A NEW METHOD FOR ESTIMATION AND MODEL SELECTION:
\( \rho \)-ESTIMATION

Y. BARAUD, L. BIRGÉ, AND M. SART

Abstract. The aim of this paper is to present a new estimation procedure that can be applied in many statistical frameworks including density and regression and which leads to both robust and optimal (or nearly optimal) estimators. In density estimation, they asymptotically coincide with the celebrated maximum likelihood estimators at least when the statistical model is regular enough and contains the true density to estimate. For very general models of densities, including non-compact ones, these estimators are robust with respect to the Hellinger distance and converge at optimal rate (up to a possible logarithmic factor) in all cases we know. In the regression setting, our approach improves upon the classical least squares from many aspects. In simple linear regression for example, it provides an estimation of the coefficients that are both robust to outliers and simultaneously rate-optimal (or nearly rate-optimal) for large class of error distributions including Gaussian, Laplace, Cauchy and uniform among others.

1. Introduction

The primary scope of this paper was to design a new and more or less universal estimation method for the regression framework where we observe independent random variables \( X_1, \ldots, X_n \) of the form \( X_i = f_i + \varepsilon_i \) where the \( f_i \) are the unknown parameters of interest and the \( \varepsilon_i \) i.i.d. errors with a partially unknown distribution which may be quite different from the usual Gaussian one. The problem arose from a question by Oleg Lepski to the first author during his visit to Nice in January 2012 about this regression framework when the errors have rather unusual distributions. That was the starting point of our study which finally resulted in a much broader approach and the design of a new estimator with several remarkable and partly unexpected properties.

To be more precise, let us first describe the framework that we shall consider here. We observe \( n \) independent random variables \( X_1, \ldots, X_n \) each \( X_i \) with an unknown distribution \( P_i \) on a measurable space \( (\mathcal{X}, \mathcal{A}) \) and our aim is to estimate their joint distribution \( P = \otimes_{i=1}^{n} P_i \), that is, to find a random approximation \( \hat{P}(X_1, \ldots, X_n) = \otimes_{i=1}^{n} \hat{P}_i(X_1, \ldots, X_n) \) of \( P \) based on the observed variables \( X_i \). To measure the quality of this approximation of \( P \) by \( \hat{P} \), we need a distance on the set of product measures on \( \mathcal{X}^n \). It is known from Le Cam’s work that a very convenient one is that derived from the Hellinger distance \( h \) and introduced in Le Cam (1975):

\[
    h^2 \left( \bigotimes_{i=1}^{n} P_i, \bigotimes_{i=1}^{n} Q_i \right) = \sum_{i=1}^{n} h^2(P_i, Q_i) = \frac{1}{2} \sum_{i=1}^{n} \int (\sqrt{dP_i} - \sqrt{dQ_i})^2.
\]

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We recall that the Hellinger distance is the bounded distance on the set of all probabilities on $\mathcal{X}$ given by

$$h^2(R,T) = \frac{1}{2} \int \left( \sqrt{dR/d\mu} - \sqrt{dT/d\mu} \right)^2 d\mu \leq 1,$$

where $\mu$ is an arbitrary positive measure which dominates both $R$ and $T$, the result being independent of the choice of $\mu$. This is why one writes symbolically $h^2(R,T) = (1/2) \int (\sqrt{dR} - \sqrt{dT})^2$.

We shall measure the quality of an estimator $\hat{\theta}$ by its quadratic risk with respect to the distance $h$: $E_P[h^2(\hat{\theta}(X_1, \ldots, X_n), P)]$, which is a bounded function of $P$ since $h \leq \sqrt{n}$, the notation $E_P$ meaning that the $X_i$ have the joint distribution $P$. We shall put a special emphasis on regression models on $\mathbb{R}^n$, for which $\mathcal{X} = \mathbb{R}$ and either

$$X_i = f_i + \varepsilon_i \quad \text{or} \quad X_i = (W_i, Y_i) \quad \text{with} \quad Y_i = f(W_i) + \varepsilon_i.$$ 

In both cases the $\varepsilon_i$ are assumed to be i.i.d. with density $p$ with respect to the Lebesgue measure $\mu$. The first case corresponds to fixed design regression for which $X_i$ has density $p(\cdot - f_i)$, the second case to random design regression with i.i.d. random variables $W_i$ independent of the $\varepsilon_i$, the random variables $f(W_i)$ replacing the deterministic values $f_i$.

1.1. The translation model. The simplest case of fixed design regression occurs when the function $i \mapsto f_i$ is constant and equal to $\theta \in \Theta \subset \mathbb{R}$. It corresponds to the case of i.i.d. variables $X_i$ with density $p(\cdot - \theta)$ and to a parametric model with a single translation parameter $\theta$. The problem is to find an estimator $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$ for $\theta$ so that

$$\hat{\theta} = \hat{P} = P^\otimes_n \theta \quad \text{with} \quad \frac{dP^\otimes_n \theta}{d\mu} = p(\cdot - \hat{\theta}).$$

If $\theta_0$ obtains, that is $P = P^\otimes_n \theta_0$, the quadratic risk of $\hat{\theta}$ or, equivalently, of $\hat{\theta}$, writes

$$E_{\theta_0} \left[ h^2 \left( P^\otimes_n \theta, P^\otimes_n \theta_0 \right) \right] = E_{\theta_0} \left[ nh^2 \left( P^\otimes \theta, P^\otimes \theta_0 \right) \right],$$

where $E_\theta$ stands for $E_{P^\otimes_n \theta}$. Since the risk is a function of the unknown parameter $\theta_0$, a common way (although not the only one) of evaluating the performance of an estimator $\hat{\theta}$ is via its maximum quadratic risk $R_M(\hat{\theta}) = \sup_{\theta \in \Theta} E_{\theta} [nh^2 (P^\otimes \theta, P^\otimes \theta)]$. This leads to the notion of minimax risk for the problem at hand:

$$R_M(\Theta) = \inf_{\hat{\theta}} \sup_{\theta \in \Theta} E_{\theta} [nh^2 (P^\otimes \theta, P^\otimes \theta)] ,$$

where the infimum runs over all possible estimators $\hat{\theta}$ of $\theta$. An optimal estimator $\hat{\theta}$ is therefore one that minimizes $R_M(\hat{\theta})$. Unfortunately, computing $R_M(\Theta)$ exactly is generally an intractable optimization problem and we shall merely look for approximately optimal estimators $\tilde{\theta}$ satisfying

$$\sup_{\theta \in \Theta} E_{\theta} [nh^2 (P^\otimes \theta, P^\otimes \theta)] \leq CR_M(\Theta),$$

where $C$ is a constant which does not depend on $n$. 

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1.2. **The search for a universal estimator.** In the simple situation of a translation model, as described above, one would like to find a general method (the same for all densities \( p \)) for designing \( \tilde{\theta} \) rather than an ad hoc strategy for each \( p \). In the past, various procedures have been considered for solving this problem, in particular empirical methods based on moments or quantiles, or the celebrated maximum likelihood estimator (MLE for short). Unfortunately, none of them is really satisfactory in the sense that each one may behave quite poorly for some densities \( p \). For instance methods based on moments do not work when \( p \) does not admit moments; the MLE only behaves well and achieves optimal convergence rates when \( n \) goes to infinity if \( p \) is nice enough but, when \( p \) is unbounded, it simply does not exist; as to quantile methods they may fail to get the right (optimal) order of convergence to \( \theta \) when \( n \) tends to infinity as in the case of \( p = I_{[0,1]} \).

From a more realistic point of view, there are also additional problems to be solved that we did not mention yet: typically, \( p \) is in the best cases only approximately known and may even be completely unknown; the assumptions we make on the \( f_i \) (like \( f_i = \theta \) for all \( i \)) are likely to be (at least slightly) inaccurate, which means that the assumption that the function \( i \mapsto f_i \) is constant might only be an approximation of the truth. A good estimation procedure should therefore be robust, that is not too sensitive to these types of misspecifications. One major drawback of the MLE lies in the fact that it is not robust: if the assumption that the \( X_i \) are i.i.d. with density \( p(\cdot - \theta) \) is only slightly violated, the MLE may perform quite poorly. As it is impossible to know whether the translation model really holds or not, the MLE should be used with great caution as emphasized by Le Cam (1990).

Is it, nevertheless, possible to build an estimator that can cope with all these difficulties? Even for the simple case of a translation parameter, this is definitely not obvious. A major result in this direction is from P. Huber (1964) and there have been a number of papers in this direction published in the 70’s and we refer the reader to Huber (1981).

Attempts to design “universal” estimators in general settings have been made by Le Cam ((1973) and (1975)), Birgé (1983) and (2006) Yang and Barron (1999) or Baraud (2011) and the construction that we shall present here is in the line of these previous papers. Actually, the problem of estimating \( \theta \) in the translation model as well as many other problems in density estimation can essentially be solved by using the methods developed in these papers.

Things become much more delicate when we turn to the regression framework for which the number of unknown parameters is “a priori” equal to the number of observations. The construction which we present here is an attempt, not only to answer the above mentioned question of Oleg Lepski but also to solve this search for a “universal” estimator, at least in the case of independent observations.

1.3. **What can be expected.** The scope of the present paper actually goes far beyond regression and we consider more generally the problem of statistical estimation from the observation of \( n \) independent random variables whose distribution \( P_s \) on \( \mathcal{X}^n \) depends on a parameter \( s \) belonging to some set \( \mathcal{S} \) which can either be a Euclidean or a functional space. We always assume that the parametrization \( s \mapsto P_s \) is one to one so that the metric \( h \) on \( \{P_s, s \in \mathcal{S}\} \) can be transferred to \( \mathcal{S} \) and we shall write indifferently \( h(P_t, P_u) \) or \( h(t, u) \).

Following Birgé (2006) and Baraud (2011) but also much earlier contributions including the sieves approach of Grenander (1981) or the one in Barron, Birgé and Massart (1999) among many others, our approach is based on models, that is subsets \( \mathcal{S} \) of \( \mathcal{S} \). This means
that we do as if \( s \) did belong to \( \overline{S} \) and design estimators \( \widehat{s}(X_1, \ldots, X_n) \) with values in \( \overline{S} \), although we do not assume that this is true. As already mentioned, we shall evaluate the performance of \( \widehat{s} \) by its quadratic risk \( \mathbb{E}_s [h^2(\mathbf{P}_s, \mathbf{P}_{\widehat{s}})] \) (\( \mathbb{E}_s [h^2(s, \widetilde{s})] \) for short) where \( \mathbb{E}_s \) denotes the expectation when \( s \) obtains. When the data are i.i.d. with values in a measurable space \((\mathcal{X}, \mathcal{A})\) and \( s \) is their common density with respect to some dominating measure \( \mu \), \( \mathbf{P}_s = (s \cdot \mu)^\otimes n \) and, for \( t, u \in \mathcal{X} \),

\[
\frac{1}{n} h^2(t, u) = h^2(t, u) = \frac{1}{2} \int_{\mathcal{X}} \left( \sqrt{t} - \sqrt{u} \right)^2 d\mu = 1 - \rho(t, u),
\]

where \( \rho(t, u) = \int_{\mathcal{X}} \sqrt{tu} d\mu \) is called the Hellinger affinity of \( t \) and \( u \).

A good indicator of the quality of the estimator \( \widetilde{s} \) under the assumption that \( s \in \overline{S} \) is its maximal risk \( \sup_{s \in \overline{S}} \mathbb{E}_s [h^2(s, \widetilde{s})] \) as compared to the minimax risk over \( \overline{S} \), \( R_M(\overline{S}) = \inf_{s \in \overline{S}} \sup_{s \in \overline{S}} \mathbb{E}_s [h^2(s, \widetilde{s})] \), and a good estimator \( \widetilde{s} \) should satisfy

\[
\sup_{s \in \overline{S}} \mathbb{E}_s [h^2(s, \widetilde{s})] \leq C_0 R_M(\overline{S}),
\]

where \( C_0 \) (as well as all \( C_j \)'s with \( j \in \mathbb{N} \) that we shall introduce below) denotes a positive universal constant (independent of \( n \) and \( \overline{S} \)).

Nevertheless, since there is no way to check precisely whether the true parameter value \( s \) does actually belong to \( \overline{S} \), one would like that the previous result remains approximately true if the model \( \overline{S} \) is slightly misspecified, that is when \( s \notin \overline{S} \) but \( h(s, \overline{S}) = \inf_{t \in \overline{S}} h(s, t) \) is small, in which case we shall say that the estimator is robust. It is clear that whatever \( \widetilde{s} \in \overline{S} \), \( \mathbb{E}_s [h^2(s, \widetilde{s})] \geq \inf_{t \in \overline{S}} h^2(s, t) \). In view of this and the definition of the minimax risk, a good and robust estimator \( \widetilde{s} \) based on the model \( \overline{S} \) should satisfy

\[
\mathbb{E}_s [h^2(s, \widetilde{s})] \leq C_1 \max \left\{ R_M(\overline{S}), \inf_{t \in \overline{S}} h^2(s, t) \right\} \quad \text{for all } s \in \mathcal{X}.
\]

As already mentioned, several popular methods of estimation, like those based on moments estimation or the MLE, are not robust and minimum contrast estimators based on the \( \mathbb{L}_2 \)-contrast suffer from the same weaknesses. As to those which do possess some robustness properties with respect to misspecification, like methods based on the \( \mathbb{L}_1 \)-contrast or quantile estimation, they may unfortunately lead to sub-optimal rates of estimation.

1.4. T-estimators. Birgé (2006), following ideas by Le Cam (1973) and (1975) and earlier constructions of Birgé (1983) and (1984) derived estimators (called T-estimators) that satisfy (3) under some assumption on the model \( \overline{S} \). He introduced the notion of metric dimension \( D_{\overline{S}} \) of a subset \( \overline{S} \) of the metric space \((\mathcal{X}, h)\), as a function from \((0, +\infty)\) into \([1/2, +\infty]\) and proved that a conveniently chosen T-estimator satisfies an analogue of (3), namely that, for all \( s \in \mathcal{X} \),

\[
\mathbb{E}_s [h^2(s, \widetilde{s})] \leq C_5 \max \{ n \eta_n^2, \inf_{t \in \overline{S}} h^2(s, t) \} \quad \text{for all } \eta_n \text{ such that } n \eta_n^2 \geq C_2 D_{\overline{S}}(\eta_n).
\]

This implies in particular that \( R_M(\overline{S}) \leq C_3 \eta_n^2 \) and that (3) holds provided that \( R_M(\overline{S}) \geq C_4 \eta_n^2 > 0 \). In particular, if the function \( D_{\overline{S}} \) is bounded by the constant \( D_{\overline{S}} \) (the so-called finite dimensional case), one can set \( \eta_n = C_2 D_{\overline{S}}/n \) and the minimax risk \( R_M(\overline{S}) \) is bounded by \( C_5 D_{\overline{S}} \) independently of \( n \).

It happens, as shown in Birgé (1983), that in many situations the inequality \( R_M(\overline{S}) \geq C_4 \eta_n^2 \) actually holds. Unfortunately it is not always the case. There are situations for
which $\tilde{D}_\mathcal{S}(\eta) = +\infty$ for all $\eta > 0$ and this is typically the case when the diameter of $\mathcal{S}$ is not smaller than $\sqrt{n}$. This happens automatically when $\mathcal{S}$ is a translation model with parameter space $\Theta = \mathbb{R}$ so that the use of a T-estimator requires that $\theta$ belong to some known interval $[a, a + M]$ and its risk bound would unfortunately deteriorate as $M$ becomes larger. This difficulty can be fixed via the use of a preliminary estimator, a quantile estimator for instance, allowing to locate the parameter $\theta$ approximately but this solution does not extend to the regression framework. There are also cases where the ratio $R_M(\mathcal{S})/\left(n\eta_0^2\right)$ tends to zero when $n$ tends to infinity which means that the risk bound derived from the metric dimension has not the right order of magnitude.

1.5. Dimensions. The notion of metric dimension $\tilde{D}_\mathcal{S}$ actually applies to subsets of any metric space, not only $\mathcal{S}$, and is actually the right notion that is needed to control the performance of T-estimators. It is one possible way of measuring the massiveness of a model $\mathcal{S}$ but definitely not the only one. Others have been developed earlier, the simplest one being the ordinary dimension of a Euclidean space, but one can also mention Kolmogorov’s entropy — see Kolmogorov and Tikhomirov (1961) — among other possible notions. The fact that there are often some close relationship between the minimax risk over $\mathcal{S}$ and some notion of dimension of $\mathcal{S}$ has been known for a long time, the simplest example being the estimation of the mean of a Gaussian vector with identity covariance matrix when this mean is assumed to belong to a $D$-dimensional linear space $\mathcal{S}$. Similar results hold for parametric statistical estimation problems which are regular enough. Upper bounds for the minimax risk based on some earlier (more restrictive) version of metric dimension were developed by Le Cam (1973) and (1975) and generalized by Birgé (1983) together with the connection to lower bounds previously developed by Ibragimov and Has’minskii (1980). The performance of the MLE on a parameter set $\mathcal{S}$ may also be deduced from some suitable notion of dimension, namely entropy with bracketing — see van de Geer (1995) and Birgé and Massart (1993)— and the concentration of the posterior distribution in Bayesian frameworks as well, see Ghosal, Gosh and van der Vaart (2000).

The superiority of the notion of metric dimension is due to the fact that it is a weaker notion than entropy. For instance, the metric dimension of a Euclidean space is roughly equal to its ordinary dimension while its entropy is infinite. The entropy of a compact set automatically allows to control its metric dimension while the reciprocal is not true. Nevertheless, as already mentioned in the previous section, it is not possible to characterize the minimax risk over a model $\mathcal{S}$ by its metric dimension and we do not know of any notion $\bar{D}_\mathcal{S}$ such that

$$c\varphi(D_\mathcal{S}, n) \leq R_M(\mathcal{S}) \leq C\varphi(D_\mathcal{S}, n) \quad \text{with } 0 < c < C,$$

for some suitable function $\varphi$, at least under very mild assumptions on $\mathcal{S}$.

1.6. $\rho$-estimators. Our paper, although initially motivated by Oleg Lepski’s question and the will of finding a generic treatment of fixed-design regression under very weak assumptions (in particular no boundedness restrictions and no moment conditions), also results in both an improvement over T-estimators and a path in the direction of solving the problem summarized by (5).

The construction of estimators from tests between balls centered on the points of some finite set, which is due to Le Cam (1973) has been extended to countable sets and developed at length in the form of T-estimators by Birgé (2006) and taken back later by Mathieu Sart.
in (2011) and (2012) in a context of dependent data. A generalization of T-estimators, which leads to T-estimators when applied to discretized models, but allows to relax some of the assumptions needed for the use of T-estimators, was built by Baraud (2011). A modification of Baraud’s construction, following an idea of Mathieu Sart, finally led to our new method of estimation.

In order to give a brief account of our procedure, let us consider the problem of density estimation on a model $\mathcal{S}$ containing only two distinct densities: $t_0$ and $t_1$ (that may be different from the true one $s$). The difference $\rho(s,t_1) - \rho(s,t_0) = h^2(s,t_0) - h^2(s,t_1)$ (see (2) for the definition of $\rho$) tells us which of the points $t_0$ or $t_1$ is closer to $s$ with respect to the Hellinger distance. If we have at hand a good estimator $T_n(t_0,t_1)$ of $\rho(s,t_1) - \rho(s,t_0)$, it can be used not only to decide which of $t_0$ and $t_1$ is closer to $s$ but also, considering $\sup_{t \in S} T_n(t)$ as an estimator of

$$T(t_0) = \sup_{t \in S} \left[ h^2(s,t_0) - h^2(s,t) \right] = h^2(s,t_0) - \inf_{t \in S} h^2(s,t),$$



to see whether $t_0$ is likely to be almost a closest point to $s$ in $\overline{S}$. Indeed, the smaller the quantity $T(t_0)$, the better $t_0$ as an approximation of $s$ in $S$. It seems therefore natural to try to minimize $\sup_{t \in S} T_n(t)$ with respect to $t_0$ in order to derive a good estimator of $s$ within $\overline{S}$ provided that $T_n(t_0,t_1)$ is close enough to $\rho(s,t_1) - \rho(s,t_0)$ for all $t_0,t_1$. These are, roughly speaking, the ideas behind the construction of what we shall call a $\rho$-estimator since it is based on a suitable estimation of the Hellinger affinities between the true density and the points in the model. While T-estimators are based on special nets of $\overline{S}$ making them quite difficult to use in practice, $\rho$-estimators result from the optimization of a criterion over $\overline{S} \times \overline{S}$ and can therefore be computed practically, at least when $\overline{S}$ is a low-dimensional parametric model.

1.7. **What’s new here?** While $\rho$-estimators retain all the nice properties of T-estimators, in particular their robustness, their risk is controlled by new notions of dimensions which improve the one of metric dimension as shown by Corollary 3 below. These dimensions have the nice property to remain finite for many unbounded models which is an essential property for the statistical problems we want to solve. Furthermore in Section 6.5, we shall provide an example of a compact parameter space for which the metric dimension fails to provide the right order of magnitude for the minimax risk while our approach does.

An additional attractive feature of $\rho$-estimators in density estimation lies in the fact that when $n$ is large enough, they recover the usual MLE at least when the model is parametric, regular enough and contains the true density to estimate. Some simulations developed by Sart show that this occurs even for moderate values of $n$. Another connection with the MLE lies in the fact that the risk bounds obtained for the MLE under bracketing entropy assumptions are still valid (up to possible numerical constants) for bounding the risks of $\rho$-estimators.

In the regression framework, our procedure improves upon the classical least squares from numerous aspects. First of all, we can deal with errors bearing no finite moments of any order such as the Cauchy distribution while the least squares approach cannot. Besides, we can handle various types of errors possibly leading to faster rates of estimation of the parameters than the parametric ones reached by the least squares. Finally, our procedure guarantees robustness properties for the resulting estimator that the use of least squares does not. More generally, apart from T-estimators which require to know a bound on the
The supremum norm of the regression function (in order to ensure that the metric dimension of the parameter set is finite), which is a rather restrictive assumption for regression models, we are not aware of any statistical procedure that leads to a rate-optimal estimator (up to a possible logarithmic factor) when the distribution of the errors is only assumed to belong to a large family of possible ones, including the Gaussian, Cauchy, uniform, etc. Even in the case of the simple linear regression, our method may estimate at a much faster rate than the least squares (when the errors are uniformly distributed on \([-1, 1]\) for instance).

Moreover, although initially built to handle complicated situations of regression with fixed design, our procedure also allows to deal with various random design problems, including non-linear ones, which, to our knowledge, have never been considered earlier in the literature from a theoretical point of view.

1.8. Organization of the paper. We present in Section 2 three statistical settings to which our procedure can be applied and the basic ideas underlying our approach in Section 3. The construction of the estimator and the main results about its performance on a single model can be found in Section 4. In Section 5 we show that, in favorable cases, the MLE is a particular case of \(\rho\)-estimator. We also show that the assumptions which are used to analyze the performance of the MLE in favorable situations can also be used to derive similar risk bounds for \(\rho\)-estimators. In Section 6, we illustrate the performance of \(\rho\)-estimators in the regression setting (with either fixed or random design) and provide an example where its risk remains controlled in a situation where the metric dimension of the model can be made arbitrary large and even infinite. In Section 7, we consider the problem of model selection. We establish there an oracle-type inequality and provide an application in view of estimating a regression function when the distribution of the errors belongs to a large class of densities including Laplace, Gaussian, uniform among others. We provide an annex on VC-subgraph classes in Section 8 since models \(\mathcal{F}\) of these types play a special role in our results. Finally, Section 9 is devoted to the proofs.

2. The statistical setting and examples

2.1. Main notations and conventions. In the sequel we shall use the following notations and conventions. We set \(\mathbb{N}^* = \mathbb{N} \setminus \{0\}\), \(\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}\), \(\log_+ x = \max\{\log x, 0\}\) for \(x > 0\) and \(\log_0 = 0\). For \(x, y \in \mathbb{R}\), \(x \wedge y\) and \(x \vee y\) denote \(\min\{x, y\}\) and \(\max\{x, y\}\) respectively, \(\delta_x\) denotes the Dirac measure at point \(x\) and \(|A|\) the cardinality of the set \(A\). Except if otherwise specified (in Section 3 below), we shall use the conventions sup \(\emptyset = 0\), \(0/0 = 1\), \(0 \times (+\infty) = 0\) and \(x/0 = +\infty\) for all \(x > 0\). The word countable always means finite or countable. Throughout the paper, \(C, C', \ldots\) denote positive numerical positive constants that may vary from line to line. The notations \(C(\cdot), C'(\cdot), \ldots\) mean that \(C, C', \ldots\) are positive functions depending on the argument specified in the parenthesis (when the number of arguments is too large, the dependency is specified in the text). We shall also often use the fact that

\[
(x + y)^2 \leq (1 + \alpha)x^2 + (1 + \alpha^{-1})y^2 = (1 + \alpha)(x^2 + \alpha^{-1}y^2) \quad \text{for all } \alpha > 0.
\]

Our definitions and results will actually involve a number of numerical constants. In order to avoid complicated formulas, we shall give specific names to the numerical constants that will be systematically used in the sequel.

\[
c_0 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right); \quad c_1 = 2\left(7 + 4\sqrt{2}\right); \quad c_2 = 1 + \frac{1}{\sqrt{2}}; \quad \kappa = 357; \quad c_3 = 8c_2\kappa; \quad c_4 = 2.5c_3.
\]
2.2. The general statistical setting. Let \( X = (X_1, \ldots, X_n) \) be a vector of independent random variables with values in a product of measured spaces \( \prod_{i=1}^n \mathcal{X}_i, \otimes_{i=1}^n \mathcal{F}_i, \otimes_{i=1}^n \mu_i \). We assume that for each \( i \), \( X_i \) admits a density \( s_i \) with respect to \( \mu_i \) and our aim is to estimate \( s = (s_1, \ldots, s_n) \) from the observation of \( X \). To avoid trivialities, we shall always assume that \( n \geq 3 \). We shall emphasize the dependence of the distribution of \( X \) with respect to the unknown parameter \( s \) by writing \( \mathbb{P}_s[X \in A] \) for a measurable set \( A \) and \( \mathbb{E}_s[g(X)] \) for an integrable function \( g \).

Let \( \mathcal{L}_1 \) be the set of probability densities \( u \) with respect to \( \mu_i \) so that \( u \) is a nonnegative measurable function on \( (\mathcal{X}_i, \mathcal{F}_i) \) such that \( \int_{\mathcal{X}_i} u \, d\mu_i = 1 \) and let us metricize \( \mathcal{L}_i \) by the Hellinger distance \( h \) given, according to (1), by

\[
h^2(u, u') = \frac{1}{2} \int_{\mathcal{X}_i} \left( \sqrt{u} - \sqrt{u'} \right)^2 \, d\mu_i \quad \text{for all } u, u' \in \mathcal{L}_i.
\]

We define \( (\mathcal{L}_0, h) \) as the product space \( \prod_{i=1}^n \mathcal{L}_i \) equipped with the distance \( h(t, u) = \left[ \sum_{i=1}^n h^2(t_i, u_i) \right]^{1/2} \) and, for \( u \in \mathcal{L}_0 \) and more generally for \( u \in \prod_{i=1}^n \mathbb{L}_1(\mathcal{X}_i, \mathcal{F}_i, \mu_i) \), we shall use the notations

\[
u(X) = \sum_{i=1}^n u_i(X_i) \quad \text{and} \quad \int u \, d\mu = \sum_{i=1}^n \int_{\mathcal{X}_i} u_i \, d\mu_i.
\]

We metricize \( \mathcal{L}_0 \), following Le Cam, by the distance \( h \) given, by analogy with (2), by

\[
h^2(t, t') = \frac{1}{2} \int \left( \sqrt{t} - \sqrt{t'} \right)^2 \, d\mu = \sum_{i=1}^n h^2(t_i, t'_i) \quad \text{for } t, t' \in \mathcal{L}_0.
\]

Hereafter, we shall deal with estimators with values in \( \mathcal{L}_0 \) and measure their performances by the risk induced by the loss function \( h^2 \). For simplicity we shall also call \( h \) the Hellinger distance and define the Hellinger affinity \( \rho \) between two elements \( t \) and \( t' \) of \( \mathcal{L}_0 \) as

\[
\rho(t, t') = \int \sqrt{tt'} \, d\mu = \sum_{i=1}^n \int_{\mathcal{X}_i} \sqrt{t_i t'_i} \, d\mu_i = n - h^2(t, t').
\]

For \( t \in \mathcal{L}_0 \) and \( y > 0 \), we shall denote by \( B(t, y) \) the closed ball of center \( t \) and radius \( y \) in the metric space \( (\mathcal{L}_0, h) \) and, given some subset \( S \) of \( \mathcal{L}_0 \), by \( B^S(s, y) = B(s, y) \cap S \) the closed Hellinger ball in \( S \) centered at \( s \) with radius \( y \). Finally, \( h(t, S) = \inf_{t' \in S} h(t, t') \).

The general setting that we presented earlier will allow us to deal in particular with the three following frameworks.

2.3. The density framework. In this framework, we assume that the random variables \( X_1, \ldots, X_n \) are i.i.d. with values in a measured space \( (\mathcal{X}, \mathcal{F}, \mu) \) and common density \( s \) with respect to \( \mu \). To cope with our general framework, we shall take \( \mathcal{X}_i = \mathcal{X}, \mu_i = \mu \) for all \( i \) and \( s = (s, \ldots, s) \). In this particular context, it will be convenient to identify a density \( t \) on \( (\mathcal{X}, \mathcal{F}, \mu) \) with the element \( t = (t, t, \ldots, t) \) of \( \mathcal{L}_0 \), which we shall do in the sequel. Given two densities \( t, t' \) on \( (\mathcal{X}, \mathcal{F}, \mu) \), we have the relations

\[
h^2(t, t') = nh^2(t, t') \quad \text{and} \quad \rho(t, t') = n\rho(t, t').
\]

The risk of an estimator \( \tilde{s} = (\tilde{s}, \ldots, \tilde{s}) \) of \( s \) is therefore \( \mathbb{E}_s \left[ h^2(s, \tilde{s}) \right] = n\mathbb{E}_s \left[ h^2(s, \tilde{s}) \right] \).
2.4. The homoscedastic regression framework with fixed design. In this framework, we assume that the \( X_i \) are real-valued random variables satisfying equations of the form

\[
X_i = f_i + \lambda \varepsilon_i \quad \text{for} \quad i = 1, \ldots, n, \quad \lambda > 0,
\]

where the vector \( f = (f_1, \ldots, f_n) \) belongs to \( \mathbb{R}^n \), the \( \varepsilon_i \) are real-valued i.i.d. random variables with density \( p \) with respect to the Lebesgue measure \( \mu \) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) and \( \mathcal{B}(A) \) denotes the Borel \( \sigma \)-algebra on the topological space \( A \). It follows that the density of \( X_i \) is \( s_i(x) = \lambda^{-1} p(\lambda^{-1}(x - f_i)) \) so that estimating \( s_i \) amounts to estimating \( \lambda, p \) and \( f_i \). Our aim is therefore to estimate \( f, p \) and \( \lambda \) from the observation of \( X_1, \ldots, X_n \). To deal with this statistical framework, it will be convenient to introduce the following notations: for \( f \) and \( x \) in \( \mathbb{R} \), \( \lambda \) in \( \mathbb{R}^*_+ \), \( f \) and \( x = (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \), we set

\[
p_{f,\lambda}(x) = \frac{1}{\lambda^p} \left( \frac{x - f}{\lambda} \right) \quad \text{for all} \quad x \in \mathbb{R}; \quad p_f = p_{f,1};
\]

\[
p_{f,\lambda}(\mathbf{x}) = \left( \frac{1}{\lambda^p} \left( \frac{x_1 - f_1}{\lambda} \right), \ldots, \frac{1}{\lambda^p} \left( \frac{x_n - f_n}{\lambda} \right) \right) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n \text{ and } p_f = p_{f,1}.
\]

It follows that \( p_f(x) = p(x - f), p_{0,\lambda}(x) = \lambda^{-1} p(x/\lambda), etc. \) In this statistical framework we shall take, for \( i = 1, \ldots, n, \mathcal{F}_i = \mathbb{R}, \mathcal{A}_i = \mathcal{B}(\mathbb{R}), \mu_i = \mu \) and \( s_i = p_{f,\lambda} \) so that the density of \( \mathbf{X} \) is \( \otimes_{i=1}^n p_{f_i,\lambda} \) and therefore entirely determined by \( p_{f,\lambda} \).

2.5. The homoscedastic regression framework with random design. Let \((W, Y)\) be a pair of random variables with values in \((\mathcal{W} \times \mathbb{R}, \mathcal{W} \otimes \mathcal{B}(\mathbb{R}))\) linked by the relation

\[
Y = f(W) + \varepsilon
\]

where \( f \) is unknown in a set \( \mathcal{F} \) of measurable functions from \((\mathcal{W}, \mathcal{W})\) into \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) and \( \varepsilon \) is an unobservable random variable, independent of \( W \) and admitting a known (or approximately known) density \( p \) with respect to the Lebesgue measure \( \mu \). In contrast, the distribution \( \nu \) of \( W \) is possibly unknown.

Our aim is to estimate \( f \), or equivalently the conditional distribution of \( Y \) given \( W \), from the observation of \( \mathbf{X} = (X_1, \ldots, X_n) \) where the \( X_i \) are i.i.d. with the same distribution on \((\mathcal{W} \times \mathbb{R})\) as that of the pair \((W, Y)\) so that \( \mathbf{X} \) can be identified to \((\mathcal{W}, \mathbf{Y})\) with \( \mathbf{W} = (W_1, \ldots, W_n) \) and \( \mathbf{Y} = (Y_1, \ldots, Y_n) \). Since the density of \((W, Y)\) with respect to the dominating measure \( \nu \otimes \mu \) is \( s(w, y) = p(y - f(w)) = p_f(w, y) \) and \( p_f(W, \cdot) \) is also the conditional density of \( Y \) given \( W \) with respect to \( \mu \), estimating \( s \) by an estimator of the form \( \hat{s} = p_{\hat{f}} \) leads to the estimator \( \hat{f} = \hat{f}(\mathbf{W}, \mathbf{Y}) \) of \( f \) and also provides an estimator \( p_{\hat{f}}(w, \cdot) \) of this conditional density. At this stage, it is important to emphasize the fact that the construction of \( \hat{s} \) should not involve \( \nu \) in order to provide genuine estimators of \( f \) and \( p_f \). As we shall see in Section 6.3, the \( p \)-estimator of \( s \) derived from our general method does satisfy this requirement.

In this framework, to evaluate the performance of \( \hat{f}(\mathbf{W}, \mathbf{Y}) \), we use the risk \( \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \) of \( \hat{s}(\mathbf{W}, \mathbf{Y}) \) or, equivalently the risk of \( p_{\hat{f}} \) which writes

\[
\mathbb{E}_s \left[ h^2(p_{\hat{f}}, p_{\hat{f}}) \right] = n \mathbb{E}_s \left[ h^2(p_f, p_{\hat{f}}) \right] = n \mathbb{E}_s \left[ \int_{\mathcal{W}} h^2(\left( p_f(w, \cdot), p_{\hat{f}}(w, \cdot) \right)) d\nu(w) \right].
\]
3. Basic ideas underlying our approach

3.1. The density framework. The aim of this section is to present the basic ideas and formulas underlying our approach. For the sake of simplicity, we shall restrict this introduction to the density framework described in Section 2.3 where the observations \( X_1, \ldots, X_n \) are i.i.d. with an unknown density \( s \) with respect to \( \mu \).

Given two candidate densities \( t, t' \) for \( s \), one should prefer \( t' \) to \( t \) if it is closer to \( s \), that is, if \( h^2(s, t') \) is smaller than \( h^2(s, t) \) or equivalently if \( \rho(s, t') - \rho(s, t) > 0 \). Deciding whether \( t' \) is preferable to \( t \) amounts thus to estimating the difference \( \rho(s, t') - \rho(s, t) \) in a suitable way. To do so, we start by an approximation of the affinity \( \rho \). For two densities \( t \) and \( t' \), we set

\[
(11) \quad r = \frac{t + t'}{2} \quad \text{and} \quad g(s, t, t') = \frac{1}{2} \left[ \rho(t, r) + \int_X \sqrt{\frac{t}{r}} s \, d\mu \right] < +\infty,
\]

using the special convention that \( t/r = 0 \) when \( t = t' = r = 0 \). It was proved in Proposition 1 of Baraud (2011) that

\[
(12) \quad 0 \leq g(s, t, t') - \rho(s, t) \leq \frac{h^2(s, t) + h^2(s, t')}{\sqrt{2}}.
\]

The important point about (12) lies in the fact that the constant \( \sqrt{2} \) is larger than 1. This makes it possible to use the sign of the difference

\[
T(s, t, t') = g(s, t', t) - g(s, t, t')
\]

as an alternative benchmark to find which of \( t \) and \( t' \) is closer to \( s \) (up to a multiplicative constant). It actually follows from (12), as shown in Corollary 1 in Baraud (2011), that

\[
(13) \quad T(s, t, t') \leq \left( 1 + \frac{1}{\sqrt{2}} \right) h^2(s, t) - \left( 1 - \frac{1}{\sqrt{2}} \right) h^2(s, t'),
\]

\[
(14) \quad T(s, t, t') \geq \left( 1 - \frac{1}{\sqrt{2}} \right) h^2(s, t) - \left( 1 + \frac{1}{\sqrt{2}} \right) h^2(s, t').
\]

Given some subset \( S \) of \( \mathcal{L}_0 \), (13) and the fact that \( T(s, t, t) = 0 \) also imply that, for \( t \in S \)

\[
0 \leq \sup_{t' \in S} T(s, t, t') \leq \left( 1 + \frac{1}{\sqrt{2}} \right) h^2(s, t) - \left( 1 - \frac{1}{\sqrt{2}} \right) h^2(s, S),
\]

hence

\[
0 \leq \inf_{t \in S} \sup_{t' \in S} T(s, t, t') \leq \sqrt{2} h^2(s, S).
\]

If \( u \in S \) is such that \( h^2(s, u) = ah^2(s, S) \), it follows from (14) that

\[
\sup_{t' \in S} T(s, u, t') \geq \left( 1 - \frac{1}{\sqrt{2}} \right) h^2(s, u) - \left( 1 + \frac{1}{\sqrt{2}} \right) h^2(s, S)
= \frac{h^2(s, S)}{\sqrt{2}} \left[ a \left( \sqrt{2} - 1 \right) - \left( \sqrt{2} + 1 \right) \right].
\]

If \( a > 5 + 4\sqrt{2} \), then \( \sup_{t' \in S} T(s, u, t') > \sqrt{2} h^2(s, S) \) and \( u \) cannot be a minimizer of \( t \mapsto \sup_{t' \in S} T(s, t, t') \). It follows that any minimizer \( \overline{s} \) of this function does satisfy \( h^2(s, \overline{s}) \leq (5 + 4\sqrt{2}) h^2(s, S) \). Therefore, minimizing over \( S \) the function \( t \mapsto \sup_{t' \in S} T(s, t, t') \) leads to some point \( \overline{s} \in S \) which, up to a constant factor, is the closest to \( s \), that is, the best approximation of \( s \) in \( S \). In particular, if \( s \in S \), \( \overline{s} = s \).
Unfortunately, $T(s, t, t')$ depends on $\rho(s, t, t')$ which depends on the unknown $s$. Our interest for the quantity $\varrho(s, t, t')$ rather than $\rho(s, t)$ lies in the fact that the former can be estimated by its empirical counterpart, namely

$$
\varrho(X, t, t') = \frac{1}{2n} \sum_{i=1}^{n} \left[ \rho(t, r) + \sqrt{\frac{r}{r(X_i)}} \right] \quad \text{with} \quad r = \frac{t + t'}{2},
$$

which is an unbiased estimator of $\varrho(s, t, t')$. A natural way of deciding which of the densities $t$ or $t'$ is the closest to $s$ is therefore to replace the unknown $T(s, t, t')$ by an unbiased estimator, namely the statistic

$$
T(X, t, t') = \varrho(X, t', t) - \varrho(X, t, t').
$$

Note that

$$
T(X, t, t') = \frac{1}{2} \left[ \rho(t', r) - \rho(t, r) \right] + \frac{1}{\sqrt{2}} \sqrt{\frac{r}{r(X_i)}} \psi \left( \sqrt{\frac{t'}{t(X_i)}} \right),
$$

where $\psi$ is the Lipschitz, increasing function from $[0, +\infty]$ to $[-1, 1]$ (with Lipschitz constant not larger than 1.143) given by

$$
\psi(u) = \sqrt{\frac{1}{1 + u^2}} - \sqrt{\frac{1}{1 + u^2}} = \frac{u - 1}{\sqrt{1 + u^2}} \quad \text{for} \quad u \in [0, +\infty) \quad \text{and} \quad \psi(+\infty) = 1.
$$

Here we use the convention that $t'(X_i)/t(X_i) = 1$ when $t(X_i) = t'(X_i) = 0$ as indicated in Section 2.1. This convention is indeed consistent with the one we started from on the ratio $t/r$ since when $t(X_i) = t'(X_i) = r(X_i) = 0$ for some $i$,

$$
\left[ \sqrt{\frac{t'}{r(X_i)}} - \sqrt{\frac{t}{r(X_i)}} \right] = 0 - 0 = 0 \quad \text{and} \quad \psi \left( \sqrt{\frac{t'}{t(X_i)}} \right) = \psi \left( \frac{0}{0} \right) = \psi(1) = 0.
$$

Replacing the “ideal” statistic $T(s, t, t')$ by its empirical counterpart $T(X, t, t')$ leads to an estimation error given by the process $Z(X, t, t')$ defined on $\mathcal{L}_0^2$ by

$$
Z(X, t, t') = \sqrt{2} \left[ T(X, t, t') - T(s, t, t') \right] = \sqrt{2} \left[ \left[ \varrho(X, t', t) - \varrho(s, t', t) \right] - \left[ \varrho(X, t, t') - \varrho(s, t, t') \right] \right] = \frac{1}{n} \sum_{i=1}^{n} \left[ \psi \left( \sqrt{\frac{t'}{t}(X_i)} \right) - \varrho \left( \frac{t'}{t}(X_i) \right) \right].
$$

3.2. The general framework. We may similarly apply the previous reasoning to the more general context of independent but not necessarily i.i.d. data $X_i$, $1 \leq i \leq n$. To do so, we shall extend the previous notations to elements of $\mathcal{L}_0$, not only densities, and, in view of the application to the regression setting, we shall not renormalize the sums by $1/n$. This leads to the following notations to be used throughout this paper: for $t, t' \in \mathcal{L}_0$ and
\[ r = (t + t')/2, \text{ we set} \]
\[
\varrho(X, t, t') = \frac{1}{2} \left[ \rho(t, r) + \sqrt{\frac{r}{t}}(X) \right] = \frac{1}{2} \sum_{i=1}^{n} \left[ \rho(t, r_i) + \sqrt{\frac{r_i}{t}}(X_i) \right];
\]
(19) \[ T(X, t, t') = \varrho(X', t) - \varrho(X, t, t') \]
\[
= \frac{1}{2} \sum_{i=1}^{n} [\rho(t'_i, r_i) - \rho(t_i, r_i)] + \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \psi \left( \sqrt{\frac{t'_i}{r_i}}(X_i) \right); \]
(20) \[ Z(X, t, t') = \psi \left( \sqrt{\frac{t'_i}{t_i}}(X_i) \right) - \mathbb{E}_{\psi} \left[ \psi \left( \sqrt{\frac{t'_i}{t_i}}(X_i) \right) \right] \]
\[
= \sum_{i=1}^{n} \left[ \psi \left( \sqrt{\frac{t'_i}{t_i}}(X_i) \right) - \mathbb{E}_{\psi_i} \left[ \psi \left( \sqrt{\frac{t'_i}{t_i}}(X_i) \right) \right] \right]. \]

With these notations, \( \varrho(X, t, t') \) is in fact the analogue of \( n\varrho(X, t, t') \) given by (15), and so on.

4. Estimation on a model

4.1. Models. As already mentioned, our construction of estimators will be based on “models”. A model \( S \subset \mathcal{L}_0 \) should be viewed as an approximation set for the true unknown parameter \( s \) which is used to build an estimator. It does not necessarily contain \( s \), although we shall occasionally assume so. Typical models are either the parametric models that are used in Statistics or more general subsets of \( \mathcal{L}_0 \) with well-known approximation properties that are derived from Approximation Theory in order to get a control on the approximation error \( h(s, S) \). For measurability reasons to be explained later, we shall adopt the following definition for a model.

**Definition 1.** A model \( S \) is a nonempty separable subset of the metric space \((\mathcal{L}_0, h)\).

In typical situations, \((\mathcal{L}_0, h)\) itself is separable so that any nonempty subset of \( \mathcal{L}_0 \) can be used as a model. In the density framework described in Section 2.3, we have identified a density \( t \) on \((\mathcal{X}, \mathcal{A}, \mu)\) with the element \( t = (t, t, \ldots, t) \) of \( \mathcal{L}_0 \). Similarly, we shall identify a separable set \( S \) of densities to the subset \{ \( t = (t, \ldots, t), \ t \in S \} \subset \mathcal{L}_0 \) and for simplicity denote the same way both sets.

4.2. Construction of the estimators. In order to avoid measurability issues, the construction of our estimator will be performed over countable sets \( S \subset \mathcal{L}_0 \) only. By the way, this corresponds to the practical point of view since numerical optimization will always be done over a finite set. Replacing the original model \( S \) by a countable and dense subset \( S \) does not involve any additional approximation error since then \( h(s, S) = h(s, \overline{S}) \). If instead we replace \( S \) by \( S \subset \mathcal{L}_0 \) which only satisfies \( \sup_{u \in S} h(u, S) = \sup_{u \in S} \inf_{t \in S} h(u, t) \leq \eta \) this replacement may involve an additional error \( |h(s, S) - h(s, \overline{S})| \) which is not larger than \( \eta \). We delay the discussion about what should be suitable choice of \( S \) for a given model \( S \) in Section 4.5.

Given \( S \), we noticed in the previous section that an almost best approximation of the density \( s \) in \( S \) can be obtained by minimizing over \( S \) the function \( t \mapsto \sup_{t' \in S} T(s, t, t') \).
If we assume that \( T(X, t, t') \) provides a good approximation of \( T(s, t, t') \), it looks natural to minimize \( \sup_{t' \in S} T(X, t, t') \) with respect to \( t \in S \) to derive a good estimation \( \hat{s}(X) \) of \( s \). This suggests the following construction in the general situation of independent random variables. For each \( t \in S \) we define

\[
\Upsilon(S, t) = \sup_{t' \in S} T(X, t, t'),
\]

which is always non-negative since \( T(X, t, t) = 0 \), and

\[
\mathcal{E}(X, S) = \left\{ \hat{s} \in S \mid \Upsilon(S, \hat{s}) \leq \inf_{t \in S} \Upsilon(S, t) + \frac{\kappa}{10} \right\}
\]

where \( \kappa \) is given by (7).

Finally, we define our estimator of \( s \) as any element (chosen in a measurable way) \( \hat{s} \) in the closure of \( \mathcal{E}(X, S) \) with respect to \( h \). We shall call such an estimator a \( \rho \)-estimator, the greek letter \( \rho \) referring to the Hellinger affinity. Although it depends on our choice of \( S \), we shall, for simplicity, omit to make this dependence of \( \hat{s} \) with respect to \( S \) explicit in our notations. Though the risk bounds we shall establish remain valid for any choice of \( \hat{s} \) in the closure of \( \mathcal{E}(X, S) \), we recommend in practice to choose \( \hat{s} \) as a minimizer of \( \Upsilon(S, \cdot) \) over \( S \) whenever it exists.

4.3. Main theorem. The properties of our estimator follow from those of the empirical process \( Z(X, \ldots) \) defined by (20). Given an element \( \bar{s} \in \mathcal{L}_0 \) and a positive number \( y \), we set

\[
\mathcal{B}^S(s, \bar{s}, y) = \left\{ t \in S \mid h^2(s, t) + h^2(s, \bar{s}) \leq y^2 \right\}
\]

and

\[
w^S(s, \bar{s}, y) = \mathbb{E}_s \left[ \sup_{t \in \mathcal{B}^S(s, \bar{s}, y)} |Z(X, s, t)| \right]
\]

with \( w^S(s, \bar{s}, y) = 0 \) if \( \mathcal{B}^S(s, \bar{s}, y) \) is empty, according to our convention. When \( s \) belongs to \( S \) and one takes \( \bar{s} = s \),

\[
w^S(s, s, y) = \mathbb{E}_s \left[ \sup_{t \in \mathcal{B}^S(s, s, y)} |Z(X, s, t)| \right]
\]

measures, in some sense, the massiveness of \( S \) in a neighborhood of \( s \). Since \( -1 \leq \psi \leq 1 \), the process \( |Z| \) is bounded by \( 2n \) and the non-decreasing mapping \( y \mapsto w^S(s, \bar{s}, y) \) as well. This implies that the number

\[
D^S(s, \bar{s}) = y_0^2 \lor 1 \quad \text{with} \quad y_0 = \sup \left\{ y \geq 0 \mid w^S(s, \bar{s}, y) > c_0 y^2 / 4 \right\}
\]

belongs to the interval \([1, 8nc_0^{-1}]\). It follows that

\[
w^S(s, s, y) \leq c_0 y^2 / 4 \quad \text{for all} \quad y > \sqrt{D^S(s, \bar{s})}.
\]

**Theorem 1.** Let \( S \) be a countable subset of \( \mathcal{L}_0 \). The estimation procedure described in Section 4.2 leads to the following bound which is valid for any \( \rho \)-estimator \( \hat{s} \) based on \( S \), \( \bar{s} \) in \( \mathcal{L}_0 \) and \( \xi > 0 \)

\[
\mathbb{P}_s \left[ h^2(s, \hat{s}) \leq \inf_{\bar{s} \in S} \left\{ c_1 h^2(s, \bar{s}) - h^2(s, S) + c_2 D^S(s, \bar{s}) \right\} + c_3 (1.45 + \xi) \right] \geq 1 - e^{-\xi}.
\]

This implies in particular that

\[
\mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq \inf_{\bar{s} \in S} \left\{ c_1 h^2(s, \bar{s}) - h^2(s, S) + c_2 D^S(s, \bar{s}) \right\} + 2.45 c_3
\]
and, more generally,

\[ \mathbb{E}_s \left[ h^\ell(s, \hat{s}) \right] \leq C(\ell) \inf_{\mathbb{P} \in \mathcal{S}} \{ h^\ell(s, \mathbb{P}) + (D^S(s, \mathbb{P}))^{\ell/2} \} \quad \text{for all } \ell \geq 1. \]  

In the sequel, we shall often content ourselves to provide our results in the form of exponential deviations similar to (24) looking like

\[ \mathbb{P}_s \left[ h^2(s, \hat{s}) \leq \Gamma + c\xi \right] \geq 1 - e^{-\xi} \quad \text{for all } \xi > 0, \]

where \( \Gamma \) depends on various parameters involved in our assumptions. Such a deviation bound immediately leads to \( \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq \Gamma + c \) and implies bounds similar to (26) for the moments of \( h(s, \hat{s}) \) as well as risk bounds for more general loss functions.

In the forthcoming sections, we shall use this central Theorem to establish risk bounds for our estimator over more general models than just countable ones. Before turning to these bounds, let us note here that we can already deduce from (25) (taking \( s = s \)) that

\[ \text{the estimator } \hat{s} \text{ satisfies:} \]

(27) \( \sup_{s \in \mathcal{S}} \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq Cd(S) \) with \( d(S) = \sup_{\mathbb{P} \in \mathcal{S}} D^S(s, \mathbb{P}) \geq 1 \).

For the inequality above, the assumption that \( s \) belongs to \( \mathcal{S} \) is quite restrictive, nevertheless we shall see in the next section that a bound of this type is not only true for the elements \( s \) lying in \( \mathcal{S} \) but also for those which are close enough to \( \mathcal{S} \) with respect to the Kullback-Leibler divergence.

4.4. Robustness properties with respect to the Kullback-Leibler divergence. We recall that the Kullback-Leibler divergence (KL-divergence for short) between two probabilities \( P, Q \) is given by

\[ K(P, Q) = \int_X \log \left( \frac{dP}{dQ} \right) dP \in [0, +\infty) \quad \text{if } P \ll Q \quad \text{and} \quad K(P, Q) = +\infty \quad \text{otherwise.} \]

For \( t, t' \in \mathcal{L}_0 \), we shall set for simplicity

\[ K(t, t') = K \left( \bigotimes_{i=1}^n (t_i \cdot \mu_i), \bigotimes_{i=1}^n (t'_i \cdot \mu_i) \right) = \sum_{i=1}^n K(t_i \cdot \mu_i, t'_i \cdot \mu_i). \]

It is well-known that \( 2h^2(t, t') \leq K(t, t') \) for all \( t, t' \in \mathcal{L}_0 \).

**Theorem 2.** Let \( \mathcal{S} \) be a model and \( S \) a countable subset of \( \mathcal{S} \) satisfying

(28) \( \inf_{\mathbb{P} \in \mathcal{S}} K(s, \mathbb{P}) = \inf_{\mathbb{P} \in \mathcal{S}} K(s, \mathbb{P}) = K(s, \mathcal{S}) \) for all \( s \in \mathcal{L}_0 \).

Then, any \( \rho \)-estimator \( \hat{s} \) based on \( S \) satisfies, for all \( s \in \mathcal{L}_0 \) and \( d(S) \) given by (27),

(29) \( \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq C (K(s, \mathcal{S}) + d(S)) \)

for some universal constant \( C \).

**Proof.** It relies on Theorem 1 and the following proposition (to be proved in Section 9.1) which is a variant of the lemma (Section 5.3) in Barron (1991) and of independent interest since it applies to many other situations.
Proposition 1. Let \( s \) and \( \bar{s} \) belong to \( \mathcal{L}_0 \) and \( T(X) \) be a random variable such that
\[
\mathbb{P}_\bar{s}[T(X) \geq z] \leq ae^{-z} \quad \text{for all } z \geq 0 \quad \text{and some } a > 0.
\]
Then, if \( c = \log(1 + a) + K(s, \bar{s}) \),
\[
\mathbb{E}_s[T(X)] \leq 1 + c + \log \left(1 + c + \sqrt{2}\right) < 1 + c + \sqrt{2c}.
\]
If \( \mathbb{P}_\bar{s}[T(X) \geq z] \leq ae^{-bz} \) for all \( z \geq z_0 \geq 0 \) with \( a, b > 0 \) and \( c' = \log(1 + ae^{-bz_0}) + K(s, \bar{s}) \), then
\[
\mathbb{E}_s[T(X)] \leq z_0 + b^{-1} \left(1 + c' + \sqrt{2c'}\right).
\]

It follows from (24) with \( s \) replaced by \( \bar{s} \in S \) that
\[
\mathbb{P}_\bar{s}[h^2(s, \bar{s}) > c_2 D^S(s, \bar{s}) + 1.45c_3 + \xi] \leq e^{-\xi/c_3} \quad \text{for all } \xi > 0
\]
and we may therefore apply (31) to \( T(X) = h^2(s, \bar{s}) - c_2 D^S(s, \bar{s}) - 1.45c_3 \) which corresponds to \( a = 1, b = 1/c_3 \) and \( z_0 = 0 \). Then \( c' = \log 2 + K \) (with \( K = K(s, \bar{s}) \)), leading to
\[
\mathbb{E}_s[h^2(s, \bar{s})] \leq c_2 D^S(s, \bar{s}) + 1.45c_3 + c_3 \left(1 + \log 2 + K + \sqrt{2(log 2 + K)}\right).
\]
We finally derive from the triangular inequality and (6) with \( \alpha = 1/50 \) that
\[
\mathbb{E}_s[h^2(s, \bar{s})] \leq \frac{51}{50} \left[c_2 D^S(s, \bar{s}) + c_3 \left(2.45 + \log 2 + K + \sqrt{2(log 2 + K)}\right) + 50h^2(s, \bar{s})\right].
\]
Since \( \bar{s} \) is arbitrary in \( S \) and \( h^2 \leq K/2 \), we can obtain that
\[
\mathbb{E}_s[h^2(s, \bar{s})] \leq \frac{51}{50} \inf_{\bar{s} \in S} \left\{c_2 D^S(s, \bar{s}) + c_3 \left(2.45 + \log 2 + K + \sqrt{2(log 2 + K)}\right) + 25K\right\}
\]
leading to
\[
\mathbb{E}_s[h^2(s, \bar{s})] \leq C \left[\inf_{\bar{s} \in S} K(s, \bar{s}) + d(S)\right]
\]
for some universal constant \( C > 0 \) (since \( d(S) \geq 1 \)) and we conclude by using (28).

Inequality (29) shows that (27) is not only true when \( s \) belongs to \( S \) but also when it belongs to \( \overline{S} \) and that this risk bound deteriorates by at most the additional term \( K(s, \overline{S}) \) when \( s \) does not belong to \( \overline{S} \). The estimator \( \bar{s} \) is therefore robust with respect to the KL-divergence.

Similar results actually hold for any estimator \( \bar{s} \) and any nonnegative loss function \( \ell \) such that an analogue of (32) is satisfied, precisely
\[
\mathbb{P}_\bar{s}[\ell(s, \bar{s}) > C(\bar{s}) + \xi] \leq e^{-b\xi} \quad \text{for all } \xi > 0 \quad \text{and } \bar{s} \in S.
\]
Indeed, this implies by (31) that
\[
\mathbb{E}_s[\ell(s, \bar{s})] \leq C(\bar{s}) + (C'/b) \left[1 + K(s, \bar{s})\right]
\]
and, if the loss function \( \ell \) satisfies \( \ell(s, t) \leq A \left[\ell(s, u) + \ell(u, t)\right] \) for some constant \( A \) and all \( s, t, u \), then
\[
\mathbb{E}_s[\ell(s, \bar{s})] \leq A \left[C(\bar{s}) + (C'/b) \left[1 + K(s, \bar{s})\right] + \ell(\bar{s}, s)\right] \quad \text{for all } \bar{s} \in \overline{S}
\]
and finally
\[
\mathbb{E}_s[\ell(s, \bar{s})] \leq A \sup_{\bar{s} \in \overline{S}} C(\bar{s}) + C_0 \left[1 + K(s, \overline{S}) + \ell(s, \overline{S})\right].
\]
This means that, if one allows bias terms depending on KL-divergences, which is often the case in density estimation when one uses likelihood-based methods, one can always assume that the true parameter belongs to the model $S$ and then extend the result to all $s$ satisfying $\inf_{\pi \in S} K(s, \bar{S}) < +\infty$.

4.5. **Robustness properties with respect to the Hellinger distance.** Unfortunately, if $h^2(s, \bar{S}) \leq K(s, \bar{S})/2$, the reciprocal $h^2(s, \bar{S}) \geq cK(s, \bar{S})/2$ for some positive $c$ is definitely not true in general and we cannot use the previous results to get robustness properties with respect to the Hellinger distance, which is actually a much stronger property. Hopefully, this robustness property is already included in our Theorem 1. We do need this robustness property in order to work with general models, not only countable ones. Since our models, as described in Section 4.1 (for instance the classical sets that are used in Approximation Theory), are typically uncountable, we have to replace them by countable approximations.

We shall therefore apply the following strategy: given a model $\bar{S}$ for $s$ we shall replace it by a countable approximating set $S$ to build our estimator. The natural question at this stage is then: “given $\bar{S}$, how to choose $S$?”. First, $S$ should approximate $\bar{S}$ within some (typically small) $\eta$, according to the following definition.

**Definition 2.** Given a model $\bar{S}$ in the metric space $(\mathcal{L}_0, h)$ and $\eta \geq 0$, we say that a countable subset $S[\eta]$ of $\mathcal{L}_0$ (not necessarily included in $\bar{S}$) is an $\eta$-net for $\bar{S}$ if $\sup_{t \in S} h(t, S[\eta]) \leq \eta$. In particular a countable and dense subset of $\bar{S}$ is a 0-net.

Setting

$$D^S = \sup_{(s, \bar{S}) \in \mathcal{L}_0 \times \mathcal{S}} D^S(s, \bar{S}) \quad \text{and} \quad \overline{D}^S = \sup_{(s, \bar{S}) \in \mathcal{L}_0 \times \mathcal{S}} D^S(s, \bar{S}),$$

we derive the following corollary:

**Corollary 1.** Let $S = S[\eta]$ be a countable $\eta$-net for $\bar{S}$, $\hat{s}$ a $\rho$-estimator based on $S$ and $c_1' = c_1 - 1$. Then, for all $\xi > 0$,

$$\mathbb{P}_s \left[ h^2(s, \hat{s}) \leq 2c_1'h^2(s, \bar{S}) + \left( 2c_1'\eta^2 + c_2D^S[\eta] \right) + c_3(1.45 + \xi) \right] \geq 1 - e^{-\xi},$$

hence

$$\mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq 2c_1'h^2(s, \bar{S}) + \left( 2c_1'\eta^2 + c_2D^S[\eta] \right) + c_4.$$

**Proof.** These bounds are straightforward consequences of (24), (25) and the inequality $h^2(s, S[\eta]) \leq 2h^2(s, \bar{S}) + 2\eta^2$. \hfill $\square$

We see that (35) corresponds to a decomposition of the risk into the sum of three terms among which only one, namely $2c_1'\eta^2 + c_2D^S[\eta]$ depends on the chosen net $S[\eta]$. We shall now focus our attention on this quantity, introducing the two following notions of dimension.

**Definition 3.** Given a model $\bar{S}$ in $\mathcal{L}_0$, we define its dimension $D(\bar{S})$ and its uniform dimension $\overline{D}(\bar{S})$ by

$$D(\bar{S}) = \inf_{S} \left[ 2c_1' \sup_{u \in \bar{S}} h^2(u, S) + c_2D^S \right] \quad \text{and} \quad \overline{D}(\bar{S}) = \inf_{S} \left[ 2c_1' \sup_{u \in \bar{S}} h^2(u, S) + c_2\overline{D}^S \right],$$

where $D^S$ and $\overline{D}^S$ have been defined in (33) and, in both cases, the infimum is taken over all countable subsets $S$ of $\mathcal{L}_0$. 

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It follows from these definitions and the bound $D^S(s, \mathcal{S}) \leq 8n/c_0$ that

$$1 < c_2 \leq D(\mathcal{S}) \leq \overline{D}(\mathcal{S}) \leq 2c'_1 n + 8n(c_2/c_0) < 142n. \tag{37}$$

If $\mathcal{S} \subset \mathcal{S}'$, then $D(\mathcal{S}) \leq D(\mathcal{S}')$ and $\overline{D}(\mathcal{S}) \leq \overline{D}(\mathcal{S}')$ which means that $D$ and $\overline{D}$ are non-decreasing with respect to the inclusion. These two dimensions have actually different purposes: we shall use $D(\mathcal{S})$ to bound the risk of a $\rho$-estimator on a given model $\mathcal{S}$ while $\overline{D}(\mathcal{S})$ will be used for model selection purposes in Section 7.

Since the separability of $\overline{D}$ implies the existence of $\eta$-nets $S[\eta]$ for $\mathcal{S}$ whatever $\eta \geq 0$, (36) can be reformulated as

$$D(\mathcal{S}) = \inf_{\eta > 0} \inf_{S[\eta]} \left[ 2c'_1 \eta^2 + c_2 D^{S[\eta]} \right] \quad \text{and} \quad \overline{D}(\mathcal{S}) = \inf_{\eta > 0} \inf_{S[\eta]} \left[ 2c'_1 \eta^2 + c_2 \overline{D}^{S[\eta]} \right],$$

where the infima now run over all possible $\eta$-nets $S[\eta]$ of $\mathcal{S}$.

The important property of these dimensions lies in the fact that they allow to replace the model $\mathcal{S}$ by a suitable $\eta$-net $S[\eta]$ ($\eta \geq 0$) to which Theorem 1 applies. In particular, choosing $S[\eta]$ such that

$$2c'_1 \eta^2 + c_2 D^{S[\eta]} \leq D(\mathcal{S}) + c_3/20,$$

we get from (34) and an integration with respect to $\xi$ the following risk bounds.

**Corollary 2.** Given a model $\mathcal{S}$, there exists a $\rho$-estimator $\hat{s}$ such that for all $s \in \mathcal{L}_0$

$$\mathbb{P}_s \left[ h^2(s, \hat{s}) \leq 2c'_1 h^2(s, \mathcal{S}) + D(\mathcal{S}) + c_3(1.5 + \xi) \right] \geq 1 - e^{-\xi} \quad \text{for all } \xi > 0 \tag{38}$$

and

$$\mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq 2c'_1 h^2(s, \mathcal{S}) + D(\mathcal{S}) + c_4. \tag{39}$$

The replacement of $\mathcal{S}$ by a suitable pair ($\eta, S[\eta]$) leads to a risk bound depending on $\mathcal{S}$ only. This means that, given a model $\mathcal{S}$, the risk of $\hat{s}$ breaks down, up to numerical constants, into the bias term $h^2(s, \mathcal{S})$ which depends on the quality of the approximation of $s$ by the model $\mathcal{S}$ and the dimensional term $D(\mathcal{S})$ which measures in some sense the massiveness of $\mathcal{S}$. In particular, in the i.i.d. case, we get

$$\mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq 2c'_1 h^2(s, \mathcal{S}) + n^{-1} \left[ D(\mathcal{S}) + c_4 \right],$$

as expected.

4.5.1. **Models which are VC-subgraph classes.** The first situation that we shall consider deals with models $\mathcal{S}$ such that any countable subset $S$ of $\mathcal{S}$ satisfies $D^S \leq D'$ where $D'$ only depends on $\mathcal{S}$ but not on the choice of the subset $S$. In such a case it is natural to choose for $S$ a countable and dense subset of $\mathcal{S}$ which is a 0-net for $\mathcal{S}$ so that (35) leads to

$$\mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq 2c'_1 h^2(s, \mathcal{S}) + c_2 D' + c_4. \tag{40}$$

In order to deal with this situation, we first need to prove an auxiliary result and for this we shall consider an element $t = (t_1, \ldots, t_n)$ in $\mathcal{L}_0$ as a real-valued function on $\mathcal{X} = \bigcup_{i=1}^n \{i\} \times \mathcal{X}_i$ defined by

$$t(\mathcal{X}) = t_i(x) \quad \text{for all } \mathcal{X} = (i, x) \in \mathcal{X}. \tag{41}$$
Replacing the $X_i$ by the random variables $\overline{X}_i = (i, X_i)$ so that $t(\overline{X}_i) = t_i(X_i)$, we see that $w^S(s, \mathcal{S}, y)$ can be written as

$$w^S(s, \mathcal{S}, y) = \mathbb{E}_s \left[ \sup_{f \in \mathcal{F}^S(s, \mathcal{S}, y)} \left\{ \sum_{i=1}^n (f(\overline{X}_i) - \mathbb{E}_s[f(\overline{X}_i)]) \right\} \right],$$

where the supremum runs among the class $\mathcal{F}^S(s, \mathcal{S}, y)$ of real-valued functions $f$ on $\mathcal{S}$ given by

$$\mathcal{F}^S(s, \mathcal{S}, y) = \left\{ \psi \left( \sqrt{t/\mathcal{S}} \right) \mid t \in \mathcal{B}^S(s, \mathcal{S}, y) \right\}.$$
and
\[ C E_\mathcal{S} \left[ h^2(\mathbf{s}, \mathcal{S}) \right] \leq h^2(\mathbf{s}, \mathcal{S}) + \nabla \left[ 1 + \log_+(n/\nabla) \right] + 1 \]

for some universal constant \( C > 0 \). If \( \mathcal{X}_i = \mathcal{X} \) for all \( i \) and \( \overline{\mathcal{S}} \) is of the form \( \{ t = (t_i, \ldots, t_i), t_i \in \Sigma \} \) for a set \( \Sigma \) of real valued functions on \( \mathcal{X} \) which is VC-subgraph with index \( \nabla \), the previous bound still holds.

**Proof.** Let \( S \) be any countable subset of \( \overline{\mathcal{S}} \). Then it is VC-subgraph with index not larger than \( \nabla \). Since for all \( \mathbf{s} \) and \( \mathbf{s} \) in \( \mathcal{L}_0 \) and \( y > 0 \), \( \mathcal{F}^S(\mathbf{s}, \mathbf{s}, y) \subset \psi(\sqrt{\mathcal{S}/\mathcal{S}}) \), it follows from (vii) of Proposition 11 that \( \mathcal{F}^S(\mathbf{s}, \mathbf{s}, y) \) is VC-subgraph with index not larger than \( \nabla \). Then, by (87) below, there exists a universal constant \( A \) such that, for all \( \mathbf{s}, \mathbf{s} \in \mathcal{L}_0 \), \( y, z > 0 \) and probability \( Q \) on \( \overline{\mathcal{F}} \),
\[ \log N \left( \mathcal{F}^S(\mathbf{s}, \mathbf{s}, y), Q, z \right) \leq 2 \nabla \log_+(A/z). \]

Proposition 2 therefore applies with \( \mathcal{F}(x) = 2 \nabla \log_+(Ax) \) and we may assume that \( A \geq 2e \). An integration by parts then shows that \( L \leq 2 \) and (45) follows from (44). Inequalities (46) and (47) derive from (34) and (35) respectively with \( S = S[\eta] \) and \( \eta = 0 \). \( \square \)

4.5.2. **Models which are totally bounded.** Of course, not all models are VC-subgraph classes but there exists another type of models for which we are able to bound \( D^S[\eta] \) for suitable \( \eta \)-nets of \( \overline{\mathcal{S}} \). When \( \overline{\mathcal{S}} \) is totally bounded, one can take \( S[\eta] \) finite for all \( \eta > 0 \) and so are the subsets \( B^S[\eta](\mathbf{s}, \mathbf{s}, y) \) of \( S[\eta] \) for all positive \( y \) and \( \eta \). Conversely, if, for all \( y, \eta > 0 \), one can choose \( S[\eta] \) so that the sets \( B^S[\eta](\mathbf{s}, \mathbf{s}, y) \) are finite, this is in particular true for \( y = \sqrt{2n} \) and, since the distance \( h \) is bounded by \( \sqrt{n} \), \( S[\eta] = B^S[\eta](\mathbf{s}, \mathbf{s}, \sqrt{2n}) \) is finite for all \( \eta > 0 \). This implies that \( \overline{\mathcal{S}} \) is totally bounded so that this approach based on the cardinality of \( B^S[\eta](\mathbf{s}, \mathbf{s}, y) \) is restricted to totally bounded models only. It nevertheless has the advantage to require the control of the supremum of the process \(|Z(X, \mathbf{s}, \cdot)|\) over a finite set which can be done via the following result to be proved in Section 9.6.

**Proposition 3.** Let \( \mathbf{s} \) and \( \mathbf{s} \) belong to \( \mathcal{L}_0 \), \( y > 0 \) and \( S \) be a countable subset of \( \mathcal{L}_0 \). Assume that \( |B^S(\mathbf{s}, y)| < +\infty \), then
\[ w^S(\mathbf{s}, \mathbf{s}, y) \leq 2 \left[ y \sqrt{3 \log_+(N) + \log_+(N)} \right] \quad \text{with} \quad N = 2 |B^S(\mathbf{s}, y)|. \]

With such a result at hand, bounding \( D^S[\eta](\mathbf{s}, \mathbf{s}) \) amounts to controlling \( |S[\eta] \cap B(\mathbf{s}, y)| \) when \( S[\eta] \) is minimal. Since \( \mathbf{s} \) is unknown, we need to bound the number of points of \( S[\eta] \) lying in an arbitrary Hellinger ball of radius \( y \). It is then natural to introduce the following entropy bounds.

**Definition 4.** Given a totally bounded model \( \overline{\mathcal{S}} \) of \( \mathcal{L}_0 \) and \( 0 < \eta \leq y \), we set
\[ \mathcal{H}^S(\eta, y) = \inf_{S[\eta]} \sup_{S[\eta] \in \mathcal{L}_0} \log |S[\eta] \cap S(\mathbf{s}, y)| \geq 0, \]

where the infimum runs among all \( \eta \)-nets \( S[\eta] \) for \( \overline{\mathcal{S}} \). We shall say that \( \overline{\mathcal{S}} \) has an entropy dimension bounded by \( V \geq 0 \) if
\[ \mathcal{H}^S(\eta, y) \leq V \log(y/\eta) \quad \text{for all} \quad \eta > 0 \quad \text{and} \quad y \geq 2\eta. \]
Let \( \tilde{D} \) be a right-continuous function from \( (0, \sqrt{n}] \) into \([1/2, +\infty)\) with \( \tilde{D} (\sqrt{n}) = 1/2 \). We shall say that \( \overline{S} \) has a metric dimension bounded by \( \tilde{D}(\cdot) \) if

\[
(50) \quad \mathcal{H}^{\overline{S}}(\eta, y) \leq (y/\eta)^2 \tilde{D}(\eta) \quad \text{for all } \eta > 0 \quad \text{and} \quad y \geq 2\eta.
\]

The definition of the metric dimension is due to Birgés (1984) (Definition 6 p. 293). Since the distance we use is bounded by \( \sqrt{n} \), \( \mathcal{H}^{\overline{S}}(\sqrt{n}, 2\sqrt{n}) = 0 \) and the additional condition \( \tilde{D}(\sqrt{n}) = 1/2 \) is not restrictive. The logarithm being a slowly varying function, it is not difficult to see that the notion of metric dimension is more general than the entropy one in the sense that if \( \overline{S} \) has an entropy dimension bounded by some \( V \), then it also has a metric dimension bounded by \( \tilde{D}(\cdot) \) with

\[
(51) \quad \tilde{D}(\eta) \leq (1/2) \vee [V(\log 2)/4] \quad \text{for all } \eta > 0.
\]

**Proposition 4.** Let \( \overline{S} \) be a totally bounded nonempty subset of \( L_0 \) with metric dimension bounded by \( \tilde{D}(\cdot) \) and \( n \geq 131c_0^{-2} \). Let \( \overline{\eta} \) be defined by

\[
\overline{\eta} = \inf \left\{ \eta > 0 \left| \eta^{-2} \tilde{D}(\eta) \leq c_0^2/262 \right. \right\}.
\]

Then one can find an \( \overline{\eta} \)-net \( S[\overline{\eta}] \) for \( \overline{S} \) which satisfies \( \overline{D}^{S[\overline{\eta}]} \leq 4\overline{\eta}^2 \). Hence \( \overline{D}(\overline{S}) \leq 2(a_1' + 2c_2) \overline{\eta}^2 \) and any \( \rho \)-estimator \( \overline{s} \) based on \( S[\overline{\eta}] \) satisfies

\[
P_s \left[ h^2(s, \overline{s}) \leq 2a_1' h^2(s, \overline{S}) + 2(a_1' + 2c_2) \overline{\eta}^2 + c_3 (1.45 + \xi) \right] \geq 1 - e^{-\xi} \quad \text{for all } \xi > 0.
\]

**Proof.** For \( \eta > 0 \), let \( S[\eta] \) be a minimal \( \eta \)-net for \( \overline{S} \). Using (50) and the fact that \( \tilde{D}(\eta) \geq 1/2 \), we derive that

\[
\log_{+} \left( 2 |B^{S[\eta]}(s, y)| \right) \leq \log 2 + \frac{y^2 \tilde{D}(\eta)}{\eta^2} \leq \left( 1 + \frac{\log 2}{\eta} \right) \frac{y^2 \tilde{D}(\eta)}{\eta^2} \quad \text{for all } y \geq 2\eta.
\]

If, moreover, \( y \geq 2\overline{\eta} \), using the fact that \( \tilde{D}(\eta) \) is right-continuous and the definition of \( \overline{\eta} \), we see that

\[
\frac{y^2 \tilde{D}(\overline{\eta})}{\overline{\eta}^2} \leq \frac{c_0^2 y^2}{262},
\]

so that we can apply Proposition 3 with \( \log_{+} (2 |B^{S[\eta]}(s, y)|) \leq [1 + (\log 2)/2] c_0^2 y^2/262 = ay^2 \) and get

\[
w^{S[\eta]}(s, \overline{s}, y) \leq 2 \left( a + \sqrt{3a} \right) y^2 < c_0 y^2/4,
\]

which leads to \( D^{S[\eta]}(s, \overline{s}) \leq 4\overline{\eta}^2 \), hence to the bounds for \( \overline{D}^{S[\eta]} \) and \( \overline{D}(\overline{S}) \). The final bound then follows from (34). \( \square \)

### 4.5.3. Minimax risk on a model.

Let us now focus on the specific case of the risk of \( \rho \)-estimators over a model \( \overline{S} \) when \( s \) is an arbitrary point in \( \overline{S} \) or equivalently on the maximal risk of \( \rho \)-estimators over a model \( \overline{S} \) in \( L_0 \) since it provides an upper bound for the minimax risk on \( \overline{S} \). We recall that the minimax risk over a subset \( \overline{S} \) of \( L_0 \) and the maximal risk of an estimator \( \overline{s} \) over \( \overline{S} \) are given respectively by

\[
(52) \quad R_M(\overline{S}) = \inf_{\overline{s}} \sup_{s \in \overline{S}} E_s \left[ h^2(s, \overline{s}) \right] \leq \sup_{s \in \overline{S}} E_s \left[ h^2(s, \overline{s}) \right],
\]
where the infimum runs among all possible estimators \( \tilde{s} \) of \( s \). In order to bound \( R_M(\overline{S}) \), we shall restrict our attention to \( \rho \)-estimators that satisfy (39). The maximal risk \( \mathcal{R}_\rho(\overline{S}) \) of any such \( \rho \)-estimator \( \tilde{s} \) over \( \overline{S} \) can be bounded from (39) and (37) by

\[
\mathcal{R}_\rho(\overline{S}) \leq D(\overline{S}) + c_4 \leq \overline{D}(\overline{S}) + c_4.
\]

It follows that it suffices to bound \( D(\overline{S}) \) (or \( \overline{D}(\overline{S}) \)) from above in order to control the minimax risk over \( \overline{S} \). We can therefore derive from the results of the previous sections the following bounds for the minimax risk.

**Corollary 3.** If \( \overline{S} \), viewed as a set of real-valued functions on \( \mathcal{X} \) as defined by (41) is VC-subgraph with index \( \nu \), then

\[
R_M(\overline{S}) \leq \mathcal{R}_\rho(\overline{S}) \leq c_4 + C\nu \left[ 1 + \log_+ \left( n/\nu \right) \right].
\]

If \( \overline{S} \) is a totally bounded nonempty subset of \( \mathcal{L}_0 \) with metric dimension bounded by \( D(\cdot) \), then

\[
R_M(\overline{S}) \leq \mathcal{R}_\rho(\overline{S}) \leq c_4 + 2 \left( c'_4 + 2c_2 \right) \eta^2 \text{ with } \eta = \inf \left\{ \eta > 0 \mid \eta^{-2} D(\eta) \leq c_0^2/262 \right\}.
\]

The bound (54) for \( \mathcal{R}_\rho(\overline{S}) \) that we derived from Theorem 3 involves a logarithmic factor although one would rather expect a bound of the form \( \mathcal{R}_\rho(\overline{S}) \leq C\nu \). If we compare this result to (49) (with \( \eta = \sqrt{n} \) in order to make \( h \) and the \( L_2(Q) \)-distance comparable), we see that this phenomenon is due to the entropy bound (48) which is uniform with respect to \( y \). An entropy bound of the form

\[
\log N(\mathcal{F}^S(s, s, y), Q, z) \leq C\nu \log_+ \left( \frac{A_n/\sqrt{n}}{z} \right) \text{ for all } y \text{ and } z > 0
\]

would lead to the expected inequality \( \mathcal{R}_\rho(\overline{S}) \leq C\nu \). Unfortunately, we do not know whether a bound such as (55) is true or not.

There exists at least one situation where this extra logarithmic factor can be removed: when \( \overline{S} \) consists of piecewise constant functions. For the sake of simplicity, let us only consider the density framework described in Section 2.3. Given a finite partition \( \mathcal{I} \) of \( \mathcal{X} \) such that \( \mu(I) > 0 \) for all \( I \in \mathcal{I} \), we consider the set \( \overline{S} = \overline{S}_{\mathcal{I}} \) of all densities on \( (\mathcal{X}, \mathcal{A}, \mu) \) which are piecewise constant on each element \( I \) of \( \mathcal{I} \), which means that

\[
\overline{S}_{\mathcal{I}} = \left\{ t = \sum_{I \in \mathcal{I}} \frac{t_I}{\mu(I)} \mathbb{1}_I \mid t_I \geq 0 \text{ for all } I \in \mathcal{I} \text{ and } \sum_{I \in \mathcal{I}} t_I = 1 \right\}.
\]

If we choose for \( S \) a subset of \( \overline{S}_{\mathcal{I}} \), the resulting estimator \( \tilde{s} \) will therefore be an histogram-type estimator.

As a subset of a linear space with dimension not larger than \(|\mathcal{I}| + 2\), \( \overline{S}_{\mathcal{I}} \) is VC-subgraph with index not larger than \(|\mathcal{I}| + 2\) and the following holds.

**Proposition 5.** For any finite partition \( \mathcal{I} \) of \( \mathcal{X} \), such that \( \mu(I) > 0 \) for all \( I \in \mathcal{I} \),

\[
D(\overline{S}_{\mathcal{I}}) \leq 96c_2c_0^2 \left\{ I \in \mathcal{I} \mid \int_I s \, d\mu > 0 \right\} \leq 96c_2c_0^2 |\mathcal{I}|.
\]

Compared to Theorem 3, the extra logarithmic factor has disappeared. Nevertheless, Proposition 5 only provides an upper bound on \( D(\overline{S}_{\mathcal{I}}) \) and not on \( \overline{D}(\overline{S}_{\mathcal{I}}) \). The proof of this proposition is postponed to Section 9.7.
5. Connection with the Maximum Likelihood Estimator

Throughout this section, we consider the problem of density estimation from \( n \) i.i.d. observations \( X_1, \ldots, X_n \) as described in Section 2.3. Our aim is to show that \( \rho \)-estimation may recover the classical MLE in various situations.

5.1. Regular parametric models. We consider here a parametric set of densities \( \{t_\theta, \theta \in \Theta'\} \) on the measured space \((\mathcal{X}, \sigma, \mu)\) indexed by some open subset \( \Theta' \) of \( \mathbb{R}^d \) and such that the mapping \( \theta \mapsto P_\theta = t_\theta \cdot \mu \) is one-to-one. Our model is \( S = \{t_\theta, \theta \in \Theta\} \) for some \( \Theta \subset \Theta' \).

Assumption 1.

(i) The parameter set \( \Theta \) is a compact and convex subset of \( \Theta' \), \( \inf_{(x, \theta) \in \mathcal{X} \times \Theta} t_\theta(x) > 0 \) and the true parameter \( s \) is an element \( t_\vartheta \in \Theta \) such that \( \vartheta \in \mathring{\Theta} \).

(ii) The mapping \( \theta \mapsto \sqrt{t_\theta(x)} \) from \( \Theta' \rightarrow \mathbb{R} \) is differentiable and all its partial derivatives are continuous and bounded uniformly with respect to \( x \in \mathcal{X} \).

(iii) The Fisher Information matrix is continuous and invertible on \( \Theta \).

(iv) With probability tending to one when \( n \) goes to infinity, there exists a maximum likelihood estimator \( \tilde{\theta}_n \) which is \( \sqrt{n} \)-consistent.

One can then prove (in Section 9.4):

Theorem 4. Let \( \mathcal{S} \) be a parametric model of densities satisfying Assumption 1 and \( S \) an arbitrary countable and dense subset of \( \mathcal{S} \). With probability tending to 1 as \( n \) tends to infinity, \( t_{\tilde{\theta}_n} \) belongs to the closure of \( E(X, S) \) and is therefore a \( \rho \)-estimator.

This result shows that when the model is regular enough and contains the true density, \( \rho \)-estimation allows to recover the MLE, at least when \( n \) is large enough. The numerical study of Mathieu Sart on very simple statistical models \( \mathcal{S} \) seems to indicate that our procedure allows to recover the MLE in almost all simulations even when the number of observations \( n \) is small. Consequently, there seems to be some space for improvement in Theorem 4. At least, as we shall see in the next section, Assumption 1 could be weakened.

5.2. A direct computation on a non-regular model. In this section, we give an example of a non-regular statistical model (in the usual statistical sense) on which we also recover the MLE with probability 1. This means that the connections between the MLE and \( \rho \)-estimators are not restricted to situations where the parameter is estimated at the usual parametric rate \( n^{-1/2} \).

Let us consider the density model \( \mathcal{S} = \{q_\theta = 1_{[-1/2+\theta,1/2+\theta]}, \theta \in \mathbb{R}\} \) and deal with the problem of estimating \( \theta \) from the observation of an \( n \)-sample \( X_1, \ldots, X_n \) of density \( s \in \mathcal{S} \). For all \( \theta, \theta' \in \mathbb{R} \), easy calculations show that

\[
  h^2(q_\theta, q_{\theta'}) = |\theta - \theta'| \wedge 1
\]

and \( S = \{q_\theta, \theta \in \mathbb{Q}\} \) provides thus a countable and dense subset of \( \mathcal{S} \).

Proposition 6. Assume that \( s \in \mathcal{S} \) and let \( X^{(1)} < \ldots < X^{(n)} \) be the order statistics corresponding to our sample. The estimator \( \tilde{\theta}_n = (X^{(1)} + X^{(n)}) / 2 \) of \( \theta \) maximizes the likelihood and \( q_{\tilde{\theta}_n} \) is a \( \rho \)-estimator of \( s \).
Proof. The fact that the likelihood \( \theta \mapsto \prod_{i=1}^n \mathbb{1}_{X_i-1/2,X_i+1/2}(\theta) \) is maximal for \( \theta = \tilde{\theta}_n \) is easy to check. It remains to show that if \( s \in \overline{S} \), \( q_{\tilde{\theta}_n} \) belongs to the closure of \( \mathcal{E}(X,S) \) with probability 1.

In the sequel, we only consider points \( \theta \) and \( \theta' \) that belong to \( Q \). Since the density \( q_0 \) is even, \( \rho(q_\theta, (q_{\theta'} + q_\theta)/2) = \rho(q_{\theta'}, (q_{\theta'} + q_\theta)/2) \) and therefore

\[
T(X, q_\theta, q_{\theta'}) = \frac{1}{\sqrt{2}} \sum_{i=1}^n \psi \left( \sqrt{\frac{q_{\theta'}}{q_\theta}}(X_i) \right) \quad \text{for all } \theta, \theta' \in Q.
\]

For all \( \theta' \), \( q_{\theta'}(\cdot) \) takes its values in \( \{0, 1\} \) and for all \( i \in \{1, \ldots, n\} \), \( q_{\theta'}(X_i) = 1 \) if and only if \( \theta' \in [X_{i-1/2}, X_i + 1/2] \). It follows that \( \theta' \in Q \) and \( q_{\theta'}(X_i) = 1 \) for all \( i \) if and only if \( \theta' \in \tilde{\Theta} \) with

\[
\tilde{\Theta} = [X_{(n)} - 1/2, X_{(1)} + 1/2] \cap Q.
\]

This random subset of \( Q \) is non-void since, when \( s \in \overline{S} \), the diameter of \( \tilde{\Theta} \) is \( \Delta(X) = 1 - (X_{(n)} - X_{(1)}) > 0 \) \( \ell_s \)-a.s. For all \( \theta' \in \tilde{\Theta} \) and \( i \in \{1, \ldots, n\} \)

\[
\psi \left( \sqrt{\frac{q_{\theta'}}{q_\theta}}(X_i) \right) = \psi \left( \sqrt{\frac{1}{q_\theta}(X_i)} \right) = \mathbb{1}_{\{q_{\theta'}=0\}}(X_i),
\]

which implies that \( T(X, q_\theta, q_{\theta'}) \geq 1/\sqrt{2} > 0 \) if \( \theta \not\in \tilde{\Theta} \), hence,

\[
\Upsilon(S, q_\theta) = \sup_{\theta' \in Q} T(X, q_\theta, q_{\theta'}) \geq \sup_{\theta' \in \tilde{\Theta}} T(X, q_\theta, q_{\theta'}) \geq \frac{1}{\sqrt{2}} \quad \text{for } \theta \not\in \tilde{\Theta}.
\]

For \( \theta \in \tilde{\Theta} \), \( q_{\theta}(X_i) = 1 \) for all \( i \) so that \( q_{\theta'}(X_i) \leq q_{\theta}(X_i) \) for all \( i \) and \( T(X, q_\theta, q_{\theta'}) \leq 0 \) whatever \( \theta' \). It follows that

\[
\Upsilon(S, q_\theta) = \sup_{\theta' \in Q} T(X, q_\theta, q_{\theta'}) = 0.
\]

Hence, \( \theta \mapsto \Upsilon(S, q_\theta) \) is minimum for the elements \( \theta \in \tilde{\Theta} \) and \( \{q_\theta, \theta \in \tilde{\Theta}\} \subset \mathcal{E}(X,S) \). Since \( \tilde{\theta}_n \) belongs to the closure of \( \tilde{\Theta} \) (with respect to the Euclidean distance) and since for any sequence \( (\theta_j)_{j \geq 1} \) converging towards \( \tilde{\theta}_n \), \( q_{\theta_j} \) converges towards \( q_{\tilde{\theta}_n} \) with respect to the Hellinger distance, \( q_{\tilde{\theta}_n} \) belongs to the closure of \( \mathcal{E}(X,S) \) and is therefore a \( \rho \)-estimator. \( \square \)

5.3. Risk bounds under entropy with bracketing. Since, for some specific models of densities \( \overline{S} \) that contain the true density \( s \), our estimator and the MLE coincide with probability close to 1, it is natural to wonder how to compare the performance of these two estimators on more general models \( \overline{S} \), possibly not containing \( s \). One way to do so is to compare their risk bounds. In the literature, the risk bounds which are established for the MLE usually take the following form

\[
(57) \quad C \mathbb{E}_s [h^2(s, \tilde{s})] \leq K(s, \overline{S}) + \tau_n^2 \vee n^{-1},
\]

where \( C \) is a positive universal constant and \( K(s, \overline{S}) = \inf_{t \in \overline{S}} K(s, t) \). As to the number \( \tau_n^2 \), which usually corresponds to the maximal risk over \( \overline{S} \), it is obtained by solving an equation depending on the bracketing entropy of \( \overline{S} \). In this section we shall establish a similar risk bound for our estimator (up to universal constants) with the same value of \( \tau_n \) and under similar assumptions than those leading to \((57)\) in the literature (we shall use those given in Massart (2007) Theorem 7.11).
Assumption 2. The following holds.

(i) There exists a countable subset \( S \) of \( \overline{S} \) such that for all densities \( s \) on \( (\mathcal{X}, \mathcal{A}, \mu) \), \( K(s, S) = K(s, \overline{S}) \).

(ii) Let \( \sigma > 0 \) and \( \overline{s} \in S \). There exists a non-increasing mapping \( z \mapsto \mathcal{H}^S(\overline{s}, \sigma, z) \) from \( (0, +\infty) \) into \( (0, +\infty) \) such that for all \( z > 0 \) there exists a family \( \mathcal{I}(\overline{s}, \sigma, z) \) of pairs of nonnegative measurable functions on the measured space \( (\mathcal{X}, \mathcal{A}, \mu) \) such that
\[
\log 2 \leq \log |\mathcal{I}(\overline{s}, \sigma, z)| \leq \mathcal{H}^S(\overline{s}, \sigma, z)
\]
and for all \( t \in \mathcal{B}^S(\overline{s}, \sigma \sqrt{n}) \) one can find a pair \((t_L, t_U) \in \mathcal{I}(\overline{s}, \sigma, z)\) such that \( t_L \leq t \leq t_U \) and
\[
\frac{1}{2} \int (\sqrt{t_U} - \sqrt{t_L})^2 d\mu \leq z^2.
\]

(iii) There exists a non-decreasing function \( \phi \) from \( (0, +\infty) \) into \( (0, +\infty) \) such that \( x \mapsto \phi(x)/x \) is non-increasing on \( (0, +\infty) \) and for which
\[
\sup_{\pi \in S} \int_0^\sigma \sqrt{\mathcal{H}^S(\overline{s}, \sigma, z)} \, dz \leq \phi(\sigma).
\]

From these assumptions, we can derive the following result to be proved in Section 9.5.

Theorem 5. Let \( X_1, \ldots, X_n \) be an \( n \)-sample with values in \( (\mathcal{X}, \mathcal{A}, \mu) \) and density \( s \) with respect to \( \mu \). Let \( \overline{S} \) be a model of densities satisfying Assumption 2 and
\[
\tau_n = \inf \{ \sigma > 0, \phi(\sigma) \leq \sqrt{n} \sigma^2 \}.
\]
Then there exist universal constants \( C, C'> 0 \) such that
\[
(58) \quad d(S) = \sup_{\pi \in \overline{S}} D^S(\overline{s}, \pi) \leq (Cn\tau_n^2) \lor 1
\]
and for any \( \rho \)-estimator \( \widehat{s} \) of \( s \)
\[
(59) \quad C' \mathbb{E}_\overline{s} [h^2(s, \overline{s})] \leq K(s, \overline{S}) + \tau_n^2 \lor n^{-1}.
\]

5.4. Histogram estimators. Going back to the framework that we already considered at the end of Section 4.5.3, we consider here the problem of estimation of a density \( s \) with respect to some dominating measure \( \mu \) on \( \mathcal{X} \) using a model of piecewise constant functions. More precisely we consider a finite or countable partition \( \mathcal{I} \) of \( \mathcal{X} \) with \( \mu(I) > 0 \) for all \( I \in \mathcal{I} \). The set of piecewise constant densities with pieces belonging to the partition is the model \( \overline{\mathcal{S}}_{\mathcal{I}} \) of Section 4.5.3 which can be identified to the unit simplex \( \mathcal{S} \) in \( [0, 1]^{|\mathcal{I}|} \) by setting for each density \( t \in \overline{\mathcal{S}}_{\mathcal{I}} \)
\[
t = \sum_{I \in \mathcal{I}} \frac{t_I}{\mu(I)} 1_I \quad \text{with} \quad t_I = \int_I t(x) \, d\mu(x) \quad \text{and} \quad t_I = \{t_I, I \in \mathcal{I}\} \in \mathcal{S}.
\]
Since the Hellinger metric is topologically equivalent to the \( L_1 \)-metric, \( \overline{\mathcal{S}}_{\mathcal{I}} \) is a separable metric space for the distance \( h \). We finally set \( \overline{S} = \{t = (t, t, \ldots, t), t \in \overline{\mathcal{S}}_{\mathcal{I}}\} \subset \overline{\mathcal{S}}_{\mathcal{I}}^d \).

Given \( n \) i.i.d. observations \( X_1, \ldots, X_n \) with values on \( \mathcal{X} \) and density \( t \) with respect to \( \mu \) and \( N_I = \sum_{i=1}^n 1_I(X_i) \) for \( I \in \mathcal{I} \), the vector \((N_1, N_2, \ldots) \in [0, n]^{|\mathcal{I}|}\) is a multinomial vector with parameter \( t_I \), the MLE over \( \mathcal{S} \) is then given by \((\hat{t}_I, I \in \mathcal{I})\) with \( \hat{t}_I = N_I/n \) and the corresponding density estimator \( \hat{t} = \sum_{I \in \mathcal{I}} \hat{t}_I/\mu(I) 1_I \) of \( t \) is the MLE on the model \( \overline{\mathcal{S}}_{\mathcal{I}} \). It is also the histogram estimator of the true density \( s \) with respect to the partition \( \mathcal{I} \) of \( \mathcal{X} \).
Proposition 7. The histogram estimator $\hat{t}$ is a $p$-estimator built on the model $\overline{S}_I$.

Proof. One first observes that for $t, u \in \overline{S}_I$, $\rho(t, u) = \sum_{I \in \mathcal{I}} \sqrt{t_I u_I}$ and $(du/dt)(x) = u_I/t_I$ for $x \in I$ with the convention $0/0 = 1$. The definition (19) of the test function $T$ implies that

$$T(X, t, u) = \frac{n}{2\sqrt{2}} \sum_{I \in \mathcal{I}} \left[ \sqrt{t_I u_I + u_I^2 - \sqrt{t_I u_I + t_I^2}} + \frac{1}{\sqrt{2}} \sum_{I \in \mathcal{I}} \psi \left( \sqrt{\frac{u_I}{t_I}} \right) N_I \right]$$

$$= \frac{n}{2\sqrt{2}} \sum_{I \in \mathcal{I}} \left[ \sqrt{t_I + u_I (\sqrt{u_I} - \sqrt{t_I})} + 2\hat{t}_I \sqrt{\frac{u_I/t_I - 1}{1 + (u_I/t_I)}} \right].$$

If $\mathcal{J} = \{I \in \mathcal{I} \mid \hat{t}_I > 0\}$, it then follows that

$$T(X, \hat{t}, u) = \frac{n}{2\sqrt{2}} \sum_{I \in \mathcal{I}} \left[ \sum_{I \in \mathcal{J}} \hat{t}_I G(x_I) + \sigma(u) \right] \text{ with } G(x) = \frac{\sqrt{x} - 1}{\sqrt{1 + x}} (3 + x) \text{ for all } x \geq 0.$$

Since

$$G'(x) = \frac{2x^2 - x^{3/2} + 3x + \sqrt{x} + 3}{2\sqrt{x}(1 + x)^{3/2}} > 0 \text{ for all } x > 0,$$

$G'(1) = \sqrt{2}$ and $(1 - x) (G'(x) - \sqrt{2}) > 0$ for all $x \neq 1$. It follows that if $u' \neq u$ and $x_I = u_I/\hat{t}_I$ for $I \in \mathcal{J}$ we get

$$\hat{t}_I \left[ G(x_I) - G(x_I') \right] < \sqrt{2} (u_I - u_I') \text{ if either } x_I > x_I' \geq 1 \text{ or } x_I < x_I' \leq 1.$$  

We now consider two cases.

— If $\sigma(u) > 0$ there exists some $u' > 0$ for $I' \in \mathcal{J}^c$ and some $I \in \mathcal{J}$ with $x_I < 1$. It is therefore possible to decrease $u_I$ to $u_I + \epsilon > 0$ and increase $u_I$ to $u_I + \epsilon \leq \hat{t}_I$ for $\epsilon > 0$ small enough which implies for $T$ an increase larger than $n2^{-3/2} \sqrt{2} - 1/\epsilon > 0$. It follows that

$$T(X, \hat{t}, u) < \sup_{t \in \overline{S}} T(X, \hat{t}, t) \text{ for all } u = (u, u, \ldots, u) \text{ such that } \sigma(u) > 0.$$

— If $\sigma(u) = 0$ and $u \neq \hat{t}$, one can find $J, J' \in \mathcal{J}$ and $\epsilon > 0$ such that $x_J' = x_J + \epsilon/\hat{t}_J \leq 1$ and $x_{J'} = x_{J'} - \epsilon/\hat{t}_J \geq 1$. For such an $u$, we define $u'$ by $u'_J = u_J + \epsilon$, $u'_J = u_J - \epsilon$ and $u'_I = u_I$ for all other $I \in \mathcal{I}$ so that $\sum_{I \in \mathcal{I}} u'_I = \sum_{I \in \mathcal{I}} u_I = 1$ and $u'_I \in \mathcal{J}'$ as required. It then follows from (60) that

$$T(X, \hat{t}, u) - T(X, \hat{t}, u') = (n/2) \left[ u_J - u'_J + u_{J'} - u'_{J'} \right] = 0.$$

It follows that, for all $u \neq \hat{t}$, $T(X, \hat{t}, u) < \sup_{t \in \overline{S}} T(X, \hat{t}, t)$ and finally

$$\sup_{t \in \overline{S}} T(X, \hat{t}, t) = T(X, \hat{t}, \hat{t}) = 0.$$  

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Since the mapping
\[ y \mapsto \sqrt{u_I - \sqrt{t_I + y}} \frac{\sqrt{t_I + y + u_I}}{\sqrt{t_I + y + u_I}} (3t_I + y + u_I) \]
is continuous at 0 uniformly with respect to \( u_I \in [0,1] \) for \( \hat{t}_I > 0 \) and \( J \) is finite, replacing \( S \) by a dense subset \( S \) and \( \tilde{t} \) by a close enough approximation \( \tilde{t}_\varepsilon \) with \( \varepsilon > 0 \) and \( \sigma (\tilde{t}_\varepsilon) = 0 \) leads to
\[ \Upsilon (S, \tilde{t}_\varepsilon) = \sup_{u \in S} T(X, \tilde{t}_\varepsilon, u) < \varepsilon. \]
Since \( \varepsilon \) is arbitrary, this proves that the MLE \( \hat{t} \) belongs to the closure of \( \mathcal{E}(X, S) \). \( \square \)

This applies in particular to the example of Section 4.5.3. It also applies to the case of a finite or countable set \( \mathcal{X} \) with \( J = \{ \{j\}, j \in \mathcal{X} \} \) that is to a multimodal (or generalized multimodal when \( \mathcal{X} \) is infinite) estimation problem with \( \mu \) the counting measure, for which the MLE is again a \( \rho \)-estimator.

### 6. Examples

#### 6.1. Homoscedastic regression with fixed design

In this section, we consider the statistical framework described in Section 2.4. Our aim is therefore to estimate the function \( f \) from the observation of the \( X_i \).

The choice of a model \( \overline{S}_{q,F} \) corresponds here to those of a density \( q \) (with respect to the Lebesgue measure \( \mu \)) to approximate \( p \) and of a subset \( F \) of \( \mathbb{R}^n \) to approximate \( f \). More precisely, given \( q \) and \( F \), we define the model \( \overline{S}_{q,F} \) as the set of functions from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) given by
\[ \overline{S}_{q,F} = \{ x \mapsto q_g(x) = (q(x_1 - g_1), \ldots, q(x_n - g_n)) \mid g \in F \}. \]

**Assumption 3.** The density \( q \) is unimodal.

**Theorem 6.** Let Assumption 3 be satisfied. If \( F \), viewed as a class of functions on \( \{1, \ldots, n\} \), is VC-subgraph with index \( \nabla \), then
\[ \overline{D}(\overline{S}_{q,F}) \leq C \nabla \left[ 1 + \log_+ (n/\nabla) \right] \]

and the estimator \( \hat{s} = q_f \) built in Section 4.2 satisfies, for any density \( p \) and vector \( f \in \mathbb{R}^n \),
\[ \mathbb{P}_s \left[ \text{Ch}^2 (p_r, q_f) \right] \leq h^2 (p_r, \overline{S}_{q,F}) + \nabla \left[ 1 + \log_+ (n/\nabla) \right] + \xi \geq 1 - e^{-\xi} \quad \text{for all } \xi > 0. \]

**Proof.** A vector \( g \in F \) and the element \( q_g \in \overline{S}_{q,F} \) can both be viewed as functions on \( \mathcal{X} = \{1, \ldots, n\} \times \mathbb{R} \) defined respectively, for \( \pi = (i, x) \in \mathcal{X} \), by \( g(\pi) = g_i \) and \( q_g(\pi) = q(x - g_i) \). Under Assumption 3, it follows from the properties (iii), (i), (vi) of Proposition 11, that \( \overline{S}_{q,F} \) is VC-subgraph with index not larger than \( C \nabla \). We conclude with Theorem 3. \( \square \)

Since this proof relies on the fact that \( \overline{S}_{q,F} \) is VC-subgraph, Assumption 3 can be replaced by “\( q \) is multimodal with no more than \( k \) modes”. Indeed, in this case the set \( \overline{S}_{q,F} \) is still VC-subgraph but with index bounded by \( C'(k) \nabla \) as noticed in the remark at the end of Section 8. It follows that the constants \( C \) appearing in (61) and (62) now depend on \( k \).

There are various ways of applying the previous theorem according to the type of bound we would like to get. Let us first note that, by the triangular inequality, \( h (p_r, q_g) \leq \)
h(p_f, p_{q_f}) + h(p_R, q_{q_R}) and, by translation invariance, h^2(p_R, q_{q_R}) = nh^2(p, q) so that h^2(pr, S_{q, F}) \leq 2h^2(p, S_{p, F}) + 2nh^2(p, q). Therefore (62) implies that, for all \(\xi > 0\),
\[
P_s \left[ \text{Ch}^2(p_f, q_{q_f}) \leq nh^2(p, q) + \inf_{g \in F} h^2(q_f, q_{q_f}) + \nu \left(1 + \log_+ \left(n/\nu\right)\right) + \xi \right] \geq 1 - e^{-\xi}.
\]
and, by the same argument,
\[
P_s \left[ \text{Ch}^2(p_f, q_{q_f}) \leq nh^2(p, q) + \inf_{g \in F} h^2(q_f, q_{q_f}) + \nu \left(1 + \log_+ \left(n/\nu\right)\right) + \xi \right] \geq 1 - e^{-\xi}.
\]
Noticing that h^2(q_f, q_{q_f}) \leq 2h^2(q_f, p_f) + 2h^2(p_f, q_{q_f}), we also derive similarly that
\[
(63) P_s \left[ \text{Ch}^2(q_f, q_{q_f}) \leq nh^2(p, q) + \inf_{g \in F} h^2(q_f, q_{q_f}) + \nu \left(1 + \log_+ \left(n/\nu\right)\right) + \xi \right] \geq 1 - e^{-\xi}.
\]
This last formula provides a risk bound for the estimation of \(f\) by \(\hat{f}\) when the loss function takes the special form \(\ell(f, \hat{f}) = h^2(q_f, q_{q_f})\):
\[
C E_s \left[ \ell(f, \hat{f}) \right] \leq nh^2(p, q) + \inf_{g \in F} \ell(f, g) + \nu \left(1 + \log_+ \left(n/\nu\right)\right).
\]
If \(p\) is known so that we can set \(q = p\) we find the usual “bias plus variance” risk bound without the \(nh^2(p, q)\) term, \(\nu \left(1 + \log_+ \left(n/\nu\right)\right)\) playing here the role of a variance term. This shows that the price to pay for not knowing the density \(p\) of the errors and replacing it by \(q\) is an additional bias term of order \(nh^2(p, q)\).

To make this last risk bound more precise, we introduce the following definition.

**Definition 5.** We shall say that a density \(q\) is of order \(\alpha \in (-1, 1]\) if it satisfies
\[
(64) a_q \left| u - v \right|^{1+\alpha} A_q^{-1} \leq h^2(q_u, q_v) \leq A_q \left| u - v \right|^{1+\alpha} A_q^{-1}
\]
and some constants \(A_q \geq a_q > 0\) depending on \(q\).

The reader can find in Ibragimov and Has’minskii (1981) Chapter VI p. 281 some sufficient conditions on the density \(q\) to ensure that (64) holds. For illustration, we present here some examples borrowed from these authors. The density \(q(x) = 1_{[-1/2, 1/2]}\) is of order 0. For \(\alpha \in (-1, 1]\), the density \(q(x) = [2(1 + |z|)]^{-1}(1 - |z|)\alpha 1_{[-1, 1]}(x)\) is of order \(\alpha\) and so is \(q(x) = C(\alpha)e^{-|z|\alpha}/2\) for \(\alpha \in (0, 1)\). For \(\alpha > 1\), this latter density is of order 1. If the translation model \(\theta \mapsto q(\cdot - \theta)\) is regular with Fisher information bounded away from 0 and infinity, it is of order 1.

Let us now set, for \(\alpha \in (-1, 1]\), \(g, g' \in \mathbb{R}^n\) and \(G \subset \mathbb{R}^n\),
\[
\text{d}_{1+\alpha}(g, g') = \sum_{i=1}^{n} \left| g_i - g'_i \right|^{1+\alpha} A_q^{-1} \quad \text{and} \quad \text{d}_{1+\alpha}(g, G) = \inf_{g' \in G} \text{d}_{1+\alpha}(g, g').
\]
Applying Theorem 6 with the bound (63) leads to the following result.

**Corollary 4.** Let \(F\) be a subset of \(\mathbb{R}^n\) which is VC-subgraph with index \(\nu\) and \(q\) be a density on \(\mathbb{R}\) of order \(\alpha \in (-1, 1]\) which satisfies Assumption 3. Then the estimator \(\hat{s} = q_{\hat{f}}\) satisfies, for all \(f \in \mathbb{R}^n\) and all \(\xi > 0\),
\[
(65) P_s \left[ d_{1+\alpha}(f, \hat{f}) \leq C'(q) (d_{1+\alpha}(f, F) + nh^2(p, q) + \nu \left(1 + \log_+ \left(n/\nu\right)\right) + \xi) \right] \geq 1 - e^{-\xi}.
\]
To comment on this result, let us consider the example of the shift model for i.i.d. observations. Assume that the $X_i$ are i.i.d. with common density $p(\cdot - \theta)$ for some unknown parameter $\theta \in \mathbb{R}$ but a known density $p$ that we assume to be of order $\alpha \in (-1, 1]$ and to satisfy Assumption 3. In this case $f_i = \theta$ for all $i$ and it is natural to fix $q = p$ and consider as a model for $f$ the linear span $F$ of $(1, \ldots, 1)$ in $\mathbb{R}^n$. The distance $d_{1+\alpha}(f, g)$ between $f = (\theta, \ldots, \theta)$ and an element $g = (\theta', \ldots, \theta')$ of $F$ becomes
\[ d_{1+\alpha}(f, g) = n \left( |\theta - \theta'|^{1+\alpha} \wedge A_p^{-1} \right) \]
and we can deduce from (65) that our estimator $\hat{f} = (\hat{\theta}, \ldots, \hat{\theta})$ satisfies, for $n$ large enough and with a probability close to 1,
\[ |\theta - \hat{\theta}| \leq C(p, \alpha)[(\log n)/n]^{1/(1+\alpha)}. \]

As soon as $\alpha \in (-1, 1)$, the rate we get improves the usual parametric one $1/\sqrt{n}$ achieved by the classical least-squares estimator (under suitable moment conditions on the $\varepsilon_i$). Though faster, this rate can still be improved by a logarithmic factor, as, in fact, the maximum likelihood estimator can achieve the rate $n^{-1/(1+\alpha)}$ (we refer to Theorem 6.3 p. 314 of the book by Ibragimov and Has’minskiĭ (1981)). We do not know whether this extra logarithmic factor is due to our techniques or to the fact that our estimator is robust since its risk will only be slightly increased if $p \neq q$ but $h(p, q)$ is small, as shown by (65), which will not necessarily be the case for the MLE, for instance.

6.2. Simple linear regression. As an illustration of the superiority of $p$-estimators over the least squares method in some regression frameworks, we consider the very simple situation of observations $Y_i = a + bx_i + \varepsilon_i$, $1 \leq i \leq n$, that is, a simple linear regression where the errors $\varepsilon_i$ are i.i.d. with a unimodal density $p$ and satisfy $\mathbb{E}[\varepsilon_i] = 0$ and $\text{Var}(\varepsilon_i) = 1$. We moreover assume that $n = 2r - 1$ is odd with $r \geq 2$ and $x_i = n^{-1}[2i - n - 1]$ ($x_1 = -1+1/n, \ldots, x_{r-1} = -2/n, x_r = 0, x_{r+1} = 2/n, \ldots, x_n = 1-1/n$) so that $x_i \in (-1, 1)$ for all $i = 1, \ldots, n$,
\[ \sum_{i=1}^n x_i = 0 \quad \text{and} \quad \sum_{i=1}^n x_i^2 = \frac{n^2 - 1}{3n}. \]

It is well known that, in this case, the least squares estimator $(\hat{a}, \hat{b})$ of the parameter $(a, b)$ satisfies
\[ \mathbb{E}[(\hat{a} - a)^2] = \frac{1}{n} \quad \text{and} \quad \mathbb{E}[(\hat{b} - b)^2] = \frac{3n}{n^2 - 1} > \frac{3}{n}. \]

Let us now assume that the density $p$ satisfies
\[ h^2 \left( p(\cdot - \theta), p(\cdot - \theta') \right) \geq c \left[ |\theta - \theta'|^{\gamma} \wedge 1 \right] \]
for some $\gamma \in (0, 2)$ and $c > 0$. The joint density of the observations $Y_i$ is $\prod_{i=1}^n p(y_i - a - bx_i) = \prod_{i=1}^n p_{a+bx_i}(y_i)$ and can be estimated by a $p$-estimator resulting in the estimated density $\prod_{i=1}^n p_{\hat{a} + \hat{b}x_i}(y_i)$ and we know that, with large probability,
\[ \sum_{i=1}^n h^2 \left( p_{\hat{a} + \hat{b}x_i}, p_{a + bx_i} \right) \leq B \log n, \]
for some constant $B$ independent of $n$. Then (66) implies, with $\alpha = \hat{a} - a$, $\beta = \hat{b} - b$, that
\[ \sum_{i=1}^n \left[ |\hat{a} - a + (\hat{b} - b) x_i|^{\gamma} \wedge 1 \right] = \sum_{i=1}^n [|\alpha + \beta x_i|^{\gamma} \wedge 1] \leq B \log n/c. \]
When the density \( \ell \) function is of order \( p \), for analyzing the performance of the least squares, we shall rather stick to Hellinger-type losses since then, according to (2.5), Audibert and Catoni (2011) and the references therein). Our point of view is different. We shall consider the \( L_2 \)-norms of the elements of \( \mathcal{F} \) on \( \nu \) for all \( i \) large enough and finally
\[
\max \{|\alpha|, |\beta|\} < 1 \text{ for } n \text{ large enough and finally}
\]
\[
|\alpha| + |\beta| = |\tilde{a} - a| + |\tilde{b} - b| = O_P \left( (\log n/n)^{1/\gamma} \right),
\]
which improves the estimation by least-squares at least when \( n \) is large.

### 6.3. Homoscedastic regression with random design

In this section, we consider the regression framework with random design described in Section 2.5. Least squares or penalized least squares are the classical estimators which are used in this context and many efforts have been made to analyse their performances under suitable conditions on the moment of the errors and the distribution of the design (see Baraud (2002), Audibert and Catoni (2011) and the references therein). Our point of view is different. We shall consider that the design is completely unknown and rather assume that the distribution of the errors is approximately known and symmetric but possibly without moments. Furthermore, while the \( L_2 \)-norm with respect to the law of the design is the usual loss function that is used for analyzing the performance of the least squares, we shall rather stick to Hellinger-type losses. More precisely, we evaluate the performance of an estimator \( \hat{f} \) of \( f \) by the risk
\[
E_n \left[ h^2(p_{\hat{f}}, p_f) \right] = nE_n \left[ \int_{\mathbb{W}} h^2(p_{\hat{f}}(w, \cdot), p_f(w, \cdot)) \, d\nu(w) \right],
\]
with \( p_g(w, \cdot) = p(\cdot - g(w)) \) for all \( g \in \mathcal{F} \). This actually corresponds to the use of the loss function \( \ell(g, g') \) on \( \mathcal{F} \) with
\[
\ell(g, g') = h^2(p_g, p_{g'}) = \int_{\mathbb{W}} h^2(p_g(w, \cdot), p_{g'}(w, \cdot)) \, d\nu(w).
\]

When the density \( p \) is of order \( \alpha \in (-1, 1) \), as given by Definition 5, one can relate \( \ell \) to some power of a more classical \( L_{1+\alpha} \)-loss since then, according to (64),
\[
a_p \int_{\mathbb{W}} |g - g'|^{1+\alpha} \, d\nu \leq \ell(g, g') \leq A_p \int_{\mathbb{W}} |g - g'|^{1+\alpha} \, d\nu \quad \text{for all } g, g' \in \mathcal{F}.
\]

If, moreover, the \( L_\infty \)-norms of the elements of \( \mathcal{F} \) are uniformly bounded by some number \( b > 0 \), then
\[
\frac{1}{A_p(2b)^{1+\alpha}} |g - g'|^{1+\alpha} \leq |g - g'|^{1+\alpha} \quad \text{for all } g, g' \in \mathcal{F}.
\]
and \( \ell(g, g') \) becomes of the same order as \( \|g - g'\|_{1+\alpha, \nu}^{1+\alpha} = \int_W |g - g'|^{1+\alpha} d\nu \) since

\[
\frac{a_p}{[A_p(2b)^{1+\alpha}]^{1/2}} \|g - g'\|_{1+\alpha, \nu}^{1+\alpha} \leq A_p \|g - g'\|_{1+\alpha, \nu}^{1+\alpha}.
\]

In particular, we recover the usual \( L_2 \)-loss when \( p \) is of order 1 which is the case for the Gaussian, Cauchy and Laplace distributions among others.

To estimate \( f \) we proceed as follows: we choose a candidate density \( q \) for \( p \) which we assume to be symmetric and unimodal and consider a model \( F \subset \mathcal{F} \), which is VC-subgraph with index \( \overline{V}(F) \) to approximate \( f \). To \( F \), we associate the model of densities (with respect to \( \nu \otimes \mu \)) given by

\[
\overline{S}_F = \{q_\mathcal{S} = (q_g, \ldots, q_g) \mid g \in F\} \quad \text{where} \quad q_g(w, y) = q(y - g(w))
\]

and estimate the density \( s \) of \( (W, Y) \) from the observation of \((W_1, Y_1), \ldots,(W_n, Y_n)\) building the corresponding \( \rho \)-estimator from a countable and dense subset \( S \) of \( \overline{S}_F \). We can apply this procedure without knowing \( \nu \) since, under the assumptions that \( q \) is symmetric and \( \mu \) is the Lebesgue measure, for all \( g, g' \) and \( w \in \mathcal{W} \),

\[
\int_{\mathcal{R}} \sqrt{q_g(w, y)r(w, y)} \, d\mu(y) = \int_{\mathcal{R}} \sqrt{q_{g'}(w, y)r(w, y)} \, d\mu(y) \quad \text{with} \quad r = \frac{q_g + q_{g'}}{2},
\]

so that by integration with respect to \( \nu \), \( \rho(q_g, r) = \rho(q_{g'}, r) \). Therefore \( T((W, Y), p_g, p_{g'}) \) simply becomes

\[
T((W, Y), p_g, p_{g'}) = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \psi \left( \frac{p_{g'}(W_i, Y_i)}{p_g(W_i, Y_i)} \right).
\]

As in the proof of Theorem 6, it follows from Proposition 11 that, under Assumption 5, \( \{q_g, g \in F\} \) is VC-subgraph with index not larger than \( C'\overline{V}(F) \). Applying Theorem 3 then leads to the following result.

**Theorem 7.** If \( q \) is unimodal and symmetric and \( F \) is a model for \( f \) which is VC-subgraph of index \( \overline{V}(F) \) there exists a \( \rho \)-estimator \( \hat{s} = q_{\hat{f}} \) of \( s = p_f \) such that for all \( \xi > 0 \), with probability at least \( 1 - e^{-\xi} \),

\[
Ch^2(p_f, q_{\hat{f}}) \leq \inf_{g \in F} h^2(p_f, q_g) + \overline{V}(F) \left[ 1 + \log_+ \left( \frac{n}{\overline{V}(F)} \right) \right] + \frac{\xi}{n}
\]

\[
\leq 2h^2(p, q) + \inf_{g \in F} 2h^2(p_f, p_g) + \overline{V}(F) \left[ 1 + \log_+ \left( \frac{n}{\overline{V}(F)} \right) \right] + \frac{\xi}{n}.
\]

In particular, for all \( \xi > 0 \),

\[
P_s \left[ C\ell(f, \hat{f}) \leq \ell(f, F) + \overline{V}(F) \left[ 1 + \log_+ \left( \frac{n}{\overline{V}(F)} \right) \right] + \frac{\xi}{n} \right] \geq 1 - e^{-\xi}.
\]

If, besides, (64) holds and max \( \sup_{g \in F} \|g\|_{\infty}, \|f\|_{\infty} \) \( \leq b < +\infty \), for all \( \xi > 0 \),

\[
P_s \left[ C' \|f - \hat{f}\|_{1+\alpha, \nu}^{1+\alpha} \leq \inf_{g \in F} \|f - g\|_{1+\alpha, \nu}^{1+\alpha} + \overline{V}(F) \left[ 1 + \log_+ \left( \frac{n}{\overline{V}(F)} \right) \right] + \frac{\xi}{n} \right] \geq 1 - e^{-\xi},
\]

for some constant \( C' \) depending only on \( A_p, a_p, b \) and \( \alpha \).
6.4. **Further examples for regression problems.** Let us recall that we want to estimate the unknown function \( f \) on \( \mathcal{W} \) in both the random design framework where \( Y = f(W) + \varepsilon \) and the fixed design framework which corresponds, with analogous notations, to \( X = f(w) + \varepsilon \) with \( w \in \mathcal{W} = \{1, \ldots, n\} \).

As we noticed in the previous sections, when dealing with both regression frameworks, when the model \( F \) for the regression function \( f \) is VC-subgraph, the performance of the estimator \( \hat{f} \) depends on the VC-index \( \mathbb{V}(F) \). A common practice to design regression models is to choose for \( F \) a \( D \)-dimensional linear space of functions which, according to Section 8, is VC-subgraph with index bounded by \( D + 2 \). This includes the celebrated "linear model" in the fixed design framework when \( F \) is the linear span of \( D \) linear independent vectors \( g^1, \ldots, g^D \) in \( \mathbb{R}^n \) or, equivalently, of \( D \) functions \( g^1, \ldots, g^D \) on \( \{1, \ldots, n\} \).

Let us, for a moment, focus on this situation of \( F \) being a \( D \)-dimensional linear space. Classical least squares estimators in the fixed design case lead to risk bounds of order \( \log(n/D) \) factor, with the loss function

\[
d_2(g, g') = \frac{1}{n} \sum_{i=1}^{n} \left( |g_i - g_i'|^2 \wedge A_q^{-1} \right).
\]

When the errors have a uniform distribution, we derive a bound of order \((D/n) \log(n/D)\) for the loss

\[
d_1(g, g') = \frac{1}{n} \sum_{i=1}^{n} \left( |g_i - g_i'| \wedge A_q^{-1} \right),
\]

while unbounded errors of the form \( q(x) = [2(1-\beta)]^{-1}(1-|x|)^{-\beta} \mathbb{I}_{[-1,1]}(x) \) with \( 0 < \beta < 1 \) lead to the same bound with the loss

\[
d_{1-\beta}(g, g') = \frac{1}{n} \sum_{i=1}^{n} \left( |g_i - g_i'|^{1-\beta} \wedge A_q^{-1} \right).
\]

Since subsets of VC-subgraph classes are also VC-subgraph one can restrict \( F \) to be a bounded subset of such a linear space and get similar results for the random design situation, according to Theorem 7, with loss functions of the form \( \|f - \hat{f}\|_2^{2}, \|f - \hat{f}\|_1, \|f - \hat{f}\|_1^{1-\beta} \) respectively.

An alternative way of building models that still satisfy the assumptions which are needed to apply our results is as follows. We start from a \( D \)-dimensional linear space \( G \) of functions on \( \mathcal{W} \) and consider some monotone function \( \Psi \). Finally we take for \( F \) the set \( \{\Psi \circ g, g \in G\} \). It follows from (ii) of Proposition 11 that \( F \) is still VC-subgraph with index not larger than \( D + 2 \) and the previous results still holds. We may replace "monotone" by "unimodal" and get similar results according to (vi) of the same proposition. This allows, given \( D \) independent functions \( g^1, \ldots, g^D \), to use for instance models \( F \) of the following forms:

\[
\left\{ \exp \left[ \sum_{j=1}^{D} \beta_j g^j \right], \beta_j \in \mathbb{R} \text{ for } 1 \leq j \leq D \right\} \quad \text{or} \quad \left\{ \sum_{j=1}^{D} \beta_j g^j, \beta_j \in \mathbb{R} \text{ for } 1 \leq j \leq D \right\}.
\]
6.5. A parametric bound over a set with infinite metric dimension. In this section, we want to show that, unlike T-estimators, the construction and performance of which heavily depend on the metric dimension of the model that is used, our estimator can, in some cases, achieve a parametric bound which is not connected to its metric dimension. The following illustration given for the density framework described in Section 2.3 is borrowed from Birgé (1983) (Section 6).

Let \( \Lambda \) be any nonvoid subset of \( \mathbb{N} \) and \( \Theta = \Lambda^{\mathbb{N}^r} \), \( \mathcal{K} = \bigcup_{j \geq 1} \Lambda^j \) be respectively the sets of infinite and finite sequences with entries in \( \Lambda \). Note that the set \( \mathcal{K} \) is countable and that we may introduce on \( \mathcal{K} \) the family of probabilities \( \{P_\theta, \theta \in \Theta\} \) given by

\[
P_\theta = \sum_{j \geq 1} 2^{-j} \delta_{(\theta_1, \ldots, \theta_j)} \quad \text{for all} \quad \theta = (\theta_1, \ldots, \theta_k, \ldots) \in \Theta.
\]

In the sequel, we denote by \( s_\theta \) the density of \( P_\theta \) with respect to the counting measure on \( \mathcal{K} \), that is, for all \( x \in \mathcal{K} \), \( s_\theta(x) = P_\theta(\{x\}) \), and set \( \overline{S} = \{s_\theta, \theta \in \Theta\} \). Our aim is to estimate \( s_\theta \) from the observation of a sample \( X_1, \ldots, X_n \).

It will be convenient to define the following operators: \( \ell(x) \) is the length of an element \( x \in \mathcal{K} \), that is, \( \ell(x) = j \) if \( x \in \Lambda^j \); \( \pi_j \) is the operator from \( \Theta \) to \( \Lambda^j \) such that \( \pi_j(\theta) = (\theta_1, \ldots, \theta_j) \); finally \( \pi_1 \) is an operator from \( \mathcal{K} \) to \( \Theta \) such that if \( x \in \Lambda^j \), \( \pi_1 \circ \pi_j(x) = x \) or, equivalently, \( \pi_1(x) = x \) for all \( x \in \mathcal{K} \). It follows that \( \pi_1(\mathcal{K}) \) is a countable subset of \( \Theta \) and \( S = \{s_\theta, \theta \in \pi_1(\mathcal{K})\} \) is a countable subset of \( \overline{S} \).

Let \( J \) be the mapping from \( \Theta^2 \) to \( \mathbb{N} \cup \{+\infty\} \) defined by

\[
J(\theta, \theta') = \sup \{ j \in \mathbb{N} \mid \theta_k = \theta'_k \quad \text{for} \quad 1 \leq k \leq j \} \quad \text{with} \quad \sup \mathbb{N} = +\infty, \quad \sup \varnothing = 0.
\]

Since \( s_\theta(x) = 2^{-\ell(x)} \) if \( \pi_1(x) = x \) and \( s_\theta(x) = 0 \) otherwise, the Hellinger distance between two densities \( s_\theta \) and \( s_{\theta'} \) with \( \theta, \theta' \in \Theta \) is given by

\[
h^2(s_\theta, s_{\theta'}) = 1 - \rho(s_\theta, s_{\theta'}) = 1 - \sum_{j=1}^{J(\theta, \theta')} 2^{-j} = 2^{-\infty} = 0.
\]

For every \( y \in (0, 1) \), \( \eta \in (0, y/\sqrt{2}) \) and \( \theta \in \Theta \), any Hellinger ball centered at \( P_\theta \) of radius \( y \) contains at least \( |\Lambda| \) elements of \( \Theta \) such that \( h(P_\theta, P_{\theta'}) > \eta \) (it suffices to change for a suitable \( j \geq 1 \) the coordinates \( \theta_j \) of \( \theta \) into an arbitrary element of \( \Lambda \setminus \{\theta_j\} \)). This shows that the metric dimension of \( \overline{S} \) can be made arbitrarily large or even infinite by playing with the cardinality of \( \Lambda \).

Though \( \overline{S} \) can be massive, the parameter \( \theta \) is not difficult to estimate. Let us first observe that, if \( \ell(X_i) = j \), then \( X_i = \pi_j(\theta) \) \( P_\theta \) a.s. Therefore, if \( k = \ell(X_{i_0}) = \sup_{1 \leq i \leq n} \ell(X_i) \) and \( \hat{\Theta} = \pi_{k-1}(X_{i_0}) \), for all \( \hat{\theta} \in \hat{\Theta} \),

\[
s_{\hat{\theta}}(X_i) = 2^{-\ell(X_i)} = s_\theta(X_i), \quad \text{for} \quad i \in \{1, \ldots, n\}, \quad P_\theta \text{-a.s.}
\]

while for all \( \theta' \notin \hat{\Theta} \) \( s_{\theta'}(X_i) \leq s_{\theta}(X_i) \) for \( 1 \leq i \leq n \) and \( 0 = s_{\theta'}(X_{i_0}) < s_{\theta}(X_{i_0}) \). It follows that the likelihood reaches its maximum over the elements \( \hat{\theta} \in \hat{\Theta} \) \( P_\theta \) a.s. It is proven in Birgé (1983) that these maximum likelihood estimators satisfy

\[
\mathbb{E}_\theta [h^2(s_\theta, s_{\hat{\theta}})] \leq C n^{-1}, \quad \text{for all} \quad \theta \in \Theta
\]

and some numerical constant \( C > 0 \). In particular, the minimax rate over \( \overline{S} \) is parametric.
Let us now observe that if \( \theta \in \Theta \), \( x = \pi_j(\theta) \) and \( \theta' = \pi_{-1}(x) \), it follows from (68) that 
\[
h^2(s_\theta, s_{\theta'}) \leq 2^{-j}
\]
so that \( S \) is a dense subset of \( \mathcal{S} \). We can therefore use \( S \) to apply our 
procedure and by doing so we obtain the following result.

**Proposition 8.** For all \( \theta \in \Theta \) and any choice of a maximum likelihood estimator \( \hat{\theta} \) in \( \hat{\Theta} \), 
\( s_\hat{\theta} \) is a \( \rho \)-estimator of \( s_\theta \), \( \rho_\theta \)-a.s.

**Proof.** Since \( S \) is dense in \( \mathcal{S} \) with respect to the Hellinger distance, it suffices to prove that 
the elements of \( \hat{S} = \{ s_{\hat{\theta}}, \hat{\theta} \in \hat{\Theta} \} \cap S \) minimize \( \Upsilon(S, \cdot) \) over \( S \). First note that for \( \theta, \theta' \in \Theta \) with \( \theta \neq \theta' \),
\[
\rho \left( s_\theta, \frac{s_\theta + s_{\theta'}}{2} \right) \leq \sum_{x \in X} \frac{s_\theta (x) + s_{\theta'} (x)}{2} = \sum_{j=1}^{J(\theta, \theta')} 2^{-j} + \sum_{j > J(\theta, \theta')} \frac{2^{-j}}{\sqrt{2}}
\]
\[
= 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) 2^{-J(\theta, \theta')} = \rho \left( s_\theta, \frac{s_\theta + s_{\theta'}}{2} \right).
\]

Let us now fix some element \( s_{\hat{\theta}} \in \hat{S} \). Using (69), we have \( T(X, s_{\hat{\theta}}, s_{\hat{\theta}}) = 0 \) for \( s_{\hat{\theta}} \in \hat{S} \) and 
for \( s_{\theta'} \in S \setminus \hat{S} \)
\[
T(X, s_{\hat{\theta}}, s_{\theta'}) = \frac{n}{\sqrt{\frac{s_{\theta'} (X_i) - s_{\hat{\theta}} (X_i)}{\sqrt{s_{\theta'} (X_i) + s_{\hat{\theta}} (X_i)}}} \leq \frac{\sqrt{s_{\theta'} (X_{i_0}) - s_{\hat{\theta}} (X_{i_0})}}{\sqrt{\frac{s_{\theta'} (X_{i_0}) + s_{\hat{\theta}} (X_{i_0})}} / 2 = -\sqrt{2}.
\]

Consequently \( \Upsilon(S, s_{\hat{\theta}}) = 0 \) and therefore \( s_{\hat{\theta}} \) minimizes \( \Upsilon(S, \cdot) \) over \( S \). \( \square \)

We shall prove in Section 9.8, the following result.

**Proposition 9.** For all \( s \in \mathcal{S} \) and \( \bar{s} \in S \),
\[
D^S(s, \bar{s}) \leq 64c_0^{-2} \left[ 2\sqrt{2} + n^2h^2(s, \bar{s}) \right]^2.
\]

Applying (25) (we recall that \( h^2(\cdot, \cdot) = nh^2(\cdot, \cdot) \)), we obtain that for all \( s \in \mathcal{S} \) and \( \bar{s} \in S \)
\[
\mathbb{E}_s \left[ h^2(s, \bar{s}) \right] \leq c_1 h^2(s, \bar{s}) + n^{-1} \left[ 64c_2 c_0^{-2} \left( 2\sqrt{2} + n^2h^2(s, \bar{s}) \right)^2 + 2.45c_3 \right]
\]
and by choosing \( \bar{s} \) arbitrarily close to \( s \in \mathcal{S} \) we derive that any \( \rho \)-estimator \( \hat{s} \) (and therefore 
any maximum likelihood estimator) satisfies
\[
\mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq \left[ 512 c_2 c_0^{-2} + 2.45 c_3 \right] n^{-1} \text{ for all } s \in \mathcal{S}
\]
and thus achieves a parametric rate of convergence.

**7. Model selection**

In this section, we consider the problem of model selection. More precisely, we assume 
that we have at disposal an at most countable collection \( S \) of models \( \mathcal{S} \) for the parameter 
\( s \) and not a single one as in the previous sections. We may therefore associate to each \( \mathcal{S} \) a 
\( \rho \)-estimator \( \hat{s}(\mathcal{S}) \) and our aim is to select from the data a model \( \hat{S} \subseteq S \) or, equivalently, an 
estimator \( \tilde{s}(\mathcal{S}) \) among the family of candidates \( \{ \tilde{s}(\mathcal{S}), \mathcal{S} \subseteq S \} \), in such a way that its risk 
is as close as possible to the infimum of those over the family.
7.1. Estimation procedure and main result. Let \( \mathcal{S} \) be a countable family of models in \( \mathcal{L}_0 \), endowed with a mapping \( \Delta \) from \( \mathcal{S} \) into \( \mathbb{R}_+ \) satisfying
\[
\sum_{\mathcal{S} \in \mathcal{S}} \exp \left[ -\Delta (\mathcal{S}) \right] \leq 1.
\]
To each \( \mathcal{S} \in \mathcal{S} \) we attach either a countable \( \eta \)-net or a dense subset \( S \), as we did for a single model in Section 4.5, the connection between \( \mathcal{S} \) and \( S \) being emphasized by the notations. This results in a new collection \( \mathcal{S} \) of subsets \( S \) of \( \mathcal{L}_0 \), each \( S \in \mathcal{S} \) corresponding to a model \( \mathcal{S} \). When \( \mathcal{S} \) satisfies \( (\mathcal{S}) \), we set \( \mathcal{S} = \bigcup_{\mathcal{S} \in \mathcal{S}} S = \bigcup_{\mathcal{S} \in \mathcal{S}} S \), which results in a countable subset \( \mathcal{S} \) of \( \mathcal{L}_0 \). Let
\[
\mathcal{Y}(S, t) = \sup_{t' \in \mathcal{S}} [T(X, t, t') - \text{pen}(t')] + \text{pen}(t)
\]
and, with \( \kappa \) given by (7),
\[
\mathcal{E}(X, S) = \left\{ \hat{s} \in \mathcal{S} \mid \mathcal{Y}(S, \hat{s}) \leq \inf_{t \in \mathcal{S}} \mathcal{Y}(S, t) + (\kappa/10) \right\}.
\]
As in Section 4.2, we define our estimator \( \hat{s} \) of \( S \) as any element of the closure of \( \mathcal{E}(X, S) \) with respect to the Hellinger distance \( h \). When \( \mathcal{S} \) reduces to a single element \( \mathcal{S} \) and \( \text{pen} \) is constant on \( S \), the estimator \( \hat{s} \) coincides with the one defined in Section 4.2.

**Theorem 8.** If the penalty function \( \text{pen} \) satisfies
\[
\text{pen}(t) \geq \inf \left\{ (1/8) D^{\mathcal{S}} + \kappa \Delta(\mathcal{S}) \mid \mathcal{S} \in \mathcal{S}, S \ni t \right\}, \quad \text{for all } t \in \mathcal{S},
\]
where \( D^{\mathcal{S}} \) is defined by (33), the estimator \( \hat{s} \) satisfies, for all \( s \in \mathcal{L}_0 \) and \( \xi > 0 \),
\[
P_s \left[ h^2(s, \hat{s}) \leq \inf_{\mathcal{S} \in \mathcal{S}} \left\{ c_1 h^2(s, \mathcal{S}) + 8 c_2 \text{pen}(\mathcal{S}) + c_3 (1.45 + \xi) \right\} \right] \geq 1 - e^{-\xi}.
\]
In particular, if equality holds in (71) and the sets \( S \in \mathcal{S} \) are chosen to satisfy
\[
2 c_1 \sup_{u \in \mathcal{S}} h^2(u, S) + c_2 D^{\mathcal{S}} \leq D(\mathcal{S}) + \frac{c_3}{20},
\]
then for all \( s \in \mathcal{L}_0 \) and \( \xi > 0 \),
\[
P_s \left[ h^2(s, \hat{s}) \leq \inf_{\mathcal{S} \in \mathcal{S}} \left\{ 2 c_1 h^2(s, \mathcal{S}) + D(\mathcal{S}) + c_3 (\Delta(\mathcal{S}) + 1.5 + \xi) \right\} \right] \geq 1 - e^{-\xi}.
\]
When the models \( \mathcal{S} \) are VC-subgraph with respective indices \( V(\mathcal{S}) \) we have seen in Theorem 3 that
\[
D^{\mathcal{S}} \leq C V(\mathcal{S}) \left[ 1 + \log_+ \left( n/V(\mathcal{S}) \right) \right]
\]
for all choices of a countable and dense subset \( S \) of \( \mathcal{S} \). For such choices of \( S \) and of a penalty \( \text{pen} \) satisfying (71) with equality, we derive from (72) that the estimator \( \hat{s} \) satisfies, for all \( \xi > 0 \),
\[
P_s \left[ Ch^2(s, \hat{s}) \leq \inf_{\mathcal{S} \in \mathcal{S}} \left\{ h^2(s, \mathcal{S}) + V(\mathcal{S}) \left[ 1 + \log_+ \left( n/V(\mathcal{S}) \right) \right] + \Delta(\mathcal{S}) + \xi \right\} \right] \geq 1 - e^{-\xi}.
\]
As we have seen in Proposition 5, we are not always able to bound from above the uniform dimension \( D(\mathcal{S}) \) of a model \( \mathcal{S} \) but sometimes only its dimension \( D(\mathcal{S}) \). In this case, model selection is still possible under the following alternative assumption.
Assumption 4. Let $\tilde{D}$ be a mapping from $\mathbb{S}$ into $[1, +\infty)$ such that for all $\mathbb{S}, \mathbb{S}' \in \mathbb{S}$,
$$D_{\mathbb{S}, \mathbb{S}'} \leq \tilde{D}(\mathbb{S}) + \tilde{D}(\mathbb{S}')$$.

Theorem 9. Let Assumption 4 hold and the penalty function $\text{pen}$ satisfy
\begin{equation}
\text{pen}(t) \geq \inf \left\{ (1/8) \tilde{D}(\mathbb{S}) + \kappa \Delta(\mathbb{S}) \right\} \mathbb{S}, \mathbb{S} \ni t, \text{ for all } t \in \mathbb{S}.
\end{equation}
The estimator $\hat{s}$ then satisfies for all $s \in \mathcal{L}_0$ and $\xi > 0$,
\begin{equation}
P_s \left[ h_s^2(s, \hat{s}) \leq \inf_{\mathbb{S} \in \mathbb{S}} \left\{ c_1 h_s^2(s, \mathbb{S}) + 16c_2 \text{pen}(\mathbb{S}) \right\} + c_3(1.45 + \xi) \right] \geq 1 - e^{-\xi}.
\end{equation}
If, moreover, Assumption 4 holds with $\tilde{D}(\mathbb{S}) \leq aD_{\mathbb{S}}$ for some $a \geq 1/2$ and the sets $S \in \mathbb{S}$ are chosen to satisfy,
\begin{equation}
2c_1 \sup_{u \in \mathbb{S}} h_s^2(u, S) + c_2 D_{\mathbb{S}} \leq D(\mathbb{S}) + \frac{c_3}{80a} \text{ for all } S \in \mathbb{S}
\end{equation}
and equality holds in (75), then for all $s \in \mathcal{L}_0$ and $\xi > 0$,
\begin{equation}
P_s \left[ h_s^2(s, \hat{s}) \leq \inf_{\mathbb{S} \in \mathbb{S}} \left\{ 2c_1 h_s^2(s, \mathbb{S}) + 2aD(\mathbb{S}) + c_3(2\Delta(\mathbb{S}) + 1.5 + \xi) \right\} \right] \geq 1 - e^{-\xi}.
\end{equation}

We shall now turn to examples in the next sections. Throughout these sections, we shall assume that $\hat{s}$ is built for a choice of the penalty function satisfying equality in (71). Finally, given a countable set $T$, we shall say that $\pi$ is a positive sub-probability on $T$, if $\pi(t) > 0$ for all $t$ in $T$ and $\sum_{t \in T} \pi(t) \leq 1$. Given such a $\pi$, we shall set $\Delta_\pi(t) = -\log (\pi(t))$.

7.2. Homoscedastic regression with unknown scaling. We consider here the regression setting described in Section 2.4 where $\mu$ is the Lebesgue measure and both $p$ and $f$ are unknown. Throughout this section, we shall consider a family $\mathbb{F}$ of subsets $\mathcal{F} \subset \mathbb{R}^n$ to approximate $f$ and a family $\mathcal{Q}$ of densities $q$ together with a scaling parameter $\lambda > 0$ to approximate $p$ by densities of the form $q_{0,\lambda}$. We recall from (8) that $q_{0,\lambda}(x) = \lambda^{-1}q(x/\lambda)$ for $\lambda > 0$ and $x \in \mathbb{R}$ and make the following assumptions.

Assumption 5. The family $\mathbb{F}$ is a countable family of VC-subgraph classes $\mathcal{F}$ with respective VC-indices $V(\mathcal{F})$ and $\mathbb{F}$ is endowed with a positive sub-probability $\pi$.

Assumption 6. The family $\mathcal{Q}$ is countable and endowed with a positive sub-probability $\gamma$. For each $q \in \mathcal{Q}$, Assumption 3 is satisfied and there exists a non-decreasing function $w_q$ from $[1, 2]$ into $\mathbb{R}_+$ such that $w_q(1) = 0$ and
$$h_s^2(q, 0, \lambda) \leq w_q(\lambda) \text{ for all } \lambda \in [1, 2].$$

Given a density $q$ on $\mathbb{R}$, a vector $g \in \mathbb{R}^n$ and $\lambda > 0$, we define the density $q_{g, \lambda}(x_1, \ldots, x_n)$ with respect to the Lebesgue measure on $\mathbb{R}^n$ according to (9), that is,
$$q_{g, \lambda}(x_1, \ldots, x_n) = (q_{0,\lambda}(x_1 - g_1), \ldots, q_{0,\lambda}(x_n - g_n)).$$
We consider the family of models $\mathbb{S}$ defined as follows. For $i \in \mathbb{N}, j \in \mathbb{Z}, k \in \{0, \ldots, 2^i - 1\}, F \in \mathbb{F}$ and $q \in \mathcal{Q}$, let
$$\mathbb{S}_{q,F}^{i,j,k} = \{ q_{g, \lambda} | g \in F, \lambda = \lambda_{i,j,k} \} \text{ with } \lambda_{i,j,k} = 2^j(1 + k2^{-i})$$
and define the family $\mathbb{S}$ as
\begin{equation}
\mathbb{S} = \left\{ \mathbb{S}_{q,F}^{i,j,k} | q \in \mathcal{Q}, F \in \mathbb{F}, (i, j, k) \in \mathbb{N} \times \mathbb{Z} \times \{0, \ldots, 2^i - 1\} \right\}.
\end{equation}
We endow $\mathcal{S}$ with the weights $\Delta$ given by
\[
\Delta \left( S_{q,F}^{i,j,k} \right) = \Delta_\gamma(q) + \Delta_\pi(F) + |j| + i + 2 + i \log 2
\]
and we check that
\[
\sum_{S \in \mathcal{S}} e^{-\Delta(S)} = \sum_{q \in \mathcal{Q}} \gamma(q) \sum_{F \in \mathcal{F}} \pi(F) \sum_{j \in \mathbb{Z}} e^{-|j|+1} \sum_{i \in \mathbb{N}} e^{-i(1+1)} \sum_{k=0}^{2i-1} e^{-i \log 2} \leq 1.
\]
Then, the following holds.

**Theorem 10.** Let $F$ be a family of models satisfying Assumption 5 and $\mathcal{Q}$ a family of densities satisfying Assumption 6. For the collection $\mathcal{S}$ and weight function $\Delta$ defined by (78) and (79) respectively, the estimator $\hat{s}$ satisfies for all densities $p$ and parameters $f \in \mathbb{R}^n$,
\[
C \mathbb{E}_s [h^2(s,\hat{s})] \leq \inf_{F \in \mathcal{F}} \left[ \inf_{g \in F} h^2(p_f, p_g) + \nabla(F) \left( 1 + \log_+ \left( n/\nabla(F) \right) \right) + \Delta_\pi(F) \right]
\]
\[
+ \inf_{q \in \mathcal{Q}, \lambda > 0} \left[ nh^2(p, q_{0,\lambda}) + \inf_{i \geq 0} \left[ \sum_{\gamma > 0} \gamma \left( 1 + 2^{-i} \right) + i \right] + \Delta_\gamma(q) + |\log \lambda| \right].
\]

**Proof.** Let us consider some given model $S_{q,F}^{i,j,k} \in \mathcal{S}$. Applying Theorem 8 to the family $\mathcal{S}$ we have that, up to a universal constant factor, $\mathbb{E}_s [h^2(s,\hat{s})] = \mathbb{E}_s [h^2(p_f, \hat{s})]$ is not larger than
\[
A = h^2(p_f, q_{0,\lambda,i,j,k}) + \Delta_\pi(S_{q,F}^{i,j,k}) + \Delta_\gamma(q) + \Delta_\pi(F) + |j| + i + 1
\]
whenever the choices of $F \in \mathcal{F}, g \in F, i \in \mathbb{N}, j \in \mathbb{Z}$ and $k \in \{0, \ldots, 2^i - 1\}$. By the triangular inequality, for $\lambda > 0$ and $\lambda = \lambda_i,j,k$,
\[
h(p_f, q_{0,\lambda,i,j,k}) \leq h(p_f, p_g) + h(p_g, q_{0,\lambda,i,j,k}) + h(q_{0,\lambda,i,j,k}, q_{0,\lambda,i,j,k}).
\]
Since under the Lebesgue measure the Hellinger distance is translation and scale invariant,
\[
h^2(p_g, q_{0,\lambda,i,j,k}) \leq nh^2(p, q_{0,\lambda}) \quad \text{and} \quad h(q_{0,\lambda,i,j,k}, q_{0,\lambda,i,j,k}) = \sqrt{n} h(q_{0,\lambda,i,j,k}, q_{0,\lambda,i,j,k}).
\]
It remains to bound $h(q_{0,\lambda,i,j,k}, q)$. Let $j \in \mathbb{Z}$ be such that $2^j \leq \lambda < 2^{j+1}$. Then $|j| \leq |\log \lambda|/\log 2 + 1$ and for all $i \in \mathbb{N}$ one can find $k \in \{0, \ldots, 2^i - 1\}$ such that
\[
\lambda = \lambda_i,j,k = 2^i (1 + 2^{-i}) \leq 1 \leq \lambda < \lambda_i,j,k+1 = 2^i (1 + (k+1)2^{-i}), \quad \text{hence} \quad 1 \leq 1 \leq 1 + 2^{-i} \leq 2
\]
It then follows from Assumption 6 that
\[
h^2(q_{0,\lambda,i,j,k}, q) \leq w_q(\lambda_i,j,k) \leq w_q(1 + 2^{-i})
\]
since $w_q$ is nondecreasing. Putting all these bounds together for these choices of $i, j, k$ we derive that, for some universal constant $C$ and all $i \in \mathbb{N}$,
\[
CA \leq h^2(p_f, p_g) + nh^2(p, q_{0,\lambda}) + w_q(1 + 2^{-i}) + \sum_{i \geq 0} \sum_{k=0}^{2i-1} e^{-i \log 2} \leq C \nabla(F) \left[ 1 + \log_+ \left( n/\nabla(F) \right) \right],
\]
which concludes the proof. \quad \square
Let us now comment on this result. To fix up the ideas, let us take for \( q \) and \( \lambda \) the values that provide the best approximation of \( p \) by \( q_{0,\lambda} \) among all choices in \( \mathcal{Q} \times (0, +\infty) \), even though this choice might not be the optimal one in view of minimizing our risk bound. The quantity \( nh^2(p, q_{0,\lambda}) \) corresponds to the usual bias term resulting from the approximation of \( p \) by the family of densities \( q'_{0,\lambda'} \) for \( q' \in \mathcal{Q} \) and \( \lambda' > 0 \). The quantity

\[
\inf_{F \in \mathcal{F}} \left[ \inf_{g \in \mathcal{G}} h^2(p, p_g) + \bar{V}(F) \left[ 1 + \log_+ \left( n/\bar{V}(F) \right) \right] + \Delta_\pi(F) \right]
\]

is the bound that we would get, if \( p \) were known, for estimating \( f \) by model selection among the family \( \bigcup_{F \in \mathcal{F}} F \). Finally, the quantity \( \Delta_\gamma(q) + \inf_{i \geq 0} \left[ nw_q \left( 1 + 2^{-i} \right) + i \right] + |\log \lambda| \) comes from our estimation of \( q \) and \( \lambda \).

This regression model includes in particular the case of a known form of the errors corresponding to \( p = \mathcal{P}_{0,\tau} \) where \( \mathcal{P} \) is known and \( \tau \) unknown, in which case it is natural to take \( \mathcal{Q} = \{ \mathcal{P} \} \). The risk bound then becomes after a proper rescaling:

\[
C_{\mathcal{E}_s} \left[ h^2(s, s) \right] \leq \inf_{F \in \mathcal{F}} \left[ \inf_{g \in \mathcal{G}} h^2(p, p_g) + \bar{V}(F) \left[ 1 + \log_+ \left( n/\bar{V}(F) \right) \right] + \Delta_\pi(F) \right]
\]

\[
+ \inf_{\lambda > 0} \left[ nh^2(q_{0,\tau}, q_{0,\lambda}) + |\log \lambda| \right] + \inf_{i \geq 0} \left[ nw_{\mathcal{P}} \left( 1 + 2^{-i} \right) + i \right]
\]

\[
= \inf_{F \in \mathcal{F}} \left[ \inf_{g \in \mathcal{G}} h^2(p, p_g) + \bar{V}(F) \left[ 1 + \log_+ \left( n/\bar{V}(F) \right) \right] + \Delta_\pi(F) \right]
\]

\[
+ \inf_{\sigma > 0} \left[ nh^2(q_{0,\sigma}, q_{0,\lambda}) + |\log \sigma| \right] + \inf_{i \geq 0} \left[ nw_{\mathcal{P}} \left( 1 + 2^{-i} \right) + i \right].
\]

7.3. An example. Let us consider the family of densities \( \mathcal{Q} = \{ p^\beta \mid \beta \geq 0 \} \) indexed by the parameter \( \beta \in [0, +\infty) \) and given by

\[
p^\beta(x) = \Lambda(\beta) e^{-|x|^{1/\beta}}, \quad \Lambda(\beta) = \left[ \int_{\mathbb{R}} e^{-|x|^{1/\beta}} \, dx \right]^{-1} \text{ for } \beta > 0 \text{ and } p^0 = \frac{1}{2} \mathbb{I}_{[-1,1]}.
\]

It follows from symmetry and a change of variables that

\[
[\Lambda(\beta)]^{-1} = 2 \int_0^\infty e^{-x^{1/\beta}} \, dx = 2 \beta \Gamma(\beta) \text{ for } \beta > 0.
\]

The family \( \mathcal{Q} \) contains the Laplace and Gaussian distributions and the uniform distribution on \([−1, 1]\) which corresponds to the limit of the densities \( p^\beta \) when \( \beta \) tends to 0. We recall from Section 6.1 that these densities are of order \( 2/\beta \) for \( \beta > 2 \) and of order \( 1 \) for \( \beta < 2 \). In particular, for such densities inequality (64) is satisfied for \( \bar{\alpha} = (2/\beta) \land 1 \) in place of \( \alpha \). We shall consider \( \mathcal{Q} \) as a model for our unknown density \( p \). In order to apply Theorem 10, which only holds for a countable family \( \mathcal{Q} \), we have to discretize \( \mathcal{Q} \). To do so, we need the following approximation result.

**Proposition 10.** For all \( \beta > \beta' > 0 \),

\[
h^2 \left( p^\beta, p^{\beta'} \right) \leq \begin{cases} 
(5/2) \left( (\beta/\beta') - 1 \right)^2 & \text{ if } 0 < \beta' < \beta \leq 1, \\
(3/2) (\beta - \beta')^2 & \text{ if } 1 < \beta' < \beta \leq 3, \\
[(3/4)(\beta - \beta')(\log \beta)]^2 & \text{ if } 3 < \beta' < \beta. 
\end{cases}
\]

Moreover, for \( 0 < \beta \leq 1 \),

\[
h^2 \left( p^\beta, p^0 \right) \leq (3/4)\beta
\]
and, for $\lambda \in [1, 2]$ and $\beta \geq 0$,

$$w_p^\beta(\lambda) \leq (3/5)(\lambda - 1).$$

The proof is postponed to Section 9.9. We are now in a position to prove the following result.

**Corollary 5.** There exists a countable subset $Q$ of $\Omega$ and a positive sub-probability $\gamma$ on $Q$ with the following properties: for all $p^\beta \in Q$ there exists $p^{\beta'} \in Q$ such that

$$\beta' \leq \beta, \quad h^2(p^\beta, p^{\beta'}) \leq n^{-1} \quad \text{and} \quad \gamma(p^{\beta'}) = \begin{cases} \frac{c(\sqrt{n} \log n)^{-1}}{cn^{-1}(\beta' - 3)^{-2}} & \text{if } 0 \leq \beta' \leq 3, \\ c(n^{-1} + \beta' - n^{-1})^{-2} & \text{if } \beta' > 3, \end{cases}$$

for a suitable value of the constant $c$.

**Proof.** We define $Q$ as the image by the application $\beta \mapsto p^\beta$ of a countable subset $B = B_1 \cup B_2 \cup B_3$ of $\mathbb{R}$. First we build $B_1 = \{b_0 < b_1 < \ldots < b_m \} \subset [0, 1]$ with $b_0 = 0, b_1 = 2/\sqrt{3n}$,

$$b_m = 1, \quad b_{i+1} = b_i \left(1 + \sqrt{2/(5n)}\right) \quad \text{for } 1 \leq i \leq m - 2 \quad \text{and} \quad b_m \leq b_{m-1} \left(1 + \sqrt{2/(5n)}\right).$$

It follows from (83) and (84) that for any $\beta \leq 1$, there exists $b_i \leq \beta$ with $h^2(p^\beta, p^{b_i}) \leq n^{-1}$ and, since $b_{m-1} = (2/\sqrt{3n}) \left(1 + \sqrt{2/(5n)}\right)^{m-2}$, $m \leq \kappa_1 \sqrt{n} \log n$ for some constant $\kappa_1$. We then build $B_2 = \{b_{m+1} < \ldots < b_{m+l}\} \subset (1, 3]$ in a similar way with $b_{i+1} = b_i + \sqrt{2/(3n)}$, $b_{m+l} = 3 \leq b_{m+l-1} + \sqrt{2/(3n)}$, which implies that $l \leq \kappa_2 \sqrt{n}$. Finally we build $B_3 \subset (3, +\infty)$ as the infinite sequence $(b_{m+l+j})_{j \geq 1}$ with

$$b_{m+l+j} = 3 + \frac{j}{\sqrt{n} \alpha_j}, \quad \alpha_j = \log \left(3 + \frac{j}{\sqrt{n}}\right) \quad \text{for } j \geq 1.$$ 

Then $b_{m+l+j}$ goes to infinity with $j$ and, since $\alpha_j > 1$,

$$\sqrt{n}(b_{m+l+j+1} - b_{m+l+j}) = \frac{j + 1}{\alpha_{j+1}} - \frac{j}{\alpha_j} < \frac{1}{\log b_{m+l+j+1}}.$$

It follows from (83) that, for $\beta > 3$, there exists $b_i \leq \beta$ with $h^2(p^\beta, p^{b_i}) \leq n^{-1}$. Since $m + l + 1 < (\kappa_1 + \kappa_2 + 1) \sqrt{n} \log n$ and

$$\frac{1}{n} \sum_{j \geq 1} (b_{m+l+j} - 3)^{-2} < \sum_{j \geq 1} j^{-2} \log^2(3 + j) = \kappa_3 < +\infty,$$

it follows that

$$\sum_{j=0}^{m+l} (\sqrt{n} \log n)^{-1} + \frac{1}{n} \sum_{j \geq 1} (b_{m+l+j} - 3)^{-2} < (\kappa_1 + \kappa_2 + \kappa_3 + 1)$$

so that $\gamma$ is a sub-probability for a large enough value of $c$ independent of $n$. \hfill \Box

We may now apply Theorem 10 to our example with the family $Q$ and the sub-probability $\gamma$ provided by Corollary 5, which leads to the following result.

**Corollary 6.** Let $\Omega$ be the family of densities defined by (81) and $\mathbb{F}$ be a family of models satisfying Assumption 5. Let $Q$ and $\gamma$ be given by Corollary 5. For $\mathbb{F}$ and $\Delta$ defined by (78)
and (79) respectively, the estimator \( \hat{s} \) satisfies, for all densities \( p \) and vectors \( f \in \mathbb{R}^n \),

\[
C \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq \inf_{F \in \mathcal{F}} \left[ \inf_{g \in F} h^2(p_F, p_g) + \mathcal{V}(F) \left( 1 + \log_+ \left( \frac{n}{\mathcal{V}(F)} \right) \right) + \Delta_\pi(F) \right] + \inf_{\beta \geq 0, \lambda > 0} \left[ nh^2(p, p_0^\beta) + \log(\beta \vee 1) + |\log \lambda| \right].
\]

In particular, if \( p = p_0^\beta \) for some \( \beta \in \mathcal{Q} \) with \( \beta \neq 2 \) and \( \tau > 0 \),

\[
C(\beta) \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq \inf_{F \in \mathcal{F}} \left[ \inf_{g \in F} d_{1+(2/\beta)\lambda_1} (f, g) + \mathcal{V}(F) \left( 1 + \log_+ \left( \frac{n}{\mathcal{V}(F)} \right) \right) + \Delta_\pi(F) \right] + \inf_{\beta > 0} \left[ nh^2(p_0^\beta, p_0^\beta) + \log(\beta \vee 1) + |\log \tau\sigma| \right].
\]

**Proof.** Clearly, the family \( \mathcal{Q} \) satisfies Assumption 6, the last requirement deriving from Proposition 10 with \( w_q(\lambda) = (3/5)(\lambda - 1) \) for all \( \lambda \in [1, 2] \) and \( q \in \mathcal{Q} \). Under Assumption 5 on the family \( \mathcal{F} \), we may apply Theorem 10 and we get that, for \( q \in \mathcal{Q}, \lambda \in \mathbb{R}^*_+ \) and \( f \in \mathbb{R}^n \), the risk of \( \hat{s} \) is bounded by \( C(R_1 + R_2) \) with

\[
R_1 = \inf_{F \in \mathcal{F}} \left[ \inf_{g \in F} h^2(p_F, p_g) + \mathcal{V}(F) \left( 1 + \log_+ \left( \frac{n}{\mathcal{V}(F)} \right) \right) + \Delta_\pi(F) \right] + \inf_{i \geq 0} \left[ n2^{-i} + i \right]
\]

and

\[
R_2 = \inf_{q \in \mathcal{Q}, \lambda > 0} \left[ nh^2(p, q_0, \lambda) + \Delta_\gamma(q) + |\log \lambda| \right].
\]

Let us first observe that, since \( \mathcal{V}(F) \geq 1, \mathcal{V}(F) \left( 1 + \log_+ \left( n/\mathcal{V}(F) \right) \right) \geq 1 + \log n \), so that the term \( \inf_{i \geq 0} \left[ n2^{-i} + i \right] \leq 2(1 + \log n) \) in \( R_1 \) can be ignored at the price of the modification of the universal constant \( C \). The Hellinger distance being unchanged by scale changes,

\[
h^2(p, q_0, \lambda) \leq 2h^2(p, p_0^\beta) + 2 + \Delta_\gamma(p^\beta) + |\log \lambda| \quad \text{for any } p^\beta \in \mathcal{Q} \text{ and } q \in \mathcal{Q},
\]

so that

\[
R_2 \leq \inf_{\beta \geq 0, \lambda > 0} \left[ 2nh^2(p_0^\beta, p_0^\beta) + 2 + \Delta_\gamma(p^\beta) + |\log \lambda| \right].
\]

In view of (85), \( \Delta_\gamma(p^\beta) \leq C + \log n + 2 \log(\beta \vee 1) \) and the first risk bound follows since we may again omit terms of ordre \( \log n \). The second one then derives from the fact that \( p^\beta \) is of order \( (2/\beta) \wedge 1 \) for \( \beta \neq 2 \) and (64), arguing as we did to get (80). \( \square \)

### 7.4 Random design regression

We now turn back to the framework of Section 6.3. The same arguments with Theorem 8 replacing Theorem 1 lead to the following generalization of Theorem 7.

**Theorem 11.** If \( q \) is unimodal and symmetric and \( \mathcal{F} \) is a family of models for \( f \) satisfying Assumption 5, there exists a \( \rho \)-estimator \( \hat{s} = q^{-1}_f \) of \( s = pf \) such that for all \( \xi > 0 \), with probability at least \( 1 - e^{-\xi} \),

\[
Ch^2(pf, q_f)
\]

\[
\leq \inf_{F \in \mathcal{F}} \left\{ \inf_{g \in F} h^2(pf, q_g) + \frac{\mathcal{V}(F)}{n} \left[ 1 + \log_+ \left( \frac{n}{\mathcal{V}(F)} \right) \right] + \frac{\Delta_\pi(F)}{n} \right\} + \frac{\xi}{n}
\]

\[
\leq 2h^2(p, q) + \inf_{F \in \mathcal{F}} \left\{ \inf_{g \in F} 2\ell(f, g) + \frac{\mathcal{V}(F)}{n} \left[ 1 + \log_+ \left( \frac{n}{\mathcal{V}(F)} \right) \right] + \frac{\Delta_\pi(F)}{n} \right\} + \frac{\xi}{n}.
\]
with \( \ell \) given by (67). In particular, if \( p \) is known, unimodal and symmetric and \( q = p \),

\[
C\ell(f, \hat{f}) \leq \inf_{F \in \mathcal{F}} \left\{ \inf_{g \in F} \ell(f, g) + \frac{\nabla(F)}{n} \left[ 1 + \log_+ \left( \frac{n}{V(F)} \right) \right] + \frac{\Delta_\pi(F)}{n} \right\} + \frac{\xi}{n}
\]

with probability at least \( 1 - e^{-\xi} \).

If, besides, (64) holds and \( \max \{ \sup_{g \in \mathcal{F}} \| g \|_\infty, \| f \|_\infty \} \leq b < +\infty \), then

\[
C' \left\| f - \hat{f} \right\|_{1+\alpha, \nu} \leq \inf_{F \in \mathcal{F}} \left\{ \inf_{g \in F} \| f - g \|_{1+\alpha, \nu} + \frac{\nabla(F)}{n} \left[ 1 + \log_+ \left( \frac{n}{V(F)} \right) \right] + \frac{\Delta_\pi(F)}{n} \right\} + \frac{\xi}{n}
\]

with probability at least \( 1 - e^{-\xi} \) for some constant \( C' \) depending only on \( A_p, a, b \) and \( \alpha \).

We are not aware of any other procedure that leads to a comparable result. To illustrate this fact, let us consider the following example. We assume that \( p \) is approximately known (approximately equal to \( q \)) and that the regression function \( f \) takes the form

\[
f = \Psi(\zeta) \quad \text{with} \quad \zeta = \sum_{j=1}^{M} \beta_j \zeta_j, \quad \beta = (\beta_1, \ldots, \beta_M) \in \mathbb{R}^M,
\]

where the \( \zeta_j \) are \( M \) given functions on \( \mathcal{Y} \), \( \Psi \) is an unknown non-decreasing function on \( \mathbb{R} \) and \( M \) may be larger than \( n \) but most of the coefficients \( \beta_j \) are equal to zero, which means that we ignore which functions \( \zeta_j \) are really influential. We also choose a finite set \( \{ \Psi_k, 1 \leq k \leq K \} \) of non-decreasing functions to approximate \( \Psi \).

Given \( k \in \{1, \ldots, K\} \) and a non-void subset \( m \) of \( M = \{1, \ldots, M\} \), we consider the model

\[
F_{k,m} = \left\{ \Psi_k \left( \sum_{j \in m} \beta_j \zeta_j \right), \beta \in \mathbb{R}^M \right\}.
\]

This leads to the family \( \mathcal{F} = \{ F_{k,m}, 1 \leq k \leq K \text{ and } m \subset M \} \) of models for \( f \) and we may set \( \pi(F_{k,m}) = [K M (eM/|m|)^{|m|}]^{-1} \), so that

\[
\sum_{k=1}^{K} \sum_{m \in M} \pi(F_{k,m}) = \sum_{k=1}^{K} \sum_{m \in M} \sum_{|m|=\ell} \frac{1}{K M} \left( \frac{eM}{\ell} \right)^{-l} \leq 1
\]

since \( (\frac{M}{l}) \leq (eM/l)^l \). It follows after some simplifications, that the quadratic risk of the corresponding \( \rho \)-estimator can be bounded in the following way (for \( n > 1 \)):

\[
C\mathbb{E}_n \left[ h^2(f, \hat{f}) \right] \leq \inf_{1 \leq k \leq K} \inf_{m \in M} \left\{ \inf_{g \in F_{k,m}} \ell(f; g) + \frac{|m|}{n} \log \left( \frac{nM}{|m|} \right) \right\} + h^2(p, q) + \frac{\log(KM)}{n}.
\]

8. VC-CLASSES AND SUBGRAPHS

We recall, following Dudley (1984) that

**Definition 6.** Let \( \mathcal{C} \) be a non-empty class of subsets of a set \( \Xi \). If \( A \subset \Xi \) with \( |A| = n \), then

\[
\Delta_n(\mathcal{C}, A) = \left| \{ A \cap B, B \in \mathcal{C} \} \right| \quad \text{and} \quad \Delta_n(\mathcal{C}) = \max_{A \subset \Xi, |A| = n} \Delta_n(\mathcal{C}, A).
\]
If $V = \sup \{ n \in \mathbb{N} | \Delta_n(C) = 2^n \} < +\infty$, then $C$ is a VC-class with VC-dimension $V$ and VC-index $\overline{V} = \inf \{ n \in \mathbb{N} | \Delta_n(C) < 2^n \} = V + 1$.

A class $\mathcal{F}$ of functions from a set $\mathcal{X}$ with values in $(-\infty, +\infty]$ is VC-subgraph with dimension $V$ and index $\overline{V}$ if the class of subgraphs $\{(x, u) \in \mathcal{X} \times \mathbb{R}, f(x) > u \}$ as $f$ varies among $\mathcal{F}$ is a VC-class of sets in $\mathcal{X} \times \mathbb{R}$ with dimension $V$ and index $\underline{V}$.

It immediately follows from this definition that any subset of a VC-subgraph class with index $\overline{V}$ is VC-subgraph with index not larger than $\overline{V}$, a property that we shall repeatedly use. Other known properties of VC-subgraph classes directly derive from the properties of VC-classes as described in van der Vaart and Wellner (1996), Lemma 2.6.17.

If $\mathcal{F}$ is VC-subgraph with index $\overline{V}$ on a set $\mathcal{X}$ and $\|f\|_\infty \leq 1$ for all $f \in \mathcal{F}$, it follows from Theorem 2.6.7 in van der Vaart and Wellner (1996) that, for some numerical constant $K$ and all probability measures $Q$ on $\mathcal{X}$,

$$N(\mathcal{F}, Q, \epsilon) \leq K\overline{V}(16\epsilon)^{\overline{V}+1}\epsilon^{-2(\overline{V}-1)} \quad \text{for } 0 < \epsilon < 1.$$  \hfill (86)

Noticing that $N(\mathcal{F}, Q, 1) = 1$ since the closed ball of center 0 and radius 1 contains $\mathcal{F}$, we derive, since $\overline{V} \geq 1$, that there exists a universal constant $A$ such that

$$\log N(\mathcal{F}, Q, \epsilon) \leq 2\overline{V}\log_+(A/\epsilon) \quad \text{for all } \epsilon > 0.$$  \hfill (87)

When the functions lying in $\mathcal{F}$ are all nonnegative, it is not difficult to see that we can restrict the class of subgraphs to that of “nonnegative subgraphs” gathering the sets of the form $\{(x, u) \in \mathcal{X} \times \mathbb{R}, f(x) > u \geq 0 \}$. If $S$ is a subset of a linear space of dimension $D$ then $S$ is VC-subgraph with index $\overline{V} \leq D + 2$ (see Lemma 2.6.15 in van der Vaart and Wellner (1996)). We shall repeatedly use the following properties of VC-subgraph classes.

**Proposition 11.** Let $\mathcal{F}$ be VC-subgraph with dimension $V$ on a set $\mathcal{X}$.

(i) For all functions $g$ on $\mathcal{X}$, $\mathcal{F} + g = \{ f + g, f \in \mathcal{F} \}$ is VC-subgraph with dimension not larger than $V$.

(ii) For all monotone function $\varphi$ on $\mathbb{R}$, $\varphi(\mathcal{F}) = \{ \varphi \circ f, f \in \mathcal{F} \}$ is VC-subgraph with dimension not larger than $V$.

(iii) The class $-\mathcal{F}$ is VC-subgraph with dimension not larger than $V$.

(iv) The class $\mathcal{F}_+ = \{ f \vee 0, f \in \mathcal{F} \}$ is VC-subgraph with dimension not larger than $V$.

(v) If $\mathcal{F}$ and $\mathcal{G}$ are VC-subgraph with respective dimensions $V$ and $V'$, $\mathcal{F} \lor \mathcal{G} = \{ f \vee g, f \in \mathcal{F}, g \in \mathcal{G} \}$ is VC-subgraph with dimension not larger than $4.701(V + V')$ and the same holds for $\mathcal{F} \land \mathcal{G} = \{ f \land g, f \in \mathcal{F}, g \in \mathcal{G} \}$.

(vi) If $q$ is unimodal, the class $q(\mathcal{F}) = \{ q \circ f, f \in \mathcal{F} \}$ is VC-subgraph with dimension not larger than $9.41V$.

(vii) Let $\psi$ be given by (17), $g$ be some nonnegative function on $\mathcal{X}$ and all functions in $\mathcal{F}$ be nonnegative. The class of functions $\psi(\sqrt{f/g}) = \{ \psi(\sqrt{f/g}) | f \in \mathcal{F} \}$ is VC-subgraph with dimension not larger than $V$.

**Proof.** For a proof of (i)–(iv), we refer to Lemma 2.6.18 in van der Vaart and Wellner (1996) and for (v) to the bound (1.2) from van der Vaart and Wellner (2009) together with the relationship between VC-classes and VC-subgraph classes as explained in van der Vaart and Wellner (1996), Section 2.6.5.

For (vi) we argue as follows: $q$ can be written as $\varphi_1 \land \varphi_2$ where $\varphi_1$ is nondecreasing and $\varphi_2$ nonincreasing so that $q \circ f = (\varphi_1 \circ f) \land (\varphi_2 \circ f)$. It follows that $q(\mathcal{F}) \subset \varphi_1(\mathcal{F}) \land \varphi_2(\mathcal{F})$. The bound then follows from (v).
Let us finally prove (vii). It will be useful here and later on to introduce the function $\phi$ from $[0, +\infty]$ to $[-1, 1]$ given by

\begin{equation}
\phi(x) = \psi(\sqrt{x}), \quad \phi(0/0) = \phi(1) = 0 \quad \text{and} \quad \phi(x/0) = \phi(+\infty) = 1 \quad \text{for all } x > 0,
\end{equation}

according to the conventions of Section (2.1). Note that $\phi$ is continuous and increasing, hence one-to-one.

Let $(x_1, u_1), \ldots, (x_m, u_m)$ be $m \geq 1$ points in $\mathcal{X} \times \mathbb{R}$ shattered by the subgraphs of $\phi(\mathcal{F}/g) = \psi(\sqrt{\mathcal{F}/g})$. It suffices to prove that $m \leq V$. First note that we necessarily have $u_i < 1$ for all $i$ since $\phi$ is bounded by 1. In particular for all $i$, $\phi^{-1}(u_i) < +\infty$. Besides, because of our convention, we also have $u_i \geq 0$ for those $i$ such that $g(x_i) = 0$, since otherwise there would be no $f$ in $\mathcal{F}$ such that $\phi(f/g)(x_i) \leq u_i < 0$. For all $I \subset \{1, \ldots, m\}$ there exists an element $f$ of $\mathcal{F}$, depending on $I$, such that $i$ belongs to $I$ if and only if $\phi(f/g)(x_i) > u_i$. This is equivalent to $f(x_i) > g(x_i)\phi^{-1}(u_i)$ if $g(x_i) > 0$ and equivalent to $f(x_i) > 0$ when $g(x_i) = 0$ since $u_i \geq 0$. In both cases, this is equivalent to $f(x_i) > g(x_i)\phi^{-1}(u_i)$. This means that the subgraphs of $\mathcal{F}$ shatter the set $\{(x_i, g(x_i)\phi^{-1}(u_i)), i = 1, \ldots, m\}$, which is possible only when $m \leq V$.

Remark: The proof of (vi) extends to multimodal functions with a given number $k$ of modes recursively by noticing that a function with $k$ modes can be seen as the supremum of a unimodal function and a multimodal one with $k - 1$ modes. It follows that if $q$ is multimodal with $k$ modes, $q(\mathcal{F})$ is VC-subgraph with dimension not larger than $C(k)V$.

9. Proofs

9.1. Proof of Proposition 1. It actually follows from the next one:

**Proposition 12.** If $s$ and $\xi \in \mathcal{L}_0$ and $T(X)$ is such that $\mathbb{P}_s[T(X) \geq z] \leq ae^{-z}$ for all $z \geq 0$ and some $a > 0$, then

\begin{equation}
\mathbb{E}_s[T(X)] \leq (1 + \zeta^{-1}) \left[ \log(1 + a\zeta) + K \right] \quad \text{for all } \zeta > 0 \quad \text{and} \quad K \geq K(s, \xi).
\end{equation}

If, in particular, $K > 0$, then $\mathbb{E}_s[T(X)] \leq 1 + \zeta_0$ where $\zeta_0$ is the unique solution of $\zeta_0 = \log(1 + a\zeta_0) + K$ in $(0, +\infty]$.

**Proof.** We start with the following lemma which appears in a slightly different form in Barron (1991).

**Lemma 1.** Let $P$ and $Q$ two probabilities on $(\mathcal{X}, \mathcal{A})$ and $f$ a function from $(\mathcal{X}, \mathcal{A})$ into $\mathbb{R}$ such that $\int_\mathcal{X} (f \wedge 0) dP > -\infty$. Then

\begin{equation}
\int_\mathcal{X} f dP \leq \log \left( \int_\mathcal{X} e^f dQ \right) + K(P, Q) \leq \log \left( 1 + \int_0^{+\infty} e^\xi Q(f > \xi) d\xi \right) + K(P, Q).
\end{equation}

**Proof.** First note that our assumption implies that $\int_\mathcal{X} f dP$ is well-defined (possibly equal to $+\infty$) and that the result is obvious if either $K(P, Q)$ or $\int_\mathcal{X} e^f dQ$ is infinite. It therefore suffices to prove the result when $P \ll Q$ and $\int_\mathcal{X} e^f dQ < +\infty$. For all $u \in \mathbb{R}$ and $v > 0$, $uv \leq v\log v - v + e^v$. Applying this inequality with $u = f(x) - \log \int_\mathcal{X} e^f dQ$ and $v = (dP/dQ)(x)$ for all $x \in \mathcal{X}$ such that $(dP/dQ)(x) > 0$, we get

\begin{equation}
\left( f(x) - \log \int_\mathcal{X} e^f dQ \right) \frac{dP}{dQ}(x) \leq \frac{dP}{dQ}(x) \log \left( \frac{dP}{dQ}(x) \right) - \frac{dP}{dQ}(x) + \int_\mathcal{X} e^{f(x)} e^f dQ.
\end{equation}
The first inequality follows by integration with respect to $Q$ and the second from the inequalities
\[
\int_{\mathcal{X}} e^t dQ = \int_{0}^{+\infty} Q[e^t > t] dt = \int_{-\infty}^{+\infty} Q[f > \xi] e^\xi d\xi \leq 1 + \int_{0}^{+\infty} e^\xi Q[f > \xi] d\xi.
\]
\[\square\]

To prove Proposition 12 we apply the lemma with $f = \lambda T$, $0 < \lambda < 1$, $P = \mathbb{P}_s$ and $Q = \mathbb{P}_\Xi$, getting
\[
\lambda \mathbb{E}_s[T(X)] \leq \log \left(1 + \int_{0}^{+\infty} e^\xi \mathbb{P}_\Xi[\lambda T(X) > \xi] d\xi\right) + K.
\]
Hence, setting $\zeta = \lambda/(1 - \lambda) > 0$ so that $\lambda = \zeta/(\zeta + 1)$, we get
\[
\frac{\zeta}{\zeta + 1} \mathbb{E}_s[T(X)] \leq \log \left(1 + \int_{0}^{+\infty} e^\xi \mathbb{P}_\Xi[T(X) > \xi/\lambda] d\xi\right) + K
\]
\[
\leq \log \left(1 + a \int_{0}^{+\infty} \exp[-\xi/\zeta] d\xi\right) + K = \log(1 + a\zeta) + K,
\]
which proves (89). The last statement is clear. \[\square\]

If $K > 0$, let $\overline{K} = K$. Then $\mathbb{E}_s[T(X)] \leq 1 + \zeta_0$ which we bound in the following way, setting $f(x) = x - \log(1 + x)$. We first observe that, since $\log(1+uv) \leq \log(1+u) + \log(1+v)$ for all $u, v \geq 0$,
\[
f(\zeta_0) = \zeta_0 - \log(1 + \zeta_0) \leq \log(1 + a) + K
\]
and, since $f$ is increasing on $[0, +\infty[$, $\zeta_0 \leq f^{-1}(c)$. Moreover,
\[
f \left(1 + c + \log \left(1 + c + \sqrt{2c}\right)\right) - c = \log \left(1 + c + \sqrt{2c}\right) - \log \left(1 + c + \log \left(1 + c + \sqrt{2c}\right)\right)
\]
has the same sign as
\[(90) \quad \left(1 + c + \sqrt{2c}\right) - \left(1 + c + \log \left(1 + c + \sqrt{2c}\right)\right) = \sqrt{2c} - \log \left(1 + c + \sqrt{2c}\right),
\]
which has the sign of $\exp[\sqrt{2c}] - (1 + c + \sqrt{2c}) > 0$ since $e^x > 1 + x + (x^2/2)$ for $x > 0$. It follows that $c < f \left(1 + c + \log \left(1 + c + \sqrt{2c}\right)\right)$ and finally, using again the fact that the right-hand side of (90) is positive,
\[
1 + \zeta_0 \leq 1 + f^{-1}(c) < 1 + c + \log \left(1 + c + \sqrt{2c}\right) < 1 + c + \sqrt{2c},
\]
which completes the proof of (30) in the case of $K > 0$. If $K = 0$ we replace it by $\overline{K} > 0$ and let it go to zero to derive (30). To get (31) we apply (30) to the random variable $T'(X) = b[T(X) - z_0]$.

9.2. Proof of Proposition 2. We start with the following lemma.

Lemma 2. Let $X_1, \ldots, X_n$ be independent random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathcal{X}$, $\mathcal{F}$ a class of functions on $\mathcal{X}$ bounded by 1 and $\mathcal{F}$ a mapping from $[0, +\infty)$ satisfying the assumptions of Proposition 2. If
\[
(91) \quad \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \mathbb{E} \left[f^2(X_i)\right] \leq v^2
\]
\[
\sup_{f \in \mathcal{F}} \left(\sum_{i=1}^{n} \mathbb{E} \left[f^2(X_i)\right]\right) \leq v^2
\]
and

\[ \log N \left( \mathcal{F}, \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(\omega)}, z \right) \leq \mathcal{H} \left( \frac{1}{z} \right) \quad \text{for all } z > 0 \text{ and } \omega \in \Omega, \]

then there exists a universal constant \( C \) such that,

\[
E \left[ \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} (f(X_i) - E[f(X_i)]) \right| \right] \leq C \left[ v L \sqrt{\mathcal{H} \left( \frac{\sqrt{n}}{v} \lor 1 \right)} + L^2 \mathcal{H} \left( \frac{\sqrt{n}}{v} \lor 1 \right) \right].
\]

We prove (43) for a given value of \( y \) by applying the lemma to the family \( \mathcal{F}^S(s, \mathfrak{S}, y) \), the elements of which are bounded by 1 and satisfy (91) with \( v^2 = 6y^2 \) (because of Proposition 14), and to the function \( \mathcal{H}_y \) with \( L = L_y \). Using \( 2ab \leq a \alpha^2 + \alpha^{-1}b^2 \) for all \( a, b \in \mathbb{R} \) and \( \alpha = c_0/(4C_0) \), we have

\[
w^S(s, \mathfrak{S}, y) \leq \frac{C_0}{2} \left[ \alpha y^2 + (2 + 6\alpha^{-1}) L^2 \mathcal{H}_y \left( \frac{\sqrt{n}}{6y^2} \lor 1 \right) \right]
\leq \frac{c_0 y^2}{8} + C_0 \left( 1 + \frac{12C_0}{c_0} \right) L^2 \mathcal{H}_y \left( \frac{\sqrt{n}}{6y^2} \lor 1 \right).
\]

Then (44) follows from the definition of \( D^S(s, \mathfrak{S}) \).

Let us now turn to the proof of Lemma 2. It is a simpler version of Theorem 3.1 of Giné and Koltchinskii (2006), the only difference lying in the fact that Lemma 2 applies to variables \( X_i \) that are independent but not necessarily i.i.d. We shall therefore only give a sketch of proof.

By a symmetrization argument,

\[
E \left[ \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} (f(X_i) - E[f(X_i)]) \right| \right] \leq 2E = 2E \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \epsilon_i f(X_i) \right],
\]

where the \( \epsilon_i \) are Rademacher random variables independent of the \( X_i \). Arguing as in Giné and Koltchinskii with \( F = 1 \), we get

\[
E \leq C n^{1/2} \mathbb{E} \left[ \int_0^{2\tilde{\sigma}_n} \sqrt{\mathcal{H}(1/z)} \, dz \right] \quad \text{with } \tilde{\sigma}_n^2 = \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} f^2(X_i) \leq 1.
\]

The function \( u \mapsto \mathcal{H}(1/u) \) being nonincreasing, \( u \mapsto \int_0^{u} \mathcal{H}(1/z) \, dz \) is concave and therefore

\[
E \leq C n^{1/2} \int_0^{2E[\tilde{\sigma}_n]} \sqrt{\mathcal{H}(1/z)} \, dz \leq C n^{1/2} \int_0^{2\sqrt{E[\tilde{\sigma}_n]}} \sqrt{\mathcal{H}(1/z)} \, dz.
\]

Symmetrization and contraction arguments together with the fact that \( |f| \leq 1 \) for all \( f \in \mathcal{F} \) lead to

\[
E[\tilde{\sigma}_n^2] \leq B^2 = \frac{v^2 + 8E}{n} \land 1.
\]
Using a change of variables, the definition of $L$, the monotonicity of $\mathcal{H}$ and the fact that $B \geq (v/\sqrt{n}) \wedge 1$ we obtain that

$$
E \leq Cn^{1/2} \int_{0}^{2B} \sqrt{\mathcal{H}(1/z)} \, dz = Cn^{1/2} \int_{1/(2B)}^{\infty} \sqrt{\mathcal{H}(u)} \, du
$$

$$
\leq 2CLn^{1/2} B \sqrt{\mathcal{H}\left(\frac{1}{2B}\right)} \leq C' \max \{v, \sqrt{E}\} \sqrt{\mathcal{H}\left(\frac{\sqrt{n}}{v} \vee 1\right)}
$$

for some universal constant $C' \geq 1$. Finally, resolving this inequality with respect to $E$, we get

$$
E \leq C \left[vL \sqrt{\mathcal{H}\left(\frac{\sqrt{n}}{v} \vee 1\right)} + L^2 \mathcal{H}\left(\frac{\sqrt{n}}{v} \vee 1\right)\right].
$$

9.3. Proofs of Theorem 1, 8 and 9. All three theorems actually follow from the following (slightly) stronger result. Let

$$
c_1 = \frac{2 \left(1+1/\sqrt{2} + c_0\right)}{1-1/\sqrt{2} - c_0} = 2 \left(7 + 4\sqrt{2}\right) < 25.5;
$$

$$
c_2 = \left[4 \left(1-1/\sqrt{2} - c_0\right)\right]^{-1} = \left(2 + \sqrt{2}\right)/2 < 1.8 \quad \text{and} \quad c_3 = 8c_2\kappa.
$$

Theorem 12. Let $s \in \mathcal{Z}_0$, $\bar{s} \in \mathcal{S}$ and the penalty function $\text{pen}$ satisfy

(93) $\text{pen}(t) \geq \text{pen}_0(t) = \inf \{(1/8)D^S(s, \bar{s}) + \kappa \Delta(\mathcal{S}) | \bar{s} \in \mathcal{S}, s \ni t\}$ for all $t \in \mathcal{S}$.

The estimator $\hat{s}$ satisfies for all $\xi > 0$

(94) $\mathbb{P}_s\left[h^2(s, \bar{s}) \leq c_1 h^2(s, \bar{s}) - h^2(s, S) + 8c_2 \text{pen}(\bar{s}) + c_3(1.45 + \xi)\right] \geq 1 - e^{-\xi}$.

In particular, if (93) is an equality and $\bar{s} \in S$ then, for all $\xi > 0,$

(95) $\mathbb{P}_s\left[h^2(s, \bar{s}) \leq c_1 h^2(s, \bar{s}) - h^2(s, S) + c_2 D^S(s, \bar{s}) + c_3 \left(\Delta(\mathcal{S}) + 1.45 + \xi\right)\right] \geq 1 - e^{-\xi}$

and

$$
\mathbb{E}_s\left[h^2(s, \bar{s})\right] \leq c_1 h^2(s, \bar{s}) + c_2 D^S(s, \bar{s}) + c_3 \left(\Delta(\mathcal{S}) + 2.45\right).
$$

9.3.1. Proof of Theorem 1. Taking $\mathcal{S} = \{\bar{s}\}$ hence $\mathcal{S} = S$, $\Delta(\mathcal{S}) = 0$ and $\text{pen}(t) = (1/8)D^S(s, \bar{s})$ for all $t \in S$ we derive from (95) that

$$
\mathbb{P}_s\left[h^2(s, \bar{s}) \leq c_1 h^2(s, \bar{s}) - h^2(s, S) + c_2 D^S(s, \bar{s}) + c_3(1.45 + \xi)\right] \geq 1 - e^{-\xi} \quad \text{for all } \xi > 0.
$$

Theorem 1 follows from the facts that this inequality is true for all choices of $\bar{s} \in S$.

9.3.2. Proof of Theorem 8. Inequality (72) is a straightforward consequence of Theorem 12 since, for all $(s, \bar{s}) \in \mathcal{L}_0 \times \mathcal{L}_0$, $D^S(s, \bar{s}) \leq \mathcal{D}^S$ and therefore (71) implies that (93) holds.
9.3.3. Proof of Theorem 9. Let us fix some \( \bar{s} \in S \). There exists \( \overline{S'} \in S \) such that \( \bar{s} \in S' \) and by definition of \( D^S(\cdot, \cdot) \) and Assumption 4 for all \( S \in S \),

\[
D^S(s, \bar{s}) \leq D^{S, S'}(s, \bar{s}) \leq D^{S, S'} \leq \tilde{D}(S) + \tilde{D}(S').
\]

Consequently, the penalty function \( \text{pen}' \) defined for \( t \in S \) by

\[
\text{pen}'(t) = (1/8)\tilde{D}(S') + \text{pen}(t)
\]

satisfies (93). Besides, the estimator based on \( \text{pen}' \) is the same as that based on \( \text{pen} \) because these two penalties differ by a fixed quantity independent of \( t \). We get from Theorem 12 that the estimator \( \hat{s} \) satisfies for all \( \xi > 0 \), with probability at least \( 1 - e^{-\xi} \)

\[
\begin{align*}
\text{pen}(t) &\leq c_1 h^2(s, \bar{s}) + 8c_2 \text{pen}(s) + c_3(1.45 + \xi) \\
&= c_1 h^2(s, \bar{s}) + 8c_2 \text{pen}(s) + c_2 \tilde{D}(S') + c_3(1.45 + \xi).
\end{align*}
\]

Then (76) follows from the facts that \( \text{inf}\{ (1/8)\tilde{D}(S') \mid S \in S, S' \supseteq \bar{s} \} \leq \text{pen}(s) \) and \( \bar{s} \) is arbitrary in \( S \). Let us fix some model \( \bar{S} \in S \), choose \( S' \in \bar{S} \) such that \( h^2(s, s') \leq h^2(s, \bar{s}) + c_3/(160c_1) \) and \( \bar{s} \in S \) such that \( h^2(s', \bar{s}) \leq h^2(s', S) + c_3/(160c_1) \). Using (76) with equality in (75), we derive that, with probability at least \( 1 - e^{-\xi} \),

\[
\begin{align*}
\text{pen}(t) &\leq c_1 h^2(s, \bar{s}) + 16c_2 \text{pen}(s) + c_3(1.45 + \xi) \\
&\leq 2c_1 h^2(s, \bar{s}) + 2c_1 h^2(s', \bar{s}) + 2ac_2 D^S + 16c_2 \kappa \Delta(\bar{s}) + c_3(1.475 + \xi) \\
&= 2c_1 h^2(s, \bar{s}) + 2c_1 h^2(s', S) + 2ac_2 D^S + c_3(2\Delta(\bar{s}) + 1.475 + \xi) \\
&\leq 2c_1 h^2(s, \bar{s}) + 2ac_2 D^S + c_3(2\Delta(\bar{s}) + 1.5 + \xi)
\end{align*}
\]

where the last inequality derives from (77). Finally, we conclude by using that \( \bar{S} \) is arbitrary in \( \bar{S} \).

9.3.4. Proof of Theorem 12. It is divided into 2 steps.

Step 1. Here we prove the following fundamental lemma.

Lemma 3. Provided that the function \( \text{pen} \) satisfies (93), for all \( \bar{s} \in \mathcal{L}_0 \) and \( \xi > 0 \),

\[
\mathbb{P}_s \left[ \frac{1}{\sqrt{2}} Z(X, \bar{s}, t) \leq c_0 \left( h^2(s, t) + h^2(s, \bar{s}) \right) + \text{pen}(t) + \kappa(1.4 + \xi) \right] \geq 1 - e^{-\xi}.
\]

Proof. The proof relies on two propositions. The first one presents a version of Talagrand’s result on the suprema of empirical processes that is proved in Massart (2007). The second one is proved in Baraud (2011) (more precisely, we refer to the proof of his Proposition 3 on page 386 with the difference that in this paper his function \( \psi \) is equal to \( 1/\sqrt{2} \) our function \( \psi \) involving an additional factor 2 for the control of the function \( \psi^2 \) as defined by (17)).

Proposition 13. Let \( T \) be some finite set, \( U_1, \ldots, U_n \) be independent centered random vectors with values in \( \mathbb{R}^T \) and

\[
Z = \sup_{t \in T} \left| \sum_{i=1}^n U_{i,t} \right|.
\]
If for some positive numbers $b$ and $v$,
\[
\max_{i=1, \ldots, n} |U_{i,t}| \leq b \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{E}_{s}[U_{i,t}^2] \leq v^2 \quad \text{for all } t \in T,
\]
then,
\[
\mathbb{P} \left[ Z \leq (1 + 16c)\mathbb{E}(Z) + 2v^2c^{-1} + (2 + c^{-1})bx \right] \geq 1 - e^{-x^2} \quad \text{for all } c > 0, x > 0.
\]
\begin{equation}
\mathbb{P} \left[ Z \leq \mathbb{E}(Z) + 2\sqrt{2v^2 + 16b\mathbb{E}(Z)x + 2bx} \right] \geq 1 - e^{-x^2}.
\end{equation}

\begin{proof}
The second displayed formula on page 170 of Massart (2007) tells us that
\[
\mathbb{P} \left[ Z \leq \mathbb{E}(Z) + 2\sqrt{2v^2 + 16b\mathbb{E}(Z)x + 2bx} \right] \geq 1 - e^{-x^2}.
\]
To derive (96) we use the fact that
\[
2\sqrt{2v^2 + 16b\mathbb{E}(Z)x} \leq 2v^2c^{-1} + 16c\mathbb{E}(Z) + bc^{-1}x.
\]
\end{proof}

Even though the result is stated for finite $T$, it can easily be extended to countable sets by monotone convergence.

\begin{proposition}
Let $X = (X_1, \ldots, X_n)$ be a vector of independent random variables and $t, s \in \mathcal{L}_0$.
\[
\mathbb{E}_s \left[ \psi^2 \left( \sqrt{\frac{t}{s}}(X) \right) \right] = \sum_{i=1}^{n} \mathbb{E}_s \left[ \psi^2 \left( \sqrt{\frac{t_i}{s_i}}(X_i) \right) \right] \leq 6 \left[ h^2(s, t) + h^2(s, s) \right].
\]
\end{proposition}

Let us now turn to the proof of Lemma 3. We fix $\xi > 0$, $S$ in $\mathbb{S}$, $\tau = a/(2c_0^2) > 0$ and set and for $j \in \mathbb{N}$,
\[
y_j^2 = (5/4)^j \left[ D^S(s, s) + \tau \left( \Delta(s) + \xi + 1.4 \right) \right],
\]
\[
B_j^S(s, s) = \left\{ t \in S \mid y_j^2 < h^2(s, t) + h^2(s, s) \leq y_{j+1}^2 \right\}
\]
\[
Z_j^S(X, s) = \left\{ \sup_{t \in B_j^S(s, s)} |Z(X, s, t)| \right\} \quad \text{and} \quad x_j = \tau^{-1}y_j^2 \geq \Delta(s) + \xi + 1.4 \times (5/4)^j.
\]
For each $j \geq 0$, we may apply Theorem 13 to the supremum $Z_j^S(X, s)$ by taking $T = B_j^S(s, s)$ and
\begin{equation}
U_{i,t} = \frac{1}{\sqrt{2}} \left\{ \psi \left( \sqrt{\frac{t_i}{s_i}}(X_i) \right) - \mathbb{E}_s \left[ \psi \left( \sqrt{\frac{t_i}{s_i}}(X_i) \right) \right] \right\} \quad \text{for all } i = 1, \ldots, n.
\end{equation}
For such a choice, the assumptions of the Proposition 13 are met with $b = \sqrt{2}$ (since $\psi$ is bounded by 1) and $v^2 = 3y_{j+1}^2$ (by Proposition 14 and the definition of the set $B_j^S(s, s)$).
It follows from (96) that, with probability at least $1 - e^{-x_j}$, for all $t \in B_j^S(s, s)$,
\[
\frac{1}{\sqrt{2}}Z(X, s, t) \leq Z_j^S(X, s) \leq (1 + 16c)\mathbb{E}_s[Z_j^S(X, s)] + 3\sqrt{2}cy_{j+1}^2 + \sqrt{2} \left( 2 + c^{-1} \right)x_j.
\]
Since $B_j^S(s, s) \subset B_j^S(s, y_{j+1})$, it follows from the definition of $D^S(s, s)$ and the fact that $y_{j+1}^2 > D^S(s, s)$ that,
\[
\mathbb{E} \left[ Z_j^S(X, s) \right] \leq (\sqrt{2})^{-1}w^S(s, s, y_{j+1}) \leq (\sqrt{2})^{-1}c_0y_{j+1}^2/4,
\]

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which leads, since \( x_j = 4y_{j+1}/(5\tau) \), to
\[
\frac{1}{\sqrt{2}} Z(X, \bar{s}, t) \leq \frac{y_{j+1}}{\sqrt{2}} \left(1 + 16c\frac{c_0}{4} + 6c + \frac{8(2 + e^{-1})}{5\tau}\right)
\]
and, setting \( c = (2\tau)^{-1/2} \), since \( t \in B_j^S(s, \bar{s}) \),
\[
\frac{1}{\sqrt{2}} Z(X, \bar{s}, t) - c_0 \left[h^2(s, t) + h^2(s, \bar{s})\right]
\leq \frac{y_{j+1}}{\sqrt{2}} \left[\frac{c_0}{4} \left(1 + \frac{16}{\sqrt{2}\tau}\right) + \frac{6}{\sqrt{2}\tau} + \frac{16}{5\tau} + \frac{16}{5\sqrt{2}\tau} - \frac{4c_0\sqrt{2}}{5}\right].
\]
The bracketed factor writes
\[
\frac{c_0}{4} + \frac{4c_0^2}{\sqrt{a}} + \frac{6c_0}{\sqrt{a}} + \frac{32c_0^2}{5a} + \frac{16c_0}{\sqrt{5a}} - \frac{4c_0\sqrt{2}}{5} = \frac{c_0}{5} \left[\frac{5}{4} - 4\sqrt{2} + \frac{46}{\sqrt{a}} + \left(2 - \sqrt{2}\right)\left(\frac{5}{\sqrt{a}} + \frac{8}{a}\right)\right],
\]
which is negative for \( a = 125.4 \). Hence, with probability \( 1 - e^{-x_0} \), for all \( t \in B_j^S(s, \bar{s}) \),
\[
\frac{1}{\sqrt{2}} Z(X, \bar{s}, t) - c_0 \left[h^2(s, t) + h^2(s, \bar{s})\right] < 0.
\]
Then we define
\[
Z^S(X, \bar{s}) = \frac{1}{\sqrt{2}} \sup_{t \in B^S(s, \bar{s}, y_0)} |Z(X, \bar{s}, t)|
\]
and we apply Proposition 13 in a similar way to \( Z^S(X, \bar{s}) \) with \( x = x_0 = \frac{y_0^2}{\tau} \). We then deduce that, with probability at least \( 1 - e^{-x_0} \), for all \( t \in B^S(s, \bar{s}, y_0) \),
\[
\frac{1}{\sqrt{2}} Z(X, \bar{s}, t) \leq Z^S(X, \bar{s}) \leq \frac{y_0^2}{\sqrt{2}} \left[c_0(4c + 1/4) + 6c + 2(2 + e^{-1})\tau^{-1}\right].
\]
Hence, with \( c = (10\tau/3)^{-1/2} = c_0/\sqrt{209} \),
\[
\frac{1}{\sqrt{2}} Z(X, \bar{s}, t) \leq c_0\frac{y_0^2}{\sqrt{2}} \left[4c_0 \left(\frac{1}{\sqrt{209}} + \frac{2}{125.4}\right) + \frac{1}{4} + \frac{6}{\sqrt{209}} + \frac{4\sqrt{209}}{125.4}\right] < 0.122y_0^2.
\]
Since \( \{B^S_j(s, \bar{s}, y_0)\}, \{B^S_j(s, \bar{s}), \ j \geq 0\} \) provides a partition of \( S \), by putting all these inequalities together, we get that for all \( t \in S \),
\[
\frac{1}{\sqrt{2}} Z(X, \bar{s}, t) - c_0 \left(h^2(s, t) + h^2(s, \bar{s})\right) < 0.122y_0^2 < (1/8)D^S(s, \bar{s}) + \kappa \left[\Delta(S) + \xi + 1.4\right],
\]
except on a set of probability not larger than
\[
2e^{-x_0} + \sum_{j \geq 1} e^{-x_j} \leq e^{-\xi - \Delta(S)} \left[2e^{-1.4} + \sum_{j \geq 1} e^{-1.4 \times (5/4)^j}\right] < e^{-\xi - \Delta(S)}.
\]
The result extends to all \( t \in S \) by summing these bounds over \( S \in \mathbb{S} \). \( \square \)

**Step 2.** Let us now set for \( s, t, t' \in \mathcal{L}_0 \)
\[
T(s, t, t') = \mathbb{E}_s [T(X, t, t')] = \sum_{i=1}^n T(s_i, t_i, t'_i).
\]
Applying inequality (13) to each coordinate \( s_i, \bar{s}_i \) and \( t_i \) of \( s, \bar{s} \) and \( t \) respectively and by summing these inequalities over \( i \in \{1, \ldots, n\} \) we obtain that for all \( s \in \mathcal{L}_0, \bar{s} \in \mathcal{S} \) and \( t \in \mathcal{S} \)

\[
T(s, \bar{s}, t) \leq \left( 1 + \frac{1}{\sqrt{2}} \right) h^2(s, \bar{s}) - \left( 1 - \frac{1}{\sqrt{2}} \right) h^2(s, t).
\]

Let us fix \( s \in \mathcal{L}_0 \) and \( \bar{s} \in \mathcal{S} \). Recalling that \( Z(X, s, t) / \sqrt{2} = T(X, \bar{s}, t) - T(s, \bar{s}, t) \) for all \( t \in \mathcal{S} \), we deduce from Lemma 3 that with probability at least \( 1 - e^{-\xi} \),

\[
T(X, \bar{s}, t) - T(s, \bar{s}, t) \leq c_0 \left( h^2(s, t) + h^2(s, \bar{s}) \right) + \text{pen}(t) + \kappa(1.4 + \xi)
\]

which, together with (98), leads to

\[
T(X, \bar{s}, t) - \text{pen}(t)
\]

\[
\leq \left( 1 + \frac{1}{\sqrt{2}} + c_0 \right) h^2(s, \bar{s}) - \left( 1 - \frac{1}{\sqrt{2}} - c_0 \right) h^2(s, \bar{s}) + \kappa(1.4 + \xi)
\]

for all \( t \in \mathcal{S} \). Hence,

\[
\overline{\mathcal{Y}}(s, \bar{s}) = \sup_{t \in \mathcal{S}} |T(X, s, t) - \text{pen}(t)| + \text{pen}(\bar{s})
\]

satisfies

\[
\overline{\mathcal{Y}}(s, \bar{s}) \leq \left( 1 + \frac{1}{\sqrt{2}} + c_0 \right) h^2(s, \bar{s}) - \left( 1 - \frac{1}{\sqrt{2}} - c_0 \right) h^2(s, \bar{s}) + \text{pen}(\bar{s}) + \kappa(1.4 + \xi).
\]

In particular, it follows from the definition of \( \hat{s} \) that

\[
\overline{\mathcal{Y}}(s, \hat{s}) \leq \overline{\mathcal{Y}}(s, \bar{s}) + \frac{\kappa}{10}
\]

(101)

\[
\leq \left( 1 + \frac{1}{\sqrt{2}} + c_0 \right) h^2(s, \bar{s}) - \left( 1 - \frac{1}{\sqrt{2}} - c_0 \right) h^2(s, \bar{s}) + \text{pen}(\bar{s}) + \kappa(1.5 + \xi).
\]

Using (100) with \( t = \hat{s} \), inequality (101) and the fact that \( T(X, s, \hat{s}) = -T(X, \hat{s}, s) \), we obtain that with probability at least \( 1 - e^{-\xi} \)

\[
\left( 1 - \frac{1}{\sqrt{2}} - c_0 \right) h^2(s, \hat{s}) \leq \left( 1 + \frac{1}{\sqrt{2}} + c_0 \right) h^2(s, \bar{s}) - T(X, \hat{s}, \bar{s}) + \text{pen}(\bar{s}) + \kappa(1.4 + \xi)
\]

\[
\leq \left( 1 + \frac{1}{\sqrt{2}} + c_0 \right) h^2(s, \bar{s}) + \overline{\mathcal{Y}}(s, \hat{s}) + \text{pen}(\bar{s}) + \kappa(1.4 + \xi)
\]

\[
\leq 2 \left( 1 + \frac{1}{\sqrt{2}} + c_0 \right) h^2(s, \bar{s}) - \left( 1 - \frac{1}{\sqrt{2}} - c_0 \right) h^2(s, \bar{s}) + 2 \text{pen}(\bar{s}) + 2\kappa(1.45 + \xi),
\]

which leads to the result.

9.4. **Proof of Theorem 4.** In order to simplify the notations, when using the Hellinger distance on our model, we shall write \( h(\theta, \theta') \) instead of \( h(t_{\theta}, t_{\theta'}) \). We shall denote by \( |\cdot| \) the Euclidean distance on \( \mathbb{R}^d \) (as well as the absolute value) and given a function \( t \) on \( \mathcal{X} \), \( \|t\|_{\infty} = \sup_{x \in \mathcal{X}} |t(x)| \). All along this proof, we shall denote by \( A_i, i \in \mathbb{N} \), constants that only depend on the structure of the parametric model \( \overline{\mathcal{S}} \) as described by Assumption 1.
A first consequence of Assumption 1-(ii) is that the parametric family \( \{ t_\theta, \theta \in \Theta' \} \) is regular with continuous Fisher Information \( I(\theta) \). Since \( I \) is invertible on the compact \( \Theta \) its eigenvalues are bounded away from zero and infinity on \( \Theta \) which implies that

\[
A_0 |\overline{\theta} - \theta| \leq h(\overline{\theta}, \theta) \leq A_1 |\overline{\theta} - \theta| \quad \text{for all } \overline{\theta}, \theta \in \Theta, \quad \text{with } 0 < A_0 < A_1.
\]

These inequalities derive from (7.20) p.82 of the book by Ibragimov and Has’minski˘ı (1981).

It also follows from Assumption 1-(ii) and -(i) that

\[
\| \sqrt{\frac{t_\theta}{t_{\theta'}}} - \sqrt{\frac{t_\pi}{t_{\pi'}}} \|_{\infty} \leq A_3 |\overline{\theta} - \theta| \leq \frac{A_3}{A_0} h(\theta, \overline{\theta}) = A_4 h(\theta, \overline{\theta}) \quad \text{for all } \overline{\theta}, \theta, \theta' \in \Theta.
\]

Using the triangular inequality together with the facts that \( \psi \) is 1.15-Lipschitz and satisfies \( \psi(1/x) = -\psi(x) \) for all \( x > 0 \), we get for all \( \theta, \overline{\theta}, \theta', \overline{\theta} \in \Theta,

\[
\| \psi \left( \sqrt{\frac{t_{\theta'}}{t_\theta}} \right) - \psi \left( \sqrt{\frac{t_{\theta'}}{t_\pi}} \right) \|_{\infty} \leq 1.15 A_4 \left[ h(\theta, \overline{\theta}) + h(\theta', \overline{\theta}) \right].
\]

Moreover

**Lemma 4.** The function

\[
(t, t') \mapsto \rho\left( t', \frac{t + t'}{2} \right) - \rho\left( t, \frac{t + t'}{2} \right)
\]

is uniformly continuous on \( \overline{\mathcal{S}} \times \overline{\mathcal{S}} \) with respect to the Hellinger distance.

**Proof:** It is clearly enough to show the continuity of \( (t, t') \mapsto h\left( t, (t + t')/2 \right) \) and, since

\[
|h \left( t, (t + t')/2 \right) - h \left( u, (u + u')/2 \right)| \leq h(t, u) + h \left( (t + t')/2, (u + u')/2 \right),
\]

it is enough to bound the second term. By the classical inequalities between the Hellinger and variation distances,

\[
h^2 \left( \frac{t + t'}{2}, \frac{u + u'}{2} \right) \leq \frac{1}{2} \int \left| \frac{t + t'}{2} - \frac{u + u'}{2} \right| \, d\mu
\]

\[
\leq \frac{1}{4} \int \left[ |t - u| + |t' - u'| \right] \, d\mu \leq \frac{1}{\sqrt{2}} [h(t, u) + h(t', u')],
\]

which concludes the proof. \( \square \)

Together with (102) and (104) the lemma shows that \( (\theta, \theta') \mapsto T(X, t_\theta, t_{\theta'}) \) is continuous from \( \Theta \times \Theta \) into \( \mathbb{R} \) with probability 1, uniformly with respect to \( X \).

Recalling that \( s = t_\theta \in \overline{\mathcal{S}} \), let us set, for \( \Gamma \geq 1 \), \( J \in \mathbb{N} \) and \( n \geq 1 \),

\[
\delta = \sqrt{\Gamma d/n} \quad \text{and} \quad \mathcal{C}(\Gamma, J) = \left\{ (\theta, \theta') \in \Theta^2 \mid \text{with } h(\theta, \theta) \leq \delta \quad \text{and} \quad h(\theta, \theta') > 2^{J/2} \delta \right\}.
\]

We want to establish the following intermediate result.

**Proposition 15.** Under Assumption 1, there exist a positive constant \( C \) and a positive integer \( J_0 \), both depending on \( \overline{\mathcal{S}} \) only, such that, for all \( J \geq J_0 \) and \( \Gamma \in [1, n/d] \),

\[
\mathbb{P}_s \left[ \sup_{(\theta, \theta') \in \mathcal{C}(\Gamma, J)} T(X, t_\theta, t_{\theta'}) < 0 \right] \geq 1 - \exp[-C \sqrt{\Gamma d}].
\]
Proof: First of all, let us note that \( \sup_{(\theta, \theta') \in \mathcal{C}(\Gamma, J)} T(\mathbf{X}, t_\theta, t_{\theta'}) \) is measurable since \( \mathcal{C}(\Gamma, J) \) is separable and \((\theta, \theta') \mapsto T(\mathbf{X}, t_\theta, t_{\theta'})\) is continuous. Let us then set
\[
B = \{ \theta \in \Theta \mid h(\theta, \theta) \leq \delta \};
\]
and, for \(k \in \mathbb{N}\), let \(B_k \subset B\) and \(C_{j,k} \subset C_j\) be \(2^{-k/2}\delta\)-nets for \(B\) and \(C_j\) respectively. Since, by \((102)\), \((S, h)\) and \((\Theta, | |) \subset (\mathbb{R}^d, | |)\) are isometric (up to constants), we may choose \(B_k\) and \(C_{j,k}\) in such a way that
\[
(105) \quad \log |B_k| \leq A_5 dk \quad \text{and} \quad \log |C_{j,k}| \leq A_5 d(j + j + 1 + k) \quad \text{for all } k, j \in \mathbb{N},
\]
as it would be the case for Euclidean balls.

For \(\theta \in \Theta\) and \(j, k \in \mathbb{N}\), we denote by \(\theta_k\) and \(\theta_{j,k}\) minimizers of the function \(\theta' \mapsto h(\theta, \theta')\) over \(B_k\) and \(C_{j,k}\) respectively. For all \(j \in \mathbb{N}\) and \((\theta_0, \theta'_{j,0}) \in B_0 \times C_{j,0}\), by Proposition 14,
\[
\mathbb{E}_s \left[ \psi^2 (\sqrt{\frac{t_{\theta'_{j,0}}}{t_{\theta_0}}} / \sqrt{\frac{t_{\theta_0}}{t_{\theta_0}}}) \right] \leq 6 \left[ h^2 (\theta, \theta_0) + h^2 (\theta, \theta'_{j,0}) \right] \leq 6\delta^2 (1 + 2^{J+j+1}) \leq 2^{J+j+4}\delta^2.
\]
Since \(\left| \psi \left( \sqrt{\frac{t_{\theta'_{j,0}}}{t_{\theta_0}}} / \sqrt{\frac{t_{\theta_0}}{t_{\theta_0}}} \right) \right| \leq 1\), we may use Bernstein’s inequality with \(x_{j,0} = 2^{J+j}\Gamma d / 100\), then \((105)\) to derive that
\[
\mathbb{P}_s \left[ \sup_{(\theta_0, \theta'_{j,0}) \in B_0 \times C_{j,0}} T \left( X, t_{\theta_0}, t_{\theta'_{j,0}} \right) - \mathbb{E}_s \left[ T \left( X, t_{\theta_0}, t_{\theta'_{j,0}} \right) \right] > x_{j,0} \right]
\leq \mathbb{P}_s \left[ T \left( X, t_{\theta_0}, t_{\theta'_{j,0}} \right) - \mathbb{E}_s \left[ T \left( X, t_{\theta_0}, t_{\theta'_{j,0}} \right) \right] > x_{j,0} \right]
\leq \mathbb{P}_s \left[ \sum_{i=1}^{n} \left( \psi \left( \sqrt{\frac{t_{\theta'_{j,0}}}{t_{\theta_0}}} (X_i) \right) - \mathbb{E}_s \left[ \psi \left( \sqrt{\frac{t_{\theta'_{j,0}}}{t_{\theta_0}}} (X_i) \right) \right] \right) > \sqrt{2} x_{j,0} \right]
\leq \exp \left[ A_5 d (J + j + 1) - \frac{2x_{j,0}^2}{2^{J+j+4}\delta^2} \right]
\leq \exp \left[ -C 2^{J+j+1}\Gamma d \right]
\]
for some \(C > 0\) and \(J_0\) large enough (depending on the \(A_i\), which means on \(S\)) since \(J \geq J_0\). For \((\theta, \theta') \in B \times C_j\) and \(k \in \mathbb{N}\), let
\[
\Delta T \left( X, t_{\theta_k}, t_{\theta'_{j,k+1}}, t_{\theta_{k+1}}, t_{\theta'_{j,k+1}} \right)
= \left\{ T \left( X, t_{\theta_{k+1}}, t_{\theta'_{j,k+1}} \right) - \mathbb{E}_s \left[ T \left( X, t_{\theta_{k+1}}, t_{\theta'_{j,k+1}} \right) \right] \right\}
- \left\{ T \left( X, t_{\theta_k}, t_{\theta'_{j,k}} \right) - \mathbb{E}_s \left[ T \left( X, t_{\theta_k}, t_{\theta'_{j,k}} \right) \right] \right\}
= \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \left[ \psi \left( \sqrt{\frac{t_{\theta'_{j,k+1}}}{t_{\theta_{k+1}}} (X_i) \right) - \psi \left( \sqrt{\frac{t_{\theta_{j,k+1}}}{t_{\theta_k}}} (X_i) \right) \right]\]
- \left[ \psi \left( \sqrt{\frac{t_{\theta'_{j,k+1}}}{t_{\theta_{k+1}}} (X_i) \right) - \psi \left( \sqrt{\frac{t_{\theta_{j,k+1}}}{t_{\theta_k}}} (X_i) \right) \right].
\]
It follows from (104) and (102) that
\[
\left\| \psi \left( \sqrt{\frac{t_{\theta_{j,k+1}}}{t_{\theta_{k+1}}}} \right) - \psi \left( \sqrt{\frac{t_{\theta_{j,k}}}{t_{\theta_k}}} \right) \right\|_\infty \leq A_6 \left[ h \left( \theta_{j,k+1}, \theta'_{j,k} \right) + h \left( \theta_{k+1}, \theta_k \right) \right] \\
\leq A_6 2^{1-k/2} \left( 2^{-1/2} + 1 \right) < 7A_6 2^{-k/2-1}\delta,
\]
therefore,
\[
\mathbb{E}_s \left[ \left( \psi \left( \sqrt{\frac{t_{\theta_{j,k+1}}}{t_{\theta_{k+1}}}} \right) - \psi \left( \sqrt{\frac{t_{\theta_{j,k}}}{t_{\theta_k}}} \right) \right)^2 \right] < A_7 2^{-k} \delta^2.
\]

For \( x_{j,k} = (k+1)2^{-k/2+J+j} \Gamma d/100 \) and \( \Gamma d \leq n \), we deduce from Bernstein’s inequality and (105) that
\[
P_s \left[ \begin{array}{c}
sup_{(\theta_k, \theta_{k+1}) \in B_k \times B_{k+1}} \Delta T(X, t_{\theta_k}, t_{\theta_{j,k}}, t_{\theta_{j,k+1}}, t_{\theta'_{j,k+1}}) > x_{j,k} \\
\end{array} \right]
\]
\[
\leq \exp \left[ -C \left( (k+1)2^{J+j+1} \Gamma d \right) \right] \leq \exp \left[ -(k+1) - (j+1) - C2^J \Gamma d \right],
\]
for some \( C > 0 \) and \( J_0 \) large enough (depending on \( S \)).

Putting all these bounds together, we get for some \( C > 0 \) and \( J \) large enough, with probability at least
\[
1 - e^{-C2^J \Gamma d} \left( \sum_{j \geq 1} e^{-j} + \sum_{j \geq 1} e^{-j} \sum_{k \geq 1} e^{-k} \right) \geq 1 - e^{-C2^J \Gamma d},
\]
for all \( j \in \mathbb{N}, \theta \in B \) and \( \theta' \in C_j \),
\[
T(X, t_{\theta}, t_{\theta'}) = \mathbb{E}_s \left[ T(X, t_{\theta}, t_{\theta'}) \right] + T(X, t_{\theta}, t_{\theta'}) - \mathbb{E}_s \left[ T(X, t_{\theta}, t_{\theta'}) \right] \\
= \mathbb{E}_s \left[ T(X, t_{\theta}, t_{\theta'}) \right] + \lim_{k \to +\infty} \left\{ T \left( X, t_{\theta_k}, t_{\theta'_{j,k}} \right) - \mathbb{E}_s \left[ T \left( X, t_{\theta_k}, t_{\theta'_{j,k}} \right) \right] \right\} \\
= \mathbb{E}_s \left[ T(X, t_{\theta}, t_{\theta'}) \right] + T \left( X, t_{\theta_0}, t_{\theta'_{j,0}} \right) - \mathbb{E}_s \left[ T \left( X, t_{\theta_0}, t_{\theta'_{j,0}} \right) \right] \\
+ \sum_{k \in \mathbb{N}} \Delta T \left( X, t_{\theta_k}, t_{\theta'_{j,k}}, t_{\theta_{j,k+1}}, t_{\theta'_{j,k+1}} \right) \\
\leq \mathbb{E}_s \left[ T(X, t_{\theta}, t_{\theta'}) \right] + x_{j,0} + \sum_{k \in \mathbb{N}} x_{j,k} \\
\leq \mathbb{E}_s \left[ T(X, t_{\theta}, t_{\theta'}) \right] + \frac{2^{J+j} \Gamma d}{100} \left( 1 + \sum_{k \geq 0} (k+1)2^{-k/2} \right) \\
< \mathbb{E}_s \left[ T(X, t_{\theta}, t_{\theta'}) \right] + 0.13 \times 2^{J+j+\rho \delta^2}.\]
We conclude by using (12) which implies that, if \((\theta, \theta') \in B \times C_j\),

\[
\begin{align*}
    n^{-1}E_s \left[ T(X, t_\theta, t_{\theta'}) \right] &= \phi(s, t_{\theta'}, t_\theta) - \phi(s, t_\theta, t_{\theta'}) \\
    &\leq \phi(s, t_{\theta'}, t_\theta) - \rho(s, t_{\theta'}) + \rho(s, t_\theta) - \rho(s, t_\theta) \\
    &\leq \frac{1}{\sqrt{2}} \left[ h^2(\vartheta, \vartheta') + h^2(\vartheta, \theta) \right] + h^2(\vartheta, \theta) - h^2(\vartheta, \theta') \\
    &\leq - \left[ \left( 1 - \frac{1}{\sqrt{2}} \right) h^2(\vartheta, \vartheta') - \left( 1 + \frac{1}{\sqrt{2}} \right) h^2(\vartheta, \theta) \right] \\
    &\leq - \left[ 2^J + \frac{1}{\sqrt{2}} \right] \left( 1 - \frac{1}{\sqrt{2}} \right) - \left( 1 + \frac{1}{\sqrt{2}} \right) \delta^2 < -0.255 \times 2^J \delta^2
\end{align*}
\]

provided that \(J_0\) is large enough since \(J \geq J_0\).

Let us now proceed with the proof of Theorem 4. Under Assumption 1, the MLE \(\tilde{\theta}_n\) converges towards the true parameter \(\vartheta\) at rate \(1/\sqrt{n}\). Therefore, given \(\varepsilon > 0\), for \(\Gamma\) large enough depending on \(\varepsilon\), \(h \left( \vartheta, \tilde{\theta}_n \right) \leq \delta = \sqrt{\Gamma d/n}\) with probability larger than \(1 - \varepsilon/2\). We may now apply Proposition 15 with this particular value of \(\Gamma\), provided that \(n \geq \Gamma d\). It follows that, for a suitable choice of \(J \geq J_0\),

\[
\mathbb{P}_s \left[ \sup_{(\theta, \theta') \in \mathcal{E}(\Gamma, J)} T(X, t_\theta, t_{\theta'}) \right. \left. < 0 \right] \geq 1 - \exp\left[ -C2^J \Gamma d \right] \geq 1 - \varepsilon/2.
\]

Therefore

\[
\mathbb{P}_s \left[ \sup_{\theta' \in B} T(X, t_{\tilde{\theta}_n}, t_{\theta'}) < 0 \right] \geq 1 - \varepsilon \quad \text{with} \quad B = \left\{ \theta' \in \Theta \text{ such that } h \left( \vartheta, \theta' \right) \leq 2^{J/2} \delta \right\}.
\]

From now on, we shall work on the event of probability larger than \(1 - \varepsilon\) on which

\[
(106) \quad h \left( \vartheta, \tilde{\theta}_n \right) \leq \delta \quad \text{and} \quad \sup_{\theta' \in B} T\left( X, t_{\tilde{\theta}_n}, t_{\theta'} \right) < 0.
\]

It remains to evaluate \(\sup_{\theta' \in B} T\left( X, t_{\tilde{\theta}_n}, t_{\theta'} \right)\) on this event. For all \(\theta' \in B\), \(h \left( \tilde{\theta}_n, \theta' \right) \leq 2^{(J+2)/2} \delta\). Moreover, using the inequalities

\[
0 \leq \frac{\sqrt{a + b}}{2} - \frac{\sqrt{a} + \sqrt{b}}{2} \leq \frac{\left( \sqrt{b} - \sqrt{a} \right)^2}{4\sqrt{a}} \quad \text{for all } a, b > 0,
\]

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which derive from \(2xy \leq x^2 + y^2\), we get
\[
\left[ \rho \left( t_{\theta'}, \frac{1}{2} \left( t_{\theta_n} + t_{\theta'} \right) \right) - \rho \left( t_{\theta_n'} \frac{1}{2} \left( t_{\theta_n} + t_{\theta'} \right) \right) \right]
\]
\[
= \int \left( \sqrt{t_{\theta'}} - \sqrt{t_{\theta_n'}} \right) \left( \frac{1}{2} \left( t_{\theta_n} + t_{\theta'} \right) - \frac{1}{2} \left( t_{\theta_n} + t_{\theta'} \right) \right) d\mu \nonumber \\
= \int \left( \sqrt{t_{\theta'}} - \sqrt{t_{\theta_n'}} \right) \left[ \sqrt{\frac{1}{2} \left( t_{\theta_n} + t_{\theta'} \right)} - \frac{1}{2} \left( t_{\theta_n} + t_{\theta'} \right) \right] d\mu 
\]
\[
\leq \frac{1}{4} \int \left( \frac{t_{\theta'}}{t_{\theta'}} - \frac{t_{\theta_n'}}{t_{\theta_n'}} \right) \left( \sqrt{t_{\theta'}} - \sqrt{t_{\theta_n'}} \right)^2 d\mu 
\]
\[
\leq \frac{1}{2} \left\| \sqrt{\frac{t_{\theta'}}{t_{\theta'}}} - \sqrt{\frac{t_{\theta_n'}}{t_{\theta_n'}}} \right\|_{\infty} h^2 \left( \tilde{\theta}_n, \theta' \right) \leq \frac{A_4}{2} h^3 \left( \tilde{\theta}_n, \theta' \right)
\]
(107)

by (103) and (102). Besides, when \(u\) converges to 0, \(\psi(1 + u) = (1/\sqrt{2}) \log(1 + u) + O(u^3)\). Setting \(u = \sqrt{t_{\theta'}/t_{\theta_n}} - 1\) so that by (103) \(|u| \leq A_4 h \left( \tilde{\theta}_n, \theta' \right)\), then using the fact that \(\tilde{\theta}_n\) maximizes the likelihood, we derive that
\[
2\sqrt{2} \sum_{i=1}^{n} \psi \left( \sqrt{\frac{t_{\theta'}}{t_{\theta_n}}} (X_i) \right) \leq \sum_{i=1}^{n} \log t_{\theta'} (X_i) - \sum_{i=1}^{n} \log t_{\theta_n} (X_i) + A_8 nh^3 \left( \tilde{\theta}_n, \theta' \right) 
\]
\[
\leq A_8 nh^3 \left( \tilde{\theta}_n, \theta' \right) .
\]

Together with (106) and (107) this shows that, with probability larger than \(1 - \varepsilon\),
\[
\sup_{\theta' \in B} T \left( X, t_{\theta_n}, t_{\theta'} \right) \leq \frac{A_4 + A_8}{4} nh^3 \left( \tilde{\theta}_n, \theta' \right),
\]
hence
\[
\sup_{\theta' \in \Theta} T \left( X, t_{\theta_n}, t_{\theta'} \right) < A_9 \left( 2^J \Gamma d \right)^{3/2} \frac{1}{n}. \tag{108}
\]

Since the mapping \((\theta, \theta') \mapsto T (X, t_{\theta}, t_{\theta'})\) is uniformly continuous on \(\Theta \times \Theta\),
\[
\mathcal{E} = \left\{ t_{\theta} \in \mathcal{S}, \sup_{\theta' \in \Theta} T (X, t_{\theta}, t_{\theta'}) < A_9 \left( 2^J \Gamma d \right)^{3/2} \frac{1}{n} \right\}
\]
is an open subset of \(\mathcal{S}\) hence \(S \cap \mathcal{E}\) is also dense in \(\mathcal{E}\). Besides, \(S \cap \mathcal{E} \subset \mathcal{E}(X, S)\) for \(n\) large enough. Setting \(\text{Cl}(B)\) for the closure of a subset \(B\) in \((\mathcal{L}_0, h)\) and using (108) we get with probability at least \(1 - \varepsilon\)
\[
t_{\theta_n} \in \mathcal{E} = \mathcal{E} \cap \text{Cl}(S \cap \mathcal{E}) \subset \text{Cl}(S \cap \mathcal{E}) \subset \text{Cl}(\mathcal{E}(X, S)),
\]
showing that \(t_{\theta_n}\) is a \(\rho\)-estimator.

9.5. **Proof of Theorem 5.** Inequality (59) is obtained by combining (29) (Assumption 2-(i) corresponds to (28) in the density context) and (58). Consequently, it suffices to prove (58) and to do so we may assume with no loss of generality that \(s = \bar{s} \in S\), which we shall do in the remaining part of this proof.
Let us consider the symmetric family $\mathcal{F} = \mathcal{F}(y)$ defined for $y = \sigma\sqrt{n} > 0$ by

$$\mathcal{F} = \mathcal{F}(y) = \left\{ \psi(\sqrt{t/s}) | t \in \mathcal{B}^S(s,y) \right\} \cup \left\{ -\psi(\sqrt{t/s}) | t \in \mathcal{B}^S(s,y) \right\}.$$  

For all $f \in \mathcal{F}$, $|f| \leq 1$ and it follows from Proposition 14 that for all integers $k \geq 2$, $E_s \left[ |f(X_1)|^k \right] \leq E_s \left[ f^2(X_1) \right] \leq (6\sigma^2)^{k}$. Since $\psi$ is increasing and Lipschitz with Lipschitz constant $L < \sqrt{3}$, it follows from Assumption 2-(ii), (iii) that the family of pairs $\mathcal{T}^S(s, \sigma, \epsilon)$ given by

$$\left\{ \left( \psi(\sqrt{t_L/s}), \psi(\sqrt{t_U/s}) \right), \left( -\psi(\sqrt{t_L/s}), -\psi(\sqrt{t_U/s}) \right), (t_L, t_U) \in I(s, \sigma, \epsilon/\sqrt{2L}) \right\}$$

covers $\mathcal{F}$ with at most $2\exp \left[ \mathcal{U}^S_1(s, \sigma, \epsilon/L\sqrt{2}) \right] \leq \exp \left[ 2\mathcal{U}^S_1(s, \sigma, \epsilon/L\sqrt{2}) \right]$ brackets and that for all integers $k \geq 2$

$$E_s \left[ \left( \psi(\sqrt{t_U/s})(X_1) - \psi(\sqrt{t_L/s})(X_1) \right)^k \right]$$

$$\leq 2^{k-2}E_s \left[ \left( \psi(\sqrt{t_U/s})(X_1) - \psi(\sqrt{t_L/s})(X_1) \right)^2 \right]$$

$$\leq 2^{k-2}L^2 \int_{\mathcal{F}} (\sqrt{t_U} - \sqrt{t_L})^2 d\mu \leq \epsilon^2 \times 2^{k-2}.$$ 

Note that

$$w^S(s, s, y) \leq E_s \left[ \sup_{f \in \mathcal{F}} \left( \sum_{i=1}^{n} (f(X_i) - E_s[f(X_i)]) \right) \right] = E_s \left[ \sup_{f \in \mathcal{F}} \left( \sum_{i=1}^{n} (f(X_i) - E_s[f(X_i)]) \right) \right]$$

and we may therefore apply Theorem 6.8 in Massart (2007) (with $\sigma^2$ replaced by $6\sigma^2 \wedge 1$, $b = 2$, $\delta = \epsilon$ and $\epsilon = L/\sqrt{3} \in (0,1]$) and get

$$w^S(s, s, y) \leq 27L^{-1}\sqrt{6n} \int_{0}^{\sigma L/\sqrt{2}} \sqrt{\mathcal{H}^S_1(s, \sigma, \epsilon/(L\sqrt{2}))} \, d\epsilon + 12\mathcal{H}^S_1(s, \sigma, \sigma\sqrt{3}/L)$$

$$= 54\sqrt{3n} \int_{0}^{\sigma} \sqrt{\mathcal{H}^S_1(s, \sigma, z)} \, dz + 12\mathcal{H}^S_1(s, \sigma, \sigma\sqrt{3}/L).$$

Since $\epsilon \mapsto \mathcal{H}^S_1(s, \sigma, \epsilon)$ is non-increasing, $\mathcal{H}^S_1(s, \sigma, \sigma\sqrt{3}/L) \leq \mathcal{H}^S_1(s, \sigma, \sigma) \leq 2\sigma^2\phi^2(\sigma)$. Let us now choose some $\lambda_0 > 1$. It follows from the definition of $\tau_n$ and the monotonicity of $\sigma \mapsto \phi(\sigma)/\sigma$ that for all $\lambda' \in \lfloor 1, \lambda_0 \rfloor$ and $\sigma \geq \lambda_0\tau_n$

$$\frac{\phi(\sigma)}{\sigma} \leq \frac{\phi(\lambda'\tau_n)}{\lambda'\tau_n} \leq \lambda'\tau_n\sqrt{n} \leq \lambda'\sqrt{n}.$$ 

Letting $\lambda'$ tend to 1 we get $\phi(\sigma) \leq \sigma^2\sqrt{n}/\lambda_0$. Putting these bounds together we get that, for all $y = \sqrt{n}\sigma \geq \lambda_0\sqrt{n}\tau_n$ with $\lambda_0 = 2555$,

$$w^S(s, s, y) \leq 54\sqrt{3n}\phi(\sigma) + 12\sigma^{-2}\phi^2(\sigma) \leq \left( \frac{54\sqrt{3}}{\lambda_0} + \frac{12}{\lambda_0^2} \right) n\sigma^2 \leq c_0y^2/4.$$ 

Finally, $\sup_{s \in S} D^S(s, s) \leq (\lambda_0^2n\sigma_n^2) \vee 1$.  

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9.6. Proof of Proposition 3. It actually derives from the next one applied with \( T = \mathcal{B}^S(s, \Xi, y) \subset \mathcal{B}^S(s, y) \) (so that \(|T| \leq |\mathcal{B}^S(s, y)|\)) and the \( U_{i,t} \) defined by (97). For such a choice, one can take \( H = \log_+(2|\mathcal{B}^S(s, y)|), b = \sqrt{2} \) and \( v^2 = 3y^2 \) (because of Proposition 14).

**Proposition 16.** Let \( T \) be a finite set and \( U_1, \ldots, U_n \) independent random variables with values in \( \mathbb{R}^T \) satisfying for all \( t \in T \)

\[
\max_{i=1, \ldots, n} |U_{i,t}| \leq b \text{ a.s.; } \sum_{i=1}^n \mathbb{E}[U_{i,t}^2] \leq v^2 \quad \text{and} \quad \log_+(2|T|) \leq H
\]

for some positive numbers \( b, v \) and \( H \). Then,

\[
\mathbb{E}\left[ \sup_{t \in T} \left| \sum_{i=1}^n (U_{i,t} - \mathbb{E}[U_{i,t}]) \right| \right] \leq bH + v\sqrt{2H}.
\]

**Proof.** Since the \( U_{i,t} \) are independent for \( i = 1, \ldots, n \) and satisfy (109) for all \( t \in T \), classical computations on the Laplace transform of \( S_{n,t} = \sum_{i=1}^n U_{i,t} - \mathbb{E}[U_{i,t}] \) give, for \( \lambda \in (0, 1/b) \),

\[
\mathbb{E}[\exp(\lambda|S_{n,t}|)] \leq 2 \exp\left( \frac{\lambda^2 v^2}{2(1 - \lambda b)} \right), \quad \text{for all } t \in T.
\]

For a proof of this inequality we refer to inequality (2.21) in Massart (2007). Applying Jensen’s inequality to this bound leads to

\[
\mathbb{E}\left[ \sup_{t \in T} |S_{n,t}| \right] = \frac{1}{\lambda} \log \mathbb{E}\left[ \exp\left( \sup_{t \in T} |S_{n,t}| \right) \right] \leq \frac{1}{\lambda} \log \mathbb{E}\left[ \exp\left( \lambda \sup_{t \in T} |S_{n,t}| \right) \right]
\]

\[
\leq \frac{1}{\lambda} \log \left[ \sum_{t \in T} \mathbb{E}[\exp(\lambda|S_{n,t}|)] \right] \leq \frac{H}{\lambda} + \frac{\lambda v^2}{2(1 - \lambda b)}.
\]

Minimizing the right-hand side with respect to \( \lambda \in (0, 1/b) \) leads to \( \lambda = \sqrt{2H}/\left( v + b\sqrt{2H} \right) \) and finally \( \mathbb{E}\left[ \sup_{t \in T} |S_{n,t}| \right] \leq bH + v\sqrt{2H} \). \( \square \)

9.7. Proof of Proposition 5. Let us denote by \( P_s \) the probability associated to \( s \) on \((\mathcal{F}, \mathcal{A}, \mu)\), by \( J \) the set \( \{ I \in \mathcal{I} \mid \int_I s d\mu > 0 \} \) and by \( S \) an at most countable and dense subset of \( \overline{S}_T \). Let us fix an element \( \Xi = \sum_{I \in \mathcal{I}} \mathbb{I}_I \) in \( S \). For \( t = \sum_{I \in \mathcal{I}} t_I \mathbb{I}_I \) with \( t \in \mathcal{B}^S(s, \Xi, y) \)

\[
\psi\left( \sqrt{\frac{t}{s}}(X_i) \right) = \sum_{I \in J} \psi\left( \sqrt{\frac{t_I}{s_I}} \mathbb{I}_I(X_i) \right) \quad \text{for all } i = 1, \ldots, n
\]

and by Cauchy-Schwarz inequality

\[
S_n(t) = \sum_{i=1}^n \left[ \psi\left( \sqrt{\frac{t}{s}}(X_i) \right) - \mathbb{E}_s \left[ \psi\left( \sqrt{\frac{t}{s}}(X_i) \right) \right] \right] \]

\[
= \sum_{I \in J} \psi\left( \sqrt{\frac{t_I}{s_I}} \right) \sum_{i=1}^n \left[ \mathbb{I}_I(X_i) - \mathbb{E}_s [\mathbb{I}_I(X_i)] \right]
\]

\[
\leq \left( \sum_{I \in J} \psi^2\left( \sqrt{\frac{t_I}{s_I}} \right) P_s(I) \right)^{1/2} \left( \sum_{I \in J} \left[ \sum_{i=1}^n \frac{\mathbb{I}_I(X_i) - \mathbb{E}_s [\mathbb{I}_I(X_i)]}{\sqrt{P_s(I)}} \right]^2 \right)^{1/2}.
\]
By Proposition 14, for all \( t \in \mathcal{B}(s, \bar{s}, y) \)

\[
n \mathbb{E}_s \left[ \psi^2 \left( \sqrt{\frac{t}{s}} X_1 \right) \right] = n \sum_{i \in I} \psi^2 \left( \frac{t_i}{s_i} \right) P_s(I) \leq 6y^2.
\]

Hence,

\[
\sup_{t \in \mathcal{B}(s, \bar{s}, y)} S_n(t) \leq \frac{y \sqrt{6}}{\sqrt{n}} \left| \sum_{i \in J} \left( \sum_{i=1}^n \frac{\mathbb{I}_I(X_i) - \mathbb{E}_s[\mathbb{I}_I(X_i) / s_i]}{\sqrt{P_s(I)}} \right) \right|^{1/2}
\]

and, taking expectations on both sides and using the concavity of the square-root, we get

\[
w^S(s, \bar{s}, y) \leq \frac{y \sqrt{6}}{\sqrt{n}} \sqrt{\sum_{i \in J} \sum_{i=1}^n \frac{P_s(I)}{P_s(I)}} = y \sqrt{6|J|},
\]

which leads to \( D^S(s, \bar{s}) \leq 96c_0^{-2} |J| \). The bound on \( D(S_\xi) \) follows.

9.8. **Proof of Proposition 9.** If \( s \neq \bar{s} \), let \( J \in \mathbb{N} \) such that \( h^2(s, \bar{s}) = 2^{-J} \), \( \Omega_J(X) = \{ \ell(X_i) \leq J \} \) for \( i = 1, \ldots, n \). Since \( s \in \bar{S} \), there exists \( \theta^* \in \Theta \) such that \( s = s_{\theta^*} \) and for \( y \geq 1 \), let us us set

\[
\Theta[\theta^*, y] = \{ \theta \in \Theta \mid s_{\theta} \in \mathcal{B}(s, \bar{s}, y) \}.
\]

We decompose \( w^S(s, \bar{s}, y) \) as

\[
w^S(s, \bar{s}, y) = \mathbb{E}_s \left[ \sup_{t \in \mathcal{B}(s, \bar{s}, y)} |Z(X, s, t)| \right] = \mathbb{E}_s \left[ \sup_{\theta \in \Theta[\theta^*, y]} |Z(X, s, s_{\theta})| \right]
\]

with

\[
w^S_1(s, \bar{s}, y) = \mathbb{E}_s \left[ \sup_{\theta \in \Theta[\theta^*, y]} |Z(X, s, s_{\theta})| \mathbb{I}_{\Omega_J(X)} \right]
\]

and

\[
w^S_2(s, \bar{s}, y) = \mathbb{E}_s \left[ \sup_{\theta \in \Theta[\theta^*, y]} |Z(X, s, s_{\theta})| \mathbb{I}_{\Omega_J(X)^c} \right].
\]

Let us bound from above each of these two quantities, starting with \( w^S_1(s, \bar{s}, y) \).

On the event \( \Omega_J(X) \), \( s(X_i) = \bar{s}(X_i) \) for all \( i = 1, \ldots, n \) and hence,

\[
w^S_1(s, \bar{s}, y) \leq \mathbb{E}_s \left[ \sup_{\theta \in \Theta[\theta^*, y]} \left\| \sum_{i=1}^n \left( \psi \left( \sqrt{\frac{s_{\theta}(X_i)}{s(X_i)}} \right) - \mathbb{E}_s \left[ \psi \left( \sqrt{\frac{s_{\theta}(X_i)}{s(X_i)}} \right) \right] \right) \right\| \right].
\]

For \( \theta \in \Theta \) and \( i = 1, \ldots, n \), either \( J(\theta, \theta^*) \geq \ell(X_i) \) and \( s_{\theta}(X_i) = s(X_i) = 2^{-\ell(X_i)} \) in which case, \( \psi \left( \sqrt{s_{\theta}(X_i)/s(X_i)} \right) = \psi(1) = 0 \), or \( J(\theta, \theta^*) < \ell(X_i) \), \( s_{\theta}(X_i) = 0 \) and then \( \psi \left( \sqrt{s_{\theta}(X_i)/s(X_i)} \right) = \psi(0) = -1 \). Hence, in any case \( -\psi \left( \sqrt{s_{\theta}(X_i)/s(X_i)} \right) = -\psi(-1) = 1 \).
Let us now introduce \( n \) Rademacher random variables \( \varepsilon_1, \ldots, \varepsilon_n \), independent of the \( X_i \). By a symmetrization argument,

\[
\mathbf{w}^S_1(s, \overline{s}, y) \leq 2 \mathbb{E}_s \left[ \sup_{\theta \in \Theta[\theta^*, y]} \left( \sum_{i=1}^n \varepsilon_i \psi \left( \frac{\sqrt{\theta(X_i)}}{s(X_i)} \right) \right) \right] = 2 \mathbb{E}_s \left[ \sup_{\theta \in \Theta[\theta^*, y]} \left( \sum_{i=1}^n \varepsilon_i \mathbb{1}_{J(\theta, \theta^*) < \ell(X_i)} \right) \right].
\]

Let us now work conditionally to \( X_1, \ldots, X_n \) and denote by \( \mathbb{E}_\varepsilon \) the corresponding conditional expectation. Up to a re-ordering of the \( \varepsilon_i \) with no loss of generality, we may assume that \( \ell(X_1) \geq \ell(X_2) \geq \ldots \geq \ell(X_n) \). Then \( \sum_{i=1}^n \varepsilon_i \mathbb{1}_{J(\theta, \theta^*) < \ell(X_i)} \) is necessarily of the form \( \sum_{i=1}^k \varepsilon_i \) for some nonnegative integer \( k = k(\theta, \theta^*, \textbf{X}) \) corresponding to the number of \( X_i \) of length \( \ell(X_i) \) larger than \( J(\theta, \theta^*) \). By (68), for \( \theta' \in \Theta[\theta^*, y] \), \( J(\theta, \theta') \geq \log_2(n/y^2) \), hence \( k \) cannot exceed the number \( \hat{N} \) of \( X_i \) of length not smaller than \( \log_2(n/y^2) \). We deduce that

\[
\max_{0 \leq k \leq \hat{N}} \left| \sum_{i=1}^n \varepsilon_i \mathbb{1}_{J(\theta, \theta^*) < \ell(X_i)} \right| \leq \max_{0 \leq k \leq \hat{N}} \left| \sum_{i=1}^k \varepsilon_i \right|
\]

with the convention \( \sum_{i=1}^0 = 0 \). Taking the average over \( \varepsilon_i \) and using Doob’s maximal inequality, we get

\[
\mathbb{E}_\varepsilon \left[ \sup_{\theta \in \Theta[\theta^*, y]} \left( \sum_{i=1}^n \varepsilon_i \mathbb{1}_{J(\theta, \theta^*) < \ell(X_i)} \right) \right] \leq \mathbb{E}_\varepsilon \left[ \max_{0 \leq k \leq \hat{N}} \left| \sum_{i=1}^k \varepsilon_i \right|^2 \right]^{1/2} \leq 2 \left( \mathbb{E}_\varepsilon \left[ \sum_{i=1}^{\hat{N}} \varepsilon_i^2 \right] \right)^{1/2} = 2 \sqrt{\hat{N}}.
\]

Taking the expectation with respect to \( X_1, \ldots, X_n \), we get

\[
\mathbf{w}^S_1(s, \overline{s}, y) \leq 4 \mathbb{E}_s \left[ \sqrt{\hat{N}} \right] \leq 4 \sqrt{\mathbb{E}_s \left( \sum_{i=1}^n \mathbb{1}_{J(\theta, \theta^*) < \ell(X_i)} \right)} \leq 4 \sqrt{n \mathbb{P}_s [\ell(X_1) > \log_2(n/y^2)]} \leq 4 \sqrt{2^{J(X_1)} - J} \leq 4 \sqrt{n^2 2^{-J}} \leq 4 \sqrt{2y^2}.
\]

As to the quantity \( \mathbf{w}^S_2(s, \overline{s}, y) \), since \( \psi \) is bounded by one and \( y \geq 1 \),

\[
\mathbf{w}^S_2(s, \overline{s}, y) \leq 2n \mathbb{P}_s [\Omega_j(X)] \leq 2n^2 \mathbb{P}_s [\ell(X_1) > J] \leq 2n^2 2^{-J} \leq 2n^2 2^{-J} y = 2n^2 h^2(s, \overline{s}) y.
\]

Putting these bounds together, we get that for all \( y \geq 1 \)

\[
(110) \quad \mathbf{w}(s, \overline{s}, y) = \mathbf{w}^S_1(s, \overline{s}, y) + \mathbf{w}^S_2(s, \overline{s}, y) \leq 2 \left( 2\sqrt{2 + n^2 h^2(s, \overline{s})} \right) y,
\]

which leads to the bound on \( D^S(s, \overline{s}) \).
If \( s = \pi \), we proceed in the same way with \( J = +\infty \) which means that \( \Omega_J(X) = \emptyset \), \( w^S_2(s, \bar{s}, y) = 0 \) and (110) remains valid.

### 9.9. Proof of Proposition 10.

Note that the set of densities \( \{p^\beta, \beta > 0\} \) is a regular statistical model with respect to the parameter \( \beta \). Following Theorem 2.1 p. 121 (equation 2.9) and Section 5 p.133 of the book by Ibragimov and Has’minski˘ı (1981), for \( \beta' > \beta > 0 \)

\[
(111) \quad h^2 \left( p^\beta, p^{\beta'} \right) \leq \left[ \sup_{\beta \leq b \leq \beta'} I(b)(\beta' - \beta)^2 / 8 \right] \wedge 1,
\]

where \( I \) denotes the Fisher Information of the parametric model given by

\[
I(b) = \int_{\mathbb{R}} \left( \frac{p^b(x)}{p^b(x)} \right)^2 dx = 2 \int_{0}^{\infty} \left( \frac{\dot{p}^b(x)}{p^b(x)} \right)^2 dx
\]

and \( \dot{p}^b(x) \) the derivative of \( p^b(x) \) with respect to \( b \). It follows from (82) that

\[
\dot{p}^b(x) = p^b(x) \left[ -\frac{1}{b} - \frac{\Gamma'(b)}{\Gamma(b)} + \frac{x^{-1/b}}{b^2} \log x \right] \quad \text{for } x > 0,
\]

hence

\[
\left( \frac{\dot{p}^b(x)}{p^b(x)} \right)^2 \leq 3p^b(x) \left[ \frac{x^{-1/b}}{b^2} + \left( \frac{\Gamma'(b)}{\Gamma(b)} \right)^2 + \frac{1}{b^2} x^{2/b} (\log x)^2 \right]
\]

and

\[
I(b) \leq 3 \left[ \frac{x^{-1/b}}{b^2} + \left( \frac{\Gamma'(b)}{\Gamma(b)} \right)^2 + \frac{1}{b^2} \Gamma(b) \int_0^\infty e^{-x^{1/b}} x^{2/b} (\log x)^2 dx \right].
\]

Using again a change of variables we get

\[
\int_0^\infty e^{-x^{1/b}} x^{2/b} (\log x)^2 dx = b^3 \int_0^\infty e^{-b u^{b+1}} (\log u)^2 du = b^3 \Gamma''(b+2),
\]

so that finally,

\[
I(b) \leq 3J \quad \text{with} \quad J \leq \frac{1}{b^2} + \left( \frac{\Gamma'(b)}{\Gamma(b)} \right)^2 + \frac{\Gamma''(b+2)}{b^2 \Gamma(b)} = \frac{1}{b^2} + \left( \frac{\Gamma'(b)}{\Gamma(b)} \right)^2 + \frac{(b+1)\Gamma''(b+2)}{b \Gamma(b+2)}.
\]

Binet’s formula for \( \log \Gamma \) (see Whittaker and Watson (1996) page 251) tells us that

\[
\frac{\Gamma'(b)}{\Gamma(b)} = \log b - \frac{1}{2b} - 2 \int_0^\infty \frac{x}{(x^2 + b^2)(e^{2\pi x} - 1)} dx
\]

hence

\[
\frac{\Gamma''(b)}{\Gamma(b)} - \left( \frac{\Gamma'(b)}{\Gamma(b)} \right)^2 = \frac{1}{b} + \frac{1}{2b^2} + 4b \int_0^\infty \frac{x}{(x^2 + b^2)^2 (e^{2\pi x} - 1)} dx.
\]

One should then observe that, since \( e^u \geq 1 + u \),

\[
\int_0^\infty \frac{x}{(x^2 + b^2)(e^{2\pi x} - 1)} dx \leq \frac{1}{2\pi} \int_0^\infty \frac{dx}{(x^2 + b^2)} = \frac{1}{4b}
\]

and

\[
\int_0^\infty \frac{x}{(x^2 + b^2)^2 (e^{2\pi x} - 1)} dx \leq \frac{1}{2\pi} \int_0^\infty \frac{dx}{(x^2 + b^2)^2} = \frac{1}{2\pi b^3} \int_0^\infty \frac{dx}{(x^2 + 1)^2} \leq \frac{1}{4b^2}.
\]
It follows that

\[
\left| \frac{\Gamma'(b)}{\Gamma(b)} \right| \leq \begin{cases} 
\frac{3}{2(b)} & \text{if } 0 < b \leq 1 \\
1 & \text{if } 1 < b \leq 3 \\
\log b & \text{if } b > 3
\end{cases}
\]

and

\[
J \leq \frac{1}{b^2} + \left( \frac{\Gamma'(b)}{\Gamma(b)} \right)^2 + \frac{b+1}{b} \left[ \left( \frac{\Gamma'(b+2)}{\Gamma(b+2)} \right)^2 + \frac{1}{b+2} + \frac{3}{2(b+2)^2} \right].
\]

For \( b \leq 1 \), hence \( 2 < b+2 \leq 3 \), we get

\[
J \leq \frac{13}{4b^2} + \frac{b(b+1)}{b^2} \left[ 1 + \frac{1}{b+2} + \frac{3}{2(b+2)^2} \right] \leq \frac{25}{4b^2};
\]

for \( 1 < b \leq 3 \), hence \( 3 < b+2 \), we get

\[
J \leq \frac{1}{b^2} + 1 + \frac{b+1}{b} \left[ \log b + \frac{1}{b+2} + \frac{3}{2(b+2)^2} \right] \leq 3.812
\]

and for \( b > 3 \),

\[
J \leq \frac{1}{b^2} + (\log b)^2 + \frac{b+1}{b} \left[ (\log(b+2))^2 + \frac{1}{b+2} + \frac{3}{2(b+2)^2} \right] \leq \frac{17}{4}(\log b)^2.
\]

Finally

\[
I(b) \leq \begin{cases} 
\frac{75}{32b^2} < \frac{5b^2}{2} & \text{if } 0 < b \leq 1 \\
1.43 < \frac{3}{2} & \text{if } 1 < b \leq 3 \\
(9/16)(\log b)^2 & \text{if } b > 3
\end{cases}
\]

and our first bound then follows from (111).

Let us now turn to the second inequality.

\[
h^2 \left( p^\beta, p^0 \right) = 1 - \int_{-1}^{1} \sqrt{p^\beta(x)/2} \, dx = 1 - \frac{1}{\sqrt{\Gamma(\beta)}} \int_{0}^{1} e^{-(1/2)x^{1/\beta}} \, dx.
\]

Since

\[
\int_{0}^{1} e^{-(1/2)x^{1/\beta}} \, dx \geq 1 - \int_{0}^{1} (1/2)x^{1/\beta} \, dx = 1 - \frac{\beta}{2(\beta+1)}
\]

and, for \( \beta \leq 1 \),

\[
\Gamma(\beta) = \int_{0}^{\infty} x^{\beta-1}e^{-x} \, dx \leq \int_{0}^{1} x^{\beta-1} \, dx + \int_{1}^{\infty} e^{-x} \, dx = \beta^{-1} + e^{-1},
\]

we get

\[
h^2 \left( p^\beta, p^0 \right) \leq 1 - \left( 1 - \frac{\beta}{2(\beta+1)} \right) \left[ \beta \left( \beta^{-1} + e^{-1} \right) \right]^{-1/2}
\]

\[
\leq 1 - \left( 1 - \frac{\beta}{2(\beta+1)} \right) \left( 1 - \frac{\beta}{2e} \right) \leq \frac{\beta}{2} \left( \frac{1}{\beta+1} + \frac{1}{e} \right)
\]

and our bound follows.

To control of \( w_{p^\beta} \) we observe that for \( \beta > 0 \),

\[
h^2 \left( p^\beta, \frac{1}{\lambda} p^\beta \left( \frac{\lambda}{\lambda} \right) \right) = 1 - \frac{1}{\beta \Gamma(\beta) \sqrt{\lambda}} \int_{0}^{+\infty} e^{-(1/2)x^{1/\beta}} \left( 1 + \lambda^{-1/\beta} \right) \, dx
\]
Using the change of variables \( z = x \left( (1 + \lambda^{-1/\beta})/2 \right) \), and the assumption \( \lambda \in [1, 2] \), we get

\[
\begin{align*}
\frac{1}{\lambda^\beta} \left( \frac{\dot{x}}{\lambda} \right)^{\beta - 1} & = 1 - \frac{1}{\beta \Gamma(\beta) \sqrt{\lambda}} \int_0^{+\infty} 2^\beta \left( 1 + \lambda^{-1/\beta} \right)^{-\beta} e^{-z^\beta} dz \\
& = 1 - \frac{2^\beta}{\sqrt{\lambda} (1 + \lambda^{-1/\beta})\beta} = \left( \frac{2}{\lambda^{1/\beta} + 1} \right)^\beta \left( \left( \frac{\lambda^{1/\beta} + 1}{2} \right)^\beta - \sqrt{\lambda} \right) \\
& \leq \left( \frac{\lambda^{1/\beta} + 1}{2} \right)^\beta - \sqrt{\lambda} \leq \lambda - \sqrt{\lambda} \leq (3/5)(\lambda - 1).
\end{align*}
\]

The particular case of \( \beta = 0 \) is straightforward.

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Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06100 Nice, France.

E-mail address: *baraud@unice.fr*

Univ. Paris VI, CNRS, LPMA, UMR 7599, 75005 Paris, France.

E-mail address: *lucien.birge@upmc.fr*

Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06100 Nice, France.

E-mail address: *mathieu.sart@unice.fr*