Stieltjes electrostatic model interpretation for bound state problems

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Abstract. In this paper, it is shown that Stieltjes electrostatic model and quantum Hamilton Jacobi formalism are analogous to each other. This analogy allows the bound state problem to mimic as \( n \) unit moving imaginary charges \( i \bar{\hbar} \), which are placed in between the two fixed imaginary charges arising due to the classical turning points of the potential. The interaction potential between \( n \) unit moving imaginary charges \( i \bar{\hbar} \) is given by the logarithm of the wave function. For an exactly solvable potential, this system attains stable equilibrium position at the zeros of the orthogonal polynomials depending upon the interval of the classical turning points.

Keywords. Orthogonal polynomials; quantum Hamilton Jacobi and zeros of orthogonal polynomials.

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1. Introduction

Stieltjes [1,2] considered the following problem with \( n \) moving unit charges, interacting through a logarithmic potential, which are placed between two fixed charges \( p \) and \( q \) at \(-1\) and \(1\), respectively, on a real line. It was then proved that the system attains a stable equilibrium position at the zeros of the Jacobi polynomial \( P_n^{(\alpha,\beta)}(x) \). The proof is given in Szego’s book (§6.7) [3]. If the interval is changed on the real line for the fixed charges, then the system attains stable equilibrium position at the zeros of the orthogonal polynomial with the respective intervals. For example, in the interval \([0; \infty)\) one gets the Laguerre polynomials \( L_n^{(b)}(x) \) and for the interval \((-\infty; \infty)\) one gets the Hermite polynomials \( H_n(x) \). This model has been extended to the zeros of general orthogonal polynomials in ref. [4].

The quantum Hamilton Jacobi (QHJ) formalism was formulated for the bound state problems by Leacock and Padgett [5,6] and, later on, was successfully applied to several exactly solvable models (ESMs) [7–11] in one dimension, the quasiexactly solvable (QES) models [12], the periodic potentials [13] and the PT symmetric potentials [14] in
quantum mechanics. In QHJ, the central role is played by the quantum momentum function (QMF). This function, in general, contains fixed poles that arise due to the classical turning points of the potential. In general, for most of the potentials in quantum mechanics, there will be only two fixed poles, and $n$ moving poles arise due to the zeros of the wave function. Thus, one can immediately see the connection between the two scenarios presented above. The fixed poles of the potential are like the two fixed charges and $n$ moving poles on the real line are like $n$ moving charges.

1.1 Electrostatic model

Stieltjes considered the interaction forces for the $n$ moving unit charges arising from a logarithmic potential that are between the fixed charges $p$ and $q$ at $-1$ and $1$, respectively, on a real line as

\[ L = -\log D_n(x_1, x_2, \ldots, x_n) + p \sum_{i=1}^{n} \log \left( \frac{1}{|1-x_i|} \right) + q \sum_{i=1}^{n} \log \left( \frac{1}{|1+x_i|} \right), \]

where

\[ -\log D_n(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i < j \leq n} \log \left( \frac{1}{|x_i - x_j|} \right). \]

The expression (1) becomes a minimum by taking $(\partial L/\partial x_k) = 0$, then one has

\[ \sum_{i=1, i \neq k}^{n} \frac{1}{x_i - x_k} - \frac{p}{x_k - 1} - \frac{q}{x_k + 1} = 0. \]

Then, in refs [1,2] it was proved that the system attains stable equilibrium when \{x_k\} are zeros of the Jacobi polynomial given by

\[ (1-x^2)P_n''(x) + 2[q-p-(p+q)x]P_n'(x) = n[n+2(p+q)-1]P_n(x), \]

where $P_n^{(2p-1,2q-1)}(x)$ are the Jacobi polynomials (for the proof, refer to Szego’s book (§6.7) [3]). The zeros of the Laguerre and the Hermite polynomials admit the same interpretation.

1.2 Quantum Hamilton Jacobi

In this section, a brief review of quantum Hamilton Jacobi formalism is presented (for details, see refs [9,11]). The Schrödinger equation is given by

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi(x,y,z) + V(x,y,z)\psi(x,y,z) = E\psi(x,y,z). \]

One defines a function $S$ analogous to the classical characteristic function by the relation \( \psi(x,y,z) = \exp(iS/\hbar) \), which, when substituted in (5), gives

\[ (\vec{\nabla}S)^2 - i\hbar \vec{\nabla} \cdot (\vec{\nabla}S) = 2m(E - V(x,y,z)). \]
The quantum momentum function $p$ is defined in terms of the function $S$ as $\vec{p} = \vec{\nabla} S$. Substituting this in eq. (6) gives the QHJ equation for $\vec{p}$ as

$$(\vec{p})^2 - i\hbar \vec{\nabla} \cdot \vec{p} = 2m(E - V(x, y, z))$$

and from eq. (5) and the QMF $\vec{p} = \vec{\nabla} S$, one can see that $\vec{p}$ is the logarithmic derivative of $\psi(x, y, z)$, i.e., $\vec{p} = -i\hbar \vec{\nabla} \ln \psi(x, y, z)$. The above discussion of the QHJ formalism is done in three dimensions, and the same equation in one dimension takes the following form:

$$p^2 - i\hbar \frac{dp}{dx} = 2m(E - V(x)),$$

which is also known as the Riccati equation. In one dimension the QMF takes the form $p = -i\hbar \frac{d}{dx} \ln \psi(x)$. It is shown by Leacock and Padgett [5,6] that the action angle variable gives rise to the exact quantization condition

$$J(E) = \frac{1}{2\pi} \oint_C p \, dx = n\hbar.$$

2. Model

Starting with the QMF, the analogue between the two models is established. The fact that only the residues of the QMF are required for finding the eigenvalues is studied in refs [7,8]. The formalism for effectively obtaining both the eigenfunctions and the eigenvalues from the singularity structure of the quantum momentum function is given in ref. [9]. Then the quantum momentum function is given by [7–12]

$$p(x) = \sum_{k=1}^{n} \frac{-i}{x - x_k} + Q(x),$$

where the moving poles are simple poles with residue $-i\hbar$ (we take here $\hbar = m = 1$) [9,11] and $Q(x)$ is the residue of fixed poles arising due to the exactly solvable potentials. This equation resembles eq. (3). Thus, the quantum momentum function can be interpreted as a system of equations for $n$ moving poles arising from the logarithmic derivative of wave function and fixed poles arising from the classical turning points. By asking the following question: When does this system come to stable equilibrium? From the above discussion it is clear that the answer can be obtained using the Stieltjes electrostatic model. It can be shown that the same wave function can be obtained from both the models. Thus, there exists an analogy between the Stieltjes electrostatic interpretation for zeros of orthogonal polynomials and the quantum Hamilton Jacobi formalism.

In the quantum Hamilton Jacobi formalism, if one has complete information about the pole structure of the quantum momentum function, then by calculating the integral in eq. (9), one gets the exact quantization condition. One can also get the quantization condition by converting the quantum momentum function into a differential equation. Therefore, the connection between the Stieltjes electrostatic interpretation and the quantum Hamilton Jacobi formalism is established by solving the quantum momentum
function as a differential equation. This is achieved by taking the limit \( \lim_{x \to x_j} ip(x) = 0 \) and thus eq. (10) is given by

\[
\lim_{x \to x_j} \left[ \sum_{k=1}^{n} \frac{1}{x - x_k} + i Q(x) \right] = 0. \tag{11}
\]

By introducing the polynomial

\[
f_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n), \tag{12}
\]

one has

\[
\frac{1}{x - x_k} = \frac{f'_n(x_k)}{f_n(x_k)}. \tag{13}
\]

By taking the limit on both sides one gets

\[
\lim_{x \to x_i} \left[ \frac{f'_n(x_k)}{f_n(x_k)} - \frac{1}{x - x_k} \right] = \sum_{j=1, i \neq k}^{n} \frac{1}{x_j - x_k}. \tag{14}
\]

As \( Q(x) \) does not have any poles at \( x_k \), eq. (11) is given as

\[
\sum_{j=1, i \neq k}^{n} \frac{1}{x_j - x_k} + i Q(x) = 0. \tag{15}
\]

Using the formula \([4,15]\)

\[
2 \sum_{j=1, i \neq k}^{n} \frac{1}{x_j - x_k} = -\frac{f''_n(x_k)}{f'_n(x_k)}, \tag{16}
\]

eq. (15) becomes

\[
-\frac{1}{2} \frac{f''_n(x_k)}{f'_n(x_k)} + i Q(x_k) = 0, \quad 1 \leq k \leq n \tag{17}
\]

the solution for the differential eq. (17), for an exactly solvable potential that is for certain \( Q(x_k) \), are the zeros of appropriate orthogonal polynomials. The interval is fixed by the fixed poles of the potential. It is well known that the classical orthogonal polynomials arise as solutions to the bound state problems. Thus, the classical orthogonal polynomials are classified into three different categories depending upon the range of the polynomials. The polynomials in the intervals \((-\infty; \infty)\) are the Hermite polynomials, in the intervals \([0; \infty)\) are the Laguerre polynomials and in the intervals \([-1; 1]\) are the Jacobi polynomials. Their singularity structure is as follows:

\[
Q(x) = x, \quad Q(x) = \frac{b}{x} + C
\]

and

\[
Q(x) = -\frac{a}{x - 1} - \frac{b}{x + 1}
\]
for the Hermite, the Laguerre and the Jacobi polynomials respectively. Hence, the differential equation can be obtained by examining the singularity structure of the quantum momentum function. This can be seen by rewriting eq. (17) as

\[- f''_n(x) + 2iQ(x_k)f'_n(x) = 0, \quad 1 \leq k \leq n.\] (18)

The function \(Q(x)\), which has the information of fixed pole singularity structure, appears as the coefficient of \(f'_n(x_k)\). By examining the differential equations of the Hermite, the Laguerre and the Jacobi polynomials, the coefficients of \(Q(x)\) are fixed.

As an illustration, we consider the Coulomb potential in natural units

\[V(r) = \frac{\lambda}{r} - \frac{1}{4} - \frac{l(l + 1)}{r^2}.\] (19)

The Coulomb problem is solved using quantum Hamilton Jacobi formalism in ref. [11]. The quantum momentum function for the Coulomb problem [11] is given as

\[p(r) = \sum_{k=1}^{n} \frac{-i}{r - r_k} + \frac{b}{r} + C.\] (20)

Comparing QMF with eq. (10) one finds \(Q(r) = (b/r) + C\). Using eq. (16) and following the Stieltjes model procedure, we convert the QMF (20) into a differential equation

\[- f''_n(r) + 2i \left( \frac{b}{r} + C \right) f'_n(r) = 0, \quad 1 \leq k \leq n.\] (21)

This system attains stable equilibrium when \(\{r_k\}\) are the zeros of the Laguerre polynomial, as the interval is between \([0, \infty]\)

\[- \frac{d^2}{dr^2} f_n(r) + 2i \left( \frac{b}{r} + C \right) \frac{d}{dr} f_n(r) = \lambda f_n(r).\] (22)

Here \(f_n(r) = L^l_{\lambda}(r)\) satisfies the Laguerre differential equation

\[r \frac{d^2}{dr^2} f_n(r) + (l + 1 - r) \frac{d}{dr} f_n(r) + \lambda f_n(r) = 0,\] (23)

where \(\lambda\) is an integer. By examining the first two terms of the differential eqs (22) and (23) one gets \(2b = -i(l + 1)\) and \(2C = i\).

By following the method adopted in QHJ, writing the quantum momentum function as

\[p = \sum_{k=1}^{n} i \frac{f'(r)}{f(r)} + Q(r),\] (24)

and substituting in eq. (8), one gets

\[f''_n(r_k) + 2iQ(r) f'_n(r_k) + [Q^2(r) - iQ'(r) - E + V(r)] f(r) = 0.\] (25)

The search for the polynomial solutions leads to quantization. This is equivalent to demanding \([Q^2(r) - iQ'(r) - E + V(r)]\) to be constant, i.e.,

\[V(r) - E = \lambda - Q^2(r) + i Q'(r).\] (26)
This gives
\[ Q(r) = \frac{1}{2i} \left( \frac{l + 1}{r} - 1 \right). \]  

(27)

Thus, the same wave function is obtained from both the methods. It can also be seen that the wave function can be obtained from the quantum momentum function itself, and one need not solve the QHJ equation. It can also be observed from eq. (26) that \( Q(r) \) resembles the superpotential. If we change \( i \) to \(-i\) in eq. (26) one gets the partner potentials. Similar analysis can be done for the harmonic oscillation and the Scarf potential whose solutions are the Hermite and the Jacobi polynomials.

3. Discussion

From the previous discussion, it is clear that the Stieltjes electrostatic model and the QHJ formalism are analogous to each other. Therefore, this analogy allows the bound state problem to mimic as \( n \) unit moving imaginary charges \( i\bar{h} \), which are placed in between the two fixed imaginary charges arising due to the classical turning points of the potential. The interaction potential between \( n \) unit moving imaginary charges \( i\bar{h} \) is given by the logarithm of the wave function. For an exactly solvable potential, this system attains stable equilibrium position at the zeros of the orthogonal polynomials depending upon the interval of the classical turning points. Once charges arise in any model, they satisfy the continuity equation of the form
\[ \frac{\partial}{\partial t} \rho + \nabla \cdot J = 0, \]  

(28)

as, eqs (18) and (25) are nothing but different forms of the Schrödinger equation. Therefore, there exists a continuity equation of this form for these imaginary charges with \( \rho = \int_V \psi^* \psi \, dV \) is the probability density function and \( J = \frac{\hbar}{i} \left[ \psi^* (\nabla \psi) - \psi (\nabla \psi^*) \right] \) is the probability current density function. Hence, the conservation of probability leading to conservation of imaginary charge and probability current leads to current density for imaginary charge. In this model \( \rho \) is the amount of imaginary charge and \( J \) is the current density for imaginary charge. Thus, this model is consistent with quantum mechanics.

4. Conclusion

In this paper, the two different models, the Stieltjes electrostatic model and the quantum Hamilton Jacobi formalism, are examined, except that one is a classical model and the other is a quantum model. It is shown that Stieltjes electrostatic model and quantum Hamilton Jacobi formalism are analogous to each other. A new feature that emerges from this study is that the wave function can be obtained from the quantum momentum function itself and one need not solve the quantum Hamilton Jacobi equation. The Stieltjes electrostatic model gives good insights to the methodology of the quantum Hamilton Jacobi formalism. It is interesting to note that the Stieltjes electrostatic model existed almost 30 years before quantum mechanics came into existence.
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