Joint backward extension property for weighted shifts

Piotr Pikul

Abstract. Weighted shifts on directed trees are a decade-old generalisation of classical shift operators in the sequence space $\ell^2$. In this paper we introduce the joint backward extension property (JBEP) for classes of weighted shifts on directed trees. If a class satisfies JBEP, the existence of a common backward extension within the class for a family of weighted shifts on rooted directed trees does not depend on the additional structure of the big tree (of fixed depth). We decide whether several classes of operators have JBEP. For subnormal or power hyponormal weighted shifts the property is satisfied, while it fails for completely hyperexpansive or quasinormal. Nevertheless some positive results on joint backward extensions of completely hyperexpansive weighted shifts are provided.

Contents

1. Introduction 1
2. Weighted shifts on directed trees 3
3. Joint backward extension property 8
4. Power hyponormality 13
5. Complete hyperexpansivity 19
References 32

1. Introduction

The weighted shifts on $\ell^2$ are long known objects in the operator theory. A survey on them can be found e.g. in [22]. The study of backward extensions (sometimes called back-step extensions) was originally about whether or not the sequence $\{a_n\}_{n=0}^\infty$ of weights can be extended by a prefix $\{a_{-n}\}_{n=1}^k$ so that the weighted shift with weights $\{a_{n-k}\}_{n=0}^\infty$ has specified property (e.g. if it is subnormal). For the results on this topic we refer the reader to e.g. [6, 10, 16, 14, 17, 18].

2020 Mathematics Subject Classification. Primary 47B37, 47B20; Secondary 47A20, 47B02, 05C20.

Key words and phrases. Weighted shift, directed tree, backward extensions, Hilbert space, power hyponormal, completely hyperexpansive.
In [13] there was introduced a generalisation of the concept of a weighted shift, namely a weighted shift on a directed tree. To state the idea briefly, dealing with trees we allow an element of orthonormal basis to have more than one successor (child). Formula defining action of such operator takes the form (cf. (2.4))

\[ S_\lambda e_v = \sum_{u \in \text{Chi}(v)} \lambda_u e_u, \]

where \( \text{Chi}(v) \) is the set of children of the vertex \( v \).

The study of backward extensions of weighted shifts on directed trees was initiated in [21] and this paper extends some results presented therein. In the class of weighted shifts on directed trees we have great freedom in defining a backward extension. Especially we can ask whether a family of weighted shifts on rooted directed trees has a common backward extension with certain property. The simplest case of such a joint extension involves only one new vertex, a parent of all the former roots (see rooted sum, Definition 2.5), and weights corresponding to new edges.

We can also consider more complex additional structure of the “extended tree”. In case of several classes of operators, given at most countable uniformly bounded family of weighted shifts on rooted directed trees admitting “\( k \)-step backward extensions” (see Definition 2.4) we can extend them all to a weighted shift on a single rooted directed tree, regardless of the first \( k \) “levels” of the extended tree (see Theorem 3.7). This behaviour is a consequence of the joint backward extension property (see Section 3). It is the case e.g. for power hyponormal or subnormal weighted shifts while it fails to hold for completely hyperexpansive or quasinormal weighted shifts.

Section 2 recalls basic facts about weighted shifts on directed trees and basic operations on said trees. While the notation is based mainly on the directed forest approach presented in [21], it should not be confusing to the reader familiar with [13].

In Section 3 we introduce concepts related to backward extensions. Especially the formulation of joint backward extension property (JBEP) is given (see Definition 3.4) and its major consequence is proven. While we can extend the given collection of trees backwards in \( k \) steps in various different ways, the choice of the resulting tree is irrelevant for the question of the existence of a weighted shift extending the given collection of weighted shifts belonging to a class that has the JBEP (see Theorem 3.7). There are given some examples of well known classes for which it can be easily verified whether they satisfy JBEP.

Then we answer the questions whether the classes of subnormal, power hyponormal (Section 4) and completely hyperexpansive (Section 5) weighted shifts have the JBEP. The answer is affirmative for the first two classes (see Theorems 3.10 and 4.5) but not for the third one (see Example 5.13). Nevertheless, in the last section we provide some positive results about the existence of joint backward extensions for completely hyperexpansive weighted shifts (see Theorems 5.15 and 5.18).

We denote the set of non-negative integers by \( \mathbb{Z}_+ \) and reserve the symbol \( \mathbb{N} \) for the set of positive ones. The sets of real and complex numbers are denoted by \( \mathbb{R} \) and \( \mathbb{C} \) respectively. We will use the disjoint union symbol “\( \sqcup \)”. 
to emphasise the disjointness of considered sets. Given a topological space $X$, by $\mathcal{B}(X)$ we mean the $\sigma$-algebra of Borel subsets of $X$. For $t \in X$ we denote by $\delta_t$ the Borel probability measure on $X$ concentrated on $\{t\}$ (Dirac measure). We follow the conventions that $0^0 = 1$ and $\sum_{v \in \emptyset} x_v = 0$. If $f : A \to A$ is any function, then $f^0 = \text{id}_A$. For any $k \in \mathbb{N}$, by convention the function $t \mapsto t - k = 1/t$ attains $+\infty$ at $0$.

Throughout this paper all Hilbert spaces will be over the complex numbers. For a Hilbert space $H$ we denote the algebra of all bounded linear operators on $H$ by $B(H)$. We recall that an operator $T \in B(H)$ is said to be subnormal if it is a restriction of some normal operator $N \in B(K)$ (on a possibly larger Hilbert space $K$) to $H$. Clearly $H$ is an invariant subspace for $N$. It is called hyponormal if $[T^*, T] := T^*T - TT^* \geq 0$. The following property of hyponormal operators is well known (see [5, Proposition 4.4 (e)]).

If $T \in B(H)$ is hyponormal and $K$ is a closed invariant subspace for $T$, then $T\mid_K$ is hyponormal.

(1.1)

Notions of subnormality and hyponormality originate from the work of Halmos [7]. We refer the reader to [5] for the fundamentals of the theory of subnormal and hyponormal operators.

We will also study the question of when an operator $T$ is power hyponormal, i.e. whether $T^n$ is hyponormal for every $n \geq 1$. It is well known that every subnormal operator is power hyponormal but the converse is not true. The early counterexample is due to Stampfli [25].

2. Weighted shifts on directed trees

Weighted shifts on directed trees were introduced in [13]. Foundations given therein are sufficient to understand the operators studied in this paper. However, we will use notation focused on directed forests from [21]. In this section we recall basic facts. The proofs can be found either in [21, Section 2] or derived from the very similar arguments presented in [13].

2.1. Directed trees. This subsection contains graph-theoretical part of the paper. We recall definitions and basic properties. While throughout this paper we will consider directed trees exclusively, we start with the notion of a directed forest.

2.1. Definition. Pair $\mathcal{T} = (V, p)$ is called a directed forest if $V$ is a non-empty set and $p : V \to V$ is a function satisfying the following condition:

if $n \geq 1$, $v \in V$ and $p^n(v) = v$, then $p(v) = v$.

(2.1)

Elements of the set $\text{root}(\mathcal{T}) := \{u \in V : p(u) = u\}$ are called roots of the forest $\mathcal{T}$. We call elements of the set $V$ vertices of the forest and $p$ is called the parent function.

Later in this section we assume that $\mathcal{T} = (V, p)$ is a directed forest.

If $u, v \in V$ are vertices such that $p(v) = u \neq v$, then we call $u$ the parent of $v$. We also set $V^\circ := V \setminus \text{root}(\mathcal{T})$. The elements of $\text{Chi}(v) := p^{-1}(v) \setminus \{v\}$
are called children of the vertex $v$. In general, we define the set of $k$-th children of $v$ as

$$\text{Chi}^{(k)}(v) := \{u \in V : p^k(u) = v \neq p^{k-1}(u)\}, \quad k \geq 1,$$

and $\text{Chi}^{(0)}(v) := \{v\}$. Clearly $\text{Chi}(v) = \text{Chi}^{(1)}(v)$. We also denote the set of all descendants of a vertex $v$ by $\text{Des}(v) := \bigcup_{n=0}^{\infty} \text{Chi}^{(n)}(v)$ and use the symbol $\text{Des}^+(v)$ for $\text{Des}(v) \setminus \{v\}$. The cardinality of the set $\text{Chi}(v)$ is called the degree of the vertex $v$ and denoted by $\deg(v)$.

If there is a risk of ambiguity we write $\text{Chi}_T(v)$, $\text{Des}_T(v)$, etc. to make the dependence on $T$ explicit.

An element of $V \setminus p(V)$ is called a leaf. If there is no leaf (i.e. $p(V) = V$) then the forest is called leafless.

In the following lemma we list some basic facts. Their elementary proofs are omitted.

2.2. LEMMA. Let $T = (V, p)$ be a directed forest. Then

(a) $\text{Chi}^{(k)}(v) \cap \text{Chi}^{(k)}(w) = \emptyset$ for $k \in \mathbb{Z}_+$ and distinct $v, w \in V$,

(b) $\text{Chi}^{(k)}(v) \cap \text{Chi}^{(l)}(v) = \emptyset$ for $k, l \in \mathbb{Z}_+$, $v \in V$, provided that $k \neq l$,

(c) for any $k \in \mathbb{N}$,

$$\text{Chi}^{(k)}(v) = \bigcup_{u \in \text{Chi}(v)} \text{Chi}^{(k-1)}(u) = \bigcup_{u \in \text{Chi}^{(k-1)}(v)} \text{Chi}(u),$$

(d) if $v \in V$ and $w \in \text{Des}(v)$, then $\text{Des}(w) \subseteq \text{Des}(v)$,

(e) $v \in V$ is a leaf if and only if $v \in V^0$ and $\deg(v) = 0$.

A directed tree is a directed forest containing only one tree. In other words, it is a directed forest which is connected as a graph. The forest is called degenerate if $V^0 = \emptyset$ (i.e. $p = \text{id}_V$). Otherwise it is non-degenerate. It may be worth noting that a non-degenerate directed tree is leafless if and only if every vertex has a positive degree.

2.3. REMARK. Our definition of a directed tree is consistent with the “traditional” one (see e.g. [13]). The corresponding set of edges can be defined as (cf. [13] (3.2.1))

$$E_T := \{(u, v) \in V^2 : u \neq v, \quad u = p(v)\}.$$

In particular, the notions of a root, a child, a $k$-th child coincide (cf. Lemma 2.2 (b)). Main differences regard the parent function (in our case defined also on a root) and the definition of leaves (we never call the root a leaf).

For every tree $T \subseteq V$ in a directed forest $T = (V, p)$, the pair $(T, p|_T)$ is a directed tree and the set root($((T, p|_T))$ has at most one element. A directed tree with a root is called a rooted directed tree. We usually denote the root of a directed tree $T$ by $\omega_T$, or simply $\omega$ if there is no ambiguity. A directed tree without a root is called rootless.
For \( W \subseteq V \) we can define a subforest \( \mathcal{T}_{|W} = (W, p_W) \), where

\[
p_W(v) := \begin{cases} p(v) & \text{if } p(v) \in W, \\ v & \text{if } p(v) \notin W, \end{cases} \quad v \in W.
\]

If \( W = \text{Des}(v) \) for some \( v \in V \) we write \( \mathcal{T}(v \rightarrow) := \mathcal{T}_{|W} \). Then \( \mathcal{T}(v \rightarrow) \) is a rooted directed tree and \( \text{root}(\mathcal{T}(v \rightarrow)) = \{v\} \). In this case we will use the name subtree.

For directed forests a natural notion of isomorphism can be introduced (see [21, Definition 2.8]). In most cases we are not making any distinction between isomorphic forests.

Forests are said to be disjoint if so are their sets of vertices. We can always assume that the considered forests are disjoint by taking appropriate isomorphic forests if necessary.

### 2.2. Backward extensions of directed trees

In the following subsection we recall operations on directed trees that are essential for the study of backward extensions of associated weighted shift operators.

Let us start with the simplest backward extension of a directed tree.

2.4. DEFINITION. Given a directed tree \( \mathcal{T} = (V, p) \) with root \( \omega \) and \( k \geq 0 \) we define its \( k \)-step backward extension as \( \mathcal{T}_{(k)} := (V_{(k)}, p_{(k)}) \), where \( V_{(k)} := V \sqcup \{\omega_j\}_{j=1}^k \), \( \omega_j \)'s are new distinct vertices, \( \omega_0 := \omega \) and

\[
p_{(k)}(v) := \begin{cases} p(v) & \text{if } v \in V^0 \\ \omega_j & \text{if } v = \omega_{j-1}, \ j = 1, \ldots, k, \\ \omega_k & \text{if } v = \omega_k \end{cases} \quad v \in V_{(k)}.
\]

![Diagram of a 3-step backward extension of a directed tree](image)

**Figure 1.** 3-step backward extension of a directed tree.

We will call any directed tree isomorphic to the one constructed above a \( k \)-step backward extension of \( \mathcal{T} \).

Another important operation on rooted directed trees is joining them by adding a new root as a parent for their original roots. Below you can find the formal definition. See also Figure 2.

2.5. DEFINITION. The rooted sum of a family \( \{\mathcal{T}_j = (V_j, p_j) : j \in J\} \) of pairwise disjoint rooted directed trees is the directed tree defined by

\[
\bigsqcup_{j \in J} \mathcal{T}_j := (V_{\bigsqcup J}, p_{\bigsqcup J}),
\]

where

\[
V_{\bigsqcup J} := \bigcup_{j \in J} V_j, \quad p_{\bigsqcup J}(v) := \begin{cases} p_j(v) & \text{if } v \in V_j \end{cases} \quad v \in V_{\bigsqcup J}.
\]
where \( V_J := \{ \omega \} \sqcup \bigcup_{j \in J} V_j \) and

\[
p_J(v) := \begin{cases} p_j(v) & \text{if } v \in V_j^o \\ \omega & \text{otherwise} \end{cases}, \quad v \in V_J.
\]

![Figure 2](image-url)

**Figure 2.** The rooted sum of the family \( \{T_1, T_2, T_3\} \).

Note that if the given family is empty, its rooted sum contains only one vertex and is a degenerate directed forest (tree).

Similarly as before, we will call any directed tree isomorphic to \( \bigcup_{j \in J} T_j \) a rooted sum of the family \( \{T_j\}_{j \in J} \). In particular

\[
\text{for any directed forest } T \text{ and its vertex } v, \text{ the directed tree } T(v \rightarrow) \text{ is a rooted sum of the family } \{T(u \rightarrow) : u \in \text{Chi}(v)\}. \tag{2.3}
\]

### 2.3. Weighted shift operators.

This subsection recalls the weighted shift operators on directed forests as defined in [21, Section 2]. Most of fundamental facts is analogous to the case of weighted shifts on directed trees described in [13].

Given a set \( V \) we consider the Hilbert space \( \ell^2(V) \) of all square-summable complex functions on \( V \) with the standard inner product

\[
\langle f, g \rangle := \sum_{v \in V} f(v)\overline{g(v)}, \quad f, g \in \ell^2(V).
\]

By \( \{e_v : v \in V\} \) we denote the canonical orthonormal basis of \( \ell^2(V) \) consisting of functions such that \( e_v \mid_{V \setminus \{v\}} \equiv 0 \) and \( e_v(v) = 1 \).

For a set \( W \subseteq V \), we will identify \( \ell^2(W) \) with a closed subspace of \( \ell^2(V) \) obtained by taking extensions of elements from \( \ell^2(W) \) by 0 on \( V \setminus W \). In particular \( \ell^2(\emptyset) = \{0\} \). With this convention we have \( \ell^2(W_1) \perp \ell^2(W_2) \), provided that \( W_1 \) and \( W_2 \) are disjoint subsets of \( V \). In most cases in this paper \( V \) will be the set of vertices in a directed tree.

### 2.6. Definition (Weighted shift).

For a directed forest \( T = (V, p) \) and a set of complex weights \( \lambda = \{\lambda_v\}_{v \in V} \) such that \( \lambda_{\omega} = 0 \) for \( \omega \in \text{root}(T) \), we
define the operator \( S_\lambda \) in \( \ell^2(V) \), called the \textit{weighted shift on} \( T \) \textit{with weights} \( \lambda \), as follows:

\[
\mathcal{D}(S_\lambda) := \{ f \in \ell^2(V) : \Gamma_\lambda(f) \in \ell^2(V) \},
\]

\[
S_\lambda(f) := \Gamma_\lambda(f), \quad f \in \mathcal{D}(S_\lambda),
\]

where \( \Gamma_\lambda : \mathbb{C}^V \to \mathbb{C}^V \) is defined by \( \Gamma_\lambda(f)(v) := \lambda_v f(p(v)) \).

We say a weighted shift \( S_\lambda \) is \textit{proper} if \( \{ v \in V : \lambda_v = 0 \} = \text{root}(T) \).

The above definition of a weighted shift operator is almost the same as [13] Definition 3.1.1. For a directed tree both definitions of \( S_\lambda \) agree up to the requirement of specifying the zero weight for the root.

Later in this paper the statement “\( \lambda \) is a system of weights on \( T \)” includes the assumption that \( \lambda_\omega = 0 \) for \( \omega \in \text{root}(T) \).

A weight can be understood as being assigned to an edge joining a vertex with its parent. As stated in the equality (2.4), the operator \( S_\lambda \) shifts a value at given vertex through all outgoing edges.

In view of Definition 2.6, a classical unilateral weighted shift on \( \ell^2(\mathbb{Z}_+) \) is a weighted shift on the directed tree \( T = (\mathbb{Z}_+, p) \), where \( p(0) = 0 \) and \( p(n + 1) = n \) for \( n \geq 0 \). For a bilateral weighted shift (acting on \( \ell^2(\mathbb{Z}) \)) we use the tree \( T = (\mathbb{Z}, p) \) with \( p(n) = n - 1 \) for \( n \in \mathbb{Z} \).

Proofs of the following simple facts does not differ from the case of directed trees described in [13] (cf. Propositions 3.1.3 and 3.1.8 therein).

2.7. \textsc{Proposition.} Let \( S_\lambda \) be a weighted shift on a directed forest \( T = (V, p) \). Then

(a) if \( v \in V \) and \( e_v \in \mathcal{D}(S_\lambda) \), then

\[
S_\lambda e_v = \sum_{u \in \text{Chi}(v)} \lambda_u e_u,
\]

(b) \( S_\lambda \) is a bounded operator on \( \ell^2(V) \) if and only if

\[
\sup \left\{ \sum_{u \in \text{Chi}(v)} |\lambda_u|^2 : v \in V \right\} < \infty;
\]

moreover, if \( S_\lambda \in \mathcal{B}(\ell^2(V)) \), then

\[
||S_\lambda||^2 = \sup \left\{ ||S_\lambda e_v||^2 : v \in V \right\} = \sup \left\{ \sum_{u \in \text{Chi}(v)} |\lambda_u|^2 : v \in V \right\},
\]

(c) \( S_\lambda \) is bounded from below on \( \mathcal{D}(S_\lambda) \) if and only if

\[
\inf \left\{ \sum_{u \in \text{Chi}(v)} |\lambda_u|^2 : v \in V \right\} > 0,
\]

(d) if \( S_\lambda \) is proper and densely defined, then \( T \) is locally countable (i.e. every tree in \( T \) is countable),

(e) if \( \ell^2(\text{Des}(v)) \subseteq \mathcal{D}(S_\lambda) \) for some \( v \in V \), then

\[
S_\lambda(\ell^2(\text{Des}(v))) \subseteq \ell^2(\text{Des}(v)).
\]

To describe powers of a weighted shift we introduce the following notation

\[
\lambda_v^{(k)} := \begin{cases} 1 & \text{if } k = 0 \\ \prod_{j=0}^{k-1} \lambda_{p^j(v)} & \text{if } k \geq 1 \end{cases}, \quad v \in V.
\]
Then the following fact holds (cf. [21] Lemma 3.6). 

2.8. Lemma. Suppose that $S_\lambda$ is a bounded weighted shift on a directed forest $\mathcal{T} = (V, p)$. Let $k \in \mathbb{N}$. Then for $f \in \ell^2(V)$,

$$S^{k}_\lambda e_v = \sum_{u \in \text{Chi}^{(k)}(v)} \lambda_{u}^{(k)} e_u,$$

(2.5)

$$S^{*k}_\lambda e_v = \lambda_{v}^{(k)} e_{p^{k}(v)}.$$  

(2.6)

A benefit from considering weighted shifts on directed forests is that this class is closed on taking powers of operator. In fact, the operator $S^{k}_\lambda$ is the weighted shift on the $k$-th power $\mathcal{T}^k$ of the forest $\mathcal{T}$ with weights $\lambda^{(k)} = \{\lambda_{v}^{(k)}\}_{v \in V}$. The reader is referred to [21] Definition 2.12 for the notion of $k$-th power of a directed forest. According to [21] Lemma 2.14 (c), if $S_\lambda$ is a weighted shift on an infinite directed tree, $S^{k}_\lambda$ is a weighted shift on a directed forest consisting of at least $k$ trees. In particular, for $k \geq 2$, it is no longer a weighted shift on a directed tree.

As a consequence of the formula (2.5) and Lemma 2.2 (a) we obtain the following useful property.

2.9. Corollary. Let $S_\lambda$ be a bounded weighted shift on a directed forest $\mathcal{T} = (V, p)$, $k \in \mathbb{Z}^+$ and $\sum_{v \in V} a_v e_v$ be a vector in $\ell^2(V)$. Then

$$\|S^{k}_\lambda \sum_{v \in V} a_v e_v\|^2 = \sum_{v \in V} |a_v|^2 \|S^{k}_\lambda e_v\|^2.$$ 

(2.7)

Similarly as for the classical weighted shifts (see [22] Corollary 1, p.52), the study of weighted shifts on directed trees can be restricted to the case when all the weights are non-negative (see [13] Theorem 3.2.1, [21] Theorem 3.7). In view of [21] Proposition 3.2, considering directed forests allows us to assume that all weights (except those associated to the roots) are strictly positive (cf. [13] Proposition 3.1.6).

3. Joint backward extension property

3.1. Introduction to backward extensions. Given a directed tree $\mathcal{T} = (V, p)$ and complex numbers $\{\lambda_v\}_{v \in \text{Des}^+(u)}$ for some $u \in V$, we will use the following notation

$$\lambda^+_u := \{\tilde{\lambda}_v\}_{v \in \text{Des}(u)}; \quad \tilde{\lambda}_v := \begin{cases} \lambda_v & \text{if } v \neq u, \\ 0 & \text{if } v = u, \end{cases} \quad v \in \text{Des}(u). \quad (3.1)$$

The so defined $\lambda^+_u$ is a system of weights on the rooted directed tree $\mathcal{T}_{(u \rightarrow)}$ (see page 5). If $S_\lambda$ is a bounded weighted shift on $\mathcal{T}$, then $S_{\lambda^+_u}$, the weighted shift on $\mathcal{T}_{(u \rightarrow)}$, is equal to the restriction $S_{\lambda}|_{\ell^2(\text{Des}(u))}$ of $S_{\lambda}$ to the invariant subspace $\ell^2(\text{Des}(u))$ (see Proposition 2.7 [13]).

We will consider whether weighted shifts on rooted directed trees can be extended to shifts on larger trees (for which the former root has a parent) with the same properties (e.g. subnormality). The basic case of such extension is the subject of the following definition.

Let $\mathcal{C}$ be a class of operators, e.g. subnormal. We will focus on classes closed with respect to restrictions to closed invariant subspaces.
3.1. Definition. Let $S_\lambda \in B(\ell^2(V))$ be a weighted shift on a rooted directed tree $T$. Given $k \in \mathbb{N}$, we say $S_\lambda$ admits a $C^k$-step backward extension if it belongs to the class $C$ and there exist non-zero weights $\lambda'_0, \ldots, \lambda'_{k-1}$ such that the weighted shift $S'_{\lambda'}$ on $T_{(k)}$ (see Definition 2.4) is bounded and belongs to $C$, where $\lambda' = \{\lambda'_v\}_{v \in V(k)}$, $\lambda'_v = 0$ and $\lambda'_v = \lambda_v$ whenever $v \in V^\circ$.

The weighted shift $S'_{\lambda'}$ is then called a $C^k$-step backward extension of $S_\lambda$. We say that $S_\lambda$ admits a $C^0$-step backward extension if $S_\lambda$ belongs to $C$.

In the case of a concrete class of operators we will replace “$C^k$” in the above definition by the name of members of the class, e.g. we will write that a weighted shift admits a subnormal $k$-step backward extension.

Assuming the class $C$ is closed on taking restrictions to invariant subspaces implies that every weighted shift on a directed tree admitting $C^k$-step backward extension admits also $C^{(k-1)}$-step backward extension ($k \geq 1$). Indeed, $\ell^2(V_{(k-1)})$ is an invariant subspace for a weighted shift on $T_{(k)}$ (cf. Proposition 2.7 [b]).

It is worth mentioning that if a weighted shift on a directed tree admits a subnormal (resp. power hyponormal) $k$-step backward extension, then any its nonzero scalar multiple does. It is not the case in the class of completely hyperexpansive operators (see Section 5), because due to $A_1(T) \leq 0$ any contractive completely hyperexpansive operator is automatically an isometry.

3.2. Proposition ([13 Proposition 8.1.3]). Let $T$ be a countable rooted and leafless directed tree. Then there exists $\lambda$, a system of weights on $T$, such that $S_\lambda$ is proper and isometric.

Proof. As usual, we denote the tree by $T = (V, p)$ and its root by $\omega$. To construct an isometry it suffices to take nonzero weights $\{\lambda_u\}_{u \in Chi(v)}$ such that $\sum_{u \in Chi(v)} |\lambda_u|^2 = 1$ for each $v \in V$. \qed

For several classes of weighted shifts the property of admitting $k$-step backward extensions is always satisfied.

3.3. Proposition. Let $C$ be one of the following classes of operators (in all cases we assume boundedness)

- all bounded,
- contractive,
- bounded from below,
- expansive,
- isometric,
- scalar multiples of isometries,
- non-zero hyponormal.

Then any weighted shift in the class $C$ admits a $C^k$-step backward extension for arbitrary $k \in \mathbb{N}$.

Proof. In case of $S_\lambda$ belonging to any of the aforementioned classes, setting $\lambda'_{uj} = \|S_\lambda e_u\|$ for $j = 0, \ldots, k-1$ gives a desired $C^k$-step backward extension (cf. Definition 3.1). Indeed, for bounded and contractive operators one can use Proposition 2.7 [b]. For operators bounded from below or expansive c.f. Proposition 2.7 [c]. Scalar multiples of isometries can be
resolved similarly as in the proof of Proposition 3.2. For the case of hyponormal operators see [13, Theorem 5.1.2] (c.f. [21, Theorem 4.1] with $k = 1$ or Theorem 4.1 in this paper). □

It is worth recalling that according to [13, Proposition 8.1.7.] all quasi-normal proper weighted shifts on directed trees are scalar multiples of isometries.

3.2. JBEP and its consequences. A more complex case of a backward extension is passing from a family of weighted shifts on rooted directed trees to a weighted shift on the rooted sum of given trees altogether in the same class of operators. For some classes the following phenomenon can happen.

3.4. Definition. Let $\mathcal{C}$ be a class of weighted shifts on directed trees.
We say that $\mathcal{C}$ has the joint backward extension property (JBEP) if the following statement is true.

For every $k \in \mathbb{Z}_+$ there exists a constant $c_k > 0$ such that for any non-empty at most countable family $\{S_j\}_{j \in J}$ of bounded proper weighted shifts on rooted directed trees $T_j = (V_j, p_j)$, the following conditions are equivalent:

(a) there exists a system $\{\theta_j\}_{j \in J} \subseteq \mathbb{C} \setminus \{0\}$ for which the weighted shift $S_\lambda$ on $\bigsqcup_{j \in J} T_j = (V_{\mathcal{J}}, p)$ with weights $\lambda = \{\lambda_v\}_{v \in V_{\mathcal{J}}}$ such that (with $\omega_j \in \text{root}(T_j)$)

$$
\lambda_{\omega_j} = \theta_j \quad \text{and} \quad S_\lambda \omega_j = S_j \quad \text{for} \quad j \in J,
$$

(3.2)

is bounded and admits a $\mathcal{C}$ $k$-step backward extension,

(b) the system $\{\theta_j\}_{j \in J}$ in (a) can be chosen so that additionally

$$
\|S_\lambda\| \leq c_k \sup\{\|S_j\| : j \in J\},
$$

(c) $S_j$ admits a $\mathcal{C}$ $(k+1)$-step backward extension for each $j \in J$ and $\sup\{\|S_j\| : j \in J\} < \infty$.

For several standard classes of operators it is immediate that they have the joint backward extension property.

3.5. Proposition. The following classes of weighted shifts have the joint backward extensions property with $c_k = 1$:

(a) bounded,
(b) contractive
(c) isometric,
(d) hyponormal.

Proof. Thanks to Proposition 3.3 it is enough to prove that a family of weighted shifts belonging to one of the above classes has an extension onto the rooted sum. In the case of hyponormal operators it can be easily deduced from the condition given in [13, Theorem 5.1.2]. Note that the assumption of admitting hyponormal $(k+1)$-step backward extension implies that operators in (c) are non-zero. Solutions for the remaining classes are straightforward. □

One could expect that for every class each member of which admits $k$-step backward extension (see Proposition 3.3) the joint backward extension
property would be satisfied. Such intuition is misleading and an example are operators bounded from below.

3.6. Proposition. The class of bounded from below weighted shifts on directed trees does not have the joint backward extensions property.

Proof. Consider a countable family \( \{ S_n \}_{n \in \mathbb{N}} \) of unilateral weighted shifts \( (T_n \simeq (\mathbb{N}, p)) \) for \( n \in \mathbb{N} \), with \( p(n) := \max\{1, n - 1\} \) such that \( \|S_n e_v\| = \frac{1}{n} \) for each vertex \( v \) of \( T_n \). According to Proposition 3.3, \( S_n \) admits a bounded from below \( k \)-step backward extension for every \( n, k \in \mathbb{N} \). On the other hand, the operators \( \{S_n\}_{n \in \mathbb{N}} \) clearly cannot be extended to a single operator bounded from below. \( \square \)

The above proof justifies that also the class of scalar multiples of isometries (i.e. quasinormal weighted shifts) does not satisfy JBEP.

The joint backward extension property has an impressive consequence. It is related to extending at most countable family of weighted shifts on directed trees to a weighted shift on any completion of the corresponding trees by a finite-depth \( 2 \) structure preserving the class in question.

3.7. Theorem. Let \( \mathcal{C} \) be a class with the joint backward extension property, \( T = (V, p) \) be a leafless directed tree with the root \( \omega \) and \( k \geq 1 \). Assume

\[ \text{Chi}^{(4)}(\omega) = \{\omega_1, \omega_2, \omega_3, \ldots\} \]

and vertices from \( W_4 \) are black while those outside \( W_4 \) are gray.

\[ \text{Figure 3.} \text{ Subtrees joined "at the depth } k = 4\text{"; here } \text{Chi}^{(4)}(\omega) = \{\omega_1, \omega_2, \omega_3, \ldots\} \text{ and vertices from } W_4 \text{ are black while those outside } W_4 \text{ are gray.} \]

\[ \text{3.7. Theorem. Let } \mathcal{C} \text{ be a class with the joint backward extension property, } T = (V, p) \text{ be a leafless directed tree with the root } \omega \text{ and } k \geq 1. \text{ Assume} \]

\[ \text{The term depth appears here without a formal definition, as it is not necessary in the statement of the theorem. Figures 3 and 4 should give an intuition.} \]
that weights \( \{ \lambda_v \}_{v \in V} \subseteq \mathbb{C} \setminus \{0\} \) are given, where

\[
W_k := \bigcup_{v \in \text{Chi}^{(k)}(\omega)} \text{Des}^0(v).
\]

Then the following conditions are equivalent:

(a) there exist \( \{ \lambda_v \}_{v \in V \setminus W_k} \subseteq \mathbb{C} \) such that the weighted shift \( S_\lambda \) with weights \( \lambda = \{ \lambda_v \}_{v \in V} \) is bounded proper and belongs to \( \mathcal{C} \),

(b) the set \( \text{Chi}^{(k)}(\omega) \) is at most countable, the weighted shift \( S_{\lambda^0} \) (see (3.1)) is bounded proper and admits a \( \mathcal{C} \) \( k \)-step backward extension for every \( v \in \text{Chi}^{(k)}(\omega) \), and \( \sup \{ \| S_{\lambda^0} \| : v \in \text{Chi}^{(k)}(\omega) \} < \infty \).

It may be worth noticing that \( \mathcal{C} \) is bounded proper and belongs to \( \mathcal{C} \).

\[
\text{Des}^0(v) := \bigcup_{\omega \in \text{Chi}(v)} \text{Des}^0(\omega).
\]

Proof. By Proposition 2.7 (4) and the assumption that \( \mathcal{T} \) is leafless any of the conditions (a) or (b) implies that the entire tree (i.e. the set \( V \)) is at most countable.

We will proceed by induction on \( k \).

The case \( k = 1 \) is covered by equivalence (a) \( \Leftrightarrow b \) (c) in Definition 3.4 with \( k = 0 \). Now assume that (a) and (b) are equivalent for a fixed \( k \geq 1 \). We will show that (a) \( \Leftrightarrow b \) \( k + 1 \)

(b) \( \Rightarrow \) (a) Assume that for each \( v \in \text{Chi}^{(k+1)}(\omega) \), the weighted shift \( S_{\lambda^0} \) is bounded by a common constant \( C > 0 \) and admits a \( \mathcal{C} \) \( (k+1) \)-step backward extension. For a fixed \( u \in \text{Chi}^{(k)}(\omega) \) we can apply the joint backward extension property to the family \( \{ S_{\lambda^0} \}_{v \in \text{Chi}(u)} \) in order to get nonzero weights \( \{ \lambda_v \}_{v \in \text{Chi}(u)} \) such that the weighted shift \( S_{\lambda^0} \) admits a \( \mathcal{C} \) \( k \)-step backward extension (see (2.3)). According to the condition (b) in Definition 3.4 the weights can be chosen such that the norm of \( S_{\lambda^0} \) is bounded by \( c_k \exp(C) \) for all \( u \in \text{Chi}^{(k)}(\omega) \).

Summarising, \( \{ S_{\lambda^0} \}_{u \in \text{Chi}^{(k)}(\omega)} \) is a uniformly bounded family of proper weighted shifts, each of which admits a \( \mathcal{C} \) \( k \)-step backward extension. Using the induction hypothesis completes the proof of (a).

(a) \( \Rightarrow \) (b) Now assume that there exist weights \( \{ \lambda_v \}_{v \in V \setminus W_{k+1}} \) such that \( S_\lambda \) is bounded proper and belongs to \( \mathcal{C} \). We want to show that each weighted shift \( S_{\lambda^0} \) with \( v \in \text{Chi}^{(k+1)}(\omega) \) admits \( \mathcal{C} \) \( (k+1) \)-step backward extension. They are uniformly bounded as restrictions of the bounded operator \( S_\lambda \) to invariant subspaces (see Proposition 2.7 (4)).

The whole system \( \lambda \) can be seen as an extension of the system \( \{ \lambda_v \}_{v \in W_k} \) since \( W_k \supseteq W_{k+1} \). Then, by induction hypothesis, each weighted shift \( S_{\lambda^0} \) for \( u \in \text{Chi}^{(k)}(\omega) \) admits a \( \mathcal{C} \) \( k \)-step backward extension. If \( u \in \text{Chi}^{(k)}(\omega) \), then \( S_{\lambda^0} \) is a weighted shift on a rooted sum of the family \( \{ \mathcal{T}(v \rightarrow w) \}_{v \in \text{Chi}(u)} \) (cf. (2.3)). It follows from the joint backward extension property that \( S_{\lambda^0} \) admits a \( \mathcal{C} \) \( (k + 1) \)-step backward extension for every \( v \in \text{Chi}^{(k)}(\omega) \). By Lemma 2.2 (4),

\[
\text{Chi}^{(k+1)}(\omega) = \bigcup_{u \in \text{Chi}^{(k)}(\omega)} \text{Chi}(u),
\]

which shows that (b) is valid. This completes the proof. \( \square \)
Figure 4. An example of two non-isomorphic directed trees completing a family \( \{T_1, T_2, T_3, T_4\} \) at the depth \( k = 3 \). According to Theorem 3.7, an extension in class \( C \) exists either in both cases or in neither.

3.8. Remark. Theorem 3.7 can be interpreted in a different way. Namely, given a family of rooted directed trees we can complete them into one directed tree by adding some new vertices such that the roots of initial trees become the \( k \)-th children of the root of the new tree. The value of this theorem is that the condition (b) does not depend on the structure of the first \( k \) generations of the enveloping tree. It is worth pointing out that for \( k \geq 2 \) the same family of trees can be completed in a variety of ways with enveloping directed trees being not-isomorphic (see Figure 4).

3.9. Remark. From the proof of Theorem 3.7 one can deduce, that the construction of joint backward extension is possible also if the roots of the trees from the predefined family are not necessarily all on the same level. It suffices that if for some \( v \in \text{Chi}_{j}^{(j)}(\omega) \) the weighted shift \( S_{\lambda_{v}} \) is fixed, then it admits \( C \) \( j \)-step backward extension.

3.10. Theorem. The class of bounded subnormal weighted shifts on directed trees has the joint backward extension property.

The above theorem follows from [21, Lemma 5.6]. Theorem 5.7 therein was a particular case of Theorem 3.7 above.

In the forthcoming section we investigate whether classes of power hyponormal and completely hyperexpansive weighted shifts satisfy JBEP.

4. Power hyponormality

All hyponormal classical weighted shifts on \( \ell^{2} \) are automatically power hyponormal. This is no longer the case for weighted shifts on directed trees, as shown by [13, Example 5.3.2]. In [21, Section 4] a description of all those directed trees on which every bounded hyponormal and proper weighted shift is power hyponormal is given.

The characterisation of hyponormality of weighted shifts on directed trees was presented in [13, Theorem 5.1.2]. The theorem below is a generalisation to the case of arbitrary powers of proper weighted shifts on directed forests. It can be deduced from the result for trees or found in [21, Section 4].
4.1. Theorem. A bounded proper weighted shift \( S_\lambda \) on a directed forest \( T = (V, p) \) is power hyponormal if and only if the following two conditions are satisfied:

(a) the forest \( T \) is leafless,
(b) for all \( v \in V \) and \( k \in \mathbb{N} \)

\[
h^{(k)}(v) = h^{(k)}_{\lambda}(v) := \sum_{u \in \text{Chi}^{(k)}(v)} \frac{|\lambda_u|^2}{\|S^k e_u\|^2} \leq 1. \tag{4.1}
\]

The following lemma reduces the question about admitting a power hyponormal \( k \)-step backward extension for a weighted shift on a rooted directed tree to the existence of a single number. For the notion of the \( k \)-step backward extension of a directed tree see Definition 2.4.

4.2. Lemma. Fix \( k \in \mathbb{N} \). Suppose that \( \mathcal{T}_{(k)} = (V_{(k)}, p_{(k)}) \) is a \( k \)-step backward extension of a rooted directed tree \( T = (V, p) \) and \( S_\lambda \) is a bounded power hyponormal proper weighted shift on \( \mathcal{T}_{(k)} \) with weights \( \lambda = \{\lambda_v\}_{v \in V_{(k)}} \). Let \( \tilde{\lambda} = \{\tilde{\lambda}_v\}_{v \in V_{(k)}} \) be a system of weights such that \( \tilde{\lambda}_v := \lambda_v \) for \( v \in V^0 \), \( \tilde{\lambda}_{v,j} := \lambda_{v_{k-1}} \) for \( j = 0, \ldots, k-1 \) and \( \tilde{\lambda}_{v,k} := 0 \). Then the weighted shift \( S_{\tilde{\lambda}} \) on \( \mathcal{T}_{(k)} \) is proper and power hyponormal.

Note that both \( S_\lambda \) and \( S_{\tilde{\lambda}} \) are power hyponormal \( k \)-step backward extensions of the weighted shift \( S_\lambda|_{\ell^2(V)} \) on \( T \).

Proof. The leaflessness of \( \mathcal{T}_{(k)} \) follows from the hyponormality of \( S_\lambda \). In the view of Theorem 4.1 all we have to prove is that \( h^{(n)}_{\lambda}(v) \leq 1 \) for all \( v \in V_{(k)} \) and \( n \in \mathbb{N} \). Since \( \tilde{\lambda}_v = \lambda_v \) for \( v \in V^0 = \text{Des}^0(\omega) \), we see that \( h^{(n)}_{\lambda}(v) = h^{(n)}_{\tilde{\lambda}}(v) \leq 1 \) for \( v \in \text{Des}(\omega) \) and \( n \in \mathbb{N} \). It remains to estimate \( h^{(n)}_{\tilde{\lambda}}(\omega_m) \) for \( m = 1, \ldots, k \) and \( n \in \mathbb{N} \).

Set \( \theta = |\tilde{\lambda}_{\omega_0}|^2 = |\lambda_{\omega_{k-1}}|^2 \). Since

\[
1 \geq h^{(1)}_{\lambda}(\omega_{j+2}) = \frac{|\lambda_{\omega_{j+1}}|^2}{|\lambda_{\omega_j}|^2}, \quad j = 0, \ldots, k-2,
\]

we conclude that

\[
|\tilde{\lambda}_{\omega_{m}}|^2 = \frac{1}{\theta} \leq |\lambda_{\omega_m}|^2, \quad m = 0, \ldots, k-1. \tag{4.2}
\]

If \( 2n \leq m \), then

\[
h^{(n)}_{\lambda}(\omega_m) = \frac{|\tilde{\lambda}_{\omega_{m-n}}|^2}{|\lambda_{\omega_{m-n}}|^2} = \theta^n = 1.
\]

When \( n \leq m < 2n \) we have

\[
h^{(n)}_{\lambda}(\omega_m) = \frac{|\tilde{\lambda}_{\omega_{m-n}}|^2}{\|S^n_{\lambda} e_{\omega_{m-n}}\|^2} = \frac{|\tilde{\lambda}_{\omega_{m-n}}|^2}{\|S^n_{\tilde{\lambda}} e_{\omega_{m-n}}\|^2} = \frac{\theta^{2n-m}}{\|S^{2n-m} e_{\omega_0}\|^2}.
\]

Since \( 1 \leq 2n - m \leq m \), it follows that

\[
1 \geq h^{(2n-m)}_{\lambda}(\omega_{2n-m}) = \frac{|\tilde{\lambda}_{\omega_{2n-m}}|^2}{\|S^{2n-m}_{\lambda} e_{\omega_0}\|^2} = \frac{1}{\theta^{2n-m}} \geq \frac{\theta^{2n-m}}{\|S^{2n-m} e_{\omega_0}\|^2} = h^{(n)}_{\lambda}(\omega_m).
\]
We are left with the case when \( m < n \), but then
\[
\begin{align*}
\quad h^{(n)}_\lambda(\omega_m) &= \sum_{v \in \text{Chi}^{(n)}(\omega_m)} \frac{|\lambda_v^{(n)}|^2}{\|S^n_\lambda e_v\|^2} = \sum_{v \in \text{Chi}^{(n-m)}(\omega_0)} \frac{|\lambda_v^{(m)}|^2|\lambda_v^{(n-m)}|^2}{\|S^m_\lambda e_v\|^2} \\
\quad \leq & \sum_{v \in \text{Chi}^{(n-m)}(\omega_0)} \frac{|\lambda_v^{(m)}|^2|\lambda_v^{(n-m)}|^2}{\|S^m_\lambda e_v\|^2} = h^{(n)}_\lambda(\omega_m) \leq 1.
\end{align*}
\]

This completes the proof of the power hyponormality of \( S^m_\lambda \).

\[ \Box \]

The following characterisation of weighted shifts admitting power hyponormal backward extension holds.

4.3. Lemma. Let \( S_\lambda \) be a nonzero bounded and power hyponormal proper weighted shift on a directed tree \( T = (V, p) \) with a root \( \omega \) and \( k \) be a positive integer. Then the following conditions are equivalent:

(a) \( S_\lambda \) admits power hyponormal \( k \)-step backward extension,
(b) there exists \( C > 0 \) such that
\[
\sum_{u \in \text{Chi}^{(n)}(\omega)} \frac{|\lambda_u^{(n)}|^2}{\|S_{\lambda}^{n+m} e_u\|^2} \leq C, \quad n \in \mathbb{Z}_+, \ 1 \leq m \leq k.
\] (4.3)

Recall that from properness and power hyponormality of \( S_\lambda \) we conclude that \( \|S_{\lambda}^{n} e_u\|^2 > 0 \) for all \( n \in \mathbb{Z}_+ \) and \( u \in V \).

Proof. (a)⇒(b) Assume that the weighted shift \( S_\lambda \) on \( T(k) \) is a power hyponormal \( k \)-step backward extension of \( S_\lambda \) (cf. Definition 3.1).

Denote \( \theta = |\lambda_\omega|^{2} \). It is strictly positive, since we do not accept zero weights for backward extensions. In view of the previous lemma, we can assume that \( \lambda_{\omega_m} = \lambda_\omega \) for \( m = 1, \ldots, k - 1 \).

Since \( S_\lambda \) is power hyponormal, by Theorem 4.1 we have
\[
\begin{align*}
1 &\geq h^{(n+m)}_\lambda(\omega_m) = \sum_{u \in \text{Chi}^{(n+m)}(\omega_m)} \frac{|\lambda_u^{(n+m)}|^2}{\|S_{\lambda}^{n+m} e_u\|^2} = \sum_{u \in \text{Chi}^{(n)}(\omega_0)} \frac{\theta^m |\lambda_u^{(n)}|^2}{\|S_{\lambda}^{n+m} e_u\|^2} \\
&\text{for } n \in \mathbb{Z}_+ \text{ and } 1 \leq m \leq k. \quad \text{Consequently}
\end{align*}
\]
\[
\sum_{u \in \text{Chi}^{(n)}(\omega_0)} \frac{|\lambda_u^{(n)}|^2}{\|S_{\lambda}^{n+m} e_u\|^2} \leq \frac{1}{\theta^m}, \quad n \in \mathbb{Z}_+, \ 1 \leq m \leq k.
\]

Hence, setting \( C = \max\{1, 1/\theta^k\} \) is sufficient for (b).

(b)⇒(a) We can find a constant \( \theta \in (0, 1) \) such that
\[
\frac{1}{\theta^m} \geq \sum_{u \in \text{Chi}^{(n)}(\omega)} \frac{|\lambda_u^{(n)}|^2}{\|S_{\lambda}^{n+m} e_u\|^2}, \quad n \in \mathbb{Z}_+, \ 1 \leq m \leq k \quad \text{(4.4)}
\]
and also
\[
\|S_{\lambda}^{m} e_u\|^2 \geq \theta^m, \quad 1 \leq m \leq k. \quad \text{(4.5)}
\]

Using this constant we set \( \tilde{\lambda}_{\omega_m} := \sqrt{\theta} \) for \( m = 0, \ldots, k - 1 \), \( \tilde{\lambda}_{\omega_0} := 0 \) and \( \tilde{\lambda}_v := \lambda_v \) for \( v \in V^0 \). We claim that the weighted shift \( S_{\lambda} \) with weights \( \tilde{\lambda} = \{\tilde{\lambda}_v\}_{v \in V(k)} \) is power hyponormal.
Clearly, $h^{(n)}(v) = h^{(n)}_\lambda(v) \leq 1$ for all $v \in V$ and $n \in \mathbb{N}$. Fix $1 \leq m \leq k$.

If $2n < m$, then $h^{(n)}_\lambda(\omega_m) = \theta^n = 1$.

In the case $n \leq m < 2n$ we have

$$h^{(n)}_\lambda(\omega_m) = \frac{|\tilde{\lambda}^{(n)}_{m-n}|^2}{\|S^2_{\lambda}e_{\omega_{m-n}}\|^2} = \frac{\theta^n}{\|S^2_{\lambda}e_{\omega_0}\|^2} \leq 1.$$

We are left with the case $m < n$. Then

$$h^{(n)}_\lambda(\omega_m) = \sum_{u \in \text{Chi}^{(n)}(\omega_m)} \frac{|\tilde{\lambda}^{(n)}_u|^2}{\|S^2_{\lambda}e_u\|^2} \leq \frac{\theta^m |\tilde{\lambda}^{(n-m)}_u|^2}{\|S^2_{\lambda}e_u\|^2} \leq 1.$$

Since, by Theorem 4.1, $\mathcal{T}$ is leafless and by $S_\lambda \neq 0$ it is non-degenerate, $\mathcal{T}(k)$ is leafless. By the same theorem the weighted shift $S_\lambda$ on $\mathcal{T}(k)$ is a power hyponormal $k$-step backward extension of $S_\lambda$, so (a) is proven.

From the above characterisation we can deduce the following fact about backward extensions of power hyponormal weighted shifts.

**4.4. Corollary.** If a bounded power hyponormal proper weighted shift on a rooted directed tree is bounded from below, then it admits a power hyponormal $k$-step backward extension for every $k \in \mathbb{N}$.

**Proof.** Denote the considered weighted shift on a rooted directed tree $\mathcal{T} = (V, p)$ by $S_\lambda$. Let $C \in (0, 1]$ be such that

$$\|S_\lambda f\|^2 \geq C\|f\|^2, \quad f \in \ell^2(V).$$

If $n \in \mathbb{N}$ and $m = 1, \ldots, k$, then by Theorem 4.1

$$\sum_{u \in \text{Chi}^{(n)}(\omega)} \frac{|\lambda^{(n)}_u|^2}{\|S^{m+n}_{\lambda}e_u\|^2} \leq \sum_{u \in \text{Chi}^{(n)}(\omega)} \frac{|\lambda^{(n)}_u|^2}{\|S^{m}_{\lambda}e_u\|^2C^m} \leq \frac{h^{(n)}(\omega)}{C^m} \leq C^{-k}.$$

For $n = 0$, we obtain

$$\sum_{u \in \text{Chi}^{(0)}(\omega)} \frac{|\lambda^{(0)}_u|^2}{\|S^{m}_{\lambda}e_u\|^2} = \frac{1}{\|S^{m}_{\lambda}e_\omega\|^2} \leq \frac{1}{C^m} \leq C^{-k}.$$

Applying Lemma 1.3 finishes the proof.

Now we are ready to prove that the class of power hyponormal weighted shifts satisfies the joint backward extension property (with $c_k = 1$). For the reader’s convenience, statement of the theorem contains details of the JBEP. See Definition 2.5 for the notion of rooted sum of directed trees.

**4.5. Theorem.** Let $J$ be a non-empty at most countable set and $k \in \mathbb{Z}_+$. Suppose that for $j \in J$, $S_j$ is a bounded proper weighted shift on a directed tree $\mathcal{T}_j = (V_j, p_j)$ with root $\omega_j$. Then the following conditions are equivalent:
Lemma 4.3 we have to prove that for every $\lambda\omega_j = \theta_j$ and $S_{\lambda\omega_j} = S_j$ for all $j \in J$.

This leads to

$$\lambda\omega_j = \theta_j \quad \text{and} \quad S_{\lambda\omega_j} = S_j \quad \text{for all} \quad j \in J.$$ (4.6)

Moreover, if $S_\lambda$ is a bounded proper weighted shift on $\mathcal{L}_{\lambda} = \sum_{j \in J} \mathcal{T}_j$ that admits a power hyponormal $k$-step backward extension, then $\|S_\lambda\| = \sup\{\|S_j\| : j \in J\}$.

**Proof.** Set $\mathcal{T} = \mathcal{L}_{\lambda} = \sum_{j \in J} \mathcal{T}_j$.

(a)⇒(b) The uniform boundedness part of (b) is obvious. According to Lemma 4.3 we have to prove that for every $j \in J$, there exists a constant $C_j > 0$ such that

$$\sum_{u \in \mathrm{Chi}_{\mathcal{T}_j}^{(n)}(\omega_j)} \left| \lambda_u^{(n+1)} \right|^2 \cdot \frac{||S_j^m e_u||^2}{||S_j^{n+m} e_u||^2} \leq C_j, \quad n \in \mathbb{Z}_+, \ 1 \leq m \leq k + 1.$$ (4.7)

Fix $j_0 \in J$. Note that assuming $C \geq 1$, the inequality (4.3) is satisfied also for $m = 0$, since $h^{(n)}(\omega) \leq 1 \leq C$. By the assumption on $S_\lambda$ and Lemma 4.3 we have

$$C \geq \sum_{u \in \mathrm{Chi}_{\mathcal{T}_j}^{(n+1)}(\omega)} \frac{||S_j^{n+1} e_u||^2}{||S_j^{n+1+m} e_u||^2} \left| \lambda_u^{(n+1)} \right|^2$$

$$= \sum_{j \in J} \sum_{u \in \mathrm{Chi}_{\mathcal{T}_j}^{(n)}(\omega_j)} \frac{||S_j^{n+m} e_u||^2}{||S_j^{n+m} e_u||^2} \left| \lambda_u^{(n)} \right|^2 \geq \left| \theta_{j_0} \right|^2 \sum_{u \in \mathrm{Chi}_{\mathcal{T}_j_0}^{(n)}(\omega_{j_0})} \frac{\left| \lambda_u^{(n)} \right|^2}{||S_j^{n+m} e_u||^2} \quad 1 \leq m \leq k \leq k + 1, \ n \in \mathbb{Z}_+.$$ (4.7)

This leads to

$$\sum_{u \in \mathrm{Chi}_{\mathcal{T}_{j_0}}^{(n)}(\omega_{j_0})} \frac{||S_{j_0}^m e_u||^2}{||S_{j_0}^{n+m} e_u||^2} = \sum_{u \in \mathrm{Chi}_{\mathcal{T}_{j_0}}^{(n)}(\omega_{j_0})} \frac{||S_j^{n+m} e_u||^2}{||S_j^{n+m} e_u||^2} \leq \frac{C}{\left| \theta_{j_0} \right|^2} =: C_{j_0}$$

for $m = 1, \ldots, k + 1$ and $n \in \mathbb{Z}_+$, which completes the proof of (b).

(b)⇒(a) Since operators in question cannot be zero, we may and do assume that $\sup_{j \in J} \|S_j\| = 1$.

It follows from Lemma 4.3 that for every $j \in J$ there exists a constant $C_j \geq 1$ such that

$$\sum_{u \in \mathrm{Chi}_{\mathcal{T}_j}^{(n)}(\omega_j)} \frac{||S_j^m e_u||^2}{||S_j^{n+m} e_u||^2} \leq C_j, \quad n \in \mathbb{Z}_+, \ 1 \leq m \leq k + 1.$$ (4.7)
Pick a system of positive real numbers \( \{a_j\}_{j \in J} \) such that
\[
(*) \quad \sum_{j \in J} a_j C_j^3 < \infty, \quad (**) \quad \sum_{j \in J} a_j C_j = 1.
\]
Indeed, as \( J \) is at most countable, there exist positive real numbers \( \{a'_j\}_{j \in J} \) such that \( \sum_{j \in J} a'_j = 1 \). Then, by setting \( a_j := a'_j \min\{1, C_j^{-3}\} \) for \( j \in J \setminus \{j_0\} \), where \( j_0 \in J \) is fixed, and defining
\[
a_{j_0} := \left( 1 - \sum_{j \in J \setminus \{j_0\}} a_j C_j \right) / C_{j_0},
\]
we obtain (\( * \)) and (\( ** \)).

Take \( \theta_j \)'s such that \( |\theta_j|^2 = a_j \) for all \( j \in J \) and define \( \lambda = \{\lambda_v\}_{v \in V_j} \) by (4.6). We claim that this system meets the condition (a). Clearly
\[
\sum_{u \in \text{Chi}(\omega)} |\lambda_u|^2 = \sum_{j \in J} a_j \leq \sum_{j \in J} a_j C_j \quad (**) \equiv 1.
\]
According to Proposition 2.7 (b), this together with uniform boundedness of \( S_j \)'s proves that \( S_\lambda \) is bounded.

Let \( n \) be a positive integer. Then
\[
\mathfrak{h}^{(n)}(\omega) = \sum_{u \in \text{Chi}^{(n)}(\omega)} \frac{\|S_\lambda^ne_u\|^2}{\|S_\lambda^n e_u\|^2}
= \sum_{j \in J} a_j \sum_{u \in \text{Chi}^{(n-1)}(\omega_j)} \frac{\|S_\lambda^{n-1}e_u\|^2}{\|S_\lambda^n e_u\|^2}
\leq \sum_{j \in J} a_j C_j = 1.
\]

Since all \( S_j \)'s are power hyponormal, we infer from Theorem 4.1 that \( S_\lambda \) is power hyponormal. If \( k = 0 \), then we are done.

In the case \( k \geq 1 \), according to Lemma 4.3 it suffices to prove that there exists a constant \( C > 0 \) satisfying (4.3). We will consider two possible cases.

Case 1. \( n = 0 \).

First, observe that for \( j \in J \),
\[
C_j \frac{\|S_j^m e_{\omega_j}\|^2}{\|S_j^m e_u\|^2} = \sum_{u \in \text{Chi}^{(0)}(\omega_j)} C_j \frac{\|S_\lambda^{n+1}e_u\|^2}{\|S_\lambda^n e_u\|^2} \leq C_j^2, \quad m = 0, \ldots, k.
\]
Hence, for \( m \in \{1, \ldots, k\} \),
\[
\sum_{u \in \text{Chi}^{(0)}(\omega)} \frac{\|S_\lambda^m e_u\|^2}{\|S_\lambda^m e_u\|^2} = \frac{1}{\|S_\lambda^m e_{\omega}\|^2} = \frac{1}{\sum_{j \in J} a_j \|S_j^{m-1}e_{\omega_j}\|^2}
= \frac{1}{\sum_{j \in J} a_j C_j \frac{\|S_j^{m-1}e_{\omega_j}\|^2}{C_j}} \quad (1) \leq \sum_{j \in J} a_j C_j \frac{C_j}{\|S_j^{m-1}e_{\omega_j}\|^2}
\]
One can observe that

\[ \| S^m e_u \|^2 = \sum_{j \in J} a_j C_j \leq \sum_{j \in J} \| S^m e_u \|^2, \]

where (†) follows from (**) and the Jensen inequality applied to the convex function \( x \mapsto \frac{1}{x} \).

Case 2. \( n \in \mathbb{N} \).

For \( m \in \{1, \ldots, k\} \), we have

\[
\sum_{u \in \text{Chi}_{Jm}^{(n)}(\omega)} \frac{\| S_{\lambda}^m u \|^2}{\| S_{\lambda}^{n+m} e_u \|^2} = \sum_{j \in J} \sum_{u \in \text{Chi}_{Jj}^{(n-1)}(\omega_j)} \frac{\| S_{\lambda}^n u \|^2}{\| S_{\lambda}^{n+m} e_u \|^2} \\
= \sum_{j \in J} a_j \sum_{u \in \text{Chi}_{Jj}^{(n-1)}(\omega_j)} \frac{\| S_{\lambda}^{n-1} e_u \|^2}{\| S_{\lambda}^{n+m} e_u \|^2} \\
\leq \sum_{j \in J} a_j C_j = 1.
\]

Together with the previous estimation we obtain that the condition (b) in Lemma 4.3 is satisfied with constant \( C := \sum_{j \in J} a_j C_j \geq 1 \). Hence \( S_{\lambda} \) admits a power hyponormal \( k \)-step backward extension.

For the “moreover” part observe that

\[
\| S_{\lambda}^n(S_{\lambda} e_w) \|^2 = \left\| \sum_{j \in J} \lambda_{\omega_j} S_{\lambda} e_{\omega_j} \right\|^2 \leq \sum_{j \in J} \| \lambda_{\omega_j} \|^2 \| S_{\lambda} e_{\omega_j} \|^2 = \| S_{\lambda} e_w \|^4,
\]

and

\[
\| S_{\lambda}(S_{\lambda} e_w) \|^2 = \sum_{j \in J} |\lambda_{\omega_j}|^2 \| S_{\lambda} e_{\omega_j} \|^2 \leq \sum_{j \in J} |\lambda_{\omega_j}|^2 \| S_j \|^2 \\
\leq \| S_{\lambda} e_w \|^2 \sup \{ \| S_j \|^2 : j \in J \}.
\]

By hyponormality we have \( \| S_{\lambda}^n(S_{\lambda} e_w) \|^2 \leq \| S_{\lambda}(S_{\lambda} e_w) \|^2 \), and consequently \( \| S_{\lambda} \| \leq \sup \{ \| S_j \| : j \in J \} \).

As stated in Theorem 3.7 the joint backward extension property applies not only to extensions to weighted shifts onto rooted sums.

4.6. Remark. According to [21] Proposition 3.9, if some countable leafless directed subtree \( T_{(v \rightarrow)} \) is given without predefined weights, we can set them in such a way that the associated weighted shift admits isometric (subnormal, power hyponormal) \( k \)-step backward extension, and hence such “blank directed tree” does not affect the existence of a subnormal (power hyponormal) extension.

5. Complete hyperexpansivity

Given an operator \( T \in B(H) \) we define

\[
A_n(T) := \sum_{j=0}^n (-1)^j \binom{n}{j} T^{\circ j} T_j, \quad n \geq 0.
\]

One can observe that \( A_{n+1}(T) = A_n(T) - T^* A_n(T) T \) for every \( n \in \mathbb{Z}_+ \) and \( A_0(T) = I \). We call \( T \in B(H) \) completely hyperexpansive if \( A_n(T) \leq 0 \)
for all \( n \in \mathbb{N} \). The notion was introduced by Athavale in [2]. Due to the following result of Agler [1] the complete hyperexpansivity is in a sense ‘dual’ to subnormality (and contractivity).

5.1. Theorem (Agler). An operator \( T \in B(H) \) is a subnormal contraction if and only if \( A_n(T) \geq 0 \) for every \( n \in \mathbb{N} \).

From the aforementioned recurrence easily follows that every isometry is a completely hyperexpansive operator, such that

\[
A_n(T) = I - T^*T = 0, \quad n \geq 1.
\]

It is worth noting that for a completely hyperexpansive operator \( T \in B(H) \) also \( T^n \) is completely hyperexpansive for all \( n \in \mathbb{N} \). Proof of this fact can be found in [12, Theorem 2.3].

Hyperexpansivity is closely related to the following notion.

5.2. Definition. A sequence \( \{a_n\}_{n=0}^\infty \) is called completely alternating if

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{m+j} \leq 0, \quad n \in \mathbb{N}, \; m \in \mathbb{Z}_+.
\]

Some basic properties of these sequences are listed below. Their elementary proofs are omitted.

5.3. Proposition. Let \( \{a_n\}_{n=0}^\infty \) be a sequence of real numbers. Define

\[
A_m^n := \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{m+j}, \quad m,n \in \mathbb{Z}_+.
\]

Then

(a) \( A_m^0 = a_m \) and \( A_m^{n+1} = A_m^n - A_{m+1}^n \) for \( m,n \in \mathbb{Z}_+ \),

(b) if \( \{a_n\}_{n=0}^\infty \) is completely alternating, then so is \( \{a_{n+1}\}_{n=0}^\infty \),

(c) if \( \{a_n\}_{n=0}^\infty \) is completely alternating, then the sequence \( \{A_m^n\}_{m=0}^\infty \) is monotonically increasing for every \( n \in \mathbb{Z}_+ \) (in particular \( \{a_n\}_{n=0}^\infty \) is monotonically increasing),

(d) if \( \{a_n\}_{n=0}^\infty \) is completely alternating sequence with positive terms, then the quotients of consecutive terms form a monotonically decreasing sequence, i.e. \( \frac{a_{n+1}}{a_n} \geq \frac{a_{n+2}}{a_{n+1}} \) for \( n \geq 0 \),

(e) completely alternating sequences form a convex cone in \( \mathbb{R}^{\mathbb{Z}_+} \) closed with respect to pointwise convergence.

More facts about completely alternating sequences can be found in e.g. [3].

The relation between completely hyperexpansive operators and completely alternating sequences is presented in the following lemma. In [13] the criterion appearing in the lemma is used as a definition of complete hyperexpansivity.

5.4. Lemma. An operator \( T \in B(H) \) is completely hyperexpansive if and only if the sequence \( \{\|T^n f\|^2\}_{n=0}^\infty \) is completely alternating for every \( f \in H \).

Proof. It is a consequence of (5.1), (5.2) and the definition of a negative definite operator. \( \square \)
Completely alternating sequences are somehow similar to the moment sequences mentioned in the section about subnormality. The following proposition provides a handy measure-theoretic characterisation. Its proof can be found in [3, Proposition 4.6.12].

5.5. Theorem. A sequence \( \{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R} \) is completely alternating if and only if there exists a positive Borel measure \( \tau \) on \([0, 1]\) such that
\[
a_n = a_0 + \int_{[0, 1]} \sum_{j=0}^{n-1} t^j \, d\tau(t), \quad n \in \mathbb{N}.
\] (5.3)

5.6. Remark. The formulation of [3, Proposition 4.6.12] mentions a Radon measure \( \mu \) on \([0, 1)\) such that
\[
a_n = a + nb + \int_{[0, 1)} (1 - x^n) \, d\mu(x), \quad n \in \mathbb{Z}_+,
\]
for some \( a, b \in \mathbb{R} \) with \( b \geq 0 \). Measure \( \tau \) from (5.3) is then given by the following one-to-one correspondence
\[
\tau(A) = b\delta_1(A) + \int_A (1 - x) \, d\mu(x), \quad A \in \mathcal{B}([0, 1]).
\]
A more detailed comment can be found in [2, Remark 1].

5.7. Corollary. Measure \( \tau \) appearing in Theorem 5.5 is unique for a completely alternating sequence \( \{a_n\}_{n=0}^{\infty} \).

Proof. Consider the values \( A_m^1 \) defined by (5.2). Using the recursive formula (see Proposition 5.3) we conclude from (5.3) that
\[
A_m^1 = -\int_{[0, 1]} t^m \, d\tau(t), \quad m \in \mathbb{Z}_+.
\]
As a consequence we obtain that given two Borel measures \( \tau \) and \( \tau' \) satisfying (5.3) we have \( \int_{[0, 1]} p(t) \, d\tau(t) = \int_{[0, 1]} p(t) \, d\tau'(t) \) for any polynomial \( p \), and hence \( \tau = \tau' \) by uniform density of polynomials among continuous functions on \([0, 1]\). \( \square \)

Measure \( \tau \) satisfying (5.3) will be called the representing measure of the completely alternating sequence \( \{a_n\}_{n=0}^{\infty} \). It can be easily observed that the map assigning representing measures to completely alternating sequences is “linear”.

Using (2.7), Lemma 5.4 and Proposition 5.3 (c) we obtain the following characterisation (we use the notation \( \int_0^1 = \int_{[0, 1]} \)):

5.8. Corollary. Let \( S_\lambda \) be a bounded weighted shift on a directed forest \( \mathcal{T} = (V, p) \). Then the following conditions are equivalent:
(a) \( S_\lambda \) is completely hyperexpansive,
(b) the sequence \( \{\|S_\lambda^n v\|^2\}_{n=0}^{\infty} \) is completely alternating for every \( v \in V \),
(c) for every \( v \in V \), there exists a (unique) Borel measure \( \tau_v \) on \([0, 1]\) such that
\[
\|S_\lambda^{n+1} v\|^2 - \|S_\lambda^n v\|^2 = \int_0^1 t^n \, d\tau_v(t), \quad n \in \mathbb{Z}_+.
\]
Moreover, \( \tau_v \) from (c) is the representing measure for the completely alternating sequence \( \{\|S_\lambda^n v\|^2\}_{n=0}^{\infty} \).
Moreover, if $S_\lambda$ is completely hyperexpansive. Consequently, if $S_\lambda$ is a completely hyperexpansive weighted shift on a directed forest $T$, then $T$ contains neither leaves nor degenerate trees (i.e. $\text{Chi}(v) \neq \emptyset$ for every vertex $v$ of $T$).

Knowing the characterisation given in Corollary 5.8 a natural starting point for the study of backward extensions of completely hyperexpansive weighted shifts is the question about extensions of completely alternating sequences. The following lemma addresses this problem. It generalises [13 Lemma 7.1.2].

5.9. Lemma. Let $\{a_n\}_{n=0}^\infty$ be a completely alternating sequence with the representing measure $\tau$ and $k$ be a positive integer. Then the following conditions are equivalent:

(a) there exist $\{a_{n-k}\}_{n=0}^\infty \subseteq \mathbb{R}$ such that the sequence $\{a_{n-k}\}_{n=0}^\infty$ is completely alternating with $a_{-k} = 1$,

(b) $\tau(\{0\}) = 0$ and

$$\int_0^1 \left(\frac{1}{t} + \ldots + \frac{1}{t^k}\right) d\tau(t) \leq a_0 - 1.$$  \hfill (5.4)

Moreover, if (a) is satisfied, then the Borel measure $\rho$ defined by

$$\rho(A) := \int_A \frac{1}{t^k} d\tau(t) + \left( a_0 - 1 - \int_0^1 \left(\frac{1}{t} + \ldots + \frac{1}{t^k}\right) d\tau(t) \right) \delta_0(A),$$  \hfill (5.5)

is representing for $\{a_{n-k}\}_{n=0}^\infty$ and $a_n \geq 1$ for all $n \geq -k$.

Proof. (a)$\Rightarrow$(b) Let $\rho_0$ be the representing measure of a completely alternating sequence $\{a_{n-k}\}_{n=0}^\infty$ (see Theorem 5.5). Define the Borel measure $\tau_0$ on $[0, 1]$ by the formula $\tau_0(A) := \int_A t^k \rho_0(t)$ for $A \in \mathcal{B}([0, 1])$. Then

$$a_n = a_{-k} + \int_0^1 \left(1 + \ldots + t^{n+k-1}\right) d\rho_0(t)$$

$$= a_{-k} + \int_0^1 \left(1 + \ldots + t^{k-1}\right) d\rho_0(t) + \int_0^1 \left(t^k + \ldots + t^{n+k-1}\right) d\rho_0(t)$$

$$= a_0 + \int_0^1 \left(1 + \ldots + t^{n-1}\right) t^k d\rho_0(t)$$

$$= a_0 + \int_0^1 \left(1 + \ldots + t^{n-1}\right) d\tau_0(t), \quad n \in \mathbb{N},$$

and by the uniqueness of representing measure, $\tau_0 = \tau$ and consequently $\tau(\{0\}) = \tau_0(\{0\}) = 0$.

Using (5.3) to $\rho_0$ and $\{a_{n-k}\}_{n=0}^\infty$ with $n = k$ we obtain

$$a_0 = a_{-k} + \int_0^1 \left(1 + \ldots + t^{k-1}\right) d\rho_0(t)$$

$$\geq 1 + \int_{(0, 1]} \left(1 + \ldots + t^{k-1}\right) d\rho_0(t)$$

$$= 1 + \int_{(0, 1]} \left(\frac{1}{t^k} + \ldots + \frac{1}{t}\right) d\tau(t)$$
\[= 1 + \int_0^1 \left( \frac{1}{1} + \frac{1}{t} \right) d\tau(t).\]

This proves the inequality (5.4).

For the ‘moreover’ part we need to verify that \( \rho \) given by (5.5) is a representing measure for \( \{a_{n-k}\}_{n=0}^{\infty} \), i.e. \( \rho = \rho_0 \). If \( A \in \mathcal{B}([0,1]) \) and \( 0 \notin A \), then
\[
\rho(A) \overset{\text{(5.5)}}{=} \int_A \frac{1}{t} d\tau(t) = \int_A \frac{1}{tk} d\tau_0(t) = \int_A \frac{1}{tk} d\rho_0(t) = \rho_0(A).
\]

To complete the proof of \( \rho = \rho_0 \) it suffices to compute \( \rho_0(\{0\}) \). Arguing as in the previous paragraph, we get
\[
a_0 - 1 = a_{-k} - 1 + \int_0^1 \left( 1 + \ldots + t^{k-1} \right) d\rho_0(t)
= \rho_0(\{0\}) + \int_{(0,1]} \left( \frac{1}{t} + \ldots + \frac{1}{t} \right) t^k d\rho_0(t)
= \rho_0(\{0\}) + \int_0^1 \left( \frac{1}{t} + \ldots + \frac{1}{t} \right) d\tau(t).
\]

Hence
\[
\rho_0(\{0\}) = a_0 - 1 - \int_0^1 \left( \frac{1}{1/t} + \ldots + \frac{1}{7} \right) d\tau(t) \overset{\text{(5.5)}}{=} \rho(\{0\}).
\]

That \( \{a_{n-k}\}_{n=0}^{\infty} \subseteq [1, \infty) \) follows from Proposition (5.3) (c) and \( a_{-k} = 1 \).

(b)⇒(a) Define the Borel measure \( \rho \) by (5.5). By (5.4) it is a well defined positive measure. Put \( a_{-k} := 1 \) and
\[
a_{-j} := a_{-k} + \int_0^1 \left( 1 + \ldots + t^{j-1} \right) d\rho(t), \quad j = 1, \ldots, k - 1.
\]

According to Theorem (5.5) it suffices to prove that \( \rho \) satisfies (5.3) for the sequence \( \{a_{n-k}\}_{n=0}^{\infty} \). With convention that \( \sum_{j=N}^{N-1} f(j) = 0 \), we obtain
\[
a_{-k} + \int_0^1 \sum_{j=0}^{n-1} t^j d\rho(t)
= 1 + \rho(\{0\}) + \int_{(0,1]} \sum_{j=0}^{n-1} t^j d\rho(t)
= 1 + \rho(\{0\}) + \int_{(0,1]} \sum_{j=k}^{k-1} \frac{1}{t^{j+1}} d\tau(t) + \int_{(0,1]} \sum_{j=k}^{n-1} t^{j-k} d\tau(t)
\overset{\text{(5.5)}}{=} a_0 + \int_0^1 \sum_{j=0}^{n-k-1} t^j d\tau(t) = a_{n-k}, \quad n \geq k.
\]

This, combined with (5.6), completes the proof. \( \square \)

Having the above lemma we can pass to a characterisation of those weighted shifts that admit completely hyperexpansive \( k \)-step backward extensions. This is a generalisation of [14, Theorem 4.2.(1°)] to the case of weighted shifts on directed trees.
5.10. Theorem. For a bounded completely hyperexpansive weighted shift $S_\lambda$ on a directed tree $T = (V, p)$ with the root $\omega$ and a number $k \geq 1$, the following conditions are equivalent:

(a) $S_\lambda$ admits a completely hyperexpansive $k$-step backward extension,

(b) $\int_0^1 (\frac{1}{t} + \ldots + \frac{1}{t^k}) d\tau(t) < 1$, where $\tau$ is the representing measure for the completely alternating sequence $\{||S_{\lambda}^n e_\omega||^2\}_{n=0}^\infty$.

Proof. Clearly, $S_\lambda \neq 0$ because $A_1(0) = I \not\succeq 0$ (see (5.1)).

(a) $\Rightarrow$ (b) Let $S_{\lambda}'$ be a completely hyperexpansive $k$-step backward extension of $S_\lambda$, i.e. a weighted shift on $T(k)$. Denote the root of $T(k)$ by $\omega_k$ (cf. Definition 2.4). By the complete hyperexpansivity of $S_{\lambda}'$ and Lemma 5.4, the sequence $\{||S_{\lambda}' e_{\omega_k}||^2\}_{n=0}^\infty$ is a completely alternating sequence with the representing measure, say $\tau_k$.

Since $S_{\lambda}' e_{\omega_k} = \lambda^{(k)} e_\omega \neq 0$, we have

$$||S_{\lambda}' e_\omega||^2 = \frac{||S_{\lambda}^{n+k} e_{\omega_k}||^2}{||\lambda^{(k)}||^2}, \quad n \geq 0.$$  \hspace{1cm} (5.7)

Then the Borel measure $\tau$ given by

$$\tau(A) := \int_A t^k |\lambda^{(k)}|^{-2} d\tau_k(t), \quad A \in \mathcal{B}([0,1]),$$  \hspace{1cm} (5.8)

is a representing measure for $\{||S_{\lambda}^n e_\omega||^2\}_{n=0}^\infty$ such that $\tau(\{0\}) = 0$. Indeed, according to Corollary 5.8 (c),

$$||S_{\lambda}^{n+1} e_\omega||^2 - ||S_{\lambda}^n e_\omega||^2 \overset{\text{(5.8)}}{=} \left( ||S_{\lambda}^{n+k+1} e_{\omega_k}||^2 - ||S_{\lambda}^{n+k} e_{\omega_k}||^2 \right) |\lambda^{(k)}|^{-2}
$$

$$= \left| \lambda^{(k)} \right|^{-2} \int_0^1 t^{n+k} d\tau_k(t)
$$

$$= \int_0^1 t^n t^k |\lambda^{(k)}|^{-2} d\tau_k(t) \overset{\text{(5.8)}}{=} \int_0^1 t^n \, d\tau(t)$$

for $n \in \mathbb{Z}_+$. Further we have

$$|\lambda^{(k)}|^2 = ||S_{\lambda}^k e_{\omega_k}||^2 = 1 + \int_0^1 \left( 1 + \ldots + t^{k-1} \right) d\tau_k(t)
$$

$$\geq 1 + \int_{(0,1)} \left( \frac{1}{t^k} + \ldots + \frac{1}{t} \right) t^k d\tau(t)
$$

$$= 1 + |\lambda^{(k)}|^2 \int_{(0,1)} \left( \frac{1}{t^k} + \ldots + \frac{1}{t} \right) d\tau(t)
$$

$$> |\lambda^{(k)}|^2 \int_{(0,1)} \left( \frac{1}{t^k} + \ldots + \frac{1}{t} \right) d\tau(t).$$

Dividing both sides by $|\lambda^{(k)}|^2$ yields (b).

(b) $\Rightarrow$ (a) Set $C_0 := \int_0^1 \left( \frac{1}{t^k} + \ldots + \frac{1}{t} \right) d\tau(t)$. Fix an arbitrary real number $C \geq \frac{1}{1-C_0}$. Consider the completely alternating sequence $\{a_n\}_{n=0}^\infty$, where $a_n := C ||S_{\lambda}^n e_\omega||^2$. Its representing measure is $C \tau$ and

$$\int_0^1 \left( \frac{1}{t^k} + \ldots + \frac{1}{t} \right) dC \tau(t) \leq a_0 - 1.$$
It follows from Lemma 5.9 that the sequence can be extended to a completely alternating sequence \( \{a_{n-k}\}_{n=0}^{\infty} \) with positive terms in such a way that \( a_{-k} = 1 \).

We can find weights \( \lambda'_{w_l} \) for \( l = 0, \ldots, k-1 \) such that
\[
|\lambda'_{w_l}|^2 = \|S_{X_{\omega}}^l e_{\omega_l}\|^2 = a_{j-k}, \quad j = 0, \ldots, k,
\]
where \( \omega_0 = \omega, X' = \{\lambda'_v\}_{v \in V(\omega)}, \lambda'_{w_l} = 0 \) and \( \lambda'_v = \lambda_v \) for \( v \in V^o \). This can be done e.g. by defining \( \lambda'_{w_l} := \sqrt{\frac{a_{j-k}}{a_j}} \) for \( l = 0, \ldots, k-1 \). Clearly \( S_{\lambda} \subset S_{X'} \).

Observe that by (5.9) we have
\[
|\lambda_{w_l}|^2 = a_0 = C\|S_{\lambda}^0 e_{\omega}\|^2 = C,
\]
and consequently
\[
\|S_{X_{\omega}}^{k+n} e_{\omega_l}\|^2 = \|\lambda_{w_l}^{(k)} S_{X_{\omega}}^n e_{\omega_l}\|^2 = C\|S_{\lambda}^n e_{\omega}\|^2 = a_n, \quad n \geq 0.
\]
This combined with (5.9) yields
\[
\|S_{X_{\omega}}^n e_{\omega_l}\|^2 = a_{n-k}, \quad n \geq 0,
\]
which implies that for \( j = 0, \ldots, k \) and \( n \geq 0 \),
\[
\|S_{X_{\omega}}^n e_{\omega_j}\|^2 \leq \frac{\|S_{X_{\omega}}^{k+n} e_{\omega_l}\|^2}{a_j}=\frac{a_{n-j}}{a_{j}},
\]
This shows that the sequence \( \{\|S_{X_{\omega}}^n e_{\omega}\|^2\}_{n=0}^{\infty} \) for \( j = 1, \ldots, k \) is completely alternating being a scaled tail of the sequence \( \{a_{n-k}\}_{n=0}^{\infty} \) (see Proposition 5.3).

Since \( \{\|S_{\lambda}^n e_{\nu}\|^2\}_{n=0}^{\infty} = \{\|S_{X_{\omega}}^n e_{\omega}\|^2\}_{n=0}^{\infty} \) for \( v \in V \), from Corollary 5.8 we conclude that the constructed k-step backward extension is completely hyperexpansive.

Isometric weighted shifts are very special case of the completely hyperexpansive ones. They clearly admit completely hyperexpansive k-step backward extensions for any \( k \in \mathbb{N} \). In fact this makes them unique in the whole class.

5.11. COROLLARY. Let \( S_{\lambda} \) be a bounded completely hyperexpansive proper weighted shift on a rooted directed tree. Then the following conditions are equivalent:
(a) \( S_{\lambda} \) is an isometry,
(b) \( S_{\lambda} \) admits a completely hyperexpansive k-step backward extension for every \( k \in \mathbb{N} \).

PROOF. (a)⇒(b) It is immediate consequence of Proposition 3.3 and the fact that isometries are completely hyperexpansive.

(b)⇒(a) Let \( \tau \) be the representing measure of a completely alternating sequence \( \{\|S_{X_{\omega}}^n e_{\omega}\|^2\}_{n=0}^{\infty} \), where \( \omega \) is the root of underlying tree \((V,p)\). From Theorem 5.10 we see that
\[
\int_0^1 t \, d\tau \leq \int_0^1 (\frac{1}{t} + \ldots + \frac{1}{t^n}) \, d\tau(t) < 1, \quad n \in \mathbb{N}.
\]
This implies that \( \tau \equiv 0 \) and hence
\[
\|S_{\lambda}^n e_{\omega}\| = 1, \quad n \in \mathbb{Z}_+.
\]
Complete hyperexpansivity of $S_λ$ implies that
\[ \sum_{u \in \text{Chi}(v)} |λ_u|^2 = \|S_λ e_v\|^2 \geq \|e_v\|^2 = 1, \quad v \in V. \]
(In particular $\text{Chi}(v) \neq \emptyset$ for all $v \in V$.) It suffices to prove that $\|S_λ e_v\|^2 \leq 1$ for every $v \in V$. Pick $v \in V = \text{Des}(ω)$. There exists unique $n \in \mathbb{Z}_+$ such that $v \in \text{Chi}^{(n)}(ω)$ (cf. Lemma 2.2 (b)). Denote also $B := \text{Chi}^{(n)}(ω) \setminus \{v\}$. Then by (5.10),
\[ \|S_λ^n e_ω\|^2 = \|S_λ^{n+1} e_ω\|^2 \geq \sum_{u \in \text{Chi}^{(n)}(ω)} |λ_u(n)|^2 \|S_λ e_u\|^2 \]
\[ = \sum_{u \in B} |λ_u(n)|^2 \|S_λ e_u\|^2 + |λ_v(n)|^2 \|S_λ e_v\|^2 \]
\[ \geq \sum_{u \in B} |λ_u(n)|^2 + |λ_v(n)|^2 \|S_λ e_v\|^2 \]
\[ = \|S_λ^n e_ω\|^2 - |λ_v(n)|^2 + |λ_v(n)|^2 \|S_λ e_v\|^2 \]
\[ = \|S_λ^n e_ω\|^2 + |λ_v(n)|^2 (\|S_λ e_v\|^2 - 1). \]
By properness, $|λ_v(n)|^2 > 0$ and hence $\|S_λ e_v\|^2 - 1 \leq 0$, which completes the proof.

5.12. REMARK. Without the assumption of properness, we can easily construct a non-isometric weighted shift admitting a completely hyperexpansive $k$-step backward extension for arbitrary $k \in \mathbb{N}$. It suffices to start from one isometric and one non-isometric completely hyperexpansive weighted shift on disjoint rooted directed trees. Then modify the trees so that the root $ω_2$ of the tree with non-isometric weighted shift has a parent belonging to the other tree and preserve all the weights, in particular $λ_{ω_2} = 0$ (see Figure 5). The weighted shift on the new tree is in fact an orthogonal sum of the original weighted shifts.

Knowing the characterisation form Theorem 5.10 we are ready to prove that the class of completely hyperexpansive bounded weighted shifts on directed trees does not have the joint backward extension property. This is shown in the following example.

5.13. EXAMPLE. Let $T = (V_{2,0}, p_2)$ be a directed tree consisting of two infinite branches originating from the root (see $\mathcal{T}_{2,0}$ in [13] (6.2.10)] and

![Figure 5. Illustration to Remark 5.12.](image-url)
Pick \( k \in \mathbb{N} \) and \( \alpha \in (0, \frac{1}{k}) \) and define the weights \( \lambda = \{\lambda_v\}_{v \in V_{2,0}} \) by (cf. Figure 6)

\[
\lambda_0 = 0, \quad \lambda_{(j,n)} = \begin{cases} 
\sqrt{\frac{n}{n-1}} & \text{if } j = 1, \; n \geq 2 \\
\sqrt{\alpha} & \text{if } n = j = 1 \\
1 & \text{if } j = 2
\end{cases}, \quad (j, n) \in V_{2,0}.
\]

Note that \( T \) is a rooted sum of the directed trees \( T_j := T((j, 1) \rightarrow) \) for \( j = 1, 2 \).

The weighted shift \( S_\lambda \) admits a completely hyperexpansive \( k \)-step backward extension, while the weighted shift \( S_{\lambda \rightarrow (1, 1)} \) on \( T_1 \) does not admit a 1-step backward extension.

**Proof.** It can be easily verified that \( \|S_\lambda^n e_{(1,1)}\|^2 = n + 1 \) for \( n \in \mathbb{Z}_+ \), and hence the sequence \( \{\|S_\lambda^n e_{(1,1)}\|^2\}_{n=0}^\infty \) is completely alternating with the representing measure \( \delta_1 \) (see (5.3)). Note that for any other vertex \( v \) of \( T_{((1,1) \rightarrow)} \), the sequence \( \{\|S_\lambda^n e_v\|^2\}_{n=0}^\infty \) is a scaled tail of \( \{\|S_\lambda^n e_{(1,1)}\|^2\}_{n=0}^\infty \), hence it is completely alternating (see Proposition [5.3]), and consequently the weighted shift \( S_{\lambda \rightarrow (1,1)} \) is completely hyperexpansive (see Corollary [5.8]). According to Theorem [5.10] \( S_{\lambda \rightarrow (1,1)} \) does not admit completely hyperexpansive 1-step backward extension.

Clearly, \( \|S_\lambda^n e_{(2,1)}\|^2 = 1 \) for \( n \in \mathbb{Z}_+ \), so the sequence \( \{\|S_\lambda^n e_{(2,1)}\|^2\}_{n=0}^\infty \) is completely alternating with the zero representing measure. The equalities

\[
\|S_\lambda^n e_{(0,0)}\|^2 = \alpha n + 1, \quad n \in \mathbb{Z}_+, \\
\int_0^1 \left( \frac{1}{t^k} + \ldots + \frac{1}{t^k} \right) \; d\alpha \delta_1(t) = k\alpha < 1.
\]

Using Corollary 5.8 one can easily verify that \( S_\lambda \) is completely hyperexpansive and by Theorem [5.10] we obtain that it admits a completely hyperexpansive \( k \)-step backward extension. \( \Box \)

The following fact, whose standard measure-theoretic proof will be skipped, is going to be useful in further considerations regarding representing measures.
5.14. Lemma. Let \( \{\mu_j\}_{j \in J} \) be a family of probability measures on a fixed measure space \((X, \mathcal{A})\) and \( \sum_{j \in J} a_j \) be a convergent series of non-negative real numbers. Then \( \bar{\mu} \) defined by

\[
\bar{\mu}(A) := \sum_{j \in J} a_j \mu_j(A), \quad A \in \mathcal{A},
\]

is a finite measure on \((X, \mathcal{A})\) and for every \( \mathcal{A} \)-measurable function \( f : X \to [0, +\infty) \) the following equality holds

\[
\int_X f \, d\bar{\mu} = \sum_{j \in J} a_j \int_X f \, d\mu_j.
\] (5.11)

Even though the extendability of a weighted shift on the rooted sum does not possess such elegant properties as in the case of subnormal and power hyponormal operators, some positive results can be proven. For example, the existence of \((k + 1)\)-step backward extensions for all members of a given family of weighted shifts implies the existence of a \( k \)-step backward extension of the enveloping operator on the rooted sum.

5.15. Theorem. Let \( J \) be a non-empty at most countable set and \( k \in \mathbb{Z}_+ \).

For \( j \in J \), let \( S_j \) be a bounded weighted shift on a directed tree \( T_j = (V_j, p_j) \)
with root \( \omega_j \). Assume that \( S_j \)'s are uniformly bounded and each of them admits a completely hyperexpansive \((k + 1)\)-step backward extension.

Then there exists a system \( \{\theta_j\}_{j \in J} \subseteq \mathbb{C} \setminus \{0\} \) such that the weighted shift \( S_\lambda \) on \( \bigcup_{j \in J} T_j = (V, p) \) with weights \( \lambda = \{\lambda_v\}_{v \in V} \), determined by

\[
\lambda_{\omega_j} = \theta_j \quad \text{and} \quad S_{\lambda_{\omega_j}} = S_j \quad \text{for all} \quad j \in J,
\] (5.12)

is bounded and admits a completely hyperexpansive \( k \)-step backward extension. Moreover, if \( k \geq 1 \), then

\[
\|S_\lambda\| \leq \max \left\{ \sup_{j \in J} \|S_j\|, \sqrt{\frac{k+1}{k}} \right\}.
\] (5.13)

Proof. By Theorem 5.10, the representing measure of the completely alternating sequence of \( \{\|S_j e_{\omega_j}\|_2\}_{n=0}^{\infty} \), say \( \tau_j \), satisfies

\[
D_j := \int_0^1 \left( \frac{1}{t} + \ldots + \frac{1}{k+1} \right) \, d\tau_j(t) < 1, \quad j \in J.
\]

Consequently, also

\[
C_j := \int_0^1 \frac{1}{t} \, d\tau_j(t) \leq \frac{1}{k+1} \left( \frac{1}{k+1} \right) D_j < \frac{1}{k+1}, \quad j \in J.
\]

Therefore \( \tau_j(\{0\}) = 0 \) for \( j \in J \). Pick positive real numbers \( \{a_j\}_{j \in J} \) such that

\[
(\ast) \ \sum_{j \in J} a_j (1 - C_j) = 1, \quad (**) \ \sum_{j \in J} a_j < \infty.
\]

Take \( \{\theta_j\}_{j \in J} \) such that \( |\theta_j|^2 = a_j \) for \( j \in J \). Let \( \lambda \) be as in (5.12) with \( \lambda_\omega = 0 \). Then, by assumption,

\[
\sum_{v \in \text{Chi}(v)} |\lambda_v|^2 = \|S_j e_v\|_2^2 \leq \sup_{j \in J} \|S_j\|_2^2 < \infty, \quad j \in J, \ v \in V_j.
\] (5.14)

\[^3\]Except for the case \( k = 0 \), the inequality (**) follows from (*).
It follows from (***) that \( \sum_{u \in \text{Ch}i(\omega)} |\lambda_u|^2 < \infty \), hence, by Proposition 2.7 (b), the operator \( S_\lambda \) is bounded.

Define the Borel measure \( \tau \) on \([0, 1]\) by

\[
\tau(A) := \sum_{j \in J} a_j \int_A \frac{1}{t} \, d\tau_j(t), \quad A \in \mathcal{B}([0, 1]).
\]

Since \( C_j < 1 \) for \( j \in J \), (***) implies that \( \tau \) is a finite measure.

By Lemma 5.14 we get (recall that \( \tau_j(\{0\}) = 0 \) for every \( j \in J \))

\[
\|S^{n+1}_\lambda e_\omega\|^2 = \sum_{j \in J} a_j \|S^n e_{\omega_j}\|^2 \\
= \sum_{j \in J} a_j \left( 1 + \int_0^1 \frac{1}{t} \, d\tau_j(t) \right) \\
= \sum_{j \in J} a_j \left( 1 - \int_0^1 \frac{1}{t} \, d\tau_j(t) + \int_0^1 \frac{1}{t} \, d\tau_j(t) \right) \\
= \sum_{j \in J} a_j (1 - C_j) + \sum_{j \in J} a_j \int_0^1 \frac{1}{t} \, d\tau_j(t) \\
\overset{(\ast)}{=} 1 + \int_0^1 \frac{1}{t^n} \, d\tau(t), \quad n \geq 1.
\]

Observe also that

\[
\|S_\lambda e_\omega\|^2 = \sum_{j \in J} a_j = 1 + \sum_{j \in J} a_j C_j = 1 + \int_0^1 t^0 \, d\tau(t).
\]

Summarising, we have proven that \( \{\|S^n_\lambda e_\omega\|^2\}_{n=0}^\infty \) is a completely alternating sequence with the representing measure \( \tau \). For \( j \in J, v \in V_j \) and \( n \in \mathbb{Z}_+ \), we have \( \|S^n_\lambda e_v\|^2 = \|S^n_j e_v\|^2 \), and since \( S_j \) is completely hyperexpansive, \( \{\|S^n_\lambda e_v\|^2\}_{n=0}^\infty \) is a completely alternating sequence. According to Corollary 5.8 we see that \( S_\lambda \) is completely hyperexpansive. In the case \( k = 0 \) the proof is complete.

Now suppose that \( k \geq 1 \). Then

\[
\sum_{j \in J} a_j = \sum_{j \in J} a_j (1 - C_j) \frac{1}{1 - C_j} \overset{(\ast)}{=} \frac{1}{1 - \frac{1}{k+1}} = \frac{k+1}{k}.
\]

This gives \( \|S^n_\lambda e_\omega\|^2 \leq \frac{k+1}{k} \), which combined with (5.14) proves the inequality (5.13).

Using Lemma 5.14 and the fact that \( \tau(\{0\}) = 0 \), we get

\[
\int_0^1 \left( \frac{1}{t} + \ldots + \frac{1}{t^k} \right) \, d\tau(t) = \sum_{j \in J} a_j \int_0^1 \left( \frac{1}{t} + \ldots + \frac{1}{t^k} \right) \frac{1}{t} \, d\tau_j(t) \\
= \sum_{j \in J} a_j \int_0^1 \left( \frac{1}{t^2} + \ldots + \frac{1}{t^{k+1}} \right) \, d\tau_j(t) \\
= \sum_{j \in J} a_j (D_j - C_j)
\]
This together with Theorem 5.10 proves that $S_\lambda$ admits a completely hyperexpansive $k$-step backward extension. □

5.16. REMARK. Observe that the above theorem is precisely the implication (c) $\Rightarrow$ (a) $\land$ (b) with $c_k = \sqrt{\frac{k+1}{k}}$ from Definition 3.4 (JBEP). Since the proof of (b)$\Rightarrow$ (a) in Theorem 3.7 does not use the other part of JBEP, the implication is valid for the class of completely hyperexpansive weighted shifts.

5.17. REMARK. Note that in the case of weighted shifts bounded from below the implication (a) $\Rightarrow$ (c) in the joint backward extension property is still true, unlike the opposite one (see Proposition 3.6). This is a different way of violating JBEP from what we have proven about completely hyperexpansive weighted shifts.

As shown in Example 5.13 even if not all members of a given family of weighted shifts admit completely hyperexpansive backward extensions they may still be jointly extended to a rooted sum of the underlying trees. In fact, at least one of them has to admit appropriate backward extension, as described in the following theorem.

5.18. THEOREM. Let $S_\lambda$ be a bounded proper weighted shift on a directed tree $T$ with root $\omega$ admitting a completely hyperexpansive $k$-step backward extension for some $k \in \mathbb{Z}_+$. Then for every $n \in \mathbb{Z}_+$, there exists $v \in \text{Chi}^{(n)}(\omega_k)$ such that the weighted shift $S_{\lambda \rightarrow v}$ on $T_{(v \rightarrow)}$ admits a completely hyperexpansive $(n+k)$-step backward extension.

PROOF. Since $\text{Chi}^{(n)}_T(\omega) = \text{Chi}^{(n+k)}_T(\omega_k), \ n \in \mathbb{Z}_+$, we can reduce ourselves to the case when $k = 0$, replacing $S_\lambda$ by its completely hyperexpansive extension on $T_k$ if necessary.

We proceed by induction on $n \in \mathbb{Z}_+$. Clearly, for $n = 0$ the claim is satisfied.

Assume $n \geq 0$ and there exists $u \in \text{Chi}^{(n)}(\omega)$ such that the weighted shift $S_{\lambda \rightarrow u}$ on $T_{(u \rightarrow)}$ admits a completely hyperexpansive $n$-step backward extension. We want to prove that for some $v \in \text{Chi}^{(n+1)}(\omega)$ the weighted shift $S_{\lambda \rightarrow v}$ admits a completely hyperexpansive $(n+1)$-step backward extension. We will find such $v$ in $\text{Chi}(u) \subseteq \text{Chi}^{(n+1)}(\omega)$. Denote by $\tau$ the representing measure of the completely alternating sequence $\{\|S_{\lambda e_u}^j\|^2\}_{j=0}^\infty$ (see Lemma 5.4). This sequence admits an extension to a completely alternating sequence $\{\|S_{\lambda e_u}^j\|^2\}_{j=0}^\infty$, satisfying $\|S_{\lambda e_u}^j\|^2 = 1$, hence by Lemma 5.9 $\tau(\{0\}) = 0$,

$$\int_0^1 \frac{1}{t} \, d\tau(t) \leq \|S_{\lambda e_u}\|^2 - 1 \quad (5.15)$$
and the representing measure $\rho$ for the sequence \{\|S^j_\lambda e_u\|^2\}_{j=0}^\infty is given by the formula:

$$\rho(A) = \int_A \frac{1}{t} \ d\tau(t) + \left(\|S_\lambda e_u\|^2 - 1 - \int_0^1 \frac{1}{t} \ d\tau(t)\right)\delta_0(A), \ A \in \mathfrak{B}([0,1]).$$

For $v \in \text{Chi}(u)$, let $\tau_v$ be the representing measure of the completely alternating sequence \{\|S^j_\lambda e_v\|^2\}_{j=0}^\infty. We can define the Borel measure $\tau_0$ on [0, 1] as follows

$$\tau_0(A) := \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \tau_v(A), \ A \in \mathfrak{B}([0,1]).$$

Observe that

$$\tau_v([0,1]) = \int_0^1 \ 1 \ d\tau_v = \|S_\lambda e_v\|^2 - 1 \leq \|S_\lambda\|^2 - 1, \ v \in \text{Chi}(u).$$

This fact, combined with convergence of the series $\sum_{v \in \text{Chi}(u)} |\lambda_v|^2$ gives that the measure $\tau_0$ is finite. Then

$$\|S^j_\lambda e_u\|^2 \geq \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \|S^j_\lambda e_v\|^2 \geq \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \left(1 + t^j\right) \ d\tau_v(t), \ j \geq 1.$$

This means that $\tau_0$ is a representing measure for the completely alternating sequence \{\|S^j_\lambda e_u\|^2\}_{j=0}^\infty and, by Corollary \[5.7\] $\tau_0 = \tau$. This in particular means that $\tau_0(\{0\}) = 0$, and consequently $\tau_v(\{0\}) = 0$ for $v \in \text{Chi}(u)$.

If $n \geq 1$, then we use the existence of a completely hyperexpansive $n$-step backward extension of $S^j_\lambda e_u$ and Theorem \[5.10\] to obtain that $\rho(\{0\}) = 0$, which implies $\rho(A) = \int_A \frac{1}{t} \ d\tau(t)$ for any $A \in \mathfrak{B}([0,1])$. By the same theorem

$$1 > \int_0^1 \left(\frac{1}{t} + \ldots + \frac{1}{t^n}\right) \ d\rho(t) = \int_0^1 \left(\frac{1}{t^2} + \ldots + \frac{1}{t^{n+1}}\right) \ d\tau(t). \quad (5.16)$$

Set $D_v := \int_0^1 \left(\frac{1}{t} + \ldots + \frac{1}{t^n}\right) \ d\tau_v(t)$ for $v \in \text{Chi}(u)$. According to Theorem \[5.10\] it suffices to prove that $D_v < 1$ for any $v_0 \in \text{Chi}(u)$. Note that

$$0 \geq \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_0^1 \left(\frac{1}{t} + \ldots + \frac{1}{t^n}\right) \ d\tau_v(t) - \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \quad \text{(cf. Lemma \[5.14\])}$$

$$\geq \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 (D_v - 1),$$

where \((\dagger)\) is a consequence of \[5.16\] if $n \geq 1$ and is obvious for $n = 0$. At least one of the summands $|\lambda_v|^2 (D_v - 1)$ has to be negative, hence, by properness

$\text{Note that } S^{j-1}_\lambda e_u = S^j_\lambda e_u \text{ for all } j \in \mathbb{Z}_+.$
of \( S_X \), \( D_{v_0} < 1 \) for some \( v_0 \in \text{Chi}(u) \subseteq \text{Chi}^{(n+1)}(\omega) \). This completes the proof. \( \square \)

We finish the article listing all classes of weighted shifts on directed trees for which we have decided whether they satisfy JBEP:

| Classes having JBEP          | Classes without JBEP                |
|------------------------------|-------------------------------------|
| all bounded                 | bounded from below                  |
| isometric                   | quasinormal                         |
| contractive                 | completely hyperexpansive           |
| expansive                   |                                     |
| hyponormal                  |                                     |
| power hyponormal            |                                     |
| subnormal                   |                                     |

**Acknowledgements.** Results presented in this paper are a substantial part of my dissertation written under supervision of prof. Jan Stochel, to whose guidance I am very grateful.

**References**

[1] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory 13 (1985), 203–217.
[2] A. Athavale, *On completely hyperexpansive operators*, Proc. Amer. Math. Soc. 124 (1996), 3745–3752.
[3] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, Springer, Berlin, 1984.
[4] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, *Subnormality of unbounded composition operators over one-circuit directed graphs: Exotic examples*, Adv. Math. 310 (2017), 484–556.
[5] J. B. Conway, *The theory of subnormal operators*, Mathematical Surveys and Monographs, 36, Amer. Math. Soc., Providence, RI, 1991.
[6] R. E. Curto, *Quadratically hyponormal weighted shifts*, Integr. Equ. Oper. Theory 13 (1990), 49–66.
[7] P. R. Halmos, *Normal dilations and extensions of operators*, Summa Brasil. Math. 2 (1950), 125–134.
[8] P. R. Halmos, *A Hilbert space problem book*, Springer-Verlag, New York Inc., 1982.
[9] M. Hazarika, B. Kalita, *Back-step extension of weighted shifts*, Bull. Calcutta Math. Soc. 106 (2014), no. 3, 169–188.
[10] T. Hoover, I. B. Jung, A. Lambert, *Moment sequences and backward extensions of subnormal weighted shifts*, J. Austral. Math. Soc. 73 (2002), 27–36.
[11] T. Ito, T. K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. 34 (1972), 157-164.
[12] Z. J. Jabłoński, *Complete hyperexpansivity, subnormality and inverted boundedness conditions*, Integr. equ. oper. theory 44 (2002), 316–336.
[13] Z. Jabłoński, I. B. Jung, J. Stochel, *Weighted shifts on directed trees*, Mem. Amer. Math. Soc. 216 (2012), no. 1017.
[14] Z. J. Jabłoński, I. B. Jung, J. Stochel, *Backward Extensions of Hyperexpansive Operators*, Studia Math. 173 (2006), 233–257.
[15] Z. J. Jabłoński, I. B. Jung, J. Stochel, *Normal extensions escape from the class of weighted shifts on directed trees*, Complex Anal. Oper. Theory 7 (2013), 409–419.
[16] I. B. Jung, A. Lambert, J. Stochel, *Backward extensions of subnormal operators*, Proc. Amer. Math. Soc. 132 (2004), 2291–2302.
[17] I. B. Jung, C. Li, *Backward extensions of hyponormal weighted shifts*, Math. Japon. 52 (2000), no. 2, 267–278.
[18] I.B. Jung, C. Li, A formula for $k$-hyponormality of backstep extensions of subnormal weighted shifts, Proc. Amer. Math. Soc. 129 (2001), 2343–2351.

[19] A. Lambert, Subnormality and weighted shifts, J. London Math. Soc. 14 (1976), 476–480.

[20] S. McCullough, V. Paulsen, A note on joint hyponormality, Proc. Amer. Math. Soc. 107 (1989), 187–195.

[21] P. Pikul, Backward extensions of weighted shifts on directed trees, Integr. Equ. Oper. Theory 94 (2022), #26.

[22] A.L. Shields, Weighted shift operators and analytic function theory, Topics in operator theory, Math. Surveys, No 13, Amer. Math. Soc., Providence, R.I., 1974, pp. 49–128.

[23] J.A. Shohat, J.D. Tamarkin, The problem of moments, Math. Surveys 1, Amer. Math. Soc., Providence, R.I., 1943.

[24] B. Simon, The classical moment problem as a self-adjoint finite difference operator, Adv. Math. 137 (1998), 82–203.

[25] J.G. Stampfli, Which weighted shifts are subnormal?, Pacific J. Math. 17 (1966), no. 2, 367–379.

P. Pikul, Instytut Matematyki, Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, 30-348 Kraków, Poland

Email address: Piotr.Pikul@im.uj.edu.pl