Quantum critical behavior of itinerant ferromagnets

D. Belitz
Department of Physics and Materials Science Institute
University of Oregon, Eugene, OR 97403

T.R. Kirkpatrick
Institute for Physical Science and Technology, and Department of Physics
University of Maryland, College Park, MD 20742
(January 21, 2022)

The quantum ferromagnetic transition of itinerant electrons is considered. We give a pedagogical review of recent results which show that zero-temperature soft modes that are commonly neglected, invalidate the standard Landau-Ginzburg-Wilson description of this transition. If these modes are taken into account, then the resulting order parameter field theory is nonlocal in space and time. Nevertheless, for both disordered and clean systems the critical behavior has been exactly determined for spatial dimensions $d > 2$ and $d > 1$, respectively. The critical exponents characterizing the paramagnetic-to-ferromagnetic transition are dimensionality dependent, and substantially different from both mean-field critical exponents, and from the classical Heisenberg exponents that characterize the transition at finite temperatures. Our results should be easily observable, particularly those for the disordered case, and experiments to check our predictions are proposed.

I. INTRODUCTION

Phase transitions that occur in a quantum mechanical system at zero temperature ($T = 0$) as a function of some non-thermal control parameter are called quantum phase transitions. In contrast to their finite-temperature counterparts, which are often referred to as thermal or classical phase transitions, the critical fluctuations one has to deal with at zero temperature are quantum fluctuations rather than thermal ones, and the need for a quantum mechanical treatment of the relevant statistical mechanics makes the theoretical description of quantum phase transitions somewhat different from that of classical ones. However, as Hertz has shown in a seminal paper, the basic theoretical concepts that have led to successfully describe and understand thermal transitions work in the quantum case as well.

Experimentally, the zero-temperature behavior of any material can of course not be studied directly, and furthermore the most obvious control parameter that drives a system through a quantum transition is often some microscopic coupling strength that is hard to change experimentally. As a result, the dimensionless distance from the critical point, $t$, which for classical transitions with a transition temperature $T_c$ is given by $t = T/T_c - 1$ and is easy to tune with high accuracy, is much harder to control in the quantum case. However, $t$ is usually dependent on some quantity that can be experimentally controlled, like e.g. the composition of the material. Also, the zero temperature critical behavior manifests itself already at low but finite temperatures. Indeed, in a system with a very low thermal transition temperature all but the final asymptotic behavior in the critical region is dominated by quantum effects. The study of quantum phase transitions is therefore far from being of theoretical interest only.

Perhaps the most obvious example of a quantum phase transition is the paramagnet-to-ferromagnet transition of itinerant electrons at $T = 0$ as a function of the exchange interaction between the electronic spins. Early theoretical work on this transition suggested that the critical behavior in the physical dimensions $d = 2$ and $d = 3$ was not dominated by fluctuations, and mean-field like, as is the thermal ferromagnetic transition in dimensions $d > 4$.

The reason for this is a fundamental feature of quantum statistical mechanics, namely the fact that statics and dynamics are coupled. As a result, a quantum mechanical system in $d$ dimensions is very similar so the corresponding classical system in $d + z$ dimensions, where the so-called dynamical critical exponent $z$ can be thought of as an extra dimensionality that is provided to the system by time or temperature. The $d + z$-dimensional space relevant for the statistical mechanics of the quantum system bears some resemblance to $d + 1$-dimensional Minkowski space, but $z$ does not need to be equal to 1 in nonrelativistic systems. For clean and disordered itinerant quantum ferromagnets, one finds $z = 3$ and $z = 4$, respectively, in mean-field theory. This appears to reduce the upper critical dimension $d^c_z$, above which fluctuations are unimportant and simple mean-field theory yields the correct critical behavior, from $d^c_z = 4$ in the classical case to $d^c_z = 1$ and $d^c_z = 0$, respectively, in the clean and disordered quantum cases. If this were true, then this quantum phase transition would be rather uninteresting from a critical phenomena point of view.

It has been known for some time that, for the case of
disordered systems, this conclusion cannot be correct. It is known that in any system with quenched disorder that undergoes a phase transition, the critical exponent $\nu$ that describes the divergence of the correlation length, $\xi \sim t^{-\nu}$ for $t \to 0$, must satisfy the inequality $\nu \geq 2/d$. However, mean-field theory yields $\nu = 1/2$, which is incompatible with this inequality for $d < 4$. Technically, this implies that the disorder must be a relevant perturbation with respect to the mean-field fixed point. The mean-field fixed point must therefore be unstable, and the phase transition must be governed by some other fixed point that has a correlation length exponent $\nu \geq 2/d$.

Recently such a non-mean field like fixed point has been discovered, and the critical behavior has been determined exactly for all dimensions $d > 2$. It was found that both the value $d^c = 0$ for the upper critical dimension, and the prediction of mean-field critical behavior for $d > d^c$ were incorrect. Instead, $d^c = 2$, and while both the quantum fluctuations and the disorder fluctuations are irrelevant with respect to the new fixed point for all $d > d^c$, there are two other “upper critical dimensionalities”, $d^{c+} = 4$ and $d^{c+} = 6$. The critical behavior for $d^c < d < d^{c+}$ is governed by a non-standard Gaussian fixed point with non-mean field like exponents, and only for $d > d^{c+}$ does one obtain mean-field exponents. In addition, the clarification of the physics behind this surprising behavior has led to the conclusion that very similar effects occur in clean systems. In that case, $d^c = 1$ in agreement with the early result, but again the critical behavior is nontrivial in a range of dimensionalities $d^c < d < d^{c+} = 3$, and only for $d > d^{c+}$ does one obtain mean-field critical behavior. In addition, we have found that Hertz’s $1 - \epsilon$ expansion for the clean case is invalid. This explains an inconsistency between this expansion and an exact exponent relation that was noted earlier by Sachdev. In order to keep our discussion focused, in what follows we will restrict ourselves to the disordered case, where the effects are more pronounced, and will only quote results for the clean case where appropriate.

The basic physical reason behind the complicated behavior above the upper critical dimensionality $d^{c+}$, i.e. in a regime in parameter space where the critical behavior is not dominated by fluctuations, is simple. According to our general understanding of continuous phase transitions or critical points, in order to understand the critical singularities at any such transition, one must identify all of the slow or soft modes near the critical point, and one must make sure that all of these soft modes are properly included in the effective theory for the phase transition. This is obvious, since critical phenomena are effects that occur on very large length and time scales, and hence soft modes, whose excitation energies vanish in the limit of long wavelengths and small frequencies, will in general influence the critical behavior. In previous work on the ferromagnetic transition it was implicitly assumed that the only relevant soft modes are the fluctuations of the order parameter, i.e. the magnetization. For finite temperatures this is correct. However, at $T = 0$ there are additional soft modes in a disordered electron system, namely diffusive particle-hole excitations that are distinct from the spin density excitations that determine the magnetization. In many-body perturbation theory these modes manifest themselves as products of retarded and advanced Green’s functions, and in field theory they can be interpreted as the Goldstone modes that result from the spontaneous breaking of the symmetry between retarded and advanced correlation functions, or between positive and negative imaginary frequencies. In a different context, namely the transport theory for disordered electron systems, these diffusive excitations are sometimes referred to as ‘diffusons’ and ‘Cooperons’, respectively, and they are responsible for what is known as ‘weak localization effects’ in disordered electron systems. For our purposes, their most important feature is their spatial long-range nature in the zero frequency limit. This long-range nature follows immediately from the diffusion equation

$$\left(\partial_t - D \partial^2_{\mathbf{q}}\right) f(\mathbf{x}, t) = 0 \quad ,$$

(1.1a)

for some diffusive quantity $f$, with $D$ the diffusion constant. Solving this equation by means of a Fourier-Laplace transform to wavevectors $\mathbf{q}$ and complex frequencies $\omega$, one obtains in the limit of zero frequency,

$$f(\mathbf{q}, \omega = 0) = \frac{1}{D |\mathbf{q}|^2} f(\mathbf{q}, t = 0) \quad .$$

(1.1b)

Long-range static correlations are thus an immediate consequence of the diffusive nature of the density dynamics in disordered systems.

The fact that we are concerned with the zero frequency or long-time limit is due to the order parameter, i.e. the magnetization, being a conserved quantity. Since the only way to locally change the order parameter density is to transport this conserved quantity from one region in space to another, in order to develop long-range order over arbitrarily large distances the systems needs an infinitely long time. This in turn means that criticality can be reached only if the frequency is taken to zero before the wavenumber. This feature would be lost if there were some type of spin-flip scattering mechanism present, and our results hold only in the absence of such processes. For the same reason, they do not apply to quantum antiferromagnets, which show a quite different behavior.

It is important that the long-range static correlations mentioned above are distinct from the order parameter fluctuations. For instance, the latter are soft only at the critical point and in the ordered phase, while the former are soft even in the paramagnetic phase, and they do not change their nature at the critical point. However, since they couple to the conserved order parameter, they influence the critical behavior. If one integrates out these diffusive modes in order to obtain an effective theory or Landau-Ginzburg-Wilson (LGW) functional in terms of the order parameter only, then their long-range nature
leads to infrared singular integrals, which in turn results in singular vertices in the LGW functional, or diverging coupling constants for couplings between the order parameter fluctuations. The usual LGW philosophy of deriving an effective local field theory entirely in terms of the order parameter field does not lead to a well behaved field theory in this case. The situation is analogous to a well known phenomenon in high energy physics: Suppose some interaction between, say, fermions, is mediated by the exchange of some other particles, e.g. gauge bosons of mass $M$. If the bosons are integrated out, then the resulting theory will be nonrenormalizable, i.e. it will be ill-behaved on momentum scales larger than the mass $M$. The nonrenormalizable theory corresponds to the order parameter LGW theory, except that in statistical mechanics one runs into infrared problems rather ultraviolet ones. Nevertheless, it turns out that in our case the critical behavior can still be determined exactly even after having integrated out the additional soft modes. The point is that the diffusive modes lead to an effective long-range interaction between the order parameter fluctuations that falls off in real space like $r^{2-d}$. It is known that in general long-range interactions suppress fluctuation effects. In our case they are strong enough to not only suppress quantum fluctuations, but also any remaining disorder fluctuations. The critical behavior is thus neither dominated by quantum fluctuations (since we work above the upper critical dimension $d_\text{c}^+$), nor by the disorder fluctuations, but rather is given by a simple, though non-standard (because of the long-range interactions) Gaussian theory. The resulting Gaussian fixed point allows for a correlation length exponent that satisfies $\nu \geq 2/d$ as required, and the exponents are dimensionality dependent for all $d < 6$. In $d = 3$ they are substantially different from either the mean-field exponents, or from those for a classical Heisenberg ferromagnet. This has striking observables consequences, as we will discuss.

The outline of this paper is as follows. In Sec. I we first discuss some general aspects of itinerant ferromagnets, and then we give our results for the critical exponents and for the equation of state near the critical point. Since the purpose of this paper is to give an exposition and discussion of these results that is as nontechnical as possible, they will be presented without any derivations. In Sec. II we discuss these results as well as several possible experiments that could be performed to test our predictions. Finally, in Sec. III we sketch the derivation of our theoretical results.

II. RESULTS

In order to put the phase transition we are going to consider in perspective, let us first discuss the qualitative phase diagram that one expects for a disordered itinerant electron system in $d = 3$. Let $F_0^a < 0$ be the Fermi liquid parameter that characterizes the strength of the system’s tendency towards ferromagnetism: For $|F_0^a| < 1$ the system is paramagnetic with a spin susceptibility $\chi_\lambda \sim 1/(1 + F_0^a)$, while for $|F_0^a| > 1$ the clean Fermi liquid has a ferromagnetic ground state. In Fig. 1 we show the qualitative phase diagram one expects for a disordered system at $T = 0$ in the $F_0^a-\lambda$ plane, where $\lambda$ is some dimensionless measure of the disorder. For $\lambda = 0$, we have the transition from a paramagnetic metal (PM) to a ferromagnetic metal (FM) at $F_0^a = -1$. At small but nonzero $\lambda$ this transition will occur at somewhat smaller values of $|F_0^a|$, since the disorder effectively increases the spin triplet electron-electron interaction amplitude, and hence $|F_0^a|$. This accounts for the downward curvature of the PM-FM transition line. At $|F_0^a| = 0$, a metal-insulator transition of Anderson type is known to occur at a critical disorder value $\lambda_c$. At nonzero $|F_0^a|$ such a transition from a paramagnetic metal to a paramagnetic insulator (PI) still occurs, albeit it now is what is called an Anderson-Mott transition that takes place at a somewhat larger value of the disorder. The two transition lines will meet at a multicritical point $\lambda_c$, and for large values of $\lambda$ and $|F_0^a|$ one expects a ferromagnetic insulator (FI). The transitions from the FM and PI phases, respectively, to the FI phase have not been studied theoretically, which is why we denote them by dashed lines in the figure. We will be mostly interested in the phase transition that occurs across the PM-FM transition line at finite disorder, but far away from the metal-insulator transition. However, in Sec. III below we will come back to the remaining regions in this phase diagram.

In Fig. 2 we show the same phase diagram in the $F_0^a-T$ plane for some value of the disorder $0 < \lambda \ll \lambda_c$. With increasing temperature $T$, the critical value of $|F_0^a|$ increases, since in order to achieve long-range order, a

![FIG. 1. Schematic phase diagram for a 3-d disordered itinerant electron system in the plane spanned by the Landau parameter $F_0^a$ and the disorder $\lambda$ at $T = 0$. See the text for further explanations.](image-url)
larger $|F_0^a|$ is needed to compensate for the disordering effect of the thermal fluctuations. The inset shows schematically the boundary of the critical region (dashed line) and the crossover line (dotted line) that separates classical critical behavior (cc) from quantum critical behavior (qc).

Our theoretical results for the zero temperature paramagnet-to-ferromagnet transition can be summarized as follows. Let $t$ be the dimensionless distance from the line separating the regions PM and FM in Fig. 1. Then the equation of state, which determines the magnetization $m$ as a function of $t$ and the magnetic field $h$, can be written

$$tm + m^{d/2} + m^3 = h,$$

where we have left out all prefactors of the various terms. Equation (2.1) is valid for all dimensions $d > 2$. Notice the term $m^{d/2}$, which occurs in addition to what otherwise is an ordinary mean-field equation of state. It is a manifestation of the soft particle-hole excitations mentioned in the Introduction. For $d < 6$ it dominates the $m^3$-term, and hence we have for the exponent $\beta$, which determines the vanishing of the zero-field magnetization via $m(t, h = 0) \sim t^\beta$,

$$\beta = \begin{cases} 2/(d - 2) & \text{for } 2 < d < 6 \\ 1/2 & \text{for } d > 6 \end{cases}.$$  

Similarly, the exponent $\delta$, defined by $m(t = 0, h) \sim h^{1/\delta}$, is obtained as

$$\delta = \begin{cases} d/2 & \text{for } 2 < d < 6 \\ 3 & \text{for } d > 6 \end{cases}.$$  

Now let us consider the order parameter field $M(x, t)$ as a function of space and time, i.e. the field whose average yields the magnetization, $\langle M(x, t) \rangle = m$. Here the angular brackets $\langle \rangle$ denote a trace with the full statistical operator, i.e. they include a quantum mechanical expectation value, a disorder average, and at nonzero temperature also a thermal average. We first consider the case of $T = 0$, and Fourier transform to wave vectors $\mathbf{q}$ (with modulus $q = |\mathbf{q}|$) and frequencies $\omega$. For the order parameter correlation function $G(q, \omega) = \langle M(\mathbf{q}, \omega) M(-\mathbf{q}, -\omega) \rangle$ we find in the limit of small $q$ and $\omega$,

$$G(q, \omega) = \frac{1}{t + q^{d-2} + q^2 - i\omega/q^2}.$$  

Here we have again omitted all prefactors of the terms in the denominator, since they are of no relevance for our discussion. The most interesting feature in Eq. (2.3) is the term $q^{d-2}$. It is again an immediate consequence of the additional soft modes discussed in the first section, and Eq. (2.3), like Eq. (2.1), is valid for $d > 2$. For $q = \omega = 0$, the correlation function $G$ determines the magnetic susceptibility $\chi_m \sim G(q = 0, \omega = 0)$ in zero magnetic field. Hence we have $\chi_m(t) \sim t^{-1} \sim t^{-\gamma}$, where the last relation defines the critical exponent $\gamma$. This yields

$$\gamma = 1,$$  

which is valid for all $d > 2$. $\gamma$ thus has its usual mean-field value. However, for nonzero $q$ the anomalous $q^{d-2}$ term dominates the usual $q^2$ dependence for all $d < 4$. The correlation function at zero frequency can then be written

$$G(q, \omega = 0) \sim \frac{1}{1 + (q\xi)^{d-2}},$$  

with the correlation length $\xi \sim t^{1/(d-2)} \sim t^{-\nu}$. For $d > 4$ the $q^2$ term is dominant, and we have for the correlation length exponent $\nu$,

$$\nu = \begin{cases} 1/(d - 2) & \text{for } 2 < d < 4 \\ 1/2 & \text{for } d > 4 \end{cases}.$$  

Note that $\nu \geq 2/d$, as it must be according to the discussion in the introduction. The wavenumber dependence of $G$ at criticality, i.e. at $t = 0$, is characterized by the exponent $\eta$: $G(q, \omega = 0) \sim q^{-2+\eta}$. From Eq. (2.3) we obtain,

$$\eta = \begin{cases} 4 - d & \text{for } 2 < d < 4 \\ 0 & \text{for } d > 4 \end{cases}.$$  

Finally, consider the correlation function at a wavenumber such that $q\xi = 1$. Then it can be written

\[ G(q, \omega) = \frac{1}{t + q^{d-2} + q^2 - i\omega/q^2}. \]
\( G(q = \xi^{-1}, \omega) \sim \frac{1}{1 - i \omega \tau} \),

(2.7a)

with the relaxation or correlation time \( \tau \sim \xi^2/t \sim \xi^{2+1/\nu} \sim \xi^2 \), where the last relation defines the dynamical critical exponent \( z \). From Eq. (2.5) we thus obtain,

\[
\begin{align*}
    z &= \{ 
        \begin{array}{ll}
            d & \text{for } 2 < d < 4 \\
            4 & \text{for } d > 4
        \end{array}
    \}
\end{align*}
\]

(2.7b)

Notice that with increasing dimensionality \( d \), the exponents \( \nu \), \( \eta \), and \( z \) "lock into" their mean-field values at \( d = d_{c}^{++} = 4 \), while \( \beta \) and \( \delta \) do so only at \( d = d_{\ast}^{++} = 6 \). In the special dimensions \( d = 4 \) and \( d = 6 \) the power-law scaling behavior quoted above holds only up to additional multiplicative logarithmic dependences on the variables \( t \), \( h \), and \( T \). Since these corrections to scaling occur only in unphysical dimensions they are of academic interest only, and we refer the interested reader to Refs. 4 for details.

The results for the clean case are qualitatively similar, but the anomalous term in the equation of state, Eq. (2.3), is \( m^d \) instead of \( m^{d/2} \). This is because the additional soft modes in that case are ballistic instead of diffusive, so their frequency scales with wavenumber like \( \omega \sim q \) rather than \( \omega \sim q^{2} \). As a result, the two special dimensions \( d_{c}^{++} \) and \( d_{\ast}^{++} \) coincide, and are now \( d_{c}^{++} = 3 \), while the upper critical dimension proper, above which fluctuations are irrelevant, is \( d_{c}^{\ast} = 1 \). For \( 1 < d < 3 \), the exponent values are \( \beta = \nu = 1/(d - 2) \), \( \delta = z = d \), \( \eta = 3 - d \), and \( \gamma = 1 \). For \( d > 3 \), all exponents take on their mean-field values as they do in the disordered case for \( d > 3 \), and in \( d = 3 \) there are logarithmic corrections to power-law scaling.

We now turn to the behavior at nonzero temperatures. Then the equation of state acquires temperature corrections, and it is helpful to distinguish between the cases \( m >> T \) and \( m << T \), with \( m \) and \( T \) measured in suitable units. Taking into account the leading corrections in either limit, the equation of state reads

\[
\begin{align*}
    tm + m^{d/2} \left( 1 + T/m \right) &= h \quad \text{(for } m >> T \text{)} \quad ,
    \\
    \left( t + T^{(d-2)/2} \right) m + m^{3} &= h \quad \text{(for } T >> m \text{)}
\end{align*}
\]

(2.8)

Equation (2.8) shows that for any nonzero temperature the asymptotic critical behavior is not given by the quantum critical exponents. Since Eq. (2.8) takes temperature into account only perturbatively, it correctly describes only the initial deviation from the quantum critical behavior, and approximates the classical critical behavior by the mean-field result. A full crossover calculation would yield instead the classical Heisenberg critical behavior in the asymptotic limit. Also, we are considering only the saddle point contribution to the magnetization. For models with no additional soft modes it has been shown that fluctuations that act as dangerous irrelevant variables introduce another temperature scale that dominates the one obtained from the saddle point.

In the present case, however, fluctuations are suppressed by the long-range nature of the effective field theory, and the fluctuation temperature scale is subdominant. The behavior described by Eq. (2.3) can be summarized by means of a generalized homogeneity law,

\[
m(t,H) = b^{-\beta/\nu} m(t^{1/\nu}, T^{\phi/\nu}, H^{b/\nu}) \quad .
\]

(2.9a)

Here \( \beta \), \( \nu \), and \( \delta \) have the values given above, and \( b \) is an arbitrary scale factor.

\[
\phi = 2\nu
\]

(2.9b)

is the crossover exponent that describes the deviation from the quantum critical behavior due to the relevant perturbation provided by the nonzero temperature. The entry \( T^{\phi/\nu} = T^{2} \) in the scaling function in Eq. (2.9a) reflects the fact that the temperature dependence of the saddle point solution is determined by that of the diffusive modes, i.e. frequency or temperature scales like \( T \sim q^{2} \sim b^{-2} \). The critical temperature scale, \( T \sim b^{-2} \), would be dominant if it were present, but since the leading behavior of the magnetization is not determined by critical fluctuations, it is suppressed.

By differentiating Eq. (2.9a) with respect to the magnetic field \( h \), one obtains an analogous homogeneity law for the magnetic susceptibility, \( \chi_{m} \),

\[
\chi_{m}(t,H) = b^{\gamma/\nu} \chi_{m}(t^{1/\nu}, T^{\phi/\nu}, H^{b/\nu}) \quad ,
\]

(2.10a)

with

\[
\gamma = \beta(\delta - 1) = 1
\]

(2.10b)

in agreement with Eq. (2.4). This result is in agreement with a more direct calculation of \( \chi_{m} \) using a more direct calculation of \( \chi_{m} \): The same temperature corrections that modify the equation of state, Eq. (2.3), lead to a replacement of the term \( q^{d-2} \) in the denominator of Eq. (2.3) by \( q^{2} + T^{(d-2)/2} \). Since the homogeneous order parameter correlation function determines the spin or order parameter susceptibility, this yields

\[
\chi_{m}(t,H) = \frac{1}{t + T^{1/2\nu}}
\]

(2.10c)

in agreement with Eqs. (2.10a, 2.10b).

Finally, the critical behavior of the specific heat \( c_{V} \) has been calculated. It is most convenient to discuss the specific heat coefficient, \( \gamma_{V} = \lim_{T \rightarrow 0} c_{V}/T \), which in a Fermi liquid would simply be a constant. Its behavior at criticality, \( t = 0 \), is adequately represented by the integral

\[
\gamma_{V} = \int_{0}^{\Lambda} dq \frac{q^{d-1}}{T + q^{2} + q^{4} + h^{4} - 1/3q^{2}}
\]

(2.11a)

Remarkably, in zero magnetic field, \( \gamma_{V} \) diverges logarithmically as \( T \rightarrow 0 \) for all dimensions \( 2 < d < 4 \). This can be shown to be a consequence of the dynamical exponent \( z \) being exactly equal to the spatial dimensionality.
leading scaling behavior of homogeneity law with a logarithmic correction for the leading scaling behavior of $\gamma_V$,

$$
\gamma_V(t, T, H) = \Theta(4 - d) \ln b + F_{\gamma}(t b^{1/\nu}, T b^{\nu}, H b^{\nu \delta/\nu}) \ .
$$

(2.11b)

Here $\Theta(x)$ denotes the step function, and $F_{\gamma}$ is an unknown scaling function. Note that $\gamma_V$ is determined by Gaussian fluctuations and depends on the critical temperature scale, i.e. $T$ scales like $t^\nu$ in Eq. (2.11b). This is the leading temperature scale, and whenever it is present it dominates the diffusive temperature scale that shows dependence of $d$ in that range of dimensionalities. If one restores the dependence of $\gamma_V$ on $t$, then one obtains a generalized homogeneity law with a logarithmic correction for the leading scaling behavior of $\gamma_V$,

$$
\gamma_V(t, T, H) = \Theta(4 - d) \ln b + F_{\gamma}(t b^{1/\nu}, T b^{\nu}, H b^{\nu \delta/\nu}) \ .
$$

(2.11b)

III. EXPERIMENTAL IMPLICATIONS, AND DISCUSSION

A. Experimental Implications

Let us now discuss the experimental implications of the results presented in the preceding section. Obviously, one needs a material that shows a transition from a paramagnetic state to a ferromagnetic one at zero temperature as a function of some experimentally tunable parameter $x$. Obvious candidates are magnetic alloys of the stoichiometry $P_xF_{1-x}$, with $P$ a paramagnetic metal and $F$ a ferromagnetic one. Such materials show the desired transition as a function of the composition parameter $x$; examples include Ni for the ferromagnetic component, and Al or Ga for the paramagnetic one. At the critical concentration $x_c$, they also are substantially disordered, but due to the fact that both constituents are metals they are far from any metal-insulator transition. Our theory should therefore be applicable to these systems. The schematic phase diagram at $T = 0$ in the $T$-$x$ plane is shown in Fig. 3. Notice that this is a realistic phase diagram, as opposed to the ‘theoretical’ one in Figs. 2 and 3. A change of the composition parameter $x$ leads, besides a change of $F_0^+$, to many other changes in the microscopic parameters of the system. As $x$ is varied, the system will therefore move on a complicated path in the diagram shown in, say, Fig. 3. However, since the critical behavior near the transition is universal, it is independent of the exact path traveled.

One possible experiment would consist in driving the system at a low, fixed temperature through the transition by changing the composition $x$. While this involves the preparation of many samples, this way of probing a quantum phase transition has been used to observe the metal-insulator transition in P doped Si. It might also be possible to use the stress tuning technique that has been used for the same purpose. Either way one will cross the transition line along a more or less vertical path in Fig. 3, and for a sufficiently low temperature this path will go through both the classical and the quantum critical region indicated in the inset in Fig. 3. Due to the large difference between the quantum critical exponents quoted in Sec. II and the corresponding exponents for classical Heisenberg magnets, the resulting crossover should be very pronounced and easily observable. For instance, for 3-d systems our Eq. (2.2a) predicts $\beta = 2$, while the value for the thermal transition is $\beta_{class} \approx 0.37$. The resulting crossover in the critical behavior of the magnetization is schematically shown in Fig. 4. Alternatively, one could prepare a sample with a value of $x$ that is as close as possible to $x_c$, and measure the magnetic field dependence of the magnetization, extrapolated to $T = 0$, to obtain the exponent $\delta$. Again, there is a large difference between our prediction of $\delta = 1.5$ in $d = 3$, and the classical value $\delta_{class} \approx 4.86$.

Yet another possibility is to measure the zero-field magnetic susceptibility as a function of both $t = |x - x_c|$ and $T$. Equation (2.10a) predicts

$$
\chi_m(t, T) = T^{-1/2} f_\chi(T/t^2) \ .
$$

(3.1)

Here $f_\chi$ is a scaling function that has two branches, $f_\chi^+$ for $x > x_c$, and $f_\chi^-$ for $x < x_c$. Both branches approach a constant for large values of their argument, $f_\chi^+(y \to \infty) = \text{const}$. For small arguments, we have $f_\chi^+(y \to 0) \sim \sqrt{y}$, while $f_\chi^-$ diverges at a nonzero value $y^*$ of its argument that signalizes the classical transition, $f_\chi^-(y \to y^*) \sim (y - y^*)^{-\gamma_{class}}$, with $\gamma_{class} \approx 1.39$ the susceptibility exponent for the classical transition.
FIG. 4. Schematic critical behavior of the magnetization \( m \) at nonzero temperature, showing the crossover from the quantum critical behavior \((\beta = 2, \text{dashed line})\) to the classical critical behavior \((\beta \approx 0.37, \text{dotted line})\). Notice that the actual transition is classical in nature.

The prediction is then that a plot of \( \chi m T^{1/2} \) versus \( T/t_0 \) will yield a universal function the shape of which is schematically shown in Fig. 5. Notice that the exponents are known exactly, so the only adjustable parameter for plotting experimental data will be the position of the critical point. This is on sharp contrast to some other quantum phase transitions, especially metal-insulator transitions, where the exponent values are not even approximately known, which makes scaling plots almost meaningless.

Finally, one can consider the low-temperature behavior of the specific heat. According to Eq. (2.11b), as the temperature is lowered for \( x > x_c \) the leading temperature dependence of the specific heat will be

\[
c_V(T) \sim T \ln T.
\]  

(3.2a)

At criticality this behavior will continue to \( T = 0 \), while for \( x > x_c \) it will cross over to

\[
c_V(T) \sim (\ln t) T^{-\alpha}.
\]  

(3.2b)

For \( x \lesssim x_c \) one will encounter the classical Heisenberg transition where the specific heat shows a finite cusp (i.e., the exponent \( \alpha \), defined by \( c_V \sim (T-T_c)^{-\alpha} \), is negative).

B. Theoretical Discussion

There are also various theoretical implications of the results presented in Sec. I. One aspect is the general message that the usual LGW philosophy must not be applied uncritically to quantum phase transitions, because of the large number of soft modes that exist at zero temperature in a generic system. If any of these couple to the order parameter, then an effective theory entirely in terms of the order parameter will not be well behaved. In the present case we have actually been able to use this to our advantage, since the long-ranged interaction that the additional soft modes induce in the order parameter theory suppress the disorder fluctuations, which is the reason for the remarkable fact that we are able to exactly determine the critical behavior of a three-dimensional, disordered system. In general, however, the presence of soft modes in addition to the order parameter fluctuations will call for the derivation of a more complete low-energy effective theory that keeps all of the soft modes explicitly.

Another very interesting aspect is a connection between our results on the ferromagnetic transition, and a substantial body of literature on a problem that appears in the theory of the metal-insulator transition in interacting disordered electron systems, i.e. the transition from PM to PI in Fig. 1. This problem has been known ever since the metal-insulator transition of interacting disordered electrons was first considered, and it has led to substantial confusion in that field. Early work on the metal-insulator transition showed that in two-dimensional systems without impurity spin-flip scattering, the spin-triplet interaction amplitude scaled to large values under renormalization group iterations. This problem was interpreted, incorrectly as it turned out later, as a signature of local moment formation in all dimensions. Subsequently, the present authors studied this problem in some detail. We were able to explicitly resum the perturbation theory and show that at a critical value of the interaction strength, or of the disorder, there is a bulk, thermodynamic phase transition in \( d > 2 \) that is not the metal-insulator transition. While this ruled out local moments (which would not
lead to to phase transition), the physical meaning of this transition was obscure at the time since no order parameter had been identified, and its description was entirely in terms of soft diffusion modes. However, the critical exponents obtained are identical to those given in Sec. 1 for the quantum ferromagnetic phase transition, and in both cases logarithmic corrections to scaling are found. Because the exponents in the two cases are identical, we conclude that the transition found earlier by us, whose physical nature was unclear, is actually the ferromagnetic transition. One also concludes that our speculations in Ref. 19 about the nature of the ordered phase as an ‘incompletely frozen spin phase’ with no long-range magnetic order, were not correct; this phase is actually the metallic ferromagnetic phase. On the other hand, the techniques used in that reference allowed for a determination of the qualitative phase diagram as a function of dimensionality, which our present analysis is not capable of. This analysis showed the existence of yet another interesting dimensionality above \( d = 2 \), which we denote by \( d^* \). With the appropriate reinterpretation of the ‘incompletely frozen spin phase’ as the ferromagnetic phase, the qualitative phase diagram for \( 2 < d < d^* \) is shown in Fig. 6. Compared to Fig. 5, the following happens as \( d \) is lowered from \( d = 3 \): The multicritical point \( M \) moves to downward, and at \( d = d^* \) it reaches the \( \lambda \)-axis. \( d^* \) was estimated in Ref. 19 to be approximately \( d^* = 2.03 \). For \( d < d^* \), the insulator phase can therefore not be reached directly from the paramagnetic metal. This explains why in the perturbative renormalization group calculations in \( d = 2 + \epsilon \) one necessarily encounters the ferromagnetic transition first, and it should finally put to rest the long discussion about the physical meaning of the runaway flow that is encountered in these theories. It also shows that none of these theories are suitable for studying the metal-insulator transition in the absence of spin-flip mechanisms, as they start out in the wrong phase.

It should also be pointed out that our earlier theory depended crucially on there being electronic spin conservation. This feature would be lost if there were some type of impurity spin flip scattering process. In that case, the soft modes that lead to the long-range order parameter interactions acquire a mass or energy gap, and at sufficiently large scales the interactions are effectively of short range. The asymptotic critical phenomena in this case are described by a short-range, local order parameter field theory with a random mass, or temperature, term. Such a term is present in the case of a conserved order parameter also, but due to the long ranged interaction it turns out to be irrelevant with respect to the nontrivial Gaussian fixed point. In the absence of the conservation law, however, the random mass term is relevant with respect to the Gaussian fixed point analogous to the one discussed here. This underscores the important role that is played by the order parameter being conserved in our model. The quantum phase transition in a model where it is not has been discussed in Ref. 8.

We finally discuss why some of our results are in disagreement with Sachdev's general scaling analysis of quantum phase transitions with conserved order parameters. For instance, it follows from our Eqs. 2.11 and 2.11b that the Wilson ratio, defined as \( W = (m/H)/(C_v/T) \), diverges at criticality rather than being a universal number as predicted in Ref. 8. Also, for \( 2 < d < 4 \) the function \( F_\gamma \) in Eq. (2.11b), for \( t = 0 \) and neglecting corrections to scaling, is a function of \( T/H \), in agreement with Ref. 8 but for \( d > 4 \) this is not the case. The reason for this breakdown of general scaling is that we work above an upper critical dimensionality, and hence dangerous irrelevant variables appear that prevent a straightforward application of the results of Ref. 8 to the present problem. These dangerous irrelevant variables have to be considered very carefully, and on a case by case basis. This caveat is particularly relevant for quantum phase transitions since they tend to have a low upper critical dimension. It is well known that a given irrelevant variable can be dangerous with respect to some observables but not with respect to others. Specifically, in our case there is a dangerous irrelevant variable that affects the leading scaling behavior of the magnetization, but not that of the specific heat coefficient, and this leads to the divergence of the Wilson ratio. This dangerous irrelevant variable is also the reason why the exponents \( \beta \) and \( \delta \), which describe the critical behavior of the magnetization, remain dimensionality dependent up to \( d = 6 \), while all other exponents ‘lock into’ their mean-field values already at \( d = 4 \).
IV. THEORETICAL OUTLINE

Here we sketch the derivation of the results that were presented in Sec. II. We do so for completeness only, and will be very brief. A detailed account of the derivation can be found in Ref. 3 for the disordered case, and in Ref. 2 for the clean case.

Hertz has shown how to derive an LGW functional for a quantum ferromagnet. One starts by separating the spin-triplet part of the electron-electron interaction, i.e., the interaction between spin density fluctuations, from the rest of the action, writing

\[ S = S_0 + S_{\text{int}}^{(t)} \]

with

\[ S_{\text{int}}^{(t)} = \frac{\Gamma_t}{2} \int dx \mathbf{n}_s(x) \cdot \mathbf{n}_s(x) \].

Here \( S_{\text{int}}^{(t)} \) is the spin-triplet interaction part of the action, and \( S_0 \) contains all other parts, in particular the electron-electron interaction in all other channels. \( \Gamma_t \) is the spin triplet interaction amplitude, which is related to the Landau parameter \( F_0^a \) used above by \( \Gamma_t = -F_0^a/(1 + F_0^a) \). \( \mathbf{n}_s(x) \) is the electron spin density vector, \( x = (x, \tau) \) denotes space and imaginary time, and \( \int dx = \int dx_0^{1/T} \int \). In the critical region near a quantum phase transition, imaginary time scale like a length to the power \( z \), and the space-time nature of the integrals in the action accounts for the system’s effective dimension \( d + z \) that was mentioned in the Introduction.

Now \( S_{\text{int}}^{(t)} \) is decoupled by means of a Hubbard-Stratonovich transformation. The partition function, apart from a noncritical multiplicative constant, can then be written

\[ Z = \int D[M] \exp \left( -\Phi[M] \right) \],

with the LGW functional

\[ \Phi[M] = \frac{\Gamma_t}{2} \int dx \mathbf{M}(x) \cdot \mathbf{M}(x) \]

\[ -\ln \left( \exp \left[ -\Gamma_t \int dx \mathbf{M}(x) \cdot \mathbf{n}_s(x) \right] \right)_{S_0} \].

Here \( \langle \ldots \rangle_{S_0} \) denotes an average taken with the action \( S_0 \). If the LGW functional \( \Phi \) is formally expanded in powers of \( \mathbf{M} \), then the term of order \( \mathbf{M}^n \) obviously has a coefficient that is given by a connected \( n \)-point spin density correlation function of the ‘reference system’ defined by the action \( S_0 \).

At this point we need to remember that our reference system \( S_0 \) contains quenched disorder, which has not been averaged over yet. The \( n \)-point correlation functions that form the coefficients of the LGW functional therefore still depend explicitly on the particular realization of the randomness in the system. The average over the quenched disorder, which we denote by \( \{ \ldots \}_{\text{dis}} \), requires averaging the free energy, i.e., we are interested in \( \langle \ln Z \rangle_{\text{dis}} \). This is most easily done by means of the replica trick i.e., one writes

\[ \langle \ln Z \rangle_{\text{dis}} = \lim_{n \to 0} \frac{1}{n} \left( \{ Z^n \}_{\text{dis}} - 1 \right) \],

where the index \( \alpha \) labels \( n \) identical replicas of the system. The disorder average is now easily performed by expanding the exponential in Eq. (4.3). Upon reexponentiation, the coefficients in the replicated LGW functional are disorder averaged correlation functions of the reference system that are cumulants with respect to the disorder average. The Gaussian part of \( \Phi^\alpha \) is simply

\[ \Phi^\alpha_{(2)}[\mathbf{M}^\alpha] = \frac{1}{2} \int dx_1 dx_2 \mathbf{M}^\alpha(x_1) \left( \delta(x_1 - x_2) - \Gamma_t \chi(x_1 - x_2) \right) \cdot \mathbf{M}^\alpha(x_2) \].

Here \( \chi(x) \) is the disorder averaged spin susceptibility or 2-point spin density correlation function of the reference system. The cubic term, \( \Phi^\alpha_{(3)} \), has a coefficient given by the averaged 3-point spin density correlation function. For the quartic term, the cumulant nature of these correlation functions leads to two terms with different replica structures, and higher order terms have correspondingly more complicated structures.

The next step is to calculate the spin density correlation functions for the reference system. It now becomes important that we have kept in our action \( S_0 \) the electron-electron interaction in all channels except for the spin-triplet one that has been decoupled in deriving the LGW functional. At this point our treatment deviates from that of Hertz, who took the reference ensemble to describe noninteracting electrons. This was generally considered an innocent approximation that should not have any qualitative effects. However, this belief was mistaken, since the spin density correlations of interacting electrons are qualitatively different from those of noninteracting ones. The spin susceptibility can be easily calculated in perturbation theory. The result shows that the static spin susceptibility as a function of the wavenumber \( q \) is nonanalytic at \( q = 0 \). For small \( q \) it has the form

\[ \chi(q) = \text{const} - q^{d-2} - q^2 \].

The nonanalyticity is a consequence of the presence of soft particle-hole excitations in the spin-triplet channel, and it occurs only in an interacting electron system. That is, the prefactor of the \( q^{d-2} \) term, which we have suppressed in Eq. (4.3), vanishes for vanishing interaction amplitudes. Renormalization group arguments can then be used to ascertain that this perturbative result indeed represents the exact behavior of \( \chi \) in the long-wavelength
limit. If one also considers the frequency dependence of $\chi$, one obtains the Gaussian part of the LGW functional in the form
\[ \Phi^0(M) = \frac{1}{2} \sum_q \sum_{\omega_n} M^\alpha(q, \omega_n) \left[ t_0 + q^{d-2} + q^2 + |\omega_n|/q^2 \right] \cdot M^\alpha(-q, -\omega_n), \]
(4.6a)
where
\[ t_0 = 1 - \Gamma_t \chi_s(q \to 0, \omega_n = 0) \]
(4.6b)
is the bare distance from the critical point, and the $\omega_n = 2\pi T n$ are bosonic Matsubara frequencies.

The Gaussian theory, Eqs. (4.4), can be analyzed using standard renormalization group techniques. Such an analysis reveals the existence of a Gaussian fixed point whose critical properties are the ones given in Sec. II. The remaining question is whether this fixed point is stable with respect to the higher, non-Gaussian terms in the action. These higher terms also need to be considered in order to obtain the critical behavior of the magnetization.

A calculation of the higher correlation functions that determine the non-Gaussian vertices of the field theory shows that the nonanalyticity that is analogous to the quartic cumulant contribution that is the disorder coefficient $\Gamma_t \cdot M^4$ in the LGW functional, considered at vanishing frequency as a function of a representative wavenumber $q$, is
\[ \chi^{(4)}(q \to 0) \sim q^{d+2-2n} \]
(4.7)
As a result, the coefficients cannot, as usual, be expanded about zero wavenumber, and the theory is nonlocal. Despite this unpleasant behavior of the field theory, it is easy to see by power counting that all of these terms except for one are irrelevant with respect to the Gaussian fixed point in all dimensions $d > 2$. The one exception is the quartic cumulant contribution that is the disorder average of the square of the spin susceptibility, which is marginal in $d = 4$, but irrelevant in all other dimensions. This term is physically of interest, since it represents the random mass or random temperature contribution that one would expect in a theory of disordered magnets, and that was mentioned in Sec. II above.

The conclusion from these considerations is that apart from logarithmic corrections to scaling in certain special dimensions, the Gaussian theory yields the exact critical behavior, and the only remaining question pertains to the form of the equation of state. Since the quartic coefficient $\chi^{(4)}$ is a dangerous irrelevant variable for the magnetization, this requires a scaling interpretation of the infrared divergence of $\chi^{(4)}$. In Ref. [1] it has been shown that for scaling purposes the wavenumber $q$ in Eq. (1.7) can be identified with the magnetization $m^{1/2}$. This is physically plausible, since the divergence stems from an integration over soft modes that are rendered massive by an external magnetic field. Since a nonzero magnetization acts physically like a magnetic field, it cuts off the singularity in Eq. (1.7). With this interpretation of the singular coefficients, the term of order $m^n$ in the saddle point solution of the LGW theory has the structure $m^{n-1} (m^{1/2})^{d+2-2n} = m^{d/2}$, which leads to the equation of state as given in Eq. (2.1). One might wonder why the magnetic fluctuations in the paramagnetic phase do not also cut off the singularity in Eq. (1.7), and thus weaken or even destroy the effects discussed above. While such a cutoff mechanism does exist, it enters the theory only via the fluctuations, which are RG irrelevant with respect to the Gaussian fixed point. It therefore shows only in the corrections to scaling, but not in the leading critical behavior.

Again, all of these arguments can be repeated for the case without disorder. The only changes one encounters pertain to the values of various exponents due to the different character of the soft modes. This leads to the results quoted in Sec. II.

ACKNOWLEDGMENTS

This work was supported by the NSF under grant numbers DMR-96-32978 and DMR-95-10185. Part of the work reviewed here was performed during the 1995 Workshop on Quantum Phase Transitions at the TSRC in Telluride, CO. We thank the participants of that workshop for stimulating discussions. We also would like to acknowledge our collaborators on the clean case, Thomas Vojta and Rajesh Narayanan.

1 J. A. Hertz, Phys. Rev. B 14, 1165 (1976), and references therein.
2 A. J. Millis, Phys. Rev. B 48, 7183 (1993); S. Sachdev, A. V. Chubukov, and A. Sokol, Phys. Rev. B 51, 14874 (1995).
3 This follows from the Harris criterion, A. B. Harris, J. Phys. C 7, 1671 (1974), and has been rigorously proven by J. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. 57, 2999 (1986).
4 D. Belitz and T.R. Kirkpatrick, Europhys. Lett. 35, 201 (1996); T. R. Kirkpatrick and D. Belitz, Phys. Rev. B 53, 14364 (1996).
5 T. Vojta, D. Belitz, R. Narayanan, and T.R. Kirkpatrick, Europhys. Lett. 36, 191 (1996).
6 S. Sachdev, Z. Phys. B 94, 469 (1994).
The term ‘weak localization’ is ill-defined, and often misunderstood. Here we use it to refer to the nonanalytic behavior of electronic correlation functions in the limit of zero momentum and/or frequency that is induced by quenched disorder, or by a combination of interactions and quenched disorder, and that occurs even if the system is far from any kind of metal-insulator transition. The physical reason for these nonanalyticities is the diffusive motion of the electrons in the presence of quenched disorder; for a detailed discussion see, e.g., Refs. 10 and 11.

8 T. R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. 76, 2571 (1996).
9 M. E. Fisher, S.-K. Ma, and B. G. Nickel, Phys. Rev. Lett. 29, 917 (1972).
10 For a review, see, P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).
11 For a review, see, D. Belitz and T. R. Kirkpatrick, Rev. Mod. Phys. 66, 261 (1994).
12 S. Sachdev, Phys. Rev. B 55, 142 (1997).
13 See, e.g., N. F. Mott, Metal-Insulator Transitions, Taylor & Francis (New York 1990).
14 H. Stupp, M. Hornung, M. Lakner, O. Madel, and H. von Löhneysen, Phys. Rev. Lett. 71, 2634 (1993).
15 M. A. Paalanen, T. F. Rosenbaum, G. A. Thomas, and R. N. Bhatt, Phys. Rev. Lett. 48, 1284 (1982).
16 D. Belitz and T. R. Kirkpatrick, Phys. Rev. B 52, 13922 (1995).
17 C. Castellani, C. DiCastro, P. A. Lee, M. Ma, S. Sorella, and E. Tabet, Phys. Rev. B 30, 1596 (1984); A. M. Finkel’stein, Z. Phys. B 56, 189 (1984).
18 A. M. Finkel’stein, Pis’ma Zh. Eksp. Teor. Fiz. 86, 367 (1984) [JETP Lett. 40, 796 (1984)]; C. Castellani, C. DiCastro, P. A. Lee, M. Ma, S. Sorella, and E. Tabet, Phys. Rev. B 33, 6169 (1986).
19 T. R. Kirkpatrick and D. Belitz, J. Phys. Cond. Matt. 2, 5269 (1990); Phys. Rev. B 45, 3187 (1992).
20 The logarithmic corrections to scaling in the two theories are not quite the same. We believe this is due to the fact that of the two integral equations in Ref. 11 only one was exact.
21 For a definition and discussion of the concept of dangerous irrelevant variables, see, M. E. Fisher, in Advanced Course on Critical Phenomena, edited by F. W. Halbe, Springer (New York 1983).
22 For a pedagogical explanation of the replica trick, see, e.g., G. Grinstein in Fundamental Problems in Statistical Mechanics VI, edited by E. G. D. Cohen, North Holland (Amsterdam 1985).
23 See, e.g., S. K. Ma, Modern Theory of Critical Phenomena, (Benjamin, Reading, MA 1976); or M. E. Fisher, Ref. 21.