Unsteady radiative-convective heat transfer on a radiating surface

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Abstract. Research of radiation-convective heat exchange on radiating surfaces at natural and forced convection is complex mathematical task and here we obtain approximate analytical formulations for this process. We consider two dimensional unsteady heat transfer between solid surface and fluid under the natural laminar convection within optically transparent grey media. Also we assume constant thermo-physical properties except density which is decreasing linearly with temperature. Complex radiative-convective unsteady heat transfer approximately can be considered as a multi-stage process. At the beginning heat transfer coefficient is time dependent but almost independent on longitudinal coordinate. Afterwards heat transfer coefficient becomes dependent on longitudinal coordinate but does not change over time. Analytic formulations obtained for those two stages could be merged along the “time-space” characteristic basing on the equality of heat flows and temperatures there. Solutions are constructed using asymptotic expansions. Theoretical analysis of the solutions revealed the following: effect of radiation leads to a change in the heat transfer coefficient from the values that are characteristic to the second order boundary conditions to the values that are characteristic for the first order boundary conditions. The rate of this transition depends on $\beta$ radiation coefficient. Experimental research confirmed correctness of the simplifications introduced.

1. Introduction

The problems of unsteady radiation-convective heat transfer on radiating surfaces are characterized by significant mathematical difficulty and only recently they started being considered mainly by the numerical methods. For instance, they are topical in application to the gas-cooled nuclear reactors, high-speed aircrafts, cryogenic systems and other important technical applications.

Detailed analysis of radiative-convective heat transfer confirms interaction between radiation and convection. Interaction not only exists in a case of radiating-absorbing media but also by the influence of a boundary condition for the flow of diathermic fluids [1]. For all boundary conditions, except the first order ones, there is always a dynamic interaction between the radiative and convective heat fluxes. As a result, when calculating the Nusselt number convective heat flux should be determined taking into account the radiation, the proportion of which, in turn, depends on the temperature of the surface, i.e. on convective heat transfer.

Publications [2], [3] were related to the particular aspects of this process. In [2] Rosenstock provided solution for unsteady laminar forced convection with the first order boundary conditions. It is shown that with a steep jump in velocity and temperature the transition from transient to steady regime
occurs at the front of the thermal and dynamic waves moving from the edge of the plate. The same character of the process was found by experimental research by Gebhart [3] at natural convection.

Transient free convection flow with periodic change in wall temperature and constant heat flux was researched by Das et al [4]. Recently, a theoretical analysis of free convection with mass transfer was done by Narahari and Dutta [5].

Mebine P. and Adigio E.M. [6] constructed numerical solution for unsteady free convection near emitting vertical porous plate. Rosseland heat conductivity approximation is used for radiation. Parameter influence on convective heat transfer is considered.

Large set of radiative-convective heat transfer problems on a plain surface was researched by N.A. Rubtsov et al. In [7] Rubtsov and Sinitsyn numerically solved the boundary layer problem of unsteady radiative-convective heat transfer of selectively emitting and absorbing medium on a flat plate. It is shown that the main contribution to the radiative flux is made by intrinsic radiation of the plate. In [8] Rubtsov, Sinitsyn, Timofeev studied numerically the conjugate problem of radiative-convective heat transfer on the opaque plate by means of averaged radiative fluxes. Same authors in [9] analyzed the effect of conjugation, injection, scattering, reflection from the surface, etc on the heat exchange. Numerical simulation showed that optical thickness of the boundary layer is the main factor influencing the plate radiative heating. Growing optical thickness leads to increased time-to-steady state.

Summing up the cited works, we conclude that analytical solutions for radiative-convective heat transfer are extremely important as they allow for detailed parametric analysis of this process and also help in verifying accuracy of numerical simulations, etc. In the present work we analytically solved one of non-linear problems of second order for unsteady convection on heat emitting surface.

Complex unsteady radiation-convective heat transfer can be studied in some approximation as the multistage process. In such a process the moving forces of different nature (molecular and molar ones) act at different stages with different intensity. At the beginning of the process the molecular mechanism of transfer is predominant, and at the final stage the molar exchange has the main importance. This means that in the initial period the main role in heat transfer attributes to intrinsic radiation. At this, the heat transfer coefficient assumed to be not dependent on the longitudinal coordinate. When the perturbation passes, i.e., when liquid at the inlet reaches the considered cross-section, certain heat transfer stabilization occurs. Now the main role in heat transfer is attributed to the convective mechanism. At this, the heat transfer coefficient depends on the longitudinal coordinate and does not change in time. For the transitional area we can merge the obtained solutions along the “space-time” characteristic on the basis of heat fluxes and temperatures equality. Using this principle, the approximate-analytical problem of unsteady radiation-convective heat transfer on a flat vertical heat-releasing surface is solved below for the stratified flow.

2. Theoretical study for the condition of free convection

The problem of natural unsteady convection of the emitting (or heated by thermal radiation) vertical surface in a diathermal medium with constant thermal-physical properties, excluding viscosity, is studied. Convection starts at stepped supply of constant heat flux to the surface $q_w$: this flux is scattered by convection $q_k = \lambda_{liq} \text{Pr} \Delta T_w$ of washing liquid with temperature $T_{liq}$ and radiation into the ambient medium $q_{rad} = \epsilon \sigma_0 (T_w^4 - T_\infty^4)$ with temperature $T_\infty$. Here $q_w$, $q_k$, $q_{rad}$ – densities of the surface, convective and radiative heat flux, respectively; $T_w$, $T_{liq}$, $T_\infty$ – absolute temperatures of liquid, wall, ambient medium, and washing liquid, respectively; $\lambda_{liq}$ – heat conductivity of liquid; $\epsilon$ – emissivity of the wall surface; $\sigma_0$ – Stefan-Boltzmann constant.

As it is presented in [10], laminar flow can be sufficiently approximated by the boundary layer equations. Due to low stability of laminar boundary layer at significant natural convection speeds viscosity and pressure gradient as it is shown by Yang [1] does not have major impact on laminar flow structure.
With the above assumptions from the mathematical approach the stated problem is reduced to the solution of the followings system of differential equations of mass, momentum and energy conservation in the boundary layer [10] with Boussinesq hypothesis [11]:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  \hfill (1)

\[ \frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{g(T - T_\infty)}{T_\infty} \]  \hfill (2)

\[ \frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} \]  \hfill (3)

with initial at \( t \leq 0 \)

\[ u = 0, \quad T = T_\infty \]  \hfill (4)

and boundary conditions when \( t > 0 \)

at \( y = 0 \)

\[ u = v = 0, \quad \mu \frac{\partial u}{\partial y} = \tau_w \]  \hfill (5)

\[ \lambda_{1q} \frac{\partial T}{\partial y} = q_w - \varepsilon \sigma_0 (T_w^4 - T_e^4) \]  \hfill (6)

at \( y = \delta \)

\[ \frac{\partial u}{\partial y} = 0, \quad \frac{\partial T}{\partial y} = 0 \]  \hfill (6)

\[ u = 0, \quad T = T_\infty \]  \hfill (6)

Here \( x, y, t \) – spatial coordinates and time; \( u, v \) – velocities; \( \mu, \nu \) – dynamic and kinematic viscosity; \( g \) – standard gravity, \( a, \lambda \) - coefficient of thermal diffusivity and thermal conductivity of the fluid; \( \tau_w \) – wall friction; \( \delta \) – boundary layer thickness.

Solutions to equations (2), (3) are found separately for unsteady stage

\[ \frac{D}{Dt} \equiv \frac{\partial}{\partial t} \]  \hfill (7)

and stabilized heat transfer

\[ \frac{D}{Dt} \equiv u \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial y} \]  \hfill (8)

The following temperature profile meets boundary conditions (5), (6)

\[ T - T_\infty = \frac{q_w \delta}{2 \lambda} \left( \frac{\nu^2}{\delta^2} - 2 \frac{\nu}{\delta} + 1 \right) \]  \hfill (9)

From this profile at \( y=0 \) we obtain the expression for the boundary layer thickness

\[ \delta = \frac{2 \lambda (T_w - T_\infty)}{q_w} = \frac{2 \lambda (\Theta_w - 1)}{2 \sigma_0 T_\infty^3 (\beta - \Theta_w^4)} \]  \hfill (10)
Where \( \beta = \frac{q_w + \varepsilon \sigma T_w^4}{\varepsilon \sigma T_w^4} \) — radiation parameter and \( \Theta_w = T_w/T_{\infty} \) — dimensionless temperature of the wall. Using temperature profile (9), (10) with adherence to boundary conditions (5), (6), we obtain from (2) distribution for velocity component \( u \)

\[
u = \frac{gq_w \delta^3}{6\lambda v T_w^3} \left( -\frac{y^4}{4\delta^2} + \frac{y^3}{\delta^3} - \frac{5y^2}{4\delta^2} + \frac{y}{2\delta} \right) \tag{11}
\]

Then integrating (3) from \( y=0 \) to \( y=\delta \) with the use of (1), (5), (6), we obtain equation for the heat flux

\[
\frac{\partial}{\partial t} q_w \delta^2 + \frac{g}{140\lambda v T_w} \frac{\partial}{\partial x} q_w \delta^5 = 6\alpha q_w \tag{12}
\]

Using (10), we write down characteristic system \([12]\) of obtained equation (12)

\[
2(\Theta_w-1) d\Theta_w - (\Theta_w-1)^2 d\Theta_w^4 = 6\alpha \left( \frac{\varepsilon \sigma T_w^3}{2\lambda} \right)^2 dt, \tag{13}
\]

\[
\frac{5(\Theta_w - 1)\delta^4}{(\beta - \Theta_w^4)} + \frac{3(\Theta_w - 1)^5}{(\beta - \Theta_w^4)} = \frac{420\alpha v}{g} \left( \frac{\varepsilon \sigma T_w^3}{2\lambda} \right)^4 dx, \tag{14}
\]

\[
\frac{5(\beta - \Theta_w^4)^4 + 12(\Theta_w - 1)(\Theta_w - 1)^3}{2(\beta - \Theta_w^4) + 4(\Theta_w - 1)\Theta_w^4} \frac{d\Theta_w}{(\beta - \Theta_w^4)^3} = \frac{70\nu}{g} \left( \frac{\varepsilon \sigma T_w^3}{2\lambda} \right)^2 dx. \tag{15}
\]

The integral of equation (13) for initial conditions \( \Theta_w = 1 \) gives the transcendent expression for calculation of the surface temperature change at the unsteady stage of heat transfer in the dimensionless form

\[
\frac{(\Theta_w-1)^2}{(\beta - \Theta_w^4)} + 2 \int_1^\Theta_w (\Theta-1)d\Theta = 3\delta \frac{Sk T_w^3}{\lambda} \tag{16}
\]

Where \( Sk = \frac{\varepsilon \sigma T_w^3 x}{\lambda} \) — Stark number and \( Fo = at/\lambda^2 \) — Fourier number.

With consideration of (16) we obtain the formulation for Nusselt number

\[
\text{Nu}(Fo) = \frac{1}{3\sqrt{Fo}} \left[ 1 + 2 \left( \frac{\beta - \Theta_w^4}{(\Theta_w - 1)^2} \right) \int_1^\Theta_w \frac{(\Theta-1)d\Theta}{(\beta - \Theta_w^4)^3} \right]^{1/2} \tag{17}
\]

where \( \text{Nu} = \frac{q_w x}{\lambda(T_w - T_{\infty})} \) - Nusselt number.

Integral in (16), (17) is expressed by elementary functions

\[
\int \frac{(\Theta-1)d\Theta}{(\beta - \Theta_w^4)^2} = \frac{\Theta(\Theta-1)}{4\beta(\beta - \Theta_w^4)} + \frac{1}{8\beta\sqrt{\beta}} \ln \left| \frac{\beta + \Theta^2}{\beta - \Theta^2} \right| - \frac{3\sqrt{\beta}}{16\beta^2} \left[ \ln \left( \frac{\sqrt{\beta} + \Theta}{\sqrt{\beta} - \Theta} \right) + 2\arctg \left( \frac{\Theta}{\sqrt{\beta}} \right) \right] \tag{18}
\]

Let’s estimate contribution of integral (18) for the limiting cases of low and high \( Fo \) numbers. For low \( Fo \) numbers thermal resistance of the heated layer is low, therefore, \( \Theta_w \to 1 \). In the opposite case of high \( Fo \) it is significant and heat transfer occurs mainly via radiation, hence, \( \Theta_w \to \sqrt{\beta} \). Using the latter, we obtain Nusselt expression for the limiting cases
\[ \Theta_w \to 1 \quad \text{Nu}_q(Fo) = \sqrt[3]{\frac{2}{3}} Fo^{-1/2} \quad (19) \]
\[ \Theta_w \to \sqrt[3]{\beta} \quad \text{Nu}_T(Fo) = \sqrt[3]{\frac{1}{3}} Fo^{-1/2} \quad (20) \]

Finally, a change in heat transfer law occurs from the formulation that is typical for the boundary condition of the second order (low Fo), to the formulation that is characteristic for the boundary condition of the first order (high Fo). The speed of this transition depends on radiation parameter \( \beta \).

With a rise of \( \beta \) the transition process to the boundary conditions of the first order accelerates. The error of expressions (19), (20) does not exceed 7% [13].

Integration of equation (14) for the stage of steady convective heat transfer leads to expression for determination of surface temperature

\[ \int_0^\Theta_r (\Theta-1)d\Theta \Big/ (\beta-\Theta^4) \Bigg] \quad 1/4 \]
\[ = \frac{35Sk^4}{Ra} \quad (21) \]

Where \( Ra = \frac{g\beta x^3(T-T_\infty)}{va} \) – Rayleigh number, and \( \beta \) – linear expansion coefficient.

Then Nusselt number is calculated by

\[ \text{Nu}(x) = \sqrt[4]{\frac{Ra}{35}} \left( 1 + \frac{5}{3} \int_0^\Theta \frac{(\Theta-1)d\Theta}{(\beta-\Theta^4)} \right)^{1/4} \quad (22) \]

We should note that integral in (21), (22) is expressed via the elementary functions, and namely

\[ \int_0^\Theta \frac{\Theta(\Theta-1)^4}{12\beta(\beta-\Theta^4)} + 2\sqrt{\beta} \frac{77-270\sqrt{\beta}-7\beta}{512\beta^4} \text{arctg} \frac{\Theta}{\sqrt[3]{\beta}} + \]
\[ + \frac{21\beta^2\Theta+7\beta\Theta^3-128\beta^5+486\beta^6-2700\theta^7-400\beta^2+2400\theta+121\beta-77\Theta^5}{384\beta^2(\beta-\Theta_w)^3} + \]
\[ + \sqrt{\beta} \frac{(77+270\sqrt{\beta}-7\beta)}{512\beta^3} \ln \frac{\Theta}{\sqrt[3]{\beta-\Theta^2}} + \frac{5}{16\beta^2} \ln \frac{\sqrt[3]{\beta+\Theta^2}}{\sqrt[3]{\beta-\Theta^2}} \quad (23) \]

Let’s analyze the contribution of integral to (21), (22). At low \( x \), when convective heat transfer is predominant, \( \Theta_w \to 1 \), and if \( x \) is growing, the main role belongs to radiation heat transfer, and \( \Theta_w \to \sqrt[3]{\beta} \). Using the rule of L’Hospital, we obtain for

\[ \lim_{0}^{\Theta} \frac{\Theta(\Theta-1)^4}{12\beta(\beta-\Theta^4)} = \begin{cases} \frac{1}{5}, & \Theta_w \to 1 \\ 0, & \Theta_w \to \sqrt[3]{\beta} \end{cases} \]

Then \( \text{Nu}(x) \) numbers are calculated by the following expressions:

\[ \Theta_w \to 1, \quad \text{Pr} = 0.72 \text{ (air)}, \quad \text{Nu}(x) = 0.408Gr^{1/4}, \quad [10] \quad (24) \]

which is typical for the boundary conditions of the second order,

\[ \Theta_w \to \sqrt[3]{\beta}, \quad \text{Pr} = 0.72, \quad \text{Nu}(x) = 0.374Gr^{1/4}, \quad [11] \quad (25) \]
which is typical for the boundary conditions of the first order. Here $Pr = \frac{v}{a}$ – Prandtl number; $Gr = \frac{g \alpha^3 \beta (T_w - T_\infty)}{\nu^2}$ – Grashof number.

As for unsteady stage of heat transfer, the law of steady convective heat transfer changes here from the boundary conditions of the second order to the boundary conditions of the first order with development of the boundary layer. The rate of this transition depends on radiation parameter $\beta$. With $\beta$ increasing this transition accelerates. The obtained solutions have the error not higher than 4% from the test data [3].

The solution to equation (15), describing the motion of thermal wave front (the boundary of transition from unsteady solution to the steady one), can be found only for the limiting cases

$$\Theta_w \rightarrow 1 \quad x_T = \frac{0.105 \sqrt{\beta}}{\frac{a}{Pr} \tau^{5/2}}$$  \hspace{1cm} (26)

$$\Theta_w \rightarrow \sqrt{\beta} \quad x_T = \frac{0.257 (T_w - T_\infty)}{Pr T_\infty} \tau^2$$  \hspace{1cm} (27)

Dependences (26), (27) are proved by experimental data [3].

3. Experimental investigation under the conditions of free convection

To verify the assumptions of the mathematical model and reliability of theoretical results, we have carried out a series of experiments together with E.M. Puzyrev. The layout of experimental setup is shown in figure 1.

Figure 1. Layout of experimental setup. 1 – tank; 2 – tank lid; 3 – operating surface; 4 – transformer; 5 – voltammeter D128; 6 – potentiometer PP-63; 7 – amplifier F-116/2; 8 – thermocouples; 9 – thermocouple probe; 10 – automatic electronic bridge EVM-211A; 11 – resistance thermometer; 12 – movable resistance thermometer; 13 – self-recording potentiometer EPP-09.2M; 14 – electric hot-wire anemometer ETAM-3; 15 – sensor of hot-wire anemometer; 16 – Oscilloscope C1–8A and A–700.
Atmospheric air was used as the medium transparent for thermal radiation. To exclude outer perturbations, the air volume was isolated from the ambient medium by the walls of steel tank with the diameter of 850 mm and height of 1500 mm. To imitate the infinitely large sizes of the surrounding walls the inner surface of the tank was shaded with lamp soot with emissivity $\varepsilon = 0.96$. Different types of the heated working surface were tested in experiments. The plates made of electroconductive paper and foiled glass-fiber plastic were the most convenient and reliable in operation. Heating was performed by the electric current reduced by transformer (4). The supplied power was determined by voltameter (5) with accuracy class 1.0. The working surface temperature was measured by potentiometer (6) of class 0.05 or amplifier (7) of class 1.5 by electromotive force of copper-constantan thermocouples of 0.1-mm diameter. The initial stable operation of setup was registered with electron potentiometer (13). The flow regime was determined by thermal anemometer (14) and oscilloscope (16). Experimental data was gathered for all processes in dimensionless coordinates, obtained at theoretical research. Comparisons of experimental and theoretical values of convective heat transfer are shown in figure 2.

Experiments prove the correctness of simplifications made at theoretical analysis of the processes of radiation-convective heat transfer on the emitting surface.

![Figure 2](image)

Figure 2. Comparison of theoretical (solid line) and experimental data (crosses) at steady convective heat exchange with natural convection ($\beta = 2.8$)

All these prove again the validity of mathematical model and justify simplifications introduced at theoretical analysis of the complex process of radiation-convective heat transfer.

In conclusion, it should be noted that:

- The approximate-analytical method for investigation of unsteady heat transfer in the regime of free convection was developed. This method allows determination of the main regularities of surface radiation effect on convective heat transfer before numerical calculations.
- Solutions for unsteady and steady heat transfer stages were obtained. These two solutions were merged on the “coordinate-time” characteristic. The following key findings were discovered by theoretical analysis of these solutions: radiation leads to degeneration of the law of convective heat transfer from its formulation typical for the boundary conditions of the
second order to expressions characteristic to the boundary conditions of the first order. The rate of this transition depends significantly on radiation parameter $\beta$.

- There was the experimental investigation, which proved validity of assumptions made at development of the mathematical model. The compliance between experimental and theoretical data is sufficient for engineering applications.

References
[1] Sess R D 1967 Problems of Heat Transfer (Moscow, Atomizdat)
[2] Rosenshtok U L 1965 Series. Heat and mass transfer (Minsk) 1
[3] Gebkhard G 1967 J. Heat transfer Series C 89 3
[4] Das U N et al 1999 J. Heat Transfer 121
[5] Narahari M and Dutta K 2009 ASME Conference Proceedings 63
[6] Mebiade P and Adigio E M 2009 Turk. J. Phys. 33 2
[7] Rubcov N A and Sinicin V A 2001 J. Appl. Mechanics and technical phys. 42 1
[8] Rubcov N A, Sinicin V A and Timofeev A M 1998 Thermophysics of high temperatures 36 4
[9] Rubcov N A, Sinicin V A and Timofeev A M 1998 J. Appl. Mechanics and technical phys. 39 5
[10] Schlichting H. 1951 Grenzschicht (Karlsruhe, Theory)
[11] Jaluria Y 1980 Natural convection Heat and Mass Transfer (New York, Pergamon Press)
[12] Kamke E 1965 Handbook on standard differential equations (Moscow, Nauka)
[13] Lykov A V 1972 Theory of Heat Conductivity (Moscow, Vysshaya shkola)