COHERENT SHEAVES ON RIBBONS AND THEIR MODULI

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Abstract. A ribbon is a non-reduced curve modelled on the first infinitesimal neighbourhood of a smooth curve in a surface. This paper is devoted to describe some properties of coherent sheaves on such a curve and their Simpson moduli space. In particular we give necessary and sufficient conditions for the existence of semistable quasi locally free sheaves (in the sense of Drézet) of a fixed complete type and we compute the dimension of the Zariski closure in the moduli space of the locus of semistable quasi locally free sheaves of a fixed complete type. We also show when vector bundles on the reduced subcurve deform to sheaves supported on the ribbon. We find a special kind of non quasi locally free sheaves which, as generalized line bundles, are direct images of quasi locally free sheaves on an appropriate blow up of the ribbon. Finally, we give a conjectural description of the irreducible components of the Simpson moduli space, explaining precisely which parts have already been proved and what lacks for the complete result.

Introduction

This paper is devoted to the study of pure coherent sheaves on a ribbon and their moduli spaces. A ribbon $X$ is a non-reduced projective $k$-scheme of dimension 1, where $k$ is an algebraically closed field, such that its reduced subcurve $X_{\text{red}}$ is a smooth $k$-curve and its nilradical $N \subset O_X$ is locally generated by a single non-zero square-zero element. In other words it is a primitive double curve, or a primitive multiple curve of multiplicity 2, in the sense of [D1]. Ribbons have been classified in [BE, §1].

Coherent sheaves on ribbons and their moduli have been studied by Drézet in various articles (the main ones are [D1], [D2], [D3] and [D4]), sometimes only as a special case of sheaves on primitive multiple curves of any multiplicity. Various basic properties which will be recalled with precise references in the first section are due to him.

A special kind of sheaves on ribbons, the so-called generalized line bundles (i.e. pure sheaves which are generically invertible or, in other words, generalized divisors in the sense of Hartshorne), has been introduced by Bayer and Eisenbud in [BE] and has been studied by Eisenbud and Green in [EG] in order to deal with the Green conjecture about the Clifford index of smooth curves. Some of their properties have been treated also in [D1, §8.2]. The moduli space of pure sheaves of generalized rank 2 (see Definition 2010 Mathematics Subject Classification. 14D20; 14H60.

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which are either generalized line bundles or direct images of rank 2 vector bundles on $X_{\text{red}}$, has been studied in some special cases by Donagi, Ein and Lazarsfeld in [DEL] §3 and in full generality by Chen and Kass in [CK], although the latter left an open question about the irreducible components ([CK, Question 4.8]) which has been answered in [Sa2].

Other papers studying coherent sheaves on some kinds of non-reduced schemes comprehending ribbons as particular cases are [I] by Inaba and [Y] by Yang.

We now describe the structure of the article and to enunciate its main results.

Section 1 recalls some of the known properties of coherent sheaves on a ribbon and is principally based on the already cited articles by Drézet.

Section 2 is devoted to the study of various properties of quasi locally free sheaves on $X$ (see Definition 1.11). The main result of this section is Theorem 2.5, giving necessary and sufficient conditions for the existence of a semistable quasi locally free sheaf of a fixed complete type on $X$ (for the definition of the complete type, see the end of Definition 1.2(i)). Its statement is the following:

**Theorem A.** Let $X$ be a ribbon such that $\bar{g} \geq 2$, where $\bar{g}$ is the genus of $X_{\text{red}}$. There exists a semistable quasi locally free sheaf $\mathcal{F}$ on $X$ of complete type $((r_0, r_1), (d_0, d_1))$, with $r_0 > r_1 > 0$, if and only if

$$\frac{d_0 + (r_0 + r_1) \deg(\mathcal{N})}{r_0} \leq \frac{d_1}{r_1} \leq \frac{d_0}{r_0},$$

where $\mathcal{N}$ is the conormal sheaf of $X_{\text{red}}$ in $X$ (or, equivalently, the nilpotent ideal of $\mathcal{O}_X$).

There exists a stable sheaf as above if and only if the inequalities are strict.

In particular, this theorem improves, in the case of ribbons, the sufficient conditions for the existence of stable quasi locally free sheaves of rigid type obtained by Drézet in [D3] and described in Fact 1.23(i).

Another significant result of this section is Theorem 2.8 computing the dimension of the loci of stable quasi locally free sheaves of fixed complete type:

**Theorem B.** Let $X$ be a ribbon such that $\deg(\mathcal{N}) < 0$ and $\bar{g} \geq 2$, where, as above, $\mathcal{N}$ is the conormal sheaf of $X_{\text{red}}$ in $X$ and $\bar{g}$ is the genus of $X_{\text{red}}$. Let $((r_0, r_1), (d_0, d_1))$ be integers verifying the hypotheses of Theorem A with strict inequalities. The locus of semistable quasi locally free sheaves on $X$ of complete type $((r_0, r_1), (d_0, d_1))$ has dimension $1 + (r_0^2 + r_1^2)(\bar{g} - 1) - r_0 r_1 \deg(\mathcal{N})$.

Within quasi locally free sheaves on $X$, there are also vector bundles on $X_{\text{red}}$. In this section we study also the conditions under which a vector bundle on $X_{\text{red}}$ deforms to a sheaf supported on $X$ (see Propositions 2.12 and 2.13).

The next section, i.e. Section 3, is concerned with some special kinds of non quasi locally free sheaves on $X$, namely generalized vector bundles and pure sheaves generically isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_{X_{\text{red}}}^\oplus n$, with $n$ a positive integer. About the latter, the main result is Theorem 3.5 which extends
[EG] Theorem 1.1] from generalized line bundles (which are the case \( n = 0 \)) to a wider class of coherent sheaves:

**Theorem C.** Let \( n \) be a non-negative integer and let \( \mathcal{F} \) be a pure sheaf on \( X \) generically isomorphic to \( \mathcal{O}_X \oplus \mathcal{O}^{\oplus n}_{X_{\text{red}}} \). Then there is a unique divisor \( D \subset X_{\text{red}} \) and a unique quasi locally free sheaf \( \mathcal{F}' \), locally isomorphic to \( \mathcal{O}_{X'} \oplus n\mathcal{O}_{X_{\text{red}}} \), on the blow-up \( q : X' \to X \) of \( X \) at \( D \) such that \( q_*\mathcal{F}' \simeq \mathcal{F} \).

This Theorem is quite useful in the study of semistability conditions and loci in the moduli space of this kind of sheaves. Indeed, it allows to prove that a sheaf \( \mathcal{F} \) as in the statement is semistable if and only the quasi locally free sheaf on the blow up \( \mathcal{F}' \) is semistable (see Corollary 3.8). This fact permits to describe some properties of the loci that have such sheaves as generic elements in the moduli space of these sheaves. Indeed, it allows to prove that a sheaf \( \mathcal{F} \) as in the statement is semistable if and only the quasi locally free sheaf on the blow up \( \mathcal{F}' \) is semistable (see Corollary 3.8).

The aim of the last section is to justify Conjecture D, i.e. the following conjectural description of the irreducible components of \( M \) of stable sheaves on a ribbon:

**Conjecture D.** Let \( X \) be a ribbon of arithmetic genus \( g \) such that \( \bar{g} \geq 2 \); where \( \bar{g} \) is the genus of \( X_{\text{red}} \), let \( \delta = -\deg \mathcal{N} \), where \( \mathcal{N} \) is the conormal sheaf of \( X_{\text{red}} \) in \( X \) or, equivalently, the nilradical of \( \mathcal{O}_X \) and let \( M = M_\delta(X, R, D) \) be the moduli space of stable sheaves of generalized rank \( R \) and generalized degree \( D \) on \( X \).

(i) Assume \( 0 < \delta \leq 2\bar{g} - 2 \), equivalently \( g \leq 4\bar{g} - 3 \). The irreducible components are the closures of the following loci:

- For any sequence of integers \( (r_0, r_1), (d_0, d_1) \) such that \( r_0 > r_1 \geq 0 \), \( r_0 + r_1 = R \), \( d_0 + d_1 = D \) and, if \( r_1 > 0 \), \( (d_0 - (r_0 + r_1)\delta)/r_0 < d_1/r_1 < d_0/r_0 \), the locus of quasi locally free stable sheaves of complete type \( (r_0, r_1), (d_0, d_1) \).

- If \( R \) is even, the locus of stable generalized vector bundles of generalized rank \( R \) and degree \( D \) and fixed index \( b \), for any positive integer \( b < r\delta \) where \( r = R/2 \). All these components have dimension \( 1 + (r_0^2 + r_1^2)(\bar{g} - 1) + r_0r_1\delta \) (with \( r_0 = r_1 = r \) in the case of generalized vector bundles). Distinct complete types correspond to distinct irreducible components.

(ii) If \( \delta > 2\bar{g} - 2 \), equivalently \( g > 4\bar{g} - 3 \), then we have to distinguish two cases.

- (a) If \( R = 2r \) is even, then the only irreducible components of \( M \) are the closures of the loci of stable generalized vector bundles of generalized rank \( R \) and degree \( D \) and fixed index \( b < r\delta \) and they have dimension \( 1 + 2r^2(\bar{g} - 1) + r^2\delta \).

- (b) If \( R = 2a + 1 \) is odd, then the only irreducible components of \( M \) are the closures of the loci \( N(a, d_0, d_1) \) of stable quasi locally free sheaves of rigid type of generalized rank \( R \) and generalized degree \( D \) with \( (d_0 - (2a + 1)\delta)/(a + 1) < d_1/a < d_0/(a + 1) \). They have dimension \( 1 + (a^2 + a)\delta + (2a^2 + 2a + 1)(\bar{g} - 1) \).

As we will explain in detail, in the first point the conjectural parts are only the irreducibility of the loci of generalized vector bundles of fixed index and the fact that the cited loci, which are irreducible of that dimension and not contained one in the other, are really irreducible components (they are
surely irreducible components for \( r_0 = r_1 + 1 > 1 \) and, also, that there are no other irreducible components. On the other hand, the second part is much more conjectural. The whole conjecture surely holds for generalized rank 2 (it is a part of [CK Theorem 4.7] together with [Sa2 Corollary 1]).

1. First properties

This section collects from literature the properties of coherent sheaves on ribbons that we will need in the article. In this paper only coherent sheaves are considered, hence this attribute will be usually omitted in the following. Moreover, vector bundle will be used as a synonym of locally free sheaf of finite rank. Let us begin recalling precisely what we mean by a ribbon.

Let \( \mathbb{K} \) be an algebraically closed field. A ribbon \((X, \mathcal{O}_X)\) (in the following simply \( X \)) is a non-reduced projective \( \mathbb{K} \)-scheme whose reduced subscheme \((X_{\text{red}}, \mathcal{O}_{X_{\text{red}}})\) (in the following just \( X_{\text{red}} \)) is a smooth \( \mathbb{K} \)-curve and whose nilradical \( \mathcal{N} \subset \mathcal{O}_X \) is locally generated by a single non-zero square-zero element. In particular, this definition implies immediately that \( \mathcal{N} \) can be seen as a line bundle on \( X_{\text{red}} \) and that it coincides with the conormal sheaf of \( X_{\text{red}} \) in \( X \). It holds that \( \deg(\mathcal{N}) \), i.e. the degree of \( \mathcal{N} \) on \( X_{\text{red}} \), equals \( 2\bar{g} - 1 - g \), where \( \bar{g} \) is the genus of \( X_{\text{red}} \) and \( g = 1 - \chi(\mathcal{O}_X) \) is the (arithmetic) genus of \( X \).

Remark 1.1. This definition of ribbon is more restrictive than that given in [BE, §1], where a ribbon is any scheme of finite type over a field such that its reduced subscheme is connected and its nilradical verifies the same properties as above.

Using our definition, a ribbon is just a projective primitive multiple curve of multiplicity 2 (or primitive double curve; see, e.g., [D2, §2.1]).

In this paper, following Drézet’s use and being less precise than [CK], we will not distinguish between a sheaf on \( X_{\text{red}} \) and its direct image on \( X \), in order to lighten notation and exposition. A reason for which this is not too confusing will be given in Remark 1.6(ii).

1.1. Canonical filtrations.

A sheaf on \( X \) has two canonical filtrations, that we now recall.

Definition 1.2. Let \( \mathcal{F} \) be a sheaf on \( X \).

(i) The first canonical filtration of \( \mathcal{F} \) is

\[
0 \subseteq \mathcal{N}\mathcal{F} \subset \mathcal{F}.
\]

It is immediate to check that \( \mathcal{F}/(\mathcal{N}\mathcal{F}) = \mathcal{F}|_{X_{\text{red}}} \).

The graduate object associated to this filtration is denoted by \( \text{Gr}_1(\mathcal{F}) \) and is called the first graduate object of \( \mathcal{F} \).

The complete type of \( \mathcal{F} \) is \((r_0(\mathcal{F}), r_1(\mathcal{F})), (d_0(\mathcal{F}), d_1(\mathcal{F})) = ((\text{rk}(\mathcal{F}|_{X_{\text{red}}}), \text{rk}(\mathcal{N}\mathcal{F})), (\deg(\mathcal{F}|_{X_{\text{red}}}), \deg(\mathcal{N}\mathcal{F}))) \). If the sheaf in question is clear, we denote it simply as \((r_0, r_1), (d_0, d_1)\).

(ii) The second canonical filtration of \( \mathcal{F} \) is defined as

\[
0 \subset \mathcal{F}^{(1)} \subset \mathcal{F},
\]
where $\mathcal{F}(1)$ is the subsheaf of $\mathcal{F}$ annihilated by $\mathcal{N}$. It is immediate to check that $\mathcal{F}(1) = \mathcal{F}$ if and only if $\mathcal{F}$ is a sheaf on $X_{\text{red}}$.

The graduate object associated to this filtration is denoted by $\text{Gr}_2(\mathcal{F})$ and is called the second graduate object of $\mathcal{F}$.

The following fact collects some interesting properties, mainly about the two canonical filtrations and their mutual relations:

**Fact 1.3.** Let $\mathcal{F}$ be a sheaf on $X$.

(i) (See [D4, §3.2.3]) There is a canonical isomorphism

$$\mathcal{N} \mathcal{F} \simeq (\mathcal{F}/\mathcal{F}(1)) \otimes \mathcal{N}.$$  

Moreover there is a canonical exact sequence on $X_{\text{red}}$:

$$0 \to \mathcal{N} \mathcal{F} \to \mathcal{F}(1) \to \mathcal{F}|_{X_{\text{red}}} \to \mathcal{N} \mathcal{F} \otimes \mathcal{N}^{-1} \to 0.$$  

(1.1)

(ii) (See [D4, §3.2.4]) A subsheaf of $\mathcal{F}$, $\mathcal{F}$ is defined on $X_{\text{red}}$ if and only if $\mathcal{F} \subseteq \mathcal{F}(1)$.

On the other hand, $\mathcal{F}/\mathcal{F}$ is a sheaf on $X_{\text{red}}$ if and only if $\mathcal{N} \mathcal{F} \subseteq \mathcal{F}$. In this case there is a canonical morphism $\mathcal{F}/\mathcal{F} \otimes \mathcal{N}^{-1} \to \mathcal{F}$, which is surjective if and only if $\mathcal{F} = \mathcal{N} \mathcal{F}$, while it is injective if and only if $\mathcal{F} = \mathcal{F}(1)$.

(iii) (See [D4, §3.4]) Let $\mathcal{F}$ be a sheaf on $X_{\text{red}}$ and let $\mathcal{E}$ be a vector bundle on it; then there exists the following canonical exact sequence:

$$0 \to \text{Ext}^1_{\mathcal{O}_{X_{\text{red}}}}(\mathcal{E}, \mathcal{F}) \to \text{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \to \text{Hom}(\mathcal{E} \otimes \mathcal{N}, \mathcal{F}) \to 0.$$  

By the previous point, if $\mathcal{F}$ is a sheaf on $X$ sitting in a short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$ represented by $\sigma \in \text{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$, then $\mathcal{F} = \mathcal{N} \mathcal{F}$ if and only if $\pi(\sigma)$ is surjective, while $\mathcal{F} = \mathcal{F}(1)$ if and only if $\pi(\sigma)$ is injective.

(iv) (See [D2, Proposition 3.5]) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on $X$. Then:

(a) $\varphi$ is surjective if and only if the induced morphism $\varphi|_{X_{\text{red}}} : \mathcal{F}|_{X_{\text{red}}} \to \mathcal{G}|_{X_{\text{red}}}$ is surjective. In this case, also the induced morphism $\mathcal{N} \mathcal{F} \to \mathcal{N} \mathcal{G}$ is surjective.

(b) $\varphi$ is injective if and only if the induced morphism $\mathcal{F}(1) \to \mathcal{G}(1)$ is injective. If this is the case, the induced morphism $\mathcal{F}/\mathcal{F}(1) \to \mathcal{G}/\mathcal{G}(1)$ is injective, too.

(v) (See [D1, Proposition 7.3.1]) $r_0(\mathcal{F})$ is upper semicontinuous, while $r_1(\mathcal{F})$ is lower semicontinuous.

**Remark 1.4.** Thanks to Fact [LX(1)], the complete type of a sheaf $\mathcal{F}$ can be characterized also in terms of the second canonical filtration of $\mathcal{F}$. It holds that $r_0 = \text{rk}(\mathcal{F}(1))$, $r_1 = \text{rk}(\mathcal{F}/\mathcal{F}(1))$, $d_0 = \text{deg}(\mathcal{F}(1)) + r_1 \text{deg}(\mathcal{N})$ and $d_1 = \text{deg}(\mathcal{F}/\mathcal{F}(1)) - r_1 \text{deg}(\mathcal{N})$.

### 1.2. Generalized rank and degree.

Now we introduce two fundamental invariants of a sheaf on $X$: the generalized rank and the generalized degree.
Definition 1.5. Let $\mathcal{F}$ be a sheaf on $X$. The **generalized rank** of $\mathcal{F}$, $R(\mathcal{F})$, is its generic length, i.e. the length of its generic stalk $\mathcal{F}_\eta$ as an $\mathcal{O}_{X,\eta}$ module (here and throughout the paper $\eta$ denotes the generic point of $X$). The **generalized degree** of $\mathcal{F}$ is $\text{Deg}(\mathcal{F}) := \chi(\mathcal{F}) - R(\mathcal{F})\chi(\mathcal{O}_{X,\eta})$.

Remark 1.6.

(i) These are not the original definitions given by Drézet (see e.g. [D1, §§4.1.3-4.1.4]), i.e. the rank and the degree (as a sheaf on $X_{\text{red}}$) of the first graded object $\text{Gr}_1(\mathcal{F})$, but it is quite easy to check that $R(\mathcal{F}) = r_0(\mathcal{F}) + r_1(\mathcal{F})$ and $\text{Deg}(\mathcal{F}) = d_0(\mathcal{F}) + d_1(\mathcal{F})$; hence, the two definitions are equivalent. More generally, it is immediate to verify that, if $0 \to \mathcal{F} \to \mathcal{F}$ is any filtration of $\mathcal{F}$ such that both $\mathcal{F}$ and $\mathcal{F}/\mathcal{F}$ are sheaves on $X_{\text{red}}$, then $R(\mathcal{F}) = \text{rk}(\mathcal{F} \oplus (\mathcal{F}/\mathcal{F}))$ and $\text{Deg}(\mathcal{F}) = \text{deg}(\mathcal{F} \oplus (\mathcal{F}/\mathcal{F}))$.

(ii) If $\mathcal{F}$ is defined on $X_{\text{red}}$, meaning that it is the direct image on $X$ of a sheaf on $X_{\text{red}}$ or, equivalently, it is annihilated by $\mathcal{N}$, then its generalized rank and degree are equal to its classical rank and degree as a sheaf on $X_{\text{red}}$ (so, for such sheaves we will often speak simply of rank and degree).

Thus, avoiding the distinction between a sheaf on $X_{\text{red}}$ and its direct image on $X$ does not make confusion when we consider its generalized rank and degree.

This is also one of the main reasons for which it is more convenient to use generalized rank and degree instead of classical ones, which can be defined also for sheaves on ribbons (the rank is $\text{rk}(\mathcal{F}) = R(\mathcal{F})/2$ and the degree is $\text{deg}(\mathcal{F}) = \chi(\mathcal{F}) - \text{rk}(\mathcal{F})\chi(\mathcal{O}_X)$ which is equal to $\text{Deg}(\mathcal{F}) - \text{rk}(\mathcal{F})\text{deg}(\mathcal{N})$).

The following fact collects the basic properties of these invariants

Fact 1.7.

(i) (See [D1, §4.2.2]) Let $\mathcal{O}_X(1)$ be a very ample line bundle on $X$, let $\mathcal{O}_{X_{\text{red}}}(1)$ be its restriction to $X_{\text{red}}$ and let $d = \text{deg}(\mathcal{O}_{X_{\text{red}}}(1))$. If $\mathcal{F}$ is a sheaf on $X$, its Hilbert polynomial with respect of $\mathcal{O}_X(1)$ is $P(\mathcal{F}(T) = \text{Deg}(\mathcal{F}) + R(\mathcal{F})\chi(\mathcal{O}_{X_{\text{red}}}) + R(\mathcal{F})dT$.

(ii) (See [D1, Corollaire 4.3.2]) The generalized rank and degree are additive: if $0 \to \mathcal{F} \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves on $X$, then $R(\mathcal{F}) = R(\mathcal{F}') + R(\mathcal{F}'')$ and $\text{Deg}(\mathcal{F}) = \text{Deg}(\mathcal{F}') + \text{Deg}(\mathcal{F}'')$.

(iii) (See [D1, Proposition 4.3.3]) The generalized rank and degree of sheaves on $X$ are invariant by deformation.

(iv) (See [D1, §§4.1.3-4.1.4].) Let $\mathcal{F}$ be a vector bundle of rank $n$ over $X$, then $R(\mathcal{F}) = 2n = 2\text{rk}(\mathcal{F}|_{X_{\text{red}}})$ and $\text{Deg}(\mathcal{F}) = 2\text{deg}(\mathcal{F}|_{X_{\text{red}}}) + n\text{deg}(\mathcal{N})$ (in particular $\text{Deg}(\mathcal{O}_X) = \text{deg}(\mathcal{N})$). This implies, in particular, that $\text{Deg}(\mathcal{F})$ must have the same parity of $n\text{deg}(\mathcal{N})$.

1.3. Purity and duality.

We now introduce the notions of pure, torsion-free and reflexive sheaves. The distinction between pure and torsion-free is taken from [CK] Definition
2.1; Drézet speaks only of reflexive and torsion-free sheaves (faisceaux sans torsion in French), but he defines the latter as Chen and Kass define pure ones (cf. [D2, §3.3]).

**Definition 1.8.** Let \( F \) be a sheaf on \( X \). Its *dimension* \( d(F) \) is the dimension of its support.

A sheaf \( F \) on \( X \) is *pure* if it has dimension 1 and does not contain torsion sheaves (i.e. sheaves of finite support), in other words if \( d(F) = d(F) = 1 \) for any non-zero subsheaf \( G \subset F \).

Let \( U \) be an open subscheme of \( X \), a regular function \( f \in H^0(U, \mathcal{O}_X) \) is a non-zerodivisor on \( F \) if the multiplication map \( f \cdot : F|_U \to F|_U \) is injective and the sheaf \( F \) is torsion-free if every non-zerodivisor on \( \mathcal{O}_X \) is a non-zerodivisor also on \( F \).

The *dual* sheaf of \( F \) is \( F^\vee := \text{Hom}(F, \mathcal{O}_X) \).

The sheaf \( F \) is *reflexive* if the canonical morphism \( F \to F^{\vee\vee} \) is an isomorphism.

**Remark 1.9.**

(i) Our definition of pure sheaf is not exactly that of [CK], which is more similar to that used in arbitrary dimension: they give it in general and not only for dimension 1 sheaves. Using their definition, a torsion sheaf would be a pure sheaf of dimension 0.

(ii) If \( F \) is a sheaf on \( X_{\text{red}} \), then there is a small ambiguity because its dual \( F^\vee \) on \( X \) is obviously different from its dual \( F^* \) on \( X_{\text{red}} \), defined as \( \text{Hom}(F, \mathcal{O}_X) \). But, by [D2, Lemme 4.1], there is a canonical isomorphism \( F^\vee \simeq F^* \otimes N \), hence reflexiveness on \( X \) is equivalent to reflexiveness on \( X_{\text{red}} \).

The relationship between the above introduced notions is described in the following fact.

**Fact 1.10.** Let \( F \) be a sheaf on \( X \). Then:

(i) (See [D2, Corollaire 4.6]) It holds that \( \text{Ext}^{i}_{\mathcal{O}_X}(F, \mathcal{O}_X) = 0 \) for any \( i \geq 2 \).

(ii) (See [CK, Lemma 2.2] and [D2, Proposition 3.8 and Théorème 4.4]) The following are equivalent:

(a) \( F \) is pure;
(b) \( F \) is torsion-free;
(c) \( F^{(1)} \) is a vector bundle on \( X_{\text{red}} \);
(d) \( F \) is reflexive;
(e) it holds that \( \text{Ext}^{1}_{\mathcal{O}_X}(F, \mathcal{O}_X) = 0 \).

Moreover, if the above conditions hold, \( F/F^{(1)} \) and \( N/F \) are vector bundles on \( X_{\text{red}} \).

(iii) (See [D2, Proposition 4.2]) We have that \( (F^\vee)^{(1)} = (F|_{X_{\text{red}}})^\vee \).

(iv) (See [D3, Proposition 4.4.1]) It holds that \( \text{R}(F^\vee) = \text{R}(F) \) and \( \text{Deg}(F^\vee) = -\text{Deg}(F) + \text{R}(F) \cdot \deg(N) + h^0(T(F)) \), where \( T(F) \) is the torsion subsheaf of \( F \), i.e. its greatest subsheaf with finite support.

(v) Assume, moreover, that \( F \) is torsion-free. Then:
(a) (See [D3] Proposition 4.3.1(i)) There is a canonical isomorphism between $T(F|_{X_{\text{red}}})$ and $\text{Ext}^1_{\mathcal{O}_X}(T(F|_{X_{\text{red}}}), \mathcal{O}_{X}) \otimes \mathcal{N}$, where $T(F|_{X_{\text{red}}})$ and $T(F|_{X_{\text{red}}})$ are, respectively, the torsion subsheaves of $F|_{X_{\text{red}}}$ and $F|_{X_{\text{red}}}$.

(b) (See [D3] Proposition 4.3.1(ii)) There is a canonical isomorphism between $(\ker(F \to (F|_{X_{\text{red}}})|^{\otimes}))^{\otimes}$ and $\mathcal{N}^{\otimes} \otimes \mathcal{N}^{-1}$.

(vi) (See [D2] Corollaire 4.5) Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of sheaves on $X$ with $\mathcal{G}$ torsion-free, then also the dual sequence $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{E} \to 0$ is exact.

1.4. Quasi locally free sheaves and pure sheaves.

There is a special type of torsion-free sheaves that plays a major role in the theory of sheaves over a ribbon: the so-called quasi locally free sheaves.

**Definition 1.11.** (Cf. [D1] §5.1.) Let $F$ be a sheaf on $X$. It is quasi locally free in a closed point $P$ if there exists an open neighbourhood $U$ of $P$ and integers $a, b$ such that $F$ is isomorphic to $\mathcal{O}_{X_{\text{red}}}^{\oplus a} \oplus \mathcal{O}_{X_{\text{red}}}^{\oplus b}$. It is quasi locally free if it is such in any closed point. The type of a quasi locally free sheaf $F$ is $(a, b)$.

The following fact contains some significant results.

**Fact 1.12.** Let $F$ be a sheaf on $X$.

(i) (See [D2] Théorème 3.9 and Corollaire 3.10) Let $P$ be a closed point of $X_{\text{red}}$. The following assertions are equivalent:

(a) $F$ is quasi locally free (resp. quasi locally free in $P$);
(b) $\mathcal{N} F$ and $F|_{X_{\text{red}}}$ are locally free on $X_{\text{red}}$ (resp. are free in $P$);
(c) $F^{(1)}/\mathcal{N} F$ and $F|_{X_{\text{red}}}$ are locally free on $X_{\text{red}}$ (resp. are free in $P$);
(d) $F$ and $F|_{X_{\text{red}}}$ are torsion-free (resp. are torsion-free in $P$).

(ii) (See [D2] §4.1.1) If $F$ is quasi locally free, then there exists a canonical isomorphism $\mathcal{N} (F|^{\otimes}) \simeq (\mathcal{N} F)^{*}$ (equivalently $F|^{(1)}\simeq (F/F^{(1)})^{*}$).

(iii) (See [D1] Théorème 5.1.6) $F$ is generically quasi locally free, i.e. there exists an open $0 \neq U \subseteq X$ such that $F$ is quasi locally free in each point of $U$.

Within quasi locally free sheaves there are those of rigid type:

**Definition 1.13.** A sheaf $F$ on $X$ is said to be quasi locally free of rigid type if there exists a positive integer $a$ such that $F$ is locally isomorphic to $\mathcal{O}_{X_{\text{red}}}^{\oplus a} \oplus \mathcal{O}_{X_{\text{red}}}^{\oplus a}$; in other words $F$ is a quasi locally free sheaf of type $(1, a)$.

These are relevant because being a quasi locally free sheaf of rigid type is an open condition in flat families (see [D2] Proposition 6.9).

Now we turn our attention to sheaves which are not quasi locally free. Before passing to pure ones, we give a definition that holds for any sheaf on $X$.

**Definition 1.14.** Let $F$ be a sheaf on a ribbon $X$ and let $U$ be an open on which $F$ is quasi locally free (see Fact 1.12(iii)). The type of $F$ is, by definition, the type of $F|_{U}$.
In other words, if \( \eta \) is the generic point of \( X \), the type of a sheaf \( \mathcal{F} \) is the pair of non-negative integers \( (a, b) \) such that \( \mathcal{F}_\eta \cong \mathcal{O}_{X_{\text{red}}}^{\oplus a} \oplus \mathcal{O}_{X_{\text{red}}}^{\oplus b} \). In particular, a sheaf is a torsion sheaf, i.e. it has finite support, if and only if its type is \( (0, 0) \).

**Remark 1.15.** Let \( \mathcal{F} \) be a sheaf of type \( (a, b) \) on \( X \). It holds that \( R(\mathcal{F}) = a + 2b \), while \( r_0(\mathcal{F}) = a + b \) and \( r_1(\mathcal{F}) = b \).

The following definition of index is taken from [D1, §6.3.7], while those of local index and of local index sequence are inspired by [CK, Definition 2.7], which is about generalized line bundles:

**Definition 1.16.** Let \( \mathcal{F} \) be a pure sheaf on \( X \). The index of \( \mathcal{F} \) is \( b(\mathcal{F}) = h^0(\mathcal{T}) \), where \( \mathcal{T} \) is the torsion part of \( \mathcal{F}|_{X_{\text{red}}} \). For any closed point \( p \), the local index of \( \mathcal{F} \) at \( p \), denoted by \( b_p(\mathcal{F}) \), is the length of \( \mathcal{T}_p \) as an \( \mathcal{O}_{X_{\text{red}}, p} \)-module. The local index sequence of \( \mathcal{F} \), denoted by \( b(\mathcal{F}) \), is the collection \( \{b_p(\mathcal{F}) : p \in \text{Supp}(\mathcal{T})\} \).

**Remark 1.17.** Let \( \mathcal{F} \) be a pure sheaf. It follows immediately from the definition that \( b(\mathcal{F}) \) is a non-negative integer which vanishes if and only if \( \mathcal{F} \) is quasi locally free, by Fact 1.12(i).

The following fact relates pure sheaves with positive index to quasi locally free ones.

**Fact 1.18.** (See [D1 Lemme 6.3.4 and Corollaire 6.4.2]) Let \( \mathcal{F} \) be a pure sheaf with positive index on \( X \). There exist two quasi locally free sheaves \( \mathcal{E} \) and \( \mathcal{G} \) on \( X \) (not necessarily unique and of the same type of \( \mathcal{F} \), in other words generically isomorphic to it) such that the following exact sequences are exact:

\[
0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{T} \to 0,
\]

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{T} \to 0,
\]

where \( \mathcal{T} \) is the torsion part of \( \mathcal{F}|_{X_{\text{red}}} \). Moreover it holds that \( \mathcal{N}\mathcal{E} = \mathcal{N}\mathcal{F} \), while \( \mathcal{G}(1) = \mathcal{F}(1) \).

A kind of sheaves that seems particularly significant for the study of the moduli space is the following generalization of generalized line bundle:

**Definition 1.19.** A **generalized vector bundle** \( \mathcal{F} \) is a pure sheaf on \( X \) such that \( \mathcal{F}_\eta \) is a free \( \mathcal{O}_{X_{\eta}} \)-module of finite rank \( r \), where \( \eta \) is the generic point of \( X \), i.e. it is a pure sheaf of type \( (0, r) \).

This is equivalent to require that \( \text{rk}(\mathcal{N}\mathcal{F}) = \text{rk}(\mathcal{F}|_C) \), or, in other words, that the complete type of \( \mathcal{F} \) is \( ((r, r), (d_0, d_1)) \).

A generalized vector bundle being quasi locally free (or, equivalently, with index 0) is just a vector bundle.

**Remark 1.20.**

(i) According to my knowledge, this definition of generalized vector bundle is new. It generalizes that of generalized line bundle, which appeared in [BE] and is equivalent, in this context, to the notion of generalized divisor in the sense of Hartshorne (as already pointed out in [EG]).
(ii) If \( \mathcal{F} \) is a generalized vector bundle of generalized rank \( 2r \), generalized degree \( D \) and index \( b \), it holds that 
\[
d_0(\mathcal{F}) = (D + b - r \deg(\mathcal{N})/2) \text{ and } d_1(\mathcal{F}) = (D - b + r \deg(\mathcal{N}))/2.
\]
It follows from the fact that \( \mathcal{N} \mathcal{F} = (\mathcal{F}/\mathcal{F}(1)) \otimes \mathcal{N} \), from the fact that, for a 
generalized vector bundle, \( (\mathcal{F}|_{X_{\text{red}}})^{\vee} = \mathcal{F}/\mathcal{F}/(1) \) and from the 
definition of the index.

Conjecturally, generalized vector bundles are the only non quasi locally 
free sheaves which appear as generic components of the moduli space of 
semistable sheaves on \( X \). We will return on this point extensively in Section 4.

1.5. Semistability and moduli space.

First of all, we need to recall the definition of semistability for pure sheaves 
on a ribbon.

**Definition 1.21.** Let \( \mathcal{F} \) be a pure sheaf on \( X \). Its **slope** is 
\[
\mu(\mathcal{F}) = \text{Deg}(\mathcal{F})/R(\mathcal{F}).
\]
We say that \( \mathcal{F} \) is **(slope)-semistable** if \( \mu(\mathcal{G}) \leq \mu(\mathcal{F}) \) for any \( \mathcal{G} \subset \mathcal{F} \). If the 
inequality is always strict, \( \mathcal{F} \) is **stable**.

A semistable sheaf \( \mathcal{F} \) is **polystable** if it is isomorphic to the direct sum of 
stable sheaves (clearly of its same slope).

Let \( \mathcal{F} \) be a semistable sheaf, then a **Jordan-Holder filtration** of \( \mathcal{F} \) is 
a filtration whose associated graded object \( \text{Gr}_{\text{JH}}(\mathcal{F}) \) is polystable of the 
same slope of \( \mathcal{F} \). It is well known that any semistable sheaf admits a 
(non-necessarily unique) Jordan-Holder filtration and that \( \text{Gr}_{\text{JH}}(\mathcal{F}) \) is 
independent of the choice of the filtration (cf., e.g., [HL, Proposition 1.5.2]).

Clearly, if \( \mathcal{F} \) is stable, \( \text{Gr}_{\text{JH}}(\mathcal{F}) = \mathcal{F} \).

Two semistable sheaves \( \mathcal{F} \) and \( \mathcal{G} \) are said to be **S-equivalent** if \( \text{Gr}_{\text{JH}}(\mathcal{F}) \) and 
\( \text{Gr}_{\text{JH}}(\mathcal{G}) \) are isomorphic.

**Remark 1.22.**

(i) As usual, it is possible to check semistability considering only satu-
rated subsheaves of \( \mathcal{F} \) (i.e. subsheaves \( \mathcal{G} \) such that the quotient 
\( \mathcal{F}/\mathcal{G} \) is pure) and it is equivalent to the reverse inequalities for quo-
tients of \( \mathcal{F} \) (these are well-known basic properties of semistability, 
see e.g. [HL, Proposition 1.2.6]).

(ii) It follows from Facts [L,e][iv] and [L,e][vi] and from the previous 
point that a pure sheaf \( \mathcal{F} \) is (semi)stable if and only if its dual \( \mathcal{F}^{\vee} \) is 
(semi)stable.

(iii) By Fact [L,e]i, slope-semistability is equivalent, on a ribbon, to 
Gieseker semistability (which is defined in terms of the leading co-
efficient of the reduced Hilbert polynomial, for its precise definition 
see [HL]).

(iv) As pointed out, e.g., in [D3, §1.2], there exist stable sheaves on \( X \) 
different from stable vector bundles on \( X_{\text{red}} \) if and only if \( \deg(\mathcal{N}) < 0 \).

Indeed, let \( \mathcal{F} \) be a pure sheaf not defined over \( X_{\text{red}} \). Then, 
\( \ker(\mathcal{F} \twoheadrightarrow (\mathcal{F}|_{X_{\text{red}}})^{\vee}) \) and \( \mathcal{F}/\mathcal{F}(1) \) are non-trivial. This implies
that $\mathcal{F}$ can be stable only if $\mu(\ker(\mathcal{F} \to (\mathcal{F}|_{X_{\text{red}}})^{\vee})) = \mu(\mathcal{N}\mathcal{F}) + b(\mathcal{F})/R(\mathcal{N}\mathcal{F}) < \mu(\mathcal{F}) < \mu(\mathcal{F}/\mathcal{F}^{(1)}) = \mu(\mathcal{N}\mathcal{F}) - \deg(\mathcal{N})$. Hence, the index $b(\mathcal{F})$ (which is non-negative, see Remark 1.17) must be less than $-R(\mathcal{N}\mathcal{F})\deg(\mathcal{N})$.

Similarly, $\mathcal{F}$ semistable implies $b(\mathcal{F}) \leq -R(\mathcal{N}\mathcal{F})\deg(\mathcal{N})$ and, in particular, $\deg(\mathcal{N}) \leq 0$. The case $\deg(\mathcal{N}) = 0$ is not particularly interesting, because in this case a sheaf $\mathcal{F}$ can be only strictly semistable (but not stable); indeed, in this case, its subsheaf $\mathcal{N}\mathcal{F}$ and its quotient $\mathcal{F}/\mathcal{F}^{(1)} = \mathcal{N}\mathcal{F} \otimes \mathcal{N}^{-1}$ have the same slope.

(v) The so-called Simpson moduli space (see [Si] and also the textbook [HL]) is a projective good moduli space $M(X, P)$, whose $\mathbb{k}$-valued points parametrize $S$-equivalence classes of semistable sheaves of fixed Hilbert polynomial $P$ on $X$. It has an open subscheme, denoted by $M_\mathfrak{s}(X, P)$, parametrizing stable sheaves. By the fact the Hilbert polynomial of a sheaf on a ribbon $X$ is completely determined by the generalized rank $R$ and the generalized degree $D$ (see (1.2)), in the following $M(X, P)$ (resp. $M_\mathfrak{s}(X, P)$) will be denoted also by $M(X, R, D)$ (resp. $M_\mathfrak{s}(X, R, D)$). Often we will omit the $X$ in the previous notation.

We end this introductory section recalling what Drézet proved in his articles about some relevant loci in the Simpson moduli space (his original results are about sheaves on primitive multiple curves of any multiplicity). The first point of the following fact is an adaptation of [D2 Proposition 6.12] and [D3 Théorème 5.3.3] to the case of ribbons, while the second one is essentially [D3 §5.2.2].

**Fact 1.23.**

(i) Let $a$ be a positive integer, let $d_0$ and $d_1$ be two integers and let $N(a, d_0, d_1) \subset M_\mathfrak{s}(X, 2a + 1, d_0 + d_1)$ be the locus of stable quasi locally free sheaves of rigid type of complete type $(a+1, a, (d_0, d_1))$. The locus $N(a, d_0, d_1)$ is open and irreducible. If it is non-empty, it has dimension $1 + (a^2 + a)\delta + (2a^2 + 2a + 1)(\bar{g} - 1)$, where $\delta = -\deg(\mathcal{N})$ and $\bar{g}$ is the genus of $X_{\text{red}}$.

If $\bar{g} \geq 2$, then it is non-empty if $d_0/(a + 1) - \delta < d_1/a < (d_0 - a\delta)/(a + 1)$.

(ii) The locus of stable vector bundle of rank $r$ (i.e. generalized rank $2r$) and generalized degree $D$ is non-empty if and only if $\delta > 0$ and $D = 2d - r\delta$ for some integer $d$ (as above, $\delta = -\deg(\mathcal{N})$). In this case, it is a smooth irreducible open of $M_\mathfrak{s}(X, 2r, D)$ of dimension $1 + r^2\delta + (2r^2)(\bar{g} - 1) = 1 + r^2(g - 1)$.

**Remark 1.24.** The assumption $\bar{g} \geq 2$ in the first part of the fact is needed (although not explicitly stated in the cited articles) because non-emptiness is proved applying the so-called Lange’s conjecture on $X_{\text{red}}$, that is about the existence of exact sequences of (semi)stable vector bundles on smooth projective curves of genus greater than or equal to 2 (for a brief and very clear introduction to it see [D]; there it is assumed that the characteristic of the base field is 0, but in the article in which the conjecture is proved, i.e. [RT], this hypothesis does not appear).
2. Quasi locally free sheaves

This section is divided into two subsections. The first one is about quasi locally free sheaves supported on \( X \) while the second one is about deformations of vector bundles of rank \( r \) on \( X_{\text{red}} \), which are quasi locally free sheaves of type \((r,0)\) on \( X \), to sheaves supported on \( X \).

2.1. Quasi locally free sheaves supported on \( X \).

We begin this subsection with an existence result which will be useful in order to prove the semistability conditions of Theorem 2.5. It extends [D3, Proposition 3.4.1] from quasi locally free sheaves of rigid type to all quasi locally free sheaves, in the case of ribbons (the cited result is about primitive multiple curves of any multiplicity). The method of proof is inspired by [D3, §3.2].

**Proposition 2.1.** Let \((r_0, r_1)\) be a pair of positive integers with \( r_0 > r_1 \) and let

\[
0 \to F \xrightarrow{f} E \xrightarrow{e} G \to F \otimes N^{-1} \to 0 \quad (*)
\]

be an exact sequence of vector bundles on \( X_{\text{red}} \), with \( \text{rk}(F) = r_1 \) and \( \text{rk}(G) = r_0 \). Then there exists a quasi locally free sheaf \( \mathcal{F} \) on \( X \) such that its associated canonical exact sequence (1.1) is isomorphic to (1).

**Proof.** In this proof we use the same notation of Fact 1.3(iii). Let \( F \) be a sheaf over \( X \) corresponding to an element \( \sigma_F \in \text{Ext}^1_{O_X}(G, F) \) such that \( \pi(\sigma_F) = g \otimes \text{id}_N \). Hence, \( N \mathcal{F} = F \) and \( \mathcal{F}|_{X_{\text{red}}} = G \), by the surjectivity of \( g \) and by Fact 1.3(iii). Moreover, by Fact 1.12(i) it holds that \( \mathcal{F}/\mathcal{F}^{(1)} = F \otimes N^{-1} \) and by Fact 1.12(i) such an \( \mathcal{F} \) is quasi locally free.

For all these sheaves it is also fixed \( K = \ker(\mathcal{F}|_{X_{\text{red}}} \to \mathcal{F}/\mathcal{F}^{(1)}) = \ker(g) = \text{im}(e) \), which is also equal to \( \mathcal{F}^{(1)}/N \mathcal{F} \) (see Fact 1.3(i)). Therefore, \( \mathcal{F}^{(1)} \) is represented by an element \( \sigma_{\mathcal{F}} \in \text{Ext}^1_{O_{X_{\text{red}}}}(K, F) \). Thus, we need \( \sigma'_{\mathcal{F}} = \sigma_E \), where \( \sigma_E \) is the element in \( \text{Ext}^1_{O_{X_{\text{red}}}}(K, F) \) associated to the short exact sequence \( 0 \to F \xrightarrow{f} E \xrightarrow{e} K \to 0 \).

The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Ext}^1_{O_{X_{\text{red}}}}(G, F) & \xrightarrow{\pi} & \text{Hom}(G \otimes N, F) \\
\downarrow & & \downarrow \\
\text{Ext}^1_{O_{X_{\text{red}}}}(K, F) & \xrightarrow{p} & \text{Hom}(K \otimes N, F)
\end{array}
\]

By definition of \( K \), \( p(\sigma_{\mathcal{F}}) \) belongs to \( \text{Ext}^1_{O_{X_{\text{red}}}}(K, F) \) for any \( \mathcal{F} \) as above (because \( \pi(\sigma_{\mathcal{F}}) = g \otimes \text{id}_N \)). Moreover, by Fact 1.3(ii) it holds that \( p(\sigma_{\mathcal{F}}) = \sigma'_{\mathcal{F}} \).

Hence, there exists an \( \mathcal{F} \) such that \( \sigma'_{\mathcal{F}} = \sigma_E \), by the surjectivity of the first vertical arrow of the commutative diagram (this surjectivity can be easily checked looking at the long exact sequence of \( \text{Ext} \)'s on \( X_{\text{red}} \) associated to the short exact sequence \( 0 \to K \to G \to F \otimes N^{-1} \to 0 \) and remembering that both \( F \) and \( F \otimes N^{-1} \) are locally free on \( X_{\text{red}} \) so that \( \text{Ext}^2_{O_{X_{\text{red}}}}(F \otimes N^{-1}, F) = 0 \)).

q.e.d.
Corollary 2.2. For any pair of positive integers \( r_0 > r_1 \) and any pair of integers \( (d_0, d_1) \), there exists a quasi locally free sheaf on \( X \) of complete type \( ((r_0, r_1), (d_0, d_1)) \).

Before stating the promised necessary and sufficient conditions for the existence of a semistable quasi locally free sheaf of a fixed complete type, it is useful to give two easy numerical lemmata which will be used in order to prove them. The first one is a small improvement of a simple but useful lemma by Drezet (i.e. [D3, Lemme 5.1.1], which is not stated in numerical terms but in terms of slope of sheaves).

Lemma 2.3. Let \( R_1, R_2, R_3, R_4, R_5 \) and \( R_6 \) be positive real numbers and let \( D_1, D_2, D_3, D_4, D_5 \) and \( D_6 \) be real numbers such that

\[
R_1 = R_2 + R_3, \quad R_4 = R_5 + R_6, \\
D_1 = D_2 + D_3, \quad D_4 = D_5 + D_6.
\]

Let \( \mu_i = R_i / D_i \) for \( i = 1, \ldots, 6 \). Assume that \( \mu_2 \geq \mu_3 \) (resp. \( \mu_5 \geq \mu_6 \)) and that \( \mu_6 \geq \mu_3, \mu_5 \geq \mu_2 \) and \( R_4 / R_1 \geq R_6 / R_3 \). Then it holds that \( \mu_4 \geq \mu_1 \).

If, moreover, \( \mu_2 > \mu_3 \) (resp. \( \mu_5 > \mu_6 \)) or \( \mu_6 > \mu_3 \) or \( \mu_5 > \mu_2 \), then \( \mu_4 > \mu_1 \).

Proof. The case \( \mu_2 \geq \mu_3 \) is, essentially, [D3, Lemme 5.1.1]. The proof of the case \( \mu_5 \geq \mu_6 \) is almost identical to that of the cited result, so we give only a sketch of it.

Under our hypotheses, \( \mu_4 - \mu_1 \geq (R_1 R_4)^{-1}(R_5 R_3 - R_2 R_6)(\mu_5 - \mu_6) \geq 0 \).

The last inequality is due to the fact that, in our case, \( R_4 / R_1 \geq R_6 / R_3 \) is equivalent to \( R_5 / R_2 \geq R_6 / R_3 \).

The last assertion of the statement holds because the first inequality is strict if \( \mu_6 > \mu_3 \) or \( \mu_5 > \mu_2 \) while the second is strict if \( \mu_5 > \mu_6 \). q.e.d.

Lemma 2.4. Let \( m_1 > m_2 > m_3 \) and \( m_1' > m_2' > m_3' \) be non-negative integers and let \( q_1, q_2, q_3 \) and \( q_1', q_2', q_3' \) be real numbers. Assume \( q_1 \leq q_1', q_2 \leq q_2', q_3 \leq q_3' \), \( q_1' \leq q_1 + q_2, m_1 m_3 - m_1' m_3 \leq 0 \) and \( m_2 m_1' - m_2' m_1 \leq 0 \).

Then \( w \leq w' \), where \( w = [m_3 q_1 + (m_2 - m_3) q_2 + (m_1 - m_2) q_3] / m_1 \) and \( w' = [m_3' q_1' + (m_2' - m_3') q_2' + (m_1' - m_2') q_3'] / m_1' \). If one of the inequalities in the hypotheses is strict, then \( w < w' \).

Proof. It is an easy calculation: \( w' - w = \frac{m_3' - m_3}{m_1'} q_1 - \frac{m_3}{m_1} q_1 + \frac{m_2' - m_2}{m_1'} q_2 - \frac{m_2 - m_2}{m_1} q_2 + \frac{m_1' - m_1}{m_1} q_3 - \frac{m_1 - m_1}{m_1} q_3 \geq \frac{1}{m_1 m_1'} [q_1' (m_1 m_3' - m_1' m_3) + q_2' (m_2' m_1 - m_2 m_1') + m_3 m_1'] q_3' (m_1' m_1 - m_1 m_1') + m_2 m_1] (q_1' - q_2') + (m_2' m_1 - m_2 m_1') (q_3' - q_3) \geq 0 \). If one of the inequalities in the hypotheses is strict then \( w' - w > 0 \) because, then, one of the two above inequalities has to be strict, too. q.e.d.

Now, we state and prove the promised theorem about semistability conditions:

Theorem 2.5. Let \( X \) be a ribbon such that \( \bar{g} \geq 2 \). There exists a semistable quasi locally free sheaf \( \mathcal{F} \) on \( X \) of complete type \( ((r_0, r_1), (d_0, d_1)) \), with \( r_0 > r_1 > 0 \), if and only if
Indeed, if $F$ is semistable, then $\mu(F^{(1)}) \leq \mu(F) \leq \mu(F|_{X_{\text{red}}})$ and this inequalities are equivalent to (2.1), because $\mu(F^{(1)}) = (d_0 - a\delta)/r_0$, while $\mu(F|_{X_{\text{red}}}) = d_0/r_0$ and $\mu(F) = (d_0 + d_1)/(r_0 + r_1)$, by definition. In the stable case both the inequalities are strict.

In order to prove the sufficiency part we want to make use of Proposition 2.1. So, we need to find an appropriate exact sequence $0 \rightarrow F \rightarrow E \rightarrow \mathcal{G} \rightarrow 0$ of vector bundles on $X_{\text{red}}$, with $F$ of rank $r_1$ and degree $d_1$ and $\mathcal{G}$ of rank $r_0$ and degree $d_0$, such that an associated quasi locally free sheaf $\mathcal{F}$ on $X$ is (semi)stable (recall that $F = N|_X$, $E = F^{(1)}$ and $\mathcal{G} = F|_{X_{\text{red}}}$).

We can always work with a stable vector bundle $F$ and we can also assume that $\mathcal{K} = \ker(g) = \operatorname{coker}(f)$ is a stable vector bundle of rank $r_0 - r_1$ and degree $d_0 - d_1 - r_1\delta$.

It is useful to distinguish the three following cases:

(i) $d_0/r_0 - \delta < d_1/r_1 < (d_0 - r_1\delta)/r_0$; in this case both $E$ and $\mathcal{G}$ can be stable, because the right inequality is $\mu(E) < \mu(\mathcal{G})$ and the left inequality is equivalent to $\mu(\mathcal{G}) < \mu(F \otimes \mathcal{N}^{-1})$;

(ii) $(d_0 - r_1\delta)/r_0 \leq d_1/r_1 \leq d_0/r_0$; this time only $\mathcal{G}$ can be stable while $E$ is surely unstable and can be strictly semistable only if the right inequality is an equality.

(iii) $[d_0 - (r_0 + r_1)\delta]/r_0 \leq d_1/r_1 \leq d_0/r_0 - \delta$; in this case only $E$ can be stable while $\mathcal{G}$ is surely unstable and can be strictly semistable only if the right inequality holds as an equality.

If $\mathcal{F}$ verifies the hypotheses of (i), then its dual $\mathcal{F}^\vee$ verifies that of (iii) hence, by Remark 1.22(ii), it is sufficient to handle only one of the two cases.

Let us start with case (i). In this case, the numerical data allow to assume that both $\mathcal{G} = F|_{X_{\text{red}}}$ and $E = F^{(1)}$ are stable and this is really possible thanks to Lange’s conjecture.

Let $\mathcal{G} \subset \mathcal{F}$ be a saturated subsheaf. If $\mathcal{G} \subset F^{(1)}$ or $F|_{X_{\text{red}}} \rightarrow (\mathcal{F}/\mathcal{G})$, we have done by hypothesis. So, assume that nor $\mathcal{G}$ neither $\mathcal{F}/\mathcal{G}$ are defined over $X_{\text{red}}$.

In this case, we have, by Fact 1.3(iv)(b) that $0 \subset \mathcal{G}^{(1)} \subset \mathcal{F}^{(1)}$ and $0 \subset \mathcal{G}/\mathcal{G}^{(1)} \subset \mathcal{F}/\mathcal{F}^{(1)}$; hence, $\mu(\mathcal{G}^{(1)}) \leq \mu(\mathcal{F}^{(1)})$ and $\mu(\mathcal{G}/\mathcal{G}^{(1)}) \leq \mu(\mathcal{F}/\mathcal{F}^{(1)})$.

We can conclude that $\mu(\mathcal{G}) < \mu(\mathcal{F})$ by Lemma 2.3 if we have that $r_0 + r_1)/r_0$ is the rank of $\mathcal{G}$ and $r_1$ is the rank of $\mathcal{G}/\mathcal{G}^{(1)}$; this condition is equivalent to $r_0(\mathcal{F})r_1 \geq r_0(\mathcal{G})$.

We can cover the remaining cases looking at $\mathcal{F}/\mathcal{G}$: indeed, it holds that, by Fact 1.3(iv)(a) $\mathcal{N}|_X \rightarrow \mathcal{N}(\mathcal{F}/\mathcal{G})$ and $\mathcal{F}|_{X_{\text{red}}} \rightarrow (\mathcal{F}/\mathcal{G})|_{X_{\text{red}}}$. Moreover, under our hypothesis about $\mathcal{F}/\mathcal{G}$, $\operatorname{rk}(\mathcal{N}|_X) = r_1(\mathcal{F}/\mathcal{G}) > 0$ and $\operatorname{rk}(\mathcal{F}/\mathcal{G}|_{X_{\text{red}}}) = r_0(\mathcal{F}/\mathcal{G}) > 0$; thus, $\mu(\mathcal{N}|_X) \leq \mu(\mathcal{N}(\mathcal{F}/\mathcal{G}))$ and $\mu(\mathcal{F}|_{X_{\text{red}}}) \leq \mu((\mathcal{F}/\mathcal{G})|_{X_{\text{red}}})$. Therefore, we can conclude that $\mu(\mathcal{F}) < \mu(\mathcal{F}/\mathcal{G}) < \mu(\mathcal{F}/\mathcal{G})$. 

\[
\frac{d_0 - (r_0 + r_1)\delta}{r_0} \leq \frac{d_1}{r_1} \leq \frac{d_0}{r_0}, 
\] where, as usual, $\delta = -\deg(\mathcal{N})$. 

Proof. The necessity is quite trivial, for both semistability and stability. Indeed, if $\mathcal{F}$ is semistable, then $\mu(\mathcal{F}^{(1)}) \leq \mu(\mathcal{F}) \leq \mu(\mathcal{F}|_{X_{\text{red}}})$ and this inequalities are equivalent to (2.1), because $\mu(\mathcal{F}^{(1)}) = (d_0 - a\delta)/r_0$, while $\mu(\mathcal{F}|_{X_{\text{red}}}) = d_0/r_0$ and $\mu(\mathcal{F}) = (d_0 + d_1)/(r_0 + r_1)$, by definition. In the stable case both the inequalities are strict.
$\mu(\mathcal{F}/\mathcal{G})$, again by Lemma 2.3 if it holds that $(r_0(\mathcal{F}/\mathcal{G}) + r_1(\mathcal{F}/\mathcal{G}))/r_0 + r_1 \geq r_1(\mathcal{F}/\mathcal{G})/r_1$, equivalently if $r_1r_0(\mathcal{F}/\mathcal{G}) \geq r_0r_1(\mathcal{F}/\mathcal{G})$. The last inequality is implied by $r_1(r_0 - r_0(\mathcal{G})) \geq r_0(r_1 - r_1(\mathcal{G}))$, which is equivalent to $r_0(\mathcal{G})/r_1 \leq r_0r_1(\mathcal{G})$.

We can turn our attention to case (iii). In this case we assume $E$ stable (it is possible by Lange’s conjecture), while we choose $G = K \oplus (F \otimes N^{-1})$. Let $\mathcal{I}$ be a saturated subsheaf of $\mathcal{F}$. If $\mathcal{I} \subseteq \mathcal{F}^{(1)} = \mathcal{E}$, we have done by hypothesis (and it is possible that $\mu(\mathcal{I}) = \mu(\mathcal{F})$ only if $\mathcal{I} = \mathcal{E}$ and $\mu(\mathcal{E}) = \mu(\mathcal{F})$, i.e. if $[d_0 - (r_0 + r_1)]/r_0 = d_1/r_1$. If $\mathcal{F}/\mathcal{G}$ is defined over $X_{\text{red}}$, we have also done, because in this case $\mu(\mathcal{F}/\mathcal{G}) \geq \mu(\mathcal{F}/\mathcal{F}^{(1)}) = \mu(F \otimes N^{-1}) \geq \mu(\mathcal{F})$ (this time, the equalities are equivalent to $\mathcal{F}/\mathcal{G} = \mathcal{F}/\mathcal{F}^{(1)}$ and $[d_0 - (r_0 + r_1)]/r_0 = d_1/r_1$); the first inequality is due to the fact $\mathcal{G} = \mathcal{F}/\mathcal{F}^{(1)} \oplus K$, with both the addends stable and $\mu(\mathcal{F}/\mathcal{F}^{(1)}) \leq \mu(K)$ (with the equality if and only if $d_1/r_1 = d_0/r_0 - \delta$).

Therefore, the only case that remains to be handled is that of $\mathcal{I} \subset \mathcal{F}^{(1)}$ such that both $\mathcal{I}$ and $\mathcal{F}/\mathcal{I}$ are not supported over $X_{\text{red}}$. In this case, by Fact 1.3(iv)(b) 0 $\notin \mathcal{G}^{(1)} \subset \mathcal{F}^{(1)}$ and 0 $\notin \mathcal{G}/\mathcal{G}^{(1)} \subset \mathcal{F}/\mathcal{F}^{(1)}$; so, by the stability of $\mathcal{F}^{(1)}$ and of $\mathcal{F}/\mathcal{F}^{(1)}$, it holds that $\mu(\mathcal{G}^{(1)}) \leq \mu(\mathcal{F}^{(1)})$ and $\mu(\mathcal{G}/\mathcal{G}^{(1)}) \leq \mu(\mathcal{F}/\mathcal{F}^{(1)})$ (with the equalities if and only if the sheaves are equal, and this cannot happen for both the sheaves at the same time). As above, we can conclude that $\mu(\mathcal{G}) < \mu(\mathcal{F})$ by Lemma 2.3 if we have $(r_0 + r_1)/(r_0(\mathcal{G}) + r_1(\mathcal{G})) \geq r_0/r_0(\mathcal{G})$ or, equivalently, $r_0(\mathcal{G})/r_1 \geq r_0r_1(\mathcal{G})$. Thus, only the case in which $0 < r_0(\mathcal{G})/r_1 < r_0r_1(\mathcal{G})$ remains open.

The following diagram is commutative:

$$
\begin{array}{ccc}
N\mathcal{F} & \longrightarrow & \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(1)}/N\mathcal{G} \\
\downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \mathcal{F} \\
N \mathcal{F}^{(1)} & \longrightarrow & \mathcal{F}^{(1)} \longrightarrow \mathcal{F}^{(1)}/N \mathcal{F}^{(1)}
\end{array}
$$

It implies, by snake’s lemma, that $\mathcal{I} := \ker(\mathcal{F}) \subset (N\mathcal{F})/(N\mathcal{G})$ and $\mathcal{H} := \im(\mathcal{F}) \subset (\mathcal{F}^{(1)})/(N \mathcal{F})$. Let $\mathcal{J} := \ker(N \mathcal{F} \rightarrow (N \mathcal{F})/(N \mathcal{G})/\mathcal{I})$. It holds that $\rk(\mathcal{J}) = \rk(\ker(\mathcal{G}^{(1)} \rightarrow \mathcal{H}))$ and $\deg(\mathcal{J}) = \deg(\ker(\mathcal{G}^{(1)} \rightarrow \mathcal{H}))$. Hence, we have that

$$
\mu(\mathcal{G}) = \frac{r_0(\mathcal{G})}{r_0(\mathcal{G})} \left( \frac{r(\mathcal{J})}{r_0(\mathcal{G})} \mu(\mathcal{J}) + \frac{\rk(\mathcal{H})}{r_0(\mathcal{G})} \mu(\mathcal{H}) \right) + \frac{r_1(\mathcal{G})}{r_0(\mathcal{G})} \mu(\mathcal{G}/\mathcal{G}^{(1)}).
$$

By the stability of $\mathcal{F} = N\mathcal{F}$ and of $\mathcal{K}$, it holds that $\mu(\mathcal{J}) \leq \mu(\mathcal{F})$, $\mu(\mathcal{H}) \leq \mu(\mathcal{K})$ and $\mu(\mathcal{G}/\mathcal{G}^{(1)}) \leq \mu(\mathcal{F} \otimes N^{-1})$. We have also that $\mu(\mathcal{K}) \geq \mu(\mathcal{F} \otimes N^{-1})$ (by hypothesis of case (iii)) and, so, $\mu(\mathcal{K}) \geq \mu(\mathcal{F})$, too. Moreover, we are under the condition $0 < r_0(\mathcal{G})/r_1 < r_0r_1(\mathcal{G})$, which implies that $R(\mathcal{G})r_1 - R(\mathcal{F}) r_0(\mathcal{F}) < 0$ (because $\rk(\mathcal{J}) \geq r_1(\mathcal{G})$) and $R(\mathcal{F}) r_0(\mathcal{G}) - R(\mathcal{G})r_0 < 0$. Therefore, we can conclude that $\mu(\mathcal{G}) < \mu(\mathcal{F})$ by Lemma 2.1 q.e.d.

The following corollary is a straightforward consequence of the Theorem, improving Fact 1.2(iv) (the notation is the same there used):

**Corollary 2.6.** Assume $\bar{g} \geq 2$. The locus $N(a, d_0, d_1)$ is non-empty if and only if $(d_0 - (2a + 1)\delta)/(a + 1) < d_1/a < d_0/(a + 1)$. 
Remark 2.7.

(i) In order to avoid any possibility of misunderstanding, it is better to point out explicitly that the Theorem does not mean that any quasi locally free sheaf of a complete type verifying the inequalities (2.1) is (semi)stable. It is extremely easy to find counterexamples, e.g. using split sheaves. The statement is just that there exist a quasi locally free sheaf of that complete type which is (semi)stable. This implies that a generic (in some suitable sense) quasi locally free sheaf of that complete type is (semi)stable.

(ii) As pointed out also in Remark 1.24, the hypothesis $\bar{g} \geq 2$ is due to the use of Lange’s conjecture, which holds for these genera. For the elliptic case (i.e. when the reduced subcurve is elliptic), it can be replaced looking at short exact sequences of indecomposable vector bundles (recall that on smooth elliptic curves indecomposable is equivalent to semistable and that the indecomposable vector bundles are completely classified, see, e.g., [1]), at least in the external cases (i.e. cases (ii) and (iii) in the proof of the Theorem), in which we need only one short exact sequence of semistable vector bundles, whose existence is guaranteed by [BR, Theorem 0.1]. But also in the elliptic case, the existence of one such exact sequence should be sufficient to conclude that for generic semistable bundles the generic extension is semistable, then it can be used also for the central case (i.e. case (i) in the proof of the Theorem); hence, the Theorem (and, hence, the Corollary) should hold also in the elliptic case, with some small modifies in the proof.

For the rational case, i.e. when $X_{\text{red}}$ is a rational curve, it is well-known that there are not stable bundle of rank greater than or equal to 2 and that the only semistable bundles are polystable ones. These sheaves probably could be used to do alternative computations. I did by hand some explicit computations only in the case of generalized rank 3. I omit them, because they are quite tedious, but the result is the following: there exists a stable quasi locally free sheaf of generalized rank 3 if and only if

$$
\begin{align*}
\delta &\geq 3 \text{ and } \frac{d_0-3\delta+3}{2} < d_1 < \frac{d_0-3}{2} \text{ or } d_1 = \frac{d_0-3\delta}{2} + 1, \frac{d_0-\delta}{2} - 1; \\
\delta & = 2 \text{ and } d_1 = \frac{d_0}{2} - \delta, \frac{d_0}{2} + 1 - \delta.
\end{align*}
$$

On the other hand there exists a strictly semistable such sheaf if and only if

$$
\begin{align*}
\delta &\geq 3 \text{ and } 2d_1 = d_0 - 3\delta, d_0 - 3\delta + 3, d_0 - 3, d_0; \\
\delta & = 2 \text{ and } 2d_1 = d_1 - 2\delta - 2, d_0 - 2\delta + 1, d_0 - 2\delta + 4; \\
\delta & = 1 \text{ and } 2d_1 = d_0 - 2\delta - 1, d_0 - 2\delta + 2; \\
\delta & = 0 \text{ and } 2d_1 = d_0 - 2\delta.
\end{align*}
$$

The next result is a computation of the dimension of the locus of quasi locally free sheaves of fixed complete type (for $\bar{g} \geq 2$). Observe that it generalizes the dimensional part of Fact 1.24(i).
**Theorem 2.8.** Let $X$ be a ribbon such that $\delta = -\deg(\mathcal{N}) > 0$ and $\bar{g} \geq 2$ and let $((r_0, r_1), (d_0, d_1))$ be integers verifying the hypotheses of Theorem 2.7, with the inequalities strict. The locus of semistable quasi locally free sheaves on $X$ of complete type $((r_0, r_1), (d_0, d_1))$ has dimension $1 + (r_0^2 + r_1^2)(\bar{g} - 1) + r_0r_1\delta$.

**Proof.** First of all, observe that we can restrict our attention to the range in which $\mathcal{K}$ can be stable, because the other cases are covered by duality. So we can assume $[d_0 - (r_0 + r_1)\delta]/r_0 < d_1/r_1 < (d_0 - r_1\delta)/r_0$.

Observe also that there are not conditions about $\mathcal{F} = \mathcal{N}\mathcal{F}$ and about $\mathcal{K} = \mathcal{F}(1)/\mathcal{N}\mathcal{F}$; so, we can start with these two vector bundles over $X_{\text{red}}$ generic. They give rise to $r_0^2(\bar{g} - 1) + 1$ and $(r_0 - r_1)^2(\bar{g} - 1) + 1$ moduli, respectively. Then, we have to compute how many vector bundles over $X_{\text{red}}$ are extensions of $\mathcal{K}$ by $\mathcal{F}$ and then look at the extensions over $X$ of these vector bundles by $\mathcal{F} \otimes \mathcal{N}^{-1}$.

The possible $\mathcal{F}(1)$'s have $\text{ext}^1_{X_{\text{red}}}(\mathcal{K}, \mathcal{F})$ moduli; so, we have to compute: $\text{ext}^1_{X_{\text{red}}}(\mathcal{K}, \mathcal{F}) = h^1(K^* \otimes \mathcal{F}) = -\deg(K^* \otimes \mathcal{F}) + h^0(K^* \otimes \mathcal{F}) + r_1(r_0 - r_1)(\bar{g} - 1) = (r_0d_1 + r_1(d_0 - d_1 - r_1\delta) + r_1(r_0 - r_1)(\bar{g} - 1) = -r_0d_1 + r_1d_0 - r_0^2\delta + r_1(r_0 - r_1)(\bar{g} - 1)$; observe that the $h^0$ vanishes because $d_1/r_1 < (d_0 - r_1\delta)/r_0$.

Now, it seems that it remains to compute $\text{ext}^1(\mathcal{F} \otimes \mathcal{N}^{-1}, \mathcal{F}(1))$, which is equal to $\text{ext}^1_{X_{\text{red}}}(\mathcal{F} \otimes \mathcal{N}^{-1}, \mathcal{F}(1)) + \text{hom}(\mathcal{F}, \mathcal{F}(1))$ by Fact 2.3(iii). The extensions corresponding to sheaves of the desired complete type are those whose associated morphism from $\mathcal{F}$ to $\mathcal{F}(1)$ is injective, again by Fact 2.3(ii), but the endomorphism has been fixed when constructing $\mathcal{F}(1)$ as an extension of $\mathcal{K}$ by $\mathcal{F}$, apart from automorphisms of $\mathcal{F}$. So, the only remaining moduli are given by $\text{ext}^1_{X_{\text{red}}}(\mathcal{F} \otimes \mathcal{N}^{-1}, \mathcal{F}(1)) = h^1(F^* \otimes \mathcal{N} \otimes \mathcal{F}(1)) = -\deg(F^* \otimes \mathcal{N} \otimes \mathcal{F}(1)) + h^0(F^* \otimes \mathcal{N} \otimes \mathcal{F}(1) + r_0r_1(\bar{g} - 1) = r_0d_1 + r_0r_1\delta - r_1(d_0 - d_1\delta) + r_0r_1(\bar{g} - 1)$; indeed, the $h^0$ vanishes because under our hypotheses $[d_0 - (r_0 + r_1)\delta]/r_0 < d_1/r_1$, which is equivalent to $\deg(F^* \otimes \mathcal{N} \otimes \mathcal{F}(1)) < 0$.

It remains to sum up these moduli: $r_0^2(\bar{g} - 1) + 1 + (r_0 - r_1)^2(\bar{g} - 1) + 1 - r_0d_1 + r_1d_0 - r_0^2\delta + r_1(r_0 - r_1)(\bar{g} - 1) + r_0d_1 + r_0r_1\delta - r_1(d_0 - d_1\delta) + r_0r_1(\bar{g} - 1) = (r_0^2 + r_1^2)(\bar{g} - 1) + r_0r_1\delta + 1$, as wanted. 

q.e.d.

**Remark 2.9.**

(i) The loci studied in the previous Proposition are irreducible by [D2, Théorème 6.8]. Moreover, we expect that they are irreducible components for $0 < \delta < 2\bar{g} - 2$ (see Section 4).

(ii) It follows from [D2, Proposition 3.12] that, as in the case of quasi locally free sheaves of rigid type, the dimension obtained in the Theorem equals $h^1(\text{End}(\mathcal{F}))$, for any stable quasi locally free $\mathcal{F}$ of that complete type. This implies that these loci are not smooth, out of the locally free case, because the tangent space has dimension $\text{ext}^1(\mathcal{F}, \mathcal{F})$, i.e., by the Ext-spectral sequence, $h^1(\text{End}(\mathcal{F})) + h^0(\text{Ext}^1(\mathcal{F}, \mathcal{F}))$, and, if $\mathcal{F}$ is not locally free, $h^0(\text{Ext}^1(\mathcal{F}, \mathcal{F})) \neq 0$ (although I have not computed it explicitly).

We end this subsection showing that to be quasi locally free is an open condition in families of sheaves of fixed type.
Proposition 2.10. Let $Z$ be a $\mathbb{K}$-scheme and let $\mathfrak{F}$ be a family of sheaves on $X$ of fixed type $(m_1, m_2)$ parametrized by $Z$. Then the set of closed points $z \in Z$ where $\mathfrak{F}_z$ is quasi locally free is open.

Proof. Let $z_0 \in Z$ be a point where $\mathfrak{F}_{z_0}$ is quasi locally free on $X$. If such a $z_0$ exists, then the set of points $(z, P) \in Z \times \mathbb{K} X$ such that there exists a surjective morphism $\mathcal{O}_{X,P}^{\oplus m_0} \to \mathfrak{F}_{z,P}$, with $m_0 := m_1 + m_2$, is non-empty. For any such $(z, P)$ there exists a neighbourhood $U \subset Z \times \mathbb{K} X$ such that $\mathcal{O}_{U}^{\oplus m_0} \to \mathfrak{F}|_U$. Hence, there exists an open $W \subset Z \times \mathbb{K} X$ such that for any $(z, P) \in W$ there is an epimorphism $\mathcal{O}_{X_{\text{red}},P}^{\oplus m_0} \to M/(y_p M)$, where $M = \mathfrak{F}_{z,P}$ and $0 \subset y_p M \subset M$ is its first canonical filtration (the surjective morphism is induced restricting to $X_{\text{red}}$ the previous one). By the fact the family is of sheaves of fixed type, $M/(y_p M)$ has to be of the form $\mathcal{O}_{X_{\text{red}},P}^{\oplus m_0} \otimes N$, where $N$ is a torsion module. Therefore, it follows that the epimorphism is an isomorphism, i.e. that $\mathcal{O}_{X_{\text{red}},P}^{\oplus m_0} \cong M/(y_p M)$. This implies that $\mathfrak{F}_{z,P}$ is quasi free of type $(m_1, m_2)$.

If $T$ denotes the projection of $(Z \times \mathbb{K} X) \setminus W$ in $Z$, then the desired open is $Z \setminus T$. q.e.d.

Remark 2.11. The hypothesis that the sheaves in the family are of fixed type cannot be removed, at least in general (it is not necessary only in some special cases as that of quasi locally free sheaves of rigid type, cf. [D2 Proposition 6.9]). An example in which without this hypothesis the Proposition would fail is given by those families that deform rank 2 vector bundles over $X_{\text{red}}$ to generalized line bundles over $X$ (see [D1 Théorème 7.2.3] and [Sa2 Theorem 1]).

2.2. Deformations of vector bundles on $X_{\text{red}}$.

In this short subsection we give two propositions about deformations of vector bundles of arbitrary rank $r > 1$ on $X_{\text{red}}$ (in other words of pure sheaves of type $(r,0)$ on $X$) to sheaves supported on $X$. The first one, which requires $r \geq 3$ is inspired by the first part of [D1 Théorème 7.2.3], which is about rank 2 vector bundles on $X_{\text{red}}$.

Proposition 2.12. Let $\mathcal{E}$ be a vector bundle of rank $r \geq 3$ on $X_{\text{red}}$. If there exists a non-trivial subsheaf $\mathcal{F} \subset \mathcal{E}$ of rank $r' < r$ such that $\text{Hom}(\mathcal{E} / \mathcal{F} \otimes \mathcal{N}, \mathcal{F}) \neq 0$, then $\mathcal{E}$ deforms to pure sheaves having schematic support equal to $X$.

If, moreover, the generic element of this homomorphism group has maximal rank, i.e. $\min\{r', r - r'\}$, then $\mathcal{E}$ deforms to pure sheaves of type $(|r - 2r'|, \min\{r', r - r'\})$.

Proof. We can restrict our attention to the case in which $\mathcal{F}$ is a saturated subsheaf of $\mathcal{E}$, because if $\mathcal{F}$ is not saturated and $\mathcal{F}_{\text{sat}}$ is its saturation, then $\text{Hom}(\mathcal{E} / \mathcal{F} \otimes \mathcal{N}, \mathcal{F}) \neq 0$ implies that $\text{Hom}(\mathcal{E} / \mathcal{F}_{\text{sat}} \otimes \mathcal{N}, \mathcal{F}_{\text{sat}}) \neq 0$.

In the saturated case, both the assertions are trivial consequences of Fact [L3(iii)] indeed, the latter implies that the generic element of the universal family of extensions of $\mathcal{E} / \mathcal{F}$ by $\mathcal{F}$ is defined over $X$ and that it is of the asserted type if the generic element in $\text{Hom}(\mathcal{E} / \mathcal{F} \otimes \mathcal{N}, \mathcal{F})$ has maximal rank. q.e.d.
The following proposition is an extension of [Sa2 Theorem 1] from rank 2 to arbitrary rank, although it is less precise.

**Proposition 2.13.** Let $X$ be a ribbon such that $\delta = -\deg(\mathcal{N}) > 2\bar{g} - 2$ and $\bar{g} \geq 2$. Any vector bundle of rank $r \geq 2$ and degree $d$ on $X_{\text{red}}$ deforms to pure sheaves over $X$ of type $(r - 2, 1)$ (hence, of generalized rank $r$) and generalized degree $d$, with the possible exception of the case in which $\delta = 2\bar{g} - 1$, $r = 3$ and $3$ divides both $d$ and $\bar{g}$.

**Proof.** The case $r = 2$ is [Sa2 Theorem 1].

The point is to show that any vector bundle $\mathcal{E}$ of rank $r \geq 2$ and degree $d$ as in the statement verifies the hypothesis of Proposition 2.12 for $r' = n - 1$ or for $r' = 1$ (these $r'$'s are due to the hypothesis about the type).

Throughout the proof, we will denote by $s_{r'}$ the $r'$-Segre invariant of $\mathcal{E}$, i.e. the number $r'd - r \text{ max}\{\deg(\mathcal{E}')| \text{rk}(\mathcal{E}') = r', \mathcal{E}' \subset \mathcal{E}\}$, for any $0 < r' < r$.

For any $\mathcal{F}$ saturated subbundle of $e\mathcal{E}$, it holds that $\text{Hom}((\mathcal{E}/\mathcal{F}) \otimes \mathcal{N}, \mathcal{F}) \neq 0$ if and only if $h^0((\mathcal{E}/\mathcal{F})^* \otimes \mathcal{N}^{-1} \otimes \mathcal{F}) > 0$. Let $\mathcal{F} \subset \mathcal{E}$ be a subbundle of rank $r'$ of maximal degree. In this case, $\text{deg}((\mathcal{E}/\mathcal{F})^* \otimes \mathcal{N}^{-1} \otimes \mathcal{F}) = r'(r - r')\delta - s_{r'}$.

Therefore, $h^0((\mathcal{E}/\mathcal{F})^* \otimes \mathcal{N}^{-1} \otimes \mathcal{F}) = r'(r - r')\delta - s_{r'} + h^1((\mathcal{E}/\mathcal{F})^* \otimes \mathcal{N}^{-1} \otimes \mathcal{F}) - r'(r - r')(\bar{g} - 1)$. By the basic properties of the Segre invariants, the right hand term is always positive for any $r'$ if $\delta \geq 2\bar{g} + r - 1$. But we need that the right hand term is positive only in the case $r' = r - 1$. This is the case if $\delta \geq 2\bar{g}$ or if $\delta = 2\bar{g} - 1$ and $d$ is not congruent to $\bar{g}$ modulo $r$. Also the case $\delta = 2\bar{g} - 1$, $r' = 2$ and $r \geq 4$ (and also $r' = r - i$ and $r \geq i + 2$ for any $2 \leq i \leq r - 2$) follows from an almost trivial calculation.

Only the case in which $\delta = 2\bar{g} - 1$, $r = 3$ and $3$ divides both $d$ and $\bar{g}$ remains open. In this case, for $\mathcal{E}$ generic, one obtains, for both $r' = 1$ and $2$, that $(\mathcal{E}/\mathcal{F})^* \otimes \mathcal{N}^{-1} \otimes \mathcal{F}$ is a stable rank 2 vector bundle of degree $2(\bar{g} - 1)$ and that there is a 2-dimensional family of $\mathcal{F}$ of maximal degree (by [RT Theorem 0.2]). If one were able to show that the bundles $(\mathcal{E}/\mathcal{F})^* \otimes \mathcal{N}^{-1} \otimes \mathcal{F}$ are distinct for different $\mathcal{F}$’s, one could conclude by [Si Theorem III.2.4], which asserts, in particular, that the Brill-Noether locus of stable rank 2 vector bundles of degree $2(\bar{g} - 1)$ with at least one global section is a divisor in the moduli space of stable vector bundles of rank 2 and degree $2(\bar{g} - 1)$. q.e.d.

### 3. Non quasi locally free sheaves

In this section we study some properties of two special kinds of pure sheaves with positive index. It is divided into two subsections, a quite short one about generalized vector bundles and a second one about sheaves of type $(n, 1)$, with $n$ a non-negative integer.

#### 3.1. Generalized vector bundles

The first result of this short subsection about generalized vector bundles is the following semistability condition:

**Proposition 3.1.** Let $X$ be a ribbon, let $\delta = -\deg(\mathcal{N})$, let $b$ be a non-negative integer and let $r$ be a positive integer. There exists a semistable (resp.stable) generalized vector bundle $\mathcal{F}$ of generalized rank $2r$ and of index $b$ if and only if $b \leq r\delta$ (resp. $b < r\delta$).
Proof. The necessity is trivial: indeed, the inequality \( b \leq r\delta \) (resp. \(<\)) is equivalent to \( \mu(\mathcal{F}^{(1)}) \leq \mu(\mathcal{F}/\mathcal{F}^{(1)}) \) (resp. \(<\)).

Also the sufficiency can be easily checked directly, but it is an immediate consequence of Fact 1.18 and of [DB Théorème 5.4.2]. q.e.d.

Remark 3.2. In the case of generalized line bundles, i.e. of \( r = 1 \), it holds something more. Indeed, any generalized line bundle on a ribbon of index \( b \) such that \( b \leq \delta \) is semistable and it is stable if and only if the inequality is strict (see [CK Lemma 3.2]).

The following proposition is a computation of the dimension of the locus of semistable generalized vector bundles in the Simpson moduli space.

**Proposition 3.3.** Let \( X \) be a ribbon and let, as usual, \( \delta = -\deg(\mathcal{N}) \). Assume also \( \delta > 0 \). The locus of stable generalized vector bundles of generalized rank \( 2r \) and of fixed index less than \( r\delta \) has dimension \( 1 + 2r^2(\bar{g} - 1) + r^2\delta \), where as usual \( \bar{g} \) is the genus of \( X_{\text{red}} \).

**Proof.** It is a trivial consequence of [I Proposition 2.1 and Remark 2.7(i)]. q.e.d.

**Remark 3.4.** The above Proposition includes, as special cases, the dimensional parts of Fact 1.23(ii), about vector bundles, i.e. generalized vector bundles with index 0, and of [CK Theorem 4.6], about irreducible components of generalized line bundles.

3.2. **Sheaves of type** \((n, 1)\).

In this subsection we turn our attention to pure sheaves of type \((n, 1)\), for a non-negative integer \( n \). They are interesting because they are the pushforward of quasi locally free sheaves on a blow up of \( X \) (this assertion is made precise in the following theorem) and this fact allows to derive easily many of their properties from those of quasi locally free sheaves.

**Theorem 3.5.** Let \( n \) be a non-negative integer and let \( \mathcal{F} \) be a pure sheaf on \( X \) of type \((n, 1)\), i.e. generically isomorphic to \( \mathcal{O}_X \oplus \mathcal{O}_{X_{\text{red}}}^{\oplus n} \), and let \( \mathcal{T} \) be the torsion part of \( \mathcal{F}|_{X_{\text{red}}} \). There is a unique divisor \( D \subset X_{\text{red}} \) such that \( \mathcal{T} \) is isomorphic to \( \mathcal{O}_D \) and a unique quasi locally free sheaf \( \mathcal{F}' \) of type \((n, 1)\), i.e. locally isomorphic to \( \mathcal{O}_{X'} \oplus \mathcal{O}_{X_{\text{red}}}^{\oplus n} \), on the blow up \( q : X' \to X \) of \( X \) at \( D \) such that \( q_*\mathcal{F}' \simeq \mathcal{F} \).

This theorem generalizes [EG Theorem 1.1] which deals with generalized line bundles (i.e. with the case \( n = 0 \)).

**Proof.** The proof is essentially the same of the cited place. Indeed, the key point of that proof is that \( \mathcal{N}\mathcal{F} \) and \( \mathcal{K} = \ker(\mathcal{F} \to \mathcal{F}) \), where \( \mathcal{F} \) is the locally free part of \( \mathcal{F}|_{X_{\text{red}}} \), are line bundles on \( X_{\text{red}} \) (such that \( \mathcal{N}\mathcal{F} \subset \mathcal{K} \)), which implies that \( \mathcal{K}/\mathcal{N}\mathcal{F} \), isomorphic to \( \mathcal{T} \) by snake’s lemma, can be written as \( \mathcal{O}_D \) for a unique effective divisor \( D \) of \( X_{\text{red}} \).

The fact that \( \mathcal{N}\mathcal{F} \) and \( \mathcal{K} \) are line bundles on \( X_{\text{red}} \) is trivial: they are pure because are subsheaves of \( \mathcal{F} \) and they have generalized rank 1 by additivity of the generalized rank (see Fact 1.14(ii)); hence, e.g. by Fact 1.14(iii), they are pure sheaves of rank 1 on \( X_{\text{red}} \), i.e. line bundles on it.

At this point the proof is verbatim the same of [EG Theorem 1.1]: it is possible to give to \( \mathcal{F} \) a structure of \( \mathcal{O}_{X'} \)-module (which is unique because it
is derived only from the $\mathcal{O}_X$-module structure of $\mathcal{F}$) and, writing $\mathcal{F}'$ for $\mathcal{F}$ with this structure, it is clear that $q_*\mathcal{F}' \simeq \mathcal{F}$. Also the uniqueness of the divisor follows as there. Let us recall how to define such a structure.

Let $f \in H^0(\mathcal{O}_{X_{\text{red}}}(D))$ be a section vanishing on $D$, let $\sigma'$ be a section of $\mathcal{O}_X$ defined over an open set $U$ of $X$ (recall that $X$ and $X'$ are homeomorphic) and $m$ a section of $\mathcal{F}(U)$. Shrinking $U$, if necessary, it is possible to find a section $\sigma$ of $\mathcal{O}_X(U)$ with the same image of $\sigma'$ in $\mathcal{O}_{X_{\text{red}}}(U)$. Hence, $\sigma' = \sigma + f^{-1}\tau$, where $\tau$ is an appropriate section of $\mathcal{N}(U)$. The sheaf $\mathcal{F}$ admits a structure of $\mathcal{O}_X$-module if we can define $\sigma'm$ as $\sigma m + f^{-1}(\tau m)$; the latter is well defined because $\tau m \in \mathcal{N}/\mathcal{F}$ and $\mathcal{K} = \mathcal{O}_{X_{\text{red}}}((\mathcal{F}) \otimes \mathcal{N}/\mathcal{F}$. It is possible to verify that this definition is independent of the choice of $\sigma$.

A similar result cannot hold for any pure sheaf on $X$: e.g., the blow up $q : X' \to X$ associated to $\mathcal{F} \oplus \mathcal{O}_X$, where $\mathcal{F}$ is a generalized line bundle with positive index, is the same associated to $\mathcal{F}$ and it is impossible to find a quasi locally free sheaf on $X'$ such that $\mathcal{F} \oplus \mathcal{O}_X$ is its direct image via $q$. It is easy to see it looking at the local descriptions: if $P$ is a closed point where $\mathcal{F}$ is not free, $\mathcal{F}_P \cong (x^b, y)$, where $b$ is the index of $\mathcal{F}$ in $P$, $y$ is a generator of the nilradical of $A = \mathcal{O}_{X,P}$ and $x$ is a nonzerodivisor whose image in $\mathcal{O}_{X_{\text{red}},P}$ is a generator of the maximal ideal, while $\mathcal{F}'_{X',P} = A[y/x^b] = A'$. The module $\mathcal{F}_P \oplus A$ is the direct image of a module on $A'$ if it is closed under multiplication by $y/x^b$ (indeed $A'$ and $A$ have the same ring of fractions) but this is impossible, e.g., for the element $(y, 1)$ that is mapped to $(0, y/x^b)$ which does not belong to $\mathcal{F}_P \oplus A$.

The sheaves involved in the above Theorem have a quite nice behaviour with respect to semistability. Indeed, we have the following result which generalizes [D1, Lemme 9.1.2], which deals only with quasi locally free sheaves of generalized rank 3.

**Proposition 3.6.** Let $n$ be a positive integer and let $\mathcal{F}$ be a pure sheaf on $X$ of type $(n, 1)$. Then $\mathcal{F}$ is semistable if and only if the following two conditions are verified:

(i) for any subbundle $\mathcal{E} \subseteq \mathcal{F}^{(1)}$ of rank less than $n$ we have $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$;

(ii) for any pure quotient $(\mathcal{F}_{X_{\text{red}}})^{\vee \vee} \to \mathcal{G}$ of rank less than $n$ it holds that $\mu(\mathcal{G}) \geq \mu(\mathcal{F})$.

Furthermore, $\mathcal{F}$ is stable if and only if the inequalities in (i) and (ii) are strict.

**Proof.** Necessity is obvious, we have to prove only sufficiency.

Throughout the proof we will denote $(\mathcal{F}_{X_{\text{red}}})^{\vee \vee}$ by $\mathcal{F}$.

We will prove only the semistable case, because the stable one is essentially identical.

Let $\mathcal{E} \subseteq \mathcal{F}$ be a saturated subsheaf. If $\mathcal{E}$ is defined over $X_{\text{red}}$, then $\mathcal{E} \subseteq \mathcal{F}^{(1)}$. If it has rank $\leq n$, then $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$ by (i). On the other hand, if it has rank $n + 1$, it has the same rank of $\mathcal{F}^{(1)}$ and it is contained in it. Hence, $\mu(\mathcal{E}) \leq \mu(\mathcal{F}^{(1)})$ and it suffices to check that $\mu(\mathcal{F}^{(1)}) \leq \mu(\mathcal{F})$. This follows from condition (ii) because $\mathcal{F}/\mathcal{F}^{(1)}$ is a pure quotient of $\mathcal{F}$ of rank 1 and, thus, $\mu(\mathcal{F}/\mathcal{F}^{(1)}) \geq \mu(\mathcal{F})$. q.e.d.
So, assume that $\mathcal{E}$ is not defined over $X_{\text{red}}$; this means that $\mathcal{E}$ is generically isomorphic to $O_X \oplus O_{X_{\text{red}}}^{\oplus m}$ with $0 \leq m < n$. Hence, $\mathcal{F}/\mathcal{E}$ is generically isomorphic to $O_{X_{\text{red}}}^{\oplus (n-m)}$. Furthermore, being pure ($\mathcal{E}$ is saturated), this quotient is a rank $n - m$ vector bundle on $X_{\text{red}}$. Thus, $\mathcal{F}/\mathcal{E}$ is a pure quotient of $\mathcal{F}$ of rank $\leq n$ and so, by (iii) $\mu(\mathcal{F}/\mathcal{E}) \geq \mu(\mathcal{F})$, which is equivalent to $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$.

**Remark 3.7.**

(i) The case $n = 0$, i.e. that of generalized line bundles, is not covered by the Proposition. In order to cover also their case, one should drop the hypothesis of rank $\leq n$ in the two conditions. Indeed if $\mathcal{I}$ is a generalized line bundle, it holds that $(\mathcal{I}|_{X_{\text{red}}})^{\vee\vee} = \mathcal{I}/\mathcal{I}^{(1)}$ and the two conditions (without the cited hypothesis) are both equivalent to $b(\mathcal{I}) \leq -\deg(\mathcal{N})$, which is equivalent to the semistability of $\mathcal{I}$ (see [CK, Lemma 3.2]).

(ii) The hypothesis $\deg(\mathcal{N}) < 0$, which, as pointed out in Remark 1.22(iv), is necessary for the existence of stable sheaves not defined over $X_{\text{red}}$, does not appear in the statement of the Proposition because it would be redundant. Indeed, it follows from the two conditions and from the observation that they cover the two sheaves used in the cited remark: $\ker(\mathcal{F} \to (\mathcal{F}|_{X_{\text{red}}})^{\vee\vee})$ is a line subbundle of $\mathcal{F}^{(1)}$ while $\mathcal{F}/\mathcal{F}^{(1)} = \mathcal{N} / \mathcal{N}^{-1}$ is a pure quotient of $(\mathcal{F}|_{X_{\text{red}}})^{\vee\vee}$ of rank 1.

**Corollary 3.8.** Let $\mathcal{F}$ be a pure sheaf of type $(n,1)$ with $n$ a non-negative integer, let $q : X' \to X$ be the blow up of $X$ with respect to the divisor associated to the torsion part of $\mathcal{F}|_{X_{\text{red}}}$ and let $\mathcal{F}'$ be the quasi locally free sheaf on $X'$ such that $q_* (\mathcal{F}') = \mathcal{F}$ (see Theorem 3.7). Then $\mathcal{F}$ is (semi)stable if and only if $\mathcal{F}'$ is (semi)stable.

**Proof.** It holds by definition that $\text{Deg}(\mathcal{F}) = \text{Deg}(\mathcal{F}')$ and $R(\mathcal{F}) = R(\mathcal{F}')$; hence, $\mu(\mathcal{F}) = \mu(\mathcal{F}')$. The construction of $\mathcal{F}'$ implies that $\mathcal{F}^{(1)} = \mathcal{F}'^{(1)}$ and $(\mathcal{F}|_{X_{\text{red}}})^{\vee\vee} = \mathcal{F}'|_{X_{\text{red}}}$. Therefore, the assertion follows from the Proposition for $n$ positive and from Remark 3.7(iv) for $n = 0$.

Let $n$ be a positive integer, $b$ a non-negative one and $d_0$ and $d_1$ two integers and let $L(n,b,d_0,d_1) \subset M_s(X, (n+2, d_0 + d_1)$ be the locus of stable sheaves $\mathcal{F}$ of complete type $((n+1,1), (d_0,d_1))$ with index $b$.

The above Corollary allows to describe $L(n,b,d_0,d_1)$ in terms of loci of quasi locally free sheaves over appropriate blow ups.

More precisely, set $S_b := \text{Sym}^b X_{\text{red}}$ (which is, as well-known, isomorphic to the Hilbert scheme of zero dimensional subschemes of $X_{\text{red}}$ of length $b$) and let $\mathcal{D}$ be the tautological divisor of $X_{\text{red}} \times S_b$. By the fact that $\mathcal{D}$ is also a subscheme of $X \times S_b$, we can consider $\rho : \mathcal{X} \to X \times S_b$, the blow up of $X \times S_b$ along $\mathcal{D}$.

It is clear that $\mathcal{X}$ can be seen as an $S_b$-scheme. Furthermore, for any closed point $s \in S_b$ corresponding to an effective divisor $D$ of $X_{\text{red}}$ of length $b$, the fibre $\mathcal{X}_s$ is isomorphic to the blow up $X'_D$ of $X$ along $D$.

We can consider the relative moduli space of semistable sheaves of fixed Hilbert polynomial of $\mathcal{X}/S_b$. By the properties of this moduli space (see,
e.g., [HL, Theorem 4.3.7]), its fibre at any closed point \( s \in S_b \) is isomorphic to the moduli space of semistable sheaves of the same Hilbert polynomial on \( X' \) (where, as above, \( D \) is the divisor corresponding to \( s \)).

Combining Theorem 3.5 and Corollary 3.8 it holds that \( L(n, b, d_0, d_1) \subset M_\delta(X, n + 2, d_0 + d_1) \) is the direct image of the sublocus of the relative moduli space whose fibre in \( s \) is isomorphic to \( \text{L}(X'_D, n, 0, d_0, d_1) \). This method leads also to a decomposition of \( L(n, b, d_0, d_1) \) as the disjoint union \( \bigcup L(n,(b_1, \ldots, b_j),d_0,d_1) \), where the union is taken over all the (unordered and integral) partitions \( (b_1, \ldots, b_j) \) of \( b \) and \( L(n,(b_1, \ldots, b_j),d_0,d_1) \) is the locus parametrizing the sheaves of \( L(n,b,d_0,d_1) \) having local index sequence \( b_i = \{b_1, \ldots, b_j\} \). Indeed, in order to describe \( L(n,(b_1, \ldots, b_j),d_0,d_1) \) it is sufficient to look at the appropriate diagonal in \( S_b \).

This gives a quite precise description of \( L(n,b,d_0,d_1) \), thanks to the knowledge of \( L(X'_D, n, 0, d_0, d_1) \) (which are irreducible of dimension \( 1 + (n^2 + 2k + 2)(\bar{g} - 1) + (n + 1)(-\deg(N(X'))) \)), see Proposition 2.8 and Remark 2.9 (i).

**Theorem 3.9.** Let \( X \) be a ribbon such that \( \delta = -\deg(N) > 0 \) and \( \bar{g} \geq 2 \). Let \( n \) and \( b \) be positive integers and \( d_0 \) and \( d_1 \) integers.

The locus \( L(n,b,d_0,d_1) \) is non-empty if and only if \( b < \delta \) and \( (d_0 - b - (n + 2)\delta)/(n + 1) < d_1 < (d_0 - b)/(n + 1) \).

In this case, it is irreducible in \( M_\delta(n + 2, d_0 + d_1) \) and has dimension \( 1 + (n^2 + 2k + 2)(\bar{g} - 1) + (n + 1)(\delta - b) \).

Under the same hypotheses, for any (unordered) partition \( b_1, \ldots, b_j \) of \( b \) (with all the \( b_i \) positive integers) \( L(n,(b_1, \ldots, b_j),d_0,d_1) \) is non-empty and irreducible of dimension \( 1 + (n^2 + 2k + 2)(\bar{g} - 1) + (n + 1)(\delta - b) + j \).

**Proof.** The assertion follows from the above discussion together with Proposition 2.9 and Remark 2.9 and from the easy observation that if \( N' \) is the nilradical of \( \mathcal{O}_{X'} \), where \( q:X' \rightarrow X \) is the blow up of \( X \) along an effective divisor \( D \) of \( X_{\text{red}} \) of length \( b \), then \( \deg(N') = \deg(N) + b \).

q.e.d.

**Remark 3.10.**

(i) The dimension of \( L(n,b,d_0,d_1) \), for \( b > 0 \), is strictly less than that of the locus of quasi locally free sheaves of the same complete type.

(ii) This method can be applied to generalized line bundles, using the relative Picard scheme instead of the relative moduli space, in order to give an alternative demonstration of [CK, Lemma 4.4 and Theorem 4.6] (the only difference is that in the case of generalized rank greater than or equal to 3 there are not conditions about the parity of the index).

4. CONJECTURES AND OPEN PROBLEMS

In this section, we state and justify some conjectures about sheaves on a ribbon. They imply the conjectural description of the irreducible components of the Simpson moduli space anticipated in the introduction. We also explain the state of the art of the various conjectures.

The first one explains why generalized vector bundles and quasi locally free sheaves seem to be particularly relevant.
Conjecture 4.1. Let \( \mathcal{F} \) be a pure sheaf on \( X \) which is not a generalized vector bundle. Then \( \mathcal{F} \) generizes to a quasi locally free sheaf.

If the conjecture held, then the only kind of sheaves on a ribbon that can be generic elements of an irreducible component of the moduli space would be quasi locally free sheaves and generalized vector bundles. Another consequence is that the closure of the loci of semistable quasi locally free sheaves studied in Theorem 2.8 would be irreducible components of the moduli space of semistable sheaves over \( X \) when \( 0 < \delta \leq 2\bar{g} - 2 \) (where, as usual, \( \delta = -\deg(\mathcal{N}) \) and \( \bar{g} \) is the genus of \( X_{\text{red}} \)). This would be due to dimensional reasons and to the upper (resp. lower) semicontinuity of \( r_0 \) (resp. \( r_1 \)) (see Fact 1.3(v)). Indeed, if \( (r_0, r_1) \) and \( (s_0, s_1) \) are two pairs of non-negative integers such that \( r_0 + r_1 = s_0 + s_1 \) and \( s_1 < s_0 < r_0 \), then, using that \( 0 < \delta \leq 2\bar{g} - 2 \), \( 1 + (r_0^2 + r_1^2)(\bar{g} - 1) + r_0 r_1 \delta > 1 + (s_0^2 + s_1^2)(\bar{g} - 1) + s_0 s_1 \delta \) (cf. Theorem 2.8); hence, a locus of sheaves with \( (r_0, r_1) \) as rank-part of the complete type cannot be contained in the closure of a locus of those with \( (s_0, s_1) \) as rank-part of the complete type. On the other hand, by the above cited semicontinuity, any sheaf with \( (s_0, s_1) \) as rank-part of the complete type is not contained in the closure of a locus of sheaves with \( (r_0, r_1) \) as rank-part of the complete type.

There are two possible strategies to demonstrate the above Conjecture. The first one is the following: let \( \mathcal{F} \) be a pure sheaf on \( X \) of complete type \( ((r_0, r_1), (d_0, d_1)) \) with \( r_0 > r_1 \). If \( \mathcal{F} \) is quasi locally free, there is nothing to prove; so, in particular, we can assume \( \mathcal{F} \) of index \( b > 0 \) and then, in particular, \( r_1 > 0 \). Look at \( \mathcal{F}(1)/\mathcal{F} \); it is a sheaf defined over \( X_{\text{red}} \) with a locally free part of rank \( r_0 - r_1 > 0 \) and a torsion part of length \( b \). By Proposition 4.2 below, there is a flat family of sheaves over \( X_{\text{red}} \) with a fibre isomorphic to \( \mathcal{F}(1)/\mathcal{N} \mathcal{F} \) and the generic fibre locally free. This is the point for which it is required that \( \mathcal{F} \) is not a generalized vector bundle; indeed, for generalized vector bundles \( r_0 = r_1 \) and such a family cannot exist.

An idea for the proof would be to obtain from this family a flat family of short exact sequences of sheaves on \( X_{\text{red}} \) with a fibre isomorphic to the exact sequence \( \mathcal{N} \mathcal{F} \hookrightarrow \mathcal{F}(1) \twoheadrightarrow \mathcal{F}(1)/\mathcal{N} \mathcal{F} \) and with generic fibre such that all the terms were vector bundles on \( X_{\text{red}} \). Then we would have to get from it another flat family of short exact sequences, this time of sheaves on \( X \), with a fibre isomorphic to \( \mathcal{F}(1) \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}(1) \) and the generic fibre with central term quasi locally free.

A possible method to do that could be using the relative ext sheaf of the first family by the constant family over the same base with fibres isomorphic to \( \mathcal{N} \mathcal{F} \); if attached to this relative ext sheaf there were an universal family of extensions (it is true if this relative ext sheaf commutes with base change, see [Lan]), then the universal family would be an appropriate family of short exact sequences on \( X_{\text{red}} \). After that one should repeat a similar argument with the relative ext sheaf of the constant family with fibre isomorphic to \( \mathcal{F}/\mathcal{F}(1) \) by the previous family and then using the universal family of extensions (which exists if this second relative ext commutes with base change, see again [Lan]). Unfortunately, until now I was able neither to control these relative ext sheaves nor to check if they commute with base change.
Before explaining the second strategy, we give the previously cited result (which is probably well-known to the expert) about sheaves with torsion over $X_{\text{red}}$, or rather, on any smooth projective curve:

**Proposition 4.2.** Let $C$ be a smooth projective curve on $\mathbb{K}$ and let $\mathcal{G}$ be a sheaf of degree $d$ and rank $r > 0$ on it. There exists a flat family of sheaves on $C$ with a fibre isomorphic to $\mathcal{G}$ and generic fibre a vector bundle of the same rank.

**Proof.** Let $\mathcal{G} = \mathcal{E} \oplus \mathcal{T}$ with $\mathcal{E}$ a vector bundle of rank $r$ and $\mathcal{T}$ a torsion sheaf. Assume $\mathcal{T}$ is generated by $s$ global sections. It holds that $\mathcal{E}(m)$ is generated by global sections for any $m > 0$. Hence, $\mathcal{O}_{C}^{\oplus r+1} \oplus \mathcal{O}_{C}^{\oplus s} \to \mathcal{E}(m) \oplus \mathcal{T}$. For $m$ sufficiently large, Quot$_{r,d}^{\oplus s}(\mathcal{O}_{C}^{\oplus r+1})$, i.e. the Quot scheme parametrizing quotients of $\mathcal{O}_{C}^{\oplus r+1}$ of rank $r$ and degree $d$, is irreducible and the generic quotient is a vector bundle, by [PR, Theorems 6.2 and 6.4]. Hence, the universal quotient family of Quot$_{r,d}^{\oplus s}(\mathcal{O}_{C}^{\oplus r+1})$ twisted by $-m$ is a family with the desired properties. 

Now we explain the second strategy for Conjecture 4.1. First of all, one could try to show that the locus of pure sheaves of fixed complete type is irreducible. If this were the case, then the conjecture would follow from the fact that for any complete type with $r_0 > r_1$ there exists a quasi locally free sheaf by Corollary 2.2 and from Proposition 2.10, asserting that being quasi locally free is an open condition in families of fixed type (and sheaves of the same complete type are also of the same type).

Conjecture 4.1 is suggested by Theorems 2.8 and 3.9 too. Indeed, they imply that, at least for sheaves of type $(n,1)$ with $n > 0$, the dimension of moduli spaces of non quasi locally free pure sheaves is strictly less than the dimension of loci of quasi locally free sheaves of the same type; hence, the loci of sheaves of positive index could really be in the closure of the locus of quasi locally free sheaves of the same complete type.

Also for the second strategy of proof the hypothesis $r_0 > r_1$ is really needed there are many complete types for which vector bundles do not exist. Indeed, a vector bundle of generalized rank $2r$ and generalized degree $D$ on $X$ has complete type $((r,r),((D - r \deg(\mathcal{N}))/2, (D + r \deg(\mathcal{N}))/2) (it is a particular case of Remark 1.20(ii)). Hence, for any choice of a pair of integers $(d_0,d_1)$ such that $d_0 - d_1 \neq -r \deg(\mathcal{N})$, there is not a vector bundle of complete type $((r,r),(d_0,d_1))$, while there are generalized vector bundles of that complete type. Indeed, if $\mathcal{F}$ is a generalized vector bundle of generalized rank $2r$, generalized degree $D$ and index $b$, its complete type is $((r,r),((D - r \deg(\mathcal{N}) + b)/2, (D + r \deg(\mathcal{N}) - b)/2)$ (again, by Remark 1.20(ii)).

About generalized vector bundles, the case of generalized line bundles (for which, see [CK]) and the case of index 0 (see Fact 1.28(ii)) suggest the following conjecture:

**Conjecture 4.3.** The locus of semistable generalized vector bundles of generalized rank $2r$ and of fixed index less than $r\delta$ on a ribbon $X$ is irreducible in the moduli space.
This conjectures implies immediately, by upper semicontinuity of \( r_0 \) (see Fact 1.13(v)), that the closures of these loci are irreducible components of the Simpson moduli space.

The conjecture seems to hold also by the fact that, as recalled few lines above, the index of a generalized vector bundle determines, together with its generalized rank and generalized degree, its complete type and, as said also in the second strategy for Conjecture 4.1, we think that the locus of fixed complete type should be irreducible. We have not yet justified this idea. First of all, we know that it holds for quasi locally free sheaves (see Remark 2.9(i)) and for generalized line bundles. Moreover, in the case of a split ribbon \( X \) (i.e. having a retraction to \( X_{\text{red}} \)) with \( \mathcal{N} \) isomorphic to the canonical bundle of \( X_{\text{red}} \) this fact holds by the known results about the nilpotent cone of the moduli space of Higgs bundles on \( X_{\text{red}} \) (see [Sa1, Appendix A])

The last conjecture before the resuming one is suggested by Proposition 2.13 together with Conjecture 4.1:

**Conjecture 4.4.** If \( \delta > 2\bar{g} - 2 \) (where, as usual \( \delta = -\text{deg}(\mathcal{N}) \) and \( \bar{g} \) is the genus of \( X_{\text{red}} \)), the only irreducible components of the moduli space of coherent sheaves on \( X \) are those whose generic elements are either quasi locally free sheaves of rigid type (for generalized rank odd) or generalized vector bundles (for generalized rank even).

Indeed, if one were able to show that deformations of subsheaves or quotients of the canonical filtrations of a quasi locally sheaf \( \mathcal{F} \) on \( X \) induce deformations of \( \mathcal{F} \) itself (maybe using the relative ext sheaves), the Proposition could be used to prove the conjecture. Indeed, one could proceed by induction on the generalized rank, starting from the first interesting case, i.e. generalized rank 3 (for generalized rank 2 the conjecture holds: it is [Sa2, Corollary 1]). In generalized rank 3 there are only sheaves of type \( (3,0) \) and \( (1,1) \): the first are rank 3 vector bundles on \( X_{\text{red}} \), all of which deform, under our hypotheses, to sheaves on \( X \) by Proposition 2.13 (with that possible exception cited in its statement, but we think that it is not a real exception). Therefore, the only possible generic elements of an irreducible component are sheaves of type \( (1,1) \). The quasi locally free sheaves of this type are of rigid type; hence, Conjecture 4.4 reduces to Conjecture 4.1. In generalized rank 4, again by Proposition 2.13, one has to consider only sheaves of type \( (2,1) \) and \( (0,2) \), the latter being generalized vector bundles. Within sheaves of type \( (2,1) \) we have to consider only quasi locally free ones, assuming Conjecture 4.1. If \( \mathcal{F} \) is a sheaf of type \( (2,1) \), \( \mathcal{F}|_{X_{\text{red}}} \) is a rank 3 vector bundle on \( X_{\text{red}} \). Hence, again by Proposition 2.13, it deforms to a sheaf of type \( (1,1) \). If the generic extension of a sheaf of this type by the line bundle (on \( X_{\text{red}} \)) \( \mathcal{N} \mathcal{F} \) is a generalized vector bundle, one would have done. If it is not the case, one could look to a rank 2 quotient of \( \mathcal{F}|_{X_{\text{red}}} \) and to the kernel of the composed morphism from \( \mathcal{F} \) to it, which is either a rank 2 vector bundle on \( X_{\text{red}} \) or a generalized line bundle on \( X \). In the first case the rank 2 vector bundles deform, again by Proposition 2.13, to two generalized line bundles on \( X \), whose extensions are generalized vector bundle on \( X \); in the second case, only one of them has to be deformed to a generalized line bundle and the conclusion is the same. This idea could
be formalized by induction for any $n$, if one were able to prove Conjecture 4.1 and to control when and how deformations of subsheaves and quotients related to the canonical filtrations induce deformations of the sheaf itself.

We conclude collecting Conjectures 4.3, 4.4 and 4.1 in a unique statement about the irreducible components of the moduli space of stable sheaves on $X$.

**Conjecture 4.5.** Let $X$ be a ribbon of genus $g$ such that $\bar{g} \geq 2$ (where, as usual, $\bar{g}$ is the genus of $X_{\text{red}}$), let $\delta = -\deg \mathcal{N}$ and let $M = M_s(X, R, D)$ be the moduli space of stable sheaves of generalized rank $R$ and generalized degree $D$ on $X$.

(i) Assume $0 < \delta \leq 2\bar{g} - 2$, equivalently $g \leq 4\bar{g} - 3$. The irreducible components are the closures of the following loci:

- For any sequence of integers $((r_0, r_1), (d_0, d_1))$ such that $r_0 > r_1 \geq 0$, $r_0 + r_1 = R$, $d_0 + d_1 = D$ and, if $r_1 > 0$, $(d_0 - (r_0 + r_1)\delta)/r_0 < d_1/r_1 < d_0/r_0$, the locus of quasi locally free stable sheaves of complete type $((r_0, r_1), (d_0, d_1))$.
- If $R$ is even, the locus of stable generalized vector bundles of generalized rank $R$ and degree $D$ and fixed index $b$, for any positive integer $b < r\delta$, where $r = R/2$. All these component have dimension $(1 + (r_0^2 + r_1^2)(\bar{g} - 1) + r_0r_1\delta)$. Distinct complete types correspond to distinct irreducible components.

(ii) If $\delta > 2\bar{g} - 2$, equivalently $g > 4\bar{g} - 3$, then we have to distinguish two cases.

(a) If $R = 2r$ is even, then the only irreducible components of $M$ are the closures of the loci of stable generalized vector bundles of generalized rank $R$ and degree $D$ and fixed index $b < r\delta$ and they have dimension $1 + 2r^2(\bar{g} - 1) + r^2\delta$.

(b) If $R = 2a + 1$ is odd, then the only irreducible components of $M$ are the closures of the loci $N(a, d_0, d_1)$ of stable quasi locally free sheaves of rigid type of generalized rank $R$ and generalized degree $D$ with $(d_0 - (2a + 1)\delta)/(a + 1) < d_1/a < d_0/(a + 1)$. They have dimension $1 + (a^2 + a)\delta + (2a^2 + 2a + 1)(\bar{g} - 1)$.

Observe that, in particular, the Simpson moduli space is expected to have pure dimension if and only if $\delta \geq 2\bar{g} - 2$ and that in the case $\delta = 2\bar{g} - 2$ the dimension is precisely the dimension of the moduli space of rank $R$ stable vector bundles on $X_{\text{red}}$, i.e. $1 + R^2(\bar{g} - 1)$.

The dimensional results are all known (see Fact 1.24(i), Theorem 2.8 and Proposition 3.3 for sheaves supported on $X$; the dimension of the moduli space of stable vector bundles on $X_{\text{red}}$, which is the case $r_1 = 0$, is well-known). Also the irreducibility of the loci of quasi locally free sheaves is known (see Remark 2.4(i)). In the first part of the conjecture (i.e. $\delta \leq 2\bar{g} - 2$), the only conjectural parts are that the loci of generalized vector bundles of fixed index are irreducible (which is Conjecture 4.3), hence irreducible components (see the discussion after Conjecture 4.3), and that stable sheaves of positive index which are not generalized vector bundles belong to the closures of the loci of quasi locally free sheaves. Indeed, this would imply immediately that the closures of the cited loci of quasi locally
free sheaves are irreducible components, as explained after Conjecture 4.1. The inequalities on the complete type and the index in the statement are the stability conditions given by Theorem 2.5 and Proposition 3.1.

The second part, i.e. $\delta > 2g - 2$, is a reformulation of Conjecture 4.4 with the addition of the stability conditions, which are, again, Theorem 2.5 and Proposition 3.1.

The conjecture holds for generalized rank 2: it is [CK, Theorem 4.7] with the addition of [Sa2, Corollary 1]. In generalized rank 3 it almost holds: the only fact that is conjectural is that the loci in the statement are all the irreducible components. Indeed, the quasi locally free sheaves of type $(1,1)$ are of rigid type; hence, the closures of their loci with fixed complete type verifying stability conditions are irreducible components. Moreover, the locus of stable rank 3 vector bundles on $X_{\text{red}}$ cannot be a component for $\delta > 2g - 1$ or for $\delta = 2g - 1$ and 3 not dividing both $D$ and $\bar{g}$, by Proposition 2.13.

Therefore, for $R = 3$, Conjecture 4.1 is essentially equivalent to Conjecture 4.1 (they would be really equivalent if one were able to cover also the possibly exceptional case in the statement of Proposition 2.13).

The conjecture holds for any generalized rank in the case $X$ admits a retraction to $X_{\text{red}}$ and $\mathcal{M}$ is the canonical sheaf of $X_{\text{red}}$ by what is known for the nilpotent cone of the moduli space of stable Higgs bundles on $X_{\text{red}}$ (see [Sa1, Appendix A]).

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Ad maiorem Dei gloriam

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