Evolving Maximally Symmetric Spacetime Bubbles from Spontaneous $Z_2$—Violation at Electroweak Symmetry Breaking Scale

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Abstract

The Einstein-Gordon equations for Friedmann-Robertson-Walker (FRW) geometries in feedback reaction with the quartically self-interacting physical field, arisen from the “inner parity” spontaneous breaking, are explicitly formulated. The Hamiltonian density non-positive extrema would classically forbid both spatially closed and flat homogeneous and isotropic worlds if these were to allow the physical field to (repeatedly) go through and to (finally) settle down in a ground state. In this respect, the fixed point exact solutions of the spontaneous $Z_2$—symmetry breaking Einstein-Gordon equations (mandatory) describe $(k = -1)$—FRW manifolds which actually are either Milne or anti-de Sitter Universes. Setting the $Z_2$—invariance breaking scale at the one of the electroweak symmetry, we speculate on the cosmological implications of the Higgs-anti-de Sitter bubbles and derive a set of particular closed-form solutions to the $S^2$—cobordism with a spatially-flat FRW Universe.

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1 Introduction

The resourceful M Theory [1-10] and the celebrated AdS/CFT correspondence [11,12] have released a high tide of (new) investigations on the anti-de Sitter (AdS) spacetime, its extended versions to more than four dimensions and on the features of the (quantum) matter-fields evolving therein [13-22]. The applications to High Energy Physics, General Relativity and Cosmology [14,23-32], covering topics like 4D gauge field theories via M Theory, “conformal Higgs”, mass hierarchy (problem), dimensional compactification, either small or large extra dimensions, new unified theories, etc., together with the late observational data on the Universe (cosmological) acceleration [33-37], determinedly go far beyond the (so-called) Standard Model(s). Along with the growing interest in Pre-Big-Bang and other new types of Inflation [38-49]—which could also solve the cosmological constant problem—and in the models with various “dark energy” contents [48,50], there is also a need for a better understanding of the geometrodynamical link between the spacetime structure and the nature of spontaneously symmetry breaking vacua. A comprehensive account on this matter, emphasizing the False Vacuum Physics “subtleties” and quoting most of the previous papers in the field, has been given by T. Banks [51]; somewhat similarly, for an external continuous symmetry, J. W. Moffat [52] has explicitly worked out the intriguing, particularly astrophysical, consequences of the $SO(3,1)$—invariance spontaneous breaking. Last but not least, with respect to the subject we would like to speculate on, there is also the recent paper of A. Dev et al. [53], where the conclusion is drawn, based on a careful analysis of the late(st) gravity lensing and high-z supernova data [54], that the possibility of a (quasi)Milne stage, i.e. $(k = -1)$—FRW spacetime with linearly expanding scale function (so that the Universe could undergo a uniform expansion), has actually not been completely ruled out.

Formally, what we are dealing with, in the present paper, is a geometro-
dynamical analysis of the “extremal” spacetime structures derived as exact solutions of the Einstein’s field equations, for (initially) a generic FRW background with a quartically self-interacting scalar field as matter-content, in the system fixed points, i.e. at the three $Z_2$—symmetric extrema of the (fourth-degree polynomial) Hamiltonian density. The “catch” is that, while the central fixed point, the local maximum, is “gravitationally” inconsistent only with the spatially-closed FRW geometries, each of the other two fixed points, the absolute minima representing the matter-source degenerate vacua, is (geometrically and gravitationally) consistent (in the sense of an $R$—valued solution to the corresponding Einstein equations) only with the $(k = -1)$—family of FRW manifolds. As a matter of fact, once the vacuum has been set in one of the (two possible) ground states, the spontaneous $Z_2$—symmetry breaking does “instantly” create an anti-de Sitter Universe; when slightly perturbed, it gets filled with massive particles representing the physical field (quantum) excitations around the settled ground state. Considering therefore “$k = -1$” as a compulsory condition, the previously mentioned central fixed point corresponds to a Milne phase, which, being unstable against the coherent field fluctuations, does primarily turn into an anti-de Sitter one of (very) small curvature (in absolute value).

Informally from the rigour perspective, we set the $Z_2$—invariance breaking scale at the one of the electroweak symmetry, “taking” the Higgs-boson mass somewhere inbetween 115 and 300 $GeV$, and analyzing the respective values for the “gravitationally sustained” proper-pulsation, energy and power of the Higgs—anti-de Sitter bubbles, we speculate on some of their cosmological implications, such as a stronger CBR anisotropy on the frequencies ranging from (about) 190 $MHz$ upto (some) 1.4 $GHz$, more prominent towards the Giant Void(s), seizable deviations from the “whole sky”—averaged intensity-level of the 21 $cm(s)$ Hydrogen line, inner parity violating seeds of galaxy formation and Higgs-vacuum-based anti-de Sitter power-sources feeding the quasars’ cores.
A particular set of closed-form solutions to the $S^2$-cobordism between an anti-de Sitter bubble and a spatially-flat Universe is readily worked out in the final part of the paper. It (generically) points out that, as seen from the $(k = 0)$-FRW spacetime, the coordinate-radius of the small anti-de Sitter bubbles, as well as of the ones (not necessarily small) that might have existed in the course of Inflation, does asymptotically vanish at some high exponential rate. However, analytically, similar conclusions on the large bubbles evolution, in a subexponentially expanding conformally-flat Universe, cannot be drawn so easily, because of the highly nonlinear character of the respective $S^2$-cobordering equation(s).

# 2 Spontaneously Broken $Z_2$-Symmetry

Let us consider the inner parity (i.e. the field reflection $\Phi \rightarrow -\Phi$) invariant Lagrangian density

$$\mathcal{L}[\Phi] = -\frac{1}{2} \eta^{ab} \Phi_{|a} \Phi_{|b} + \frac{1}{2} \mu^2 \Phi^2 - \frac{\lambda}{24} \Phi^4$$

(1)

of a quartically self-interacting real scalar field $\Phi$, where, $\eta = \text{diag}[1, 1, 1, -1]$ is the fundamental metric tensor for a pseudo-orthonormal tetrad $\{e_a = e^i_a \partial_i\}_{i=1,4}$ whose dual $\{\omega^a = \omega^a_i \, dx^i \mid \langle \omega^a , e_b \rangle = \delta^a_b \}_{i=1,4}$ generates the spacetime metric

$$ds^2 = \eta_{ab} \omega^a \omega^b ,$$

(2)

the $(\cdot)_{|a}$ notation stands for the tetradic derivative with respect to $e_a$, i.e. $(\cdot)_{|a} = e_a(\cdot) = e^i_a \partial_i(\cdot)$, and $\mu^2$ — with mass$^2$ dimension — and $\lambda$ (dimensionless) are the two positive parameters that “accommodate” the spontaneously (discrete) symmetry breaking mechanism. Working out, from the functional expression

$$T = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g} \left[ \sqrt{-g} \mathcal{L} \right] ,$$

(3)
the covariant components (with respect to the “rigid” tensor basis 
\{ω^a \otimes ω^b\}_{a,b=1,4}^d) of the conservative stress-energy-momentum tensor, for the considered scalar field, it results in full

\[ T_{ab} = Φ|aΦ|b - \frac{1}{2} η_{ab} \left[ Φ^{|c}Φ^{|c} - μ^2 Φ^2 + \frac{λ}{12} Φ^4 \right], \tag{4} \]

so that, the extrema of the Hamiltonian density

\[ \mathcal{H} = T_{44} = \frac{1}{2} δ^{ab} Φ|aΦ|b - \frac{μ^2}{2} Φ^2 + \frac{λ}{24} Φ^4 \tag{5} \]

are given by the equation

\[ \frac{∂\mathcal{H}}{∂Φ}(Φ_0) = Φ_0 \left( \frac{λ}{6} Φ_0^2 - μ^2 \right) = 0, \tag{6} \]

i.e. \{Φ_0^α\}_{α=−0,+} = \{-μ\sqrt{\frac{λ}{6}}, 0, μ\sqrt{\frac{λ}{6}}\}. Inspecting the sign of the second derivative

\[ \frac{∂^2\mathcal{H}}{∂Φ^2}(Φ_0) = \frac{λ}{2} Φ_0^2 - μ^2 \tag{7} \]

for each of the three roots \{Φ_0^α\}_{α=−0,+}, it instantly results that Φ_0^0, where \frac{∂^2\mathcal{H}}{∂Φ^2} = −μ^2, is an unstable fixed point, while Φ_0^±, where \frac{∂^2\mathcal{H}}{∂Φ^2} = 2μ^2, are the “real” minima which correspond to the two possible ground states of the initially fictitious (i.e. apparently deprived of direct particle interpretation) scalar field Φ.

Choosing \( v = Φ_0^+ = μ\sqrt{\frac{λ}{6}} \) as the vacuum expectation value of Φ, in its ground state, and accordingly shifting the field

\[ Φ = v + χ, \text{ where } χ : M_4 → \mathbb{R}, \tag{8} \]

such that χ = 0 represents the true vacuum of the theory, one ends up with the spontaneously \( Z_2 \) broken Lagrangian density

\[ \mathcal{L}[χ] = -\frac{1}{2} χ^{|c}χ^{|c} - \frac{1}{2}(2μ^2)χ^2 - μ\sqrt{\frac{λ}{6}} χ^3 - \frac{λ}{24} χ^4 + \frac{3μ^4}{2λ} \tag{9} \]
of the physically observable massive, $m_\chi = \sqrt{2} \mu$, (real) scalar field $\chi$, subsequently obeying the inner parity violating ternary nonlinear (generalized) Gordon equation

$$\Box \chi - \left(2 \mu^2\right) \chi = 3 \mu \sqrt{\frac{\lambda}{6}} \chi^2 + \frac{\lambda}{6} \chi^3,$$

where

$$\Box \chi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left[ \sqrt{-g} g^{ik} \frac{\partial \chi}{\partial x^k} \right]$$

is the d’Alembert operator on $M_4$, in terms of some local coordinates $\{x^i\}_{i=1,4}$.

Therefore, either from (9) and (3), or straightly from (4) with the shift (8), it yields for the components of the energy-momentum tensor $T$ (of the physical field $\chi$, and with respect to the tensor basis $\{\omega^a \otimes \omega^b\}_{a,b=1,4}$) the actual expression

$$T_{ab} = \chi_{[a} \chi_{b]} - \frac{1}{2} \eta_{ab} \left[ \chi^{[c} \chi_{c]} + 2 V(\chi) - \frac{3 \mu^4}{\lambda} \right],$$

where the total, semi-classical (i.e. without quantum corrections) potential

$$V(\chi) = \mu^2 \chi^2 + \mu \sqrt{\frac{\lambda}{6}} \chi^3 + \frac{\lambda}{24} \chi^4$$

is clearly no longer invariant under the discrete transformation $\chi \rightarrow - \chi$.

In the Minkowski spacetime, which keeps on being flat whatever the matter energy-momentum is, the time-translation isometry, and the respective action-functional invariance, accounting for energy conservation, allows gauging the energy scale by any constant amount. Hence, any constant, field-independent, contribution to the (44)-component of the conservative energy-momentum tensor, such as $-3 \mu^4/(2 \lambda)$ in (12), does actually leave no observable signature in the field dynamics and so, it can just simply be thrown away. However, in general and physically more realistic situations, where gravity cannot be neglected, the matter stress-energy-momentum tensor does clearly affect the metric of the Lorentzian base manifold, so that any
of its additional terms, even a constant one, cannot be omitted any longer unless there are some serious, both mathematical and physical, reasons.

3 Inner Parity Non-Invariant Einstein-Gordon Equations in FRW Cosmologies

We have come to the point where we can address the question of what type of cosmic-time evolving homogeneous and isotropic 3-geometry “fits” the massive scalar source-field $\chi$ in such a way to produce an exact solution to the Einstein-(generalized) Gordon equations. Since both homogeneity and isotropy require a maximal $G_6$—group of motion on the 3-dimensional (sub)manifold $N_3$ (of $M_4$), this must possess constant curvature, i.e. $k = \{1,0, -1\}$, and thus, can only be the sphere $S^3$, the Euclidian $R^3$, or the disconnected wings of the hyperboloid $H^3$ defined, in the flat $R^4$ of Cartesian coordinates $(X^\alpha, T^\alpha) = 1,3$ and metric signature 2, by the “typical” equation $T^2 - \delta_{\alpha\beta} X^\alpha X^\beta = 1$. Hence, in terms of dimensionless Euler-like coordinates

\[
(\alpha, \beta, \theta) = \begin{cases}
(0, 2\pi) \times (0, 2\pi) \times (0, \frac{\pi}{2} \text{ or } \pi) \text{ on } S^3, k = 1 \\
R^3, k = 0 \\
R \times (0, 2\pi) \times R \text{ on } H^3, k = -1
\end{cases}
\] (14)

the metric on $N_3$ does respectively read

\[
dl^2_{N_3} = \begin{cases}
\cos^2 \theta (d\alpha)^2 + \sin^2 \theta (d\beta)^2 + (d\theta)^2, k = 1 \\
(d\alpha)^2 + (d\beta)^2 + (d\theta)^2, k = 0 \\
\cosh^2 \theta (d\alpha)^2 + \sinh^2 \theta (d\beta)^2 + (d\theta)^2, k = -1
\end{cases}
\] (15)

and therefore, considering the spacetime $M_4 = N_3 \times R$ as a continuous “tower” of $\{t = \text{cst} \mid \forall \text{cst} \in R\}$—cosmic-time orthogonal foliations $N_3$ homothetic to $N_3$, the metric on $M_4$ gets the well-known Friedmann-Robertson-Walker (FRW) form

\[ds^2 = a^2 e^{2f} dl^2_{N_3} - (dt)^2,\] (16)
where \( a \) is a scale parameter with dimension of length and the modified metric function \( f : \mathbb{R} \to \mathbb{R} \) does actually express the primitive

\[
f(t) = \int^t H(t') \, dt'
\]  

(17)

of the celebrated Hubble function

\[
H(t) \triangleq \frac{e^{-f} \, d}{a \, dt} \left( a \, e^f \right) \equiv \frac{df}{dt}
\]  

(18)

Consequently, with respect to (16) and (2), the dually related pseudo-orthonormal bases \( \{ \omega; e \} \) are respectively given by the concrete expressions

\[
\begin{align*}
\omega^1 &= a e^f \cos(\theta) \, d\alpha, \quad \omega^2 = a e^f \sin(\theta) \, d\beta, \quad \omega^3 = a e^f \, d\theta, \quad \omega^4 = dt \\
(\text{a}) & \quad \omega^1 = a e^f \, d\alpha, \quad \omega^2 = a e^f \, d\beta, \quad \omega^3 = a e^f \, d\theta, \quad \omega^4 = dt \\
& \quad \omega^1 = a e^f \cosh(\theta) \, d\alpha, \quad \omega^2 = a e^f \sinh(\theta) \, d\beta, \quad \omega^3 = a e^f \, d\theta, \quad \omega^4 = dt \\
& \quad e_1 = \frac{e^{-f}}{a} \, \sec(\theta) \partial_\alpha, \quad e_2 = \frac{e^{-f}}{a} \, \cosec(\theta) \partial_\beta, \quad e_3 = \frac{e^{-f}}{a} \, \partial_\theta, \quad e_4 = \partial_t \\
(\text{b}) & \quad e_1 = \frac{e^{-f}}{a} \, \partial_\alpha, \quad e_2 = \frac{e^{-f}}{a} \, \partial_\beta, \quad e_3 = \frac{e^{-f}}{a} \, \partial_\theta, \quad e_4 = \partial_t \\
& \quad e_1 = \frac{e^{-f}}{a} \, \sech(\theta) \partial_\alpha, \quad e_2 = \frac{e^{-f}}{a} \, \cosech(\theta) \partial_\beta, \quad e_3 = \frac{e^{-f}}{a} \, \partial_\theta, \quad e_4 = \partial_t
\end{align*}
\]

(19)

which move the exterior-forms formalism, through the Cartan’s Equations

\[
(\text{a}) \quad d\omega^a = \Gamma^a_{bc} \omega^b \wedge \omega^c, \quad \text{without torsion}, \\
(\text{b}) \quad R_{ab} = d\Gamma_{ab} + \Gamma_{ac} \wedge \Gamma^c_b,
\]

(20)

where

\[
\Gamma_{ab} = \eta_{ad} \Gamma^d_b = \Gamma_{abc} \omega^c,
\]

all the way down to the essential components \( R_{abcd} \) of the curvature 2-forms,

\[
R_{ab} = \frac{1}{2} R_{abcd} \omega^c \wedge \omega^d,
\]
namely,

\[(a) \quad R_{1212} = R_{1313} = R_{2323} = \left( f_{|4} \right)^2 + \frac{k}{a^2} e^{-2f}, \]
\[(b) \quad R_{1414} = R_{2424} = R_{3434} = -\left[ f_{|44} + \left( f_{|4} \right)^2 \right]. \quad (21)\]

Thus, the Ricci tensor gets no off-diagonal components and therefore it reads

\[(a) \quad R_{\alpha\beta} = \left[ f_{|44} + 3 \left( f_{|4} \right)^2 + \frac{2k}{a^2} e^{-2f} \right] \delta_{\alpha\beta}, \]
\[(b) \quad R_{44} = -3 \left[ f_{|44} + \left( f_{|4} \right)^2 \right], \quad (22)\]

where \(\alpha, \beta = 1, 3\), leading to the scalar curvature

\[R = 6 \left[ f_{|44} + 2 \left( f_{|4} \right)^2 + \frac{k}{a^2} e^{-2f} \right] \quad (23)\]

and altogether to the algebraically essential components of the Einstein tensor

\[(a) \quad G_{\alpha\beta} = -\left[ 2f_{|44} + 3 \left( f_{|4} \right)^2 + \frac{k}{a^2} e^{-2f} \right] \delta_{\alpha\beta}, \]
\[(b) \quad G_{44} = 3 \left[ \left( f_{|4} \right)^2 + \frac{k}{a^2} e^{-2f} \right] \quad (24)\]

As it can be noticed, for both \(k = \{0, 1\}\) the \(G_{44}\)-component does always stand non-negative and that is an important restriction, through the Einstein equation \(G_{44} = \kappa_0 T_{44}\), on the type of matter-sources that can be fit (in the sense of an exact solution) into such geometries: excepting the Minkowskian \((k = 0)\)-vacuum case, all the other matter-sources — if combined — should have a positive total energy density, \(w = T_{44} > 0\), i.e. they should behave on the whole, for \(k = 0\) or 1, as conventional matter, fulfilling the Hawking’s weak energy condition, \(T_{ab} u^a u^b \geq 0\) (for any non-spacelike 4-vector \(u\)). On the contrary, and completely nontrivial from a geometrodynamical perspective, in the hyperbolic case, \(k = -1\), the sign of \(G_{44}\) and (together with it) the
one of the (resulting) energy density gets undefined, unless \( f|_4 = 0 \) when the Einstein equations demand a deeply exotic kind of matter, of state-equation

\[
P = -\frac{1}{3} w, \quad \text{where } w = -\frac{3/\kappa_0}{a^2},
\]
i.e.

\[
P = \frac{1}{3} |w|,
\]
which can be comprehended as a sort of ghost-black-body radiation. Anyway, in the general situation, of evolving (time-orthogonal) \( H^3 \)-foliations, the geometrodynamics gets much more involved, since it can accommodate some reasonably mixed matter-sources, made both of “ordinary” and “exotic” matter. Thus, letting apart for the moment the well-known conventional sources, such as the thermalized electromagnetic (or other massless-field) radiation and the baryonic dust, we deal with the case where the \((k = -1)\)-FRW geometry is driven by the spontaneously inner parity breaking massive scalar field, \( \chi \), alone. As all of the essential Einstein tensor components are on-diagonal and only time-dependent, the source field \( \chi \) can only be coherent and therefore, its conservative stress-energy-momentum tensor does also become diagonal, exhibiting the components — subsequently derived from (12) —

\[
(a) \quad T_{\alpha\beta} = -\frac{1}{2} \left[ -\left(\chi|_4\right)^2 + 2 V(\chi) - \frac{3\mu^4}{\lambda} \right] \delta_{\alpha\beta} \\
(b) \quad T_{44} = \frac{1}{2} \left[ (\chi|_4)^2 + 2 V(\chi) - \frac{3\mu^4}{\lambda} \right] 
\]

(25)
Hence, the whole set of “quartically generalized” Einstein-Gordon equations

\[
G_{ab}[f] = \kappa_0 T_{ab}[\chi]
\]
immediately goes down to the following functionally 2-dimensional nonlinear differential system

\[
(a) \quad 2 f|_{44} + 3 \left(f|_4\right)^2 - \frac{e^{-2f/a^2}}{a^2} = \frac{\kappa_0}{2} \left[ -\left(\chi|_4\right)^2 + 2 V(\chi) - \frac{3\mu^4}{\lambda} \right], \\
(b) \quad 3 \left[ (f|_4)^2 - \frac{e^{-2f/a^2}}{a^2} \right] = \frac{\kappa_0}{2} \left[ (\chi|_4)^2 + 2 V(\chi) - \frac{3\mu^4}{\lambda} \right], 
\]

(26)
where the semi-classical potential $V$ is given by (13). We have not included in (26) the generalized Gordon equation (10), explicitly worked out for the considered ($k = -1$)–FRW spacetime dynamically sustained by the coherent massive scalar $\chi$, since all three of them, i.e. (26.a, b) and (10), taken together, are not functionally independent because of the twice contracted Second Bianchi Identity

$$0 \equiv \left( R^{ab} - \frac{1}{2} g^{ab} R \right)_{;b} = G_{;b}^{ab} = \kappa_0 T_{;b}^{ab} \Rightarrow T_{;b}^{ab} = 0,$$

i.e. if the energy-momentum tensor $T$ is correctly derived from the field Lagrangian density $L$ then its 4-divergenceless property (the one of being conservative) does accurately account for the field dynamics prescribed by the Euler-Lagrange equations

$$\frac{\delta L}{\delta \chi} = 0.$$ 

To put it shortly, the source-field equation (10), particularized to the form

$$\chi_{|44} + 3 f_{|4} \chi_{|4} + 2 \mu^2 \chi = -3\mu \sqrt{\frac{\lambda}{6}} \chi^2 - \frac{\lambda}{6} \chi^3,$$ 

must spring out from (26) just by taking first-order derivatives and subsequently doing algebraic manipulations. Indeed, taking the time-derivative of (26.b) it yields

$$2 \left[ f_{|44} + \frac{e^{-2f}}{a^2} \right] + 3 \left[ (f_{|4})^2 - \frac{e^{-2f}}{a^2} \right] = \frac{\kappa_0}{3} \frac{\chi_{|4}}{f_{|4}} \left[ \chi_{|44} + \frac{dV}{d\chi} \right]$$

and, by insertion in (26.a), written as

$$2 \left[ f_{|44} + \frac{e^{-2f}}{a^2} \right] + 3 \left[ (f_{|4})^2 - \frac{e^{-2f}}{a^2} \right] = \frac{\kappa_0}{2} \left( \chi_{|4} \right)^2 + \kappa_0 V(\chi) - \kappa_0 \frac{3\mu^4}{2\lambda},$$

i.e., using (26.b),

$$2 \left[ f_{|44} + \frac{e^{-2f}}{a^2} \right] + \frac{\kappa_0}{2} \left( \chi_{|4} \right)^2 + \kappa_0 V(\chi) - \kappa_0 \frac{3\mu^4}{2\lambda}$$

$$= - \frac{\kappa_0}{2} \left( \chi_{|4} \right)^2 + \kappa_0 V(\chi) - \kappa_0 \frac{3\mu^4}{2\lambda},$$
it gives
\[
\frac{1}{3} \frac{\chi_{|4}}{f_{|4}} \left[ \chi_{|44} + \frac{dV}{d\chi} \right] + \left( \chi_{|4} \right)^2 = 0,
\]
i.e.
\[
\chi_{|44} + 3f_{|4} \chi_{|4} = -\frac{dV}{d\chi}, \quad \text{with } \chi_{|4} \neq 0 \neq f_{|4},
\]
where the last equation does exactly come to the Gordon one (28) just by plugging in the \( \chi \)-derivative of the fourth-degree polynomial potential (13).

Nevertheless, is good to know that we can play all three equations, (26.a, b) and (28), since, in some concrete calculations, the result might be got easier in some particular combination of them, instead of working only with (26) as they stand.

In the above given proof of the compatibility of Euler-Lagrange equation (28) with the Einstein’s ones (26), we have asked for \( f_{|4} \neq 0 \) and \( \chi \neq 0 \). If \( f_{|4} = 0 \) then \( f \) can be scaled to zero and the system (26) becomes
\[
\begin{align*}
\frac{1}{a^2} &= \frac{\kappa_0}{2} \left( \chi_{|4} \right)^2 - \kappa_0 V(\chi) + \frac{3\kappa_0 \mu^4}{2\lambda}, \\
\frac{3}{a^2} &= -\frac{\kappa_0}{2} \left( \chi_{|4} \right)^2 - \kappa_0 V(\chi) + \frac{3\kappa_0 \mu^4}{2\lambda},
\end{align*}
\]
Subtracting the first equation from the second one, it yields
\[
\left( \chi_{|4} \right)^2 = -\frac{2/\kappa_0}{a^2} \Rightarrow \chi = \chi_0 \pm i \sqrt{\frac{2}{\kappa_0 a}},
\]
so that, the massive scalar \( \chi \) would be actually a ghost; in addition, since \( \chi_{|44} \equiv 0 \) (in this case), the nonlinear Gordon equation (28) just turns into the algebraic equation
\[
\chi^2 + 3\mu \sqrt{\frac{\lambda}{6}} \chi + \frac{12\mu^2}{\lambda} = 0, \quad \text{with } \chi \sim \frac{t}{a}, \, (\forall) \, t \in \mathbb{R},
\]
which obviously cannot be satisfied as \( \chi \) is clearly time-dependent. Hence, the \( (f = \text{const}) \)-particular case is definitely forbidden for the considered matter-source.
4 Maximally Symmetric Fixed Points

The other particular case, \( \chi_{|4} = 0 \), is by far of much interest for it reveals the simplest \((k = -1)\)–FRW spacetime dynamics in the fixed points of the non-linear Gordon equation for the physical field \( \chi \) left-over by the spontaneous breaking of the discrete inner-symmetry \( \phi \rightarrow -\phi \). The shortest path to the solution(s) is paved by the observation that for \( \chi_{|4} = 0 \) the right-hand-side of the two Einstein equations (26) gets the same and therefore, subtracting them, it yields the modified metric function essential equation

\[
f_{|44} + \frac{e^{-2f}}{a^2} = 0
\]  

which can be inserted back into (26.a), with \( \chi_{|4} = 0 \), getting at once the same equation (26.b) (with \( \chi_{|4} = 0 \)). Hence, among the three Einstein-(generalized) Gordon equations we have just to solve, in this particular case, the very simple system

\[
\begin{align*}
(a) & \quad (f_{|4})^2 - \frac{e^{-2f}}{a^2} = \frac{\kappa_0}{3} V(\chi) - \frac{\kappa_0 \mu^4}{2\lambda} \\
(b) & \quad \chi \left[ 2\mu^2 + 3\mu \sqrt{\frac{\lambda}{6}} \chi + \frac{\lambda}{6} \chi^2 \right] = 0
\end{align*}
\]  

where \( V(\chi) \) is given by (13), i.e.

\[
V(\chi) = \frac{\lambda}{24} \chi^2 \left[ \chi^2 + 4\mu \sqrt{\frac{6}{\lambda}} \chi + \frac{24\mu^2}{\lambda} \right]
\]  

Ordered by their magnitudes, the roots of (30.b) – meaning the matter-field fixed-point values – do respectively read

\[
\chi_L = -2\mu \sqrt{\frac{6}{\lambda}}, \quad \chi_M = -\mu \sqrt{\frac{6}{\lambda}}, \quad \chi_R = 0,
\]  

where the indices \( L, M, R \) come from “left, Milne, right”, respectively. By their “turn by turn” insertion in (31), one immediately gets the corresponding
values of the semi-classical (4-nominal self-interaction) potential, namely

\[ V_L = 0, \quad V_M = \frac{3\mu^4}{2\lambda}, \quad V_R = 0, \quad (33) \]

so that, the nonlinear first-order differential equation (30.a) of the metric function \( f \) does only take the following two particular forms

\[
(a) \quad \left( f_{|4} \right)^2 - \frac{e^{-2f}}{a^2} = -\frac{\kappa_0 \mu^4}{2\lambda}, \quad \text{for} \quad V_L = V_R = 0, \\
(b) \quad \left( f_{|4} \right)^2 = 0, \quad \text{for} \quad V_M = \frac{3\mu^4}{2\lambda}, \quad (34)
\]

which will be correspondingly leading to the only two types of \( k = -1 \) homogeneous and isotropic universes that can accommodate and are also physically supported by the massive, \( m_\chi = \sqrt{2}\mu \), real scalar field \( \chi \) in any of its three 4-dimensionally constant main states. It is worth noticing that without integrating the equation (34) we can already say, by the help of equation (29), what the two generated spacetimes are. Indeed, in the simpler case (34.b), because of (29), it also results that

\[
 f_{|44} + \left( f_{|4} \right)^2 = 0 \quad (35)
\]

and thus, having a look at the components (21) of the curvature tensor, we instantly realize that all of them vanish. Hence, the spacetime corresponding to (34.b), supported by the static physical field \( \chi_M = -\mu \sqrt{6/\lambda} \), is flat, being basically (geometrically) the Minkowski spacetime. However, especially from a cosmological perspective, the difference is that this spacetime is patched in a different coordinate-system, namely the Milne’s one, which sharply presents the evolution of the \( H^3 \)—spacelike-foliation, instead of the static picture of the Cartesian \( R^3 \)-foliations (of constant Minkowskian time, \( x^4 = t \)). Concerning the other \( (k = -1) \)—FRW model, the one related to the equation (34.a), we get, using again the essential equation (29), that

\[
 f_{|44} + \left( f_{|4} \right)^2 = -\frac{\kappa_0 \mu^4}{2\lambda} \quad (36)
\]
and so, inserting it, together with (34.a), in the expressions (21) of the Riemann tensor components and in the expression (23), with $k = -1$, of the scalar curvature, it yields

\[(a) \quad R_{\alpha\beta\alpha\beta} = - R_{\alpha4\alpha4} = - \frac{\kappa_0 \mu^4}{2\lambda}, \quad \alpha = \overline{1,3}, \beta \neq \alpha, \beta = \overline{1,3},\]

\[(b) \quad R = - 12 \left( \frac{\kappa_0 \mu^4}{2\lambda} \right), \quad (37)\]

which clearly points out that the solution to the “basic” equation (34.a) does surely sustain a ($k = -1$)–FRW Universe of constant negative (4D) curvature and that can only be the anti-de Sitter spacetime. Although investigating it, and some of its possibly observable cosmological consequences, mostly related to the spontaneous breaking of the field-reflection inner symmetry, is the main goal of the present paper, we would like (first) to make some remarks on the model described by the solution of (34.b), better known as the Milne Universe, and to comment a bit its linear stability within the context of the coherent linear perturbations of the massive source-field $\chi$ around its fixed-point value $\chi_M$.

5 Milne Spacetime Coherent Perturbations

As it is almost obvious, because of its very simple form, the equation (34.b) can be immediately written as

\[\left( \frac{dS}{dt} \right)^2 = 1, \quad (38)\]

where

\[S = a e^t \quad (39)\]

is the “standard” cosmological scale function — casting the FRW-metric into the “history making” form

\[ds^2 = S^2(t) dl_{N_3}^2 - (dt)^2, \quad (40)\]
(where $N_3$ is one of the $S^3$, $R^3$, $H^3$ manifolds) — and so, independently of both the sign we choose for the solution and the corresponding integration constant, the scale function is basically reading

$$S(t) = |t|,$$  \hfill (41)

so that, in $(k = -1)$—spherical (physically dimensionless) coordinates $\{r, \theta, \varphi\}$ on the upper wing (let’s say) of $H^3$, the generic metric (40) does explicitly turn into the one of the Milne Universe

$$ds^2_{\text{Mil}} = t^2 \left[ \frac{(dr)^2}{1 + r^2} + r^2 d\Omega^2 \right] - (dt)^2,$$  \hfill (42)

where

$$d\Omega^2 = (d\theta)^2 + \sin^2 \theta (d\varphi)^2$$  \hfill (43)

is the well-known metric on the unit sphere $S^2$. Since, as we have already shown, this spacetime is properly flat, there must be a globally defined local coordinates transformation that takes the Minkowskian metric,

$$ds^2_{\text{Mnk}} = (dR)^2 + R^2 d\Omega^2 - (dT)^2,$$  \hfill (44)

into the Milne’s one, (42), and vice versa. In this respect, as the angular $(\theta, \varphi)$—coordinates are the same (in the two metrics and on the two manifolds), it is quite obvious that the usual radial coordinate $R$ is given in terms of the Milne coordinates “$r, t$” by the simple relation

$$R(r, t) = r |t|, \text{ with } r \geq 0.$$  \hfill (45)

Then, the Minkowskian time $T$ must also be a function of the two coordinates $(r, t)$, i.e.

$$T = T(r, t),$$  \hfill (46)

such that, plugging in (44) the square of its differential and the one of (45), it should equate the two (1+1)-metrics, i.e.

$$(dR)^2 - (dT)^2 = \frac{t^2}{1 + r^2} (dr)^2 - (dt)^2,$$  \hfill (47)
which concretely comes to the self-embedding condition

\[
\left( |t| \frac{dr}{dt} + r \frac{d|t|}{dt} \right)^2 - \left( \frac{\partial T}{\partial r} \frac{dr}{dt} + \frac{\partial T}{\partial t} \frac{dt}{dt} \right)^2 = \frac{t^2}{1 + r^2} (dr)^2 - (dt)^2 , \tag{48}
\]

leading therefore to the following set of equations that (46) must satisfy:

\[
\begin{align*}
(a) & \quad \left( \frac{\partial T}{\partial r} \right)^2 = \frac{t^2 r^2}{1 + r^2} \\
(b) & \quad \frac{\partial T}{\partial r} \frac{\partial T}{\partial t} = r t \\
(c) & \quad \left( \frac{\partial T}{\partial t} \right)^2 = 1 + r^2
\end{align*}
\tag{49}
\]

Hence, setting to zero the integration constants, the two branches that spring out from the equations (49.a, c), compatible with (49.b), are respectively defined by

\[
\begin{align*}
(a) & \quad \frac{\partial T}{\partial t} = -\sqrt{1 + r^2} \Rightarrow \frac{\partial T}{\partial r} = -\frac{rt}{\sqrt{1 + r^2}} \Rightarrow T = -t \sqrt{1 + r^2} , \\
(b) & \quad \frac{\partial T}{\partial t} = \sqrt{1 + r^2} \Rightarrow \frac{\partial T}{\partial r} = \frac{rt}{\sqrt{1 + r^2}} \Rightarrow T = t \sqrt{1 + r^2} , \tag{50}
\end{align*}
\]

where only (50.b) preserves the same orientation on the considered manifold, i.e.

\[
\left| \frac{\partial R}{\partial t} \frac{\partial R}{\partial r} \right| = \frac{|t|}{\sqrt{1 + r^2}} > 0 , \quad \forall t \in \mathbb{R} - \{0\} . \tag{51}
\]

Thus, collecting the two results (45) and (50.b), we have actually got the celebrated orientation-preserving Milne transformation,

\[
\begin{align*}
(a) & \quad R = r |t| , \\
(b) & \quad T = t \sqrt{1 + r^2} , \tag{52}
\end{align*}
\]

where \( r \geq 0 \) and \( t \in \mathbb{R} \), which casts (44) into the form (42) and, with a bit of care (again, about orientation), we can immediately write down, from (52),
its (proper) inverse

\[
\begin{align*}
(a) \quad r &= \frac{r}{\sqrt{T^2 - R^2}} \\
(b) \quad t &= \text{sgn}(T) \sqrt{T^2 - R^2},
\end{align*}
\]

with \( T^2 - R^2 \geq 0 \), \( R \geq 0 \), \( T \in \mathbb{R} \), which actually defines the Milne coordinates \((r, t)\) in terms of the Minkowskian ones \((R, T)\) and accordingly turns (42) into (44). Although the bulk of the Milne’s Universe properties and structure is very well known since quite long ago, its connection to the spacetime geometry erected by the “remnant” massive scalar field \( \chi \), “after” the \( \mathbb{Z}_2 \)-invariance got spontaneously broken, has not been investigated to a large extent in the literature. It might be shedding a brand new light on the real (physical) significance of the Milne Universe — previously considered just as a “toy model” — and on a seemingly not yet extensively explored bunch of cosmological implications of the decaying remnant-field blown up Milne bubbles in the observable Universe. That is why it is worth studying their behaviour at least with respect to the linear perturbations of the source-field \( \chi \) around its static value \( \chi_M \).

Hence, let us consider the coherently evolving small field-perturbation \( \psi \), such that

\[
\chi = \chi_M + \psi, \text{ where } \lvert \psi \rvert \ll \lvert \chi_M \rvert = \sqrt{\frac{6}{\lambda}},
\]

whose dynamics is effectively controlled by (28), which concretely becomes

\[
\frac{d^2 \psi}{dt^2} + \frac{3}{t} \frac{d\psi}{dt} - \mu^2 \psi = 0,
\]

once the expression

\[
f = \ln \left( \frac{|t|}{a} \right),
\]

derived from (39) and (41), and the identity

\[
\frac{\lambda}{6} \chi_M^2 + 3\mu \sqrt{\frac{\lambda}{6}} \chi_M + 2\mu^2 = 0,
\]
stated by (30.b), are taken into account. Because of the matter-field fluctuations encoded in (55), the initial Milne background starts exhibiting metric perturbations which no longer fulfill the unperturbed equations (34.b) and (35); instead, their evolution is controlled by the two Einstein equations (26.a, b). Fortunately, since the last two terms on the right hand side of (26) are exactly the same, one gets a single essential equation for the perturbed modified metric function, \( f \), namely

\[
f_{\rho\rho} + \frac{e^{-2f}}{a^2} = -\frac{\kappa_0}{2} \left( \chi_{\|4} \right)^2 ,
\]

just by subtracting (26.b) from (26.a). As it can be noticed, although one can reduce the order of differentiability in (58), by the well-known substitutions

\[
p = \frac{df}{dt}, \quad dt = \frac{df}{p} \Rightarrow \frac{d}{dt} = p \frac{d}{df}, \quad q = p^2 ,
\]

casting it into the form of the linear, non homogeneous, first-order differential equation

\[
\frac{dq}{df} + \kappa_0 \left( \frac{d\psi}{df} \right)^2 q = -\frac{2}{a^2} e^{-2f} - \kappa_0 \left( \chi_{\|4} \right)^2 = \kappa_0 p^2 \left( \frac{d\chi}{df} \right)^2 q
\]

because of (54) and (59) — whose solution reads

\[
q = e^{-\kappa_0 \int f'(\frac{d\chi}{df})^2 d\xi} \left[ C - \frac{2}{a^2} \int f e^{-2\left[ \frac{\xi - \frac{m}{2} \int f'(\frac{d\chi}{df})^2 d\eta} \right] d\xi} \right] ,
\]

where \( C = \text{const.} \geq 0 \), so that

\[
\frac{df}{dt} = \pm e^{\frac{m}{a^2} f'(\frac{d\chi}{df})^2 d\xi} \left[ C - \frac{2}{a^2} \int f e^{-2\left[ \frac{\xi - \frac{m}{2} \int f'(\frac{d\chi}{df})^2 d\eta} \right] d\xi} \right]^{1/2}
\]

and therefore

\[
t = t_0 \pm \int e^{\frac{m}{a^2} f'(\frac{d\chi}{df})^2 d\xi} \frac{df}{\left[ C - \frac{2}{a^2} \int f e^{-2\left[ \frac{\xi - \frac{m}{2} \int f'(\frac{d\chi}{df})^2 d\eta} \right] d\xi} \right]^{1/2}} , \quad t_0 \in \mathbb{R} ,
\]
one cannot actually get an explicit closed-form solution quite because of the matter-source contribution $\frac{\kappa_0}{2} \left( \frac{d\psi}{df} \right)^2$ which, in principle, should be worked out from the awkward (although linear) second-order differential equation of the linear field-fluctuation around $\chi_M$

$$\frac{d^2 \psi}{df^2} + \left[ 3 + \frac{d}{df} (\ln \sqrt{q}) \right] \frac{d\psi}{df} - \frac{\mu^2}{q} \psi = 0 ,$$

representing the image of (55), softly generalized as

$$\frac{d^2 \psi}{dt^2} + 3 \frac{df}{dt} \frac{d\psi}{dt} - \mu^2 \psi = 0 ,$$

under the whole set of substitutions (59). Hence, even for small perturbations around the Milne value of the physical field $\chi$, it is clearly unlikely to get an exact solution $\{\psi, q\} (f) : \mathbb{R} \rightarrow \{\mathbb{R}, \mathbb{R}_+\}$, i.e., by (59), $\{\psi, f\} (t) : \mathbb{R} \rightarrow \mathbb{R}$, to the highly nonlinearly coupled differential equations (60) and (61) and therefore, analytically, the best one can do, at least for tackling the subject, is to consider the linear fluctuation $h$ in the modified metric function $f$, around its Milne form (56), i.e.

$$f = \ln \left| \frac{t}{a} \right| + h , \text{ with } e^h \approx 1 + h$$

and to subsequently linearize the essential (inhomogeneous) Einstein equation (58), getting the far much simpler form

$$\frac{d^2 h}{dt^2} - 2 \frac{\kappa_0}{t^2} h = - \frac{\kappa_0}{2} \left( \frac{d\psi}{dt} \right)^2 ,$$

where the source-term on the right hand side is going to be evaluated, in the first-order approximation, by integrating the fluctuating remnant field equation (55). In this respect, setting

$$\tau = \mu t \text{ and } \psi = \tau^\nu U(\tau) , \nu \in \mathbb{R} ,$$

it yields the differential equation

$$\frac{d^2 U}{d\tau^2} + \frac{2\nu + 3}{\tau} \frac{dU}{d\tau} - \left( 1 - \frac{\nu(\nu + 2)}{\tau^2} \right) U = 0 ,$$
for the new function $U$, so that, for

$$\nu = -1 ,$$  \hfill (66)

it concretely gets the form of the “modified Bessel functions” equation,

$$\frac{d^2 U}{d\tau^2} + \frac{1}{\tau} \frac{dU}{d\tau} - \left(1 + \frac{1}{\tau^2}\right) U = 0 ,$$  \hfill (67)

and therefore, referring to the perturbation field $\psi$ in (64) with (66), one instantly gets the two (functionally) linearly independent modes

$$\psi_+(t) = \frac{N_+}{\mu t} I_1(\mu t) , \quad \psi_-(t) = \frac{N_-}{\mu t} K_1(\mu t) ,$$  \hfill (68)

where the real constants $N_\pm$ – of renormalization dimension $D = 1$ – set the amplitude of the physical field fluctuation at some reference-moment “$t_0$”. In spite of the way it heads into the future (timelike) infinity, decaying extremely fast, the mode $\psi_-$ is strongly singular at the Milne-time origin, $t = 0$, and takes very large values around it, so that, at such early epochs, it hardly can be considered as a small perturbation that should fulfill the requirement $|\psi_-(0 < \mu t << 1)| << \mu \sqrt{6/\lambda} = |\chi_M|$. Therefore, only the $(t = 0)$—nonsingular growing mode $\psi_+$ contributes to the source term in (63) and drives the evolution of the modified metric function perturbation, $h$. With respect to the dimensionless time-like parameter $\tau$ (defined in (64)) the equation (63) does simply read

$$\frac{d^2 h}{d\tau^2} - \frac{2}{\tau^2} h = -\frac{\kappa_0}{2} \left(\frac{d\psi_+}{d\tau}\right)^2$$  \hfill (69)

and we work out its general solution(s) by starting with the function substitution

$$h = \tau^\gamma F(\tau) ,$$  \hfill (70)

which turns (69) into the inhomogeneous Euler equation, for the function $F$,

$$\frac{d^2 F}{d\tau^2} + \frac{2\gamma}{\tau} \frac{dF}{d\tau} + \frac{\gamma(\gamma - 1) - 2}{\tau^2} F = -\frac{\kappa_0}{2} \left(\frac{d\psi_+}{d\tau}\right)^2 \tau^{-\gamma} .$$  \hfill (71)
As it can be noticed regarding the term in F, this can be nontrivially vanished by taking \( \gamma \) as each of the two roots of the very simple 2-nd degree equation

\[
\gamma^2 - \gamma - 2 = 0 \Rightarrow \gamma = \{-1, 2\}
\]  

(72)

and therefore we get — in principle — two branches of solutions, respectively stated by the mathematical relations

\[
\begin{align*}
(a) \quad & h_- = \frac{1}{\tau} F_-(\tau), \quad \frac{dF_-}{d\tau} = G_-(\tau), \\
(b) \quad & \frac{dG_-}{d\tau} = -\frac{2}{\tau} G_- = -\frac{\kappa_0}{2} \left(\frac{d\psi_+}{ds}\right)^2 \tau, \text{ for } \gamma = -1; \\
(c) \quad & h_+ = \tau^2 F_+(\tau), \quad \frac{dF_+}{d\tau} = G_+(\tau), \\
(d) \quad & \frac{dG_+}{d\tau} + \frac{4}{\tau} G_+ = -\frac{\kappa_0}{2} \frac{1}{\tau^2} \left(\frac{d\psi_+}{ds}\right)^2, \text{ for } \gamma = 2.
\end{align*}
\]  

(73)

Thus, by (73.a, c), we have reduced the second-order differential equation(69) to the inhomogeneous first-order differential equations (73.b, d) whose solutions can be easily derived as

\[
\begin{align*}
(a) \quad & G_-(\tau) = G^0_- \tau^2 - \frac{\kappa_0}{2} \tau^2 \int^\tau \frac{ds}{s} \left(\frac{d\psi_+}{ds}\right)^2, \quad G^0_- = \text{cst} \in \mathbb{R}, \\
(b) \quad & G_+(\tau) = \frac{G^0_+}{\tau^4} - \frac{\kappa_0}{2} \frac{1}{\tau^4} \int^\tau \frac{ds}{s^2} \left(\frac{d\psi_+}{ds}\right)^2 ds, \quad G^0_+ = \text{cst} \in \mathbb{R},
\end{align*}
\]  

(74)

where \( \psi_+ \) is given by (68), with \( \mu t \) replaced by \( s \), i.e.

\[
\psi_+(s) = \frac{N_+}{s} I_1(s) \Rightarrow \left\{ \frac{d\psi_+}{ds} = \frac{N_+}{s} \left[ \frac{dl_1}{ds} - \frac{l_1}{s} \right], \quad \frac{dl_1}{ds} = \frac{1}{2} [I_0(s) + I_2(s)] \right\}.
\]  

(75)

Making (a respective) use of (73.a, c), once (74.a, b) have been gotten, we come to the concrete expression(s) of the — “formally” two — (most) general
solutions,

\[(a) \quad h_-(\tau) = F^0_\tau^{-1} + \frac{1}{3} G^0_\tau^2 - \frac{\kappa_0}{2} \tau^{-1} \int \tau^2 \left( \int s \left( \frac{d\psi_+}{ds} \right)^2 \right) d\tau,\]

\[(b) \quad h_+(\tau) = -\frac{G^0_\tau}{3} \tau^{-1} + F^0_\tau^2 - \frac{\kappa_0}{2} \tau^2 \int \frac{d\tau}{\tau^4} \left( \int s^2 \left( \frac{d\psi_+}{ds} \right)^2 \right) ds,\]

with \(F^0_\pm = \text{cst} \in \mathbb{R}\), of the linearized Einstein-Gordon equation (69) \(\Leftrightarrow\) (63), which describes, envisaging (62), the coherent Milne’s Universe metric fluctuations induced by an initially small perturbation –

\[|\psi_+(0)| = \frac{1}{2} |N_+| \ll \mu \sqrt{\frac{6}{\lambda}} = |\chi_M|,\]

right on the singular, but free of “real” geometrical singularities, \(\{t = 0\}\) – foliation – in the physical field \(\chi\), leftover around \(\chi_M\) by the spontaneous breaking of the inner parity invariance. Nevertheless, in a straight computational manner, it is not quite trivial to realize that the two branches do actually coincide.

The difference in the way they look is only apparent and comes from an additional part-by-part integral that has been subtly performed – “by itself”, actually – in the switch from (73.a, b) to (73.c,d). Indeed, one can notice first that

\[\int s \left( \frac{d\psi_+}{ds} \right)^2 \right) d\tau = \int \frac{d\tau}{s^3} s^2 \left( \frac{d\psi_+}{ds} \right)^2 = \int \frac{1}{s^3} \left[ s^2 \left( \frac{d\psi_+}{ds} \right)^2 \right] ds \quad (77)\]

and setting

\[u = \frac{1}{s^3}, \quad dv = s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds,\]

\[du = -\frac{3ds}{s^4}, \quad v = \int s \xi^2 \left( \frac{d\psi_+}{d\xi} \right)^2 d\xi,\]

(78)
it yields (integrating “by parts”)

\[
\int \frac{d\tau}{\tau} \left( \frac{d\psi_+}{d\tau} \right)^2 = \frac{1}{\tau^3} \int s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds + 3 \int \frac{d\tau}{\tau^4} \left( \int \tau^2 s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds \right).
\]

That is to be used in the part-by-part integral

\[
\int \tau^2 \left( \int \frac{d\tau}{s} \left( \frac{d\psi_+}{ds} \right)^2 \right) d\tau = \frac{\tau^3}{3} \int \tau^2 s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds - \frac{1}{3} \int \tau^2 \left( \frac{\psi_+}{d\tau} \right)^2 d\tau,
\]

i.e.

\[
\int \tau^2 \left( \int \frac{d\tau}{s} \left( \frac{d\psi_+}{ds} \right)^2 \right) d\tau = \frac{1}{3} \int \tau^2 s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds + \tau^3 \int \frac{d\tau}{\tau^4} \left( \int \tau^2 s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds \right) - \frac{1}{3} \int \tau^2 s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds = \tau^3 \int \frac{d\tau}{\tau^4} \left( \int \tau^2 s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds \right),
\]

explicitly stating that

\[
\frac{1}{\tau} \int \tau^2 \left( \int \frac{d\tau}{s} \left( \frac{d\psi_+}{ds} \right)^2 \right) d\tau = \tau^2 \int \frac{d\tau}{\tau^4} \left( \int \tau^2 s^2 \left( \frac{d\psi_+}{ds} \right)^2 ds \right)
\]

and therefore, with

\[
- \frac{G^0_+}{3} = F_0^- = F_0, \\
F_+^0 = \frac{G_0^0}{3} = - \frac{1}{6} G_0,
\]

it has been entirely proven that the two seemingly different branches (76) are actually the same, being subsequently described by the modified metric
function perturbation

\[ h = \frac{F_0}{\tau} - \frac{G_0}{6} \tau^2 - \frac{\kappa_0}{2} \tau^{-1} \int \tau^2 \left( \int \frac{ds}{s} \left( \frac{d\psi_+}{ds} \right)^2 \right) \, d\tau. \]  

(82)

With that and (62), the proper scale function (39) becomes

\[ S(\tau) = \frac{F_0}{\mu} + \frac{\tau}{\mu} \left[ 1 - \frac{G_0}{6} \tau^2 - \frac{\kappa_0}{2} \tau^{-1} \int \tau^2 \left( \int \frac{ds}{s} \left( \frac{d\psi_+}{ds} \right)^2 \right) \, d\tau \right], \]

so that, the seemingly divergent term \( F_0/\tau \), in (82), brings nothing more than a constant (universal) shift in the Milne’s cosmic-time, making no contribution at all to the curvature disturbances produced by the source-field (linear) fluctuations \( \psi_+ \). Hence, it simply can be dropped away just by setting \( F_0 = 0 \).

Quite on the contrary, the seemingly arbitrary constant \( G_0 \) makes an important contribution to the curvature of the coherently \( \psi_+ \) perturbed Milne Universe, mostly with respect to its stability.

To give the details of this matter, let us first work out the curvature perturbations straight from the relations (82), (62) and (21, with \( k = -1 \)). It primarily results

\[ R_{\alpha\beta\alpha\beta} = \frac{2\mu^2}{\tau^2} \left[ \tau \frac{dh}{d\tau} + h \right] = \mu^2 \frac{2}{\tau^2} \frac{d}{d\tau} (\tau h), \quad \alpha, \beta = 1, 3, \]

\[ R_{\alpha4\alpha4} = - \mu^2 \left[ \frac{d^2h}{d\tau^2} + 2 \frac{dh}{d\tau} \right] = - \mu^2 \frac{d}{d\tau} \left( \tau^2 \frac{d}{d\tau} \left( \frac{dh}{d\tau} \right) \right), \quad \alpha = 1, 3, \]

with no summation on the repeated indices, and plugging the (82) in, it yields

\[ R_{\alpha\beta\alpha\beta} = - \mu^2 \left[ G_0 + \kappa_0 \int \frac{d\tau}{\tau} \left( \frac{d\psi_+}{d\tau} \right)^2 \right] \]

\[ R_{\alpha4\alpha4} = - R_{\alpha\beta\alpha\beta} + \frac{\kappa_0}{2} \mu^2 \left( \frac{d\psi_+}{d\tau} \right)^2 \]  

(83)

26
On the other hand, completely independent of the form of \( h \), the same curvature components can be derived from the exact form (26) of the Einstein-(generalized) Gordon equations, where the potential (13) reads, within the linear approximation assumption,

\[
V(\chi) = V(\chi_M + \psi) = V(\chi_M) + \frac{dV}{d\chi} \bigg|_{\chi_M} \psi + \frac{1}{2} \frac{d^2V}{d\chi^2} \bigg|_{\chi_M} \psi^2 + O(\psi^{n>2})
\]

i.e., since \( \chi_M \) is an extremum of \( V \),

\[
V(\psi) = \frac{3\mu^4}{2\lambda} - \frac{\mu^2}{2} \psi^2
\]

Thus, one gets the relations

\[
\left( f_{|4} \right)^2 - \frac{e^{-2f}}{a^2} = \frac{\kappa_0}{6} \left[ (\psi_{|4})^2 - \mu^2 \psi^2 \right]
\]

\[
2 \left[ f_{|44} + \left( f_{|4} \right)^2 \right] = -\frac{\kappa_0}{2} \left[ (\psi_{|4})^2 + \mu^2 \psi^2 \right] - \left[ (f_{|4})^2 - \frac{e^{-2f}}{a^2} \right] \tag{84}
\]

which straightforwardly lead, because of (21, with \( k = -1 \)), to the entirely \( \psi^+ \)-dependent expressions of the essential curvature components,

\[
R_{\alpha \beta \alpha \beta} = \frac{\kappa_0}{6} \left[ \left( \frac{d\psi^+}{d\tau} \right)^2 - \mu^2 \psi^+_+ \right]
\]

\[
R_{\alpha 4 \alpha 4} = \frac{\kappa_0}{6} \left[ 2 \left( \frac{d\psi^+}{d\tau} \right)^2 + \mu^2 \psi^+_+ \right]
\]

(no summation, \( \alpha \neq \beta \in \{1, 2, 3\} \)), which can obviously be written, in terms of the dimensionless variable \( \tau \), as

\[
(a) \quad R_{\alpha \beta \alpha \beta} = \frac{\kappa_0}{6} \mu^2 \left[ \left( \frac{d\psi^+}{d\tau} \right)^2 - \psi^+_+ \right]
\]

\[
(b) \quad R_{\alpha 4 \alpha 4} = \frac{\kappa_0}{6} \mu^2 \left[ 2 \left( \frac{d\psi^+}{d\tau} \right)^2 + \psi^+_+ \right] \tag{85}
\]
Adding the two expressions, we instantly get the second result (83) and that is a very good cheek-out since, basically, the formulae (86) and (83) have been independently derived, speaking of the concretely employed methods. Hence, it is quite sufficient to refer the calculations just to the spatial sectional curvature components and, if we worked well, the two expressions (86.a) and (the first of) (83) should produce the same result. As, because of (75), we would be dealing with the three Bessel functions, \( \{ I_n(\tau) \}_{n=0,2} \), the analytical closed-form estimation of the integral involved in (83) would certainly be out of (the normal) reach. So that, we are going to consider only the first two terms, i.e.

\[
I_1(\tau) \cong \frac{\tau}{2} + \frac{\tau^3}{16},
\]

from the power-series expression of the modified Bessel function \( I_1 \). Consequently,

\[
\psi_+ = \frac{N_+}{2} \left[ 1 + \frac{\tau^2}{8} \right], \quad \frac{d\psi_+}{d\tau} = \frac{N_+}{8} \tau
\]

and therefore, the first of (83) becomes

\[
R_{\alpha\beta\alpha\beta} = -\mu^2 G_0 - \frac{\kappa_0\mu^2}{128} N_+^2 \tau^2,
\]

while (86.a) does concretely read

\[
R_{\alpha\beta\alpha\beta} = -\frac{\kappa_0\mu^2}{24} N_+^2 - \frac{\kappa_0\mu^2}{128} N_+^2 \tau^2
\]

Hence, as we have said, the integration constant \( G_0 \), in (82), is just seemingly arbitrary for it must actually equate the static contribution \( \frac{\kappa_0}{24} N_+^2 \) of the perturbation field \( \psi_+ \). Subsequently, either from (83) or straightly from (86.b), the mixed components of the perturbed curvature, close to the singular epoch \( t = 0 \), are given by

\[
R_{\alpha4\alpha4} = \frac{\kappa_0\mu^2}{24} N_+^2 + \frac{\kappa_0\mu^2}{64} N_+^2 \tau^2
\]
and, just for the sake of completeness, the metric perturbation function reads

\[ h = -\frac{\kappa_0 N^2}{144} \left[ 1 + \frac{9\tau^2}{80} \right] \tau^2 \]  

(90)

It can be concluded so far, inspecting the “early” coherently perturbed components (88), (89) of the curvature tensor, that the spontaneously inner-parity breaking generated Milne phase is clearly unstable, no matter how small the source-field perturbation is, and it primarily runs into an anti-de Sitter phase of scalar curvature \( R[0] = -\frac{\mu}{2} (\mu N^2)^2 \).

6 Higgs—anti-de Sitter Spacetime Bubbles

That is nicely closing the circle, for it brings us back to the only Einstein equation (34.a) characterizing the spacetime “supported” by the other two fixed point values of the field \( \chi \). Written with respect to the cosmological scale function (39) and introducing the notation

\[ \omega_0^2 = \frac{\kappa_0 \mu^4}{2\lambda} \quad \Leftrightarrow \quad \omega_0 = \mu \sqrt{\frac{\kappa_0}{2\lambda}}, \]  

(91)

the above equation, (34.a), becomes extremely simple

\[ \left( \frac{dS}{dt} \right)^2 = 1 - \omega_0^2 S^2 \quad \Rightarrow \quad \frac{dS}{dt} = \pm \sqrt{1 - (\omega_0 S)^2}, \]  

(92)

so that, its general solution reads, “by the book”,

\[ S(t) = \omega_0^{-1} \sin(\omega_0 t + \gamma_0), \]  

(93)

where the constant phase-factor \( \gamma_0 \) accounts for both the sign-choices (±). Actually, considering a positive scale factor “a” — with physical dimension of length — and because \( f : \mathbb{R} \to \mathbb{R} \), the scale function defined by (39) must be non-negative, reading therefore

\[ S(t) = \frac{1}{\omega_0} |\sin(\omega_0 t + \gamma_0)|, \quad \text{such that} \quad f = \ln |\sin(\omega_0 t + \gamma_0)| \]  

(94)
Hence, for the other two fixed-point values $\chi_{L,R}$ given by (32), double roots of the potential (31), we have been through quite fast with the non-linear Einstein-Gordon system (30), once we had (34.a) integrated, getting its exact solution(s) as a pair of anti-de Sitter Universes, whose metric does explicitly read (in terms of $(k = -1)$–spherical coordinates $(r, \theta, \varphi)$

$$ds^2 = \frac{1}{\omega_0^2} \sin^2(\omega_0 t + \gamma_0) \left[ \frac{(dr)^2}{1 + r^2} + r^2 d\Omega^2 \right] - (dt)^2,$$  

(95)

actually representing two harmonically oscillating $(k = -1)$–bubbles, that go through an eternal sequence of cosmic Bangs and Crunches, one of them filled up with the remnant field $\chi_L = -2\mu \sqrt{6/\lambda}$ and the other seemingly empty as the massive source-field $\chi$ vanishes everywhere inside, but not before it left an enormous amount of exotic vacuum-energy. In this respect, let us see what the numbers would be if one took $\lambda \approx 6$ and considered the smallest symmetry breaking scale, namely the one involved in the Higgs sector of the Standard Model, where (probably, for now, as the Higgs has not been experimentally detected yet) its mass, $m_H = \sqrt{2}\mu$, lies somewhere inbetween 115 and 300 GeV, i.e. (in Kilos)

$$2 \cdot 10^{-25} \, (kg) \leq m_H = \sqrt{2}\mu < 5.3 \cdot 10^{-25} \, (kg)$$  

(96)

First, with the fundamental (universal) constants $c$ and $\hbar$ plugged in, the vacuum-energy density

$$\mathcal{H}_0 = T_{44}[0] = -\frac{3\mu^4}{2\lambda},$$  

(97)

sustaining the anti-de Sitter “bubble” where the Higgs cools down to its undynamized ground state $\chi = 0$, does explicitly read

$$\mathcal{H}_0 = -\frac{3m_H^4 c^5}{8\lambda \hbar^3} = -\frac{1.246}{\lambda} \cdot 10^{45} \, (J/m^3),$$  

(98)

yielding in modulus, for $\lambda \approx 6$, the impressive values

$$|\mathcal{H}_0| \approx 2 \cdot 10^{44} \div 10^{46} \, (J/m^3)$$  

(99)
which, nevertheless, have been frequently encountered in the Domain Walls Theory. Similarly, the proper pulsation (91), measured in $s^{-1}$, is given by the formula

$$\omega_0 = \frac{cm^2_H}{h} \sqrt{\frac{\pi Gc}{\hbar \lambda}} \approx \frac{2.94 \div 20.82}{\sqrt{\lambda}} \cdot 10^9 \ (s^{-1}) \quad (100)$$

that subsequently leads to the proper frequency

$$\nu_0 = \frac{cm^2_H}{2\pi h} \sqrt{\frac{\pi Gc}{h \lambda}} \approx \frac{0.468 \div 3.314}{\sqrt{\lambda}} \ (GHz) \quad (101)$$

and to the geometrical period of the Bang-Crunch cycles in these Higgs-anti-de Sitter spacetime bubbles,

$$T = \frac{\pi}{\omega_0} \approx (0.151 \div 1.071)\sqrt{\lambda} \ (ns) \ . \quad (102)$$

For the considered $\lambda$, their respective values are

(a) $\omega_0 \approx (1.2 \div 8.5) \cdot 10^9 \ (s^{-1})$,  
(b) $\nu_0 \approx 0.19 \div 1.35 \ (GHz)$,  
(c) $T \approx 0.35 \div 2.626 \ (ns)$ \ . \quad (103)

Concerning the cosmological length-scale parameter $\omega_0^{-1}$, which is nothing else but the amplitude of the anti-de Sitter scale function oscillation, it reads (in international units)

$$\omega_0^{-1} = \frac{h}{m^2_H} \sqrt{\frac{h \lambda}{\pi Gc}} \approx \frac{1.44 \div 10.2}{\sqrt{\lambda}} \ (cm) \quad (104)$$

and is getting, as $\omega_0$ did in (103.a), the numerical values

$$\omega_0^{-1} \approx 3.53 \div 25 \ (cm) \quad (105)$$

Based on these data, we can speculate a bit, in a sort of what if...-manner, on the possible cosmological consequences of the existence, in some regions
of our Universe, of some (2+1)-dimensional windows towards the bulk-space extra-dimensions where such Higgs–anti-de Sitter (harmonically oscillating) bubbles might be living. For instance, the upper limit of the proper frequency $\nu_0$ is pretty close to the famous 21 cm(s) Hydrogen-line so that, inspecting the whole sky, if the Higgs-boson mass were around 300 GeV, there would (presumably) be some conventionally unexplained deviations from the averaged level of the electromagnetic radiation received from the known and ordinary excited baryonic astrophysical objects. Similarly, referring to the rest of the $\nu_0$–values, as the present thermalized-photons temperature is too small to significantly dynamize the Higgs-like field $\chi$ around its ground state, $\chi = 0$, one can presume that, watching for instance the Giant Voids, which are pretty much deprived of the other forms of conventional matter, there might be detected some disturbances in (or, eventually fluctuating, anisotropy of) the \{n $\nu_0$\}_{n=T,3} channels of the Cosmic Microwave Background Radiation, coming from the junction with (such) an electroweak spontaneously broken–anti-de Sitter “small” scale Universe. In some respect, the situation is very much alike the one in Chaotic Inflation – where inflating Baby Universes pop up (chaotically) from the spacetime foam – except that now we deal with harmonically oscillating ($k = -1$)–regions, geometrodynamically exactly accommodating the initial self-interacting field $\Phi$ in one of the absolute minima of its quartic potential, that pop up in an already inflated bubble, which is our own Universe. The reason why we have included the third harmonic of $\nu_0$ among the frequencies on which there might be some deviations from the black body radiation law of the Cosmic Microwave Background, lies in the manner the total energy of an anti-de Sitter three-dimensional ball depends upon time. Indeed, considering the well-known formula (for the energy of a \{t = cst.\}–compact filled in by the matter-density $\mathcal{H}_0$)

$$E = \int_{N_\Phi(t=cst.)} \sqrt{-g} \, \mathcal{H}_0 \, d^3x,$$

(106)

where $d^3x = dr \, d\theta \, d\varphi$, $\sqrt{-g} = |\frac{\sin^3(\omega_0 t)}{\omega_0^3}| \frac{\omega_0^2 \sin \theta}{\sqrt{1 + r^2}}$ (derived from (95), discarding
\( \gamma_0 \) and \( \mathcal{H}_0 \) being given by (98), it yields for the instantaneous (total) energy of the \( \{ t = \text{cst.} \} - H^3 \)-ball, of dimensionless radius \( r_0 \), the expression

\[
E(t) = - \mathcal{E} F(r_0) |\sin(\omega_0 t)|^3 , \tag{107}
\]

where the radial volume-function \( F(r_0) \) and the (physically dimensional) energy amplitude \( \mathcal{E} \) are respectively given by

\[
(a) \quad F(r_0) = = \frac{1}{2} \left[ r_0 \sqrt{1 + r_0^2} - \ln \left( r_0 + \sqrt{1 + r_0^2} \right) \right] ,
\]

\[
(b) \quad \mathcal{E} = \frac{3hc^4}{2Gm_H^2} \sqrt{\frac{h\lambda}{\pi Gc}} , \quad [\mathcal{E}] = \text{Joule} \tag{108}
\]

Since \( \sin^3 x \equiv \frac{3}{4} \sin x - \frac{1}{4} \sin(3x) \), it clearly results a 25% energy-distribution on the \( 3\nu_0 \)-channel. With (108) and (107), the mean-energy during an anti-de Sitter cycle (102),

\[
\langle E \rangle_T = \frac{1}{T} \int_0^T E(t) \, dt = - \frac{\mathcal{E} F(r_0)}{\pi} \int_0^\pi \sin^3 \gamma \, d\gamma ,
\]

i.e. — formally —

\[
\langle E \rangle_T = - \frac{4\mathcal{E}}{3\pi} F(r_0) , \tag{109}
\]

does actually read

\[
\langle E \rangle_T = - \frac{2hc^4}{\pi Gm_H^2} \sqrt{\frac{h\lambda}{\pi Gc}} F(r_0) , \tag{110}
\]

and, in terms of absolute values, it already gives an idea about the effectively involved power

\[
\mathcal{P}_\text{ef} = \frac{\langle E \rangle_T}{T} ,
\]

namely, in watts,

\[
\mathcal{P}_\text{ef} = \frac{2c^5}{\pi^2 G} F(r_0) . \tag{111}
\]

Of course, in a rigorous manner, the instantaneous power \( \mathcal{P}(t) \) comes being expressed from (107) as

\[
\mathcal{P} = \frac{dE}{dt} = - 3\omega_0 \mathcal{E} F(r_0) \sin^2(\omega_0 t) \cos(\omega_0 t) , \quad (\forall) \ t \in \left[ 0, \frac{\pi}{\omega_0} \right] ,
\]
i.e. — inserting (100) and (108.b) —

\[ P = -\frac{3c^5}{2G} \sin^2(\omega_0 t) \cos(\omega_0 t) F(r_0), \]  

so that it takes symmetric values during an anti-de Sitter cycle, being negative in its first half, when the bubble blows to its maximum size \( \omega_0^{-1} r_0 \), and respectively positive, on the second half, while the deflating bubble goes into the T-crunch. Hence, although the sooth averaged power is zero, yet one can meaningfully define the *anti-de Sitter semi cycle mean-power*, (in absolute value),

\[ \langle P \rangle_{1/2} = \frac{3c^5}{2G} \left[ \frac{2\omega_0}{\pi} \int_0^{\pi/\omega_0} \sin^2(\omega_0 t) \cos(\omega_0 t) dt \right] F(r_0), \]  

i.e.

\[ \langle P \rangle_{1/2} = \frac{c^5}{\pi G} F(r_0) (W), \]  

which is released for instance in the crunch-directed decaying phase; compared to \( P_{ef} \), it is \( \frac{\pi}{2} \)-times larger but, nevertheless, of the same order of magnitude.

At the electroweak symmetry breaking scale, that has been considered here, the *energy amplitude* “alone” already achieves *intriguing* numerical values,

\[ E \cong (0.7 \div 4.8) \cdot 10^{43} \ (J), \]  

which are — “astrophysically speaking” — of the same order (of magnitude) with the ones of a *medium size* galaxy. Hence, speculating again, envisaging the modulus of the mean-energy (110),

\[ |\langle E \rangle_T| \cong (0.3 \div 2) F(r_0) \cdot 10^{43} \ (J), \]  

it might turn out that decaying Higgs-vacuum—anti-de Sitter bubbles, no larger than few tens of \( \omega_0^{-1} \) (given by (105)), could (in principle) provide enough energy to act as *seeds* in the galaxy formation process. In what it concerns the power (114),

\[ \langle P \rangle_{1/2} \cong F(r_0) \cdot 10^{52} \ (W), \]  

34
which would be released if the bubble stopped growing again after it crunched, that might account for the one emitted by quasars, if some understandable anti-de Sitter-Higgs–electromagnetic conversion (mechanism), acting in the core of the quasar, could be figured out. Nevertheless, it should exist, since the derived power expressions (111-114) are completely independent not only of the electroweak breaking scale parameters, but also of the universal Planck constant, being therefore entirely of generally relativistic gravitational origin.

7 \( S^2 \)-Cobordism and “Wick Companions”

Finally, there are two more features (of the topic we are discussing) that we would like to address in the remaining part of the paper.

The first concerns the \( \{ r = \text{cst.}\} - (2 + 1) \)-dimensional cobordism of the anti-de Sitter sphere of coordinate-radius \( r_0 \) to a spatially flat FRW-Universe of scale function \( a(T) \). It comes about by equating the corresponding metrics,

\[
\frac{r_0^2}{\omega^2_0} \sin^2(\omega_0 t) \, d\Omega^2 - (dt)^2 = a^2(T)(dR)^2 + a^2(T)R^2 \, d\Omega^2 - (dT)^2 ,
\]

such that the first cobordering equation reads

\[
a(T)R = \frac{r_0}{\omega_0} |\sin(\omega_0 t)| ,
\]

leading therefore, to the second one

\[
(dt)^2 - a^2(T)(dR)^2 = (dT)^2 ,
\]

i.e.

\[
\left(\frac{dT}{dt}\right)^2 - a^2(T) \left(\frac{dR}{dt}\right)^2 = 1 .
\]

Extracting \( R \) from (119) and taking its derivative with respect to \( t \), then plugging the result back into (120), the latter becomes

\[
\left[ 1 - \frac{r_0^2}{\omega^2_0} \frac{\sin^2(\omega_0 t)}{a^2(T)} \left(\frac{da}{dt}\right)^2 \right] \left(\frac{dT}{dt}\right)^2 + \frac{r_0^2}{\omega_0} \frac{\sin(2\omega_0 t)}{a(T)} \frac{da}{dt} \frac{dT}{dt} - \left[ 1 + \frac{r_0^2}{\omega^2_0} \cos^2(\omega_0 t) \right] = 0
\]

(121)
and, as it can be noticed, although is just a first-order differential equation, it actually is a highly nonlinear one, especially when general forms of the \((k = 0)\)–scale function \(a(T)\) are to be considered. Moreover, because of the trigonometric functions involved in each of the three terms, (the dimensionless coordinate-radius) \(r_0\) alone is getting us in trouble, even for simple forms of \(a(T)\), particularly when it achieves large values. Hence, a closed form exact solution to the second cobordering equation (121) does not come easy.

However, some particular — but not trivial — cases can be worked out completely, even if they might be looking a bit nasty, and we are talking here about the “critical” case where

\[
\frac{1}{a} \frac{da}{dt} = \frac{\omega_0}{r_0} |\sin(\omega_0 t)|^{-1},
\]  

such that it reduces the degree of (121), regarded as an algebraic equation in \(\frac{dT}{dt}\), yielding the far much simpler equation

\[
\frac{dT}{dt} = \frac{1 + r_0^2 \cos^2(\omega_0 t)}{2r_0 \cos(\omega_0 t)},
\]

whose solution

\[
T = T_0 + \omega_0^{-1} \left[ \frac{1}{2r_0} \ln \left| \frac{1 + \tan \left(\frac{\omega_0 t}{2}\right)}{1 - \tan \left(\frac{\omega_0 t}{2}\right)} \right| + \frac{r_0}{2} \sin(\omega_0 t) \right], \quad T_0 = \text{cst.} \in \mathbb{R},
\]

gives the concrete dependence (in this case) of the \((k = 0)\)–RW universal time on the one in the anti-de Sitter bubble. Because of (123), the \((k = 0)\)–scale function equation (122) reads

\[
\frac{1}{a} \frac{da}{dt} = \left( \frac{1}{r_0^2} + \frac{1}{2} \right) \frac{d(\omega_0 t)}{\sin(2\omega_0 t)} + \frac{1}{2} \csc(2\omega_0 t) d(\omega_0 t),
\]

getting therefore the solution

\[
a(t) = a_0 \left[1 - \cos(2\omega_0 t)\right]^{\frac{1}{2}} \left|\tan(\omega_0 t)\right|^{-\frac{1}{2}} \omega_0, \quad a_0 = \text{cst.} \in \mathbb{R}_+,
\]
which, together with (124), give the complete parametric representation of
the scale function \(a(T)\). Consequently, the first cobordering equation (119)
does explicitly set the behaviour of the “true” radius \(R\) of the anti-de Sitter
sphere as it is actually seen from (within) the corresponding spatially flat
Universe (of scale function derived from) (125) and (124); that is
\[
R(t) = \frac{\omega_0^{-1} r_0}{2^{1/4} a_0} \left| \sin(\omega_0 t) \right|^{\frac{3}{4}} \left| \tan(\omega_0 t) \right|^{-\frac{1}{2r_0^2}} . \tag{126}
\]
These are non-perturbative (exact) results. Considering now a small enough
anti-de Sitter bubble, such that \(r_0^2 \ll 1\), they got major simplification since
(124) becomes
\[
2r_0 \omega_0 (T - T_0) \approx \ln \left| \frac{1 + \tan(\frac{\omega_0 t}{2})}{1 - \tan(\frac{\omega_0 t}{2})} \right| , \tag{127}
\]
so that
\[
\tan(\frac{\omega_0 t}{2}) = \tanh[r_0 \omega_0 (T - T_0)] ,
\]
which leads, after a short calculation, to the scale function expression
\[
a(T) = 2^{1/4} a_0 \left| \tanh[2r_0 \omega_0 (T - T_0)] \right|^{\frac{1}{2}} \left| \sinh[2r_0 \omega_0 (T - T_0)] \right|^{\frac{1}{2r_0^2}} \tag{128}
\]
and to the \((k = 0)\)-radial coordinate evolution law (in RW-time)
\[
R(T) = \frac{\omega_0^{-1} r_0}{2^{1/4} a_0} \left| \tanh[2r_0 \omega_0 (T - T_0)] \right|^{\frac{1}{2}} \left| \sinh[2r_0 \omega_0 (T - T_0)] \right|^{-\frac{1}{2r_0^2}} \tag{129}
\]
Late into the future, for \(T - T_0 >> (2r_0 \omega_0)^{-1}\), each of them does respectively
go as
\[
(a) \quad a(T) \approx b_0 e^{\omega_0 T} , \\
(b) \quad R(T) \approx \frac{\omega_0^{-1} r_0}{b_0} e^{-\omega_0 T} , \tag{130}
\]
with \(b_0 = a_0 / 2^{1/2} r_0\), lighting up clearly a strongly decaying Higgs-vacuum
(small scale) bubble, \(S^2\)-connected to an extremely fast inflating universe.
As a matter of fact, this beautiful picture can also be obtained as an "uncritical" exact solution of the cobordering equation (121) in the case where the physical radius $\omega_0^{-1}r_0$ of the anti-de Sitter sphere does sharply equate the inverse, $H_0^{-1}$, of the Hubble constant of a de Sitter Steady-State Universe,

$$a(T) = e^{H_0 T}, \quad H_0 > 0.$$ (131)

So, using (131) and the aforementioned coordinate-radius constraint,

$$r_0 = \frac{\omega_0}{H_0},$$ (132)

the equation (121) gets the much more tractable form

$$\left[ \cos \tau \frac{dT}{d\tau} + \frac{\omega_0}{H_0} \sin \tau \right]^2 - \left( 1 + \frac{\omega_0^2}{H_0^2} \right) = 0; \quad \tau = \omega_0 t, \quad T = \omega_0 T;$$ (133)

whose time-orientation preserving positive branch,

$$\cos \tau \frac{dT}{d\tau} + \frac{\omega_0}{H_0} \sin \tau = \sqrt{1 + \left( \frac{\omega_0}{H_0} \right)^2},$$ (134)

does immediately lead to the de Sitter–anti-de Sitter synchronization law

$$T - T_0 = \frac{\omega_0}{H_0} \ln |\cos \tau| + \sqrt{1 + \left( \frac{\omega_0}{H_0} \right)^2} \ln \left| \frac{1 + \tan(\tau/2)}{1 - \tan(\tau/2)} \right|,$$

i.e., in physically dimensional quantities,

$$T - T_0 = H_0^{-1} \left[ \ln |\cos(\omega_0 t)| + \sqrt{(\frac{H_0}{\omega_0})^2 + 1} \ln \left| \frac{1 + \tan(\omega_0 t/2)}{1 - \tan(\omega_0 t/2)} \right| \right]$$ (135)

and subsequently, through (131) and the first cobordering equation (119), to the variation law of the de Sitter-radial coordinate,

$$R(t) = H_0^{-1} |\tan(\omega_0 t)| \cdot \left[ \frac{|\cos(\omega_0 t)|}{1 + \sin(\omega_0 t)} \right]^{\sqrt{1 + (H_0/\omega_0)^2}}.$$ (136)
As it can be noticed, using (135) written as
\[
\frac{|\cos(\omega_0 t)|\sqrt{1+(H_0/\omega_0)^2-1}}{[1 + \sin(\omega_0 t)]\sqrt{1+(H_0/\omega_0)^2}} = e^{-H_0 T},
\] (137)
where we have set (for convenience) \(T_0 = 0\), the expression (136) reads
\[R = H_0^{-1}|\sin(\omega_0 t)|e^{-H_0 T}\] (138)
and asymptotically achieves the (130.b)-like behaviour,
\[R(T) \approx H_0^{-1}e^{-H_0 T},\] (139)
at late events (into the future), where, as \(T \to \infty\), \(|\sin(\omega_0 t)| \to 1\).

However, in the general case, where \(r_0^2 << 1\) is clearly invalidated, we could not find other exact solutions, in closed-form, besides the ones given above; eventually, a numerical study of the cobordering equations (121), (119) for power-like scale functions, \(a \sim T^\nu\), with \(\nu > 0\), such as the ones in the radiation dominated era, \(\nu = 1/2\), or in the one of “dusty”-matter, \(\nu = 2/3\), even supplied with an accelerating cosmological term, might be quite important for it could point out some sort of bifurcations in the \((k = 0)\)-cosmological evolution of large Higgs–anti-de Sitter spacetime bubbles. What else could be done in this respect, would be to look for the proper simultaneous embeddings of the two \(S^2\)-connected universes, so that to get a clear and “very pictural” understanding of the resulting spacetime global structure; that can further be used as the unperturbed background in similar — but seriously improved — models with conventional matter-sources and dynamical “remnant” field.

The second matter we would have liked to refer to had regarded the instanton companion (gotten by a Wick-rotation) of the Higgs–anti-de Sitter spacetime, which is precisely the “never-started — never-ending” \((k = 1)\)-de Sitter Universe,
\[
d s^2 = \frac{1}{\omega_0^2} c h^2(\omega_0 t) d l^2_{S^3} - (d t)^2, \ \omega_0 = \mu^2 \sqrt{\frac{\kappa_0}{2\lambda}},\] (140)
in the (static source-field) fixed point case, and does beautifully turn into the Linde’s Inflationary Universe with quadratic driven-source, $\mu^2 \phi^2$, where $\phi$ is a genuine inflaton, when the spontaneous $Z_2$–invariance breaking resulting field gets excited.

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References

[1] E. Bergshoeff, E. Sezgin and P.K. Townsend, Phys. Lett. B189 (1987) 75; Ann. Phys. 185 (1988) 330.

[2] E. Witten, Nucl. Phys. B443 (1995) 85; Nucl. Phys. B471 (1996) 135.

[3] P.K. Townsend, Phys. Lett. B350 (1995) 184; hep-th/9507048.

[4] J.H. Schwarz, hep-th/9510086.

[5] P. Horava and E. Witten, Nucl. Phys. B460 (1996) 506; Nucl. Phys. B475 (1996) 94.

[6] A. Lukas et al., Phys. Rev. D59 (1999) 086001.

[7] K. Benakli, Int. J. Mod. Phys. D8 (1999) 153; Phys. Lett. B447 (1999) 51.

[8] S. W. Hawking and H. S. Reall, Phys. Rev. D59 (1999) 023502.

[9] H.S. Reall, Phys. Rev. D59 (1999) 103506.
[10] J. E. Lidsey, Class. Quant. Grav. 17 (2000) L39.

[11] J. Maldacena, Adv. Theor. Phys. 2 (1998) 231; Phys. Rev. Lett. 80 (1998) 4859.

[12] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253; Phys. Rev. Lett. 81 (1998) 2862.

[13] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505; JHEP 9807 (1998) 006.

[14] L. Randall and R.S. Sundrum, Phys. Rev. Lett. 83 (1999) 3370, 4690.

[15] L. Randall and M.D. Schwartz, JHEP 0111 (2001) 003; Phys. Rev. Lett. 88 (2002) 081801.

[16] L. Randall et al., hep-th/0208120.

[17] S. Nojiri and S.D. Odintsov, Phys. Lett. B444 (1998) 92, B519 (2001) 145, B531 (2002) 143.

[18] S. W. Hawking et al., Phys. Rev. D59 (1999) 044033, 064005; JHEP 0105 (2001) 001; Phys. Rev. D62 (2000) 043501.

[19] I. Brevik and S.D. Odintsov, Phys. Lett. B475 (2000) 247.

[20] A. Karch et al., JHEP 0112 (2001) 016.

[21] R. Bousso and L. Randall, JHEP 0204 (2002) 057.

[22] J. Quiroga et al., gr-qc/0211007.

[23] E. Witten, Nucl. Phys. B500 (1997) 3, B507 (1997) 658; JHEP 9707 (1997) 003, 9802 (1998) 006.

[24] N. Arkani-Hamed et al., Phys. Lett. B429 (1998) 263; Phys. Rev. D59 (1999) 086004, D63 (2001) 064020.
[25] A. Donini and S. Rigolin, Nucl. Phys. B550 (1999) 59.

[26] J.L. Hewett, Phys. Rev. Lett. 82 (1999) 4765.

[27] K. Cheung and W. Y. Keung, Phys. Rev. D60 (1999) 112003.

[28] K. Cheung, Phys. Rev. D61 (2000) 015005; Phys. Lett. B460 (1999) 383.

[29] A.K. Gupta et al., hep-ph/9904234.

[30] S. Cullen and M. Perelstein, Phys. Rev. Lett. 83 (1999) 268.

[31] A. Sen, hep-ph/9810356; Nucl. Phys. Proc. Suppl. 94 (2001) 35.

[32] J.H. Schwarz, Phys. Rept. 315 (1999) 107; hep-th/9807195; hep-th/9812037.

[33] M. Hamuy et al., Astron. J. 109 (1995) 1, 112 (1996) 2391.

[34] S. Perlmutter et al., Astrophys. J. 483 (1997) 565; Nature 391 (1998) 51; Astrophys. J. 517 (1999) 565.

[35] P.M. Garnovich et al., Astrophy. J. 493 (1998) L53.

[36] B.P. Schmidt et al., Astrophys. J. 507 (1998) 46.

[37] R. Pain et al., Astrophys. J. 557 (2002) 120.

[38] M.S. Turner and E. Weinberg, Phys. Rev. D56 (1997) 4604.

[39] M. Gasperini, Phys. Rev. D61 (2000) 087301.

[40] A. Feinstein et al., Phys. Lett. B491 (2000) 190.

[41] M. Sloth, hep-ph/0208241.

[42] S.W. Hawking and N. Turok, Phys. Lett. B425 (1998) 25.
[43] N. Turok and S.W. Hawking, Phys. Lett. B432 (1998) 271.

[44] S.W. Hawking et al., Phys. Rev. D63 (2001) 083504.

[45] S.W. Hawking and T. Hertog, Phys. Rev. D66 (2002) 123509.

[46] A. Linde, hep-th/0205259; hep-th/0211048.

[47] R. Kallosh et al., Phys. Rev. D66 (2002) 123503.

[48] R. Kallosh and A. Linde, hep-th/0208157; hep-th/0301087.

[49] P.P. Fiziev, gr-qc/0202074.

[50] M.C. Bento et al., Phys. Rev. D66 (2002) 043507; astro-ph/0210375; astro-ph/0210468.

[51] T. Banks, hep-th/0211160.

[52] J.W. Moffat, hep-th/0211167.

[53] A. Dev et al., Phys. Lett. B548 (2002) 12.

[54] A. Dev et al., Phys. Lett. B504 (2001) 207.