DOOB’S INEQUALITY FOR NON-COMMUTATIVE MARTINGALES

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Abstract. Let $1 \leq p < \infty$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence of positive elements in a non-commutative $L_p$ space and $(E_n)_{n \in \mathbb{N}}$ be an increasing sequence of conditional expectations, then

$$\left\| \sum_n E_n(x_n) \right\|_p \leq c_p \left\| \sum_n x_n \right\|_p.$$ 

This inequality is due to Burkholder, Davis and Gundy in the commutative case. By duality, we obtain a version of Doob’s maximal inequality for $1 < p \leq \infty$.

Introduction:

Inspired by quantum mechanics and probability, non-commutative probability has become an independent field of mathematical research. We refer to P.A. Meyer’s exposition \[Me\], the successive conferences on quantum probability \[AvW\], the lecture notes by Jajte \[Ja1, Ja2\] on almost sure and uniform convergence and finally the work of Voiculescu, Dykema, Nica \[VDN\] and of Biane, Speicher \[BS\] concerning the recent progress in free probability and free Brownian motion. Doob’s inequality is a classical tool in probability and analysis. Transferring classical inequalities into the non-commutative setting theory often requires an additional insight. Pisier, Xu \[PX, Ps3\] use functional analytic and combinatorial methods to establish the non-commutative versions of the Burkholder-Gundy square function inequality. The absence of stopping time arguments, at least until the time of this writing, imposes one of the main difficulties in this recent branch of martingale theory.

The formulation of Doob’s inequality for non-commutative martingales faces the following problem. For an increasing sequence of conditional expectations $(E_n)_{n \in \mathbb{N}}$ and a positive operator $x$ in $L_p$, there is no reason why sup$_n E_n(x)$ or sup$_n |E_n(x)|$ should be an element in $L_p$ or represent a (possibly unbounded) operator at all. Using Pisier’s non-commutative vector-valued $L_p$-space $L_p(N; \ell_\infty)$ we can overcome this problem and at least guess the right formulation of Doob’s inequality. However, Pisier’s definition is restricted to von Neumann algebras with a $\sigma$-weakly dense net of finite dimensional subalgebras, so-called hyperfinite

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von Neumann algebras. But maximal inequalities are also interesting for free stochastic processes where the underlying von Neumann algebra is genuinely not hyperfinite. All these obstacles disappear for the so-called dual version of Doob’s inequality: For every sequence \((x_n)_{n \in \mathbb{N}}\) of positive operators

\[
\left\| \sum_n E_n(x_n) \right\|_p \leq c_p \left\| \sum_n x_n \right\|_p .
\]

\((DD_p)\)

In the commutative case this inequality is due to Burkholder, Davis and Gundy \([\text{BDG}]\) (even in the more general setting of Orlicz norms). Since it is crucial to understand our approach to Doob’s inequality, let us indicate the duality argument relating \((DD_p)\) and Doob’s inequality in the commutative case. Indeed, \((DD_p)\) implies that \(T(x_n) = \sum_n E_n(x_n)\) defines a continuous linear map between \(L_p(\ell_1)\) and \(L_p\). The norm of \(\|T^*\|\) yields the best constant in Doob’s inequality for the conjugate index \(p' = \frac{p}{p-1}\).

\[
\left\| \sup_n |E_n(x)| \right\|_{p'} = \|T^*(x)\|_{L_{p'}(\ell_\infty)} \leq \|T\| \|x\|_{p'} = c_p \|x\|_{p'} .
\]

Personally, I learned this argument after reading Dilworth’s paper \([\text{Dj}]\). But I am sure it is known to experts in the field, see Garcia’s \([\text{Ga}]\) for the general theory (and \([\text{AMS}]\) for the explicit equivalence). \((DD_p)\) admits an entirely elementary proof in the commutative case (see again \([\text{AMS}]\)). This elementary proof still works in the non-commutative case for \(p = 2\), see Lemma \([3.1]\). It is the starting point of our investigation. We recommend the reader (not familiar with modular theory) to start in section 3 where interpolation is used to extend \((DD_p)\) to \(1 \leq p \leq 2\) and suitable norms are introduced to make the above duality argument work in the non-commutative case. In section 4, we establish the dual version \((DD_p)\) in the range \(2 \leq p < \infty\) using duality arguments which rely on Pisier/Xu’s version of Stein’s inequality in combination with techniques from Hilbert \(C^*\)-modules. By duality, we obtain the non-commutative Doob inequality in the more delicate range \(1 < p \leq 2\). The heart of our arguments rely on the (apparently new) connection between Hilbert \(C^*\)-modules and non-commutative \(L_p\) spaces presented in section 2. These duality techniques are necessary because \(p > 1\) and \(0 \leq a \leq b\) implies \(0 \leq a^p \leq b^p\) only for commuting operators. This is very often used in the ‘elementary’ approach to commutative martingales inequalities as in Garsia’s book \([\text{Ga}]\).

Let us formulate our main results for finite von Neumann algebras. If \(\tau : N \to \mathbb{C}\) is a normal, tracial state, i.e. \(\tau(xy) = \tau(yx)\), then the space \(L_p(N, \tau)\) is defined by the completion of \(N\) with respect to the norm

\[
\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}} .
\]
We refer to the first section for more precise definitions and references. Given a subalgebra $M \subset N$, the embedding $\iota : L_1(M, \tau) \subset L_1(N, \tau)$ is isometric because $|x| = \sqrt{x^*x} \in M$ for all $x \in M$. The dual map $E = \iota^* : N \to M$ yields a conditional expectation satisfying

$$E(axb) = aE(x)b$$

for all $a, b \in M$ and $x \in N$, see [TR, Theorem 3.4.]. Since $E$ is trace preserving, $E$ extends to a contraction $E : L_p(N, \tau) \to L_p(M, \tau)$ with range $L_p(M, \tau)$. In the following, we consider an increasing sequence $(N_n)_{n \in \mathbb{N}}$ of von Neumann subalgebras with conditional expectations $(E_n)_{n \in \mathbb{N}}$. We recall that an element $x$ is positive if it is of the form $x = y^*y$.

**Theorem 0.1.** Let $1 \leq p < \infty$, then there exists a constant $c_p$ depending only on $p$ such that for every sequence of positive elements $(x_n)_{n \in \mathbb{N}} \subset L_p(N, \tau)$

$$\left\| \sum_n E_n(x_n) \right\|_p \leq c_p \left\| \sum_n x_n \right\|_p .$$

(DD$_p$)

Note the close relation to the non-commutative Stein inequality, see [PX, Theorem 2.3.],

$$\left\| \sum_n E_n(x_n)^*E_n(x_n) \right\|_p \leq \gamma^2 \sum_n x_n^*x_n \right\|_p .$$

Using Kadison’s inequality $E_n(x_n)^*E_n(x_n) \leq E_n(x_n^*x_n)$, it turns out that (DD$_p$) is stronger than Stein’s inequality. However, Stein’s inequality combined with the theory of Hilbert $C^*$-modules yields one of the fundamental inequalities in the proof of Theorem 0.1. Using a Hahn-Banach separation argument à la Grothendieck-Pietsch, we deduce Doob’s maximal inequality.

**Theorem 0.2.** [Doob’s maximal inequality] Let $1 < p \leq \infty$ and $x \in L_p(N, \tau)$, then there exist $a, b \in L_{2p}(N, \tau)$ and a sequence of contractions $(y_n) \subset N$ such that

$$E_n(x) = ay_nb \quad \text{and} \quad \|a\|_{2p} \|b\|_{2p} \leq c_{p'} \|x\|_p$$

Here $c_{p'}$ is the constant in Theorem 0.1 for the conjugate index $p' = \frac{p}{p-1}$. In particular, for every positive $x \in L_p(N, \tau)$, there exists a positive $b \in L_p(N, \tau)$ such that

$$E_n(x) \leq b$$

for all $n \in \mathbb{N}$.

In the case of hyperfinite von Neumann algebras this is equivalent to the corresponding vector-valued inequality, see [PS2], and therefore justifies the name ‘maximal inequality’. 
In the semi-commutative case where \( N_n = L_\infty(\Omega, \Sigma_n, \mu) \otimes M \) and \( \Sigma_n \) are increasing \( \sigma \)-algebras this inequality is stronger than the vector-valued Doob inequality, see Remark 3.3. In particular, this applies for random matrices. The inequality can be extended to a continuous index set under suitable density assumptions, for examples for Clifford martingales or free stochastical processes. Since these modifications are rather obvious, we omit the details.

Clearly, maximal inequalities immediately imply almost sure convergence. Therefore it is not surprising that Theorem 0.2 implies almost uniform convergence convergence of the martingale truncations \( (E_n(x)) \) for \( p \geq 2 \) and bilateral convergence for \( 1 < p \leq 2 \) in case of a tracial state. We refer to [Ja1, Ja2] for the definition of these notions and more details. In the tracial case the bilateral convergence of the martingale truncations is known by a result of Cuculescu [Cu] even for martingales in \( L_1(N, \tau) \). Therefore Theorem 0.2 provides a alternative approach to these results but only for \( p > 1 \). However, the maximal inequality discussed in [Ja1] cannot easily be interpolated to obtain Theorem 0.2 as in the real case. In a subsequent paper [DJ2], we will apply the maximal inequality of Theorem 0.2 in Haagerup \( L_p \) spaces in order to obtain (bilateral) almost sure convergence for all states thus underlining the strength of these maximal inequalities.

Preliminary results and notation are contained in section 1. Section 5 contains immediate applications to submartingales and conditional expectations associated to actions of groups.

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1. Notation and preliminary results

As a shortcut, we use \( (x_n), (x_{nk}) \) instead of \( (x_n)_{n \in \mathbb{N}}, (x_{nk})_{n, k \in \mathbb{N}} \) for sequences indexed by the natural numbers \( \mathbb{N} \) or its cartesian product \( \mathbb{N}^2 \). We use standard notation in operator algebras, as in [LR, KR]. In particular, \( B(H) \) denotes the algebra of bounded operators on a Hilbert space \( H \) and \( \mathcal{K}(H) \), \( \mathcal{K} \) denote the subalgebra of compact operators on \( H \), \( \ell_2 \), respectively. The letters \( N, M \) will be used for von Neumann algebras, i.e. subalgebras of some \( B(H) \) which are closed with respect to the \( \sigma \)-weak operator topology. We refer to [DX, LR] for the different locally convex topologies relevant to operator algebras. For
Let us denote by $M_n(N)$ the von Neumann algebra of $n \times n$ matrices with values in $N$. We will briefly use $M_n$ for $M_n(\mathbb{C})$. Given $C^*$-algebras $A$, $B$, we denote by $A \otimes B$ the minimal tensor product. For von Neumann algebras $N \subset B(H_1)$, $M \subset B(H_2)$, we use $N \bar{\otimes} M$ for the closure of $N \otimes M \subset B(H_1 \otimes H_2)$ in the $\sigma$-weak operator topology. Let us recall that a von Neumann algebra is semifinite if there exists a normal, semifinite faithful trace. A trace is a positive homogeneous, additive function on $N_+ = \{x^*x \mid x \in N\}$, the cone of positive elements of $N$, such that for all increasing nets $(x_i)_i$ with supremum in $N$ and for all $x \in N$

1) $\tau(\sup_i x_i) = \sup_i \tau(x_i)$.
2) For every $0 < x$ there exists $0 < y < x$ such that $0 < \tau(y) < \infty$.
3) $\tau(x) = 0$ implies $x = 0$.
4) For all unitaries $u \in N$: $\tau(uxu^*) = \tau(x)$.

A positive homogeneous, additive function $w : N_+ \to [0, \infty]$ satisfying 1), 2), 3), but not the last property 4), is called a n.s.f. (normal, semifinite, faithful) weight.

It will be worthwhile to clarify the different notions of non-commutative $L_p$-spaces. If $\tau$ is a trace then

$$m(\tau) = \left\{ \sum_{i=1}^n y_i x_i \mid n \in \mathbb{N}, \sum_{i=1}^n [\tau(y_i^* y_i) + \tau(x_i^* x_i)] < \infty \right\}$$

is the definition ideal on which there exists a unique linear extension $\tau : m(\tau) \to \mathbb{C}$ satisfying $\tau(xy) = \tau(yx)$. The $L_p$-(quasi)-norm is defined for $x \in m(\tau)$ by

$$\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}.$$

Then $L_p(N, \tau)$ is the completion of $m(\tau)$ with respect to the $L_p$-norm. (For $p < 1$ smaller ideal is needed in order to guarantee that $\tau(|x|^p)$ is finite.) We refer to [Ne, Le, Kr, Ye] for more on this and the fact that $L_p(N, \tau)$ can be realized as unbounded operators affiliated to $N$.

The starting point of Kosaki’s [Ko] definition of an $L_p$-space is a normal faithful state $\phi$ on a von Neumann algebra $N$. Then $N$ acts on the Hilbert space $L_2(N, \phi)$ obtained by completing $N$ with respect to the norm

$$\|x\|_{L_2(N, \phi)} = \phi(x^*x)^{\frac{1}{2}}.$$

The modular operator $\Delta$ is an (unbounded) operator obtained from the polar decomposition $S = J\Delta^{\frac{1}{2}}$ of the antilinear operator $S(x) = x^*$ on $L_2(N, \phi)$, see [Kr, Section 9.2.]. We
denote by $\sigma_t^\phi : N \to N$ the modular automorphism group defined by $\sigma_t^\phi(x) = \Delta^{it}x\Delta^{-it}$.

Let us recall the standard notation
\[
x.\phi(y) = \phi(xy).
\]

For each $t$ there is a natural map $I_t : N \to N_*$ ($N_*$ the unique predual of $N$) defined by
\[
I_t(x) = \sigma_t(x).\phi.
\]

According to [Ko, Theorem 2.5.], there is a unique extension $I_z : N \to N_*$ such that for fixed $x$ the function $f_x : \{z \mid -1 \leq \text{Im}(z) \leq 0\} \to N_*$, $f_z(x) = I_z(x)$ is analytic and satisfies
\[
f_x(t)(y) = \phi(y\sigma_t^\phi(x)) \quad \text{and} \quad f_x(-i+t)(y) = \phi(\sigma_t^\phi(x)y).
\]

The density of the algebra of analytic elements shows that for $0 \leq \eta \leq 1$ the map $I_{-i\eta}$ is injective. By complex interpolation, the Banach space
\[
L_p(N, \phi, \eta) = [I_{-i\eta}(N), N_*]_p
\]

is defined by specifying \(\|x\|_0 = \|I_{-i\eta}^{-1}(x)\|_N\) and \(\|x\|_1 = \|x\|_{N_*}\). We refer to [Le, Fi] for further information.

Haagerup’s abstract $L_p$ space [Ha1, Te] is defined for every von Neumann algebra $N$ using the crossed product $N \rtimes_{\sigma^w} \mathbb{R}$ with respect to the modular automorphism group of a n.s.f. weight $w$. (In our applications, we can assume $w = \phi$ for a n.f. state.) If $N$ acts faithfully on a Hilbert space $H$, then the crossed product $N \rtimes_{\sigma^w} \mathbb{R}$ is defined as the von Neumann algebra defined on $L_2(\mathbb{R}, H)$ and generated by
\[
\pi(x)(\xi(t)) = \sigma_{-it}^w(\xi(t)) \quad \text{and} \quad \lambda(s)\xi(t) = \xi(t-s).
\]

Then $N \rtimes_{\sigma^w} \mathbb{R}$, see [PTA], is semifinite and admits a unique trace $\tau$ such that the dual action
\[
\theta_s(x) = W(s)xW(s)^*
\]
satisfies $\tau(\theta_s(x)) = e^{-s}\tau(x)$. Here $W(s)$ is defined by the phase shift
\[
W(s)\xi(t) = e^{-ists}\xi(t).
\]

The dual action satisfies $\theta_s(\pi(x)) = \pi(x)$ and moreover
\[
(1.1) \quad \pi(N) = \{x \in N \rtimes_{\sigma^w} \mathbb{R} \mid \theta_s(x) = x, \text{ for all } s \in \mathbb{R}\}
\]
Let us agree to identify $N$ with $\pi(N)$ in the following. $L_p(N)$ is defined to be the space of unbounded, $\tau$-measurable operators affiliated to $N \rtimes_{\sigma^w} \mathbb{R}$ such that for all $s \in \mathbb{R}$

$$\theta_s(x) = e^{-\frac{s}{p}} x.$$ 

Note that the intersection $L_p(N) \cap L_q(N)$ is $\{0\}$ for different values $p \neq q$. There is a natural isomorphism between $N_*$ and $L_1(N)$ such that for every normal functional $\phi \in N_*$

$$\tau(a_\phi x) = \phi\left(\int_\mathbb{R} \theta_s(x)\right)$$

for all positive $x \in N \rtimes_{\sigma^w} \mathbb{R}$. The key point in this construction is the definition of the trace function $tr : L_1(N) \to \mathbb{C}$ (corresponding to the integral in the commutative case) given by

$$tr(a_\phi) = \phi(1).$$

Let $\frac{1}{p} + \frac{1}{p'} = 1$ and $x \in L_p(N), y \in L_{p'}(N)$. Then we have the trace property

$$tr(xy) = tr(yx).$$

The polar decomposition $x = u|x|$ of $x \in L_p(N)$ satisfies $u \in N$ and

$$\|x\|_p = tr(|x|^p)^{\frac{1}{p'}}.$$ 

$N$ acts as a left and right module on $L_p(N)$ and more generally Hölder’s inequality

$$\|xy\|_r \leq \|x\|_p \|y\|_q$$

holds whenever $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. As for semifinite von Neumann algebras, there is a positive cone $L_p(N)_+$ in $L_p(N)$ consisting of elements in $L_p(N)$ which are positive as unbounded operators affiliated to $N \rtimes_{\sigma^w} \mathbb{R}$. Following [13, Proposition 33, Theorem 32], we deduce for $0 \leq x \leq y \in L_p(N)$ and $\frac{1}{p} + \frac{1}{p'} = 1$

$$\|x\|_p = \sup_{z \in L_{p'}(N)_+, \|z\|_{p'} \leq 1} tr(zz�x) \leq \sup_{z \in L_{p'}(N)_+, \|z\|_{p'} \leq 1} tr(zy) = \|y\|_p.$$ 

In the sequel, we will often use the following simple observation.

**Lemma 1.1.** Let $0 < p \leq \infty$ and $x, y \in L_p(N)$ such that $x^* x \leq y^* y$. If $p$ is the left support projection of $y$, then $xy^{-1}p$ is a well-defined element in $N$ of norm less than one.

**Proof:** We note that $a = xy^{-1}p$ is affiliated with $N \rtimes_{\sigma^w} \mathbb{R}$ and

$$p(y^{-1})^* x^* xy^{-1}p \preceq p(y^{-1})^* y^* yy^{-1}p \preceq p$$
shows that $a$ is a contraction and in particular $\tau$-measurable. Moreover, we have
\[ p = \theta_s(p) = \theta_s(yy^{-1}p) = e^{-\frac{t}{n}}y\theta_s(y^{-1}p) \]
and hence
\[ (1.4) \quad \theta_s(y^{-1}p) = e^{\frac{t}{n}}y^{-1}p. \]
Therefore, the equality
\[ \theta_s(a) = \theta_s(x)\theta_s(y^{-1}p) = e^{-\frac{t}{n}}xe^{\frac{t}{n}}y^{-1}p = a \]
shows with (1.1) that $a$ is a contraction in $N$. \qed

In the $\sigma$-finite case, Kosaki’s $L_p$-space is isomorphic to Haagerup’s $L_p$ space, see [Ko, section 8]. Indeed, given a n. f. state $\phi$ with corresponding density $D$ in $L_1(N) \cong N_*$, then
\[ \sigma^\phi_t(x) = D^itxD^{-it} \]
and for all $x \in N$
\[ \|I_{-in}(x)\|_{[I_{-in}(N), N_*]} \sim \|D^txD^{1-i\theta}\|_{L_p(N)}. \]
We recall that an element $N$ is analytic, if $t \mapsto \sigma^\phi_t(x)$ extends to an analytic function with values in $N$. The *-closed subalgebra of analytic elements will be denoted by $A$. The following Lemma is probably well-known, see [Ko], [JX, Lemma 1.1]. We add a short proof for the convenience of the reader.

**Lemma 1.2.** Let $0 < p < \infty$, then $D^{\frac{1}{p}}A_+D^{\frac{1}{p}}$ is dense in $L_p(N)_+$ and $ND^{\frac{1}{p}}$ is dense in $L_p(N)$ and for $1 \leq p \leq \infty$ the map $J_p : L_p(N) \to L_1(N)$, $J_p(x) = xD^{1-\frac{1}{p}}$ is injective.

**Proof:** $D$ is a $\tau$-measurable operator with support projection 1. Therefore, $xD^{1-\frac{1}{p}} = 0$ implies that $x = xD^{1-\frac{1}{p}}D^{\frac{1}{p}-1}$ is a well-defined $\tau$-measurable operator and equals 0. Hence, $J_p$ is injective for $1 \leq p < \infty$. Let $\frac{1}{p} + \frac{1}{p'} = 1$. We show the density of $D^{\frac{1}{p}}A_+D^{\frac{1}{p}}$ in $L_p(N)_+$ for $1 \leq p < \infty$. If this is not the case, the Hahn-Banach theorem implies the existence of $x \in L_p(N)_+$ and $y \in L_{p'}(N)_{sa}$ such that $tr(yD^{\frac{1}{p}}aD^{\frac{1}{p'}}) = tr(D^{\frac{1}{p'}}yD^{\frac{1}{p}}a) \leq 0$ for all $a \in A_+$ and $tr(xy) > 0$. Using the $\sigma$-strong density of $A_+$ in $N_+$, see [PTA], we deduce that $y = D^{\frac{1}{p'}}D^{\frac{1}{p}}yD^{\frac{1}{p}}D^{-\frac{1}{p}}$ is negative and hence $tr(xy) \leq 0$, a contradiction. Let $\frac{1}{2} \leq p \leq 1$ and $y \in L_p(N)_+$, then we can approximate $y^{\frac{1}{p}}$ by an element $x = D^{\frac{1}{p'}}aD^{\frac{1}{p}}$, $a \in A_+$ and hence Hölder’s inequality implies that $x^2 - y = x(x - y^{\frac{1}{p}}) + (x - y^{\frac{1}{p}})y^{\frac{1}{p}}$ approximates $y$. Since $a$ is analytic, we observe that $x^2 = D^{\frac{1}{p'}}aD^{\frac{1}{p}}D^{\frac{1}{p}}aD^{\frac{1}{p}} = D^{\frac{1}{p'}}\sigma_\frac{1}{p'}(a)^*\sigma_\frac{1}{p'}(a)D^{\frac{1}{p}}$
is in $D_{2p}^\frac{1}{2}A_+D_{2p}^\frac{1}{2}$. By induction, we deduce the density of $D_{2p}^\frac{1}{2}A_+D_{2p}^\frac{1}{2}$ in $L_p(N)_+$ for all $0 < p < \infty$. Since $AD_{2p}^\frac{1}{2} = D_{2p}^\frac{1}{2}AD_{2p}^\frac{1}{2}$ and every element is a linear combination of 4 positive elements, we obtain the assertion.

2. Hilbert $C^*$-modules and $L_p$-spaces

In this section, we will analyze some Banach spaces related to the dual version of Doob’s inequality and combine Kasparov’s stabilisation theorem for Hilbert $C^*$-modules with Stein’s inequality for non-commutative martingales proved by Pisier and Xu [PX]. Let $N \subset M$ be a von Neumann subalgebra and $E : M \to M$ be a normal conditional expectation onto $N$. $E$ is normal if the dual map $E^*$ satisfies $E^*(M_*) \subset M_*$ and hence has a predual map $E_* : M_* \to M_*$ such that $E = (E_*)^*$. In order to simplify the exposition, we will assume in addition that $\phi : M \to \mathbb{C}$ is a normal faithful state satisfying $\phi = \phi|_N \circ E$. Using Kosaki’s interpolation spaces [Ko, Proposition 4.1], it is very easy to check that $E$ extends to certain $L_p$-spaces. However, in our context it is more convenient to work with the Haagerup $L_p$-spaces.

**Lemma 2.1.** Under the previous assumptions let $D$ be the density of $\phi$ in $L_1(M)$ and $\sigma_t$ be the modular group, then $E_*(D) = D$, and for all $x \in M$, $y \in M_*$

$$E(\sigma_t(x)) = \sigma_t(E(x)) \quad \text{and} \quad tr(E_*(y)) = tr(y).$$

**Proof:** The first assertion follows from

$$tr(Dx) = \phi(x) = \phi(E(x)) = tr(DE(x)) = tr(E_*(D)x)$$

which is valid for all $x \in M$. For $E(\sigma_t(x)) = \sigma_t(E(x))$ see [C, Lemme 1.4.3]. In order to prove the third assertion let $\psi \in M_*$ and $a_\psi$ be the corresponding density then

$$tr(E_*(a_\psi)) = \psi \circ E(1) = \psi(E(1)) = \psi(1) = tr(a_\psi).$$

We will need several approximation results.

**Lemma 2.2.** Let $A$ be a $^*$-closed (not necessarily norm closed) but $\sigma$-strongly dense subalgebra of $M$. For $0 < p < \infty$ the space $AD_{2p}^\frac{1}{2}$ is norm dense in $L_p(N)$.
**Proof:** According to Lemma 1.2, $MD^\frac{1}{p}$ is norm dense in $L_p(M)$. According to Kaplansky’s density theorem [11], Theorem II.4.8], every element in the unit ball of $M$ is the strong*-limit of elements in the unit ball of $A$. An application of the following Lemma 2.3 yields the assertion.

**Lemma 2.3.** Let $0 < p < \infty$, $M$ be a von Neumann algebra, $a \in L_1(M)_+$, $(x_\alpha)_\alpha \subset N$ be a bounded net and $x \in N$. If $x_\alpha a^\frac{1}{p}$ converges to $xa^\frac{1}{p}$ in norm for some $1 \leq p < \infty$, then this is true for all $0 < p < \infty$. In particular, if $(x_\alpha)_\alpha \subset N$ converges to $x$ strongly, then for all $b \in L_p(N)$, $x_\alpha b$ converges to $xb$ in norm.

**Proof:** We consider the set $I = \{ 0 < p < \infty \mid \lim_\alpha x_\alpha a^\frac{1}{p} = xa^\frac{1}{p} \}$. Here, we refer to convergence in norm. Let us first observe that $0 < q < p \in I$ implies $q \in I$. Indeed, according to Hölder’s inequality (1.2), the linear map $M_r : L_p(N) \to L_q(N)$ defined by $M_r(x) = xa^\frac{1}{q} - x$ is continuous and therefore, the convergence of $x_\alpha a^\frac{1}{p}$ implies the convergence of $x_\alpha a^\frac{1}{p} = M_r(x_\alpha a^\frac{1}{p})$. Let us now show that $1 \leq p \in I$ implies $2p \in I$, i.e.

$$\| (x_\alpha - x)a^\frac{1}{2p} \|_{2p} = 0.$$ We note that

$$\| (x_\alpha - x)a^\frac{1}{2p} \|^2_{2p} = \| a^\frac{1}{2p}(x_\alpha^* - x^*)(x_\alpha - x)a^\frac{1}{2p} \|_p$$

and define the analytic function $f : \{ 0 \leq Im(z) \leq 1 \} \to L_p(N)$ given by

$$f_\alpha(z) = \exp((z - \frac{1}{2})^2) a^\frac{1}{p}(x_\alpha^* - x^*)(x_\alpha - x)a^\frac{1}{p}.$$ Since $a^{it}xa^{-it} \in L_p(N)$ for every $x \in L_p(N)$ this is well defined and analytic in the interior. For $z = 1 + it$, we get

$$\| f_\alpha(1 + it) \|_p \leq \exp(\frac{1}{4}) \left\| a^\frac{1}{2p}(x_\alpha^* - x^*)(x_\alpha - x)a^\frac{1}{p} \right\|_p$$

$$\leq \exp(\frac{1}{4}) \left\| a^\frac{1}{p}(x_\alpha^* - x^*) \right\|_p \| x_\alpha - x \|_\infty$$

$$\leq \exp(\frac{1}{4}) \left\| (x_\alpha - x)a^\frac{1}{p} \right\|_p \sup_\alpha \| x_\alpha - x \|_\infty.$$ Similarly,

$$\| f_\alpha(it) \|_p \leq \exp(\frac{1}{4}) \left\| a^\frac{1}{p}(x_\alpha^* - x^*)(x_\alpha - x)a^\frac{1}{p} \right\|_p$$

$$\leq \exp(\frac{1}{4}) \sup_\alpha \| x_\alpha^* - x^* \|_\infty \left\| (x_\alpha - x)a^\frac{1}{p} \right\|_p.$$
Therefore, \( p \in I \) implies with the three line Lemma

\[
\lim_{\alpha} \left\| a^{\frac{1}{p}}(x_{\alpha} - x^*)(x_{\alpha} - x)a^{\frac{1}{p}} \right\|_p = \lim_{\alpha} \left\| f_{\alpha}(\frac{1}{2}) \right\|_p \\
\leq \lim_{\alpha} \sup_t \max \{ \left\| f_{\alpha}(it) \right\|_p, \left\| f_{\alpha}(1 + it) \right\|_p \} \\
\leq \exp(\frac{1}{4}) \sup_{\alpha} \| x_{\alpha} - x \|_\infty \lim_{\alpha} \left\| (x_{\alpha} - x)a^{\frac{1}{p}} \right\|_p = 0.
\]

Hence, if \( I \cap [1, \infty) \) is non empty, then \( I = (0, \infty) \) and the first assertion is proved. Let \( b \in L_p(N) \). We use the polar decomposition of \( b^* \) to find a partial isometry \( u \) such that \( b = |b^*|u \). Then \( a = |b^*|^p \) is in \( L_1(N)_+ \). If \( x_{\alpha} \) converges to \( x \) strongly then \( x_{\alpha}a^{\frac{1}{2}} \) converges in norm to \( xa^{\frac{1}{2}} \). This means \( 2 \in I \) and hence \( I = (0, \infty) \). In particular, \( p \in I \) and hence \( x_{\alpha}|b^*| = x_{\alpha}a^{\frac{1}{2}} \) converges to \( x|b^*| \). This immediately implies that \( x_{\alpha}b = x_{\alpha}|b^*|u \) converges to \( xb = x|b^*|u \).

In this paper, we will frequently use the space \( L_p(M; \ell^2) \subset L_p(B(\ell_2) \otimes M) \) of column matrices with values in \( L_p(M) \).

**Corollary 2.4.** Let \( 0 < p < \infty \) and \( A \) \( \sigma \)-strongly dense \( ^\ast \)-closed subalgebra of \( M \), then the space of finite sequences \((a_nD_x^\theta)\) such that \( a_n \in A \) is dense in \( L_p(M; \ell^2) \).

**Proof:** Let \( p_n \) be the orthogonal projection onto the first \( n \) unit vectors in \( \ell_2 \), then \( p_n \otimes 1 \) converges to \( 1 \otimes 1 \) in the strong\(^*\) topology. Hence, the space of finite sequences is dense in \( L_p(M; \ell^2) \). We can apply Lemma 2.2 for each coordinate to obtain the assertion.

In the next step we refer to [JX, Proposition 2.3] to understand how a state preserving conditional expectation extends to a contraction \( E_p \) on \( L_p(M) \) and in fact \( E_s = E_1 \). We will later often use these facts without further reference and drop the index \( p \) or \( s \) because it is easily determined by the context.

**Proposition 2.5.** Let \( 1 \leq p \leq \infty \), then there is a contraction \( E_p : L_p(M) \rightarrow L_p(M) \) such that for all \( x \in M \) and \( 0 \leq \theta \leq 1 \)

\[
E_p(D^{\frac{1}{p}}(D_x^{\theta})) = D^{\frac{1}{p}}(E(x)D_x^{\theta})
\]

\( E_p \) satisfies \( E_p(x^*) = E_p(x) \) and \( E_p(x) \) is positive for all positive \( x \). Moreover, if \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1 \), \( a \in L_p(N) \), \( b \in L_q(N) \) and \( x \in L_r(M) \), then

\[
E_s(axb) = aE_r(x)b.
\]
Conditional expectations (or, more generally, positive operator-valued weights) are closely connected to Hilbert $C^*$-modules. An excellent reference for the few facts and the notation we need in this paper is Lance’s book [La]. We recall that a Hilbert $C^*$-module $M$ over a $C^*$-algebra $A$ is a right $A$-module $M$ together with an $A$-valued sesquilinear form $\langle \cdot, \cdot \rangle : M \times M \to A$ such that for all $a \in A$ and $x, y \in M$

$$
\langle x, ya \rangle = \langle x, y \rangle a ,
$$
$$
\langle x, y \rangle^* = \langle y, x \rangle .
$$

The norm in $M$ is given by $\|x\|_M = \|\langle x, x \rangle\|_A^{\frac{1}{2}}$. We should note that the sesquilinear form is assumed to be linear in its second and antilinear in its first component. The standard example of a Hilbert $C^*$-module over a $C^*$-algebra $A$ is the space $H_A$, the closure of finite sequences $(x_n) \subset A$ with respect to the norm

$$
\| (x_n) \| = \left\| \sum_n x_n^* x_n \right\|_A^{\frac{1}{2}} .
$$

We can identify $H_A$ with the space of column matrices $H_A \subset \mathcal{K} \otimes A$. In the context of von Neumann algebras $A = N$, we derive a similar example using the space of column matrices $C(N) \subset B(\ell_2) \otimes N$ which consists of sequences $(x_n)$ such that $\sum_n x_n^* x_n$ converges in the $\sigma$-weak operator topology. Then the sesquilinear form

$$
\langle (x_n), (y_n) \rangle = (x_1^*, x_2^*, \cdots) (y_1, y_2, \cdots)^\perp = \sum_n x_n^* y_n
$$

converges in the $\sigma$-weak operator topology and satisfies the axioms of a Hilbert $C^*$-module. Let us indicate how this concept applies to non-commutative $L_p$-spaces. Indeed, $Ses : L_p(N; \ell_2^C) \times L_q(N; \ell_2^C) \to L_r(N)$ defined by

$$
Ses((x_n), (y_n)) = \sum_n x_n^* y_n , \quad \text{where} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}
$$

is still a sesquilinear form. We will omit $Ses$ and simply write $(x_n)^*(y_n)$ to remind the natural matrix multiplication. Hölder’s inequality immediately implies

$$
\left( \sum_n x_n^* y_n \right)_r \leq \left( \sum_n x_n^* x_n \right)^{\frac{1}{2}}_p \left( \sum_n y_n^* y_n \right)^{\frac{1}{2}}_q \quad (2.1)
$$

whenever $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Our aim is to extend this type of inequalities to the sesquilinear form

$$
\langle x, y \rangle = E(x^* y)
$$
given by a conditional expectation \( E : M \to N \). We denote by \( L_\infty(M, E) \) the completion of \( M \) with respect to the norm
\[
\|x\|_{L_\infty(M, E)} = \|E(x^*x)\|_{\infty}^{\frac{1}{2}}.
\]
In analogy with the case \( p = \infty \), we introduce the following notation:

**Definition 2.6.** Let \( 0 \leq \frac{1}{p} \leq \infty \) and \( x = aD_1^\frac{1}{p} \in MD_1^\frac{1}{p} \), then
\[
\|x\|_{L_p(M, E)} = \left\|D_1^\frac{1}{p}E(a^*a)D_1^\frac{1}{p}\right\|_{\frac{1}{p}}^{\frac{1}{2}}.
\]
The completion with respect to this 'norm' is denoted by \( L_p(M, E) \).

**Remark 2.7.** Let us note that for \( p \geq 2 \), the precaution \( x \in MD_1^\frac{1}{p} \) is unnecessary because for all \( x \in L_p(N) \) the conditional expectation \( E(x^*x) \) is well-defined and
\[
\|x\|_{L_p(N, E)} = \left\|E(x^*x)\right\|_{\frac{1}{p}}^{\frac{1}{2}} \leq \left\|x^*x\right\|_{\frac{1}{p}}^{\frac{1}{2}} = \|x\|_p .
\]
By norm density of \( MD_1^\frac{1}{p} \) in \( L_p(M) \), we obtain the same closure. However, for \( p < 2 \) we no longer dispose of the continuity of \( E \) on \( L_\infty(N) \) (even in the commutative case). This justifies our slightly artificial definition.

Our next step will be to prove the triangle inequality for \( p \geq 1 \) using Kasparov’s stabilisation result.

**Proposition 2.8.** Let \( N \subset M \), \( E \) and \( \phi \) as above and assume in addition that \( M_\phi \) is separable. For all \( 0 < p \leq \infty \), there exists an isometric right \( N \)-module map \( u_p : L_p(M, E) \to L_p(N; \ell_2^C) \). Moreover,

i) For all \( x \in L_p(M, E) \) and \( y \in L_q(M, E) \)
\[
u_p(x)^*u_q(y) = E(x^*y).
\]

ii) Let \( \frac{1}{r} = \frac{1}{q} + \frac{1}{p} \). For all \( x \in L_p(M, E) \) and \( y \in L_q(M, E) \)
\[
\left\|E(x^*y)\right\|_r \leq \|x\|_{L_p(M, E)} \|y\|_{L_q(M, E)} .
\]

iii) If \( 0 < p < \infty \) there exists a contractive projection \( Q_p \) onto the image of \( u_p \) such that for all \( z \in L_p(N; \ell_2^C) \)
\[
Q_p(z)^*Q_p(z) \leq z^*z .
\]
iv) If $1 < p, p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then
\[ Q_p' = Q_{p'} . \]

In particular, $L_p(M, E)$ is a normed space for $1 \leq p < \infty$ and $p$-normed for $0 < p \leq 1$.

**Proof:** Let $X \subset M$ be a σ-strongly dense separable $C^*$-algebra and let $F$ be the Hilbert $C^*$-module generated by $N$ and $X$. Then $F$ is a countably generated Hilbert $N$-module and according to [4, Theorem 6.2.] there exists a unitary $w : F \oplus H_N \to H_N$ such that
\[ w(f, x)^* w(g, y) = \langle f, g \rangle + x^* y . \]

Let $\tilde{Q} : F \oplus H_N \to F$ be the projection onto the first coordinate. This projection carries over to $Q = w\tilde{Q}w^{-1} : H_N \to w(F)$. We define $u : F \to H_N$ to be the restriction of $w$ to the first component. Clearly, $u$ preserves the $N$-valued sesquilinear form. For $0 < p \leq \infty$, we define the map $u_p : FD_1^\frac{1}{p} \to L_p(N; \ell_2^p)$ by
\[ u_p(aD_1^\frac{1}{p}) = u(a)D_1^\frac{1}{p} . \]

Then for all $a, b \in F$
\[ D_1^\frac{1}{p} E(a^* b) D_1^\frac{1}{p} = D_1^\frac{1}{p} u(a)^* u(b) D_1^\frac{1}{p} = u_p(aD_1^\frac{1}{p})^* u_q(bD_1^\frac{1}{p}) . \]

This justifies $i)$ for the subsets $FD_1^\frac{1}{p}, \; FD_2^\frac{1}{p}$, respectively. In particular, we obtain
\[ \left\| aD_1^\frac{1}{p} \right\|_{L_p(M, E)}^2 = \left\| D_1^\frac{1}{p} E(a^* a) D_1^\frac{1}{p} \right\|_{\frac{1}{p}} = \left\| u_p(aD_1^\frac{1}{p})^* u_p(aD_1^\frac{1}{p}) \right\|_{\frac{1}{p}} = \left\| u_p(xD_1^\frac{1}{p}) \right\|_p^2 , \]

which implies that $u_p$ is isometric when restricted to $FD_1^\frac{1}{p}$. According to the following Lemma 2.10 and Proposition 2.9, $FD_1^\frac{1}{p}$ is dense and hence $i)$ follows. Assertion $ii)$ is an immediate consequence of (2.1). Indeed,
\[ \left\| D_1^\frac{1}{p} E(a^* b) D_1^\frac{1}{p} \right\|_r = \left\| u_p(aD_1^\frac{1}{p})^* u_q(bD_1^\frac{1}{p}) \right\|_r \leq \left\| u_p(aD_1^\frac{1}{p}) \right\|_p \left\| u_q(bD_1^\frac{1}{p}) \right\|_q \]
\[ = \left\| aD_1^\frac{1}{p} \right\|_{L_p(M, E)} \left\| bD_1^\frac{1}{p} \right\|_{L_q(M, E)} . \]

Since $L_p(N; \ell_2^p)$ is a Banach space, respectively a complete $p$-normed space, and $u_p$ is isometric, we deduce the last assertion. The projection $Q_p : L_p(N; \ell_2^p) \to L_p(N; \ell_2^p)$ is densely defined for a sequence $(z_nD_1^\frac{1}{p})$ such that $z_n = 0$ for $n \geq n_0$ by
\[ Q_p[(z_nD_1^\frac{1}{p})] = Q[(z_n)]D_1^\frac{1}{p} . \]

(Here and in the following we will use the brackets $[\cdot]$ to indicate that $Q$ is applied to the sequence $(z_n)$.) Since $w$ is a unitary in the sense of Hilbert $N$-modules, we deduce for
In particular,

\[ w^{-1}([z_n]) = f + (a_n) \]

\[ Q([z_n])^*Q([z_n]) = \langle \tilde{Q}(f + (a_n)), \tilde{Q}(f + (a_n)) \rangle = \langle f, f \rangle + \sum_n a_n^* a_n \]

\[ = \langle f + (a_n), f + (a_n) \rangle = \langle w(f + (a_n)), w(f + (a_n)) \rangle = \sum_n z_n^* z_n. \]

In particular,

\[ \left\| Q_p([z_n D\frac{1}{n}]) \right\|_p^2 = \left\| D\frac{1}{n} Q([z_n])^*Q([z_n]) D\frac{1}{n} \right\|_{\frac{1}{n}} \leq \left\| D\frac{1}{n} \sum_n z_n^* z_n D\frac{1}{n} \right\|_{\frac{1}{n}} \]

\[ = \left\| (z_n D\frac{1}{n}) \right\|_{L_p(N; \ell_2^z)}. \]

Hence, \( Q_p \) extends by density, see Corollary 2.4, to a contraction on \( L_p(N; \ell_2^z) \). This proves \textit{iii}). In order to prove \textit{iv}) it suffices again by density to consider finite sequences \( z = (z_n D\frac{1}{n}) \) and \( \tilde{z} = (\tilde{z}_n D\frac{1}{n}) \) with \( z_n, \tilde{z}_n \in F \). Using again \( w^{-1}([z_n]) = f + (a_n), w^{-1}([\tilde{z}_n]) = \tilde{f} + (\tilde{a}_n) \), we deduce from the fact that \( w \) is a unitary module map

\[ tr(Q_p(z)^* \tilde{z}) = tr(D\frac{1}{n} Q([z_n])^*(\tilde{z}_n) D\frac{1}{n}) \]

\[ = \phi(\langle Q([z_n]), (\tilde{z}_n) \rangle) \]

\[ = \phi(\langle \tilde{Q}(f + (y_n)), \tilde{f} + (\tilde{a}_n) \rangle) \]

\[ = \phi(E(f^* \tilde{f})) \]

\[ = \phi(\langle f + (a_n), \tilde{Q}(\tilde{f} + (\tilde{a}_n)) \rangle) \]

\[ = tr(z^* Q_{p'}(\tilde{z})). \]

Let us briefly clarify how the unitary module map \( w : F \oplus H_N \rightarrow H_N \) extends to an isometry onto \( C(N) \). We recall that \( M \) acts naturally on \( L_2(M) \) and that \( p_E = E_2 : L_2(M) \rightarrow L_2(M) \) is the orthogonal projection onto \( L_2(N) \). Hence, we obtain an isometric isomorphism

\[ L_\infty(M, E) = M p_E \subset B(L_2(N), L_2(M)) \subset B(L_2(M)). \]

Note that in general \( p_E \) is not in \( M \). Via this inclusion, we dispose of all the relevant locally convex operator topologies on \( L_\infty(M, E) \) and on

\[ L_\infty(M, E) \oplus C(N) \subset B(L_2(N), L_2(M) \oplus \ell_2(L_2(N)) \).

We denote by \( L_\infty^{st}(M, E) \) the closure with respect to the \( \sigma \)-strong topology. The following example due to O. Ramcke shows that this might be different from \( L_\infty(M, E) \). Let \( M = \)
$L_\infty([0,1]^2)$ and $E(f)(t,s) = \int_0^1 f(t,r)\,dr$. Then, the norm on $L_\infty(M,E)$ is the norm in $L_\infty(L_2)$. Given disjoint sets $I_k$ of measure $2^{-k}$ the function

$$f = \sum_k 1_{I_k} \otimes 2^{\frac{k}{2}} 1_{I_k}$$

yields an element in $\sigma$-strong closure not belonging to $L_\infty(M,E)$. The next proposition shows that $L^{st}_\infty(M,E)$ is isomorphic to a complemented module in $C(N)$. In particular, we will find a sequence $(x_j) \subset L^{st}_\infty(M,E)$ and completely contractive module maps $u_j : M \to N$ such that for all $x \in M$

$$x = \sum_j x_j u_j(x)$$

For inclusions with finite index, a finite sum of this form suffices and more can be said about the coefficients $E(x_i^* x_i)$, see [PP]. In any case

$$1 = E(1) = \sum_j u_j(1)^* E(x_j^* x_j) u_j(1)$$

shows that we are dealing with a 'partition of unity'.

**Proposition 2.9.** Under the above assumptions, $w$ extends to an isometric isomorphism between the space $L^{st}_\infty(M,E) \oplus C(N)$ and $C(N)$. Moreover, for every element $x$ in the unit ball of $L_\infty(M,E)$ there exists a net $x_\alpha$ in the unit ball of $L_\infty(M,E) \cap F$ converging to $x$ in the strong$^*$ topology and $Q$ extends to a projection onto the image of $L^{st}_\infty(M,E)$.

**Proof:** We note that for $h \in L_2(N)$ and $x \in F \oplus C(N)$

$$\|w(x)(h)\|^2 = (w(x)^* w(x)(h),h) = (x^* x(h),h) = \|x(h)\|_2^2.$$  

Hence, $w$ preserves the strong and $\sigma$-strong topology. Therefore, we obtain a natural extension, also denoted by $w$, to the $\sigma$-strong closure $L^{st}_\infty(M,E) \oplus C(N)$ of $F \oplus H_N$ with values in $C(N)$. To show that $w$, $w^{-1}$ remain contractions, it suffices to show that every element in the unit ball of $L^{st}_\infty(M,E) \oplus C(N)$, $C(N)$, can be approximated with respect to the strong topology by elements in the unit ball of $F \oplus H_N$, $H_N$, respectively. This follows immediately from Kaplansky’s density theorem [TK, Theorem II.4.8] for $C(N)$. To obtain the assertion for the $\sigma$-strong closure of $Mp_E$, we note that $F$ is $\sigma$-strong$^*$ dense in $M$, hence $Fp_E$ is $\sigma$-strong$^*$ dense in $L^{st}_\infty(M,E)$. Then, we follow the proof of Kaplansky’s density theorem, see [TK, Theorem II.4.8], and note that the function $f : F \to B(L_2(N))$ defined there by

$$f(xp_E) = 2xp_E(1 + (p_E x^* xp_E))^{-1} = 2xp_E(1 + (E(x^*)^{-1}) \in FN \subset F$$
satisfies $f(F) \subset F$. Hence, the proof of Kaplansky’s density theorem applies and therefore every element in the unit ball of

$$L^*_\infty(M, E) = M_{pE}^{\sigma-\text{strong}} = M_{pE}^{\sigma-\text{strong}^*}$$

([13], Theorem 2.6) can be approximated by a net $f(x_\alpha)$ in the unit ball of $F$. Since $Q = w^{-1} \tilde{Q} w$ corresponds to the projection onto the first component in $L^*_\infty(M, E) \oplus C(N)$, the last assertion follows easily.

Lemma 2.10. Under the previous assumptions $FD^\frac{1}{p}$ is dense in $L_p(M, E)$. The Cauchy-Schwarz inequality 2.8 ii) and i) also holds for $p = \infty$ or $q = \infty$.

Proof: Let us assume $2 \leq p < \infty$ first. Using Kaplansky’s density theorem [13], Theorem II.4.8], the unit ball of $F$ is strong*-dense in the unit ball of $M$. Using Lemma 2.3, we deduce that $FD^\frac{1}{p}$ is dense in $L_p(M)$. As observed in Remark 2.7, the inclusion $L_p(M) \subset L_p(M, E)$ is contractive and dense, hence the assertion follows. For $p < 2$, we observe that the inclusion $i : L_2(M, E) \subset L_p(M, E)$, $i(aD^\frac{1}{p}) = aD^\frac{1}{p}$ is contractive using Hölder’s inequality

$$\left\| aD^\frac{1}{p} \right\|_p = \left\| D^\frac{1}{p} E(a^*a) D^\frac{1}{p} \right\|_2^{\frac{p}{2}} \leq \left\| D^\frac{1}{2} E(a^*a) D^\frac{1}{2} \right\|_1^{\frac{1}{2}} = \left\| aD^\frac{1}{2} \right\|_{L_2(M, E)}.$$

By definition of $L_p(M, E)$, the image of $i$ is dense. Hence, $FD^\frac{1}{p} = i(FD^\frac{1}{2})$ is dense in $L_p(M, E)$. To prove the Cauchy-Schwarz inequality if $p = r < \infty$ and $q = \infty$, we fix $x = bD^\frac{1}{p}$ and $y \in L^*_\infty(M, E)$ of norm less than one. Then there exists a net $a_\alpha pE$ in the unit ball of $F$ such that $a_\alpha$ converges to $y$ with respect to the strong* topology. Then the strong* convergence of $b^*a_\alpha pE$ implies the strong* convergence of $E(b^*a_\alpha) = pE b^*a_\alpha pE$. Therefore the norm convergence of $D^\frac{1}{p} E(b^*a_\alpha)$ to $D^\frac{1}{p} E(b^*y)$ follows from Lemma 2.3. In particular according to Proposition 2.13 ii),

$$\left\| D^\frac{1}{p} E(b^*y) \right\|_p = \lim_{\alpha} \left\| D^\frac{1}{p} E(b^*a_\alpha) \right\|_p \leq \lim_{\alpha} \left\| D^\frac{1}{2} E(b^*b) D^\frac{1}{2} \right\|_2^{\frac{p}{2}} \left\| E(a_\alpha^*a_\alpha) \right\|_\infty^{\frac{1}{2}} \leq \left\| bD^\frac{1}{2} \right\|_{L_p(M, E)}.$$
Thus, by density, every norm one element in $L^\infty_p(M, E)$ induces a contractive ‘multiplier’ on $L_p(M, E)$. The case $p = \infty$ and $q < \infty$ is similar. The case $p = \infty$ and $q = \infty$ is classical and follows from the fact that $E : M \to B(L_2(M))$ is completely positive and therefore admits a dilation $E(x) = v\pi(x)v^*$ for a contraction $v$ and a $^*$-representation $\pi$. Hence, we obtain
\[
\|E(x^*y)\|_{B(H)} = \|v\pi(x^*)v^*\| \leq \|v\pi(x^*)\pi(x)v^*\|^\frac{1}{2} \|v\pi(y^*)\pi(y)v^*\|^\frac{1}{2} = \|E(x^*x)\|^\frac{1}{2} \|E(y^*y)\|^\frac{1}{2}.
\]

Now we turn our attention to the duality between $L_p(N, E)$ and $L_{p'}(N, E)$. Let us point out that we use the antilinear duality bracket
\[(x, y) = tr(x^*y)\]
between $L_p(N; \ell_2^C)$ and $L_{p'}(N; \ell_2^C)$. The following Lemma is a standard application of the Hahn-Banach theorem.

**Lemma 2.11.** Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $X_p \subset L_p(M; \ell_2^C)$ and $X_{p'} \subset L_{p'}(M; \ell_2^C)$ be subspaces and $Q_p : L_p(M; \ell_2^C) \to L_p(M; \ell_2^C)$ be a projection onto $X_p$ such that $Q_p^*$ is a projection onto $X_{p'}$, then
\[X_p^* = X_{p'}.
\]
and for every dense subset $X \subset X_{p'}$ and $x \in X_p$
\[\|x\| \leq \|Q\| \sup \left\{tr(y^*x) \mid y \in X, \|y\|_{X_{p'}} < 1\right\}.
\]

**Proof:** Since $(L_p(M; \ell_2^C))^* = L_{p'}(M; \ell_2^C)$, the antilinear map $\iota : X_{p'} \to X_p^*$
\[\iota(y)(x) = tr(y^*x)
\]
is obviously contractive. Let $f : X \to \mathbb{C}$ be a norm one functional, then we can apply the Hahn-Banach extension theorem and $(L_p(M; \ell_2^C))^* = L_{p'}(M; \ell_2^C)$ to obtain $z \in L_{p'}(M; \ell_2^C)$ such that
\[f(x) = tr(z^*x) = (z, x).
\]
Clearly,
\[(z, x) = (z, Q_p(x)) = (Q_p^*(z), x) = tr(Q_p^*(z)^*x).
\]
(Defined in this way $Q_p^\ast$ is linear.) Since, $Q_p^\ast(z) \in X'_p$ we deduce that $\iota$ is surjective. The
last formula follows from the Hahn-Banach theorem, i.e.
\[
\|x\| = \sup_{\|f\| \leq 1} |f(x)| = \sup_{\|z\| \leq 1} |tr(z^\ast x)| = \sup_{\|z\| \leq 1} |tr(Q_p^\ast(z^\ast x))|
\leq \|Q_p\| \sup_{\|y\| \leq 1, y \in X'_p} |tr(y^\ast x)|
\]
The supremum is unchanged if restricted to a dense subset.

\[\textbf{Corollary 2.12.} \text{ Let } 1 < p, p' < \infty, \frac{1}{p} + \frac{1}{p'} = 1 \text{ and } M, \text{ be separable, then } L_p(M, E)^* = L_{p'}(M, E) \text{ holds isometrically.}\]

In the last part of this section, we investigate the space $L_p(M, (E_n); \ell^C_2)$, a generalization
of $L_p(M, E)$, which is important for the dual version of Doob’s inequality. We feel that
the space $L_p(M, E)$ is easier to understand and more directly connected to Hilbert $C^*$-
mODULES. Let $(E_n)$ be a sequence of conditional expectations $E_n : M \to M$ onto von
Neumann subalgebras $N_n$ such that $\phi \circ E_n = \phi$ for all $n \in \mathbb{N}$. Pisier, Xu proved a non-
commutative version of Stein’s inequality. To formulate the version we need here, we
consider the subspace $L_p^{\text{cond}}(M; \ell^C_2) \subset L_p(M; \ell^C_2(N^2))$ of double indexed sequences $(x_{nk})$
such that $x_{nk} \in L_p(N_n)$ for all $k \in \mathbb{N}$. We refer to [PX, JX] for the proof of the following
theorem.

\[\textbf{Theorem 2.13 (Stein’s inequality). Let } 1 < p, p' < \infty \text{ such that } \frac{1}{p} + \frac{1}{p'} = 1 \text{ and } (E_n) \text{ be a sequence of conditional expectations such that } E_n E_m = E_{\min\{n,m\}}. \text{ Then the linear map } ST_p : L_p(M; \ell^C_2(N^2)) \to L_p^{\text{cond}}(M; \ell^C_2) \text{ defined by } ST_p[(x_{nk})] = (E_n(x_{nk})) \text{ is a bounded projection onto } L_p^{\text{cond}}(M; \ell^C_2) \text{ and satisfies } ST_p^* = ST_{p'}^p. \text{ In particular,}
\]
\[L_p^{\text{cond}}(M; \ell^C_2)^* = L_{p'}^{\text{cond}}(M; \ell^C_2)
\]
with equivalent norms depending only on $p$.

\[\textbf{Definition 2.14. Let } 0 < p \leq \infty, \text{ the space } L_p(M, (E_n); \ell^C_2) \text{ is the completion of the space}
\]
of sequences $(a_n D_{\frac{1}{p}}), a_n \in M$ with respect to the norm
\[
\left\| (a_n D_{\frac{1}{p}}) \right\|_{L_p(M, (E_n); \ell^C_2)} = \left\| D_{\frac{1}{p}} \sum_n E_n(a_n^* a_n) D_{\frac{1}{p}} \right\|_{\frac{1}{p}}.
\]
Proposition 2.15. Let \(0 < p \leq \infty\) and in addition \(M\) separable, then there is an isometric embedding \(u_p : L_p(M, (E_n); \ell_p^2) \rightarrow L_p^\text{cond}(M; \ell_p^2)\) and for \(p < \infty\) a norm one projection \(R_p\) onto the image of \(u_p\). Moreover,

i) if \(\frac{1}{p} + \frac{1}{q} = \frac{1}{r}\), then for all \((x_n) \in L_p(M, (E_n); \ell_p^2)\) and \((y_n) \in L_q(M, (E_n); \ell_q^2)\)

\[u_p(x_n)^*u_q(y_n) = \sum_n E_n(x_n^*y_n)\]

and

\[\left\| \sum_n E_n(y_n^*x_n) \right\|_r \leq \|y_n\|_{L_q(M, (E_n); \ell_q^2)} \left\| (x_n) \right\|_{L_p(M, (E_n); \ell_p^2)} .\]

ii) if \(1 < p, p' < \infty\) such that \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(ST_p, ST_{p'}\) denote the projection onto \(L_p^\text{cond}(M; \ell_p^2), L_{p'}^\text{cond}(M; \ell_{p'}^2)\), respectively, then

\[ (R_pST_p)^* = R_{p'}ST_{p'} .\]

**Proof:** The proof of the Cauchy-Schwarz inequality for \(p = q = \infty\) and the isometric embedding is similar to the last part of the proof in Lemma 2.10. Since we will not need it, we omit the details. For each \(n \in \mathbb{N}\), we fix the isometric isomorphism \(u^n_p : L_p(M, E_n) \rightarrow L_p(N_n; \ell_2^p)\) and define \(u_p : L_p(M, (E_n); \ell_p^2) \rightarrow L_p^\text{cond}(M; \ell_p^2)\) by

\[u_p[(x_n)] = (u^n_p(x_n)_{kn}) ,\]

i.e. we apply \(u^n_p\) to \(x_n\) and obtain a double indexed sequence. For finite sequences, we apply Proposition 2.8 i) and obtain

\[(2.2) \quad u_p[(x_n)]^*u_q[(y_n)] = \sum_n u^n_p(x_n)^*u^n_q(x_n) = \sum_n E_n(x_n^*y_n) .\]

In particular, \(u_p\) is isometric when restricted to finite sequences. For \(p < \infty\) an easy Cauchy sequence argument implies that \(u_p\) extends isometrically to \(L_p(M, (E_n); \ell_p^2)\). Similarly as in the proof of Proposition 2.8, we deduce from (2.2) the Cauchy-Schwarz inequality. If \(Q^n_p : L_p(N_n; \ell_2^p) \rightarrow L_p(N_n; \ell_2^p)\) denotes the projection onto the image of \(u^n_p\), then

\[R_p[(x_{nk})] = (Q^n_p[(x_{nk})_{kn}]\]

is certainly well-defined for finite sequences. However, we deduce from Proposition 2.8 iii)

\[R_p[(x_{nk})]^*R_p[(x_{nk})] = \sum_n \sum_k Q^n_p[(x_{nk})_{kn}]^*Q^n_p[(x_{nk})_{kn}] \leq \sum_n \sum_k x_{nk}^*x_{nk} \]
and therefore $R_p$ extends to a contraction for all $p < \infty$. Finally, we observe that Proposition 2.8 iii) implies assertion ii):

$$
\text{tr}(Q_p[(E_n(x_{nk})])^*(y_{nk})) = \sum_n \sum_k \text{tr}(Q^n_p[(E_n(x_{nk}))_k^*y_{nk}])
$$

$$
= \sum_n \sum_k \text{tr}(Q^n_p[(E_n(x_{nk}))_k^*E_n(y_{nk})])
$$

$$
= \sum_n \sum_k \text{tr}(E_n(x_{nk})^*Q^n_p[(E_n(y_{nk}))_k])
$$

$$
= \sum_{nk} \text{tr}(x^*_{nk}Q^n_p[(E_n(y_{nk}))_k])
$$

$$
= \text{tr}((x_{nk})^*Q^n_p[(E_n(y_{nk})])).
$$

We want to remove the additional assumption that $M$ has separable predual.

**Lemma 2.16.** Let $M$ be a von Neumann algebra, $\phi$ a normal faithful state, $E_n$ a sequence of normal conditional expectations onto von Neumann subalgebras $N_n$. For every separable subalgebra $A \subset M$ there exists a subalgebra $A \subset B \subset M$ with separable dual, for all $n \in \mathbb{N}$ $E_n(B) \subset B$, and a normal conditional expectation $\tilde{E} : M \to B$ such that $\phi \circ \tilde{E} = \phi$.

**Proof:** The proof is a modification of [Ki, Corollary 3.5]. Let $A_1$ be separable, *-closed (but not necessarily norm closed) algebra such that for all $x \in A$,

$$
\sup_{y \in N, \|y\| \leq 1} |\phi(yx)| = \sup_{y \in A_1, \|y\| \leq 1} |\phi(yx)|
$$

and moreover $E_n(A) \subset A_1$ for all $n \in \mathbb{N}$. Repeating this process, we obtain a separable *-closed subalgebra $A_\infty = \bigcup_k A_k$ such that the embedding $i : L_1(A_\infty, \phi) \to L_1(N, \phi)$, $i(x.\phi) = x.\phi$ is isometric. The dual map $\tilde{E} = i^* : M \to M$ is a normal conditional expectation onto the $\sigma$-weak operator closure $B$ of $A_\infty$. By construction, we have $\phi \circ \tilde{E} = \phi$ and $E_n(A_\infty) \subset A_\infty \cap N_n$. Since, $E_n$ is $\sigma$-strongly continuous, $E_n(B)$ is a von Neumann subalgebra of $N_n \cap B$ for all $n \in \mathbb{N}$.  

\[\square\]
Theorem 2.17. Let $1 \leq p \leq \infty$, then $L_p(M, (E_n); \ell^C_2)$ is a Banach space and the Cauchy-Schwarz inequality 2.15 holds. Let $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\gamma_p$ the constant from 2.13. If the sequence $(N_n)$ is either increasing or decreasing and $(x_n)$ a sequence in $L_p(M, (E_n); \ell^C_2)$, then

$$\|\langle x_n \rangle\|_{L_p(M, (E_n); \ell^C_2)} \leq \gamma_p \sup \left\{ \left| \sum_n tr(D^{\frac{1}{p'}}b^*_n x_n) \right|, \left| \|b_n D^{\frac{1}{p'}}\|_{L_{p'}(M,(E_n);\ell^C_2)} \right| \leq 1 \right\}.$$

Proof: Since the triangle inequality and the Cauchy-Schwarz inequality are checked for two sequences, it suffices to consider a countably generated subalgebra $B$ of $M$. Indeed, according to Lemma 2.16 we can even assume that $E_n(B) \subset B$ and there exists a $\phi$-preserving conditional expectation $\tilde{E} : M \to B$. Then $L_{p}^\tilde{E}(B)$, is a subspace of $L_{\tilde{E}}^p(M)$. Hence, $L_p(B, (E_n); \ell^C_2)$ is a subspace of $L_p(M, (E_n); \ell^C_2)$ and hence there is no loss of generality to assume that $M_*$ is separable. Then first assertions follow from Proposition 2.15. For the last inequality, we apply Lemma 2.11 to the image $u_p(L_p(M, (E_n); \ell^C_2))$, where $u_p$ is the isometric embedding from Proposition 2.15. Indeed, the projection $R_p ST_p$ satisfies the assumptions of Lemma 2.11 and the norm is bounded by the universal constant $\gamma_p$ from Theorem 2.13. Finally, we note that $u_p$ preserves the duality bracket

$$tr(u_p[(b_n D^{\frac{1}{p'}})b^*_n [(a_n D^{\frac{1}{p'}})]) = \sum_n tr(D^{\frac{1}{p'}}E_n(b^*_n a_n) D^{\frac{1}{p'}}) = \sum_n \phi(E_n(b^*_n a_n))$$

$$= \sum_n \phi(b^*_n a_n) = \sum_n tr(D^{\frac{1}{p'}}b^*_n a_n D^{\frac{1}{p'}}).$$

Therefore the duality formula from Lemma 2.11 is also valid for $L_p(M, (E_n); \ell^C_2)$.

3. The dual version of Doob’s inequality for $1 \leq p \leq 2$

In this section, we start with an elementary proof of the dual version of Doob’s inequality for $p = 2$ and show how the complex interpolation method can be used to extend the inequality to the interval $1 \leq p \leq 2$. Then we provide the duality argument which justifies the name ‘dual version of Doob’s inequality’. In the following, we consider a normal faithful state $\phi$, a von Neumann algebra $N$ and a sequence of von Neumann subalgebras $N_n$ with conditional expectations $E_n : N \to N$ such that

$$\phi \circ E_n = \phi$$
for all \( n \in \mathbb{N} \). (Note that in case of a tracial state such conditional expectations always exist \([1k]\).) Let us stress that in addition to the last section we also always assume that the sequence \( N_n \) is increasing.

**Lemma 3.1.** Let \((x_n)\) be a sequence of positive elements in \( L_2(N) \), then

\[
\left\| \sum_n E_n(x_n) \right\|_2 \leq 2 \left\| \sum_n x_n \right\|_2.
\]

**Proof:** By monotonicity, (1.3) and Corollary 2.4, it suffices to prove this inequality for finite sequences. Using Proposition 2.5, Lemma 2.1 and positivity as in (1.3), see \([Te, \text{Proposition 33}]\), we deduce from Hölder’s inequality

\[
\left\| \sum_n E_n(x_n) \right\|_2^2 = \sum_{nk} tr(E_n(x_n)E_k(x_k))
\]

\[
= \sum_{n \leq k} tr(E_n(x_n)E_k(x_k)) + \sum_{n > k} tr(E_n(x_n)E_k(x_k))
\]

\[
= \sum_{n \leq k} tr(E_k(E_n(x_n)x_k)) + \sum_{n > k} tr(E_n(x_n)E_k(x_k))
\]

\[
= \sum_{n \leq k} tr(\left( \sum_{n \leq k} E_n(x_n) \right) x_k) + \sum_{n > k} tr(x_n \left( \sum_{k < n} E_k(x_k) \right))
\]

\[
\leq \sum_k tr(\left( \sum_n E_n(x_n) \right) x_k) + \sum_n tr(x_n \left( \sum_k E_k(x_k) \right))
\]

\[
= 2tr(\left( \sum_n x_n \right) \left( \sum_k E_k(x_k) \right)) \leq 2 \left\| \sum_n x_n \right\|_2 \left\| \sum_n E_n(x_n) \right\|_2.
\]

Hence, we get

\[
\left\| \sum_n E_n(x_n) \right\|_2 \leq 2 \left\| \sum_n x_n \right\|_2.
\]
Lemma 3.2. If \((DD_2)\) holds with constant \(c_2\) and \(1 \leq p \leq 2\), then for all sequences \((x_n)\) and \((y_n)\) in \(L_{2p}(N)\)

\[
\left\| \sum_n E_n(x_n^*y_n) \right\|_p \leq c_2 \left( \frac{2(p-1)}{p} \right) \left( \left\| \sum_n x_n^*x_n \right\|_p \left\| \sum_n y_n^*y_n \right\|_p \right)^{\frac{1}{2}}.
\]

Proof: Let us first prove the assertion for finite sequences and \(p = 2\) or \(p = 1\). We start with \((DD_1)\). Using Lemma 2.1, we get

\[
\left\| \sum_n E_n(x_n^*x_n) \right\|_1 = tr(\sum_n E_n(x_n^*x_n)) = \sum_n tr(E_n(x_n^*x_n)) = \sum_n tr(x_n^*x_n) = \left\| \sum_n x_n^*x_n \right\|_1.
\]

By the density of elements of the form \(x_n = a_n D^\frac{1}{p}\), the Cauchy-Schwarz inequality, see Theorem 2.17, implies with Lemma 3.1 for \(p \in \{1, 2\}\)

\[
\left\| \sum_n E_n(x_n^*y_n) \right\|_p \leq \left( \left\| \sum_n E_n(x_n^*x_n) \right\|_p \left\| \sum_n E_n(y_n^*y_n) \right\|_p \right)^{\frac{1}{2}} \leq c_p \left( \left\| \sum_n x_n^*x_n \right\|_p \left\| \sum_n y_n^*y_n \right\|_p \right)^{\frac{1}{2}},
\]

where \(c_2\) is the constant given in the assumption and \(c_1 = 1\). To use interpolation, we consider finite sequences \((x_n)\) and \((y_n)\) such that

\[
\left\| \sum_n x_n^*x_n \right\|_p = 1 = \left\| \sum_n y_n^*y_n \right\|_p.
\]

We define \(X = \sum_n x_n^*x_n, Y = \sum_n y_n^*y_n\). Their support projections are denoted by \(q_X\) and \(q_Y\), and are in \(N\), see [18, Proposition 4. 2) c), Proposition 12]. Since \(X^{-\frac{1}{2}}q_X, q_YY^{-\frac{1}{2}}\) are well-defined unbounded operators, we can define

\[ v_n = x_nX^{-\frac{1}{2}}q_X, \quad w_n = y_nY^{-\frac{1}{2}}q_Y \]

Note that \(x_n^*x_n \leq X\) and \(y_n^*y_n \leq Y\) implies \(x_n = x_nq_X, y_n = y_nq_Y\), respectively and according to Lemma [11] we have \(v_n \in N, w_n \in N\). Then, we observe

\[
\sum_n v_n^*v_n = q_XX^{-\frac{1}{2}}XX^{-\frac{1}{2}}q_X \leq q_X \leq 1,
\]

\[
\sum_n w_n^*w_n = q_YY^{-\frac{1}{2}}YY^{-\frac{1}{2}}q_Y \leq q_Y \leq 1.
\]
Let $\theta$ be determined by $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$. According to Kosaki’s interpolation \cite{Ko} \[ [L_{2,L}(N, \phi), L_{4,L}(N, \phi)]_{\theta} = L_{2p,L}(N, \phi) \quad \text{and} \quad [L_{2,R}(N, \phi), L_{4,R}(N, \phi)]_{\theta} = L_{2p,R}(N, \phi) \] with respect to the state $\phi(x) = \text{tr}(Dx)$. By approximation, we may assume that there are continuous functions $X(z)$, $Y(z)$ on the strip $\{0 \leq \text{Re}(z) \leq 1\}$ with values in $N$, analytic in the interior, such that $X^\frac{1}{2} = D^\frac{1}{2}X(\theta)$, $Y^\frac{1}{2} = Y(\theta)D^\frac{1}{2}$ and
\[
\sup_t \max\{\|D^\frac{1}{2}X(it)\|_2, \|D^\frac{1}{2}X(1 + it)\|_4\} \leq 1,
\]
\[
\sup_t \max\{\|Y(it)D^\frac{1}{2}\|_2, \|Y(1 + it)D^\frac{1}{2}\|_4\} \leq 1.
\]
Now, we note Kosaki’s symmetric interpolation result \[ [L_{1,sym}(N, \phi), L_{2,sym}(N, \phi)]_{\theta} = L_{p,sym}(N, \phi). \]
Hence, by (3.1) and Hölder’s inequality we get
\[
\left\| \sum_n E_n(X_{i_n}^* v_n^* w_n Y_n^\frac{1}{2}) \right\|_p = \left\| \sum_n D^\frac{1}{2} E_n(X(\theta)v_n^* w_n Y(\theta))D^\frac{1}{2} \right\|_p
\]
\[
\leq \sup_t \left\| \sum_n D^\frac{1}{2} E_n(X(it)v_n^* w_n Y(it))D^\frac{1}{2} \right\|_1^{1-\theta}
\]
\[
\sup_t \left\| \sum_n D^\frac{1}{2} E_n(X(1 + it)v_n^* w_n Y(1 + it))D^\frac{1}{2} \right\|_2^\theta
\]
\[
\leq \sup_t \left\| \sum_n D^\frac{1}{2} X(it)v_n^* v_n X(it)^* D^\frac{1}{2} \right\|_1^{1-\frac{\theta}{2}} \left\| \sum_n D^\frac{1}{2} Y(it)^* w_n^* w_n Y(it)^* D^\frac{1}{2} \right\|_1^{\frac{\theta}{2}}
\]
\[
\leq \sup_t \left\| \sum_n D^\frac{1}{2} X(1 + it)v_n^* v_n X(1 + it)^* D^\frac{1}{2} \right\|_1^\frac{\theta}{2}
\]
\[
\sup_t \left\| \sum_n D^\frac{1}{2} Y(1 + it)^* w_n^* w_n Y(1 + it) D^\frac{1}{2} \right\|_2^\frac{\theta}{2}
\]
\[
\leq \sup_t \left\| D^\frac{1}{2} X(it)Y(it)^* D^\frac{1}{2} \right\|_1^\frac{\theta}{2} \sup_t \left\| D^\frac{1}{2} Y(it)^* Y(it) D^\frac{1}{2} \right\|_1^\frac{\theta}{2}
\]
\[
\leq \sup_t \left\| D^\frac{1}{2} X(1 + it)Y(1 + it)^* D^\frac{1}{2} \right\|_1^\frac{\theta}{2} \sup_t \left\| D^\frac{1}{2} Y(1 + it)^* Y(1 + it) D^\frac{1}{2} \right\|_1^\frac{\theta}{2}
\]
\[
\leq c_2^\theta = c_2^2 \frac{2(p-1)}{p}.
\]
The assertion is proved.\[ \square \]
Proof of Theorem 0.1 in the case $1 \leq p \leq 2$: For $1 \leq p \leq 2$ and a sequence of positive elements $(z_n) \subset L_p(N)$, we can apply Lemma 3.2 to $x_n = y_n = z_n^\frac{1}{2}$ and deduce the assertion for the sequence $(z_n)$.

The duality argument relies on the following norm for sequences $(x_n) \subset L_p(N)$

$$\|(x_n)\|_{L_p(N; \ell_1)} = \inf \left\{ \left\| \sum_{n} v_{nj} v_{nj}^* \right\|^\frac{1}{p} \left\| \sum_{n} w_{jn} w_{jn}^* \right\|^\frac{1}{p} \right\}.$$ 

Here the infimum is taken over all (double indexed) sequences $(v_{nj})$ and $(w_{nj})$ such that for all $n$

$$x_n = \sum_{j} v_{nj}w_{jn}.$$ 

We require norm convergence for $p < \infty$ and convergence in the $\sigma$-weak operator topology for $p = \infty$. In fact, we think of $x_n$ being obtained by matrix multiplication of a row with a column vector. We denote by $L_p(N; \ell_1)$ the set of all sequences admitting a decomposition as above.

Remark 3.3. This norm is motivated by the following characterization of a normal, decomposable map $T : \ell_\infty \to N$, see [Pa]. Indeed, a normal map is decomposable if and only if there are sequences $(x_n) \subset N$, $(y_n) \subset N$ such that $T(e_n) = y_n x_n$ and

$$\left\| \sum_{n} y_n y_n^* \right\|_N \left\| \sum_{n} x_n x_n \right\|_N < \infty.$$ 

Lemma 3.4. If $(DD_p)$ holds, then

$$\left\| \sum_{n} E_n(x_n^* y_n) \right\|_p \leq c_p \left\| \sum_{n} x_n^* x_n \right\|^\frac{1}{2}_p \left\| \sum_{n} y_n^* y_n \right\|^\frac{1}{2}_p,$$

The linear map $T : L_p(N; \ell_1) \to L_p(N; \ell_1)$, $T((x_n)) = (E_n(x_n))$ satisfies

$$\|T\| \leq c_p.$$ 

Proof: We have seen in (3.1) that the first inequality is an immediate consequence of the Cauchy-Schwarz inequality, see Theorem 2.17. As for the second assertion, we can
assume that \( N_\ast \) is separable and use the Kasparov maps \( u_p^n : L_p(N, E_n) \to L_p(N_n; \ell_2^C) \) from Proposition 2.8. Let \( x_n = \sum_j v_{nj} w_{nj} \), then

\[
\|(E_n(x_n))\|_{L_p(N; \ell_1)} = \left\| \left( \sum_{nj} u_p^n(v_{nj}^*) u_p^n(w_{nj}) \right) \right\|_{L_p(N; \ell_1)} \\
\leq \left\| \sum_{nj} u_p^n(v_{nj}^*) u_p^n(v_{nj}^*) \right\|_{L_p}^{\frac{1}{2}} \left\| \sum_{nj} u_p^n(w_{nj}) u_p^n(w_{nj}) \right\|_{L_p}^{\frac{1}{2}} \\
= \left\| \sum_{nj} E_n(v_{nj} v_{nj}^*) \right\|_{L_p}^{\frac{1}{2}} \left\| \sum_{nj} E_n(w_{nj}^* w_{nj}) \right\|_{L_p}^{\frac{1}{2}} \\
\leq c_p \left\| \sum_{nj} v_{nj} v_{nj}^* \right\|_{L_p}^{\frac{1}{2}} \left\| \sum_{nj} w_{nj}^* w_{nj} \right\|_{L_p}^{\frac{1}{2}}.
\]

Taking the infimum over all these decompositions, we obtain the assertion. \( \square \)

Let us state some elementary properties of the space \( L_p(N; \ell_1) \).

**Lemma 3.5.** For \( 1 \leq p \leq \infty \) the set \( L_p(N; \ell_1) \) is a Banach space. For \( 1 \leq p < \infty \), the set \( L^0_p \) of elements

\[ x_n = \sum_j v_{nj} w_{jn} \]

such that \( \text{card}\{(j, n)| v_{nj} \neq 0 \text{ or } w_{jn} \neq 0\} < \infty \) is dense in \( L_p(N; \ell_1) \). If \( (x_n) \) is a sequence then

\[ \left\| \sum_n x_n \right\|_p \leq \| (x_n) \|_{L_p(N; \ell_1)} \]

and equality holds if all the \( x_n \)'s are positive.

**Proof:** The proof of the triangle inequality is completely elementary, see also [Ps2], but essential. Indeed, let \( \varepsilon > 0 \) and

\[ x_n = \sum_{j_1} v_{nj_1} w_{j_1 n} \quad \text{and} \quad y_n = \sum_{j_2} v_{nj_2} w_{j_2 n}, \]
such that

\[
\left\| \sum_{n_1} v_{n_1} v^*_{n_1} \right\|_p = \left\| \sum_{n_1} w^*_{n_1} w_{n_1} \right\|_p \leq (1 + \varepsilon) \left\| (x_n) \right\|_{\mathcal{L}_p(N; \ell_1)} ,
\]

\[
\left\| \sum_{n_2} v_{n_2} v^*_{n_2} \right\|_p = \left\| \sum_{n_2} w^*_{n_2} w_{n_2} \right\|_p \leq (1 + \varepsilon) \left\| (y_n) \right\|_{\mathcal{L}_p(N; \ell_1)} .
\]

We have

\[
x_n + y_n = \sum_{n_1} v_{n_1} w_{n_1} + \sum_{n_2} v_{n_2} w_{n_2}
\]

and the triangle inequality in \( L_p(N) \) implies

\[
\left\| \sum_{n_1} v_{n_1} v^*_{n_1} + \sum_{n_2} v_{n_2} v^*_{n_2} \right\|_p \leq \left\| \sum_{n_1} v_{n_1} v^*_{n_1} \right\|_p + \left\| \sum_{n_2} v_{n_2} v^*_{n_2} \right\|_p \leq (1 + \varepsilon)(\left\| (x_n) \right\|_{\mathcal{L}_p(N; \ell_1)} + \left\| (y_n) \right\|_{\mathcal{L}_p(N; \ell_1)}) .
\]

Similarly,

\[
\left\| \sum_{n_1} w^*_{n_1} w_{n_1} + \sum_{n_2} w^*_{n_2} w_{n_2} \right\|_p \leq (1 + \varepsilon)(\left\| (x_n) \right\|_{\mathcal{L}_p(N; \ell_1)} + \left\| (y_n) \right\|_{\mathcal{L}_p(N; \ell_1)})
\]

and the assertion follows with \( \varepsilon \to 0 \). We consider the spaces of column matrices, row matrices, \( L_p(N; \ell_2^c(N^2)) \), \( L_p(N; \ell_2^f(N^2)) \subset L_p(B(\ell_2^c(N^2)) \otimes N) \), respectively. Using,

\[
\Phi((x_{nk}) \otimes (y_{nk})) = (\sum_k x_{nk} y_{nk})_{n \in N} ,
\]

we deduce that \( \mathcal{L}_p(N; \ell_1) \) is isomorphic to a quotient space of the projective tensor product \( L_p(N; \ell_2^c(N^2)) \otimes_\pi L_p(N; \ell_2^f(N^2)) \), see [DF] for a definition and basic properties of the projective tensor product. Hence \( \mathcal{L}_p(N; \ell_1) \) is complete. The image under \( \Phi \) of pairs of finite sequences generates \( \mathcal{L}_p^0 \). According to Corollary 2.4, finite sequences are dense in the column and row spaces and hence \( \mathcal{L}_p^0 \) is dense in \( \mathcal{L}_p(N; \ell_1) \). Finally, let \((x_n)\) be a sequence of positive elements. Clearly, \( x_n = x_n^* x_n^\perp \) and hence

\[
\left\| (x_n) \right\|_{\mathcal{L}_p(N; \ell_1)} \leq \left\| \sum_n x_n \right\|_p .
\]
On the other hand if \( x_n = \sum_j v_{nj}w_{nj} \), we deduce from Hölder’s inequality
\[
\left\| \sum_n x_n \right\|_p = \left\| \sum_j (e_{1,nj} \otimes v_{nj})(e_{nj,1} \otimes w_{nj}) \right\|_p \\
\leq \left\| \sum_j e_{1,nj} \otimes v_{nj} \right\|_2 \left\| \sum_j e_{nj,1} \otimes w_{nj} \right\|_2 \\\n= \left\| \sum_j v_{nj}v_{nj}^* \right\|_p \left\| \sum_j w_{nj}^*w_{nj} \right\|_p^{\frac{1}{2}} .
\]
Taking the infimum yields the assertion.

Inspired by Pisier’s vector-valued \( L_p(N, \tau; \ell_\infty) \) space, we define for \( 0 < p \leq \infty \)
\[
\left\| \sup_n |x_n| \right\|_p = \|(x_n)\|_{L_p(N; \ell_\infty)} = \inf_{x_n = ay_nb} \|a\|_{2p} \|b\|_{2p} \sup_n \|y_n\|_N ,
\]
where the infimum is taken over all \( a, b \in L_{2p}(N) \) and all bounded sequences \( (y_k) \). If \( N \) is a hyperfinite, finite von Neumann algebra, this space coincides with \( L_p(N, \tau; \ell_\infty) \) in the sense of Pisier \(^{Ps2}\). The first (formal) notation is suggestive and facilitates understanding our inequalities in view of the commutative theory. For positive elements, we will drop the absolute value. Let us note that Haagerup’s work \(^{Ha2}\) shows that the equality \( \mathcal{L}_1(N; \ell_\infty) = L_1(N) \otimes_\pi \ell_\infty \) (operator space projective tensor product) only holds for injective von Neumann algebras. However, this does not affect the following factorization result which is, nowadays, a standard application of the Grothendieck-Pietsch version of the Hahn-Banach theorem, see \(^{Ps1}, \ Ps2\).

**Proposition 3.6.** Let \( 1 \leq p < \infty \). If \( p = 1 \), then \( \mathcal{L}_1(N; \ell_1) = \ell_1(L_1(N)) \) holds with equal norms. If \( 1 < p, p' < \infty \) satisfy \( \frac{1}{p} + \frac{1}{p'} = 1 \), then
\[
\mathcal{L}_p(N; \ell_1)^* = \mathcal{L}_{p'}(N; \ell_\infty)
\]
holds isometrically.
Proof: If $z_n = ay_nb$ and $(y_n)$ is a bounded sequence, we deduce from Hölder’s inequality for all $(v_{nj}) \subset L_{2p}(N)$, $(w_{jn}) \subset L_{2p}(N)$

$$\left| \sum_n \text{tr}(z_n \sum_j v_{nj} w_{jn}) \right| = \left| \sum_n \text{tr}(ay_nb v_{nj} w_{jn}) \right| = \left| \sum_n \text{tr}(y_nb v_{nj} w_{jn}a) \right|$$

$$\leq \sup_n \| y_n \|_\infty \sum_n \| b v_{nj} w_{jn}a \|_1$$

$$\leq \sup_n \| y_n \|_\infty \left( \sum_n \| b v_{nj} \|_2^\frac{1}{2} \right)^\frac{1}{2} \left( \sum_n \| w_{jn}a \|_2 \right)^\frac{1}{2}$$

$$= \sup_n \| y_n \|_\infty \| b \|_{2p'} \left( \sum_n \| v_{nj} v_{nj}^* \|_1 \right)^\frac{1}{2} \left( \sum_n \| a^* w_{nj} w_{jn}a \|_1 \right)^\frac{1}{2}$$

$$\leq \sup_n \| y_n \|_\infty \| b \|_{2p'} \left( \sum_n \| v_{nj} v_{nj}^* \|_p \right)^\frac{1}{2} \left( \sum_n \| a \|_{2p'} \left( \sum_n \| w_{nj} w_{jn} \|_p \right) \right)^\frac{1}{2}.$$

Hence, $L_{p'}(N; \ell_\infty) \subset L_p(N; \ell_1)^*$. Using $\ell_1(L_1(N)) \subset L_1(N; \ell_1)$, we deduce the equality $\ell_1(L_1(N)) = L_1(N; \ell_1)$. Now we show that for $1 < p < \infty$ all the functionals are in $L_{p'}(N; \ell_\infty)$. Let $\psi : L_p(N; \ell_1) \to \mathbb{C}$ be a norm one functional. Using $\ell_1(L_p(N)) \subset L_p(N; \ell_1)$, we can assume that there exists a sequence $(z_n) \subset L_{p'}(N)$ such that

$$\psi[(z_n)] = \psi((z_n)) = \sum_n \text{tr}(z_n x_n).$$

Let us denote by $B = B_{L_{p'}(N)}^+$ the positive part of the unit ball in $L_{p'}(N)$. $B$ is compact when equipped with the $\sigma(L_{p'}(N), L_p(N))$-topology. The definition of $L_p(N; \ell_1)$ implies with the geometric/arithmetic mean inequality

$$\left| \sum_n \text{tr}(z_n v_{nj} w_{jn}) \right| = \left| \psi\left(\sum_j v_{nj} w_{jn}\right)n\right| \leq \left| \sum_n v_{nj} v_{nj}^* \right|_p \left| \sum_n w_{nj}^* w_{jn} \right|_p$$

$$\leq \frac{1}{2} \sup_{c,d \in B} \left[ \sum_n \text{tr}(v_{nj} v_{nj}^* c) + \sum_n \text{tr}(w_{jn}^* w_{jn} d) \right].$$

Since the right hand side remains unchanged under multiplication with signs $\varepsilon_{nj}$, we deduce

$$\sum_n |\text{tr}(z_n v_{nj} w_{jn})| \leq \frac{1}{2} \sup_{c,d \in B} \left[ \sum_n \text{tr}(v_{nj} v_{nj}^* c) + \sum_n \text{tr}(w_{jn}^* w_{jn} d) \right].$$
Following the Grothendieck-Pietsch separation argument as in [Ps1], we observe that the cone
\[ C = \{ g | \sup g < 0 \} \]
is disjoint from the cone \( C_- = \{ g | \sup g < 0 \} \). Here \( v = (v_n) \) and \( w = (w_n) \) are finite sequences and hence \( f_{v,w} \) is continuous with respect to the product topology on \( B \times B \). Since \( f_{v,w} + f_{\tilde{v},\tilde{w}} \) can be obtained by taking the \((\tilde{v}_n, \tilde{w}_n)\)'s to the right of the finite sequence \((v_n, w_n)\), we deduce that \( C \) is a cone. Hence, there exist a measure \( \mu \) on \( B \times B \) and a scalar \( t \) such that for all \( g \in C_- \) and \( f \in C \)
\[ \int_{B \times B} g \, d\mu < t \leq \int_{B \times B} f \, d\mu . \]
Since we are dealing with cones, it turns out that \( t = 0 \) and \( \mu \) is positive. Therefore, we can and will assume that \( \mu \) is a probability measure. We define the positive elements \( c \) and \( d \) by their projections
\[ a = \int_{B \times B} c \, d\mu(c,d) , \quad b = \int_{B \times B} d \, d\mu(c,d) . \]
By convexity of \( B \), we deduce \( a, b \in B \). Hence, we obtain
\[ \sum_n 2 |tr(z_n v_n w_n)| \leq \int_{B \times B} \sum_n [tr(v_n v_n^* c) + tr(w_n w_n^* d)] \, d\mu(c,d) \]
\[ = \sum_n \int_{B \times B} tr(v_n v_n^* c) \, d\mu(c,d) + \int_{B \times B} tr(w_n w_n^* d) \, d\mu(c,d) \]
\[ = \sum_n [tr(v_n v_n^* a) + tr(w_n w_n^* b)] . \]
Using once more \( 2st = \inf_{r>0} (rs)^2 + (r^{-1}t)^2 \), we get
\[ \sum_n |tr(z_n v_n w_n)| \leq \left( \sum_n tr(v_n v_n^* a) \right)^{\frac{1}{2}} \left( \sum_n tr(w_n w_n^* b) \right)^{\frac{1}{2}} \]
\[ = \left( \sum_n \| a^{\frac{1}{2}} v_n \|_2^2 \right)^{\frac{1}{2}} \left( \sum_n \| b^{\frac{1}{2}} w_n \|_2^2 \right)^{\frac{1}{2}} . \]
(3.2)
Let \( q_a, q_b \in N \) be the support projections of \( a, b \), respectively. Consider, \( d_a = a^{\frac{1}{2}} + (1 - q_a)D(1 - q_a), D \) the density of \( \phi \). Then \( \phi_{d_a}(x) = tr(d_a x) \) is a normal, faithful state on \( N \) and according to Lemma \([L2]\), \( d_a^{\frac{1}{2}} N \) is dense in \( L_2(N) \). Hence,
\[ q_a d_a^{\frac{1}{2}} N = a^{\frac{1}{2}} N = a^{\frac{1}{2}} a^{\frac{1}{2}} N \subset a^{\frac{1}{2}} L_{2p}(N) \]
shows that $a^{\frac{1}{2}} L_{2p}(N)$ is dense in $q_a L_2(N)$. Similarly, $b^{\frac{1}{2}} L_{2p}(N)$ is dense in $q_b L_2(N)$ and therefore (3.2) implies that for every $n \in \mathbb{N}$ there is a contraction $T_n : q_a L_2(N) → q_b L_2(N)$ such that for all $v, w ∈ L_{2p}(N)$

$$tr(wz_n v) = (b^{\frac{1}{2}} w^*, T_n(a^{\frac{1}{2}} v)) = tr(wb^{\frac{1}{2}} T_n(a^{\frac{1}{2}} v)).$$

This means $T_n$ is a bounded extension of the densely defined hermitian form

$$(b^{\frac{1}{2}} q_b (h'), z_n a^{-\frac{1}{2}} q_a(h)) = (b^{\frac{1}{2}} q_b h', z_n a^{-\frac{1}{2}} q_a h)$$

Using the density of $a^{\frac{1}{2}} L_{2p}(N)$ and $b^{\frac{1}{2}} L_{2p}(N)$ it is easily checked that $q_b T_n q_a$ is affiliated with $N$. Since $T_n$ is bounded, we deduce $q_b T_n q_a ∈ N$. On the other hand, we have for $v ∈ L_{2p}(N)$ and $w ∈ L_{2p}(N)$

$$|tr(z_n (1 − q_a) vw)| ≤ tr(a(1 − q_a) vv^*)^{\frac{1}{2}} tr(w^* wb)^{\frac{1}{2}} = 0$$

and

$$|tr(z_n vw (1 − q_b)| ≤ tr(awv^*)^{\frac{1}{2}} tr(w^* w (1 − q_b)b)^{\frac{1}{2}} = 0 .$$

This shows $z_n = q_b z_n q_a$ and therefore

$$z_n = q_b z_n q_a = q_b b^{\frac{1}{2}} q_b b^{\frac{1}{2}} z_n a^{-\frac{1}{2}} q_a a^{\frac{1}{2}} q_a = b^{\frac{1}{2}} y_n a^{\frac{1}{2}} .$$

The assertion is proved because $L^0_p$ is dense in $L_p(N; ℓ_1)$ and hence the functional $ψ$ is uniquely determined by the sequence $(z_n)$.

Remark 3.7. Let $1 ≤ p < ∞$ and $(z_n) ⊂ L_{2p}(N)$ a sequence of positive elements, then

$$\left\|\sup_n z_n\right\|_{p'} = \sup \left\{ \left\| \sum_n tr(z_n x_n) \right\|_{p} \mid x_n ≥ 0 , \left\| \sum_n x_n \right\|_p ≤ 1 \right\} .$$

Moreover, there exists a positive element $a ∈ L_{2p'}(N)$ and a sequence of positive elements $y_n$ such that

$$z_n = ay_n a \text{ and } \left\| a \right\|_{2p'} \sup_n \left\| y_n \right\|_∞ = \left\| \sup_n z_n \right\|_{p'} .$$

For positive elements $(x_n) ⊂ L_p(N)_+$, we also have

$$\left\| \sum_n x_n \right\|_p = \sup \left\{ \left\| \sum_n tr(x_n z_n) \right\|_p \mid z_n ≥ 0 , \left\| (z_n) \right\|_{L_{p'}(N; ℓ_∞)} ≤ 1 \right\}$$

and therefore the cones of positive sequences in $L_p(N; ℓ_1)$ respectively $L_{p'}(N; ℓ_∞)$ are in duality.
Proof: For positive elements \((z_n)\) satisfying

\[
\left\| \sum_n \text{tr}(z_n x_n) \right\| \leq \left\| \sum_n x_n \right\|_p,
\]

for all sequences of positive elements \((x_n) \subset L_p(N)_+\), we deduce

\[
\sum_{j,n} |\text{tr}(z_n v_{nj}^* v_{nj})| \leq \left\| \sum_n \text{tr}(z_n (\sum_j v_{nj}^* v_{nj})) \right\| \leq \left\| \sum_{nj} v_{nj}^* v_{nj} \right\|_p = \sup_{c \in B} \sum_{j,n} \text{tr}(v_{nj}^* v_{nj}^* c).
\]

Using the Hahn-Banach separation argument in the space of continuous functions on \(B\), we obtain a positive element \(a\) in the unit ball of \(L_{p'}(N)\) such that

\[
\sum_n |\text{tr}(z_n v_n v_n^*)| \leq \sum_n \text{tr}(v_n v_n^* a).
\]

Since \(a^{-\frac{1}{2}} z_n a^{-\frac{1}{2}}\) is positive, this inequality still ensures that all the \(y_n = a^{-\frac{1}{2}} z_n a^{-\frac{1}{2}}\) are positive contractions in \(N\). The last equality follows immediately from Lemma 3.5 and the duality between the positive parts of \(L_p(N)\) and \(L_{p'}(N)\).

Remark 3.8. Proposition 3.6 and Remark 3.7 can easily be modified for uncountable (ordered) index sets by requiring the inequality for all countable (ordered) subsets or for an essential supremum. This is helpful in the context of continuous filtrations.

The required duality argument is now very simple.

Lemma 3.9. Let \(1 < p \leq \infty\) and \(\frac{1}{p} + \frac{1}{p'} = 1\) assume that \((DD_{p'})\) holds with constant \(c_{p'}\), then for every \(y \in L_p(N)\)

\[
\left\| \sup_n |E_n(y)| \right\|_p \leq c_{p'} \|y\|_p.
\]

Moreover, for every sequence of positive elements \((y_n)\)

\[
\left\| \sup_{n} \left| \sum_{j \geq n} E_n(y_j) \right| \right\|_p \leq c_{p'} \left\| \sum_n y_n \right\|_p.
\]

Proof: Indeed, as observed in Lemma 3.4 and using Lemma 3.5, we deduce that \((DD_{p'})\) implies that the linear map \(T : \mathcal{L}_{p'}(N; \ell_1) \to L_p(N), T((x_n)) = \sum_n E_n(x_n)\) satisfies
\( \|T\| \leq c_{p'} \). By duality and Proposition 3.6, we deduce for all \( y \in L_p(N) \)
\[
\sup_n |E_n(y)| \leq \|T^*(y)\|_{L'_p(N;\mathbb{C})^*} \leq \|T^*\|_{p'} \|y\|_p \leq c_{p'} \|y\|_p .
\]
Given a sequence of positive elements \((y_n)\), we consider \( y = \sum_j y_j \) and obtain
\[
\sup_n |E_n(y)| \leq c_{p'} \|y\|_p = c_{p'} \sum_j y_j .
\]
However, for positive elements \((x_n) \subset L_{p'}(N)\), we deduce by positivity
\[
\sum_n \text{tr}(E_n(\sum_{j \geq n} y_j)x_n) \leq \sum_n \sum_j \text{tr}(E_n(y_j)x_n) \leq \|\sum_n x_n\|_{p'} .
\]
Hence, Remark 3.7 implies
\[
\sup_n \sum_{j \geq n} E_n(y_j) \leq \sup_n \sum_{n \in \mathbb{N}} \|E_n(y)\|_{p'} \leq c_{p'} \|y\|_p = c_{p'} \sum_j y_j .
\]
The assertion is proved.

**Theorem 3.10.** For \( 1 < p \leq \infty \) there exists a constant \( c_p \) such that for every sequence \((N_n)\) of von Neumann subalgebras with sequence of \( \phi \)-invariant conditional expectations \((E_n)\) satisfying \( E_nE_m = E_{\min(n,m)} \) and for every \( x \in L_p(N) \) there exist \( a, b \in L_{2p}(N) \) and a bounded sequence \((y_n) \subset N\) such that
\[
E_n(x) = ay_nb \quad \text{and} \quad \|a\|_{2p} \|b\|_{2p} \sup_n \|y_n\|_\infty \leq c_p \|x\|_p .
\]
If \( x \) is positive, one can in addition assume that \( b = a^* \) and all the \( y_n \)'s are positive.

**Proof for** \( 2 \leq p \leq \infty \): This follows immediately from Lemma 3.3, Lemma 3.2 and Lemma 3.1. Using that \( E_n(x) \) is positive for positive \( x \), the addition follows from Remark 3.7.

\[\boxed{3.10}\]

4. The dual version of Doob’s inequality for \( 2 \leq p < \infty \)

In our approach to \((DD_p)\) in the range \( 2 \leq p < \infty \), our aim is to obtain the same kind of inequalities for the maximal function as in Garsia’s book [Ga]. As mentioned in the introduction, we are forced to use more duality arguments because \( 0 \leq a \leq b \) implies \( a^\beta \leq b^\beta \) only for \( 0 \leq \beta \leq 1 \) and therefore most of the elementary proofs in Garsia’s book are no longer valid in the non-commutative case. We will make the same assumptions
about \( N, (N_n), (E_n) \) and \( \phi, D \) as in the previous section. In particular, the sequence \( N_n \) is supposed to be increasing.

**Lemma 4.1.** Let \( 0 \leq x \leq z \in L_1(N) \) such that the support projection of \( z \) is 1. Let \( 1 \leq \alpha \leq 2 \), then

\[
\text{tr}(z^{\frac{1-\alpha}{2}}(z^\alpha - x^\alpha)z^{\frac{1-\alpha}{2}}) \leq 2\text{tr}(z - x).
\]

**Proof:** We define \( \beta = \alpha - 1 \in (0,1) \) and observe that \( 0 \leq x \leq z \) implies

\[
x^{1-\beta} \leq z^{1-\beta}.
\]

We apply Lemma [1] to \( x^{\frac{\beta}{2}} \) and \( z^{\frac{\beta}{2}} \) and deduce from \( x^{\beta} \leq z^{\beta} \) that \( v = x^{\frac{\beta}{2}}z^{\frac{\beta}{2}} \) is a contraction in \( N \). In particular \( v^* = z^{-\frac{\beta}{2}}x^{\frac{\beta}{2}} \in N \) and

\[
a = z^{-\frac{\beta}{2}}x^{\frac{1+\beta}{2}} = v^*x^{\frac{1}{2}} \in L_2(N).
\]

For all elements \( a, b \in L_2(N) \), we note that \( (a - b)^* (a - b) \geq 0 \) implies with tracial property of the trace

\[
\text{tr}(a^*b) + \text{tr}(b^*a) \leq \text{tr}(a^*a) + \text{tr}(b^*b) = \text{tr}(a^*) + \text{tr}(b^*)
\]

We define \( b = z^{\frac{\alpha}{2}}x^{\frac{1}{2} - \frac{\alpha}{2}} \) which is in \( L_2(N) \) by Hölder’s inequality. Then, we observe

\[
\begin{align*}
\text{tr}(a^*b) &= \text{tr}(x^{\frac{1+\beta}{2}}z^{-\frac{\beta}{2}}z^{\frac{\beta}{2}}x^{\frac{1}{2} - \frac{\beta}{2}}) = \text{tr}(x), \\
\text{tr}(b^*a) &= \text{tr}(x^{\frac{1}{2} - \frac{\beta}{2}}z^{\frac{\beta}{2}}z^{-\frac{\beta}{2}}x^{\frac{1+\beta}{2}}) = \text{tr}(x).
\end{align*}
\]

Hence, we deduce

\[
2\text{tr}(x) = \text{tr}(a^*b) + \text{tr}(b^*a) \leq \text{tr}(a^*) + \text{tr}(b^*)
\]

\[
= \text{tr}(x^{\frac{1+\beta}{2}}z^{-\frac{\beta}{2}}z^{-\frac{\beta}{2}}x^{\frac{1+\beta}{2}}) + \text{tr}(z^{\frac{\beta}{2}}x^{\frac{1}{2} - \frac{\beta}{2}}z^{-\frac{\beta}{2}})
\]

\[
= \text{tr}(x^{\frac{1+\beta}{2}}z^{-\beta}x^{\frac{1+\beta}{2}}) + \text{tr}(z^{\frac{\beta}{2}}x^{1-\beta}z^{\frac{\beta}{2}})
\]

\[
\leq \text{tr}(x^{\frac{1+\beta}{2}}z^{-\beta}x^{\frac{1+\beta}{2}}) + \text{tr}(z^{\frac{\beta}{2}}z^{1-\beta}z^{\frac{\beta}{2}})
\]

\[
= \text{tr}(x^{\frac{1+\beta}{2}}z^{1-\alpha}x^{\frac{\beta}{2}}) + \text{tr}(z).
\]

This implies

\[
-\text{tr}(x^{\frac{\alpha}{2}}z^{1-\alpha}x^{\frac{\alpha}{2}}) + \text{tr}(z) \leq -2\text{tr}(x) + \text{tr}(z) + \text{tr}(z) = 2\text{tr}(z - x).
\]

The assertion follows from \( z^{\frac{1-\alpha}{2}}x^{\alpha}z^{\frac{1-\alpha}{2}} = aa^* \in L_1(N) \) and

\[
\text{tr}(x^{\frac{\alpha}{2}}z^{1-\alpha}x^{\frac{\alpha}{2}}) = \text{tr}(a^*) = \text{tr}(aa^*) = \text{tr}(z^{\frac{1-\alpha}{2}}x^{\alpha}z^{\frac{1-\alpha}{2}}).
\]
Moreover, by continuity, we can assume that there is an \( \varepsilon > 0 \) such that
\[
\left\| \sum_n \text{tr}(y_n^* x_n) \right\| \leq \sqrt{2} \left\| \varepsilon \right\|_{L^r(N; c_0^C)} \left\| \sum_j x_j^* x_j \right\|^{\frac{1}{r}}.
\]
Moreover,
\[
\left\| \sum_n \text{tr}(y_n^* x_n) \right\| \leq \sqrt{2} \left\| \varepsilon \right\|_{L^r(N; c_0^C)} \left( \sup_n \left\| E_n(x_j^* x_j) \right\| \right)^{\frac{1}{r}}.
\]

**Proposition 4.2.** Let \( 1 \leq r' < 2 < r \leq \infty \), and \( \frac{1}{r'} + \frac{1}{r} = 1 \), then for all \( (x_j) \subseteq L_r(N) \) and \( (y_n) \subseteq L_{r'}(N; (E_n); c_0^C) \)
\[
\left| \sum_n \text{tr}(y_n^* x_n) \right| \leq \sqrt{2} \left\| \varepsilon \right\|_{L^r(N; c_0^C)} \left\| \sum_j x_j^* x_j \right\|^{\frac{1}{r}}.
\]

**Proof:** Let us assume that both sequences are finite, i.e. \( x_j = 0 = y_j \) for \( j \geq m \). By density, we can moreover assume that \( y_j = a_j D^\frac{1}{r'} \) with \( a_j \in N \) and
\[
\left\| \sum_{j=1}^m D^\frac{1}{r'} E_j(a_j^* a_j) D^\frac{1}{r'} \right\| < 1.
\]
By continuity, we can assume that there is an \( \varepsilon > 0 \) such that
\[
\left\| \varepsilon D^\frac{1}{r'} + \sum_{j=1}^m D^\frac{1}{r'} E_j(a_j^* a_j) D^\frac{1}{r'} \right\| \leq 1.
\]
Let \( 1 \leq q \leq \infty \) such that \( \frac{1}{q} + \frac{2}{r} = 1 \). For \( n \in N \), we define
\[
S_n = \left( \varepsilon D^\frac{1}{r'} + \sum_{j=1}^n D^\frac{1}{r'} E_j(a_j^* a_j) D^\frac{1}{r'} \right)^{\frac{1}{2}} \in L_q(N_n) \subseteq L_q(N).
\]
The support projection of \( S_n \) is 1 and \( \varepsilon D^\frac{1}{r'} \leq S_n \). According to Lemma 4.1, we deduce that \( w_n := D^\frac{1}{r'} S_n^{-\frac{1}{2}} \in N_n \). Hence
\[
y_n S_n^{-\frac{1}{2}} = a_n D^\frac{1}{r'} S_n^{-\frac{1}{2}} = a_n D^\frac{1}{r'} D^\frac{1}{r'} S_n^{-\frac{1}{2}} = a_n D^\frac{1}{r'} w_n \in L_2(N).
\]
In particular, \( S_n^{-\frac{1}{2}} y_n S_n^{-\frac{1}{2}} \in L_2(N) \) and
\[
S_n^{-\frac{1}{2}} y_n S_n^{-\frac{1}{2}} = w_n^* D^\frac{1}{r'} a_n^* a_n D^\frac{1}{r'} w_n \in L_1(N).
\]
Moreover,
\[
E_n(S_n^{-\frac{1}{2}} y_n S_n^{-\frac{1}{2}}) = E_n(w_n^* D^\frac{1}{r'} a_n^* a_n D^\frac{1}{r'} w_n) = w_n^* D^\frac{1}{r'} E_n(a_n^* a_n) D^\frac{1}{r'} w_n
\]
\[
= S_n^{-\frac{1}{2}} D^\frac{1}{r'} E_n(a_n^* a_n) D^\frac{1}{r'} S_n^{-\frac{1}{2}}.
\]
This implies with the Cauchy-Schwarz inequality
\[
\left| \sum_n tr(y_n^*x_n) \right| = \left| \sum_n tr(x_ny_n^*) \right| = \left| \sum_n tr((x_nS_n^{\frac{1}{2}})(S_n^{-\frac{1}{2}}y_n^*)) \right|
\]
\[
= \left| \sum_n tr(S_n^{-\frac{1}{2}}y_n^*x_nS_n^{\frac{1}{2}}) \right|
\]
\[
\leq \left( \sum_n tr(S_n^{-\frac{1}{2}}y_n^*y_nS_n^{-\frac{1}{2}}) \right)^{\frac{1}{2}} \left( \sum_n tr(x_n^*x_nS_n) \right)^{\frac{1}{2}}
\]
\[
= \left( \sum_n tr(E_n(S_n^{-\frac{1}{2}}y_n^*y_nS_n^{-\frac{1}{2}})) \right)^{\frac{1}{2}} \left( \sum_n tr(x_n^*x_nS_n) \right)^{\frac{1}{2}}
\]
\[
= \left( \sum_n tr(S_n^{-\frac{1}{2}}D^\frac{1}{q}E_n(a_n^*a_n)D^\frac{1}{q}S_n^{-\frac{1}{2}}) \right)^{\frac{1}{2}} \left( \sum_n tr(E_n(x_n^*x_n)S_n) \right)^{\frac{1}{2}}.
\]

To estimate the first term, we define \( \alpha = \frac{2}{r'} \in [1, 2] \) and notice that
\[
1 - \alpha = 1 - \frac{2}{r'} = 1 - 2 + \frac{2}{r} = -\frac{1}{q}.
\]
For fixed \( n \), we define \( x = S_{n-1}^q \) and \( z = S_n^q \). Since \( \frac{r'}{2} \leq 1 \), we have
\[
x = \left( \varepsilon D^\frac{1}{q} + \sum_{j=1}^{n-1} D^\frac{1}{q}E_j(a_j^*a_j)D^\frac{1}{q} \right)^{\frac{r'}{2}} \leq \left( \varepsilon D^\frac{1}{q} + \sum_{j=1}^{n} D^\frac{1}{q}E_j(a_j^*a_j)D^\frac{1}{q} \right)^{\frac{r'}{2}} = z.
\]

Then, we note that \( z^{\frac{1-\alpha}{q}} = z^{-\frac{1}{q}} = S_n^{-\frac{1}{q}} \). Hence Lemma 4.1 implies
\[
tr(S_n^{-\frac{1}{2}}D^\frac{1}{q}E_n(a_n^*a_n)D^\frac{1}{q}S_n^{-\frac{1}{2}}) = tr(z^{\frac{1-\alpha}{q}}(z^\alpha - x^\alpha)z^{\frac{1-\alpha}{q}}) \leq 2tr(z - x)
\]
\[
= 2tr(S_n^q - S_{n-1}^q).
\]

Therefore, we obtain
\[
\sum_n tr(S_n^{-\frac{1}{2}}D^\frac{1}{q}E_n(a_n^*a_n)D^\frac{1}{q}S_n^{-\frac{1}{2}}) = \sum_n 2tr(S_n^q - S_{n-1}^q) = 2tr(S_n^q)
\]
\[
= 2 \left\| \varepsilon D^\frac{1}{q} + \sum_{j=1}^{m} D^\frac{1}{q}E_j(a_j^*a_j)D^\frac{1}{q} \right\|^{\frac{r'}{2}} \leq 2.
\]

Now, we want to estimate the second term. Let us define
\[
\theta_j = S_j - S_{j-1}.
\]
and note that $\theta_j \in L_q(N_j)$. As usual, we set $S_{-1} = 0$. Moreover, $r' \leq 2q$ implies that $\theta_j$ is positive. Then, we deduce with $E_j E_n = E_{\min(j,n)}$

$$\sum_n tr(E_n(x_n^* x_n) S_n) = \sum_{j \leq n} tr(E_n(x_n^* x_n) \theta_j)$$

$$= \sum_{j} tr(\sum_{n \geq j} E_n(x_n^* x_n) \theta_j)$$

$$= \sum_{j} tr(E_j(\sum_{n \geq j} E_n(x_n^* x_n)) \theta_j)$$

$$= \sum_{j} tr(\sum_{n \geq j} E_j(x_n^* x_n) \theta_j)$$

Now, we can continue in two different ways

$$\sum_j tr(\sum_{n \geq j} E_j(x_n^* x_n) \theta_j) \leq \left\| \sup_j E_j(\sum_{n \geq j} x_n^* x_n) \right\|_{\frac{r}{q}} \left\| \sum_j \theta_j \right\|_q$$

$$= \left\| \sup_j E_j(\sum_{n \geq j} x_n^* x_n) \right\|_{\frac{r}{q}} \left\| S_m \right\|_q$$

$$\leq \left\| \sup_j E_j(\sum_{n \geq j} x_n^* x_n) \right\|_{\frac{r}{q}}.$$ 

By homogeneity, we obtain the second assertion

$$\left\| \sum_n tr(y_n^* x_n) \right\| \leq \sqrt{2} \left\| \sup_j E_j(\sum_{n \geq j} x_n^* x_n) \right\|_{\frac{r}{q}} \left\| (y_n) \right\|_{L_{r'}(N,E;\ell_2^c)}.$$
Corollary 4.4. Let $1 \leq q < \infty$ and $\gamma_{2q}$ the constant in Stein’s inequality, then for all $(x_n) \subset L_{2q}(N)$

$$\left\| \sum_n E_n(x_n^*x_n) \right\|_q \leq 2\gamma_{2q}^2 \left\| \sup_n E_n(\sum_{j \geq n} x_j^*x_j) \right\|_q .$$

Proof: We define $r = 2q$ and note

$$\left\| \sum_n E_n(x_n^*x_n) \right\|_r^{\frac{1}{r}} = \left\| (x_n) \right\|_{L_p(N,(E_n);\ell_2^n)} .$$
Therefore the assertion follows from Proposition 4.2 and Theorem 2.17.

Proof of \((DD_p)\) for \(1 < p < \infty\): We define \(r = 2p > 2\) Let \((z_n) \subset L_p(N)\) be a sequence of positive elements and define \(x_n = z_n^\frac{1}{r}\). Hence by Theorem 2.17 and Proposition 4.2, we deduce

\[
\| \sum_n E_n(z_n) \|_p^\frac{1}{2} = \| \sum_n E_n(x_n^*x_n) \|_p^\frac{1}{2} \\
\leq \gamma_r \| (y_n) \|_{L_{r'}(N, (E_n); \ell_2)} \sup_n \left| \sum_n tr(y_n^*x_n) \right| \\
\leq \gamma_r \sqrt{2} \| \sum_n x_n^*x_n \|_p^\frac{1}{2}.
\]

The assertion follows with \(c_p \leq 2\gamma_{2p}^2\).

Proof of Theorem 0.2 and 3.10 for \(1 < p \leq 2\): This follows from \((DD_p)\) via Lemma 3.9 and Remark 3.7.

Remark 4.5. Let \(N^*\) be separable and \(\psi\) be a functional on \(L_{p'}(N, (E_n); \ell_2^r)\), then there exists a sequence \((x_n)\) such that

\[
\psi((y_n)) = \sum_n tr(x_n^*y_n) \quad \text{and} \quad \| \sup_n E_n(\sum_{j \geq n} x_j^*x_j) \|_p^\frac{1}{2} \leq d_{\frac{p}{2}} \| \psi \|_{L_{p'}(N, (E_n); \ell_2^r)^*}.
\]

Here \(d_{\frac{p}{2}}\) is the constant in Doob’s inequality from Theorem 0.2. The assertion yields an extension of the \(BMO_C-H_1^C\) duality for \(2 < p < \infty\) and fails for \(p = 2\). The proof uses the Kasparov isomorphism from Proposition 2.15. We leave it to the interested reader.

Answering a question by G. Pisier, we can even produce an asymmetric version of Doob’s inequality. Indeed, let \(1 < p \leq \infty\) and consider \(1 \leq r, s < \infty\) such that \(\frac{2}{p} = \frac{1}{s} + \frac{1}{t}\). Given \(x \in L_p(N)\) and sequences \((v_{nj}) \subset L_r(N), (w_{jn}) \subset L_s(N)\), we deduce from the
Cauchy-Schwartz inequality (2.13) \((DD_s)\) and \((DD_t)\) that

\[
\left| \sum_{n,j} tr(E_n(x)v_{nj}w_{jn}) \right| = \left| \sum_{n,j} tr(xE_n(v_{nj}w_{jn})) \right| \\
\leq \|x\|_p \left\| \sum_{n,j} E_n(v_{nj}w_{jn}) \right\|_{p'} \\
\leq \|x\|_p \left\| \sum_{n,j} E_n(v_{nj}v_{nj}^*) \right\|_s^{\frac{1}{2}} \left\| \sum_{n,j} E_n(w_{jn}^*w_{jn}) \right\|_t^{\frac{1}{2}} \\
\leq c_s^t c_t^s \|x\|_p \left\| \sum_{n,j} v_{nj}v_{nj}^* \right\|_s^{\frac{1}{2}} \left\| \sum_{n,j} w_{jn}^*w_{jn} \right\|_t^{\frac{1}{2}}.
\]

Similar as in Proposition 3.4, we deduce the existence of bounded a sequence \((z_n)\) and elements \(a \in L_{2q'}(N), b \in L_{2r'}(N)\) such that \(E_n(x) = bz_n a\) and

\[
\|a\|_{2q'} \|b\|_{2r'} \sup_n \|z_n\|_{\infty} \leq c_s^t c_t^s \|x\|_p.
\]

Note that \(\frac{1}{2q'} + \frac{1}{2r'} = \frac{1}{p}\) and therefore we have proved the following asymmetric version of Theorem 0.2. (The assumption \(2 < q, r\) is indeed necessary \([DJ1]\).)

**Corollary 4.6.** Let \(1 < p \leq \infty\) and \(2 < q, r \leq \infty\) such that \(\frac{1}{q} + \frac{1}{r} = \frac{1}{p}\). Then for every \(x \in L_p(N)\) there exists a sequence \((z_n) \subset N\) and \(a \in L_{q}(N), b \in L_{r}(N)\) such that \(E_n(x) = az_n b\) and

\[
\|a\|_q \|b\|_r \sup_n \|z_n\|_{\infty} \leq c(p, q, r) \|x\|_p.
\]

5. Applications

In this section, we present first applications of Doob’s inequality in terms of submartingales, Doob decomposition. We make the same assumptions about \(N, (N_n), (E_n), \phi\) and \(D\) as in the previous section and start with almost immediate consequences of the dual version of Doob’s inequality.

**Corollary 5.1.** Let \(1 < p \leq \infty\) and \((z_n)\) be an adapted sequence of positive elements, i.e. \(z_n \in L_p(N_n)_+\). If for all \(n \in \mathbb{N}\)

\[
z_n \leq E_n(z_{n+1}) \quad \text{and} \quad \sup_m \|z_m\|_p < \infty,
\]
then there exist a positive element \( a \in L_{2p}(N)_+ \) and a sequence of positive contractions \( (y_n) \subset N \) such that

\[
z_n = ay_n a \quad \text{and} \quad \|a\|_{2p}^2 \leq c_{p'} \sup_m \|z_m\|_p.
\]

**Proof:** It suffices to consider \( p < \infty \). We note that for \( n \leq m \)

\[
z_n \leq E_n(z_{n+1}) \leq E_n(E_{n+1}(z_{n+2})) = E_n(z_{n+2}) \leq \cdots \leq E_n(z_m).
\]

Let \( \frac{1}{p} + \frac{1}{p'} = 1 \). In order to estimate the norm in \( L_p(N; \ell_\infty) \), we refer to Remark 3.7. Given a finite sequence \( (x_n) \) of positive elements such that \( x_n = 0 \) for \( n \geq m \), we deduce from (DD)

\[
\left| \sum_n tr(z_n x_n) \right| = \sum_n tr(E_n(z_n x_n)) = \sum_n tr(E_n(z_n) E_n(x_n))
\]

\[
\leq \sum_n tr(E_n(z_m) E_n(x_n)) = \sum_n tr(E_n(z_m E_n(x_n)))
\]

\[
= \sum_n tr(z_m E_n(x_n)) \leq tr(z_m \sum_n E_n(x_n))
\]

\[
\leq \|z_m\|_p \left\| \sum_n E_n(x_n) \right\|_{p'} \leq c_{p'} \|z_m\|_p \left\| \sum_n x_n \right\|_{p'}
\]

Hence, the assertion follows from Remark 3.7. \( \blacksquare \)

**Corollary 5.2.** Let \( 2 < p \leq \infty \) and \( (z_n) \) be an adapted sequence, i.e. \( z_n \in L_p(N_n) \). If for all \( n \in \mathbb{N} \)

\[
z^*_n z_n \leq E_n(z^*_{n+1} z_{n+1}),
\]

then there exist a positive element \( a \in L_p(N) \) and a sequence \( (y_n) \) of contractions such that

\[
z_n = y_n a \quad \text{and} \quad \|a\|_p \leq c_{\frac{1}{p} + \frac{1}{p'}} \sup_m \|z_m\|_p.
\]

**Proof:** We apply Corollary 5.1 to \( z^*_n = z^*_n z_n \in L_{p'}(N_n) \) and obtain positive contractions \( (v_n) \subset N \) and \( a \in L_p(N) \) such that

\[
z^*_n z_n = av_n a \quad \text{and} \quad \|a\|_p^2 \leq c_{\frac{1}{p} + \frac{1}{p'}} \sup_m \|z^*_m z_m\|_{p'}.
\]

If \( q_a \) is the support projection of \( a \), we see that \( y_n = v_n a^{-1} q_a \) satisfies the assertion. \( \blacksquare \)
Let us mention an immediate application of \((DD_p)\) in terms of the Doob decomposition of the square function. Given a martingale sequence \(x = \sum_k d_k(x)\), where \(d_k(x) = E_k(x) - E_{k-1}(x)\) (and \(E_0(x) = 0\)), we recall that one of the square functions is given by

\[ s_c(x) = \sum_k d_k(x)^*d_k(x) \]

The square function is the discrete analogue of the quadratic variation term; see [BS, Kö] for more details. The Doob decomposition of \(s_c(x)\) is given by the martingale part

\[ V_n(x) = \sum_{k=1}^n d_k(x)^*d_k(x) - E_{k-1}(d_k(x)^*d_k(x)) \]

and the predictable part

\[ W_n(x) = \sum_{k=2}^n E_{k-1}(d_k(x)^*d_k(x)) \in L_p(M_{k-1}) \text{.} \]

Note that

\[ V_n(x) + W_n(x) = \sum_{k=1}^n d_k(x)^*d_k(x) \text{.} \]

**Corollary 5.3.** Let \(2 < p < \infty\) and \(x \in L_p(N)\); then

\[
\sup_n \max\{\|W_n(x)\|_{\frac{p}{2}}, \|V_n(x)\|_{\frac{p}{2}}\} \leq \alpha_p (1 + c_{\frac{p}{2}})^{\frac{1}{2}} \|x\|_p \text{.}
\]

Here \(\alpha_p\) is an absolute constant. In particular, there exists an element \(V_\infty(x)\) in \(L_{\frac{p}{2}}(N)\) such that \(V_n(x) = E_n(V_\infty(x))\).

**Proof:** Using \((DD_{\frac{p}{2}})\) for the sequence \((E_{k-1})\) and the non-commutative Burkholder-Gundy inequality [PX, JX], we deduce

\[
\|W_n(x)\|_{\frac{p}{2}} \leq c_{\frac{p}{2}} \left\| \sum_{k=2}^n d_k(x)^*d_k(x) \right\|_{\frac{p}{2}} \leq c_{\frac{p}{2}} \alpha^2_p \|x\|^2_p \text{.}
\]

Hence the triangle implies

\[
\|V_n(x)\|_{\frac{p}{2}} \leq \|W_n(x)\|_{\frac{p}{2}} + \left\| \sum_{k=1}^n d_k(x)^*d_k(x) \right\|_{\frac{p}{2}} \leq (c_{\frac{p}{2}} \alpha^2_p + 1) \|x\|^2_p \text{.}
\]

By uniform convexity of \(L_{\frac{p}{2}}(N)\), we obtain the limit value \(V_\infty(x) = \lim_n V_n(x)\) with the desired properties. \(\blacksquare\)
The next application yields norm estimates for \( \sum_n p_n E_n(x)q_n \) with respect to a sequence \((p_n), (q_n)\) of disjoint projections. This corresponds to a double sided non-adapted stopping time.

**Corollary 5.4.** Let \( 1 < p \leq \infty \), \((v_n), (w_n)\) be sequences of bounded elements, then for all \( x \in L_p(N) \)

\[
\left\| \sum_n v_n E_n(x)w_n \right\|_p \leq c_p \|x\|_p \max \left\{ \left\| \sum_n v_n v_n^* \right\|_\infty^{1/2}, \left\| \sum_n v_n^* v_n \right\|_\infty^{1/2} \right\}
\]

\[
\max \left\{ \left\| \sum_n w_n w_n^* \right\|_\infty^{1/2}, \left\| \sum_n w_n^* w_n \right\|_\infty^{1/2} \right\}.
\]

**Proof:** The case \( p = \infty \) is obvious. Hence, we assume \( 1 < p < \infty \). Let \( y \in L_{p'}(N) \) and choose \( y_1 \in L_{2p'}(N), y_2 \in L_{2p'}(N) \) such that \( y = y_1y_2 \) and

\[
\|y\|_{p'} = \|y_1\|_{2p'}^2 = \|y_2\|_{2p'}^2.
\]

Then, according to Lemma 3.4, we deduce from \((DD_{p'})\):

\[
\left| tr(\sum_n v_n E_n(x)w_n y) \right| = \left| \sum_n tr(xE_n(w_ny v_n)) \right| \leq \|x\|_p \left\| \sum_n E_n(w_n y_1 y_2 v_n) \right\|_{p'}
\]

\[
\leq \|x\|_p c_p \| \sum_n w_n y_1 y_1^* w_n^* \|_{p'}^{1/2} \| \sum_n v_n^* y_2 y_2^* w_n \|_{p'}^{1/2}.
\]

To conclude, we use \([PX\text{, Lemma 1.1}],\text{ see also }[JX]\):

\[
\left\| \sum_n w_n y_1 y_1^* w_n^* \right\|_{p'} \leq \|y_1 y_1^*\|_{p'} \max \left\{ \left\| \sum_n w_n w_n^* \right\|_\infty^{1/2}, \left\| \sum_n w_n^* w_n \right\|_\infty^{1/2} \right\}
\]

\[
\max \left\{ \left\| \sum_n w_n w_n^* \right\|_\infty , \left\| \sum_n w_n^* w_n \right\|_\infty \right\}.
\]

A similar argument applies for the other term and therefore taking the supremum over all \( y \) of norm 1 implies the assertion.

**Remark 5.5.** Let \( 1 \leq p < \infty \). The non-commutative Doob inequality from Theorem \( [7,\text{ Theorem 7.3}]\) implies the vector-valued Doob inequality in \( L_p(\Omega, \Sigma, \mu; L_p(N)) \).
**Proof:** It suffices to treat the discrete case. Let \( \mathcal{N} = L_\infty(\Omega, \Sigma, \mu) \otimes \mathcal{N} \) and \((\Sigma_n)_{n \in \mathbb{N}}\) be an increasing sequence of \(\sigma\)-subalgebras with conditional expectations \((E_n)\). Let \( \mathcal{N}_n = L_\infty(\Omega, \Sigma, \mu) \otimes \mathcal{N} \). Then the conditional expectation \( E_n \) onto \( \mathcal{N}_n \) is given by \( E_n = E_n \otimes \text{id} \). Let \( f \in L_p(\Omega, \Sigma, \mu; L_p(\mathcal{N})) = L_p(\mathcal{N}) \). According to Theorem 0.2 there exist \( a \in L_{2p}(\mathcal{N}) \), \( b \in L_{2p}(\mathcal{N}) \) and contractions \((z_n) \subset \mathcal{N}\) such that

\[
E_n \otimes \text{id}_{L_p(\mathcal{N})}(f) = az_nb \quad \text{and} \quad \|a\|_{2p} \|b\|_{2p} \leq c_{p'} \|f\|_p.
\]

Hence, for every \( \omega \in \Omega \) and \( n \in \mathbb{N} \)

\[
\|E_n(f)(\omega)\|_{L_p(\mathcal{N})} = \|a(\omega)z_n(\omega)b(\omega)\|_{L_p(\mathcal{N})} \leq \|a(\omega)\|_{L_{2p}(\mathcal{N})} \|z(\omega)\|_{\mathcal{N}}\|b(\omega)\|_{L_{2p}(\mathcal{N})} \leq \|a(\omega)\|_{L_{2p}(\mathcal{N})} \|b(\omega)\|_{L_{2p}(\mathcal{N})}.
\]

Hölder’s inequality implies the assertion

\[
\left( \int_{\Omega} \sup_n \|E_n(f)(\omega)\|_{L_p(\mathcal{N})}^p \, d\mu(\omega) \right)^{\frac{1}{p}} \leq \left( \int_{\Omega} \|a(\omega)\|_{L_{2p}(\mathcal{N})}^p \|b(\omega)\|_{L_{2p}(\mathcal{N})}^p \, d\mu(\omega) \right)^{\frac{1}{p}} \leq \|a\|_{L_{2p}(\Omega, \Sigma, \mu; L_{2p}(\mathcal{N}))} \|b\|_{L_{2p}(\Omega, \Sigma, \mu; L_{2p}(\mathcal{N}))} = \|a\|_{2p} \|b\|_{2p} \leq c_{p'} \|f\|_p.
\]

In the next application we want to relate group actions with \((DD_p)\). To illustrate this, we consider a finite von Neumann algebra \( \mathcal{N} \) and an increasing sequence \((A_n) \subset \mathcal{N}\) of finite dimensional subalgebras with \(1_N \in A_n\). Let \( N_n = A_n'\) be the relative commutant of \( A_n \) in \( \mathcal{N} \). If \( G_n \) denotes the unitary group of \( A_n \), we have a natural action \( \alpha_n : G_n \to B(L_p(\mathcal{N})) \)

\[
\alpha_n(u)(x) = u xu^*
\]

such that the conditional expectation on the commutant \( N_n \) is given by

\[
E_n(x) = E_{N_n}(x) = \int_{G_n} u x u^* d\mu_n(u).
\]

Let \( G = \prod_n G_n \) and \( \mu \) the product measure, then

\[
\left\| \sum_n E_n(x_n) \right\|_p \leq \left( \int_{G} \left\| \sum_n \alpha_n(u_n)(x_n) \right\|_p^p \, d\mu(u_1, u_2, \ldots) \right)^{\frac{1}{p}}.
\]

We will show that for a sequence \((x_n)\) of positive elements even the right hand side can be estimated by \( \left\| \sum_n x_n \right\|_p \). For simplicity let us use the random variables \( \alpha_n(x) : G \to L_p(\mathcal{N}) \), given for \( \omega = (u_1, u_2, \ldots) \) by

\[
\alpha_n(x)(\omega) = u_n x u_n^*.
\]
For the special case of tensor products of finite dimensional von Neumann algebras the following theorem implies \((DD_p)\).

**Theorem 5.6.** Let \(1 < p < \infty\) and \(N, (A_n), (N_n)\) be as above and \((x_n)\) be a sequence of positive elements, then

\[
\left( \int_{G} \left\| \sum_{n} \alpha_n(x_n) \right\|^p d\mu \right)^{\frac{1}{p}} \leq \kappa_p \left\| \sum_{n} x_n \right\|^p.
\]

Here \(\kappa_p\) is a constant which only depends on \(p\).

**Proof:** The assertion is obvious for \(p = 1\) and by interpolation as in Lemma \(3.2\) it suffices to prove the assertion for \(p \geq 4\). Let \((x_n)\) be a finite sequence of positive elements. Then, we observe

\[
\left( \int_{G} \left\| \sum_{n} \alpha_n(x_n) \right\|^p d\mu \right)^{\frac{1}{p}} \leq \left\| \sum_{n} E_n(x_n) \right\|^p + \left( \int_{G} \left\| \sum_{n} \alpha_n(x_n) - E_n(x_n) \right\|^p d\mu \right)^{\frac{1}{p}} \leq c_p \left\| \sum_{n} x_n \right\|^p + \left( \int_{G} \left\| \sum_{n} \alpha_n(x_n) - E_n(x_n) \right\|^p d\mu \right)^{\frac{1}{p}}.
\]

Let \(\Sigma_n\) be the \(\sigma\)-algebra generated by the first \(n\) coordinates in \(G = \prod_k G_k\) and \(\mathcal{N}_n = L_\infty(G, \Sigma_n, \mu) \otimes N \subset L_\infty(G, \Sigma, \mu) \otimes N\) with the corresponding conditional expectation \(\mathcal{E}_n\). Then, we note

\[
\mathcal{E}_{n-1}(\alpha_n(x_n)) = \int_{\prod_{k \geq n} G_k} u_n x_n u_n^* d\mu_n(u_n) d\mu_{n+1}(u_{n+1}) \cdots = E_n(x_n).
\]

Hence the \(n\)-th martingale difference of \(\sum_n \alpha_n(x_n)\) satisfies

\[
d_n = \mathcal{E}_n(\sum_{k} \alpha_k(x_k)) - \mathcal{E}_{n-1}(\sum_{k} \alpha_k(x_k)) = \alpha_n(x_n) - E_n(x_n).
\]
We apply the non-commutative Rosenthal inequality, see [JX], in this case and obtain

\[
\frac{1}{r_p} \left( \mathbb{E} \left\| \sum_n \alpha_n(x_n) - E_n(x_n) \right\|^p \right)^{\frac{1}{p}} \leq \left( \sum_n \left\| \alpha_n(x_n) - E_n(x_n) \right\|^p \right)^{\frac{1}{p}} + \left\| \sum_n \mathcal{E}_{n-1}(d_n^*d_n + d_n^*d_n) \right\|^{\frac{1}{p}}
\]

\[
\leq 2 \left( \sum_n \left\| x_n \right\|^p \right)^{\frac{1}{p}} + 2 \left\| \sum_n E_n(x_n x_n^*) + E_n(x_n^* x_n) \right\|^{\frac{1}{p}}
\]

\[
\leq 2 \left( \sum_n \left\| x_n \right\|^p \right)^{\frac{1}{p}} + 2 \frac{1}{2} \left\| \sum_n x_n x_n^* + x_n^* x_n \right\|^{\frac{1}{p}}
\]

Let \((\varepsilon_n)\) be a sequence of independent Rademacher variables. Using the triangle inequality and the orthogonality of the \((\varepsilon_n)\)'s, we deduce as in [LP]

\[
\max \left\{ \left\| \sum_n x_n^* x_n \right\|^{\frac{1}{p}}, \left\| \sum_n x_n x_n^* \right\|^{\frac{1}{p}} \right\} \leq \left( \mathbb{E} \left\| \sum_n \varepsilon_n x_n \right\| \right)^{\frac{2}{p}}.
\]

By interpolation, we have

\[
\left( \sum_n \left\| x_n \right\|^p \right)^{\frac{1}{p}} \leq \left\| \sum_n x_n x_n^* \right\|^{\frac{1}{2}}.
\]

However, since the \(x_n\) are positive, we deduce for any choice of signs \(\varepsilon_n\) with positivity

\[
\left\| \sum_n \varepsilon_n x_n \right\| \leq \left\| \sum_{\varepsilon_n=1} x_n \right\| + \left\| \sum_{\varepsilon_n=-1} x_n \right\| \leq 2 \left\| \sum_n x_n \right\|.
\]

Hence, we obtain

\[
\left( \mathbb{E} \left\| \sum_n \alpha_n(x_n) - E_n(x_n) \right\|^p \right)^{\frac{1}{p}} \leq r_p(2 + 4c_\frac{3}{4}) \left\| \sum_n x_n \right\|.
\]

The assertion is proved. \(\blacksquare\)

**Remark 5.7.** These methods can also be used to show that for every \(f \in L_p(G; L_p(N))\) there exist \(a, b \in L_{2p}(\Omega; L_{2p}(N))\) and a sequence of contractions \((y_n) \subset L_\infty(\Omega)\widehat{\otimes} N\) such
that

\[ F_n(f) = ay_n b \quad \text{and} \quad \|a\|_{2p} \|b\|_p \leq c_{2p}' \|f\|_p \]

where

\[ F_n(f)(g_1, g_2, \ldots) = \alpha_n(g_n)f(g_1, \ldots, g_n, \ldots). \]

Note that the crossed product \( N \rtimes (\alpha_n) \prod G_n \) acts on \( L_p(\Omega; L_p(N)) \) and \( F_n \) somehow removes the action of \( G_n \) on \( f \). Similar results hold for a von Neumann algebra with a faithful normal state \( \phi \) and \( \phi \)-invariant, strongly continuous group actions \( \alpha_n : G_n \to Aut(N) \) of compact groups such that the centralizer algebras are increasing or decreasing.

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