On the Class of Infinitely Divisible Exponential Distribution

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Abstract. The class of infinitely divisible exponential distribution is determined by property of Levy measure from its canonical representation of characteristic functions. Levy measure of the exponential distribution is obtained belonging to a class of completely monotones and measurable function. The class of this Levy measure is governed from exponential distribution similarly to the class of Thorin.

1. Introduction
The infinitely divisible distribution has played fundamental role in probability theory for characterizing the behavior of probability density function. The role is described by using a characteristic function as Fourier-Stieltjes transform of a random variable, then the characteristic function has some important properties that is introduced for the first time as the canonical representation of characteristic function by de Finneti, and then it is developed by Levy and Khintchine for infinite mean and variance [1]. Definition of characteristic function is referred from Lukacs [2] or Satō [3] as transformation of random variable $X$ into $\varphi(t) = E[\exp(itX)]$ for real $t$ and $\exp(itX) = \cos tX + i \sin tX$ and $i$ as imaginary unit. The random variable $X$ with distribution function $F$ is said to be infinitely divisible if for every positive integer number $n$, there exists independent and identically random variables $X_i$ for $i = 1, 2, ..., n$ with distribution function $F_n$ such that $X = X_1 + X_2 + ... + X_n$ where $F$ is $n$-fold convolution of $F_n$ in the form of $F = F_1 * F_2 * ... * F_n$. An equivalent definition of infinitely divisible distribution is that by using characteristic function, let a distribution function $F$ has characteristic function $\varphi(t)$ for every positive integer $n$, the characteristic function $\varphi(t)$ is infinitely divisible if there exists a characteristic function $\varphi_n(t)$ such that $\varphi(t) = (\varphi_n(t))^t$. This definition of infinitely divisible distribution by using characteristic function has led to the canonical form of characteristic function as a complex valued function.

Identification of infinitely divisible distribution is completely solved by canonical representation of characteristic function, where in this canonical form has Levy measure [4]. The property of Levy measure has introduced as the class of infinitely divisible distribution in the Thorin class [5,6], Jurek class [7], Goldie-Steutel-Bondesson class [8,9,10], Selfdecomposable class [11], Type $G$ class [12] and $M (R^d)$ class [13]. Recently Mselmi et al. [14] have also introduced the characterization of a class of infinitely divisible mixture distributions from the property of Levy measure. The generated class from Levy measure is still important to develop for certain distribution. The exponential distribution with memoryless property has attracted to be solved on its convolution and infinite divisibility. Ma and Liu [13] and Akkouchi [15] have introduced directly the convolution of exponential distribution with difference parameters, while the property of hypoexponential distribution has introduced by Smaili and Kadry [16] and Devianto et al. [17]. Furthermore, Devianto et al. [18] and Devianto [19]
has introduced a convolution of exponential distribution with stabilizer constant and its characteristic function property.

The exponential distribution with memoryless property has an important role on mathematical modeling for stochastic processes. This role has to identify its characteristic function property, so that it is necessary to give Levy measure and its class of infinitely divisible distribution by employing the canonical representation of characteristic function. This paper will give exploration of infinite divisibility of exponential distribution and the property of its Levy measure on governing the class of infinitely divisible distribution.

2. The Class of Infinitely Divisible Distribution

The class of infinitely divisible distribution is determined by using behavior of its characteristic function on the canonical representation. The following theorems are referred from [1], [2], [3] as the structures of infinitely divisible distribution on the terms of its canonical representation of characteristic function.

**Theorem 2.1 (de Finetti canonical representation).** A characteristic function is infinitely divisible, if and only if, it has the form

\[
\phi(t) = \lim_{n \to \infty} \exp[p_n(\phi_n(t) - 1)]
\]

where the \(p_n\) are real positive numbers and \(\phi_n(t)\) are characteristic function.

**Theorem 2.2 (Kolmogorov canonical representation).** The function \(\phi(t)\) is the characteristic function of an infinitely divisible distribution with a finite variance, if and only if, it admits the representation

\[
\phi(t) = \exp \left[ i\gamma t + \int_{-\infty}^{\infty} \left( \frac{\exp[itx] - 1 - itx}{x^2} \right) dK(x) \right]
\]

where \(\gamma\) is a real constant, \(K(x)\) is a bounded non-decreasing function, and the function under the integral sign is equal to \(-i^2/2\) at the point \(x = 0\). The canonical triple \((\gamma, u, K)\) is unique.

**Theorem 2.3 (Levy-Khintchine canonical representation).** The function \(\phi(t)\) is the characteristic function of an infinitely divisible distribution, if and only if, it admits the representation

\[
\phi(t) = \exp \left[ i\gamma t + \int_{-\infty}^{\infty} \left( \frac{\exp[itx] - 1 - itx}{1 + x^2} \right) \frac{1 + x^2}{x^2} dG(x) \right]
\]

where \(\gamma\) is a real constant, \(G(x)\) is a bounded non-decreasing function such that \(G(-\infty) = 0\). The integrand is defined to be equal to \(-i^2/2\) at the point \(x = 0\).

**Theorem 2.4 (Levy canonical representation).** The function \(\phi(t)\) is the characteristic function of an infinitely divisible distribution, if and only if, it admits the representation

\[
\phi(t) = \exp \left[ i\gamma t - \frac{t^2 \sigma^2}{2} + \int_{|x| > 0} \left( \frac{\exp[itx] - 1 - itx}{1 + x^2} \right) d\nu(x) \right]
\]

where \(\gamma\) is a real constant, \(\sigma^2\) is a real and non-negative constant and the function \(\nu(x)\) is non-decreasing on the intervals \((-\infty, 0)\) and \((0, \infty)\), and satisfies the conditions

\[
\lim_{x \to -\infty} \nu(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} \nu(x) = 0
\]

and

\[
\int_{|x| < \delta} x^2 d\nu(x) \quad \text{for every finite } \delta > 0.
\]

The canonical triple \((\gamma, \sigma^2, \nu)\) is unique.

The triple \((\gamma, \sigma^2, \nu)\) of infinitely divisible characteristic function is a unique canonical representation, the function \(\nu(x)\) is called Levy spectral function or Levy measure. The canonical representation on Theorem 2.3 is similar to Levy canonical representation as in Theorem 2.4 by setting
\[
\nu(x) = \begin{cases} 
\int_{-1}^{1} \frac{1 + y^2}{y^2} dG(y) & \text{for } x > 0 \\
\int_{-\infty}^{x} \frac{1 + y^2}{y^2} dG(y) & \text{for } x < 0.
\end{cases}
\]  

(7)

The class of infinitely divisible distribution is based on the characterization of Levy measure from canonical representation of the characteristic function. Thorin [5,6] has introduced the class \( T(\mathbb{R}^d) \) from the process of a convolution from gamma distribution. The Levy measure of Thorin class has the following form \( \nu(dx) = k(x)x^d\, dx \) where \( k(x) \) measurable and completely monotone on \((0, \infty)\). On the other hand, when \( k(t) \) measurable and non-increasing on \((0, \infty)\) then this class is noted as \( L(\mathbb{R}^d) \) or Selfdecomposable which is introduced in Sato [3].

Furthermore, Bondesson [8,9] presented the class \( B(\mathbb{R}^d) \) when he studied a generalized convolution of mixture of exponential distribution on \((0, \infty)\), then it is developed by Barndorff-Nielsen et al. [8] by setting this class as Goldie-Steutel-Bondesson class with Levy measure \( \nu(dx) = l(x)\, dx \) where \( l(x) \) measurable and completely monotone on \((0, \infty)\). However, Jurek [7] has introduced the class \( U(\mathbb{R}^d) \) where the Levy measure \( \nu(dx) = l(x)\, dx \) where \( l(x) \) measurable but non-increasing on \((0, \infty)\).

Maejima and Rosinski [12] have introduced the class \( G(\mathbb{R}^d) \) for infinitely divisible distribution with Levy measure \( \nu(dx) = g(x^2)\, dx \) for \( g(x) \) measurable and completely monotone on \((0, \infty)\). The last is the class \( M(\mathbb{R}^d) \) by Aoyama et al. [11] with Levy measure \( \nu(dx) = g(x^2)x^d\, dx \) for \( g(x) \) measurable and completely monotone on \((0, \infty)\).

3. Canonical Representation of Characteristic Function and the Class of Infinitely Divisible Exponential Distribution

The probability density function of exponential distribution is defined as

\[ f(x) = \theta \exp(-\theta x) \]

for \( x \in (0, \infty) \) and \( \theta > 0 \). It is used the Fourier-Stieltjes transform to obtain characteristic function from exponential distribution in the following term

\[ \phi(i\tau) = E[\exp(i\tau X)] = \int_{0}^{\infty} \exp(i\tau x) \theta \exp(-\theta x) \, dx = \frac{1}{1 + \frac{i\tau}{\theta}} \]

for every \( \tau \in (-\infty, \infty) \).

**Theorem 3.1.** Let the random variable \( X \) has exponential distribution, then canonical representation of characteristic function from exponential distribution is unique in the term of triple \((\gamma, \sigma^2, \nu)\), that is in the form

\[ \phi(i\tau) = \exp \left[ i\tau - \frac{\sigma^2}{2} + \int_{|x| > 0} \left( \exp[i\tau x] - 1 - \frac{i\tau x}{1 + x^2} \right) \, d\nu(x) \right] \]

for \( \gamma = \int_{0}^{\infty} \frac{\exp(-\theta x)}{1 + x^2} \, dx \), \( \sigma^2 = 0 \) and \( \nu(x) = x^{-1} \exp(-\theta x) \).

**Proof.** Let us consider following expression of mathematical equation

\[ \log \left( 1 + \frac{t}{\theta} \right) = \int_{0}^{\frac{t}{\theta}} \frac{1}{\theta + y} \, dy = \int_{0}^{\frac{t}{\theta}} dy \int_{0}^{\frac{t}{\theta} - y} \exp(-\theta x) \, dx = \int_{0}^{\infty} \exp(-\theta x) \, dx \int_{0}^{\infty} \exp(-yx) \, dy, \]

and we have the Laplace-Stieltjes transform for exponential distribution as
\[ \zeta(s) = E[\exp(sX)] = \int_0^\infty \exp(sx) \theta \exp(-\theta x) \, dx = \frac{1}{1 + \frac{s}{\theta}} \]  

for \( s \in (0, \infty) \), the equation (11) can be rewritten in the following form

\[ \zeta(s) = \exp \left[ \int_0^\infty (\exp(itx) - 1) \frac{\exp(-\theta x)}{x} \, dx \right]. \]  

We define this Laplace-Stieltjes transform for exponential distribution on the half of complex planes as continuous and analytical function on \( \{ w \in \mathbb{C} : \text{Re } w \leq 0 \} \), so that we can write

\[ \int_{-\infty}^\infty \exp(wx) f(x) \, dx = \exp \left[ \int_0^\infty (\exp(itx) - 1) \frac{\exp(-\theta x)}{x} \, dx \right]. \]  

Now, let us define \( w = it \) and \( t \in (-\infty, \infty) \), then we have the characteristic function of exponential distribution in the following form

\[ \varphi(t) = \frac{1}{1 + \frac{it}{\theta}} = \exp \left[ \int_0^\infty (\exp(itx) - 1) \frac{\exp(-\theta x)}{x} \, dx \right]. \]  

The uniqueness representation of infinitely divisible characteristic function from exponential distribution is obtained by using relation of Theorem 2.3 and Theorem 2.4 in the triple \( (\gamma, \sigma^2, \nu) \) for

\[ \nu(x) \, dx = \frac{1 + x^2}{x^2} \, dG(x) \]  

so that, we have the following terms

\[ \nu(x) = \frac{\exp(-\theta x)}{x}, \]  

\[ \sigma^2 = G(+0) - G(-0) = 0, \]  

\[ \gamma = \int_0^\infty \frac{\exp(-\theta x)}{1 + x^2} \, dx. \]  

The conditions on equation (16), (17) and (18) have established the uniqueness of canonical representation of characteristic function from exponential distribution in the term of triple \( (\gamma, \sigma^2, \nu) \).

**Theorem 3.2.** Let the random variable \( X \) has exponential distribution with canonical representation of characteristic function in the term of triple \( (\gamma, \sigma^2, \nu) \) where the Levy measure is

\[ \nu(x) = \frac{\exp(-\theta x)}{x}, \]  

for \( x \in (0, \infty) \) and \( \theta > 0 \). The Levy measure \( \nu(x) \) is measurable and completely monotone on \( (0, \infty) \).

**Proof.** Let us define an interval \( (a,b) \) where \( a, b \in (0, \infty) \) as the Borel set, so that the combined interval is also Borel set for any \( \sigma \)-algebra. Then the support of \( \nu(x) \) is also the Borel set, since

\[ (a, a + 1, a + 2, \ldots, \infty) = \bigcup_{n=1}^\infty (a, a + n), \]  

so that the Levy measure is measurable. Now let us take any number \( x_1, x_2 \in (0, \infty) \) and \( x_1 < x_2 \), then we will show that \( \nu(x_1) > \nu(x_2) \). It is easily to see that for every \( x_1 < x_2 \), then we have
so that the Levy measure from exponential distribution is non-increasing function in term of completely monotone on $(0,\infty)$.

The Levy measure of exponential distribution is measurable and completely monotone on $(0,\infty)$. This property is governed the class of infinitely divisible distribution similarly to the class of Thorin. Then it can be concluded that the class of infinitely divisible distribution generated from exponential distribution has the same class with the distribution from the convolution of gamma distribution.

The property of characteristic function and Levy measure of exponential distribution is describe graphically in Figure 1. is the shape of the parametric curve of characteristic function from exponential distribution as a smooth curve of a circle, that is continuous and never vanish on the complex plane. Figure 2. has shown the graphs of Levy measure of exponential distribution with various parameter $\theta$, this is to confirm the Levy measure from exponential distribution is measurable and non-increasing function on $(0,\infty)$, this measurable property and its completely monotone function governed the similar class of infinitely divisible distribution by Thorin [5,6] when he studied the process of a convolution from gamma distribution on infinitely divisible Pareto distribution and lognormal distribution.

4. Conclusion
The exponential distribution with memoryless property is infinitely divisible distribution, so that it can be written in the form of canonical representation of characteristic function. The canonical representation is contained Levy measure, that is measurable and non-increasing function on positive real line. This Levy measure of exponential distribution is obtained belonging to a class of measurable and completely monotone function with similar to the class of Thorin.

5. References
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