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Insights into neutrino decoupling gleaned from considerations of the role of electron mass

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Abstract

We present calculations showing how electron rest mass influences entropy flow, neutrino decoupling, and Big Bang Nucleosynthesis (BBN) in the early universe. To elucidate this physics and especially the sensitivity of BBN and related epochs to electron mass, we consider a parameter space of rest mass values larger and smaller than the accepted vacuum value. Electromagnetic equilibrium, coupled with the high entropy of the early universe, guarantees that significant numbers of electron–positron pairs are present, and dominate over the number of ionization electrons to temperatures much lower than the vacuum electron rest mass. Scattering between the electrons–positrons and the neutrinos largely controls the flow of entropy from the plasma into the neutrino seas. Moreover, the number density of electron–positron-pair targets can be exponentially sensitive to the effective in-medium electron mass. This entropy flow influences the phasing of scale factor and temperature, the charged current weak-interaction-determined neutron-to-proton ratio, and the spectral distortions in the relic neutrino energy spectra. Our calculations show the sensitivity of the physics of this epoch to three separate effects: finite electron mass, finite-temperature quantum electrodynamic (QED) effects on the plasma equation of state, and Boltzmann neutrino energy transport. The ratio of neutrino to plasma–component energy scales manifests in Cosmic Microwave Background (CMB) observables, namely the baryon density and the radiation energy density, along with the primordial helium and deuterium abundances. Our results demonstrate how the treatment of in-medium electron mass (i.e., QED effects) could translate into an important source of uncertainty in extracting neutrino and beyond-standard-model physics limits from future high-precision CMB data.

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1. Introduction

In this paper we dissect a key aspect of the physics operating in the epoch of the early universe where neutrinos cease to efficiently exchange energy with the photon–electron–positron plasma and evolve into freely falling particles. By unphysically varying the bare vacuum mass of the electron we can gain insight into the timelike entropy flow from the plasma into the decoupling neutrino component and the evolution of the neutron-to-proton ratio through the epoch of Big Bang Nucleosynthesis (BBN). We do not consider the timelike variation of the electron rest mass as possible beyond-standard-model physics in this work. By keeping all other fundamental constants fixed, we merely consider the role the electron mass has during BBN. See Ref. [1] for time variation of fundamental constants and particle masses.

The early universe possesses high entropy-per-baryon (alternatively low baryon to photon ratio $\eta = 6.1 \times 10^{-10}$) as derived from the Cosmic Microwave Background (CMB) anisotropies [2] and as independently inferred from the primordial deuterium abundance [3,4] and BBN calculations. The high entropy and concomitant disorder set the stage for the key features of neutrino decoupling and BBN.

In particular, high entropy in the early universe implies significant equilibrium electron–positron pair densities, even at temperatures well below the electron rest mass. Neutrino–electron/positron scattering in large part controls the exchange of energy and entropy between the plasma and neutrino components. In turn, the equilibrium number densities of electron–positron pairs can be exponentially sensitive to the electron mass, especially in the later stages of neutrino decoupling and BBN.

Though the baryon rest mass density is low, e.g., $\sim 10^{-4}$ g/cm$^3$ at a temperature $\sim 1$ keV, the early universe is nevertheless a polarizable, high energy density, relativistic plasma. The effective in-medium masses of the electron and positron are the relevant determinants of the number density of neutrino scattering targets. In analogy with the vacuum case, we can take the form of the in-medium dispersion relation to be $E = \sqrt{p^2 + m_{\text{eff}}^2}$ for three-momentum $p$. Here, $m_{\text{eff}}$ represents the effective mass of an electron or positron in medium. We will discuss how this term can differ from the bare vacuum mass $m_e$ as a function of temperature, momentum, and vacuum electron mass itself. There are many studies of the dispersion relations for the electron, positron, and also photon quasi-particles in this medium. These quantum electrodynamic (QED) effects or “plasma corrections” [5–8] are commonly employed in the more sophisticated treatment of BBN (see for example, Refs. [9–11]).

Observational cosmology is entering an era of two distinct, high-precision measurements: CMB observables, e.g., the effective number of relativistic degrees of freedom, $N_{\text{eff}}$, and the primordial helium abundance, $Y_P$; and high-redshift astronomical observables, e.g., the primordial deuterium abundance. The proper interpretation of these measurements necessitates detailed calculations of the epochs surrounding primordial nucleosynthesis. By examining the role of electron mass in these calculations, we are led to conclude that the in-medium electron/positron mass corrections take on a new and heightened significance.

We use natural units $\hbar = c = k_B = 1$ and assume neutrinos are massless throughout this paper. In Section 2, we discuss the statistical mechanics and thermodynamics particular to the early
universe. We give an exposition of how the vacuum electron rest mass perturbs the early universe thermodynamics in Sec. 3. Section 4 discusses the finite-temperature QED corrections to the plasma equation of state. In Sec. 5, we describe the effect of electron rest mass on Boltzmann neutrino transport and nuclear reactions. We give conclusions in Sec. 6. A note on our notation: we will predominantly use the scale factor \( a \) as the independent variable in our equations. For the sake of clarity, we will denote dependent variable values at specific epochs as \( Q(a) \) where the quantity \( Q \) is a function of \( a \).

2. Overview of statistical mechanics in the early universe

A salient feature of the evolution of the early universe is that the Hubble expansion rate, driven by gravitation, is inherently slow. An inevitable result of the slow expansion is to conduct strong and electromagnetic interactions (and even the weak interaction at substantially high temperatures) to maintain the constituents of the early universe in thermal and chemical equilibrium. Eventually the weak interaction is not strong enough to maintain equilibrium between the photon–electron–positron plasma and the neutrino seas. This event, sometimes termed weak decoupling, occurs nearly simultaneously with the epoch where the nuclear reactions — strong, electromagnetic and weak — also drop out of equilibrium. Nevertheless, the electron, positron, and photon components of the plasma (with the thermal coupling to baryons) remain well described by equilibrium thermodynamics over the vast range of BBN epochs. In thermal and chemical equilibrium, along with a homogeneous and isotropic geometry, the comoving entropy is conserved [12].

2.1. Comoving temperature parameter

We will focus on high-precision calculations of ratios involving photons, neutrinos, and baryons. At early times/high temperatures \( (T \gtrsim 10 \text{ MeV}) \), the neutrinos are thermally and chemically coupled to the electromagnetic plasma of photons, electrons, positrons, and a small abundance of baryons. Eventually, the neutrinos decouple from both the plasma and baryons and free stream. The epoch of neutrino decoupling roughly ceases at \( T \sim 10 \text{ keV} \). In order to calculate the energy density in radiation, we need to consider two scales. First, the plasma temperature \( T \) dictates the energy density in photons. As the universe expands, the plasma temperature decreases while the scale factor, \( a \), increases. The product of \( T \) with \( a \) is not a comoving invariant during the BBN epoch. The relic positrons annihilate with electrons to form photons. These photons scatter on the remaining charged leptons and quickly thermalize. If the photons have a high enough energy, they can create an electron–positron (denoted \( e^\pm \) ) pair, keeping the charged leptons in thermal and chemical equilibrium with the photons. As the universe expands and cools, the thermally distributed photons do not have the energies required to pair create \( e^\pm \). The result is that the heat in the form of charged leptons is transferred to photons. As this all occurs in thermal equilibrium, there is no change in the entropy within the plasma and so the product \( Ta \) must increase. The elimination of statistical degrees of freedom is not the only way that the product \( Ta \) can increase. The entropy density is proportional to the product of the number of degrees of freedom and the cube of the temperature. If the entropy rises, and the number of degrees of freedom remains the same, then the temperature will rise. We will consider both mechanisms in this paper.

To compare with the plasma temperature, we will introduce another energy scale called the comoving temperature parameter, \( T_{cm} \), whose product with \( a \) is a comoving invariant
\[ T_{\text{cm}}(a) = T_{\text{cm}}(a_{\text{in}}) \left[ \frac{a_{\text{in}}}{a} \right], \]  

(1)

where we have written \( T_{\text{cm}} \) as a function of \( a \). \( T_{\text{cm}}(a_{\text{in}}) \) is the plasma temperature at an initial epoch of our choosing, which we label \( a_{\text{in}} \). We normally would choose \( a_{\text{in}} \) at an early enough epoch so that the neutrinos are in thermal equilibrium with the charged leptons. As neutrino decoupling proceeds, the neutrinos maintain occupation numbers close to Fermi–Dirac (FD) equilibrium with a temperature-like parameterization close to \( T_{\text{cm}} \). In one sense, \( T_{\text{cm}} \) can be considered a neutrino temperature. We caution against this interpretation as the neutrinos are no longer in thermal equilibrium with each other at late times and so the strict thermodynamic definition of temperature is not an applicable quantity. Nevertheless, the neutrinos have an energy density and that energy density is described by the \( T_{\text{cm}} \) scale.

2.2. Standard equilibrium value of the temperature ratio

We are interested in epochs of the early universe where the energy density (and by extension entropic density) is dominated by radiation. To begin, we give the standard explanation of how to calculate the ratio of \( T_{\text{cm}}/T \) at freeze-out — these arguments can be found in many textbooks (see Refs. [12–14]). To compare with the more sophisticated treatments in Secs. 3–5, we first give the equilibrium argument. We will carefully consider the assumptions the equilibrium argument rely on later.

\( T_{\text{cm}} \) and \( T \) give the two energy scales for neutrinos and photons, respectively. We can characterize the difference in scales using entropy arguments. We describe the entropy, \( S \), of an ideal gas using its extensive property [15]:

\[ S = \frac{E + PV - \sum \mu_i N_i}{T}, \]  

(2)

where \( E \) is the total internal energy, \( P \) is the pressure, \( V \) is the volume, \( \mu_i \) is the chemical potential for a given species \( i \), and \( N_i \) is the number of particles of species \( i \). We will assume the chemical potentials are small for the plasma constituents which carry the bulk of the entropy at any given time. If we write the total internal energy in terms of the energy density and the volume, \( E = \rho V \), we can rearrange terms to find the entropic density

\[ \frac{S}{V} = \frac{\rho + P}{T}, \]  

(3)

in terms of the energy density, pressure, and temperature. Very simply, since the entropy is conserved in equilibrium, we can relate the initial \( (i) \) temperature and the final \( (f) \) temperature as follows

\[ S(a_i) = S(a_f) \implies \frac{T(a_f)}{T(a_i)} = \left\{ \frac{[S/V](a_i)}{[S/V](a_f)} \right\}^{1/3} \left[ \frac{a_i^3}{a_f^3} \right], \]  

(4)

where we have used the fact that \( V \propto a^3 \). For radiation, the energy density scales as \( \rho \propto gT^4 \), and the equation of state is \( P = \rho/3 \). The statistical weight in relativistic particles is \( g \). From Eq. (3), we are left with an expression for the entropic density \( S/V \propto gT^3 \). If our gas has multiple components with differing temperatures and statistical weights \( g \), we can write the entropic density as
\[ S/V = \frac{2\pi^2}{45} T^3 \left\{ \sum_m g_m \left[ \frac{T_m}{T} \right]^4 + \frac{7}{8} \sum_n g_n \left[ \frac{T_n}{T} \right]^4 \right\} \]

\[ = \frac{2\pi^2}{45} g_* S T^3, \]  

where \( g_* S \) is an effective-entropic-spin statistic \([12]\). The sum over \( m \) is for bosonic species, and the sum over \( n \) is for fermionic species. In the definition of \( g_* S \), we allow for the plasma constituents to have different temperatures or temperature parameters, although this expressions rests on assumed Fermi–Dirac or Bose–Einstein shaped energy spectra.

Using the expression in Eq. (6), we can solve for the ratio of temperatures in Eq. (4)

\[
\frac{T(a_i)}{T(a_f)} = \left[ \frac{g_* S(a_f)}{g_* S(a_i)} \right]^{1/3}.
\]

If we pick an initial epoch in Eq. (7) to match our choice for the initial epoch in Eq. (1), i.e., \( T(a_i) = T_{cm}(a_{in}) \), we can write \( T(a_i) = T_{cm}(a_f/a_i) \) to find

\[
\frac{T_{cm}(a_f)}{T(a_f)} = \left[ \frac{g_* S(a_f)}{g_* S(a_i)} \right]^{1/3}.
\]

Given the parameterization of \( g_* S \) in Eq. (6), Eq. (8) is true at any temperatures so long as the particles have equilibrium-shaped energy distribution functions.

Completely decoupled neutrinos have a fixed product of scale factor and neutrino temperature parameter, reflecting how the three-momentum magnitude of a free-falling neutrino redshifts with inverse scale factor. If neutrinos were taken to decouple instantaneously, then their energy and momentum distribution functions at that epoch would have a FD-shaped energy distribution characterized by two quantities: the chemical potential, and the temperature parameter. That temperature parameter is the same as the definition of \( T_{cm} \) in Eq. (1). Using Eq. (8) and taking \( T(a_f) \) after the electron–positron annihilation epoch, we have

\[
\left[ \frac{T_{cm}}{T} \right]_{f.o.} = \left[ \frac{4}{11} \right]^{1/3},
\]

where we have dropped the \( a_f \) arguments and replaced them with a “freeze-out” subscript f.o. In deriving this value, we have made the following assumptions:

1. the neutrinos, independent of energy, decouple with a FD shaped energy-distribution function and a temperature parameter synchronized with the plasma temperature at an epoch designated by \( a_{dec} \);
2. the contribution at \( T(a_{dec}) \) to \( g_* S \) from the charged leptons is \((7/8) \times 4\), which neglects effects of a nonzero value of \( m_e \);
3. the comoving entropy in the plasma is conserved;
4. finite-temperature QED effects on the equation of state for electrons, positrons, and photons are negligible;
5. the electrons and positrons have negligible chemical potentials.

In this paper, we will evaluate the sensitivity of \([T_{cm}/T]_{f.o.}\) to items 1–4. We will not consider how \([T_{cm}/T]_{f.o.}\) changes with item 5, although we will investigate how the chemical potential affects the \( e^\pm \) pair density.
2.3. Pair density of electrons and positrons

The number density for electrons or positrons is

\[ n_{e^\pm} = \frac{g}{(2\pi)^3} \int_0^\infty d^3p \frac{1}{e^{\frac{E^{\pm\mu_e}}{T}} + 1}, \tag{10} \]

where \( d^3p \) is the momentum phase-space density, \( E \) is the energy, and \( \mu_e \) is the chemical potential of the electron. We assume chemical equilibrium between the electron and positron seas, i.e., \( \mu_e \equiv \mu_{e^-} = -\mu_{e^+} \), and take \( g = 2 \), implying

\[ n_{e^-} - n_{e^+} = \frac{1}{\pi^2} \left\{ \int_0^\infty dp \frac{p^2}{e^{\frac{p}{T}} + 1} - \int_0^\infty dp \frac{p^2}{e^{\frac{p}{T}} + 1} \right\} \]
\[ \approx \frac{T^3}{6\pi^2} \left\{ \pi^2 \left[ \frac{\mu_e}{T} \right] + \left[ \frac{\mu_e}{T} \right]^3 \right\}, \tag{11} \]

where the last approximation assumes temperatures high enough that electrons have extreme relativistic kinematics. If we restrict ourselves to epochs where the only charge–carrier constituents of the universe are electrons, positrons, and protons, then the left hand side of Eq. (11) is equal to the number density of protons by charge neutrality

\[ n_{e^-} - n_{e^+} = n_p, \tag{13} \]

We term the excess of electrons over positrons “ionization electrons”, which are equal to the number of protons. The number density of protons is much smaller than the number density of a plasma particle (photons and \( e^{\pm} \)). As very high temperature (\( T >> m_e \)), we can ignore the cubic term on the right-hand side of Eq. (11) to show \( \mu_e / T \propto n_p / T^3 \sim \eta \), where \( \eta \approx 6 \times 10^{-10} \) is the baryon-to-photon ratio.

Fig. 1 shows the ratio of number of ionization electrons (equal to the difference \( n_{e^-} - n_{e^+} \)) to total charged leptons (equal to the sum \( n_{e^-} + n_{e^+} \)) versus the comoving temperature parameter for five different assumed electron vacuum rest-mass values. The third value, denoted \( m_e = 0.511 \text{ MeV} \), is the true value of the electron vacuum rest mass. At all temperatures, the number of ionization electrons is equal to the number of protons which is the same order of magnitude as the baryon number. At high temperature, the total number of charged leptons is the same order of magnitude as the number of photons. Therefore, for \( T_{cm} >> 1 \text{ MeV} \), all five curves should converge to a value similar to \( \eta \). Furthermore, as we assume all positrons eventually annihilate with electrons, all five curves will converge to unity at low temperatures. Changing the mass changes when the pairs disappear, or equivalently, when the epoch of \( e^{\pm} \) annihilation occurs. All five curves in Fig. 1 show that the number of pairs (equal to \( n_{e^+} \)) dominate over the number of ionization electrons until late times, specifically \( T_{cm} << m_e \).

3. Changing \( m_e \) in the instantaneous weak decoupling scenario

In order to calculate how a nonzero \( m_e \) changes the ratio \( T_{cm} / T \), we must start with Eq. (4) and calculate the changes to the entropy. Equation (8) no longer applies since the charged lepton energy density is not proportional to the fourth power of temperature, i.e., \( \rho \propto g T^4 \).

In the case of a non-degenerate (\( \mu = 0 \)) fermionic species, the energy density for an ideal gas at temperature \( T \) is
Fig. 1. The difference in electron and positron number densities normalized by the sum of e\textsuperscript{\pm} number densities, plotted against T\textsubscript{cm}. The five curves correspond to five different assumed values of the vacuum electron rest mass, m\textsubscript{e}.

\begin{equation}
\rho = \frac{g}{(2\pi)^3} \int d^3 p \frac{E(p)}{e^{E(p)/T} + 1}.
\end{equation}

If the particles are massless, then the dispersion relation is E(p) = p, and the energy density reduces to

\begin{equation}
\rho^{(m=0)} = \frac{7\pi^2 g}{240} T^4.
\end{equation}

If the particles have small nonzero masses, i.e. m \ll T, the dispersion relation is

\begin{equation}
E(p) = \sqrt{p^2 + m^2}
\end{equation}

\begin{equation}
\approx p + \frac{m^2}{2p},
\end{equation}

to second order in m. Substituting Eq. (17) into Eq. (14), we find

\begin{equation}
\rho^{(m\ll T)} = \rho^{(m=0)} \left\{ 1 - \frac{5}{7\pi^2} \left[ \frac{m}{T} \right]^2 \right\},
\end{equation}

where we have only kept terms to order m\textsuperscript{2}. We will define x such that x \equiv m/T. We find a similar expression to Eq. (18) for the pressure

\begin{equation}
P = P^{(m=0)} \left[ 1 - \frac{15}{7\pi^2} x^2 \right],
\end{equation}

and the entropic density

\begin{equation}
S/V = (S/V)^{(m=0)} \left[ 1 - \frac{15}{14\pi^2} x^2 \right],
\end{equation}

where we have dropped the superscript label (m \ll T) on the left-hand-side of Eqs. (19) and (20) for ease in notation. If we re-examine the statistics of e\textsuperscript{\pm} annihilation, but drop the assumption that m\textsubscript{e} = 0, then conservation of comoving entropy implies

\begin{equation}
\frac{2\pi^2}{45} T^3 (a_i) a_i^3 \left\{ 2 + \frac{7}{8} [2 + 2] \left[ 1 - \frac{15}{14\pi^2} x^2(a_i) \right] \right\} = \frac{2\pi^2}{45} T^3 (a_f) a_f^3 \left\{ 2 \right\},
\end{equation}
where we have written $x$ at the initial epoch as a function of $a_i$. The curly brackets in the left-hand-side of Eq. (21) show the change in $g_{*S}(a_i)$ to second order in $m_e$, whereas the right-hand-side is simply the case when only photons contribute to the entropy. Solving for the ratio of temperatures in Eq. (21) yields

$$\left[\frac{T_{\text{cm}}}{T}\right]_{f.o.} = \left[\frac{4}{11}\right]^{1/3} \left[1 + \frac{5}{22\pi^2}x^2(a_i)\right].$$ (22)

We will write $x(a_i)$ as $x(a_{\text{dec}}) = m_e/T(a_{\text{dec}})$ in accordance with item 1 of the list in Sec. 2.2. Fig. 2 shows contours of constant $100 \times \delta[T_{\text{cm}}/T]_{f.o.}$ in the $T(a_{\text{dec}})$ versus $m_e$ parameter space, where we take

$$\delta[T_{\text{cm}}/T]_{f.o.} \equiv \frac{T_{\text{cm}}/T}_{f.o.} - \frac{[4/11]^{1/3}}{[4/11]^{1/3}}.$$ (23)

The curves in Fig. 2 were calculated using our code BURST [11]. The contour locations agree to high precision with Eq. (22) over the entire parameter space. The agreement is the best for small $x(a_{\text{dec}})$ and slightly degrades for increasing $m_e$ and decreasing $T(a_{\text{dec}})$ as expected from Eq. (22). As $T(a_{\text{dec}})$ decreases, there are fewer $e^\pm$ pairs remaining once the neutrinos decouple. The result is that the photons do not heat up as much as when $T(a_{\text{dec}})$ is large, and so $T_{\text{cm}}/T$ decreases. Similarly, as $m_e$ increases, there is a smaller energy density of $e^\pm$ pairs and $T_{\text{cm}}/T$ increases by the same logic as the $T(a_{\text{dec}})$ dependence. For a given contour value, Eq. (22) states $T(a_{\text{dec}}) \propto m_e/[T_{\text{cm}}/T]_{f.o.}^{1/2}$, implying that the slope of the contour will increase with decreasing $[T_{\text{cm}}/T]_{f.o.}$. The contour where $\delta[T_{\text{cm}}/T]_{f.o.}$ is identically zero is reached if $m_e$ is set to zero.

Fig. 2 shows that $[T_{\text{cm}}/T]_{f.o.}$ is always slightly larger than $[4/11]^{1/3}$. Although this is useful for calculating the energy density of neutrinos given the plasma temperature, $T_{\text{cm}}$ and $T_{\text{cm}}/T$ themselves are not physical observables. We can use the baryon number density as another physical observable to tease out the value of $[T_{\text{cm}}/T]_{f.o.}$. To accomplish this task, we will utilize the baryon density $\omega_b$. $\omega_b$ is related to the contribution of baryon rest mass to the closure density of the universe, $\Omega_b$, and the Hubble parameter, $h$

$$\omega_b = \Omega_b h^2,$$ (24)

where $h$ is used to parameterize the Hubble expansion rate at the current epoch, $H_0$. 

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Fig. 2. Contours of $100 \times \delta[T_{\text{cm}}/T]_{f.o.} = 100 \times ([T_{\text{cm}}/T]_{f.o.} - [4/11]^{1/3})/[4/11]^{1/3}$ are plotted in the assumed electron rest mass $m_e = T(a_{\text{dec}})$ plane. The contours are calculated using the full BURST code, however, they agree with the analytic estimate in Eq. (22).
\[ H_0 = 100 \times h \text{ km/s/Mpc}. \] (25)

By using \( \omega_b \), we can calculate the proper baryon number density. We can relate the baryon number density at the current epoch \( n_b(a_0) \) to the baryon number density at any epoch by the following

\[ n_b(a_0) = n_b(a) \left[ \frac{a}{a_0} \right]^3, \] (26)

where we have assumed the product of number density and the cube of the scale factor is a comoving invariant. \( a_0 \) is the scale factor at the current epoch. The baryon density at the current epoch is then measured by CMB experiments using the baryon density, \( \omega_b \)

\[ n_b(a_0) = \frac{3m_{pl}^2}{8\pi m_b} \omega_b \times [10^2 \text{ km/s/Mpc}]^2, \] (27)

where \( m_{pl} \) is the Planck mass and \( m_b \) is the baryon rest mass. In this work, we will use \( \omega_b = 0.022068 \) from Ref. [2] which is identical within statistical precision to the updated value in Ref. [2]. To calculate the baryon number density at any epoch, we can use Eqs. (26), (27), and our definition for \( T_{cm} \)

\[ n_b(a) = \frac{3m_{pl}^2}{8\pi m_b} \omega_b \left[ \frac{T_{cm}(a)}{T_{cm}(a_0)} \right]^3 \times [10^2 \text{ km/s/Mpc}]^2. \] (28)

As \( T_{cm} \) is a construct and not a physical observable, we must write the ratio of comoving temperature parameters in Eq. (28) in terms of plasma temperatures. As a zeroth approximation for our purposes, we use Eq. (9) to write

\[ n_b^{(z)}(a) = \frac{3m_{pl}^2}{8\pi m_b} \omega_b \frac{11}{4} \left[ \frac{T(a)}{T(a_0)} \right]^3 \times [10^2 \text{ km/s/Mpc}]^2. \] (29)

We adorn \( n_b \) with a superscript \((z)\) to denote that Eq. (29) is a zeroth approximation since we have ignored the contribution from the nonzero electron rest mass. When we run BURST with a finite nonzero value of \( m_e \), we find slight discordance between our calculated value of \( n_b(a_0) \) and that of the true value in Eq. (27). To correct for the discrepancy, we run another iteration of BURST with a corrected baryon number density

\[ n_b(a) = \frac{3m_{pl}^2}{8\pi m_b} \omega_b \frac{11}{4} C \left[ \frac{T(a)}{T(a_0)} \right]^3 \times [10^2 \text{ km/s/Mpc}]^2, \] (30)

where \( C \) is our correction factor. It is possible to analytically calculate an estimate of \( C \) using entropy conservation. However, there is a slight subtlety we need to address to do so.

We can incorporate the baryon density into our previous nomenclature if we depart from using the entropic density and instead use the ratio of entropic density to baryon number density

\[ s = \frac{S}{V n_b} \] (31)

which we will call the entropy per baryon. The comoving invariant quantity, \([S/V]a^3\), we first employed in Eq. (4) becomes the entropy per baryon quantity \( s \). Up until this point, we have used entropy conservation in the plasma (or equivalently conservation of entropy per baryon in the plasma) to calculate ratios of quantities before and after certain epochs. As we vary both
\( m_e \) and \( T(a_{\text{dec}}) \), we are changing the entropy per baryon in the plasma at the epoch \( a_{\text{dec}} \) via Eq. (20). Because the entropy per baryon in the plasma is proportional to the quotient of plasma temperature cubed to baryon number density, a different plasma entropy per baryon at neutrino decoupling is equivalent to a different baryon density at the current epoch. Therefore, entropy conservation in the plasma is not the relevant quantity to investigate. Alternatively, we will consider the total entropy of the universe, i.e., the sum of the plasma and neutrino components

\[
s_{\text{tot}} = s_{\text{pl}} + s_{\nu}
\]

\[
= \frac{1}{n_b} \left[ \rho + P \right]_{\text{pl}} + \frac{[S/V]_\nu}{n_b}.
\]  

(32) (33)

Neutrinos will thermodynamically decouple from the plasma, implying that we cannot use Eqs. (3) and (31) to determine the entropy per baryon in the neutrino seas in general. However, at this point we are continuing to operate under the assumption of FD-shaped distributions as discussed previously (we will relax this constraint in Sec. 5), implying we can write \( s_{\nu} \) as

\[
s_{\nu} = \frac{2\pi^2}{45} \frac{7}{8} 6T_\nu^3,
\]

(34)

where the factor of 6 comes from 3 flavors of neutrinos, and 3 flavors of antineutrinos all at the same temperature parameter. We have used the symbol \( T_{\nu} \) to denote the neutrino temperature parameter

\[
T_{\nu}(a) = \begin{cases} 
T(a) & a > a_{\text{dec}} \\
T_{\text{cm}}(a) & a < a_{\text{dec}}.
\end{cases}
\]

(35)

In practice, we will refrain from using the symbol \( T_{\nu} \) and instead use either \( T(a) \) or \( T_{\text{cm}}(a) \) to denote the energy scale at a particular epoch.

We begin executing BURST at a temperature higher than \( T(a_{\text{dec}}) \). This is for computational reasons only: our code must initialize a time step before we consider the cosmological epochs relevant to whatever physics we wish to study. As a corollary, we normalize the baryon number density at the starting temperature \( T(a_{s}) = T(a_i) \) in Eq. (29). At \( T(a_{s}) \), the total entropy in the universe is

\[
s_{\text{tot}}(a_{s}) = \frac{1}{n_b(a_{s})} \frac{2\pi^2}{45} \left[ 2 + \frac{7}{8} \left[ 2 + 2 \left[ 1 - \frac{15}{14}x^2(a_{s}) \right] + \frac{7}{8} \right] T(a_{s})^3, \right.
\]

(36)

where \( x(a_{s}) \equiv m_e/T(a_{s}) \). The total entropy at the current epoch (well after electrons and positrons have annihilated) is

\[
s_{\text{tot}}(a_{0}) = \frac{1}{n_b(a_{0})} \frac{2\pi^2}{45} \left[ 2 + \frac{7}{8} \left[ \frac{T_{\text{cm}}}{T} \right]_{a_0}^3 \right] T_0^3.
\]

(37)

If we use Eq. (9) and equate Eqs. (36) to (37), we find

\[
\left[ T(a_{s}) \right]^3 = \left[ T(a_{0}) \right]^3 - \frac{15}{43}x^2(a_{s}) - \frac{315}{946}\pi^2 x^2(a_{\text{dec}}).
\]

(38)

Fig. 3 shows contours of constant \( 100 \times [1 - C] \) in the \( T(a_{\text{dec}}) \) versus \( m_e \) parameter space. The contours were calculated using BURST, and agree exceedingly well with the prediction

\[
C = 1 - \frac{15}{43}x^2(a_{s}) - \frac{315}{946}\pi^2 x^2(a_{\text{dec}}).
\]

(39)
of Eq. (39). If we only used the correction from \([T_{cm}/T]_{e.o.}\), and neglected the contribution from \(x(\alpha_e)\), the contours from the calculation would have diverged from Eq. (39) at large \(m_e\).

4. Finite temperature QED corrections

Thus far, we have taken the gas of photons, electrons, and positrons to behave like an ideal gas, with energy density for fermions from Eq. (14) and pressure for fermions and bosons given by

\[
P_j^{(0)} = \frac{g_j}{2\pi^2} \int_0^\infty dp \frac{p^4}{3E(p)} \frac{1}{e^{E(p)/T} \pm 1},
\]

where \(E(p) = \sqrt{p^2 + m_j^2}\) for species \(j\). The “+” sign in the occupation number refers to fermions (electrons or positrons), whereas the “−” sign refers to bosons (photons). We have ignored the chemical potential for electrons and positrons. The superscript \((0)\) on the pressure symbol denotes that Eq. (40) is the pressure for the ideal gas. In the early universe, charge screening and self-interaction energies will change the pressure quantity. Reference [6] gives the change to the pressure as

\[
P_j = P_j^{(0)} - P_j^{(\text{int})}
\]

\[
P_j^{(\text{int})} = \frac{1}{4\pi^2} \int_0^\infty dp \frac{p^2}{E(p)} \frac{\delta m_j^2(p, T)}{e^{E(p)/T} \pm 1},
\]

which introduces the shift in the square of the particle mass, \(\delta m_j^2(p, T)\), as a function of \(p\) and \(T\). Reference [6] calculates the shift from QED self-interactions for electrons or positrons as

\[
\delta m_e^2(p, T) = \frac{2\pi \alpha T^2}{3} + \frac{4\alpha}{\pi} \int_0^\infty dk \frac{k^2}{E(k)} \frac{1}{e^{E(k)/T} + 1}
\]

\[
- \frac{2m_e^2\alpha}{\pi p} \int_0^\infty dk \frac{k}{E(k)} \log \left| \frac{p+k}{p-k} \right| \frac{1}{e^{E(k)/T} + 1}.
\]
$m_e$ in the above expression is still the vacuum mass and $E(k) = \sqrt{k^2 + m^2_e}$. $\alpha \simeq 1/137$ is the fine structure constant. As we do not include the electron chemical potential, the shift in the square of the electron and positron masses are identical. Initially, we will ignore the third term in Eq. (43); this is tantamount to $\delta m^2_e(p, T) \rightarrow \delta m^2_e(T)$. Fig. 4 shows how Eq. (43) changes in the $T$ versus $m_e$ parameter space. Note that $\delta m^2_e$ becomes larger with increasing temperature. $\delta m^2_e$ is equal to the vacuum value of $m^2_e$ at a temperature of a few MeV. We have shown that the ratio $[T_{cm}/T]_{f.o.}$ is sensitive to the thermodynamics in this temperature range. For high precision calculations of neutrino energy density, an accurate description of this epoch is imperative.

Photons in medium are plasmons with effective mass squared given by [7]

$$\delta m^2_\gamma(T) = \frac{8\alpha}{\pi} \int_0^\infty dk \frac{k^2}{E(k)} \frac{1}{e^{E(k)/T} + 1}. \quad (44)$$

Note that the shift in mass from zero is solely a function of temperature.

With the relevant in-medium particle masses inferred from these shifts, the entropy per baryon at $a_{dec}$ will be slightly different, which will lead to a change in $[T_{cm}/T]_{f.o.}$. We can analytically estimate the contribution of $P_{(int)}$ to the total pressure if we assume $E(p) \simeq p$ in Eq. (42) and $E(k) \simeq k$ in Eqs. (43) and (44). After applying the approximations, the shifts in the masses are

$$\delta m^2_e(T) = \pi \alpha T^2, \quad (45)$$

$$\delta m^2_\gamma(T) = \frac{2\pi \alpha T^2}{3}, \quad (46)$$

and the interacting pressures are

$$P_{e_{(int)}} = \frac{\pi \alpha T^4}{48}, \quad (47)$$

$$P_{\gamma_{(int)}} = \frac{\pi \alpha T^4}{36}, \quad (48)$$

where $P_{e_{(int)}}$ is the interacting pressure for either electrons or positrons. Now that we have the expressions for the various $P_{i_{(int)}}$, we need an expression for the energy density. To calculate the total energy density, we minimize the Gibbs free energy to find [17]

$$\rho = -P + T \frac{dP}{dT}. \quad (49)$$
If we use the above approximations, the derivatives are trivial and the interacting energy densities become

\[
\rho_e^{(\text{int})} = \frac{\pi \alpha T^4}{16},
\]

\[
\rho_\gamma^{(\text{int})} = \frac{\pi \alpha T^4}{12}.
\]

(50)

(51)

The interacting pressure and energy density components give us the conserved entropy per baryon in terms of \( s_{pl}^{(0)} \)

\[
s_{pl} = s_{pl}^{(0)} \left[ 1 - \frac{25\alpha}{22\pi} \right].
\]

(52)

To calculate a new \([T_{cm}/T]_{f.o.}\) ratio, we use Eq. (22) and the correction from Eq. (52) to find

\[
\left[ \frac{T_{cm}}{T} \right]_{f.o.} = \left[ \frac{4}{11} \right]^{1/3} \left[ 1 + \frac{5}{22\pi^2} x^2(\alpha_{\text{dec}}) + \frac{25\alpha}{66\pi} \right].
\]

(53)

The QED contribution to the change in \([T_{cm}/T]_{f.o.}\) is identical to that of Eq. (41) in Ref. [8].

In Sec. 3, we estimated the change in the correction to the baryon density from the electron rest mass. As an alternative to baryon density, we can use the radiation energy density parameterized by \( N_{\text{eff}} \)

\[
\rho_{\text{rad}}(a) = \left\{ 2 + \frac{7}{4} \left[ \frac{4}{11} \right]^{4/3} N_{\text{eff}} \right\} \frac{\pi^2}{30} T^4(a).
\]

(54)

\( \rho_{\text{rad}} \) is the radiation energy density, with photon and neutrino components. We could use Eq. (54) at any value of the scale factor and monitor how \( N_{\text{eff}} \) evolves as the neutrinos decouple from the electromagnetic plasma [e.g., see Fig. (5) of Ref. [11]]. Generally, \( N_{\text{eff}} \) is considered a constant and Eq. (54) is used once \([T_{cm}/T]_{f.o.}\) nears its asymptotic value. In this paper, we will adopt the traditional approach and consider how \([T_{cm}/T]_{f.o.}\) and \( N_{\text{eff}} \) change in the \( T(\alpha_{\text{dec}}) \) versus \( m_e \) parameter space. Equation (31) of Ref. [11] provides an expression for \( N_{\text{eff}} \) which we can solve when the ratio of \([T_{cm}/T]_{f.o.}\) differs from \([4/11]^{1/3}\)

\[
N_{\text{eff}} = 1 + \delta[T_{cm}/T]_{f.o.} \times \left[ 3 + \delta\rho_{\nu_e} + \delta\rho_{\nu_\mu} + \delta\rho_{\nu_\tau} \right],
\]

(55)

where \( \delta\rho_{\nu_i} \) is the relative change in the neutrino energy density of species \( i \) at freeze-out and we have assumed that neutrinos and antineutrinos of any flavor change identically. If neutrinos were to preserve FD-shaped distributions, \( \delta\rho_{\nu_i} = 0 \). Therefore, the change in \( N_{\text{eff}} \) stemming from nonzero rest mass and QED corrections is

\[
\Delta N_{\text{eff}} \equiv N_{\text{eff}} - 3 = 3 \times \delta[T_{cm}/T]_{f.o.} \times \left[ 3 + \delta\rho_{\nu_e} + \delta\rho_{\nu_\mu} + \delta\rho_{\nu_\tau} \right] \times \frac{30}{11\pi^2} x^2(\alpha_{\text{dec}}) + \frac{50\alpha}{11\pi}.
\]

(56)

(57)

**Fig. 5** shows the changes in \([T_{cm}/T]_{f.o.}\) and \( N_{\text{eff}} \) as calculated in **BURST**. The top panel gives \( 100 \times [T_{cm}/T]_{f.o.} \) in the \( T(\alpha_{\text{dec}}) \) versus \( m_e \) parameter space. The contours agree well with the analytic estimate of Eq. (53). As compared to **Fig. 2**, the contours do not cover as large a range of values. For small \( m_e \), the QED correction dominates and the contours become spaced further apart, never reaching the \( 10^{-4} \) level as they do in **Fig. 2**. Conversely, for large \( m_e \), the contribution from the vacuum rest mass becomes dominant and the contours begin to look identical to those
Fig. 5. Plots of freeze-out quantities with finite-temperature QED effects. (Top) Contours of $100 \times \delta[T_{\text{cm}}/T]_{\text{f.o.}}$ in the $T(a_{\text{dec}})$ versus $m_e$ plane. The contours are calculated using BURST, however, they agree with the analytic estimate in Eq. (53). (Bottom) $100 \times \Delta N_{\text{eff}}$ in the $T(a_{\text{dec}})$ versus $m_e$ plane. The contours are calculated using BURST, however, they agree with the analytic estimate in Eq. (57).

of 2. The bottom panel shows how $100 \times \Delta N_{\text{eff}}$ changes in the same parameter space. If we use the small-value approximation for a power function, i.e., $[1 + y]^n \simeq 1 + ny$ for $y << 1$, Eq. (55) shows $\Delta N_{\text{eff}} \simeq 12 \delta[T_{\text{cm}}/T]_{\text{f.o.}}$. We chose the contours of the bottom plot of Fig. 5 to be 12 times the value of the contours in the top plot. Clearly, corresponding contours appear in the identical parts of the parameter space.

In this section we ignored how $\delta m_e^2$ depends on momentum $p$ in Eq. (43) when deriving the correction to $[T_{\text{cm}}/T]_{\text{f.o.}}$. We have done the full calculation with the momentum dependence included and have found a relative change in $\Delta N_{\text{eff}}$ of less than one part in $10^3$. A change this small would only be discernible in Fig. 5 for small $m_e$.

5. Full neutrino transport and nonequilibrium energy distributions

Here we consider the timelike heat flow engendered by out-of-equilibrium neutrino scattering on electrons and positrons and the associated alterations in comoving entropy conservation, $N_{\text{eff}}$, the neutron-to-proton ratio, and primordial nucleosynthesis. As the universe expands and the temperature decreases, the equilibrium between photons and $e^\pm$ pairs shifts to fewer of the later, ultimately increasing the product of scale factor and plasma temperature. However, while the annihilation of electrons and positrons proceeds, neutrinos are going from completely-thermally coupled particles to freely-falling decoupled particles. Electrons and positrons are the princi-
pal scattering targets that facilitate energy exchange between the decoupling neutrinos and the plasma. These scattering processes occur out of equilibrium, thereby generating a timelike heat flow, i.e., transfer of entropy, from the plasma into the decoupling neutrino component.

The heat transfer between $e^\pm$ pairs and neutrinos results in a decrease of $s_{pl}$ and an increase in $s_v$. Overall, the total entropy per baryon of the universe, $s_v + s_{pl}$, increases [see Fig. (9) of Ref. [11]]. The expressions we derived in sections 3 and 4 all assumed comoving entropy conservation in either the special case of the plasma or the general case of the universe in total. In either case, those expressions are not valid during the weak decoupling process.

As the neutrinos do not maintain FD equilibrium in this case, we must calculate the effective entropy per baryon in the neutrino seas using a energy-distribution based definition of entropy density for fermions

$$s_v(a) = -\frac{1}{n_b(a)} \frac{T_{cm}^3(a)}{2\pi^2} \sum_{j=1}^{6} \int_0^\infty d\epsilon \epsilon^2 \{ f_j \ln f_j + [1 - f_j] \ln[1 - f_j] \}, \quad (58)$$

where we have implicitly assumed that $a > a_{dec}$. $\epsilon$ is a dummy variable such that $\epsilon = E/T_{cm}$ and $f_j$ is the occupation number as a function of $\epsilon$ for the six individual neutrino species $j$. In addition, the $f_j$ evolve and are functions of $a$. The Quantum Kinetic Equations (QKEs) dictate the evolution of the neutrino occupation numbers [18]

$$\frac{\partial \hat{F}}{\partial t} = -i[H, \hat{F}] + \hat{C}, \quad (59)$$

where $i = \sqrt{-1}$ and we have taken Eq. (24) of [18] and simplified to a homogeneous and isotropic geometry. We have departed from our convention of using $a$ as the independent variable in order to be consistent with QKE literature (see Refs. [18–34] for a discussion on QKEs). $\hat{F}$ is a $6 \times 6$ generalized density matrix where the occupation numbers of the 3 neutrinos and 3 antineutrinos fall along the diagonal of the matrix. The coherent evolution of $\hat{F}$ is given by the commutator of a Hamiltonian-like potential with $\hat{F}$. We will ignore this term and focus on the term which can affect the entropy, mainly the collision term $\hat{C}$, which encodes incoherent neutrino scattering collisions. As we ignore the coherent evolution, the off-diagonal elements of $\hat{F}$ remain zero thereby lessening the need for a full QKE treatment. We employ a collision integral, $C_j$, originally derived in Refs. [36,37] and modified in Ref. [11]

$$\frac{df_j(\epsilon)}{dt} = C_j(\{f_k\}), \quad (60)$$

where the collision integral for $f_j(\epsilon)$ is a functional of the entire set of occupation numbers $\{f_k\}$. The change in the energy density of the neutrino component is

$$\frac{d\rho_v}{dt} = \frac{T_{cm}^4}{2\pi^2} \sum_j \int_0^\infty d\epsilon \epsilon^3 \frac{df_j(\epsilon)}{dt} \quad (61)$$

$$= -\frac{d\rho_e}{dt}. \quad (62)$$

---

1 Neutrino oscillations, via the Hamiltonian-like potential, can indeed induce changes in the entropy. We ignore those contributions. See Eq. (5.12) in Ref. [35] for details.
Table 1
Various quantities of interest for each $m_e$ run with neutrino transport. Column 2 gives $[T_{cm}/T]_{\text{f.o.}}$. For reference: $[4/11]^{1/3} = 0.7138$. Column 3 gives the correction factor for the baryon number density in Eq. (30). Column 4 gives the change in $N_{\text{eff}}$ from Eq. (55). Column 5 is the primordial mass fraction of $^4\text{He}$. $T_{cm}(a_{\text{in}}) = 10 \text{ MeV}$ for all runs.

| $m_e$ [MeV] | $[T_{cm}/T]_{\text{f.o.}}$ | $100 \times [1 - \mathcal{C}]$ | $\Delta N_{\text{eff}}$ | $Y_P$ |
|------------|-----------------|---------------------------------|-----------------|-------|
| 0.1        | 0.7144          | 0.2808                          | 0.0120          | 0.3668|
| 0.25       | 0.7146          | 0.3655                          | 0.0190          | 0.3313|
| 0.511      | 0.7154          | 0.6610                          | 0.0442          | 0.2479|
| 0.75       | 0.7164          | 1.069                           | 0.0793          | 0.1510|
| 1.0        | 0.7178          | 16.06                           | 0.1265          | 0.0441|

Here $d\rho_e/dt$ represents the instantaneous decrement in the energy density residing in the electron and positron components. Neutrino–electron scattering, for example, might result in a higher energy neutrino and a lower energy electron, leaving a nonthermal energy distribution for the electrons. We assume that thermal and chemical equilibrium in the electromagnetic plasma is instantaneously reattained. To properly follow the evolution of the temperature, we add this change in energy density from Eq. (62) into the plasma temperature derivative [38]

$$
\frac{dT}{dt} \simeq -3H \rho + P - \frac{1}{3H} \frac{d\rho_e}{dt},
$$

(63)

where $\rho$ and $P$ are the energy density and pressure, respectively, of the electromagnetic plasma. We note that Eq. (63) also contains terms for the baryon components. We have ignored these terms when writing Eq. (63) for ease in notation, but we include them in the actual calculation [see Eq. (D.28) in Ref. [38]]. In summary, the change in the plasma energy density from neutrino scattering decreases the temperature and hence raises the ratio $[T_{cm}/T]_{\text{f.o.}}$.

Once we have the derivative for the plasma temperature, we can evolve through the BBN epoch to determine $[T_{cm}/T]_{\text{f.o.}}$. Table 1 gives quantities related to the topics previously discussed for five different assumed values of $m_e$. In the third row, $m_e = 0.511 \text{ MeV}$ corresponds to the true vacuum value. The calculations include the integration of the full Boltzmann neutrino transport network of Ref. [11] and the finite-temperature QED corrections, including the momentum-dependent term in Eq. (43). We initiate the Boltzmann neutrino transport calculation at a temperature sufficiently high such that the neutrino component is in equilibrium. We take this epoch to be the same as the one designated in Eq. (1). For all values of $m_e$, we take $T_{cm}(a_{\text{in}}) = 10 \text{ MeV}$. We give the ratio $[T_{cm}/T]_{\text{f.o.}}$ in column 2, where we calculate the evolution of $T$ with the $d\rho_e/dt$ term in Eq. (63). Column 3 relates to the correction factor for the baryon number density, defined in Eq. (30), and column 4 gives the calculation of $\Delta N_{\text{eff}}$. Here, $\Delta N_{\text{eff}}$ includes not only the finite electron mass and QED corrections discussed above, but also the scattering-induced, nonthermal neutrino energy distribution functions. All three of the quantities in columns 2–4 involve a set of three corrections: nonzero electron mass; finite-temperature

---

2 The third row in the table shows $N_{\text{eff}} = 3.044$ in the standard case. This is different than the previous results of Refs. [11] and [39]. This difference stems from an alternate implementation of the finite-temperature QED effects. The new implementation is an attempt to conform to Ref. [17]. However, our implementation handles the derivative of temperature with respect to time differently than is done in Ref. [17] [see Eq. (63) and Eq. (17) in that work]. Note that our calculation neglects all neutrino flavor oscillations. As a consequence of these considerations, we would not expect to
QED effects on the plasma equation of state; and entropy flow between the plasma and neutrino seas. The contribution of the entropy flow to $\delta [T_{\text{cm}}/T]_{\text{f.o.}}, 1 - C$, and $\Delta N_{\text{eff}}$ is on order a factor of 3–50 times larger than the contribution from the other two effects, i.e., nonzero electron mass and finite-temperature QED effects. Appendix B of Ref. [42] gives an analytic estimate for how the entropy flow changes $[T_{\text{cm}}/T]_{\text{f.o.}}$. Finally, column 5 gives the primordial abundance of $^4\text{He}, Y_P$, from nucleosynthesis.

Fig. 6 shows quantities involving the plasma temperature as a function of $T_{\text{cm}}$ plotted for various values of $m_e$. The top panel shows how $T$ cools as the universe expands, while the bottom panel shows how the neutrinos cool relative to the plasma via the ratio $T_{\text{cm}}/T$. The horizontal axis in both plots is $T_{\text{cm}}$. The five colored lines show the evolution of either $T$ or $T_{\text{cm}}/T$ for various values of $m_e$. At the level of precision in the top panel, it appears that all five lines for $T$ converge at the end of the epoch of $e^\pm$ annihilation, implying identical values of $[T_{\text{cm}}/T]_{\text{f.o.}}$. The second column of Table 1 shows that there are slight differences due to the effects discussed previously. This is more apparent at the level of precision in the bottom panel, where we plot the ratio $T_{\text{cm}}/T$ versus $T_{\text{cm}}$. As the mass increases, the ratio at freeze-out also increases. Although agree with Refs. [9] ($N_{\text{eff}} = 3.046$) and [40] ($N_{\text{eff}} = 3.045$) to the $10^{-3}$ level of precision. However, we note that our value of $N_{\text{eff}}$ does agree with Ref. [41] to the $10^{-3}$ level.
the mass changes \([T_{cm}/T]_{t.o.}\), both panels vividly show how the mass changes the location of the epoch of \(e^\pm\) annihilation.

Note that Fig. 6, consistent with Fig. 1, shows the large number of \(e^\pm\) pairs in equilibrium even at temperatures well below the electron rest mass. Additionally, Fig. 6 shows how the large \(e^\pm\) density, a consequence of the high entropy, facilitates transfer of entropy from the electromagnetic plasma into the decoupling neutrino component. As we assume FD equilibrium for the \(e^\pm\) occupation numbers, a larger mass will precipitate an earlier epoch/higher temperature when the \(e^\pm\) pairs disappear. This leads to a different phasing of \(T_{cm}\) and \(T\) clearly shown in the bottom panel of Fig. 6.

The phasing of \(T_{cm}\) and \(T\) is important as both energy scales are inputs into the weak-interaction rates which interconvert neutrons and protons. Before we present results related to nucleosynthesis, we note that only the assumed electron vacuum mass is used in the calculations of the weak-interaction rates; we do not use the finite-temperature QED modifications for the weak-interaction rates [43–45]. Fig. 7 shows the neutron-to-proton ratio (denoted \(n/p\)) as a function of \(T_{cm}\) for the various masses. As the mass increases, the epoch of \(e^\pm\) annihilation moves earlier in time, higher in \(T_{cm}\). This leads to reduced efficiency in scattering-induced transfer of entropy from the electromagnetic plasma into the neutrino seas. In turn, this effect leads to generally higher plasma temperatures at a given epoch \(T_{cm}\). Conversely, we could say that the earlier epoch of \(e^\pm\) annihilation leads to smaller \(T_{cm}\) at earlier times, but this is not the correct way to think about this problem/effect. The product of \(T_{cm}\) and scale factor is a comoving invariant, i.e., \(T_{cm}a = \text{constant}\). When we compare the neutron to proton interconversion rates [see Eqs. (19)–(24) in Ref. [46]] at equivalent times/scale-factors, we are comparing at the same \(T_{cm}\), so larger \(T\) or smaller \(T_{cm}\) are not equivalent statements. If the temperature is larger, there is an enhancement in the two charged-lepton capture rates: \(e^+ + n \rightarrow p + \bar{\nu}_e\) and \(e^- + p \rightarrow n + \nu_e\).

(Note that \(m_e\) is also larger which would suppress the charged-lepton capture rates; however, the increase in temperature is more important at higher kinetic energies.) Both of these rates are enhanced and keep \(n/p\) in equilibrium to lower temperatures

\[
(n/p)^{\text{eq}} = e^{-\delta m_{np}/T},
\]

where \(\delta m_{np} \simeq 1.3\) MeV is the mass difference between a neutron and a proton, and we have neglected electron and neutrino degeneracies. The black dashed line in Fig. 7 gives \((n/p)^{\text{eq}}\) for the case where we evaluate the evolution of the temperature \(T\) with assumed electron mass \(m_e = 1.0\) MeV. The curve for \(m_e = 1.0\) MeV is the last to depart the equilibrium track as the higher plasma temperature enhances the neutron-to-proton rates. Note that \((n/p)^{\text{eq}}\) depends on the evolution of \(T\) which is different for each mass case. However, the differences are small at the level of precision of Fig. 7. The changes in the out-of-equilibrium evolution of \(n/p\) are much starker.

Fig. 8 shows the evolution of the entropy per baryon in the plasma versus the comoving temperature parameter. Electrons can annihilate with positrons to produce neutrino–antineutrino pairs. If the charged leptons have larger masses, then each such annihilation event will produce more energetic neutrinos, enabling a larger entropy transfer from the plasma into the neutrino seas. Conversely, there are fewer \(e^\pm\) pairs in equilibrium for larger-mass charged leptons, implying fewer total annihilation events and a smaller flow of entropy from the plasma into the neutrino seas. Fig. 8 clearly shows that the first effect dominates over the second.

Two pieces of evidence support this result. First, weak decoupling involves the competition between the weak interaction rates and the Hubble expansion rate. We neither changed \(G_F\) (the weak coupling constant) nor \(m_{pl}\) (the Planck mass) implying that weak decoupling will occur
Fig. 7. Neutron-to-proton ratio as a function of $T_{cm}$. The dashed black line is the equilibrium evolution of $n/p$ in the case $m_e = 1.0$ MeV [see Eq. (64)].

Fig. 8. Entropy per baryon in the plasma as a function of $T_{cm}$. Roughly at the same time/$T_{cm}$ for different masses, as verified by Fig. 8. It is true that the dynamics of weak decoupling depend on the electron rest mass through the pair density. This leads to the second piece of evidence: weak decoupling occurs during pair domination. This is supported by Fig. 1, which shows that even for $m_e = 1.0$ MeV there are orders of magnitude more pairs than ionization electrons during the range of $T_{cm}$ particular to weak decoupling. Therefore, the location of the entropy flow is independent of the rest mass. The magnitude of the flow increases with increasing rest mass. To precipitate an earlier epoch for the entropy flow, the rest mass would need to be larger than 10 MeV so that neutrinos would fall out of equilibrium because of a lack of scattering targets and not from the low strength of the weak interaction. Electron masses that large would dictate a nonperturbative treatment incongruent with sections 3 and 4.

6. Conclusion

In this paper we have examined the role of charged lepton mass in the epoch of weak decoupling and nucleosynthesis in the early universe. We unphysically vary the input vacuum electron rest mass as a means to dissect the complicated and coupled nonlinear physics in this epoch. Our goal was to gain deeper and finer-scale insights into the physics of this epoch, with a subsidiary goal to identify potential problems in high-precision calculations. Clearly, the subject of neutrino decoupling and BBN is quite correctly regarded as well understood, with the basic ideas and calculations in place since the time of Ref. [47] (see Ref. [48] for a detailed review).
However, there is a renewed push for higher precision in calculations, driven by the prospect of higher precision in future CMB experiments, i.e., CMB Stage-IV [49], and the advent of 30-m class telescopes with the possibility that we can obtain higher-precision primordial deuterium measurements [4]. If the uncertainty in the observational value of $N_{\text{eff}}$ can be reduced to less than 1%, then the predicted value of $N_{\text{eff}} = 3.046$ (in line with our value 3.044) would be statistically different from 3.0 at the $2\sigma$ level. Moreover, ideas for beyond standard model (BSM) physics that might subtly alter the physics of this epoch are ubiquitous [50–53]. As an example, BSM physics which contains nonstandard neutrino–electron interactions are sensitive to the dynamics of electrons and positrons [40,54]. For these reasons, it is important to understand the outstanding issues in the calculations that impede better computational precision. Artificially changing the electron mass and examining the consequences allowed us to leverage the overwhelming physical facts of the early universe, large entropy and slow expansion rate, into a deeper understanding of the interplay of the weak interaction and thermodynamics at this epoch. This study has also helped to underline the importance of an improved treatment of in-medium corrections to electron and positron masses.

The electron rest mass directly plays a role in the thermodynamics, neutrino energy transport, and neutron-to-proton weak-interaction rates. Our calculations show how $e^\pm$ pairs dominate over ionization electrons even at the low energy scales expected at the end of the decoupling and BBN epoch. There are two key implications of the prolonged epoch of pair existence. First, there are more charged-lepton targets for neutrino scattering at lower temperatures, implying more transfer of entropy from the electromagnetic plasma to the decoupling neutrinos. The result of the entropy flow is an altered mixing of plasma temperature with scale factor, as well as non-thermal distortions in the decoupling neutrino energy spectra and a concomitant alteration to the ratio $[T_{\text{cm}}/T]_{\text{f.o.}}$. Nonzero electron rest mass, finite-temperature QED effects, and out-of-equilibrium neutrino transport all increase the temperature ratio from the standard equilibrium value of $[4/11]^{1/3}$. Second, at lower temperature scales more positrons induce an enhanced destruction of neutrons through the no-threshold, lepton-capture process $e^+ + n \rightarrow p + \bar{\nu}_e$. The enhanced destruction of neutrons alters the primordial helium yield, $Y_p$, and the primordial deuterium yield.

The third, fourth, and fifth columns of Table 1 address the way in which three unique cosmological observables change with varying electron rest mass. These are the baryon density, $\omega_b$, a measure of relativistic energy density, $N_{\text{eff}}$, and the primordial mass fraction of helium, $Y_p$. For $\omega_b$ and $N_{\text{eff}}$, the changes range from a few tenths of a percent to ten percent. The changes in the primordial helium abundance are much more drastic; $Y_p$ changes at the $\sim \pm 50\%$ level over the range of $m_e$ considered. Most intriguing, even small corrections to in-medium electron mass produce potentially observable nucleosynthesis effects. For example, if we perturb $m_e$ by 1% from the true vacuum value (in the range of finite-temperature QED corrections), we find changes in $Y_p$ at the 0.7% level. This is indeed a small change, but Fig. 4 showed that $\delta m_e^2$ can be as large as 10% of $m_e^2$ at $T \sim 1$ MeV. Future CMB experiments [49] will achieve 1% precision in cosmological observables such as $N_{\text{eff}}$ and $Y_p$. Our work shows that nucleosynthesis considerations ($Y_p$ and possibly deuterium) has the potential to break degeneracies in beyond-standard-model physics. Insights into neutrino physics gleaned from nucleosynthesis would be complementary to those of $N_{\text{eff}}$. The frontier of precision in weak-decoupling-nucleosynthesis calculations lies in an accurate treatment of neutrino physics, including neutrino–electron scattering and ultimately neutrino flavor quantum kinetics. Our work shows how neutrino physics is tightly coupled within the physics of the electromagnetic plasma, and thereby underscores the looming importance of improved plasma and QED corrections to charge-lepton properties.
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