bifurcations, but we have just seen that only orbits with the same transition time bifurcate. □

Therefore homoclinic bifurcations only take place between double neighbors with the same transition time, i.e., core length. Close to the AI limit the horseshoe is still complete. In this situation it is possible to find all neighbors:

**Lemma 11.** Two homoclinic orbits on $U$ are neighbors in the complete horseshoe if and only if they are of the form $+\infty - _u (s+) - +\infty$ and $+\infty - _u (s-) - +\infty$.

**Proof:** We have to show that there is no homoclinic orbit with core $\delta$ such that $(o+) <_u (\delta) <_u (o-) \text{ or } (e-) <_u (\delta) <_u (e+)$, where $e = s$ if $s$ has an even number of minus signs or $o = s$ if this number is odd. If the initial string in $\delta$ differs from $s$ then $\delta$ cannot be between the sequences $(s-)$ and $(s+)$, therefore $\delta = s . . .$. It is simple to see that $e + \cdots \geq_u e + - +\infty$, $e - \cdots \leq_u e - - +\infty$.
and similarly
\[ o + \cdots \leq u \cdot o + +\infty, \quad o - \cdots \geq u \cdot o - -\infty. \]
Since \( \delta = s \ldots \) it must be of one of the forms on the left hand sides, but then the inequalities show that it is not between \((s+)\) and \((s-)\) hence they must be neighbors. Conversely, suppose we have two neighboring homoclinic orbits (on \(U\)) \(a\) and \(b\) with \(a < u \cdot b\). They must differ in at least one symbol so call the first such difference \(x\). Their leading common symbols are denoted by \(s\), so that \(a = sx\alpha\) and \(b = \bar{s}x\beta\) for some sequences \(\alpha\) and \(\beta\), where \(\bar{x}\) is the opposite symbol to \(x\). Applying the ordering relation to the possible combinations of \(s\) and \(x\) gives either
\[ e - \alpha < u \cdot (ey) - +\infty < u \cdot e + \beta \quad \text{or} \quad o + \alpha < u \cdot (oy) - +\infty < u \cdot o - \beta, \]
where the choice of the symbol \(y\) depends on whether \(s\) is even or odd and whether \(\alpha\) and \(\beta\) are \(-+\infty\). Specifically, choose \(y = +\) if either \(s = e\) and \(\beta \neq +\infty\) or \(s = o\) and \(\alpha \neq -\infty\). Choose \(y = -\) if either \(s = e\) and \(\alpha \neq +\infty\) or \(s = o\) and \(\beta \neq -\infty\). If neither \(\alpha\) nor \(\beta\) are \(-+\infty\) either choice for \(y\) works. When either \(\alpha\), \(\beta\), or both differ from \(-+\infty\) we have constructed an orbit \((sy) - +\infty\) which is between \(a\) and \(b\)—hence \(a\) and \(b\) are not neighbors. But this is a contradiction so \(\alpha, \beta = -+\infty\). \(\blacksquare\)

For a bifurcation to occur it is not enough that the orbits be neighbors on \(U\), but they must be double neighbors. In the reversible case this almost gives the proof of Conjecture \(\mathcal{E}\) but here we are working in the smaller class of orbits homoclinic to \(p\), the hyperbolic fixed point. So far we did not make use of the reversibility of the map, i.e., the results are valid for all \(b\). From this point on we will always only talk about the area preserving case. Note that the ordering is used in a range of parameters before the first bifurcation occurs, so the horseshoe ordering is still valid.

**Theorem 12.** In the area preserving Hénon map the first homoclinic bifurcation of the invariant manifolds of the fixed point \((+)\) is
\[ \text{sn}\{+\infty - (+) - +\infty, +\infty - (-) - +\infty\}. \]

**Proof:** By Lemma \([\mathcal{E}]\) we know that all neighbors on \(U\) in the complete horseshoe are of the form \((s\pm)\). For these sequences to be double neighbors they must be neighbors on \(S\) as well. By reversibility this is equivalent to the sequence and its reverse being neighbors on \(U\). But this only true if \(s\) is empty. The only double neighbors in the complete horseshoe are therefore the two orbits \(+\infty - (+) - +\infty\) and \(+\infty - (-) - +\infty\). Therefore they must bifurcate first. \(\blacksquare\)

To approximate a homoclinic orbit, which possesses an infinite number of points in phase space by a periodic orbit with only a finite number of points
Orbits & $k_{\infty}$ & $k_{\infty}$ & $\delta$
\hline
$(- + + 2)_{\infty}$ & 5.5517014388520 & 5.699160106302 & \\
$(- + + 3)_{\infty}$ & 5.6793695105731 & 5.699306445540 & 7.45095 \\
$(- + + 4)_{\infty}$ & 5.6965039879058 & 5.69931069970 & 7.11409 \\
$(- + + 5)_{\infty}$ & 5.6989125149379 & 5.699310783741 & 7.04922 \\
$(- + + 6)_{\infty}$ & 5.6992541878224 & 5.699310786628 & 7.03706 \\
$(- + + 7)_{\infty}$ & 5.6993027411880 & 5.699310786699 & 7.03489 \\
$(- + + 8)_{\infty}$ & 5.6993096429803 & 5.699310786700 & 7.03452 \\
$(- + + 9)_{\infty}$ & 5.6993106241120 & 5.699310786700 & 7.03446 \\
$(- + + 10)_{\infty}$ & 5.6993107635871 & 7.03445 & \\
$(- + + 11)_{\infty}$ & 5.6993107834145 & 7.03445 & \\

Table 3: Bifurcations in periodic approximations to the homoclinic type 1 orbit, which is the first orbit destroyed for $b = 1$. Here we use a * to denote both $+$ and $-$, giving both orbits involved in the bifurcation.

we require that the Hausdorff distance of these two point sets vanishes as the period approaches infinity. Thus for an orbit homoclinic to $(+)^{\infty}$, we study a sequence of approximating periodic orbits with an increasingly long string of $+$ symbols. In particular the rotational orbits given in Eq. (9) converge to $\zeta$ and $\alpha$ in the limit.

In Table 3 we list the first 11 members of the sequence approximating the transit time 1 homoclinic orbit, and the corresponding sequence of values, $\epsilon_{\text{sn}}$, at which these orbits undergo a saddle-node bifurcation when $b = 1$. These values converge geometrically to the parameter at which the homoclinic orbits bifurcate, and the ratio of successive differences (a "Feigenbaum ratio") is computed in the fourth column of the table. As is known theoretically for $b < 1$ [41, 22] the convergence rate, $\delta$, approaches $\lambda$, the multiplier of the fixed point $p$. From our data, the convergence rate $\delta$ agrees up to 6 digits with the multiplier

$$\lambda \approx 7.0344478$$

of the fixed point $p$ when $k \approx 5.699310786700$. Thus, our observations indicate that the convergence rate is given by the multiplier in the area preserving case as well, where to our knowledge no proof exists.

The third column in the table is the extrapolation for the converged $k$ value, given by Aitken’s $\Delta^2$ method

$$k_{\infty} = k_n - \frac{\Delta(k_n)^2}{\Delta^2(k_n)},$$

where $\Delta$ is the forward discrete difference operator. Thus we see that there is a saddle-node bifurcation of the type 1 homoclinic orbits,

$$\text{sn}\{+^{\infty} - (+) - +^{\infty}, +^{\infty} - (-) - +^{\infty}\},$$
at

\[ \epsilon_{sn}(1) \approx 0.418879233367 \quad \text{or} \quad k_{sn}(1) \approx 5.699310786700 \]

This also corresponds to the parameter value at which the topological horseshoe for the Hénon map is destroyed, and is the value in Fig. 1 at \( b = 1 \).

Since there is a sequence of saddle-node bifurcations that limit on the homoclinic bifurcation, there are elliptic islands arbitrarily close to the destruction of the horseshoe. This corresponds to an area preserving version of the results of Gavrilov and Silnikov [20, 21] and Newhouse [42, 19].

Since we can in principle follow every finite orbit from the anti-integrable limit we can begin to study the sequence of bifurcations that occur after the horseshoe is destroyed, see Table 4. For example, the bifurcation diagram for all of the homoclinic orbits of type three or less is sketched in Fig. 5.

The vertical ordering in this sketch is the same as that on the segment \( U \) with \( \alpha \) and \( \zeta \) shown. The bifurcation diagram is highly influenced by the time-reversal symmetry of the area preserving Hénon map—we will discuss this symmetry in the next section. As expected from the general theory [43], we observe three kinds of bifurcations:

**Symmetric saddle-node** bifurcations resulting in the creation of a pair of type \( t \) homoclinic orbits with opposite signatures. For example, in Fig. 5, the type 3 orbits with cores \((+++)\) and \((-+-)\) are born in such a saddle-node at \( k \approx 0.386 \).

**Pitchfork** bifurcations of type \( t \) symmetric homoclinic orbits, creating a pair of type \( t \) asymmetric orbits that are related by time reversal. For example, the \((-+-)\) orbit pitchforks at \( k \approx 0.720 \) creating the orbits \((-+++)\) and \((++-+)\). A pitchfork bifurcation requires triple neighbors to occur. The parent orbit of a homoclinic pitchfork bifurcations is always created in a symmetric saddle-node bifurcation. Up to type 11 there are are only 9 symmetric saddle-node bifurcations which do not undergo a homoclinic pitchfork bifurcation on their way to the AI limit.

**Asymmetric saddle-node** bifurcations creating two symmetry related pairs of asymmetric orbits. This bifurcation first occurs at type 4. For example the two pairs \{\((-+-+),(--++)\)\} and \{\((-+-+),(+-+++)\)\} are created at \( k \approx 5.18 \). Generically, asymmetric saddle-node bifurcations require two pairs of double neighbors to occur because of the symmetry.

The shaded region in Fig. 5 represents the range of \( k \) for which the area preserving Hénon map exhibits a horseshoe. Along the left edge we label each orbit with its core symbol sequence.

The first type \( t \) homoclinic orbits are created by a saddle-node bifurcation when the segment \( f^{-1}(S) \) first intersects \( U \). We denote this parameter
Figure 5: Sketch of bifurcations in the homoclinic orbits up to type 3 ($b = 1$).

Table 4: Homoclinic bifurcations up to core length 4.
value by $k_{sn}(t)$. This marks the creation of the subset of the incoming lobe of the turnstile with transition time $t \[26\]$. We observe that when $b = 1$, this homoclinic saddle-node bifurcation is

$$\text{sn}\{(+t), (-+t-2-\}) \text{ at } k_{sn}(t).$$

Following this, the orbit $(-+t-2-) \text{ undergoes a homoclinic pitchfork bifurcation at } k_{pf}(t)$, creating the pair

$$(-+t-2-) \rightarrow \text{pf}\{(+t-1-), (-+t-1-}\} \text{ at } k_{pf}(t).$$

However, when $b \neq 1$, the initial symmetric bifurcation and the following symmetry breaking pitchfork are replaced by a pair of nonsymmetric saddle-node bifurcations. In this case the first type $t$ bifurcation is the homoclinic saddle-node

$$\text{sn}\{(+t), (+t-1-\}) \text{ at } k_{sn}(t).$$

According to the double neighbor lemma, certain bifurcations cannot occur prior to other homoclinic bifurcations because the corresponding sequences block other sequences from being neighbors. In order to determine which orbits are neighbors even beyond the first bifurcation we make the assumption that the following symbolic ordering conjecture holds:

**Conjecture 4.** The symbolic horseshoe ordering on the invariant manifolds given in Lemma 7 persists.

The ordering relations give a unique construction of the order of the points on $U$ and $S$, and this implies that a schematic construction of the intersections of $f^{-t}(S)$ with $U$ can be constructed solely from a list of which orbits exist at a given parameter value. Such a schematic manifold plot is shown in Fig. [3] for all homoclinic orbits that exist at $k = 5.53$ up to type 5.

We can also construct a schematic bifurcation diagram for homoclinic orbits, as in Fig. [4], by drawing a horizontal line from $k = \infty$ to the $k$-value at which a particular homoclinic orbit is destroyed—actually we stop the figure at $k = 6$, since there are no bifurcations for larger $k$-values. We order the homoclinic orbits vertically according to their unimodal ordering on $U$ as usual. In this bifurcation diagram the vertical connections indicate which orbits eventually do become neighbors and bifurcate. So as to avoid artificially crossing lines, we connect pairs of asymmetric saddle-nodes by lines at the right edge of the figure to indicate that they must bifurcate at the same $k$-value.

We say that a bifurcation *straddles* the centerline if the pair of orbits involved are on either side of center of the $U$ ordering, or if one of the two pairs of an asymmetric saddle-node straddles the center line.

Through type 6, each symmetric saddle node is followed by a pitchfork bifurcation, just as we observed in Fig. [3], with the exception of the very first
bifurcation, sn \{(+), (-)\}, which corresponds to the smallest loop straddling
the center in the figure. That this is in fact the smallest loop and therefore
the first bifurcation is the content of Theorem 12.

Moreover, it is remarkable, but perhaps misleading, that through type 6
every bifurcation straddles the center. Therefore all homoclinic bifurcations
up to type 6 are forced by nesting around the center. In particular this
means that their unimodal ordering gives the bifurcation ordering, like in
unimodal maps.

This simple forcing relation is destroyed with the appearance of a sym-
metric saddle-node without pitchfork of type 7 (see Table 6). Also at type 7,
there is an asymmetric saddle-node quadruple which does not straddle the
center. Interestingly enough, this is the same bifurcation that marks the
upper $k$ endpoint of one of the gaps that we discuss in §10.

It is difficult to visualize the full homoclinic bifurcation diagram for
larger type orbits. To do so, we plot only the horizontal lines, to indicate
the range of existence of an orbit; this diagram up to transition time 11 is
given in Fig. 8. The approximate self-similarity in this picture seems to be
related to some of the gaps we discuss in §10, namely those that are related
to symmetric saddle-nodes without accompanying pitchforks of type 7, 9
and 11.
Figure 6: Schematic drawing of $U$ (dashed line) and $f^{-t}(S)$ (solid line) up to type 5 for $k = 5.53$.

Figure 7: Bifurcation diagram of homoclinic orbits up to type 5 ($b = 1$). Types 1,2,3 are shown as dotted lines (recall Fig. 5); type 4 is dashed; and type 5 is solid.
Figure 8: Existence plot of homoclinic orbits up to type 11. For each homoclinic orbit a line is drawn from large $k$ to the parameter value where this orbit is destroyed. The vertical position of each line is its formal position on $U$ according to the unimodal ordering.
9 Symmetric homoclinic bifurcations

As we mentioned above, the bifurcation diagram of the area preserving Hénon map is restricted by the fact that the map has a time-reversal symmetry. Here we briefly recall a few well known facts about reversible maps [44], and apply them to the study of homoclinic bifurcations.

A map \( f \) has a time-reversal symmetry when it is diffeomorphic to its inverse by:

\[
Rf = f^{-1}R.
\]

We call the map \( R \) a reversor for \( f \). Often, as in our case, the reversor is an involution, \( R^2 = I \). Note that each of the maps \( f^tR \) is also a reversor, in particular, we call \( fR \) the complementary reversor to \( R \). The fixed set of a reversor

\[
\text{fix}(R) = \{ x : Rx = x \},
\]

is of particular interest. For the case when \( R \) is an orientation reversing involution of the plane \( \text{fix}(R) \) is always a curve that goes through infinity, thus dividing the plane into two pieces [45].

A reversor maps an orbit \( \ldots, z_{t-1}, z_t, z_{t+1}, \ldots \) of the map onto another orbit \( \ldots, Rz_{t+1}, Rz_t, Rz_{t-1}, \ldots \). A symmetric orbit is defined as one that is mapped onto itself by \( R \). It is easy to see that any symmetric orbit must have points on \( \text{fix}(R) \cup \text{fix}(fR) \) and conversely. Moreover, if the orbit is not periodic, it has a unique point on one of these fixed sets, and if it is periodic it has exactly two points on the fixed sets [46].

Reversible maps need not be area preserving, though the multipliers of an orbit and its symmetric partner must be reciprocals of one another. Application of this to the fixed points gives that the Hénon map is reversible only when \( b = \pm 1 \). For a symmetric orbit reversibility implies that the product of the multipliers must be one. For the case \( b = \pm 1 \) a reversor is \( R(x, y) = (-y, -x) \), and a complementary reversor \( fR(x, y) = (-x - k + y^2, y) \). The fixed curves are

\[
\text{fix}(R) = \{ (x, y) : x = -y \},
\]

\[
\text{fix}(fR) = \{ (x, y) : x = \frac{1}{2}(y^2 - k) \}.
\]

Suppose that \( p \) is a symmetric, hyperbolic fixed point of a reversible map. Then, as pointed out by Devaney [47], the stable and unstable manifolds of the map are related by \( R \):

**Lemma 13.** Let \( W^u \) and \( W^s \) be the stable and unstable manifolds of a symmetric fixed point \( p \). Then \( RW^u(p, \beta) = W^s(p, R\beta) \).

**Proof:** By definition, when \( \beta \in W^u \), then \( f^{-t}(\beta) \to p \) as \( t \to \infty \). Then \( Rf^{-t}(\beta) = f^t(R\beta) \to Rp = p \). Thus, \( R\beta \in W^s \). Since \( R \) is a diffeomorphism, \( RW^u(p, \beta) = W^s(p, R\beta) \). \( \square \)
Corollary 14. If $W^u$ intersects the fixed set of a reversor, then the intersection point is homoclinic.

Homoclinic orbits of symmetric periodic orbits either come in symmetric pairs or are symmetric, and there must exist symmetric homoclinic orbits:

Lemma 15. Let $p$ be a symmetric, hyperbolic fixed point, and $\beta$ a homoclinic point, and suppose that $R$ is an orientation reversing involution. Then $R\beta$ is also a homoclinic point. Moreover, there exist symmetric homoclinic points on $\text{fix}(R)$ and $\text{fix}(fR)$.

Proof: By Lemma 13, since $\beta \in W^s \cap W^u$ then $R\beta \in W^u \cap W^s$, so it is homoclinic as well. Since $\text{fix}(R)$ divides the plane and $\beta$ and $R\beta$ are on opposite sides of this curve, the segment $W^u[\beta, R\beta]$ must cross $\text{fix}(R)$, and the crossing point is necessarily homoclinic and symmetric. We can argue similarly for $fR$. □

As is well known, pitchfork bifurcations occur with codimension one in maps with a symmetry [43]. This occurs for homoclinic bifurcations as well, as was suggested in [48]. We observed such pitchfork bifurcations in Fig. 3. A pitchfork typically occurs after a symmetric, type $t > 1$, saddle-node bifurcation creates a “tip” of $W^s$ inside the entry lobe of the turnstile. As this tip grows, one would normally expect it to bend around, as sketched in Fig. 4, creating more type $t$ homoclinic points by saddle-node bifurcation. In fact, it is a simple consequence of the linear ordering along $W^u$ and $W^s$ combined with reversibility that a single saddle-node bifurcation like that sketched in Fig. 4 is impossible:

Theorem 16. Suppose that $f$ is an orientation preserving, reversible map, with a symmetric fixed point $p$, and $S$ and $U = RS$ are segments of its stable and unstable manifold bounded by adjacent primary homoclinic orbits. Suppose that a pair of symmetric homoclinic points on $U$, $\beta_s < \gamma$ are created in a saddle-node bifurcation. Then it is impossible for there to be a single saddle-node bifurcation as a tangency of $W^s(\beta, \gamma)$ with either piece of $U \setminus W^u[\beta, \gamma]$.

Proof: Since $\beta < \gamma$, and $RW^s = W^u$, we have $R\beta < \gamma$. Suppose that $\beta$ and $\gamma$ have transition time $t$. Then $f^t(\beta) \in S$, but since $\beta$ is symmetric this point must be the same as $R\beta$. Thus $f^{-t}(R\beta) = \beta$, and similarly for $\gamma$. Since the ordering is preserved by iteration, then $\beta < \gamma$. Now suppose there is a tangency at a point $\delta = W^s(\beta, \gamma) \cap (U \setminus W^u[\beta, \gamma])$, i.e., not between $\beta$ and $\gamma$. We sketch such a configuration in Fig. 4. Thus $\beta < \gamma$. By symmetry, $R\beta < \gamma$. Since the ordering is preserved by iteration, we have $\beta < f^{-t}(R\delta) < \gamma$. Thus $f^{-t}(R\delta) \in W^u(\beta, \gamma)$ and so this point is not $\delta$ (consequently the orbit of $\delta$ is not symmetric). Since the manifolds are tangent at $\delta$, they are also tangent at $R\delta$ by symmetry. Thus there is a second, simultaneous tangency, on $U$ at $f^{-t}(R\delta)$ which contradicts the
There are three possible resolutions: first one of the two orbits, $\beta$ or $\gamma$ could undergo a pitchfork bifurcation creating a symmetry related pair of homoclinic orbits. For example, Fig. 10 shows part of the homoclinic tangle at a parameter value where the type two homoclinic orbit with core sequence $(- -)$ pitchforks. As $k$ increases this results in the creation of a pair of type 2 orbits with cores $(- +)$ and $(+ -)$, see Fig. 11. Note that the new orbits are not symmetric, but that the reversal of $(- +)$ is $(+ -)$, so they form a symmetric pair.

The second possible bifurcation is a single-saddle node on the segment $W^u(\beta, \gamma)$; this happens, for example, whenever a “tip” of an iterate of $S$ returns to $U$. This first occurs at type 3, for the bifurcation sn $\{(\ast - \ast)\}$. We sketch a similar case, at type 4, sn $\{(\ast - - \ast)\}$, in Fig. 12 which occurs at $k \approx 3.982$.

The third possible bifurcation is a pair of asymmetric saddle-node bifurcations. This first occurs for homoclinic orbits of type 4. For example, the bifurcations sn $\{(+ - + -), (- - + -)\}$ and its time-reverse, sn $\{(- + - +), ( + + - -)\}$ occur at $k \approx 5.1886$. We sketch $U$ and $f^{-4}(S)$ at this bifurcation in Fig. 12. This bifurcation also corresponds to the lower endpoint of an apparently hyperbolic parameter interval for the Hénon map, as we
Figure 10: Stable and unstable manifolds for the $(+)\infty$ fixed point of the Hénon map at $b = 1$ and $k = 3.09151$, where there is a cubic tangency of the manifolds at the $+\infty - (-) - +\infty$ homoclinic orbit.

Figure 11: Type 2 homoclinic orbits of the Hénon Map at $k = 3.5$. 
9 SYMMETRIC HOMOCLINIC BIFURCATIONS

Figure 12: Sketch of two possible homoclinic saddle-node bifurcations of type four. A symmetric saddle-node creating (* − − *) occurs on \( W^u((-++, +++) ) \) in (a). An asymmetric saddle-node creating (* − + −) and (− + − *) occurs with one point on \( W^u((-^4, (+−+)) \) in (b).

Note that the antimonotonic bifurcations shown to exist in the area contracting case [13] are exactly forbidden by this theorem.

A symmetric saddle-node followed by a pitchfork is a common bifurcation. For example, the parameter values, \( k_{sn}(t) \), at which the first type \( t \) homoclinic orbit is created decrease monotonically with \( t \). Thus at \( k_{sn}(t) \) the first type \( t \) orbit is born and there are no homoclinic orbits with type less than \( t \). For \( t > 1 \), at \( k_{sn}(t−1) \) the segment \( f^{t−1}(U) \) must intersect with \( S \), so that \( f^t(U) \) intersects with \( f(S) \). In order for this to happen (when \( b = 1 \)), there is a pitchfork bifurcation for \( k_{pf}(t) ∈ [k_{sn}(t), k_{sn}(t−1)] \) of the type \( t \) homoclinic orbit \( +∞ − (− +t−2 −) − +∞ \) giving rise to the pair of orbits with symbol sequences

\[- +t−2 − → pf\{−(+t−1), (+t−1−)\}.

We see that the children of this bifurcation differ from their parent in a single symbol and they differ from each other in two symbols. Table 5 lists the first few such homoclinic bifurcation values obtained by extrapolation of the first few members of the approximating orbit sequence.

The distance (in \( k \)) between the birth of the type \( t \) orbit and its pitchfork bifurcation shrinks to zero as the type increases.
10 Intervals with no Bifurcations

Davis, MacKay and Sannami (DMS) [3] used the numerical method of Biham and Wenzel [23] to compute the periodic orbits for the area preserving Hénon map. They showed that up to period 20, there are intervals of parameter where there appear to be no orbits created or destroyed. They studied a particular parameter interval near the destruction of the horseshoe, and elucidated the symbolic dynamics of the corresponding homoclinic tangle. We will refer to this interval as the DMS gap. Though the method of Biham and Wenzel is guaranteed to work close enough to the AI limit [11], it can fail [2]. We tested the DMS results using our continuation technique. The use of parallel computation allowed us to extend the original experiment by an order of magnitude in size so that we followed all orbits up to period 24—recall from Table 2 that there are a total of 1,465,020 possible orbits.

We verify the DMS results and identify the symbol sequences of the orbits that form the boundaries of the DMS gap.

In our experiment we follow all orbits up to period 24 and record the minimal parameter values at which they are destroyed. We then assume that each orbit exists only up to that value of \( k \); this procedure is not entirely correct, because a few orbits loop back and forth in parameter under continuation. This is related to the vanishing of twist in the neighborhood of a period tripling bifurcation [14]. However, the number of low period orbits for which this happens is very small.

In Fig. 13 we show the number of orbits that exist as a function of \( k \), with the caveat that no value is plotted if the number of orbits does not change from the previously plotted point. This plot is equivalent to that of DMS, except that we leave gaps in the intervals where there are no bifurcations.

At the anti-integrable limit the map exhibits a horseshoe so all of the periodic orbits are present. As we move away from the anti-integrable limit we see a decline in the number of periodic orbits as orbits collide and are destroyed. Flat intervals in Fig. 13 represent intervals of parameter where

| \( t \) | Core | \( k_{sn}(t) \) | Pitchfork Children | \( k_{pf}(t) \) |
|---|---|---|---|---|
| 1 | (*) | 5.69931078670 | \((-+)\), \((+-)\) | 3.09150542113 |
| 2 | (**) | 1.6277931098 | \((-+2)\), \((+2-)\) | 0.71963023592 |
| 3 | \((**+*)\) | 0.3855621701 | \((-+3)\), \((+3-)\) | -0.04427324816 |
| 4 | \((**+2*)\) | -0.13347378530 | \((-+6)\), \((+6-)\) | -0.04427324816 |
| 5 | \((**+3*)\) | -0.54918558488 | \((-+5)\), \((+5-)\) | -0.53740149261 |
| 6 | \((**+4*)\) | -1.7362572399 | \((-+7)\), \((+7-)\) | -0.70916824264 |
| 7 | \((**+5*)\) | -1.7605576670 | \((-+8)\), \((+8-)\) | -0.75830622014 |
| 8 | \((**+6*)\) | -0.79501732767 | \((-+9)\), \((+9-)\) | -0.79501732767 |

Table 5: Pitchfork bifurcations from the first type \( t \) orbits up to type 10.
very few bifurcations occur. Gaps in the plot indicate intervals of parameter where there are no bifurcations. The creation of the first type \( t \) homoclinic orbits gives rise to flat intervals. We observe that the left endpoint of each of the larger flat intervals for \( k < 3 \) corresponds to \( k_{sn}(t) \) for the saddle-node bifurcation of the first type \( t \) homoclinic orbits; these are marked in Fig. 13 and in the enlargement, Fig. 14. Similarly, the parameter values \( k_{pf}(t) \) are also marked; note that these pitchfork bifurcations are located well beyond the right endpoints of the flat intervals. Each of the gaps in the flat intervals for \( k < 3 \) must eventually fill in if we go to high enough period because in this range of \( k \) the area preserving Hénon map has an elliptic fixed point. Recall that an \( m/n \) bifurcation from the elliptic fixed point occurs at the parameter values \( k_{m/n} \), given in Eq. (10), and these values are dense in the interval \(-1 \leq k \leq 3\). Moreover, invariant circles bifurcate from the elliptic fixed point for each \( k_{\omega} \) for sufficiently irrational \( \omega \). The same argument can be used up to the end of the period doubling cascade of the fixed point at \( k \approx 4.13616680392 \), since each period doubling creates an elliptic orbit.
There are a number of distinct gaps in Fig. 13; the 3 larger gaps were studied by DMS, especially the largest one, near \( k = 5.5 \) indicated by \( L \) and \( R \) in Fig. 13. DMS conjecture that the dynamics in each gap is hyperbolic, and consequently there are no bifurcations in a gap. Our numerical evidence, which extends their study by an order of magnitude, supports this conjecture. Upon examining the orbits that limit on the endpoints of the gap up to period 24, we can extrapolate and find that each of the five largest gaps is bounded by a homoclinic bifurcation, see Table 6. Thus we see that the gaps do not fill-in with orbits converging on the homoclinic bifurcations, but we cannot rule out that there are other, unrelated period orbits with period larger than 24 that are created at parameter values in the middle of a gap.

We observe that there are two types of bifurcations bounding the gaps: symmetric and asymmetric saddle-node bifurcations. The asymmetric saddle-nodes result in the creation of two pairs of homoclinic orbits, the one listed in the table, and its time-reverse. Typically we observe that symmetric saddle node bifurcations in homoclinic orbits are followed by pitchfork bifurcations. In fact we observe that among all of the homoclinic orbits through type 11, there are only 9 special saddle-node bifurcations which are not followed by a pitchfork bifurcation. We believe that each of these special bifurcations corresponds to the endpoint of a gap. For example, the first type 1 bifurcation is not followed by a pitchfork, and it gives the left endpoint of the gap corresponding to the horseshoe. The four gap endpoints in Table 6 that correspond to symmetric saddle nodes are each of this special type. The four remaining special pairs are each of type 11, and of these at least two bound gaps of widths \( \Delta k \approx 6(10)^{-3} \). With our resolution it is not possible to clearly identify the final two as gap boundaries.

In Fig. 14 we show an enlargement of Fig. 13 but also include the data from the subshift, \( \Sigma_F \). In the upper right corner of Fig. 14 we see the tail end of the exit time 2 plateau. We also labeled the first large gap in \( \Sigma_F \) after the subshift is destroyed—this is the subshift analog of the DMS gap.

As in the DMS gap, the left and right boundaries, denoted \( L \) and \( R \), correspond to a pair of homoclinic saddle-node bifurcations with the core

| Left Endpoint Core | \( k_L \) | Right Endpoint Core | \( k_R \) |
|--------------------|--------|--------------------|--------|
| \((- + - * - + -)\) | 4.55931896797 | \((- + - * - + -)\) | 4.59567964802 |
| \((+3 - + - *)\) | 4.84317164217 | \((+3 - * - + - + -)\) | 4.86795762007 |
| \((- + - *)\) | 5.18851121215 | \((+ + - * - + +)\) | 5.53765692812 |
| \((- + + - * - + +)\) | 5.56490867348 | \((+ + - * - +3)\) | 5.60872105039 |
| \((- + + - *)\) | 5.63190980280 | \((+3 - * - +3)\) | 5.67769222229 |

Table 6: Homoclinic bifurcations bounding the gaps in Fig. 13.
sequences

\[ \text{asn}\{(\ast ++--), (---++\ast)\} \quad \text{at} \quad k_{sn}(L) \approx 1.533898312 , \]
\[ \text{sn}\{(+3 -- + + 3), (+3 -- 4 + 3)\} \quad \text{at} \quad k_{sn}(R) \approx 1.583387630 . \]

Note that the right endpoint of the gap corresponds to an orbit whose partner is not in the subshift!

The curves for all orbits and for the subshift are remarkably similar and it appears that the growth of orbits in the subshift gives an accurate representation of the overall growth of orbits in the full shift for this range of parameters. This is especially remarkable given that when all the orbits exist, the subshift contains less than 1% of the orbits in the full shift up to period 24. The figure shows that for small \( k \), the number of orbits in the full shift is nearly a constant multiple of that in the subshift.

Observing that the gaps are bounded by homoclinic orbits, we regenerated the orbit growth plot using only homoclinic orbits. As expected, the gap structure and overall shape of Fig. 13 is almost completely captured by the homoclinic orbits alone.
11 Conclusions

Continuation from an anti-integrable limit is an effective technique for studying orbits providing that there are no isolated bubbles in the bifurcation diagram. In [11] we applied the anti-integrable theory to the Hénon map to obtain a new proof of the well-known analytical bound of Devaney and Nitecki [9]. In Theorem 3 we apply a similar argument to a restricted set of orbits to find an analytical bound for the existence of a subshift of finite type. We present both analytical bounds together with the optimal bounds generated numerically using our continuation method. We observe that the horseshoe is destroyed by a type one bifurcation that is homoclinic in the orientation preserving case, and heteroclinic otherwise. In either case we conjecture that this bifurcation is the first bifurcation among all orbits of the Hénon map as we recede from the anti-integrable limit.

With our continuation method, we are able to assign a “global code” to each orbit, by fixing the designation to that at the AI limit. In the Hénon map, we demonstrate that this AI code is equivalent to the standard horseshoe code (when it exists), but it also gives a consistent way of assigning symbols to orbits beyond the destruction of the complete horseshoe. Remarkably, there appears to be a relationship between the AI codes for a number of systems including billiards, twist maps of the cylinder and the Hénon map. We will explore this relationship in a forthcoming paper [29].

We relate the properties transition time, type, and signature of homoclinic orbits to properties of the core sequence. We also demonstrate that the ordering of the homoclinic orbits on the manifold segments $U$ and $S$ is the standard unimodal ordering. The notion of double neighbors and lemmas 8 and 9 give a necessary condition for a pair of homoclinic orbits to bifurcate. Surprisingly, these also give a forcing relation that tells us which homoclinic bifurcations have to occur before other ones. Showing that homoclinic bifurcations can only take place between double neighbors with the same core length, Lemma 11 gives a symbolic criterion for a pair of homoclinic orbits to be neighbors. The ordering is certainly valid until the complete horseshoe is destroyed, which leads to the theorem that the first homoclinic bifurcation of the hyperbolic fixed point in the area preserving Hénon map occurs between the pair of type one orbits. When $b = \pm 1$, the Hénon map has a symmetry and we discuss the mechanism by which pitchfork and asymmetric saddle node bifurcations occur. The key ingredients to Theorem 16 are the ordering on the manifolds and the existence of a reversor for the map. As a result, the scenario of a tip of the manifold just repeatedly piercing through the other manifold (which is most natural in the case without symmetry) is impossible. Among the possible alternatives are the occurrence of a pitchfork bifurcation or the creation of an asymmetric saddle-node pair, which are both not generic in the non-reversible case.

With our continuation technique we compute numerical values for various bifurcations of the homoclinic orbits up to type eleven. We sketch
the bifurcation diagram at type three, and then use a simple algorithm to construct the much more complex figures for higher core length.

In contrast to our method for finding periodic orbits, the Biham and Wenzel method [24, 49] is known to fail in certain cases [2], and can only be justified in the neighborhood of the AI limit [11]. Nevertheless, for the area preserving Hénon map, we observe precisely the same number of orbits in the main DMS gap using our technique as was reported by DMS using the Biham-Wenzel method [3]. We extend the original experiment of DMS in two ways. First, we study an order of magnitude more orbits than the original experiment and yet the gaps originally reported by DMS persist. Second, we observe that homoclinic bifurcations are responsible for these gaps and we list the symbolic labels of the orbits that form the gap endpoints in Table 6. These gaps correspond to the creation and destruction of parameter intervals where the dynamics of the area preserving Hénon map appears to be conjugate to a subshift of finite type. We find a similar gap structure for a particular subshift, $\Sigma_F$, and list the symbolic labels for the homoclinic orbits that form the endpoints of the gap that is the analog of the DMS gap. The role of the special symmetric saddle-node bifurcations without an accompanying pitchfork bifurcation in this scenario remains to be elucidated.

Many of our analytical results could be transferred from the area preserving case $b = 1$ to the orientation reversing case $b = -1$. In this case there also exists a reversor, so that the theorem about the first bifurcation and about impossibility of pitchfork bifurcations could be generalized to this case. The main difference is that now we are not studying homoclinic orbits of a fixed point which is invariant under the reversor, but instead heteroclinic orbits connecting fixed points that are mapped into each other by the reversor. Correspondingly the action of the reversor on symbol sequences is not just reading them backwards, but it is reading backwards and flipping each symbol.

Let us finally remark that we can extend the antimonotonicity result [13] to the non-dissipative case $b = \pm 1$ because there are bifurcations that do occur in the “wrong” direction, i.e., orbits that are created when $k$ is decreased. This is described in detail in a separate paper [14] where it is related to the fact that in the neighborhood of the period tripling bifurcation the twist generically vanishes.
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