Stable Reduced-Order Model for Index-3 Second-Order Systems †

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Abstract: A new technique for preserving the stability of a reduced-order model (ROM) for index-3 second-order systems (SOSs) in a limited frequency interval is discussed in this paper. This technique is implemented by making indefinite terms of algebraic Lyapunov equations definite, which can be used in applications such as signal reconstruction controller design and filter design. The index-3 form is first converted into index-0 form, and then the Lyapunov equations are solved to compute the limited frequency Gramians. The terms that are indefinite can be made definite by assigning them the nearest possible positive eigenvalues. Gramians are balanced to obtain Hankel singular values, which are used later to obtain the ROM using balanced truncation.

Keywords: dynamic systems; model reduction; frequency-limited Gramians; Hankel singular values

1. Introduction

In classical multibody dynamic systems, mathematical modeling is the key analysis and design parameter for the systems. The equation representing a classical multibody dynamic system with a second-order structure can be represented as:

\[ M\ddot{\rho}(t) + D\dot{\rho}(t) + K\rho(t) = B_2^T u(t) \]  
(1a)

\[ C_2^T \dot{\rho}(t) + C_1^T \rho(t) = y(t) \]  
(1b)

where \( M \in \mathbb{R}^{n \times n}, D \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^{n \times n}, B_2^T \in \mathbb{R}^{n \times m}, C_1^T, C_2 \in \mathbb{R}^{p \times n}, \rho(t) \in \mathbb{R}^{n}, u(t) \in \mathbb{R}^{m} \) and \( y(t) \in \mathbb{R}^{p}, n \) is the order of the system, \( m \) is the number of input(s) and \( p \) is the number of output(s) of the system. In a mechanical system \( M, D \) and \( K \) are the mass, damping and stiffness matrices, respectively. The system defined in (1) is called the second-order form system. While modeling systems such as (1), the system's order becomes very large, and it becomes difficult to simulate, analyze or design a controller for such system; moreover, large storage and processing costs are the main constraints in implementing such systems [1]. To process such system, a model order reduction (MOR) technique is advised, which reduces the complexity by developing an approximate model of the system that has an approximately similar response as the original system. The reduced system requires less storage, can easily be managed, and is easier to analyze. The reduced model must preserve certain original system properties such as stability, regularity, and passivity, etc. The error in reduction must be small, and the technique must be converging and efficient [1–3].

Different reduction techniques for large-scale systems, such as balanced truncation (BT) [4–8] and interpolatory methods through the iterative rational Krylov algorithm [9–11], are used. Each method has pros and cons. The later technique is computationally efficient;
however, it does not guarantee stability nor give a prior error bound. The former technique (BT) gives a global bound on error stability preservation of the reduced system. The only drawback of this technique is solving two Lyapunov equations for Gramian factors with computational efficiency.

MOR using BT is gaining wide acceptance these days. Certain applications, such as filter design, feedback controllers, etc., need a minimum reduction error only over a limited frequency intervals. MOR using BT for the second-order system in the limited interval frequency is presented in [12]. For limited time/frequency interval, ROMs are not guaranteed to be stable. Extending the discussion to second-order index-3 systems, a Gramian-based MOR technique for a second-order index-3 system using BT has been discussed in the literature [10]; however, in the case of a limited interval frequency, no such work exists in the literature to the best of the authors’ knowledge, which is discussed in this paper. Moreover, the stability of an ROM over a finite frequency interval has been preserved in our proposed work.

The second-order index-3 system is first transformed into index-0 generalized form, and then, using the Lyapunov equation, limited interval Gramians are calculated. The terms that are becoming indefinite in Lyapunov equations can be made definite, and then they are used in limited frequency applications such as controller design signal reconstruction, etc. This is achieved by assigning the indefinite terms in the Lyapunov equation the nearest positive eigenvalues. Hankel singular values are obtained by balancing the Gramians. The proposed method is useful for limited frequency interval applications of index-3 SOSs.

The paper is organized as follows. Section 1 presents the introduction, Section 2 presents the proposed technique, and Section 3 is based on the numerical results.

2. Proposed Technique
2.1. Continuous Time Index-3 Second-Order System

A linearized holonomically constrained equation of motion [2] with index-3 structure is represented as:

\[
\begin{align*}
M \ddot{\rho}(t) + D \dot{\rho}(t) + K \rho(t) + G^T \sigma'(t) &= B_2 u(t) \\
G \rho(t) &= 0 \\
y(t) &= C_2 \rho(t)
\end{align*}
\]  

(2a),(2b),(2c)

where \( \rho \in \mathbb{R}^{n_\rho} \) is a position vector with coordinates related to the degree of freedom of the individual masses, and \( \sigma \in \mathbb{R}^{n_\sigma} \) is a vector with \( n_\sigma < n_\rho \) unknown parameters, called a Lagrange multiplier. The matrices \( M, D, K \in \mathbb{R}^{n_\rho \times n_\rho} \) are mass, stiffness, and damping, respectively, and \( G \in \mathbb{R}^{n_\sigma \times n_\rho} \) is known as a constraint matrix. \( B_2 \in \mathbb{R}^{n_\rho \times m} \) is the input matrix related to the input vector \( u(t) \in \mathbb{R}^m \), and \( C_2 \in \mathbb{R}^{n \times m} \) is the output matrix related to output vector \( y(t) \in \mathbb{R}^n \), \( n \) is the order of the system, \( m \) is the number of input(s) and \( p \) is the number of output(s) of the system.

Before applying BT to the system in (2), first, it is converted into generalized form. By applying mathematical operations, the index-3 second-order system is converted into an index-0 second-order system, which is given as:

\[
\begin{align*}
\Theta M \Theta^T \ddot{\rho}(t) + \Theta D \Theta^T \dot{\rho}(t) + \Theta K \Theta^T \rho(t) &= \Theta B_2 u(t) \\
y(t) &= C_2 \Theta^T \rho(t)
\end{align*}
\]  

(3a),(3b)

The next step index-0 second-order system is converted into a generalized first-order form:

\[
\begin{align*}
\Phi E_1 \Phi^T \dot{x}(t) &= \Phi A_1 \Phi^T x(t) + \Phi B_1 u(t) \\
y(t) &= C_1 \Phi^T x(t)
\end{align*}
\]  

(4a),(4b)

where
\[ \Phi = \begin{bmatrix} 0 & 0 \\ 0 & \Theta \end{bmatrix}, \quad E_1 = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & M \\ -K & -D \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & C_2 \end{bmatrix} \text{ and } x(t) = \begin{bmatrix} \rho(t) \\ \dot{\rho}(t) \end{bmatrix} \]

The equations in (4) can also be written as:

\[ E \dot{x}(t) = A \, x(t) + Bu(t) \quad (5a) \]
\[ y(t) = C \, x(t) \quad (5b) \]

where

\[ E = \Phi E_1 \Phi^T, \quad A = \Phi A_1 \Phi^T, \quad B = \Phi B_1 u(t) \quad C = C_1 \Phi^T \]

Further, we define frequency-limited Gramians for the obtained first-order system as:

\[ G_{c,\delta} = \frac{1}{2\pi} \int_{\delta} (i\omega E - A)^{-1} B B^T (i\omega E - A)^{-T} d\omega \quad (6) \]
\[ G_{o,\delta} = \frac{1}{2\pi} \int_{\delta} (i\omega E - A)^{-T} C^T (i\omega E - A)^{-1} d\omega \quad (7) \]

where \( \delta = [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2] \) is chosen such that the limited frequency interval Gramians are real, symmetric, and positive definite.

These Gramians are solutions to continuous time algebraic generalized Lyapunov equations:

\[ E G_{c,\delta} A^T + A G_{c,\delta} E^T = -E H B B^T - B B^T H^T E^T \quad (8) \]
\[ E G_{o,\delta} A^T + A G_{o,\delta} E^T = -E^T H^T C^T C - C^T C H E \quad (9) \]

where \( E, A, B \) and \( C \) are of second-order structure.

2.2. Balanced Truncation for Frequency Limited Second-Order System

Controllability and observability of Gramians factors are obtained by solving the Lyapunov equations. The controllability and observability of the Gramians factors are:

\[ G_{c,\delta} \approx \bar{R}_{\delta} \bar{R}_{\delta}^T \quad G_{o,\delta} \approx L_{\delta} L_{\delta}^T \]

Applying a singular value decomposition, we obtain:

\[ L_{\delta}^T \Phi E \Phi \bar{R}_{\delta} = \begin{bmatrix} U_{\delta 1} & U_{\delta 2} \end{bmatrix} \begin{bmatrix} \Sigma_{\delta 1} & 0 \\ 0 & \Sigma_{\delta 2} \end{bmatrix} \begin{bmatrix} V_{\delta 1}^T \\ V_{\delta 2}^T \end{bmatrix} \]

Right and left balancing and truncation transformation can be obtained as:

\[ \mathbb{R}_{\delta} = \bar{R}_{\delta} \quad V_{\delta 1} \Sigma_{\delta 1} \left( \frac{1}{2} \right) \quad U_{\delta} = \bar{L}_{\delta} \quad U_{\delta 1} \Sigma_{\delta 1} \left( \frac{1}{2} \right) \]

By applying the left and right transformation to the system in (15), we obtain:

\[ E_{r,\delta} = L_{\delta}^T E \mathbb{R}_{\delta}, \quad A_{r,\delta} = L_{\delta}^T A \mathbb{R}_{\delta}, \quad B_{r,\delta} = L_{\delta}^T B, \quad C_{r,\delta} = C \mathbb{R}_{\delta} \quad (10) \]

Equation (10) provides the reduced-order model in limited frequency intervals of the second-order index-3 system defined in (2).

3. Results and Discussions

The method described above is tested on a stable second-order index-3 system in (Appendix A) to check the efficiency and accuracy of the proposed scheme. The system has been reduced for different reduction orders. Consider a system with \( n = 12, m = 1 \) and \( p = 1 \). The system is reduced to the order of \( r = 4 \) and 6, in interval \( \delta = [0.1, 10] \) rad/sec using the method described above. The bode plot for the original system, the reduced system for unlimited frequency interval and reduced system for limited frequency interval
are shown in Figures 1 and 2 for the fourth and sixth orders of reduction, respectively. Regarding stability preservation in ROM, eigenvalues are shown for the limited frequency interval $\delta = [0.1, 10]$ rad/sec in Table 1 for the fourth and sixth orders. These eigenvalues show that the ROM is stable in the frequency interval $\delta = [0.1, 10]$ rad/sec for the fourth and sixth orders.

| Reduction Order | Eigen Values |
|-----------------|--------------|
| 4th | Indefinite/unstable $8.1052$ $-1.8231$ $-0.8951$ $-0.2564$ |
| | Definite/Stable $14.6433 + 0.0000i$ $-1.0199 + 0.6451i$ $-1.0199 + 0.6451i$ $-0.1161 + 0.0000i$ |
| 6th | Indefinite/unstable $14.1601 + 0.0000i$ $-1.0383 + 0.6254i$ $-1.0383 + 0.6254i$ $-0.0745 + 0.1726i$ $-0.0745 + 0.1726i$ $-0.1325 + 0.0000i$ |
| | Definite/Stable $13.6459 + 0.0000i$ $-1.0232 + 0.6456i$ $-1.0232 + 0.6456i$ $-0.0352 + 0.1794i$ $-0.0352 + 0.1794i$ $-0.0990 + 0.0000i$ |

4. Conclusions

In this research paper, stability preservation of ROM for a continuous second-order index-3 system in a limited band of frequency is proposed. Linearized systems with holonomic constraints exist in mechanical, multibody, and other fields of science. The first step is the conversion of an index-3 system to an index-0 system, and then the limited frequency interval Gramians are computed by solving Lyapunov equations. Stability is preserved by
allocating the indefinite terms in Lyapunov equations to definite by using nearest positive eigenvalues. The controllability and observability Gramians are balanced to obtain Hankel singular values based on which truncation is applied to obtain stable ROMs. The proposed technique is extensively useful for limited frequency interval applications such as building block models, the International Space Station model, CD ROM models, heat disc models, etc. The proposed technique is non-structure-preserving and can be further extended to structure-preserving in the future. The proposed work could also be further extended to time-limited applications and combined time and frequency applications.

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**Appendix A**

**Example**

\[
G = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \]

\[
M = \begin{bmatrix} 7.3 & 5.1 & 3.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5.1 & 7.3 & 5.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3.3 & 5.1 & 7.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.3 & 5.1 & 3.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.1 & 7.3 & 5.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.3 & 5.1 & 7.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7.3 & 5.1 & 3.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5.1 & 7.3 & 5.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.3 & 5.1 & 7.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7.3 & 5.1 & 3.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.1 & 7.3 & 5.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.3 & 5.1 & 7.3 \end{bmatrix}

\[
C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \]

\[
B_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T
\]

\[
D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \end{bmatrix}
\]

\[
K = \begin{bmatrix} 2.4 & 0.5 & -0.03 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 2.4 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.03 & -0.5 & 2.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.4 & -0.5 & -0.03 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.4 & -0.5 & -0.03 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.4 & -0.5 & -0.03 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.4 & -0.5 & -0.03 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.4 & -0.5 & -0.03 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.4 & -0.5 & -0.03 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.4 & -0.5 & -0.03 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.4 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

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