DEGENERATION OF POLE ORDER SPECTRAL SEQUENCES
FOR HYPERPLANE ARRANGEMENTS OF 4 VARIABLES

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Abstract. For essential reduced hyperplane arrangements of 4 variables, we show that the pole order spectral sequence degenerates almost at $E_2$, and completely at $E_3$, generalizing the 3 variable case where the complete $E_2$-degeneration is known. These degenerations are useful to determine the roots of Bernstein-Sato polynomials supported at the origin. For the proof we improve an estimate of the Castelnuovo-Mumford regularity of logarithmic vector fields which was studied by H. Derksen and J. Sidman.

Introduction

Let $Z \subset \mathbb{P}^{n-1}$ be a reduced hyperplane arrangement with $n \geq 3$. Let $f$ be a defining polynomial. We have the pole order spectral sequence which is associated with a double complex whose differentials are given by $d$ and $df \wedge$, see for instance [DiSa], [Sa3]. The abutting filtration is the pole order filtration, which determines the roots of Bernstein-Sato polynomials supported at the origin under some condition, see [Sa1, Theorem 2].

In the case $n = 3$, it is proved that the spectral sequence degenerates at $E_2$, and the $E_1$-term of the spectral sequence can be obtained from the Hilbert series of the Milnor algebra $R/(\partial f)$ using the self-duality of the $E_1$-term, where $(\partial f) \subset R := \mathbb{C}[x]$ is the Jacobian ideal generated by the partial derivatives of $f$, see [DiSa], [Sa2]. We have moreover the injectivity of the differential $d_1$ except for certain places which are irrelevant to the computation of the roots supported at the origin. These can be used to determine the roots of Bernstein-Sato polynomials except for certain special cases, see loc. cit.

Assume from now on $n = 4$. We can still calculate the $E_1$-term of the spectral sequence quite explicitly and swiftly using computer programs like Macaulay2 or Singular, see [Sa3]. However, there is no method to prove the $E_2$-degeneration for this case although there is no counterexample. In the strongly free divisor case, it is shown in [Sa3, Theorem 4.10] that the spectral sequence degenerates almost at $E_2$, and completely at $E_3$. This was motivated by a conjecture in [DiSt]. In this paper we show that this theorem also holds (with $m = 2d-1$) in a hyperplane arrangement case; more precisely, we prove the following.

Theorem 1. For essential reduced hyperplane arrangements in $\mathbb{P}^3$, the pole order spectral sequence degenerates completely at $E_3$ and almost at $E_2$; more precisely, we have

$$\mu_k^{(2)} = 0 \quad (k > 2d-2), \quad \nu_k^{(2)} = 0 \quad (k > 3d-1), \quad \rho_k^{(2)} = 0 \quad (k > 4d-2).$$

Here $\mu_k^{(2)}$, $\nu_k^{(2)}$, $\rho_k^{(2)}$ denote the dimensions of certain $E_2$-terms of the spectral sequence, see [Sa3]. The assertion for $\nu_k^{(2)}$ is slightly weaker than (4.3.4), loc. cit. for $m = 2d-2$, where the range of $k$ is given by $k > 3d-2$. This comes from the difficulty to control the $\mu_{d2,k}^{m}$ in the notation of loc. cit. For the proof of Theorem 1 we use the Castelnuovo-Mumford regularity.

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In Section 1 we improve an estimate of the regularity of logarithmic vector fields shown in \cite{DeSi}. In Section 2 we prove Theorem 1 using the estimate of the Castelnuovo-Mumford regularity.

1. Regularity of logarithmic vector fields

In this section we review some basics of Castelnuovo-Mumford regularity, and improve an estimate of the regularity of logarithmic vector fields shown in \cite{DeSi}.

1.1. Castelnuovo-Mumford Regularity (see \cite{Ei}). Let $M$ be a finitely generated graded $R$-module with $R := \mathbb{C}[x_1, \ldots, x_n]$. Take a minimal graded free resolution
\[
\to F_1 \to \cdots \to F_0 \to M \to 0,
\]
where $F_j = \bigoplus_k R(-c_{j,k})$ with $c_{j,k} \in \mathbb{Z}$. The Castelnuovo-Mumford regularity is defined by
\[
\text{reg } M := \max_{j,k} (c_{j,k} - j).
\]

Remarks 1.1 (i). By the graded version of Nakayama’s lemma, the minimality of the graded free resolution is equivalent to the vanishing of the differential of the complex
\[
\to \mathbb{C} \otimes_R F_1 \to \cdots \to \mathbb{C} \otimes_R F_0 \to 0,
\]
where $\mathbb{C} = R/\mathfrak{m}$ with $\mathfrak{m} \subset R$ the graded maximal ideal, see loc. cit. This implies that
\[
\text{Tor}_j^R(\mathbb{C}, M) = \bigoplus_k \mathbb{C}(-c_{j,k}) \quad (j \in \mathbb{N}),
\]
hence
\[
\text{reg } M = \max_j \left( \max \deg \left( \text{Tor}_j^R(\mathbb{C}, M) \right) - j \right),
\]
with $\max \deg(E) := \max\{ k \in \mathbb{Z} \mid E_k \neq 0 \}$.

(ii) We have a formula similar to (1.1.3) using local cohomology (see Theorem 4.3, loc. cit.)
\[
\text{reg } M = \max_j \left( \max \deg \left( \text{H}_j^m M \right) + j \right).
\]

Note that $H_j^m R = 0$ for $j \neq n$, and $H_n^m R = \mathbb{C}[\partial_1, \ldots, \partial_n](-n)$. (The assertion (1.1.4) is much more difficult than (1.1.3)).

1.2. Logarithmic vector fields. Let $X$ be a smooth complex algebraic variety or a complex manifold, and $D \subset X$ be an effective divisor. We denote by $\text{Der}_X$ the sheaf of algebraic or holomorphic vector fields on $X$. The subsheaf $\text{Der}_X(-\log D)$ of logarithmic vector fields along $D$ can be defined locally by the exact sequence
\[
0 \to \text{Der}_X(-\log D) \to \text{Der}_X \to \mathcal{O}_X/\mathcal{O}_X(-D),
\]
where the last morphism is given by $\text{Der}_X \ni \theta \mapsto [\theta(f)] \in \mathcal{O}_X/\mathcal{O}_X(-D)$ with $f$ a local defining function of $D$. This subsheaf is independent of a non-reduced structure of $D$ by the Leibniz law for derivations. (This is quite different from the case of logarithmic differential forms.) From now on we assume $D$ reduced.

The following is needed to improve the estimate of the Castelnuovo-Mumford regularity of logarithmic vector fields by \cite{DeSi}.

Lemma 1.2. Assume $X = X_1 \times X_2$ and $D = \text{pr}_1^* D_1 + \text{pr}_2^* D_2$ with $\text{pr}_i : X \to X_i$ natural projections ($i = 1, 2$). Then we have the canonical isomorphism
\[
\text{Der}_X(-\log D) = \text{pr}_1^* \text{Der}_{X_1}(-\log D_1) \oplus \text{pr}_2^* \text{Der}_{X_2}(-\log D_2),
\]
Proof. Since we have the canonical isomorphism
\[(1.2.3) \quad \text{Der}_X = \text{pr}_1^* \text{Der}_{X_1} \oplus \text{pr}_2^* \text{Der}_{X_2},\]
any \(\theta \in \text{Der}_X\) is written locally as \(\theta_1 + \theta_2\) with \(\theta_i \in \text{pr}_i^* \text{Der}_{X_i}\) \((i = 1, 2)\). We denote also by \(f_i\) the pull-back of a local defining function \(f_i\) of \(D_i\). The condition
\[\theta(f_1 f_2) = \theta_1(f_1 f_2) + f_1 \theta_2(f_2) \in (f_1 f_2)\]
is equivalent to that \(\theta_i(f_i) \in (f_i)\) \((i = 1, 2)\), since \(\mathcal{O}_{X,x}\) is a unique factorization domain and \(f_1, f_2\) are mutually prime. So the isomorphism (1.2.2) follows taking the pull-back of the exact sequence (1.2.1) for \((X_i, D_i)\) by \(\text{pr}_i^*\) \((i = 1, 2)\), since the latter is an exact functor of \(\mathcal{O}\)-modules. This finishes the proof of Lemma (1.2).

1.3. Regularity of logarithmic vector fields. Let \(D\) be a hyperplane arrangement in \(X = \mathbb{C}^n\) (as an algebraic subvariety). Assume \(D\) central, that is, \(f\) homogeneous. Set \(R = \mathbb{C}[x_1, \ldots, x_n]\) with \(x_1, \ldots, x_n\) the coordinates of \(X\), and
\[\text{Der}_R(-\log D) := \Gamma(X, \text{Der}_X(-\log D)).\]
This is the graded \(R\)-module of algebraic logarithmic vector fields on \(X\), where the grading is given by \(\deg x_i = 1\) and moreover \(\deg \partial / \partial x_i = -1\) in this paper. So the degree (as well as the regularity) is shifted by 1, compared with [DeSi, Sc].

The following is due to [Sc] in the case \(n = 3\), and is shown essentially in [DeSi].

Proposition 1.3. For an essential central reduced hyperplane arrangement in \(X = \mathbb{C}^n\), we have the inequality
\[(1.3.1) \quad \text{reg} \text{Der}_R(-\log D) \leq \deg D - n.\]

Proof. This can be proved by induction on \(n\) and \(\deg D\) using [DeSi] Corollary 3.7 where each hyperplane is deleted for induction. If we get a non-essential one by deleting a hyperplane from an arrangement, then the latter is defined by \(f = x_n g\) with \(g \in \mathbb{C}[x_1, \ldots, x_{n-1}]\) replacing the coordinates \(x_1, \ldots, x_n\) appropriately, where the deleted hyperplane is defined by \(x_n = 0\). (Recall that an arrangement which cannot be defined by a polynomial of fewer variables is called essential.) In this case Lemma (1.2) implies the canonical isomorphism
\[(1.3.2) \quad \text{Der}_R(-\log D) = \text{Der}_R(-\log D') \otimes_R R \oplus \text{Der}_{\mathbb{C}[x_n]}(-\log 0) \otimes_{\mathbb{C}[x_n]} R,\]
with \(D' := g^{-1}(0) \subset \mathbb{C}^{n-1}\). The regularity of the right-hand side of (1.3.2) is bounded by
\[\deg D' - (n-1) = \deg D - n\]
by inductive hypothesis (where \(\deg D \geq n\) since \(D\) is essential). So we can use (1.3.2) instead of Corollary 3.7 for this case, and proceed by induction on \(n\) and \(\deg D\). This finishes the proof of Proposition (1.3).

1.4. Regularity of Milnor algebras. In the notation of (1.3), we have a well-known decomposition
\[(1.4.1) \quad \text{Der}_R(-\log D) = \text{Der}_R(-\log D)^0 \oplus R\theta_0.\]
Here \(\theta_0 := \sum_{i=1}^n \frac{1}{i} x_i \partial / \partial x_i\) so that \(\theta_0(f) = f\), and
\[\text{Der}_R(-\log D)^0 := \{\theta \in \text{Der}_R \mid \theta(f) = 0\}.\]
Set
\[A^p_f := \text{Ker}(df \wedge : \Omega^p \to \Omega^{p+1}),\]
with $\Omega^p$ the algebraic differential $p$-forms on $\mathbb{C}^n$. Put $d := \deg D = \deg f$. We have the isomorphism of graded $R$-modules
\begin{equation}
\text{Der}_R(- \log D)^0 = A_f^{n-1}(n),
\end{equation}
since both are identified (up to a shift of grading) with
\[\{ g_i \in R \ (i \in [1,n]) \mid \sum_i g_i \partial f / \partial x_i = 0 \} .\]
By definition there are two short exact sequence of graded $R$-modules
\begin{equation}
0 \to df \wedge \Omega^{n-1} \to \Omega^n \to M \to 0,
\end{equation}
where $M$ is as in [Sa3], and is isomorphic to the shifted Milnor algebra $(R/(\partial f))(-n)$, see also [DIM]. Note that the $\Omega^p$ are free $R$-modules with $\text{reg} \Omega^p = p \ (p = n-1, n)$. There are no $\mathbb{C}$-linear relations among the $\partial f / \partial x_i$ by the essentiality of $D$. So the above two short exact sequences can be extended to a minimal graded free resolution of $M$, and we get
\begin{equation}
\text{reg } M = \text{reg } df \wedge \Omega^{n-1} - 1 = A_f^{n-1}(-d) - 2,
\end{equation}
since we have by (1.4.2)
\[\text{reg } A_f^{n-1}(-d) = \text{reg } \text{Der}_R(- \log D)^0(-d - n) \geq d + n .\]
Proposition (1.3) then implies the following.

**Corollary 1.4.** For an essential central reduced hyperplane arrangement $D$ in $\mathbb{C}^n$, we have
\begin{equation}
\text{reg } M \leq 2d - 2.
\end{equation}

### 2. Proof of the main theorem

In this section we prove Theorem 1 using the estimate of the Castelnuovo-Mumford regularity.

Since the notation of [Sa3] is used without further explanations, the reader is advised to read first few pages of the introduction as well as Sections 3.1 and 3.5 in [Sa3] before reading this section.

#### 2.1. Consequences of Corollary (1.4)

We first deduce upper estimates of the supports of $\text{Diff}^2(\mu_{\max}^{m''})$, $\text{Diff}(\mu'')$, $\mu''$, $\mu'^{m''}_{\text{def}}$ and lower estimates for $\text{Diff}^2(\rho)$, $\text{Diff}(\nu'')$, $\nu'^{m''}_{\text{def}}$, $\nu'$ from Corollary (1.4) using the self-duality [Sa3, Corollary 3.6].

**Proposition 2.1.** In the notation of [Sa3] Section 3.5, we have
\begin{equation}
\text{Diff}^2(\mu_{\max}^{m''})_k = \text{Diff}(\mu'')_k = \mu'_k = 0 \ (k > 2d-2), \quad \mu'^{m''}_{\text{def},k} = 0 \ (k > 2d-3) .
\end{equation}

**Proof.** We have the spectral sequence associated with the filtration $G$ in [Sa3] Section 3.1
\begin{equation}
E_1^{p,q} = H_{m}^{p+q} \text{Gr}_G^p M \Longrightarrow H_{m}^{p+q} M ,
\end{equation}
where $m := (x_1, \ldots, x_4) \subset R$, and $\text{Gr}_G^p M = M'''$, $\text{Gr}_G^1 M = M''$, $\text{Gr}_G^2 M = M'$. We can use generic projections $\pi_i : \mathbb{C}^4 \to \mathbb{C}^i$ to calculate the local cohomology, see also the proof of [Ei, Corollary 4.8]. We then get (using the short exact sequence (2.3.8) below for $M'''$
\begin{equation}
E_1^{p,q} = 0 \text{ unless } (p,q) \in \{(0,1),(0,2),(1,0),(2,-2)\} .
\end{equation}

So the spectral sequence degenerates at $E_1$, and the regularity of $M'$, $M''$, $M'''$ is at most $2d-2$ by Corollary (1.4) and Remark (1.1)(ii). This implies (2.1.1) except for the assertion
about $\mu_{\text{def},k}''$. As for the latter, the short exact sequence (2.3.8) below implies the isomorphisms

\begin{align*}
H^1_{m}M''' &= H^0_{m}M''_{\text{def}}, \quad H^2_{m}M''' = H^2_{m}M''_{\text{max}},
\end{align*}

where $H^j_{m}M''' = 0 \ (j \neq 1, 2)$, $H^j_{m}M''_{\text{max}} = 0 \ (j \neq 2)$, $H^j_{m}M''_{\text{def}} = 0 \ (j \neq 0)$. Proposition (2.1) then follows from Remark (1.1)(ii).

By [Sa3, Corollary 3.6], we can deduce the following from Proposition (2.1).

**Corollary 2.1.** In the notation of [Sa3, Section 3.5], we have

\begin{align*}
\text{Diff}^2(\rho)_k &= 0 \ (k < 2d+4), \quad \text{Diff}(\nu'')_k = 0 \ (k < 2d+3), \\
\nu'''_{\text{def},k} &= 0 \ (k < 2d+2), \quad \nu''_k = 0 \ (k < 2d+3).
\end{align*}

**2.2. Other estimates.** For the proof of Theorem 1, we also need lower estimates of the supports of $\mu''$, $\mu'$ and upper estimates for $\text{Diff}(\nu'')$, $\nu'''$, $\text{Diff}(\nu''_{\text{max}})$ as follows.

**Proposition 2.2.** In the notation of [Sa3, Section 3.5], we have

\begin{align*}
\text{Diff}(\mu'')_k &= \mu''_k = \mu'_k = 0 \ (k < d+3).
\end{align*}

*Proof.* Let $J \subset R$ be the saturation of the Jacobian ideal $(\partial f) \subset R$ for the product of the associated primes of $(\partial f)$ with codimension 3 in $\mathbb{C}^4$ so that $R/J = M'''(4)$ in the notation of Section 3.1, *loc. cit.* Then Proposition (2.2) is reduced to Lemma (2.2) below, since the latter implies that $M'_k = M''_k = 0 \ (k < d+3)$. (Here it is quite difficult to get an assertion about $\mu''_{\text{def}}$.)

**Lemma 2.2.** In the above notation, we have

\begin{align*}
J \subset m^{d-1}, \quad \text{that is,} \quad J_k = 0 \ (k < d-1).
\end{align*}

*Proof.* We can define the ideal $J \subset R$ also by the condition that $g^\sim|_U \in J_f(k)|_U$ for $g \in R_k$.

Here $g^\sim$ is a section of $\mathcal{O}_{\mathbb{P}^3}(k)$ defined by $g$, $J_f \subset \mathcal{O}_{\mathbb{P}^3}$ is the ideal generated by the Jacobian ideal $(\partial f) \subset R$, and $U \subset \mathbb{P}^3$ is the complement of some finite subset of $\mathbb{P}^3$. Taking a general hyperplane section, the assertion (2.2.2) is then reduced to the case $n = 3$, where it is more or less known. Indeed, the assertion for $n = 3$ is equivalent to that $\mu'_k = 0$ for $k < d+2$ (since $\deg \partial f/\partial x_i = d-1$), or equivalently, for $k > 2d-2$ by the symmetry of the $\mu'_k$, see [DiSa]. But the latter follows from (1.4.4) for $n = 3$, where we have

\begin{align*}
\nu_k &= 0 \ (k < d+2), \quad \mu'_k + \nu_{3d-k} = \tau \ (k \in \mathbb{Z}),
\end{align*}

see *loc. cit.* and also [DIM]. So Lemma (2.2) and Proposition (2.2) follow.

**Corollary 2.2.** In the notation of [Sa3, Section 3.5], we have

\begin{align*}
\text{Diff}(\nu'')_k &= 0 \ (k > 3d-2), \quad \nu'''_{\text{def},k} = 0 \ (k > 3d-3), \\
\text{Diff}(\nu''_{\text{max}})_k &= 0 \ (k \notin [d+3, 3d-1]).
\end{align*}

*Proof.* By [Sa3, Corollary 3.6], the first two inequalities follows from Proposition (2.2), and the last one is reduced to the vanishing of $\text{Diff}(\nu''_{\text{max}})_k$ for $k < d+3$, which follows from that of $\Omega''_k \ (k < 3)$, since $\nu'''_{\text{def},k} = 0 \ (k < 2d+2)$ by (2.1.1) using *loc. cit.* This finishes the proof of Corollary (2.2).

**2.3. Proof of Theorem 1.** Set

\begin{align*}
\tilde{\mu}_k := \mu_k, \quad \tilde{\nu}_k := \nu_k - \nu'_k, \quad \tilde{\rho}_k := \rho_k - \nu'_k, \\
\tilde{\chi}_f,k := \tilde{\mu}_k - \tilde{\nu}_{k+d} + \tilde{\rho}_{k+2d} \ (k \in \mathbb{Z}).
\end{align*}
Define $\check{\Eu}_i^{\leq m}$ in the same way as $\Eu_i^{\leq m}$ in [Sa3, (4.2.9)] by replacing $\chi_f$ with $\check{\chi}_f$. Put $\check{\Eu}_i := \check{\Eu}_i^{\leq m}$ ($m \gg 0$) as in loc. cit. We have by definition

$$\check{\Eu}_i^{\leq m} = \Eu_i^{\leq m} (m \gg 0), \text{ that is, } \check{\Eu}_i = \Eu_i.$$ 

From Proposition (2.1) and Corollary (2.2) together with the relation $\mu_k - \nu_k + \rho_k = 0$ ($k > 4d - 4$), we can deduce that

$$(2.3.1) \quad \text{Diff}(\check{\mu})_k = \frac{1}{2} \text{Diff}(\check{\nu})_{k+d} = \text{Diff}(\check{\rho})_{k+2d} = \tau_{Z'} (k \geq 2d),$$

where $\tau_{Z'}$ is the global Tjurina number of a general hyperplane section $Z'$ of $Z$. We thus get

$$(2.3.2) \quad \text{Diff}(\check{\chi}_f)_k = 0 \quad (k \geq 2d).$$

Since $\check{\chi}_{f,k} = \chi_{f,k} = 0$ ($k \gg 0$), this implies the vanishing

$$(2.3.3) \quad \check{\chi}_{f,k} = 0 \quad (k \geq 2d - 1),$$

and then the inequality

$$(2.3.4) \quad \Eu_d + 1 = \check{\Eu}_d + 1 = \check{\chi}_{f,d} = \chi_{f,d} - \nu'_{3d} \leq \chi_{f,d} = -\chi(U) + 1.$$ 

using Corollary (2.1) (which implies that $\nu'_{3d} = 0$) and also [Sa3, Corollary 4.8] (which proves the last equality of (2.3.4)), where [WiYu, Corollary 6.3] is used in an essential way).

For $m \gg 0$ and $i = d$, we thus get the following inequalities similar to [Sa3, (4.3.6)]

$$(2.3.5) \quad -\chi(U) \leq (\check{\Eu}_i^{\leq m}) \leq \Eu_i^{\leq m} = \Eu_i \leq -\chi(U),$$

where $(\check{\Eu}_i^{\leq m})$ is as in loc. cit. (The condition $m \gg 0$ is used for the equality $\Eu_d^{\leq m} = \Eu_d$.) All the inequalities in (2.3.5) then become equalities, and we get in particular

$$(2.3.6) \quad \nu'_{3d} = 0, \text{ that is, } \check{\mu}_{\text{def},d}^{(2)} = 0,$$

using [Sa3, Corollary 3.6]. By Lemma (2.3) below together with (2.3.1), this implies that

$$(2.3.7) \quad \nu'_k = 0 \quad (k \geq 3d), \quad \check{\mu}_{\text{def},k}^{(2)} = 0 \quad (k \leq d), \quad \frac{1}{2} \text{Diff}(\nu)_{k+d} = \text{Diff}(\rho)_{k+2d} = \tau_{Z'} (k > 2d), \quad \chi_{f,k} = 0 \quad (k \geq 2d).$$

Theorem 1 now follows from [Sa3, Theorems 4.3] for $m = 2d - 1$ using Theorem 4.9, loc. cit., since $\mu_k - \rho_k$ is independent of $k \gg 0$. We have $\rho_{3d-1}^{(2)} = 0$ by [Sa3, Proposition 1], and $\mu_{2d-1}^{(2)} = 0$, since $\nu_k^{(2)} = 0$ ($k \geq 3d$) and $\mu_{2d-1}^{(\infty)} = 0$ by (15), loc. cit. This finishes the proof of Theorem 1.

**Lemma 2.3.** In the above notation, let $k$ be an integer at most $d$. Then $\check{\mu}_{\text{def},k} = 0$ if $\mu_{\text{def},k+1} = 0$.

**Proof.** We have the short exact sequence of graded $R$-modules as in [Sa3, (3.1.8)]

$$(2.3.8) \quad 0 \rightarrow M''_{\text{def}} \rightarrow M''_{\text{max}} \overset{\phi}{\rightarrow} M''_{\text{def}} \rightarrow 0.$$ 

Take any $v \in M''_{\text{def}}$. Let $v' \in M''_{\text{max}}$ such that $\phi(v') = v$. Let $x, y$ be the pull-backs of the coordinates of $\mathbb{C}^2$ by the generic projection $\pi_2 : \mathbb{C}^4 \rightarrow \mathbb{C}^2$ in [Sa3, Section 3.1]. Since $xv = yv = 0$, there are $v_1, v_2 \in M''_{k+1}$ such that

$$\iota(v_1) = xv', \quad \iota(v_2) = yv'.$$

We have

$$yv_1 = xv_2 \quad \text{in} \quad M''_{k+2},$$

$$M_{\text{max}}^{\leq m} \overset{\phi}{\rightarrow} M_{\text{max}}^{\leq m} \rightarrow M_{\text{max}}^{\leq m} \rightarrow 0,$$
since \( \iota(yv_1) = \iota(xv_2) \). By Lemma (2.2) there are canonical isomorphisms

\[
M''^j = R_{j-4} \quad (j \leq k + 2).
\]

(Here the assumption \( k \leq d \) is used.) We then get \( v_0 \in M''^k \) such that

\[
xv_0 = v_1, \quad yv_0 = v_2,
\]
since \( R \) is a unique factorization domain. Moreover \( \iota(v_0) = v' \) (hence \( v = 0 \)), since \( M''_{\text{max}} \) is a free \( \mathbb{C}[x, y] \)-module (in particular, it is \( x \)-torsion-free). So Lemma (2.3) follows.

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