Efficient Decoding of Partial Unit Memory Codes of Arbitrary Rate

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Abstract—Partial Unit Memory (PUM) codes are a special class of convolutional codes, which are often constructed by means of block codes. Decoding of PUM codes may take advantage of existing decoders for the block code. The Dettmar–Sorger algorithm is an efficient decoding algorithm for PUM codes, but allows only low code rates. The same restriction holds for several known PUM code constructions. In this paper, an arbitrary rate construction, the analysis of its distance parameters and a generalized decoding algorithm for PUM codes of arbitrary rate are provided. The correctness of the algorithm is proven and it is shown that its complexity is cubic in the length.

Index Terms—Convolutional codes, Partial Unit Memory Codes, Bounded Minimum Distance Decoding

I. INTRODUCTION

The algebraic description and the distance calculation of convolutional codes is often difficult. By means of block codes, special convolutional codes of memory $m = 1$ can be constructed, which enable the estimation of the distance parameters. Moreover, the existing efficient block decoders can be taken into account in order to decode the convolutional code. There are constructions of these so-called Partial Unit Memory (PUM) codes based on Reed–Solomon (RS) codes, BCH codes and – in rank metric – Gabidulin codes. Decoding of these PUM codes uses the algebraic structure of the underlying RS, BCH or Gabidulin codes.

In [10], Dettmar and Sorger constructed low-rate PUM codes and decoded them up to the minimal extended row distance. Such a decoder is called Bounded Minimum Distance (BMD) decoder for convolutional codes. Winter [11] gave first ideas of an arbitrary rate construction.

In this contribution, we construct PUM codes of arbitrary rate, prove their distance properties and generalize the Dettmar–Sorger algorithm to PUM codes of arbitrary rate. We prove the correctness of the decoding algorithm and show that the complexity is cubic in the length. To our knowledge, no other construction and efficient decoding of PUM codes of arbitrary rate exist. Due to space limitations, we consider only PUM codes, but all results apply also to Unit Memory codes.

This paper is organized as follows. In Section II, we give basic definitions, Section III provides the arbitrary rate construction and calculates its parameters. In Section IV, we explain and prove the BMD decoding algorithm. Section V concludes this contribution.

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II. DEFINITIONS AND NOTATIONS

Let $q$ be a power of a prime and let $\mathbb{F}$ denote the finite field of order $q$. We denote by $\mathbb{F}^n = \mathbb{F}^{1 \times n}$ the set of all row vectors of length $n$ over $\mathbb{F}$ and the elements of a vector $\mathbf{a}_j \in \mathbb{F}^n$ by $\mathbf{a}_j = (a_{j}^{(1)}, a_{j}^{(2)}, \ldots, a_{j}^{(n)})$.

Let us define a zero-forced terminated convolutional code $C$ for some integer $L$ by the following $Lk \times (n(L+m))$ generator matrix $G$ over the finite field $\mathbb{F}$

$$G = \begin{pmatrix}
G_0 & G_1 & \cdots & G_m \\
G_0 & G_1 & \cdots & G_m \\
\vdots & \vdots & \ddots & \vdots \\
G_0 & G_1 & \cdots & G_m
\end{pmatrix}, \quad (1)$$

where $G_i$, $i = 0, \ldots, m$ are $k \times n$–matrices and $m$ denotes the memory of $C$ as in [12]. In the following, $N \overset{\text{def}}{=} L + m$.

The error-correcting capability of convolutional codes is determined by extended (or active) distances.

Let $C^c(j)$ denote the set of all codewords corresponding to paths in the minimal code trellis that diverge from the zero state at depth $0$ and return to the zero state for the first time at depth $j$. The extended row distance of order $j$ is defined as the minimum Hamming weight of all codewords in $C^c(j)$:

$$d^*_j \overset{\text{def}}{=} \min_{\mathbf{c} \in C^c(j)} \{\text{wt}(\mathbf{c})\}.$$  

Similarly, let $C^r(j)$ denote the set of all codewords leaving the zero state at depth $0$ and ending in any state at depth $j$ and let $\overline{C^r}(j)$ denote the set of all codewords starting in any state at depth $0$ and ending in the zero state in depth $j$, both without states in between. The extended column distance and the extended reverse column distance are:

$$d^c_j \overset{\text{def}}{=} \min_{\mathbf{c} \in C^c(j)} \{\text{wt}(\mathbf{c})\}, \quad d^{cr}_j \overset{\text{def}}{=} \min_{\mathbf{c} \in \overline{C^r}(j)} \{\text{wt}(\mathbf{c})\}.$$

The free distance is the minimum (Hamming) weight of any non-zero codeword of $C$ and can be determined by $d_{\text{free}} = \min_j \{d^*_j\}$. The extended row distance $d^*_j$ can be lower bounded by a linear function with slope $\alpha$:

$$\alpha = \lim_{j \to \infty} \left\{ \frac{d^*_j}{j} \right\}.$$  

PUM codes are convolutional codes of memory $m = 1$. Therefore, the semi-infinite generator matrix consists of two $k \times n$ sub-matrices $G_0$ and $G_1$. Both matrices have full rank.
if we construct an \((n, k)\) UM code. For an \((n, k \mid k_1)\) PUM code, rank\((G_0) = k\) and rank\((G_1) = k_1 < k\) hold, such that:
\[
G_0 = \begin{pmatrix} G_{00} \\ G_{01} \end{pmatrix}, \quad G_1 = \begin{pmatrix} G_{10} \\ 0 \end{pmatrix},
\]
(2)
where \(G_{00}\) and \(G_{10}\) are \(k \times n\) matrices and \(G_{01}\) is a \((k - k_1) \times n\)-matrix. The encoding rule for a code block of length \(n\) is given by 
\[c_j = \begin{pmatrix} 1_j \end{pmatrix} \cdot G_0 + \begin{pmatrix} 1_j-1 \end{pmatrix} \cdot G_1, \text{ for } 1_j, 1_j-1 \in \mathbb{F}_q^k.\]

The free distance of UM codes is upper bounded by 
\[d_{\text{free}} \leq 2n - k + 1\]
and of PUM codes by 
\[d_{\text{free}} \leq n - k + k_1 + 1.\]
For both the slope is upper bounded by 
\[\alpha \leq n - k \text{ \cite{4, 13}}.\]

As notation, let the generator matrices
\[
G_0, \begin{pmatrix} G_{01} \\ G_{10} \end{pmatrix}, \quad G_{01} \text{ and } G_{00}
\]
define the block codes \(C_0, C_1, C_{01}\) and \(C_{00}\) with the minimum Hamming distances \(d_0, d_1, d_{01}\) and \(d_{\alpha}\) and the BMD block decoders \(BMD(C_0), BMD(C_1), BMD(C_{01})\) and \(BMD(C_{00})\), which correct errors up to half their minimum distance.

III. CONSTRUCTING PUM CODES OF ARBITRARY RATE

A. Construction

Since each code block of length \(n\) of the PUM code can be seen as a codeword of the block code \(C_{\alpha}\), a great \(d_{\alpha}\) is important for the distance parameters of the convolutional code as well as for the decoding capability. One approach is to define by \(G_{\alpha}\) a Maximum Distance Separable (MDS) code and \(d_{\alpha} = n-k-k_1+1\). This is basically the construction from \cite{6, 10}, which designs low-rate PUM codes since the \((k+k_1)\times n\) matrix \(G_{\alpha}\) can define an MDS code only if \(k + k_1 \leq n\).

Otherwise (as observed by \cite{11}), there are linear dependencies between the rows of \(G_{\alpha}\), what we have to consider when constructing PUM codes of arbitrary rate. In the following, we provide a construction of arbitrary \(k_1 < k\) and calculate its distance parameters.

Let \(k + k_1 - \varphi \leq n\), for some \(\varphi < k_1\), and let the \((k + k_1 - \varphi) \times n\) matrix
\[
G_{\text{tot}} = \begin{pmatrix} A \\ \Phi \\ G_{01} \end{pmatrix}
\]
with the sub-sizes
\[
A : (k_1 - \varphi) \times n, \quad \Phi : \varphi \times n, G_{01} : (k - k_1) \times n
\]
(3)
\[
B : (k_1 - \varphi) \times n
\]
define an MDS (e.g. RS) code. We define the sub-matrices of the semi-infinite generator matrix of the PUM code as follows in order to enable arbitrary code rates.

**Definition 1 (PUM Code of Arbitrary Rate)** Let \(k_1 < k < n\) and let \(G_{\text{tot}}\) be defined as in \cite{3}. Then, we define the PUM code by the following submatrices \cite{2}:
\[
G_0 = \begin{pmatrix} G_{00} \\ G_{01} \end{pmatrix} = \begin{pmatrix} A \\ \Phi \\ G_{01} \end{pmatrix}, \quad G_1 = \begin{pmatrix} G_{10} \\ 0 \end{pmatrix} = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}.
\]
(4)

Since \(G_{\text{tot}}\) defines an MDS code, \(C_0, C_1\) and \(C_{10}\) (compare Section \cite{11} for the notations) are also MDS codes. We restrict \(\varphi < k_1\) since otherwise all rows in \(G_1\) are rows of \(G_0\). Note that any rate \(k/n\) in combination with any \(k_1\) is feasible with this restriction since \(k + 1 \leq k + k_1 - \varphi \leq n\) and hence, we have only the trivial restriction \(k < n\).

B. Calculation of Distances

We calculate the extended row distance of the construction from Definition \cite{11} by cutting the semi-infinite generator matrix into parts. Each code block of length \(n\) can be seen as a codeword of \(C_{\alpha}\) with minimum distance
\[d_{\alpha} = d(G_{\alpha}) = d(G_{\text{tot}}) = n - k - k_1 + \varphi + 1.\]

However, due to the linear dependencies between the sub-generator matrices, a non-zero information block can result in a zero code block. The following lemma bounds the maximum number of such consecutive zero code blocks.

**Lemma 1 (Consecutive Zero Code Blocks)** The maximum number \(\ell\) of zero code blocks \(c_j, c_{j+1}, \ldots, c_{j+\ell-1}\) which have no edge in common with the zero state, is
\[\ell = \left\lceil \frac{\varphi}{k_1 - \varphi} \right\rceil.\]

**Proof:** If \(\varphi = 0\), there is no zero code block obtained from a non-zero information block and \(\ell = 0\).

For \(0 < \varphi < k_1\), let
\[
i_{j-1} = (i_{0j}, \ldots, i_{kj-1}, 0, \ldots, 0, i_{k_j}, \ldots, i_{k_1-1})_{k_1 - \varphi},
\]
\[
i_j = (0, \ldots, 0, i_{0j}, \ldots, i_{kj-1}, 0, \ldots, 0, 0, \ldots, 0)_{k_1 - \varphi},
\]
\[
i_{j+k-2} = (0, \ldots, 0, i_{0j}, \ldots, i_{kj-1-1}, 0, \ldots, 0, 0, \ldots, 0)_{k_1 - \varphi},
\]
\[
i_{j+k-1} = (0, \ldots, 0, i_{0j}, \ldots, i_{kj-1-1}, 0, \ldots, 0, 0, \ldots, 0)_{k_1 - \varphi}.
\]
In the non-binary case, each second block \(i_j, i_{j+2}, \ldots\) has to be multiplied by \(1\). Then,
\[
c_{j+h} = i_{j+h-1} \cdot G_1 + i_{j+h} \cdot G_0 = 0, \quad \forall h = 0, \ldots, \ell - 1.
\]

In each step, we shift the information vector to the right by \(k_1 - \varphi\) positions, where this shift size is determined by the size of \(A\). Since \(\Phi\) has \(\varphi\) rows, this right-shifting can be done \([\varphi/(k_1 - \varphi)]\) times. We ceil the fraction since the last block \(i_{j+\ell-1}\) can contain less than \(k_1 - \varphi\) information symbols. ■

Therefore, after \(\ell\) zero code blocks there is at least one block of weight \(d_{\alpha}\) and the slope can be lower bounded by:
\[\alpha \geq \frac{d_{\alpha}}{\ell + 1} = \frac{d_{\alpha}}{\left\lceil \frac{\varphi}{k_1 - \varphi} \right\rceil + 1} = \frac{n - k - k_1 + \varphi + 1}{k_1 - \varphi}.\]

The extended distances can be estimated as follows.

**Theorem 1 (Extended Distances)** The extended distances of order \(j\) for the PUM code of Definition \cite{11} are:
\[
d_j^r \geq d_j^r = d_{01}, \quad d_j^r \geq d_j^r = d_0 + (j - 2) \cdot \alpha + d_1, \quad j > 1,
\]
\[
d_j^r \geq d_j^r = d_0 + (j - 1) \cdot \alpha, \quad j > 0,
\]
\[
d_j^r \geq d_j^r = d_j^r = (j - 1) \cdot \alpha + d_1, \quad j > 0,
\]
with \(d_{01} = n - k + k_1 + 1, d_0 = d_1 = n - k + 1\) and \(\alpha\) as in \cite{5} and \(\hat{d}_j^r, \tilde{d}_j^r, \text{ and } \bar{d}_j^r\) denote the designed extended distance.
Proof: For the calculation of the extended row distance, we start in the zero state, hence, the previous information is $i_0 = 0$. We obtain $d^r_0$ for an information block $i_1 = (0, \ldots, 0, i_{k_1}^{(1)}, \ldots, i_{k_1}^{(l-1)})$, then $c_1 \in C_{01}$. The extended row distance of order $j$ follows from (5) and a last information block $i_j = (0, \ldots, 0, i_{k_j}^{(j)}, \ldots, i_{k_j}^{(j-l)})$. The second-last block $i_{j-1}$ is arbitrary and thus $c_j = i_j \cdot G_0 + i_{j-1} \cdot G_1$ is in $C_1$.

The calculation of the extended column distance starts in the zero state, hence, $i_0 = 0$, but we end in any state, thus, $d^c_0 \geq d_0$. For higher orders, each other block is in $C_\alpha$. The reverse extended column distances considers all code blocks starting in any state, hence there is no restriction on $i_0, i_1 \text{ and } c_j \in C_\alpha$. In order to end in the zero state, $i_j = (0, \ldots, 0, i_{k_j}^{(j)}, \ldots, i_{k_j}^{(j-l)})$ and as for the extended row distance $c_j \in C_1$.

The free distance is then the minimum, i.e.,

$$d_{\text{free}} \geq \min_{i=1,2,\ldots} \{d^c_i\} = \min\{n - k + k_1 + 1, 2 \cdot (n - k + 1)\}.$$ 

Note that if $d_{\text{free}} = n - k + k_1 + 1$, then the free distance is optimal since the upper bound is achieved [4].

IV. BMD Decoding Algorithm

A. BMD Condition and Idea

Let the received sequence $r = c + e = (r_0, r_1, \ldots, r_{N-1})$ be given, where $r_h = c_h + e_h$, $h = 0, \ldots, N - 1$ is in $\mathbb{F}^n$, $c = (c_0, c_1, \ldots, c_{N-1})$ is a codeword of the (terminated) PUM code as in Definition 1 and $e_h$ is an error block of Hamming weight $wt(e_h)$. A BMD decoder for convolutional codes is defined as follows.

Definition 2 (BMD Decoder for Convolutional Codes [10])

A BMD decoder for convolutional codes guarantees to find the Maximum Likelihood (ML) path as long as

$$\sum_{h=j}^{j+i-1} wt(e_h) < \frac{d^r_i}{2}$$

holds for all $j = 0, \ldots, N - 1$ and $i = 1, \ldots, N - j$.

Algorithm 1 shows the basic principle of our generalization of the Dettmar–Sorger algorithm to arbitrary rate.

Algorithm 1: Arbitrary-Rate Decoder for PUM codes

Input: Received sequence $r$ of length $N \cdot n$

1. Decode block $r_0$ with $BMD(C_0)$,
   decode blocks $r_j$ for $j = 1, \ldots, N - 2$ with $BMD(C_\alpha)$,
   decode block $r_{N-1}$ with $BMD(C_1)$,
   calculate $i_j$ if $\ell + 1$ consecutive blocks were decoded successfully and assign metric as in (7)
2. From all found blocks $i_j$, decode $\ell_B^{(j)}$ steps forwards with $BMD(C_0)$ and $\ell_B^{(j)}$ steps backwards with $BMD(C_1)$
3. From all found blocks $i_j$, decode next block with $BMD(C_{01})$ and assign metric as in (14)
4. Search the complete path of smallest weight with the Viterbi algorithm

Output: Information sequence $i$ of length $(N - 1) \cdot k$

The main idea of the algorithm is to take advantage of the efficient BMD block decoders for $C_\alpha$, $C_0$, $C_1$ and $C_{01}$. With the results of the block decoders, we build a reduced trellis and finally use the Viterbi algorithm to find the ML path. Since this trellis has only very few edges, the overall decoding complexity is only cubic in the length. Figure 1 illustrates the decoding principle for $\ell = 1$.

Since each code block of the PUM code of length $n$ is a codeword of the block code $C_\alpha$, the first step of the algorithm is decoding with $BMD(C_\alpha)$. Due to the termination, the first and the last block can be decoded with $BMD(C_0)$, respectively $BMD(C_1)$. The decoding result of $BMD(C_\alpha)$ is $\Phi_j$. Assume it is correct, then $\Phi_j = c_j = i_j \cdot G_0 + i_{k_1}^{[j]} \cdot G_{10}$, where $i_{k_1}^{[j]} = (i_{0}^{[j]}, \ldots, i_{k_1-1}^{[j]})$ is a part of the previous information block. Now, we want to reconstruct the information $i_j = (i_{0}^{[j]}, \ldots, i_{k_j-1}^{[j]})$ and $i_{k_j}^{[j]}$. For this, we need $\ell + 1$ consecutive decoded code blocks since the linear dependencies “spread” to the next $\ell$ blocks as shown in Example 1.

Example 1 (Reconstructing the Information)

Let $\varphi = 2/3k_1$, where $\ell = 2$ and $\Phi$ has twice as much rows as $A$. Assume, we have decoded $c_0$, $c_1$ and $c_2$ and we want to reconstruct $i_1$. Decompose $i_0, i_1, i_2$ into: $i_j = (i_{0}^{[j]}, i_{1}^{[j]}, i_{2}^{[j]}, i_{3}^{[j]}, i_{4}^{[j]})$ for $j = 0, 1, 2$, where the first three sub-blocks have length $k_1 - \varphi$ and the last $k - k_1$. Then,

$$c_1 = (i_1^{[1]} | i_1^{[2]} | i_1^{[3]} | i_1^{[4]} | i_1^{[5]}) \begin{pmatrix} A \\ \Phi_1 \\ \Phi_2 \\ G_{01} \\ B \end{pmatrix} = i_1 \cdot G_{\text{tot}},$$

where $\Phi = (\Phi_2, \Phi_1)$ and $\Phi_1, \Phi_2$ have $k_1 - \varphi$ rows. Since we know $c_1$ and $G_{\text{tot}}$ defines an MDS code, we can reconstruct the vector $i_1$. This directly gives us $i_1^{[1]}$ and $i_1^{[5]}$. This can be done in the same way for $c_0$ and we also directly obtain (among others) $i_0^{[1]}$. To obtain $i_1^{[2]}$, we subtract $i_0^{[1]}$ from the known sum $i_1^{[2]} + i_0^{[0]}$. For $c_2$, this reconstruction provides $i_1^{[3]}$ and we have the whole $i_1$. This principle also gives us $i_{k_1}^{[j]} = i_{0}^{[j]} | i_{1}^{[2]} | i_{0}^{[3]}$. This is why $\ell + 1$ consecutive decoded code blocks are necessary to reconstruct an information block. Note that it does not matter if the other decoded blocks precede or succeed the wanted information, this principle works the same way.
After this decoding and reconstruction, we build an edge in a reduced trellis for each block with the metric:

$$m_j = \begin{cases} \text{wt}(r_j - \bar{c}_j) & \text{if Step 1 finds } \bar{c}_j \text{ and } \bar{i}_j, \\ [(d_{\alpha} + 1)/2] & \text{else.} \end{cases} \quad (7)$$

**Remark 1** The error of minimum weight causing a sequence of non-reconstructed information blocks in Step 1 is as follows:

$$\ell + 1 \text{ blocks, } \ell + 1 \text{ blocks, } \ell + 1 \text{ blocks, } \ell \text{ blocks},$$

where the $\times$ marks blocks with at least $d_{\alpha}/2$ errors. Also, the information of the error-free blocks cannot be reconstructed, since we need $\ell + 1$ consecutive decoded blocks. The last $\ell$ error-free blocks are the reason why we subtract $\ell$ additional decoding steps in forward direction. The (minimum) average weight in a sequence of non-reconstructed information blocks (without the last $\ell$ blocks) is therefore $d_{\alpha}/(2(\ell + 1))$.

Assume, in Step 1, we decoded $c_j$ and reconstructed $i_j$ and a part of the previous information $i_{j-1}^k$, then we calculate:

$$r_j + 1 - \left( (i_{j-1}^0, \ldots, i_{j-1}^k) \cdot G_{10} = i_j + 1 \cdot G_0 + e_{j+1} \right)$$

$$r_j - 1 - \left( (i_{j-2}^0, \ldots, i_{j-2}^k) \cdot G_{00} = (i_{j-1}^0, \ldots, i_{j-1}^k) \cdot (G_0 + e_{j-1}) \right). \quad (8)$$

Hence, as a second step, we decode $\ell_F^{(j)}$ blocks forward with BMD$(C_0)$ respectively $\ell_B^{(j)}$ blocks backward in BMD$(C_1)$. These codes have higher minimum distances than $d_{\alpha}$ and close (most of) the gaps between two sequences of correctly decoded blocks in $C_\alpha$. The values $\ell_F^{(j)}$ and $\ell_B^{(j)}$ are defined by:

$$\ell_F^{(j)} = \min_{i=1,2,\ldots} \left( i \left| \sum_{h=1}^{i} \frac{d_{\alpha} - m_{j+h}}{\ell + 1} \geq \frac{d_{\alpha}}{2} \right. \right), \quad (9)$$

$$\ell_B^{(j)} = \min_{i=1,2,\ldots} \left( i \left| \sum_{h=1}^{i} \frac{d_{\alpha} - m_{j-h}}{\ell + 1} \geq \frac{d_{\alpha}}{2} \right. \right). \quad (10)$$

**Lemma 2** in Section IV-B proves that after Step 2, the size of the gap between two correctly reconstructed blocks is at most one block.

**Lemma 2** The length of any gap between two correct reconstructions in Step 1, $i_j$, $i_{j+1}$, is less than $\min(L_F^{(j)}, L_B^{(j+i)})$ if $[6]$ holds, with

$$L_F^{(j)} = \min_{i=1,2,\ldots} \left( i \left| \sum_{h=1}^{i} \frac{d_{\alpha} - m_{j+h}}{\ell + 1} \geq \frac{d_{\alpha}}{2} \right. \right),$$

$$L_B^{(j)} = \min_{i=1,2,\ldots} \left( i \left| \sum_{h=1}^{i} \frac{d_{\alpha} - m_{j-h}}{\ell + 1} \geq \frac{d_{\alpha}}{2} \right. \right).$$

**Proof:** Step 1 fails if there occur at least $d_{\alpha}/2$ errors in every $(\ell + 1)$-th block, followed by $\ell$ correct ones (compare Remark 1). Assume there is a gap of at least $L_F^{(j)}$ blocks after Step 1. Then,

$$\sum_{h=1}^{L_F^{(j)}-\ell} \frac{d_{\alpha} - m_{j+h}}{\ell + 1} \geq \frac{d_{\alpha}}{2},$$

contradicting $[6]$. We prove this similarly for $L_B^{(j)}$ without subtracting $\ell$ in the limit of the sum, since we directly start left of the $\ell$ correct blocks on the right. Therefore, the gap size is less than $\min(L_F^{(i)}, L_B^{(i)})$.

**Lemma 3** Let $i_j$ and $i_{j+1}$ be reconstructed in Step 1. Let Step 2 decode $\ell_F^{(j)}$ blocks in forward and $\ell_B^{(j+i)}$ blocks in backward direction (see (9), (10)). Then, except for at most one block, the ML path is in the reduced trellis if $[5]$ holds.

**Proof:** First, we prove that the ML path is in the reduced trellis if $[5]$ holds and in each block less than $\min\{d_{\alpha}/2, d_{\alpha}/2\}$ errors occurred. In this case, BMD$(C_0)$ and BMD$(C_1)$ will always yield the correct decision. The ML path is in the reduced trellis if $\ell_F^{(j)} + \ell_B^{(j+i)} \geq i - 1$, since the gap is then closed. Assume that $\ell_F^{(j)} + \ell_B^{(j+i)} < i - 1$ and at least $d_{\alpha}/2$
errors occur in every \((\ell + 1)\)-th block in the gap, since Step 1 was not successful (compare Remark [1]). Then,
\[
\sum_{h=1}^{i-1} \text{wt}(e_{j+h}) \geq \frac{d_{01}^h}{2} + \frac{d_{10}^h}{2} + \frac{(i - 3) - d_{01}^h}{2(\ell + 1)} + \frac{d_{11}^h}{2} = \frac{d_{01}^i}{2} + \frac{i - 3}{2(\ell + 1)} + \frac{d_{11}^i}{2} = \frac{\ell}{2} - 1,
\]
which is a contradiction to \([6]\).

Second, we prove that at most one error block \(e_h, j < h < j + 1\) has weight at least \(d_{01}/2\) or \(d_{11}/2\). To fail in Step 1, there are at least \(d_{01}/2\) errors in every \((\ell + 1)\)-th block. If two error blocks have weight at least \(d_{01}/2 = d_{11}/2\), then
\[
\sum_{h=1}^{i-1} \text{wt}(e_{j+i}) \geq 2 \cdot \frac{d_{01}^i}{2} + \frac{i - 3}{2(\ell + 1)} + \frac{d_{11}^i}{2} \geq \frac{\ell}{2} - 1,
\]
in contradiction to \([6]\). Thus, the ML path is in the reduced trellis except for a gap of one block. \(\blacksquare\)

**Theorem 2** If \([6]\) holds, the ML path is in the reduced trellis.

**Proof:** Lemma [3] guarantees that after Step 2, the gap length is at most one block. This gap can be closed in Step 3 with \(C_{01}\), which is always able to find the correct solution since \(d_{01} \geq \ell_1 = d_{\text{free}}\). \(\blacksquare\)

### C. Decoding of a Single Block

Similar to \([10]\), we give a weaker BMD condition to guarantee ML decoding of a single block. This condition shows how fast the algorithm returns to the ML path after a sequence where \([6]\) is not fulfilled. A BMD decoder for convolutional codes guarantees the correct decoding of a block \(r_j\) of a received sequence \(r = c + e\) if the error \(e\) satisfies
\[
\sum_{h=k}^{k+i-1} \text{wt}(e_h) < \frac{\ell_1^j}{2}, \forall i, k \text{ with } k \leq j \leq j + i - 1. \quad (12)
\]

To guarantee \([12]\) for a certain block if \([6]\) is not fulfilled for the whole sequence, we introduce an *erasure node* in each step \(j\) as in \([7]\), representing all nodes which are not in the reduced trellis. Let \(e_{j-1}\) denote erasure nodes at time \(j, j-1\) and let \(s_j, s_{j-1}\) be nodes found by BMD decoding in Steps 1 and 2. Let \(t_F, t_B\) denote the minimum number of errors of any edge starting from \(s_{j-1}\) and \(s_j\) in forward, respectively backward direction. \(t_a\) denotes the minimum number errors of any edge between nodes at time \(j - 1\) and \(j\). We set the metric of the connections with the erasure nodes as follows.

| Connect | Metric |
|---------|--------|
| \(s_{j-1}, e_j\) | \(m(e_j) = m(s_{j-1}) + \max((d_{01} + 1)/2, d_{11} - t_F)\) |
| \(e_{j-1}, s_j\) | \(m(s_j) = m(e_{j-1}) + \max((d_{11} + 1)/2, d_{01} - t_B)\) |
| \(e_{j-1}, e_j\) | \(m(e_j) = m(e_{j-1}) + \left\{ \begin{array}{ll}
(d_{01} - t_a) & \text{if an edge between } s_{j-1}, s_j \\
(d_{01} + 1)/2 & \text{else}
\end{array} \right.\) |

**Theorem 3** If \([12]\) holds for \(r_j\), the Viterbi algorithm for the reduced trellis with erasure nodes finds the correct block \(c_j\).

**Proof:** The metric of the erasure nodes is always at least \(d_{01}/2\). All nodes of a state are connected with the erasure nodes of the previous and the next state. As soon as \([12]\) is fulfilled, the metric of a correct edge is better than all other edges and the ML path will be chosen. \(\blacksquare\)

### D. Complexity Analysis

The complexity is determined by the complexity of the BMD block decoders, which are all in the order \(O(n^2)\), if the construction is based on RS codes of length \(n\). Similar as Dettmar and Sorger \([10]\), we can give the following bound on the complexity. Due to space restrictions, the proof is omitted here.

**Theorem 4** Let \(C\) be a PUM code as in Definition [7] where \(G_{tot}\) is the generator matrix of an RS code. Then, the decoding complexity of Algorithm [7] of one block is upper bounded by
\[
C_{PUM} \leq O ((\ell + 1)d_m n^2) \sim O ((\ell + 1)n^3).
\]

### V. Conclusion

We presented a construction of PUM codes of arbitrary rate and provided and proved an efficient decoding algorithm. The algorithm corrects all error patterns up to half the designed extended row distance, where the complexity is cubic in the length of a block. For \(\ell = 0\), the Dettmar–Sorger algorithm \([10]\) is a special case of Algorithm [1].

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