Power Domination and Zero Forcing

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“Zero Forcing and Power Domination for Graph Products.”
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Outline

- Power Domination (PD)
- Zero Forcing (ZF)
- Connection between PD and ZF processes
- Computing PD and ZF numbers
POWER DOMINATION
Monitoring Electrical Networks

Electric power companies need to monitor the state of their networks continuously.
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- **Solution:** Place Phase Measurement Units (PMUs) at electrical nodes, where transmission lines, loads, and generators are connected.
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Monitoring Electrical Networks

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- **Solution:** Place Phase Measurement Units (PMUs) at electrical nodes, where transmission lines, loads, and generators are connected.
- A PMU placed at an electrical node measures the voltage at the node and all current phasors at the node.
- **Problem:** PMUs are costly, so it is important to minimize the number of PMUs used.
- Where should those PMUs be placed to observe the entire system?
Modeling the Problem

An electric power network modeled by a graph
The electrical nodes modeled by graph vertices
Transmission lines joining graph edges

two electrical nodes

http://kk.org/thetechnium/Electricity Network.jpg
The Power Domination Problem in Graphs

Find a minimum set of vertices from where the entire graph can be observed according to certain propagation rules.

First studied by

Haynes at al. ("Domination in graphs applied to electric power networks" (2002))
- Start with a graph $G$ whose vertices are colored either white or black.

- Let $S$ be the set of all vertices colored black. Color all neighbors of vertices in $S$ black.

- Apply the following color-change rule as many times as possible.

  **Color-change Rule:**
  - If there is a black vertex that has exactly one white neighbor - color that neighbor black.
The set $S$ is called a **power dominating set** of a graph $G$ if at the end of applying the propagation rule all vertices in $G$ are colored black.

- A **minimum power dominating set** is a power dominating set with minimum number of vertices.

- **Power domination number** for $G$, denoted $\gamma_P(G)$, is the number of vertices in a minimum power domination set.
Examples
Path $P_4$
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$$\gamma_{P}(P_4) = 1$$
Circle $C_6$
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$$\gamma_P(C_6) = 1$$
Grid
Power Domination
Zero Forcing
Connection between PD and ZF
Computing PD and ZF #s

Real-world Applications
Modeling the Problem
Examples

Grid

Grid

Grid

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Power Domination and Zero Forcing

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Real-world Applications
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Grid

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Power Domination and Zero Forcing
Grid

\[ \gamma_P(G) = 2 \]
Cylinder

\[ P(G) = 3 \]
Cylinder

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Cylinder

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Cylinder

Power Domination and Zero Forcing
Cylinder

\[ \gamma_P(G) = 3 \]
ZERO FORCING
A **zero forcing set** for a graph $G$ is a subset of vertices $B$ such that if initially the vertices in $B$ are colored black and the remaining vertices are colored white, repeated application of the color change rule can color all vertices of $G$ black.

A **minimum zero forcing set** is a zero forcing set with minimum number of vertices.

**Zero forcing number** for $G$, denoted $Z(G)$, is the number of vertices in a minimum zero forcing set.
The zero forcing number was introduced

- by mathematicians Hogben et al. ("Zero forcing sets and the minimum rank of graphs," (2008))
- and independently by mathematical physicists studying control of quantum systems
- and later by computer scientists studying graph search algorithms.
Examples
Power Domination
Zero Forcing
Connection between PD and ZF
Computing PD and ZF #s

Path $P_4$

Path $P_4$
Path $P_4$

$Z(P_4) = 1$
Path $P_4$

- $Z(P_4) = 1$

Where $Z$ denotes the Zero Forcing number and $P_4$ is a path with 4 vertices.
Path $P_4$

$Z(P_4) = 1$
Circle $C_6$
Circle $C_6$
Circle $C_6$

$Z(C_6) = 2$
Connection between PD and ZF Numbers

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Power Domination and Zero Forcing
Observation

The power domination process on a graph $G$ can be described as

1. choosing a set $S \subseteq V(G)$ and
2. applying the zero forcing process to the closed neighborhood $N[S]$ of $S$. 
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- choosing a set $S \subseteq V(G)$ and
- applying the zero forcing process to the closed neighborhood $N[S]$ of $S$.

The set $S$ is a power dominating set of $G$ if and only if $N[S]$ is a zero forcing set for $G$. 
The power domination number of several families of graphs has been determined using the following two-step process:

- **find an upper bound:**
  The upper bound is usually obtained by providing a pattern to construct a set, together with a proof that constructed set is a power dominating set;

- **find a lower bound:**
  The lower bound is usually found by exploiting structural properties of the particular family of graphs, and it usually consists of a very technical and lengthy process.
Research Problem

Finding good general lower bounds for the power domination number.

An effort in that direction is the work by Stephen et al. ("Power domination in certain chemical structures," (2015))
Theorem (1)

Let $G$ be a graph that has an edge. Then

$$\left\lceil \frac{Z(G)}{\Delta(G)} \right\rceil \leq \gamma_P(G),$$

where $\Delta(G) = \max \{\deg v : v \in V(G)\}$ is the maximum degree of $G$.

This bound is tight.
Sketch of Proof:

Choose a minimum PD set \( \{u_1, u_2, \ldots, u_t\} \). Hence \( t = \gamma_P(G) \).
Then \( \sum_{i=1}^{t} \deg u_i \leq t \Delta(G) \).

- If \( G \) has no isolated vertices: Dean et al. ("On the power dominating sets of hypercubes," (2011)):
  \[
  Z(G) \leq \sum_{i=1}^{t} \deg u_i.
  \]

- If \( G \) has isolated vertices, they contribute one to each ZF number and PD number.
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- If \( G \) has isolated vertices, they contribute one to each ZF number and PD number.

Bound is tight:

\[
Z(K_n) = \Delta(K_n) = n - 1 \quad \text{and} \quad \gamma_P(K_n) = 1. \quad \square
\]
Computing PD numbers for Tensor Products

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Power Domination and Zero Forcing
Tensor Products

Let

\[ G = (V(G), E(G)) \] and \[ H = (V(H), E(H)) \]

be disjoint graphs.

The tensor product (also called the direct product) of \( G \) and \( H \) is denoted by \( G \times H \):

- vertex set: \( V(G) \times V(H) \)
- edge set:
  a vertex \( (g, h) \) is adjacent to a vertex \( (g', h') \) in \( G \times H \) if
  \[ \{g, g'\} \in E(G) \] and \[ \{h, h'\} \in E(H). \]
Dorbec et al. ("Power domination in product graphs," (2008)) (the power domination problem for the tensor product of two paths)
Dorbec et al. ("Power domination in product graphs," (2008)) (the power domination problem for the tensor product of two paths)

**Question:**

*What is the power domination number for a tensor product of a path and a complete graph and of a cycle and a complete graph?*
Theorem

Let \( t \geq 3 \) and \( G = P_t \) or \( G = C_t \).

Suppose \( t \) is odd and \( n \geq t \), or suppose \( t \) is even and either

1. \( G = P_t \) and \( n \geq \frac{t}{2} + 2 \), or
2. \( G = C_t \) and \( n \geq \frac{t}{2} \).

Then

\[
\gamma_P(G \times K_n) = \begin{cases} 
\left\lceil \frac{t}{2} \right\rceil & \text{if } t \not\equiv 2 \mod 4, \\
\frac{t}{2} \text{ or } \frac{t}{2} + 1 & \text{if } t \equiv 2 \mod 4.
\end{cases}
\]
Sketch of Proof:

- Upper bound on $\gamma_P(G \times K_n)$:

**Theorem**

Let $n \geq 3$. If $G = P_t$ with $t \geq 2$ or $G = C_t$ with $t \geq 3$, then

$$\gamma_P(G \times K_n) \leq \begin{cases} 
\left\lfloor \frac{t}{2} \right\rfloor & \text{if } t \not\equiv 2 \pmod{4}, \\
\frac{t}{2} + 1 & \text{if } t \equiv 2 \pmod{4}.
\end{cases}$$
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**Theorem**

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- A lower bound on $\gamma_P(G \times K_n)$:
Sketch of Proof:

- Upper bound on $\gamma_P(G \times K_n)$:

**Theorem**

Let $n \geq 3$. If $G = P_t$ with $t \geq 2$ or $G = C_t$ with $t \geq 3$, then

$$\gamma_P(G \times K_n) \leq \begin{cases} \left\lceil \frac{t}{2} \right\rceil & \text{if } t \not\equiv 2 \pmod{4}, \\ \frac{t}{2} + 1 & \text{if } t \equiv 2 \pmod{4}. \end{cases}$$

- A lower bound on $\gamma_P(G \times K_n)$:

Use known ZF #s...
### Theorem

1. **(Fernandes, da Fonseca)**
   
   If $t \geq 1$ is odd and $n \geq 2$, then $Z(P_t \times K_n) = (n - 2)t + 2$.

2. **(V. et al.)**
   
   If $t \geq 2$ is even and $n \geq 3$, then $Z(P_t \times K_n) = (n - 2)t$.

### Theorem

If $n, t \geq 3$, then

$$Z(C_t \times K_n) = \begin{cases} 
(n - 2)t + 2 & \text{if } t \text{ is odd}, \\
(n - 2)t + 4 & \text{if } t \text{ is even}. 
\end{cases}$$
Sketch of Proof - continue

Observation:
\[ \deg(g, h) = \deg_G(g) \deg_H(h) \quad \text{for} \quad (g, h) \in E(G \times H). \]

Hence,
\[ \Delta(G \times H) = \Delta(G) \Delta(H). \]
Observation:

\[ \text{deg}(g, h) = \text{deg}_G(g) \text{deg}_H(h) \quad \text{for} \quad (g, h) \in E(G \times H). \]

Hence,

\[ \Delta(G \times H) = \Delta(G)\Delta(H). \]

\[ \Delta(G \times K_n) = \Delta(G)\Delta(K_n) = 2(n - 1) \]
Sketch of Proof - continue

Consider two cases depending on the parity of \( t \) and use Theorem 1.

- \( t = 2k + 1 \) for some positive integer \( k \)

\[
\gamma_P(G \times K_n) \geq \left\lceil \frac{(n-2)(2k+1)+2}{2(n-1)} \right\rceil
\]

\[
= \left\lceil k + \frac{n - 2k}{2(n - 1)} \right\rceil \geq k + 1 \text{ if } n - 2k > 0.
\]

Hence, \( \left\lceil \frac{t}{2} \right\rceil \leq \gamma_P(G \times K_n) \) if \( t \) is odd and \( n \geq t \).
Sketch of Proof - continue

- \( t = 2k \) for some positive integer \( k \).
  Take \( c = 0 \) for \( G = P_t \) and \( c = 2 \) for \( G = C_t \).

\[
\gamma_P(G \times K_n) \geq \left\lceil \frac{(n-2)(2k)+2c}{2(n-1)} \right\rceil
= \left\lceil k - \frac{k - c}{n-1} \right\rceil = k \text{ if } n - 1 > k - c.
\]

Hence,

\[
\frac{t}{2} \leq \gamma_P(G \times K_n) \text{ if } G = P_t \text{ and } n \geq \frac{t}{2} + 2, \text{ or if } G = C_t \text{ and } n \geq \frac{t}{2}. \quad \square
\]
Computing PD #s for Tensor Products

Computation of ZF #s for Lexicographic Products

Computing PD and ZF #s
Lexicographic Product

Let

\[ G = (V(G), E(G)) \text{ and } H = (V(H), E(H)) \]

be disjoint graphs.

The *lexicographic product* of \( G \) and \( H \) is denoted by \( G \star H \):

- vertex set: \( V(G) \times V(H) \)
- edge set:
  - two vertices \((g, h)\) and \((g', h')\) are adjacent in \( G \star H \) if either
    - \({g, g'} \in E(G)\), or
    - \(g = g'\) and \({h, h'} \in E(H)\).
Domination Number

A vertex $v$ in a graph $G$ is said to dominate itself and all of its neighbors in $G$.

A set of vertices $S$ is a dominating set of $G$ if every vertex of $G$ is dominated by a vertex in $S$.

The minimum cardinality of a dominating set is the domination number of $G$ (denoted by $\gamma(G)$).
Theorem

Let $G$ and $H$ be regular graphs with degree $d_G$ and $d_H$, respectively.

If $\gamma_P(H) = 1$ and $\gamma(G) = 1$, then

$$Z(G \ast H) = d_G |V(H)| + d_H.$$
Theorem

Let $G$ and $H$ be regular graphs with degree $d_G$ and $d_H$, respectively.

If $\gamma_P(H) = 1$ and $\gamma(G) = 1$, then

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Corollary

For $n \geq 2$ and $m \geq 3$,

$$Z(K_n \ast C_m) = (n - 1)m + 2.$$
Sketch of Proof

- Dorbec et al., (“Power Domination in Product Graphs,” (2008))
  \[ \gamma_P(G \ast H) = \gamma(G) \text{ if } \gamma_P(H) = 1. \]

- For lexicographic product
  \[ \deg_{G\ast H}(g, h) = (\deg_G g)|V(H)| + \deg_H h \]
  for any vertex \((g, h) \in V(G \ast H)\), hence
  \[ \Delta(G \ast H) = \Delta(G)|V(H)| + \Delta(H). \]
By Theorem 1: $Z(G \ast H) \leq \gamma_P(G \ast H) \Delta(G \ast H)$. Hence

$$Z(G \ast H) \leq \gamma(G)\left(\Delta(G)|V(H)| + \Delta(H)\right) \quad \text{if } \gamma_P(H) = 1.$$  

- Since $\gamma(G) = 1$, $G$ is $d_G$-regular, and $H$ is $d_H$-regular:

$$Z(G \ast H) \leq d_G|V(H)| + d_H.$$  

- $G \ast H$ is $(d_G|V(H)| + d_H)$-regular, hence

$$d_G|V(H)| + d_H = \delta(G \ast H) \leq Z(G \ast H),$$  

where $\delta(G) = \min\{\deg v : v \in V\}$ is the minimum degree of $G$. □
THANK YOU!

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