On the structure of the power graph and the enhanced power graph of a group

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Abstract

Let $G$ be a group. The power graph of $G$ is a graph with the vertex set $G$, having an edge between two elements whenever one is a power of the other. We characterize nilpotent groups whose power graphs have finite independence number. For a bounded exponent group, we prove its power graph is a perfect graph and we determine its clique/chromatic number. Furthermore, it is proved that for every group $G$, the clique number of the power graph of $G$ is at most countably infinite. We also measure how close the power graph is to the commuting graph by introducing a new graph which lies in between. We call this new graph as the enhanced power graph. For an arbitrary pair of these three graphs we characterize finite groups for which this pair of graphs are equal.

1 Introduction

We begin with some standard definitions from graph theory and group theory.

Let $G$ be a graph with vertex set $V(G)$. If $x \in V(G)$, then the number of vertices adjacent to $x$ is called the degree of $x$, and denoted by $\deg(x)$. The distance between two vertices in a graph is the number of edges in a shortest path connecting them. The diameter of a connected graph $G$, denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices of $G$. If $G$ is disconnected, then $\text{diam}(G)$ is defined to be

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infinite. A star is a graph in which there is a vertex adjacent to all other vertices, with no further edges. The center of a star is a vertex that is adjacent to all other vertices. Let $U \subseteq V(G)$. The induced subgraph on $U$ is denoted by $\langle U \rangle$. An independent set is a set of vertices in a graph, no two of which are adjacent; that is, a set whose induced subgraph is null. The independence number of a graph $G$ is the cardinality of the largest independent set and is denoted by $\alpha(G)$. A subset $S$ of the vertex set of $G$ is called a dominating set if for every vertex $v$ of $G$, either $v \in S$ or $v$ is adjacent to a vertex in $S$. The minimum size of dominating sets of $G$, denoted by $\gamma(G)$, is called the domination number of $G$. A clique in a graph is a set of pairwise adjacent vertices. The supremum of the sizes of cliques in $G$, denoted by $\omega(G)$, is called the clique number of $G$. By $\chi(G)$, we mean the chromatic number of $G$, i.e., the minimum number of colours which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colours.

The cyclic group of order $n$ is denoted by $C_n$. A group $G$ is called periodic if every element of $G$ has finite order. For every element $g \in G$, the order of $g$ is denoted by $o(g)$. If there exists an integer $n$ such that for all $g \in G$, $g^n = e$, where $e$ is the identity element of $G$, then $G$ is said to be of bounded exponent. If $G$ is of bounded exponent, then the exponent of $G$ is the least common multiple of the orders of its elements; that is, the least $n$ for which $g^n = e$ for all $g \in G$. A group $G$ is said to be torsion-free if apart from the identity every element of $G$ has infinite order. Let $p$ be a prime number. The $p$-quasicyclic group (known also as the Prüfer group) is the $p$-primary component of $\mathbb{Q}/\mathbb{Z}$, that is, the unique maximal $p$-subgroup of $\mathbb{Q}/\mathbb{Z}$. It is denoted by $C_{p^\infty}$. The center of a group $G$, denoted by $Z(G)$, is the set of elements that commute with every element of $G$. A group $G$ is called locally finite if every finitely generated subgroup of $G$ is finite. A group is locally cyclic if any finitely generated subgroup is cyclic.

Other concepts will be defined when needed.

Now, we define the object of interest to us in this paper.

Let $G$ be a group. The power graph of $G$, denoted by $\mathcal{G}(G)$, is the graph whose vertex set is $G$, two elements being adjacent if one is a power of the other. This graph was first introduced for semigroups in [8] and then was studied in [6] and [7] for groups. It was shown that, for a finite group, the undirected power graph determines the directed power graph up to isomorphism. As a consequence, two finite groups which have isomorphic undirected power graphs have the same number of elements of each order. The authors in [7] have shown that the only finite group whose automorphism group is the same as that of its power graph is the Klein group of order 4.

Our results about the power graph fall into four classes.

• In Section 2.1 we consider the independence number $\alpha(\mathcal{G}(G))$. We show that if the independence number is finite then $G$ is a locally finite group whose centre has finite index. Using this we are able to give precise characterizations of nilpotent
groups $G$ for which $\alpha(G)$ is finite – such a group (if infinite) is the direct product of a $p$-quasicyclic group and a nilpotent $p'$-group.

- In Section 2.2 we show that the power graph of every group has clique number at most countable. A group with finite clique number must be of bounded exponent. Hence we obtain a structure theorem for abelian groups with this property, as well as showing that it passes to subgroups and supergroups of finite index.

- We do not know whether the chromatic number of every group is at most countable; in Section 2.2.1 we prove this for periodic groups and for free groups. We show that, if $G$ has bounded exponent, then $\mathcal{G}(G)$ is perfect.

- Finally, in Section 2.3 there are some miscellaneous results. A group is periodic if and only if its power graph is connected, and in this case its diameter must be at most 2. Also we show that, if all vertex degrees in $\mathcal{G}(G)$ are finite, then $G$ is finite.

In the recent paper [10], the authors prove that the power graph of every finite group is perfect. We acknowledge that our result on the perfectness along with all results in the Section 2 were proved independently in 2011.

Another well-studied graph associated to a group $G$ is the commuting graph of $G$. This graph appears to be first studied by Brauer and Fowler in 1955 in [5] as a part of classification of finite simple groups. As the elements of the centre are adjacent to all other vertices, usually the vertices are assumed to be non-central. For more information on the commuting graph, see [2, 14, 21] and the references therein.

In Section 3 we relate the power graph to the commuting graph and characterize when they are equal for finite groups. A new graph pops up while considering these graphs, a graph whose vertex set consists of all group elements, in which two vertices $x$ and $y$ are adjacent if they generate a cyclic group. We call this graph as the enhanced power graph of $G$ and we denote it by $\mathcal{G}_e(G)$. The enhanced power graph contains the power graph and is a subgraph of the commuting graph. We further study some properties of this graph in the Section 3.

We characterize the finite groups for which equality holds for any two of these three graphs, and the solvable groups for which the power graph is equal to the commuting graph. Other results are as follows:

- If the power graphs of $G$ and $H$ are isomorphic, then their enhanced power graphs are isomorphic.

- A maximal clique in the enhanced power graph is either a cyclic or a locally cyclic subgroup.

- $\mathcal{G}_e(G)$ has finite clique number if and only if $G$ has finite exponent; if this holds, then the clique number of $\mathcal{G}_e(G)$ is equal to the largest order of an element of $G$. Also, for any group $G$, the clique number of $\mathcal{G}_e(G)$ is at most countable.
2 Power graphs of groups

2.1 Independent sets in power graphs

In this section we provide some results on the finiteness of the independence number of the power graphs. In the proof of the our first theorem, we need the following definition. Let $G$ be a group and associate with $G$ a graph $\Gamma(G)$ as follows: the vertices of $\Gamma(G)$ are the elements of $G$ and two vertices $g$ and $h$ of $\Gamma(G)$ are joined by an edge if and only if $g$ and $h$ do not commute, see [1] and [16] for more details. Now, we have the following result.

**Theorem 1.** Let $G$ be a group and $\alpha(\mathcal{G}(G)) < \infty$. Then

(i) $[G : Z(G)] < \infty$.

(ii) $G$ is locally finite.

**Proof.** (i) First we note that if $x$ and $y$ are adjacent in $\Gamma(G)$, then $x$ and $y$ are not adjacent in $\mathcal{G}(G)$. Thus $\omega(\Gamma(G)) \leq \omega(\mathcal{G}(G)) < \infty$. Hence [16, Theorem 6] implies that $[G : Z(G)] < \infty$.

(ii) Let $H$ be a finitely generated subgroup of $G$. Then by (i) and [17, 1.6.11], $Z(H)$ is finitely generated, too. So by the fundamental theorem for finitely generated abelian groups we find that $Z(H) \cong \mathbb{Z}^n \times C_{q_1} \times \cdots \times C_{q_k}$, where $n$ and $k$ are non-negative integers and every $q_i$, $1 \leq i \leq k$, is a power of a prime number. Since $\alpha(\mathcal{G}(\mathbb{Z})) = \infty$, we deduce that $H$ is a finite group and so the proof is complete. \(\square\)

Now, we characterize those abelian groups whose power graphs have finite independence number. First we need the following theorem.

**Theorem 2.** ([17, 4.3.11]) If $G$ is an abelian group which is not torsion-free, then it has a non-trivial direct summand which is either cyclic or quasicyclic.

**Theorem 3.** Let $G$ be an abelian group such that $\alpha(\mathcal{G}(G)) < \infty$. Then either $G$ is finite or $G \cong C_{p^\infty} \times H$, where $H$ is a finite group and $p \nmid |H|$.

**Proof.** If $G$ is torsion-free, then $G$ contains $\mathbb{Z}$ and so $\alpha(\mathcal{G}(G)) \geq \alpha(\mathcal{G}(\mathbb{Z})) = \infty$, a contradiction. Thus by Theorem 2 $G = G_1 \oplus H_1$, where $G_1$ is either cyclic or quasicyclic. If $H_1$ is trivial, then we are done. Otherwise, $\alpha(\mathcal{G}(H_1)) < \infty$ implies that $H_1 = G_2 \oplus H_2$, where $G_2$ is either cyclic or quasicyclic. So $G = G_1 \oplus G_2 \oplus H_2$. By repeating this procedure and using $\alpha(\mathcal{G}(G)) < \infty$, we deduce that there exists a positive integer $n$ such that $G \cong \bigoplus_{i=1}^n G_i$, where every $G_i$ is either cyclic or quasicyclic. We show that at most one $G_i$ is quasicyclic. By the contrary, suppose that $G$ contains
the group $C_p^\infty \times C_q^\infty$. It is not hard to see that for every positive integer $n$, $I_n = \{(1/p^i + \mathbb{Z}, 1/q^{n-i+1} + \mathbb{Z}) : 1 \leq i \leq n\}$ is an independent set of size $n$, a contradiction. So either $G \cong C_p^\infty \times \prod_{i=1}^n C_{p_i}$ or $G \cong \prod_{i=1}^n C_{p_i}$, where $p$ and $p_i$ are prime numbers.

Now, suppose that the first case occurs. To complete the proof, we show that $p \neq p_i$, for every $i$, $1 \leq i \leq n$. By contrary, suppose that $p = p_i$, for some $i$. Then $C_p^\infty \times C_p$ is a subgroup of $G$. Since $C_p^\infty \times \{1\}$ is an independent set in $G(C_p^\infty \times C_p)$, we get a contradiction. So the proof is complete.

**Theorem 4.** Let $p$ be a prime number and $G$ be a $p$-group such that $\alpha(G(G)) < \infty$. Then either $G$ is finite or $G \cong C_p^\infty$.

**Proof.** Since $\alpha(G(G)) < \infty$, we deduce that $\alpha(G(Z(G))) < \infty$. Thus by Theorem 3, either $Z(G)$ is finite or $Z(G) \cong C_p^\infty$, for some prime number $p$. If $Z(G)$ is finite, then by Theorem 6, $G$ is finite. Now, suppose that $Z(G) \cong C_p^\infty$. To complete the proof, we show that $G$ is abelian. To the contrary, suppose that there exists $a \in G \setminus Z(G)$. Let $H = \langle Z(G) \cup \{a\} \rangle$. Clearly, $H$ is an abelian $p$-subgroup of $G$ and $\alpha(G(H)) < \infty$. So, by Theorem 3, $H \cong C_p^\infty \cong Z(G)$. Since every proper subgroup of $C_p^\infty$ is finite, we get a contradiction. Hence $G$ is abelian and $G = Z(G) \cong C_p^\infty$.

Now, we exploit Theorem 4 to extend Theorem 3 to nilpotent groups.

**Remark 5.** Let $H$ and $K$ be two subgroups of $G$. If $H \cap K = \{e\}$, $G = HK$ and $H \subseteq Z(G)$, then $G \cong H \times K$.

**Theorem 6.** Let $G$ be an infinite nilpotent group. Then $\alpha(G(G)) < \infty$ if and only if $G \cong C_p^\infty \times H$, for some prime number $p$, where $H$ is a finite group and $p \nmid |H|$.

**Proof.** First suppose that $G \cong C_p^\infty \times H$, where $H$ is a finite group and $p \nmid |H|$. Suppose to the contrary, $\{(s_n/p^{\alpha_n} + \mathbb{Z}, g_n) : n \geq 1, s_n \in \mathbb{Z}, p \nmid s_n, g_n \in H\}$ is an infinite independent set of $G(G)$. Since $H$ is a finite group, there exists $g \in H$ such that the infinite set

$$\{(s_n/p^{\alpha_n} + \mathbb{Z}, g) : n \geq 1, s_n \in \mathbb{Z} \text{ and } p \nmid s_n\}$$

forms an independent set. Since $o(g) < \infty$, there exist $\alpha_i$ and $\alpha_j$ such that $p^{\alpha_i} \equiv p^{\alpha_j} \pmod{o(g)}$ and $\alpha_i > \alpha_j$. On the other hand, we know that $\gcd(s_i, p) = 1$. So, let $t_i$ be the multiplicative inverse of $s_i$ in $C_{p^{\alpha_i}}$. Thus by Chinese Reminder Theorem, there exists a positive integer $x$ such that $x \equiv t_is_j \pmod{p^{\alpha_j}}$ and $x \equiv p^{\alpha_j-\alpha_i} \pmod{o(g)}$.

Therefore, we have

$$p^{\alpha_i-\alpha_j}x s_i - s_j - s_i x - s_j \equiv 0 \pmod{p^{\alpha_j}} \in \mathbb{Z}, \quad g^{xp^{\alpha_i-\alpha_j}} = g.$$

Thus, $(s_i/p^{\alpha_i} + \mathbb{Z}, g)$ and $(s_j/p^{\alpha_j} + \mathbb{Z}, g)$ are adjacent, a contradiction. 

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Conversely, suppose that $\alpha(\mathcal{G}(G)) < \infty$. Then by Theorem 1 $[G : Z(G)] < \infty$ and so $G = Z(G)H$, where $H$ is a finitely generated subgroup of $G$. Now, Theorem 1 implies that $H$ is finite. By Theorem 2 $Z(G) = AB$, where $A \cong C_{p^\infty}$ and $B$ is a finite group such that $p \nmid |B|$. Also, since $H$ is nilpotent, we have $H \cong H_pH_{p_1}\cdots H_{p_t}$, where $H_p$ and $H_{p_i}$ ($1 \leq i \leq t$) are sylow $p$-subgroup and sylow $p_i$-subgroup of $H$, respectively. We show that $H_p \subseteq Z(G)$. To the contrary, suppose that $x \in H_p \setminus A$. Then $\langle A, x \rangle$ is a $p$-group and so by Theorem 4, $\langle A, x \rangle \cong C_{p^\infty} \cong \langle A \rangle$. Since every proper subgroup of $C_{p^\infty}$ is finite, we get a contradiction. Thus $H_p \subseteq Z(G)$ and so $G = ABH_{p_1}\cdots H_{p_t}$. Since $G$ is nilpotent, $BH_{p_1}\cdots H_{p_t}$ is a finite subgroup of $G$ and $p \nmid |BH_{p_1}\cdots H_{p_t}|$. Hence by Remark 5 $G \cong A \times BH_{p_1}\cdots H_{p_t}$, as desired. \hfill \Box

2.2 The colouring of power graphs

Let $G$ be a group. In this section, we first show that the chromatic number of the power graph of $G$ is finite if and only if the clique number of the power graph of $G$ is finite and this statement is also equivalent to that the exponent of $G$ is finite. Then it is proved that the clique number of the power graph of $G$ is at most countable. Finally, it is shown that the power graph of every bounded exponent group is perfect.

Lemma 7. Let $G$ be a group. If $\omega(\mathcal{G}(G))$ is finite, then $G$ is of bounded exponent.

Proof. By the contrary, suppose that $G$ is not of bounded exponent. Then for every positive integer $k$, there is an element $g_k \in G$ such that $o(g_k) > 2^k$. So one can easily show that $\{g_k^i \mid 0 \leq i \leq k\}$ is a clique of size $k + 1$ in $\mathcal{G}(G)$. This implies that $\omega(\mathcal{G}(G)) = \infty$, a contradiction. The proof is complete. \hfill \Box

Remark 8. The proof uses the Axiom of Choice for families of finite sets.

Corollary 9. Let $G$ be an abelian group and $\omega(\mathcal{G}(G)) < \infty$. Then there are some positive integers $r$ and $n_i$ and sets $I_i$, $1 \leq i \leq r$ such that

$$G \cong \prod_{i=1}^r \prod_{I_i} C_{n_i}.$$

Proof. Since $\omega(\mathcal{G}(G)) < \infty$, by Lemma 7 $G$ is bounded exponent. So the assertion follows from Prüfer-Baer Theorem (see [17, 4.3.5]). \hfill \Box

Theorem 10. The clique number of the power graph of any group is at most countably infinite.

Proof. Let $C$ be a clique in the power graph of $G$, and take $x \in C$. Then the remaining vertices $y$ of $C$ are of two types:


• \( y = x^n \) for some \( n \);
• \( x = y^n \) for some \( n \).

Clearly, there are at most countably many of the first type. We denote the set of vertices of the second type by \( C(n) \). We show that \( C(n) \) is at most countably infinite. If there is only one \( y \) in \( C(n) \), then there is nothing to prove; so suppose there are at least two elements in \( C(n) \). We claim that every element in \( C(n) \) has finite order. Choose \( y, y' \in C(n) \). With no loss of generality, one can assume that \( y' = y^k \), for some positive integer \( k \). So \( y^{(k-1)n} = 1 \). This implies that the orders of both \( y \) and \( y' \) are finite. Thus the claim is proved. Now, for every positive integer \( k \), define \( C(n,k) = \{ y \in C(n) \mid o(y) = k \} \). By the claim, \( C(n) = \bigcup_{k \geq 1} C(n,k) \). It is not hard to show that for every \( a, b \in C(n,k) \), \( \langle a \rangle = \langle b \rangle \) and so \( C(n,k) \) is finite. Therefore, \( C(n) \) is at most countably infinite.

We wonder if the same result holds for the chromatic number: Does the power graph of every group have a countable chromatic number? A group is called a \textit{pcc-group} if its power graph has at most countable chromatic number. Free groups have this property by the next theorem.

\textbf{Theorem 11.} Every free group is a pcc-group.

\textbf{Proof.} By [19, Corollary, p.51], in a free group, every non-identity element lies in a unique maximal cyclic subgroup, generated by an element which is not a proper power (when written as a reduced word). So, the power graph of a free group consists of many copies of the power graph of an infinite cyclic group, with the identity in all these copies identified.

Next, we show that every abelian group is a pcc-group. First, we need the following result.

\textbf{Lemma 12.} Every periodic group is a pcc-group.

\textbf{Proof.} Suppose that \( G \) is a periodic group. For every positive integer \( n \), let \( G_n \) be the set of all elements of \( G \) of order \( n \). If \( g, h \in G_n \) and \( g \) and \( h \) are adjacent in the power graph, then \( g \) and \( h \) generate the same cyclic group. Hence, the induced subgraph on \( G_n \) is a disjoint union of cliques of size \( \phi(n) \). So one can colour the induced subgraph on \( G_n \) with \( \phi(n) \) colours. Clearly, \( G = \bigcup_{n \geq 1} G_n \) and so the chromatic number of \( G(G) \) is at most countable.

Now, we prove that the class of pcc-groups contains the class of abelian groups.
Theorem 13. Every abelian group is a pcc-group.

Proof. Let $G$ be abelian. Then $G$ can be embedded in a divisible abelian group ([17, Theorem 4.1.6, p.98]). It is known that every divisible abelian group is of the form $H \times K$, where $H$ is a periodic group and $K$ is a direct sum of many copies of $\mathbb{Q}$, see [17, Theorem 4.1.5, p.97]. By Lemma 12, $H$ is a pcc-group. Now, the next two claims prove the assertion of the theorem.

Claim 1. If $M$ and $N$ are two pcc-groups, then $M \times N$ is a pcc-group.

Proof of Claim 1: Let $f$ and $g$ be the proper countable colouring $G(M)$ and $G(N)$, respectively. Then the new map $\phi : M \times N \to Image(f) \times Image(g)$ defined by $\phi(x, y) = (f(x), g(y))$ is a proper countable colouring for $G(M \times N)$. This completes the proof of Claim 1.

Claim 2. Every torsion-free abelian group is a pcc-group.

Proof of Claim 2: Let $A$ be a torsion-free abelian group. We show that the connected components of $G(A)$ are at most countable. First, we show that the degree of any vertex is at most countable. Now, the identity element $0$ is an isolated vertex. Suppose that $x \neq 0$. There are only countably many multiples of $x$. Also, for each natural number $n$, there is at most one element $y$ such that $ny = x$. For, if $ny = nz = x$, then $n(y - z) = 0$, so $y - z = 0$ since $A$ is torsion-free. So there are at most countably many elements of which $x$ is a multiple. Hence, the neighbourhood of $x$ is countable. This implies that the set of vertices of distance 2 from $x$ is countable as well. By induction, we conclude that the set of vertices of distance $k$ from $x$ is countable. So, the component of $G(A)$ which contains $x$ is countable. Now, we can colour each connected component with countably many colours. The proof of Claim 2 is complete.

Other classes that could be looked at would include solvable groups. One could also ask whether the class of pcc-groups is extension-closed.

2.2.1 Perfectness of the power graph

A graph $G$ is called perfect if for every finite induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$. The Strong Perfect Graph Theorem states that a finite graph $G$ is perfect if and only if neither $G$ nor $\overline{G}$ (the complement of $G$) contains an induced odd cycle of length at least 5, see [4, Theorem 14.18]. However, this is a deep theorem, and we do not need it to prove our results.

Utilizing Lemma to colour the power graph with a finite set of colours we require the group to be bounded exponent. Here we show that for such groups the resulting
power graph is always perfect and can be finitely coloured. To prove this result we facilitate the concepts of comparability graph.

Let \( \leq \) be a binary relation on the elements of a set \( P \). If \( \leq \) is reflexive and transitive, then \( (P, \leq) \) is called a pre-ordered set. All partially ordered sets are pre-ordered. The comparability graph of a pre-ordered set \( (P, \leq) \) is the simple graph \( \Upsilon(P) \) with the vertex set \( P \) and two distinct vertices \( x \) and \( y \) are adjacent if and only if either \( x \leq y \) or \( y \leq x \) (or both).

**Theorem 14.** Let \( m \) be a positive integer and \( P \) be a pre-ordered set (not necessarily finite) whose maximum chain size is \( m \). Then the comparability graph \( \Upsilon(P) \) is perfect and

\[
\omega(\Upsilon(P)) = \chi(\Upsilon(P)) = m.
\]

**Proof.** The result is well known for comparability graphs of partial orders; our proof is a slight extension of this. Since the class of comparability graphs is closed under taking induced subgraphs, it is enough to prove that the comparability graph of \( P \) has equal clique number and chromatic number. Clearly, a clique in a \( \Upsilon(P) \) is a chain in \( P \), while a colouring is a partition into antichains.

First we show that \( \omega(\Upsilon(P)) = m \). Let \( C \) be a clique in \( \Upsilon(P) \). Then \( C \) is a chain in \( P \), and so \( |C| \leq m \). Thus \( \omega(\Upsilon(P)) = m \).

Now, we show that \( \chi(\Upsilon(P)) \leq m \). We form a directed graph by putting an arc from \( x \) to \( y \) whenever \( x \leq y \) but \( y \not\leq x \); and, if \( C \) is an equivalence class of the relation \( \equiv \) defined by \( x \equiv y \) if \( x \leq y \) and \( y \leq x \), then take an arbitrary directed path on the elements of \( C \). Clearly, the longest directed path contains \( m \) vertices. Let \( P_i \) be the set of elements \( x \) for which the longest directed path ending at \( x \) contains \( i \) vertices. It is easy to see that \( P_i \) is an independent set; these sets partition \( P \) into \( m \) classes. \( \square \)

Now, we show that the power graph of a group is the comparability graph of a pre-ordered set. First, we define some notations. Let \( n \) be a positive integer and \( D(n) \) be the set of all divisors of \( n \) in \( \mathbb{N} \). Define a relation \( \preceq \) on \( D(n) \) by \( r \preceq s \) if and only if \( r | s \). Clearly, \( (D(n), \preceq) \) is a partially ordered set. Denote the set of all chains of \( (D(n), \preceq) \) by \( C(n) \). Using this convention we are able to determine the clique/chromatic number of the power graph of a group of bounded exponent (see Lemma 7).

**Theorem 15.** Let \( G \) be a group of exponent \( n \). Then \( \mathcal{G}(G) \) is a perfect graph and

\[
\chi(\mathcal{G}(G)) = \omega(\mathcal{G}(G)) = \max \left\{ \sum_{d \in C} \phi(d) : C \in C(n) \right\} \leq n,
\]

where \( G_d \) is the set of elements of \( G \) of order \( d \), for some divisor \( d \) of \( n \).
Proof. First we consider two following facts:

Fact 1. Suppose that $d$ is a divisor of $n$ and $G_d \neq \emptyset$. If $g, h \in G_d$ and $g$ and $h$ are adjacent in the power graph, then $g$ and $h$ generate the same cyclic group. So $\langle G_d \rangle$ is a disjoint union of cliques of size $\phi(d)$. Therefore; if $x$ is an element of a clique $H$ of $\langle G_d \rangle$ adjacent to an element $y$ of a clique $K$ of $\langle G_{d'} \rangle$, then every element of $H$ is adjacent to every element of $K$ and moreover, $d \mid d'$ or $d' \mid d$.

Fact 2. Note that if $z$ is an element of order $d$, then for each divisor $d'$ of $d$, $z^d$ is of order $d'$. So for each clique $T$ of $\langle G_d \rangle$, every element of $T$ is adjacent to every element of a clique $S$ of $\langle G_{d'} \rangle$.

Since $\{G_d : G\}$ has an element of order $d$ forms a partition for $G$, Fact 1 implies that every maximal clique of $\mathcal{G}(G)$ has the form $\bigcup_{i} Cl_i \bigcup_{i} Cl_{m}$, where $Cl_i$ is a clique of $\langle G_{d_i} \rangle$ of size $\phi(d_i)$ and $\{d_1, \ldots, d_m\}$ is a chain of length $m$ belonging to $\mathcal{C}(n)$. Moreover; by Fact 2, we deduce that for every chain $\{d_1, \ldots, d_m\}$ in $\mathcal{C}(n)$, there exists a clique for $\mathcal{G}(G)$ of this form. Now, by $|Cl_i| = \phi(d_i)$, we conclude that

$$\omega(\mathcal{G}(G)) = \max \left\{ \sum_{d \in C} \phi(d) : C \in \mathcal{C}(n) \right\} \leq \sum_{d \mid n} \phi(d) = n.$$ 

Define a pre-ordering $\leq$ on $G$ by $x \leq y$ if and only if $x$ is a power of $y$. Clearly, the power graph of $G$ is the comparability graph of $\leq$ and so by Theorem 14, the power graph of $G$ is perfect. Thus $\chi(\mathcal{G}(G)) = \omega(\mathcal{G}(G))$ and the proof is complete. \quad \square

The next two corollaries are direct consequences of Lemma 7 and Theorem 15.

Corollary 16. For every group $G$, the following statements are equivalent:

(i) $\chi(\mathcal{G}(G)) < \infty$;

(ii) $\omega(\mathcal{G}(G)) < \infty$;

(iii) $G$ is bounded exponent.

Moreover, the chromatic number of $\mathcal{G}(G)$ does not exceed the exponent of $G$.

Corollary 17. Let $G$ be an abelian group of exponent $n$. Then

$$\chi(\mathcal{G}(G)) = \omega(\mathcal{G}(G)) = \max \left\{ \sum_{d \in C} \phi(d) : C \in \mathcal{C}(n) \right\}.$$
**Corollary 18.** Let $H$ be a subgroup of $G$ and $[G : H] < \infty$. Then $\omega(G(H)) < \infty$ if and only if $\omega(G(G)) < \infty$.

The following example shows that a similar assertion does not hold for the independence number.

**Example 19.** Let $G = C_2 \times C_2^{\infty}$ and $H = \{0\} \times C_2^{\infty}$. Thus $[G : H] = 2$. Since $G(H)$ is a complete graph, $\alpha(H) = 1$. Clearly, the set $\{1\} \times C_2^{\infty}$ is independent and so $\alpha(G) = \infty$.

### 2.3 Miscellaneous properties

We conclude this section with three miscellaneous properties of the power graph of a group.

**Theorem 20.** If $G(G)$ is a triangle-free graph, then $G$ is isomorphic to a direct product of $C_2$ and $G(G)$ is a star.

**Proof.** First we show that the order of every element of $G$ is at most 2. Let $a \in G$. If $o(a) \geq 3$, then $\{e, a, a^2\}$ is a triangle, a contradiction. So $G$ is an elementary abelian 2-group. Therefore, by Pr"ufer-Baer Theorem, $G$ is isomorphic to a direct product of $C_2$ and so $G(G)$ is a star with the center $e$. □

The following theorem characterizes those groups whose power graphs are connected.

**Theorem 21.** Let $G$ be a group. The following statements are equivalent.

1. $G(G)$ is connected;
2. $G$ is periodic;
3. $\gamma(G(G)) = 1$;
4. $\text{diam}(G(G)) \leq 2$.

**Proof.** (i)$\implies$(ii) Let $x (x \neq e)$ be a vertex of $G(G)$. We show that $x$ is of finite order in $G$. Since $G(G)$ is connected, there is a path from $x$ to $e$. Let $y$ be the adjacent vertex to $e$ in this path. So the order of $y$ is finite. Now, suppose that $t$ is the adjacent vertex to $y$ in this path. Then the order of $t$ is finite, too. By repeating this procedure, we deduce that the order of $x$ is finite. So $G$ is periodic.

(ii)$\implies$(iii) Since every element in $G$ has a finite order, $\{e\}$ is a dominating set.

The parts (iii)$\implies$(iv) and (iv)$\implies$(i) are clear. □
Theorem 22. If \( \text{deg}(g) < \infty \), for every \( g \in G \), then \( G \) is a finite group.

Proof. Let \( g \in G \). Since \( \text{deg}(g) < \infty \), \( g \) has a finite order in \( G \). Thus \( G \) is a periodic group and so \( e \) is adjacent to every other vertices of \( \mathcal{G}(G) \). Since \( \text{deg}(e) < \infty \), we deduce that \( G \) is finite. \( \square \)

Remark 23. Let \( G \cong \prod_{i \geq 1} C_2 \). Then \( \mathcal{G}(G) \) is an infinite star with the center 0. This shows that in the previous theorem the condition \( \text{deg}(g) < \infty \), for every \( g \in G \) is necessary.

3 Power graph and commuting graph

Let \( G \) be a group. If the vertices \( x \) and \( y \) are joined in the power graph of \( G \), then they are joined in the commuting graph; so the power graph is a spanning subgraph of the commuting graph.

Question 24. For which groups is it the case that the power graph is equal to the commuting graph?

The identity is joined to all others in the commuting graph; so if the two graphs are equal, then \( G \) is a periodic group.

Theorem 25. Let \( G \) be a finite group with power graph equal to commuting graph. Then one of the following holds:

- \( G \) is a cyclic \( p \)-group;
- \( G \) is a semidirect product of \( C_p \) by \( C_q \), where \( p \) and \( q \) are primes with \( a, b > 0 \), \( q^b \mid p - 1 \) and \( C_q \) acts faithfully on \( C_p \);
- \( G \) is a generalized quaternion group.

Proof. Let \( G \) have power graph equal to commuting graph; that is, if two elements commute, then one is a power of the other. Then \( G \) contains no subgroup isomorphic to \( C_p \times C_q \), where \( p \) and \( q \) are primes, since this group fails the condition.

A theorem of Burnside [15, Theorem 12.5.2] says that a \( p \)-group containing no \( C_p \times C_p \) subgroup is cyclic or generalized quaternion. So all Sylow subgroups of \( G \) are of one of these two types.

Suppose that all Sylow subgroups are cyclic. Then \( G \) is metacyclic [15, Theorem 9.4.3]. The cyclic normal subgroup of \( G \) has order divisible by one prime only, say \( p \).
Its centraliser in $G$ is a Sylow $p$-subgroup $P$ of $G$, since it contains no elements of order coprime to $p$. Hence $G$ is a semidirect product of $P$ and a cyclic group $Q$ of order coprime to $p$, necessarily a cyclic $q$-group for some prime $q$. If $|Q| = q^b$, then we have $q^b \mid p - 1$.

So we may suppose that $G$ has generalized quaternion Sylow 2-subgroups. By Glauberman’s Z*-Theorem [11], $G/O(G)$ has a central involution, where $O(G)$ is the maximal normal subgroup of $G$ of odd order. This involution must act fixed-point-freely on $O(G)$, so $O(G)$ is abelian, and hence cyclic of prime power order. But a generalized quaternion group cannot act faithfully on such a group. So $O(G) = 1$. Then the involution in $G$ is central, so $G$ is a 2-group, necessarily a generalized quaternion group. □

Remark 26. In the infinite case, there are other examples, such as Tarski monsters, which are infinite groups whose non-trivial proper subgroups are all cyclic of prime order $p$. Probably no classification is possible.

In the next theorem, we extend Theorem 25 to solvable groups.

**Theorem 27.** Let $G$ be a solvable group with power graph equal to commuting graph. Then one of the following holds:

- $G$ is a cyclic $p$-group;
- $G$ is a semidirect product of $C_{p^a}$ by $C_{q^b}$, where $p$ and $q$ are primes with $a, b > 0$, $q^b \mid p - 1$ and $C_{q^b}$ acts faithfully on $C_{p^a};$
- $G$ is a generalized quaternion group;
- $G$ is the $p$-quasicyclic group $C_{p^\infty};$
- $G$ is a semidirect product of $p$-quasicyclic group $C_{p^\infty}$ and a finite cyclic group.

**Proof.** We use this fact that every finitely generated periodic solvable group is finite. We know that $G$ is periodic. We show that there are no three elements whose order are distinct primes. Suppose that $o(a) = p$, $o(b) = q$ and $o(c) = r$, where $p$, $q$ and $r$ are distinct primes. Let $H$ be the subgroup generated by $a$, $b$ and $c$. Then $H$ is finite. Clearly, the power graph and the commuting graph of $H$ are equal. This contradicts Theorem 25. Thus the order of every finite subgroup of $G$ is $p^\alpha q^\beta$, for some non-negative integers $\alpha$ and $\beta$. By the second part of Theorem 25, we may assume that $\beta$ is bounded, because $q^\beta \mid p - 1$. Also, by Theorem 25, the order of every element of $G$ is a $p$-power or a $q$-power, because every cyclic subgroup is a $p$-group or a $q$-group. If $\alpha, \beta > 0$, then by the second part of Theorem 25, $\langle a, b \rangle$ is semidirect product of $\langle a \rangle$ and $\langle b \rangle$. So, $\langle a \rangle$ and $\langle b \rangle$ are both cyclic (even in the case $q = 2$). Let $N$ be the set .
of all elements of $G$ whose orders are $p$-power. We show that $N$ is an abelian normal
subgroup of $G$. To see this first we show that if $x$ and $y$ are two elements of $N$, then
$xy = yx$. Let $S$ be the subgroup generated by $x$ and $y$. Then $S$ is a finite group of
order $p^α q^β$. If $β = 0$, then by Theorem 25, $S$ is a cyclic $p$-group and we are done. So
assume that $β > 0$. Let $N_1$ and $Q$ be Sylow $p$-subgroup and Sylow $q$-subgroup of $S$,
respectively. Then by Theorem 25 both are cyclic. Now, by [15, Theorem 6.2.11], $Q$ has a normal complement. So, $N_1 ≪ S$. This implies that $x, y ∈ N_1$ and so $xy = yx$.
Thus we conclude that $N$ is an abelian $p$-subgroup of $G$. Now, by the definition of $N$,
$N ≺ G$.

Now, let $Q$ be a $q$-subgroup of $G$ which has maximum size. We prove that $G = NQ$.

Let $a ∈ G$ be an element of $G$ whose order is $q$-power. Let $M$ be the subgroup
generated by $a$ and $Q$. Then $M = P_1Q_1$, where $P_1$ and $Q_1$ are Sylow $p$-subgroup and
Sylow $q$-subgroup of $M$, respectively and $P_1 ≺ M$. But $Q_1$ is a conjugate of $Q$ and so
$M = P_1Q$. Since $a ∈ M$, we have $a = bc$, where $b ∈ P_1$ and $c ∈ Q$. But $P_1 ⊆ P$ and
this implies that $a ∈ PQ$, as desired. So $G = NQ$.

Since $N$ is an abelian group, the commuting graph of $N$ and so the power graph of
$N$ is a complete graph. Now, Theorem 4 yields that $N$ is finite or $N = Z_{p^∞}$.

4 The enhanced power graph

4.1 Definition and properties

In the Section 2 we investigated some properties of power graphs of groups. In The-
orem 25 we characterized finite groups for which the power graph is the same as the
commuting graph. Now, it is natural to ask if they are not equal, how close these
graphs are. To tackle this problem we introduce an intermediate graph. This graph
can be regarded as a measurement for this difference. Given a group $G$, the enhanced
power graph of $G$ denoted by $G_e(G)$ is the graph with vertex set $G$, in which $x$ and $y$
are joined if and only if there exists an element $z$ such that both $x$ and $y$ are powers of
$z$.

The power graph and commuting graph behave well on restriction to a subgroup
(that is, if $H ≤ G$, then the power graph of $H$ is the induced subgraph of the power
graph of $G$ on the set $H$, and similarly for the commuting graph). Because of the
existential quantifier in the definition, it is not obvious that the same holds for the
enhanced graph. That this is so is a consequence of the fact that $x$ and $y$ are joined in
the enhanced power graph if and only if $(x, y)$ is cyclic. Note that

- the power graph is a spanning subgraph of the enhanced power graph;
- the enhanced power graph is a spanning subgraph of the commuting graph.
In the next remark, we use the concept of graph squares. For a graph $H$, the square of $H$ is a graph with the same vertex set as $H$ in which two vertices are adjacent if their distance in $H$ is at most two.

**Remark 28.** If we assume that the (undirected or directed) power graph has a loop at each vertex, then the enhanced power graph lies between the power graph and its square. We already saw that it contains the power graph (as a spanning subgraph). Now, let $x$ and $y$ be two vertices joined by a path $(x, z, y)$ of length 2 in the power graph. There are four cases in the directed power graph $D = \tilde{G}(G)$:

- $(x, z), (z, y) \in E(D)$. Then $x$ is a power of $z$, and $z$ a power of $y$; so $x$ is a power of $y$, and $(x, y) \in E(D)$.
- $(z, x), (y, z) \in E(D)$. Dual to the first case.
- $(x, z), (y, z) \in E(D)$. Then $x$ and $y$ are powers of $z$, so they are joined in the square of the power graph.
- $(z, x), (z, y) \in E(D)$. In this case there is nothing we can say.

Also the following holds:

**Theorem 29.** Let $G$ and $H$ be finite groups. If the power graphs of $G$ and $H$ are isomorphic, then their enhanced power graphs are also isomorphic.

**Proof.** Note that $x$ and $y$ are joined in the enhanced power graph if and only if there is a vertex $z$ which dominates both in the directed power graph. So the theorem follows from the main theorem of [6].

### 4.2 Comparing to the power graph and commuting graph

**Question 30.** For which (finite) groups is the power graph equal to the enhanced power graph?

This question connects with another graph associated with a finite group, the prime graph, defined by Gruenberg and Kegel [13]: the vertices of the prime graph of $G$ are the prime divisors of $|G|$, and vertices $p$ and $q$ are joined if and only $G$ contains an element of order $pq$. To state the next result we need a definition. The group $G$ is a 2-Frobenius group if it has normal subgroups $F_1$ and $F_2$ such that $F_1 < F_2$, $F_2$ is a Frobenius group with Frobenius kernel $F_1$, and $G/F_1$ is a Frobenius group with Frobenius kernel $F_2/F_1$.

In the statement of the following theorem, $p$ and $q$ denote distinct primes.

**Theorem 31.** For a finite group $G$, the following conditions are equivalent:
(a) the power graph of $G$ is equal to the enhanced power graph;
(b) every cyclic subgroup of $G$ has prime power order;
(c) the prime graph of $G$ is a null graph.

A group $G$ with these properties is one of the following: a $p$-group; a Frobenius group whose kernel is a $p$-group and complement a $q$-group; a 2-Frobenius group where $F_1$ and $G/F_2$ are $p$-groups and $F_2/F_1$ is a $q$-group; or $G$ has a normal 2-subgroup with quotient group $H$, where $S \leq H \leq \text{Aut}(S)$ and $S \cong A_5$ or $A_6$.

All these types of group exist. Examples include $S_3$ and $A_4$ (Frobenius groups); $S_4$ (a 2-Frobenius group); $A_5$, $A_6$ and $2^4 : A_5$.

Proof. Let $p$ and $q$ be distinct primes. The cyclic group of order $pq$ does not have property (a); so a group satisfying (a) must also satisfy (b). Conversely, suppose that (b) holds. If $x$ and $y$ are adjacent in the enhanced power graph, then $\langle x, y \rangle$ is cyclic, necessarily of prime power order; so it must be generated by one of $x$ and $y$, and so $x$ and $y$ are adjacent in the power graph.

Clearly, (b) and (c) are equivalent.

Now, let $G$ be a group satisfying these conditions. Either $G$ is a $p$-group for some prime $p$, or the prime graph of $G$ is disconnected. Now, we use the result of Gruenberg and Kegel \[13\] (stated and proved in Williams \[20\]), asserting that a finite group with disconnected prime graph is Frobenius or 2-Frobenius, simple, $\pi_1$ by simple, simple by $\pi_1$-solvable, or $\pi_1$ by simple by $\pi_1$. Here $\pi_1$ is the set of primes in the connected component of the prime graph containing 2, assuming that $|G|$ is even; and a 2-Frobenius group is a group $G$ with normal subgroups $F_1 < F_2$ such that $F_2$ is a Frobenius group with kernel $F_1$, and $G/F_1$ is a Frobenius group with kernel $F_2/F_1$.

It follows from the work of Frobenius that a Frobenius complement either has all Sylow subgroups cyclic (and so is metacyclic) or has $\text{SL}(2,3)$ or $\text{SL}(2,5)$ as a normal subgroup. These last two cases cannot occur, since the central involution commutes with elements of order 3. In the first case, the results of Gruenberg and Kegel (see the first corollary in Williams \[20\]) show that the Frobenius complement has only one prime divisor.

In the case of a 2-Frobenius group, an element of the Frobenius complement in the top group centralises some element of $F_1$; so $F_1$ and $G/F_2$ must be $p$-groups for the same prime $p$.

In the remaining cases, it can be read off from the tables in Williams \[20\] that the simple group can only be $A_5$ or $A_6$, and the conclusions of the theorem follow since $\pi_1 = \{2\}$. □
**Question 32.** For which (finite) groups is the enhanced power graph equal to the commuting graph?

Again, we have a lot of information about such a group.

**Theorem 33.** For a finite group $G$, the following conditions are equivalent:

(a) the enhanced power graph of $G$ is equal to its commuting graph;

(b) $G$ has no subgroup $C_p \times C_p$ for $p$ prime;

(c) the Sylow subgroups of $G$ are cyclic or (for $p = 2$) generalized quaternion.

A group satisfying these conditions is either a cyclic $p$-group for some prime $p$, or satisfies the following: if $O(G)$ denotes the largest normal subgroup of $G$ of odd order, then $O(G)$ is metacyclic, $H = G/O(G)$ is a group with a unique involution $z$, and $H/\langle z \rangle$ is a cyclic or dihedral 2-group, a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ for $q$ an odd prime power, or $A_7$.

An example of a group for the second case is the direct product of the Frobenius group of order 253 by $SL(2, 5)$.

**Proof.** The group $C_p \times C_p$ has commuting graph not equal to its enhanced power graph, so cannot be a subgroup of a group satisfying (a); thus (a) implies (b). Conversely, suppose that (b) holds. Let $x$ and $y$ be elements of $G$ which are adjacent in the commuting graph. Then $\langle x, y \rangle$ is abelian, and hence is the direct product of two cyclic groups, say $C_r \times C_s$. Under hypothesis (b), we must have $\gcd(r, s) = 1$, and so $\langle x, y \rangle \cong C_{rs}$; thus $x$ and $y$ are joined in the enhanced power graph.

Conditions (b) and (c) are equivalent by a theorem of Burnside [15, Theorem 12.5.2].

Suppose that a Sylow 2-subgroup $P$ of $G$ is cyclic or generalized quaternion. If $P$ is cyclic, then by Burnside’s transfer theorem [15, Section 14.3], $G$ has a normal 2-complement: that is, if $O(G)$ is the largest normal subgroup of $G$ of odd order, then $G/O(G) \cong P$. If $P$ is generalized quaternion, then by Glauberman’s $Z^*$-Theorem, $H = G/O(G)$ has a unique central involution $z$. Put $Z = \langle z \rangle$. Then the Sylow 2-subgroups $Q$ of $H/Z$ are dihedral; the Gorenstein–Walter theorem [12] shows that $H/Z$ is isomorphic to a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ (for odd $q$), or to the alternating group $A_7$, or to $Q$. For any such group $H^* = H/Z$, an argument of Glauberman (which can be found in [3]) shows that there is a unique double cover $H$ with a single involution. □

**Question 34.** What can be said about the difference of the enhanced power graph and the power graph, or the difference of the commuting graph and the enhanced power graph? In particular, for which groups is either of these graphs connected?
4.3 Maximal cliques in the enhanced power graph

We will now look at maximal cliques in the enhanced power graph. This requires a lemma which looks trivial, but we couldn’t find an easier proof of it than the one below.

**Lemma 35.** Let \( x, y, z \) be elements of a group \( G \), and suppose that \( \langle x, y \rangle \), \( \langle x, z \rangle \) and \( \langle y, z \rangle \) are cyclic. Then \( \langle x, y, z \rangle \) is cyclic.

**Proof.** The result clearly holds if one of \( x, y, z \) is the identity; so suppose not. Now, a cyclic group cannot contain elements of both finite and infinite order, so either all three elements have finite order, or all three have infinite order.

**Case 1:** \( x, y, z \) have finite order. Then they generate a finite abelian group \( A \).

We first note that it suffices to do the case where the orders of \( x, y, z \) are powers of a prime \( p \). For \( A \) is the direct sum of \( p \)-groups for various primes \( p \); each \( p \)-group is generated by certain powers of \( x, y, z \); and if each \( p \)-group is cyclic, then so is \( A \). With this assumption, suppose that \( A \) is not cyclic. Since \( \langle x, y \rangle \) is cyclic, \( A \) is the sum of two cyclic groups, and so contains a subgroup \( Q \cong C_p \times C_p \). Each of \( \langle x \rangle \), \( \langle y \rangle \) and \( \langle z \rangle \) intersects \( Q \) in a subgroup of order \( p \); let these subgroups be \( X, Y, Z \). Since \( \langle x, y \rangle \) is cyclic, it meets \( Q \) in a subgroup of order \( p \); so \( X = Y \). Similarly \( X = Z \). So \( \langle x, y, z \rangle \) meets \( Q \) in a subgroup of order \( p \), contradicting the assumption that \( Q \leq \langle x, y, z \rangle \).

**Case 2:** \( x, y, z \) have infinite order.

Then they generate a free abelian group; since \( \langle x, y \rangle \) is cyclic, we see that \( A = \langle x, y, z \rangle \) has rank at most 2. Consider the \( Q \)-vector space \( A \otimes_\mathbb{Z} Q \), which has dimension at most 2. Since \( \langle x, y \rangle \) is cyclic, the 1-dimensional subspaces \( \langle x \rangle \otimes_\mathbb{Z} Q \) and \( \langle y \rangle \otimes_\mathbb{Z} Q \) have non-empty intersection, and so are equal. Similarly for \( \langle z \rangle \otimes_\mathbb{Z} Q \). Thus \( A \otimes_\mathbb{Z} Q \) is 1-dimensional, so \( A \) is cyclic.

Now, we have the following characterization of the maximal cliques in the enhanced power graph.

**Lemma 36.** A maximal clique in the enhanced power graph is either a cyclic subgroup or a locally cyclic subgroup.

**Proof.** Clearly, a cyclic or locally cyclic subgroup is a clique. Suppose that \( C \) is a maximal clique. If \( x, y \in C \), then by Lemma 35 every element of \( \langle x, y \rangle \) is joined to every element \( z \in C \); so \( C \cup \langle x, y \rangle \) is a clique. By maximality of \( C \), we have \( \langle x, y \rangle \subseteq C \); so \( C \) is a subgroup. Now, a simple induction shows that any finite subset of \( C \) generates a cyclic group, so that \( C \) is locally cyclic. \( \square \)
Remark 37. Locally cyclic groups include the additive group of \( \mathbb{Q} \) (or the subgroup consisting of rationals whose denominators only involve primes from a prescribed set), and direct sums of copies of the \( p \)-quasicyclic groups (the Prüfer groups) \( C_{p^\infty} \) for distinct primes \( p \).

Now, we have two immediate corollaries:

Corollary 38. Let \( G \) be a group. Then \( \omega(\mathcal{G}_e(G)) < \infty \) if and only if \( G \) is a group of finite exponent. If these conditions hold, then

\[
\omega(\mathcal{G}_e(G)) = \max\{o(g) : g \in G\}.
\]

Remark 39. Note that this may be smaller than the exponent of \( G \).

Proof. Clearly, if \( G \) is not a bounded exponent group, then \( \mathcal{G}(G) \) as a subgraph of \( \mathcal{G}_e(G) \) has infinite clique number by Lemma 7. Now, let \( G \) be a periodic group. Then the subsets \( G_n = \{g \in G : o(g) = n\} \), for \( n \in \mathbb{N} \), partition \( G \) into at most countably many subsets. On each of these subsets the power graph and the enhanced power graph coincide. In particular, if \( G \) is bounded exponent, then there are only finitely many classes. It is clear that, if \( x \) and \( y \) have the same order and generate a cyclic group, then each is a power of the other. \( \square \)

Corollary 40. A clique in the enhanced power graph of a group is at most countable.

Proof. For a locally cyclic group is isomorphic to a subgroup of \( \mathbb{Q} \) or \( \mathbb{Q}/\mathbb{Z} \) and hence countable, see [17, Exercise 5, p.105]. \( \square \)

Open problems

This paper concerns several graph theoretical parameters of the power graph of a group. In Section 2.1 we studied groups whose power graph has a finite independence number. In Theorem 6, we proved that if \( G \) is a nilpotent group and \( \alpha(G) < \infty \), then either \( G \) is a finite group or \( G \cong C_{p^\infty} \times H \), for some prime number \( p \), where \( H \) is a finite group and \( p \nmid |H| \). This result motivates us to pose the following question.

Question 41. Let \( G \) be an infinite group. Is it true that \( \alpha(\mathcal{G}(G)) < \infty \) if and only if \( G \cong C_{p^\infty} \times H \), where \( H \) is a finite group and \( p \nmid |H| \).

In Section 2.2, we showed that the chromatic number of the power graph of \( G \) is finite if and only if the clique number of the power graph of \( G \) is finite and this statement is also equivalent to the finiteness of the exponent of \( G \). We proved that the clique number of the power graph of \( G \) is at most countable. We also introduced the concept of pcc groups and we posed the following question.
Question 42. Is it true that the chromatic number of the power graph of any group is at most countably infinite?

It might be interesting to ask how much of Lemma 7 can be proved without the Axiom of Choice. Is there any way of showing that the chromatic number of a group of finite exponent is finite? A good test case for this question would be an abelian group of exponent 3. Colouring the non-identity elements with two colours requires choosing one of each pair \( \{x, x^{-1}\} \), which requires AC (as Bertrand Russell famously pointed out).

In the study of the commuting graph, it is normal to delete vertices which lie in the centre of the group, since they would be joined to all other vertices. Similarly, in the study of the generating graph of a 2-generator group, the identity is an isolated vertex and is usually excluded. This convention is not used for the power graph. So any problem we raise will have two different forms, depending on which convention we use. For the power graph, the question of whether to include or exclude the identity is more interesting. Some of the results will be completely different in the two cases especially those dealing with connectedness. For example, if the identity is excluded, Theorem 21 fails, since indeed the power graph of the infinite cyclic group is connected when the identity is discarded. The next question seems interesting.

Question 43. Which groups do have the property that the power graph is connected when the identity is removed?

Or more generally:

Question 44. Which groups do have the property that the power graph is connected when the set of vertices which dominate the graph is removed?

The following question is the second version for Question 24.

Question 45. For which groups, are the induced subgraphs of the power graph and the commuting graph on \( G \setminus \{e\} \) are equal. Note that free groups have this property.

Question 46. Consider the difference of the power graph and commuting graph, the graph in which \( x \) and \( y \) are joined if they commute but neither is a power of the other. What can be said about this difference graph? In particular, for which groups is it connected? Again this question can be asked with or without the identity. Note that in a periodic group the identity is isolated in the difference graph, but this is not true for arbitrary infinite groups.
References

[1] A. Abdollahi, S. Akbari, H.R. Maimani, Non-commuting graph of a group, *J. Algebra* **298** (2) (2006), 468–492.

[2] J. Araújo, W. Bentz, J. Konieczny, The commuting graph of the symmetric inverse semigroup, *Israel Journal of Mathematics* **207** (1) (2015), 103-149.

[3] L. Babai, P.J. Cameron, Automorphisms and enumeration of switching classes of tournaments, *Electronic J. Combinatorics* **7** (1) (2000), article #R38.

[4] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer-Verlag, 2008.

[5] R. Brauer, K.A. Fowler, On groups of even order, *The Annals of Mathematics* **62** (3) (1955), 567–583.

[6] P.J. Cameron, The power graph of a finite group II, *J. Group Theory* **13** (2010), 779–783.

[7] P.J. Cameron, S. Ghosh, The power graph of a finite group, *Discrete Math.* **311** (13) (2011), 1220–1222.

[8] I. Chakrabarty, S. Ghosh, M.K. Sen, Undirected power graphs of semigroups, *Semigroup Forum* **78** (2009), 410–426.

[9] N.G. De Bruijn, P. Erdös, A colour problem for infinite graphs and a problem in the theory of relations, *Indagationes Math.* **13** (1951), 369–373.

[10] M. Feng, X. Ma, K. Wang, The structure and metric dimension of the power graph of a finite group, *European Journal of Combinatorics* **43** (2015), 82–97.

[11] G. Glauberman, Central elements in core-free groups, *J. Algebra* **4** (1966), 403–420.

[12] D. Gorenstein, J.H. Walter, The characterization of groups with dihedral Sylow 2-subgroups, *J. Algebra* **2** (1965), 85–151, 218–270, 354–393.

[13] K.W. Gruenberg, O.H. Kegel, unpublished manuscript, 1975.

[14] M. Giudici, A. Pope, On bounding the diameter of the commuting graph of a group, *Journal of Group Theory* **17** (1) (2014), 131–149.

[15] M. Hall, Jr., *The Theory of Groups*, Macmillan, New York, 1959.

[16] B.H. Neumann, A problem of Paul Erdös on groups, *J. Austral. Math. Soc.* **21** (4) (1976), 467–472.

[17] D.J.S. Robinson, A Course in the Theory of Groups, Second edition, Springer-Verlag, New York, 1995.

[18] W.R. Scott, *Group Theory*, Dover Publ., New York, 1987.

[19] M. Takashi, On partitions of free products of groups, *Osaka Math. J.* **1** (1) (1949), 49–51.

[20] J.S. Williams, Prime graph components of finite groups, *J. Algebra* **69** (1981), 487–513.
[21] T.J. Woodcock, Commuting Graphs of Finite Groups. PhD thesis, University of Virginia, 2010.