EXTREME-STRIKE ASYMPTOTICS FOR GENERAL GAUSSIAN
STOCHASTIC VOLATILITY MODELS

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Abstract. We consider a stochastic volatility stock price model in which the volatility is a
non-centered continuous Gaussian process with arbitrary prescribed mean and covariance.
By exhibiting a Karhunen-Loève expansion for the integrated variance, and using sharp
estimates of the density of a general second-chaos variable, we derive asymptotics for the
stock price density and implied volatility in these models in the limit of large or small strikes.
Our main result provides explicit expressions for the first three terms in the expansion of
the implied volatility, based on three basic spectral-type statistics of the Gaussian process:
the top eigenvalue of its covariance operator, the multiplicity of this eigenvalue, and the $L^2$
norm of the projection of the mean function on the top eigenspace. Strategies for using this
expansion for calibration purposes are discussed.

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chi-squared variates.

1. Introduction

In this article, we characterize the extreme-strike behavior of implied volatility curves for
fixed maturity for uncorrelated Gaussian stochastic volatility models. This introduction con-
tains a careful description of the problem’s background and of our motivations. Before going
into details, we summarize some of the article’s specificities; all terminology in the next two
paragraphs is referenced, defined, and/or illustrated in the remainder of this introduction.

We hold calibration of volatility smiles as a principal motivator. Cognizant of the fact
that non-centered Gaussian volatility models can be designed in a flexible and parsimonious
fashion, we adopt that class of models, imposing no further conditions on the marginal dis-
tribution of the volatility process itself, beyond pathwise continuity. The spectral structure
of second-Wiener chaos variables allows us to work at that level of generality. We find that
the first three terms in the extreme-strike implied volatility asymptotics – which is typically
amply sufficient in applications – can be determined explicitly thanks to three parameters
characterizing the top of the spectral decomposition of the integrated variance. In order to
prove such a precise statement while relying on a moderate amount of technicalities, we make
use of the the simplifying assumption that the stochastic volatility is assumed independent
of the stock price’s driving noise.

When considering the trade-off between this restriction and calibration considerations,
we observe that our model flexibility combined with known explicit spectral expansions

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and numerical tools may allow practitioners to compute the said spectral parameters in a straightforward fashion based on smile features, while also allowing them to select their favorite Gaussian volatility model class. Specific examples of Gaussian volatility processes are non-centered Brownian motion, Brownian bridge, and Ornstein-Uhlenbeck models. This last sub-class can be particularly appealing to practitioners since it contains stationary volatilities, and includes the well-known Stein-Stein model. We also mention how any Gaussian model specification, including long-memory ones, can be handled, thanks to the numerical ability to determine its spectral elements. We understand that the assumption of the stochastic volatility model being uncorrelated implies the symmetry of the implied volatility in the wings, which in some applications, is not a desirable feature; on the other hand, in many option markets, liquidity considerations limit the ability to calibrate using the large-strike wing (see the calibration study on SPX options in [15, Section 5.4]). The case of a correlated Gaussian stochastic volatility model is more complicated, but constitutes an interesting mathematical challenge, which we will investigate separately from this article, since one may need to develop completely new methods and techniques. An important step toward a better understanding of the asymptotic behavior of the implied volatility in some correlated stochastic volatility models is found in the articles [10, 11]. Another problem which is mathematically interesting and important in practice is the asymptotics for implied volatility in small or large time to maturity, on which we report in separate works.

1.1. Background and heuristics. Studies in quantitative finance based on the Black-Scholes-Merton framework have shown awareness of the inadequacy of the constant volatility assumption, particularly after the crash of 1987, when practitioners began considering that extreme events were more likely than what a log-normal model will predict. Propositions to exploit this weakness in log-normal modeling systematically and quantitatively have grown ubiquitous to the point that implied volatility (IV), or the volatility level that market call option prices would imply if the Black-Scholes model were underlying, is now a bona fide and vigorous topic of investigation, both at the theoretical and practical level. The initial evidence against constant volatility simply came from observing that IV as a function of strike prices for liquid call options exhibited non-constance, typically illustrated as a convex curve, often with a minimum near the money as for index options, hence the term ‘volatility smile’.

Stock price models where the volatility is a stochastic process are known as stochastic volatility models; the term ‘uncorrelated’ is added to refer to the submodel class in which the volatility process is independent of the noise term driving the stock price. In a sense, the existence of the smile for any uncorrelated stochastic volatility model was first proved mathematically by Renault and Touzi in [24]. They established that the IV as a function of the strike price decreases on the interval where the call is in the money, increases on the interval where the call is out of the money, and attains its minimum where the call is at the money. Note that Renault and Touzi did not prove that the IV is locally convex near the money, but their work still established stochastic volatility models as a main model class for studying IV; these models continued steadily to provide inspiration for IV studies.

A current emphasis, which has become fertile mathematical ground, is on IV asymptotics, such as large/small-strike, large-maturity, or small-time-to-maturity behaviors. These are helpful to understand and select models based on smile shapes. Several techniques are
used to derive IV asymptotics. For instance, by exploiting a method of moments and the representation of power payoffs as mixtures of a continuum of calls with varying strikes, in a rather model-free context, R. Lee proved in [22] that, for models with positive moment explosions, the squared IV’s large strike behavior is of order the log-moneyness \( \log \left( \frac{K}{S_0} \right) \) times a constant which depends explicitly on supremum of the order of finite moments. A similar result holds for models with negative moment explosions, where the squared IV behaves like \( K \mapsto \log \left( \frac{S_0}{K} \right) \) for small values of \( K \).

More general formulas describing the asymptotic behavior of the IV in the ‘wings’ (\( K \to 0 \) or \( +\infty \)) were obtained in [3, 4, 5, 17, 18, 20, 12] (see also the book [16]).

From the standpoint of modeling, one of the advantages of Lee’s original result is the dependence of IV asymptotics merely on some simple statistics, namely as we mentioned, in the notation in [22], the maximal order \( \tilde{p} \) of finite moments for the underlying \( S_T \), i.e.

\[
\tilde{p}(T) := \sup \{ p \in \mathbb{R} : \mathbb{E} \left[ (S_T)^{p+1} \right] < \infty \}.
\]

This allows the author to draw appropriately strong conclusions about model calibration. A typical class in which \( \tilde{p} \) is positive and finite is that of Gaussian volatility models, which we introduce next.

We consider the stock price model of the following form:

\[
dS_t = rS_t dt + |X_t|S_t dW_t : t \in [0, T],
\]

where the short rate \( r \) is constant, \( X(t) = m(t) + \bar{X}(t) \) with \( m \) an arbitrary continuous deterministic function on \([0, T]\) (the mean function), and \( \bar{X} \) is a continuous centered Gaussian process on \([0, T]\) independent of \( W \), with arbitrary covariance \( Q \). Note that it is not assumed in (35) that the process \( X \) is a solution to a stochastic differential equation as is often assumed in classical stochastic volatility models. A well-known special example of a Gaussian volatility model is the Stein-Stein model introduced in [26], in which the volatility process \( X \) is the so-called mean-reverting Ornstein-Uhlenbeck process satisfying

\[
dx_t = \alpha (m - X_t) dt + \beta dZ_t
\]
simple change of variable, equals

$$
\mathbb{E}[(S_1)^p] = \frac{1}{2\pi \sqrt{1 + p\sigma^2}} \int_{\mathbb{R}^2} dx \, dy \, \exp \left( -\frac{1}{2} \left( y^2 + w^2 - 2 \frac{p\sigma}{\sqrt{1 + p\sigma^2}} wy \right) \right)
$$

which by an elementary computation is finite, and equal to \((1 + p\sigma^2) / (1 + p\sigma^2 - p^2\sigma^2)\), if and only if

$$
p < \tilde{p} + 1 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\sigma^2}}.
$$

In the cases where the random volatility model \(X\) above is non-centered and is correlated with \(W\), a similar calculation can be performed, at the essentially trivial expenses of invoking affine changes of variables, and the linear regression of one normal variate against another.

The above example illustrates heuristically that, by Lee’s moment formula, the computation of \(\tilde{p}\) might be the quickest path to obtain the leading term in the large-strike expansion of the IV, for more complex Gaussian volatility models, namely ones where the volatility \(X\) is time-dependent. However, computing \(\tilde{p}\) is not necessarily an easy task, and appears, perhaps surprisingly, to have been performed rarely. For the Stein-Stein model, the value of \(\tilde{p}\) can be computed using the sharp asymptotic formulas for the stock price density near zero and infinity, established in [19] for the uncorrelated Stein-Stein model, and in [11] for the correlated one. These two papers also provide asymptotic formulas with error estimates for the IV at extreme strikes in the Stein-Stein model. Beyond the Stein-Stein model, little was known about the extreme strike asymptotics of general Gaussian stochastic volatility models. In the present paper, we extend the above-mentioned results from [19] and [11] to such models.

1.2. Motivation and summary of main result. Adopting the perspective that an asymptotic expansion for the IV can be helpful for model selection and calibration, our objective is to provide an expansion for the IV in a Gaussian volatility model relying on a minimal number of parameters, which can then be chosen to adjust to observed smiles. The restriction of non-correlated volatility means that the stock price distribution is a mixture model of geometric Brownian motions with time-dependent volatilities, whose mixing density at time \(T\) is that of the square root of a variable in the second-chaos of a Wiener process. That second-chaos variable is none other than the integrated variance

$$
\Gamma_T := \int_0^T X_t^2 ds.
$$

By relying on a general Hilbert-space structure theorem which applies to the second Wiener chaos, we prove that, in the most general case of a non-centered Gaussian stochastic volatility with a possible degeneracy in the eigenstructure of the covariance \(Q\) of \(X\) viewed as a linear operator on \(L^2([0, T])\) (i.e. when the top eigenvalue \(\lambda_1\) is allowed to have a multiplicity \(n_1\) larger than 1), the large-strike IV asymptotics can be expressed with three terms which depend explicitly on \(T\) and on the following three parameters: \(\lambda_1\), \(n_1\), and the ratio

$$
\delta = \|P_{E_1}m\|^2 / \lambda_1
$$

where \(\|P_{E_1}m\|\) is the norm in \(L^2([0, T])\) of the orthogonal projection of the mean function \(m\) on the first eigenspace of \(Q\). Specifically, with \(I(K)\) the IV as a function of strike \(K\),
letting \( k := \log \frac{K}{s_0} - rT \) be the discounted log-moneyness, as \( k \to +\infty \), we prove

\[
I(K) = L(T, \lambda_1) \sqrt{k} + M(T, \lambda_1, \delta) - \frac{n_1 - 1}{4} \frac{\log k}{\sqrt{k}} + O \left( \frac{1}{\sqrt{k}} \right)
\]

where the constants \( L \) and \( M \) depend explicitly on \( T \) and \( \lambda_1 \), and \( M \) also depends explicitly on \( \delta \). The details of these constants are in Theorem 4 on page 15. A similar asymptotic formula is obtained in the case where \( k \to -\infty \), using symmetry properties of uncorrelated stochastic volatility models (see formula (49) on page 18).

1.3. Practical implications. The first-order constant \( L \) is always strictly positive. The second-order term (the constant \( M \)) vanishes if and only if \( m \) is orthogonal to the first eigenspace of \( Q \), which occurs for instance when \( m \equiv 0 \). The third-order term vanishes if and only if the top eigenvalue has multiplicity one, which is typical. The behavior of \( L \) and \( M \) as functions of \( T \) is determined partly by how the top eigenvalue \( \lambda_1 \) depends on \( T \), which can be non-trivial. In the present paper, we assume \( T \) is fixed.

For fixed maturity \( T \), assuming that \( Q \) has lead multiplicity \( n_1 = 1 \) for instance, a practitioner will have the possibility of determining a value \( \lambda_1 \) and a value \( \delta \) to match the specific root-log-moneyness behavior of small- or large-strike IV; moreover in that case, choosing a constant mean function \( m \), one obtains \( \delta = m^2 \lambda_1^{-1} \int_0^T e_1(t) \, dt \) where \( e_1 \) is the top eigenfunction of \( Q \). Market prices may not be sufficiently liquid at extreme strikes to distinguish between more than two parameters; this is typical of calibration techniques for implied volatility curves for fixed maturity, such as the ‘stochastic volatility inspired’ (SVI) parametrization disseminated by J. Gatheral: see [13, 14] (see also [15] and the references therein). Our result shows that Gaussian volatility models with non-zero mean are sufficient for this flexibility, and provide equivalent asymptotics irrespective of the precise mean function and covariance eigenstructure, since modulo the disappearance of the third-order term in the unit top multiplicity case \( n_1 = 1 \), only \( \lambda_1 \) and \( \delta \) are relevant. Moreover, our Gaussian parametrization is free of arbitrage, since it is based on a semi-martingale model [11]. In the case of SVI parametrization, the absence of arbitrage can be non-trivial, as discussed in [13] and [15].

Modelers wishing to stick to well-known classes of processes for \( X \) may then adjust the value of \( \lambda_1 \) by exploiting any available invariance properties for the desired class. For example, if \( X \) is standard Brownian motion, or the Brownian bridge, on \([0, T]\), we have \( \lambda_1 = 4T^2/\pi \) or \( \lambda_1 = T^2/\pi \) respectively, and these values scale quadratically with respect to a multiplicative scaling constant for \( X \), beyond which an arbitrary mean value \( m \) may be chosen. If \( X \) is the mean-zero stationary OU process, we have \( \lambda_1 = \beta^2 / (\omega_T + \alpha^2) \) where \( \omega_T \) is the smallest positive solution of \( 2\alpha \omega \cos(\omega T) + (\alpha^2 - \omega) \sin(\omega T) = 0 \), in which case, for a fixed arbitrarily selected rate of mean reversion \( \alpha \), a scaling of \( \lambda_1 \) is then equivalent to selecting the variance of \( X \), while a constant mean value \( m \) can then be selected independently. [8, Chapter 1] can be consulted for the eigenstructure of the covariance of Brownian motion and the Brownian bridge, which are classical results, and for a proof of the eigenstructure of the OU covariance. The top eigenfunctions in all three of these cases are known explicit trigonometric functions (see [8, Chapter 1]), and need to be referenced when selecting \( m \). Propositions for Gaussian \( X \) which do not fall in these three classes include non-Markov processes such as fractional Brownian motion which allow for long memory. For example, OU processes
driven by fractional Brownian motion were proposed early on for option pricing, and recently analyzed in [9, 7]. Explicit expressions for $\lambda_1$ are not known in these cases, but efficient numerical techniques exist to compute the eigenfunctions and eigenvalues: see [8, Chapter 2].

The remainder of this article is structured as follows. Section 2 sets up a convenient second-chaos representation for the model’s integrated volatility. Section 3 uses calculations in a proof in [6] to derive precise asymptotics for the density of the mixing distribution. Section 4 converts these asymptotics into two-sided estimates of the density of the stock price $S_T$, thanks to the analytic tools developed in [19, 16]. Sharper asymptotic formulas for the density can also be obtained, but they are not needed to derive sharp asymptotic formulas for the IV with three terms and an error estimate. Finally, in Section 5 we characterize the wing behavior of the implied volatility in Gaussian stochastic volatility models.

2. General setup and second-chaos expansion of the integrated variance

Let $X$ be an almost-surely continuous Gaussian process on a filtered complete probability space $(\Omega, {\mathcal F}, \{F_t\}, P)$ with mean and covariance functions denoted by $m(t) = \mathbb{E}[X_t]$ and

$$Q(t, s) = \text{cov}(X_t, X_s) = \mathbb{E}[(X_t - m(t))(X_s - m(s))],$$

respectively. While such processes used in a jump-free quantitative finance context for volatility modeling will require, in addition, that $X$ be adapted to filtration of the Wiener process $W$ driving the stock-price (as in (1)), under our simplifying assumption that $X$ and $W$ be independent, this adaptability assumption can be considered as automatically satisfied, or equivalently, as unnecessary, since the filtration of $W$ can be augmented by the natural filtration of $X$. Define the centered version of $X$:

$$\tilde{X}_t := X_t - m(t), \quad t \geq 0.$$

Fix a time horizon $T > 0$. It is not hard to see that $Q(s, s) > 0$ for all $s > 0$. Since the Gaussian process $X$ is almost surely continuous, the mean function $t \mapsto m(t)$ is a continuous function on $[0, T]$, and the covariance function $(t, s) \mapsto Q(t, s)$ is a continuous function of two variables on $[0, T]^2$. This is a consequence of the Dudley-Fernique theory of regularity, which also implies that $m$ and $Q$ boast moduli of continuity bounded above by the scale $h \mapsto \log^{-1/2}(h^{-1})$ (see [2]), but this can also be established by elementary means.

In our analysis, it will be convenient to refer to the Karhunen-Loève expansion of $\tilde{X}$. Applying the so-called classical Karhunen-Loève theorem to $\tilde{X}$ (see, e.g., [27, Section 26.1]), we obtain the existence of a non-increasing sequence of non-negative summable reals $\{\lambda_n : n = 1, 2, \ldots\}$, an i.i.d. sequence of standard normal variates $\{Z_n : n = 1, 2, \ldots\}$, and

1 The continuity of the process $X$ implies its continuity in probability on $\Omega$. Hence, the process $X$ is continuous in the mean-square sense (see, e.g., [24, Lemma 1 on p. 5], or invoke the equivalence of $L^p$ norms on Wiener chaos, see [23]). Mean-square continuity of $X$ implies the continuity of the mean function on $[0, T]$. In addition, the autocorrelation function of the process $X$, that is, the function $R(t, s) = \mathbb{E}[X_tX_s]$, $(t, s) \in [0, T]^2$, is continuous (see, e.g., [11, Lemma 4.2]). Finally, since $Q(t, s) = R(t, s) - m(t)m(s)$, the covariance function $Q$ is continuous on $[0, T]^2$. 

a sequence of functions \( \{e_n : n = 1, 2, \ldots \} \) which form an orthonormal system in \( L^2 ([0, T]) \), such that
\[
\tilde{X}_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(t) Z_n.
\]
(3)

In (3), \( \{e_n = e_{n,T} \} \) are the eigenfunctions of the covariance \( Q \) acting on \( L^2 ([0, T]) \) as the following operator
\[
\mathcal{K}(f)(t) = \int_0^T f(s) Q(t, s) ds, \quad f \in L^2 ([0, T]), \quad 0 \leq t \leq T,
\]
and \( \{\lambda_n = \lambda_{n,T} \}, n \geq 1, \) are the corresponding eigenvalues (counting the multiplicities). We always assume that the orthonormal system \( \{e_n\} \) is rearranged so that
\[
\lambda_1 = \lambda_2 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \lambda_{n_1+2} = \cdots = \lambda_{n_1+n_2} > \cdots
\]
In particular, \( \lambda_1 \) is the top eigenvalue, and \( n_1 \) is its multiplicity.

Using (3), we obtain
\[
\int_0^T \tilde{X}_t^2 dt = \int_0^T \left( \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(t) Z_n \right)^2 dt = \sum_{n=1}^{\infty} \lambda_n Z_n^2.
\]
(4)

It is worth pointing out that this expression for the integrated variance of the centered volatility \( \int_0^T \tilde{X}_t^2 dt \) is in fact the most general form of a random variable in the second Wiener chaos, with mean adjusted to ensure almost-sure positivity of the integrated variance. This is established using a classical structure theorem on separable Hilbert spaces, as explained in [23, Section 2.7.4]. In other words (also see [23, Section 2.7.3] for additional details), any prescribed mean-adjusted integrated variance in the second chaos is of the form
\[
V(T) := \int_{[0,T]^2} G(s, t) dZ(s) dZ(t) + 2 \|G\|_{L^2([0,T]^2)}^2
\]
for some standard Wiener process \( Z \) and some function \( G \in L^2 ([0, T]^2) \), and moreover one can find a centered Gaussian process \( \tilde{X} \) such that \( V(T) = \int_0^T \tilde{X}_t^2 dt \) and one can compute the coefficients \( \lambda_n \) in the Karhunen-Loève representation \( 4 \) as the eigenvalues of the covariance of \( \tilde{X} \). When using the non-centered process \( X \), this analysis immediately yields
\[
\int_0^T X_t^2 dt = \int_0^T \left( \tilde{X}_t + m(t) \right)^2 dt
\]
\[
= \sum_{n=1}^{\infty} \lambda_n Z_n^2 + 2 \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left[ \int_0^T m(t) e_n(t) dt \right] Z_n
\]
\[
+ \int_0^T m(t)^2 dt.
\]
(5)

Set
\[
\delta_n = \delta_{n,T} = \int_0^T m(t) e_n(t) dt, \quad n \geq 1,
\]
(6)
and
\[
s = s(T) = \int_0^T m(t)^2 dt.
\]
Then Bessel’s inequality implies that
\[ \sum_{n=1}^{\infty} \delta_n^2 \leq s. \] (8)
Denote
\[ \tau = s - \sum_{n=1}^{\infty} \delta_n^2. \] (9)
It is not hard to see that if the function \( t \mapsto m(t) \) belongs to the image space \( \mathcal{K}(L^2[0, T]) \), then
\[ \sum_{n=1}^{\infty} \delta_n^2 = s, \] (10)
and hence, \( \tau = 0 \). For instance, the previous equality holds for a centered Gaussian process \( X \).
Equality (5) can be rewritten as follows:
\[
\int_0^T X_t^2 \, dt = \sum_{n=1}^{\infty} \lambda_n \left[ Z_n^2 + 2 \frac{\delta_n}{\sqrt{\lambda_n}} Z_n \right] + s
\]
\[
= \sum_{n=1}^{\infty} \lambda_n \left[ Z_n + \frac{\delta_n}{\sqrt{\lambda_n}} \right]^2 + \left( s - \sum_{n=1}^{\infty} \delta_n^2 \right). \] (11)
It follows from (10) and (11) that if the function \( t \mapsto m(t) \) belongs to the image space \( \mathcal{K}(L^2[0, T]) \), then
\[
\int_0^T X_t^2 \, dt = \sum_{n=1}^{\infty} \lambda_n \left[ Z_n + \frac{\delta_n}{\sqrt{\lambda_n}} \right]^2.
\] (12)

Let us denote the noncentral chi-square distribution with the number of degrees of freedom \( k \) and the parameter of noncentrality \( \lambda \) by \( \chi^2(k, \lambda) \) (more information on such distributions can be found in [16] or in any probability textbook). Define a random variable \( \tilde{Z}_n \) by
\[
\tilde{Z}_n = \left[ Z_n + \frac{\delta_n}{\sqrt{\lambda_n}} \right]^2.
\]
It is clear that \( \tilde{Z}_n \) is distributed as \( \chi^2(1, \frac{\delta_n^2}{\lambda_n}) \). Set
\[
\rho_1 = \lambda_1, \quad \rho_2 = \lambda_{n_1+1}, \quad \rho_3 = \lambda_{n_1+n_2+1}, \quad \ldots,
\]
and
\[
\Lambda_T = \frac{1}{\lambda_1} \left( \int_0^T X_t^2 \, dt - \tau \right). \] (13)
Then (11) can be rewritten as follows, where the repeated chi-squared notation is used abusively to denote independent chi-squared random variables:
\[
\Lambda_T = \chi^2 \left( n_1, \frac{1}{\lambda_1} \sum_{n=1}^{n_1} \delta_n^2 \right) + \frac{\rho_2}{\lambda_1} \chi^2 \left( n_2, \frac{1}{\rho_2} \sum_{n=n_1+1}^{n_1+n_2} \delta_n^2 \right) + \ldots. \] (14)
3. Asymptotics of the mixing density

The asymptotic behavior of a random variable such as the one on the right-hand side of equality (14) was studied in [6]. Theorem 2 in [6] provides an asymptotic formula for the complementary distribution function of the random variable mentioned above. A sharper formula for the distribution density $q_T$ of this random variable can be extracted from the proof of Theorem 2 in [6] (see the very end of that proof). Adapting this result to our case and taking into account estimate (8), we see that

$$\left| \frac{q_T(x)}{p_{\chi^2}(x; n_1, 1/\lambda_1 \sum_{n=1}^{n_1} \delta_n^2)} - A \right| = O \left( x^{-\frac{3}{2}} \right)$$

as $x \to \infty$. In (15), the number $A$ is given by

$$A = \prod_{k=2}^{\infty} \left( \frac{\lambda_1}{\lambda_1 - \rho_k} \right)^{n_k} \times \exp \left\{ \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{\lambda_1 - \rho_k} \left( \sum_{n=n_1 + \cdots + n_{k-1} + 1}^{n_1 + \cdots + n_k} \delta_n^2 \right) \right\}. \quad (16)$$

Set

$$\delta = \frac{1}{\lambda_1} \sum_{n=1}^{n_1} \delta_n^2. \quad (17)$$

Then (15) gives

$$q_T(x) = Ap_{\chi^2}(x; n_1, \delta) \left( 1 + O \left( x^{-\frac{3}{2}} \right) \right). \quad (18)$$

For $\lambda > 0$, the following formula is known:

$$p_{\chi^2}(x; n, \lambda) = \frac{1}{\sqrt{\pi n}} \frac{\lambda}{\lambda + \frac{x}{2}} e^{-\frac{\lambda x}{2}} I_{n-1}(\sqrt{\lambda x}), \quad x > 0, \quad (19)$$

where $I_\nu$ is the modified Bessel function of the first kind (see, e.g., [16], Theorem 1.31). It is easy to see that formula (19) and the formula

$$I_\nu(t) = \frac{e^t}{\sqrt{2\pi t}} \left( 1 + O \left( t^{-1} \right) \right), \quad t \to \infty,$$

describing the asymptotic behavior of the I-Bessel function, imply that

$$p_{\chi^2}(x; n, \lambda) = \frac{1}{2\sqrt{\pi}} \lambda^{-\frac{n+1}{2}} e^{\lambda x/2} e^{-\frac{x^2}{4}} \left( 1 + O \left( x^{-\frac{3}{2}} \right) \right) \quad (20)$$

as $x \to \infty$. On the other hand, if $\lambda = 0$, then

$$p_{\chi^2}(x; n, 0) = \frac{1}{2\pi^2} \frac{x^{n/2}}{\Gamma \left( \frac{n}{2} \right)} e^{\frac{x^2}{2}} \exp \left\{ -\frac{x}{2} \right\}, \quad x > 0, \quad (21)$$

(see, e.g., Lemma 1.27 in [16]).
Recall that we denoted by $q_T$ the distribution density of the random variable $\Lambda_T$ defined by (13). It follows from (18) and (20) that

$$q_T(x) = \frac{A}{2\sqrt{2\pi}} \frac{\delta^{-\frac{n_1-3}{4}}}{\lambda_1} x^{-\frac{n_1-3}{2}} e^{\sqrt{\delta} x} e^{-x^2} \left( 1 + O(x^{-\frac{1}{2}}) \right)$$

(22)
as $x \to \infty$. The constants $A$ and $\delta$ in (22) are defined by (16) and (17), respectively. For a centered Gaussian process $X$, (18) and (21) imply that

$$q_T(x) = \frac{A}{2\pi \Gamma\left(\frac{n_1}{2}\right)} x^{\frac{n_1-2}{2}} e^{-\frac{x^2}{\lambda_1}} \left( 1 + O(x^{-\frac{1}{2}}) \right), \quad x > 0,$n

(23)
as $x \to \infty$, where

$$A = \prod_{k=2}^{\infty} \left( \frac{\lambda_1}{\lambda_1 - \rho_k} \right)^{\frac{n_k}{2}}.$$n

(24)

Indeed, in this case, we have $s = 0$, $\delta_n = 0$ for all $n \geq 1$, $\delta = 0$, and $\tau = 0$.

Our main goal is to characterize the asymptotic behavior of the distribution density $p_T$ of the random variable

$$\Gamma_T = \int_0^T X_t^2 dt.$$n

(25)

It follows from (9) and (13) that $\Gamma_T = \lambda_1 \Lambda_T + \tau$. Therefore,

$$p_T(x) = \frac{1}{\lambda_1} q_T \left( \frac{1}{\lambda_1} (x - \tau) \right).$$n

(26)

**Theorem 1.** Let $p_T$ be the distribution density of the random variable $\Gamma_T$ given by (25). If $\delta > 0$, then the following asymptotic formula holds:

$$p_T(x) = C x^{\frac{n_1-3}{2}} \exp \left\{ \sqrt{\delta} \frac{\lambda_1}{x} \right\} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \times \left( 1 + O(x^{-\frac{1}{2}}) \right)$$n

(27)
as $x \to \infty$, where

$$C = \frac{A}{2\sqrt{2\pi}} \lambda_1^{\frac{n_1}{2}} \left( \sum_{n=1}^{n_1} \delta_n \right)^{-\frac{n_1-1}{4}} \exp \left\{ \frac{s - \sum_{n=1}^{\infty} \delta_n^2 - \sum_{n=1}^{n_1} \delta_n^2}{2\lambda_1} \right\}.$$n

(28)

In (27) and (28), the constants $\delta_n$, $s$, $A$, and $\delta$ are defined by (6), (7), (16), and (17), respectively. On the other hand, for a centered Gaussian process $X$, we have

$$p_T(x) = C x^{\frac{n_1-2}{2}} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \left( 1 + O(x^{-\frac{1}{2}}) \right)$$n

(29)
as $x \to \infty$, where

$$C = \frac{A}{2\pi \Gamma\left(\frac{n_1}{2}\right)} \lambda_1^{\frac{n_1}{2}}.$$n

(30)
Proof. Formulas (22) and (26) imply that
\[
p_T(x) = \frac{A}{2\sqrt{2\pi}} \lambda_1^{n_1+1} \delta^{n_1-1} \delta_n^n \exp \left\{ \frac{\tau - \sum_{n=1}^{n_1} \delta_n^2}{2\lambda_1} \right\} \\
\times \left( \sum_{n=1}^{n_1} \delta_n^2 \right)^{-\frac{n_1-1}{4}} \lambda_1^{\frac{n_1-3}{4}} \exp \left\{ \frac{\tau}{\lambda_1} \right\} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \\
\times \left( 1 + O \left( x^{-\frac{1}{2}} \right) \right)
\]
as \[x \to \infty\].

Next, taking into account the formulas
\[
(x - \tau)^{\frac{n_1-3}{4}} = x^{\frac{n_1-3}{4}} (1 + O(x^{-1}))
\]
and
\[
\exp \left\{ \frac{\sqrt{\delta(x - \tau)}}{\lambda_1} \right\} = \exp \left( \frac{\sqrt{\delta}}{\lambda_1^{\frac{1}{2}}} \right) \left( 1 + O(x^{-\frac{1}{2}}) \right),
\]
and simplifying the expression on the right-hand side of (31), we obtain formula (27). The proof of formula (29) is similar. Here we use (23) and (26).

The next assertion follows from Theorem 1.

Corollary 2. Let \(p_T\) be the distribution density of the random variable \(\Gamma_T\) given by (25). Then the following are true:

1. If \(n_1 = 1\), then
\[
p_T(x) = C x^{\frac{1}{2}} \exp \left\{ \frac{\sqrt{\delta}}{\lambda_1} \sqrt{x} \right\} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \\
\times \left( 1 + O \left( x^{-\frac{1}{2}} \right) \right)
\]
as \[x \to \infty\].

2. Suppose \(X\) is a centered Gaussian process with \(n_1 = 1\). Then
\[
p_T(x) = C x^{\frac{1}{2}} \exp \left\{ -\frac{x}{2\lambda_1} \right\} \left( 1 + O \left( x^{-\frac{1}{2}} \right) \right)
\]
as \[x \to \infty\].

4. Stock price asymptotics

In this section, we study stochastic volatility models, for which the volatility is described by the absolute value of a Gaussian process.

Recall that in the present paper we assume that the asset price process \(S\) satisfies the following linear stochastic differential equation:
\[
dS_t = rS_t dt + |X_t|S_t dW_t,
\]

(34)
where $X$ is a continuous Gaussian process on $(\Omega, F, \{F_t\}, P)$. In (34), $W$ is a standard Brownian motion on $(\Omega, F, P)$ with respect to the filtration $\{F_t\}$, and the symbol $r$ stands for the constant interest rate. It will be assumed that the processes $X$ and $W$ are independent.

The initial price of the asset will be denoted by $s_0$.

Since (34) is a linear stochastic differential equation, we have

$$S_t = s_0 \exp \left\{ r t - \frac{1}{2} \int_0^t X_s^2 ds + \int_0^t |X_s| dW_s \right\}.$$  

The previous equality follows from the Doléans-Dade formula (see [25]). Therefore, the discounted asset price process is given by the following stochastic exponential:

$$\tilde{S}_t = e^{-rt}S_t = s_0 \exp \left\{ -\frac{1}{2} \int_0^t X_s^2 ds + \int_0^t |X_s| dW_s \right\}.$$  \hspace{1cm} (35)

The next assertion states that under the restrictions on the volatility process used in the present paper, we are in a risk-neutral environment.

**Lemma 3.** Under the restrictions on the volatility process $X$ in (34), the discounted asset price process $\tilde{S}$ is a $\{F_t\}$-martingale.

**Proof.** Fix a time horizon $T > 0$. It suffices to prove that there exists $\delta > 0$ such that

$$L = \sup_{0 < t \leq T} E[\exp\{\delta X_t^2\}] < \infty$$  \hspace{1cm} (36)

(see, e.g., [16], Corollary 2.11). It will be shown that (36) holds provided that

$$\delta < \frac{1}{2 \max_{0 \leq t \leq T} Q(t, t)}.$$  \hspace{1cm} (37)

We have

$$L = \frac{1}{\sqrt{2\pi}} \sup_{0 < t \leq T} \frac{1}{\sqrt{Q(t, t)}} \int_\mathbb{R} \exp \left\{ \delta y^2 - \frac{1}{2Q(t, t)} (y - m(t))^2 \right\} \exp \left\{ - \left( \frac{1}{2Q(t, t)} - \delta \right) y^2 + \frac{m(t)}{Q(t, t)} y \right\} dy.$$
Set $\alpha(t) = \frac{1}{2Q(t,t)} - \delta$ and $\beta(t) = \frac{m(t)}{Q(t,t)}$. Then, completing the square in the previous exponential, we obtain

$$L \leq \frac{1}{\sqrt{2\pi}} \sup_{0 < t \leq T} \frac{\exp \left\{ -\frac{m(t)^2}{2Q(t,t)} \right\}}{\sqrt{Q(t,t)}} \exp \left\{ \frac{\beta(t)^2}{4\alpha(t)} \right\} \int_{\mathbb{R}} \exp \left\{ -\alpha(t) \left( y - \frac{\beta(t)}{2\alpha(t)} \right)^2 \right\} dy$$

$$= \frac{1}{\sqrt{2\pi}} \sup_{0 < t \leq T} \frac{1}{\sqrt{Q(t,t)\alpha(t)}} \exp \left\{ \frac{\beta(t)^2}{4\alpha(t)} - \frac{m(t)^2}{2Q(t,t)} \right\} \int_{\mathbb{R}} \exp \left\{ -\alpha(t)y^2 \right\} dy$$

$$= a_1 \sup_{0 < t \leq T} \frac{1}{\sqrt{Q(t,t)\alpha(t)}} \exp \left\{ \frac{m(t)^2}{2Q(t,t)} \left[ \frac{1}{1 - 2\delta Q(t,t)} - 1 \right] \right\}$$

$$= a_2 \sup_{0 < t \leq T} \frac{1}{\sqrt{1 - 2\delta Q(t,t)}} \exp \left\{ \frac{\delta m(t)^2}{1 - 2\delta Q(t,t)} \right\}$$

$$\leq a_2 \frac{1}{\sqrt{1 - 2\delta \max_{0 \leq t \leq T} Q(t,t)}} \exp \left\{ \frac{\delta m(t)^2}{1 - 2\delta \max_{0 \leq t \leq T} Q(t,t)} \right\}.$$

Finally, using (37) and the estimate $|m(t)| < M$ for all $0 \leq t \leq T$, we obtain (36).

This completes the proof of Lemma 3. \qed

Since the processes $X$ and $W$ are independent, the following formula holds for the distribution density $D_t$ of the asset price $S_t$:

$$D_t(x) = \sqrt{\frac{S_0 e^{rt}}{2\pi t}} x^{-\frac{3}{2}} \int_{0}^{\infty} y^{-1} \exp \left\{ - \left[ \frac{\log^2 \frac{x}{S_0 e^{rt}}}{2ty^2} + \frac{ty^2}{8} \right] \right\} \tilde{p}_t(y) dy. \quad (38)$$

In (38), $\tilde{p}_t$ is the distribution density of the random variable

$$\tilde{Y}_t = \left\{ \frac{1}{t} \int_{0}^{t} X_s^2 ds \right\}^{\frac{1}{2}}.$$ 

The function $\tilde{p}_t$ is called the mixing density. The proof of formula (38) can be found in [16] (see (3.5) in [16]).

It is not hard to see that

$$\tilde{p}_t(y) = 2typ_t(ty^2), \quad (39)$$
where the symbol $p_t$ stands for the density of the realized volatility $Y_t = \int_0^t X_s^2 ds$. It follows from formula (27) that

$$\tilde{p}_t(y) = \tilde{A} y^{\frac{n-1}{2}} \exp \left\{ \tilde{B} y \right\} \exp \left\{ -\tilde{C} y^2 \right\} \times (1 + O(y^{-1}))$$

as $y \to \infty$, where

$$\tilde{A} = 2 C t^{\frac{n+1}{2}}, \quad \tilde{B} = \sqrt{\frac{\delta t}{\lambda_1}}, \quad \tilde{C} = \frac{t}{2 \lambda_1}.$$  (41)

Our next goal is to estimate the function $D_t$. The asymptotic behavior as $x \to \infty$ of the integral appearing in (38) was studied in [19] (see also Section 5.3 in [19]). It is explained in [19] how to get an asymptotic formula for the integral in (38) in the case where an asymptotic formula for the mixing density is similar to formula (40). We refer the reader to the discussion of Theorem 6.1 in [16], which is based on formula (5.133) in Section 5.6 of [16] and Theorem 5.5 in [16]. The latter theorem concerns the asymptotic behavior of integrals with lognormal kernels. Having obtained an asymptotic formula for the distribution density of the asset price, we can find a similar asymptotic formula for the call pricing function $C$ at large strikes, and then obtain an asymptotic formula for the implied volatility $I$ (see Section 10.5 in [16]). However, the coefficients in the fourth term (and the higher order terms) in the asymptotic expansion of the implied volatility $I$ are complicated and not suitable for computations. For that reason, we will restrict ourselves to asymptotic formulas for the implied volatility with three terms and an error estimate. To obtain such formulas, it is not necessary to use sharp asymptotic for $D_t$ or $C$. It suffices to establish two-sided estimates for $D_t$ and $C$ (see [16], Chapters 9 and 10). The discussion below exploits the following two-sided estimate for the mixing density $\tilde{p}_t$ (this estimate follows from (40):

$$\tilde{p}_t(y) \approx y^{\frac{n-1}{2}} \exp \left\{ \tilde{B} y \right\} \exp \left\{ -\tilde{C} y^2 \right\}$$

as $y \to \infty$. Using formula (42) and reasoning as was described above, we see that for $T > 0$,

$$D_T(x) \approx \left( \log \frac{x}{s_0 e^{rt}} \right)^{\frac{n-1}{2}} \exp \left\{ \frac{\tilde{B} \sqrt{2}}{T^2 (8 \tilde{C} + T)^{\frac{1}{4}}} \sqrt{\log \frac{x}{s_0 e^{rt}}} \right\} \times \left( x^{\frac{1}{2} + \frac{\sqrt{8 \tilde{C} + T}}{2 \sqrt{T}}} \right)$$

as $x \to \infty$. In (43),

$$\tilde{B} = \frac{\sqrt{\delta T}}{\lambda_1} \quad \text{and} \quad \tilde{C} = \frac{T}{2 \lambda_1}.$$  (44)

The proof of (43) uses (42) and is similar to the proof of formula (6.49) on pages 186-187 of [16].

5. Asymptotics of the implied volatility

The call pricing function in the stochastic volatility model described by (34) will be denoted by $C$. We have

$$C(T, K) = e^{-rt} \mathbb{E} \left[ (S_T - K)^+ \right]$$
where $T$ is the maturity and $K$ is the strike price. We will fix $T$, and consider $C$ as the function $K \mapsto C(K)$ of only the strike price $K$. The Black-Scholes implied volatility associated with the pricing function $C$ will be denoted by $I$. More information on the implied volatility can be found in [14, 16].

The next assertion provides an asymptotic formula for the implied volatility in the stochastic volatility model given by (34).

**Theorem 4.** The following formula holds for the implied volatility $K \mapsto I(K)$ as $K \to \infty$:

\[
I(K) = \frac{1}{T^{\frac{3}{4}}} \left[ \left( \sqrt{8\tilde{C} + T + \sqrt{T}} \right)^{\frac{1}{2}} - \left( \sqrt{8\tilde{C} + T - \sqrt{T}} \right)^{\frac{1}{2}} \right] \sqrt{\log \frac{K}{s_0 e^{rT}}} \\
+ \frac{\sqrt{2}\tilde{B}}{T^{\frac{1}{2}(8\tilde{C} + T)^{\frac{1}{4}}}} \left[ \frac{1}{\left( \sqrt{8\tilde{C} + T - \sqrt{T}} \right)^{\frac{1}{2}}} - \frac{1}{\left( \sqrt{8\tilde{C} + T + \sqrt{T}} \right)^{\frac{1}{2}}} \right] \\
+ \frac{1 - n_1 \log \log \frac{K}{s_0 e^{rT}}}{4 \sqrt{\log \frac{K}{s_0 e^{rT}}}} + O \left( \left( \log \frac{K}{s_0 e^{rT}} \right)^{-\frac{1}{2}} \right). \tag{45}
\]

The constants in (45) are defined by

\[
\tilde{B} = \sqrt{T \sum_{n=1}^{n_1} \delta_n^2} \quad \text{and} \quad \tilde{C} = \frac{T}{2\lambda_1}. \tag{46}
\]

**Proof.** The equalities in (46) follow from [17] and [14]. We will use the following assertion which was established in [16] (see Theorem 10.17 in [16]).

**Theorem 5.** Suppose that the density of the asset price $D_T$ satisfies

\[
D_T(x) \approx x^\alpha h(x)
\]

as $x \to \infty$, where $\alpha < -2$ and $h$ is a slowly varying function. Then

\[
I(K) = \frac{\sqrt{2}}{\sqrt{T}} \sqrt{\log K + \log \frac{1}{K^{\alpha+2}h(K)} - \frac{1}{2} \log \log \frac{1}{K^{\alpha+2}h(K)}} \\
- \frac{\sqrt{2}}{\sqrt{T}} \sqrt{\log \frac{1}{K^{\alpha+2}h(K)} - \frac{1}{2} \log \log \frac{1}{K^{\alpha+2}h(K)}} \\
+ O \left( \left( \log K \right)^{-\frac{1}{2}} \right)
\]

as $K \to \infty$.

Set $\alpha = -\left( \frac{3}{2} + \frac{\sqrt{8\tilde{C} + T}}{2\sqrt{T}} \right)$ and

\[
h(x) = \left( \log \frac{x}{s_0 e^{rT}} \right)^{\frac{n_1 - 3}{4}} \exp \left\{ \frac{\tilde{B} \sqrt{2}}{T^{\frac{1}{4}}(8\tilde{C} + T)^{\frac{1}{4}}} \sqrt{\log \frac{x}{s_0 e^{rT}}} \right\}.
\]
It is easy to see that $\alpha < -2$ and $h$ is a slowly varying function. Next, taking into account (43) and applying Theorem 5, we obtain

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}} \sqrt{(-\alpha - 1) \log \frac{K}{s_0 e^r T} + \log \frac{1}{h(K)} - \frac{1}{2} \log \log \frac{1}{K^{\alpha + 2} h(K)}}$$

$$- \frac{\sqrt{2}}{\sqrt{T}} \sqrt{(-\alpha - 2) \log \frac{K}{s_0 e^r T} + \log \frac{1}{h(K)} - \frac{1}{2} \log \log \frac{1}{K^{\alpha + 2} h(K)}}$$

$$+ O \left( \left( \log \frac{K}{s_0 e^r T} \right)^{-\frac{1}{2}} \right)$$

as $K \to \infty$. Now, using the mean value theorem, we get

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}} \sqrt{(-\alpha - 1) \log \frac{K}{s_0 e^r T} + \log \frac{1}{h(K)} - \frac{1}{2} \log \log \frac{K}{s_0 e^r T}}$$

$$- \frac{\sqrt{2}}{\sqrt{T}} \sqrt{(-\alpha - 2) \log \frac{K}{s_0 e^r T} + \log \frac{1}{h(K)} - \frac{1}{2} \log \log \frac{K}{s_0 e^r T}}$$

$$+ O \left( \left( \log \frac{K}{s_0 e^r T} \right)^{-\frac{1}{2}} \right)$$

$$= \frac{\sqrt{2}}{\sqrt{T}} \sqrt{-\alpha - 1} \sqrt{\log \frac{K}{s_0 e^r T} \sqrt{1 + s_1(K)}}$$

$$- \frac{\sqrt{2}}{\sqrt{T}} \sqrt{-\alpha - 2} \sqrt{\log \frac{K}{s_0 e^r T} \sqrt{1 + s_2(K)}}$$

$$+ O \left( \left( \log \frac{K}{s_0 e^r T} \right)^{-\frac{1}{2}} \right)$$

(47)

as $K \to \infty$, where

$$s_1(K) = \frac{\log \frac{1}{h(K)} - \frac{1}{2} \log \log \frac{K}{s_0 e^r T}}{(-\alpha - 1) \log \frac{K}{s_0 e^r T}}$$

and

$$s_2(K) = \frac{\log \frac{1}{h(K)} - \frac{1}{2} \log \log \frac{K}{s_0 e^r T}}{(-\alpha - 2) \log \frac{K}{s_0 e^r T}}.$$
Next, using the formula $\sqrt{1+s} = 1 + \frac{1}{2}s + O(s^2)$ as $s \to 0$ in (47), we obtain

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}} \left[ \sqrt{-\alpha - 1} - \sqrt{-\alpha - 2} \right] \sqrt{\log \frac{K}{S_0 e^{rT}}}$$

$$+ \frac{1}{\sqrt{2T}} \left[ \frac{1}{\sqrt{-\alpha - 1}} - \frac{1}{\sqrt{-\alpha - 2}} \right] \log \frac{1}{\eta(K)} - \frac{1}{2} \log \log \frac{K}{S_0 e^{rT}}$$

$$+ O \left( \left( \log \frac{K}{S_0 e^{rT}} \right)^{-\frac{1}{2}} \right)$$

$$= \frac{\sqrt{2}}{\sqrt{T}} \left[ \sqrt{-\alpha - 1} - \sqrt{-\alpha - 2} \right] \sqrt{\log \frac{K}{S_0 e^{rT}}}$$

$$+ \left[ \frac{1}{\sqrt{-\alpha - 2}} - \frac{1}{\sqrt{-\alpha - 1}} \right] \frac{\widetilde{B}}{T^4 (8\widetilde{C} + T)^{\frac{3}{2}}}$$

$$+ \frac{1 - n_1}{4} \log \log \frac{K}{S_0 e^{rT}} + O \left( \left( \log \frac{K}{S_0 e^{rT}} \right)^{-\frac{1}{2}} \right)$$

as $K \to \infty$. Finally, taking into account the definition of the parameter $\alpha$, we see that (45) holds.

The proof of Theorem 4 is thus completed. \(\square\)

In terms of the log-moneyness $k = \log \frac{K}{S_0 e^{rT}}$, Theorem 4 can be formulated as follows.

**Theorem 6.** The following formula holds for the implied volatility $k \mapsto I(k)$ as $k \to \infty$:

$$I(k) = \frac{1}{T^4} \left[ \left( \sqrt{8\widetilde{C} + T + \sqrt{T}} \right)^{\frac{1}{2}} - \left( \sqrt{8\widetilde{C} + T - \sqrt{T}} \right)^{\frac{1}{2}} \right] \sqrt{k}$$

$$+ \frac{\sqrt{2} \widetilde{B}}{T^4 (8\widetilde{C} + T)^{\frac{3}{2}}} \left[ \frac{1}{\left( \sqrt{8\widetilde{C} + T - \sqrt{T}} \right)^{\frac{1}{2}}} - \frac{1}{\left( \sqrt{8\widetilde{C} + T + \sqrt{T}} \right)^{\frac{1}{2}}} \right]$$

$$+ \frac{1 - n_1}{4} \log \frac{k}{\sqrt{k}} + O \left( k^{-\frac{1}{2}} \right). \quad (48)$$

Note that if $n_1 = 1$, then the third term in the asymptotic expansion of the implied volatility in formula (45) vanishes. On the other hand, if the Gaussian process $X$ is centered, then $\widetilde{B} = 0$, and the second term on the tight-hand side of (45) vanishes. The next corollary takes those facts into account.
Corollary 7. The following are true:

(i) If $n_1 = 1$, then as $K \to \infty$,

$$I(K) = \frac{2\lambda_1^{\frac{1}{2}}}{T^{\frac{1}{2}} \left[ (\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} \right]} \sqrt{\log \frac{K}{s_0e^{rT}}}$$

$$+ \frac{\sqrt{2}\delta_1}{\sqrt{T}(4 + \lambda_1)^{\frac{1}{2}}} \left[ \frac{1}{(\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}}} - \frac{1}{(\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}}} \right]$$

$$+ O \left( \left( \log \frac{K}{s_0e^{rT}} \right)^{-\frac{1}{2}} \right).$$

(ii) If $X$ is a centered Gaussian process, then as $K \to \infty$,

$$I(K) = \frac{2\lambda_1^{\frac{1}{2}}}{T^{\frac{1}{2}} \left[ (\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} \right]} \sqrt{\log \frac{K}{s_0e^{rT}}}$$

$$+ \frac{1 - n_1}{4} \frac{\log \log \frac{K}{s_0e^{rT}}}{\sqrt{\log \frac{K}{s_0e^{rT}}}} + O \left( \left( \log \frac{K}{s_0e^{rT}} \right)^{-\frac{1}{2}} \right).$$

(iii) If $X$ is a centered Gaussian process and $n_1 = 1$, then as $K \to \infty$,

$$I(K) = \frac{2\lambda_1^{\frac{1}{2}}}{T^{\frac{1}{2}} \left[ (\sqrt{4 + \lambda_1} + \sqrt{\lambda_1})^{\frac{1}{2}} + (\sqrt{4 + \lambda_1} - \sqrt{\lambda_1})^{\frac{1}{2}} \right]} \sqrt{\log \frac{K}{s_0e^{rT}}}$$

$$+ O \left( \left( \log \frac{K}{s_0e^{rT}} \right)^{-\frac{1}{2}} \right).$$

The constants in Corollary 7 are obtained from those in Theorem 4 by taking into account (46). These constants depend only on the following characteristics of the process $X$: The largest eigenvalue $\lambda_1$ of the covariance operator $K$ on $[0, T]$, and the Fourier coefficient $\delta_1$ of the mean function $m$ with respect to the corresponding eigenfunction $e_1$. We refer to Section 1.3 for further discussion.

Since the processes $X$ and $W$ in (34) are independent, the stochastic volatility model described in (34) belongs to the class of the so-called symmetric models (see Section 9.8 in [16]). It is known that for symmetric models,

$$I(K) = I \left( \frac{(s_0e^{rT})^2}{K} \right) \quad \text{for all} \quad K > 0. \quad (49)$$

The next statements can be established using Theorem 4, Corollary 7 and formula (49).
Theorem 8. The following formula holds for the function $K \mapsto I(K)$ as $K \to 0$:

$$I(K) = \frac{1}{T^{\frac{1}{2}}} \left[ \left( \sqrt{8C + T + \sqrt{T}} \right)^{\frac{3}{2}} - \left( \sqrt{8C + T - \sqrt{T}} \right)^{\frac{3}{2}} \right] \sqrt{\log \frac{s_0 e^{rT}}{K}}$$

$$+ \frac{\sqrt{2B}}{T^{\frac{1}{2}}(8C + T)^{\frac{1}{2}}} \left[ \frac{1}{\left( \sqrt{8C + T - \sqrt{T}} \right)^{\frac{1}{2}}} - \frac{1}{\left( \sqrt{8C + T + \sqrt{T}} \right)^{\frac{1}{2}}} \right]$$

$$+ \frac{1}{4} \frac{n_1 \log \log \frac{s_0 e^{rT}}{K}}{\sqrt{\log \frac{s_0 e^{rT}}{K}}} + O \left( \left( \log \frac{s_0 e^{rT}}{K} \right)^{-\frac{1}{2}} \right).$$

(50)

The constants in (50) are the same as in Theorem 4.

Corollary 9. The following are true:

(i) If $n_1 = 1$, then as $K \to 0$,

$$I(K) = \frac{2\lambda_1^{\frac{1}{2}}}{T^{\frac{1}{2}}} \left[ \left( \sqrt{4 + \lambda_1} + \sqrt{\lambda_1} \right)^{\frac{3}{2}} + \left( \sqrt{4 + \lambda_1} - \sqrt{\lambda_1} \right)^{\frac{3}{2}} \right] \sqrt{\log \frac{s_0 e^{rT}}{K}}$$

$$+ \frac{\sqrt{2\lambda_1}}{\sqrt{T}(4 + \lambda_1)^{\frac{1}{2}}} \left[ \frac{1}{\left( \sqrt{4 + \lambda_1} - \sqrt{\lambda_1} \right)^{\frac{1}{2}}} - \frac{1}{\left( \sqrt{4 + \lambda_1} + \sqrt{\lambda_1} \right)^{\frac{1}{2}}} \right]$$

$$+ \frac{1}{4} \frac{n_1 \log \log \frac{s_0 e^{rT}}{K}}{\sqrt{\log \frac{s_0 e^{rT}}{K}}} + O \left( \left( \log \frac{s_0 e^{rT}}{K} \right)^{-\frac{1}{2}} \right).$$

(ii) If $X$ is a centered Gaussian process, then as $K \to 0$,

$$I(K) = \frac{2\lambda_1^{\frac{1}{2}}}{T^{\frac{1}{2}}} \left[ \left( \sqrt{4 + \lambda_1} + \sqrt{\lambda_1} \right)^{\frac{3}{2}} + \left( \sqrt{4 + \lambda_1} - \sqrt{\lambda_1} \right)^{\frac{3}{2}} \right] \sqrt{\log \frac{s_0 e^{rT}}{K}}$$

$$+ \frac{1}{4} \frac{n_1 \log \log \frac{s_0 e^{rT}}{K}}{\sqrt{\log \frac{s_0 e^{rT}}{K}}} + O \left( \left( \log \frac{s_0 e^{rT}}{K} \right)^{-\frac{1}{2}} \right).$$

(iii) If $X$ is a centered Gaussian process and $n_1 = 1$, then as $K \to 0$,

$$I(K) = \frac{2\lambda_1^{\frac{1}{2}}}{T^{\frac{1}{2}}} \left[ \left( \sqrt{4 + \lambda_1} + \sqrt{\lambda_1} \right)^{\frac{3}{2}} + \left( \sqrt{4 + \lambda_1} - \sqrt{\lambda_1} \right)^{\frac{3}{2}} \right] \sqrt{\log \frac{s_0 e^{rT}}{K}} + O \left( \left( \log \frac{s_0 e^{rT}}{K} \right)^{-\frac{1}{2}} \right).$$

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EXTREME-STRIKE ASYMPTOTICS FOR GENERAL GAUSSIAN STOCHASTIC VOLATILITY MODELS

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