ON A QUESTION OF EKEDAHL AND SERRE

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Abstract. In this paper we study the Ekedahl-Serre conjecture over number fields. The main result is the existence of an upper bound for the genus of curves whose Jacobians admit isogenies of bounded degrees to self-products of a given elliptic curve over a number field satisfying the Sato-Tate equidistribution, and the technique is motivated by similar results over function field due to Kukulies. A few variants are considered and questions involving more general Shimura subvarieties are discussed.

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1. Introduction

In this paper, we consider a question of Ekedahl and Serre [ES93] on the non-existence of algebraic curves with completely decomposable Jacobians when the genus tends to infinity. Here a Jacobian, or more generally an abelian variety $A$, is said to be completely decomposable (also called totally split in the literature) if it is isogeneous to a product of elliptic curves. In [ES93] Ekedahl and Serre has constructed various examples of algebraic curves with completely decomposable Jacobians, and they asked that whether the genera of the algebraic curves with completely decomposable Jacobians are bounded from above.

The question can be reformulated in terms of Shimura varieties. One considers $M$ the Shimura subvariety in $A_g$ parametrizing abelian varieties isomorphic to a product of elliptic curves respecting the principal polarizations, which is isomorphic to a $g$-fold product of modular curves upon the choice of a suitable level structure imposed on $A_g$.

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An abelian variety isogeneous to a product of elliptic curves thus corresponds to a point in some $T_g(M)$, where $T_g$ stands for the Hecke translation by $q \in \text{Sp}_{2g}(\mathbb{Q})$ in $\mathcal{A}_g$. The question of Ekedahl and Serre is concerned with the existence of a lower bound for $g$ such that the intersection

$$T_g^o \bigcap \left( \bigcup_{q \in \text{Sp}_{2g}(\mathbb{Q})} T_q(M) \right)$$

be finite or even empty, where $T_g^o$ is the open Torelli locus parametrizing Jacobians of smooth projective curves of genus $g$. Note that the degrees of isogenies could be computed explicitly using coordinates of the elements $q$ upon the choice of an integral symplectic bases for $\mathbb{Q}^{2g}$, and only finitely many $q$’s are involved if the isogeny degrees are bounded à priori. If the intersection above could be reduced to the case using only finitely many Hecke translates

$$T_g^o \bigcap \left( T_{q_1}(M) \cup \cdots \cup T_{q_N}(M) \right),$$

then the problem becomes closer to the Coleman-Oort conjecture, cf. [MO13], which predicts that the intersection of $T_g^o$ with a Shimura variety of dimension $> 0$ should, at least when $g$ is large enough, contain at most finitely many CM points, although the Ekedahl-Serre question asks for finer information in the intersection. We will return to this reduction later in Section 4.

The present work focuses, however, on the arithmetic aspects of such intersection. Recall that Kukulies [Kuk10] has studied the Ekedahl-Serre question for a family of Jacobians isogeneous to a self-product of one single family of elliptic curves, namely the finiteness of intersections of the form $T_g^o \cap T_q(Y)$, where $Y$ is the “diagonal” curve in $\mathcal{M}$ parametrizing $g$-fold self-products of elliptic curves. He has made use of the Sato-Tate equidistribution for elliptic curves over functional fields in positive characteristics to deduce a bound in characteristic $p$, which motivates the similar treatment in this note for Jacobians over number fields using known results of Sato-Tate type.

The main result in this paper is the following:

**Theorem 1.1.** Let $c$ be a fixed constant and $E$ be an elliptic curve over a number field $F$ admitting a semi-stable model $\mathcal{E}$ over $O_F$ and satisfying the Sato-Tate equidistribution law as in Theorems 2.2 and 2.3. Then there exists a constant $G = G(E, F, c) > 0$ such that $g < G$ holds whenever there exists a smooth projective curve $C$ of genus $g$ over $F$ with a semi-stable model $\mathcal{C}$ over $O_F$ and with an isogeny over $F$ from $J = \text{Jac}(C)$ to the $g$-fold self-product $A = E^g$ of degree at most $c$.

Here the bound on degrees of isogeny is imposed, and instead of the original Ekedahl-Serre question we are only studying a weakened version over number fields involving only finitely many Hecke translates.

It should be pointed out that the general situation of Ekedahl-Serre question is still open. The recent work of Paulhus etc. has updated the list of values of $g$ for which there exists totally decomposable Jacobians of genus $g$, cf. [Pau08, Pau13, Pau16].

Motivated by this theorem, we proceed further to generalizations of questions of Ekedahl-Serre type: for $M$ a Shimura subvariety in $\mathcal{A}_g$, under what condition does one find that the intersection of $T_g^o$ with the total Hecke orbit of $M$ be finite or even empty? We also
propose a conjecture that should reduce the study involving a total Hecke orbit to the intersection with finitely many Hecke translates, which leads to a question in the style of Coleman-Oort conjecture. The main theorem announced above could be viewed as supporting evidence in an arithmetic set-up.

The paper is organized as follows. Section 2 collects necessary facts on equidistribution of Sato-Tate type, and Section 3 proves the main result in analogy with the approach of Kukulies in positive characteristic, using results from Arakelov geometry. Section 4 includes some elementary variants of the main theorem. Finally in Section 5 we formulate questions of Ekedahl-Serre type for more general Shimura varieties, propose a conjecture that should transform it into a question of Coleman-Oort type, and present some examples established through the numerical properties of surface fibration.

2. THE SATO-TATE EQUIDISTRIBUTION

Following [Ser98], the equidistribution of Sato-Tate type on Frobenius eigenvalues is often transformed into the analyticity of L-functions:

**Theorem 2.1** (equidistribution and L-functions). Let \( F \) be a fixed number field. Let \( G \) be a compact Lie group, and \( X \) its space of conjugacy classes, endowed the probability measure \( \mu \) deduced from the normalized Haar measure on \( G \). Let \( \{ x_\wp \} \) be a sequence of points in \( X \) indexed by the prime ideals of \( \mathcal{O}_F \), and assume that for any non-trivial irreducible \( \mathbb{C} \)-linear representation \( \rho \) of \( G \), the Euler product

\[
L(\rho, s) = \prod_\wp \det(1 - \rho(x_\wp) q_\wp^{-s})^{-1}
\]

is meromorphic for \( \Re(s) > 1 \) and has no zero and pole in this half plane except possibly at \( s = 1 \). Then \( \{ x_\wp \} \) is equidistributed with respect to the measure \( \mu \) if and only if for any non-trivial irreducible representation \( \rho \) as above, the function \( L(\rho, s) \) extends to a holomorphic function on an open neighborhood of \( \Re(s) \geq 1 \) and does not vanish at \( s = 1 \).

In practice the compact Lie group \( G \) is often the maximal compact subgroup of \( G(\mathbb{C}) \), where \( G \) is a reductive \( \mathbb{Q} \)-group serving as the geometric monodromy group for some smooth projective algebraic variety \( Y \) defined over a number field \( F \). Fix an embedding \( F \hookrightarrow \mathbb{C} \), and write \( V \) for the Betti cohomology \( H^m(Y(\mathbb{C}), \mathbb{Q}) \), which is an algebraic representation of \( G \). Taking a prime \( \wp \) of good reduction for \( Y \) over \( F \), we have the special fiber \( Y_{k_\wp} \), and the étale cohomology \( H^m_{\text{ét}}(Y_{k_\wp}) := H^m(Y_{k_\wp} \otimes_{k_\wp} \bar{k_\wp}, \mathbb{Q}_\ell) \) is isomorphic to \( V \otimes \mathbb{Q}_\ell \). Moreover, one may view the Frobenius eigenvalues on \( H^m_{\text{ét}}(Y_{k_\wp}) \), after suitable normalization, as the evaluation of elements in \( G \) on \( V_\mathbb{C} \) through the natural action of \( G \) on \( V \). The invariance under conjugacy reduces the setting from \( G \) to \( X \). The recently developed theory of Sato-Tate groups and generalizations of Sato-Tate conjecture, cf. [Fit15], are not touched upon further in this note.

The Sato-Tate equidistribution is established in various cases over number fields:

**Theorem 2.2** (non-CM case). Let \( F \) be a totally real number field, and let \( E \) be an elliptic curve over \( F \) of non-CM type admitting bad reduction of multiplicative type at some finite place of \( F \). For \( \wp \) a prime of good reduction of \( E \) over \( F \), write \( a_\wp = 2 \sqrt{\wp} \cos \theta_\wp \) for the normalized Frobenius trace on \( V_\ell = H^1(E_{k_\wp}, \mathbb{Q}_\ell) \), with \( \theta_\wp \in [0, \pi] \). Then the angles \( \theta_\wp \) are
equidistributed on the interval \([0, \pi]\) with respect to the Sato-Tate measure \(\mu = \frac{2}{\pi} \sin^2 \theta d\theta\) when \(p\) runs through primes of good reduction for \(E\) over \(F\). In particular
\[
\lim_{N \to \infty} \frac{\# \{ p : q_p < N, \theta_p \in [\alpha, \beta] \}}{\# \{ p : q_p < N \}} = \int_{[\alpha, \beta]} \mu(\theta) d\theta
\]
for any interval \([\alpha, \beta] \subset [0, \pi]\), where \(q_p\) is the cardinality of the residue field at the prime \(p\).

This is proved by Clozel, Harris, Shepherd-Barron and Taylor by establishing the analytic properties of the \(L\)-functions of interest through potential automorphy lifting, see [Car08] for a brief introduction.

Note that in this case the geometric monodromy group \(G\) for \(E\) is \(SL_2\), the maximal compact subgroup \(G\) is \(SU(2)\) (inside \(SL_2(\mathbb{C})\)), whose space of conjugacy classes is identified with the interval \([0, \pi]\) (as the circle modulo reflection). If \(A\) is an abelian variety over \(F\) isogeneous to \(E\), then the geometric monodromy group is simply reduced to that of \(E\), namely \(SL_2\), and the decomposition
\[
H^1(A_{\bar{k}_p}, \mathbb{Q}_\ell) = (H^1(E_{\bar{k}_p}, \mathbb{Q}_\ell))^{\otimes g}
\]
implies that the Frobenius eigenvalues for \(A_{\bar{k}_p}\) is the \(g\)-tuple of copies of the corresponding Frobenius eigenvalue for \(E_{\bar{k}_p}\).

On the other hand the CM case for elliptic curves is known, going back to the classical works of Deuring on Hecke \(L\)-series, cf. [Deu53]:

**Theorem 2.3** (CM case). Let \(E\) be an elliptic curve over a number field admitting CM by the integer ring \(O_K\) of some imaginary quadratic number field \(K\). Then the Sato-Tate equidistribution holds for the Frobenius eigenvalues of \(H^1(E_{\bar{k}_p}, \mathbb{Q}_\ell)\).

Note that in this case, the geometric monodromy group is \(T = \text{Ker}(\text{Res}_{K/\mathbb{Q}} \mathbb{G}_m \to \mathbb{G}_m)\), and \(T(\mathbb{R})\) is the maximal compact subgroup in \(T(\mathbb{C})\), which coincides with its space of conjugacy due to the commutativity of \(T\). The Sato-Tate measure might take different form depending on whether \(F\) contains \(K\) or not, due to the different formula of Hecke \(L\)-functions in these two cases. However this does not matter for our later discussion, because only a positive density result is needed on the asymptotic distribution of primes of good reduction similar to the non-CM case, and the exact form of the Sato-Tate measure is not specified in what follows.

3. **Bounding singularities by heights**

We briefly recall the strategy of Kukulies in [Kuk10] in positive characteristic before developing the number-theoretic analogue. One starts with a function field \(F\) in characteristic \(p\), a smooth projective curve \(S_F\) over \(F\) whose Jacobian \(J_F = \text{Jac}(S_F)\) is isogeneous to the \(g\)-fold product of some elliptic curve \(E_F\) over \(F\). Assume that these structures admit non-isotrivial semi-stable models: \(F\) is the function field of some smooth projective geometrically connected curve \(C\) over \(\mathbb{F}_q\), \(S_F\) is the generic fiber of some semi-stable surface fibration \(f : S \to C\), \(E_F\) admits a semi-stable model \(E \to C\), and the isogeny \(J_F \to E_F^g\) also extends to \(J = \text{Pic}^0(S/C) \to E^g\) over \(C\). The estimation of Kukulies in this case goes as follows:
• the height inequality of Szpiro for $S \to C$ bounds the number $\delta$ of the singular points from all the (singular) fibers of $S \to C$ as

$$\delta \leq 12 \deg(\omega_{S/C})$$

with $\omega_{S/C}$ the dualizing sheaf for $S \to C$, and $\deg(\omega_{S/C}) = h(J/C) = gh(E/C)$ with $h(\bullet/C)$ the height of a group $C$-scheme, i.e. degree of the invariant differential sheaf along the neutral section;

• the Sato-Tate equidistribution for $E \to C$ shows for $g > q^n + 1$, the number of singular fibers of $S \to C$ over points from $C(\mathbb{F}_{q^n})$ is at least $\lfloor \frac{g}{2q^n} \rfloor$, and some counting rearrangements lead to $\delta \geq c(E/C)g\log g\log\log g$, which bounds $g$ in constants determined by $E \to C$.

The counterpart over number fields of the height inequality of Szpiro is the following:

**Theorem 3.1.** Let $C$ be a smooth projective curve of genus $g \geq 1$ over $\overline{\mathbb{Q}}$, with $C$ a semi-stable minimal model of $X$ over $B = \text{Spec} \mathcal{O}_F$ for the integer ring of some number field $F \subset \overline{\mathbb{Q}}$. Write $J = \text{Jac}(C/F)$ and $\mathcal{J} = \text{Pic}^0(C/B)$ for the integral model of $J$ over $B$ which is a semi-abelian $B$-scheme. Then holds the inequality

$$\Delta(C/B) \leq h_{\text{Fal}}(J) + g(10 + 4\log(2\pi))$$

where:

- $\Delta(C/B) = \frac{1}{[F:\mathbb{Q}]} \sum_p \#\text{Sing}(C(\overline{k}_p)) \log q_p$ is the weighted sum of singularities for $C \to B$, with $\text{Sing}(C(\overline{k}_p))$ the set of singular points in $C(\overline{k}_p)$ over a prime $p$ of bad reduction for $C \to B$,

- $h_{\text{Fal}}(C)$ is the Faltings height of $C$, also equal to the Faltings height $h_{\text{Fal}}(J)$ of $J$ (which can be computed explicitly using their semi-stable models).

The proof of this proposition is immediate after combining results in Arakelov geometry by Faltings with some recent improvement by Wilms, kindly explained to us by Prof. Ariyan Javanpeykar:

**Theorem 3.2** (Faltings, cf. [Fal83, Fal84]). For $C \to B$ the arithmetic surface in $\mathcal{J}J$ holds the Noether formula

$$h_{\text{Fal}}(C) = \Delta(C/B) + \omega^2(C) + \delta_{\text{Fal}}(C) - 4g \log(2\pi)$$

where $\delta_{\text{Fal}}(C/B)$ is the archimedean discriminant computed using the Riemann surface structures on $C(\mathbb{C})$ obtained from different embeddings $F \hookrightarrow \mathbb{C}$. Moreover $\omega^2(C) \geq 0$.

The improvement from Wilms [Wil16] provides a link between the $\delta$-invariant of Faltings and the $\varphi$-invariant of Kawazumi-Zhang, cf. [Kaw] and [Zh10]. For simplicity we only mention the following less precise consequence, which is sufficient for the analogue of Szpiro inequality:

**Theorem 3.3** (Wilms, cf. [Wil16]). For $C \to B$ the arithmetic surface as above holds the inequality $\delta_{\text{Fal}}(C) \geq -10g$.

The proof of the main theorem is thus reduced to an estimation of singular points, along the idea in [Kuk10]:
Proof of Theorem 1.1. For \( p \) a prime of good reduction for \( E \) over \( F \), let \( q_p \) be the cardinality of the residue field at the prime \( p \), and \( a_p \) be the trace of the \( q_p \)-Frobenius on \( H^1(E_{k_p}, \mathbb{Q}_\ell) \). The assumption on equidistribution of Sato-Tate type implies that the following subset of primes of good reduction for \( E \) over \( F \)

\[
P = \{ p : a_p \in [\sqrt{q_p}, 2\sqrt{q_p}] \}
\]

is of density \( \rho > 0 \) in the set of primes of \( F \). On the other hand, a prime \( p \) of good reduction for \( C \) is also of good reduction for \( E \) via the isogeny between \( J \otimes_F F_p \) and \( E^g \otimes_F F_p \) using Serre-Tate’s criterion, cf. [ST60]. The isomorphism

\[
H^1(C_{k_p}, \mathbb{Q}_\ell) \simeq H^1(J_{k_p}, \mathbb{Q}_\ell) \simeq H^1(E_{k_p}, \mathbb{Q}_\ell)^{\oplus g}
\]

leads to the point counting

\[
\#C(k_p) = 1 + q_p - g a_p,
\]

which together with the inequality \( \#C(k_p) \geq 0 \) implies that

\[
1 + q_p - g \sqrt{q_p} \geq 0, \quad \forall \ p \in P.
\]

In particular, \( q_p \geq g \) for such primes of good reduction.

Let

\[
P(g) := \{ p \in P \mid q_p < g \} \subseteq P.
\]

Since the set \( P \) defined above is of density \( \rho > 0 \), the subset \( P(g) \) is of size at least \( \frac{1}{2} \rho \cdot \frac{g}{\log g} \) when \( g \) is large enough. Moreover, for \( p \in P(g) \), although \( J \), or equivalently \( E \), is of good reduction, the fiber \( C_{k_p} \) must be singular due to the counting inequality \( q_p \geq g \) established above for primes of good reduction of \( C \to B \).

We claim that the number of singular points in such a singular fiber is at least \( \frac{g}{2q_p} \).

The argument is the same as in [Kuk10]: when \( q_p < g \), the Jacobian is either a torus, or isogeneous to the \( g \)-fold product of a single elliptic curve. In the toric case the curve \( C_{k_p} \) has at least \( g \) singular points; in the compact case the curve is a chain of smooth curves each of genus less at most \( q_p \), and at least \( \lfloor \frac{g}{2q_p} \rfloor \) singular points are found in the fiber.

Taking the summation over these primes in \( P(g) \) gives the following inequality:

\[
\Delta(C/B) \geq \frac{1}{d} \sum_{p \in P(g)} \frac{g}{2q_p} \log q_p \geq \rho_1 g \log g,
\]

for some \( \rho_1 \in (0, \rho/2) \) with \( d = \lceil F : \mathbb{Q} \rceil \), using some standard estimation in analytic number theory (cf. Lemma 3.5 below).

It is also known from [Fal83] (or [Ray85]) that for given \( E \) and \( F \), the Faltings height of an abelian variety \( A' \) over \( F \) admitting an \( F \)-isogeny to \( E^g \) of degree \( f \) only differs from \( gh_F(E/O_F) \) by a quantity bounded by \( \log f \). Since we have required the degree of isogeny between \( J \) and \( E^g \) be of bounded by \( c \), we get from Theorem 3.1 that

\[
\Delta(C/B) \leq h_{\text{Fal}}(J) + gO(1) = g(h_{\text{Fal}}(E) + O(1)) + O(1),
\]

where \( O(1) \) stands for some constant determined by \( E, F \) and \( c \), independent of \( g \). Thus

\[
\frac{1}{d} \rho_1 g \log g \leq g(h_{\text{Fal}}(E/O_F) + O(1)) + O(1).
\]
Eliminating the linear factor \( g \) one obtains an upper bound of \( g \) in terms of \( h_{\text{Fal}}(E) \), \( d = [F : \mathbb{Q}] \) and \( c \).

\[ \text{Remark 3.4.} \quad \text{Note that the bound in the theorem is not explicit. We should remark that, in the functional field case over complex number, an explicit upper bound } g \leq 11 \text{ is found in [LZ14] if there is a family of curves of genus } g \text{ with an isogeny between the Jacobians and the self-fiber-product of one single family of elliptic curves. However, the proof loc. cit. relies heavily on certain results such as the logarithmic Miyaoka-Yau inequality only valid in characteristic zero at the moment.} \]

In the proof of Theorem 1.1 above we have made use of some standard calculation from analytic number theory, based on the useful fact that for \( \{a_n\} \) a sequence of numbers and \( b(x) \) a function of \( C^1\)-class on \( \mathbb{R}_{\geq 0} \), holds the following identity (see for example [MV07]):

\[
\sum_{n \leq x} a_n b(n) = A(x)b(x) - \int_1^x A(t)b'(t)dt,
\]

with \( A(t) = \sum_{n \leq t} a_n \).

\[ \text{Lemma 3.5. (1). Let } P \text{ be the set of prime numbers in } \mathbb{N}, \text{ and } Q \text{ a subset of natural density } c, \text{ i.e.,} \]

\[
\lim_{x \to \infty} \frac{\#Q \cap P(x)}{\#P(x)} = c,
\]

where \( P(x) = \{p \in P : p \leq x\} \). Then asymptotically, one has

\[
\sum_{p \in Q \cap P(x)} \frac{\log p}{p} \sim c \cdot (\log x - \log \log x), \quad x \to \infty.
\]

\[ \text{(2). Let } F \text{ be a number field, } P_F \text{ its set of prime ideals and } Q_F \subset P_F \text{ a subset of density } c > 0, \text{ namely} \]

\[
\lim_{x \to \infty} \frac{\#Q_F \cap P_F(x)}{\#P_F(x)} = c,
\]

where similarly as above \( P_F(x) = \{\wp \in P_F : q_{\wp} \leq x\} \) with \( q_{\wp} \) being the cardinality of the residue field at the prime \( \wp \). Then asymptotically, it holds that

\[
\sum_{\wp \in Q_F \cap P_F(x)} \frac{\log q_{\wp}}{q_{\wp}} \geq c_1 \log x, \quad x \to \infty,
\]

for some constant \( c_1 \) depending only on \( Q_F \) and \( F \).

\[ \text{Proof. (1). Consider the sequence } \{a_n\} \text{ given by } a_p = 1 \text{ for } p \in Q \text{ and zero otherwise. Then} \]

\[
A(x) := \sum_{n \leq x} a_n \sim c \frac{x}{\log x}.
\]
The summation formula (3.1) above gives
\[
\sum_{p \in \mathbb{Q} \cap P} \frac{\log p}{p} = \sum_{n \leq x} \frac{a_n}{n} \log n \\
\sim c \frac{x}{\log x} \cdot \frac{\log x}{x} - \int_2^x c \frac{t}{\log t} d\left(\frac{\log t}{t}\right) = c \log x - c \log \log x + O(1).
\]

(2). Write \( p_F(n) \) for the number of ways representing \( n \in \mathbb{N} \) as the norm of a prime ideal from \( P_F \). Then Landau’s prime number theorem \( \sum_{\mathfrak{p} \in P_F} 1 \sim \frac{x}{\log x} \) for \( F \) is the same as
\[
\sum_{n \leq x} p_F(n) \sim \frac{x}{\log x}.
\]
Similarly one writes \( q_F(n) \) for the number of ways representing \( n \) as the norm of a prime ideal from \( Q_F \), and the assumption on natural density is
\[
\lim_{x \to \infty} \frac{\sum_{n \leq x} q_F(n)}{\sum_{n \leq x} p_F(n)} = c.
\]
Applying the summation formula mentioned above we obtain
\[
\sum_{n \leq x} q_F(n) \frac{\log n}{n} \sim c(\log x - \log \log x),
\]
and one may simply take \( c_1 = \frac{c}{2} < c \). □

4. Variants

Note that the Sato-Tate equidistribution is also known for simple CM abelian varieties, as long as the base number field is Galois over \( \mathbb{Q} \) and contains the CM field, cf. [Fit15]. Naturally one may consider similar questions of Ekedahl-Serre type in this setting. One should, however, specify the meaning of the genus \( g \) tending to infinity, and we single out two naive examples, where the growth of genera is explicitly described:

Case 1 Fix a CM abelian variety \( A \) defined over some number field, and consider Jacobians isogeneous to products of \( A \), with the degree of isogeny bounded a priori; here it is the number of copies of \( A \) in the Jacobian that grows;

Case 2 Fix a number field \( F \), and consider a sequence of CM abelian varieties \( A_n \) over \( F \) satisfying Sato-Tate equidistribution with Faltings height growing linearly according to their dimensions.

In either case similar arguments as in [Kuk10] are expected to apply when the Faltings heights of the Jacobians grow essentially linearly along the genera.

**Proposition 4.1** (Case 1). Let \( A \) be a simple abelian variety over \( \overline{\mathbb{Q}} \) with CM by some CM field \( L \). Let \( F \) be a Galois number field containing \( L \) over the integer ring of which \( A \) admits a semi-stable model \( \mathcal{A} \). Fix a constant \( c \), then there exists a constant \( G = G(A, c) \) such that if a smooth projective curve \( C \) over \( F \) admitting a semi-stable model over \( \mathcal{O}_F \) whose Jacobian \( J \) is isogeneous to a product of copies of \( A \), with the isogeny degree bounded by \( c \), then \( C \) is of genus at most \( G \).
Proof. Write \( r \) for the dimension of \( A \) and \( d \) for the degree of \( F \). The Sato-Tate group of \( A \) is a \( d \)-dimensional compact torus, which we identify with a product of circles \( T = S^1 \times \cdots \times S^1 \) (\( r \)-fold), and the Sato-Tate measure is the Haar measure \( \mu_T \) on \( T \), normalized to be of total mass \( 1 \). The Frobenius trace for a prime \( p \) of good reduction of \( A \) over \( B = \text{Spec} O_F \) is, using a basis of \( H^1(A_{k_F}, \mathbb{Q}_\ell) \), of the form

\[
t_p = \sum_{j=1, \ldots, r} 2\sqrt{q_p} \cos (\theta_p(j))
\]

with \( \theta_p = (\theta_p(1), \ldots, \theta_p(r)) \in T \) equidistributed with respect to \( \mu_T \). In particular there exists a subset \( P \) of primes in \( F \) with density \( \rho > 0 \) such that \( A \) is of good reduction at \( p \) and \( \cos \theta_p(j) \in [\frac{1}{2}, 1] \).

Let \( C \) be a smooth algebraic curve over \( F \) admitting a semi-stable model \( \mathcal{C} \) over \( B = \text{Spec} O_F \) as before, such that its Jacobian \( J \) admits an isogeny to \( A^{\oplus m} \) of degree at most \( c \). The isogeny from \( J \) to \( A^{\oplus m} \) leads to the counting

\[
\#C(k_F) = 1 + q_p - m t_p
\]

with \( t_p \in [r\sqrt{q_p}, 2r\sqrt{q_p}] \). This forces the inequality \( g \leq q_p \) with \( g = mr \) the genus of \( C \). When \( m \) grows to infinity, one finds at least \( \frac{1}{2} \rho \cdot \frac{2}{\log r} \) primes of bad reduction for \( C \) over \( O_F \) inside \( P(g) = \{ p \in P : q_p < g \} \), and it remains to argue as in the case of self-products of elliptic curves to bound \( m \) and also \( g = mr \). \hfill \Box

Note that the current state of the Sato-Tate conjecture does not yet permit the similar arguments applied to Jacobians isogeneous to the product of copies of a fixed finite set of abelian varieties \( \{ A_1, \ldots, A_N \} \): in general one is not able to deduce the equidistribution of Frobenius eigenvalues on an abelian variety of the form \( A_1^{\oplus m_1} \oplus \cdots \oplus A_N^{\oplus m_N} \). The Sato-Tate measure involves the density of primes with prescribed behavior, and the notion of density of infinite subsets of primes is not well-behaved under intersection, as long as no further input is available. If one considers the case of two elliptic curves some positive results are possible after [Har13], such as the product of two non-CM elliptic curves non-isogeneous to each other.

The following result, namely Case 2 as mentioned above, is included only to illustrate how restrictive the present treatment is when dealing with the growth of genera.

**Proposition 4.2** (Case 2). Let \( (A_n) \) be a sequence of CM abelian varieties over a number field \( F \) with dimensions tending to infinity, each of which admits a semi-stable model \( A_n \) over \( B = \text{Spec} O_F \), and all of them are of good reduction over some open subscheme \( U \subset B \) in. Assume further that:

(i) The Faltings heights \( h_{\text{Fal}}(A_n) \) are bounded with respect to their dimensions, i.e., \( h_{\text{Fal}}(A_n) = O(1) \dim(A_n) \);

(ii) the sets \( P_n = \{ p : \tau_p(n) \in [\sqrt{q_p}, 2\sqrt{q_p}] \} \) contains some fixed subset \( P \) of density \( \rho > 0 \), independent of the choice of \( n \), where \( \tau_p(n) = \frac{1}{\dim A_n} t_p \) is the Frobenius trace \( t_p(n) \) on \( H^1(A_{n,k_F}, \mathbb{Q}_\ell) \) averaged over the dimension.

Then for any given constant \( c > 0 \), there exists a constant \( G = G((A_n), c) \) such that any smooth projective curve \( C \) over \( F \) is of genus at most \( G \), provided that it admits a
semi-stable model over $O_F$ such that its Jacobian $J$ admits an isogeny to one of the $A_n$’s of degree at most $c$.

Proof. For $C$ a curve over $F$ with a semi-stable model $\mathcal{C}$ over $O_F$ such that the Jacobian $J = \text{Jac}(C)$ is isogeneous to one of the $A_n$‘s, the point counting at a prime of good reduction $p$ in $P$ again gives $g \leq q_p$, and one concludes by the singularity counting plus the bound in Faltings heights, similar to the previous cases. □

Unfortunately the conditions in the proposition above are way too far from manipulable. The natural field of definition of a simple abelian variety with CM by a CM number field $E$ is, after the fundamental construction of Shimura and Taniyama, a ray class field of the reflex field of $E$, and that a sequence of simple CM abelian varieties involving infinitely many CM type be subject to the condition in averaged trace is rarely encountered in practice.

5. FURTHER QUESTIONS

As is mentioned in the beginning of this paper, the Ekedahl-Serre question involves the Shimura subvariety $M \subset A_g$, which parametrizes $g$-fold products of elliptic curves, and all the Hecke translates of $M$. One naturally considers similar questions for other type of Hodge symmetry in terms of general Shimura subvarieties and its possible relation with the Coleman-Oort conjecture.

Recall that once a principally polarized abelian variety $A$ over $\mathbb{C}$ is given, an isomorphism of symplectic $\mathbb{Q}$-space $H^1(A, \mathbb{Q}) \simeq \mathbb{Q}^{2g}$ (the latter being the standard symplectic $\mathbb{Q}$-space of dimension $2g = 2 \dim A$) is well understood, and its Mumford-Tate group $\text{MT}(A)$ is the smallest reductive $\mathbb{Q}$-group in $\text{GSp}_{2g}$ fixing all the Hodge classes of $A$ (cf. [DM82]). The $\mathbb{Q}$-group $G = \text{MT}(A)$ gives rise to a Shimura subvariety $M$ defined by $(G, X)$, where $X$ is the $G(\mathbb{R})$-orbit of the homomorphism $x_A : S \to \text{GSp}_{2g}(\mathbb{R})$ defining the $\mathbb{C}$-structure on $H^1(A, \mathbb{R})$ (in fact one only needs the connected Shimura subvariety associated to $X^+$ the $G(\mathbb{R})^+$-orbit of $x_A$, cf. the definition used in [CLZ16]). Note that here $M$ is also the smallest Shimura subvariety in $A_g$ containing the point corresponding to $A$, and the point in $A_g$ is contained in a given Shimura variety $M'$ defined by $(G', X')$ if and only if its Mumford-Tate group $\text{MT}(A)$ is a $\mathbb{Q}$-subgroup of $G'$.

In particular, for $C$ a smooth projective algebraic curve over $\mathbb{C}$ of genus $g$, its Jacobian $J = \text{Jac}(C)$ admits the principal polarization by the theta divisor, and it is this symplectic structure that gives rise to the principal polarization on $H^1(C, \mathbb{Q}) = H^1(J, \mathbb{Q})$ and thus Torelli morphism $M_g \to A_g$; here we actually mean the corresponding moduli problems with suitable level structures (for the sake of representability), although we omit the commonly used subscripts such as $A_{g,N}$ etc. We may thus talk about closed subvarieties in $\mathcal{M}_g$ of the form $\mathcal{M}_g^G := \mathcal{M}_g \times_{A_g} M$ which corresponds to curves whose Jacobians are of Mumford-Tate groups contained in $G$.

The following generalizations might be of interest from the viewpoint of Ekedahl-Serre and Coleman-Oort:

**Question 5.1.** Fix a Shimura subvariety $M$ in $A_g$ defined by some Mumford-Tate subgroup $G \subset \text{GSp}_{2g}$. 
Write $H(M)$ for the total Hecke orbit of $M$ in $\mathcal{A}_g$, namely the union

$$H(M) = \bigcup_{q \in \text{Sp}_{2g}(\mathbb{Q})} T_q(M)$$

where $T_q(M)$ is the Hecke translate of $M$ by $q$, i.e. the Shimura subvariety associated to $qGq^{-1}$. Under what condition for $G$ and $g$ could one find $T_g^0 \cap H(M)$ being finite or even empty?

What kind of Mumford-Tate subgroups $G \subset \text{GSp}_{2g}$ could produce a zero-dimensional $M^G_g$, in other words, a finite intersection $T_g^0 \cap M$? When could the intersection $T_g^0 \cap M$ be even empty?

Note that Question (CO) is of different flavor from (ES), as only one fixed Shimura variety is considered instead of a total Hecke orbit, and (CO) slightly strengthens the original Coleman-Oort conjecture which only predicts finiteness of CM points in $T_g^0$ for $g$ large.

The main results studied in Section 3 deals with finiteness over number fields for such intersection under suitable integral constraints with bounded isogeny degrees. The bound on degrees of isogenies actually rules out all but finitely many Hecke translates, and it is expected as an example for the following conjecture, which should serve as a bridge from (ES) to (CO):

**Conjecture 5.2.** For $g$ large enough and $M$ a Shimura subvariety in $\mathcal{A}_g$, the intersection $T_g^0 \cap H(M)$ can be reduced to only finitely many Hecke translates of $M$, namely there exists $q_1, \ldots, q_N \in \text{Sp}_{2g}(\mathbb{Q})$ such that

$$T_g^0 \cap H(M) = T_g^0 \cap \left( T_{q_1}(M) \cup \cdots \cup T_{q_N}(M) \right).$$

Note that Questions (ES) and (CO) and the conjecture above make sense for zero-dimensional Shimura subvarieties, namely CM points, with examples supporting (CO) provided by equidistribution of Sato-Tate type.

One does not expect (CO) to be true for arbitrary Shimura subvarieties. The cyclic covers of $\mathbb{P}^1$ already produces positive dimensional Shimura subvarieties contained generically in the Torelli locus, cf. [Mo10]. Moreover Möller has communicated to us a counter-example in each dimension: the Hilbert modular variety $M$ in $\mathcal{A}_g$ associated to a totally real number field of degree $g$ always contains a Teichmüller curve in $T_g^0$, and of course infinitely many points are found in the intersection; the proof relies on [BM10, §6].

On the other hand, our previous work [CLTZ] has shown that Question (CO) is true for Shimura varieties whose Mumford-Tate groups contain “large” compact factors, which we reformulate as follows for reader’s convenience:

**Proposition 5.3 ([CLTZ] Proposition 2.4 and Corollary 2.5).** Let $M \subset \mathcal{A}_g$ be a Shimura subvariety defined by a connected Shimura datum $(G, X)$, such that $G^{\text{der}} = \text{Res}_{F/\mathbb{Q}} H$ for some semi-simple $F$-group with $F$ a totally real number field of degree $d$. Assume that $H$ gives rise to a compact Lie group along $r$ real embeddings $F \rightarrow \mathbb{R}$, and gives non-compact Lie groups along the other $d - r$ real embeddings. Then the open Torelli locus $T_g^0$ only
meets $M$ in at most finitely many $\bar{\mathbb{Q}}$-points when

$$\frac{r}{d} > \frac{5}{6} + \frac{1}{6g}$$

The main idea behind this criterion is a slope inequality of Xiao which allows us to exclude Shimura varieties with sufficiently many compact factors in the Mumford-Tate groups, using numerical properties of the semi-stable surface fibration associated to a curve in $T_g$. In particular the property of possessing “large” compact factors is invariant under Hecke translation for a given Mumford-Tate group, which makes (ES) more hopeful via the reduction to (CO) through the conjecture.

**Sketch of proof.** The open Torelli locus can be also defined over $\bar{\mathbb{Q}}$, it suffices to show that the intersection $T_g \cap M$ is zero-dimensional. If the intersection were of positive dimension, it would contain a curve $C$, which lifts to a curve $B$ inside $\mathcal{M}_g$ and the construction in [CLZ16] and [CLTZ] completes it into a semi-stable surface fibration $f : S \to B$, whose Hodge bundle $\omega = \overline{\mathcal{F}}_0 = \mathcal{F}_0$, a vector bundle on $B$ of rank $g$, is determined by the $(1,0)$-part in the Hodge decomposition for $C$ using the universal family of abelian varieties over $C$ given by the modular interpretation $C \subset M \subset A_g$.

The Hodge bundle $\omega$ on $B$ admits a decomposition into $F_0 \oplus F_1$, with $F_0$ the flat part (cf. [CLZ16]) of degree zero, and the slope inequality of Xiao for $f : S \to B$ implies that

$$\frac{\text{rank} F_0}{g} \leq \frac{5}{6} + \frac{1}{6g}.$$  

The direct sum $F_0 \oplus F_1$ is induced from a similar decomposition $E_0 \oplus E_1$ on $C$ by subbundles of the same ranks respectively, using the $(1,0)$-part of the Hodge decomposition for the universal family of abelian varieties by the modular interpretation of $C \subset M \subset A_g$. The refinement $C \subset M \subset A_g$ implies that $\frac{\text{rank} E_0}{g} \geq \frac{\text{rank} E_0}{g} \geq \frac{5}{6} + \frac{1}{6g}$, and a contradiction arises when

$$\frac{r}{d} > \frac{5}{6} + \frac{1}{6g}.$$  

The proposition above only requires information from the “portion” of compact factors. The paper [CLTZ] has passed on to Shimura varieties of $\text{SU}(n,1)$-type, i.e. uniformized by the Hermitian symmetric space associated to the simple Lie group $\text{SU}(n,1)$, and its argument through the slope filtration of the Higgs bundles could be adapted to more general Shimura varieties of unitary type:

**Definition 5.4.** A Shimura datum of unitary type is a Shimura datum $(G, X)$ such that $G_{\text{der}}$ is a simple $\mathbb{Q}$-group admitting a decomposition in Lie groups of the following form

$$G_{\text{der}} \otimes_{\mathbb{Q}} \mathbb{R} \simeq \text{SU}(p_1, q_1) \times \cdots \times \text{SU}(p_r, q_r)$$

with $p_i \geq q_i \geq 0$ and $p_i + q_i = n$ constant. A connected component $X^+$ of $X$ is thus the direct product of the Hermitian symmetric domains $X_i^+$ associated to those $\text{SU}(p_i, q_i)$ with $q_i > 0$.

A Shimura subvariety of unitary type in $A_g$ is the one associated to a Shimura subdatum in $(\text{GSp}_{2g}, \mathcal{H}_g^+)$ of unitary type in the sense above. (One may also consider $(G, X)$ such
that $G^\mathrm{ad}_R$ admits a decomposition in adjoint groups associated to such unitary groups, but this case is not needed in the sequel and is thus omitted.)

**Theorem 5.5.** Let $M \subset A_g$ be a Shimura subvariety of unitary type associated to some Shimura subdatum $(G, X)$ of $(\text{GSp}_{2g}, \mathcal{H}_g^2)$, with $G^\mathrm{der} \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{i=1}^{r} \text{SU}(p_i, q_i)$, $p_i + q_i = n$ such that:

- $p_i \geq q_i \geq 0$ and $\max_{i=1,\ldots,r} \left\{ \frac{q_i}{p_i} \right\} < \frac{1}{12}$,
- $q_{i_0} \geq 2$ for some $i_0 \in \{1, \ldots, r\}$.

Then $M$ only meets $T_g^\circ$ in at most finitely many points.

The main idea of the proof is similar to [CLTZ, Section 3]:

**Proof.** Using the Satake classification cf. [Sa67], the condition $q_i \geq 2$ for some $i$ forces that the symplectic representation $G^\mathrm{der} \rightarrow \text{Sp}_{2g}$ of the simple $\mathbb{Q}$-group $G^\mathrm{der}$ decomposes, after the base change from $\mathbb{Q}$ to $\mathbb{R}$, into the following form:

$$\mathbb{R}^{2g} = V_0 \bigoplus (V_1 \oplus \cdots \oplus V_r)^{\oplus m}$$

with $V_0$ a trivial representation of even dimension $2n_0$ and $V_i$ the $2n$-dimensional standard $\mathbb{R}$-linear representation of $\text{SU}(p_i, q_i)$ on $\mathbb{C}^n = \mathbb{C}^{p_i} \oplus \mathbb{C}^{q_i}$ preserving an Hermitian form of signature $(p_i, q_i)$. In particular, $g = n_0 + rmn$.

Fine information for the Hodge decomposition of the complex vector bundle associated to the locally constant sheaf associated to the $\mathbb{C}$-linearized representation $G^\mathrm{der}(\mathbb{R}) \rightarrow \text{GL}_{2g}(\mathbb{C})$ is needed. Write $E = E^{1,0} \oplus E^{0,1}$ for the Hodge decomposition on $M$ given by the modular interpretation of $M \hookrightarrow A_g$, with $E^{1,0}$ and $E^{0,1}$ both complex vector bundles of rank $g$:

- $G^\mathrm{der}(\mathbb{R})$ acts on $V_0$ trivially, and $V_0$ contributes to both $E^{1,0}$ and $E^{0,1}$ a trivial vector bundle of rank $\dim V_0$.
- $G^\mathrm{der}(\mathbb{R})$ acts on $V_i = \mathbb{C}^n$ through $\text{SU}(p_i, q_i)$, preserving an Hermitian form of signature $(p_i, q_i)$. The complexification $V_i \otimes_{\mathbb{R}} \mathbb{C}$ gives rise to a locally constant sheaf in $\mathbb{C}^{2n}$, which underlies a PVHS with a decomposition of the form

$$E_i = E_i^{1+} \oplus E_i^{1-} \oplus E_i^{0+} \oplus E_i^{0-}$$

such that

- $E_i^{1+} \oplus E_i^{0+}$ is the complexification of the homogeneous vector bundle on $X_i^+$ (the Hermitian symmetric domain of $\text{SU}(p_i, q_i)$) given by the action of the maximal compact subgroup of $\text{SU}(p_i, q_i)$ on $\mathbb{C}^{p_i}$ (through $U(p_i)$), and the complex conjugation on $\mathbb{C}^{p_i}$ induces a permutation of $E_i^{1+}$ and $E_i^{0+}$;
- similarly, $E_i^{1-} \oplus E_i^{0-}$ is associated to the negative part $\mathbb{C}^{q_i}$ on which the maximal compact subgroup of $\text{SU}(p_i, q_i)$ acts through $U(q_i)$.

Clearly the 1st Chern class of $E_i$, namely the sum of Chern classes of the summands described above, is zero as it is associated to a locally constant sheaf. The symmetry of complex polarization and the signature of Hermitian form imply that:

- $c_1(E_i^{1+}) + c_1(E_i^{0+}) = 0$
- $E_i^{1,0} = E_i^{1+} \oplus E_i^{0-}$ is the direct sum of two subbundles of equal Chern classes and rank $p_i$, $q_i$ respectively.
Assume that a curve $C$ (not necessarily projective) is contained in $M \cap T_g^\circ$. The construction in [CLZ16] produces a semi-stable surface fibration $\overline{f} : \overline{S} \to \overline{B}$, where $\overline{B}$ is the compactification of the lifting $B \subset \mathcal{M}_g$ from $C \subset T_g^\circ$. The Hodge bundle $E^{1,0}_{\overline{B}} := f^* \omega_{S/B}$ admits a decomposition obtained by pulling back and extending the direct sum $E^{1,0}_{\overline{B}}$ over $M$:

$$E^{1,0}_{\overline{B}} = E^{1,0}_{0,\overline{B}} \bigoplus \left( E^{i,+}_{i,\overline{B}} \oplus E^{i,-}_{i,\overline{B}} \oplus \cdots \oplus E^{r,+}_{r,\overline{B}} \oplus E^{r,-}_{r,\overline{B}} \right)^{\oplus m}$$

where $E^{i,+}_{i,\overline{B}}$ and $E^{i,-}_{i,\overline{B}}$ are of equal degree $d_i > 0$ and rank $p_i, q_i$ respectively, and $E^{1,0}_{0,\overline{B}}$ is trivial of rank $n_0$.

We thus have $\deg E^{1,0}_{\overline{B}} = 2m(d_1 + \cdots + d_r)$ and the maximal slope in the Harder-Narasimhan filtration of $E^{1,0}_{\overline{B}}$ is at least $\mu := \max\{\frac{d_i}{q_i} : 1 \leq i \leq r, q_i \neq 0\}$. For $\overline{f} : \overline{S} \to \overline{B}$ Xiao’s slope inequality implies (cf. [Xi87])

$$12 \deg E^{1,0}_{\overline{B}} \geq (2g - 2)\mu.$$  

We have assumed that $\max_i q_i < \frac{12}{n}$, which implies that $q_i < \frac{n}{12}$ for some $1 \leq i \leq r$. Thus $\mu > \frac{12d}{n}$, where $d = \max_i d_i$. Therefore,

$$12 \cdot 2mrn \geq 12 \deg E^{1,0}_{\overline{B}} \geq (2g - 2)\mu > 2(n_0 + rmn) \cdot \frac{12d}{n},$$

henceforth $12rn > 12(n_0 + rmn)$ which is absurd. \hfill \Box

Motivated by the theorem above, we would like to propose further the following:

**Conjecture 5.6.** Let $M \subset A_g$ be a Shimura variety of the type in Theorem 5.5. Then the intersection $T_g^\circ \cap M$ is empty.

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**References**

[BM10] Irene Bouw and Martin Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Annals of Mathematics (2) 172(2010), no.1, 139-185, MR 2680418

[Car08] Henri Carayol, *La conjecture de Sato-Tate (d’après Clozel, Harris, Shepherd-Barron, Taylor)*, Astérisque (2008), no. 317, Exp. No. 977, ix, 345–391, Séminaire Bourbaki. Vol. 2006/2007. MR 2487739

[CLZ16] Ke Chen, Xin Lu and Kang Zuo, *On the Oort conjecture for Shimura varieties of unitary and orthogonal types*, Compositio Mathematica, 152(2016), no.5, 889-917, MR 3505642

[CLTZ] Ke Chen, Xin Lu, Shengli Tan and Kang Zuo, *On Higgs bundles over Shimura varieties of ball quotient type*, arXiv:1610.07845.

[DM82] Pierre Deligne, *Hodge cycles on abelian varieties*, notes taken by J. Milne, in Hodge Cycles, Motives, and Shimura varieties, Lecture Notes in mathematics 900, pp. 9-100, Springer Verlag 1982 MR 0654325
[Deu53] Max Deuring, *Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins*, Nachr. Akad. Wiss. Göttingen, (1953), 85-94; (1955), 13-42; (1956), 37-76; (1957), 55-80. MR 0061133, 0070666, 0079611, 0089927

[ES93] Torsten Ekedahl and Jean-Pierre Serre, *Exemples de courbes algébriques à jacobienne complètement décomposable*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 5, 509-513. MR 1239039

[Fal83] Gerd Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349-366. MR 718935

[Fal84] Gerd Faltings, *Calculus on arithmetic surfaces* Annals of Mathematics (2) **119**(1984), no.2, 387-424. MR 0740897

[Fit15] Francesc Fité, *Equidistribution, L-functions, and Sato-Tate groups*, Trends in number theory, 63-88, Contemporary Mathematics 649, AMS 2015. MR 3415267

[Har14] Michael Harris, *Galois representations, automorphic forms, and the Sato-Tate conjecture*, Indian Journal of Pure and Applied Mathematics, 45(2014), no.5, 707-746. MR 3286083

[Kaw] Nariya Kawazumi, *Johnson’s homomorphism and the Arakelov-Green function*, preprint, arXiv:0801.4218

[Kuk10] Stefan Kukulies, *On Shimura curves in the Schottky locus*, J. Algebraic Geom. **19** (2010), no.2, 371-397. MR 2580860

[MO13] Ben Moonen and Frans Oort, *The Torelli locus and special subvarieties*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 549-594. MR 3184184

[ST60] Jean-Pierre Serre, John Tate, *Good reduction of abelian varieties*, Annals of Mathematics (2), **88**(1968), 492-517. MR 0236190

[Zh10] Shouwu Zhang, *Gross-Schoen cycles and dualizing sheaves*, Inventiones Mathematicae **179**(2010), 1-73, MR 2563759
