Corrections to the fine structure constant in \( D \)-dimensional space from the generalized uncertainty principle

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Abstract

In this letter we compute the corrections to the fine structure constant in \( D \)-dimensional space. These corrections stem from the generalized uncertainty principle. We also discuss in three-space dimension.

1 Introduction

Recent observations of spectral lines of distant quasars have suggested that the fine structure constant \( \alpha = e^2/(4\pi\varepsilon_0\hbar c) \) may have been slightly smaller in the very early universe [1]. Although these claims are still tentative and rather controversial, they have helped rekindle interest in Dirac’s old idea [2] that the fundamental “constants” of physics may vary in time.

Varying fine structure constant has been studied in many papers, for example see [3, 4, 5, 6, 7, 8, 9, 10]. The authors of [11] derived the formulae for the time variation of the gravitational constant \( G \) and of the fine structure constant \( \alpha \) in various models with extra dimensions and analyzed their consistency with the available observational data for distant supernovae, see also [12].

On the other hand, it is of interest to define an effective fine structure constant \( \alpha_{\text{eff}} \) in terms of an effective Planck constant \( \hbar_{\text{eff}} \) so that we define \( \alpha_{\text{eff}} \equiv e^2/(4\pi\varepsilon_0\hbar_{\text{eff}} c) \). In this letter we obtain \( \hbar_{\text{eff}} \) based on the generalized uncertainty principle.

Based on simple and general considerations with either Newtonian gravitational theory or general relativity theory, the generalized uncertainty principle has been obtained in [13]. Explicit expressions for the generalized uncertainty principle in extra dimensions are given and their holographic
properties investigated [14]. According to [15], in $D$ spatial dimensions the dimensionless constant of nature is proportional to

$$h^{2-D}e^{D-1}G^{(3-D)/2}c^{D-4}. \tag{1}$$

As argued in [15] our universe appears to possess a collection of fundamental or natural units of mass, length and time which can be constructed from the physical constants $G$, $h$ and $c$. A dimensionless constant can only be constructed if the electron charge, $e$, is also admitted and then we obtain the dimensionless quantity $e^2/(hc)$, first emphasized in [16]. In a world with $D$ spatial dimensions the units of $h$ and $c$ remain $ML^2T^{-1}$ and $LT^{-1}$ in mass (M), length (L) and time (T). The units of $G$ become $M^{-1}L^D T^{-2}$. Gauss’ theorem relates $e$ to the spatial dimension and the units of $e^2$ are $ML^D T^{-2}$ because the electric force changes in accord with $F = qq'/r^2$. It is important to note that the system of units that has been used in the above argument is not the international system of units which we use in this letter. As we know, in the international system of units, the fine structure constant is defined $\alpha = e^2/(4\pi\epsilon_0hc)$ and in the Heaviside-Lorentz system $\alpha = e^2/(hc)$.

We show here that when one considers the generalized uncertainty principle, the corrections to the fine structure constant in $D$-dimensional space can be drawn.

The plan of this paper is as follows. In section 2, we review to obtain the fine structure constant in $D$-dimensional space. In section 3, we present the generalized uncertainty principle in $D$-dimensional space and then obtain the corrected fine structure constant in $D$-dimensional space due to the generalized uncertainty principle. Finally, we discuss and conclude in section 4.

## 2 Fine structure constant in $D$-dimensional space

In this section, we derive the fine structure constant in $D$-dimensional space, see [15]. First of all, we need to use the Newton’s gravitational constant in $D$-dimensional space. Let us first derive the exact relationship between Newton’s gravitational constants $G_D$ and $G_3 = G$ in $D$ and 3-dimensional space. Using the force laws in $D$ and 3-dimensional space, which are defined
by

\[ F_D = G_D \frac{m_1 m_2}{r^{D-1}}, \]  
\[ F_3 = G_3 \frac{m_1 m_2}{r^2}, \]

and the \( D \)-dimensional space Gauss’ law, one can derive the exact relationship between the gravitational constants \( G_D \) and \( G_3 \)

\[ G_3 = \frac{\Omega_{D-1}}{4\pi} \frac{G_D}{V_{(D-3)}}, \]

where \( V_{(D-3)} \) is the volume of \((D - 3)\) extra spatial dimensions and \( \Omega_{D-1} \) is the area of a unit \((D - 1)\)-sphere

\[ \Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma \left( \frac{D}{2} \right)}. \]

For \( D = 3 \), we have \( \Omega_2 = 4\pi \) which is the surface area of a unit 2-sphere. To obtain a general formula for the fine structure constant in \( D \)-dimensional space we propose that in \( D \)-dimensional space the fine structure constant can be written as

\[ \alpha_D = \hbar^\beta \epsilon^\gamma [\Omega_{D-1} \epsilon_{D,0}]^\eta c^\xi G_D^\tau, \]

where \( \epsilon_{D,0} \) is the permittivity constant of the vacuum in \( D \)-dimensional space and its units can be obtained by the electric force law in \( D \)-dimensional space

\[ F_D = \frac{1}{\Omega_{D-1} \epsilon_{D,0} r^{D-1}}. \]

From Eq. (7) we obtain the units of \( \epsilon_{D,0} \) to be equal to \( Q^2 M^{-1} L^{-D} T^2 \). In a world with \((D + 1)\)-spacetime the units of \( \hbar \), \( c \) and \( e \) remain \( ML^2 T^{-1} \), \( LT^{-1} \) and \( Q \) in Mass (\( M \)), length (\( L \)), time (\( T \)) and electric charge (\( Q \)). Using Eq. (4) one can obtain the units of the Newton’s gravitational constant in \( D \)-dimensional space \( G_D \) to be equal to \( M^{-1} L^D T^{-2} \). Thus in \( D \)-dimensional space the dimensionless constant of nature, i.e. the fine structure constant has the units of

\[ \left( ML^2 T^{-1} \right)^\beta Q^\gamma \left( Q^2 M^{-1} L^{-D} T^2 \right)^\eta \left( LT^{-1} \right)^\xi \left( M^{-1} L^D T^{-2} \right)^\tau. \]
Because the fine structure constant in $D$-dimensional space is a dimensionless quantity, the sum of powers of mass ($M$), length ($L$), time ($T$), electric charge ($Q$) must be vanished. Therefore we have

\begin{align}
\beta - \eta - \tau &= 0, \\
2\beta - D\eta + \xi + D\tau &= 0, \\
-\beta + 2\eta - \xi - 2\tau &= 0, \\
\gamma + 2\eta &= 0.
\end{align}

We know that in 3-dimensional space we have $\alpha = e^2/(4\pi\epsilon_0 \bar{h} c)$. So for $D = 3$ we have the following conditions

\begin{align}
\beta &= -1, \\
\gamma &= 2, \\
\eta &= -1, \\
\xi &= -1, \\
\tau &= 0.
\end{align}

Using Eqs. (9)-(12) and conditions (13)-(17) one can obtain

\begin{align}
\beta &= 2 - D, \\
\gamma &= D - 1, \\
\eta &= \frac{1 - D}{2}, \\
\xi &= D - 4, \\
\tau &= \frac{3 - D}{2}.
\end{align}

Thus in $D$-dimensional space the fine structure constant is equal to

\[\alpha_D = \hbar^{2-D} e^{D-1} \Omega_{D-1} \epsilon_{D,0}^{(1-D)/2} e^{D-4} c^{(3-D)/2} G_D^{(3-D)/2}.\]

This equation is a general formula for the fine structure constant in $D$-dimensional space. For $D = 3$, Eq. (23) leads us to $\alpha = e^2/(4\pi\epsilon_0 \bar{h} c)$.

### 3 Corrections to the Fine Structure Constant

In $D$-dimensional space, the Heisenberg uncertainty principle is written as

\[\Delta x_i \Delta p_j \geq \hbar \delta_{ij},\]
where \( x_i \) and \( p_j \), \( i, j = 1 \ldots D \), are the spatial coordinates and momenta, respectively. In \( D \)-dimensional space, the maximum uncertainty in the position of an electron in the ground state in hydrogen atom is equal to the radius of the first Bohr orbit, \( r_B \),

\[
\Delta x = r_B = \left( \frac{\Omega_{D-1} \epsilon_{D,0} \hbar^2}{m e^2} \right)^{1/2}, \tag{25}
\]

where \( m \) is the mass of the electron. Eq. (25) in the case of \( D = 3 \) yields the radius of the first Bohr orbit, so-called Bohr radius, \( r_B = 4\pi\epsilon_0\hbar^2/(me^2) = 5.29 \times 10^{-11} \text{m} \). The general form of the generalized uncertainty principle is

\[
\Delta x \geq \frac{\hbar}{\Delta p_i} + \hat{\beta}^2 \frac{L_P^2}{\hbar} \Delta p_i, \tag{26}
\]

where \( L_P = (\hbar G_D/c^3)^{1/(D-1)} \) is the Planck length and \( \hat{\beta} \) is a dimensionless constant of order one. There are many derivations of the generalized uncertainty principle, some heuristic and some more rigorous. Eq. (26) can be derived in the context of string theory and non-commutative quantum mechanics. The exact value of \( \hat{\beta} \) depends on the specific model. The second term in r.h.s of Eq.(26) becomes effective when momentum and length scales are of the order of Planck mass and of the Planck length, respectively. This limit is usually called quantum regime. From Eq.(26) we solve for the momentum uncertainty in terms of the distance uncertainty, which we again take to be the radius of the first Bohr orbit. From Eq.(26), we are led to the following momentum uncertainty

\[
\frac{\Delta p_i}{\hbar} = \frac{\Delta x_i}{2\hat{\beta}^2 L_P^2} \left( 1 / \sqrt{1 - \frac{4\hat{\beta}^2 L_P^2}{\Delta x_i^2}} \right). \tag{27}
\]

Eq.(25) gives

\[
\frac{\Delta x}{L_P} = \left( \frac{\Omega_{D-1} \epsilon_{D,0} M_P^3 G_D}{m e^2} \right)^{1/2}, \tag{28}
\]

where \( M_P = [\hbar^{D-2}/(c^{D-4}G_D)]^{1/(D-1)} \) is the Planck mass. Recalling the standard uncertainty principle \( \Delta x_i \Delta p_i \geq \hbar \), we define an “effective” Planck constant \( \Delta x_i \Delta p_i \geq h_{eff} \). From Eq.(26), we can write

\[
\Delta x_i \Delta p_i \geq \hbar \left[ 1 + \hat{\beta}^2 L_P^2 \left( \frac{\Delta p_i}{\hbar} \right)^2 \right]. \tag{29}
\]
So we can define the effective Planck constant as

\[ \hbar_{\text{eff}} = h \left[ 1 + \beta^2 \left( \frac{\Delta p_i}{h_R} \right)^2 \right]. \] (30)

From Eqs.(27,28) and (30) we get

\[ \hbar_{\text{eff}} = h \left[ 1 + \frac{1}{4\beta^2} \left( \frac{m \epsilon_D}{\Omega_D - 1} \right)^2 \right]. \] (31)

Because the value of the expression \( \frac{m \epsilon_D}{\Omega_D - 1} \) is much less than one, we can expand Eq.(31). Therefore, we have

\[ \hbar_{\text{eff}} \simeq h \left[ 1 + \beta^2 \left( \frac{m \epsilon_D}{\Omega_D - 1} \right)^2 \right]. \] (32)

So the effect of the generalized uncertainty principle can be taken into account by substituting \( \hbar_{\text{eff}} \) with \( \hbar \). For three-space dimension, from Eq.(32) we obtain

\[ \hbar_{\text{eff}} \simeq h \left[ 1 + \beta^2 \times 9.30 \times 10^{-50} \right]. \] (33)

Because the factor \( 10^{-50} \) is much less than one, we conclude that the value of the effective Planck constant is very close to the value of the standard Planck constant. Substituting the effective Planck constant \( \hbar_{\text{eff}} \) from Eq.(31) into Eq.(23) we obtain the effective and corrected fine structure constant due to the generalized uncertainty principle

\[ \alpha_{D,\text{eff}} = \hbar_{\text{eff}}^{2-D} e^{D-1} [\Omega_{D-1} \epsilon_D,0]^{(1-D)/2} c^{D-4} G_D^{(3-D)/2}. \] (34)

This equation shows the corrections to the fine structure constant from the generalized uncertainty principle in \( D \)-dimensional space. For \( D = 3 \), from (34) we obtain

\[ \alpha_{\text{eff}} = e^2 e^{2[4\pi \epsilon_0]^{-1} c^{-1}} = \frac{e^2}{4\pi \epsilon_0 \hbar_{\text{eff}} c}. \] (35)
From (33), we are led to

\[ \alpha_{\text{eff}} \simeq \frac{e^2}{4\pi\epsilon_0\hbar c} \left[ 1 - \beta^2 \times 9.30 \times 10^{-50} \right]. \]  

(36)

This equation shows the corrections to the fine structure constant in three-space dimension from the generalized uncertainty principle.

4 Conclusions

In this paper, we have examine the effects of the generalized uncertainty principle in the fine structure constant in \( D \)-dimensional space. We also discuss our calculations in three-space dimension. The general form of the generalized uncertainty principle is given by (26). The Planck constant \( \hbar \) undergoes corrections from the generalized uncertainty principle, and changes into \( \hbar_{\text{eff}} \) as given in Eq.(31) or approximately (32). Then we obtain the corrections to the fine structure constant in \( D \)-dimensional space as Eq.(34). As seen from Eq.(33), the value of \( \hbar \) and \( \hbar_{\text{eff}} \) is very close to each other because the factor \( 10^{-50} \) is much smaller than one. For this reason the value of the fine structure constant due to the generalized uncertainty principle is very close to the value of the fine structure constant in the standard uncertainty principle which is \( \frac{1}{137} \). The numerical results of Eq.(34) are in progress by the author.

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