An HDG Method for Distributed Control of Convection Diffusion PDEs

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Abstract

We propose a hybridizable discontinuous Galerkin (HDG) method to approximate the solution of a distributed optimal control problem governed by an elliptic convection diffusion PDE. We derive optimal a priori error estimates for the state, adjoint state, their fluxes, and the optimal control. We present 2D and 3D numerical experiments to illustrate our theoretical results.

1 Introduction

We consider the following distributed control problem: Minimize the functional

\[ \min_{u} J(u) = \frac{1}{2} \| y - y_d \|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \| u \|_{L^2(\Omega)}^2, \quad \gamma > 0, \]  

subject to

\[ -\Delta y + \beta \cdot \nabla y = f + u \quad \text{in } \Omega, \]
\[ y = g \quad \text{on } \partial \Omega, \]  

where \( \Omega \subset \mathbb{R}^d \ (d \geq 2) \) is a Lipschitz polyhedral domain with boundary \( \Gamma = \partial \Omega \), \( f \in L^2(\Omega) \), and the vector field \( \beta \) satisfies

\[ \nabla \cdot \beta = 0. \]  

It is well known that the optimal control problem (1)-(2) is equivalent to the optimality system

\[ -\Delta y + \beta \cdot \nabla y = f + u \quad \text{in } \Omega, \]  
\[ y = g \quad \text{on } \partial \Omega, \]  
\[ -\Delta z - \nabla \cdot (\beta z) = y_d - y \quad \text{in } \Omega, \]  
\[ z = 0 \quad \text{on } \partial \Omega, \]  
\[ z - \gamma u = 0 \quad \text{in } \Omega. \]  

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Optimal control problems for convection diffusion equations arise in applications \cite{21} and are also an important step towards optimal control problems for fluid flows. Therefore, researchers have developed many different numerical methods for this type of problem including approaches based on finite differences \cite{3}, standard finite element discretizations \cite{14,16}, stabilized finite elements \cite{2,19}, the symmetric stabilization method \cite{4}, the SUPG method \cite{13,17}, the edge-stabilization method \cite{5,28}, mixed finite elements \cite{16,29,31}, and discontinuous Galerkin (DG) methods \cite{17,20,26,27,30,32,33}.

DG methods are well suited for problems with convection, but they often have a higher computational cost compared to other methods. Hybridizable discontinuous Galerkin (HDG) methods keep the advantages of DG methods, but have a lower number of globally coupled unknowns. HDG methods were introduced in \cite{9}, and now have been applied to many different problems \cite{6,8,10,12,22,23,24,25}.

HDG methods have recently been successfully applied to two PDE optimal control problems. Zhu and Celiker \cite{34} obtained optimal convergence rates for an HDG method for a distributed optimal control problem governed by the Poisson equation. The authors have also studied an HDG method for a difficult Dirichlet optimal boundary control problem for the Poisson equation in \cite{18}. We proved an optimal superlinear convergence rate for the control in polygonal domains. Despite the large amount of work on this problem, a superlinear convergence result of this type had only been previously obtained for one other numerical method on a special class of meshes \cite{1}.

Due to these recent results and the favorable properties of HDG methods, we continue to investigate HDG for optimal control problems for PDEs in this work. Specifically, we consider the above distributed control problem for the elliptic convection diffusion equation, and apply an HDG method with polynomials of degree \( k \) to approximate all the variables of the optimality system (4), i.e., the state \( y \), dual state \( z \), the numerical traces, and the fluxes \( q = -\nabla y \) and \( p = -\nabla z \). We describe the HDG method and its implementation in \textbf{Section 2} in \textbf{Section 3}, we obtain the error estimates

\[
\| y - y_h \|_{0, \Omega} = O(h^{k+1}), \quad \| z - z_h \|_{0, \Omega} = O(h^{k+1}), \\
\| q - q_h \|_{0, \Omega} = O(h^{k+1}), \quad \| p - p_h \|_{0, \Omega} = O(h^{k+1}),
\]

and

\[
\| u - u_h \|_{0, \Omega} = O(h^{k+1}).
\]

We present 2D and 3D numerical results in \textbf{Section 4} and then briefly discuss future work.

\section{HDG scheme for the optimal control problem}

We begin by setting notation.

Throughout the paper we adopt the standard notation \( W^{m,p}(\Omega) \) for Sobolev spaces on \( \Omega \) with norm \( \| \cdot \|_{m,p,\Omega} \) and seminorm \( | \cdot |_{m,p,\Omega} \). We denote \( W^{m,2}(\Omega) \) by \( H^m(\Omega) \) with norm \( \| \cdot \|_{m,\Omega} \) and seminorm \( | \cdot |_{m,\Omega} \). Specifically, \( H^1_0(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial \Omega \} \). We denote the \( L^2 \)-inner products on \( L^2(\Omega) \) and \( L^2(\Gamma) \) by

\[
(v, w) = \int_{\Omega} vw \quad \forall v, w \in L^2(\Omega), \\
\langle v, w \rangle = \int_{\Gamma} vw \quad \forall v, w \in L^2(\Gamma).
\]
Define the space $H(\text{div}, \Omega)$ as

$$H(\text{div}, \Omega) = \{ v \in [L^2(\Omega)]^d, \nabla \cdot v \in L^2(\Omega) \}.$$ 

Let $\mathcal{T}_h$ be a collection of disjoint elements that partition $\Omega$. We denote by $\partial \mathcal{T}_h$ the set $\{ \partial K : K \in \mathcal{T}_h \}$. For an element $K$ of the collection $\mathcal{T}_h$, let $e = \partial K \cap \Gamma$ denote the boundary face of $K$ if the $d-1$ Lebesgue measure of $e$ is non-zero. For two elements $K^+$ and $K^-$ of the collection $\mathcal{T}_h$, let $e = \partial K^+ \cap \partial K^-$ denote the interior face between $K^+$ and $K^-$ if the $d-1$ Lebesgue measure of $e$ is non-zero. Let $\varepsilon_h^o$ and $\varepsilon_h^\partial$ denote the set of interior and boundary faces, respectively. We denote by $\varepsilon_h$ the union of $\varepsilon_h^o$ and $\varepsilon_h^\partial$. We finally introduce

$$(w, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w, v)_K, \quad \langle \zeta, \rho \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K}. \tag{5}$$

Let $\mathcal{P}^k(D)$ denote the set of polynomials of degree at most $k$ on a domain $D$. We introduce the discontinuous finite element spaces

$$V_h := \{ v \in [L^2(\Omega)]^d : v|_K \in [\mathcal{P}^k(K)]^d, \forall K \in \mathcal{T}_h \}, \tag{6}$$

$$W_h := \{ w \in L^2(\Omega) : w|_K \in \mathcal{P}^k(K), \forall K \in \mathcal{T}_h \}, \tag{7}$$

$$M_h := \{ \mu \in L^2(\varepsilon_h) : \mu|_e \in \mathcal{P}^k(e), \forall e \in \varepsilon_h \}. \tag{8}$$

Let $M_h(o)$ and $M_h(\partial)$ denote the subspaces of $M_h$ containing each $e \in \varepsilon_h^o$ and $e \in \varepsilon_h^\partial$, respectively. Note that $M_h$ consists of functions which are continuous inside the faces (or edges) $e \in \varepsilon_h$ and discontinuous at their borders. In addition, for any function $w \in W_h$, we use $\nabla w$ to denote the piecewise gradient on each element $K \in \mathcal{T}_h$. A similar convention applies to the divergence $\nabla \cdot r$ for all $r \in V_h$.

### 2.1 The HDG Formulation

The mixed weak form of the optimality system (4a)-(4e) is given by

$$(q, r_1) - (y, \nabla \cdot r_1) + \langle y, r_1 \cdot n \rangle = 0, \tag{9a}$$

$$(\nabla \cdot (q + \beta y), w_1) = (f + u, w_1), \tag{9b}$$

$$(p, r_2) - (z, \nabla \cdot r_2) + \langle z, r_2 \cdot n \rangle = 0, \tag{9c}$$

$$(\nabla \cdot (p - \beta z), w_2) = (y_d - y, w_2), \tag{9d}$$

$$(z - \gamma u, v) = 0, \tag{9e}$$

for all $(r_1, w_1, r_2, w_2, v) \in H(\text{div}, \Omega) \times L^2(\Omega) \times H(\text{div}, \Omega) \times L^2(\Omega) \times L^2(\Omega)$. Recall we assume $\beta$ is divergence free; this allows us to rewrite the convection term $\beta \cdot \nabla y$ in (4a) as $\nabla \cdot (\beta y)$ in (4b).

To approximate the solution of this system, the HDG method seeks approximate fluxes $q_h, p_h \in V_h$, states $y_h, z_h \in W_h$, interior element boundary traces $\gamma_{\partial}^{\partial}, \gamma_{\varepsilon}^{\partial} \in M_h(o)$, and control $u_h \in W_h$ satisfying

$$(q_h, r_1)_{T_h} - (y_h, \nabla \cdot r_1)_{T_h} + \langle \gamma_{\partial}^{\partial}, r_1 \cdot n \rangle_{\partial T_h \setminus \varepsilon_h^{\partial}} = -\langle g, r_1 \cdot n \rangle_{\varepsilon_h^{\partial}}, \tag{9a}$$

$$-(q_h + \beta y_h, \nabla w_1)_{T_h} + \langle q_h, n \cdot w_1 \rangle_{\partial T_h} + \langle \beta \cdot n \gamma_{\partial}^{\partial}, w_1 \rangle_{\partial T_h \setminus \varepsilon_h^{\partial}} - (u_h, w_1)_{T_h} = -\langle \beta \cdot n g, w_1 \rangle_{\varepsilon_h^{\partial}} + (f, w_1)_{T_h} \tag{9b}$$

for all $(r_1, w_1, r_2, w_2, v) \in H(\text{div}, \Omega) \times L^2(\Omega) \times H(\text{div}, \Omega) \times L^2(\Omega) \times L^2(\Omega)$.
for all \((r_1, w_1) \in V_h \times W_h\).

\[
(p_h, r_2)_{T_h} - (z_h, \nabla \cdot r_2)_{T_h} + \langle \gamma u_h, r_2 \cdot n \rangle_{\partial T_h \setminus e_h^0} = 0,
\]

\[
- (p_h - \beta z_h, \nabla w_2)_{T_h} + \langle \beta \cdot n, w_2 \rangle_{\partial T_h}
- \langle \beta \cdot n z_h^0, w_2 \rangle_{\partial T_h \setminus e_h^0} + (y_h, w_2)_{T_h} = (y_d, w_2)_{T_h},
\]

for all \((r_2, w_2) \in V_h \times W_h\).

\[
\langle \hat{q}_h \cdot n + \beta \cdot n \hat{y}_h^0, \mu_1 \rangle_{\partial T_h \setminus e_h^0} = 0,
\]

\[
\langle \hat{p}_h \cdot n - \beta \cdot n z_h^0, \mu_2 \rangle_{\partial T_h \setminus e_h^0} = 0,
\]

for all \(\mu_1, \mu_2 \in M_h(o)\), and the optimality condition

\[
(z_h - \gamma u_h, w_3)_{T_h} = 0,
\]

for all \(w_3 \in W_h\). The numerical traces on \(\partial T_h\) are defined as

\[
\hat{q}_h \cdot n = q_h \cdot n + \tau_1 (y_h - \hat{y}_h^0) \quad \text{on} \ \partial T_h \setminus e_h^0,
\]

\[
\hat{q}_h \cdot n = q_h \cdot n + \tau_1 (y_h - g) \quad \text{on} \quad e_h^0,
\]

\[
\hat{p}_h \cdot n = p_h \cdot n + \tau_2 (z_h - \hat{z}_h^0) \quad \text{on} \ \partial T_h \setminus e_h^0,
\]

\[
\hat{p}_h \cdot n = p_h \cdot n + \tau_2 z_h \quad \text{on} \quad e_h^0,
\]

where \(\tau_1\) and \(\tau_2\) are positive stabilization functions defined on \(\partial T_h\). We specify these functions in the next section.

### 2.2 Implementation

For the numerical implementation, we follow a similar procedure to our earlier work \cite{18}. First, we perform some basic manipulations to the above system \((9a)-(9k)) to find that

\[
(q_h, p_h, y_h, z_h, \hat{y}_h^0, \hat{z}_h^0) \in V_h \times V_h \times W_h \times W_h \times M_h(o) \times M_h(o)
\]

is the solution of the following weak formulation:

\[
(q_h, r_1)_{T_h} - (y_h, \nabla \cdot r_1)_{T_h} + \langle \gamma u_h, r_1 \cdot n \rangle_{\partial T_h \setminus e_h^0} = - \langle g, r_1 \cdot n \rangle_{e_h^0},
\]

\[
(p_h, r_2)_{T_h} - (z_h, \nabla \cdot r_2)_{T_h} + \langle \gamma u_h, r_2 \cdot n \rangle_{\partial T_h \setminus e_h^0} = 0,
\]

\[
(\nabla \cdot q_h, w_1)_{T_h} - (\beta y_h, \nabla w_1)_{T_h} + \langle \tau_1 y_h, w_1 \rangle_{\partial T_h}
- (\gamma^{-1} z_h, w_1)_{T_h} + \langle \beta \cdot n - \tau_1 \hat{y}_h^0, w_1 \rangle_{\partial T_h \setminus e_h^0} = - \langle \beta \cdot n - \tau_1 g, w_1 \rangle_{e_h^0}
+ \langle f, w_1 \rangle_{T_h},
\]

\[
(\nabla \cdot p_h, w_2)_{T_h} + (y_h, w_2)_{T_h} + (\beta z_h, \nabla w_2)_{T_h}
+ \langle \tau_2 z_h, w_2 \rangle_{\partial T_h}
- \langle \tau_2 + \beta \cdot n \hat{z}_h^0, w_2 \rangle_{\partial T_h \setminus e_h^0} = (y_d, w_2)_{T_h},
\]

\[
\langle q_h \cdot n, \mu_1 \rangle_{\partial T_h \setminus e_h^0} + \langle \tau_1 y_h, \mu_1 \rangle_{\partial T_h \setminus e_h^0}
+ \langle \beta \cdot n - \tau_1 \hat{y}_h^0, \mu_1 \rangle_{\partial T_h \setminus e_h^0} = 0,
\]

\[
\langle p_h \cdot n, \mu_2 \rangle_{\partial T_h \setminus e_h^0} + \langle \tau_2 z_h, \mu_2 \rangle_{\partial T_h \setminus e_h^0}
- \langle \beta \cdot n + \tau_2 \hat{z}_h^0, \mu_2 \rangle_{\partial T_h \setminus e_h^0} = 0,
\]
for all \((r_1, r_2, w_1, w_2, \mu_1, \mu_2) \in V_h \times V_h \times W_h \times \cdots \times W_h \times M_h(\alpha) \times M_h(\alpha)\).

Note that we have used the optimality condition (9g) to eliminate \(u_h\) from the discrete equations. Once the above system (10) is solved numerically, \(u_h\) can be easily found using the optimality condition: \(u_h = \gamma^{-1}z_h\).

2.3 Matrix equations

Assume \(V_h = \text{span}\{\phi_i\}_{i=1}^{N_1}, W_h = \text{span}\{\phi_i\}_{i=1}^{N_2}, M_h^o = \text{span}\{\psi_i\}_{i=1}^{N_3}\). Then

\[
\begin{align*}
q_h &= \sum_{j=1}^{N_1} q_j \phi_j, \quad y_h = \sum_{j=1}^{N_2} y_j \phi_j, \quad \tilde{y}_h = \sum_{j=1}^{N_3} \alpha_j \psi_j, \\
p_h &= \sum_{j=1}^{N_1} p_j \phi_j, \quad z_h = \sum_{j=1}^{N_2} z_j \phi_j, \quad \tilde{z}_h = \sum_{j=1}^{N_3} \gamma_j \psi_j.
\end{align*}
\]

Substitute (11) into (10a)-(10f) and use the corresponding test functions to test (10a)-(10f), respectively, to obtain the matrix equation

\[
\begin{bmatrix}
A_1 & 0 & -A_2 & 0 & A_{15} & 0 \\
0 & A_1 & 0 & -A_2 & 0 & A_{15} \\
A_2^T & 0 & A_{12} & -A_2 & 0 & A_{16} & 0 \\
0 & A_2^T & 0 & A_{13} & 0 & A_{17} & 0 \\
A_4^T & 0 & A_{18} & 0 & A_{20} & 0 & A_{21} \\
0 & A_{15}^T & 0 & A_{19} & 0 & A_{21} & 0
\end{bmatrix}
\begin{bmatrix}
q \\
p \\
y \\
j \\
\hat{y} \\
\hat{z}
\end{bmatrix}
= \begin{bmatrix}
-b_1 \\
0 \\
-b_5 \\
b_4 \\
0 \\
0
\end{bmatrix},
\]

where \(q, p, y, j, \hat{y}, \hat{z}\) are the coefficient vectors for \(q_h, p_h, y_h, \tilde{y}_h, \tilde{z}_h\), respectively, and

\[
\begin{align*}
A_1 &= \langle \phi_j, \phi_i \rangle \tau_h, \quad A_2 = \langle \phi_j, \nabla \cdot \phi_i \rangle \tau_h, \quad A_3 = \langle \psi_j, \phi_i \cdot n \rangle \tau_h, \\
A_4 &= \langle \phi_j, \phi_i \rangle \tau_h, \quad A_5 = \langle \beta \phi_j, \nabla \phi_i \rangle \tau_h, \quad A_6 = \langle \tau_1 \phi_j, \phi_i \rangle \partial \tau_h, \\
A_7 &= \langle \beta \cdot n \phi_j, \phi_i \rangle \partial \tau_h, \quad A_8 = \langle \tau_1 \psi_j, \phi_i \rangle \partial \tau_h, \quad A_9 = \langle \beta \cdot n \psi_j, \phi_i \rangle \partial \tau_h, \\
A_{10} &= \langle \tau_1 \psi_j, \psi_i \rangle \partial \tau_h, \quad A_{11} = \langle \beta \cdot n \psi_j, \psi_i \rangle \partial \tau_h, \quad A_{12} = A_6 - A_5, \\
A_{13} &= A_4 + A_6 - A_7, \quad b_1 = \langle g, \phi_i \cdot n \rangle \psi^0, \quad b_2 = \langle (\beta \cdot n - \tau_1) g, \phi_i \rangle \psi^0, \\
b_3 &= \langle f, \phi_i \rangle \tau_h, \quad b_4 = \langle y_d, \phi_i \rangle \tau_h, \quad b_5 = b_3 - b_2.
\end{align*}
\]

The remaining matrices are constructed by extracting the corresponding rows and columns from linear combinations of \(A_3, A_8, A_9, A_{10},\) and \(A_{11}\).

2.4 Local solver

Next, we use the discontinuous nature of the approximation spaces \(V_h\) and \(W_h\) to eliminate all unknowns except the coefficient vectors of the numerical traces.

The matrix equation (12) can be rewritten as

\[
\begin{bmatrix}
B_1 & B_2 & B_3 \\
-B_2^T & B_4 & B_5 \\
B_6 & B_7 & B_8
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
0
\end{bmatrix},
\]

where \(\alpha = [q; p], \beta = [y; j], \gamma = [\hat{y}; \hat{z}], b_1 = [-b_1; 0],\) and \(b_2 = [-b_5; b_4],\) and also \(B_i\) are the corresponding blocks of the coefficient matrix of (12).
In the appendix, we show how the first two equations of (13) can be used to eliminate both $\alpha$ and $\beta$ in an element-by-element fashion. We obtain
\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} =
\begin{bmatrix}
G_1 & H_1 \\
G_2 & H_2
\end{bmatrix}
\begin{bmatrix}
\gamma \\
\beta
\end{bmatrix}
\]  
(14)
and
\[B_6 \alpha + B_7 \beta + B_8 \gamma = 0,
\]  
(15)
where $G_1, G_2, H_1, H_2$ are sparse. This gives a globally coupled equation for $\gamma$ only:
\[K \gamma = F,
\]  
(16)
where
\[K = B_6 G_1 + B_7 G_2 + B_8 \quad \text{and} \quad F = B_6 H_1 + B_7 H_2.
\]
Once $\gamma$ is computed, $\alpha$ and $\beta$ can be quickly and easily computed using (14).

3 Error Analysis

Next, we provide a convergence analysis of the above HDG method for the optimal control problem. Throughout this section, we assume $\beta \in [W^{1,\infty}(\Omega)]^d$, $\Omega$ is a bounded convex polyhedral domain, $h \leq 1$, and the solution of the optimality system (4) is smooth enough.

3.1 Main result

For our theoretical results, we require the stabilization functions $\tau_1$ and $\tau_2$ are chosen to satisfy
(A1) $\tau_1$ is piecewise constant on $\partial T_h$.
(A2) $\tau_1 = \tau_2 + \beta \cdot n$.
(A3) For any $K \in T_h$, $\min (\tau_1 - \frac{1}{2} \beta \cdot n)|_{\partial K} > 0$.

We note that (A2) and (A3) imply
\[\min (\tau_2 + \frac{1}{2} \beta \cdot n)|_{\partial K} > 0 \quad \text{for any } K \in T_h.
\]  
(17)

Theorem 1. We have
\[
\begin{align*}
\|q - q_h\|_{T_h} & \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \\
\|p - p_h\|_{T_h} & \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \\
\|y - y_h\|_{T_h} & \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \\
\|z - z_h\|_{T_h} & \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \\
\|u - u_h\|_{T_h} & \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}).
\end{align*}
\]
3.2 Preliminary material

Next, we introduce the projection operators $\Pi_V$ and $\Pi_W$ defined in [7] that we use frequently in our proof. The value of the projection on each element $K \in \mathcal{T}_h$ is determined by requiring that the components satisfy the equations

\begin{align}
(\Pi_V q + \beta \Pi_W y, r)_K &= (q + \beta y, r)_K, \\
(\Pi_W y, w)_K &= (y, w)_K, \\
(\Pi_V q \cdot n + \beta \cdot n P_M y + \tau_1 \Pi_W y, \mu)_e &= (q \cdot n + \beta \cdot n y + \tau_1 y, \mu)_e,
\end{align}

for all $(r, w, \mu) \in \mathcal{P}_{k-1}(K) \times \mathcal{P}_{k-1}(K) \times \mathcal{P}_k(e)$ and for all faces $e$ of the simplex $K$. Here, $P_M$ denotes the $L^2$-orthogonal projection from $L^2(\varepsilon_h)$ into $M_h$ satisfying

\begin{equation}
⟨P_M y - y, \mu⟩_e = 0, \quad \forall e \in \varepsilon_h, \; \forall \mu \in M_h.
\end{equation}

The following lemma from [7] provides the approximation properties of the projection operator [18].

**Lemma 1.** Suppose $k \geq 0$, and $\tau_1$ satisfies (A3). Then the system (18) is uniquely solvable for $\Pi_V q$ and $\Pi_W y$. Moreover, we have the following approximation properties

\begin{align}
\|\Pi_V q - q\|_K &\leq Ch^{k+1}q|_{k+1,K} + Ch^{k+1}y|_{k+1,K}, \\
\|\Pi_W y - y\|_K &\leq Ch^{k+1}q|_{k+1,K} + Ch^{k+1}y|_{k+1,K},
\end{align}

where $C$ is a constant depending on the polynomial degree and the shape-regularity parameters of the elements.

For the convection diffusion optimal control problem, we introduce another projection operator associated to the dual problem. The projection $\Pi_V$ and $\Pi_W$ is determined by the following equations

\begin{align}
(\Pi_V p - \beta \Pi_W z, r)_K &= (p - \beta z, r)_K, \\
(\Pi_W z, w)_K &= (z, w)_K, \\
(\Pi_V p \cdot n - \beta \cdot n P_M z + \tau_2 \Pi_W z, \mu)_e &= (p \cdot n - \beta \cdot n z + \tau_2 z, \mu)_e,
\end{align}

for all $(r, w, \mu) \in \mathcal{P}_{k-1}(K) \times \mathcal{P}_{k-1}(K) \times \mathcal{P}_k(e)$ and for all faces $e$ of the simplex $K$.

Again, results from [7] give the following estimates.

**Lemma 2.** Suppose $k \geq 0$, and $\tau_2$ satisfies (17). Then the system (21) is uniquely solvable for $\Pi_V p$ and $\Pi_W z$, and

\begin{align}
\|\Pi_V p - p\|_K &\leq Ch^{k+1}p|_{k+1,K} + Ch^{k+1}z|_{k+1,K}, \\
\|\Pi_W z - z\|_K &\leq Ch^{k+1}p|_{k+1,K} + Ch^{k+1}z|_{k+1,K},
\end{align}

where $C$ is a constant depending on the polynomial degree and the shape-regularity parameters of the elements.

Next, we present a basic approximation of the function $\beta$. Let $P_0$ be the vectorial piecewise-constant $L^2$ projection. We have the following estimate:

\[\|\beta - P_0 \beta\|_{0,\infty,\Omega} \leq Ch\|\beta\|_{1,\infty,\Omega}.\]
Lemma 3. For any \( e \in \partial K \), define \( \tau_2|_e = \tau_1|_e - P_0 \beta|_K \cdot n_e \), we have

\[
\|\tau_2 - \tau_2\|_{0, \infty, \partial T_h} \leq C \beta h\|\beta\|_{1, \infty, \Omega}.
\]

Proof.

\[
\|\tau_2 - \tau_2\|_{0, \infty, \partial T_h} = \sum_{K \in T_h} \|\tau_2 - \tau_2\|_{0, \infty, \partial K}
\]

\[
= \sum_{K \in T_h} \|\tau_1 - \beta \cdot n - \tau_1 + P_0 \beta \cdot n\|_{0, \infty, \partial K}
\]

\[
= \sum_{K \in T_h} \|\beta \cdot n - P_0 \beta \cdot n\|_{0, \infty, K}
\]

\[
\leq \|\beta - P_0 \beta\|_{0, \infty, \Omega}
\]

\[
\leq C h\|\beta\|_{1, \infty, \Omega}.
\]

We define the following HDG operators \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \).

\[
\mathcal{B}_1(q_h, y_h, \tilde{y}_h^0; r_1, w_1, \mu_1)
\]

\[
= (q_h, r_1)_{\mathcal{T}_h} - (y_h, \nabla \cdot r_1)_{\mathcal{T}_h} + \langle \tilde{y}_h^0, r_1 \cdot n \rangle_{\partial \mathcal{T}_h \setminus \partial K}
\]

\[
- \langle q_h + \beta y_h, \nabla w_1 \rangle_{\mathcal{T}_h} + \langle q_h \cdot n + \tau_1 y_h, w_1 \rangle_{\partial \mathcal{T}_h}
\]

\[
+ \langle (\beta \cdot n - \tau_1) \tilde{y}_h^0, w_1 \rangle_{\partial \mathcal{T}_h \setminus \partial \mathcal{T}_h}
\]

\[
- \langle q_h \cdot n + \beta \cdot n \tilde{y}_h^0 + \tau_1 (y_h - \tilde{y}_h^0), \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \partial K}
\]

\[
\mathcal{B}_2(p_h, z_h, \tilde{z}_h^0; r_2, w_2, \mu_2)
\]

\[
= (p_h, r_2)_{\mathcal{T}_h} - (z_h, \nabla \cdot r_2)_{\mathcal{T}_h} + \langle \tilde{z}_h^0, r_2 \cdot n \rangle_{\partial \mathcal{T}_h \setminus \partial K}
\]

\[
- \langle p_h - \beta z_h, \nabla w_2 \rangle_{\mathcal{T}_h} + \langle p_h \cdot n + \tau_2 z_h, w_2 \rangle_{\partial \mathcal{T}_h}
\]

\[
- \langle (\beta \cdot n + \tau_2) \tilde{z}_h^0, w_2 \rangle_{\partial \mathcal{T}_h \setminus \partial \mathcal{T}_h}
\]

By the definition in (23), we can rewrite the HDG formulation of the optimality system (9) as follows: find \((q_h, p_h, y_h, z_h, u_h, \tilde{y}_h^0, \tilde{z}_h^0) \in V_h \times V_h \times W_h \times W_h \times M_h(o) \times M_h(o)\) such that

\[
\mathcal{B}_1(q_h, y_h, \tilde{y}_h^0; r_1, w_1, \mu_1) = (f + u_h, w_1)_{\mathcal{T}_h}
\]

\[
- \langle g, (\beta \cdot n - \tau_1) w_1 + r_1 \cdot n \rangle_{\mathcal{T}_h \setminus \partial K},\]

(24a)

\[
\mathcal{B}_2(p_h, z_h, \tilde{z}_h^0; r_2, w_2, \mu_2) = (y_d - y_h, w_2)_{\mathcal{T}_h}
\]

(24b)

\[
(z_h - \gamma u_h, w_3)_{\mathcal{T}_h} = 0,
\]

(24c)

for all \((r_1, r_2, w_1, w_2, w_3, \mu_1, \mu_2) \in V_h \times V_h \times W_h \times W_h \times M_h(o) \times M_h(o)\).

Next, we present a basic property of the operators \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \), and show the HDG equations (24) have a unique solution.
Lemma 4. For any \((v_h, w_h, \mu_h) \in V_h \times W_h \times M_h(o)\), we have

\[
\mathcal{B}_1(v_h, w_h, \mu_h; v_h, w_h, \mu_h) = (v_h, v_h)_T + \langle \tau_1 - \frac{1}{2} \beta \cdot n \rangle (w_h - \mu_h), w_h - \mu_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

\[
= \langle (\tau_1 - \frac{1}{2} \beta \cdot n)w_h, w_h \rangle _\epsilon^0
\]

\[
+ \frac{1}{2} \beta \cdot n \rangle (w_h - \mu_h), w_h - \mu_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

\[
\mathcal{B}_2(v_h, w_h, \mu_h; v_h, w_h, \mu_h) = (v_h, v_h)_T + \langle (\tau_2 + \frac{1}{2} \beta \cdot n)w_h, w_h \rangle _\epsilon^0
\]

\[
+ \frac{1}{2} \beta \cdot n \rangle (w_h - \mu_h), w_h - \mu_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

Proof. We only prove the first identity; the second can be obtained by the same argument.

\[
\mathcal{B}_1(v_h, w_h, \mu_h; v_h, w_h, \mu_h)
\]

\[
= (v_h, v_h)_T - (w_h, \nabla \cdot v_h)_T + \langle \mu_h, v_h \cdot n \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

\[
- (v_h + \beta w_h, \nabla w_h)_T + \langle v_h \cdot n + \tau_1 w_h, w_h \rangle \mid_{\partial T_h}^0
\]

\[
+ \langle (\beta \cdot n - \tau_1) \mu_h, w_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

\[
- \langle v_h \cdot n + \beta \cdot n \mu_h + \tau_1 (w_h - \mu_h), \mu_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

\[
= (v_h, v_h)_T - (\beta w_h, \nabla w_h)_T + \langle \tau_1 w_h, w_h \rangle \mid_{\partial T_h}^0
\]

\[
+ \langle (\beta \cdot n - \tau_1) \mu_h, w_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0 - \langle \beta \cdot n \mu_h + \tau_1 (w_h - \mu_h), \mu_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

Moreover,

\[
(\beta w_h, \nabla w_h)_T = (\nabla \cdot (\beta w_h), w_h)_T = (\beta \cdot n w_h, w_h \rangle \mid_{\partial T_h}^0 - (\beta w_h, \nabla w_h)_T,
\]

which implies

\[
(\beta w_h, \nabla w_h)_T = \frac{1}{2} \langle \beta \cdot n w_h, w_h \rangle \mid_{\partial T_h}^0.
\]

Then we obtain

\[
\mathcal{B}_1(v_h, w_h, \mu_h; v_h, w_h, \mu_h)
\]

\[
= (v_h, v_h)_T + \langle (\tau_1 - \frac{1}{2} \beta \cdot n)w_h - \mu_h \rangle, w_h - \mu_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

\[
+ \langle (\tau_1 - \frac{1}{2} \beta \cdot n)w_h, w_h \rangle _\epsilon^0 - \frac{1}{2} \langle \beta \cdot n \mu_h, \mu_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0
\]

Since \(\mu_h\) is single-valued across the interfaces, we have

\[
- \frac{1}{2} \langle \beta \cdot n \mu_h, \mu_h \rangle \mid_{\partial T_h \setminus \epsilon_h}^0 = 0.
\]

This completes the proof. □

Next, we give a property of the HDG operators \(\mathcal{B}_1\) and \(\mathcal{B}_2\) that is critical to our error analysis of the method.
Lemma 5. If (A2) holds, then
\[ B_1(q_h, y_h, \hat{y}_h^0; p_h, -z_h, -\tilde{z}_h^0) + B_2(p_h, z_h, \tilde{z}_h^0; -q_h, y_h, \hat{y}_h^0) = 0. \]

Proof. By the definition of \( B_1 \) and \( B_2 \),
\[
B_1(q_h, y_h, \hat{y}_h^0; p_h, -z_h, -\tilde{z}_h^0) + B_2(p_h, z_h, \tilde{z}_h^0; -q_h, y_h, \hat{y}_h^0)
= (q_h, p_h)_{\mathcal{T}_h} - (y_h, \nabla \cdot p_h)_{\mathcal{T}_h} + (\hat{y}_h^0, p_h \cdot n)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
+ (q_h + \beta y_h, \nabla z_h)_{\mathcal{T}_h} - (q_h \cdot n + \tau_1 y_h, z_h)_{\partial \mathcal{T}_h} - ((\beta \cdot n - \tau_1) \hat{y}_h^0, z_h)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
+ (q_h \cdot n + \beta \cdot n \tilde{z}_h^0 + \tau_1 (y_h - \hat{y}_h^0), \tilde{z}_h^0)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
- (p_h, q_h)_{\mathcal{T}_h} + (z_h, \nabla \cdot q_h)_{\mathcal{T}_h} - (\tilde{z}_h^0, q_h \cdot n)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
- (p_h - \beta z_h, \nabla y_h)_{\mathcal{T}_h} + (p_h \cdot n + \tau_2 z_h, y_h)_{\partial \mathcal{T}_h} - ((\beta \cdot n + \tau_2) \tilde{z}_h^0, y_h)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
- (p_h \cdot n - \beta \cdot n \tilde{z}_h^0 + \tau_2 (z_h - \tilde{z}_h^0), \tilde{y}_h^0)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}.
\]
Integration by parts gives
\[
B_1(q_h, y_h, \hat{y}_h^0; p_h, -z_h, -\tilde{z}_h^0) + B_2(p_h, z_h, \tilde{z}_h^0; -q_h, y_h, \hat{y}_h^0)
= ((\tau_2 + \beta \cdot n - \tau_1) y_h, z_h)_{\partial \mathcal{T}_h} + ((\tau_2 + \beta \cdot n - \tau_1) \hat{y}_h^0, \tilde{z}_h^0)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}.
\]
The proof is complete by assumption (A2).

Proposition 1. There exists a unique solution of the HDG equations (24).

Proof. Since the system (24) is finite dimensional, we only need to prove the uniqueness. Therefore, we assume \( y_d = f = 0 \) and we show the system (24) only has the trivial solution.

Take \((r_1, w_1, \mu_1) = (p_h, -z_h, -\tilde{z}_h^0)\), \((r_2, w_2, \mu_2) = (-q_h, y_h, \hat{y}_h^0)\), and \(w_3 = z_h - \gamma u_h\) in the HDG equations (24a), (24b), and (24c), respectively, and sum to obtain
\[
B_1(q_h, y_h, \hat{y}_h^0; p_h, -z_h, -\tilde{z}_h^0) + B_2(p_h, z_h, \tilde{z}_h^0; -q_h, y_h, \hat{y}_h^0)
= \gamma (y_h, y_h)_{\mathcal{T}_h} + (z_h, z_h)_{\mathcal{T}_h}
\]
Since \(\gamma > 0\), Lemma 5 implies \(y_h = u_h = z_h = 0\).

Next, take \((r_1, w_1, \mu_1) = (q_h, y_h, \hat{y}_h^0)\) and \((r_2, w_2, \mu_2) = (p_h, z_h, \tilde{z}_h^0)\) in the HDG equations (24a)-(24b). Lemma 4 and (A2) and (A3) give \(q_h = p_h = 0\) and \(\hat{y}_h^0 = \tilde{z}_h^0 = 0\).

3.3 Proof of Main Result

To prove the main result, we follow the strategy of our earlier work [18] and split the proof into five steps. We consider the following auxiliary problem: find
\[
(q_h(u), p_h(u), y_h(u), z_h(u), \hat{y}_h^0(u), \tilde{z}_h^0(u)) \in V_h \times V_h \times W_h \times M_h(o) \times M_h(o)
\]
such that
\[
B_1(q_h(u), y_h(u), \hat{y}_h(u); r_1, w_1, \mu_1) = (f + u, w_1)_{\mathcal{T}_h} - \langle g, (\beta \cdot n - \tau_1) w_1 + r_1 \cdot n \rangle_{\mathcal{E}_h^0}, \quad (25a)
B_2(p_h(u), z_h(u), \tilde{z}_h(u); r_2, w_2, \mu_2) = (y_d - y_h(u), w_2)_{\mathcal{T}_h}, \quad (25b)
\]
for all \((r_1, r_2, w_1, w_2, \mu_1, \mu_2) \in V_h \times V_h \times W_h \times M_h(o) \times M_h(o)\). We begin by bounding the error between the solutions of the auxiliary problem and the mixed form (8a)-(8d) of the optimality system.
3.3.1 Step 1: The error estimates for \( \|q - q_h(u)\|_{T_h} \) and \( \|y - y_h(u)\|_{T_h} \).

The auxiliary HDG equation (25a) is precisely the standard HDG discretization of the convection diffusion PDE (4a)-(4b) for \( y \) since the exact optimal control \( u \) is fixed in (25a). The HDG error estimates for this problem have already been obtained in [7].

**Lemma 6** ([7]). If conditions (A1) and (A2) hold, we have

\[
\|y - y_h(u)\|_{T_h} + \|q - q_h(u)\|_{T_h} \leq C T^{k+1} (|q|_{k+1} + |y|_{k+1}).
\]

(26)

3.3.2 Step 2: The error equation for part 2 of the auxiliary problem (25b).

Next, we bound the error between the solution of the dual convection diffusion equation (4c)-(4d) for \( z \) and the auxiliary HDG equation (25b). We split the errors in the variables using the HDG projections. Define

\[
\begin{align*}
\delta & = p - \bar{\Pi}_V p, \\
\delta^z & = z - \bar{\Pi}_W z, \\
\delta^\tau & = z - P_M z, \\
\tilde{\delta} & = \delta^z \cdot n + \tau \delta^\tau, \\
\delta_h & = \delta^p + \delta^z, \\
\delta_h^z & = \delta^z, \\
\delta_h^\tau & = \delta^\tau, \\
\delta_h \tau & = \delta^\tau - \tilde{\delta}_h(u),
\end{align*}
\]

(27)

where \( \tilde{\delta}_h(u) = \delta_h(u) \) on \( \delta_h \) and \( \tilde{\delta}_h(u) = 0 \) on \( \delta_h^0 \). This gives \( \delta_h^z = 0 \) on \( \delta_h^0 \).

**Lemma 7.** We have

\[
\mathcal{B}_2(\delta_h^p, \delta_h^z, \delta_h^\tau; r_2, w_2, \mu_2)
\]

\[
= (\delta^p, r_2)_{T_h} + (y_h(u) - y, w_2)_{T_h} + ((\tau_2 - \tilde{\tau}_2) \delta^\tau, w_2 - \mu_2)_{\partial T_h}.
\]

(28)

**Proof.** By the definition of operator \( \mathcal{B}_2 \) (23), we have

\[
\mathcal{B}_2(\bar{\Pi}_V p, \bar{\Pi}_W z, P_M z; r_2, w_2, \mu_2)
\]

\[
= (\bar{\Pi}_V p, r_2)_{T_h} - (\bar{\Pi}_W z, \nabla \cdot r_2)_{T_h} + (P_M z, r_2 \cdot n)_{\partial T_h} \chi_h^0
\]

\[
- (\bar{\Pi}_V p - \beta \bar{\Pi}_W z, \nabla w_2)_{T_h} + (\bar{\Pi}_V p \cdot n + \tau_2 \bar{\Pi}_W z, w_2)_{\partial T_h}
\]

\[
- (\beta \cdot n + \tau_2) P_M z \cdot w_2)_{\partial T_h} \chi_h^0
\]

\[
- (\bar{\Pi}_V p \cdot n - \beta \cdot n P_M z + \tau_2 (\bar{\Pi}_W z - P_M z), \mu_2)_{\partial T_h} \chi_h^0.
\]

By properties of the HDG projections \( \bar{\Pi}_V \) and \( \bar{\Pi}_W \) in (21c) and the \( L^2 \) projection \( P_M \) in (19), we have

\[
(\bar{\Pi}_V p \cdot n + \tau_2 \bar{\Pi}_W z, w_2)_{\partial T_h} = (p \cdot n + \beta \cdot n P_M z - \beta \cdot n z + \tau_2 z, w_2)_{\partial T_h},
\]

\[
(\bar{\Pi}_V p \cdot n - \beta \cdot n P_M z + \tau_2 \bar{\Pi}_W z, \mu)_{\partial T_h} \chi_h^0 = (p \cdot n - \beta \cdot n z + \tau_2 z, \mu)_{\partial T_h} \chi_h^0.
\]

By (21a)-(21b), we have

\[
\mathcal{B}_2(\bar{\Pi}_V p, \bar{\Pi}_W z, P_M z; r_2, w_2, \mu_2)
\]

\[
= (p, r_2)_{T_h} - (\delta^p, r_2)_{T_h} - (z, \nabla \cdot r_2)_{T_h} + (z, r_2 \cdot n)_{\partial T_h} \chi_h^0
\]

\[
- (p - \beta z, \nabla w_2)_{T_h} + (p \cdot n - \beta \cdot n z, w_2)_{\partial T_h} + (\beta \cdot n P_M z + \tau_2 z, w_2)_{\partial T_h}
\]

\[
- ((\beta \cdot n + \tau_2) P_M z \cdot w_2)_{\partial T_h} \chi_h^0
\]

\[
- (\tau_2 z - \beta P_M z, \mu_2)_{\partial T_h} \chi_h^0.
\]
Note that the exact solution \( p \) and \( z \) satisfies
\[
(p, r_2)_{\Omega_h} - (z, \nabla \cdot r_2)_{\Omega_h} + \langle z, r_2 \cdot n \rangle_{\partial \Omega_h} = 0,
\]
\[-(p - \beta z, \nabla w_2)_{\Omega_h} + \langle p \cdot n - \beta \cdot nz, w_2 \rangle_{\partial \Omega_h} = (y_d - y, w_2)_{\Omega_h},
\]
\[
\langle p \cdot n - \beta \cdot nz, \mu_2 \rangle_{\partial \Omega_h} = 0,
\]
for all \((r_2, w_2, \mu_2) \in V_h \times W_h \times M_h (o)\). Since \( z = 0 \) on \( \varepsilon_h^0 \), we have
\[
\mathcal{B}_2(\hat{\Pi}_V p, \hat{\Pi}_W z, P_M z; r_2, w_2, \mu_2)
\]
\[
= -(\delta^p, r_2)_{\Omega_h} + (y_d - y, w_2)_{\Omega_h} + \langle \tau_2 \delta^z, w_2 - \mu_2 \rangle_{\partial \Omega_h}.
\]
By the definition of \( P_M \) in (19) and since \( \tau_2 \) from Lemma 7 is piecewise constant on \( \partial \Omega_h \), we have
\[
\langle \tau_2 \delta^z, w_2 - \mu_2 \rangle_{\partial \Omega_h} = \langle (\tau_2 - \tau_2) \delta^z, w_2 - \mu_2 \rangle_{\partial \Omega_h}.
\]
This gives
\[
\mathcal{B}_2(\hat{\Pi}_V p, \hat{\Pi}_W z, P_M z; r_2, w_2, \mu_2)
\]
\[
= -(\delta^p, r_2)_{\Omega_h} + (y_d - y, w_2)_{\Omega_h} + \langle (\tau_2 - \tau_2) \delta^z, w_2 - \mu_2 \rangle_{\partial \Omega_h}.
\]
Subtract part 2 of the auxiliary problem (25b) from the above equality to obtain the result. \( \square \)

3.3.3 Step 3: Estimates for \( \varepsilon_h^p \) and \( \varepsilon_h^z \) by an energy and duality argument.

**Lemma 8.** We have
\[
\| \varepsilon_h^p \|_{\Omega_h} + \| \varepsilon_h^z - \varepsilon_h^z \|_{\partial \Omega_h} \leq E + \kappa \| \varepsilon_h \|_{\Omega_h},
\]
where
\[
E = C \| \delta^p \|_{\Omega_h} + \frac{C}{\kappa} \| y_h(u) - y \|_{\Omega_h} + C \| \tau_2 - \tau_2 \|_{0, \infty, \partial \Omega_h} \| \delta^z \|_{\partial \Omega_h}
\]
and \( \kappa \) is any positive constant and \( C \) does not depend on \( \kappa \).

**Proof.** Taking \((r_2, w_2, \mu_2) = (\varepsilon_h^p, \varepsilon_h^z, \varepsilon_h^z)\) in (28) in Lemma 7 gives
\[
\mathcal{B}_2(\varepsilon_h^p, \varepsilon_h^z, \varepsilon_h^z; \varepsilon_h^p, \varepsilon_h^z, \varepsilon_h^z)
\]
\[
= (\delta^p, \varepsilon_h^p)_{\Omega_h} + (y_h(u) - y, \varepsilon_h^z)_{\Omega_h} + (\tau_2 - \tau_2) \delta^z, \varepsilon_h^z - \varepsilon_h^z)_{\partial \Omega_h}
\]
\[
\leq \| \delta^p \|_{\Omega_h} \| \varepsilon_h^p \|_{\Omega_h} + \| y_h(u) - y \|_{\Omega_h} \| \varepsilon_h \|_{\Omega_h}
\]
\[
+ \| \tau_2 - \tau_2 \|_{0, \infty, \partial \Omega_h} \| \delta^z \|_{\partial \Omega_h} \| \varepsilon_h^z - \varepsilon_h^z \|_{\partial \Omega_h}.
\]

**Lemma 4** gives
\[
\| \varepsilon_h^p \|_{\Omega_h} + \| \varepsilon_h^z - \varepsilon_h^z \|_{\partial \Omega_h}
\]
\[
\leq C \| \delta^p \|_{\Omega_h} + \frac{C}{\kappa} \| y_h(u) - y \|_{\Omega_h} + C \| \tau_2 - \tau_2 \|_{0, \infty, \partial \Omega_h} \| \delta^z \|_{\partial \Omega_h} + \kappa \| \varepsilon_h \|_{\Omega_h},
\]
where \( \kappa \) is any positive constant. \( \square \)
Next, we introduce the dual problem for any given $\Theta$ in $L^2(\Omega)$:
\[
\begin{align*}
\Phi - \nabla \Psi &= 0 & \text{in } \Omega, \\
-\nabla \cdot \Phi + \beta \cdot \nabla \Psi &= \Theta & \text{in } \Omega, \\
\Psi &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

Since the domain $\Omega$ is convex, we have the following regularity estimate
\[
\|\Phi\|_{1,\Omega} + \|\Psi\|_{2,\Omega} \leq C_{\text{reg}} \|\Theta\|_{\Omega},
\]

Before we estimate $\varepsilon^p_h$ and $\varepsilon^z_h$, we introduce the following notation, which is similar to the earlier notation in (27):
\[
\delta^\Phi = \Phi - \tilde{\Pi}_V \Phi, \quad \delta^\Psi = \Psi - \tilde{\Pi}_W \Psi, \quad \delta^\tilde{\Psi} = \Psi - P_M \Psi.
\]

**Lemma 9.** We have
\[
\begin{align*}
\|\varepsilon^p_h\|_{T_h} &\leq h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \\
\|\varepsilon^z_h\|_{T_h} &\leq h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}).
\end{align*}
\]

**Proof.** Consider the dual problem (30) and let $\Theta = \varepsilon^z_h$. Take $(r_2, w_2, \mu_2) = (\tilde{\Pi}_V \Phi, \tilde{\Pi}_W \Psi, P_M \Psi)$ in (28) in Lemma 7. Since $\Psi = 0$ on $\varepsilon^p_h$, we have
\[
\mathcal{B}_2(\varepsilon^p_h, \varepsilon^z_h, \varepsilon^z_h; \tilde{\Pi}_V \Phi, \tilde{\Pi}_W \Psi, P_M \Psi)
\]
\[
= (\varepsilon^p_h, \tilde{\Pi}_V \Phi)_{T_h} - (\varepsilon^z_h, \nabla \cdot \tilde{\Pi}_V \Phi)_{T_h} + (\varepsilon^z_h, \tilde{\Pi}_V \Phi \cdot n)_{\partial T_h}
\]
\[
- (\varepsilon^p_h - \beta \varepsilon^z_h, \nabla \tilde{\Pi}_W \Psi)_{T_h} + (\varepsilon^p_h \cdot n - \beta \cdot n \varepsilon^z_h + \tau_2(\varepsilon^z_h - \varepsilon^z_h), \tilde{\Pi}_W \Psi - P_M \Psi)_{\partial T_h}
\]
\[
= (\varepsilon^p_h, \Phi)_{T_h} - (\varepsilon^p_h, \delta^\Phi)_{T_h} - (\varepsilon^z_h, \nabla \cdot \Phi)_{T_h} + (\varepsilon^z_h, \nabla \cdot \delta^\Phi)_{T_h}
\]
\[
- (\varepsilon^z_h, \delta^\Phi \cdot n)_{\partial T_h} + (\varepsilon^p_h - \beta \varepsilon^z_h, \nabla \Psi)_{T_h} + (\varepsilon^p_h - \beta \varepsilon^z_h, \nabla \tilde{\Psi})_{T_h}
\]
\[
- (\varepsilon^p_h - \beta \varepsilon^z_h, \nabla \tilde{\Psi})_{T_h} + (\varepsilon^z_h, \delta^\Phi \cdot n)_{\partial T_h}
\]
\[
= - (\varepsilon^p_h, \delta^\Phi)_{T_h} + \|\varepsilon^z_h\|_{T_h}^2 + (\varepsilon^z_h, \nabla \cdot \delta^\Phi)_{T_h} - (\varepsilon^z_h, \delta^\Phi \cdot n)_{\partial T_h}
\]
\[
+ (\varepsilon^p_h - \beta \varepsilon^z_h, \nabla \tilde{\Psi})_{T_h} - (\varepsilon^p_h - \beta \varepsilon^z_h, \nabla \tilde{\Psi})_{T_h} + \tau_2(\varepsilon^z_h - \varepsilon^z_h, \delta^\Phi - \tilde{\Psi})_{T_h}.
\]

Here, we used $\langle \varepsilon^z_h, \Phi \cdot n \rangle_{\partial T_h} = 0$, which holds since $\varepsilon^z_h$ is single-valued function on interior edges and $\varepsilon^z_h = 0$ on $\varepsilon^p_h$.

Next, integration by parts gives
\[
(\varepsilon^z_h, \nabla \cdot \delta^\Phi)_{T_h} = (\varepsilon^z_h, \delta^\Phi \cdot n)_{\partial T_h} - (\nabla \varepsilon^z_h, \delta^\Phi)_{T_h} = (\varepsilon^z_h, \delta^\Phi \cdot n)_{\partial T_h} - (\nabla \varepsilon^z_h, \beta \delta^\Psi)_{T_h},
\]
\[
(\varepsilon^p_h, \nabla \tilde{\Psi})_{T_h} = (\varepsilon^p_h \cdot n, \delta^\Psi)_{\partial T_h} - (\nabla \cdot \varepsilon^p_h, \delta^\Psi)_{T_h} = (\varepsilon^p_h \cdot n, \delta^\Psi)_{\partial T_h},
\]
\[
(\beta \varepsilon^z_h, \delta^\Psi)_{T_h} = (\beta \cdot n \varepsilon^z_h, \delta^\Psi)_{\partial T_h} - (\beta \nabla \varepsilon^z_h, \delta^\Psi)_{T_h}.
\]

We have
\[
\mathcal{B}_2(\varepsilon^p_h, \varepsilon^z_h, \varepsilon^z_h; \tilde{\Pi}_V \Phi, \tilde{\Pi}_W \Psi, P_M \Psi)
\]
\[
= - (\varepsilon^p_h, \delta^\Phi)_{T_h} + \|\varepsilon^z_h\|_{T_h}^2 + (\varepsilon^z_h, \delta^\Phi \cdot n)_{\partial T_h} - (\varepsilon^z_h, \delta^\Phi \cdot n)_{\partial T_h} - (\varepsilon^z_h, \beta \delta^\Psi)_{T_h}
\]
\[
- (\beta \cdot n \varepsilon^z_h, \delta^\Psi)_{\partial T_h} - (\beta \cdot n \varepsilon^z_h, \delta^\Psi - \tilde{\Psi})_{\partial T_h}.
\]
Remembering that $\varepsilon_h^z$ is single-valued function on interior edges and $\varepsilon_h^0 = 0$ on $\partial \Omega$ gives

$$\langle \beta \cdot n, \varepsilon_h^z \rangle_{\partial \Omega} = 0 = \langle \beta \cdot n, \varepsilon_h^z \rangle_{\partial \Omega}.$$ 

This implies

$$\mathcal{B}_2(e_h^p, e_h^z, e_h^0, \Pi_V \Psi, \Pi_W \Psi, P_M \Psi)$$

$$= -(e_h^p, \delta \Phi)_{\partial \Omega} + \|e_h^z\|_{\partial \Omega}^2 + \langle e_h^z, \delta \Phi \cdot n \rangle_{\partial \Omega}$$

$$- \langle \beta \cdot n (e_h^z - e_h^0), \delta \Psi \rangle_{\partial \Omega} - \tau_2 (e_h^z - e_h^0, \delta \Psi - \delta \hat{\Psi})_{\partial \Omega}$$

$$= -(e_h^p, \delta \Phi)_{\partial \Omega} + \|e_h^z\|_{\partial \Omega}^2 + \langle e_h^z, \delta \Phi \cdot n - \beta \cdot n \delta \Psi - \tau_2 (\delta \Psi - \delta \hat{\Psi}) \rangle_{\partial \Omega}.$$ 

On the other hand,

$$\mathcal{B}_2(e_h^p, e_h^z, e_h^0, \Pi_V \Phi, \Pi_W \Psi, P_M \Psi)$$

$$= (\delta \Phi, \Pi_V \Phi)_{\partial \Omega} + \langle y_h(u) - y, \Pi_W \Psi \rangle_{\partial \Omega} + \langle (\tau_2 - \bar{\tau}_2) \delta \hat{\Psi}, \Pi_W \Psi - P_M \Psi \rangle_{\partial \Omega}.$$ 

Comparing the above two equalities gives

$$\|e_h^z\|_{\partial \Omega}^2 = (e_h^p, \delta \Phi)_{\partial \Omega} + \|e_h^z\|_{\partial \Omega}^2 + \langle e_h^z, \delta \Phi \cdot n - \beta \cdot n \delta \Psi - \tau_2 (\delta \Psi - \delta \hat{\Psi}) \rangle_{\partial \Omega}$$

$$= \sum_{i=1}^7 R_i.$$ 

Let $C_0 = \max\{C, 1\}$, where $C$ is the constant defined in Lemma 1. For the terms $R_1$ and $R_2$, Lemma 2 gives

$$R_1 = -(e_h^p, \delta \Phi)_{\partial \Omega} \leq \|e_h^p\|_{\partial \Omega} \|\delta \Phi\|_{\partial \Omega} \leq (E + \kappa \|e_h^z\|_{\partial \Omega}) C_0 (\|\Phi\|_1 + \|\Psi\|_1)$$

$$\leq C_0 C_{\text{reg}} (E + \kappa \|e_h^z\|_{\partial \Omega}) \|e_h^z\|_{\partial \Omega},$$

$$R_2 = -(e_h^p - e_h^z, \delta \Phi \cdot n - \beta \cdot n \delta \Psi - \tau_2 (\delta \Psi - \delta \hat{\Psi}) \rangle_{\partial \Omega}$$

$$\leq \|e_h^p - e_h^z\|_{\partial \Omega} (\|\delta \Phi\|_{\partial \Omega} + \|\tau_1\|_{0, \infty, \partial \Omega} \|\delta \Psi\|_{\partial \Omega} + \|\tau_2\|_{0, \infty, \partial \Omega} ^2 C_0 (\|\Phi\|_1 + \|\Psi\|_1)$$

$$\leq 3 (E + \kappa \|e_h^z\|_{\partial \Omega}) (1 + \|\tau_1\|_{0, \infty, \partial \Omega} + \|\tau_2\|_{0, \infty, \partial \Omega} ) C_0 (\|\Phi\|_1 + \|\Psi\|_1)$$

$$\leq 3 C_0 C_{\text{reg}} (E + \kappa \|e_h^z\|_{\partial \Omega}) (1 + \|\tau_1\|_{0, \infty, \partial \Omega} + \|\tau_2\|_{0, \infty, \partial \Omega} ) \|e_h^z\|_{\partial \Omega}.$$ 

For the terms $R_3$, $R_4$ and $R_5$, we use the triangle inequality, the regularity estimate (31), and the
assumption $h \leq 1$ to give

$$R_3 = (\delta^p, \Pi_V \Phi)_{\tau_h} \leq \|\delta^p\|_{\tau_h} (\|\Pi_V \Phi - \Phi\|_{\tau_h} + \|\Phi\|_{\tau_h})$$
$$\leq C_0 \|\delta^p\|_{\tau_h} (\|\Phi\|_{1,\Omega} + \|\Psi\|_{1,\Omega} + \|\Phi\|_{\tau_h})$$
$$\leq 2C_0C_{\text{reg}} \|\delta^p\|_{\tau_h} \varepsilon_{\text{reg}}^{\tau_h},$$
$$R_4 = (y - y_h(u), \Pi_W \Psi)_{\tau_h} \leq \|y - y_h(u)\|_{\tau_h} \|\Pi_W \Psi\|_{\tau_h}$$
$$\leq C_0 \|y - y_h(u)\|_{\tau_h} (\|\Pi_W \Psi - \Psi\|_{\tau_h} + \|\Psi\|_{\tau_h})$$
$$\leq C_0 \|y - y_h(u)\|_{\tau_h} (\|\Psi\|_{1,\Omega} + \|\Phi\|_{1,\Omega} + \|\Psi\|_{\tau_h})$$
$$\leq 2C_0C_{\text{reg}} \|y - y_h(u)\|_{\tau_h} \varepsilon_{\text{reg}}^{\tau_h},$$
$$R_5 = \langle (\tau_2 - \tilde{\tau}_2) \delta^z, \Pi_W \Psi - P_M \Psi \rangle_{\partial \tau_h}$$
$$\leq \|\tau_2 - \tilde{\tau}_2\|_{0,\infty, \partial \tau_h} \|\delta^z\|_{\partial \tau_h} \|\delta^\Psi - \delta^\tilde{\Psi}\|_{\partial \tau_h}$$
$$\leq C \beta \|\beta\|_{1,\infty, \Omega} h^{1/2} \|\delta^z\|_{\partial \tau_h} C_0 (\|\Psi\|_{1,\Omega} + \|\Phi\|_{1,\Omega} + \|\Psi\|_{\tau_h})$$
$$\leq 2C_0C_{\text{reg}} \|\beta\|_{1,\infty, \Omega} h^{1/2} \|\delta^z\|_{\partial \tau_h} \varepsilon_{\text{reg}}^{\tau_h}.$$

Summing $R_1$ to $R_5$ gives

$$\|\varepsilon_h\|_{\tau_h} \leq C(\varepsilon + \kappa \varepsilon_{\text{reg}}^{\tau_h}) + C(\|\delta^p\|_{\tau_h} + \|y - y_h(u)\|_{\tau_h} + h^{1/2} \|\delta^z\|_{\partial \tau_h}),$$

where

$$C = 4C_0C_{\text{reg}} (1 + \|\tau_1\|_{0,\infty, \partial \tau_h} + \|\tau_2\|_{0,\infty, \partial \tau_h}).$$

Choose $\kappa = \frac{1}{2C}$ gives

$$\|\varepsilon_h\|_{\tau_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}).$$

Finally, (29) and (33b) imply (33a). \hfill $\square$

As a consequence, a simple application of the triangle inequality gives optimal convergence rates for $\|p - p_h(u)\|_{\tau_h}$ and $\|z - z_h(u)\|_{\tau_h}$:

**Lemma 10.**

$$\|p - p_h(u)\|_{\tau_h} \leq \|\delta^p\|_{\tau_h} + \varepsilon_{\text{reg}}^{\tau_h}$$
$$\lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}),$$

(34a)

$$\|z - z_h(u)\|_{\tau_h} \leq \|\delta^z\|_{\tau_h} + \varepsilon_{\text{reg}}^{\tau_h}$$
$$\lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}).$$

(34b)

### 3.3.4 Step 4: Estimate for $\|u - u_h\|_{\tau_h}$, $\|y - y_h\|_{\tau_h}$ and $\|z - z_h\|_{\tau_h}$.

Next, we bound the error between the solutions of the auxiliary problem and the HDG discretization of the optimality system (24). We use these error bounds and the error bounds in Lemmas 6 and 10 to obtain the main result.

For the remaining steps, we denote

$$\zeta_g = q_h(u) - q_h, \quad \zeta_y = y_h(u) - y_h, \quad \zeta_g = y_h(u) - \tilde{y}_h,$$

$$\zeta_p = p_h(u) - p_h, \quad \zeta_z = z_h(u) - z_h, \quad \zeta_z = \tilde{z}_h(u) - \tilde{z}_h.$$
Subtracting the auxiliary problem and the HDG problem gives the following error equations

\[ \mathcal{B}_1(\zeta_q, \zeta_y, \zeta_y^i; r_1, w_1, \mu_1) = (u - u_h, w_1)_{T_h} \]  
\[ \mathcal{B}_2(\zeta_p, \zeta_z, \zeta_z^i; r_2, w_2, \mu_2) = -(\zeta_y, w_2)_{T_h}. \]  

(35a) (35b)

**Lemma 11.** We have

\[ \gamma \| u - u_h \|^2_{T_h} + \| y_h(u) - y_h \|^2_{T_h} = (z_h - \gamma u_h, u - u_h)_{T_h} - (z_h(u) - \gamma u, u - u_h)_{T_h}. \]

(36)

**Proof.** First, we have

\[ (z_h - \gamma u_h, u - u_h)_{T_h} - (z_h(u) - \gamma u, u - u_h)_{T_h} = -(\zeta_z, u - u_h)_{T_h} + \gamma \| u - u_h \|^2_{T_h}. \]

Next, Lemma 5 gives

\[ \mathcal{B}_1(\zeta_q, \zeta_y, \zeta_y^i; \zeta_p, -\zeta_z, -\zeta_z^i) + \mathcal{B}_2(\zeta_p, \zeta_z, \zeta_z^i; -\zeta_q, \zeta_y, \zeta_y^i) = 0. \]

On the other hand, using the definition of \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) gives

\[ \mathcal{B}_1(\zeta_q, \zeta_y, \zeta_y^i; \zeta_p, -\zeta_z, -\zeta_z^i) + \mathcal{B}_2(\zeta_p, \zeta_z, \zeta_z^i; -\zeta_q, \zeta_y, \zeta_y^i) = -(u - u_h, \zeta_z)_{T_h} - \| \zeta_y \|^2_{T_h}. \]

Comparing the above two equalities gives

\[ -(u - u_h, \zeta_z)_{T_h} = \| \zeta_y \|^2_{T_h}. \]

This completes the proof. \( \square \)

**Theorem 2.** We have

\[ \| u - u_h \|_{T_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \]  
\[ \| y - y_h \|_{T_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \]  
\[ \| z - z_h \|_{T_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}). \]  

(37a) (37b) (37c)

**Proof.** Recall the continuous and discretized optimality conditions (4e) and (24c) gives \( \gamma u = z \) and \( \gamma u_h = z_h \). These equations and the previous lemma give

\[ \gamma \| u - u_h \|^2_{T_h} + \| \zeta_y \|^2_{T_h} = (z_h - \gamma u_h, u - u_h)_{T_h} - (z_h(u) - \gamma u, u - u_h)_{T_h} = -(z_h(u) - z, u - u_h)_{T_h} \leq \| z_h(u) - z \|_{T_h} \| u - u_h \|_{T_h} \leq \frac{1}{2\gamma} \| z_h(u) - z \|^2_{T_h} + \frac{\gamma}{2} \| u - u_h \|^2_{T_h}. \]

By Lemma 10 we have

\[ \| u - u_h \|_{T_h} + \| \zeta_y \|_{T_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}). \]

(38)

Then, by the triangle inequality and Lemma 3 we obtain

\[ \| y - y_h \|_{T_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}). \]

Finally, since \( z = \gamma u \) and \( z_h = \gamma u_h \) we have

\[ \| z - z_h \|_{T_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}). \]

\( \square \)
3.3.5 Step 5: Estimate for $\|q - q_h\|_{\mathcal{T}_h}$ and $\|p - p_h\|_{\mathcal{T}_h}$.

Lemma 12. We have

\[
\|\xi_q\|_{\mathcal{T}_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \quad (39a)
\]
\[
\|\xi_p\|_{\mathcal{T}_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}). \quad (39b)
\]

Proof. By Lemma 4, the error equation (35a), and the estimate (38) we have

\[\|\xi_q\|^2_{\mathcal{T}_h} \lesssim \mathcal{B}_1(\xi_q, \xi_y, \xi_q, \xi_y, \xi_q) = (u - u_h, \xi_y)_{\mathcal{T}_h} \]
\[\lesssim \|u - u_h\|_{\mathcal{T}_h} \|\xi_y\|_{\mathcal{T}_h} \lesssim h^{2k+2}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1})^2. \]

Similarly, by Lemma 4 the error equation (35b), Lemma 10 and Theorem 2 we have

\[\|\xi_p\|^2_{\mathcal{T}_h} \lesssim \mathcal{B}_2(\xi_p, \xi_z, \xi_z, \xi_p, \xi_z, \xi_z) = -(\xi_y, \xi_z)_{\mathcal{T}_h} \]
\[\lesssim \|\xi_y\|_{\mathcal{T}_h} \|\xi_z\|_{\mathcal{T}_h} \lesssim h^{2k+2}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1})^2. \]

The above lemma along with the triangle inequality, Lemma 6 and Lemma 10 complete the proof of the main result:

Theorem 3. We have

\[
\|q - q_h\|_{\mathcal{T}_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}), \quad (40a)
\]
\[
\|p - p_h\|_{\mathcal{T}_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+1} + |p|_{k+1} + |z|_{k+1}). \quad (40b)
\]

4 Numerical Experiments

In this section, we present three numerical examples to confirm our theoretical results. We consider two 2D problems on a square domain $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, and a 3D problem on a cubic domain $\Omega = [0, 1] \times [0, 1] \times [0, 1] \subset \mathbb{R}^3$. For the three examples, we take $\gamma = 1$ and specify the exact state, dual state, and function $\beta$. The data $f$, $g$, and $y_d$ is generated from the optimality system (4).

Also, we chose $\tau_1 = 1$ and set $\tau_2$ using (A2). For all three examples, conditions (A1)-(A3) are satisfied.

Numerical results for $k = 0$ and $k = 1$ for the three examples are shown in Table 1. The observed convergence rates exactly match the theoretical results.

Example 1. We take $\beta = [1, 1]$, state $y(x_1, x_2) = \sin(\pi x_1)$, and dual state $z(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$.

Example 2. We take $\beta = [x_2, x_1]$, state $y(x_1, x_2) = \sin(\pi x_1)$, and dual state $z(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$.

Example 3. We take $\beta = [1, 1, 1]$, state $y(x_1, x_2, x_3) = \sin(\pi x_1)$, and dual state $z(x_1, x_2, x_3) = \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3)$. 

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
k/\sqrt{2} & 1/8 & 1/16 & 1/32 & 1/64 & 1/128 \\
\hline
\|q - q_h\|_{0,\Omega} & 1.7818e-01 & 8.6412e-02 & 4.2357e-02 & 2.0048e-02 & 1.0415e-02 \\
order & - & 1.04 & 1.03 & 1.02 & 1.00 \\
\|p - p_h\|_{0,\Omega} & 4.2057e-01 & 2.1839e-01 & 1.1116e-01 & 5.6062e-02 & 2.8151e-02 \\
order & - & 0.94 & 0.97 & 0.99 & 1.00 \\
\|y - y_h\|_{0,\Omega} & 1.6300e-01 & 8.4087e-02 & 4.2612e-02 & 2.1437e-02 & 1.0750e-02 \\
order & - & 0.95 & 0.98 & 0.99 & 1.00 \\
\|z - z_h\|_{0,\Omega} & 2.1310e-01 & 1.0803e-01 & 5.4219e-02 & 2.7138e-02 & 1.3573e-02 \\
order & - & 0.98 & 0.99 & 1.00 & 1.00 \\
\hline
\end{array}
\]

Table 1: Example 1 Errors for the state \(y\), adjoint state \(z\), and the fluxes \(q\) and \(p\) when \(k = 0\).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
k/\sqrt{2} & 1/8 & 1/16 & 1/32 & 1/64 & 1/128 \\
\hline
\|q - q_h\|_{0,\Omega} & 1.3708e-02 & 3.5192e-03 & 8.8851e-04 & 2.2301e-04 & 5.5850e-05 \\
order & - & 2.00 & 2.00 & 2.00 & 2.00 \\
\|p - p_h\|_{0,\Omega} & 3.4995e-02 & 8.9472e-03 & 2.2581e-03 & 5.6694e-04 & 1.4202e-04 \\
order & - & 2.00 & 2.00 & 2.00 & 2.00 \\
\|y - y_h\|_{0,\Omega} & 1.1705e-02 & 2.9528e-03 & 7.4012e-04 & 1.8519e-04 & 4.6315e-05 \\
order & - & 2.00 & 2.00 & 2.00 & 2.00 \\
\|z - z_h\|_{0,\Omega} & 2.3361e-02 & 5.9059e-03 & 1.4810e-03 & 3.7059e-04 & 9.2676e-05 \\
order & - & 2.00 & 2.00 & 2.00 & 2.00 \\
\hline
\end{array}
\]

Table 2: Example 2 Errors for the state \(y\), adjoint state \(z\), and the fluxes \(q\) and \(p\) when \(k = 1\).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
k/\sqrt{2} & 1/8 & 1/16 & 1/32 & 1/64 & 1/128 \\
\hline
\|q - q_h\|_{0,\Omega} & 1.7838e-01 & 8.6412e-02 & 4.2357e-02 & 2.0957e-02 & 1.0419e-02 \\
order & - & 1.04 & 1.03 & 1.02 & 1.00 \\
\|p - p_h\|_{0,\Omega} & 4.2050e-01 & 2.1848e-01 & 1.1123e-01 & 5.6101e-02 & 2.8171e-02 \\
order & - & 0.95 & 0.97 & 0.99 & 0.99 \\
\|y - y_h\|_{0,\Omega} & 1.6285e-01 & 8.4032e-02 & 4.2588e-02 & 2.1426e-02 & 1.0744e-02 \\
order & - & 0.95 & 0.98 & 0.99 & 1.00 \\
\|z - z_h\|_{0,\Omega} & 2.1223e-01 & 1.0773e-01 & 5.4094e-02 & 2.7081e-02 & 1.3546e-02 \\
order & - & 0.98 & 0.99 & 1.00 & 1.00 \\
\hline
\end{array}
\]

Table 3: Example 2 Errors for the state \(y\), adjoint state \(z\), and the fluxes \(q\) and \(p\) when \(k = 0\).
Table 4: Example 2 Errors for the state $y$, adjoint state $z$, and the fluxes $q$ and $p$ when $k = 1$.  

| $h/\sqrt{2}$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
|---------------|-----|------|------|------|-------|
| $\|q - q_h\|_{0,\Omega}$ | 1.3713e-02 | 3.5195e-03 | 8.8853e-04 | 2.2301e-04 | 5.5850e-05 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 |
| $\|p - p_h\|_{0,\Omega}$ | 3.5010e-02 | 8.9481e-03 | 2.2581e-03 | 5.6694e-04 | 1.4202e-04 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 |
| $\|y - y_h\|_{0,\Omega}$ | 1.1712e-02 | 2.9532e-03 | 7.4015e-04 | 1.8520e-04 | 4.6315e-05 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 |
| $\|z - z_h\|_{0,\Omega}$ | 2.3368e-02 | 5.9064e-03 | 1.4810e-03 | 3.7059e-04 | 9.2676e-05 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 |

Table 5: Example 3 Errors for the state $y$, adjoint state $z$, and the fluxes $q$ and $p$ when $k = 0$.  

| $h/\sqrt{2}$ | 1/2 | 1/4 | 1/8 | 1/16 | 1/32 |
|---------------|-----|-----|-----|------|------|
| $\|q - q_h\|_{0,\Omega}$ | 6.3167e-01 | 3.4472e-01 | 1.7715e-01 | 8.9373e-02 | 4.4778e-02 |
| order | - | 0.87 | 0.96 | 0.99 | 1.00 |
| $\|p - p_h\|_{0,\Omega}$ | 4.9907e-01 | 2.9505e-01 | 1.5339e-01 | 7.7393e-02 | 3.8724e-02 |
| order | - | 0.76 | 0.94 | 0.99 | 1.00 |
| $\|y - y_h\|_{0,\Omega}$ | 1.7959e-01 | 1.0026e-01 | 5.3061e-02 | 2.7275e-02 | 1.3646e-02 |
| order | - | 0.84 | 0.92 | 0.96 | 1.00 |
| $\|z - z_h\|_{0,\Omega}$ | 2.3121e-01 | 1.3646e-01 | 7.2318e-02 | 3.7004e-02 | 1.8587e-02 |
| order | - | 0.76 | 0.92 | 0.97 | 1.00 |

Table 6: Example 3 Errors for the state $y$, adjoint state $z$, and the fluxes $q$ and $p$ when $k = 1$.  

| $h/\sqrt{2}$ | 1/2 | 1/4 | 1/8 | 1/16 | 1/32 |
|---------------|-----|-----|-----|------|------|
| $\|q - q_h\|_{0,\Omega}$ | 9.2498e-02 | 2.7594e-02 | 7.4959e-03 | 1.9486e-03 | 4.8720e-04 |
| order | - | 1.75 | 1.90 | 1.94 | 2.00 |
| $\|p - p_h\|_{0,\Omega}$ | 1.8360e-01 | 5.3637e-02 | 1.3921e-02 | 3.5138e-03 | 8.7857e-04 |
| order | - | 1.80 | 1.95 | 1.99 | 2.00 |
| $\|y - y_h\|_{0,\Omega}$ | 4.4822e-02 | 1.1780e-02 | 2.9545e-03 | 7.3644e-04 | 1.8423e-04 |
| order | - | 1.93 | 2.00 | 2.00 | 2.00 |
| $\|z - z_h\|_{0,\Omega}$ | 9.1413e-02 | 2.7594e-02 | 7.3061e-02 | 3.7004e-02 | 1.8587e-02 |
| order | - | 1.73 | 1.92 | 1.97 | 2.00 |
5 Conclusions

We proposed an HDG method to approximate the solution of an optimal distributed control problems for an elliptic convection diffusion equation. We obtained optimal a priori error estimates for the control, state, dual state, and their fluxes. The next step is to study optimal control problems governed by more complicated PDEs governing fluids. It would also be of interest to investigate if postprocessing gives superconvergence for this optimal control problem.

Appendix

Before we investigate the local elimination, we give the following proposition.

**Proposition 2.** The matrices $A_{12}$ and $A_{13}$ in (12) are positive definite.

**Proof.** We only prove $A_{12}$ is positive definite; a similar argument applies to $A_{13}$. The matrix $A_{12}$ is positive definite if and only if $x^T A_{12} x > 0$ for any $x = [x_1, x_2, \ldots, x_{N_2}] \in \mathbb{R}^{N_2}$. For $x = \sum_{j=1}^{N_2} x_j \phi_j$, we have

$$x^T A_{12} x = \langle \tau_1 x, x \rangle_{\partial T_h} - (\beta x, \nabla x)_{T_h}.$$ 

Moreover

$$(\beta x, \nabla x)_{T_h} = \langle \beta \cdot n x, x \rangle_{\partial T_h} - (\beta x, \nabla x)_{T_h},$$

this implies

$$(\beta x, \nabla x)_{T_h} = \frac{1}{2} \langle \beta \cdot n x, x \rangle_{\partial T_h}.$$ 

Then,

$$x^T A_{12} x = \langle (\tau_1 - \frac{1}{2} \beta \cdot n) x, x \rangle_{\partial T_h} > 0,$$

by the assumption concerning $\tau_1$. \hfill \Box

By simple algebraic operations in equation (13), we obtain the following formulas for the matrices $G_1$, $G_2$, $H_1$, and $H_2$ in (14):

$$G_1 = B_1^{-1} B_2 (B_4 + B_2^T B_1^{-1} B_2)^{-1} (B_5 + B_2^T B_1^{-1} B_3) - B_1^{-1} B_3,$$
$$G_2 = -(B_4 + B_2^T B_1^{-1} B_2)^{-1} (B_5 + B_2^T B_1^{-1} B_3),$$
$$H_1 = -B_1^{-1} B_2 (B_4 + B_2^T B_1^{-1} B_2)^{-1},$$
$$H_2 = (B_4 + B_2^T B_1^{-1} B_2)^{-1}.$$ 

We briefly describe how these matrices can be easily computed using the HDG method described in this work.

Since the spaces $V_h$ and $W_h$ consist of discontinuous polynomials, some of the system matrices are block diagonal and each block is small and symmetric positive definite. The matrix $B_1$ is this type, and therefore $B_1^{-1}$ is easily computed and is also a matrix of the same type. Therefore, the matrices $G_1$, $G_2$, $H_1$, and $H_2$ are easily computed if $B_4 + B_2^T B_1^{-1} B_2$ is also easily inverted.
It can be checked that $B_2^T B_1^{-1} B_2$ is block diagonal with small nonnegative definite blocks. Next, $B_4 = \begin{bmatrix} A_{12} & -\gamma^{-1} A_4 \\ A_4 & A_{13} \end{bmatrix}$, where $A_4$ is symmetric positive block diagonal, $A_{12}$ and $A_{13}$ are positive block diagonal. Due to the structure of $B_1$ and $B_2$, the matrix $B_2^T B_1^{-1} B_2 + B_4$ has the form $\begin{bmatrix} C_1 & -\gamma^{-1} A_4 \\ A_4 & C_2 \end{bmatrix}$, where $C_1$ and $C_2$ are symmetric positive block diagonal. The inverse can be easily computed using the formula

$$\begin{bmatrix} C_1 & -\gamma^{-1} A_4 \\ A_4 & C_2 \end{bmatrix}^{-1} = \begin{bmatrix} C_1^{-1} - \gamma^{-1} C_1^{-1} A_4 D^{-1} A_4 C_1^{-1} & \gamma^{-1} C_1^{-1} A_4 D^{-1} \\ -D^{-1} A_4 C_1^{-1} & D^{-1} \end{bmatrix},$$

where $D = C_2 + \gamma^{-1} A_4 C_1^{-1} A_4$. Furthermore, $C_1^{-1}$ and $D^{-1}$ are both symmetric positive block diagonal.

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