Flow-time Optimization For Concurrent Open-Shop and Precedence Constrained Scheduling Models

Janardhan Kulkarni *  Shi Li †

Abstract

Scheduling a set of jobs over a collection of machines is a fundamental problem that needs to be solved millions of times a day in various computing platforms: in operating systems, in large data clusters, and in data centers. Along with makespan, flow-time, which measures the length of time a job spends in a system before it completes, is arguably the most important metric to measure the performance of a scheduling algorithm. In recent years, there has been a remarkable progress in understanding flow-time based objective functions in diverse settings such as unrelated machines scheduling, broadcast scheduling, multi-dimensional scheduling, to name a few.

Yet, our understanding of the flow-time objective is limited mostly to the scenarios where jobs have simple structures; in particular, each job is a single self contained entity. On the other hand, in almost all real world applications, think of MapReduce settings for example, jobs have more complex structures. In this paper, we consider two classical scheduling models that capture complex job structures: 1) concurrent open-shop scheduling (COSSP) and 2) precedence constrained scheduling (PCSP). Our main motivation to study these problems specifically comes from their relevance to two scheduling problems that have gained importance in the context of data centers: co-flow scheduling and DAG scheduling. We design almost optimal approximation algorithms for COSSP and PCSP, and show hardness results.

1 Introduction

Scheduling a set of jobs over a collection of machines is a fundamental problem that needs to be solved millions of times a day in various computing platforms: in operating systems, in large data clusters, and in data centers. Along with makespan, flow-time, which measures the length of time a job spends in a system before completing, is arguably the most important metric to measure the performance of a scheduling algorithm. In recent years, there has been a remarkable progress in understanding flow-time related objective functions in diverse settings such as unrelated machines scheduling [5, 7, 22, 27, 31], broadcast scheduling [11, 12, 28], multi-dimensional scheduling [30, 32], to name a few.

Yet, our understanding of the flow-time based objective functions is mostly limited to the scenarios where jobs have simple structures; in particular, each job is a single self contained entity. On the other hand, in almost all real world applications, jobs have more complex structures. Consider the MapReduce model for example. Here, each job consists of a set of Map tasks and a set of Reduce tasks. Reduce tasks cannot be processed unless Map tasks are completely processed¹. A MapReduce job is complete only when all Map and Reduce tasks are completed. Motivated by these considerations, in this paper, we consider two classical

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¹Microsoft Research, Redmond, jakul@microsoft.com

¹University at Buffalo, Buffalo, NY, shil@buffalo.edu. The work is in part supported by NSF grants CCF-1566356 and CCF-1717134.

¹In some MapReduce applications, Reduce tasks can begin after the completion of a subset of Map tasks.
scheduling models that capture more complex job structures: 1) concurrent open-shop scheduling (COSSP), and 2) precedence constrained scheduling (PCSP). Our main reason to study these problems specifically comes from their relevance to two scheduling problems that have gained importance in the context of data centers: co-flow scheduling and DAG scheduling. We discuss more about how these problems relate to COSSP and PCSP in Section 1.3.

The objective function we consider in this paper is minimizing the sum of general delay costs of jobs, first introduced in an influential paper by Bansal and Pruhs [8] in the context of single machine scheduling. In this objective, for each job \( j \) we are given a non-decreasing function \( g_j : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \), which gives the cost of completing the job at time \( t \). The goal is to minimize \( \sum_j g_j(C_j) \), where \( C_j \) is the completion time of \( j \). A desirable aspect of the general delay cost functions is that they capture several widely studied flow-time and completion time based objective functions.

- Minimizing the sum of weighted flow-times of jobs. This is captured by the function \( g_j(t) = w_j \cdot (t - r_j) \), where \( r_j \) is the release time of \( j \).
- Minimizing the sum of weighted \( p \)th power of flow-times of jobs. This is captured by the function \( g_j(t) = w_j \cdot (t - r_j)^p \).
- Minimizing the sum of weighted tardiness. This is captured by the function \( g_j(t) = w_j \cdot \max\{0, (t - d_j)\} \), where \( d_j \) is the deadline of \( j \).

In this paper, we design approximation algorithms for minimizing the sum of general delay costs of jobs for the concurrent open-shop scheduling and the precedence constrained scheduling problems.

### 1.1 Concurrent Open-shop Scheduling Problem (COSSP)

In COSSP, we are given a set of \( m \) machines and a set of \( n \) jobs. Each job \( j \) has a release time \( r_j \). The main feature of COSSP is that a job consists of \( m \) operations \( O_{1j}, O_{2j}, \ldots, O_{mj} \), one for each machine \( i \in [m] \). Each operation \( O_{ij} \) needs \( p_{ij} \) units of processing on machine \( i \). We allow operations to have zero processing lengths. Throughout the paper, we assume without loss of generality that all our input parameters are positive integers. A job is complete only when all its operations are complete. That is, if \( C_j \) denotes the completion time of \( j \), then \( p_{ij} \) units of operation \( O_{ij} \) must be processed in the interval \([r_j, C_j]\) on machine \( i \). In the concurrent open-shop scheduling model, multiple operations of the same job can be processed simultaneously across different machines.

The problem has a long history due to its applications in manufacturing, automobile and airplane maintenance and repair [53], etc., and has been extensively studied both in operations research and approximation algorithm communities [3, 6, 16, 23, 39, 43, 51, 52]. As minimizing makespan in COSSP model is trivial, much of the research has focused on the objective of minimizing the total weighted completion times of jobs. The problem was first considered by Ahmadi and Bagchi [3], who showed the NP-hardness of the problem. Later, several groups of authors, Chen and Hall [16], Garg et al. [23], Leung et al. [39], and Mastroiilli et al. [43], designed 2-approximation algorithms for the problem. Under Unique Games Conjecture, Bansal and Khot showed that this approximation factor cannot be improved for the problem [6].

Garg, Kumar, and Pandit [23] studied the more difficult objective of minimizing the total flow-time of jobs, and showed that the problem cannot be approximated better than \( \Omega(\log m) \), by giving a reduction from the set cover problem. However, they did not give any approximation algorithm to the problem, and left it as an open problem. To the best of our knowledge, the problem has remained open ever since. In this paper, we make progress on this problem.

Let \( P \) denote the ratio of the maximum processing length among all the operations to the minimum non-zero processing length of among all the operations; that is, \( P := \frac{\max_{i,j} \{p_{ij}\}}{\min_{i,j} \{p_{ij} : p_{ij} \neq 0\}} \).
Theorem 1.1. For the objective of minimizing the sum of general delay cost functions of jobs in the concurrent open-shop scheduling model, there exists a polynomial time $O(\log(m \log P))$ approximation algorithm.

We obtain the above result by generalizing the algorithm in Bansal and Pruhs [8]. Note that when $m = 1$, our result gives a $O(\log P)$ approximation algorithm to the problem, matching the best known polynomial time result in [8]. Recently, for the special case of total weighted flow-time, a constant factor approximation algorithm was obtained by Batra, Garg, and Kumar [13] when $m = 1$. However, running time of their algorithm is pseudo-polynomial. Since the approach in [13] is very different from the one in [8], our result does not generalize [13].

As we discussed earlier, the general delay cost functions capture several widely studied performance metrics. Thus, we get:

Corollary 1.2. There is a polynomial time $O(\log(m \log P))$ approximation algorithm in the concurrent open-shop scheduling model for the following objective functions: 1) Minimizing the sum of weighted flow-times of jobs; 2) Minimizing the weighted $\ell_k$-norms of flow-times of jobs; the approximation factor becomes $O((\log(m \log P)^\frac{1}{k}))$; 3) Minimizing the sum of weighted tardiness of jobs.

We give the proof Theorem 1.1 in Section 2.

1.2 Precedence Constrained Scheduling Problem (PCSP)

More complex forms of job structures are captured by the precedence constrained scheduling problem (PCSP), another problem that has a long history dating back to the seminal work of Graham [24]. Here, we have a set of $m$ identical machines and a set of $n$ jobs; each job $j$ has a processing length $p_j > 0$, a release time $r_j > 0$. Each job $j$ must be scheduled on exactly one of the $m$ machines. The important feature of the problem is that they are precedence constraints between jobs that capture the computational dependencies across jobs. The precedence constraints are given by a partial order “$\prec$”, where a constraint $j \prec j'$ requires that job $j'$ can only start after job $j$ is completed. Our goal is to schedule (preemptively) each job on exactly one machine to minimize $\sum_j g_j(C_j)$.

Precedence constrained scheduling on identical machines to minimize the makespan objective is perhaps the most well-known problem in scheduling theory. Already in 1966, Graham showed that list scheduling gives a $2$-approximation algorithm to the problem. Since then several attempts have been made to improve the approximation factor [21, 37]. However, Svensson [49] showed that problem does not admit a $2 - \epsilon$ approximation under a strong version of the Unique Games Conjecture introduced by Bansal and Khot [6]. An unconditional hardness of $(4/3 - \epsilon)$ is also known due to Lenstra and Rinnooy Kan [38]. Recently, Levey and Rothvoss [40] showed that it is possible to overcome these lowerbounds for the case when $m$ is fixed. An LP-hierarchy lift of the time-index LP with a slightly super poly-logarithmic number of rounds provides a $(1 + \epsilon)$ approximation to the problem.

Another problem that is extensively studied in the precedence constrained scheduling model is the problem of minimizing the total weighted completion times of jobs. Note that this problem strictly generalizes the makespan problem, hence all the lowerbounds also extend to this problem. The current best approximation factor of 3.387 is achieved by a very recent result of Li [41]. The work builds on a $4$-approximation algorithm due to Munier, Queyranne and Schulz ([45], [47]).

In a recent work, Agrawal et al [2] initiated the study of minimizing the total flow-time objective in DAG (Directed Acyclic Graphs) parallelizability model. In this model, each job is a DAG, and a job completes only when all the nodes in the DAG are completed. For this problem, they showed greedy online algorithms that are constant competitive when given $(1 + \epsilon)$-speed augmentation. The DAG parallelizability model is a
special case of PCSP. However, as there are no dependencies between jobs, and individual nodes of the DAG
do not contribute to the total flow-time unlike in PCSP, complexity of the problem is significantly different
from PCSP. For example, when there is only one machine, the DAG structure of individual jobs does not
change the cost of the optimal solution, and hence the problem reduces to the standard single machine
scheduling problem. Therefore, scheduling jobs using Shortest Remaining Processing Time (SRPT), where
processing length of a DAG is its total work across all its nodes, is an optimal algorithm. (Within a DAG,
the nodes can be processed in any order respecting the precedence constraints.)

On the other hand, we show a somewhat surprising result for the PCSP problem. We show that the
problem of minimizing the total flow-times of jobs does not admit any reasonable approximation factor
even on a single machine. This is in sharp contrast to makespan and the sum of weighted completion times
objective functions that admit \(O(1)\)-approximation algorithms even on multiple machines. Our hardness
proof is based on a recent breakthrough work of Manurangsi [42] on approximating the Densest-k-Subgraph
(DKS) problem.

**Theorem 1.3.** In the precedence constrained scheduling model, for the objective of minimizing the total
flow-times of jobs on a single machine, no polynomial time algorithm can achieve an approximation factor
better than \(n^{\frac{1}{1+\log \log n}}\), for some universal constant \(c > 0\), assuming the exponential time hypothesis (ETH).

To circumvent this hardness result, we study the problem in the speed augmentation model, which can
also be thought of as a bi-criteria analysis. In the speed augmentation model, each machine is given some
small extra speed compared to the optimal solution. The speed augmentation model was introduced in the
seminal work of Kalyanasundaram and Pruhs [34] to analyze the effectiveness of various scheduling heuris-
tics in the context of online algorithms. However, the model has also been used in the offline setting to over-
come strong lowerbounds on the approximability of various scheduling problems; see for example results
on non-preemptive flow-time scheduling that use \(O(1)\)-speed augmentation to obtain \(O(1)\)-approximation
ratio [10, 33].

Our second main result is an \(O(1)\)-approximation algorithm for the problem in the speed augmentation
model. Previously, no results were known for the flow-time related objective functions for PCSP.

**Theorem 1.4.** For the objective of minimizing the sum of general delay cost of jobs in the precedence
constrained scheduling model on identical machines, there exists a polynomial time \(O(1)\)-speed \(O(1)\)
approximation algorithm. Furthermore, the speed augmentation required to achieve an approximation factor
better than \(n^{1-c}\), for any \(c > 0\), has to be at least the best approximation factor of the makespan minimiza-
tion problem. The lowerbound on speed augmentation extends to any machine environment, such as related
and unrelated machine environments.

We give the proofs of above theorems in Section 3.

### 1.3 Applications of COSSP and PCSP in Data Center Scheduling

Besides being fundamental optimization problems, COSSP and PCSP models are very closely related to
the scheduling problems that arise in the context of data centers. In particular, COSSP is a special case of
the Coflow Scheduling problem introduced in a very influential work of Choudary and Stoica [17, 19]. On
the other hand, PCSP generalizes the DAG scheduling problem, again a widely studied problem in systems
literature. In fact, the DAG scheduling model has been adopted by Yarn, the resource manager of Hadoop
[1]. See [25, 26] and references there-in for more details.
Besides being fundamental optimization problems, COSSP and PCSP models are very closely related to the scheduling problems that arise in the context of data centers. We briefly describe the relevance of COSSP and PCSP to scheduling in data centers.

Coﬂow Scheduling The COSSP problem is a special case of the coﬂow scheduling abstraction introduced in a very inﬂuential work of Choudary and Stoica [17, 19], in the context of scheduling data ﬂows. They deﬁned coﬂow as a collection of parallel ﬂows with a common performance goal. Their main motivation to introduce coﬂow was that in big-data systems, MapReduce systems for example, communication is structured and takes place across machines in successive computational stages. In most cases, communication stage of jobs cannot ﬁnish until all its ﬂows have completed. For example, in MapReduce jobs, a reduce task cannot begin until all the map tasks ﬁnish. Although coﬂow abstraction was introduced to model scheduling ﬂows in big data networks, it can also be applied to job scheduling in clusters; see [25] for example.

Therefore, we describe a slightly more general version of the coﬂow abstraction. Here, we are given a set of m machines (or m resources). Each job j consists of a set of operations {Oij} for i = 1, 2, …. Associated with each operation Oij is a demand vector Dij = (dij(1), dij(2), ..., dij(m)), where each dij(k) ∈ {0, 1}. The demand vector indicates the subset of machines or resources the operation requires. An operation can be executed only when all the machines in the demand vector are allocated to it. Moreover, for each operation we are also given a processing length pij. The goal is to schedule all operations such that at any time instant the capacity constraints on machines are not violated: that is, each machine is allocated to exactly one operation. A job completes only when all its operations have ﬁnished.

The coﬂow problem studied by Choudary and Stoica [17, 19] corresponds to the case where the demand vectors of all jobs have exactly two 1s. That is, every operation needs two machines to execute. The machines typically correspond to input and output ports in a communication link. On the other hand, if the demand vector Dij of each operation Oij consists of exactly one non-zero entry, then the coﬂow scheduling is equivalent to COSSP.

In the past few years, coﬂow scheduling has attracted a lot of research both in theory and systems communities. In practice, several heuristics are known to perform well [17–19, 25] for the problem. The theoretical study of coﬂow scheduling was initiated by Qiu, Stein, and Zhong [46]. By exploiting its connections to COSSP, they designed a constant factor approximation algorithm to the objective of minimizing the total weighted completion times of jobs. Building on this work, better approximation algorithms were designed in [4, 36, 48]. Unfortunately, the techniques developed in these works do not seem to extend to the ﬂow-time related objectives.

DAG Scheduling Another problem that has attracted a lot of research in practice is the DAG scheduling problem. In this problem, we are given a set of machines (or clusters), and a set of jobs. Each job has a weight that captures the priority of a job. Each job j is represented by a directed acyclic graph (DAG). Each node of a DAG represents a task – a single unit of computation – that needs to executed on a single machine. Each task has a release time and a processing length. An edge j → j’ in the DAG indicates that the task j’ depends on the task j, and j’ can not begin its execution unless j ﬁnishes. The goal is to schedule jobs/DAGs on the machines so as to minimize the total weighted ﬂow-time of jobs. Interestingly, this is one of the models of job scheduling that has been adopted by Yarn, the resource manager of Hadoop [1]. (Hadoop is a popular implementation of MapReduce framework.) Because of this, DAG scheduling has been a very active area of research in practice; see [25, 26] and references there-in for more details.

It is not hard to see that DAG scheduling problem is a special case of the precedence constrained scheduling problem. The union of the individual DAGs of jobs can be considered as one DAG, with appropriately deﬁned release times and weights for each node. Furthermore, if jobs have no weight, and the release times of all tasks in the same DAG are equal, then the DAG scheduling model described above is same as the DAG
parallelizability model studied by Agrawal et al [2]. In fact, they design \((1 + \epsilon)\)-speed \(O(1/\epsilon)\)-competitive online algorithm for the problem. On the other hand, our approximation algorithm for PCSP gives an approximation algorithm for the DAG scheduling problem even with weights and arbitrary release times for individual tasks.

1.4 Overview of the Algorithms and Techniques

Both our algorithms are based on rounding linear programming relaxations of the problems. However, individual techniques are quite different, and hence we discuss them separately.

**Open-shop Scheduling Problem** Our algorithm for COSSP is based on the geometric view of scheduling developed by Bansal and Pruhs [8] for the problem of minimizing the sum of general delay costs of jobs on a single machine, which is a special case of our problem when \(m = 1\). The key observation that leads to this geometric view is that minimizing general delay costs is equivalent to coming-up with a set of feasible deadlines for all jobs. Moreover, testing the feasibility of deadlines further boils down to ensuring that for every interval of time, the total volume of jobs that have (arrival time, deadline) windows within that interval is not large compared to the length. By a string of nice arguments, the authors show that this deadline scheduling problem can be viewed as a capacitated geometric set cover problem called R2C, which stands for capacitated rectangle covering problem in 2-dimensional space. Further, they argue that an \(\alpha\)-approximation algorithm for R2C problem can be used to obtain an \(\alpha\)-approximation algorithm for the scheduling problem.

In R2C, we are given a set \(P\) of points in 2 dimensions, where each \(p \in P\) is specified by coordinates \((x_p, y_p)\). Associated with each point \(p \in P\) is a demand \(d_p > 0\). We are also given a set \(R\) of rectangles, where \(r \in R\) has the form \((0, x_r) \times (y_r^1, y_r^2)\). Each rectangle \(r\) has a capacity \(c(r) > 0\) and a cost \(w(r) > 0\). The goal is to choose a minimum-cost set of rectangles, such that for every \(p \in P\), the total capacity of selected rectangles covering \(p\) is at least \(d_p\).

Our problem, which we call PR2C, can be seen as a parallel version of R2C. In PR2C, we have \(m\) instances of R2C problem with a common set of rectangles. Namely, the \(i\)th instance is defined by \((P_i, R)\), where each \(p \in P_i\) is associated with a demand \(d_p\), each \(r \in R\) is associated with a capacity \(c(i, r)\) and cost \(w(r)\). Notice that a rectangle \(r \in R\) has the same cost across the \(m\) instances, but has different capacities in different instances. The goal is to find a minimum cost set \(X \subseteq R\) of rectangles that is a valid solution for every R2C instance \((P_i, R)\). Using the arguments similar to [8], we also show that if there is an \(\alpha\)-approximation to PR2C problem, it gives an \(\alpha\)-approximation for the COSSP problem.

Thus, much of our work is about designing a good approximation algorithm for the PR2C problem. Our algorithm for PR2C is a natural generalization of the algorithm for R2C of Bansal and Pruhs in [8]. In [8], Bansal and Pruhs formulated an LP relaxation for R2C based on knapsack cover (KC) inequalities. In the LP, \(x_r \in [0, 1]\) indicates the fraction of the rectangle \(r\) that is selected. If the LP solution picks a rectangle to some constant fraction, then we can also select the rectangle in our solution without increasing the cost by too much. After selecting these rectangles, some points in \(P\) are covered, and some other points \(p\) will still have residual demands \(d_p\). These not-yet-covered points are divided into two categories: light and heavy points. Roughly speaking, a point \(p\) is heavy if it is mostly covered by rectangles \(r\) with \(c(r) \geq d_p\) in the LP solution; a point \(p\) is light if it is mostly covered by rectangles \(r\) with \(c(r) < d_p\). Heavy points and light points are handled separately. The problem of covering heavy points will be reduced to the R3U

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2C” stands for the capacitated version in which rectangles have capacities and points have demands. Later, we shall use “M” for the multi-cover version, where rectangles are uncapacitated (or have capacity 1) and points have different demands. We use “U” for the uncapacitated version, where rectangles are uncapacitated and all points have demand 1.
problem, a geometric weighted set-cover problem where elements are points in 3D space and sets are axis-
parallel cuboids. On the other hand, the problem of covering light points can be reduced to $O(\log_2 P)$ R2M
instances. Each R2M instance is a geometric weighted set multi-cover instance in 2D-plane. By appealing to
the geometry of the objects produced by the scheduling instance, [8] prove that union complexity of objects
in R3U and R2M instances is small. In particular, R3U instance has union complexity $O(\log P)$ and each
R2M instance has union complexity $O(1)$. Using the technique of quasi-uniform sampling introduced in
[15, 50] for solving geometric weighted set cover instances with small union complexity, Bansal and Pruhs
obtain $O(\log \log P)$ and $O(1)$ approximation ratios for the problems of covering heavy and light points,
respectively.

In our problem, we have $m$ parallel R2C instances with a common set of rectangles. As in [8], for
each instance, we categorize the points into heavy and light points based on the LP solution. The problem
of handling heavy points then can be reduced to $m$ R3U instances, with the $m$ sets of cuboids identified.
However, we cannot solve these $m$ R3U instances separately, because such a solution cannot be mapped
back to a valid schedule for COSSP. Therefore, we combine the $m$ instances of R3C into a single instance
of a 4 dimensional problem. So, in the combined instance, our geometric objects, which we call hyper-
cuboids, contain (at most) one 4-cuboid (a cuboid in 4 dimensions) from each of the $m$ instances. The goal
is to choose a minimum cost set of objects to cover all the points. On the other hand, for the light points, [8]
reduced the problem to $O(\log P)$ R2M instances. Again, this is approach is not viable for our case as we
need to solve all the instances in parallel. By a simple trick, we first merge the $O(\log P)$ instances into one
R2M instance. We then have $m$ R2M instances with a new sets of rectangles identified, which we map into
a single 3-dimensional geometric multi-set cover problem.

In both cases we show that the union complexity of the objects in our geometric problems increase
at most by a factor of $m$ compared to the objects in [8]. Thus, we have $O(m \log P)$ union complexity
for the problem for heavy points and $O(m)$ union complexity for the problem for light points. Using
the technique of [8], we obtain $O(\log (m \log P))$ and $O(\log m)$ approximation ratios for heavy and light points
respectively, resulting in an $O(\log (m \log P))$ overall approximation ratio.

**Precedence Constrained Scheduling** Our algorithm for the precedence constrained scheduling problem
works in two steps. In the first step, we construct a migratory schedule, in which a job may be processed on
multiple machines. For migratory schedules, we can assume that all jobs have unit size by replacing each job
$j$ of size $p_j$ with a precedence chain of $p_j$ unit length jobs. Solving a natural LP relaxation for the problem
gives us a completion time vector $(C_j)_j$. Then we run the list-scheduling algorithm of Munier, Queyranne
and Schulz [45], and Queyranne and Schulz [47], that works for the problem with weighted completion time objective. Specifically, for each job \( j \) in non-decreasing order of \( C_j \) values, we insert \( j \) to the earliest available slot after \( r_j \) without violating the precedence constraints.

To analyze the completion time of job \( j \), we focus on the schedule constructed by our algorithm after the insertion of \( j \). A simple lemma is that at any time slot after \( r_j \), we are making progress towards scheduling \( j \) in the schedule: either all machines are busy in the time slot, or we are processing a set of jobs whose removal will decrease the “depth” of \( j \) in the precedence graph. This lemma was used in [45, 47] to give their 3-approximation for the problem of minimizing weighted completion time (for unit-size jobs), and recently by Li [41] to give an improved 2.415-approximation for the same problem. With 3-speed augmentation, this leads to a schedule that completes every job \( j \) by the time \( C_j \). With additional speed augmentation, this leads to an \( O(1) \)-approximation for the problem with general delay cost functions. In the second step, we convert the migratory schedule into a non-migratory one, using some known techniques (e.g. [35], [20], [29]). The conversion does not increase the completion times of jobs, but requires some extra \( O(1) \)-speed augmentation.

## 2 Concurrent Open-shop Scheduling

In this section we consider the concurrent open-shop scheduling problem. Recall that in COSSP, we are given a set of \( m \) machines and a set of \( n \) jobs. Each job has a release time \( r_j \). A job consists of \( m \) operations \( \{O_{ij}\}_i \), one for each machine \( i \in [m] \). Each operation \( O_{ij} \) needs \( p_{ij} \) units of processing on machine \( i \). A job finishes only when all its operations are completed. The goal is to construct a preemptive schedule that minimizes the sum of costs incurred by jobs:

\[
\sum_{j} g_j(C_j).
\]

As we mentioned earlier, our algorithm for COSSP is based on the geometric view of scheduling developed in the work of Bansal and Pruhs [8]. Similar to [8], we first reduce our problem to a geometric covering problem that we call Parallel Capacitated Rectangle Covering Problem (PR2C). We argue that an \( \alpha \)-approximation to PR2C will give an \( \alpha \) approximation to our problem. Then, we design a \( O(\log(m \log P)) \)-factor approximation algorithm to PR2C.

### 2.1 Reduction to PR2C problem.

In PR2C, we have \( m \) instances of R2C problem with a common set of rectangles. Input to the problem consists of sets \( P_1, P_2, \ldots, P_m \), where each \( P_i \) is a set of points in 2-dimensional space. Each point \( p \in P_i \) is specified by its coordinates \((xp, yp)\) and has a demand \( d_p > 0 \). The input also consists of a set \( R \) of rectangles. Each rectangle \( r \in R \) has the form \((0, xr) \times (yr_1, yr_2)\), and has a cost \( w(r) \). Each rectangle \( r \) also has a capacity \( c(i, r) \) which depends on the point set \( P_i \). Notice that a rectangle \( r \in R \) has the same cost across the \( m \) instances, but has different capacities in different instances. The goal is to find a minimum cost set \( X \subseteq R \) of rectangles that is a valid solution for every R2C instance \((P_i, R)\). Recall that a set of rectangles is a valid solution to an instance of R2C, if for every point the total capacity of rectangles that cover the point is at least the demand of the point.

We can capture the PR2C problem using an integer program. Let \( x_r \) be a binary variable that indicates if the rectangle \( r \in R \) is picked in the solution or not. Then, the following integer program IP (1 - 3) captures the PR2C problem.

\[
\text{Minimize } \sum_{r \in R} w(r)x_r
\]
\[ \forall i, \forall p \in P_i : \quad \sum_{r \geq p} c(i, r) x_r \geq d_p \]  
\[ \forall r \in R : \quad x_r \in \{0,1\} \]  

To see the connection between COSSP and PR2C, we need to understand the structure of feasible solutions to COSSP. Consider a feasible schedule \( S \) to an instance of COSSP. Suppose the completion time of a job \( j \) is \( C_j \) in \( S \). This implies that on every machine \( i \), the job \( j \) completes its operation \( O_{ij} \) in the interval \([r_j, C_j]\). If \( S \) is a feasible schedule and \( \{C_j\}_j \) is the completion times of jobs, then processing jobs using the Earliest Deadline First (EDF) algorithm on each machine ensures that all jobs finish by their deadline \( C_j \). Thus, one of the main observations behind our reduction is that minimizing the sum of costs incurred by jobs is equivalent to coming up with a deadline \( C_j \) for each job \( j \). Thus a natural approach is to formulate COSSP as a deadline scheduling problem. However, this raises the question: Is there a way to test if a set of deadlines \( \{C_j\}_j \) is a feasible solution to COSSP? An answer to the question is given by the characterization of when EDF will find a feasible schedule (one where all jobs meet their deadlines) on a single machine. To proceed, we need to set up some notation.

**Definition 2.1.** For a given interval \( I \) and a machine \( i \), the excess of \( I \) on machine \( i \), denoted by \( \xi(i, I) \), is defined as \( \max\{0, P(i,J(I)) - |I|\} \).

Following lemma states the condition under which EDF can schedule all the jobs within their deadlines.

**Lemma 2.2.** Given a set of jobs with release times and deadlines, scheduling operations on each machine according to EDF is feasible if and only if, for every machine \( i \) and every interval \( I = [t_1, t_2] \), the total processing lengths of operations corresponding to jobs in \( J(I) \) that have deadlines greater than \( t_2 \) is at least \( \xi(i, I) \).

Clearly, if the total processing lengths of operations corresponding to jobs in \( J(I) \) that have deadlines greater than \( t_2 \) is less than \( \xi(i, I) \), then no algorithm can meet all the deadlines, as the length of operations scheduled in the interval \( I \) is greater than \( |I| \). Sufficiency of the above lemma follows from a bipartite graph matching argument, and we refer the reader to [8] for more details.

Lemma 2.2 leads to an integer programming formulation for COSSP, and we shall use this IP to define the PR2C instance. Let \( L = \max_i \{\sum_j p_{ij}\} \); note that every reasonable schedule finishes by \( L \). For every job \( j \), and every integer \( q \in [-1, \lceil \log_2 g_j(L) \rceil] \), let \( t_{j,q} \) be the largest integer \( t \in (r_j, L) \) such that \( g_j(t) \leq 2^q \) (if such \( t \) does not exist, then \( t_{j,q} = r_j \)). Let \( t_{j,-2} = r_j \). For every \( j \) and \( q \in [-1, \lceil \log_2 g_j(L) \rceil] \), we have a variable \( x_{j,q} \) indicating whether \( C_j \in (t_{j,q-1}, t_{j,q}] \). Therefore, for a job \( j \), the total number of \( x_{j,q} \) variables in our IP will be at most \( O(\log g_j(L)) \). For every \( j \) and every \( t > r_j \), let \( q_{j,t} \) be the integer \( q \) such that \( t \in (t_{j,q_{j,t}-1}, t_{j,q_{j,t}}] \).

Our IP for COSSP is as follows:

Minimize \( \sum_j \sum_q \lfloor 2^q \rfloor x_{j,q} \) \hspace{1cm} (4)  
\[ \forall i, \forall I = [t_1, t_2] : \quad \sum_{j \in J(I)} p_{ij} x_{j,q_{i,j,t_2}} \geq \xi(i, I) \]  
\[ \forall j, \forall q : \quad x_{j,q} \in \{0,1\} \]  

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We shall argue that the value of above IP is within a factor of $O(1)$ times the optimum cost of the scheduling problem. First, given a feasible schedule $S$ for COSSP with completion time vector $(C_j)_{j}$, we construct a solution to the above integer program with a small cost. For each $j$ and integer $q$ such that $2^q-1 \leq g_j(C_j)$, we let $x_{j,q} = 1$; set all other $x$ variables to 0. Clearly, the cost of the solution to the IP is at most $O(1)$ times the cost of $S$. Moreover, Constraint (5) is satisfied: we have that $\sum_{j \in J(I)} C_j \geq \xi(i,I)$. If $C_j > t_2$ for some $j \in J(I)$, then $x_{j,q,t_2} = 1$, as $2^{q,t_2} \leq g_j(t_2) \leq g_j(C_j)$.

On the other hand, if we are given an optimum solution to the above IP, we can convert it into a schedule for COSSP of cost at most the cost of the IP. For every $j$, let the completion time $C_j$ of the job to be $t_{j,q}$, where $q$ is the largest number such that $x_{j,q} = 1$; such a $q$ must exist in order to satisfy the Constraint (5) for the interval $t_1 = t_2 = r_j$. Then the cost of our schedule is at most the cost of IP. On the other hand, Constraint (5) says that $\sum_{j \in J(I)} p_{ij} x_{j,q,t_2} \geq \xi(i,I)$, implying $\sum_{j \in J(I)}, C_j > t_2 p_{ij} \geq \xi(i,I)$, as $x_{j,q,t_2} = 1$ implies $C_j \geq t_2$.

Remark 2.3. We will not discuss the representation of arbitrary delay functions $g_j(t)$ and how to compute $t_{j,q}$ here. We refer the readers to [8] for more details. Running time of all our algorithms will be polynomial in $n, m, \log L$ and max$_j \{\log g_j(1)\}$.

The above integer program hints towards how we can interpret COSSP geometrically as PR2C. Indeed, it is equivalent to PR2C problem. For every machine $i$ in COSSP, we create a set $P_i$ in PR2C. For every interval $I = [t_1, t_2]$ with $\xi(i,I) > 0$, we associate a point $p_I = (t_1, t_2)$ in 2-dimensional space. The demand $d_{p_I}$ of a point $p_I \in P_i$ is equal to $\xi(i,I)$, the excess of interval $I$ on machine $i$. This completes the description of the point sets $P_1, P_2, \ldots, P_m$ in PR2C. Now we define the rectangle set $R$. We shall create a rectangle for each job $j$ and each $q$ for which $x_{j,q}$ is defined. Notice that $x_{j,q}$ appears on the left side of Constraint (5) if $r_j \in [t_1, t_2]$ and $t_2 \in [t_{j,q-1}, t_{j,q}]$. Thus, we shall let the rectangle for the variable $x_{j,q}$ to be $[0, r_j] \times [t_{j,q-1}, t_{j,q}]$. This rectangle has cost $2^q$; for the instance $P_i$, it has a capacity $p_{ij}$. Notice that the rectangle for $x_{j,q}$ covers a point $(t_1, t_2)$ if and only if $t_1 \leq r_j \leq t_{j,q-1} < t_2 \leq t_{j,q}$; in other words, the job $j$ is released in the interval $[t_1, t_2]$, and its completion time is greater than $t_2$, which is exactly what we want. Thus, the IP (4 - 6) is equivalent to our PR2C problem. We now forget about the COSSP problem and focus exclusively on designing a good algorithm for the PR2C problem.

2.2 Algorithm For PR2C Problem

Our algorithm for PR2C is based on rounding a linear programming relaxation of the IP (1 - 3). As pointed out in [8], the linear programming relaxation of the IP (1 - 3) obtained by relaxing the variables $x_r \in [0,1]$ has a large integrality gap even when there is only one set of points. We strengthen our LP by adding the so called KC inequalities, introduced first in the work of Carr et al ([14]). Towards that we need to define $c(i,S)$, which indicates the total capacity of rectangles in $S \subseteq R$ with respect to a point set $P_i$: $c(i,S) = \sum_{r \in S} c(i,r)$. We are ready to write the LP.

Minimize $\sum_{r \in R} w(r)x_r \tag{7}$

$\forall i, \forall p \in P_i, \forall S \subseteq R :$ $\sum_{r \in R \setminus S : p \in r} (\min\{c(i,r), \max\{0, d_p - c(i,S)\}\}) \cdot x_r \geq d_p - c(i,S) \tag{8}$

$\forall r \in R :$ $x_r \geq 0 \tag{9}$

Let us focus on the KC inequalities Eq. (8) for a point set $P_i$. Fix a point $p \in P_i$ and a set of rectangles $S \subseteq R$. Recall that $p$ has a demand of $d_p$. Suppose, in an integral solution all the rectangles in $S$ are
chosen. Then, they contribute at most \(c(i, S)\) towards satisfying the demand of the point \(p\). The remaining rectangles still need to account for \(d_p - c(i, S)\). Notice also that in the KC constraints we truncate the capacity of a rectangle \(r\) to \(\min\{c(i, r), \max\{0, d_p - c(i, S)\}\}\). This ensures that the LP solution does not cheat by picking a rectangle with a large capacity to a tiny fraction. Clearly, the truncation has no effect on the integral solution.

There are exponentially many KC constraints in our LP. However, using standard arguments, we can solve the LP in polynomial time to get a \((1 + \epsilon)\)-approximate solution, for any \(\epsilon > 0\), which suffices for our purposes to obtain a logarithmic approximation to COSSP; see [8, 14] for more details. The rest of this section is devoted to rounding the LP solution.

**Weighted Geometric Set Multi-Cover Problem.** The main tool used in our rounding algorithm is the result by [9] for the weighted geometric set multi-cover problem. Similar to the standard set cover problem, in this problem, we are given a set \(U\) of points and a set \(M\) consisting of subsets of \(U\); typically, the sets in \(M\) are defined by geometric objects. Further, each point \(p \in U\) has a demand \(d_p\), and a set \(r \in M\) has a weight \(w(r)\). The goal is to find the minimum weight set \(S \subseteq M\) such that every point \(p\) is covered by at least \(d_p\) number of subsets in \(S\). Note crucially that the sets in \(M\) do not have capacities. If the sets have capacities, then the problem becomes similar to PR2C. The interesting aspect of geometric set cover problems is that sets in \(M\) are defined by geometric objects, hence subsets in \(M\) have more structure than in the standard set cover problem. In particular, if the sets in \(M\) have low union complexity, then the problem admits a better than \(O(\log |M|)\) approximation algorithm. Now we introduce the concept of union complexity of sets.

**Definition 2.4.** Given a set \(S\) of \(n\) geometric objects, the union complexity of \(S\) is the number of edges in the arrangement of the boundary of \(S\).

We will not get into the details of the definition, and we refer the readers interested in knowing more to [44] for an excellent introduction or to [9, 15, 50]. For our purposes, it suffices to know that the union complexity of 3-dimensional objects is the total number of vertices, edges, and faces on the boundary. It turns out that the geometric objects in our rounding algorithm are 4-dimensional objects. However, by appropriate projection, we reduce our problem of bounding the union complexity of 4-dimensional objects to that of 3-dimensional ones.

For geometric objects with low union complexity, building on the breakthrough works of Chan et al [15] and Varadarajan [50], Bansal and Pruhs [9] showed the following theorem.

**Theorem 2.5.** Suppose the union complexity of every \(S \subseteq M\) of size \(k\) is at most \(kh(k)\). Then, there is a polynomial time \(O(\log h(n))\) approximation algorithm for the weighted geometric set multi-cover problem. Further, the approximation factor holds even against a feasible fractional solution.

**Rounding** Our rounding algorithm is based on reducing our problem to several instances of the geometric set cover problem, and then appealing to Theorem 2.5. Consider an optimal solution \(x = \{x_r\}_r\) to the LP (7 - 9). Let \(\beta > 1/20\) be some constant. We first scale the solution by a factor of \(1/\beta\), and let \(x' = \{x'_r\}_r\) be this scaled solution. Clearly, scaling increases the cost of LP solution at most by a factor of \(1/\beta\). Our rounding consists of three steps.

In the first step, we pick all rectangles \(r\) for which \(x'_r \geq 1\). Let \(S\) denote this set. Let \(P_i(S)\) denote the set of points that are covered by rectangles in \(S\). We modify the point sets \(P_i\) for \(i = 1, 2, \ldots, m\), by removing the points that are already covered by \(S\). For all \(i \in [m]\), let \(P'_i = P_i \setminus S\). In the second step, for each \(P'_i\) we classify the points in the set into two types, heavy or light, based on the LP solution. For heavy points, we create an instance of the geometric set cover problem, and then appeal to Theorem 2.5. The main technical difficulty here is to bound the union complexity of the geometric objects in our instance. For the
light points, we create another separate instance of the geometric set multi-cover problem, and then apply the Theorem 2.5. Finally, we obtain our solution by taking the union of the rectangles picked in all three steps.

Fix a set $P'_i$ of uncovered points. For a point $p \in P'_i$, let $S(p)$ denote the set of rectangles in $S$ that contain $p$. That is, $S(p) = S \cap \{r : r \in R, p \in r\}$. For a point $p$, define the residual demand of $p$ as $d_p - c(i, S_p)$. From the definition of set $P'_i$, for every point $p \in P'_i$, we note that $d_p - c(i, S_p) > 0$. We now apply the KC inequalities on the set $S(p)$ for each point $p$. From Eq.(8) we have, for all $p \in P'_i$:

$$\sum_{r \in R \setminus S(p): p \in r} \left(\min\{c(i, r), d_p - c(i, S(p))\}\right) \cdot x_r \geq d_p - c(i, S(p))$$

which implies that our scaled solution $x'$ satisfies for all $p \in P'_i$:

$$\sum_{r \in R \setminus S(p): p \in r} \left(\min\{c(i, r), d_p - c(i, S(p))\}\right) \cdot x'_r \geq \frac{d_p - c(i, S(p))}{\beta}. \quad (11)$$

Note that for all $r$, $x'_r \in [0, 1]$; otherwise, we would have picked those rectangles in $S$. Next we round the residual demands of points and the capacities of rectangles as follows: For a point $p \in P'_i$, let $\tilde{d}_p$ denote the residual demand of $p$ rounded up to the nearest power of 2. On the other hand, we round down the capacities $c(i, r)$ of rectangles $r \in R$ to the nearest power of 2; let $\tilde{c}(i, r)$ denote this new rounded down capacities. Since $x'$ is scaled by a factor of $1/\beta$ we still have that for all $p \in P'_i$

$$\sum_{r \in R \setminus S(p): p \in r} \left(\min\{\tilde{c}(i, r), \tilde{d}_p\}\right) \cdot x'_r \geq \frac{\tilde{d}_p}{4\beta} \geq 3\tilde{d}_p. \quad (12)$$

To classify the points into heavy and light, we need the notion of class.

**Definition 2.6.** Let $\tilde{c}_{\min}(i) = \min_{r \in R} \{\tilde{c}(i, r)\}$. A rectangle $r \in R$ is a class $k$ rectangle with respect to a point set $P_i$ if $\tilde{c}(i, r) = 2^k \cdot \tilde{c}_{\min}(i)$. We say that a point $p \in P'_i$ is a class $k$ point if $\tilde{d}_p = 2^k \cdot \tilde{c}_{\min}(i)$.

Recall that the capacities of rectangles for different point sets can be different. Therefore, the class of a rectangle depends on the point set $P'_i$. Now we categorize points in $P'_i$ into heavy and light as follows.

**Definition 2.7.** A point $p \in P'_i$ belonging to class $k$ is heavy if its demand is satisfied by the rectangles of class at least $k$ in the LP solution. Otherwise, we say that the point is light.

From the definition, for all heavy points $p \in P'_i$ we have

$$\sum_{r \in R \setminus S, \tilde{c}(i, r) \geq \tilde{d}_p, p \in r} x'_r \geq 1, \quad (11)$$

and from Eq.(10) all light points $p \in P'_i$ satisfy

$$\sum_{r \in R \setminus S, \tilde{c}(i, r) < \tilde{d}_p, p \in r} \tilde{c}(i, r)x'_r \geq (1/4\beta - 1)\tilde{d}_p = 2\tilde{d}_p. \quad (12)$$

Let $P'_1 \subseteq P'_i$ denote the set of heavy points and let $P'_2 \subseteq P'_i$ denote the set of light points. Let $R' = R \setminus S$. We create two separate instances, a heavy instance and a light instance of the PR2C problem,
corresponding to the set of heavy points and the set of light points. The capacities of rectangles in these instances will be their rounded down values: that is, the capacity of a rectangle \( r \in \mathcal{R}' \) for the point sets \( \mathcal{P}^*_i \) will be \( \overline{c}(i, r) \). Similarly, the demand of a point \( p \in \mathcal{P}^*_i \) will be its residual demand \( d_p - c(\ast, S(p)) \). Notice that the LP solution \( x' \) restricted to \( \mathcal{R}' \) is a feasible solution for both \( \{\mathcal{P}^*_i\}_i \) and \( \{\mathcal{P}^2_i\}_i \). We round the solution \( x' \) for these two instances of PR2C problem separately, and take their union.

### 2.2.1 Heavy Instance and Hyper-4cuboid Covering Problem

We round the heavy instance \( \{\mathcal{P}^1_i\}_i, \mathcal{R}' \) by reducing it to a weighted geometric set cover problem called hyper-4cuboid covering problem (HCCP). HCCP is a generalization of the R3U problem considered in [8]. In HCCP, we are given a set \( U \) of points \( p = (p_1, p_2, p_3, p_4) \) in 4-dimensional space. We are also given a set \( M \) of geometric objects. Each object \( v \in M \) is a set of \( m \) disjoint 4-dimensional cuboids. Formally, \( v = \{r_1, r_2, \ldots, r_m\} \), where each \( r_i \) is a 4-dimensional cuboid (4-cuboid) of the form \( [2i, 2i+1] \times [0, x] \times [y_1, y_2] \times [0, z] \). We call \( v \) a hyper-4cuboid. Each hyper-4cuboid has a cost \( w(v) \). The goal is to find the minimum cost set \( S \subseteq M \) such that \( S \) covers all the points in \( U \). We say that a hyper-4cuboid \( v \) covers a point \( p \) if \( \exists r_i \in v, p \in r_i \).

Now we show that there is a reduction from the PR2C problem on the heavy instance to the HCCP problem. For a point \( p \in \mathcal{P}^1_i \) with coordinates \( (x, y) \), we create a point \( p' \in U \) with coordinates \( (2i + 1/2, x, y, \tilde{d}_p) \). Note that the last coordinate of \( p' \) is determined by \( \tilde{d}_p \), which is the residual demand of point \( p \). And the first coordinate of \( p' \) is determined by the index of the set \( \mathcal{P}^1_i \). For every rectangle \( r \in \mathcal{R}' \), we create a hyper-4cuboid \( v_r \in M \), which contains exactly one 4cuboid for each point set \( \mathcal{P}^1_i \). For a given index \( i \in [m] \) and a rectangle \( r = [0, x] \times [y_1, y_2] \), there is a 4cuboid \( r_i \) of the form \( [2i, 2i+1] \times [0, x] \times [y_1, y_2] \times [0, \overline{c}(i, r)] \). Note that the last coordinate is determined by the capacity of rectangle \( r \) towards the point set \( \mathcal{P}^1_i \). The cost of hyper-4cuboid \( v_r \) is same as the cost of rectangle \( r \).

**Lemma 2.8.** Suppose there is a feasible solution of cost \( \alpha \) to the PR2C problem on the heavy instance, where for each \( i \in [m] \) and for each \( p \in \mathcal{P}^1_i \), the demand of the point \( p \) is completely satisfied by the rectangles of class at least the class of \( p \). Then there is a feasible solution of cost at most \( \alpha \) for the corresponding instance of HCCP. Similarly, a solution of cost \( \alpha \) to the HCCP problem gives a solution of the same cost for the heavy instance of PR2C.

**Proof.** Suppose \( S \subseteq \mathcal{R}' \) is a feasible solution of cost \( \alpha \) satisfying the condition stated in the lemma. Then, we claim that the set of hyper-4cuboids \( S' \) corresponding to the rectangles \( r \in S \) is a feasible solution to HCCP. Consider a point \( p' \in U \) in the instance of HCCP produced by our reduction. Suppose \( p' \) has coordinates \( (2i + 1/2, x, y, \tilde{d}_p) \). Then, there must be a point \( p \in \mathcal{P}^1_i \) with coordinates \( (x, y) \) and the demand \( \tilde{d}_p \). Suppose \( r \in S \) covers this point \( p \), and has dimensions \( r = [0, x] \times [y_1, y_2] \). Then, we claim that \( v_r \in S' \) covers the point \( p' \). This is true, since \( v_r \) contains a 4-cuboid with coordinates \( h = [2i, 2i+1] \times [0, x] \times [y_1, y_2] \times [0, \overline{c}(i, r)] \). It is easy to verify that \( p' \in h \) as \( \overline{c}(i, r) > \tilde{d}_p \), which follows from the condition of the lemma.

The opposite direction also follows from the one-to-one correspondence between the rectangles in \( \mathcal{R} \) and the hyper-4cuboids. Suppose \( S' \) is a feasible solution to the instance of HCCP problem. Then, it is easy to verify that picking the rectangles \( r \in \mathcal{R} \) corresponding to the hyper-4cuboids \( v_r \in S \) defines a feasible solution to \( \{\mathcal{P}^1_i\}_i, \mathcal{R}' \). \( \square \)

Now, observe crucially that there is a fractional solution of cost at most the cost of LP solution to the heavy instance satisfying the requirements of Lemma 2.8. This is true because the LP solution \( x' \) satisfies
the inequality Eq. (11). Therefore to apply Theorem 2.5, it remains to quantify the union complexity of hyper-4-cuboids in our reduction.

**Lemma 2.9.** The union complexity of any \( q \) hyper-4 cuboids in \( M \) is at most \( O(qm \log P) \), where \( P = \frac{\max\{p_{ij}\}}{\min\{p_{ij}\}} \).

**Proof.** Consider a set \( S \subseteq M \) of \( q \) hyper-4 cuboids. For \( i = 1, 2, \ldots, m \), define \( \mathcal{L}_i \) as the set of all 4-cuboids at level \( i \); formally, \( \mathcal{L}_i \subseteq \bigcup_{v \in S} t_{\mathcal{L}_i}(v) \), such that every 4-cuboid \( r \in \mathcal{L}_i \) has the first dimension equal to \([2i, 2i + 1]\). In other words, \( \mathcal{L}_i \) is the set of all 4-cuboids which have the same first dimension \([2i, 2i + 1]\). Now, observe from our construction that for any pair \( i, i' \), the objects in \( \mathcal{L}_i \) and \( \mathcal{L}_{i'} \) do not intersect. This is because 4-cuboids at different levels are separated by a distance of 1 in the first dimension. Therefore, the union complexity of the subset \( S \) is equal to \( m \) times the maximum of union complexity of 4-cuboids in \( \{\mathcal{L}_i\}_i \). However, 4-cuboids in the sets \( \mathcal{L}_i \) all have the same form; thus, it suffices to bound the union complexity for any \( \mathcal{L}_i \).

Fix some \( i \), and consider the 4-cuboids in the set \( \mathcal{L}_i \). Notice carefully that all the 4-cuboids in \( \mathcal{L}_i \) share the same first dimension \([2i, 2i + 1]\). This implies that we can ignore the first dimension as it does not add any edges to the arrangement of objects in \( \mathcal{L}_i \). Now consider the projection of 4-cuboids to the remaining 3 dimensions; these are cuboids of the form \([0, x] \times [y_1, y_2] \times [0, z]\). For these type of cuboids, [8] proves that the union complexity of 4-cuboids is at most \( O(q\theta) \), where \( \theta \) is the number of distinct values taken by \( z \). In our reduction, the values of \( z \) correspond to the capacities of rectangles. Recall that the capacities of rectangles in PR2C are defined based on the processing lengths of operations. As we round down the capacities to the nearest power of 2, the number of distinct values taken by \( z \) is at most \( \log P \), where \( P = \frac{\max\{p_{ij}\}}{\min\{p_{ij}\}} \). Putting everything together, we complete the proof.

From Lemma 2.8, 2.9 and Theorem 2.5, we get the following.

**Lemma 2.10.** There is a solution of cost \( O(\log(m \log P)) \) times the cost of LP solution for the HCCP problem, and hence for the PR2C on the heavy instance.

### 2.3 Covering Light Points and Geometric Multi Cover by Hyper-cuboids

Here, we design a \( O(\log m) \) approximation algorithm to PR2C problem restricted to the light instance: \((\{P^2_i\}_i, R')\). There is one main technical difference between our algorithm for the light case from the algorithm of Bansal and Pruhs [8]. Bansal and Pruhs divide the light instance into \( \log P \) different levels, where in each level they solve 2-dimensional geometric multi-cover problem, and show a \( O(1) \) approximation to each level. This in turn leads to a constant factor approximation algorithm to the light instance. However, in our problem we cannot solve the instances separately. Therefore, using a simple trick, we map our problem into a single instance of 3-dimensional problem, and show a \( \log m \) approximation to it. This is precisely the reason we lose the factor \( O(\log m) \) for the light case, as the union complexity of our objects becomes \( m \) times the union complexity of the objects in [8].

Recall that for every point \( p \in P^2_i \), the LP solution satisfies the inequality Eq. (12). Again, our idea is to reduce the instance to a geometric uncapacitated set multi-cover problem, and then appeal to Theorem 2.5. The geometric objects produced by our reduction are sets of cuboids. Hence we abbreviate this problem as GMCC.

In GMCC, we are given a set \( U \) of points in 3-dimensional space, and a set \( M \) of geometric objects. Each geometric object \( v \in M \) is a collection of \( m \) disjoint cuboids, which we call as hyper-cuboids. Each point \( p \in U \) has a demand \( e_p \), and each hyper-cuboid \( v \in M \) has a cost \( b_v \). The individual cuboids constituting a
The cuboids that can contain \( y \) this cuboid by \( h \) dimension \( H \) that in our reduction different points cuboids that constitute \( v \).

Lemma 2.11. If there is an integral solution to the instance of \( \mathbb{P} \mathbb{R} \mathbb{C} \) to an instance of GMCC. Fix some \( i \in [m] \). Let \( T > 0 \) be some large constant, and recall \( P = \max\{p_{ij}\} / \min\{p_{ij}\} \). For every light point \( p \in \mathcal{P}^2 \) with coordinates \((x_p, y_p)\), we create \( \log P \) different points, each of them shifted in the second dimension by a distance of \( T \).

The \( \log P \) different points corresponding to a point \( p \in \mathcal{P}^2 \) have coordinates \((2i + 1/2, kT + x_p, y_p)\), for \( k = 1, 2, ... \log P \). To complete the description of point set \( U \), we need to define the demands of each point, which we will do after describing the cuboids in \( M \).

For every rectangle \( r \in \mathcal{R}'' \), with dimensions \([0, r_j] \times [t_1, t_2]\), we create one hyper-cuboid \( v_r \in \mathcal{M} \). The cost of hyper-cuboid \( v_r \) is same the as the cost of rectangle \( r \). The hyper-cuboid \( v_r \) contains \( m \) different cuboids, one for each point set \( \mathcal{P}^2 \). Fix an index \( i \in [m] \). The cuboid corresponding to the set \( \mathcal{P}^2 \) has dimensions \([2i, 2i + 1] \times [0, kT + r_j] \times [t_1, t_2]\), where \( k \) denotes the class of rectangle \( r \) with respect to \( \mathcal{P}^2 \). Note that there is one-to-one correspondence between rectangles \( r \in \mathcal{R}'' \) to hyper-cuboids \( v_r \in \mathcal{M} \).

Let \( \mathcal{H}^{-1}(v_r) = r \); that is, the rectangle \( r \in \mathcal{R}'' \) that corresponds to the hyper-cuboid \( v_r \in \mathcal{M} \). We say that \( v \) is picked to an extent of \( x_r \) in the LP solution \( x' \) to mean the extent to which the rectangle \( r \) is picked in the LP solution.

Fix a point \( p \in U \). We say that \( p \) is contained in the hyper-cuboid \( v \in \mathcal{M} \) if it is contained in any of the \( m \) cuboids that constitute \( v \). Now we are ready to define the demand \( e_p \) of a point \( p \) as \( e_p = \left\lceil \sum_{\mathcal{W}:p \in \mathcal{W}} x_{\mathcal{H}^{-1}(v)} \right\rceil \).

To understand the above definition, let us consider a point \( p \in U \) with coordinates \((2i + 1/2, kT + x, y)\). The cuboids that can contain \( p \) should have form \([2i, 2i + 1] \times [0, kT + r_j] \times [t_1, t_2]\), where \( kT + x < r_j \) and \( y \in [t_1, t_2] \). These cuboids correspond to the class \( k \) rectangles in \( \mathcal{R} \). Therefore, demand of a point \( p \in U \) with coordinates \((2i + 1/2, kT + x, y)\) is exactly equal to the number of class \( k \) rectangles that cover the point \( p \in \mathcal{P}^2 \) in the LP solution.

This immediately tells us that the LP solution \( x' \) is a feasible fractional solution to the instance of GMCC problem produced by our reduction \( \mathcal{H} \). Now, we show the opposite direction.

Lemma 2.11. If there is an integral solution to the instance of GMCC produced by our reduction \( \mathcal{H} \) of cost at most \( \alpha \) times the LP cost, then there is an integral solution of the same cost to the instance of \( \mathbb{P} \mathbb{R} \mathbb{C} \) on the light instance.

Proof. Suppose \( S \subseteq \mathcal{M} \) is a solution to GMCC problem. We show that there is a corresponding solution to \( \mathbb{P} \mathbb{R} \mathbb{C} \) problem of exactly the same cost. Our solution is simple. For every hyper-cuboid \( v \in \mathcal{M} \), we pick the corresponding rectangle \( r \) in our solution \( S' \) to \( \mathbb{P} \mathbb{R} \mathbb{C} \). From our reduction \( \mathcal{H} \), the costs of hyper-cuboids is same as the costs of rectangles, which implies that solution costs of the two problems are the same. It remains to show that \( S' \) is a feasible solution for the light instance.

Fix a point set \( \mathcal{P}_i \), and consider any point \( p \in \mathcal{P}_i \). Let \( p = (x, y) \). In our reduction, there are \( \log P \) different points \( p_1', p_2', ..., p_{\log P}' \), where the point \( p_i' \) has coordinates \((2i + 1/2, kT + x, y)\). The demand of \( d_{p_i'} \) is exactly equal to the total number of class \( k \) rectangles that cover the point \( p \) in the LP solution. Recall that in our reduction \( \mathcal{H} \), each rectangle \( r \in \mathcal{R} \) produces exactly one cuboid at the level \( i \) (that is with first dimension \([2i, 2i + 1]\)) in the corresponding hyper-cuboid \( v_r \in S' \). By abusing notation, let us call denote this cuboid by \( h(r, i) \).

Let class of point \( p \) be \( k^* \). The total capacity of rectangles that cover point \( p \in \mathcal{P}_i \) in our solution \( S' \) is

\[
\sum_{2 \leq k \leq k^*} \sum_{r \in S': r \in \text{cls}(k)} 2^k \cdot \bar{c}_{\text{min}}(i) \tag{13}
\]
From the one-to-one correspondence between the rectangles and the cuboids, the term \( \sum_{k \leq k^*} \sum_{r \in S':r \in \text{cls}(k)} 2^{k} \bar{c}_{\min}(i) \) is exactly equal to the number of cuboids that cover the point \( p'_k \), which is at least the demand \( d'_{p'_k} \). Now, in our reduction, the the demand of the point \( p'_k \) (which has coordinates \((2i + 1/2, kT + x, y)\)) is exactly equal to the rounded down value of the total number of class \( k \) rectangles that cover the point \( p \) in the LP solution. Therefore,

\[
\sum_{k \leq k^*} \sum_{r \in S':r \in \text{cls}(k)} 2^{k} \cdot \bar{c}_{\min}(i) \geq \sum_{k \leq k^*} 2^{k} \cdot \bar{c}_{\min}(i) \cdot \left| \sum_{r \in S':r \in \text{cls}(k)} x'_r \right|.
\]

Simplifying the right-hand side of the above equation,

\[
\sum_{k \leq k^*} 2^{k} \cdot \bar{c}_{\min}(i) \cdot \left| \sum_{r \in S':r \in \text{cls}(k)} x'_r \right| \geq \sum_{k \leq k^*} 2^{k} \cdot \bar{c}_{\min}(i) \cdot \left( \left( \sum_{r \in S':r \in \text{cls}(k)} x'_r \right) - 1 \right)
\]

\[
\geq \sum_{k \leq k^*} 2^{k} \cdot \bar{c}_{\min}(i) \cdot \left( \sum_{r \in S':r \in \text{cls}(k)} x'_r \right) - 2^{k} \bar{c}_{\min}(i)
\]

\[
\geq 2d'(p) - d'(p) = d'(p),
\]

where the last inequality follows from the Eq. (12), and the fact that \( p \) is a light point. Therefore, \( S' \) is a feasible solution to GMCC.

Thus, to complete our algorithm for the light instance, it remains to design an \( O(\log m) \) approximation algorithm to GMCC. For that we once again plan to rely on the Theorem 2.5, which requires to us to bound the union complexity of objects in \( M \).

**Lemma 2.12.** The union complexity of any \( q \) hyper-cuboids in \( M \) is at most \( O(mq) \).

**Proof.** Consider any \( S \subseteq M \) of \( q \) hyper-cuboids. From our definition, each hyper-cuboid \( v \) is a set of \( m \) cuboids. For \( i = 1, 2, \ldots, m \), define \( \mathcal{L}_i \) as the set of all cuboids with first dimension \([2i, 2i + 1]\). Clearly, for any two indices \( i, i' \), the cuboids in \( \mathcal{L}_i \) and \( \mathcal{L}_{i'} \) do not intersect, as they are separated by a distance of \( 1 \) in the first dimension. Moreover, the cuboids in different \( \mathcal{L}_i \) have same form except that they are shifted in the first dimension. Thus, the union complexity of \( q \) hyper-cuboids in \( S \) is at most \( m \) times the union complexity of \( q \) cuboids in \( \mathcal{L}_i \), for any \( i \in [m] \).

Fix some \( i \) and consider the set of \( q \) cuboids in \( \mathcal{L}_i \). Since all of the \( q \) cuboids have exactly the same side in the first dimension, \([2i, 2i + 1]\), the union complexity of \( q \) cuboids in \( \mathcal{L}_i \) is equal to the union complexity of the projection of the cuboids to the last two dimensions. This projection of the cuboids produces a set of axis parallel rectangles. Furthermore, these \( q \) axis parallel rectangles partition into \( \log P \) sets, \( Z_1, Z_2, \ldots Z_{\log P} \), such that no two rectangles from different sets intersect. This is true as these rectangles have coordinates \([0, kT]\) in the second dimension. In each set \( Z_k \), the rectangles are abutting the \( Y \)-axis (or the axis of second dimension). Bansal and Pruhs [8] showed that the union of complexity such axis parallel rectangles abutting \( Y \)-axis is at most \( O(|Z_k|) \). Therefore, the union complexity all the \( q \) axis parallel rectangles is at most \( \sum_k |Z_k| = O(q) \). Thus we conclude that the union complexity \( q \) hyper-cuboids is equal to \( mq \), which completes the proof.

Thus, from Theorem 2.5, we get a \( O(\log m) \) approximation algorithm for the light instance of PR2C.
2.3.1 Proof of Theorem 1.1

Our final solution for PR2C problem is obtained by taking the union of all the rectangles picked in our solutions for the heavy and the light instance, and the set \( S \). Recall that in \( S \), we pick all the rectangles for which \( x_r > 1/\beta \), for \( \beta = 12 \). The cost of our solution is at most \( (O(\log(m \log P)) + O(\log m) + 1/\beta)) \) times the cost of LP (7 - 9), which implies an \( O(\log(m \log P)) \) approximation algorithm. This completes the proof.

3 Precedence Constrained Scheduling

In this problem, each job \( j \) has a processing length \( p_j > 0 \), a release time \( r_j > 0 \). We have a set of \( m \) identical machines; Each job \( j \) must be scheduled on exactly one of the \( m \) machines. The important feature of the problem is that they are precedence constraints between jobs that capture the computational dependencies across jobs. The precedence constraints are given by a partial order “\( \prec \)”, where a constraint \( j \prec j' \) requires that job \( j' \) can only start after job \( j \) is completed. Without loss of generality we assume that \( r_{j'} > r_j \) whenever \( j \prec j' \). Our goal is to minimize the sum of arbitrary cost functions of completion times of jobs. Let \( g_j(t) \) denote the cost of completing a job \( j \) at time \( t \).

Our algorithm for PCSP consists of two main steps: In the first step, using a linear programming relaxation for the problem, we find a migratory schedule in which a job may be processed on multiple machines. In the second step, using some known results and some extra speed augmentation, we convert the migratory schedule into a non-migratory schedule in which every job is completely processed on a single machine.

3.1 Migratory Schedule

First we write a time-indexed linear programming relaxation for the problem; using standard arguments, this can be converted into a polynomial sized LP by increasing the cost of solution by an \( \epsilon \)-factor for any \( \epsilon > 0 \). See [41] for more details. Consider the LP (14 - 19) given below. The variables of our LP are \( x_{jt} \), which indicate if a job \( j \) is processed at time \( t \). Note that we do not create \( x_{ijt} \) variables indicating on which machine \( j \) got scheduled at each time step. This is not required as machines are identical.

Minimize \[ \sum_j \sum_{t:t \geq r_j} x_{jt} \cdot g_j(t)/p_j + \sum_j g_j(p_j) \]  
(14)

\( \forall j : \) \[ \sum_{t:t \geq r_j} x_{jt}/p_j \geq 1 \]  
(15)

\( \forall t : \) \[ \sum_{j:r_j \geq t} x_{jt} \leq m \]  
(16)

\( \forall j : \) \[ c_j \geq \sum_{t:t \geq r_j} x_{jt} \cdot \left( \frac{t}{p_j} + \frac{1}{2} \right) \]  
(17)

\( \forall j, j', j \prec j' : \) \[ c_{j'} \geq c_j + p_j \]  
(18)

\( \forall j, t : \) \[ x_{jt} \in [0,1] \]  
(19)

The constraints (15) ensure that every job is completely processed. At any time \( t \), at most \( m \) units of jobs can be processed by \( m \) machines; this is ensured by the constraints (16). The variables \( c_j \) are axillary,
and indicate the completion time of job $j$. The constraints (18) ensure that if $j < j'$, the completion time of $j'$ has to be at least the completion time of $j + p_{j'}$. Now consider the objective function. The first term lowerbounds the cost incurred by any job $j$ in a feasible schedule by its fractional cost. Suppose completion time of a job $j$ in an integral solution is $C_j$. Then, $\sum_{t:t \geq r_j} x_{jt} \cdot g_j(t)/p_j \leq \sum_{t:t \geq r_j} x_{jt} \cdot g_j(C_j)/p_j \leq g(C_j) \cdot \sum_{t:t \geq r_j} x_{jt}/p_j = g_j(C_j)$. The second term in the objective function is also clearly a lowerbound on the cost of any feasible schedule. Therefore, the cost of LP solution is at most twice the optimal solution.

**Rounding.** Let $x^* := \{x^*_{ij}\}_{j,t}$ denote an optimal solution to LP (14 - 19). Define $C_j$ as the earliest time in $x^*$ when at least $p_j/2$ units of job $j$ is processed. We first produce a schedule where we only process $p_j/2$ units of each job $j$; then using speed augmentation we convert this into a valid schedule by processing twice the amount of each job that is scheduled at each time slot. Let $p'_j = p_j/2$.

**Claim 3.1.** The cost incurred by a job $j$ in $x^*$ is at least $g_j(C_j)/2$.

**Proof.** Proof follows from a simple observation that $\sum_{t:t \geq C_j} x_{jt} \cdot g_j(t)/p_j \geq \sum_{t:t \geq r_j} x_{jt} \cdot g_j(C_j)/p_j = g_j(C_j)/2$. \hfill $\Box$

Next, we prove the following crucial property about the LP solution $x^*$.

**Lemma 3.2.** The vector of $(C_j)_{j \in J}$ values satisfies the following property. For any time interval $[a, b]$.

$$\sum_{j:C_j \leq b, r_j + p'_j > a} \min \{p'_j, r_j + p'_j - a\} \leq 2 \cdot (b - a).$$

**Proof.** Consider LHS of the above equation. We classify the jobs in the set $\{j : C_j \leq b, r_j + p'_j > a\}$ into two types: Let $J_1$ be the set of jobs such that $\forall j \in J_1, [r_j, C_j] \in [a, b]$. Let $J_2$ denote the remaining jobs: $J_2 = \{j : C_j \leq b, r_j + p'_j > a\} \setminus J_1$. Since $C_j \leq b$ for a job $j \in J_1$, it implies that $p'_j$ units of the job $j$ is completely processed in the interval $[a, b]$. Therefore, from the capacity constraints (16), $\sum_{j \in J_1} p'_j \leq (b - a)$. Now let us focus on the jobs in $J_2$. For every job $j \in J_2$, it is true that $r_j \leq a$. Further, since $r_j + p'_j > a$, it must be the case that at least $(r_j + p'_j - a)$ units of job $j$ is processed in the interval $[a, b]$. Again, from the capacity constraints (16), total units of processing done on jobs in $J_2$ is at most $b - a$. Thus, we conclude that $\sum_{j \in J_2} (r_j + p'_j - a) \leq (b - a)$. Now,

$$\sum_{j:C_j \leq b, r_j + p'_j > a} \min \{p'_j, r_j + p'_j - a\} = \sum_{j \in J_1} p'_j \leq (b - a) + \sum_{j \in J_2} (r_j + p'_j - a) \leq 2(b - a),$$

and the proof is complete. \hfill $\Box$

Our algorithm for the precedence constrained scheduling problem is simple: We schedule the jobs in the increasing order of $C_j$ values respecting the precedence and the capacity constraints. Below, we give a formal description the algorithm. Let the speed augmentation factor be $\alpha \geq 1$. In our algorithm, $\bar{S}_j$ will be the time when $j$ is ready for scheduling: the earliest time when $j$ has arrived and all predecessors of $j$ have completed. $\bar{C}_j$ is the completion time of $j$.

**Lemma 3.3.** Suppose the vector $C$ satisfies condition (3.2). Then the algorithm above with $\alpha = 3$ constructs a schedule with $\bar{C}_j \leq C_j$ for every $j \in J$. 

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Algorithm 1 List-scheduling For Identical Machines \((\alpha, (C_j)_{j \in J})\)

1: let \(c : [0, \infty) \rightarrow \{0, 1, 2, \cdots, m\}\) be identically 0
2: for \(j \in J\) in non-decreasing order of \(C_j\) values, breaking ties so as to maintain precedence constraints
3: \(\text{let } S_j \leftarrow \max \{ r_j, \max_{j' < j} C_{j'} \} \)
4: \(\text{let } C_j \geq S_j \) be the minimum \(t'\) such that \(\int_{t=S_j}^{t'} 1 \{ c(t) < m \} \, dt = p_j' / \alpha \)
5: schedule \(j\) preemptively at time points \(\{ t \in (S_j, C_j) : c(t) < m \} \)
6: \(\text{let } c(t) \leftarrow c(t) + 1 \) for every \(t \in (S_j, C_j)\) with \(c(t) < m\)

Till the end of this section, we fix an arbitrary \(j^* \in J\). We shall show that \(\bar{C}_{j^*} \leq C_{j^*}\). Let \(\text{SOL}\) be the schedule constructed by the algorithm after the iteration \(j^*\); let \(J'\) be the set of jobs scheduled in \(\text{SOL}\); i.e., the set of jobs considered before \(j^*\) (including \(j^*\)). We say a time point \(t\) is \emph{busy} if all machines are processing jobs at time \(t\) in the schedule \(\text{SOL}\); otherwise \(t\) is \emph{idle}.

For every \(j \in J'\), let \(p_j(t)\) be the amount of job \(j\) that has been processed by time \(t\) in \(\text{SOL}\). Let \(R_t = \{ j \in J' : t \in (S_j, C_j) \}\) be the set of unfinished jobs at time \(t\) that are ready for scheduling, i.e., the set of arrived and incomplete jobs whose predecessors are completed.

For every \(t \in [0, \infty)\), let \(f(t) = \min \{ t, \min_{j \in R_t} (r_j + p_j(t)) \}\).

\textbf{Claim 3.4.} \(f\) is a non-decreasing function.

\textbf{Proof.} We consider the change of \(\min \{ t, \min_{j \in R_t} (r_j + p_j(t)) \}\) as \(t\) increases. The term \(r_j + p_j(t)\) can only increase as \(t\) increases. Jobs maybe added to and removed from \(R_t\). Removing jobs from \(R_t\) can only increase the minimum. So, it suffices to consider adding jobs to \(R_t\). If some job \(j'\) has \(S_{j'} = t\), then \(j' \not\in A_t\) but \(j' \in A_{t'}\) for \(t' \in (t, t + 1]\). We have either \(t = r_{j'}\) or \(t = \bar{C}_{j''}\) for some \(j'' < j'\); notice that \(j'' \in R_t\) in the latter case. In the former case, \(r_{j'} + p_{j'}(t) = t\); in the latter case, \(r_{j'} + p_{j'}(t) = r_{j''} + p_{j''}(t)\), where the inequality is guaranteed by the preprocessing of the instance. So, we always have \(\min \{ t, \min_{j \in R_t \cup J'(j')} (r_j + p_j(t)) \} = \min \{ t, \min_{j \in R_t} (r_j + p_j(t)) \}\), i.e., adding \(j'\) to \(R_t\) does not change the quantity. \(\square\)

\textbf{Definition 3.5.} For any \(t \geq 0\), we say \(t\) is a fresh point if \(f(t) = t\).

\textbf{Lemma 3.6.} Let \(a\) be a fresh point. Let \(b > a\) be an integer such that there are no fresh points in \((a, b)\). The total length of idle slots in \((a, b)\) is at most \((b - a) / \alpha\).

\textbf{Proof.} Since there are no fresh points in \((a, b)\), we have that \(f(t) = \min_{j \in R_t} (r_j + p_j(t))\) for every \(t \in (a, b)\). If \(t \in (a, b)\) is idle, then all the jobs in \(R_t\) must be processed at time \(t\): a job \(j \in R_t\) is not being processed at time \(t\) will contradict the fact that \(t\) is an idle slot. Let \((a', b') \subseteq (a, b)\) be a subset of idle slots such that \(R_t\) is the same for all \(t \in (a', b')\). Then \(f(t)\) will increase at a rate of \(\alpha\) within the interval \((a', b')\). Since \(R_t\) will be changed only finite number of times as \(t\) goes from \(a\) to \(b\), we have that \(f(b) - f(a)\) is at least \(\alpha\) times the total length of idle slots in \((a, b)\). Since \(f(b) - f(a) = f(b) - a \leq b - a\), we have that the total length of idle slots in \((a, b)\) is at most \((b - a) / \alpha\). \(\square\)

\textbf{Lemma 3.7.} We have \(\bar{C}_{j^*} \leq C_{j^*}\).
Let $t$ be the last fresh point before $\tilde{C}_j^*$. By Lemma (3.6), the total length of idle slots in $(t, \tilde{C}_j^*)$ is at most $(\tilde{C}_j^* - t)/\alpha$. The total length of busy slots is at most

$$\frac{1}{m} \cdot \left( \sum_{j : j \in \mathcal{J}, C_j \leq C_j^*, r_j + p_j > t} \min \{p_j, r_j + p_j - t\} \right) \leq 2 \cdot (C_j^* - t)/\alpha.$$ 

When $\alpha = 3$ we get $(\tilde{C}_j^* - t) \leq (\tilde{C}_j^* - t)/3 + 2(C_j^* - t)/3$. This implies $\tilde{C}_j^* \leq C_j^*$, which completes the proof.

The above lemma and Claim (3.1) imply the following.

**Lemma 3.8.** There is a migratory 6-speed $O(1)$-approximation algorithm for the PCSP problem.

**Proof.** Note that our algorithm when given a speed augmentation of 3 produces a migratory schedule where the completion time of every job is at most $C_j$. Therefore, from Claim (3.1), the cost incurred by our algorithm is at most twice the cost of LP solution. However, in our schedule we only process $p_j'$ units of a job $j$. To convert this into a valid schedule, we need another round of speed augmentation of factor 2. Therefore, the total speed augmentation required is 6. This completes the proof.

### 3.2 Non-Migratory Schedule

The schedule produced by our algorithm processes each job in the interval $[S_j, \tilde{C}_j]$, but a single job may be processed on multiple machines. To convert this migratory schedule into non-migratory schedule, we use the algorithm either in [29, 35] or in [20]. The result in [20] is more suitable for our purposes.

**Theorem 3.9** ([20, 29]). Given a set of jobs where each job $j$ has an associated interval $[s_j, d_j]$, if there is a migratory schedule on $m$ machines that completely processes the jobs in their intervals, then there is a non-migratory schedule that schedules all the jobs within their intervals that processes each job exactly on one machine if each machine is given an $O(1)$ speed augmentation.

Using the above theorem, we complete the proof of Theorem (1.4).

**Proof of Theorem (1.4).** Let $S$ be the migratory scheduled produced by Lemma (3.8). We invoke the Theorem (3.9) on $S$, where we set the interval for each job $j$ to be $[S_j, \tilde{C}_j]$. Therefore, with an overall speed augmentation of $O(1)$, we get a schedule that completely processes each job in the interval $[S_j, \tilde{C}_j]$ on a single machine. Further, the cost of the schedule is at most twice the cost of the LP solution. Combining all the pieces, we complete the proof of Theorem (1.4) regarding the upper bound. We show the lowerbound on the speed augmentation required in the next section.

### 3.3 Necessity of Speed-Augmentation and Proof of Theorem (1.3)

Here we show that our analysis of PCSP is essentially tight, and in particular $O(1)$ speed augmentation is necessary. First we show the necessity of speed augmentation to the case of single machine. Our reduction is from the Densest-k-Subgraph (DkS) problem. In the DkS problem, we are given a graph $G$ with $n$ vertices and $m$ edges, and integers $k > 0, L > 0$. The goal is to find a subgraph of $G$ on $k$ vertices that contains at least $L$ edges. Recently, Manurangsi [42] showed the following result.
Theorem 3.10. There exists a constant $c > 0$ such that, assuming exponential time hypothesis, no polynomial time algorithm can distinguish between the following two cases:

- There exists a set of $k$ vertices of $G$ that contains an induced subgraph with $L$ edges.
- Every subset of $k$ vertices of $G$ contains at most $L/n^{1/(\log \log n)^c}$ induced edges.

For our reduction, it will be more convenient to consider the following vertex version of the DKS problem. In this problem, given $G, L$, the goal is to approximate the number of vertices $S$ of $G$ that contains at least $L$ edges. An $\alpha$-approximation to the vertex version of DKS problem gives an $1/2\alpha^2$-approximation to the DKS problem. To see this, assume that there is an $\alpha$ approximation to vertex version of DKS. Given a DKS instance, we assume we know the number $L$ of edges in the densest $k$-subgraph of $G$ (this can be achieved by guessing). Thus we can find a set of vertices of size $\alpha \cdot k$ that contains at least $L$ edges. We then randomly choose $k$ out of the $\alpha k$ vertices. It is easy to see that in expectation, at least $Lk(k-1)/(\alpha k)(\alpha k - 1)$ edges will be contained in the subgraph induced by the $k$ chosen vertices. This is at least $L/2\alpha^2$ when $k \geq 2$. This process can be easily derandomized.

Corollary 3.11. There exists a constant $c > 0$ such that, assuming exponential time hypothesis, no polynomial time algorithm can distinguish between the following two cases:

- There exists a set of $k$ vertices of $G$ that contains an induced subgraph with $L$ edges.
- Every subset of size $k \cdot n^{1/(\log \log n)^c}/2$ vertices of $G$ contains at most $L$ induced edges.

Now we give a reduction from the vertex version of DKS to the PCSP problem on a single machine.

**Proof of Theorem 1.3.** To keep the notation simple, we allow jobs to have weights in our proof. However, it easy to see that in the precedence constrained scheduling model, weighted and unweighted cases are equivalent. To see this, suppose there is a job $j$ of weight $w_j$. Then, one can create $w_j$ dummy jobs of zero processing length that all depend on job $j$. This ensures that until $j$ finishes, it pays $w_j$ cost towards the flow-time, which is same as job $j$ having a weight of $w_j$.

Consider an instance $I$ of the vertex version of DKS problem. Let $\mathcal{H}(I)$ denote the instance of PCSP produced by our reduction. For every vertex $v \in G$, we create a job $j_v$ in $\mathcal{H}(I)$. The processing length of job $j_v$ is zero, and weight is 1. For every edge $e \in G$, we have a job $j_e \in \mathcal{H}(I)$. The processing length of job $j_e$ is 1 and its weight is zero. All these jobs have release time of 0. If $v \in e$ in the graph, we have $j_e < j_v$. Let $\delta = 1/n^2$, and $T > 0$ be an arbitrarily large number. In the interval $(m-L,T]$, at every time step $m-L+k\delta$, for $k = 1, 2, \ldots$, we release a job of size $\delta$ and weight 1. Now consider the following two cases.

**Case 1:** Suppose there exists a set of $k$ vertices $S$ of $G$ such that $G[S]$ contains at least $L$ edges. Then, we claim that there is a solution of cost at most $(n-k)(m-L)+k(T+L)+T-(m-L)$. We can schedule the edge jobs correspondent to the edges not in $G[S]$ in interval $[0,m-L]$. Then the $n-k$ vertex jobs $\{j_v: v \notin S\}$ can be scheduled at time $m-L$. All the edge jobs can be completed by time $T+L$; thus, the $k$ vertex jobs $\{j_v: v \in S\}$ can be scheduled at time $T+L$. The total weighted completion time for $\delta$-sized jobs is $T-(m-L)$.

**Case 2:** Every subset of size $k \cdot n^{1/2(\log \log n)^c}/2$ vertices of $G$ contains at most $L$ induced edges. This implies that in any algorithm at least $k \cdot n^{1/2(\log \log n)^c}/2$ vertex jobs are alive at time $t = m-L$. This further implies that at all times in the interval $(m-L,T]$ at least $k \cdot n^{1/2(\log \log n)^c}/2$ jobs of weight 1 are alive in any algorithm’s schedule. This is because, if any job corresponding to an edge is scheduled, it will
take at least one time unit to complete that job. During that period at least \( n^2 \) jobs arrive which can not be scheduled. Note that \( k \cdot n^{1/2(\log \log n)^c}/2 < n^2 \). Therefore, cost incurred by any algorithm in this case is at least \( (k \cdot n^{1/2(\log \log n)^c}/2) \cdot (T - (m - L)) \).

Suppose \( T \) is some large polynomial in \( n \). For our purposes \( T = n^4 \) suffices. In this case, the cost of schedule in case 1 is dominated by term \( Tk \) and the cost the schedule in case 2 is dominated by \( (k \cdot n^{1/2(\log \log n)^c}/2) \cdot T \). From Corollary (3.11), no polynomial time algorithm can distinguish between these two cases. Clearly, our reduction \( \mathcal{H} \) is of size polynomial in \( n \) and takes polynomial time. This completes the proof.

We now turn our attention to the multiple machines setting. Here we give a simple reduction that shows to obtain a non-trivial approximation factor speed augmentation has to be at least the approximation factor of the underlying makespan minimization problem. Our result holds not just for the identical machines case but also for any machine environment such as related machines setting, unrelated machines setting etc.

**Theorem 3.12.** Suppose the precedence constrained scheduling to minimize makespan problem cannot be approximated better than \( \gamma > 1 \) for a machine environment \( \mathcal{M} \). Then, unless given a speed augmentation of at least \( \gamma \), the problem of minimizing the total unweighted flow-time cannot be approximated better than \( \Omega(n^{1-c}) \) for any constant \( c \in [0, 1) \).

**Proof.** Consider an instance \( I \) of precedence constrained scheduling to minimize makespan problem. Let \( m \) be the number of machines and \( n' \) be the number of jobs in \( I \). Let \( J_1 \) denote the set of jobs in \( I \). Without loss of generality, assume that the optimal makespan is 1; this can be achieved by scaling the processing lengths of jobs by the optimal makespan value. With this assumption, we know that it is NP-hard to distinguish if optimal makespan is 1 or \( \gamma \). We create an instance \( I' \) for the flow-time problem as follows: In \( I' \), the machines are same as that in \( I \). Now we describe the job set: At time \( t = 0 \), we release the jobs in the makespan instance \( J_1 \) with same precedence constraints.

Let \( T = \gamma^t \) and let \( \delta > 0 \) be a very tiny constant. For each time instant \( t = (1 + k\delta) \in [1, T) \) a single job \( j \) arrives. The processing length of \( j \) is such that it takes exactly of \( \delta \) time units to complete the job on any machine. Further, \( j \) can be processed only if all the jobs preceding it are completed; i.e., \( j' \prec j \) for all \( j' \) with \( r_{j'} < r_j \). This completes the description of our instance \( I' \). Let \( J_2 \) denote the set of jobs that arrive in the interval \( [1, T] \).

Now, let us calculate the total flow-time of jobs in the optimal solution. The total flow-time of jobs in \( J_1 \) is at most \( n' \) since we assumed that optimal makespan is 1. The total flow-time of jobs that are released in the interval \( [1, T] \) is exactly equal to \( T \) as there is exactly one job alive at any time \( t \in [1, T] \). Thus, the total flow-time of jobs in the optimal solution is at most \( n' + T \).

Let \( A \) be any polynomial time algorithm that is given a speed of \( \gamma^{1-c} \) for any \( c > 0 \). Let us calculate the total flow-time of jobs in the schedule produced by \( A \). Towards that let us focus on the number of jobs that are alive at time \( t = (\gamma^t - 1) \). We claim that exactly \( (\gamma^t - 1)/\delta \) jobs from the set \( J_2 \) are alive at that time. This is true since \( A \) can not complete the jobs in the makespan instance \( J_1 \) earlier than \( \gamma^t \), and the jobs that are released in the interval \( [1, \gamma^t] \) cannot begin their execution until all the jobs in \( J_1 \) are completed. The total number of jobs that arrive in \( [1, \gamma^t] \) is equal to \( (\gamma^t - 1)/\delta \). Due to precedence constraints, \( A \) has to process jobs in \( J_2 \) only one at time. This implies that at all times \( t \in (T - 1/2, T) \) at least \( (\gamma^t - 1)/2\delta \) jobs are alive. Thus, the total flow-time of \( A \) is at least \( (\gamma^t - 1)^2/2\delta \).

Suppose \( 1/\delta = n'^{1/c} \), where \( n' \) is the number of jobs in the makespan instance. Since \( c \) is a constant, this ensures the our reduction is of size \( O(n'^{1/c}) \). The ratio of the cost of \( A \) to that of the optimal solution is at least \( ((\gamma^t - 1)^2/2\delta + n')/(n' + \gamma^t) = \Omega(n'^{1/(1-c)}) \). Now, let us express this in terms of the number of
jobs in the flow-time instance. The total number of jobs in the flow-time instance \( n \) is equal to \( n' + T/\delta \approx T/\delta = T \cdot n^{(1/c)} \). Thus, writing our lowerbound in terms \( n \) we get a gap of \( n^{1-c} \) on the approximation factor, which completes the proof.

**Proof of Theorem (1.4).** Above theorem immediately gives the lowerbound on the speed augmentation factor claimed in the Theorem (1.4).

4 Conclusion and Open Problems

In this paper, we considered two classical scheduling models and gave nearly optimal approximation algorithms. However, plenty of interesting open problems remain. In the open-shop scheduling model, the most interesting open question is if we can generalize our result to the non-concurrent case. Another interesting open problem is to obtain a non-trivial approximation algorithm for the co-flow scheduling problem. For the precedence constrained scheduling model, a more open-ended question is if we can show results without any speed augmentation for some interesting special cases.

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