The Bratteli diagram is an infinite graph which reflects the structure of projections in a $C^*$-algebra. We prove that every strictly ergodic unimodular Bratteli diagram of rank $2g + m - 1$ gives rise to a minimal geodesic lamination with the $m$-component principal region on a surface of genus $g \geq 1$. The proof is based on the Morse theory of the recurrent geodesics on the hyperbolic surfaces.

Key words and phrases: Bratteli diagrams, geodesic laminations

AMS (MOS) Subj. Class.: 19K, 46L, 57M.

Introduction

The paper deals with three apparently independent topics: the Morse theory of the recurrent geodesics, the Nielsen-Thurston theory of the geodesic laminations and, finally, a piece of the $C^*$-algebra theory, known as the Bratteli diagrams. Our goal is to show that the Bratteli diagrams imply the geodesic laminations via the Morse theory. Such a result links geometry to the operator algebras.
Recall that the simplest $C^*$-algebra can be written as

$$A_n = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C}),$$  \hfill (1)

where $M_{n_i}$ are the square matrices with the complex entries and $r, n_1, \ldots, n_r$ some non-negative integers. The closure of an infinite sequence

$$A_1 \subseteq A_2 \subseteq \ldots$$  \hfill (2)

of such $C^*$-algebras is a $C^*$-algebra denoted by

$$\mathfrak{a} = \lim_{n \to \infty} A_n.$$

The Bratteli diagram is an infinite connected graph, which reflects an embedding of the finite-dimensional algebras $A_n$ in the inductive limit $\mathfrak{a}$ ([1]). Such an embedding is described by a matrix of partial multiplicities $M_n$ whose entries are non-negative integers. If $\text{Rank } M_n = r = \text{Const}$ through all the embeddings, we say that the Bratteli diagram has rank $r$. If $\det M_n = \pm 1$ for all $n$, the Bratteli diagram is called unimodular. The Bratteli diagram is strictly ergodic if the linear space

$$L = \bigcap_{n=1}^{\infty} M_1 \ldots M_n (\mathbb{R}_+^r)$$  \hfill (4)

has dimension 1.

Let $S$ be a connected complete hyperbolic surface of genus $g$. Recall that a geodesic on $S$ is the maximal arc consisting of locally shortest sub-arcs. The study of geodesics on surfaces goes back to Birkhoff, Hadamard, Morse and Hedlund. A geodesic is simple if it has no self-crossing or self-tangent points. The periodic geodesic is an elementary example of simple geodesic. Birkhoff conjectured and Morse proved existence of simple recurrent non-periodic geodesics. The set of such geodesics turns to be uncountable on any hyperbolic surface.

The problem of classification of the recurrent non-periodic geodesics leads to the concept of a geodesic lamination. The geodesic lamination $\lambda$ on $S$ is a disjoint union of all recurrent non-periodic geodesics which lie in the closure of each other. One can think of $\lambda$ as an uncountable set of non-periodic geodesics running in the “same direction” on surface $S$. The geodesic
lamination proved to be fundamental in the dynamics, complex analysis, and topology ([2]).

The set $S - \lambda$ is a principal region of the geodesic lamination $\lambda$. It is an important combinatorial invariant of $\lambda$ and essentially a finite union of the “ideal polygons” $U_1, \ldots, U_m$. Each $U_i$ has a type described by a positive integer $k_i$. By a singularity data of $\lambda$ one understands a set $\Delta = (k_1, \ldots, k_m)$ such that $\Sigma k_i = 2g - 2$. In the present paper we prove the following theorem.

**Theorem 1** Let $B$ be a strictly ergodic unimodular Bratteli diagram of rank $r \geq 2$ and $\Delta$ a singularity data. Then the pair $(B, \Delta)$ defines a geodesic lamination $\lambda$ on the hyperbolic surface $S$ of the genus $g = g(\Delta)$ such that:

(i) the principal region of $\lambda$ has $r - 2g + 1$ connected components;

(ii) the singularity data of $\lambda$ coincides with $\Delta$.

The paper is organized as follows. In Section 1 we introduce the notation and lemmas which will be used to prove our main theorem. For other facts, we refer to the relevant bibliography. Theorem 1 is proved in Section 2. An example of the Bratteli diagram of an irrational rotation algebra with the “golden mean” Rieffel parameter is considered in Section 3.

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1 Preliminaries

In this section we bring together some useful facts on the geodesic laminations, Morse theory and combinatorics of the $C^*$-algebras. Our exposition is sketchy, and we refer the reader to ([1]), ([2] and ([4]) for a complete treatment. We omit an introduction to the interval exchange transformations, which are technically important for the proof of Theorem 1. However, we hope that the reader can recover the details by reading ([5]) and ([6]). Our notation is borrowed from there.

1.1 Geodesic laminations

Let $S$ be a connected complete hyperbolic surface. By a geodesic on $S$ we understand the maximal arc consisting of locally shortest sub-arcs. The geodesic is called simple if it has no self-crossing or self-tangent points.
A geodesic lamination on $S$ is a closed subset $\lambda$ of $S$ which is a disjoint union of simple geodesics of $S$. The geodesics of $\lambda$ are called leaves of $\lambda$. The lamination $\lambda$ is called minimal if no proper subset of $\lambda$ is a geodesic lamination. The following lemma gives a classification of the minimal laminations.

**Lemma 1** A minimal lamination $\lambda$ in a closed orientable hyperbolic surface $S$ is either a singleton (simple closed geodesic) or an uncountable nowhere dense subset of $S$.

*Proof.* See Lemma 3.3 of ([2]). □

The laminations $\lambda, \lambda'$ on $S$ are (topologically) equivalent if there exists a homeomorphism $\varphi : S \to S$ such that each leaf of $\lambda$ through point $x \in S$ goes to the leaf of $\lambda'$ through point $\varphi(x)$. Clearly, the set of all laminations, $\Lambda(S)$, on $S$ splits into the equivalence classes under this relation.

If $\lambda \in \Lambda(S)$, then a component of $S - \lambda$ is called a principal (complementary) region for $\lambda$. (Note that $S - \lambda$ may have several connected components.) The leaves of $\lambda$ which form the boundary of a principal region are called boundary leaves. If $\lambda$ is minimal (which we always assume to be), then each boundary leaf is a dense leaf of $\lambda$, isolated from one side.

Note that by Lemma 1

$$\text{Area } (S - \lambda) = \text{Area } S,$$

and therefore principal region is a complete hyperbolic surface of area $-2\pi \chi(S)$, where $\chi(S) = 2 - 2g$ is the Euler characteristic of surface $S$. If $U$ is a component of the preimage of the principal region in $D$, then $U$ is the union of the ideal polygons $U_i$ in $D$ (see Fig. 1).

The hyperbolic area of an ideal $n$-gon $U_i$ is equal to $(n - 2)\pi$, see [2]. Since

$$\sum \text{Area } U_i = (4g - 4)\pi,$$

the number of ideal polygons in $U$ is finite.

By a singularity data we mean the number and shape of the ideal polygons $U_i$ which cover the principal region of $\lambda$. Since $\text{Area } U_i = (n - 2)\pi$ and $\sum \text{Area } U_i = (4g - 4)\pi$, there exists only finite number of opportunities for such data with fixed $g$. 

4
(i) Maximal number of ideal polygons  
(ii) Minimal number of ideal polygons

Figure 1: Region $U = \bigcup U_i$ for surface $g = 2$

Let $\Delta = (k_1, \ldots, k_m)$ be a set of positive integers and half-integers such that $\sum k_i = 2g - 2$. To each ideal $n$-gon $U_i$ we assign number $k_i$ such that

$$k_i = \frac{n - 2}{2}.$$  

(7)

The reader can verify that condition $\sum \text{Area } U_i = (4g - 4)\pi$ is equivalent to $\sum k_i = 2g - 2$.

**Definition 1** Given lamination $\lambda \in \Lambda(S)$, the unordered tuple $\Delta = (k_1, \ldots, k_m)$ is called a singularity data of $\lambda$.

### 1.2 Morse coding of the geodesic lines

The idea of the method is to dissect the surface along $r = 2g + m - 1$ loops so that it becomes simply connected. Each loop gets a label and the geodesic line becomes a bi-infinite sequence of the labels accordingly the order it intersects the loops. Conversely, every bi-infinite sequence of the labels defines a geodesic (Morse’s theorem). Let us pass to the construction whose details can be found in ([4]).

Let $S$ be a connected complete hyperbolic surface of genus $g$ with $m$ boundary components. With no restriction, we can assume that the boundary components

$$h_1, h_2, \ldots, h_m$$

(8)

are closed geodesics of $S$. To render $S$ simply connected, choose a point $P$ on one of the boundary components, say $h_m$. First, using geodesic arcs one connects $P$ with a point lying on each of the remaining boundary components

$$h_1, h_2, \ldots, h_{m-1},$$

(9)
and then dissects $S$ along the arcs. The new surface will have a unique boundary.

Let

$$c_1, c_2, \ldots, c_{2g} \quad (10)$$

be the geodesic loops based in the point $P \in h_m$ and such that they dissect $S$ into a simply connected plane region $T$.

For simplicity, let us assume that the boundary components $p_i$ are punctures. In this case $T$ is a polygon bounded by an even number of the geodesic segments

$$c_1, c'_1, c_2, c'_2, \ldots, c_{2g}, c'_{2g}; h_1, h'_1, h_2, h'_2, \ldots, h_{m-1}, h'_{m-1} \quad (11)$$

where slash denotes the opposite sides of the cut.

Suppose we have an infinite stock of copies of $T$. If $l$ is a geodesic on $S$ which is disjoint from the boundary of $S$, we label each copy of $T$ with a symbol $\sigma$ from the set

$$c_1, c_2, \ldots, c_{2g}; h_1, h_2, \ldots, h_{m-1} \quad (12)$$

if $l \cap \sigma \neq \emptyset$. In this way one constructs an infinite cyclic cover of $S$ by gluing the copies of $T$ along the sides of $T$ hit by $l$. Such a cover is uniquely defined by a bi-infinite sequence of symbols

$$\ldots \sigma_{-1}, \sigma_0, \sigma_1 \ldots \quad (13)$$

with the values in the set (12). The above bi-infinite sequence is called a reduced (symbolic) curve.

Clearly, every geodesic on $S$ which does not intersect the boundary can be turned into a “symbolic curve”. An amazing fact proved by M. Morse is that the converse is true.

**Lemma 2** There is a one to one correspondence between the set of all geodesics on $S$ which does not intersect the boundary of $S$ and the set of all reduced curves.

**Proof.** This is essentially Theorem 3 of ([4]). □
1.3 AF $C^*$-algebras and Bratteli diagrams

An AF (approximately finite-dimensional) algebra is defined to be a norm closure of an ascending sequence of the finite dimensional algebras $M_n$’s, where $M_n$ is an algebra of $n \times n$ matrices with the entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents a multi-matrix algebra $M_n = M_{n_1} \oplus \ldots \oplus M_{n_k}$.

Let

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots \quad (14)$$

be a chain of algebras and their homomorphisms. A set-theoretic limit $A = \lim M_n$ has a natural algebraic structure given by the formula

$$a_m \rightarrow a, b_k \rightarrow b$$

for the sequences $a_m \in M_m, b_k \in M_k$. The homeomorphisms of the above (multi-matrix) algebras admit a canonical description ([3]). Suppose that $p, q \in \mathbb{N}$ and $k \in \mathbb{Z}^+$ are such numbers that $kq \leq p$. Let us define a homomorphism $\varphi : M_q \rightarrow M_p$ by the formula

$$a \mapsto a \oplus \ldots \oplus a \oplus 0_h, \quad (15)$$

where $p = kq + h$. More generally, if $q = (q_1, \ldots, q_s), p = (p_1, \ldots, p_r)$ are vectors in $\mathbb{N}^s, \mathbb{N}^r$, respectively, and $\Phi = (\phi_{kl})$ is a $r \times s$ matrix with the entries in $\mathbb{Z}^+$ such that $\Phi(q) \leq p$, then the homomorphism $\varphi$ is defined by the formula:

$$a_1 \oplus \ldots \oplus a_s \rightarrow \underbrace{(a_1 \oplus a_1 \oplus \ldots)}_{\phi_{11}} \oplus \underbrace{(a_2 \oplus a_2 \oplus \ldots)}_{\phi_{12}} \oplus \ldots \oplus 0_{h_1} \quad (16)$$

$$\oplus \underbrace{(a_1 \oplus a_1 \oplus \ldots)}_{\phi_{21}} \oplus \underbrace{(a_2 \oplus a_2 \oplus \ldots)}_{\phi_{22}} \oplus \ldots \oplus 0_{h_2} \oplus \ldots$$

where $\Phi(q) + h = p$. We say that $\varphi$ is a canonical homomorphism between $M_p$ and $M_q$. Any homomorphism $\varphi : M_q \rightarrow M_p$ can be rendered canonical ([3]).

Graphical presentation of the canonical homomorphism is called a Bratteli diagram. Every “block” of such diagram is a bipartite graph with $r \times s$ matrix $\Phi = (\phi_{kl})$.

In general, Bratteli diagram is given by a vertex set $V$ and edge set $E$ such that $V$ is an infinite disjoint union $V_1 \sqcup V_2 \sqcup \ldots$, where each $V_i$ has cardinality $n$. Any pair $V_{i-1}, V_i$ defines a non-empty set $E_i \subset E$ of edges with a pair of range and source functions $r, s$ such that $r(E_i) \subseteq V_i$ and $s(E_i) \subseteq V_{i-1}$. The
non-negative integral matrix of “incidences” \( M = (\phi_{ij}) \) shows how many edges there are between the \( k \)-th vertex in row \( V_{i-1} \) and \( l \)-th vertex in row \( V_i \).

### 2 Proof of Theorem 1

Let us outline the main steps of the proof. Let \( B \) be a strictly ergodic unimodular Bratteli diagram of rank \( r \). Then \( B \) defines a simple dimension group \( G \) of rank \( r \) with a unique state. Any such group can be realized as a dense subgroup \( \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_r \) of the real line. For a canonical state, we have \( \sum \lambda_i = 1 \). Given the singularity data \( \Delta \), we can construct an interval exchange transformation \( (\lambda, \pi) \) on the intervals \( \lambda_1, \ldots, \lambda_r \). The infinite sequence of induced transformations \( (\lambda, \pi) \supset (\lambda', \pi') \supset \ldots \) contracts to a point \( \theta \in [0, 1] \). It is possible to associate to \( \theta \) an infinite sequence of symbols taking values in the set \( \lambda_1, \ldots, \lambda_r \). We show that such a sequence is a recurrent non-periodic symbolic geodesic sequence. By the Morse theorem, we get a recurrent geodesic on the surface of genus \( g \). Let us pass to the detailed construction.

**Part I.** Let \((B, \Delta)\) be as in the theorem. We wish to construct an interval exchange transformation \( (\lambda, \pi) \) from the pair \((B, \Delta)\). For that consider a dimension group

\[
G(B) = \lim_{n \to \infty} (\mathbb{Z}^r, M_n),
\]

where \( r \) is the rank of \( B \) and \( M_n \) are matrices of partial multiplicities of \( B \). The unimodularity of \( B \) implies that \( G \) is a simple dimension group of the rank \( r \) without infinitesimal elements. The strict ergodicity of \( B \) is equivalent to the state space \( S(G) \) of \( G \) is a point ([3], Ch. 4). Let us recall the following lemma.

**Lemma 3** Suppose that \( G \) is a simple dimension group of rank \( r \) without infinitesimal elements and \( \dim S(G) = d - 1 \). Then \( G \) is order isomorphic to a dense subgroup of \( \mathbb{R}^d \), provided with the relative strict order.

**Proof.** This is essentially a Corollary 4.7 p. 25 of ([3]). \( \Box \)

Note that in our case \( \dim S(G) = 0 \) and therefore \( d = 1 \). Thus by Lemma 3 we have a dense subgroup of rank \( r \) of the real line. Let us fix generators
\(\lambda_1, \ldots, \lambda_r\) of the subgroup to be positive reals. Note that \(\lambda_i\) are linearly independent over \(\mathbb{Q}\) except the normalization condition \(\lambda_1 + \ldots + \lambda_r = 1\) which comes from the unique standard state on \(G\).

We set \(\lambda = (\lambda_1, \ldots, \lambda_r)\) and we wish to construct a permutation \(\pi\) on the above intervals from the singularity data \(\Delta\). Recall that every element \(\pi \in \Sigma_r\) of the permutation group on \(r\) elements decomposes into the elementary cycles \(\pi_1 \circ \ldots \circ \pi_s\). The decomposition is unique up to a cyclic permutation.

Let \(\Delta = (k_1, \ldots, k_m)\) be a singularity data. The Veech’s “zippered rectangles” construction (Section 6 of [6]) implies that the total number of elementary cycles

\[s = r - 2g + 1 = m.\]  

(18)

The length of the elementary cycle \(\pi_i\) is also determined (up to an isomorphism) by the corresponding singularity \(k_i\), see Veech, ibid. Therefore, we get a permutation \(\pi \in \Sigma_r\) such that

\[\pi = \pi_1 \circ \ldots \circ \pi_m.\]  

(19)

**Part II.** Let \((\lambda, \pi)\) be an interval exchange transformation obtained from the pair \((B, \Delta)\). We will assign to \((\lambda, \pi)\) an infinite sequence of symbols taking value in the finite set \(\lambda_1, \ldots, \lambda_r\). To achieve this goal, we will use the concept of “induced transformations” developed by Keane, Rauzy and Veech ([5]).

Recall that an interval

\[\Gamma = [\xi, \eta), \quad 0 \leq \xi < \eta \leq |\lambda|,\]  

(20)

where \(\lambda = \lambda_1 + \ldots + \lambda_r\), is called *admissible* for the interval exchange transformation \(\varphi = \varphi(\lambda, \pi)\) if \(\Gamma\) splits on \(r\) parts \(\lambda_1', \ldots, \lambda_r'\) such that \(\varphi^n(\lambda_i')\) is continuous on each of \(\lambda_i'\). The corresponding interval exchange transformation on \(\Gamma\) is called *induced*. The positive vectors \(\lambda = (\lambda_1, \ldots, \lambda_r)\) and \(\lambda' = (\lambda_1', \ldots, \lambda_r')\) are connected by the formula

\[\lambda = M\lambda',\]  

(21)

where \(M\) is a non-negative integral matrix of determinant \(\pm 1\), see ([5], Section 3). The matrix \(M\) coincides with the matrix of the partial multiplicity \(M_1\) which occurs at the first position in the Bratteli diagram \(B\).
Let
\[ \Gamma_1 \supset \Gamma_2 \supset \ldots \]  
be an infinite sequence of admissible intervals. Clearly, \( |\Gamma_n| \to 0 \) as \( n \to \infty \).
Then the set
\[ \theta = \cap_{n=1}^{\infty} \Gamma_n, \]  
(23)
is either empty or consists of a point. Assuming that the admissible intervals have only a finite number of common (right) endpoints, we get that \( \theta \) is a point such that \( 0 < \theta < 1 \).

Denote by \( \lambda_{ij}^{(j)} \), \( 1 \leq i \leq r \) a part of admissible interval \( \Gamma_j \) such that \( \theta \in \lambda_{ij}^{(j)} \). A sequence of subintervals
\[ S = \{ \lambda_{ij}^{(j)} \}_{j=1}^{\infty}, \]  
(24)
we call a pre-code.

To construct a code \( S^* \) from the pre-code \( S \), we insert a finite number of symbols between any two symbols \( \lambda_{ij}^{(j-1)} \) and \( \lambda_{ij}^{(j)} \) of \( S \) as follows. Recall that
\[ \lambda_{ij}^{(j-1)} = a_{ij1} \lambda_1^j + \ldots + a_{ijr} \lambda_r^j, \]  
(25)
where \( a_{mn} \) are entries of the matrix \( M_j \). We insert \( a_{ij1} \) symbols \( \lambda_1^j \), \( a_{ij2} \) symbols \( \lambda_2^j \), etc, between the symbols \( \lambda_{ij}^{(j-1)} \) and \( \lambda_{ij}^{(j)} \) of \( S \) in the order the orbit of the point \( \theta \) under the induced transformation \( \varphi_j = \varphi_j(\lambda_j, \pi_j) \) hits the admissible interval \( \Gamma_j \). We have therefore:
\[ \ldots \lambda_{ij-1}^{(j-1)} \underbrace{\lambda_1^j \ldots \lambda_{ij}^j}_{a_{ij1}} \ldots \underbrace{\lambda_1^j \ldots \lambda_{ij}^j}_{a_{ijr}} \lambda_{ij}^{(j)} \ldots \]  
(26)

Part III. Let \( S^* \) be a code associated to the pair \( (B, \Delta) \) as described above. \( S^* \) can be converted to a symbolic geodesic \( \Sigma = \Sigma(S^*) \) by “forgetting” the upper indices in the sequence \( S^* \). Clearly, the symbols of \( \Sigma \) take values in a finite set of cardinality \( r \).

**Lemma 4** \( \Sigma \) is a recurrent non-periodic symbolic geodesic.
Proof. The idea is to identify $\Sigma$ with a recurrent trajectory of a suspension flow over the interval exchange transformation $\varphi$ constructed in Part I. Indeed, let $v_t$ be such a flow obtained by the “zippered rectangles” method ([6]). Consider a trajectory $l = v_t(\theta)$ through the point $\theta$ defined in Part II. Since flow $v_t$ is minimal, the closure of $l$ is the entire surface $S$. In particular, $l$ is a recurrent non-periodic trajectory.

The intervals

$$\lambda_1, \ldots, \lambda_r$$

(27)
give a dissection of $S$ into a simply connected domain as follows. For $i = 1, \ldots, r$, one takes a rectangle with the opposite sides $\lambda_i, \lambda'_i$ and $f_i(a), f_i(b)$, where $\lambda'_i$ is the image of $\lambda_i$ under the Poincaré (first return) mapping and $a, b$ are the ends of the interval $\lambda_i$. In this way, the recurrent trajectory $l$ becomes a symbolic trajectory $\Sigma$ with the desired property. □

To finish the proof, let $S$ be a hyperbolic surface. Take a standard dissection of $S$ by the $r$ curves $c_1, \ldots, c_{2g}; h_1, \ldots, h_{m-1}$ as described in Section 1.2. Then the Morse theorem says that there exists a recurrent non-periodic geodesic $l_{\Sigma}$ on $S$. The closure $\lambda = \overline{l_{\Sigma}}$ is a minimal geodesic lamination with $m$ principal regions. Theorem follows. □

3 An example

In this section we consider an example of the “golden mean” Bratteli diagram. We construct a symbolic geodesic $\Sigma$ in this case, and show that $\Sigma$ coincides with an example of Morse.

Example. Let $B$ be a Bratteli diagram presented in Fig. 2.

The incidence matrix is a constant unimodular matrix

$$M_n = M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$  

(28)

Such a diagram is known to be strictly ergodic, see Effros ([3]), Theorem 6.1. The singularity data $\Delta = (0)$, i.e. there is a unique singular point of index $O$ (a fake saddle).

We wish to construct a recurrent geodesic $l_{\Sigma}$ from the pair $(B, \Delta)$. First, notice that the formula

$$r = 2g + m - 1,$$  

(29)
implies \( g = 1 \) since \( r = 2, m = 1 \). Therefore, our surface is a torus \( T^2 \).

The canonical state on the dimension group \( G = G(B) \) gives us

\[
\lambda_1 = \frac{\sqrt{5} - 1}{2}, \quad \lambda_2 = \frac{3 - \sqrt{5}}{2}.
\]  

(30)

The sequence of admissible intervals \( \Gamma_1 \supset \Gamma_2 \supset \ldots \) becomes

\[
[\lambda_1 - \frac{\lambda_1}{\epsilon}, \lambda_1 + \frac{\lambda_2}{\epsilon}] \supset [\lambda_1 - \frac{\lambda_1}{\epsilon^2}, \lambda_1 + \frac{\lambda_2}{\epsilon^2}] \supset \ldots
\]

(31)

where \( \epsilon = \frac{3 + \sqrt{5}}{2} \) is the Perron-Frobenius eigenvalue of the matrix \( M^2 \). Such a sequence contracts to the point \( \theta = \lambda_1 = \frac{\sqrt{5} - 1}{2} \).

Denote by \( a \) and \( b \) the points of the upper and lower row of the Bratteli diagram on Fig.2. Then the pre-code of \( \theta \):

\[
S = b \ a \begin{array}{c} 1 \ b \ a a \ b \ a a a b \ldots \end{array}
\]

(32)

The code of \( \theta \) is obtained from \( S \) by inserting \( a \) between any \( ba \), \( b \) between any \( aa \) and nothing between any \( ab \):

\[
S^* = b a \begin{array}{c} 1 \ b a \ a b a \ b a b a b \ldots \end{array}
\]

(33)

Up to a notation, \( S^* \) coincides with the Morse example of a forward symbolic non-periodic recurrent geodesic on \( T^2 \), see ([4]), §14.
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