SIMPLICITY OF EIGENVALUES AND NON-VANISHING OF EIGENFUNCTIONS OF A QUANTUM GRAPH

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Abstract. We prove that after an arbitrarily small adjustment of edge lengths, the spectrum of a compact quantum graph with δ-type vertex conditions can be simple. We also show that the eigenfunctions, with the exception of those living entirely on a looping edge, can be made to be non-vanishing on all vertices of the graph.

As an application of the above result, we establish that the secular manifold (also called “determinant manifold”) of a large family of graphs has exactly two smooth connected components.

1. Introduction

A quantum graph is a metric graph equipped with a self-adjoint differential operator (usually of Schrödinger type) defined on the edges and matching conditions specified at the vertices. Every edge of the graph has a length assigned to it.

One of the fundamental questions of the spectral theory is that of presence in the spectrum of degenerate (or repeated) eigenvalues. In particular, it is usually the case that within a rich enough set of problems, the problems with degenerate eigenvalues form a small subset. In other words, unless a system has symmetries (which usually force degeneracy in the spectrum, see, for example, [20]), it is highly unlikely to have degenerate eigenvalues.

Mathematically, a classical result by Uhlenbeck [18] (see also [19] for a generalization) establishes generic simplicity of eigenvalues of the Laplace-Beltrami operator on compact manifolds, with respect to the set of all possible metrics on the manifold. Some generic properties of eigenfunctions are also established. Since then, various extensions and generalizations of this result have been proven for different circumstances (see, for example, [14] and references therein).

On graphs, the question of simplicity of eigenvalues was considered by Friedlander in [13], who proved that the eigenvalues are simple generically with respect to the perturbation of the edge lengths of the graph. The proof is based on perturbation theory and applies to graphs with Neumann–Kirchhoff (NK) conditions only (see Section 2 for the definitions). When this article was in preparation, an outline of a shorter proof, under the same conditions, was released by Colin de Verdière [11].

In this work we consider a wider range of vertex conditions, namely the δ-type conditions on vertices of the graph. Furthermore, we also investigate the eigenfunctions, showing that generically they do not vanish on vertices, unless this is unavoidable due to presence of looping edges. Both of these results are important in applications, in particular all recent results on the number of zeros of graph eigenfunctions assume both the simplicity of eigenvalues and non-vanishing of eigenfunctions on vertices as a precondition (see [6,4,2,1,10] and references therein).

In the proof, the simplicity of eigenvalues and non-vanishing of eigenfunctions are tightly interconnected; each property is assisting in the proof of the other (the proof is done by
induction). The proof is geometric in nature and uses local modifications of the graph to reduce it to previously considered case. In Section 6 of the paper we also consider an application of the result to the study of the secular manifold of a graph, showing that for large classes of graphs, the set of smooth points of the manifold has exactly two connected components.

We remark that from the general consideration one can deduce the result for generic choices of the vertex conditions. The challenge is to obtain it for a fixed choice of vertex conditions (and a generic choice of edge lengths). The existing proofs cannot be readily re-used for this purpose. While the original proof due to Friedlander [13] is very technical, the simpler proof by Colin de Verdière [11] relies on the properties of the so-called “secular manifold” for quantum graphs which does not exist for general \( \delta \)-type conditions. Finally, we mention a result of Exner and Jex, where the simplicity of the ground state eigenvalue and positivity of the corresponding eigenfunction was established for graph with non-repulsive \( \delta \)-type conditions [12].

2. Quantum graph Hamiltonian

We start by defining the quantum graph, following the notational conventions of [9]. Let \( \Gamma = (V, E) \) be a connected metric graph with a set of vertices \( V = \{v_j\} \) and edges \( E = \{e_j\} \). Both sets \( V \) and \( E \) are assumed to be finite and the edges are of bounded length. We allow multiple edges between a given pair of vertices and the edges that loop from a vertex to itself (see also Remark 2.1 below).

A function \( f \) on \( \Gamma \) is a collection of functions \( f_e(x) \) defined on each edge \( e \). Consider the Laplace operator \( H \) defined by

\[
H : f \mapsto -\frac{d^2 f}{dx^2},
\]

acting on the functions that belong to the Sobolev \( H^2(e) \) space on each edge \( e \) and satisfy the \( \delta \)-type boundary conditions with coefficients \( \alpha_v \) at the vertices of the graph,

\[
\begin{align*}
  f(x) & \text{ is continuous at } v \\
  \sum_{e \in E_v} \frac{df}{dx_e}(v) & = \alpha_v f(v),
\end{align*}
\]

where for each vertex \( v \), the corresponding vertex condition \( \alpha_v \) is a fixed real number. The set \( E_v \) is the set of edges joined at the vertex \( v \); by convention, each derivative at a vertex is taken into the corresponding edge. We will often encounter the special case with \( \alpha_v = 0 \), which is known as the Neumann–Kirchhoff (NK) condition. The special value \( \alpha_v = \infty \) should be taken to mean the Dirichlet condition \( f(v) = 0 \). Such condition will only be allowed at vertices of degree 1, as it effectively disconnects the edges if imposed at a vertex of degree 2 or higher. Conditions with \( \alpha_v \neq 0, \infty \) will be called Robin-type.

Remark 2.1. NK condition (equation (2.1) with \( \alpha_v = 0 \)) at a vertex of degree 2 is equivalent to \( f \) being continuously differentiable at \( v \). Therefore, a graph with an NK vertex of degree 2 is equivalent to a graph which has no vertex at this location, just a continuous edge, see Fig. 1. We will often use this fact in reverse, choosing a point on an edge and declaring it to be a vertex of degree 2 with NK condition. We will call such a vertex a trivial vertex.

Note that by introduction of such trivial vertices, a graph with multiple or looping edges may be converted into a simple graph.
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\[ \alpha_{v_1} = 0 \quad \iff \quad \alpha_{v_2} = 0 \]

Figure 1. If the vertex conditions \( \alpha_{v_1} = \alpha_{v_2} = 0 \) with \( \text{deg}(v_1) = \text{deg}(v_2) = 2 \), the loop is equivalent to a looping edge.

3. MAIN RESULTS

The question we address here is when is it typical (with respect to variation of edge lengths) for a graph to have simple spectrum and to have eigenfunctions that do not vanish on vertices of the graph. To motivate our results, we first consider examples which turn out to be the only cases one needs to take special care about.

**Definition 3.1.** A loop is a chain of vertices

\[ v, v_1, \ldots, v_n, v, \]

connected by edges, with each of the intermediate vertices \( v_1, \ldots, v_n \) having degree 2. We include the possibility of having \( n = 0 \), in which case \( v \) is connected to itself by a looping edge.

By Remark 2.1, a looping edge is equivalent to a loop with intermediate vertices with \( \alpha_{v_j} = 0 \), see Fig. 1.

**Example 3.2.** Let \( L \) be a graph consisting of one looping edge with no vertices. We will call such a graph a circle. By Remark 2.1 it can be equivalently represented as a cycle graph (a number of vertices connected into a closed chain) with all vertices having \( \alpha_v = 0 \). It is easy to see that the spectrum of the graph is

\[ 0, \left( \frac{2\pi}{\ell} \right)^2, \left( \frac{2\pi}{\ell} \right)^2, \left( \frac{4\pi}{\ell} \right)^2, \left( \frac{4\pi}{\ell} \right)^2, \left( \frac{6\pi}{\ell} \right)^2, \ldots \]

where \( \ell \) is the length of the looping edge. We note that the double degeneracies in the spectrum cannot be resolved by changing the edge length.

The eigenfunctions can be represented as

\[(3.1) \quad C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x) = A \sin(\sqrt{\lambda} x + \theta), \]

with constants \( C_1 \) and \( C_2 \) (or \( A \) and \( \theta \)) arbitrary. Here \( \lambda \) is the eigenvalue and the origin \( x = 0 \) can be put in an arbitrary location on the graph. It is important to note that for any eigenvalue \( \lambda \neq 0 \) and any point on the graph, there is an eigenfunction which vanishes at that point.

**Example 3.3.** Consider a graph \( \Gamma \) with a looping edge \( L \), see Fig. 1. We assume that there are no other (non-trivial) vertices on the loop. The condition at the attachment point \( v \) is of \( \delta \)-type with arbitrary \( \alpha_v \).

If \( \ell \) is the length of the loop, then \( (2\pi n/\ell)^2 \) is an eigenvalue of \( \Gamma \) for any integer \( n > 0 \). We demonstrate this by constructing an eigenfunction of \( \Gamma \). On the loop we take the function \( f \)
to be equal to the eigenfunction of the corresponding circle, equation (3.1), chosen to vanish at the attachment point \( v \). The function \( f \) is extended to the rest of the graph \( \Gamma \) by setting it to 0 identically. This obviously makes \( f \) continuous and, since \( f \) is an eigenfunction with respect to the loop,

\[
(3.2) \quad \sum_{e \in E(\Gamma)} \frac{df}{dx_e}(v) = \sum_{e \in E(L)} \frac{df}{dx_e}(v) = 0 = \alpha f(v),
\]

for any \( \alpha \); the second summation is performed only over the edge-ends that belong to the loop \( L \). We thus have an eigenfunction of \( \Gamma \) which is supported exclusively on the loop \( L \); in particular it is zero on all vertices of \( \Gamma \). Moreover, such an eigenfunction cannot be destroyed by changing the lengths of graph \( \Gamma \). We also note, that the eigenfunction that is supported exclusively on a given loop is unique (for a given value of \( \lambda \)). This can be easily seen as the eigenfunction satisfies the Dirichlet problem on the looping edge.

It turns out that having no other vertices on the loop is an essential feature of Example 3.3.

**Lemma 3.4.** Let \( \Gamma \) be a graph with \( \delta \)-type conditions at vertices. Suppose \( L \) is loop in \( \Gamma \) which has at least one vertex with \( \alpha_v \neq 0 \) on it, other than the attachment vertex. Then there is a small modification of edge lengths of \( \Gamma \), after which \( \Gamma \) has no eigenfunctions \( f \) supported exclusively on the loop \( L \).

This lemma, proved in Section 5.1, motivates the following definition.

**Definition 3.5.** A pure loop is a loop with no vertices \( v_j \) having \( \alpha_{v_j} \neq 0 \), other than, possibly, the attachment point \( v \). In fact, in what follows, by a “loop” we will always mean a pure loop, unless explicitly stated otherwise. As mentioned already, a graph consisting of one pure loop is called a circle; a graph consisting of an impure loop will be called an impure loop graph.

Now we are able to formulate our main result.

**Theorem 3.6.** Let \( \Gamma \) be a connected graph with \( \delta \)-type conditions at vertices. If \( \Gamma \) is not equivalent to a circle, then, after a small modification of edge lengths, the new graph \( \tilde{\Gamma} \) will satisfy the following genericity conditions

(i) \( \sigma(\tilde{\Gamma}) \) is simple, and
(ii) for each eigenfunction \( f \) of \( \tilde{\Gamma} \),

(a) either \( f(v) \neq 0 \) for each vertex \( v \), or
(b) \( \text{supp } f = L \) for only one loop \( L \) of \( \tilde{\Gamma} \).

More precisely, in the space of all possible edge lengths, the set on which the above conditions are satisfied is residual (comeagre).

**Remark 3.7.** A residual or comeagre set is a countable intersection of sets with dense interiors. Informally, a residual set is “large”. In particular, since all spaces we will be dealing with (namely, the space of all possible lengths or the space of all points on a graph) are complete metric spaces, by Baire Category Theorem a residual set is dense. In particular, it implies that the adjustments in the edge lengths, promised in the Theorem, can be chosen arbitrarily small.

In general, the converse is not true, a dense set is not necessarily residual (for example, the set of rationals is dense yet meagre). But in our case, it is in fact enough to establish the Theorem for a dense set of lengths. Indeed, once that has been established, one can argue as...
follows. For any $N$, the set of lengths on which the conclusions of the Theorem are true for all $n < N$ is still dense but it is also open: both eigenvalues and eigenfunctions are continuous under change of lengths \([8, 9]\) and a small perturbation will preserve strict inequalities between the eigenvalues, and between values of $f(v)$ and 0. Taking the intersection of open dense sets over countably many $N$ we obtain a residual set.

4. Preliminary observations

Before we start the proof of the main theorem, we make some observations.

**Remark 4.1.** If $\lambda = 0$ and a graph has no Robin-type vertices (i.e. only $\alpha_v = 0$ or $\alpha_v = \infty$ are allowed), it can be easily shown (see, e.g. [17, Thm 1]) that the corresponding eigenfunction $f$ of $\Gamma$ is a constant on every edge. Therefore the multiplicity of 0 in the spectrum is at most the number of connected components in the graph, which is one in our case. If there is a vertex with Dirichlet condition $\alpha_v = 0$, the value 0 is not an eigenvalue. Hence, Theorem 3.6 holds for $\lambda = 0$ on graphs with no Robin-type conditions.

The same proof does not apply to graphs with some $\alpha_v \neq 0, \infty$, which introduces a layer of complication into the proof of our main result.

The proof of Theorem 3.6 is built around modifications made to the structure of a graph. The following theorem describes one of the modifications we find useful and its effect on the spectrum. We denote by $\Gamma_\alpha$ a compact quantum graph with a distinguished vertex $v$. Arbitrary self-adjoint conditions are fixed at all vertices other than $v$, while $v$ is endowed with the $\delta$-type condition with coefficient $\alpha$.

**Theorem 4.2** (Berkolaiko–Kuchment [8] and [9 Thm 3.1.8]). Let $\Gamma_\alpha'$ be the graph obtained from the graph $\Gamma_\alpha$ by changing the coefficient of the condition at vertex $v$ from $\alpha$ to $\alpha'$. If $-\infty \leq \alpha < \alpha' < \infty$ (where $\alpha' = \infty$ corresponds to the Dirichlet condition), then

\[
\lambda_{n-1}(\Gamma_\alpha') \leq \lambda_n(\Gamma_\alpha) \leq \lambda_n(\Gamma_\alpha').
\]

If the eigenvalue $\lambda_n(\Gamma_\alpha)$ is simple and its eigenfunction $f$ is such that either $f(v)$ or $\sum f'(v)$ is non-zero, then the inequalities can be made strict,

\[
\lambda_{n-1}(\Gamma_\alpha') < \lambda_n(\Gamma_\alpha) < \lambda_n(\Gamma_\alpha').
\]

If $\alpha' < \alpha$, the inequalities are adjusted accordingly,

\[
\lambda_n(\Gamma_\alpha') < \lambda_n(\Gamma_\alpha) < \lambda_{n+1}(\Gamma_\alpha').
\]

Another modification we will use is splitting a vertex into two. A vertex $v$ is replaced by two vertices, $v_1$ and $v_2$ which, among them, split the set of edges originally incident to the vertex $v$: $E_v$ is a disjoint union of $E_{v_1}$ and $E_{v_2}$. The $\delta$-type constant at the new vertices is chosen such that $\alpha_{v_1} + \alpha_{v_2} = \alpha_v$ (usually $\alpha_{v_1}$ will be taken to be 0). The key observation here is that if an eigenfunction $f$ satisfies the sum of derivatives condition (2.1) with respect to the subset $E_n \subset E_v$, it will automatically satisfy it with respect to $E_{v_2}$ and will therefore be an eigenfunction of the modified graph with the additional property that $f(v_1) = f(v_2)$. To arrive to a contradiction, we will need to show that the latter is unlikely to happen.

**Lemma 4.3.** Let $x$ be a point on the edge $e$. Then, in any neighborhood of $x$ there is a residual set of $y$ such that for all eigenfunctions $f_n$ either $f_n(y) \neq 0$ or $f_n \equiv 0$ on the edge $e$.

Similarly, given a sequence of values $\{\phi_n\}$, all of them non-zero, there is a residual set of $y$ such that for all normalized eigenfunctions $f_n$ with $\lambda_n > 0$ we have $f_n(y) \neq \phi_n$. 

\[
\lambda_{n-1}(\Gamma_\alpha') \leq \lambda_n(\Gamma_\alpha) \leq \lambda_n(\Gamma_\alpha').
\]
Figure 2. A "figure of 8" graph. The spectrum is simple if and only if the lengths $\ell_1$ and $\ell_2$ are rationally independent.

Proof. Fix a neighborhood of $x$. Any eigenfunction $f$ that is not identically zero on $e$ has only finitely many zeros in the neighborhood: otherwise there is an accumulation point for zeros at which $f = f' = 0$ and therefore $f \equiv 0$. The union of the zero points over all possible $n$ is a countable set; its complement is residual.

The second part of the lemma is proved analogously, only one need not worry about $f_n$ vanishing identically on the edge. \hfill \Box

There is one example of a graph for which our modifications fail. We consider this example separately.

Example 4.4. Consider the "figure of 8" graph with Neumann–Kirchhoff condition at the vertex of degree 4, shown in Fig. 2. The lengths of the two loops will be denoted by $\ell_1$ and $\ell_2$. From Example 3.3 we know that there are two classes of "loop eigenfunctions" with eigenvalues $\{(2\pi n/\ell_1)^2\}_{n \geq 1}$ and $\{(2\pi n/\ell_2)^2\}_{n \geq 1}$. One can also construct an eigenfunction by starting with an eigenfunction on a circle of length $\ell_1 + \ell_2$ and pinching the circle at the points of distance $\ell_1/2$ from a maximum (the eigenfunction takes equal values at these points) to create an eigenfunction of the figure of 8 graph. Therefore $\{(2\pi n/(\ell_1 + \ell_2))^2\}_{n \geq 0}$ are also eigenvalues. Symmetry considerations (there is a basis of eigenfunctions which are either even or odd with respect to the flip of a loop) show that the above choices exhaust the set of eigenfunctions.

Now, if the numbers $\ell_1$ and $\ell_2$ are rationally independent, the three sets above are mutually disjoint and the spectrum of the figure of 8 graph is simple. Conclusion (ii) of Theorem 3.6 also holds with the third class of eigenfunctions.

5. Proofs of the main results

5.1. Proof of Lemma 3.4. We begin by establishing the following auxiliary result.

Lemma 5.1. Let $\Gamma$ be an impure loop graph, i.e. a graph consisting of one loop with at least one vertex with coefficient $\alpha_v \neq 0$. Then, for a residual set of edge lengths, the eigenvalues of the graph are simple.

Proof. We prove the lemma by induction on the number of vertices with $\alpha_v \neq 0$.

Let $\Gamma_1$ be a loop with one vertex $v$ and $\ell$ be the length of the loop. Parametrize the loop with a coordinate $x$ such that $x = 0$ corresponds to $v$, $x > 0$ in the clockwise direction and $x < 0$ in the anticlockwise direction, with $\ell/2$ ad $-\ell/2$ corresponding to the same point. Since $\Gamma_1$ has reflection symmetry, every eigenfunction is either odd or even, see Fig. 3

If $f$ is odd, it satisfies $f(-x) = -f(x)$ for each point $x$. In particular, $f(0) = 0$ and, by continuity, $f(\ell/2) = f(-\ell/2) = -f(\ell/2) = 0$. Solving the equation $Hf = \lambda f$, we have
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Figure 3. A loop with one vertex and the structure of the odd (left) and even (right) eigenfunctions.

\[ f(x) = \sin(\sqrt{x}x) \]

(5.1) \[ \sqrt{x} = 2k\pi/\ell, \quad k \in \mathbb{N}. \]

If \( f \) is even, i.e. \( f(-x) = f(x) \) for every point \( x \). Since \( f'((\ell)/2) = -f'(-\ell)/2 \) by symmetry and \( f'((\ell)/2) - f'(-\ell)/2 = 0 \) by NK vertex condition at \( \ell/2 \), we have \( f'((\ell)/2) = 0 \). At \( x = 0 \),

\[ \sum_{e \in E_0} \frac{df}{dx_e}(0) = \alpha_0 f(0), \]

i.e. \( 2f'_+(0) = \alpha_0 f(0) \), where \( f'_+ \) denotes the one-sided derivative taken at 0 in the positive direction. Solving the equation \( Hf = \lambda f \), we have

(5.2) \[ 2\sqrt{x}\sin(\sqrt{x}\ell/2) = \alpha_0 \cos(\sqrt{x}\ell/2). \]

The roots of equation (5.2) cannot coincide with (5.1): the substitution of \( \sqrt{x} = 2k\pi/\ell \) into (5.2) results in \( 0 = \pm \alpha_0 \), which contradicts our assumptions. Hence we proved the base case for the induction.

Suppose the statement is true for any impure loop graph with \( n \) Robin-type vertex conditions. Consider \( \Gamma \), an impure loop graph with \( n + 1 \) nonzero vertex conditions. Pick any vertex \( v \) and change \( \alpha_v \) to 0; using inductive hypothesis, adjust the edge lengths to obtain a graph \( \Gamma' \) with simple spectrum. By Lemma 4.3 it is now possible to pick a point near the former position of the vertex \( v \), where none of the eigenfunctions are zero. Note that the eigenfunctions cannot vanish on an open subset of the graph, since the unique continuation holds for the impure loop graph. Now we change the vertex condition at the new \( v \) back to \( \alpha_v \) and use the strict inequalities in Theorem 4.2 to conclude that the spectrum is still simple.

We have established that there is an arbitrarily small perturbation of edge lengths which will make the spectrum simple. Therefore, the set of admissible lengths is dense and, by the argument in Remark 3.7, we deduce that the set of admissible lengths is residual. \( \square \)

Remark 5.2. The proof of Lemma 5.1 allows for a slightly stronger statement: the length modifications actually preserve the total length of the graph.

We are now ready to prove Lemma 3.4.

Proof of Lemma 3.4. Split the loop \( L \) at the attachment point \( v \) from \( \Gamma \). Assign NK vertex condition to the former attachment point on the loop, which we now call \( v_1 \), see Fig. 4, and keep other vertex conditions unchanged.
Apply Lemma 5.1 to $L$ so that $\sigma(L')$ is simple for the changed length loop $L'$. Furthermore, by Lemma 4.3, we can pick a point $v'_1$ arbitrarily close to the former attachment point so that each eigenfunction $f$ of $L'$ is nonzero at $v'_1$. Attach $L'$ back to $\Gamma \setminus L$ at $v'_1$. Then the new graph $\Gamma'$ satisfies the same vertex conditions everywhere as $\Gamma$.

If there exists an eigenfunction $g$ of $\Gamma'$ with $\text{supp } g = L'$, then necessarily $g(v_0) = 0$ and $g$ is an eigenfunction of the loop $L'$ (see equation (3.2)), which is a contradiction. \hfill \Box

Now, we are ready to prove the main theorem.

Proof of theorem 3.6. We will prove the result by an induction on the number of edges of the graph. We will also employ a sub-induction on the number of vertices with Robin-type conditions.

If $\Gamma$ consists of one edge which is not a loop, the statement holds by the classical Sturm-Liouville theory. The case of a loop with no non-trivial $\delta$-type vertices is specifically excluded by the assumptions of the Theorem. The case of a loop with one vertex $v$ with a non-zero $\delta$-type condition is covered by Lemma 5.1 (part (iib) of the Theorem is true automatically). A loop with more than one non-zero condition is already a graph with at least two edges.

The plan for the inductive step is as follows. First we establish part (i) for a graph $\Gamma$ if both parts of the Theorem hold for every graph with a smaller number of edges or with the same number of edges and a smaller number of vertices with Robin-type conditions. Then we will establish part (ii) assuming, in addition to the above, that the spectrum of $\Gamma$ is simple.

Consider $\Gamma$, a connected graph with $n$ edges, satisfying $\delta$-type conditions with coefficients $\alpha_v$ for each vertex $v$. For the proof of part (i) we consider three cases.

Part (i), case 1. $\Gamma$ has no loops or cycles, i.e. it is a tree.

Choose an edge $e$ leading to a leaf (a vertex of degree 1) of the tree and split it from the tree. The new one-edge graph we denote by $\Gamma_1$ while the rest of the tree is denoted by $\Gamma_2$. 

Figure 4. Modifications to graph $\Gamma$ for the proof of Lemma 3.4

Figure 5. Part (i), case 1: splitting away an edge from a tree.
The attachment point $v$ of that edge is split into two vertices, $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$, see Fig. 6. We assign NK condition to vertex $v_1$. The vertex $v_2$ inherits the $\delta$-type condition with the constant $\alpha_v$ (which may also be 0), while all other vertices keep their previous conditions.

Adjust the edge lengths of $\Gamma_1$ and $\Gamma_2$ so that

1. the graph $\Gamma_2$ satisfies (i) and (ii), and
2. $\sigma(\Gamma_1) \cap \sigma(\Gamma_2) = \{0\}$.

Note that $\sigma(\Gamma_1)$ is a set of strictly decreasing functions of the edge length. Once condition (1) has been satisfied (which is possible on a residual set by the inductive hypothesis) and the discrete set $\sigma(\Gamma_2)$ has been fixed, the intersection $\sigma(\Gamma_1) \cap \sigma(\Gamma_2)$ is non-empty on a countable set of lengths of $\Gamma_1$, and its complement is dense. Altogether, conditions (1) and (2) above are satisfied on a dense set of edge lengths of the graph $\Gamma$. We remark that the graph $\Gamma_2$ satisfies properties (i) and (ii) of the Theorem automatically.

Glue $v_1$ and $v_2$ back together and call the resulting vertex $\tilde{v}$. The new graph $\tilde{\Gamma}$ has the same vertex conditions as $\Gamma$. We claim that the spectrum of $\Gamma$ is simple. Assume the contrary, $\lambda \in \sigma(\tilde{\Gamma})$ is multiple with corresponding eigenfunctions $f_i$. Then we can find a non-zero linear combination $f = \sum a_i f_i$, which is still an eigenfunction of $\tilde{\Gamma}$, such that it is also an eigenfunction with respect to the graph $\Gamma_1$, i.e. $f'(\tilde{v}) = 0$ along the edge $e$. Since

$$\alpha_v f(\tilde{v}) = \sum_{e' \in E_{\tilde{v}}} \frac{df}{dx_{e'}}(\tilde{v}) = \sum_{e' \in E_{\tilde{v}} \setminus \{e\}} \frac{df}{dx_{e'}}(\tilde{v}),$$

the function $f$ is also an eigenfunction with respect to the graph $\Gamma_2$.

If $f$ is non-zero on both $\Gamma_1$ and $\Gamma_2$, condition (2) is violated. If $f$ is zero on one of them, it is zero on $\tilde{v}$ which violates condition (1). Therefore, the spectrum of $\tilde{\Gamma}$ is simple.

Part (i), case 2. $\Gamma$ contains at least one loop.

For each loop $L$ of $\Gamma$, we repeat the steps of the previous case, namely,

1. split $L$ away from the rest of $\Gamma$ at the attachment point $v$, see Fig. 6. For $\Gamma \setminus L$, keep $\alpha_v$ vertex condition for the attachment point $v_2$ and keep all other vertex conditions unchanged. Adjust the lengths of $\Gamma \setminus L$ so that $\Gamma \setminus L$ satisfies (i) and (ii),

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{Part (i), case 2: splitting away a loop.}
\end{figure}
(2) assign NK vertex condition to the attachment point $v_1$ on $L$; adjust $L$ so that

$$\sigma(L) \cap \sigma(\Gamma \setminus L) = \{0\},$$

and glue $L$ back to $\Gamma$.

For each loop the above condition (1) is satisfied on a residual set of lengths of the graph $\Gamma \setminus L$ and for each $\Gamma \setminus L$, condition (2) is satisfied on a dense set of lengths of $L$. Altogether they are satisfied on a dense set of lengths of $\Gamma$, but repeating the argument of Remark 3.7 we conclude that the set is actually residual.

For all loops $L$ simultaneously, the conditions are satisfied on an intersection of residual sets which is also residual.

Let us first consider $\lambda$ which is an eigenvalue of some loop $L$. By Example 3.3, $\lambda = (2\pi n/\ell)^2 \in \sigma(\Gamma)$, where $\ell$ is the length of the loop graph $L$. As explained in Example 3.3, the eigenfunction supported on the loop is unique.

We want to show that the above $\lambda$ is simple in the spectrum $\sigma(\Gamma)$. Assume the contrary, there is at least one eigenfunction $f$ which is not identically zero on $\Gamma \setminus L$. Transform $f$ by flipping the loop; this is still an eigenfunction which we will denote by $\tilde{f}$. The function $g = (f + \tilde{f})/2$ has the following properties: it is an eigenfunction of $\Gamma$, not identically zero on $\Gamma \setminus L$, and it is even with respect to flipping the loop. The latter implies that its derivative at the midpoint of the loop is zero

$$g'_L(\ell/2) = 0.$$

Parametrizing the loop as $[0, \ell]$, we know that $g$ on the loop takes the form $g_L = A \cos(2\pi n(x-\ell/2)/\ell)$ which by direct computation implies that

$$g'_L(0) = g'_L(\ell) = 0.$$

Therefore, the function $g$ satisfies the $\delta$-type conditions at the attachment point $v$ also with respect to the graph $\Gamma \setminus L$. Thus we get $\lambda \in \sigma(\Gamma \setminus L)$ in contradiction with condition (2) above.

Now, if $\lambda$ is not an eigenvalue of any loop of $\Gamma$, and is multiple, we can find $f = \sum_{i=1}^2 a_i f_i$ such that $f$ satisfies NK vertex condition with respect to some loop $L$. It must be identically zero on $L$ (otherwise $\lambda$ is an eigenvalue of $L$), therefore at the attachment point $f(v) = 0$. Since $f$ is an eigenfunction on the graph $\Gamma \setminus L$, which satisfies (ii), it must be supported on some other loop $L'$, resulting in a contradiction.

We remark that there is one case for which the graph $\Gamma \setminus L$ is not covered by the inductive hypothesis because it is a circle: the “figure of 8” graph considered explicitly in Example 4.4.

Part (i), case 3. $\Gamma$ contains at least one cycle and no loops.

Now we also employ the induction on the number of Robin-type (i.e. $\alpha_v \neq 0, \infty$) vertices. First consider the base case of no Robin-type vertices. For this case we can assume that $\lambda \neq 0$, as the case $\lambda = 0$ is already covered by Remark 4.1.

Pick a vertex $v$ on the cycle such that $\deg(v) \geq 3$. Split $\Gamma$ at $v$ so that $\deg(v_1) = 2$, $\deg(v_2) \geq 1$, and the graph is still connected (this is possible precisely because $v$ is on a cycle). For the new graph $\Gamma'$, assign NK vertex condition to $v_1$ and $v_2$, and keep other vertex conditions unchanged, see Fig. 7. The vertex $v_1$ is now trivial, hence $\Gamma'$ has effectively one edge less than $\Gamma$. We can thus use induction and adjust $\Gamma'$ to satisfy conditions (i) and (ii).

First, we show that we can always find a new vertex $v'_1$ near $v_1$ so that $f(v'_1) \neq f(v_2)$ for each non-constant eigenfunction $f$ of $\Gamma'$. Because our modification of $\Gamma$ may have created loops, we need to consider three possibilities:
(1) neither \( v_1 \) nor \( v_2 \) is on a loop. Then \( \Gamma' \) has no loops. Since \( \Gamma' \) satisfies (i) and (ii), \( f(v_2) \neq 0 \). By Lemma 4.3, there exists \( v'_1 \) so that \( f(v'_1) \neq f(v_2) \).

(2) only one of \( v_1 \) and \( v_2 \) is on a loop. After relabeling, we may assume that \( v_1 \) is on a loop we denote \( L \) and \( v_2 \) is not on a loop. Since \( \Gamma' \) satisfies (i) and (ii), for each eigenfunction, is not identically 0 in the neighborhood of \( v_1 \) and we can again apply Lemma 4.3 whether \( f(v_2) \) is zero or not.

(3) both of \( v_1 \) and \( v_2 \) are on loops, see Fig. 8. Then both \( v_1 \) and \( v_2 \) may be adjusted. Since the eigenfunction cannot vanish identically around both \( v_1 \) and \( v_2 \) at the same time, we can again use Lemma 4.3 to make adjustments until \( f(v'_1) \neq f(v'_2) \).

Now glue \( v'_1 \) and \( v_2 \) (or \( v'_2 \), if appropriate) together and call the vertex \( \tilde{v} \). Note that the new graph \( \tilde{\Gamma} \) has the same vertex conditions as \( \Gamma \). Assume that \( \lambda \in \sigma(\tilde{\Gamma}) \) is multiple with eigenfunctions \( f_i \). Similarly to before, we can find \( f = \sum a_i f_i \), still an eigenfunction of \( \tilde{\Gamma} \), such that it satisfies Neumann–Kirchhoff condition with respect to edges that were connected to \( v'_1 \),

\[ \sum_{e \in E_{v'_1}} \frac{df}{dx_e}(v'_1) = 0. \]

Then \( f \) also satisfies the NK vertex condition with respect to the edges connected to \( v_2 \). Therefore, \( f \) is an eigenfunction of \( \Gamma' \), which means \( f(v'_1) \neq f(v_2) \), contradicting the fact that \( f \) is continuous at \( \tilde{v} \) for \( \tilde{\Gamma} \). Hence \( \lambda \) is simple.

Finally we consider the case of graphs without cycles but with some Robin-type vertices. Choosing an arbitrary vertex \( v \) with \( \alpha_v \neq 0 \), we set \( \alpha_v = 0 \) and obtain a graph \( \Gamma' \) with
less Robin-type vertices for which, as we assumed, the Theorem holds. We adjust the edge lengths to obtain a graph \( \tilde{\Gamma}' \) both properties (i) and (ii), in particular we achieve that \( f(v) \neq 0 \) for every eigenfunction of \( f \) of the graph (using, if necessary, Lemma 4.3). Changing the parameter \( \alpha_v \) back to its original non-zero value we use Theorem 4.2 in its strict form (equation (4.2)) to conclude that the spectrum of the graph \( \tilde{\Gamma} \) is simple.

Proof of part (ii). We will show the statement on a single vertex basis, that is we fix a vertex \( v \) and show that each eigenfunction is either non-zero at \( v \) or is supported on a loop. This is achieved by small modifications of the graph and holds on a residual set of lengths. As a result it will hold at every \( v \) on an intersection of residual sets, which is also residual.

First, we assume that the vertex \( v \) has \( \alpha_v = 0 \). We may assume that the spectrum of \( \Gamma \) is simple. Also, after a series of modifications we may assume that for each edge \( e \) of the graph, \( \Gamma \setminus e \) satisfies (i) and (ii). If removing \( e \) disconnects the graph, we assume that each of the two components satisfies the assumptions. In the special case when removing an edge creates a new loop, we also ask that the loop states do not vanish at the point where the edge was attached (again achievable by a small movement of the attachment point). Each of the conditions can be fulfilled on a residual set.

Now let \( (\lambda, f) \) be the \( n \)-th eigenpair of \( \Gamma \) such that \( f \) is the first eigenfunction that vanishes at the chosen vertex \( v \), \( f(v) = 0 \).

Case 1: \( f|_e \equiv 0 \) for some edge \( e \) incident to \( v \).

Now \( f(u) = f(v) = 0 \) for the end points \( u, v \) of the edge \( e \) and \( f \) is also an eigenfunction of \( \Gamma \setminus e \). Since \( \Gamma \setminus e \) satisfies (i) and (ii), \( \text{supp} f = L \) for a loop \( L \) in \( \Gamma \setminus e \). If this loop is present in \( \Gamma \), we have nothing further to prove. If the loop is not present in \( \Gamma \), then either \( u \) or \( v \) lies on the loop and we get a contradiction with the conditions imposed on \( \Gamma \setminus e \) above.

Case 2: \( f|_e \neq 0 \) for each edge \( e \) incident to \( v \).

Parametrizing the edges incident to \( v \) so that \( x = 0 \) at \( v \), we have

\[
- \frac{d^2}{dx^2} f = \lambda f, \quad \begin{cases} 
\sum_{e \in E_v} \frac{df}{dx_e} (0) = 0, \\
f(0) = 0,
\end{cases}
\]

and, denoting \( k = \sqrt{|\lambda|} \), we get

\[
f_e(x) = \begin{cases} 
A_e \sin(kx), & \lambda > 0, \\
A_e x, & \lambda = 0, \\
A_e \sinh(kx), & \lambda < 0
\end{cases} \quad \text{and} \quad \sum_{e \in E_v} A_e = 0.
\]
Note that each coefficient $A_e \neq 0$ since $f|_e \neq 0$. At $v$, we will shorten the edges with $f'_e = A_e > 0$ and lengthen edges with $f'_e < 0$ in a way which will be controlled by a (small) parameter $\epsilon$. Namely, we ask that $f_e(\tilde{v}) = \epsilon$, where $\tilde{v}$ denotes the new position of the vertex $v$ and the function $f_e$ is kept as before, see Fig. 9.

In this way, $f$ is still continuous at $\tilde{v}$ and satisfies $\delta$-type vertex condition for some parameter $\alpha'$, which we compute as follows. The new position of the vertex $\tilde{v}$ on edge $e$ is determined by the equation $f_e(x_e) = \epsilon$, or

$$x_e = \begin{cases} 1/k \arcsin(\epsilon/A_e), & \lambda > 0, \\ \epsilon/A_e, & \lambda = 0, \\ 1/k \arcsinh(\epsilon/A_e), & \lambda < 0. \end{cases}$$

We now find the coefficient $\alpha'_v$ from the condition

$$\sum_{e \in E_v} \frac{df}{dx_e} (x_e) = \alpha'_v f(x_e) = \alpha'_v \epsilon,$$

leading to, by Taylor expansion in each of the three cases,

$$\alpha'_v = \frac{1}{\epsilon} \sum_{e \in E_v} f'_e(x_e) = O(\epsilon).$$

We now consider two families of graphs, continuously depending on the parameter $\epsilon$: $\Gamma'$ which has the modified edge lengths and the NK condition at the vertex $v$ and $\Gamma''$ which in addition to changed lengths has $\alpha'_v = \alpha'_v(\epsilon)$ condition computed above.

If the parameter $\epsilon$ is small enough, the eigenfunctions below $n$-th are still non-zero at $v$. Furthermore, the eigenvalues of three graphs $\Gamma$, $\Gamma'(\epsilon)$ and $\Gamma''(\epsilon)$ are still in correspondence. More precisely, for any $k$ and small enough $\epsilon$, the eigenvalue $\lambda_k(\Gamma_1)$ is closer to $\lambda_k(\Gamma_2)$ than to any other eigenvalue of $\Gamma_2$ for any two of the above three graphs.

We now claim that the $n$-th eigenfunction of $\Gamma'$ doesn’t vanish on $v$ for any $\epsilon > 0$, provided it is small enough. Indeed, if $f(v) = 0$, then it is also an eigenfunction of $\Gamma''(\epsilon)$ (condition (2.1) will be satisfied for any $\alpha_v$) and must have index $n$ due to the correspondence of eigenvalues, but we explicitly constructed the $n$-th eigenfunction of $\Gamma''$ above to have value $\epsilon$ on the vertex $v$. This completes the proof of case 2.

Finally, we have to consider the vertex $v$ with a Robin-type condition: $\alpha_v \neq 0$. In this case we make edge length adjustments to the graph $\Gamma_0$ obtained from $\Gamma$ by setting $\alpha_v = 0$. Once condition (ii) is satisfied for $\Gamma_0$ at $v$, it is also satisfied for $\Gamma$ at $v$. Indeed, if an eigenfunction of $\Gamma$ vanishes at $v$, it automatically becomes the eigenfunction of $\Gamma_0$, still vanishing at $v$, which is a contradiction.

6. An application: connectedness of the secular manifolds

In this section we will deal only with graphs with NK or Dirichlet vertex conditions. For such a graph $\Gamma$ it is possible to find the eigenvalues $\lambda = k^2$ as the solutions of the equation

$$F_\Gamma(k) := C \det \left( e^{-ikL/2} I - e^{ikL/2} S \right) = 0,$$

where all matrices have dimension $2E$ with $E$ being the number of edges of the graph; they should be thought as operating on vectors indexed by the directed edges of the graph (each edge corresponds to two directed edges). The matrix $I$ is the identity matrix, $L$ is the diagonal matrix populated with the edge lengths and $S$ is a unitary matrix with real entries.
of known form [16] (the precise form is irrelevant to our discussion). The constant \( C \) can be chosen so that \( F_\Gamma \) is real for real \( k \).

Each length appears in the matrix \( L \) twice: once for each direction of the edge. As a consequence, the diagonal matrix \( e^{ikL} \) has two entries \( e^{ik\ell_e} \) for each edge \( e \). Substituting \( k\ell_e \) with the torus variables \( \kappa_e \in [0, 2\pi) \), we get the function \( \Phi_\Gamma(\kappa_1, \ldots, \kappa_E) \) such that

\[
\Phi_\Gamma(k\ell_1, \ldots, k\ell_E) = F_\Gamma(k).
\]

The solutions \( \vec{\kappa} \) of \( \Phi_\Gamma(\vec{\kappa}) = 0 \) form an algebraic subvariety \( \Sigma_\Gamma \) of the torus \( \mathbb{T}^E \). We call \( \Sigma_\Gamma \) the secular manifold of the graph \( \Gamma \). The study of \( \Sigma_\Gamma \) as a tool of understanding eigenvalues of a quantum graphs was pioneered by Barra and Gaspard [5].

It has been conjectured by Colin de Verdière [11] that the secular manifold is irreducible if the graph has a symmetry which is preserved under any change of edge lengths. It can be shown (see [11] for a partial proof) that in this case, the graph is an interval, a circle, a mandarin [3] (also called “pumpkin graph” by other authors [15]) or has some loops. It is also conjectured that in this case the set of the non-smooth points has co-dimension 2 with respect to the manifold \( \Sigma_\Gamma \).

In this section we prove a related result for a family of quantum graphs. We start with some terminology from [11] (whose term for \( \Sigma_\Gamma \) is “determinant manifold”). A point of \( \Sigma_\Gamma \) is smooth if the differential of \( \Phi_\Gamma \) at this point is non-zero. A point \( \vec{\kappa} \neq 0 \) is smooth if and only if 1 is a non-degenerate eigenvalue of the graph \( \Gamma \) with edge lengths set to \( \ell_e = \kappa_e \) or, more generally, if \( \lambda = k^2 \) is an eigenvalue of \( \Gamma \) with lengths \( \ell_e \) such that \( \vec{\kappa} = k\ell_e \mod 2\pi \).

In what follows we will omit the “modulo 2\( \pi \)” from the description of points on the torus, to keep the notation compact.

**Theorem 6.1.** Let the graph \( \Gamma \) have no loops and have a vertex of degree one. Then the set of smooth points of \( \Sigma_\Gamma \) has two connected components.

**Example 6.2.** Consider a star graph with three edges \((v_0, v_1), (v_0, v_2)\) and \((v_0, v_3)\) (see Fig. 10) with NK condition at the central vertex \( v_0 \) and Dirichlet conditions at the leaves \( v_1, v_2, v_3 \). There are three edges and therefore three torus variables. It can be shown [5, 7] that the secular function of such a graph has the form

\[
\Phi_{\text{star,D}}(\vec{\kappa}) = \sum_{j=1}^{3} \sin(\kappa_j) \sin(\kappa_{j+1}) \cos(\kappa_{j+2}),
\]

where indices of \( \kappa \) are taken modulo 3. The secular manifold for the graph is shown in Figure [11]. There are four sheets visible but they match pairwise under the torus periodicity. The two sheets touch each other through the conical points of non-smoothness.
Figure 11. Secular manifold of the star graph shown over $[-\pi, \pi]^3$ (left). Of the four sheets visible, the first and third are parts of the same sheet by the torus periodicity; same for the sheets two and four. A detail of the plot over $[-\pi/2, \pi/2]^3$ is shown on the right.

Similar pictures result if we consider the star graph with Neumann conditions at the leaves, which results in the secular determinant

$$\Phi_{\text{star}, N}(\vec{\kappa}) = \sum_{j=1}^{3} \cos(\kappa_j) \cos(\kappa_{j+1}) \sin(\kappa_{j+2}) = \Phi_{\text{star}, D}(\vec{\kappa} - \pi/2).$$

**Proof of Theorem 6.1.** Choose the edge lengths in such a way that

1. the eigenvalue spectrum of $\Gamma$ is simple and eigenfunctions do not vanish at vertices,
2. the edge lengths are rationally independent.

Denote the vector of the edge lengths by $\vec{\ell}_0$.

Condition (2) implies that the flow $k \mapsto k\vec{\ell}_0$ is ergodic on the torus, and its intersections with the secular manifold are dense in it. The closure of the odd-numbered intersections (within the set of smooth points of $\Sigma_\Gamma$) forms one component and the even-numbered intersections, the other. We will prove that they are mutually disjoint and connected.

It is known (see, for example, [1] or [11]) that the gradient of $\Phi_\Gamma$ has either all non-negative components or all non-positive components. Since $k_n = \sqrt{\lambda_n}$ are simple roots of the real-valued function $F_\Gamma(k) = \Phi_\Gamma(k\vec{\ell}_0)$, the derivatives $F_\Gamma'(k_n)$ alternate in sign and therefore the gradient is non-negative on one of the components we defined and non-positive on the other. At a point of intersection of the two components, the gradient must vanish which contradicts the definition of the components.

Let now $\vec{\kappa}_1$ and $\vec{\kappa}_2$ be the two points on the same component (without loss of generality, take the component of even-numbered eigenvalues). Surround them by small open neighbourhoods $U_1$ and $U_2$, such that $U_j \cap \Sigma_\Gamma$ are connected and contain only smooth points of the same component. From definition of the components we can find two eigenvalues $\lambda_{2n_1} = k_1^2$ and $\lambda_{2n_2} = k_2^2$, $n_1 < n_2$, such that $k_j\vec{\ell}_0 \in U_j \cap \Sigma_\Gamma$, $j = 1, 2$. We now need to show that the points $k_1\vec{\ell}_0$ and $k_2\vec{\ell}_0$ can be connected by a path on $\Sigma_\Gamma$ which does not pass through any singular points.
Denote by \( v \) the vertex of degree 1 and by \( e_1 \) the edge leading to it. The vertex condition at \( v \) can be written as

\[
\cos(\theta/2) \frac{df}{dx_{e_1}}(v) = \sin(\theta/2) f(v),
\]

with \( \theta = \theta_v \) equal to 0 for Neumann and \( \pi \) for Dirichlet. Now we start with the eigenvalue \( \lambda = k_2^2 \) and continue it analytically by changing \( \theta \). According to [8, Thm 6.1] (see also [9, Thm 3.1.13]), the eigenvalue \( \lambda(\theta) \) is an analytic function unless there exists an eigenfunction of the graph \( \Gamma \) which satisfies both Dirichlet and Neumann conditions (and thus any other \( \delta \)-type conditions) at the vertex \( v \). Such an eigenfunction would have to be identically zero on the edge \( e_1 \), which we ruled out in condition (1) above. Therefore, \( \lambda(\theta) \) is analytic and passes through every eigenvalue of \( \Gamma \) at the points \( \theta = 2\pi n + \theta_v \), \( n \in \mathbb{Z}, n \geq n_0 \). In particular, for \( n = \tilde{n} := 2(n_1 - n_2) < 0 \) we will have \( \lambda(\theta) = k_2^2 \).

We will now map this \( \lambda \)-path, parametrized by \( \theta \) decreasing from \( \theta_v \) to \( 2\pi \tilde{n} + \theta_v \), to a path on the secular manifold \( \Sigma_\Gamma \). Starting with an eigenfunction \( f \) on \( \Gamma \) with the vertex condition at \( v \) specified by \( \theta \), we prolong the edge \( e_1 \) to have the length

\[
\ell(e_1)(\theta) := \ell_{0,e_1} + \frac{\theta_v - \theta}{2\sqrt{\lambda(\theta)}}.
\]

A direct calculation shows that the eigenfunction \( f \) continued as a sine wave with the same amplitude past the old location of the vertex \( v \) will satisfy condition (6.4) with \( \theta = \theta_v \) at the new location of \( v \).

Define the vector-function \( \vec{\ell}(\theta) \) by using (6.5) for the component corresponding to \( e_1 \) and keeping all other components equal to the corresponding components of \( \vec{\ell}_0 \). The above discussion shows that the point \( \vec{\kappa}(\theta) = \sqrt{\lambda(\theta)} \vec{\ell}(\theta) \) will remain on the secular manifold for all \( \theta \) between \( \theta_v \) and \( 2\pi \tilde{n} + \theta_v \) and will pass only through points of multiplicity 1, which are exactly the points of smoothness. This path connects the points \( k_1 \vec{\ell}_0 \) and \( k_2 \vec{\ell}_0 \) as required. The seeming mismatch of lengths at \( \theta = 2\pi \tilde{n} + \theta_v \) is due to the modular arithmetic on the torus; here we use the fact that \( \tilde{n} \) is even. \( \square \)

**Remark 6.3.** An identical theorem can be proved for a graph with a bridge, i.e. an edge whose removal disconnects the graph. The method of proof is the same with a point an the bridge being the location where the variable \( \delta \)-type condition is introduced and the resulting eigenfunction is related to the eigenfunction of the original graph with a longer edge. Whether the same result holds for graphs without such edges (such as the tetrahedron graph — the complete graph on 4 vertices) is still unknown.

**Example 6.4.** The following example shows that there is no direct link between reducibility of the secular manifold and its connectedness. For the mandarin graph with three edges (see Fig. 10), the secular determinant decomposes into a product of the secular determinants of Dirichlet and Neumann stars due to the reflection symmetry,

\[
\Phi_{\text{mandarin}}(\vec{\kappa}) = \Phi_{\text{star,D}}(\vec{\kappa}/2) \cdot \Phi_{\text{star,N}}(\vec{\kappa}/2),
\]

see equations (6.2) and (6.3). Note that while the conditions of Theorem 6.1 (or Remark 6.3) are not satisfied, the secular manifold still has two connected components, see Fig. 12.
Figure 12. Secular manifold of the mandarin graph shown over $[0, 2\pi]^3$ (left) and, to provide another perspective, over $[-\pi/2, 3\pi/2]^3$ (right).

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