CONGRUENCE OF HYPER_SURFACES
OF A PSEUDO-EUCLIDEAN SPACE

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R. S. Kulkarni has proved [1] that the so-called bending of a hypersurface in an Euclidean space determines the congruence class of the hypersurface. In the present paper we show that a similar result holds for hypersurfaces of a pseudo-Euclidean space $\mathbb{R}^{n+1}_s$, $n > 2$. We prove also a corresponding theorem, which accounts for the behaviour of the second fundamental form on isotropic vectors.

1. PRELIMINARIES

Let $M$ be a Riemannian or a pseudo-Riemannian manifold with a metric tensor $g$. A tangent vector $\xi$ is said to be isotropic, it is nonzero and $g(\xi, \xi) = 0$. Of course, for isotropic vectors one speaks only when the manifold is pseudo-Riemannian, i.e. when $g$ is an indefinite metric. The values of a symmetric tensor of type (0,2) on isotropic vectors give a good information about this tensor, as it is shown by the following

**Lemma 1** [2]. Let $M$ be a pseudo-Riemannian manifold. If $L$ is a symmetric tensor of type (0,2) on a tangent space $T_pM$, such that $L(\xi, \xi) = 0$ for every isotropic vector $\xi$ in $T_pM$, then $L = cg$, where $c$ is a real number.

Let $\nabla$ and $R$ denote the covariant differentiation and the curvature tensor of $M$, respectively. The Ricci tensor and the scalar curvature will be denoted by $S$ and $\tau$, respectively. Then the Weil conformal curvature tensor $C$ for $M$ is defined by

$$C = R - \frac{1}{n-2} \varphi + \frac{\tau}{(n-1)(n-2)} \pi_1,$$
where $n = \dim M$, $\varphi$ is defined by

$$\varphi(T)(x, y, z, u) = g(x, u)T(y, z) - g(x, z)T(y, u) + g(y, z)T(x, u) - g(y, u)T(x, z),$$

for any symmetric tensor $T$ of type $(0,2)$ and $\pi_1 = \frac{1}{2}\varphi(g)$. As it is well known [3], if $n > 3$, then $M$ is conformally flat if and only if the Weil conformal curvature tensor vanishes identically. If $n = 3$ a necessary and sufficient condition for $M$ to be conformally flat is [3]

$$\nabla_X \left( S - \frac{r}{4} g \right)(Y, Z) - \nabla_Y \left( S - \frac{r}{4} g \right)(X, Z) = 0.$$  

If $\overline{M}$ is another Riemannian or pseudo-Riemannian manifold, we denote the corresponding objects for $\overline{M}$ by a bar overhead. Assume that $f$ is a conformal diffeomorphism of $M$ onto $\overline{M}$: $f^*\overline{g} = \varepsilon e^{2\sigma} g$, where $\varepsilon = \pm 1$ and $\sigma$ is a smooth function. Then we have [3]

$$f^*\overline{R} = \varepsilon e^{2\sigma} \{ R + \varphi(Q) \},$$

where

$$Q(X, Y) = X\sigma Y - g(\nabla_X \nabla \sigma, Y) - \frac{1}{2} ||\nabla \sigma||^2 g(X, Y),$$

$\nabla \sigma$ denoting the gradient of $\sigma$ and $||\nabla \sigma||^2 = g(\nabla \sigma, \nabla \sigma)$.

In [4] we have proved the following

**Lemma 2.** Let $M$ and $\overline{M}$ be pseudo-Riemannian manifolds of dimension $> 2$ and $f$ be a diffeomorphism of $M$ onto $\overline{M}$. Assume that at a point $p$ of $M$ there exists an isotropic vector $\xi$, such that every isotropic vector, which is sufficiently close to $\xi$, is mapped by $f_*$ in an isotropic vector in $f(p)$. Then $f_*$ is a homothety at $p$.

In what follows $M$ will be a hypersurface of an Euclidean space $\mathbb{R}^{n+1}$ or of a pseudo-Euclidean space $\mathbb{R}^{n+1}_s$, such that the restriction $g$ of the usual metric of $\mathbb{R}^{n+1}$ to $M$ is nondegenerate. Denote the second fundamental form of $M$ by $h$. Then we have the equation of Gauss

$$R(X, Y, Z, U) = h(X, U)h(Y, Z) - h(X, Z)h(Y, U)$$

and the equation of Codazzi

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = 0.$$ 

Recall also, that a point $p$ of $M$ is said to be quasi-umbilic, if

$$h = \alpha g + \beta \omega \otimes \omega$$

in $p$, where $\alpha, \beta$ are real functions and $\omega$ is an 1-form. In particular, if $\beta$ is zero the point $p$ is called umbilic.

The bending [1] $K_h$ of $M$ is said to be the function, assigning to each nonisotropic nonzero tangent vector $x$ at a point of $M$ the number

$$K_h(x) = \frac{h(x, x)}{g(x, x)}.$$
Two hypersurfaces $M$ and $\overline{M}$ being defined a diffeomorphism $f$ of $M$ onto $\overline{M}$ is said to be bending preserving [1], if

$$(1.4) \quad K_{\overline{h}}(f_*x) = K_h(x)$$

for each nonisotropic nonzero vector $x$ on $M$, whose image is also nonisotropic. The analogue of (1.4) for isotropic vectors is

$$(1.5) \quad \lim_{x \to \xi} \frac{K_{\overline{h}}(f_*x)}{K_h(x)} = 1 ,$$

where the isotropic vector $\xi$ is approximated by nonisotropic nonzero vectors, whose images are also nonisotropic. We shall prove:

**Theorem 1.** Let $M$ and $\overline{M}$ be hypersurfaces with indefinite metrics in $\mathbb{R}^{n+1}_s$, $n > 2$, and let $f$ be a diffeomorphism of $M$ onto $\overline{M}$, satisfying (1.5) for each isotropic vector $\xi$ on $M$. If the nonquasi-umbilic points are dense in $M$ and if $M$ is not conformally flat, then $f$ is a congruence.

We recall that $f$ is said to be a congruence if it can be extended to a motion of $\mathbb{R}^{n+1}_s$.

**Theorem 2.** Let $M$ and $\overline{M}$ be hypersurfaces with indefinite metrics in $\mathbb{R}^{n+1}_s$, $n > 2$, and let $f$ be a bending preserving diffeomorphism of $M$ onto $\overline{M}$. If the nonumbilic points are dense in $M$ and the curvature tensor of $M$ does not vanish identically in a point $p$, then there exists a neighbourhood $V$ of $p$ such that $f|_V$ is a congruence of $V$ onto $f(V)$.

**Remark.** The proof of the congruence theorem in [1] can be applied for hypersurfaces with definite metrics (i.e. spacelike hypersurfaces) in $\mathbb{R}^{n+1}_1$.

## 2. BASIC RESULTS

In this section we prove two lemmas, which will be useful in the proofs of Theorems 1 and 2.

**Lemma 3.** Let $M$ and $\overline{M}$ be hypersurfaces with indefinite metrics in $\mathbb{R}^{n+1}_s$, $n > 2$, and let $f$ be a diffeomorphism of $M$ onto $\overline{M}$, satisfying (1.5) for each isotropic vector $\xi$ on $M$. If the nonumbilic points are dense in $M$, then

a) $f$ is conformal: $f^*\overline{g} = e^{2\sigma}g$;

b) $f^*\overline{h} = e^{2\sigma}\{h + \lambda g\}$, where $\lambda$ is a smooth function;

c) $f^*\overline{R} = e^{4\sigma}\{R + \lambda\varphi(h) + \lambda^2 \pi_1\}$;

d) the following equations hold:

$$\begin{align*}
(2.1) \quad & X\sigma B(Y, Z) - Y\sigma B(X, Z) + \frac{1}{n}\{X\lambda g(Y, Z) - Y\lambda g(X, Z)\} = 0 , \\
(2.2) \quad & B(Y, \nabla\sigma) = \frac{n-1}{n}Y\lambda ,
\end{align*}$$

where $B = h - \frac{\text{tr} h}{n}g$. 
Proof. Let \( p \) be a nonumbilic point of \( M \), i.e. \( h \) is not proportional to \( g \) in \( p \). Then by Lemma 1 there exists an isotropic vector \( \xi \) in \( T_pM \), such that \( h(\xi, \xi) \neq 0 \). Hence \( h(\xi', \xi') \neq 0 \) for each isotropic vector \( \xi \), which is sufficiently close to \( \xi \). Then (1.5) implies that \( f_\ast \xi' \) is isotropic. According to Lemma 2, \( f_\ast \) is a homothety at \( p \). Since the nonumbilic points are dense in \( M \), \( f \) is conformal and then a) is proved.

From a) and (1.5) it follows \((f^\ast \bar h)(\xi, \xi) = \varepsilon e^{2\sigma} h(\xi, \xi)\). Applying again Lemma 1, we obtain b). Then c) follows from b) and from the equations of Gauss for \( M \) and \( \overline{M} \).

To simplify the notations in the proof of d), we identify \( M \) with \( \overline{M} \) via \( f \), and omit \( f^\ast \) from the formulas. Then we have [3]

\[
\nabla_X Y = \nabla_X Y + X\sigma Y + Y\sigma X - g(Y, Y)\nabla\sigma .
\]

Hence, using b) and the equations of Codazzi for \( g \) and \( \bar g \), we find

\[
(2.3) \quad X\sigma h(Y, Z) - Y\sigma h(X, Z) + X\lambda g(Y, Z) - Y\lambda g(X, Z) + g(Y, Z)h(Y, \nabla\sigma) - g(Y, Z)h(X, \nabla\sigma) = 0 ,
\]

which implies immediately

\[
(2.4) \quad h(Y, \nabla\sigma) = \frac{n-1}{n} Y\lambda + \frac{\text{tr} h}{n} Y\sigma ,
\]

i.e. (2.2). From (2.3) and (2.4) we obtain (2.1). This proves the lemma.

We note that the conditions of Lemma 3 are fulfilled in Theorem 1, as well as in Theorem 2.

**Lemma 4.** If in Lemma 3 \( \|\sigma\|^2 = 0 \) and \( U \) denotes the open set \( \{ p \in M : \nabla\sigma \neq 0 \} \), then

a) each point of \( U \) is quasi-umbilic;

b) \( R = 0 \) in \( U \).

**Proof.** We shall use a connected component \( U_1 \) of \( U \). Let in (2.1) \( X = Z = \nabla\sigma, Y = \nabla\lambda \). By (2.2), we obtain

\[
(2.5) \quad (\nabla\sigma)\lambda = 0 .
\]

Now we put \( X = \nabla\sigma \) in (2.1) and get use of (2.2) and (2.5). The result is \( Y\sigma Z\lambda = 0 \) for arbitrary vector fields \( Y, Z \). Since \( \nabla\sigma \) can not vanish in \( U_1 \), it follows \( \lambda = \text{const} \) (in \( U_1 \)). Then (2.1) reduces to

\[
X\sigma B(Y, Z) - Y\sigma B(X, Z) = 0 ,
\]

which implies

\[
(2.6) \quad B = \mu d\sigma \otimes d\sigma ,
\]

where \( \mu \) is a smooth function. Equivalently, we may write

\[
(2.6') \quad h = \frac{\text{tr} h}{n} g + \mu d\sigma \otimes d\sigma ,
\]

thus proving a). From the equation of Gauss for \( g \) it follows

\[
S(x, x) = \text{tr} h . h(x, y) - \sum_{i=1}^n h(x, e_i)h(y, e_i)g(e_i, e_i) ,
\]
where \{e_i; i, \ldots, n\} is an orthogonal frame. Hence, using (2.6'), we obtain

\begin{equation}
S = \frac{n-2}{n} \mu \operatorname{tr} h \, d\sigma \otimes d\sigma + \frac{n-2}{n^2} (\operatorname{tr} h)^2 g .
\end{equation}

Thus we get

\begin{equation}
\tau = \frac{n-1}{n} (\operatorname{tr} h)^2 .
\end{equation}

From (2.7) and (2.8) we compute for \( P = S - \frac{\tau}{n} g \):

\begin{equation}
P = \frac{n-2}{n} \mu \operatorname{tr} h \, d\sigma \otimes d\sigma .
\end{equation}

By Lemma 3 c) we find immediately

\begin{equation}
 f^* S = \varepsilon e^{2\sigma} \{ S + \lambda (n-2) h + \lambda \operatorname{tr} h . g + (n-1)\lambda^2 g \} ,
\end{equation}

\begin{equation}
 f^* \tau = \tau + 2(n-1) \lambda \operatorname{tr} h + n(n-1)\lambda^2 ,
\end{equation}

\begin{equation}
 f^* P = \varepsilon e^{2\sigma} \{ P + (n-2) \lambda B \} .
\end{equation}

Analogously, (1.2) yields

\begin{equation}
 f^* P = P + (n-2) Q - \frac{n-2}{2n(n-1)} (\varepsilon \bar{\tau} e^{2\sigma} - \tau) g .
\end{equation}

From the last two equations we obtain

\begin{equation}
 Q = \frac{\varepsilon e^{2\sigma} - 1}{n-2} P + \varepsilon \lambda e^{2\sigma} B + \frac{\varepsilon \bar{\tau} e^{2\sigma} - \tau}{2n(n-1)} g .
\end{equation}

Hence, using (2.6) and (2.9), we find

\begin{equation}
 Q = \nu d\sigma \otimes d\sigma + \frac{\varepsilon \bar{\tau} e^{2\sigma} - \tau}{2n(n-1)} g ,
\end{equation}

where

\[ \nu = \mu \left( \frac{\varepsilon e^{2\sigma} - 1}{n} \operatorname{tr} h + \varepsilon \lambda e^{2\sigma} \right) . \]

Since \( \nabla \sigma \) is isotropic, (1.3) yields \( Q(X, \nabla \sigma) = 0 \). Thus, applying (2.11) we conclude that

\begin{equation}
 \varepsilon \bar{\tau} e^{2\sigma} - \tau = 0 .
\end{equation}

Then, (2.11) reduces to

\begin{equation}
 Q = \nu d\sigma \otimes d\sigma
\end{equation}

or, according to (1.3) -

\[ g(\nabla_X \nabla \sigma, Y) = (1 - \nu) X \sigma Y \sigma . \]

Hence, using the equation of Codazzi for \( g \) and (2.6), we derive

\[ (X \mu Y \sigma - Y \mu X \sigma) Z \sigma + \frac{1}{n} \{ X \operatorname{tr} h g(Y, Z) - Y \operatorname{tr} h g(X, Z) \} = 0 . \]

Here we assume that \( Z \) is orthogonal to \( \nabla \sigma \) and \( X \), and \( Y \) is not orthogonal to \( Z \). The result is \( X \operatorname{tr} h = 0 \). i.e. \( \operatorname{tr} h \) is a constant. Thus, by (2.8) and (2.10), \( \tau \) and \( \bar{\tau} \) are also
constants. If $\bar{\tau} \neq 0$, (2.12) implies $d\sigma = 0$, which is a contradiction. Let $\bar{\tau} = 0$. According to (2.12), (2.8) and (2.10), $\tau = \text{tr}_h = \lambda = 0$. By Lemma 3 c)

\[(2.13) \quad \bar{R} = e^{4\sigma} R \]

On the other hand, from $\text{tr}_h = \lambda = 0$ and (1.2), (2.11'), we obtain

\[(2.14) \quad \bar{R} = \varepsilon e^{2\sigma} R \]

From (2.13) and (2.14) we find $(e^{2\sigma} - \varepsilon) R = 0$ in $U_1$ and hence this holds on $U$. Since $\sigma$ can not vanish in an open subset of $U$, it follows $R = 0$ in $U$, which proves our assertion.

3. PROOF OF THEOREM 1

First we assume that there exists a point $p$ of $M$ such that $\|\nabla \sigma\|^2 \neq 0$ in $p$. Then $\|\nabla \sigma\|^2 \neq 0$ in a neighbourhood $V$ of $p$. In (2.1) we assume that $X = Z = \nabla \sigma$ and that $Y$ is orthogonal to $\nabla \sigma$. Using (2.2), we obtain $Y \lambda = 0$ in $V$. Hence $\nabla \lambda = \rho \nabla \sigma$ on $V$, where $\rho$ is a smooth function. Using again (2.1) with $X = \nabla \sigma$ and applying (2.2), we find

\[B = \rho \left\{ \frac{1}{\|\nabla \sigma\|^2} d\sigma \otimes d\sigma - \frac{1}{n} g \right\} \]

in $V$. However, this contradicts the assumption that the set of nonquasi-umbilic points is dense.

So $\|\nabla \sigma\|^2 = 0$. Now, let us assume that $\nabla \sigma$ does not vanish at a point $p$ and hence, in an open set $U$. By Lemma 4 a) each point of $U$ is quasi-umbilic, which is impossible.

Consequently $\nabla \sigma$ vanishes identically in $M$, i.e. $\sigma$ is a constant. Then $\lambda$ is also a constant. Indeed, assuming in (2.1) that $V$ is orthogonal to $X$ and that $Y = Z$, $g(Y, Y) \neq 0$, we obtain $X \lambda = 0$.

Since $\sigma$ is a constant, (1.2) implies

\[(3.1) \quad f^* \bar{R} = \varepsilon e^{2\sigma} R . \]

Let us assume that $f$ is not an isometry, i.e. $(\sigma, \varepsilon) \neq (0, 1)$. Then (3.1) and Lemma 3 c) yield

\[(3.2) \quad R = \alpha \varphi(h) + \beta \pi_1 , \]

where

\[\alpha = \frac{\lambda}{\varepsilon e^{-2\sigma} - 1} , \quad \beta = \frac{\lambda^2}{\varepsilon e^{-2\sigma} - 1} \]

are constants. From (3.2), by a standard way (see e.g. [5] or [6], Example 4), we conclude that the Weyl conformal curvature tensor of $M$ vanishes identically. So, if $n > 3$ then $M$ is conformally flat, which is a contradiction. Let $n = 3$. Using (3.2) we find

\[(3.3) \quad S - \frac{\tau}{4} g = \alpha h + \frac{\beta}{2} g . \]
Since $\alpha, \beta$ are constants, the equation of Codazzi and (3.3) imply (1.1). Thus $M$ is conformally flat, which is not the case. Consequently $f$ is an isometry, i.e. $\sigma = 0, \varepsilon = 1$. Then by (3.1) and Lemma 3 c) we obtain

$$\lambda \varphi(h) + \lambda^2 \pi_1 = 0,$$

which implies

$$(3.4) \quad \lambda \{(n - 2)h + g \text{tr} h\} + (n - 1)\lambda^2 g = 0.$$

But $M$ can not be totally umbilic. So (3.4) yields $\lambda = 0$. Hence $f_* \bar{h} = h$. Since $f$ is an isometry, this proves the theorem.

4. PROOF OF THEOREM 2

By Lemma 3 a), b) and (1.4) we conclude that $\lambda = 0$. Putting $X = \nabla \sigma$ in (2.1) and using (2.2), we obtain

$$\|\nabla \sigma\|^2 B(Y, Z) = 0.$$

Since the nonumbilic points are dense, this implies $\|\nabla \sigma\|^2 = 0$. According to Lemma 4 b), $R = 0$ in the open set $U$, in which $\nabla \sigma \neq 0$. Thus the point $p$, in which $R \neq 0$, can not lie in the closure $\overline{U}$ of $U$. Consequently, the open set $M \setminus \overline{U}$ is nonempty. Note that $d\sigma = 0$ in $M \setminus \overline{U}$. Let $V$ be the connected component of $p$ in $M \setminus \overline{U}$. Since $\sigma$ is a constant in $V$, (1.2) reduces to

$$(4.1) \quad f^* R = \varepsilon e^{2\sigma} R$$

in $V$. On the other hand, applying Lemma 3 c) with $\lambda = 0$, we obtain

$$(4.2) \quad f^* R = e^{4\sigma} R.$$

From (4.1) and (4.2) we find $(e^{4\sigma} - \varepsilon)R = 0$. Since $p$ lies in $V$, this implies $\sigma = 0$ (in $V$) and $\varepsilon = 1$. So we have $f^* \bar{g} = g, f^* \bar{h} = h$ in $V$. Consequently, $f$ is a congruence of $V$ onto $f(V)$, which completes the proof.

Remark. If the manifolds in Theorem 2 are analytic or the set of points, in which $R$ is not zero, is dense, then $f$ is a congruence of $M$ onto $\overline{M}$.

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