The Existence and Uniqueness of Global Admissible Conservative Weak Solution for the Periodic Single-Cycle Pulse Equation

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Abstract. This paper is devoted to studying the existence and uniqueness of global admissible conservative weak solution for the periodic single-cycle pulse equation without any additional assumptions. Firstly, introducing a new set of variables, we transform the single-cycle pulse equation into an equivalent semilinear system. Using the standard ordinary differential equation theory, the global solution of the semilinear system is studied. Secondly, returning to the original coordinates, we get a global admissible conservative weak solution for the periodic single-cycle pulse equation. Finally, choosing some vital test functions which are different from [Bressan (Discrete Contin. Dyn. Syst 35:25-42, 2015), Brunelli (Phys. Lett. A 353:475-478, 2006)], we find an equation to single out a unique characteristic curve through each initial point. Moreover, the uniqueness of global admissible conservative weak solution is obtained.

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1. Introduction

In this paper, we consider the following Cauchy problem for the periodic integrable single-cycle pulse equation

\[\begin{align*}
    u_{xt} & = u + \frac{1}{2}u(u^2)_{xx}, \\
    u(t, x) |_{t=0} & = u_0(x), \\
    u(t, x+1) & = u(t, x),
\end{align*}\]  

(1.1)

Indeed, Eq. (1.1) is a generalized short pulse eq. [28]

\[u_{xt} = u + au^2u_{xx} + buu_x^2,\]  

(1.2)

with $a/b = 1$. In [28], Sakovich stated the generalized short pulse equation (1.2) is integrable in two cases of its coefficients. The first one is the short pulse (SP) equation [29]

\[u_{xt} = u + \frac{1}{6}(u^3)_{xx}\]  

(SP)

with $a/b = 1/2$. The SP equation is a model as an alternative equation to the cubic nonlinear Schrödinger equation to describe the evolution of ultra-short optical pulses in nonlinear media [29]. It is integrable [26] and has a bi-Hamiltonian structure [4]. The local well-posedness, global well-posedness and blow-up phenomenon were studied in [23,25]. There are many other lectures for SP equation, see [24,27].

The second one is the above single-cycle pulse equation (1.1) for the reason the smooth envelope soliton of Eq. (1.1) is as short as one cycle of its carrier frequency [28]. Thanks to the soliton theory, there are many interesting properties of (1.1) to investigate. For instance, the well-posedness, blow-up solutions, periodic solutions, weak solutions, and so on. In effect, Li and Yin [22] established the local well-posedness of (1.1) in $H^s(\mathbb{S})$ with $s \geq 2$ by the Kato’s theory. Moreover, they derived a relationship...
between the single-cycle pulse equation and the sine-Gordon equation and finally got a global existence result. However, they didn’t give any results about weak solutions for Eq. (1.1).

Recently, Hone, Novikov and Wang [19] displayed a more general equation than Eq. (1.2) by classifying the nonlinear partial differential equations of second order

$$u_{xt} = u + c_0 u^2 + c_1 uu_x + c_2 uu_{xx} + c_3 u^2_x + d_0 u^3 + d_1 u^2 u_x + d_2 u^2 u_{xx} + d_3 uu_x^2.$$  

(1.3)

In fact, Eq. (1.3) not only includes Eq. (1.1) and SP, but also contains another vital equation that we called the dispersive Hunter-Saxton equation

$$u_{xt} = u + 2uu_{xx} + u_x^2$$  

(1.4)

for the reason Eq. (1.4) has one more dispersive term $u$ than the following Hunter-Saxton (HS) eq. [20]

$$\left(u_t + uu_x\right)_x = \frac{1}{2}u_x^2.$$  

(1.5)

The HS equation (1.5) as an asymptotic model of liquid crystals is locally well-posed and has global strong solutions, global weak solutions and global dissipative solutions, see [2,21].

The HS equation (1.5) can as well arise in a different physical context as the high-frequency limit of the following classical Camassa-Holm (CH) equation [12]

$$u_t - u_{xxt} + 3uu_x = 2uu_xu_{xx} + uu_{xxx}. $$  

(1.6)

The CH equation is completely integrable [10] and has bi-Hamiltonian structure [15]. The local well-posedness and ill-posedness, global strong solutions, blow-up solutions and global weak solutions of the CH equation were discussed in [7–9,11,13,17,31,32]. Furthermore, the existence and uniqueness of global conservative solutions on the line and on the circle were studied in [1,3,18]. However, their unique results depend on a Lipschitz metric $d_p$ from $\mathcal{H}$ to $\mathcal{D}$, where $\mathcal{H}$ satisfies $\int_0^1 y(\xi) d\xi = 0$ and $y_\xi + H_\xi = 1 + \|H_\xi\|_{L^1}$, which means that some additional assumptions are needed for the uniqueness.

As far as we know, the existence and uniqueness of global conservative weak solution for (1.1) has not been investigated yet. In this paper, we aim to study it by following the idea of [1,3]. Unlike [16,18], our proof does not need any additional assumptions.

We first give the definitions of admissible conservative weak solutions.

**Definition 1.1.** Let $u_0 \in H^1(S)$. We say $u(t,x) \in L^\infty(\mathbb{R}^+; H^1(S))$ is a global conservative weak solution to the Cauchy problem (2.3), if $u(t,x)$ satisfies

$$\int_{\mathbb{R}^+} \int_{S} (u\phi_x + u^2u_x\phi_x)(t,x) dx dt = \int_{\mathbb{R}^+} \int_{S} \left[(u - uu_x^2 - u - uu_x^2)\phi\right](t,x) dx dt + \int_{S} u_0(x)\phi(0,x) dx$$  

(1.7)

and

$$\int_{\mathbb{R}^+} \int_{S} (u\phi_t - \frac{1}{3} u^3 \phi_x)(t,x) dx dt = -\int_{\mathbb{R}^+} \int_{S} \left(\frac{1}{2}(u - uu_x^2) + f(t)\phi(t,x)\right) dx dt - \int_{S} u_0(x)\phi(0,x) dx$$  

(1.8)

for all $\phi, \phi \in C^\infty_0(\mathbb{R}^+; D(S))$ and $f(t)$ is the boundary term choosed by (2.4). Moreover, the quantities $\int_{S} u_x^2(t,x) dx$, $\int_{S} (u - uu_x^2)(t,x) dx$ are conserved in time.

**Definition 1.2.** Let $u_0 \in H^1(S)$. We say $u(t,x) \in L^\infty(\mathbb{R}^+; H^1(S))$ is a global admissible conservative weak solution to the Cauchy problem (2.3), if $u(t,x)$ satisfies the following properties:

1. The function $u$ provides a solution to the Cauchy problem (2.3) in the sense of Definition 1.1.
2. For all $\psi \in C^\infty_0(\mathbb{R}^+; D(S))$, $u(t,x)$ satisfies

$$\int_{\mathbb{R}^+} \int_{S} (u_x^2 \psi_t - u_x^2 u_x^2 \psi_x) dx dt = -\int_{\mathbb{R}^+} \int_{S} (u_x^2 - 2uu_x^2) \psi dx dt - \int_{S} u_0^2(z) \psi(0,z) dz.$$  

(1.9)

Now, we present our main results.
Theorem 1.3. Let \( u_0(x) \in H^1(S) \). Suppose that \( u_0(x) \) satisfies
\[
\int_S u_0 - u_0 u_0^2 \, dx = 0. \tag{1.10}
\]
Then the problem (1.1) has a unique global admissible conservative weak solution in the sense of Definition 1.2.

Remark 1.4. The key in the proof is to prove the characteristic is unique and Lipschitz continuous with an adapted variables \((t, \beta)\), since the test functions on the line constructed by Bressan et al. are not applicable to the period case. To overcome it, we construct some new test functions on the circle which are different from the ones constructed by Bressan et al. The construction method is technically skillful and is also suitable for the periodic Camassa-Holm equation.

The rest of our paper is as follows. In the second section, we give some basic equations about the periodic single-pulse equation. In the third section, we deduce an equivalent semilinear system and then establish the global solutions to the semilinear system. In the fourth section, returning to the original variables, we obtain the global admissible conservative weak solution of the original equation. In the last section, by constructing some new test functions, we prove that the global admissible conservative weak solution for (1.1) is unique.

2. The Basic Equations

In this section, we give some basic equations about Eq. (1.1). Before that, let’s introduce some definitions.

**Definition 2.1** [30]. Let \( T_n \) denote a circle of unit length. We say that \( \mathcal{D}(T_n) \) is the collection of all complex-valued infinitely differentiable functions on \( T_n \) if the locally convex topology in it is generated by the semi-norms
\[
\|v\|_\beta = \sup_{x \in T_n} |D^\beta v(x)|,
\]
where
\[
T_n = \{ x \mid x \in \mathbb{R}_n, x = (x_1, \cdots, x_n), |x_i| \leq \pi, i = 1, \cdots, n \}
\]
and \( \beta = (\beta_1, \cdots, \beta_n) \) is an arbitrary multi-index with non-negative components.

**Remark 2.2** [14,30]. Any function \( v(x) \in \mathcal{D}(T_n) \) can be represented as
\[
v(x) = \sum_{k \in \mathbb{Z}_n} a_k e^{ikx} \quad \text{(convergence in } \mathcal{D}(T_n) \text{),}
\]
where \( \{a_k\}_{k \in \mathbb{Z}_n} \) is a sequence of complex numbers such that
\[
|a_k| \leq c_m (1 + |k|)^{-m}, \quad k \in \mathbb{Z}_n, \tag{2.1}
\]
for all \( m \in 0, 1, 2, \cdots \). Here \( c_m \) is an appropriate positive constant. It holds \( a_k = \hat{v}(k), k \in \mathbb{Z}_n \). Conversely, if \( \{a_k\}_{k \in \mathbb{Z}_n} \) satisfies (2.1) then \( \sum_{k \in \mathbb{Z}_n} a_k e^{ikx} \) convergence in \( \mathcal{D}(T_n) \). If \( v(x) \) is its limit function then we have \( \hat{v}(k) = a_k, k \in \mathbb{Z}_n \).

For the sake of simplicity, we hereafter assume that the period \( T_1 \) is the unit period \( S = [0,1] \) in one dimension.

**Notation.**
\[
\bar{w} = \int_S w(x) \, dx : \text{the mean value of the real function } w(x) \text{ over } S.
\]
\[
Pw(x) = w(x) - \bar{w} : \text{the orthogonal projection onto mean zero functions.}
\]
\[
\partial_x^{-1} w(x) = \int_0^x P(w(y)) \, dy : \text{the inverse of the differential operator.}
\]
By applying the orthogonal projection to (1.1), we obtain
\[
\begin{align*}
(u_t - u^2 u_x)_x &= u - uu^2 - u - uu^2_x, & t \geq 0, \ x \in \mathbb{R}, \\
u(t, x)|_{t=0} &= u_0(x), & x \in \mathbb{R}, \\
u(t, x + 1) &= u(t, x), & t \geq 0, \ x \in \mathbb{R}.
\end{align*}
\]

Integrating both sides of the first equation in (2.2) with respect to \(x\), and choosing a specific boundary term, we have
\[
\begin{align*}
\frac{d}{dt} \left( u^2 \right)_x &= \frac{1}{1 - h} \int_{\mathbb{R}} (1 - u^2) \frac{\partial}{\partial x} \left( u - uu^2_x \right) dy \\
f(t) &= \frac{1}{1 - h} \int_{\mathbb{R}} (1 - u^2) \frac{\partial}{\partial x} \left( u - uu^2_x \right)(y) dy.
\end{align*}
\]

For smooth solutions, multiplying the first equation of (2.2) by 2\(u_x\), we get
\[
(u^2)_t + (-u^2 u_x)_x = (u^2)_x - 2u_x u - uu^2_x.
\]

Combining (2.3) and (2.5), we deduce that the following quantities
\[
E(t) := \int_{\mathbb{R}} u^2_x(t, x) dx,
\]
\[
F(t) := \int_{\mathbb{R}} (u - uu^2_x)(t, x) dx
\]
are constants in time.

3. An Equivalent Semilinear System and Global Solution of the System

3.1. An Equivalent Semilinear System

Define the spaces \(I, \ X\) as
\[
I = \left\{ g \in H^1_{\text{loc}}(\mathbb{R}) \mid g(\eta + 1) = g(\eta) + 1, \ \text{for all } \eta \in \mathbb{R} \right\},
\]
\[
X = I \times H^1(\mathbb{S}) \times L^\infty(\mathbb{S}) \times L^\infty(\mathbb{S}) \times L^\infty(\mathbb{S})
\]
with the norms \(\|g\|_I = \|g\|_{H^1}\) and \(\|(y, U, V, W, Q)\|_X = \|y\|_{H^1} \times \|U\|_{L^\infty} \times \|V\|_{L^\infty} \times \|W\|_{L^\infty} \times \|Q\|_{L^\infty}\).

Assume that \(u\) is smooth and periodic. Let \(y: \mathbb{R} \rightarrow I, t \mapsto y(t, \cdot)\) be the characteristic as the solutions of
\[
\frac{d}{dt} y(t, \xi) = -u^2(t, y(t, \xi)), \quad y(0, \xi) = y_0(\xi)
\]
where \(y_0(\xi)\) will be given below.

Introduce some new variables
\[
U(t, \xi) = u(t, y(t, \xi)), \quad V(t, \xi) = \frac{1}{1 + u^2_x \circ y},
\]
\[
W(t, \xi) = \frac{u_x \circ y}{1 + u^2_x \circ y}, \quad Q(t, \xi) = (1 + u^2_x \circ y) y_{\xi}.
\]

Owing to (3.1) and the first eq. of (2.3), we deduce
\[
U_t(t, \xi) = K(\xi) - P(1)y(\xi) - \frac{1}{1 - h} \int_{\mathbb{S}} (2QV - Q) (\eta) (K(\eta) - P(1)y(\eta)) d\eta.
\]
where \(K(\xi) = \int_{y^{-1}(t, 0)} (2UQV - UQ)(\eta)d\eta\) and \(P(1) = \int_{\mathbb{S}} (2UQV - UQ)(\xi)d\xi\) is a function of time \(t\).
Similarly, using (2.3), (2.5) and (3.1), we obtain
\[
\begin{align*}
V_t(t, \xi) &= -2UV + 2WVP(1), \\
W_t(t, \xi) &= 2UV - U - 2V^2P(1) + VP(1), \\
Q_t(t, \xi) &= -2WQP(1).
\end{align*}
\]

In a word, we formally derive an equivalent semilinear system to (2.3)
\[
\begin{align*}
y_t(t, \xi) &= -U^2, \\
U_t(t, \xi) &= K(\xi) - P(1)y(\xi) - \frac{1}{1-h} \int_\gamma (2QV - Q)(\eta)(K(\eta) - P(1)y(\eta)) \, d\eta, \\
V_t(t, \xi) &= -2UV + 2WVP(1), \\
W_t(t, \xi) &= 2UV - U - 2V^2P(1) + VP(1), \\
Q_t(t, \xi) &= -2WQP(1).
\end{align*}
\] (3.2)

It follows that
\[
\begin{align*}
y_{\xi t}(t, \xi) &= -2UU_{\xi}, \\
U_{\xi t}(t, \xi) &= UQ(2V - 1) - P(1)y_{\xi}, \\
V_{\xi t}(t, \xi) &= -2UV + 2WVP(1), \\
W_{\xi t}(t, \xi) &= 2UV - U - 2V^2P(1) + VP(1), \\
Q_{\xi t}(t, \xi) &= -2WQP(1).
\end{align*}
\] (3.3)

3.2. Global Solution of the Semilinear System

In this subsection, we turn our attention to finding a global solution for (3.2). Since the spaces \( I, X \) are not Banach spaces, in order to get the global solution, a suitable Banach space should be constructed. Set \( \gamma = y - Id \) where \( Id \) denotes the identity. Then, the map \( y \mapsto \gamma \) is obviously a bijection between \( I \) and \( H^1(S) \). Therefore, there is a bijection \( (y, U, V, W, Q) \mapsto (\gamma, U, V, W, Q) \) between \( X \) and \( Y := H^1(S) \times H^1(S) \times L^\infty(S) \times L^\infty(S) \times L^\infty(S) \) which is a Banach space. Hence, system (3.2) becomes the following equivalent form
\[
\begin{align*}
\gamma_t(t, \xi) &= -U^2, \\
U_t(t, \xi) &= K(\xi) - P(1)(\gamma + Id)(\xi) - \frac{1}{1-h} \int_S (2QV - Q)(\eta)(K(\eta) - P(1)(\gamma + Id)(\eta)) \, d\eta, \\
V_t(t, \xi) &= -2UV + 2WVP(1), \\
W_t(t, \xi) &= 2UV - U - 2V^2P(1) + VP(1), \\
Q_t(t, \xi) &= -2WQP(1).
\end{align*}
\] (3.4)

We will prove that system (3.4) as the ODEs is locally well-posed in the Banach space \( Y \). Before that, let’s choose an appropriate initial data \( (y_0, U_0, V_0, W_0, Q_0) \) as follows
\[
\begin{align*}
y_0(\xi) + \int_0^{\gamma_0(\xi)} u_0^2 \, dx &= (1 + h)\xi, \\
U_0(\xi) &= u_0 \circ y_0(\xi), \\
V_0(\xi) &= \frac{1}{1+u_0^2 \circ y_0(\xi)} \\
W_0(\xi) &= \frac{1}{1+u_0^2 \circ y_0(\xi)} \\
Q_0(\xi) &= (1 + u_0^2 \circ y_0) y_0^2(\xi) = 1 + h.
\end{align*}
\] (3.5)

Noting that \( u_0 \in H^1(S) \), the function \( y_0(\xi) \) is well-defined as the map \( y_0 \mapsto y_0(\xi) + \int_0^{\gamma_0(\xi)} u_0^2 \, dx \) is continuous and strictly increasing. It is straightforward to verify that \( y_0(0) = 0, \ y_0^\xi > 0 \) and \( y_0(\xi) \in I \), which follows that \( U_0 \in W^{1,\infty}(S), \ V_0 \in L^\infty(S), \ W_0 \in L^\infty(S), \ Q_0 \in L^\infty(S) \). Therefore,
\[
(\gamma_0, U_0, V_0, W_0, Q_0) \in [W^{1,\infty}(S)]^2 \times [L^\infty(S)]^3 \subset Y.
\]

The following lemma gives the local existence of solutions to system (3.4) with initial data (3.5).
Lemma 3.1. Let $u_0 \in H^1(S)$. Then there exists a time $T > 0$ such that the problem (3.4)–(3.5) has a solution $(\gamma(t), U(t), V(t), W(t), Q(t))$ in $L^\infty(0; T; Y)$.

Proof. Note that the initial data $(\gamma_0, U_0, V_0, W_0, Q_0) \in [W^{1, \infty}(S)]^2 \times [L^\infty(S)]^3 \subset Y$. In order to prove the local existence, we only need to demonstrate that the right side of (3.4) is Lipschitz continuous on every bounded set $B_M \subset Y$ with

$$B_M = \left\{(\gamma(t), U(t), V(t), W(t), Q(t)) \in Y \mid \|(\gamma(t), U(t), V(t), W(t), Q(t))\|_Y \leq M\right\}.$$  

Here we just verify the second equation of the right side of (3.4) is Lipschitz continuous, since the others are similar and more easier. For $\xi \in S$, it is obvious that $UQV + \frac{1}{2}Q(1 - V)$ is Lipschitz continuous from $B_M$ to $L^2(S)$. Then $K(\xi)$ is Lipschitz continuous from $B_M$ to $H^1(S)$. In a similar way, $P(\xi)(\gamma + \operatorname{Id})$ and $\int_S[K(\eta) - P(\xi)(\gamma + \operatorname{Id})(\eta)](QV)(\eta)d\eta$ are Lipschitz continuous from $B_M$ to $H^1(S)$. Therefore, the solution Lipschitz continuity is true. Thanks to the standard theory of ODEs in Banach spaces, there exists a solution $(\gamma(t), U(t), V(t), W(t), Q(t))$ to problem (3.4)–(3.5) on a time interval $[0, T]$ with $T > 0$. \hfill $\square$

Remark 3.2. Recall that the map $y \mapsto \gamma$ is a bijection between $I$ and $H^1(S)$. Then we know that system (3.2) as well has a local solution $(y(t), U(t), V(t), W(t), Q(t))$ in $L^\infty(0; T; X)$ from Lemma 3.1.

Theorem 3.3. Following the similar proof of [18], we see that the solution $(\gamma(t), U(t), V(t), W(t), Q(t))$ belongs to $L^\infty(0; T; [W^{1, \infty}(S)]^2 \times [L^\infty(S)]^3)$ with the initial data $(\gamma_0, U_0, V_0, W_0, Q_0)$, so that

$$(\gamma_\xi(t), U_\xi(t), V(t), W(t), Q(t)) \in [L^\infty(S)]^5$$

is a solution to (3.3). Moreover, we can assert that $y_\xi \geq 0$ and $\operatorname{meas}(A) = 0$, $\operatorname{meas}(\mathcal{N}) = 0$ where

$$A = \left\{(t, \xi) \in [0, T] \times \mathbb{R} \mid y_\xi(t, \xi) = 0\right\},$$

$$\mathcal{N} = \left\{t \in [0, T] \mid y_\xi(t, \xi) = 0, \quad \xi \in \mathbb{R}\right\}.$$  

Next, we will extend the local solution to the global solution.

Theorem 3.4. Let $u_0 \in H^1(S)$. Then the local solution $(\gamma(t), U(t), V(t), W(t), Q(t))$ to the problem (3.4)–(3.5) is global. Moreover,

$$(\gamma(t), U(t), V(t), W(t), Q(t)) \in [W^{1, \infty}(S)]^2 \times [L^\infty(S)]^3$$

for all time $t \geq 0$.

Proof. To obtain the global existence, without loss of generality, it suffices to demonstrate the solution $(\gamma(t), U(t), V(t), W(t), Q(t))$ is uniformly bounded on any bounded time interval $[0, T]$ with $T > 0$.

We first claim that

$$W^2 + V^2 = V, \quad \text{for a.e. } \xi, \quad (3.6)$$

$$y_\xi = VQ, \quad U_\xi = WQ, \quad \text{for a.e. } \xi. \quad (3.7)$$

Taking advantage of (3.2) and (3.3), we see

$$(W^2 + V^2)_t = V_t, \quad (3.8)$$

$$(U_\xi - WQ)_t = P(1)(VQ - y_\xi), \quad (3.9)$$

$$(VQ - y_\xi)_t = WQ - U_\xi. \quad (3.10)$$

Observing that $W^2_\xi + V^2_\xi = V_\xi$, $U_\xi = W_\xi Q_0$ and $y_\xi = V_\xi Q_0$ at initial time $t = 0$, then (3.6) holds by (3.8). If $P(1) = 0$, (3.7) is obviously true. Otherwise, differentiating (3.9) with respect to $t$ and then using the elliptic equations theory, we prove (3.7).
Next, set \( \tilde{E}(t) := \int_S Q - QV \, d\xi \) and \( \tilde{F}(t) := \int_S 2UQV - UQ \, d\xi \). It’s easy to deduce that \( \frac{\partial}{\partial t} \tilde{E} = \frac{\partial}{\partial t} \tilde{F} = 0 \), which means \( \tilde{E}(t) \) and \( \tilde{F}(t) \) remain constants in time. So

\[
\tilde{E}(t) = \tilde{E}(0) := \tilde{E}_0 = \int_S Q_0 - Q_0V_0 \, d\xi = \int_S (u_{0x}^2 \circ y_0) y_0 \, d\xi = \int_S u_{0x}^2 \, dx, \tag{3.11}
\]

\[
P(1) = \tilde{F}(t) = \tilde{F}(0) := \tilde{F}_0 = \int_S (1 - u_{0x}^2 \circ y_0)(u_0 \circ y_0) y_0 \, d\xi = \int_S u_0 - u_0 u_{0x}^2 \, dx. \tag{3.12}
\]

We finally prove that the local solution is uniformly bounded on any bounded time interval. (3.6) implies that

\[
0 \leq V \leq 1 \quad \text{and} \quad |W| \leq \frac{1}{2}, \tag{3.13}
\]

whence \( V(t, \xi), W(t, \xi) \) are uniformly bounded in \( L^\infty(0, T; L^\infty(S)) \).

Notice that \( 0 < 1 + h = Q_0(\xi) \in L^\infty(S) \). By solving the fifth equation of (3.4), we find

\[
0 < Q(t, \xi) = Q_0(\xi)e^{\int_0^t (\gamma_2 W P(1)) \, d\tau} \leq (1 + h)e^{\tilde{F}_0T}. \tag{3.14}
\]

Hence, \( Q(t, \xi) \in L^\infty(0, T; L^\infty(S)) \).

(3.7), (3.13) and (3.14) infer that

\[
\|U_\xi\|_{L^\infty} \leq \frac{1}{2}(1 + h)e^{\tilde{F}_0T},
\]

\[
\|\gamma_\xi\|_{L^\infty} \leq \|y_\xi\|_{L^\infty} + 1 \leq (1 + h)e^{\tilde{F}_0T} + 1. \tag{3.15}
\]

Moreover, using the conservative quantities \( \tilde{E}(t) \) and \( \tilde{F}(t) \), we get the uniform boundedness of \( u \). In fact, for any fixed \( \xi, \eta \in S \),

\[
\int_S (U(\xi) - U(\eta))(2QV - Q)(\eta) \, d\eta = U(\xi) \int_S (2QV - Q)(\eta) \, d\eta - \int_S (2UQV - UQ)(\eta) \, d\eta,
\]

\[
= U(\xi)(1 - h) - \tilde{F}_0.
\]

On the other hand, we discover

\[
\left| \int_S (U(\xi) - U(\eta))(2QV - Q)(\eta) \, d\eta \right| \leq \left| \int_S \int_\eta^\xi U_\xi d\xi(2QV - Q) \, d\eta \right|
\]

\[
\leq \|U_\xi\|_{L^\infty} \left( \|2QV\|_{L^\infty} + \|Q\|_{L^\infty} \right)
\]

\[
\leq \frac{3}{2}(1 + h)^2 e^{\tilde{F}_0T} \triangleq B.
\]

Thus, we conclude that

\[
|U(\xi)| \leq \frac{B + |\tilde{F}_0|}{1 - h}, \ a.e. \ \xi \in S. \tag{3.16}
\]

It follows that

\[
|\gamma(t, \xi)| \leq |y(t, \xi) - \xi| \leq |y_0(\xi) - \xi| + \int_0^t U^2(\tau, \xi) \, d\tau \leq h + T \frac{(B + |\tilde{F}_0|)^2}{(1 - h)^2}, \ a.e. \ \xi \in S. \tag{3.17}
\]

From (3.13)–(3.17), we deduce that the solution \((\gamma(t), U(t), V(t), W(t), Q(t))\) remains bounded on any bounded time interval \([0, T]\). This proves the theorem. \( \square \)

Remark 3.5. Similarly, for \( u_0 \in H^1(S) \), we know that system (3.2) with initial data \((y_0, U_0, V_0, W_0, Q_0)\) has a global solution \((y(t), U(t), V(t), W(t), Q(t))\) in \( X \) for any time \( t \geq 0 \).
4. Global Admissible Conservative Weak Solution for the Original Equation

In this section, we are going to prove the global existence of admissible conservative weak solution for (2.3).

Theorem 4.1. Let \( u_0(x) \in H^1(S) \). Then the problem (2.3) has a global admissible conservative weak solution in the sense of Definition 1.2.

Proof. From Remark 3.5, we get a global solution \((y, U, V, W, Q)\) to system (3.2). Hence, for each fixed \( \xi \in \mathbb{S} \), the map \( t \mapsto y(t, \xi) \) gives a solution to problem (3.1).

Write \( u(t, x) = U(t, \xi) \) if \( x = y(t, \xi) \). (4.1)

We have to explain the definition makes sense. Indeed, by Theorem 3.3 we deduce that \( y_\xi(t, \xi) \geq 0 \) for all \( t \geq 0 \) and a.e. \( \xi \). Therefore, the map \( \xi \mapsto y(t, \xi) \) is nondecreasing. If \( \xi_1 < \xi_2 \) but \( y(t, \xi_1) = y(t, \xi_2) \), we have

\[
0 = \int_{\xi_1}^{\xi_2} y_\xi(t, \eta)d\eta = \int_{\xi_1}^{\xi_2} (QV)(t, \eta)d\eta.
\]

If \( Q \neq 0 \), we discover \( V = 0 \) in \([\xi_1, \xi_2]\), which implies \( W = 0 \) in \([\xi_1, \xi_2]\). It follows that

\[
U(t, \xi_2) - U(t, \xi_1) = \int_{\xi_1}^{\xi_2} U_\xi(\eta)d\eta = \int_{\xi_1}^{\xi_2} (WQ)(\eta)d\eta = 0.
\]

Otherwise, if \( Q = 0 \), the above equality also makes sense. Hence, the map \( (t, x) \mapsto u(t, x) \) is well-defined for any \( t \geq 0 \) and \( x \in \mathbb{S} \).

(3.7) and (4.1) infer that

\[
u_x(t, y(t, \xi)) = \frac{W}{V}, \quad \text{as} \quad y_\xi \neq 0.
\]

(4.2)

Changing the variables and applying (3.7) and (4.2), we find

\[
E(t) = \int_{\mathbb{S}} u_x^2(t, x)dx = \int_{\mathbb{S}\cap\{y_\xi \neq 0\}} u_x^2(t, y(t, \xi))y_\xi d\xi
\]

\[
= \int_{\mathbb{S}\cap\{y_\xi \neq 0\}} (Q - VQ)(t, \xi)d\xi = \int_{\mathbb{S}} (Q - VQ)(t, \xi)d\xi
\]

\[
= \bar{E}(t) = \bar{E}_0 = \int_{\mathbb{S}} u_0^2 dx.
\]

(4.3)

Similarly, we gain

\[
F(t) = \int_{\mathbb{S}} (u - uu_x^2)(t, x)dx = \int_{\mathbb{S}\cap\{y_\xi \neq 0\}} (u - uu_x^2)(t, y(t, \xi))y_\xi d\xi
\]

\[
= \int_{\mathbb{S}\cap\{y_\xi \neq 0\}} (2UVQ - UQ)(t, \xi)d\xi = \int_{\mathbb{S}} (2UVQ - UQ)(t, \xi)d\xi
\]

\[
= \bar{F}(t) = \bar{F}_0 = \int_{\mathbb{S}} u_0(1 - u_{0x}^2) dx.
\]

(4.4)

(4.1) and (4.3) infer that \( u \) belongs to \( L^\infty(\mathbb{R}^+; H^1(\mathbb{S})) \). We have to prove that \( u \) satisfies Eq. (2.3). In light of (3.3), we discover \( y_{\xi t} = U_\xi(t, \xi) \). Therefore, for any \( \varphi \in C_0^\infty(\mathbb{R}^+; D(\mathbb{S})) \), applying the change of
variables, we see
\[
\int_{\mathbb{R}^+} \int_S (u \varphi_{tx} + u^2 u_x \varphi_x) (t, x) dx dt = \int_{\mathbb{R}^+} \int_S (u \varphi_{tx} + u^2 u_x \varphi_x) (t, y(t, \xi)) y_\xi d\xi dt
\]
\[
= \int_{\mathbb{R}^+} \int_S U \varphi_{tx}(t, y(t, \xi)) y_\xi + U^2 U \varphi_x(t, y(t, \xi)) d\xi dt
\]
\[
= \int_{\mathbb{R}^+} \int_S U (\varphi(t, y(t, \xi)))_t + \left(U^3 (\varphi(t, y(t, \xi)))_t \right) d\xi dt
\]
\[
= -\int_{\mathbb{R}^+} \int_S U_\xi (\varphi(t, y(t, \xi)))_t d\xi dt
\]
\[
= \int_{\mathbb{R}^+} \int_S U_\xi \varphi(t, y(t, \xi)) d\xi dt + \int_S U_0 \xi (\varphi(0, y_0(\xi))) d\xi
\]
\[
= \int_{\mathbb{R}^+} \int_S (2UQV - UQ) \varphi(t, y(t, \xi)) d\xi dt + \int_S u_0 x (y_0(\xi)) y_0 \varphi(0, y_0(\xi)) d\xi
\]
\[
= \int_{\mathbb{R}^+} \int_S (u - uu_x^2) \varphi(t, x) dx dt + \int_S u_0 x (\varphi(0, x)) dx
\]
where we use
\[
(\varphi(t, y(t, \xi)) y_\xi)_t = \varphi_{xt}(t, y(t, \xi)) y_\xi - U^2 (\varphi_x(t, y(t, \xi)))_t - 2UU_\xi \varphi_x(t, y(t, \xi))
\]
in the third equality.

Similarly, for any \( \phi, \psi \in C_0^\infty(\mathbb{R}_+; D(S)) \), we have
\[
\int_{\mathbb{R}^+} \int_S \left(u \phi_t - \frac{1}{3} u^3 \phi_x \right) (t, x) dx dt = -\int_{\mathbb{R}^+} \int_S H(t, x) \phi(t, x) dx dt - \int_S u_0 (x) \phi(0, x) dx
\]
and
\[
\int_{\mathbb{R}^+} \int_S \left(u_x^2 \psi_t - u^2 u_x^2 \psi_x \right) (t, z) dz dt = -\int_{\mathbb{R}^+} \int_S \left[ (u_x^2)_t - 2P(1) u_x \right] (t, z) dz dt - \int_S u_0^2 (z) \psi(0, z) dz
\]
where \( H(t, x) = \partial^{-1}_x (u - uu_x^2) - f(t) \) is a bounded variable.

In a word, we verify that \( u \) is indeed a global admissible conservative weak solution to the Cauchy problem (2.3) in the sense of Definition 1.2. This completes the proof of Theorem 4.1.

\[\square\]

5. Uniqueness of the Global Admissible Conservative Weak Solution

5.1. Useful Lemmas

Since \( u(t, x) \) is a global admissible conservative weak solution to the Cauchy problem (2.3) in the sense of Definition 1.2, we have \( |u|_{H^1} \leq C |u_0|_{H^1} \).

For smooth case, owing to (2.5) and (3.1), we can easily deduce that
\[
\frac{d}{dt} \int_0^t u_x^2 (t, z) dz = \int_0^t \left[ (u_x^2)_x - 2P(1) u_x \right] (t, z) dz - \int_0^t u_x u_x^2 (t, 0), \quad y(0) = y_0(\xi).
\](5.1)

However, in the weak sense, some special test functions need to be selected to solve (3.1). The key idea is to combine (3.1) and (5.1) in the weak sense to get a unique solution of (3.1).
Instead of the variables \((t, x)\), it is convenient to work with an adapted set of variables \((t, \beta)\), where \(\beta\) is implicitly defined as

\[
y(t, \beta) + \int_0^y u_x^2(t, z)dz = (1 + h)\beta.
\] (5.2)

Now, we present some useful lemmas.

**Lemma 5.1** [5]. Let the map \(x \mapsto \phi(x)\) be an absolutely continuous from \([a, b]\) to \([c, d]\). Suppose \(\phi(x)\) is strictly monotonic. Then for any measurable set \(A \subseteq [a, b]\), we have \(\text{meas}(\phi(A)) = \int_A \phi_x dx\).

**Lemma 5.2.** Let the map \(x \mapsto f(x)\) be a bijection from \([a, b]\) to \([c, d]\). Suppose that \(f(x)\) is absolutely continuous and strictly monotonic. Then for any \(E \subseteq [c, d]\) and \(\text{meas}(E) = d - c\), we have \(\text{meas}(f^{-1}(E)) = b - a\).

**Proof.** Let \(A = f^{-1}(E^c)\) where \(E^c\) is the complement of \(E\). If \(\text{meas}(f^{-1}(E^c)) = \delta > 0\), we see from Lemma 5.1 for \(\phi(x) = f(x)\) that

\[
0 = \text{meas}(E^c) = \int_{E^c} dy = \int_{f^{-1}(E^c)} f_x dx.
\]

This means \(f_x \big|_{a,c}^0 = 0\) in \(f^{-1}(E^c)\), which contradicts the strict monotonicity. Therefore, we have \(\text{meas}(f^{-1}(E^c)) = 0\), and thus \(\text{meas}(f^{-1}(E)) = b - a\). \(\square\)

**Lemma 5.3.** Let \(u = u(t, x)\) be a global admissible conservative weak solution of (2.3). Then, for every \(t \geq \tau > 0\), we have

\[
\lim_{\epsilon \to 0} \int_\tau^t \int_\frac{8}{5} \frac{8}{\epsilon} (u^2 u_x^2)(s, z)dzds = \lim_{\epsilon \to 0} \int_\tau^t \int_\frac{1}{\frac{8}{5}} \frac{8}{\epsilon} (u^2 u_x^2)(s, z)dzds = \lim_{\epsilon \to 0} \int_\tau^t \int_\frac{1}{\frac{8}{5}} \frac{8}{\epsilon} (u^2 u_x^2)(s, z)dzds.
\]

**Proof.** For \(\epsilon > 0\) small enough, let

\[
p_\epsilon(s, z) = \begin{cases} 0 & 0 \leq z < \frac{1}{8}\epsilon, \\ \frac{8}{\epsilon}(z - \frac{1}{8}) & \frac{1}{8}\epsilon \leq z < \frac{2}{8}\epsilon, \\ 1 & \frac{2}{8}\epsilon \leq z < \frac{3}{8}\epsilon, \\ \frac{1}{1 - \frac{2}{3}}(1 - \frac{8}{3} - z) & \frac{3}{8}\epsilon \leq z < 1 - \frac{1}{8}\epsilon, \\ 0 & 1 - \frac{1}{8}\epsilon \leq z < 1, \end{cases} \quad (5.3)
\]

\[
\chi_\epsilon(s) = \begin{cases} 0 & 0 \leq s < \tau - \epsilon, \\ \frac{1}{\epsilon}(s - \tau + \epsilon) & \tau - \epsilon \leq s < \tau, \\ 1 & \tau \leq s < t, \\ 1 - \frac{1}{\epsilon}(s - t) & t \leq s < t + \epsilon, \\ 0 & t + \epsilon \leq s. \end{cases} \quad (5.4)
\]

Define

\[
\psi_\epsilon(s, z) := \min\{p_\epsilon(s, z), \chi_\epsilon(s)\}. \quad (5.5)
\]

By an approximation argument, (1.9) remains valid for any test function \(\psi\) which is Lipschitz continuous with compact support. Using \(\psi_\epsilon\) as the test function in (1.9) we obtain

\[
\int_{S^+} \int_S (u_x^2 \psi_\epsilon t - u_x^2 \psi_\epsilon x)(s, z)dzds = -\int_{S^+} \int_S ((u_x^2)_{x} - 2P(1)u_x)(s, z)dzds. \quad (5.6)
\]
Taking the limit of (5.6) as $\epsilon \to 0$, we find
\[
- \lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{\frac{\xi}{\epsilon}}^{1} \frac{8}{\epsilon} (u^2 u_x^2)(s, z)dzds + \lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{\frac{\xi}{\epsilon}}^{1} \frac{1}{1 - \frac{\epsilon}{2}} (u^2 u_x^2)(s, z)dzds
\]
\[
= - \lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{\frac{\xi}{\epsilon}}^{1} \frac{1}{1 - \frac{\epsilon}{2}} (1 - \frac{\epsilon}{8} - z) [(u^2)_x - 2P(1)u_x](s, z)dzds.
\]

This completes the proof of the lemma. □
Lemma 5.4. Let \( u = u(t, x) \) be a global admissible conservative weak solution of (2.3). Then, for every \( t \geq 0 \), the maps \( \beta \mapsto y(t, \beta) \) and \( \beta \mapsto u(t, y(t, \beta)) \) implicitly defined by (5.2) are Lipschitz continuous with constant \( 1 + h \).

Proof. 1. For fixed time \( t \geq 0 \), it’s easy to deduce that \( y(t, \beta) \) is continuous and strictly monotonic. Moreover, we have \( y(t, 0) = 0 \), \( y(t, 1) = 1 \) and \( y(t, \beta + 1) = y(t, \beta) + 1 \). If \( \beta_1 < \beta_2 \), then
\[
y(t, \beta_2) - y(t, \beta_1) = -\int_{y(t, \beta_1)}^{y(t, \beta_2)} u_x^2(z)dz + (1 + h)(\beta_2 - \beta_1) \leq (1 + h)(\beta_2 - \beta_1). \tag{5.12}
\]

2. To prove the Lipschitz continuous of the map \( \beta \mapsto u(t, y(t, \beta)) \), let’s suppose that \( \beta_1 < \beta_2 \). By virtue of (5.12), we see
\[
|u(t, y(t, \beta_2)) - u(t, y(t, \beta_1))| \leq \int_{y(t, \beta_1)}^{y(t, \beta_2)} |u_x|dz \leq \int_{y(t, \beta_1)}^{y(t, \beta_2)} \frac{1}{2}(1 + u_x^2)dz
\]
\[
\leq \frac{1}{2} \left[ y(t, \beta_2) - y(t, \beta_1) + \int_{y(t, \beta_1)}^{y(t, \beta_2)} u_x^2(z)dz \right]
\]
\[
\leq \frac{1}{2}(1 + h)(\beta_2 - \beta_1). \tag{5.13}
\]

Lemma 5.5. Let \( u = u(t, x) \) be a global admissible conservative weak solution of (2.3). Then, for \( y_0(\xi) \) satisfying \( y_0(\xi) + \int_{0}^{y_0(\xi)} u_0^2(z)dz = (1 + h)\xi \), there exists a unique Lipschitz continuous map \( t \mapsto y(t) := y(t, \beta(t, \xi)) \) which solves (3.1).

Proof. 1. Since \( u(t, x) \) belongs to \( L^\infty(\mathbb{R}^+; H^1 \rightarrow C^2) \), then \( |u(t, x)| \leq M \). For any \( b > 0 \), suppose that \( a > 0 \) is small enough such that \( a \leq \frac{b}{2M} \), then \( u(t, x) \in L^\infty(0, a; C[y_0(\xi) - b, y_0(\xi) + b]) \). Applying the Arezela-Ascoli theorem and the Schauder fixed point theorem, the Cauchy problem (3.1) has a solution \( y(t) \) on \([0, a]\). Moreover, \( y(t) \) is Lipschitz continuous with \( t \) on \([0, a]\). For other cases, we can use the continuous method. Thus, \( y(t) \) is Lipschitz continuous with time \( t \) for any \( t \geq 0 \).

2. Define
\[
\psi_2(\epsilon, z) := \min\{p_2(\epsilon, z), \chi(\epsilon)\} \tag{5.14}
\]
where
\[
p_2(\epsilon, z) = \left\{
\begin{array}{ll}
0 & 0 \leq z < \frac{1}{8} \epsilon, \\
\frac{8}{\epsilon} (z - \frac{\epsilon}{8}) & \frac{1}{8} \epsilon \leq z < \frac{3}{8} \epsilon, \\
1 & \frac{3}{8} \epsilon \leq z < \frac{3}{8} \epsilon + y(s), \\
1 - \frac{8}{\epsilon} (z - \frac{\epsilon}{8} - y(s)) & \frac{3}{8} \epsilon + y(s) \leq z < \frac{5}{8} \epsilon + y(s), \\
0 & \frac{5}{8} \epsilon + y(s) \leq y < 1,
\end{array}
\right.
\]
and \( \chi(\epsilon) \) as (5.4).

Using \( \psi_2 \) as the test function in (1.9) and taking the limit as \( \epsilon \to 0 \), we find
\[
\int_{0}^{u(t)} u_x^2(z)dz - \int_{0}^{u(\tau)} u_x^2(z)dz = \int_{\tau}^{t} \int_{0}^{y(s)} ((u^2)_x - 2P(1)u_x)dzds - \lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{\frac{8}{\epsilon} + y(s)}^{\frac{8}{\epsilon} + y(s)} \frac{8}{\epsilon} (u^2 u_x^2)dzds
\]
\[
- \lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{\frac{8}{\epsilon} + y(s)}^{\frac{8}{\epsilon} + y(s)} \frac{8}{\epsilon} u_x^2 (u^2(s, y(s)) - u^2(s, z))dzds. \tag{5.15}
\]
Since \( u_x \in L^2(\mathbb{S}) \), similar to [1], taking advantage of the Cauchy’s inequality and the Dominated Convergence Theorem, we deduce
\[
\lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{\frac{8}{\epsilon} + y(s)}^{\frac{8}{\epsilon} + y(s)} \frac{8}{\epsilon} u_x^2 (u^2(s, y(s)) - u^2(s, z))dzds = 0 \quad \text{for a.e.} \ y(s) \in \mathbb{S}.
\]
By Lemma 5.2 and the fact that the map \( \xi \mapsto y(t) := y(t, \beta(t, \xi)) \) is strictly monotonic and Lipschitz continuous from \([0, 1]\) to \([0, 1]\), we see

\[
\lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{\frac{\epsilon}{2} + y(s)}^{\frac{\epsilon}{2} + y(s)} \frac{8}{c} u_x^2 \left( u^2(s, y(s)) - u^2(s, z) \right) dz \, ds = 0 \quad \text{for a.e. } \xi \in \mathbb{S}.
\]

Thus, for almost everywhere \( \xi \in \mathbb{S} \), we discover

\[
\int_{0}^{y(t)} u_x^2(z) \, dz = \int_{0}^{y(\tau)} u_x^2(z) \, dz = \int_{\tau}^{t} \int_{0}^{y(s)} \left( (u^2)_x - 2P(1) u_x \right) \, dz \, ds - \lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{8}{c} (u^2 u_x^2) \, dz \, ds. \quad (5.17)
\]

Taking \( \tau = 0 \) in (5.17), we get that for almost everywhere \( \xi \in \mathbb{S} \),

\[
\int_{0}^{y(t)} u_x^2(z) \, dz = \int_{0}^{y_0(\xi)} u_x^2(z) \, dz + \int_{0}^{t} \int_{0}^{y(s)} \left( (u^2)_x - 2P(1) u_x \right) \, dz \, ds - \lim_{\epsilon \to 0} \int_{0}^{t} \int_{\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{8}{c} (u^2 u_x^2) \, dz \, ds.
\quad (5.18)

3. Consider the integral equation:

\[
(1 + h) \beta(t, \xi) = y(t) + \int_{0}^{y(t)} u_x^2(z) \, dz
\]

\[
= y_0(\xi) + \int_{0}^{y_0(\xi)} u_x^2(z) \, dz + \int_{0}^{t} \int_{0}^{y(s)} -u^2(s, y(s)) + \int_{0}^{y(s)} (u^2)_x - 2P(1) u_x \, dz \, ds
\]

\[
- \lim_{\epsilon \to 0} \int_{0}^{t} \int_{\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{8}{c} (u^2 u_x^2) \, dz \, ds
\]

\[
= (1 + h) \xi + \int_{0}^{t} \int_{0}^{y(s)} -2P(1) u_x \, dz \, ds - \int_{0}^{t} u^2(s, 0) \, ds - \lim_{\epsilon \to 0} \int_{0}^{t} \int_{\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{8}{c} (u^2 u_x^2) \, dz \, ds
\]

\[
= (1 + h) \xi + \int_{0}^{t} \int_{0}^{y(s)} -2P(1) u_x \, dz \, ds - \int_{0}^{t} u^2(s, 0) \, ds - A(t) \quad (5.19)
\]

where \( A(t) := \lim_{\epsilon \to 0} \int_{0}^{t} \int_{\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{8}{c} (u^2 u_x^2) \, dz \, ds \) is bounded by Lemma 5.3.

Introducing the function

\[
G(s, \beta(s, \xi)) = \int_{0}^{y(s, \beta(s, \xi))} -2P(1) u_x \, dz,
\]

we rewrite (5.19) in the form

\[
\beta(t, \xi) = \xi + \frac{1}{1 + h} \int_{0}^{t} G(s, \beta(s, \xi)) \, ds - \frac{1}{1 + h} \int_{0}^{t} u^2(s, 0) \, ds - \frac{1}{1 + h} A(t). \quad (5.21)
\]

For each fixed \( t \geq 0 \), since \( \|u\|_{H_1} \leq C \|u_0\|_{H_1} \) and the map \( \beta \mapsto y(t, y(t, \beta)) \) is Lipschitz continuous, we deduce that the function \( \beta \mapsto G(s, \beta) \) is uniformly bounded and Lipschitz continuous:

\[
|G(s, \beta_1) - G(s, \beta_2)| \leq C_{u_0} |\beta_1 - \beta_2|.
\]

Moreover, for each fixed \( t \geq 0 \), the map \( \xi \mapsto \beta(t, \xi) \) is strictly monotonic and Lipschitz continuous. Indeed, owing to (5.22), we see

\[
|\beta(t, \xi_2) - \beta(t, \xi_1)| \leq |\xi_2 - \xi_1| + \frac{1}{1 + h} \int_{0}^{t} |G(s, \beta(s, \xi_2)) - G(s, \beta(s, \xi_1))| \, ds
\]

\[
\leq |\xi_2 - \xi_1| + C_{u_0} \int_{0}^{t} |\beta(s, \xi_2) - \beta(s, \xi_1)| \, ds
\]

\[
\leq e^{C_{u_0} t} |\xi_2 - \xi_1|
\]

where the last inequality is obtained by using the Gronwall lemma.
Assuming \( \xi_2 > \xi_1 \), we have
\[
\beta(t, \xi_2) - \beta(t, \xi_1) = \xi_2 - \xi_1 + \frac{1}{1 + h} \int_0^t G(s, \beta(s, \xi_2)) - G(s, \beta(s, \xi_1)) ds \geq (\xi_2 - \xi_1)(1 - C_{w_0} t).
\]
This implies that the monotonicity makes sense when \( t \) is sufficiently small. Without loss of generality, we assume that \( t \) is small enough, otherwise we use the continuous method. For each fixed \( t \geq 0 \), being the composition of two Lipschitz functions, the maps \( \xi \mapsto G(t, y(t, \beta(t, \xi))) \) and \( \xi \mapsto u(t, y(t, \beta(t, \xi))) \) are Lipschitz continuous.

4. Thanks to the Lipschitz continuity of the function \( G \), the integral equation (5.21) has a unique solution by a standard fixed point theory. As a matter of fact, consider the Banach space of all continuous function \( \beta : \mathbb{R}^+ \to \mathbb{R} \) with weighted norm
\[
||\beta|| := \sup_{t \geq 0} e^{-2Ct}|\beta(t)|.
\]
On this space, we claim that the Picard map
\[
(P\beta)(t) = \xi + \int_0^t G(s, \beta(s, \xi)) ds - \int_0^t u^2(s, 0) ds - A(t)
\]
is a strict contraction. Indeed, assume \( ||\beta_2 - \beta_1|| = a > 0 \). This implies
\[
|\beta_2(s) - \beta_1(s)| \leq ae^{2Cs}.
\]
Hence,
\[
|(P\beta_2)(t) - (P\beta_1)(t)| = \left| \int_0^t G(s, \beta_1(s)) - G(s, \beta_2(s)) ds \right| \leq C \int_0^t |\beta_1(s) - \beta_2(s)| ds
\]
\[
\leq \int_0^t C ae^{2Cs} ds \leq \frac{a}{2} e^{2Ct},
\]
that is \( ||P\beta_2 - P\beta_1|| \leq \frac{a}{2} \).

Since the map \( \beta \mapsto y(t, \beta) \) is strictly monotonic increasing and Lipschitz continuous, we obtain a unique solution \( y(t) := y(t, \beta(t, \xi)) \) from (5.19). Being the composition of two Lipschitz functions, the map \( t \mapsto y(t, \beta(t, \xi)) \) is Lipschitz continuous and provides a unique solution to (5.19).

5. Finally, we prove the uniqueness of (3.1). Assume there are two different functions \( y_1(\cdot) \) and \( y_2(\cdot) \), both satisfying (5.18) together with the characteristic equation (3.1). Choose the measurable functions \( \beta_1 \) and \( \beta_2 \) such that \( y_1(t) = y(t, \beta_1(t)) \) and \( y_2(t) = y(t, \beta_2(t)) \). Then \( y_1(t) \) and \( y_2(t) \) satisfy (5.19) with the same initial data. This contradicts the uniqueness of (5.19) proved in step 4.

**Lemma 5.6.** For any \( 0 \leq \tau \leq t \), the map \( t \mapsto u(t, y(t)) \) is Lipschitz continuous and the following equality holds
\[
u(t, y(t)) - u(\tau, y(\tau)) = \int_\tau^t H(s, y(s)) ds,
\]
where \( H(s, x) = \partial_x^{-1}(u - uu_x^2) - f(s) \) is a bounded variable.

**Proof.** By virtue of (1.8), for any test function \( \phi \in C_0^\infty(\mathbb{R}^+; D(S)) \), one has
\[
\int_{\mathbb{R}^+} \int_S \left( u \phi_t - \frac{1}{3} u^3 \phi_x \right)(s, z) dz ds = - \int_{\mathbb{R}^+} \int_S (H \phi)(s, z) dz ds - \int_S u_0(z) \phi(0, z) dz.
\]
Given any \( \psi \in C_0^\infty(\mathbb{R}^+; D(S)) \), let \( \phi = \psi_x \). Since the map \( x \mapsto u(t, x) \) is absolutely continuous, integrating by parts, we obtain
\[
\int_{\mathbb{R}^+} \int_S (u_x \psi_t - u^2 u_x \psi_x)(s, z) dz ds = \int_{\mathbb{R}^+} \int_S (H \psi_x)(s, z) dz ds - \int_S u_{0x}(z) \psi(0, z) dz.
\]
By an approximation argument, the identity (5.25) remains valid for any test function \( \psi \) which is Lipschitz continuous with compact support. Let \( \psi = \psi_{2\epsilon} \) in (5.14). Taking the limit of (5.25) as \( \epsilon \to 0 \), we have

\[
\int_0^y u_x(t, z)dz - \int_0^y u_x(\tau, z)dz = \int_\tau^t H(s, y(s)) - H(s, 0)ds - \lim_{\epsilon \to 0} \int_\tau^t \frac{8}{\epsilon} \frac{(u^2 u_x)(s, z)}{\epsilon}dzds
\]

\[
- \lim_{\epsilon \to 0} \int_\tau^t \frac{\epsilon \psi + y(s)}{\epsilon} u_x(s, z) \left( u^2(s, y(s)) - u^2(s, z) \right)dzds. \tag{5.26}
\]

Applying the Lebesgue Differentiation Theorem, we obtain

\[
\lim_{\epsilon \to 0} \int_\tau^t \frac{\epsilon \psi + y(s)}{\epsilon} u_x(s, z) \left( u^2(s, y(s)) - u^2(s, z) \right)dzds = 0 \text{ for a.e. } y(s) \in S. \tag{5.27}
\]

Combining Lemma 5.2 and the fact that the map \( \xi \mapsto y(s) := y(s, \beta(s, \xi)) \) is strictly monotonic and Lipschitz continuous from \([0, 1]\) to \([0, 1]\), we deduce

\[
\lim_{\epsilon \to 0} \int_\tau^t \frac{\epsilon \psi + y(s)}{\epsilon} u_x(s, z) \left( u^2(s, y(s)) - u^2(s, z) \right)dzds = 0 \text{ for a.e. } \xi \in S. \tag{5.28}
\]

Thus, from (5.26)–(5.28), we see for almost everywhere \( \xi \in S \),

\[
\int_0^y u_x(t, z)dz - \int_0^y u_x(\tau, z)dz = \int_\tau^t H(s, y(s)) - H(s, 0)ds - \lim_{\epsilon \to 0} \int_\tau^t \frac{8}{\epsilon} \frac{(u^2 u_x)(s, z)}{\epsilon}dzds. \tag{5.29}
\]

To prove that the map \( t \mapsto u(t, y(t)) \) is Lipschitz continuous, we modify the previous test functions. For any \( k \in [0, 1] \), let \( \chi_\epsilon(s) \) as (5.4) and

\[
p_{3\epsilon}(s, z) = \begin{cases} 
0 & 0 \leq z < \frac{1}{8} \epsilon, \\
\frac{8}{\epsilon}(z - \frac{1}{8} \epsilon) & \frac{1}{8} \epsilon \leq z < \frac{2}{8} \epsilon, \\
\frac{8}{\epsilon} & \frac{2}{8} \epsilon \leq z < \frac{3}{8} \epsilon + k, \\
1 - \frac{8}{\epsilon}(z - \frac{3}{8} \epsilon - k) & \frac{3}{8} \epsilon + k \leq z < \frac{4}{8} \epsilon + k, \\
0 & \frac{4}{8} \epsilon + k \leq z < 1,
\end{cases} \tag{5.30}
\]

Define

\[
\psi_{3\epsilon}(s, z) := \min\{p_{3\epsilon}(s, z), \chi_\epsilon(s)\}. \tag{5.31}
\]

Using the test function \( \psi = \psi_{3\epsilon} \) in (5.25) and taking the limit as \( \epsilon \to 0 \), we gain

\[
\int_0^k u_x(t, z)dz - \int_0^k u_x(\tau, z)dz = \int_\tau^t H(s, k) - H(s, 0)ds - \lim_{\epsilon \to 0} \int_\tau^t \frac{8}{\epsilon} \frac{(u^2 u_x)(s, z)}{\epsilon}dzds
\]

\[
+ \lim_{\epsilon \to 0} \int_\tau^t \frac{4}{\epsilon} \frac{u^2 u_x}{\epsilon}dzds. \tag{5.32}
\]

In addition, using \( \phi = \psi_{3\epsilon} \) in (5.24) and taking the limit as \( \epsilon \to 0 \), we obtain

\[
\int_0^k u(t, z)dz - \int_0^k u(\tau, z)dz = \int_\tau^t \int_0^k H(s, z)dzds - \frac{1}{3} \lim_{\epsilon \to 0} \int_\tau^t \frac{8}{\epsilon} \frac{u^3}{\epsilon}dzds
\]

\[
+ \frac{1}{3} \lim_{\epsilon \to 0} \int_\tau^t \frac{4}{\epsilon} \frac{u^3}{\epsilon}dzds. \tag{5.33}
\]
Differentiating (5.33) with the variable $k$ and using the fact that $u$ is bounded and absolutely continuous, we have

$$u(t, k) - u(\tau, k) = \int_{\tau}^{t} H(s, k)ds + \lim_{\epsilon \to 0} \int_{\tau}^{t} \frac{1}{\epsilon} \int_{\tau}^{s+\epsilon} 8 \frac{u^2}{\epsilon} u_x dz ds. \quad (5.34)$$

Subtracting (5.34) from (5.32), we receive

$$u(t, 0) - u(\tau, 0) = \int_{\tau}^{t} H(s, 0)ds + \lim_{\epsilon \to 0} \int_{\tau}^{t} \frac{1}{\epsilon} \int_{\tau}^{s+\epsilon} 8 (u^2 u_x)(s, z)dz ds. \quad (5.35)$$

Subtracting (5.35) from (5.29), we get

$$u(t, y(t)) - u(\tau, y(\tau)) = \int_{\tau}^{t} H(s, y(s))ds.$$

Thus,

$$|u(t, y(t)) - u(\tau, y(\tau))| = \left| \int_{\tau}^{t} H(s, y(s))ds \right| \leq C_{u_0} (t - \tau),$$

that is the map $t \mapsto u(t, y(t))$ is Lipschitz continuous. $\square$

### 5.2. Uniqueness of the Global Admissible Conservative Weak Solution

In this subsection, we investigate the uniqueness of the global admissible conservative weak solution to (1.1). Before that, we prove the uniqueness of the global admissible conservative weak solution to (2.3):

**Theorem 5.7.** Let $u(t, x)$ be a global admissible conservative weak solution to the problem (2.3) in the sense of Definition 1.2, then $u(t, x)$ is unique.

**Proof.** The proof will be worked out in several steps, which is similar to [1].

**Step 1.** From Lemmas 5.4–5.6, the maps $(t, \beta) \mapsto (y, u)(t, \beta)$, $\beta \mapsto G(t, \beta)$ and $\beta \mapsto H(t, \beta)$ are Lipschitz continuous. Thanks to Rademacher’s theorem, the partial derivatives $y_t, y_x, u_t, u_x, G_{\beta}$ and $H_{\beta}$ exist almost everywhere. Since $t \mapsto \beta(t, \xi)$ is the unique solution to (5.19), the following holds:

(GC) For a.e. $\xi$ and a.e. $t > 0$, the point $\beta(t, \xi)$ is a Lebesgue point for the partial derivatives $y_t, y_{\beta}, u_t, u_{\beta}, G_{\beta}$ and $H_{\beta}$. Moreover, $y_{\beta}(t, \xi) > 0$ for a.e. $t > 0$.

If (GC) holds, we then say that $t \mapsto \beta(t, \xi)$ is a good characteristic.

**Step 2.** We seek an ODE describing how the quantities $u_{\beta}$ and $y_{\beta}$ vary along a good characteristic. Assume $\tau$, $t \notin \mathcal{N}$ and $\beta(\tau, t, \xi)$ be a good characteristic, where $\beta(\tau, t, \xi)$ is a more general definition of (5.21):

$$\beta(\tau, t, \xi) = \xi + \frac{1}{1 + h} \int_{\tau}^{t} G(s, \beta(s, \xi))ds - \frac{1}{1 + h} \int_{\tau}^{t} u^2(s, 0)ds - \frac{1}{1 + h} \lim_{\epsilon \to 0} \int_{\tau}^{t} \frac{1}{\epsilon} \int_{\tau}^{s+\epsilon} 8 (u^2 u_x^2)dz ds. \quad (5.36)$$

Differentiating (5.36) with respect to $\xi$, we find

$$\frac{d}{d\xi} \beta(\tau, t, \xi) = 1 + \frac{1}{1 + h} \int_{\tau}^{t} G_{\beta}(s, \beta(s, \xi)) \frac{d\beta}{d\xi} ds. \quad (5.37)$$

Next, differentiating with $\xi$ the identity

$$y(t, \beta(t, \xi)) = y(\tau, \xi) - \int_{\tau}^{t} u^2(s, y(s, \beta(s, \xi)))ds, \quad (5.38)$$

we have

$$y_{\xi}(t, \beta(t, \xi)) \frac{d}{d\xi} \beta(\tau, t, \xi) = y_{\xi}(\tau, \xi) - \int_{\tau}^{t} (u^2)_{\beta}(s, y(s, \beta(s, \xi))) \frac{d}{d\xi} \beta(s, t, \xi)ds. \quad (5.39)$$
Finally, differentiating with $\xi$ the identity (5.23), we get
\[
 u_\beta(t, y(t)) \frac{d}{d\xi} \beta(t, t, \xi) = u_\beta(t, y(t)) + \int_\tau^t H_\beta(s, y(s)) \frac{d}{d\xi} \beta(s, t, \xi) \, ds. \tag{5.40}
\]

Combining with (5.37)–(5.40), we thus obtain the following ODEs
\[
 \begin{align*}
 \frac{d}{dt} (u_\beta(t, y(t), t, \xi)) &= \frac{1}{1+h} G_\beta(s, \beta(s, \xi)) \frac{d\beta}{dx}, \\
 \frac{d}{dt} (u_\beta(t, y(t), t, \xi)) &= -(u^2)_\beta(s, y(s, y(t, \xi))) \frac{d}{dx} \beta(t, t, \xi), \\
 \frac{d}{dt} (u_\beta(t, y(t), t, \xi)) &= H_\beta(s, y(s)) \frac{d}{dx} \beta(t, t, \xi). 
\end{align*} \tag{5.41}
\]

In particular, it’s easy to verify that the quantities within square brackets on the left hand sides of (5.41) are absolutely continuous. After some calculations, we find
\[
 \begin{align*}
 \frac{d}{dt} y_\beta + \frac{1}{1+h} G_\beta y_\beta &= -(u^2)_\beta, \\
 \frac{d}{dt} u_\beta + \frac{1}{1+h} G_\beta u_\beta &= H_\beta. \tag{5.42}
\end{align*}
\]

**Step 3.** We now return to the original coordinate $(t, x)$ and derive an evolution equation for the partial derivative $u_x$ along a “good” characteristic.

Fix a point $(\tau, x)$ with $\tau \not\in \mathcal{N}$. Assume that $x$ is a Lebesgue point for the map $x \mapsto u_x(\tau, x)$. Let $\xi$ be such that $x = y(\tau, \xi)$ and assume that $t \mapsto \beta(t, t, \xi)$ is a good characteristic, so that (GC) holds. From (5.2) we observe that
\[
y_\beta(t, \beta) = \frac{1 + h}{1 + u_x^2(\tau, y(\beta))} > 0. \tag{5.43}
\]

Then the partial derivative $u_x$ can be computed as
\[
u_x(t, y(t, \beta(t, t, \xi))) = \frac{u_\beta(t, y(t, \beta(t, t, \xi)))}{y_\beta(t, \beta(t, t, \xi))}.
\]

Using (5.39) and (5.40) describing the evolution of $u_\beta$ and $y_\beta$, we can easily conclude that the map $t \mapsto u_x(t, y(t, \beta(t, t, \xi)))$ is absolutely continuous and satisfies
\[
\frac{d}{dt} u_x(t, y(t, \beta(t, t, \xi))) = \frac{1}{1 + u_x^2} \frac{u_x(t, y(t, \beta(t, t, \xi)))}{y_\beta(t, \beta(t, t, \xi))} = \frac{y_\beta H_\beta + 2 u u_x^2}{y_\beta^2}. \tag{5.44}
\]

**Step 4.** Given $u = u(t, x)$ is a admissible conservative weak solution, define
\[
\begin{align*}
 U(t, \xi) &= u(t, y(t, \xi)), \\
 V(t, \xi) &= \frac{1}{1 + u_x^2} \circ y, \\
 W(t, \xi) &= \frac{u_x \circ y}{1 + u_x^2} \circ y, \\
 Q(t, \xi) &= (1 + u_x^2) y_\xi.
\end{align*}
\]

After some calculations, we get the following semilinear system
\[
\begin{align*}
 y_\xi(t, \xi) &= -U^2, \\
 U_\xi(t, \xi) &= K(\xi) - P(1) y(\xi) - \frac{1}{1-h} \int_\xi (2QV - Q)(\eta)(K(\eta) - P(1) y(\eta)) \, d\eta, \\
 V_\xi(t, \xi) &= -2UW + 2WV P(1), \\
 W_\xi(t, \xi) &= 2UV - U - 2V^2 P(1) + VP(1), \\
 Q_\xi(t, \xi) &= -2WQP(1).
\end{align*} \tag{5.45}
\]

For every $\xi \in S$, we take the following initial condition
\[
\begin{align*}
 \int_0^{y_\xi(0)} u_\xi^2 dx + y_\xi(0) &= (1 + h)\xi, \\
 U_0(\xi) &= u_0 \circ y_0(\xi), \\
 V_0(\xi) &= \frac{1}{1 + u_{x_0}^2 \circ y_0(\xi)}, \\
 W_0(\xi) &= \frac{1}{1 + u_{x_0}^2 \circ y_0(\xi)}, \\
 Q_0(\xi) &= (1 + u_{x_0}^2 \circ y_0) y_{0\xi}(\xi) = 1 + h. \tag{5.46}
\end{align*}
\]
By the Lipschitz continuity of all coefficients and the previous steps, we conclude that the Cauchy problem (5.45)–(5.46) has a unique global solution.

**Step 5.** To prove the uniqueness, consider two admissible conservative weak solutions $u_1$ and $u_2$ of (2.3) with the same initial data $u_0 \in H^1(\mathbb{R})$. For a.e. $t \geq 0$, the corresponding Lipschitz continuous maps $\xi \mapsto y_1(t, \xi)$, $\xi \mapsto y_2(t, \xi)$ are strictly increasing. Hence, they have continuous inverses, say $x \mapsto (y_1)^{-1}(t, x)$, $x \mapsto (y_2)^{-1}(t, x)$.

As we have deduced that $y_1(t, \xi) = y_2(t, \xi)$, $u_1(t, y_1(t, \xi)) = u_2(t, y_2(t, \xi))$.

In turn, for a.e. $t \geq 0$ and a.e. $x \in \mathbb{S}$, this implies

$$u_1(t, x) = u_1(t, y_1(t, \xi)) = u_2(t, y_2(t, \xi)) = u_2(t, x).$$

This completes the proof of Theorem 5.7. $\Box$

**The proof of Theorem 1.3.** Combining Theorems 4.1 and 5.7, we obtain a unique global admissible conservative weak solution of (2.3). Since the initial data satisfies $\int_\mathbb{S} u_0 - u_0 u_0^2 dx = 0$, it follows that $\int_\mathbb{S} u - u_0^2 dx = 0$ by the conservative quantities. Hence, (2.3) is equivalent to (1.1). Thereby we finally gain a unique global admissible conservative weak solution of (1.1). $\Box$

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**Declarations
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